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Nonlinear Water Waves: Comparison of different variational methods

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"Nonlinear Water Waves: Comparison of different variational methods"

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Summary

Modelling of water waves is an extensive domain of research which plays an important role in many engineering applications such as wave run-up on coasts and structures or wave-body interaction (especially ship dynamics). Water waves (the terms surface waves and gravity waves are also in use) are created normally by a gravitational force in the presence of a free surface along which the pressure is constant. An important feature of this problem is that propagation phenomena take place in horizontal directions, and non-local couplings (wave-wave and seabed-wave) exist through the vertical structure of the flow field.

In this thesis we consider the problem of evolution of waves on the surface of an inviscid, incompressible fluid over an arbitrary seabed topography under the influence of gravity. The Eulerian description of the fluid is adopted, i.e. the motion of the fluid particles is determined by the velocity field in the domain occupied by water at every moment of time.

In the first chapter the equations of the physical problem are presented. The flow is considered irrotational. The unknown physical quantities involved are the velocity potential and the free surface elevation. The equations governing the physical problem are, a linear partial differential equation (Laplace), for the velocity potential, in a time dependent domain bounded from above by the free surface, and a fixed bottom surface, from below. A homogenous Neumann condition have to be satisfied on the bottom surface and two non linear boundary conditions on the free surface. Time differentiation appears only in the two equations on the free surface.

In the second chapter the Hamiltonian formulation of the problem is presented. The Hamiltonian which equals the total energy is written in terms of the free surface elevation and the trace of the velocity potential on the free surface, by means of an appropriate Dirichlet to Neumann operator. The evolution of the system is described by means of surface quantities which play the role of generalized coordinates and momenta. This is possible only if we take into account a priori, the fact that the flow is incompressible and irrotational and the bottom surface impermable. We then derive Hamilton's equations by varying the Hamiltonian. The resulting system is two non-linear and non-local evolution equations in terms of surface quantities.

In the third chapter another variational principle of the water wave problem (Luke's Variational Principle), is presented. In this formulation, apart from irrotationality, no other kinematic conditions have to be satisfied a priori since all the equations presented in Chapter 1 are derived by the variational principle. The

Lagrangian density involved is the integral of the pressure over the whole time-dependent domain. In the last section of this chapter the connection between Luke's variational principle and Hamilton's principle is established.

In the fourth chapter we apply Luke's variational principle in conjunction with an accurate consistent representation of the velocity potential. The representation is an infinite series expansion in terms of vertical functions with fast decaying coefficients which is compatible with the bottom and free-surface conditions. The main feature is that it permit us to represent exactly the velocity potential in the variational principle. The end result is a reformulation of the fully non-linear water wave problem as a system of two evolution equations and an infinite coupled mode system of ODE's (2D-case) or PDE's (3D-case). A biproduct of this analysis is a novel representation of the DtN operator, which is exact, general and convenient for the case of general bathymetry.

Contents

1 Physical context - Differential formulation of the water wave problem

In this chapter we summarize briefly the formulation of the problem of water waves which can be found in many books (see eg. [Wit74, Ch. 13, s. 1], [Sto57, Ch. 1, s. 1], [Lam75, Ch. 9, s. 227]) in literature.

Consider an inviscid incompressible fluid (water) in a constant gravitational field over a general bathymetry. The horizontal and vertical space coordinates are denoted by $\boldsymbol{x} := (x_1, x_2)$ and z respectively and the corresponding components of the velocity vector $\boldsymbol{u}$ by (u_1, u_2, v). The gravitational acceleration $\boldsymbol{g}$ is in the negative z direction. We assume in addition that the density ρ remains constant and that there is an external force $\boldsymbol{F} = -\rho g(0, 0, 1)$. We can write the equations of motion (Euler Equations)

$$\frac{D\boldsymbol{u}}{Dt} = \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} = -\frac{1}{\rho}\nabla p - g(0, 0, 1), \tag{1.1}$$

which together with the *continuity equation* (recall that $\partial_t \rho = 0$)

$$\operatorname{div}\boldsymbol{u} = \partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_z v = 0, \tag{1.2}$$

constitute a system of partial differential equations that once appropriate boundary and initial conditions are specified can provide us the velocity $\boldsymbol{u} = (u_1, u_2, v)$ and pressure p.

In the rest of the thesis the fluid flow is assumed to be *irrotational* (i.e. $\operatorname{curl}\boldsymbol{u} = \nabla \times \boldsymbol{u} = 0$), so the existence of a singled valued potential $\Phi(\boldsymbol{x}, z, t)$ is assured from which the velocity field is derived

$$\boldsymbol{u} = \nabla\Phi \equiv (\partial_{x_1}\Phi, \partial_{x_2}\Phi, \partial_z\Phi). \tag{1.3}$$

From (1.2), Φ satisfies the Laplace equation

$$\Delta\Phi = \partial^2_{x_1x_1}\Phi + \partial^2_{x_2x_2}\Phi + \partial^2_{zz}\Phi = 0. \tag{LE}$$

Substituting (1.3) in (1.1) and integrating over the liquid domain we obtain the so called *Bernoulli's Principle* for irrotational flow

$$\partial_t\Phi + \frac{1}{2}(|\nabla\Phi|^2) + \frac{p}{\rho} + gz = f(t), \tag{1.4}$$

where $f(t)$ is an arbitrary function of time. The potential equation (LE) can be used to determine the velocity components and then *Bernoulli's principle* (1.4) will give the pressure p. We note that the pressure p depends on $f(t)$ which is function only of time and does not depend on the space variables of the fluid domain. This function added to the pressure does not change the pressure gradient, hence, does not effect the motion of the fluid. $f(t)$ can be absorbed in Φ simply by introducing a potential $\Phi' = \Phi - \int f(t)dt$. Φ' is harmonic and satisfies $\nabla\Phi' = \nabla\Phi$. Taking (1.4) for Φ' the right hand side vanishes, thus we may take $f(t) \equiv 0$ without loss of generality.

At this point we have to define precisely the fluid domain in order to proceed to the statements of the boundary conditions. We consider the case of two fluids (water - air), separated by an interface described by a function $\eta(\mathbf{x}, t)$, over a bottom fixed boundary described by $h(\mathbf{x})$. For $t \geq t_0$ denote by $\mathcal{D}_h^\eta(t)$ the (unknown) time dependent fluid domain defined by

$$\mathcal{D}_h^\eta = \left\{(\mathbf{x}, z) \in S \times \mathbb{R} : z \in (-h(\mathbf{x}), \eta(\mathbf{x}, t))\right\}, \tag{1.5}$$

where $S \subseteq \mathbb{R}^2$ is the horizontal projection of the free surface. The bottom boundary (bathymetry) and the unknown free surface that bound the fluid domain vertically are denoted by

$$\Gamma_h = \left\{(\mathbf{x}, z) \in S \times \mathbb{R} : z = -h(\mathbf{x})\right\}, \tag{1.6}$$

$$\Gamma_\eta(t) = \left\{(\mathbf{x}, z) \in S \times \mathbb{R} : z = \eta(\mathbf{x}, t)\right\}. \tag{1.7}$$

We denote by $\boldsymbol{N}_- = \frac{1}{\sqrt{1+|\nabla_\mathbf{x} h|^2}}(-\nabla_\mathbf{x} h, -1)^T$ and $\boldsymbol{N}_+ = \frac{1}{\sqrt{1+|\nabla_\mathbf{x}\eta|^2}}(-\nabla_\mathbf{x}\eta, 1)^T$ the outward unit normal vectors to Γ_h and $\Gamma_\eta(t)$ respectively, where $\nabla_\mathbf{x} = (\partial_{x_1}, \partial_{x_2})$ is the horizontal gradient. On the fixed bottom boundary Γ_h the normal velocity of the fluid must vanish

$$\boldsymbol{N}_- \cdot [\boldsymbol{u}]_{z=-h} = 0 \qquad \text{for} \quad t \geq 0, \quad \mathbf{x} \in S. \tag{1.8}$$

The interface between two fluids is defined by the property that no particles cross it. This means that the velocity of the fluid normal to the $\Gamma_\eta(t)$ equals the normal velocity of $\Gamma_\eta(t)$. Hence,

$$\boldsymbol{N}_+ \cdot [\boldsymbol{u}]_{z=\eta} = \frac{-\partial_t\eta}{\sqrt{|\nabla_\mathbf{x}\eta|^2 + 1}} \qquad \text{for} \quad t \geq 0, \quad \mathbf{x} \in S. \tag{1.9}$$

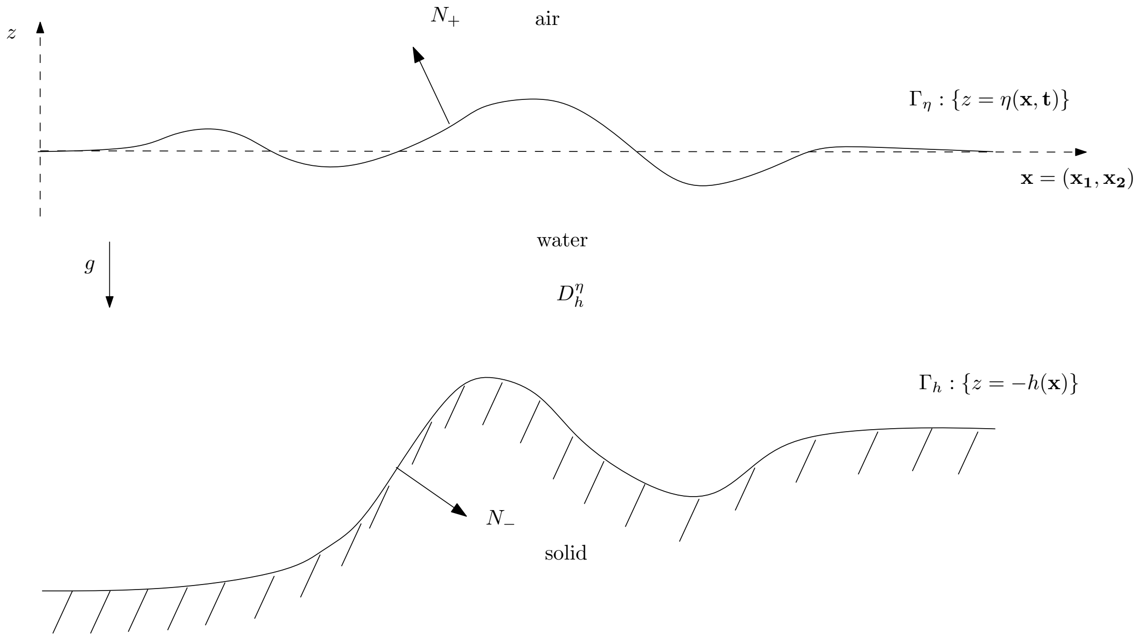

Figure 1.1: The configuration of the fluid domain

Using $\boldsymbol{u} = \nabla\Phi$ the above conditions yield

$$\text{(KC1)} \qquad \nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}} \Phi + \partial_z \Phi = 0, \qquad \text{on} \quad \Gamma_h, \qquad t \geq t_0,$$

$$\text{(KC2)} \qquad \partial_t \eta + \nabla_{\mathbf{x}} \Phi \cdot \nabla_{\mathbf{x}} \eta - \partial_z \Phi = 0, \qquad \text{on} \quad \Gamma_\eta(t), \qquad t \geq t_0.$$

Equation (KC1) is a Neumann boundary condition while (KC2) shows that the particles of the fluid that are initially in the free surface remain there and it is a *kinematic boundary condition*. These conditions are not sufficient to determine both η and Φ and another boundary condition on the free surface is needed. Since the free surface is massless, the forces in the fluids on the two sides must be equal, hence, (neglecting surface tension) the pressure in the water and the pressure in the air must be equal at the surface. Using the fact that the density of air is very small compared with that of water we can assume that the pressure in air does not change as the free surface evolves. This way we can prescribe the value $\bar{p}$ for the externally applied pressure (due to air) and state the boundary condition $p = \bar{p}$ on $\Gamma_\eta(t)$. Hence, (1.4) yields the condition

$$\text{(DC)} \qquad \partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + g\eta = -\frac{\bar{p}}{\rho} = 0 \qquad \text{on} \quad \Gamma_\eta(t), \quad t \geq t_0,$$

where we chose $\bar{p} = 0$. Eq. (DC) is called *Bernoulli* or *dynamic boundary condition*. In the horizontal dimensions, $\mathcal{D}_h^\eta$ is unbounded so a condition at infinity is needed. A natural one is that the derivatives of Φ and η stay bounded as $|\mathbf{x}| \to \infty$. To sum up, the formulation of the water wave problem can be stated as follows: *Given the externally applied pressure $\bar{p}$ and the bottom fixed boundary $h(\mathbf{x})$ find the free surface elevation $\eta(\mathbf{x}, t)$ and the velocity potential $\Phi(\mathbf{x}, z, t)$, $(\mathbf{x}, z) \in \mathcal{D}_h^\eta$, $t > t_0$ that satisfy the following system*

$$(\mathcal{P}_{WW}) \qquad \begin{cases} \Delta\Phi = 0, & \text{in} \quad \mathcal{D}_h^\eta(t), \\ \nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}} \Phi + \partial_z \Phi = 0, & \text{on} \quad \Gamma_h, \\ \partial_t \eta + \nabla_{\mathbf{x}} \Phi \cdot \nabla_{\mathbf{x}} \eta - \partial_z \Phi = 0, & \text{on} \quad \Gamma_\eta(t), \\ \partial_t \Phi + \dfrac{1}{2} |\nabla \Phi|^2 + g\eta = 0, & \text{on} \quad \Gamma_\eta(t). \end{cases}$$

The above system should be supplemented with appropriate initial conditions or conditions at infinity in the case of an horizontally unbounded liquid or specific lateral conditions in the case there are lateral boundaries.

Remark 1. Instantaneously, Φ satisfies a boundary value problem which depends non-linearly on the shape of the domain $\mathcal{D}_h^\eta$ which unknown. Thus the geometry cannot be solved straightforward from the linear equation. Time differentiation is not applied to the field equation (LE) but in the boundary conditions (KC2) and (DC). The velocity potential Φ satisfies a linear equation in $\mathcal{D}_h^\eta$ and two non-linear boundary conditions on the free surface, which is in contrast with the usual linear boundary condition for an elliptic equation, such as (KC1).

2 Hamiltonian formulation for the water wave problem

In this chapter we describe the variational structure of the hydrodynamic problem presented in Chapter 1. In the first section we recall the elementary facts about the reformulations of *Classical mechanics*, namely the Lagrangian and Hamiltonian formalism and give their generalization for systems with a continuous number of degrees of freedom.

2.1 General background

Some standard books on the theoretical framework of *Classical Mechanics* are [Gol80, AM78, MRA01, LL60, Arn78, Gan75]. At first we recall the basic facts about an ideal classical mechanical dynamical systems with finite degrees of freedom.

Lagrangian Formalism: In the case of a classical mechanical dynamical system, a discrete set of functions of time, denoted by $q_i(t)$, $i = 1, ..., N$ can be considered as a set of *generalized coordinates* if the following two requirements are satisfied

1. The geometric configuration of the system at any time instant t is fully described by means of the values of $q_i(t)$, $i = 1, ..., N$ and

2. The set $\{q_i(t),\ i = 1, ..., N\}$ is minimal, in the sense that there is no any functional relationship of the form

$$f(q_1(t), q_2(t), ..., q_n(t)) = 0, \quad t \geq t_0$$

   compatible with the totality of the geometric configuration of the system for all $t \geq t_0$.

The time derivatives of the generalized coordinates q_i, are denoted by $\dot{q}_i$ and are called the *generalized velocities* of the system. The system of generalized coordinates and velocities $\{q_i(t), \dot{q}_i(t),\ i = 1, 2, ..., N\}$ provides a complete geometric and kinematic description of the dynamical system. A function

$$L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{2.1}$$

of $2N+1$ arguments is called a *Lagrangian function* of the mechanical system. The evolution of the *generalized coordinates* is governed by the system of equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, ..., N \tag{2.2}$$

called *Lagrange Equations.* In classical mechanics the Lagrangian function is given by the equation

$$L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = K(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) - V(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{2.3}$$

where K is the *kinetic energy*, V is the *potential energy* and we assumed the absence of external forces (i.e forces that are not described by the potential energy V).

Hamiltonian Formalism: Assuming that a classical mechanical dynamical system admits of a Lagrangian formalism, one can define the *generalized momenta* $\boldsymbol{p}(t) = \{p_i(t)\}_{i=1}^N$ corresponding to the generalized coordinates by the equations

$$p_i(t) = \frac{\partial}{\partial \dot{q}_i} L(t, \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)), \quad i = 1, ...N, \tag{2.4}$$

where $\partial/\partial_{\dot{q}_i}$ is the partial derivative with respect to $\dot{q}_i$, $i = 1...N$. The *Hamiltonian* of the system is defined by the equation

$$H(t, \boldsymbol{q}, \boldsymbol{p}) = \sum_{i=1}^{N} p_i \dot{q}_i - L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{2.5}$$

The space of all possible states $(\mathbf{q}, \boldsymbol{\pi})$ of the system is called the *phase space* of the system. The basic object of the theory is the *Hamiltonian* which is a function $(\boldsymbol{q}, \boldsymbol{\pi}) \to H(\boldsymbol{q}, \boldsymbol{\pi})$ of the positions $\boldsymbol{q}$ and momenta $\boldsymbol{p}$ of the system, which equals the sum of the kinetic and potential energy. i.e the total energy.

$$H = K + V \tag{2.6}$$

The system then evolves according to Hamilton's equations

$$\begin{aligned} \partial_t q_i &= \frac{\partial H}{\partial p_i}, \\ \partial_t p_i &= -\frac{\partial H}{\partial q_i}. \end{aligned} \tag{2.7}$$

Defining the *action functional*

$$S = \int_{t_0}^{T} \big(\sum_{i=1}^{N} p_i \dot{q}_i - H(t, \boldsymbol{q}, \boldsymbol{p})\big) \tag{2.8}$$

one can derive Hamilton's equations variationally by examining the stationarity of S.

In the case of a system of classical mechanics the configuration space is an N-dimensional manifold representing all the kinematically admissible positions $\mathbf{q} \in \mathcal{M} \subseteq \mathbb{R}^N$. For $q \in M$, $\dot{q} = \partial_t q$ is an element of the tangent space $T_q(M)$. The corresponding Langrangian function $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is a function on the *tangent bundle* $T(\mathcal{M}) = \{(\mathbf{q}, \dot{\mathbf{q}}) | \mathbf{q} \in \mathcal{M}, \dot{\mathbf{q}} \in T_{\mathbf{q}}(\mathcal{M})\} \subseteq \mathbb{R}^{2N}$. The generalized momenta $\boldsymbol{p}$ lies in the cotangent space $T_q^*(M)$ and the evolution $t \mapsto (q(t), p(t))$ is described in terms of *Hamiltonian function* $H : T^*(M) \to \mathbb{R}$, where $T^*(M) = \big\{(q, p) : q \in M,\, p \in T_q^*(M)\big\}$ is the *cotangent bundle.* $T^*(M)$ is the *phase space* in which the system evolves.

The above classical mechanical formalism can be extended in the case of generalized coordinates that are continuously distributed i.e $q_x(t) = q(x, t)$, $x \in D \subseteq \mathbb{R}^d$. We then say that the system admits of a Lagrangian formalism if there is a function $L = L(t, q(x, t), \dot{q}(x, t))$ such that the governing equations (Lagrange equations) take the form

$$\frac{\partial}{\partial t} \frac{\delta L}{\delta \dot{q}} - \frac{\delta L}{\delta q} = 0, \quad x \in D \tag{2.9}$$

where $\delta / \delta q$ and $\delta / \delta \dot{q}$ are the variational derivatives. The configuration space is an infinite dimensional manifold M and the generalized momenta are defined by

$$p_{\mathbf{x}}(t) = \frac{\delta}{\delta \dot{q}_{\mathbf{x}}(t)} L(q_{\mathbf{x}}(t), \dot{q}_{\mathbf{x}}(t)) \tag{2.10}$$

where $\delta / \delta \dot{q}_{\mathbf{x}}(t)$ is the variational derivative with respect to $\dot{q}_{\mathbf{x}}(t)$. Accordingly *Hamilton's equations* are

$$\partial_t q = \frac{\delta H}{\delta p} \tag{2.11}$$

$$\partial_t p = -\frac{\delta H}{\delta q} \tag{2.12}$$

or writing $u = (q,p)^T$, $\nabla H = (\frac{\delta H}{\delta q}, \frac{\delta H}{\delta p})^T$

$$\begin{cases} \partial_t u = J\nabla H(u) \\ u(t_0) = u_0 \in M \end{cases} \tag{2.13}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The solution map $t \mapsto S_t(u_0)$ is a mapping that describes the evolution of the system from an initial state $(q(t_0), \pi(t_0))$ to $(q(t), \pi(t))$ in the phase space. When M is finite dimensional and the *Hamiltonian field* $X_H(u) \in C^1(M \to \mathbb{R})$ (i.e. $H \in C^2(M \to \mathbb{R})$) then by the usual theory of ODE's there exists a unique solution for the initial value problem (2.13) locally in time.

2.2 Generalized coordinates and kinetic energy of water waves

In this section we review the Hamiltonian structure of the water wave problem. i.e the fact that Eqs. ($\mathcal{P}_{WW}$) can be put in Hamiltonian form (2.11). For the case of a fluid with infinite depth Zakharov [Zak68] was the first that provided us the Hamilton's equations for the free surface evolution equations (infinite depth water). The observation of Zakharov was that the potential on the free surface and the free surface elevation are enough to define the fluid flow since the boundary value problem for the Laplace equation, given the free surface and the value of the potential on it, is well-posed. Craig and Sulem [CS93] expressed Zakharov's equations by means of an appropriate Dirichlet to Neumann (DtN) operator which is a an operator that maps the potential on the free surface (Dirichlet data) to the normal derivative of the potential on the free surface (Neumann data) multiplied by a scalar function. Later Craig et al.[CGS09] generalized this formulation in the case of variable bathymetry. The Hamiltonian H is regarded as a functional of (η, φ) where $\eta(\mathbf{x}, t)$ is the height of the free surface, and $\varphi(\mathbf{x}, t)$ is the trace of the harmonic function Φ on the free surface, with homogenous Neumann condition on the bottom. The evolution takes place in the space of harmonic functions on $\mathcal{D}_h^\eta$.

Recall that the free surface $\Gamma_\eta(t)$ is fully described by the equation, $z = \eta(\mathbf{x}, t)$, as a graph $\Gamma_\eta(t) = \{(\mathbf{x}, z) \in \mathbb{R}^2 \times \mathbb{R} : z = \eta(\mathbf{x}, t)\}$. The configuration space M is the space of all possible free surface elevations $t \mapsto q_{\mathbf{x}}(t) = \eta(\mathbf{x}, t), \quad \mathbf{x} \in S \subset \mathbb{R}^2$ where S is the projection of Γ_η on the horizontal plane (x_1, x_2). Notice that the generalized co-ordinates and velocities, $q_{\mathbf{x}}(t) = \eta(\mathbf{x}, t)$, $\dot{q}_{\mathbf{x}}(t) = \dot{\eta}(\mathbf{x}, t)$, respectively, are continuously distributed on $S \subset \mathbb{R}^2$. To proceed further in developing a Hamiltonian formalism for the water wave problem, we need to express the Lagrangian density $\mathcal{L} = \mathcal{K} - \mathcal{V}$ with respect to the tangent variables $(\eta, \dot{\eta})$. The kinetic energy of the fluid domain is given by

$$\mathcal{K} = \frac{1}{2} \int_{\mathcal{D}_h^\eta} |\nabla \Phi|^2 \, dz \, d\mathbf{x}. \tag{2.14}$$

This is a domain functional since the domain of integration changes as the system evolves. Via Green's formula the kinetic energy functional can be written as

$$\mathcal{K} = \frac{1}{2}\Big(- \int_{\mathcal{D}_h^\eta} \Phi \, \Delta \Phi \, d\mathbf{x} dz + \int_{\Gamma_h} \Phi \, \partial_{\boldsymbol{N}_-} \Phi \, d\Gamma_h + \int_{\Gamma_\eta} \Phi \, \partial_{\boldsymbol{N}_+} \Phi \, d\Gamma_\eta \Big). \tag{2.15}$$

Introducing the trace of Φ on the free surface Γ_η

$$\varphi(\mathbf{x}, t) = \Phi(\mathbf{x}, \eta(\mathbf{x}, t), t) =: [\Phi]_{z=\eta}, \tag{2.16}$$

and taking into account the kinematics ,(LE), (KC1) (i.e irrotationality, incombressibility and bottom boundary condition), of the problem, we see that the first two terms are zero while the third can be expressed as

$$\mathcal{K} = \frac{1}{2} \int_S \varphi \, \partial_{\boldsymbol{N}_+} \Phi R_+ \, d\mathbf{x}. \tag{2.17}$$

where $R_+ = (1 + |\nabla_{\mathbf{x}} \eta|^2)^{1/2}$. Notice that the integrand depends on surface quantities. The next step is to write the normal derivative of the potential on Γ_η, that appears in the above expression, with respect to the potential on Γ_η. Zakharov (in the case of deep water) linked the two quantities by means of the Green's function associated associated with the boundary value problem for the Laplace equation. Here we follow [CS93] where an appropriate Dirichlet to Neumann operator is used which maps the velocity potential on the free surface to the normal derrivative of the potential velocity on the free surface multiplied by the scalar function $R_+ = (1 + |\nabla_{\mathbf{x}} \eta|^2)^{1/2}$.

2.2.1 The DtN Operator

Consider the following elliptic boundary value problem in the fluid domain $\mathcal{D}_h^\eta$

$$(\mathcal{P}_{DtN}) \qquad \begin{cases} \Delta\Phi = 0 \quad \text{in} \quad \mathcal{D}_h^\eta \\ [\Phi]_{z=\eta} = \varphi, \\ \boldsymbol{N}_- \cdot [\nabla\Phi]_{z=-h} = 0 \end{cases}$$

together with appropriate lateral conditions. Supposing that η and h are smooth enough then there exists a unique Φ in $\in H^{k+2}(\mathcal{D}_h^\eta)$, satisfying the essential condition $[\Phi]_{z=\eta} = \varphi$, solution (variational, classical) to ($\mathcal{P}_{DtN}$) (for a proof of this statement, see [Lan05, Lem. 2.9, Th . 2.9]).[1] The well-posedness of problem ($\mathcal{P}_{DtN}$) determines an one-to-one correspondance between the Dirichlet and Neumann data.

This motivates the definition of an operator which maps the Dirichlet data ($\varphi = \left[\Phi\right]_{z=\eta}$) onto the Neumann data ($\partial_{\boldsymbol{N}_+}\Phi$) multiplied with a the scalar function R_+ that are involved in the expression of the kinetic energy functional.

Definition 2.1 (DtN Operator)**.** For the fluid domain $\mathcal{D}_h^\eta$ defined by the instantaneous surface elevation $\eta(\cdot, t), h \in C^1(\mathbb{R}^2)$ for $t \geq t_0$ and the unique solution Φ of problem ($\mathcal{P}_{DtN}$) define

$$G(\eta, h) : \ \varphi \ \mapsto \ R_+\partial_{\boldsymbol{N}_+}\Phi|_{\Gamma_\eta}, \tag{2.18}$$

where $R_+ = \left(1 + |\nabla_{\mathbf{x}}\eta|^2\right)^{1/2}$.

Clearly, the DtN operator is linear in φ and possitive since

$$\begin{aligned} (\varphi, G(\eta, h)\varphi) &= \int\limits_S [\Phi]_{z=\eta} R_+ \partial_{\boldsymbol{N}_+}\Phi \, d\mathbf{x} \\ &= \int_{\Gamma_\eta} \Phi \partial_{\boldsymbol{N}_+}\Phi \, d\Gamma_\eta \\ &= \int_{\mathcal{D}_h^\eta} |\nabla\Phi|^2 \, dV \ > 0. \end{aligned}$$

Apparently this is not exactly the DtN operator since it maps the Dirichlet data to the Neumann data multiplied by the scalar function R_+. Thus defined, DtN

[1] Thus the behaviour of φ as $|\mathbf{x}| \to \infty$ uniquely determines the behaviour of Φ near infinity. Additionaly if we consider horizontal bottoms $h(\mathbf{x})$ and periodic $\eta(\mathbf{x}, t)$ and φ then Φ will be periodic too.

operator is self-adjoint.[2] Indeed let u, v sufficiently smooth functions on $\mathbb{R}^2$ (e.g. Schwartz) and consider their harmonic extensions U, V in $\mathcal{D}_h^\eta$ satisfying ($\mathcal{P}_{DtN}$). Green's identity for U, V gives

$$\int\limits_{\Gamma_\eta} \left\{ U(\partial_{N_+} V) - V(\partial_{N_+} U) \right\} d\Gamma_\eta = 0.$$

Hence

$$(u, G(\eta, h)v) = (v, G(\eta, h)u)\,, \qquad \forall u, v \in \mathcal{S}(\mathbb{R}^2), \tag{2.19}$$

where $\mathcal{S}(\mathbb{R}^2)$ is the Schwartz space.

It is known that such operators depend analytically on the parametrization of the surface η. Coifman and Meyer in [CM85] considered small Lipschitz perturbations of a line or plane, and Craig et al. [HN05] C^1 perturbations of hyperplanes in any dimension. See also [NR01].

Kinetic energy

From the definition of DtN and (2.17), the kinetic energy functional $\mathcal{K}$ can be written in terms of η and φ as

$$\mathcal{K}[\varphi, \eta] \;=\; \frac{1}{2} \int\limits_S \varphi\, G(\eta, h) \varphi\, d\mathbf{x}, \tag{2.20}$$

which is an integral functional quadratic in φ that depends non-linearly and non-locally on η and h through the DtN operator. Note that till this point the only information from the fluid interior used, in order to write the kinetic energy in terms of surface variables, is irrotationality and incombressibility (LE) and impermability of the bottom surface (KC1). The kinematic free boundary condition (KC2) can be written as

$$\partial_t \eta = G(\eta, h)\varphi. \tag{2.21}$$

[2] See also [Mil90] where instead of the DtN operator the operator $\hat{K}$ is used, defined by $\hat{K} = R_+^2 \hat{D} - \nabla_{\mathbf{x}}\eta \cdot \nabla_{\mathbf{x}}$, where $\hat{D}$ is the operator that reproduces the vertical derivative of the potential. i.e. $\hat{D}\varphi = [\partial_z \Phi]_{z=\eta}$. The operator $\hat{K}$ equals the DtN operator and eventually is self-adjoint.

Inverting the above linear equation we can write the Lagrangian with the respect to the tangent variables $(\eta, \partial_t \eta)$ as

$$\mathcal{L}[\eta, \partial_t \eta] \; = \; \mathcal{K}[\eta, \partial_t \eta] - \mathcal{V}[\eta] \; = \; \frac{1}{2} \int\limits_S \dot{\eta}\, G^{-1}(\eta, h) \dot{\eta} \; d\mathbf{x} - \frac{1}{2} \int\limits_S \eta^2 \; d\mathbf{x}, \tag{2.22}$$

where integration by parts have been performed for the potential energy term. This expression permits us to derive the canonical coordinates of the water wave problem from first principles of mechanics ([Cra07, CGK00]). Similar expressions as the above are derived by Milder in [Mil77a] by means of a suitable Green function and by Athanassoulis in [Ath97] by means of a Neumann function in the case of the presence of mouving bodies in the fluid.

2.3 Hamilton's Equations for the water wave problem

From (2.22) we can derive the generalised momentum

$$\frac{\delta \mathcal{L}}{\delta \dot{\eta}} = G^{-1}(\eta, h) \dot{\eta} = \varphi. \tag{2.23}$$

We then construct the Hamiltonian functional using the Legendre transform

$$\mathcal{H}[\eta, \varphi] = \frac{1}{2} \int\limits_S \varphi G(\eta, h) \varphi d\mathbf{x} + \frac{g}{2} \int\limits_S \eta^2 d\mathbf{x}, \tag{2.24}$$

It is clear that we need the shape derivative of DtN operator in order to compute the variation of the kinetic energy functional $\mathcal{K}$. To that direction we use the formulla from [Lan05].

Theorem 2.1. *[Lan05, Th. 3.20] The mapping*

$$\eta \;\mapsto\; G(\eta, h)\varphi,$$

is well defined and differentiable. For all $\delta\eta$ one has

$$\begin{aligned} \delta_\eta G(\eta, h)\varphi \cdot \delta\eta \; &= \; - G(\eta, h)\Big(R_+^{-2}\big(G(\eta, h)\varphi + \nabla_{\mathbf{x}}\eta \cdot \nabla_{\mathbf{x}}\varphi\big)\delta\eta\Big) - \\ &\quad - \nabla_{\mathbf{x}} \cdot \left[\left(\nabla_{\mathbf{x}}\varphi - \Big(R_+^{-2}\big(G(\eta, h)\varphi + \nabla_{\mathbf{x}}\eta \cdot \nabla_{\mathbf{x}}\varphi\big)\nabla_{\mathbf{x}}\eta\Big)\right)\delta\eta\right]. \end{aligned}$$

Lemma 2.1. *The first variation of the kinetic energy functional*

$$\mathcal{K}[\varphi,\eta] \;=\; \frac{1}{2}\int_S \varphi\, G(\eta,h)\varphi\, d\mathbf{x},$$

in the direction $\delta\eta$ reads

$$\delta_\eta\mathcal{K}[\varphi,\eta;\delta\eta] \;=\; \int_S \Big\{ -\frac{1}{2}|\nabla_{\mathbf{x}}\varphi|^2 + \frac{1}{2R_+^2}\big((G(\eta,h)\varphi) + \nabla_{\mathbf{x}}\varphi\cdot\nabla_{\mathbf{x}}\eta\big)^2 \Big\}\delta\eta\, d\mathbf{x}.$$

Proof. From the preceding theorem we have

$$\begin{aligned}
\delta_\eta\mathcal{K}[\varphi,\eta;\delta\eta] \;&=\; \frac{1}{2}\int_S \varphi\left(\delta_\eta G(\eta,h)\varphi\cdot\delta\eta\right) d\mathbf{x}\\
&=\; \frac{1}{2}\int_S \Big\{ -\varphi\, G(\eta,h)\Big(R_+^{-2}\big(G(\eta,h)\varphi + \nabla_{\mathbf{x}}\eta\cdot\nabla_{\mathbf{x}}\varphi\big)\delta\eta\Big) -\\
&\qquad -\varphi\,\nabla_{\mathbf{x}}\cdot\Big[\Big(\nabla_{\mathbf{x}}\varphi - \Big(R_+^{-2}\big(G(\eta,h)\varphi + \nabla_{\mathbf{x}}\eta\cdot\nabla_{\mathbf{x}}\varphi\big)\nabla_{\mathbf{x}}\eta\Big)\Big)\delta\eta\Big]\Big\} d\mathbf{x}.
\end{aligned}$$

Using the self-adjointness of DtN (2.19) for the first term and integration by parts for the second we obtain

$$\begin{aligned}
\delta_\eta\mathcal{K}[\varphi,\eta;\delta\eta] \;&=\; \frac{1}{2}\int_S -G(\eta,h)\varphi\,\Big(R_+^{-2}\big(G(\eta,h)\varphi + \nabla_{\mathbf{x}}\eta\cdot\nabla_{\mathbf{x}}\varphi\big)\delta\eta\Big) -\\
&\qquad -\nabla_{\mathbf{x}}\varphi\cdot\left[\left(\nabla_{\mathbf{x}}\varphi - \Big(R_+^{-2}\big(G(\eta,h)\varphi + \nabla_{\mathbf{x}}\eta\cdot\nabla_{\mathbf{x}}\varphi\big)\nabla_{\mathbf{x}}\eta\Big)\right)\delta\eta\right] d\mathbf{x},
\end{aligned}$$

and after some elementary algebra we obtain the result □

Taking the variational derivatives of the Hamiltonian $H[\varphi,\eta] = K[\varphi,\eta] + V[\eta]$ (2.24) one verifies that Hamilton's Equations (2.11) read

(HE)

$$\begin{cases}
\partial_t\eta \;=\dfrac{\delta\mathcal{K}}{\delta\varphi} \;=\; G(\eta,h)\varphi,\\[2ex]
\partial_t\varphi \;=\; -\,g\eta - \dfrac{\delta\mathcal{K}}{\delta\eta} \;=\; -g\eta + \dfrac{1}{2}|\nabla_{\mathbf{x}}\varphi|^2 - \dfrac{1}{2R_+^2}\big((G(\eta,h)\varphi) + \nabla_{\mathbf{x}}\varphi\cdot\nabla_{\mathbf{x}}\eta\big)^2.
\end{cases}$$

which is an evolution Hamiltonian system of equations for the elevation of the free surface $\eta(\mathbf{x},t)$ and the trace of the velocity potential on the free surface $\varphi(\mathbf{x},t)$. These equations are equivalent with the boundary conditions (KC2), (DC) while (LE) and (KC1) are expressed through the definition of the DtN operator.

Given a procedure for estimating the DtN operator in the instantaneous fluid domain $\mathcal{D}_h^\eta$ (subject to lateral boundary conditions), the water-wave problem is reduced to the simultaneous time-integration of the kinematic and the dynamic free surface boundary conditions. Craig and Sulem in [CS93], used a Taylor series expansion of the DtN operator, for the case of an horizontal bottom. The same method was extended in the variable bottom case in [CGS09] and applied to small-amplitude long waves propagating over a randomly varying bottom. Recently Craig et al. [CLS11] studied these equations in the case of rapidly varying periodic bottom boundary in the shallow water scaling regime. For a brief review of the DtN approach see [Sch05].

3 An unconditional variational principle for the water wave problem $(\mathcal{P}_{WW})$(Luke's Principle) and its connection with Hamilton's Principle

The physical situation presented in Chapter 1 is classically described by means of the *fields* $\Phi(\mathbf{x}, z, t)$ and $\eta(\mathbf{x}, t)$ with $\mathbf{x} \in S \subseteq \mathbb{R}^2$, $(\mathbf{x}, z) \in \mathcal{D}_h^\eta$ and $t \geq t_0$ and the system of equations $(\mathcal{P}_{WW})$ together with inital-latteral conditions. This system was obtained by a variational principle from the observation of Luke in [Luk67], that the action functional that gives the correct field equation (Laplace equation) together with all the important boundary conditions is

$$S[\Phi, \eta] \;=\; \int_I \int_{\mathcal{D}_h^\eta} \Big(\partial_t \Phi + \frac{1}{2}|\nabla \Phi|^2 + gz\Big)\, dV\, dt, \tag{3.1}$$

that is the integral of the pressure over the whole time-dependent domain $I \times \mathcal{D}_h^\eta$. Here no a priori separation of the kinematic part of the problem is needed as in the derivation of Hamilton's equations (Chapter 2). We refer also to [Pet64] where the case of waves in a finite container is treated (with surface tension). In the following sections Luke's principle is rederived as a special case of a more general functional.

3.1 Notation and Preliminaries

In what follows let $\Phi \in C^1\big([t_0, T) \to C^2(\mathcal{D}_h^\eta)\big) \cap C^1\big([t_0, T) \to C^1(\overline{\mathcal{D}_h^\eta})\big) \equiv \mathcal{M}_\Phi$ be a field with $\eta \in C^1\big([t_0, T] \to C_0^1(S)\big) \equiv \mathcal{M}_\eta$ and $S \subset \mathbb{R}^2$ open. The space C^k is the space of k-differentiable functions and the indice 0 means compact support. The space of all $\big(\Phi, \eta\big)$ satisfying the above conditions with the additional requirement that they vanish as $|\mathbf{x}| \to \infty$, is called the *configuration space* and is denoted by

$$\mathcal{M} := \Big\{\big(\Phi, \eta\big) \in \mathcal{M}_\Phi \times \mathcal{M}_\eta : \lim_{|\mathbf{x}| \to \infty} \Phi = \lim_{|\mathbf{x}| \to \infty} \eta = 0\Big\},$$

where ∂S is the lateral boundary of the domain. Given a *Lagrangian function*

$$G : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times J_h^\eta \to \mathbb{R},$$

continuously differentiable with respect to all of its arguments we write

$$(1, \vec{2}, 3, 4) \to G(1, \vec{2}, 3, 4) \equiv G$$

and denote the corresponding partial derivatives, with respect to its functional arguments, by

$$\begin{aligned}
D_1 G &:= D_1 G(1, \vec{2}, 3, 4) &:=& \frac{\partial G}{\partial 1}(1, \vec{2}, 3, 4), \\
\vec{D}_2 G &:= \vec{D}_2 G(1, \vec{2}, 3, 4) &:=& \Big(\frac{\partial G}{\partial 2_1}(1, \vec{2}, 3, 4), \frac{\partial G}{\partial 2_2}(1, \vec{2}, 3, 4)\Big), \quad \vec{2} = (2_1, 2_2), \\
D_3 G &:= D_3 G(1, \vec{2}, 3, 4) &:=& \frac{\partial G}{\partial 3}(1, \vec{2}, 3, 4).
\end{aligned}$$

We will assume that the function $G(1, \vec{2}, 3, 4)$ is linear in the first argument and more specifically $D_1 G = 1$. The *Lagrangian density functional* $\mathcal{G}$ is the spatial integral of the *Lagrangian function* G

$$\mathcal{G}[\Phi(\cdot, \cdot, t), \eta(\cdot, t)] = \int_{\mathcal{D}_{h(\cdot)}^{\eta(\cdot, t)}} G(\partial_t \Phi(\cdot, \cdot, t), \nabla_{\mathbf{x}} \Phi(\cdot, \cdot, t), \partial_z \Phi(\cdot, \cdot, t), z)\, dV. \tag{3.2}$$

We define the corresponding *action functional* $S : \mathcal{M} \to \mathbb{R}$ as the time integral of the *Lagrangian density functional* $\mathcal{G}$

$$S[\Phi, \eta] = \int_{t_0}^{T} \mathcal{G}[\Phi(\cdot, \cdot, t), \eta(\cdot, t)] dt. \tag{3.3}$$

In the rest of the thesis, we refer to that functional as the generalized Luke's functional, or simply *g-Luke's functional*. It is a real number dependent on the field Φ and the shape of the domain $\mathcal{D}_h^\eta$ which is fully determined by the field η. The system evolves from the point $(\Phi(\cdot, t_0), \eta(\cdot, t_0))$ to the point $(\Phi(\cdot, T), \eta(\cdot, T))$ in the configuration space $\mathcal{M}$ along the path $t \to (\Phi(\cdot, t), \eta(\cdot, t))$ that renders the action functional stationary. Stationarity means that the first variation of S at $\big(\Phi, \eta\big)$ is zero. i.e.

$$\delta S[\Phi, \eta; \delta\Phi, \delta\eta] \quad = \quad \delta_\Phi S[\Phi, \eta; \delta\Phi] \; + \; \delta_\eta S[\Phi, \eta; \delta\eta] \quad = \quad 0, \tag{3.4}$$

where $\delta_\Phi S[\Phi,\eta;\delta\Phi]$ and $\delta_\eta S[\Phi,\eta;\delta\eta]$ are the partial variations of S w.r.t Φ and η, respectively. In calculating the first variation of the action functional S the space of admissible variations $\big(\delta\Phi,\delta\eta\big)$ is also needed. We define that space as the subset of $\mathcal{M}$ characterized by the property of isochronality

(3.5)
$$\delta\mathcal{M} = \delta\mathcal{M}_\Phi \times \delta\mathcal{M}_\eta = \big\{\big(\Psi,\zeta\big) \in \mathcal{M} : \Psi(\cdot,t_0) = \Psi(\cdot,T) = 0,\, \zeta(\cdot,t_0) = \zeta(\cdot,T) = 0\big\}.$$

3.1.1 Calculation of the first variation of the g-Luke's functional $S[\Phi,\eta]$

By definition, the partial variation of S at $\big(\Phi,\eta\big)$ in the direction $\delta\Phi$ is

$$\begin{aligned}
\delta_\Phi S[\Phi,\eta;\delta\Phi] &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[S[\Phi(\cdot,t)+\epsilon\delta\Phi(\cdot,t),\eta] - S_{\mathcal{G}}[\Phi(\cdot,t),\eta]\Big] = \\
&= \int_{t_0}^{T}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[\mathcal{G}[\Phi(\cdot,t)+\epsilon\delta\Phi(\cdot,t),\eta(\cdot,t)] - \mathcal{G}[\Phi(\cdot,t),\eta(\cdot,t)]\Big]dt \\
&= \int_{t_0}^{T}\delta_\Phi\mathcal{G}[\Phi(\cdot,\cdot,t),\eta(\cdot,t);\delta\Phi(\cdot,\cdot,t)]dt
\end{aligned}$$

and similarly

$$\delta_\eta S[\Phi,\eta;\delta\eta] = \int_{t_0}^{T}\delta_\eta\mathcal{G}[\Phi(\cdot,\cdot,t),\eta(\cdot,t);\delta\Phi(\cdot,\cdot,t)]dt.$$

It then suffices to calculate the variations of (dropping the notation $(\cdot,\cdot,t)$, $(\cdot,t)$)

$$\mathcal{G}[\Phi,\eta] = \int_S\int_{-h}^{\eta} G\big(\partial_t\Phi,\nabla_{\mathbf{x}}\Phi,\partial_z\Phi,z\big)\,d\mathbf{x}\,dz. \tag{3.6}$$

To calculate the variation $\delta_\Phi\mathcal{G}[\Phi,\eta;\delta\Phi]$ and $\delta_\eta\mathcal{G}[\Phi,\eta;\delta\eta]$, we classically define the functions

$$i(\epsilon) := \mathcal{G}[\Phi+\epsilon\delta\Phi,\eta] = \int_{\mathcal{D}_h^\eta} G(\partial_t\Phi+\epsilon\partial_t\delta\Phi,\nabla_{\mathbf{x}}\Phi+\epsilon\nabla_{\mathbf{x}}\delta\Phi,\partial_z\Phi+\epsilon\partial_z\delta\Phi,z)dV,$$

$$j(\epsilon) := \mathcal{G}[\Phi,\eta+\epsilon\delta\eta] = \int_{\mathcal{D}_h^{\eta+\epsilon\delta\eta}} G(\partial_t\Phi,\nabla_{\mathbf{x}}\Phi,\partial_z\Phi,z)dV.$$

Taking the ϵ-derivative of $i(\epsilon)$ we obtain (recall that we have assumed that $D_1 G = 1$.)

$$i'(\epsilon) = \int_{\mathcal{D}_h^\eta} \Big\{ \partial_t \delta\Phi + \overrightarrow{D}_2 G(\partial_t\Phi + \epsilon\partial_t\delta\Phi, \nabla_{\mathbf{x}}\Phi + \epsilon\nabla_{\mathbf{x}}\Phi, \partial_z\Phi + \epsilon\partial_z\delta\Phi, z) \ \cdot \nabla_{\mathbf{x}}\delta\Phi +$$
$$+ D_3 G(\partial_t\Phi + \epsilon\partial_t\delta\Phi, \nabla_{\mathbf{x}}\Phi + \epsilon\nabla_{\mathbf{x}}\Phi, \partial_z\Phi + \epsilon\partial_z\delta\Phi, z) \ \partial_z\delta\Phi \Big\} dV.$$

Since the bounds of integration depend on the function of variation, in order to calculate $j'(\epsilon)$ we use Leibniz integral rule

$$\begin{aligned} j'(\epsilon) &= \int_S \Big\{ \frac{d}{d\epsilon} \int_{-h}^{\eta+\epsilon\delta\eta} G\big(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \partial_z\Phi, z\big)\, dz \Big\} d\mathbf{x} \\ &= \int_S \Big[G\big(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \partial_z\Phi, z\big)\Big]_{z=\eta+\epsilon\delta\eta} \delta\eta d\mathbf{x}. \end{aligned}$$

Taking $\epsilon = 0$ we have

(3.7)
$$\delta_\Phi \mathcal{G}[\Phi, \eta; \delta\Phi] =$$
$$= \int_{\mathcal{D}_h^\eta} \Big\{ \partial_t\delta\Phi + \overrightarrow{D}_2 G\big(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \partial_z\Phi, z\big) \ \cdot \nabla_{\mathbf{x}}\delta\Phi + D_3 G\big(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \partial_z\Phi, z\big) \ \partial_z\delta\Phi \Big\} dV,$$

$$\delta_\eta \mathcal{G}[\Phi, \eta; \delta\eta] = \int_S \Big[G\big(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \partial_z\Phi, z\big)\Big]_{z=\eta} \delta\eta d\mathbf{x}. \tag{3.8}$$

The second and third terms of the r.h.s of (3.7) can be written, using Green's Identity and recalling that $\delta\Phi = 0$ on the lateral boundaries, as

$$\begin{aligned} &\int_{\mathcal{D}_h^\eta} \Big\{ \overrightarrow{D}_2 G\big(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \partial_z\Phi, z\big) \ \cdot \nabla_{\mathbf{x}}\delta\Phi + D_3 G\big(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \partial_z\Phi, z\big) \ \partial_z\delta\Phi \Big\} dV = \\ &\quad = \int_{\Gamma_\eta} \Big(\overrightarrow{D}_2 G\big(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \partial_z\Phi, z\big)\, ,\, D_3 G\big(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \partial_z\Phi, z\big) \Big) \cdot \boldsymbol{N}_+ \delta\Phi \, d\Gamma_\eta + \\ &\quad + \int_{\Gamma_h} \Big(\overrightarrow{D}_2 G\big(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \partial_z\Phi, z\big)\, ,\, D_3 G\big(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \partial_z\Phi, z\big) \Big) \cdot \boldsymbol{N}_- \delta\Phi \, d\Gamma_h - \\ &\quad - \int_{\mathcal{D}_h^\eta} \Big(\mathrm{div}_{\mathbf{x}} \overrightarrow{D}_2 G\big(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \partial_z\Phi, z\big) + \partial_z D_3 G\big(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \partial_z\Phi, z\big) \Big) \ \delta\Phi dV. \end{aligned}$$

The first term of the r.h.s of (3.7) is tranformed using Leibniz integral rule as follows

$$\int_{\mathcal{D}_h^\eta} \partial_t\delta\Phi \ dV = \partial_t \int_S \int_{-h}^{\eta} \delta\Phi dz d\mathbf{x} - \int_S \partial_t\eta \ [\delta\Phi]_{z=\eta} d\mathbf{x}. \tag{3.9}$$

Due to the isochronality of $\delta\Phi$ the first term of the r.h.s of the last equality will integrate out to the boundaries of $[t_0, T]$. Hence we obtain

$$\begin{aligned}\delta_\Phi S[\Phi,\eta;\delta\Phi] \quad = \quad & \int_{t_0}^{T}\int_S \Big\{ -\int_{-h}^{\eta} \big(\mathrm{div}(\overrightarrow{D}_2 G, D_3 G)\,\delta\Phi\,\big)\, dz + \\ & + \big[(\overrightarrow{D}_2 G, D_3 G)\big]_{z=\eta}\cdot R_+ \boldsymbol{N}_+ - \partial_t\eta\Big)\big[\delta\Phi\big]_{z=\eta} \\ & + \Big(\big[(\overrightarrow{D}_2 G, D_3 G)\big]_{z=-h}\cdot R_- \boldsymbol{N}_-\Big)\big[\delta\Phi\big]_{z=-h}\Big\} d\mathbf{x}dt. \end{aligned} \tag{3.10}$$

Finally from (3.8) we obtain

$$\delta_\eta S[\Phi,\eta;\delta\eta] \quad = \quad \int_{t_0}^{T}\int_S \Big[G\big(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \partial_z\Phi, z\big)\Big]_{z=\eta} \delta\eta d\mathbf{x}dt. \tag{3.11}$$

Equations (3.10) and (3.11), in conjuction with (3.4) and the fundamental lemma of calculus of variations lead to the following

Theorem 3.1. *The pair of fields $(\Phi,\eta)\in\mathcal{M}$ satisfies the variational equation*

$$\delta S[\Phi,\eta;\delta\Phi,\delta\eta] := \delta_\Phi S[\Phi,\eta;\delta\Phi] + \delta_\eta S[\Phi,\eta;\delta\eta] = 0, \tag{3.12}$$

for all $\big(\delta\Phi,\delta\eta\big)\in\delta\mathcal{M}$ if and only if it is a solution of the following system of equations.

$$div\big(\overrightarrow{D}_2 G, D_3 G\big) = \nabla_{\mathbf{x}}\cdot(\overrightarrow{D}_2 G) \;+\; \partial_z(D_3 G) \quad = 0 \quad in \quad \mathcal{D}_h^\eta, \tag{3.13a}$$

$$\big(\vec{D}_2 G, D_3 G\big)\cdot R_+ \boldsymbol{N}_+ - \partial_t\eta = 0 \quad on \quad \Gamma_\eta, \tag{3.13b}$$

$$\big(\vec{D}_2 G, D_3 G\big)\cdot R_- \boldsymbol{N}_- = 0 \quad on \quad \Gamma_h, \tag{3.13c}$$

$$G = 0 \quad on \quad \Gamma_\eta. \tag{3.13d}$$

3.2 Application to the water wave problem

To recover Luke's variational principle we define the action functional

$$S[\Phi,\eta] = \int_I \mathcal{L}[\Phi,\eta]\, dt, \quad \mathcal{L}[\Phi,\eta] = \int_{\mathcal{D}_h^\eta} \big(\partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + gz\big) dV. \tag{3.14}$$

Theorem 3.2 (Luke's Variational Principle)**.** *Let $(\Phi,\eta)\in\mathcal{M}$ satisfy*

$$\delta S[\Phi,\eta;\delta\Phi,\delta\eta] = \delta_\Phi S[\Phi,\eta;\delta\Phi] + \delta_\eta S[\Phi,\eta;\delta\eta] = 0, \tag{3.15}$$

for all $(\delta\Phi, \delta\eta) \in \delta\mathcal{M}$*, where* S *is given by* (3.14)*. Then* (Φ, η) *is a solution of* $(\mathcal{P}_{WW})$*. That is,*

$$\Delta\Phi = 0, \quad in \quad \mathcal{D}_h^\eta, \tag{3.16a}$$

$$\nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}}\Phi + \partial_z\Phi = 0, \quad on \quad \Gamma_h, \tag{3.16b}$$

$$\partial_t\eta + \nabla_{\mathbf{x}}\Phi \cdot \nabla_{\mathbf{x}}\eta - \partial_z\Phi = 0, \quad on \quad \Gamma_\eta, \tag{3.16c}$$

$$\partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + g\eta = 0, \quad on \quad \Gamma_\eta. \tag{3.16d}$$

Although this set of equations results as an application of Theorem 3.1 for $G = \partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + gz$, we present here the original proof by Luke.

Proof. Following the usual procedure of calculus of variations we calculate

$$\delta_\Phi S[\Phi, \eta; \delta\Phi] = \int_{t_0}^{T} \int_{\mathcal{D}_h^\eta} \Big\{ \partial_t\delta\Phi + \nabla_{\mathbf{x}}\Phi \cdot \nabla_{\mathbf{x}}\delta\Phi + \partial_z\Phi\, \partial_z\delta\Phi \Big\} dV\, dt.$$

Green's identity and Leibniz integral rule imply

$$\begin{aligned} \delta_\Phi S[\Phi, \eta; \delta\Phi] = \int_{t_0}^{T} \Big\{ \partial_t \int_{\mathcal{D}_h^\eta} \delta\Phi\ dV + \int_{\Gamma_\eta} \left(-\frac{\partial_t\eta}{R} + N_+ \cdot \nabla\Phi \right) \delta\Phi\ d\Gamma_\eta + \\ + \int_{\Gamma_h} N_- \cdot \nabla\Phi\delta\Phi\ d\Gamma_h - \int_{\mathcal{D}_h^\eta} \Delta\Phi\delta\Phi\ dV \Big\} dt. \end{aligned}$$

Using the isochronality of $\delta\Phi$ (i.e. $\delta\Phi(\cdot, t_0) = \delta\Phi(\cdot, T) = 0$) the first term vanishes hence,

$$\begin{aligned} \delta_\Phi S[\Phi, \eta; \delta\Phi] = \int_{t_0}^{T} \int_S \Big\{ \Big(\partial_t\eta - R_+ N_+ \cdot \big[\nabla\Phi\big]_{z=\eta} \Big) \big[\delta\Phi\big]_{z=\eta} + \\ + \int_S R_- N_- \cdot \big[\nabla\Phi\big]_{z=-h} \big[\delta\Phi\big]_{z=-h} - \int_{-h}^{\eta} \Delta\Phi\, \delta\Phi\, dz \Big\} d\mathbf{x}\, dt, \end{aligned} \tag{3.17}$$

The partial variation of $S[\Phi, \eta]$ w.r.t η is calculated exactly as in the previous more general case; see (3.11)

$$\delta_\eta S[\Phi, \eta; \delta\eta] = \int_I \int_S \Big(\big[\partial_t\Phi\big]_{z=\eta} + \frac{1}{2}\big[\nabla\Phi\big]_{z=\eta}^2 + g\eta \Big) \delta\eta\, d\mathbf{x}\, dt.$$

Substituting the above results, into equation (3.15) we obtain the variational equation

(3.18)
$$\int_{t_0}^{T}\int_{S}\Big\{\Big(\partial_t\eta - R_+N_+\cdot\big[\nabla\Phi\big]_{z=\eta}\Big)\big[\delta\Phi\big]_{z=\eta} + R_-N_-\cdot\big[\nabla\Phi\big]_{z=-h}\big[\delta\Phi\big]_{z=-h} - \\ - \int_{-h}^{\eta}\Delta\Phi\,\delta\Phi\,dz + \big[\partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + g\eta\big]_{z=\eta}\delta\eta\Big\}\,d\mathbf{x}\,dt = 0.$$

Choosing first $\delta\eta = 0$ and $\delta\Phi = 0$ on Γ_η and Γ_h we obtain (3.16a). Choosing $\delta\Phi = 0$ on Γ_η we obtain (3.16b) and consequently by the arbitariness of $\delta\Phi$ we obtain (3.16c). Finally since $\delta\eta$ are also arbitary we obtain (3.16d) □

This formulation uses the fields Φ and η and produces all the equations of the water wave problem ($\mathcal{P}_{WW}$); equations (3.16a)-(3.16d). Especially, annuling the first variation of Luke's action functional S with respect to Φ in the direction $\delta\Phi$ gives all the equations of the kinematical part of the problem (LE), (KC1), (KC2) while, annuling the variation in η implies (DC). Luke's variational principle is an unconstrained principle. Besides irrotationality, none of the kinematic conditions are considered as a priori conditions. This should be contrasted with the Hamiltonian formulation, which is possible only if one assumes a priori (besides irrotationality) that the fluid is incompressible (i.e. Φ solves (LE)), the bottom impermable (i.e Φ satisfies (KC1)) and the kinematic free surface condition (KC1) holds.

3.3 Recovering Hamilton's Equations by means of Luke's Principle

Consider the action functional introduced by Luke in [Luk67]

$$S[\Phi,\eta] = \int_I \mathcal{L}[\Phi,\eta]\,dt, \quad \mathcal{L}[\Phi,\eta] = \int_{\mathcal{D}_h^\eta}\big(\partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + gz\big)dV. \tag{3.19}$$

The stationarity of the above functional with respect to independent variations $\delta\Phi$ and $\delta\eta$ that vanish on the lateral boundaries, implies all the equations of ($\mathcal{P}_{WW}$), that we repeat here for convenience

$$\Delta\Phi = 0, \quad \text{in} \quad \mathcal{D}_h^\eta, \tag{3.20a}$$

$$\nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}}\Phi + \partial_z\Phi = 0, \quad \text{on} \quad \Gamma_h, \tag{3.20b}$$

$$\partial_t\eta = -\nabla_{\mathbf{x}}\Phi \cdot \nabla_{\mathbf{x}}\eta + \partial_z\Phi, \quad \text{on} \quad \Gamma_\eta, \tag{3.20c}$$

$$\partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + g\eta = 0, \quad \text{on} \quad \Gamma_\eta. \tag{3.20d}$$

Miles in [Mil77b] begins his analysis from an equivalent action functional, obtained the following way. Applying Leibniz integral rule to the first term of Luke's functional (3.19) we obtain

$$\int_{-h}^{\eta} \partial_t\Phi dz = \partial_t \int_{-h}^{\eta} \Phi\, dz - \partial_t\eta \big[\Phi\big]_{z=\eta}, \tag{3.21}$$

Integration by parts for the last term gives

$$\int_{-h}^{\eta} gz\, dz = \frac{1}{2}g\eta^2 - \frac{1}{2}gh^2. \tag{3.22}$$

By virtue of (3.21) and (3.22), the action functional (3.19) (after neglecting the terms that contribute only on the temporal boundaries) takes the equivalent form

$$S[\Phi,\eta] = \int_{t_0}^{T} \mathcal{L}[\Phi,\eta]\, dt, \quad \mathcal{L}[\Phi,\eta] = \int_S \partial_t\eta\, [\Phi]_{z=\eta}\, d\mathbf{x} - \mathcal{H}[\Phi,\eta], \tag{3.23}$$

where $\mathcal{H}$ is the Hamiltonian functional (total energy) (see Section 2.2)

$$\mathcal{H}[\Phi,\eta] = \frac{1}{2}\int_{\mathcal{D}_h^\eta} \big(\nabla\Phi\big)^2 dV + \frac{1}{2}\int_S g\eta^2\, d\mathbf{x}. \tag{3.24}$$

Observe that the kinetic energy in (3.24) is not expressed in surface variables. Since stationarity of S imply the full set of equations of the water wave problem (3.20a)-(3.20d), for the dependent fields Φ and η, we may suppose that solving the boundary value problem (3.20a), (3.20b) with $\big[\Phi\big]_{z=\eta} = \varphi$, we can replace the solution to (3.24) and invoke Green's identity to obtain an action functional on the surface variables φ and η [Mil77b, Sec. 2]:

$$S[\varphi,\eta] = \int_{t_0}^{T} \mathcal{L}[\varphi,\eta]\, dt, \quad \mathcal{L}[\varphi,\eta] = \int_S \partial_t\eta\, \varphi\, d\mathbf{x} - \mathcal{H}[\varphi,\eta], \tag{3.25}$$

where

$$\mathcal{H}[\varphi,\eta] = \frac{1}{2}\int_S \Big(\varphi R_+ \boldsymbol{N}_+ \cdot [\nabla\Phi]_{z=\eta} + g\eta^2\Big)\, d\mathbf{x}, \tag{3.26}$$

and $\Phi = \Phi[\varphi,\eta]$ is the solution of the boundary value problem ($\mathcal{P}_{DtN}$). Equation (3.26) is the total energy and can be written more explicitely using the chain rule to transform the spatial derivative of the potential Φ on the free surface

$$[\nabla\Phi]_{z=\eta} = \begin{pmatrix} \nabla_{\mathbf{x}}\varphi - \big[\partial_z\Phi\big]_{z=\eta}\nabla_{\mathbf{x}}\eta \\ [\partial_z\Phi]_{z=\eta} \end{pmatrix}, \tag{3.27}$$

$$[\partial_t\Phi]_{z=\eta} = \quad \partial_t\varphi - \big[\partial_z\Phi\big]_{z=\eta}\partial_t\eta. \tag{3.28}$$

Hence, substituting (3.27) in (3.26) the Hamiltonian energy functional takes the form

$$\mathcal{H}[\varphi,\eta] = \frac{1}{2}\int_S \Big(\varphi\big(-\nabla_{\mathbf{x}}\varphi\cdot\nabla_{\mathbf{x}}\eta + \zeta[\varphi,\eta]R_+^2\big) + g\eta^2\Big)\, d\mathbf{x}, \tag{3.29}$$

where $\zeta[\varphi,\eta] = \big[\partial_z\Phi\big]_{z=\eta}$acts linearly on φ and nonlinearly and non-locally on η. Stationarity of $S[\varphi,\eta]$ means

(3.30)

$$\delta S[\varphi,\eta] \quad = \quad \int_I\int_S \Big\{ \Big(\partial_t\eta - \frac{\delta\mathcal{H}}{\delta\varphi}\Big)\delta\varphi \ - \ \Big(\partial_t\varphi + \frac{\delta\mathcal{H}}{\delta\eta}\Big)\delta\eta \Big\} d\mathbf{x}\, dt \quad = \quad 0.$$

where $\frac{\delta\mathcal{H}}{\delta\varphi}$ and $\frac{\delta\mathcal{H}}{\delta\eta}$ are the variational derivatives of $\mathcal{H}$ with respect to φ and η correspondingly. They are defined by the following equations

$$\delta_\varphi\mathcal{H}[\varphi,\eta;\delta\varphi] = \int_S \frac{\delta\mathcal{H}}{\delta\varphi}\delta\varphi\, d\mathbf{x}, \quad \delta_\eta\mathcal{H}[\varphi,\eta;\delta\eta] = \int_S \frac{\delta\mathcal{H}}{\delta\eta}\delta\eta\, d\mathbf{x} \tag{3.31}$$

and they have to be determined. This implies the evolution equations

$$\partial_t\eta = \frac{\delta\mathcal{H}}{\delta\varphi}, \qquad \partial_t\varphi = -\frac{\delta\mathcal{H}}{\delta\eta}, \tag{3.32}$$

and thus the second form of Hamilton's principle is recovered. Miles in [Mil77b] did not calculate directly this functional derivatives. Instead he deduces the explicit form of (3.32) by reformulating (3.20c) and (3.20d). This is done by substituting (3.27) and (3.28) in (3.20c) and (3.20d), obtaining

$$\begin{aligned} \partial_t\eta &= \big[\partial_z\Phi\big]_{z=\eta}R_+^2 - \nabla_{\mathbf{x}}\eta\cdot\nabla_{\mathbf{x}}\varphi \\ \partial_t\varphi &= -g\eta - \frac{1}{2}\big(\nabla_{\mathbf{x}}\varphi\big)^2 + \frac{1}{2}\big[\partial_z\Phi\big]_{z=\eta}^2 R_+^2 \end{aligned} \tag{3.33}$$

where $\left[\partial_z \Phi\right]_{z=\eta}$ is found by the solution Φ of ($\mathcal{P}_{DtN}$).

Eqs. (3.33) are exactly Hamilton's equations derived by Craig and Sulem in [CS93], with the use of the DtN operator (see Chapter 2). In terms of the DtN operator the action functional whose stationarity gives Eqs. (HE) reads

$$S[\varphi,\eta] = \int_{t_0}^{T} \mathcal{L}[\varphi,\eta]\, dt, \quad \mathcal{L}[\varphi,\eta] = \int_S \partial_t \eta\, \varphi\, d\mathbf{x} - \mathcal{H}[\varphi,\eta], \tag{3.34}$$

where

$$\mathcal{H}[\varphi,\eta] = \frac{1}{2}\int_S \left(\varphi\, G(\eta,h)\varphi + g\,\eta^2\right) d\mathbf{x}. \tag{3.35}$$

4 Application of Luke's Principle in conjunction with the consistent, coupled, local-mode representation of the wave potential

The usefulness of Luke's variational principle is that, being unconditional, it allows us to use any convenient reprentation of the velocity potential, avoiding the a priori consideration of the kinematics which is neccessary in the Hamiltonian formulation. Naturally, the representation one chooses, has to satisfy all analytical and geometrical prerequisities, implied by the a priori smoothness assumptions and by the peculiarities of the domain.

For the case of linearized waves over general bathymetry regions, the idea of the sloping-bottom mode has been first introduced by Athanassoulis and Belibassakis in [AB99] in order to enable the consistent satisfaction of the Neuman boundary condition on the general bottom topography. This method was further applied - with the addition of the free surface mode - to weakly nonlinear wave interaction with large floating structures in [BA06] and to non-linear water waves in [BA11].

In the first section of this chapter we present the consistent, coupled, local-mode representation of the wave potential and state the essential properties. In the second section we introduce the represention in the variational principle for the generalised Luke's action functional presented in Chapter 3. In the third section we derive these equations for Luke's functional, making use of the properties of the representation. In the two last sections we show the connection of the derived equations with Hamilton's equations and express DtN operator in terms of the local-mode representation.

4.1 Vertical Expansion of the wave potential

Here we present in detail the consistent local-mode series expansion, in order to represent the velocity potential field ([AB02]):

(4.1)
$$\begin{aligned}\Phi(\mathbf{x},z,t) &= \varphi_{-2}(\mathbf{x},t)Z_{-2}(z;\eta,h)+\varphi_{-1}(\mathbf{x},t)Z_{-1}(z;\eta,h)+\sum_{n\geq 0}\varphi_n(\mathbf{x},t)Z_n(z;\eta,h)\\ &= \sum_{n\geq -2}\varphi_n(\mathbf{x},t)Z_n(z;\eta,h) =: \Phi[\boldsymbol{\varphi},\eta],\end{aligned}$$

where we denoted $\boldsymbol{\varphi}=(\varphi_{-2},\varphi_{-1},\varphi_0,...)=\{\varphi_n\}_{n\geq -2}$. The field φ_{-2} will be reffered as the *free surface mode*, φ_{-1} as *bottom surface mode*, φ_0 as the *propagating mode* and $\{\varphi_n\}_{n\geq 1}$ the *evanecent modes*. The vertical functions $Z_n(z;\eta,h)$ are chosen such that the $\{Z_n\}_{n\geq 0}$ is an $L^2\big((-h,\eta)\big)$ basis which can be obtained as a solution of a Sturm-Liouville boundary eigenvalue problem ([CL72]) for the self-adjoint operator d^2/dz^2 on the family of intervals $J_h^\eta=[-h(\mathbf{x}),\eta(\mathbf{x},t)]$, $t\geq t_0$

$$\frac{d^2}{dz^2}Z_n-k_n^2Z_n=0, \tag{4.2a}$$

$$\big[\partial_z Z_n-\mu_0 Z_n\big]_{z=\eta}=0, \tag{4.2b}$$

$$\big[\partial_z Z_n\big]_{z=-h}=0. \tag{4.2c}$$

The solution of the above above eigenvalue problem leads to a sequence of eigenfunctions $\{Z_n\}_{n\geq 0}$ given by

$$Z_0(z;\eta,h)=\frac{\cosh[k_0(z+h)]}{\cosh[k_0(\eta+h)]},\ \ Z_n(z;\eta,h)=\frac{\cos[k_n(z+h)]}{\cos[k_n(\eta+h)]},\ \ n=1,2,...,$$

The numerical parameters $\mu_0=\omega^2/g$, h_0 are positive constants, not subjected to any a-priori restrictions. Moreover, the z-independent quantities $k_0=k_0(\eta,h)$ and $k_n=k_n(\eta,h)$ are defined as the positive roots of the transcendental equations,

$$\mu_0-k_0\tanh(k_0(\eta+h))=0,\quad \mu_0+k_n\tan(k_n(\eta+h))=0. \tag{4.3}$$

Note that for $n\geq 0$ the functions $Z_n=Z_n(z;\eta,h)=Z_n(z;\eta,h,k_n(\eta,h))$ depend both explixitely and implicitely on η and h. The expressions of $Z_{-2}(z;\eta,h)$, $Z_{-1}(z;\eta,h)$ are given by

$$Z_{-2}(z;\eta,h)=\frac{\mu_0h_0+1}{2h_0(\eta+h)}(z+h)^2-\frac{\mu_0h_0+1}{2h_0}(\eta+h)+1, \tag{4.4}$$

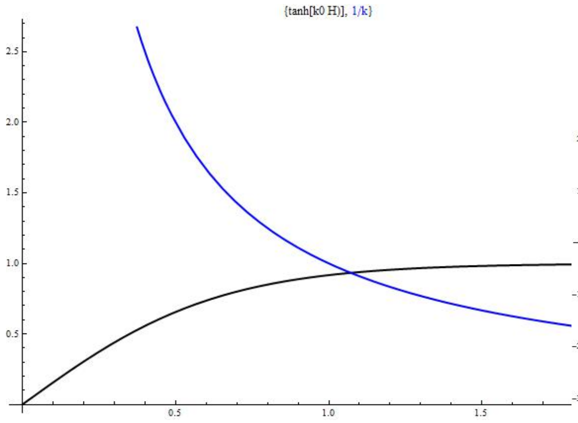

Figure 4.1: Eigenvalue k_0

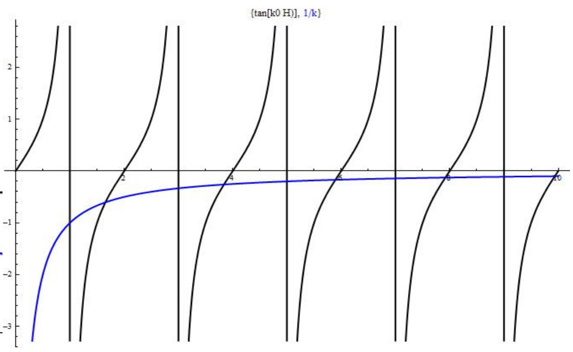

Figure 4.2: Eigenvalues k_n

$$Z_{-1}(z;\eta,h) = \frac{\mu_0 h_0 - 1}{2h_0(\eta+h)}(z+h)^2 + \frac{1}{h_0}(\eta+h) + \frac{2h_0 - (\eta+h)(\mu h_0 + 1)}{2h_0}, \tag{4.5}$$

The functions Z_{-2}, Z_{-1} are second order polynomials in z, satisfying the following boundary conditions

$$\begin{aligned} \big[\partial_z Z_{-2} - \mu_0 Z_{-2}\big]_{z=\eta} = \frac{1}{h_0}, \quad [\partial_z Z_{-2}]_{z=-h} = 0, \\ [\partial_z Z_{-1}]_{z=\eta} = 0, \quad [\partial_z Z_{-1}]_{z=-h} = \frac{1}{h_0}. \end{aligned} \tag{4.6}$$

Notice that the eigenfunctions spanning $L^2(-h,\eta)$ and the additional functions Z_{-2}, Z_{-1} have been selected so that

$$[Z_n]_{z=\eta} = 1, \quad n \geq -2. \tag{4.7}$$

More details about the construction and validity of the expansion (4.1) can be found in [BA11]. Equation (4.7) implies

$$[\Phi]_{z=\eta} = [\Phi[\boldsymbol{\varphi},\eta]]_{z=\eta} = \sum_{n\geq -2} \varphi_n := \varphi. \tag{4.8}$$

It is easy to verify that the vertical derivatives on the free and bottom surfaves are given by

$$[\partial_z \Phi - \mu\Phi]_{z=\eta} = \frac{\varphi_{-2}}{h_0}, \tag{4.9a}$$

$$[\partial_z \Phi]_{z=-h} = \frac{\varphi_{-1}}{h_0}. \tag{4.9b}$$

The spatial and temporal derivatives of Φ involved in the Lagrangian density take the following form (denoting $\partial/\partial\eta = \partial_\eta$ etc.)

$$
\begin{aligned}
\partial_t\Phi &= \sum_{n\geq -2} \partial_t\varphi_n Z_n + \varphi_n \partial_t Z_n \\
&= \sum_{n\geq -2} \partial_t\varphi_n Z_n + \varphi_n (\partial_\eta Z_n)(\partial_t \eta),
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
\nabla\Phi &= \sum_{n\geq -2} \begin{pmatrix} \nabla_{\mathbf{x}}\varphi_n Z_n + \varphi_n \nabla_{\mathbf{x}} Z_n \\ \varphi_n \partial_z Z_n \end{pmatrix} \\
&= \begin{pmatrix} \nabla_{\mathbf{x}}\varphi_n Z_n + \varphi_n \Big(\nabla_{\mathbf{x}}\eta(\partial_\eta Z_n) + \nabla_{\mathbf{x}} h(\partial_h Z_n)\Big) \\ \varphi_n \partial_z Z_n \end{pmatrix}
\end{aligned} \tag{4.11}
$$

It is usefull to calculate the partial derrivatives of Z_n with respect to the surface elevation η, the bottom surface h and the parameters k_n, $n \geq 0$. Note that the functions $\{Z_n\}_{n=-2,1}$ depend explicitelly on η and h, while $\{Z_n\}_{n\geq 0}$ depend also implicitelly on η and h through $k_n(\eta, h)$. Their values for $z = \eta$, after taking into account the dispersion relationships (see Apendix (A)) become

$$
\begin{aligned}
\big[\partial_\eta Z_{-2}\big]_{z=\eta} &= -\frac{1}{h_0} - \mu_0, & \big[\partial_h Z_{-2}\big]_{z=\eta} &= 0, \\
\big[\partial_\eta Z_{-1}\big]_{z=\eta} &= -\mu_0, & \big[\partial_h Z_{-1}\big]_{z=\eta} &= 0, \\
\big[\partial_\eta Z_{0}\big]_{z=\eta} &= -\mu_0, & \big[\partial_h Z_{0}\big]_{z=\eta} &= \big[\partial_{k_0} Z_0\big]_{z=\eta} = 0, \\
\big[\partial_\eta Z_{n}\big]_{z=\eta} &= -\mu_0, & \big[\partial_h Z_{n}\big]_{z=\eta} &= \big[\partial_{k_n} Z_n\big]_{z=\eta} = 0,
\end{aligned} \tag{4.12}
$$

The above formulae can be written in compact form as follows ([BA11])

$$
\big[\partial_\eta Z_n\big]_{z=\eta} = -[\partial_z Z_n]_{z=\eta} = -\big(\frac{\delta_{-2n}}{h_0} + \mu_0\big), \quad n \geq -2, \tag{4.13}
$$

where δ_{mn} denotes Kronecker's delta. Using the properties of $\{Z_n\}$ given by (4.7)-(4.12) we obtain the following expressions of $\partial_t\Phi$ and $\nabla\Phi$ on the free surface

$$
[\nabla\Phi]_{z=\eta} = \begin{pmatrix} \big[\nabla_{\mathbf{x}}\Phi\big]_{z=\eta} \\ [\partial_z\Phi]_{z=\eta} \end{pmatrix} = \begin{pmatrix} \nabla_{\mathbf{x}}\varphi + \Big(-\varphi_{-2}/h_0 - \mu_0\varphi\Big)\nabla_{\mathbf{x}}\eta \\ \varphi_{-2}/h_0 + \mu_0\varphi \end{pmatrix}, \tag{4.14}
$$

$$
[\partial_t\Phi]_{z=\eta} = \partial_t\varphi - \Big(\varphi_{-2}/h_0 + \mu_0\varphi\Big)\partial_t\eta. \tag{4.15}
$$

4.2 g-Luke's action functional expressed in terms of the local mode representation

Introducing the representation (4.1) in the variational principle, we exchange the unknown field $\Phi : [t_0, T) \times \mathcal{D}_h^\eta \to \mathbb{R}$ with the infinite unknown coefficients of the expansion i.e. the fields $\varphi_n : [t_0, T) \times S \to \mathbb{R}, \quad n \geq -2$. In the next subsection we treat the general functional $\tilde{S}$ as a composition of S with $\Phi[\boldsymbol{\varphi}, \eta]$.

First Variation of Composite Functionals

We briefly recall here some standard facts about differentiation of composite functionals. Let as set

$$\tilde{\mathcal{M}}_{\boldsymbol{\varphi}} \times \mathcal{M}_\eta := C^1\big([t_0,T) \to C^2(S)\big) \times C^1\big([t_0,T) \to C^2(S)\big)... \times C^1\big([t_0,T) \to C^1(S)\big),$$

and assume that we have $\Phi = \Phi[\boldsymbol{\varphi}, \eta]$ as a mapping $\tilde{\mathcal{M}}_{\boldsymbol{\varphi}} \times \mathcal{M}_\eta \to \mathcal{M}$, which is also Fréchet differentiable. Consider the action functional $\tilde{S} : \tilde{\mathcal{M}}_{\boldsymbol{\varphi}} \times \mathcal{M}_\eta \to \mathbb{R}$ given by,

$$\tilde{S}[\boldsymbol{\varphi}, \eta] = S[\Phi[\boldsymbol{\varphi}, \eta], \eta] = \int_{t_0}^{T} \mathcal{G}[\Phi[\boldsymbol{\varphi}, \eta], \eta] dt. \tag{4.16}$$

Composing the Lagrangian density functional $\mathcal{G}[\Phi, \eta]$ with $\Phi = \Phi[\boldsymbol{\varphi}, \eta]$, we obtain the Lagrangian density $\tilde{\mathcal{G}}$ as a functional on $\boldsymbol{\varphi} = \{\varphi_n\}$, $n \geq -2$ and η, that is,

$$\tilde{\mathcal{G}}[\boldsymbol{\varphi}, \eta] = \mathcal{G}[\Phi[\boldsymbol{\varphi}, \eta], \eta] = \int_{\mathcal{D}_h^\eta} G\big(\partial_t \Phi[\boldsymbol{\varphi}, \eta], \nabla_{\mathbf{x}} \Phi[\boldsymbol{\varphi}, \eta], \partial_z \Phi[\boldsymbol{\varphi}, \eta], z\big)\, dV. \tag{4.17}$$

We also assume that the variations $\delta\boldsymbol{\varphi}(\mathbf{x}, t) = (\delta\varphi_{-2}, \delta\varphi_{-1}, \delta\varphi_0, ...)$ are isochronal. The variation of $\tilde{S}$ is by definition

$$\delta\tilde{S}[\boldsymbol{\varphi}, \eta; \delta\boldsymbol{\varphi}, \delta\eta] = \sum_{m \geq -2} \delta_{\varphi_m} \tilde{S}[\boldsymbol{\varphi}, \eta; \delta\varphi_m] + \delta_\eta \tilde{S}[\boldsymbol{\varphi}, \eta; \delta\eta] \tag{4.18}$$

where $\delta_{\varphi_m} \tilde{S}[\boldsymbol{\varphi}, \eta; \delta\varphi_m]$ is the partial variation of $\tilde{S}$ in the direction $\delta\varphi_m$ and $\delta_\eta \tilde{S}[\boldsymbol{\varphi}, \eta; \delta\eta]$ is the partial variation of $\tilde{S}$ in the direction $\delta\eta$. We proceed by considering $\tilde{S}$ as the composition of the functional S with $\Phi = \Phi[\boldsymbol{\varphi}, \eta]$. The composition rule for such a functional reads (see [LV00, Lem. 3.1.1], [MRA01,])

$$\delta_{\varphi_m} \tilde{S}[\boldsymbol{\varphi}, \eta; \delta\varphi_m] \;=\; \delta_\Phi S\big[\, \Phi[\boldsymbol{\varphi}, \eta]\,,\, \eta\,;\, \delta_{\varphi_m} \Phi[\boldsymbol{\varphi}, \eta; \delta\varphi_m] \,\big], \quad m \geq -2, \tag{4.19a}$$

$$(4.19b) \qquad \delta_\eta \tilde{S}[\boldsymbol{\varphi}, \eta; \delta\eta] = \delta_\Phi S\big[\,\Phi[\boldsymbol{\varphi}, \eta]; \delta_\eta \Phi[\boldsymbol{\varphi}, \eta; \delta\eta]\,\big] + \delta_\eta S\big[\Phi[\boldsymbol{\varphi}, \eta]; \delta\eta\,\big],$$

where $\delta_{\varphi_m}\Phi[\boldsymbol{\varphi}, \eta; \delta\varphi_m]$ and $\delta_\eta\Phi[\boldsymbol{\varphi}, \eta; \delta\eta]$ denote the partial variations of $\Phi[\boldsymbol{\varphi}, \eta]$ with respect to φ_m and η, correspondingly.

Calculation of the partial variations of g-Luke's functional expressed in terms of the local-mode representation

In this subsection we apply the previous facts in order to calculate the partial variations of g-Luke's functional expressed in terms of the local-mode representation given by (4.1). From the form of the partial variations (4.19a) and (4.19b) we already see that the condition of stationarity of the functional $\tilde{S}$:

$$(4.20) \qquad \delta\tilde{S} = \sum_{m\geq -2} \delta_{\varphi_m}\tilde{S}[\boldsymbol{\varphi}, \eta; \delta\varphi_m] + \delta_\eta \tilde{S}[\boldsymbol{\varphi}, \eta; \delta\eta] = 0,$$

contains the variation of the representation $\Phi[\boldsymbol{\varphi}, \eta]$ with respect to η. In what follows we use the notation

$$(4.21) \qquad \big\langle f, g \big\rangle = \int_{-h(\mathbf{x})}^{\eta(\mathbf{x},t)} f(\mathbf{x}, z) g(\mathbf{x}, z) dz.$$

We easily see from (4.1) that for every $m \geq -2$ the partial Fréchet derivatives of $\Phi[\boldsymbol{\varphi}, \eta]$ with respect to φ_m are

$$(4.22) \qquad \delta_{\varphi_m}\Phi[\boldsymbol{\varphi}, \eta; \delta\varphi_m] = Z_m \delta\varphi_m, \quad m \geq -2$$

which implies that the partial Fréchet derivative of $\Phi[\boldsymbol{\varphi}, \eta]$ with respect to $\boldsymbol{\varphi}$ is

$$(4.23) \qquad \delta_{\boldsymbol{\varphi}}\Phi[\boldsymbol{\varphi}, \eta; \delta\boldsymbol{\varphi}] = \sum_{m\geq -2} Z_m \delta\varphi_m.$$

and the partial Fréchet derivative of $\Phi[\boldsymbol{\varphi}, \eta]$ with respect to η

$$(4.24) \qquad \delta_\eta\Phi[\boldsymbol{\varphi}, \eta; \delta\eta] = \Big(\sum_{m\geq -2} \varphi_m \, \partial_\eta Z_m \Big) \delta\eta.$$

The partial derivatives of the functions Z_n with respect to η and k_n are calculated in the Appendix (A). Substituting (4.13) in (4.24), we also obtain

$$(4.25) \qquad \big[\delta_\eta \Phi[\boldsymbol{\varphi}, \eta; \delta\eta]\big]_{z=\eta} = (\varphi_{-2}/h_0 + \mu\varphi)\delta\eta.$$

Lemma 4.1. *Let $\tilde{S} : \tilde{\mathcal{M}} \to \mathbb{R}$ be a functional such that*

$$\tilde{S}[\boldsymbol{\varphi}, \eta] := S\big[\Phi[\boldsymbol{\varphi}, \eta], \eta\big] = \int_{t_0}^{T} \int_{\mathcal{D}_h^\eta} G\big(\partial_t \Phi[\boldsymbol{\varphi}, \eta], \nabla_{\mathbf{x}} \Phi[\boldsymbol{\varphi}, \eta], \partial_z \Phi[\boldsymbol{\varphi}, \eta], z\big) dV dt.$$

Then the for $m \geq -2$ the partial variation of $\tilde{S}$ at $(\boldsymbol{\varphi}, \eta) = (\varphi_{-2}, \varphi_{-1}, \varphi_0, ..., \eta)$ in the direction $\delta\varphi_m$ is

$$\begin{aligned} \delta_{\varphi_m} \tilde{S}[\boldsymbol{\varphi}, \eta; \delta\varphi_m] = \int_{t_0}^{T} \int_S \Big\{ &- \partial_t \eta + \big[\big(\overrightarrow{D}_2 G, D_3 G\big) \cdot R_+ \boldsymbol{N}_+\big]_{z=\eta} + \\ &+ \Big\langle - \mathit{div}(\overrightarrow{D}_2 G, D_3 G)\, , Z_m \Big\rangle + \\ &+ \big[\big(\overrightarrow{D}_2 G, D_3 G\big) \cdot \ R_- \boldsymbol{N}_- Z_m\big]_{z=-h} \Big\} \delta\varphi_m d\mathbf{x} dt, \end{aligned} \tag{4.26}$$

where $G \equiv G\big(\partial_t \Phi[\boldsymbol{\varphi}, \eta], \nabla_{\mathbf{x}} \Phi[\boldsymbol{\varphi}, \eta], \partial_z \Phi[\boldsymbol{\varphi}, \eta], z\big)$, $\boldsymbol{N}_+ = R_+^{-1}\big(- \nabla_{\mathbf{x}} \eta, 1\big)$ and $\boldsymbol{N}_- = R_-^{-1}\big(- \nabla_{\mathbf{x}} h, -1\big)^T$ are the outward unit normal vectors on Γ_η and Γ_h respectively and $R_+ = \big((\nabla_{\mathbf{x}} \eta)^2 + 1\big)^{1/2}$, $R_- = \big((\nabla_{\mathbf{x}} h)^2 + 1\big)^{1/2}$ are the corresponding scalar functions.

Proof. From (3.10) we know that

$$\begin{aligned} \delta_\Phi S[\Phi, \eta; \delta\Phi] = \int_{t_0}^{T} \int_S \Big\{ &- \int_{-h}^{\eta} \big(\mathrm{div}(\overrightarrow{D}_2 G, D_3 G)\, \delta\Phi\,\big)\, dz + \\ &+ \big[(\overrightarrow{D}_2 G, D_3 G)\big]_{z=\eta} \cdot R_+ \boldsymbol{N}_+ - \partial_t \eta\Big) \big[\delta\Phi\big]_{z=\eta} \\ &+ \Big(\big[(\overrightarrow{D}_2 G, D_3 G)\big]_{z=-h} \cdot R_- \boldsymbol{N}_- \Big) \big[\delta\Phi\big]_{z=-h} \Big\} d\mathbf{x} dt. \end{aligned} \tag{4.27}$$

From the composition rule (4.19a) we obtain

$$\begin{aligned} \delta_{\varphi_m} \tilde{S}[\boldsymbol{\varphi}, \eta\,;\, \delta\varphi_m] &= \delta_\Phi S\big[\, \Phi[\boldsymbol{\varphi}, \eta], \eta; \delta_{\varphi_m} \tilde{\Phi}[\boldsymbol{\varphi}, \eta; \delta\varphi_m]\,\big] \\ &= \int_{t_0}^{T} \int_S \Big\{ - \int_{-h}^{\eta} \big(\mathrm{div}(\overrightarrow{D}_2 G, D_3 G)\big) \big(\delta_{\varphi_m} \Phi[\boldsymbol{\varphi}, \eta; \delta\varphi_m]\big)\, dz + \\ &\quad + \Big(\big[(\overrightarrow{D}_2 G, D_3 G)\big]_{z=\eta} \cdot R_+ \boldsymbol{N}_+ - \partial_t \eta\Big) \big[\delta_{\varphi_m} \Phi[\boldsymbol{\varphi}, \eta; \delta\varphi_m]\big]_{z=\eta}\Big) d\mathbf{x} dt \\ &\quad + \Big(\big[(\overrightarrow{D}_2 G, D_3 G)\big]_{z=-h} \cdot R_- \boldsymbol{N}_- \Big) \big[\delta_{\varphi_m} \Phi[\boldsymbol{\varphi}, \eta; \delta\varphi_m]\big]_{z=-h} \Big\} d\mathbf{x} dt \\ &= \int_{t_0}^{T} \int_S \Big\{ - \Big\langle \mathrm{div}(\overrightarrow{D}_2 G, D_3 G), Z_m \Big\rangle + \\ &\quad + \Big(\big[(\overrightarrow{D}_2 G, D_3 G)^T\big]_{z=\eta} \cdot R_+ \boldsymbol{N}_+ - \partial_t \eta\Big) \big[Z_m\big]_{z=\eta} \\ &\quad + \Big(\big[(\overrightarrow{D}_2 G, D_3 G)^T\big]_{z=-h} \cdot R_- \boldsymbol{N}_- \Big) \big[Z_m\big]_{z=-h} \Big\} \delta\varphi_m d\mathbf{x} dt \end{aligned}$$

Where (4.22) have been used. Using the fact that $[Z_n]_{z=\eta} = 1$ we obtain the result. $\square$

Lemma 4.2. *The partial variation of $\tilde{S}$ at $(\boldsymbol{\varphi}, \eta) = (\varphi_{-2}, \varphi_{-1}, \varphi_0, ..., \eta)$ in the direction $\delta\eta$ is*

$$\begin{aligned}\delta_\eta \tilde{S}[\boldsymbol{\varphi}, \eta; \delta\eta] = \int_{t_0} \int_S \Big\{ \sum_{l \geq -2} \Big(&- \langle div(\overrightarrow{D}_2 G, D_3 G), \partial_\eta Z_l \rangle + \\ &+ \Big(\big[(\overrightarrow{D}_2 G, D_3 G) \big]_{z=\eta} \cdot R_+ \boldsymbol{N}_+ - \partial_t \eta \Big) \big[\partial_\eta Z_l \big]_{z=\eta} \\ &+ \big[(\overrightarrow{D}_2 G, D_3 G) \big]_{z=-h} \cdot R_- \boldsymbol{N}_- \big[\partial_\eta Z_l \big]_{z=-h} \Big) \varphi_l + [G]_{z=\eta} \Big\} \delta\eta d\mathbf{x} dt\end{aligned}$$

where $G \equiv G\big(\partial_t \Phi[\boldsymbol{\varphi}, \eta], \nabla_{\mathbf{x}} \Phi[\boldsymbol{\varphi}, \eta], \partial_z \Phi[\boldsymbol{\varphi}, \eta], z\big)$, $\boldsymbol{N}_+ = R_+^{-1}\big(- \nabla_{\mathbf{x}} \eta, 1\big)$ and $\boldsymbol{N}_- = R_-^{-1}\big(- \nabla_{\mathbf{x}} h, -1\big)^T$ are the outward unit normal vectors on Γ_η and Γ_h respectively, $R_+ = \big((\nabla_{\mathbf{x}} \eta)^2 + 1\big)^{1/2}$, $R_- = \big((\nabla_{\mathbf{x}} h)^2 + 1\big)^{1/2}$ are the corresponding scalar functions and $\partial_\eta Z_l = \partial Z_l / \partial \eta$ are given in the Appendix (A).

Proof. From (3.10) and (4.19b)

$$\begin{aligned}\delta_\eta \tilde{S}[\boldsymbol{\varphi}, \eta; \delta\eta] = \int_{t_0}^T \int_S \Big\{ \int_{-h}^{\eta} &-\big(\mathrm{div}(\overrightarrow{D}_2 G, D_3 G)\big) \delta_\eta \Phi[\boldsymbol{\varphi}, \eta; \delta\eta] \, dz + \\ &+ \Big(\big[(\overrightarrow{D}_2 G, D_3 G)^T \big]_{z=\eta} \cdot R_+ \boldsymbol{N}_+ - \partial_t \eta \Big) \big[\delta_\eta \Phi[\boldsymbol{\varphi}, \eta; \delta\eta] \big]_{z=\eta} \Big) + \\ &+ \Big(\big[(\overrightarrow{D}_2 G, D_3 G)^T \big]_{z=-h} \cdot R_- \boldsymbol{N}_- \Big) \big[\delta_\eta \Phi[\boldsymbol{\varphi}, \eta; \delta\eta] \big]_{z=-h} + \big[G \big]_{z=\eta} \delta\eta \Big\} d\mathbf{x} dt \\ = \int_{t_0} \int_S \Big\{ \sum_{m \geq -2} \Big(&- \langle \mathrm{div}(\overrightarrow{D}_2 G, D_3 G), \partial_\eta Z_m \rangle \varphi_m + \\ &+ \Big(\big[(\overrightarrow{D}_2 G, D_3 G)^T \big]_{z=\eta} \cdot R_+ \boldsymbol{N}_+ - \partial_t \eta \Big) \big[\partial_\eta Z_m \big]_{z=\eta} \varphi_m \\ &+ \big[(\overrightarrow{D}_2 G, D_3 G)^T \big]_{z=-h} \cdot R_- \boldsymbol{N}_- \big[\partial_\eta Z_m \big]_{z=-h} \varphi_m \Big) + \big[G \big]_{z=\eta} \Big\} \delta\eta \, d\mathbf{x} \, dt.\end{aligned}$$

where (4.24) have been used. $\square$

4.2.1 Stationarity of the g-Luke's action functional expressed in terms of the local-mode representation

We can write the variational equation (4.18) as,

$$\delta \tilde{S}[\boldsymbol{\varphi}, \eta; \delta\boldsymbol{\varphi}, \delta\eta] = \int_{t_0}^T \int_S \Big\{ \sum_{m \geq -2} \frac{\delta \tilde{\mathcal{G}}}{\delta \varphi_m} \delta\varphi_m + \frac{\delta \tilde{\mathcal{G}}}{\delta \eta} \delta\eta \Big\} \, d\mathbf{x} \, dt = 0, \tag{4.28}$$

for all $\delta\varphi_m$, $m \geq -2$ and for all $\delta\eta$, where $\delta\tilde{\mathcal{G}}/\delta\varphi_m$ and $\delta\tilde{\mathcal{G}}/\delta\eta$ denote the variational derivatives of $\tilde{\mathcal{G}}[\boldsymbol{\varphi},\eta] = \mathcal{G}\big[\Phi[\boldsymbol{\varphi},\eta]\big]$ and are given by the expressions in curly brackets in the statements of Lemma (4.1) and Lemma (4.2) correspondigly. Choosing first $\delta\eta = 0$ and all $\delta\varphi_m$ to vanish expect one we obtain

$$\frac{\delta\tilde{\mathcal{G}}}{\delta\varphi_{m_*}} = 0, \quad \text{for some} \quad m_* \geq -2 \tag{4.29}$$

Repeating this procedure we obtain the following infinite system

$$\frac{\delta\tilde{\mathcal{G}}}{\delta\varphi_m} = 0, \quad m \geq -2. \tag{4.30}$$

Choosing, now, all $\delta\varphi_m$ to be zero and $\delta\eta$ arbitary we obtain

$$\frac{\delta\tilde{\mathcal{G}}}{\delta\eta} = 0. \tag{4.31}$$

Finally, the Euler-Lagrange equations for $\tilde{S}$ are

$$\frac{\delta\tilde{\mathcal{G}}}{\delta\varphi_m} = 0, \quad m \geq -2 \quad \text{and} \quad \frac{\delta\tilde{\mathcal{G}}}{\delta\eta} = 0. \tag{4.32}$$

4.3 Stationarity of Luke's functional expressed in terms of the local mode representation and the Coupled Mode System (CMS)

Using the expansion (4.1) into the Lagrangian density function introduced in [Luk67], we obtain

$$L = L(\partial_t\Phi[\boldsymbol{\varphi},\eta], \nabla\Phi[\boldsymbol{\varphi},\eta], z) = \partial_t\Phi[\boldsymbol{\varphi},\eta] + \frac{1}{2}\big(\nabla\Phi[\boldsymbol{\varphi},\eta]\big)^2 + gz. \tag{4.33}$$

The Lagrangian density functional then reads

$$\tilde{\mathcal{L}}[\boldsymbol{\varphi},\eta] = \mathcal{L}[\Phi[\boldsymbol{\varphi},\eta],\eta] = \int_{\mathcal{D}_h^\eta} L(\partial_t\Phi[\boldsymbol{\varphi},\eta], \nabla\Phi[\boldsymbol{\varphi},\eta], z)\, dV. \tag{4.34}$$

Now $\tilde{\mathcal{L}}$ can be written

(4.35)

$$\tilde{\mathcal{L}}[\boldsymbol{\varphi},\eta] = \int_S \Big\{ \int_{-h}^{\eta} \sum_{n\geq -2} \big(\partial_t\varphi_n\, Z_n + \varphi_n\, \partial_t Z_n\big) dz +$$
$$+ \frac{1}{2} \sum_{n\geq -2} \sum_{m\geq -2} \big(\mathcal{A}_{mn} \nabla_{\mathbf{x}}\varphi_n \cdot \nabla_{\mathbf{x}}\varphi_m + 2\vec{\mathcal{B}}_{mn} \cdot \nabla_{\mathbf{x}}\varphi_n\varphi_m + \mathcal{C}_{mn}\varphi_n\varphi_m + \mathcal{D}_{mn}\varphi_n\varphi_m\big) + g\eta^2 \Big\} d\mathbf{x},$$

where $\mathcal{A}_{mn}$, $\vec{\mathcal{B}}_{mn}$, $\mathcal{C}_{mn}$ and $\mathcal{D}_{mn}$ are vertical integrals given by

$$\mathcal{A}_{mn} = \left\langle Z_n, Z_m \right\rangle = \mathcal{A}_{nm}, \tag{4.36a}$$

$$\vec{\mathcal{B}}_{mn} = \left\langle Z_n, \nabla_{\mathbf{x}} Z_m \right\rangle, \tag{4.36b}$$

$$\mathcal{C}_{mn} = \left\langle \nabla_{\mathbf{x}} Z_n, \nabla_{\mathbf{x}} Z_m \right\rangle = \mathcal{C}_{nm}, \tag{4.36c}$$

$$\mathcal{D}_{mn} = \left\langle \partial_z Z_n, \partial_z Z_m \right\rangle = \mathcal{D}_{nm}. \tag{4.36d}$$

Observe that the Hamiltonian energy, appearing in the Lagrangian (4.35), in terms of the local-mode representation, is given by

$$\begin{aligned} &\tilde{\mathcal{H}}[\boldsymbol{\varphi}, \eta] = \\ &= \int_S \left\{ \frac{1}{2} \big(\mathcal{A}_{mn} \nabla_{\mathbf{x}} \varphi_n \cdot \nabla_{\mathbf{x}} \varphi_m + 2\vec{\mathcal{B}}_{mn} \cdot \nabla_{\mathbf{x}} \varphi_n \varphi_m + \mathcal{C}_{mn} \varphi_n \varphi_m + \mathcal{D}_{mn} \varphi_n \varphi_m \big) + g\eta^2 \right\} d\mathbf{x} \end{aligned} \tag{4.37}$$

or alternativelly by,

$$\begin{aligned} &\tilde{\mathcal{H}}[\boldsymbol{\varphi}, \eta] = \\ &= \int_S \left\{ \frac{1}{2} \big(\mathcal{A}_{mn} \nabla_{\mathbf{x}} \varphi_n \cdot \nabla_{\mathbf{x}} \varphi_m + 2\vec{\mathcal{B}}_{mn} \cdot \nabla_{\mathbf{x}} \varphi_n \varphi_m + \mathcal{C}'_{mn} \varphi_n \varphi_m \big) + g\eta^2 \right\} d\mathbf{x} \end{aligned} , \tag{4.38}$$

where

$$\mathcal{C}'_{mn} = \left\langle \nabla Z_n \cdot \nabla Z_m \right\rangle . \tag{4.39}$$

The action functional is the time integral of $\tilde{\mathcal{L}}$, i.e.

$$\tilde{S}[\boldsymbol{\varphi}, \eta] = S[\Phi[\boldsymbol{\varphi}, \eta], \eta] = \int_{t_0}^{T} \mathcal{L}[\Phi[\boldsymbol{\varphi}, \eta], \eta] dt. \tag{4.40}$$

Stationarity means that the first variation of $\tilde{S}$ is zero. i.e.

$$\delta \tilde{S}[\boldsymbol{\varphi}, \eta; \delta\boldsymbol{\varphi}, \delta\eta] = \sum_{m \geq -2} \delta_{\varphi_m} \tilde{S}[\boldsymbol{\varphi}, \eta; \delta\varphi_m] + \delta_\eta \tilde{S}[\boldsymbol{\varphi}, \eta; \delta\eta] = 0. \tag{4.41}$$

We could proceed by calculating the variations using (4.35). This, indeed, would be the only way to find the variation $\delta\tilde{S}$, in the case where simplifications are made concerning the horizontal velocity $\nabla_{\mathbf{x}}\Phi$ or the bottom. For this simplified representation, $\tilde{S}$ would not be exactly equal to S and we can only say that $\tilde{S} \approx S \circ \Phi[\boldsymbol{\varphi}, \eta]$ and we proceed by varying the simplified form of (4.35). For example, Klopman et al. in [KGD10], use a simplified Hamiltonian, after introducing a general series representation for the velocity potential, which contains vertical integrals of the form of $\mathcal{A}_{mn}$ and $\mathcal{D}_{mn}$. However the integrals given by $\vec{\mathcal{B}}_{mn}$ and $\mathcal{C}_{mn}$ have simplified forms. Specifically the terms containing the derivatives of the vertical functions with respect to the depth h and the parameters k_n are supressed. A similar situation is occured in [CD12] where several approximate models are derived. There, the authors use ansatzes to represent both the potential and the velocity. In the case of arbitary depth a product of a vertical hyperbolic cosine function (Z_0 in the present notation) with the trace of the potential is chosen to represent the velocity potential. The partial derivatives with respect to the free surface elevation, depth and the parameter are not kept in the representation of the horizontal velocity. This can be seen if one takes the horizontal derivative of the ansatz for the potential and compares it with the ansatz used for the velocity. We mention that these terms are important when modelling flows over steep bottoms and for wave reflections as is described in [DK09] for the case of linearised waves.

Here, since no simplifications are made, and the involved arguments in the Lagrangian equal exactly their corresponding representations through (4.1), we can apply directly the previous results, when we vary $\tilde{S} = S \circ \Phi[\boldsymbol{\varphi}, \eta]$. The resulting Euler-Lagrange equations were first appeared in [AB02]. First we calculate the partial variations of $\tilde{S}$ at $(\boldsymbol{\varphi}, \eta)$ in the direction $\delta\varphi_m$, $m \geq -2$ and $\delta\eta$.

Lemma 4.3. *The partial variation of the functional $\tilde{S}[\boldsymbol{\varphi}, \eta]$ at $(\boldsymbol{\varphi}, \eta)$ in the direction $\delta\varphi_m$, for $m \geq -2$ is given by*

$$\begin{aligned}\delta_{\varphi_m}\tilde{S}[\boldsymbol{\varphi}, \eta; \delta\varphi_m] = \int_I\int_S \Big\{ &- \partial_t\eta - \nabla_{\mathbf{x}}\eta \cdot \nabla_{\mathbf{x}}\varphi + R_+^2\Big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\Big) - \\ &- \sum_{n\geq -2} A_{mn}(\eta, h)\nabla^2_{\mathbf{x}}\varphi_n + \vec{B}_{mn}(\eta, h)\cdot\nabla_{\mathbf{x}}\varphi_n + C_{mn}(\eta, h)\varphi_n\Big\}\delta\varphi_m\, d\mathbf{x}\, dt,\end{aligned}$$

where $R_+ = \sqrt{\big(\nabla_{\mathbf{x}}\eta\big)^2 + 1}$ and

$$\begin{aligned}A_{mn}(\eta, h) =& \langle Z_n, Z_m\rangle = A_{nm},\\ \vec{B}_{mn}(\eta, h) =& 2\,\langle \nabla_{\mathbf{x}}Z_n, Z_m\rangle + \nabla_{\mathbf{x}}h[Z_nZ_m]_{z=-h},\\ C_{mn}(\eta, h) =& \langle \Delta Z_n, Z_m\rangle + \Big[\big(\nabla_{\mathbf{x}}h\cdot\nabla_{\mathbf{x}}Z_n + \partial_z Z_n\big)Z_m\Big]_{z=-h}\\ =& \langle \Delta Z_n, Z_m\rangle - R_-\boldsymbol{N}_-\cdot[(\nabla Z_n)Z_m]_{z=-h}.\end{aligned}$$

Note that, the matrix A_{mn} is symmetric and for $m, n \geq -2$ and the submatrix $\big(A_{mn}\big)_{m,n\geq 0}$ is diagonal. Furthermore, the elements of $\vec{B}_{mn}$ are two dimensional vectors depending on the horizontal derivatives of the vertical functions and the bottom. The matrix C_{mn} contains the full Laplacian of the vertical functions as well as the their outward normal derivative on the bottom surface.

Proof. We apply Lemma (4.1) to the functional

$$\tilde{S}[\boldsymbol{\varphi}, \eta] = \int_I \int_{\mathcal{D}_h^\eta} L\, dV\, dt, \quad L = \partial_t \Phi[\boldsymbol{\varphi}, \eta] + \frac{1}{2}\big(\nabla\Phi[\boldsymbol{\varphi}, \eta]\big)^2 + gz,$$

and we proceed by calculating the terms involved

$$\begin{aligned}
\Big\langle -\operatorname{div}(\overrightarrow{D}_2 L, D_3 L), Z_m \Big\rangle &= -\Big\langle \Delta\Phi[\boldsymbol{\varphi}, \eta], Z_m \Big\rangle \\
&= -\Big\langle \sum_{n\geq -2} \nabla_{\mathbf{x}}^2 \varphi_n Z_n + 2\nabla_{\mathbf{x}}\varphi_n \cdot \nabla_{\mathbf{x}} Z_n + \varphi_n \Delta Z_n \,,\, Z_m \Big\rangle \\
&= -\sum_{n\geq -2} \big\langle Z_n, Z_m \big\rangle \nabla_{\mathbf{x}}^2 \varphi_n + 2\big\langle \nabla_{\mathbf{x}} Z_n, Z_m \big\rangle \cdot \nabla_{\mathbf{x}}\varphi_n + \big\langle \Delta Z_n, Z_m \big\rangle \varphi_n
\end{aligned}$$

The bottom surface term reads

$$\begin{aligned}
\big[(\overrightarrow{D}_2 L, D_3 L) \cdot R_- \boldsymbol{N}_- Z_m\big]_{z=-h} =& \big[\nabla\Phi[\boldsymbol{\varphi}, \eta]\big]_{z=-h} \cdot R_- \boldsymbol{N}_- Z_m \\
=& -\nabla_{\mathbf{x}} h \cdot \sum_{n\geq -2} \Big(\nabla_{\mathbf{x}}\varphi_n \big[Z_n Z_m\big]_{z=-h} + \varphi_n \big[\nabla_{\mathbf{x}} Z_n Z_m\big]_{z=-h} \Big) - \\
& - \sum_{n\geq -2} \varphi_n \big[\partial_z Z_n Z_m\big]_{z=-h} \\
=& -\sum_{n\geq -2} \nabla_{\mathbf{x}} h \big[Z_n Z_m\big]_{z=-h} \cdot \nabla_{\mathbf{x}}\varphi_n - \\
& - \big[(\nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}} Z_n + \partial_z Z_n) Z_m\big]_{z=-h} \varphi_n
\end{aligned}$$

For the free surface term we use (4.14)

$$[\nabla\Phi[\boldsymbol{\varphi}, \eta]]_{z=\eta} = \begin{pmatrix} \nabla_{\mathbf{x}}\varphi + \Big(-\varphi_{-2}/h_0 - \mu_0\varphi \Big) \nabla_{\mathbf{x}}\eta \\ \varphi_{-2}/h_0 + \mu_0\varphi \end{pmatrix},$$

to obtain

$$\begin{aligned}\big[(\overrightarrow{D}_2L, D_3L)\cdot R_+\boldsymbol{N}_+Z_m\big]_{z=\eta} &= \big[\nabla\Phi[\boldsymbol{\varphi},\eta]\big]_{z=\eta}\cdot R_+\boldsymbol{N}_+\\ &= -\nabla_{\mathbf{x}}\eta\cdot\nabla_{\mathbf{x}}\varphi - \big(\nabla_{\mathbf{x}}\eta\big)^2\sum_{n\geq -2}[\frac{\partial Z_n}{\partial\eta}]_{z=\eta}\varphi_n + \frac{\varphi_{-2}}{h_0} + \mu\varphi\\ &= -\nabla_{\mathbf{x}}\eta\cdot\nabla_{\mathbf{x}}\varphi + \big(\nabla_{\mathbf{x}}\eta\big)^2\Big(\frac{\varphi_{-2}}{h_0}+\mu\varphi\Big) + \frac{\varphi_{-2}}{h_0} + \mu\varphi\\ &= -\nabla_{\mathbf{x}}\eta\cdot\nabla_{\mathbf{x}}\varphi + \Big(\big(\nabla_{\mathbf{x}}\eta\big)^2+1\Big)\Big(\frac{\varphi_{-2}}{h_0}+\mu\varphi\Big).\end{aligned}$$

□

We also have

Lemma 4.4. *The partial variation of the functional $\tilde{S}[\boldsymbol{\varphi},\eta]$ on $(\boldsymbol{\varphi},\eta)$ in the direction $\delta\eta$ is given by*

$$\delta_\eta\tilde{S}[\boldsymbol{\varphi},\eta;\delta\eta] = \int_I\int_S\Bigg\{\partial_t\varphi + g\eta + \frac{1}{2}(\nabla_{\mathbf{x}}\varphi)^2 - \frac{1}{2}R_+^2\Big(\frac{\varphi_{-2}}{h_0}+\mu\varphi\Big)^2 + \\ + \sum_{n\geq -2}\Big(\sum_{l\geq -2} -a_{nl}(\eta,h)\nabla_{\mathbf{x}}^2\varphi_n - \vec{b}_{nl}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n - c_{nl}(\eta,h)\varphi_n\Big)\varphi_l\Bigg\}\delta\eta d\mathbf{x}dt,$$

where

$$\begin{aligned} a_{nl}(\eta,h) &= \langle Z_n, \partial_\eta Z_l\rangle\,,\\ \vec{b}_{nl}(\eta,h) &= 2\,\langle\nabla_{\mathbf{x}}Z_n,\partial_\eta Z_l\rangle + \nabla_{\mathbf{x}}h[Z_n(\partial_\eta Z_l)]_{z=-h},\\ c_{nl}(\eta,h) &= \langle\Delta Z_n,\partial_\eta Z_l\rangle + \big[\big(\nabla_{\mathbf{x}}h\cdot\nabla_{\mathbf{x}}Z_n + \partial_z Z_n\big)\big(\partial_\eta Z_l\big)\big]_{z=-h},\end{aligned}$$

and $\partial_\eta Z_l = \partial Z_l/\partial\eta$ are given in Apendix (A).

Proof. From Lemma (4.1)

$$\begin{aligned}\delta_\eta\tilde{S}[\boldsymbol{\varphi},\eta;\delta\eta] = \int_{t_0}\int_S\Big\{\sum_{l\geq -2}\Big(&-\big\langle\mathrm{div}(\overrightarrow{D}_2L, D_3L),\partial_\eta Z_l\big\rangle +\\ &+\Big(\big[(\overrightarrow{D}_2L,D_3L)\big]_{z=\eta}\cdot R_+\boldsymbol{N}_+ - \partial_t\eta\Big)\big[\partial_\eta Z_l\big]_{z=\eta}\\ &+\big[(\overrightarrow{D}_2L,D_3L)\big]_{z=-h}\cdot R_-\boldsymbol{N}_-\big[\partial_\eta Z_l\big]_{z=-h}\Big)\varphi_l\\ &+\Big[L\big(\partial_t\Phi[\boldsymbol{\varphi},\eta],\nabla_{\mathbf{x}}\Phi[\boldsymbol{\varphi},\eta],\partial_z\Phi[\boldsymbol{\varphi},\eta],z\big)\Big]_{z=\eta}\Big\}\,\delta\eta\,d\mathbf{x}\,dt.\end{aligned}$$

We proceed by calculating the terms involved in the above equation

$$\begin{aligned}\Big\langle \operatorname{div}(\overrightarrow{D}_2L, D_3L), \partial_\eta Z_l \Big\rangle &= \Big\langle \Delta\Phi[\boldsymbol{\varphi},\eta], \partial_\eta Z_l \Big\rangle \\ &= \sum_{n\geq -2} \langle Z_n, \partial_\eta Z_l\rangle \nabla^2_{\mathbf{x}}\varphi_n + 2\langle \nabla_{\mathbf{x}} Z_n, \partial_\eta Z_l\rangle \cdot \nabla_{\mathbf{x}}\varphi_n + \langle \Delta Z_n, \partial_\eta Z_l\rangle \varphi_n.\end{aligned}$$

For the bottom surface term we have

$$\begin{aligned}\big[(\overrightarrow{D}_2L, D_3L)^T\big]_{z=-h} \cdot R_- \boldsymbol{N}_- \big[(\partial_\eta Z_l)\big]_{z=-h} = \\ = \sum_{n\geq -2} -[Z_n(\partial_\eta Z_l)]_{z=-h}\nabla_{\mathbf{x}} h\cdot\nabla_{\mathbf{x}}\varphi_n - \\ -[(\nabla_{\mathbf{x}} Z_n)(\partial_\eta Z_l)]_{z=-h}\cdot\nabla_{\mathbf{x}} h\varphi_n - [(\partial_z Z_n)(\partial_\eta Z_l)]_{z=-h}\varphi_n.\end{aligned}$$

For the free surface terms we use the fact that (see (4.14))

$$[\nabla\Phi]_{z=\eta} = \begin{pmatrix} \nabla_{\mathbf{x}}\varphi + \Big(-\varphi_{-2}/h_0 - \mu_0\varphi\Big)\nabla_{\mathbf{x}}\eta \\ \varphi_{-2}/h_0 + \mu_0\varphi \end{pmatrix},$$

and

$$\sum_{l\geq -2} \varphi_l \big[\partial_\eta Z_l\big]_{z=\eta} = -\partial_z\big[\Phi[\boldsymbol{\varphi},\eta]\big]_{z=\eta} = -\varphi_{-2}/h_0 - \mu_0\varphi,$$

to obtain

$$\sum_{l\geq -2} \partial_t\eta[\partial_\eta Z_l]_{z=\eta}\varphi_l \quad = \quad -\partial_t\eta\big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\big),$$

$$\begin{aligned}\sum_{l\geq -2} \big[(\overrightarrow{D}_2L, D_3L)^T\big]_{z=\eta} \cdot R_+\boldsymbol{N}_+\big[\partial_\eta Z_l\big]_{z=\eta}\varphi_l &= \big[\nabla\Phi[\boldsymbol{\varphi},\eta]\big]_{z=\eta}\cdot R_+\boldsymbol{N}_+ \sum_{l\geq -2}\varphi_l\big[\partial_\eta Z_l\big]_{z=\eta} \\ &= \Big(-\nabla_{\mathbf{x}}\varphi\cdot\nabla_{\mathbf{x}}\eta + \big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\big)(\nabla_{\mathbf{x}}\eta)^2 + \big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\big)\Big)\Big(-\frac{\varphi_{-2}}{h_0} - \mu_0\varphi\Big) \\ &= \nabla_{\mathbf{x}}\varphi\cdot\nabla_{\mathbf{x}}\eta\big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\big) - \big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\big)^2\big((\nabla_{\mathbf{x}}\eta)^2 + 1\big),\end{aligned}$$

$$
\begin{aligned}
[L]_{z=\eta} &= [\partial_t \Phi[\boldsymbol{\varphi},\eta]]_{z=\eta} + \frac{1}{2}\big[\nabla\Phi[\boldsymbol{\varphi},\eta]\big]^2_{z=\eta} + g\eta \\
&= \partial_t\varphi - \big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\big)\partial_t\eta + \\
&\quad + \frac{1}{2}\big(\nabla_{\mathbf{x}}\varphi\big)^2 - \nabla_{\mathbf{x}}\varphi\cdot\nabla_{\mathbf{x}}\eta\big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\big) + \frac{1}{2}\big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\big)^2\big(\nabla_{\mathbf{x}}\eta\big)^2 + \\
&\quad + \frac{1}{2}\big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\big)^2 + g\eta \\
&= \partial_t\varphi - \big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\big)\partial_t\eta + \\
&\quad + \frac{1}{2}\big(\nabla_{\mathbf{x}}\varphi\big)^2 - \nabla_{\mathbf{x}}\varphi\cdot\nabla_{\mathbf{x}}\eta\big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\big) + \\
&\quad + \frac{1}{2}\big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\big)^2\Big(\big(\nabla_{\mathbf{x}}\eta\big)^2 + 1\Big) + g\eta.
\end{aligned}
$$

Taking the sum of the above terms we see that the terms $\big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\big)\partial_t\eta$ and $\nabla_{\mathbf{x}}\varphi\cdot\nabla_{\mathbf{x}}\eta\big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\big)$ cancel out and after some rearrangement we obtain the result $\square$

4.3.1 Stationarity of $\tilde{S}[\boldsymbol{\varphi},\eta]$

After the calculation of the partial variations of $\tilde{S}$, the variational equation (4.41) takes the form, for all $\delta\boldsymbol{\varphi}$, $\delta\eta$

(4.42)

$$
\begin{aligned}
\delta\tilde{S}[\boldsymbol{\varphi},\eta;\delta\boldsymbol{\varphi},\delta\eta] = \int_{t_0}\int_S \Big\{ \sum_{m\geq -2} \Big(&- \partial_t\eta - \nabla_{\mathbf{x}}\eta\cdot\nabla_{\mathbf{x}}\varphi + R_+^2\Big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\Big) - \\
&- \sum_{n\geq -2} A_{mn}(\eta,h)\nabla^2_{\mathbf{x}}\varphi_n + \vec{B}_{mn}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + C_{mn}(\eta,h)\varphi_n\Big)\delta\varphi_m + \\
&+ \Big(\partial_t\varphi + g\eta + \frac{1}{2}(\nabla_{\mathbf{x}}\varphi)^2 - \frac{1}{2}R_+^2\Big(\frac{\varphi_{-2}}{h_0} + \mu\varphi\Big)^2 + \\
+ \sum_{l\geq -2}\big(\sum_{n\geq -2} - a_{nl}(\eta,h)\nabla^2_{\mathbf{x}}\varphi_n &- \vec{b}_{nl}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n - c_{nl}(\eta,h)\varphi_n\big)\,\varphi_l\Big)\delta\eta\Big\}\,d\mathbf{x}\,dt = 0.
\end{aligned}
$$

Choosing first $\delta\eta = 0$ and all $\delta\varphi_m$ to vanish except one, say $\delta\varphi_{m_*}$, we obtain (using Lemma 4.3)

$$\delta_{\varphi_m} S[\boldsymbol{\varphi},\eta;\delta\varphi_m] = \int_I\int_S \Big\{ -\partial_t\eta - \nabla_{\mathbf{x}}\eta\cdot\nabla_{\mathbf{x}}\varphi + R_+^2\Big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\Big) - \\ - \sum_{n\geq -2}\Big(A_{mn_*}(\eta,h)\nabla_{\mathbf{x}}^2\varphi_n + \vec{B}_{mn_*}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + C_{mn_*}(\eta,h)\varphi_n\Big)\Big\}\delta\varphi_{m_*}d\mathbf{x}dt = 0,$$

which implies the equation

$$\partial_t\eta = -\nabla_{\mathbf{x}}\eta\cdot\nabla_{\mathbf{x}}\varphi + R_+^2\Big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\Big) - \\ - \sum_{n\geq -2} A_{mn_*}(\eta,h)\nabla_{\mathbf{x}}^2\varphi_n + \vec{B}_{mn_*}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + C_{mn_*}(\eta,h)\varphi_n.$$

Repeating this procedure consecutively for all m_* we arrive at the following infinite system

(4.43)
$$\partial_t\eta = -\nabla_{\mathbf{x}}\eta\cdot\nabla_{\mathbf{x}}\varphi + R_+^2\Big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\Big) - \\ - \sum_{n\geq -2} A_{mn}(\eta,h)\nabla_{\mathbf{x}}^2\varphi_n + \vec{B}_{mn}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + C_{mn}(\eta,h)\varphi_n, \quad m\geq -2.$$

Choosing now, $\delta\varphi_m = 0$, for all $m\geq -2$, equation (4.42) becomes

(4.44)
$$\int_I\int_S \Big\{\partial_t\varphi + g\eta + \frac{1}{2}(\nabla_{\mathbf{x}}\varphi)^2 - \frac{1}{2}R_+^2\Big(\frac{\varphi_{-2}}{h_0} + \mu\varphi\Big)^2 - \\ - \sum_{l\geq -2}\Big(\sum_{n\geq -2} a_{nl}(\eta,h)\nabla_{\mathbf{x}}^2\varphi_n + \vec{b}_{nl}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + c_{nl}(\eta,h)\varphi_n\Big)\varphi_l\Big\}\,\delta\eta\,d\mathbf{x}\,dt = 0.$$

By the arbitariness of $\delta\eta$ we obtain

$$\begin{aligned}\partial_t\varphi \;=&\; -g\eta - \frac{1}{2}(\nabla_{\mathbf{x}}\varphi)^2 + \frac{1}{2}R_+^2\Big(\frac{\varphi_{-2}}{h_0} + \mu\varphi\Big)^2 + \\ &+ \sum_{l\geq -2}\Big(\sum_{n\geq -2} a_{ln}(\eta,h)\nabla_{\mathbf{x}}^2\varphi_n + \vec{b}_{nl}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + c_{nl}(\eta,h)\varphi_n\Big)\varphi_l.\end{aligned} \tag{4.45}$$

The nonlinear CMS, Eqs. (4.43) and (4.45), has been derived by Luke's variational principle, and thus it is equivalent to the conventional description of the water wave problem defined by Eqs. (3.16a)-(3.16b). This system of equations was first derived in [AB02]. The difference with the present form, is that the free-surface terms involved in the definitions of the matrix coefficients A_{mn}, $\vec{B}_{mn}$, C_{mn}, a_{nl}, $\vec{b}_{nl}$, c_{nl} in [AB02], are summed out of the series using Eqs. (4.12). We state this fact in the following corollary.

Corollary 4.1. *Let A'_{mn}, B'_{mn}, C'_{mn}, $a^{(0,2)}_{nl}$, $a^{(1,1)}_{nl}$, $b^{(1,1)}_{mn}$, $c^{(0,0)}_{mn}$ be the matrix coeeficients defined in [AB02]. Then*

$$\begin{aligned}\sum_{n\geq -2} A'_{mn}\nabla^2_{\mathbf{x}}\varphi_n + B'_{mn}\nabla_{\mathbf{x}}\varphi_n + C'_{mn}\varphi_n =&\\ = \nabla_{\mathbf{x}}\eta\cdot\nabla_{\mathbf{x}}\varphi - R^2_+\Big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\Big) +&\\ + \sum_{n\geq -2} A_{mn}(\eta,h)\nabla^2_{\mathbf{x}}\varphi_n + \vec{B}_{mn}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + C_{mn}(\eta,h)\varphi_n&\end{aligned}$$

$$\begin{aligned}\sum_{l\geq -2}\sum_{n\geq -2} a^{(0,2)}_{nl}\nabla^2_{\mathbf{x}}\varphi_n\varphi_l - a^{(1,1)}_{nl}\nabla_{\mathbf{x}}\varphi_n\cdot\nabla_{\mathbf{x}}\varphi_l + b^{(1,1)}_{mn}\nabla_{\mathbf{x}}\varphi_n\varphi_l + c'_{mn}\varphi_n\varphi_l =&\\ = \frac{1}{2}(\nabla_{\mathbf{x}}\varphi)^2 - \frac{1}{2}R^2_+\Big(\frac{\varphi_{-2}}{h_0} + \mu\varphi\Big)^2 +&\\ + \sum_{l\geq -2}\Big(\sum_{n\geq -2} a_{nl}(\eta,h)\nabla^2_{\mathbf{x}}\varphi_n + \vec{b}_{nl}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + c_{nl}(\eta,h)\varphi_n\Big)\varphi_l&\end{aligned}$$

We note that the CMS has been obtained without any essential assumptions concerning the vertical structure of the wave potential. Furthermore no simplifications (mild-slope etc) were made and thus, the present CMS, being equivalent to the complete formulation, is expected to be able to fully account for wave nonlinearity and dispersion.

Further study of the subsystem (4.43)-Reformulation of the CMS as two evolution equations

The subsystem (4.43) is of peculiar form, since the time derivative of η, $\partial_t\eta$, appears as the left hand side of all equations. Writting (4.43) as

$$\begin{aligned}\sum_{n\geq -2} A_{mn}(\eta,h)\nabla^2_{\mathbf{x}}\varphi_n + \vec{B}_{mn}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + C_{mn}(\eta,h)\varphi_n =&\\ = \quad \partial_t\eta + \nabla_{\mathbf{x}}\eta\cdot\nabla_{\mathbf{x}}\varphi - R^2_+\Big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\Big),&\end{aligned} \tag{4.46}$$

we see that the r.h.s terms cannot, in fact, be dependent on m. We resolve this controversy as follows [AB02] : Choose the m_*^{th} equation and substract it from the others to obtain the following system

(4.47)
$$\sum_{n\geq -2} \hat{A}_{mn}(\eta,h)\nabla_{\mathbf{x}}^2\varphi_n + \hat{\vec{B}}_{mn}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + \hat{C}_{mn}(\eta,h)\varphi_n = 0, \quad m\geq -2, \quad m\neq m_*.$$

where

$$\begin{aligned}
\hat{A}_{mn}(\eta,h) &= A_{mn}(\eta,h) - A_{m_*n}(\eta,h),\\
\hat{\vec{B}}_{mn}(\eta,h) &= \vec{B}_{mn}(\eta,h) - \vec{B}_{m_*n}(\eta,h),\\
\hat{C}_{mn}(\eta,h) &= C_{mn}(\eta,h) - C_{m_*n}(\eta,h),
\end{aligned}$$

together with the remaning equation

(4.48)
$$\begin{aligned}
\partial_t\eta &= -\nabla_{\mathbf{x}}\eta\cdot\nabla_{\mathbf{x}}\varphi + R_+^2\Big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\Big) -\\
&\quad - \sum_{n\geq -2} A_{m_*n}(\eta,h)\nabla_{\mathbf{x}}^2\varphi_n + \vec{B}_{m_*n}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + C_{m_*n}(\eta,h)\varphi_n.
\end{aligned}$$

Theorem 4.1. *($\mathcal{P}_{WW}$) is equivalent with the following system of evolution equations for φ and η*

$$\begin{aligned}
\partial_t\eta &= -\nabla_{\mathbf{x}}\eta\cdot\nabla_{\mathbf{x}}\varphi + R_+^2\Big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\Big) -\\
&\quad - \sum_{n\geq -2} A_{m_*n}(\eta,h)\nabla_{\mathbf{x}}^2\varphi_n + \vec{B}_{m_*n}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + C_{m_*n}(\eta,h)\varphi_n,
\end{aligned} \tag{4.49}$$

$$\begin{aligned}
\partial_t\varphi &= -g\eta - \frac{1}{2}(\nabla_{\mathbf{x}}\varphi)^2 + \frac{1}{2}R_+^2\Big(\frac{\varphi_{-2}}{h_0} + \mu_0\varphi\Big)^2 +\\
&\quad + \sum_{l\geq -2}\sum_{n\geq -2}\Big(a_{nl}(\eta,h)\nabla_{\mathbf{x}}^2\varphi_n + \vec{b}_{nl}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + c_{nl}(\eta,h)\varphi_n\Big)\varphi_l,
\end{aligned} \tag{4.50}$$

where $\{\varphi_n\}_{n\geq -2}$ are obtained by the infinite coupled mode system

(4.51)
$$\sum_{n\geq -2} \hat{A}_{mn}(\eta,h)\nabla_{\mathbf{x}}^2\varphi_n + \hat{\vec{B}}_{mn}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + \hat{C}_{mn}(\eta,h)\varphi_n = 0, \quad m\geq -2, \quad m\neq m_0,$$

$$\sum_{n\geq -2}\varphi_n = \varphi. \tag{4.52}$$

where

$$\hat{A}_{mn}(\eta,h) = A_{mn}(\eta,h) - A_{m_*n}(\eta,h),$$
$$\hat{\vec{B}}_{mn}(\eta,h) = \vec{B}_{mn}(\eta,h) - \vec{B}_{m_*n}(\eta,h),$$
$$\hat{C}_{mn}(\eta,h) = C_{mn}(\eta,h) - C_{m_*n}(\eta,h).$$

Theorem 4.2. *The following identity holds*

$$\begin{aligned}\sum_{n\geq -2} A_{mn}(\eta,h)\nabla^2_{\mathbf{x}}\varphi_n + \vec{B}_{mn}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + C_{mn}(\eta,h)\varphi_n = \\ = \int_{-h}^{\eta} \Delta\Phi[\boldsymbol{\varphi},\eta]\, Z_m\, dz - R_-\boldsymbol{N}_-\cdot\left[\nabla\Phi[\boldsymbol{\varphi},\eta]\, Z_m\right]_{z=-h}.\end{aligned} \tag{4.53}$$

Theorem 4.3. *The following identity holds*

$$\begin{aligned}\sum_{n\geq -2}\Big(\sum_{l\geq -2} a_{nl}(\eta,h)\nabla^2_{\mathbf{x}}\varphi_n + \vec{b}_{nl}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + c_{nl}(\eta,h)\varphi_n\Big)\varphi_l = \\ = \int_{-h}^{\eta}\Delta\Phi[\boldsymbol{\varphi},\eta]\,\big(\sum_{l\geq -2}(\partial_\eta Z_l)\varphi_l\big)\,dz - R_-\boldsymbol{N}_-\cdot\big[\nabla\Phi[\boldsymbol{\varphi},\eta]\,\big(\sum_{l\geq -2}(\partial_\eta Z_l)\varphi_l\big)\big]_{z=-h}.\end{aligned} \tag{4.54}$$

Proof. Recall that

$$\Delta\Phi[\boldsymbol{\varphi},\eta] = \sum_{n\geq -2}\nabla^2_{\mathbf{x}}\varphi_n Z_n + 2\nabla_{\mathbf{x}}\varphi_n\cdot\nabla_{\mathbf{x}}Z_n + \varphi_n\Delta Z_n, \tag{4.55}$$

Using the above equation, the first term of the r.h.s (4.54) becomes

$$\begin{aligned}&\Big\langle\Delta\Phi[\boldsymbol{\varphi},\eta],(\partial_\eta Z_l)\varphi_l\Big\rangle = \\ &= \sum_{l\geq -2}\Big(\sum_{n\geq -2}\langle Z_n,\partial_\eta Z_l\rangle\,\nabla^2_{\mathbf{x}}\varphi_n + 2\,\langle\nabla_{\mathbf{x}}Z_n,\partial_\eta Z_l\rangle\cdot\nabla_{\mathbf{x}}\varphi_n + \langle\Delta Z_n,\partial_\eta Z_l\rangle\,\varphi_n\Big)\varphi_l.\end{aligned}$$

where we denoted $\partial_\eta Z_l = \partial Z_l/\partial\eta$. Similarly, for the second term in the r.h.s of (4.54), we have

$$\begin{aligned}&-R_-\boldsymbol{N}_-\cdot\big[\nabla\Phi[\boldsymbol{\varphi},\eta]\Big(\sum_{l\geq -2}(\partial_\eta Z_l)\varphi_l\Big)\big]_{z=-h} = \\ &= \sum_{l\geq -2}\Big(\sum_{n\geq -2}\nabla_{\mathbf{x}}h\cdot\nabla_{\mathbf{x}}\varphi_n\big[Z_n(\partial_\eta Z_l)\big]_{z=-h} + \varphi_n\nabla_{\mathbf{x}}h\cdot\big[(\nabla_{\mathbf{x}}Z_n)(\partial_\eta Z_l)\big]_{z=-h} + \\ &\qquad + \varphi_n\big[(\partial_z Z_n)(\partial_\eta Z_l)\big]_{z=-h}\Big)\varphi_l.\end{aligned}$$

The sum of the r.h.s of the last two equations is

$$\sum_{l\geq -2}\Big(\sum_{n\geq -2}\langle Z_n,\partial_\eta Z_l\rangle\,\nabla^2_{\mathbf{x}}\varphi_n+\big(2\,\langle\nabla_{\mathbf{x}}Z_n,\partial_\eta Z_l\rangle+\nabla_{\mathbf{x}}h\big[Z_n(\partial_\eta Z_l)\big]_{z=-h}\big)\cdot\nabla_{\mathbf{x}}\varphi_n+$$
$$+\big(\,\langle\Delta Z_n,\partial_\eta Z_l\rangle+\nabla_{\mathbf{x}}h\cdot\big[\nabla_{\mathbf{x}}Z_n(\partial_\eta Z_l)\big]_{z=-h}+\big[\partial_z Z_n(\partial_\eta Z_l)\big]_{z=-h}\big)\varphi_n\Big)\varphi_l.$$

Using the definitions of a_{nl}, $\vec{b}_{nl}$, c_{nl} found in Lemma (4.4), we obtain the result. □

Theorems 4.3 and 4.2 show, that if $\Phi[\boldsymbol{\varphi},\eta]$ solves the Laplace equation and the bottom boundary condition, then the double series in the r.h.s of (4.83) vanishes.

4.4 Expression of DtN operator in terms of the coupled-mode representation

It is interesting to note that the system (4.84) with (4.85), has a specific physical meaning. In fact, it gives an indirect representation of the DtN operator (see Chapter 2, Definition 2.1).

Theorem 4.4. *Let η be the instataneous surface elevation defining the corresponding fluid domain $\mathcal{D}_h^\eta$. Let $\Phi(\mathbf{x},z)$ be a function defined on $\mathcal{D}_h^\eta$, such that $\Phi\in C^2(\mathcal{D}_h^\eta)\cap C^1(\overline{\mathcal{D}_h^\eta})$ and $\lim\limits_{|\mathbf{x}|-\rightarrow\infty}\Phi(\mathbf{x},z)=0$. Assuming that $\varphi_n=O(n^{-4})$ uniformly in $\mathbf{x}$, the velocity potential field $\Phi(\mathbf{x},z)$ can be represented in the form*

$$\Phi(\mathbf{x},z)=\sum_{n\geq -2}\varphi_n(\mathbf{x})\,Z_n(z;\eta,h) \tag{4.56}$$

and the series can be termwise differentiated up to two times (at least). Then,

1. *If Φ is the solution of $(\mathcal{P}_{DtN})$ then $\boldsymbol{\varphi}=(\varphi_{-2},\varphi_{-1},\varphi_0,...)$ is the solution of (4.84) and (4.85).*
2. *If $\boldsymbol{\varphi}=(\varphi_{-2},\varphi_{-1},\varphi_0,...)$ is the solution of (4.84) and (4.85) then Φ is the solution of $(\mathcal{P}_{DtN})$.*

Proof. 1. Let Φ be the solution of $(\mathcal{P}_{DtN})$. Then it satisfies $[\Phi]_{z=\eta}=\varphi$. By the construction of Z_n we have

$$[Z_n]_{z=\eta}=1,\quad \text{for all}\quad n\geq -2,$$

and this implies $\sum_{m\geq -2} \varphi_m = \big[\Phi\big]_{z=\eta} = \varphi$. Furthermore, Φ is the unique solution to the following variational equation

$$\delta\mathcal{I}[\Phi;\delta\Phi] = 0, \quad \text{for all} \quad \delta\Phi: \ [\delta\Phi]_{z=\eta} = 0,$$

where

$$\mathcal{I}[\Phi] = \frac{1}{2}\int_{\mathcal{D}_h^\eta} \big(\nabla\Phi\big)^2 dV. \tag{4.57}$$

Performing the variation in the functional $\mathcal{I}[\Phi]$, we obtain that Φ is the solution of the following variational equation

$$\delta\mathcal{I}[\Phi;\delta\Phi] = \int_{\mathcal{D}_h^\eta} \big(\nabla\Phi\big)\cdot\big(\nabla\delta\Phi\big) dV = 0, \quad \text{for all} \quad \delta\Phi: \ [\delta\Phi]_{z=\eta} = 0 \tag{4.58}$$

Integration by parts shows

$$\delta\mathcal{I}[\Phi;\delta\Phi] = \int_{\Gamma_h} \partial_{N_-}\Phi\,\delta\Phi\,d\Gamma_h - \int_{\mathcal{D}_h^\eta} \Delta\Phi\,\delta\Phi\,dV = 0, \tag{4.59}$$

or,

$$\delta\mathcal{I}[\Phi;\delta\Phi] = \int_S N_-\cdot\big[\nabla\Phi\big]_{z=-h}\big[\delta\Phi\big]_{z=-h} R_- d\mathbf{x} - \int_S\int_{-h}^{\eta} \Delta\Phi\,\delta\Phi\,dV = 0. \tag{4.60}$$

Replace

$$\Phi = \Phi[\boldsymbol{\varphi}] = \sum_{n\geq -2} \varphi_n\,Z_n, \tag{4.61}$$

in the functional $\mathcal{I}$ to obtain

$$\begin{aligned}
\tilde{\mathcal{I}}[\boldsymbol{\varphi}] &= \mathcal{I}\big[\Phi[\boldsymbol{\varphi}]\big] \\
&= \frac{1}{2}\int_{\mathcal{D}_h^\eta} \big(\nabla\Phi[\boldsymbol{\varphi}]\big)^2 dV \\
&= \frac{1}{2}\int_{\mathcal{D}_h^\eta} \Big(\nabla\big(\sum_{n\geq -2}\varphi_n\,Z_n\big)\Big)^2 dV.
\end{aligned}$$

The variation $\delta\Phi$ becomes

$$\delta\Phi = \delta\Phi[\boldsymbol{\varphi};\delta\boldsymbol{\varphi}] = \sum_{n\geq -2} \delta\varphi_n\,Z_n. \tag{4.62}$$

and $[\delta\Phi]_{z=\eta} = 0$ implies $\sum\limits_{m\geq -2} \delta\varphi_m = 0$. Stationarity of $\tilde{\mathcal{I}}[\boldsymbol{\varphi}]$ means

$$\delta\tilde{\mathcal{I}}[\boldsymbol{\varphi};\delta\boldsymbol{\varphi}] = 0, \quad \text{for all} \quad \delta\boldsymbol{\varphi} : \sum_{m\geq -2} \delta\varphi_m = 0.$$

The Fréchet derivative of $\tilde{I}[\boldsymbol{\varphi}]$ with respect to $\boldsymbol{\varphi}$, $\delta\tilde{\mathcal{I}}[\boldsymbol{\varphi};\delta\boldsymbol{\varphi}]$, can be calculated using the formula

$$\delta\tilde{\mathcal{I}}[\boldsymbol{\varphi};\delta\boldsymbol{\varphi}] = \delta\mathcal{I}\big[\Phi[\boldsymbol{\varphi}];\delta\Phi[\boldsymbol{\varphi};\delta\boldsymbol{\varphi}]\big]. \tag{4.63}$$

Hence, substituting (4.61), (4.62) in (4.60) we obtain

(4.64)

$$\delta\tilde{\mathcal{I}}[\boldsymbol{\varphi};\delta\boldsymbol{\varphi}] =$$
$$= \int_S \sum_{m\geq -2} \Big\{ \sum_{n\geq -2} \Big(A_{mn}(\eta,h)\nabla^2_{\mathbf{x}}\varphi_n + \vec{B}_{mn}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + C_{mn}(\eta,h)\varphi_n \Big) \Big\}\delta\varphi_m\, d\mathbf{x} = 0,$$

where the matrix coefficients are given by

$$\begin{aligned}
A_{mn}(\eta,h) &= \langle Z_n, Z_m\rangle\,,\\
\vec{B}_{mn}(\eta,h) &= 2\,\langle \nabla_{\mathbf{x}} Z_n, Z_m\rangle + \nabla_{\mathbf{x}} h [Z_n Z_m]_{z=-h},\\
C_{mn}(\eta,h) &= \langle \Delta Z_n, Z_m\rangle + \Big[\big(\nabla_{\mathbf{x}} h\cdot\nabla_{\mathbf{x}} Z_n + \partial_z Z_n\big) Z_m\Big]_{z=-h}.
\end{aligned}$$

The variational equation (4.64) holds for all $\delta\varphi_m$ such that

$$\sum_{m\geq -2} \delta\varphi_m = 0. \tag{4.65}$$

Using (4.65) we can write for $m_0 \geq -2$

$$\delta\varphi_{m_0} = -\sum_{m\geq -2} \delta\varphi_m, \quad m\neq m_0. \tag{4.66}$$

Now (4.64) can be written for $m\neq m_0$

$$\int_S \sum_{n\geq -2} \Big(A_{mn_0}(\eta,h)\nabla^2_{\mathbf{x}}\varphi_n + \vec{B}_{mn_0}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + C_{mn_0}(\eta,h)\varphi_n \Big)\,\delta\varphi_{m_0} +$$
$$+ \sum_{m\geq -2} \left(\sum_{n\geq -2} \Big(A_{mn}(\eta,h)\nabla^2_{\mathbf{x}}\varphi_n + \vec{B}_{mn}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + C_{mn}(\eta,h)\varphi_n \Big) \right) \delta\varphi_m\, d\mathbf{x} = 0.$$

Substituting (4.66) in the above equation we obtain,

$$\int_S \sum_{n\geq -2} \Big(A_{mn_0}(\eta,h)\nabla^2_{\mathbf{x}}\varphi_n + \vec{B}_{mn_0}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + C_{mn_0}(\eta,h)\varphi_n\Big)\Big(-\sum_{m\geq -2}\delta\varphi_m\Big)+$$
$$+\sum_{m\geq -2}\left(\sum_{n\geq -2}\Big(A_{mn}(\eta,h)\nabla^2_{\mathbf{x}}\varphi_n + \vec{B}_{mn}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + C_{mn}(\eta,h)\varphi_n\Big)\right)\delta\varphi_m\, d\mathbf{x} = 0,$$

and a factorizing shows that

$$\int_S \sum_{m\geq -2}\left(\sum_{n\geq -2}\hat{A}_{mn}(\eta,h)\nabla^2_{\mathbf{x}}\varphi_n + \hat{\vec{B}}_{mn}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + \hat{C}_{mn}(\eta,h)\varphi_n\right)\delta\varphi_m\, d\mathbf{x} = 0,$$

where

$$\begin{aligned}
\hat{A}_{mn}(\eta,h) &= A_{mn}(\eta,h) - A_{mn_0}(\eta,h),\\
\hat{\vec{B}}_{mn}(\eta,h) &= \vec{B}_{mn}(\eta,h) - \vec{B}_{mn_0}(\eta,h),\\
\hat{C}_{mn}(\eta,h) &= C_{mn}(\eta,h) - C_{mn_0}(\eta,h).
\end{aligned}$$

This implies the system

$$\sum_{n\geq -2}\hat{A}_{mn}(\eta,h)\nabla^2_{\mathbf{x}}\varphi_n + \hat{\vec{B}}_{mn}(\eta,h)\cdot\nabla_{\mathbf{x}}\varphi_n + \hat{C}_{mn}(\eta,h)\varphi_n = 0, \quad m\geq -2, \quad m\neq m_0,$$

Of cousre the fields $\boldsymbol{\varphi} = (\varphi_{-2},\varphi_{-1},...)$ should satisfy the same latteral conditions as Φ i.e. $\lim_{|\mathbf{x}|\to\infty}\varphi_m = 0$, for all $m\geq -2$.

2. For the inverse, Let $\boldsymbol{\varphi} = (\varphi_{-2},\varphi_{-1},...)$ satisfy the system (4.84) and (4.85). Observe that the coefficients $\hat{A}_{mn}$, $\hat{\vec{B}}_{mn}$, $\hat{C}_{mn}$ are given by

$$\hat{A}_{mn}(\eta,h) = \langle Z_n, Z_m - Z_{m_0}\rangle, \tag{4.67a}$$

$$\hat{\vec{B}}_{mn}(\eta,h) = 2\langle\nabla_{\mathbf{x}}Z_n,(Z_m - Z_{m_0})\rangle + \nabla_{\mathbf{x}}h[Z_n(Z_m - Z_{m_0})]_{z=-h}, \tag{4.67b}$$

$$\hat{C}_{mn}(\eta,h) = \langle\Delta Z_n,(Z_m - Z_{m_0})\rangle - R_-\boldsymbol{N}_-\cdot\big[\big(\nabla Z_n\big)(Z_m - Z_{m_0})\big]_{z=-h}. \tag{4.67c}$$

Multiplying (4.84) with arbitary $\delta\varphi_m$ and integrating over the horizontal domain S and denoting $\hat{A}_{mn}\equiv\hat{A}_{mn}(\eta,h)$ etc. we can write

$$\sum_{n\geq -2}\int_S\Big\{\hat{A}_{mn}\nabla^2_{\mathbf{x}}\varphi_n\delta\varphi_m + \hat{\vec{B}}_{mn}\cdot\nabla_{\mathbf{x}}\varphi_n\delta\varphi_m + \hat{C}_{mn}\varphi_n\delta\varphi_m\Big\}d\mathbf{x} = 0, \tag{4.68}$$

and denote the three terms of (4.68) as

$$I_{mn} = \int_S \Big\{ \hat{A}_{mn} \nabla_{\mathbf{x}}^2 \varphi_n \delta\varphi_m \Big\} d\mathbf{x}, \tag{4.69a}$$

$$II_{mn} = \int_S \Big\{ \hat{\vec{B}}_{mn} \cdot \nabla_{\mathbf{x}} \varphi_n \delta\varphi_m \Big\} d\mathbf{x}, \tag{4.69b}$$

$$III_{mn} = \int_S \Big\{ \hat{C}_{mn} \varphi_n \delta\varphi_m \Big\} d\mathbf{x}. \tag{4.69c}$$

The first term of (4.68) can be written via integration by parts as

$$I_{mn} = \sum_{n\geq -2} \int_S \Big\{ -(\nabla_{\mathbf{x}} \hat{A}_{mn}) \cdot (\nabla_{\mathbf{x}} \varphi_n) \delta\varphi_m - \hat{A}_{mn} (\nabla_{\mathbf{x}} \varphi_n) \cdot (\nabla_{\mathbf{x}} \delta\varphi_m) \Big\} d\mathbf{x}. \tag{4.70}$$

Using the definition of the matrix coefficient $\hat{A}_{mn}$ (Eqs. (4.67)), and invoking Leibnitz's integral rule, we see that

$$\nabla_{\mathbf{x}} \hat{A}_{mn} = \nabla_{\mathbf{x}} \eta [Z_n \hat{Z}_m]_{z=\eta} + \nabla_{\mathbf{x}} h [Z_n \hat{Z}_m]_{z=-h} + \big\langle \nabla_{\mathbf{x}} Z_n, \hat{Z}_m \big\rangle + \big\langle Z_n, \nabla_{\mathbf{x}} \hat{Z}_m \big\rangle, \tag{4.71}$$

where we denoted $\hat{Z}_m = Z_m - Z_{m_0}$. Using the fact that $[\hat{Z}_m]_{z=\eta} = 0$, (4.70) can further be written as

$$\begin{aligned} I_{mn} = \int_S \Big\{ \big(-\nabla_{\mathbf{x}} h [Z_n \hat{Z}_m]_{z=-h} - \big\langle \nabla_{\mathbf{x}} Z_n, \hat{Z}_m \big\rangle - \big\langle Z_n, \nabla_{\mathbf{x}} \hat{Z}_m \big\rangle \big) \cdot \big(\nabla_{\mathbf{x}} \varphi_n \big) \delta\varphi_m - \\ - \big\langle Z_n, \hat{Z}_m \big\rangle \big(\nabla_{\mathbf{x}} \varphi_n \big) \cdot \nabla_{\mathbf{x}} \delta\varphi_m \Big\} d\mathbf{x}. \end{aligned} \tag{4.72}$$

The third term of (4.68) is treated similarly. First observe that

$$\begin{aligned} \big\langle \nabla_{\mathbf{x}}^2 Z_n, \hat{Z}_m \big\rangle = \nabla_{\mathbf{x}} \cdot \big\langle \nabla_{\mathbf{x}} Z_n, \hat{Z}_m \big\rangle - \\ - \nabla_{\mathbf{x}} \eta \cdot [\nabla_{\mathbf{x}} Z_n \hat{Z}_m]_{z=\eta} - \nabla_{\mathbf{x}} h [\nabla_{\mathbf{x}} Z_n \hat{Z}_m]_{z=-h} - \big\langle \nabla_{\mathbf{x}} Z_n, \nabla_{\mathbf{x}} \hat{Z}_m \big\rangle, \end{aligned} \tag{4.73}$$

and

$$\big\langle \partial_{zz}^2 Z_n, Z_m \big\rangle = [\partial_z Z_n \hat{Z}_m]_{z=\eta} - [\partial_z Z_n \hat{Z}_m]_{z=-h} - \big\langle \partial_z Z_n, \partial_z \hat{Z}_m \big\rangle. \tag{4.74}$$

Recall also that

$$R_- \boldsymbol{N}_- \cdot \big[\big(\nabla Z_n \big) \hat{Z}_m \big]_{z=-h} = -\nabla_{\mathbf{x}} h \cdot \big[\nabla_{\mathbf{x}} Z_n \hat{Z}_m \big]_{z=-h} - \big[\partial_z Z_n \hat{Z}_m \big]_{z=-h}. \tag{4.75}$$

Now the third term of (4.68), in virtue of the last three equations and the fact that $[\hat{Z}_m]_{z=\eta} = 0$, can be written as

$$III_{mn} = \int_S \Big\{ \nabla_{\mathbf{x}} \cdot \langle \nabla_{\mathbf{x}} Z_n, \hat{Z}_m \rangle \varphi_n \delta\varphi_m - \\ - \langle \nabla_{\mathbf{x}} Z_n, \nabla_{\mathbf{x}} \hat{Z}_m \rangle \varphi_n \delta\varphi_m - \langle \partial_z Z_n, \partial_z \hat{Z}_m \rangle \varphi_n \delta\varphi_m \Big\} d\mathbf{x},$$

and an integration by parts for the first term shows

$$III_{mn} = \int_S \Big\{ - \langle \nabla_{\mathbf{x}} Z_n, \hat{Z}_m \rangle \cdot \nabla_{\mathbf{x}} (\varphi_n \delta\varphi_m) \\ - \langle \nabla_{\mathbf{x}} Z_n, \nabla_{\mathbf{x}} \hat{Z}_m \rangle \varphi_n \delta\varphi_m - \langle \partial_z Z_n, \partial_z \hat{Z}_m \rangle \varphi_n \delta\varphi_m \Big\} d\mathbf{x} \tag{4.76}$$

We can now compose Eq. (4.68) as

$$\sum_{n \geq -2} \Big(I_{mn} + II_{mn} + III_{mn} \Big) = 0 \tag{4.77}$$

Substituting the expressions (4.72), (4.69b), (4.76) in (4.77) we obtain the following variational equation

$$\sum_{n \geq -2} \int_S \Big\{ \langle Z_n, \hat{Z}_m \rangle \nabla_{\mathbf{x}} \varphi_n \cdot \nabla_{\mathbf{x}} \delta\varphi_m + \langle Z_n, \nabla_{\mathbf{x}} \hat{Z}_m \rangle \cdot \nabla_{\mathbf{x}} (\varphi_n \delta\varphi_m) + \\ + \langle \nabla_{\mathbf{x}} Z_n, \nabla_{\mathbf{x}} \hat{Z}_m \rangle \varphi_n \delta\varphi_m + \langle \partial_z Z_n, \partial_z \hat{Z}_m \rangle \varphi_n \delta\varphi_m \Big\} d\mathbf{x} = 0 \tag{4.78}$$

Taking the sum over m, one can verify that

$$\int_S \Big\{ \int_{-h}^{\eta} \nabla \Big(\sum_{n \geq -2} \varphi_n Z_n \Big) \cdot \nabla \Big(\sum_{m \geq -2} \hat{Z}_m \delta\varphi_m \Big) dz \Big\} d\mathbf{x} = 0. \tag{4.79}$$

Which is exactly the variational equation (4.58) where $\Phi = \sum_{n \geq -2} \varphi_n Z_n$ and $\delta\Phi = \sum_{m \geq -2} \hat{Z}_m \delta\varphi_m$. Furthermore for all $\delta\varphi_m$ one has

$$[\delta\Phi]_{z=\eta} = \Big[\sum_{m \geq -2} \hat{Z}_m \delta\varphi_m \Big]_{z=\eta} = \sum_{m \geq -2} [\hat{Z}_m]_{z=\eta} \delta\varphi_m = 0. \tag{4.80}$$

and also the following holds

$$[\Phi]_{z=\eta} = [\sum_{n \geq -2} \varphi_n Z_n]_{z=\eta} = \sum_{n \geq -2} \varphi_n = \varphi. \tag{4.81}$$

Now Green's theorem and the assumption that $\delta\varphi_n$ vanish outside S for all n, yields

$$\int_S \Big\{ \int_{-h}^{\eta} \Delta\Big(\sum_{n\geq -2} \varphi_n Z_n \Big)\Big(\sum_{m\geq -2} \hat{Z}_m \delta\varphi_m \Big) dz - \\ - R_- \boldsymbol{N}_- \cdot \big[\nabla\big(\sum_{n\geq -2} \varphi_n Z_n \big)\big(\sum_{m\geq -2} \hat{Z}_m \delta\varphi_m \big)\big]_{z=-h} \Big\} d\mathbf{x} = 0.$$

We deduce that $\Phi = \sum_{n\geq -2} \varphi_n Z_n$ solves $(\mathcal{P}_{DtN})$. □

By virtue of Theorems (4.3) and (4.2) in conjunction with Theorem (4.4) we can state the following result

Theorem 4.5. *$(\mathcal{P}_{WW})$ is equivalent with the following system of evolution equations for φ and η*

$$\partial_t \eta = -\nabla_{\mathbf{x}}\eta \cdot \nabla_{\mathbf{x}}\varphi + R_+^2 \Big(\frac{\varphi_{-2}}{h_0} + \mu_0 \varphi \Big) \tag{4.82}$$

$$\partial_t \varphi = -g\eta - \frac{1}{2}(\nabla_{\mathbf{x}}\varphi)^2 + \frac{1}{2} R_+^2 \Big(\frac{\varphi_{-2}}{h_0} + \mu_0 \varphi \Big)^2 \tag{4.83}$$

where $\{\varphi_n\}_{n\geq -2}$ are obtained by the infinite coupled mode system

$$\sum_{n\geq -2} A_{mn}(\eta,h)\nabla_{\mathbf{x}}^2 \varphi_n + \hat{B}_{mn}(\eta,h)\cdot \nabla_{\mathbf{x}}\varphi_n + C_{mn}(\eta,h)\varphi_n = 0, \quad m \geq -2, \tag{4.84}$$

$$\sum_{n\geq -2} \varphi_n = \varphi. \tag{4.85}$$

4.5 Implication to the representation of the DtN operator

In the previous subsection we treated the free suraface elevation η and the trace of the velocity potential φ, as data for the problem $(\mathcal{P}_{DtN})$ expressed in terms of the local mode representation. When $\varphi(\cdot,t)$ and $\eta(\cdot,t)$ are known at an instant t, then we can solve the (CMS) Eqs. (4.84) and (4.85), and obtain a solution $\boldsymbol{\varphi} = \boldsymbol{\varphi}[\varphi,\eta,h] = (\varphi_{-2},\varphi_{-1},\varphi_0,...)$. The dependence of the solution on the free surface η and the fixed bottom surface h is expressed through the coefficients $\hat{A}_{mn}$, $\hat{\tilde{B}}_{mn}$, $\hat{C}_{mn}$. Recall the definition of the DtN operator

$$G(\eta, h)\varphi = R_+ \partial_{\boldsymbol{N}_+} \Phi|_{\Gamma_\eta}, \tag{4.86}$$

where Φ solves ($\mathcal{P}_{DtN}$). Having in hand a solution $\boldsymbol{\varphi} = \boldsymbol{\varphi}[\varphi, \eta, h] = (\varphi_{-2}, \varphi_{-1}, \varphi_0, ...)$ of Eqs. (4.84) and (4.85) we can reconstruct the wave potential $\Phi = \sum_{n \geq -2} \varphi_n Z_n$ and express the DtN operator in terms of the free surface mode φ_{-2} as follows

$$\tilde{G}(\eta, h)\varphi = -\nabla_{\mathbf{x}}\eta \cdot \nabla_{\mathbf{x}}\varphi + R_+^2 \Big(\frac{\varphi_{-2}}{h_0} + \mu_0 \varphi \Big). \tag{4.87}$$

In order to compare the expression of the DtN operator (4.87), derived through the local mode representation of the velocity potential, we briefly recall the approach developed in a series of works (see [CS93, Cra07, CLS11, CGNS05]), where a functional Taylor expansion of the DtN is used. For the case of variable bottom of the form $h(\mathbf{x}) = -h_0 + \beta(\mathbf{x})$ where $\beta(\mathbf{x})$ denotes the variation of the bottom of the fluid domain from its mean value $-h_0$, Craig et al. [CGNS05] obtained a Taylor expansion of the DtN operator given by

$$G(\eta, h) = \sum_{l=0}^{\infty} G^{(l)}(\eta, \beta), \tag{4.88}$$

where

$$G^{(0)} = |D| \tanh(h_0 |D|) + |D| L(\beta), \qquad D = -i\nabla_{\mathbf{x}}, \tag{4.89}$$

and for l odd,

$$G^{(l)} = |D|^{l-1} D \frac{\eta^l}{l!} \cdot D - \sum_{j=2,even}^{l-1} |D|^j \frac{\eta^j}{j!} G^{(l-j)} - \sum_{j=1,odd}^{l} |D|^{j-1} G^{(0)} \frac{\eta^j}{j!} G^{(l-j)}, \tag{4.90}$$

and for l even,

(4.91)

$$G^{(l)} = |D|^{l-2} G^{(0)} D \frac{\eta^l}{l!} \cdot D - \sum_{j=2,even}^{l-1} |D|^j \frac{\eta^j}{j!} G^{(l-j)} - \sum_{j=1,odd}^{l-1} |D|^{j-1} G^{(0)} \frac{\eta^j}{j!} G^{(l-j)}.$$

The bottom variation is expressed through the operator $L(\beta)$ which also can be expanded in Taylor series, with the first few terms given by

$$
\begin{aligned}
L_0 &= 0,\\
L_1 &= -\frac{D}{|D|}\mathrm{sech}(h_0|D|)\cdot \beta D \mathrm{sech}(h|D|),\\
L_2 &= \frac{D}{|D|}\mathrm{sech}(h|D|)\cdot \beta D \sinh(h_0|D|)L_1.
\end{aligned}
$$

A Partial derivatives of the vertical functions

The vertical functions Z_n, $n \geq -2$ are given by

$$Z_{-2}(z, \eta, h) = \frac{\mu_0 h_0 + 1}{2h_0(\eta + h)}(z + h)^2 - \frac{\mu_0 h_0 + 1}{2h_0}(\eta + h) + 1, \tag{A.1}$$

$$Z_{-1}(z; \eta, h) = \frac{\mu_0 h_0 - 1}{2h_0(\eta + h)}(z + h)^2 + \frac{1}{h_0}(\eta + h) + \frac{2h_0 - (\eta + h)(\mu h_0 + 1)}{2h_0}, \tag{A.2}$$

$$Z_0(z; \eta, h) = \frac{\cosh\left[k_0(z + h)\right]}{\cosh\left[k_0(\eta + h)\right]}, \tag{A.3}$$

$$Z_n(z; \eta, h) = \frac{\cos\left[k_n(z + h)\right]}{\cos\left[k_n(\eta + h)\right]}, \quad \text{n} \geq 0, \tag{A.4}$$

where $k_0 = k_0(\eta, h)$ and $k_n = k_n(\eta, h)$ satisfy the dispersion relations

$$\mu_0 - k_0 tanh(k_0(\eta + h)) = 0, \quad \mu_0 + k_n tan(k_n(\eta + h)) = 0. \tag{A.5}$$

Note that Z_{-2} and Z_{-1} depend explicitely on η an h, while Z_n , $n \geq 0$ depend both explicitely and implicitely (through (A.5)) on η and h. Their partial derivatives are given by

$$\partial_\eta Z_{-2} = \frac{1 + h_0\mu_0}{2h_0} - \frac{1 + h_0\mu_0}{2h_0(\eta + h)^2}(z + h)^2,$$

$$\partial_h Z_{-2} = -\frac{1 + h_0\mu_0}{2h_0} + \frac{1 + h_0\mu_0}{h_0(\eta + h)}(z + h) - \frac{(1 + h_0\mu_0)(z + h)^2}{2h_0(\eta + h)^2}.$$

$$\partial_\eta Z_{-1} = \frac{-1 - h_0\mu_0}{2h_0} - \frac{-1 + h_0\mu_0}{2h_0(\eta + h)^2}(z + h)^2,$$

$$\partial_h Z_{-1} = \frac{1}{h_0} - \frac{-1 - h_0\mu_0}{2h_0} + \frac{-1 + h_0\mu_0}{h_0(\eta + h)}(z + h) - \frac{(1 + h_0\mu_0)(z + h)^2}{2h_0(\eta + h)^2}.$$

Recalling that for $n \geq 0$, the vertical functions $Z_n = Z_n(z;\eta,h)$, depend implicitely on $k_n = k_n(\eta,h)$, we also obtain

$$\partial_\eta Z_0 = \left\{ \begin{array}{r} (z+h)\sinh\big[k_0(z+h)\big](\partial_\eta k_0)- \\ -\cosh\big[k_0(z+h)\big]\tan\big[k_0(\eta+h)\big]\,\big(k_0+(\eta+h)(\partial_\eta k_0)\big) \end{array} \right\} \frac{1}{\cosh\big[k_0(\eta+h)\big]},$$

$$\partial_{k_0} Z_0 = \left\{ \begin{array}{l} \sinh\big[k_0(z+h)\big]\cosh\big[k_0(\eta+h)\big](z+h)- \\ -\sinh\big[k_0(\eta+h)\big]\cosh\big[k_0(z+h)\big](\eta+h) \end{array} \right\} \frac{1}{\cosh^2\big[k_0(\eta+h)\big]},$$

$$\partial_h Z_0 = \left\{ \begin{array}{l} \sinh\big[k_0(z+h)\big]\cosh\big[k_0(\eta+h)\big]\big((\partial_h k_0)(z+h)+k_0\big)- \\ \quad -\sinh\big[k_0(\eta+h)\big]\cosh\big[k_0(z+h)\big]\big(\dfrac{\partial k_0}{\partial h}(z+h)+k_0\big) \end{array} \right\} \frac{1}{\cosh^2\big[k_0(\eta+h)\big]},$$

$$\partial_\eta Z_n = \left\{ \begin{array}{r} -(z+h)\sin\big[k_n(z+h)\big](\partial_\eta k_n)+ \\ +\cos\big[k_n(z+h)\big]\tan\big[k_n(\eta+h)\big]\,\big(k_n+(\eta+h)(\partial_\eta k_n)\big) \end{array} \right\} \frac{1}{\cos\big[k_n(\eta+h)\big]},$$

$$\partial_{k_n} Z_n = \left\{ \begin{array}{l} -\sin\big[k_n(z+h)\big]\cos\big[k_n(\eta+h)\big](z+h)+ \\ \quad +\sin\big[k_n(\eta+h)\big]\cos\big[k_n(z+h)\big](\eta+h) \end{array} \right\} \frac{1}{\cos^2\big[k_n(\eta+h)\big]},$$

$$\partial_h Z_n = \left\{ \begin{array}{l} -\sin\big[k_n(z+h)\big]\cos\big[k_n(\eta+h)\big]\big((\partial_h k_n)(z+h)+k_n\big)+ \\ \quad +\sin\big[k_n(\eta+h)\big]\cos\big[k_n(z+h)\big]\big((\partial_h k_n)(z+h)+k_n\big) \end{array} \right\} \frac{1}{\cos^2\big[k_n(\eta+h)\big]}.$$

For the values, of the above derivatives, on the free surface we easily obtain

$$[\partial_h Z_n]_{z=\eta} = 0, \quad n \geq -2,$$

$$[\partial_{k_n} Z_n]_{z=\eta} = 0, \quad n \geq 0,$$

$$[\partial_\eta Z_{-2}]_{z=\eta} = -\frac{1}{h_0} - \mu_0,$$

$$[\partial_\eta Z_{-1}]_{z=\eta} = -\mu_0, \quad \left[\frac{\partial Z_{-1}}{\partial h}\right]_{z=\eta} = 0$$

$$
\begin{aligned}
[\partial_\eta Z_0]_{z=\eta} &= (\eta+h)\tanh\big[k_0(\eta+h)\big]\,(\partial_\eta k_0) - \tanh\big[k_0(\eta+h)\big]\,\big(k_0 + (\eta+h)(\partial_\eta k_0)\big) \\
&= -k_0 \tanh\big[k_0(\eta+h)\big],
\end{aligned}
$$

$$
\begin{aligned}
[\partial_\eta Z_n]_{z=\eta} &= -(\eta+h)\tan\big[k_n(\eta+h)\big]\,(\partial_\eta k_n) + \tan\big[k_n(\eta+h)\big]\,\big(k_n + (\eta+h)(\partial_\eta k_n)\big) \\
&= k_n \tan\big[k_n(\eta+h)\big].
\end{aligned}
$$

Using the dispersion relations Eqs. (A.5) in the two last equations, we finally obtain

$$
[\partial_\eta Z_0]_{z=\eta} = -\mu_0,
$$

$$
[\partial_\eta Z_n]_{z=\eta} = -\mu_0, \quad n \geq 0.
$$

B Calculation of the variation of $\tilde{S}[\boldsymbol{\varphi}, \eta]$

The velocity potential is represented by a series expansion in terms of vertical functions (4.1) that depend implicitely by the free surface elevation and the depth. Substituting the representation into (3.14) the action functional $S[\Phi, \eta]$ becomes a functional on the boundary fields $\boldsymbol{\varphi}(\mathbf{x}, t) := \{\varphi_n(\mathbf{x}, t)\}_{n\geq -2}$ and $\eta(\mathbf{x}, t)$ given by

$$\begin{aligned}
\tilde{S}[\boldsymbol{\varphi}, \eta] \quad &= \quad \int_I \int_{\mathcal{D}_h^\eta} \Big\{ \sum_n \partial_t \varphi_n Z_n + \varphi_n \partial_t Z_n + \\
&\quad + \frac{1}{2}\Big(\sum_{n\geq -2} \nabla_{\mathbf{x}} \varphi_n Z_n + \varphi_n \nabla_{\mathbf{x}} Z_n \Big)^2 + \frac{1}{2}\Big(\sum_{n\geq -2} \varphi_n \partial_z Z_n \Big)^2 + gz \Big\} \, dV \, dt \\
&= \quad \tilde{F}[\boldsymbol{\varphi}, \eta] \; + \; \tilde{K}[\boldsymbol{\varphi}, \eta] \; + \; V[\eta],
\end{aligned}$$

where

$$\tilde{F}[\boldsymbol{\varphi}, \eta] = \sum_{n\geq 2} \int_I \int_{\mathcal{D}_h^\eta} \big\{ \partial_t \varphi_n Z_n + \varphi_n \partial_t Z_n \big\} dV dt,$$

$$\tilde{K}[\boldsymbol{\varphi}, \eta] = \frac{1}{2} \int_I \int_{\mathcal{D}_h^\eta} \Big\{ \big(\sum_{n\geq 2} \nabla_{\mathbf{x}} \varphi_n Z_n + \varphi_n \nabla_{\mathbf{x}} Z_n \big)^2 + \big(\sum_{n\geq 2} \varphi_n \partial_z Z_n \big)^2 \Big\} \, dV dt,$$

and

$$V[\eta] = \int_I \int_{\mathcal{D}_h^\eta} gz dV dt. \tag{B.1}$$

These three terms will be reffered as the time derrivative term, the kinetic energy term and the potential energy term correspondigly and will be treated seperately for convenience of the reader. The variation of the action functional $\tilde{S}[\{\varphi_n\}_{n\geq -2}, \eta]$ is the sum of the variations of the three terms, with respect to all functional arguments. The following identities simplify the presentation of the calculations of the variations of the three terms $\tilde{F}$, $\tilde{K}$ and V. They are known as Leibnitz's integral rule ([Die69]).

Proposition B.1. *Let $a, b \in C^1(I \to C_0^1(\mathbb{R}^2))$ be two functions with graphs Γ_a, Γ_b that define the open bounded domain $\mathcal{D}_b^a$. Given the functions $f, g \in C^1(I \to$*

$C^2(\mathcal{D}_b^a)) \cap C^1(I \to C_0^1(\overline{\mathcal{D}_b^a}))$ *the following identities hold*

$$\text{(B.2a)} \qquad \big\langle f, \partial_t g \big\rangle = \frac{d}{dt}\big\langle f, g \big\rangle - \partial_t a [fg]_{z=a} + \partial_t b [fg]_{z=b} - \big\langle \partial_t f, g \big\rangle$$

$$\text{(B.2b)} \quad \langle f, \partial_{x_i} g \rangle = \partial_{x_i} \langle f, g \rangle - \langle \partial_{x_i} f, g \rangle - [fg]_{z=a} \partial_{x_i} a + [fg]_{z=b} \partial_{x_i} b, \quad i = 1, 2,$$

(B.2c)

$$\langle \nabla_{\mathbf{x}} f, \nabla_{\mathbf{x}} g \rangle = \nabla_{\mathbf{x}} \cdot \langle \nabla_{\mathbf{x}} f, g \rangle - \big\langle \nabla_{\mathbf{x}}^2 f, g \big\rangle - [\nabla_{\mathbf{x}} f g]_{z=a} \cdot \nabla_{\mathbf{x}} a + [\nabla_{\mathbf{x}} f g]_{z=b} \cdot \nabla_{\mathbf{x}} b$$

with the notation $\big\langle f, g \big\rangle = \displaystyle\int_{b(\mathbf{x},t)}^{a(\mathbf{x},t)} f(\mathbf{x}, z, t)\, g(\mathbf{x}, z, t)\, dz.$

The partial variation of the kinetic energy term $\tilde{K}$ w.r.t $\delta\varphi_m$ reads

$$\begin{aligned}
\delta_{\varphi_m} \tilde{K}[\boldsymbol{\varphi}, \eta; \delta\varphi_m] \; &= \frac{d}{d\epsilon}\big[\tilde{\mathcal{K}}[\varphi_1, ..., \varphi_m + \epsilon\delta\varphi_m, ..., \eta]\big]_{\epsilon=0} \\
&= \int_I \int_{\mathcal{D}_h^\eta} \Big\{ \Big(\sum_{n \geq -2} \nabla_{\mathbf{x}} \varphi_n Z_n + \varphi_n \nabla_{\mathbf{x}} Z_n \Big) \cdot \Big(\nabla_{\mathbf{x}} \delta\varphi_m Z_m + \delta\varphi_m \nabla_{\mathbf{x}} Z_m \Big) + \\
&\quad + \Big(\sum_{n \geq -2} \varphi_n \partial_z Z_n \Big) \delta\varphi_m \partial_z Z_m \Big\} dV\, dt \\
&= \int_I \int_S \Big\{ \int_{-h}^{\eta} \Big(\sum_{n \geq -2} \nabla_{\mathbf{x}} \varphi_n Z_n + \varphi_n \nabla_{\mathbf{x}} Z_n \Big) \cdot \Big(\nabla_{\mathbf{x}} \delta\varphi_m Z_m + \delta\varphi_m \nabla_{\mathbf{x}} Z_m \Big)\, dz + \\
&\quad + \int_{-h}^{\eta} \Big(\sum_{n \geq -2} \partial_z Z_n \Big) \delta\varphi_m \partial_z Z_m \, dz \Big\} d\mathbf{x}\, dt.
\end{aligned}$$

For the first term we use (B.2c) with $f = \nabla_{\mathbf{x}}(\sum \varphi_n Z_n)$ and $g = \sum \delta\varphi_n Z_n$ and for the second we integrate by parts. Hence, we obtain

$$\begin{aligned}
\delta_{\varphi_m} \tilde{K}[\boldsymbol{\varphi}, \eta; \delta\varphi_m] = \int_I \int_S \Big\{ &- \sum_{n \geq -2} \nabla_{\mathbf{x}}^2 \varphi_n \langle Z_n, Z_m \rangle + 2\nabla_x \varphi_n \langle \nabla_{\mathbf{x}} Z_n, Z_m \rangle + \varphi_n \big\langle \nabla_{\mathbf{x}}^2 Z_n, Z_m \big\rangle - \\
&- \sum_{n \geq -2} \nabla_{\mathbf{x}} \varphi_n \cdot \nabla_{\mathbf{x}} h [Z_n Z_m]_{z=h} - \varphi_n \nabla_{\mathbf{x}} h \cdot [\nabla_{\mathbf{x}} Z_n Z_m]_{z=h} - \\
&- \sum_{n \geq -2} \nabla_{\mathbf{x}} \varphi_n \cdot \nabla_{\mathbf{x}} \eta [Z_n Z_m]_{z=\eta} + \varphi_n \nabla_{\mathbf{x}} \eta \cdot [\nabla_{\mathbf{x}} Z_n Z_m]_{z=\eta} + \\
&+ \sum_{n \geq -2} \varphi_n \Big([\partial_z Z_n Z_m]_{z=\eta} + [\partial_z Z_n Z_m]_{z=h} - \big\langle \partial_{zz}^2 Z_n, Z_m \big\rangle \Big) \Big\} \delta\varphi_m d\mathbf{x} dt.
\end{aligned}$$

For the time derivative term one has

$$\delta_{\varphi_m}\tilde{F}[\varphi,\eta;\delta\varphi_m] = \int_{t_0}^{T}\int_S -\partial_t\eta\delta\varphi_m d\mathbf{x}dt.$$

and obviously, for the potential emergy term

$$\delta_{\varphi_m}V[\eta;\delta\varphi_m] = 0 \tag{B.3}$$

After reordering the terms, summing the partial variations of the three terms and denoting $\Delta = \nabla_{\mathbf{x}}^2 + \partial_{zz}^2$ we obtain the result of Lemma 4.3. Simililarly, for the variation in η, one has

$$\begin{aligned}
\delta_\eta\tilde{F}\big[\boldsymbol{\varphi},\eta;\delta\eta\big] =& \frac{d}{d\epsilon}\tilde{F}[\boldsymbol{\varphi},\eta+\epsilon\delta\eta]\big|_{\epsilon=0} \\
=& \frac{d}{d\epsilon}\Bigg[\int_I\int_{\mathcal{D}_h^{\eta+\epsilon\delta\eta}}\Big(\sum_{n\geq -2}\partial_t\varphi_n Z_n\big(z;\eta+\epsilon\delta\eta,h\big)+\varphi_n\partial_t Z_n\big(z;\eta+\epsilon\delta\eta,h\big)\Big)dV\,dt\Bigg]_{\epsilon=0} \\
=& \int_I\int_S\Big\{\Big(\sum_{n\geq -2}\partial_t\varphi_n\big[Z_n\big]_{z=\eta}+\varphi_n\big[\partial_t Z_n\big]_{z=\eta}\Big)\delta\eta+ \\
&+\int_{-h}^{\eta}\Big(\sum_{n\geq -2}\partial_t\varphi_n\,(\partial_\eta Z_n)\delta\eta+\varphi_n\,\partial_t\big(\,(\partial_\eta Z_n)\,\delta\eta\big)\Big)dz\,\Big\}d\mathbf{x}dt.
\end{aligned}$$

Using idendity (B.2a), we can write

$$\begin{aligned}
\int_{-h}^{\eta}\partial_t\varphi_n\,(\partial_\eta Z_n)\delta\eta\,dz =& \,\partial_t\int_{-h}^{\eta}\varphi_n\,(\partial_\eta Z_n)\delta\eta\,dz- \\
&-\int_{-h}^{\eta}\varphi_n\partial_t\big((\partial_\eta Z_n)\delta\eta\big)\,dz-\partial_t\eta\varphi_n[\partial_\eta Z_n]_{z=\eta}\delta\eta
\end{aligned} \tag{B.4}$$

The first term of the right hand side of the last equation contributes only to the temporal boundaries and can be neglected. Using also the fact that

$$\varphi_n\partial_t Z_n = \varphi_n\big(\partial_\eta Z_n+(\partial_{k_n}Z_n)(\partial_\eta k_n)\big)\partial_t\eta = \varphi_n(\partial_\eta Z_n)(\partial_t\eta), \tag{B.5}$$

we obtain

$$\delta_\eta\tilde{F}\big[\boldsymbol{\varphi},\eta;\delta\eta\big] = \int_I\int_S\Big(\sum_{n\geq -2}\partial_t\varphi_n[Z_n]_{z=\eta}\Big)\delta\eta. \tag{B.6}$$

For the kinetic energy term we compute

$$
\begin{aligned}
\delta_\eta \tilde{K}\big[\boldsymbol{\varphi}, \eta; \delta\eta\big] &= \frac{d}{d\epsilon}\big[\tilde{K}[\boldsymbol{\varphi}, \eta+\epsilon\delta\eta]\big]_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\Bigg[\frac{1}{2}\int_I\int_{\mathcal{D}_h^{\eta+\epsilon\delta\eta}}\Big(\sum_{n\geq -2}\nabla_{\mathbf{x}}\varphi_n Z_n\big(z;\eta+\epsilon\delta\eta,h\big)+\varphi_n\nabla_{\mathbf{x}}Z_n\big(z;\eta+\epsilon\delta\eta,h\big)\Big)^2 \\
&\qquad + \Big(\sum_{n\geq -2}\varphi_n\partial_z Z_n\big(z;\eta+\epsilon\delta\eta,h\big)\Big)^2 dV\,dt\Bigg]_{\epsilon=0} \\
&= \int_I\int_S\Bigg\{\Big\{\frac{1}{2}\Big(\sum_{n\geq -2}\nabla_{\mathbf{x}}\varphi_n[Z_n]_{z=\eta}+\varphi_n[\nabla_{\mathbf{x}}Z_n]_{z=\eta}\Big)^2+\frac{1}{2}\Big(\sum_{n\geq -2}\varphi_n[\partial_z Z_n]_{z=\eta}\Big)^2\Big\}\delta\eta \\
&+\int_{-h}^{\eta}\Big\{\Big(\sum_{n\geq -2}\nabla_{\mathbf{x}}\varphi_n Z_n+\varphi_n\nabla_{\mathbf{x}}Z_n\Big)\cdot\Big(\sum_{l\geq -2}\nabla_{\mathbf{x}}\varphi_l(\partial_\eta Z_l)\delta\eta+\varphi_l\nabla_{\mathbf{x}}\big((\partial_\eta Z_l)\delta\eta\big)\Big)+ \\
&\qquad + \Big(\sum_{n\geq -2}\varphi_n\partial_z Z_n\Big)\Big(\sum_{l\geq -2}\varphi_l\,\partial_z(\partial_\eta Z_l)\delta\eta\Big)\Big\}dz\Bigg\}d\mathbf{x}dt
\end{aligned}
$$

The first term is computed, using (4.13), as follows

$$
\begin{aligned}
&\frac{1}{2}\Big(\sum_{n\geq -2}\nabla_{\mathbf{x}}\varphi_n[Z_n]_{z=\eta}+\varphi_n[\nabla_{\mathbf{x}}Z_n]_{z=\eta}\Big)^2+\frac{1}{2}\Big(\sum_{n\geq -2}\varphi_n[\partial_z Z_n]_{z=\eta}\Big)^2 = \\
&=\frac{1}{2}(\nabla_{\mathbf{x}}\varphi)^2-\nabla_{\mathbf{x}}\varphi\cdot\nabla_{\mathbf{x}}\eta\big(\frac{\varphi_{-2}}{h_0}+\mu_0\varphi\big)+\frac{1}{2}\big((\nabla_{\mathbf{x}}\eta)^2+1\big)\big(\frac{\varphi_{-2}}{h_0}+\mu_0\varphi\big)^2
\end{aligned}
\tag{B.7}
$$

For the second term we use (B.2c) with $f=\sum\varphi_n Z_n$ and $g=\sum\varphi_l(\partial_\eta Z_l)\delta\eta$ to obtain

(B.8)

$$
\begin{aligned}
\int_{-h}^{\eta}\nabla_{\mathbf{x}}f\cdot\nabla_{\mathbf{x}}g\,dz = &-\Big\langle\nabla_{\mathbf{x}}^2\big(\sum\varphi_n Z_n\big),\big(\sum\varphi_l(\partial_\eta Z_l)\delta\eta\big)\Big\rangle - \\
&-\big[\nabla_{\mathbf{x}}\big(\sum\varphi_n Z_n\big)\big(\sum\varphi_l(\partial_\eta Z_l)\delta\eta\big)\big]_{z=\eta}\cdot\nabla_{\mathbf{x}}\eta- \\
&-\big[\nabla_{\mathbf{x}}\big(\sum\varphi_n Z_n\big)\big(\sum\varphi_l(\partial_\eta Z_l)\delta\eta\big)\big]_{z=-h}\cdot\nabla_{\mathbf{x}}h \\
= &-\Big\langle\nabla_{\mathbf{x}}^2\big(\sum\varphi_n Z_n\big),\big(\sum\varphi_l(\partial_\eta Z_l)\delta\eta\big)\Big\rangle - \\
&-(\nabla_{\mathbf{x}}\eta)^2\big(\frac{\varphi_{-2}}{h_0}+\mu_0\varphi\big)^2\delta\eta+\nabla_{\mathbf{x}}\varphi\cdot\nabla_{\mathbf{x}}\eta\big(\frac{\varphi_{-2}}{h_0}+\mu_0\varphi\big)\delta\eta- \\
&-\big[\nabla_{\mathbf{x}}\big(\sum\varphi_n Z_n\big)\big(\sum\varphi_l(\partial_\eta Z_l)\delta\eta\big)\big]_{z=-h}\cdot\nabla_{\mathbf{x}}h,
\end{aligned}
$$

where the term contributing only on the latteral boundaries is neglected and also (4.13) have been used. Finally for the third term we perform an integration by parts

$$
\begin{aligned}
\int_{-h}^{\eta}\Big(\sum_{n\geq -2}\varphi_n\partial_z Z_n\Big)\Big(\sum_{l\geq -2}\varphi_l\,\partial_z(\partial_\eta Z_l)\delta\eta\Big)\,dz = -\big(\frac{\varphi_{-2}}{h_0}+\mu_0\varphi\big)^2\delta\eta -& \\
-\Big(\sum_{n\geq -2}\varphi_n[\partial_z Z_n]_{z=-h}\Big)\Big(\sum_{l\geq -2}\varphi_l\,[\partial_\eta Z_l]_{z=-h}\delta\eta\Big) -& \\
-\big\langle\partial_z\Big(\sum_{n\geq -2}\varphi_n\partial_z Z_n\Big),\sum_{l\geq -2}\varphi_l(\partial_\eta Z_l)\delta\eta\big\rangle.&
\end{aligned}
\tag{B.9}
$$

For the potential energy term one has

$$
\delta_\eta V[\boldsymbol{\varphi},\eta;\delta\eta] = \int_I\int_S g\eta\delta\eta\,d\mathbf{x}\,dt. \tag{B.10}
$$

Taking the sum of (B.6), (B.8), (B.9), (B.10) one can verify the result of Lemma 4.4.

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