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## 





#### Abstract

The Method of Auxiliary Sources (MAS) is an approximate method for the solution of scattering problems. In the case of interest in the present thesis, that of scattering by an acoustically soft sphere, excited externally, one assumes MxN fictitious sources of acoustic field (to be referred to here as 'MAS currents') located on an auxiliary spherical surface inside the sphere-scatterer, for finite $\mathrm{M}, \mathrm{N}$. The 'MAS currents' are such that the boundary condition of the vanishing acoustic field is satisfied on MxN collocation points on the soft scatterer. A [(P-1) Q]x[(M-1) N] system of linear algebraic equations thus results. Once the MAS currents are found and calculated, the acoustic field ('MAS field') due to them can be easily determined.

What is shown in the numerical investigations of this thesis (by means of MATLAB) is that, in the case of 3-D problems, for the placement of the mentioned auxiliary sources in a certain area (which is found in this thesis and determined by a 'critical' radius), the auxiliary currents may oscillate, but we obtain a convergent field despite these oscillations; furthermore the oscillations are neither due to round-off nor matrix ill-conditioning. It is also demonstrated that, as M and N go to infinity, it is possible to have a 'MAS field' convergent to the true, correct field (for all points outside the sphere) together with divergent 'MAS currents'.

The thesis describes therefore a difficulty (namely oscillations) associated with the implementing of 'MAS'. The main advantages of illustrating a difficulty via a simple problem are two: (1) if the difficulty occurs in a simple problem, it is also likely to occur in more complicated problems and (2) it is less likely to confuse the said difficulty with other difficulties (namely, effects due to round-off, matrix ill-conditioning or shape elongation).


## Keywords

Method of Auxiliary Sources, Convergence of numerical methods, acoustic scattering problems, boundary value problems, solvability.

## Euxapıoties







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# Convergence and Oscillations in the Method of Auxiliary Sources Applied to 3-D Acoustic Scattering Problems (Diploma Thesis) 

Author<br>Gerasimos Palaiopanos<br>Assessment Committee<br>G. Fikioris (Associate Professor, NTUA, supervisor),<br>N. Tsitsas (Assistant Professor, AUTH, co-supervisor)<br>I. Tsalamengas (Professor, NTUA)

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## Chapter 1

## Introduction to Acoustic Wave Scattering theory

In this chapter we follow the textbook of [3], which unifies the theories of acoustic, electromagnetic and elastic waves and discusses the many physical and geometric aspects of their interactions with obstacles (one of which we will study in our problem later in the next chapters). According to this, we will make an introduction to scattering theory, giving the main differential equations, boundary conditions and physical interpretations of problems about scattering within acoustics.

This book also contains discussions on low frequency scattering in particular as well as a significant number of results previously unpublished. We will not present such details here, but the extended bibliography included in this textbook can be used as reference for any researcher.

### 1.1 DEFINITION OF THE ACOUSTIC WAVE

In this work, the quantity which we are interested in is the excess acoustic pressure field. An acoustic wave in an irrotational, homogeneous, isotropic, compressible fluid medium is characterized by a variation in the ambient pressure field. This variation is what is called "the excess acoustic pressure field" and is denoted by $\mathcal{U}(\mathbf{r}, t)$, a scalar function of position $\mathbf{r}=\left(x_{1}, x_{2}, x_{3}\right)$ and time $t$. In cartesian coordinates $\mathbf{r}=(x, y, z)$ and in spherical (which we are going to constantly use in the following chapters) $\mathbf{r}=(r, \theta, \phi)$. The excess pressure is measured in units of force per unit area.

The velocity of propagation of an acoustic wave is denoted by $\mathbf{V}(\mathbf{r}, t)$, a vector valued function of position and time.

### 1.2 THE ESTABLISHMENT OF THE WAVE EQUATION FOR ACOUSTIC WAVES

Supposing that the propagation of sound waves we are studying is linearized, the basic equations which relate the above defined quantities $\mathcal{U}$ and $\mathbf{V}$ are:

$$
\begin{gather*}
\dot{\mathcal{U}}=-\frac{1}{\gamma} \nabla \mathbf{V}  \tag{1.1}\\
\rho \dot{\mathbf{V}}=-\nabla \mathcal{U}+\delta \nabla \nabla \cdot \mathbf{V} \tag{1.2}
\end{gather*}
$$

where a dot over a letter indicates differentiation with respect to $t$.
Equation of continuity and the equation of state are merged to produce equation (1.1): it demonstrates the physical principle that the material flow out of any volume element will reduce the pressure, and the divergence of the velocity field is proportional to the rate of the pressure change (for small compression). The proportionality constant $\frac{1}{\gamma}$ is called the compressibility modulus, while $\gamma$ is called the mean compressibility and expresses relative volume reduction per unit increase in surface pressure. The parameter $\gamma$ is measured in units of inverse pressure.

Equation (1.2) is the irrotational part of the linearized Navier-Stokes equation of a fluid.

- $\rho$ : this positive real constant $\rho$ is the mass density
- $\delta$ : the non-negative real constant $\delta$ is the compressional viscosity which describes the rate of change of mass per unit length and respresents the losses, the conversion of mechanical energy to heat.
$-\delta=0$ : for lossless media while
$-\delta>0$ : characterizes lossy media
Introducing (1.1) into (1.2) leads to the equation:

$$
\begin{equation*}
\rho \dot{\mathbf{V}}=-\nabla \mathcal{U}-\gamma \delta \nabla \dot{\mathcal{U}} \tag{1.3}
\end{equation*}
$$

which, when substituted into the time derivative of (1.1), yields the equation that governs sound wave propagation:

$$
\begin{equation*}
\ddot{\mathcal{U}}=\frac{1}{\gamma \rho} \nabla^{2} \mathcal{U}+\frac{\delta}{\rho} \nabla^{2} \dot{\mathcal{U}} \tag{1.4}
\end{equation*}
$$

In the case of lossless media, we obtain the classical wave equation:

$$
\begin{equation*}
\ddot{\mathcal{U}}=\frac{\delta}{\rho} \nabla^{2} \mathcal{U} \tag{1.5}
\end{equation*}
$$

Besides, the time derivative of (1.2) by means of (1.1) leads to

$$
\begin{equation*}
\ddot{\mathbf{V}}=\frac{1}{\gamma \rho} \nabla(\nabla \cdot \mathbf{V})+\frac{\delta}{\rho} \nabla(\nabla \cdot \dot{\mathbf{V}}) \tag{1.6}
\end{equation*}
$$

Due to our irrotational medium:

$$
\begin{equation*}
\nabla \times \mathbf{V}=\mathbf{0} \tag{1.7}
\end{equation*}
$$

so we can assume the existence of a scalar velocity potential $\Phi(\mathbf{r}, t)$ (time dependent function) such that

$$
\begin{equation*}
\mathbf{V}=\nabla \Phi \tag{1.8}
\end{equation*}
$$

also:

$$
\begin{equation*}
\nabla(\nabla \cdot \mathbf{V})=\nabla^{2} \mathbf{V} \tag{1.9}
\end{equation*}
$$

The physical meaning of the velocity potential $\Phi$ is the rate of change of the area. Introducing (1.8) into (1.6) yields

$$
\begin{equation*}
\ddot{\mathbf{V}}=\frac{1}{\gamma \rho} \nabla^{2} \mathbf{V}+\frac{\delta}{\rho} \nabla^{2} \dot{\mathbf{V}} \tag{1.10}
\end{equation*}
$$

With the use of (1.8), 1.10) is trasformed to

$$
\begin{equation*}
\nabla \ddot{\Phi}=\nabla\left(\frac{1}{\gamma \rho} \nabla^{2} \phi+\frac{\delta}{\rho} \nabla^{2} \dot{\Phi}\right) \tag{1.11}
\end{equation*}
$$

Any function independent of position $\mathbf{r}$ may be added to the velocity potential, as it is a function not uniquely defined. Under the assumption that the appropriate function is added so that not only is (1.11) satisfied but also that

$$
\begin{equation*}
\ddot{\Phi}=\frac{1}{\gamma \rho} \nabla^{2} \phi+\frac{\delta}{\rho} \nabla^{2} \dot{\Phi} \tag{1.12}
\end{equation*}
$$

Therefore one can observe the obvious similarity in the equation fulfilled by the excess pressure, velocity and velocity potential (nearly the same equation: (1.4), (1.10) and (1.12)). Substituting (1.1) equation to (1.8) we obtain the relation between velocity and excess pressure:

$$
\begin{equation*}
\dot{\mathcal{U}}=-\frac{1}{\gamma} \nabla^{2} \Phi \tag{1.13}
\end{equation*}
$$

Supposing that all field quantities have a harmonic time dependence with angular or circular frequency $\omega$ :

$$
\begin{align*}
\mathcal{U}(\mathbf{r}, t) & =u(\mathbf{r}) \cdot e^{-i \omega t}  \tag{1.14}\\
\mathbf{V}(\mathbf{r}, t) & =v(\mathbf{r}) \cdot e^{-i \omega t}  \tag{1.15}\\
\Phi(\mathbf{r}, t) & =\phi(\mathbf{r}) \cdot e^{-i \omega t} \tag{1.16}
\end{align*}
$$

where $\mathcal{U}, \mathbf{V}, \Phi$ take on complex values dependent of $\mathbf{r}$ and $\omega$ but not of t . The above definitions require some further clarification based on the fact that the physical quantities, pressure $\mathcal{U}$ and velocity $\mathbf{V}$, are real whereas in (1.14)-(1.16) are defined as complex valued functions. Throughout this thesis we follow the widely accepted convention of working with the complex pressure, velocity and velocity potential with the understanding that physical quantities will correspond to the real parts, e.g. $\operatorname{Re}\{\mathcal{U}\}=\operatorname{Re}\left\{u \cdot e^{-i \omega t}\right\}$.
Substituting (1.14) and (1.16) into (1.12) we find the governing field equations

$$
\left(\nabla^{2}+k^{2}\right)\left\{\begin{array}{l}
u  \tag{1.17}\\
\phi
\end{array}\right\}=0
$$

where

$$
\begin{equation*}
k^{2}=\frac{\omega^{2} \gamma \rho}{1-i \omega \gamma \delta} \tag{1.18}
\end{equation*}
$$

By (1.18) the parameter $k$ is defined as the wave number or the propagation constant, yet it is needed to specify the branch of the complex square root (respecting physical conventions).

What is more, this relation is called the dispersion or characteristic relation for the medium of propagation. For the case of a plane wave $u$ propagating in the direction $\hat{\mathbf{k}}$, the complex time-dependent field is expressed as

$$
\begin{align*}
\mathcal{U}(\mathbf{r}, t) & =e^{i \hat{k} \cdot \mathbf{r}-i \omega t} \\
& =\exp \left[i\left(\hat{\mathbf{k}} \cdot \mathbf{r}-\frac{\omega}{\operatorname{Re}\{k\}} t\right) \operatorname{Re}\{k\}-\hat{\mathbf{k}} \cdot \mathbf{r} \cdot \operatorname{Im}\{k\}\right] \tag{1.19}
\end{align*}
$$

For increasing $\mathbf{r}$ in the direction of propagation, we see from (1.19) that the appropriate branch of the square root of $(1.18)$ to be chosen is $\operatorname{Im}\{k\} \geq 0$ so that physically the wave decreases in intensity as it traverses through the lossy medium (in the opposite case it would grow as it transverses a medium). Thus the $\operatorname{Im}\{k\}$ is related to power measuring and particularly it demonstrates the rate at which energy is attenuated. Equation (1.19) is also used to define phase velocity as follows:

$$
\begin{equation*}
\hat{\mathbf{k}} \cdot \mathbf{r}-\frac{\omega}{\operatorname{Re}\{k\}} t=\text { const } \tag{1.20}
\end{equation*}
$$

The phase fronts are defined to be the surfaces of constant phase which are planes (therefore plane waves) having $\hat{\mathbf{k}}$ as unit normal to the phase front. If we differentiate 1.20 with respect to time, $\hat{\mathbf{k}} \cdot \frac{\mathrm{d}}{\mathrm{d} x} \mathbf{r}$, we receive the velocity with which the front moves in the direction of the normal:

$$
\begin{equation*}
\hat{\mathbf{k}} \cdot \frac{d \mathbf{r}}{d t}=\frac{\omega}{\operatorname{Re}\{k\}}=c \tag{1.21}
\end{equation*}
$$

where the quantity $c$ is called the phase velocity. By virtue of (1.18), the phase velocity may be rewritten as

$$
\begin{equation*}
c=\frac{1}{\sqrt{\gamma \rho}} \sqrt{\frac{2\left(1+\omega^{2} \delta^{2} \gamma^{2}\right)}{1+\sqrt{1+\omega^{2} \delta^{2} \gamma^{2}}}} \tag{1.22}
\end{equation*}
$$

which is simplified to:

$$
\begin{equation*}
c=\frac{1}{\sqrt{\gamma \rho}} \tag{1.23}
\end{equation*}
$$

for lossless media. The angular frequency $\omega$ is related to the temporal period $T$ by

$$
\begin{equation*}
\omega=\frac{2 \pi}{T} \tag{1.24}
\end{equation*}
$$

The spatial period or wavelength $\lambda$ is the distance the phase front travels in one time period and is acquired by:

$$
\begin{equation*}
\lambda=c T=\frac{2 \pi c}{\omega}=\frac{2 \pi}{\operatorname{Re}\{k\}} \tag{1.25}
\end{equation*}
$$

Generalizing, for time harmonic waves:

$$
\begin{equation*}
\mathcal{U}(\mathbf{r} ; t)=|\mathcal{U}(\mathbf{r} ; t)| e^{i \Theta(\mathbf{r})-i \omega t} \tag{1.26}
\end{equation*}
$$

The surfaces obtained by:

$$
\begin{equation*}
\Theta(\mathbf{r})-\omega t=\text { constant } \tag{1.27}
\end{equation*}
$$

are the phase fronts with unit normal vector the $\nabla \Theta /|\nabla \Theta|$. The phase velocity $c$, the velocity in the direction of the normal, is acquired by differentiating with respect to time

$$
\begin{equation*}
\frac{\mathrm{d} \Theta}{\mathrm{~d} t}=\nabla \Theta \cdot \frac{\mathrm{d} \mathbf{r}}{\mathrm{~d} t}-\omega=0 \tag{1.28}
\end{equation*}
$$

hence

$$
\begin{equation*}
c=\frac{\nabla \Theta}{|\nabla \Theta|} \cdot \frac{\mathrm{d} \mathbf{r}}{\mathrm{~d} t}=\frac{\omega}{|\nabla \Theta|} \tag{1.29}
\end{equation*}
$$

Suppressing the time factor in 1.13 we get

$$
\begin{equation*}
i \omega u=\frac{1}{\gamma} \nabla^{2} \phi \tag{1.30}
\end{equation*}
$$

which, combined with 1.17 and 1.18 yields the following equation for excess pressure and velocity potential

$$
\begin{equation*}
u=\frac{i \omega \rho}{1-i \omega \delta \gamma} \phi=\frac{i k^{2}}{\omega \gamma} \phi \tag{1.31}
\end{equation*}
$$

We can express the time harmonic velocity field, using the above relation, as

$$
\begin{equation*}
v=\nabla \phi=\frac{1-i \omega \delta \gamma}{i \omega \rho} \nabla u \tag{1.32}
\end{equation*}
$$

Having derived the field equations in a homogeneous isotropic medium, we now consider the case in which waves propagate in a region consisting of two such media. We assume that the field quantities in both media have the same harmonic time dependence. This is possible since the material parameters of both media are taken to be independent of time.
The scattering problem we will examine in the next chapters belongs to a class of problems which concern the manner in which a bounded obstacle, denoted by $V^{-}$with boundary $S=$ $\partial V^{-}$, perturbs an acoustic wave originating in $V^{+}$, the unbounded exterior of $\bar{V}^{-}=V^{-} \cup S$. The obstacle $V^{-}$is a nonempty bounded open set, not necessarily simply connected, with boundary $S$ sufficiently smooth so as to allow the applicability of the Gauss-Green theorems and the existence of boundary values of field quantities in the classical sense. When the scattering obstacle is

- impenetrable: the acoustic field exists only in $V^{+}$and the boundary conditions must be imposed on $S$
- penetrable: we consider it to be another homogeneous fluid characterized by different constitutive parametres $\rho, \delta$ and $\gamma$ as well as the derived expressions, $c$ and $k$.

Note: We will affix superscripts + or - to these parametres as well as to the field quantities to distinguish between those in $V^{+}$and those in $V^{-}$. One notable exception is the wave number of the exterior $V^{+}$which, because it appears so often, will remain the unsuperscripted $k$. Moreover, the medium in $V^{+}$is considered to be lossless, i.e.

$$
\begin{equation*}
\delta^{+}=0 \tag{1.33}
\end{equation*}
$$

hence we have in $V^{+}$

$$
\begin{align*}
k=k^{+} & =\omega \sqrt{\gamma^{+} \rho^{+}}>0  \tag{1.34}\\
u^{+} & =i \omega \rho^{+} \phi^{+}  \tag{1.35}\\
v^{+} & =\frac{1}{i \omega \rho^{+}} \nabla u^{+} \tag{1.36}
\end{align*}
$$

and the Helmholtz equation is satisfied by both velocity potential and excess pressure:

$$
\left(\nabla^{2}+k^{2}\right)\left\{\begin{array}{l}
u^{+}  \tag{1.37}\\
\phi^{+}
\end{array}\right\}=0
$$

The medium in $V^{-}$is, in general, not assumed to be lossless. Hence

$$
\begin{equation*}
\delta^{-} \geq 0 \tag{1.38}
\end{equation*}
$$

We introduce the dimensionless relative index of refraction $\eta$ to be the ratio of wave numbers in $V^{-}$and $V^{+}$

$$
\begin{equation*}
\eta=\frac{k^{-}}{k}=\frac{\gamma^{-} \rho^{-}}{\gamma^{+} \rho^{+}} \frac{1}{\sqrt{1-i \omega \delta^{-} \gamma^{-}}} \tag{1.39}
\end{equation*}
$$

where the choice of the branch of the square root is such that $\operatorname{Im}\{\eta\} \geq 0$ and $\operatorname{Im}\left\{k^{-}\right\} \geq 0$. For relatively small dissipation, 1.39 implies that

$$
\begin{equation*}
\eta=\sqrt{\frac{\gamma^{-} \rho^{-}}{\gamma^{+} \rho^{+}}}\left(1+\frac{i \omega \delta^{-} \gamma^{-}}{2}\right)+O\left(\left(\delta^{-}\right)^{2}\right), \delta^{-} \rightarrow 0^{+} \tag{1.40}
\end{equation*}
$$

while for lossless scattering $\left(\delta^{-}=0\right)$

$$
\begin{equation*}
\eta=\sqrt{\frac{\gamma^{-} \rho^{-}}{\gamma^{+} \rho^{+}}}=\frac{c^{+}}{c^{-}} \tag{1.41}
\end{equation*}
$$

Now we write the governing equations in $V^{-}$as

$$
\begin{align*}
& \left(\nabla^{2}+\eta^{2} k^{2}\right)\left\{\begin{array}{l}
u^{-} \\
\phi^{-}
\end{array}\right\}=0  \tag{1.42}\\
& u^{-}=\frac{i \omega \rho^{-}}{1-i \omega \delta^{-} \gamma^{-}} \phi^{-} \tag{1.43}
\end{align*}
$$

and

$$
\begin{equation*}
v^{-}=\frac{1-i \omega \delta^{-} \gamma^{-}}{i \omega \rho^{-}} \nabla u^{-} \tag{1.44}
\end{equation*}
$$

Of course, the case of lossless scatterers corresponds to $\delta^{-}=0$.
In summary the governing field equations are the wave equation in the time domain:

$$
\begin{equation*}
\ddot{\mathcal{U}}(\mathbf{r}, t)=\frac{1}{\gamma \rho} \nabla^{2} \mathcal{U}(\mathbf{r}, t)+\frac{\delta}{\rho} \dot{\mathcal{U}}(\mathbf{r}, t) \tag{1.45}
\end{equation*}
$$

and the Helmholtz equation in the frequency or spectral domain

$$
\begin{equation*}
\left(\nabla^{2}+\frac{\omega^{2} \gamma \rho}{1-\omega \delta \gamma}\right) u(\mathbf{r})=0 \tag{1.46}
\end{equation*}
$$

and the same equations hold for the velocity potential as well as each Cartesian component of the velocity. The relation between frequency, wave number and constitutive parametres is given by the dispersion relation which may also be obtained by substituting the time harmonic plane wave, $e^{i k \hat{\mathbf{k r}}-i \omega t}$ into equation (1.45). In the following study of this thesis we will mainly consider the excess pressure $u$ in the frequency domain although every relation involving $u$ may be straight interpreted into a relatioon involving $\phi$ through (1.31) or into a relation for $v$ through (1.32).

### 1.3 BOUNDARY CONDITIONS

The class of scattering problems we are concerned with involves the determination of how $V^{-}$perturbs some known incident wave. We denote this incident wave by $u^{i}$ and its precise nature is discussed below. The total excess pressure field that exists in $V^{+}$is denoted by $u^{+}$. When $V^{-}$consists of another fluid medium there will also be a field in $V^{-}$denoted by $u^{-}$. The conditions relating $u^{+}$and $u^{-}$on $S$ are called transmission conditions and they are discussed in the next section. When no field exists in $V^{-}$we say that the scatterer is impenetrable and a variety of boundary conditions on $S$ are used to model situations in which the effect of the incident field in not felt in $V^{-}$. We confine attention to three different boundary conditions:

## - The Dirichlet or soft or pressure release surface:

$$
\begin{equation*}
u^{+}(r)=0, \mathbf{r} \in S \tag{1.47}
\end{equation*}
$$

## - The Neumann or hard surface:

$$
\begin{equation*}
\frac{\partial}{\partial n} u^{+}(\mathbf{r})=0, \mathbf{r} \in S \tag{1.48}
\end{equation*}
$$

where $\frac{\partial}{\partial n}$ denotes the derivative in the direction of the normal on $S$ into $V^{+}$.

- The Robin or impedance surface:

$$
\begin{equation*}
\left(\frac{\partial}{\partial n}+i \frac{\omega \rho^{+}}{Z^{+}}\right) u^{+}(\mathbf{r})=0, \mathbf{r} \in S \tag{1.49}
\end{equation*}
$$

where $Z^{+}$is the acoustic impedance measured in units of pressure per unit velocity. Equation (1.49) may also be written:

$$
\begin{equation*}
\left(\frac{\partial}{\partial n}+i k \nu\right) u^{+}(\mathbf{r})=0, \mathbf{r} \in S \tag{1.50}
\end{equation*}
$$

where the dimensionless parameter $\nu$ is given by

$$
\begin{equation*}
\nu=\frac{1}{Z^{+}} \sqrt{\frac{\rho^{+}}{\gamma^{+}}} \tag{1.51}
\end{equation*}
$$

These boundary conditions have the following translation: The function $u^{+}$denotes the excess pressure field of the medium surrounding the scatterer.

- A soft surface offers no resistance to pressure; thus the scatterer yields in such a way as to maintain zero pressure on its boundary, whereas
- a hard surface admits no local displacements and therefore the normal component of the velocity field (being proportional to $\hat{\mathbf{n}} \cdot \nabla u_{+}$) should vanish.
- Finally, a surface with finite impedance has an intermediate behaviour between the soft and the hard surface. In fact, the impedance boundary condition can be written as

$$
\begin{equation*}
Z^{+} \frac{\partial}{\partial n}\left(\frac{u^{+}}{i \omega \rho^{+}}\right)=-u^{+} \tag{1.52}
\end{equation*}
$$

or, using

$$
\begin{equation*}
Z^{+} \hat{\mathbf{n}} \cdot \nabla \phi^{+}=-u^{+} \tag{1.53}
\end{equation*}
$$

or

$$
\begin{equation*}
Z^{+} v_{n}^{+}=-u^{+} \tag{1.54}
\end{equation*}
$$

where $v_{n}^{+}$stands for the normal component of the velocity field on $S$. Relation (1.54) indicates that the normal velocity is proportional to the decrease of pressure at any point on $S$, the proportionality constant being the coefficient of acoustic impedance. Hence, an impedance boundary condition describes a balance between the pressure and the normal velocity field. In other words the surface stress, caused by the normal velocity, compensates for the excess pressure.

In fact, the physical meaning of the acoustic impedance is:

$$
\begin{equation*}
Z=\frac{\text { excess pressure }}{\text { normal velocity }} \tag{1.55}
\end{equation*}
$$

and hence it is measured in units of pressure per unit velocity.
As $Z^{+} \rightarrow 0$ the normal velocity is incapable of producing any pressure on the surface, which is therefore a 'soft' surface. On the other hand, as $1 / Z^{+} \rightarrow 0$ the pressure produces no normal velocity, i.e. the surface undergoes no local displacements and therefore is a 'hard' surface.

### 1.4 TRANSMISSION CONDITIONS

When the disturbance or incident wave in $V^{+}$is transmitted in to $V^{-}$the scatterer is penetrable and the excess pressure $u^{-}$, the velocity potential $\phi^{-}$and the velocity are governed by relations (1.38)-(1.44). However, the two fluids meet at the boundary $S$ and the conditions relating the excess pressure in $V^{+}$and $V^{-}$at $S$, the transmission conditions, are

$$
\begin{align*}
u^{+}(\mathbf{r}) & =u^{-}(\mathbf{r}), \mathbf{r} \in S  \tag{1.56}\\
\frac{\partial}{\partial n} u^{+}(\mathbf{r}) & =\beta \frac{\partial}{\partial n} u^{-}(\mathbf{r}), \mathbf{r} \in S \tag{1.57}
\end{align*}
$$

where $\beta$ is complex for lossy and real for lossless scatterers. Condition (1.56) states that the excess pressure field is continuous across $S$. Condition (1.57) results from requiring continuity of the normal component of the velocity field across $S$. This may be seen from the expressions relating velocity and excess pressure (1.36) and (1.44), as follows

$$
\begin{align*}
\hat{\mathbf{n}} \cdot v^{+}=\frac{1}{i \omega \rho^{+}} \frac{\partial}{\partial n} u^{+} & =\hat{\mathbf{n}} \cdot v^{-}=\frac{1-i \omega \delta^{-} \gamma^{-}}{i \omega \rho^{-}} \frac{\partial}{\partial n} u^{-}  \tag{1.58}\\
\Leftrightarrow \frac{\partial}{\partial n} u^{+} & =\beta \frac{\partial}{\partial n} u^{-} \tag{1.59}
\end{align*}
$$

where

$$
\begin{equation*}
\beta=\frac{\rho^{+}}{\rho^{-}}\left(1-i \omega \delta^{-} \gamma^{-}\right) \tag{1.60}
\end{equation*}
$$

is a dimensionless constant.
Note that for lossless scatterers only, $\beta$ represents the ratio of the mass densities. Nevertheless, for both lossy and lossless scatterers embedded in a lossless medium the product of $\beta$ and $\eta^{2}$ is always real and it represents the ratio of mean compressibilities as follows

$$
\begin{align*}
\beta \eta^{2} & =\frac{\rho^{+}}{\rho^{-}}\left(1-i \omega \delta^{-} \gamma^{-}\right) \frac{\rho^{-} \gamma^{-}}{\rho^{+} \gamma^{+}\left(1-i \omega \delta^{-} \gamma^{-}\right)} \\
& =\frac{\gamma^{-}}{\gamma^{+}} \tag{1.61}
\end{align*}
$$

We remark that if the exterior medium were lossy, then this ratio would no longer be real and $\beta$ would also have to be redefined. However, we always consider $V^{+}$to be lossless.
If the transmission conditions (1.57) and (1.56) are written in terms of the velocity potential $\phi$, then they read as follows

$$
\begin{align*}
\beta \phi^{+}(\mathbf{r}) & =\phi^{-}(\mathbf{r}), \mathbf{r} \in S  \tag{1.62}\\
\frac{\partial}{\partial n} \phi^{+}(\mathbf{r}) & =\frac{\partial}{\partial n} \phi^{-}(\mathbf{r}), \mathbf{r} \in S \tag{1.63}
\end{align*}
$$

Consequently, the excess pressure is continuous across $S$ but its normal derivative is continuous.
Remark: The general transmission problem is a two-parametre problem involiving $\beta$ and $\eta$. For lossless scatterers, the special cases of equal densities, where $\beta=1$, or equal wave numbers, where $\eta=1$, or equal compressibilities, where $\beta \eta^{2}=1$ furnish one-parametre transmission problems. Of course, if both $\beta=\eta=1$, then the medium exhibits no discontinuity in its physical parametres and therefore no scattering occurs.
Observe that parametre $\beta$ occurs in the trasmission conditions (1.56) and (1.62). We now discuss the limiting cases when $\beta \rightarrow 0, \infty$ or $\rho^{-} \rightarrow \infty, 0$. Rewriting equation (1.43) which relates excess pressure and velocity potential in $V^{-}$with the help of (1.60) as

$$
\begin{equation*}
u^{-}=\frac{i \omega \rho^{+}}{\beta} \phi^{-} \tag{1.64}
\end{equation*}
$$

we observe that if we assume that $\phi^{-}$is bounded in $V^{-}$(no sources in $V^{-}$), then

$$
\begin{equation*}
\lim _{\beta \rightarrow \infty} u^{-}=0, \text { in } V^{-} \tag{1.65}
\end{equation*}
$$

On the other hand, under the assumption that $u^{-}$is bounded in $V^{-}$, we infer that:

$$
\begin{equation*}
\lim _{\beta \rightarrow 0} \phi^{-}=0, \quad \text { in } V^{-} \tag{1.66}
\end{equation*}
$$

In the limiting case as $\beta \rightarrow \infty\left(\right.$ or $\left.\rho^{-} \rightarrow 0\right),(\sqrt{1.64})$ implies that $u^{-} \rightarrow 0$ and the boundary conditions (1.56), (1.57), (1.62) and (1.63) become

$$
\begin{align*}
u^{+} & =0, \phi^{+}=0 \text { on } S  \tag{1.67}\\
\frac{\partial}{\partial n} u^{-} & =0, \frac{\partial}{\partial n} \phi^{+}=\frac{\partial}{\partial n} \phi^{-}=0 \text { on } S \tag{1.68}
\end{align*}
$$

where we have assumed that $\phi^{-}$and the normal derivative of $u^{+}$remain bounded on $S$ as $\beta \rightarrow \infty$. Therefore, the problem of determining $u^{+}$or $\phi^{+}$in $V^{+}$becomes one of solving an exterior Dirichlet problem for the total field $u^{+}$or $\phi^{+}$. While $u^{-}=0$ in $V^{-}$, the velocity potential $\phi^{-}$is nonzero. This interior velocity potential is the solution of the interior Helmholtz equation (1.42) with boundary condition

$$
\begin{equation*}
\frac{\partial}{\partial n} \phi^{-}=\frac{\partial}{\partial n} \phi^{+} \text {on } S \tag{1.69}
\end{equation*}
$$

### 1.5 RADIATION CONDITIONS

Scattering problems always involve an unbounded domain which has infinity as part of its boundary. Any condition on that particular part of the boundary has to be given in an asymptotic form, as $r \rightarrow \infty$, where r is the magnitude of the position vector, $r=|r|$. For the development of the acoustic scattering problem we use the following condition, due to Sommerfeld (1912):

$$
\begin{equation*}
\lim _{r \rightarrow \infty} r\left(\frac{\partial}{\partial r} u(\mathbf{r})-i k u(\mathbf{r})\right)=0, \hat{\mathbf{r}} \in S^{2} \tag{1.70}
\end{equation*}
$$

where $u$ stands for the scattered pressure field and the convergence is taken to be uniform over all directions $\hat{\mathbf{r}}=\mathbf{r} / r$. The set $S^{2}$ denotes the surface of the unit ball in $\mathbb{R}$. The Sommerfeld radiation condition specifies the appropriate geometric attenuation of the scattered field and imposes the outgoing character of the scattered wave. It provides necessary condition when formulating the scattering problem as a well-posed exterior boundary value problem. The velocity potential and all Cartesian components of the velocity field satisfy the same radiation condition as the excess pressure.

### 1.6 INCIDENT FIELDS AND THE FUNDAMENTAL SOLUTION

The incident field in a scattering problem is the field that would exist in $\mathbb{R}^{3}$ if there were no scatterer present. We consider incident fields which are plane waves or point sources or superpositions of plane waves and/or point sources. Let $A \subset V^{+}$be a bounded domain which contains all point sources. If there are no point sources, then $A$ is empty. All incident fields, $u^{i}$, that we consider are solutions of

$$
\begin{equation*}
\left(\nabla^{2}+k^{2}\right) u^{i}(\mathbf{r})=0, \quad \mathbf{r} \in \mathbb{R}^{3} \backslash A \tag{1.71}
\end{equation*}
$$

In particular a plane incident field will be a wave of the form

$$
\begin{equation*}
u^{i}(\mathbf{r})=e^{i k \hat{\mathbf{k}} \cdot \mathbf{r}}, \quad \mathbf{r} \in \mathbb{R}^{3} \tag{1.72}
\end{equation*}
$$

which propagates in the direction of $\hat{\mathbf{k}}$.
Plane waves satisfy the Helmholtz equation (1.37) at all points in $\mathbb{R}^{3}$ but they do not satisfy the radiation condition (1.70). On the other hand the field due to a point source at $\mathbf{r}_{0}$ will satisfy the radiation condition but will be a solution of the Helmholtz equation only in $\mathbb{R}^{3} \backslash \mathbf{r}_{0}$. Such a field will be a function of two points: $G\left(\mathbf{r}, \mathbf{r}_{0}\right)$ and is a fundamental solution of the Helmholtz equation. Explicitly:

$$
\begin{equation*}
G\left(\mathbf{r}, \mathbf{r}_{0}\right)=\frac{e^{i k\left|\mathbf{r}-\mathbf{r}_{0}\right|}}{i k\left|\mathbf{r}-\mathbf{r}_{0}\right|}=h\left(k\left|\mathbf{r}-\mathbf{r}_{0}\right|\right) \tag{1.73}
\end{equation*}
$$

where $h=h_{0}^{(1)}$ is the spherical Hankel function of the first kind and order zero. This function is a solution of the equation

$$
\begin{equation*}
\left(\nabla_{r}^{2}+k^{2}\right) G\left(\mathbf{r}, \mathbf{r}_{0}\right)=-\frac{4 \pi}{i k} \delta\left(\mathbf{r}-\mathbf{r}_{0}\right) \tag{1.74}
\end{equation*}
$$

where $\delta$ denotes the Dirac point measure. The reason for considering this particular form of $G$ is that is coincides with the spherical Hankel function $h_{0}^{(1)}$ but, even more importantly, because it fulfills the radiation condition and moreover defines a dimensionlesss fundamental solution. Note that the dimensions of the Dirac measure are considered to be inverse volumes so that its volume integral is dimensionless.
All the acoustic incident fields considered in this thesis are of the form (1.72) or (1.73) or a linear combination of such fields for different incident directions $\hat{\mathbf{k}}$ or source locations $\mathbf{r}_{0}$. Remark that

$$
\begin{align*}
G\left(\mathbf{r}, \mathbf{r}_{0}\right) & =\frac{e^{i k r_{0}-i k \hat{\mathbf{r}}_{0} \cdot \mathbf{r}}}{i k r_{0}}+O\left(\frac{1}{\mathbf{r}_{0}^{2}}\right) \\
& =h\left(k \mathbf{r}_{0}\right) e^{-i k \hat{\mathbf{r}}_{0} \cdot \mathbf{r}}+O\left(\frac{1}{\mathbf{r}_{0}^{2}}\right), \quad \mathbf{r} \rightarrow \infty \tag{1.75}
\end{align*}
$$

As

$$
\begin{equation*}
e^{-i k \hat{\mathbf{r}}_{0} \cdot \mathbf{r}}=\lim _{r_{0} \rightarrow \infty} i k r_{0} e^{-i k r_{0}} G\left(\mathbf{r}, \mathbf{r}_{0}\right) \tag{1.76}
\end{equation*}
$$

we may consider the plane wave propagating in the direction $-\hat{\mathbf{r}}_{0}$ as a modified point source at $\mathbf{r}_{0}$ as $r_{0} \rightarrow \infty$.

### 1.7 THE BASIC SCATTERING PROBLEMS

Here we enumerate the outcomes of the previous sections and define the basic mathematical problems encountered in acoustic scattering. The physical meaning of the parametres involved has been explained in preceding sections and is not repeated here. In all the following problems
we specify an incident field $u^{i}$ with an exterior wave number $k$ and a surface $S$ which bounds the scatterer, $V^{-}$. Additionally, we seek functions $u^{+}$and $u$, the total and scattered fields respectively, related by

$$
\begin{equation*}
u=u^{+}-u^{i} \text { in } V^{+} \tag{1.77}
\end{equation*}
$$

where the function $u$ fulfills the Helmholtz equation

$$
\begin{equation*}
\left(\nabla^{2}+k^{2}\right) u(\mathbf{r})=0, \text { in } V^{+} \tag{1.78}
\end{equation*}
$$

and the radiation condition

$$
\begin{equation*}
\lim _{r \rightarrow \infty} r\left(\frac{\partial}{\partial r} u(\mathbf{r})-i k u(\mathbf{r})\right)=0, \text { uniformly in } \hat{\mathbf{r}} \in S^{2} \tag{1.79}
\end{equation*}
$$

In addition, one of the following conditions must be satisfied:

## - The Dirichlet problem:

$$
\begin{equation*}
u^{+}(\mathbf{r})=0, \mathbf{r} \in S \tag{1.80}
\end{equation*}
$$

- The Neumann problem:

$$
\begin{equation*}
\frac{\partial}{\partial n} u^{+}(\mathbf{r})=0, \mathbf{r} \in S \tag{1.81}
\end{equation*}
$$

where $\frac{\partial}{\partial n}$ denotes the derivative in the direction of the normal on $S$ into $V^{+}$.

- The Robin problem: For a given $\nu$

$$
\begin{equation*}
\left(\frac{\partial}{\partial n}+i k \nu\right) u^{+}(\mathbf{r})=0, \mathbf{r} \in S \tag{1.82}
\end{equation*}
$$

- The transmission problem For given $\eta$ and $\beta$, in addition to $u^{+}$we need to find $u^{-}$ which fulfills the Helmholtz equation:

$$
\begin{equation*}
\left(\nabla^{2}+k^{2} \eta^{2}\right) u^{-}(\mathbf{r})=0, \quad \mathbf{r} \in V^{-} \tag{1.83}
\end{equation*}
$$

with boundary conditions:

$$
\begin{align*}
u^{+}(\mathbf{r}) & =u^{-}(\mathbf{r}), \mathbf{r} \in S  \tag{1.84}\\
\frac{\partial}{\partial n} u^{+}(\mathbf{r}) & =\beta \frac{\partial}{\partial n} u^{-}(\mathbf{r}), \mathbf{r} \in S \tag{1.85}
\end{align*}
$$

Here we have formulated the problems in terms of excess pressure. Equivalent mathematical formulations in terms of the velocity potential may be obtained easily using the expressions relating $u$ and $v$. All four of these problems are well-posed boundary valued problems for any $C^{2}$-surface $S$, i.e. each one possesses a unique and stable classical solution (Colton and Kress, 1983, Jones 1986).
By classical solutions we mean

1. $u^{+} \in C^{2}\left(V^{+}\right) \cap C^{0}\left(V^{+} \cup S\right)$ for the Dirichlet problem
2. $u^{+} \in C^{2}\left(V^{+}\right) \cap C^{1}\left(V^{+} \cup S\right)$ for the Neumann and Robin problems
3. $u^{+} \in C^{2}\left(V^{+}\right) \cap C^{1}\left(V^{+} \cup S\right), u^{-} \in C^{2}\left(V^{-}\right) \cap C^{1}\left(V^{-} \cup S\right)$ for the transmission problem

For surfaces with a finite number of corners and edges an additional condition of local finite energy must be imposed. This is written as an energy condition:

$$
\begin{equation*}
\int_{\Omega}\left(|\nabla u(\mathbf{r})|^{2}+|k u(\mathbf{r})|^{2}\right) d v(\mathbf{r})<+\infty \tag{1.86}
\end{equation*}
$$

where $\Omega$ is a bounded subdomain of $V^{+}$(Meixner 1949). Condition 1.86) replaces the assumption of differentiability up to the boundary and all the boundary conditions involving differentiation on the boundary can be applied only when these normal derivatives exist (i.e. almost everywhere). More about this can be found in Van Bladel (1995) and the references therein.

## Chapter 2

## The Method of Auxiliary Sources (MAS)

### 2.1 INTRODUCTION

According to [5] the Method of Auxiliary Sources (MAS) is an advanced and highly promising numerical technique for solving elliptic boundary-value problems. It constitutes a powerful alternative to the standard surface integral-equation formulation, and possesses significant advantages concerning numerical stability, computational accuracy, and ease of implementation. These features make it attractive for the numerical solution of problems occurring in electromagnetic (EM) scattering analysis, antenna modeling, waveguide structures, etc. MAS was introduced, named, and developed by several researchers in the Republic of Georgia. It is important to emphasize that the same method has been independently developed by other research groups, elsewhere in the world, under different names, such as the "Current Model Method" or the "Discrete Singularity Method". This has been mainly for treating EM-scattering problems. The common basic concept of all these methods is that the EM boundary-value problem is not formulated in terms of continuous equivalent surface currents flowing on the surfaces - where the corresponding boundary conditions are enforced - but in terms of discrete fictitious currents, the "auxiliary sources" (ASs), located at some distance away from the physical boundaries.

It is true that in typical integral-equation techniques, by applying the equivalence principle, the EM field inside a homogeneous, 48 isotropic, and linear regionidomain of the structure under investigation can be expressed in terms of a known impressed field (the excitation) and unknown equivalent electric and magnetic continuous currents distributed over its boundary surface. These are employed in order to model the field discontinuity across this boundary surface. Then, by expressing the corresponding boundary conditions in terms of the impressed field and the equivalent continuous currents, different types of surface integral equations are obtained, which, in the general case, are numerically solved via the Method of Moments (MOM). Unlike MOM, MAS does not account for currents that yield field discontinuities on the boundaries. Instead, it directly constructs the unknown EM fields in each domain with the assistance of fictitious, equivalent point sources, the auxiliary sources, displaced with respect to the boundaries. These auxiliary sources are chosen so that their fields are elementary analytical solutions to the boundary-value problem. The actual EM fields in each domain
are then expressed as weighted superpositions of these analytical solutions, and the unknown expansion coefficients are determined through point-wise invocation of the relevant boundary conditions. It should be noted that the concept of field approximation by means of a linear combination of analytically known field functions is not unique to the MAS approach, alone. The thin-wire approximation, where the current is modeled as a filament flowing along the axis of the wire, as well as the classical Mie solution for a sphere, with the unknowns being discrete multipole sources at the center of the sphere, have conceptual similarities to MAS.

These two methods are restricted to specific geometries. The innovation that enables application to general geometries is the use of multiple origins. The idea has been successfully employed for years in electrostatics. It is known as "the Charge Simulation Method", according to which fictitious, discrete, line charges are distributed at multiple origins outside the region where the electrostatic field is to be computed. The potentials of the fictitious charges are particular solutions of the Laplace and Poisson equations, and their magnitudes are determined by satisfying the boundary conditions at a discrete set of points on the boundary. Accordingly, in electrodynamics, various numerical methods -which are based on the "Extended Boundary Condition Method" (EBCM), and are often known as "Generalized Multipole Techniques" (GMTs) or "Multiple Multipole Techniques" (MMTs) - simulate EM fields by means of cylindrical- and spherical-wave multipole functions, up to some specified order, centered at multiple origins, for treating two-dimensional (2D) and three-dimensional (3D) problems, respectively. The theoretical background of the GMT was established by Kupradze and Vekua, and independently by Yasuura. Although having independently evolved, MAS could, in a sense, be considered as a special case of the GMT in which only poles of zero order are activated, forming a set of fictitious, but otherwise physically interpretable and analytically simpler, sources. However, it is sometimes considered preferable to over-determine the linear system of equations.


Figure 2.1: A PEC scatterer with a smooth surface $S$ illuminated by a known external field $\mathbf{E}^{\text {inc }}$ inside an infinite homogeneous and linear space with dielectric permittivity $\epsilon$ and magnetic permeability $\mu_{0}$.

### 2.2 FUNDAMENTALS

In what we discuss below, we will call as "standard" MAS the most widely used version of MAS. In this method, the radiating auxiliary sources are chosen to be either current filaments for two dimensional problems, generating fields proportional to a Hankel function (two-dimensional Green's function), or pairs of elementary dipoles for three-dimensional problems, generating fields proportional to the three-dimensional Green's function. The members of each pair are perpendicular to each other and, simultaneously, tangential to the auxiliary surface, to account for the magnetic-field discontinuity across the auxiliary surface. In a standard MAS formulation, the auxiliary sources are homogeneously distributed on auxiliary surfaces, conformal to the physical boundaries. The EM fields in each domain are expressed as weighted superpositions of the EM fields generated by all the auxiliary sources. These superpositions have unknown expansion coefficients, to be determined by point-matching the relevant boundary conditions at a discrete set of collocation points (CPs) on the physical boundaries. The distribution of the collocation points is, again, homogeneous, and their number is usually equal to the number of auxiliary sources.


Figure 2.2: MAS model equivalent to 2.1): the PEC scatterer does not exist. The auxiliary sources radiate inside an infinite homogeneous and linear space. They are located on an auxiliary surface $S^{\prime}$ enclosed by the fictitious physical surface $S$. The collocation points at which the boundary condition is satisfied are located on the fictitious surface $S$.

For a better understanding of the fundamentals of standard MAS, two generalized problems of EM scattering of an external known electric field, $\mathbf{E}^{\text {inc }}$, by a perfect electric conductor (PEC) and by a homogeneous isotropic dielectric scatterer are considered. The corresponding geometries are shown in Figures (2.1) and (2.2), respectively.

In the first problem, the PEC, with a smooth external surface $S$, is located inside an infinite homogeneous isotropic and linear space with dielectric permittivity $\epsilon$ and magnetic permeability $\mu_{0}$ (Figure (2.1)). The auxiliary sources radiate, in the absence of the PEC, inside an infinite homogeneous isotropic and linear space with dielectric permittivity $\epsilon$ and
magnetic permeability $\mu_{0}$. They are located on an auxiliary surface, $S^{\prime}$, enclosed by the (fictitious) physical surface, $S$ (Figure (2.2). Then, the unknown scattered field, $\mathbf{E}^{s}$, is described by

$$
\begin{equation*}
\mathbf{E}^{s}=\sum_{n} \mathbf{E}_{n}^{s}=\sum_{n} \mathbf{G}_{n} \cdot \mathbf{a}_{n} \tag{2.1}
\end{equation*}
$$

where $\mathbf{E}_{n}^{s}$ denotes the electric field of the $n$th auxiliary source, the $\mathbf{G}_{n}$ are the known analytic solutions of the corresponding wave equation, and the $\mathbf{a}_{n}$, are the unknown coefficients, to be determined. (Equation (2.1), as well as Equations (2.2) and (2.3), are given in the most general format of the three-dimensional problem, where the unknown coefficients are vectors and the known analytic solutions of the wave equation (Green's functions) are dyadics.) By imposing the satisfaction of the boundary condition (a vanishing total electric field tangential to $S$ ) at a discrete set of collocation points on the (fictitious) physical surface $S$, a system of linear equations is derived in terms of $\mathbf{a}_{n}$. The solution of this system yields the unknown coefficients and, consequently, the unknown scattered field, $E^{\prime}$. Existence and uniqueness issues of the MAS solution have been explicitly addressed in relevant literature [5].


Figure 2.3: A homogeneous isotropic and linear dielectric scatterer with dielectric permittivity $\epsilon$, magnetic permeability $\mu_{0}$ and a smooth surface $S$ illuminated by a known external field $\mathbf{E}^{i n c}$ in free space.

In the second problem, a homogeneous isotropic and linear dielectric scatterer, with dielectric permittivity $\epsilon$ and magnetic permeability $\mu_{0}$ and a smooth external surface $S$, is located in free space (Figure (2.3)). Now, two sets of auxiliary sources are required for the MAS formulation. One is a set of auxiliary sources radiating in free space in the absence of the dielectric scatterer, and located on an auxiliary surface, $S^{\prime}$, enclosed by the (fictitious) physical surface, $S$ (Figure (2.4)). The second is a set of auxiliary sources radiating again in the absence of the dielectric scatterer, but inside an infinite space filled by the material of the dielectric scatterer, and located on an auxiliary surface, $S "$, enclosing the (fictitious) physical surface, $S$ (Figure
(2.5). Then, the unknown scattered field, $\mathbf{E}^{s}$, in Region I is described as a sum of the fields of the first set of auxiliary sources,

$$
\begin{equation*}
\mathbf{E}^{s}=\sum_{n} \mathbf{E}_{n}^{s}=\sum_{n} \mathbf{G}_{n}^{I} \cdot \mathbf{a}_{n}^{I} \tag{2.2}
\end{equation*}
$$

where $\mathbf{E}^{s}$ denotes the electric field of the $n$th auxiliary source of the first set, with the $\mathbf{a}_{n}^{I}$ being unknown coefficients to be determined, and the $\mathbf{G}_{n}^{I}$ being known analytic solutions of the wave equation for $\mathbf{E}$. The unknown field, $\mathbf{E}^{I I}$, in Region II is described as a sum of the fields of the second set of auxiliary sources,

$$
\begin{equation*}
\mathbf{E}^{I I}=\sum_{n} \mathbf{E}_{n}^{I I}=\sum_{n} \mathbf{G}_{n}^{I I} \cdot \mathbf{a}_{n}^{I I} \tag{2.3}
\end{equation*}
$$

where $\mathbf{E}_{n}^{I I}$ denotes the electric field of the $n$th auxiliary source of the second set, with the $\mathbf{a}_{n}^{I I}$ being unknown coefficients to be determined, and the $\mathbf{G}_{n}^{I I}$ being known analytic solutions of the wave equation for $\mathbf{E}^{I I}$. By imposing the satisfaction of the boundary conditions for the total EM field tangential to $S$ at discrete collocation points on the (fictitious) physical surface $S$, a coupled system of linear equations is derived in terms of $\mathbf{a}_{n}^{I}$ and $\mathbf{a}_{n}^{I I}$. The solution to this system of equations yields the unknown coefficients and, consequently, the unknown fields, $\mathbf{E}^{s}$ and $\mathbf{a}^{I I}$.

It is worth highlighting that MAS can improve in terms of efficiency and accuracy based on its attribute of the non-vanishing distance between source and observation points (i.e., between auxiliary sources and collocation. points). This displacement with respect to the boundaries practically eliminates the Green's-function singularity problem of a typical MOM kernel, forming a set of smooth functions on the boundaries. Moreover, the implementation of the technique is conceptually very simple: by choosing a finite number of auxiliary sources and matching the boundary conditions at a discrete set of collocation points, a matrix equation is automatically derived instead of an integral equation, avoiding the necessity of any MOM transformations. Furthermore, since each solution in the set is analytically known, there is no need to integrate currents in order to determine fields at any stage of the solution (i.e., filling the kernel, checking the results, calculating near and far fields, etc).

Finally, in terms of complexity, it can be proven that MAS exhibits a very low computational cost in terms of CPU time, due to the associated extremely rapid matrix filling. Depending on several parameters (geometry, electrical size and shape, integration order, etc.), the complexity of MAS may be much lower than the complexity of MOM. If the radiator/scatterer is such that the MAS discretization required for convergence is less or equally dense as for MOM, then MAS is always much more efficient. Even if this is not the case however - i.e., when the MAS discretization should be denser - there exists, in general, a threshold in geometry size under which MAS is computationally less costly than MOM. A detailed operation count for a particular scattering problem, stating the conditions under which MAS is computationally less intense than MOM, is presented in (5].


Figure 2.4: The MAS model equivalent to the situation in (2.3) for describing the scattered field in Region I. The dielectric scatterer does not exist, the auxiliary sources radiate in free space, and they are located on an auxiliary surface $S^{\prime}$ enclosed by the (fictitious) physical surface $S$. The collocation points, at which the boundary conditions are satisfied, are located on (fictitious) $S$.


Figure 2.5: The MAS model equivalent to the situation in (2.3), for describing the field in Region 11. The dielectric scatterer does not exist, the auxiliary sources radiate inside an infinite homogeneous and linear space. They are located on an auxiliary surface $S^{\prime \prime}$, enclosing the (fictitious) physical surface $S$. The collocation points, at which the boundary conditions are satisfied, are located on the (fictitious) $S$.

## Chapter 3

## Two Types of Convergence in the Method of Auxiliary Sources: 2-D Electromagnetics

### 3.1 INTRODUCTION

Following the work of professor G. Fikioris in [4] we are introducing a study on the MAS on a two dimension scattering problem on electromagnetics. Specifically, we use the geometry of an infinitely long PEC cylinder illuminated by an infinite electric current filament.

The Method of Auxiliary Sources (MAS) is an approximate method for the solution of electromagnetic scattering problems. In the case of interest in the problem of this chapter, that of scattering by a closed, smooth, perfect electric conductor (PEC), illuminated externally, one assumes electric and/or magnetic currents (to be referred to here as "MAS currents") on $N$ fictitious sources located on an auxiliary surface inside the PEC (which is assumed to possess an interior region). The MAS currents are such that the boundary condition of vanishing tangential electric field on $N$ collocation points on the PEC scatterer is satisfied. A $N \times N$ system of linear algebraic equation thus results. Once the MAS currents are found, the field ("MAS field") due to them can be easily determined. The aforementioned method is often referred to by names other than MAS; our use of the term MAS is consistent with the recent article [5], which is an overview of MAS. There, the origins of MAS are discussed, as are variations of MAS and relations of MAS to other methods. Such discussions can also be found in the comprehensive works [3] and [4].

For a growing $N$, one hopes for convergence of the MAS field to the true field. Furthermore, when the auxiliary surface is smooth and closed (this is usually the case in the literature), it is natural to anticipate that - when properly normalized - the MAS currents should remain unchanged. As $N$ grows, that is, one hopes for a better approximation to the field with only small changes in the corresponding normalized MAS currents. In other words, one hopes that the normalized MAS currents converge to corresponding (i.e. electric or magnetic) surface current densities.

We demonstrate in this chapter that it is possible for the MAS field to converge to the true field without having the normalized MAS currents converge. We show this through a study of a simple two-dimensional problem, in which the scatterer is an infinitely long,

PEC circular cylinder illuminated by an infinitely long, constant-current line source. The electric MAS currents are equispaced and lie on a circle, and so do the $N$ collocation points. For this simple case, the matrix in the $N \times N$ system of algebraic equations is circulant and an analytical, explicit solution for the MAS currents and fields can be obtained. Our main conclusions then follow by examining the limit of currents and fields as $N \rightarrow \infty$. Very helpful to us is the "continuous version" of MAS, in which an unknown surface current density, located on the auxiliary surface, is determined from an integral equation arising from the exact satisfaction of the relevant boundary condition; like the aforementioned circulant system, for our simple problem the integral equation can be solved explicitly. The fact that twodimensional, cylindrical PEC cylinders and equispaced MAS currents and collocation points lead to explicit MAS solutions has been exploited in a number of recent works referenced in [4]. Circulant matrices and explicit solutions also arise in refs., which discuss the accuracy of moment-method solutions in the context of a simple scattering problem like ours. As we point out in the main body of this work, some of our intermediate results (including results about the "continuous" version of MAS) are consistent with, or can be viewed as consequences of, theorems on the behavior of MAS solutions for general geometries (referenced in 4]. But we do not rely on such theorems. Rather, we derive all our results from first principles, using relatively simple mathematics and (apart from a brief reference to the notion of non-radiating currents) only fundamental concepts from electromagnetic theory. This approach enables us to shed light on various aspects of our problem, including certain similarities/differences between MAS and its continuous version.

### 3.2 ADDITION THEOREM AND ASYMPTOTIC APPROXIMATIONS

We will make use of the following identities, large- $n$ approximations of some functions and formulas:

$$
\begin{gather*}
H_{0}^{(1)}\left(\sqrt{x_{1}^{2}+x_{2}^{2}-2 x_{1} x_{2} \cos \theta}\right)=\sum_{n=-\infty}^{\infty} J_{n}\left(\min \left\{x_{1}, x_{2}\right\}\right) H_{n}^{(1)}\left(\max \left\{x_{1}, x_{2}\right\}\right) e^{i n \theta}  \tag{3.1}\\
J_{n}(x)=(-1)^{n} J_{-n}(x) \sim \frac{1}{\sqrt{2 \pi n}}\left(\frac{e x}{2 n}\right)^{n}, n \rightarrow \infty  \tag{3.2}\\
H_{n}^{(1)}(x)=(-1)^{n} H_{-n}(x) \sim-i \frac{2}{\sqrt{\pi n}}\left(\frac{e x}{2 n}\right)^{-n}, n \rightarrow \infty  \tag{3.3}\\
\frac{d}{d x} H_{n}^{(1)}(x)=(-1)^{n} \frac{d}{d x} H_{-n}(x) \sim-i \frac{2 n}{\sqrt{\pi}} \frac{1}{x}\left(\frac{e x}{2 n}\right)^{-n}, n \rightarrow \infty \tag{3.4}
\end{gather*}
$$

### 3.3 THE SCATTERING PROBLEM AND ITS EXACT SOLUTION

The geometry is pictured in Fig. (3.1). Our scatterer is an infinitely long, PEC cylinder surrounded by free space, with axis along the $z$-axis and radius $\rho_{c y l}$; let ( $\rho_{c y l}, \phi_{c y l}$ ) denote the
polar coordinates of a point on the PEC surface. The source is an electric current filament $I$, on the $x$-axis, with polar coordinates $\left(\rho_{f i l}, 0\right)$; it is located outside the cylinder so that $\rho_{f i l}>\rho_{c y l}$. Let ( $\rho_{o b s}, \phi_{o b s}$ ) denote the polar coordinates of the observation point. We will use the notation $R_{A, B}$ to denote the distance from the point $\left(\rho_{A}, \phi_{A}\right)$ to the point $\left(\rho_{B}, \phi_{B}\right)$; for example,

$$
\begin{equation*}
R_{f i l, c y l}=\sqrt{\rho_{f i l}^{2}+\rho_{c y l}^{2}-2 \rho_{f i l} \rho_{c y l} \cos \phi_{c y l}} \tag{3.5}
\end{equation*}
$$

The electric field is $z$-directed, $\mathbf{E}=\hat{\mathbf{z}} E_{z}$, and the exact solution outside the PEC cylinder $\left(\rho_{o b s}>\rho_{c y l}\right)$ is

$$
\begin{equation*}
E_{z}=H_{0}^{(1)}\left(k R_{f i l, o b s}\right)-\sum_{n=-\infty}^{n=\infty} \frac{J_{n}\left(k \rho_{c y l}\right) H_{n}^{(1)}\left(k \rho_{f i l}\right)}{H_{n}^{(1)}\left(k \rho_{f i l}\right)} H_{n}^{(1)}\left(k \rho_{o b s}\right) e^{i n \phi_{o b s}} \tag{3.6}
\end{equation*}
$$



Figure 3.1: Geometry and the three regions of our scattering problem
In the RHS of (3.6), the overall factor $-k^{2} I /\left(4 \omega \epsilon_{0}\right)$, which is unimportant for our purposes, has been omitted. Formula (3.6) is readily verified: The first term of the RHS is the incident
field. The second term, the scattered field, is a Fourier series, each term of which satisfies the wave equation (in the cylindrical coordinate system $\left(\rho_{o b s}, \phi_{o b s}, z\right)$ ), as well as the outgoingwave condition. Furthermore, the (total) field $E_{z}$ vanishes on the PEC surface. This is seen by applying the addition theorem (3.1) to the $H_{0}^{(1)}\left(k R_{\text {obs,fil }}\right)$ in (3.6) and combining the two resulting series. For $\rho_{o b s}<\rho_{f i l}$, the result is

$$
\begin{align*}
E_{z}= & \sum_{n=-\infty}^{n=\infty} H_{n}^{(1)}\left(k \rho_{f i l}\right) \\
& \times \frac{J_{n}\left(k \rho_{o b s}\right) H_{n}^{(1)}\left(k \rho_{c y l}\right)-J_{n}\left(k \rho_{c y l}\right) H_{n}^{(1)}\left(k \rho_{o b s}\right)}{H_{n}^{(1)}\left(k \rho_{c y l}\right)} \\
& \times e^{i n \phi_{o b s}}, \quad\left(\rho_{o b s}<\rho_{f i l}\right) . \tag{3.7}
\end{align*}
$$

The form in (3.7) obviously vanishes when $\rho_{o b s}=\rho_{c y l}$. For $\rho_{o b s}<\rho_{f i l}$, 3.7) is a useful alternative to (3.6).

We will be more concerned with the derivative with respect to $\rho_{o b s}$ viz.:

$$
\begin{align*}
\frac{\partial E_{z}}{\partial \rho_{o b s}}=\frac{\partial}{\partial \rho_{o b s}} H_{0}^{(1)}\left(k R_{f i l, o b s}\right) & -\sum_{n=-\infty}^{n=\infty} \frac{J_{n}\left(k \rho_{c y l}\right) H_{n}^{(1)}\left(k \rho_{f i l}\right)}{H_{n}^{(1)}\left(k \rho_{c y l}\right)} \\
& \times \frac{\partial}{\partial \rho_{o b s}} H_{n}^{(1)}\left(k \rho_{o b s}\right) e^{i n \phi_{o b s}} \tag{3.8}
\end{align*}
$$

A particularly simple expression can be obtained on the surface of the cylinder $\left(\rho_{o b s}=\rho_{\text {cyl }}\right)$ by differentiating (3.7) and using the Wroskian relation [1], 9.1.16] to obtain:

$$
\begin{equation*}
\frac{\partial E_{z}}{\partial \rho_{o b s}}=\frac{-2 i}{\pi \rho_{o b s}} \sum_{n=-\infty}^{n=\infty} \frac{H_{n}^{(1)}\left(k \rho_{f i l}\right)}{H_{n}^{(1)}\left(k \rho_{c y l}\right)} e^{i n \phi_{o b s}}, \quad\left(\rho_{o b s}=\rho_{c y l}\right) \tag{3.9}
\end{equation*}
$$

### 3.4 ANALYTIC CONTINUATION OF THE EXACT SOLUTION

Convergence of the series in (3.6) (for the scattered electric field) can be examined by using (3.2) and (3.3). For large $|n|$, the $n$th term is seen to behave like $(1 / n)\left(\rho_{\text {cri }} / \rho_{o b s}\right)^{|n|} e^{i|n| \phi_{o b s}}$, with the distance $\rho_{\text {cri }}$ defined by:

$$
\begin{equation*}
\rho_{c r i}=\frac{\rho_{c y l}^{2}}{\rho_{f i l}} \tag{3.10}
\end{equation*}
$$

Thus, the $n$-th term of the series in (3.6) behaves like the $n$-th term of the Taylor series for $\ln (1+x)$ a series which converges for $|x|<1$ and diverges for $|x|>1$. Therefore, the series in (3.6) converges for all $\rho_{o b s}>\rho_{c r i}$ and diverges for all $\rho_{o b s}<\rho_{c r i}$.

Because of (3.4)-(3.2), the $n$-th term of the series in (3.8) (for the scattered electric field derivative) behaves like $\frac{\left(\rho_{c r i}\right.}{\left.\rho_{o b s}\right)^{n n} e^{i n| |_{o b s}}}$, i.e. like that of a geometric series. Therefore, the series in (3.8) also converges when $\rho_{o b s}>\rho_{c r i}$ and diverges when $\rho_{o b s}<\rho_{c r i}$.

The "critical" radius $\rho_{\text {cri }}$ is smaller than the radius $\rho_{\text {cyl }}$ of the PEC cylinder. Also, in the magnetostatic limit, the meaning of $\rho_{\text {cri }}$ is well known: It is the distance from the originmeasured along the segment joining origin and filament-where the image (with current $-I$ ) must be located, as shown in Fig. 2. This is true because for any point $\left(\rho_{c y l}, \phi_{c y l}\right)$ on the PEC cylinder, the distance from $\left(\rho_{c r i}, 0\right)$ to $\left(\rho_{c y l}, \phi_{c y l}\right)$ is a constant multiple of $R_{\text {filccyl }}$, and this amounts to a constant magnetostatic potential on the surface of the PEC cylinder.

The series solution (3.6), originally found for points $\rho_{\text {obs }}>\rho_{c y l}$ (Region 3 of Fig. (3.1)), is thus also convergent and meaningful inside the PEC surface, until the critical radius ( $\rho_{\text {obs }}>$ $\rho_{c r i}$ ). We have thus extended our solution (3.6) to Region 2 of Fig. (3.1). Since we found that the derivative (3.8) of (3.6) is also well-defined until the critical radius, the extended solution is in fact the analytic continuation (with respect to the single complex variable $\rho_{o b s}$ ) of the original solution.

### 3.5 CONTINUOUS MAS

Our first auxiliary source is a continuous cylindrical source radius $\rho_{\text {aux }}$, located in the inside of the PEC surface, so that $\rho_{a u x}<\rho_{c y l}$. It carries a $z$-directed electric surface current density $J^{s}\left(\phi_{a u x}\right)$ (to be precise the electric surface current density is found by multiplying $J^{s}\left(\phi_{a u x}\right)$ by $I$ ), which is to be determined from the boundary condition on $E_{z}$. (Other auxiliary sources, which include a magnetic surface current density, are also suitable.) With a normalization consistent with (3.6), the (total) field $E_{z}$ is:

$$
\begin{equation*}
E_{z}=H_{0}^{(1)}\left(k R_{f i l, o b s}\right)+\rho_{a u x} \int_{-\pi}^{\pi} H_{0}^{(1)}\left(k R_{a u x, o b s}\right) J^{s}\left(\phi_{a u x}\right) d \phi_{a u x} \tag{3.11}
\end{equation*}
$$

where, as in (3.6), the first term is the incident field. The second term, the scattered field, is the field due to our auxiliary source, expressed as an integral involving the Green's function. By enforcing the boundary condition $E_{z}=0$ when $\left(\rho_{o b s}, \phi_{o b s}\right)=\left(\rho_{c y l}, \phi_{c y l}\right)$, (3.11) gives

$$
\begin{align*}
\int_{-\pi} \pi H_{0}^{(1)}\left(k R_{a u x, c y l}\right) J^{s}\left(\phi_{a u x}\right) d \phi_{a u x}= & -\frac{1}{\rho_{a u x}} H_{0}^{(1)}\left(k R_{f i l, c y l}\right) \\
& -\pi<\phi_{c y l}<\pi \tag{3.12}
\end{align*}
$$

which is a Fredholm integral equation of the first kind with unknown $J^{s}\left(\phi_{\text {aux }}\right)$ and kernel $H_{0}^{(1)}\left(k R_{\text {aux,obs }}\right)$. Since

$$
\begin{equation*}
R_{a u x, c y l}=\sqrt{\rho_{a u x}^{2}+\rho_{c y l}^{2}-2 \rho_{a u x} \rho_{c y l} \cos \left(\phi_{a u x}-\phi_{c y l}\right)} \tag{3.13}
\end{equation*}
$$

the kernel is a "difference kernel", i.e. it depends only on the difference $\phi_{a u x}-\phi_{c y l}$ of the two variables, not the two variables separately; we denote it by $K\left(\phi_{a u x}-\phi_{c y l}\right)$. Furthemore, $K(\phi)$ is a periodic function of $\phi$ with period $2 \pi$.
The solution to integral equations of the above type is found in closed form in relevant theory. The solution is given as the Fourier series

$$
\begin{equation*}
J^{s}(\phi)=\frac{1}{2 \pi} \sum_{n=-\infty}^{\infty} \frac{g_{n}}{K_{-n}} e^{i n \phi} \tag{3.14}
\end{equation*}
$$

in which the coefficients involve the Fourier-series coefficients $K_{n}, g_{n}$ of the kernel $K(\phi)$ and the RHS $g\left(\phi_{c y l}\right)$. Both these latter coefficients can be found immediately from the addition theorem (3.1), because

$$
\begin{equation*}
K(\phi)=H_{0}^{(1)}\left(k \sqrt{\rho_{a u x}^{2}+\rho_{c y l}^{2}-2 \rho_{a u x} \rho_{c y l} \cos \phi}\right) \tag{3.15}
\end{equation*}
$$

and

$$
\begin{equation*}
g\left(\phi_{c y l}\right)=-\frac{1}{\rho_{a u x}} H_{0}^{(1)}\left(k \sqrt{\rho_{f i l}^{2}+\rho_{c y l}^{2}-2 \rho_{f i l} \rho_{c y l} \cos \phi_{c y l}}\right) \tag{3.16}
\end{equation*}
$$

We thus find the solution to 3.12 to be

$$
\begin{equation*}
J^{s}\left(\phi_{a u x}\right)=-\frac{1}{2 \pi \rho_{a u x}} \sum_{n=-\infty}^{\infty} \frac{J_{n}\left(k \rho_{c y l}\right) H_{n}^{(1)}\left(k \rho_{f i l}\right)}{J_{n}\left(k \rho_{a u x}\right) H_{n}^{(1)}\left(k \rho_{c y l}\right)} e^{i n \phi_{a u x}} \tag{3.17}
\end{equation*}
$$

Note that this current density is not well-defined when $J_{n}\left(\rho_{a u x}\right)=0$ for some $n$, i.e., in cases where the electrical radius of the auxiliary source coincides with a zero of any Bessel function. These are exceptional cases. It is natural that $J_{n}\left(k \rho_{\text {aux }}\right)$ appears in the denominator, because the corresponding $J_{s}\left(\phi_{a u x}\right)=e^{i n \phi_{a u x}}$ is identically zero.

Excluding the aforementioned exceptional cases, when does the Fourier series in (3.17) converge? Use of (3.3) and (3.2) shows that, for large $|n|$, the $n$th term behaves like

$$
\left(\rho_{c r i} / \rho_{a u x}\right)^{|n|} e^{i|n| \phi_{a u x}}
$$

with $\rho_{\text {cri }}$ defined in (3.10). Thus, the series in (3.17) behaves a geometric series.
It converges when $\rho_{\text {cri }}<\rho_{\text {aux }}<\rho_{\text {cyl }}$ (where the latter inequality was assumed to start with) or when the auxiliary cylindrical surface lies within Region 2 of figure (3.1), 4 ; it diverges when the auxiliary surface lies within Region 1 of the same figure ( $\rho_{\text {aux }}<\rho_{\text {cri }}$ ).

Conversely, if the field of an auxiliary cylindrical current source is to cancel the incident field at $\rho_{\text {obs }}=\rho_{\text {cyl }}$, the radius of the auxiliary source must be larger than the critical radius $\rho_{\text {cri }}$. This result is consistent with general theorems referenced in [4] which state that the auxiliary surface must envelope all singularities for the integral equation to be solvable.

Remark: Note that Region 2 shrinks as the filament approaches the PEC cylinder and grows as the filament moves to infinity so that, for an incident plane wave, there is a convergent $J^{s}\left(\phi_{a u x}\right)$ for any $\rho^{a u x}$ with $0<\rho_{a u x}<\rho_{c y l}$, using the previous definition (see also the figure below):

$$
\rho_{c r i}=\frac{\rho_{c y l}^{2}}{\rho_{f i l}}
$$

The aforementioned divergence of the Fourier series and the ensuing nonsolvability of (3.12) when $\rho_{\text {aux }}<\rho_{\text {cri }}$ reminds us of a situation encountered in center-driven linear antennas of infinite length and specifically, in Hallen's integral equation with the approximate kernel and the delta-function generator: One seeks a solution in the form of a Fourier transform,


Figure 3.2: Geometry and the three regions of our scattering problem
only to find that the Fourier inversion integral diverges [4]. Therefore, that integral equation is nonsolvable [4, as is (3.12) when $\rho_{\text {aux }}<\rho_{\text {cri }}$.

Let us return to our PEC cylinder and focus on the case $\rho_{c r i}<\rho_{a u x}<\rho_{c y l}$, where the continuous auxiliary source is meaningful. Since $J^{s}\left(\phi_{a u x}\right)$ was derived solely from the condition that $E_{z}$ vanishes on the surface of the PEC cylinder, it is of interest to verify that $J^{s}\left(\phi_{a u x}\right)$ - as given in (3.17) - yields the correct field everywhere. This can be done by using (3.1) to re-write (3.11) as

$$
\begin{gather*}
E_{z}=H_{0}^{(1)}\left(k R_{f i l, o b s}\right)+\rho_{a u x} \sum_{n=-\infty}^{\infty} J_{n}\left(k \rho_{a u x}\right) H_{n}^{(1)}\left(k \rho_{o b s}\right) \\
\times e^{i n \phi_{o b s}} \int_{-\pi}^{\pi} e^{-i n \phi_{a u x}} J^{s}\left(\phi_{a u x}\right) d \phi_{a u x} \tag{3.18}
\end{gather*}
$$

and recognizing the integral in (3.18) as $2 \pi$ times the $n$th Fourier-series coefficient of $J^{s}\left(\phi_{a u x}\right)$. When this coefficient is found from (3.17) and substituted into (3.18), the result is found to be independent of $\rho_{a u x}$ and precisely equal to the RHS of (3.6).

When $\rho_{c r i}<\rho_{\text {aux }}<\rho_{c y l}$, the auxiliary surface current density (3.17) yields the true electric field at all points outside the PEC cylinder $\left(\rho_{\text {obs }}>\rho_{\text {cyl }}\right)$ and its analytic continuation - as defined in section (3.4) - for all points between the auxiliary surface and the PEC cylinder ( $\rho_{\text {aux }}<\rho_{\text {obs }}<\rho_{c y l}$ ).

In fact, the above statement gives physical significance to the aforementioned analytic continuation: The analytic continuation is actually a field produced by a current - namely, the exterior $\left(\rho_{o b s}>\rho_{a u x}\right)$ field produced by the surface current density $J^{s}\left(\phi_{a u x}\right)$ of radius $\rho_{a u x}$. The analytic continuation therefore satisfies Maxwell's equations for all $\rho_{o b s}>\rho_{a u x}$, something not obvious beforehand.

The auxiliary surface is not backed up by a perfect conductor and the (total) electric field on the auxiliary surface is nonzero. This is seen by setting $\rho_{o b s}=\rho_{a u x}$ in (3.7). One can also determine the field at all points interior to the surface $\rho_{a u x}$. Without showing this calculation in detail, let us mention that the interior field also turns out to be nonzero. This is not surprising: As explained by means of the equivalence principle in [4], to produce a zero interior field, one should generally choose a $J^{s}\left(\phi_{\text {aux }}\right)$ equal to the tangential magnetic field (proportional, in our case, to (3.8)) together with an auxiliary magnetic surface current density equal to the tangential electric field (given here by (3.6) or (3.7)). Our choice, which consists solely of the electric surface current density $J_{s}\left(\phi_{a u x}\right)$, is different.

Nor is $J^{s}\left(\phi_{a u x}\right)$ in (3.17) equal to the tangential magnetic field, compare to (3.8). But in the limiting case $\rho_{\text {aux }} \rightarrow \rho_{\text {cyl }}$, the $J^{s}\left(\phi_{\text {aux }}\right)$ in (3.17) does reduce to the tangential magnetic field in (3.8), compare (3.17) to the specialized expression in (3.9). In other words, as the auxiliary surface approaches the PEC cylinder, the auxiliary surface current density $J^{s}\left(\phi_{a u x}\right)$ reduces to the true surface current density on the scatterer.

### 3.6 DISCRETE MAS

$$
\begin{align*}
R_{a u x, o b s}=R_{l, o b s}=\sqrt{\rho_{a u x}^{2}+\rho_{o b s}^{2}-2 \rho_{a u x} \rho_{o b s} \cos \left(\phi_{o b s}-\frac{2 \pi l}{N}\right)} \\
l=0,1,2, \ldots, N-1 \tag{3.19}
\end{align*}
$$

With a normalization consistent with the previous sections, the (total) field $E_{z}$ is:

$$
\begin{equation*}
E_{z}=H_{0}^{(1)}\left(k R_{f i l, o b s}\right)+\sum_{l=0}^{N-1} I_{l} H_{0}^{(1)}\left(k R_{l, o b s}\right) \tag{3.20}
\end{equation*}
$$

Here, the scattered field has been written as a sum over the $N$ MAS currents.
We now take $N$ equispaced collocation points on the PEC cylinder. Collocation point $\# p$ is located at $\left(\rho_{c y l}, 2 \pi p / N\right)$; for simplicity, the angles are the same as those of the auxiliary sources.

By enforcing the boundary condition $E_{z}=0$ when $\left(\rho_{o b s}, \phi_{o b s}\right)=\left(\rho_{c y l}, 2 \pi p / N\right)$, 3.20) yields:

$$
\begin{equation*}
\sum_{l=0}^{N-1} H_{0}^{(1)}\left(k b_{l, p}\right) I_{l}=-H_{0}^{(1)}\left(k d_{p}\right), \quad p=0,1,2, \ldots, N-1 \tag{3.21}
\end{equation*}
$$



Figure 3.3: MAS current $\# l$ is located at $\left(\rho_{a u x}, \phi_{l}\right)$ and collocation point $\# p$ is located at $\left(\rho_{c y l}, \phi_{p}\right)$, where $\phi_{l}=2 \pi(l-1) / N, l=0,1, \ldots, N-1$. The distance between MAS current $\# l$ and collocation point $\# p$ is $b_{l, p}$, and the distance between filament $I$ and the collocation point $p$ is $d_{p}$. In the figure, $N=8$ and we show $b_{1,2}$ and $d_{7}$.
where

$$
\begin{equation*}
b_{p, l}=b_{l, p}=\sqrt{\rho_{a u x}^{2}+\rho_{c y l}^{2}-2 \rho_{a u x} \rho_{c y l} \cos \frac{2 \pi(p-l)}{N}}, p, l=0,1,2, \ldots, N-1 \tag{3.22}
\end{equation*}
$$

is the distance between auxiliary filament $l$ and collocation point $p$, and

$$
\begin{equation*}
d_{p, l}=d_{p}=\sqrt{\rho_{f i l}^{2}+\rho_{c y l}^{2}-2 \rho_{f i l} \rho_{c y l} \cos \frac{2 \pi p}{N}}, \quad p=0,1,2, \ldots, N-1 \tag{3.23}
\end{equation*}
$$

is the distance between filament $I$ and collocation point $p$.
Equation (3.21) is $N \times N$ system of linear algebraic equatoins with unknowns the MAS currents $I_{l}$. The matrix on the LHS is circulant. That is, each row is a cyclic permutation of the first row, while the last element of each row is the first element of the next row; These facts are a consequence of the equalities $b_{p, l}=b_{0, l-p}$ and $b_{0, q N+l}=b_{0, l}$ ( $q$ integer).

The integral equation of the previous section was solved using Fourier series; in an analogous manner, a system lioke (3.21) with a circulant matrix can be solved in closed form using discrete Fourier transforms (DFTs). Although ths is very well known (see references of 4], we derive the relevant formulas below in summary: In a circulant system with periodic sequence:

$$
\begin{equation*}
B_{l}=B_{l+N}, \quad l=0, \pm 1, \pm 2, \ldots \tag{3.24}
\end{equation*}
$$

in the form:

$$
\begin{equation*}
\sum_{l=0}^{N-1} B_{l-p} I_{l}=D_{p}, \quad p=0,1,2, \ldots, N-1 \tag{3.25}
\end{equation*}
$$

we introduce the expression (by [4]):

$$
\begin{align*}
B^{(m)} & =\frac{1}{N} \sum_{p=0}^{N-1} B_{p} e^{\frac{-i 2 \pi m p}{N}} \\
D^{(m)} & =\frac{1}{N} \sum_{p=0}^{N-1} D_{p} e^{\frac{-i 2 \pi m p}{N}} \\
I^{(m)} & =\frac{1}{N} \sum_{p=0}^{N-1} I_{p} e^{\frac{-i 2 \pi m p}{N}} \tag{3.26}
\end{align*}
$$

Note: We stress here that a closed-form solution is possible because the MAS currents are equispaced and so are the collocation points. In view of the importance of the spacing of collocation points to the convergence of collocation methods, some of the results that follows might only hold for equispaced points).

Using the above relations (3.24)-(3.26), the solution to (3.21) is:

$$
\begin{equation*}
I_{l}=\sum_{m=0}^{N-1} I^{(m)} e^{i 2 \pi l m / N}, \quad l=0,1,2, \ldots, N-1 \tag{3.27}
\end{equation*}
$$

where

$$
\begin{equation*}
I^{(m)}=\frac{1}{N} \frac{D^{(m)}}{B^{N-m}} \tag{3.28}
\end{equation*}
$$

with

$$
\begin{align*}
B^{(m)}=\frac{1}{N} \sum_{l=0}^{N-1} H_{0}^{(1)}\left(k b_{0, l}\right) e^{-i 2 \pi l m / N}, & \\
& m=0,1,2, \ldots, N-1 \tag{3.29}
\end{align*}
$$

and

$$
\begin{align*}
D^{(m)}=-\frac{1}{N} \sum_{p=0}^{N-1} H_{0}^{(1)}\left(k d_{p}\right) e^{-i 2 \pi p m / N} & , \\
& m=0,1,2, \ldots, N-1 \tag{3.30}
\end{align*}
$$

The quantity given by (3.28) is the DFT of the solution, while the quantities given by (3.30) and (3.29) are the DFTs of the RHS vector and first row of the matrix, respectively. Useful expressions for the latter two quantities are obtained by substituting (3.22) and (3.23) into (3.29) and (3.30), respectively applying the addition theorem (3.1) to the resulting Hankel functions, interchanging the order of summation, and using the identity from the first section:

$$
\sum_{p=0}^{N-1} e^{i 2 \pi p(n-m) / N}=\left\{\begin{array}{cc}
N, & \text { if } n-m=\text { multiple of } N  \tag{3.31}\\
0, & \text { otherwise }
\end{array}\right.
$$

The results are given by the convergent series

$$
\begin{align*}
B^{(m)}=\sum_{q=-\infty}^{\infty} J_{q N+m}\left(k \rho_{a u x}\right) H_{q N+m}^{(1)}\left(k \rho_{c y l}\right) & \\
& m=0,1,2, \ldots, N-1 \tag{3.32}
\end{align*}
$$

and

$$
\begin{align*}
& D^{(m)}=\sum_{q=-\infty}^{\infty} J_{q N+m}\left(k \rho_{c y l}\right) H_{q N+m}^{(1)}\left(k \rho_{f i l}\right) \\
& m=0,1,2, \ldots, N-1 \tag{3.33}
\end{align*}
$$

Note from (3.32) and (3.33) that $D^{(m)}=D^{(N-m)}$ and $B^{(m)}=B^{(N-m)}$ so that (3.28) implies

$$
\begin{equation*}
I^{(m)}=I^{(N-m)}=\frac{1}{N} \frac{D^{(m)}}{B^{m}} \tag{3.34}
\end{equation*}
$$

In what follows we assume for simplicity that $N$ is odd. In that case, 3.27) can be replaced by

$$
\begin{align*}
& I_{l}=I^{(0)}+2 \sum_{m=1}^{(N-1) / 2} I^{(m)} \cos \frac{2 \pi l m}{N} \\
& l=0,1,2, \ldots, N-1(\text { Nodd }) \tag{3.35}
\end{align*}
$$

### 3.7 BEHAVIOUR OF MAS CURRENTS: A LARGE NUMBER OF SOURCES

Together with (3.32)-(3.34), (3.35) is an explicit expression for the $N$ discrete MAS currents. We now determine the asymptotic behaviour of this expression in the limit $N \rightarrow \infty$; in this limit, the discrete MAS currents become a surface current density $J_{\text {limit }}^{S}\left(\phi_{\text {aux }}\right)$. For $\phi_{\text {aux }}=$ $2 \pi l / N, J_{\text {limit }}^{s}\left(\phi_{\text {aux }}\right)$ equals the limit of the ratio of the MAS current $I_{l}$ to the arc-length distance between adjacent currents. That is

$$
\begin{equation*}
J_{l i m i t}^{s}\left(\phi_{a u x}\right)=\lim _{N \rightarrow \infty} \frac{N I_{l}}{2 \pi \rho_{a u x}} \quad\left(\phi_{a u x}=\frac{2 \pi l}{N}\right) \tag{3.36}
\end{equation*}
$$

In what follows, the N quantities $N I_{l} /\left(2 \pi \rho_{a u x}\right) l=0,1, \ldots, N-1$ will be referred to as "normalized MAS currents". In view of the results of section (3.5) one should expect the limit in 3.36 to:

- case 1: exist when the discrete sources lie within Region 2 of Fig. (3.1), 4], ( $\rho_{\text {cri }}<$ $\left.\rho_{\text {aux }}<\rho_{\text {cyl }}\right)$, giving $J_{\text {limit }}^{s}\left(\phi_{\text {aux }}\right)=J^{s}\left(\phi_{\text {aux }}\right)$, and
- case 2: diverge when the discrete sources lie within Region $1\left(\rho_{a u x}<\rho_{c r i}\right)$

Note: One should also expect divergence in the aforementioned exceptional cases $J_{n}\left(k \rho_{a u x}\right)$ for some $n$.

We continue to assume that $N$ is odd (the formulation for $N$ even is similar). Because of (3.2), (3.2) and the fact that $0 \leq m \leq(N-1) / 2$ in the RHS of (3.35), in the limit $N \rightarrow \infty$, only the $q=0$ term in each sum (3.32) and (3.33) needs to be kept. Note: two terms ( $q=0$ and $q=-1$ ) must be kept when $m=N / 2$ and this is why the case $N=$ even is slightly more complicated.

For $N$ odd, therefore, $(3.32)-(3.34)$ give:

$$
\begin{align*}
I^{(m)}=-\frac{1}{N} & \frac{J_{m}\left(k \rho_{c y l}\right) H_{m}^{(1)}\left(k \rho_{f i l}\right)}{J_{m}\left(k \rho_{a u x}\right) H_{m}^{(1)}\left(k \rho_{c y l}\right)} \\
& m=0,1,2, \ldots, \frac{N-1}{2},(N \rightarrow \infty) \tag{3.37}
\end{align*}
$$

With (3.35) and (3.37) we can immediately calculate the limit in (3.36) as follows:

$$
\begin{align*}
J_{\text {limit }}^{s}\left(\phi_{a u x}\right) & =-\frac{1}{2 \pi \rho_{a u x}}\left[\frac{J_{0}\left(k \rho_{c y l}\right) H_{0}^{(1)}\left(k \rho_{f i l}\right)}{J_{0}\left(k \rho_{a u x}\right) H_{0}^{(1)}\left(k \rho_{c y l}\right)}\right. \\
& \left.+2 \sum_{m=1}^{\infty} \frac{J_{m}\left(k \rho_{c y l}\right) H_{m}^{(1)}\left(k \rho_{f i l}\right)}{J_{m}\left(k \rho_{a u x}\right) H_{m}^{(1)}\left(k \rho_{c y l}\right)} \cos \left(m \phi_{a u x}\right)\right] . \tag{3.38}
\end{align*}
$$

Because the coefficients of $\cos \left(m \phi_{a u x}\right)$ in this expression are even in $m, J_{\text {limit }}^{s}$ is equal to $J_{\text {limit }}^{s}$, the surface current density in (3.17). Therefore, we have verified what we started out to show. In particular, we have divergence when $\rho_{a u x}<\rho_{c r i}$.

### 3.8 MAS FIELD: BEHAVIOUR FOR A LARGE NUMBER OF SOURCES

Using the relations (3.24)-(3.26), for finite N , one can find a convenient expression for the electric field $E_{z}$, to be referred to as "MAS field": Apply (3.1) to the second Hankel function in (3.20), interchange the order of summation and introduce the expression (3.26), for the DFT $I^{(n)}$. The result is:

$$
\begin{equation*}
E_{z}=H_{0}^{(1)}\left(k R_{f i l, o b s}\right)+N \sum_{n=-\infty}^{\infty} I^{(n)} J_{n}\left(k \rho_{a u x}\right) \times H_{0}^{(1)}\left(k \rho_{o b s}\right) e^{i n \phi_{o b s}}, \quad\left(\rho_{o b s}>\rho_{a u x}\right) . \tag{3.39}
\end{equation*}
$$

Note that the quantities $I^{(n)}$ were originally defined in (3.27) for $n$ between 0 and $N-1$; when $n$ does not lie within these limits, the $I^{n}$ in (3.39) must be understood as the periodic extension, with period N , of the original $I^{(n)}$. This implies, in particular, that $I^{(-n)}=I^{(n)}$ (as also seen from (3.34)).

Equation (3.39) is exact. We determin its asymptotic behaviour as $N \rightarrow \infty$, irrespective of whether $\rho_{\text {aux }}<\rho_{\text {cri }}$ or not. All $I^{(n)}$ 's in (3.39) can be replaced by the large- $N$ forms in
(3.37). When this is done, the factor $J_{n}\left(k \rho_{a u x}\right)$ cancels out and one is left with

$$
\begin{align*}
E_{z} \sim H_{0}^{(1)}\left(k R_{o b s, f i l}\right) & -\sum_{n=-\infty}^{\infty} \frac{J_{n}\left(k \rho_{c y l}\right) H_{n}^{(1)}\left(k \rho_{f i l}\right)}{H_{n}^{(1)}\left(k \rho_{c y l}\right)} \\
& \times H_{n}^{(1)}\left(k \rho_{o b s}\right) e^{i n \phi_{a u x}} \quad(N \rightarrow \infty) \tag{3.40}
\end{align*}
$$

which is presicely (3.6). From section (3.4) it is known that series in (3.40) converges when and diverges when.

We can distinguish the following cases:
$\diamond$ Case $1 \rho_{c r i}<\rho_{a u x}<\rho_{c y l}$ : We have found the expected result that the limit as $N \rightarrow \infty$ of the MAS field is the

- true (exact) field (3.6) (just as we caclulated analytically) outside the PEC cylindrical surface $\left(\rho_{o b s} \in\left(\rho_{c y l}, \infty\right)\right)$ and
- its analytic continuation between the auxiliary surface and the PEC cylinder $\left(\rho_{o b s} \in\left(\rho_{a u x}, \rho_{c y l}\right)\right)$.
$\diamond$ Case $2 \rho_{\text {aux }}<\rho_{\text {cri }}$ : This case is much more interesting from its theoretical aspect: If the MAS currents are placed within this region, then the limit of the MAS field:
- case 2.1: exists and is the true field for $\rho_{o b s} \in\left(\rho_{c y l}, \infty\right)$
- case 2.2: exists and is the analytic continuation of the true field for $\rho_{o b s} \in$ ( $\rho_{\text {cri }}, \rho_{\text {cyl }}$ )
- case 2.3: does not exist in the portion of the region exterior to the auxiliary sources $\rho_{\text {obs }} \in\left(\rho_{\text {aux }}, \rho_{\text {cri }}\right)$. This is the only case in which the series in 3.40) diverges.

Below follow some remarks on the behaviour of the field:
I. For Case 2, we found in section (3.7) that the limit of the normalized MAS currents does not exist. Case 2.1 shows that it is possible, in the limit, to obtain the true electric field without having the normalized MAS currents converge to a surface current density. The analytic, explicit demonstration of this phenomenon is one of the main objectives of this section.
II. It is well known that fields behave smoothly away from their sources, so it may seem peculiar that the "field" in Case 2 behaves abruptly across the critical surface, i.e., as $\rho_{\text {obs }}$ becomes less than $\rho_{\text {cri }}$ and one moves from Case 2.2 to Case 2. But this "field" is not a field produced by a current; it is merely a mathematical limit as $N \rightarrow \infty$, of the
field of $N$ sources. There is no reason for this field to obey Maxwell's equations. For any finite $N$ (however large), the field is a true field satisfying Maxwell's equations and does not behave abruptly.

### 3.9 DISCUSSIONS ON NUMERICAL RESULTS

Below are demonstrated some numerical results which show the main conclusions of this chapter. The quantities we have calculated are:
$\diamond$ In the figures (3.4)-(3.7), the continuous surface current density $J^{s}\left(\phi_{\text {aux }}\right)$ has been calculated from (3.17)'s series using a large enough number of terms.
$\diamond$ The MAS currents $I_{l}$ have been calculated by solving the system of linear equations (3.21) using a standard package routine. For clarity, a continuous curve has been obtained by joining the values $N I_{l} /\left(2 \pi \rho_{\text {aux }}\right)$ of normalized MAS current by straight lines.

In Fig. (3.4) furthermore, to facilitate comparison with $J^{s}\left(\phi_{a u x}\right)$, the continuous variable $\phi_{\text {aux }}($ rather than $l)$ is shown on the horizontal axis, with the quantity $N I_{l} /\left(2 \pi \rho_{\text {aux }}\right)$ appearing at $\phi_{a u x}=2 \pi l / N$. The "exact" scattered electric field $E_{z s}^{\text {exact }}$ is calculated from the infinite sum in the RHS of (3.6) using a sufficiently large number of terms, while the MAS scattered electric field $E_{z s}^{M A S}$ is found from the finite sum in the RHS of (3.20), with the MAS currents $I_{l}$ determined by solving our linear system as described previously. In all results that follow, $k \rho_{c y l}=2.1$ and $k \rho_{\text {fil }}=3$ so that, $k \rho_{\text {cri }}=1.47$.

Fig. (3.4) shows the real and imaginary parts of $J^{s}\left(\phi_{\text {aux }}\right)$ for $k \rho_{\text {aux }}=1.9$ together with the corresponding continuous curves of normalized MAS currents when $N=70$. The two sets of curves coincide. Here, $\rho_{c r i}<\rho_{a u x}<\rho_{c y l}$, so that the coincidence of the two curves is simply an illustration of the convergence of the normalized MAS currents to $J^{s}\left(\phi_{\text {aux }}\right)$ (as studied in the two previous sections).

In our numerical investigations, we noticed that the aforementioned convergence was, generally, very rapid as $N$ increased. It is worth providing a partial explanation of this, based on the fact that the system of linear equations (3.21) can be viewed as a very efficient discretization of the integral equation (3.11): If we approximate the integral on the LHS of (3.11) by the rectangular rule mentioned in [4] and then satisfy the resulting equation at the quadrature points (i.e., points $\rho_{\text {cyl }}$ located at integer multiples of $2 \pi / N$ ), a system of linear equations for the normalized MAS currents $N I_{l} /\left(2 \pi \rho_{\text {aux }}\right)$ results, which is precisely (3.21). Similar observations are made in relevant literature of [4].

Fig. (3.5) shows the normalized MAS currents when $N=70$ (as in Fig. 5), but with $k \rho_{a u x}=1.3$. This time, $\rho_{a u x}<\rho_{c r i}$ and, by the discussion in the previous sections, there is no meaningful $J^{s}\left(\phi_{\text {aux }}\right)$ to which the normalized MAS currents can be compared. One sees a smooth imaginary part, but a real part that oscillates rapidly near $\phi_{a u x}=0$ and $\phi_{a u x}=2 \pi$. The initial values $\operatorname{rmRe}\left\{I_{0}\right\}, \operatorname{Re}\left\{I_{1}\right\}, \ldots$ alternate in sign. If $N$ is increased to 100 , the first oscillating value $\operatorname{Re}\left\{I_{0}\right\}=-152$ of Fig. 6 in [4], becomes $\operatorname{Re}\left\{I_{0}\right\}=-963$. These results mean that the normalized MAS currents do not converge, just as we have predicted theoretically (Case 2 of previous section).

To ensure that the results in Fig. (3.5) are free of roundoff error-and, in particular, free of effects of matrix ill-conditioning-we have obtained coincident results by an independent


Figure 3.4: Real (top) and imaginary (bottom) parts of auxiliary surface-current density $J^{s}\left(\phi_{\text {aux }}\right)$, together with the corresponding continuous curves of normalized MAS currents $\frac{N}{2 \pi \rho_{a u x}} I_{l} ; k \rho_{c y l}=2.1, k \rho_{f i l}=3$, and $k \rho_{a u x}=1.9 ; N=70$ for the normalized MAS currents. The two sets of continuous curves coincide.
way, specifically from eqns. (3.27)-3.30). (The fact that matrix ill-conditioning is important in MAS has been well-documented further in [4] and supported by our previous remark: We saw that the system (3.21) can be viewed as a discretization of (3.11).)

In Fig. (3.5), we saw that the imaginary part appears smooth whereas the real part presents rapid oscillations, believed not to be due to roundoff error. Strikingly similar phenomena occur when solving Hallen's equation with the approximate kernel, for the antenna of infinite length center-driven by a delta-function generator (see [4] for more details). This similarity is mentioned to reinforce the belief that the results in Fig. 6 have nothing to do with roundoff errors/matrix ill-conditioning.

Needless to say, the $E_{z s}^{\mathrm{MAS}}$ obtained from the MAS currents of Fig. 5 is close to the exact field $E_{z s}^{\text {exact }}$. But this remains true even for the $E_{z s}^{\mathrm{MAS}}$ obtained from the MAS currents of


Figure 3.5: Real (top) and imaginary (bottom) parts of normalized MAS currents $\frac{N}{2 \pi \rho_{a u x}} I_{l}$ as function of element number $l$. Parameters as in previous Fig. (3.4) except that $k \rho_{a u x}=1.3$. (Here, there is no meaningful $J^{s}\left(\phi_{a u x}\right)$ to show.)

Fig. (3.5). This is illustrated in Fig. (3.6), at a distance $k \rho_{o b s}=10\left(\rho_{o b s}>\rho_{c y l}\right.$; Case 2.1 of previous section) and in Fig. 8 at a distance $k \rho_{o b s}=1.8\left(\rho_{c r i}<\rho_{o b s}<\rho_{c y l}\right.$; Case 2.2 of previous section). $E_{z s}^{\mathrm{MAS}}$ coincides with $E_{z s}^{\text {exact }}$ in both cases.

In conclusion, $E_{z s}^{\text {exact }}$ is thetrue field in Fig. 7, and theanalytic continuation of the true field in Fig. 8.

- We should underline here that these"correct" results for the field are obtained from the "abnormal" MAS currents of Fig. (3.5); this is precisely what the discussion in previous section predicts.

The small field values in Figs. (3.6) and (3.7), where $N=70$, are free of roundoff errors. Those values are obtained, via the summation (3.20), from the large, oscillating currents $I_{l}$.


Figure 3.6: Real (top) and imaginary (bottom) parts of exact scattered field $E_{z s}^{\text {exact }}$ when $k \rho_{c y l}=2.1, k \rho_{\text {fil }}=3$, and $k \rho_{o b s}=10$, together with corresponding curves of MAS scattered field $E_{z s}^{\mathrm{MAS}}$ when $k \rho_{a u x}=1.3$ and $N=70$. Thus, $E_{z s}^{\mathrm{MAS}}$ is obtained from the oscillating currents of Fig. (3.5) The two sets of curves coincide.

For larger values of $N$, the values $I_{l}$ grow and the summation 3.20 becomes increasingly susceptible to roundoff. Thus - in any computer, whose wordlength is necessarily finite increasing $N$ will eventually produce results corrupted by roundoff errors.

Finally, the field obtained at a distance $k \rho_{o b s}=1.4$ (so that $\rho_{a u x}<\rho_{o b s}<\rho_{c r i}$, third case 2.3 of previous section) is shown in Fig. (3.8). Here, there is no exact field to compare to, but oscillations are apparent (at least in the imaginary part), indicating the divergence of $E_{z s}^{\text {MAS }}$ as $N \rightarrow \infty$.

To sum up, the main conclusions in this chapter are:

* When the auxiliary surface is located in Region 2 of Fig. (3.1), the continuous problem has a meaningful solution (surface current density) $J^{s}\left(\phi_{\text {aux }}\right)$. The field obtained from $J^{s}\left(\phi_{\text {aux }}\right)$ is the true field (or its analytic continuation) for all observation points outside


Figure 3.7: Like (3.6) (so that $E_{z s}^{\mathrm{MAS}}$ is obtained from the oscillating currents of Fig. 6), except that $k \rho_{o b s}=1.8$. Once again, the two sets of curves coincide.
the auxiliary source. As $N \rightarrow \infty$, the limit of the normalized, discrete MAS currents $\frac{N I_{l}}{2 \pi \rho_{a u x}}(l=0,1, \ldots, N-1)$ exists, and is equal to $J^{s}\left(\phi_{a u x}\right)$.
$\star$ When the auxiliary surface is located in Region 1 of Fig. (3.1), the continuous problem has no solution, i.e., there is no meaningful surface current density $J^{s}\left(\phi_{a u x}\right)$ that will satisfy the boundary condition. For any finite $N$, one can find the discrete MAS currents $I_{l}$ (in Region 1) and, from the $I_{l}$, subsequently determine the electric field. In the limit $N \rightarrow \infty$, the normalized MAS currents diverge, while the electric field does converge to the correct electric field. Numerical results obtained by two independent methods showed that the divergence appears as oscillations near $l=0$ and $l=N-1$ in the plot of $I_{l}$ (that is, at points closest to the filament $I$ ). These oscillations are consistent with the analytical study and similar to the oscillations observed when numerically solving Hallén's equation with the approximate kernel and the delta-function generator. Thus


Figure 3.8: MAS scattered field $E_{z s}^{\mathrm{MAS}}$. Like Fig. (3.6), but with $k \rho_{o b s}=1.4$. (Here, there is no meaningful $E_{z s}^{\text {exact }}$ to show.)
the oscillations are almost certainly not due to roundoff errors or matrix ill-conditioning (it might be possible to further investigate this point, as discussed in Section V). Despite the oscillations, one still obtains the true field (or its analytic continuation) for all observation points in Region 3 (or Region 2, respectively). When the observation point is in Region 1, however - still beyond the auxiliary surface - the electric field obtained as described above diverges. The abrupt behavior of the limiting value of the field-as the observation point moves from Region 2 into Region 1-is not peculiar, because this limiting value is not a true field satisfying Maxwell's equations.

We summarize the above in the following table:

| $\begin{aligned} & \hline \rho_{\text {aux }} \text { loca- } \\ & \text { tion } \end{aligned}$ | Continuous Solution of Integral Equation with $J^{s}\left(\phi_{a u x}\right)$ | Normalized Series Term ("Current") $I_{n}$ as solution to the linear (MAS) system | Disrete MAS Field $E^{M A S}$ |
| :---: | :---: | :---: | :---: |
| $\begin{aligned} & \rho_{\text {aux }} \quad> \\ & \rho_{\text {cri }} \end{aligned}$ | exists | $\frac{N I_{n}}{2 \pi \rho_{a u x}} \xrightarrow{N \rightarrow \infty} J^{s}\left(\phi_{\text {aux }}\right)$ | it exists and converges to the true exact field: $E^{M A S}(N) \rightarrow E^{e x c}$. |
| $\begin{array}{ll} \hline \rho_{\text {aux }} & < \\ \rho_{\text {cri }} \end{array}$ | does not exist | for $N \rightarrow \infty$ : does not converge, it oscillates rapidly as $N$ grows and takes large absolute values | it exists and converges to the true exact field: $E^{M A S}(N) \rightarrow E^{e x c}$. |

## Chapter 4

## Point Source Scattering by an Acoustically Soft Sphere: Solution by the MAS

### 4.1 3D PROBLEM STATEMENT AND GEOMETRY

We use the spherical coordinate system as our problem concerns a spherical geometry. As mentioned in [10], in this system the position of a point is specified by three numbers: the radial distance of that point from the origin $(r)$, its polar angle $(\theta)$ measured from a fixed zenith direction and the azimuth angle $(\phi)$ of its orthogonal projection on a reference plane that passes through the origin and is orthogonal to the zenith, measured from a fixed reference direction on that plane. These coordinate symbols are illustred in figures (4.1).The radial distance is also called the radius or radial coordinate. The polar angle may be called colatitude, zenith angle, normal angle or inclination angle.

Our spherical scatterer looks like the sphere of figure 4.1) and we can see some planes intersecting it on figures (4.2b) (a constant $z$-plane and two constant $\phi$-planes).

To convert spherical coordinates to cartesian ones, the following formulae can be used:

$$
\begin{aligned}
r & =\sqrt{x^{2}+y^{2}+z^{2}} \\
\theta & =\arccos \left(\frac{z}{r}\right) \\
\phi & =\arctan \left(\frac{y}{x}\right)
\end{aligned}
$$

### 4.2 THE SCATTERING PROBLEM AND ITS EXACT SOLUTION

A soft spherical scatterer $V$ of radius $r_{s p h}$ is excited by a time-harmonic spherical acoustic wave, generated by a point-source located in the exterior of $V$. The field $u^{p r}$, radiated by this point-source under the assumptions that the sphere is absent (and that $\mathbb{R}^{3}$ is filled by the material of $V$ ), constitutes the primary field of the Sommerfeld's method [7, 8]. Suppressing


Figure 4.1: Spherical Coordinates


Figure 4.2: Spherical coordinate systems and planes
the harmonic time dependence $\exp (-i \omega t)$ and using the normalization of [2] and [6], the primary spherical field at the location of the observation vector $\mathbf{r}_{\text {obs }}$ is given by:

$$
\begin{equation*}
u^{p r}\left(\mathbf{r}_{o b s}\right)=r_{p s} \exp \left(-i k r_{p s}\right) \frac{\exp \left(i k\left|\mathbf{r}_{o b s}-\mathbf{r}_{p s}\right|\right)}{\left|\mathbf{r}_{o b s}-\mathbf{r}_{p s}\right|}, \quad \mathbf{r}_{o b s} \in \mathbb{R}^{3} \backslash\left\{\mathbf{r}_{p s}\right\} \tag{4.1}
\end{equation*}
$$

so that it reduces to a plane wave with unit amplitude and direction of propagation that of the unit vector $-\hat{\mathbf{r}}_{p s}$, when the point-source recedes to infinity, i.e.

$$
\begin{equation*}
\lim _{r_{p s} \rightarrow \infty} u^{p r}\left(\mathbf{r}_{o b s}\right)=\exp \left(-i k \hat{\mathbf{r}}_{p s} \cdot \mathbf{r}_{o b s}\right) \tag{4.2}
\end{equation*}
$$

where $r_{p s}=\left|\mathbf{r}_{p s}\right|$.
We select the spherical coordinate system $\left(r_{o b s}, \theta_{o b s}, \phi_{o b s}\right)$ with the origin $O$ at the centre of $V$, so that the point-source lies at $r_{o b s}=r_{p s}, \theta_{o b s}=0$, with position vector $\mathbf{r}_{p s}=\left(0,0, r_{p s}\right)$, therefore it is located in the exterior of $V$, on $z$-axis. Thus, the primary and the scattered field are axisymmetric. Then, the primary field is expressed as in [6]:

$$
u^{p r}\left(r_{o b s}, \theta_{o b s}\right)=\frac{1}{h_{0}\left(k r_{p s}\right)}\left\{\begin{array}{l}
\sum_{n=0}^{\infty}(2 n+1) j_{n}\left(k r_{p s}\right) h_{n}\left(k r_{o b s}\right) P_{n}\left(\cos \theta_{o b s}\right),  \tag{4.3}\\
r_{o b s}>r_{p s} \\
\sum_{n=0}^{\infty}(2 n+1) h_{n}\left(k r_{p s}\right) j_{n}\left(k r_{o b s}\right) P_{n}\left(\cos \theta_{o b s}\right), \\
r_{o b s}<r_{p s}
\end{array},\right.
$$

where $j_{n}$ and $h_{n}$ are the $n$-th order spherical Bessel and Hankel function of the first kind and $P_{n}$ is a Legendre polynomial.

The generated scattered field, due to the presence of the sphere, is denoted by $u^{s c}$ and is expressed as follows:

$$
\begin{equation*}
u^{s c}\left(r_{o b s}, \theta_{o b s}\right)=\frac{1}{h_{0}\left(k r_{p s}\right)} \sum_{n=0}^{\infty}(2 n+1) a_{n} h_{n}\left(k r_{o b s}\right) P_{n}\left(\cos \theta_{o b s}\right), \quad r_{o b s}>r_{s p h} \tag{4.4}
\end{equation*}
$$

Applying Sommerfeld's method, (see e.g. [6]), the total spherical field $u_{e x}^{t}$ in the exterior of the scatterer is defined as the superposition of the primary and the scattered fields, as follows:

$$
\begin{equation*}
u_{e x}^{t}\left(\mathbf{r}_{o b s}\right)=u^{p r}\left(\mathbf{r}_{o b s}\right)+u^{s c}\left(\mathbf{r}_{o b s}\right), \quad r_{o b s}>r_{s p h}, \quad r_{o b s} \neq r_{p s} \tag{4.5}
\end{equation*}
$$

On the surface of the soft sphere, the total field must satisfy the Dirichlet boundary condition as described in Chapter 1:

$$
\begin{equation*}
u_{e x c}^{t}\left(\mathbf{r}_{o b s}\right)=0, \quad r_{o b s}=r_{s p h} \tag{4.6}
\end{equation*}
$$

By imposing the latter boundary condition and using the field expressions (4.3) and (4.4), we obtain:

$$
\begin{equation*}
a_{n}=-\frac{h_{n}\left(k r_{p s}\right) j_{n}\left(k r_{s p h}\right)}{h_{n}\left(k r_{s p h}\right)} \tag{4.7}
\end{equation*}
$$

and hence, the exact expression of the scattered field is:

$$
\begin{align*}
u^{s c}\left(r_{o b s}, \theta_{o b s}\right)=-\frac{1}{h_{0}\left(k r_{p s}\right)} \sum_{n=0}^{\infty} & (2 n+1) \frac{h_{n}\left(k r_{p s}\right) j_{n}\left(k r_{s p h}\right)}{h_{n}\left(k r_{s p h}\right)} \\
& \times h_{n}\left(k r_{o b s}\right) P_{n}\left(\cos \theta_{o b s}\right), \quad r_{o b s}>r_{s p h} . \tag{4.8}
\end{align*}
$$

So, by (4.1) and (4.8), the exact solution of the total field in 4.5) may be expressed as:

$$
\begin{align*}
u_{e x}^{t}\left(\mathbf{r}_{o b s}\right)=u^{p r}\left(\mathbf{r}_{o b s}\right)-\frac{1}{h_{0}\left(k r_{p s}\right)} \sum_{n=0}^{\infty} & (2 n+1) \frac{h_{n}\left(k r_{p s}\right) j_{n}\left(k r_{s p h}\right)}{h_{n}\left(k r_{s p h}\right)} \\
& \times h_{n}\left(k r_{o b s}\right) P_{n}\left(\cos \theta_{o b s}\right), \quad r_{o b s}>r_{s p h}, \quad r_{o b s} \neq r_{p s} . \tag{4.9}
\end{align*}
$$

### 4.3 CONVERGENCE OF THE SCATTERED FIELD'S SERIES

### 4.3.1 Spherical Bessel/Hankel functions and asymptotic approximations

Convergence of the scattered field series in (4.8) can be examined by using the large- $n$ asymptotic approximations of the spherical Bessel and Hankel functions. To this end, we remind that the spherical Bessel and Hankel functions are related to the cylindrical ones via the following identities:

$$
\begin{align*}
j_{n}(x) & =\sqrt{\frac{\pi}{2 x}} J_{n+\frac{1}{2}}(x),  \tag{4.10}\\
h_{n}(x) \equiv h_{n}^{(1)}(x) & =\sqrt{\frac{\pi}{2 x}} H_{n+\frac{1}{2}}^{(1)}(x), \tag{4.11}
\end{align*}
$$

while the cylindrical Bessel $J_{n}$ and Hankel $H_{n}^{(1)}(x)=J_{n}(x)+i Y_{n}(x)$ functions have the following large- $n$ asymptotic approximations [see e.g. [1], section 9.3]:

$$
\begin{align*}
J_{n}(x) & =(-1)^{n} J_{-n}(x) \sim \frac{1}{\sqrt{2 \pi n}}\left(\frac{e x}{2 n}\right)^{n}, \quad n \rightarrow+\infty,  \tag{4.12}\\
H_{n}^{(1)}(x) & =(-1)^{n} H_{-n}^{(1)}(x) \sim-i \sqrt{\frac{2}{\pi n}}\left(\frac{e x}{2 n}\right)^{-n}, \quad n \rightarrow+\infty . \tag{4.13}
\end{align*}
$$

### 4.3.2 Derivative of Hankel functions

Differentiating (4.13), we obtain:

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} x} H_{n}^{(1)}(x)=(-1)^{n} \frac{\mathrm{~d}}{\mathrm{~d} x} H_{-n}^{(1)}(x) \sim i \sqrt{\frac{2 n}{\pi}} \frac{1}{x}\left(\frac{e x}{2 n}\right)^{-n}, \quad n \rightarrow+\infty \tag{4.14}
\end{equation*}
$$

Thus, the derivative of the spherical Hankel function becomes:

$$
\begin{aligned}
\frac{d}{d x} h_{n}(x) & =\frac{d}{d x}\left(\sqrt{\frac{\pi}{2 x}} H_{n+\frac{1}{2}}^{(1)}(x)\right) \\
& =\frac{d \sqrt{\frac{\pi}{2 x}}}{d x} H_{n+\frac{1}{2}}^{(1)}(x)+\sqrt{\frac{\pi}{2 x}} \frac{d H_{n+\frac{1}{2}}^{(1)}(x)}{d x} \\
& =-\sqrt{\frac{\pi}{2}} \frac{1}{2 \sqrt{x^{3}}} H_{n+\frac{1}{2}}^{(1)}(x)+\sqrt{\frac{\pi}{2 x}} \frac{d H_{n+\frac{1}{2}}^{(1)}(x)}{d x}
\end{aligned}
$$

So, be means of 4.13 and 4.14 we obtain:

$$
\begin{aligned}
& \frac{d}{d x} h_{n}(x) \stackrel{n \rightarrow \infty, \sqrt{4.13}, \sqrt{4.14}}{\sim} \\
& \quad \sqrt{\frac{\pi}{2 x}} \frac{1}{2 \sqrt{x^{3}}} i \sqrt{\frac{2}{\pi\left(n+\frac{1}{2}\right)}}\left(\frac{e x}{2\left(n+\frac{1}{2}\right)}\right)^{-n-\frac{1}{2}}+\sqrt{\frac{\pi}{2 x}} i \sqrt{\frac{2\left(n+\frac{1}{2}\right)}{\pi}} \frac{1}{x}\left(\frac{e x}{2\left(n+\frac{1}{2}\right)}\right)^{-n-\frac{1}{2}} \\
& \quad=i \frac{1}{2 x^{2}} \sqrt{\frac{1}{\left(n+\frac{1}{2}\right)}}\left(\frac{e x}{2\left(n+\frac{1}{2}\right)}\right)^{-n-\frac{1}{2}}+i \sqrt{\frac{\left(n+\frac{1}{2}\right)}{x^{3}}}\left(\frac{e x}{2\left(n+\frac{1}{2}\right)}\right)^{-n-\frac{1}{2}} \\
& \quad=i \frac{1}{\sqrt{2} x^{2}}\left(\frac{e x}{2\left(n+\frac{1}{2}\right)}\right)^{-n}+i \sqrt{2} \frac{\left(n+\frac{1}{2}\right)}{\sqrt{x^{3}}}\left(\frac{e x}{2\left(n+\frac{1}{2}\right)}\right)^{-n}
\end{aligned}
$$

Thus finally

$$
\Rightarrow \frac{d}{d x} h_{n}(x)=i \frac{1}{\sqrt{2} x^{2}}\left(\frac{e x}{2\left(n+\frac{1}{2}\right)}\right)^{-n}+i \sqrt{2} \frac{\left(n+\frac{1}{2}\right)}{\sqrt{x^{3}}}\left(\frac{e x}{2\left(n+\frac{1}{2}\right)}\right)^{-n}
$$

or

$$
\begin{equation*}
\frac{d}{d x} h_{n}(x)=i \frac{1}{\sqrt{2} x^{2}}\left(\frac{e x}{(2 n+1)}\right)^{-n}+i \sqrt{2} \frac{\left(n+\frac{1}{2}\right)}{\sqrt{x^{3}}}\left(\frac{e x}{(2 n+1)}\right)^{-n} \tag{4.15}
\end{equation*}
$$

Furthermore, the Legendre polynomial function has the following large- $n$ asymptotic expression (see e.g. (3.9) of [9]):

$$
\begin{equation*}
P_{n}(\cos \theta) \sim \sqrt{\frac{2}{n \pi \sin \theta}} \cos \left(\left(n+\frac{1}{2}\right) \theta-\frac{\pi}{4}\right), \quad \theta \in(0, \pi), \quad n \rightarrow+\infty \tag{4.16}
\end{equation*}
$$

Now, by substituting (4.10) and (4.11) to (4.8), we get:

$$
\begin{aligned}
& (2 n+1) \frac{h_{n}\left(k r_{p s}\right) j_{n}\left(k r_{s p h}\right)}{h_{n}\left(k r_{s p h}\right)} h_{n}\left(k r_{o b s}\right) P_{n}\left(\cos \theta_{o b s}\right)= \\
& (2 n+1) \frac{\sqrt{\frac{\pi}{2 k r_{p s}}} H_{n+\frac{1}{2}}\left(k r_{p s}\right) \sqrt{\frac{\pi}{2 k r_{s p h}}} J_{n+\frac{1}{2}}\left(k r_{s p h}\right)}{\sqrt{\frac{\pi}{2 k r_{s p h}}} H_{n+\frac{1}{2}}\left(k r_{s p h}\right)} \sqrt{\frac{\pi}{2 k r_{o b s}}} H_{n+\frac{1}{2}}\left(k r_{o b s}\right) P_{n}\left(\cos \theta_{o b s}\right) .
\end{aligned}
$$

### 4.3.3 Convergence

Next, by using the large- $n$ asymptotic approximations (4.12)- 4.16) we obtain the following approximation of the $n$-th term of the series (4.8):

$$
\begin{aligned}
& n \rightarrow \infty \\
& \sim \\
&(2 n+1) \frac{\pi}{2}^{2} \\
& \sqrt{\frac{1}{r_{o b s} r_{p s}}} \cdot \frac{-i \sqrt{\frac{2}{\pi\left(n+\frac{1}{2}\right)}}\left(\frac{e k r_{p s}}{2\left(n+\frac{1}{2}\right)}\right)^{-\left(n+\frac{1}{2}\right)} \frac{1}{\sqrt{2 \pi\left(n+\frac{1}{2}\right)}}\left(\frac{e k r_{s p h}}{2\left(n+\frac{1}{2}\right)}\right)^{\left(n+\frac{1}{2}\right)}}{-i \sqrt{\frac{2}{\pi\left(n+\frac{1}{2}\right)}}\left(\frac{e k r_{s p h}}{2\left(n+\frac{1}{2}\right)}\right)^{-\left(n+\frac{1}{2}\right)}} . \\
& \cdot(-i) \sqrt{\frac{2}{\pi\left(n+\frac{1}{2}\right)}}\left(\frac{e k r_{o b s}}{2\left(n+\frac{1}{2}\right)}\right)^{-\left(n+\frac{1}{2}\right)} \cdot \sqrt{\frac{2}{n \pi \sin \theta}} \cos \left(\left(n+\frac{1}{2}\right) \theta-\frac{\pi}{4}\right) \\
&=(2 n+1) \frac{2 \pi}{k} \sqrt{\frac{1}{r_{o b s} r_{p s}}} \cdot \frac{-i}{\pi(2 n+1)} \frac{\left(\frac{e k r_{p s}}{2 n}\right)^{-\left(n+\frac{1}{2}\right)}\left(\frac{e k r_{s p h}}{2 n}\right)^{\left(n+\frac{1}{2}\right)}}{\left(\frac{\left.e k r_{s p h}\right)^{-\left(n+\frac{1}{2}\right)}}{2 n}\right)} \\
& \quad\left(\frac{e k r_{o b s}}{2 n}\right)^{-\left(n+\frac{1}{2}\right)} \sqrt{\frac{2}{n \pi \sin \theta}} \cos \left(\left(n+\frac{1}{2}\right) \theta-\frac{\pi}{4}\right)
\end{aligned}
$$

Therefore the $n$-th term of the series (4.8) behaves like:

$$
\begin{equation*}
-\frac{2 i}{k} \frac{r_{s p h}}{r_{p s} r_{o b s}} \sqrt{\frac{2}{\pi \sin \theta}} b_{n} \tag{4.17}
\end{equation*}
$$

where the sequence $b_{n}$ is defined by:

$$
\begin{equation*}
b_{n}=\left(\frac{r_{c r i}}{r_{o b s}}\right)^{n} \sqrt{\frac{1}{n}} \cos \left(\left(n+\frac{1}{2}\right) \theta-\frac{\pi}{4}\right) \tag{4.18}
\end{equation*}
$$

while the critical radius $r_{\text {cri }}$ is defined by:

$$
\begin{equation*}
r_{c r i}=\frac{\left(r_{s p h}\right)^{2}}{r_{p s}} \tag{4.19}
\end{equation*}
$$

Note that the critical radius is smaller than the radius of the soft sphere.
Now, we have that:

$$
\begin{equation*}
b_{n}=c_{n}+d_{n} \tag{4.20}
\end{equation*}
$$

where

$$
\begin{aligned}
c_{n} & =\frac{1}{2}\left(\frac{r_{c r i}}{r_{\text {obs }}}\right)^{n} \sqrt{\frac{1}{n}} e^{j\left(\left(n+\frac{1}{2}\right) \theta-\frac{\pi}{4}\right)} \\
d_{n} & =\frac{1}{2}\left(\frac{r_{c r i}}{r_{\text {obs }}}\right)^{n} \sqrt{\frac{1}{n}} e^{j\left(-\left(n+\frac{1}{2}\right) \theta+\frac{\pi}{4}\right)}
\end{aligned}
$$

and for the sequence $c_{n}$ holds:

$$
\begin{equation*}
\left|\frac{c_{n+1}}{c_{n}}\right|=\left|\frac{r_{c r i}}{r_{o b s}}\right| \sqrt{\frac{n}{n+1}} \xrightarrow{n \rightarrow \infty}\left|\frac{r_{c r i}}{r_{o b s}}\right| . \tag{4.21}
\end{equation*}
$$

Similarly, the same relation is found for $d_{n}$.
By the Cauchy ratio test we have that the power series $\sum_{n=n_{0}}^{\infty} c_{n}, \sum_{n=n_{0}}^{\infty} d_{n}$ converge for all $r_{o b s}$ with $\left|\frac{r_{c r i}}{r_{o b s}}\right|<1$ and diverge for all $r_{o b s}$ with $\left|\frac{r_{c r i}}{r_{o b s}}\right| \geq 1$. Hence, by 4.20 we conclude that the power series $\sum_{n=n_{0}}^{\infty} b_{n}$ converges for $r_{o b s}>r_{c r i}$ and diverges for $r_{o b s} \leq r_{c r i}$. By means of (4.17) the latter conclusion holds also for the convergence of the scattered field series (4.8). This means that the series solution (4.8), originally found for points laying outside the sphere, is also convergent and meaningful inside the sphere, and until the critical radius.

### 4.4 ANALYTIC CONTINUATION OF THE SCATTERED FIELD'S SERIES

We are studying now the convergence of the derivative of the scattered acoustic field's series.

$$
\begin{align*}
\frac{\partial}{\partial r_{o b s}} u^{t}\left(r_{o b s}, \theta_{o b s}\right) & =\frac{\partial}{\partial r_{o b s}}\left(u^{p r}\left(r_{o b s}, \theta_{o b s}\right)+u^{s c}\left(r_{o b s}, \theta_{o b s}\right)\right) \\
& =\frac{\partial}{\partial r_{o b s}} u^{p r}\left(r_{o b s}, \theta_{o b s}\right)-\frac{\partial}{\partial r_{o b s}} \frac{1}{h_{0}\left(k r_{p s}\right)} \\
& \sum_{n=0}^{\infty}(2 n+1) \frac{h_{n}\left(k r_{p s}\right) j_{n}\left(k r_{s p h}\right)}{h_{n}\left(k r_{s p h}\right)} h_{n}\left(k r_{o b s}\right) P_{n}\left(\cos \theta_{o b s}\right) \tag{4.22}
\end{align*}
$$

The series in (4.22) is:

$$
\begin{align*}
& \frac{\partial}{\partial r_{o b s}} u^{s c}\left(r_{o b s}, \theta_{o b s}\right)=-\frac{1}{h_{0}\left(k r_{p s}\right)} \sum_{n=0}^{\infty}(2 n+1) \frac{h_{n}\left(k r_{p s}\right) j_{n}\left(k r_{s p h}\right)}{h_{n}\left(k r_{s p h}\right)} \\
& \frac{\partial}{\partial r_{o b s}} h_{n}\left(k r_{o b s}\right) P_{n}\left(\cos \theta_{o b s}\right) \tag{4.23}
\end{align*}
$$

and its $n$-th term behaves as follows:

$$
\begin{aligned}
& (2 n+1) \frac{h_{n}\left(k r_{p s}\right) j_{n}\left(k r_{s p h}\right)}{h_{n}\left(k r_{s p h}\right)} \frac{\partial}{\partial r_{o b s}} h_{n}\left(k r_{o b s}\right) P_{n}\left(\cos \theta_{o b s}\right) \\
& =(2 n+1) \frac{\sqrt{\frac{\pi}{2 k r_{p s}}} H_{n+\frac{1}{2}}\left(k r_{p s}\right) \sqrt{\frac{\pi}{2 k r_{s p h}}} J_{n+\frac{1}{2}}\left(k r_{s p h}\right)}{\sqrt{\frac{\pi}{2 k r_{s p h}}} H_{n+\frac{1}{2}}\left(k r_{s p h}\right)} \frac{\partial}{\partial r_{o b s}} h_{n}\left(k r_{o b s}\right) P_{n}\left(\cos \theta_{o b s}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \stackrel{n \rightarrow \infty, \sqrt{4.15}}{ }(2 n+1) \sqrt{\frac{\pi}{2 k}} \sqrt{\frac{1}{r_{p s}}} . \\
& \frac{-i \sqrt{\frac{2}{\pi\left(n+\frac{1}{2}\right)}}\left(\frac{e k r_{p s}}{2\left(n+\frac{1}{2}\right)}\right)^{-\left(n+\frac{1}{2}\right)} \frac{1}{\sqrt{2 \pi\left(n+\frac{1}{2}\right)}}\left(\frac{e k r_{s p h}}{2\left(n+\frac{1}{2}\right)}\right)^{\left(n+\frac{1}{2}\right)}}{-i \sqrt{\frac{2}{\pi\left(n+\frac{1}{2}\right)}}\left(\frac{e k r_{s p h}}{2\left(n+\frac{1}{2}\right)}\right)^{-\left(n+\frac{1}{2}\right)}} . \\
& \cdot\left(i \frac{1}{\sqrt{2} x^{2}}\left(\frac{e x}{2\left(n+\frac{1}{2}\right)}\right)^{-n}+i \sqrt{2} \frac{\left(n+\frac{1}{2}\right)}{\sqrt{x^{3}}}\left(\frac{e x}{2\left(n+\frac{1}{2}\right)}\right)^{-n}\right) \\
& \cdot \sqrt{\frac{2}{n \pi \sin \theta}} \cos \left(\left(n+\frac{1}{2}\right) \theta-\frac{\pi}{4}\right) \\
& \sim \frac{1}{\sqrt{2 n+1}} \cdot \frac{1}{\sqrt{\pi}} \cdot \sqrt{\frac{\pi}{2 k}} \sqrt{\frac{1}{r_{p s}}} . \\
& \cdot \frac{r_{s p h}}{\sqrt{r_{p s} \cdot(2 n+1)}} \cdot \frac{\left(\frac{e k r_{p s}}{2\left(n+\frac{1}{2}\right)}\right)^{-n}\left(\frac{e k r_{s p h}}{2\left(n+\frac{1}{2}\right)}\right)^{n}}{\left(\frac{e k r_{s p h}}{2\left(n+\frac{1}{2}\right)}\right)^{-n}} \text {. } \\
& \cdot\left(i \frac{1}{\sqrt{2} x^{2}}\left(\frac{e x}{2\left(n+\frac{1}{2}\right)}\right)^{-n}+i \sqrt{2} \frac{\left(n+\frac{1}{2}\right)}{\sqrt{x^{3}}}\left(\frac{e x}{2\left(n+\frac{1}{2}\right)}\right)^{-n}\right) \\
& \cdot \sqrt{\frac{2}{n \pi \sin \theta}} \cos \left(\left(n+\frac{1}{2}\right) \theta-\frac{\pi}{4}\right) \\
& \sim \frac{1}{\sqrt{2 n+1}} \cdot \frac{1}{\sqrt{\pi}} \cdot \sqrt{\frac{\pi}{2 k}} \sqrt{\frac{1}{r_{p s}}} . \\
& \cdot \frac{r_{s p h}}{\sqrt{r_{p s} \cdot(2 n+1)}} \cdot\left(\frac{e k r_{s p h}^{2}}{r_{p s} \cdot(2 n+1)}\right)^{n} \text {. } \\
& \cdot\left(i \frac{1}{\sqrt{2} x^{2}}\left(\frac{e x}{2\left(n+\frac{1}{2}\right)}\right)^{-n}+i \sqrt{2} \frac{\left(n+\frac{1}{2}\right)}{\sqrt{x^{3}}}\left(\frac{e x}{2\left(n+\frac{1}{2}\right)}\right)^{-n}\right) \\
& \text { - } \sqrt{\frac{2}{n \pi \sin \theta}} \cos \left(\left(n+\frac{1}{2}\right) \theta-\frac{\pi}{4}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \sim \frac{1}{\sqrt{2 n+1}} \cdot \frac{1}{\sqrt{\pi}} \cdot \sqrt{\frac{\pi}{2 k}} \sqrt{\frac{1}{r_{p s}}} . \\
& \cdot \frac{r_{s p h}}{\sqrt{r_{p s} \cdot(2 n+1)}} \cdot\left(\frac{e k r_{s p h}^{2}}{r_{p s} \cdot(2 n+1)}\right)^{n} \text {. } \\
& \cdot\left(i \frac{1}{\sqrt{2}\left(k r_{o b s}\right)^{2}}\left(\frac{e k r_{o b s}}{2\left(n+\frac{1}{2}\right)}\right)^{-n}+i \sqrt{2} \frac{\left(n+\frac{1}{2}\right)}{\sqrt{\left(k r_{o b s}\right)^{3}}}\left(\frac{e k r_{o b s}}{2\left(n+\frac{1}{2}\right)}\right)^{-n}\right) \\
& \cdot \sqrt{\frac{2}{n \pi \sin \theta}} \cos \left(\left(n+\frac{1}{2}\right) \theta-\frac{\pi}{4}\right) \\
& \sim \frac{1}{\sqrt{2 n+1}} \cdot \frac{1}{\sqrt{\pi}} \cdot \sqrt{\frac{\pi}{2 k}} \sqrt{\frac{1}{r_{p s}}} . \\
& \cdot \frac{r_{s p h}}{\sqrt{r_{p s} \cdot(2 n+1)}} \cdot\left(\frac{r_{s p h}^{2}}{r_{p s} \cdot(2 n+1)}\right)^{n} . \\
& \cdot\left(i \frac{1}{\sqrt{2}\left(k r_{o b s}\right)^{2}}\left(\frac{r_{o b s}}{2 n+1}\right)^{-n}+i \sqrt{2} \frac{\left(n+\frac{1}{2}\right)}{\sqrt{\left(k r_{o b s}\right)^{3}}}\left(\frac{r_{o b s}}{2 n+1}\right)^{-n}\right) \\
& \text { - } \sqrt{\frac{2}{n \pi \sin \theta}} \cos \left(\left(n+\frac{1}{2}\right) \theta-\frac{\pi}{4}\right) \\
& \sim \frac{1}{\sqrt{2 n+1}} \cdot \frac{1}{\sqrt{\pi}} \cdot \sqrt{\frac{\pi}{2 k}} \sqrt{\frac{1}{r_{p s}}} . \\
& \cdot \frac{r_{s p h}}{\sqrt{r_{p s} \cdot(2 n+1)}} \cdot\left(\frac{r_{s p h}^{2}}{r_{p s} r_{o b s}}\right)^{n} . \\
& \cdot\left(i \frac{1}{\sqrt{2}\left(k r_{o b s}\right)^{2}}+i \frac{(2 n+1)}{\sqrt{2\left(k r_{o b s}\right)^{3}}}\right) \\
& \cdot \sqrt{\frac{2}{n \pi \sin \theta}} \cos \left(\left(n+\frac{1}{2}\right) \theta-\frac{\pi}{4}\right) \\
& \sim \frac{1}{\sqrt{2 n+1}} \cdot \frac{1}{\sqrt{\pi}} \cdot \sqrt{\frac{\pi}{2 k}} \sqrt{\frac{1}{r_{p s}}} . \\
& \cdot \frac{r_{s p h}}{\sqrt{r_{p s} \cdot(2 n+1)}} \cdot\left(\frac{r_{s p h}^{2}}{r_{p s} r_{o b s}}\right)^{n} . \\
& \cdot \frac{i}{\sqrt{2}\left(k r_{o b s}\right)^{\frac{3}{2}}}\left(\frac{1}{\sqrt{k r_{o b s}}}+(2 n+1)\right) \\
& \cdot \sqrt{\frac{2}{n \pi \sin \theta}} \cos \left(\left(n+\frac{1}{2}\right) \theta-\frac{\pi}{4}\right)
\end{aligned}
$$

$$
\begin{gathered}
\sim \frac{1}{\sqrt{2 n+1}} \cdot \frac{1}{\sqrt{\pi}} \cdot \sqrt{\frac{\pi}{2 k}} \sqrt{\frac{1}{r_{p s}}} \cdot \\
\\
\cdot \frac{r_{s p h}}{\sqrt{r_{p s} \cdot(2 n+1)}} \cdot\left(\frac{r_{s p h}^{2}}{r_{p s} r_{o b s}}\right)^{n} \cdot \\
\cdot \frac{i}{\sqrt{2}\left(k r_{o b s}\right)^{\frac{3}{2}}}(2 n+1) \\
\cdot \sqrt{\frac{2}{n \pi \sin \theta}} \cos \left(\left(n+\frac{1}{2}\right) \theta-\frac{\pi}{4}\right) \\
\sim \frac{1}{\sqrt{2 n+1}} \cdot \sqrt{\frac{1}{2 k}} \sqrt{\frac{1}{r_{p s}}} \cdot \frac{r_{s p h}}{\sqrt{r_{p s} \cdot(2 n+1)}} \\
\cdot\left(\frac{r_{s p h}^{2}}{r_{p s} r_{o b s}}\right)^{n} \cdot \frac{i(2 n+1)}{\sqrt{2}\left(k r_{o b s}\right)^{\frac{3}{2}}} \cdot \sqrt{\frac{2}{n \pi \sin \theta}} \cos \left(\left(n+\frac{1}{2}\right) \theta-\frac{\pi}{4}\right)
\end{gathered}
$$

Using the same arguments about convergence as in the previous chapter, we observe that the above power series converges for all $r_{o b s}$ with $\left|\frac{r_{c r i}}{r_{o b s}}\right|<1$ and diverges for all $r_{o b s}$ with $\left|\frac{r_{c r i}}{r_{o b s}}\right| \geq 1$. By means of $4.17 \mid$ the latter conclusion holds also for the convergence of the series (4.23) and (4.22). This means that the series solution (4.9), originally found for points laying outside the sphere, is also convergent and meaningful inside the sphere, and until the critical radius.

Since we verified the convergence of the derivative of the scattered field, we conclude:
$\triangleright$ Not only have we extended our original solution (4.9) to the region $r_{c r i}<r_{o b s}<r_{s p h}$, in the previous section, but also this extension is in fact its analytic continuation.

### 4.5 FIELD DERIVATIVE WITH RESPECT TO $r_{o b s}$ ON THE SURFACE OF THE SPHERE

On the surface of the sphere (for $r_{o b s}=r_{s p h}$ ) we obtain:

$$
\begin{aligned}
& \frac{\partial}{\partial r_{o b s}} u^{t}\left(r_{o b s}, \theta_{o b s}\right)=\frac{\partial}{\partial r_{o b s}}\left(u^{p r}\left(r_{o b s}, \theta_{o b s}\right)+u^{s c}\left(r_{o b s}, \theta_{o b s}\right)\right) \\
& \begin{aligned}
&= \frac{\partial}{\partial r_{o b s}} \frac{1}{h_{0}\left(k r_{p s}\right)} \sum_{n=0}^{\infty}(2 n+1) h_{n}\left(k r_{p s}\right) j_{n}\left(k r_{o b s}\right) P_{n}\left(\cos \theta_{o b s}\right) \\
& \quad-\frac{\partial}{\partial r_{o b s}} \frac{1}{h_{0}\left(k r_{p s}\right)} \sum_{n=0}^{\infty}(2 n+1) \frac{h_{n}\left(k r_{p s}\right) j_{n}\left(k r_{s p h}\right)}{h_{n}\left(k r_{s p h}\right)} h_{n}\left(k r_{o b s}\right) P_{n}\left(\cos \theta_{o b s}\right) \\
&\left.=\frac{\partial}{\partial r_{o b s}} \frac{1}{h_{0}\left(k r_{p s}\right)} \sum_{n=0}^{\infty}(2 n+1) \frac{h_{n}\left(k r_{p s}\right) j_{n}\left(k r_{o b s}\right) h_{n}\left(k r_{s p h}\right)}{h_{n}\left(k r_{s p h}\right)}\right) P_{n}\left(\cos \theta_{o b s}\right) \\
& \quad-\frac{\partial}{\partial r_{o b s}} \frac{1}{h_{0}\left(k r_{p s}\right)} \sum_{n=0}^{\infty}(2 n+1) \frac{h_{n}\left(k r_{p s}\right) j_{n}\left(k r_{s p h}\right)}{h_{n}\left(k r_{s p h}\right)} h_{n}\left(k r_{o b s}\right) P_{n}\left(\cos \theta_{o b s}\right) \\
&=\frac{\partial}{\partial r_{o b s}} \frac{1}{h_{0}\left(k r_{p s}\right)}\left(\sum_{n=0}^{\infty}(2 n+1) \cdot h_{n}\left(k r_{p s}\right) \frac{j_{n}\left(k r_{o b s}\right) h_{n}\left(k r_{s p h}\right)-j_{n}\left(k r_{s p h}\right) h_{n}\left(k r_{o b s}\right)}{h_{n}\left(k r_{s p h}\right)} P_{n}\left(\cos \theta_{o b s}\right)\right) \\
&=\frac{k}{h_{0}\left(k r_{p s}\right)} \sum_{n=0}^{\infty}(2 n+1) \cdot h_{n}\left(k r_{p s}\right) \frac{\frac{\mathrm{d} j_{n}\left(k r_{o b s}\right)}{\mathrm{d} k r_{o b s}} h_{n}\left(k r_{s p h}\right)-j_{n}\left(k r_{s p h} \frac{\mathrm{~d} h_{n}\left(k r_{o b s}\right)}{\mathrm{d} k r_{o b s}}\right.}{h_{n}\left(k r_{s p h}\right)} P_{n}\left(\cos \theta_{o b s}\right) \\
&=\frac{k}{h_{0}\left(k r_{p s}\right)} \sum_{n=0}^{\infty}(2 n+1) \cdot h_{n}\left(k r_{p s}\right) \\
& \frac{\mathrm{d}}{\mathrm{~d} r_{o b s}} j_{n}\left(k r_{o b s}\right) \cdot h_{n}\left(k r_{s p h}\right)-j_{n}\left(k r_{s p h}\right) \cdot \frac{\mathrm{d}}{\mathrm{~d} k r_{o b s}} h_{n}\left(k r_{o b s}\right) \\
& h_{n}\left(k r_{s p h}\right) \\
& P_{n}\left(\cos \theta_{o b s}\right)
\end{aligned}
\end{aligned}
$$

The calculations on the above fraction are as follows:

$$
\begin{aligned}
\Leftrightarrow & \frac{j_{n}{ }^{\prime}\left(k r_{o b s}\right) h_{n}\left(k r_{s p h}\right)-j_{n}\left(k r_{s p h}\right) h_{n}{ }^{\prime}\left(k r_{o b s}\right)}{h_{n}\left(k r_{s p h}\right)} \\
& =\frac{j_{n}{ }^{\prime}\left(k r_{o b s}\right)\left(j_{n}\left(k r_{s p h}\right)+i y_{n}\left(k r_{s p h}\right)\right)-j_{n}\left(k r_{s p h}\right)\left(j_{n}\left(k r_{o b s}\right)+i y_{n}\left(k r_{o b s}\right)\right)^{\prime}}{h_{n}\left(k r_{s p h}\right)} \\
& =\frac{j_{n}{ }^{\prime}\left(k r_{o b s}\right) j_{n}\left(k r_{s p h}\right)+i j_{n}{ }^{\prime}\left(k r_{o b s}\right) y_{n}\left(k r_{s p h}\right)-j_{n}\left(k r_{s p h}\right) j_{n}{ }^{\prime}\left(k r_{o b s}\right)-i j_{n}\left(k r_{s p h}\right) y_{n}{ }^{\prime}\left(k r_{o b s}\right)}{h_{n}\left(k r_{s p h}\right)}
\end{aligned}
$$

But, because we have taken the condition for $r_{o b s}=r_{s p h}$, we obtain:

$$
\begin{aligned}
& \frac{\mathbf{j}_{\mathbf{n}}{ }^{\prime}\left(\mathbf{k r}_{\mathbf{o b s}}\right) \mathbf{j}_{\mathbf{n}}\left(\mathbf{k r}_{\mathbf{s p h}}\right)+i j_{n}{ }^{\prime}\left(k r_{o b s}\right) y_{n}\left(k r_{s p h}\right)-\mathbf{j}_{\mathbf{n}}\left(\mathbf{k r}_{\mathbf{s p h}}\right) \mathbf{j}_{\mathbf{n}}{ }^{\prime}\left(\mathbf{k r}_{\mathbf{o b s}}\right)-i j_{n}\left(k r_{s p h}\right) y_{n}{ }^{\prime}\left(k r_{o b s}\right)}{h_{n}\left(k r_{s p h}\right)} \\
& =\frac{j_{n}{ }^{\prime}\left(k r_{o b s}\right) y_{n}\left(k r_{s p h}\right)-j_{n}\left(k r_{s p h}\right) y_{n}{ }^{\prime}\left(k r_{o b s}\right)}{h_{n}\left(k r_{s p h}\right)}
\end{aligned}
$$

Using [1], [10.1.6] about the Wronskian of bessel spherical functions of fractional order, we obtain:

$$
\begin{align*}
\left.\frac{\partial}{\partial r_{o b s}} u^{t}\left(r_{o b s}, \theta_{o b s}\right)\right|_{r_{o b s}=r_{s p h}} & =\frac{k \cdot i}{h_{0}\left(k r_{p s}\right)} \sum_{n=0}^{\infty}(2 n+1) \cdot h_{n}\left(k r_{p s}\right) \frac{-\left(\frac{1}{k \cdot r_{s p h}}\right)^{2}}{h_{n}\left(k r_{s p h}\right)} P_{n}\left(\cos \theta_{o b s}\right) \\
& =-\frac{i}{k \cdot r_{s p h}^{2} \cdot h_{0}\left(k r_{p s}\right)} \sum_{n=0}^{\infty}(2 n+1) \cdot \frac{h_{n}\left(k r_{p s}\right)}{h_{n}\left(k r_{s p h}\right)} P_{n}\left(\cos \theta_{o b s}\right) \tag{4.24}
\end{align*}
$$

This is the condition satisfied on the surface of the sphere.

### 4.6 DISCRETE MAS

### 4.6.1 Definition and Geometry: Placement of Discrete Auxiliary Sources

In the discretized version of the Method of the Auxiliary Sources (MAS), the auxiliary sources (ASs) are $M \times N$ discrete point-sources generating spherical acoustic waves. Their unknown amplitudes will be determined by approximately satisfying the boundary condition (4.6), as described below. The amplitude of the field radiated by the $(m, n)$-AS will be denoted by $U_{m n}, m=0,1, \ldots, M-1, n=0,1, \ldots, N-1$. We will also use the term "currents" within quotes, due to the resemblance to the electromagnetic problem.

$$
u^{p r}\left(\mathbf{r}_{o b s}\right)=r_{p s} \exp \left(-i k r_{p s}\right) \frac{\exp \left(i k\left|\mathbf{r}_{o b s}-\mathbf{r}_{p s}\right|\right)}{\left|\mathbf{r}_{o b s}-\mathbf{r}_{p s}\right|}, \quad \mathbf{r}_{o b s} \in \mathbb{R}^{3} \backslash\left\{\mathbf{r}_{p s}\right\}
$$

The ASs lie on the surface of an auxiliary sphere of radius $r_{a u x}$ in the interior of the soft one; thus $0<r_{\text {aux }}<r_{\text {sph }}$.


Figure 4.3: Partitioning of a Sphere
More precisely, the $(m, n)$-AS is located at:

$$
\begin{equation*}
\left(r_{a u x}, \theta_{m}, \phi_{n}\right)=\left(r_{a u x}, \frac{\pi m}{M}, \frac{2 \pi n}{N}\right), \quad m=1, \ldots, M-1 \quad n=1, \ldots, N \tag{4.25}
\end{equation*}
$$

and hence the $M \times N$ ASs are equi-angled.
We denote by $R_{o b s, m n}$ the distance between the $(m, n)$-AS and the observation point, namely:

$$
\begin{equation*}
R_{o b s, m n}=\left|\mathbf{r}_{o b s}-\mathbf{r}_{m n}\right| \tag{4.26}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{r}_{o b s}=\left(r_{o b s} \sin \left(\theta_{o b s}\right) \cos \left(\phi_{o b s}\right), r_{o b s} \sin \left(\theta_{o b s}\right) \sin \left(\phi_{o b s}\right), r_{o b s} \cos \left(\theta_{o b s}\right)\right) \tag{4.27}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{r}_{m n}=\left(r_{a u x} \sin \left(\theta_{m}\right) \cos \left(\phi_{n}\right), r_{a u x} \sin \left(\theta_{m}\right) \sin \left(\phi_{n}\right), r_{a u x} \cos \left(\theta_{m}\right)\right) . \tag{4.28}
\end{equation*}
$$

Hence, we get:

$$
\begin{align*}
& R_{o b s, m n} \\
& =\sqrt{r_{o b s}^{2}+r_{a u x}^{2}-2 r_{a u x} r_{o b s}\left[\sin \left(\theta_{o b s}\right) \sin \left(\theta_{m}\right) \cos \left(\phi_{o b s}-\phi_{n}\right)+\cos \left(\theta_{o b s}\right) \cos \left(\theta_{m}\right)\right]} \tag{4.29}
\end{align*}
$$

### 4.6.2 Discrete MAS - Total Acoustic Field

The total acoustic field is the superposition of the primary field and the fields generated by all the AS's:

$$
\left.\begin{array}{rl}
u_{M A S}^{s c}\left(r_{o b s}, \theta_{o b s}, \phi_{o b s}\right) & =\binom{\text { summing pressure contributions }}{\text { of each point source }} \\
& =\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} U_{m n} h_{0}^{(1)}\left(k R_{o b s, m n}\right) \\
u_{M A S}^{t}\left(\mathbf{r}_{o b s}, \theta_{o b s}, \phi_{o b s}\right)=u^{p r}\left(\mathbf{r}_{o b s}\right)+u_{M A S}^{s c}\left(\mathbf{r}_{o b s}\right)
\end{array}\right] \begin{aligned}
& \Leftrightarrow u_{M A S}^{t}\left(\mathbf{r}_{o b s}, \theta_{o b s}, \phi_{o b s}\right)=u^{p r}\left(\mathbf{r}_{o b s}\right)+\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} u^{a u x}\left(\mathbf{R}_{o b s, m n}\right) \\
& (4.30) \\
& 4.31) \tag{4.33}
\end{aligned} \Rightarrow u_{M A S}^{t}\left(r_{o b s}, \theta_{o b s}, \phi_{o b s}\right)=u^{p r}\left(r_{o b s}, \theta_{o b s}\right)+\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} U_{m n} h_{0}^{(1)}\left(k R_{o b s, m n}\right) .
$$



Figure 4.4: Spherical Coordinates

### 4.6.3 Collocation Points and Boundary Condition

Next, we take $P \times Q$ equi-angled collocation points (CP's) on the soft sphere-surface. As shown in Fig. 2, the $(p, q)$-CP is located at $\left(r_{s p h}, \frac{\pi p}{P}, \frac{2 \pi q}{Q}\right)$, where $p=1,2, \ldots, P-1$ and $q=1,2, \ldots, Q$. By enforcing the boundary condition:

$$
\begin{equation*}
u_{M A S}^{t}\left(r_{o b s}, \theta_{o b s}, \phi_{o b s}\right)=0, \quad\left(r_{o b s}, \theta_{o b s}, \phi_{o b s}\right)=\left(r_{s p h}, \theta_{p}, \phi_{q}\right)=\left(r_{s p h}, \frac{\pi p}{P}, \frac{2 \pi q}{Q}\right) \tag{4.34}
\end{equation*}
$$

from Eq. 4.33 we get for $p=1,2, \ldots, P-1$ and $q=1,2, \ldots, Q$

$$
\begin{equation*}
\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} U_{m n} h_{0}^{(1)}\left(k R_{p q, m n}\right)=-u^{p r}\left(r_{s p h}, \theta_{p}\right)=-r_{p s} \exp \left(-i k r_{p s}\right) \frac{\exp \left(i k R_{p q, r_{p s}}\right)}{R_{p q, r_{p s}}} \tag{4.35}
\end{equation*}
$$

where:

$$
\begin{align*}
& R_{p q, m n} \equiv b_{p, q, m, n} \\
& =\sqrt{r_{s p h}^{2}+r_{a u x}^{2}-2 r_{s p h} r_{a u x}\left[\sin \left(\theta_{p}\right) \sin \left(\theta_{m}\right) \cos \left(\phi_{q}-\phi_{n}\right)+\cos \left(\theta_{p}\right) \cos \left(\theta_{m}\right)\right]} \\
& =\sqrt{r_{s p h}^{2}+r_{a u x}^{2}-2 r_{s p h} r_{a u x}\left[\sin \left(\frac{\pi p}{P}\right) \sin \left(\frac{\pi m}{M}\right) \cos \left(\frac{2 \pi q}{Q}-\frac{2 \pi n}{N}\right)+\cos \left(\frac{\pi p}{P}\right) \cos \left(\frac{\pi m}{M}\right)\right]} \tag{4.36}
\end{align*}
$$

for $p=1,2, \ldots, P-1$ and $q=1,2, \ldots, Q$,

$$
m=1,2, \ldots, M-1 \text { and } n=1,2, \ldots, N, \text { as well as }
$$

$$
\begin{align*}
R_{p q, r_{p s}} \equiv d_{p, q} & =\sqrt{r_{s p h}^{2}+r_{p s}^{2}-2 r_{s p h} r_{p s}\left[\sin \left(\theta_{p}\right) \sin (0) \cos \left(\phi_{q}-\phi_{n}\right)+\cos \left(\theta_{p}\right) \cos (0)\right]} \\
& =\sqrt{r_{s p h}^{2}+r_{p s}^{2}-2 r_{s p h} r_{p s} \cos \left(\frac{\pi p}{P}\right) \equiv d_{p}} . \tag{4.37}
\end{align*}
$$

for $p=1,2, \ldots, P-1$ and $q=1,2, \ldots, Q$,

$$
m=1,2, \ldots, M-1 \text { and } n=1,2, \ldots, N .
$$

### 4.6.4 Formulation of the Linear System to be solved in terms of the amplitudes $U_{m n}$

Therefore, the linear system of equations to be solved with unknowns the amplitudes of the point sources ("MAS currents" $U_{m, n}$ ) is given by (4.35), using 4.36) and (4.37), as:

$$
\begin{equation*}
\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} U_{m n} h_{0}^{(1)}\left(k b_{p, q, m, n}\right)=-r_{p s} \exp \left(-i k r_{p s}\right) \frac{\exp \left(i k d_{p}\right)}{d_{p}} \tag{4.38}
\end{equation*}
$$

Using the above relation we form a linear system

$$
\begin{equation*}
A \cdot \overline{\mathbf{x}}=\overline{\mathbf{g}} \tag{4.39}
\end{equation*}
$$

with an "A" square matrix of dimensions $(P-1) Q \times(M-1) N$ :
$\left[\begin{array}{ccccccc}h_{0}^{(1)}\left(k b_{1,1,1,1}\right) & \ldots & h_{0}^{(1)}\left(k b_{1,1,1, N}\right) & \ldots & h_{0}^{(1)}\left(k b_{1,1, M-1,1}\right) & \ldots & h_{0}^{(1)}\left(k b_{1,1, M-1, N}\right) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{0}^{(1)}\left(k b_{1, Q, 1,1}\right) & \ldots & h_{0}^{(1)}\left(k b_{1, Q, 1, N}\right) & \ldots & h_{0}^{(1)}\left(k b_{1, Q, M-1,1}\right) & \ldots & h_{0}^{(1)}\left(k b_{1, Q, M-1, N}\right) \\ h_{0}^{(1)}\left(k b_{2,1,1,1}\right) & \ldots & h_{0}^{(1)}\left(k b_{2,1,1, N}\right) & \ldots & h_{0}^{(1)}\left(k b_{2,1, M-1,1}\right) & \ldots & h_{0}^{(1)}\left(k b_{2,1, M-1, N}\right) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{0}^{(1)}\left(k b_{2, Q, 1,1}\right) & \ldots & h_{0}^{(1)}\left(k b_{2, Q, 1, Q}\right) & \ldots & h_{0}^{(1)}\left(k b_{2, Q, M-1,1}\right) & \ldots & h_{0}^{(1)}\left(k b_{2, Q, M-1, N}\right) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ h_{0}^{(1)}\left(k b_{P-1,1,1,1}\right) & \ldots & h_{0}^{(1)}\left(k b_{P, 1,1, N}\right) & \ldots & h_{0}^{(1)}\left(k b_{P, 1, M-1,1}\right) & \ldots & h_{0}^{(1)}\left(k b_{P, 1, M-1, N}\right) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ h_{0}^{(1)}\left(k b_{P-1, Q, 1,1}\right) & \ldots & h_{0}^{(1)}\left(k b_{P-1, Q, 1, N}\right) & \ldots & h_{0}^{(1)}\left(k b_{P-1, Q, M-1,1}\right) & \ldots & h_{0}^{(1)}\left(k b_{P-1, Q, M-1, N}\right)\end{array}\right]$
where the unknowns are the $U_{m, n}$ "MAS currents" and their vector is:

$$
\bar{x}=\left(U_{1,1}, U_{1,2}, U_{1,3}, \ldots, U_{1, N-1}, U_{2,1}, U_{2,2}, U_{2,3}, \ldots, U_{M-1,1}, U_{M-1,2}, \ldots, U_{M-1, N-1}\right)^{\mathrm{T}}
$$

Also:

$$
\begin{aligned}
& \bar{g}=-r_{p s} \exp \left(-i k r_{p s}\right)\left(\frac{\exp \left(i k d_{1}\right)}{d_{1}}, \ldots, \frac{\exp \left(i k d_{1}\right)}{d_{1}},\right. \\
& \frac{\exp \left(i k d_{2}\right)}{d_{2}}, \ldots, \frac{\exp \left(i k d_{2}\right)}{d_{2}}, \\
& \left.\frac{\exp \left(i k d_{P-1}\right)}{d_{P-1}}, \ldots, \frac{\exp \left(i k d_{P-1}\right)}{d_{P-1}}\right)^{\mathrm{T}}
\end{aligned}
$$

where the dots indicate a number of $Q$ terms in total. Note that for each pair of values $\left(p_{0}, m_{0}\right)$, if $\boldsymbol{Q}=\boldsymbol{N}$, it holds that

$$
\diamond b_{p_{0}, q, m_{0}, n}=b_{p_{0}, q-n, m_{0}, 0}=b_{p_{0}, 0, m_{0}, q-n}=b_{p_{0}, n-q, m_{0}, 0} .
$$

Hence, the MAS matrix is composed of $(M-1) \times(P-1)$ circulant blocks, where each block is a $Q \times N$ matrix.

## Chapter 5

## Numerical Results and Discussion

This chapter contains representative numerical results and the main conclusion drawn from them. Included are studies of the effects of changing the main paremetres of the problem.

### 5.1 PARTITIONING OF THE SPHERICAL SURFACE

There are four fundamental parameters that we investigated using our programme:
$\triangleright M$ : the number of sources for each constant $\phi$ (and varying $\theta$ )
$\triangleright N$ : the number or sources for each constant $\theta$ (and varying $\phi$ )
$\triangleright r_{a u x}$ : the radius of the surface containing the auxiliary sources
$\triangleright r_{s p h}$ : the radius of the spherical acoustic scatterer
$\triangleright r_{p s}$ : the position of the point source of the primary field
We have used a quite simple partitioning of the spherical surface in $(M-1) \times(N-1)$ regions, by placing $M \times N$ points on it as seen in figure (5.1).


Figure 5.1: Partitioning a Spherical surface
The problem has a $\phi$-symmetry and the reason for this is the placement of the point source: it is located on position $\mathbf{r}_{\mathbf{p s}}=(x, y, z)=\left(0,0, r_{p s}\right)$, on the $z$-axis.

### 5.2 VALIDATION OF THE NUMERICAL CODE

We have developed a numerical code and have tested it by means of the following two checks:
I. We selected a set of points, called "median collocation points" (midCPs), on the auxiliary surface $r_{a u x}$. They are placed between the collocation points mentioned in the previous chapter, in section "1.4.3. Collocation points and boundary condition", i.e. at points

$$
\begin{aligned}
(r, \theta, \phi)=\left(r_{\text {aux }}, \frac{\pi m}{M}, \frac{2 \pi n}{N}\right), & m=1,2, \ldots, M-1 \\
n & =1,2, \ldots, N-1
\end{aligned}
$$

We tested to see whether the norm of the "currents" $U$ is less than a very small threshold and found that this is indeed the case.
II. We tested and finally verified that the exact field obtained from the respective formula of the previous chapter

$$
\begin{align*}
u_{e x}^{t}\left(\mathbf{r}_{o b s}\right)=u^{p r}\left(\mathbf{r}_{o b s}\right)- & \frac{1}{h_{0}\left(k r_{p s}\right)} \sum_{n=0}^{\infty}(2 n+1) \frac{h_{n}\left(k r_{p s}\right) j_{n}\left(k r_{s p h}\right)}{h_{n}\left(k r_{s p h}\right)} \\
& \times h_{n}\left(k r_{o b s}\right) P_{n}\left(\cos \theta_{o b s}\right), \quad r_{o b s}>r_{s p h}, \quad r_{o b s} \neq r_{p s} . \tag{5.1}
\end{align*}
$$

coincides with the MAS field calculated in the same chapter

$$
\begin{equation*}
u_{M A S}^{t}\left(r_{o b s}, \theta_{o b s}, \phi_{o b s}\right)=u^{p r}\left(r_{o b s}, \theta_{o b s}\right)+\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} U_{m n} h_{0}^{(1)}\left(k R_{o b s, m n}\right) \tag{5.2}
\end{equation*}
$$

### 5.3 MAS "CURRENTS"

### 5.3.1 Changing the radius-parameter: $r_{\text {aux }}$

Writing some functions in version 7.10.0.499 (R2010a) Matlab with double precision arithmetic for solving the linear system of equations

$$
\begin{equation*}
A \overline{\mathbf{x}}=\overline{\mathbf{g}} \tag{5.3}
\end{equation*}
$$

of the previous chapter, we can plot the discrete MAS "currents" $U_{m n}$ (that is the amplitudes of the acoustic waves radiated by the point sources located on the auxiliary spherical surface), where:

$$
\begin{equation*}
\frac{U_{m n}}{\Delta S}=\frac{U_{m n}}{r_{a u x}^{2} \sin \theta_{a u x} \Delta \theta \Delta \phi}=\frac{U_{m n}}{r_{a u x}^{2} \sin \theta_{\text {aux }} \frac{\pi}{M} \frac{2 \pi}{N}} \tag{5.4}
\end{equation*}
$$

In the curves that appear on figures (5.2)-(5.3) the discrete points have been joined by straight lines. Besides, the horizontal axis is expressed by the serial number $\#(m, n)$ of each source.

We consider the following placings of the auxiliary point-sources:

- Case 1: $r_{a u x} \in\left(r_{c r i}, r_{s p h}\right)$ In the figures (5.2), (5.3), (5.4), (5.5), (5.6), (5.7) where

$$
\begin{equation*}
r_{c r i}=\frac{\left(r_{s p h}\right)^{2}}{r_{p s}} \tag{5.5}
\end{equation*}
$$

the parameters have been set as follows:

$$
\begin{aligned}
& \left.\begin{array}{c}
k r_{s p h}=2.1 \\
k r_{p s}=3.0
\end{array}\right\} \xrightarrow[\text { definition }]{\text { critical radius }} k r_{c r i}=1.47 \\
& k r_{\text {aux }}=1.80 \\
& \# \text { sources }: M \times N=37 \times 14
\end{aligned}
$$



Figure 5.2: Real part of "Currents"
In the pair of figures 5.2 -(5.3) are shown the real and the imaginary parts (in each figure respectively) of normalized MAS "currents" $U_{m, n}$ as function of element number $(m, n)$.
From the beginning of each curve, we observe that every 14 "currents" have the same value (the curve is "flat" for every 14 values). This is due to the problem's $\phi$-symmetry We recall that $N$ is the number of sources for constant $\theta$ (varying $\phi$ ); we will clarify later (in Section 5.5) that there is no need to increase this number, because, as will be


Figure 5.3: Imaginary part of "Currents"


Figure 5.4: Real part of "Currents"
seen in figure (5.42) below, for 14 sources $(N=14)$ the exact and MAS field are found


Figure 5.5: Imaginary part of "Currents"
to coincide.
Moreover, in figures (5.4)-(5.5) we see the "current" values across a ring of a fixed $\theta$ value, that is, as a function of variable $\phi_{a u x}$.

We observe that these values are very close, verifying in more detail the currents' independence of $\phi$ coordinate.
Figures (5.6)-5.7) show the normalized MAS "current" values as function of $\theta_{\text {aux }}$ (the horizontal axis is expressed by the continuous variable $\theta_{a u x}$ ). The continuous curve shown has resulted by connecting all values by straight line segments. We can see that this curve is smooth, therefore adjacent currents have close values. As we will see, the scattered acoustic field resulting from these currents, according to the previous chapter's formula about the MAS field, is created with great accuracy and as the number of sources increases, this accuracy will increase (see next subsection 5.3.2).
$\triangleright$ Such stable behaviour of the currents is certainly to be expected from a valid numerical method.


Figure 5.6: Real part of "Currents"


Figure 5.7: Imaginary part of "Currents"

- Case 2: $r_{a u x} \in\left(0, r_{c r i}\right)$ In this case, we choose the parametres:

$$
\begin{aligned}
&\left.\begin{array}{rl}
k r_{s p h} & =2.1 \\
k r_{p s} & =3.0
\end{array}\right\} \xrightarrow[\text { definition }]{\text { critical radius }} k r_{c r i}=1.47 \\
& k r_{\text {aux }}=1.25
\end{aligned} \begin{aligned}
& \text { \#sources : } M \times N=25 \times 14
\end{aligned}
$$

Here, in the case of $r_{\text {aux }}$ being less than the critical radius, we solve the linear system (5.3) again and obtain these MAS currents as solution. We choose to place again no more than 14 sources on each ring of every fixed $\theta$. In figures (5.8)-(5.9) we have again all normalized MAS currents presented as function of element number $(m, n)$.


Figure 5.8: Real part of all "Currents"

Observing (5.10)-5.11), we can verify again the $\phi$-symmetry in this problem., because all currents in each ring of constant $\theta$ have the same values (they are 14 in number in each ring).
In the pair (5.10)-5.11) of figures showing the normalized MAS currents on a fixed $\theta$, all currents should be equal (we should observe horizontal lines in the real and imaginary part) due to the $\phi$-symmetry . However there are some small differences beyond the fourth significant digit, which we can consider negligible and are due to round-off errors in solving the linear system of equations (5.3) . In other words, the horizontal scale in figures (5.10-(5.11) is very fine.


Figure 5.9: Imaginary part of all "Currents"


Figure 5.10: Real part of all "Currents"
$\rightarrow$ In comparison with the case of $r_{a u x}>r_{c r i}$ of (5.6)-(5.7), we can observe in (5.12)-


Figure 5.11: Imaginary part of all "Currents"


Figure 5.12: Real part of all "Currents"
(5.13) that in each adjacent pair, the normalized MAS currents are now oscillating


Figure 5.13: Imaginary part of all "Currents"
between positive and negative values. This leads to one of the most important conclusions of this work:

The values $\operatorname{Re}\left\{U_{1, n}\right\}, \operatorname{Re}\left\{U_{2, n}\right\}, \operatorname{Re}\left\{U_{3, n}\right\}, \ldots$, as well as the $\operatorname{Im}\left\{U_{1, n}\right\}, \operatorname{Im}\left\{U_{1, n}\right\}$, $\operatorname{Im}\left\{U_{1, n}\right\}, \ldots$

- have significantly large absolute values, compared with the values in (5.6)-(5.7)
- alternate in sign
indicating that the normalized MAS "currents" do not converge (if we choose to increase the number of sources, so that $M \rightarrow \infty)$.


### 5.3.2 Changing the parameter $M$ (number of auxiliary sources on constant $\phi$ )

In the following figures for both cases $r_{a u x}>r_{c r i}$ and $r_{a u x}<r_{c r i}$ we demonstrate the variation with $M$ (total number of point sources on radius $r_{a u x}$ ). We choose to increase this number only with respect to $\theta$ variable and keep $N=14$ sources on each spherical ring.

- Case 1: $r_{a u x} \in\left(r_{c r i}, r_{s p h}\right)$ When $r_{c r i}<r_{a u x}<r_{s p h}$, we expect the normalized MAS currents on the auxiliary surface (5.4) to converge all to a true limit for $M \rightarrow \infty$. That is:

$$
\begin{equation*}
\lim _{M \rightarrow \infty} \frac{U_{m n}}{\Delta S}=J^{s} \tag{5.8}
\end{equation*}
$$

$\triangleright$ Although we do not have an explicit formula for $J_{s}$ (as we did in chapter 3) we will provide numerical evidence for the validity of (5.8).

In the figures shown, the parameters have been set as follows:

$$
\begin{aligned}
&\left.\begin{array}{rl}
k r_{s p h} & =2.1 \\
k r_{p s} & =3.0
\end{array}\right\} \xrightarrow[\text { definition }]{\text { critical radius }} k r_{c r i}=1.47 \\
& k r_{\text {aux }}=1.80 \\
& \# \text { sources }: M \times N=27 \times 14
\end{aligned}
$$



Figure 5.14: Real part of all "currents"


Figure 5.15: Imaginary part of all "currents"


Figure 5.16: Real part of all "currents"

We observe that as we add more sources, the curve showing the normalized "currents"


Figure 5.17: Imaginary part of all "currents"


Figure 5.18: Real part of all "currents"
(compare the pair (5.14)-(5.15) and (5.20)-(5.21) becomes smoother and seems to ap-


Figure 5.19: Imaginary part of all "currents"
proach a limit $J^{s}$ of (5.8), which is the curve of the continuous MAS problem.
We cannot increase the number of sources above the threshold that the condition number $r_{\text {cond }}$ of our square matrix $A$ (of our linear system) dictates (no more than $10^{16}$ as discussed in section (5.4)) below. If we do, then ill-conditioning in solving our system appears and the consequent round-off errors may spoil our conclusions.

$$
\begin{aligned}
& \left.\begin{array}{r}
k r_{s p h}=2.1 \\
k r_{p s}=3.0
\end{array}\right\} \xrightarrow[\text { definition }]{\text { critical radius }} k r_{c r i}=1.47 \\
& \begin{aligned}
& k r_{\text {aux }}=1.80 \\
& \# \text { sources }: M \times N=47 \times 14
\end{aligned}
\end{aligned}
$$



Figure 5.20: Real part of all "currents"


Figure 5.21: Imaginary part of all "currents"


Figure 5.22: Real part of all "currents"


Figure 5.23: Imaginary part of all "currents"


Figure 5.24: Real part of all "currents"


Figure 5.25: Imaginary part of all "currents"

- Case 2: $r_{\text {aux }} \in\left(0, r_{\text {cri }}\right)$ In this case, that the auxiliary surface of the discrete sources lies within the region of $\left(0, r_{c r i}\right)$, we should expect that the limit of (5.4), which is:

$$
\begin{equation*}
\lim _{M \rightarrow \infty} \frac{U_{m n}}{\Delta S}=\lim _{M \rightarrow \infty} \frac{U_{m n}}{r_{a u x}^{2} \sin \theta_{\text {aux }} \frac{\pi}{M} \frac{2 \pi}{N}} \tag{5.11}
\end{equation*}
$$

for $M \rightarrow \infty$ diverges. We provide numerical evidence for this in figures (5.30)-(5.31), (5.36)-(5.37).

Our parametres are:


Figure 5.26: Real part of all"currents"
Figures (5.32), (5.33), (5.36), (5.37), (5.34), (5.35) are believed to be free of ill-conditioning effects. It is not possible to increase $M$ much more, because the condition numbers become prohibitively large.


Figure 5.27: Imaginary part of all "currents"


Figure 5.28: Real part of all "currents"


Figure 5.29: Imaginary part of all "currents"


Figure 5.30: Real part of all "currents"


Figure 5.31: Imaginary part of all "currents"

Now, we increase the number of the sources to the largest possible value, so that the condition number does not imply any matrix ill-conditioning. Parametre setting is:

$$
\begin{aligned}
& \left.\begin{array}{r}
k r_{s p h}=2.1 \\
k r_{p s}=3.0
\end{array}\right\} \xrightarrow[\text { definition }]{\text { critical radius }} k r_{c r i}=1.47 \\
& \begin{array}{l}
k r_{\text {aux }}=1.25 \\
\# \text { sources }: M \times N=27 \times 14
\end{array}
\end{aligned}
$$



Figure 5.32: Real part of all "currents"
We observe that the MAS currents present an abnormal behaviour as the number of sources for varying $\theta$ grows (compare (5.26)-(5.27) and (5.32)-(5.33)): They oscillate more and more rapidly and at the same time they increase in value for both their real and imaginary part.
$\triangleright$ It is worth mentioning that, according to (5.30-(5.31) and (5.36)-5.37) this phenomenon appears more for $\theta=0$ (close to the 'North pole', where our point source is located).

Besides, the initial values $\operatorname{Re}\left\{I_{1, n}\right\}, \operatorname{Re}\left\{I_{2, n}\right\}, \operatorname{Re}\left\{I_{3, n}\right\}, \ldots$ etc, as well as $\operatorname{Im}\left\{I_{1, n}\right\}$, $\operatorname{Im}\left\{I_{2, n}\right\}, \operatorname{Im}\left\{I_{3, n}\right\}, \ldots$ etc alternate in sign and, also, grow in absolute value (compare (5.30)-(5.31) and (5.36)-(5.37)).


Figure 5.33: Imaginary part of all "currents"


Figure 5.34: Real part of "currents"


Figure 5.35: Imaginary part of "currents"


Figure 5.36: Real part of "currents"


Figure 5.37: Imaginary part of "currents"
$\rightarrow$ Therefore, the normalized MAS currents do not converge for increasing $M$ (ideally we should demostrate this for $M \rightarrow \infty$, but we cannot increase $M$ indefinitely, because of round-off errors).

Further evidence for the divergence of the limit in (5.11) is presented in figures (5.38) and (5.38). In these figures, the parameters have been set as follows:

$$
\left.\begin{array}{rl}
\begin{array}{c}
k \cdot r_{s p h}
\end{array}=2.1 \\
k \cdot r_{p s} & =3.0
\end{array}\right\} \xrightarrow[\text { definition }]{\text { critical radius }} k \cdot r_{c r i}=1.47 .
$$

It is seen that the curves remain almost the same as M increases.


Figure 5.38: Plottings 1


Figure 5.39: Plottings 1

### 5.4 BEHAVIOUR OF THE CONDITION NUMBER

### 5.4.1 Changing the number of sources

A measure of possible ill-conditioning of the linear system $A x=g$ is the condition number defined as $r_{\text {cond }}=\|A\|\|A\|^{-1}$, where $\|A\|$ denotes the two-norm condition number (particuarly in our Matlab implementation we have used the function "cond $(\mathrm{X})$ ", which returns as 2-norm condition number: the ratio of the largest singular value of X to the smallest). It is a quantity that measures the sensitivity of the solution of a system of linear equations to errors in the data. It gives an indication of the accuracy of the results from matrix inversion and the linear equation solution. Values of cond(X) near 1 indicate a well-conditioned matrix, but, if too high, they imply that the linear system solution will suffer from poor accuracy.

In figure (5.40), we can observe the behaviour of the condition number of our linear system as a function of the number of sources (we increase the $M$ sources for varying $\theta$ variable (constant $\phi$ ) and we keep $N=14$ for a placing of sources in $r_{a u x}=1.80>r_{c r i}$. It is obvious that for a small number of sources (from $M=3$ to $M=30$ ) the logarithmic curve of the condition number approximates a straight line. That means that the condition number increases quite rapidly (exponentially) as the number of MAS sources increases linearly.

When the number of sources increases beyond $M=130$, the condition number increases beyond the value $10^{15}$ which can be considered as a threshold for the accurate solution of our system for our implementation in version 7.10.0.499 (R2010a) Matlab with double precision arithmetic. In all of our numerical experiments we make sure that the number or discrete auxiliary sources we impose is such that the condition number remains significantly lower than this threshold.

In figure (5.41) we set $r_{a u x}=1.25<r_{c r i}$. We observe that the condition-number stops being linear above the threshold of $10^{17}$.

Again, while experimenting with the case of $r_{a u x}<r_{c r i}$ and plotting the MAS currents, we should make sure that the number of sources is such that the condition number of our matrix remains less than this threshold.


Figure 5.40: Condition Number 1


Figure 5.41: Condition Number 2

### 5.5 COMPARING EXACT vs DISCRETE SOLUTION OF THE SCATTERED FIELD

The exact scattered pressure field $u_{e x}^{t}$ is obtained by the infinite series of the previous chapter retaining a large number of terms On the other hand, the MAS scattered component of the acoustic field $u_{\text {MAS }}^{t}$ is taken from the finite sum in the same chapter where the MAS "currents" (amplitudes) $U_{m n}$ are found from the solution of our linear system (5.3).

- Case 1: $r_{a u x} \in\left(r_{c r i}, r_{s p h}\right)$

In the figures shown, the parameters have been set as follows:

$$
\begin{aligned}
& \left.\begin{array}{r}
k r_{s p h}=2.1 \\
k r_{p s}=3.0
\end{array}\right\} \xrightarrow{r_{c r i}} k r_{c r i}=1.47 \\
& k r_{a u x}=1.80 \\
& \# \text { sources }: M \times N=17 \times 14
\end{aligned}
$$



Figure 5.42: Real Part of field $u$
$\diamond$ It is seen that the analytical and MAS Fields coincide. This is an expected result. The corresponding relative errors are shown in figure (5.44).


Figure 5.43: Imaginary Part of field $u$


Figure 5.44: Error between Exact and MAS Field

- Case 2: $r_{a u x} \in\left(0, r_{c r i}\right)$

Here, the parametres are chosen as follows:

$$
\begin{aligned}
& \left.\begin{array}{r}
k \cdot r_{s p h}=2.1 \\
k \cdot r_{p s}=3.0
\end{array}\right\} \xrightarrow{r_{c r i}} r_{c r i}=1.47 \\
& \begin{array}{l}
k \cdot r_{\text {aux }}=1.25 \\
\# \text { sources }: M \times N=25 \times 14
\end{array}
\end{aligned}
$$



Figure 5.45: Real part of fields
This case of $r_{\text {aux }}<r_{\text {cri }}$ is the most interesting of the two, because as is illustrated by figures (5.45)-(5.46) the MAS-acoustic field, in the limit of $M \rightarrow \infty$
$\diamond$ exists for $r_{o b s}>r_{s p h}$. Besides, on the above figures (5.45)-(5.46) we can see that the MAS field produced by a small number of sources coincides with the exact field $u_{e x}$.
$\diamond$ has been generated by the diverging and oscillating acoustic currents of figures (5.30)-5.31), (5.36)-5.37). Thus, the correct and exact field is obtained from the abnormal MAS "currents".


Figure 5.46: Imaginary part of fields


Figure 5.47: Error between fields

We summarize the most important conclusions in the following table:

| $r_{\text {aux }}$ location | Normalized Series Term ("Current") <br> $\left(U_{m, n}\right)$ as solution to the linear (MAS) <br> system | MAS Field $u_{M A S}^{t}$ |
| :--- | :--- | :--- |
| $r_{\text {aux }}>r_{\text {cri }}$ | $\frac{U_{m n}}{r_{\text {aux }}^{2} \sin \left(\theta_{\text {aux }}\right) \frac{\pi}{M} \frac{2 \pi}{N}} \xrightarrow{M, N \rightarrow \infty} J^{s}\left(\theta_{\text {aux }}, \phi_{\text {aux }}\right)$ | it converges |
| $u_{M A S}^{t}(M, N) \quad \rightarrow$ |  |  |
| $u_{M A S}^{t} \equiv u_{e x}^{t}$ |  |  |

where $u^{e x}$ is the true (exact) field.

### 5.6 DISCUSSION ABOUT DIFFERENT GEOMETRIES AND GENERALIZATION OF THE CONCLUSIONS

In the above problem, we have applied the MAS to a soft spherical scatterer which was illuminated externally. We have seeked to approximately satisfy the boundary condition on the soft surface using $(M-1) \times N$ auxiliary sources located inside it $\left(r_{a u x}<r_{s p h}\right)$. We expect that our conclusions about the divergence of the normalized "current" density on this auxiliary surface and the convergence of the acoustic field (both in the case of the ( $r_{\text {aux }}<r_{c r i}$ ), with $r_{c r i}$ defined above) remain valid in more complicated geometries. Therefore, if we study a scattering problem with a geometry not allowing knowledge of the solution a priori, it is possible that use of MAS leads to "abnormal" oscillations or divergence of the solution, which may or may not be due to round-off. It is important that one verifies (by means of numerical methods) if this abnormal behaviour is a result of matrix ill-conditioning or not. Similar "abnormal" phenomena are likely to happen in more complicated geometries. It is important to distinguish between the abnormal results herein and abnormal results due to round-off error: Results due to matrix ill-conditioning can possibly be overcome by more powerful computers, while oscillations discussed in this thesis cannot.

## Chapter 6

## Conclusions and Prospect

### 6.1 CONCLUSIONS FROM THE 3D PROBLEM

The three main conclusions drawn from the numerical results in this diploma thesis are, for the case of 3-D problems, new (to the best of our knowledge). A "MAS" user should be considerate of where to place the auxiliary surface of sources (a spherical one, of radius $r_{\text {aux }}$, in our case) with reference to the critical spherical surface (defined by $r_{c r i}=\frac{\left(r_{s p h}\right)^{2}}{r_{p s}}$ ). According to what has been studied thoroughly in the previous section, the conclusions are:
$\diamond$ the auxiliary "currents" may oscillate;
$\diamond$ these oscillations are not due to roundoff or matrix ill-conditioning; and
$\diamond$ we obtain the correct acoustic field despite the oscillations.
Specifically:
$\star$ When the auxiliary surface is located in the region $r_{a u x} \in\left(r_{c r i}, r_{s p h}\right)$, the field obtained from the "MAS currents" is the true field (or its analytic continuation) for all observation points outside the auxiliary source. As $M, N \rightarrow \infty$, the limit of the normalized, discrete MAS currents

$$
\lim _{M, N \rightarrow \infty} \frac{U_{m n}}{\Delta S}=\lim _{M, N \rightarrow \infty} \frac{U_{m n}}{r_{a u x}^{2} \sin \theta_{\text {aux }} \frac{\pi}{M} \frac{2 \pi}{N}}
$$

exists.
$\star$ When the auxiliary surface is located in area $r_{a u x} \in\left(0, r_{c r i}\right)$, for any finite $M, N$, one can find the discrete MAS "currents" $U_{m, n}$ and, from these, subsequently determine the acoustic field. In the limit $M, N \rightarrow \infty$, the normalized MAS currents diverge, while the electric field does converge to the correct electric field.
Numerical results showed that the divergence appears as oscillations near $\theta=0$ in the plot of $U_{m, n}$ (that is, at points closest to the point source). The oscillations are almost certainly not due to roundoff errors or matrix ill-conditioning Despite the oscillations, one still obtains the true field for all observation points outside the sphere (in the area $\left.\left(r_{s p h}, \infty\right)\right)$.

The thesis therefore describes a difficulty (namely, oscillations) associated with MAS. The main advantages of illustrating a difficulty via a simple problem are two:
$\triangleright$ If the difficulty occurs in a simple problem, it is also likely to occur in more complicated problems.
$\triangleright$ It is less likely to confuse the said difficulty with other difficulties (namely, effects due to roundoff, matrix ill-conditioning, or shape elongation).

### 6.2 FUTURE WORK

For future work in our scattering boundary value problem there remains the "continuous" MAS version to be studied. Also, one can study the same geometry, but different type of boundary conditions (i.e. trasmission conditions, as described in chapter 1). Finally, an interesting work could be done on the 3D vectorial problem: the electromagnetic version of our scattering problem.

## Bibliography

[1] I. Abramowitz and I. A. Stegun. Handbook of Mathematical Functions With Formulas, Graphs and Mathematical Tables. Dover, New York, National Bureau of Standards Applied Mathematics Series, Vol. 55, Washington D.C.: U. S. Government Printing Office, 1965. .
[2] G. Dassios and G. Kamvyssas. Point source excitation in direct and inverse scattering: the soft and the hard small sphere. 67-84, 55, IMA J. Appl. Math., 1995. .
[3] Kleinman Ralph Dassios George. Low Frequency Scattering. Clarendon Press, Oxford Mathematical Monographs, Oxford University Press: Great Clarendon Street, Oxford OX2 6DP, Inc., 2000. ISBN: 019853678X, 9780198536789.
[4] Fikioris G. On Two Types of Convergence in the Method of Auxiliary Sources. 2022-2033, 54, IEEE Trans. Antennas Propag., 2006. .
[5] D. I. Kaklamani and H. T. Anastassiu. "Aspects of the Method of Auxiliary Sources (MAS) in computational electromagnetics,". IEEE Antennas Propag. Mag., vol. 44, no. 3, pp. 48-64,, Jun. 2002.
[6] Tsitsas NL and Athanasiadis C. Point-source excitation of a layered sphere: direct and far-field inverse scattering problems. 549-580, 61, Quart. J. Mech. Appl. Math., 2008. .
[7] A. Sommerfeld. Partial Differential Equations in Physics. Academic Press, 1949. .
[8] J. A. Stratton. Electromagnetic Theory. McGraw-Hill, 1941. .
[9] wikipedia. Legendre Link. 2010. .
[10] Wikipedia. Spherical coordinate system. Internet Site, 2013. .

