NATIONAL TECHNICAL UNIVERSITY OF ATHENS

DEPARTMENT OF SURVEYING ENGINEERING

A Study on Geodetic Boundary Value Problems in Ellipsoidal Geometry

by

Georgios Panou

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Supervisor:
Associate Professor D. Delikaraoglou

Athens, June 2014

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Athens, June 2014

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CONTENTS

ABSTRACT

In this thesis, special geodetic problems are treated as boundary value problems. The geodesic problem, the gravity field due to a homogeneous ellipsoid and the linear fixed altimetry-gravimetry problem are thoroughly studied in ellipsoidal geometry. The study is not limited on a biaxial ellipsoid (oblate spheroid), which is the well-known mathematical model used in geodesy, but is extended on a triaxial ellipsoid. The key issue in the current analysis is the expression of the above problems in the suitable ellipsoidal coordinate system.

The ellipsoidal coordinate system is described in some detail. For a one-to-one correspondence between ellipsoidal and Cartesian coordinates two variants of ellipsoidal coordinates are introduced. The transformation between ellipsoidal and Cartesian coordinates on a triaxial ellipsoid is presented in these two variants. Also, the element of distance and Laplace's equation are expressed in these coordinates. The classical transformation between ellipsoidal and Cartesian coordinates on a biaxial ellipsoid is presented as a degenerate case.

The geodesic problem on a triaxial ellipsoid is solved as a boundary value problem, using the calculus of variations. The boundary value problem is formulated by means of the Euler-Lagrange equation and consists of solving a non-linear second order ordinary differential equation, subject to the Dirichlet conditions. Subsequently, this problem is reduced to an initial value problem with Dirichlet and Neumann conditions. The Neumann condition is determined iteratively by solving a system of four first-order ordinary differential equations with numerical integration. The last iteration yields the solution of the boundary value problem. From the solution, the ellipsoidal coordinates and the angle between the line of constant longitude and the geodesic, at any point along the geodesic, are determined. Also, the constant in Liouville's equation is determined and the geodesic distance between the two points, as an integral, is computed by numerical integration. To demonstrate the validity of the method, numerical examples are given. The geodesic boundary value problem and its solution on a biaxial ellipsoid are obtained as a degenerate case. In this case, using a special case of the Euler-Lagrange equation, the Clairaut equation is verified and the

Clairaut constant is precisely determined. Also, the numerical tests are validated by comparison to Vincenty's method.

The exterior gravity potential and its derivative induced by a homogeneous triaxial ellipsoid are presented. Some expressions, which are used for the representation of the exterior gravitational potential, are mentioned. Subsequently, the mathematical framework using ellipsoidal coordinates is derived. In this case, the gravitational potential includes elliptic integrals which can be computed by a numerical integration method. From the gravity potential, the gravity vector components are subsequently obtained. The novel general expressions can be applied to a triaxial and a biaxial ellipsoid. Also, the gravity field due to a homogeneous oblate spheroid is obtained as a degenerate case. Numerical examples are given in order to demonstrate the validity of the general expressions.

The linear fixed altimetry-gravimetry boundary value problem is analyzed with respect to the existence and uniqueness of the solution. Nowadays, it is possible to determine very precisely points on the physical surface of the Earth by three-dimensional satellite positioning and the problem is to determine the disturbing potential in an unbounded domain representing the exterior of the Earth. In order to establish realistic boundary conditions, a Dirichlet condition is imposed at seas and an oblique derivative condition on land. Then, mathematical methods are used, within the frame of functional analysis, for attacking the problem under consideration. Specifically, the Stampacchia theorem is used to decide upon the existence and uniqueness of the weak solution of the problem in a weighted Sobolev space. Finally, it is confirmed that the condition of validity for such a theorem has a geometrical interpretation.

Lastly, a method for solving the problem of height datum unification is presented. This is essentially a problem of determining the potential differences among different height datums. The local height datums vary mainly due to different ways of their definition, methods of realization and the fact that they are based on local data. The main approaches for determining potential differences are outlined and compared, taking into account the recent developments in the theory of geodetic boundary value problems (BVPs). This allowed us to select the fixed mixed BVP as

the most suitable type for the estimation of the quasigeoid, which has the advantage that is independent of any local height datums and it can be regarded as a global height datum. The basic method of datum unification relies on the comparison of the potential differences of each local height datum with the so-determined global height datum (i.e. the quasigeoid).

Keywords: triaxial ellipsoid, ellipsoidal coordinates, geodesic problem, gravity potential, altimetry-gravimetry problem, Stampacchia theorem, numerical integration

EXTENDED ABSTRACT IN GREEK

ΠΕΡΙΛΗΨΗ

Διερεύνηση Γεωδαιτικών Προβλημάτων Συνοριακών Τιμών
σε Ελλειψοειδή Γεωμετρία

Στην παρούσα διατριβή, ειδικά γεωδαιτικά προβλήματα αντιμετωπίζονται ως προβλήματα συνοριακών τιμών. Το γεωδαισιακό πρόβλημα, το πεδίο βαρύτητας που παράγεται από ένα ομογενές ελλειψοειδές και το γραμμικό δεσμευμένο αλτιμετρικό-βαρυτημετρικό πρόβλημα μελετώνται πλήρως σε ελλειψοειδή γεωμετρία. Η μελέτη δεν περιορίζεται στο διαξονικό ελλειψοειδές (πεπλατυσμένο σφαιροειδές), που είναι το κλασικό μαθηματικό μοντέλο που χρησιμοποιείται στη γεωδαισία, αλλά επεκτείνεται και στο τριαξονικό ελλειψοειδές. Το ουσιαστικό θέμα στην ανάλυση των παραπάνω προβλημάτων είναι η έκφρασή τους στο κατάλληλο ελλειψοειδές σύστημα συντεταγμένων.

Το ελλειψοειδές σύστημα συντεταγμένων περιγράφεται λεπτομερώς. Για μια αντιστοιχία ένα προς ένα μεταξύ ελλειψοειδών και Καρτεσιανών συντεταγμένων, εισάγονται δύο παραλλαγές των ελλειψοειδών συντεταγμένων. Στις δύο αυτές παραλλαγές παρουσιάζεται ο μετασχηματισμός μεταξύ ελλειψοειδών και Καρτεσιανών συντεταγμένων στο τριαξονικό ελλειψοειδές. Επίσης, το στοιχειώδες μήκος και η εξίσωση Laplace εκφράζονται στις συντεταγμένες αυτές. Για την παραλλαγή των ελλειψοειδών συντεταγμένων που χρησιμοποιείται στους φορμαλισμούς της παρούσας μελέτης, ο μετασχηματισμός έχει τη μορφή

$$x = \sqrt{u^2 + E_x^2}\left(\cos^2\beta + \frac{E_e^2}{E_x^2}\sin^2\beta\right)^{1/2}\cos\lambda\,, \tag{1α}$$

$$y = \sqrt{u^2 + E_y^2}\cos\beta\sin\lambda\,, \tag{1β}$$

$$z = u\sin\beta\left(1 - \frac{E_e^2}{E_x^2}\cos^2\lambda\right)^{1/2}, \tag{1γ}$$

όπου $u \geq 0$, $-\pi/2 \leq \beta \leq +\pi/2$, $-\pi < \lambda \leq +\pi$ είναι οι ελλειψοειδείς συντεταγμένες. Συνεπώς, οι συντεταγμένες επιφάνειες είναι: (i) τριαξονικά ελλειψοειδή (u = σταθερό), (ii) μονόχωνα υπερβολοειδή (β = σταθερό) και (iii) δίχωνα υπερβολοειδή (λ = σταθερό). Γεωμετρικά, οι συντεταγμένες ερμηνεύονται ως ακολούθως: Σε ένα Καρτεσιανό σύστημα συντεταγμένων, ένα σημείο P έχει συντεταγμένες (x, y, z). Ορίζουμε ότι ένα τριαξονικό ελλειψοειδές που περιλαμβάνει το σημείο P, έχει ως αρχή την αρχή του συστήματος, ο πολικός άξονας του $2b$ ταυτίζεται με τον άξονα των z, ο μεγάλος ισημερινός άξονας $2a_x$ ταυτίζεται με τον άξονα των x, ο μικρός ισημερινός άξονας $2a_y$ ταυτίζεται με τον άξονα των y και οι δύο γραμμικές εκκεντρότητες έχουν σταθερές τιμές E_x και E_y. Η συντεταγμένη u είναι ο πολικός ημιάξονας του ελλειψοειδούς, β είναι το ελλειψοειδές πλάτος και λ είναι το ελλειψοειδές μήκος. Από τη σχέση

$$\beta = \tan^{-1}\left(\frac{z}{y}\right), \tag{2}$$

οδηγούμαστε στην ερμηνεία ότι το ελλειψοειδές πλάτος β χαρακτηρίζει την κλίση των ασύμπτωτων της οικογένειας των ομοέστιων κύριων υπερβολών στο επίπεδο x = 0. Όμοια, από τη σχέση

$$\lambda = \tan^{-1}\left(\frac{y}{x}\right), \tag{3}$$

το ελλειψοειδές μήκος λ, χαρακτηρίζει την κλίση των ασύμπτωτων της οικογένειας των ομοέστιων κύριων υπερβολών στο επίπεδο z = 0. Ο αντίστροφος μετασχηματισμός έχει τη μορφή

$$u = \sqrt{b^2 + s_1}\,, \tag{4α}$$

$$\beta = \tan^{-1}\sqrt{\frac{-b^2 - s_2}{a_y^2 + s_2}}\,, \tag{4β}$$

$$\lambda = \tan^{-1}\sqrt{\frac{-a_y^2 - s_3}{a_x^2 + s_3}}\,, \tag{4γ}$$

όπου οι τιμές των s_1, s_2, s_3 υπολογίζονται από την επίλυση κατάλληλης κυβικής εξίσωσης. Τέλος, ο κλασικός μετασχηματισμός μεταξύ ελλειψοειδών και Καρτεσιανών συντεταγμένων στο διαξονικό ελλειψοειδές παρουσιάζεται ως μια εκφυλισμένη περίπτωση.

Το γεωδαισιακό πρόβλημα περιλαμβάνει τον προσδιορισμό της γεωδαισιακής μεταξύ δύο σημείων $P_0(\beta_0, \lambda_0)$ και $P_1(\beta_1, \lambda_1)$ σε ένα ελλειψοειδές. Στην παρούσα μελέτη, το γεωδαισιακό πρόβλημα στο τριαξονικό ελλειψοειδές επιλύεται ως ένα πρόβλημα συνοριακών τιμών χρησιμοποιώντας το λογισμό μεταβολών. Διακρίνονται δύο περιπτώσεις όπου η γεωδαισιακή περιγράφεται ως: (i) β = β(λ) και (ii) λ = λ(β). Το συνοριακό πρόβλημα τυποποιείται σύμφωνα με την εξίσωση Euler-Lagrange και περιλαμβάνει την επίλυση μίας μη γραμμικής δεύτερης τάξης συνήθη διαφορική εξίσωση. Στην πρώτη περίπτωση, η εξίσωση έχει τη μορφή

$$2\overline{E}\,\overline{G}\beta'' + \overline{E}\,\overline{E}_\lambda(\beta')^3 - \left(2\overline{E}\,\overline{G}_\beta - \overline{E}_\beta\overline{G}\right)(\beta')^2 + \left(2\overline{E}_\lambda\overline{G} - \overline{E}\,\overline{G}_\lambda\right)\beta' - \overline{G}\,\overline{G}_\beta = 0\,, \tag{5}$$

και υπόκεινται στις συνθήκες τύπου Dirichlet

$$\beta_0 = \beta(\lambda_0), \qquad \beta_1 = \beta(\lambda_1). \tag{6}$$

Ακολούθως, το πρόβλημα ανάγεται σε ένα πρόβλημα αρχικών τιμών

$$\frac{d}{d\lambda}(\beta) = \beta'\,, \tag{7α}$$

$$\frac{d}{d\lambda}(\beta') = \left(-\frac{1}{2}\frac{\overline{E}_\lambda}{\overline{G}}\right)(\beta')^3 + \left(\frac{\overline{G}_\beta}{\overline{G}} - \frac{1}{2}\frac{\overline{E}_\beta}{\overline{E}}\right)(\beta')^2 + \left(\frac{1}{2}\frac{\overline{G}_\lambda}{\overline{G}} - \frac{\overline{E}_\lambda}{\overline{E}}\right)\beta' + \left(\frac{1}{2}\frac{\overline{G}_\beta}{\overline{E}}\right), \tag{7β}$$

με συνθήκες Dirichlet και Neumann

$$\text{D: } \beta_0 = \beta(\lambda_0), \qquad \text{N: } \beta'_0 = \beta'(\lambda_0). \tag{8}$$

Η συνθήκη Neumann προσδιορίζεται με επαναληπτική διαδικασία επιλύοντας, με αριθμητική ολοκλήρωση, το παρακάτω σύστημα τεσσάρων πρώτης τάξης συνήθων διαφορικών εξισώσεων

$$\frac{d}{d\lambda}(\beta) = \beta', \tag{9α}$$

$$\frac{d}{d\lambda}(\beta') = \left(-\frac{1}{2}\frac{\overline{E}_\lambda}{\overline{G}}\right)(\beta')^3 + \left(\frac{\overline{G}_\beta}{\overline{G}} - \frac{1}{2}\frac{\overline{E}_\beta}{\overline{E}}\right)(\beta')^2 + \left(\frac{1}{2}\frac{\overline{G}_\lambda}{\overline{G}} - \frac{\overline{E}_\lambda}{\overline{E}}\right)\beta' + \left(\frac{1}{2}\frac{\overline{G}_\beta}{\overline{E}}\right), \tag{9β}$$

$$\frac{d}{d\lambda}\left(\frac{\partial\beta}{\partial\beta'_0}\right) = \frac{\partial\beta'}{\partial\beta'_0}, \tag{9γ}$$

$$\frac{d}{d\lambda}\left(\frac{\partial\beta'}{\partial\beta'_0}\right) = \left[p_{33}(\beta')^3 + p_{22}(\beta')^2 + p_{11}\beta' + p_{00}\right]\frac{\partial\beta}{\partial\beta'_0} + \left[3p_3(\beta')^2 + 2p_2\beta' + p_1\right]\frac{\partial\beta'}{\partial\beta'_0}, \tag{9δ}$$

όπου οι συντελεστές στην εξίσωση (9δ) δίνονται από τις σχέσεις

$$p_{33} = \frac{\partial p_3}{\partial\beta} = -\frac{1}{2}\frac{\overline{E}_{\beta\lambda}\overline{G} - \overline{E}_\lambda\overline{G}_\beta}{\overline{G}^2}, \tag{10α}$$

$$p_{22} = \frac{\partial p_2}{\partial\beta} = \frac{\overline{G}\,\overline{G}_{\beta\beta} - \overline{G}_\beta\overline{G}_\beta}{\overline{G}^2} - \frac{1}{2}\frac{\overline{E}\,\overline{E}_{\beta\beta} - \overline{E}_\beta\overline{E}_\beta}{\overline{E}^2}, \tag{10β}$$

$$p_{11} = \frac{\partial p_1}{\partial\beta} = \frac{1}{2}\frac{\overline{G}\,\overline{G}_{\beta\lambda} - \overline{G}_\beta\overline{G}_\lambda}{\overline{G}^2} - \frac{\overline{E}\,\overline{E}_{\beta\lambda} - \overline{E}_\beta\overline{E}_\lambda}{\overline{E}^2}, \tag{10γ}$$

$$p_{00} = \frac{\partial p_0}{\partial\beta} = +\frac{1}{2}\frac{\overline{E}\,\overline{G}_{\beta\beta} - \overline{E}_\beta\overline{G}_\beta}{\overline{E}^2}. \tag{10δ}$$

Η τελευταία επανάληψη παράγει και τη λύση του συνοριακού προβλήματος. Από τη λύση προσδιορίζονται, σε κάθε σημείο κατά μήκος της γεωδαισιακής, οι ελλειψοειδείς συντεταγμένες και η γωνία α μεταξύ της γραμμής σταθερού ελλειψοειδούς μήκους και της γεωδαισιακής, από τη σχέση

$$\alpha = arc\cot\left(\frac{\sqrt{\overline{E}}}{\sqrt{\overline{G}}}\beta'\right). \tag{11}$$

Επίσης, προσδιορίζεται η σταθερά c στην εξίσωση του Liouville

$$\left(\cos^2\beta + \frac{E_e^2}{E_x^2}\sin^2\beta\right)\sin^2\alpha + \frac{E_e^2}{E_x^2}\cos^2\lambda\cos^2\alpha = c^2, \tag{12}$$

και υπολογίζεται, με αριθμητική ολοκλήρωση, η γεωδαισιακή απόσταση μεταξύ δύο σημείων, από τη σχέση

$$s = \int_{\lambda_0}^{\lambda_1}\sqrt{\overline{E}(\beta')^2 + \overline{G}}\,d\lambda. \tag{13}$$

Ανάλογες εκφράσεις ισχύουν και για τη δεύτερη περίπτωση. Για να αποδεχθεί η ισχύς της μεθόδου δίνονται αριθμητικά παραδείγματα. Το γεωδαισιακό πρόβλημα συνοριακών τιμών και η λύση του στο διαξονικό ελλειψοειδές λαμβάνονται ως μια εκφυλισμένη περίπτωση. Στην περίπτωση αυτή, χρησιμοποιώντας μια ειδική περίπτωση της εξίσωσης Euler-Lagrange, η εξίσωση Clairaut επαληθεύεται και η σταθερά του Clairaut προσδιορίζεται με ακρίβεια. Επίσης, οι αριθμητικοί έλεγχοι επικυρώνονται από τη σύγκριση των αποτελεσμάτων με τη μέθοδο του Vincenty.

Το εξωτερικό δυναμικό βαρύτητας U και η παραγωγός του (γ_u, γ_β, γ_λ), που παράγονται από ένα ομογενές τριαξονικό ελλειψοειδές, παρουσιάζονται σε ελλειψοειδείς συντεταγμένες (u, β, λ) από τις σχέσεις

$$U(u,\beta,\lambda) = \frac{3}{2}G\mathrm{M}\left[I_0(u) - I_1(u)x^2 - I_2(u)y^2 - I_3(u)z^2\right] + \Phi(u,\beta,\lambda), \tag{14}$$

όπου

$$I_0(u) = \int_u^{+\infty} \frac{d\sigma}{\left(\sigma^2 + E_x^2\right)^{1/2}\left(\sigma^2 + E_y^2\right)^{1/2}}, \tag{15α}$$

$$I_1(u) = \int_u^{+\infty} \frac{d\sigma}{\left(\sigma^2 + E_x^2\right)^{3/2}\left(\sigma^2 + E_y^2\right)^{1/2}}, \tag{15β}$$

$$I_2(u) = \int_u^{+\infty} \frac{d\sigma}{\left(\sigma^2 + E_x^2\right)^{1/2}\left(\sigma^2 + E_y^2\right)^{3/2}}, \tag{15γ}$$

$$I_3(u) = \int_u^{+\infty} \frac{d\sigma}{\sigma^2\left(\sigma^2 + E_x^2\right)^{1/2}\left(\sigma^2 + E_y^2\right)^{1/2}}, \tag{15δ}$$

$$\Phi(u,\beta,\lambda) = \frac{1}{2}\omega^2\left[\left(u^2 + E_y^2 + E_e^2\cos^2\lambda\right)\cos^2\beta + \left(u^2 + E_x^2\right)\frac{E_e^2}{E_x^2}\sin^2\beta\cos^2\lambda\right], \tag{15ε}$$

και

$$\gamma_u = \left[\frac{\left(u^2 + E_x^2\right)\left(u^2 + E_y^2\right)}{\left(u^2 + E_y^2\sin^2\beta\right)\left(u^2 + E_y^2 + E_e^2\sin^2\lambda\right)}\right]^{1/2}\frac{\partial U}{\partial u}, \tag{16α}$$

$$\gamma_\beta = \left[\frac{E_x^2 - E_y^2\sin^2\beta}{\left(u^2 + E_y^2\sin^2\beta\right)\left(E_y^2\cos^2\beta + E_e^2\sin^2\lambda\right)}\right]^{1/2}\frac{\partial U}{\partial\beta}, \tag{16β}$$

$$\gamma_\lambda = \left[\frac{E_x^2 - E_e^2\cos^2\lambda}{\left(u^2 + E_y^2 + E_e^2\sin^2\lambda\right)\left(E_y^2\cos^2\beta + E_e^2\sin^2\lambda\right)}\right]^{1/2}\frac{\partial U}{\partial\lambda}. \tag{16γ}$$

Αρχικά, αναφέρονται κάποιες εκφράσεις που χρησιμοποιούνται για την αναπαράσταση του εξωτερικού ελκτικού δυναμικού. Ακολούθως, το μαθηματικό πλαίσιο δημιουργείται σύμφωνα με τις ελλειψοειδείς συντεταγμένες. Όπως είναι φανερό από τις Εξισώσεις 14 και 15, το δυναμικό περιλαμβάνει ελλειπτικά ολοκληρώματα που υπολογίζονται με αριθμητική ολοκλήρωση. Από το δυναμικό βαρύτητας λαμβάνονται διαδοχικά οι συνιστώσες του διανύσματος της βαρύτητας. Οι παραπάνω νέες εκφράσεις ισχύουν σε ένα τριαξονικό και διαξονικό ελλειψοειδές. Επίσης, το πεδίο βαρύτητας που παράγεται από ένα ομογενές πεπλατυσμένο σφαιροειδές λαμβάνεται και ως μια εκφυλισμένη περίπτωση. Προκειμένου να επικυρωθεί η ισχύς των γενικών εκφράσεων δίνονται αριθμητικά παραδείγματα.

Το γραμμικό δεσμευμένο αλτιμετρικό-βαρυτημετρικό πρόβλημα συνοριακών τιμών αναλύεται ως προς την ύπαρξη και τη μοναδικότητα της λύσης του. Στις μέρες μας είναι δυνατό να προσδιοριστούν με μεγάλη ακρίβεια, μέσω δορυφορικών τεχνολογιών, σημεία στη φυσική γήινη επιφάνεια. Συνεπώς, το πρόβλημα ανάγεται στον προσδιορισμό του διαταρακτικού δυναμικού T σε ένα μη φραγμένο χωρίο Ω που αναπαριστά το εξωτερικό της Γης. Προκειμένου να σχηματιστούν ρεαλιστικές συνοριακές συνθήκες, επιβάλλεται μια συνθήκη Dirichlet στις θάλασσες $\partial\Omega_S$ και μια συνθήκη πλάγιας παραγώγου στη στεριά $\partial\Omega_L$. Το μαθηματικό μοντέλο που προκύπτει, έχει την παρακάτω μορφή

$$\Delta T = 0 \qquad \text{in } \Omega, \tag{17α}$$

$$T = f_S \qquad \text{on } \partial\Omega_S, \tag{17β}$$

$$(\boldsymbol{n}\cdot\nabla T) + (\boldsymbol{a}\cdot\nabla_{\partial\Omega_L} T) = -f_L \qquad \text{on } \partial\Omega_L, \tag{17γ}$$

$$T = O(|\boldsymbol{x}|^{-1}) \qquad \text{as } |\boldsymbol{x}| \to +\infty. \tag{17δ}$$

Κατόπιν, χρησιμοποιούνται μαθηματικές μέθοδοι εντός του πλαισίου της συναρτησιακής ανάλυσης για την αντιμετώπιση του υπό μελέτη προβλήματος. Ειδικότερα, χρησιμοποιείται το θεώρημα του Stampacchia στην απόφαση για την ύπαρξη και την μοναδικότητα της ασθενούς λύσης του προβλήματος σε ένα σταθμικό χώρο Sobolev. Τα αποτελέσματα της μελέτης συνοψίζονται στο ακόλουθο θεώρημα:

Θεώρημα. Θεωρούμε Ω ένα μη φραγμένο χωρίο και $\Omega' = \Re^3 - \overline{\Omega}$ ένα αστρόμορφο χωρίο ως προς την αφετηρία με σύνορο τύπου $C^{1,1}$. Επιπλέον, θεωρούμε $\boldsymbol{a} \in H^{1,\infty}$ τέτοιο ώστε

$$\left|\mathrm{div}_{\partial\Omega_L}(\boldsymbol{a})\right| < t, \tag{18}$$

να ισχύει στο $\partial\Omega_L$, όπου t είναι μια θετική σταθερά. Τότε, για όλα τα $f_S \in H^{1/2,2}(\partial\Omega_S)$ και $f_L \in H^{-1/2,2}(\partial\Omega_L)$ υπάρχει μία και μόνο μία ασθενή λύση $T \in W_2^{(1)}(\Omega)$ του γραμμικού δεσμευμένου αλτιμετρικού-βαρυτημετρικού προβλήματος συνοριακών τιμών. Τέλος, επιβεβαιώνεται ότι η συνθήκη ισχύος του θεωρήματος (Εξίσωση 18) έχει γεωμετρική ερμηνεία.

Καταλήγοντας, παρουσιάζεται μια μέθοδος επίλυσης του προβλήματος ενοποίησης των υψομετρικών αναφορών. Πρόκειται ουσιαστικά για ένα πρόβλημα προσδιορισμού των διαφορών δυναμικού μεταξύ των διαφόρων υψομετρικών αναφορών. Οι τοπικές υψομετρικές αναφορές διαφέρουν κυρίως λόγω των διαφορετικών τρόπων ορισμού τους, των μεθόδων υλοποίησης και του γεγονότος ότι στηρίζονται σε τοπικά δεδομένα. Οι κύριες προσεγγίσεις προσδιορισμού διαφορών δυναμικού περιγράφονται και συγκρίνονται, λαμβάνοντας υπόψη τις τρέχουσες εξελίξεις της θεωρίας των γεωδαιτικών προβλημάτων συνοριακών τιμών. Αυτό μας επιτρέπει να επιλέξουμε το δεσμευμένο μεικτό πρόβλημα συνοριακών τιμών ως το πιο κατάλληλο για την εκτίμηση του σχεδόν γεωειδούς, που είναι ανεξάρτητο από κάθε τοπική υψομετρική αναφορά και μπορεί να θεωρηθεί ως παγκόσμια υψομετρική αναφορά. Η βασική μέθοδος ενοποίησης των αναφορών στηρίζεται στη σύγκριση των διαφορών δυναμικού καθεμιάς τοπικής υψομετρικής αναφοράς με την αποκαλούμενη παγκόσμια υψομετρική αναφορά, δηλαδή το σχεδόν γεωειδές.

Λέξεις-κλειδιά: τριαξονικό ελλειψοειδές, ελλειψοειδείς συντεταγμένες, γεωδαισιακό πρόβλημα, δυναμικό βαρύτητας, αλτιμετρικό-βαρυτημετρικό πρόβλημα, θεώρημα Stampacchia, αριθμητική ολοκλήρωση

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a	major semiaxis
a_x	major equatorial semiaxis
a_y	minor equatorial semiaxis
A	bilinear form
b	polar semiaxis
c	Clairaut's constant
C	geopotential number
ds	line element
E	linear eccentricity
E_x, E_y, E_e	linear eccentricities
$\overline{E}$, $\overline{F}$, $\overline{G}$	first fundamental coefficients
$\overline{E}_\beta$, $\overline{E}_\lambda$, $\overline{G}_\beta$, $\overline{G}_\lambda$	first-order partial derivatives
$\overline{E}_{\beta\beta}$, $\overline{E}_{\lambda\lambda}$, $\overline{G}_{\beta\beta}$, $\overline{G}_{\lambda\lambda}$	second-order partial derivatives
$\overline{E}_{\beta\lambda}$, $\overline{E}_{\lambda\beta}$, $\overline{G}_{\beta\lambda}$, $\overline{G}_{\lambda\beta}$	second-order mixed derivatives
E_n^m	Lamé functions of the first kind
f_S	data on sea
f_L	data on land
F_n^m	Lamé functions of the second kind
$\boldsymbol{g}$	actual gravity vector
g	actual gravity
G	gravitational constant
h	geometric height
H	orthometric height
H^*	normal height
k	Gauss curvature
k_e	mean curvature of the ellipsoid
$k_{\partial\Omega_L}$	mean curvature of $\partial\Omega_L$
ℓ	Liouville's constant

M	mass
$P_{n,m}$	associated Legendre functions
(r, β, λ)	spherical coordinates
s	geodesic distance
(s_1, s_2, s_3)	ellipsoidal coordinates (in length squared)
T	disturbing potential
U	normal gravity potential
(u, β, λ)	ellipsoidal coordinates (parameter, latitude, longitude)
V	gravitational potential
W	actual gravity potential
$W_2^{(1)}$, $H^{1/2,2}$, $H^{-1/2,2}$	Sobolev spaces
$\lvert \boldsymbol{x} \rvert$	Euclidean norm
(x, y, z)	Cartesian coordinates
α	azimuth
α_0, α_1	angles
$\boldsymbol{\gamma}$	normal gravity vector
γ_u, γ_β, γ_λ	normal gravity vector components
γ_x, γ_y, γ_z	normal gravity vector components
γ	normal gravity
$\delta \boldsymbol{g}$	gravity disturbance vector
δg	gravity disturbance
Δ	Laplace operator
∇	gradient operator
Δg	gravity anomaly
ζ	height anomaly
$\partial\Omega$	surface of the Earth
$\partial\Omega_S$	sea
$\partial\Omega_L$	land
κ	geodesic curvature
$\boldsymbol{v}$, $\boldsymbol{h}$, $\boldsymbol{n}$, $\boldsymbol{a}$	vector fields
N	geoidal undulation

(ρ, μ, ν)	ellipsoidal coordinates (in length)
ρ_0	density
Φ	centrifugal potential
ω	angular velocity

1. INTRODUCTION

1.1. Background and motivation

Geodetic research has traditionally been motivated by the need to approximate closer and closer the physical reality. Researchers such as Burša (1971), Burša and Šíma (1980) and Burša and Fialová (1993) have shown that the Earth is better approximated by a triaxial ellipsoid than a biaxial one and have estimated parameters determining the Earth's triaxiality. Furthermore, several non-spherical celestial bodies such as planets, natural satellites, asteroids and comets are already modeled by a triaxial ellipsoid. Tables with such triaxial ellipsoid parameters are included in Seidelmann et al. (2007). Also, present day accuracy requirements and modern computational capabilities continue to push toward the study of the triaxial ellipsoid as a geometrical model in geodesy and related interdisciplinary sciences. Indeed, the transformation between geodetic (planetographic) and Cartesian coordinates on a triaxial ellipsoid has been presented by Grafarend and Krumm (2006) and recently by Feltens (2009) and Ligas (2012a, b). The Lamé surfaces as a generalization of the triaxial ellipsoid have been presented by Nádeník (2005). Zagrebin (1973) has studied the gravity field of the Earth and the Moon and Chandrasekhar (1969) the triaxial (Jacobi) ellipsoid as a figure of equilibrium. Also, an azimuthal mapping of the triaxial ellipsoid has been presented by Grafarend and Krumm (2006). Other studies concerning triaxial ellipsoids are mentioned in Feltens (2009).

Anyone acquainted with geodetic theory understands that the problems which have been solved in triaxial ellipsoidal geometry are very few compared to the corresponding problems in biaxial ellipsoidal and spherical geometry. Using a triaxial ellipsoidal geometry, in the framework of boundary value problems, some of today's most challenging problems can be studied. Namely, the geodesic problem, the gravity field due to a homogeneous ellipsoid and the linear fixed altimetry-gravimetry problem can be addressed on an ellipsoid.

The geodesic problem entails determining the geodesic between two given points on an ellipsoid. Quoting from Karney (2013) (http://en.wikipedia.org/wiki/Geodesics_on_an_ellipsoid), " … geodesics play an important role in several areas:

- for measuring distances and areas in Geographic Information Systems,
- the definition of maritime boundaries,
- in the rules of the Federal Aviation Administration for area navigation,
- the method of measuring distances in the FAI Sporting Code.

Also, by the principle of least action, many problems in physics can be formulated as a variational problem similar to that for geodesics. For this reason, geodesics on simple surfaces such as biaxial or triaxial ellipsoids are frequently used -as test cases- for exploring new methods … ".

For the representation of the Earth's external gravitational field, spherical harmonics have been extensively used in geodesy. However, since an oblate spheroid is closer to the shape of the Earth, Holota (2005, 2011) and Claessens (2006), among others, have attempted to use oblate spheroidal harmonics and to solve the geodetic boundary value problems in an oblate spheroidal boundary. In other bodies of the solar system (planets, natural satellites, asteroids and comets), whose shape can be represented under certain circumstances by a triaxial ellipsoid, it is postulated that ellipsoidal harmonics would be even more suitable for the representation of their gravitational fields. For example, Garmier and Barriot (2001) and Hu (2012) applied the classical theory of ellipsoidal harmonics (Hobson, 1931; Dassios, 2012) in modeling the gravitational field of the comet Wirtanen, the Martian moon Phobos and the asteroid 433 Eros. Today, the gravity field modeling efforts within the geodetic community are focussing on numerical and computational aspects. On the other hand this is not an optimal situation.

The linear fixed altimetry-gravimetry problem is considered to be most suitable and of great importance in the future because the quasigeoid obtained through its solution is independent of any local height datum and can be regarded as a global height datum.

Therefore, it can be used for solving the problem of height datum unification, as outlined in the works of Sacerdote and Sansò (2003) and Zhang et al. (2009). One of the present day challenges of geodesy is the unification of all local and regional height datums into one consistent height datum. The practical problem underlying such premise is to realize a global reference surface supporting geometric (e.g. from GPS) and physical heights (e.g. from levelling, sea level observations) and to integrate the existing local height systems into one global system that is compatible with international standards and enables cost-saving implementation of modern (satellite, terrestrial, airborne and shipborne) geodetic techniques.

1.2. Thesis key objectives and problems to be addressed

The main objective of this research is to derive a detailed analysis on the above geometrical and physical geodetic problems, in the framework of boundary value problems, using an ellipsoidal geometry. Since the existing geodetic methods tackling these problems can be applied exclusively in a biaxial or a triaxial ellipsoid, this study is not limited on a biaxial ellipsoid (oblate spheroid), which is the well-known mathematical model being used in geodesy, but is also extended on a triaxial ellipsoid. The key issue in this analysis is the expression of the problems in a suitable ellipsoidal coordinate system.

The complicated structure implied by the ellipsoidal system, both in the analytical and the geometrical level is described in some detail in Chapter 2. Among the different variants of ellipsoidal coordinates, it is necessary to select those that (i) provide one-to-one correspondence between ellipsoidal and Cartesian coordinates and (ii) can be applied in the case of biaxial and triaxial ellipsoids. Consequently, these coordinates must be fully described with respect to their special geometric characteristics and the transformation between them and the Cartesian coordinates. Also, it is important to show that the classical transformation between ellipsoidal and Cartesian coordinates can be derived as a degenerate case. Similarly, the development of problems in triaxial ellipsoid should be shown that also holds in the case of biaxial ellipsoid.

To better understand the geometry of the triaxial ellipsoid, it is appropriate to study the geodesics; that is, those characteristic curves that have the greatest geodetic importance and have also many interesting properties. In order for the appropriate formulations to hold in both (biaxial and triaxial) ellipsoidal geometries, it is necessary to apply mathematical methodologies that are independent of the ellipsoidal surface. Therefore, along these lines, the geodesic problem and its solution on an ellipsoid are presented in Chapter 3.

Another typical application example of the geometry of triaxial ellipsoid is the determination of the gravity field due to a homogeneous ellipsoid. The corresponding general expressions involved in this determination are developed in Chapter 4. The derived gravity field can be regarded as a mathematical model that approximates the actual gravity field and can be suitably applied in the process of linearization of the geodetic boundary value problems.

The linear fixed altimetry-gravimetry problem is important to be investigated with respect to the existence and uniqueness of the solution. The relevant theoretical and practical aspects involved in such an approach are discussed in Chapters 5 and 6.

2. ELLIPSOIDAL COORDINATES

2.1. Introduction

In potential theory it is natural to use ellipsoidal coordinates, since they allow the separation of Laplace's equation and help formulate boundary conditions in a reasonably simple way. For these reasons, ellipsoidal coordinates have been used for formulating the theory of ellipsoidal harmonics and the solution of geodetic boundary value problems, e.g. by Hobson (1931), Garmier and Barriot (2001) and, more recently, by Lowes and Winch (2012). However, the commonly used variant of ellipsoidal coordinates has two disadvantages: (i) without imposing additional rules it generally determines eight points in space and (ii) it holds solely if one of the coordinate surfaces is a triaxial, not a biaxial ellipsoid.

To overcome these problems we use an alternative variant of ellipsoidal coordinates, originally introduced by Tabanov (1999). This leads to a one-to-one correspondence between ellipsoidal and Cartesian coordinates and an ellipsoidal system which deforms continuously to an oblate spheroidal and spherical system. Consequently, these coordinates may be useful in applications of geometrical geodesy, like ellipsoidal map projections and geodesics.

In this chapter, the alternative variant of the ellipsoidal coordinates, which is used in the following formulations, is presented in some detail along with its geometrical interpretation. In the direct transformation, the Cartesian coordinates are expressed using trigonometric functions. In the inverse transformation, the ellipsoidal coordinates are computed by solving a cubic equation with three real roots. Formulas relating the variants of the ellipsoidal coordinates are developed and the element of distance and the Laplace's equation are expressed in these two variants.

2.2. Ellipsoidal coordinate system

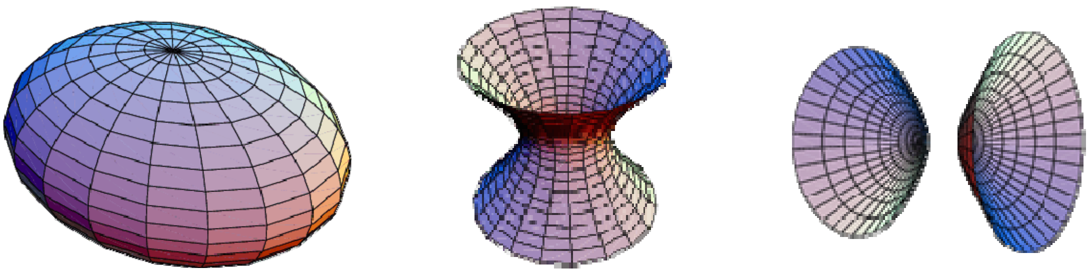
In order to introduce an ellipsoidal coordinate system, we consider a triaxial ellipsoid which, in Cartesian coordinates (*x*, *y*, *z*), is described by

$$\frac{x^2}{a_x^2}+\frac{y^2}{a_y^2}+\frac{z^2}{b^2}=1, \tag{2.1}$$

where $0 < b < a_y < a_x < +\infty$ are its three semiaxes. A family of confocal quadrics (second degree surfaces) to this ellipsoid is given as

$$\frac{x^2}{a_x^2+s}+\frac{y^2}{a_y^2+s}+\frac{z^2}{b^2+s}=1, \tag{2.2}$$

where s is a real number called the parameter of the family. For each value of s bigger than $-a_x^2$, Eq. (2.2) represents a quadric which is (i) a triaxial ellipsoid, when $-b^2 < s < +\infty$, (ii) a hyperboloid of one sheet, when $-a_y^2 < s < -b^2$, and (iii) a hyperboloid of two sheets, when $-a_x^2 < s < -a_y^2$ (see Fig. 2.1). Finally, when $s < -a_x^2$, Eq. (2.2) represents an imaginary quadric (Kellogg, 1953).



(i) (ii) (iii)

Figure 2.1. Coordinate surfaces: (i) triaxial ellipsoid, (ii) hyperboloid of one sheet and (iii) hyperboloid of two sheets.

In Dassios (2012) is proved that, for every point (*x*, *y*, *z*) in space with $xyz \neq 0$ (this excludes the Cartesian planes *x* = 0, *y* = 0 and *z* = 0) Eq. (2.2), which is a cubic equation in *s*, has three unequal real roots s_1, s_2, s_3 such that

$$-a_x^2 < s_3 < -a_y^2 < s_2 < -b^2 < s_1 < +\infty . \tag{2.3}$$

Thus, through each point (*x*, *y*, *z*) in space with $xyz \neq 0$ passes exactly one triaxial ellipsoid (s_1 = constant), one hyperboloid of one sheet (s_2 = constant) and one hyperboloid of two sheets (s_3 = constant). These variables (s_1, s_2, s_3) are known as ellipsoidal coordinates and have dimensions of length squared. Also, the ellipsoidal coordinate system (s_1, s_2, s_3) is a triply orthogonal system and the principal sections (see below) of the coordinate surfaces share three pairs of foci ($\pm\ E_x$, 0, 0), ($\pm\ E_e$, 0, 0), (0, $\pm\ E_y$, 0), where $E_x = \sqrt{a_x^2 - b^2}$, $E_y = \sqrt{a_y^2 - b^2}$ and $E_e = \sqrt{a_x^2 - a_y^2}$ are the focal lengths (linear eccentricities), i.e. the distances between the coordinate origin *O* and the focal points F_1 (or $F_1^{'}$), F_2 (or $F_2^{'}$) and F_3 (or $F_3^{'}$), respectively (see Fig. 2.2). The linear eccentricity E_e is related to E_x and E_y by $E_e^2 = E_x^2 - E_y^2$. Hence, the ellipsoidal coordinate system is entirely characterized by two parameters e.g. E_x and E_y (two-parametric system). Amongst these parameters it holds that $E_y < E_x$ and $E_e < E_x$.

Figure 2.2 displays the Cartesian planes *x* = 0, *y* = 0 and *z* = 0. These planes intersect any one of the confocal quadrics either in an ellipse or in a hyperbola which are called principal ellipses and principal hyperbolas of the corresponding quadric. When s_1 = 0, Eq. (2.2) represents the fundamental (or reference) ellipsoid (2.1) which has three principal ellipses mutually perpendicular. From Eq. (2.2) a family of confocal principal hyperbolas is obtained

$$\frac{y^2}{a_y^2 + s_2} + \frac{z^2}{b^2 + s_2} = 1,\ x = 0, \tag{2.4}$$

with foci at (0, $\pm E_y$, 0). The linear equation

$$z = \pm y \frac{\sqrt{-b^2 - s_2}}{\sqrt{a_y^2 + s_2}}, \tag{2.5}$$

represents the two asymptotes of the family of hyperbolas. Also, from Eq. (2.2) a family of confocal principal hyperbolas is obtained

$$\frac{x^2}{a_x^2 + s_3} + \frac{y^2}{a_y^2 + s_3} = 1,\ z = 0, \tag{2.6}$$

with foci at ($\pm E_e$, 0, 0). The linear equation

$$y = \pm x \frac{\sqrt{-a_y^2 - s_3}}{\sqrt{a_x^2 + s_3}}, \tag{2.7}$$

represents the two asymptotes of the family of hyperbolas. Note that, the confocal hyperboloids of two sheets do not intersect the plane $x = 0$. Finally, when the ellipsoidal coordinates (s_1, s_2, s_3) reach their limiting values, we get degenerate quadrics corresponding to parts of the planes $x = 0$, $y = 0$ and $z = 0$ (see Fig. 2.2). In this study the dominant part is played by two special curves on which two coordinates take equal values:

When $s_1 = s_2 = -b^2$, from Eq. (2.2) we obtain the focal ellipse

$$\frac{x^2}{E_x^2} + \frac{y^2}{E_y^2} = 1,\ z = 0. \tag{2.8}$$

The foci of the focal ellipse are ($\pm E_e$, 0, 0) and its semiaxes are E_x and E_y.

When $s_2 = s_3 = -a_y^2$, from Eq. (2.2) we obtain the focal hyperbola

$$\frac{x^2}{E_e^2} - \frac{z^2}{E_y^2} = 1,\ y = 0. \qquad (2.9)$$

The foci of the focal hyperbola are ($\pm\ E_x$, 0, 0) and its semiaxes are E_e and E_y.

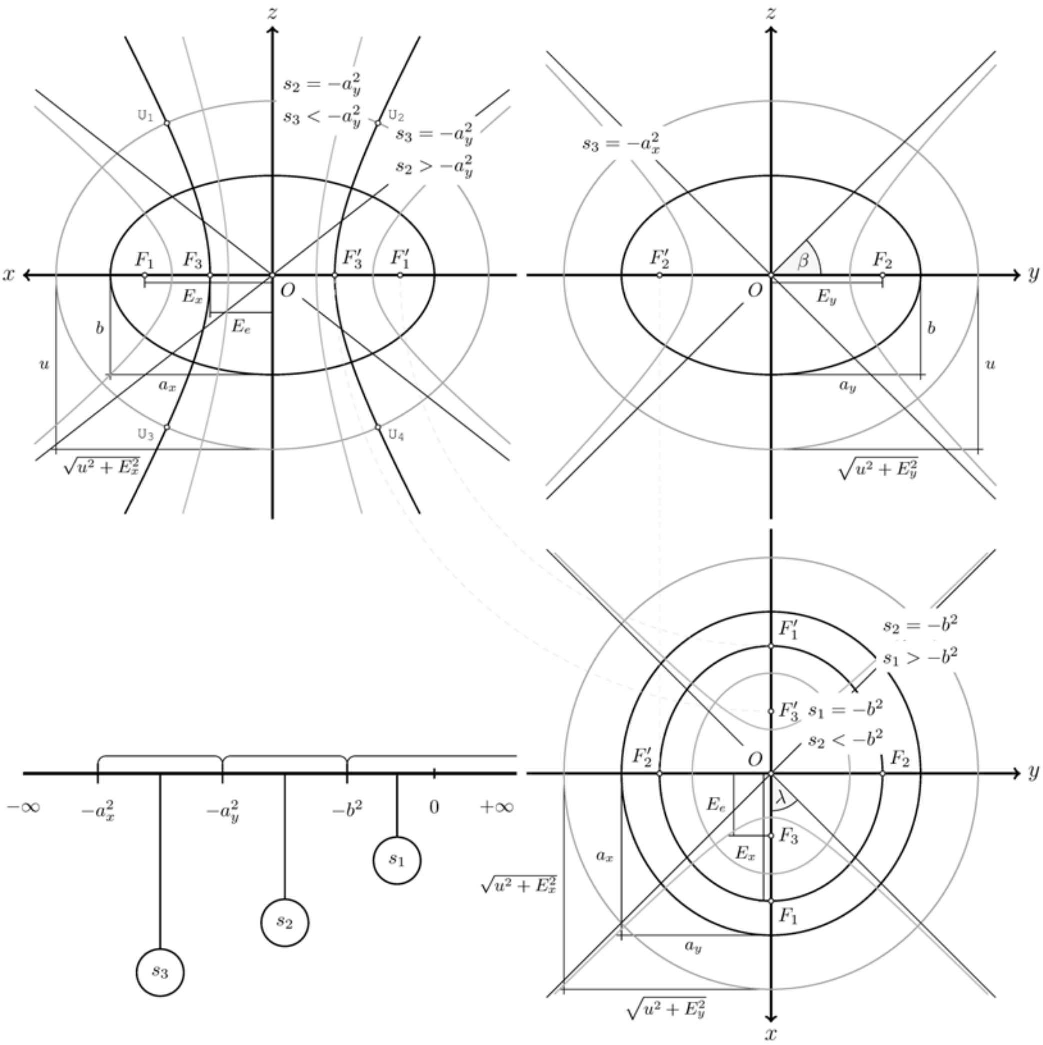


Figure 2.2. Ellipsoidal coordinates and Cartesian planes x = 0 (above right), y = 0 (above left) and z = 0 (below right).

2.3. From ellipsoidal to Cartesian coordinates

Formulas relating ellipsoidal (s_1, s_2, s_3) and Cartesian coordinates (x, y, z) are obtained by Eq. (2.2), as expressed in Kellogg (1953):

$$x^2 = \frac{(a_x^2 + s_1)(a_x^2 + s_2)(a_x^2 + s_3)}{(a_x^2 - a_y^2)(a_x^2 - b^2)}, \tag{2.10}$$

$$y^2 = \frac{(a_y^2 + s_1)(a_y^2 + s_2)(a_y^2 + s_3)}{(a_y^2 - a_x^2)(a_y^2 - b^2)}, \tag{2.11}$$

$$z^2 = \frac{(b^2 + s_1)(b^2 + s_2)(b^2 + s_3)}{(b^2 - a_x^2)(b^2 - a_y^2)}, \tag{2.12}$$

where $-a_x^2 \leq s_3 \leq -a_y^2 \leq s_2 \leq -b^2 \leq s_1 < +\infty$. According to these equations there are, in general, eight points ($\pm$ *x*, $\pm$ *y*, $\pm$ *z*) in space, symmetrically located in octants, corresponding to the same (s_1, s_2, s_3) and thus the transformation is not one-to-one. In order to have a one-to-one correspondence between ellipsoidal and Cartesian coordinates one usually has to introduce new ellipsoidal coordinates, expressing s_1, s_2, s_3 and hence *x*, *y*, *z* in terms of suitable functions of three new coordinates. For example, Byerly (1893) express the ellipsoidal coordinates in terms of elliptic functions, while Wang and Guo (1989) express them in terms of theta functions as well. Elliptic and theta functions are special kind and complicated functions to handle and, for that reason, we have avoided to represent the Cartesian coordinates in terms of such functions. Clearly, the change of variables does not affect the system, since each of the new coordinates is a function of the old ones.

2.3.1. Common variant

The theory of ellipsoidal harmonics, e.g. Hobson (1931), mostly uses the ellipsoidal coordinates (ρ, μ, ν) given by the relations

$$s_1 = \rho^2 - a_x^2, \tag{2.13}$$

$$s_2 = \mu^2 - a_x^2, \tag{2.14}$$

$$s_3 = \nu^2 - a_x^2 . \tag{2.15}$$

Substituting Eqs. (2.13)-(2.15) into Eqs. (2.10)-(2.12), yields

$$x = \frac{1}{E_x E_e} \rho\mu\nu , \tag{2.16}$$

$$y = \frac{1}{E_y E_e} \sqrt{\rho^2 - \mathrm{E}_e^2} \sqrt{\mu^2 - \mathrm{E}_e^2} \sqrt{\mathrm{E}_e^2 - \nu^2} , \tag{2.17}$$

$$z = \frac{1}{E_x E_y} \sqrt{\rho^2 - \mathrm{E}_x^2} \sqrt{\mathrm{E}_x^2 - \mu^2} \sqrt{\mathrm{E}_x^2 - \nu^2} , \tag{2.18}$$

where $0 \le \nu \le E_e \le \mu \le E_x \le \rho < +\infty$. These coordinates have dimensions of length. In this case, to ensure that a point (*x*, *y*, *z*) corresponds to the point (ρ, μ, ν) we have to impose additional rules. Specifically, ν is to be taken with the positive sign when *x* is positive and vice versa; $\sqrt{\mathrm{E}_e^2 - \nu^2}$ is to be taken with the positive sign when *y* is positive and vice versa; and $\sqrt{\mathrm{E}_x^2 - \mu^2}$ is to be taken with the positive sign when *z* is positive and vice versa. The quantities ρ, μ, $\sqrt{\rho^2 - \mathrm{E}_e^2}$, $\sqrt{\mu^2 - \mathrm{E}_e^2}$, $\sqrt{\rho^2 - \mathrm{E}_x^2}$ and $\sqrt{\mathrm{E}_x^2 - \nu^2}$ are to be taken always with the positive sign. Thus, it follows that the ellipsoidal coordinates (ρ, μ, ν) have the disadvantage that, in order to fully fix a point in space, we need to know not merely the values of its coordinates ρ, μ and ν, but the signs of ν, $\sqrt{E_e^2 - \nu^2}$ and $\sqrt{E_x^2 - \mu^2}$ as well (Byerly, 1893; Hobson, 1931).

Substituting Eqs. (2.13)-(2.15) into Eq. (2.2), the coordinate surfaces are

i) triaxial ellipsoids (ρ = constant)

$$\frac{x^2}{\rho^2} + \frac{y^2}{\rho^2 - E_e^2} + \frac{z^2}{\rho^2 - E_x^2} = 1 , \qquad E_x < \rho < +\infty , \tag{2.19}$$

ii) hyperboloids of one sheet (μ = constant)

$$\frac{x^2}{\mu^2}+\frac{y^2}{\mu^2-E_e^2}-\frac{z^2}{E_x^2-\mu^2}=1, \qquad E_e<\mu<E_x, \tag{2.20}$$

ii) hyperboloids of two sheets (ν = constant)

$$\frac{x^2}{\nu^2}-\frac{y^2}{E_e^2-\nu^2}-\frac{z^2}{E_x^2-\nu^2}=1, \qquad 0<\nu<E_e. \tag{2.21}$$

For $\rho = a_x$, from Eq. (2.19) we obtain the reference ellipsoid (2.1). For $\rho = \mu = E_x$, from Eqs. (2.19) and (2.20) we obtain the focal ellipse (2.8) and for $\mu = \nu = E_e$, from Eqs. (2.20) and (2.21) correspondingly the focal hyperbola (2.9).

The main characteristic of the previous coordinate systems is that it can be used only if the first coordinate surface (s_1 = constant or ρ = constant) is a triaxial ellipsoid. In the following section, we present an alternative variant of ellipsoidal coordinates which will overcome these problems.

2.3.2. Alternative variant

In order to have a one-to-one correspondence between ellipsoidal and Cartesian coordinates, we introduce ellipsoidal coordinates (*u*, β, λ) by the relations

$$s_1 = u^2 - b^2, \tag{2.22}$$

$$s_2 = -a_y^2 \sin^2\beta - b^2\cos^2\beta, \tag{2.23}$$

$$s_3 = -a_x^2 \sin^2\lambda - a_y^2\cos^2\lambda. \tag{2.24}$$

In a Cartesian coordinate system, a point *P* has the coordinates (*x*, *y*, *z*). We pass through *P* a triaxial ellipsoid whose centre is the origin *O*, its polar axis coincides with the z-axis, its major equatorial axis coincides with the x-axis, its minor equatorial axis coincides with the y-axis and two linear eccentricities have the constant values E_x and E_y. The coordinate u is the polar semiaxis of this ellipsoid, β is the ellipsoidal latitude and λ is the ellipsoidal longitude. Substituting Eq. (2.23) into Eq. (2.5), we obtain

$$\beta = \tan^{-1}\left(\frac{z}{y}\right), \tag{2.25}$$

which leads us to the interpretation that the ellipsoidal latitude β represents the inclination of the asymptotes of the family of confocal principal hyperbolas (2.4) on the plane $x = 0$. Similarly, substituting Eq. (2.24) into Eq. (2.7), we obtain

$$\lambda = \tan^{-1}\left(\frac{y}{x}\right), \tag{2.26}$$

and thus the ellipsoidal longitude λ represents the inclination of the asymptotes of the family of confocal principal hyperbolas (2.6) on the plane $z = 0$, (see Fig. 2.2).

Substituting Eqs. (2.22)-(2.24) into Eqs. (2.10)-(2.12), we derive the equations introduced by Tabanov (1999) and presented also by Dassios (2012)

$$x = \sqrt{u^2 + E_x^2}\left(\cos^2\beta + \frac{E_e^2}{E_x^2}\sin^2\beta\right)^{1/2}\cos\lambda\,, \tag{2.27}$$

$$y = \sqrt{u^2 + E_y^2}\cos\beta\sin\lambda\,, \tag{2.28}$$

$$z = u\sin\beta\left(1 - \frac{E_e^2}{E_x^2}\cos^2\lambda\right)^{1/2}, \tag{2.29}$$

where $u \geq 0$, $-\pi/2 \leq \beta \leq +\pi/2$, $-\pi < \lambda \leq +\pi$.

Now, substituting Eqs. (2.22)-(2.24) into Eq. (2.2), the coordinate surfaces are (i) triaxial ellipsoids (u = constant), (ii) hyperboloids of one sheet (β = constant) and (iii) hyperboloids of two sheets (λ = constant) with their respective equations given as

$$\frac{x^2}{u^2+E_x^2}+\frac{y^2}{u^2+E_y^2}+\frac{z^2}{u^2}=1, \qquad u>0, \tag{2.30}$$

$$\frac{x^2}{E_e^2+E_y^2\cos^2\beta}+\frac{y^2}{E_y^2\cos^2\beta}-\frac{z^2}{E_y^2\sin^2\beta}=1, \quad -\pi/2<\beta<+\pi/2,\ \beta\neq 0, \tag{2.31}$$

$$\frac{x^2}{E_e^2\cos^2\lambda}-\frac{y^2}{E_e^2\sin^2\lambda}-\frac{z^2}{E_x^2-E_e^2\cos^2\lambda}=1, \quad -\pi<\lambda<+\pi,\ \lambda\neq 0,\ \lambda\neq\pm\pi/2. \tag{2.32}$$

For $u = b$, from Eq. (2.30) we obtain the reference ellipsoid (2.1). For $u = 0$ and β = 0, from Eqs. (2.30) and (2.31) we obtain the focal ellipse (2.8) and for β = ±π/2 and λ = 0 (or λ = ±π), from Eqs. (2.31) and (2.32) the focal hyperbola (2.9).

When the values β = ±π/2 and λ = 0 (or λ = ±π) are substituted in Eqs. (2.27)-(2.29) we get the Cartesian coordinates

$$x=\pm\sqrt{u^2+E_x^2}\,\frac{E_e}{E_x}, \qquad y=0, \qquad z=\pm u\frac{E_y}{E_x}. \tag{2.33}$$

These coordinates correspond to umbilical points U_1, U_2, U_3, U_4 on the ellipsoid u = constant (see Fig. 2.2).

As pointed out in Dassios (2012), an important characteristic of this system is that it specifies uniquely the points in the different Cartesian octants, without having to impose additional rules, as it is the case with the (s_1, s_2, s_3) or (ρ, μ, ν) systems. In addition, when $a_x = a_y \equiv a$, i.e. $E_x = E_y \equiv E$ and $E_e = 0$, Eqs. (2.27)-(2.29) reduce to the well-known oblate spheroidal system (Heiskanen and Moritz, 1967):

$$x = \left(u^2 + E\right)^{1/2} \cos\beta \cos\lambda\,, \qquad y = \left(u^2 + E\right)^{1/2} \cos\beta \sin\lambda\,, \qquad z = u \sin\beta\,, \tag{2.34}$$

where $u \geq 0$, $-\pi/2 \leq \beta \leq +\pi/2$, $-\pi < \lambda \leq +\pi$. In this case, the triaxial ellipsoids become oblate spheroids (u = constant), the hyperboloids of one sheet become hyperboloids of revolution (β = constant) and the hyperboloids of two sheets become meridian planes (λ = constant). The focal ellipse becomes a focal circle whose radius is the linear eccentricity E and the focal hyperbola becomes the z-axis. Also, it is well-known that the oblate spheroid has two umbilical points $(x, y, z) = (0, 0, \pm u)$ which are its poles. Finally, when $E = 0$, Eqs. (2.34) degenerates to the spherical system $r \equiv u$, β (latitude) and λ (longitude).

Summing up, the fixed point (x, y, z) can be represented with respect to a continuously changing coordinate system, which gradually approaches first the oblate spheroidal system and then the spherical one. The importance of this procedure is that we avoid any degeneracy of the variables, as happens with the ellipsoidal coordinates (s_1, s_2, s_3) or (ρ, μ, ν). Conversely, the intervals of variation of the coordinates (u, β, λ) remain invariants as the system transforms first to the oblate spheroidal and then to the spherical one (Dassios, 2012).

At this point, we can show the connection between the ellipsoidal coordinates (ρ, μ, ν) and (u, β, λ). Hence, by comparing Eqs. (2.13)-(2.15) with Eqs. (2.22)-(2.24), we are lead to

$$\rho^2 = u^2 + E_x^2\,, \qquad \mu^2 = E_e^2 \sin^2\beta + E_x^2 \cos^2\beta\,, \qquad \nu^2 = E_e^2 \cos^2\lambda\,. \tag{2.35}$$

2.4. From Cartesian to ellipsoidal coordinates

The next obvious step is to compute the ellipsoidal (s_1, s_2, s_3) from the Cartesian coordinates (x, y, z). Substituting the known Cartesian coordinates (x, y, z) in Eq. (2.2), we obtain a cubic equation in s, from which we can evaluate the three real roots s_1, s_2 and s_3.

Equation (2.2) can be written equivalently as

$$s^3 + c_2 s^2 + c_1 s + c_0 = 0\,, \tag{2.36}$$

where

$$c_2 = a_x^2 + a_y^2 + b^2 - x^2 - y^2 - z^2\,, \tag{2.37}$$

$$c_1 = a_x^2 a_y^2 + a_x^2 b^2 + a_y^2 b^2 - \left(a_y^2 + b^2\right)x^2 - \left(a_x^2 + b^2\right)y^2 - \left(a_x^2 + a_y^2\right)z^2\,, \tag{2.38}$$

$$c_0 = a_x^2 a_y^2 b^2 - a_y^2 b^2 x^2 - a_x^2 b^2 y^2 - a_x^2 a_y^2 z^2\,. \tag{2.39}$$

This equation has three real roots s_1, s_2 and s_3 which are distributed according to Eq. (2.3). When a cubic equation has three real roots its solutions can be expressed as (Garmier and Barriot, 2001)

$$s_1 = 2\sqrt{p}\cos\left(\frac{\omega}{3}\right) - \frac{c_2}{3}\,, \tag{2.40}$$

$$s_2 = 2\sqrt{p}\cos\left(\frac{\omega}{3} - \frac{2\pi}{3}\right) - \frac{c_2}{3}\,, \tag{2.41}$$

$$s_3 = 2\sqrt{p}\cos\left(\frac{\omega}{3} - \frac{4\pi}{3}\right) - \frac{c_2}{3}\,, \tag{2.42}$$

where

$$p = \frac{c_2^2 - 3c_1}{9}\,, \tag{2.43}$$

$$q = \frac{9c_1 c_2 - 27c_0 - 2c_2^3}{54}\,, \tag{2.44}$$

and

$$\omega = \cos^{-1}\left(\frac{q}{\sqrt{p^3}}\right). \tag{2.45}$$

It should be mentioned that Garmier and Barriot (2001) have also applied a numerical algorithm (secant method) for the computation of the roots of Eq. (2.2). In this work, we have included only the explicit solutions (2.40)-(2.42).

2.4.1. Common variant

Inverting Eqs. (2.13)-(2.15), results in

$$\rho = \sqrt{a_x^2 + s_1}\ , \tag{2.46}$$

$$\mu = \sqrt{a_x^2 + s_2}\ , \tag{2.47}$$

$$\nu = \sqrt{a_x^2 + s_3}\ , \tag{2.48}$$

where the same conventions with regard to the proper signs hold, according to the explanations given in Section 2.3.1.

2.4.2. Alternative variant

Inverting Eqs. (2.22)-(2.24), results in

$$u = \sqrt{b^2 + s_1}\ , \tag{2.49}$$

$$\beta = \tan^{-1}\sqrt{\frac{-b^2 - s_2}{a_y^2 + s_2}}\ , \tag{2.50}$$

$$\lambda = \tan^{-1}\sqrt{\frac{-a_y^2 - s_3}{a_x^2 + s_3}}\,, \qquad (2.51)$$

where the conventions with regard to the proper quadrant for the coordinates β, λ need to be applied. In the spheroidal case, the corresponding expressions have been derived by Heiskanen and Moritz (1967) and Featherstone Claessens (2008).

2.5. Laplace's equation in ellipsoidal coordinates

The general form of the element of distance in arbitrary orthogonal coordinates η_1, η_2, η_3 is

$$ds^2 = h_1^2 d\eta_1^2 + h_2^2 d\eta_2^2 + h_3^2 d\eta_3^2 . \qquad (2.52)$$

It can be shown that Laplace's operator Δ in these coordinates and for a function *V* is

$$\Delta V = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial\eta_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial\eta_1}\right) + \frac{\partial}{\partial\eta_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial\eta_2}\right) + \frac{\partial}{\partial\eta_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial\eta_3}\right)\right]. \qquad (2.53)$$

The Laplace equation, $\Delta V = 0$, is the main representative of second-order partial differential equations of elliptic type, for which fundamental methods of solution of boundary value problems for elliptic equations have been and are being developed.

2.5.1. Common variant

In ellipsoidal coordinates (ρ, μ, ν), the element of distance *ds* is written as (Hobson, 1931)

$$ds^2 = h_\rho^2 d\rho^2 + h_\mu^2 d\mu^2 + h_\nu^2 d\nu^2 , \qquad (2.54)$$

where the scale factors (metric coefficients) h_ρ^2, h_μ^2, h_ν^2 are given by

$$h_\rho^2 = \frac{(\rho^2-\mu^2)(\rho^2-\nu^2)}{(\rho^2-E_x^2)(\rho^2-E_e^2)}, \tag{2.55}$$

$$h_\mu^2 = \frac{(\mu^2-\rho^2)(\mu^2-\nu^2)}{(\mu^2-E_x^2)(\mu^2-E_e^2)}, \tag{2.56}$$

$$h_\nu^2 = \frac{(\nu^2-\rho^2)(\nu^2-\mu^2)}{(\nu^2-E_x^2)(\nu^2-E_e^2)}. \tag{2.57}$$

Note that in Eq. (2.52) there are no terms with $d\rho d\mu$, $d\rho d\nu$ and $d\mu d\nu$ because the ellipsoidal coordinates are orthogonal: the ellipsoids ρ = constant, the hyperboloids of one sheet μ = constant and the hyperboloids of two sheets ν = constant intersect each other orthogonally.

In these coordinates, Laplace's equation, $\Delta V = 0$, can be written as (Hobson, 1931)

$$\begin{aligned}
&(\mu^2-\nu^2)\sqrt{\rho^2-E_x^2}\sqrt{\rho^2-E_e^2}\,\frac{\partial}{\partial\rho}\left(\sqrt{\rho^2-E_x^2}\sqrt{\rho^2-E_e^2}\,\frac{\partial V}{\partial\rho}\right)+\\
&(\rho^2-\nu^2)\sqrt{E_x^2-\mu^2}\sqrt{\mu^2-E_e^2}\,\frac{\partial}{\partial\mu}\left(\sqrt{E_x^2-\mu^2}\sqrt{\mu^2-E_e^2}\,\frac{\partial V}{\partial\mu}\right)+\\
&(\rho^2-\mu^2)\sqrt{E_x^2-\nu^2}\sqrt{E_e^2-\nu^2}\,\frac{\partial}{\partial\nu}\left(\sqrt{E_x^2-\nu^2}\sqrt{E_e^2-\nu^2}\,\frac{\partial V}{\partial\nu}\right)=0.
\end{aligned} \tag{2.58}$$

2.5.2. Alternative variant

Using Eqs. (2.35) we can transform Eq. (2.52) and Eq. (2.56) to the form in which the ellipsoidal coordinates (u, β, λ) are the independent variables. Hence, in ellipsoidal coordinates (u, β, λ), the element of distance ds is

$$ds^2 = h_u^2 du^2 + h_\beta^2 d\beta^2 + h_\lambda^2 d\lambda^2, \tag{2.59}$$

where

$$h_u^2 = \frac{(u^2 + E_y^2 \sin^2 \beta)(u^2 + E_y^2 + E_e^2 \sin^2 \lambda)}{(u^2 + E_x^2)(u^2 + E_y^2)}, \tag{2.60}$$

$$h_\beta^2 = \frac{(u^2 + E_y^2 \sin^2 \beta)(E_y^2 \cos^2 \beta + E_e^2 \sin^2 \lambda)}{E_x^2 - E_y^2 \sin^2 \beta}, \tag{2.61}$$

$$h_\lambda^2 = \frac{(u^2 + E_y^2 + E_e^2 \sin^2 \lambda)(E_y^2 \cos^2 \beta + E_e^2 \sin^2 \lambda)}{E_x^2 - E_e^2 \cos^2 \lambda}. \tag{2.62}$$

In these coordinates, Laplace's equation, $\Delta V = 0$, becomes

$$\begin{aligned}
&(E_y^2 \cos^2 \beta + E_e^2 \sin^2 \lambda)\sqrt{u^2 + E_x^2}\sqrt{u^2 + E_y^2}\ \frac{\partial}{\partial u}\left(\sqrt{u^2 + E_x^2}\sqrt{u^2 + E_y^2}\ \frac{\partial V}{\partial u}\right) + \\
&(u^2 + E_y^2 + E_e^2 \sin^2 \lambda)\sqrt{E_e^2 + E_y^2 \cos^2 \beta}\ \frac{\partial}{\partial \beta}\left(\sqrt{E_e^2 + E_y^2 \cos^2 \beta}\ \frac{\partial V}{\partial \beta}\right) + \\
&(u^2 + E_y^2 \sin^2 \beta)\sqrt{E_x^2 - E_e^2 \cos^2 \lambda}\ \frac{\partial}{\partial \lambda}\left(\sqrt{E_x^2 - E_e^2 \cos^2 \lambda}\ \frac{\partial V}{\partial \lambda}\right) = 0.
\end{aligned} \tag{2.63}$$

2.5.3. Spheroidal expressions

For $a_x = a_y \equiv a$, i.e. $E_x = E_y \equiv E$ and $E_e = 0$, Eq. (2.57) and Eq. (2.61) reduce to the same spheroidal expressions that have been derived by Heiskanen and Moritz (1967)

$$ds^2 = h_u^2 du^2 + h_\beta^2 d\beta^2 + h_\lambda^2 d\lambda^2, \tag{2.64}$$

where

$$h_u^2 = \frac{u^2 + E^2 \sin^2 \beta}{u^2 + E^2}, \tag{2.65}$$

$$h_\beta^2 = u^2 + E^2 \sin^2 \beta, \tag{2.66}$$

$$h_\lambda^2 = (u^2 + E^2)\cos^2\beta\,, \tag{2.67}$$

and

$$(u^2 + E^2)\frac{\partial^2 V}{\partial u^2} + 2u\frac{\partial V}{\partial u} + \frac{\partial^2 V}{\partial \beta^2} - \tan\beta\frac{\partial V}{\partial \beta} + \frac{u^2 + E^2\sin^2\beta}{(u^2 + E^2)\cos^2\beta}\frac{\partial^2 V}{\partial \lambda^2} = 0\,. \tag{2.68}$$

In the limiting case, $E \to 0$, these equations reduce to the well-known spherical expressions (Heiskanen and Moritz, 1967).

3. THE GEODESIC BOUNDARY VALUE PROBLEM AND ITS SOLUTION ON AN ELLIPSOID

3.1. Introduction

The shortest path between two points on a curved surface is along a geodesic, i.e. the analogue of a straight line on a plane. The geodesic problem entails determining the geodesic between two given points $P_0(\beta_0, \lambda_0)$ and $P_1(\beta_1, \lambda_1)$ on an ellipsoid (van Brunt, 2004).

For a triaxial ellipsoid, the explicit description of geodesics was given by Jacobi (1839). Using the ellipsoidal coordinates, Jacobi showed that the geodesics can be reduced to integrals. These integrals include a constant that also appears in Liouville's equation, the Liouville constant (see Section 3.4). A recent application of this technique with examples concerning the behavior of long geodesics was presented by Karney (GeographicLib: http://geographiclib.sourceforge.net/html/triaxial.html). As an alternative approach, Shebl and Farag (2007) used a conformal mapping between a triaxial ellipsoid and a sphere in order to approximate a geodesic on a triaxial ellipsoid.

For a biaxial ellipsoid, a historical summary of solution methods for the geodesic problem can be found in Deakin and Hunter (2010) and Karney (2013). Among these methods, Vincenty's iterative formulas based on series expansions are widely used (Vincenty, 1975). Recently, Karney (2013) gave improved series expansions for solving the problem. However, Sjöberg (2012) and Sjöberg and Shirazian (2012) solved the problem by decomposing the solutions into those on a sphere and the corrections for the ellipsoid. The spherical solutions are given in closed form, while the corrections for an ellipsoid are expressed with elliptic integrals, suitable for numerical integration. A similar approach is followed by Saito (1970). Also, part of the problem is the determination of the so-called Clairaut constant (i.e., the cosine of the maximum latitude of the geodesic), which was treated by Sjöberg (2007). Today, considering modern computational capabilities, we prefer solution methods that use numerical integration rather than a series expansion approach, because a truncated series solution makes a mathematical approximation. By comparison, numerical integration suffers only from computational errors, which can be addressed with improved computational systems and require no change in the theoretical background.

Solving the geodesics as a boundary value problem is a well-studied topic in differential geometry, but only as far as the properties of the geodesics are concerned. On the other hand, there are several studies (e.g Maekawa 1996, Chen and Chen 2011) which present computational schemes for general, free-form parametric or regular surfaces, but with no focus on the ellipsoid.

In this chapter, we present a method which solves the geodesic problem on an ellipsoid. In this method, the geodesic (i.e. ellipsoidal coordinates and derivatives) is obtained and then the angle between a line of constant longitude and the geodesic at any point along the geodesic is computed, together with the geodesic distance between the two points. Also, the Liouville constant is precisely determined, including an accuracy check. Our solution includes numerical integrations and so its accuracy is limited by the computational system used. The generalized algorithm can be applied for triaxial ellipsoids, biaxial ellipsoids, and spheres; it is particularly interesting to show how the general expressions are reduced in the biaxial case. Between two points on a biaxial ellipsoid, we can also determine the Clairaut constant without using the Clairaut equation. In addition, we do not use conformal mapping with an auxiliary sphere, as do Saito (1970), Vincenty (1975) and Karney (2013).

Finally, it would be interesting to generalize the biaxial ellipsoid solution from the (different) approaches of Sjöberg (2012), Sjöberg and Shirazian (2012) and Karney (2013) to the geodesic problem on a triaxial ellipsoid.

3.2. Geodesic boundary value problem

By setting $u = b$ in Eqs. (2.27)-(2.29), the triaxial ellipsoid which is described by Eq. (2.1) may be parameterized as (Jacobi, 1839; Tabanov, 1999; Dassios, 2012)

$$x = a_x \left(\cos^2 \beta + \frac{E_e^2}{E_x^2} \sin^2 \beta \right)^{1/2} \cos \lambda, \tag{3.1a}$$

$$y = a_y \cos \beta \sin \lambda, \tag{3.1b}$$

$$z = b \sin \beta \left(1 - \frac{E_e^2}{E_x^2} \cos^2 \lambda \right)^{1/2}, \tag{3.1c}$$

where $-\pi/2 \leq \beta \leq +\pi/2$ and $-\pi < \lambda \leq +\pi$. These parameters can be interpreted as ellipsoidal latitude and ellipsoidal longitude, respectively (see Fig. 3.1). More details about the ellipsoidal coordinates are included in Dassios (2012). In this parameterization, the first fundamental coefficients $\overline{E}$, $\overline{F}$ and $\overline{G}$ can be expressed as

$$\overline{E} = \mathrm{B}\left(E_y^2 \cos^2 \beta + E_e^2 \sin^2 \lambda\right), \tag{3.2a}$$

$$\overline{F} = 0, \tag{3.2b}$$

$$\overline{G} = \Lambda\left(E_y^2 \cos^2 \beta + E_e^2 \sin^2 \lambda\right), \tag{3.2c}$$

where

$$\mathrm{B} = \frac{a_y^2 \sin^2 \beta + b^2 \cos^2 \beta}{E_x^2 - E_y^2 \sin^2 \beta}, \tag{3.3a}$$

$$\Lambda = \frac{a_x^2 \sin^2 \lambda + a_y^2 \cos^2 \lambda}{E_x^2 - E_e^2 \cos^2 \lambda}. \tag{3.3b}$$

In Eq. (3.2b), $\overline{F} = 0$ indicates that the β-curves and λ-curves are orthogonal. Also, $\mathrm{B} \neq 0$, $\Lambda \neq 0$ for all points, and $\overline{E} = \overline{G} = 0$ at the umbilical points U_1, U_2, U_3, U_4, i.e., when $\lambda = 0$ or $+\pi$ and $\beta = \pm\pi/2$. From Eqs. (3.2a), (3.2c) and (3.3) we obtain the partial derivatives which are presented in Appendix A.1.

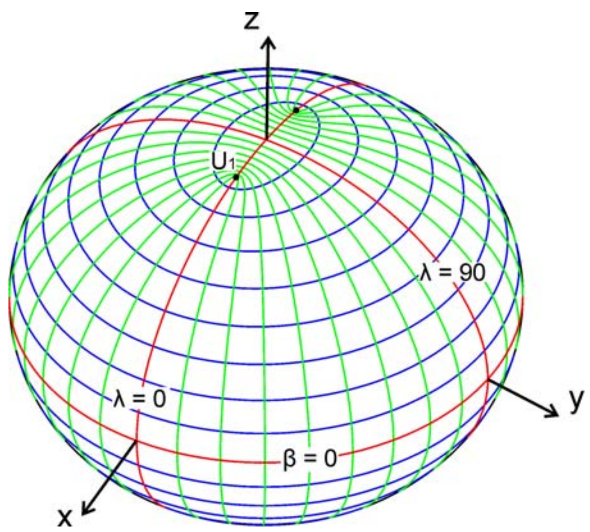


Figure 3.1. The ellipsoidal coordinates on a triaxial ellipsoid.

For an orthogonal parameterization, the line element *ds* on a triaxial ellipsoid is given by (Deakin and Hunter, 2008)

$$ds^2 = \overline{E}d\beta^2 + \overline{G}d\lambda^2 . \tag{3.4}$$

The geodesic curvature κ along the respective parametric lines is given by (Struik, 1961)

$$(\kappa)_{\beta=\text{const.}} = +\frac{1}{2}\frac{\overline{G}_\beta}{\overline{G}\sqrt{\overline{E}}} , \tag{3.5a}$$

$$(\kappa)_{\lambda=\text{const.}} = -\frac{1}{2}\frac{\overline{E}_\lambda}{\overline{E}\sqrt{\overline{G}}} . \tag{3.5b}$$

Thus, according to Eqs. (3.5), (A3b) and (A3c) only the principal ellipses are geodesics on a triaxial ellipsoid. For this reason, in order to describe the geodesics on a triaxial ellipsoid, we consider two cases: (3.2.1) $\lambda_0 \neq \lambda_1$ with the independent variable being the ellipsoidal longitude, and (3.2.2) $\lambda_0 = \lambda_1$ with the independent variable being the ellipsoidal latitude.

3.2.1. Case with $\lambda_0 \neq \lambda_1$

We consider a curve on a triaxial ellipsoid to be described by β = β(λ), i.e., with the ellipsoidal latitude a function of ellipsoidal longitude. Using Eq. (3.4) the line element is given by

$$ds = \sqrt{\overline{E}(\beta')^2 + \overline{G}}\, d\lambda , \tag{3.6a}$$

where

$$\beta' = \frac{d\beta}{d\lambda} . \tag{3.6b}$$

Hence, the length s from $\lambda = \lambda_0$ to $\lambda = \lambda_1$ ($\lambda_0 < \lambda_1$) is obtained by

$$s = \int_{\lambda_0}^{\lambda_1} f(\lambda, \beta, \beta') d\lambda , \tag{3.7a}$$

where

$$f(\lambda, \beta, \beta') = \sqrt{\overline{E}(\beta')^2 + \overline{G}} . \tag{3.7b}$$

From the calculus of variations, it is well-known that a geodesic β = β(λ) satisfies the Euler-Lagrange equation (van Brunt, 2004)

$$\frac{d}{d\lambda}\left(\frac{\partial f}{\partial \beta'}\right) - \frac{\partial f}{\partial \beta} = 0 . \tag{3.8}$$

Using Eq. (3.7b) we obtain

$$\frac{\partial f}{\partial \beta'} = \frac{\overline{E}\beta'}{\sqrt{\overline{E}(\beta')^2 + \overline{G}}} , \tag{3.9a}$$

and

$$\frac{\partial f}{\partial \beta} = \frac{\overline{E}_\beta (\beta')^2 + \overline{G}_\beta}{2\sqrt{\overline{E}(\beta')^2 + \overline{G}}}. \tag{3.9b}$$

By writing out the total derivative in Eq. (3.8) using the chain rule, the Euler-Lagrange equation becomes

$$\frac{\partial^2 f}{\partial \beta' \partial \beta'}\beta'' + \frac{\partial^2 f}{\partial \beta \partial \beta'}\beta' + \frac{\partial^2 f}{\partial \lambda \partial \beta'} - \frac{\partial f}{\partial \beta} = 0, \tag{3.10a}$$

where

$$\beta'' = \frac{d^2\beta}{d\lambda^2}. \tag{3.10b}$$

Substituting Eqs. (3.9) into Eq. (3.10a) subsequently yields

$$2\overline{E}\,\overline{G}\beta'' + \overline{E}\,\overline{E}_\lambda (\beta')^3 - \left(2\overline{E}\,\overline{G}_\beta - \overline{E}_\beta \overline{G}\right)(\beta')^2 + \left(2\overline{E}_\lambda \overline{G} - \overline{E}\,\overline{G}_\lambda\right)\beta' - \overline{G}\,\overline{G}_\beta = 0, \tag{3.11}$$

which is a non-linear second-order ordinary differential equation. The Dirichlet conditions associated with this equation are

$$\beta_0 = \beta(\lambda_0), \qquad \beta_1 = \beta(\lambda_1). \tag{3.12}$$

Hence, the geodesic between two points with $\lambda_0 \neq \lambda_1$ on a triaxial ellipsoid is described by a two-point boundary value problem.

3.2.2. Case with $\lambda_0 = \lambda_1$

Here, the curve is described by $\lambda = \lambda(\beta)$. Using Eq. (3.4) the line element is given by

$$ds = \sqrt{\overline{E} + \overline{G}(\lambda')^2}\, d\beta, \tag{3.13a}$$

where

$$\lambda' = \frac{d\lambda}{d\beta}. \tag{3.13b}$$

Hence, the length s from $\beta = \beta_0$ to $\beta = \beta_1$ $(\beta_0 < \beta_1)$ is obtained by

$$s = \int_{\beta_0}^{\beta_1} g(\beta, \lambda, \lambda')\, d\beta, \tag{3.14a}$$

where

$$g(\beta, \lambda, \lambda') = \sqrt{\overline{E} + \overline{G}(\lambda')^2}\ . \tag{3.14b}$$

Using similar reasoning as was applied in the previous case, we have

$$\frac{d}{d\beta}\left(\frac{\partial g}{\partial \lambda'}\right) - \frac{\partial g}{\partial \lambda} = 0, \tag{3.15}$$

or equivalently

$$\frac{\partial^2 g}{\partial \lambda' \partial \lambda'} \lambda'' + \frac{\partial^2 g}{\partial \lambda \partial \lambda'} \lambda' + \frac{\partial^2 g}{\partial \beta \partial \lambda'} - \frac{\partial g}{\partial \lambda} = 0, \tag{3.16a}$$

where

$$\lambda'' = \frac{d^2 \lambda}{d\beta^2}. \tag{3.16b}$$

Using the partial derivatives of Eq. (3.14b), Eq. (3.16a) yields

$$2\overline{E}\overline{G}\lambda''+\overline{G}\overline{G}_\beta(\lambda')^3-\left(2\overline{E}_\lambda\overline{G}-\overline{E}\overline{G}_\lambda\right)(\lambda')^2+\left(2\overline{E}\overline{G}_\beta-\overline{G}\overline{E}_\beta\right)\lambda'-\overline{E}\overline{E}_\lambda=0, \tag{3.17}$$

which is subject to the Dirichlet conditions

$$\lambda_0=\lambda(\beta_0), \qquad \lambda_1=\lambda(\beta_1). \tag{3.18}$$

Thus, similar to the previous case, the geodesic between two points with $\lambda_0 = \lambda_1$ on a triaxial ellipsoid is described by a two-point boundary value problem.

3.3. Numerical solution

For solving the above two-point boundary value problems, there are several numerical approaches such as shooting methods, finite differences and finite element methods (see e.g., Fox, 1990; Keller, 1992). However, in this study we develop a method based on Taylor's theorem. This method reduces the boundary value problem to an initial value problem which can be solved by well-known numerical techniques.

3.3.1. Case with $\lambda_0 \neq \lambda_1$

Equation (3.11) is written equivalently as a system of two first-order differential equations,

$$\frac{d}{d\lambda}(\beta)=f_1(\lambda,\beta,\beta'), \tag{3.19a}$$

$$\frac{d}{d\lambda}(\beta')=f_2(\lambda,\beta,\beta'), \tag{3.19b}$$

where

$$f_1(\lambda,\beta,\beta')=\beta', \tag{3.20a}$$

and

$$f_2(\lambda,\beta,\beta') = p_3(\beta')^3 + p_2(\beta')^2 + p_1\beta' + p_0\,, \tag{3.20b}$$

with

$$p_3 = -\frac{1}{2}\frac{\overline{E}_\lambda}{\overline{G}}\,, \tag{3.21a}$$

$$p_2 = \frac{\overline{G}_\beta}{\overline{G}} - \frac{1}{2}\frac{\overline{E}_\beta}{\overline{E}}\,, \tag{3.21b}$$

$$p_1 = \frac{1}{2}\frac{\overline{G}_\lambda}{\overline{G}} - \frac{\overline{E}_\lambda}{\overline{E}}\,, \tag{3.21c}$$

$$p_0 = +\frac{1}{2}\frac{\overline{G}_\beta}{\overline{E}}\,. \tag{3.21d}$$

The boundary values associated with this system are

$$\text{D: } \beta_0 = \beta(\lambda_0), \qquad \text{N: } \beta'_0 = \beta'(\lambda_0), \tag{3.22}$$

which are a Dirichlet (D) and a Neumann (N) condition, respectively. The solution $\beta = \beta(\lambda)$ depends on the values given by Eq. (3.22). Our aim is to determine the unknown value β'_0 such that

$$\beta_1 = \beta(\beta_0, \beta'_0\,; \lambda_1). \tag{3.23}$$

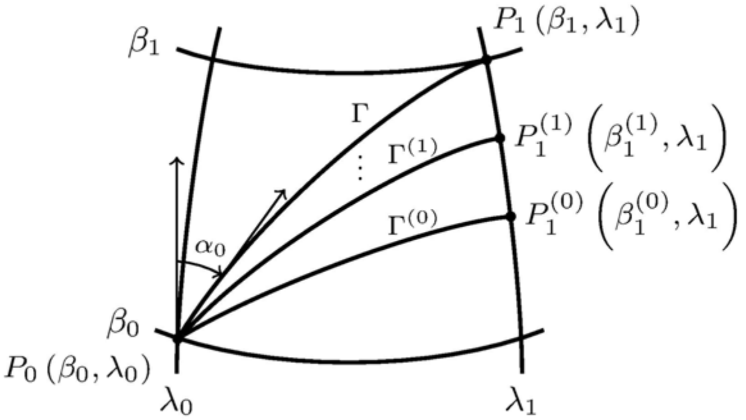


Figure 3.2. The geodesic on an ellipsoid.

We start with an approximate value $\beta'^{(0)}_0$ and we integrate the system of Eqs. (3.19) on the interval $[\lambda_0, \lambda_1]$ using any convenient numerical method. Thus, we determine the geodesic (see Fig. 3.2)

$$\Gamma^{(0)}:\ \beta = \beta\left(\beta_0, \beta'^{(0)}_0; \lambda\right), \tag{3.24}$$

with

$$\beta\left(\beta_0, \beta'^{(0)}_0; \lambda_1\right) = \beta_1^{(0)} \neq \beta_1 . \tag{3.25}$$

Therefore, we search for a correction $\delta\beta'^{(0)}_0$ such that

$$\beta\left(\beta_0, \beta'^{(0)}_0 + \delta\beta'^{(0)}_0; \lambda_1\right) = \beta_1 . \tag{3.26}$$

Using Taylor's theorem (second and higher order terms ignored), Eq. (3.26) can be written as

$$\beta_1 = \beta\left(\beta_0, \beta'^{(0)}_0; \lambda_1\right) + \left(\frac{\partial\beta}{\partial\beta'^{(0)}_0}\right)_1 \delta\beta'^{(0)}_0, \tag{3.27}$$

and from Eqs. (3.25), (3.26) and (3.27) we then obtain

$$\delta\beta'^{(0)}_0 = \frac{\beta_1 - \beta_1^{(0)}}{\left(\dfrac{\partial\beta}{\partial\beta'^{(0)}_0}\right)_1}, \tag{3.28}$$

In Eq. (3.28) the derivative has an unknown value. In order to solve this problem we apply the chain rule in Eqs. (3.19) to obtain

$$\frac{d}{d\lambda}\left(\frac{\partial\beta}{\partial\beta'_0}\right) = \frac{\partial f_1}{\partial\lambda}\frac{\partial\lambda}{\partial\beta'_0} + \frac{\partial f_1}{\partial\beta}\frac{\partial\beta}{\partial\beta'_0} + \frac{\partial f_1}{\partial\beta'}\frac{\partial\beta'}{\partial\beta'_0}, \tag{3.29a}$$

$$\frac{d}{d\lambda}\left(\frac{\partial\beta'}{\partial\beta'_0}\right) = \frac{\partial f_2}{\partial\lambda}\frac{\partial\lambda}{\partial\beta'_0} + \frac{\partial f_2}{\partial\beta}\frac{\partial\beta}{\partial\beta'_0} + \frac{\partial f_2}{\partial\beta'}\frac{\partial\beta'}{\partial\beta'_0}, \tag{3.29b}$$

where the first terms on the right side of Eqs. (3.29) are equal to zero. Hence, we can integrate the system of Eqs. (3.19) on the interval $[\lambda_0, \lambda_1]$ and obtain at λ_1 the values

$$\beta_1^{(0)}, \qquad \left(\frac{\partial\beta}{\partial\beta'^{(0)}_0}\right)_1, \tag{3.30}$$

which are required in Eq. (3.28). In other words, by integrating the system of Eqs. (3.19), we obtain the geodesic $\Gamma^{(0)}$ and the value $\delta\beta'^{(0)}_0$ which is required to start a new iteration.

Now, we start with the value

$$\beta'^{(1)}_0 = \beta'^{(0)}_0 + \delta\beta'^{(0)}_0, \tag{3.31}$$

and via numerical integration on the interval $[\lambda_0, \lambda_1]$ we determine the geodesic

$$\Gamma^{(1)}:\ \beta = \beta\left(\beta_0, \beta'^{(1)}_0; \lambda\right), \qquad (3.32)$$

with

$$\beta\left(\beta_0, \beta'^{(1)}_0; \lambda_1\right) = \beta_1^{(1)} \neq \beta_1 . \qquad (3.33)$$

Using the results at λ_1

$$\beta_1^{(1)}, \qquad \left(\frac{\partial\beta}{\partial\beta'^{(1)}_0}\right)_1, \qquad (3.34)$$

we compute the new correction $\delta\beta'^{(1)}_0$. The process is repeated *m* times until we reach a value $\beta'^{(m)}_0$ such that $\left|\beta_1^{(m)} - \beta_1\right| < \varepsilon$, where $\varepsilon > 0$ is a user-defined threshold for the desired accuracy.

Introducing the variables

$$x_1 = \beta, \qquad x_2 = \beta', \qquad x_3 = \frac{\partial\beta}{\partial\beta'_0}, \qquad x_4 = \frac{\partial\beta'}{\partial\beta'_0}, \qquad (3.35)$$

the system of Eqs. (3.19) and (3.29) can be rewritten as

$$x_1' = x_2, \qquad (3.36a)$$

$$x_2' = p_3(x_2)^3 + p_2(x_2)^2 + p_1 x_2 + p_0, \qquad (3.36b)$$

$$x_3' = x_4, \qquad (3.36c)$$

$$x_4' = \left[p_{33}(x_2)^3 + p_{22}(x_2)^2 + p_{11}x_2 + p_{00}\right]x_3 + \left[3p_3(x_2)^2 + 2p_2x_2 + p_1\right]x_4, \qquad (3.36d)$$

where

$$p_{33} = \frac{\partial p_3}{\partial \beta} = -\frac{1}{2}\frac{\overline{E}_{\beta\lambda}\overline{G} - \overline{E}_{\lambda}\overline{G}_{\beta}}{\overline{G}^2}, \tag{3.37a}$$

$$p_{22} = \frac{\partial p_2}{\partial \beta} = \frac{\overline{G}\,\overline{G}_{\beta\beta} - \overline{G}_{\beta}\overline{G}_{\beta}}{\overline{G}^2} - \frac{1}{2}\frac{\overline{E}\,\overline{E}_{\beta\beta} - \overline{E}_{\beta}\overline{E}_{\beta}}{\overline{E}^2}, \tag{3.37b}$$

$$p_{11} = \frac{\partial p_1}{\partial \beta} = \frac{1}{2}\frac{\overline{G}\,\overline{G}_{\beta\lambda} - \overline{G}_{\beta}\overline{G}_{\lambda}}{\overline{G}^2} - \frac{\overline{E}\,\overline{E}_{\beta\lambda} - \overline{E}_{\beta}\overline{E}_{\lambda}}{\overline{E}^2}, \tag{3.37c}$$

$$p_{00} = \frac{\partial p_0}{\partial \beta} = +\frac{1}{2}\frac{\overline{E}\,\overline{G}_{\beta\beta} - \overline{E}_{\beta}\overline{G}_{\beta}}{\overline{E}^2}. \tag{3.37d}$$

This system of the four first-order differential equations can be solved on the interval $[\lambda_0, \lambda_1]$ using a numerical integration method such as Runge-Kutta or a Taylor series (see Butcher, 1987). The required initial conditions are described below.

The step size $\delta\lambda$ is given by $\delta\lambda = (\lambda_1 - \lambda_0)/n$, where n is the number of steps; a greater number of steps leads to slower computation but greater accuracy, and vice versa. For the variable x_1 the initial value is always the ellipsoidal latitude β_0. For the variable x_2 the initial value can be approximated by the spherical case. Subsequently, in each iteration this value is corrected according to the previous method. For the variables x_3 and x_4 the initial values are always 0 and 1, respectively. Finally, the last iteration yields the geodesic Γ between the two points with $\lambda_0 \neq \lambda_1$ on a triaxial ellipsoid (see Fig. 3.2).

3.3.2. Case with $\lambda_0 = \lambda_1$

In a manner similar to that presented above, introducing the variables

$$y_1 = \lambda, \qquad y_2 = \lambda', \qquad y_3 = \frac{\partial \lambda}{\partial \lambda'_0}, \qquad y_4 = \frac{\partial \lambda'}{\partial \lambda'_0}, \tag{3.38}$$

allows Eq. (3.17) to be reduced to the system

$$y_1' = y_2 , \tag{3.39a}$$

$$y_2' = q_3 (y_2)^3 + q_2 (y_2)^2 + q_1 y_2 + q_0 , \tag{3.39b}$$

$$y_3' = y_4 , \tag{3.39c}$$

$$y_4' = \left[q_{33} (y_2)^3 + q_{22} (y_2)^2 + q_{11} y_2 + q_{00} \right] y_3 + \left[3 q_3 (y_2)^2 + 2 q_2 y_2 + q_1 \right] y_4 , \tag{3.39d}$$

where

$$q_3 = -\frac{1}{2} \frac{\overline{G}_\beta}{\overline{E}} , \tag{3.40a}$$

$$q_2 = \frac{\overline{E}_\lambda}{\overline{E}} - \frac{1}{2} \frac{\overline{G}_\lambda}{\overline{G}} , \tag{3.40b}$$

$$q_1 = \frac{1}{2} \frac{\overline{E}_\beta}{\overline{E}} - \frac{\overline{G}_\beta}{\overline{G}} , \tag{3.40c}$$

$$q_0 = +\frac{1}{2} \frac{\overline{E}_\lambda}{\overline{G}} . \tag{3.40d}$$

and

$$q_{33} = \frac{\partial q_3}{\partial \lambda} = -\frac{1}{2} \frac{\overline{E}\,\overline{G}_{\beta\lambda} - \overline{E}_\lambda \overline{G}_\beta}{\overline{E}^2} , \tag{3.41a}$$

$$q_{22} = \frac{\partial q_2}{\partial \lambda} = \frac{\overline{E}\,\overline{E}_{\lambda\lambda} - \overline{E}_\lambda \overline{E}_\lambda}{\overline{E}^2} - \frac{1}{2} \frac{\overline{G}\,\overline{G}_{\lambda\lambda} - \overline{G}_{\lambda\lambda} \overline{G}_\lambda}{\overline{G}^2} , \tag{3.41b}$$

$$q_{11} = \frac{\partial q_1}{\partial \lambda} = \frac{1}{2}\frac{\overline{E}\overline{E}_{\beta\lambda} - \overline{E}_\beta \overline{E}_\lambda}{\overline{E}^2} - \frac{\overline{G}\overline{G}_{\beta\lambda} - \overline{G}_{\beta\beta}\overline{G}_\lambda}{\overline{G}^2}, \tag{3.41c}$$

$$q_{00} = \frac{\partial q_0}{\partial \lambda} = +\frac{1}{2}\frac{\overline{E}_{\lambda\lambda}\overline{G} - \overline{E}_\lambda \overline{G}_\lambda}{\overline{G}^2}. \tag{3.41d}$$

This system can be integrated on the interval $[\beta_0, \beta_1]$ and the last iteration yields the geodesic between the two points with $\lambda_0 = \lambda_1$ on a triaxial ellipsoid.

3.4. Liouville's constant, angles and geodesic distance

Equation (3.4) can be rewritten as

$$ds^2 = \left(E_y^2 \cos^2\beta + E_e^2 \sin^2\lambda\right)\left(\mathrm{B}d\beta^2 + \Lambda d\lambda^2\right), \tag{3.42}$$

which, according to Klingenberg (1982, p. 305) is called a Liouville line element. Then, along a geodesic it holds that

$$E_y^2 \cos^2\beta \sin^2\alpha - E_e^2 \sin^2\lambda \cos^2\alpha = \ell, \tag{3.43}$$

where ℓ is a constant and α is the angle at which the geodesic cuts the curve λ = constant. Eq. (3.43), which is known as the Liouville equation, can be written equivalently as

$$\left(\cos^2\beta + \frac{E_e^2}{E_x^2}\sin^2\beta\right)\sin^2\alpha + \frac{E_e^2}{E_x^2}\cos^2\lambda\cos^2\alpha = c^2, \tag{3.44}$$

where c is a new constant.

The angle α at any point along the geodesic $\beta = \beta(\lambda)$ with $\lambda_0 < \lambda_1$ is computed by (Deakin and Hunter, 2008)

$$\alpha = arc\cot\left(\frac{\sqrt{E}}{\sqrt{G}}\beta'\right) = arc\cot\left(\frac{\sqrt{\mathrm{B}}}{\sqrt{\Lambda}}\beta'\right), \tag{3.45}$$

which gives $-\pi/2 \le \alpha \le +\pi/2$ and $\alpha \ne 0$. When $\alpha < 0$, the correct angle is obtained as $\alpha = \alpha + \pi$. Furthermore, since $0 < \alpha < +\pi$, Eq. (3.44) implies that $0 < c \le 1$, where $c = 1$ is on the principal ellipse xy (see Fig. 3.2). Similarly, the angle α at any point along the geodesic $\lambda = \lambda(\beta)$ with $\beta_0 < \beta_1$ is computed by

$$\alpha = \frac{\pi}{2} - arc\cot\left(\frac{\sqrt{\Lambda}}{\sqrt{\mathrm{B}}}\lambda'\right), \tag{3.46}$$

which gives $0 \le \alpha \le +\pi$ and $\alpha \ne +\pi/2$. When $\alpha > +\pi/2$, the correct angle is obtained as $\alpha = \alpha - \pi$. Since $-\pi/2 < \alpha < +\pi/2$, Eq. (3.44) implies that $0 \le c < 1$. On the principal ellipses xz and yz, Eq. (3.44) gives $c = E_e/E_x$ and $c = 0$, respectively. We also note that Eqs. (3.45) and (3.46) involve the variables $x_1 = \beta$, $x_2 = \beta'$, and $y_1 = \lambda$, $y_2 = \lambda'$, respectively, which are obtained by numerical integration. In this way, using Eq. (3.44) one can check the accuracy of the numerical integration and subsequently can compute the geodesic distance between two given points.

The distance s between two given points along the geodesic $\beta = \beta(\lambda)$ with $\lambda_0 < \lambda_1$, using Eqs. (3.7), is written as an integral

$$s = \int_{\lambda_0}^{\lambda_1} \sqrt{\overline{E}(\beta')^2 + \overline{G}}\, d\lambda, \tag{3.47}$$

which can be computed by a numerical integration method such as the Newton-Cotes formulas (see, e.g., Hildebrand, 1974). Similarly, the distance s between two points along the geodesic $\lambda = \lambda(\beta)$ with $\beta_0 < \beta_1$, using Eqs. (3.14), is computed by

$$s = \int_{\beta_0}^{\beta_1} \sqrt{\overline{E} + \overline{G}(\lambda')^2}\, d\beta. \tag{3.48}$$

There is a common misconception that a geodesic is the shortest path between two points, but this is not always the case. A discussion on this complex problem is included in Struik (1961) and Guggenheimer (1977). However, we can show under what condition a geodesic is the shortest path. According to Guggenheimer (1977, p. 265-266), a given geodesic is not the shortest path when it contains two conjugate points, and the minimum geodesic distance of two mutually conjugate points is $\pi(\max k)^{-1/2}$, where k is the Gauss curvature. On a triaxial ellipsoid, it holds that max k = $a_x^2 / (b^2 a_y^2)$, (Klingenberg, 1982, p. 311). Hence, the length s of a geodesic which does not contain conjugate points is

$$s \leq \pi b \frac{a_y}{a_x} . \tag{3.49}$$

Thus, Eq. (3.49) provides the length limit, below which a geodesic is the shortest path between two points on a triaxial ellipsoid.

3.5. The geodesic boundary value problem on a biaxial ellipsoid

In the biaxial case it holds that $a_x = a_y \equiv a$ i.e., $E_x = E_y \equiv h$ and $E_e = 0$. Under these conditions and by setting $u = b$ in Eq. (2.34), the ellipsoidal coordinates (β, λ) are related to Cartesian coordinates (x, y, z) by (e.g. Heiskanen and Moritz, 1967; Featherstone and Claessens, 2008)

$$x = a\cos\beta\cos\lambda , \qquad y = a\cos\beta\sin\lambda , \qquad z = b\sin\beta , \tag{3.50}$$

where $-\pi/2 \leq \beta \leq +\pi/2$ and $-\pi < \lambda \leq +\pi$. In this parameterization, Eqs. (3.2) and (3.3) are written as

$$\overline{E} = \mathrm{B}E^2 \cos^2 \beta , \tag{3.51a}$$

$$\overline{F} = 0 , \tag{3.51b}$$

$$\overline{G} = \Lambda E^2 \cos^2 \beta \,, \tag{3.51c}$$

where

$$\mathrm{B} = \frac{a^2 \sin^2 \beta + b^2 \cos^2 \beta}{E^2 \cos^2 \beta} \,, \tag{3.52a}$$

$$\Lambda = \frac{a^2}{E^2} \,. \tag{3.52b}$$

In Eq. (3.51b), $\overline{F} = 0$ indicates that the β-curves (parallels) and λ-curves (meridians) are orthogonal. Also, $\mathrm{B} \neq 0$, $\Lambda \neq 0$ and $\overline{E} \neq 0$ at all locations, and $\overline{G} = 0$ at the poles (i.e., when $\beta = \pm\pi/2$). The partial derivatives of Eqs. (3.51a), (3.51c) and (3.52) are presented in Appendix A.2.

In the biaxial case we study only the case with $\lambda_0 \neq \lambda_1$ since the case where $\lambda_0 = \lambda_1$ can be excluded as trivial: it is well-known that all meridians are geodesics with Clairaut's constant c equal to zero. Also, the azimuths α along the meridian are zero and the geodesic distance s between two points on the meridian is given by (Deakin and Hunter, 2008)

$$s = \int_{\beta_0}^{\beta_1} \sqrt{\overline{E}}\, d\beta \,. \tag{3.53}$$

The curve on a biaxial ellipsoid is described by $\beta = \beta(\lambda)$ with $\lambda_0 < \lambda_1$ and the geodesic boundary value problem consists of solving the equation

$$2\overline{E}\,\overline{G}\beta'' - \left(2\overline{E}\,\overline{G}_\beta - \overline{E}_\beta \overline{G}\right)(\beta')^2 - \overline{G}\,\overline{G}_\beta = 0 \,, \tag{3.54}$$

subject to the Dirichlet conditions

$$\beta_0 = \beta(\lambda_0), \qquad \beta_1 = \beta(\lambda_1). \tag{3.55}$$

3.5.1. Numerical solution

Following the method discussed in Section 3.3.1, the geodesic boundary value problem on a biaxial ellipsoid is reduced to a system of four first-order differential equations. Hence, we rewrite Eqs. (3.36) as

$$x_1' = x_2\,, \tag{3.56a}$$

$$x_2' = p_2 (x_2)^2 + p_0\,, \tag{3.56b}$$

$$x_3' = x_4\,, \tag{3.56c}$$

$$x_4' = \left[p_{22}(x_2)^2 + p_{00}\right] x_3 + 2 p_2 x_2 x_4\,. \tag{3.56d}$$

Using Eqs. (3.21),

$$p_2 = \frac{\overline{G}_\beta}{\overline{G}} - \frac{1}{2}\frac{\overline{E}_\beta}{\overline{E}}\,, \tag{3.57a}$$

$$p_0 = +\frac{1}{2}\frac{\overline{G}_\beta}{\overline{E}}\,, \tag{3.57b}$$

and, using Eqs. (3.37),

$$p_{22} = \frac{\partial p_2}{\partial \beta} = \frac{\overline{G}\,\overline{G}_{\beta\beta} - \overline{G}_\beta \overline{G}_\beta}{\overline{G}^2} - \frac{1}{2}\frac{\overline{E}\,\overline{E}_{\beta\beta} - \overline{E}_\beta \overline{E}_\beta}{\overline{E}^2}\,, \tag{3.58a}$$

$$p_{00} = \frac{\partial p_0}{\partial \beta} = +\frac{1}{2}\frac{\overline{E}\,\overline{G}_{\beta\beta} - \overline{E}_\beta \overline{G}_\beta}{\overline{E}^2}\,. \tag{3.58b}$$

Hence, this system can be integrated on the interval $[\lambda_0, \lambda_1]$ and the last iteration yields the geodesic between two points with $\lambda_0 \neq \lambda_1$ on a biaxial ellipsoid.

3.5.2. Clairaut's constant, azimuths and geodesic distance

In the biaxial case, Eq. (3.44) becomes

$$\cos\beta \sin\alpha = c\,, \tag{3.59}$$

which is the well-known Clairaut's equation. Also, the function f in Eq. (3.7b) does not contain the independent variable λ explicitly. Therefore, along any geodesic β = β(λ) it holds that

$$f - \beta' \frac{\partial f}{\partial \beta'} = ac\,, \tag{3.60}$$

which is a special case of the Euler-Lagrange equation (Eq. (3.8)), (van Brunt, 2004). Substituting Eqs. (3.7b) and (3.9a) into Eq. (3.60) we obtain

$$\frac{\overline{E}\beta'}{\sqrt{\overline{E}(\beta')^2 + \overline{G}}} = ac\,. \tag{3.61}$$

Now, substituting Eq. (3.45) into Eq. (3.61) and using Eqs. (3.51c) and (3.52b), we obtain the Eq. (3.59). Hence, Eqs. (3.59) and (3.61) are equivalent and the Clairaut constant c can be computed by Eq. (3.61) at any value of the independent variable λ. In this way, using Eq. (3.61) one can check the accuracy of the numerical integration and subsequently can compute the azimuths along the geodesic and the geodesic distance between two given points.

The azimuth α at any point along the geodesic is computed by Eq. (3.45). Also, since $0 < \alpha < +\pi$, Eq. (3.59) implies that $0 < c \leq 1$. The geodesic distance s between the two points is computed by Eq. (3.47). Finally, Eq. (3.49) becomes

$$s \leq \pi b\,, \tag{3.62}$$

which ensures that the resulting geodesic is the shortest path between two points on a biaxial ellipsoid.

3.6. Numerical examples

In order to demonstrate the validity of the algorithms presented above, numerical examples are given for each case. All algorithms were implemented in MATLAB. The numerical computations in the triaxial case were carried out using Earth's geometrical parameters a_x = 6378172 m, a_y = 6378103 m and b = 6356753 m (Burša and Šíma, 1980). For solving the system of the four first-order differential equations, the fourth-order Runge-Kutta numerical integration method was used. The number of steps n was selected as 16000 in order to cover all cases with sufficient accuracy but keep computational time within reason. The latitudes at λ_1 and the longitudes at β_1 were required to converge with an accuracy $\varepsilon = 10^{-12}$ rad, which corresponds to approximately 0.006 mm. Also, the geodesic distance between two points was computed by Simpson's rule (i.e., the three point rule) and according to Eq. (3.49) the maximum geodesic distance which ensures the shortest path property is s = 19970112.4835 m.

In the triaxial case with $\lambda_0 \neq \lambda_1$, taking into account the geometry of the triaxial ellipsoid, we use two input sets. In the first set starting points (β_0, λ_0) with λ_0 = 0° and β_0 = 0°, 1°, 5°, 30°, 60°, 75° and 80° were selected, as well as points (β_1, λ_1) with λ_1 = 0.5°, 1°, 5°, 40°, 90°, 120°, 170°, 175°, 179°, 179.5° and β_1 = 0°, ±1°, ±5°, ±30°, ±60°, ±75° and ±80° (set 1). Note that, when β_0 = 0° only the values $\beta_1 \geq 0°$ were used (symmetry). Hence, in total 850 geodesics were derived. Despite the fact that 15999 intermediate points were determined and usually the iterative procedure reached convergence in three or four iterations for each geodesic, the entire set was completed in about 2.5 h of processing time using a 2.8 GHz Intel processor, which corresponds to about 10 seconds for each complete determination of a geodesic. A sample of results is presented in Table 3.1.

In the second set starting points (β_0, λ_0) with λ_0 = –90° and β_0 = 0°, 1°, 5°, 30°, 60°, 75° and 80° were selected, as well as points (β_1, λ_1) with λ_1 = –89.5°, –89°, –85°, –50°, 0°, 30°, 80°, 85°, 89°, 89.5° and β_1 = 0°, $\pm 1°$, $\pm 5°$, $\pm 30°$, $\pm 60°$, $\pm 75°$ and $\pm 80°$ (set 2). Similarly, when β_0 = 0° only the values β_1 ≥ 0° were used (symmetry). We should point out that the iterative procedure does not convergence in the cases where the resulting geodesic includes the values β = 90° and λ = 0° (i.e., where it passes above the umbilical point). For this reason, we present the rear extreme cases in Table 3.2 which can be interpreted as follows: the geodesic with (β_0, λ_0) = (+80°, –90°) convergence in all cases with –80° ≤ β ≤ +80° and λ_1 ≈ +55°, the geodesic with (β_0, λ_0) = (+75°, –90°) convergence in all cases with –75° ≤ β ≤ +75° and λ_1 ≈ +66°, and so on. Finally, we point out that the first geodesic of Table 3.2 may not be the shortest path between those points, since its length exceeds the limit of Eq. (3.49).

In the triaxial case with $\lambda_0 = \lambda_1$, starting points (β_0, λ_0) with λ_0 = 0.5°, 5°, 30°, 45°, 60°, 85°, 89.5° and β_0 = 0°, –1°, –5°, –30°, –60°, –75°, –80° were selected, as well as points (β_1, λ_1) with $\lambda_1 = \lambda_0$ and β_1 = 0°, $\pm 1°$, $\pm 5°$, $\pm 30°$, $\pm 60°$, $\pm 75°$, $\pm 80°$ where $\beta_0 < \beta_1$ (set 3). Hence, in total 441 geodesics were derived. A sample of the results is presented in Table 3.3.

Furthermore, we have used the values for β and λ of sets 1 and 3 using the parameters of GRS80 a = 6378137 m, b = 6356752.3141 (Moritz, 1980) i.e. a biaxial ellipsoid, as input to the general algorithm of the triaxial case (Section 3.3). A sample of the results is presented in Table 3.4. In addition, we used the same β and λ values as input to the biaxial case algorithm (Section 3.5.1), which produced identical results. Also, in order to validate the algorithm which has been presented, the results of Table 3.4 were compared to those obtained using the Vincenty's method. For this study, Vincenty's algorithm (Vincenty, 1975) was implemented with the requirement that the longitude differences were to converge with an accuracy 10^{-12} rad ≈ 0.006 mm. The results between our proposed method and Vincenty's method show agreement to within 6×10^{-6} seconds of arc for azimuth and 0.21 mm in geodesic distance. These results are presented in Table 3.5.

In order to get an estimate of the difference in length of a geodesic on a biaxial or a triaxial ellipsoid, we consider the case of an equatorial geodesic. The difference between the two surfaces is represented by the equatorial flattening of the triaxial ellipsoid, which is about 10 ppm for the triaxial ellipsoid used in our computations. Therefore, the difference in geodesic length will reach a maximum value of about 200 m, for near-antipodal equatorial points.

Table 3.1. Numerical examples in the triaxial case with $\lambda_0 \neq \lambda_1$ and $\lambda_0 = 0°$.

β_0	λ_0	β_1	λ_1	c	α_0 (° ' ")	α_1 (° ' ")	s (m)
0°	0°	0°	90°	1.00000000000	90 00 00.0000	90 00 00.0000	10018754.9569
1°	0°	–80°	5°	0.05883743460	179 07 12.2719	174 40 13.8487	8947130.7221
5°	0°	–60°	40°	0.34128138370	160 13 24.5001	137 26 47.0036	8004762.4330
30°	0°	–30°	175°	0.86632464962	91 07 30.9337	91 07 30.8672	19547128.7971
60°	0°	60°	175°	0.06207487624	02 52 26.2393	177 04 13.6373	6705715.1610
75°	0°	80°	120°	0.11708984898	23 20 34.7823	140 55 32.6385	2482501.2608
80°	0°	60°	90°	0.17478427424	72 26 50.4024	159 38 30.3547	3519745.1283

Table 3.2. Numerical examples in the triaxial case with $\lambda_0 \neq \lambda_1$, $\lambda_0 = -90°$ and $\beta_0 = \beta_1$.

β_0	λ_0	β_1	λ_1	c	α_0 (° ' ")	α_1 (° ' ")	s (m)
0°	–90°	0°	89.5°	1.00000000000	90 00 00.0000	90 00 00.0000	19981849.8629
1°	–90°	1°	89.5°	0.18979826428	10 56 33.6952	169 03 26.4359	19776667.0342
5°	–90°	5°	89°	0.09398403161	05 24 48.3899	174 35 12.6880	18889165.0873
30°	–90°	30°	86°	0.06004022935	03 58 23.8038	176 02 07.2825	13331814.6078
60°	–90°	60°	78°	0.06076096484	06 56 46.4585	173 11 05.9592	6637321.6350
75°	–90°	75°	66°	0.05805851008	12 40 34.9009	168 20 26.7339	3267941.2812
80°	–90°	80°	55°	0.05817384452	18 35 40.7848	164 25 34.0017	2132316.9048

Table 3.3. Numerical examples in the triaxial case with $\lambda_0 = \lambda_1$.

β_0	λ_0	β_1	λ_1	c	α_0 (° ' ")	α_1 (° ' ")	s (m)
0°	0.5°	80°	0.5°	0.05680316848	–0 00 16.0757	0 01 32.5762	8831874.3717
–1°	5°	75°	5°	0.05659149555	–0 01 47.2105	0 06 54.0958	8405370.4947
–5°	30°	60°	30°	0.04921108945	–0 04 22.3516	0 08 42.0756	7204083.8568
–30°	45°	30°	45°	0.04017812574	–0 03 41.2461	0 03 41.2461	6652788.1287
–60°	60°	5°	60°	0.02843082609	–0 08 40.4575	0 04 22.1675	7213412.4477
–75°	85°	1°	85°	0.00497802414	–0 06 44.6115	0 01 47.0474	8442938.5899
–80°	89.5°	0°	89.5°	0.00050178253	–0 01 27.9705	0 00 16.0490	8888783.7815

Table 3.4. Numerical examples in the biaxial case.

β_0	λ_0	β_1	λ_1	c	α_0 (° ' ")	α_1 (° ' ")	s (m)
0°	0°	0°	90°	1.00000000000	90 00 00.0000	90 00 00.0000	10018754.1714
1°	0°	0°	179.5°	0.30320665822	17 39 11.0942	162 20 58.9032	19884417.8083
5°	0°	–80°	170°	0.03104258442	178 12 51.5083	10 17 52.6423	11652530.7514
30°	0°	–75°	120°	0.24135347134	163 49 04.4615	68 49 50.9617	14057886.8752
60°	0°	–60°	40°	0.19408499032	157 09 33.5589	157 09 33.5589	13767414.8267
75°	0°	–30°	0.5°	0.00202789418	179 33 03.8613	179 51 57.0077	11661713.4496
80°	0°	–5°	120°	0.15201222384	61 05 33.9600	171 13 22.0148	11105138.2902
0°	0°	60°	0°	0.00000000000	00 00 00.0000	00 00 00.0000	6663348.2060

Table 3.5. Numerical tests and comparisons with Vincenty's method.

β_0	λ_0	β_1	λ_1	$\Delta\alpha_0$ (")	$\Delta\alpha_1$ (")	Δs (mm)
0°	0°	0°	90°	0	0	-0.004
1°	0°	0°	179.5°	-	-	-
5°	0°	–80°	170°	2×10^{-7}	-2×10^{-7}	-0.21
30°	0°	–75°	120°	1×10^{-7}	2×10^{-7}	0.05
60°	0°	–60°	40°	6×10^{-7}	6×10^{-7}	0.06
75°	0°	–30°	0.5°	-1×10^{-7}	5×10^{-7}	0.05
80°	0°	–5°	120°	6×10^{-7}	3×10^{-8}	0.04
0°	0°	60°	0°	0	0	0.03

4. THE GRAVITY FIELD DUE TO A HOMOGENEOUS ELLIPSOID

4.1. Introduction

In this chapter, we present the exterior gravity field in the simple case of a homogeneous triaxial ellipsoid. In Balmino (1994) the gravitational potential is expanded in Legendre series but this creates complications, especially when we want to obtain the derivatives of the potential. For this reason, we start with the compact expressions of classical potential theory, e.g. Kellogg (1953) and MacMillan (1958). Subsequently, we show that the ellipsoidal coordinates (u, β, λ) play a special role in the construction of the gravitational potential, which differs with respect to alternatives as reported in the related literature, such as Miloh (1990), and provide expressions of the gravity field applicable in all cases: a triaxial ellipsoid, an oblate spheroid and a sphere. These expressions contain improper integrals; therefore we provide a suitable transformation in order to evaluate them by numerical integration methods. Also, we present a connection with the Lamé functions, some numerical examples and a geometrical interpretation. Finally, the gravity field due to a homogeneous oblate spheroid is obtained as a degenerate case. This leads to an equivalent and simpler expression of the gravity field of an oblate spheroid than the ones that have been discussed by Wang (1988) and Hvoždara and Kohút (2012).

4.2. Gravity potential

The gravity potential U of a triaxial ellipsoid rotating with constant angular velocity ω, as described in a co-rotating reference system, is the sum of the gravitational potential V, generated by the total mass M contained in this ellipsoid, and the centrifugal potential Φ, due to the rotational motion, i.e.

$$U = V + \Phi. \tag{4.1}$$

We assume that the centre of mass of the ellipsoid coincides with the origin O of the coordinate system and its axis of rotation coincides with the z-axis. Therefore, the centrifugal potential Φ is expressed by

$$\Phi = \frac{1}{2}\omega^2\left(x^2 + y^2\right). \tag{4.2}$$

Since we consider that the solid ellipsoid Ω is homogeneous, i.e. has a distribution of mass of constant density ρ_0, the gravitational potential V at an exterior point P is given by the special case of Newton's integral

$$V(P) = G\rho_0 \int_{\Omega} \frac{1}{l} dv, \tag{4.3}$$

where G is the gravitational constant and l the distance between the mass element $dm = \rho_0\, dv$ and the attracted point P. Note that, the total mass is M = $(4/3)\pi\, a_x\, a_y\, b\, \rho_0$.

As shown in Balmino (1994), the gravitational potential V due to a homogeneous ellipsoid can be expanded in the form of a Legendre series in spherical coordinates (r, β, λ):

$$V(r,\beta,\lambda) = \frac{GM}{r}\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\left(\frac{a_x}{r}\right)^{2n} C_{2n,2m}P_{2n,2m}(\sin\beta)\cos 2m\lambda, \tag{4.4}$$

where $P_{n,m}$ are the associated Legendre functions with

$$P_{n,m}(t) = \frac{\left(1-t^2\right)^{m/2}}{2^n\, n!}\frac{d^{\,n+m}}{dt^{\,n+m}}\left(t^2 - 1\right)^n, \tag{4.5}$$

and the coefficients $C_{2n,2m}$ are given by explicit expressions (Balmino, 1994).

Kellogg (1953) and MacMillan (1958) demonstrated that the gravitational potential V induced by a homogeneous ellipsoid can be given by a simple integral:

$$V(s_1, s_2, s_3) = \frac{3}{4} GM \int_{s_1}^{+\infty} \left(1 - \frac{x^2}{a_x^2 + s} - \frac{y^2}{a_y^2 + s} - \frac{z^2}{b^2 + s} \right) \frac{ds}{(a_x^2 + s)^{1/2} (a_y^2 + s)^{1/2} (b^2 + s)^{1/2}}, \tag{4.6}$$

where (s_1, s_2, s_3) are the ellipsoidal coordinates which correspond to (*x*, *y*, *z*) by Eqs. (2.10)-(2.12). Here, the coordinate s_1 is the positive root of Eq. (2.2). It is easy to show that Eq. (4.6) satisfies Laplace's equation in the exterior of the ellipsoid and, in addition, the regularity condition at infinity, i.e. as s_1 tends towards infinity, *V* tends towards zero.

4.2.1. General expressions

Following Kellogg (1953) and MacMillan (1958), the gravitational potential *V* due to a homogeneous ellipsoid at an exterior point *P*(*u*, β, λ), where $u \geq b$, is given by

$$V(u, \beta, \lambda) = \frac{3}{2} GM \int_{u}^{+\infty} \left(1 - \frac{x^2}{\sigma^2 + E_x^2} - \frac{y^2}{\sigma^2 + E_y^2} - \frac{z^2}{\sigma^2} \right) \frac{d\sigma}{(\sigma^2 + E_x^2)^{1/2} (\sigma^2 + E_y^2)^{1/2}}, \tag{4.7}$$

where the Cartesian coordinates (*x*, *y*, *z*) are related to the ellipsoidal (*u*, β, λ) by Eqs. (2.27)-(2.29). Thus, the gravitational potential *V* may be written as

$$V(u, \beta, \lambda) = \frac{3}{2} GM \left[I_0(u) - I_1(u) x^2 - I_2(u) y^2 - I_3(u) z^2 \right], \tag{4.8}$$

where

$$I_0(u) = \int_{u}^{+\infty} \frac{d\sigma}{(\sigma^2 + E_x^2)^{1/2} (\sigma^2 + E_y^2)^{1/2}}, \tag{4.9}$$

$$I_1(u) = \int_{u}^{+\infty} \frac{d\sigma}{(\sigma^2 + E_x^2)^{3/2} (\sigma^2 + E_y^2)^{1/2}}, \tag{4.10}$$

$$I_2(u) = \int_u^{+\infty} \frac{d\sigma}{\left(\sigma^2 + E_x^2\right)^{1/2}\left(\sigma^2 + E_y^2\right)^{3/2}}, \tag{4.11}$$

$$I_3(u) = \int_u^{+\infty} \frac{d\sigma}{\sigma^2\left(\sigma^2 + E_x^2\right)^{1/2}\left(\sigma^2 + E_y^2\right)^{1/2}}. \tag{4.12}$$

Clearly, by substituting Eqs. (2.27)-(2.29) into Eq. (4.8) the resulting potential can be fully expressed in ellipsoidal coordinates (u, β, λ). In turn, the integrals I_0, I_1, I_2 and I_3 can be expressed in terms of elliptic integrals of the first and the second kind, see e.g. MacMillan (1958). Also, Fukushima and Ishizaki (1994) provide algorithms that allow numerical computations. On the other hand, in order to evaluate these integrals, considering the modern computational capabilities, we take advantage of numerical integration methods. For this reason, the improper integrals are transformed to definite integrals by the substitution $\sigma = 1/t$. As a result, we obtain

$$I_0(u) = \int_0^{1/u} \frac{dt}{\left(1 + E_x^2 t^2\right)^{1/2}\left(1 + E_y^2 t^2\right)^{1/2}}, \tag{4.13}$$

$$I_1(u) = \int_0^{1/u} \frac{t^2 dt}{\left(1 + E_x^2 t^2\right)^{3/2}\left(1 + E_y^2 t^2\right)^{1/2}}, \tag{4.14}$$

$$I_2(u) = \int_0^{1/u} \frac{t^2 dt}{\left(1 + E_x^2 t^2\right)^{1/2}\left(1 + E_y^2 t^2\right)^{3/2}}, \tag{4.15}$$

$$I_3(u) = \int_0^{1/u} \frac{t^2 dt}{\left(1 + E_x^2 t^2\right)^{1/2}\left(1 + E_y^2 t^2\right)^{1/2}}. \tag{4.16}$$

Now, the above integrals can be computed by numerical methods such as Newton-Cotes formulas, Simpson rules, etc.

Following the theory of ellipsoidal harmonics e.g. Dassios (2012), the integrals in Eqs. (4.9)-(4.12) may be expressed in terms of Lamé functions of the second kind F_n^m, i.e.

$$F_n^m(u) = (2n+1)E_n^m(u)\int_u^{+\infty} \frac{d\sigma}{\left[E_n^m(\sigma)\right]^2 \left(\sigma^2 + E_x^2\right)^{1/2} \left(\sigma^2 + E_y^2\right)^{1/2}}, \tag{4.17}$$

where the Lamé functions of the first kind E_n^m used in this paper are

$$E_0^0(u) = 1, \qquad E_1^1(u) = \left(u^2 + E_x^2\right)^{1/2}, \; E_1^2(u) = \left(u^2 + E_y^2\right)^{1/2}, \; E_1^3(u) = u. \tag{4.18}$$

Therefore, Eq. (4.8) can be rewritten as

$$V(u,\beta,\lambda) = \frac{G\mathrm{M}}{2}\left[3F_0^0(u) - \frac{F_1^1(u)}{E_1^1(u)}x^2 - \frac{F_1^2(u)}{E_1^2(u)}y^2 - \frac{F_1^3(u)}{E_1^3(u)}z^2\right]. \tag{4.19}$$

Finally, by substituting Eqs. (2.27) and (2.28) into Eq. (4.2), we obtain

$$\Phi(u,\beta,\lambda) = \frac{1}{2}\omega^2\left[\left(u^2 + E_y^2 + E_e^2\cos^2\lambda\right)\cos^2\beta + \left(u^2 + E_x^2\right)\frac{E_e^2}{E_x^2}\sin^2\beta\cos^2\lambda\right]. \tag{4.20}$$

4.2.2. Oblate spheroidal case

The formulas which have been derived in the previous Section 4.2.1 have the advantage that they can be applied in any case, e.g. to a triaxial ellipsoid, oblate spheroid and sphere. However, it is interesting to show how these general expressions are reduced to the oblate spheroidal case, where $a_x = a_y \equiv a$, i.e. $E_x = E_y \equiv E$, $E_e = 0$ and $I_1 = I_2 \equiv I_{12}$. In this case

$$V(u,\beta) = \frac{3}{2}G\mathrm{M}\int_u^{+\infty}\left(1 - \frac{x^2 + y^2}{\sigma^2 + E^2} - \frac{z^2}{\sigma^2}\right)\frac{d\sigma}{\left(\sigma^2 + E^2\right)}, \tag{4.21}$$

or equivalently

$$V(u,\beta)=\frac{3}{2}G\mathrm{M}\left[I_0(u)-I_{12}(u)(x^2+y^2)-I_3(u)z^2\right], \tag{4.22}$$

where

$$I_0(u)=\int_u^{+\infty}\frac{d\sigma}{\sigma^2+E^2}=\frac{1}{E}\tan^{-1}\left(\frac{E}{u}\right), \tag{4.23}$$

$$I_{12}(u)=\int_u^{+\infty}\frac{d\sigma}{\left(\sigma^2+E^2\right)^2}=\frac{1}{2E^3}\left[\tan^{-1}\left(\frac{E}{u}\right)-\frac{Eu}{u^2+E^2}\right], \tag{4.24}$$

$$I_3(u)=\int_u^{+\infty}\frac{d\sigma}{\sigma^2\left(\sigma^2+E^2\right)}=\frac{1}{E^3}\left[\frac{E}{u}-\tan^{-1}\left(\frac{E}{u}\right)\right]. \tag{4.25}$$

As we see, in this case the integrals are expressible in terms of elementary functions. Also, equivalent expressions in other oblate spheroidal systems are included in Wang (1988) and Hvoždara and Kohút (2012). In a similar treatment, it can be easily shown that in the spherical case, where $E = 0$, it holds that $V = G\mathrm{M}/r$.

Finally, in the oblate spheroidal case, Eq. (4.20) is reduced to

$$\Phi(u,\beta)=\frac{1}{2}\omega^2\left(u^2+E^2\right)\cos^2\beta\,. \tag{4.26}$$

4.3. Gravity vector

4.3.1. General expressions

From the gravity potential U, the gravity vector $\boldsymbol{\gamma}$ is obtained by

$$\boldsymbol{\gamma} = \mathrm{grad}\ U. \tag{4.27}$$

Evaluating the gradient in ellipsoidal coordinates (u, β, λ), the gravity vector components γ_u, γ_β and γ_λ are related to the gravity potential by

$$\gamma_u = \frac{1}{h_u}\frac{\partial U}{\partial u} = \left[\frac{\left(u^2+E_x^2\right)\left(u^2+E_y^2\right)}{\left(u^2+E_y^2\sin^2\beta\right)\left(u^2+E_y^2+E_e^2\sin^2\lambda\right)}\right]^{1/2}\frac{\partial U}{\partial u}, \tag{4.28}$$

$$\gamma_\beta = \frac{1}{h_\beta}\frac{\partial U}{\partial \beta} = \left[\frac{E_x^2-E_y^2\sin^2\beta}{\left(u^2+E_y^2\sin^2\beta\right)\left(E_y^2\cos^2\beta+E_e^2\sin^2\lambda\right)}\right]^{1/2}\frac{\partial U}{\partial \beta}, \tag{4.29}$$

$$\gamma_\lambda = \frac{1}{h_\lambda}\frac{\partial U}{\partial \lambda} = \left[\frac{E_x^2-E_e^2\cos^2\lambda}{\left(u^2+E_y^2+E_e^2\sin^2\lambda\right)\left(E_y^2\cos^2\beta+E_e^2\sin^2\lambda\right)}\right]^{1/2}\frac{\partial U}{\partial \lambda}, \tag{4.30}$$

where, using Eq. (4.1) and inserting the gravitational potential V given by Eq. (4.8), we have

$$\frac{\partial U}{\partial u} = \frac{3}{2}\mathrm{GM}\left[\frac{\partial}{\partial u}I_0(u) - \frac{\partial}{\partial u}\left(I_1(u)x^2\right) - \frac{\partial}{\partial u}\left(I_2(u)y^2\right) - \frac{\partial}{\partial u}\left(I_3(u)z^2\right)\right] + \frac{\partial \Phi}{\partial u}, \tag{4.31}$$

$$\frac{\partial U}{\partial \beta} = -\frac{3}{2}\mathrm{GM}\left[I_1(u)\frac{\partial\left(x^2\right)}{\partial \beta} + I_2(u)\frac{\partial\left(y^2\right)}{\partial \beta} + I_3(u)\frac{\partial\left(z^2\right)}{\partial \beta}\right] + \frac{\partial \Phi}{\partial \beta}, \tag{4.32}$$

$$\frac{\partial U}{\partial \lambda} = -\frac{3}{2}\mathrm{GM}\left[I_1(u)\frac{\partial\left(x^2\right)}{\partial \lambda} + I_2(u)\frac{\partial\left(y^2\right)}{\partial \lambda} + I_3(u)\frac{\partial\left(z^2\right)}{\partial \lambda}\right] + \frac{\partial \Phi}{\partial \lambda}. \tag{4.33}$$

The above partial derivatives can be easily obtained using Eqs. (2.27)-(2.29), (4.20) and additionally

$$\frac{\partial I_0}{\partial u} = -\frac{1}{\left(u^2+E_x^2\right)^{1/2}\left(u^2+E_y^2\right)^{1/2}}, \tag{4.34}$$

$$\frac{\partial I_1}{\partial u} = -\frac{1}{\left(u^2 + E_x^2\right)^{3/2}\left(u^2 + E_y^2\right)^{1/2}}, \tag{4.35}$$

$$\frac{\partial I_2}{\partial u} = -\frac{1}{\left(u^2 + E_x^2\right)^{1/2}\left(u^2 + E_y^2\right)^{3/2}}, \tag{4.36}$$

$$\frac{\partial I_3}{\partial u} = -\frac{1}{u^2\left(u^2 + E_x^2\right)^{1/2}\left(u^2 + E_y^2\right)^{1/2}}. \tag{4.37}$$

In order to obtain the gravity vector components in the Cartesian system (x, y, z), one can use the chain rule

$$\gamma_u = \frac{1}{h_u}\left[\frac{\partial U}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial u}\right], \tag{4.38}$$

and similarly for γ_β and γ_λ, where the partial derivatives $\partial x/\partial u$ etc. are easily obtained from Eqs. (2.27)-(2.29). Since the coordinates of both systems are orthogonal, the transformation matrix is orthogonal, therefore one obtains the result

$$\gamma_x = \frac{1}{h_u}\frac{\partial x}{\partial u}\gamma_u + \frac{1}{h_\beta}\frac{\partial x}{\partial \beta}\gamma_\beta + \frac{1}{h_\lambda}\frac{\partial x}{\partial \lambda}\gamma_\lambda, \tag{4.39}$$

$$\gamma_y = \frac{1}{h_u}\frac{\partial y}{\partial u}\gamma_u + \frac{1}{h_\beta}\frac{\partial y}{\partial \beta}\gamma_\beta + \frac{1}{h_\lambda}\frac{\partial y}{\partial \lambda}\gamma_\lambda, \tag{4.40}$$

$$\gamma_z = \frac{1}{h_u}\frac{\partial z}{\partial u}\gamma_u + \frac{1}{h_\beta}\frac{\partial z}{\partial \beta}\gamma_\beta + \frac{1}{h_\lambda}\frac{\partial z}{\partial \lambda}\gamma_\lambda. \tag{4.41}$$

4.3.2. Oblate spheroidal case

In this case, Eqs. (4.28)-(4.30) are reduced to the expressions

$$\gamma_u = \left(\frac{u^2 + E^2}{u^2 + E^2 \sin^2 \beta} \right)^{1/2} \frac{\partial U}{\partial u}, \tag{4.42}$$

$$\gamma_\beta = \left(\frac{1}{u^2 + E^2 \sin^2 \beta} \right)^{1/2} \frac{\partial U}{\partial \beta}, \tag{4.43}$$

$$\gamma_\lambda = \left[\frac{1}{\left(u^2 + E^2\right)\cos^2 \beta} \right]^{1/2} \frac{\partial U}{\partial \lambda}, \tag{4.44}$$

where using Eqs. (4.1) and (4.22), we have

$$\frac{\partial U}{\partial u} = \frac{3}{2} GM \left[\frac{\partial}{\partial u} I_0(u) - \frac{\partial}{\partial u}\left[I_{12}(u)\left(x^2 + y^2\right)\right] - \frac{\partial}{\partial u}\left(I_3(u) z^2 \right) \right] + \frac{\partial \Phi}{\partial u}, \tag{4.45}$$

$$\frac{\partial U}{\partial \beta} = -\frac{3}{2} GM \left[I_{12}(u) \frac{\partial\left(x^2 + y^2\right)}{\partial \beta} + I_3(u) \frac{\partial\left(z^2\right)}{\partial \beta} \right] + \frac{\partial \Phi}{\partial \beta}, \tag{4.46}$$

$$\frac{\partial U}{\partial \lambda} = 0. \tag{4.47}$$

In this case, we have to use Eqs. (2.34), (4.26) and additionally

$$\frac{\partial I_0}{\partial u} = -\frac{1}{\left(u^2 + E^2\right)}, \qquad \frac{\partial I_{12}}{\partial u} = -\frac{1}{\left(u^2 + E^2\right)^2}, \qquad \frac{\partial I_3}{\partial u} = -\frac{1}{u^2\left(u^2 + E^2\right)}. \tag{4.48}$$

In the oblate spheroidal case one can simplify Eqs. (4.39)-(4.41) and obtain the more familiar gravity vector components in the rectangular coordinate system (x, y, z) (see Heiskanen and Moritz, 1967).

4.4. Numerical examples and interpretation

As an example, the numerical values obtained by means of the novel expressions have been computed for the best-fitting planetocentric triaxial ellipsoid representing the

Moon. Following Burša (1994) we took a_x = 1737830 m, a_y = 1737578 m, b = 1737161 m, $GM = 49028\times 10^8$ $m^3 s^{-2}$ and $\omega = 2.6616995\times 10^{-6} s^{-1}$. All algorithms are implemented in MATLAB. To emphasize the one-to-one correspondence between ellipsoidal and Cartesian coordinates eight points were selected each in a different octant. At these points the gravity potential U and the gravity vector components were computed using Eqs. (4.1), (4.8), (4.20) and (4.28)-(4.30). For the numerical integrations an adaptive Simpson rule was used and the results are presented in Table 4.1. As one can see from Table 4.1, the values obtained for the gravity potential are very close to that in Burša and Šíma (1980), i.e. 2825390 m^2/s^2 which refers to the actual gravity field of the Moon.

Table 4.1. Numerical results.

Point	Ellipsoidal coordinates (u, β, λ)	Gravity potential U (m^2/s^2)	γ_u (m/s^2)	γ_β (m/s^2)	γ_λ (m/s^2)
1	(1737161 m, 30°, 20°)	2821614.40	-1.623946	0.000169	0.000069
2	(1737261 m, 40°, 120°)	2821597.64	-1.623804	0.000158	-0.000095
3	(1737461 m, 50°, -130°)	2821292.00	-1.623438	0.000171	0.000121
4	(1737661 m, 60°, -40°)	2820982.21	-1.623070	0.000171	-0.000144
5	(1737361 m, -60°, 45°)	2821483.10	-1.623635	-0.000162	0.000142
6	(1737561 m, -70°, 145°)	2821165.93	-1.623264	-0.000147	-0.000169
7	(1737761 m, -20°, -145°)	2820640.20	-1.622824	-0.000114	0.000093
8	(1737861 m, -30°, -65°)	2820592.91	-1.622673	-0.000135	-0.000078

The equipotential surfaces of the gravitational field have an interesting geometric property. When the gravitational potential is constant, i.e. $V(u, \beta, \lambda) = c$, Eq. (4.8) can be written equivalently as

$$\frac{x^2}{\xi^2}+\frac{y^2}{\eta^2}+\frac{z^2}{\zeta^2}=1, \tag{4.49}$$

where

$$\xi = \sqrt{k/I_1}\ ,\ \eta = \sqrt{k/I_2}\ ,\ \zeta = \sqrt{k/I_3}\ \text{ and }\ k = I_0 - \frac{2}{3}\frac{c}{GM}\,. \tag{4.50}$$

Equations (4.9)-(4.12) and (4.50) show that I_1, I_2, I_3 and *k* are all functions of *u* which, in turn, is a function of the Cartesian coordinates *x*, *y*, *z*, as described in Chapter 2. Thus, Eq. (4.49) describes a level surface of an order higher than second.

5. AN ANALYSIS OF THE LINEAR FIXED ALTIMETRY-GRAVIMETRY BOUNDARY VALUE PROBLEM

5.1. Introduction

The determination of the Earth's external gravity field is usually formulated in terms of various types of Geodetic Boundary Value Problems (GBVPs) for the Laplace equation. Most investigations on GBVPs have been motivated by the need to find more accurate and reliable procedures to handle the variety of available gravity field related data. During the last thirty years, there has been a great deal of interest in studying the so-called Altimetry-Gravimetry Boundary Value Problems (AGBVPs). These take into account that the situation with available terrestrial geodetic data is different over the sea part and the land part of the Earth's surface. Three kinds of the AGBVP have been defined according to the type of input data used and these have been discussed in several papers. Some of them deal with the formulation of the problem, as well as with the existence and uniqueness of the solution, such as e.g. Holota (1983a, b), Sacerdote and Sansò (1983, 1987), Svensson (1983, 1988), Keller (1996) and Lehmann (1999).

Nowadays, with the establishment of the International Terrestrial Reference Frame (ITRF) and the development of the Global Navigation Satellite Systems (GNSS), we can determine very precisely the 3D positions of points on the physical surface of the Earth, which can be considered as a fixed boundary. In this case, the physical surface of the Earth is assumed to be known and the problem is to determine the disturbing potential in the Earth's exterior using two types of main data: (a) in land areas, we can have gravimetric data at points with precisely determined 3D positions which yield surface gravity disturbances, and (b) at seas, we are able to evaluate the disturbing potential with the help of the satellite altimetry and oceanographic data. This situation leads to the formulation of a fixed altimetry-gravimetry (mixed) boundary value problem with a Dirichlet condition at seas and an oblique derivative condition on lands which is also known as AGBVP-III (Lehmann, 1999).

In Keller (1996), this problem is treated primarily in its linearized form, using the concept of weak solutions in functions spaces. The Dirichlet-oblique derivative problem is formulated for an exterior domain and mapped by the Kelvin transform to an internal domain. In the sequel, the weak formulation of the problem is studied and standard theorems of existence and uniqueness are proven. Using various assumptions, a weak solution is shown to be also a classical solution. In the case of the spherical and constant radius approximation of the problem, which can be derived as a special case, the results are much stronger. This Dirichlet-normal derivative (Neumann) problem was treated by Sansò (1993) and it was found that if a solution is looked for in a suitable function space then the problem is well-posed.

In this chapter, we analyze the linear fixed mixed boundary value problem in an unbounded domain representing the exterior of the Earth. The Stampacchia theorem enables us to prove an existence and uniqueness result for the weak solution to the problem. Our considerations are based on the work of Holota (1997) for the linear gravimetric boundary value problem. Also, the linear fixed mixed boundary value problem was addressed numerically by Čunderlík and Mikula (2009).

5.2. Formulation of the problem

In order to define the problem under consideration, let us consider a three-dimensional Euclidean space $\Re^3$ and rectangular Cartesian coordinates x_i, $i = 1, 2, 3$, with the origin at the Earth's centre of mass. We assume that the Earth is a rigid, non-deformable body and that the system of coordinates rotates together with the Earth with a known constant angular velocity around the x_3-axis. We further assume that the problem is independent of time, i.e. not only that there are no changes relative to the Earth but also that there are no changes with respect to the Cartesian coordinate system. In addition, the space outside the Earth is assumed as being empty. For the general point $\boldsymbol{x} = (x_1, x_2, x_3)$, the Euclidean norm is denoted by $|\boldsymbol{x}|$ and the Euclidean inner product of two vectors by $(\cdot)$.

The actual gravity potential of the Earth W is composed of the gravitational potential V generated by the Earth and the centrifugal potential Φ due to the rotational motion of the Earth. The normal gravity potential U corresponds to a mathematical model of the Earth (e.g. a geocentric biaxial ellipsoid) rotating with the same angular velocity as the Earth. The small difference between the actual gravity potential W and the normal gravity potential U (known) is the disturbing potential T (unknown), so that

$$T = W - U. \tag{5.1}$$

In the mass-free exterior of the Earth, the disturbing potential T satisfies the Laplace equation $\Delta T = 0$ (harmonic function) and is regular at infinity. Moreover, the disturbing potential is the quantity to be determined from the available data on the Earth's surface. We assume that for the whole surface of the Earth there is continuous coverage with data.

With gravimetric measurements at land points with precisely determined 3D positions provided by the GNSS we can have the magnitude g of the actual gravity vector $\boldsymbol{g} = \nabla W$ and we can compute the magnitude γ of the normal gravity vector $\boldsymbol{\gamma} = \nabla U$ at the same point, where ∇ it denotes the gradient operator. Thus, we can compute the gravity disturbance δg, i.e.

$$\delta g = g - \gamma, \tag{5.2}$$

where g is corrected for gravitational interaction with the Moon, the Sun and the planets and so on.

Applying the gradient operator in Eq. (5.1), we obtain

$$\nabla T = \nabla (W - U) = \nabla W - \nabla U = \boldsymbol{g} - \boldsymbol{\gamma} = \delta \boldsymbol{g}, \tag{5.3}$$

that is, the gradient of the disturbing potential ∇T equals the gravity disturbance vector $\delta\boldsymbol{g}$.

Defining the unit vector fields

$$\boldsymbol{v} = -\frac{\boldsymbol{g}}{g} \text{ and } \boldsymbol{h} = -\frac{\boldsymbol{\gamma}}{\gamma}, \tag{5.4}$$

the difference between the directions of $\boldsymbol{v}$ and $\boldsymbol{h}$, i.e. the plumb line and the ellipsoidal normal through the same point on the Earth's surface, is the deflection of the vertical. If we now neglect the deflection of the vertical, which implies that the directions of the normals $\boldsymbol{v}$ and $\boldsymbol{h}$ coincide, the gravity disturbance is given by

$$\delta g = g - \gamma = -[(\boldsymbol{v}\cdot\boldsymbol{g}) - (\boldsymbol{h}\cdot\boldsymbol{\gamma})] = -[(\boldsymbol{h}\cdot\boldsymbol{g}) - (\boldsymbol{h}\cdot\boldsymbol{\gamma})] = -(\boldsymbol{h}\cdot\delta\boldsymbol{g}). \tag{5.5}$$

Finally, form the inner product of $\boldsymbol{h}$ and Eq. (5.3), we may write

$$(\boldsymbol{h}\cdot\nabla T) = -\delta g. \tag{5.6}$$

Comparing Eq. (5.6) with Eq. (5.3), we see that the gravity disturbance δg is the normal component of the gravity disturbance vector $\delta\boldsymbol{g}$.

Respectively, in the sea areas, a point P situated on the geoid is projected onto a point Q on the ellipsoid by means of the ellipsoidal normal $\boldsymbol{h}$. Expanding the potential U at P according to Taylor's theorem and truncating the series at the linear term we get

$$U_P \approx U_Q + (\boldsymbol{h}\cdot\nabla U)_Q N = U_Q - \gamma_Q N. \tag{5.7}$$

Here, N is the geoidal height or geoidal undulation, i.e. the distance between the geoid and the reference ellipsoid.

Using Eqs. (5.1) and (5.7) we arrive at

$$T_P = \gamma_Q N + \delta W, \tag{5.8}$$

where

$$\delta W = W_P - U_Q. \tag{5.9}$$

Since we compare the geoid W_o with a reference ellipsoid U_o of the same potential we have $W_P = U_Q = U_o = W_o$. Finally, we obtain the well-known Bruns' formula

$$T_P = \gamma_Q N, \tag{5.10}$$

which relates a physical quantity, the disturbing potential *T*, to a geometric quantity, the geoidal undulation *N* (Heiskanen and Moritz, 1967). Geoidal undulations at sea can be derived through new enhanced mappings of the mean sea surface height of the worlds oceans, derived from a combination of multi-year and multi-satellite altimetry data, in combinations with mean dynamic topography (MDT) models which provide the necessary correction that bridges the geoid and the mean sea surface constraining large-scale ocean circulation.

For the mathematical model describing this physical setting, let $\Omega \subset \Re^3$ be the exterior of the Earth whose boundary $\partial\Omega$ is the surface of the Earth. The boundary of Ω is decomposed into two parts as $\partial\Omega = \partial\Omega_S \cup \partial\Omega_L$, where $\partial\Omega_S \cap \partial\Omega_L = \emptyset$. Here $\partial\Omega_S$ represents the sea part and $\partial\Omega_L$ the land part of the Earth's surface. Under the previous assumptions, the problem is to find a function *u* (disturbing potential *T*) in Ω such that

$$\Delta u = 0 \quad \text{in } \Omega, \tag{5.11}$$

$$u = f_S \quad \text{on } \partial\Omega_S \text{ (sea)}, \tag{5.12}$$

$$(\boldsymbol{h} \cdot \nabla u) = -\delta g \quad \text{on } \partial\Omega_L \text{ (land)}, \tag{5.13}$$

$$u = O(|\boldsymbol{x}|^{-1}) \quad \text{as } |\boldsymbol{x}| \to \infty, \tag{5.14}$$

where $f_S = \gamma_Q N + \delta W$.

The problem, as formulated above, is a linear fixed boundary value problem with a Dirichlet condition on the part $\partial\Omega_S$ of the boundary and an oblique derivative condition on the remainder $\partial\Omega_L$ of the boundary. Also, the boundary $\partial\Omega$ divides the Euclidean space $\Re^3$ into an unbounded domain Ω, the exterior of the Earth, and a bounded domain $\Omega' = \Re^3 - \overline{\Omega}$, the interior of the Earth, where $\overline{\Omega}$ denotes the closure of Ω (i.e. $\overline{\Omega} = \Omega \cup \partial\Omega$). In the remainder of this paper, we shall continue using the symbolism $\boldsymbol{n}$ for the outward unit normal vector of $\partial\Omega'$.

Equation (5.13) represents an oblique derivative boundary condition because in general the normal $\boldsymbol{n}$ to the Earth's surface does not coincide with the direction of the unit vector $\boldsymbol{h}$ defined by Eq. (5.4). Therefore, the oblique boundary condition is more suitable than a normal (Neumann) boundary condition.

In this juncture, it is reasonable to make the assumption that

$$(\boldsymbol{h}\cdot\boldsymbol{n}) \geq c > 0 \qquad \text{on } \partial\Omega_L . \tag{5.15}$$

This implies that the vector field $\boldsymbol{h}$ is non tangential to $\partial\Omega_L$ for all $\boldsymbol{x} \in \partial\Omega_L$.

Let $\boldsymbol{h}$, $\boldsymbol{n}$ and $\boldsymbol{a}$ be continuous vector fields on $\partial\Omega_L$, such that

$$\frac{\boldsymbol{h}}{(\boldsymbol{h}\cdot\boldsymbol{n})} = \boldsymbol{n} + \boldsymbol{a}, \tag{5.16}$$

which, in turn, leads to the following equivalent formulation of the boundary condition given by Eq. (5.13)

$$(\boldsymbol{n}\cdot\nabla u) + (\boldsymbol{a}\cdot\nabla_{\partial\Omega_L} u) = -f_L \qquad \text{on } \partial\Omega_L , \tag{5.17}$$

where $\boldsymbol{a}$ is tangent to $\partial\Omega_L$, i.e. $(\boldsymbol{a}\cdot\boldsymbol{n}) = 0$ on $\partial\Omega_L$, $\nabla_{\partial\Omega_L}$ denotes the gradient operator on $\partial\Omega_L$ and $f_L = \delta g/(\boldsymbol{h}\cdot\boldsymbol{n})$. Furthermore, let $T(\partial\Omega_L)$ be the tangent space of $\partial\Omega_L$. We can find continuous vector fields $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ forming an orthonormal basis of the tangential plane on $\partial\Omega_L$ and generating $T(\partial\Omega_L)$. Thus, $\boldsymbol{a} = \sum_{i=1}^{2}\left[(\boldsymbol{h}\cdot\boldsymbol{e}_i)/(\boldsymbol{h}\cdot\boldsymbol{n})\right]\boldsymbol{e}_i$ and for each differentiable function, defined on $\partial\Omega_L$, we have $\nabla_{\partial\Omega_L} u$ =

$\sum_{i=1}^{2} \boldsymbol{e}_i \cdot (\partial u / \partial \boldsymbol{e}_i)$. In Eq. (5.17), the term $\boldsymbol{a} \cdot \nabla_{\partial \Omega_L}$ can be considered as a perturbation with respect to the main operator $\boldsymbol{n} \cdot \nabla$ (Rozanov and Sansò, 2003).

Finally, we obtain the following equivalent problem

$$\Delta u = 0 \qquad \text{in } \Omega, \tag{5.18}$$

$$u = f_S \qquad \text{on } \partial\Omega_S, \tag{5.19}$$

$$(\boldsymbol{n} \cdot \nabla u) + (\boldsymbol{a} \cdot \nabla_{\partial\Omega_L} u) = -f_L \qquad \text{on } \partial\Omega_L, \tag{5.20}$$

$$u = O(|\boldsymbol{x}|^{-1}) \qquad \text{as } |\boldsymbol{x}| \to \infty. \tag{5.21}$$

5.3. Mathematical background

In this section we briefly present the tools from functional analysis which we used in order to derive the main results of this chapter.

First, we will assume that Ω' is star-shaped domain with respect to the origin, i.e. every half line from the origin meets $\partial\Omega$ in exactly one point. Also, we assume that Ω' is a domain with a $C^{1,1}$ boundary, i.e. it is locally the graph of a function whose derivative is Lipschitz continuous and we can assume that its tangent space $T(\partial\Omega)$ is well defined, as shown e.g. by Raskop and Grothaus (2006). It should be noted, that a $C^{1,1}$ boundary is a special case of a Lipschitz boundary. For a detailed definition of Lipschitz boundary, see Holota (1997).

Next, we define the function spaces which play an important role in the treatment of the problem. $C^i(\Omega)$ is the space of functions, which together with derivatives up to the order i are continuous on Ω. $C^\infty(\Omega)$ is the space of functions with continuous partial derivatives of any order and $C_0^\infty(\Omega)$ is the space of $C^\infty(\Omega)$ functions with compact support in Ω. Finally, $L_2(\partial\Omega)$ is the space of square integrable functions on $\partial\Omega$.

Following Holota (1997), we work with the weighted Sobolev space $W_2^{(1)}$ that is equipped with the inner product given by

$$(u,v)_1 \equiv \int_\Omega \frac{uv}{|\boldsymbol{x}|^2} d\boldsymbol{x} + \int_\Omega (\nabla u \cdot \nabla v) d\boldsymbol{x} . \quad (5.22)$$

This product induces the norm $(u,u)_1^{1/2} \equiv \|u\|_1$. $W_2^{(1)}$ is the space containing functions which are square integrable on Ω under the weight $|\boldsymbol{x}|^{-2}$ and have derivatives of the first order, in a certain generalized sense, which are again square integrable. It should be noted that harmonic functions with their characteristic regularity at infinity belong to $W_2^{(1)}$.

Let H be a real Hilbert space with norm $\| \ \|$ and inner product (,).

The main tool that we will use for deriving an existence and uniqueness result for the weak solution of the problem is known as the *Stampacchia theorem* (Lions and Stampacchia, 1967).

Theorem 1 (Stampacchia). Let A: $H \times H \rightarrow \Re$ be a continuous and coercive bilinear form (not necessarily symmetric), i.e. there exist positive constants c_1 and c_2 such that

$$|A(u,v)| \leq c_1 \|u\| \|v\| \qquad \forall \ u, v \in H \quad (5.23)$$

and

$$A(v,v) \geq c_2 \|v\|^2 \qquad \forall \ v \in H. \quad (5.24)$$

Let K be a non empty, closed and convex subset of H and F be a continuous linear form on H. Then there exists a unique u in K such that

$$A(u, v-u) \geq F(v-u) \qquad \forall \; v \in K. \tag{5.25}$$

Note that when $K = H$, inequality (5.25) is equivalent to $A(u,v) = F(v)$ for all $v \in H$ and this is the result of the well-known Lax-Milgram theorem.

The proof of Theorem 1 can be found in Lions and Stampacchia (1967).

The theorems and lemmas that follow are useful for the purpose of showing the solvability of the problem at hand.

Theorem 2 (equivalent norms). Let Ω be an unbounded domain such that $\Omega' = \Re^3 - \overline{\Omega}$ is a star-shaped domain at the origin with Lipschitz boundary. Then the norms $\|u\|_1$ and

$$\|u\| = \left(\int_\Omega |\nabla u|^2 d\boldsymbol{x}\right)^{1/2} \tag{5.26}$$

are equivalent, i.e. there exist positive constants c_3 and C_3 such that

$$c_3 \|u\|_1 \leq \|u\| \leq C_3 \|u\|_1 \qquad \forall \; u \in W_2^{(1)}. \tag{5.27}$$

The proof of this theorem is shown in Holota (1997).

For $s \in \Re$ the Sobolev space $H^{s,2}(\Re^m)$, $m \in \aleph$ consists of all functions $u \in L_2(\Re^m)$ such that $\int_{\Re^m} (1+|\boldsymbol{x}|)^s |\hat{u}(\boldsymbol{x})|^2 d\boldsymbol{x} < +\infty$, where $\hat{u}$ is the Fourier transform of u.

This space equipped with the norm $\|u\|_{H^{s,2}} = \left(\int_{\Re^m} (1+|\mathbf{x}|)^s |\hat{u}(\mathbf{x})|^2 d\mathbf{x}\right)^{1/2}$ becomes a Hilbert space. Using $H^{s,2}(\Re^m)$ one can (via local coordinates) define the Sobolev spaces for functions and vector fields, on $\partial\Omega$, namely $H^{s,2}(\partial\Omega)$ and $H^{s,2}(\partial\Omega; T(\partial\Omega))$, respectively. $H^{-1/2,2}(\partial\Omega)$ is the dual space of $H^{1/2,2}(\partial\Omega)$ and in the sequel we shall denote by $\langle f, v\rangle$ their duality product. Note that, $L_2(\partial\Omega) \subset H^{-1/2,2}(\partial\Omega)$ and if $f \in L_2(\partial\Omega)$, then $\langle f, v\rangle = \int_{\partial\Omega} fvd\sigma$. Finally, $H^{1,\infty}(\Re^m)$ is the subspace of $L_\infty(\Re^m)$ consisting of functions which have essentially bounded derivatives in a generalized sense and again, by using local coordinates, one can define $H^{1,\infty}(\partial\Omega)$ and $H^{1,\infty}(\partial\Omega; T(\partial\Omega))$. For more detailed explanations, the interested reader is referred to Raskop and Grothaus (2006).

Theorem 3 (trace theorem). Let Ω be an unbounded domain with Lipschitz boundary. Then there exists a continuous linear operator *Z*: $W_2^{(1)} \rightarrow H^{1/2,2}(\partial\Omega)$ such that

$$Z(u) = u\big|_{\partial\Omega} \qquad \forall\ u \in C(\overline{\Omega}) \cap W_2^{(1)} \tag{5.28}$$

and

$$\|Z(u)\|_{H^{1/2,2}(\partial\Omega)} \leq c_4 \|u\|_1 \qquad \forall\ u \in W_2^{(1)}, \tag{5.29}$$

where c_4 is a positive constant. *Z*(*u*) is called the trace of *u* on $\partial\Omega$. In the sequel, in order to simplify the notation we shall use *u* instead of *Z*(*u*). This theorem is explained fully in Raskop (2009).

Lemma 1. One has that

$$\nabla_{\partial\Omega} : H^{1/2,2}(\partial\Omega) \rightarrow H^{-1/2,2}(\partial\Omega; T(\partial\Omega)) \tag{5.30}$$

is continuous, i.e. there exist positive constant c_5 such that

$$\|\nabla_{\partial\Omega} u\|_{H^{-1/2,2}(\partial\Omega;T(\partial\Omega))} \leq c_5 \|u\|_{H^{1/2,2}(\partial\Omega)} \quad \forall\ u \in H^{1/2,2}(\partial\Omega). \tag{5.31}$$

We would like to mention that the proof of this lemma needs at least a $C^{1,1}$ boundary.

Lemma 2. Let $u \in H^{1/2,2}(\partial\Omega)$ and $a \in H^{1,\infty}(\partial\Omega)$. Then for a positive constant $c_6(a)$ we have

$$\|au\|_{H^{1/2,2}(\partial\Omega)} \leq c_6(a)\, \|u\|_{H^{1/2,2}(\partial\Omega)}. \tag{5.32}$$

Lemma 3. The following inequality is valid

$$\left|\langle f, v\rangle\right| \leq \|f\|_{H^{-1/2,2}(\partial\Omega)} \|v\|_{H^{1/2,2}(\partial\Omega)} \quad \forall\, f \in H^{-1/2,2}(\partial\Omega), \quad \forall\, v \in H^{1/2,2}(\partial\Omega). \tag{5.33}$$

Lemma 4. For all u, $v \in H^{1/2,2}(\partial\Omega)$ and $\boldsymbol{a} \in H^{1,\infty}(\partial\Omega; T(\partial\Omega))$ one has that $\mathrm{div}_{\partial\Omega}(\boldsymbol{a}) \in L_\infty(\partial\Omega)$ and

$$\sum_{i=1}^{3} \left\langle v a_i, (\nabla_{\partial\Omega} u)_i \right\rangle = -\frac{1}{2} \int_{\partial\Omega} \mathrm{div}_{\partial\Omega}(\boldsymbol{a}) uv d\sigma. \tag{5.34}$$

The divergence on $\partial\Omega$ is defined by $\mathrm{div}_{\partial\Omega}(\boldsymbol{a}) = \sum_{i=1}^{2} \boldsymbol{e}_i \cdot \left(\partial \boldsymbol{a} / \partial \boldsymbol{e}_i\right)$, where $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is an orthonormal basis of $T(\partial\Omega)$. This definition is independent of the selected basis.

For the proof of the previous lemmas see Raskop and Grothaus (2006).

5.4. Solvability of the problem

A classical solution of a linear fixed mixed boundary value problem corresponding to continuous data f_S, f_L on $\partial\Omega_S$, $\partial\Omega_L$, respectively and continuous vector fields $\boldsymbol{n}$, $\boldsymbol{a}$ on $\partial\Omega_L$, is a function $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, which fulfils Eqs. (5.18)-(5.20) pointwise.

In order to allow weak assumptions on coefficients and data, we are interested in weak solutions. It what follows, we require that $\boldsymbol{a} \in H^{1,\infty}(\partial\Omega; T(\partial\Omega))$, $f_S \in$

$H^{1/2,2}(\partial\Omega_S)$ and $f_L \in H^{-1/2,2}(\partial\Omega_L)$, where all these conditions are related to the "regularity" of the boundary (the first one) and the data (the last two).

We define the sets

$$K = \{u \in W_2^{(1)} : Z(u) = f_S \text{ on } \partial\Omega_S\} \tag{5.35}$$

and

$$V = \{v \in C^\infty(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega_S\}. \tag{5.36}$$

Then the set K is closed and convex by the continuity and the linearity of Z, respectively.

We define the bilinear form for our problem

$$A(u,v) = \int_\Omega (\nabla u \cdot \nabla v) d\boldsymbol{x} - \sum_{i=1}^{3} \left\langle v a_i, (\nabla_{\partial\Omega_L} u)_i \right\rangle \qquad \forall\ u, v \in K \tag{5.37}$$

and the functional

$$F(v) = \left\langle f_L, v \right\rangle \qquad \forall\ v \in H^{1/2,2}(\partial\Omega). \tag{5.38}$$

Lemma 5. Let $u \in K$ be a solution of the variational inequality (5.25). Then

$$A(u,v) = F(v) \qquad \forall\ v \in V. \tag{5.39}$$

Proof. Let $u \in K$ be a solution of inequality (5.25) and $v \in V$. Then

$$Z(u \pm v) = Z(u) \pm Z(v) = Z(u) \pm v\big|_{\partial\Omega_S} = Z(u) = f_S$$

and thus $u \pm v \in K$. Setting $u + v$ in the inequality (5.25), results in $A(u,v) \geq F(v)$ and setting $u - v$, it follows that $A(u,v) \leq F(v)$. Consequently, we have the claimed equality.

Definition 1. The function $u \in W_2^{(1)}(\Omega)$ is called a weak solution of a linear fixed mixed boundary value problem if $u \in K$ and

$$\int_{\Omega} (\nabla u \cdot \nabla v) d\boldsymbol{x} - \sum_{i=1}^{3} \left\langle v a_i, (\nabla_{\partial\Omega_L} u)_i \right\rangle = \left\langle f_L, v \right\rangle \qquad \forall\ v \in V. \tag{5.40}$$

Lemma 6. If $u \in C^2(\overline{\Omega})$ is a weak solution, then it is a classical solution.

Proof. Since $u \in C^2(\overline{\Omega})$ is a weak solution we have that $u \in K$, $Z(u) = f_S$ on $\partial\Omega_S$ and thus $u = f_S$ on $\partial\Omega_S$, i.e. showing the validity of the boundary condition given by Eq. (5.19).

For $u \in C^2(\overline{\Omega})$ and $v \in V$, by the representation of the dual pairing in terms of integrals, we have

$$\sum_{i=1}^{3} \left\langle v a_i, (\nabla_{\partial\Omega_L} u)_i \right\rangle = \int_{\partial\Omega_L} (\boldsymbol{a} \cdot \nabla_{\partial\Omega_L} u) v d\sigma \text{ and } \left\langle f_L, v \right\rangle = \int_{\partial\Omega_L} f_L v d\sigma .$$

Hence, Eq. (5.40) becomes

$$\int_{\Omega} (\nabla u \cdot \nabla v) d\boldsymbol{x} - \int_{\partial\Omega_L} (\boldsymbol{a} \cdot \nabla_{\partial\Omega_L} u) v d\sigma = \int_{\partial\Omega_L} f_L v d\sigma \qquad \forall\ v \in V. \tag{5.41}$$

One can use the Gauss-Green theorem to transform Eq. (5.41) to

$$\int_{\Omega} v \Delta u d\boldsymbol{x} + \int_{\partial\Omega} (\boldsymbol{n} \cdot \nabla u) v d\sigma + \int_{\partial\Omega_L} [(\boldsymbol{a} \cdot \nabla_{\partial\Omega_L} u) + f_L] v d\sigma = 0 \quad \forall\ v \in V. \tag{5.42}$$

Since u has to fulfill this for all $v \in V$ in particular the equality is valid for all $v \in C_0^\infty(\Omega)$. Hence, $\Delta u = 0$ in Ω ($C_0^\infty(\Omega)$ is dense in $L_2(\Omega)$). Combining this result with Eq. (5.42) we get

$$\int_{\partial\Omega_L} [(\boldsymbol{n}\cdot\nabla u) + (\boldsymbol{a}\cdot\nabla_{\partial\Omega_L} u) + f_L] v d\sigma = 0 \qquad \forall\ v \in V. \tag{5.43}$$

Consequently, u has to fulfill also the boundary condition given by Eq. (5.20) and therefore, u is a classical solution (the restrictions of functions belonging to V, on $\partial\Omega$, are dense in $L_2(\partial\Omega)$).

Using Lemma 3 and (trace) Theorem 3 it follows that

$$|F(v)| = |\langle f_L, v\rangle| \le \|f_L\|_{H^{-1/2,2}(\partial\Omega)} \|v\|_{H^{1/2,2}(\partial\Omega)} \le c_4 \|f_L\|_{H^{-1/2,2}(\partial\Omega)} \|v\|_1 \ \forall\ v \in W_2^{(1)}, \tag{5.44}$$

and hence $F(v)$ is a continuous functional on $W_2^{(1)}$.

Using Lemmas 1, 2 and (trace) Theorem 3 we get

$$\begin{aligned}
|A(u,v)| &= \left|\int_\Omega (\nabla u\cdot\nabla v) d\boldsymbol{x} - \sum_{i=1}^{3}\langle v a_i, (\nabla_{\partial\Omega_L} u)_i\rangle\right| \\
&\le \left|\int_\Omega (\nabla u\cdot\nabla v) d\boldsymbol{x}\right| + \left|\sum_{i=1}^{3}\langle v a_i, (\nabla_{\partial\Omega_L} u)_i\rangle\right| \\
&\le \|u\|_1 \|v\|_1 + \sum_{i=1}^{3}\left|\langle v a_i, (\nabla_{\partial\Omega_L} u)_i\rangle\right| \\
&\le \|u\|_1 \|v\|_1 + \sum_{i=1}^{3}\|v a_i\|_{H^{1/2,2}(\partial\Omega)} \|(\nabla_{\partial\Omega_L} u)_i\|_{H^{-1/2,2}(\partial\Omega)} \\
&\le \|u\|_1 \|v\|_1 + \sum_{i=1}^{3} c_6(a_i)\|v\|_{H^{1/2,2}(\partial\Omega)} \|\nabla_{\partial\Omega_L} u\|_{H^{-1/2,2}(\partial\Omega;T(\partial\Omega))} \\
&\le \|u\|_1 \|v\|_1 + \sum_{i=1}^{3} c_5 c_6(a_i)\|v\|_{H^{1/2,2}(\partial\Omega)} \|u\|_{H^{1/2,2}(\partial\Omega)} \\
&\le \left(1 + c_4^2\sum_{i=1}^{3} c_5 c_6(a_i)\right) \|u\|_1 \|v\|_1,
\end{aligned}$$

or

$$\left|A(u,v)\right| \;\le\; c_1 \,\|u\|_1 \,\|v\|_1 \quad \forall\; u,v \in W_2^{(1)} . \tag{5.45}$$

Hence, $A(u,v)$ is continuous on $W_2^{(1)} \times W_2^{(1)}$.

In order to apply Stampacchia's theorem, we only need to show that $A(u,v)$ is coercive on $W_2^{(1)}$.

Using Lemma 4 and Definition 1 we obtain

$$A(v,v) = \int_\Omega \left|\nabla v\right|^2 d\boldsymbol{x} + \frac{1}{2}\int_{\partial\Omega_L} \operatorname{div}_{\partial\Omega_L}(\boldsymbol{a}) v^2 d\sigma . \tag{5.46}$$

Immediately we see that

$$A(v,v) \;\ge\; \int_\Omega \left|\nabla v\right|^2 d\boldsymbol{x} - \frac{1}{2}\left|I\right| , \tag{5.47}$$

where

$$I = \int_{\partial\Omega_L} \operatorname{div}_{\partial\Omega_L}(\boldsymbol{a}) v^2 d\sigma . \tag{5.48}$$

From the continuous and dense embedding (see Raskop and Grothaus, 2006)

$$H^{1/2,2}(\partial\Omega) \;\subset\; L_2(\partial\Omega) \;\subset\; H^{-1/2,2}(\partial\Omega) ,$$

it holds that

$$\|v\|_{L_2(\partial\Omega)} \le c\,\|v\|_{H^{1/2,2}(\partial\Omega)} , \tag{5.49}$$

where c is a positive constant. Thus, we can write the estimate

$$|I| \leq \sup_{\partial\Omega_L} \left|\operatorname{div}_{\partial\Omega_L}(\boldsymbol{a})\right| \|v\|^2_{L_2(\partial\Omega_L)} \leq \sup_{\partial\Omega_L} \left|\operatorname{div}_{\partial\Omega_L}(\boldsymbol{a})\right| c^2 \|v\|^2_{H^{1/2,2}(\partial\Omega_L)} . \tag{5.50}$$

Moreover, using (trace) Theorem 3 it follows that

$$|I| \leq \sup_{\partial\Omega_L} \left|\operatorname{div}_{\partial\Omega_L}(\boldsymbol{a})\right| c^2 \, c_4^2 \, \|v\|_1^2 . \tag{5.51}$$

Combining now the last result with inequality (5.47) and using Theorem 2 (on equivalent norms), we obtain

$$A(v,v) \geq \left(c_3^2 - \frac{1}{2} \sup_{\partial\Omega_L} \left|\operatorname{div}_{\partial\Omega_L}(\boldsymbol{a})\right| c^2 c_4^2 \right) \|v\|_1^2 . \tag{5.52}$$

Supposing that

$$\left|\operatorname{div}_{\partial\Omega_L}(\boldsymbol{a})\right| < \frac{2c_3^2}{c^2 c_4^2} , \tag{5.53}$$

we have that $A(u,v)$ is coercive on $W_2^{(1)}$.

Summing up, we have proved the following theorem:

Theorem 4. Let Ω be an unbounded domain and $\Omega' = \Re^3 - \overline{\Omega}$ be a star-shaped domain at the origin with $C^{1,1}$ boundary. Further let $\boldsymbol{a} \in H^{1,\infty}(\partial\Omega; T(\partial\Omega))$ such that

$$\left|\operatorname{div}_{\partial\Omega_L}(\boldsymbol{a})\right| < \frac{2c_3^2}{c^2 c_4^2} \tag{5.54}$$

holds on $\partial\Omega_L$, where the constants c, c_3 and c_4 are given in inequalities (5.49), (5.27) and (5.29), respectively. Then for all $f_S \in H^{1/2,2}(\partial\Omega_S)$ and $f_L \in H^{-1/2,2}(\partial\Omega_L)$ there exists one and only one weak solution $u \in W_2^{(1)}(\Omega)$ of the linear fixed mixed boundary value problem.

Condition given by inequality (5.54) has a geometrical interpretation. Generally, the direction of ***h*** does not differ too much from that of ***n***. For example, taking as $\partial\Omega$ the surface of the geoid, the angle between the vector ***h*** and the vector ***n*** can be estimated by several tens of seconds of arc. Under this assumption we have

$$\left|\mathrm{div}_{\partial\Omega_L}(\boldsymbol{a})\right| \approx \left|2\left(k_e - k_{\partial\Omega_L}\right)\right|, \tag{5.55}$$

where k_e is the mean curvature of the ellipsoid and $k_{\partial\Omega_L}$ is the mean curvature of $\partial\Omega_L$ (Holota, 1997; Rozanov and Sansò, 2003). Hence, condition given by inequality (5.54) results in

$$\left|k_e - k_{\partial\Omega_L}\right| < \frac{c_3^2}{c^2 c_4^2}. \tag{5.56}$$

Lastly, it should be noted that, if $\partial\Omega$ represents the Earth's surface (though smoothed to a certain degree) the estimate of $\mathrm{div}_{\partial\Omega_L}(\boldsymbol{a})$ is a rather difficult problem which strongly depends on the slopes and curvatures of $\partial\Omega_L$. Typically ***h*** is the opposite direction of the normal gravity and directed fairly close to the normal ***n*** to the Earth's surface. In turn the magnitude of ***a*** is small in the average (apart from some extreme cases in mountainous areas). In the case that ***h*** = ***n*** we can put ***a*** = **0** and thus, clearly, the resulting Dirichlet-Neumann problem has a unique solution. For the geometrical interpretation, a detailed analysis can be found in Holota (1997) and Rozanov and Sansò (2003).

6. AN APPROACH TO THE HEIGHT DATUM UNIFICATION PROBLEM

6.1. Introduction

For a long time, Mean Sea Level (MSL) has been regarded as the reference surface for heights. MSL expresses a state of gravitational equilibrium and is generally determined as the average height of the ocean's surface measured by long-term sea level observations in one or several tide gauges (Zhang et al., 2009). However MSL is not an equipotential surface of the Earth's gravity field, because in reality, due to currents, air pressure, temperature and salinity variations, etc., this does not occur, not even as a long term average. Therefore, different height datums refer to different equipotential surfaces, and consequently there exist various off-sets between different local height datums with respect to the chosen "reference surface". In addition, the MSL and the geoid are not the same. The geoid describes the irregular shape of the Earth and is the true zero surface for measuring elevations, since it is an equipotential surface of the Earth's gravity field that approximates the global MSL in the least squares sense. The deviation between MSL and the geoid can vary globally in as much as ±2 m and is often referred to as stationary Sea Surface Topography (SST), (Ardalan and Safari, 2005). In some oceanic regions, like the equatorial areas, the assumptions about a stationary SST do not hold, and consequently the marine geoid in these areas has to be computed separately (in patches) for different zones that cannot be directly connected. Therefore, what is defined as "zero elevation" in one region is not the same zero elevation defined in another region, which is why locally defined height datums differ from each other and need to be inter-connected, e.g. through GNSS.

Ideally, a global height datum conforming to the modern accuracy standards is required in order to serve many of the tasks of geodesy today, such as: to study SST at different tide gauges, construct regional or global geospatial information systems, monitor global climate changes by measuring long-term MSL variations, reduction in polar ice-cap volumes, post-glacial rebound and land subsidence studies, compute reliable estimates of ocean currents, etc. All of these applications require a global view of the Earth with measurements not only on land, but over the oceans as well (Fotopoulos, 2003).

In this chapter, we approach this height datum unification problem through the determination of potential difference between two (or more) local height datums based on the linear fixed altimetry-gravimetry (mixed) boundary value problem as outlined previously. This allows obtaining the quasigeoid (instead of the geoid) which, although is not a level surface (in continental areas), and therefore, has no physical meaning, is a computationally convenient reference surface that is independent of any local height datums and can be regarded as a global height datum.

6.2. Approaches for determining potential differences

In general, there are three main approaches that can be followed in order to determine potential differences: (i) the classical, (ii) the oceanographic and (iii) the Geodetic Boundary Value Problem (GBVP) approach.

In the classical approach, potential differences can be determined by spirit levelling combined with gravity measurements. This involves a process that is repeated in a leap-frog fashion to produce elevation differences between established bench marks that comprise the vertical control network in the area of interest. When considering an arbitrary point P_o at sea level and another point P connected to P_o, the potential difference between P and P_o can be determined as

$$C = C(P,\ P_o) = W(P_o) - W(P) = W_o - W_P = -\int_{P_o}^{P} dW = -\int_{P_o}^{P} g\,dn\,, \qquad (6.1)$$

where C is known as the geopotential number of P that denotes the difference between the Earth's actual potential $W_o = W(P_o)$ at the geoid and the actual potential $W_P = W(P)$ of the surface on which the point P resides; g and dn denote respectively the average value of actual gravity and the elevation increment between successive benchmarks. Being a difference between geopotential values, the geopotential number

C is independent of the levelling route along which the levelling is run in order to relate the height of point P to the sea level (at point P_o). Geopotential numbers make possible to estimate the orthometric H and the normal H^* height of a point, in the adopted local height datum, by using the following simple relations

$$H = \frac{C}{\bar{g}}, \tag{6.2}$$

$$H^* = \frac{C}{\bar{\gamma}}, \tag{6.3}$$

where $\bar{g}$ is the mean gravity along the actual plumb line from point P_o on the geoid up to point P on the surface of the Earth and $\bar{\gamma}$ is the mean value of the normal gravity from the surface of the Earth down to the quasigeoid along the normal plumb line. True orthometric heights are never achieved since their computation requires knowledge of assumptions about the behaviour of g inside the Earth (e.g. due to variations of the crustal density) where the mass distribution is unknown, and because it is also impossible to measure actual gravity along the plumb line, inside the Earth's topography. Normal heights on the other hand, do not have these problems. Normal gravity can be calculated at any point without any hypotheses, as it is a simple analytical function of position depending only on the defining parameters of the reference level ellipsoid, which generates the normal gravity field. Hence, the normal height of a point P on the physical surface of the Earth can be interpreted as the height above the quasigeoid. The quasigeoid is identical with the geoid over the oceans and is very close to the geoid anywhere else. Its main advantage is that it can be computed rigorously without the necessity to make any hypotheses about the density distribution of the topographic masses, which accompanies the task of geoid determination. Once the quasigeoid is determined, it can be transformed into a geoid by introducing the desired hypothesis about the density of the topographic masses (Heiskanen and Moritz, 1967).

In spite of their obvious shortcomings (e.g. being time consuming, costly, laborious and suffering from problems of accumulation of the errors), this type of definition of height datums might be sufficient for applications of local or regional scale but would cause significant problems, as soon as connections of the height networks of different countries or continents separated by very wide areas and/or by oceans and unification of height datums in global scale are concerned (Colombo, 1980; Rummel and Teunissen, 1988; Xu, 1992).

In the oceanographic approach, geostrophic and steric sea level variation procedures are applied to the problem of determining the potential difference between two (or more) points across widely separated oceanic areas. These potential differences on the sea surface can be estimated from analyses of historical ocean subsurface temperature and salinity observations and/or inferred, for instance, from satellite altimetry merged mean sea anomalies (since 1993) and GRACE gravimetry (more recently) or from tide gauge data (over the past decades). This type of height datum unification is based on the presumption that the ocean acts as a huge level that can connect the zero points of the height datums realized by the reference tide gauges. However, the accuracy of ocean levelling is relatively low, mainly because the phenomena affecting the measuring processes are very complex and difficult to model, but also due to many practical drawbacks, such as: the sparseness of ocean data (salinity, temperature, velocities of ocean currents), the time variability of the ocean, the inadequate knowledge of the ocean mass changes (e.g. due to change in atmospheric water, land hydrology and land ice mass), the non-validity of the geostrophic assumption about ocean currents, the problematic nature of satellite radar altimetry data close to the coast, and the lack of precise regional or local (i.e. non open-ocean) tidal models (Ardalan and Safari, 2005; Zhang et al., 2009).

Under the framework of GBVPs, the potential difference between two (or more) areas can also be applied for height datum unification by introducing the local height datum discrepancies directly into the GBVPs (Rummel and Teunissen, 1988; Lehmann, 2000; Ardalan et al., 2010). Using gravity measurements and levelling, only potential differences can be obtained, whereas the absolute value of the geopotential cannot be obtained at any point with acceptable accuracy. Consequently, the boundary values of the geopotential must be assumed to be known except for one additive constant that must be determined by imposing a suitable additional constraint (Sacerdote and Sansò, 2003). However, these methods require the use of local heights, e.g. in order to calculate the gravity anomalies. Furthermore, they can be affected by inconsistencies in the gravity data coming from different sources, which may have different datums or processed by inconsistent methods. In these cases, such uncertainties can be misinterpreted as height datum discrepancies.

This GBVP approach is the most recent one, and since it represents the starting point of our present work, it is discussed briefly in the sequel, in an effort to highlight what is the most suitable GBVP formulation for determining the sought potential differences among various height datums (i.e. local to global, local indirectly to other local), by estimating the height datum discrepancies as follow up step after the BVP solution.

6.3. Formulations of geodetic boundary value problems

GBVPs represent a well-established basis of the analysis of terrestrial and satellite-based geodetic measurements for inference of the gravity field of the Earth, as well as the quasigeoid or the geoid. The treatment of BVPs has always been used in geodesy as a suitable framework for determining the Earth's disturbing potential *T*. The classical theory of the GBVPs originated initially from the works of G. G. Stokes (Stokes, 1849) and M. S. Molodensky (Molodensky et al., 1962), and was followed, in recent years, by more complicated formulations attempting to approximate the real world more closely, while also dealing with the issues of well-posedness (i.e. existence, uniqueness and continuous dependence of the solution on boundary data). Depending on the type of data, several BVPs can be defined. However, after linearization around a suitable approximate solution, all problems are special cases of a problem for the Laplace equation in the Earth's exterior. The boundary condition associated with the GBVPs, in general, has the form of the so-called fundamental equation of physical geodesy (Heiskanen and Moritz, 1967)

$$\frac{\partial T}{\partial h} - \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} T = -\Delta g \, , \tag{6.4}$$

where T is the disturbing potential, γ is the normal gravity, h is the geometric (ellipsoidal) height, ∂h denotes the partial derivative with respect to the direction of the normal plumb line and Δg denotes the gravity anomalies defined on the boundary surface being considered. This boundary is not the Earth's physical surface, only one of its approximations, that is: the geoid, in the case of the Stokes' approach or the telluroid -a surface in close proximity (of the order of ±100 m) to the Earth's physical surface- in the Molodensky's approach, respectively.

Theoretically, the Stokes' problem requires the knowledge of reduced (to the geoid) gravity anomalies which, in turn, requires the availability of levelling and gravity measurements (i.e. orthometric heights) all over the boundary surface. Respectively, in the Molodensky's approach the telluroid must be known a priori in order to reduce the measured surface gravity anomalies on it, i.e. to compute the corresponding gravity anomaly on the telluroid as

$$\Delta g = g(P) - \gamma(Q), \tag{6.5}$$

where g is the actual gravity at point P on the Earth's surface and Q is a point on the telluroid. Hence, in order to compute the normal gravity γ at the point Q on the telluroid one needs the corresponding normal height H^*. In practice, as the gravity anomaly values Δg must be known on the whole Earth for computing the height anomaly ζ, the length of the ellipsoidal normal between the Earth's surface and the telluroid, there are errors introduced in the computation of ζ because of the off-sets of the levelling datums.

Let's consider S to be the Earth's physical surface and W and $\boldsymbol{g}$ be, respectively, the actual geopotential and gravity vector on this surface. Then there exists a relation

$$\boldsymbol{g} = F(S, W), \tag{6.6}$$

that is, the gravity vector $\boldsymbol{g}$ on S is dependent on the geometry of surface S and the value of the geopotential W on it, and this dependence is expressed by F which is a nonlinear operator.

In the Molodensky's problem the task is to determine S, the Earth's surface, if $\boldsymbol{g}$ and W are given everywhere on it. Formally, we have to solve Eq. (6.6) for S

$$S = F_1(\boldsymbol{g}, W), \tag{6.7}$$

that is, to get geometry from gravity.

Nowadays, as already explained in the previous chapter, the geometry S is considered known, and we can now solve Eq. (6.6) for W

$$W = F_2(S, \boldsymbol{g}), \tag{6.8}$$

that is, to get potential from gravity.

In spite of the similarities between the two approaches, between getting geometry from gravity or getting potential from gravity, there exists a fundamental difference between them: Eq. (6.7) solves a free-boundary problem, since the boundary S covered with boundary data is taken a priori as unknown and “free” to move only in the vertical direction, so that the information about the normal heights is already used a priori in order to fix the boundary, i.e. to obtain the telluroid. By contrast, Eq. (6.8) solves a fixed-boundary problem, since the boundary S is given, so that the realization of normal heights may be controlled by the independently determined quantities h and ζ. In mathematical terms, fixed-boundary problems are usually simpler than free ones.

Within the framework of BVP theory, the (quasi)geoid determination problem is more suitably classified as an altimetry-gravimetry boundary value problem (AGBVP). The most important relevant formulations of AGBVPs or as they are discussed in the literature under the shorter name of “Altimetry-Gravimetry Problems” (AGPs) are summarized in Table 6.1, where besides g and C, another observable at the points of measurements is considered, the geometric (ellipsoidal) heights h determined from precise GNSS positioning, and σ represents, in compact notation, the coordinate pair or solid angle.

Table 6.1. Basic formulations of AGBVPs.

Part of the Earth's surface	Treatment of Parameters	Boundary Value Problem		
		AGP-I	AGP-II	AGP-III
Land	Known	g, σ, C	g, σ, C	g, σ, h
	Unknown	h	h	W
Sea	Known	σ, h, C	g, σ, h	σ, h, C
	Unknown	g	W	g

The type of AGP-I formulation is a favourable approach for global or regional applications, whereby the ellipsoidal heights h being used are determined on the sea surface by satellite radar altimetry, when ship gravity data are not available or their coverage is poor. The AGP-II approach is often used in local areas close to coastlines where there is usually poor steric levelling data, but adequate coverage of ship gravity data and, when geopotential numbers on the sea surface are not available, ellipsoidal heights h are determined on the sea surface by satellite radar altimetry. The AGP-I and AGP-II are free-boundary problems on land and fixed-boundary problems on sea. It has been pointed out in the geodetic literature, e.g. by Lehmann (2000), that the treatment of AGPs in spherical and constant radius approximation leads to mathematically well-posed problems in the case of the AGP-I and AGP-II, while the AGP-I may exhibit features of ill-posedness in special situations. Well-posedness of AGPs is one of the most exciting (and still largely unsolved) problems in geodesy which is usually considered for mathematical analysis.

The AGP-III formulation is currently of interest for hybrid applications whereby, in the sea areas ellipsoidal heights h are determined by satellite altimetry, replacing sea gravity there, and on land, observed ellipsoidal heights h are determined by GNSS, replacing geometric levelling data. In contrast to the AGP-I and AGP-II, the AGP-III is a fixed-boundary problem. Furthermore, this is generally a well-posed BVP, as shown in the previous chapter. Overall, the treatment of a fixed AGP formulation is considered as the most important for the near future, since, in practical terms, this would mean that height information on land could be provided entirely by space techniques rather than by the costly and time consuming conventional geometric levelling procedures.

In summary, considering the distinct features of the AGP-III, that is, being a fixed BVP, suitable of utilizing the data from the modern geodetic technologies (i.e. mixed), and being also a well-posed BVP, our approach to the height datum unification problem is based on the variant formulation outlined in the next section.

6.4. A variant formulation of a fixed mixed BVP

Realization of a unified global height datum, based on the joint processing of terrestrial and satellite geodetic data, admits a variant formulation of the linear fixed mixed boundary value problem. The linear fixed mixed BVP can be mathematically described for each part of the Earth's surface by using the following form

$\Delta T = 0,$ in the 3D space outside the Earth's physical surface

$T = T^{*} + \delta W,$ on sea

$\frac{\partial T}{\partial h} = -\delta g\ ,$ on land

$T = O\left(\frac{1}{r}\right),$ as $r \rightarrow +\infty$

where Δ is the Laplace operator, T is the (unknown) disturbing potential, $\delta g = g - \gamma$ denotes the gravity disturbances that correspond to difference between the gravity data on land (i.e. on the Earth's surface) and the normal gravity from a reference ellipsoid (e.g. GRS80) that can be computed at the same point by knowing its ellipsoidal height; T^{*} represents "observed" values for the disturbing potential (e.g. from satellite altimetry, ship-borne gravimetry, etc. through the application of the well-known Bruns' formula) which requires the dynamic ocean topography to be removed e.g. by ocean levelling; δW is a perturbation of the Dirichlet boundary condition which, in this case, represents the datum disturbance parameter $\delta W = W_o -$

U_o, that is, the difference between the actual (unknown) potential and the normal potential on the surface of the reference ellipsoid (which is also used in the linearization process).

In practice, since W_o is not precisely known, the value U_o is not necessarily equal to the traditionally used theoretical or approximate values of W_o. According to Sánchez (2008), the continuously improving modern geodetic techniques, especially those involving the precise determination of geometrical coordinates by GNSS positioning and satellite altimetry, and the accurate gravity field models provided by the new satellite missions, can now facilitate the accurate estimation of a suitable W_o value by evaluating powerful theoretical approaches that 30 years ago were not applicable in practice. In short, the evaluation of δW can become part of the problem, and numerically a value for it can be obtained using, for instance, the approach shown by Čunderlík and Mikula (2009).

6.5. Outline of proposed method

Based on the previously described AGP formulation, our proposed realization of the height datum unification method can be explained with the simple example illustrated by Fig. 6.1 which shows two equipotential surfaces defined by reference stations (fundamental stations *I* and *II*) in the two local height datums *I* and *II*, respectively.

As long as we select the same reference ellipsoid, the quasigeoid determined by this method would make possible to establish a reference surface that contains middle and high frequency height components, but without reference to any local height datums. Therefore, the height anomalies ζ_0, as obtained from the solution of the previously described boundary value problem can be regarded as a “global” height datum.

On the other hand, let us assume that in the local height datum *I*, for an arbitrary point *A* we know its normal height $H_{A,I}$. The local height anomalies $\zeta_{A,I}$ can be obtained by a combination of GPS/GNSS and levelling data

$$\zeta_{A,I} = h_A - H_{A,I}\,, \tag{6.9}$$

where h_A denotes the ellipsoidal height obtained from GNSS procedures and $H_{A,I}$ corresponds to the normal height from levelling based on the local height datum I involved.

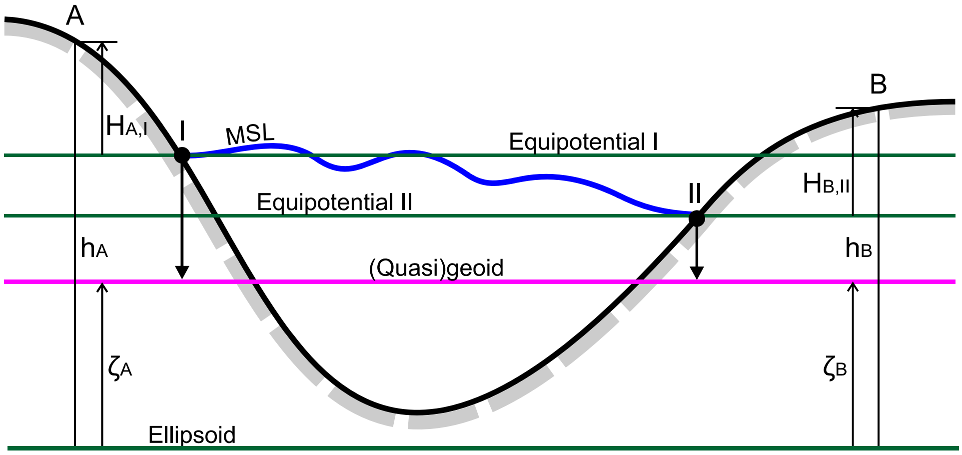


Figure 6.1. Height datum problem.

If common ellipsoidal parameters are adopted for the computation of both local and global height anomalies, we obtain the following equation

$$\delta W_I = \zeta_{A,I}\,\gamma_A - \zeta_{A0}\,\gamma_A = \left(\zeta_{A,I} - \zeta_{A0}\right)\gamma_A = \delta\zeta_{A,I}\,\gamma_A\,, \tag{6.10}$$

where δW_I is the potential difference between the global and local height datum I and ζ_{A0} is the height anomaly for point A as obtained from the solution of the BVP. Note that, local height anomalies $\zeta_{A,I}$ and the height anomalies ζ_{A0} must correspond to same point A on the Earth surface. Similar equations to Eqs. (6.9) and (6.10) hold for an arbitrary point B on local height datum II.

Considering the case of two local height datums, if we calculate their datum potential differences to the global datum individually, using Eq. (6.10), the potential difference between two local height datums shall be given as

$$W_I - W_{II} = \delta W_{I,II} = \delta\zeta_{A,I}\,\gamma_A - \delta\zeta_{B,II}\,\gamma_B\,, \tag{6.11}$$

where W_I and W_{II} represent the potential of the respective local height datums *I* and *II*, $\delta W_{I,II}$ is the potential difference between the two local height datums and $\delta\zeta_{A,I}$ and $\delta\zeta_{B,II}$ represent the height differences between global height datum and local height datum at points *A* and *B*, respectively.

In practice, this process could be applied to many points on the local height datum *I*, in order to estimate a mean value $\delta\overline{W}_I$ (the potential difference between the global height datum and the local height datum *I*) and its standard deviation. Similarly, the process could be applied to many points on the local height datum *II*, in order to estimate a mean value $\delta\overline{W}_{II}$ (the potential difference between the global height datum and the local height datum *II*) and its standard deviation. Finally, we can estimate the potential difference $\delta\overline{W}_{I,II}$ between the two local height datums *I* and *II*. This same process can be applied for many local height datums, i.e. by applying Eq. (6.10), and subsequently, the mutual relation between any pair of local height datums can be carried out by applying Eq. (6.11). Therefore, a full unification can be realized in this truly integrated way.

7. CONCLUSIONS AND RECOMMENDATIONS

7.1. Conclusions and summary of contributions

In this thesis, various fundamental geodetic problems were extensively studied, in the framework of boundary value problems and using an ellipsoidal geometry. The main conclusions resulting from this research and a summary of contributions are as follows:

In Chapter 2, the ellipsoidal coordinate system is presented. It is shown that the needed one-to-one correspondence between ellipsoidal and Cartesian coordinates of points in space can be obtained in two ways. From Eqs. (2.16)-(2.18), in order to determine a point in space we need to know not merely the values of its coordinates ρ, μ and ν, but the signs of various quantities as well. By contrast, from Eqs. (2.27)-(2.29) for the determination of a point we only need to know the ellipsoidal coordinates (u, β, λ). This underlying property of the transformation is due to the trigonometric functions which are elementary and simple, instead of the elliptic and theta functions which, as shown in the literature, are used for many similar problems. Furthermore, expressing the transformation in terms of ellipsoidal parameter u, ellipsoidal latitude β and ellipsoidal longitude λ is more relevant to applications concerning celestial bodies. Also, a way to compute the ellipsoidal coordinates from the Cartesian coordinates of a given point is presented.

In Chapter 3, the geodesic problem on an ellipsoid is solved as a boundary value problem. From its solution, the ellipsoidal coordinates at any point along the geodesic can be determined, making this method a convenient approach for plotting a geodesic between two given points on an ellipsoid. For a biaxial ellipsoid, the numerical tests show that the solutions practically agree with Vincenty's solution. Hence, our method can be used to validate Vincenty's method and, in addition, to provide an accurate solution to the geodesic problem even in extreme cases, such as between points nearly antipodal to one another.

In Chapter 4, new analytical expressions of the gravity field due to a triaxial ellipsoid of constant density are presented. Ellipsoidal coordinates (u, β, λ) give the possibility to obtain general expressions applicable to an ellipsoidal or spheroidal body. The resulting elliptic integrals can be computed using numerical methods.

In Chapter 5, Stampacchia's theorem gave us the possibility to obtain existence and uniqueness of a weak solution of our linear fixed mixed boundary value problem. As a consequence, via Theorem 4 we can solve the mixed problem for more general boundaries (not only for spherical and ellipsoidal boundaries) and for a broader set of functions.

In Chapter 6, we propose the use of a fixed mixed BVP for attacking the classic height datum unification problem. The main advantage of this approach is that it is independent of any local height datum and makes use of all modern geodetic measurements (e.g. satellite altimetry at sea and GNSS-based geometric heights on land). The main outcome of the method is the potential differences between each local height datum and the global height datum realized through the solution of the aforementioned BVP that leads to the estimation of the quasigeoid. A comparison of potential differences from different height datums will then yield information on their relative vertical positions.

The accomplishments and contributions of this study with regard to the aforementioned geodetic problems are six fold and summarized as follows:

I) For the ellipsoidal coordinates introduced by Tabanov (1999), we have given the geometrical interpretation. We have presented a way to compute the ellipsoidal from the Cartesian coordinates. Also, we have expressed the Laplace's equation in these coordinates (Panou, 2014).

II) We have treated the geodesic problem on an ellipsoid as a boundary value problem (Panou, 2013; Panou et al., 2013a).

III) We have given a method for solving the geodesic problem for an ellipsoid by directly integrating the system of ordinary differential equations for a geodesic (Panou, 2013; Panou et al., 2013a).

IV) We have developed a general formula for the gravity field due to a homogeneous ellipsoid, oblate spheroid and sphere (Panou, 2014).

V) We have analyzed the linear fixed altimetry-gravimetry boundary value problem with respect to the existence and uniqueness of the solution, using Stampacchia's theorem (Panou et al., 2013b).

VI) We have proposed the solution of the linear fixed altimetry-gravimetry boundary value problem for solving the height datum unification problem (Panou and Delikaraoglou, 2013).

7.2. Future works

In geodetic applications, it is very useful to connect the ellipsoidal coordinates with the geodetic (planetographic) coordinates on a triaxial ellipsoid. Such a connection between ellipsoidal and geodetic coordinates in the oblate spheroidal case has been presented by Featherstone and Claessens (2008). Finally, by setting $u = b$ in the transformation (2.27)-(2.29), the resulting parameterization of the ellipsoid (3.1) may be useful in many geometrical applications, such as the derivation of ellipsoidal map projections and the determination of other characteristic curves on an ellipsoid.

The presented method for solving the geodesic problem on a triaxial ellipsoid does not include some special cases, which warrant further study. These are: a) geodesics having a length that exceeds the limit of Eq. (3.49), such that there are more than one between the given two points and the shortest path must be determined; b) geodesics that pass between the umbilical points ($\beta = \pm 90°$); and c) the umbilical geodesics (see GeographicLib). Also, the method uses ellipsoidal coordinates because these constitute the orthogonal set of parametric curves on a triaxial ellipsoid. On the other hand, in the geodetic applications, the geodetic coordinates are used. Therefore, there is a need for a transformation between the two sets of coordinates.

The theory presented for the gravity field due to a homogeneous ellipsoid can be extended in many respects. One can obtain expressions for the gravity field in the interior of a triaxial ellipsoid, homogeneous or composed of confocal ellipsoidal shells of different density. In addition, the use of the ellipsoidal coordinates (u, β, λ) allows the separation of variables in Laplace's equation, so one can formulate several boundary value problems, like the gravity field of a level triaxial ellipsoid. Also, it is possible to transform other expressions of the potential theory, which involve the ellipsoidal coordinates (ρ, μ, ν), to the coordinates (u, β, λ) using the substitutions (2.35). This gives the opportunity to apply mathematical tools on a triaxial ellipsoid (e.g. ellipsoidal harmonics) to the case of an oblate spheroid, which is traditionally related to the shape of the Earth.

In the analysis of the linear fixed altimetry-gravimetry boundary value problem, we have used Stampacchia's theorem. Its main advantage is that allows us to treat directly the mixed boundary value problem, therefore it can be used in similar geodetic problems of mixed type (such as the Dirichlet-Robin problem). In addition, it can be used for bilinear forms which are not necessarily symmetric, as in the case of our problem. Finally, it should be mentioned that the regularity of the data and the resulting improvement of the solution remains an important issue that needs further attention.

The author hopes that the developments presented in this work will direct new research into the various aspects dealing with the geodetic use of a triaxial ellipsoid. Also, the generalization of geodetic solutions from a spherical or spheroidal geometry to ellipsoidal geometry would present a challenge to the global geodetic community.

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APPENDIX

A.1 Triaxial case

$$\mathrm{B}' = \frac{d\mathrm{B}}{d\beta} = \frac{a_x^2 E_y^2 \sin 2\beta}{\left(E_x^2 - E_y^2 \sin^2 \beta\right)^2} \tag{A1a}$$

$$\Lambda' = \frac{d\Lambda}{d\lambda} = -\frac{b^2 E_e^2 \sin 2\lambda}{\left(E_x^2 - E_e^2 \cos^2 \lambda\right)^2} \tag{A1b}$$

$$\mathrm{B}'' = \frac{d^2\mathrm{B}}{d\beta^2} = \frac{2a_x^2 E_y^4 \sin^2 2\beta}{\left(E_x^2 - E_y^2 \sin^2 \beta\right)^3} + \frac{2a_x^2 E_y^2 \cos 2\beta}{\left(E_x^2 - E_y^2 \sin^2 \beta\right)^2} \tag{A2a}$$

$$\Lambda'' = \frac{d^2\Lambda}{d\lambda^2} = \frac{2b^2 E_e^4 \sin^2 2\lambda}{\left(E_x^2 - E_e^2 \cos^2 \lambda\right)^3} - \frac{2b^2 E_e^2 \sin 2\lambda}{\left(E_x^2 - E_e^2 \cos^2 \lambda\right)^2} \tag{A2b}$$

First-order partial derivatives:

$$\overline{E}_\beta = \frac{\partial \overline{E}}{\partial \beta} = \mathrm{B}'\left(E_y^2 \cos^2 \beta + E_e^2 \sin^2 \lambda\right) - \mathrm{B} E_y^2 \sin 2\beta \tag{A3a}$$

$$\overline{E}_\lambda = \frac{\partial \overline{E}}{\partial \lambda} = \mathrm{B} E_e^2 \sin 2\lambda \tag{A3b}$$

$$\overline{G}_\beta = \frac{\partial \overline{G}}{\partial \beta} = -\Lambda E_y^2 \sin 2\beta \tag{A3c}$$

$$\overline{G}_\lambda = \frac{\partial \overline{G}}{\partial \lambda} = \Lambda'\left(E_y^2 \cos^2 \beta + E_e^2 \sin^2 \lambda\right) + \Lambda E_e^2 \sin 2\lambda \tag{A3d}$$

Second-order partial derivatives:

$$\overline{E}_{\beta\beta} = \frac{\partial^2 \overline{E}}{\partial \beta^2} = \mathrm{B}''\left(E_y^2 \cos^2 \beta + E_e^2 \sin^2 \lambda\right) - 2\mathrm{B}' E_y^2 \sin 2\beta - 2\mathrm{B} E_y^2 \cos 2\beta \quad \text{(A4a)}$$

$$\overline{E}_{\lambda\lambda} = \frac{\partial^2 \overline{E}}{\partial \lambda^2} = 2\mathrm{B} E_e^2 \cos 2\lambda \quad \text{(A4b)}$$

$$\overline{G}_{\beta\beta} = \frac{\partial^2 \overline{G}}{\partial \beta^2} = -2\Lambda E_y^2 \cos 2\beta \quad \text{(A4c)}$$

$$\overline{G}_{\lambda\lambda} = \frac{\partial^2 \overline{G}}{\partial \lambda^2} = \Lambda''\left(E_y^2 \cos^2 \beta + E_e^2 \sin^2 \lambda\right) + 2\Lambda' E_e^2 \sin 2\lambda + 2\Lambda E_e^2 \cos 2\lambda \quad \text{(A4d)}$$

Second-order mixed derivatives:

$$\overline{E}_{\beta\lambda} = \frac{\partial}{\partial \lambda}\left(\frac{\partial \overline{E}}{\partial \beta}\right) = \frac{\partial}{\partial \beta}\left(\frac{\partial \overline{E}}{\partial \lambda}\right) = \overline{E}_{\lambda\beta} = \mathrm{B}' E_e^2 \sin 2\lambda \quad \text{(A5a)}$$

$$\overline{G}_{\beta\lambda} = \frac{\partial}{\partial \lambda}\left(\frac{\partial \overline{G}}{\partial \beta}\right) = \frac{\partial}{\partial \beta}\left(\frac{\partial \overline{G}}{\partial \lambda}\right) = \overline{G}_{\lambda\beta} = -\Lambda' E_y^2 \sin 2\beta \quad \text{(A5b)}$$

A.2 Biaxial case

In the biaxial case where $a_x = a_y \equiv a$ i.e., $E_x = E_y \equiv E$ and $E_e = 0$, Eqs. (A1)-(A5) are written as

$$\mathrm{B}' = \frac{d\mathrm{B}}{d\beta} = \frac{a^2 \sin 2\beta}{E^2 \cos^4 \beta} \quad \text{(A6a)}$$

$$\Lambda' = \frac{d\Lambda}{d\lambda} = 0 \quad \text{(A6b)}$$

$$B'' = \frac{d^2 B}{d\beta^2} = \frac{2a^2 \sin^2 2\beta}{E^2 \cos^6 \beta} + \frac{2a^2 \cos 2\beta}{E^2 \cos^4 \beta} \tag{A7a}$$

$$\Lambda'' = \frac{d^2 \Lambda}{d\lambda^2} = 0 \tag{A7b}$$

First-order partial derivatives:

$$\overline{E}_\beta = \frac{\partial \overline{E}}{\partial \beta} = B' E^2 \cos^2 \beta - B E^2 \sin 2\beta \tag{A8a}$$

$$\overline{E}_\lambda = \frac{\partial \overline{E}}{\partial \lambda} = 0 \tag{A8b}$$

$$\overline{G}_\beta = \frac{\partial \overline{G}}{\partial \beta} = -\Lambda E^2 \sin 2\beta \tag{A8c}$$

$$\overline{G}_\lambda = \frac{\partial \overline{G}}{\partial \lambda} = 0 \tag{A8d}$$

Second-order partial derivatives:

$$\overline{E}_{\beta\beta} = \frac{\partial^2 \overline{E}}{\partial \beta^2} = B'' E^2 \cos^2 \beta - 2B' E^2 \sin 2\beta - 2B E^2 \cos 2\beta \tag{A9a}$$

$$\overline{E}_{\lambda\lambda} = \frac{\partial^2 \overline{E}}{\partial \lambda^2} = 0 \tag{A9b}$$

$$\overline{G}_{\beta\beta} = \frac{\partial^2 \overline{G}}{\partial \beta^2} = -2\Lambda E^2 \cos 2\beta \tag{A9c}$$

$$\overline{G}_{\lambda\lambda} = \frac{\partial^2 \overline{G}}{\partial \lambda^2} = 0 \tag{A9d}$$

Second-order mixed derivatives:

$$\overline{E}_{\beta\lambda} = \frac{\partial}{\partial\lambda}\left(\frac{\partial\overline{E}}{\partial\beta}\right) = \frac{\partial}{\partial\beta}\left(\frac{\partial\overline{E}}{\partial\lambda}\right) = \overline{E}_{\lambda\beta} = 0 \tag{A10a}$$

$$\overline{G}_{\beta\lambda} = \frac{\partial}{\partial\lambda}\left(\frac{\partial\overline{G}}{\partial\beta}\right) = \frac{\partial}{\partial\beta}\left(\frac{\partial\overline{G}}{\partial\lambda}\right) = \overline{G}_{\lambda\beta} = 0 \tag{A10b}$$

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September 2004 – July 2009: Undergraduate student, Department of Surveying Engineering, National Technical University of Athens.

2004: Completion of studies in High School. Graduation with "Excellent" (18.9/20).

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Diploma Thesis in the Department of Surveying Engineering of the National Technical University of Athens entitled "Determination of Astronomical Latitude Using a Self-Calibration Method". Supervisor Dr. R. Korakitis.

AWARDS – SCHOLARSHIPS – DISTINCTIONS

2013: Academy of Athens. Received the 2013 prestigious "D. Lambadarios Award", which is accompanied by a prize of 5.000 € and represents the highest honour bestowed by the Academy of Athens to the recipient who is innovating in the areas of Geodetic Science.

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2011-2014: Graduate Studies Scholarship from the National Technical University of Athens for PhD studies.

2010: "Thomaidio Award for the Best Diploma Thesis" in the National Technical University of Athens for the year 2009.

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- Member of the European Geosciences Union (E.G.U.) (2011 – to date).

PARTICIPATION IN RESEARCH PROJECTS

Principal New Scientist in the “Re-formulation of the altimetric - gravimetric Geodetic Boundary Value Problem (Altimetry - Gravimetry Geodetic Boundary Value Problem) and investigation of its solvability (in terms of existence - uniqueness) in order to improve the solution” project under the Basic Research Programme 2010, National Technical University of Athens. Duration: 1/12/2010 - 30/11/2012. Funding: 15.000 €.

FOREIGN LANGUAGES KNOWN & COMPUTER SKILLS

- English.
- Excellent knowledge of Windows - Office software suite (Word, Excel, PowerPoint, Access).
- Programming in Matlab.
- CAD Software: AutoCAD.
- GIS Software: ArcGIS.
- Very good knowledge of image processing tools (CorelDRAW, Photoshop).
- Technical Typing (TeX, LaTex).

RESEARCH INTERESTS

- Geometrical Geodesy.
- Physical Geodesy.
- Geodetic Boundary Value Problems.

PUBLICATIONS

PAPERS IN REFEREED JOURNALS

- Panou G., 2014. The gravity field due to a homogeneous triaxial ellipsoid in generalized coordinates. *Studia Geophysica et Geodaetica*, Accepted 06/05/2014.
- Panou G., 2013. The geodesic boundary value problem and its solution on a triaxial ellipsoid. *Journal of Geodetic Science*, **3**, 240–249.
- **Panou G.**, Delikaraoglou D. and Korakitis R., 2013. Solving the geodesics on the ellipsoid as a boundary value problem. *Journal of Geodetic Science*, **3**, 40–47.
- **Panou G.**, Yannakakis N. and Delikaraoglou D., 2013. An analysis of the linear fixed altimetry-gravimetry boundary-value problem. *Studia Geophysica et Geodaetica*, **57**, 203–216.

CONFERENCE PRESENTATIONS – POSTERS

- Panou G., 2014. The oblate spheroidal harmonics under coordinate system rotation and translation. European Geosciences Union, General Assembly, 27 April – 2 May, Vienna, Austria.
- **Panou G.** and Delikaraoglou D., 2013. The gravity field of the level triaxial ellipsoid. European Geosciences Union, General Assembly, 07 – 12 April, Vienna, Austria.
- **Panou G.** and Delikaraoglou D., 2012. Expansion of the gravitational potential in triaxial ellipsoidal harmonics. European Geosciences Union, General Assembly, 22 – 27 April, Vienna, Austria.
- **Panou G.**, Yannakakis N. and Delikaraoglou D., 2011. An analysis of the linear fixed altimetry-gravimetry boundary value problem. European Geosciences Union, General Assembly, 03 – 08 April, Vienna, Austria.

PAPERS IN DEDICATED HONORARY VOLUMES

- **Panou G.** and Delikaraoglou D., 2013. An approach to the height datum unification problem based on a fixed mixed boundary value problem. In: Katsampalos K. V., Rossikopoulos D., Spatalas S. and Tokmakidis K. (Eds.),

On measurements of lands and constructions: Dedicated volume in honor of Professor Emeritus D. G. Vlachos, Ziti editions, Thessaloniki, Greece, pp. 308–318.

- **Panou G.**, Korakitis R. and Lambrou E., 2013. Determination of astronomical latitude using a self-calibration method. In: Katsampalos K. V., Rossikopoulos D., Spatalas S. and Tokmakidis K. (Eds.), *On measurements of lands and constructions*: Dedicated volume in honor of Professor Emeritus D. G. Vlachos, Ziti editions, Thessaloniki, Greece, pp. 142–153, (in Greek).