NATIONAL TECHNICAL UNIVERSITY OF ATHENS
SCHOOL OF NAVAL ARCHITECTURE AND MARINE ENGINEERING
DIVISION OF SHIP & MARINE HYDRODYNAMICS

TWO-TIME RESPONSE EXCITATION THEORY FOR NON LINEAR STOCHASTIC DYNAMICAL SYSTEMS

By

Sevasti Ivi C. Tsantili

PhD Thesis

Thesis Supervisor: Professor Gerassimos A. Athanassoulis

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Στη μητέρα μου Δανάη,
στις θείες μου Αναστασία, Ασημίνα, Νάντα.

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Summary

In the context of probabilistic modeling uncertain parameters/functions are quantified as random. In general, the characterization of random functions is a difficult task involving the knowledge of the hierarchy of the probability distributions of all orders or, equivalently, the knowledge of the characteristic functional. If random parameters/functions enter either as random initial conditions or as input/excitation (external and/or parametric) to dynamical systems, then the systems output/response will also be a random function. In case that the dynamics can be modeled by means of differential equations these equations are called Random Differential Equations (RDEs). The difficulty of calculating the probabilistic characteristics of the response is drastically reduced when we assume that the stochastic excitation is delta correlated. However, this assumption is not plausible when the correlation time of the excitation is of the same order of magnitude as the system's relaxation time, as is the case for macroscopic dynamical systems, e.g. for systems excited by sea waves, wind loads, or earthquakes. In this case the excitation can be realistically modelled by smoothly correlated (colored) random functions. RDEs with colored excitation (also known as generalized Langevin equations) involve an increased amount of complexity due to the fact that in order to obtain system's response probabilistic structure one has to consider infinite dimensional differential equations. Although the general case of smoothly-correlated excitation is the most interesting case for many applications in engineering and applied sciences, existing methodologies fail to treat it in a satisfactory way.

In response to this situation, the response-excitation (RE) theory, a new method for the probabilistic characterization of any non-linear system with any type of smoothly-correlated random excitation, has been recently introduced by Athanassoulis & Sapsis (2006) and Sapsis &Athanassoulis (2008). The RE theory, proposes the joint treatment of the probabilistic structure of the response and the excitation, leaving the space for their stochastic dependence to be determined during the solution of the problem. Athanassoulis and Sapsis used the characteristic functional approach to derive an equation for the joint RE characteristic functional and showed that, by appropriately projecting this infinite dimensional equation, it is possible to obtain equations for the evolution of the joint response-excitation probability density function (REPDF). The derived joint REPDF evolution equation is a peculiar one, involving two times (one for the excitation, s, and one for the response, t), and partial derivatives only with respect to one of them (response time), whereas, after the differentiation, the limit of the excitation time $s \to t$ should be taken. I.e., the REPDF evolution equation includes the "half-time" derivative $\partial f_{x(t)y(s)}(\alpha,\beta)/\partial t\big|_{s\to t}$. This peculiarity gives rise to fundamental questions regarding both the well-posedness and the methods of its numerical solution. While working on this thesis, it became evident that the joint REPDF evolution equation of Athanassoulis and Sapsis in not a closed equation, and thus cannot provide a unique REPDF. The same finding has also been stated recently by Venturi *et al* (2012). This is due to the fact that when the half time limit $\partial f_{x(t)y(s)}(\alpha,\beta)/\partial t\big|_{s\to t}$ is considered the non-local (in time) characteristics of the problem are partially lost. The present work continues the study of the RE theory, aiming at the clarification of various obscure points, and its further development towards the implementation of efficient algorithms for numerical solutions.

In the first part of this thesis, the RE theory, introduced by Athanassoulis and Sapsis, is reviewed and generalized to second-order nonlinear systems. The joint REPDF evolution equation for non-linear dynamical systems under smoothly-correlated stochastic excitation is re-derived, using the characteristic functional approach. To verify the validity of the obtained equations, the latter have been used to re-derive the infinite system of the limiting two-time moment equations, which are also obtained directly from the dynamical system. Finally the joint REPDF evolution equation is specified to the case of the ship roll problem.

Subsequently, a well-studied, simple problem is considered in the context of the RE theory. More precisely, the two-time RE moment equations are developed for a linear scalar dynamical system under colored stochastic excitation. These equations are solved analytically and results are obtained for different stochastic input functions. For Gaussian excitation, a complete analytical solution of the studied problem, both in the transient and in the long-time statistical equilibrium state, is produced. The analytical solution of this simple problem is used in order to verify/clarify the REPDF evolution equation for linear RDEs, and prove that it can have multiple solutions. Thus, the need for an *a priori* closure of the REPDF evolution equation, providing additional information about the RE correlation structure, becomes evident. The formulation and implementation of an efficient closure of this type is one of the fundamental contributions of this Thesis.

The findings from the study of the linear/Gaussian case are generalized for the non-linear/non-Gaussian case, drawing, also, on evidence gained looking into Monte Carlo (MC) simulations results, performed by Z.G. Kapelonis. In fact, in the long-time statistical equilibrium state the joint REPDF tends to concentrate around the equilibrium curve of deterministic problems realized on the RE-phase space. Reclaiming these findings, new, auxiliary, local conditions are developed in the RE-phase space, by the use of local linearizations/Gaussianizations around the equilibrium curve of the non-linear scalar dynamical system in the long-time. These, analytically solvable, local conditions, can successfully approximate the local RE correlation structure as is verified by comparisons with results obtained by MC simulations and, therefore, can be used to form a new *a priori* closure scheme for the non-linear REPDF evolution equation. The analytically obtained local closure information for the RE correlation structure is synthesized in the REPDF evolution equation by the use of an appropriate representation of the two-time joint REPDF, consisting of a superposition of Gaussian Kernels. The reformulated REPDF evolution equation, together with the new local closure conditions, is numerically solved using a Galerkin scheme. This allows for the specific structure of the considered RDE to enter in the Galerkin coefficients both explicitly thought their dependence from the equations of the dynamical system to be solved and implicitly through the Kernel coefficients which contain information from the family of the localized problems. The Galerkin coefficients, having the form of products of polynomials with bi-dimensional Gaussian densities are analytically calculated, and the problem is solved as a constraint minimization problem. This Galerkin scheme has been used for the determination of the joint RE probabilistic characteristics of a half-oscillator, subject to asymptotically stationary, colored, Gaussian or non-Gaussian (cubic Gaussian) excitation. The obtained results are satisfactorily compared with solutions obtained from MC simulations for the same problem.

The selection of the appropriate computational domain for the numerical solution of the joint REPDF evolution equation in the long-time, initiated the development of a new methodology for the formulation and solution of a system of two-time RE moment equations. These equations can apply to any non-linear system with arbitrary polynomial non-linearities, excited by colored Gaussian or polynomially non-Gaussian processes. More precisely, moment equations for the response mean value $m_x(t)$, the two-time RE cross-covariance $C_{xy}(t,s)$, two-time response auto-covariance $C_{xx}(t,s)$ and time-diagonal response auto-covariance $C_{xx}(t,t)$ are derived directly from the dynamical system. A Gaussian closure condition is, then, applied in order to eliminate the higher order moments from the two-time moment equations. Following the Gaussian closure, considering s as a parameter, the derived equations can be considered as linear ODEs with respect to t, having coefficients depended on the time-diagonal moments. The equation for $C_{xy}(t,s)$ is used to express $C_{xy}(t,t)$ as a non-linear, non-local in time (causal) operator on the whole history of $m_x(u)$ and $C_{xx}(u,u)$, $t_0 \leq u \leq t$. Using the obtained operator for $C_{xy}(t,t)$, a closed, non-linear, causal system of evolution equations for $m_x(t)$, $C_{xx}(t,t)$ is obtained. After solving this causal system, the two-time moments can be calculated for all (t,s) pairs as well. Results obtained by the direct solution of the two-time RE moment equations in the long-time, statistical equilibrium limit are presented. Moreover, a first idea on a bi-Gaussian moment closure scheme that could extend the presented methodology to bi-stable half oscillators in the long-time limit is discussed. Obtained results are compared with MC simulations satisfactorily in the mono-stable case. In the bi-stable case the discussed bi-Gaussian moment closure scheme gives acceptable, preliminary, results only for the time-diagonal moments and the two-time RE cross-correlation, whereas, in its present form, fails to approximate the two-time response auto-correlation.

Σύνοψη

Στο πλαίσιο της πιθανοθεωρητικής μοντελοποίησης οι παράμετροι/συναρτήσεις που ενέχουν αβεβαιότητα μοντελοποιούνται ως τυχαίες. Γενικά, ο χαρακτηρισμός των τυχαίων συναρτήσεων είναι μια δύσκολη εργασία καθώς αφορά τη γνώση της ιεραρχίας των κατανομών πιθανότητας όλων των τάξεων ή, ισοδυνάμως, τη γνώση του χαρακτηριστικού συναρτησιακού. Όταν οι αρχικές συνθήκες ή/και η τυχαία είσοδος/διέγερση (εξωτερική ή/και παραμετρική) δυναμικών συστημάτων μοντελοποιούνται από τυχαίες παραμέτρους/συναρτήσεις, τότε η έξοδος/απόκριση του συστήματος θα είναι επίσης μια τυχαία συνάρτηση. Στην περίπτωση που η δυναμική του συστήματος μπορεί να μοντελοποιηθεί με τη χρήση διαφορικών εξισώσεων, τότε οι εξισώσεις αυτές ονομάζονται τυχαίες διαφορικές εξισώσεις (ΤΔΕ). Η δυσκολία του υπολογισμού των πιθανοθεωρητικών χαρακτηριστικών της απόκρισης μειώνεται δραστικά όταν υποθέσουμε ότι η συνάρτηση συσχέτισης της στοχαστικής διέγερσης μπορεί να μοντελοποιηθεί από μια συνάρτηση δέλτα. Εντούτοις, αυτή η υπόθεση δεν είναι ευλογοφανής όταν ο χρόνος συσχέτισης της διέγερσης είναι της ίδιας τάξης μεγέθους με τον χρόνο ηρεμίας του συστήματος, όπως συμβαίνει σε μακροσκοπικά δυναμικά συστήματα π.χ. συστήματα που διεγείρονται από θαλάσσια κύματα, φορτία ανέμου, ή σεισμούς. Σε αυτή την περίπτωση η διέγερση μπορεί να μοντελοποιηθεί ρεαλιστικά από τυχαίες συναρτήσεις με λείες συναρτήσεις συσχέτισης (ομαλή διέγερση). Οι τυχαίες διαφορικές εξισώσεις με ομαλή διέγερση (γνωστές και ως γενικευμένες εξισώσεις Langevin) εμπεριέχουν αυξημένη πολυπλοκότητα λόγω του ότι, προκειμένου να χαρακτηρίσει κανείς πιθανοθεωρητικά την απόκριση, πρέπει να θεωρήσει απειροδιάστατες διαφορικές εξισώσεις. Παρά το γεγονός ότι η γενική περίπτωση τυχαίων διεγέρσεων με λείες συναρτήσεις συσχέτισης παρουσιάζει μεγάλο ενδιαφέρον στη μηχανική και στις εφαρμοσμένες επιστήμες, οι υπάρχουσες μεθοδολογίες αποτυγχάνουν να την αντιμετωπίσουν ικανοποιητικά.

Σε απάντηση αυτής της κατάστασης η θεωρία απόκρισης-διέγερσης (ΑΔ), μια νέα μέθοδος για τον πιθανοθεωρητικό χαρακτηρισμό κάθε μη-γραμμικού συστήματος υπό κάθε τύπου τυχαία διέγερση με λεία συνάρτηση συσχέτισης, εισήχθη πρόσφατα από τους Αθανασούλη & Σαψή (2006) και Σαψή και Αθανασούλη (2008). Η θεωρία ΑΔ, προτείνει την από κοινού αντιμετώπιση της πιθανοθεωρητικής δομής της απόκρισης και της διέγερσης, αφήνοντας χώρο για τον καθορισμό της στοχαστικής τους εξάρτησης κατά την επίλυση του προβλήματος. Οι Αθανασσούλης και Σαψής χρησιμοποιώντας τη μέθοδο του χαρακτηριστικού συναρτησιακού έδειξαν ότι, προβάλλοντας κατάλληλα την απειροδιάστατη εξίσωση, είναι δυνατό να παραχθούν εξισώσεις για την εξέλιξη της από κοινού συνάρτηση πυκνότητας πιθανότητας (σππ) της απόκρισης και της διέγερσης. Η παραχθείσα εξίσωση εξέλιξης της από κοινού σππ απόκρισης και διέγερσης (σππΑΔ) παρουσιάζει ιδιαιτερότητες καθώς περιέχει δυο χρόνους (έναν για την διέγερση, s, και έναν για την απόκριση, t), και μια μερική παραγώγο μόνο ως προς έναν από αυτούς (χρόνο απόκρισης), ενώ, μετά την παραγώγιση, πρέπει να λαμβάνεται το όριο του χρόνου διέγερσης $s \to t$. Δηλαδή η σππΑΔ συμπεριλαμβάνει την παράγωγο «μισού χρόνου» $\partial f_{x(t)y(s)}(\alpha,\beta)/\partial t\big|_{s\to t}$. Αυτή η ιδιαιτερότητα προκάλεσε βασικά ερωτήματα σχετικά με το αν η εξίσωση είναι καλά ορισμένη αλλά και ως προς τη μέθοδο αριθμητικής της επίλυσης. Κατά τη διάρκεια εκπόνησης της παρούσας διατριβής έγινε φανερό ότι η σππΑΔ των Αθανασούλη και Σαψή δεν είναι κλειστή και άρα δεν μπορεί να προσδιορίσει κατά μοναδικό τρόπο την από κοινού σππΑΔ. Το ίδιο εύρημα διατυπώθηκε πρόσφατα από τους Venturi *et al* (2012). Αυτό οφείλεται στο γεγονός ότι όταν λαμβάνεται το όριο «μισού χρόνου» $\partial f_{x(t)y(s)}(\alpha,\beta)/\partial t\big|_{s\to t}$ τα μη-τοπικά (στο χρόνο) χαρακτηριστικά του προβλήματος μερικώς χάνονται. Η παρούσα εργασία συνεχίζει την μελέτη της θεωρίας ΑΔ, με σκοπό να ξεκαθαρίσει κάποια ασαφή σημεία και να την αναπτύξει περαιτέρω αποσκοπώντας στην εφαρμογή αποτελεσματικών αλγορίθμων για αριθμητικές λύσεις.

Στο πρώτο μέρος της εργασίας αυτής, η θεωρία ΑΔ, που εισήχθη από τους Αθανασούλη και Σαψή, επανεξετάζεται και γενικεύεται σε μη-γραμμικά συστήματα δεύτερης τάξεως. Η εξίσωση εξέλιξης της από κοινού σππΑΔ για μη-γραμμικά δυναμικά συστήματα υπό λεία στοχαστική διέγερση παράγεται ξανά με τη χρήση της μεθόδου του χαρακτηριστικού συναρτησιακού. Για να επαληθευτεί η ισχύς των παραχθεισών εξισώσεων, οι τελευταίες χρησιμοποιούνται για την παραγωγή εκ νέου του άπειρου συστήματος οριακών εξισώσεων ροπών δυο χρόνων, οι οποίες μπορούν να παραχθούν και απευθείας

από το δυναμικό σύστημα. Τέλος, η εξίσωση εξέλιξης της από κοινού σππΑΔ συγκεκριμενοποιείται για την περίπτωση του προβλήματος της κίνησης διατοιχισμού πλοίου (ship roll ploblem).

Στη συνέχεια ένα ευρέως μελετημένο, απλό πρόβλημα εξετάζεται στα πλαίσια της θεωρίας ΑΔ. Συγκεκριμένα αναπτύσσονται οι εξισώσεις ροπών ΑΔ δύο χρόνων για ένα γραμμικό βαθμωτό δυναμικό σύστημα υπό ομαλή διέγερση. Αυτές οι εξισώσεις λύνονται αναλυτικά, ενώ αποτελέσματα λαμβάνονται για διαφορετικές περιπτώσεις στοχαστικής διέγερσης. Για κανονική (Gaussian) διέγερση, λαμβάνουμε μια πλήρη αναλυτική λύση του υπό εξέταση προβλήματος, τόσο στην μεταβατική όσο και στην κατάσταση στατιστικής ισορροπίας σε μεγάλους χρόνους. Η αναλυτική λύση αυτού του απλού προβλήματος χρησιμοποιείται στη συνέχεια για να επαληθεύσει/αποσαφηνίσει την εξίσωση εξέλιξης της από κοινού σππΑΔ για γραμμικές ΤΔΕ, και να αποδείξει ότι αυτή δέχεται περισσότερες από μία λύσεις. Επομένως, αναδεικνύεται η ανάγκη συμπλήρωσης της εξίσωσης εξέλιξης της από κοινού σππΑΔ με επιπλέον συνθήκες οι οποίες είναι ικανές να παρέχουν επιπλέον πληροφορία για τη δομή συσχέτισης της απόκρισης και της διέγερσης του συστήματος. Μια από τις θεμελιώδεις συμβολές της παρούσας διατριβής αποτελεί και η ανάπτυξη και εφαρμογή ενός αποτελεσματικού σχήματος που οδηγεί σε κλειστές λύσεις της εξίσωσης εξέλιξης της από κοινού σππΑΔ.

Τα ευρήματα από τη μελέτη της γραμμικής/Gaussian περίπτωσης γενικεύονται στη μη-γραμμική/μη-Gaussian περίπτωση αξιοποιώντας, επιπλέον, ευρήματα που προέκυψαν από τη μελέτη αποτελεσμάτων προσομοιώσεων Monte Carlo (MC) οι οποίες πραγματοποιήθηκαν από τον Ζαχαρία Γ. Καπελώνη. Συγκεκριμένα, σε μεγάλους χρόνους η από κοινού σππΑΔ τείνει να συγκεντρώνεται γύρω από την καμπύλη ισορροπίας των ντετερμινιστικών προβλημάτων που πραγματοποιούνται στο χώρο ΑΔ. Λαμβάνοντας υπόψη τα ευρήματα αυτά, παράγονται νέες τοπικές (στο χώρο απόκρισης-διέγερσης), από κοινού εξισώσεις ροπών ΑΔ δύο χρόνων, χρησιμοποιώντας τοπικές γραμματικοποιήσεις/κανονικοποιήσεις (linearizations/Gaussianizations) γύρω από την καμπύλη ισορροπίας του μη-γραμμικού βαθμωτού συστήματος σε μεγάλους χρόνους. Οι τοπικές εξισώσεις επιλύονται αναλυτικά για διάφορες περιπτώσεις. Τα αποτελέσματα συγκρίνονται ικανοποιητικά με αποτελέσματα από προσομοιώσεις MC και επομένως μπορούν να χρησιμοποιηθούν για να σχηματίσουν ένα νέο σχήμα που συμπληρώνει, *a priori*, τη μη-γραμμική εξίσωση εξέλιξης της από κοινού σππΑΔ. Η πληροφορία για την τοπική δομή συσχέτισης της απόκρισης και της διέγερσης που λαμβάνεται αναλυτικά μέσω των νέων εξισώσεων συντίθεται με την εξίσωσης εξέλιξης της από κοινού σππΑΔ με τη χρήση κατάλληλης αναπαράστασης αποτελούμενης από Gaussian Kernels, η οποία μπορεί να «φέρει» την επιπλέον πληροφορία στην αρχική εξίσωση. Η αναδιαμορφωμένη εξίσωση εξέλιξης της από κοινού σππΑΔ μαζί με τις νέες συμπληρωματικές εξισώσεις επιλύονται αριθμητικά μέσω ενός σχήματος επίλυσης τύπου Galerkin. Με τον τρόπο αυτό εισάγεται η δομή της συγκεκριμένης ΤΔΕ στους συντελεστές Galerkin τόσο άμεσα, μέσω της εξάρτησης τους από την εξίσωση του προς επίλυση δυναμικού συστήματος, όσο και έμμεσα, μέσω των παραμέτρων των Kernel που περιέχουν πληροφορίες από την οικογένεια των τοπικών εξισώσεων. Οι συντελεστές του σχήματος Galerkin, έχουν τη μορφή γινομένων πολυωνύμων με δυσδιάστατες Gaussian κατανομές και μπορούν να υπολογιστούν αναλυτικά, ενώ τελικά το πρόβλημα λύνεται ως πρόβλημα ελαχιστοποίησης υπό περιορισμούς. Αυτό το σχήμα Galerkin χρησιμοποιείται για τον προσδιορισμό των από κοινού πιθανοθεωρητικών χαρακτηριστικών ΑΔ ενός βαθμωτού ταλαντωτή υπό ασυμπωτικά στάσιμη, ομαλή Gaussian ή μη-Gaussian (κυβική Gaussian) διέγερση. Τα αποτελέσματα συγκρίνονται με επιτυχία με αποτελέσματα προσομοιώσεων που παρήχθησαν μέσω MC προσομοιώσεων για το ίδιο πρόβλημα.

Η επιλογή του κατάλληλου υπολογιστικού πεδίου για την αριθμητική επίλυση της εξίσωσης εξέλιξης της από κοινού σππΑΔ σε μεγάλους χρόνους, έδωσε το έναυσμα για την ανάπτυξη μιας νέας μεθοδολογίας για τον σχηματισμό και την επίλυση ενός συστήματος εξισώσεων ροπών δυο χρόνων. Αυτές οι εξισώσεις μπορούν να εφαρμοστούν σε κάθε μη-γραμμικό σύστημα με πολυωνυμικές μη-γραμμικότητες, που διεγείρεται από ομαλές Gaussian ή (πολυωνυμικά) μη-Gaussian τυχαίες διαδικασίες. Συγκεκριμένα, εξισώσεις ροπών για τη μέση τιμή της απόκρισης $m_x(t)$, τη συνάρτηση συνδιακύμανσης ΑΔ δύο χρόνων $C_{xy}(t,s)$, τη συνάρτηση αυτοδιακύμανσης ΑΔ δυο χρόνων $C_{xx}(t,s)$ και τη συνάρτηση αυτοδιακύμανσης στη διαγώνιο των χρόνων $C_{xx}(t,t)$ παράγονται απευθείας από το δυναμικό σύστημα. Η υπόθεση ότι οι τυχαίες συναρτήσεις είναι Gaussian (Gaussian closure condition) τίθεται στη συνέχεια προκειμένου να εξαλειφθούν οι ροπές ανώτερης τάξης από τις εξισώσεις ροπών δυο χρόνων. Μετά την εφαρμογή της υπόθεσης αυτής, θεωρώντας το χρόνο s ως παράμετρο, οι εξισώσεις που παίρνουμε μπορούν να θεωρηθούν ως γραμμικές συνήθεις διαφορικές εξισώσεις ως προς το χρόνο t, έχοντας παραμέτρους που εξαρτώνται από τις ροπές στη διαγώνιο των χρόνων. Η εξίσωση για την $C_{xy}(t,s)$ χρησιμοποιείται για να εκφράσει την $C_{xy}(t,t)$ ως έναν μη-γραμμικό, μη-τοπικό στο χρόνο (αιτιατό) τελεστή πάνω σε όλη την ιστορία των $m_x(u)$ και $C_{xx}(u,u)$, για $t_0 \leq u \leq t$. Χρησιμοποιώντας τον τελεστή για το $C_{xy}(t,t)$ λαμβάνεται ένα κλειστό, μη-γραμμικό αιτιατό σύστημα από εξισώσεις εξέλιξης για τις $m_x(t)$, $C_{xx}(t,t)$. Μετά την επίλυση του αιτιατού συστήματος μπορούν να υπολογιστούν οι ροπές δυο χρόνων για όλα τα ζεύγη (t,s). Παρουσιάζονται αποτελέσματα από την επίλυση των εξισώσεων ροπών ΑΔ δύο χρόνων στην κατάσταση στατιστικής ισορροπίας, σε μεγάλους χρόνους. Επίσης, αποσκοπώντας στην επέκταση της μεθοδολογίας σε βαθμωτούς ταλαντωτές με δυο σημεία ευστάθειας (bi-stable), σε μεγάλους χρόνους, παρουσιάζονται κάποιες πρώτες ιδέες για ένα σχήμα στο οποίο τίθεται, εναλλακτικά, η υπόθεση ότι οι τυχαίες συναρτήσεις είναι μια υπέρθεση από δυο Gaussian τυχαίες συναρτήσεις (bi-Gaussian closure condition). Στην περίπτωση συστημάτων με ένα σημείο ευστάθειας τα αποτελέσματα που λαμβάνονται συγκρίνονται ικανοποιητικά με αποτελέσματα από προσομοιώσεις MC. Στην περίπτωση συστημάτων με δύο σημεία ευστάθειας το υπό συζήτηση σχήμα δίνει αποδεκτά αποτελέσματα για τις ροπές στην διαγώνιο των χρόνων και για τη συνάρτηση διασυσχέτισης ΑΔ δύο χρόνων ενώ, υπό την παρούσα μορφή του σχήματος, οι συναρτήσεις αυτοσυσχέτισης δεν προσεγγίζονται επιτυχώς.

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List of symbols

$x(t;\theta)$: random (or stochastic) response (or output)
$y(t;\theta)$: random (or stochastic) excitation o(or input)
$\boldsymbol{x}(t;\theta)=\left(x_1(t;\theta),x_2(t;\theta),...,x_N(t;\theta)\right)$: vector response random function
$\mathscr{P}_{\boldsymbol{x}(t)}(d\boldsymbol{x})$: infinite dimensional measure of the Borel sets of the sample (functional) Banach space $\mathscr{X}$ of the responses $\boldsymbol{x}(t;\theta)$
$\mathscr{F}_{\boldsymbol{x}(t)}(\boldsymbol{u})$: characteristic functional of the response $\boldsymbol{x}(t;\theta)$
$\delta_{u_1}\mathscr{F}\left(u_1,u_2\right)=\delta_{u_1}\mathscr{F}\left(\left[u_1;h_{u_1}\right],u_2\right)$: first-order Gateaux functional derivatives of $\mathscr{F}_{\boldsymbol{x}(t)}(\boldsymbol{u})$ with respect to the first variable (u_1), taken along the direction h_{u_1}
$F_{\boldsymbol{x}(t)}(\boldsymbol{\alpha})$: probability distribution the response function $\boldsymbol{x}(t)$
$f_{\boldsymbol{x}(t)}(\boldsymbol{\alpha})$: pdf of the response $\boldsymbol{x}(t;\theta)$
$\varphi_{\boldsymbol{x}(t)}(\boldsymbol{u})$: characteristic function of the response $\boldsymbol{x}(t;\theta)$
$f_{x(t+\Delta t)}(\alpha,t+\Delta t\,|\,\alpha',t)$: conditional pdf of the random variable $x(t+\Delta t;\theta)$ given that $x(t;\theta)=\alpha'$
$\varphi_{\Delta x(t)}(u,t+\Delta t\,|\,\alpha',t)$: conditional characteristic function
$a_q\left(t+\Delta t\,|\,\alpha',t\right)=E^{\theta}\left(\left(x\left(t+\Delta t;\theta\right)-x\left(t;\theta\right)\right)^q\,\Big|\,\alpha',t\right)$: conditional incremental moments of the stochastic function $x(t;\theta)$.
$A_q\left(t+\Delta t\,|\,\alpha,t\right)=\lim\limits_{\Delta t\to 0}\dfrac{1}{\Delta t}E^{\theta}\left[\left(x\left(t+\Delta t;\theta\right)-x\left(t;\theta\right)\right)^q\,\Big|\,x\left(t;\theta\right)=\alpha\right]$: *derivative moments* of the stochastic function $x(t;\theta)$

$\boldsymbol{B}(t;\theta)=\left\{B_m(t;\theta),\ m=1,2,...,M\right\}$: $M-$dimensional Wiener Process
$m_{x_0}=\mathrm{E}^{\theta}\left[x_0(\theta)\right]$: initial response mean value
$m_{y_0}=\mathrm{E}^{\theta}\left[y_0(\theta)\right]$: initial excitation mean value
$m_x(t)=\mathrm{E}^{\theta}\left[x(t;\theta)\right]$: response mean value function
$m_x^{(\infty)}=\lim\limits_{t\to\infty}\mathrm{E}^{\theta}\left[x(t;\theta)\right]$:asymptotic (or long-time) response mean value function
$m_y(t)=\mathrm{E}^{\theta}\left[y(t;\theta)\right]$: excitation mean value function
$R_{x_0x_0}=\mathrm{E}^{\theta}\left[x_0(\theta)x_0(\theta)\right]$: initial response auto-correlation function
$R_{y_0y_0}=\mathrm{E}^{\theta}\left[x_0(\theta)x_0(\theta)\right]$: initial excitation auto-correlation function
$R_{yy}(t,s)=\mathrm{E}^{\theta}\left[y(t;\theta)\,y(s;\theta)\right]$: two-time excitation auto-correlation function

$R_{xy}(t,s) = \mathrm{E}^{\theta}\left[x(t;\theta)\,y(s;\theta)\right]$: two-time RE cross-correlation function

$R_{xx}(t,s) = \mathrm{E}^{\theta}\left[x(t;\theta)\,x(s;\theta)\right]$: two-time response auto-correlation function

$R_{xy}^{(\infty)}(\tau) = \lim\limits_{\substack{t\to\infty, s\to\infty \\ t-s=\tau}} \mathrm{E}^{\theta}\left[x(t;\theta)\,y(s;\theta)\right]$: asymptotic (or long-time) RE cross-correlation function

$R_{xx}^{(\infty)}(\tau) = \lim\limits_{\substack{t\to\infty, s\to\infty \\ t-s=\tau}} \mathrm{E}^{\theta}\left[x(t;\theta)\,x(s;\theta)\right]$: asymptotic (or long-time) correlation function

$R_{x\,y}^{j_1 j_2}(t,s) = \mathrm{E}^{\theta}\left[\left(x(t;\theta)\right)^{j_1}\cdot\left(y(s;\theta)\right)^{j_2}\right]$

$R_{xy}^{11}(t,s) = R_{xy}(t,s)$

$R_{x_0\,x}^{1\;k}(t_0,t) = R_{x_0\,x}^{\;\;k}(t) = \mathrm{E}^{\theta}\left[x_0(\theta)\cdot\left(x(t;\theta)\right)^{k}\right]$

$R_{x_0\,x}^{\;\;1}(t) = R_{x_0\,x}(t)$

$C_{x\,y}^{j_1 j_2}(t,s) = \mathrm{E}^{\theta}\left[\left(x(t;\theta) - m_x(t)\right)^{j_1}\cdot\left(y(s;\theta) - m_y(s)\right)^{j_2}\right]$

Similar notation is also used for the central second-order moments (cross-covariances) $C_{x\,y}^{j_1 j_2}(t,s)$, $C_{x_0\,x}^{\;\;k}(t)$, etc, as well as for the asymptotic cross-covariance $C_{xy}^{(\infty)}(\tau), C_{xx}^{(\infty)}(\tau)$. The above notation is unambiguous except for the case $x=y$, $t=s$. In this case the moment function $C_{x\,x}^{j_1 j_2}(t,t)$ can be written by various, apparently different but fully equivalent, ways; all couples (j_1, j_2) with $j_1 + j_2 = j$ represent the same $j-$th order central moment of $x(t;\theta)$. For example,

$C_{xx}^{21}(t,t) = C_{xx}^{12}(t,t) = C_{xx}^{30}(t,t) = C_{xx}^{03}(t,t) = \mathrm{E}^{\theta}\left[\left(x(t;\theta)-m_x(t)\right)^3\right]$

$\tau_{yy}^{\mathrm{corr}} = \dfrac{1}{\sigma_y^2}\displaystyle\int_0^{\infty} |C_{yy}^{(\infty)}(u)|\,du$: correlation time of the random excitation $y(t;\theta)$

$S_{yy}(\omega)$: spectrum of the random excitation $y(t;\theta)$

$f_{x(t)y(s)}(\alpha,\beta)$: joint (two-time) response excitation probability density function (REPDF)

$f_{x(t)x(s)}(\alpha_1,\alpha_2)$: joint (two-time) response probability density function (pdf)

$f_{xx}(x_1, y_1;\tau)$: lag-time response pdf

$f_{x(t)}(\alpha_1)$: response pdf

$f_{y(s)}(\alpha,\beta)$: excitation pdf

$\lim\limits_{s\to t} \partial f_{x(t)y(s)}(\alpha,\beta)/\partial t$: half time derivative of the joint REPDF

$C_{x_{\ell oc}\,y_{\ell oc}}^{(\infty)}(t-s)$: local RE cross-covariance

$C_{x_{\ell oc}\,x_{\ell oc}}^{(\infty)}(t-s)$: local response auto-covariance

$\lim\limits_{s\to t} \partial C_{x_{\ell oc}\,y_{\ell oc}}^{(\infty)}(t-s)\big/\partial t$: half-time derivative of the local RE cross-covariance

$f_{x_{loc}(t)\;y_{loc}(s)}^{(\infty)}(\alpha,\beta)$: local two-time joint REPDF

$K\left(\alpha,\beta\,;\,\alpha_i,\beta_j\,,\Sigma_{\alpha i,\beta j}\right)$: Gaussian Kernel Density function, centered at α_i,β_j, and covariance matrix $\Sigma_{\alpha i,\,\beta j}$

$\Lambda_{\kappa,\lambda}(\alpha\,,\beta)$: Gaussian Galerkin Kernels

$G_{ij\,,\kappa\lambda}$: Galerkin coefficients

D_{ess} : The essential support of the joint REPDF

$D_{\alpha\beta}$: Computational domain of the Galerkin scheme

In Section 2.6:

$x(t;\theta)$: the roll motion (angle),

$I+A$: the inertia coefficient,

$b_1,\ b_3$: the damping coefficients,

$K_1,\ K_3$: hydrostatic coefficients.

In section 4.5:

$f_{xy}(\alpha\,,\beta)=\lim\limits_{t\to\infty} f_{x(t)y(t)}(\alpha\,,\beta)$: the time-independent (statistical equilibrium) joint REPDF.

Abbreviations

Ch.Fnl	Characteristic functional (approach)
FDE	Functional Differential Equation
FPK	Fokker-Planck-Kolmogorov (equation)
KDF	Kernel Density Function
KDR	Kernel Density Representation
KL	Karhunen-Loeve (expansion)
lpGF	Low pass Gaussian filter (a Gaussian random process)
MC	Monte Carlo (simulation)
ODE(s)	ordinary differential equation(s)
OU	Ornsten-Uhlenbeck (process)
pdf(s)	probability density function(s)
RDE(s)	random differential equation(s), i.e., ordinary differential equation(s) driven by smoothly-correlated random functions
RE	response-excitation (theory, moments, phase space)
REPDF	response excitation probability function evolution equation
SDE(s)	stochastic differential equation(s), i.e., ordinary differential equation(s) driven by delta-correlated stochastic functions
sOU	Sifted Ornstein-Uhlenbeck (process)

CHAPTER 1

INTRODUCTION

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1.1. A general survey of probabilistic methods in stochastic dynamics

Within physical sciences and engineering, attempts to explain and predict physical systems are generally based on translating the interactions among their components and the interaction with the environment into mathematical equations. However, due to the system's complexity and/or lack of access/knowledge to all of the involved scales/mechanisms of interactions, the information is often insufficient to successfully model all the involved features using deterministic mathematical modeling. Probabilistic modeling offers a powerful alternative that allows the inclusion and quantification of uncertainty for some of the system's components and/or for the external excitation. Inclusion of the latter to dynamical equations, modeling the laws of physics, enables a better understanding on how these uncertainties act and evolve in time. (see Prigogine 1997, Chapter 1).

In the context of probabilistic modeling uncertain parameters/functions are quantified as random. In general, the quantification of random functions is a difficult task since their complete probabilistic characterization requires the knowledge of the hierarchy of the probability distributions of all orders or, equivalently, the knowledge of the characteristic functional (Hopf 1952), (Kotulski & Sobczyk 1984), (Vakhania et al. 1987). When random parameters/functions enter as random initial conditions or as input/excitation (external and/or parametric) to dynamical systems then the systems output/response will also be a random function. In case that the dynamics can be modeled by means of differential equations these equations are called Random Differential Equations (RDEs). The main goal is to use all the available information on the input probabilities and on the dynamics governing the evolution of the studied system in order to probabilistically characterize the system's response. Of course there are methods that allow to derive a partial probabilistic characterization of the response, fully exploiting the available information concerning the data random functions as the solution of moment equations.

The difficulty of calculating the probabilistic response is drastically reduced when we assume that the stochastic excitation is a delta correlated process, also referred to as white noise (Pugachev & Sinitsyn 2001), (Di Paola & Falsone 1993), (Soize 1994), (Sun 2006). In this context, the response will be a Markovian process that follows the Chapman-Kolmogorov equation and can be completely characterized by its transition probability function (Van Kampen 1998) when, in the most interesting cases (diffusion processes) it takes the form of a probability density function (pdf), and it is governed by the *Fokker-Planck-Kolmogorov* (FPK) equation. Extension of this method to systems subject to other types of noises (e.g. Poisson and Lévy have also been developed (e.g.: (Grigoriu 2004), for a review of FPK equations see e.g.: (Risken 1989)). However, the assumption of delta-correlated excitation is a plausible simplification when the correlation time of the random excitation is much smaller than the system's relaxation time (Lin 1986), (Roberts & Spanos 1986), (Mokshin et al. 2005). The latter is not generally the case for macroscopic dynamical systems, for which the correlation time of the excitation is of the same order of magnitude as the system's relaxation time. This is especially true for many engineering applications, e.g., for systems excited by sea waves, wind loads, or earthquakes. Such cases can be realistically modelled by smoothly correlated random functions, also known as colored random noises. RDEs with colored excitations (also known as *generalized Langevin equations*) involve an increased amount of complexity due to the fact that in order to obtain system's response probabilistic structure one has to consider infinite dimensional differential equations (Hopf 1952)(Beran 1986)(Hanggi 1978)(Luczka 2005)(Sapsis & Athanassoulis 2008).

In cases where the correlation time of the excitation is small but not negligible, i.e. the response is nearly Markovian, it is possible to create an Itô SDE for slowly varying (compared to the fluctuation of the excitation) quantities of the oscillation as the amplitude of the response envelope that can be considered as Markovian processes and pass to the corresponding averaged FPK. This technique, known as the *stochastic averaging method*, was first introduced by Stratonovich (1963) and made rigorous, under clearly stated asymptotic assumptions by Khasminskii (1966). It has been extensively applied to problems in Physics and Engineering. See, e.g., (Lin & Cai 2000), (Luczka 2005), (Ibrahim 1985), (Roberts & Spanos 1986), (Red-Horse & Spanos 1992), (Dostal et al. 2012). This is a useful asymptotic method that can be used to treat cases where the correlation time is different from zero, but small in comparison with the system's relaxation time.

An approach which can resolve the non-Markovian characteristics of the excitation, keeping a close connection with the standard Itô SDE and the FPK equation, is the *filtering approach*. This method is implemented by augmenting the system of dynamical equations with a linear filter, excited by a delta-correlated process and providing as output a process modelling a more realistic, excitation ((Spanos 1983), (Spanos 1986), (Muscolino 1995), (Pugachev & Sinitsyn 2001), (Luczka 2005), (Hu et al. 2012), (Francescutto & Naito 2004)), (Er 2013). For instance a first-order filter has as output an Ornstein-Uhlenbeck (OU) Gaussian process, whereas a second-order filter produces a Gaussian harmonic noise. Such problems can be solved by means of the FPK equation approach or the various generalizations of it. Besides, in this case, it is possible to systematically formulate moment equations up to any order that the available information concerning the input random functions allows. This method is rather general and effective as far as the excitation is Gaussian and the appropriate filter is of low order. For non-Gaussian excitation it is not clear how this method can be applied. One should solve a system identification problem to define a non-linear filter that could have as output a successful approximation of the excitation.

The *method of moments* is another well-known and extensively used method that allows to derive a partial probabilistic characterization of the response. Moment equations can be derived either directly from the random system (Beran 1986), or by FPK and generalized FPK equations describing the evolution of the response density (Jazwinski 1970), (Soong & Grigoriu 1993), (Di Paola & Floris 2008). When the system is linear, the solution of a system of moment equations, allows one to exploit the knowledge of excitation moments up to a specific order to determine the response moments up to the same order (Di Paola & Elishakoff 1996), (Conte & Peng 1996; Lutes & Sarkani 1997; Qiu & Wu 2010). In particular when the system is linear and the excitation is Gaussian, the solution is also Gaussian, and thus, the solution of the moment system provides a complete probabilistic characterization of the problem. However, when the system is non-linear the (truncated) moment system is not closed and, thus, some closure scheme should be invoked. The simplest one is the Gaussian closure, introduced by Goodman and Whittle in the 50's and extensively used thenceforth in the study of random vibrations (Lutes & Sarkani 1997). It has been found that it works well for mono-stable oscillators, while for bi-stable ones may lead to inadequate or erroneous results (Hasofer & Grigoriu 1995; Grigoriu 2008). Also, many types of non-Gaussian closures have been devised and used for treating moment equations coming from nonlinear stochastic systems under delta-correlated excitation. Among them we mention the cumulant-neglect and the quasi-moment neglect closure (Roberts & Spanos 2003), (Wu & Lin 1984), (Lutes & Sarkani 1997), the use of specific parametric models for the underlying pdf (Crandall 1980), (Hampl & Schuëller 1989), (Pugachev & Sinitsyn 2001), (Er 2000), (Er et al. 2011), the information closure (Chang & Lin 2002; Sobczyk & Hołobut 2012) and the polynomial-Gaussian closure (Robson 1981; Anh & Hai 2000).

Another approximate method, formally applicable to any non-linear system under colored Gaussian excitation, is the *equivalent statistical linearization* (Roberts & Spanos 2003), (Socha 2008). The method is simple and easy to apply but with restricted interest. Its main drawback is that it can never give evidence of the characteristic peculiarities on the non-linearity. An interesting improvement of this method is the *local statistical linearization*, introduced by Pradlwarter (2001). In this approach the linearization is performed locally in phase space and, thus, the non-linear characteristics of the dynamical system are not superseded and the obtained pdf reflects adequately the non-linear features of the problem.

Another category of methods of uncertainty quantification are the ones that based on truncated spectral expansions of random input/output functions (Spanos & Ghanem 1989), (Spanos & Ghanem 1991), (Ghanem & Spanos 1990; Ghanem & Spanos 2003). These methods may be referred to as *spectral stochastic Galerkin methods* (other terms used in scientific literature are stochastic Galerkin methods, spectral stochastic finite elements methods). Among other spectral representations, the Polynomial Chaos (PC) expansion (Wiener 1938) (Ghanem & Spanos 2003), in terms of Hermite polynomials, is usually applied to the representation of the response random functions, whereas, the random input is usually represented by a truncated Karhunen-Loeve (KL) expansion (Karhunen 1947; Loeve 1978), (Ghanem & Spanos 2003). A system of linear algebraic equations is then obtained by a Galerkin-type projection onto a complete basis in the space of random variables. Xiu and Karniadakis (2002, 2003) proposed a generalized polynomial chaos expansion using a trial basis from the Askey family of hypergeometric polynomials to account for non-Gaussian effects. Moreover, recently, Sapsis and Lermusiaux (2009) introduced a more general expansion in which the basis is dynamically evolved. Using the condition of *dynamic orthogonality* (DO) Sapsis and Lermusiaux (2009) derived evolution equations for general stochastic systems (including partial differential equations) which can be efficiently solved numerically, see e.g. (Sapsis & Lermusiaux 2012), (Ueckermann et al. 2013), (Sapsis et al. 2013). If the same restrictions for the expansion of the response as in the generalized PC expansion are assumed on the *dynamically orthogonal field equations* the generalized PC equations can be recovered.

An interesting circle of ideas for studying the probabilistic response of non-linear dynamical systems under general excitation has also been developed, on the basis of the Karhunen-Loeve Theorem. The fundamental idea is to replace the given random functions, entering into the stochastic system, by their Karhunen-Loeve expansions, reducing the initial problem to a problem involving only stochastic variables. Then, in principle, the evolution of the joint, response-excitation pdf is governed by a high dimensional *Liouville type equation***,** also known as *Dostupov-Pugachev equation*. In Venturi et al (Venturi et al. 2012) the sparse grid collocation method (Foo & Karniadakis 2008; Foo & Karniadakis 2010) is used to find the joint response-excitation pdf by the numerical solution of the Dostupov-Pugachev equation, whereas, in Cho et al. (2013) the same equation is solved by considering the response and the excitation space separately, using the sparse-grid collocation method for excitation space and an adaptive discontinuous Galerkin method for the response space. Another, closely related approach has been recently developed by Li and co-workers see e.g. (Li et al. 2009; Li et al. 2012) who have used the Dostupov-Pugachev equation to formulate the *generalized density evolution equation* using a Lagrangian description of the random system. The numerical solution of the generalized density evolution equation requires the selection of representative points of the random parameter space. That is, the numerical solution of this includes elements from the probability domain and the physical domain (Li & Chen 2009).

Another circle of ideas for studying dynamical systems initiated and developed mainly by physicists aims at the formulation of closed equations governing the evolution of the pdf of the system's response (Hänggi & Jung 1995). In principle such equations are not closed as they involve an average that expresses the non-local, in time, correlation between response and excitation. To get closed, solvable equations one needs to invoke the statistical properties of the excitation in order to calculate the interaction between the response and the excitation. To this end, and after the application of the Furutsu-Novikov-Donsker formula (or generalizations of it), several methods have been developed as e.g. *the small correlation time expansions*, with which one can produce effective approximate FPK equations (Dekker 1982), (Hänggi & Jung 1995) (Venturi et al. 2012). *The decoupling approximation*, which does not *a priori* restrict the noise to small correlation time, but neglects correlations between response and excitation and is, therefore, valid for weak-intensity random noise excitations (Hänggi & Jung 1995). *The unified colored noise approximation*, that increases in accuracy for increasing non-linear damping and decreases in accuracy with color intensity (Jung & Hanggi 1987), (Luo & Zhu 2003), (Luczka 2005).

Apart from the above techniques, a new general approach to the probabilistic study of dynamical systems under colored (smoothly-correlated) random excitation, was introduced by Athanassoulis and Sapsis. The *response-excitation theory* (RE theory) (Athanassoulis & Sapsis 2006; Sapsis & Athanassoulis 2006; Sapsis & Athanassoulis 2008), is based on a generic approach introduced by Eberhard Hopf (1952) which treats the evolution of the underlying, infinite-dimensional, probability measure, associated with the involved processes, by means of the evolution of their joint characteristic functional (Ch.Fnl), termed the *characteristic functional approach* (Ch.Fnl approach). The Ch.Fnl approach has been extensively used in the statistical modeling and analysis of turbulent flows (see, e.g., (Lewis & Kraichnan 1962; Beran 1986), (Vishik & Furshikov 1988)). The application of this approach to treat stochastically excited Ordinary Differential Equations (ODEs) was discussed by Beran (1986), see also (Vishik & Furshikov 1988), and used by Kotulski and Sobczyk (1984), to obtain a closed form solution for the Ch.Fnl of a stochastically excited linear oscillator and other linear problems. Along the lines introduced by Sapsis and Athanassoulis (2008), the Ch.Fnl approach can be exploited in order to obtain new Partial Differential Equations (PDEs), governing the evolution of the joint, Response-Excitation pdfs (REPDFs) by appropriate projections of the Functional Differential Equation (FDE). Venturi, Karniadakis et al. (Venturi et al. 2012), (Venturi & Karniadakis 2012), elaborated further this approach, confirming the equation derived in (Athanassoulis & Sapsis 2006; Sapsis & Athanassoulis 2008) and answered in negative the question raised in (Athanassoulis & Sapsis 2006; Sapsis & Athanassoulis 2008), if this equation alone ensures uniqueness. Accordingly, it becomes evident that a kind of completion of this equation is necessary. The completion proposed by (Venturi et al. 2012), results in a complicated equation, that includes the entire history of the response process in a functional integral form, while a simplified (computable) version of the latter equation, seems to be valid only for weakly colored excitation. Alternatively, the same authors (Venturi et al. 2012), (Cho et al. 2013) have proposed numerical solutions for special cases of the REPDF evolution equation in which this coincides with the *Dostupov-Pugachev equation* (see also discussion above). The RE theory will be further discussed in the next subsection.

1.2. Motivation and scope of the present work

The motivation for the present work has been the development of technics permitting the probabilistic characterization of the solution of RDEs under general (smoothly correlated) random excitation. Since the complete probabilistic characterization of such random functions is very complicated (involving infinite dimensional mathematical tools) (Vakhania et al. 1987), one usually aims at a weaker solution, i.e. the knowledge of some pdfs of the response process, or some moment functions of the response.

As we have already discussed in the previous section, the difficulty is drastically reduced when the stochastic excitation is a delta correlated function that makes the response a Markovian random function. Existing methodologies fail to treat in a satisfactory way the general case of smoothly-correlated excitation, which is the most interesting case in engineering and applied sciences (see also the discussion below, in this section). In response to this situation the RE theory, a new method for the probabilistic characterization of any non-linear system with any type of smoothly-correlated random excitation, has been recently introduced by Athanassoulis & Sapsis (2006) and Sapsis &Athanassoulis (2008).

The RE theory proposes the joint treatment of the probabilistic structure of the response and the excitation, leaving the space for their stochastic dependence to be determined during the solution of the problem. Athanassoulis and Sapsis used the characteristic functional approach to derive an equation for the joint RE characteristic functional and showed that by appropriately projecting this infinite dimensional equation it is possible to obtain equations for the evolution of the joint REPDF. The derived joint REPDF evolution equation is a peculiar equation, involving two times (one for the excitation, s, and one for the response, t), and partial derivatives only with respect to one (response) time, whereas, after the differentiation, the limit of the excitation time $s \to t$ should be taken. I.e. the REPDF evolution equation includes the half time derivative $\partial f_{x(t)y(s)}(\alpha,\beta)/\partial t\big|_{s\to t}$. This peculiarity gives rise to fundamental questions regarding both the well-posedness and the methods of its numerical solution. While working on this thesis, it became evident that the REPDF evolution equation of Athanassoulis and Sapsis in not a closed equation, and thus cannot provide a unique REPDF. The same finding has also been stated recently by Venturi *et al* (2012). This is due to the fact that when the half time limit $\partial f_{x(t)y(s)}(\alpha,\beta)/\partial t\big|_{s\to t}$ is considered the non-local (in time) characteristics of the problem are lost.

The present work continues the study of the RE theory, aiming at the clarification of various obscure points, and its further development towards the implementation of efficient algorithms for numerical solutions. To close the joint REPDF evolution equation we need to *a priori* approximate the half-time derivative $\partial f_{x(t)y(s)}(\alpha,\beta)/\partial t\big|_{s\to t}$ in order to account for the non-local (in time) response-excitation correlation structure. For a linear RDE under Gaussian excitation, the inclusion of the two-time RE cross-correlation moment equation could provide the additional information needed for the problem to be well posed. This finding is generalized for the non-linear/non-Gaussian case reclaiming also evidence gained by looking into MC simulations results, performed by Z.G. Kapelonis. According to these observations, in the long-time statistical equilibrium state the joint REPDF tends to concentrate around the equilibrium curve of deterministic problems realized on the RE-phase space. These ideas motivated the formulation of new auxiliary local, in the RE-phase space, moment conditions that can be combined with the joint REPDF evolution equation through a Kernel density representation of the joint REPDF. The closed REPDF evolution equation can be solved numerically by a Galerkin scheme.

The definition of the appropriate computational domain of the Galerkin scheme initiated the development of a new methodology that aims at finding moments of non-linear RDEs under arbitrary, smoothly-correlated excitation. The formulation of the two-time RE moment equations for scalar RDEs, as well as the two-fold closure (Gaussian moment closure and time closure) of these equations is presented. The direct solution of the equations in the long-time statistical equilibrium state is found in (almost) closed form. Finally, some first ideas on the generalization of the methodology to bi-stable half oscillators using, instead of the Gaussian, a bi-Gaussian moment closure condition is discussed.

The RE theory can be used for the determination of the response pdfs of nonlinear systems with arbitrary polynomial non-linearities excited by colored, Gaussian or non-Gaussian, stochastic processes. The determination of the response pdfs is necessary for prediction of structural reliability, structural failures and level-crossing events that are problems of great importance in engineering sciences. Moreover, this set up, that goes beyond the delta-correlated excitation, is the natural way of modeling/study in almost every field of macroscopic stochastic dynamics, e.g. systems excited by sea waves (Francescutto & Naito 2004),(G.A. Athanassoulis et al. 2009), wind (Sura 2003), (Sapsis & Dijkstra 2013) and earthquakes (Varotsos et al. 2002)(Yulmetyeva et al. 2009). It also has interesting applications in the context of statistical physics (Luczka 2005), (Van Kampen 2007), medical physics (Wang 2009), material sciences (Liu et al. 2010), reactions involving macromolecules (Guerin et al. 2012)(Guerin et al. 2013), system's biology (Bratsun et al. 2005; Shahrezaei et al. 2008), electrical engineering and neuroscience (Galán 2009).

An important example of the above described set up is the roll motion in a realistic seaway. Rolling motion is the degree of freedom of ship dynamics that has perhaps attracted the most attention. This is justified since roll motion is easily excited in the sea, most pronounced, highly nonlinear and most dangerous; see, e.g., (Belenky & Sevastianov 2003). The complicacies of the dynamics of roll motion are due partly to the nonlinearities in the restoring moment term and the damping term, and partly to the excitation mechanisms, which include external excitation by waves and wind, as well as parametric excitation. Wave loads on ships can be considered as Gaussian or nearly Gaussian, smoothly-correlated, stochastic processes. Wind velocity and wind loads, also important for roll motion, can be considered as superposition of a steady mean and two randomly fluctuating components; one modeling the background turbulent wind flow, which is nearly stationary and nearly Gaussian with a broad-band spectrum ((Simiu & Scanlan 1986), Ch. 14; (Belenky & Sevastianov 2003), Sec. 8.2.1), and a second one, modeling squalls, which should be considered non-stationary and non-Gaussian (see, e.g., (Belenky & Sevastianov 2003), Sec. 8.2.2, (Michelacci 1983)). Therefore, in a realistic model the excitation, i.e. the roll moment due to wind and waves is in general, non-Gaussian, non-stationary, and has correlation time comparable with the relaxation time of roll motion making the roll motion a non-Markovian random function. The RE theory has been applied to the ship roll problem with both parametric and external stochastic excitation (G.A. Athanassoulis et al. 2009), (Athanassoulis et al. 2012a), however in order to obtain interesting results in this setting the solution to problems that involve full oscillators shall be obtained. Since now we have managed to solve numerically the joint REPDF evolution equation and the two-time RE moment equations for first order equations (half oscillators) under general smoothly-correlated (colored) random excitation. However the methodologies that will be developed in what follows in this thesis can be extended to full oscillators.

1.3. Preview of Chapters

The thesis is organized as follows:

In the first part of *Chapter 2* some existing methods aiming at the probabilistic characterization of systems of RDEs are briefly presented. Subsequently, the RE theory, introduced by Athanassoulis and Sapsis, is reviewed and generalized to second order nonlinear systems. The joint REPDF evolution equation for non-linear dynamical systems under smoothly-correlated stochastic excitation is re-derived, using the Characteristic functional approach. These equations generalize existing results obtained by Athanassoulis and Sapsis to systems of two equations. To verify their validity, the equations are used to re-derive the infinite system of the limiting two-time moment equations. Finally the obtained equations are applied to the ship roll problem.

Chapter 3 focuses on linear random dynamical systems. The two-time RE moment equations are developed for a linear scalar dynamical system under colored stochastic excitation. These equations are solved analytically and results are obtained for different stochastic input functions. For Gaussian excitation, a complete analytical solution of the studied problem, both in the transient and in the long-time statistical equilibrium state, is produced and used to:

i. Prove that, for linear problems under Gaussian excitation the REPDF evolution equation is verified if and only if the two-time RE moment equations are verified on the diagonal.

ii. Prove that the REPDF evolution equation, as it stands, can have multiple solutions.

iii. Demonstrate that the developed, so far, methodology fails to properly take into consideration the RE correlation structure.

iv. Discuss the need for an *a priori* closure of the REPDF evolution equation by providing additional information about the RE correlation structure, as well as how this can be accomplished by the use of two-time RE moment equations.

Chapter 4 builds on the findings from the solution of the linear problem, to develop new auxiliary local, in the RE-phase space, conditions by the use of local linearizations/ Gaussianizations around the equilibrium curve of the non-linear scalar dynamical system in the long-time. The equations are solved analytically and compared with results obtained by Monte Carlo (MC) simulations. These local conditions are used to form a new *a priori* closure scheme for the non-linear REPDF evolution equation, providing the necessary additional information regarding the RE-correlation structure. This information is synthesized in the RE-evolution equation by the use of a Gaussian Kernel representation for the joint two-time RE-density. The REPDF evolution equation, together with the new local closure conditions, is then numerically solved using a Galerkin scheme. The obtained results are discussed and compared with Monte Carlo (MC) simulations.

In *Chapter 5* a new method is developed for the formulation and solution of two-time, response-excitation moment equations for a non-linear half oscillator excited by colored, Gaussian or non-Gaussian processes. To obtain a solution, a two-fold closure (moment and time closure) is presented. For a mono-stable half oscillator the moment closure is obtained by applying the standard Gaussian closure to the two-time RE moments. The time closure is achieved by using an exact non-local (in time) condition for the one-time moments. The same moment system is also considered and solved directly in the long-time, statistical equilibrium limit. Moreover, a bi-Gaussian moment closure scheme that extends the presented methodology to bi-stable cubic non-linear half oscillators in the long-time limit is discussed. Obtained results are compared MC Simulations.

Finally, directions for future research are discussed.

1.4. Main contribution of the present work

1. Derivation of the joint REPDF evolution equation for two-dimensional non-linear dynamical systems under smoothly correlated stochastic excitation, using the Characteristic functional approach (G. A. Athanassoulis et al. 2009)(G.A. Athanassoulis et al. 2009)]. (Generalization of results obtained by Athanassoulis and Sapsis to 2D systems)
2. Analytic verification of the REPDF evolution equation for linear problems under Gaussian excitation
3. Clarification of the non-uniqueness of solutions of the REPDF evolution equation. Non-uniqueness is mainly due to the fact that the correlation length of the excitation is not properly taken into account as some of the non-local (in time) characteristics are lost when taking the limit $\partial f_{x(t)y(s)}(\alpha,\beta)/\partial t\big|_{s\to t}$.
4. Development of an *a priori* closure scheme for the REPDF evolution equation via localized/linearized problems, accounting for the local, in space, response-excitation correlation structure (Athanassoulis et al. 2012b)(Athanassoulis et al. 2012a)(Tsantili et al. 2013)
5. Numerical solution of the REPDF evolution equation in the long-time, based on a Kernel Density (KD) representation of the REPDF and a Galerkin-type numerical scheme, which can embed the acquired information about the local RE structure. (Athanassoulis et al. 2012b)(Athanassoulis et al. 2012a) (Tsantili et al. 2013)
6. Formulation and solution of two-time, response-excitation moment equations for a monostable non-linear half oscillator excited by colored, Gaussian or non-Gaussian processes; applying a moment closure and a time closure.(Athanassoulis et al. 2013a)(G.A. Athanassoulis et al. 2013b)
7. First ideas on a bi-Gaussian moment closure scheme which could work well for bi-stable half-oscillators.

1.5. On the validation of the obtained results

The experimental verification of the probabilistic solution of RDEs is either extremely difficult or, most often, impossible. To validate the numerical solution of the REPDF evolution equation and the two-time RE moment equations developed in this thesis, we need to compare the results with similar ones obtained by other methods such us:

- Analytic solutions,
- Asymptotic results,
- Results obtained by other well established methods for cases, if available,
- Results obtained by mathematical numerical experiments, in the lines of MC simulations.

Analytic solutions exist only for linear RDEs under Gaussian excitation. These have been exploited for the verification and the clarification of the non-uniqueness of solutions of the joint REPDF evolution equation. Asymptotic methods for tail probabilities are not relevant with the present work, whereas, asymptotic methods for long-time statistics have been used for the analytic calculation of long-time moments. The most promising and general validation method is by comparison with results obtained by mathematical numerical experiments in the lines of MC simulations. This method, although computationally expensive, is the generic approach with which we can probabilistically characterize any kind of RDEs. The MC simulation results that will be presented in this thesis have been obtained by an algorithm developed and implemented in Matlab® by Zacharias G. Kapelonis, which involves the following steps:

i. The 1-D random-phase model (Longuet-Higgins 1952; Pierson 1952)(Athanassoulis et al. 1991) (Athanassoulis 1990) is used to generate sample functions of the random excitation. According to this model every zero mean normally distributed random function $y(t;\theta)$ with correlation function $R_{yy}(\tau)$ can be modeled by a superposition of harmonic functions:

$$y(t;\theta)=\sum_{j=1}^{J}A_j\cos\left(\omega_j\cdot t+\Xi_j(\theta)\right)$$

where θ is the stochastic argument, $\omega_j>0,\ \ j=1,2,...,J$ are deterministic constants that model the frequencies, $A_j=\sqrt{S_+(\omega_j)\dfrac{\omega_j-\omega_{j-1}}{\pi}}$ are the deterministic constants that model the amplitudes of the corresponding terms (harmonics) defined by the one sided spectrum of the random excitation $S_+(\omega)=2\int_0^{+\infty}R_{yy}(\tau)\cdot e^{-i\omega\tau}d\tau$ and $\Xi_j(\theta),\ \ j=1,2,...,J$ are independent random variables, uniformly distributed in $(0,2\pi)$.

ii. For each sample function of the random excitation the deterministic version of the RDE is solved using ODE45, a MATLAB® implementation of the Dormant-Prince method (Dormand & Prince 1980), based on an explicit Runge-Kutta (4,5) formula.

iii. MC pdf estimations are computed using the kernel density estimation via diffusion, introduced by Botev et al. (2010) and coded in MATLAB® functions by the same author.

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CHAPTER 2

On The RE Theory In Stochastic Dynamics

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2.1. Introduction

In the first part of this chapter we present some well-known methods that aim at the probabilistic characterization of systems of RDEs. These methods involve equations that model the evolution of the system's pdfs. Firstly, the Liouville equation for the probabilistic characterization of the response of systems with random initial conditions is derived. Moreover, we discuss the extension of the Liouville equation**,** known as Dostupov-Pugachev equation, that can apply to non-linear dynamical systems under general excitation when the latter can be decomposed into a countable set of uncorrelated random variables. Secondly, we focus on systems having delta correlated random input and Markovian output. To this end, the FPK equation is derived as a special case of the Kramers-Moyal expansion. Finally, we discuss the colored noise master equation for the evolution of the response of generalized Langevin equations.

Subsequently, the RE theory, introduced by Athanassoulis and Sapsis, is reviewed and generalized to second order nonlinear systems. The joint Response Excitation probability density function (REPDF) evolution equation for non-linear dynamical systems under smoothly correlated stochastic excitation is produced, using the characteristic functional approach. To verify their validity, the equations are used to re-obtain the infinite system of the limit two-time moment equations. Finally the obtained equations are specialized in the ship roll problem.

2.2. Methods for the probabilistic characterization of systems of RDEs

Let us consider the system of RDEs of the form:

$$\dot{\boldsymbol{x}}(t;\theta) = \boldsymbol{G}\left(\boldsymbol{x}(t;\theta), \boldsymbol{y}(t;\theta)\right), \tag{1a}$$

$$\boldsymbol{x}(0;\theta) = \boldsymbol{x}_0(\theta). \tag{1b}$$

The probabilistic characterization of system (1) is equivalent with the determination of the infinite dimensional measure $\mathscr{P}_{\boldsymbol{x}(t)}(d\boldsymbol{x})$ of the Borel sets of the sample (functional) Banach space $\mathscr{X}$ of the responses $\boldsymbol{x}(t) \equiv \boldsymbol{x}(t;\theta)$. The probability measure $\mathscr{P}_{\boldsymbol{x}(t)}(d\boldsymbol{x})$ can be equivalently and more conveniently expressed by the characteristic functional $\mathscr{F}_{\boldsymbol{x}(t)}(\boldsymbol{u})$ of the response $\boldsymbol{x}(t;\theta)$, defined by:

$$\mathscr{F}_{\boldsymbol{x}(t)}(\boldsymbol{u}) = \int_{\mathscr{X}} \exp\left(i\left\langle \boldsymbol{u}, \boldsymbol{x}\right\rangle\right) d\mathscr{P}_{\boldsymbol{x}(t)}, \tag{2}$$

where $\boldsymbol{u} \in \mathscr{U} = \mathscr{X}'$ = the topological dual of $\mathscr{X}$. See (Vakhania et al. 1987)(Pugachev & Sinitsyn 2001) for the definition and the basic properties of the characteristic functional.

As discussed in Athanassoulis (2009) solving the problem (1) in the probability domain means to determine the probability measure $\mathscr{P}_{\boldsymbol{x}(t)}(d\boldsymbol{x})$ [equivalently, the characteristic functional $\mathscr{F}_{\boldsymbol{x}(t)}(\boldsymbol{u})$], of the response function $\boldsymbol{x}(t;\theta)$, in terms of the deterministic system function $\boldsymbol{G}(\cdot,\cdot)$, and the joint probability measure of all stochastic elements [initial value $\boldsymbol{x}_0(\theta)$ and stochastic functions $\boldsymbol{y}(t;\theta)$] appearing in the equation, that is to construct a mapping $\mathcal{T}$ such that:

$$\left\{\begin{matrix} \textit{Probabilistic structure} \\ \textit{of the response } \ \boldsymbol{x}(\cdot\,;\theta) \end{matrix}\right\} = \mathcal{T}\left\{\begin{matrix} \textit{Deterministic system function } \boldsymbol{G}(\cdot,\cdot) \textit{ and} \\ \textit{Probability structure of the initial state } \boldsymbol{x}_0(\theta) \\ \textit{and stochastic excitation } \boldsymbol{y}(\cdot\,;\theta) \end{matrix}\right\} \tag{3}$$

Mapping (3) can be formalized in terms of the characteristic functionals, as follows:

$$\mathscr{F}_{\boldsymbol{x}}(\bullet) = \mathcal{T}\left[G(\bullet,\bullet), \mathscr{F}_{\boldsymbol{x}_0\boldsymbol{y}}(\bullet,\bullet)\right], \tag{4}$$

where $\mathscr{F}_{\boldsymbol{x}_0\boldsymbol{y}}(\boldsymbol{u}_0,\boldsymbol{v})$ is the joint characteristic functional of the initial state and all input random elements.

In what follows, we shall review various existing approaches for the solution of problem (1) in the probability domain. We are going to present equations governing the response pdf $f_{\boldsymbol{x}(t)}(\boldsymbol{\alpha})$ and discuss about their limitations. Subsequently, we shall focus on the recently introduced RE theory (Athanassoulis & Sapsis 2006; Sapsis & Athanassoulis 2006; Sapsis & Athanassoulis 2008), where the main object of analysis is the joint REPDF $f_{\boldsymbol{x}(t)\boldsymbol{y}(s)}(\boldsymbol{\alpha},\boldsymbol{\beta})$.

2.2.1. The Liouville equation and the Dostupov-Pugachev extension

For a system of differential equations with random initial conditions

$$\dot{\boldsymbol{x}}(t;\theta) = \boldsymbol{G}\left(\boldsymbol{x}(t;\theta)\right), \tag{1}$$

$$\boldsymbol{x}(0;\theta) = \boldsymbol{x}_0(\theta), \tag{2}$$

the following theorem holds true (Soong 1973)(Risken 1996):

Theorem (Liouville-Gibbs): The pdf $f_{\boldsymbol{x}(t)}(\boldsymbol{\alpha}) = f_{x_1(t)x_2(t)\ldots x_N(t)}(\alpha_1,\alpha_2,\ldots,\alpha_N)$ of the response $\boldsymbol{x}(t;\theta) = \left(x_1(t;\theta), x_2(t;\theta),\ldots,x_N(t;\theta)\right)$ of system of Equ.(1) and Equ.(2) verifies the PDE:

$$\boxed{\frac{\partial f_{\boldsymbol{x}(t)}(\boldsymbol{\alpha})}{\partial t} + \sum_{n=1}^{N}\frac{\partial}{\partial \alpha_n}\left(G_n(\boldsymbol{\alpha},t) f_{\boldsymbol{x}(t)}(\boldsymbol{\alpha})\right) = 0}, \tag{3}$$

which is known as the **Liouville equation**.

Proof: Use is made of the Fourier transforms between the characteristic function of the response $\varphi_{\mathbf{x}(t)}(\boldsymbol{u})$ and the response pdf $f_{\mathbf{x}(t)}(\boldsymbol{\alpha})$, i.e.:

$$\varphi_{\mathbf{x}(t)}(\boldsymbol{u}) \;=\; \mathscr{F}\left[f_{\mathbf{x}(t)}(\boldsymbol{\alpha})\,;\boldsymbol{\alpha}\to\boldsymbol{u}\right] \equiv \tilde{f}_{\mathbf{x}(t)}(\boldsymbol{u}) \;=\; \int_{\mathbb{R}^N} exp\left\{ i\boldsymbol{u}^{\mathrm{T}}\cdot\boldsymbol{\alpha}\right\} f_{\mathbf{x}(t)}(\boldsymbol{\alpha})\,d\boldsymbol{\alpha}\,, \tag{4}$$

$$f_{\mathbf{x}(t)}(\boldsymbol{\alpha}) \;=\; \mathscr{F}^{-1}\left[\varphi_{\mathbf{x}(t)}(\boldsymbol{u})\,;\boldsymbol{u}\to\boldsymbol{\alpha}\right] \;=\; \frac{1}{\left(2\pi\right)^N}\int_{\mathbb{R}^N} exp\left\{ -i\boldsymbol{u}^{\mathrm{T}}\cdot\boldsymbol{\alpha}\right\} \varphi_{\mathbf{x}(t)}(\boldsymbol{u})\,d\boldsymbol{u}\,. \tag{5}$$

The characteristic function $\varphi_{\mathbf{x}(t)}(\boldsymbol{u})$ of $\boldsymbol{x}(t\,;\beta)$ is given by the equation:

$$\varphi_{\mathbf{x}(t)}(\boldsymbol{u}) = E^{\theta}\left[exp\left\{ i\sum_{n=1}^{N} u_n x_n(t\,;\theta)\right\}\right]. \tag{6}$$

Differentiating Equ.(6) with respect to time, then, using Equ.(1) to eliminate the time derivatives of the response, we obtain:

$$\begin{aligned}
\frac{\partial\varphi_{\mathbf{x}(t)}(\boldsymbol{u})}{\partial t} \;&=\; \frac{\partial}{\partial t}E^{\theta}\left[exp\left\{ i\sum_{n=1}^{N} u_n x_n(t\,;\theta)\right\}\right] \;=\; E^{\theta}\left[\frac{\partial}{\partial t} exp\left\{ i\sum_{n=1}^{N} u_n x_n(t\,;\theta)\right\}\right] \;=\\
&=\; E^{\theta}\left[exp\left\{ i\sum_{n=1}^{N} u_n x_n(t\,;\theta)\right\}\cdot i\sum_{m=1}^{N} u_m \dot{x}_m(t\,;\theta)\right] \;=\\
&=\; i\sum_{m=1}^{N} u_m E^{\theta}\left[G_m\left(\boldsymbol{x}(t\,;\theta),t\right)\cdot exp\left\{ i\sum_{n=1}^{N} u_n x_n(t\,;\theta)\right\}\right],
\end{aligned}$$

that is:

$$\frac{\partial\varphi_{\mathbf{x}(t)}(\boldsymbol{u})}{\partial t} \;=\; i\sum_{m=1}^{N} u_m \int_{\mathbb{R}^N} G_m\left(\boldsymbol{\alpha},t\right)\cdot exp\left\{ i\sum_{n=1}^{N} u_n \alpha_n\right\} f_{\mathbf{x}(t)}(\boldsymbol{\alpha})\,d\boldsymbol{\alpha}. \tag{7}$$

From Equ.(7) we can obtain a closed equation with respect to $f_{\mathbf{x}(t)}(\boldsymbol{\alpha})$, i.e.:

$$\begin{aligned}
\frac{\partial\varphi_{\mathbf{x}(t)}(\boldsymbol{u})}{\partial t} \;&=\; \sum_{m=1}^{N}\int_{\mathbb{R}^N} i u_m G_m\left(\boldsymbol{\alpha},t\right)\cdot exp\left\{ i\sum_{n=1}^{N} u_n \alpha_n\right\} f_{\mathbf{x}(t)}(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} \;=\\
&=\; \sum_{m=1}^{N}\int_{\mathbb{R}^N} G_m\left(\boldsymbol{\alpha},t\right) f_{\mathbf{x}(t)}(\boldsymbol{\alpha})\cdot\frac{\partial}{\partial\alpha_m} exp\left\{ i\sum_{n=1}^{N} u_n \alpha_n\right\} d\boldsymbol{\alpha} \;=\\
&\left[\text{integrating by parts, assuming that } \lim_{\boldsymbol{\alpha}\to\infty} G_m\left(\boldsymbol{\alpha},t\right) f_{\mathbf{x}(t)}(\boldsymbol{\alpha}) \;=\; 0\right]\\
&=\; -\sum_{m=1}^{N}\int_{\mathbb{R}^N} exp\left\{ i\sum_{n=1}^{N} u_n \alpha_n\right\}\cdot\frac{\partial}{\partial\alpha_m} G_m\left(\boldsymbol{\alpha},t\right) f_{\mathbf{x}(t)}(\boldsymbol{\alpha})d\boldsymbol{\alpha},
\end{aligned}$$

$$= -\sum_{m=1}^{N} \mathscr{F}\left[\frac{\partial}{\partial \alpha_m} G_m(\boldsymbol{\alpha}, t) f_{\mathbf{x}(t)}(\boldsymbol{\alpha}) ; \boldsymbol{\alpha} \to \boldsymbol{u}\right]. \quad (8)$$

Applying the inverse Fourier transform, $\mathscr{F}^{-1}\left[\tilde{g}(\boldsymbol{u},t) ; \boldsymbol{u} \to \boldsymbol{\alpha}\right]$, of the first and the last term of Equ.(8) we obtain Equ.(3).▪

On the basis of the Liouville-Gibbs theorem the problem of determining of the response pdf $f_{\mathbf{x}(t)}(\boldsymbol{\alpha})$ has been transformed to an initial value problem for first order pdf (Equ.(3)), with initial value the given joint pdf of the initial conditions, i.e.: $f_{\mathbf{x}(0)}(\boldsymbol{\alpha}) = f_{\mathbf{x}_0}(\boldsymbol{\alpha})$.

The Liouville equation can be generalized to systems containing time independent random variables (Soong 1973). That is, to RDEs of the type

$$\dot{\boldsymbol{x}}(t;\theta) = \mathbf{G}\left(\boldsymbol{x}(t;\theta), \mathbf{A}(\theta), t\right), \quad (9)$$

$$\boldsymbol{x}(0;\theta) = \boldsymbol{x}_0(\theta), \quad (10)$$

where $\mathbf{A}(\theta) = \left(A_1(\theta), A_2(\theta), ..., A_M(\theta)\right)$ is a known random vector.

This generalization is obtained as follows. Considering the vector process $\boldsymbol{z}(t;\theta) = \begin{bmatrix} \boldsymbol{x}(t;\theta) \\ \mathbf{A}(\theta) \end{bmatrix}$, we can write system (9)-(10) in the form:

$$\dot{\boldsymbol{z}}(t;\theta) = \overline{\mathbf{G}}\left(\boldsymbol{z}(t;\theta), t\right), \quad (11)$$

$$\boldsymbol{z}(0;\theta) = \boldsymbol{z}_0(\theta), \quad (12)$$

$$\text{where } \overline{\mathbf{G}} = \begin{bmatrix} \mathbf{G} \\ \mathbf{0} \end{bmatrix} \text{ and } \boldsymbol{z}_{\mathbf{0}}(\theta) = \begin{bmatrix} \boldsymbol{x}_{\mathbf{0}}(\theta) \\ \mathbf{A}(\theta) \end{bmatrix}, \quad (13a,b)$$

then applying the Liouville equation to the augmented version of Equs.(9,10), we get:

$$\frac{\partial f_{\mathbf{z}(t)}(\boldsymbol{\alpha})}{\partial t} + \sum_{n=1}^{N+M} \frac{\partial}{\partial z_n}\left(\overline{G}_n(\boldsymbol{\alpha}, t) f_{\mathbf{z}(t)}(\boldsymbol{\alpha})\right) = 0. \quad (14)$$

Substitution of Equ.(13) into Equ.(14) gives an evolution equation for the joint pdf of the response $\boldsymbol{x}(t;\theta)$ and the random vector $\mathbf{A}(\theta)$,

$f_{\mathbf{x}(t)\mathbf{A}}(\boldsymbol{\alpha}, \boldsymbol{A}) \equiv f_{x_1(t)x_2(t)...x_N(t)A_1A_2...A_M}(\alpha_1, \alpha_2, ..., \alpha_N, A_1, A_2, ..., A_M)$, i.e.:

$$\boxed{\frac{\partial f_{\mathbf{x}(t)\mathbf{A}}(\boldsymbol{\alpha},\boldsymbol{A})}{\partial t} + \sum_{n=1}^{N}\frac{\partial}{\partial \alpha_n}\left(G_n(\boldsymbol{\alpha},\boldsymbol{A},t) f_{\mathbf{x}(t)\mathbf{A}}(\boldsymbol{\alpha},\boldsymbol{A})\right) = 0}, \tag{15a}$$

with initial condition the given joint pdf of $\boldsymbol{x}_0$, $\mathbf{A}$, i.e.:

$$f_{\mathbf{x}(0)\mathbf{A}}(\boldsymbol{\alpha},\boldsymbol{A}) = f_{\mathbf{x_0 A}}(\boldsymbol{\alpha_0},\boldsymbol{A}). \tag{15b}$$

Equ.(15) is known as the Dostupov-Pugachev equation (Dostupov & Pugachev 1957), (Li & Chen 2008) and holds true in general dynamical systems involving random parameters (time-independent random elements). The above method can also be used for studying (approximately) more general random dynamical systems, containing time-dependent random elements. This is possible by representing the input random function into a countable set of uncorrelated random variables using e.g. the Karhunen-Loeve expansion (Karhunen 1947; Loeve 1978)(Ghanem & Spanos 2003):

$$\boldsymbol{y}(t;\theta) = m_y(t) + \sum_{m=0}^{\infty} A_m(\theta)\sqrt{\lambda_m}\, g_m(t), \tag{16}$$

where $m_{\xi_m} = 0$, $E^{\theta}[\xi_m(\theta)\cdot\xi_n(\theta)] = \delta_{mn}$ and $\sqrt{\lambda_m}$, $g_m(t)$ are the eigenvectors and eigenvalues arising from the spectral decomposition of the input covariance Kernel, i.e.

$$C_{\boldsymbol{yy}}(t,s) = \sum_{m=0}^{\infty}\lambda_m\, g_m(t)\, g_m(s), \tag{17}$$

Equ.(16) must be truncated at a certain finite level that can sufficiently approximate the infinite dimensional process $\boldsymbol{y}(t;\theta)$. Under this consideration Equ.(15) is a possibly high dimensional equation that holds true for every possible value of the random vector $\mathbf{A}(\theta)$. In Venturi et al (Venturi et al. 2012) the sparse grid collocation method (Foo & Karniadakis 2008; Foo & Karniadakis 2010) is used for the numerical solution of Equ.(15). In addition, in Cho et al. (2013), Equ.(15) is solved numerically by considering the response and the excitation space separately and using the sparse-grid collocation method for excitation space and an adaptive discontinuous Galerkin method for the response space. Moreover Li and co-workers (Li et al. 2009; Li et al. 2012) use the Dostupov-Pugachev equation (15) to formulate the generalized density evolution equation using a Lagrangian description of the random system (11), (12) aiming to uncouple the values of the physical solutions $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_N)$ from the density $f_{\mathbf{x}(t)\mathbf{A}}(\boldsymbol{\alpha},\boldsymbol{A})$ in the partial differentiation operator with respect to the state variables appearing in Equ.(15). For the numerical solution of the generalized density evolution equation requires the selection of representative points of the random parameter space. That is, the numerical solution of this includes elements from the probability domain and the physical domain (Li & Chen 2009).

2.2.2. The Kramers-Moyal expansion and the FPK equation

Let $f_{x(t)}(\alpha)$ be the pdf of the scalar stochastic function $x(t;\theta)$, $t \in T$, then it is well known that:

$$f_{x(t+\Delta t)}(\alpha) = \int_{-\infty}^{\infty} f_{x(t+\Delta t)}(\alpha, t+\Delta t \,|\, \alpha', t) f_{x(t)}(\alpha')\, d\alpha', \tag{1}$$

where $f_{x(t+\Delta t)}(\alpha, t+\Delta t \,|\, \alpha', t)$ is the conditional pdf of the random variable $x(t+\Delta t;\theta)$ given that $x(t;\theta) = \alpha'$.

The conditional pdf $f_{x(t+\Delta t)}(\alpha, t+\Delta t \,|\, \alpha', t)$ can be expressed as the inverse Fourier transform of the (conditional) characteristic function $\varphi_{\Delta x(t)}(u, t+\Delta t \,|\, \alpha', t)$ of the random variable $\Delta x(t;\theta) = x(t+\Delta t;\theta) - x(t;\theta)$, given that $x(t;\theta) = \alpha'$, i.e.:

$$f_{x(t+\Delta t)}\left(\alpha, t+\Delta t \,|\, \alpha', t\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iu\Delta\alpha} \phi_{\Delta x(t)}\left(u, t+\Delta t \,|\, \alpha', t\right) du, \tag{2}$$

where $\varphi_{\Delta x}(u, t+\Delta t \,|\, \alpha', t) = \int_{-\infty}^{\infty} e^{iu\Delta x(t;\theta)} f_{x(t+\Delta t)}(\alpha, t+\Delta t \,|\, \alpha', t)\, d\alpha$.

Expanding $\varphi_{\Delta x(t)}(u, t+\Delta t \,|\, \alpha', t)$ on a Taylor series around $u = 0$ (assuming that $\varphi_{\Delta x}\left(u, t+\Delta t \,|\, \alpha', t\right)$ is an analytic function) we get:

$$f_{x(t+\Delta t)}\left(\alpha, t+\Delta t \,|\, \alpha', t\right) = \frac{1}{2\pi} \sum_{q=0}^{\infty} \frac{a_q\left(t+\Delta t \,|\, \alpha', t\right)}{q!} \int_{-\infty}^{\infty} (iu)^q e^{-iu\Delta\alpha} du, \tag{3}$$

where $a_q\left(t+\Delta t \,|\, \alpha', t\right) = E^{\theta}\left(\Delta x^q \,|\, \alpha', t\right) = E^{\theta}\left(\left(x\left(t+\Delta t;\theta\right) - x\left(t;\theta\right)\right)^q \,\middle|\, \alpha', t\right)$, (4)

are the *conditional incremental moments* of the stochastic function $x(t;\theta)$.

Substituting Equ.(4) into Equ.(1), after integration it is obtained:

$$f_{x(t+\Delta t)}(\alpha) - f_{x(t)}(\alpha) = \sum_{q=1}^{\infty} \frac{(-1)^q}{q!} \frac{\partial}{\partial x^q} \left[a_q\left(t+\Delta t \,|\, \alpha, t\right) f_{x(t)}(\alpha) \right]. \tag{5}$$

Dividing Equ.(5) with Δt then taking the limit $\Delta t \to 0$, we have:

$$\boxed{\frac{\partial f_{x(t)}(\alpha)}{\partial t}-\sum_{q=1}^{\infty}\frac{(-1)^q}{q!}\frac{\partial}{\partial x^q}\left[A_q\left(t+\Delta t\middle|\alpha,t\right)f_{x(t)}(\alpha)\right]=0}, \tag{6}$$

where

$$A_q\left(t+\Delta t\middle|\alpha,t\right)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}E^{\theta}\left[\left(x\left(t+\Delta t;\theta\right)-x\left(t;\theta\right)\right)^q\middle|x\left(t;\theta\right)=\alpha\right]\quad q=1,2,\ldots.^{1} \tag{7}$$

are called *derivative moments* of the stochastic function $x(t;\theta)$, whereas Equ.(6) is known as the **Kramers-Moyal expansion** (Moyal 1949) (Risken 1996) as well as **kinetic equation** (Soong 1973). It is straightforward to generalize the Kramers-Moyal equation to vector processes case $\boldsymbol{x}(t;\theta)=\left(x_1(t;\theta),\ldots,x_N(t;\theta)\right)$. In this case Equ.(1) takes the form:

$$f_{\mathbf{x}(t+\Delta t)}(\boldsymbol{\alpha})=\int_{-\infty}^{\infty}f_{\mathbf{x}(t+\Delta t)}(\boldsymbol{\alpha},t+\Delta t|\boldsymbol{\alpha}',t)f_{\mathbf{x}(t)}(\boldsymbol{\alpha}')\,d\boldsymbol{\alpha}'. \tag{8}$$

Following the same steps it is obtained:

$$\frac{\partial f_{\mathbf{x}(t)}(\boldsymbol{\alpha})}{\partial t}-\sum_{q_1,q_2,\ldots,q_N=1,}^{\infty}\left[\prod_{n=1}^{N}\frac{(-1)^{q_n}}{\left(q_n\right)!}\frac{\partial^{q_n}}{\partial {x_n}^{q_n}}\right]\left[A_{q_1,q_2,\ldots,q_N}(\boldsymbol{\alpha},t)\,f_{\mathbf{x}(t)}(\boldsymbol{\alpha})\right]=0, \tag{10}$$

where

$$A_{q_1,q_2,\ldots,q_N}(\boldsymbol{\alpha},t)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}E^{\theta}\left[\prod_{n=1}^{N}\left[x_n(t+\Delta t;\theta)-x_n(t;\theta)\right]^{q_n}\middle|\boldsymbol{x}(t;\theta)=\boldsymbol{\alpha}\right]. \tag{11}$$

According to Pawula theorem (Pawula 1967)(Soong 1973), if the derivative moment $A_q\left(t+\Delta t\middle|\alpha,t\right)$ exists for every q and is zero for some even q, then $A_q\left(t+\Delta t\middle|\alpha,t\right)=0$, for all $q\geq 3$. In this case, it follows that $A_{q_1,q_2,\ldots,q_N}(\boldsymbol{\alpha},t)=0$ (with probability one) for all q_n that are such that $\sum_{n=1}^{N}q_n\geq 3$. In fact, it can be proved that when the stochastic input is a Gaussian delta correlated process all derivative moments for $q\geq 3$ are equal to zero, and the Kramers-Moyal expansion takes the form of the **Fokker-Planck Kolmogorov (FPK) equation**, i.e.:

$$\boxed{\frac{\partial f_{\mathbf{x}(t)}(\boldsymbol{\alpha})}{\partial t}=-\sum_{n=1}^{N}\frac{\partial}{\partial\alpha_n}\left[A_n(\boldsymbol{\alpha},t)\,f_{\mathbf{x}(t)}(\boldsymbol{\alpha})\right]+\frac{1}{2}\sum_{n_1,n_2=1}^{N}\frac{\partial^2}{\partial\alpha_{n_1}\partial\alpha_{n_2}}\left[A_{n_1n_2}(\boldsymbol{\alpha},t)\,f_{\mathbf{x}(t)}(\boldsymbol{\alpha})\right].} \tag{12}$$

[1] The derivation is formal. Questions concerning the existence of the derivative moments and the convergence of the infinite series appearing in the derivation are not considered here.

See e.g. (Risken 1996). Supplied with the appropriate initial and boundary conditions Equ.(12) can be uniquely solved providing the response pdf $f_{\boldsymbol{x}(t)}(\boldsymbol{\alpha})$.

For the derivation of the Kramers-Moyal expansion (6), as well as for its reduction to the FPK equation (12), we did not make use of any information about the dynamical system that governs the evolution of the random function $\boldsymbol{x}(t;\theta)$. In what follows, we shall use specific dynamical system equations in order to specify the derivative moments $A_n(\boldsymbol{\alpha},t)$, $A_{n_1n_2}(\boldsymbol{\alpha},t)$ appearing in the FPK equation (12).

Let us assume that the stochastic excitation is a delta correlated $M-$ dimensional Wiener Process $\boldsymbol{B}(t;\theta)=\left\{B_m(t;\theta),\ m=1,2,...,M\right\}$ (Pugachev & Sinitsyn 2001), (Soize 1994), (Sun 2006) with components $B_m(t;\theta)$, $m=1,2,...,M$, that have the properties:

$$E^{\theta}\left[\Delta B_m(t;\theta)\right]=E^{\theta}\left[B_m(t+\Delta t;\theta)-B_m(t;\theta)\right]=0\,, \tag{13a}$$

$$E^{\theta}\left[\Delta B_{m_1}(t;\theta)\Delta B_{m_2}(t;\theta)\right]=2D_{m_1m_2}\Delta t\,,\ m_1,m_2=1,2,...,M\,, \text{ where } D_{m_1m_2} \text{ are constants,} \tag{13b}$$

and the system of RDE's (Equ.(1)_Sec(2.2)) is a system of the Itô stochastic differential equations, i.e. Equ.(1a)_Sec(2.2.2) takes the form:

$$d\boldsymbol{x}(t;\theta)=\boldsymbol{Q}\left(\boldsymbol{x}(t;\theta)\right)dt+\boldsymbol{G}\left(\boldsymbol{x}(t;\theta)\right)d\boldsymbol{B}(t;\theta)\,, \tag{14}$$

where $\boldsymbol{Q}(\bullet)=\left\{Q_n,\ n=1,2,...,N\right\}$ is a deterministic vector, and
$\boldsymbol{G}(\bullet)=\ \left\{G_{nm},\ n=1,2,...,N,\ \ m=1,2,...,M\right\}$ is a deterministic matrix.

In this context, the response $\boldsymbol{x}(t;\theta)$ will be a Markovian process that follows the Chapman-Kolmogorov equation, i.e:

$$f_{x(t_3)}(\alpha_3,t_3|\alpha_1,t_1)=\int_{-\infty}^{\infty} f_{x(t_3)}(\alpha_3,t_3|\alpha_2,t_2)f_{x(t_2)}(\alpha_2,t_2|\alpha_1,t_1)\,d\alpha_2\,,\quad t_1<t_2<t_3\,, \tag{15}$$

and can be completely characterized by its transition probability density function $f_{\boldsymbol{x}(t)}\left(\boldsymbol{\alpha},t|\boldsymbol{\alpha}',t'\right)$. The transition pdf $f_{\boldsymbol{x}(t)}\left(\boldsymbol{\alpha},t|\boldsymbol{\alpha}',t'\right)$ will also verify Equ.(12) (this is $f_{\boldsymbol{x}(t)}(\boldsymbol{\alpha})$ for special initial condition $f_{\boldsymbol{x}(t')}(\boldsymbol{\alpha})=\delta(\boldsymbol{\alpha}-\boldsymbol{\alpha}')$, (Risken 1996)).

We shall prove (following the derivation presented in (Soong, 1973)) that the Fokker-Planck-Kolmogorov (FPK) Equ.(12) governing the evolution of the conditional pdf $f_{\boldsymbol{x}(t)}\left(\boldsymbol{\alpha},t|\boldsymbol{\alpha}',t_0\right)$ of the response $\boldsymbol{x}(t;\theta),\ t\geq t_0$, has the form:

$$\frac{\partial f_{\mathbf{x}(t)}(\boldsymbol{\alpha},t|\boldsymbol{\alpha}',t_0)}{\partial t} = -\sum_{n=1}^{N}\frac{\partial}{\partial\alpha_n}\left[Q_n(\boldsymbol{\alpha},t)f_{\mathbf{x}(t)}\left(\boldsymbol{\alpha},t|\boldsymbol{\alpha}',t_0\right)\right] + \\ + \sum_{n_1,n_2=1}^{N}\frac{\partial^2}{\partial\alpha_{n_1}\partial\alpha_{n_2}}\left(\sum_{m_1,m_2=1}^{M}G_{n_1m_1}(\boldsymbol{\alpha},t)D_{m_1m_2}G_{n_2m_2}(\boldsymbol{\alpha},t)f_{\mathbf{x}(t)}\left(\boldsymbol{\alpha},t|\boldsymbol{\alpha}',t_0\right)\right), \quad (16)$$

Proof

As already discussed the transition probability $f_{\mathbf{x}(t)}\left(\boldsymbol{\alpha},t|\boldsymbol{\alpha}',t_0\right)$ will satisfy the FPK equation (Equ.12), i.e.:

$$\frac{\partial f_{\mathbf{x}(t)}\left(\boldsymbol{\alpha},t|\boldsymbol{\alpha}',t_0\right)}{\partial t} = -\sum_{n=1}^{N}\frac{\partial}{\partial\alpha_n}\left[A_n(\boldsymbol{\alpha},t)f_{\mathbf{x}(t)}\left(\boldsymbol{\alpha},t|\boldsymbol{\alpha}',t_0\right)\right] + \\ + \frac{1}{2}\sum_{n_1,n_2=1}^{N}\frac{\partial^2}{\partial\alpha_{n_1}\partial\alpha_{n_2}}\left[A_{n_1n_2}(\boldsymbol{\alpha},t)f_{\mathbf{x}(t)}\left(\boldsymbol{\alpha},t|\boldsymbol{\alpha}',t_0\right)\right], \quad (17)$$

where

$$A_n(\boldsymbol{\alpha},t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}E^{\theta}\left[\left(x_n(t+\Delta t;\theta)-x_n(t;\theta)\right)\Big|\mathbf{x}(t;\theta)=\boldsymbol{\alpha}\right], \quad (18)$$

$$A_{n_1n_2}(\boldsymbol{\alpha},t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}E^{\theta}\left[\left(x_{n_1}(t+\Delta t;\theta)-x_{n_1}(t;\theta)\right)\left(x_{n_2}(t+\Delta t;\theta)-x_{n_2}(t;\theta)\right)\Big|\mathbf{x}(t;\theta)=\boldsymbol{\alpha}\right]. \quad (19)$$

The derivative moments $A_n(\boldsymbol{\alpha},t)$, $A_{n_1n_2}(\boldsymbol{\alpha},t)$ can be calculated by the use of the specific dynamical equations of the system. More precisely, from Equ.(14) we have:

$$\Delta x_n(t;\theta) = x_n(t+\Delta t;\theta)-x_n(t;\theta) = Q_n\left(\mathbf{x}(t;\theta)\right)\Delta t + \sum_{m=1}^{M}G_{nm}\left(\mathbf{x}(t;\theta)\right)\Delta B_m(t;\theta)+o(t), \quad (20)$$

whereas:

$$\Delta x_{n_1}(t;\theta)\Delta x_{n_2}(t;\theta) = \left(Q_{n_1}(\mathbf{x},t)\Delta t + \sum_{m_1=1}^{M}G_{n_1m_1}(\mathbf{x},t)\Delta B_{m_1}(t;\theta)\right)\times \\ \times\left(Q_{n_2}(\mathbf{x},t)\Delta t + \sum_{m_2=1}^{M}G_{n_2m_2}(\mathbf{x},t)\Delta B_{m_2}(t;\theta)\right)+o(t) = \\ = Q_{n_1}(\mathbf{x},t)Q_{n_2}(\mathbf{x},t)(\Delta t)^2 + Q_{n_1}(\mathbf{x},t)\sum_{m_2=1}^{M}G_{n_2m_2}(\mathbf{x},t)\Delta t\,\Delta B_{m_2}(t;\theta) + \\ + Q_{n_2}(\mathbf{x},t)\sum_{m_1=1}^{M}G_{n_1m_1}(\mathbf{x},t)\Delta t\,\Delta B_{m_1}(t;\theta) +$$

$$+ \sum_{m_1=1}^{M}\sum_{m_2=1}^{M} G_{n_1m_1}(\mathbf{x},t)G_{n_2m_2}(\mathbf{x},t)\,\Delta B_{m_1}(t;\theta)\Delta B_{m_2}(t;\theta) + o(t). \tag{21}$$

From Equs.(18,20) we obtain:

$$A_n(\boldsymbol{\alpha},t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}E^{\theta}\left[\left(x_n(t+\Delta t;\theta)-x_n(t;\theta)\right)\middle|\,\mathbf{x}(t;\theta)=\boldsymbol{\alpha}\right]=$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}E^{\theta}\left[Q_n(\mathbf{x},t)\Delta t+\sum_{m=1}^{M}G_{nm}(\mathbf{x},t)\Delta B_m(t;\theta)+o(t)\,\middle|\,\mathbf{x}(t;\theta)=\boldsymbol{\alpha}\right]=$$

$$= \lim_{\Delta t\to 0}E^{\theta}\left[Q_n(\mathbf{x},t)\,\middle|\,\mathbf{x}(t;\theta)=\boldsymbol{\alpha}\right] + \lim_{\Delta t\to 0}\frac{1}{\Delta t}E^{\theta}\left[\sum_{m=1}^{M}G_{nm}(\mathbf{x},t)\Delta B_m(t;\theta)\,\middle|\,\mathbf{x}(t;\theta)=\boldsymbol{\alpha}\right]=$$

$$\left[\textit{since } \Delta B_m(t;\theta) \text{ is } \textit{independent of } \mathbf{x}(t;\theta)\right]$$

$$= Q_n(\boldsymbol{\alpha},t) + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\sum_{m=1}^{M}G_{nm}(\mathbf{x},t)E^{\theta}\left[\Delta B_m(t;\theta)\right] = Q_n(\boldsymbol{\alpha},t). \tag{22}$$

Then, from Equs.(19,21) we have:

$$A_{n_1n_2}(\boldsymbol{\alpha},t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}E^{\theta}\left[\Delta x_{n_1}(t;\theta)\Delta x_{n_2}(t;\theta)\,\middle|\,\mathbf{x}(t;\theta)=\boldsymbol{\alpha}\right] =$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}E^{\theta}\left[\sum_{m_1=1}^{M}\sum_{m_2=1}^{M}G_{n_1m_1}(\mathbf{x},t)G_{n_2m_2}(\mathbf{x},t)\,\Delta B_{m_1}(t;\theta)\Delta B_{m_2}(t;\theta)\,\middle|\,\mathbf{x}(t;\theta)=\boldsymbol{\alpha}\right] =$$

$$= \left[\sum_{m_1,m_2=1}^{M}G_{n_1m_1}(\boldsymbol{\alpha},t)\cdot G_{n_2m_2}(\boldsymbol{\alpha},t)\cdot\lim_{\Delta t\to 0}\frac{1}{\Delta t}\cdot\sum_{m_1,m_2=1}^{M}E^{\theta}\left[\Delta B_{m_1}(t;\theta)\Delta B_{m_2}(t;\theta)\,\middle|\,\mathbf{x}(t;\theta)=\boldsymbol{\alpha}\right]\right]=$$

$$= \sum_{m_1,m_2=1}^{M}G_{n_1m_1}(\boldsymbol{\alpha},t)\cdot G_{n_2m_2}(\boldsymbol{\alpha},t)\,2D_{m_1m_2}\ . \tag{23}$$

Combining Equs.(17, 22, 23) we obtain Equ.(16).

In Equ.(14) the stochastic input $d\boldsymbol{B}(t;\theta)$, being the derivative of the Wiener process is a delta-correlated stochastic function that is well-known as white noise. However, in case that this process is in fact a limiting approximation of a non-white process one must consider, instead of the Itô SDE Equ.(14), the Stratonovich SDE (Stratonovich 1966)(Risken 1996), which is equivalent to Equ.(14), when each component of the drift term

$\boldsymbol{Q}\left(\boldsymbol{x}(t;\theta)\right)= \left\{Q_n\left(\boldsymbol{x}(t;\theta)\right), n=1,2,...,N\right\}$ is replaced by:

$$Q_n\left(\boldsymbol{x}(t;\theta)\right)=Q_n\left(\boldsymbol{x}(t;\theta)\right) + \frac{1}{2}\sum_{m_1=1}^{M}\sum_{m_2=1}^{M}\frac{\partial G_{nm_2}\left(\boldsymbol{x}(t;\theta)\right)}{\partial x_{m_1}}G_{m_1m_2}\left(\boldsymbol{x}(t;\theta)\right). \tag{24}$$

2.2.3. The colored noise master equation

In this section we shall present a theory aiming at the formulation of closed equations governing the system's response pdf $f_{x(t)}(\alpha)$ by the use of the system's dynamical equations and without any a priori simplifying assumptions for the involved stochastic elements. This theory was primarily developed from physicists in the context of statistical mechanics (see e.g. (Hanggi 1978)).

Let us consider a scalar dynamical system that is excited by a smoothly correlated random function $y(t;\theta)$:

$$\dot{x}(t;\theta) = Q(x(t;\theta)) + G(x(t;\theta)) \cdot y(t;\theta)\,, \tag{1a}$$

$$x(0;\theta) = x_0(\theta)\,. \tag{1b}$$

We shall formulate an equation for the evolution of the response pdf $f_{x(t)}(\alpha)$. To this end, the response pdf is represented by an average over the realizations of $y(t;\theta)$ (Hänggi & Jung 1995) (Hänggi et al. 1984), i.e.:

$$f_{x(t)}(\alpha) = \left\langle \delta\big(x(t) - \alpha \big) \right\rangle. \tag{2}$$

Differentiation of Equ.(2) with respect to time, using Equ.(1a) and exploiting the properties of the δ function, yields:

$$\boxed{\begin{aligned} \frac{\partial}{\partial t} f_{x(t)}(\alpha) &= -\frac{\partial}{\partial \alpha}\left\langle \delta\big(x(t) - \alpha \big)\dot{x}(t) \right\rangle = \\ &= -\frac{\partial}{\partial \alpha}\Big[Q(\alpha) f_{x(t)}(\alpha) \Big] - \frac{\partial}{\partial \alpha} G(\alpha) \left\langle \delta\big(x(t) - \alpha \big) y(t) \right\rangle, \end{aligned}} \tag{3}$$

Equ.(3) is not closed since it involves an average that expresses the non-local, in time, correlation between the response pdf, in terms of a functional average over the response realizations, and the excitation. One needs to invoke the statistical properties of the excitation in order to produce a closed expression for this functional average. Assuming that the excitation $y(t;\theta)$ is a Gaussian process, this can be accomplished by an application of the Furutsu-Novikov-Donsker formula (Luczka 2005) that for an arbitrary functional $W\big[y\big]$ of a Gaussian process $y(t;\theta)$ reads as follows:

$$\left\langle W[y]\, y(t) \right\rangle = \int_{t_0}^{t} \left\langle y(s) y(t) \right\rangle \left\langle \frac{\delta W[y]}{\delta y(s)} \right\rangle ds\ . \tag{4}$$

Applying Equ.(4) to Equ.(3) for the functional $\delta\left(x(t) - \alpha \right)$, it is obtained:

$$\frac{\partial}{\partial t} f_{x(t)}(\alpha) = -\frac{\partial}{\partial \alpha}\left[Q(\alpha) f_{x(t)}(\alpha)\right] - \frac{\partial}{\partial \alpha} G(\alpha) \int_{t_0}^{t} C_{yy}(t-s) \left\langle \delta\left(x(t) - \alpha \right) \frac{\delta x(t)}{\delta y(s)} \right\rangle ds \,, \quad (5)$$

where the functional derivative $\frac{\delta x(t)}{\delta y(s)}$ [2] is given by the integral equation (Hänggi et al. 1984), (Luczka 2005):

$$\frac{\delta x(t)}{\delta y(s)} = \theta(t-s) \cdot G\left(x(s)\right) \exp\left[\int_{s}^{t} \left\{ Q'(x(u)) \; + G'(x(u)) \cdot y(u) \right\} du \right], \quad (6)$$

or alternatively:

$$\frac{\delta x(t)}{\delta y(s)} = \theta(t-s) \cdot G\left(x(t)\right) \exp\left[\int_{s}^{t} \left\{ Q'(x(u)) - Q(x(u)) \cdot \frac{G'\left(x(u)\right)}{G\left(x(u)\right)} \right\} du \right], \quad (7)$$

where $\theta(t-s)$ is the unit step function expressing causality. Indeed, the function $x(t;\theta)$ depends on the noise $y(s;\theta)$ only for $s < t$.

Combining Equs.(5-7) we obtain

$$\begin{aligned} \frac{\partial}{\partial t} f_{x(t)}(\alpha) = & -\frac{\partial}{\partial \alpha}\left[Q(\alpha) f_{x(t)}(\alpha)\right] + \frac{\partial}{\partial \alpha} G(\alpha) \frac{\partial}{\partial \alpha} G(\alpha) \times \\ & \times \int_{t_0}^{t} C_{yy}(t-s) \left\langle \delta\left(x(t) - \alpha \right) \left(\exp\left[\int_{s}^{t} \left\{ Q'(x(u)) \; - Q(x(u)) \cdot \frac{G'\left(x(u)\right)}{G\left(x(u)\right)} \right\} du \right] \right) \right\rangle ds. \end{aligned} \quad (8)$$

In Equ.(8), that is known as the **colored noise master equation** (Hänggi et al. 1984), the response is no longer coupled with the excitation in the functional average appearing in the third term of its right hand side. Nevertheless, Equ.(8) remains not closed since the function $\delta\left(x(t) - \alpha \right)$ is in general dependent on the response probabilities $f_{x(t)x(s)}(\alpha_1, \alpha_2)$ for $t_0 \le s \le t$. A review of classes of closed colored noise master equations can be found in (Hänggi et al. 1984). The colored noise master equation as given by Equ.(8) can however be the starting point of approximations. In fact, several methods have been developed. The most widely used method is **the small correlation time approximation** with which one can produce the approximate FPK equation (Dekker 1982), (Hänggi & Jung 1995)(Venturi et al. 2012). More precisely, in case that the excitation function is assumed to be an Ornstein–Uhlenbeck process, i.e. the excitation auto-covariance in Equ.(8) is given by the formula:

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[2] Being the solution of Equ.(1) $x(t)$ is a function of $x(s)$ for $t_0 \le s \le t$, i.e. $x(t) = x\left[t; y(\overset{s}{\underset{t_0}{s}}) \right]$

$$C_{yy}(t-s) = \frac{D}{\tau}\exp\left(-|t-s|/\tau\right). \tag{9}$$

the approximate FPK that is valid only for small correlation time τ reads as follows (Hänggi & Jung 1995):

$$\frac{\partial}{\partial t} f_{x(t)}(\alpha) = -\frac{\partial}{\partial \alpha}\left[Q(\alpha) f_{x(t)}(\alpha)\right] + D\frac{\partial}{\partial \alpha} G(\alpha)\frac{\partial}{\partial \alpha}\left[G(\alpha)h(\alpha,t) f_{x(t)}(\alpha)\right], \tag{10}$$

where:

$$h(\alpha,t) = [1-\exp(-t/\tau)] + \tau\cdot G(\alpha)\cdot\left(\frac{Q(\alpha)}{G(\alpha)}\right)'\left\{\left[1-\exp(-t/\tau)\right] - \frac{t}{\tau}\exp(-t/\tau)\right\}. \tag{11}$$

Another method of approximation that aims at the closure of Equ.(8) is **the decoupling approximation**, that does not a priory restrict the noise to small correlation times, however it neglects correlations between the response and the excitation and is therefore valid for narrow random excitations having narrow distributions, i.e. the noise intensity $D \ll 1$ (Hänggi & Jung 1995). Moreover, the **unified colored noise approximation** is an alternative method whose accuracy increases with the system's non-linear damping and decreases with color intensity (Jung & Hanggi 1987) (Luo & Zhu 2003).

It is evident that although no a priori restriction for the systems random input were made, both Equ.(3) and the colored noise master equation (Equ.(8)) are closed and/or computable only under specific assumptions/approximations. The probabilistic characterization of the response of a dynamical system under general colored excitation remains an open problem. In what follows we shall present an alternative approach that intends to contribute to its solution.

2.3. The RE theory in stochastic dynamics

The RE theory proposes another approach to deal with the controversy stemming from the stochastic dependence between response and excitation, that is to "accept" this dependence and focus on its study. This point of view, which necessarily must lean on the joint consideration of response and excitation, was apparently initiated by Lewis & Kraishman (Lewis & Kraichnan 1962), in the context of their study on the statistical formulation of the Navier-Stokes equations, as a generalization of Eberhard Hopf's approach to statistical formulation of turbulence (Hopf 1952), (see also (Beran 1986)). The means to treat jointly the probabilistic structure of response and excitation, leaving all the space for their stochastic dependence (to be determined during the solution of the problem) is the joint, response-excitation, characteristic functional, defined as follows:

$$\mathscr{F}_{xy}(u,\upsilon) \equiv \mathscr{F}_{xy}(u,\upsilon) = \\ = \int_{\mathscr{X}}\int_{\mathscr{Y}} \exp\left\{i\left(<u,x>+<\upsilon,y>\right)\right\}\mathscr{P}_{xy}(dxdy), \tag{1}$$

where

$\mathscr{P}_{\mathbf{xy}}(dxdy)$ is the joint RE probability measure on the Borel sets of the joint sample (function) space $\mathscr{X} \times \mathscr{Y}$, and

$<u,\boldsymbol{x}> + <v,\boldsymbol{y}>$ is the duality pairing in the dual topological system : $<(\mathscr{U},\mathscr{V}),(\mathscr{X},\mathscr{Y})> = <\mathscr{U}\times\mathscr{V}\ ,\ \mathscr{X}\times\mathscr{Y}>$.

On the basis of the above discussion the solution of problem (1) in the probability domain requires to find an extended mapping $\mathcal{T}_{ext}$ that is such that:

$$\mathscr{F}_{\boldsymbol{x}\boldsymbol{y}}(\bullet,\bullet,\bullet) = \mathcal{T}_{ext}\left[G(\bullet,\bullet),\mathscr{F}_{\boldsymbol{x}_0\boldsymbol{y}}(\cdot,\cdot)\right], \tag{2}$$

Following the methodology of Lewis & Kraichnan (Lewis & Kraichnan 1962) [see (Beran 1986)], Athanassoulis and Sapsis (Athanassoulis & Sapsis 2006; Sapsis & Athanassoulis 2006; Sapsis & Athanassoulis 2008) derived functional differential equations for the joint characteristic functional $\mathscr{F}_{\boldsymbol{xy}}(u,\boldsymbol{v})$ and showed that by appropriately projecting this infinite dimensional equation it is possible to obtain equations for the evolution of the joint REPDF.

2.3.1. The REPDF evolution equation in the scalar case

Let us consider the scalar, first-order, RDE:

$$\dot{x}(t;\theta) = \sum_{q,Q_1,Q_2} A_{q\,Q_1Q_2}(t)x^q(t;\theta)y_1^{Q_1}(t;\theta)y_2^{Q_2}(t;\theta), \tag{1a}$$

with stochastic initial condition

$$x(t_0) = x_0(\theta), \tag{1b}$$

where $y_1(t;\theta),\ y_2(t;\theta)$ are given smoothly correlated random functions, both defined on the common domain $\mathbb{T}\times\Theta = \left[t_0,T_*\right]\times\Theta$, Θ is an appropriate sample space [thus, θ is the stochastic argument], $x_0(\theta)$ is a given stochastic variable, and $A_{q\,Q_1Q_2}(t)$ are known deterministic functions.

The joint REPDF of the RDE (1) will follow the joint REDPF evolution equation (Athanassoulis & Sapsis 2006; Sapsis & Athanassoulis 2006; Sapsis & Athanassoulis 2008):

$$\left.\frac{\partial}{\partial t}f_{x(t)y_1(s)y_2(s)}(\alpha,\beta_1,\beta_2)\right|_{s\to t} + \\ + \frac{\partial}{\partial\alpha}\left[\sum_{q,\,Q_1,\,Q_2} A_{q,Q_1,Q_2}(t)\alpha^q\beta_1^{\ Q_1}\beta_2^{\ Q_2}f_{x(t)y_1(s)y_2(s)}(\alpha,\beta_1,\beta_2)\right] = 0, \tag{2}$$

supplemented by the initial conditions:

$$f_{x(t_0)y_1(s)y_2(s)}(\alpha,\beta_1,\beta_2) = f_{x(t_0)}(\alpha)\cdot f_{y_1(s)y_2(s)}(\beta_1,\beta_2) = \alpha \textit{ trivariate pdf known at any time } s \geq t_0 . \tag{3a}$$

the marginal-compatibility constrain:

$$\int_{\alpha\in\mathbb{R}} f_{x(t)y_1(s)y_2(s)}(\alpha,\beta_1,\beta_2)d\alpha = f_{y_1(s)y_2(s)}(\beta_1,\beta_2) = \alpha \textit{ bivariate pdf known at any time } s \geq t_0 , \quad \forall\, \beta\in\mathbb{R} , \tag{3b}$$

and constitutive conditions:

$$f_{x(t)y_1(s)y_2(s)}(\alpha,\beta_1,\beta_2) \geq 0 , \quad \int_{\alpha\in\mathbb{R}}\int_{\beta\in\mathbb{R}} f_{x(t)y_1(s)y_2(s)}(\alpha,\beta_1,\beta_2)d\alpha = 1 . \tag{3c,3d}$$

In what follows we shall review the derivation of the REPDF evolution equation following the same steps as Athanassoulis and Sapsis (Athanassoulis & Sapsis 2006; Sapsis & Athanassoulis 2006; Sapsis & Athanassoulis 2008), generalizing and focusing on 2D random systems. Preliminary results obtained during the period I was working on this thesis have been presented in (G. A. Athanassoulis et al. 2009; G.A. Athanassoulis et al. 2009).

2.4 Derivation of the REPDF evolution equation for the general 2D-system

2.4.1. Formulation of the problem

Let us consider the system of RDEs :

$$\dot{x}_n(t) = \sum_{q_1,q_2,q_3,q_4} A^{(n)}_{q_1q_2q_3q_4}(t)\cdot x_1^{q_1}(t)x_2^{q_2}(t)y_1^{q_3}(t;\theta)y_2^{q_4}(t;\theta) , \quad n=1,2 , \tag{1}$$

with initial conditions:

$$x_n(t_0) = x_{n0}(\theta) , \qquad n=1,2 . \tag{2}$$

where $y_1(t;\theta)$, $y_2(t;\theta)$ are given random functions, both defined on the common domain $\mathbb{T}\times\Theta = [t_0, T_*]\times\Theta$, Θ is an appropriate sample space [θ denotes the stochastic argument], $x_0(\theta)$ is a given random variable, and $A^{(n)}_{q_1q_2q_3q_4}(t)$, $n=1,2$ are known deterministic functions. Clearly, if a solution $x(t)$ of Equ.(1) exists, it will also be a stochastic function. Thus $x(t)$ may equally well be denoted as $x(t;\theta)$.

The summation is over non-negative integers, non-exceeding some maximum values. Some of the terms $A^{(n)}_{q_1q_2q_3q_4}(t)\cdot x_1^{q_1}(t)x_2^{q_2}(t)y_1^{q_3}(t;\theta)y_2^{q_4}(t;\theta)$ with indices (q_1,q_2,q_3,q_4) lying within the admissible range, might be absent because the corresponding coefficients $A^{(n)}_{q_1q_2q_3q_4}(t)$ are (taken to be) zero.

In some cases it is expedient to use a different letter for the exponents of $y_1(t;\theta)$ and $y_2(t;\theta)$ and the corresponding indices. When this is the case, Equ.(1) will be written in the form

$$\dot{x}_n(t;\theta) = \sum_{q_1,q_2,r_1,r_2} A^{(n)}_{q_1,q_2,r_1,r_2}(t)\cdot x_1^{q_1}(t;\theta)x_2^{q_2}(t;\theta)y_1^{r_1}(t;\theta)y_2^{r_2}(t;\theta), \quad n=1,2. \tag{3}$$

Equ.(1) (or equivalently Equ.(3)) can model any kind of random 2D systems with polynomial non-linearities. We shall present some interesting special cases:

1. Deterministic equation with random initial condition (without excitation):

$$\dot{x}_n(t;\theta) = \sum_{q_1 q_2} A^{(n)}_{q_1 q_2}(t)\cdot x_1^{q_1}(t;\theta)x_2^{q_2}(t;\theta), \quad n=1,2, \tag{4}$$

$$x_n(t_0) = x_{n0}(\theta), \quad n=1,2. \tag{5}$$

2. Deterministic equation with random initial condition and simple external random excitation

$$\dot{x}_n(t) = \sum_{q_1,q_2} A^{(n)}_{q_1 q_2}(t)\cdot x_1^{q_1}(t;\theta)x_2^{q_2}(t;\theta) + \sum_{r_n} A^{(n)}_{r_n}(t)\cdot y_n^{r_n}(t;\theta), \quad n=1,2, \tag{6}$$

$$x_n(t_0) = x_{n0}(\theta), \quad n=1,2. \tag{7}$$

3. Deterministic system with random initial conditions and self-multiplicative external random excitations

$$\dot{x}_n(t) = \sum_{q_1,q_2} A^{(n)}_{q_1 q_2}(t)\cdot x_1^{q_1}(t;\theta)x_2^{q_2}(t;\theta) + \sum_{q_3,q_4} A^{(n)}_{q_3 q_4}(t)\cdot y_1^{q_3}(t;\theta)y_2^{q_4}(t;\theta), \quad n=1,2, \tag{8}$$

$$x_n(t_0) = x_{n0}(\theta), \quad n=1,2. \tag{9}$$

4. Stochastic equation with random excitation (including random parametric excitation) and random initial conditions

$$\begin{aligned}\dot{x}_n(t) = {} & \sum_{q_1,q_2} A^{(n)}_{q_1 q_2}(t)\cdot x_1^{q_1}(t;\theta)x_2^{q_2}(t;\theta) + \sum_{q_3,q_4} A^{(n)}_{q_3 q_4}(t)\cdot y_1^{q_3}(t;\theta)y_2^{q_4}(t;\theta) + \\ & + a_1^{(n)}(t)\cdot x_1(t;\theta)\cdot y_1(t;\theta) + a_2^{(n)}(t)\cdot x_2(t;\theta)\cdot y_1(t;\theta)\cdot y_2(t;\theta), \quad n=1,2,\end{aligned} \tag{10}$$

$$x_n(t_0) = x_{n0}(\theta), \quad n=1,2. \tag{11}$$

More complicated forms of the cross-multiplicative terms (random parametric excitation) can easily be obtained.

2.4.2. The characteristic functional(s) associated with the system's RE

In Sections 2.4.2 -2.4.4 we shall use the general random 2D dynamical system (Equs.(1,2)_Sec(2.4.1)) in order to derive functional differential equations governing the evolution of the characteristic functional associated with the system's response and excitation random functions. The derivation follows the same steps as in (Athanassoulis 2009).

Assuming that the excitation function $\boldsymbol{y}(t;\theta)=\left(y_1(t;\theta),y_2(t;\theta)\right)$ is a given 2D stochastic function, taking values in an appropriate Banach space $\mathscr{Y}^{(2)}=\mathscr{Y}\times\mathscr{Y}$. Its probabilistic structure is fully described by means of its characteristic functional

$$\mathscr{F}_{\boldsymbol{y}}(\boldsymbol{\upsilon})\equiv\mathscr{F}_{y_1y_2}(\upsilon_1,\upsilon_2)=\int\limits_{\mathscr{Y}^{(2)}}\exp\{i<<\boldsymbol{\upsilon},\boldsymbol{\beta}>>\}\mathscr{P}_{\boldsymbol{y}}(d\boldsymbol{y}), \tag{1}$$

where

$\boldsymbol{\upsilon}=(\upsilon_1,\upsilon_2)\in\mathscr{V}^{(2)}=\mathscr{V}\times\mathscr{V}=\left(\mathscr{Y}^{(2)}\right)'=\textit{The topological dual of }\mathscr{Y}^{(2)}$,

$<<\boldsymbol{\upsilon},\boldsymbol{\beta}>> \;=\; <\upsilon_1,\beta_1>+<\upsilon_2,\beta_2>$,

$<\bullet,\bullet>$ is the duality pairing in the system $<\mathscr{V}=\mathscr{Y}',\mathscr{Y}>$

$<<\boldsymbol{\upsilon},\boldsymbol{\beta}>>$ is the duality pairing in $<\mathscr{V}^{(2)},\mathscr{Y}^{(2)}>$, and

$\mathscr{P}_{\boldsymbol{y}}(d\boldsymbol{y})$ is the probability measure on the Borel sets of the sample (function) space $\mathscr{Y}^{(2)}=\mathscr{Y}\times\mathscr{Y}$.

Concerning the response function $\boldsymbol{x}(t;\theta)=\left(x_1(t;\theta),x_2(t;\theta)\right)$, we assume that it exists and belongs to another (appropriate) Banach space $\mathscr{X}^{(2)}=\mathscr{X}\times\mathscr{X}$. Its probabilistic structure is also described by means of its characteristic functional

$$\mathscr{F}_{\boldsymbol{x}}(\boldsymbol{u})\equiv\mathscr{F}_{x_1x_2}(u_1,u_2)=\int\limits_{\mathscr{X}^{(2)}}\exp\{i<<\boldsymbol{u},\boldsymbol{\alpha}>>\}\mathscr{P}_{\boldsymbol{x}}(d\boldsymbol{\alpha}), \tag{2}$$

where

$\boldsymbol{u}=\left(u_1,u_2\right)\in\mathscr{U}^{(2)}=\mathscr{U}\times\mathscr{U}=\left(\mathscr{X}^{(2)}\right)'=\textit{The topological dual of }\mathscr{X}^{(2)}$,

$<<\boldsymbol{u},\boldsymbol{\alpha}>> \;=\; <u_1,\alpha_1>+<u_2,\alpha_2>$,

$<\bullet,\bullet>$ being the duality pairing in the system $<\mathscr{U}=\mathscr{X}',\mathscr{X}>$,

$<<\boldsymbol{\upsilon},\boldsymbol{\alpha}>>$ being the duality pairing in $<\mathscr{U}^{(2)},\mathscr{X}^{(2)}>$, and

$\mathscr{P}_{\boldsymbol{x}}(d\boldsymbol{\alpha})$ is the probability measure on the Borel sets of the sample (function) space $\mathscr{X}^{(2)}=\mathscr{X}\times\mathscr{X}$.

Solving the Problem given by Equ.(1, 2)_Sec(2.4.1) in the probabilistic domain means to determine the probability measure $\mathscr{P}_{\boldsymbol{x}}(d\boldsymbol{\alpha})$ [equivalently, characteristic functional $\mathscr{F}_{\boldsymbol{x}}(\boldsymbol{u})$], of the response function $\boldsymbol{x}(t)=\left(x_1(t),x_2(t)\right)$, in terms of the system parameters $A^{(n)}_{q_1q_2q_3q_4}(t)$, the probability distribution $F_{x_{10}x_{20}}(\alpha_1,\alpha_2)$ of the initial values, and the probability measure $\mathscr{P}_{\boldsymbol{y}}(d\boldsymbol{\beta})$ [equivalently, characteristic functional $\mathscr{F}_{\boldsymbol{y}}(\boldsymbol{\upsilon})$], of the excitation function $\boldsymbol{y}(t;\theta)=\left(y_1(t;\theta),y_2(t;\theta)\right)$.

To pursue in this direction the joint, response-excitation, characteristic functional is considered

$$\mathscr{F}_{\boldsymbol{xy}}(\boldsymbol{u},\boldsymbol{\upsilon}) \equiv \mathscr{F}_{x_1x_2y_1y_2}\left(u_1,u_2,\upsilon_1,\upsilon_2\right) = \\ = \int_{\mathscr{X}^{(2)}}\int_{\mathscr{Y}^{(2)}} \exp\left\{i\left(<<\boldsymbol{u},\boldsymbol{\alpha}>>+<<\boldsymbol{\upsilon},\boldsymbol{\beta}>>\right)\right\}\mathscr{P}_{\boldsymbol{xy}}\left(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\right), \tag{3}$$

where

$\mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta})$ is the joint, response-excitation, probability measure on the Borel sets of the joint sample (function) space $\mathscr{X}^{(2)}\times\mathscr{Y}^{(2)}=\mathscr{X}\times\mathscr{X}\times\mathscr{Y}\times\mathscr{Y}$, and

$$<<\boldsymbol{u},\boldsymbol{\alpha}>>+<<\boldsymbol{\upsilon},\boldsymbol{\beta}>> \;=\; <u_1,\alpha_1>+<u_2,\alpha_2>+<\upsilon_1,\beta_1>+<\upsilon_2,\beta_2>$$

is the duality pairing in the system

$$<\left(\mathscr{U}^{(2)},\mathscr{V}^{(2)}\right),\left(\mathscr{X}^{(2)},\mathscr{Y}^{(2)}\right)> = \\ = <\mathscr{U}\times\mathscr{U}\times\mathscr{V}\times\mathscr{V}\;,\;\mathscr{X}\times\mathscr{X}\times\mathscr{Y}\times\mathscr{Y}>$$

Following the methodology of Hopf (1952) [see also Beran 1968], we shall first derive functional differential equations for the joint characteristic functional $\mathscr{F}_{\boldsymbol{xy}}(\boldsymbol{u},\boldsymbol{\upsilon})$.

2.4.3. Functional derivatives of the joint RE characteristic functional

To facilitate the symbolics in performing the functional differentiation, we write the joint characteristic functional in the following form

$$\mathscr{F}_{\boldsymbol{xy}}(\boldsymbol{u},\boldsymbol{\upsilon}) \equiv \mathscr{F}_{x_1x_2y_1y_2}\left(u_1,u_2,\upsilon_1,\upsilon_2\right) = \\ = \int_{\mathscr{X}^{(2)}}\int_{\mathscr{Y}^{(2)}} \exp\left\{i\left(<<\boldsymbol{u},\boldsymbol{\alpha}>>+<<\boldsymbol{\upsilon},\boldsymbol{\beta}>>\right)\right\}\mathscr{P}_{\boldsymbol{xy}}\left(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\right) = \\ = \int\int \exp\left\{i<u_1,\alpha_1>\right\}\exp\left\{i<u_2,\alpha_2>\right\}\exp\left\{i<\upsilon_1,\beta_1>\right\}\exp\left\{i<\upsilon_2,\beta_2>\right\}\mathscr{P}_{\boldsymbol{xy}}\left(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\right)$$

and we shall abbreviate it as follows, in accordance with our special needs:

$$\begin{aligned}\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon}) &= \int\int \exp\left\{i < u_1, \alpha_1 >\right\} \dots \mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\, d\boldsymbol{\beta}) = [\ or\ ] \\ &= \int\int \exp\left\{i < u_2, \alpha_2 >\right\} \dots \mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\, d\boldsymbol{\beta}) = [\ or\ ] \\ &= \int\int \exp\left\{i < \upsilon_1, \beta_1 >\right\} \dots \mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\, d\boldsymbol{\beta}) = [\ or\ ] \\ &= \int\int \exp\left\{i < \upsilon_2, \beta_2 >\right\} \dots \mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\, d\boldsymbol{\beta}).\end{aligned}$$

There are four first-order Gateaux functional derivatives of $\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon}) \equiv \mathscr{F}\left(u_1,u_2,\upsilon_1,\upsilon_2\right)$. The one with respect to the first variable (u_1), taken along the direction h_{u_1} is defined and calculated as follows

$$\begin{aligned}\delta_{u_1}\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon}) = \delta_{u_1}\mathscr{F}\left(\left[u_1;h_{u_1}\right],u_2,\upsilon_1,\upsilon_2\right) = \lim_{\varepsilon\to 0}\frac{\mathscr{F}\left(u_1+\varepsilon h_{u_1},u_2,\upsilon_1,\upsilon_2\right)-\mathscr{F}\left(u_1,u_2,\upsilon_1,\upsilon_2\right)}{\varepsilon} = \\ = \lim_{\varepsilon\to 0}\frac{\int\int\left(\exp\left\{i<u_1+\varepsilon h_{u_1},\alpha_1>\right\}-\exp\left\{i<u_1,\alpha_1>\right\}\right)\dots\mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta})}{\varepsilon} = \\ = \lim_{\varepsilon\to 0}\int\int\frac{\left(\exp\left\{i<u_1+\varepsilon h_{u_1},\alpha_1>\right\}-\exp\left\{i<u_1,\alpha_1>\right\}\right)}{\varepsilon}\dots\mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}) = \\ = \lim_{\varepsilon\to 0}\int\int\frac{\left(\exp\left\{i<\varepsilon h_{u_1},\alpha_1>\right\}-1\right)\exp\left\{i<u_1,\alpha_1>\right\}}{\varepsilon}\dots\mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}) = \\ \left[\textit{assuming that the } \lim_{\varepsilon\to 0} \textit{ may pass through the functional integral sign}\right] \\ = \int\int\left(\lim_{\varepsilon\to 0}\frac{\left(\exp\left\{i<\varepsilon h_{u_1},\alpha_1>\right\}-1\right)\exp\left\{i<u_1,\alpha_1>\right\}}{\varepsilon}\right)\dots\mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}) = \\ = \int\int i<h_{u_1},\alpha_1>\exp\left\{i<u_1,\alpha_1>\right\}\dots\mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}),\end{aligned}$$

that is

$$\begin{aligned}\delta_{u_1}\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon}) &= \delta_{u_1}\mathscr{F}\left(\left[u_1;h_{u_1}\right],u_2,\upsilon_1,\upsilon_2\right) = \\ &= \int\int i<h_{u_1},\alpha_1>\exp\{\cdots\}\,\mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}),\end{aligned} \tag{1}$$

where $\exp\{\cdots\}$ stands for $\exp\left\{i\left(<<\boldsymbol{u},\boldsymbol{\alpha}>>+<<\boldsymbol{\upsilon},\boldsymbol{\beta}>>\right)\right\}$.

The definition and corresponding calculations for the other three Gateaux functional derivatives are similar. The results are as follows:

$$\begin{aligned}\delta_{u_2}\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon}) &= \delta_{u_2}\mathscr{F}\left(u_1,\left[u_2;h_{u_2}\right],\upsilon_1,\upsilon_2\right) = \\ &= \int\int i<h_{u_2},\alpha_2>\exp\{\cdots\}\,\mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}),\end{aligned} \tag{2}$$

$$\delta_{\upsilon_1}\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon}) = \delta_{\upsilon_1}\mathscr{F}\left(u_1,u_2,\left[\upsilon_1;h_{\upsilon_1}\right],\upsilon_2\right) = \\ = \int\int i < h_{\upsilon_1},\beta_1 > \exp\{\cdots\}\,\mathscr{P}_{xy}\left(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\right), \tag{3}$$

$$\delta_{\upsilon_2}\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon}) = \delta_{\upsilon_2}\mathscr{F}\left(u_1,u_2,\upsilon_1,\left[\upsilon_2;h_{\upsilon_2}\right]\right) = \\ = \int\int i < h_{\upsilon_2},\beta_2 > \exp\{\cdots\}\,\mathscr{P}_{xy}\left(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\right). \tag{4}$$

Applying Equ.(1) and Equ.(2) for $h_{u_n} = \delta_t(\cdot)$, the Dirac delta functional supported at t, we obtain

$$\delta_{u_1}\mathscr{F}\left(\left[u_1;\delta_t(\cdot)\right],u_2,\upsilon_1,\upsilon_2\right) = \int\int i < \delta_t(\cdot),\alpha_1 > \exp\{\cdots\}\,\mathscr{P}_{xy}\left(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\right) = \\ = \int\int i\,\alpha_1(t)\,\exp\{\cdots\}\,\mathscr{P}_{xy}\left(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\right),$$

$$\delta_{u_2}\mathscr{F}\left(u_1,\left[u_2;\delta_t(\cdot)\right],\upsilon_1,\upsilon_2\right) = \int\int i < \delta_t(\cdot),\beta_2 > \exp\{\cdots\}\,\mathscr{P}_{xy}\left(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\right) = \\ = \int\int i\,\beta_2(t)\,\exp\{\cdots\}\,\mathscr{P}_{xy}\left(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\right).$$

Differentiating now with respect to time, assuming that time derivative $\frac{d\bullet}{dt}$ can be inter-changed with the functional integral $\int\int(\cdots)\,\mathscr{P}_{xy}\left(d\boldsymbol{x}d\boldsymbol{y}\right)$, we obtain

$$\frac{1}{i}\frac{d}{dt}\delta_{u_1}\mathscr{F}\left(\left[u_1;\delta_t(\cdot)\right],u_2,\upsilon_1,\upsilon_2\right) = \int\int \dot{\alpha}_1(t)\,\exp\{\cdots\}\,\mathscr{P}_{xy}\left(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\right), \tag{5}$$

$$\frac{1}{i}\frac{d}{dt}\delta_{u_2}\mathscr{F}\left(u_1,\left[u_2;\delta_t(\cdot)\right],\upsilon_1,\upsilon_2\right) = \int\int \dot{\alpha}_2(t)\,\exp\{\cdots\}\,\mathscr{P}_{xy}\left(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\right). \tag{6}$$

Assume now that q_1 is a non-negative integer, and apply q_1-fold Gateaux differentiation to the first functional variable u_1 in the directions $h_{u_1}^{(1)},h_{u_1}^{(2)},...,h_{u_1}^{(q_1)}$:

$$\delta_{u_1}^{(q_1)}\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon}) = \delta_{u_1}^{(q_1)}\mathscr{F}\left(\left[u_1;h_{u_1}^{(1)},h_{u_1}^{(2)},...,h_{u_1}^{(q_1)}\right],u_2,\upsilon_1,\upsilon_2\right) = \\ = \int\int i^{q_1} < h_{u_1}^{(1)},\alpha_1 >< h_{u_1}^{(2)},\alpha_1 > ... < h_{u_1}^{(q_1)},\alpha_1 > \exp\{\cdots\}\,\mathscr{P}_{xy}\left(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\right).$$

In a similar manner, assuming that all functional derivatives considered do exist, we obtain the following result for the $\left(q_1+q_2+q_3+q_4\right)-$ fold derivative with respect to $u_1,u_2,\upsilon_1,\upsilon_2$ successively, as indicated in the notation:

$$\delta_{u_1}^{(q_1)}\delta_{u_2}^{(q_2)}\delta_{\upsilon_1}^{(q_3)}\delta_{\upsilon_2}^{(q_4)}\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon}) =$$
$$= \delta_{u_1}^{(q_1)}\delta_{u_2}^{(q_2)}\delta_{\upsilon_1}^{(q_3)}\delta_{\upsilon_2}^{(q_4)}\mathscr{F}\left(\left[u_1;h_{u_1}^{(1)},...,h_{u_1}^{(q_1)}\right],\left[u_2;h_{u_2}^{(1)},...,h_{u_2}^{(q_2)}\right],\left[\upsilon_1;h_{\upsilon_1}^{(1)},...,h_{\upsilon_1}^{(q_3)}\right],\left[\upsilon_2;h_{\upsilon_2}^{(1)},...,h_{\upsilon_2}^{(q_4)}\right]\right)=$$
$$= \int\int i^{\,q_1+q_2+q_3+q_4}\Big(<h_{u_1}^{(1)},\alpha_1>...<h_{u_1}^{(q_1)},\alpha_1>\cdot<h_{u_2}^{(1)},\alpha_2>...<h_{u_2}^{(q_2)},\alpha_2>\cdot$$
$$\cdot<h_{\upsilon_1}^{(1)},\beta_1>...<h_{\upsilon_1}^{(q_3)},\beta_1>\cdot<h_{\upsilon_2}^{(1)},\beta_2>...<h_{\upsilon_2}^{(q_4)},\beta_2>\cdot\exp\{\cdots\}\,\mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}).\tag{7}$$

Applying the above general formula (Equ.(7)) to the directions

$$h_{u_1}^{(1)}=...=h_{u_1}^{(q_1)}=h_{u_2}^{(1)}=...=h_{u_2}^{(q_2)}=h_{\upsilon_1}^{(1)}=...=h_{\upsilon_1}^{(q_3)}=h_{\upsilon_2}^{(1)}=...=h_{\upsilon_2}^{(q_4)}=\delta_t(\cdot),$$

we find

$$\delta_{u_1}^{(q_1)}\delta_{u_2}^{(q_2)}\delta_{\upsilon_1}^{(q_3)}\delta_{\upsilon_2}^{(q_4)}\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon}) =$$
$$= \delta_{u_1}^{(q_1)}\delta_{u_2}^{(q_2)}\delta_{\upsilon_1}^{(q_3)}\delta_{\upsilon_2}^{(q_4)}\mathscr{F}\left(\left[u_1;\delta_t(\cdot),...,\delta_t(\cdot)\right],\left[u_2;\delta_t(\cdot),...,\delta_t(\cdot)\right],\left[\upsilon_1;\delta_t(\cdot),...,\delta_t(\cdot)\right],\left[\upsilon_2;\delta_t(\cdot),...,\delta_t(\cdot)\right]\right)=$$
$$= \int\int i^{\,q_1+q_2+q_3+q_4}\cdot\alpha_1^{q_1}(t)\,\alpha_2^{q_2}(t)\,\beta_1^{q_3}(t)\,\beta_2^{q_4}(t)\cdot\exp\{\cdots\}\,\mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}).$$

Thus

$$\frac{1}{i^{\,q_1+q_2+q_3+q_4}}\,\delta_{u_1}^{(q_1)}\delta_{u_2}^{(q_2)}\delta_{\upsilon_1}^{(q_3)}\delta_{\upsilon_2}^{(q_4)}\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon}) = \int\int\alpha_1^{q_1}(t)\,\alpha_2^{q_2}(t)\,\beta_1^{q_3}(t)\,\beta_2^{q_4}(t)\cdot\exp\{\cdots\}\,\mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}).\tag{8}$$

2.4.4. The functional differential equations

Combining now the differential system given by Equs.(1, 2)_Sec(2.4.1) with the functional derivatives given by Equs.(5, 6, 8)_Sec(2.4.3), we obtain

$$\frac{1}{i}\frac{d}{dt}\delta_{u_n}\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon}) - \sum_{q_1,q_2,q_3,q_4}\frac{1}{i^{\,q_1+q_2+q_3+q_4}}A_{q_1q_2q_3q_4}^{(n)}(t)\,\delta_{u_1}^{(q_1)}\delta_{u_2}^{(q_2)}\delta_{\upsilon_1}^{(q_3)}\delta_{\upsilon_2}^{(q_4)}\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon}) =$$
$$= \int\int\left(\dot{\alpha}_n(t)-\sum_{q_1,q_2,q_3,q_4}A_{q_1q_2q_3q_4}^{(n)}(t)\,\alpha_1^{q_1}(t)\,\alpha_2^{q_2}(t)\,\beta_1^{q_3}(t)\,\beta_2^{q_4}(t)\right)\exp\{\cdots\}\,\mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}) = 0,\tag{1}$$
$$n=1,2\,.$$

Thus, we have established that the following

a. Functional Differential Equation:

$$\frac{1}{i}\frac{d}{dt}\delta_{u_n}\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon}) - \sum_{q_1,q_2,q_3,q_4}\frac{1}{i^{q_1+q_2+q_3+q_4}}A^{(n)}_{q_1q_2q_3q_4}(t)\,\delta^{(q_1)}_{u_1}\delta^{(q_2)}_{u_2}\delta^{(q_3)}_{\upsilon_1}\delta^{(q_4)}_{\upsilon_2}\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon}) = 0, \quad (\boldsymbol{u},\boldsymbol{\upsilon})\in\left(\mathscr{U}^{(2)},\mathscr{V}^{(2)}\right), \quad n=1,2. \tag{2}$$

The above **functional differential equation** (2) should be supplemented by appropriate **compatibility conditions** and **initial conditions** for the joint characteristic functional.

b. Compatibility conditions:

Although the joint characteristic functional $\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon})=\mathscr{F}_{\boldsymbol{xy}}(\boldsymbol{u},\boldsymbol{\upsilon})$ is the principal unknown quantity, the marginal characteristic functional $\mathscr{F}_{\boldsymbol{y}}(\boldsymbol{\upsilon})=\mathscr{F}_{\boldsymbol{xy}}(\boldsymbol{0},\boldsymbol{\upsilon})=\ \mathscr{F}_{\boldsymbol{xy}}\left(0,0,\upsilon_1,\upsilon_2\right)$, related to the excitation only, is known, i.e.

$$\mathscr{F}_{\boldsymbol{y}}(\boldsymbol{\upsilon})=\mathscr{F}_{\boldsymbol{xy}}(\boldsymbol{0},\boldsymbol{\upsilon})=\mathscr{F}_{\boldsymbol{xy}}\left(0,0,\upsilon_1,\upsilon_2\right) = \textit{a known characteristic functional}, \quad \textit{where} \quad \boldsymbol{\upsilon}=\left(\upsilon_1,\upsilon_2\right)\in\mathscr{V}^{(2)}=\mathscr{V}\times\mathscr{V}. \tag{3}$$

c. Initial conditions:

The joint characteristic function- functional of the initial state – excitation of the system,

$$\left(x_1(t_0),x_2(t_0),y_1(\overset{t}{\underset{t_0}{s}}),y_2(\overset{t}{\underset{t_0}{s}})\right)=\left(x_{10}(\theta),x_{20}(\theta),y_1(\overset{t}{\underset{t_0}{s}};\theta),y_2(\overset{t}{\underset{t_0}{s}};\theta)\right),$$

is known, i.e., if we set $u_n=\tilde{u}_n\delta_{t_0}(\cdot)$, $\tilde{u}_n\in\mathbb{R}$, $n=1,2$, then

$$\mathscr{F}_{\boldsymbol{xy}}\left(\tilde{u}_1\delta_{t_0}(\cdot),\tilde{u}_2\delta_{t_0}(\cdot),\upsilon_1,\upsilon_2\right) = \textit{a known characteristic function with respect to } \left(\tilde{u}_1,\tilde{u}_2\right)\in\mathbb{R}^2 \textit{ and a known characteristic functional with respect to } \left(\upsilon_1,\upsilon_2\right)\in\mathscr{V}^{(2)}. \tag{4}$$

Making the plausible assumption that the initial state is independent from the history of the excitation, the initial condition is simplified as follows

$$\mathscr{F}_{\boldsymbol{xy}}\left(\tilde{u}_1\delta_{t_0}(\cdot),\tilde{u}_2\delta_{t_0}(\cdot),\,0,\,0\right) = \varphi_{x_1(t_0)x_2(t_0)}\left(\tilde{u}_1,\tilde{u}_2\right) = \textit{a known characteristic function} \quad \left(\tilde{u}_1,\tilde{u}_2\right)\in\mathbb{R}^2 \tag{5}$$

The derivation of Equ.(5) goes as follows:

$$\mathscr{F}\left(\tilde{u}_1\delta_{t_0}(\cdot),\tilde{u}_2\delta_{t_0}(\cdot),0,0\right) =$$
$$= \int\int \exp\left\{i < \tilde{u}_1\delta_{t_0}(\cdot), \alpha_1(\cdot) >\right\} \exp\left\{i < \tilde{u}_2\delta_{t_0}(\cdot), \alpha_2(\cdot) >\right\} \exp\left\{i < 0, \beta_1 >\right\} \exp\left\{i < 0, \beta_2 >\right\} \mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\, d\boldsymbol{\beta})$$
$$= \int \exp\left\{i < \tilde{u}_1\delta_{t_0}(\cdot), \alpha_1(\cdot) >\right\} \exp\left\{i < \tilde{u}_2\delta_{t_0}(\cdot), \alpha_2(\cdot) >\right\} \mathscr{P}_{\boldsymbol{x}}(d\boldsymbol{\alpha}) =$$
$$= \int_{\mathbb{R}\times\mathbb{R}} \exp\left\{i\left(\tilde{u}_1\alpha_1 + \tilde{u}_2\alpha_2\right)\right\} f_{x_1(t_0)x_2(t_0)}(\alpha_1,\alpha_2)\, d\alpha_1 d\alpha_2 = \varphi_{x_1(t_0)x_2(t_0)}\left(\tilde{u}_1,\tilde{u}_2\right),$$

where $\varphi_{x_1(t_0)x_2(t_0)}(\tilde{u}_1,\tilde{u}_2)$ is the (known) joint characteristic function of the initial state $\left(x_{10}(\theta), x_{20}(\theta)\right)$.

2.4.5. Projection of the FDEs to finite dimensions. Derivation of an equation for the joint RE characteristic function

We shall first recall the relation between the (infinite-dimensional) joint, response-excitation, characteristic functional and a specific, finite-dimensional joint, response-excitation, characteristic function.

Consider the joint characteristic functional

$$\mathscr{F}_{\boldsymbol{xy}}(\boldsymbol{u},\boldsymbol{v}) \equiv \mathscr{F}_{x_1x_2y_1y_2}\left(u_1,u_2,v_1,v_2\right) =$$
$$= \int_{\mathscr{X}^{(2)}} \int_{\mathscr{Y}^{(2)}} \exp\left\{i\left(<<\boldsymbol{u},\boldsymbol{\alpha}>> + <<\boldsymbol{v},\boldsymbol{\beta}>>\right)\right\} \mathscr{P}_{\boldsymbol{xy}}\left(d\boldsymbol{\alpha}\, d\boldsymbol{\beta}\right) =$$
$$= \int\int \exp\left\{i < u_1, \alpha_1 >\right\} \exp\left\{i < u_2, \alpha_2 >\right\} \exp\left\{i < v_1, \beta_1 >\right\} \exp\left\{i < v_2, \beta_2 >\right\} \mathscr{P}_{\boldsymbol{xy}}(d\boldsymbol{\alpha}\, d\boldsymbol{\beta})$$

and make the substitution

$$\mathscr{U}\times\mathscr{U} \ni \boldsymbol{u} = \left(u_1, u_2\right) = \left(\tilde{u}_1\delta_{t_1}(\cdot), \tilde{u}_2\delta_{t_2}(\cdot)\right) \equiv \tilde{\boldsymbol{u}} \underset{\mathbf{1,2}}{\otimes} \boldsymbol{\delta}_t(\cdot), \quad \text{where } \left(\tilde{u}_1, \tilde{u}_2\right) \in \mathbb{R}^2,$$

and

$$\mathscr{V}\times\mathscr{V} \ni \boldsymbol{v} = \left(v_1, v_2\right) = \left(\tilde{v}_1\delta_{s_1}(\cdot), \tilde{v}_2\delta_{s_2}(\cdot)\right) \equiv \tilde{\boldsymbol{v}} \underset{\mathbf{1,2}}{\otimes} \boldsymbol{\delta}_s(\cdot), \text{ where } \left(\tilde{v}_1, \tilde{v}_2\right) \in \mathbb{R}^2.$$

Then, we obtain

$$\mathscr{F}_{\boldsymbol{xy}}(\boldsymbol{u},\boldsymbol{v})\Big|_{\substack{\boldsymbol{u}=\tilde{\boldsymbol{u}}\otimes_{1,2}\boldsymbol{\delta}_t(\cdot) \\ \boldsymbol{v}=\tilde{\boldsymbol{v}}\otimes_{1,2}\boldsymbol{\delta}_s(\cdot)}} \equiv \mathscr{F}_{\boldsymbol{xy}}\left(\tilde{u}_1\delta_{t_1}(\cdot), \tilde{u}_2\delta_{t_2}(\cdot), \tilde{v}_1\delta_{s_1}(\cdot), \tilde{v}_2\delta_{s_2}(\cdot)\right) =$$
$$= \int_{\mathscr{X}^{(2)}} \int_{\mathscr{Y}^{(2)}} \exp\left\{i\left(\tilde{u}_1\alpha_1(t_1) + \tilde{u}_2\alpha_2(t_2) + \tilde{v}_1\beta_1(s_1) + \tilde{v}_2\beta_2(s_2)\right)\right\} \mathscr{P}_{\boldsymbol{xy}}\left(d\boldsymbol{\alpha}\, d\boldsymbol{\beta}\right) = \qquad (*)$$

$$= \int_{\mathbb{R}^{(2)}} \int_{\mathbb{R}^{(2)}} \exp\left\{ i\left(\tilde{u}_1\alpha_1 + \tilde{u}_2\alpha_2 + \tilde{\upsilon}_1\beta_1 + \tilde{\upsilon}_2\beta_2 \right) \right\} P_{x_1(t_1)x_2(t_2)y_1(s_1)y_2(s_2)}\left(d\boldsymbol{\alpha}\, d\boldsymbol{\beta} \right) \equiv$$

$$\equiv \int_{\mathbb{R}^{(2)}} \int_{\mathbb{R}^{(2)}} \exp\left\{ i\left(\tilde{u}_1\alpha_1 + \tilde{u}_2\alpha_2 + \tilde{\upsilon}_1\beta_1 + \tilde{\upsilon}_2\beta_2 \right) \right\} d^4 F_{x_1(t_1)x_2(t_2)y_1(s_1)y_2(s_2)}\left(\alpha_1, \alpha_2, \beta_1, \beta_2 \right) = (**)$$

$$= \varphi_{x_1(t_1)x_2(t_2)y_1(s_1)y_2(s_2)}\left(\tilde{u}_1, \tilde{u}_2, \tilde{\upsilon}_1, \tilde{\upsilon}_2 \right),$$

that is

$$\mathscr{F}_{\boldsymbol{xy}}\left(\boldsymbol{u}, \boldsymbol{\upsilon} \right)\Big|_{\substack{\boldsymbol{u} = \tilde{\boldsymbol{u}} \otimes_{1,2} \boldsymbol{\delta}_t(\cdot) \\ \boldsymbol{\upsilon} = \tilde{\boldsymbol{\upsilon}} \otimes_{1,2} \boldsymbol{\delta}_s(\cdot)}} \equiv \mathscr{F}_{\boldsymbol{xy}}\left(\tilde{u}_1\delta_{t_1}(\cdot), \tilde{u}_2\delta_{t_2}(\cdot), \tilde{\upsilon}_1\delta_{s_1}(\cdot), \tilde{\upsilon}_2\delta_{s_2}(\cdot) \right) = \varphi_{x_1(t_1)x_2(t_2)y_1(s_1)y_2(s_2)}\left(\tilde{u}_1, \tilde{u}_2, \tilde{\upsilon}_1, \tilde{\upsilon}_2 \right). \qquad (1)$$

If, in addition the vector $x_1(t_1)x_2(t_2)y_1(s_1)y_2(s_2)$ is continuously distributed, then

$$\mathscr{F}_{\boldsymbol{xy}}\left(\boldsymbol{u}, \boldsymbol{\upsilon} \right)\Big|_{\substack{\boldsymbol{u} = \tilde{\boldsymbol{u}} \otimes_{1,2} \boldsymbol{\delta}_t(\cdot) \\ \boldsymbol{\upsilon} = \tilde{\boldsymbol{\upsilon}} \otimes_{1,2} \boldsymbol{\delta}_s(\cdot)}} \equiv \mathscr{F}_{\boldsymbol{xy}}\left(\tilde{u}_1\delta_{t_1}(\cdot), \tilde{u}_2\delta_{t_2}(\cdot), \tilde{\upsilon}_1\delta_{s_1}(\cdot), \tilde{\upsilon}_2\delta_{s_2}(\cdot) \right) =$$

$$= \varphi_{x_1(t_1)x_2(t_2)y_1(s_1)y_2(s_2)}\left(\tilde{u}_1, \tilde{u}_2, \tilde{\upsilon}_1, \tilde{\upsilon}_2 \right) =$$

$$= \int_{\mathbb{R}^{(2)}} \int_{\mathbb{R}^{(2)}} \exp\left\{ i\left(\tilde{u}_1 x_1 + \tilde{u}_2 x_2 + \tilde{\upsilon}_1 y_1 + \tilde{\upsilon}_2 y_2 \right) \right\} f_{x_1(t_1)x_2(t_2)y_1(s_1)y_2(s_2)}\left(\alpha_1, \alpha_2, \beta_1, \beta_2 \right) d\alpha_1 d\alpha_2 d\beta_1 d\beta_2. \qquad (2)$$

Now, exploiting the above equations, we shall express the time derivatives $\frac{d}{dt}\delta_{u_1}\mathscr{F}$ and $\frac{d}{dt}\delta_{u_2}\mathscr{F}$, applied to appropriate functional derivatives of the characteristic functional, as time derivatives applied to the corresponding characteristic function.

From

$$\frac{1}{i}\frac{d}{dt}\delta_{u_1}\mathscr{F}\left(\left[u_1; \delta_t(\cdot) \right], u_2, \upsilon_1, \upsilon_2 \right) = \int \int \dot{\alpha}_1(t) \exp\{\cdots\}\, \mathscr{P}_{\boldsymbol{xy}}\left(d\boldsymbol{\alpha}\, d\boldsymbol{\beta} \right),$$

$$\frac{1}{i}\frac{d}{dt}\delta_{u_2}\mathscr{F}\left(u_1, \left[u_2; \delta_t(\cdot) \right], \upsilon_1, \upsilon_2 \right) = \int \int \dot{\alpha}_2(t) \exp\{\cdots\}\, \mathscr{P}_{\boldsymbol{xy}}\left(d\boldsymbol{\alpha}\, d\boldsymbol{\beta} \right).$$

We obtain

$$\begin{aligned}
&\frac{i\tilde{u}_1}{i}\frac{d}{dt}\delta_{u_1}\mathscr{F}\left(\left[u_1;\delta_t(\cdot)\right],u_2,\upsilon_1,\upsilon_2\right) + \frac{i\tilde{u}_2}{i}\frac{d}{dt}\delta_{u_2}\mathscr{F}\left(u_1,\left[u_2;\delta_t(\cdot)\right],\upsilon_1,\upsilon_2\right) = \\
&= \int\int i\tilde{u}_1\dot{\alpha}_1(t)\exp\{\cdots\}\,\mathscr{P}_{xy}(d\mathbf{x}d\mathbf{y}) + \int\int i\tilde{u}_2\dot{\alpha}_2(t)\exp\{\cdots\}\,\mathscr{P}_{xy}(d\mathbf{x}d\mathbf{y}) = \\
&= \int\int\left(i\tilde{u}_1\dot{\alpha}_1(t) + i\tilde{u}_2\dot{\alpha}_2(t)\right)\exp\{\cdots\}\,\mathscr{P}_{xy}(d\mathbf{x}d\mathbf{y}) = \;[\textit{see proof below}] \\
&= \frac{\partial}{\partial t}\varphi_{x_1(t)x_2(t)y_1(s)y_2(s)}\left(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2\right)
\end{aligned} \tag{3}$$

►The proof of the last equality is based on the equation $(*)=(**)$, already established above, with $t_1=t_2=t$ and $s_1=s_2=s$:

$$\begin{aligned}
&\varphi_{x_1(t)x_2(t)y_1(s)y_2(s)}\left(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2\right) = \\
&= \int\int\exp\left\{i\left(\tilde{u}_1\alpha_1(t)+\tilde{u}_2\alpha_2(t)+\tilde{\upsilon}_1\beta_1(s)+\tilde{\upsilon}_2\beta_2(s)\right)\right\}\mathscr{P}_{xy}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta})
\end{aligned} \tag{4}$$

Differentiating both sides of the above equation with respect to the current time t, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\varphi_{x_1(t)x_2(t)y_1(s)y_2(s)}\left(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2\right) = \\
&= \frac{\partial}{\partial t}\int\int\exp\left\{i\left(\tilde{u}_1\alpha_1(t)+\tilde{u}_2\alpha_2(t)+\tilde{\upsilon}_1\beta_1(s)+\tilde{\upsilon}_2\beta_2(s)\right)\right\}\mathscr{P}_{xy}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}) = \\
&= \int\int\frac{\partial}{\partial t}\exp\left\{i\left(\tilde{u}_1\alpha_1(t)+\tilde{u}_2\alpha_2(t)+\tilde{\upsilon}_1\beta_1(s)+\tilde{\upsilon}_2\beta_2(s)\right)\right\}\mathscr{P}_{xy}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}) = \\
&= \int\int\left(i\tilde{u}_1\dot{\alpha}_1(t)+i\tilde{u}_2\dot{\alpha}_2(t)\right)\exp\left\{i\left(\tilde{u}_1\alpha_1(t)+\tilde{u}_2\alpha_2(t)+\tilde{\upsilon}_1\beta_1(s)+\tilde{\upsilon}_2\beta_2(s)\right)\right\}\mathscr{P}_{xy}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta})
\end{aligned}$$

which concludes the proof of Equ.(3) ◄

Consider now the following expression, corresponding to the second term of the left hand side of Equ.(2)_ Sec(2.4.4), evaluated at

$$(u_1,u_2) = (\tilde{u}_1\delta_t(\cdot),\tilde{u}_2\delta_t(\cdot)) \text{ and } (\upsilon_1,\upsilon_2) = (\tilde{\upsilon}_1\delta_s(\cdot),\tilde{\upsilon}_2\delta_s(\cdot)),$$

$$\begin{aligned}
&i\tilde{u}_1\sum_{q_1,q_2,q_3,q_4}\frac{1}{i^{q_1+q_2+q_3+q_4}}A^{(1)}_{q_1q_2q_3q_4}(t)\,\delta^{(q_1)}_{u_1}\delta^{(q_2)}_{u_2}\delta^{(q_3)}_{\upsilon_1}\delta^{(q_4)}_{\upsilon_2}\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon})\Big|_{\substack{\boldsymbol{u}=\tilde{\boldsymbol{u}}\otimes_{1,2}\boldsymbol{\delta}_t(\cdot)\\ \boldsymbol{\upsilon}=\tilde{\boldsymbol{\upsilon}}\otimes_{1,2}\boldsymbol{\delta}_s(\cdot)}} + \\
&+ i\tilde{u}_2\sum_{q_1,q_2,q_3,q_4}\frac{1}{i^{q_1+q_2+q_3+q_4}}A^{(2)}_{q_1q_2q_3q_4}(t)\,\delta^{(q_1)}_{u_1}\delta^{(q_2)}_{u_2}\delta^{(q_3)}_{\upsilon_1}\delta^{(q_4)}_{\upsilon_2}\mathscr{F}(\boldsymbol{u},\boldsymbol{\upsilon})\Big|_{\substack{\boldsymbol{u}=\tilde{\boldsymbol{u}}\otimes_{1,2}\boldsymbol{\delta}_t(\cdot)\\ \boldsymbol{\upsilon}=\tilde{\boldsymbol{\upsilon}}\otimes_{1,2}\boldsymbol{\delta}_s(\cdot)}} =
\end{aligned}$$

$$= i\tilde{u}_1 \int\int \sum_{q_1,q_2,q_3,q_4} A^{(1)}_{q_1q_2q_3q_4}(t)\alpha_1^{q_1}(t)\alpha_2^{q_2}(t)\beta_1^{q_3}(s)\beta_2^{q_4}(s)\ \exp\{\cdots\}\ \mathscr{P}_{xy}(d\boldsymbol{\alpha}\, d\boldsymbol{\beta}) +$$

$$+\ i\tilde{u}_2 \int\int \sum_{q_1,q_2,q_3,q_4} A^{(2)}_{q_1q_2q_3q_4}(t)\alpha_1^{q_1}(t)\alpha_2^{q_2}(t)\beta_1^{q_3}(s)\beta_2^{q_4}(s)\ \exp\{\cdots\}\ \mathscr{P}_{xy}(d\boldsymbol{\alpha}\, d\boldsymbol{\beta}) =$$

$$= i\tilde{u}_1 \sum_{q_1,q_2,q_3,q_4} A^{(1)}_{q_1q_2q_3q_4}(t)\int\int \alpha_1^{q_1}(t)\alpha_2^{q_2}(t)\beta_1^{q_3}(s)\beta_2^{q_4}(s)\ \exp\{\cdots\}\ \mathscr{P}_{xy}(d\boldsymbol{\alpha}\, d\boldsymbol{\beta}) +$$

$$+\ i\tilde{u}_2 \sum_{q_1,q_2,q_3,q_4} A^{(2)}_{q_1q_2q_3q_4}(t)\int\int \alpha_1^{q_1}(t)\alpha_2^{q_2}(t)\beta_1^{q_3}(s)\beta_2^{q_4}(s)\ \exp\{\cdots\}\ \mathscr{P}_{xy}(d\boldsymbol{\alpha}\, d\boldsymbol{\beta}) = (\&)$$

Using now the equation (see below for the proof):

$$\frac{1}{i^{q_1+q_2+q_3+q_4}}\frac{\partial^{q_1}}{\partial\tilde{u}_1^{q_1}}\frac{\partial^{q_2}}{\partial\tilde{u}_2^{q_2}}\frac{\partial^{q_3}}{\partial\tilde{\upsilon}_1^{q_3}}\frac{\partial^{q_4}}{\partial\tilde{\upsilon}_2^{q_4}}\varphi_{x_1(t)x_2(t)y_1(s)y_2(s)}(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2) = \\ = \int\int \alpha_1^{q_1}(t)\alpha_2^{q_2}(t)\beta_1^{q_3}(s)\beta_2^{q_4}(s)\ \exp\{\cdots\}\ \mathscr{P}_{xy}(d\boldsymbol{\alpha}\, d\boldsymbol{\beta}) \tag{5}$$

we obtain

$$(\&) = \sum_{q_1,q_2,q_3,q_4} \frac{i\tilde{u}_1 A^{(1)}_{q_1q_2q_3q_4}(t)}{i^{q_1+q_2+q_3+q_4}}\frac{\partial^{q_1}}{\partial\tilde{u}_1^{q_1}}\frac{\partial^{q_2}}{\partial\tilde{u}_2^{q_2}}\frac{\partial^{q_3}}{\partial\tilde{\upsilon}_1^{q_3}}\frac{\partial^{q_4}}{\partial\tilde{\upsilon}_2^{q_4}}\varphi_{x_1(t)x_2(t)y_1(s)y_2(s)}(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2) + \\ + \sum_{q_1,q_2,q_3,q_4} \frac{i\tilde{u}_2 A^{(2)}_{q_1q_2q_3q_4}(t)}{i^{q_1+q_2+q_3+q_4}}\frac{\partial^{q_1}}{\partial\tilde{u}_1^{q_1}}\frac{\partial^{q_2}}{\partial\tilde{u}_2^{q_2}}\frac{\partial^{q_3}}{\partial\tilde{\upsilon}_1^{q_3}}\frac{\partial^{q_4}}{\partial\tilde{\upsilon}_2^{q_4}}\varphi_{x_1(t)x_2(t)y_1(s)y_2(s)}(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2) \tag{6}$$

►The proof of the Equ.(5) goes as follows: consider again Equ.(4) and differentiate both sides first q_1-times with respect to the parameter $\tilde{u}_1$,

$$\frac{\partial^{q_1}}{\partial\tilde{u}_1^{q_1}}\varphi_{x_1(t)x_2(t)y_1(s)y_2(s)}(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2) =$$

$$= \frac{\partial^{q_1}}{\partial\tilde{u}_1^{q_1}}\int\int \exp\{i(\tilde{u}_1\alpha_1(t)+\tilde{u}_2\alpha_2(t)+\tilde{\upsilon}_1\beta_1(s)+\tilde{\upsilon}_2\beta_2(s))\}\ \mathscr{P}_{xy}(d\boldsymbol{\alpha}\, d\boldsymbol{\beta}) =$$

$$= \int\int \frac{\partial^{q_1}}{\partial\tilde{u}_1^{q_1}}\exp\{i(\tilde{u}_1\alpha_1(t)+\tilde{u}_2\alpha_2(t)+\tilde{\upsilon}_1\beta_1(s)+\tilde{\upsilon}_2\beta_2(s))\}\ \mathscr{P}_{xy}(d\boldsymbol{\alpha}\, d\boldsymbol{\beta}) =$$

$$= \int\int i^{q_1}\alpha_1^{q_1}(t)\ \exp\{i(\tilde{u}_1\alpha_1(t)+\tilde{u}_2\alpha_2(t)+\tilde{\upsilon}_1\beta_1(s)+\tilde{\upsilon}_2\beta_2(s))\}\ \mathscr{P}_{xy}(d\boldsymbol{\alpha}\, d\boldsymbol{\beta}).$$

In a similar fashion we obtain

$$\frac{\partial^{q_1}}{\partial\tilde{u}_1^{q_1}}\frac{\partial^{q_2}}{\partial\tilde{u}_2^{q_2}}\frac{\partial^{q_3}}{\partial\tilde{\upsilon}_1^{q_3}}\frac{\partial^{q_4}}{\partial\tilde{\upsilon}_2^{q_4}}\varphi_{x_1(t)x_2(t)y_1(s)y_2(s)}\left(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2\right) =$$

$$= \frac{\partial^{q_1}}{\partial\tilde{u}_1^{q_1}}\frac{\partial^{q_2}}{\partial\tilde{u}_2^{q_2}}\frac{\partial^{q_3}}{\partial\tilde{\upsilon}_1^{q_3}}\frac{\partial^{q_4}}{\partial\tilde{\upsilon}_2^{q_4}}\int\int \exp\left\{i\left(\tilde{u}_1\alpha_1(t)+\tilde{u}_2\alpha_2(t)+\tilde{\upsilon}_1\beta_1(s)+\tilde{\upsilon}_2\beta_2(s)\right)\right\}\mathscr{P}_{xy}\left(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\right) =$$

$$= \int\int \frac{\partial^{q_1}}{\partial\tilde{u}_1^{q_1}}\frac{\partial^{q_2}}{\partial\tilde{u}_2^{q_2}}\frac{\partial^{q_3}}{\partial\tilde{\upsilon}_1^{q_3}}\frac{\partial^{q_4}}{\partial\tilde{\upsilon}_2^{q_4}}\exp\left\{i\left(\tilde{u}_1\alpha_1(t)+\tilde{u}_2\alpha_2(t)+\tilde{\upsilon}_1\beta_1(s)+\tilde{\upsilon}_2\beta_2(s)\right)\right\}\mathscr{P}_{xy}\left(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\right) =$$

$$= \int\int i^{q_1+q_2+q_3+q_4}\,\alpha_1^{q_1}(t)\,\alpha_2^{q_2}(t)\,\beta_1^{q_3}(t)\,\beta_2^{q_4}(t)\,\exp\left\{i\left(\tilde{u}_1\alpha_1(t)+\tilde{u}_2\alpha_2(t)+\tilde{\upsilon}_1\beta_1(s)+\tilde{\upsilon}_2\beta_2(s)\right)\right\}\mathscr{P}_{xy}\left(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\right)$$

which is exactly what we wanted to prove.◄

Combining now Equ.(3) and Equ.(6), we obtain an equation for the evolution of the joint RE characteristic function of Equ.(1)_Sec(2.4.1):

$$\boxed{\begin{aligned}&\left.\frac{\partial}{\partial t}\varphi_{x_1(t)x_2(t)y_1(s)y_2(s)}\left(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2\right)\right|_{s\to t} =\\ &= \sum_{q_1,q_2,q_3,q_4}\frac{i\tilde{u}_1\,A^{(1)}_{q_1q_2q_3q_4}(t)}{i^{q_1+q_2+q_3+q_4}}\;\frac{\partial^{q_1}}{\partial\tilde{u}_1^{q_1}}\frac{\partial^{q_2}}{\partial\tilde{u}_2^{q_2}}\frac{\partial^{q_3}}{\partial\tilde{\upsilon}_1^{q_3}}\frac{\partial^{q_4}}{\partial\tilde{\upsilon}_2^{q_4}}\varphi_{x_1(t)x_2(t)y_1(t)y_2(t)}\left(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2\right) +\\ &+ \sum_{q_1,q_2,q_3,q_4}\frac{i\tilde{u}_2\,A^{(2)}_{q_1q_2q_3q_4}(t)}{i^{q_1+q_2+q_3+q_4}}\;\frac{\partial^{q_1}}{\partial\tilde{u}_1^{q_1}}\frac{\partial^{q_2}}{\partial\tilde{u}_2^{q_2}}\frac{\partial^{q_3}}{\partial\tilde{\upsilon}_1^{q_3}}\frac{\partial^{q_4}}{\partial\tilde{\upsilon}_2^{q_4}}\varphi_{x_1(t)x_2(t)y_1(t)y_2(t)}\left(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2\right)\end{aligned}} \quad (7)$$

The above equation should be supplemented by:

Compatibility conditions:

The marginal characteristic function related to the excitation is known, i.e.

$$\boxed{\varphi_{y_1(s)y_2(s)}\left(\tilde{\upsilon}_1,\tilde{\upsilon}_2\right) = \varphi_{x_1(t)x_2(t)y_1(s)y_2(s)}\left(0,0,\tilde{\upsilon}_1,\tilde{\upsilon}_2\right) = known\ \ characteristic\ \ function, \quad \forall s \geq t_0} \quad (8)$$

Initial conditions:

The joint characteristic function of the initial state – excitation of the system, $\left(x_1(t_0),x_2(t_0),y_1(s),y_2(s)\right)=\left(x_{10}(\theta),x_{20}(\theta),y_1(s),y_2(s)\right)$, is probabilistically known, i.e. $\varphi_{x_1(t_0)x_2(t_0)y_1(s)y_2(s)}\left(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2\right) = known\ \ characteristic\ \ function, \quad \forall s \geq t_0$.

Making the plausible assumption that the initial state is independent from the excitation, the initial condition is simplified as follows

$$\boxed{\varphi_{x_1(t_0)x_2(t_0)}\left(\tilde{u}_1,\tilde{u}_2\right) = known\ \ characteristic\ \ function} \quad (9)$$

2.4.6. Derivation of the joint REPDF evolution equation.

We shall use the equation for the evolution of the joint RE characteristic function (Equ.(7)_ Sec(2.4.5)) to find the joint REPDF evolution equation

For any $t_1, t_2, s_1, s_2 \geq t_0$, the characteristic function $\varphi_{x_1(t_1)x_2(t_2)y_1(s_1)y_2(s_2)}(u_1,u_2,\upsilon_1,\upsilon_2)$ and the corresponding pdf $f_{x_1(t_1)x_2(t_2)y_1(s_1)y_2(s_2)}(\alpha_1,\alpha_2,\beta_1,\beta_2)$ form a Fourier transform pair, and thus being connected by means of the formula

$$\varphi_{x_1(t_1)x_2(t_2)y_1(s_1)y_2(s_2)}\left(u_1,u_2,\upsilon_1,\upsilon_2\right) = \int_{\mathbb{R}^{(2)}}\int_{\mathbb{R}^{(2)}} f_{x_1(t_1)x_2(t_2)y_1(s_1)y_2(s_2)}\left(\alpha_1,\alpha_2,\beta_1,\beta_2\right)\times$$
$$\times \exp\left\{i\left(u_1\alpha_1 + u_2\alpha_2 + \upsilon_1\beta_1 + \upsilon_2\beta_2\right)\right\} d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 . \tag{1}$$

Using the notational conventions

$\boldsymbol{u} = \left(u_1,u_2\right),\ \ \boldsymbol{\upsilon} = \left(\upsilon_1,\upsilon_2\right),\ \ \boldsymbol{\alpha} = \left(\alpha_1,\alpha_2\right),\ \ \boldsymbol{\beta} = \left(\beta_1,\beta_2\right),$

$\boldsymbol{x}(\boldsymbol{t}) = \left(x_1(t_1), x_2(t_2)\right),\ \ \boldsymbol{y}(\boldsymbol{s}) = \left(y_1(s_1), y_2(s_2)\right),$

Equ.(1) can be written in the more compact form:

$$\varphi_{\boldsymbol{x}(\boldsymbol{t})\boldsymbol{y}(\boldsymbol{s})}\left(\boldsymbol{u},\boldsymbol{\upsilon}\right) = \int_{\mathbb{R}^{(2)}}\int_{\mathbb{R}^{(2)}} f_{\boldsymbol{x}(\boldsymbol{t})\boldsymbol{y}(\boldsymbol{s})}\left(\boldsymbol{\alpha},\boldsymbol{\beta}\right) \exp\left\{i\left(\boldsymbol{u}\cdot\boldsymbol{\alpha} + \boldsymbol{\upsilon}\cdot\boldsymbol{\beta}\right)\right\} d\boldsymbol{\alpha} d\boldsymbol{\beta} . \tag{2}$$

Note that in the following analysis symbols $\boldsymbol{x}(\boldsymbol{t}),\ \boldsymbol{y}(\boldsymbol{s})$ will be given the (more restricted) meaning $\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{x}(t) = \left(x_1(t), x_2(t)\right)$ and $\boldsymbol{y}(\boldsymbol{s}) = \boldsymbol{y}(s) = \left(y_1(s), y_2(s)\right)$.

Sometimes use is made of the following symbolic form of the Fourier Transform Equ.(1) or Equ.(2):

$$\varphi_{\boldsymbol{x}(\boldsymbol{t})\boldsymbol{y}(\boldsymbol{s})}\left(\boldsymbol{u},\boldsymbol{\upsilon}\right) = \mathscr{F}_{ourier}\left[f_{\boldsymbol{x}(\boldsymbol{t})\boldsymbol{y}(\boldsymbol{s})}\left(\boldsymbol{\alpha},\boldsymbol{\beta}\right) ; \left(\boldsymbol{\alpha},\boldsymbol{\beta}\right) \to \left(\boldsymbol{u},\boldsymbol{\upsilon}\right)\right]. \tag{3}$$

Substituting now the characteristic function in Equ.(7)_Sec(2.4.5) with the Fourier transform of the pdf (Equ.2) we obtain:

$$\frac{\partial}{\partial t}\mathscr{F}_{ourier}\left[f_{\boldsymbol{x}(t)\boldsymbol{y}(s)}\left(\boldsymbol{\alpha},\boldsymbol{\beta}\right) ; \left(\boldsymbol{\alpha},\boldsymbol{\beta}\right) \to \left(\boldsymbol{u},\boldsymbol{\upsilon}\right)\right]\Bigg|_{s\to t} =$$
$$= \sum_{q_1,q_2,q_3,q_4} \frac{i\tilde{u}_1 A^{(1)}_{q_1q_2q_3q_4}(t)}{i^{q_1+q_2+q_3+q_4}} \frac{\partial^{q_1}}{\partial u_1^{q_1}}\frac{\partial^{q_2}}{\partial u_2^{q_2}}\frac{\partial^{q_3}}{\partial \upsilon_1^{q_3}}\frac{\partial^{q_4}}{\partial \upsilon_2^{q_4}} \int_{\mathbb{R}^{(2)}}\int_{\mathbb{R}^{(2)}} \exp\left\{i\left(\boldsymbol{u}\cdot\boldsymbol{\alpha} + \boldsymbol{\upsilon}\cdot\boldsymbol{\beta}\right)\right\} f_{\boldsymbol{x}(t)\boldsymbol{y}(t)}\left(\boldsymbol{\alpha},\boldsymbol{\beta}\right) d\boldsymbol{\alpha} d\boldsymbol{\beta} +$$
$$+ \sum_{q_1,q_2,q_3,q_4} \frac{i\tilde{u}_2 A^{(2)}_{q_1q_2q_3q_4}(t)}{i^{q_1+q_2+q_3+q_4}} \frac{\partial^{q_1}}{\partial u_1^{q_1}}\frac{\partial^{q_2}}{\partial u_2^{q_2}}\frac{\partial^{q_3}}{\partial \upsilon_1^{q_3}}\frac{\partial^{q_4}}{\partial \upsilon_2^{q_4}} \int_{\mathbb{R}^{(2)}}\int_{\mathbb{R}^{(2)}} \exp\left\{i\left(\boldsymbol{u}\cdot\boldsymbol{\alpha} + \boldsymbol{\upsilon}\cdot\boldsymbol{\beta}\right)\right\} f_{\boldsymbol{x}(t)\boldsymbol{y}(t)}\left(\boldsymbol{\alpha},\boldsymbol{\beta}\right) d\boldsymbol{\alpha} d\boldsymbol{\beta} =$$

[*assuming that differentiations can pass under the integral sign]*,

$$= \sum_{q_1,q_2,q_3,q_4} \frac{i\tilde{u}_1 A^{(1)}_{q_1q_2q_3q_4}(t)}{i^{q_1+q_2+q_3+q_4}} \int_{\mathbb{R}^{(2)}}\int_{\mathbb{R}^{(2)}} (i\alpha_1)^{q_1}(i\alpha_2)^{q_2}(i\beta_1)^{q_3}(i\beta_2)^{q_4} \exp\{i(\boldsymbol{u}\cdot\boldsymbol{\alpha}+\boldsymbol{\upsilon}\cdot\boldsymbol{\beta})\} f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta})d\boldsymbol{\alpha}d\boldsymbol{\beta} +$$

$$+ \sum_{q_1,q_2,q_3,q_4} \frac{i\tilde{u}_2 A^{(2)}_{q_1q_2q_3q_4}(t)}{i^{q_1+q_2+q_3+q_4}} \int_{\mathbb{R}^{(2)}}\int_{\mathbb{R}^{(2)}} (i\alpha_1)^{q_1}(i\alpha_2)^{q_2}(i\beta_1)^{q_3}(i\beta_2)^{q_4} \exp\{i(\boldsymbol{u}\cdot\boldsymbol{\alpha}+\boldsymbol{\upsilon}\cdot\boldsymbol{\beta})\} f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta})d\boldsymbol{\alpha}d\boldsymbol{\beta} =$$

$$= \sum_{q_1,q_2,q_3,q_4} A^{(1)}_{q_1q_2q_3q_4}(t) \int_{\mathbb{R}^{(2)}}\int_{\mathbb{R}^{(2)}} \alpha_1^{q_1}\alpha_2^{q_2}\beta_1^{q_3}\beta_2^{q_4}(iu_1)\exp\{i(\boldsymbol{u}\cdot\boldsymbol{\alpha}+\boldsymbol{\upsilon}\cdot\boldsymbol{\beta})\} f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta})d\boldsymbol{\alpha}d\boldsymbol{\beta} +$$

$$+ \sum_{q_1,q_2,q_3,q_4} A^{(2)}_{q_1q_2q_3q_4}(t) \int_{\mathbb{R}^{(2)}}\int_{\mathbb{R}^{(2)}} \alpha_1^{q_1}\alpha_2^{q_2}\beta_1^{q_3}\beta_2^{q_4}(iu_2)\exp\{i(\boldsymbol{u}\cdot\boldsymbol{\alpha}+\boldsymbol{\upsilon}\cdot\boldsymbol{\beta})\} f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta})d\boldsymbol{\alpha}d\boldsymbol{\beta} =$$

$$= \sum_{q_1,q_2,q_3,q_4} A^{(1)}_{q_1q_2q_3q_4}(t) \int_{\mathbb{R}^{(2)}}\int_{\mathbb{R}^{(2)}} \alpha_1^{q_1}\alpha_2^{q_2}\beta_1^{q_3}\beta_2^{q_4}\frac{\partial}{\partial\alpha_1}\left[\exp\{i(\boldsymbol{u}\cdot\boldsymbol{\alpha}+\boldsymbol{\upsilon}\cdot\boldsymbol{\beta})\}\right] f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta})d\boldsymbol{\alpha}d\boldsymbol{\beta} +$$

$$+ \sum_{q_1,q_2,q_3,q_4} A^{(2)}_{q_1q_2q_3q_4}(t) \int_{\mathbb{R}^{(2)}}\int_{\mathbb{R}^{(2)}} \alpha_1^{q_1}\alpha_2^{q_2}\beta_1^{q_3}\beta_2^{q_4}\frac{\partial}{\partial\alpha_2}\left[\exp\{i(\boldsymbol{u}\cdot\boldsymbol{\alpha}+\boldsymbol{\upsilon}\cdot\boldsymbol{\beta})\}\right] f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta})d\boldsymbol{\alpha}d\boldsymbol{\beta} =$$

[applying integration by parts in each term]

$$= -\sum_{q_1,q_2,q_3,q_4} A^{(1)}_{q_1q_2q_3q_4}(t) \int_{\mathbb{R}^{(2)}}\int_{\mathbb{R}^{(2)}} \alpha_2^{q_2}\beta_1^{q_3}\beta_2^{q_4}\exp\{i(\boldsymbol{u}\cdot\boldsymbol{\alpha}+\boldsymbol{\upsilon}\cdot\boldsymbol{\beta})\}\frac{\partial}{\partial\alpha_1}\left[\alpha_1^{q_1} f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta})\right]d\boldsymbol{\alpha}d\boldsymbol{\beta} -$$

$$-\sum_{q_1,q_2,q_3,q_4} A^{(2)}_{q_1q_2q_3q_4}(t) \int_{\mathbb{R}^{(2)}}\int_{\mathbb{R}^{(2)}} \alpha_1^{q_1}\beta_1^{q_3}\beta_2^{q_4}\exp\{i(\boldsymbol{u}\cdot\boldsymbol{\alpha}+\boldsymbol{\upsilon}\cdot\boldsymbol{\beta})\}\frac{\partial}{\partial\alpha_2}\left[\alpha_2^{q_2} f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta})\right]d\boldsymbol{\alpha}d\boldsymbol{\beta} =$$

[substituting r_1, r_2 for the exponents and subscripts q_3, q_4, and using the notational conventions $\boldsymbol{q}=(q_1,q_2)$,

$\boldsymbol{r}=(r_1,r_2)$, $\boldsymbol{\alpha}^{\boldsymbol{q}}=\alpha_1^{q_1}\alpha_2^{q_2}$, $\boldsymbol{\beta}^{\boldsymbol{r}}=\beta_1^{r_1}\beta_2^{r_2}=\beta_1^{q_3}\beta_2^{q_4}$],

$$= -\sum_{\boldsymbol{q},\boldsymbol{r}}\int_{\mathbb{R}^{(2)}}\int_{\mathbb{R}^{(2)}} A^{(1)}_{\boldsymbol{qr}}(t)\boldsymbol{\beta}^{\boldsymbol{r}}\frac{\partial}{\partial\alpha_1}\left[\boldsymbol{\alpha}^{\boldsymbol{q}} f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta})\right]\exp\{i(\boldsymbol{u}\cdot\boldsymbol{\alpha}+\boldsymbol{\upsilon}\cdot\boldsymbol{\beta})\}d\boldsymbol{\alpha}d\boldsymbol{\beta}$$

$$-\sum_{\boldsymbol{q},\boldsymbol{r}}\int_{\mathbb{R}^{(2)}}\int_{\mathbb{R}^{(2)}} A^{(2)}_{\boldsymbol{qr}}(t)\boldsymbol{\beta}^{\boldsymbol{r}}\frac{\partial}{\partial\alpha_2}\left[\boldsymbol{\alpha}^{\boldsymbol{q}} f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta})\right]\exp\{i(\boldsymbol{u}\cdot\boldsymbol{\alpha}+\boldsymbol{\upsilon}\cdot\boldsymbol{\beta})\}d\boldsymbol{\alpha}d\boldsymbol{\beta} =$$

$$= -\sum_{\boldsymbol{q},\boldsymbol{r}}\mathscr{F}_{ourier}\left[A^{(1)}_{\boldsymbol{qr}}(t)\boldsymbol{\beta}^{\boldsymbol{r}}\frac{\partial}{\partial\alpha_1}\left[\boldsymbol{\alpha}^{\boldsymbol{q}} f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta})\right];(\boldsymbol{\alpha},\boldsymbol{\beta})\to(\boldsymbol{u},\boldsymbol{\upsilon})\right]$$

$$-\sum_{\boldsymbol{q},\boldsymbol{r}}\mathscr{F}_{ourier}\left[A^{(2)}_{\boldsymbol{qr}}(t)\boldsymbol{\beta}^{\boldsymbol{r}}\frac{\partial}{\partial\alpha_2}\left[\boldsymbol{\alpha}^{\boldsymbol{q}} f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta})\right];(\boldsymbol{\alpha},\boldsymbol{\beta})\to(\boldsymbol{u},\boldsymbol{\upsilon})\right].$$

That is, Equ.(7)_Sec(2.4.5) has been given the following form

$$\frac{\partial}{\partial t}\mathscr{F}_{ourier}\left[f_{\mathbf{x}(t)\mathbf{y}(s)}(\boldsymbol{\alpha},\boldsymbol{\beta});(\boldsymbol{\alpha},\boldsymbol{\beta})\to(\boldsymbol{u},\boldsymbol{\upsilon})\right]\Bigg|_{s\to t} =$$

$$= -\sum_{n=1,2}\sum_{\boldsymbol{q},\boldsymbol{r}}\mathscr{F}_{ourier}\left[A^{(n)}_{\boldsymbol{qr}}(t)\boldsymbol{\beta}^{\boldsymbol{r}}\frac{\partial}{\partial\alpha_n}\left[\boldsymbol{\alpha}^{\boldsymbol{q}} f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta})\right];(\boldsymbol{\alpha},\boldsymbol{\beta})\to(\boldsymbol{u},\boldsymbol{\upsilon})\right]$$

Applying now the inverse Fourier Transform to both sides of the above equation, we obtain

$$\left.\frac{\partial}{\partial t} f_{\mathbf{x}(t)\mathbf{y}(s)}(\boldsymbol{\alpha},\boldsymbol{\beta})\right|_{s\to t} = -\sum_{n=1,2}\sum_{\mathbf{q},\mathbf{r}} A^{(n)}_{\mathbf{qr}}(t)\boldsymbol{\beta}^{\mathbf{r}} \frac{\partial}{\partial \alpha_n}\left[\, \boldsymbol{\alpha}^{\mathbf{q}} f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta})\right] \tag{4}$$

which is the sought-for equation in terms of the joint REPDF. Clearly, it is equivalent with Equ.(7)_Sec(2.4.5).

Theorem: If the stochastic function $x(t;\theta)$ satisfies the differential system

$$\begin{aligned}\dot{x}_n(t) &= \sum_{q_1,q_2,r_1,r_2} A^{(n)}_{q_1,q_2,r_1,r_2}(t)\cdot x_1^{q_1}(t)x_2^{q_2}(t)y_1^{r_1}(t;\theta)y_2^{r_2}(t;\theta) \equiv \\ &\equiv \sum_{q_1,q_2,r_1,r_2} A^{(n)}_{\mathbf{qr}}(t)\cdot \mathbf{x}^{\mathbf{q}}(t)\ \mathbf{y}^{\mathbf{r}}(t;\theta), \qquad n=1,2,\end{aligned} \tag{5}$$

then, the joint, response-excitation, pdf

$$f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta}) = f_{x_1(t)x_2(t)y_1(t)y_2(t)}(\alpha_1,\alpha_2,\beta_1,\beta_2) \tag{6}$$

satisfies the differential equation (4), that we repeat here for convenience:

$$\left.\frac{\partial}{\partial t} f_{\mathbf{x}(t)\mathbf{y}(s)}(\boldsymbol{\alpha},\boldsymbol{\beta})\right|_{s\to t} = -\sum_{n=1,2}\sum_{\mathbf{q},\mathbf{r}} A^{(n)}_{\mathbf{qr}}(t)\boldsymbol{\beta}^{\mathbf{r}} \frac{\partial}{\partial \alpha_n}\left[\, \boldsymbol{\alpha}^{\mathbf{q}} f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta})\right] \tag{7}$$

The above equation should be supplemented by:

Compatibility conditions ensuring that the marginal pdf related to the excitation is known at any time $t \geq t_0$, i.e.

$$\begin{aligned}f_{\mathbf{y}(t)}(\boldsymbol{\beta}) &= \int_{\mathbb{R}^2} f_{\mathbf{x}(t)\mathbf{y}(t)}(\boldsymbol{\alpha},\boldsymbol{\beta})d\boldsymbol{\alpha} \equiv \\ &\equiv \int_{\mathbb{R}^2} f_{x_1(t)x_2(t)y_1(t)y_2(t)}(\alpha_1,\alpha_2,\beta_1,\beta_2)d\alpha_1 d\alpha_2 = \textit{known at any time } t \geq t_0 .\end{aligned} \tag{8}$$

Initial conditions ensuring that, here assuming that the initial state is independent from the excitation, the joint pdf of the initial state $\big(x_1(t_0),x_2(t_0)\big)=\big(x_{10}(\theta),x_{20}(\theta)\big)$, is probabilistically known, i.e.

$$f_{x_1(t_0)x_2(t_0)}(\alpha_1,\alpha_2) = \textit{known probability density function}. \tag{9}$$

Equ.(7) must also be supplemented by **constitutive conditions** ensuring that $f_{x(t)y_1(s)y_2(s)}(\alpha,\beta_1,\beta_2)$ is a pdf:

$$f_{x(t)y_1(s)y_2(s)}(\alpha,\beta_1,\beta_2) \geq 0, \quad \int\limits_{\alpha\in\mathbb{R}} \int\limits_{\beta\in\mathbb{R}} f_{x(t)y_1(s)y_2(s)}(\alpha,\beta_1,\beta_2)\, d\alpha = 1 \quad . \tag{10a,b}$$

The derived joint REPDF evolution equation (7) is a peculiar equation, involving two times and four probability arguments (two for the excitation, and two for the response) and partial derivatives only with respect to one (response) time and the two response probability arguments. This peculiarity gives rise to fundamental questions regarding both the well-posedness of the problem (7-10) and the methods of its numerical solution. Recently, (Venturi et al. 2012) re-examined the RE theory using a different (but essentially equivalent) method. They confirmed the validity of the REPDF evolution equation derived in (Sapsis & Athanassoulis 2008) and answered in negative the question regarding its well-posedness, by presenting a simple example in which the REPDF evolution equation is valid but it does not ensure uniqueness of solutions(this will be discussed in Section 3.6). Accordingly, it becomes clear that a kind of completion of problem (7-10) is necessary. The type of completion proposed by (Venturi et al. 2012) results in a complicated equation, including the entire history of the response process in a functional integral form, which cannot be considered as an attractive alternative. In Section 3.6 we shall use a simple problem, i.e. a scalar linear RDE with Gaussian excitation, to prove that the REPDF evolution equation does not have a unique solution, since when the equation is considered only time diagonally $(s \to t)$, the non-local (in time) characteristics of the problem are lost. In Chapter 4 an *a priori* closure technique, by formulating and using localized linear problems accounting for the RE correlation structure, shall be introduced and used for the numerical solution of the long-time, steady-state REPDF evolution equation that corresponds to non-liner scalar RDEs (see also the discussion in Section 4.3)

2.5. Infinite system of limiting two-time RE moment equations

To verify the validity of the REPDF evolution equation (Equ.(8)_Sec(2.4.6)), this is used to re-obtain the infinite system of the limit two-time moment equations. More precisely, in this section the infinite system of moment equations are derived both directly from the dynamical system (Equ.(1)_Sec(2.4.1)) and from the equation for the evolution of the joint response-excitation characteristic function (Equ.(7)_Sec(2.4.5)).

Derivation of the infinite system of moment equations from the dynamical system

Let us consider the dynamical system with polynomial non-linearities:

$$\dot{x}_1(t;\theta) = \sum_{q_1,q_2,q_3,q_4} A^{(1)}_{q_1q_2q_3q_4}(t)\cdot x_1^{q_1}(t;\theta)x_2^{q_2}(t;\theta)y_1^{q_3}(t;\theta)y_2^{q_4}(t;\theta), \tag{1a}$$

$$\dot{x}_2(t;\theta) = \sum_{q_1,q_2,q_3,q_4} A^{(2)}_{q_1q_2q_3q_4}(t)\cdot x_1^{q_1}(t;\theta)x_2^{q_2}(t;\theta)y_1^{q_3}(t;\theta)y_2^{q_4}(t;\theta). \tag{1b}$$

Multiplying equation (1a) with $n_1 \cdot x_1^{n_1-1}(t;\theta)\, x_2^{n_2}(t;\theta)\, y_1^{m_1}(s;\theta) y_2^{m_2}(s;\theta)$ and equation (1b) with $n_2 \cdot x_1^{n_1}(t;\theta)\, x_2^{n_2-1}(t;\theta)\, y_1^{m_1}(s) y_2^{m_2}(s)$, respectively, we get:

$$n_1 \cdot x_1^{n_1-1}(t;\theta)\dot{x}_1(t;\theta)\; x_2^{n_2}(t;\theta)\, y_1^{m_1}(s;\theta) y_2^{m_2}(s;\theta) = n_1 \cdot x_1^{n_1-1}(t;\theta)\; x_2^{n_2}(t;\theta)\, y_1^{m_1}(s;\theta) y_2^{m_2}(s;\theta) \times \times \sum_{q_1,q_2,q_3,q_4} A^{(1)}_{q_1 q_2 q_3 q_4}(t) \cdot x_1^{q_1}(t;\theta) x_2^{q_2}(t;\theta) y_1^{q_3}(t;\theta) y_2^{q_4}(t;\theta), \tag{2a}$$

$$n_2 \cdot x_2^{n_2-1}(t;\theta)\dot{x}_2(t;\theta) x_1^{n_1}(t;\theta)\, y_1^{m_1}(s;\theta) y_2^{m_2}(s;\theta) = = n_2 \cdot x_1^{n_1}(t;\theta)\, x_2^{n_2-1}(t;\theta)\, y_1^{m_1}(s;\theta) y_2^{m_2}(s;\theta) \times \times \sum_{q_1,q_2,q_3,q_4} A^{(2)}_{q_1 q_2 q_3 q_4}(t) \cdot x_1^{q_1}(t;\theta) x_2^{q_2}(t;\theta) y_1^{q_3}(t;\theta) y_2^{q_4}(t;\theta). \tag{2b}$$

Adding equations (2a), (2b) we have:

$$\begin{aligned} \frac{d}{dt} x_1^{n_1}(t;\theta)\; x_2^{n_2}(t;\theta)\, y_1^{m_1}(s;\theta) y_2^{m_2}(s;\theta) &= \\ = n_1 \cdot x_1^{n_1-1}(t;\theta)\; x_2^{n_2}(t;\theta)\, y_1^{m_1}(s;\theta) y_2^{m_2}(s;\theta) &+ \\ \times \sum_{q_1,q_2,q_3,q_4} A^{(1)}_{q_1 q_2 q_3 q_4}(t) \cdot x_1^{q_1}(t;\theta) x_2^{q_2}(t;\theta) y_1^{q_3}(t;\theta) y_2^{q_4}(t;\theta) &+ \\ + n_2 \cdot x_1^{n_1}(t;\theta)\, x_2^{n_2-1}(t;\theta)\, y_1^{m_1}(s;\theta) y_2^{m_2}(s;\theta) &\times \\ \times \sum_{q_1,q_2,q_3,q_4} A^{(2)}_{q_1 q_2 q_3 q_4}(t) \cdot x_1^{q_1}(t;\theta) x_2^{q_2}(t;\theta) y_1^{q_3}(t;\theta) y_2^{q_4}(t;\theta)&. \end{aligned} \tag{3}$$

Considering the limit $s \to t$, we obtain:

$$\begin{aligned} \left. \frac{d}{dt} x_1^{n_1}(t;\theta)\; x_2^{n_2}(t;\theta)\, y_1^{m_1}(s;\theta) y_2^{m_2}(s;\theta) \right|_{s\to t} &= \\ = n_1 \sum_{q_1,q_2,q_3,q_4} A^{(1)}_{q_1 q_2 q_3 q_4}(t) \cdot x_1^{q_1+n_1-1}(t;\theta) x_2^{q_2+n_2}(t;\theta) y_1^{q_3+m_1}(t;\theta) y_2^{q_4+m_2}(t;\theta) &+ \\ + n_2 \sum_{q_1,q_2,q_3,q_4} A^{(2)}_{q_1 q_2 q_3 q_4}(t) \cdot x_1^{q_1+n_1}(t;\theta) x_2^{q_2+n_2-1}(t;\theta) y_1^{q_3+m_1}(t;\theta) y_2^{q_4+m_2}(t;\theta)&. \end{aligned} \tag{4}$$

Applying the mean value operator to Equ.(4) we get the infinite system of moment equations of system (1):

$$E^{\theta}\left[\frac{d}{dt}x_1^{n_1}(t;\theta)\ x_2^{n_2}(t;\theta)\, y_1^{m_1}(s;\theta) y_2^{m_2}(s;\theta)\Big|_{s\to t}\right] = \tag{5}$$

$$= n_1\cdot E^{\theta}\left[\sum_{q_1,q_2,q_3,q_4} A^{(1)}_{q_1q_2q_3q_4}(t)\cdot x_1^{q_1+n_1-1}(t;\theta)x_2^{q_2+n_2}(t;\theta)y_1^{q_3+m_1}(t;\theta)y_2^{q_4+m_2}(t;\theta)\right]+$$

$$+\, n_2\cdot E^{\theta}\left[\sum_{q_1,q_2,q_3,q_4} A^{(2)}_{q_1q_2q_3q_4}(t)\cdot x_1^{q_1+n_1}(t;\theta)x_2^{q_2+n_2-1}(t;\theta)y_1^{q_3+m_1}(t;\theta)y_2^{q_4+m_2}(t;\theta)\right],$$

that is (interchanging the mean square derivative with the mean value operator):

$$\frac{d}{dt}E^{\theta}\left[x_1^{n_1}(t;\theta)\ x_2^{n_2}(t;\theta)\, y_1^{m_1}(s;\theta) y_2^{m_2}(s;\theta)\Big|_{s\to t}\right] =$$

$$= n_1\cdot \sum_{q_1,q_2,q_3,q_4} A^{(1)}_{q_1q_2q_3q_4}(t)\cdot E^{\theta}\left[x_1^{q_1+n_1-1}(t;\theta)x_2^{q_2+n_2}(t;\theta)y_1^{q_3+m_1}(t;\theta)y_2^{q_4+m_2}(t;\theta)\right]+ \tag{6}$$

$$+\, n_2\cdot \sum_{q_1,q_2,q_3,q_4} A^{(2)}_{q_1q_2q_3q_4}(t)\cdot E^{\theta}\left[x_1^{q_1+n_1}(t;\theta)x_2^{q_2+n_2-1}(t;\theta)y_1^{q_3+m_1}(t;\theta)y_2^{q_4+m_2}(t;\theta)\right].$$

Let $R^{n_1,n_2,m_1,m_2}_{x_1,x_2,y_1,y_2}(t,t,s,s) \equiv E^{\theta}\left[x_1^{n_1}(t;\theta)\ x_2^{n_2}(t;\theta)\, y_1^{m_1}(s;\theta) y_2^{m_2}(s;\theta)\right]$, then Equ.(6) is equivalently written:

$$\boxed{\left.\frac{\partial R^{n_1,n_2,m_1,m_2}_{x_1,x_2,y_1,y_2}(t,t,s,s)}{\partial t}\right|_{s\to t} = n_1 \sum_{q_1,q_2,q_3,q_4} A^{(1)}_{q_1q_2q_3q_4}(t)\cdot R^{q_1+n_1-1,q_2+n_2,q_3+m_1,q_4+m_2}_{x_1,x_2,y_1,y_2}(t,t,t,t) + n_2 \sum_{q_1,q_2,q_3,q_4} A^{(2)}_{q_1q_2q_3q_4}(t)\cdot R^{q_1+n_1,q_2+n_2-1,q_3+m_1,q_4+m_2}_{x_1,x_2,y_1,y_2}(t,t,t,t)} \tag{7}$$

Derivation of the infinite system of moment equations from the evolution equation of the joint response-excitation characteristic function

The new equation derived for the evolution of the joint response-excitation characteristic function that corresponds to system (1) is (we repeat Equ.(7)_Sec(2.4.5) for convenience):

$$\left.\frac{\partial}{\partial t}\varphi_{x_1(t)x_2(t)y_1(s)y_2(s)}(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2)\right|_{s\to t} =$$

$$= \sum_{q_1,q_2,q_3,q_4} \frac{i\tilde{u}_1 A^{(1)}_{q_1q_2q_3q_4}(t)}{i^{q_1+q_2+q_3+q_4}}\ \frac{\partial^{q_1}}{\partial\tilde{u}_1^{q_1}}\frac{\partial^{q_2}}{\partial\tilde{u}_2^{q_2}}\frac{\partial^{q_3}}{\partial\tilde{\upsilon}_1^{q_3}}\frac{\partial^{q_4}}{\partial\tilde{\upsilon}_2^{q_4}}\varphi_{x_1(t)x_2(t)y_1(t)y_2(t)}(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2)+ \tag{8}$$

$$+ \sum_{q_1,q_2,q_3,q_4} \frac{i\tilde{u}_2 A^{(2)}_{q_1q_2q_3q_4}(t)}{i^{q_1+q_2+q_3+q_4}}\ \frac{\partial^{q_1}}{\partial\tilde{u}_1^{q_1}}\frac{\partial^{q_2}}{\partial\tilde{u}_2^{q_2}}\frac{\partial^{q_3}}{\partial\tilde{\upsilon}_1^{q_3}}\frac{\partial^{q_4}}{\partial\tilde{\upsilon}_2^{q_4}}\varphi_{x_1(t)x_2(t)y_1(t)y_2(t)}(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2)$$

Applying the differential operator $\dfrac{\partial^{n_1+n_2+m_1+m_2}}{\partial\tilde{u}_1^{n_1}\partial\tilde{u}_2^{n_2}\partial\tilde{\upsilon}_1^{m_1}\partial\tilde{\upsilon}_2^{m_2}}[\quad]$ at the point $\tilde{u}_1=\tilde{u}_2=\tilde{\upsilon}_1=\tilde{\upsilon}_2=0$ to Equ.(8), we can re-obtain Equ.(7). To this end, the following known relationships are used:

$$\frac{\partial^{n_1+n_2+m_1+m_2}}{\partial\tilde{u}_1^{n_1}\partial\tilde{u}_2^{n_2}\partial\tilde{\upsilon}_1^{m_1}\partial\tilde{\upsilon}_2^{m_2}}\varphi_{x_1(t)x_2(t)y_1(s)y_2(s)}(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2)\Bigg|_{\substack{\tilde{u}_1=0\\ \tilde{u}_2=0\\ \tilde{\upsilon}_1=0\\ \tilde{\upsilon}_2=0}}=i^{n_1+n_2+m_1+m_2}R^{n_1,n_2,m_1,m_2}_{x_1,x_2,y_1,y_2}(t,t,s,s), \quad (9)$$

along with the fact that for every C^n differential function $f(x):\mathbb{R}\to\mathbb{R}$, we have (Sapsis & Athanassoulis 2008):

$$\frac{d^n}{dx^n}[xf(x)]\Bigg|_{x=0}=n\frac{d^{n-1}}{dx^{n-1}}[f(x)]\Bigg|_{x=0} \tag{10}$$

Specifically, applying the differential operator $\dfrac{\partial^{n_1+n_2+m_1+m_2}}{\partial\tilde{u}_1^{n_1}\partial\tilde{u}_2^{n_2}\partial\tilde{\upsilon}_1^{m_1}\partial\tilde{\upsilon}_2^{m_2}}[\quad]$ in each one of the terms of Equ.(8), we obtain:

for the first term,

$$\frac{\partial^{n_1+n_2+m_1+m_2}}{\partial\tilde{u}_1^{n_1}\partial\tilde{u}_2^{n_2}\partial\tilde{\upsilon}_1^{m_1}\partial\tilde{\upsilon}_2^{m_2}}\left[\frac{\partial}{\partial t}\varphi_{x_1(t)x_2(t)y_1(s)y_2(s)}(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2)\right]\Bigg|_{s\to t}\Bigg|_{\substack{\tilde{u}_1=0\\ \tilde{u}_2=0\\ \tilde{\upsilon}_1=0\\ \tilde{\upsilon}_2=0}}=$$

$$\frac{\partial}{\partial t}\left[\frac{\partial^{n_1+n_2+m_1+m_2}}{\partial\tilde{u}_1^{n_1}\partial\tilde{u}_2^{n_2}\partial\tilde{\upsilon}_1^{m_1}\partial\tilde{\upsilon}_2^{m_2}}\varphi_{x_1(t)x_2(t)y_1(s)y_2(s)}\left(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2\right)\Bigg|_{\substack{\tilde{u}_1=0\\ \tilde{u}_2=0\\ \tilde{\upsilon}_1=0\\ \tilde{\upsilon}_2=0}}\right]\Bigg|_{s\to t}= \qquad \text{[using (9)]}$$

$$=i^{n_1+n_2+m_1+m_2}\frac{\partial R^{n_1,n_2,m_1,m_2}_{x_1,x_2,y_1,y_2}(t,t,s,s)}{\partial t}\Bigg|_{s\to t}, \tag{11}$$

for the second term:

$$\frac{\partial^{n_1+n_2+m_1+m_2}}{\partial\tilde{u}_1^{n_1}\partial\tilde{u}_2^{n_2}\partial\tilde{\upsilon}_1^{m_1}\partial\tilde{\upsilon}_2^{m_2}}\times$$

$$\times\sum_{q_1,q_2}\frac{i\tilde{u}_1A^{(1)}_{q_1q_2q_3q_4}(t)}{i^{q_1+q_2+q_3+q_4}}\frac{\partial^{q_1}}{\partial\tilde{u}_1^{q_1}}\frac{\partial^{q_2}}{\partial\tilde{u}_2^{q_2}}\frac{\partial^{q_3}}{\partial\tilde{\upsilon}_1^{q_3}}\frac{\partial^{q_4}}{\partial\tilde{\upsilon}_2^{q_4}}\varphi_{x_1(t)x_2(t)y_1(t)y_2(t)}(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2)\Bigg|_{\substack{\tilde{u}_1=0\\ \tilde{u}_2=0\\ \tilde{\upsilon}_1=0\\ \tilde{\upsilon}_2=0}}=$$

$$\sum_{q_1,q_2,q_3,q_4} \frac{A^{(1)}_{q_1q_2q_3q_4}(t)}{i^{q_1+q_2+q_3+q_4-1}} \left[\frac{\partial^{n_1+n_2+m_1+m_2}}{\partial\tilde{u}_1^{n_1}\partial\tilde{u}_2^{n_2}\partial\tilde{\upsilon}_1^{m_1}\partial\tilde{\upsilon}_2^{m_2}} \tilde{u}_1 \frac{\partial^{q_1}}{\partial\tilde{u}_1^{q_1}} \frac{\partial^{q_2}}{\partial\tilde{u}_2^{q_2}} \frac{\partial^{q_3}}{\partial\tilde{\upsilon}_1^{q_3}} \frac{\partial^{q_4}}{\partial\tilde{\upsilon}_2^{q_4}} \varphi_{x_1(t)x_2(t)y_1(t)y_2(t)}(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2) \right]\Bigg|_{\substack{\tilde{u}_1=0\\ \tilde{u}_2=0\\ \tilde{\upsilon}_1=0\\ \tilde{\upsilon}_2=0}} =$$

[using (10)]

$$\sum_{q_1,q_2,q_3,q_4} \frac{A^{(1)}_{q_1q_2q_3q_4}(t)}{i^{q_1+q_2+q_3+q_4-1}} n_1 \frac{\partial^{n_1+n_2+m_1+m_2+q_1+q_2+q_3+q_4-1}}{\partial\tilde{u}_1^{n_1+q_1-1}\partial\tilde{u}_2^{n_2+q_2}\partial\tilde{\upsilon}_1^{m_1+q_3}\partial\tilde{\upsilon}_2^{m_2+q_4}} \left[\varphi_{x_1(t)x_2(t)y_1(t)y_2(t)}\left(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2\right)\right]\Bigg|_{\substack{\tilde{u}_1=0\\ \tilde{u}_2=0\\ \tilde{\upsilon}_1=0\\ \tilde{\upsilon}_2=0}} =$$

[using (9)]

$$= n_1 i^{n_1+n_2+m_1+m_2} \sum_{q_1,q_2,q_3,q_4} A^{(1)}_{q_1q_2q_3q_4}(t)\cdot R^{q_1+n_1-1,q_2+n_2,q_3+m_1,q_4+m_2}_{x_1,x_2,y_1,y_2}(t,t,t,t)\,. \qquad (12)$$

Similarly, for the third term we obtain:

$$\frac{\partial^{n_1+n_2+m_1+m_2}}{\partial\tilde{u}_1^{n_1}\partial\tilde{u}_2^{n_2}\partial\tilde{\upsilon}_1^{m_1}\partial\tilde{\upsilon}_2^{m_2}} \sum_{q_1,q_2} \frac{i\tilde{u}_2 A^{(2)}_{q_1q_2q_3q_4}(t)}{i^{q_1+q_2+q_3+q_4}} \frac{\partial^{q_1}}{\partial\tilde{u}_1^{q_1}} \frac{\partial^{q_2}}{\partial\tilde{u}_2^{q_2}} \frac{\partial^{q_3}}{\partial\tilde{\upsilon}_1^{q_3}} \frac{\partial^{q_4}}{\partial\tilde{\upsilon}_2^{q_4}} \varphi_{x_1(t)x_2(t)y_1(t)y_2(t)}(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}_1,\tilde{\upsilon}_2)\Bigg|_{\substack{\tilde{u}_1=0\\ \tilde{u}_2=0\\ \tilde{\upsilon}_1=0\\ \tilde{\upsilon}_2=0}} =$$

$$n_2\cdot \sum_{q_1,q_2,q_3,q_4} A^{(2)}_{q_1q_2q_3q_4}(t)\cdot R^{q_1+n_1,q_2+n_2-1,q_3+m_1,q_4+m_2}_{x_1,x_2,y_1,y_2}(t,t,t,t)\,. \qquad (13)$$

Combining equation (8) with equations (11), (12), (13), equation (7) is re-obtained.

2.6. Application to the ship roll problem

The RE theory has been applied to the ship roll problem with both parametric and external stochastic excitation (G.A. Athanassoulis et al. 2009). As an example case we shall consider the roll motion equation with external stochastic excitation (see, e.g., (Belenky & Sevastianov 2003)):

$$(I+A)\,\ddot{x}(t) + b_1\,\dot{x}(t) + b_3\,\dot{x}^3(t) + K_1x(t) + K_3x^3(t) = y(t\,;\theta), \qquad (1)$$

where:

$x(t)=x(t;\theta)$ is the roll motion (angle),
$I+A$ is the inertia coefficient,
$b_1,\ b_3$ are the damping coefficients,
$y(t;\theta)$ is the external excitation,
$K_1,\ K_3$ are hydrostatic coefficients.

The external excitation $y(t;\theta)$, that is the roll moment due to wind and waves, contains a mean steady component (wind), two nearly stationary, nearly Gaussian components (gust/ waves), a non-stationary, non-Gaussian component (squall), whereas a quadratic component might also be important in some cases. In this case $y(t;\theta)$ is in general, non-Gaussian, non-stationary, and has correlation time comparable with the relaxation time of roll motion $x(t;\theta)$. Therefore, roll motion $x(t;\theta)$ will be a non-Markovian random function requiring specific modeling techniques for its probabilistic characterization. One approach is to model the excitation as the output of a filter driven by white noise (solving an inverse/identification problem) and study the augmented dynamical system in the sense of Itô (Francescutto & Naito 2004). Another approach is to use stochastic averaging techniques, formulating an approximate Itô SDE for a slowly varying motion parameter (e.g., energy) (Stratonovich 1963; Khasminskii 1966), this was applied to ship roll motion e.g. by Roberts (Roberts 1982; Roberts & Dacunha 1985; Roberts & Vasta 2000), Kreuzer & Sichermann (Kreuzer & Sichermann 2007).

The response-excitation theory allows to obtain an equation for the evolution of the joint roll motion ($x_1(t) = x(t)$), roll velocity ($x_2(t) = \dot{x}_1(t)$) and excitation $y(s)$ PDF (joint REPDF $f_{x_1(t)x_2(t)y(s)}(\alpha_1,\alpha_2,\beta)$). More precisely, applying Equ.(7)_Sec(2.4.5) to system (1) we obtain the joint REPDF evolution equation:

$$\frac{\partial}{\partial t} f_{x_1(t)x_2(t)y(s)}(\alpha_1,\alpha_2,\beta)\Big|_{s\to t} + \\ + \mathscr{L}_{\alpha_1\alpha_2} f_{x_1(t)x_2(t)y(s)}(\alpha_1,\alpha_2,\beta) + \beta \frac{f_{x_1(t)x_2(t)y(s)}(\alpha_1,\alpha_2,\beta)}{\partial\alpha_2} = 0 \tag{2}$$

where:

$$\mathscr{L}_{\alpha_1\alpha_2}(\bullet) = \left[-\frac{1}{I+A} \left(b_1 + 3\,b_3\,\alpha_2^2 - \alpha_2 \frac{\partial}{\partial\alpha_1} + K_1\,\alpha_1 \frac{\partial}{\partial\alpha_2} + K_3\,\alpha_1^3 \frac{\partial}{\partial\alpha_2} + \right.\right. \\ \left.\left. + b_1\,\alpha_2 \frac{\partial}{\partial\alpha_2} + b_3\,\alpha_2^3 \frac{\partial}{\partial\alpha_2} + b_1\alpha_2 \frac{\partial}{\partial\alpha_2} + b_3\alpha_2^3 \frac{\partial}{\partial\alpha_2} \right) \right] (\bullet) \tag{3}$$

The above equation should be supplemented by compatibility conditions (Equ.(8) in Sec(2.4.6)), initial conditions (Equ.(9)_Sec(2.4.6)) and constitutive conditions (Equs(10a,b) in Sec(2.4.6)))

As discussed in Section 2.4.6, pp.32, fundamental questions regarding both the well-posedness and the methods for the numerical solution of the REPDF evolution equation (2) have been raised. This will be discussed in the context of a simple problem (linear scalar RDE with Gaussian input) in Chapter 3, whereas, in Chapter 4 an *a priori* closure technique shall be introduced and used for the numerical solution of the long-time, steady-state REPDF evolution equation that corresponds to non-liner scalar RDEs

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CHAPTER 3

APPLICATION OF RE THEORY TO LINEAR DYNAMICAL SYSTEMS

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3.1. Introduction

Chapter 3 will explore the well-studied (Qiu & Wu 2010), simple problem of the probabilistic characterization of a linear Random Differential Equation (RDE) under smoothly correlated (colored) stochastic excitation (input) in the context of the RE theory. To this end, we develop and solve analytically two-time RE moment equations for the response mean value $m_x(t)$, the two-time response-excitation cross-correlation $R_{xy}(t,s)$, the two-time response auto-correlation $R_{xx}(t,s)$ and the one-time response auto-correlation $R_{xx}(t,t)$. The obtained formulae are implemented for the case that the two-time auto-correlation function $R_{yy}(t,s)$ of the random input is a low pass Gaussian Filter (lpGF), an Ornstein-Uhlenbeck (OU) process or, alternatively, a shifted OU (sOU) process.

Assuming that the input $y(t;\theta)$ in the linear RDE is a Gaussian random function (hereafter referred to as linear/Gaussian case) the joint REPDF $f_{x(t)y(s)}(\alpha,\beta)$ is a 2D Gaussian density, uniquely defined by the solution of the two-time RE moment equations. This analytically obtained joint REPDF $f_{x(t)y(s)}(\alpha,\beta)$ is used to verify the REPDF evolution equation. In fact, we will prove that, in the linear/Gaussian case, the REPDF evolution equation becomes equivalent with the limit two-time RE moment equations.

Thereafter, the equivalence between the REPDF evolution equation and the limit two-time RE moment equations will be invoked, in order to clarify that the REPDF evolution equation does not have a unique solution, as also stated in Venturi et al.(Venturi et al. 2012). As will be shown the non-uniqueness is due to the fact that the correlation length of the excitation is not properly taken into account since some of the non-local (in time) characteristics are lost when taking the limit $\partial f_{x(t)y(s)}(\alpha,\beta)/\partial t\Big|_{s\to t}$.

Subsequently, the need for an *a priori* closure of the REPDF evolution equation, providing additional information about the RE correlation structure, will be discussed. In the linear/Gaussian case this could be provided by the moment equation for the two-time RE cross-correlation $R_{xy}(t,s)$.

Lastly, we examine the connection between the REPDF evolution equation response-marginal with the two-time RE moment equation in the linear/Gaussian case.

3.1.1. The underlying deterministic problem. The scalar case

Consider the scalar RDE, subject to a known mean square (m.s.) continuous stochastic excitation $y(t;\theta)$, and a known stochastic initial condition $x_0(\theta)$:

$$\dot{x}(t;\theta) = A\cdot x(t;\theta) + B\cdot y(t;\theta), \tag{1a}$$

$$x(t_0;\theta) = x_0(\theta), \qquad A,B\in\mathbb{R}. \tag{1b}$$

Hereafter, the RDEs will be studied in the m.s. sense (for the m.s. calculus and its application to the study of RDEs see e.g. (Soong 1973; Loeve 1978; Saaty 1981; Sobczyk 1991)).

In order to determine the probabilistic characteristics of the unknown through the known stochastic quantities, use shall be made of the deterministic transformation that defines the sample function of the system's response for a specific realization of the excitation, and a specific value of the initial condition. We shall thus need first to consider the deterministic version of the initial value problem (1), i.e.:

$$\dot{x}(t) = A \cdot x(t) + B \cdot y(t), \tag{2a}$$
$$x(t_0) = x_0. \tag{2b}$$

and its general solution, given by the integral:

$$x(t) = e^{A\cdot(t-t_0)}\left(\int_{t_0}^{t} B\cdot y(s) e^{-A\cdot(s-t_0)} ds + x_0\right). \tag{3}$$

For $A \le 0$, the solution given by Equ. (3) is asymptotically (as $t \to \infty$) stable, whereas for $A > 0$ the solution is asymptotically unstable.

3.1.2 The underlying deterministic problem. The vector case

Some of the results presented in this chapter will also be generalized to linear dynamical systems of the form:

$$\dot{\mathbf{x}}(t;\theta) = \mathbf{A}\cdot\mathbf{x}(t;\theta) + \mathbf{B}\cdot\mathbf{y}(t;\theta), \tag{4a}$$
$$\mathbf{x}(t_0;\theta) = \mathbf{x_0}(\theta), \tag{4b}$$

where $\mathbf{A} = \left[A_{n_1 n_2}\right]_{\substack{n_1=1,2,\dots,N \\ n_2=1,2,\dots,N}}$, $\mathbf{B} = \left[B_{nm}\right]_{\substack{n=1,2,\dots,N \\ m=1,2,\dots,M}}$ are deterministic, time invariant matrices, $\mathbf{y}(t;\theta) = \left(y_1(t;\theta), y_2(t;\theta), \dots, y_M(t;\theta)\right)^{\mathrm{T}}$ is a known stochastic excitation, $\mathbf{x}(t;\theta) = \left(x_1(t;\theta), x_2(t;\theta), \dots, x_M(t;\theta)\right)^{\mathrm{T}}$ is system's response and $\mathbf{x_0}(\theta) = \mathbf{x}(0;\theta)$ is a known stochastic initial condition.

The deterministic transformation that defines the sample function of the system's response that corresponds to a specific realization of the excitation and a specific value of the initial condition are given by the deterministic version of the initial value problem (4), i.e.:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\cdot\mathbf{x}(t) + \mathbf{B}\cdot\mathbf{y}(t), \tag{5a}$$
$$\mathbf{x}(t_0) = \mathbf{x_0}. \tag{5b}$$

and its general solution (pp.80 in (Ahmad & Rao 1999)) that is given by the integral:

$$\mathbf{x}(t) = \int_{t_0}^{t} \mathbf{\Phi}(t-s_1)\cdot\mathbf{B}\cdot\mathbf{y}(s_1) ds_1 + \mathbf{\Phi}(t)\mathbf{x_0}, \tag{6}$$

where $\mathbf{\Phi}(t)=\left[\Phi_{n_1 n_2}\right]_{\substack{n_1=1,2,\ldots,N \\ n_2=1,2,\ldots,N}}$ is the fundamental matrix of the corresponding homogeneous system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\cdot\mathbf{x}(t). \tag{7}$$

The stability of the homogeneous linear system (7) is determined by the eigenvalues of the matrix $\mathbf{A}$.

3.2. Analytical solution to the moment problem. The scalar case

In this section, we shall derive moment equations for the RDE given by Equ.(1)_Sec.(3.1.1). These equations allow us to determine the response mean value $m_x(t)$, the RE cross-correlation $R_{xy}(t,s)$ and the two-time response auto-correlation $R_{xx}(t,s)$, through the known mean value $m_y(t)$ and auto-correlation $R_{yy}(t,s)$ of the excitation as well the known moments of the excitation and response initial conditions m_{x_0}, $R_{x_0 x_0}$, m_{y_0}, $R_{y_0 y_0}$. Hereafter, it will be assumed that the initial value $x_0(\theta)$ is independent from the excitation $y(t;\theta)$.

3.2.1. Analytical solution to the moment problem in the transient regime

To calculate **the response mean value function** $m_x(t)$, we take mean values in linear RDE, Equs.(1a,b)_Sec(3.1.1), i.e.:

$$E^{\theta}\left[\frac{d}{dt}x(t;\theta)\right] = A\cdot E^{\theta}\left[x(t;\theta)\right]+B\cdot E^{\theta}\left[y(t;\theta)\right], \tag{1a}$$

$$E^{\theta}\left[x(0\,;\,\theta)\right]=E^{\theta}\left[x_0(\theta)\right]. \tag{1b}$$

Interchanging the m.s.-derivative of the $x(t;\theta)$ with the mean value operator (see e.g. Soong 1973), we obtain the following **differential equation** and **initial condition** for **the response mean value**:

$$\boxed{\dot{m}_x(t)=A\cdot m_x(t)+B\cdot m_y(t)}\,, \tag{2a}$$

$$\boxed{m_x(t_o)=m_{x_0}}\,, \qquad \forall t\geq t_0. \tag{2b}$$

Applying the analytic solution given by the integral Equ.(3)_Sec(3.1.1) to the initial value problem given by Equ.(2), we obtain **the response mean value** for $t\geq t_0$, i.e. :

$$\boxed{m_x(t)=e^{A\cdot t}\cdot B\cdot\int_{t_0}^{t} m_y(s)e^{-A\cdot s}ds+e^{A\cdot(t-t_0)}\cdot m_{x_0},\ \forall t\geq t_0}\,. \tag{3}$$

Alternatively, let us denote that the response mean value (Equ.(3)) can be obtained, if we consider Equ.(3)_Sec(3.1.1) applied on the sample functions, i.e:

$$x(s;\theta)=e^{A\cdot(s-t_0)}\cdot\left(\int_{t_0}^{s} B\cdot y(s_1;\theta)e^{-A\cdot(s_1-t_0)}ds_1+x_0(\theta)\right), \tag{4}$$

then apply the mean value operator.

To calculate **the two-time RE cross-correlation function** $R_{xy}(t,s)$, Equs.(1a,b)_Sec(3.1.1) are multiplied with $y(s;\theta)$, where $s\geq t_0$, i.e.:

$$\dot{x}(t;\theta)\cdot y(s;\theta) = A\cdot x(t;\theta)\cdot y(s;\theta)+B\cdot y(t;\theta)\cdot y(s;\theta), \tag{5a}$$

$$x(t_0;\theta)\cdot y(s;\theta)=x_0(\theta)\cdot y(s;\theta). \tag{5b}$$

Applying the mean value operator to Equs.(5a,5b) and interchanging the ms-derivative with the mean value operator, we obtain:

$$\frac{\partial}{\partial t}E^{\theta}\left[x(t;\theta)\cdot y(s;\theta)\right] = A\cdot E^{\theta}\left[x(t;\theta)\cdot y(s;\theta)\right]+B\cdot E^{\theta}\left[x(t;\theta)\cdot y(s;\theta)\right], \tag{6a}$$

$$\mathrm{E}^{\theta}\left[x(t_0;\theta)\cdot y(s;\theta)\right] = \mathrm{E}^{\theta}\left[x_0(\theta)\cdot y(s;\theta)\right]. \tag{6b}$$

Since we have assumed that the response initial condition $x_0(\theta)$ is independent from excitation $y(s;\theta)$, Equ.(6b) becomes:

$$\mathrm{E}^{\theta}\left[x_0(\theta)\cdot y(s;\theta)\right] = \mathrm{E}^{\theta}\left[x_0(\theta)\right]\cdot\mathrm{E}^{\theta}\left[y(s;\theta)\right] = m_{x_0}\cdot m_y(s). \tag{7}$$

Combining Equ.(6) and Equ.(7), we obtain the following **differential equation** and **initial condition** for **the two-time RE cross-correlation function:**

$$\boxed{\frac{\partial}{\partial t}R_{xy}(t,s) = A\cdot R_{xy}(t,s)+B\cdot R_{yy}(t,s)}, \tag{8a}$$

$$\boxed{R_{xy}(t_0,s) = m_{x_0}\cdot m_y(s)} \quad , \qquad \forall t\geq t_0, \forall s\geq t_0 \ . \tag{8b}$$

Applying the analytic solution given by the integral Equ.(3)_Sec(3.1.1) to the initial value problem given by Equ.(8), we obtain **the two-time RE cross-correlation function** $R_{xy}(t,s)$ at time $t\geq t_0$, $s\geq t_0$, i.e.:

$$\boxed{R_{xy}(t,s)=e^{A\cdot t}\cdot\int_{t_0}^{t} B\cdot R_{yy}(t_1,s)e^{-A\cdot t_1}dt_1+e^{A\cdot(t-t_0)}\cdot m_{x_0}\cdot m_y(s),\ \forall t\geq t_0, \forall s\geq t_0}. \tag{9}$$

As shown in Equ.(9), in order to calculate $R_{xy}(t,s)$, integration is performed over all values of $R_{yy}(t_1,s)$ for $t_0 \le t_1 \le t$, which takes into consideration the non-local effects of the colored stochastic excitation.

Taking the limit $s \to t$ to both sides of Equ.(9), we obtain further **the one-time RE cross-correlation function** $R_{xy}(t,t)$, i.e.:

$$\boxed{R_{xy}(t,t) \quad = e^{A\cdot t}\cdot\int_{t_0}^{t} B\cdot R_{yy}(t_1,t)\cdot e^{-A\cdot t_1}dt_1 + e^{A\cdot(t-t_0)}\cdot m_{x_0}\cdot m_y(t)\ ,\ \forall t \ge t_0} \ . \tag{10}$$

Similarly, to calculate **the two-time response auto-correlation function** $R_{xx}(t,s)$, we multiply Equ.(1a) _Sec(3.1.1) and Equ.(1b) _Sec(3.1.1) with $x(s;\theta)$ and $x(t;\theta)$, respectively, i.e:

$$\dot{x}(t;\theta)\cdot x(s;\theta) = A\cdot x(t;\theta)\cdot x(s;\theta) + B\cdot y(t;\theta)\cdot x(s;\theta),$$
$$x(t_0;\theta)\cdot x(t;\theta) = x_0(\theta)\cdot x(t;\theta).$$

Applying the mean value operator, we then obtain:

$$\frac{\partial}{\partial t}E^{\theta}\left[x(t;\theta)\cdot x(s;\theta)\right] = A\cdot E^{\theta}\left[x(t;\theta)\cdot x(s;\theta)\right] + B\cdot E^{\theta}\left[y(t;\theta)\cdot x(s;\theta)\right], \tag{11a}$$
$$E^{\theta}\left[x(t_0;\theta)\cdot x(s;\theta)\right] = E^{\theta}\left[x_0(\theta)\cdot x(s;\theta)\right]. \tag{11b}$$

Since the response $x(t;\theta), \forall t \ge t_0$ depends on its initial condition $x_0(\theta)$, an additional equation for the $R_{xx}(t_0,t)$ is needed, so as to calculate the **two-time response auto-correlation** $R_{xx}(t,s)$. Multiplying Equs.(1a,1b)_Sec(3.1.1) with $x_0(\theta)$ and subsequently applying the mean value operator, we obtain:

$$\begin{aligned}\frac{\partial}{\partial t}E^{\theta}\left[x(t;\theta)\cdot x_0(\theta)\right] &= A\cdot E^{\theta}\left[x(t;\theta)\cdot x_0(\theta)\right] + B\cdot E^{\theta}\left[y(t;\theta)\cdot x_0(\theta)\right] = \\ &= A\cdot E^{\theta}\left[x(t;\theta)\cdot x_0(\theta)\right] + B\cdot m_{x_0}\cdot m_y(t),\end{aligned} \ . \tag{11c}$$
$$E^{\theta}\left[x(t_0;\theta)\cdot x_0(\theta)\right] = E^{\theta}\left[x_0(\theta)\cdot x_0(\theta)\right]. \tag{11d}$$

Combining Equ.(11a), (11c) and (11d), we get the following **initial value problem** for **the two-time response auto-correlation function** $R_{xx}(t,s)$**:**

$$\boxed{\frac{\partial}{\partial t}R_{xx}(t,s) = A\cdot R_{xx}(t,s) + B\cdot R_{xy}(s,t)}, \tag{12a}$$

$$\boxed{\frac{\partial}{\partial t}R_{x_0x}(t) = A\cdot R_{x_0x}(t) + B\cdot m_{x_0}\cdot m_y(t)}\ , \tag{12b}$$

$$\boxed{R_{xx}(t_0,t_0)= R_{x_0x_0}} \,. \qquad \forall t \geq t_0, \forall s \geq t_0 \tag{12c}$$

The solution of Equs.(12b,c) is given by the formula:

$$R_{xx}(t_0,s)=e^{A\cdot(s-t_0)}\cdot B\cdot m_{x_0}\cdot\int_{t_0}^{s} m_y(t_1)\cdot e^{-A\cdot(t_1-t_0)}dt_1+e^{A\cdot(s-t_0)}\cdot R_{x_0x_0}\,. \tag{13}$$

Combining Equ.(12a) and (13), the initial value problem for the two-time response auto-correlation function $R_{xx}(t,s)$ becomes**:**

$$\frac{\partial}{\partial t}R_{xx}(t,s)= A\cdot R_{xx}(t,s)+B\cdot R_{xy}(s,t) \quad , \tag{14a}$$

$$R_{xx}(t_0,s)=e^{A\cdot(s-t_0)}\cdot B\cdot m_{x_0}\cdot\int_{t_0}^{s} m_y(t_1)\cdot e^{-A\cdot(t_1-t_0)}dt_1+e^{A\cdot(s-t_0)}\cdot R_{x_0x_0}\,, \forall t\geq t_0, \forall s \geq t_0\,. \tag{14b}$$

Applying the analytic solution given by the integral Equ.(3)_Sec(3.1.1) to the initial value problem given by Equs.(14a,b), we get **the two-time auto- correlation function** of the **response $R_{xx}(t,s)$ as a function of the two-time RE cross-correlation function $R_{xy}(s,t)$** and the response mean value $m_y(t)$, $\forall\ t,s\geq t_0$, i.e.:

$$\boxed{\begin{aligned}R_{xx}(t,s)&=e^{A\cdot t}\cdot B\cdot\int_{t_0}^{t} R_{xy}(s,t_1)e^{-A\cdot t_1}dt_1+\\ &\quad+e^{A\cdot(t+s-t_0)}\cdot B\cdot m_{x_0}\cdot\int_{t_0}^{s} m_y(t_1)\cdot e^{-A\cdot t_1}dt_1+e^{A\cdot(t+s-2\cdot t_0)}\cdot R_{x_0x_0}\,.\end{aligned}} \tag{15}$$

Alternatively, we can derive a relationship that provides **the two-time response auto-correlation $R_{xx}(t,s)$ as a function of two-time auto-correlation function of the excitation** $R_{yy}(t,s)$,$\forall\ t,s\geq t_0$. In fact, from Equ. (9):

$$R_{xy}(s,t_1)=e^{A\cdot s}\cdot B\cdot\int_{t_0}^{s} R_{yy}(t_2,t_1)e^{-A\cdot t_2}dt_2+e^{A\cdot(s-t_0)}\cdot m_{x_0}\cdot m_y(t_1)\ . \tag{16}$$

Substituting $R_{xy}(s,t_1)$ in Equ.(15) with Equ.(16), the first integral in the right hand side of Equ.(15) becomes:

$$\int_{t_0}^{t} R_{xy}(s,t_1)\cdot e^{-A\cdot t_1}dt_1=\int_{t_0}^{t}\left(e^{A\cdot s}\cdot\int_{t_0}^{s} B\cdot R_{yy}(t_2,t_1)e^{-A\cdot t_2}dt_2+e^{A\cdot(s-t_0)}\cdot m_{x_0}\cdot m_y(t_1)\right)\cdot e^{-A\cdot t_1}dt_1=$$

$$= e^{A\cdot s}\int_{t_0}^{t}\left(e^{-A\cdot t_1}\cdot\int_{t_0}^{s} B\cdot R_{yy}(t_2,t_1)e^{-A\cdot t_2}dt_2\right)dt_1 + \qquad (17)$$
$$+e^{A\cdot(s-t_0)}B\cdot\int_{t_0}^{t} m_{x_0}\cdot m_y(t_1)e^{-A\cdot t_1}dt_1.$$

Substituting $R_{xy}(s,t_1)$ in Equ.(15) with Equ.(17), we get the **two-time response auto-correlation function** $R_{xx}(t,s)$ as a function of the two-time excitation auto-correlation function $R_{yy}(t,s)$, $m_y(t)$, $\forall\ t,s\geq t_0$, i.e.:

$$R_{xx}(t,s) = B\cdot e^{A\cdot(s+t)}\int_{t_0}^{t}\left(e^{-A\cdot t_1}\cdot\int_{t_0}^{s} B\cdot R_{yy}(t_2,t_1)e^{-A\cdot t_2}dt_2\right)dt_1 + e^{A\cdot(t+s-t_0)}B\cdot m_{x_0}\cdot\int_{t_0}^{t} m_y(t_1)e^{-A\cdot t_1}dt_1$$
$$+e^{A\cdot(t+s-t_0)}\cdot B\cdot m_{x_0}\cdot\int_{t_0}^{s} m_y(s_1)\cdot e^{-A\cdot s_1}ds_1 + e^{A\cdot(t+s-2\cdot t_0)}\cdot R_{x_0x_0}, \qquad \forall\ t,s\geq t_0. \qquad (18)$$

Moreover, taking the limit $s\to t$ in both sides of Equ.(18), we obtain the one-time **response auto-correlation** $R_{xx}(t,t)$, i.e.:

$$R_{xx}(t,t) = B^2\cdot e^{2\cdot A\cdot t}\cdot\int_{t_0}^{t}\left(e^{-A\cdot t_1}\cdot\int_{t_0}^{t} B\cdot R_{yy}(t_2,t_1)e^{-A\cdot t_2}dt_2\right)dt_1 + 2\cdot e^{A\cdot(2\cdot t-t_0)}B\cdot\int_{t_0}^{t} m_{x_0}\cdot m_y(t_1)e^{-A\cdot t_1}dt_1$$
$$+e^{2\cdot A\cdot(t-t_0)}\cdot R_{x_0x_0} \quad , \quad \forall\ t\geq t_0. \qquad (19)$$

We shall now derive an equation that describes the evolution of $R_{xx}(t,t)$. In Chapter 5, a moment equation for the evolution of $R_{xx}(t,t)$ will be used in conjunction with an evolution equation for the evolution of $R_{xy}(t,s)$, in order to obtain a time closure for the two-time RE moment equations that are developed there for non-linear RDEs.

Multiplying Equ.(1a)_Sec(3.1.1) and Equ.(1b)_Sec(3.1.1) with $2\cdot x(t;\theta)$ and $x(0;\theta)$, respectively, we obtain:

$$2\cdot x(t;\theta)\cdot\dot{x}(t;\theta) = 2\cdot A\cdot\left(x(t;\theta)\right)^2 + 2\cdot B\cdot x(t;\theta)\cdot y(t;\theta), \qquad (20a)$$
$$x(0;\theta)\cdot x(0;\theta) = x_0(\theta)\cdot x(0;\theta), \qquad (20b)$$

that is:

$$\frac{d}{dt}\left(x(t;\theta)\right)^2 = 2\cdot A\cdot\left(x(t;\theta)\right)^2 + 2\cdot B\cdot x(t;\theta)\cdot y(t;\theta), \qquad (21a)$$

$$\left(x(0;\theta)\right)^2 = \left(x_0(\theta)\right)^2 . \tag{21b}$$

Applying the mean value operator to Equs.(21a,b), we obtain an initial value problem **for the one time response auto-correlation function** $R_{xx}(t,t)$, i.e.:

$$\boxed{\frac{d}{dt}R_{xx}(t,t) = 2\cdot A\cdot R_{xx}(t,t) + 2\cdot B\cdot R_{xy}(t,t)}, \tag{22a}$$

$$\boxed{R_{xx}(t_0,t_0) = R_{x_0 x_0}} . \tag{22b}$$

Subsequently, substituting $R_{xy}(t,t)$ in Equ.(22a) with $R_{xy}(t,t)$ from Equ.(10), Equ.(22a) becomes:

$$\frac{d}{dt}R_{xx}(t,t) = 2\cdot A\cdot R_{xx}(t,t) + 2\cdot B\cdot\left(e^{A\cdot t}\cdot\int_{t_0}^{t} B\cdot R_{yy}(t_1,t)\cdot e^{-A\cdot t_1}dt_1 + e^{A\cdot(t-t_0)}\cdot m_{x_0}\cdot m_y(t)\right),$$

which is equivalent to:

$$\frac{d}{dt}x_1(t) = 2\cdot A\cdot x_1(t) + 2\cdot B\cdot y_1(t), \tag{23}$$

where

$$x_1(t) = R_{xx}(t,t), \tag{23b}$$

$$y_1(t) = 2\cdot e^{A\cdot t}\cdot\int_{t_0}^{t} B\cdot R_{yy}(t_1,t)\cdot e^{-A\cdot t_1}dt_1 + 2\cdot e^{A\cdot(t-t_0)}\cdot m_{x_0}\cdot m_y(t). \tag{23c}$$

Applying the analytic solution given by the integral Equ.(3)_Sec(3.1.1) to the initial value problem given by Equs.(23,22b), we obtain:

$$R_{xx}(t,t) = 2\cdot e^{2A\cdot t}\cdot\int_{t_0}^{t} B\cdot y_1(t_2)\cdot e^{-2A\cdot t_2}dt_2 + 2\cdot e^{2A\cdot(t-t_0)}\cdot R_{x_0 x_0}, \tag{24}$$

where:

$$\int_{t_0}^{t} B\cdot y_1(t_2)e^{-2A\cdot t_2}dt_2 = B^2\cdot\int_{t_0}^{t}\int_{t_0}^{t_2}\underbrace{R_{yy}(t_1,t_2)\cdot e^{-A\cdot(t_1+t_2)}}_{g(t_1,t_2)}dt_1dt_2 + B\cdot\int_{t_0}^{t} e^{-A\cdot(t_2-t_0)}\cdot m_{x_0}\cdot m_y(t_2)dt_2 \tag{25}$$

Notice that $g(t_1,t_2)$ is symmetric with respect to the diagonal $t_1 = t_2$, i.e.: $g(t_1,t_2) = g(t_2,t_1)$, therefore, the first integral of the right hand side of Equ.(25) can be written as:

$$\int_{t_0}^{t}\int_{t_0}^{t_2} B^2 \cdot R_{yy}\left(t_1,t_2\right)\cdot e^{-A\cdot(t_1+t_2)}dt_1dt_2 = \frac{1}{2}\int_{t_0}^{t}\int_{t_0}^{t} B^2 \cdot R_{yy}\left(t_1,t_2\right)\cdot e^{-A\cdot(t_1+t_2)}dt_1dt_2 \tag{26}$$

Combining Equ.(24) and (26), we re-obtain Equ.(19).

Finally, we shall present an alternative derivation of the one-time response auto-correlation function $R_{xx}(t,t)$. More precisely, Equ.(20) can be considered as non-linear RDE for the stochastic process $u(t;\theta)\equiv\left(x(t;\theta)\right)^2$. In this case, the initial value problem given by Equs.(20a,b) becomes:

$$\frac{du(t;\theta)}{dt} = 2\cdot A\cdot u(t;\theta) + 2\cdot B\cdot\left(u(t;\theta)\right)^{1/2}\cdot y(t;\theta), \tag{27a}$$

$$u(t_0) = \left(x_0\right)^2. \tag{27b}$$

The solution of the deterministic problem

$$\frac{du(t)}{dt} = 2\cdot A\cdot u(t) + 2\cdot B\cdot\left(u(t)\right)^{1/2}\cdot y(t), \tag{28a}$$

$$u(t_0) = \left(x_0\right)^2, \tag{28b}$$

is given by the formula

$$u(t) = \frac{1}{4}e^{2\cdot A\cdot t}\cdot\left(\int_{t_0}^{t} 2\cdot B\cdot e^{-A\cdot t_1}\cdot u(t_1)dt_1 + 2\cdot e^{-A\cdot t_0}\cdot x_0\right)^2. \tag{29}$$

Applying Equ.(29) on the sample functions of $u(t;\theta)$, we obtain:

$$\begin{aligned} u(t;\theta) = \left(x(t;\theta)\right)^2 &= \frac{1}{4}\cdot e^{2\cdot A\cdot t}\cdot\left(2\cdot e^{-A\cdot t_0}\cdot x_0 + \int_{t_0}^{t} 2\cdot B\cdot e^{-A\cdot t_1}\cdot y\left(t_1;\theta\right)dt_1\right)^2 = \\ &= e^{2\cdot A\cdot t}\cdot B^2\cdot\left(\int_{t_o}^{t} e^{-A\cdot t_1}\cdot\int_{t_o}^{t} e^{-A\cdot t_2}\cdot y\left(t_1;\theta\right)\cdot y\left(t_2;\theta\right)dt_2dt_1\right) + \\ &+ 2\cdot e^{A\cdot(2\cdot t - t_0)}\cdot x_0(\theta)\cdot B\cdot\int_{t_0}^{t} e^{-A\cdot t_1}\cdot y\left(t_1\right)dt_1 + e^{2\cdot A\cdot(t-t_0)}\cdot\left(x_0(\theta)\right)^2. \end{aligned} \tag{30}$$

Taking mean values in Equ.(30), we re-obtain Equ.(19).

3.2.3. Analytical solution of the moment problem in the long-time, statistical equilibrium limit.

In this section, we shall focus on the **asymptotic (long-time) behavior** of the linear RDE given by Equ.(1)_Sec(3.1.1). We shall consider the case that in the long-time the system reaches a statistical equilibrium limit. To this end, we shall assume that $A \le 0$, i.e. the corresponding deterministic problem is asymptotically stable, and that the random input $y(t;\theta)$ is **a wide sense stationary random process**, i.e. the following properties hold true:

a) $m_y(t) = m_y = \text{steady} < \infty \;,\quad \forall t \ge t_0 \,,$ (1a)

b) $R_{yy}(t,s) < \infty \,,\qquad \text{for } t,\, s \ge t_0 \,,$ (1b)

c) $R_{yy}(t,s) = R_{yy}(t+\tau, s+\tau),\;\; \text{for } t,\, s \ge t_0 \,, \tau \in \mathbb{R} \,.$ (1c)

We shall first apply Equs.(1a-1c) to Equs.(3,9,15,18)_Sec(3.2.1) to get **the mean value of the response** $m_x(t)$, the **two-time RE cross-correlation** $R_{xy}(t,s)$, the **two-time response auto-correlation function** $R_{xx}(t,s)$ **in terms of** $R_{xy}(t,s)$, and the **two-time response auto-correlation function** $R_{xx}(t,s)$ **in terms of** $R_{yy}(t-s)$ for **stationary random input** $y(t;\theta)$. Performing elementary algebraic manipulations, we obtain:

$$\boxed{m_x(t) = -\frac{B}{A} \cdot m_y + \frac{B}{A} e^{A\cdot(t-t_0)} \cdot m_y + e^{A\cdot(t-t_0)} \cdot m_{x_0}} \,, \tag{2}$$

$$\boxed{R_{xy}(t,s) = e^{A\cdot t} \cdot \int_{t_0}^{t} B \cdot R_{yy}(t_1 - s) \cdot e^{-A\cdot(t_1)} dt_1 + e^{A\cdot(t-t_0)} \cdot m_{x_0} \cdot m_y} \,, \tag{3}$$

$$\boxed{R_{xx}(t,s) = e^{A\cdot t} B \cdot \int_{t_0}^{t} R_{xy}(s,t_1) e^{-A\cdot t_1} dt_1 - \frac{B}{\mathrm{A}} \cdot m_{x_0} \cdot m_y \cdot e^{A\cdot(t+s)} \cdot \left(e^{-A\cdot s} - e^{-A\cdot t_0} \right) + e^{A\cdot(t+s-2\cdot t_0)} \cdot R_{x_0 x_0}} . \tag{4}$$

$$\boxed{\begin{aligned} R_{xx}(t,s) = B^2 \cdot e^{A\cdot(s+t)} \int_{t_0}^{t} \left(e^{-A\cdot t_1} \cdot \int_{t_0}^{s} R_{yy}(t_2 - t_1) e^{-A\cdot t_2} dt_2 \right) dt_1 + \\ - \frac{B}{A} \cdot m_{x_0} \cdot m_y \cdot \left(e^{A\cdot(s-t_0)} + e^{A\cdot(t-t_0)} \right) + e^{A\cdot(t+s-2\cdot t_0)} \cdot \left(R_{x_0 x_0} + 2 \cdot \frac{B}{A} \cdot m_{x_0} \cdot m_y \right) \end{aligned}} . \tag{5}$$

Let us now focus on the long-time asymptotic limit ($s,t \to \infty$) of the moments of the linear RDE given by Equs.(2-5) for finite time lag, i.e. $|t-s| = |\tau| < \infty$.

Taking the limit $t \to \infty$ of Equ.(2), we get that the asymptotic response mean value $\lim_{t\to\infty} m_x(t) \equiv m_x^{\infty}$ is time invariant and equals:

$$\boxed{m_x^{(\infty)} = -\frac{B}{A}\cdot m_y} \quad . \tag{6}$$

Moreover, applying the limit $s,t \to \infty$ for $|t-s| = |\tau| < \infty$ to Equs.(3-5), we get the asymptotic cross-correlation $R_{xy}^{(\infty)}(t,s)$

$$\boxed{R_{xy}^{(\infty)}(t,s) = \lim_{\substack{t\to\infty, s\to\infty \\ t-s=\tau}} \left[B\cdot e^{A\cdot t} \cdot \int_{t_0}^{t} R_{yy}(t_1,s) e^{-A\cdot t_1} dt_1 \right]}, \tag{7}$$

the asymptotic response auto-correlation $R_{xx}^{(\infty)}(t,s)$ in terms of $R_{xy}(t,s)$

$$\boxed{R_{xx}^{(\infty)}(t,s) = \lim_{\substack{t\to\infty, s\to\infty \\ t-s=\tau}} \left[B\cdot e^{A\cdot t} \cdot \int_{t_0}^{t} R_{xy}(s,t_1) e^{-A\cdot t_1} dt_1 \right]}, \tag{8a}$$

and the asymptotic response auto-correlation $R_{xx}^{(\infty)}(t,s)$ in terms of $R_{yy}(t-s)$:

$$\boxed{R_{xx}^{(\infty)}(t,s) = B^2 \cdot \lim_{\substack{t\to\infty, s\to\infty \\ t-s=\tau}} e^{A\cdot(t+s)} \cdot \int_{t_0}^{t} e^{-A\cdot t_1} \int_{t_0}^{s} R_{yy}(s_1 - t_1)\cdot e^{-A\cdot s_1}\, ds_1\, dt_1} \quad . \tag{8b}$$

We shall now prove the following theorem:

Theorem 1: Let $A \le 0$, if the linear RDE (Equ.(1)_Sec(3.1.1)) is excited by a stationary stochastic process, then, in the asymptotic limit that $t,s \to \infty$, both the asymptotic response-excitation cross-correlation $R_{xy}^{(\infty)}(t,s)$ and the asymptotic excitation auto-correlation $R_{xx}^{(\infty)}(t,s)$ tend to become stationary. That is:

a) $$\lim_{\substack{t\to\infty \\ s\to\infty}} R_{xy}(t+\tau, s+\tau) \equiv R_{xy}^{(\infty)}(t+\tau, s+\tau) = R_{xy}^{(\infty)}(t,s), \tag{9}$$

b) $$\lim_{\substack{t\to\infty \\ s\to\infty}} R_{xx}(t+\tau, s+\tau) \equiv R_{xx}^{(\infty)}(t+\tau, s+\tau) = R_{xx}^{(\infty)}(t,s). \tag{10}$$

Proof:

a) From Equ.(3), we have:

$$R_{xy}(t+\tau, s+\tau) = e^{A\cdot(t+\tau)}\cdot B\cdot \underbrace{\int_{t_0}^{t} R_{yy}(t_1,s) e^{-A\cdot(t_1)} dt_1}_{I_1} + e^{A\cdot(t+\tau-t_0)}\cdot m_{x_0}\cdot m_y \quad . \tag{11}$$

We shall first calculate the integral I_1.

Let $t_1 = u + \tau$ then $dt_1 = du$ and $t_0 - \tau \le u \le t$, then:

$$I_1(t,s) = \int_{t_0-\tau}^{t} R_{yy}(u+\tau, s+\tau)\cdot e^{-A\cdot(u+\tau)}du = e^{-A\cdot\tau}\cdot\int_{t_0-\tau}^{t} R_{yy}(u+\tau, s+\tau)\cdot e^{-A\cdot u}du \;. \quad (12)$$

Therefore, invoking the stationarity hypothesis for $y(t;\theta)$, we obtain:

$$I_1(t,s) = e^{-A\cdot\tau}\int_{t_0-\tau}^{t} R_{yy}(u,s)\cdot e^{-A\cdot u}du = e^{-A\cdot\tau}\left(\int_{t_0-\tau}^{t_0} R_{yy}(u,s)\cdot e^{-A\cdot u}du + \int_{t_0}^{t} R_{yy}(u,s)\cdot e^{-A\cdot u}du\right). \quad (13)$$

Substituting $I_1(t,s)$ in Equ.(11) by Equ.(13), we obtain:

$$\begin{aligned} R_{xy}(t+\tau, s+\tau) = e^{A\cdot(t+\tau)}\cdot B\cdot\left(e^{-A\cdot\tau}\cdot\left(\int_{t_0-\tau}^{t_0} R_{yy}(u,s)\cdot e^{-A\cdot u}du + \int_{t_0}^{t} R_{yy}(u,s)\cdot e^{-A\cdot u}du\right)\right) \\ + e^{A\cdot(t+\tau-t_0)}\cdot m_{x_0}\cdot m_y\,, \end{aligned} \quad (14)$$

or

$$\begin{aligned} R_{xy}(t+\tau, s+\tau) = e^{A\cdot t}\cdot B\cdot\int_{t_0-\tau}^{t_0} R_{yy}(u,s)\cdot e^{-A\cdot u}du + e^{A\cdot t}\cdot B\cdot\int_{t_0}^{t} R_{yy}(u,s)\cdot e^{-A\cdot u}du \\ + e^{A\cdot(t+\tau-t_0)}\cdot m_{x_0}\cdot m_y\,. \end{aligned} \quad (15)$$

Taking the limit $t, s \to \infty$ in Equ.(15), we obtain:

$$R_{xy}^{(\infty)}(t+\tau, s+\tau) = \lim_{t\to\infty}\left[e^{A\cdot t}\cdot B\cdot\int_{t_0}^{t} R_{yy}(u,s)\cdot e^{-A\cdot u}du\right] = R_{xy}^{(\infty)}(t,s) \quad (16)$$

b) (b) is proved in exactly the same way as (a) ■

Corollary: Let $A \le 0$. Assuming that the linear RDE is excited by a stationary stochastic process, then in the limiting case $t, s \to \infty$:

1. The response $x(t;\theta)$ is a wide sense stationary stochastic process.

2. The vector process $\mathbf{XY}(t;\theta)=\left(x(t;\theta),y(t;\theta)\right)$ is a wide sense stationary stochastic process.

In Chapter 5 (Section 5.3.1), it will be proved that the following asymptotic formulae hold true:

$$\lim_{\substack{t\to\infty\\ w=constant}} R_{xy}^{(\infty)}(t,t+w) = B\cdot\int_{w}^{\infty} e^{-A(u-w)}\cdot R_{yy}(u)\,du. \tag{17}$$

$$\lim_{\substack{t_1\to\infty,\ t_2\to\infty\\ |t_1-t_2|=w}} R_{xx}^{(\infty)}(t_1,t_2) = \frac{B^2}{2A}\cdot\int_{v=-\infty}^{v=+\infty} R_{yy}(v)\,e^{-A|v-w|}\,dv. \tag{18}$$

3.2.4. Application to specific excitation functions.

In this section, we are going to implement the obtained formulae for the two-time RE cross-correlation $R_{xy}(t,s)$, the two-time response auto-correlation $R_{xx}(t,s)$ and the one-time response auto-correlation $R_{xx}(t,t)$ of the RDE given by Equ.(1)_Sec(3.1.1), for specific input two-time auto-correlation functions $R_{yy}(t,s)$. We will consider the cases that the stochastic input is a low pass Gaussian Filter (lpGF), an Ornstein-Uhlenbeck (OU) or, alternatively, a sifted OU (sOU) process.

3.2.4.a. Low-pass Gaussian filter (lpGF)

The low-pass Gaussian filter (lpGF) two-time auto-correlation function and the spectrum are given by Equ.(1a) and Equ.(1b), respectively:

$$R_{yy}(t,s)=\sigma^2\cdot\exp\left(-a(t-s)^2\right) \quad (1a), \qquad S_{yy}(\omega)=\frac{1}{2\cdot\sqrt{\pi\cdot a}}\cdot\sigma^2\cdot\exp\left(-\frac{\omega^2}{4\cdot a}\right). \quad (1b)$$

Parameter a controls the correlation time τ_{yy}^{corr} [1] of the excitation processes which is given by Equ.(1c)

$$\tau_{yy}^{\text{corr}} = \frac{1}{\sigma_y^2}\int_0^{\infty}|C_{yy}^{(\infty)}(u)|\,du = \sqrt{\pi}\,/\,(2\sqrt{a}\,). \tag{1c}$$

[1] In general there are several ways to define the correlation time see e.g. (Hristopoulos & Žucovič 2011) for the a definition of the correlation time which also applies to covariance models having more than two parameters.

In fact, as illustrated in **Fig.1a**, τ_{yy}^{corr} decreases with a. The limiting values of τ_{yy}^{corr} with respect to a are: $\lim_{a\to 0}\tau_{yy}^{\text{corr}}=\infty$, $\lim_{a\to\infty}\tau_{yy}^{\text{corr}}=0$. In **Fig.1b-1c,** the lpGF two-time auto-correlation function and spectrum are plotted for various values of the auto-correlation time, i.e. $\tau_{yy}^{\text{corr}}=0.93, 0.51, 0.33$, corresponding to $\sigma^2=1$ and $a=0.9,\ 3,\ 7$, respectively. We can observe that, when τ_{yy}^{corr} decreases, the excitation auto-correlation $R_{yy}(t,s)$ tends to the delta correlated one, whereas the excitation spectrum is flattened.

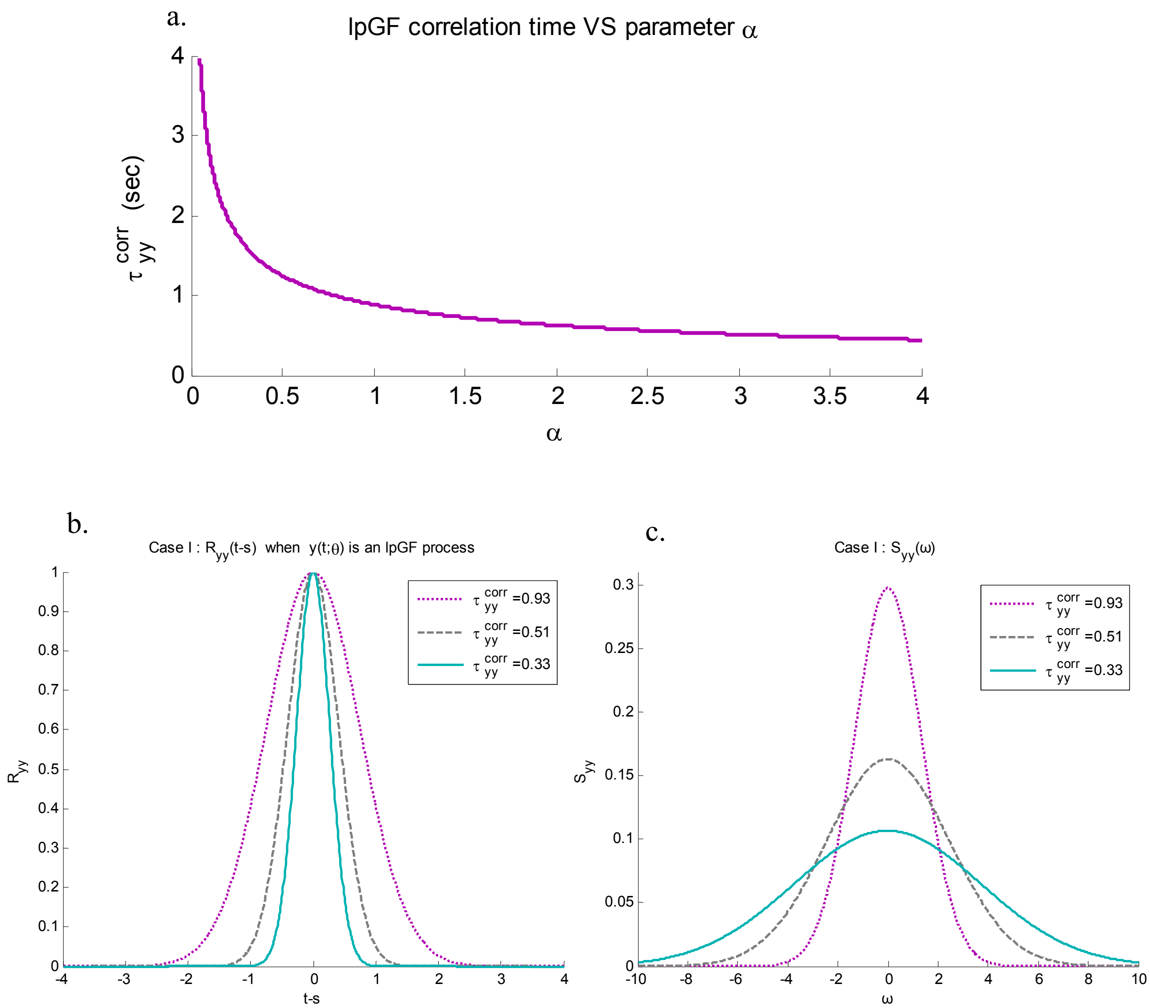


Figure 1:a. The correlation time τ_{yy}^{corr} of the lpGF stochastic excitation against parameter a. b.The lpGF auto-correlation function $R_{yy}(t,s)$.c. The lpGF spectrum $S_{yy}(\omega)$.

We shall obtain analytic formulae of the two-time response auto-correlation and response - excitation cross-correlation functions in the transient and in the long-time statistical equilibrium limit. These results will be illustrated by an example case (**Case I**) that we are going to use throughout this section when the excitation is a lpGF function. More precisely, we are going to show results for the 3 cases of correlation time of the lpGF excitation that we have just mentioned ($\tau_{yy}^{\text{corr}}=0.93, 0.51, 0.33$ and for $\sigma^2=1$) when the parameters of the RDE given by Equ.(1)_Sec(3.1.1) are $A = -1$, $B = 1$. Moreover, in this example case we willm assume that $m_y(t) = 0$ and $m_{x_0} = 0$, then from Equ.(3)_Sec(3.2.1) it is straightforward to find that $m_x(t) = 0$.

In what follows, use will be made of the following integration formulae that are derived in Appendix 1 (see Equs.(1,8)_Sec(A.1)):

$$I_2(t,s) \equiv \int_{t_0}^{t} \exp\left(-a(t_1 - s)^2\right) \cdot e^{-A \cdot t_1} dt_1 = \\ = \exp\left(-A \cdot s + \frac{A^2}{4 \cdot a}\right) \cdot \frac{\sqrt{\pi}}{2\sqrt{a}} \left(-erf\left(\sqrt{a} \cdot (t_0 - s) + \frac{A}{2\sqrt{a}}\right) + erf\left(\sqrt{a} \cdot (t - s) + \frac{A}{2\sqrt{a}}\right)\right), \tag{2}$$

and

$$I_3(t,s) = \int_{t_0}^{t} e^{-2 \cdot A \cdot t_1} \cdot erf\left(\sqrt{a} \cdot (s - t_1) + \frac{A}{2 \cdot \sqrt{a}}\right) dt_1 = \\ = -\frac{1}{2A}\left(e^{-2 \cdot A \cdot t} \cdot erf\left(\sqrt{a} \cdot (s - t) + \frac{A}{2 \cdot \sqrt{a}}\right) - e^{-2 \cdot A \cdot t_0} \cdot erf\left(\sqrt{a} \cdot (s - t_0) + \frac{A}{2 \cdot \sqrt{a}}\right) + \right. \\ \left. + e^{-2 \cdot A \cdot s}\left(erf\left(\sqrt{a} \cdot (t - s) + \frac{A}{2\sqrt{a}}\right) - erf\left(\sqrt{a} \cdot (t_0 - s) + \frac{A}{2\sqrt{a}}\right)\right)\right), \tag{3}$$

Applying integration formula (2) in Equ.(3)_Sec(3.2.3), we find the **transient two-time RE cross-correlation function for the lpGF excitation** $R_{xy}(t,s)$:

$$R_{xy}(t,s) = \frac{B \cdot \sigma^2 \cdot \sqrt{\pi}}{2\sqrt{a}} \cdot e^{\frac{A^2}{4 \cdot a}} \times \\ \times \left[\exp\left(A \cdot (t - s)\right) \cdot \left(-erf\left(\sqrt{a} \cdot (t_0 - s) + \frac{A}{2\sqrt{a}}\right) + erf\left(\sqrt{a} \cdot (t - s) + \frac{A}{2\sqrt{a}}\right)\right)\right]. \tag{4a}$$

Taking the limit $s \to t$ to Equ.(4a), we obtain the one–time RE correlation function:

$$R_{xy}(t,t) = \frac{B \cdot \sigma^2 \cdot \sqrt{\pi}}{2\sqrt{a}} \cdot e^{\frac{A^2}{4 \cdot a}} \times \left[\left(-erf\left(\sqrt{a} \cdot (t_0 - t) + \frac{A}{2\sqrt{a}}\right) + erf\left(\frac{A}{2\sqrt{a}}\right)\right)\right], \tag{4b}$$

In **Fig.2,** results obtained by using Equ.(4a) and Equ.(4b) are plotted for the considered Case I. In Fig.2a, the one-time RE cross-correlation function $R_{xy}(t,t)$ is plotted against the actual time *t*. As expected, the more correlated excitation results in more correlated RE cross-correlation function. It is also interesting to notice that in the most correlated case $\tau_{yy}^{\text{corr}} = 0.93$, $R_{xy}(t,t)$ becomes time invariant almost a second latter than the less correlated case $\tau_{yy}^{\text{corr}} = 0.33$. In Fig.2b, the transient, two-time, RE cross-correlation $R_{xy}(t,s)$ is plotted against the response time t for fixed excitation time $s=1$. Notice that in all the considered cases there is a correlation of the response with future values of the excitation. This is verified by the non-zero values of $R_{xy}(t,s)$ for $t<s=1$. The latter feature is a significant difference of the examined correlation structure from the delta-correlated one, and we can see that for the less correlated case ($\tau_{yy}^{\text{corr}} = 0.33$) this feature tends to vanish. As expected, the correlation of the response with the past values of the excitation lasts for a larger time interval, having a maximum value for $t-s \approx 0.2$ sec for the less correlated case and for $t-s \approx 0.5$ sec for the more correlated case. The response after $t \approx 6$ de-correlates with the excitation at $s=1$.

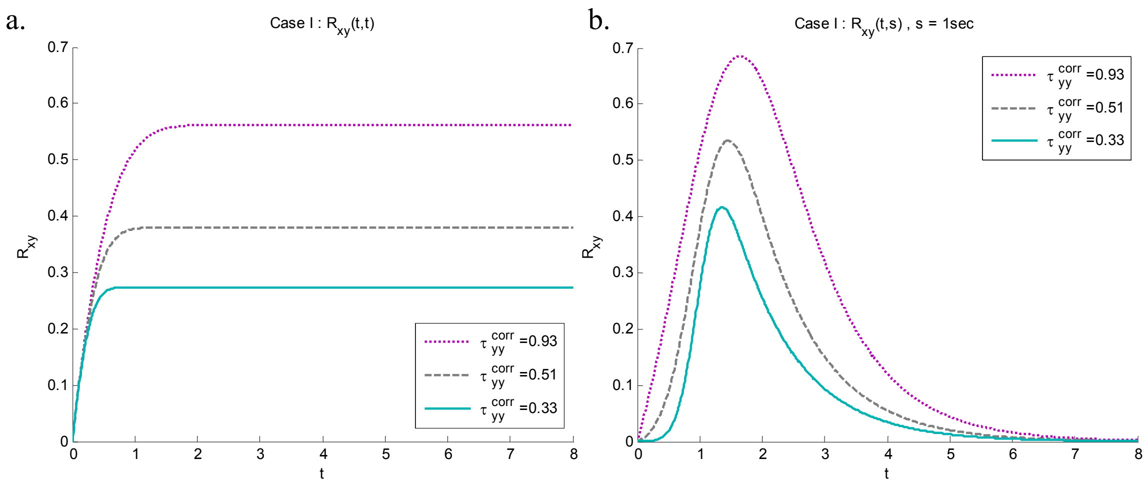


Figure 2: a. The transient diagonal (one-time) RE cross-correlation function of Case I . b. The transient (two-times) RE cross-correlation function of Case I against the response time t, for excitation time $s=1$ sec.

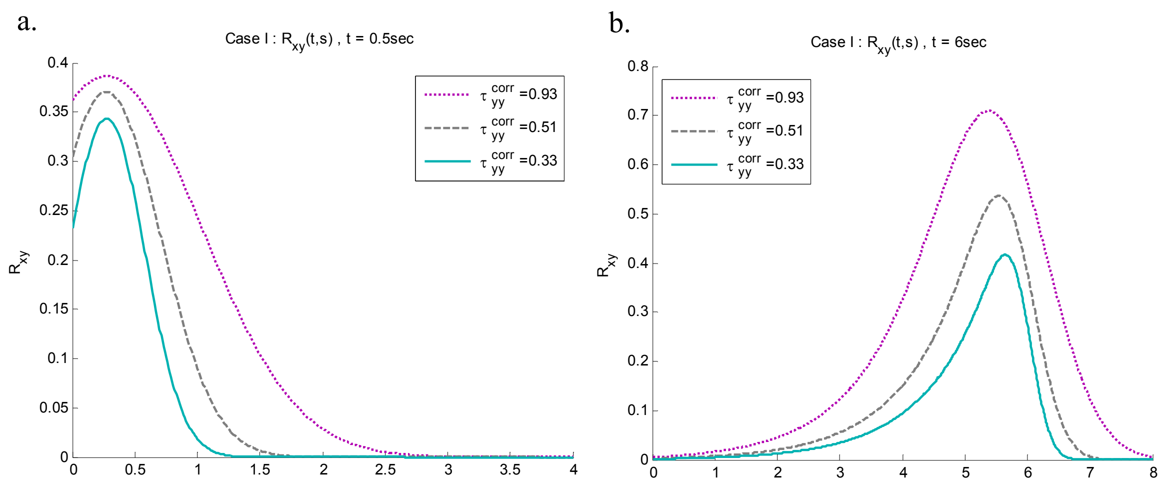


Figure 3: a. The transient two-time RE cross-correlation function of Case I against the rexcitation time s, for excitation time $t=0.5$ sec. b. The same for excitation time $t=6$ sec

In **Fig.3** the transient, two-time RE cross-correlation function is plotted against the excitation time s for fixed response time $t=0.5$ sec (Fig.3a) and $t=6$ (Fig.3b), when the system has already reached the long-time statistical equilibrium state. The comments reported in relation to Fig.2b hold true in this case as well.

Let us now proceed to the calculation of the **transient two-time response auto-correlation for the lpGF excitation** using the obtained results that are summarized in Equ.(5)_ Sec(3.2.3), that we repeat here for convenience:

$$R_{xx}(t,s)=B^2\cdot e^{A\cdot(s+t)}\int_{t_0}^{t}\left(e^{-A\cdot t_1}\cdot\int_{t_0}^{s}R_{yy}(t_2,t_1)e^{-A\cdot t_2}dt_2\right)dt_1-$$
$$-\frac{B}{A}\cdot m_{x_0}\cdot m_y\cdot\left(e^{A\cdot(s-t_0)}+e^{A\cdot(t-t_0)}\right)+e^{A\cdot(t+s-2\cdot t_0)}\cdot\left(R_{x_0x_0}+2\cdot\frac{B}{A}\cdot m_{x_0}\cdot m_y\right), \tag{5}$$

Assuming that $R_{yy}(t,s)$ is given by Equ.(1a), we apply subsequently the integration formulae (2), (3) to Equ.(5), to obtain:

$$R_{xx}(t,s)=B^2\cdot\sigma^2\cdot\frac{\sqrt{\pi}}{2\sqrt{a}}\cdot\exp\left(\frac{A^2}{4}\right)\times$$
$$\times e^{A\cdot(t+s)}\cdot\int_{t_0}^{t}e^{-2A\cdot t_1}\cdot\left(-erf\left(\sqrt{a}\cdot(t_0-t_1)+\frac{A}{2\cdot\sqrt{a}}\right)+erf\left(\sqrt{a}\cdot(s-t_1)+\frac{A}{2\cdot\sqrt{a}}\right)\right)dt_1-$$
$$-\frac{B}{A}\cdot m_{x_0}\cdot m_y\cdot\left(e^{A\cdot(s-t_0)}+e^{A\cdot(t-t_0)}\right)+e^{A\cdot(t+s-2\cdot t_0)}\cdot\left(R_{x_0x_0}+2\cdot\frac{B}{A}\cdot m_{x_0}\cdot m_y\right)$$
$$=B^2\cdot\sigma^2\cdot\frac{\sqrt{\pi}}{2\sqrt{a}}\cdot\exp\left(\frac{A^2}{4}\right)\times\left(e^{A\cdot(t+s)}\cdot I_3(t,s)-e^{A\cdot(t+s)}\cdot I_3(t,t_0)\right)$$
$$-\frac{B}{A}\cdot m_{x_0}\cdot m_y\cdot\left(e^{A\cdot(s-t_0)}-e^{A\cdot(t-t_0)}\right)+e^{A\cdot(t+s-2\cdot t_0)}\cdot R_{x_0x_0}, \tag{6a}$$

where

$$e^{A\cdot(t+s)}\cdot I_3(t,s)=-\frac{1}{2A}\cdot\left(e^{A\cdot(s-t)}\cdot erf\left(\sqrt{a}\cdot(s-t)+\frac{A}{2\cdot\sqrt{a}}\right)-e^{A\cdot(t+s-2\cdot A\cdot t_0)}\cdot erf\left(\sqrt{a}\cdot(s-t_0)+\frac{A}{2\cdot\sqrt{a}}\right)+\right.$$
$$\left.+e^{A\cdot(t-s)}\cdot\left(erf\left(\sqrt{a}\cdot(t-s)+\frac{A}{2\sqrt{a}}\right)-erf\left(\sqrt{a}\cdot(t_0-s)+\frac{A}{2\sqrt{a}}\right)\right)\right), \tag{6b}$$

and

$$e^{A\cdot(t+s)}\cdot I_3(t,t_0)=-\frac{1}{2A}e^{A\cdot(t+s)}\cdot\left(e^{-2\cdot A\cdot t}\cdot erf\left(\sqrt{a}\cdot(t_0-t)+\frac{A}{2\cdot\sqrt{a}}\right)-e^{-2\cdot A\cdot t_0}\cdot erf\left(\frac{A}{2\cdot\sqrt{a}}\right)+\right.$$
$$\left.+e^{-2\cdot A\cdot t_0}\left(erf\left(\sqrt{a}\cdot(t-t_0)+\frac{A}{2\sqrt{a}}\right)-erf\left(\frac{A}{2\sqrt{a}}\right)\right)\right)$$
$$=-\frac{1}{2A}\cdot\left(e^{A\cdot(s-t)}\cdot erf\left(\sqrt{a}\cdot(t_0-t)+\frac{A}{2\cdot\sqrt{a}}\right)+\right.$$
$$\left.+e^{A\cdot(t+s-2\cdot A\cdot t_0)}\left(erf\left(\sqrt{a}\cdot(t-t_0)+\frac{A}{2\sqrt{a}}\right)-2\cdot erf\left(\frac{A}{2\sqrt{a}}\right)\right)\right). \tag{6c}$$

After some algebraic manipulations to Equ. (6), we obtain the following formula for **the transient, two-time auto correlation function of the response**:

$$\begin{aligned}
R_{xx}(t,s) = & -\frac{B^2}{A}\sigma^2 \cdot \frac{\sqrt{\pi}}{4\sqrt{a}} \cdot \exp\left(\frac{A^2}{4\cdot a}\right) \times \\
& \times\Bigg(\Bigg(e^{A\cdot(s-t)} \cdot \left(erf\left(\sqrt{a}\cdot(s-t)+\frac{A}{2\cdot\sqrt{a}}\right) - erf\left(\sqrt{a}\cdot(t_0-t)+\frac{A}{2\cdot\sqrt{a}}\right)\right)+ \\
& - e^{A\cdot(t+s-2\cdot t_0)}\left(erf\left(\sqrt{a}\cdot(s-t_0)+\frac{A}{2\cdot\sqrt{a}}\right)+erf\left(\sqrt{a}\cdot(t-t_0)+\frac{A}{2\sqrt{a}}\right)-2\cdot erf\left(\frac{A}{2\sqrt{a}}\right)\right)+ \\
& + e^{A\cdot(t-s)}\cdot\left(erf\left(\sqrt{a}\cdot(t-s)+\frac{A}{2\sqrt{a}}\right)-erf\left(\sqrt{a}\cdot(t_0-s)+\frac{A}{2\sqrt{a}}\right)\right)\Bigg)\Bigg)- \\
& -\frac{B}{A}\cdot m_{x_0}\cdot m_y\cdot\left(e^{A\cdot(s-t_0)}+e^{A\cdot(t-t_0)}\right)+e^{A\cdot(t+s-2\cdot t_0)}\cdot\left(R_{x_0x_0}+2\cdot\frac{B}{A}\cdot m_{x_0}\cdot m_y\right).
\end{aligned} \tag{7a}$$

As expected, $R_{xx}(t,s)$ is symmetric function of the two time arguments t,s. Taking the limit $s\to t$ to Equ.(7a), we obtain the one–time (diagonal) response auto-correlation function:

$$\begin{aligned}
R_{xx}(t,t) = & -\frac{B^2}{A}\sigma^2 \cdot \frac{\sqrt{\pi}}{2\sqrt{a}} \cdot \exp\left(\frac{A^2}{4\cdot a}\right) \times \\
& \times\Bigg(\left(erf\left(\frac{A}{2\cdot\sqrt{a}}\right)-erf\left(\sqrt{a}\cdot(t_0-t)+\frac{A}{2\cdot\sqrt{a}}\right)\right)+ \\
& -e^{2\cdot A\cdot(t-t_0)}\left(erf\left(\sqrt{a}\cdot(t-t_0)+\frac{A}{2\cdot\sqrt{a}}\right)-erf\left(\frac{A}{2\sqrt{a}}\right)\right)\Bigg)-\frac{B}{A}\cdot m_{x_0}\cdot m_y\cdot e^{A\cdot(t-t_0)}+ \\
& +e^{2\cdot A\cdot(t-t_0)}\cdot\left(R_{x_0x_0}+2\cdot\frac{B}{A}\cdot m_{x_0}\cdot m_y\right).
\end{aligned} \tag{7b}$$

In **Figs.4-5**, results obtained by the use of Equs.(7a,7b) are plotted for Case I. In **Fig.4**, the diagonal response auto-correlation functions $R_{xx}(t,t)$ is plotted against the actual time t for initial value $R_{x_0x_0}=0$ (Fig.4a) and $R_{x_0x_0}=1$ (Fig.4b). Again as expected the more correlated excitation results in a more correlated response auto-correlation function. In **Fig.5,** the transient, two-time, auto-correlation function is plotted against the response time t for fixed excitation time $s=1$ and for two different initial values $R_{x_0x_0}=0$ (Fig.5a) and $R_{x_0x_0}=1$ (Fig.5b).

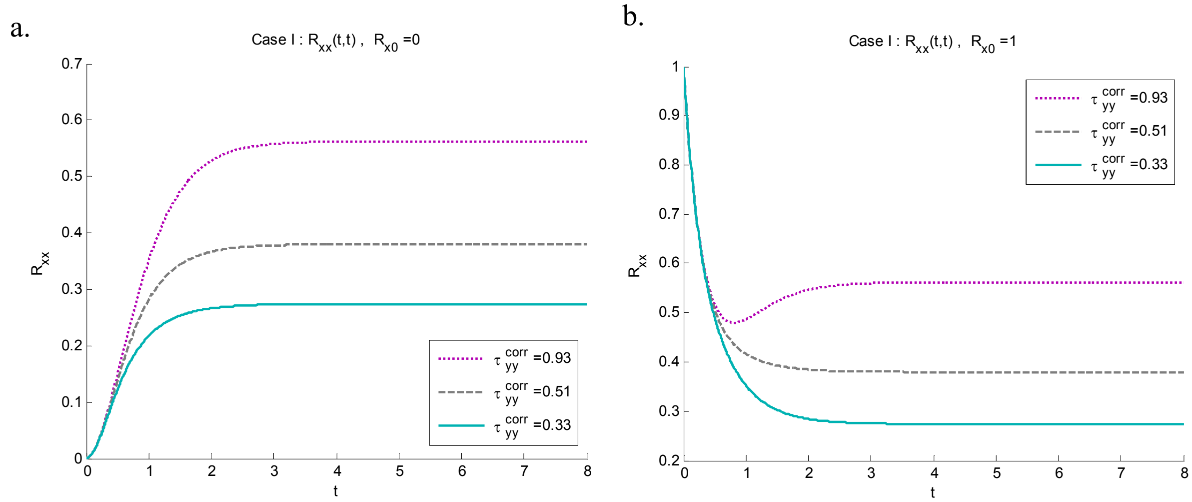


Figure 4: a.b The transient diagonal (one-time) response auto-correlation function of Case I. In a. the initial value of the response auto-correlation function $R_{x_0 x_0} = 0$, in b. $R_{x_0 x_0} = 1$.

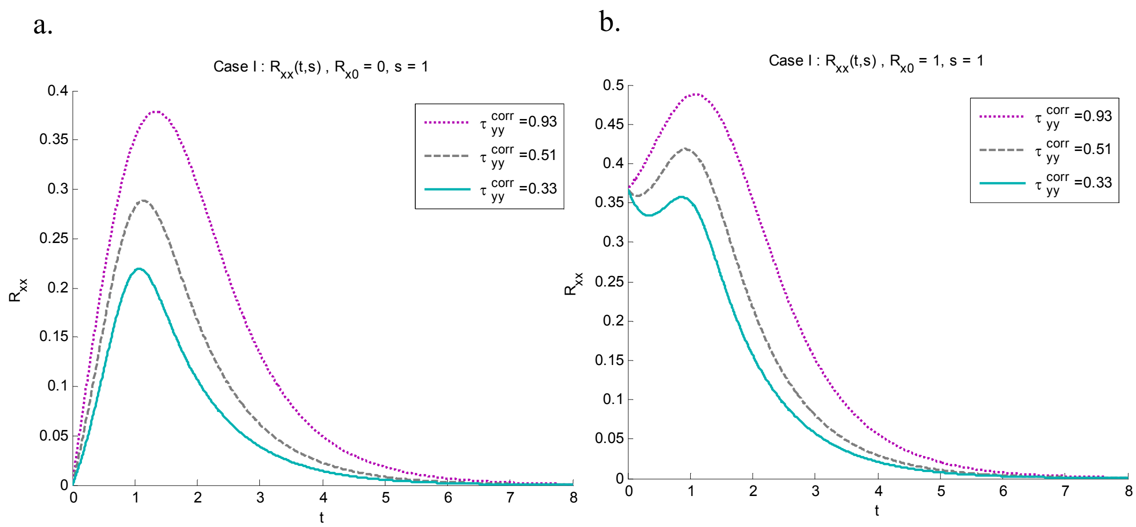


Figure 5:a.,b.: The transient (two-time) RE cross-correlation function of Case I against the response time t, for fixed excitation time $s = 1\sec$. In a. the initial value of the response auto-correlation function $R_{x_0 x_0} = 0$, in b $R_{x_0 x_0} = 1$.

We shall now study **the long-time statistical equilibrium limit** of the two-time response and RE cross-correlation functions of the RDE given by Equ.(1)_Sec(3.1.1) under lpGF excitation.

In fact, applying the integration formula (2) in Equ. (7)_Sec.(3.2.3) for the **long-time limit of the two-time RE cross-correlation function** under excitation that has lpGF auto-correlation function (given by Equ.(1a)), we obtain:

$$R_{xy}^{(\infty)}(t,s) = \frac{B\cdot\sigma^2\cdot\sqrt{\pi}}{2\sqrt{a}}\cdot e^{\frac{A^2}{4\cdot a}} \times$$
$$\times \lim_{\substack{t\to\infty \\ s\to\infty}} \left[\exp\left(A\cdot(t-s)\right)\cdot \left(-erf\left(\sqrt{a}\cdot(t_0 - s) + \frac{A}{2\sqrt{a}}\right) + erf\left(\sqrt{a}\cdot(t-s) + \frac{A}{2\sqrt{a}}\right)\right)\right]. \tag{8}$$

Assuming that $\tau = t - s$ remains finite, the following asymptotic formulae hold true:

$$\lim_{s\to\infty}\left(\sqrt{a}\cdot\left(t_0 - s\right) + \frac{A}{2\sqrt{a}}\right) = -\infty \;\Rightarrow\; \lim_{s\to\infty} erf\left(u\left(s,t_0\right)\right) = -1, \tag{9}$$

and

$$\lim_{\substack{t\to\infty\\ s\to\infty\\ t-s=\tau}}\left(\sqrt{a}\cdot\left(t - s\right) + \frac{A}{2\sqrt{a}}\right) \;\Rightarrow\; \lim_{\substack{t\to\infty\\ s\to\infty\\ t-s=\tau}} erf\left(u\left(s,t\right)\right) = erf\left(\sqrt{a}\cdot\tau + \frac{A}{2\sqrt{a}}\right). \tag{10}$$

Using Equ.(9) and Equ.(10), Equ.(8) becomes:

$$\boxed{R_{xy}^{(\infty)}\left(t-s\right) = B\cdot\sigma^2\cdot\frac{\sqrt{\pi}}{2\sqrt{a}}\cdot\exp\left(\frac{A^2}{4\cdot a}+A\cdot\left(t-s\right)\right)\cdot\left(erf\left(\sqrt{a}\cdot\left(t-s\right)+\frac{A}{2\cdot\sqrt{a}}\right)+1\right)}\,. \tag{11}$$

As expected, in the limiting case the RE cross-correlation does not depend on the initial time t_0. Moreover, $R_{xy}^{(\infty)}\left(t-s\right)$ is a non-symmetric function, that is, the degree of correlation depends on whether the time lag $\tau = t - s$ is positive or negative.

To find **the long-time limit of the response auto-correlation function** $R_{xx}^{(\infty)}\left(t,s\right)$, we consider the limit $s,t\to\infty$ of Equ.(7a), for finite time lag $\left|t-s\right| = \left|\tau\right| < \infty$, i.e:

$$\begin{aligned} R_{xx}^{(\infty)}\left(t,s\right) = -\frac{B^2}{2A}\sigma^2\cdot\frac{\sqrt{\pi}}{2\sqrt{a}}\cdot\exp\left(\frac{A^2}{4\cdot a}\right)\times \\ \lim_{\substack{t\to\infty\\ s\to\infty\\ t-s=\tau}}\left(\left(e^{A\cdot(s-t)}\cdot\left(erf\left(\sqrt{a}\cdot\left(s-t\right)+\frac{A}{2\cdot\sqrt{a}}\right)+erf\left(\sqrt{a}\cdot\left(t_0-t\right)+\frac{A}{2\cdot\sqrt{a}}\right)\right)+\right.\right. \\ +e^{A\cdot(t+s-2\cdot A\cdot t_0)}\left(\cdot erf\left(\sqrt{a}\cdot\left(s-t_0\right)+\frac{A}{2\cdot\sqrt{a}}\right)-erf\left(\sqrt{a}\cdot\left(t-t_0\right)+\frac{A}{2\sqrt{a}}\right)\right)+ \\ \left.\left.+e^{A\cdot(t-s)}\cdot\left(erf\left(\sqrt{a}\cdot\left(t-s\right)+\frac{A}{2\sqrt{a}}\right)-erf\left(\sqrt{a}\cdot\left(t_0-s\right)+\frac{A}{2\sqrt{a}}\right)\right)\right)\right). \end{aligned} \tag{12}$$

The following asymptotic formulae hold true:

$$\lim_{t\to\infty} u\left(t_0,t\right) \equiv \lim_{t\to\infty}\left(\sqrt{a}\cdot\left(t-t_0\right)+\frac{A}{2\sqrt{a}}\right) = \infty \;\Rightarrow\; \lim_{s\to\infty} erf\left(u\left(s,t_0\right)\right) = 1,$$

(13a)

$$\lim_{s\to\infty} u\left(s,t_0\right) \equiv \lim_{s\to\infty}\left(\sqrt{a}\cdot\left(t_0-s\right)+\frac{A}{2\sqrt{a}}\right) = -\infty \;\Rightarrow\; \lim_{s\to\infty} erf\left(u\left(s,t_0\right)\right) = -1, \tag{13b}$$

$$\lim_{\substack{t\to\infty \\ s\to\infty \\ t-s=\tau}} u(s,t) \equiv \lim_{\substack{t\to\infty \\ s\to\infty \\ t-s=\tau}} \left(\sqrt{a}\cdot(t-s)+\frac{A}{2\sqrt{a}}\right) \Rightarrow \lim_{\substack{t\to\infty \\ s\to\infty \\ t-s=\tau}} erf\left(u(s,t)\right) = erf\left(\sqrt{a}\cdot\tau+\frac{A}{2\sqrt{a}}\right) \quad (13c)$$

.

Using Equs.(13), Equ.(12) becomes:

$$R_{xx}^{(\infty)}(t,s) = -\frac{B^2\cdot\sigma^2}{2A}\cdot\frac{\sqrt{\pi}}{2\sqrt{a}}\cdot\exp\left(\frac{A^2}{4\cdot a}\right)\times$$
$$\left(e^{A\cdot(s-t)}\cdot erf\left(\sqrt{a}\cdot(s-t)+\frac{A}{2\cdot\sqrt{a}}\right)+e^{A\cdot(t-s)}\cdot\left(erf\left(\sqrt{a}\cdot(t-s)+\frac{A}{2\sqrt{a}}\right)+1\right)+e^{A\cdot(s-t)}\right)$$
$$=-\frac{B^2\cdot\sigma^2}{2A}\cdot\frac{\sqrt{\pi}}{2\sqrt{a}}\cdot\exp\left(\frac{A^2}{4}\right)\times$$
$$\left(e^{A\cdot(s-t)}\cdot\left(erf\left(\sqrt{a}\cdot(s-t)+\frac{A}{2\cdot\sqrt{a}}\right)+1\right)+e^{A\cdot(t-s)}\cdot\left(erf\left(\sqrt{a}\cdot(t-s)+\frac{A}{2\sqrt{a}}\right)+1\right)\right), \quad (14)$$

That is, for $A\le 0$,

$$R_{xx}^{(\infty)}(t-s) = \\ = \frac{\sqrt{\pi}}{4\sqrt{a}}\cdot\frac{B^2\cdot\sigma^2}{(-A)}\cdot e^{\frac{A^2}{4\cdot a}}\times\left(e^{A\cdot(s-t)}\cdot\left(erf\left(\sqrt{a}\cdot(s-t)+\frac{A}{2\cdot\sqrt{a}}\right)+1\right)+e^{A\cdot(t-s)}\left(erf\left(\sqrt{a}\cdot(t-s)+\frac{A}{2\cdot\sqrt{a}}\right)+1\right)\right). \quad (15)$$

In **Fig.6**, the long-time RE cross-correlation $R_{xy}^{(\infty)}(t-s)$ (Fig.6a) and the long-time response auto-correlation function $R_{xx}^{(\infty)}(t-s)$ (Fig.6b) are plotted for Case I.

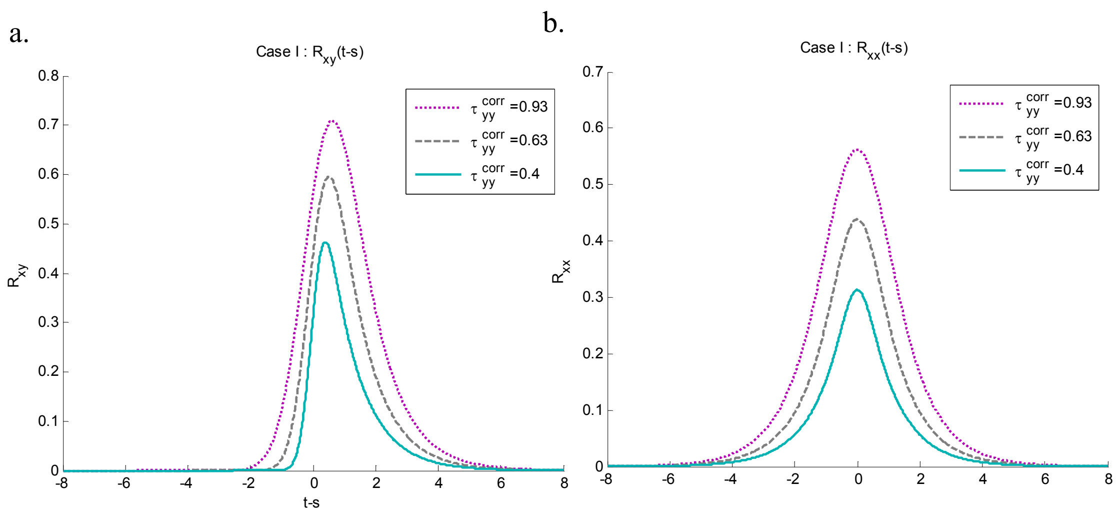


Figure 6: a.The long time RE cross-correlation function $R_{xy}^{(\infty)}(t-s)$, b. The long time response auto-correlation function $R_{xx}^{(\infty)}(t-s)$, of Case I plotted against the time lag $t-s$

Comparing Figs.1a, 6a and 6b, an interesting feature emerges: for $t-s<0$, the long-time RE cross-correlation function $R_{xy}^{(\infty)}(t-s)$ resembles more the excitation auto-correlation function, whereas for $t-s>0$ resembles the response auto-correlation function.

In Appendix 2, the formulae for the long-time statistical equilibrium RE cross-correlation function $R_{xy}^{(\infty)}(t-s)$ and the response auto-correlation function $R_{xx}^{(\infty)}(t-s)$ are re-obtained treating the same problem in the frequency domain. In fact, using the transfer function $H_{xy}(\omega)=\dfrac{B}{i\cdot\omega-\mathrm{A}}$ of the linear RDE under stochastic excitation with the lpGF spectrum $S_{yy}(\omega)$, we find that its stationary RE spectrum $S_{xy}(\omega)$ and stationary response spectrum $S_{xx}(\omega)$ are given by

$$S_{xy}(\omega)=\sigma^2\cdot\frac{-B\cdot\mathrm{A}}{2\cdot\sqrt{\pi\cdot a}}\cdot\frac{1}{\omega^2+\mathrm{A}^2}\cdot\exp\left(-\frac{\omega^2}{4\cdot a}\right)+i\cdot\sigma^2\cdot\frac{B}{2\cdot\sqrt{\pi\cdot a}}\cdot\frac{\omega}{\omega^2+\mathrm{A}^2}\cdot\exp\left(-\frac{\omega^2}{4\cdot a}\right)= \\ =G_{xy}(\omega)+i\cdot Q_{xy}(\omega), \tag{16}$$

$$S_{xx}(\omega)=\left|\frac{B}{i\cdot\omega-\mathrm{A}}\right|^2\cdot\frac{1}{2\cdot\sqrt{\pi\cdot a}}\cdot\sigma^2\cdot\exp\left(-\frac{\omega^2}{4\cdot a}\right). \tag{17}$$

from which, applying the inverse Fourier transform, we can re-obtain Equs.(11) and (15)

In **Figs.7-8,** Equs.(16),(17) are used to plot the stationary RE cross-spectrum $S_{xy}(\omega)$ (Fig.7) and stationary response auto-spectrum $S_{xx}(\omega)$ (Fig.8) for Case I. In contrast to the auto-spectrum, cross-spectrum is a complex function. In **Fig.7** the co-spectrum $G_{xy}(\omega)$ = $\mathrm{Re}\{S_{xy}(\omega)\}$ (Fig.7a) and the (quadrature) quad-spectrum $Q_{xy}(\omega)$ = $\mathrm{Im}\{S_{xy}(\omega)\}$ (Fig.7b) are plotted separately.

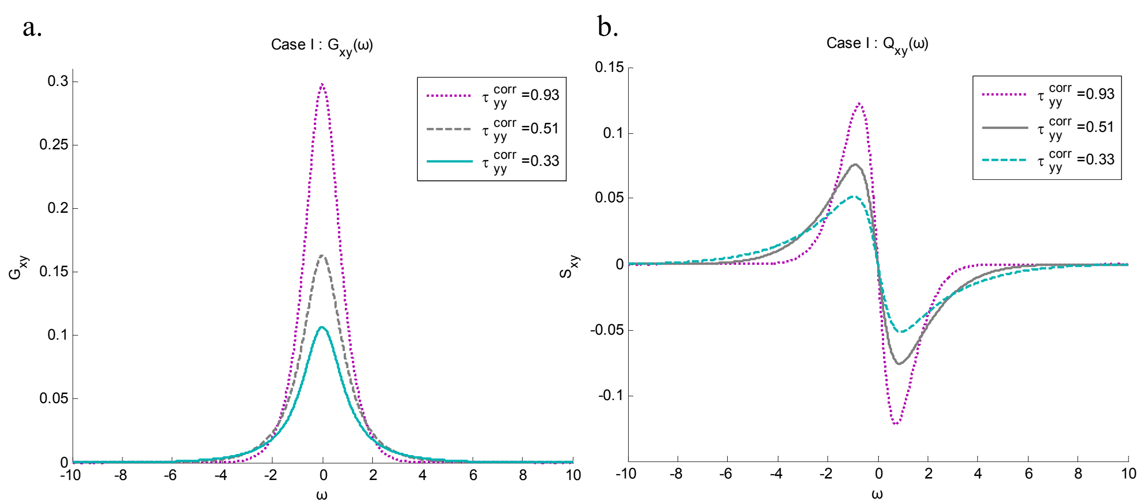


Figure 7: The real and the imaginary part of the RE spectrum $S_{xy}(\omega)$ of Case I in the long time statistical equilibrium state. a. The co-spectrum $G_{xy}(\omega)$ = $\mathrm{Re}\{S_{xy}(\omega)\}$, b. The quad-spectrum $Q_{xy}(\omega)$ = $\mathrm{Im}\{S_{xy}(\omega)\}$

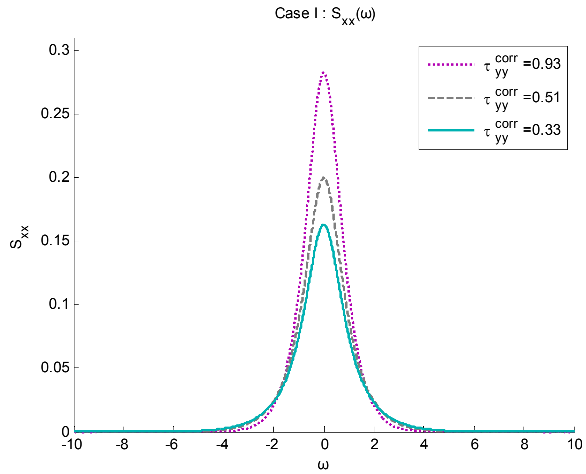


Figure 8: The Response spectrum $S_{xy}(\omega)$ for the lpGF excitation in the long time statistical equilibrium state

3.2.4.b. Ornstein-Uhlenbeck (OU) excitation

We shall now implement the formulae obtained in Sec.(3.2.1) for the two-time RE moments to another case study where the stochastic excitation of the RDE given by Equ.(1)_Sec(3.1.1). is a sifted or centered Ornstein-Uhlenbeck(OU) random function. The formulae that we are going to obtain will be for the more general case that the excitation is a shifted Ornstein-Uhlenbeck process (sOU). The auto-correlation function of the sOU process is given by Equ.(18a) and the spectrum of the sOU by Equ.(18b)

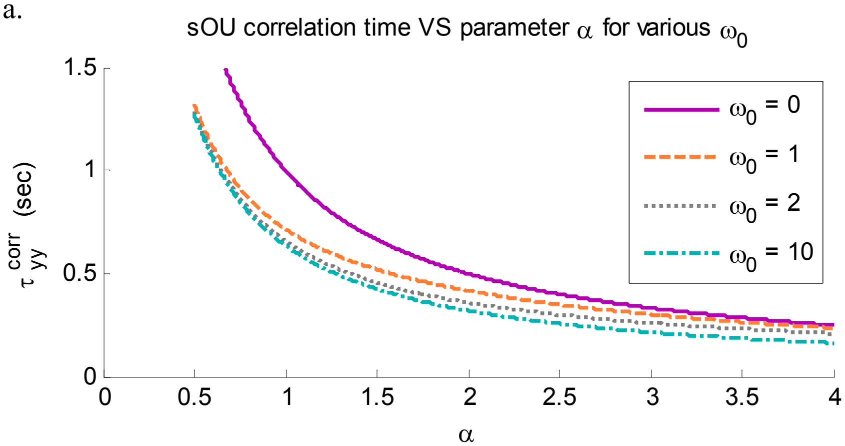
$$R_{yy}(t,s) = \sigma^2 \cdot \exp(-a \cdot |t-s|) \cdot \cos(\omega_0 \cdot (t-s)), \tag{18a}$$

$$S_{yy}(\omega) = \frac{\sigma^2}{2\pi}\left\{\frac{a}{a^2+(\omega_0+\omega)^2} + \frac{a}{a^2+(\omega_0-\omega)^2}\right\}. \tag{18b}$$

Parameters a and ω_0 control the correlation time τ_{yy}^{corr} of the excitation processes, which is given by Equ.(18c):

$$\tau_{yy}^{corr} = \frac{a}{a^2+\omega_0^2} + \frac{e^{-a\pi/(2\omega_0)}}{1-e^{-a\pi/(\omega_0)}} \cdot \frac{2\omega_0}{a^2+\omega_0^2}, \quad \omega_0 > 0. \tag{18c}$$

a.



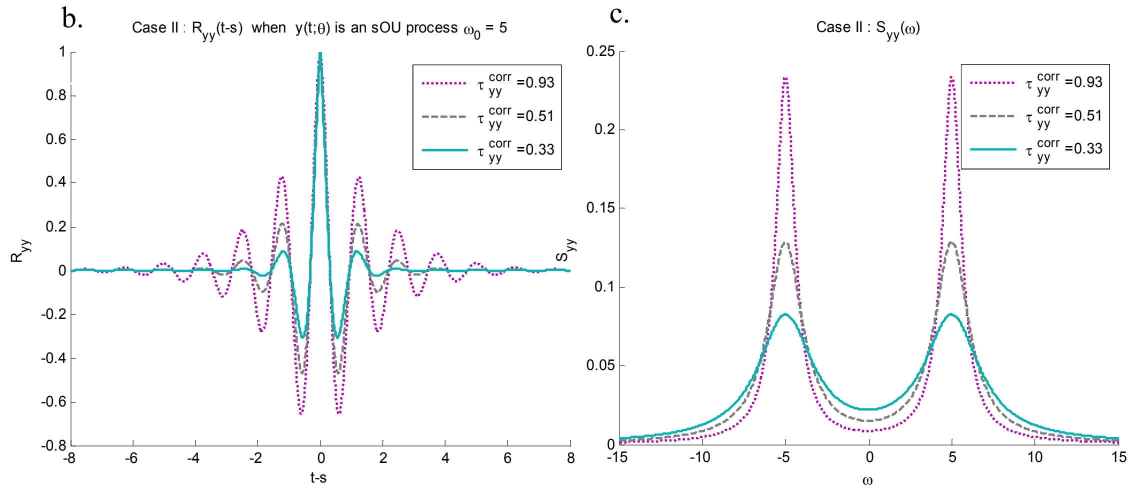


Figure 9: a. The correlation time τ_{yy}^{corr} of the sOU stochastic excitation against parameter a and for various values of the central spectral frequency ω_0. b. The sOU input correlation function $R_{yy}(t,s)$ and c. the sOU spectrum $S_{yy}(\omega)$ for the study Case II

In fact, as we can see in **Fig.9** τ_{yy}^{corr} decreases with a and increases with ω_0. The limiting values of τ_{yy}^{corr} with respect to a, ω_0 are $\lim_{a\to 0}\tau_{yy}^{\text{corr}} = \infty$, $\lim_{a\to\infty}\tau_{yy}^{\text{corr}} = 0$ and $\lim_{\omega_0\to\infty}\tau_{yy}^{\text{corr}} = 0$.

When the central frequency ω_0 of the sOU process (Equ.(18a)) is equal to zero, we get the centered Ornstein-Uhlenbeck (cOU) process. This is generally known as the Ornstein-Uhlenbeck (OU) process, and that is how it is going to be referred hereafter. More precisely, the OU auto-correlation function is given by Equ.(19a), the OU spectrum by Equ.(19b) and the OU correlation time by Equ.(19c):

$$R_{yy}(t,s) = \sigma^2 \cdot \exp\left(-a\cdot|t-s|\right), \qquad S_{yy}(\omega) = \frac{\sigma^2}{\pi}\cdot\frac{a}{a^2+1}, \qquad \tau_{yy}^{\text{corr}} = 1/a.$$

(19a,b,c)

The OU excitation is a very well-studied stochastic process, being also the solution of the Itô SDE.

$$\dot{y}(t;\theta) = -a\, y(t;\theta) + \xi(t;\theta), \tag{20}$$

where $\xi(t;\theta)$ is a Gaussian with noise, with $\sigma_\xi^2 = 2a\,\sigma_y^2$. Following the filtering approach ((Benfratello & Muscolino 1999),(Francescutto & Naito 2004; Weiss & van de Beld 2007)(Di Paola & Floris 2008)) it is possible to consider Equ.(20) along with Equ.(1)_Sec.(3.1.1) as a system of two Itô Stochastic Differential Equations (SDEs) for the stochastic process $x(t;\theta)$ and $y(t;\theta)$, and derive moment equations from these (Soong & Grigoriu 1993). Hereafter, this approach will be referred to as the **Itô/filtering approach**.

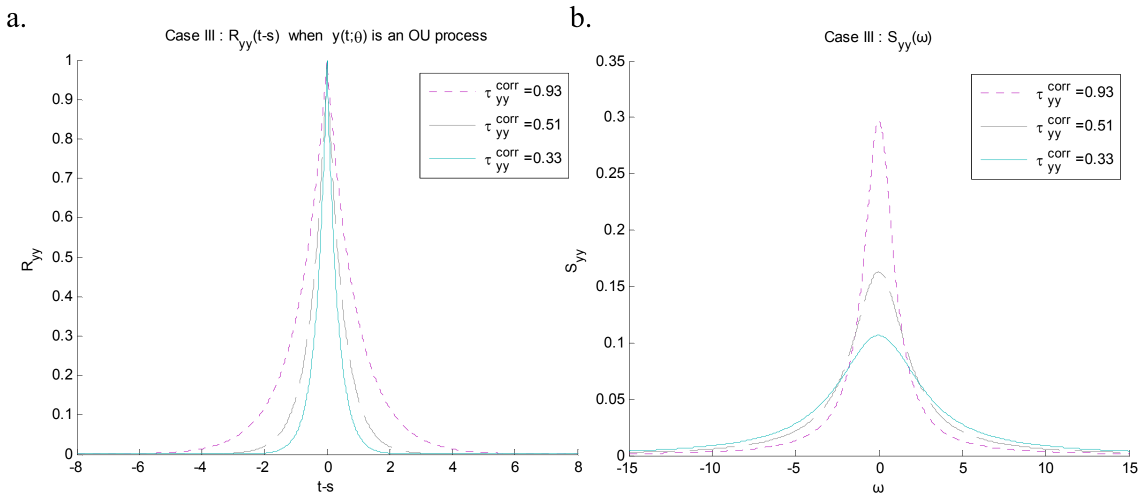


Figure 10:a.The OU correlation function $R_{yy}(t,s)$ and the OU spectrum $S_{yy}(\omega)$ (b.) for Case III.

We shall now proceed to the implementation of the formulae obtained in Section 3.2.1. Results will be illustrated for two cases. In **Case II**, all parameters are same as in case I except from the stochastic input that is a sOU process, with $a = 0.68, 1.26, 2$, $\omega = 5$, $\tau_{yy}^{\text{corr}} = 0.93$, 0.51, 0.33 and all other parameters are as in case I. In **Case III,** the stochastic input is an OU process with $a = 1.07, 1.95, 3$ and correlation time $\tau_{yy}^{\text{corr}} = 0.93, 0.51, 0.33$. All other paramet-ers are as in Case I. The values of parameter a have been chosen to be such that Case I, Case II and Case III have the same correlation time, so as to be able to compare cases. The auto-correlation function $R_{yy}(t,s)$ and the spectrum $S_{yy}(\omega)$ for Cases II and III are plotted in **Fig.9**, **Fig.10** respectively.

Considering that the two-time input auto-correlation is given by Equ.(19a), from Equ.(3)_ Sec(3.2.2) we have that the **transient two-time RE cross-correlation function** $R_{xy}(t,s)$ **for the sOU excitation** is given by the formula:

$$R_{xy}(t,s) = e^{A\cdot t}\cdot B\cdot\sigma^2\cdot\int_{t_0}^{t}\exp\left(-a\cdot\left|t_1 - s\right|\right)\cdot\cos\left(\omega_0\cdot\left(t_1 - s\right)\right)\cdot e^{-A\cdot t_1}dt_1 + e^{A\cdot(t-t_0)}\cdot m_{x_0}\cdot m_y. \tag{21}$$

Due to the presence of the absolute value of the time lag in Equ.(21), two separate cases have to be considered, i.e.: $t \geq s$, the response follows the excitation and $t < s$, the response is in advance of the excitation. In the first case ($t \geq s$), we obtain:

$$R_{xy}(t,s)\Big|_{t\geq s} = e^{A\cdot(t-s)}\cdot B\cdot\sigma^2\cdot\left(\int_{t_0}^{s}e^{(a-A)\cdot(t_1-s)}\cdot\cos\left(\omega_0\cdot\left(t_1-s\right)\right)dt_1 + \right.$$

$$\left. + \int_{s}^{t}e^{-(a+A)\cdot(t_1-s)}\cdot\cos\left(\omega_0\cdot\left(t_1-s\right)\right)dt_1\right) + e^{A\cdot(t-t_0)}\cdot m_{x_0}\cdot m_y\cdot \tag{22}$$

Applying the integration formula given by Equ.(1)_App.(3) to Equ.(22), we have:

$$R_{xy}(t,s)\Big|_{t\geq s} = B\cdot\sigma^2\cdot\left(\frac{a-A}{(a-A)^2+\omega_0^2}+\frac{A+a}{(A+a)^2+\omega_0^2}\right)\cdot e^{A\cdot(t-s)}+ \\ -\frac{e^{A\cdot(t-t_0)-a\cdot(s-t_0)}}{(a-A)^2+\omega_0^2}\cdot\left[(a-A)\cos(\omega_0\cdot(t_0-s))+\omega_0\cdot\sin(\omega_0\cdot(t_0-s))\right]+ \\ +\frac{e^{-a\cdot(t-s)}}{(A+a)^2+\omega_0^2}\cdot\left[-(A+a)\cdot\cos(\omega_0\cdot(t-s))+\omega_0\cdot\sin(\omega\cdot(t-s))\right]+ \\ +e^{A\cdot(t-t_0)}\cdot m_{x_0}\cdot m_y. \tag{23}$$

In the second case ($t < s$), Equ.(21) takes the form:

$$R_{xy}(t,s)\Big|_{t<s} = B\cdot\sigma^2\cdot\left[e^{A\cdot(t-s)}\cdot\int_{t_0}^{t} e^{(a-A)\cdot(t_1-s)}\cdot\cos(\omega_0\cdot(t_1-s))\,dt_1\right]+e^{A\cdot(t-t_0)}\cdot m_{x_0}\cdot m_y. \tag{24a}$$

Applying Equ.(1)_App(3) to Equ.(24a), we obtain:

$$R_{xy}(t,s)\Big|_{t<s} = B\cdot\sigma^2\cdot\left[\left[\frac{e^{a\cdot(t-s)}}{(a-A)^2+\omega^2}\cdot\left((a-A)\cdot\cos(\omega\cdot(t-s))+\omega\cdot\sin(\omega\cdot(t-s))\right)- \right.\right. \\ \left.\left. -\frac{e^{A(t-t_0)-a(s-t_0)}}{(a-A)^2+\omega^2}\cdot\left((a-A)\cdot\cos(\omega\cdot(t_0-s))+\omega\cdot\sin(\omega\cdot(t_0-s))\right)\right]\right]+ \\ +e^{A\cdot(t-t_0)}\cdot m_{x_0}\cdot m_y. \tag{24b}$$

Comparing Equ.(23) with Equ.(24b), we notice that the terms that depend on the initial conditions are the same. Hereafter, we shall denote these common terms with $TR_{xy}(t,s,t_0,m_{x_0})$, whereas those that are independent from the initial conditions will be denoted by $SR_{xy}(t,s)\Big|_{t\geq s}$ and $SR_{xy}(t,s)\Big|_{t<s}$ in $R_{xy}(t,s)\Big|_{t\geq s}$ and $R_{xy}(t,s)\Big|_{t<s}$, respectively. Then, Equs. (23,24b) can be rewritten in the more compact form that will simplify the calculations for the two-time response auto-correlation function $R_{xx}(t,s)$, i.e.:

$$R_{xy}(t,s)\Big|_{t<s} = SR_{xy}(t,s)\Big|_{t<s} + TR_{xy}\left(t,s,t_0,m_{x_0}\right), \tag{25a}$$

$$R_{xy}(t,s)\Big|_{t\geq s} = SR_{xy}(t,s)\Big|_{t\geq s} + TR_{xy}\left(t,s,t_0,m_{x_0}\right), \tag{25b}$$

where:

$$TR_{xy}\left(t,s,t_0,m_{x_0}\right) = -B\cdot\sigma^2\cdot\frac{e^{A\cdot(t-t_0)-a\cdot(s-t_0)}}{(a-A)^2+\omega_0^2}\cdot\left[(a-A)\cos\left(\omega_0\cdot(t_0-s)\right)+\omega_0\cdot\sin\left(\omega_0\cdot(t_0-s)\right)\right]+ + e^{A\cdot(t-t_0)}\cdot m_{x_0}\cdot m_y, \tag{26a}$$

$$SR_{xy}(t,s)\Big|_{t<s} = B\cdot\sigma^2\cdot\frac{e^{a\cdot(t-s)}}{(a-A)^2+\omega^2}\cdot\left((a-A)\cdot\cos\left(\omega\cdot(t-s)\right)+\omega\cdot\sin\left(\omega\cdot(t-s)\right)\right), \tag{26b}$$

$$SR_{xy}(t,s)\Big|_{t\geq s} = B\cdot\sigma^2\cdot\left[\frac{2\cdot a\cdot\left((A+a)\cdot(a-A)+{\omega_0}^2\right)}{\left((a-A)^2+(\omega_0)^2\right)\cdot\left((A+a)^2+(\omega_0)^2\right)}\cdot e^{A\cdot(t-s)} + \right. \left. + \frac{e^{-a\cdot(t-s)}}{(A+a)^2+(\omega_0)^2}\cdot\left[-(A+a)\cdot\cos\left(\omega_0\cdot(t-s)\right)+\omega_0\cdot\sin\left(\omega_0\cdot(t-s)\right)\right]\right]. \tag{26c}$$

The obtained results given by Equs.(25,26) are illustrated in **Figs.11-14**. The description and general remarks made in Figs. 2-3 for Case I apply also in Figs.11-12 for Case II and Figs. 13-14 for Case III. Moreover, in Figs.11-12 that the one-time and two-time RE cross-correlation functions are plotted for the sOU random input (Case II) that the excitation has a non-zero central frequency, $\omega_0 = 5$, we notice that the frequency of both the one- and two-time RE cross-correlation functions increase with the correlation time of the excitation. In Figs.13-14 results for the case that the excitation is an OU are plotted. These are given by Equs.(25-26) for $\omega_0 = 0$. The obtained results are indicated with diamond marker (♦), in order to distinguish these from the case that the excitation is a lpGF random function (Case I) (are also illustrated in Figs.13-14) for comparison reasons. The most important finding here is that the two-time RE moments are significantly affected by the shape of the input function (lpGF vs OU, see Figs.(1,10)), especially for the more correlated case $\tau_{yy}^{\text{corr}} = 0.93$, despite the fact that all other parameters (σ_y^2, τ_{yy}^{corr}) are the same. The response auto-correlation obtained under lpGF random input is always higher than the response auto-correlation obtained under OU input. Around $t-s=0$ the difference is as high as 15%. Moreover, in all cases the response stays correlated with the input for more time, when the stochastic input is an OU(Case III in figures) process than when it is a lpGF(Case I in figures).

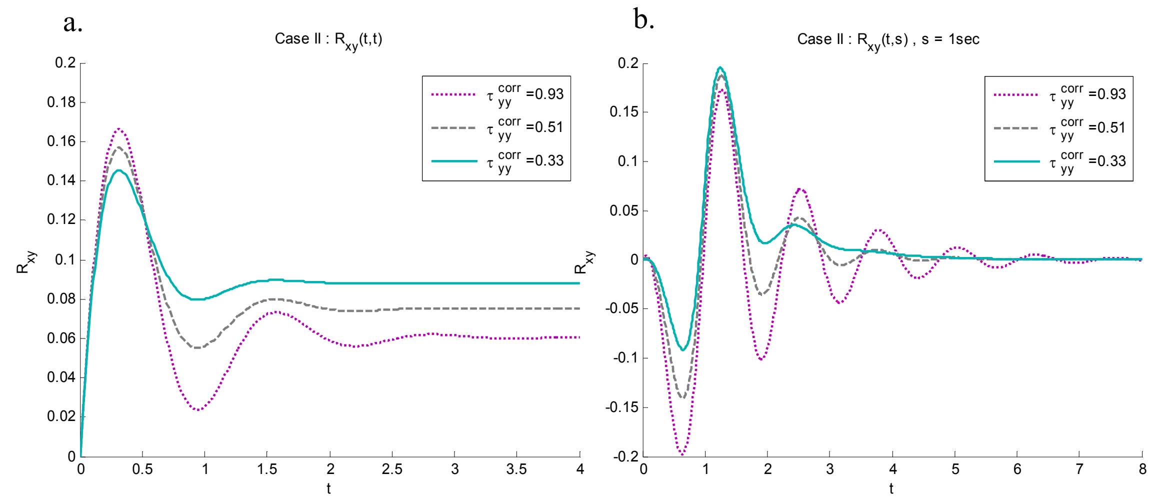


Figure 11: a. The transient diagonal (one-time) RE cross-correlation function of Case II. b. The transient (two-times) RE correlation function of Case II against the response time t, for excitation time $s = 1$ sec.

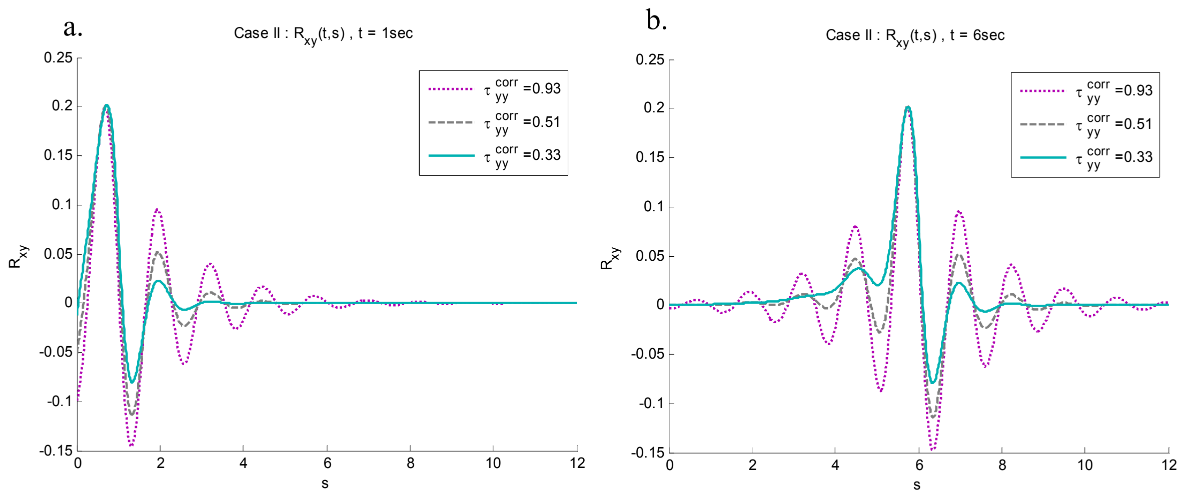


Figure 12: a. The transient two-time RE cross-correlation function of Case II against the excitation time s, for excitation time $t = 1$ sec. b. The same for excitation time $t = 6$ sec

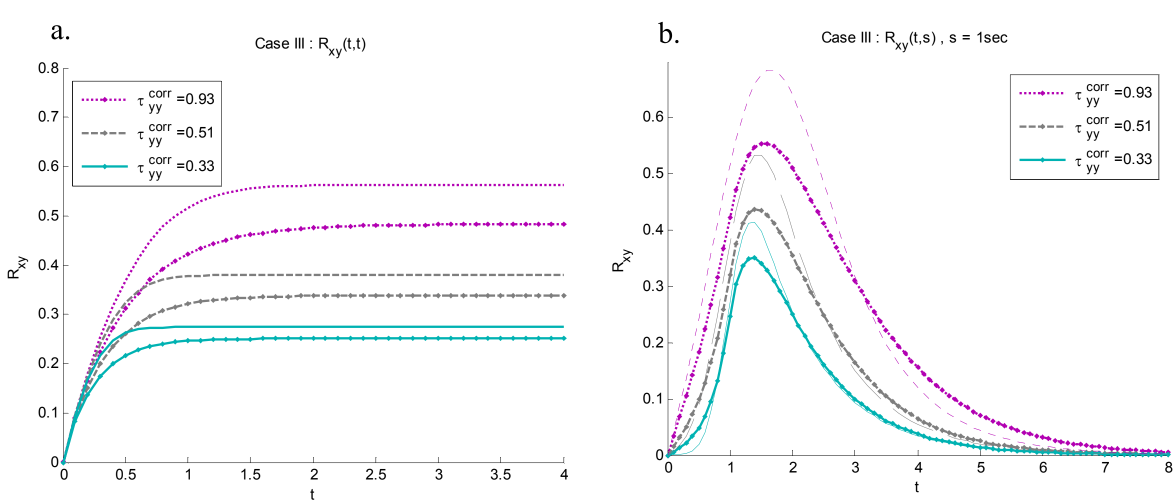


Figure 13: The same as in Fig.11 for Case III (lines with diamond markers). Same results for Case I are also depicted (lines with no markers)

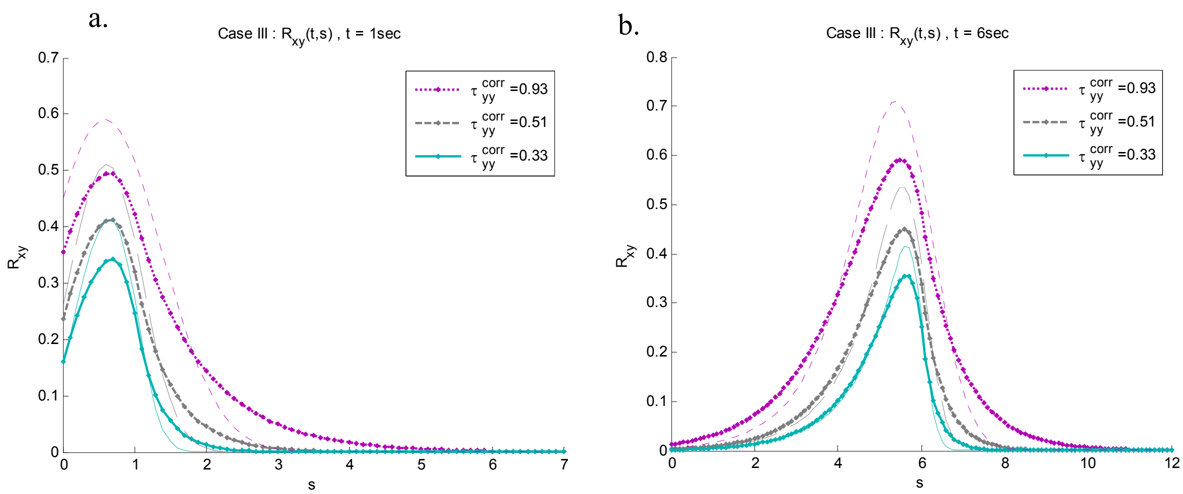


Figure 14: The same as in Fig.12 for Case III (lines with diamond markers). Same results for Case I are also depicted (lines with no markers)

Proceeding now to the calculation of the **transient two-time response auto-correlation** $R_{xx}(t,s)$, for the sOU excitation using Equ.(4)_Sec(3.2.3), we have:

$$R_{xx}(t,s) = e^{A\cdot t}\cdot B\cdot \int_{t_0}^{t} R_{xy}(s,t_1) e^{-A\cdot t_1} dt_1 + \\ -\frac{B}{\mathrm{A}}\cdot m_{x_0}\cdot m_y \cdot e^{A\cdot(t+s-t_0)}\cdot\left(e^{-A\cdot s} - e^{-A\cdot t_0}\right) + e^{A\cdot(t+s-2\cdot t_0)}\cdot R_{x_0 x_0} \quad . \tag{27}$$

Since $R_{xx}(t,s)$ is a function of $R_{xy}(s,t)$ we shall calculate separately $R_{xx}(t,s)$ for $t<s$ and $t \geq s$. More precisely, from Equs.(25,27) we obtain:

$$R_{xx}(t,s)\Big|_{t<s} = e^{A\cdot t}\cdot B\cdot \int_{t_0}^{t} SR_{xy}(s,t_1)\Big|_{s\geq t_1}\cdot e^{-A\cdot t_1} dt_1 + e^{A\cdot t}\cdot B\cdot \int_{t_0}^{t} TR_{xy}\left(s,t_1,t_0,m_{x_0}\right)\cdot e^{-A\cdot t_1} dt_1 \\ -\frac{B}{\mathrm{A}}\cdot m_{x_0}\cdot m_y \cdot e^{A\cdot(t+s-t_0)}\cdot\left(e^{-A\cdot s} - e^{-A\cdot t_0}\right) + e^{A\cdot(t+s-2\cdot t_0)}\cdot R_{x_0 x_0} \quad , \tag{28a}$$

$$R_{xx}(t,s)\Big|_{t\geq s} = B\cdot e^{A\cdot t}\cdot \int_{t_0}^{s} SR_{xy}(s,t_1)\Big|_{s\geq t_1}\cdot e^{-A\cdot t_1} dt_1 + B\cdot e^{A\cdot t}\cdot \int_{s}^{t} SR_{xy}(s,t_1)\Big|_{s<t_1}\cdot e^{-A\cdot t_1} dt_1 + \\ + B\cdot e^{A\cdot t}\cdot \int_{t_0}^{t} TR_{xy}\left(s,t_1,t_0,m_{x_0}\right)\cdot e^{-A\cdot t_1} dt_1 \; - \\ -\frac{B}{\mathrm{A}}\cdot m_{x_0}\cdot m_y \cdot e^{A\cdot(t+s-t_0)}\cdot\left(e^{-A\cdot s} - e^{-A\cdot t_0}\right) + e^{A\cdot(t+s-2\cdot t_0)}\cdot R_{x_0 x_0}, \tag{28b}$$

where:

$$SR_{xy}\left(s,t_1\right)\Big|_{s\geq t_1} = B\cdot\sigma^2\cdot\left[\frac{2\cdot a\cdot\left((a^2-A^2)+\omega^2\right)}{\left((a-A)^2+\omega^2\right)\cdot\left((A+a)^2+\omega^2\right)}\cdot e^{A\cdot(s-t_1)}+\right.$$
$$\left.+\frac{e^{-a\cdot(s-t_1)}}{(A+a)^2+\omega_0^2}\cdot\left(-(A+a)\cdot\cos\left(\omega_0\cdot(s-t_1)\right)+\omega_0\cdot\sin\left(\omega_0\cdot(s-t_1)\right)\right)\right], \quad (29a)$$

$$SR_{xy}\left(s,t_1\right)\Big|_{s<t_1} = B\cdot\sigma^2\cdot\frac{e^{a\cdot(s-t_1)}}{(a-A)^2+\omega^2}\cdot\left((a-A)\cdot\cos\left(\omega\cdot(s-t_1)\right)+\omega\cdot\sin\left(\omega\cdot(s-t_1)\right)\right), \quad (29b)$$

$$TR_{xy}\left(s,t_1,t_0,m_{x_0}\right) = B\cdot\sigma^2\cdot\left[-\frac{e^{A\cdot(s-t_0)-a\cdot(t_1-t_0)}}{(a-A)^2+\omega_0^2}\cdot\left((a-A)\cos\left(\omega_0\cdot(t_0-t_1)\right)+\right.\right. \quad (29c)$$
$$\left.\left.+\omega_0\cdot\sin\left(\omega_0\cdot(t_0-t_1)\right)\right)+e^{A\cdot(s-t_0)}\cdot m_{x_0}\cdot m_y\right].$$

All integrals appearing in Equs.(28a,b) are computed in Appendix 3. Combining Equs.(28a) and Equs.(5,7)_App(3), we obtain:

$$R_{xx}(t,s)\Big|_{t<s} = \frac{B^2\cdot\sigma^2}{\left((a-A)^2+\omega^2\right)\cdot\left((A+a)^2+\omega^2\right)}\times\left[-\left(e^{A\cdot(s-t)}-e^{A\cdot(t+s)-2A\cdot t_0}\right)\cdot\frac{a\cdot\left((a^2-A^2)+\omega^2\right)}{A}+\right.$$
$$+e^{a\cdot(t-s)}\cdot\left[\left(-(a^2-A^2)+(\omega_0)^2\right)\cdot\cos\left(\omega_0\cdot(t-s)\right)-2\cdot\omega_0\cdot a\cdot\sin\left(\omega_0\cdot(t-s)\right)\right]+$$
$$+e^{A\cdot t-a\cdot s+(a-A)\cdot t_0}\cdot\left[\left((a^2-A^2)-(\omega_0)^2\right)\cdot\cos\left(\omega_0\cdot(s-t_0)\right)-2\cdot\omega_0\cdot a\cdot\sin\left(\omega_0\cdot(s-t_0)\right)\right]+ \quad (30a)$$
$$+e^{A\cdot s-a\cdot t+(a-A)\cdot t_0}\cdot\left[\left((a^2-A^2)-(\omega_0)^2\right)\cdot\cos\left(\omega_0(t-t_0)\right)-2\cdot\omega_0\cdot a\cdot\sin\left(\omega_0(t-t_0)\right)\right]-$$
$$\left.-e^{A\cdot(t+s-2\cdot t_0)}\cdot\left((a^2-A^2)-(\omega_0)^2\right)\right]-\frac{B}{A}\cdot e^{A\cdot(t+s-t_0)}\cdot m_{x_0}\cdot m_y\cdot\left(e^{-A\cdot t}+e^{-A\cdot s}-2\cdot e^{-A\cdot t_0}\right)+e^{A\cdot(t+s-2\cdot t_0)}\cdot R_{x_0x_0}.$$

Combining Equ.(28b) and Equs.(7,8,10)_App.(3) we get:

$$R_{xx}(t,s)\Big|_{t\geq s} = \frac{B^2\cdot\sigma^2}{\left((a-A)^2+\omega^2\right)\cdot\left((A+a)^2+\omega^2\right)}\times\left[-\left(e^{A\cdot(t-s)}-e^{A\cdot(t+s)-2A\cdot t_0}\right)\frac{a\cdot\left((a^2-A^2)+\omega^2\right)}{A}+\right.$$
$$+e^{-a\cdot(t-s)}\cdot\left[(-(a^2-A^2)+(\omega_0)^2)\cdot\cos\left(\omega_0\cdot(t-s)\right)-2\cdot\omega_0\cdot a\cdot\sin\left(\omega\cdot(s-t)\right)\right]$$
$$+e^{A\cdot t-a\cdot s+(a-A)\cdot t_0}\cdot\left[\left((a^2-A^2)-(\omega_0)^2\right)\cdot\cos\left(\omega_0\cdot(t_0-s)\right)-2\cdot\omega_0\cdot a\cdot\sin\left(\omega_0\cdot(s-t_0)\right)\right]+ \quad (30b)$$
$$+e^{A\cdot s-a\cdot t+(a-A)\cdot t_0}\cdot\left[\left((a^2-A^2)-(\omega_0)^2\right)\cdot\cos\left(\omega_0(t-t_0)\right)-2\cdot\omega_0\cdot a\cdot\sin\left(\omega_0(t-t_0)\right)\right]-$$
$$\left.-e^{A\cdot(t+s-2\cdot t_0)}\cdot((a^2-A^2)-(\omega_0)^2)\right]-\frac{B}{A}\cdot e^{A\cdot(t+s-t_0)}\cdot m_{x_0}\cdot m_y\cdot(e^{-A\cdot t}+e^{-A\cdot s}-2\cdot e^{-A\cdot t_0})+e^{A\cdot(t+s-2\cdot t_0)}\cdot R_{x_0x_0}.$$

Comparing Equs.(30a), (30b) we see that as expected $R_{xx}(t,s)\Big|_{t<s} = R_{xx}(s,t)\Big|_{s\geq t}$.

Moreover, for the response auto-correlation $R_{xx}(t,t)$ on the diagonal, i.e. for $s \to t$, we have that $R_{xx}(t,t) = R_{xx}(t,t)\big|_{t \geq s} = R_{xx}(t,t)\big|_{t \leq s}$, that is:

$$
\begin{aligned}
R_{xx}(t,t) = {} & \frac{B^2 \cdot \sigma^2}{\left((a-A)^2 + \omega^2\right)\cdot\left((A+a)^2 + \omega^2\right)} \times \\
& \times \Bigg[-\left(1 - e^{2A\cdot(t-t_0)}\right) \frac{a \cdot \left((a^2 - A^2) + \omega^2\right)}{A} + \left((-(a^2 - A^2) + (\omega_0)^2\right) + \\
& + 2 \cdot e^{-(a-A)\cdot(t-t_0)} \cdot \left(\left((a^2 - A^2) - (\omega_0)^2\right)\cdot \cos\left(\omega_0 \cdot (t-t_0)\right) - 2\omega_0 a \cdot \sin\left(\omega_0 \cdot (t-t_0)\right)\right) - \\
& - e^{\,2\cdot A(t-t_0)} \cdot \left((a^2 - A^2) - (\omega_0)^2\right)\Bigg] - \frac{B}{A} \cdot m_{x_0} \cdot m_y \cdot e^{A(2t-t_0)} \cdot \left(2 \cdot e^{-A t} - 2 \cdot e^{-A t_0}\right) + e^{2\cdot A(t-t_0)} \cdot R_{x_0 x_0}.
\end{aligned}
\tag{31}
$$

The one- and two-time response auto-correlation functions, $R_{xx}(t,t)$ and $R_{xx}(t,s)$, respectively, of Case II as calculated by the use of Equs.(30,31) are illustrated in Fig.(15). In Fig.(16), results for the Case study III are illustrated together with the results from case I, for comparison reasons.

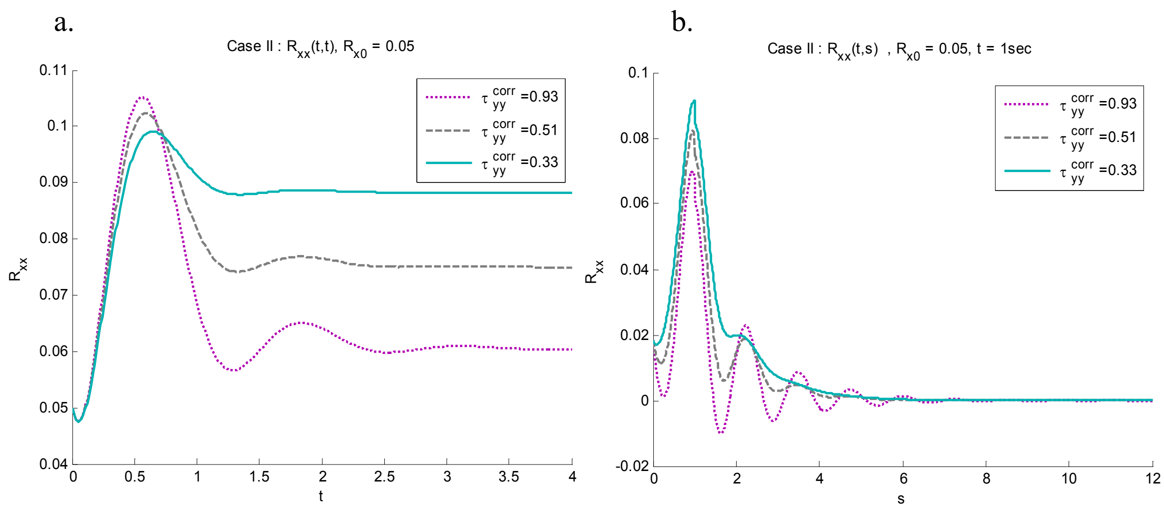


Figure 15: a. The transient diagonal (one-time) response correlation function of Case II. b. The transient (two-time) RE cross-correlation function of Case II against the response time s, for fixed excitation time $t = 1\,\text{sec}$. The initial value of the response correlation function $R_{x_0 x_0} = 0.05$,

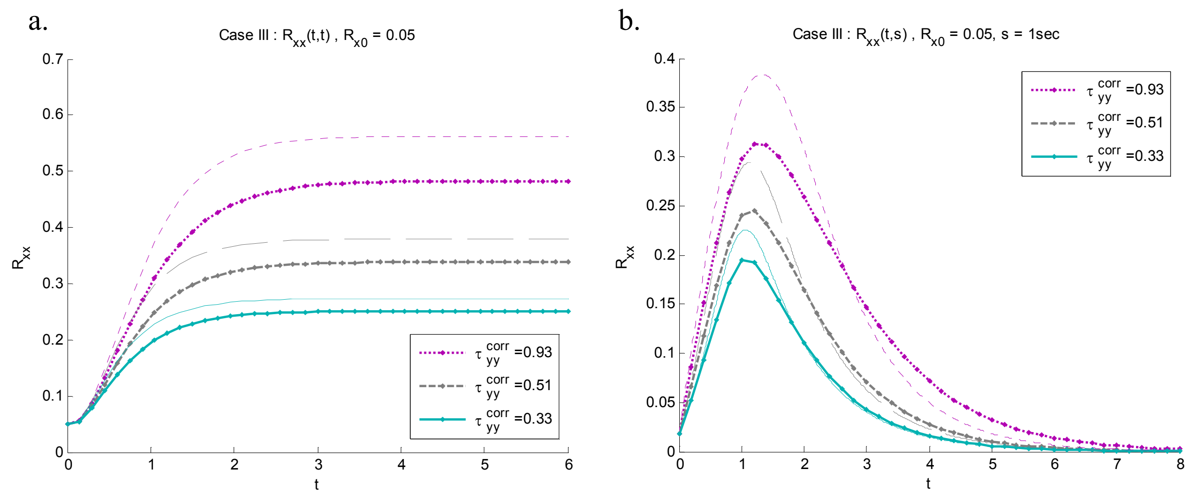


Figure 16: The same as in Fig.15 for Case III (lines with diamond markers). Same results for Case I are also depicted (lines with no markers)

Having found the transient two-time moments for the random initial value problem given by Equ.(1)_Sec(3.1.1) for a sOU random input, it is straightforward to find the long-time limits of the two-time response-excitation cross-correlation function $R_{xy}^{(\infty)}(t,s)$ and the two-time response auto-correlation function $R_{xx}^{(\infty)}(t,s)$. In fact, considering the limits $s,t \to \infty$ and for finite time lag $|t-s| = |\tau| < \infty$ in Equs. (25b), (25a), (30b) and (30a), respectively, we have:

$$R_{xy}^{(\infty)}(t,s)\Big|_{t\geq s} = \lim_{\substack{t\to\infty \\ s\to\infty \\ t-s=\tau}} R_{xy}(t,s)\Big|_{t\geq s} = B\cdot\sigma^2\cdot\Bigg[\left(\frac{a-A}{(a-A)^2+\omega_0^2}+\frac{A+a}{(A+a)^2+\omega_0^2}\right)\cdot e^{A\cdot(t-s)} + + \frac{e^{-a\cdot(t-s)}}{(A+a)^2+\omega_0^2}\cdot\Big(-(A+a)\cdot\cos\big(\omega_0\cdot(t-s)\big) + \omega_0\cdot\sin\big(\omega\cdot(t-s)\big)\Big)\Bigg], \quad (32)$$

$$R_{xy}^{(\infty)}(t,s)\Big|_{t<s} = \lim_{\substack{t\to\infty \\ s\to\infty \\ t-s=\tau}} R_{xy}(t,s)\Big|_{t<s} = = B\cdot\sigma^2\cdot\frac{e^{a\cdot(t-s)}}{(a-A)^2+\omega^2}\cdot\Big((a-A)\cdot\cos\big(\omega\cdot(t-s)\big) + \omega\cdot\sin\big(\omega\cdot(t-s)\big)\Big). \quad (33)$$

$$R_{xx}^{(\infty)}(t,s)\Big|_{t<s} = \lim_{\substack{t\to\infty \\ s\to\infty \\ t-s=\tau}} R_{xx}(t,s)\Big|_{t<s} = \frac{B^2\cdot\sigma^2}{\big((a-A)^2+\omega^2\big)\cdot\big((A+a)^2+\omega^2\big)}\times \times\Bigg[-e^{A(s-t)}\,\frac{a\cdot\big((a^2-A^2)+\omega^2\big)}{A} - -e^{a\cdot(t-s)}\cdot\Big[\big((a^2-A^2)-(\omega_0)^2\big)\cdot\cos\big(\omega_0\cdot(t-s)\big) + 2\cdot\omega_0\cdot a\cdot\sin\big(\omega_0\cdot(t-s)\big)\Big], \quad (34)$$

$$R_{xx}^{(\infty)}(t,s)\Big|_{t\geq s}=\frac{B^2\cdot\sigma^2}{\left((a-A)^2+\omega^2\right)\cdot\left((A+a)^2+\omega^2\right)}\times$$
$$\times\left[-e^{A\cdot(t-s)}\frac{a\cdot\left((a^2-A^2)+\omega^2\right)}{A}-e^{-a\cdot(t-s)}\cdot\left[(-(a^2-A^2)+(\omega_0)^2)\cdot\cos\left(\omega_0\cdot(t-s)\right)-\right.\right.$$
$$\left.\left.-2\cdot a\cdot\omega_0\cdot\sin\left(\omega\cdot(t-s)\right)\right]\right]. \qquad (35)$$

Combining Equ.(34) and Equ.(35) we have

$$R_{xx}^{(\infty)}(t,s)=\frac{B^2\cdot\sigma^2}{\left((a-A)^2+\omega^2\right)\cdot\left((A+a)^2+\omega^2\right)}\times\left[-e^{A|t-s|}\frac{a\cdot\left((a^2-A^2)+\omega^2\right)}{A}+\right.$$
$$\left.+e^{-a\cdot|t-s|}\cdot\left[(-(a^2-A^2)+(\omega_0)^2)\cdot\cos\left(\omega_0\cdot|t-s|\right)+2\cdot a\cdot\omega_0\cdot\sin\left(\omega\cdot|t-s|\right)\right]\right]. \qquad (36)$$

The long-time RE cross-correlation function $R_{xy}^{(\infty)}(t,s)$ and response auto-correlation function $R_{xx}^{(\infty)}(t,s)$ for Case Study II as computed by Equs.(32,33) and Equ.(35) respectively are plotted in **Fig.17**.

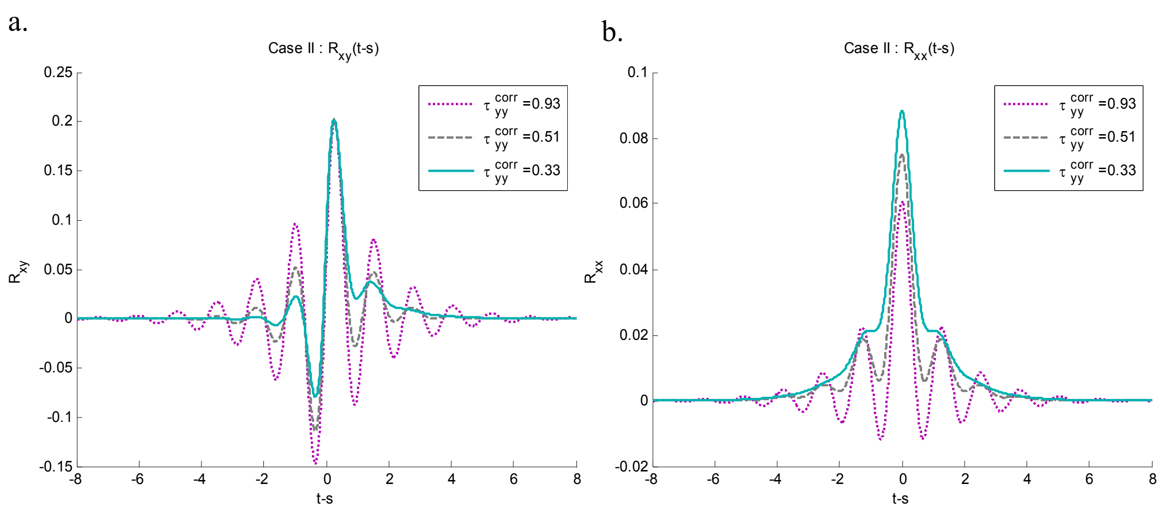


Figure 17: a.The long time RE correlation funcrion $R_{xy}^{(\infty)}(t-s)$, b. The long time response correlation function $R_{xx}^{(\infty)}(t-s)$, of Case II plotted against the time lag $t-s$

Same results for the Case III are illustrated in **Fig.18**. In fact, in **Fig.18** results are also compared with Case I that the excitation is a lpGF process. Differences between the two Cases due to the shape of the stochastic input are significant, reaching a 10% at the peaks of the most correlated case. These finding highlights the importance of methods that treat random ODEs under general random excitation, beyond the limitations of Itô/Filtering approach.

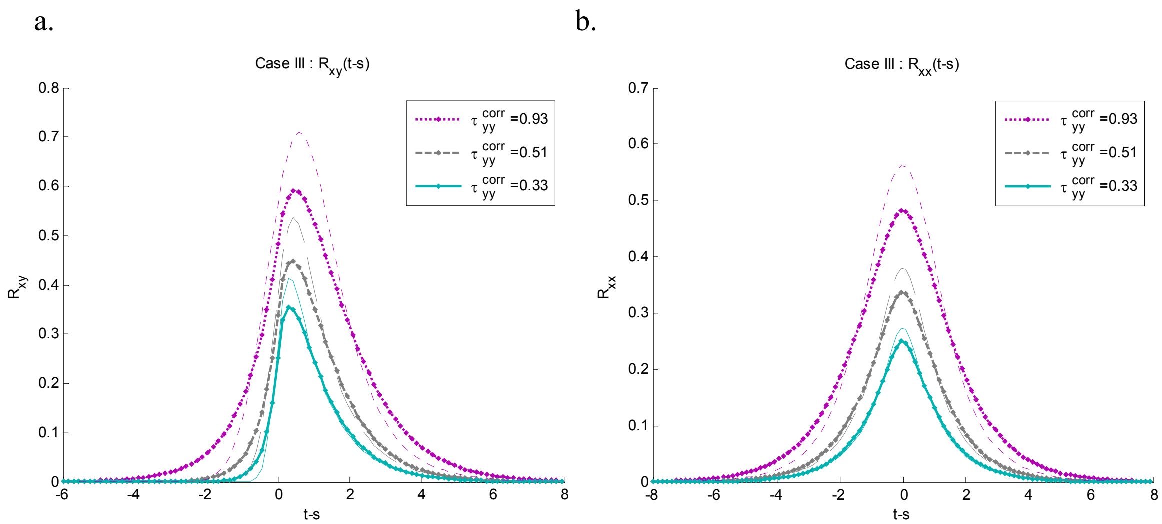


Figure 18: The same as in Fig.17 for Case III (lines with diamond markers). Same results for Case I are also depicted (lines with no markers)

3.3. Analytical solution to the moment problem. The vector case

Let us consider the stochastic system that is given by Equ.(4)_Sec.(3.1.1), we repeat here for convenience:

$$\dot{\mathbf{x}}(t;\theta) = \mathbf{A}\cdot\mathbf{x}(t;\theta)+\mathbf{B}\cdot\mathbf{y}(t;\theta), \quad (1a)$$

$$\mathbf{x}(t_0;\theta)=\mathbf{x_0}(\theta). \quad (1b)$$

where $\mathbf{A}=\left[A_{n_1 n_2}\right]_{\substack{n_1=1,2,\dots,N\\ n_2=1,2,\dots,N}}$ and $\mathbf{B}=\left[B_{nm}\right]_{\substack{n=1,2,\dots,N\\ m=1,2,\dots,M}}$ are deterministic, time invariant matrices, $\mathbf{y}(t;\theta)=\left(y_1(t;\theta),y_2(t;\theta),\dots,y_M(t;\theta)\right)^{\mathrm{T}}$ is a known stochastic excitation, $\mathbf{x}(t;\theta)=\left(x_1(t;\theta),x_2(t;\theta),\dots,x_M(t;\theta)\right)^{\mathrm{T}}$ is the system's response and $\mathbf{x_0}(\theta)=\mathbf{x}(0;\theta)$ is a known stochastic initial condition.

In this section we shall find integral formulae from which we can obtain the first and second order moments of the stochastic system (Equ.(1)), i.e. the unknown mean value of the response $\boldsymbol{m}_{\mathbf{x}}(t)=E^{\theta}\left[\mathbf{x}(t;\theta)\right]$, the two-time RE cross-covariance $\boldsymbol{C}_{\mathbf{xy}}(t,s)=E^{\theta}\left[\left(\mathbf{x}(t;\theta)-\boldsymbol{m}_{\mathbf{x}}(t)\right)\left(\mathbf{y}(s;\theta)-\boldsymbol{m}_{y}(s)\right)^{\mathrm{T}}\right]$, and the two-time response auto-covariance $\boldsymbol{C}_{\mathbf{xx}}(t,s)=E^{\theta}\left[\left(\mathbf{x}(t;\theta)-\boldsymbol{m}_{\mathbf{x}}(s)\right)\left(\mathbf{x}(t;\theta)-\boldsymbol{m}_{\mathbf{x}}(s)\right)^{\mathrm{T}}\right]$. The calculations are made in terms of the known excitation mean value $\boldsymbol{m}_{\mathbf{y}}(t)=E^{\theta}\left[\mathbf{y}(t;\theta)\right]$, the excitation auto-covariance $\boldsymbol{C}_{\mathbf{yy}}(t,s)=E^{\theta}\left[\left(\mathbf{y}(t;\theta)-\boldsymbol{m}_{\mathbf{y}}(t)\right)\left(\mathbf{y}(s;\theta)-\boldsymbol{m}_{\mathbf{y}}(s)\right)^{\mathrm{T}}\right]$, the initial response mean value $\boldsymbol{m}_0=\mathrm{E}^{\theta}\left[\mathbf{x_0}(\theta)\right]$ and the initial response auto-covariance

$\mathbf{C}_{\mathbf{x}_0\mathbf{x}_0} = E^{\theta}\left[\left(\mathbf{x_0}(\theta)-\boldsymbol{m}_0\right)\left(\mathbf{x_0}(\theta)-\boldsymbol{m}_0\right)^{\mathrm{T}}\right]$. In what follows it will be assumed that the initial value $\mathbf{x_0}(\theta)$ is independent from the excitation $\mathbf{y}(t;\theta)$. The realizations of the stochastic function of the response $\mathbf{x}(t;\theta)$ are defined thought the realizations of the excitation by means of the relationship:

$$\mathbf{x}(t;\theta) = \int_{t_0}^{t} \mathbf{\Phi}(t-s_1)\cdot\mathbf{B}\cdot\mathbf{y}(s_1;\theta)\,ds_1 \;+\; \mathbf{\Phi}(t)\,\mathbf{x_0}(\theta). \tag{2}$$

Taking mean values in Equ. (2) we obtain:

$$E^{\theta}\left[\mathbf{x}(t;\theta)\right] = \int_{t_0}^{t} \mathbf{\Phi}(t-s_1)\cdot\mathbf{B}\cdot E^{\theta}\left[\mathbf{y}(s_1;\theta)\right]ds_1 \;+\; \mathbf{\Phi}(t)\,E^{\theta}\left[\mathbf{x_0}(\theta)\right]. \tag{3}$$

That is the **mean value of the response** is given by the integral formula:

$$\boldsymbol{m}_{\mathbf{x}}(t) = \int_{t_0}^{t} \mathbf{\Phi}(t-s_1)\cdot\mathbf{B}\cdot\boldsymbol{m}_{\mathbf{y}}(s_1)\,ds_1 \;+\; \mathbf{\Phi}(t)\,\boldsymbol{m}_0. \tag{4}$$

Simarly, to find an integral formula for the **two-time RE cross covariance** $\boldsymbol{C}_{\mathbf{xy}}(t,s)$ we subtract Equ.(4) from Equ.(2).

$$\mathbf{x}(t;\theta) - \boldsymbol{m}_{\mathbf{x}}(t) = \int_{t_0}^{t} \mathbf{\Phi}(t-s_1)\cdot\mathbf{B}\cdot\left(\mathbf{y}(s_1;\theta)-\boldsymbol{m}_{\mathbf{y}}(s_1)\right)ds_1 \;+\; \mathbf{\Phi}(t)\left(\mathbf{x_0}(\theta)-\boldsymbol{m}_0\right), \tag{5}$$

then multiply Equ. (5) with $\left(\mathbf{y}(s;\theta)-\boldsymbol{m}_{\mathbf{y}}(s)\right)^{\mathrm{T}}$, where $s \geq t_0$, and apply the mean value operator :

$$\begin{aligned} &E^{\theta}\left[\left(\mathbf{x}(t;\theta)-\boldsymbol{m}_{\mathbf{x}}(t)\right)\left(\mathbf{y}(s;\theta)-\boldsymbol{m}_{\mathbf{y}}(s)\right)^{\mathrm{T}}\right] = \\ &\quad = \int_{t_0}^{t} \mathbf{\Phi}(t-s_1)\cdot\mathbf{B}\cdot E^{\theta}\left[\left(\mathbf{y}(s_1;\theta)-\boldsymbol{m}_{\mathbf{y}}(s_1)\right)\left(\mathbf{y}(s;\theta)-\boldsymbol{m}_{\mathbf{y}}(s)\right)^{\mathrm{T}}\right]ds_1 \;+ \\ &\qquad\qquad +\; \mathbf{\Phi}(t)\,E^{\theta}\left[\left(\mathbf{x_0}(\theta)-\boldsymbol{m}_0\right)\right]E^{\theta}\left[\left(\mathbf{y}(s;\theta)-\boldsymbol{m}_{\mathbf{y}}(s)\right)^{\mathrm{T}}\right] \end{aligned} \tag{6}$$

That is:

$$\boldsymbol{C}_{\mathbf{xy}}(t,s) = \int_{t_0}^{t} \mathbf{\Phi}(t-s_1)\cdot\mathbf{B}\cdot\boldsymbol{C}_{\mathbf{yy}}(s_1,s)\,ds_1 \tag{7}$$

To find an integration formula that provides the **two time-response auto- covariance** $\boldsymbol{C}_{\mathbf{xx}}(t,s)$, we multiply Equ. (5) with $\left(\mathbf{x}(s;\theta)-\boldsymbol{m}_{\mathbf{x}}(s)\right)^{\mathrm{T}}$, then apply the mean value operator, i.e. :

$$E^{\theta}\left[\left(\mathbf{x}(t;\theta)-\boldsymbol{m}_{\mathbf{x}}(t)\right)\left(\mathbf{x}(s;\theta)-\boldsymbol{m}_{\mathbf{x}}(s)\right)^{\mathrm{T}}\right] =$$

$$= \int_{t_0}^{t}\int_{t_0}^{s} \mathbf{\Phi}(t-s_1)\cdot\mathbf{B}\cdot \mathrm{E}^{\theta}\left[\left(\mathbf{y}(s_1;\theta)-\boldsymbol{m}_{\mathbf{y}}(s_1)\right)\left(\mathbf{y}(s_2;\theta)-\boldsymbol{m}_{\mathbf{y}}(s_2)\right)^{\mathrm{T}}\right]\cdot\mathbf{B}^{\mathrm{T}}\cdot\mathbf{\Phi}(t-s_2)^{\mathrm{T}}\,ds_1 ds_2 \;+$$

$$+\;\mathbf{\Phi}(t)\,\mathrm{E}^{\theta}\left[\left(\mathbf{x_0}(\theta)-\boldsymbol{m}_0\right)\right]\int_{t_0}^{s}\mathrm{E}^{\theta}\left[\mathbf{y}(s_2;\theta)-\boldsymbol{m}_{\mathbf{y}}(s_2)\right]^{\mathrm{T}}\cdot\mathbf{B}^{\mathrm{T}}\cdot\mathbf{\Phi}(t-s_2)^{\mathrm{T}}\,ds_2 +$$

$$+\int_{t_0}^{t}\mathbf{\Phi}(t-s_1)\cdot\mathbf{B}\cdot\mathrm{E}^{\theta}\left[\mathbf{y}(s_1;\theta)-\boldsymbol{m}_{\mathbf{y}}(s_1)\right]ds_1\,\mathrm{E}^{\theta}\left[\mathbf{x_0}(\theta)-\boldsymbol{m}_0\right]^{\mathrm{T}}\mathbf{\Phi}(s)^{\mathrm{T}} \;+$$

$$+\;\mathbf{\Phi}(t)\,\mathrm{E}^{\theta}\left[\left(\mathbf{x_0}(\theta)-\boldsymbol{m}_0\right)\left(\mathbf{x_0}(\theta)-\boldsymbol{m}_0\right)^{\mathrm{T}}\right]\mathbf{\Phi}(s)^{\mathrm{T}}.$$

That is:

$$\boldsymbol{C}_{\mathbf{xx}}(t,s)=\int_{t_0}^{t}\int_{t_0}^{s}\mathbf{\Phi}(t-s_1)\cdot\mathbf{B}\cdot\boldsymbol{C}_{\mathbf{yy}}(s_1,s_2)\cdot\mathbf{B}^{\mathrm{T}}\cdot\mathbf{\Phi}(t-s_2)^{\mathrm{T}}\,ds_1 ds_2 \;+\;\mathbf{\Phi}(t)\boldsymbol{C}_{\mathbf{x_0 x_0}}\mathbf{\Phi}(s)^{\mathrm{T}}. \quad (8)$$

Taking the limit ($s \to t$) to Equ.(7) and Equ.(8), respectively we obtain integration formulae for the **one-time RE-cross covariance** $\boldsymbol{C}_{\mathbf{xy}}(t,t)$ and the **one-time response variance** $\boldsymbol{C}_{\mathbf{xx}}(t,t)$, i.e.:

$$\boldsymbol{C}_{\mathbf{xy}}(t,t)=\int_{t_0}^{t}\mathbf{\Phi}(t-s_1)\cdot\mathbf{B}\cdot\boldsymbol{C}_{\mathbf{yy}}(s_1,t)ds_1 \;, \quad (9)$$

$$\boldsymbol{C}_{\mathbf{xx}}(t,t)=\int_{t_0}^{t}\int_{t_0}^{t}\mathbf{\Phi}(t-s_1)\cdot\mathbf{B}\cdot\boldsymbol{C}_{\mathbf{yy}}(s_1,s_2)\cdot\mathbf{B}^{\mathrm{T}}\cdot\mathbf{\Phi}(t-s_2)^{\mathrm{T}}\,ds_1 ds_2 \;+\;\mathbf{\Phi}(t)\boldsymbol{C}_{\mathbf{x_0 x_0}}\mathbf{\Phi}(t)^{\mathrm{T}}. \quad (10)$$

3.4. The two-time joint REPDF of the scalar linear stochastic problem under Gaussian excitation.

Assuming that the input $y(t;\theta)$ of the linear RDE (Equ.(1)_Sec(3.2.1)) is a Gaussian random function then the joint two-time REPDF $f_{x(t)y(s)}(\alpha,\beta)$ is a 2D Gaussian density. In this case, the solution of the two-time RE moment equations (given by Equ.(3,9,19)_Sec.(3.2.1), here repeated for convenience) :

$$m_x(t)=e^{A\cdot t}\cdot B\cdot\int_{t_0}^{t} m_y(s)e^{-A\cdot s}ds+e^{A\cdot(t-t_0)}\cdot m_{x_0},\;\; \forall t\ge t_0 \;, \quad (1)$$

$$R_{xy}(t,s)=e^{A\cdot t}\cdot\int_{t_0}^{t} B\cdot R_{yy}(t_1,s)e^{-A\cdot(t_1)}dt_1+e^{A\cdot(t-t_0)}\cdot m_{x_0}\cdot m_y,\;\; \forall t\ge t_0 \;, \quad (2)$$

$$R_{xx}(t,t) = B^2 \cdot e^{2\cdot A\cdot t} \cdot \int_{t_0}^{t} \left(e^{-A\cdot t_1} \cdot \int_{t_0}^{t} B \cdot R_{yy}(t_2,t_1) e^{-A\cdot t_2} dt_2 \right) dt_1 + \\ + 2 \cdot e^{A(2\cdot t - t_0)} B \cdot \int_{t_0}^{t} m_{x_0} \cdot m_y(t_1) e^{-A\cdot t_1} dt_1 + e^{2\cdot A\cdot(t-t_0)} \cdot R_{x_0 x_0} \quad , \quad \forall\, t \geq t_0 \,. \tag{3}$$

uniquely defines the joint REPDF $f_{x(t)y(s)}(\alpha,\beta)$. More precisely, for each selection of the stochastic excitation $y(t;\theta)$ and of the response initial condition $x_0(\theta)$, which are assigned to the problem by the moments: $m_y(t)$, $R_{yy}(t,s)$, $t,s \geq t_0$ and m_{x_0}, $R_{x_0 x_0}$, the joint REPDF $f_{x(t)y(s)}(\alpha,\beta)$ is given by the formula:

$$f_{x(t)y(s)}(\alpha,\beta) = \frac{1}{2\pi\sqrt{C_{xx}(t,t)C_{yy}(s,s) - \left(C_{xy}(t,s)\right)^2}} \times \tag{4}$$

$$\times \left[\exp\left[-\frac{C_{yy}(s,s)\cdot(\alpha - m_x(t))^2 - 2C_{xy}(t,s)(\alpha - m_x(t))\left(\beta - m_y(s)\right) + C_{xx}(t,t)\cdot\left(\beta - m_y(s)\right)^2}{2\left(C_{xx}(t,t)C_{yy}(s,s) - \left(C_{xy}(t,s)\right)^2\right)} \right] \right],$$

where: $C_{xx}(t,t) = R_{xx}(t,t) - (m_x(t))^2$, $C_{yy}(s,s) = R_{xx}(s,s) - (m_y(s))^2$,
$C_{xy}(t,s) = R_{xx}(t,s) - m_x(t)\cdot m_y(s)$.
(5a,b,c)

In **Fig.19**, the two-time joint REPDFs $f_{x(t)y(s)}(\alpha,\beta)$ of the linear/Gaussian case are plotted in the RE-space for lpGF stochastic input (Case I studied in Section 3.2.4), with $R_{x_0 x_0} = 1$. Two different cases of input correlation time, i.e. $\tau_{yy}^{\text{corr}} = 0.93$ sec (left column) and $\tau_{yy}^{\text{corr}} = 0.33$ sec (right column), are considered. The time evolution of $f_{x(t)y(s)}(\alpha,\beta)$ for each case of input correlation time is illustrated in each column. More precisely, three different values of the time variable t are considered, i.e. $t = 1.5$ sec (Figs.19c,19d) and $t = 2$ sec (Figs.19e,19f), whereas in all cases s remains constant, i.e. $s = 1$ sec. Results are in accordance with the results obtained for the two-time RE moments in Section 3.2.4.a, for example, the two-time RE cross-correlation reaches its maximum at $t = 1.5$ sec in the pdf plotted in Fig.19c (for $\tau_{yy}^{\text{corr}} = 0.93$ sec) that is in agreement with the results plotted in Fig.2b. It is apparent, that for different input auto-correlation the joint REPDFs evolve differently in time, having increased one-time response auto-correlation and two-time RE cross-correlation in the case that $\tau_{yy}^{\text{corr}} = 0.93$ sec (left column). In **Fig.20,** the transient time diagonal ($t \to s$) joint REPDF $f_{x(t)y(t)}(\alpha,\beta)$ is plotted for the same parameters of the linear/Gaussian problem as in Fig.19 and for the same values of correlation time, i.e. $\tau_{yy}^{\text{corr}} = 0.93$ sec (left column) and $\tau_{yy}^{\text{corr}} = 0.33$ (right column). Here, joint REPDFs are illustrated at $t = 0.5$ sec (Figs.20a, 20b), $t = 1.5$ sec (Figs.20c,20d) and $t = 4$ sec (Figs.20e,20f). It becomes evident that, as time evolves, the correlation of the joint REPDF increases until it reaches the long-time statistical equilibrium state, in accordance with the time evolution of $R_{xy}(t,t)$, $R_{xx}(t,t)$ (see also Figs.2a,4a).

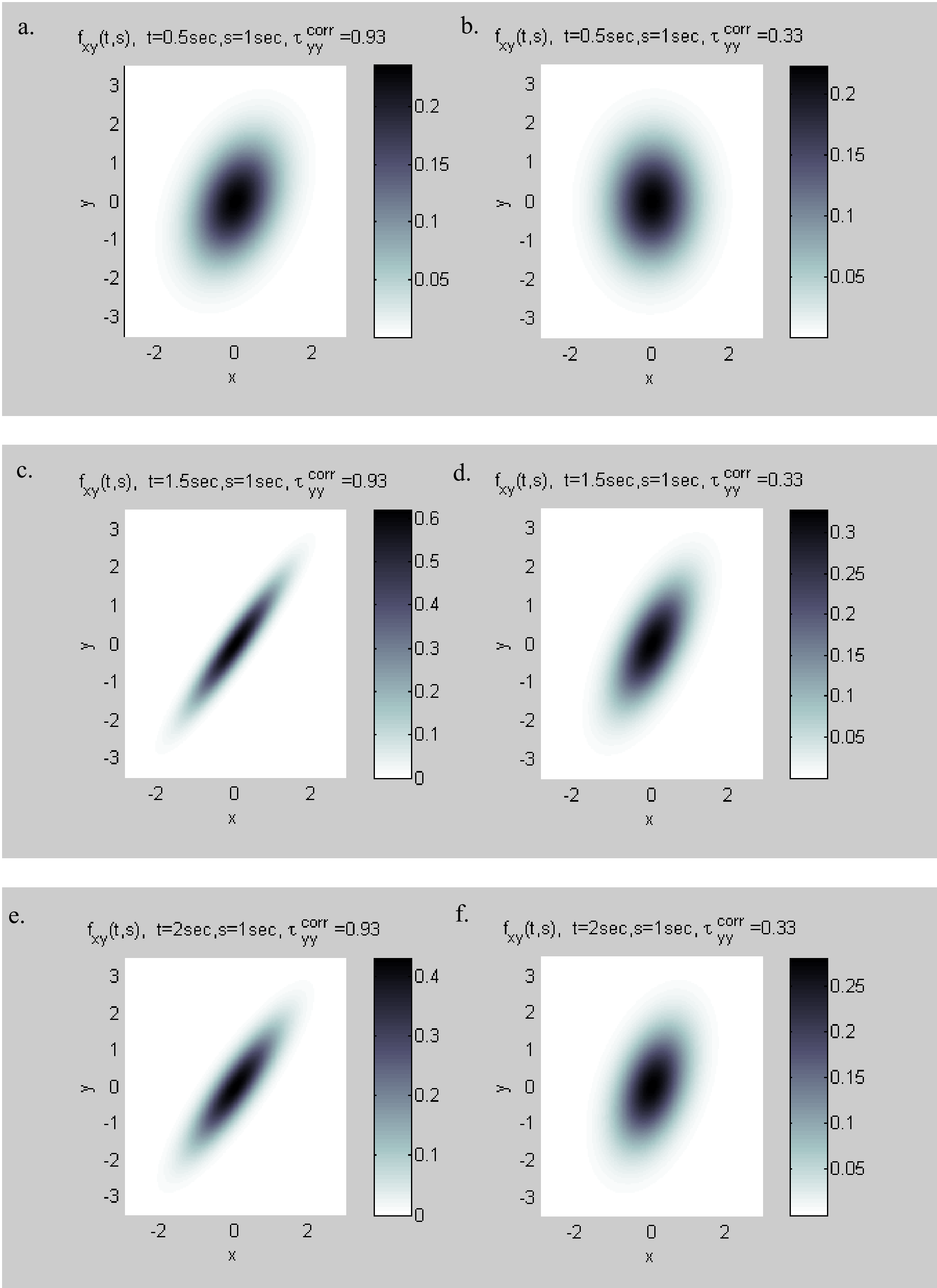


Figure 19: The time evolution of the joint two-time REPDF $f_{x(t)y(s)}(\alpha,\beta)$ for Case I. The time variable s is constant, i.e. $s=1$ sec, whereas t evolves in time, i.e.: $t=0.5, 1.5, 2$ sec. In the left column (Figs.19a,c,e) the most correlated case is considered, i.e.: stochastic input correlation time, $\tau_{yy}^{corr}=0.93$ sec. In the right column (Figs.19b,d,f) the less correlated one, $\tau_{yy}^{corr}=0.33$ sec

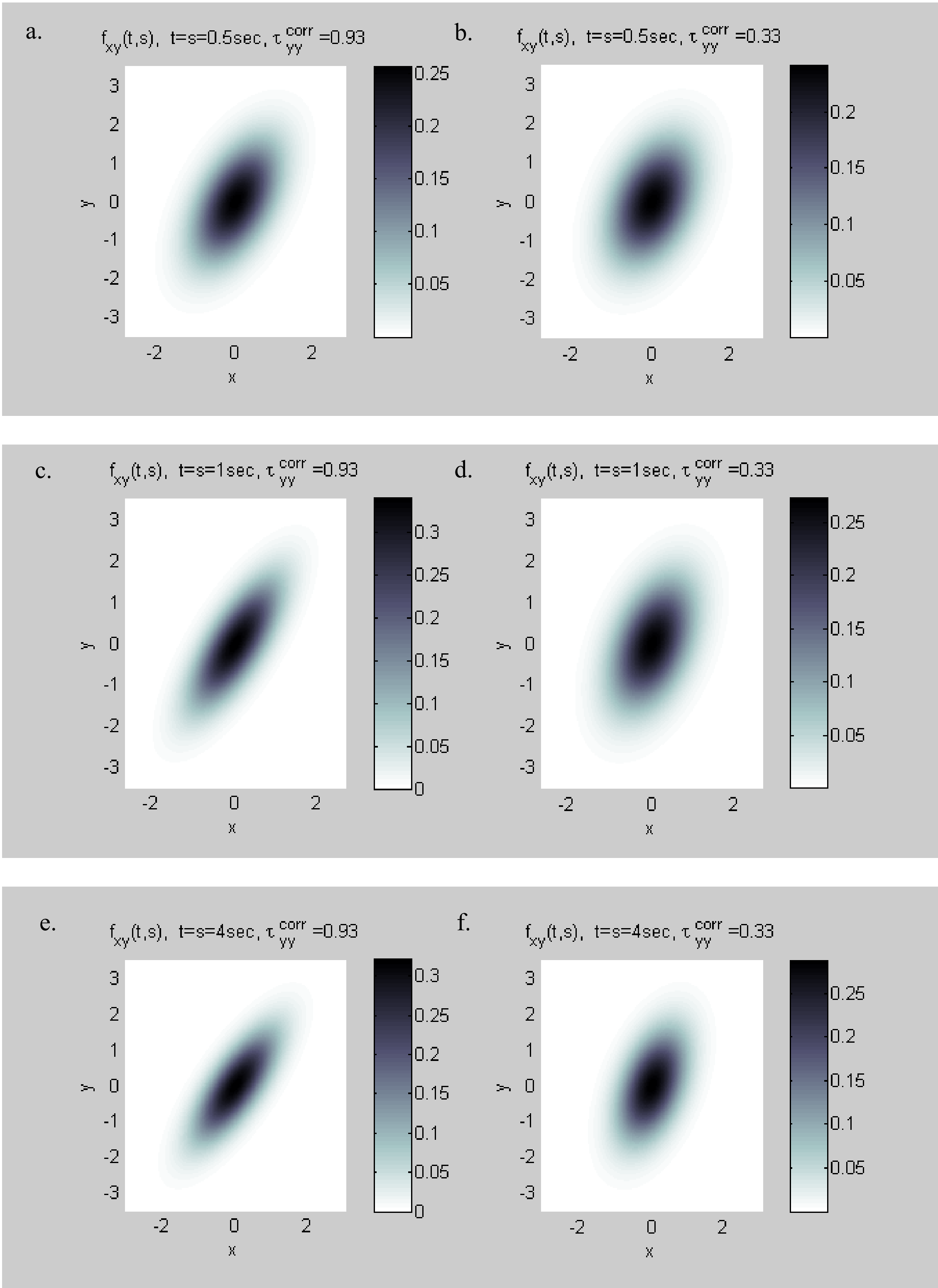


Figure 20: The time evolution of the joint diagonal REPDF $f_{x(t)y(t)}(\alpha,\beta)$ for Case I. The time variable t evolves in time, i.e.: $t = 0.5, 1, 4$ sec. In the left column (Figs.20a,c,e) the most correlated case is considered, i.e.: stochastic input correlation time, $\tau_{yy}^{corr} = 0.93$ sec. In the right column (Figs.19b,d,f) the less correlated one, $\tau_{yy}^{corr} = 0.33$ sec

3.5. Verification of the REPDF evolution equation.

In Section 3.4 the solution of the two-time RE moment equations was used to analytically define the joint REPDF of the RDE (Equ.(1)_Sec.(3.1.1)) under smoothly correlated Gaussian stochastic excitation. In this section the analytically obtained joint REPDF, given by Equ.(4)_Sec(3.4), is going to be used in order to:

1. Examine the connection between the two–time RE moment equations and the joint REPDF evolution equation in the linear/Gaussian case.
2. Verify the joint REPDF evolution equation in the linear/Gaussian case.

Following the methodology developed in Athanassoulis & Sapsis (Athanassoulis & Sapsis 2006) and Sapsis & Athanassoulis (Sapsis & Athanassoulis 2006; Sapsis & Athanassoulis 2008) (see also Section 2.3.1), the joint REPDF evolution equation in the linear/Gaussian case reads as follows:

$$\left[\frac{\partial f_{x(t)y(s)}(\alpha,\beta)}{\partial t}+A\cdot f_{x(t)y(s)}(\alpha,\beta)+\left[A\cdot\alpha\cdot+B\cdot\beta\right]\frac{\partial f_{x(t)y(s)}(\alpha,\beta)}{\partial\alpha}\right]_{s\to t}=0\,,\qquad \forall\ \alpha\in\mathbb{R},\ \beta\in\mathbb{R} \tag{1}$$

Supplemented by the initial conditions:

$$f_{x(t_0)y(s)}(\alpha,\beta)\ =\ f_{x(t_0)}(\alpha)\cdot f_{y(s)}(\beta)\ =\ \alpha\ \ pdf\ \ known\ at\ any\ time\ s\geq t_0\ , \tag{2a}$$

the marginal-compatibility constrain:

$$\int\limits_{\alpha\in\mathbb{R}} f_{x(t)y(s)}(\alpha,\beta)d\alpha\ =\ f_{y(s)}(\beta)\ =\ \text{a known pdf},\qquad \forall\ \beta\in\mathbb{R}\,, \tag{2b}$$

and the constitutive conditions:

$$f_{x(t)y(s)}(\alpha,\beta)\geq 0\,,\qquad \int\limits_{\beta\in\mathbb{R}}\int\limits_{\alpha\in\mathbb{R}} f_{x(t)y(s)}(\alpha,\beta)d\beta\, d\alpha\ =\ 1\ . \tag{2c,2d}$$

The REPDF evolution equation (1) before applying the limit $s\to t$ will be called hereafter **"off-diagonal REPDF differential constraint".**

Let us assume that $\forall t\geq t_0$, $m_y(t)=m_x(t)=0$. Replacing the joint REPDF $f_{x(t)y(s)}(\alpha,\beta)$ in the off-diagonal REPDF differential constraint with the Gaussian joint REPDF given by Equ.(4)_Sec(3.4), after some extensive algebraic manipulations included in Appendix 5, we find that the left hand side of the off-diagonal REPDF differential constraint can be equivalently written as (we repeat Equs.(3b,3c,9,12, 15, 17)_App.(5) for convenience):

$$\frac{\partial f_{x(t)y(s)}(\alpha,\beta)}{\partial t}+A\cdot f_{x(t)y(s)}(\alpha,\beta)+\left[A\cdot\alpha\cdot+B\cdot\beta\right]\frac{\partial f_{x(t)y(s)}(\alpha,\beta)}{\partial\alpha}=$$
$$\frac{\exp\{E(t,t;\alpha,\beta)\}}{2\cdot\pi\cdot W(t,s)}\cdot\left[-Q_{00}(t,s)+\left(\frac{R_{yy}(s,s)}{(W(t,s))^2}Q_{00}(t,s)\right)\cdot\alpha^2-Q_{11}(t,s)\cdot\alpha\cdot\beta+Q_{02}(t,s)\cdot\beta^2\right], \quad (3)$$

where:

$$W(t,s)\equiv\sqrt{R_{xx}(t,t)\cdot R_{yy}(s,s)-\left(R_{xy}(t,s)\right)^2}\,, \quad (4a)$$

$$E(\alpha,\beta;\,t,s)\equiv R_{yy}(s,s)\cdot\alpha^2-2\cdot R_{xy}(t,s)\cdot\alpha\cdot\beta+R_{xx}(t,t)\cdot\beta^2\,, \quad (4b)$$

$$Q_{00}(t,s)=\frac{1}{W(t,s)^2}\cdot\left[\frac{R_{yy}(s,s)}{2}\cdot\left(\frac{d}{dt}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,s)\right)-\right.$$
$$\left.-R_{xy}(t,s)\cdot\left(\frac{\partial}{\partial t}R_{xy}(t,s)-A\cdot R_{xy}(t,s)-B\cdot R_{yy}(s,s)\right)\right], \quad (4c)$$

$$Q_{11}(t,s)=\frac{1}{(W(t,s))^4}\left[R_{xy}(t,s)R_{yy}(s,s)\cdot\left(\frac{d}{dt}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,s)\right)-\right.$$
$$\left.-\left(\left(R_{xy}(t,s)\right)^2+R_{xx}(t,t)\cdot R_{yy}(s,s)\right)\cdot\left(\frac{\partial}{\partial t}R_{xy}(t,s)-A\cdot R_{xy}(t,s)-B\cdot R_{yy}(s,s)\right)\right], \quad (4d)$$

$$Q_{02}(t,s)=\frac{1}{(W(t,s))^4}\cdot\left[-R_{xy}(t,s)\cdot R_{xx}(t,t)\cdot\left(\frac{\partial}{\partial t}R_{xy}(t,s)-A\cdot R_{xy}(t,s)-B\cdot R_{yy}(s,s)\right)+\right.$$
$$\left.+\frac{1}{2}\left(R_{xy}(t,s)\right)^2\left(\frac{d}{dt}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,s)\right)\right]. \quad (4e)$$

It is evident that Equs.(4c-4e) include two differential expressions of the moments $R_{xy}(t,s)$ and $R_{xx}(t,t)$ that resemble the differential expressions appearing in the two-time moment equations Equs.(8,22)_Sec.(3.2.1) with different time arguments.

Let us now drop the zero mean value assumption for the stochastic excitation $y(t;\theta)$. The joint density REPDF $f_{\tilde{x}(t)\tilde{y}(s)}(\alpha,\beta)$ of the Gaussian random functions $\tilde{x}(t;\theta)=x(t;\theta)-m_x(t)$, $\tilde{y}(s;\theta)=y(s;\theta)-m_y(s)$ will verify Equ.(3). Substituting $f_{\tilde{x}(t)\tilde{y}(s)}(\alpha,\beta)$ in Equ.(3) and following algebraic manipulations (see Equs.(18-27) in Appendix 5), we find that the left hand of the off-diagonal REPDF differential constraint is equivalently written as:

$$\frac{\partial f_{x(t)y(s)}(\alpha,\beta)}{\partial t}+A\cdot f_{x(t)y(s)}(\alpha,\beta)+\left[A\cdot\alpha\cdot+B\cdot\beta\right]\frac{\partial f_{x(t)y(s)}(\alpha,\beta)}{\partial\alpha}= \tag{5}$$

$$=\frac{\exp\left\{\tilde{E}\left(t,t;\alpha+m_x(t),\beta+m_y(s)\right)\right\}}{2\cdot\pi\cdot\tilde{W}(t,s)}\times\left[-\tilde{Q}_{00}(t,s)+\left(\frac{C_{yy}(s,s)}{\left(\tilde{W}(t,s)\right)^2}\tilde{Q}_{00}(t,s)\right)\cdot(\alpha+m_x(t))^2-\right.$$
$$\left.-\tilde{Q}_{11}(t,s)\cdot(\alpha+m_x(t))\cdot(\beta+m_y(s))+\tilde{Q}_{02}(t,s)\cdot(\beta+m_y(s))^2\right],$$

where:

$$\tilde{W}(t,s)=\sqrt{C_{xx}(t,t)\cdot C_{yy}(s,s)-\left(C_{xy}(t,s)\right)^2}\,, \tag{6a}$$

$$\tilde{E}\left(t,s;\alpha+m_x(t),\beta+m_y(s)\right)=C_{yy}(s,s)\cdot(\alpha+m_x(t))^2-2\cdot C_{xy}(t,s)\cdot(\alpha+m_x(t))\cdot\left(\beta+m_y(s)\right)+$$
$$+C_{xx}(t,t)\cdot\left(\beta+m_y(s)\right)^2\,, \tag{6b}$$

$$\tilde{Q}_{00}(t,s)=\frac{1}{\left(\tilde{W}(t,s)\right)^2}\cdot\left[\frac{1}{2}\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot\left(\frac{d}{dt}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,s)\right)-\right.$$
$$-\left(m_x(t)\cdot R_{\tilde{y}\tilde{y}}(s,s)-m_y(s)\cdot R_{\tilde{x}\tilde{y}}(t,s)\right)\cdot\left(\frac{d}{dt}m_x(t)-A\cdot m_x(t)-B\cdot m_y(s)\right)+ \tag{6c}$$
$$\left.-R_{\tilde{x}\tilde{y}}(t,s)\cdot\left(\frac{\partial}{\partial t}R_{xy}(t,s)-A\cdot R_{xy}(t,s)-B\cdot R_{yy}(s,s)\right)\right],$$

$$\tilde{Q}_{11}(t,s)=\frac{1}{\left(\tilde{W}(t,s)\right)^4}\cdot\left[R_{\tilde{x}\tilde{y}}(t,s)R_{\tilde{y}\tilde{y}}(s,s)\cdot\left(\frac{d}{dt}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,s)\right)+\right.$$
$$+\left(m_y(s)\cdot\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2+m_y(s)\cdot R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)-\right. \tag{6d}$$
$$\left.-2\cdot m_x(t)\cdot R_{\tilde{x}\tilde{y}}(t,s)R_{\tilde{y}\tilde{y}}(s,s)\right)\cdot\left(\frac{d}{dt}m_x(t)-A\cdot m_x(t)-B\cdot m_y(s)\right)-$$
$$\left.-\left(\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2+R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\right)\cdot\left(\frac{\partial}{\partial t}R_{xy}(t,s)-A\cdot R_{xy}(t,s)-B\cdot R_{yy}(s,s)\right)\right],$$

$$\tilde{Q}_{02}(t,s)=\frac{1}{\left(\tilde{W}(t,s)\right)^4}\cdot\left[\frac{1}{2}\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2\left(\frac{d}{dt}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,s)\right)-\right.$$
$$-\left(m_x(t)\cdot\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2-R_{\tilde{x}\tilde{y}}(t,s)\cdot R_{\tilde{x}\tilde{x}}(t,t)m_y(s)\right)\left(\frac{d}{dt}m_x(t)-A\cdot m_x(t)-B\cdot m_y(s)\right)- \tag{6e}$$
$$\left.-R_{\tilde{x}\tilde{y}}(t,s)\cdot R_{\tilde{x}\tilde{x}}(t,t)\cdot\left(\frac{\partial}{\partial t}R_{xy}(t,s)-A\cdot R_{xy}(t,s)-B\cdot R_{yy}(s,s)\right)\right]\,,$$

$$R_{\tilde{x}\tilde{x}}(t,t) = C_{xx}(t,t) = R_{xx}(t,t) - \left(m_x(t)\right)^2, \tag{6f}$$

$$R_{\tilde{y}\tilde{y}}(s,s) = C_{yy}(s,s) = R_{yy}(s,s) - \left(m_y(s)\right)^2, \tag{6g}$$

$$R_{\tilde{x}\tilde{y}}(t,s) = C_{xy}(t,s) = R_{xy}(t,s) - m_x(t)\cdot m_y(s). \tag{6h}$$

That is, the off-diagonal REPDF differential constraint has been transformed to an equivalent form that contains differential expressions of the moments $m_x(t), m_y(s)$, $R_{xy}(t,s)$ and $R_{xx}(t,t)$. The connection between the off-diagonal REPDF differential constraint and these differential expressions is given by the following theorem:

Theorem 1: The joint two-time RE Gaussian pdf $f_{x(t)y(s)}(\alpha,\beta)$, $t,s \geq t_0$, of the linear RDE (Equ.(1)_Sec.(3.1.1)) under Gaussian excitation verifies the **off-diagonal REPDF differential constraint**:

$$\frac{\partial f_{x(t)y(s)}(\alpha,\beta)}{\partial t} + A\cdot f_{x(t)y(s)}(\alpha,\beta) + [A\cdot\alpha\cdot + B\cdot\beta]\frac{\partial f_{x(t)y(s)}(\alpha,\beta)}{\partial \alpha} = 0, \forall\ \alpha \in \mathbb{R}, \beta \in \mathbb{R}, \tag{7}$$

if and only if $\forall\ t,s \geq t_0$ the **auxiliary two-time RE moment constraints hold true**, i.e:

$$\frac{\partial}{\partial t}R_{xy}(t,s) - A\cdot R_{xy}(t,s) - B\cdot R_{yy}(s,s) = 0, \tag{8a}$$

$$\frac{d}{dt}R_{xx}(t,t) - 2\cdot A\cdot R_{xx}(t,t) - 2\cdot B\cdot R_{xy}(t,s) = 0, \tag{8b}$$

$$\frac{d}{dt}m_x(t) - A\cdot m_x(t) - B\cdot m_y(s) = 0. \tag{8c}$$

Proof

Since the left hand side of Equ.(7) can be equivalently expressed by Equ.(5), it follows that if Equs. (8a-8c) holds true, so does Equ.(7).

To prove the converse, two cases are considered, i.e. $m_y(t) = m_x(s) = 0$ and $m_y(t) \neq 0$.

Let $m_y(t) = m_y(s) = 0$. Equation (8c) is *a priori* verified.

Moreover, the left hand of Equ.(7) can be equivalently expressed by Equ.(3) that holds true $\forall\ \alpha \in \mathbb{R}, \beta \in \mathbb{R}$. Letting $\alpha = \beta = 0$, from Equ.(3) we obtain that $Q_{00}(t,s) = 0$. Similarly, for $\alpha = 0, \beta = 1$, from Equ.(3) we also have that $Q_{02}(t,s) = 0$. Lastly, for $\alpha = 1$, $\beta = 1$, since $Q_{00}(t,s) = Q_{02}(t,s) = 0$, from Equ.(3) we get that, if Equ.(7) holds true, $Q_{11}(t,s) = 0$.

Since we have proved that $Q_{00}(t,s)=Q_{11}(t,s)=0$, then:

$$Q_{11}(t,s)-\frac{R_{xy}(t,s)}{\left(W(t,s)\right)^2}\cdot Q_{00}(t,s)=0, \tag{9}$$

where

$$Q_{11}(t,s)-\frac{R_{xy}(t,s)\cdot Q_{00}(t,s)}{\left(W(t,s)\right)^2} = -\frac{R_{xx}(t,t)\cdot R_{yy}(s,s)}{W(t,s)^4}\cdot\left(\frac{\partial}{\partial t}R_{xy}(t,s)-A\cdot R_{xy}(t,s)-B\cdot R_{yy}(s,s)\right). \tag{10}$$

Since $\frac{R_{xx}(t,t)\cdot R_{yy}(s,s)}{W(t,s)^4}\neq 0$, from Equ.(10) it is straightforward that Equ.(8a) holds true. Then, it easy to conclude e.g. from Equ.(4c), that Equ.(8b) also holds true.

Let us now assume that $m_y(t)\neq 0$. In this case, the left hand of Equ.(7) can be equivalently expressed by Equ.(5). Since Equ.(5) holds true $\forall\ \alpha\in\mathbb{R}, \beta\in\mathbb{R}$, then for $\alpha=-m_x(t)$, $\beta=-m_y(s)$ from Equ.(5) we have that, if Equ.(7) holds true, $\tilde{Q}_{00}(t,s)=0$.Similarly, for $\alpha=-m_x(t)$, $\beta=-m_y(s)+1$ we obtain that $\tilde{Q}_{02}(t,s)=0$. Finally, for $\alpha=-m_x(t)+1$, $\beta=-m_y(s)+1$ similarly, we get that $\tilde{Q}_{11}(t,s)=0$.

Since, $Q_{00}(t,s)=Q_{11}(t,s)=0$ then:

$$\tilde{Q}_{02}(t,s)-\frac{R_{\tilde{x}\tilde{x}}(t,t)}{\left(\tilde{W}(t,s)\right)^2}\cdot\tilde{Q}_{00}(t,s)=0. \tag{11}$$

Subsequently, considering the system of equations:

$$\tilde{Q}_{02}(t,s)-\frac{R_{\tilde{x}\tilde{x}}(t,t)}{\left(\tilde{W}(t,s)\right)^2}\cdot\tilde{Q}_{00}(t,s)=0,\qquad \tilde{Q}_{11}(t,s)=0,\qquad \tilde{Q}_{00}(t,s)=0. \tag{12a-c}$$

in terms of the variables:

$$x_1=\frac{d}{dt}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,s), \tag{13a}$$

$$x_2=\frac{d}{dt}m_x(t)-A\cdot m_x(t)-B\cdot m_y(s), \tag{13b}$$

$$x_3=\frac{\partial}{\partial t}R_{xy}(t,s)-A\cdot R_{xy}(t,s)-B\cdot R_{yy}(s,s), \tag{13c}$$

we get the homogeneous linear system of equations:

$$-\frac{1}{2}x_1+m_x(t)\cdot x_2=0, \tag{14a}$$

$$\begin{aligned} &R_{\tilde{x}\tilde{y}}(t,s)R_{\tilde{y}\tilde{y}}(s,s)\cdot x_1+\\ &+\Big(m_y(s)\cdot\big(R_{\tilde{x}\tilde{y}}(t,s)\big)^2+m_y(s)\cdot R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)-2\cdot m_x(t)\cdot R_{\tilde{x}\tilde{y}}(t,s)R_{\tilde{y}\tilde{y}}(s,s)\Big)\cdot x_2-\\ &\qquad -\Big(\big(R_{\tilde{x}\tilde{y}}(t,s)\big)^2+R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\Big)\cdot x_3=0, \end{aligned} \tag{14b}$$

$$\frac{1}{2}\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot x_1-\big(m_x(t)\cdot R_{\tilde{y}\tilde{y}}(s,s)-m_y(s)\cdot R_{\tilde{x}\tilde{y}}(t,s)\big)\cdot x_2-R_{\tilde{x}\tilde{y}}(t,s)\cdot x_3=0\ . \tag{14c}$$

The determinant D of the linear system of Equs.(14) is analytically calculated in Appendix 5 (see Equs.(28-34)_App(5)) and is equal to:

$$D=m_y(s)\cdot R_{\tilde{x}\tilde{y}}(t,s)\Big(\big(R_{\tilde{x}\tilde{y}}(t,s)\big)^2+R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\Big)\neq 0, \tag{15}$$

therefore, the homogeneous system of Equs.(14) only has the zero solution

$x_1=x_2=x_3=0$. ■

Corollary 1: The REPDF evolution equation for the linear RDE under Gaussian excitation:

$$\left.\frac{\partial f_{x(t)y(s)}(\alpha,\beta)}{\partial t}\right|_{s\to t}+A\cdot f_{x(t)y(t)}(\alpha,\beta)+\left[A\cdot\alpha\cdot+B\cdot\beta\right]\frac{\partial f_{x(t)y(t)}(\alpha,\beta)}{\partial\alpha}=0, \tag{16}$$
$$\forall\ \alpha\in\mathbb{R},\ \beta\in\mathbb{R}$$

holds true if and only if the moments of the sought for density $f_{x(t)y(t)}(\alpha,\beta)$ verify the limit two-time RE moment equations:

$$\left.\frac{\partial}{\partial t}R_{xy}(t,s)\right|_{s\to t}-A\cdot R_{xy}(t,t)-B\cdot R_{yy}(t,t)=0, \tag{17a}$$

$$\frac{d}{dt}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,t)=0, \tag{17b}$$

$$\frac{d}{dt}m_x(t)-A\cdot m_x(t)-B\cdot m_y(t)=0. \tag{17c}$$

Theorem 2: The auxiliary two-time RE moment constraints (Equs.(8a-8c)) of the linear RDE:
i) Do not hold true $\forall\ t,s\geq t_0$
ii) Hold true in the limiting case $s\to t$

Proof

The moments of $R_{xy}(t,s)$ and $R_{xx}(t,t)$ of the linear RDE follow the two-time RE moment equations (Equs.(8a,22a,2a)_Sec.(3.2.1)):

$$\frac{\partial}{\partial t}R_{xy}(t,s)-A\cdot R_{xy}(t,s)-B\cdot R_{yy}(t,s)=0\,, \qquad (18a)$$

$$\frac{d}{dt}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,t)=0\,, \qquad (18b)$$

$$\frac{d}{dt}m_x(t)-A\cdot m_x(t)-B\cdot m_y(t)=0\,, \qquad (18c)$$

where $R_{yy}(t,s)$ is the given input auto-correlation function and $m_y(t)$ is the given input mean value.

Let us suppose that Equs.(8a-8c) hold true, then from Equs.(18a-18c) we obtain that :

$$R_{yy}(t,s)=R_{yy}(s,s)\,,\ R_{xy}(t,s)=R_{xy}(t,t)\ \text{and}\ m_y(t)=m_y(s)\,,\forall\ t,s\geq t_0\,,$$

that is not true, therefore part (i) has been proved.

Taking the limit $s\rightarrow t$ of Equs.(18a-18c) we obtain the limit two-time RE moment equations, i.e. Equs.(17a-17c), respectively, therefore Equs.(17a-17c) hold true. Moreover, taking the limit $s\rightarrow t$ of Equs.(8a-8c) we re-obtain Equs.(17a-17c). Part (ii) has also been proved. ■

Corollary 2: The Gaussian pdf $f_{x(t)y(s)}(\alpha,\beta)$ (see Equ.(4)_Sec(3.4)) of the linear RDE (Equ.(1)_Sec.(3.1.1)) verifies the REPDF evolution equation (Equs.(1a)) together with conditions (2a-2d).

Proof

The moments of the Gaussian pdf $f_{x(t)y(s)}(\alpha,\beta)$ are the solutions of the two-time RE moment equations. It follows that these will also verify the limit two-time RE moment equations and therefore according to Corollary 1 the REPDF evolution equation. Moreover, in the case that the sought for density is Gaussian, the initial conditions of the two-time REPDF evolution equation, Equ.(2a), and the initial conditions of the two-time RE moment equations, Equs.(8b,22b,2b)_Sec.(3.2.1), are equivalent so Equ.(2a) is verified. In addition, the marginal constraint Equ.(2b) is also verified, since the marginal moments $m_y(t)$, $R_{yy}(t,t)$ is the given input of the moment problem. Finally, it is obvious that the constitutive conditions Equs.(2c-2d) are verified too.

3.6. On the non-uniqueness of solutions of the REPDF evolution equation

In Section 3.5, we have proved that the jointly Gaussian REPDF $f_{x(t)y(s)}(\alpha,\beta)$ of the linear RDE under Gaussian excitation verifies the REPDF evolution equation if and only if the limit two-time RE moment equations are verified (see Corollary 1). In this section this equivalence will be used in order to clarify that the REPDF evolution equation does not have a unique solution, as also stated in Venturi et al.(2012).

As we have seen in Section 3.4, the solution of the two-time RE moment equations i.e:

$$m_x(t) = e^{A\cdot t}\cdot B\cdot\int_{t_0}^{t} m_y(s)\, e^{-A\cdot s}ds + e^{A\cdot(t-t_0)}\cdot m_{x_0},\ \forall t\geq t_0 \tag{19a}$$

$$R_{xy}(t,s) = e^{A\cdot t}\cdot\int_{t_0}^{t} B\cdot R_{yy}(t_1,s)\, e^{-A(t_1)}dt_1 + e^{A\cdot(t-t_0)}\cdot m_{x_0}\cdot m_y,\ \forall t\geq t_0 \tag{19b}$$

$$\begin{aligned} R_{xx}(t,t) = {} & e^{A\cdot t}\cdot B\cdot\int_{t_0}^{t} R_{xy}(t,t_1)e^{-A\cdot t_1}dt_1 + \\ & + e^{A\cdot(2t-t_0)}\cdot B\cdot m_{x_0}\cdot\int_{t_0}^{t} m_y(t_1)\cdot e^{-A\cdot t_1}dt_1 + e^{2A\cdot(t-t_0)}\cdot R_{x_0x_0}\end{aligned} \tag{19c}$$

can uniquely define the joint Gaussian REPDF $f_{x(t)y(s)}(\alpha,\beta)$ (see Equ.(4)_Sec(3.4)).

However, differentiating equations (19a)-(19c) with respect to time t, then taking the limit $s\to t$ it is easy to confirm that Equs. (19a)-(19c) verify the limit two-time RE moment equations, and according to Corollary 1, so is the case for the joint REPDF evolution equation, regardless of the correlation time of the stochastic input. Therefore the solution to both the limit two-time RE moment equations and the joint REPDF evolution equation is not unique.

In fact, when the moment equation for the two-time RE cross-correlation $R_{xy}(t,s)$ is considered time diagonally ($s\to t$) the two-time auto-correlation $R_{yy}(t,s)$ of the stochastic input, appearing in the left hand of Equ.(18a), is replaced by the one-time auto-correlation of the stochastic input $R_{yy}(t,t)$. Thus parameters of the input two-time auto-correlation $R_{yy}(t,s)$ that control the correlation time (e.g. the parameter a for lpGF or OU input and a,ω_0 for sOU stochastic input) are not taken into account. On the contrary, when $R_{xy}(t,t)$ is calculated based on the whole history of $R_{yy}(t_1,t)$ for $t_0\leq t_1\leq t$ (i.e. taking the limit $s\to t$ of Equ.(19)) the non-local effects of the colored stochastic excitation that are controlled by the correlation time of the input are taken into account.

It becomes evident that some of the non-local (in time) characteristics of the problem are lost when we also take the limit $\partial f_{x(t)y(s)}(\alpha,\beta)/\partial t\big|_{s\to t}$ in the REPDF evolution. Therefore, additional information about the joint RE correlation structure should be supplied to the REPDF evolution equation so that this has a unique solution. In the linear/Gaussian case this additional information could be provided by the equation for the evolution of the two-time RE cross-correlation $R_{xy}(t,s)$, i.e. Equ.(18a).

Finally in this section, we shall discuss the counter-example presented by Venturi et al.(2012) in order to demonstrate the non-uniqueness of solutions of the REPDF evolution equation. More specifically, in the following RDE was considered:

$$\dot{x}(t;\theta)+x(t;\theta)=y(t;\theta), \tag{24a}$$

$$y(t;\theta)=\sin(t)+\xi(\theta), \tag{24b}$$

$$x(t_0;\theta)=x_0(\theta), \tag{24c}$$

where $\xi(\theta)$ is a Gaussian random variable, $x_0(\theta)$ is a Gaussian zero mean random variable that is independent from $\xi(\theta)$.

The analytical solution of Equ.(24) (this is given by Equ.(2.18) of (Venturi et al. 2012)) reads as follows:

$$x(t;\theta)=e^{-(t-t_0)}\left(\int_{t_0}^{t} y(s;\theta)\cdot e^{(s-t_0)}ds+x_0(\theta)\right)= \tag{25}$$

$$=e^{-(t-t_0)}\cdot[\xi(\theta)\cdot(e^{t-t_0}-1)+x_0(\theta)+\frac{1}{2}\cdot e^{(t-t_0)}(\sin(t)-\cos(t))-\frac{1}{2}\cdot(\sin(t_0)-\cos(t_0))].$$

In the RDE given by Equ.(24) the time dependence is present only in the deterministic part of the stochastic excitation that is given by Equ.(24b), whereas all the other stochastic quantities of the RDE are time invariant. Therefore, Equ.(25) can be considered as a mapping between the random variables

$$x(t;\theta)=A(t)\cdot\xi(\theta)+B(t)\cdot x_0(\theta)+C(t), \tag{26a}$$
$$y(t;\theta)=\sin(t)+\xi(\theta)\ , \tag{26b}$$

where

$$A(t)\equiv 1-e^{-(t-t_0)},\ B(t)\equiv e^{-(t-t_0)}, \tag{26c,d}$$

$$C(t)\equiv\frac{1}{2}\cdot\left[e^{(t-t_0)}(\sin(t)-\cos(t))-(\sin(t_0)-\cos(t_0))\right]. \tag{26e}$$

Using the mapping approach (pp.142 in (Papoulis 1991)) from Equ.(25), the joint pdf that corresponds to the RDE (Equ.24) is obtained, i.e.:

$$f_{x(t)y(s)}(\alpha,\beta)=\frac{1}{2\pi\cdot B(t)}\exp\left[-\frac{1}{2}(\beta-\sin(s))^2-\frac{(\alpha-A(t)\cdot\beta+A(t)\cdot\sin(s)-C(t))^2}{2\cdot B(t)^2}\right]. \tag{27}$$

It is straightforward that $f_{x(t)y(s)}(\alpha,\beta)$ verifies the REPDF-evolution equation that corresponds to the RDE Equ.(24), i.e.:

$$\left.\frac{\partial f_{x(t)y(s)}(\alpha,\beta)}{\partial t}\right|_{s\to t} = \frac{\partial}{\partial \alpha}(\alpha \cdot f_{x(t)y(t)}(\alpha,\beta)) - \beta \cdot \frac{\partial f_{x(t)y(t)}(\alpha,\beta)}{\partial \alpha}, \quad \forall\ \alpha \in \mathbb{R},\ \beta \in \mathbb{R}\ . \tag{28}$$

In Ventury et al. (2012) it is argued that $C(t)$ could be instead any function $C(t) = a(t)$, with $a(t_0) = 0$ and the density given by Equ.(27) would still satisfy the REPDF evolution equation (28). The latter argument seems not to be correctly stated as, in order for $C(t)$ to be such that Equ.(27) satisfies Equ.(28), there should be some information on the relationship between $C(t)$ and $C'(t)$ (the latter appears in the right hand of Equ.(28) after the time differentiation of the joint REPDF $f_{x(t)y(s)}(\alpha,\beta)$ given by Equ.(27)). Nevertheless, it is true that Equ.(28) followed by conditions given by Equs.(2a-2d) admits more than one solution and, therefore, the boundary value problem of Equs.(28, 2a-2d) is indeed not well-posed. We shall now prove that when the two-time RE moment equation that corresponds to the RDE (Equ.24) is also considered, $C(t)$ is uniquely defined and given by Equ.(26e).

Multiplying Equ.(24a-c) with $y(s;\theta) = \sin(s) + \xi(\theta)$ and taking mean values we obtain the two-time RE moment equation that governs the RE cross-correlation $R_{xy}(t,s)$ of the random differential equation (24), i.e.:

$$\frac{\partial}{\partial t} R_{xy}(t,s) + R_{xy}(t,s) - R_{yy}(t,s) = 0\ , \tag{29a}$$

$$R_{xy}(t_0,s) = 0\ \ , \tag{29b}$$

where

$$R_{yy}(t,s) = E^{\theta}\left[(\sin(t) + \xi(\theta)) \cdot (\sin(s) + \xi(\theta))\right] = \sin(t) \cdot \sin(s) + \sin(t) \cdot m_{\xi} + \sin(s) \cdot m_{\xi} + R_{\xi\xi}\ . \tag{30}$$

We shall use the dynamical system equations (26a) and (26b) to express $R_{xy}(t,s)$ in terms of $C(t)$, then, we shall use the two-time RE moment equation to find the deterministic function $C(t)$. More precisely, multiplying Equ.(26a) with $y(s;\theta) = \sin(s) + \xi(\theta)$ (as given by Equ.(26b)) and taking mean values we get:

$$\begin{aligned} R_{xy}(t,s) &= E^{\theta}[(A(t)\cdot\xi(\theta) + B(t)\cdot x_0(\theta) + C(t))\cdot(\sin(s) + \xi(\theta))] = \\ &= E^{\theta}[A(t)\cdot\xi(\theta)\cdot\sin(s) + B(t)\cdot x_0(\theta)\cdot\sin(s) + C(t)\cdot\sin(s) + \\ &\qquad + A(t)\cdot\xi(\theta)\cdot\xi(\theta) + B(t)\cdot x_0(\theta)\cdot\xi(\theta) + C(t)\cdot\xi(\theta)] = \\ &= (1 - e^{-(t-t_0)})\cdot(\sin(s)\cdot m_{\xi} + R_{\xi\xi}) + C(t)\cdot(\sin(s) + m_{\xi}), \end{aligned} \tag{31a}$$

$$R_{xy}(t_0,s) = C(t_0)\cdot(\sin(s) + m_{\xi})\ . \tag{31b}$$

Subsequently, replacing Equ.(31a) in Equ.(29) we get an equation for the evolution of $C(t)$:

$$\frac{dC(t)}{dt}\cdot(\sin(t)+m_{\xi})+e^{-(t-t_0)}\cdot(\sin(t)\cdot m_{\xi}+R_{\xi\xi})+(1-e^{-(t-t_0)})\cdot(\sin(t)\cdot m_{\xi}+R_{\xi\xi})+ \\ +C(t)\cdot(\sin(t)+m_{\xi})-\sin(t)\cdot\sin(t)-\sin(t)\cdot m_{\xi}-\sin(t)\cdot m_{\xi}-R_{\xi\xi}=0, \tag{32a}$$

$$C(t_0)\cdot(\sin(t)+m_{\xi})=0. \tag{32b}$$

All the terms that depend on the time variable s are simplified here, and we get:

$$\frac{dC(t)}{dt}+C(t)-\sin(t)=0, \tag{33a}$$

$$C(t_0)=0. \tag{33b}$$

The solution of the initial value problem of Equs.(33) is exactly Equ.(26e)

3.7. Equation for the evolution of response pdf in the linear/Gaussian case.

As discussed in Section 2.2.3 in order to find the pdf of the response of stochastic systems excited by colored noise, several methods have been developed focusing on the solution of an equation for the response density (Hänggi & Jung 1995). In fact, in Ventury et al. (2012) the consistence of the response-marginal REPDF with Equ.(8)_Sec(2.2.3) has been established. In this section we are going to discuss the connection of the response-marginal REPDF evolution equation with the two-time RE moment equations. We focus on two points:

1. We shall show the connection of the response-marginal REPDF evolution equation with the one-time RE moment equation (see Equ.(22)_Sec.(3.2.a)) in the linear/Gaussian case.
2. We shall use this simple case in order to demonstrate how the system of the two-time RE moment equations could be used as an alternative way to approximate the non-local term appearing in the response-marginal REPDF evolution equation.

3.7.1. Connection with the one-time response moment equation

For convenience and without this being restrictive, in this section, we shall assume that $m_y(t)=0$, $m_{x_0}=0$ and therefore, from Equ.(1)_Sec(3.4), $m_x(t)=0$. To find an equation for the evolution of the response we integrate with respect to the excitation variable the REPDF evolution equation Equ(1)_Sec(3.5). Subsequently, assuming that the integration can be performed before the partial differentiation, we obtain the following equation for the evolution of the response density:

$$\frac{\partial}{\partial t}f_{x(t)}(\alpha)+\frac{\partial}{\partial\alpha}\alpha\cdot A\cdot f_{x(t)}(\alpha)+B\cdot\frac{\partial}{\partial\alpha}\int\beta\cdot f_{x(t)y(s)}(\alpha,\beta)d\beta=0, \quad \forall\,\alpha\in\mathbb{R},\beta\in\mathbb{R}. \tag{1}$$

When the excitation is a Gaussian stochastic function, as discussed in Section 3.4, the joint REPDF will be a 2D Gaussian density given by Equ.(4)_Sec(3.4). Moreover, all terms in Equ.(1) can be analytically computed, i.e.:

$$\int \beta \cdot f_{x(t)y(s)}(\alpha,\beta)\,d\beta = \\ = \frac{R_{xy}(t,s)}{R_{xx}(t,t)} \cdot \frac{1}{\sqrt{2\cdot\pi}\cdot\sqrt{R_{xx}(t,t)}} \cdot \exp\left[-\frac{\alpha^2}{2\cdot R_{xx}(t,t)}\right] = \alpha \cdot \frac{R_{xy}(t,t)}{R_{xx}(t,t)} \cdot f_{x(t)}(\alpha). \tag{2}$$

Then,

$$\frac{\partial}{\partial\alpha}\int \beta \cdot f_{x(t)y(s)}(\alpha,\beta)\,d\beta = \frac{R_{xy}(t,t)}{R_{xx}(t,t)} \cdot f_{x(t)}(\alpha) + \alpha \cdot \frac{R_{xy}(t,t)}{R_{xx}(t,t)} \cdot \frac{d\, f_{x(t)}(\alpha)}{d\alpha} \quad . \tag{3}$$

Moreover, since:

$$\frac{\partial\, f_{x(t)}(\alpha)}{\partial\alpha} = \frac{1}{\sqrt{2\cdot\pi}\cdot\sqrt{R_{xx}(t,t)}} \frac{\partial}{\partial\alpha} \exp\left[-\frac{\alpha^2}{2\cdot R_{xx}(t,t)}\right] = -\frac{\alpha}{R_{xx}(t,t)} \cdot f_{x(t)}(\alpha) \ . \tag{4}$$

Combining Equs.(3,4) we obtain:

$$\frac{\partial}{\partial\alpha}\int \beta \cdot f_{x(t)y(s)}(\alpha,\beta)\,d\beta = \left(\frac{R_{xy}(t,t)}{R_{xx}(t,t)} - \alpha^2 \cdot \frac{R_{xy}(t,t)}{\left(R_{xx}(t,t)\right)^2} \right) \cdot f_{x(t)}(\alpha). \tag{5}$$

Moreover for the time derivative of the Gaussian pdf we have:

$$\frac{\partial}{\partial t} f_{x(t)}(\alpha) = -\frac{1}{2\cdot R_{xx}(t,t)} \cdot \frac{\partial}{\partial t} R_{xx}(t,t) \cdot f_{x(t)}(\alpha) + \frac{\alpha^2}{2\cdot\left(R_{xx}(t,t)\right)^2} \cdot \frac{\partial}{\partial t} R_{xx}(t,t) \cdot f_{x(t)}(\alpha) \ . \tag{6}$$

Introducing Equs.(4-6) in Equ.(1) we get:

$$-\frac{1}{2\cdot R_{xx}(t,t)} \cdot \frac{d}{dt} R_{xx}(t,t) \cdot f_{x(t)}(\alpha) + A \cdot f_{x(t)}(\alpha) + B \cdot \frac{R_{xy}(t,t)}{R_{xx}(t,t)} \cdot f_{x(t)}(\alpha) + \\ + \frac{\alpha^2}{2\cdot\left(R_{xx}(t,t)\right)^2} \cdot \frac{d}{dt} R_{xx}(t,t) \cdot f_{x(t)}(\alpha) - A \cdot \frac{\alpha^2}{R_{xx}(t,t)} \cdot f_{x(t)}(\alpha) - B \cdot \alpha^2 \cdot \frac{R_{xy}(t,t)}{\left(R_{xx}(t,t)\right)^2} \cdot f_{x(t)}(\alpha) = 0, \tag{7}$$

that is:

$$-\frac{1}{2\cdot R_{xx}(t,t)}\left(\frac{d}{dt}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,t)\right)\cdot f_{x(t)}(\alpha)+ \\ +\frac{\alpha^2}{2\cdot\left(R_{xx}(t,t)\right)^2}\left(\frac{d}{dt}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,t)\right)\cdot f_{x(t)}(\alpha)=0. \quad (8)$$

Since Equ.(8) holds true $\forall\alpha$ it is easy to verify that Equ.(7) holds true if and only if

$$\frac{d}{dt}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,t)=0\,. \quad (9)$$

3.7.2. Approximation of the non-local term using the two-time-RE moment equations

In Section 3.7.1 we wrote the non-local term of Equ.(1)_Sec(3.7.2) in terms of the diagonal moments $R_{xy}(t,t),\ R_{xx}(t,t)$, i.e.:

$$\frac{\partial}{\partial\alpha}\int\beta\cdot f_{x(t)y(s)}(\alpha,\beta)d\beta=\left(\frac{R_{xy}(t,t)}{R_{xx}(t,t)}-\alpha^2\cdot\frac{R_{xy}(t,t)}{\left(R_{xx}(t,t)\right)^2}\right)\cdot f_{x(t)}(\alpha)\,, \quad (1)$$

Equ.(1) can be used in conjunction with the solution of the two-time RE moment equations in order to obtain a closed form of Equ.(1)_Sec(3.7.2). That is, Equ.(1)_Sec(3.7.2) can be rewritten as:

$$\frac{\partial}{\partial t}f_{x(t)}(\alpha)+\frac{\partial}{\partial\alpha}\alpha\cdot A\cdot f_{x(t)}(\alpha)+B\cdot\left(\frac{R_{xy}(t,t)}{R_{xx}(t,t)}-\alpha^2\cdot\frac{R_{xy}(t,t)}{\left(R_{xx}(t,t)\right)^2}\right)\cdot f_{x(t)}(\alpha)=0\,, \quad (2)$$

$$\forall\ \alpha\in\mathbb{R},\ \beta\in\mathbb{R},$$

where:

$$R_{xy}(t,t)=e^{A\cdot t}\cdot B\cdot\int_{t_0}^{t}R_{yy}(t_1,t)\cdot e^{-A\cdot t_1}dt_1\ ,\qquad \forall\, t\ge t_0, \quad (3)$$

$$R_{xx}(t,t)=e^{A\cdot t}\cdot B\cdot\int_{t_0}^{t}R_{xy}(t,t_1)e^{-A\cdot t_1}dt_1+e^{2A\cdot(t-t_0)}\cdot R_{x_0x_0}\,. \quad (4)$$

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CHAPTER 4

APPLICATION OF RE THEORY TO NON LINEAR DYNAMICAL SYSTEMS: SOLUTION OF THE REPDF EVOLUTION EQUATION IN THE LONG-TIME

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4.1. Introduction

In the previous chapter the joint REPDF of the linear Random Differential Equation (RDE) (Equ.(1)_Sec(3.1.1)) under smoothly correlated (colored) Gaussian excitation was obtained analytically through the solution of two-time RE moment equations. The obtained solution was used to verify the REPDF evolution equation as well as to demonstrate that the REPDF evolution equation, as it stands, can have multiple solutions. In fact, the correlation length of the excitation constitutes a degree of freedom that is not properly taken into consideration when the equation is considered only time diagonally $(s \to t)$, as the non-local (in time) characteristics of the problem are lost. Nevertheless, in the linear/Gaussian case, the inclusion of an additional constraint for the two-time RE cross-correlation could provide the additional information needed for the problem to be well posed.

In this chapter, we show how these ideas take shape and are generalized for the probabilistic characterization of a steady state non-linear half oscillator under non-Gaussian excitation. The problem is treated in the context of response-excitation theory, introduced by Athanassoulis & Sapsis (Athanassoulis & Sapsis 2006) and Sapsis & Athanassoulis (Sapsis & Athanassoulis 2006; Sapsis & Athanassoulis 2008).

More precisely, taking into consideration the findings from the solution of the linear problem we develop auxiliary local conditions, in the RE-phase space, that provide the necessary additional information regarding the RE correlation structure of the non-linear random problem. The local information is synthesized in the REPDF evolution equation by the use of an appropriate representation of the two-time joint REPDF, consisting of a superposition of Gaussian Kernels. The REPDF evolution equation, together with the new local closure conditions, is numerically solved using a Galerkin scheme. Let it be noted that, in the linear case under Gaussian excitation, the additional constraints coincide with the global two-time RE moment equations discussed in Chapter 3. Some preliminary results of this method have been presented in Athanassoulis, Tsantili & Kapelonis (Athanassoulis et al. 2012b), (Athanassoulis et al. 2012a).

An important question, related with the process of the numerical solution of the joint REPDF evolution equation in the long-time, is how to define the appropriate computational domain. A methodology for the approximate, *a priori*, determination of the computational domain has been developed based on the solution of a system of two-time RE moment equations for the non-linear/non-Gaussian problem. The derivation and solution of these equations is presented in Chapter 5.

The development of the presented methodology drew on results obtained by MC simulations performed by Z.G. Kapelonis. The evidence gained by looking into these results, made clear that, in the long-time statistical-equilibrium state the joint REPDF tends to concentrate around the equilibrium curve of deterministic problems which are naturally realized on the RE-phase space. In accordance with this observation, it was made clear that the implementation of the numerical solution of REPDF evolution equation can exploit information coming from the stability analysis of the corresponding deterministic problems.

4.2. Formulation of the problem

Consider a non-linear half oscillator, of the form:

$$\dot{x}(t\,;\theta) \;=\; H\big(x(t\,;\theta)\big) + \Psi\big(y(t\,;\theta)\big), \tag{1a}$$

$$x(0\,;\theta) = x_0(\theta)\,, \tag{1b}$$

where $H(\bullet)$, $\Psi(\bullet)$ are non-linear functions, θ is the stochastic argument (the sample-point indicator), $y(t\,;\theta)$ is a given, smoothly correlated, asymptotically stationary, Gaussian stochastic process (regular colored noise) with continuous path functions. The initial condition $x_0(\theta)$ is a given random variable, independent from the process $y(t;\theta)$. In what follows we shall assume that $H(\bullet)$, $\Psi(\bullet)$ are polynomial function of $x(t\,;\theta)$, $y(t\,;\theta)$, respectively, i.e.

$$H\big(x(t\,;\theta)\big) = \sum_{\nu=0}^{N} A_{\nu}(t)\cdot x^{\nu}(t\,;\theta)\,, \tag{2a}$$

$$\Psi\big(y(t\,;\theta)\big) = \sum_{m=0}^{M} B_{m}(t)\cdot y^{m}(t\,;\theta)\,. \tag{2b}$$

The polynomial excitation $\Psi\big(y(t\,;\theta)\big)$ can model (strongly) non-Gaussian processes.

A special case of the stochastic initial problem given by Equ.(1) is the cubic half oscillators under non Gaussian Excitation. The probabilistic description of cubic half-oscillator excited by delta-correlated processes has been studied by many authors. Hasofer and Grigoriu (Hasofer & Grigoriu 1995), and Grigoriu (Grigoriu 2008) studied the case of Gaussian white-noise excitation, solving the corresponding moment problem and commenting on the properties of the moment closures. Wojtkiewicz, Grigoriu et al. (Wojtkiewicz et al. 1999), (Grigoriu 1995) generalized various techniques developed for the case of Gaussian excitation to systems driven by Poisson or Gaussian plus Poisson white-noises, and studied the cubic half oscillator as an example. The same problem, under OU excitation, has been treated by Jung & Risken, and others ((Jung & Risken 1985), (Debnath et al. 1990)), in the context of the filtering approach (augmented state space, two-dimensional FPK equation). Furthermore, the same problem has been also studied using approximate one-dimensional FPK equation in conjunction with the short relaxation time approximation (Hänggi et al. 1984). Here we consider the following special form of Equ.(1):

$$\dot{x}(t\,;\theta) \;=\; \mu_1\cdot x(t\,;\theta) + \mu_3\cdot x^3(t\,;\theta) + \kappa_1\cdot y(t\,;\theta) + \kappa_3\cdot y^3(t\,;\theta)\,, \tag{3a}$$

$$x(t_0\,;\theta) \;=\; x_0(\theta)\,, \tag{3b}$$

In this case the excitation

$$z(t\,;\theta) \;=\; \kappa_1\cdot y(t\,;\theta) + \kappa_3\cdot y^3(t\,;\theta)\,, \tag{4}$$

is a linear-plus-cubic-Gaussian process, having a bimodal first-order pdf in the case $\kappa_1\cdot\kappa_3 < 0$. Without any loss of generality, we shall assume that $\kappa_1 > 0$. The signs of

μ_1, μ_3, κ_3 may be either $+1$, or -1, affecting the structure of the solution to both the deterministic and the stochastic problem. The case $\mu_1 < 0$, $\mu_3 < 0$ will be referred as the monostable case and the case $\mu_1 > 0$, $\mu_3 < 0$ as the bistable case, in accordance with the stability properties of the homogeneous equation.

The condition of asymptotic stationarity means that there exist a constant $m_y^{(\infty)}$ and a bounded stationary covariance function $C_{yy}^{(\infty)}(\tau)$, such that

$$(\forall \varepsilon > 0)\left(\exists t_\varepsilon > t_0\right): \quad \left[t \geq t_\varepsilon\right] \Rightarrow \left|m_y(t) - m_y^{(\infty)}\right| < \varepsilon, \tag{5a}$$

and

$$(\forall \varepsilon > 0)\left(\exists t_\varepsilon > t_0\right): \quad \left[t \geq t_\varepsilon \wedge s \geq t_\varepsilon\right] \Rightarrow \left|C_{yy}(t,s) - C_{yy}^{(\infty)}(t-s)\right| < \varepsilon. \tag{5b}$$

In what follows we will consider the system for time instances $t, s \geq T$, where time T is advanced enough, so that the statistical equilibrium state has been reached. In this state the system no longer depends on the initial condition. In this connection we are going to ignore the initial condition (1b). Through the development of the theory four special cases of monostable non-linear half oscillators (Equ.3a) will be considered, i.e.

1. A cubic non-linear half oscillator under Gaussian excitation (the non-linear/Gaussian Case)

$$\dot{x}(t;\theta) = -x(t;\theta) - x^3(t;\theta) + y(t;\theta). \tag{6}$$

2. A linear half oscillator under cubic Gaussian excitation (the linear/non-Gaussian Case)

$$\dot{x}(t;\theta) = -x(t;\theta) + y^3(t;\theta). \tag{7}$$

3. A cubic non-oscillator under a superposition of a Gaussian and a cubic Gaussian excitation (the non-linear/non-Gaussian Case)

$$\dot{x}(t;\theta) = -x(t;\theta) - x^3(t;\theta) + y(t;\theta) + 0.2 \cdot y^3(t;\theta). \tag{8}$$

A linear half oscillator under Gaussian excitation will also be considered for comparison purposes (the linear/Gaussian Case)

$$\dot{x}(t;\theta) = -x(t;\theta) + y(t;\theta). \tag{9}$$

Moreover, the non-linear bi-stable half oscillator under Gaussian excitation (bi-stable/Gaussian) will also be discussed in Chapter 5

$$\dot{x}(t;\theta) = x(t;\theta) - 0.4 \cdot x^3(t;\theta) + y(t;\theta), \tag{10}$$

The considered excitation functions $y(t;\theta)$ follow three different correlation structures, either lpGF, given by Equ.(1)_Sec(3.2.4), sOU given by Equ.(18)_Sec(3.2.4), or OU, given by Equ.(19)_Sec(3.2.4) and three different correlation times (colors), i.e. $\tau_{yy}^{corr} = 0.93, 0.51, 0.33$ (the same as in the linear case studied in Chapter 3). In all cases the correlation time of the excitation is of the same order of magnitude with the **relaxation time** of the non-linear half oscillator, i.e. the characteristic time for the system to reach its equilibrium position after being perturbed. The latter is smaller than that of the linear oscillator, for which $\tau_{\text{relax}}^{(\text{lin})} = 1$, since the nonlinearity contributes to the damping term.

In **Fig.(1a)** we can see the trajectory of the system of the non-linear/non-Gaussian Case (see Equ.(8)) in the RE phase space. The considered excitation is a strongly colored lpGF ($\tau_{yy}^{corr} = 0.93$). The time evolution of the response and the excitation along with their cross-covariance function $C_{xy}(t,s)$, are shown in **Fig.(1b)** and **Fig.(1c)**. The RE cross-covariance has been computed by the Monte Carlo simulation data up to the current time (i.e. 250 sec), which is long enough so that the system has reached the long-time statistical equilibrium state and it is plotted against the time lag $\tau = t - s$. Negative time lag ($\tau < 0$) corresponds to future lag values (excitation in advance of response). As already discussed in the linear case, and in contrast to cases of delta-correlated excitation, there is a correlation between the current response value and the future excitation. These results were obtained by Z.G. Kapelonis, from Monte Carlo simulation of the RDE given by Equ.(8).

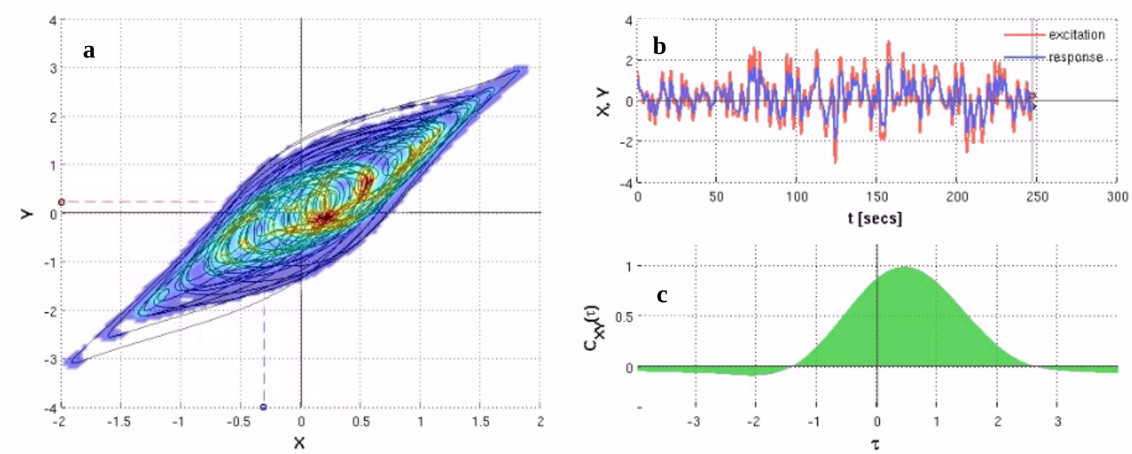

Figure 1: Results obtained by MC simulation. **a.** The trajectory of the half oscillator in the RE-phase space. **b.** The time evolution of the response and the excitation. **c.** The RE cross covariance $C_{xy}(t,s)$

We shall now proceed to the description of the RE method for the probabilistic characterization of the considered non-linear half oscillators in the long-time statistical equilibrium state.

4.3. The REPDF evolution equation in the long-time

Using the procedure developed by Athanassoulis & Sapsis (Athanassoulis & Sapsis 2006) and Sapsis & Athanassoulis (Sapsis & Athanassoulis 2008), that is also explained in Chapter 2 of this thesis, a partial (response) time, evolution equation is obtained for the joint REPDF $f_{x(t)y(s)}(\alpha, \beta)$, which is valid in the limit $s \to t$. More precisely, for the RDE given by Equ.(1)_Sec(4.2), the REPDF evolution equation takes the form:

$$\boxed{\frac{\partial}{\partial t} f_{x(t)y(s)}(\alpha,\beta)\Big|_{s\to t} + \frac{\partial}{\partial\alpha}\Big[\left(H(\alpha)+\Psi(\beta)\right)\cdot f_{x(t)y(s)}(\alpha,\beta)\Big|_{s\to t}\Big] = 0\,,\forall(\alpha,\beta)\in\mathbb{R}^2}. \quad (1)$$

Since the REPDF evolution equation is valid for all t, it is, thus, valid when t is considered in the long-time regime. In this case the system is no longer affected by the initial condition, therefore no initial conditions need to be considered for the sought for joint REPDF. Nevertheless, the REPDF evolution equation is supplemented by the marginal-compatibility constraint,

$$\int_{\alpha\in\mathbb{R}} f_{x(t)y(s)}(\alpha,\beta)d\alpha = f_{y(s)}(\beta) = \text{ a known pdf}, \qquad \forall\ \alpha\in\mathbb{R},\ \beta\in\mathbb{R}, \quad (2a)$$

ensuring that $f_{x(t)y(s)}(\alpha,\beta)$ has the appropriate (given) marginal, as well as by the constitutive conditions:

$$f_{x(t)y(s)}(\alpha,\beta)\geq 0\,, \quad (2b)$$

$$\int_{\beta\in\mathbb{R}}\int_{\alpha\in\mathbb{R}} f_{x(t)y(s)}(\alpha,\beta)d\beta\, d\alpha = 1. \quad (2c)$$

Equ.(1) is of a very peculiar type and has, as analytically shown in Chapter 3 for the linear case, multiple solutions. More precisely, Equ.(1) contains two times, the excitation time s and the response time t and it is crucial that the time derivative is considered only with respect to the response time t (half-time derivative). The distinction is essential since:

$$\frac{\partial}{\partial t} f_{x(t)y(s)}(\alpha,\beta)\Big|_{s\to t} \neq \frac{\partial}{\partial t} f_{x(t)y(t)}(\alpha,\beta)$$

and especially in the considered long-time regime:

$$\lim_{t\to\infty}\frac{\partial}{\partial t} f_{x(t)y(t)}(\alpha,\beta) = 0\,, \quad \text{while} \quad \lim_{t\to\infty}\frac{\partial}{\partial t} f_{x(t)y(s)}(\alpha,\beta)\Big|_{s\to t} \neq 0\,.$$

Notice that, the presence of the half-time derivative is a significant difference between the steady state REPDF-evolution equation and steady state FPK equation. The latter does not contain a time derivative, being stationary (Soize 1994; Risken 1996; Wang et al. 2000).

However, the half-time derivative $\partial f_{x(t)y(s)}(\alpha,\beta)/\partial t$ cannot be properly evaluated (implemented), without knowing the specific form of $f_{x(t)y(s)}(\alpha,\beta)$, since there is no way to separate the effect of the response time t from the effect of the excitation time s in the unknown function $f_{x(t)y(s)}(\alpha,\beta)$. The essence of this difficulty stems from the fact that, behind of the half-time differentiation, some non-local in time effects are hidden. Accordingly, the term $\lim_{s \to t} \partial f_{x(t)y(s)}(\alpha, \beta) / \partial t$, should somehow be approximated and introduced in Equ.(1), before any attempt to formulating a numerical scheme for solving. This will be taken on in the next section, where a local (in state variables α, β) approximation of $\lim_{s \to t} \partial f_{x(t)y(s)}(\alpha, \beta) / \partial t$ will be constructed, by formulating and solving a localized (linear/Gaussian) differential equation at each α, β.

4.4. A priori closure conditions: Local linear equations with local Gaussian excitation

4.4.1. Formulation and solution of the localized problem

In case that the oscillator is linear and the excitation Gaussian the RE correlation structure can be fully determined by the solution of the two-time, linear, RE moment equations (Chapter 3). Nevertheless, in case that the oscillator is non-linear and/or the excitation non-Gaussian, the correlation structure of the REPDF needs also higher order moments to be defined. In this section we present how this complicacy can be overcome by approximating the non-linear and/or non-Gaussian random problem by many locally linear, locally Gaussian sub-problems. To this end we define and solve localized linear problems, providing information concerning the local RE-correlation structure, in the long-time statistical equilibrium state. These problems are going to be used as an *a priori* closure scheme for the REPDF evolution equation (Equ.(1)_Sec.(4.3)).

Focusing on a (any) specific point y_0 of the excitation state space, we find the corresponding (deterministic) long-time equilibrium point x_0 in the response state space of the RDE given by Equ.(1a)_Sec(4.2), by solving the equation

$$\dot{x}(t;\theta) = H\big(x(t;\theta)\big) + \Psi\big(y(t;\theta)\big) = 0. \tag{1}$$

The curve $H(x_0) = -\Psi(y_0)$ will be called (x_0, y_0) - or RE-equilibrium curve. For instance, for the cubic half-oscillator described by Equ.(3a)_Sec(4.2), the **RE-equilibrium curve** is given by the equation $\mu_1 \cdot x_0 + \mu_3 \cdot x_0^3 + \kappa_1 \cdot y_0 + \kappa_3 \cdot y_0^3 = 0$. In agreement with results obtained by extensive MC simulations for various cases, the RE-equilibrium curve is around where the joint REPDF is concentrated when the system reaches the long-time statistical equilibrium state. For example, in **Fig.2a and Fig.2b** the equilibrium curves $x_0 + x_0^3 = y_0$ of the non-linear/Gaussian Case (see Equ.(6)_Sec(4.2)) and $x_0 + x_0^3 = y_0 + 0.2 \cdot y_0^3$ of the non-linear/non-Gaussian Case (see Equ.(8)_Sec(4.2)) are, respectively, plotted.

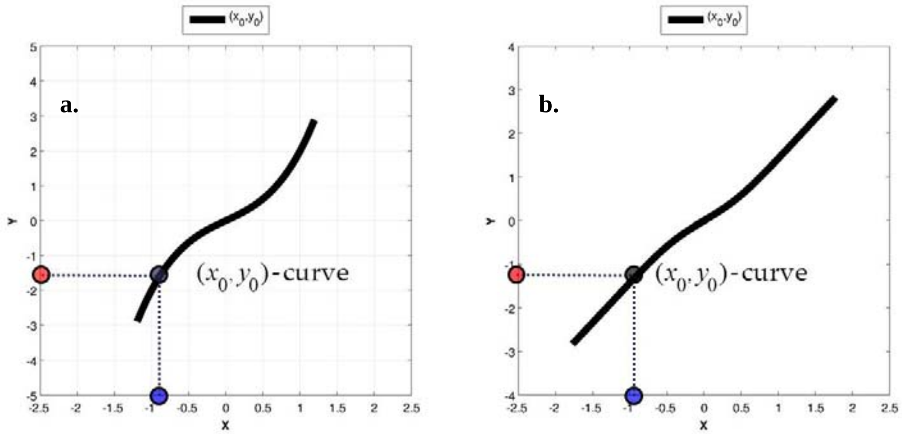

Figure 2: The (x_0, y_0) - or RE-equilibrium curve plotted in the RE-phase space: **a.** For the non-linear/Gaussian case (Equ.(6)_Sec(4.2)), **b.** for the non-linear/non-Gaussian case (Equ.(8)_Sec(4.2)).

To find the localized sub-problems, we introduce a linear approximation of the right-hand side of Equ.(1a)_Sec(4.2) around each point (x_0, y_0), i.e.

$$\dot{x}(t;\theta) \approx H'(x_0)\cdot\big(x(t;\theta)-x_0\big)+\Psi'(y_0)\cdot\big(y(t;\theta)-y_0\big), \tag{2}$$

and formulate the following ***localized version*** of Equ.(1a)_Sec(4.2):

$$\dot{x}_{\ell oc}(t;\theta) = H'(x_0)\cdot\Big(x_{\ell oc}(t;\theta)-x_0\Big)+\Psi'(y_0)\cdot\big(y_{\ell oc}(t;\theta)-y_0\big). \tag{3}$$

The subscript ($_{\ell oc}$) has been introduced in order to remind us that Equ.(3) is just a ***localized version*** of Equ.(1a)_Sec(4.2). The localized random excitation, $y_{\ell oc}(t;\theta)-y_0$, is taken Gaussian (as the global one), with zero mean value and appropriate two-time response auto-covariance function $C_{y_{\ell oc}y_{\ell oc}}(t,s)$. Since we are interested in the long-time, steady-state solution, we can choose $C_{y_{\ell oc}y_{\ell oc}}(t,s)=C^{(\infty)}_{y_{\ell oc}y_{\ell oc}}(t-s)$. Compatible with the localization is a scaled version of the long-time limit of the global autocovariance function $C^{(\infty)}_{yy}(t-s)$, that is $C^{(\infty)}_{y_{\ell oc}y_{\ell oc}}(t-s) = \sigma^2_{y_{\ell oc}}\,C^{(\infty)}_{yy}(t-s)/\sigma^2_y$. The solution of Equ.(3) will provide us with the long-time local correlation structure between $x(t;\theta)$ and $y(s;\theta)$ in the vicinity of $(\alpha,\beta)=(x_0,y_0)$, i.e. the elements of the long-time covariance matrix

$$\Sigma^{(\infty)}_{\ell oc}(t;s) = \begin{pmatrix} C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(t-s) & C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s) \\ C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s) & C^{(\infty)}_{y_{\ell oc}y_{\ell oc}}(t-s) \end{pmatrix} \tag{4}$$

where

$$C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)=\lim_{t\to\infty,\ s\to\infty} C_{x_{\ell oc}y_{\ell oc}}(t,s), \qquad C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(t-s)=\lim_{t\to\infty,\ s\to\infty} C_{x_{\ell oc}x_{\ell oc}}(t,s)$$

Let it be noted that the correlation time is not affected by the choice of the scaling factor of the localized excitation, i.e., will have the same correlation time (see Equ.(1c)_Sec(3.2.4)) with the global one.

Since the localised stochastic problems (Equs.(3)) are linear with Gaussian excitation, they can be solved analytically, following the procedure discussed in Sections 2 and 4 of Chapter 3. More precisely, taking mean values to Equ.(3) it is straightforward to find that in the long-time statistical equilibrium state the response mean value $m_{x_{\ell oc}}^{(\infty)} = x_0$, a fact that is compatible with the considered localization. Subsequently, multiplying Equ.(3) first by $\left(y_{loc}(s;\theta) - y_0\right)$ then with $\left(x_{loc}(s;\theta) - x_0\right)$ we get

$$\begin{aligned}\dot{x}_{\ell oc}(t;\theta)\cdot\left(y_{loc}(s;\theta)-y_0\right) = {} & H'(x_0)\cdot\left(x_{\ell oc}(t;\theta)-x_0\right)\cdot\left(y_{loc}(s;\theta)-y_0\right) + \\ & + \Psi'(y_0)\cdot\left(y_{loc}(t;\theta)-y_0\right)\cdot\left(y_{loc}(s;\theta)-y_0\right)\end{aligned} \tag{5}$$

and

$$\begin{aligned}\dot{x}_{\ell oc}(t;\theta)\cdot\left(x_{loc}(s;\theta)-x_0\right) = {} & H'(x_0)\cdot\left(x_{\ell oc}(t;\theta)-x_0\right)\cdot\left(x_{loc}(s;\theta)-x_0\right) + \\ & + \Psi'(y_0)\cdot\left(y(t;\theta)-y_0\right)\cdot\left(x_{loc}(s;\theta)-x_0\right).\end{aligned} \tag{6}$$

Applying the mean value operator to Equs.(5,6) we obtain the local, two-time, RE moment equations. Consider now that both $t,\, s \to \infty$, and write the moment equations in terms of the elements of the covariance matrix given by Equ.(4), i.e.

$$\frac{\partial C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)}{\partial t} = H'(x_0)\cdot C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s) + \Psi'(y_0)\cdot C^{(\infty)}_{y_{\ell oc}y_{\ell oc}}(t-s)\,, \tag{7}$$

$$\frac{\partial C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(t-s)}{\partial t} = H'(x_0)\cdot C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(t-s) + \Psi'(y_0)\cdot C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(s-t)\,. \tag{8}$$

From Equs.(7,8) it is deduced that the local correlation matrix (Equ.4) is scaled uniformly by $\sigma^2_{y_{\ell oc}} / \sigma^2_y$. Moreover, Equs.(7,8) are, respectively, of the same type as Equ.(8a,12a)_Sec (3.2.1) whose long-time solution is given by Equs.(7,8b)_Sec.(3.2.3). Applying them to the localized moment problem (Equs.(7,8)) we obtain

$$C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s) = \lim_{\substack{t\to\infty \\ s\to\infty \\ t-s=\tau}} \left[\Psi'(y_0)e^{H'(x_0)\cdot t}\int_{t_0}^{t} C^{(\infty)}_{y_{\ell oc}y_{\ell oc}}(s_1-s)\cdot e^{-H'(x_0)\cdot s_1}ds_1\right], \tag{9}$$

and

$$C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(t-s) = \lim_{\substack{t\to\infty \\ s\to\infty \\ t-s=\tau}} \left[e^{H'(x_0)\cdot(t+s)}\left(\Psi'(y_0)\right)^2\int_{t_0}^{s}\int_{t_0}^{t} C^{(\infty)}_{y_{\ell oc}y_{\ell oc}}(s_1-s_2)\cdot e^{-H'(x_0)\cdot(s_1+s_2)}ds_1\,ds_2\right]. \tag{10}$$

Alternatively the localized moment problem could be directly solved in the lag-time domain, taking into consideration that

$$\frac{\partial C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(t-s)}{\partial t} \underset{t-s=\tau}{=} \frac{d\, C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(\tau)}{d\tau}, \qquad \frac{\partial C^{(\infty)}_{xx}(t,s)}{\partial t} \underset{t-s=\tau}{=} \frac{dC^{(\infty)}_{xx}(\tau)}{d\tau} \quad (11a,11b)$$

and subsequently write Equs.(7,8) in the form

$$\frac{dC^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(\tau)}{d\tau} = H'(x_0)\cdot C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(\tau) + \Psi'(y_0)\cdot C^{(\infty)}_{y_{\ell oc}\, y_{\ell oc}}(\tau) \quad (12)$$

and

$$\frac{dC^{(\infty)}_{x_{\ell oc}\, x_{\ell oc}}(\tau)}{dt} = H'(x_0)\cdot C^{(\infty)}_{x_{\ell oc}\, x_{\ell oc}}(\tau) + \Psi'(y_0)\cdot C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(-\tau). \quad (13)$$

These equations are of the same type as Equs.(14,15)_Sec(5.3.1), the latter are solved analytically and their solution is given by Equs.(19,23)_Sec(5.3.1) respectively. Applying these to Equs.(12,13) we find

$$C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(\tau) = \lim_{\substack{t\to\infty \\ \tau=\text{const.}}} C_{x_{\ell oc}\, y_{\ell oc}}(t, t-\tau) = \Psi'(y_0)\cdot \int_{-\tau}^{\infty} e^{H'(x_0)\cdot(u+\tau)}\cdot C_{y_{\ell oc}\, y_{\ell oc}}(u)\, du \quad (14)$$

$$C^{(\infty)}_{x_{\ell oc}\, x_{\ell oc}}(\tau) = \frac{\left(\Psi'(y_0)\right)^2}{2\cdot\left(-H'(x_0)\right)} \times \int_{v=-\infty}^{v=+\infty} C_{y_{\ell oc}\, y_{\ell oc}}(v)\cdot e^{H'(x_0)\cdot|v-\tau|}\, dv \quad (15)$$

or equivalently for $\tau = t-s$

$$\boxed{C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(t-s) = \lim_{\substack{t\to\infty \\ t-s=\text{const.}}} C_{x_{\ell oc}\, y_{\ell oc}}(t,s) = \Psi'(y_0)\cdot \int_{-t+s}^{\infty} e^{H'(x_0)\cdot(u+t-s)}\cdot C^{(\infty)}_{y_{\ell oc}\, y_{\ell oc}}(u)\, du} \quad (16)$$

$$\boxed{C^{(\infty)}_{x_{\ell oc}\, x_{\ell oc}}(t-s) = \frac{\left(\Psi'(y_0)\right)^2}{2\cdot\left(-H'(x_0)\right)} \times \int_{v=-\infty}^{v=+\infty} C^{(\infty)}_{y_{\ell oc}\, y_{\ell oc}}(v)\cdot e^{H'(x_0)\cdot|v-t+s|}\, dv} \quad (17)$$

In Athanassoulis, Tsantili & Kapelonis(Athanassoulis et al. 2013), it is proved that the two approaches coincide, that is the long-time limits described by Equs(9,10) are exactly Equs.(16,17).

The half-time derivative of the local RE cross-covariance

Combining Equs.(7,16) we calculate the half-time derivative of $C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(t-s)$ with respect to the excitation time t,

$$\frac{\partial C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(t-s)}{\partial t} = H'(x_0)\cdot \Psi'(y_0) \int_{-t+s}^{\infty} e^{H'(x_0)\cdot(u+t-s)} \cdot C_{y_{\ell oc}\, y_{\ell oc}}(u)\, du + \Psi'(y_0)\cdot C^{(\infty)}_{y_{\ell oc}\, y_{\ell oc}}(t-s)\,, \tag{18}$$

as well as its limit as $s \to t$,

$$\boxed{\begin{aligned} \left.\frac{\partial C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(t-s)}{\partial t}\right|_{s\to t} &= H'(x_0)\cdot C^{(\infty)}_{y_{\ell oc}\, x_{\ell oc}}(0) + \Psi'(y_0)\cdot C^{(\infty)}_{y_{\ell oc}\, y_{\ell oc}}(0) = \\ &= H'(x_0)\cdot\left(\Psi'(y_0)\cdot \int_{0}^{\infty} e^{H'(x_0)\cdot u} \cdot C_{y_{\ell oc}\, y_{\ell oc}}(u)\, du\right) + \Psi'(y_0)\cdot C^{(\infty)}_{y_{\ell oc}\, y_{\ell oc}}(0) \end{aligned}}\,. \tag{19}$$

The above result will be used, in conjunction with an appropriate representation for the joint REPDF, to implement the correlation structure of the sought for joint REPDF, especially the long-time limit $\lim_{s\to t} \partial f_{x(t)y(s)}(\alpha,\beta)/\partial t$. Clearly, the obtained results can apply to any choice of the functions $H(\bullet)$, $\Psi(\bullet)$ (see Section 4.2 for their definition). For instance, setting $H'(x_0) = \mu_1 + 3\cdot\mu_3\, x_0^2$, $\Psi'(y_0) = \kappa_1 + 3\cdot\kappa_3\, y_0^2$, Equs.(18,19) specialize for the cubic half oscillator given by Equ.(3a)_Sec(4.2). We shall now see how these formulae further particularize in case that the correlation structure of the random excitation is a lpGF or an sOU process (see Equ.(1)_Sec(3.2.4.a) and Equ.(18)_Sec(3.2.4.b) for the definition of the two processes).

4.4.2. The case of lpGF excitation

Since the moment Equs.(7,8)_Sec(4.4.1) are linear we can use the analytic calculations performed in Sec.(3.2.4) for the linear RDF (Equ.(1)_Sec(3.1.1)) under lpGF input to calculate the local RE cross-covariance $C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(t-s)$ and the local response auto-covariance $C^{(\infty)}_{x_{\ell oc}\, x_{\ell oc}}(t-s)$ of the non-linear RDE Equ.(3a)_Sec(4.2) for the same input. More precisely, assuming that the local input is a lpGF random process, i.e.

$$C_{y_{\ell oc}\, y_{\ell oc}}(t-s) = (\sigma^2_{y_{\ell oc}} / \sigma^2_y)\cdot C^{(\infty)}_{yy}(t-s) = \sigma^2_{y_{\ell oc}} \cdot \exp\left(-a\cdot(t-s)^2\right), \tag{1}$$

and setting $A = H'(x_0)$ and $B = \Psi'(y_0)$ to Equs.(11,15)_Sec(3.2.4), we obtain the following formulae for that the local RE cross-covariance and the local auto-covariance of the local problem:

$$C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(t-s) = \frac{\sqrt{\pi}}{2\sqrt{a}} \cdot \sigma^2_{y_{\ell oc}} \cdot \Psi'(y_0) \cdot e^{\frac{(H'(x_0))^2}{4\cdot a}} \times \times e^{H'(x_0)\cdot(t-s)} \cdot \left\{ \operatorname{erf}\left(\sqrt{a}\cdot(t-s) + \frac{H'(x_0)}{2\cdot\sqrt{a}} \right) + 1 \right\}, \tag{2}$$

$$C^{(\infty)}_{x_{\ell oc}\, x_{\ell oc}}(t-s) = \frac{\sqrt{\pi}\cdot\left(\Psi'(y_0)\right)^2 \cdot \sigma^2_{y_{\ell oc}}}{4\sqrt{a}\cdot\left(-H'(x_0)\right)} \cdot e^{\frac{H'(x_0)^2}{4a}} \times \times \left\{ e^{H'(x_0)\cdot(s-t)} \left[\operatorname{erf}\left(\sqrt{a}\cdot(s-t) + \frac{H'(x_0)}{2\cdot\sqrt{a}} \right) + 1 \right] + + e^{H'(x_0)\cdot(t-s)} \left[\operatorname{erf}\left(\sqrt{a}\cdot(t-s) + \frac{H'(x_0)}{2\cdot\sqrt{a}} \right) + 1 \right] \right\}. \tag{3}$$

Taking the limit $s \to t$, Equs.(2-3) reduce to the corresponding local covariances on the diagonal $s = t$

$$C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(0) = \frac{\sqrt{\pi}}{2\sqrt{a}} \cdot \sigma^2_{y_{\ell oc}} \cdot \Psi'(y_0) \cdot \mathrm{e}^{\frac{H'(x_0)^2}{4\cdot a}} \cdot \left[\operatorname{erf}\left(\frac{H'(x_0)}{2\cdot\sqrt{a}} \right) + 1 \right], \tag{4}$$

$$C^{(\infty)}_{x_{\ell oc}\, x_{\ell oc}}(0) = \frac{\sqrt{\pi}\cdot\left(\Psi'(y_0)\right)^2 \cdot \sigma^2_{y_{\ell oc}}}{2\sqrt{a}\cdot\left(-H'(x_0)\right)} \cdot e^{\frac{H'(x_0)^2}{4a}} \times \left[\operatorname{erf}\left(\sqrt{a}\cdot + \frac{H'(x_0)}{2\cdot\sqrt{a}} \right) + 1 \right]. \tag{5}$$

Now, using Equs.(19)_Sec(4.4.1) and Equs.(1,4), we can explicitly compute the half-time derivative $\lim_{s\to t} \partial C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(t-s) \big/ \partial t$, when the excitation is a lpGF:

$$\left. \frac{\partial C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(t-s)}{\partial t} \right|_{s\to t} = H'(x_0) \cdot \frac{\sqrt{\pi}}{2\sqrt{a}} \cdot \sigma^2_{y_{\ell oc}} \cdot \Psi'(y_0) \cdot \mathrm{e}^{\frac{H'(x_0)^2}{4\cdot a}} \cdot \left[\operatorname{erf}\left(\frac{H'(x_0)}{2\cdot\sqrt{a}} \right) + 1 \right] + \Psi'(y_0) \cdot \sigma^2_{y_{\ell oc}}. \tag{6}$$

In **Figs.3-5** the local correlation characteristics $C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(0)$, $C^{(\infty)}_{x_{\ell oc}\, x_{\ell oc}}(0)$ and $\lim_{s\to t} \partial C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(t-s) \big/ \partial t$ are plotted against the "centers" y_0 for the four half-oscillators considered in Section 4.2 (Equs.(6-9)_Sec(4.2)), and for 3 values of correlation time, $\tau^{\mathrm{corr}}_{yy} = 0.93, 0.51, 0.33$, of the lpGF input process. For the local input variance we have assumed that $\sigma^2_{y_{\ell oc}} = 0.1$. As we can see, and as expected, in the case that a linear half oscillator is considered (Case a in Figs.3-5), for the same input correlation time, all the three quantities have the same value at any "center y_0" in the excitation space. In the linear/Non-

Gaussian case (Case b in Figs.3-5) all three quantities exhibit local characteristics following the pattern of $\Psi'(y_0) = \kappa_1 + 3 \cdot y_0^2$ in the right hand side of Equs.(4-6). More precisely, the three quantities have a minimum at $y_0 = 0$ as does $\Psi'(y_0)$ (the minimum value is zero in this case (Equ.(7)_Sec(4.2) since $\kappa_1 = 0$) . On the contrary, in the non-linear/Gaussian case (Case c in Figs.3-5) a maximum is developed at $y_0 = 0$, due to the varying values of $H'(x_0) = -1 - 0.6 \cdot x_0^2$ in Equs.(4-6). Let it be noted that the excitation "centers" y_0 form a monoton (increasing) function of x_0 on the RE curve (Fig.2a).

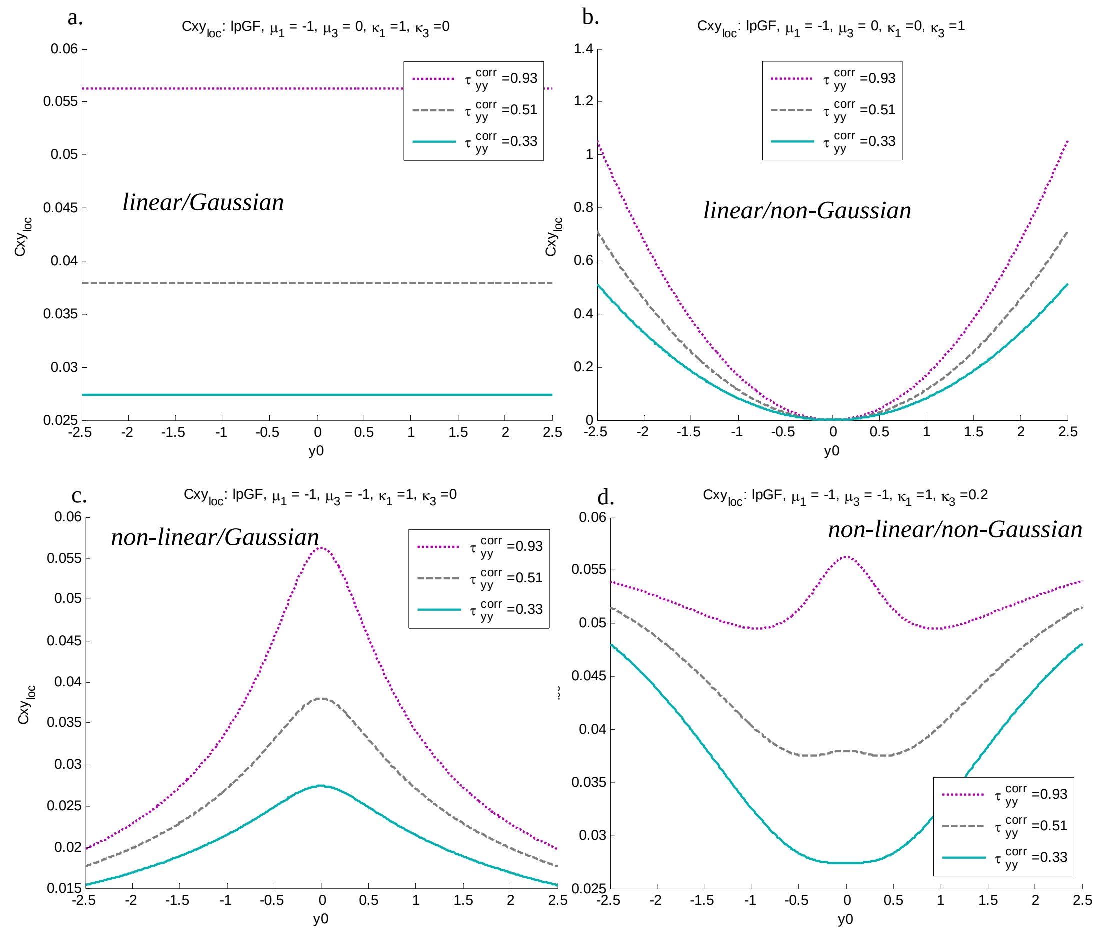

Figure 3: The long time response-excitation covariance $C^{(\infty)}_{x_{\ell oc} y_{\ell oc}}(0)$ for different values of correlation time of the lpGF stochastic excitation against the excitation state space for: **a.** the linear/ Gaussian case **b.** the Linear/non-Gaussian Case. **c.** the non-linear/Gaussian case. **d.** the non-linear/non-Gaussian case.

In the non-linear/non-Gaussian case (Case d in Figs.3-5) the above effects are combined, depending on the correlation time of the excitation. For instance, in Fig.3d we can see that the RE cross-covariance in the least correlated case, $\tau_{yy}^{\text{corr}} = 0.33$, is minimized at $y_0 = 0$, similar to the linear/non-Gaussian case. On the contrary, in the most correlated case, $\tau_{yy}^{\text{corr}} = 0.93$, the non-linearity prevails over the non-Gaussianity resulting in the creation of a local maximum at $y_0 = 0$. In all examined cases in Figs.3-5, the local (co)variances $C^{(\infty)}_{x_{\ell oc} y_{\ell oc}}(0)$, $C^{(\infty)}_{x_{\ell oc} x_{\ell oc}}(0)$, increase with the correlation time of the lpGF input, as expected.

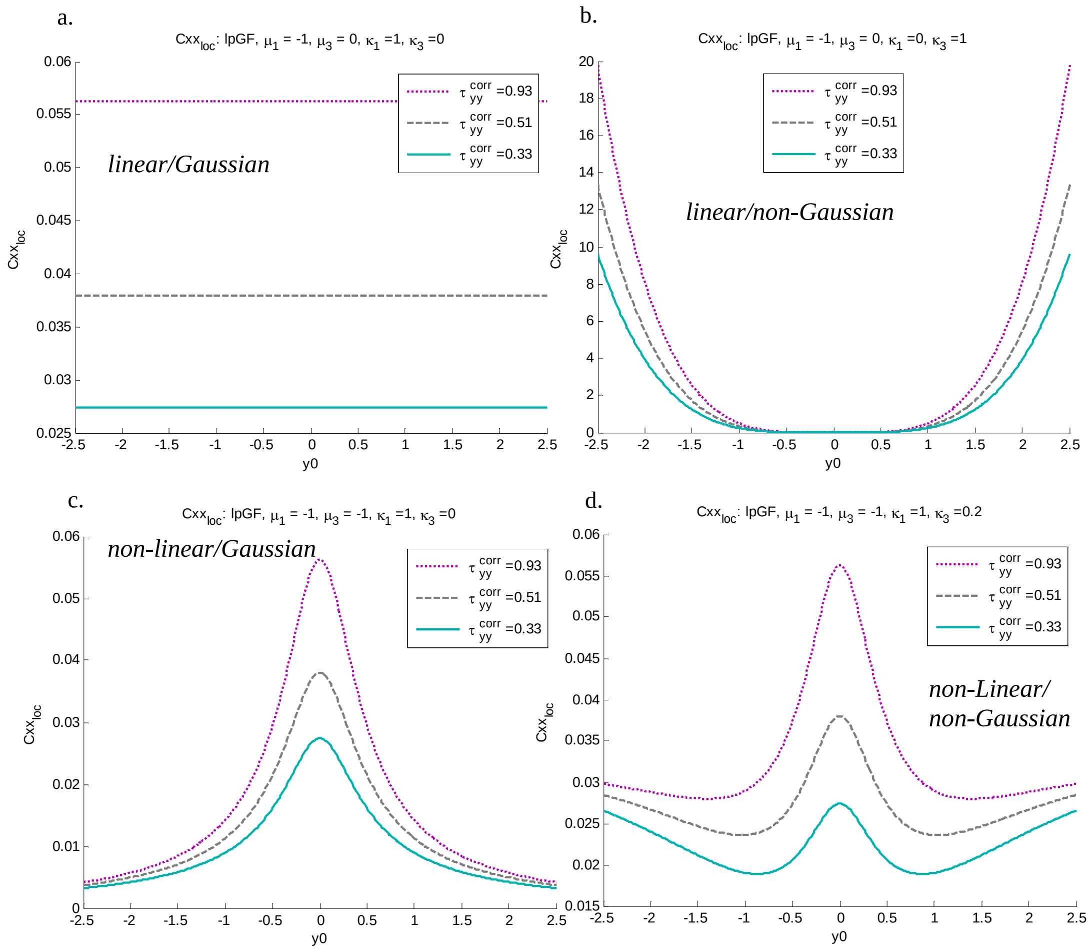

Figure 4: The same as in Fig.3 for $C^{(\infty)}_{x_{\ell oc} x_{\ell oc}}(0)$.

On the contrary, the half-time derivative $\partial C^{(\infty)}_{x_{\ell oc} y_{\ell oc}}(t-s)/\partial t\big|_{s\to t}$, Fig.5, decreases with the correlation time in the considered cases of non-linear half oscilators. The latter is due to the fact that, while $\partial C^{(\infty)}_{x_{\ell oc} y_{\ell oc}}(t-s)/\partial t\big|_{s\to t}$ is an increasing function of $C^{(\infty)}_{y_{\ell oc} y_{\ell oc}}(0)$, it is also a decreasing function of $C^{(\infty)}_{x_{\ell oc} y_{\ell oc}}(0)$ and a linear combination of the two (co)variances (see Equ.(19)_Sec(4.4.1)).

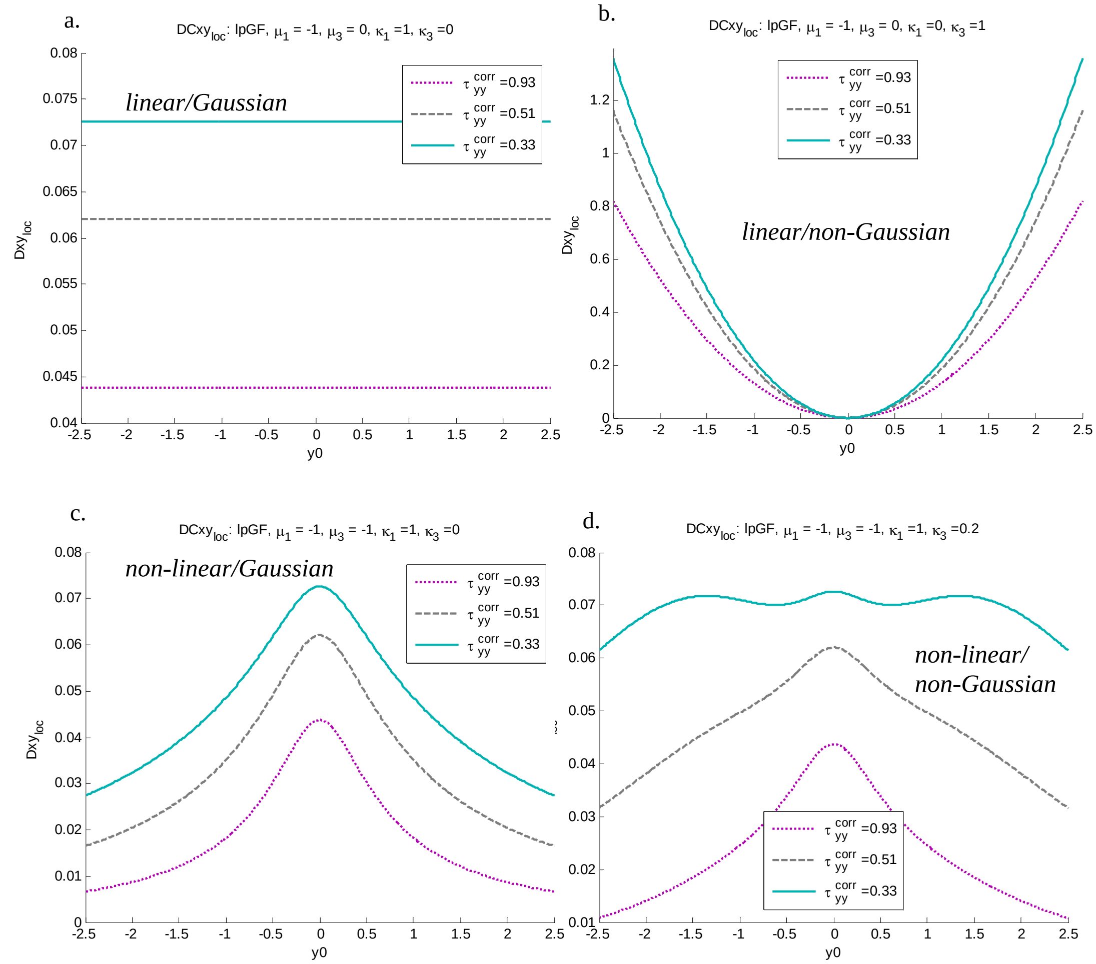

Figure 5: The same as in Fig.3 for $\partial C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(t-s)/\partial t\Big|_{s\to t}$

4.4.3. The case of sOU and OU excitation

Similarly, we shall now use the results obtained in Section 3.2.4 to calculate the local RE cross-covariance, $C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(t-s)$, and local response variance, $C^{(\infty)}_{x_{\ell oc}\, x_{\ell oc}}(t-s)$, of the non-linear RDE (Equ.(3a)_Sec(4.2)) when the input is an sOU process, i.e.

$$C^{(\infty)}_{y_{\ell oc}\, y_{\ell oc}}(t-s) = (\sigma^2_{y_{\ell oc}}/\sigma^2_{y})\cdot C^{(\infty)}_{yy}(t-s) = \sigma^2_{y_{\ell oc}}\cdot \exp\left(-a\cdot|t-s|\right)\cdot \cos\left(\omega_0\cdot(t-s)\right) \qquad (1)$$

For instance, setting $A = H'(x_0)$, $B = \Psi'(y_0)$ to Equs.(32,33,36)_Sec(3.2.4) we have:

$$C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)\Big|_{t\geq s} = \Psi'(y_0)\cdot\sigma^2_{y_{\ell oc}}\cdot\left[\left(\frac{a-H'(x_0)}{\left(a-H'(x_0)\right)^2+\omega_0^2}+\frac{H'(x_0)+a}{\left(H'(x_0)+a\right)^2+\omega_0^2}\right)\cdot e^{H'(x_0)\cdot(t-s)}+\right.$$
$$\left.+\frac{e^{-a\cdot(t-s)}}{\left(H'(x_0)+a\right)^2+\omega_0^2}\cdot\left(-\left(H'(x_0)+a\right)\cdot\cos\left(\omega_0\cdot(t-s)\right)+\omega_0\cdot\sin\left(\omega_0\cdot(t-s)\right)\right)\right]$$

$$C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)\Big|_{t<s} = \Psi'(y_0)\cdot\sigma^2_{y_{\ell oc}}\times$$
$$\times\frac{e^{a\cdot(t-s)}}{\left(a-H'(x_0)\right)^2+\omega_0^2}\cdot\left((a-H'(x_0))\cdot\cos\left(\omega_0\cdot(t-s)\right)+\omega_0\cdot\sin\left(\omega\cdot(t-s)\right)\right)$$

(2,3)

$$C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(t-s)=\frac{(\Psi'(y_0))^2\cdot\sigma^2_{y_{\ell oc}}\cdot e^{H'(x_0)\cdot t}}{\left(\left(a-H'(x_0)\right)^2+\omega_0^2\right)\cdot\left(\left(H'(x_0)+a\right)^2+\omega_0^2\right)}\cdot\left[-e^{H'(x_0)\cdot|t-s|}\frac{a\cdot\left(a^2-\left(H'(x_0)\right)^2+\omega_0^2\right)}{H'(x_0)}+\right.$$
$$\left.+e^{-a\cdot|t-s|}\cdot\left[\left(-a^2+\left(H'(x_0)\right)^2+\omega_0^2\right)\cdot\cos\left(\omega_0\cdot|t-s|\right)+2\cdot a\cdot\omega_0\cdot\sin\left(\omega\cdot|t-s|\right)\right]\right]$$

(4)

Then taking the limit $s\to t$ to Equs.(2-4), since the left and the right limit of $C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)$ (limits of Equs.(2-3), respectively) coincide, Equs.(2-4), reduce to:

$$C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(0)=\Psi'(y_0)\cdot\sigma^2_{y_{\ell oc}}\cdot\frac{\left(a-H'(x_0)\right)}{\left(a-H'(x_0)\right)^2+\omega_0^2}, \tag{5}$$

$$C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(0)=\frac{(\Psi'(y_0))^2\cdot\sigma^2_{y_{\ell oc}}}{\left(\left(a-H'(x_0)\right)^2+\omega_0^2\right)\cdot\left(\left(H'(x_0)+a\right)^2+\omega_0^2\right)}\times$$
$$\times\left[-\frac{a\cdot\left(a^2-H'(x_0)^2+\omega_0^2\right)}{H'(x_0)}-a^2+\left(H'(x_0)\right)^2+\omega_0^2\right]. \tag{6}$$

Subsequently combining Equs.(19)_Sec(4.4.1) and Equs.(5,6), we get that the half-time derivative is given by the formula

$$\left.\frac{\partial C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)}{\partial t}\right|_{s\to t} = \Psi'(y_0)\cdot\sigma^2_{y_{\ell oc}}\cdot\frac{a-H'(x_0)}{\left(a-H'(x_0)\right)^2+\omega_0^2}+\Psi'(y_0)\cdot\sigma^2_{y_{\ell oc}}. \tag{7}$$

Finally, setting $\omega_0=0$ in Equs.(2-7) we obtain the corresponding results for the case that the excitation is an OU process. We write the ones that correspond to the limit case $s\to t$, (i.e. setting $\omega_0=0$ to Equs.(5-7))

$$C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(0)=\frac{\Psi'(y_0)\cdot\sigma^2_{y_{\ell oc}}}{\left(a-H'(x_0)\right)},\qquad C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(0)=-\frac{(\Psi'(y_0))^2\cdot\sigma^2_{y_{\ell oc}}}{H'(x_0)\cdot\left(a-H'(x_0)\right)},$$

$$\left.\frac{\partial C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)}{\partial t}\right|_{s\to t}=\Psi'(y_0)\cdot\sigma^2_{y_{\ell oc}}\cdot\frac{a-H'(x_0)}{\left(a-H'(x_0)\right)^2}+\Psi'(y_0)\cdot\sigma^2_{y_{\ell oc}}. \qquad (8\text{-}10)$$

In **Figs.6-8**, the local correlation characteristics $C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(0)$, $C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(0)$, $\lim_{s\to t}\partial C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)\big/\partial t$ are plotted against the "centers" y_0, for the four cases of half-oscillators considered in section 4.2. (Equs.(6-9)_Sec.(4.2)), and three values of the correlation time ($\tau^{\text{corr}}_{yy}=0.93, 0.51, 0.33$) of the OU process. The local input variance is $\sigma^2_{y_{\ell oc}}=0.1$. All parameters have been selected to be identical with the ones discussed for lpFG input (Sec.(4.4.2)) for comparison reasons. In agreement with findings discussed previously, the local correlation characteristics develop a minimum at zero in the linear/non-Gaussian case (Case b, Figs.6-8), a maximum at zero in the non-linear/Gaussian (Case c, Figs.6-8). These effects are combined in the non-linear/non-Gaussian case following a similar pattern as in the case that the input is a lpGF stochastic process to both $C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(0)$, $C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(0)$, (Case d, Figs.6,7), this compatibility is lost in $\lim_{s\to t}\partial C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)\big/\partial t$ (Case d, Figs.8), especially for large absolute values of the excitation "centers" y_0. In general all the local correlation characteristics are significantly affected by the shape of the input function (lpGf vs OU). The local covariances $C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(0)$ and $C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(0)$ obtained under OU random input are always lower than the ones obtained under lpGF input and around $y_0=0$, this difference is as high as 18%. On the contrary, the local half-time derivatives $\lim_{s\to t}\partial C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)\big/\partial t$ obtained under OU random input have always higher values than the ones obtained by the lpGF excitation with these differences being as high as 400% (in the non-linear/non-Gaussian case) for large absolute values of the excitation "centers" y_0.

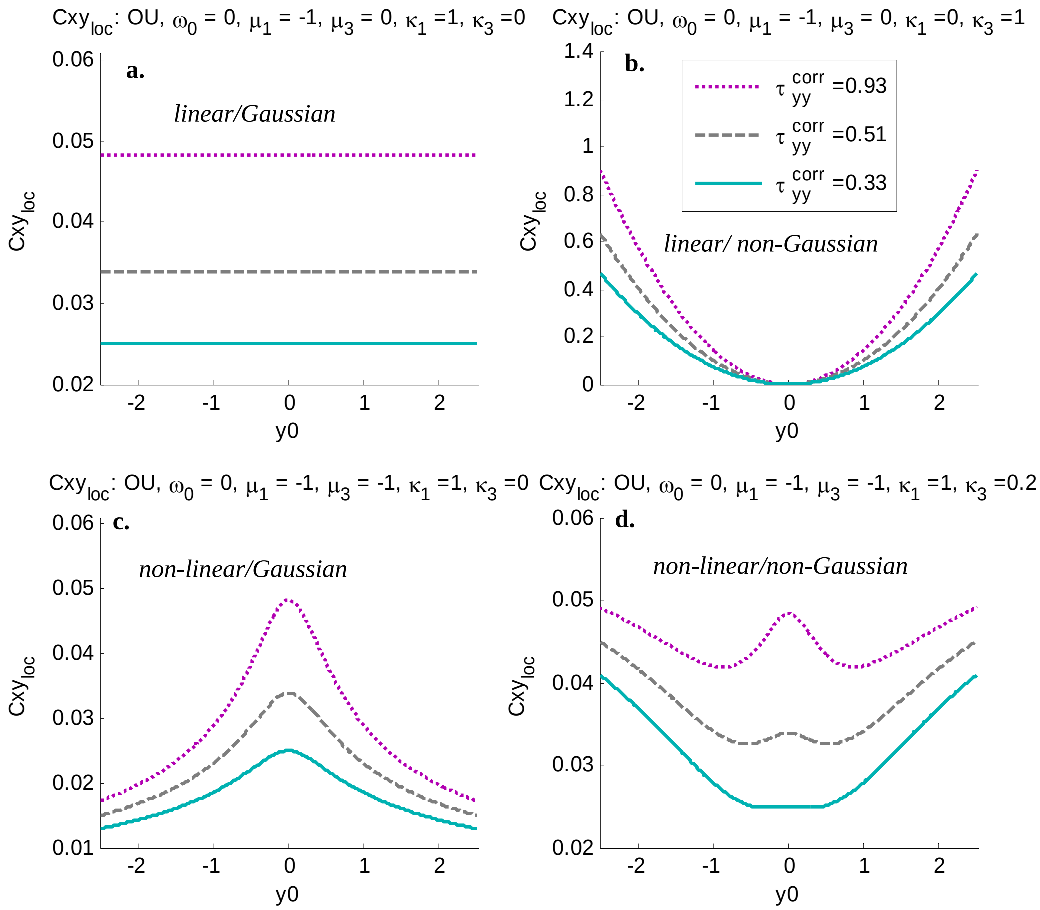

Figure 6: $C^{(\infty)}_{x_{loc}y_{loc}}(0)$ for different values of correlation time of the OU stochastic excitation against the excitation state space for: **a.** the linear/ Gaussian case **b.** the linear/non-Gaussian Case. **c.** the non-linear/Gaussian case. **d.** the non-linear/non-Gaussian case.

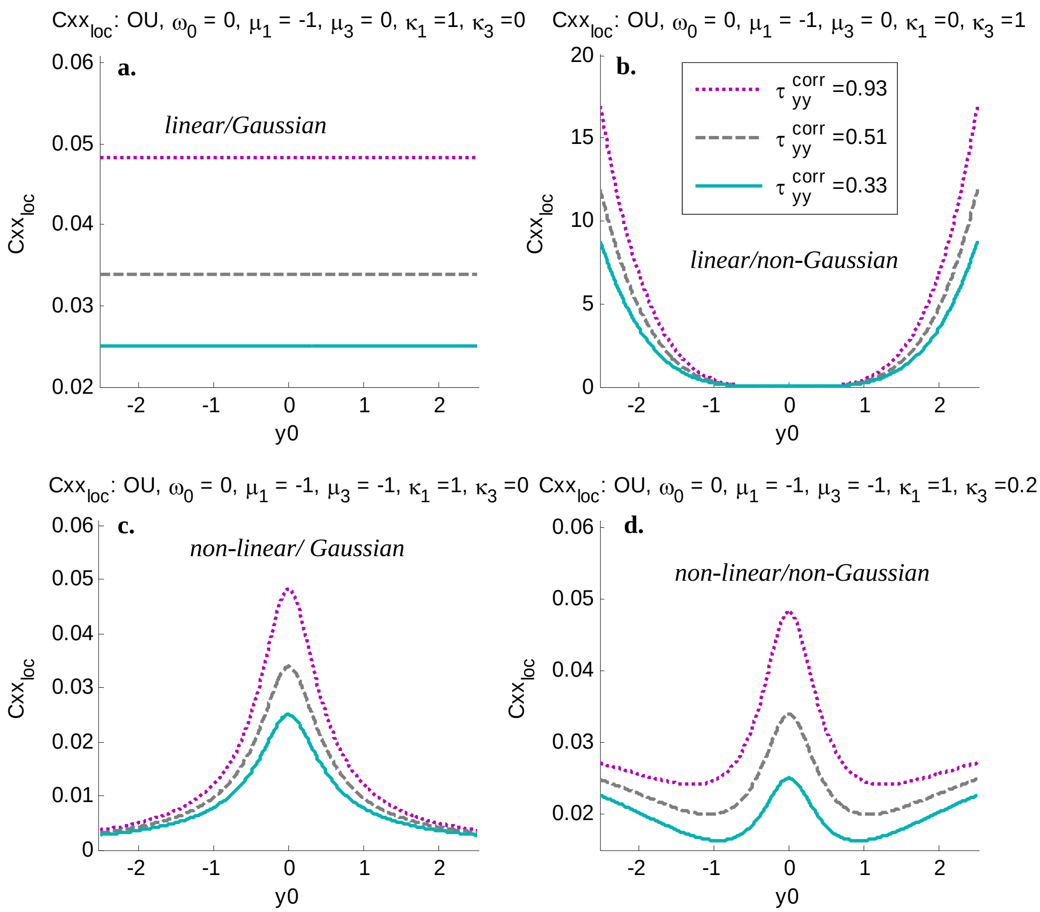

Figure 7: The same as in Fig.6 for $C^{(\infty)}_{x_{loc}x_{loc}}(0)$.

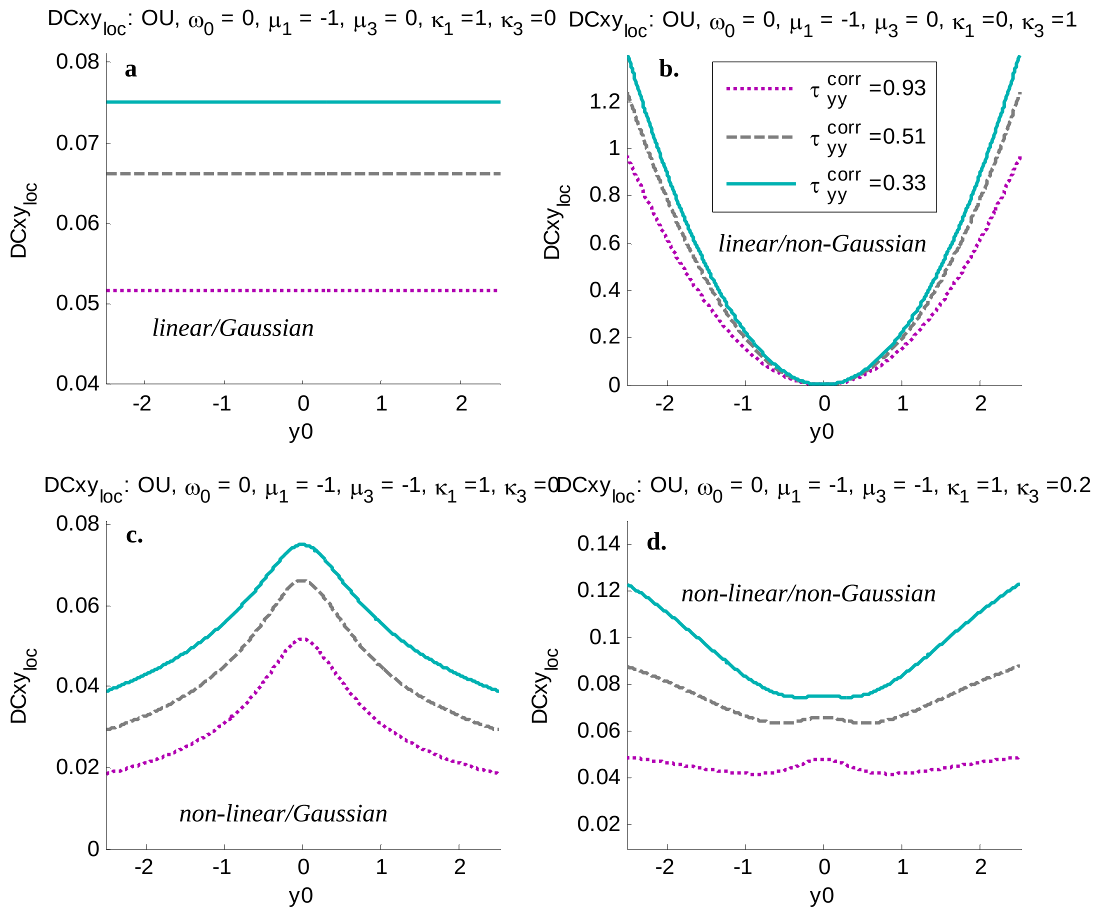

Figure 8: The same as in Fig.6 for $\lim_{s \to t} \partial C^{(\infty)}_{x_{\ell oc}\, y_{\ell oc}}(t-s) \big/ \partial t$.

4.4.4. Local Gaussian REPDFs

The formulae for the two-time moments in the long-time regime obtained in Sections 4.4.1, allow us to have a complete probabilistic characterisation of the linearized/localised RDE given by Equ.(3)_Sec(4.4.1) (we repeat here for convenience):

$$\dot{x}_{\ell oc}(t;\theta) \;=\; H'(x_0)\cdot\left(x_{\ell oc}(t;\theta)-x_0\right) + \Psi'(y_0)\cdot\left(y_{\ell oc}(t;\theta)-y_0\right)$$

where: $(x_0\, , y_0)$ are points of the RE-equilibrium curve defined in Section 4.4.1

More precisely, if $C^{(\infty)}_{yy}(t-s)$ is the long-time covariance of the random input of the non-linear half-oscillator given by Equ.(1)_Sec(4.2) (we repeat here for convenience):

$$\dot{x}(t;\theta) = H\left(x(t;\theta)\right) + \Psi\left(y(t;\theta)\right) = 0\,,$$

then, for each choice of the scaling parameter $\sigma^2_{y_{\ell oc}}$, the local, long-time, joint, REPDFs $f^{(\infty)}_{x_{loc}(t)\ y_{loc}(s)}(\alpha,\beta)$ of the linearized/localised RDE, will be Gaussian pdfs centered around the (x_0, y_0)-points (see also Section 3.4), i.e:

$$f^{(\infty)}_{x_{\ell oc}(t)\ x_{\ell oc}(s)}(\alpha,\beta) = \frac{1}{2\pi\sqrt{C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(0)\cdot\sigma^2_{y_{\ell oc}} - C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)}} \times \qquad (1)$$

$$\times \exp\left[-\frac{\sigma^2_{y_{\ell oc}}\cdot(\alpha-x_0)^2 - 2C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)\cdot(\alpha-x_0)\cdot(\beta-y_0) + C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(0)\cdot(\beta-y_0)^2}{2\left(C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(0)\cdot\sigma^2_{y_{\ell oc}} - C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)\right)}\right],$$

where

$$C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s) = \Psi'(y_0)\cdot(\sigma^2_{y_{\ell oc}}/\sigma^2_y)\int_{-t+s}^{\infty} e^{H'(x_0)\cdot(u+t-s)}\cdot C^{(\infty)}_{yy}(u)\,du\,, \qquad (2)$$

$$C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(0) = \frac{\left(\Psi'(y_0)\right)^2}{2\cdot\left(-H'(x_0)\right)} \times (\sigma^2_{y_{\ell oc}}\ /\ \sigma^2_y)\cdot\int_{v=-\infty}^{v=+\infty} C^{(\infty)}_{yy}(v)\cdot e^{H'(x_0)\cdot|v|}\,dv\,. \qquad (3)$$

Moreover, the local RE cross-covariance $C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)$ and local auto-covariance $C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(0)$, given by Equs.(2,3), further particularize to Equs.(2,5)_Sec(4.4.2) when the stochastic input $C^{(\infty)}_{yy}(t-s)$ is a lpGF and to Equs.(2,3,6)_Sec(4.4.3) when the input is a sOU or OU process.

In **Figs.9-11** the local joint REPDFs $f^{(\infty)}_{x_{loc}(t)\ y_{loc}(t)}(\alpha,\beta)$ (in the limiting case $t \to s$) are plotted in the RE space for three half-oscillators considered in Section (4.2), i.e. the linear/Gaussian case (Fig.9), the non-linear/Gaussian case (Fig.10) and the non-linear/non-Gaussian case (Fig.11), with lpGf excitation function and for two values of the excitation correlation time ($\tau^{\text{corr}}_{yy} = 0.93,\ 0.33$). The scaling parameter, $\sigma^2_{y_{\ell oc}}$, that also defines the excitation variance of each local REPDF, has been selected to be equal to 1/10 of the global excitation variance σ^2_y, i.e. $\sigma^2_{y_{\ell oc}} = 0.1$. In each of the cases (Cases a, b of each Figure), five local REPDF's are plotted, centered at five different points (x_0, y_0) of the equilibrium curve. Let it be noted that the "centers y_0" are all points of the essential support of the input function $y(t;\theta)$. The characteristics of the local REPDFs, in the RE space, follow the local characteristics of the local (co)variances $C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(0)$, $C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(0)$ discussed in Figs.(3,4). More precisely, in the linear/Gaussian case (Fig.9) all the five Gaussian REPDFs are the same for each excitation correlation time, whereas, in both the non-linear/Gaussian case (Fig.10) and the non-linear/non-Gaussian case (Fig.(11)) the (co)variances of each local REPDF are different, e.g. the local REPDF's that are centered at zero have larger response variance and RE cross-covariance than the other four local joint REPDFs. Another important feature, developed in all the three cases of the half oscillators illustrated in Figs.(9-11), is that the (co)variances of the local REPDFs increase with the excitation time of the stochastic input τ_{yy}^{corr}.

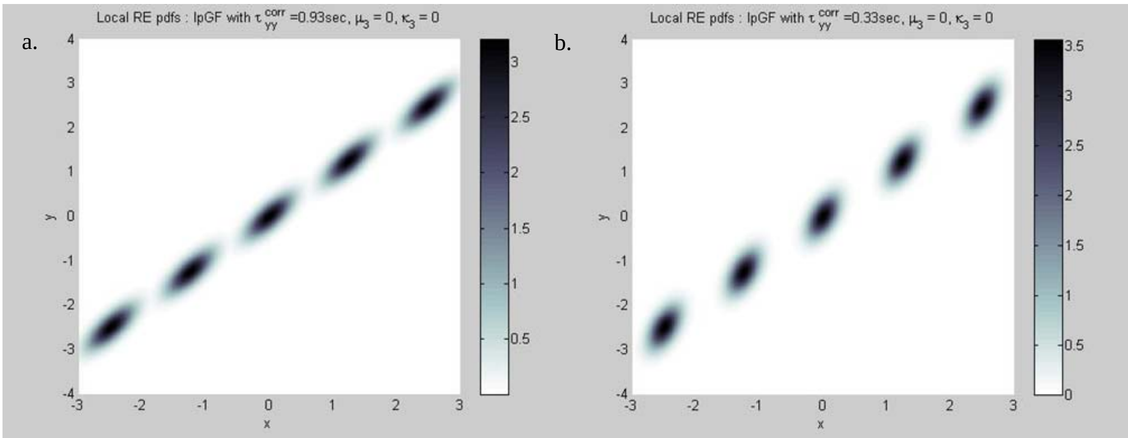

Figure 9: Local REPDFs plotted in the RE-phase space for the linear/Gaussian case with lpGF excitation and two cases of excitation correlation time $\tau_{yy}^{\text{corr}} = 0.93$ (**a**) and $\tau_{yy}^{\text{corr}} = 0.33$ (**b**)

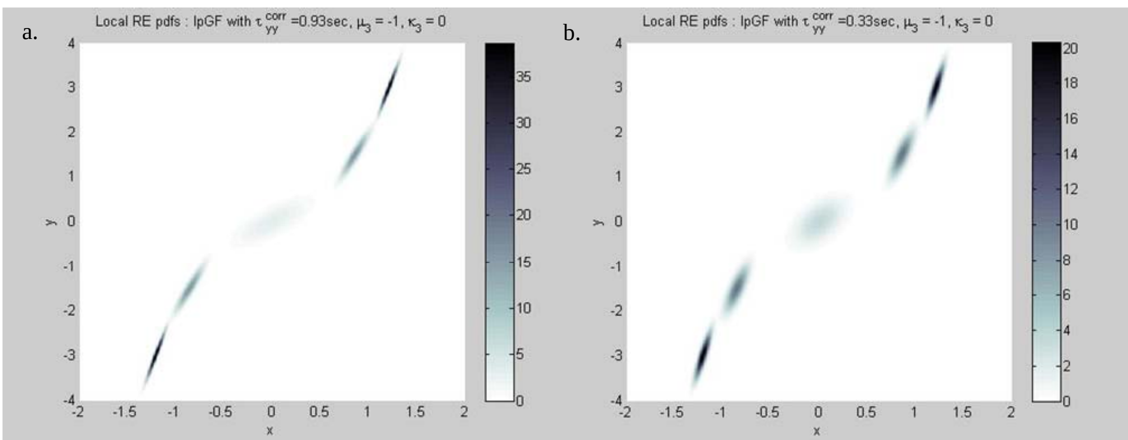

Figure 10: The same as in Fig.9 for the non-linear/Gaussian case.

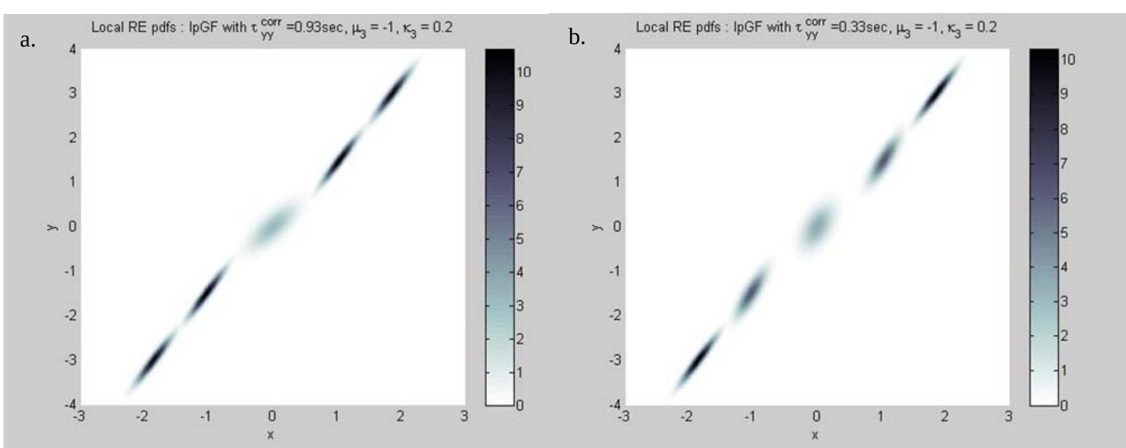

Figure 11: The same as in Fig.8 for the non-linear/non-Gaussian case.

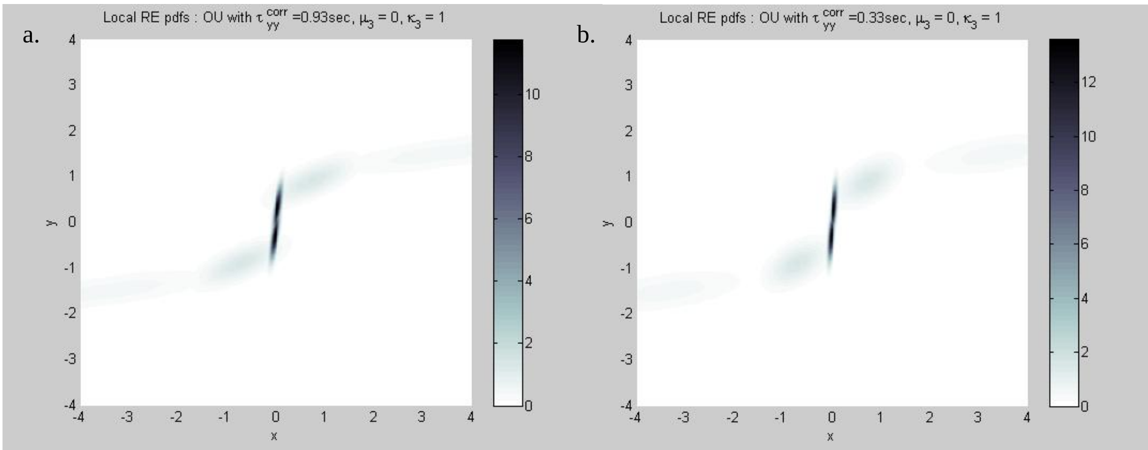

Figure 12: Local REPDFs plotted in the RE-phase space for the linear/non-Gaussian case with OU excitation and two cases of excitation correlation time $\tau_{yy}^{corr} = 0.93$ (**a**) and $\tau_{yy}^{corr} = 0.33$ (**b**)

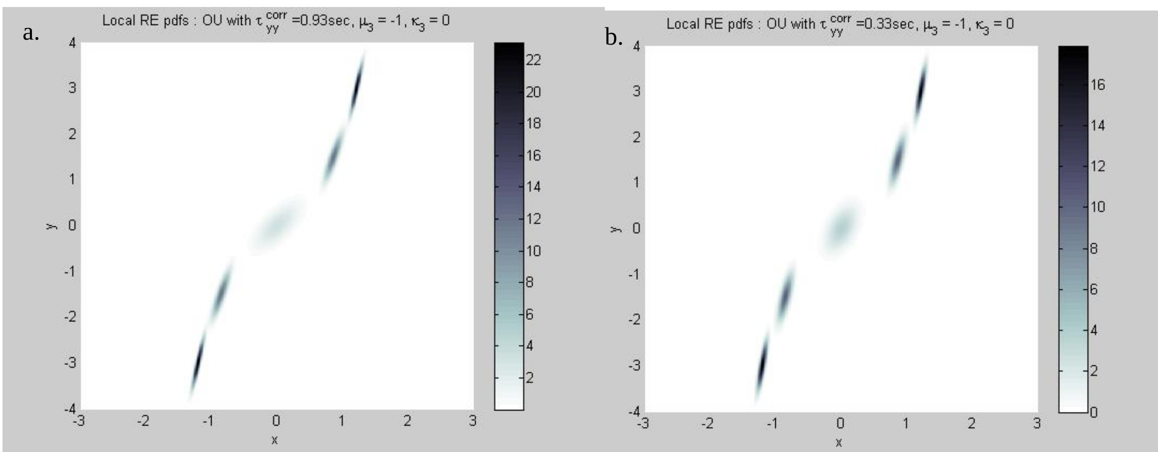

Figure 13: The same as in Fig.12 for the non-linear/Gaussian case

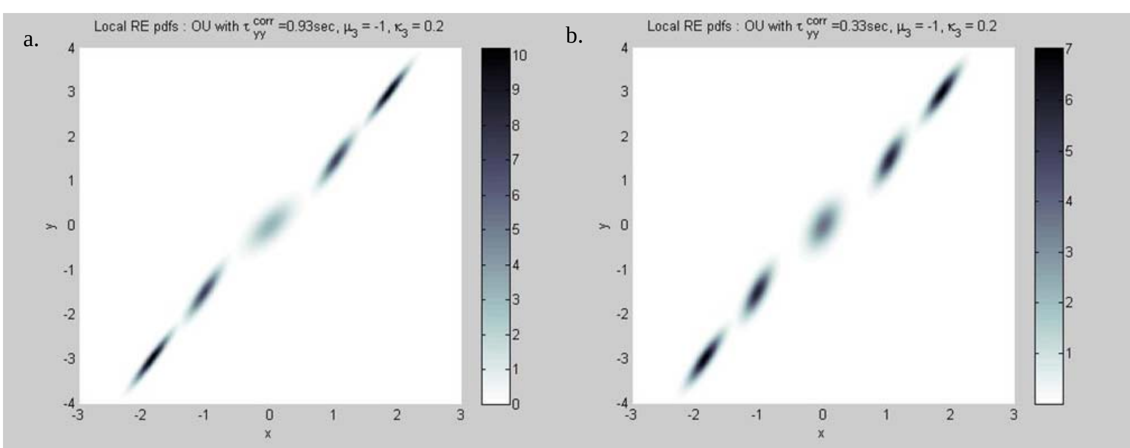

Figure 14: The same as in Fig.12 for the non-linear/non-Gaussian case.

In **Figs.12-14** the local REPDFs $f^{(\infty)}_{x_{loc}(t)\; y_{loc}(t)}(\alpha,\beta)$ (in the limiting case $t \to s$) are plotted in the RE space for three non-linear half-oscillators considered in Section (4.2), i.e. the linear/non-Gaussian case (Fig.12), the non-linear/Gaussian case (Fig.13) and the non-linear/non-Gaussian case (Fig.14), with OU excitation function and for two values of the excitation correlation time ($\tau^{\text{corr}}_{yy} = 0.93,\ 0.33$). The scaling parameter has been selected to be the same as in Figs.(9-11), i.e. $\sigma^2_{y_{\ell oc}} = 0{,}1$. In fact, comments for Figs. (10, 11) also apply to **Figs.13,14** respectively. In the linear/non-Gaussian case, Fig.(12), the local REPDF given by Equ.(1) is not defined at "$y_0 = 0$ center" since both the local (co)variances $C^{(\infty)}_{x_{\ell oc} y_{\ell oc}}(0)$, $C^{(\infty)}_{x_{\ell oc} x_{\ell oc}}(0)$ are zero (see case b in Figs.(6-7)). Nevertheless, in **Fig.12**, we can see that the two local REPDFs centered close to zero have significantly decreased (co)variances in comparison with the four other local REPDFs plotted for each case of the correlation time (a,b in Fig.12).

4.4.5. Comparison of local REPDFs with MC simulation results

The analytically calculated local REPDFs $f^{(\infty)}_{x_{loc}(t)\; y_{loc}(t)}(\alpha,\beta)$, given by Equs.(1-3)_Sec(4.4.4), of the linearized /localised RDE given by Equ.(3)_Sec (4.4.1), which is repeated herewith for easy reference,

$$\dot{x}_{\ell oc}(t;\theta) = H'(x_0)\cdot\left(x_{\ell oc}(t;\theta) - x_0\right) + \Psi'(y_0)\cdot\left(y_{\ell oc}(t;\theta) - y_0\right), \tag{1}$$

have been compared with by MC simulation results performed by Z. Kapelonis. The results of the MC simulation are obtained by generating samples functions $y_{\ell oc}(t)$, of the local excita-tion $y_{\ell oc}(t;\theta)$, using the 1-D random-phase model. For instance, the local excitation pdf is centered at the "centers y_0" and the local excitation covariance is a scaled version of the global, i.e., $C_{y_{\ell oc}\, y_{\ell oc}}(t,s) = \sigma^2_{y_{\ell oc}}\, C_{yy}(t,s)/\sigma^2_y$. Subsequently, the non-linear equation,

$$\dot{x}_{\ell oc}(t) = H\left(x_{\ell oc}(t)\right) + \Psi\left(y_{\ell oc}(t)\right), \tag{2a}$$
$$x_{\ell oc}(0) = 0, \tag{2b}$$

(that is, the deterministic version of the RDE, Equ.(1)_Sec.(4.4.1), with zero initial condition), is solved using ODE45, a MATLAB® implementation of the Dormant-Prince method (Dormand & Prince 1980), an explicit Runge-Kutta (4,5) formula. The MC pdf estimations are computed using the kernel density estimation via diffusion, introduced by Botev et al. (Botev et al. 2010) and coded in MATLAB® functions by the same author.

In **Figs.15,16** the local Gaussian REPDFs $f^{(\infty)}_{x_{loc}(t)\; y_{loc}(t)}(\alpha,\beta)$ obtained by the analytical solution of locally linear/ locally Gaussian problem (left column, cases a,c,e in Figs.15,16) are compared with local non-Gaussian REPDF's obtained by MC simulations (right column, cases b,d,f in Figs.15,16). For each method of solution, three local REPDFs are plotted for three different "centers y_0". In **Fig.15** the non-linear/Gaussian case is considered, whereas in **Fig.16** the non-linear/non-Gaussian case. In all cases the stochastic input is a lpGF with

$\sigma^2_{y_{loc}} = 0.1$ and $\tau^{corr}_{yy} = 0.93$. In general, results obtained by the two methods compare pretty satisfactorily.

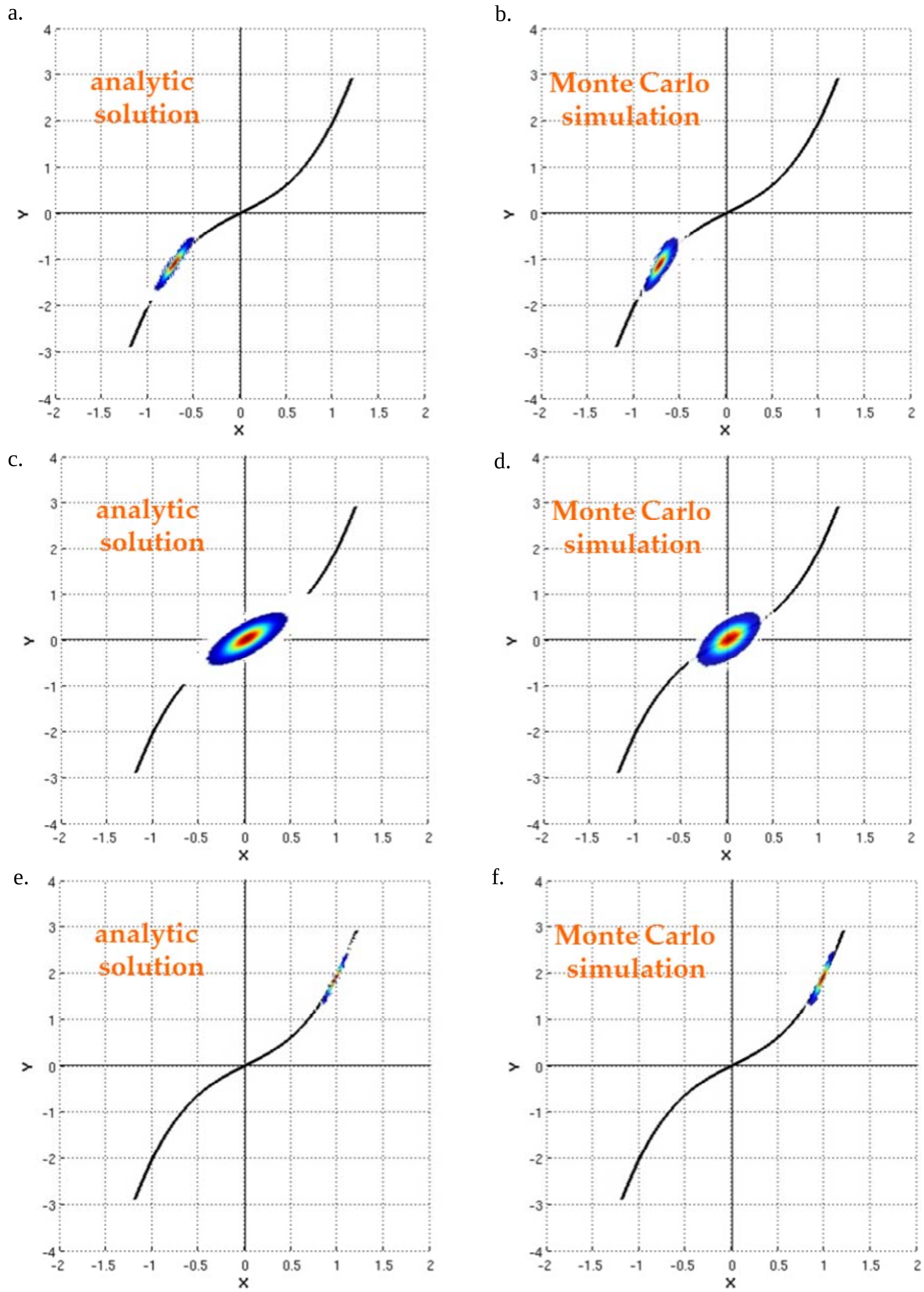

Figure 15: Local REPDF's obtained by the analytic solution of the local linear problem (left column) and MC simulations (right column) on the non-linear problem. Here the non-linear/Gaussian case in considered.

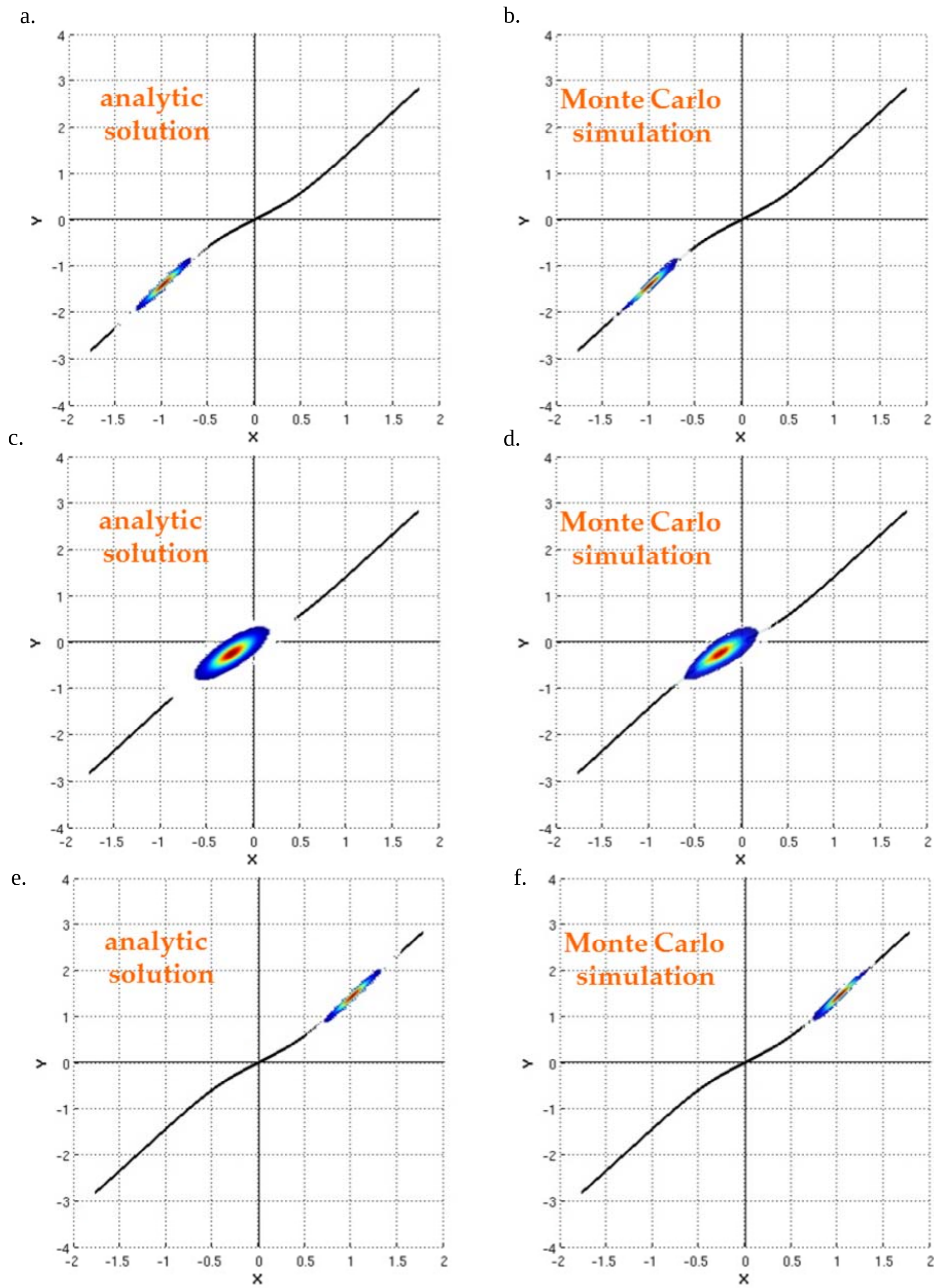

Figure 16: The same as in Fig.15 for the non-linear/non-Gaussian case.

4.5. Numerical Solution of the REPDF evolution equation in the long-time-statistical equilibrium regime

4.5.1. Kernel density representation for the joint response-excitation and marginal pdfs.

The target of the numerical solution to the REPDF evolution equation, Equ.(1,2a,b)_Sec(4.3), supplemented by all appropriate auxiliary conditions about the RE correlation structure, is to find the time-independent (statistical equilibrium) joint REPDF $f_{xy}(\alpha,\beta) = \lim_{t\to\infty} f_{x(t)y(t)}(\alpha,\beta)$. However, in order to cope with the appearance of the unusual, response-time (half-time) derivative in $\partial f_{x(t)y(s)}(\alpha,\beta)/\partial t\Big|_{s\to t}$ we have to introduce a suitable representation of the lag-time dependent joint REPDF $f_{x(t)y(s)}(\alpha,\beta)$. The selection that has been made aims to provide the ability to embed the additional information concerning the local RE-correlation structure, acquired in Sec.(4.4), to the REPDF evolution equation.

Setting $\varphi(\alpha,\beta;\tau) = f_{x(t)y(t-\tau)}(\alpha,\beta)$, the representation problem concerns functions of the type $(\alpha,\beta;\tau) \to \varphi(\alpha,\beta;\tau)$, defined on $\mathbb{R}\times\mathbb{R}\times V(0)$, where $V(0)$ is a neighborhood of zero, and having non-negative values. In addition, we assume that, for each $\tau\in V(0)$, $(\alpha,\beta) \to \varphi(\alpha,\beta;\tau)$ is continuous, it has continuous partial derivative with respect to α, tends uniformly to zero as $\sqrt{\alpha^2+\beta^2} \to \infty$, and satisfy the integral constraint $\iint_{\mathbb{R}^2} \varphi_\tau(\alpha,\beta;\tau)\,d\alpha\,d\beta = 1$ (in order to be a pdf). As a function of the lag time, $\tau \to \varphi(\alpha,\beta;\tau)$ should be continuously differentiable.

The implementation of an efficient representation of functions $\varphi(\alpha,\beta;\tau)$, preserving all above stated properties, is a difficult problem, without any supporting theoretical background. On the basis of previous (successful) experience in representing bivariate pdfs by superposition of Kernel Density Functions (KDFs) (Athanassoulis & Belibassakis 2002; Athanassoulis & Gavriliadis 2002), and in view of the fact that $f_{x(t)y(s)}(\alpha,\beta)$ is locally (in $(\alpha,\beta)-$space) approximated and investigated in terms of a Gaussian pdf (see Section 4.4) we adopt the following representation, which will be subsequently called Kernel Density Representation (KDR):

$$f_{x(t)y(s)}(\alpha,\beta) = \sum_{i,j} p_{ij}\cdot K\Big(\alpha,\beta;\alpha_i,\beta_j,\Sigma_{\alpha i,\beta j}(t-s)\Big), \qquad (1)$$

where (α_i,β_j), $(i,j)\in\mathbb{Z}\times\mathbb{Z}$, is a grid of points in the phase space $\mathbb{R}\times\mathbb{R}$, each (α_i,β_j) serving as the center of a Gaussian kernel density function $K(\alpha,\beta;\cdot,\cdot,\bullet)$, while

$$\Sigma_{\alpha i,\beta j}(t-s) = \begin{pmatrix} C_{\alpha_i\alpha_i}(0) & C_{\alpha_i\beta_j}(t-s) \\ C_{\alpha_i\beta_j}(t-s) & C_{\beta_j\beta_j}(0) \end{pmatrix} \tag{2}$$

is the covariance matrix of $K(\alpha,\beta;\cdot,\cdot,\bullet)$. To ensure that (1) will always be a legitimate pdf, the following constraints are imposed on the unknown coefficients p_{ij}, $(i,j)\in\mathbb{Z}\times\mathbb{Z}$.

$$0 \le p_{ij} \quad \text{and} \quad \sum_{i,j} p_{ij} = 1. \tag{3a,b}$$

In principle $f_{x(t)y(t-\tau)}(\alpha,\beta)$, $\tau\in V(0)$, is supported on the whole plane $\mathbb{R}\times\mathbb{R}$. In the present work we focus on the main-mass part of $f_{xy}(\alpha,\beta) = f_{x(t)y(t)}(\alpha,\beta)$, that is, on its form in its essential support D_{ess}. The latter is conventionally defined as the subset of $\mathbb{R}\times\mathbb{R}$ where $f_{xy}(\alpha,\beta) > \varepsilon \approx 10^{-3}\cdot\max\left\{ f_{xy}(\alpha,\beta)\right\}$.

$$D_{ess} = \left\{ (\alpha,\beta)\in\mathbb{R}\times\mathbb{R} : \ f_{xy}(\alpha,\beta) > \varepsilon \approx 10^{-3}\cdot\max\left\{ f_{xy}(\alpha,\beta)\right\}\right\}$$

Thus, the approximation will be restricted in a compact subdomain

$$D_{\alpha\beta} = \left[\alpha_{\min},\alpha_{\max}\right]\times\left[\beta_{\min},\beta_{\max}\right],$$

of $\mathbb{R}\times\mathbb{R}$, such that $D_{ess}\subseteq D_{\alpha\beta}$ (tail questions are not considered herewith), and the indices (i,j) will run over the finite set $N(I)\times N(J)$, where $N(I)=\{1,2,\dots,I\}$ and $N(J)$ is similarly defined. Since D_{ess} is not known *a priori*, some preliminary information is necessary in order to choose the computational domain $D_{\alpha\beta}$. This information is provided by the long-time solution of the two-time RE moment equations (studied in Section 5.3), in conjunction with the essential support of the known excitation pdf $f_y(\beta)$.

The known marginal pdf of the excitation $f_y(\beta)$ also admits a KDR which reserves the marginal compatibility (Athanassoulis & Belibassakis 2002). In fact, the excitation pdf admits the following marginal KDR, when the grid points (α_i,β_j), $(i,j)\in\mathbb{Z}\times\mathbb{Z}$ are regularly spaced:

$$f_{y(s)}(\beta) = \sum_{i,j} p_{ij}\cdot\int_{\alpha\in\mathbb{R}} K\left(\alpha,\beta;\alpha_i,\beta_j,\bar{\Sigma}_{\alpha i,\beta j}(t-s)\right)d\alpha = \sum_{j} p_j\cdot K_{\beta_j}\left(\beta;\beta_j,\sigma_{\beta j}\right), \tag{4}$$

where $p_j = \sum_i p_{ij}$.

The above representation generalizes to:

$$f_{y(s)}(\beta) = \sum_{i,j} p_{ij} \cdot \int_{\alpha\in\mathbb{R}} K\left(\alpha,\beta;\alpha_i,\beta_j,\bar{\Sigma}_{\alpha i,\beta j}(t-s)\right) d\alpha = \sum_{i,j} p_{ij} \cdot K_{\beta_{j(i)}}(\beta;\beta_{j(i)},\sigma_{\beta j\,(i)}), \tag{5}$$

when the 2D-Kernels span a non-regularly spaced grid, i.e. when to each response grid point α_i correspond $j(i)$ Kernels in the RE space.

4.5.2. Reformulation of the long-time limit form of the joint REPDF evolution equation using the KDR representation

Introducing the KDR, given by Equ(1)_Sec(4.5.1) in Equ.(1,2a)_Sec(4.3), (the constitutive conditions, Equ.(2b,2c)_Sec(4.3) are automatically satisfied thanks to the defining properties of the KDR), we obtain the following reformulation of problem Equ.(1,2)_Sec(4.3):

$$\sum_{i,j} p_{ij} \left[\frac{\partial}{\partial t} K\left(\alpha,\beta;\alpha_i,\beta_j,\Sigma_{\alpha i,\beta j}(t-s)\right)\Big|_{s\to t} + \right.$$
$$\left. + \frac{\partial}{\partial\alpha}\left\{ \left(H(\alpha) + \Psi(\beta)\right)\cdot K\left(\alpha,\beta;\alpha_i,\beta_j,\Sigma_{\alpha i,\beta j}(0)\right)\Big|_{s\to t} \right\} \right] = 0, \tag{1}$$
$$\forall(\alpha,\beta)\in\mathbb{R}\times\mathbb{R}$$

under the marginal compatibility constraint

$$\sum_{i,j} p_{ij} K_{\beta_{j(i)}}\left(\beta;\beta_{j(i)},\sigma_{\beta j\,(i)}\right) - f_y(\beta) = 0, \tag{2}$$

where

$$K_{\beta_{j(i)}}\left(\beta;\beta_{j(i)},\sigma_{\beta j\,(i)}\right) = \int_{\alpha\in\mathbb{R}} K\left(\alpha,\beta;\alpha_i,\beta_j,\Sigma_{\alpha i,\beta j}(t-s)\right) d\alpha, \tag{3}$$

is the induced marginal kernel density function.

The local character of the introduced KDR allows us to supplement Equ.(1) with previously obtained information about the local correlation structure. Assuming Gaussian Kernels (although other choices are possible), i.e.

$$K\left(\alpha,\beta;\alpha_i,\beta_j,\bar{\Sigma}_{\alpha i,\beta j}(t-s)\right) = \frac{1}{2\cdot\pi\cdot W_{ij}(t-s)} \cdot \exp\left[-\frac{E_{ij}(\alpha,\beta;t-s)}{2\cdot\left(W_{ij}(t-s)\right)^2} \right], \tag{5}$$

where:

$$W_{ij}(t-s) = \sqrt{C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j} - \left(C_{\alpha_i\beta_j}(t-s)\right)^2}, \tag{6a}$$

$$E_{ij}(\alpha,\beta;t-s) = C_{\beta_j\beta_j}\cdot\left(\alpha-\alpha_i\right)^2 - 2\cdot C_{\alpha_i\beta_j}(t-s)\cdot\left(\alpha-\alpha_i\right)\cdot\left(\beta-\beta_j\right) + C_{\alpha_i\alpha_i}\cdot\left(\beta-\beta_j\right)^2. \tag{6a}$$

Then, after performing some algebra (see Appendix 6) in the first term of Equ.(1), we have:

$$\frac{\partial}{\partial t}K\left(\alpha,\beta;\alpha_i,\beta_j,\bar{\Sigma}_{\alpha i,\beta j}(t-s)\right)\bigg|_{s\to t} = \frac{\partial}{\partial t}C_{\alpha_i\beta_j}(t-s)\bigg|_{s\to t}\cdot\frac{C_{\alpha_i\beta_j}\cdot}{2\cdot\pi\cdot\left(W_{ij}\right)^5}\times$$
$$\times\left[\left(W_{ij}\right)^2 - C_{\beta_j\beta_j}\cdot\left(\alpha-\alpha_i\right)^2 - C_{\alpha_i\alpha_i}\cdot\left(\beta-\beta_j\right)^2 + \left(\frac{C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}}{C_{\alpha_i\beta_j}} + C_{\alpha_i\beta_j}\right)\cdot\left(\alpha-\alpha_i\right)\cdot\left(\beta-\beta_j\right)\right]\times$$
$$\times\exp\left[-\frac{E_{ij}\left(\alpha,\beta;t-s\right)}{2\cdot\left(W_{ij}\right)^2}\right], \qquad (7)$$

where $C_{\alpha_i\alpha_i} \equiv C_{\alpha_i\alpha_i}(0)$, $C_{\beta_j\beta_j} \equiv C_{\beta_j\beta_j}(0)$.

The kernel variances $C_{\alpha_i\alpha_i}, C_{\beta_j\beta_j}$ are adjusted to $D_{\alpha\beta}$ and the resolution of the grid, aiming at a certain degree of overlapping between contiguous Kernels. The details of the implementation of the KDF parameters will be further discussed in Section 4.5.5. However, it is important to mention that the RE cross-covariance $C_{\alpha_i\beta_j}$ and the half-time derivative $\partial C_{\alpha_i\beta_j}(t-s)/\partial t\big|_{s\to t}$, appearing in Equs.(5-7), are estimated (approximated) using obtained information about the local correlation structure from the closest to (α_i,β_j) point of the equilibrium curve, which will be denoted as $(x_0,y_0)\big|_{\alpha_i,\beta_j}$. That is, the solution of the localized/linearized problem at $(x_0,y_0)\big|_{\alpha_i,\beta_j}$ (Equ.(3)_Sec(4.4.1)) is used to a priori approximate $\partial C_{\alpha_i\beta_j}(t-s)/\partial t\big|_{s\to t}$ and $C_{\alpha_i\beta_j}$, rendering the localized/linearized problems *a priori* closure conditions to the reformulated REPDF evolution equation (given by Equs.(1,2)).

On the basis of the KDR given by Equ.(1)_Sec(4.5.1), and the a priori approximation of the Gaussian KDF coefficients by the solution of the localized/linearized problem, the determination of the sought-for joint REPDF has been reduced to the determination of the coefficients p_{ij}, $(i,j)\in N(I)\times N(J)$, from the system of Equs.(1,2).

4.5.3. Galerkin discretization of the problem

Since the reformulated joint REPDF evolution equation, Equs.(1,2)_Sec(4.5.2), should be satisfied for every $(\alpha,\beta)\in\mathbb{R}\times\mathbb{R}$, a discretization is necessary in order to obtain numerical solutions. We shall use a Galerkin type, weighted-residual method (Kantorovich & Krylov 1964)(Zeidler 1990) to find a discrete system of equations, approximately equivalent to Equs.(1,2)_Sec(4.5.2). Similar methods have been used by various authors for solving the steady state FPK equation. see e.g. (Bhandari & Sherrer 1968; Langley 1985; McWilliam et al. 2000).

Let us define the residuals

$$R_{2d}\left(\alpha,\beta;\left\{p_{ij}\right\}\right) = \sum_{i,j} p_{ij}\left[\frac{\partial}{\partial t}K\left(\alpha,\beta;\alpha_i,\beta_j,\bar{\Sigma}_{\alpha i,\beta j}(t-s)\right)\right|_{s\to t} + $$
$$+ \frac{\partial}{\partial\alpha}\left[\left(H(\alpha)+\Psi(\beta)\right)\cdot K\left(\alpha,\beta;\alpha_i,\beta_j,\bar{\Sigma}_{\alpha i,\beta j}(0)\right)\right], \tag{1a}$$
$$R_{1d}\left(\beta;\left\{p_{ij}\right\}\right) = \sum_{i,j} p_{ij} K_{\beta_{j(i)}}\left(\beta;\beta_{j(i)},\sigma_{\beta j(i)}\right) - f_y(\beta). \tag{1b}$$

According to the weighted-residual method, the unknowns p_{ij}, $(i,j)\in N(I)\times N(J)$, are evaluated by imposing the condition that the projection of the residuals on a system of linearly independent functions is zero:

$$\iint_{\mathbb{R}^2} R_{2d}\left(\alpha,\beta;\left\{p_{ij}\right\}\right)\cdot\Lambda_{\kappa,\lambda}(\alpha,\beta)\,d\alpha\,d\beta = 0, \quad \forall\,(\kappa,\lambda)\in N(K)\times N(L), \tag{2a}$$
$$\int_{\mathbb{R}} R_{1d}\left(\beta;\left\{p_{ij}\right\}\right)\cdot\tilde{\Lambda}_{\tilde{\lambda}}(\beta)\,d\beta = 0, \quad \forall\,\tilde{\lambda}\in N(\tilde{L}), \tag{2b}$$

where $N(K)=\left\{1,2,\ldots,K\right\}$, etc. There is quite a flexibility in choosing the functions $\Lambda_{\kappa,\lambda}(\alpha,\beta)$ and $\tilde{\Lambda}_{\tilde{\lambda}}(\beta)$, which we shall call subsequently ***Galerkin kernels***. In the present work the Galerkin kernels are chosen to be Gaussian kernels, similar to the representation kernels $K(\alpha,\beta;\cdot,\cdot,\bullet)$. Combining Equs.(1) and (2), we obtain

$$\sum_{i,j} p_{ij}\cdot G_{ij,\kappa\lambda} = 0, \quad \forall\,(\kappa,\lambda)\in N(K)\times N(L), \tag{3a}$$
$$\sum_{i,j} p_{ij}\cdot\tilde{G}_{j(i),\tilde{\lambda}} = g_{\tilde{\Lambda}_{\tilde{\lambda}}}(f_y), \quad \forall\,\tilde{\lambda}\in N(\tilde{L}), \tag{3b}$$

where

$$G_{ij,\kappa\lambda} = G^{(1)}_{ij,\kappa\lambda} + G^{(2)}_{ij,\kappa\lambda}, \tag{4a}$$
$$G^{(1)}_{ij,\kappa\lambda} = \iint_{\mathbb{R}^2}\frac{\partial}{\partial t}K\left(\alpha,\beta;\alpha_i,\beta_j,\Sigma_{\alpha i,\beta j}(t-s)\right)\Big|_{s\to t}\cdot\Lambda_{\kappa,\lambda}(\alpha,\beta)\,d\alpha\,d\beta, \tag{4b}$$
$$G^{(2)}_{ij,\kappa\lambda} = \iint_{\mathbb{R}^2}\frac{\partial}{\partial\alpha}\left[\left(H(\alpha)+\Psi(\beta)\right)\cdot K\left(\alpha,\beta;\alpha_i,\beta_j,\Sigma_{\alpha i,\beta j}(0)\right)\right]\cdot\Lambda_{\kappa,\lambda}(\alpha,\beta)\,d\alpha\,d\beta =$$
$$= -\iint_{\mathbb{R}^2}\left(H(\alpha)+\Psi(\beta)\right)\cdot K\left(\alpha,\beta;\alpha_i,\beta_j,\Sigma_{\alpha i,\beta j}(0)\right)\cdot\Lambda^{(1)}_{\kappa,\lambda}(\alpha,\beta)\,d\alpha d\beta, \tag{4c}$$
$$\Lambda^{(1)}_{\kappa,\lambda}(\alpha,\beta) = \frac{\partial\Lambda_{\kappa,\lambda}(\alpha,\beta)}{\partial\alpha}, \tag{4d}$$
$$\tilde{G}_{j(i),\tilde{\lambda}} = \int_{\mathbb{R}} K_{\beta_{j(i)}}\left(\beta;\beta_{j(i)},\sigma_{\beta j(i)}\right)\cdot\tilde{\Lambda}_{\tilde{\lambda}}(\beta)\,d\beta, \tag{4e}$$

$$g_{\tilde{\Lambda}_{\tilde{\lambda}}}\left(f_y\right) = \int_{\mathbb{R}} f_y(\beta)\cdot\tilde{\Lambda}_{\tilde{\lambda}}(\beta)\, d\beta . \tag{4f}$$

The specific structure of the equation to be solved enters in the coefficients $\tilde{G}_{j(i),\tilde{\lambda}}, G^{(1)}_{ij,\kappa\lambda}, G^{(2)}_{ij,\kappa\lambda}$ implicitly through Kernel coefficients which contain information from the family of the localized problems. Moreover $G^{(2)}_{ij,\kappa\lambda}$ is also explicitly dependent from the structure to be solved, e.g. for the case study that the RDE is a cubic half-oscillator described by Equ.(3a)_Sec(4.2), $G^{(2)}_{ij,\kappa\lambda}$ specifies to

$$G^{(2)}_{ij,\kappa\lambda} = -\iint_{\mathbb{R}^2} (\mu_1\cdot\alpha + \mu_3\cdot\alpha^3 + \kappa_1\cdot\beta + \kappa_3\cdot\beta^3)\times \times K\left(\alpha,\beta;\alpha_i,\beta_j,\Sigma_{\alpha i,\beta j}(0)\right)\cdot\Lambda^{(1)}_{\kappa,\lambda}(\alpha,\beta)\, d\alpha d\beta. \tag{5}$$

On the basis of the above discussion, the problem of calculating the expansion coefficients p_{ij} of the joint REPDF takes the following form:

Find p_{ij}, $(i,j)\in N(I)\times N(J)$, satisfying the homogeneous equation

$$\sum_{i,j} p_{ij}\cdot G_{ij,\kappa\lambda} = 0, \quad \forall(\kappa,\lambda)\in N(K)\times N(L), \tag{6a}$$

under the marginal compatibility constraint

$$\sum_{i,j} p_{ij}\cdot\tilde{G}_{j(i),\tilde{\lambda}} = g_{\tilde{\Lambda}_{\tilde{\lambda}}}(f_y), \qquad \forall\tilde{\lambda}\in N(\tilde{L}), \tag{6b}$$

and the constitutive constraints

$$\sum_{i,j} p_{ij} = 1, \qquad p_{ij} \geq 0, \ \forall(i,j)\in N(I)\times N(J). \tag{6c}$$

Recall that the problem, defined by Equ.(6), is supplemented by the family of the linearized/ Gaussianized problems, that embed information about the RE correlation structure to the Galerkin coefficient $\tilde{G}_{j(i),\tilde{\lambda}}, G_{ij,\kappa\lambda}$.

4.5.4. Analytic Computation of the Galerkin Coefficients.

Assuming that the Galerkin kernels are Gaussian pdfs, i.e. $\tilde{\Lambda}_{\tilde{\lambda}}(\beta)$, $\Lambda_{\kappa,\lambda}(\alpha,\beta)$, are given by the formulae

$$\tilde{\Lambda}_{\tilde{\lambda}} = \frac{1}{\sqrt{2\cdot\pi\cdot C_{\beta_{\tilde{\lambda}}\beta_{\tilde{\lambda}}}}}\cdot\exp\left[-\frac{(\beta-\beta_{\tilde{\lambda}})^2}{2\cdot C_{\beta_{\tilde{\lambda}}\beta_{\tilde{\lambda}}}}\right], \tag{1}$$

$$\Lambda_{\kappa,\lambda}(\alpha,\beta) = K\left(\alpha,\beta;\alpha_\kappa,\beta_\lambda,\Sigma_{\alpha\kappa,\beta\lambda}\right) = \frac{1}{2\cdot\pi\cdot W_{\kappa\lambda}(t-s)}\cdot\exp\left[-\frac{E_{\kappa\lambda}(\alpha,\beta;t-s)}{2\cdot\left(W_{\kappa\lambda}(t-s)\right)^2}\right], \quad (2)$$

where

$$W_{\kappa\lambda}(t-s) = \sqrt{C_{\alpha_\kappa\alpha_\kappa}\cdot C_{\beta_\lambda\beta_\lambda} - \left(C_{\alpha_\kappa\beta_\lambda}(t-s)\right)^2}, \quad (3a)$$

$$E_{\kappa\lambda}(\alpha,\beta;t-s) = C_{\beta_\lambda\beta_\lambda}\cdot(\alpha-\alpha_\kappa)^2 - 2\cdot C_{\alpha_\kappa\beta_\lambda}(t-s)\cdot(\alpha-\alpha_\kappa)\cdot(\beta-\beta_\lambda) + C_{\alpha_\kappa\alpha_\kappa}\cdot(\beta-\beta_\lambda)^2 \quad (3b)$$

and, therefore, (applying Equ.(7)_App(4))

$$\Lambda^{(1)}_{\kappa,\lambda}(\alpha,\beta) = \frac{\partial\Lambda_{\kappa,\lambda}(\alpha,\beta)}{\partial\alpha} = \quad (4)$$

$$= \frac{-C_{\beta_\lambda\beta_\lambda}\cdot(\alpha-\alpha_\kappa) + C_{\alpha_\kappa\beta_\lambda}\cdot(\beta-\beta_\lambda)}{2\cdot\pi\cdot\left(W_{\kappa\lambda}(t-s)\right)^3}\cdot\exp\left[-\frac{E_{\kappa\lambda}(\alpha,\beta;t-s)}{2\cdot\left(W_{\kappa\lambda}(t-s)\right)^2}\right].$$

The Galerkin coefficients, $G^{(1)}_{ij,\kappa\lambda}$ (Equ.(4b)_Sec(4.5.3)), $G^{(2)}_{ij,\kappa\lambda}$ ((Equ.(4c)_Sec(4.5.3) or Equ.(5)_Sec(4.5.3) for the cubic half oscillator) and $\tilde{G}_{j,\lambda(i)}$ (Equs.(4e)_Sec(4.5.3)), can be analytically computed. The marginal Galerkin coefficient $\tilde{G}_{j,\lambda(i)}$, after some algebraic calculations, takes the form:

$$\tilde{G}_{j(i),\lambda} = \frac{1}{\sqrt{2\cdot\pi}\cdot\sqrt{C_{\beta_{j(i)}\beta_{j(i)}} + C_{\beta_\lambda\beta_\lambda}}}\cdot\exp\left[-\frac{(\beta_{j(i)}-\beta_\lambda)^2}{2\cdot(C_{\beta_{j(i)}\beta_{j(i)}} + C_{\beta_\lambda\beta_\lambda})}\right], \quad (5)$$

where $\beta_{j(i)}$, $C_{\beta_{j(i)}\beta_{j(i)}}$ are, respectively, the marginal mean value and variance of the representation Kernel $K\left(\alpha,\beta;\alpha_i,\beta_j,\Sigma_{\alpha i,\beta j}(t-s)\right)$ (see Equ.(5)_Sec(4.5.1)). Let it be noted that the marginal Galerkin coefficients $\tilde{G}_{j,\lambda}$, are compatible with the locality of the marginal KDR, since, as we can see in Equ.(5), $\tilde{G}_{j(i),\lambda}\to 0$, as $\left|\beta_j-\beta_\lambda\right|\to\infty$.

Moreover, as shown in Appendix 7, the Galerkin coefficients $G^{(1)}_{ij,\kappa\lambda}$, $G^{(2)}_{ij,\kappa\lambda}$, can be written in the equivalent form (see Equs.(4,15)_ App(7))

$$G^{(1)}_{ij,\kappa\lambda} = \frac{1}{4\cdot\pi^2}\cdot\frac{\partial C_{\alpha_i\beta_j}}{(W_{ij})^5\cdot W_{\kappa\lambda}}\cdot\iint_{\mathbb{R}^2}\left(\Pi_{1,20}\cdot\alpha^2 + \Pi_{1,02}\cdot\beta^2 + \Pi_{1,11}\cdot\alpha\cdot\beta + \Pi_{1,00}\right)\times \quad (6)$$

$$\times\exp\left[-\left(Q_{1,20}\cdot\alpha^2 + Q_{1,11}\cdot\alpha\cdot\beta + Q_{1,02}\cdot\beta^2 + Q_{1,10}\cdot\alpha + Q_{1,01}\cdot\beta + Q_{1,00}\right)\right]d\alpha d\beta$$

and

$$G^{(2)}_{ij,\kappa\lambda} = \frac{1}{4\cdot\pi^2\cdot W_{ij}\cdot(W_{\kappa\lambda})^3}\int\int_{\mathbb{R}^2}\Big(\Pi_{2,10}\cdot\alpha+\Pi_{2,01}\cdot\beta+\Pi_{2,20}\cdot\alpha^2+\Pi_{2,11}\cdot\alpha\cdot\beta+\Pi_{2,02}\beta^2+ \\ +\Pi_{2,30}\cdot\alpha^3+\Pi_{2,21}\cdot\alpha^2\cdot\beta++\Pi_{2,21}\cdot\alpha^2\cdot\beta+\Pi_{2,03}\beta^2+ \\ +\Pi_{2,40}\cdot\alpha^4+\Pi_{2,31}\cdot\alpha^3\cdot\beta++\Pi_{2,13}\cdot\alpha\cdot\beta^3+\Pi_{2,04}\beta^4\Big)\times \\ \times\exp\left\{-\left(Q_{2,20}\cdot\alpha^2+Q_{2,11}\cdot\alpha\cdot\beta+Q_{2,02}\cdot\beta^2+Q_{2,10}\cdot\alpha+Q_{2,01}\cdot\beta+Q_{2,00}\right)d\alpha d\beta,\right. \quad (7)$$

where:

$$W_{ij} = \sqrt{C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}-(C_{\alpha_i\beta_j})^2},\ W_{\kappa\lambda}=\sqrt{C_{\alpha_\kappa\alpha_\kappa}\cdot C_{\beta_\lambda\beta_\lambda}-(C_{\alpha_\kappa\beta_\lambda})^2}, \quad (8a,8b)$$

$$\Pi_{1,20} = -C_{\alpha_i\beta_j}\cdot C_{\beta_j\beta_j}, \qquad \Pi_{1,02} = -C_{\alpha_i\beta_j}\cdot C_{\alpha_i\alpha_i}, \quad (8c,8d)$$

$$\Pi_{1,11} = \left(C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}+(C_{\alpha_i\beta_j})^2\right), \qquad \Pi_{1,00} = C_{\alpha_i\beta_j}\cdot(W_{ij})^2, \quad (8e,8f)$$

$$\Pi_{2,10} = -C_{\beta_\lambda\beta_\lambda}\cdot(\mu_3\cdot\alpha_\kappa{}^3+\mu_1\cdot\alpha_\kappa+\kappa_1\cdot\beta_\lambda+\kappa_3\cdot\beta_\lambda{}^3), \quad (8g)$$

$$\Pi_{2,01} = C_{\alpha_\kappa\beta_\lambda}\cdot(\mu_3\cdot\alpha_\kappa{}^3+\mu_1\cdot\alpha_\kappa+\kappa_1\cdot\beta_\lambda+\kappa_3\cdot\beta_\lambda{}^3), \quad (8h)$$

$$\Pi_{2,11} = 3\cdot\mu_3\cdot C_{\alpha_\kappa\beta_\lambda}\cdot\alpha_\kappa{}^2+\mu_1\cdot C_{\alpha_\kappa\beta_\lambda}-\kappa_1\cdot C_{\beta_\lambda\beta_\lambda}-3\cdot\kappa_3\cdot C_{\beta_\lambda\beta_\lambda}\cdot\beta_\lambda{}^2, \quad (8i)$$

$$\Pi_{2,20} = -(3\cdot\mu_3\cdot C_{\beta_\lambda\beta_\lambda}\cdot\alpha_\kappa{}^2+\mu_1\cdot C_{\beta_\lambda\beta_\lambda}), \qquad \Pi_{2,02} = (\kappa_1+3\cdot\kappa_3\cdot C_{\alpha_\kappa\beta_\lambda}\cdot\beta_\lambda{}^2)\cdot C_{\alpha_\kappa\beta_\lambda}, \quad (8k,8l)$$

$$\Pi_{2,21} = 3\cdot\mu_3\cdot C_{\alpha_\kappa\beta_\lambda}\cdot\alpha_\kappa, \quad \Pi_{2,12} = -3\cdot\kappa_3\cdot C_{\beta_\lambda\beta_\lambda}\cdot\beta_\lambda, \quad \Pi_{2,30} = -3\cdot\mu_3\cdot C_{\beta_\lambda\beta_\lambda}\cdot\alpha_\kappa, \quad (8m\text{-}8o)$$

$$\Pi_{2,03} = 3\cdot\kappa_3\cdot C_{\alpha_\kappa\beta_\lambda}\cdot\beta_\lambda, \qquad \Pi_{2,31} = \mu_3\cdot C_{\alpha_\kappa\beta_\lambda}, \qquad \Pi_{2,13} = -\kappa_3\cdot C_{\beta_\lambda\beta_\lambda}, \quad (8p\text{-}8s)$$

$$\Pi_{2,04} = \kappa_3\cdot C_{\alpha_\kappa\beta_\lambda}, \qquad \Pi_{2,40} = -\mu_3\cdot C_{\beta_\lambda\beta_\lambda} \quad (8u,8v)$$

$$Q_{1,20} = \frac{1}{2}\cdot\left(\frac{C_{\beta_j\beta_j}}{(W_{ij})^2}+\frac{C_{\beta_\lambda\beta_\lambda}}{(W_{\kappa\lambda})^2}\right), \qquad Q_{1,11} = -\left(\frac{C_{\alpha_i\beta_j}}{(W_{ij})^2}+\frac{C_{\alpha_\kappa\beta_\lambda}}{(W_{\kappa\lambda})^2}\right), \quad (9a,9b)$$

$$Q_{1,02} = \frac{1}{2}\cdot\left(\frac{C_{\alpha_i\alpha_i}}{(W_{ij})^2}+\frac{C_{\alpha_\kappa\alpha_\kappa}}{(W_{\kappa\lambda})^2}\right), \qquad Q_{1,10} = \frac{C_{\beta_\lambda\beta_\lambda}\cdot(\alpha_i-\alpha_\kappa)-C_{\alpha_\kappa\beta_\lambda}\cdot(\beta_j-\beta_\lambda)}{(W_{\kappa\lambda})^2}, \quad (9c,9d)$$

$$Q_{1,01} = \frac{C_{\alpha_\kappa\alpha_\kappa}\cdot(\beta_j-\beta_\lambda)-C_{\alpha_\kappa\beta_\lambda}\cdot(\alpha_i-\alpha_\kappa)}{(W_{\kappa\lambda})^2}, \quad (9e)$$

$$Q_{1,00} = \frac{1}{2\cdot(W_{\kappa\lambda})^2}\cdot\left(C_{\beta_\lambda\beta_\lambda}\cdot(\alpha_i-\alpha_\kappa)^2-2\cdot C_{\alpha_\kappa\beta_\lambda}\cdot(\alpha_i-\alpha_\kappa)\cdot(\beta_j-\beta_\lambda)+C_{\alpha_\kappa\alpha_\kappa}\cdot(\beta_j-\beta_\lambda)\right) \quad (9f)$$

$$Q_{2,10} = \frac{1}{(W_{ij})^2}\cdot\left(C_{\beta_j\beta_j}\cdot(\alpha_\kappa-\alpha_i)-C_{\alpha_i\beta_j}\cdot(\beta_\lambda-\beta_j)\right) \quad (9g)$$

$$Q_{2,01} = \frac{1}{(W_{ij})^2}\cdot\left(C_{\alpha_i\alpha_i}\cdot(\beta_\lambda-\beta_j)-C_{\alpha_i\beta_j}\cdot(\alpha_\kappa-\alpha_i)\right) \quad (9h)$$

$$Q_{2,00} = \frac{1}{2\cdot(W_{ij})^2}\cdot\left(C_{\beta_j\beta_j}\cdot(\alpha_\kappa-\alpha_i)^2+C_{\alpha_i\alpha_i}\cdot(\beta_\lambda-\beta_j)^2-2\cdot C_{\alpha_i\beta_j}\cdot(\alpha_\kappa-\alpha_i)\cdot(\beta_\lambda-\beta_j)\right) \quad (9i)$$

The explicit calculation of the Galerkin coefficients is tedious. The details of calculations are presented in Appendices 8-9. Herewith we present the results:

Explicit Calculation of the Galerkin coefficient $G^{(1)}_{ij,\kappa\lambda}$

The explicit calculation of the Galerkin coefficient $G^{(1)}_{ij,\kappa\lambda}$ is performed by the application of integration formulas for the up to 2nd order polynomial exponential integral derived in Appendix 8. More precisely, in Appendix 8 it is proved that when

$$Q_{20} > 0 \qquad \text{and} \qquad \frac{4\cdot Q_{02}\cdot Q_{20}-\left(Q_{11}\right)^2}{4\cdot Q_{20}} > 0\,, \tag{10a,b}$$

the following integration formula holds true (see Equ.(1)_App(8)):

$$\begin{aligned}
&\iint_{\mathbb{R}^2}\left(\Pi_{20}\cdot\alpha^2+\Pi_{11}\cdot\alpha\cdot\beta+\Pi_{02}\beta^2+\Pi_{10}\cdot\alpha+\Pi_{01}\cdot\beta+\Pi_{00}\right)\times\\
&\qquad\times\exp\left\{-\left(Q_{20}\cdot\alpha^2+Q_{11}\cdot\alpha\cdot\beta+Q_{02}\cdot\beta^2+Q_{10}\cdot\alpha+Q_{01}\cdot\beta+Q_{00}\right)d\alpha d\beta=\right.\\
&=\frac{2\cdot\pi}{\sqrt{4\cdot Q_{02}\cdot Q_{20}-\left(Q_{11}\right)^2}}\cdot\left(c_{20}\cdot\Pi_{20}+c_{11}\cdot\Pi_{11}+c_{02}\cdot\Pi_{02}+c_{10}\cdot\Pi_{10}+c_{01}\cdot\Pi_{01}+\Pi_{00}\right)\times\\
&\qquad\times\exp\left(-\frac{Q_{02}\cdot\left(Q_{10}\right)^2-Q_{11}\cdot Q_{10}\cdot Q_{01}+\left(Q_{11}\right)^2\cdot Q_{00}+Q_{20}\cdot\left(Q_{01}\right)^2-4\cdot Q_{20}\cdot Q_{02}\cdot Q_{00}}{-4\cdot Q_{20}\cdot Q_{02}+\left(Q_{11}\right)^2}\right),
\end{aligned}\tag{11}$$

where

$$c_{10}=\frac{-2\cdot Q_{02}\cdot Q_{10}+Q_{11}\cdot Q_{01}}{4\cdot Q_{20}\cdot Q_{02}-Q_{11}^2}\,, \tag{12a}$$

$$c_{01}=\frac{-2\cdot Q_{20}\cdot Q_{01}+Q_{11}\cdot Q_{10}}{4\cdot Q_{20}\cdot Q_{02}-Q_{11}^2}\,, \tag{12b}$$

$$c_{20}=\frac{2\cdot Q_{02}}{4\cdot Q_{02}\cdot Q_{20}-\left(Q_{11}\right)^2}+\frac{\left(Q_{11}\cdot Q_{01}-2\cdot Q_{02}\cdot Q_{10}\right)^2}{\left(4\cdot Q_{02}\cdot Q_{20}-\left(Q_{11}\right)^2\right)^2}, \tag{12c}$$

$$c_{11}=-\frac{Q_{11}}{4\cdot Q_{02}\cdot Q_{20}-\left(Q_{11}\right)^2}+\frac{\left(Q_{11}\cdot Q_{10}-2\cdot Q_{20}\cdot Q_{01}\cdot\right)\left(Q_{11}\cdot Q_{01}-2\cdot Q_{02}\cdot Q_{10}\right)}{\left(4\cdot Q_{02}\cdot Q_{20}-\left(Q_{11}\right)^2\right)^2}\,, \tag{12d}$$

$$c_{02}=\frac{2\cdot Q_{20}}{\left(4\cdot Q_{02}\cdot Q_{20}-\left(Q_{11}\right)^2\right)^2}+\frac{\left(-2\cdot Q_{20}\cdot Q_{01}+Q_{11}\cdot Q_{10}\right)^2}{\left(4\cdot Q_{02}\cdot Q_{20}-\left(Q_{11}\right)^2\right)^2}\,. \tag{12e}$$

Applying Equ.(11) to Equ.(6), after some extensive algebraic manipulations that we skip in this work, we find that the constrains given by Equ.(10a) and Equ.(10b), respectively, reduce to:

$$\frac{C_{\beta_j\beta_j}}{(W_{ij})^2}+\frac{C_{\beta_\lambda\beta_\lambda}}{(W_{\kappa\lambda})^2}\geq 0 \text{ and}$$

$$C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}+C_{\alpha_\kappa\alpha_\kappa}\cdot C_{\beta_\lambda\beta_\lambda}+C_{\alpha_i\alpha_i}\cdot C_{\beta_\lambda\beta_\lambda}+C_{\beta_j\beta_j}\cdot C_{\alpha_\kappa\alpha_\kappa}-\left(C_{\alpha_i\beta_j}+C_{\alpha_\kappa\beta_\lambda}\right)^2=$$
$$=\left(\sqrt{C_{\alpha_i\alpha_i}}\cdot\sqrt{C_{\beta_\lambda\beta_\lambda}}-\sqrt{C_{\beta_j\beta_j}}\cdot\sqrt{C_{\alpha_\kappa\alpha_\kappa}}\right)^2\geq 0,$$

whereas the Galerkin coefficient $G^{(1)}_{ij,\kappa\lambda}$ is given by the explicit formula:

$$G^{(1)}_{ij,\kappa\lambda}=\frac{1}{2\cdot\pi}\cdot\frac{\partial C_{\alpha_i\beta_j}\cdot(C_{\alpha_\kappa\beta_\lambda}+C_{\alpha_i\beta_j})}{\left(W_{ij,\kappa\lambda}\right)^5}\times$$
$$\times\Big[(W_{ij,\kappa\lambda})^2-(\alpha_i-\alpha_k)^2\cdot(C_{\beta_j\beta_j}+C_{\beta_\lambda\beta_\lambda})-(\beta_j-\beta_\lambda)^2\cdot(C_{\alpha_i\alpha_i}+C_{\alpha_\kappa\alpha_\kappa})+$$
$$+(\alpha_i-\alpha_k)\cdot(\beta_j-\beta_\lambda)\cdot\left(C_{\alpha_\kappa\beta_\lambda}+C_{\alpha_i\beta_j}+\frac{C_{\beta_j\beta_j}\cdot C_{\alpha_\kappa\alpha_\kappa}+C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}+C_{\alpha_\kappa\alpha_\kappa}\cdot C_{\beta_\lambda\beta_\lambda}+C_{\alpha_i\alpha_i}\cdot C_{\beta_\lambda\beta_\lambda}}{C_{\alpha_\kappa\beta_\lambda}+C_{\alpha_i\beta_j}}\right)\Bigg]\times$$
$$\times\exp\Bigg[-\frac{1}{2\cdot(W_{ij,\kappa\lambda})^2}\cdot\Big((C_{\beta_j\beta_j}+C_{\beta_\lambda\beta_\lambda})\cdot(\alpha_i-\alpha_\kappa)^2-2\cdot(C_{\alpha_\kappa\beta_\lambda}+C_{\alpha_i\beta_j})\cdot(\alpha_i-\alpha_\kappa)\cdot(\beta_j-\beta_\lambda)+$$
$$+(C_{\alpha_i\alpha_i}+C_{\alpha_\kappa\alpha_\kappa})\cdot(\beta_j-\beta_\lambda)^2\Bigg],\qquad(13a)$$

where

$$W_{ij,\kappa\lambda}=\sqrt{C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}+C_{\alpha_\kappa\alpha_\kappa}\cdot C_{\beta_\lambda\beta_\lambda}+C_{\alpha_i\alpha_i}\cdot C_{\beta_\lambda\beta_\lambda}+C_{\beta_j\beta_j}\cdot C_{\alpha_\kappa\alpha_\kappa}-(C_{\alpha_i\beta_j}+C_{\alpha_\kappa\beta_\lambda})^2}\,.\qquad(13b)$$

Let it be noted that as we can see in Equ.(13a) the Galerkin coefficients $G^{(1)}_{ij,\kappa\lambda}$ reserve a local character, since $G^{(1)}_{ij,\kappa\lambda}\to 0$, as $\left|\alpha_i-\alpha_\kappa\right|\to\infty$, $\left|\beta_j-\beta_\lambda\right|\to\infty$.

Explicit Calculation of the Galerkin coefficient $G^{(2)}_{ij,\kappa\lambda}$

Similarly, for the explicit calculation of the Galerkin coefficient $G^{(2)}_{ij,\kappa\lambda}$, integration formulae for the calculation of up to 4th order quadratic exponential integrals are applied to Equ.(7). More precisely, Equ.(1)_App(8) is used for the calculation of the quadratic exponential integrals and Equ.(2)_App.(9) for the calculation of 3,4-polynomial/quadratic-exponential integrals. Being laborious the involved substitutions are directly performed in Matlab®. Let it be noted that the local character of the KDR is retained to the Galerkin coefficient $G^{(2)}_{ij,\kappa\lambda}$ also.

4.5.5. Solution of the half oscillator problem

The problem $\mathscr{P}$ defined by Equs.(6)_Sec(4.5.3) is solved as a constraint minimization problem, using the LSQLIN function of MATLAB®. As a rule, it is assumed that $K\geq I$ and $L\geq J$, and thus, the number of equations $K\times L$ is greater than the number of unknowns $I\times J$. The numerical solution involves three steps. In the **first step** representation kernels and Galerkin kernels are identified. Concerning the representation kernels, their centers $\alpha_i,\beta_j,\ i\in N(I),\ j\in N(J)$, are placed on a regularly spaced grid over the computational domain $D_{\alpha\beta}$. The latter is defined as the Cartesian product of the essential supports of the two marginal density functions $D_{\alpha\beta} = D_\alpha \times D_\beta$. The essential support of the excitation density (D_β) is known, whereas the essential support of the response density (D_α) is initially estimated by means of the long-time equilibrium state global variance of the response $C_{xx}^{\infty}(0)$. This is accomplished by the direct solution of the two-time RE moment equations in the long-time, presented in Chapter 5. The essential support, D_α, of the excitation is assumed to be

$$D_\alpha \approx \left(-4.5 \cdot \sqrt{C_{xx}^{\infty}(0)}\, ,\, 4.5 \cdot \sqrt{C_{xx}^{\infty}(0)} \right). \tag{1}$$

$C_{xx}^{\infty}(0)$ can be calculated using Equs.(24)_Sec(5.3.1) for any smoothly correlated stochastic input. Explicit formulas providing $C_{xx}^{\infty}(0)$ when the excitation is a lpGF or an sOU process or are provided by Equs.(6,14)_Sec(5.3.2) (for $m_y^{\infty} = 0$), respectively.

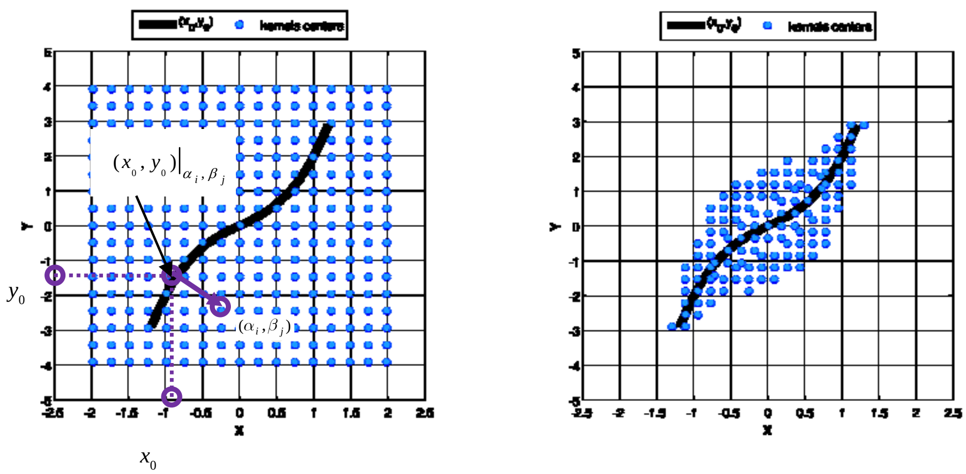

Figure 17: Non-linear/Gaussian case: a. The regularly spaced grid over the computational domain $D_{\alpha\beta}$ where the representation Kernels are placed. Each Kernel's RE correlation structure is defined by the local correlation structure of the closest point of the equilibrium curve. b. After one iteration the grid densifies around the high probability mass area.

The kernel variances $C_{\alpha_i\alpha_i}, C_{\beta_j\beta_j}$ are adjusted to $D_{\alpha\beta}$ and the resolution of the grid, aiming at a certain degree of overlapping between contiguous Kernels. For each examined case the RE equilibrium curve is found by solving Equ.(1)_Sec(4.4.1). The Kernel covariances and half-time derivatives take values from the analytically calculated local correlation structure of the closest point of the RE equilibrium curve as it is shown in **Fig.17a** for the non-linear/Gaussian case (see Equ.(6)_Sec(4.2)), and in **Fig.21a** for the non-linear/non-Gaussian case (see Equ.(8)_Sec(4.2)). More precisely, the kernel covariances $C_{\alpha_i\beta_j}$, $i \in N(I)$, $j \in N(J)$, are defined by means of the formula

$$C_{\alpha_i\beta_j} = \rho_{loc}^{(\infty)}(0) \cdot \sqrt{C_{\alpha_i\alpha_i}} \cdot \sqrt{C_{\beta_j\beta_j}}\, , \tag{2}$$

where the local correlation coefficient $\rho_{loc}^{(\infty)}(0)$ is calculated from the localized problem studied in Section 4.4, at the point $(x_0, y_0)\big|_{\alpha_i, \beta_j}$ and is given by the equation:

$$\rho_{loc}^{(\infty)}(0) = \frac{C_{x_{\ell oc}\, y_{\ell oc}}^{(\infty)}(0)}{\sqrt{C_{x_{\ell oc}\, x_{\ell oc}}^{(\infty)}(0) \cdot C_{y_{\ell oc}\, y_{\ell oc}}^{(\infty)}(0)}} \quad . \tag{3}$$

$C_{x_{\ell oc}\, y_{\ell oc}}^{(\infty)}(0)$, $C_{x_{\ell oc}\, x_{\ell oc}}^{(\infty)}(0)$ are obtained from Equs.(16,17)_Sec(4.4.1) by setting $\tau = 0$, i.e

$$C_{x_{\ell oc}\, y_{\ell oc}}^{(\infty)}(0) = \Psi'(y_0) \cdot \int_0^\infty e^{H'(x_0) \cdot u} \cdot C_{y_{\ell oc}\, y_{\ell oc}}(u)\, du \,, \tag{4}$$

$$C_{x_{\ell oc}\, x_{\ell oc}}^{(\infty)}(0) = \frac{\left(\Psi'(y_0)\right)^2}{2 \cdot \left(-H'(x_0)\right)} \times \int_{v=-\infty}^{v=+\infty} C_{y_{\ell oc}\, y_{\ell oc}}(v) \cdot e^{\,H'(x_0) \cdot |v|}\, dv \,, \tag{5}$$

and as explained in Section 4.2 particularize to Equs.(4,5)_Sec(4.4.2) for a lpGF excitation and Equs.(5,6)_Sec(4.4.3) for OU excitation. Let it be noted that the correlation coefficient $\rho_{loc}^{(\infty)}(0)$ is independent from the scaling parameter $\sigma_{y_{\ell oc}}^2$, that defines the variance of the localized excitation , i.e. $C_{y_{\ell oc}\, y_{\ell oc}}^{(\infty)} = \sigma_{y_{\ell oc}}^2\, C_{yy}^{(\infty)}(t-s) / \sigma_y^2$.

The long-time limit $\partial C_{\alpha_i \beta_j} = \lim_{s \to t} \partial C_{\alpha_i \beta_j}(t-s) / \partial t$, necessary in order to fully specify the coefficients $G_{i\,j,\kappa\lambda}$, is estimated from the localized approximation given by Equs.(19)_Sec(4.4.1) for any smoothly correlated stochastic excitation

$$\left. \frac{\partial C_{x_{\ell oc}\, y_{\ell oc}}^{(\infty)}(t-s)}{\partial t} \right|_{s \to t} = H'(x_0) \cdot \left(\Psi'(y_0) \cdot \int_0^\infty e^{H'(x_0) \cdot u} \cdot C_{y_{\ell oc}\, y_{\ell oc}}(u)\, du \right) + \Psi'(y_0) \cdot C_{y_{\ell oc}\, y_{\ell oc}}^{(\infty)}(0) \,, \tag{6}$$

by Equ.(6)_Sec.(4.4.2) for lpGF and by Equ.(7)_Sec.(4.4.3) for sOU stochastic excitation. For the calculation of the half-time derivative the scaling parameter defining the local excitation variance $C_{y_{\ell oc}\, y_{\ell oc}}^{(\infty)}(0) = \sigma_{y_{\ell oc}}^2$, is independent from the Kernel variances that are defined in terms of the of the overlapping between contiguous Kernels and is selected to be equal with the global excitation variance, i.e. $\sigma_{y_{\ell oc}}^2 = \sigma_y^2$.

In the present work the Galerkin Kernels have been selected to be identical with the representation Kernels. Other choices are also possible, and some of them have been tested successfully in various numerical experiments performed. The marginal Galerkin Kernels were selected to span the computational domain of the excitation density, with standard deviation ensuring the necessary overlapping between adjacent kernel, enabling an accurate and smooth approximation of the known density $f_y(\beta)$. Having defined all parameters appearing in the representation and Galerkin kernels, all Galerkin coefficients $G_{ij,\kappa\lambda}$, $\tilde{G}_{j(i),\tilde{\lambda}}$, $g_{\tilde{\Lambda}_{\tilde{\lambda}}}(f_y)$, can be calculated, and thus we can proceed to the **second step**, namely, the numerical solution of problem Equ.(6)_Sec(4.5.3). This is performed using LSQLIN, the constrained least squares MATLAB® function. The solution yields p_{ij}, from which a first estimate of the joint REPDF is obtained. Then, we proceed to the **third step**. Now, the solution obtained in the second step is exploited in order to estimate the essential support D_{ess}, to redistribute the kernels, and redefine the kernel parameters. More precisely, kernel centers densify where most probability mass is concentrated, space out at the low-mass areas, and vanish completely outside D_{ess} (see Figs.17b,21b) for the non-linear/Gaussian and non-linear/non-Gaussian case, respectively). The kernel variances $C_{\alpha_i\alpha_i}, C_{\beta_j\beta_j}$ are adjusted to the new grid, and Kernel parameters $C_{\alpha_i\beta_j}$, $\partial C_{\alpha_i\beta_j}$ are calculated again in the same way as in step one. Similarly, Galerkin Kernels are redefined using the new centers and new parameters of the representation kernels. With the new parameter-set, problem Equ.(6)_Sec(4.5.3)., is solved again. Within usually one or two iterations, the essential support converges, and the final solution $f_{xy}(\alpha,\beta)$ is extracted. This solution compares pretty good with the corresponding one calculated by means of the MC method.

The final p_{ij} estimations can be used in the representation formula, Equ.(1)_Sec(4.5.1), for the evaluation of the joint REPDF. The marginal pdfs can be calculated by the integration of the joint density.

4.5.6. Results

Numerical results are presented for three cases of the half oscillator discussed in Section 4.2., namely, the non-linear/Gaussian, the non-linear/non-Gaussian and the linear/non-Gaussian case (see Equs.(6,8,7)_Sec(4.2), respectively). The correlation structure of the excitation is lpGF and two different correlation times are considered for each half oscillator. The joint REPDFs $f_{xy}(\alpha,\beta)$ and the marginal pdfs $f_x(\alpha)$ derived by the numerical solution of the constraint optimization problem $\mathscr{P}$ (hereby referred to as RE solution) are systematically compared with the results obtained by a conventional Monte Carlo scheme developed by Z.G. Kapelonis (hereby referred to as MC simulation).

In **Figs**.**18-19**, the joint REPDF are shown, as using RE theory and MC simulations, respectively, for the non-linear/Gaussian case. Two different correlation times of the lpGF are considered, i.e. $\tau_{yy}^{corr} = 0.93$ sec (case a) and $\tau_{yy}^{corr} = 0.4$ sec (case b). Figs.18-19 also depict the marginals obtained by the two methods. The absolute difference between the joint REPDFs obtained by the MC and RE method for cases a, b is shown in **Fig.20** (case a, b, respectively). This difference is, in general, less than 5% in all cases, except for the high probability areas in the strongly colored case where it locally reaches a maximum of 15%. In general, the pdfs shapes, as obtained by the two methods, are very similar. In addition, the response pdfs calculated by the two methods, compare very satisfactorily, regardless of the colour strength with the difference in the response marginal been less than 2%. An interesting and somewhat

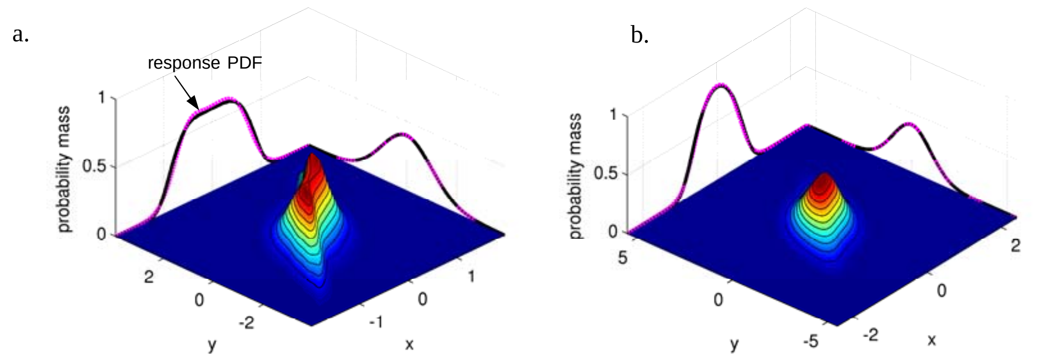

Figure 18: Non-linear/Gaussian case: REPDFs as calculated using the RE solution for lpGF stochastic input with correlation time $\tau_{yy}^{corr} = 0.93$ (case **a**) and $\tau_{yy}^{corr} = 0.4$ (case **b**). The marginal projections depict both MC (solid lines) and RE solutions (dashed lines).

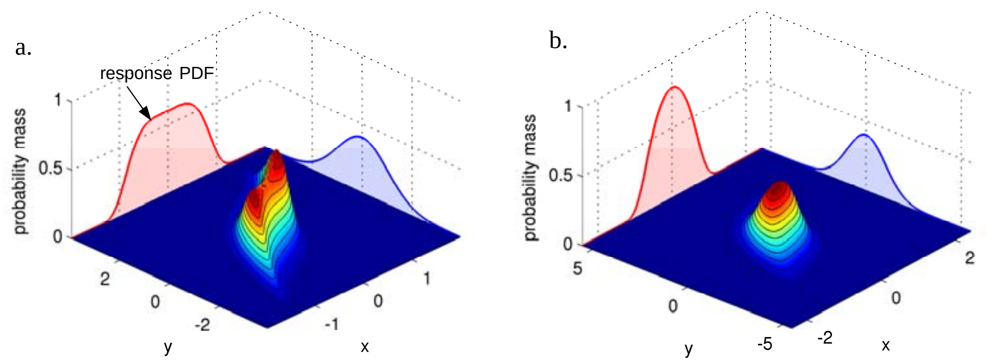

Figure 19: Non-linear/Gaussian case: REPDFs as calculated the MC solution for the same cases as in Fig.18. The projections depict the marginal pdfs.

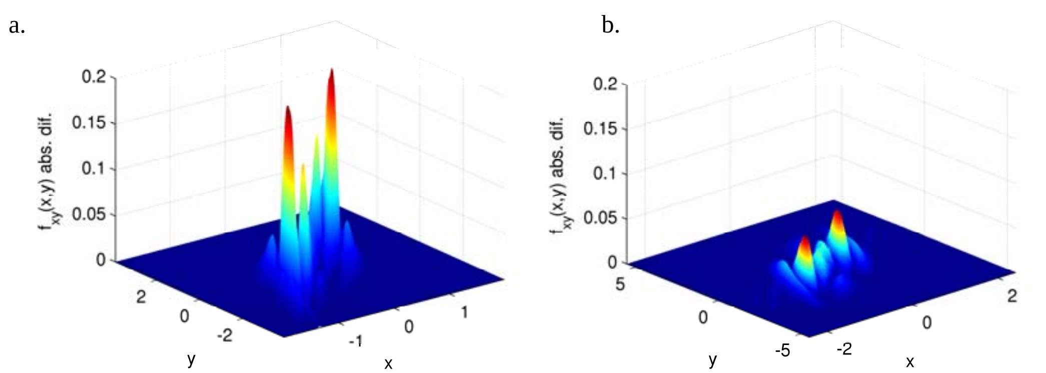

Figure 20: Non-linear/Gaussian case: The absolute difference between RE and MC solutions illustrated in Figs.18 and 19, respectively. Cases a, b are defined as in Fig.18.

surprising feature, confirmed by both methods, is that, in the case of high values of τ_{yy}^{corr}, the joint REPDF becomes bimodal, although the examined system is mono-stable (the corresponding potential is $-x^2/2 - x^4/4$). This leads to a quite flattened marginal response PDF. Note that the bi-modality of the joint REPDF, as well as the response PDF, is a well-documented feature of the bi-stable systems (Jung & Risken 1985)(Grigolini et al. 1988).

Similar results have been obtained in the non-linear/non-Gaussian case. In **Figs.22-24**, the joint REPDF are shown, as using RE theory and MC simulations for the non-linear/non-Gaussian case under lpGF input with $\tau_{yy}^{corr} = 0.93$ (case a) and $\tau_{yy}^{corr} = 0.4$ (case b). The RE solution is provided by a superposition of the representation Kernels whose RE correlation structure is defined by the correlation structure of the closest point in the equilibrium curve as shown in **Fig.21**. Comments on Figs.18-20 also apply here. Moreover, we can see that in this case the presence of non-Gaussian excitation results in joint REPDFs (Figs.22,23) with enhanced tales in comparison to the previous case that the excitation is Gaussian (Figs.18,19**).**

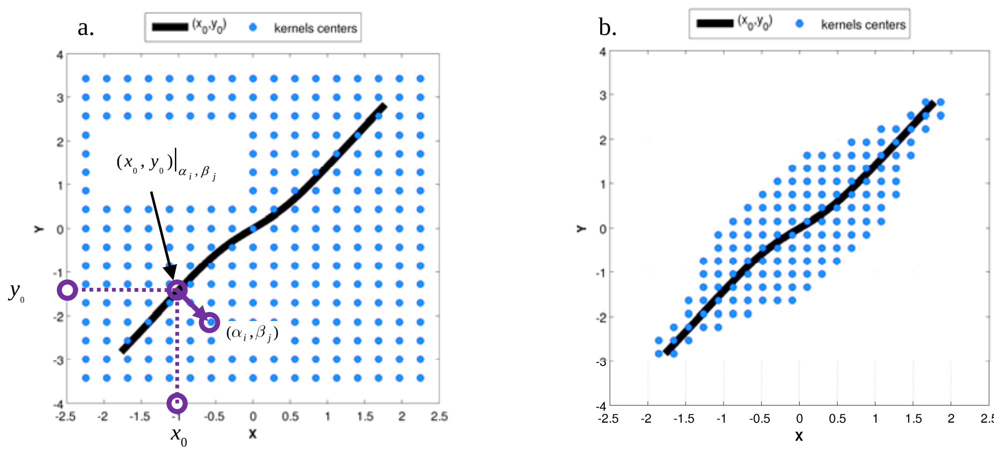

Figure 21: Non-linear/non-Gaussian case: **a.** The regularly spaced grid over the computational domain $D_{\alpha\beta}$ where the representation Kernels are placed. Each Kernels RE correlation structure is defined by the local correlation structure of the closest point of the equilibrium curve. **b.** After on iteration the grid densifies around the high probability mass area.

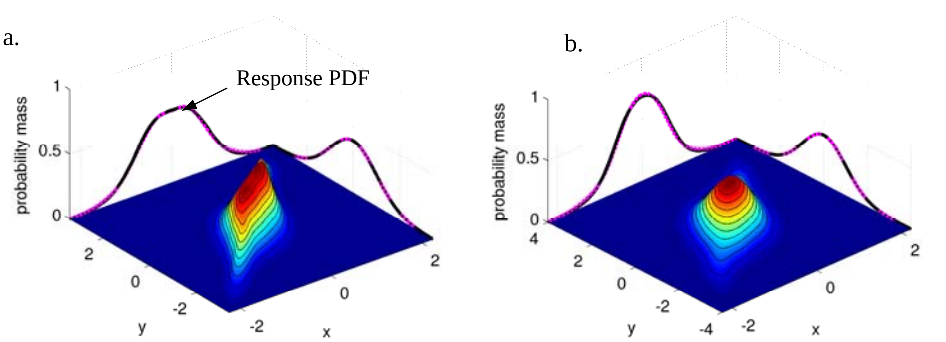

Figure 22: Non-linear/non-Gaussian case: REPDFs as calculated using the RE solution , for lpGF stochastic input with correlation time $\tau_{yy}^{corr} = 0.93$ (case **a**)and $\tau_{yy}^{corr} = 0.4$ (case **b**). The marginal projections depict both MC (solid lines) and RE solutions (dashed lines)

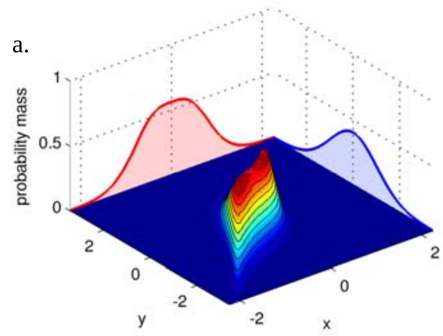

Figure 23: Non-linear/non- Gaussian case: REPDFs as calculated the MC solution for the same case as in Fig.22. The projections depict the marginal pdfs.

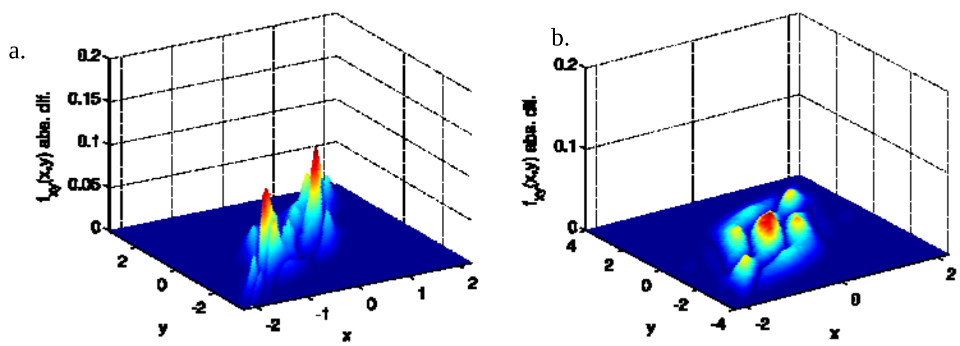

Figure 24: Non-linear/non-Gaussian case: The absolute difference between RE and MC solutions illustrated in Figs.22 and 23 respectively, for lpGF input with correlation time $\tau_{yy}^{corr} = 0.93$ (case **a**) and $\tau_{yy}^{corr} = 0.4$ (case **b**).

In **Figs.25-27** results are demonstrated for the linear/non-Gaussian case for lpGF input with excitation correlation time $\tau_{yy}^{corr} = 0.51$ (case a) and $\tau_{yy}^{corr} = 0.33$ (case b). The absolute difference between the RE (Fig.25) and MC method (Fig.26) is in general, less than 5% in all cases, except for the high probability areas in the strongly colored case where it locally reaches a maximum of 20%. The latter, local, high mismatch should be associated, with the local steepness of the corresponding PDFs. It is interesting to notice the strong deviations of the calculated PDFs from the "equivalent" 2D Gaussian distributions, for all the examined cases. The same also holds for the response densities, as intuitively expected.

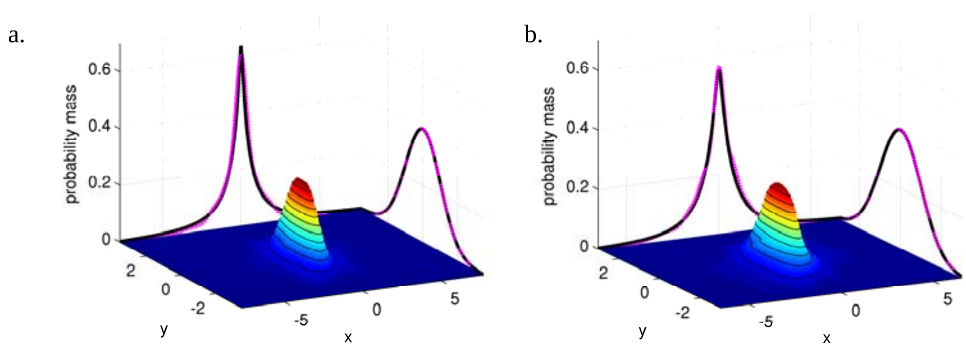

Figure 25: Linear/non-Gaussian case: REPDFs as calculated using the RE solution for lpGF stochastic input with correlation time $\tau_{yy}^{corr} = 0.51$ (case **a**) and $\tau_{yy}^{corr} = 0.33$ (case **b**). The marginal projections depict both MC (solid lines) and RE solutions (dashed lines).

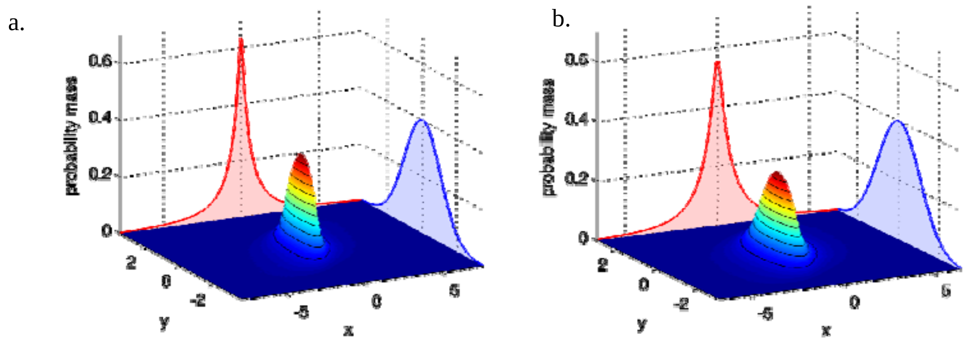

Figure 26: Linear/non-Gaussian case: REPDFs as calculated the MC solution for the same case as in Fig.25. The projections depict the marginal pdfs.

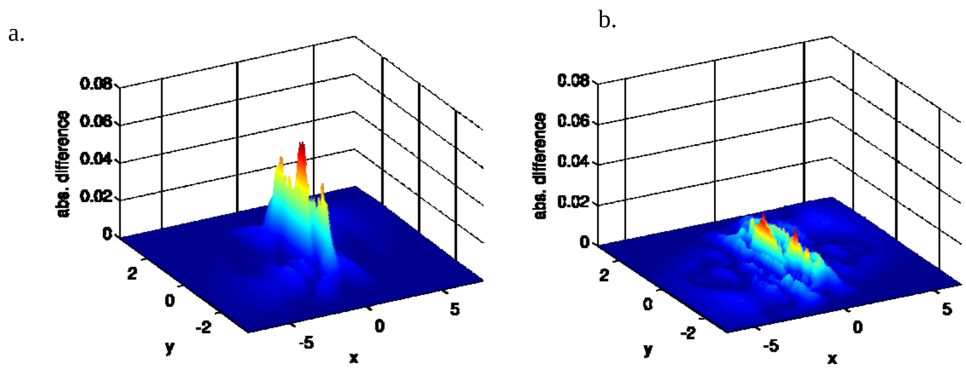

Figure 27: Non-linear/non-Gaussian case: The absolute difference between RE and MC solutions illustrated in Figs.25 and 26, respectively, for lpGF input with correlation time $\tau_{yy}^{corr} = 0.51$ (case **a**) and $\tau_{yy}^{corr} = 0.33$ (case **b**).

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CHAPTER 5

APPLICATION OF RE THEORY TO NON LINEAR DYNAMICAL SYSTEMS: TWO-TIME RE MOMENT EQUATIONS FOR NON LINEAR DYNAMICAL SYSTEMS

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5.1. Introduction

In this Chapter, we will present the formulation and solution of two-time (t,s), response-excitation (RE) moment equations for the computation of the evolution of the response mean value $m_x(t)$, response-excitation cross-covariance $C_{xy}(t,s)$ and response auto-covariance $C_{xx}(t,s)$ of a cubic half oscillator, excited by colored (Gaussian or non-Gaussian) noise.

The derivation of these moment equations is, similarly to the linear case, straightforward based on the repeated use of the dynamical equation multiplied by the excitation or the response function and been averaged. However, unlike the linear case, the non-linearity makes the moment system an infinite hierarchy of moment equations that must be closed at a certain level. Moreover, the presence of two time variables that is essential in order to capture the non-local in time effects of the colored stochastic input, makes the system not closed in time as well.

In the monostable case, to obtain a moment closure, a standard Gaussian closure assumption will be invoked. The latter will lead us to a system of two-time RE moment equations for $C_{xy}(t,s)$, $C_{xx}(t,s)$, $m_x(t)$ that is closed in terms of moments but not closed in terms of time. To obtain a time closure an exact time-closure condition will be used for the one-time moments $C_{xy}(t,t)$, $C_{xx}(t,t)$, $m_x(t)$. Let it be noted that the extra condition for the one-time moments (see Equ.(22)_Sec(3.2.1)) is not required in order to obtain a time closure in the linear case.

The two-fold closure of the two-time RE moment equations for the non-linear half oscillator will be presented both in the transient state as well as directly in the long-time limit. Subsequently, the solution is presented in the long-time only. As we have already discussed in Chapter 4, the long-time solution of the two-time RE moment equations is used to define the computational domain of the numerical scheme, for the solution of the REPDF evolution equation in the long-time. However, the two-time RE moment equations have also been numerically solved in the time domain, by the use of a two-scale iterative scheme. (see Athanassoulis, Tsantili & Kapelonis, 2013a).

Finally, a first idea for a bi-Gaussian moment closure scheme that generalizes the two-time RE moment equations for the bi-stable case will be discussed. The obtained equations will be solved directly in the long-time for an example case. The solution requires the use of auxiliary information, obtained by MC simulations, concerning the mean values of the two stable modes around which the probability mass is concentrated. Results are subsequently discussed.

5.2. Two-time RE moment equations. The monostable case

In this section we shall develop two-time RE moment equations for a monostable non-linear half oscillator given by Equ.(3)_Sec(4.2). In consistence with the linear case we shall first develop a set of differential equation, for up to second order moments of the non-linear half oscillator i.e. $m_x(t)$, $R_{xy}(t,s)$, $R_{xx}(t,s)$. Here moment equations will be also developed in terms of the central moments $C_{xy}(t,s), C_{xx}(t,s)$.

5.2.1. Derivation of the two-time RE moment equations

Applying the mean value operator to Equ.(3)_Sec(4.2), we obtain a differential equation for the mean value of the response

$$\frac{d m_x(t)}{dt} = \mu_1 \cdot m_x(t) + \mu_3 \cdot R^{21}_{xx}(t,t) + \kappa_1 \cdot m_y(t) + \kappa_3 \cdot R^{21}_{yy}(t,t), \tag{1a}$$

$$m_x(t_0) = m_0. \tag{1b}$$

Multiplying Equ.(3)_Sec(4.2) subsequently by $y(s;\theta)$, and $x(s;\theta)$, then applying the mean value operator, we get the following differential equation for the correlation functions $R_{xy}(t,s)$ and $R_{xx}(t,s)$:

$$\frac{\partial R_{xy}(t,s)}{\partial t} = \mu_1 \cdot R_{xy}(t,s) + \mu_3 \cdot R^{31}_{xy}(t,s) + \kappa_1 \cdot R_{yy}(t,s) + \kappa_3 \cdot R^{31}_{yy}(t,s), \tag{2a}$$

$$R_{xy}(t_0,s) = \mathrm{E}^{\theta}\left[x_0(\theta) \cdot y(s;\theta)\right] = m_{x_0} \cdot m_y(s) \tag{2b}$$

and

$$\frac{\partial R_{xx}(t,s)}{\partial t} = \mu_1 \cdot R_{xx}(t,s) + \mu_3 \cdot R^{31}_{xx}(t,s) + \kappa_1 \cdot R_{xy}(s,t) + \kappa_3 \cdot R^{13}_{xy}(s,t), \tag{3a}$$

$$R_{xx}(t_0,s) \equiv R_{x_0 x}(s) = \mathrm{E}^{\theta}\left[x_0(\theta) \cdot x(s;\theta)\right]. \tag{3b}$$

The initial condition (3b) is not known, since it depends on the unknown response $x(s;\theta)$. We have, thus, to derive an equation permitting us to calculate the one-time moment $R_{xx}(t_0,t) = R_{x_0 x}(t)$. Such an equation is easily obtained by multiplying Equ.(3)_Sec(4.2) by $x_0(\theta)$ and taking mean values:

$$\frac{d R_{x_0 x}(t)}{dt} = \mu_1 \cdot R_{x_0 x}(t) + \mu_3 \cdot R_{x_0 x}^{\ 3}(t) + \kappa_1 \cdot m_y(t)\, m_{x_0} + \kappa_3 \cdot R^{21}_{yy}(t,t)\, m_{x_0}. \tag{3c}$$

The initial condition for the latter equation is the known quantity

$$R_{x_0 x}(t_0) = \mathrm{E}^{\theta}\left[x_0(\theta) \cdot x_0(\theta)\right] = R_{x_0 x_0}. \tag{3d}$$

Since, $\partial R_{xx}(t,s)/\partial t \,\Big|_{s=t} \neq dR_{xx}(t,t)/dt$, differential equations (2) and (3) cannot be applied (as they stand) to the time-diagonal case $s = t$. It is possible, however, to obtain a differential equation for $R_{xx}(t,t)$.This can be done by multiplying Equ.(3)_Sec(4.2) by

$2x(t;\theta)$ and then applying the mean value operator. The resulting equation and the corresponding initial conditions are:

$$\frac{\partial R_{xx}(t,t)}{\partial t} = 2\cdot\mu_1\cdot R_{xx}(t,t) + 2\cdot\mu_3\cdot R_{xx}^{31}(t,t) + 2\cdot\kappa_1\cdot R_{xy}(t,t) + 2\cdot\kappa_3\cdot R_{xy}^{13}(t,t)\,, \tag{4a}$$

$$R_{xx}(t_0,t_0) = R_{x_0x_0}\,. \tag{4b}$$

Let it be noted that Equ.(4a) will be proved essential in order to obtain a time closure of the two-time RE moment equations (in Section 5.2.3). Since, after the moment closure (performed in Section 5.2.2.), the equations that will be obtained for the two-time RE moments will include the (also unknown) time-diagonal moment. The corresponding equation was not required to obtain a time closure in the linear case (see Equ.(22)_Sec(3.2.1), since in this case such a dependence between the two-time and the diagonal moments, does not exist.

The two-time RE cross-covariance $C_{xy}(t,s)$ and two-time response auto-covariance $C_{xx}(t,s)$, as well as the response variance $C_{xx}(t,t)$ can be calculated from the corresponding correlations and the system's mean values using the equations:

$$C_{xy}(t,s) = R_{xy}(t,s) - m_x(t)m_y(s)\,, \tag{5a}$$

$$C_{xx}(t,s) = R_{xx}(t,s) - m_x(t)m_x(s)\,. \tag{5b}$$

Nevertheless, we shall also develop equations for the evolution of the mean value of the response, the two-time RE cross-covariance, the two-time response auto-covariance.

It can be easily proved that the following identities hold true:

$$x^3(t;\theta) = \left(x(t;\theta) - m_x(t)\right)^3 + 3m_x(t)\left(x(t;\theta) - m_x(t)\right)^2 + 3m_x^2(t)\;\left(x(t;\theta) - m_x(t)\right) + m_x{}^3(t)\,, \tag{6a}$$

$$y^3(t;\theta) = \left(y(t;\theta) - m_y(t)\right)^3 + 3m_y\left(y(t;\theta) - m_y(t)\right)^2 + 3m_y^2(t)\left(y(t;\theta) - m_x(t)\right) + m_y^3(t)\,. \tag{6b}$$

Substituting (6a), (6b), in Equs.(3)_Sec(4.2) we get:

$$\begin{aligned}\dot{x}(t;\theta) &= \mu_1\cdot x(t;\theta) + \kappa_1\cdot y(t;\theta) + \\ &+ \mu_3\cdot\left[\left(x(t;\theta) - m_x(t)\right)^3 + 3m_x(t)\left(x(t;\theta) - m_x(t)\right)^2 + 3m_x^2(t)\;\left(x(t;\theta) - m_x(t)\right) + m_x{}^3(t)\right] + \\ &+ \kappa_3\cdot\left[\left(y(t;\theta) - m_y(t)\right)^3 + 3m_y\left(y(t;\theta) - m_y(t)\right)^2 + 3m_y^2(t)\left(y(t;\theta) - m_x(t)\right) + m_y^3(t)\right],\end{aligned} \tag{7a}$$

$$x(t_0;\theta) = x_0(\theta)\,. \tag{7b}$$

Applying the mean value operator in Equ.(7) we get a differential equation for the response mean value in terms of central moments:

$$\begin{aligned}\frac{dm_x(t)}{dt} &= \left(\mu_1 + \mu_2 m_x(t)^2\right)\cdot m_x(t) + \mu_3\cdot\left(C_{xx}^{30}(t,t) + 3m_x(t)\,C_{xx}(t,t)\right) + \\ &+ \left(\kappa_1 + \kappa_2 m_y(t)^2\right)\cdot m_y(t) + \kappa_3\cdot\left(C_{yy}^{30}(t,t) + 3m_y(t)\,C_{yy}(t,t)\right),\end{aligned} \tag{8a}$$

$$m_x(t_0) = m_0. \tag{8b}$$

To obtain a differential equation for the two-time RE-cross covariance $C_{xy}(t,s)$ we first multiply Equ.(7) with $y(s;\theta)-m_y(s)$ to obtain:

$$\begin{aligned}
\dot{x}(t;\theta)\big(y(s;\theta)-m_y(s)\big) &= \mu_1\cdot x(t;\theta)\big(y(s;\theta)-m_y(s)\big)+\kappa_1\cdot y(t;\theta)\big(y(s;\theta)-m_y(s)\big)\\
&+\mu_3\cdot\Big(\big(x(t;\theta)-m_x(t)\big)^3\big(y(s;\theta)-m_y(s)\big)+\\
&\quad+3m_x(t)\big(x(t;\theta)-m_x(t)\big)^2\big(y(s;\theta)-m_y(s)\big)+\\
&\quad+3\big(m_x(t)\big)^2\big(x(t;\theta)-m_x(t)\big)\big(y(s;\theta)-m_y(s)\big)+m_x^{\ 3}(t)\big(y(s;\theta)-m_y(s)\big)\Big)\\
&+\kappa_3\cdot\Big(\big(y(t;\theta)-m_y(t)\big)^3\big(y(s;\theta)-m_y(s)\big)+\\
&\quad+3m_y(t)\big(y(t;\theta)-m_y(t)\big)^2\big(y(s;\theta)-m_y(s)\big)+\\
&\quad+3\big(m_y(t)\big)^2\big(y(t;\theta)-m_y(t)\big)\big(y(s;\theta)-m_y(s)\big)+m_y^{\ 3}(t)\big(y(s;\theta)-m_y(s)\big)\Big),
\end{aligned} \tag{9a}$$

$$x(t_0;\theta)\big(y(s;\theta)-m_y(s)\big) = x_0(\theta)\big(y(s;\theta)-m_y(s)\big). \tag{9b}$$

Taking mean values in Equ. (9) and making use of the formula:

$$\frac{\partial C_{xy}(t,s)}{\partial t} = E^\theta\Big[\big(\dot{x}(t;\theta)-m_x(t)\big)\big(y(s;\theta)-m_y(s)\big)\Big] = E^\theta\Big[\dot{x}(t;\theta)\big(y(s;\theta)-m_y(s)\big)\Big], \tag{10}$$

we have:

$$\begin{aligned}
\frac{\partial C_{xy}(t,s)}{\partial t} &= \mu_1\cdot C_{xy}(t,s)+\kappa_1\cdot C_{yy}(t,s)+\\
&+\mu_3\cdot\Big(C_{xy}^{31}(t,s)+3m_x(t)\,C_{xy}^{21}(t,s)+3m_x^2(t)\,C_{xy}(t,s)\Big)+\\
&+\kappa_3\cdot\Big(C_{yy}^{31}(t,s)+3m_y(t)\,C_{yy}^{21}(t,s)+3m_y^2(t)\,C_{yy}(t,s)\Big),
\end{aligned} \tag{11a}$$

$$C_{xy}(t_0,s) = 0. \tag{11b}$$

Similarly, multiplying Equ.(7) with $x(s;\theta)-m_x(s)$, then taking mean value we get a differential equation for the two-time response auto-covariance, i.e:

$$\begin{aligned}
\frac{\partial C_{xx}(t,s)}{\partial t} &= \mu_1\cdot C_{xx}(t,s)+\kappa_1\cdot C_{xy}(s,t)+\\
&+\mu_3\cdot\Big(C_{xx}^{31}(t,s)+3m_x(t)\,C_{xx}^{21}(t,s)+3m_x^2(t)\,C_{xx}(t,s)\Big)+\\
&+\kappa_3\cdot\Big(C_{xy}^{13}(s,t)+3m_y(t)\,C_{xy}^{12}(s,t)+3m_y^2(t)\,C_{xy}(s,t)\Big),
\end{aligned} \tag{12a}$$

$$C_{xx}(t_0,s) \equiv C_{x_0 x}(s) = \mathrm{E}^{\theta}\left[x_0(\theta)\cdot x(s;\theta)\right] - m_{x_0}\cdot m_x(s)\,, \tag{12b}$$

In order to calculate the one-time moment $C_{x_0 x}(t_0,t) = C_{x_0 x}(t)$ we multiply Equ.(7) with $x_0(\theta) - m_{x_0}$. Then taking mean values we obtain:

$$\frac{dC_{x_0 x}(t)}{dt} = \left(\mu_1 + 3\mu_3 m_x^2(t)\cdot\right)C_{x_0 x}(t) + \mu_3\cdot\left(C_{x_0 x}^{\;\;3}(t) + 3m_x(t)\,C_{x_0 x}^{\;\;2}(t)\right), \tag{12c}$$

$$C_{x_0 x}(t_0) = C_{x_0 x_0}\,, \tag{12d}$$

Multiplying Equ.(7) with $2\cdot(x(t;\theta) - m_x(t))$, then take mean values we obtain an equation for the evolution of the one-time response variance, i.e:

$$\begin{aligned}\frac{dC_{xx}(t,t)}{dt} &= 2\cdot\left(\mu_1 + 3\mu_3 m_x^2(t)\right)\cdot C_{xx}(t,t) + 2\cdot\left(\kappa_1 + 3\kappa_3 m_y^2(t)\right)\cdot C_{xy}(t,t) + \\ &+ 2\cdot\mu_3\cdot\left(C_{xx}^{31}(t,t) + 3m_x(t)\,C_{xx}^{21}(t,t)\right) + 2\cdot\kappa_3\cdot\left(C_{xy}^{13}(t,t) + 3m_y(t)\,C_{xy}^{12}(t,t)\right),\end{aligned} \tag{13a}$$

$$C_{xx}(t_0,t_0) = C_{x_0 x_0} \tag{13b}$$

The five differential equations (8), (11a), (12a), (12c) and (13a) will be considered as a system of equations for the five moment functions $m_x(t)$, $C_{xy}(t,s)$, $C_{xx}(t,s)$, $C_{xx}(t,t)$. Unlike the linear case, these contain higher-order moments, like $C_{xx}^{31}(t,s)$, $C_{xx}^{21}(t,s)$. On the other hand, as in the linear case, two of them, namely Eqs. (11a) and (12a), being differential equations only with respect to the response time t, contain a second time variable s, acting as a parameter. This fact renders them in the examined here non-linear case not closed in time as well. We shall elaborate closure schemes in both respects. As far as the time closure is concerned we shall present two approaches, one for the transient state and one that closes the problem directly in the long-time statistical equilibrium state.

Lastly, in this section we shall comment on an apparently controversial feature of Equ. (12a), describing the evolution of $C_{xx}(t,s)$. In this equation, terms containing $C_{xy}(s,t)$, $C_{xy}^{12}(s,t)$ and $C_{xy}^{13}(s,t)$ are included in the last term of right-hand side, bringing into play an apparently non-causal dependence: the effect of $y(t;\theta)$ on $x(s;\theta)$, for $s<t$. This apparent controversy is resolved in the next section, by appropriately combining Equs. (11a) and (12a), resulting in a dependence of $C_{xx}(t,s)$ upon the whole history of the data moment $C_{yy}(\tau,t)$, for all $\tau\in\left[t_0,\,t\right]$. (See more on this on Athanassoulis, G. A., Tsantili, I. C., Kapelonis 2013).

5.2.2. Moment Closure of the two-time RE moment equations

To eliminate higher-order moments from the right-hand side of Equs.(8,11-13)_Sec(5.2.1), use will be made of the Gaussian closure assumption. This is the simplest moment closure assumption, introduced by Goodman and Whittle in the 50's and extensively used thenceforth in the study of random vibrations and more general stochastic dynamical systems (Lutes & Sarkani 1997; Socha 2008).

More precisely, assuming that the joint two-time REPDF $f_{x(t)y(s)}(\alpha,\beta)$ and the joint two-time response density $f_{x(t)x(s)}(\alpha_1,\alpha_2)$ are Gaussian all the third-order central moments vanishes, and the forth-order ones are expressed by means of second-order central moments, in accordance to Isserlis' Theorem (Isserlis 1918):

$$C^{31}_{xx}(t,s) = 3\cdot C_{xx}(t,t)\cdot C_{xx}(t,s), \tag{1a}$$

$$C^{31}_{xy}(t,s) = 3\cdot C_{xx}(t,t)\cdot C_{xy}(t,s), \tag{1b}$$

$$C^{13}_{xy}(s,t) = 3\cdot C_{yy}(t,t)\cdot C_{xy}(s,t), \tag{1c}$$

$$C^{31}_{yy}(t,s) = 3\cdot C_{yy}(t,t)\cdot C_{yy}(t,s), \tag{1d}$$

$$C_{x_0 x}^{3}(t) = 3\cdot C_{xx}(t,t)\cdot C_{x_0 x}(t). \tag{1e}$$

Introducing the approximations (1) into Equs.(8, 11-13)_Sec(5.2.1), we obtain:

$$\frac{dm_x(t)}{dt} = \left(\mu_1 + \mu_3 m_x^2(t) + 3\mu_3 C_{xx}(t,t)\right)\cdot m_x(t) + + \left(\kappa_1 + \kappa_3 m_y^2(t) + 3\kappa_3 C_{yy}(t,t)\right)\cdot m_y(t), \tag{2}$$

$$\frac{\partial C_{xy}(t,s)}{\partial t} = \left(\mu_1 + 3\mu_3 m_x^2(t) + 3\mu_3 C_{xx}(t,t)\right) C_{xy}(t,s) + +\left(\kappa_1 + 3\kappa_3 m_y^2(t) + 3\kappa_3 C_{yy}(t,t)\right) C_{yy}(t,s), \tag{3}$$

$$\frac{\partial C_{xx}(t,s)}{\partial t} = \left(\mu_1 + 3\mu_3 m_x^2(t) + 3\mu_3 C_{xx}(t,t)\right) C_{xx}(t,s) + + \left(\kappa_1 + 3\kappa_3 m_y^2(t) + 3\kappa_3 C_{yy}(t,t)\right)\cdot C_{xy}(s,t), \tag{4}$$

$$\frac{dC_{x_0 x}(t)}{dt} = \left(\mu_1 + 3\mu_3 m_x^2(t) + 3\mu_3 C_{xx}(t,t)\right) C_{x_0 x}(t), \tag{5}$$

$$\frac{dC_{xx}(t,t)}{dt} = 2\cdot\left(\mu_1 + 3\mu_3 m_x^2(t) + 3\mu_3 C_{xx}(t,t)\right)\cdot C_{xx}(t,t) + + 2\cdot(\kappa_1 + 3\kappa_3 m_y^2(t) + 3\kappa_3 C_{yy}(t,t))\cdot C_{xy}(t,t). \tag{6}$$

Having been working off the higher-order moments, it remains to elaborate on the second special feature of the two-time moment equations, the simultaneous appearance of the two time variables t,s.

5.2.3. Time Closure of the two-time RE moment equations

We shall now implement a time closure, obtaining an one-time, closed, causal subsystem of two equations for the time-diagonal moments $m_x(t)$ and $C_{xx}(t,t)$. Setting

$$A_x(t) \equiv A_x\left[m_x(t), C_{xx}(t,t)\right] = \mu_1 + 3\mu_3\, m_x^2(t) + 3\mu_3 C_{xx}(t,t), \tag{1a}$$

$$B_y(t) = \kappa_1 + 3\kappa_3\, m_y^2(t) + 3\kappa_3\, C_{yy}(t,t) \tag{1b}$$

we rewrite Equ.(3)_Sec(5.2.2) in the form

$$\frac{\partial C_{xy}(t,s)}{\partial t} = A_x(t)\cdot C_{xy}(t,s) + B_y(t)\cdot C_{yy}(t,s). \tag{2}$$

Although the function $A_x(t)$ is not known, it possesses two important properties: First, it is not dependent on the $C_{xy}(t,s)$ itself and, second, in the monostable case studied herewith ($\mu_1, \mu_3 < 0$), it is always negative:

$$A_x(t) < 0, \qquad \text{for all } t \geq t_0. \tag{3}$$

On the basis of the first property, we can consider Equ.(2) as a linear, first-order ODE for the two-time cross-covariance $C_{xy}(t,s)$ with respect to t (s being considered as a parameter). In accordance with the standard theory of first-order ODEs (Teschl 2012), the solution of Equ. (2) with initial condition Equ.(11b)_Sec(5.2.1), can be expressed by the following integral formula, in terms of the unknown function $A_x(t)$:

$$C_{xy}(t,s) = \int_{t_0}^{t} B_y(\tau)\, C_{yy}(\tau,s) \exp\left(\int_{\tau}^{t} A_x(u)\, du\right) d\tau. \tag{4}$$

This solution is valid for any $t \geq t_0$ and for any $s \neq t$. However, looking closer at the structure of the right-hand side of Equ.(4), we observe that it is a continuous function on s for all $s \geq t_0$ (since it depends on s only through the continuous data function $C_{yy}(\tau,s)$). Thus, taking the limit of both sides of Equ.(4) for $s \to t$, we obtain

$$C_{xy}(t,t) = \int_{t_0}^{t} B_y(\tau)\, C_{yy}(\tau,t) \exp\left(\int_{\tau}^{t} A_x\left[m_x(u), C_{xx}(u,u)\right] du\right) d\tau. \tag{5}$$

Equ.(5) expresses the time-diagonal cross-covariance $C_{xy}(t,t)$ as a non-linear, causal operator on the whole history of the (unknown) response mean value $m_x(u)$, $t_0 \le u \le t$, and response, time-diagonal auto-covariance $C_{xx}(u,u)$, $t_0 \le u \le t$.

Substituting $C_{xy}(t,t)$ from Equ. (5) in Equ.(6)_Sec(5.2.2), we obtain

$$\frac{dC_{xx}(t,t)}{dt} = 2\cdot\left(\mu_1 + 3\mu_3\, m_x^2(t) + 3\mu_3\, C_{xx}(t,t)\right)\cdot C_{xx}(t,t) + \\ + 2\cdot B_y(t)\cdot \int_{t_0}^{t} B_y(\tau)\, C_{yy}(\tau,t)\, \exp\left(\int_{\tau}^{t} \mu_1 + 3\mu_3\, m_x^2(u) + 3\mu_3\, C_{xx}(u,u)\, du \right) d\tau. \tag{6}$$

Equ.(2)_Sec(5.2.2) and Equ.(6), with initial conditions Equ.(8b)_Sec(5.2.1) and Equ.(13b)_Sec(5.2.1), respectively, form a closed, non-linear, causal system of evolution equations for the moment functions $m_x(t), C_{xx}(t,t)$.

More precisely, as discussed in Athanassoulis et al (Athanassoulis et al. 2013a), system of Equ.(6) and Equ.(2)_Sec(5.2.2) belongs to the family of causal differential systems, for which an extensive literature has been developed in the last two decades; see the books (Corduneanu 2002; Lakshmikantham et al. 2009; Burton 2005; Gripenberg et al. 1990), and the references cited there. In Athanassoulis et al (Athanassoulis et al. 2013a) discussion about local existence and uniqueness of this system is presented as well as a robust and efficient, iterative, numerical solution of this system in the whole time domain $[t_0, +\infty)$.

Representation of the off-diagonal moments $C_{xy}(t,s)$ and $C_{xx}(t,s)$ in terms of the diagonal ones

Having solved the system of Equ.(2)_Sec(5.2.2) and Equ.(6) and obtained the functions $m_x(t)$ and $C_{xx}(t,t)$, Equ.(4) provides us with the two-time cross-covariance $C_{xy}(t,s)$ in the interval $R_{ts}(T) = \left\{ (t,s) :\ t_0 \le t \le T,\ t_0 \le s \le T \right\}$. Furthermore, substituting $C_{xy}(s,t)$ in the right-hand side of Equ.(4)_Sec(5.2.2) from Equ.(4) (with arguments t and s interchanged), Equ.(4)_Sec(5.2.2) becomes a first-order ODE with known variable coefficients of the form

$$\frac{\partial C_{xx}(t,s)}{\partial t} = A_x(t)\, C_{xx}(t,s) + F_{xy}(t,s), \tag{8a}$$

where

$$F_{xy}(t,s) = \int_{\tau=t_0}^{\tau=s} B_y(\tau)\, C_{yy}(\tau,t)\, B_y(t)\, \exp\left(\int_{u=\tau}^{u=s} A_x(u)\, du \right) d\tau. \tag{8b}$$

For any $s \geq t_0$, the initial condition $C_{xx}(t_0, s) = C_{x_0 x}(s)$ is calculated by solving Equ.(5)_Sec(5.2.2) with initial condition $C_{xx}(t_0, t_0) = C_{x_0 x_0}$:

$$C_{x_0 x}(s) = C_{x_0 x_0} \cdot \exp\left(\int_{t_0}^{s} A_x(u)\, du \right). \tag{8c}$$

The solution of Equ.(8a) with initial condition Equ.(8c) is given by the formula

$$C_{xx}(t, s) = C_{x_0 x_0} \cdot \exp\left(\int_{t_0}^{t} A_x(u)\, du + \int_{t_0}^{s} A_x(u)\, du \right) +$$
$$+ \int_{\tau_1 = t_0}^{\tau_1 = t} \int_{\tau_2 = t_0}^{\tau_2 = s} G_{yy}(\tau_1, \tau_2) \exp\left(\int_{u_1 = \tau_1}^{u_1 = t} A_x(u_1)\, du_1 \right) \exp\left(\int_{u_2 = \tau_2}^{u_2 = s} A_x(u_2)\, du_2 \right) d\tau_1 d\tau_2\,, \tag{9a}$$

where

$$G_{yy}(\tau_1, \tau_2) = B_y(\tau_1)\, C_{yy}(\tau_1, \tau_2)\, B_y(\tau_2). \tag{9b}$$

The symmetry relation $C_{xx}(t, s) = C_{xx}(s, t)$ is clearly revealed by Equs.(9).

In Athanassoulis et al.(2013) it is shown that in the case where the input random process can be obtained as the solution of an Itô equation (as, e.g., happens with an Ornstein-Uhlenbeck process), the proposed non-local system is localized, leading to moment equations identical with the usual ones. The latter indicates that the presented approach consistently generalizes the Itô/filtering approach, remaining valid for any kind of covariance $C_{yy}(t,s)$. Furthermore, in (Athanassoulis et al. 2013a) the two-time RE moment system Equs.(2-6)_Sec(5.2.2) is numerically solved by means of an appropriate, two-scale, iterative scheme. The solution of the latter allows to also numerically calculate the two-time moments $C_{xy}(t, s)$ and $C_{xx}(t, s)$ using Equs.(4,9a). Numerical results are presented for lpGF, OU and sOU colored stochastic excitations. It is found that both the correlation time and the details of the shape of the input random function affect appreciably the response covariance which in general cannot be taken into account in the Itô/FPK/filtering approaches. The obtained results compare satisfactorily with extensive MC simulations results for all the examined cases except when a bi-stable half oscillator is considered, the latter is attributed to the failure of the Gaussian closure. In fact, as it will be discussed in Section 5.4., in the bi-stable case the structure of the joint REPDF, $f_{x(t)\, y(t)}(\alpha, \beta)$, becomes bi-modal, making the Gaussian closure inappropriate. In Section 5.4. a first idea for the generalization of the two-time RE moment equations for the bi-stable case, in the long-time, will be discussed. In the next section we shall present how the derived two-time moment equations can be directly solved in the long-time.

5.3. Two-time moment equations in the long-time

5.3.1. Direct solution in the long-time

In the long-time that the system reaches an equilibrium state the mean values become time independent whereas the second order moments depend only on the time lag $t - s = \tau$, i.e:

$$\lim_{t\to\infty} m_x(t) \equiv m_x^{(\infty)}, \lim_{s\to\infty} m_y(s) \equiv m_y^{(\infty)},$$

$$\lim_{\substack{t\to\infty,\ s\to\infty \\ t-s=\tau}} C_{xy}(t,s) \equiv C_{xy}^{(\infty)}(t-s) \quad \text{and} \quad \lim_{\substack{t\to\infty,\ s\to\infty \\ t-s=\tau}} C_{xx}(t,s) \equiv C_{xx}^{(\infty)}(t-s).$$

Then the two time moment equations given by Equs.(2-6)_Sec(5.2.2) take the form :

$$0 = \left(\mu_1 + \mu_3 m_x^{(\infty)} + 3\mu_3 C_{xx}^{(\infty)}(0)\right)\cdot m_x^{(\infty)} + \\ + \left(\kappa_1 + \kappa_3 m_y^{(\infty)} + 3\kappa_3 C_{yy}^{(\infty)}(0)\right)\cdot m_y^{(\infty)}, \tag{1}$$

$$\frac{\partial}{\partial t}C_{xy}^{(\infty)}(t-s) = \left(\mu_1 + 3\cdot\mu_2\cdot C_{xx}^{(\infty)}(0) + 3\mu_2 (m_x^{(\infty)})^2\right)\cdot C_{xy}^{(\infty)}(t-s) + \\ + \left(\kappa_1 + 3\cdot\kappa_2\cdot C_{yy}^{(\infty)}(0) + 3\kappa_2 \left(m_y^{(\infty)}\right)^2\right)\cdot C_{yy}^{(\infty)}(t-s), \tag{2}$$

$$\frac{\partial}{\partial t}C_{xx}^{(\infty)}(t-s) = \left(\mu_1 + 3\cdot\mu_2\cdot C_{xx}^{(\infty)}(0) + 3\mu_2 (m_x^{(\infty)})^2\right)\cdot C_{xx}^{(\infty)}(t-s) + \\ + \left(\kappa_1 + 3\cdot\kappa_2\cdot C_{yy}^{(\infty)}(0) + 3\kappa_2 (m_y^{(\infty)})^2\right)\cdot C_{xy}^{(\infty)}(s-t), \tag{3}$$

$$0 = \left(\mu_1 + 3\mu_3 (m_x^{(\infty)})^2 + 3\mu_3 C_{xx}(0)\right)\cdot C_{xx}(0) + \\ + \left(\kappa_1 + 3\kappa_3 (m_y^{(\infty)})^2 + 3\kappa_3 C_{yy}(0)\right)\cdot C_{xy}(0), \tag{4}$$

or, equivalently :

$$0 = \tilde{A}_x^{(\infty)}\cdot m_x^{(\infty)} + \tilde{B}_y^{(\infty)}\cdot m_y^{(\infty)}, \tag{5}$$

$$\frac{\partial}{\partial t}C_{xy}^{(\infty)}(t-s) = A_x^{(\infty)}\cdot C_{xy}^{(\infty)}(t-s) + B_y^{(\infty)}\cdot C_{yy}^{(\infty)}(t-s), \tag{6}$$

$$\frac{\partial}{\partial t}C_{xx}^{(\infty)}(t-s) = A_x^{(\infty)}\cdot C_{xx}^{(\infty)}(t-s) + B_y^{(\infty)}\cdot C_{xy}^{(\infty)}(s-t), \tag{7}$$

$$0 = A_x^{(\infty)}\cdot C_{xx}(0) + B_y^{(\infty)}\cdot C_{xy}(0), \tag{8}$$

where

$$\tilde{A}_x^{(\infty)} = \mu_1 + \mu_3 m_x^{(\infty)} + 3\mu_3 C_{xx}^{(\infty)}(0), \tag{9a}$$

$$B_y^{(\infty)} \equiv \kappa_1 + 3\kappa_3 (m_y^{(\infty)})^2 + 3\kappa_3 C_{yy}^{(\infty)}(0), \tag{9b}$$

$$A_x^{(\infty)} \equiv \mu_1 + 3\mu_3 (m_x^{(\infty)})^2 + 3\mu_3 C_{xx}^{(\infty)}(0), \tag{9c}$$

$$\tilde{B}_y^{(\infty)} \equiv \kappa_1 + \kappa_3 (m_y^{(\infty)})^2 + 3\kappa_3 C_{yy}^{(\infty)}(0). \tag{9d}$$

We observe that in the long-time steady state the system of two-time RE moment equations becomes a system of algebraic-differential equations containing four unknowns: the numbers $m_x^{(\infty)}$ and $C_{xx}^{(\infty)}(0)$ (appearing also in $\tilde{A}_x^{(\infty)}$ and $A_x^{(\infty)}$), and the functions $C_{xy}^{(\infty)}(t-s)$ and $C_{xx}^{(\infty)}(t-s)$.

For the monostable half oscillator $\mu_1 < 0$, $\mu_3 < 0$ therefore $A_x^{(\infty)} < 0$. Since Equs.(2,3) are linear we can apply the formulae obtained from our study for the long-time behavior of the linear half oscillator. More precisely applying Equs.(7,8b)_Sec.(3.2.3) to Equs.(2,3) we can obtain the long-time moments as limits of the transient solution:

$$C_{xy}^{(\infty)}(t-s) = \lim_{\substack{t\to\infty \\ s\to\infty \\ t-s=\tau}} \left[\cdot e^{A_x^{(\infty)}\cdot t} \cdot B_y^{(\infty)} \cdot \int_{t_0}^{t} C_{yy}^{(\infty)}(s_1 - s)\cdot e^{-A_x^{(\infty)}\cdot s_1} ds_1 \right] , \tag{10}$$

and

$$C_{xx}^{(\infty)}(t-s) = \lim_{\substack{t\to\infty \\ s\to\infty \\ t-s=\tau}} \left[e^{A_x^{(\infty)}\cdot(t+s)} \cdot (B_y^{(\infty)})^2 \cdot \int_{t_0}^{s}\int_{t_0}^{t} C_{yy}^{(\infty)}(s_1 - s_2)\cdot e^{-A_x^{(\infty)}\cdot(s_1+s_2)} ds_1\, ds_2 \right] . \tag{11}$$

Alternatively we can consider the solution of the long-time problem in the lag time domain, taking into consideration that

$$\lim_{\substack{t\to\infty,\ s\to\infty \\ t-s=\tau}} \frac{\partial C_{xy}(t,s)}{\partial t} = \frac{d C_{xy}^{(\infty)}(\tau)}{d\tau}, \tag{12a}$$

$$\lim_{\substack{t\to\infty,\ s\to\infty \\ t-s=\tau}} \frac{\partial C_{xx}(t,s)}{\partial t} = \frac{d C_{xx}^{(\infty)}(\tau)}{d\tau} . \tag{12b}$$

Equs.(5-8) can be equivalently written as:

$$0 = \tilde{A}_x^{(\infty)}\cdot m_x^{(\infty)} + \tilde{B}_y^{(\infty)}\cdot m_y^{(\infty)}, \tag{13}$$

$$\frac{\partial C_{xy}^{(\infty)}(\tau)}{\partial\tau} = A_x^{(\infty)}\cdot C_{xy}^{(\infty)}(\tau) + B_y^{(\infty)}\cdot C_{yy}^{(\infty)}(\tau), \tag{14}$$

$$\frac{\partial C_{xx}^{(\infty)}(\tau)}{\partial\tau} = A_x^{(\infty)}\cdot C_{xx}^{(\infty)}(\tau) + B_y^{(\infty)}\cdot C_{xy}^{(\infty)}(-\tau), \tag{15}$$

$$0 = A_x^{(\infty)}\cdot C_{xx}^{(\infty)}(0) + B_y^{(\infty)}\cdot C_{xy}^{(\infty)}(0). \tag{16}$$

The general solution of Equ. (14) can be written in the form

$$C_{xy}^{(\infty)}(\tau) = C_{xy}^{(\infty)}(\tau_*) \cdot \exp\left\{A_x^{(\infty)} \cdot (\tau - \tau_*)\right\} + \\ + B_y^{(\infty)} \cdot \int_{\tau_1=\tau_*}^{\tau_1=\tau} C_{yy}^{(\infty)}(\tau_1) \cdot \exp\left\{A_x^{(\infty)} \cdot (\tau - \tau_1)\right\} d\tau_1, \tag{17}$$

where τ_* is any fixed time in the lag domain. Despite the explicit appearance of τ_* in the right-hand side of Equ.(17), the latter is independent from τ_*. Since $A_x^{(\infty)} < 0$ and $C_{xy}^{(\infty)}(\tau)$ is bounded, the limit of the first term of the right-hand side of Equ.(17), as $\tau_* \to -\infty$, is zero. Thus, the choice $\tau_* = -\infty$ is legitimate, and Equ.(17) can be written in the more convenient form

$$C_{xy}^{(\infty)}(\tau) = B_y^{(\infty)} \cdot \int_{\tau_1=-\infty}^{\tau_1=\tau} C_{yy}^{(\infty)}(\tau_1) \cdot \exp\left\{A_x^{(\infty)} \cdot (\tau - \tau_1)\right\} d\tau_1, \tag{18}$$

or, equivalently, by reversing the limits of integration and performing the change of variable $u = -\tau_1$, in the form

$$C_{xy}^{(\infty)}(\tau) = B_y^{(\infty)} \cdot \int_{u=-\tau}^{u=\infty} C_{yy}^{(\infty)}(u) \cdot \exp\left\{A_x^{(\infty)} \cdot (\tau + u)\right\} du. \tag{19}$$

Equ.(15) is of the same type as Equ.(14), thus performing similar steps we obtain

$$C_{xx}^{(\infty)}(\tau) = B_y^{(\infty)} \cdot \int_{\tau_1=-\infty}^{\tau_1=\tau} C_{xy}^{(\infty)}(-\tau_1) \cdot \exp\left\{A_x^{(\infty)} \cdot (\tau - \tau_1)\right\} d\tau_1, \tag{20}$$

where $C_{xy}^{(\infty)}(-\tau_1)$ follows Equ.(19), that is

$$C_{xx}^{(\infty)}(\tau) = \left(B_y^{(\infty)} \right)^2 \cdot \int_{\tau_1=-\infty}^{\tau_1=\tau} \int_{\tau_2=\tau_1}^{\tau_2=+\infty} C_{yy}^{(\infty)}(\tau_2) \cdot \exp\left\{A_x^{(\infty)} \cdot (\tau - 2 \cdot \tau_1 + \tau_2)\right\} d\tau_2 \, d\tau_1. \tag{21}$$

The domain of integration of the repeated integrals appearing in the right-hand side of Equ. (21) is shown in Fig. (1).

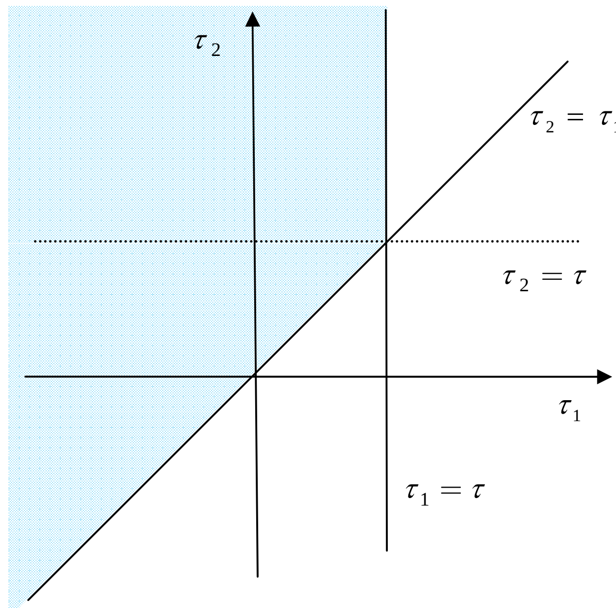

Fig. 1. The domain of integration in the $(\tau_1, \tau_2)-$plane

The τ_1 integration can be explicitly performed by changing the order of integration:

$$\int_{\tau_1=-\infty}^{\tau_1=\tau} \int_{\tau_2=\tau_1}^{\tau_2=+\infty} \{\ldots\} = \int_{\tau_2=-\infty}^{\tau_2=\tau} \int_{\tau_1=-\infty}^{\tau_1=\tau_2} \{\ldots\} + \int_{\tau_2=\tau}^{\tau_2=+\infty} \int_{\tau_1=-\infty}^{\tau_1=\tau} \{\ldots\}. \qquad (22)$$

Changing the order of integration, as shown in Equ.(22), Equ.(21) becomes

$$C_{xx}^{(\infty)}(\tau) = \left(B_y^{(\infty)} \right)^2 \cdot \left(\int_{\tau_2=-\infty}^{\tau_2=\tau} C_{yy}^{(\infty)}(\tau_2) \cdot e^{A_x^{(\infty)} \cdot (\tau+\tau_2)} \int_{\tau_1=-\infty}^{\tau_1=\tau_2} e^{A_x^{(\infty)} \cdot (-2\cdot\tau_1)} \, d\tau_1 \, d\tau_2 + \right.$$

$$\left. + \int_{\tau_2=\tau}^{\tau_2=+\infty} C_{yy}^{(\infty)}(\tau_2) \cdot e^{A_x^{(\infty)} \cdot (\tau+\tau_2)} \int_{\tau_1=-\infty}^{\tau_1=\tau} e^{A_x^{(\infty)} \cdot (-2\cdot\tau_1)} \, d\tau_1 \, d\tau_2 \right) =$$

$$= -\frac{\left(B_y^{(\infty)} \right)^2}{2\cdot A_x^{(\infty)}} \cdot \left(\int_{\tau_2=-\infty}^{\tau_2=\tau} C_{yy}^{(\infty)}(\tau_2) \cdot e^{A_x^{(\infty)} \cdot (\tau-\tau_2)} d\tau_2 + \right.$$

$$\left. \int_{\tau_1=\tau}^{\tau_2=+\infty} C_{yy}^{(\infty)}(\tau_2) \cdot e^{A_x^{(\infty)} \cdot (-\tau+\tau_2)} \, d\tau_2 \right) =$$

$$= \frac{\left(B_y^{(\infty)} \right)^2}{2\cdot A_x^{(\infty)}} \cdot \int_{\tau_2=+\infty}^{\tau_2=-\infty} C_{yy}^{(\infty)}(\tau_2) \cdot e^{A_x^{(\infty)} \cdot |\tau+\tau_2|} d\tau_2 \; .$$

That is:

$$C_{xx}^{(\infty)}(\tau) = \left(B_y^{(\infty)} \right)^2 \frac{1}{\left(-2 \cdot A_x^{(\infty)}\right)} \int_{u=-\infty}^{u=+\infty} C_{yy}^{(\infty)}(u) \cdot \exp\left\{ A_x^{(\infty)} \cdot \left| u + \tau \right| \right\} du . \tag{23}$$

Integral representations (19) and (23) together with Equ.(13) cannot completely solve the limiting form of the two -time RE moment system (Equs.(13–16)) since the unknown values of the response long-time moments $m_x^{(\infty)}$ and $C_{xx}^{(\infty)}(0)$ are contained in the constant $A_x^{(\infty)}$. However, we can obtain two equations for these unknowns $m_x^{(\infty)}$ and $C_{xx}^{(\infty)}(0)$ from Equ.(13) and Equ.(23). More precisely setting $\tau = 0$ Equ.(23) becomes

$$\begin{aligned} C_{xx}^{(\infty)}(0) \cdot \left(\mu_1 + 3\mu_3 \, m_x^{(\infty)} + 3\mu_3 \, C_{xx}^{(\infty)}(0) \right) = \\ = -\left(B_y^{(\infty)} \right)^2 \cdot \int_{u=0}^{u=+\infty} C_{yy}^{(\infty)}(\tau_2) \cdot \exp\left\{ A_x^{(\infty)} \cdot u \right\} du . \end{aligned} \tag{24}$$

Remark: Setting $\tau = 0$ to Equ.(19) we have

$$C_{xy}^{(\infty)}(0) = B_y^{(\infty)} \cdot \int_{u=0}^{u=+\infty} C_{yy}^{(\infty)}(\tau_1) \cdot \exp\left\{ A_x^{(\infty)} \cdot u \right\} du , \tag{25}$$

then using Equ.(16), Equ.(24) is re-obtained. That is, in order to solve the system of two-time RE moment equations directly in the long-time, the time closure can be obtained with two equivalent ways, one of which does not involve the use of Equ.(15).

Combining Equs. (13), (19), (23) and (24) we can have a closed long-time solution. In fact:

- If $\mu_3 = 0$ (linear half oscillator) the system of Equs. (13), (19), (23) is trivialized since the constant $A_x^{(\infty)}$ does not contain the unknown $C_{xx}^{(\infty)}(0)$, providing explicit solutions directly, i.e. from Equ.(13)

$$m_x^{(\infty)} = -\frac{\left(\kappa_1 + \kappa_3 \, m_y^{(\infty)} + 3\kappa_3 \, C_{yy}^{(\infty)}(0) \right)}{\mu_1} \cdot m_y^{(\infty)} , \tag{26}$$

  then, from Equ.(19) and Equ.(23) it is straightforward to obtain $C_{xy}^{(\infty)}(\tau)$, $C_{xx}^{(\infty)}(\tau)$.

- If $\mu_3 \neq 0$, it is convenient to distinguish two cases. When $m_y^{(\infty)} = 0$, then it is easy to see from Equ.(13) that $m_x^{(\infty)} = 0$ as well, then Equ.(24) is simplified to

$$C_{xx}^{(\infty)}(0)\cdot\left(\mu_1+3\mu_3\,C_{xx}^{(\infty)}(0)\right)= \\ =-\left(B_y^{(\infty)}\right)^2\cdot\int_{u=0}^{u=+\infty}C_{yy}^{(\infty)}(u)\cdot\exp\left\{-\left(\;\mu_1+3\mu_3\,C_{xx}^{(\infty)}(0)\;\right)\cdot u\right\}du, \tag{27}$$

which has a unique solution in the positive real axis. (The uniqueness can be theoretically proved in some cases, and has always been confirmed numerically). Having defined $C_{xx}^{(\infty)}(0)$ it is straightforward to obtain $C_{xy}^{(\infty)}(\tau)$, $C_{xx}^{(\infty)}(\tau)$ from Equ.(19) and Equ.(23).

- If $\mu_3\neq 0$ and $m_y^{(\infty)}\neq 0$, we can use Equs.(9a, 13) to express $C_{xx}^{(\infty)}(0)$, and therefore $A_x^{(\infty)}$, in terms of $m_x^{(\infty)}$, i.e

$$C_{xx}^{(\infty)}(0)\;=\;-\frac{1}{3\cdot\mu_3}\left[\frac{\tilde{B}_y^{(\infty)}\cdot m_y^{(\infty)}}{m_x^{(\infty)}}+\mu_1+\mu_3\left(m_x^{(\infty)}\right)^2\right], \tag{28}$$

then, from Equ.(9c) and Equ.(28) we have

$$A_x^{(\infty)}\;=\;2\mu_3\cdot\left(m_x^{(\infty)}\right)^2-\frac{\tilde{B}_y^{(\infty)}}{m_x^{(\infty)}}\cdot m_y^{(\infty)}. \tag{29}$$

Using Equ.(28) and Equ.(29) we can then eliminate $C_{xx}^{(\infty)}(0)$ from Equ.(24), obtaining:

$$\left(\frac{\tilde{B}_y^{(\infty)}\cdot m_y^{(\infty)}}{3\mu_3\,m_x^{(\infty)}}-\frac{2}{3}\left(m_x^{(\infty)}\right)^2\right)\cdot\left(\frac{\tilde{B}_y^{(\infty)}\cdot m_y^{(\infty)}}{m_x^{(\infty)}}+\mu_1+\mu_3\left(m_x^{(\infty)}\right)^2\right)+ \\ +\left(B_y^{(\infty)}\right)^2\int_{u=0}^{u=\infty}C_{yy}^{(\infty)}(u)\,\exp\left\{\left(-\frac{\tilde{B}_y^{(\infty)}\cdot m_y^{(\infty)}}{m_x^{(\infty)}}+2\mu_3\left(m_x^{(\infty)}\right)^2\right)\cdot u\right\}du\;=\;0. \tag{30}$$

If multiple solutions occur in Equ. (30), we keep the one leading to positive $C_{xx}^{(\infty)}(0)$ (using Equ. (27)). Having found $m_x^{(\infty)}$, $C_{xx}^{(\infty)}(0)$ it is straightforward to obtain $C_{xy}^{(\infty)}(\tau)$, $C_{xx}^{(\infty)}(\tau)$ from Equ.(19) and Equ.(23).

5.3.2. Analytic computation of the long-time moments for lpGH, OU and sOU stochastic input correlation function

In this paragraph we are going to implement the obtained formulae for the **long-time RE cross- covariance** $C_{xy}^{(\infty)}(t-s)$ **and long-time response auto-covariance** $C_{xx}^{(\infty)}(t-s)$ for the case that the random input, $y(t;\theta)$, is a lpGF, a sOU or an OU process. Use shall be made of corresponding results obtained in Sec(3.2.4). in the linear case for lpGF, sOU or OU random input. The latter is justified since the integrals that provide the long-time moments (see Equs.(10,11)_Sec(5.3.1) or Equs(19,23)_Sec(5.3.1)) are of the same type as the ones given for the non-central moments of the linear problem (see Equs.(7,8b)_Sec(3.2.3) or Equs.(17,18)_Sec(3.2.3), respectively).

Low-pass Gaussian filter (lpGF)

In case that the excitation is a lpGF

$$C_{yy}^{(\infty)}(t-s) = \sigma_y^2 \exp\left(-a(t-s)^2\right), \tag{1}$$

setting $A = A_x^{(\infty)}$, $B = B_y^{(\infty)}$ to Equs.(11,15)_Sec(3.2.4) we get

$$C_{xy}^{(\infty)}(t-s) = \\ = B_y^{(\infty)} \sigma_y^2 \frac{\sqrt{\pi}}{2\sqrt{a}} \exp\left(\frac{(A_x^{(\infty)})^2}{4 \cdot a} + A_x^{(\infty)} \cdot (t-s) \right) \cdot \left(erf\left(\sqrt{a} \cdot (t-s) + \frac{A_x^{(\infty)}}{2 \cdot \sqrt{a}} \right) + 1 \right), \tag{2}$$

and

$$C_{xx}^{(\infty)}(t-s) = -\frac{\sqrt{\pi}}{4\sqrt{a}} \cdot \frac{(B_y^{(\infty)})^2 \cdot \sigma_y^2}{A_x^{(\infty)}} \cdot e^{\frac{(A_x^{(\infty)})^2}{4 \cdot a}} \times \\ \times \left(e^{A_x^{(\infty)} \cdot (s-t)} \cdot \left(erf\left(\sqrt{a} \cdot (s-t) + \frac{A_x^{(\infty)}}{2 \cdot \sqrt{a}} \right) + 1 \right) + e^{A_x^{(\infty)} \cdot (t-s)} \left(erf\left(\sqrt{a} \cdot (t-s) + \frac{A_x^{(\infty)}}{2 \cdot \sqrt{a}} \right) + 1 \right) \right), \tag{3}$$

where

$$B_y^{(\infty)} \equiv \kappa_1 + 3\kappa_3 (m_y^{(\infty)})^2 + 3\kappa_3 C_{yy}^{(\infty)}(0), \tag{4a}$$

$$A_x^{(\infty)} \equiv \mu_1 + 3\mu_3 (m_x^{(\infty)})^2 + 3\mu_3 C_{xx}^{(\infty)}(0). \tag{4b}$$

In order to determine the unknowns $m_x^{(\infty)}$ and $C_{xx}^{(\infty)}(0)$ appearing in Equs.(2,3) three cases are distinguished:

- If $\mu_3 = 0$, $m_x^{(\infty)}$ is given by Equ.(26)_Sec(5.3.1), (we repeat here for convenience)

$$m_x^{(\infty)} = -\frac{\left(\kappa_1 + \kappa_3 m_y^{(\infty)} + 3\kappa_3 C_{yy}^{(\infty)}(0)\right)}{\mu_1} \cdot m_y^{(\infty)} \quad . \tag{5}$$

  In this case $C_{xx}^{(\infty)}(0)$ does not need to be *a priori* computed.

- If $\mu_3 \neq 0$, $m_y^{(\infty)} = 0$ then $m_x^{(\infty)} = 0$ and the long-time response variance $C_{xx}^{(\infty)}(0)$ is given by Equ.(3) setting $t = s$ i.e.

$$C_{xx}^{(\infty)}(0) = -\frac{\sqrt{\pi}}{2\sqrt{a}} \cdot \frac{(B_y^{(\infty)})^2 \cdot \sigma_y^2}{A_x^{(\infty)}} \cdot e^{\frac{(A_x^{(\infty)})^2}{4 \cdot a}} \cdot \left(erf\left(\frac{A_x^{(\infty)}}{2 \cdot \sqrt{a}} \right) + 1 \right). \tag{6}$$

- If $\mu_3 \neq 0$ and $m_y^{(\infty)} \neq 0$, using Equ(28,29)_Sec(5.3.1) we can express $C_{xx}^{(\infty)}(0)$ and $A_x^{(\infty)}$ in terms of $m_x^{(\infty)}$ (we repeat her for convenience)

$$C_{xx}^{(\infty)}(0) = -\frac{1}{3\cdot\mu_3}\left[\frac{\tilde{B}_y^{(\infty)}\cdot m_y^{(\infty)}}{m_x^{(\infty)}} + \mu_1 + \mu_3\left(m_x^{(\infty)}\right)^2\right], \tag{7}$$

$$A_x^{(\infty)} = 2\mu_3\cdot\left(m_x^{(\infty)}\right)^2 - \frac{\tilde{B}_y^{(\infty)}}{m_x^{(\infty)}}\cdot m_y^{(\infty)}. \tag{8}$$

then combining Equs(6-8) we can obtain a closed equation for the (non-zero) response mean value $m_x^{(\infty)}$:

$$\begin{aligned}
&-\frac{1}{3\cdot\mu_3}\left(\frac{\tilde{B}_y^{(\infty)}\cdot m_y^{(\infty)}}{m_x^{(\infty)}} + \mu_1 + \mu_3\left(m_x^{(\infty)}\right)^2\right)\cdot\left(2\mu_3\cdot\left(m_x^{(\infty)}\right)^2 - \frac{\tilde{B}_y^{(\infty)}}{m_x^{(\infty)}}\cdot m_y^{(\infty)}\right) + \\
&\quad + \frac{\sqrt{\pi}}{2\sqrt{a}}\cdot(B_y^{(\infty)})^2\cdot\sigma^2\cdot\exp\{-4\cdot a\}\cdot\exp\left\{2\mu_3\cdot\left(m_x^{(\infty)}\right)^2 - \frac{\tilde{B}_y^{(\infty)}}{m_x^{(\infty)}}\cdot m_y^{(\infty)}\right\}\times \\
&\quad \times\left[\left(erf\left(2\mu_3\cdot\left(m_x^{(\infty)}\right)^2 - \frac{\tilde{B}_y^{(\infty)}}{m_x^{(\infty)}}\cdot m_y^{(\infty)}\right)\cdot\frac{1}{2\cdot\sqrt{a}}\right) + 1\right] = 0,
\end{aligned} \tag{9}$$

where $\tilde{B}_y^{(\infty)}$, $B_y^{(\infty)}$ are given by Equs.(9b,9d)_Sec(5.3.1) respectively .

Sifted Ornstein-Uhlenbeck (sOU) stochastic process

In the case that the excitation is a of a sOU process:

$$C_{yy}^{(\infty)}(t-s) = \sigma^2\cdot\exp\left(-a\cdot\left|t-s\right|\right)\cdot\cos\left(\omega_0\cdot(t-s)\right), \tag{10}$$

setting $A = A_x^{(\infty)}$, $B = B_y^{(\infty)}$ to Equs.(32,33,36)_Sec(3.2.4) we get

$$\begin{aligned}
C_{xy}^{(\infty)}(t-s)\Big|_{t\geq s} &= B_y^{(\infty)}\cdot\sigma_y^2\cdot\Bigg[\left(\frac{a - A_x^{(\infty)}}{(a - A_x^{(\infty)})^2 + \omega_0^2} + \frac{A_x^{(\infty)} + a}{(A_x^{(\infty)} + a)^2 + \omega_0^2}\right)\cdot e^{A_x^{(\infty)}\cdot(t-s)} + \\
&\quad + \frac{e^{-a\cdot(t-s)}}{(A_x^{(\infty)} + a)^2 + \omega_0^2}\cdot\left(-(A_x^{(\infty)} + a)\cdot\cos\left(\omega_0\cdot(t-s)\right) + \omega_0\cdot\sin\left(\omega_0\cdot(t-s)\right)\right)\Bigg],
\end{aligned} \tag{11a}$$

$$\begin{aligned}
&C_{xy}^{(\infty)}(t-s)\Big|_{t<s} = \\
&\quad = B_y^{(\infty)}\cdot\sigma_y^2\cdot\frac{e^{a\cdot(t-s)}}{(a-A_x^{(\infty)})^2 + \omega_0^2}\cdot\left((a-A_x^{(\infty)})\cdot\cos\left(\omega_0\cdot(t-s)\right) + \omega_0\cdot\sin\left(\omega_0\cdot(t-s)\right)\right),
\end{aligned} \tag{11b}$$

and

$$C_{xx}^{(\infty)}(t-s)=$$

$$=\frac{\left(B_y^{(\infty)}\right)^2\cdot\sigma_y^2}{\left((a-A_x^{(\infty)})^2+\omega_0^2\right)\cdot\left((A_x^{(\infty)}+a)^2+\omega_0^2\right)}\cdot\left[\frac{-e^{A\cdot|t-s|}}{A_x^{(\infty)}}\cdot a\cdot\left(\left(a^2-(A_x^{(\infty)})^2\right)+\omega_0^2\right)+\right.$$
$$\left.+e^{-a\cdot|t-s|}\cdot\left[\left(-\left(a^2-(A_x^{(\infty)})^2\right)+\omega_0^2\right)\cdot\cos\left(\omega_0\cdot|t-s|\right)+2\cdot a\cdot\omega_0\cdot\sin\left(\omega_0\cdot|t-s|\right)\right]\right],\tag{12}$$

where $A_x^{(\infty)}$, $B_y^{(\infty)}$ are given by Equs.(9c,9b)_Sec(5.3.1).

In order to determine the unknowns $m_x^{(\infty)}$ and $C_{xx}^{(\infty)}(0)$ appearing in Equs.(11,12) three cases are distinguished:

- If $\mu_3=0$, the response mean value $m_x^{(\infty)}$ is given by Equ.(26)_Sec(5.3.1) :

$$m_x^{(\infty)}=-\frac{\left(\kappa_1+\kappa_3\,m_y^{(\infty)}+3\kappa_3\,C_{yy}^{(\infty)}(0)\right)}{\mu_1}\cdot m_y^{(\infty)},\tag{13}$$

  In this case $C_{xx}^{(\infty)}(0)$ does not need to be *a priori* computed.

  If $\mu_3\neq 0$, $m_y^{(\infty)}=0$ then $m_x^{(\infty)}=0$, whereas setting $t=s$ in Equ.(12) we obtain:

$$C_{xx}^{(\infty)}(0)=\left(B_y^{(\infty)}\right)^2\cdot\sigma_y^2\,\frac{a-A_x^{(\infty)}}{\left(-A_x^{(\infty)}\right)\left\{\left(a-A_x^{(\infty)}\right)^2+\omega_0^2\right\}}\tag{14}$$

- If $\mu_3\neq 0$, $m_y^{(\infty)}\neq 0$ using Equ(28,29)_Sec(5.3.1) we can express $C_{xx}^{(\infty)}(0)$ and $A_x^{(\infty)}$ in terms of $m_x^{(\infty)}$:

$$C_{xx}^{(\infty)}(0)=-\frac{1}{3\cdot\mu_3}\left[\frac{\tilde{B}_y^{(\infty)}\cdot m_y^{(\infty)}}{m_x^{(\infty)}}+\mu_1+\mu_3\left(m_x^{(\infty)}\right)^2\right],\tag{15}$$

$$A_x^{(\infty)}=2\mu_3\cdot\left(m_x^{(\infty)}\right)^2-\frac{\tilde{B}_y^{(\infty)}}{m_x^{(\infty)}}\cdot m_y^{(\infty)},\tag{16}$$

  then combining Equs(14-16) we can obtain a closed equation for the (non-zero) response mean value $m_x^{(\infty)}$

$$-\frac{1}{3\cdot\mu_3}\cdot\left(\frac{\tilde{B}_y^{(\infty)}\cdot m_y^{(\infty)}}{m_x^{(\infty)}}+\mu_1+\mu_3\,(m_x^{(\infty)})^2\right)\cdot\left(2\,\mu_3\,(m_x^{(\infty)})^2-\frac{\tilde{B}_y^{(\infty)}}{m_x^{(\infty)}}\cdot m_y^{(\infty)}\right)+$$

$$+\left(B_y^{(\infty)}\right)^2\cdot\sigma_y^2\cdot\frac{a-2\,\mu_3\,(m_x^{(\infty)})^2+\dfrac{\tilde{B}_y^{(\infty)}}{m_x^{(\infty)}}\cdot m_y^{(\infty)}}{\left\{\left(a-2\,\mu_3\cdot(m_x^{(\infty)})^2+\dfrac{\tilde{B}_y^{(\infty)}}{m_x^{(\infty)}}\cdot m_y^{(\infty)}\right)^2+\omega_0^2\right\}}=0 \quad (17),$$

where $\tilde{B}_y^{(\infty)}$, $B_y^{(\infty)}$ are given by Equs.(9b,9d)_Sec(5.2.4) respectively.

In case that the excitation is an OU process, i.e. $C_{yy}^{(\infty)}(t-s)=\sigma^2\cdot\exp\left(-a\cdot\left|t-s\right|\right)$, the cor-responding long-time moments are obtained setting $\omega_0=0$ to Equs.(11-12, 14,17)

5.3.3. Results

The direct solution in the long-time of the two-time RE moment equations is used to calculate the long-time RE cross-covariance $C_{xy}^{(\infty)}(t-s)$ and the long-time response auto-covariance $C_{xx}^{(\infty)}(t-s)$. The calculations are performed by the use of Equs.(2-9)_Sec(5.3.2) for lpGF input and of Equs.(11-17)_Sec(5.3.2) for either sOU or OU input. The solution of the transcendental equations Equs.(9,17)_Sec(5.3.2), required in the non-linear cases ($\mu_3\neq 0$) when the excitation has non-zero asymptotic mean value ($m_y^{(\infty)}\neq 0$), was performed numerically in Matlab.

In **Figs.2-10** results are illustrated for three half-oscillators, more specifically, for the non-linear/Gaussian, the linear/non-Gaussian and the non-linear/non-Gaussian considered in Section 4.2.(see Equs.(6-8)_Sec(4.2)). The input function $y(t;\theta)$ has been assumed to be either lpGF (Figs.2,5), OU (Figs.3,6,8-10) or sOU with central frequency $\omega_0=5$ (Figs.4,7). In all cases the input variance $\sigma_y^2=1$, whereas three different cases of input correlation time have been considered i.e. $\tau_{yy}^{\text{corr}}=0.93, 0.51, 0.33$. For the input mean value, we also consider three cases, $m_y^{(\infty)}=0$ (Figs.2-8), $m_y^{(\infty)}=0.3$ (Fig.9) and $m_y^{(\infty)}=1.5$ (Fig.10). The parameters in Figs.2-8 have been selected to be the same as in the linear/Gaussian case discussed in Chapter 3 in order to examine the effects of the non-linearity or/and the non-Gaussianity in the asymptotic covariances.

In **Figs.2-4** the long-time RE cross-covariance $C_{xy}^{(\infty)}(t-s)$ (case a in Figs(2-4)) and the long-time response auto-covariance $C_{xx}^{(\infty)}(t-s)$ (case b in Figs(2-4)) for the non-linear/Gaussian case are illustrated. Comparing these results with the ones obtained in Chapter 3, and more precisely by comparing Fig.2 with Fig.6_Sec(3.2.4), Fig.3 with Fig.18_Sec(3.2.4) and Fig.4 with Fig.17_Sec(3.2.4), we observe that, in all cases, the non-linearity leads to smaller values of the response moments. This is expected since the non-linear term has negative coefficient, amplifying the restoring force of the linear term.

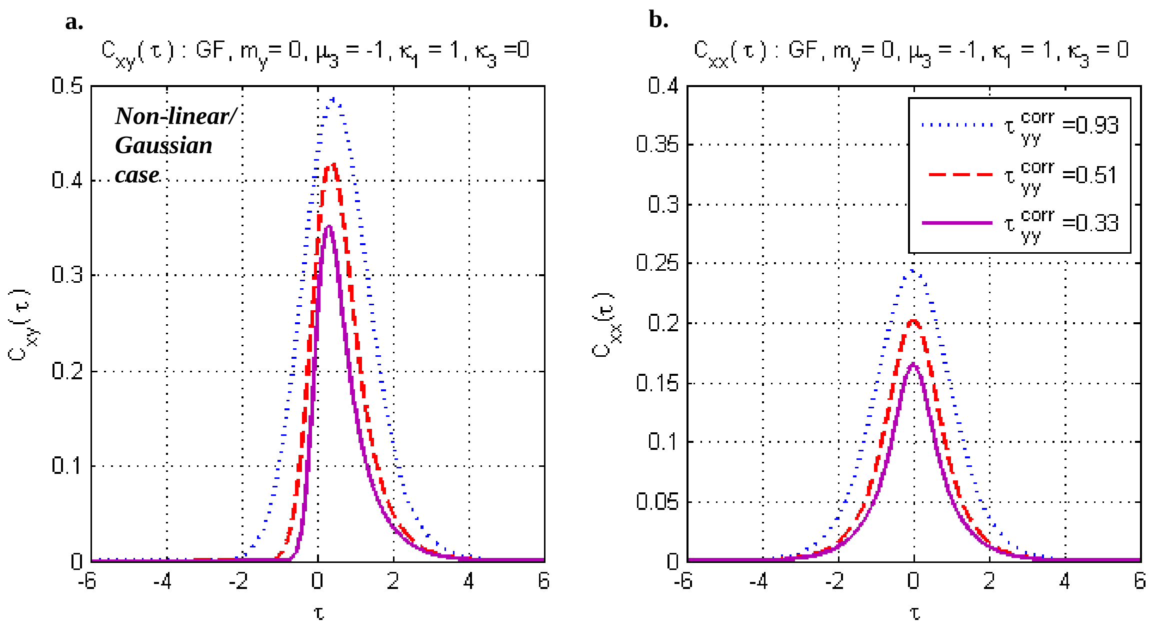

Figure 2: The long time response-excitation cross-covariance $C_{xy}^{(\infty)}(\tau)$ (where $\tau = t - s$) (a) and response covariance $C_{xx}^{(\infty)}(\tau)$ (b), of the non-linear/Gaussian case under Gaussian lpGF excitation with $m_y^{(\infty)} = 0$, results are plotted for three cases of the excitation correlation time $\tau_{yy}^{corr} = 0.93, 0.51, 0.33$.

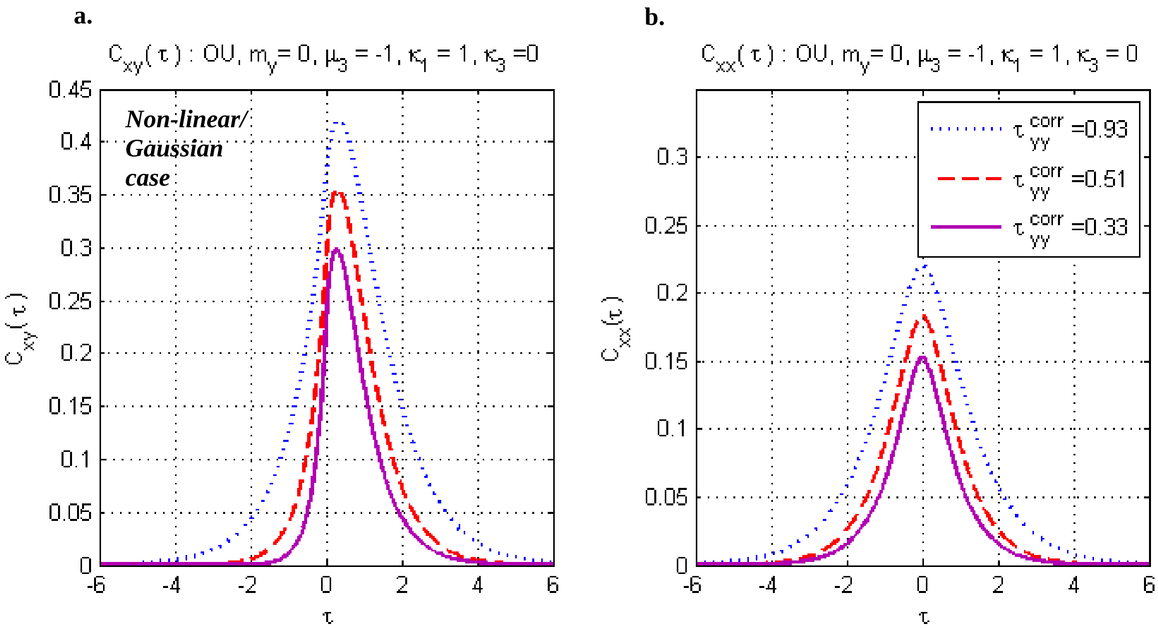

Figure 3: The same as Fig.2 but now the system has OU excitation.

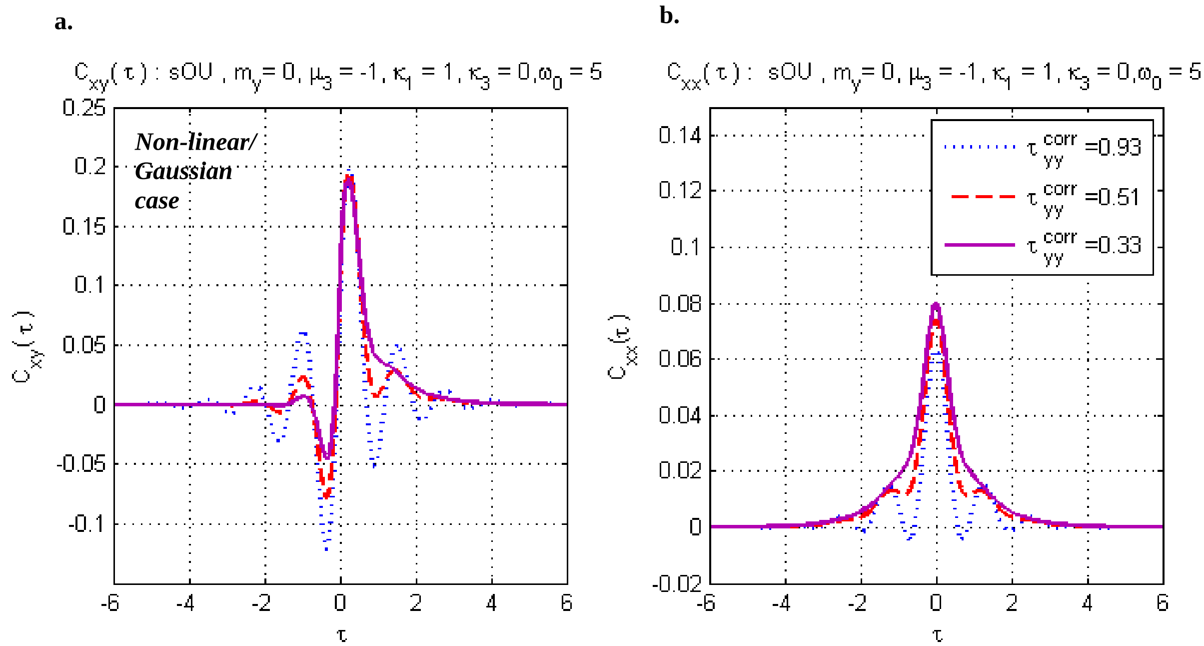

Figure 4: The same as Fig.2 but now the system has sOU excitation with central frequency $\omega_0 = 5$

a. b.

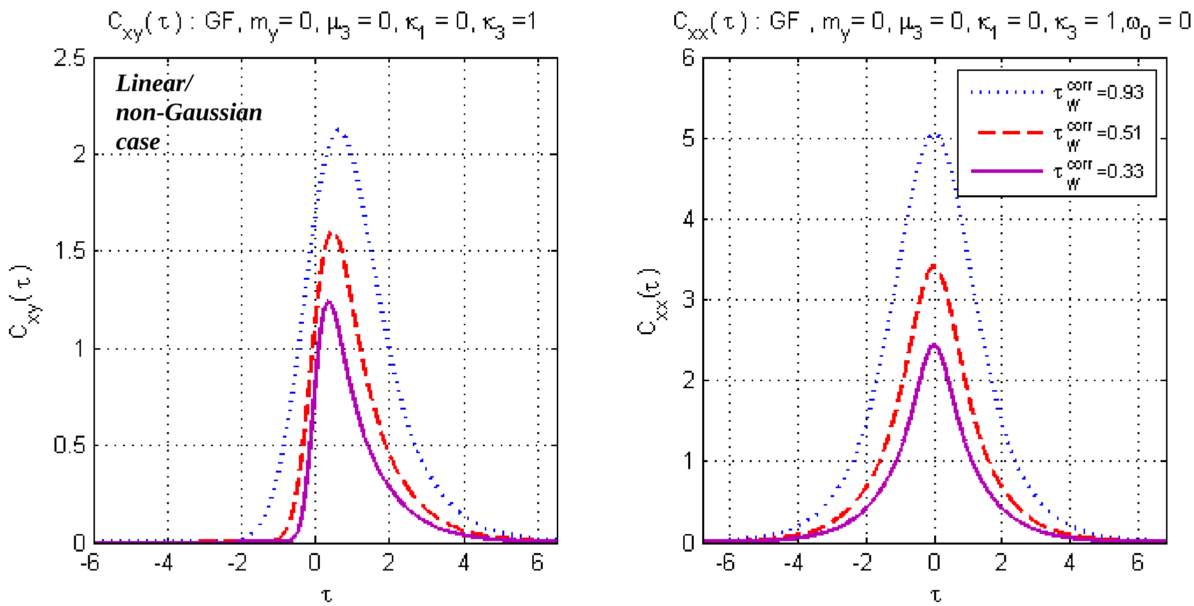

Figure 5: Long time response-excitation cross-covariance $C_{xy}^{(\infty)}(\tau)$ (where $\tau = t - s$) (a) and response covariance $C_{xx}^{(\infty)}(\tau)$ (b), of the lineal/non-Gaussian case linear under lpGF excitation with $m_y^{(\infty)} = 0$, results are plotted for three cases of the excitation correlation time $\tau_{yy}^{corr} = 0.93, 0.51, 0.33$.

a. **b.**

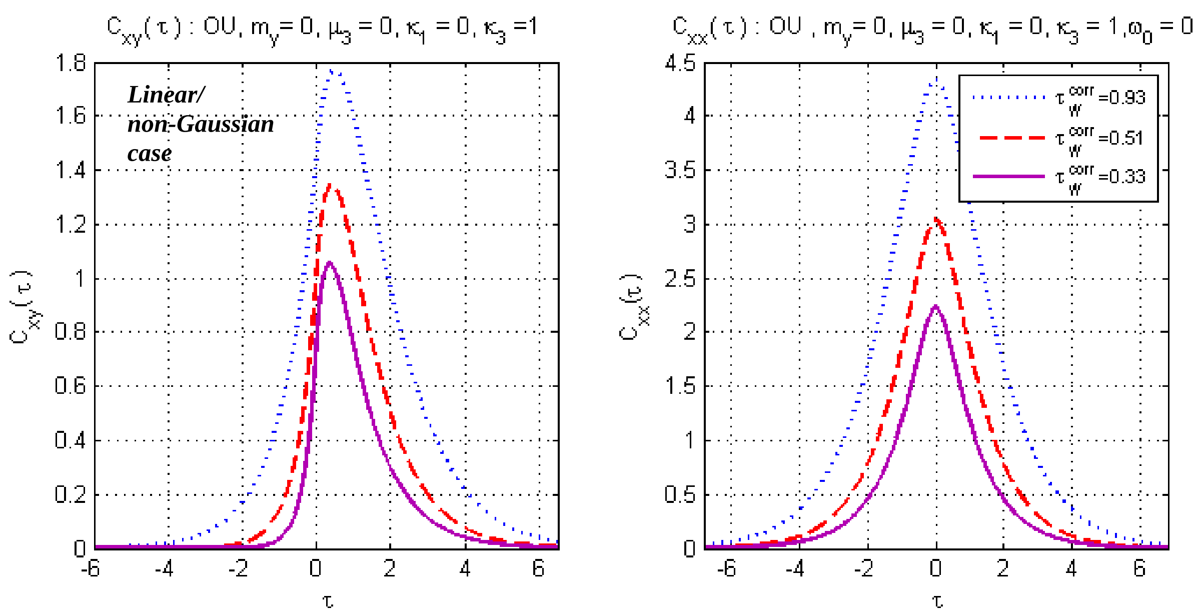

Figure 6: Same as Fig.5 but now the system has OU excitation.

On the contrary the results for the linear/non-Gaussian case, illustrated in **Figs.5-7** indicate that the non-Gaussianity has opposite effects than the non-linearity leading to significantly larger values of the response moments. In the non-linear/non-Gaussian case illustrated in **Fig.8**, for OU stochastic input, we can also observe the effect of non-Gaussianity to the non-linear system, the latter resulting to moments with increased values comparing to the non-linear/Gaussian case Fig.3

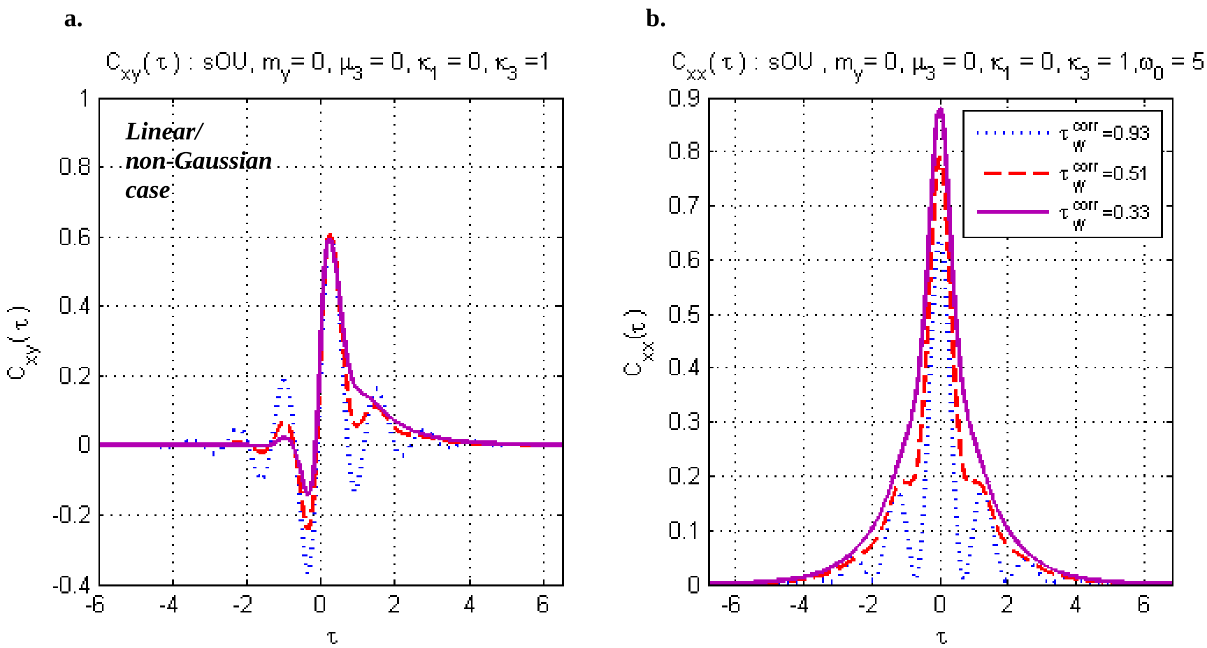

Figure 7: The same as Fig.5 but now the system has sOU excitation with central frequency $\omega_0 = 5$.

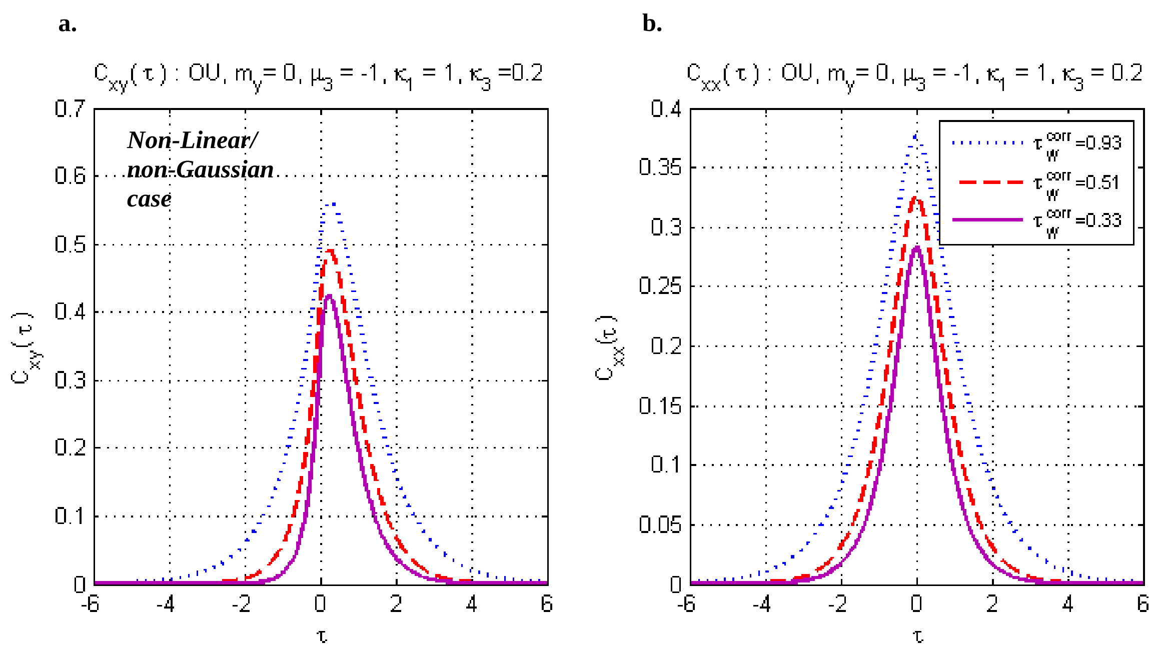

Figure 8: Long time response-excitation cross-covariance $C_{xy}^{(\infty)}(\tau)$ (where $\tau = t - s$) (a) and response covariance $C_{xx}^{(\infty)}(\tau)$ (b), of the non-linear/non-Gaussian case under OU excitation with $m_y^{(\infty)} = 0$, results are plotted for three cases of the excitation correlation time $\tau_{yy}^{\text{corr}} = 0.93, 0.51, 0.33$.

In all the examined cases in Figs.2-8, the general rule is that the values of the asymptotic RE cross-covariance $C_{xy}^{(\infty)}(t-s)$ (case a) and the asymptotic response auto-covariance $C_{xx}^{(\infty)}(t-s)$ (case b) increase with the excitation correlation time. Moreover, in all cases, for the same central frequency of the excitation, the frequency of both $C_{xy}^{(\infty)}(t-s)$ and $C_{xx}^{(\infty)}(t-s)$ increase with the correlation time of the excitation (see Figs.4,7). In addition, in all the examined cases there is a correlation of the response with future values of the excitation (for $\tau = t - s < 0$), a feature that tends to vanish as the correlation time of the excitation decreases. Another, important finding here is that the two-time RE moments are significantly affected by the shape of the input function (lpGF vs OU vs sOU), especially for the most correlated case when all other parameters are the same. The latter is an essential advantage of the introduced two-time RE moment equations since that kind of details, cannot, in general, be taken into account by the Itô/filtering approach.

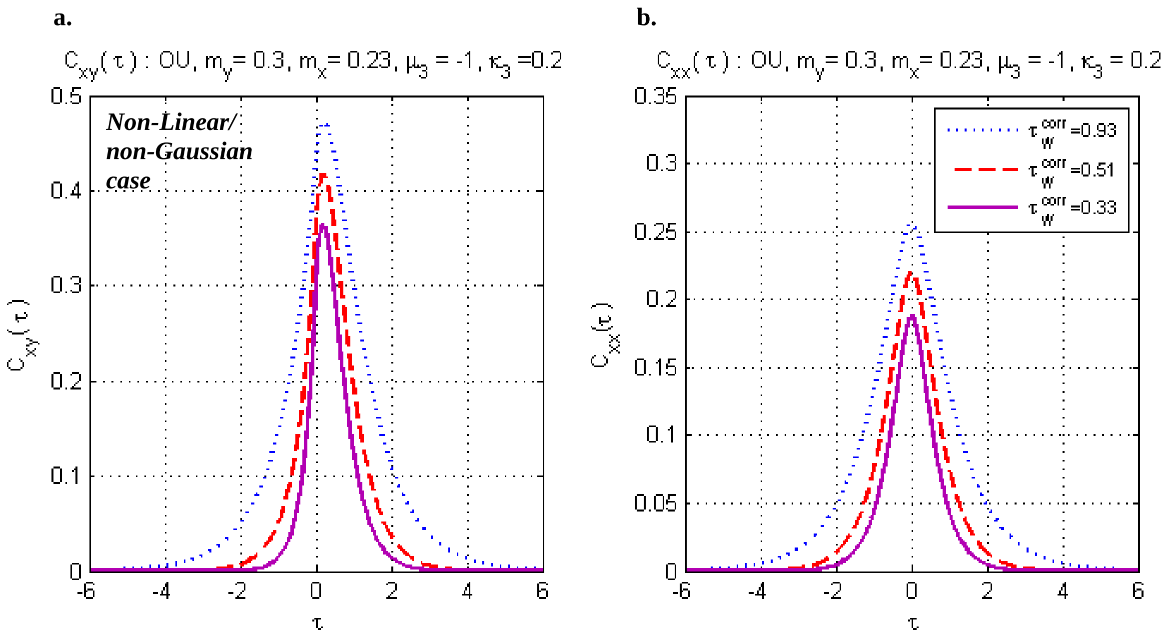

Figure 9: The same as Fig.8 but now the system's input has mean value $m_y^{(\infty)} = 0.3$

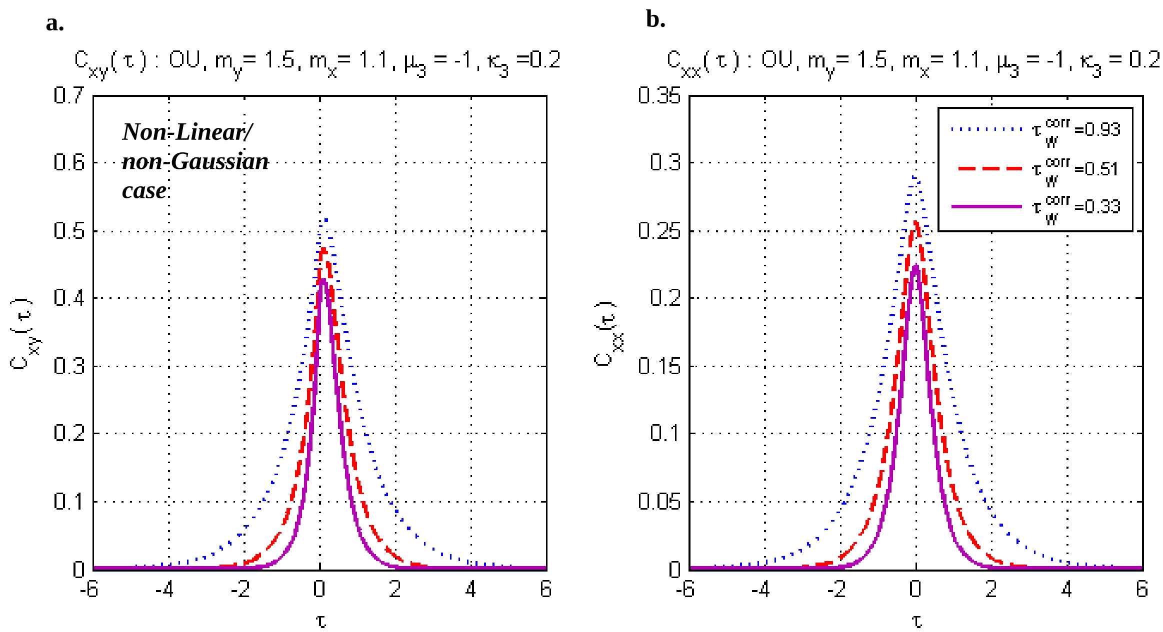

Figure 10: The same as Fig.9 but now the system's input has mean value $m_y^{(\infty)} = 1.5$

In **Figs.9,10** the same case as in Fig. 8 is considered for different input mean values, $m_y^{(\infty)} = 0.3$ (in Fig.9) and $m_y^{(\infty)} = 1.5$ (in Fig.10). As we can see, the long-time excitation mean values does affect the long-time (co)variances.

In order to confirm the obtained results MC simulations have also been performed by Z. Kapelonis. In **Fig.11-12** the covariances $C_{xy}^{(\infty)}(\tau)$ and $C_{xx}^{(\infty)}(\tau)$ are plotted, along with corresponding MC results for various cases of the half-oscillator (Equ.(3)_Sec(4.2)), under OU input functions $y(t;\theta)$, with $m_y^{(\infty)} = 0$, $\sigma_y^2 = 1.0$, $\tau_{yy}^{\text{corr}} = 1.0$ and $\omega_0 = 0$ (case a in figures) or $\omega_0 = 2$ (case b figures). In **Fig.11** a mildly nonlinear case ($\mu_1 = -1.0$, $\mu_3 = -0.2$) is considered under Gaussian excitation ($\kappa_1 = 1.0$, $\kappa_3 = 0$). In **Fig.12** similar results are shown for stronger nonlinearity ($\mu_1 = -1.0$, $\mu_3 = -0.5$) under non-Gaussian excitation (with $\kappa_1 = 1.0$, $\kappa_3 = 0.5$). The analytical results presented in Figs.11 and 12 are confirmed by MC simulations (shown by bullets and open triangles in the figures), displaying excellent agreement in the mildly nonlinear case (Fig. 11) and slight discrepancies (especially near the local extremes) in the stronger nonlinear/ non-Gaussian case (Fig. 12). The latter can be attributed to the combined effect of parameters μ_3 and κ_3, which makes the Gaussian closure assumption less accurate. More comparisons of the obtained results with MC simulations can be found in (G.A. Athanassoulis et al. 2013b).

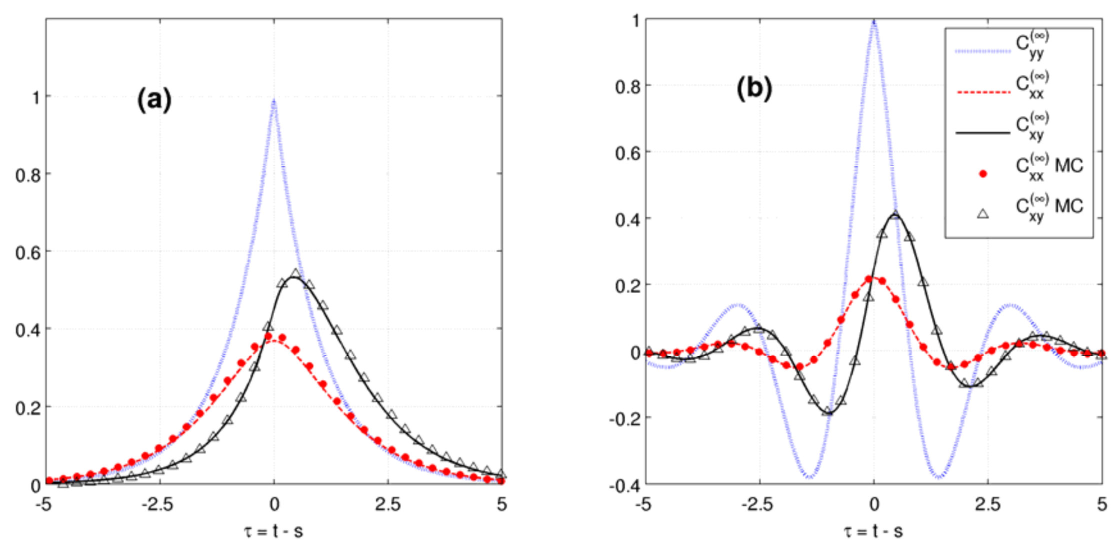

Figure 11: Excitation auto-covariance $C_{yy}^{(\infty)}(\tau)$, response-excitation cross-covariance $C_{xy}^{(\infty)}(\tau)$, and response auto-covariance $C_{xx}^{(\infty)}(\tau)$ for a non-linear/Gaussian case ($\mu_1 = -1.0$, $\mu_3 = -0.2$, $\kappa_1 = 1.0, \kappa_3 = 0$) excited by an OU random function $y(t;\theta)$ with $m_y^{(\infty)} = 0$, $\sigma_y^2 = 1.0$, $\tau_{yy}^{\text{corr}} = 1.0$, and central spectral frequency $\omega_0 = 0$ (a), and $\omega_0 = 2$ (b). Bullets and open triangles denote MC simulation results.

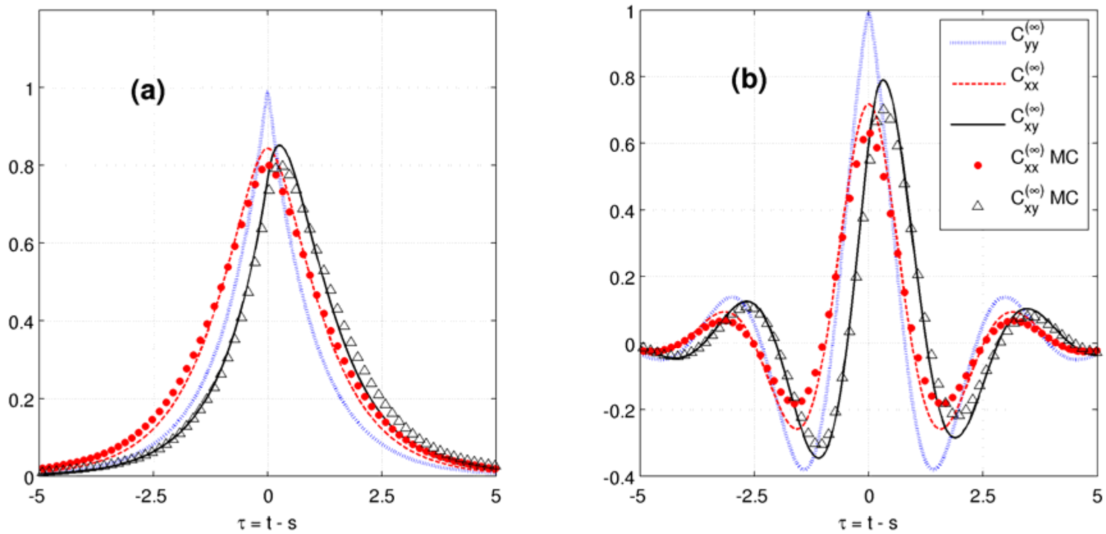

Fig. 12. The same as Fig. 11, but with $\mu_3 = -0.5$, $\kappa_3 = 0.5$. In this case the excitation is non-Gaussian. Central spectral frequency is again $\omega_0 = 0$ (a), and $\omega_0 = 2$ (b).

5.4. Two-time RE moment equations. The bi-stable Case

The introduced system of the two-time, RE moment equations for a cubic half-oscillator, excited by colored (Gaussian or non-Gaussian) noise, was made solvable by the use of a moment closure and time closure. Time closure is exact (given the moment closure), however moment closure is approximate and valid as long as the joint two-time REPDF $f_{x(t)y(s)}(\alpha,\beta)$ and the joint two-time response pdf $f_{x(t)x(s)}(\alpha_1,\alpha_2)$, of the studied system, remain close to the Gaussian structure.

Several cases where the Gaussian closure has led to inadequate results have been extensively discussed for stochastic differential equations excited by white or second order filtered white noises. More precisely, this method is considered unsuitable when the system is strongly non-linear, has multiplicative random excitations (Er 1998) or has more than one stable points (Hasofer & Grigoriu 1995), (Grigoriu 2008). A case where the strong non-Gaussian character of the excitation made the Gaussian assumption incompatible with the system's REPDFs and response pdfs is also discussed in Athanassoulis et al. (2013).

In this section we shall make an attempt to generalize the presented methodology to treat problem that the Gaussian closure is not a valid assumption. To this end, we shall consider the closure of the two-time RE moment equations for a bi-stable half oscillator under Gaussian excitation. The closure and solution of the moment system will be performed directly in the long-time limit. As a test case we shall use the bi-stable/Gaussian half oscillator that is described by Equ.(10)_Sec(4.2), assuming that this is excited by an OU random function $y(t;\theta)$, with $m_y^{(\infty)} = 0$, $\sigma_y^2 = 2.0$, $\tau_{yy}^{\text{corr}} = 0.5$. In Figs.13-15 joint densities corresponding

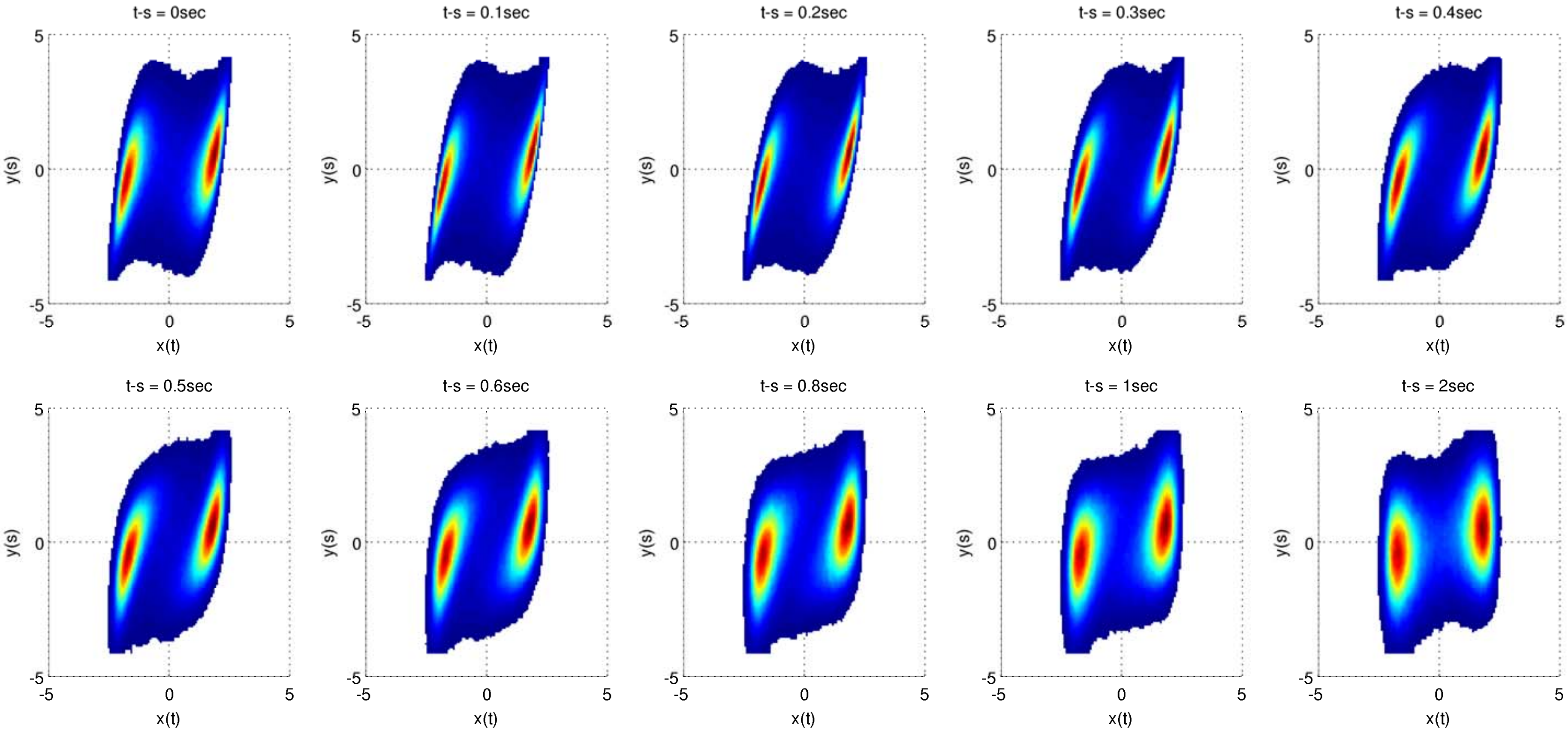

Figure 13: The long time statistical equilibrium joint REPDF $f_{xy}(x_1, y_1; \tau)$ for a bistable/Gaussian half oscillator with OU input functions $y(t;\theta)$, with $m_y^{(\infty)} = 0$, $\sigma_y^2 = 2.0$, $\tau_{yy}^{\text{corr}} = 0.5$, obtained by MC simulations, for positive time lags $\tau = t - s$ between the response and the excitation

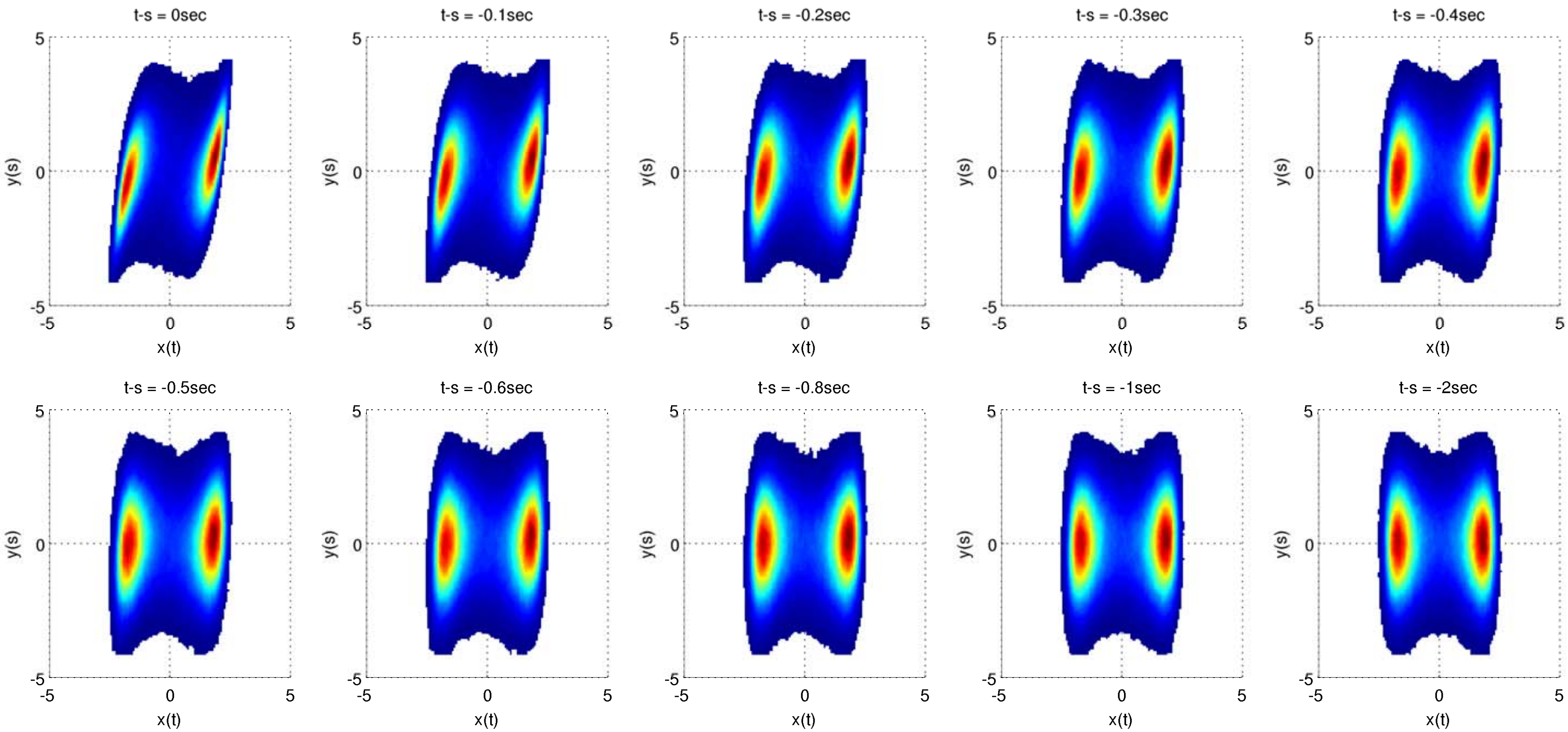

Figure 14: The same as Fig.13 now for negative time lags

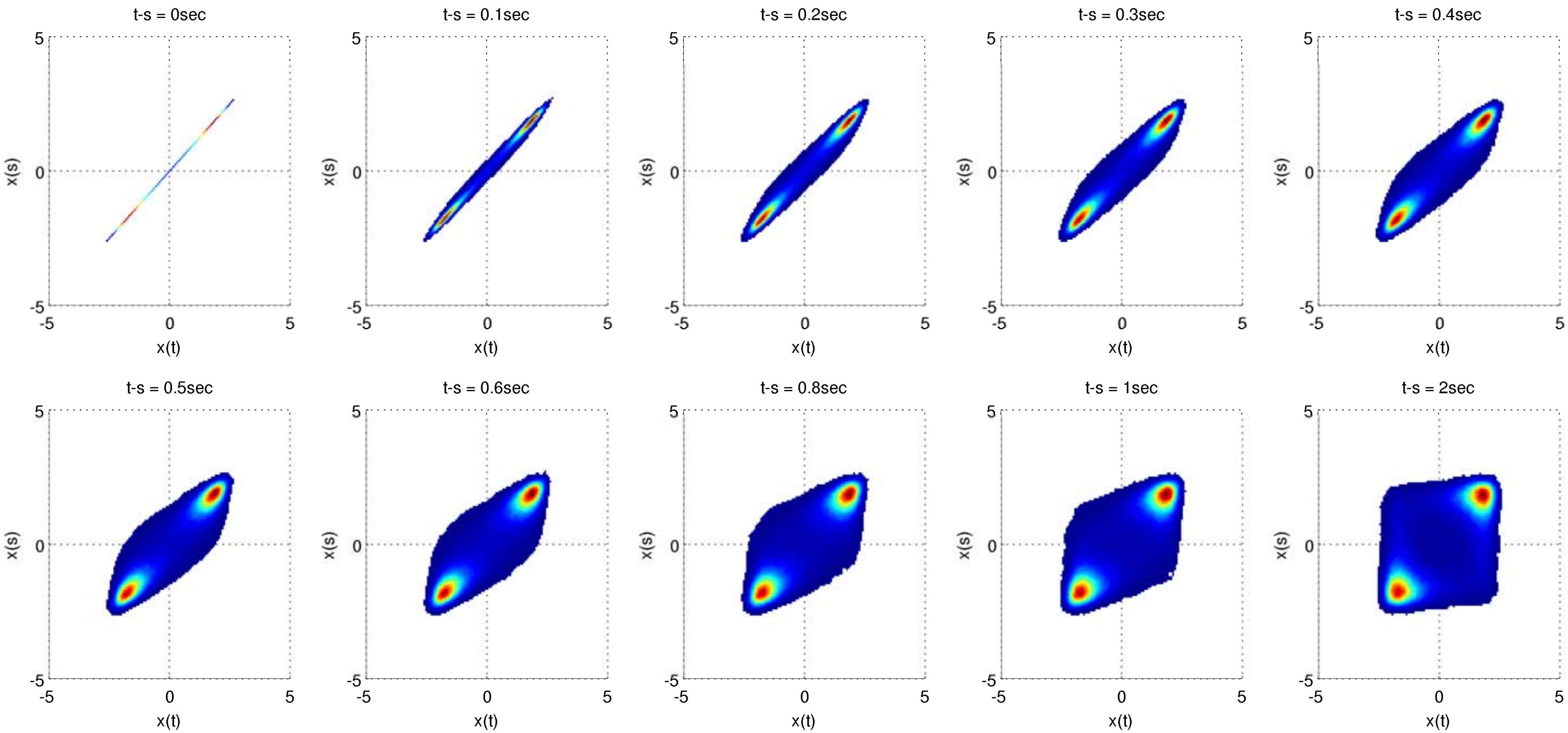

Figure 15: The same as in Fig 13 for the long time statistical equilibrium joint response pdf $f_{xx}(x_1, y_1; \tau)$

to this test case are plotted as obtained by MC simulations performed by Z.G. Kapelonis. More precisely, in Figs.13,14 the joint lag-time asymptotic REPDF $f_{xy}(x_1, y_1; \tau)$ is plotted for positive ($\tau \geq 0$, Fig.13) and negative ($\tau \leq 0$ Fig.14) time lag, whereas, in Fig.15 the joint asymptotic response pdf $f_{xx}(x_1, y_1; \tau)$ for the same problem is shown. Looking into this figures we can notice that:

i) The joint REPDF is concentrated around two skew symmetric stable modes

ii) The mean values of the probability mass that is concentrated around each one of the stable modes depends on the time lag

These remarks allow to understand why the Gaussian assumption is inappropriate for this case. Taking into consideration the above we will present some first ideas on a bi-Gaussian moment closure scheme. For its solution we will use auxiliary information obtained from MC simulations, concerning the mean values of the two stable modes around which the probability mass is concentrated.

Let us rewrite the two-time RE moment equations for $m_x(t)$, $R_{xy}(t,s)$, $R_{xx}(t,s)$ and $R_{xx}(t,t)$, that where obtained in section 5.2, for the case of a non-linear half oscillator under Gaussian excitation (i.e. setting $\kappa_3 = 0$, to Equs.(1-4)_Sec(5.2.1)). Since our focus is on the long-time limit, we are going to ignore the initial conditions. Under these assumptions, the differential equations for $m_x(t)$, $R_{xy}(t,s)$, $R_{xx}(t,s)$ and $R_{xx}(t,t)$ reduce to

$$\frac{dm_x(t)}{dt} = \mu_1 \cdot m_x(t) + \mu_3 \cdot R_{xx}^{21}(t,t) + \kappa_1 \cdot m_y(t) \ , \tag{1}$$

$$\frac{\partial R_{xy}(t,s)}{\partial t} = \mu_1 \cdot R_{xy}(t,s) + \mu_3 \cdot R_{xy}^{31}(t,s) + \kappa_1 \cdot R_{yy}(t,s) \ , \tag{2}$$

$$\frac{\partial R_{xx}(t,s)}{\partial t} = \mu_1 \cdot R_{xx}(t,s) + \mu_3 \cdot R_{xx}^{31}(t,s) + \kappa_1 \cdot R_{xy}(s,t), \tag{3}$$

and

$$\frac{dR_{xx}(t,t)}{dt} = 2 \cdot \mu_1 \cdot R_{xx}(t,t) + 2 \cdot \mu_3 \cdot R_{xx}^{31}(t,t) + 2 \cdot \kappa_1 \cdot R_{xy}(t,t) \ , \tag{4}$$

5.4.1. Direct solution of the non-central two-time RE moment equations

We shall solve the reformulated system of two-time RE moment equations directly in the long-time, therefore we assume that the joint, RE, stationarity has been achieved, i.e.

$$\lim_{t \to \infty} \frac{dm_x(t)}{dt} = 0 = \lim_{t \to \infty} \frac{dR_{xx}^{ij}(t,t)}{dt}, \tag{1a}$$

and

$$\lim_{\substack{t\to\infty,\ s\to\infty \\ t-s=\tau}} \frac{\partial R_{xy}(t,s)}{\partial t} = \frac{d R_{xy}^{(\infty)}(\tau)}{d\tau}, \quad \lim_{\substack{t\to\infty,\ s\to\infty \\ t-s=\tau}} \frac{\partial R_{xx}(t,s)}{\partial t} = \frac{d R_{xx}^{(\infty)}(\tau)}{d\tau} . \text{(1b,1c)}$$

$$R_{xx}^{(ij,\infty)}(t,s) = R_{xx}^{(ij,\infty)}(\tau),\ R_{xy}^{(ij,\infty)}(t,s) = R_{xy}^{(ij,\infty)}(\tau), \text{(for } \tau = t-s) \quad \text{(1d)}$$

Taking the limits of both sides of Equs.(1-4)_Sec(5.4) as $t\to\infty$, $s\to\infty$ and applying Equ.(1), we obtain

$$0 = \mu_1 \cdot m_x^{(\infty)} + \mu_3 \cdot R_{xx}^{(21,\infty)}(0) + \kappa_1 \cdot m_y^{(\infty)} \quad (2)$$

$$\frac{d R_{xy}^{(\infty)}(\tau)}{d\tau} = \mu_1 \cdot R_{xy}^{(\infty)}(\tau) + \mu_3 \cdot R_{xy}^{(31,\infty)}(\tau) + \kappa_1 \cdot R_{yy}(\tau) \quad (3)$$

$$\frac{d R_{xx}^{(\infty)}(\tau)}{d\tau} = \mu_1 \cdot R_{xx}^{(\infty)}(\tau) + \mu_3 \cdot R_{xx}^{(31,\infty)}(\tau) + \kappa_1 \cdot R_{xy}^{(\infty)}(-\tau) \quad (4)$$

$$0 = \mu_1 \cdot R_{xx}^{(\infty)}(0) + \mu_3 \cdot R_{xx}^{(31,\infty)}(0) + \kappa_1 \cdot R_{xy}^{(\infty)}(0) \quad (5)$$

Assuming that the asymptotic mean value of the excitation $m_y^{(\infty)} = 0$, since the response density is a symmetric function, for the third order moments of the response appearing in Equ.(2) it is straightforward that $R_{xx}^{(21,\infty)}(0) = 0$. Therefore, from Equ.(2) the response mean value $m_x^{(\infty)} = 0$. In the next section, we are going to introduce a bi-Gaussian moment closure aiming to approximate the asymptotic RE cross-correlation $R_{xy}^{(\infty)}(\tau)$ and the asymptotic response auto-covariance $R_{xx}^{(\infty)}(\tau)$ using Equs.(3-5) .

5.4.2. Bi-Gaussian Closure

In line with the observations discussed in the previous section, we are going to assume that the asymptotic joint REPDF $f_{xy}(x_1, y_1; \tau)$ and the asymptotic response pdf $f_{xx}(x_1, x_2; \tau)$ can be expressed as sums of Gaussian densities

$$f_{xy}(x_1, y_1; \tau) = \frac{1}{2} \cdot \left\{ G\left(m_x^+(\tau), m_y^+(\tau); \boldsymbol{\Sigma}_{xy}(\tau) \right) + G\left(m_x^-(\tau), m_y^-(\tau); \boldsymbol{\Sigma}_{xy}(\tau) \right) \right\} \quad (1)$$

$$f_{xx}(x_1, x_2; \tau) = \frac{1}{2} \cdot \left\{ G\left(m_x^+(\tau), m_x^+(\tau); \boldsymbol{\Sigma}_{xx}(\tau) \right) + G\left(m_x^-(\tau), m_x^-(\tau); \boldsymbol{\Sigma}_{xx}(\tau) \right) \right\} \quad (2)$$

where

$$m_x^+(\tau) = E^\theta\left[\, x(t+\tau;\theta) > 0 \middle| x(t) > 0 \,\right],\;\; m_y^+(\tau) = E^\theta\left[\, y(t+\tau;\theta) \middle| x(t) > 0 \,\right] \tag{3a,b}$$

$$m_x^+(\tau) = -m_x^-(\tau)\,, \quad m_y^+(\tau) = -m_y^-(\tau) \tag{4a,b}$$

and

$$\mathbf{\Sigma}_{xy}(\tau) = \begin{pmatrix} \Sigma_{xx}(0) & \Sigma_{xy}(\tau) \\ \Sigma_{xy}(\tau) & \Sigma_{yy}(0) \end{pmatrix}, \quad \mathbf{\Sigma}_{xx}(\tau) = \begin{pmatrix} \Sigma_{xx}(0) & \Sigma_{xx}(\tau) \\ \Sigma_{xx}(\tau) & \Sigma_{xx}(0) \end{pmatrix}. \tag{5a,b}$$

Consequently, the long-time moments appearing in Equs (3-5)_Sec(5.4.1) can be written as functions of the introduced variables $m_y^+(\tau)$, $m_x^+(\tau)$, $\Sigma_{yy}(0)$, $\Sigma_{xx}(0)$, $\Sigma_{xx}(\tau), \Sigma_{xy}(\tau)$. More precisely, for the asymptotic RE cross-correlation $R_{xy}^{(\infty)}(\tau)$ we have:

$$R_{xy}^{(\infty)}(\tau) = \iint\limits_{\mathbb{R}\times\mathbb{R}} x\,y\, f_{xy}(x_1, y_1;\tau)\, dx\, dy =$$

$$= \frac{1}{2} \iint\limits_{\mathbb{R}\times\mathbb{R}} x\,y \left\{ G\Big(m_x^+(\tau), m_y^+(\tau); \mathbf{\Sigma}_{xy}(\tau) \Big) + G\Big(m_x^-(\tau), m_y^-(\tau); \mathbf{\Sigma}_{xy}(\tau) \Big) \right\} dx\, dy =$$

[setting $u = x - m_x^+(\tau)$, $v = y - m_y^+(\tau)$]

$$= \frac{1}{2} \iint\limits_{\mathbb{R}\times\mathbb{R}} \Big(u + m_x^+(\tau)\Big)\Big(v + m_y^+(\tau)\Big) G\Big(u, v; \mathbf{\Sigma}_{xy}(\tau) \Big) du\, dv +$$

$$+ \frac{1}{2} \cdot \iint\limits_{\mathbb{R}\times\mathbb{R}} \Big(u + m_x^-(\tau)\Big)\Big(v + m_y^-(\tau)\Big) G\Big(u, v; \mathbf{\Sigma}_{xy}(\tau) \Big) du\, dv =$$

$$= \frac{1}{2} \iint\limits_{\mathbb{R}\times\mathbb{R}} \Big(uv + m_x^+(\tau)v + u m_y^+(\tau) + m_x^+(\tau) m_y^+(\tau)\Big) G\Big(u, v; \mathbf{\Sigma}_{xy}(\tau) \Big) du\, dv +$$

$$+ \frac{1}{2} \iint\limits_{\mathbb{R}\times\mathbb{R}} \Big(uv + m_x^-(\tau)v + u m_y^-(\tau) + m_x^-(\tau) m_y^-(\tau)\Big) G\Big(u, v; \mathbf{\Sigma}_{xy}(\tau) \Big) du\, dv \;.$$

That is,

$$R_{xy}^{(\infty)}(\tau) = \iint\limits_{\mathbb{R}\times\mathbb{R}} \Big(uv + m_x^+(\tau) m_y^+(\tau)\Big) G\Big(u, v; \mathbf{\Sigma}_{xy} \Big) du\, dv = \Sigma_{xy}(\tau) + m_x^+(\tau) m_y^+(\tau)\,. \tag{6}$$

Similarly we find that:

$$R_{xx}^{(\infty)}(\tau) = \Sigma_{xx}(\tau) + m_x^+(\tau) m_x^+(\tau)\,, \qquad R_{xx}^{(\infty)}(0) = \Sigma_{xx}(0) + m_x^+(0) m_x^+(0) \tag{7, 8}$$

$$R_{xy}^{(\infty)}(0) = \Sigma_{xy}(0) + m_x^+(0) m_y^+(0)\,. \tag{9}$$

Subsequently,

$$R_{xy}^{(31,\infty)}(\tau) = \iint_{\mathbb{R}\times\mathbb{R}} x^3\, y\, f_{xy}(x_1, y_1;\tau)\, dx\, dy =$$

$$= \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} x^3\, y \left\{ G\Big(m_x^+(\tau), m_y^+(\tau); \boldsymbol{\Sigma}_{xy}(\tau) \Big) + G\Big(m_x^-(\tau), m_y^-(\tau); \boldsymbol{\Sigma}_{xy}(\tau) \Big) \right\} dx\, dy$$

$$[\text{ setting } \ u = x - m_x^+(\tau) \ , \ \ v = y - m_y^+(\tau) \ ]$$

$$= \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} \left(u + m_x^+(\tau)\right)^3 \left(v + m_y^+(\tau)\right) G\Big(u, v; \boldsymbol{\Sigma}_{xy}(\tau) \Big) du\, dv +$$

$$+ \frac{1}{2}\cdot \iint_{\mathbb{R}\times\mathbb{R}} \left(u + m_x^-(\tau)\right)^3 \left(v + m_y^-(\tau)\right) G\Big(u, v; \boldsymbol{\Sigma}_{xy}(\tau) \Big) du\, dv =$$

$$= \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} \left(u^3 + 3u^2\left(m_x^+(\tau)\right) + 3u\left(m_x^+(\tau)\right)^2 + \left(m_x^+(\tau) \right)^3\right)\left(v + m_y^+(\tau)\right) G\Big(u, v; \boldsymbol{\Sigma}_{xy}(\tau) \Big) du\, dv +$$

$$+ \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} \left(u^3 + 3u^2\left(m_x^-(\tau)\right) + 3u\left(m_x^-(\tau)\right)^2 + \left(m_x^-(\tau)\right)^3\right)\left(v + m_y^-(\tau)\right) G\Big(u, v; \boldsymbol{\Sigma}_{xy}(\tau) \Big) du\, dv =$$

$$= \iint_{\mathbb{R}\times\mathbb{R}} \left(u^3 v + 3u\left(m_x^+(\tau) \right)^2 v + 3u^2\, m_x^+(\tau)\, m_y^+(\tau) + \left(m_x^+(\tau) \right)^3 m_y^+(\tau)\right) G\Big(u, v; \boldsymbol{\Sigma}_{xy}(\tau) \Big) du\, dv,$$

that is:

$$R_{xy}^{(31,\infty)}(\tau) = \Sigma_{xy}^{31}(\tau) + 3\left(m_x^+(\tau)\right)^2 \Sigma_{xy}(\tau) + 3 m_x^+(\tau)\, m_y^+(\tau) \Sigma_{xx}(0) + \left(m_x^+(\tau) \right)^3 m_y^+(\tau)\,. \tag{10}$$

Similarly,

$$R_{xx}^{(31,\infty)}(\tau) = \iint_{\mathbb{R}\times\mathbb{R}} (x_1)^3\, x_2\, f_{xx}(x_1, x_2;\tau)\, dx\, dy =$$

$$= \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} (x_1)^3\, x_2 \left\{ G\Big(m_x^+, m_y^+; \boldsymbol{\Sigma}_{xx}(\tau) \Big) + G\Big(m_x^-, m_y^-; \boldsymbol{\Sigma}_{xx}(\tau) \Big) \right\} dx\, dy =$$

$$[\text{ setting } \ u = x_1 - m_x^+(\tau) \ , \ \ v = x_2 - m_x^+(\tau) \ ]$$

$$= \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} \left(u + m_x^+(\tau)\right)^3 \left(v + m_x^+(\tau)\right) G\Big(u, v; \boldsymbol{\Sigma}_{xx}(\tau) \Big) du\, dv +$$

$$+ \frac{1}{2}\cdot \iint_{\mathbb{R}\times\mathbb{R}} \left(u + m_x^-(\tau)\right)^3 \left(v + m_x^-(\tau)\right) G\Big(u, v; \boldsymbol{\Sigma}_{xx}(\tau) \Big) du\, dv =$$

$$= \frac{1}{2} \iint_{\mathbb{R}\times\mathbb{R}} \left(u^3 + 3u^2 m_x^+(\tau) + 3u\left(m_x^+(\tau)\right)^2 + \left(m_x^+(\tau)\right)^3\right)\left(v + m_x^+(\tau)\right) G\left(u, v; \boldsymbol{\Sigma}_{xx}(\tau)\right) du\, dv +$$

$$+ \frac{1}{2} \iint_{\mathbb{R}\times\mathbb{R}} \left(u^3 + 3u^2 m_x^-(\tau) + 3u\left(m_x^-(\tau)\right)^2 + \left(m_x^-(\tau)\right)^3\right)\left(v + m_x^-(\tau)\right) G\left(u, v; \boldsymbol{\Sigma}_{xx}(\tau)\right) du\, dv =$$

$$= \iint_{\mathbb{R}\times\mathbb{R}} \left(u^3 v + 3uv\left(m_x^+(\tau)\right)^2 + 3u^2\left(m_x^+(\tau)\right)^2 m_x^+ + \left(m_x^+(\tau)\right)^3 m_x^+\right) G\left(u, v; \boldsymbol{\Sigma}_{xx}(\tau)\right) du\, dv \ .$$

that is,

$$R_{xx}^{(31,\infty)}(\tau) = \Sigma_{xx}^{31}(\tau) + 3(m_x^+(\tau))^2 \Sigma_{xx}(\tau) + 3(m_x^+(\tau))^2 \Sigma_{xx}(0) + (m_x^+(\tau))^4. \tag{11}$$

Since $G\left(m_x^+(\tau), m_y^+(\tau); \Sigma_{xy}(\tau)\right)$, $G\left(m_x^+(\tau), m_x^+(\tau); \Sigma_{xx}(\tau)\right)$ are Gaussian, we can apply the Isserlis theorem in order to express the 4th order moments, appearing in Equs.(10,11), in terms of 2nd order moments. More precisely from Equs.(1a,1b)_Sec(5.2.2) we have:

$$\Sigma_{xy}^{31}(\tau) = 3\cdot\Sigma_{xx}(0)\cdot\Sigma_{xy}(\tau), \quad \Sigma_{xx}^{31}(\tau) = 3\cdot\Sigma_{xx}(0)\cdot\Sigma_{xx}(\tau). \tag{12), (13}$$

Substituting Equs.(12, 13) to Equs.(10, 11) we obtain:

$$\begin{aligned} R_{xy}^{(31,\infty)}(\tau) &= 3\cdot\Sigma_{xx}(0)\cdot\Sigma_{xy}(\tau) + 3\left(m_x^+(\tau)\right)^2 \Sigma_{xy}(\tau) + \\ &\quad + 3m_x^+(\tau)m_y^+(\tau)\Sigma_{xx}(0) + \left(m_x^+(\tau)\right)^3 m_y^+(\tau) = \\ &= 3\cdot\Sigma_{xx}(0)\cdot\left(\Sigma_{xy}(\tau) + m_x^+(\tau)m_y^+(\tau)\right) + \\ &\quad + 3\left(m_x^+(\tau)\right)^2\left(\Sigma_{xy}(\tau) + m_x^+(\tau)m_y^+(\tau)\right) - 2\left(m_x^+(\tau)\right)^3 m_y^+(\tau) = \\ &= 3\cdot\left(R_{xx}^{(\infty)}(0) - \left(m_x^+(0)\right)^2\right) R_{xy}^{(\infty)}(\tau) + \\ &\quad + 3\left(m_x^+(\tau)\right)^2 R_{xy}^{(\infty)}(\tau) - 2\left(m_x^+(\tau)\right)^3 m_y^+(\tau) = \\ &= 3\cdot\left(R_{xx}^{(\infty)}(0) - \left(m_x^+(0)\right)^2 + \left(m_x^+(\tau)\right)^2\right) R_{xy}^{(\infty)}(\tau) - 2\left(m_x^+(\tau)\right)^3 m_y^+(\tau), \end{aligned} \tag{14}$$

and

$$R_{xx}^{(31,\infty)}(\tau) = 3\cdot\left(R_{xx}^{(\infty)}(0) - \left(m_x^+(0)\right)^2 + \left(m_x^+(\tau)\right)^2\right)\cdot R_{xx}^{(\infty)}(\tau) - 2\cdot\left(m_x^+(\tau)\right)^4, \tag{15}$$

setting $\tau = 0$ to Equ.(15) we also obtain

$$R_{xx}^{(31,\infty)}(0) = 3\cdot\left(R_{xx}^{(\infty)}(0)\right)^2 - 2\cdot\left(m_x^+(0)\right)^4, \tag{16}$$

Replacing Equs.(14-16) to Equs(3-5)_Sec(5.4.1) we get a closed (in terms of moments) version of the latter, i.e.

$$\frac{dR_{xy}^{(\infty)}(\tau)}{d\tau} = \left(\mu_1 + 3\cdot\mu_3\cdot\left(R_{xx}^{(\infty)}(0) - \left(m_x^+(0)\right)^2 + \left(m_x^+(\tau)\right)^2\right)\right) R_{xy}^{(\infty)}(\tau) - \\ - 2\mu_3\cdot(m_x^+(\tau)\ {}^3 m_y^+(\tau) \ + \kappa_1\cdot R_{yy}(\tau), \quad (17)$$

$$\frac{dR_{xx}^{(\infty)}(\tau)}{d\tau} = \left(\mu_1 + 3\cdot\mu_3\cdot\left(R_{xx}^{(\infty)}(0) - \left(m_x^+(0)\right)^2 + \left(m_x^+(\tau)\right)^2\right)\right)\cdot R_{xx}^{(\infty)}(\tau) - \\ - 2\cdot\mu_3\cdot\left(m_x^+(\tau)\right)^4 + \kappa_1\cdot R_{xy}^{(\infty)}(-\tau), \quad (18)$$

$$0 = \left(\mu_1 + 3\cdot\mu_3\cdot R_{xx}^{(\infty)}(0)\right)\cdot R_{xx}^{(\infty)}(0) - 2\cdot\mu_3\cdot\left(m_x^+(0)\right)^4 + \kappa_1\cdot R_{xy}^{(\infty)}(0)\ . \quad (19)$$

Let:

$$A_x^{(\infty)}(\tau) = \mu_1 + 3\cdot\mu_3\cdot\left(R_{xx}^{(\infty)}(0) - \left(m_x^+(0)\right)^2 + \left(m_x^+(\tau)\right)^2\right), \quad (20)$$

then Equs.(17-19) can be written in the more compact form

$$\frac{dR_{xy}^{(\infty)}(\tau)}{d\tau} = A_x^{(\infty)}(\tau)\cdot R_{xy}^{(\infty)}(\tau) - 2\mu_3\cdot(m_x^+(\tau))^3 m_y^+(\tau) + \kappa_1\cdot R_{yy}(\tau)\ , \quad (21)$$

$$\frac{dR_{xx}^{(\infty)}(\tau)}{d\tau} = A_x^{(\infty)}(\tau)\cdot R_{xx}^{(\infty)}(\tau) - 2\cdot\mu_3\cdot\left(m_x^+(\tau)\right)^4 + \kappa_1\cdot R_{xy}^{(\infty)}(-\tau)\ , \quad (22)$$

$$0 = A_x^{(\infty)}(0)\cdot R_{xx}^{(\infty)}(0) - 2\cdot\mu_3\cdot\left(m_x^+(0)\right)^4 + \kappa_1\cdot R_{xy}^{(\infty)}(0)\ . \quad (23)$$

The general solution of Equs.(21, 22) is given by the formula:

$$R_{xy}^{(\infty)}(\tau) = R_{xy}^{(\infty)}(\tau_*)\cdot\exp\left\{\int_{u=\tau_*}^{u=\tau} A_x^{(\infty)}(u)\,du\right\} + \\ + \int_{\tau_1=\tau_*}^{\tau_1=\tau}\left(-2\mu_3\cdot(m_x^+(\tau))^3 m_y^+(\tau) + \kappa_1\cdot R_{yy}^{(\infty)}(\tau)\right)\cdot\exp\left\{\int_{u=\tau_1}^{u=\tau} A_x^{(\infty)}(u)\,du\right\} d\tau_1 \quad (24)$$

and

$$R_{xx}^{(\infty)}(\tau) = R_{xx}^{(\infty)}(\tau_*) \cdot \exp\left\{ \int_{u=\tau_*}^{u=\tau} A_x^{(\infty)}(u)\,du \right\} + \\ + \int_{\tau_1=\tau_*}^{\tau_1=\tau} \left(-2\mu_3 \cdot (m_x^+(\tau_1))^4 + \kappa_1 \cdot R_{xy}^{(\infty)}(-\tau_1)\right) \cdot \exp\left\{ \int_{u=\tau_1}^{u=\tau} A_x^{(\infty)}(u)\,du \right\} d\tau_1, \tag{25}$$

where τ_* is any fixed time in the lag domain.

Following the procedure presented in the solution of the two-time RE moment equations in the long-time (see Section 5.3.1), under the assumption that $\mu_1 > 0$, $\mu_3 < 0$ are such that $A_\xi^{(\infty)}(\tau) < 0$, Equs.(24 ,25) can be written in the more convenient form:

$$R_{xy}^{(\infty)}(\tau) = \int_{\tau_1=-\infty}^{\tau_1=\tau} \left(-2\mu_3 \cdot (m_x^+(\tau_1))^3 m_y^+(\tau_1) + \kappa_1 \cdot R_{yy}^{(\infty)}(\tau_1)\right) \cdot \exp\left\{ \int_{u=\tau_1}^{u=\tau} A_x^{(\infty)}(u)\,du \right\} d\tau_1, \tag{26}$$

$$R_{xx}^{(\infty)}(\tau) = \int_{\tau_1=-\infty}^{\tau_1=\tau} \left(-2\mu_3 \cdot (m_x^+(\tau_1))^4 + \kappa_1 \cdot R_{xy}^{(\infty)}(-\tau_1)\right) \cdot \exp\left\{ \int_{u=\tau_1}^{u=\tau} A_x^{(\infty)}(u)\,du \right\} d\tau_1, \tag{27}$$

where

$$R_{xy}^{(\infty)}(-\tau_1) = \int_{\tau_2=-\infty}^{\tau_2=-\tau_1} \left(-2\mu_3 \cdot (m_x^+(\tau_2))^3 m_y^+(\tau_2) + \kappa_1 \cdot R_{yy}^{(\infty)}(\tau_2)\right) \cdot \exp\left\{ \int_{u=\tau_2}^{u=\tau_1} A_x^{(\infty)}(u)\,du \right\} d\tau_2. \tag{28}$$

Since $A_x^{(\infty)}(\tau)$ is a function of $R_{xx}^{(\infty)}(0)$, to obtain a closed form of the system of Equs.(26-28) we need an extra constraint for $R_{xx}^{(\infty)}(0)$. Combining Equs.(23, 26) we obtain an extra condition, that allows us to compute $R_{xx}^{(\infty)}(0)$:

$$0 = A_x^{(\infty)}(0) \cdot R_{xx}^{(\infty)}(0) - 2 \cdot \mu_3 \cdot \left(m_x^+(0) \right)^4 + \\ + \kappa_1 \cdot \int_{\tau_1=-\infty}^{\tau_1=0} \left(-2\mu_3 \cdot (m_x^+(\tau_1))^3 m_y^+(\tau_1) + \kappa_1 \cdot R_{yy}^{(\infty)}(\tau_1)\right) \cdot \exp\left\{ \int_{u=\tau_1}^{u=0} A_x^{(\infty)}(u)\,du \right\} d\tau_1 \tag{29}$$

The conditional mean values $m_x^+(\tau)$, $m_y^+(\tau)$, defined by Equs.(3,4), are auxiliary data provided by MC simulations.

In conclusion, the closed in terms of both moments and time system is given by the equations (for convenience we repeat here Equs.(29, 26, 27):

$$0 = A_x^{(\infty)}(0)\cdot R_{xx}^{(\infty)}(0) - 2\cdot\mu_3\cdot\left(m_x^+(0)\right)^4 +$$

$$+\ \kappa_1\cdot\int_{\tau_1=-\infty}^{\tau_1=0}\left(-2\mu_3\cdot(m_x^+(\tau_1))^3 m_y^+(\tau_1)+\kappa_1\cdot R_{yy}^{(\infty)}(\tau_1)\right)\cdot\exp\left\{\int_{u=\tau_1}^{u=0}A_x^{(\infty)}(u)\,du\right\}d\tau_1, \quad (30)$$

$$R_{xy}^{(\infty)}(\tau) = \int_{\tau_1=-\infty}^{\tau_1=\tau}\left(-2\mu_3\cdot(m_x^+(\tau_1))^3 m_y^+(\tau_1)+\kappa_1\cdot R_{yy}^{(\infty)}(\tau_1)\right)\cdot\exp\left\{\int_{u=\tau_1}^{u=\tau}A_x^{(\infty)}(u)\,du\right\}d\tau_1 \quad (31)$$

$$R_{xx}^{(\infty)}(\tau) = \int_{\tau_1=-\infty}^{\tau_1=\tau}\left(-2\mu_3\cdot(m_x^+(\tau_1))^4+\kappa_1\cdot R_{xy}^{(\infty)}(-\tau_1)\right)\cdot\exp\left\{\int_{u=\tau_1}^{u=\tau}A_x^{(\infty)}(u)\,du\right\}d\tau_1, \quad (32)$$

where:

$$A_x^{(\infty)}(\tau) = \mu_1 + 3\cdot\mu_3\cdot\left(R_{xx}^{(\infty)}(0) - \left(m_x^+(0)\right)^2 + \left(m_x^+(\tau)\right)^2\right)$$

$m_x^+(\tau)$, $m_y^+(\tau)$ are provided by MC simulations

5.4.3. Preliminary results- Discussion

In **Fig.16,17** the long-time RE cross-correlation $R_{xy}^{(\infty)}(\tau)$ and response auto-correlation $R_{xx}^{(\infty)}(\tau)$ of the bi-stable/Gaussian half oscillator obtained by the solution of the system of Equs.(30-32) are plotted and compared with results obtained by MC simulations. As we can see in **Fig.16** the two methods compare pretty satisfactorily in the approximation of $R_{xy}^{(\infty)}(\tau)$, having a local mismatch that reaches 10%, around $\tau = 0$. On the contrary, as we can see in **Fig.17**, the proposed scheme fails as the time lag increases, providing an acceptable approximation of $R_{xx}^{(\infty)}(\tau)$ only around $\tau = 0$. The latter is attributed to the failure of the bi-Gaussian closure condition to approximate the joint asymptotic response density $f_{xx}(x_1, y_1; \tau)$ since as the lag time τ increases $f_{xx}(x_1, y_1; \tau)$ develops two extra modes making the bi-Gaussian closure inadequate, as shown in **Fig.18**. In this case the long-time statistical equilibrium joint response pdf $f_{xx}(x_1, y_1; \tau)$ would be more efficiently approximated by the use of a superposition of four Gaussian densities.

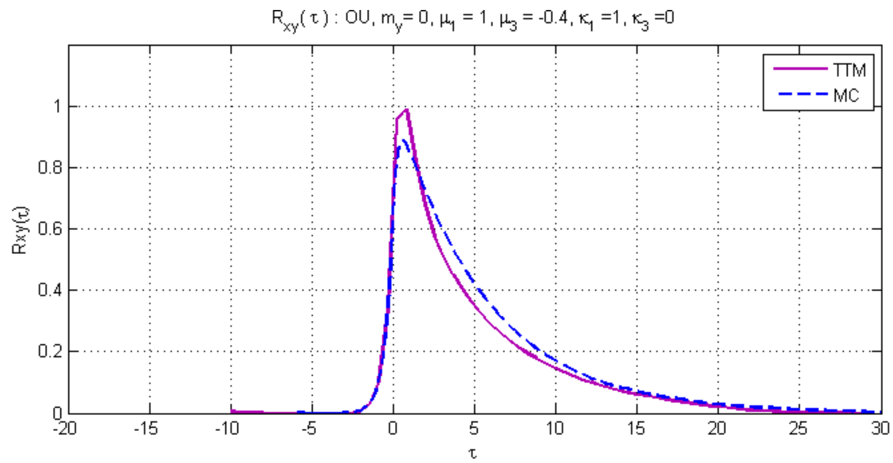

Figure 16: The asymptotic RE cross-correlation $R_{xx}^{(\infty)}(\tau)$ for the bi-stable/Gaussian half oscillator with the same parameters as in Fig13 obtained by the solution of the two-tome moment problem (solid lines) compared with results obtained with MC simulations (dashed lines).

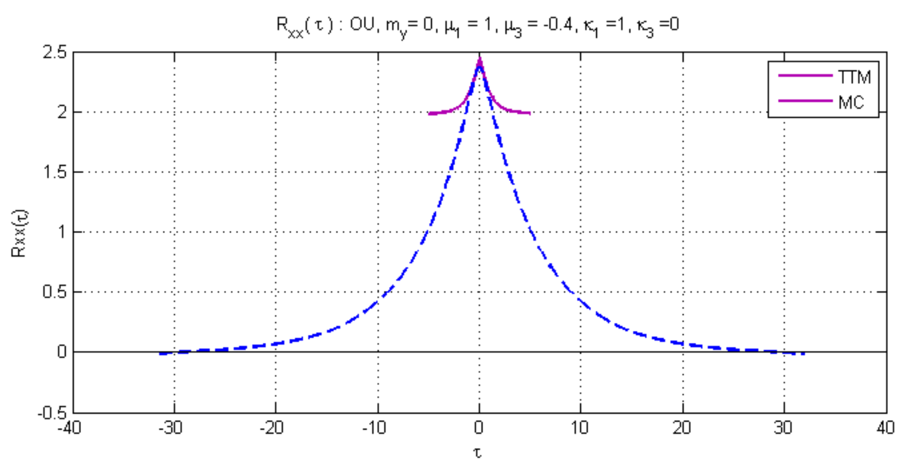

Figure 17: The shame as in Fig.16 for the asymptotic response auto-correlation $R_{xx}^{(\infty)}(\tau)$).

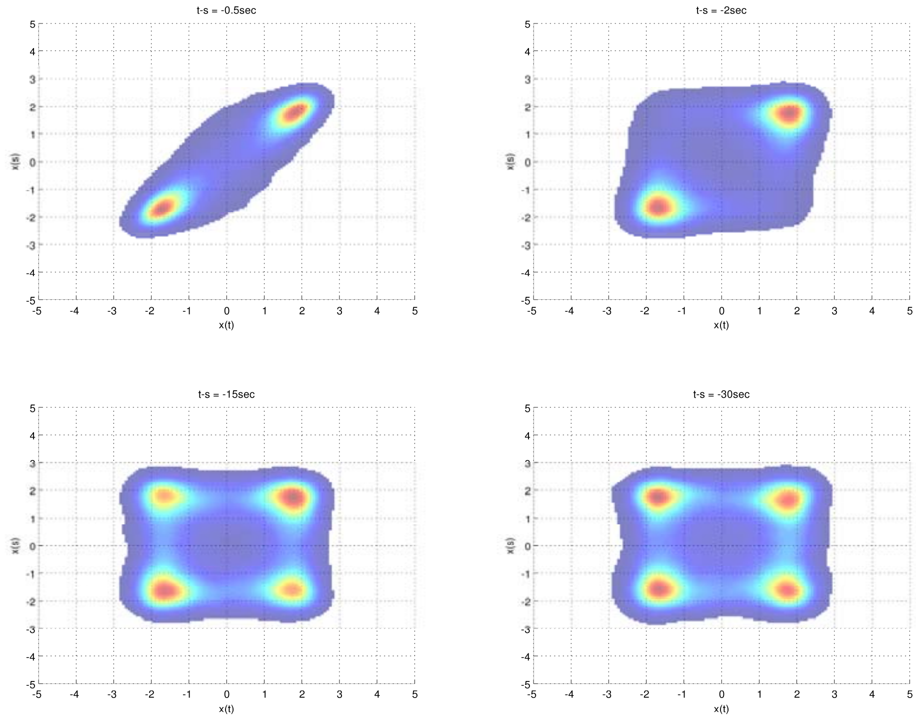

Figure 18: For the same case as in Fig.13, the asymptotic (statistical equilibrium) joint response pdf $f_{xx}(x_1, y_1; \tau)$. As the time lag $\tau = t - s$ increases the joint density develops 2 extra modes

5.5. References

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Directions for future work

1. Extension and further development of the numerical solution of the joint REPDF evolution equation in the steady state to bi-stable and to second order random oscillators.
2. Numerical solution of the transient joint REPDF evolution equation of scalar RDEs by the use of transient local two-time RE moment equations.
3. Use the characteristically functional approach, presented in Section 2.4, to find equations for the evolution of the joint two-time RE pdf $f_{x(t)y(s)}(\alpha,\beta)$ and the joint two-time response pdf $f_{x(t)x(s)}(\alpha_1,\alpha_2)$. Then, solve the system of the new two-time equations for the evolution of $f_{x(t)y(s)}(\alpha,\beta)$ and $f_{x(t)x(s)}(\alpha_1,\alpha_2)$, together with the REPDF evolution equation (for $f_{x(t)y(t)}(\alpha,\beta)$), applying a time closure similar with the one used for the solution of the two-time RE moment equations in Section 5.2.3.
4. Extensions of the two-time RE moment equations to bi-stable and second order stochastic oscillators
5. Use the system of the two-time RE moment equations obtained in Chapter 5 to close the response-marginal REPDF (Equ.(3)_Sec(2.2.3)) for non-linear generalized Langevin equations.

List of publications [1]

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Other Publications/Talks

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Athanassoulis, G. A., Tsantili, I. C., & Kapelonis, Z. G. (2012).“The Joint Response-Excitation PDF Evolution Equation. Numerical Solutions for the Long-Time, Steady-State Response of a Half Oscillator under strongly Colored, Gaussian or non-Gaussian”. Poster presented on workshop of 'Uncertainty Quantification', October 9-13, ICERM, Brown University, Providence, US.

APPENDICES

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A.1. Auxiliary integrals used in Section 3.2.4. for lpGF input

We shall prove the following integration formula:

$$I_2(t,s) \equiv \int_{t_0}^{t} \exp\left(-a(t_1-s)^2 - A\cdot t_1\right) dt_1 =$$
$$= \exp\left(-A\cdot s + \frac{A^2}{4\cdot a}\right)\cdot \frac{\sqrt{\pi}}{2\sqrt{a}}\left(-erf\left(\sqrt{a}\cdot(t_0-s)+\frac{A}{2\sqrt{a}}\right) + erf\left(\sqrt{a}\cdot(t-s)+\frac{A}{2\sqrt{a}}\right)\right). \quad (1)$$

Proof

$$I_2(t,s) = \int_{t_0}^{t} \exp\left(-a(t_1-s)^2 - A\cdot t_1\right) dt_1 = \int_{t_0}^{t} \exp\left(-a\cdot t_1^2 + 2\cdot a\cdot s\cdot t_1 - a\cdot s^2 - A\cdot t_1\right) dt_1 =$$
$$= \exp\left(-a\cdot s^2\right)\cdot \int_{t_0}^{t} \exp\left(-a\cdot t_1^2 + 2\cdot t_1\left(a\cdot s - \frac{A}{2}\right)\right) dt_1 =$$
$$= \exp\left(-a\cdot s^2\right)\cdot \int_{t_0}^{t} \exp\left(-a\cdot t_1^2 + 2\cdot t_1\left(a\cdot s - \frac{A}{2}\right) - \frac{(a\cdot s - A/2)^2}{a} + \frac{(a\cdot s - A/2)^2}{a}\right) dt_1 =$$
$$= \exp\left(\frac{(a\cdot s - A/2)^2}{a} - a\cdot s^2\right)\cdot \int_{t_0}^{t} \exp\left(-\left(\sqrt{a}\cdot(t_1 - s) + \frac{A}{2\cdot\sqrt{a}}\right)^2\right) dt_1 =$$
$$= \exp\left(\frac{a^2\cdot s^2 - a\cdot s\cdot A + A^2/4}{a} - a\cdot s^2\right)\cdot \int_{t_0}^{t} \exp\left(-\left(\sqrt{a}\cdot(t_1 - s) + \frac{A}{2\cdot\sqrt{a}}\right)^2\right) dt_1 =$$
$$= \exp\left(-A\cdot s + \frac{A^2}{4\cdot a}\right)\cdot \underbrace{\int_{t_0}^{t} \exp\left(-\left(\sqrt{a}\cdot t_1 - \frac{a\cdot s - A/2}{\sqrt{a}}\right)^2\right) dt_1}_{I_{21}(t,s)} . \quad (2)$$

Let us define:

$$u = u(s,t) \equiv \sqrt{a}\cdot t - \frac{a\cdot s - A/2}{\sqrt{a}} = \sqrt{a}\cdot(t-s) + \frac{A}{2\sqrt{a}}, \quad (3a)$$

$$u_0 = u(s,t_0) \equiv \sqrt{a}\cdot(t_0-s) + \frac{A}{2\sqrt{a}},\ u_1 = u(s,t_1) \equiv \sqrt{a}\cdot(t_1-s) + \frac{A}{2\sqrt{a}}, \quad (3b,c)$$

then:

$$u_0 = u(s,t_0) \le u_1 = u(s,t_1) \le u = u(s,t), \quad (4)$$

and

$$I_{21}\left(u(s,t)\right)=\frac{1}{\sqrt{a}}\int_{u_0}^{u}\exp\left(-u_1^2\right)du_1=\frac{1}{\sqrt{a}}\left[-\int_{0}^{u_0}\exp\left(-u_1^2\right)du_1+\int_{0}^{u}\exp\left(-u_1^2\right)du_1\right]. \quad (5)$$

The integrals of normal distributions, appearing in Equ.(5), can be approximated by error functions. Specifically, the following formula holds true:

$$\int_0^x \exp\left[-s^2\right]ds=\frac{\sqrt{\pi}}{2}erf\left(x\right), \quad (6)$$

therefore:

$$I_{21}\left(u(s,t)\right)=\frac{\sqrt{\pi}}{2\sqrt{a}}\left(-erf\left(u\left(s,t_0\right)\right)+erf\left(u\left(s,t\right)\right)\right). \quad (7)$$

From Equ.(2) and Equ.(7) we obtain the integration formula (1) . ■

We shall also prove that:

$$I_3(t,s)=\int_{t_0}^{t}e^{-2\cdot A\cdot t_1}\cdot erf\left(\sqrt{a}\cdot(s-t_1)+\frac{A}{2\cdot\sqrt{a}}\right)dt_1=$$
$$=-\frac{1}{2A}\left(e^{-2\cdot A\cdot t}\cdot erf\left(\sqrt{a}\cdot(s-t)+\frac{A}{2\cdot\sqrt{a}}\right)+e^{-2\cdot A\cdot t_0}\cdot erf\left(\sqrt{a}\cdot(s-t_0)+\frac{A}{2\cdot\sqrt{a}}\right)+\right. \quad (8)$$
$$\left.+e^{-2\cdot A\cdot s}\left(erf\left(\sqrt{a}\cdot(t-s)+\frac{A}{2\sqrt{a}}\right)-erf\left(\sqrt{a}\cdot(t_0-s)+\frac{A}{2\sqrt{a}}\right)\right)\right).$$

Proof

$$I_3(t,s)=\int_{t_0}^{t}e^{-2\cdot A\cdot t_1}\cdot erf\left(\sqrt{a}\cdot(s-t_1)+\frac{A}{2\cdot\sqrt{a}}\right)dt_1=$$
$$=-\frac{1}{2A}\left(\left[e^{-2\cdot A\cdot t_1}\cdot erf\left(\sqrt{a}\cdot(s-t_1)+\frac{A}{2\cdot\sqrt{a}}\right)\right]_{t_0}^{t}-\int_{t_0}^{t}e^{-2\cdot A\cdot t_1}\cdot\frac{\partial}{\partial t_1}erf\left(\sqrt{a}\cdot(s-t_1)+\frac{A}{2\cdot\sqrt{a}}\right)dt_1\right)=$$
$$=-\frac{1}{2A}\cdot e^{-2\cdot A\cdot t}\cdot erf\left(\sqrt{a}\cdot(s-t)+\frac{A}{2\cdot\sqrt{a}}\right)+\frac{1}{2A}\cdot e^{-2\cdot A\cdot t_0}\cdot erf\left(\sqrt{a}\cdot(s-t_0)+\frac{A}{2\cdot\sqrt{a}}\right)+$$

$$+\frac{1}{2A}\cdot\underbrace{\int_{t_0}^{t} e^{-2\cdot A\cdot t_1}\cdot\frac{\partial}{\partial t_1}\left(erf\left(\sqrt{a}\cdot(s-t_1)+\frac{A}{2\cdot\sqrt{a}}\right)\right)dt_1}_{I_{31}(t,s)} \quad . \qquad (9)$$

The integral $I_{31}(s,t)$, appearing in Equ.(9), shall be calculated separately. The derivative of the error function follows from its definition, that is:

$$\frac{d}{dz}erf(z)=\frac{2}{\sqrt{\pi}}\cdot\exp(-z^2).$$

Therefore, by the chain rule, for the derivative appearing in the integral $I_{31}(s,t)$ we have:

$$\begin{aligned}&\frac{\partial}{\partial t_1}\left(erf\left(\sqrt{a}\cdot(s-t_1)+\frac{A}{2\cdot\sqrt{a}}\right)\right)=\\ &=\frac{2}{\sqrt{\pi}}\cdot\exp\left(-\left(\sqrt{a}\cdot(s-t_1)+\frac{A}{2\cdot\sqrt{a}}\right)^2\right)\cdot\frac{\partial\left(\left(\sqrt{a}\cdot(s-t_1)+\frac{A}{2\cdot\sqrt{a}}\right)\right)}{\partial t_1}=\\ &=-\frac{2\sqrt{a}\cdot}{\sqrt{\pi}}\cdot\exp\left(-\left(\sqrt{a}\cdot(s-t_1)+\frac{A}{2\cdot\sqrt{a}}\right)^2\right)=\\ &=-\frac{2\sqrt{a}\cdot}{\sqrt{\pi}}\cdot\exp\left(-\left(a\cdot(s-t_1)^2+A\cdot(s-t_1)+\frac{A^2}{4\cdot a}\right)\right),\end{aligned} \qquad (10)$$

therefore, combining Equs.(9,10), we obtain:

$$\begin{aligned}I_{31}(t,s)&=-\frac{2\sqrt{a}}{\sqrt{\pi}}\cdot\int_{t_0}^{t}\exp\left(-\left(a\cdot(s-t_1)^2+A\cdot(s-t_1)+\frac{A^2}{4\cdot a}+2\cdot A\cdot t_1\right)\right)dt_1=\\ &=-\frac{2\sqrt{a}}{\sqrt{\pi}}\cdot\int_{t_0}^{t}\exp\left(-a\cdot(s-t_1)^2-A\cdot s+A\cdot t_1-\frac{A^2}{4\cdot a}-2\cdot A\cdot t_1\right)dt_1=\\ &=-\frac{2\sqrt{a}}{\sqrt{\pi}}\cdot\exp\left(-A\cdot s-\frac{A^2}{4\cdot a}\right)\cdot\int_{t_0}^{t}\exp\left(-\left(a\cdot(s-t_1)^2-A\cdot t_1\right)\right)dt_1=\\ &=-\frac{2\sqrt{a}}{\sqrt{\pi}}\cdot\exp\left(-A\cdot s-\frac{A^2}{4\cdot a}\right)\cdot I_2(t,s)=\\ &=\exp(-2\cdot A\cdot s)\cdot\left(erf\left(\sqrt{a}\cdot(t_0-s)+\frac{A}{2\sqrt{a}}\right)-erf\left(\sqrt{a}\cdot(t-s)+\frac{A}{2\sqrt{a}}\right)\right). \quad (11)\end{aligned}$$

A.2. Frequency Domain Analysis of the linear RDE with lpGf input

Purpose of this Appendix is to calculate the response auto-spectrum $S_{xx}(\omega)$ and the joint (stationary) RE cross-spectrum $S_{xy}(\omega)$ of the linear RDE (Equ(1)_Sec(3.1.1)) under lpGF excitation. Subsequently, by applying the inverse Fourier transform we shall re-obtain the long-time RE cross-correlation $R_{xy}^{(\infty)}(t-s)$ and the long-time response auto-correlation $R_{xx}^{(\infty)}(t-s)$ (these were also obtained in Section 3.2.4.a by the solution of the two-time RE moment equations).

Let us assume that the stationary excitation has zero mean value, i.e. $m_y = 0$, then:

$$C_{yy}(t,s) = R_{yy}(t,s) = \sigma^2 \cdot \exp\left(-a(t-s)^2\right) \; , \tag{1}$$

or, for $\tau = t - s$, $C_{yy}(\tau) = \sigma^2 \cdot \exp\left(-a \cdot \tau^2\right)$. (2)

From Equ.(6)_Sec(3.2.3) we have :

$$m_x^\infty = -\frac{B}{A} \cdot m_y = 0 . \tag{3}$$

The spectrum of the stationary excitation can be found using the formula below (see (Athanassoulis 2000)):

$$S_{yy}(\omega) = \frac{1}{\pi}\int_0^\infty C_{yy}(\tau) \cdot \cos(\omega \cdot \tau)\, d\omega = \frac{1}{\pi}\sigma^2 \cdot \int_0^\infty \exp\left(-a \cdot \tau^2\right) \cdot \cos(\omega \cdot \tau)\, d\omega . \tag{4}$$

From Equ.(4) we obtain (using the integration formula of pp. 480 in (Gradshteyn & Ryzhik 1965)):

$$S_{yy}(\omega) = \frac{1}{\pi} \cdot \frac{\sqrt{\pi}}{2 \cdot \sqrt{a}} \cdot \sigma^2 \cdot \exp\left(-\frac{\omega^2}{4 \cdot a}\right) = \frac{1}{2 \cdot \sqrt{\pi \cdot a}} \cdot \sigma^2 \cdot \exp\left(-\frac{\omega^2}{4 \cdot a}\right). \tag{5}$$

Let:

$$Y(t) = \mathrm{Re}\left[Y_0 \cdot \exp(i \cdot \omega \cdot t)\right], \qquad X(t) = \mathrm{Re}\left[X_0 \cdot \exp(i \cdot \omega \cdot t)\right], \tag{6a,b}$$

then from Equ.(2a)_Sec(3.1.1) we have:

$$X_0 \cdot i \cdot \omega \cdot \exp(i \cdot \omega \cdot t) = A \cdot X_0 \cdot \exp(i \cdot \omega \cdot t) + B \cdot Y_0 \cdot \exp(i \cdot \omega \cdot t), \tag{7}$$

or

$$X_0 \cdot (i \cdot \omega - A) = B \cdot Y_0 , \tag{8}$$

and the transfer function $H_{xy}(\omega)$ of system given by Equ.(2)_Sec(3.1.1) is given by the formula:

$$H_{xy}(\omega)=\frac{X_0}{Y_0}=\frac{B}{i\cdot\omega-\mathrm{A}}. \tag{9}$$

We shall use $H_{xy}(\omega)$ to calculate the joint RE spectrum $S_{xy}(\omega)$, i.e.

$$\begin{aligned}S_{xy}(\omega)&=H_{xy}(\omega)\cdot S_{yy}(\omega)=\frac{B}{-\mathrm{A}+i\cdot\omega}\cdot\frac{1}{2\cdot\sqrt{\pi\cdot a}}\cdot\sigma^2\cdot\exp\left(-\frac{\omega^2}{4\cdot a}\right)=\\&=\frac{B(-\mathrm{A}+i\cdot\omega)}{\omega^2+\mathrm{A}^2}\cdot\frac{1}{2\cdot\sqrt{\pi\cdot a}}\cdot\sigma^2\cdot\exp\left(-\frac{\omega^2}{4\cdot a}\right)=\\&=\sigma^2\cdot\frac{-B\cdot\mathrm{A}}{2\cdot\sqrt{\pi\cdot a}}\cdot\frac{1}{\omega^2+\mathrm{A}^2}\cdot\exp\left(-\frac{\omega^2}{4\cdot a}\right)+i\cdot\sigma^2\cdot\frac{B}{2\cdot\sqrt{\pi\cdot a}}\cdot\frac{\omega}{\omega^2+\mathrm{A}^2}\cdot\exp\left(-\frac{\omega^2}{4\cdot a}\right).\end{aligned} \tag{10}$$

Then, from $S_{xy}(\omega)$, applying the inverse Fourier transform, we find the long-time RE cross-correlation $R^{(\infty)}_{xy}(\tau)$:

$$\begin{aligned}R^{(\infty)}_{xy}(\tau)&=\int_{-\infty}^{+\infty}S_{xy}(\omega)\cdot\exp(i\cdot\omega\cdot\tau)d\omega=\int_{-\infty}^{+\infty}S_{xy}(\omega)\cdot\cos(\omega\cdot\tau)\cdot d\omega+i\cdot\int_{-\infty}^{+\infty}S_{xy}(\omega)\cdot\sin(\omega\cdot\tau)\cdot d\omega=\\&=\sigma^2\cdot\frac{B}{\sqrt{\pi\cdot a}}\cdot\left(-\mathrm{A}\cdot\int_0^{+\infty}\frac{1}{\omega^2+\mathrm{A}^2}\cdot\exp\left(-\frac{\omega^2}{4\cdot a}\right)\cdot\cos(\omega\cdot\tau)\cdot d\omega-\right.\\&\left.\qquad-\int_0^{+\infty}\frac{\omega}{\omega^2+\mathrm{A}^2}\cdot\exp\left(-\frac{\omega^2}{4\cdot a}\right)\cdot\sin(\omega\cdot\tau)\cdot d\omega\right).\end{aligned} \tag{11}$$

To calculate the sin and cosine transforms appearing in Equ.(11) we use the following integration formulae (from (Magnus et al. 1954) (see Equ.15, pp.15 and Equ.26, pp.74)):

$$\begin{aligned}&\int_0^{+\infty}\frac{1}{\omega^2+\mathrm{A}^2}\cdot\exp\left(-\frac{\omega^2}{4\cdot a}\right)\cdot\cos(\omega\cdot\tau)\cdot d\omega=\\&=\frac{\pi}{4\cdot(-A)}\cdot e^{\frac{A^2}{4\cdot a}}\cdot\left[e^{-A\cdot\tau}\left(1+erf\left(\sqrt{a}\cdot\tau+\frac{1}{2\cdot\sqrt{a}}\cdot A\right)\right)+e^{A\cdot\tau}\left(1+erf\left(-\sqrt{a}\cdot\tau+\frac{1}{2\cdot\sqrt{a}}\cdot A\right)\right)\right],\end{aligned} \tag{12}$$

and

$$\int_0^{+\infty}\frac{\omega}{\omega^2+\mathrm{A}^2}\cdot\exp\left(-\frac{\omega^2}{4\cdot a}\right)\cdot\sin(\omega\cdot\tau)\cdot d\omega=$$

$$= \frac{\pi}{4}\cdot e^{\frac{A^2}{4\cdot a}}\cdot\left[e^{-A\cdot\tau}\left(1+erf\left(\sqrt{a}\cdot\tau+\frac{1}{2\cdot\sqrt{a}}\cdot A\right)\right)-e^{A\cdot\tau}\left(1+erf\left(-\sqrt{a}\cdot\tau+\frac{1}{2\cdot\sqrt{a}}\cdot A\right)\right)\right]. \qquad (13)$$

Substituting Equs.(12) and (13) in Equ. (11), we get:

$$R_{xy}^{(\infty)}(\tau)=\int_{-\infty}^{+\infty}S_{xy}(\omega)\cdot\exp(i\cdot\omega\cdot\tau)\,d\omega=\int_{-\infty}^{+\infty}S_{xy}(\omega)\cdot\cos(\omega\cdot\tau)\cdot d\omega+i\cdot\int_{-\infty}^{+\infty}S_{xy}(\omega)\cdot\sin(\omega\cdot\tau)\cdot d\omega=$$

$$=\sigma^2\cdot\frac{B}{\sqrt{\pi\cdot a}}\cdot\left((-\mathrm{A})\cdot\int_0^{+\infty}\frac{1}{\omega^2+\mathrm{A}^2}\cdot\exp\left(-\frac{\omega^2}{4\cdot a}\right)\cdot\cos(\omega\cdot\tau)\cdot d\omega-\int_0^{+\infty}\frac{\omega}{\omega^2+\mathrm{A}^2}\cdot\exp\left(-\frac{\omega^2}{4\cdot a}\right)\cdot\sin(\omega\cdot\tau)\cdot d\omega\right)$$

$$=\sigma^2\cdot\frac{B\cdot\sqrt{\pi}}{4\cdot\sqrt{a}}\cdot e^{\frac{A^2}{4\cdot a}}\cdot\left[\frac{(-\mathrm{A})}{(-A)}\cdot\left[e^{-A\cdot\tau}\left(1+erf\left(\sqrt{a}\cdot\tau+\frac{1}{2\cdot\sqrt{a}}\cdot A\right)\right)+e^{A\cdot\tau}\left(1+erf\left(-\sqrt{a}\cdot\tau+\frac{1}{2\cdot\sqrt{a}}\cdot A\right)\right)\right]-\right.$$

$$\left.-\left[e^{-A\cdot\tau}\left(1+erf\left(-\sqrt{a}\cdot\tau+\frac{1}{2\cdot\sqrt{a}}\cdot A\right)\right)-e^{A\cdot\tau}\left(1+erf\left(\sqrt{a}\cdot\tau+\frac{1}{2\cdot\sqrt{a}}\cdot A\right)\right)\right]\right],$$

that is:

$$R_{xy}^{(\infty)}(\tau)\;=\;\sigma^2\cdot\frac{B\cdot\sqrt{\pi}}{2\cdot\sqrt{a}}\cdot e^{\frac{A^2}{4\cdot a}}\cdot e^{A\cdot\tau}\left(1+erf\left(\sqrt{a}\cdot\tau+\frac{1}{2\cdot\sqrt{a}}\cdot A\right)\right). \qquad (14)$$

We shall now calculate the spectrum of the response $S_{xx}(\omega)$:

$$S_{xx}(\omega)=\left|H_{xy}(\omega)\right|^2\cdot S_{yy}(\omega)=\left|\frac{B}{j\cdot\omega-\mathrm{A}}\right|^2\cdot\frac{1}{2\cdot\sqrt{\pi\cdot a}}\cdot\sigma^2\cdot\exp\left(-\frac{\omega^2}{4\cdot a}\right). \qquad (15)$$

Then, applying the inverse Fourier transform to $S_{xx}(\omega)$ we obtain the long-time response auto-correlation $R_{xx}^{(\infty)}(\tau)$:

$$R_{xx}^{(\infty)}(\tau)=2\cdot\int_0^{\infty}S_{xx}(\omega)\cdot\cos(\omega\cdot\tau)\,d\omega=\frac{B^2}{\sqrt{\pi\cdot a}}\cdot\sigma^2\int_0^{\infty}\frac{\cos(\omega\cdot\tau)}{A^2+\omega^2}\cdot\exp\left(-\frac{\omega^2}{4\cdot a}\right)d\omega\quad. \qquad (16)$$

Applying Equ.(12) to Equ.(16) we obtain:

$$R_{xx}^{(\infty)}(t,s)\;=$$

$$=\frac{B^2}{\sqrt{\pi\cdot a}}\cdot\sigma^2\,\frac{\pi}{4\cdot(-A)}\cdot e^{\frac{A^2}{4\cdot a}}\cdot\left[e^{-A\cdot\tau}\left(1+erf\left(\sqrt{a}\cdot\tau+\frac{1}{2\cdot\sqrt{a}}\cdot A\right)\right)+e^{A\cdot\tau}\left(1+erf\left(-\sqrt{a}\cdot\tau+\frac{1}{2\cdot\sqrt{a}}\cdot A\right)\right)\right]=$$

$$=\frac{\sqrt{\pi}}{4\sqrt{a}}\cdot\frac{B^2\cdot\sigma^2}{(-A)}\cdot e^{\frac{A^2}{4\cdot a}}\times\left(e^{A\cdot(s-t)}\cdot\left(erf\left(\sqrt{a}\cdot(s-t)+\frac{A}{2\cdot\sqrt{a}}\right)+1\right)+e^{A\cdot(t-s)}\left(erf\left(\sqrt{a}\cdot(t-s)+\frac{A}{2\cdot\sqrt{a}}\right)+1\right)\right).$$

(17)

A.3. Auxiliary integrals used in Section 3.2.4. for OU input

In what follows we shall make use of the formulae:

$$I_4(a,\omega,t_0,t,s) = \int_{t_0}^{t} e^{a\cdot(t_1-s)}\cdot\cos(\omega\cdot(t_1-s))\,dt_1 \overset{\tau=t_1-s}{=} \int_{t_0-s}^{t-s} e^{a\cdot\tau}\cdot\cos(\omega\cdot\tau)\,d\tau = \\ = \frac{e^{a\tau}}{a^2+\omega^2}\cdot\left(a\cdot\cos(\omega\tau)+\omega\cdot\sin(\omega\tau)\right)\Big|_{x=t_0-s}^{x=t-s} \tag{1}$$

$$I_5(a,\omega,t_0,t,s) = \int_{t_0}^{t} e^{a(t_1-s)}\cdot\sin(\omega(t_1-s))\,dt_1 \overset{\tau=t_1-s}{=} \int_{t_0-s}^{t-s} e^{a\tau}\cdot\sin(\omega\tau)\,d\tau \\ = \frac{e^{a\tau}}{a^2+\omega^2}\cdot\left(-\omega\cdot\cos(\omega\tau)+a\cdot\sin(\omega\tau)\right)\Big|_{\tau=t_0-s}^{\tau=t-s} \tag{2}$$

and

$$I_6(c_4,c_5,a,\omega,t_0,t,s) = c_4\cdot I_4(a,\omega,t_0,t,s)+c_5\cdot I_5(a,\omega,t_0,t,s) = \\ = \frac{e^{a\tau}}{a^2+\omega^2}\cdot\left((c_4\cdot a-c_5\cdot\omega)\cdot\cos(\omega\tau)+(c_4\cdot\omega+c_5\cdot a)\cdot\sin(\omega\tau)\right)\Big|_{\tau=t_0-s}^{\tau=t-s} \tag{3}$$

We shall first consider the integrals appearing in Equ.(28a)_Sec(3.2.4). For the first integral appearing in Equ.(28a)_Sec(3.2.4) we have:

$$\int_{t_0}^{t} SR_{xy}(s,t_1)\Big|_{s\geq t_1}\cdot e^{-A\cdot t_1}dt_1 = \qquad (\text{Substituting from Equ.(29a)\_Sec(3.2.4)})$$

$$= B\cdot\sigma^2\cdot\left[\frac{2\cdot a\cdot\left((a^2-A^2)+\omega^2\right)}{\left((a-A)^2+\omega^2\right)\cdot\left((A+a)^2+\omega^2\right)}\cdot e^{A\cdot s}\int_{t_0}^{t} e^{-2\cdot A\cdot t_1}dt_1 + \right.$$

$$\left. -\frac{e^{-A\cdot s}}{(A+a)^2+\omega_0^2}\cdot\left[\int_{t_0}^{t} e^{(a-A)\cdot(t_1-s)}\left((A+a)\cdot\cos(\omega_0\cdot(t_1-s))+\omega_0\cdot\sin(\omega_0\cdot(t_1-s))\right)dt_1\right] = \right.$$

$$= B\cdot\sigma^2\cdot\left[-e^{A\cdot s}\cdot\left(e^{-2A\cdot t}-e^{-2A\cdot t_0}\right)\frac{a\cdot\left((a^2-A^2)+\omega^2\right)}{A\left((a-A)^2+\omega^2\right)\cdot\left((A+a)^2+\omega^2\right)}-\right.$$
$$\left.-\frac{e^{-A\cdot s}}{(a-A)^2+\omega_0^2}\cdot\left[I_6\left(A+a,\omega_0,a-A,\omega_0,t_0,t,s\right)\right]\right]. \quad (4)$$

Applying Equ.(3) to Equ.(4), we obtain:

$$\int_{t_0}^{t} SR_{xy}(s,t_1)\Big|_{s\geq t_1}\cdot e^{-A\cdot t_1}dt_1 = \frac{B\cdot\sigma^2}{\left((a-A)^2+\omega^2\right)\cdot\left((A+a)^2+\omega^2\right)}\times$$
$$\times\left[-e^{A\cdot s}\cdot\left(e^{-2A\cdot t}-e^{-2A\cdot t_0}\right)\frac{a\cdot\left((a^2-A^2)+\omega^2\right)}{A}-\right. \quad (5)$$
$$-e^{-A\cdot s+(a-A)\cdot(t-s)}\cdot\left[\left((a^2-A^2)-(\omega_0)^2\right)\cdot\cos\left(\omega_0\cdot(t-s)\right)+2\cdot\omega_0\cdot a\cdot\sin\left(\omega_0\cdot(t-s)\right)\right]$$
$$\left.+e^{-A\cdot s+(a-A)\cdot(t_0-s)}\cdot\left[(a^2-A^2)\cdot\cos\left(\omega_0\cdot(t_0-s)\right)+2\cdot\omega_0\cdot a\cdot\sin\left(\omega_0\cdot(t_0-s)\right)\right]\right].$$

For the second integral of Equ.(28a)_Sec(3.2.4) we have:

$$\int_{t_0}^{t} TR_{xy}\left(s,t_1,t_0,m_{x_0}\right)\cdot e^{-A\cdot t_1}dt_1 = \quad (\text{Substituting from Equ.29c \_ Sec(3.2.4)}) \quad =$$
$$= -\frac{B\cdot\sigma^2\cdot e^{A\cdot(s-2\cdot t_0)}}{(a-A)^2+\omega_0^2}\cdot\left[\int_{t_0}^{t} e^{-(a+A)\cdot(t_1-t_0)}\left((a-A)\cdot\cos\left(\omega_0\cdot(t_1-t_0)\right)-\omega_0\cdot\sin\left(\omega_0\cdot(t_1-t_0)\right)\right)dt_1\right]+$$
$$+e^{A\cdot(s-t_0)}\cdot m_{x_0}\cdot m_y\cdot\int_{t_0}^{t} e^{-A\cdot t_1}dt_1$$
$$= -B\cdot\sigma^2\cdot\frac{e^{A\cdot(s-2\cdot t_0)}}{(A+a)^2+\omega_0^2}\cdot\left[I_6\left(a-A,-\omega_0,-(a+A),\omega_0,t_0,t,t_0\right)\right]+ \quad (6)$$
$$-\frac{B}{A}\cdot e^{A\cdot(s-t_0)}\cdot m_{x_0}\cdot m_y\cdot\left(e^{-A\cdot t}-e^{-A\cdot t_0}\right).$$

Applying Equ.(3) to Equ.(6) we obtain:

$$\int_{t_0}^{t} TR_{xy}\left(s,t_1,t_0,m_{x_0}\right)\cdot e^{-A\cdot t_1}dt_1 = \frac{B\cdot\sigma^2}{\left((A+a)^2+\omega_0^2\right)\cdot\left((A-a)^2+\omega_0^2\right)}\times$$
$$\times\left[-e^{A\cdot(s-2\cdot t_0)-(a+A)\cdot(t-t_0)}\cdot\left(\left(-(a^2-A^2)+(\omega_0)^2\right)\cdot\cos\left(\omega_0(t-t_0)\right)+\right.\right.$$
$$\left.\left.+\,2\cdot\omega_0\cdot a\cdot\sin\left(\omega_0(t-t_0)\right)\right)-e^{A\cdot(s-2\cdot t_0)}\cdot\left(((a^2-A^2)+(\omega_0)^2\right)\right] \tag{7}$$
$$-\frac{B}{A}\cdot e^{A\cdot(t+s-t_0)}\cdot m_{x_0}\cdot m_y\cdot\left(e^{-A\cdot t}-e^{-A\cdot t_0}\right).$$

Let us now calculate the integrals appearing in Equ.(28b)_Sec(3.2.4). The first integral of Equ.(28b)_Sec(3.2.4), using Equ.(5), is equivalently written :

$$\int_{t_0}^{s} SR_{xy}\left(s,t_1\right)\Big|_{s\geq t_1}\cdot e^{-A\cdot t_1}dt_1 = \frac{B\cdot\sigma^2}{\left((a-A)^2+\omega^2\right)\cdot\left((A+a)^2+\omega^2\right)}\times$$
$$\times\left[-e^{A\cdot s}\cdot\left(e^{-2A\cdot s}-e^{-2A\cdot t_0}\right)\frac{a\cdot\left((a^2-A^2)+\omega^2\right)}{A}-e^{-A\cdot s}\cdot\left((a^2-A^2)-(\omega_0)^2\right)+\right. \tag{8}$$
$$\left.+\,e^{-A\cdot s+(a-A)\cdot(t_0-s)}\cdot\left[\left((a^2-A^2)-(\omega_0)^2\right)\cdot\cos\left(\omega_0\cdot(t_0-s)\right)+2\cdot\omega_0\cdot a\cdot\sin\left(\omega_0\cdot(t_0-s)\right)\right]\right].$$

whereas the second integral appearing in Equ.(28b) becomes:

$$\int_{s}^{t} SR_{xy}\left(s,t_1\right)\Big|_{s<t_1}\cdot e^{-A\cdot t_1}dt_1 = \qquad (\text{Substituting from Equ.(29b) \_Sec(3.2.4)})$$
$$=\frac{B^2\cdot\sigma^2\cdot e^{-A\cdot s}}{(a-A)^2+\omega^2}\cdot\int_{s}^{t} e^{-(a+A)\cdot(t_1-s)}\left((a-A)\cdot\cos\left(\omega\cdot(t_1-s)\right)-\omega\cdot\sin\left(\omega\cdot(t_1-s)\right)\right)dt_1 \tag{9}$$
$$=\frac{B^2\cdot\sigma^2\cdot e^{-A\cdot s}}{(a-A)^2+\omega^2}\cdot I_6\left((a-A),-\omega_0,-(a+A),\omega_0,s,t,s\right).$$

Applying Equ.(4) in Equ.(9) we get:

$$\int_{s}^{t} SR_{xy}\left(s,t_1\right)\Big|_{s<t_1}\cdot e^{-A\cdot t_1}dt_1 = \frac{B^2\cdot\sigma^2}{\left((a-A)^2+\omega^2\right)\cdot\left((a+A)^2+(\omega_0)^2\right)}\cdot\left[e^{-A\cdot s-(a+A)\cdot(t-s)}\times\right.$$
$$\times\left((-(a^2-A^2)+(\omega_0)^2)\cdot\cos\left(\omega_0\cdot(t-s)\right)+2\cdot a\cdot\omega_0\cdot\sin\left(\omega\cdot(t-s)\right)\right)- \tag{10}$$
$$\left.-e^{-A\cdot s}\cdot(-(a^2-A^2)+(\omega_0)^2)\right].$$

A.4. Some auxiliary formulae concerning the Gaussian joint REPDF.

The joint REPDF $f_{x(t)y(s)}(\alpha,\beta)$ that corresponds to the linear RDE given by Equ.(1) _Sec(3.1.1), when the excitation is assumed to Gaussian, will be a 2D-Gaussian pdf:

$$f_{x(t)y(s)}(\alpha,\beta)=\frac{1}{2\pi\cdot\sqrt{C_{xx}(t,t)\cdot C_{yy}(s,s)-\left(C_{xy}(t,s)\right)^2}}\times$$
$$\times\exp\left[-\frac{1}{2\cdot\left(C_{xx}(t,t)\cdot C_{yy}(s,s)-\left(C_{xy}(t,s)\right)^2\right)}\times\right. \tag{1}$$
$$\left.\times\left(C_{yy}(s,s)\cdot\left(\alpha-m_x(t)\right)^2-2\cdot C_{xy}(t,s)\cdot\left(\alpha-m_x(t)\right)\cdot\left(\beta-m_y(s)\right)+C_{xx}(t,t)\cdot\left(\beta-m_y(s)\right)^2\right)\right],$$

where s, t represent the excitation and the response time respectively.

Let:

$$W(t,s)\equiv\sqrt{C_{xx}(t,t)\cdot C_{yy}(s,s)-\left(C_{xy}(t,s)\right)^2}\ , \tag{2}$$

$$E(\alpha,\beta;\,t,s)\equiv C_{yy}(s,s)\cdot\left(\alpha-m_x(t)\right)^2-2\cdot C_{xy}(t,s)\cdot\left(\alpha-m_x(t)\right)\cdot\left(\beta-m_y(s)\right)+ \\ +\,C_{xx}(t,t)\cdot\left(\beta-m_y(s)\right)^2, \tag{3}$$

then $f_{x(t)y(s)}(\alpha,\beta)$ can equivalently be written:

$$f_{x(t)y(s)}(\alpha,\beta)\;=\;\frac{1}{2\pi\cdot W(t,s)}\cdot\exp\left[-\frac{E(\alpha,\beta;\,t,s)}{2\cdot\left(W(t,s)\right)^2}\right]. \tag{4}$$

The partial derivative of $f_{x(t)y(s)}(\alpha,\beta)$ with respect to the response time t will be:

$$\frac{\partial}{\partial t}f_{x(t)y(s)}(\alpha,\beta)=$$
$$=\frac{1}{2\cdot\pi}\cdot\frac{\partial}{\partial t}\left(\frac{1}{W(t,s)}\cdot\exp\left[-\frac{E(\alpha,\beta;\,t,s)}{2\cdot\left(W(t,s)\right)^2}\right]\right)=$$
$$=\frac{1}{2\cdot\pi}\cdot\left[-\frac{\frac{\partial}{\partial t}W(t,s)}{\left(W(t,s)\right)^2}+\frac{\frac{\partial}{\partial t}\left[-\frac{E(\alpha,\beta;\,t,s)}{2\cdot\left(W(t,s)\right)^2}\right]}{W(t,s)}\right]\cdot\exp\left[-\frac{E(\alpha,\beta;\,t,s)}{2\cdot\left(W(t,s)\right)^2}\right]=$$

$$=\frac{1}{2\cdot\pi}\cdot\left[-\frac{\frac{\partial}{\partial t}W(t,s)}{\left(W(t,s)\right)^2}+\frac{\frac{\partial}{\partial t}\left[W(t,s)\right]\cdot E(\alpha,\beta;\,t,s)}{\left(W(t,s)\right)^4}-\frac{\frac{\partial}{\partial t}E(\alpha,\beta;\,t,s)}{2\cdot\left(W(t,s)\right)^3}\right]\times$$
$$\times\exp\left[-\frac{1}{2\cdot\left(W(t,s)\right)^2}\cdot E(\alpha,\beta;\,t,s)\right],$$

where:

$$\frac{\partial}{\partial t}W(t,s)=\frac{\partial}{\partial t}\left(\sqrt{C_{xx}(t,s)\cdot C_{yy}(s,s)-\left(C_{xy}(t,s)\right)^2}\right)$$
$$=\frac{C_{yy}(s,s)\cdot\frac{\partial}{\partial t}C_{xx}(t,t)-2\cdot C_{xy}(t,s)\cdot\frac{\partial}{\partial t}C_{xy}(t,s)}{2\cdot W(t,s)},$$

and

$$\frac{\partial}{\partial t}E(\alpha,\beta;\,t,s)=C_{yy}(s,s)\cdot\frac{\partial}{\partial t}\left(\alpha-m_x(t)\right)^2-2\cdot\left(\beta-m_y(s)\right)\cdot\frac{\partial}{\partial t}\left[C_{xy}(t,s)\cdot\left(\alpha-m_x(t)\right)\right]+$$
$$+\left(\beta-m_y(s)\right)^2\cdot\frac{\partial}{\partial t}\left[C_{xx}(t,t)\right]=$$
$$=2\cdot\left(\alpha-m_x(t)\right)\cdot C_{yy}(s,s)\cdot\frac{\partial m_x(t)}{\partial t}-$$
$$-2\cdot\left(\beta-m_y(s)\right)\cdot\left(\left(\alpha-m_x(t)\right)\frac{\partial}{\partial t}\left[C_{xy}(t,s)\cdot\right]-C_{xy}(t,s)\cdot\frac{\partial m_x(t)}{\partial t}\right)$$
$$+\left(\beta-m_y(s)\right)^2\cdot\frac{\partial}{\partial t}\left[C_{xx}(t,t)\right].$$

That is:

$$\frac{\partial}{\partial t}f_{x(t)y(s)}(\alpha,\beta)=\frac{1}{2\cdot\pi}\cdot\left[-\frac{\frac{\partial}{\partial t}W(t,s)}{\left(W(t,s)\right)^2}+\frac{\frac{\partial}{\partial t}\left[W(t,s)\right]\cdot E(\alpha,\beta;\,t,s)}{\left(W(t,s)\right)^4}-\frac{\frac{\partial}{\partial t}E(\alpha,\beta;\,t,s)}{2\cdot\left(W(t,s)\right)^3}\right]\times$$
$$\times\exp\left[-\frac{1}{2\cdot\left(W(t,s)\right)^2}\cdot E(\alpha,\beta;\,t,s)\right],\qquad(5a)$$

where:

$$\frac{\partial}{\partial t}W(t,s)=\frac{C_{yy}(s,s)\cdot\frac{\partial}{\partial t}C_{xx}(t,t)-2\cdot C_{xy}(t,s)\cdot\frac{\partial}{\partial t}C_{xy}(t,s)}{2\cdot W(t,s)}, \tag{5b}$$

$$\begin{aligned}\frac{\partial}{\partial t}E(\alpha,\beta;t,s)&=2\cdot(\alpha-m_x(t))\cdot C_{yy}(s,s)\cdot\frac{\partial m_x(t)}{\partial t}-\\&-2\cdot(\beta-m_y(s))\cdot\left((\alpha-m_x(t))\frac{\partial}{\partial t}\left[C_{xy}(t,s)\cdot\right]-C_{xy}(t,s)\cdot\frac{\partial m_x(t)}{\partial t}\right)\\&+(\beta-m_y(s))^2\cdot\frac{\partial}{\partial t}\left[C_{xx}(t,t)\right],\end{aligned} \tag{5c}$$

Consequently, the partial derivative of $f_{x(t)y(s)}(\alpha,\beta)$, with respect to the response variable α will be:

$$\begin{aligned}\frac{\partial}{\partial\alpha}f_{x(t)y(s)}(\alpha,\beta)&=\frac{\partial}{\partial\alpha}\left[\frac{1}{2\cdot\pi\cdot W(t,s)}\cdot\exp\left[-\frac{E(\alpha,\beta;t,s)}{2\cdot(W(t,s))^2}\right]\right]=\\&=-\frac{1}{4\cdot\pi\cdot(W(t,s))^3}\cdot\frac{\partial E(\alpha,\beta;t,s)}{\partial\alpha}\cdot\exp\left[-\frac{E(\alpha,\beta;t,s)}{2\cdot(W(t,s))^2}\right],\end{aligned} \tag{6a}$$

where:

$$\frac{\partial E(\alpha,\beta;t,s)}{\partial\alpha}=2\cdot C_{yy}(s,s)\cdot(\alpha-m_x(t))-2\cdot C_{xy}(t,s)\cdot(\beta-m_y(s)). \tag{6b}$$

That is:

$$\frac{\partial}{\partial\alpha}f_{x(t)y(s)}(\alpha,\beta)=-\frac{C_{yy}(s,s)\cdot(\alpha-m_x(t))-C_{xy}(t,s)\cdot(\beta-m_y(s))}{2\cdot\pi\cdot(W(t,s))^3}\cdot\exp\left[-\frac{E(\alpha,\beta;t,s)}{2\cdot(W(t,s))^2}\right]. \tag{7}$$

In case that the long-time statistical equilibrium state of Equ.(1)_Sec(3.1.1) is considered, the partial derivative with respect t (half time derivative) given by Equ.(5) simplifies to:

$$\begin{aligned}\frac{\partial}{\partial t}f_{x(t)y(s)}(\alpha,\beta)&=\frac{1}{2\cdot\pi}\cdot\left[-\frac{\frac{\partial}{\partial t}W(t,s)}{(W(t,s))^2}+\frac{\frac{\partial}{\partial t}\left[W(t,s)\right]\cdot E(\alpha,\beta;t,s)}{(W(t,s))^4}-\frac{\frac{\partial}{\partial t}E(\alpha,\beta;t,s)}{2\cdot(W(t,s))^3}\right]\times\\&\times\exp\left[-\frac{1}{2\cdot(W(t,s))^2}\cdot E(\alpha,\beta;t,s)\right],\end{aligned} \tag{8a}$$

where:

$$\frac{\partial}{\partial t}W(t,s)=-\frac{2\cdot C_{xy}(t,s)\cdot\frac{\partial}{\partial t}C_{xy}(t,s)}{2\cdot W(t,s)}, \tag{8b}$$

$$\frac{\partial}{\partial t}E(\alpha,\beta;t,s)=-2\cdot\left(\beta-m_y(s)\right)\cdot\left(\alpha-m_x(t)\right)\frac{\partial C_{xy}(t,s)}{\partial t}. \tag{8}$$

A.5. Equivalent expression for the off - diagonal REPDF evolution constrain

Let us consider the left hand side of the off - diagonal REPDF evolution constrain in the linear/ Gaussian case Equ.(1) _Sec(3.5) :

$$LH=\frac{\partial f_{x(t)y(s)}(\alpha,\beta)}{\partial t}+A\cdot f_{x(t)y(s)}(\alpha,\beta)+[A\cdot\alpha\cdot+B\cdot\beta]\frac{\partial f_{x(t)y(s)}(\alpha,\beta)}{\partial\alpha}. \tag{1}$$

We shall first assume that $m_x(t)=m_y(s)=0$. In this case the joint REPDF $f_{x(t)y(s)}(\alpha,\beta)$ IS given by the formula:

$$f_{x(t)y(s)}(\alpha,\beta)=\frac{1}{2\pi\cdot\sqrt{R_{xx}(t,t)\cdot R_{yy}(s,s)-\left(R_{xy}(t,s)\right)^2}}\times \times\exp\left\{-\frac{R_{yy}(s,s)\cdot\alpha^2-2\cdot R_{xy}(t,s)\cdot\alpha\cdot\beta+R_{xx}(t,t)\cdot\beta^2}{2\cdot\left(R_{xx}(t,t)\cdot R_{yy}(s,s)-\left(R_{xy}(t,s)\right)^2\right)}\right\}. \tag{2}$$

For $m_x(t)=m_y(s)=0$ the first term of the right hand side of Equ.(1) (see Equ(5)_App.4) reduces to:

$$\frac{\partial}{\partial t}f_{x(t)y(s)}(\alpha,\beta)=\frac{1}{2\cdot\pi\cdot\left(W(t,s)\right)^2}\times \times\left[-\frac{\partial}{\partial t}W(t,s)+\frac{\frac{\partial}{\partial t}[W(t,s)]\cdot E(\alpha,\beta;t,s)}{\left(W(t,s)\right)^2}-\frac{\frac{\partial}{\partial t}E(\alpha,\beta;t,s)}{2\cdot\left(W(t,s)\right)}\right] \times\exp\left[-\frac{1}{2\cdot\left(W(t,s)\right)^2}\cdot E(\alpha,\beta;t,s)\right], \tag{3a}$$

where:

$$W(t,s)\equiv\sqrt{R_{xx}(t,t)\cdot R_{yy}(s,s)-\left(R_{xy}(t,s)\right)^2}, \tag{3b}$$

$$E(\alpha,\beta;t,s) \equiv R_{yy}(s,s)\cdot\alpha^2 - 2\cdot R_{xy}(t,s)\cdot\alpha\cdot\beta + R_{xx}(t,t)\cdot\beta^2, \tag{3c}$$

$$\frac{\partial}{\partial t}\left[W(t,s)\right] = \frac{R_{yy}(s,s)\cdot\frac{\partial}{\partial t}R_{xx}(t,t) - 2\cdot R_{xy}(t,s)\cdot\frac{\partial}{\partial t}R_{xy}(t,s)}{2\cdot W(t,s)}, \tag{3d}$$

$$\frac{\partial}{\partial t}E(\alpha,\beta;t,s) = -2\cdot\alpha\cdot\beta\cdot\frac{\partial}{\partial t}R_{xy}(t,s) + \beta^2\cdot\frac{\partial}{\partial t}R_{xx}(t,t). \tag{3e}$$

From Equs.(3c, 3d), the 2nd term appearing in the summation in the right hand of Equ.(3a), becomes:

$$\frac{\partial}{\partial t}W(t,s)\cdot\frac{E(\alpha,\beta;t,s)}{\left(W(t,s)\right)^2} = \frac{\partial}{\partial t}W(t,s)\cdot\frac{R_{yy}(s,s)\cdot\alpha^2 - 2\cdot R_{xy}(t,s)\cdot\alpha\cdot\beta + R_{xx}(t,t)\cdot\beta^2}{\left(W(t,s)\right)^2} =$$

$$= \frac{\partial}{\partial t}W(t,s)\cdot\frac{R_{yy}(s,s)}{\left(W(t,s)\right)^2}\cdot\alpha^2 - 2\cdot\frac{\partial}{\partial t}W(t,s)\cdot\frac{R_{xy}(t,s)}{\left(W(t,s)\right)^2}\cdot\alpha\cdot\beta + \frac{\partial}{\partial t}W(t,s)\cdot\frac{R_{xx}(t,t)}{\left(W(t,s)\right)^2}\cdot\beta^2. \tag{4}$$

Then, from Equs.(4,3e) for the 2nd and 3rd term of Equ.(3a) we obtain:

$$\frac{\partial}{\partial t}W(t,s)\cdot\frac{E(\alpha,\beta;t,s)}{\left(W(t,s)\right)^2} - \frac{\frac{\partial}{\partial t}E(\alpha,\beta;t,s)}{2\cdot\left(W(t,s)\right)} =$$

$$= \frac{\partial}{\partial t}W(t,s)\cdot\frac{R_{yy}(s,s)}{\left(W(t,s)\right)^2}\cdot\alpha^2 - \left(2\cdot\frac{\partial}{\partial t}W(t,s)\cdot\frac{R_{xy}(t,s)}{\left(W(t,s)\right)^2} - \frac{\partial}{\partial t}R_{xy}(t,s)\cdot\frac{1}{W(t,s)}\right)\cdot\alpha\cdot\beta +$$

$$+\left(\frac{\partial}{\partial t}W(t,s)\cdot\frac{R_{xx}(t,t)}{\left(W(t,s)\right)^2} - \frac{\partial}{\partial t}R_{xx}(t,t)\cdot\frac{1}{2\cdot W(t,s)}\right)\cdot\beta^2. \tag{5}$$

Applying Equ.(5) in Equ.(3a), Equ.(3a) becomes:

$$\frac{\partial}{\partial t}f_{x(t)y(s)}(\alpha,\beta) = \frac{1}{2\cdot\pi\cdot\left(W(t,s)\right)^2}\times\left[-\frac{\partial}{\partial t}W(t,s) + \frac{\partial}{\partial t}W(t,s)\cdot\frac{R_{yy}(s,s)}{\left(W(t,s)\right)^2}\cdot\alpha^2 -\right.$$

$$-\left(2\cdot\frac{\partial}{\partial t}W(t,s)\cdot\frac{R_{xy}(t,s)}{\left(W(t,s)\right)^2} - \frac{\partial}{\partial t}R_{xy}(t,s)\cdot\frac{1}{W(t,s)}\right)\cdot\alpha\cdot\beta +$$

$$\left.+\left(\frac{\partial}{\partial t}W(t,s)\cdot\frac{R_{xx}(t,t)}{\left(W(t,s)\right)^2} - \frac{\partial}{\partial t}R_{xx}(t,t)\cdot\frac{1}{2\cdot W(t,s)}\right)\cdot\beta^2\right]\times$$

$$\times \exp\left[-\frac{1}{2\cdot\left(W\left(t,s\right)\right)^2}\cdot E\left(\alpha,\beta;\,t,s\right)\right]. \qquad (6)$$

We shall now compute the partial derivative of $f_{x(t)y(s)}(\alpha,\beta)$ with respect to α. Setting $m_x\left(t\right)=m_y\left(s\right)=0$ in Equ.(7)_App(4), we have:

$$\frac{\partial}{\partial\alpha} f_{x(t)y(s)}\left(\alpha,\beta\right)=-\frac{\alpha\cdot R_{yy}\left(s,s\right)-\beta\cdot R_{xy}\left(t,s\right)}{2\cdot\pi\cdot\left(W\left(t,s\right)\right)^3}\cdot\exp\left[-\frac{E\left(\alpha,\beta;\,t,s\right)}{2\cdot\left(W\left(t,s\right)\right)^2}\right]. \qquad (7)$$

Then, the 3rd term of the right hand side of Equ.(1) is equivalently written:

$$\left[A\cdot\alpha+\ B\cdot\beta\right]\cdot\frac{\partial}{\partial\alpha} f_{x(t)y(s)}\left(\alpha,\beta\right)=$$

$$=-\left[A\cdot\alpha+\ B\cdot\beta\right]\cdot\frac{\alpha\cdot R_{yy}\left(s,s\right)-\beta\cdot R_{xy}\left(t,s\right)}{2\cdot\pi\cdot\left(W\left(t,s\right)\right)^3}\cdot\exp\left[-\frac{E\left(\alpha,\beta;\,t,s\right)}{2\cdot\left(W\left(t,s\right)\right)^2}\right]=$$

$$=-\frac{\exp\left[E\left(t,s;\alpha,\beta\right)\right]}{2\cdot\pi\cdot W^3\left(t,s\right)}\cdot\left[A\cdot\alpha+\ B\cdot\beta\right]\cdot\left(\left(R_{yy}\left(s,s\right)\cdot\alpha-R_{xy}\left(t,s\right)\cdot\beta\right)\right)=$$

$$=-\frac{\exp\left[E\left(t,s;\alpha,\beta\right)\right]}{2\cdot\pi\cdot W^3\left(t,t\right)}\times$$

$$\times\left[R_{yy}\left(s,s\right)\cdot A\cdot\alpha^2-R_{xy}\left(t,s\right)\cdot A\cdot\alpha\cdot\beta+R_{yy}\left(s,s\right)\cdot B\cdot\alpha\cdot\beta-B\cdot R_{xy}\left(t,s\right)\cdot\beta^2\right],$$

that is:

$$\left[A\cdot\alpha+\ B\cdot\beta\right]\cdot\frac{\partial}{\partial\alpha} f_{x(t)y(s)}\left(\alpha,\beta\right)=\frac{1}{2\cdot\pi\cdot W\left(t,s\right)}\times \qquad (8)$$

$$\times\left[-\frac{A\cdot R_{yy}\left(s,s\right)}{W^2\left(t,s\right)}\cdot\alpha^2+\frac{A\cdot R_{xy}\left(t,s\right)-B\cdot R_{yy}\left(s,s\right)}{W^2\left(t,s\right)}\alpha\cdot\beta+\frac{B\cdot R_{xy}\left(t,s\right)}{W^2\left(t,s\right)}\cdot\beta^2\right]\times\exp\left[E\left(t,s;\alpha,\beta\right)\right].$$

From Equs.(6,8) we get that the right hand side of Equ.(1) becomes:

$$\frac{\partial f_{x(t)y(s)}\left(\alpha,\beta\right)}{\partial t}+A\cdot f_{x(t)y(s)}\left(\alpha,\beta\right)+\left[A\cdot\alpha\cdot+B\cdot\beta\right]\frac{\partial f_{x(t)y(s)}\left(\alpha,\beta\right)}{\partial\alpha}=$$

$$\frac{\exp\left\{E(t,t;\alpha,\beta)\right\}}{2\cdot\pi\cdot W(t,s)}\cdot\left[-Q_{00}(t,s)+\left(\frac{R_{yy}(s,s)}{\left(W(t,s)\right)^2}Q_{00}(t,s)\right)\cdot\alpha^2-Q_{11}(t,s)\cdot\alpha\cdot\beta+Q_{02}(t,s)\cdot\beta^2\right],\tag{9}$$

where:

$$Q_{00}(t,s)=\frac{1}{W(t,s)}\cdot\frac{\partial}{\partial t}W(t,s)-A,\tag{10a}$$

$$Q_{11}(t,s)=\frac{1}{\left(W(t,s)\right)^2}\left[2\cdot\frac{\partial}{\partial t}W(t,s)\cdot\frac{R_{xy}(t,s)}{W(t,s)}-\frac{\partial}{\partial t}R_{xy}(t,s)-A\cdot R_{xy}(t,s)+B\cdot R_{yy}(s,s)\right],\tag{10b}$$

$$Q_{02}(t,s)=\frac{1}{\left(W(t,s)\right)^2}\left(\frac{\partial}{\partial t}W(t,s)\cdot\frac{R_{xx}(t,t)}{\left(W(t,s)\right)}-\frac{1}{2}\frac{\partial}{\partial t}R_{xx}(t,t)+B\cdot R_{xy}(t,s)\right).\tag{10c}$$

We shall treat each one of Equs.(10a-c) separately, trying to express them in terms of differential expressions that look like the two-time RE moment equations (Equs.(8,22) _Sec.(3.2.1)).

For $Q_{00}(t,s)$ from Equ.(3d, 10a), we have:

$$Q_{00}(t,s)=\frac{R_{yy}(s,s)\cdot\frac{\partial}{\partial t}R_{xx}(t,t)-2\cdot R_{xy}(t,s)\cdot\frac{\partial}{\partial t}R_{xy}(t,s)-2\cdot A\cdot W(t,s)}{2\cdot W(t,s)^2}=$$
$$=\frac{1}{W(t,s)^2}\cdot\left[\frac{R_{yy}(s,s)}{2}\cdot\frac{\partial}{\partial t}R_{xx}(t,t)-R_{xy}(t,s)\cdot\frac{\partial}{\partial t}R_{xy}(t,s)-A\cdot R_{xx}(t,t)\cdot R_{yy}(s,s)+A\cdot\left(R_{xy}(t,s)\right)^2\right]=$$
$$=\frac{1}{W(t,s)^2}\cdot\left[\frac{R_{yy}(s,s)}{2}\cdot\left(\frac{\partial}{\partial t}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)\right)-R_{xy}(t,s)\cdot\left(\frac{\partial}{\partial t}R_{xy}(t,s)-A\cdot\left(R_{xy}(t,s)\right)\right)\right].\tag{11}$$

Adding and subtracting $B\cdot R_{yy}(s,s)\cdot R_{xy}(t,s)$ in Equ.(11), we obtain:

$$Q_{00}(t,s)=\frac{1}{W(t,s)^2}\cdot\left[\frac{R_{yy}(s,s)}{2}\cdot\left(\frac{\partial}{\partial t}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,s)\right)\right.$$
$$\left.-R_{xy}(t,s)\cdot\left(\frac{\partial}{\partial t}R_{xy}(t,s)-A\cdot R_{xy}(t,s)-B\cdot R_{yy}(s,s)\right)\right].\tag{12}$$

Let us now proceed to $Q_{11}(t,s)$. From Equs.(3d, 10b) we have:

$$Q_{11}(t,s) = \frac{1}{\left(W(t,s)\right)^4}\cdot\left[R_{xy}(t,s)R_{yy}(s,s)\cdot\frac{\partial}{\partial t}R_{xx}(t,t) - 2\cdot R_{xy}(t,s)\cdot R_{xy}(t,s)\frac{\partial}{\partial t}R_{xy}(t,s) + \right.$$
$$+\left(-\frac{\partial}{\partial t}R_{xy}(t,s) - A\cdot R_{xy}(t,s) + B\cdot R_{yy}(s,s)\right)\times$$
$$\left.\times\left(R_{xx}(t,t)\cdot R_{yy}(s,s) - \left(R_{xy}(t,s)\right)^2\right)\right] =$$
$$= \frac{1}{\left(W(t,s)\right)^4}\cdot\left[R_{xy}(t,s)R_{yy}(s,s)\cdot\frac{\partial}{\partial t}R_{xx}(t,t) - 2\cdot\left(R_{xy}(t,s)\right)^2\cdot\frac{\partial}{\partial t}R_{xy}(t,s) - \right.$$
$$-R_{xx}(t,t)\cdot R_{yy}(s,s)\cdot\frac{\partial}{\partial t}R_{xy}(t,s) + \left(R_{xy}(t,s)\right)^2\cdot\frac{\partial}{\partial t}R_{xy}(t,s) +$$
$$-A\cdot R_{xy}(t,s)\cdot R_{xx}(t,t)\cdot R_{yy}(s,s) + A\cdot R_{xy}(t,s)\cdot\left(R_{xy}(t,s)\right)^2 +$$
$$\left. + B\cdot R_{xx}(t,t)\cdot\left(R_{yy}(s,s)\right)^2 - B\cdot R_{yy}(s,s)\cdot\left(R_{xy}(t,s)\right)^2\right]. \qquad (13)$$

Adding and subtracting $A\cdot R_{xy}(t,s)\cdot R_{xx}(t,t)\cdot R_{yy}(s,s) + B\cdot\left(R_{xy}(t,s)\right)^2\cdot R_{yy}(s,s)$ to Equ.(13) we obtain:

$$Q_{11}(t,s) = \frac{1}{\left(W(t,s)\right)^4}\cdot\left[R_{xy}(t,s)R_{yy}(s,s)\cdot\frac{\partial}{\partial t}R_{xx}(t,t) - \left(R_{xy}(t,s)\right)^2\cdot\frac{\partial}{\partial t}R_{xy}(t,s) - \right.$$
$$-R_{xx}(t,t)\cdot R_{yy}(s,s)\cdot\frac{\partial}{\partial t}R_{xy}(t,s) +$$
$$-A\cdot R_{xy}(t,s)\cdot R_{xx}(t,t)\cdot R_{yy}(s,s) + A\cdot R_{xy}(t,s)\cdot\left(R_{xy}(t,s)\right)^2 +$$
$$+B\cdot\left(R_{yy}(s,s)\right)^2\cdot R_{xx}(t,t) \quad - B\cdot\left(R_{xy}(t,s)\right)^2\cdot R_{yy}(s,s) -$$
$$-A\cdot R_{xy}(t,s)\cdot R_{xx}(t,t)\cdot R_{yy}(s,s) - B\cdot\left(R_{xy}(t,s)\right)^2\cdot R_{yy}(s,s) +$$
$$\left. + A\cdot R_{xy}(t,s)\cdot R_{xx}(t,t)\cdot R_{yy}(s,s) + B\cdot\left(R_{xy}(t,s)\right)^2\cdot R_{yy}(s,s)\right]. \qquad (14)$$

That is:

$$Q_{11}(t,s) = \frac{1}{\left(W(t,s)\right)^4}\left[R_{xy}(t,s)R_{yy}(s,s)\cdot\left(\frac{\partial}{\partial t}R_{xx}(t,t) - 2\cdot A\cdot R_{xx}(t,t) - 2\cdot B\cdot R_{xy}(t,s)\right) - \right.$$
$$\left. -\left(\left(R_{xy}(t,s)\right)^2 + R_{xx}(t,t)\cdot R_{yy}(s,s)\right)\cdot\left(\frac{\partial}{\partial t}R_{xy}(t,s) - A\cdot R_{xy}(t,s) - B\cdot R_{yy}(s,s)\right)\right]. \qquad (15)$$

For $Q_{02}(t,s)$, from Equ.(3d,10c) we have:

$$Q_{02}(t,s)=\frac{1}{\left(W(t,s)\right)^2}\cdot\left(\frac{\partial}{\partial t}W(t,s)\cdot\frac{R_{xx}(t,t)}{\left(W(t,s)\right)}-\frac{1}{2}\frac{\partial}{\partial t}R_{xx}(t,t)+B\cdot R_{xy}(t,s)\right)=$$

$$=\frac{1}{\left(W(t,s)\right)^4}\cdot\left[\frac{1}{2}R_{xx}(t,t)\cdot R_{yy}(s,s)\cdot\frac{\partial}{\partial t}R_{xx}(t,t)-R_{xx}(t,t)\cdot R_{xy}(t,s)\cdot\frac{\partial}{\partial t}R_{xy}(t,s)+\right.$$

$$\left.+\left(-\frac{1}{2}\frac{\partial}{\partial t}R_{xx}(t,t)+B\cdot R_{xy}(t,s)\right)\cdot\left(R_{xx}(t,t)\cdot R_{yy}(s,s)-\left(R_{xy}(t,s)\right)^2\right)\right]=$$

$$=\frac{1}{\left(W(t,s)\right)^4}\cdot\left[-R_{xy}(t,s)\cdot R_{xx}(t,t)\cdot\frac{\partial}{\partial t}R_{xy}(t,s)+B\cdot R_{xy}(t,s)\cdot R_{xx}(t,t)\cdot R_{yy}(s,s)+\right.$$

$$\left.+\frac{1}{2}\left(R_{xy}(t,s)\right)^2\frac{\partial}{\partial t}R_{xx}(t,t)-B\cdot R_{xy}(t,s)\cdot\left(R_{xy}(t,s)\right)^2\right].\quad(16)$$

Finally, adding and subtracting $A\cdot R_{xx}(t,t)\cdot\left(R_{xy}(t,s)\right)^2$ in Equ.(16), we obtain:

$$Q_{02}(t,s)=\frac{1}{\left(W(t,s)\right)^4}\cdot\left[-R_{xy}(t,s)\cdot R_{xx}(t,t)\cdot\left(\frac{\partial}{\partial t}R_{xy}(t,s)-A\cdot R_{xy}(t,s)-B\cdot R_{yy}(s,s)\right)+\right.$$

$$\left.+\frac{1}{2}\left(R_{xy}(t,s)\right)^2\left(\frac{\partial}{\partial t}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,s)\right)\right].\ (17)$$

Comparing Equs.(12,15,17) we notice that these are all written in terms of two differential expressions of the two-time moments $R_{yy}(s,s),R_{xy}(t,s),R_{xx}(t,t)$. In what follows we shall drop the zero mean value assumption to show that in this case, an additional differential expression for the mean value also appears.

Let us assume that $m_x(t)\neq m_y(s)\neq 0$. Then, the Gaussian random functions: $\tilde{x}(t;\theta)=x(t;\theta)-m_x(t)$, $\tilde{y}(s;\theta)=y(s;\theta)-m_y(s)$ will have zero mean values, i.e. $m_{\tilde{y}}(s)=0$, $m_{\tilde{x}}(t)=0$, therefore $f_{\tilde{x}(t)\tilde{y}(s)}(\alpha,\beta)$ will verify Equ.(9). Thereafter we shall use the equivalent expression for the parameters $Q_{00}(t,s)$, $Q_{11}(t,s)$, $Q_{02}(t,s)$ given by Equs.(12,15,17), when we refer to the random functions $\tilde{x}(t;\theta)$, $\tilde{y}(s;\theta)$ these parameters are denoted by $\tilde{Q}_{00}(t,s)$, $\tilde{Q}_{11}(t,s)$, $\tilde{Q}_{02}(t,s)$. More precisely for $f_{\tilde{x}(t)\tilde{y}(s)}(\alpha,\beta)$ will verify the following equation:

$$\frac{\partial f_{\tilde{x}(t)\tilde{y}(s)}(\alpha,\beta)}{\partial t}+A\cdot f_{\tilde{x}(t)\tilde{y}(s)}(\alpha,\beta)+\left[A\cdot\alpha\cdot+B\cdot\beta\right]\frac{\partial f_{\tilde{x}(t)\tilde{y}(s)}(\alpha,\beta)}{\partial\alpha}=\quad(18)$$

$$\frac{\exp\left\{\tilde{E}(t,t;\alpha,\beta)\right\}}{2\cdot\pi\cdot\tilde{W}(t,s)}\cdot\left[-\tilde{Q}_{00}(t,s)+\left(\frac{R_{\tilde{y}\tilde{y}}(s,s)}{\left(\tilde{W}(t,s)\right)^2}\tilde{Q}_{00}(t,s)\right)\cdot\alpha^2-\tilde{Q}_{11}(t,s)\cdot\alpha\cdot\beta+\tilde{Q}_{02}(t,s)\cdot\beta^2\right],$$

where:

$$R_{\tilde{x}\tilde{x}}(t,t) = C_{xx}(t,t) = R_{xx}(t,t) - \left(m_x(t)\right)^2, \tag{19a}$$

$$R_{\tilde{y}\tilde{y}}(s,s) = C_{yy}(s,s) = R_{yy}(s,s) - \left(m_y(s)\right)^2, \tag{19b}$$

$$R_{\tilde{x}\tilde{y}}(t,s) = C_{xy}(t,s) = R_{xy}(t,s) - m_x(t)\cdot m_y(s), \tag{19c}$$

$$\tilde{W}(t,s) \equiv \sqrt{R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s) - \left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2}, \tag{19d}$$

$$\tilde{E}(\alpha,\beta;t,s) \equiv R_{\tilde{y}\tilde{y}}(s,s)\cdot\alpha^2 - 2\cdot R_{\tilde{x}\tilde{y}}(t,s)\cdot\alpha\cdot\beta + R_{\tilde{x}\tilde{x}}(t,t)\cdot\beta^2, \tag{19f}$$

$$\tilde{Q}_{00}(t,s) = \frac{1}{\left(\tilde{W}(t,s)\right)^2}\cdot\left[\frac{1}{2}\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot\left(\frac{\partial}{\partial t}R_{\tilde{x}\tilde{x}}(t,t) - 2\cdot A\cdot R_{\tilde{x}\tilde{x}}(t,t) - 2\cdot B\cdot R_{\tilde{x}\tilde{y}}(t,s)\right) - \right.$$
$$\left. - R_{\tilde{x}\tilde{y}}(t,s)\cdot\left(\frac{\partial}{\partial t}R_{\tilde{x}\tilde{y}}(t,s) - A\cdot R_{\tilde{x}\tilde{y}}(t,s) - B\cdot R_{\tilde{y}\tilde{y}}(s,s)\right)\right], \tag{19g}$$

$$\tilde{Q}_{11}(t,s) = \frac{1}{\left(\tilde{W}(t,s)\right)^4}\cdot\left[R_{\tilde{x}\tilde{y}}(t,s)R_{\tilde{y}\tilde{y}}(s,s)\cdot\left(\frac{\partial}{\partial t}R_{\tilde{x}\tilde{x}}(t,t) - 2\cdot A\cdot R_{\tilde{x}\tilde{x}}(t,t) - 2\cdot B\cdot R_{\tilde{x}\tilde{y}}(t,s)\right) - \right.$$
$$\left. - \left(\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2 + R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\right)\cdot\left(\frac{\partial}{\partial t}R_{\tilde{x}\tilde{y}}(t,s) - A\cdot R_{\tilde{x}\tilde{y}}(t,s) - B\cdot R_{\tilde{y}\tilde{y}}(s,s)\right)\right], \tag{19h}$$

$$\tilde{Q}_{02}(t,s) = \frac{1}{\left(\tilde{W}(t,s)\right)^4}\times\left[-R_{\tilde{x}\tilde{y}}(t,s)\cdot R_{\tilde{x}\tilde{x}}(t,t)\cdot\left(\frac{\partial}{\partial t}R_{\tilde{x}\tilde{y}}(t,s) - A\cdot R_{\tilde{x}\tilde{y}}(t,s) - B\cdot R_{\tilde{y}\tilde{y}}(s,s)\right) + \right.$$
$$\left. + \frac{1}{2}\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2\left(\frac{\partial}{\partial t}R_{\tilde{x}\tilde{x}}(t,t) - 2\cdot A\cdot R_{\tilde{x}\tilde{x}}(t,t) - 2\cdot B\cdot R_{\tilde{x}\tilde{y}}(t,s)\right)\right]. \tag{19i}$$

The joint REPDF $f_{x(t)y(s)}(\alpha,\beta)$ can be expressed through $f_{\tilde{x}(t)\tilde{y}(s)}(\alpha,\beta)$, i.e.:
$f_{x(t)y(s)}(\alpha,\beta) = f_{\tilde{x}(t)\tilde{y}(s)}(\alpha + m_x(t), \beta + m_y(t))$.

Then, from Equ.(18) we obtain:

$$\frac{\partial f_{\tilde{x}(t)\tilde{y}(s)}(\alpha + m_x(t), \beta + m_y(t))}{\partial t} + A\cdot f_{\tilde{x}(t)\tilde{y}(s)}(\alpha + m_x(t), \beta + m_y(t)) +$$
$$+\left[A\cdot(\alpha + m_x(t))\cdot + B\cdot(\beta + m_y(t))\right]\frac{\partial f_{\tilde{x}(t)\tilde{y}(s)}(\alpha + m_x(t), \beta + m_y(t))}{\partial\tilde{\alpha}} =$$
$$= \frac{\exp\left\{\tilde{E}\left(t,t;\alpha + m_x(t), \beta + m_y(t)\right)\right\}}{2\cdot\pi\cdot\tilde{W}(t,s)}\times\left[-\tilde{Q}_{00}(t,s) + \left(\frac{R_{\tilde{y}\tilde{y}}(s,s)}{\left(\tilde{W}(t,s)\right)^2}\tilde{Q}_{00}(t,s)\right)\cdot(\tilde{\alpha} + m_x(t))^2 - \right.$$
$$\left. - \tilde{Q}_{11}(t,s)\cdot(\alpha + m_x(t))\cdot(\beta + m_y(t)) + \tilde{Q}_{02}(t,s)\cdot(\beta + m_y(t))^2\right], \tag{20}$$

that is:

$$\frac{\partial f_{x(t)y(s)}(\alpha,\beta)}{\partial t}+A\cdot f_{x(t)y(s)}(\alpha,\beta)+[A\cdot\alpha\cdot+B\cdot\beta]\frac{\partial f_{x(t)y(s)}(\alpha,\beta)}{\partial\alpha}=$$
$$=\frac{\exp\left\{\tilde{E}\left(t,t;\alpha+m_x(t),\beta+m_y(t)\right)\right\}}{2\cdot\pi\cdot\tilde{W}(t,s)}\times\Bigg[-\tilde{Q}_{00}(t,s)+\left(\frac{R_{\tilde{y}\tilde{y}}(s,s)}{\left(\tilde{W}(t,s)\right)^2}\tilde{Q}_{00}(t,s)\right)\cdot(\tilde{\alpha}+m_x(t))^2-$$
$$-\tilde{Q}_{11}(t,s)\cdot(\alpha+m_x(t))\cdot(\beta+m_y(t))+\tilde{Q}_{02}(t,s)\cdot(\beta+m_y(t))^2\Bigg],\quad (21)$$

under the constrains given by Equs.(19a-19i).

The differential expressions for the moments $R_{\tilde{x}\tilde{x}}(t,t)$, $R_{\tilde{y}\tilde{y}}(s,s)$, $R_{\tilde{x}\tilde{y}}(t,s)$ that appear in Equs. (19g-19i) can be written in terms of differential expressions for the moments $R_{xx}(t,t)$, $R_{xy}(t,s)$, $R_{yy}(s,s)$. In fact from Equs.(19a-c) we have:

$$\frac{\partial}{\partial t}R_{\tilde{x}\tilde{x}}(t,t)-2\cdot A\cdot R_{\tilde{x}\tilde{x}}(t,t)-2\cdot B\cdot R_{\tilde{x}\tilde{y}}(t,s)=\frac{\partial}{\partial t}C_{xx}(t,t)-2\cdot A\cdot C_{xx}(t,t)-2\cdot B\cdot C_{xy}(t,s)=$$
$$=\frac{\partial}{\partial t}\left(R_{xx}(t,t)-\left(m_x(t)\right)^2\right)-2\cdot A\cdot\left(R_{xx}(t,t)-\left(m_x(t)\right)^2\right)-2\cdot B\cdot\left(R_{xy}(t,s)-m_x(t)\cdot m_y(s)\right).$$

That is:

$$\frac{\partial}{\partial t}R_{\tilde{x}\tilde{x}}(t,t)-2\cdot A\cdot R_{\tilde{x}\tilde{x}}(t,t)-2\cdot B\cdot R_{\tilde{x}\tilde{y}}(t,s)=$$
$$=\frac{\partial}{\partial t}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,s)-2\cdot m_x(t)\cdot\left(\frac{\partial}{\partial t}m_x(t)-A\cdot m_x(t)-B\cdot m_y(s)\right),\quad (22)$$

and

$$\frac{\partial}{\partial t}R_{\tilde{x}\tilde{y}}(t,s)-A\cdot R_{\tilde{x}\tilde{y}}(t,s)-B\cdot R_{\tilde{y}\tilde{y}}(s,s)=\frac{\partial}{\partial t}C_{xy}(t,s)-A\cdot C_{xy}(t,t)-B\cdot C_{yy}(s,s)=$$
$$=\frac{\partial}{\partial t}\left(R_{xy}(t,s)-m_x(t)\cdot m_y(s)\right)-A\cdot\left(R_{xy}(t,s)-m_x(t)\cdot m_y(s)\right)-B\cdot\left(R_{yy}(s,s)-\left(m_y(s)\right)^2\right).$$

That is:

$$\frac{\partial}{\partial t}R_{\tilde{x}\tilde{y}}(t,s)-A\cdot R_{\tilde{x}\tilde{y}}(t,s)-B\cdot R_{\tilde{y}\tilde{y}}(s,s)=$$
$$=\frac{\partial}{\partial t}R_{xy}(t,s)-A\cdot R_{xy}(t,s)-B\cdot R_{yy}(s,s)-m_y(s)\cdot\left(\frac{\partial}{\partial t}m_x(t)-A\cdot m_x(t)-B\cdot m_y(s)\right).\quad (23)$$

Replacing the differential expressions for the moments $R_{\tilde{x}\tilde{x}}(t,t), R_{\tilde{y}\tilde{y}}(s,s), R_{\tilde{x}\tilde{y}}(t,s)$ that appear in Equs.(19g-19i), from Equs.(22,23) we obtain:

$$\tilde{Q}_{00}(t,s) = \frac{1}{\left(\tilde{W}(t,s)\right)^2}\cdot\Bigg[\frac{1}{2}\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot\left(\frac{\partial}{\partial t}R_{xx}(t,t) - 2\cdot A\cdot R_{xx}(t,t) - 2\cdot B\cdot R_{xy}(t,s)\right) -$$
$$-\left(m_x(t)\cdot R_{\tilde{y}\tilde{y}}(s,s) - m_y(s)\cdot R_{\tilde{x}\tilde{y}}(t,s)\right)\cdot\left(\frac{\partial}{\partial t}m_x(t) - A\cdot m_x(t) - B\cdot m_y(s)\right) +$$
$$-R_{\tilde{x}\tilde{y}}(t,s)\cdot\left(\frac{\partial}{\partial t}R_{xy}(t,s) - A\cdot R_{xy}(t,s) - B\cdot R_{yy}(s,s)\right)\Bigg], \qquad (24)$$

$$\tilde{Q}_{11}(t,s) = \frac{1}{\left(\tilde{W}(t,s)\right)^4}\cdot\Bigg[R_{\tilde{x}\tilde{y}}(t,s)R_{\tilde{y}\tilde{y}}(s,s)\cdot\left(\frac{\partial}{\partial t}R_{xx}(t,t) - 2\cdot A\cdot R_{xx}(t,t) - 2\cdot B\cdot R_{xy}(t,s)\right) +$$
$$+\Big(m_y(s)\cdot\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2 + m_y(s)\cdot R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s) -$$
$$-2\cdot m_x(t)\cdot R_{\tilde{x}\tilde{y}}(t,s)R_{\tilde{y}\tilde{y}}(s,s)\Big)\cdot\left(\frac{\partial}{\partial t}m_x(t) - A\cdot m_x(t) - B\cdot m_y(s)\right) -$$
$$-\left(\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2 + R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\right)\cdot\left(\frac{\partial}{\partial t}R_{xy}(t,s) - A\cdot R_{xy}(t,s) - B\cdot R_{yy}(s,s)\right)\Bigg], \qquad (25)$$

$$\tilde{Q}_{02}(t,s) = \frac{1}{\left(\tilde{W}(t,s)\right)^4}\cdot\Bigg[\frac{1}{2}\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2\left(\frac{\partial}{\partial t}R_{\tilde{x}\tilde{x}}(t,t) - 2\cdot A\cdot R_{\tilde{x}\tilde{x}}(t,t) - 2\cdot B\cdot R_{\tilde{x}\tilde{y}}(t,s)\right) -$$
$$-\left(m_x(t)\cdot\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2 - R_{\tilde{x}\tilde{y}}(t,s)\cdot R_{\tilde{x}\tilde{x}}(t,t)m_y(s)\right)\left(\frac{\partial}{\partial t}m_x(t) - A\cdot m_x(t) - B\cdot m_y(s)\right) -$$
$$-R_{\tilde{x}\tilde{y}}(t,s)\cdot R_{\tilde{x}\tilde{x}}(t,t)\cdot\left(\frac{\partial}{\partial t}R_{\tilde{x}\tilde{y}}(t,s) - A\cdot R_{\tilde{x}\tilde{y}}(t,s) - B\cdot R_{\tilde{y}\tilde{y}}(s,s)\right)\Bigg] \;. \qquad (26)$$

Moreover, from Equs.(24,26) we obtain:

$$-\frac{R_{\tilde{x}\tilde{x}}(t,t)}{\left(\tilde{W}(t,s)\right)^2}\cdot\tilde{Q}_{00}(t,s) + \tilde{Q}_{02}(t,s) =$$
$$\frac{1}{\left(\tilde{W}(t,s)\right)^4}\cdot\Bigg[-\frac{1}{2}\cdot\left(R_{\tilde{x}\tilde{x}}(t,t)R_{\tilde{y}\tilde{y}}(s,s) - \left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2\right)\cdot\left(\frac{\partial}{\partial t}R_{xx}(t,t) - 2\cdot A\cdot R_{xx}(t,t) - 2\cdot B\cdot R_{xy}(t,s)\right) -$$
$$+\left[R_{\tilde{x}\tilde{x}}(t,t)\left(m_x(t)\cdot R_{\tilde{y}\tilde{y}}(s,s) - m_y(s)\cdot R_{\tilde{x}\tilde{y}}(t,s)\right) - \left(m_x(t)\cdot\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2 - R_{\tilde{x}\tilde{y}}(t,s)\cdot R_{\tilde{x}\tilde{x}}(t,t)m_y(s)\right)\right]$$
$$\times\left(\frac{\partial}{\partial t}m_x(t) - A\cdot m_x(t) - B\cdot m_y(s)\right)\Bigg],$$

that is:

$$\tilde{Q}_{02}(t,s)-\frac{R_{\tilde{x}\tilde{x}}(t,t)}{\left(\tilde{W}(t,s)\right)^2}\cdot\tilde{Q}_{00}(t,s)=$$
$$=\frac{1}{\left(\tilde{W}(t,s)\right)^4}\cdot\left[-\frac{1}{2}\cdot\left(\tilde{W}(t,s)\right)^2\cdot\left(\frac{\partial}{\partial t}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,s)\right)+\right.$$
$$\left.+\left(m_x(t)\cdot\left(\tilde{W}(t,s)\right)^2\times\left(\frac{\partial}{\partial t}m_x(t)-A\cdot m_x(t)-B\cdot m_y(s)\right)\right)\right]. \tag{27}$$

Let us now consider the system of Equations (27), (25), (24):

$$\tilde{Q}_{02}(t,s)-\frac{R_{\tilde{x}\tilde{x}}(t,t)}{\left(\tilde{W}(t,s)\right)^2}\cdot\tilde{Q}_{00}(t,s)=0\,,\qquad \tilde{Q}_{11}(t,s)=0\,,\qquad \tilde{Q}_{00}(t,s)=0\,, \tag{28a-c}$$

in terms of the variables:

$$x_1=\frac{\partial}{\partial t}R_{xx}(t,t)-2\cdot A\cdot R_{xx}(t,t)-2\cdot B\cdot R_{xy}(t,s)\,, \tag{29a}$$

$$x_2=\frac{\partial}{\partial t}m_x(t)-A\cdot m_x(t)-B\cdot m_y(s)\,, \tag{29b}$$

$$x_3=\frac{\partial}{\partial t}R_{xy}(t,s)-A\cdot R_{xy}(t,s)-B\cdot R_{yy}(s,s)\,. \tag{29c}$$

Combining Equs.(27, 25, 24,29), the linear system (28a-c) is written as:

$$-\frac{1}{2}x_1+m_x(t)\cdot x_2=0\,, \tag{30a}$$

$$R_{\tilde{x}\tilde{y}}(t,s)R_{\tilde{y}\tilde{y}}(s,s)\cdot x_1+$$
$$+\left(m_y(s)\cdot\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2+m_y(s)\cdot R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)-2\cdot m_x(t)\cdot R_{\tilde{x}\tilde{y}}(t,s)R_{\tilde{y}\tilde{y}}(s,s)\right)\cdot x_2- \tag{30b}$$
$$-\left(\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2+R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\right)\cdot x_3=0,$$

$$\frac{1}{2}\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot x_1-\left(m_x(t)\cdot R_{\tilde{y}\tilde{y}}(s,s)-m_y(s)\cdot R_{\tilde{x}\tilde{y}}(t,s)\right)\cdot x_2-R_{\tilde{x}\tilde{y}}(t,s)\cdot x_3=0\,. \tag{30c}$$

The determinant of the linear system of Equs.(30) is:

$$D=\begin{vmatrix} -\frac{1}{2} & m_x(t) & 0 \\ R_{\tilde{x}\tilde{y}}(t,s)\,R_{\tilde{y}\tilde{y}}(s,s) & \begin{matrix} m_y(s)\cdot\left(\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2+R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\right)- \\ -2\cdot m_x(t)\cdot R_{\tilde{x}\tilde{y}}(t,s)\,R_{\tilde{y}\tilde{y}}(s,s)\end{matrix} & -\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2-R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s) \\ \frac{1}{2}\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot & -\left(m_x(t)\cdot R_{\tilde{y}\tilde{y}}(s,s)+m_y(s)\cdot R_{\tilde{x}\tilde{y}}(t,s)\right) & -R_{\tilde{x}\tilde{y}}(t,s) \end{vmatrix}=$$

$$=-\frac{1}{2}D_1-m_x(t)D_2 \quad , \tag{31}$$

where:

$$D_1=\begin{vmatrix} \begin{matrix} m_y(s)\cdot\left(\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2+R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\right)- \\ -2\cdot m_x(t)\cdot R_{\tilde{x}\tilde{y}}(t,s)\,R_{\tilde{y}\tilde{y}}(s,s)\end{matrix} & -\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2-R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s) \\ -m_x(t)\cdot R_{\tilde{y}\tilde{y}}(s,s)-m_y(s)\cdot R_{\tilde{x}\tilde{y}}(t,s) & -R_{\tilde{x}\tilde{y}}(t,s) \end{vmatrix}=$$

$$\begin{aligned}
&=-m_y(s)\cdot\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^3-m_y(s)R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot R_{\tilde{x}\tilde{y}}(t,s)+2\cdot m_x(t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2- \\
&\quad -m_x(t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2-m_y(s)\cdot\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^3-m_x(t)\cdot R_{\tilde{x}\tilde{x}}(t,t)\cdot\left(R_{\tilde{y}\tilde{y}}(s,s)\right)^2- \\
&\quad -m_y(s)\cdot R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot R_{\tilde{x}\tilde{y}}(t,s)= \\
&=-m_y(s)\left(\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^3+R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot R_{\tilde{x}\tilde{y}}(t,s)+\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^3+R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot R_{\tilde{x}\tilde{y}}(t,s)\right) \\
&\quad +m_x(s)\cdot\left(R_{\tilde{y}\tilde{y}}(s,s)\cdot\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2-R_{\tilde{x}\tilde{x}}(t,t)\cdot\left(R_{\tilde{y}\tilde{y}}(s,s)\right)^2\right),
\end{aligned}$$

that is:

$$\begin{aligned}
D_1 &=-2\cdot m_y(s)\left(\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^3+R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot R_{\tilde{x}\tilde{y}}(t,s)\right)+ \\
&\quad +m_x(s)\cdot\left(R_{\tilde{y}\tilde{y}}(s,s)\cdot\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2-R_{\tilde{x}\tilde{x}}(t,t)\cdot\left(R_{\tilde{y}\tilde{y}}(s,s)\right)^2\right),
\end{aligned} \tag{32}$$

$$D_2=\begin{vmatrix} R_{\tilde{x}\tilde{y}}(t,s)\,R_{\tilde{y}\tilde{y}}(s,s) & -\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2-R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s) \\ \frac{1}{2}\cdot R_{\tilde{y}\tilde{y}}(s,s) & -R_{\tilde{x}\tilde{y}}(t,s) \end{vmatrix}=$$

$$=-R_{\tilde{y}\tilde{y}}(s,s)\cdot\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2+\frac{1}{2}\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2+\frac{1}{2}\cdot R_{\tilde{x}\tilde{x}}(t,t)\cdot\left(R_{\tilde{y}\tilde{y}}(s,s)\right)^2=$$

$$=-\frac{1}{2}\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot\left(R_{\tilde{x}\tilde{y}}(t,s)\right)^2+\frac{1}{2}\cdot R_{\tilde{x}\tilde{x}}(t,t)\cdot\left(R_{\tilde{y}\tilde{y}}(s,s)\right)^2. \tag{33}$$

From Equs.(31-33) we obtain:

$$D = -\frac{1}{2}\cdot D_1 - m_x(t)\cdot D_2 = m_y(s)\cdot\Big(\big(R_{\tilde{x}\tilde{y}}(t,s)\big)^3 + R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot R_{\tilde{x}\tilde{y}}(t,s)\Big) -$$
$$-\frac{1}{2}\cdot m_x(t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\cdot\big(R_{\tilde{x}\tilde{y}}(t,s)\big)^2 + \frac{1}{2}\cdot R_{\tilde{x}\tilde{x}}(t,t)\cdot\big(R_{\tilde{y}\tilde{y}}(s,s)\big)^2 +$$
$$+\frac{1}{2}\cdot m_x(t)\cdot\big(R_{\tilde{x}\tilde{y}}(t,s)\big)^2 R_{\tilde{y}\tilde{y}}(s,s) - m_x(t)\cdot\frac{1}{2}\cdot\big(R_{\tilde{y}\tilde{y}}(s,s)\big)^2 R_{\tilde{x}\tilde{x}}(t,t) =$$
$$= m_y(s)\cdot R_{\tilde{x}\tilde{y}}(t,s)\Big(\big(R_{\tilde{x}\tilde{y}}(t,s)\big)^2 + R_{\tilde{x}\tilde{x}}(t,t)\cdot R_{\tilde{y}\tilde{y}}(s,s)\Big) \neq 0\ . \tag{34}$$

A.6. Some auxiliary formulae concerning lag-time 2D Gaussian Kernels

Let us assume lag time dependent Gaussian Kernels :

$$K\big(\alpha,\beta;\alpha_i,\beta_j,\bar{\bar{\Sigma}}_{\alpha i,\beta j}(t-s)\big) = \frac{1}{2\cdot\pi\cdot W_{ij}}\cdot\exp\left[-\frac{E_{ij}(\alpha,\beta;\,t-s)}{2\cdot\big(W_{ij}\big)^2}\right], \tag{1}$$

where:

$$W_{ij}(t-s) = \sqrt{C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j} - \big(C_{\alpha_i\beta_j}(t-s)\big)^2}\ ,$$
$$E_{ij}(\alpha,\beta;\,t-s) = C_{\beta_j\beta_j}\cdot\big(\alpha-\alpha_i\big)^2 - 2\cdot C_{\alpha_i\beta_j}(t-s)\cdot\big(\alpha-\alpha_i\big)\cdot\big(\beta-\beta_j\big) + C_{\alpha_i\alpha_i}\cdot\big(\beta-\beta_j\big)^2 .$$

Applying Equs.(8)_App(4) to the Gaussian Kernels given Equ.(1) we get:

$$\frac{\partial}{\partial t}K\big(\alpha,\beta;\alpha_i,\beta_j,\bar{\bar{\Sigma}}_{\alpha i,\beta j}(t-s)\big) =$$
$$= \frac{1}{2\cdot\pi}\cdot\left[-\frac{\frac{\partial}{\partial t}W_{ij}(t-s)}{\big(W_{ij}(t-s)\big)^2} + \frac{\frac{\partial}{\partial t}\big[W_{ij}(t-s)\big]\cdot E_{ij}(t-s)}{\big(W_{ij}(t-s)\big)^4} - \frac{\frac{\partial}{\partial t}E_{ij}(\alpha,\beta;\,t-s)}{2\cdot\big(W_{ij}(t-s)\big)^3}\right]\times$$
$$\times\exp\left[-\frac{E_{ij}(\alpha,\beta;\,t-s)}{2\cdot\big(W_{ij}(t-s)\big)^2}\right], \qquad \forall i\in\mathbb{N},\ \forall j\in\mathbb{N}. \tag{2}$$

where:

$$\frac{\partial}{\partial t}W_{ij}(t-s) = \frac{\partial}{\partial t}\left(\sqrt{C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j} - \big(C_{\alpha_i\beta_j}(t-s)\big)^2}\right) = -\frac{C_{\alpha_i\beta_j}(t-s)\cdot\frac{\partial}{\partial t}C_{\alpha_i\beta_j}(t-s)}{W_{ij}(t-s)}\ , \tag{3}$$

and

$$\frac{\partial}{\partial t}E_{ij}(\alpha,\beta;t-s)=$$
$$=\frac{\partial}{\partial t}\left(C_{\beta_j\beta_j}\cdot(\alpha-\alpha_i)^2\cdot\alpha^2-2\cdot C_{\alpha_i\beta_j}(t-s)\cdot(\alpha-\alpha_i)\cdot(\beta-\beta_j)+C_{\alpha_i\alpha_i}\cdot(\beta-\beta_j)^2\right)= \tag{4}$$
$$=-2\cdot\frac{\partial C_{\alpha_i\beta_j}(t-s)}{\partial t}\cdot(\alpha-\alpha_i)\cdot(\beta-\beta_j).$$

That is:

$$\frac{\partial}{\partial t}K\left(\alpha,\beta;\alpha_i,\beta_j,\bar{\Sigma}_{\alpha i,\beta j}(t-s)\right)=$$
$$=\frac{1}{2\cdot\pi}\cdot\frac{1}{\left(W_{ij}(t-s)\right)^3}\left[C_{\alpha_i\beta_j}(t-s)\cdot\frac{\partial}{\partial t}C_{\alpha_i\beta_j}(t-s)-\frac{C_{\alpha_i\beta_j}(t-s)\cdot\frac{\partial}{\partial t}C_{\alpha_i\beta_j}(t-s)\cdot E_{ij}(\alpha,\beta;t-s)}{\left(W_{ij}(t-s)\right)^2}+\right.$$
$$\left.+(\alpha-\alpha_i)\cdot(\beta-\beta_j)\cdot\frac{\partial}{\partial t}C_{\alpha_i\beta_j}(t-s)\right]\cdot\exp\left[-\frac{E_{ij}(\alpha,\beta;t-s)}{2\cdot\left(W_{ij}(t-s)\right)^2}\right]$$
$$=\frac{1}{2\cdot\pi}\cdot\frac{C_{\alpha_i\beta_j}(t-s)\cdot\frac{\partial}{\partial t}C_{\alpha_i\beta_j}(t-s)}{\left(W_{ij}(t-s)\right)^3\left(C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}-\left(C_{\alpha_i\beta_j}(t-s)\right)^2\right)^{3/2}}\times$$
$$\times\left[1+\frac{-C_{\beta_j\beta_j}\cdot(\alpha-\alpha_i)^2+2\cdot C_{\alpha_i\beta_j}(t-s)\cdot(\alpha-\alpha_i)\cdot(\beta-\beta_j)-C_{\alpha_i\alpha_i}\cdot(\beta-\beta_j)^2}{C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}-\left(C_{\alpha_i\beta_j}(t-s)\right)^2}+\frac{(\alpha-\alpha_i)\cdot(\beta-\beta_j)}{C_{\alpha_i\beta_j}(t-s)}\right]\times$$
$$\times\exp\left[-\frac{C_{\beta_j\beta_j}\cdot(\alpha-\alpha_i)^2-2\cdot C_{\alpha_i\beta_j}(t-s)\cdot(\alpha-\alpha_i)\cdot(\beta-\beta_j)+C_{\alpha_i\alpha_i}\cdot(\beta-\beta_j)^2}{2\cdot\left(C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}-\left(C_{\alpha_i\beta_j}(t-s)\right)^2\right)}\right], \tag{5}$$

or:

$$\frac{\partial}{\partial t}K\left(\alpha,\beta;\alpha_i,\beta_j,\bar{\Sigma}_{\alpha i,\beta j}(t-s)\right)=\frac{1}{2\cdot\pi}\cdot\frac{C_{\alpha_i\beta_j}(t-s)\cdot\frac{\partial}{\partial t}C_{\alpha_i\beta_j}(t-s)}{\left(C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}-\left(C_{\alpha_i\beta_j}(t-s)\right)^2\right)^{3/2}}\times \tag{6}$$
$$\times\left[1-\frac{C_{\beta_j\beta_j}\cdot(\alpha-\alpha_i)^2+C_{\alpha_i\alpha_i}\cdot(\beta-\beta_j)^2}{C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}-\left(C_{\alpha_i\beta_j}(t-s)\right)^2}+\frac{\left(\left(C_{\alpha_i\beta_j}(t-s)\right)^2+C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}\right)\cdot(\alpha-\alpha_i)\cdot(\beta-\beta_j)}{\left(C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}-\left(C_{\alpha_i\beta_j}(t-s)\right)^2\right)\cdot C_{\alpha_i\beta_j}(t-s)}\right]\times$$
$$\times\exp\left[-\frac{C_{\beta_j\beta_j}\cdot(\alpha-\alpha_i)^2-2\cdot C_{\alpha_i\beta_j}(t-s)\cdot(\alpha-\alpha_i)\cdot(\beta-\beta_j)+C_{\alpha_i\alpha_i}\cdot(\beta-\beta_j)^2}{2\cdot\left(C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}-\left(C_{\alpha_i\beta_j}(t-s)\right)^2\right)}\right].$$

A.7. Computation of Galerkin coefficients

For Gaussian representation and the Galerkin Kernels, substituting Equ.(7)_Sec(4.5.2) and Equ.(2)_Sec(4.5.4) in Equ.(4b)_Sec(4.5.3), we obtain that the first Galerkin coefficient, $G^{(1)}_{ij,\kappa\lambda}$ will be given by the formula:

$$G^{(1)}_{ij,\kappa\lambda} = \iint_{\mathbb{R}^2} \frac{\partial}{\partial t} \mathrm{K}\left(\alpha,\beta;\alpha_i,\beta_j,\bar{\Sigma}_{\alpha_i,\beta_j}(t-s)\right)\Big|_{s\to t} \cdot \Lambda_{\kappa,\lambda}(\alpha,\beta)d\alpha d\beta =$$

$$= \frac{1}{4\cdot\pi^2}\cdot\frac{C_{\alpha_i\beta_j}\cdot\partial C_{\alpha_i\beta_j}}{(W_{ij})^5\cdot W_{\kappa\lambda}}\times$$

$$\times\iint_{\mathbb{R}^2}\left[-C_{\beta_j\beta_j}\cdot(\alpha-\alpha_i)^2+\left(\frac{C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}}{C_{\alpha_i\beta_j}}+C_{\alpha_i\beta_j}\right)\cdot(\alpha-\alpha_i)\cdot(\beta-\beta_j)-C_{\alpha_i\alpha_i}\cdot(\beta-\beta_j)^2+(W_{ij})^2\right]\times$$

$$\times\exp\left[-\frac{C_{\beta_j\beta_j}\cdot(\alpha-\alpha_i)^2-2\cdot C_{\alpha_i\beta_j}\cdot(\alpha-\alpha_i)\cdot(\beta-\beta_j)+C_{\alpha_i\alpha_i}\cdot(\beta-\beta_j)^2}{2\cdot(W_{ij})^2}-\right.$$

$$\left.-\frac{C_{\beta_\lambda\beta_\lambda}\cdot(\alpha-\alpha_\kappa)^2-2\cdot C_{\alpha_\kappa\beta_\lambda}\cdot(\alpha-\alpha_\kappa)\cdot(\beta-\beta_\lambda)+C_{\alpha_\kappa\alpha_\kappa}\cdot(\beta-\beta_\lambda)^2}{2\cdot(W_{ij})^2}\right]d\alpha d\beta\equiv$$

$$\equiv\frac{1}{4\cdot\pi^2}\cdot\frac{C_{\alpha_i\beta_j}\cdot\partial C_{\alpha_i\beta_j}}{(W_{ij})^5\cdot W_{\kappa\lambda}}\cdot\iint_{\mathbb{R}^2}\Pi_1(\alpha,\beta)\cdot\exp\left(Q_1(\alpha,\beta)\right)d\alpha d\beta \ . \tag{1}$$

We shall write Equ.(1) in an equivalent form that allows the application of the integration formula that is given by Equ(1)_App(8). More precisely, after some algebraic manipulation we obtain that $\Pi_1(\alpha,\beta)$ in Equ.(1) can be equivalently written as:

$$\Pi_1(\bar{\alpha},\bar{\beta}) = \Pi_{1,20}\cdot\bar{\alpha}^2+\Pi_{1,11}\cdot\bar{\alpha}\cdot\bar{\beta}+\Pi_{1,02}\cdot\bar{\beta}^2+\Pi_{1,00}\,, \tag{2a}$$

where

$\bar{\alpha}=\alpha-\alpha_i\,,\ \bar{\beta}=\beta-\beta_j$

$$\Pi_{1,20}=-C_{\beta_j\beta_j}\,,\ \ \Pi_{1,11}=\frac{C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}}{C_{\alpha_i\beta_j}}+C_{\alpha_i\beta_j}\,,\ \ \Pi_{1,02}=C_{\alpha_i\alpha_i}\,,\ \ \Pi_{1,00}=(W_{ij})^2\ , \tag{2b-e}$$

whereas for the exponent $Q_1(\alpha,\beta)$ appearing in right hand side of Equ.(1), similarly, we get:

$$Q_1(\bar{\alpha},\bar{\beta}) = Q_{1,20}\cdot\bar{\alpha}^2+Q_{1,11}\cdot\bar{\alpha}\cdot\bar{\beta}+Q_{1,02}\cdot\bar{\beta}^2+Q_{1,01}\cdot\bar{\beta}+Q_{2,00}\,, \tag{3a}$$

where:

$\bar{\alpha}=\alpha-\alpha_i\,,\ \bar{\beta}=\beta-\beta_j$

$$Q_{1,20}=\frac{1}{2}\cdot\left(\frac{C_{\beta_j\beta_j}}{(W_{ij})^2}+\frac{C_{\beta_\lambda\beta_\lambda}}{(W_{\kappa\lambda})^2}\right),\quad Q_{1,11}=-\left(\frac{C_{\alpha_i\beta_j}}{(W_{ij})^2}+\frac{C_{\alpha_\kappa\beta_\lambda}}{(W_{\kappa\lambda})^2}\right), \tag{3b,c}$$

$$Q_{1,02}=\frac{1}{2}\cdot\left(\frac{C_{\alpha_i\alpha_i}}{(W_{ij})^2}+\frac{C_{\alpha_\kappa\alpha_\kappa}}{(W_{\kappa\lambda})^2}\right), \tag{3d}$$

$$Q_{1,10}=\frac{C_{\beta_\lambda\beta_\lambda}\cdot(\alpha_i-\alpha_\kappa)-C_{\alpha_\kappa\beta_\lambda}\cdot(\beta_j-\beta_\lambda)}{(W_{\kappa\lambda})^2},Q_{1,01}=\frac{C_{\alpha_\kappa\alpha_\kappa}\cdot(\beta_j-\beta_\lambda)-C_{\alpha_\kappa\beta_\lambda}\cdot(\alpha_i-\alpha_\kappa)}{(W_{\kappa\lambda})^2}, \tag{3e,f}$$

$$Q_{1,00}=\frac{1}{2\cdot(W_{\kappa\lambda})^2}\cdot\left(C_{\beta_\lambda\beta_\lambda}\cdot(\alpha_i-\alpha_\kappa)^2-2\cdot C_{\alpha_\kappa\beta_\lambda}\cdot(\alpha_i-\alpha_\kappa)\cdot(\beta_j-\beta_\lambda)\ +C_{\alpha_\kappa\alpha_\kappa}\cdot(\beta_j-\beta_\lambda)\right). \tag{3g}$$

Applying Equs.(2-3) in Equ.(1) we get that $G^{(1)}_{ij,\kappa\lambda}$ can be equivalently written as:

$$\begin{aligned}G^{(1)}_{ij,\kappa\lambda}=\mathrm{M}_1\cdot\iint_{\mathbb{R}^2}&\Pi_{1,20}\cdot\alpha^2+\Pi_{1,02}\cdot\beta^2+\Pi_{1,11}\cdot\alpha\cdot\beta+\Pi_{1,00}\times\\&\times\exp[-\left(Q_{1,20}\cdot\alpha^2+Q_{1,11}\cdot\alpha\cdot\beta+Q_{1,02}\cdot\beta^2+Q_{1,10}\cdot\alpha+Q_{1,01}\cdot\beta+Q_{1,00}\right)d\alpha d\beta\ ,\end{aligned} \tag{4a}$$

where:

$$\mathrm{M}_1=\frac{1}{4\cdot\pi^2}\cdot\frac{C_{\alpha_i\beta_j}\cdot\partial C_{\alpha_i\beta_j}}{(W_{ij})^5\cdot W_{\kappa\lambda}}, \tag{4b}$$

$$W_{ij}=\sqrt{C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}-(C_{\alpha_i\beta_j})^2},\qquad W_{\kappa\lambda}=\sqrt{C_{\alpha_\kappa\alpha_\kappa}\cdot C_{\beta_\lambda\beta_\lambda}-(C_{\alpha_\kappa\beta_\lambda})^2}\ , \tag{4c,4d}$$

$$\Pi_{1,20}=-C_{\beta_j\beta_j},\quad \Pi_{1,11}=\frac{C_{\alpha_i\alpha_i}\cdot C_{\beta_j\beta_j}}{C_{\alpha_i\beta_j}}+C_{\alpha_i\beta_j},\quad \Pi_{1,02}=C_{\alpha_i\alpha_i},\quad \Pi_{1,00}=(W_{ij})^2, \tag{4e-4f}$$

$$Q_{1,20}=\frac{1}{2}\cdot\left(\frac{C_{\beta_j\beta_j}}{(W_{ij})^2}+\frac{C_{\beta_\lambda\beta_\lambda}}{(W_{\kappa\lambda})^2}\right),\quad Q_{1,11}=-\left(\frac{C_{\alpha_i\beta_j}}{(W_{ij})^2}+\frac{C_{\alpha_\kappa\beta_\lambda}}{(W_{\kappa\lambda})^2}\right), \tag{4g,4h}$$

$$Q_{1,02}=\frac{1}{2}\cdot\left(\frac{C_{\alpha_i\alpha_i}}{(W_{ij})^2}+\frac{C_{\alpha_\kappa\alpha_\kappa}}{(W_{\kappa\lambda})^2}\right), \tag{4i}$$

$$Q_{1,10}=\frac{C_{\beta_\lambda\beta_\lambda}\cdot(\alpha_i-\alpha_\kappa)-C_{\alpha_\kappa\beta_\lambda}\cdot(\beta_j-\beta_\lambda)}{(W_{\kappa\lambda})^2},\ Q_{1,01}=\frac{C_{\alpha_\kappa\alpha_\kappa}\cdot(\beta_j-\beta_\lambda)-C_{\alpha_\kappa\beta_\lambda}\cdot(\alpha_i-\alpha_\kappa)}{(W_{\kappa\lambda})^2}, \tag{4k,4l}$$

$$Q_{1,00}=\frac{1}{2\cdot(W_{\kappa\lambda})^2}\cdot\left(C_{\beta_\lambda\beta_\lambda}\cdot(\alpha_i-\alpha_\kappa)^2-2\cdot C_{\alpha_\kappa\beta_\lambda}\cdot(\alpha_i-\alpha_\kappa)\cdot(\beta_j-\beta_\lambda)\ +C_{\alpha_\kappa\alpha_\kappa}\cdot(\beta_j-\beta_\lambda)\right). \tag{4m}$$

We shall now consider the second Galerkin coefficient $G^{(2)}_{ij,\kappa\lambda}$ for the case study that the RDE is a cubic half-oscillator described by Equ.(3a)_Sec(4.2). Substituting Equ.(5)_Sec(4.5.2) and Equ.(4)_Sec(4.5.4) in Equ.(5)_Sec(4.5.3) we obtain:

$$G^{(2)}_{ij,\kappa\lambda} = -\frac{1}{4\cdot\pi^2\cdot W_{ij}\cdot(W_{\kappa\lambda})^3}\times$$

$$\times\iint_{\mathbb{R}^2}\left(\underbrace{\left(\mu_1\cdot\alpha+\kappa_1\cdot\beta\right)\cdot\left(-C_{\beta_\lambda\beta_\lambda}\cdot(\alpha-\alpha_\kappa)+C_{\alpha_\kappa\beta_\lambda}\cdot(\beta-\beta_\lambda)\right)}_{\Pi^1_2(\alpha,\beta)}+\right.$$

$$\underbrace{+\mu_3\cdot\alpha^3\cdot\left(-C_{\beta_\lambda\beta_\lambda}\cdot(\alpha-\alpha_\kappa)+C_{\alpha_\kappa\beta_\lambda}\cdot(\beta-\beta_\lambda)\right)}_{\Pi^2_2(\alpha,\beta)}+$$

$$\left.\underbrace{+\kappa_3\cdot\beta^3\cdot\left(-C_{\beta_\lambda\beta_\lambda}\cdot(\alpha-\alpha_\kappa)+C_{\alpha_\kappa\beta_\lambda}\cdot(\beta-\beta_\lambda)\right)}_{\Pi^3_2(\alpha,\beta)}\right)\times$$

$$\times\exp\left[-\frac{\left(C_{\beta_j\beta_j}\cdot(\alpha-\alpha_i)^2-2\cdot C_{\alpha_i\beta_j}\cdot(\alpha-\alpha_i)\cdot(\beta-\beta_j)+C_{\alpha_i\alpha_i}\cdot(\beta-\beta_j)^2\right)}{2\cdot(W_{ij})^2}-\right.$$

$$\left.-\frac{C_{\beta_\lambda\beta_\lambda}\cdot(\alpha-\alpha_\kappa)^2-2\cdot C_{\alpha_\kappa\beta_\lambda}\cdot(\alpha-\alpha_\kappa)\cdot(\beta-\beta_\lambda)+C_{\alpha_\kappa\alpha_\kappa}\cdot(\beta-\beta_\lambda)^2}{2\cdot(W_{\kappa\lambda})^2}\right]d\alpha\, d\beta\,. \qquad (5)$$

The terms $\Pi^1_2(\alpha,\beta)$, $\Pi^2_2(\alpha,\beta)$, $\Pi^3_2(\alpha,\beta)$ appearing in Equ.(5) can be equivalently written as:

$$\Pi^1_2(\alpha,\beta)=\left(\mu_1\cdot(\alpha-\alpha_\kappa+\alpha_\kappa)+\kappa_1\cdot(\beta-\beta_\lambda+\beta_\lambda)\right)\cdot\left(-C_{\beta_\lambda\beta_\lambda}\cdot(\alpha-\alpha_\kappa)+C_{\alpha_\kappa\beta_\lambda}\cdot(\beta-\beta_\lambda)\right), \quad (6)$$

$$\Pi^2_2(\alpha,\beta)=\mu_3\cdot(\alpha-\alpha_\kappa+\alpha_\kappa)^3\cdot\left(-C_{\beta_\lambda\beta_\lambda}\cdot(\alpha-\alpha_\kappa)+C_{\alpha_\kappa\beta_\lambda}\cdot(\beta-\beta_\lambda)\right)\,, \quad (7)$$

$$\Pi^3_2(\alpha,\beta)=\kappa_3\cdot(\beta-\beta_\lambda+\beta_\lambda)^3\cdot\left(-C_{\beta_\lambda\beta_\lambda}\cdot(\alpha-\alpha_\kappa)+C_{\alpha_\kappa\beta_\lambda}\cdot(\beta-\beta_\lambda)\right)\,. \quad (8)$$

After some algebraic manipulations from Equs.(6-8) we obtain that $\Pi^1_2(\alpha,\beta)$, $\Pi^2_2(\alpha,\beta)$, $\Pi^3_2(\alpha,\beta)$ can be equivalently given by Equs.(9-11), respectively, i.e.:

$$\Pi^1_2(\bar\alpha,\bar\beta) = \Pi^1_{2,20}\cdot\bar\alpha^2+\Pi^1_{2,11}\cdot\bar\alpha\cdot\bar\beta+\Pi^1_{2,10}\cdot\bar\alpha+\Pi^1_{2,01}\cdot\bar\beta+\Pi^1_{2,02}\cdot\bar\beta^2, \quad (9a)$$

where $\alpha-\alpha_\kappa=\bar\alpha$, $\beta-\beta_\lambda=\bar\beta$,

$$\Pi^1_{2,20}=-C_{\beta_\lambda\beta_\lambda}\cdot\mu_1,\qquad \Pi^1_{2,11}=C_{\alpha_\kappa\beta_\lambda}\cdot\mu_1-\kappa_1\cdot C_{\beta_\lambda\beta_\lambda},\qquad \Pi^1_{2,10}=\kappa_1\cdot C_{\alpha_\kappa\beta_\lambda}, \quad (9b\text{-}9e)$$

$$\Pi^1_{2,01}=-C_{\beta_\lambda\beta_\lambda}\cdot(\mu_1\cdot\alpha_\kappa+\kappa_1\cdot\beta_\lambda),\qquad \Pi^1_{2,02}=C_{\alpha_\kappa\beta_\lambda}\cdot(\mu_1\cdot\alpha_\kappa+\kappa_1\cdot\beta_\lambda), \quad (9f,9g)$$

$$\Pi^2_2(\bar\alpha,\bar\beta)=\Pi^2_{2,40}\bar\alpha^4+\Pi^2_{2,31}\bar\alpha^3\cdot\bar\beta+\Pi^2_{2,30}\bar\alpha^3+\Pi^2_{2,21}\cdot\bar\alpha^2\cdot\bar\beta+\Pi^2_{2,20}\cdot\bar\alpha^2+$$
$$+\Pi^2_{2,11}\bar\alpha\cdot\bar\beta+\Pi^2_{2,10}\bar\alpha+\Pi^2_{2,01}\bar\beta, \quad (10a)$$

where,

$$\Pi^2_{2,40}=-C_{\beta_\lambda\beta_\lambda},\ \Pi^2_{2,31}=C_{\alpha_\kappa\beta_\lambda},\ \Pi^2_{2,30}=-3\cdot C_{\beta_\lambda\beta_\lambda}\cdot\alpha_\kappa, \quad (10b\text{-}10d)$$

$$\Pi^2_{2,21}=3\cdot C_{\alpha_\kappa\beta_\lambda}\cdot\alpha_\kappa,\ \Pi^2_{2,20}=-3\cdot C_{\beta_\lambda\beta_\lambda}\cdot{\alpha_\kappa}^2,\ \Pi^2_{2,11}=3\cdot C_{\alpha_\kappa\beta_\lambda}\cdot{\alpha_\kappa}^2\,, \quad (10e\text{-}10g)$$

$$\Pi^{2}_{2,10} = -\,C_{\beta_\lambda\beta_\lambda}\cdot\alpha_\kappa^{\;3},\;\; \Pi^{2}_{2,01} = C_{\alpha_\kappa\beta_\lambda}\cdot\alpha_\kappa^{\;3}, \tag{10h, 10i}$$

and

$$\begin{aligned}\Pi^{3}_{2}(\overline{\alpha},\overline{\beta}) = {} & \Pi^{3}_{2,11}\cdot\overline{\alpha}\cdot\overline{\beta} + \Pi^{3}_{2,02}\cdot\overline{\beta}^{2} + \Pi^{3}_{2,10}\cdot\overline{\alpha} + \Pi^{3}_{2,01}\cdot\beta_\lambda^{\;3}\cdot\overline{\beta} + \\ & + \Pi^{3}_{2,03}\cdot\overline{\beta}^{3} + \Pi^{3}_{2,12}\cdot\overline{\alpha}\cdot\overline{\beta}^{2} + \Pi^{3}_{2,13}\cdot\overline{\alpha}\cdot\overline{\beta}^{3} + \overline{\beta}^{4}\;,\end{aligned} \tag{11a}$$

where

$$\Pi^{3}_{2,11} = -3\cdot\kappa_3\cdot C_{\beta_\lambda\beta_\lambda}\cdot\beta_\lambda^{\;2},\qquad \Pi^{3}_{2,02} = 3\cdot\kappa_3\cdot C_{\alpha_\kappa\beta_\lambda}\cdot\beta_\lambda^{\;2},\qquad \Pi^{3}_{2,10} = -\kappa_3\cdot C_{\beta_\lambda\beta_\lambda}\cdot\beta_\lambda^{\;3}, \tag{11b-11d}$$

$$\Pi^{3}_{2,01} = \kappa_3\cdot C_{\alpha_\kappa\beta_\lambda}\cdot\beta_\lambda^{\;3},\qquad \Pi^{3}_{2,03} = 3\cdot\kappa_3\cdot C_{\alpha_\kappa\beta_\lambda}\cdot\beta_\lambda,\qquad \Pi^{3}_{2,12} = -3\cdot\kappa_3\cdot C_{\beta_\lambda\beta_\lambda}\cdot\beta_\lambda, \tag{11e-11g}$$

$$\Pi^{3}_{2,13} = -\kappa_3\cdot C_{\beta_\lambda\beta_\lambda},\qquad \Pi^{3}_{2,04} = \kappa_3\cdot C_{\alpha_\kappa\beta_\lambda}\,. \tag{11k,11l}$$

Finally let $Q_2(\alpha,\beta)$ be the exponent in Equ.(5), i.e.

$$\begin{aligned}Q_2(\alpha,\beta) \equiv {} & -\frac{\left(C_{\beta_j\beta_j}\cdot(\alpha-\alpha_i)^2 - 2\cdot C_{\alpha_i\beta_j}\cdot(\alpha-\alpha_i)\cdot(\beta-\beta_j) + C_{\alpha_i\alpha_i}\cdot(\beta-\beta_j)^2\right)}{2\cdot(W_{ij})^2} - \\ & -\frac{C_{\beta_\lambda\beta_\lambda}\cdot(\alpha-\alpha_\kappa)^2 - 2\cdot C_{\alpha_\kappa\beta_\lambda}\cdot(\alpha-\alpha_\kappa)\cdot(\beta-\beta_\lambda) + C_{\alpha_\kappa\alpha_\kappa}\cdot(\beta-\beta_\lambda)^2}{2\cdot(W_{\kappa\lambda})^2}\,.\end{aligned} \tag{12}$$

Then, Equ.(12) can be equivalently written as:

$$\begin{aligned}Q_2(\overline{\alpha},\overline{\beta}) = {} & \frac{1}{2\cdot\left(W_{ij}\right)^2}\cdot\Big(C_{\beta_j\beta_j}\cdot\left(\overline{\alpha}+\alpha_\kappa-\alpha_i\right)^2 - 2\cdot C_{\alpha_i\beta_j}\cdot\left(\overline{\alpha}+\alpha_\kappa-\alpha_i\right)\cdot\left(\overline{\beta}+\beta_\lambda-\beta_j\right) + \\ & + C_{\alpha_i\alpha_i}\cdot\left(\overline{\beta}+\beta_\lambda-\beta_j\right)^2\Big) + \frac{1}{2\cdot\left(W_{\kappa\lambda}\right)^2}\left(\cdot C_{\beta_\lambda\beta_\lambda}\cdot\overline{\alpha}^2 - 2\cdot C_{\alpha_\kappa\beta_\lambda}\cdot\overline{\alpha}\cdot\overline{\beta} + C_{\alpha_\kappa\alpha_\kappa}\cdot\overline{\beta}^2\right),\end{aligned} \tag{13}$$

where, $\alpha-\alpha_\kappa = \overline{\alpha}$, $\beta-\beta_\lambda = \overline{\beta}$.

After some algebraic manipulations Equ.(13) yields:

$$Q_2(\overline{\alpha},\overline{\beta}) = Q_{2,20}\cdot\overline{\alpha}^2 + Q_{2,10}\cdot\overline{\alpha} + Q_{2,11}\cdot\overline{\alpha}\cdot\overline{\beta} + Q_{2,02}\cdot\overline{\beta}^2 + 2\cdot Q_{2,01}\cdot\overline{\beta} + Q_{2,00}\;\;, \tag{14a}$$

where:

$$Q_{2,20} = \frac{1}{2}\cdot\left(\frac{C_{\beta_j\beta_j}}{(W_{ij})^2} + \frac{C_{\beta_\lambda\beta_\lambda}}{(W_{\kappa\lambda})^2}\right),\qquad Q_{2,11} = -\left(\frac{C_{\alpha_i\beta_j}}{(W_{ij})^2} + \frac{C_{\alpha_\kappa\beta_\lambda}}{(W_{\kappa\lambda})^2}\right), \tag{14b, 14c}$$

$$Q_{2,02} = \frac{1}{2}\cdot\left(\frac{C_{\alpha_i\alpha_i}}{(W_{ij})^2} + \frac{C_{\alpha_\kappa\alpha_\kappa}}{(W_{\kappa\lambda})^2}\right), \tag{14d}$$

$$Q_{2,10} = \frac{1}{(W_{ij})^2}\cdot\left(C_{\beta_j\beta_j}\cdot(\alpha_\kappa-\alpha_i) - C_{\alpha_i\beta_j}\cdot(\beta_\lambda-\beta_j)\right), \tag{14e}$$

$$Q_{2,01} = \frac{1}{(W_{ij})^2} \cdot \left(C_{\alpha_i \alpha_i} \cdot (\beta_\lambda - \beta_j) - C_{\alpha_i \beta_j} \cdot (\alpha_\kappa - \alpha_i) \right), \tag{14f}$$

$$Q_{2,00} = \frac{1}{2 \cdot (W_{ij})^2} \cdot \left(C_{\beta_j \beta_j} \cdot (\alpha_\kappa - \alpha_i)^2 + C_{\alpha_i \alpha_i} \cdot (\beta_\lambda - \beta_j)^2 - 2 \cdot C_{\alpha_i \beta_j} \cdot (\alpha_\kappa - \alpha_i) \cdot (\beta_\lambda - \beta_j) \right). \tag{14g}$$

Combining Equs.(5-14), performing some algebra, we obtain that the second Galerkin coefficient $G^{(2)}_{ij,\kappa\lambda}$ is equivalently written as:

$$G^{(2)}_{ij,\kappa\lambda} = \frac{1}{4 \cdot \pi^2 \cdot W_{ij} \cdot (W_{\kappa\lambda})^3} \int\int_{\mathbb{R}^2} \Big(\Pi_{2,10} \cdot \alpha + \Pi_{2,01} \cdot \beta + \Pi_{2,20} \cdot \alpha^2 + \Pi_{2,11} \cdot \alpha \cdot \beta + \Pi_{2,02} \beta^2 +$$

$$+ \Pi_{2,30} \cdot \alpha^3 + \Pi_{2,21} \cdot \alpha^2 \cdot \beta + + \Pi_{2,21} \cdot \alpha^2 \cdot \beta + \Pi_{2,03} \beta^2 +$$

$$+ \Pi_{2,40} \cdot \alpha^4 + \Pi_{2,31} \cdot \alpha^3 \cdot \beta + + \Pi_{2,13} \cdot \alpha \cdot \beta^3 + \Pi_{2,04} \beta^4 \Big) \times$$

$$\times \exp\left\{ -\left(Q_{2,20} \cdot \alpha^2 + Q_{2,11} \cdot \alpha \cdot \beta + Q_{2,02} \cdot \beta^2 + Q_{2,10} \cdot \alpha + Q_{2,01} \cdot \beta + Q_{2,00} \right) d\alpha d\beta, \right. \tag{15a}$$

where:

$$W_{ij} = \sqrt{C_{\alpha_i \alpha_i} \cdot C_{\beta_j \beta_j} - (C_{\alpha_i \beta_j})^2}, \; W_{\kappa\lambda} = \sqrt{C_{\alpha_\kappa \alpha_\kappa} \cdot C_{\beta_\lambda \beta_\lambda} - (C_{\alpha_\kappa \beta_\lambda})^2}, \tag{15b,15c}$$

$$\Pi_{2,10} = -C_{\beta_\lambda \beta_\lambda} \cdot (\mu_3 \cdot \alpha_\kappa{}^3 + \mu_1 \cdot \alpha_\kappa + \kappa_1 \cdot \beta_\lambda + \kappa_3 \cdot \beta_\lambda{}^3), \tag{15d}$$

$$\Pi_{2,01} = C_{\alpha_\kappa \beta_\lambda} \cdot (\mu_3 \cdot \alpha_\kappa{}^3 + \mu_1 \cdot \alpha_\kappa + \kappa_1 \cdot \beta_\lambda + \kappa_3 \cdot \beta_\lambda{}^3), \tag{15e}$$

$$\Pi_{2,11} = 3 \cdot \mu_3 \cdot C_{\alpha_\kappa \beta_\lambda} \cdot \alpha_\kappa{}^2 + \mu_1 \cdot C_{\alpha_\kappa \beta_\lambda} - \kappa_1 \cdot C_{\beta_\lambda \beta_\lambda} - 3 \cdot \kappa_3 \cdot C_{\beta_\lambda \beta_\lambda} \cdot \beta_\lambda{}^2, \tag{15f}$$

$$\Pi_{2,20} = -(3 \cdot \mu_3 \cdot C_{\beta_\lambda \beta_\lambda} \cdot \alpha_\kappa{}^2 + \mu_1 \cdot C_{\beta_\lambda \beta_\lambda}), \qquad \Pi_{2,02} = (\kappa_1 + 3 \cdot \kappa_3 \cdot C_{\alpha_\kappa \beta_\lambda} \cdot \beta_\lambda{}^2) \cdot C_{\alpha_\kappa \beta_\lambda}, \tag{15g,15h}$$

$$\Pi_{2,21} = 3 \cdot \mu_3 \cdot C_{\alpha_\kappa \beta_\lambda} \cdot \alpha_\kappa, \quad \Pi_{2,12} = -3 \cdot \kappa_3 \cdot C_{\beta_\lambda \beta_\lambda} \cdot \beta_\lambda, \quad \Pi_{2,30} = -3 \cdot \mu_3 \cdot C_{\beta_\lambda \beta_\lambda} \cdot \alpha_\kappa, \tag{15i-15l}$$

$$\Pi_{2,03} = 3 \cdot \kappa_3 \cdot C_{\alpha_\kappa \beta_\lambda} \cdot \beta_\lambda, \qquad \Pi_{2,31} = \mu_3 \cdot C_{\alpha_\kappa \beta_\lambda}, \qquad \Pi_{2,13} = -\kappa_3 \cdot C_{\beta_\lambda \beta_\lambda}, \tag{15m-15o}$$

$$\Pi_{2,04} = \kappa_3 \cdot C_{\alpha_\kappa \beta_\lambda}, \qquad \Pi_{2,40} = -\mu_3 \cdot C_{\beta_\lambda \beta_\lambda}, \tag{15p,15q}$$

$$Q_{2,10} = \frac{1}{(W_{ij})^2} \cdot \left(C_{\beta_j \beta_j} \cdot (\alpha_\kappa - \alpha_i) - C_{\alpha_i \beta_j} \cdot (\beta_\lambda - \beta_j) \right), \tag{15r}$$

$$Q_{2,01} = \frac{1}{(W_{ij})^2} \cdot \left(C_{\alpha_i \alpha_i} \cdot (\beta_\lambda - \beta_j) - C_{\alpha_i \beta_j} \cdot (\alpha_\kappa - \alpha_i) \right), \tag{15s}$$

$$Q_{2,00} = \frac{1}{2 \cdot (W_{ij})^2} \cdot \left(C_{\beta_j \beta_j} \cdot (\alpha_\kappa - \alpha_i)^2 + C_{\alpha_i \alpha_i} \cdot (\beta_\lambda - \beta_j)^2 - 2 \cdot C_{\alpha_i \beta_j} \cdot (\alpha_\kappa - \alpha_i) \cdot (\beta_\lambda - \beta_j) \right). \tag{15t}$$

A8. Calculation of 2-polynomial/quadratic-exponential integrals

In this Appendix we shall prove that for $Q_{20} > 0$, $4 \cdot Q_{02} \cdot Q_{20} - (Q_{11})^2 > 0$ the following integration formula holds true:

$$I = \iint_{\mathbb{R}^2} \left(\Pi_{20}\cdot\alpha^2 + \Pi_{11}\cdot\alpha\cdot\beta + \Pi_{02}\beta^2 + \Pi_{10}\cdot\alpha + \Pi_{01}\cdot\beta + \Pi_{00}\right)\times$$

$$\times\exp\left\{-\left(Q_{20}\cdot\alpha^2 + Q_{11}\cdot\alpha\cdot\beta + Q_{02}\cdot\beta^2 + Q_{10}\cdot\alpha + Q_{01}\cdot\beta + Q_{00}\right)d\alpha d\beta =\right.$$

$$= \frac{2\cdot\pi}{\sqrt{4\cdot Q_{02}\cdot Q_{20} - \left(Q_{11}\right)^2}}\cdot\left(c_{20}\cdot\Pi_{20} + c_{11}\cdot\Pi_{11} + c_{02}\cdot\Pi_{02} + c_{10}\cdot\Pi_{10} + c_{01}\cdot\Pi_{01} + \Pi_{00}\right)\times \qquad (1)$$

$$\times\exp\left(-\frac{Q_{02}\cdot\left(Q_{10}\right)^2 - Q_{11}\cdot Q_{10}\cdot Q_{01} + \left(Q_{11}\right)^2\cdot Q_{00} + Q_{20}\cdot\left(Q_{01}\right)^2 - 4\cdot Q_{20}\cdot Q_{02}\cdot Q_{00}}{-4\cdot Q_{20}\cdot Q_{02} + \left(Q_{11}\right)^2}\right),$$

where

$$c_{10} = \frac{-2\cdot Q_{02}\cdot Q_{10} + Q_{11}\cdot Q_{01}}{4\cdot Q_{20}\cdot Q_{02} - Q_{11}^2}\ , \qquad (2a)$$

$$c_{01} = \frac{-2\cdot Q_{20}\cdot Q_{01} + Q_{11}\cdot Q_{10}}{4\cdot Q_{20}\cdot Q_{02} - Q_{11}^2}\ , \qquad (2b)$$

$$c_{20} = \frac{2\cdot Q_{02}}{4\cdot Q_{02}\cdot Q_{20} - \left(Q_{11}\right)^2} + \frac{\left(Q_{11}\cdot Q_{01} - 2\cdot Q_{02}\cdot Q_{10}\right)^2}{\left(4\cdot Q_{02}\cdot Q_{20} - \left(Q_{11}\right)^2\right)^2}, \qquad (2c)$$

$$c_{11} = -\frac{Q_{11}}{4\cdot Q_{02}\cdot Q_{20} - \left(Q_{11}\right)^2} + \frac{\left(Q_{11}\cdot Q_{10} - 2\cdot Q_{20}\cdot Q_{01}\cdot\right)\left(Q_{11}\cdot Q_{01} - 2\cdot Q_{02}\cdot Q_{10}\right)}{\left(4\cdot Q_{02}\cdot Q_{20} - \left(Q_{11}\right)^2\right)^2}\ , \qquad (2d)$$

$$c_{02} = \frac{2\cdot Q_{20}}{\left(4\cdot Q_{02}\cdot Q_{20} - \left(Q_{11}\right)^2\right)^2} + \frac{\left(-2\cdot Q_{20}\cdot Q_{01} + Q_{11}\cdot Q_{10}\right)^2}{\left(4\cdot Q_{02}\cdot Q_{20} - \left(Q_{11}\right)^2\right)^2}. \qquad (2e)$$

To this end, we shall use two (alternative) general closed-form expressions for the integrals

$I_n(p,q,c) = \int_{x\in\mathbb{R}} x^n \exp\left[-px^2 + 2qx - c\right]dx$, for any value of $n\in\mathbb{N}$, reported by Gradshteyn and Ryzhik (1965). These read as follows:

$$I_n(p,q,c) = \int_{x\in\mathbb{R}} x^n \exp\left[-px^2 + 2qx - c\right]dx =$$

$$= \frac{1}{2^{n-1}p}\cdot\frac{\sqrt{\pi}}{\sqrt{p}}\cdot\exp(-c)\cdot\frac{d^{n-1}}{dq^{n-1}}\left(q\cdot\exp\left[\frac{q^2}{p}\right]\right) =$$

$$= n!\exp\left[\frac{q^2}{p} - c\right]\cdot\frac{\sqrt{\pi}}{\sqrt{p}}\cdot\left(\frac{q}{p}\right)^n\cdot\sum_{k=0}^{[n/2]}\frac{1}{(n-2k)!(k)!}\left(\frac{p}{4q^2}\right)^k, \quad \textit{for } p > 0. \qquad (3)$$

That is, for:

$$n=0:\int_{x\in\mathbb{R}}\exp\left[-px^2+2qx-c\right]dx = \frac{\sqrt{\pi}}{\sqrt{p}}\cdot\exp\left[\frac{q^2}{p}-c\right], \tag{4a}$$

$$n=1:\int_{x\in\mathbb{R}}x\exp\left[-px^2+2qx-c\right]dx = \frac{\sqrt{\pi}}{\sqrt{p}}\cdot\frac{q}{p}\cdot\exp\left[\frac{q^2}{p}-c\right], \tag{4b}$$

$$n=2:\int_{x\in\mathbb{R}}x^2\exp\left[-px^2+2qx-c\right]dx = \frac{\sqrt{\pi}}{\sqrt{p}}\cdot\left(\frac{1}{2\cdot p}+\frac{q^2}{p^2}\right)\cdot\exp\left[\frac{q^2}{p}-c\right], \tag{4c}$$

$$n=3:\int_{x\in\mathbb{R}}x^3\exp\left[-px^2+2qx-c\right]dx = \frac{\sqrt{\pi}}{2\cdot p^{7/2}}\cdot q\cdot(3\cdot p+2\cdot q^2)\cdot\exp\left[\frac{q^2}{p}-c\right]=$$

$$=\frac{\sqrt{\pi}}{\sqrt{p}}\cdot\left(\frac{3q}{2p^2}+\frac{q^3}{p^3}\right)\cdot\exp\left[\frac{q^2}{p}-c\right], \tag{4d}$$

$$n=4:\int_{x\in\mathbb{R}}x^4\exp\left[-px^2+2qx-c\right]dx = \frac{\sqrt{\pi}}{4\cdot p^{9/2}}\cdot(3\cdot p^2+12\cdot p\cdot q^2+4\cdot q^4)\cdot\exp\left[\frac{q^2}{p}-c\right]=$$

$$=\frac{\sqrt{\pi}}{\sqrt{p}}\cdot(\frac{3}{4\cdot p^2}+\frac{3\cdot q^2}{p^3}+\frac{q^4}{p^4})\cdot\exp\left[\frac{q^2}{p}-c\right], \tag{4e}$$

$$n=5:\int_{x\in\mathbb{R}}x^5\exp\left[-px^2+2qx-c\right]dx = \frac{\sqrt{\pi}q}{4p^{11/2}}\cdot(15p^2+20pq^2+4q^4)\cdot\exp\left[\frac{q^2}{p}-c\right]=$$

$$=\frac{\sqrt{\pi}}{\sqrt{p}}\cdot(\frac{15}{4}\cdot\frac{q}{p^3}+5\cdot\frac{q^3}{p^4}+\frac{q^5}{p^5})\cdot\exp\left[\frac{q^2}{p}-c\right]. \tag{4f}$$

Proof

The integral I (given by Equ.(1)) can be equivalently written as:

$$I=\iint_{\mathbb{R}^2}\left(\Pi_{20}\cdot\alpha^2+\Pi_{11}\cdot\alpha\cdot\beta+\Pi_{02}\beta^2+\Pi_{10}\cdot\alpha+\Pi_{01}\cdot\beta+\Pi_{00}\right)\times$$

$$\times\exp\left\{-\left(Q_{20}\cdot\alpha^2+Q_{11}\cdot\alpha\cdot\beta+Q_{02}\cdot\beta^2+Q_{10}\cdot\alpha+Q_{01}\cdot\beta+Q_{00}\right)d\alpha d\beta=\right.$$

$$=\iint_{\mathbb{R}^2}\left(\Pi_{20}\cdot\alpha^2+\left(\Pi_{11}\cdot\beta+\Pi_{10}\right)\cdot\alpha+\Pi_{02}\beta^2+\Pi_{01}\cdot\beta+\Pi_{00}\right)\times$$

$$\times\exp\left\{-Q_{20}\cdot\alpha^2+2\cdot\left(-\frac{\left(Q_{11}\cdot\beta+Q_{10}\right)}{2}\right)\cdot\alpha-\left(Q_{02}\cdot\beta^2+Q_{01}\cdot\beta+Q_{00}\right)\right\}d\alpha d\beta=$$

$$=\iint_{\mathbb{R}^2}\left(R_2\cdot\alpha^2+R_1\left(\beta\right)\cdot\alpha+R_0\left(\beta\right)\right)\cdot\exp\left\{-p\cdot\alpha^2+2\cdot q\left(\beta\right)\cdot\alpha-c\left(\beta\right)\right\}d\alpha d\beta \quad (*)$$

Let us assume that $Q_{20}>0$, then applying Equs.(4a), (4b) and (4c) from (*) we obtain:

$$I = \frac{\sqrt{\pi}}{\sqrt{p}} \cdot \int_{\mathbb{R}} \underbrace{\left(R_2 \cdot \left(\frac{1}{2p} + \frac{q^2(\beta)}{p^2} \right) + R_1(\beta) \cdot \frac{q(\beta)}{p} + R_0(\beta) \right)}_{\Pi(\beta)} \cdot \exp\left(\underbrace{\frac{q^2(\beta)}{p} - c(\beta)}_{Q(\beta)} \right) d\beta \quad , \tag{5a}$$

where:

$$R_2 = \Pi_{20}, \quad R_1(\beta) = \Pi_{11} \cdot \beta + \Pi_{10}, \qquad R_0(\beta) = \Pi_{02}\beta^2 + \Pi_{01} \cdot \beta + \Pi_{00}, \tag{5b-5d}$$

$$p = Q_{20}, \quad q(\beta) = -\frac{Q_{11} \cdot \beta + Q_{10}}{2}, \qquad c(\beta) = Q_{02} \cdot \beta^2 + Q_{01} \cdot \beta + Q_{00}. \tag{5e-5g}$$

The term $\Pi(\beta)$, denoted in Equ.(5), after some elementary algebraic calculations can be equivalently written as:

$$\Pi(\beta) = R_2^{'} \cdot \beta^2 + R_1^{'} \cdot \beta + R_0^{'}, \tag{6a}$$

where:

$$R_2^{'} = \Pi_{20} \cdot \left(\frac{Q_{11}}{2 \cdot Q_{20}} \right)^2 - \Pi_{11} \cdot \frac{Q_{11}}{2 \cdot Q_{20}} + \Pi_{02}, \tag{6b}$$

$$R_1^{'} = \Pi_{20} \cdot \frac{Q_{11} \cdot Q_{10}}{2 \cdot (Q_{20})^2} - \Pi_{11} \cdot \frac{Q_{10}}{2 \cdot Q_{20}} - \Pi_{10} \frac{Q_{11}}{2 \cdot Q_{20}} + \Pi_{01}, \tag{6c}$$

$$R_0^{'} = \Pi_{20} \cdot \left(\left(\frac{Q_{10}}{2 \cdot Q_{20}} \right)^2 + \frac{1}{2 \cdot Q_{20}} \right) - \Pi_{10} \cdot \frac{Q_{10}}{2 \cdot Q_{20}} + \Pi_{00}, \tag{6d}$$

Similarly, for the exponent $Q(\beta)$, denoted in Equ.(5), we obtain:

$$Q(\beta) = -p^{'} \cdot \beta^2 + 2 \cdot q^{'} \cdot \beta - c^{'}, \tag{7a}$$

where:

$$p^{'} = Q_{02} - \frac{(Q_{11})^2}{4 \cdot Q_{20}} = \frac{4 \cdot Q_{02} \cdot Q_{20} - (Q_{11})^2}{4 \cdot Q_{20}}, \tag{7b}$$

$$q^{'} = \frac{Q_{11} \cdot Q_{10}}{4 \cdot Q_{20}} - \frac{Q_{01}}{2} = \frac{Q_{11} \cdot Q_{10} - 2 \cdot Q_{01} \cdot Q_{20}}{4 \cdot Q_{20}}, \tag{7c}$$

$$c^{'} = -\frac{(Q_{10})^2}{4 \cdot Q_{20}} + Q_{00} = \frac{4 \cdot Q_{20} \cdot Q_{00} - (Q_{10})^2}{4 \cdot Q_{20}}, \tag{7e}$$

Combining Equs.(5-7) we obtain:

$$I = \frac{\sqrt{\pi}}{\sqrt{p}} \cdot \int_{\mathbb{R}} \left(R_2^{'} \cdot \beta^2 + R_1^{'} \cdot \beta + R_0^{'} \right) \cdot \exp\left(-p^{'} \cdot \beta^2 + 2 \cdot q^{'} \cdot \beta - c^{'} \right) d\beta \, . \tag{8}$$

Let us now assume that $p^{'}=\dfrac{4\cdot Q_{02}\cdot Q_{20}-\left(Q_{11}\right)^{2}}{4\cdot Q_{20}}>0$, then we can apply integration formulae (4a), (4b) and (4c) to Equ.(8). This will yield:

$$I=\frac{\pi}{\sqrt{p}\cdot\sqrt{p^{'}}}\cdot\underbrace{\left(R_{2}^{'}\cdot\left(\frac{1}{2\cdot p^{'}}+\left(\frac{q^{'}}{p^{'}}\right)^{2}\right)+R_{1}^{'}\cdot\frac{q^{'}}{p^{'}}+R_{0}^{'}\right)}_{\Pi'}\cdot\exp\underbrace{\left(\frac{q^{'}}{p^{'}}-c^{'}\right)}_{Q'}. \qquad (9)$$

where the terms appearing in Equ.(9) are given by Equs.(7).

Substituting from Equs.(7) and performing some algebraic calculations, the term Π', denoted in Equ.(9), can be equivalently written as:

$$\Pi'=c_{20}\cdot\Pi_{20}+c_{11}\cdot\Pi_{11}+c_{02}\cdot\Pi_{02}+c_{10}\cdot\Pi_{10}+c_{01}\cdot\Pi_{01}+c_{00}\cdot\Pi_{00}\,, \qquad (10a)$$

where

$$c_{20}=\frac{2\cdot Q_{02}}{4\cdot Q_{02}\cdot Q_{20}-\left(Q_{11}\right)^{2}}+\frac{\left(Q_{11}\cdot Q_{01}-2\cdot Q_{02}\cdot Q_{10}\right)^{2}}{\left(4\cdot Q_{02}\cdot Q_{20}-\left(Q_{11}\right)^{2}\right)^{2}}, \qquad (10b)$$

$$c_{11}=-\frac{Q_{11}}{4\cdot Q_{02}\cdot Q_{20}-\left(Q_{11}\right)^{2}}+\frac{\left(Q_{11}\cdot Q_{10}-2\cdot Q_{20}\cdot Q_{01}\cdot\right)\left(Q_{11}\cdot Q_{01}-2\cdot Q_{02}\cdot Q_{10}\right)}{\left(4\cdot Q_{02}\cdot Q_{20}-\left(Q_{11}\right)^{2}\right)^{2}}\,, \qquad (10\,c)$$

$$c_{02}=\frac{2\cdot Q_{20}}{4\cdot Q_{02}\cdot Q_{20}-\left(Q_{11}\right)^{2}}+\frac{\left(-2\cdot Q_{20}\cdot Q_{01}+Q_{11}\cdot Q_{10}\right)^{2}}{\left(4\cdot Q_{02}\cdot Q_{20}-\left(Q_{11}\right)^{2}\right)^{2}}, \qquad (10d)$$

$$c_{10}=\frac{Q_{11}\cdot Q_{01}-2\cdot Q_{02}\cdot Q_{10}}{4\cdot Q_{20}\cdot Q_{02}-Q_{11}^{2}}, \qquad (10e)$$

$$c_{01}=\frac{-2\cdot Q_{20}\cdot Q_{01}+Q_{11}\cdot Q_{10}}{4\cdot Q_{20}\cdot Q_{02}-Q_{11}^{2}}, \qquad (10f)$$

whereas, similarly, the exponent Q', denoted in Equ.(9), can be equivalently written as:

$$Q'=-\frac{Q_{02}\cdot\left(Q_{10}\right)^{2}-Q_{11}\cdot Q_{10}\cdot Q_{01}+Q_{20}\cdot\left(Q_{01}\right)^{2}}{-4\cdot Q_{20}\cdot Q_{02}+\left(Q_{11}\right)^{2}}-Q_{00}\,. \qquad (11)$$

Substituting Equs.(10,11) in Equ.(9) we obtain Equ.(1).

A9. Calculation of 3,4-polynomial/quadratic-exponential integrals

In this Appendix we shall make use of the following formulae that can be easily obtained, combining Equs.(4a-4e), Equ.(5) and Equ.(8) of Appendix 8.

$$I_{\beta\_0} = \int_{\mathbb{R}} \exp\left(\frac{q^2(\beta)}{p} - c(\beta)\right) d\beta = \int_{\mathbb{R}} \exp\left[-p^{'} \cdot \beta^2 + 2 \cdot q^{'} \cdot \beta - c^{'}\right] d\beta = \exp\left[Q'\right], \tag{1a}$$

$$I_{\beta\_1} = \int_{\mathbb{R}} \beta \cdot \exp\left[-p^{'} \cdot \beta^2 + 2 \cdot q^{'} \cdot \beta - c^{'}\right] d\beta = \frac{q^{'}}{p^{'}} \cdot \exp\left[Q'\right], \tag{1b}$$

$$I_{\beta\_2} = \int_{\mathbb{R}} \beta^2 \exp\left[-p^{'} \cdot \beta^2 + 2 \cdot q^{'} \cdot \beta - c^{'}\right] d\beta = \left(\frac{1}{2 \cdot p^{'}} + \frac{\left(q^{'}\right)^2}{\left(p^{'}\right)^2}\right) \cdot \exp\left[Q'\right], \tag{1c}$$

$$I_{\beta\_3} = \int_{\mathbb{R}} \beta^3 \cdot \exp\left[-p^{'} \cdot \beta^2 + 2 \cdot q^{'} \cdot \beta - c^{'}\right] d\beta = \left(\frac{3 \cdot q'}{2\left(p'\right)^2} + \frac{\left(q'\right)^3}{p'^3}\right) \cdot \exp\left[Q'\right], \tag{1d}$$

$$I_{\beta\_4} = \int_{\mathbb{R}} \beta^4 \exp\left[-p^{'} \cdot \beta^2 + 2 \cdot q^{'} \cdot \beta - c^{'}\right] d\beta = \left(\frac{3}{4 \cdot (p')^2} + \frac{3 \cdot (q')^2}{(p')^3} + \frac{(q')^4}{(p')^4}\right) \cdot \exp\left[Q'\right]. \tag{1e}$$

where:

$$p = Q_{20}, \qquad q(\beta) = -\frac{Q_{11} \cdot \beta + Q_{10}}{2}, \qquad c(\beta) = Q_{02} \cdot \beta^2 + Q_{01} \cdot \beta + Q_{00}. \tag{1f-1h}$$

$$Q' = \frac{\left(q'\right)^2}{\left(p'\right)} - c' = -\frac{Q_{02} \cdot \left(Q_{10}\right)^2 - Q_{11} \cdot Q_{10} \cdot Q_{01} + Q_{20} \cdot \left(Q_{01}\right)^2}{-4 \cdot Q_{20} \cdot Q_{02} + \left(Q_{11}\right)^2} - Q_{00}, \tag{1i}$$

$$p^{'} = Q_{02} - \frac{\left(Q_{11}\right)^2}{4 \cdot Q_{20}} = \frac{4 \cdot Q_{02} \cdot Q_{20} - \left(Q_{11}\right)^2}{4 \cdot Q_{20}}, \tag{1j}$$

$$q^{'} = \frac{Q_{11} \cdot Q_{10}}{4 \cdot Q_{20}} - \frac{Q_{01}}{2} = \frac{Q_{11} \cdot Q_{10} - 2 \cdot Q_{01} \cdot Q_{20}}{4 \cdot Q_{20}}, \tag{1k}$$

$$c^{'} = -\frac{\left(Q_{10}\right)^2}{4 \cdot Q_{20}} + Q_{00} = \frac{4 \cdot Q_{20} \cdot Q_{00} - \left(Q_{10}\right)^2}{4 \cdot Q_{20}}, \tag{1l}$$

Then, we shall prove the following integration formula:

$$\begin{aligned} \tilde{I} = \iint_{\mathbb{R}^2} &\left(\Pi_{30} \cdot \alpha^3 + \Pi_{21} \cdot \alpha^2 \cdot \beta + \Pi_{12} \cdot \alpha \cdot \beta^2 + \Pi_{03} \beta^2 + \right. \\ &\left. + \Pi_{40} \cdot \alpha^4 + \Pi_{31} \cdot \alpha^3 \cdot \beta + \Pi_{13} \cdot \alpha \cdot \beta^3 + \Pi_{04} \cdot \beta^4\right) \times \\ &\times \exp\left\{-\left(Q_{20} \cdot \alpha^2 + Q_{11} \cdot \alpha \cdot \beta + Q_{02} \cdot \beta^2 + Q_{10} \cdot \alpha + Q_{01} \cdot \beta + Q_{00}\right) d\alpha d\beta = \right. \\ &= \frac{\pi}{\sqrt{Q_{20} \cdot p'}} \cdot \left(\Pi_{30} \cdot c_{30} + \Pi_{21} \cdot c_{21} + \Pi_{12} \cdot c_{12} + \Pi_{03} \cdot c_{03} + \right. \\ &\left. + \Pi_{40} \cdot c_{40} + \Pi_{31} \cdot c_{31} + \Pi_{13} \cdot c_{13} + \Pi_{04} \cdot c_{04}\right) \times \exp\left[Q'\right], \end{aligned} \tag{2}$$

where:

$$Q' = \frac{(q')^2}{(p')} - c' = -\frac{Q_{02}\cdot(Q_{10})^2 - Q_{11}\cdot Q_{10}\cdot Q_{01} + Q_{20}\cdot(Q_{01})^2}{-4\cdot Q_{20}\cdot Q_{02} + (Q_{11})^2} - Q_{00}\ , \tag{3a}$$

$$p' = Q_{02} - \frac{(Q_{11})^2}{4\cdot Q_{20}} = \frac{4\cdot Q_{02}\cdot Q_{20} - (Q_{11})^2}{4\cdot Q_{20}}\ , \tag{3b}$$

$$q' = \frac{Q_{11}\cdot Q_{10}}{4\cdot Q_{20}} - \frac{Q_{01}}{2} = \frac{Q_{11}\cdot Q_{10} - 2\cdot Q_{01}\cdot Q_{20}}{4\cdot Q_{20}}\ , \tag{3c}$$

$$c' = -\frac{(Q_{10})^2}{4\cdot Q_{20}} + Q_{00} = \frac{4\cdot Q_{20}\cdot Q_{00} - (Q_{10})^2}{4\cdot Q_{20}}\ , \tag{3c}$$

$$c_{12} = \frac{1}{2\cdot Q_{20}}\left(-Q_{10}\cdot\left(\frac{1}{2\cdot p'} + \frac{q'^2}{p'^2}\right) - Q_{11}\cdot\left(\frac{3}{2(p')^2} + \frac{(q')^2}{(p')^3}\right)\right), \tag{4a}$$

$$c_{03} = \left(\frac{3}{2(p')^2} + \frac{(q')^2}{(p')^3}\right), \tag{4b}$$

$$c_{13} = \frac{1}{2\cdot Q_{20}}\cdot\left(-Q_{10}\cdot\left(\frac{3}{2(p')^2} + \frac{(q')^2}{p'^3}\right) - Q_{11}\cdot\left(\frac{3}{4\cdot(p')^2} + \frac{3\cdot(q')^2}{(p')^3} + \frac{(q')^4}{(p')^4}\right)\right), \tag{4c}$$

$$c_{04} = \left(\frac{3}{4\cdot(p')^2} + \frac{3\cdot(q')^2}{(p')^3} + \frac{(q')^4}{(p')^4}\right), \tag{4d}$$

$$c_{31} = \frac{-1}{8\cdot(Q_{20})^3}\cdot\left(Q_{11}{}^3\cdot I_{\beta\_4} + 3\cdot Q_{10}\cdot Q_{11}{}^2\cdot I_{\beta\_3} + \right.$$
$$\left. + \left(6\cdot Q_{11}\cdot Q_{20} + 3\cdot Q_{10}{}^2\cdot Q_{11}\right)\cdot I_{\beta\_2} + \left(6\cdot Q_{10}\cdot Q_{20} + Q_{10}{}^3\right)\cdot I_{\beta\_1}\right), \tag{4e}$$

$$c_{30} = \frac{-1}{4}\frac{1}{(Q_{20})^2}\cdot\left(\frac{1}{2\cdot Q_{20}}\cdot Q_{11}{}^3\cdot I_{\beta\_3} + \frac{3}{2\cdot Q_{20}}\cdot Q_{10}\cdot Q_{11}{}^2\cdot I_{\beta\_2} + \right.$$
$$\left. + \left(3\cdot Q_{11} + \frac{3}{2\cdot Q_{20}}\cdot Q_{10}{}^2\cdot Q_{11}\right)\cdot I_{\beta\_1} + \left(3\cdot Q_{10} + \frac{Q_{10}{}^3}{2\cdot Q_{20}}\right)\cdot I_{\beta\_0}\right), \tag{4f}$$

$$c_{21} = \frac{1}{4\cdot(Q_{20})^2}\cdot\left(\left(2\cdot Q_{20} + (Q_{10})^2\right)\cdot I_{\beta\_1} + 2\cdot Q_{10}\cdot Q_{11}\cdot I_{\beta\_2} + (Q_{11})^2\cdot I_{\beta\_3}\right), \tag{4g}$$

$$c_{40} = \frac{1}{4\cdot(Q_{20})^4}\times$$
$$\times\left(\left(3\cdot(Q_{20})^2 + 3\cdot Q_{20}\cdot(Q_{10})^2 + \frac{1}{4}(Q_{10})^4\right)\cdot I_{\beta\_0} + \left(6\cdot Q_{20}\cdot Q_{10}\cdot Q_{11} + Q_{11}\cdot(Q_{10})^3\right)I_{\beta\_1}\right.$$
$$\left. + \left(3\cdot Q_{20}\cdot(Q_{11})^2 + \frac{3}{2}\cdot(Q_{11})^2\cdot(Q_{10})^2\right)\cdot I_{\beta\_2} + (Q_{11})^3\cdot Q_{10}\cdot I_{\beta\_3} + \frac{1}{4}\cdot(Q_{11})^4\cdot I_{\beta\_4}\right). \tag{4h}$$

Proof

We shall break down the integral $\tilde{I}$ in **eight integrals** $I_{30}, I_{21}, I_{12}, I_{03}, I_{40}, I_{31}, I_{13}, I_{04}$, where:

$$I_{ij} = \Pi_{ij} \cdot \iint_{\mathbb{R}^2} \alpha^i \, \beta^j \exp\left\{-\left(Q_{20} \cdot \alpha^2 + Q_{11} \cdot \alpha \cdot \beta + Q_{02} \cdot \beta^2 + Q_{10} \cdot \alpha + Q_{01} \cdot \beta + Q_{00}\right) d\alpha d\beta \right. .$$

Each integral shall be calculated separately.

The first integral I_{30} will be given by the formula:

$$I_{30} = \Pi_{30} \cdot \iint_{\mathbb{R}^2} \alpha^3 \cdot \exp\left\{-\left(Q_{20} \cdot \alpha^2 + Q_{11} \cdot \alpha \cdot \beta + Q_{02} \cdot \beta^2 + Q_{10} \cdot \alpha + Q_{01} \cdot \beta + Q_{00}\right) d\alpha d\beta = \right.$$

$$= \Pi_{30} \cdot \iint_{\mathbb{R}^2} \alpha^3 \cdot \exp\left\{-p \cdot \alpha^2 + 2 \cdot q(\beta) \cdot \alpha - c(\beta)\right\} d\alpha d\beta , \quad (5)$$

where:

$$p = Q_{20}, \qquad q(\beta) = -\frac{Q_{11} \cdot \beta + Q_{10}}{2}, \qquad c(\beta) = Q_{02} \cdot \beta^2 + Q_{01} \cdot \beta + Q_{00} . \qquad \text{(6a-6c)}$$

Applying Equ (4d)_App(8) to Equ.(5) we obtain:

$$I_{30} = \Pi_{30} \cdot \frac{\sqrt{\pi}}{p^2 \sqrt{p}} \cdot \int_{\mathbb{R}} \left(\frac{3}{2} \cdot q(\beta) + \frac{\left(q(\beta)\right)^3}{p} \right) \cdot \exp\left[\frac{q(\beta)^2}{p} - c(\beta) \right] d\beta \; . \qquad (7)$$

Then, substituting Equ.(6b) in Equ.(7) and using Equs.(1), after some algebraic manipulations, we get:

$$\begin{aligned} I_{30} = -\Pi_{30} \cdot \frac{1}{4} \cdot \frac{\pi}{\sqrt{Q_{20} \cdot p'}} \frac{1}{\left(Q_{20}\right)^2} \cdot \Bigg(& \frac{1}{2 \cdot Q_{20}} \cdot {Q_{11}}^3 \cdot I_{\beta\_3} + \frac{3}{2 \cdot Q_{20}} \cdot Q_{10} \cdot {Q_{11}}^2 \cdot I_{\beta\_2} + \\ & + \left(3 \cdot Q_{11} + \frac{3}{2 \cdot Q_{20}} \cdot {Q_{10}}^2 \cdot Q_{11} \right) \cdot I_{\beta\_1} + \left(3 \cdot Q_{10} + \frac{{Q_{10}}^3}{2 \cdot Q_{20}} \right) \cdot I_{\beta\_0} \Bigg), \end{aligned} \qquad (8)$$

where, $I_{\beta\_0}, I_{\beta\_1}, I_{\beta\_2}, I_{\beta\_3}$, q' are given by Equ.(1)

The second integral I_{21}

$$I_{21} = \Pi_{21} \cdot \iint_{\mathbb{R}^2} \alpha^2 \cdot \beta \cdot \exp\left\{-\left(Q_{20} \cdot \alpha^2 + Q_{11} \cdot \alpha \cdot \beta + Q_{02} \cdot \beta^2 + Q_{10} \cdot \alpha + Q_{01} \cdot \beta + Q_{00}\right) d\alpha d\beta = \right.$$

$$=\Pi_{21}\cdot\frac{\sqrt{\pi}}{\sqrt{p}}\cdot\int_{\mathbb{R}}\beta\cdot\left(\frac{1}{2\cdot p}+\frac{\left(q(\beta)\right)^2}{(p)^2}\right)\cdot\exp\left[\frac{q(\beta)^2}{p}-c(\beta)\right]d\beta=$$

$$=\Pi_{21}\cdot\frac{\sqrt{\pi}}{\sqrt{p}}\cdot\int_{\mathbb{R}}\left(\frac{1}{2\cdot p}\cdot\beta+\frac{1}{4\cdot p^2}\left((Q_{11})^2\cdot\beta^3+2\cdot Q_{10}\cdot Q_{11}\cdot\beta^2+(Q_{10})^2\cdot\beta\right)\right)\cdot\exp\left[\frac{q(\beta)^2}{p}-c(\beta)\right]d\beta=$$

$$=\Pi_{21}\cdot\frac{\pi}{2\cdot\sqrt{p\cdot p'}}\cdot\left(\left(\frac{1}{p}+\frac{1}{2\cdot p^2}\cdot(Q_{10})^2\right)\cdot I_{\beta\_1}+\frac{1}{p^2}\cdot Q_{10}\cdot Q_{11}\cdot I_{\beta\_2}+\frac{1}{2\cdot p^2}\cdot(Q_{11})^2\cdot I_{\beta\_3}\right),$$

that is:

$$I_{21}=\Pi_{21}\cdot\frac{\pi}{\sqrt{Q_{20}\cdot p'}}\cdot\frac{1}{4\cdot(Q_{20})^2}\cdot\left(\left(2\cdot Q_{20}+(Q_{10})^2\right)\cdot I_{\beta\_1}+2\cdot Q_{10}\cdot Q_{11}\cdot I_{\beta\_2}+(Q_{11})^2\cdot I_{\beta\_3}\right)\ ,\qquad (9)$$

where $I_{\beta\_1}, I_{\beta\_2}, I_{\beta\_3}, p'$ are given by Equ.(1).

The third integral I_{12}

$$I_{12}=\Pi_{12}\cdot\int_{\mathbb{R}}\beta^2\int_{\mathbb{R}}\alpha\cdot\exp\left\{-\left(Q_{20}\cdot\alpha^2+Q_{11}\cdot\alpha\cdot\beta+Q_{02}\cdot\beta^2+Q_{10}\cdot\alpha+Q_{01}\cdot\beta+Q_{00}\right)d\alpha d\beta=\right.$$

$$=\Pi_{12}\cdot\frac{\sqrt{\pi}}{\sqrt{p}\cdot p}\int_{\mathbb{R}}\beta^2\cdot\left(-\frac{Q_{11}\cdot\beta+Q_{10}}{2}\right)\cdot\exp\left[-p'\cdot\beta^2+2\cdot q'\cdot\beta-c'\right]d\beta\ =$$

$$=-\Pi_{12}\cdot\frac{\pi}{2\cdot\sqrt{Q_{20}\cdot p'}\cdot Q_{20}}\cdot\left(Q_{10}\cdot I_{\beta\_2}+Q_{11}\cdot I_{\beta\_3}\right),\qquad (10)$$

where $I_{\beta\_2}, I_{\beta\_3}, p'$ are given by Equ.(1).

The forth integral I_{03} will be given by the formula:

$$I_{03}=\Pi_{03}\cdot\iint_{\mathbb{R}^2}\beta^3\cdot\exp\left\{-\left(Q_{20}\cdot\alpha^2+Q_{11}\cdot\alpha\cdot\beta+Q_{02}\cdot\beta^2+Q_{10}\cdot\alpha+Q_{01}\cdot\beta+Q_{00}\right)d\alpha d\beta\right.$$

$$=\Pi_{03}\cdot\frac{\pi}{\sqrt{Q_{20}\cdot p'}}\cdot I_{\beta\_3},\qquad (11)$$

where I_{β_3} , p', are given by Equ.(1).

The fifth integral I_{40} will be given by the formula:

$$I_{40} = \Pi_{40} \cdot \iint_{\mathbb{R}^2} \alpha^4 \cdot \exp\left\{-\left(Q_{20}\cdot\alpha^2 + Q_{11}\cdot\alpha\cdot\beta + Q_{02}\cdot\beta^2 + Q_{10}\cdot\alpha + Q_{01}\cdot\beta + Q_{00}\right) d\alpha d\beta = \right.$$

$$= \Pi_{04}\cdot\frac{\sqrt{\pi}}{\sqrt{p}}\cdot\int_{\mathbb{R}}\left(\frac{3}{4\cdot p^2} + \frac{3\cdot q(\beta)^2}{p^3} + \frac{q(\beta)^4}{p^4}\right)\exp\left[\frac{q(\beta)^2}{p} - c(\beta)\right]d\beta =$$

$$= \Pi_{04}\cdot\frac{\pi}{\sqrt{p\cdot p'}}\cdot\left(\frac{3}{4\cdot p^2}\cdot I_{\beta\_0} + \frac{3}{p^3}\cdot\int_{\mathbb{R}} q(\beta)^2\cdot\exp\left[\frac{q(\beta)^2}{p} - c(\beta)\right] + \right.$$

$$\left. + \frac{1}{p^4}\int_{\mathbb{R}} q(\beta)^4\cdot\exp\left[\frac{q(\beta)^2}{p} - c(\beta)\right]\right), \tag{12}$$

then since:

$$q(\beta)^2 = \frac{1}{4}\cdot\left(\left(Q_{11}\right)^2\cdot\beta^2 + 2\cdot Q_{10}\cdot Q_{11}\cdot\beta + \left(Q_{10}\right)^2\right), \tag{13a}$$

$$q(\beta)^4 = \frac{1}{16}\cdot\left(\left(Q_{11}\right)^4\cdot\beta^4 + 4\cdot\left(Q_{11}\right)^3\cdot\beta^3\cdot Q_{10} + 6\cdot\left(Q_{11}\right)^2\cdot\left(Q_{10}\right)^2\cdot\beta^2 + 4\cdot Q_{11}\cdot\left(Q_{10}\right)^3\cdot\beta + \left(Q_{10}\right)^4\right), \tag{13b}$$

substituting Equs.(13a,b) to Equ.(12) we obtain:

$$I_{40} = \Pi_{04}\cdot\frac{\pi}{\sqrt{p\cdot p'}}\cdot\left(\frac{3}{4\cdot p^2}\cdot I_{\beta\_0} + \int_{\mathbb{R}}\frac{3}{4\cdot p^3}\cdot\left(\left(Q_{11}\right)^2\cdot\beta^2 + 2\cdot Q_{10}\cdot Q_{11}\cdot\beta + \left(Q_{10}\right)^2\right) + \right.$$

$$+ \frac{1}{p^4}\cdot\frac{1}{16}\cdot\left(\left(Q_{11}\right)^4\cdot\beta^4 + + 4\cdot\left(Q_{11}\right)^3\cdot\beta^3\cdot Q_{10} + 6\cdot\left(Q_{11}\right)^2\cdot\left(Q_{10}\right)^2\cdot\beta^2 + \right.$$

$$\left.\left. + 4\cdot Q_{11}\cdot\left(Q_{10}\right)^3\cdot\beta + \left(Q_{10}\right)^4\right)\cdot\exp\left[\frac{q(\beta)^2}{p} - c(\beta)\right]d\beta\right) =$$

[performing some algebra and applying Equ.(1)]

$$= \Pi_{04}\cdot\frac{\pi}{\sqrt{p\cdot p'}}\cdot\frac{1}{4}\cdot\left(\left(\frac{3}{p^2} + \frac{3}{p^3}\cdot\left(Q_{10}\right)^2 + \frac{1}{p^4}\cdot\frac{1}{4}\left(Q_{10}\right)^4\right)\cdot I_{\beta\_0} + \right.$$

$$+ \left(\frac{3}{p^3}\cdot 2\cdot Q_{10}\cdot Q_{11} + \frac{1}{p^4}\cdot Q_{11}\cdot\left(Q_{10}\right)^3\right)I_{\beta\_1} + \left(\frac{3}{p^3}\cdot\left(Q_{11}\right)^2 + \frac{3}{2}\cdot\frac{1}{p^4}\cdot\left(Q_{11}\right)^2\cdot\left(Q_{10}\right)^2\right)\cdot I_{\beta\_2} +$$

$$\left. + \frac{1}{p^4}\cdot\left(Q_{11}\right)^3\cdot Q_{10}\cdot I_{\beta\_3} + \frac{1}{4}\cdot\frac{1}{p^4}\cdot\left(Q_{11}\right)^4\cdot I_{\beta\_4}\right).$$

That is:

$$I_{40} = \Pi_{04} \cdot \frac{\pi}{\sqrt{Q_{20} \cdot p'}} \cdot \frac{1}{4 \cdot (Q_{20})^4} \times$$
$$\left(\left(3 \cdot (Q_{20})^2 + 3 \cdot Q_{20} \cdot (Q_{10})^2 + \frac{1}{4} (Q_{10})^4 \right) \cdot I_{\beta\_0} + \left(6 \cdot Q_{20} \cdot Q_{10} \cdot Q_{11} + Q_{11} \cdot (Q_{10})^3 \right) I_{\beta\_1} + \right. \tag{14}$$
$$\left. + \left(3 \cdot Q_{20} \cdot (Q_{11})^2 + \frac{3}{2} \cdot (Q_{11})^2 \cdot (Q_{10})^2 \right) \cdot I_{\beta\_2} + (Q_{11})^3 \cdot Q_{10} \cdot I_{\beta\_3} + \frac{1}{4} \cdot (Q_{11})^4 \cdot I_{\beta\_4} \right),$$

where $I_{\beta\_0}, I_{\beta\_1}, I_{\beta\_2}, I_{\beta\_3}, I_{\beta\_4}$, $p^{'}$ are given by Equ.(1).

The sixth integral I_{13} will be given by the formula:

$$I_{13} = \Pi_{13} \cdot \iint_{\mathbb{R}^2} \alpha \cdot \beta^3 \cdot \exp\left\{ -\left(Q_{20} \cdot \alpha^2 + Q_{11} \cdot \alpha \cdot \beta + Q_{02} \cdot \beta^2 + Q_{10} \cdot \alpha + Q_{01} \cdot \beta + Q_{00} \right) d\alpha d\beta \right.$$
$$= \Pi_{13} \cdot \int_{\mathbb{R}} \beta^3 \cdot \int_{\mathbb{R}} \alpha \cdot \exp\left[-Q_{20} \cdot \alpha^2 + 2 \cdot \left(-\frac{(Q_{11} \cdot \beta + Q_{10})}{2} \right) \cdot \alpha - \left(Q_{02} \cdot \beta^2 + Q_{01} \cdot \beta + Q_{00} \right) \right\} d\alpha d\beta$$
$$= \Pi_{13} \cdot \int_{\mathbb{R}} \beta^3 \cdot \int_{\mathbb{R}} \alpha \cdot \exp\left\{ -p \cdot \alpha^2 + 2 \cdot q(\beta) \cdot \alpha - c(\beta) \right\} d\alpha d\beta \,. \tag{15}$$

Applying Equ (4b)_App(8) to Equ.(15) we obtain:

$$I_{13} = \Pi_{13} \cdot \int_{\mathbb{R}} \beta^3 \cdot \frac{\sqrt{\pi}}{\sqrt{p}} \cdot \frac{q(\beta)}{p} \cdot \exp\left[\frac{q(\beta)^2}{p} - c(\beta) \right] d\beta \,, \tag{16}$$

or from Equ (7a)_App(8):

$$I_{13} = \Pi_{13} \cdot \frac{\sqrt{\pi}}{\sqrt{p} \cdot p} \int_{\mathbb{R}} \beta^3 \cdot \left(-\frac{Q_{11} \cdot \beta + Q_{10}}{2} \right) \cdot \exp\left[-p^{'} \cdot \beta^2 + 2 \cdot q^{'} \cdot \beta - c^{'} \right] d\beta, \tag{17}$$

that is:

$$I_{13} = -\Pi_{13} \cdot \frac{\sqrt{\pi}}{2 \cdot \sqrt{Q_{20}} \cdot Q_{20}} Q_{10} \cdot I_{\beta\_3} - \Pi_{13} \cdot \frac{\sqrt{\pi}}{2 \cdot \sqrt{Q_{20}} \cdot Q_{20}} \cdot Q_{11} \cdot I_{\beta\_4} \,, \tag{18}$$

where, I_{β_3}, I_{β_4} are given by Equ.(1).

The seventh integral I_{31} will be given by the formula:

$$I_{31} = \Pi_{13} \cdot \int_{\mathbb{R}} \beta \cdot \int_{\mathbb{R}} \alpha^3 \cdot \exp\left\{ -Q_{20} \cdot \alpha^2 + 2 \cdot \left(-\frac{(Q_{11} \cdot \beta + Q_{10})}{2} \right) \cdot \alpha - \left(Q_{02} \cdot \beta^2 + Q_{01} \cdot \beta + Q_{00} \right) \right\} d\alpha d\beta$$

$$= \Pi_{31} \cdot \int_{\mathbb{R}} \beta \cdot \int_{\mathbb{R}} \alpha^3 \cdot \exp\left\{ -p \cdot \alpha^2 + 2 \cdot q(\beta) \cdot \alpha - c(\beta) \right\} d\alpha d\beta$$

$$= \Pi_{31} \cdot \frac{\sqrt{\pi}}{p^2 \cdot \sqrt{p}} \int_{\mathbb{R}} \beta \cdot \left(\frac{3}{2} \cdot q(\beta) + \frac{\left(q(\beta)\right)^3}{p} \right) \cdot \exp\left[\frac{q(\beta)^2}{p} - c(\beta) \right] d\beta\,, \qquad (19)$$

however, substituting from Equ.(6b), we obtain

$$\frac{3}{2} \cdot q(\beta) + \frac{\left(q(\beta)\right)^3}{p} =$$

$$= -\frac{3}{4} \cdot \left(Q_{11} \cdot \beta + Q_{10} \right) - \frac{1}{8 \cdot Q_{20}} \cdot \left(Q_{11}{}^3 \cdot \beta^3 + 3 \cdot Q_{10} \cdot Q_{11}{}^2 \cdot \beta^2 + 3 \cdot Q_{10}{}^2 \cdot Q_{11} \cdot \beta + Q_{10}{}^3 \right) =$$

$$= -\frac{1}{8 \cdot Q_{20}} \cdot Q_{11}{}^3 \cdot \beta^3 - \frac{3}{8 \cdot Q_{20}} \cdot Q_{10} \cdot Q_{11}{}^2 \cdot \beta^2 - \left(\frac{3}{4} \cdot Q_{11} + \frac{3}{8 \cdot Q_{20}} \cdot Q_{10}{}^2 \cdot Q_{11} \right) \cdot \beta - \left(\frac{3}{4} \cdot Q_{10} + \frac{Q_{10}{}^3}{8 \cdot Q_{20}} \right),$$

(20)

therefore, replacing Equ.(20) in Equ.(19) we obtain:

$$I_{31} = \Pi_{31} \cdot \frac{\sqrt{\pi}}{p^2 \cdot \sqrt{p}} \cdot \int_{\mathbb{R}} \left(-\frac{1}{8 \cdot Q_{20}} \cdot Q_{11}{}^3 \cdot \beta^4 - \frac{3}{8 \cdot Q_{20}} \cdot Q_{10} \cdot Q_{11}{}^2 \cdot \beta^3 - \left(\frac{3}{4} \cdot Q_{11} + \frac{3}{8 \cdot Q_{20}} \cdot Q_{10}{}^2 \cdot Q_{11} \right) \cdot \beta^2 + \right.$$
$$\left. - \left(\frac{3}{4} \cdot Q_{10} + \frac{Q_{10}{}^3}{8 \cdot Q_{20}} \right) \cdot \beta \right) \cdot \exp\left[\frac{q(\beta)^2}{p} - c(\beta) \right] d\beta =$$

$$= -\Pi_{31} \cdot \frac{\pi}{\sqrt{p \cdot p'}} \cdot \frac{1}{4 \cdot p^2} \cdot \left(\frac{1}{2 \cdot Q_{20}} \cdot Q_{11}{}^3 \cdot I_{\beta\_4} + \frac{3}{2 \cdot Q_{20}} \cdot Q_{10} \cdot Q_{11}{}^2 \cdot I_{\beta\_3} + \right.$$
$$\left. + \left(3 \cdot Q_{11} + \frac{3}{2 \cdot Q_{20}} \cdot Q_{10}{}^2 \cdot Q_{11} \right) \cdot I_{\beta\_2} + \left(3 \cdot Q_{10} + \frac{Q_{10}{}^3}{2 \cdot Q_{20}} \right) \cdot I_{\beta\_1} \right).$$

That is:

$$I_{31} = -\Pi_{31} \cdot \frac{\pi}{\sqrt{Q_{20} \cdot p'}} \cdot \frac{1}{8 \cdot \left(Q_{20}\right)^3} \cdot \left(Q_{11}{}^3 \cdot I_{\beta\_4} + 3 \cdot Q_{10} \cdot Q_{11}{}^2 \cdot I_{\beta\_3} + \right.$$
$$\left. + \left(6 \cdot Q_{11} \cdot Q_{20} + 3 \cdot Q_{10}{}^2 \cdot Q_{11} \right) \cdot I_{\beta\_2} + \left(6 \cdot Q_{10} \cdot Q_{20} + Q_{10}{}^3 \right) \cdot I_{\beta\_1} \right). \qquad (21)$$

where $I_{\beta\_1}, I_{\beta\_2}, I_{\beta\_3}$, p' are given by Equ.(1).

The eighth integral I_{04} will be given by the formula:

$$I_{04} = \Pi_{04} \cdot \iint_{\mathbb{R}^2} \beta^4 \cdot \exp\left\{-\left(Q_{20} \cdot \alpha^2 + Q_{11} \cdot \alpha \cdot \beta + Q_{02} \cdot \beta^2 + Q_{10} \cdot \alpha + Q_{01} \cdot \beta + Q_{00}\right) d\alpha d\beta = \right.$$
$$= \Pi_{04} \cdot \frac{\sqrt{\pi}}{\sqrt{p}} \cdot \int_{\mathbb{R}} \beta^4 \cdot \exp\left[\frac{q(\beta)^2}{p} - c(\beta)\right] d\beta = \Pi_{04} \cdot \frac{\pi}{\sqrt{Q_{20} \cdot p'}} \cdot I_{\beta\_4}. \tag{22}$$

where I_{β_4} , $p^{'}$, are given by Equ.(1).

Adding Equs.(8,9,10,11,14,18,21,22) Equ.(2) is obtained.

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