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ΔΠΜΣ Εφαρμοσμένες Μαθηματικές Επιστήμες

Natural Deduction with General Elimination Rules and a Proof of Hauptsatz without multicut

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του

Ιωάννη Γεωργαρά

Επιβλέπων: Γεώργιος Κολέτσος
Καθηγητής Ε.Μ.Π.

Αθήνα, Ιούλιος 2014

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Περίληψη

Στην παρούσα διπλωματική παρουσιάσαμε το σύστημα φυσικής απαγωγής και την έννοια της κανονικοποίησης μιας απόδειξης φυσικής απαγωγής. Ακολούθως παρουσιάσαμε το σύστημα ακολουθητών Gentzen και το πολύ σημαντικό θεώρημα απαλοιφής της τομής. Στη συνέχεια περάσαμε σε ένα καινούριο σύστημα Φυσικής Απαγωγής με Γενικευμένους Κανόνες Απαλοιφής. Χάρη στους καινούριους κανόνες πλέον ισχύει μια ισοδυναμία μεταξύ Φυσικής Απαγωγής και συστήματος ακολουθητών χωρίς τα προβλήματα που υπήρχαν με το σύνηθες σύστημα Φυσικής Απαγωγής. Εξετάσαμε τις κανονικές και τις μη-κανονικές αποδείξεις σε αυτό το σύστημα και δείξαμε ότι ισχύει η κανονικοποίηση ως ισοδύναμη πράξη με την απαλοιφή της τομής. Στη συνέχεια παρουσιάσαμε μια διαφορετική απόδειξη του Hauptsatz χωρίς τον κανόνα της πολυτομής που είχε εισάγει ο Gentzen στην περίπτωση που η δεξιά υπόθεση της τομής είχε προέλθει από συστολή.

Abstract

In this thesis we presented the Natural Deduction System and the concept of normalization of Natural Deduction derivation. Next, we presented Gentzen's Sequent Calculus and the very important cut elimination theorem. We continued with a new system of Natural Deduction with General Elimination Rules. Thanks to the new rules, we now have a correspondence between Natural Deduction and Sequent Calculus without the problems that existed with the standard Natural Deduction system. We examined normal and non-normal derivations in this system and we showed that normalization applies as an equivalent procedure to Cut Elimination. Last, we presented a different proof of Hauptsatz without the Multicut rule that was introduce by Gentzen in the case that the right premiss of the Cut rule was derived by contraction.

Ευχαριστίες

Θέλω να ευχαριστήσω τον επιβλέποντα της διπλωματικής μου, τον κ.Κολέτσο, χάρη στον οποίο διδάχθηκα πολλά όμορφα μαθήματα όλα μου τα χρόνια ως προπτυχιακός και μεταπτυχιακός στη Σχολή Εφαρμοσμένων Μαθηματικών και Φυσικών Επιστημών με τα οποία γνώρισα τον όμορφο κόσμο της Λογικής. Η διδασκαλία του και η στήριξή του προς εμένα ήταν καθοριστικός παράγοντας στην επιλογή μου να ασχοληθώ με αυτό το πεδίο. Επίσης, θέλω να ευχαριστήσω τα υπόλοιπα μέλη της τριμελούς μου επιτροπής, τον κ.Στεφανέα και τον κ.Αρβανιτάκη, που μου έδωσαν τη δυνατότητα να μάθω ακόμα περισσότερα πράγματα στη Λογική.

Θέλω επίσης να ευχαριστήσω κάθε έναν καθηγητή της ΣΕΜΦΕ που έκαναν τόσο ευχάριστο το πέρασμά μου από τη σχολή -και τα 70 συνολικά μαθήματα που πέρασα- για την απόκτηση του προπτυχιακού και μεταπτυχιακού στη συνέχεια διπλώματος.

Τέλος, δεν μπορώ να παραλείψω τους φίλους μου που ήταν δίπλα μου όλα αυτά τα χρόνια και την κοπέλα μου, Δανάη, που ήταν μαζί μου στα χρόνια του μεταπτυχιακού και η στήριξή της ήταν ανεκτίμητη!

Ιωάννης Γεωργαράς

Περιεχόμενα

Κεφάλαιο 1

Natural Deduction

1.1 The calculus

We shall use the notation

$$\begin{array}{c} \vdots \\ A \end{array}$$

to designate a *deduction* of A, that is, ending at A. The deduction will be written as a finite tree, and in particular, the tree will have leaves labelled by sentences. For these sentences, there are two possible states, dead or alive.

In the usual state, a sentence is alive, that is to say it takes an active part in the proof: we say it is a hypothesis. The typical case is illustrated by the first rule of natural deduction, which allows us to form a deduction consisting of a single sentence:

$$A$$

Here A is both the leaf and the root; logically, we deduce A, but that was easy because A was assumed!

Now a sentence at a leaf can be dead, when it no longer plays an active part in the proof. Dead sentences are obtained by killing live ones. The typical example is the $\to$-introduction rule:

$$\frac{\begin{array}{c} [A] \\ \vdots \\ B \end{array}}{A \to B} \to \mathrm{I}$$

It must be understood thus: starting from a deduction of B, in which we choose a certain number of occurrences of A as hypotheses (the number is arbitrary: $0, 1, 250, ...$), we form a new deduction of which the conclusion is $A \to B$, but in which all these occurrences of A have been discharged, i.e. killed. There may be other occurrences of A which we have chosen not to discharge.

This rule illustrates very well the illusion of the tree-like notation: it is of critical importance to know when a hypothesis was discharged, and so it is essential to record this. But if we do this in the example above, this means we have to link the crossed A with the line of the $\to I$ rule; but it is no longer a genuine tree we are considering!

1.1.1 The rules

Hypothesis: A

Introductions:

$$\dfrac{\begin{matrix}\vdots & \vdots \\ \mathrm{A} & \mathrm{B}\end{matrix}}{\mathrm{A}\wedge\mathrm{B}}\,\wedge\mathrm{I} \qquad \dfrac{\begin{matrix}[\mathrm{A}] \\ \vdots \\ \mathrm{B}\end{matrix}}{\mathrm{A}\to\mathrm{B}}\,\to\mathrm{I} \qquad \dfrac{\begin{matrix}\vdots \\ \mathrm{A}\end{matrix}}{\mathrm{A}\vee\mathrm{B}}\,\vee\mathrm{I}_\mathrm{R} \qquad \dfrac{\begin{matrix}\vdots \\ \mathrm{B}\end{matrix}}{\mathrm{A}\vee\mathrm{B}}\,\vee\mathrm{I}_\mathrm{L}$$

$$\dfrac{\begin{matrix}\vdots \\ \mathrm{A}\end{matrix}}{\forall x\mathrm{A}}\,\forall\mathrm{I} \qquad \dfrac{\begin{matrix}\vdots \\ A[t/x]\end{matrix}}{\exists x\mathrm{A}}\,\exists\mathrm{I}$$

Eliminations:

$$\dfrac{\begin{matrix}\vdots & \vdots \\ A \to B & B\end{matrix}}{B}\,\to\mathrm{E} \qquad \dfrac{\begin{matrix} & [A] & [B] \\ \vdots & \vdots & \vdots \\ A \vee B & C & C\end{matrix}}{C}\,\vee\mathrm{E}$$

$$\dfrac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{A}\,\wedge\mathrm{E}_\mathrm{R} \qquad \dfrac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{B}\,\wedge\mathrm{E}_\mathrm{L}$$

$$\dfrac{\begin{matrix}\vdots \\ \forall x A\end{matrix}}{A[t/x]}\,\forall\mathrm{E} \qquad \dfrac{\begin{matrix} & [A] \\ \vdots & \vdots \\ \exists x A & C\end{matrix}}{C}\,\exists\mathrm{E} \qquad \dfrac{\begin{matrix}\vdots \\ \bot\end{matrix}}{C}\,\bot\mathrm{E}$$

The rule →E is traditionally called *modus ponens.*

The above rules are for the intuitionistic system Ni. To obtain the classical system Nc, the $\bot_c$ rule is added:

$$\dfrac{\begin{array}{c}[\neg A]\\ \vdots\\ \bot\end{array}}{A}\ \bot_c$$

Some remarks:

All the rules, except →I, preserve the stock of hypotheses: for example, the hypotheses in the deduction above which ends in →E, are those of the two immediate sub-deductions.

For well-known logical reasons, it is necessary to restrict ∀I to the case where the variable x is not free in any hypothesis (it may, on the other hand, be free in a dead leaf). The variable x must no longer be free in the hypotheses or the conclusion after use of the rule ∃E. There is, of course, no rule ⊥I.

The fundamental symmetry of the system is the *introduction/elimination* symmetry, which replaces the *hypothesis/conclusion* symmetry that cannot be implemented in this context.

1.2 Normal Deductions

In this section we shall study the process of normalization for Natural Deductions in Intuitionistic Logic.

We shall assume, unless stated otherwise, that applications of ⊥E have atomic conclusions in the deductions we consider.

Normalizations aim at removing local maxima of complexity, i.e. formula occurrences which are first introduced and immediately afterwards eliminated. However, an introduced formula may be used as a minor premise of an application of ∨E or ∃E, then stay the same throughout a sequence of applications of these rules, being eliminated at the end. This also constitutes a local maximum, which we should like to eliminate; for that we need the so-called permutation conversions. First we give a precise definition.

NOTATION. In order to be able to generalize conveniently later on, we introduce the term *del-rule* (from "disjunction-elimination-like"): the del-rules of Natural Deduction are ∃E, ∨E.

Definition 1.1. *A segment (of length n) in a deduction $\mathcal{D}$ is a sequence $A_1, \ldots, A_n$ of consecutive occurrences of a formula A in $\mathcal{D}$ such that* *(i) for* $1 < n, i < n, A_i$ *is a minor premise of a del-rule application in* $\mathcal{D}$*, with conclusion* A_{i+1}*,*

(ii) A_n *is not a minor premise of a del-rule application,*

(iii) A_1 *is not the conclusion of a del-rule application.*

(Note: An f.o. which is neither a minor premise nor the conclusion of an application of ∨E or ∃E always belongs to a segment of length 1.) A segment is maximal, or a cut (segment) if A_n is the major premise of an E-rule, and either $n > 1$, or $n = 1$ and $A_1 \equiv A_n$ is the conclusion of an I-rule. The cutrank $\mathrm{cr}(\sigma)$ of a maximal segment σ with formula A is $|A|$. The cutrank $\mathrm{cr}(\mathcal{D})$ of a deduction $\mathcal{D}$ is the maximum of the cutranks of cuts in $\mathcal{D}$. If there is no cut, the cutrank of $\mathcal{D}$ is zero. A critical cut of $\mathcal{D}$ is a cut of maximal cutrank among all cuts in $\mathcal{D}$. We shall use σ, σ' for segments.

We shall say that σ is a subformula of σ' if the formula A in σ is a subformula of B in σ'. A deduction without critical cuts is said to be normal.

REMARK. The obvious notion for a cut segment of length greater than 1 which comes to mind stipulates that the first formula occurrence of the segment must be the conclusion of an I-rule; but it turns out we can handle our more general notion of cut in our normalization process without extra effort. Note that a formula occurrence can belong to more than one segment of length greater than 1, due to the ramifications in ∨E-applications.

1.2.1 Detour conversions

We first show how to remove cuts of length 1. We write "conv" for "converts to".

∧-conversion:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1 & \vdots\,\mathcal{D}_2\\ A_1 & A_2\end{matrix}}{A_1\wedge A_2}}{A_i} \qquad \text{conv} \qquad \begin{matrix}\vdots\,\mathcal{D}_i\\ A_i\end{matrix} \qquad \text{for } i\in 1,2.$$

∨-conversion:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}\\ A_i\end{matrix}}{A_1\vee A_2} \quad \begin{matrix}[A_1]\\ \vdots\,\mathcal{D}_1\\ C\end{matrix} \quad \begin{matrix}[A_2]\\ \vdots\,\mathcal{D}_2\\ C\end{matrix}}{C} \qquad \text{conv} \qquad \begin{matrix}\vdots\,\mathcal{D}\\ [A_i]\\ \vdots\,\mathcal{D}_i\\ C\end{matrix} \qquad \text{for } i\in 1,2.$$

→-conversion:

$$\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\,\mathcal{D}\\ B\end{matrix}}{A \to B} \quad \begin{matrix}\vdots\,\mathcal{D}_1\\ A\end{matrix}}{B} \quad \text{conv} \quad \begin{matrix}\vdots\,\mathcal{D}_1\\ [A]\\ \vdots\,\mathcal{D}\\ B\end{matrix}$$

∀-conversion:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}\\ A\end{matrix}}{\forall x A}}{A[t/x]} \quad \text{conv} \quad \begin{matrix}\vdots\,\mathcal{D}[t/x]\\ A[t/x]\end{matrix}$$

∃-conversion:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}\\ A[t/x]\end{matrix}}{\exists x A} \quad \begin{matrix}[A]\\ \vdots\,\mathcal{D}'\\ C\end{matrix}}{C} \quad \text{conv} \quad \begin{matrix}\vdots\,\mathcal{D}\\ [A[t/x]]\\ \vdots\,\mathcal{D}'[t/x]\\ C\end{matrix}$$

1.2.2 Permutation conversions

In order to remove cuts of length > 1, we permute E-rules upwards over minor premises of ∨E, ∃E.

∨-perm conversion:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}\\ A\vee B\end{matrix}\quad \begin{matrix}\vdots\,\mathcal{D}_1\\ C\end{matrix}\quad \begin{matrix}\vdots\,\mathcal{D}_2\\ C\end{matrix}}{C}\quad \vdots\,\mathcal{D}'}{D}\,\text{E-rule} \quad\text{conv}\quad \dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix}\quad \dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ C\end{matrix}\quad \vdots\,\mathcal{D}'}{D}\quad \dfrac{\begin{matrix}\vdots\,\mathcal{D}_2\\ C\end{matrix}\quad \vdots\,\mathcal{D}'}{D}}{D}$$

∃-perm conversion:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}\\ \exists x A\end{matrix}\quad \begin{matrix}\vdots\,\mathcal{D}'\\ C\end{matrix}}{C}\quad \vdots\,\mathcal{D}''}{D}\,\text{E-rule} \quad\text{conv}\quad \dfrac{\begin{matrix}\vdots\,\mathcal{D}\\ \exists x A\end{matrix}\quad \dfrac{\begin{matrix}\vdots\,\mathcal{D}'\\ C\end{matrix}\quad \vdots\,\mathcal{D}''}{D}}{D}$$

1.2.3 Simplification conversions

Applications of ∨E with major premise $A_1 \vee A_2$, where at least one of $[A_1], [A_2]$ is empty in the deduction of the first or second minor premise, are redundant; we accordingly introduce simplifying conversions. Similarly, an application of $\exists$E with major premise $\exists x A$, where the assumption class $[A]$ in the derivation of the minor premise is empty, is redundant. Redundant applications of $\vee$E or $\exists$E can be removed by the following conversions:

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{D}\\ A\vee B\end{array}\quad \begin{array}{c}\vdots\,\mathcal{D}_1\\ C\end{array}\quad \begin{array}{c}\vdots\,\mathcal{D}_2\\ C\end{array}}{C}\qquad \text{conv}\qquad \begin{array}{c}\vdots\,\mathcal{D}_i\\ C\end{array}$$

where no assumptions are discharged by $\vee$E in $\mathcal{D}_i$, and

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{D}\\ \exists x A\end{array}\quad \begin{array}{c}\vdots\,\mathcal{D}'\\ C\end{array}}{C}\qquad \text{conv}\qquad \begin{array}{c}\vdots\,\mathcal{D}'\\ C\end{array}$$

where no assumptions of $\mathcal{D}'$ are discharged at the final rule application. The simplification for $\vee$E introduces a non-deterministic element if both discharged assumption classes $[A_i]$ are empty.

Theorem 1.2. *Each derivation $\mathcal{D}$ in Natural Deduction reduces to a normal derivation.*

Απόδειξη. In applications of E-rules we always assume that the major premise is to the left of the minor premise(s), if there are any minor premises. We use a main induction on the cutrank n of $\mathcal{D}$, with a subinduction on m, the sum of lengths of all critical cuts (= cut segments) in $\mathcal{D}$.

By a suitable choice of the critical cut to which we apply a conversion we can achieve that either n decreases (and we can appeal to the main induction hypothesis), or that n remains constant but m decreases (and we can appeal to the subinduction hypothesis). Let us call σ a t.c.c. (top critical cut) in $\mathcal{D}$ if no critical cut occurs in a branch of $\mathcal{D}$, above σ. Now apply a conversion to the rightmost t.c.c. of $\mathcal{D}$; then the resulting $\mathcal{D}'$ has a lower cutrank (if the segment treated has length 1, and is the only critical segment in $\mathcal{D}$), or has the same cutrank, but a lower value for m.

To see this in the case of an implication conversion, suppose we apply a conversion to the rightmost t.c.c. consisting of a formula occurrence $A \to B$

$$\dfrac{\dfrac{\begin{array}{c}[\mathrm{A}]\\ \vdots\,\mathcal{D}'\\ \mathrm{B}\end{array}}{A\to B}\quad \begin{array}{c}\vdots\,\mathcal{D}''\\ A\end{array}}{B}\qquad \text{conv}\qquad \begin{array}{c}\vdots\,\mathcal{D}''\\ [A]\\ \vdots\,\mathcal{D}'\\ B\end{array}$$

Then the repeated substitution of $\mathcal{D}''$ at each f.o. of $[A]$ cannot increase the value of m, since $\mathcal{D}$, does not contain a t.c.c. cut in $\mathcal{D}''$ above the minor premise A of $\to$E (such a cut would have to occur to the right of $A \to B$, contrary to our assumption). □

Theorem 1.3. *Deductions in Natural Deduction are strongly normalizing w.r.t. the conversions listed, that is all reduction sequences terminate (every strategy produces normal forms).*

1.2.4 Normal Deductions in CL

The system Nc is not as well-behaved w.r.t. to normalization as Ni. In particular, no obvious "formulas-as-types" parallel is available. Nevertheless, as shown by Prawitz, a form of normalization for Nc w.r.t. the $\bot \wedge \to \forall$-language is possible, by observing that $\bot_c$ for this language may be restricted to instances with atomic conclusions. For example, the left tree below may be transformed into the tree on the right hand side:

$$
\dfrac{\begin{array}{c}[\neg(B \to C)] \\ \vdots\,\mathcal{D} \\ \bot\end{array}}{B \to C}
\qquad\qquad
\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{(B \to C) \quad B}{C} \quad \neg C}{\bot}}{[\neg(B \to C)]} \\ \vdots\,\mathcal{D} \\ \bot\end{array}}{C}}{B \to C}
$$

Κεφάλαιο 2

Sequent Calculus

The sequent calculus, due to Gentzen, is the prettiest illustration of the symmetries of Logic. It presents numerous analogies with natural deduction, without being limited to the intuitionistic case. This calculus is generally ignored by computer scientists. Yet it underlies essential ideas: for example, PROLOG is an implementation of a fragment of sequent calculus, and the "tableaux" used in automatic theorem-proving are just a special case of this calculus. In other words, it is used unwittingly by many people, but mixed with control features, i.e. programming devices. What makes everything work is the sequent calculus with its deep symmetries, and not particular tricks. So it is difficult to consider, say, the theory of PROLOG without knowing thoroughly the subtleties of sequent calculus.

2.1 The calculus

Definition 2.1. *A sequent is an expression* $\Gamma \Rightarrow \Delta$ *where* Γ *and* Δ *are finite sequences of formulae* $\Gamma_1, \ldots, \Gamma_n$ *and* $\Delta_1, \ldots, \Delta_n$.

The naive (denotational) interpretation is that the conjunction of the Γ_i implies the disjunction of the Δ_j. In particular,

- if Γ is empty, the sequent asserts the disjunction of the Δ_j ;
- if Γ is empty and Δ is just Δ_1, it asserts Δ_1;
- if Δ is empty, it asserts the negation of the conjunction of the Γ_i;
- if Γ and Δ are empty, it asserts contradiction.

2.1.1 Structural rules

These rules, which seem not to say anything at all, impose a certain way of managing the "slots" in which one writes formulae. They are:

1. The weakening rules

$$\frac{\Gamma \Rightarrow \Delta}{A,\, \Gamma \Rightarrow \Delta}\,\text{LW} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A,\, \Delta}\,\text{RW}$$

as their name suggests, allow replacement of a sequent by a weaker one.

2. The contraction rules

$$\frac{A,\, A,\, \Gamma \Rightarrow \Delta}{A,\, \Gamma \Rightarrow \Delta}\,\text{LC} \qquad \frac{\Gamma \Rightarrow A,\, A,\, \Delta}{\Gamma \Rightarrow A,\, \Delta}\,\text{RC}$$

express the idempotence of conjunction and disjunction.

In fact, contrary to popular belief, these rules are the most important of the whole calculus, for, without having written a single logical symbol, we have practically determined the future behaviour of the logical operations. Yet these rules, if they are obvious from the denotational point of view, should be examined closely from the operational point of view, especially the contraction. It is possible to envisage variants on the sequent calculus, in which these rules are abolished or extremely restricted. That seems to have some very beneficial effects, leading to linear logic [5]. But without going that far, certain well-known restrictions on the sequent calculus seem to have no purpose apart from controlling the structural rules, as we shall see in the following sections.

2.1.2 The intuitionistic case

Essentially, the intuitionistic sequent calculus is obtained by restricting the form of sequents: an intuitionistic sequent is a sequent $\Gamma \Rightarrow \Delta$ where Δ is a sequence formed from at most one formula. In the intuitionistic sequent calculus, the only structural rule on the right is RW since RC assumes several formulae on the right.

The intuitionistic restriction is in fact a modification to the management of the formulae –the particular place distinguished by the symbol $\Rightarrow$ is a place where contraction is forbidden –and from that, numerous properties follow. On the other hand, this choice is made at the expense of the left/right symmetry. A better result is without doubt obtained by forbidding contraction (and weakening) altogether, which allows the symmetry to reappear.

Otherwise, the intuitionistic sequent calculus will be obtained by restricting to the intuitionistic sequents, and preserving –apart from one exception –the classical rules of the calculus.

2.1.3 The "identity" group

1. For every formula A there is the identity axiom $A \Rightarrow A$. In fact one could limit it to the case of atomic A, but this is rarely done. 2. The cut rule

$$\frac{\Gamma \Rightarrow A, \Delta \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \text{Cut}$$

is another way of expressing the identity. The identity axiom says that A (on the left) is stronger than A (on the right); this rule states the converse truth, i.e. A (on the right) is stronger than A (on the left).

The identity axiom is absolutely necessary to any proof, to start things off. That is undoubtedly why the cut rule, which represents the dual, symmetric aspect can be eliminated, by means of a difficult theorem which is related to the normalisation theorem. The deep content of the two results is the same; they only differ in their syntactic dressing.

2.1.4 Logical rules

There is tradition which would have it that Logic is a formal game, a succession of more or less arbitrary axioms and rules. Sequent calculus (and natural deduction as well) shows this is not at all so: one can amuse oneself by inventing one's own logical operations, but they have to respect the left/right symmetry, otherwise one creates a logical atrocity without interest. Concretely, the symmetry is the fact that we can eliminate the cut rule.

1. *Conjunction*: on the left, two unary rules; on the right, one binary rule:

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ \text{L}\wedge_{\text{L}} \qquad \frac{B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ \text{L}\wedge_{\text{R}}$$

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma' \Rightarrow B, \Delta'}{\Gamma, \Gamma' \Rightarrow A \wedge B, \Delta, \Delta'}\ \text{R}\wedge$$

2. *Disjunction*: obtained from conjunction by interchanging right and left:

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma' \Rightarrow \Delta'}{A \vee B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \text{L}\vee \qquad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\ \text{R}\vee_{\text{L}} \qquad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\ \text{R}\vee_{\text{R}}$$

Special case: The intuitionistic rule L∨ is written:

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma' \Rightarrow \Delta}{A \vee B, \Gamma, \Gamma' \Rightarrow \Delta}\ \text{L}\vee$$

where Δ contains zero or one formula. This rule is not a special case of its classical analogue, since a classical L$\vee$ leads to Δ, Δ on the right. This is the only case where the intuitionistic rule is not simply a restriction of the classical one.

3. *Implication*: here we have on the left a rule with two premises and on the right a rule with one premise. They match again, but in a different way from the case of conjunction: the rule with one premise uses two occurrences in the premise:

$$\frac{\Gamma \Rightarrow A,\, \Delta \qquad B,\, \Gamma' \Rightarrow \Delta'}{A \to B,\, \Gamma,\, \Gamma' \Rightarrow \Delta,\, \Delta'}\,\text{L}\!\to \qquad \frac{A,\, \Gamma \Rightarrow B,\, \Delta}{\Gamma \Rightarrow A \to B,\, \Delta}\,\text{R}\!\to$$

4. *Universal quantification*: two unary rules which match in the sense that one uses a variable and the other a term:

$$\frac{A[t/x],\, \Gamma \Rightarrow \Delta}{\forall x A,\, \Gamma,\, \Rightarrow \Delta}\,\text{L}\forall \qquad \frac{\Gamma \Rightarrow A[y/x],\, \Delta}{\Gamma \Rightarrow \forall x A,\, \Delta}\,\text{R}\forall$$

R$\forall$ is subject to a restriction: x must not be free in Γ, Δ.

5. *Existential quantification*: the mirror image of 4:

$$\frac{A[y/x],\, \Gamma \Rightarrow \Delta}{\exists x A,\, \Gamma,\, \Rightarrow \Delta}\,\text{L}\exists \qquad \frac{\Gamma \Rightarrow A[t/x],\, \Delta}{\Gamma \Rightarrow \exists x A,\, \Delta}\,\text{R}\exists$$

L$\exists$ is subject to the same restriction as R$\forall$: x must not be free in Γ, Δ.

2.2 Cut Elimination

Closure under Cut just says that the Cut rule is admissible: if $\vdash \Gamma \Rightarrow \Delta, A$ and $A, \Gamma' \Rightarrow \Delta'$ in the system considered, then also $\vdash \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. This in itself does not give us an algorithm for constructing a deduction of $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ from given deductions of $\Gamma \Rightarrow \Delta, A$ and $A, \Gamma' \Rightarrow \Delta'$. In the systems studied here the deductions are recursively enumerable. So, if we know that the system is closed under Cut, there exists, trivially, an uninteresting algorithm for finding a deduction of $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ given the fact that $\Gamma \Rightarrow \Delta, A$ and $A, \Gamma' \Rightarrow \Delta'$ are deducible: just search through all deductions until one arrives at a deduction for $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. For such a trivial algorithm we cannot find a bound on the depth of the cutfree proof in terms of the depth of the original proof.

We shall say that cut elimination holds for a system, if there is a "non-trivial" algorithm for transforming a deduction in this system into a deduction with the same conclusion in the same system without the cut rule.

Definition 2.2. *The level of a cut is defined as the sum of the depths of the deductions of the premises; the rank of a cut on A is $|A|+1$. The cut rank of a deduction $\mathcal{D}$, cr($\mathcal{D}$), is the maximum of the ranks of the cutformulas occurring in $\mathcal{D}$.*

Theorem 2.3. *Cut elimination holds for sequent calculus.*

The strategy is to successively remove cuts which are topmost among all cuts with rank equal to the rank of the whole deduction, i.e. topmost maximal-rank cuts. It suffices to show how to replace a subdeduction $\mathcal{D}$ of the form

$$\dfrac{\overset{\vdots\, \mathcal{D}_0}{\Gamma \Rightarrow \Delta, A} \qquad \overset{\vdots\, \mathcal{D}_1}{A, \Gamma' \Rightarrow \Delta'}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \text{Cut}$$

where $\text{cr}(\mathcal{D}_i) \leq |A| = \text{cr}(\mathcal{D}) - 1$ for $i \in \{0, 1\}$, by a $\mathcal{D}^*$ with the same conclusion, such that $\text{cr}(\mathcal{D}^*) \leq |A|$. The proof proceeds by a main induction on the cutrank, with a subinduction on the level of the cut at the bottom of $\mathcal{D}$. If we try to prove cut elimination directly for the system we have introduced, we encounter difficulties with the Contraction rule. We should like to transform a deduction

$$\dfrac{\overset{\vdots\, \mathcal{D}'}{\Gamma \Rightarrow A} \qquad \dfrac{\overset{\vdots\, \mathcal{D}''}{\Gamma', A, A \Rightarrow B}}{\Gamma', A \Rightarrow B}\ \text{LC}}{\Gamma, \Gamma' \Rightarrow B}\ \text{Cut}$$

into

$$\dfrac{\dfrac{\overset{\vdots\, \mathcal{D}'}{\Gamma \Rightarrow A} \qquad \dfrac{\overset{\vdots\, \mathcal{D}'}{\Gamma \Rightarrow A} \qquad \overset{\vdots\, \mathcal{D}''}{\Gamma', A, A \Rightarrow B}}{\Gamma, \Gamma', A \Rightarrow B}\ \text{Cut}}{\Gamma, \Gamma, \Gamma' \Rightarrow B}\ \text{Cut}}{\Gamma, \Gamma' \Rightarrow B}\ \text{LC}$$

but this does not give a reduction in the height of the subtrees above the lowest new cut. The solution is to replace Cut by a derivable generalization of the Cut rule:

$$\text{Multicut} \qquad \dfrac{\Gamma \Rightarrow \Delta, A^n \qquad A^m, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \qquad (n, m > 0)$$

where $A^k, k \in \mathbb{N}$, stands for k copies of A. Multicut, also called "Mix", can then be eliminated from this modified calculus.

Rank and level of a Multicut application (a multicut) are defined as rank and level of a cut. We can apply either the strategy of removing topmost cuts, or the strategy of removing topmost maximal-rank cuts. Under both strategies we use an induction on the rank of the multicut, with a subinduction on the level of the multicut, in showing how to get rid of a multicut of rank $k+1$ applied to two proofs with cutrank 0 (on the first strategy) or less than $k+1$ (on the second strategy). In the example above, the upper deduction is simply replaced by

$$\dfrac{\Gamma \Rightarrow A \qquad \begin{matrix}\vdots\, \mathcal{D}''\\ \Gamma', A, A \Rightarrow B\end{matrix}}{\Gamma, \Gamma' \Rightarrow B}\,\text{Multicut}$$

Instructive is the following case, the most complicated one: let $\mathcal{D}$ be obtained by a multicut on the following two cutfree deductions:

$$\dfrac{\begin{matrix}\vdots\, \mathcal{D}_{00}\\ \Gamma, A \Rightarrow B, (A \to B)^m, \Delta\end{matrix}}{\Gamma \Rightarrow (A \to B)^{m+1}, \Delta}\,\text{R}\to \qquad \dfrac{\begin{matrix}\vdots\, \mathcal{D}_{10}\\ \Gamma', (A \to B)^n \Rightarrow A, \Delta'\end{matrix} \qquad \begin{matrix}\vdots\, \mathcal{D}_{11}\\ \Gamma', (A \to B)^n, B \Rightarrow \Delta'\end{matrix}}{\Gamma'(A \to B)^{n+1} \Rightarrow \Delta'}\,\text{L}\to$$

In the case where $m, n > 0$ we construct $\mathcal{D}_a, \mathcal{D}_b, \mathcal{D}_c$:

$$\mathcal{D}_a \equiv \left\{ \dfrac{\begin{matrix}\vdots\, \mathcal{D}_{00}\\ \Gamma, A \Rightarrow B, (A \to B)^m, \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\, \mathcal{D}_{10}\\ \Gamma', (A \to B)^n \Rightarrow A, \Delta'\end{matrix} \qquad \begin{matrix}\vdots\, \mathcal{D}_{11}\\ \Gamma', (A \to B)^n, B \Rightarrow \Delta'\end{matrix}}{\Gamma'(A \to B)^{n+1} \Rightarrow \Delta'}}{\Gamma, \Gamma', A \Rightarrow B, \Delta, \Delta'} \right.$$

$$\mathcal{D}_b \equiv \left\{ \dfrac{\dfrac{\begin{matrix}\vdots\, \mathcal{D}_{00}\\ \Gamma, A \Rightarrow B, (A \to B)^m, \Delta\end{matrix}}{\Gamma \Rightarrow (A \to B)^{m+1}, \Delta} \qquad \begin{matrix}\vdots\, \mathcal{D}_{10}\\ \Gamma', (A \to B)^n \Rightarrow A, \Delta'\end{matrix}}{\Gamma, \Gamma' \Rightarrow A, \Delta, \Delta'} \right.$$

$$\mathcal{D}_c \equiv \left\{ \dfrac{\dfrac{\begin{matrix}\vdots\, \mathcal{D}_{00}\\ \Gamma, A \Rightarrow B, (A \to B)^m, \Delta\end{matrix}}{\Gamma \Rightarrow (A \to B)^{m+1}, \Delta} \qquad \begin{matrix}\vdots\, \mathcal{D}_{11}\\ \Gamma', (A \to B)^n, B \Rightarrow \Delta'\end{matrix}}{\Gamma, \Gamma', B \Rightarrow \Delta, \Delta'} \right.$$

In each of these deductions the multicut on $A \to B$ has a lower level than in $\mathcal{D}$. Therefore we can construct by the IH their transforms $\mathcal{D}'_a, \mathcal{D}'_b, \mathcal{D}'_c$ of cutrank

$\leq |A \to B|$ and combine these in

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}'_a\\ \Gamma,\Gamma',A\Rightarrow B,\Delta,\Delta'\end{array}\qquad\begin{array}{c}\vdots\,\mathcal{D}'_b\\ \Gamma,\Gamma'\Rightarrow A,\Delta,\Delta'\end{array}}{\Gamma,\Gamma,\Gamma',\Gamma'\Rightarrow B,\Delta,\Delta,\Delta',\Delta'}\qquad\begin{array}{c}\vdots\,\mathcal{D}'_c\\ \Gamma,\Gamma',B\Rightarrow\Delta,\Delta'\end{array}}{(\Gamma,\Gamma')^3\Rightarrow(\Delta,\Delta')^3}}{\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,\mathrm{C}$$

The multicuts are now all of lower rank.

Κεφάλαιο 3

Natural Deduction with General Elimination Rules

The structure of derivations in natural deduction is analyzed through isomorphism with a suitable sequent calculus, with twelve hidden convertibilities revealed in usual natural deduction. A general formulation of conjunction and implication elimination rules is given, analogous to disjunction elimination. Normalization through permutative conversions now applies in all cases. Derivations in normal form have all major premisses of elimination rules as assumptions. Conversion in any order terminates.

Through the condition that in a cut-free derivation of the sequent $\Gamma \Rightarrow C$, no inactive weakening or contraction formulas remain in Γ, a correspondence with the formal derivability relation of natural deduction is obtained: All formulas of Γ become open assumptions in natural deduction, through an inductively defined translation. Weakenings are interpreted as vacuous discharges, and contractions as multiple discharges. In the other direction, non-normal derivations translate into derivations with cuts having the cut formula principal either in both premisses or in the right premiss only.

3.1 Introduction

We shall analyze the structure of derivations in natural deduction through isomorphic correspondence with derivations in a suitable sequent calculus. The key insight is to formulate all elimination rules of natural deduction in the manner of disjunction elimination. The standard conjunction and implication elimination rules come out as special cases: it is seen that these rules stand behind the failure of unique correspondence between natural deduction and sequent calculus derivations. In particular, twelve cases of failure of normalization in propositional logic are identified. When conjunction and implication elimination rules are formulated as general elimination rules, derivations permit conversion to full normal form. The characteristic of this form is that all major premisses of elimination rules are assumptions. Normalization holds for any order of conversions.

In full normal form for intuitionistic logic, also premisses of falsity elimination, or the rule "ex falso quodlibet", are assumptions. Thus, a normal intuitionistic derivation of a formula C begins with assumptions and inferences of the form $\frac{\bot}{A}$, followed by subderivations in minimal logic. The usual conjunction and implication elimination rules do not permit this, which created a discrepancy between natural deduction and sequent calculus. In the former, falsity elimination can occur in the middle of a derivation, but in the latter, falsity elimination always is in the beginning of a derivation.

The concept of full normal form is extended to intuitionistic predicate logic by a general elimination rule for the universal quantifier, analogous to the elimination rule for the existential quantifier. This will bring forth twelve more cases of hidden convertibilities in natural deduction.

Our analysis is based on translations establishing isomorphism between natural deduction derivations and suitable sequent calculus derivations. The formal derivability relation of sequent calculus, written $\Gamma \Rightarrow C$, is usually related to a meta-level derivability relation for natural deduction, written $\Gamma \vdash C$. This latter is defined through the existence of a natural deduction derivation of C from open assumptions contained in Γ. We give a correspondence with the formal derivability relation of natural deduction: If in the derivation of $\Gamma \vdash C$ there remain no inactive weakening or contraction formulas in the context Γ, all formulas of Γ become open assumptions in the translation to a natural deduction derivation. Equivalence between natural deduction and sequent calculus only obtains when inactive weakenings and contractions are absent in the latter.

In the sequent calculus we use, weakening is an explicit structural rule. Weakening by a formula that is active in a logical rule in a sequent calculus derivation corresponds to a vacuously discharged formula in natural deduction. To study contraction, we treat contexts as multisets. A sequent calculus derivation has contractions whenever more formula occurrences are discharged in a natural deduction rule than is indicated in the schematic rule, say, more than one in implication introduction. It was not possible to see fully what weakening and contraction amount to in terms of natural deduction before the general elimination rules were available.

The proof of cut elimination for the sequent calculus corresponding to natural deduction with general elimination rules is a straightforward induction on length of cut formula and height of derivation of the premisses of cut. When contexts are treated as multisets, a case of cut elimination is encountered in which the right premiss has been derived by contraction. To obtain cut elimination for this case, a multi-cut rule, as in Gentzen's original proof, can be used. But a direct proof is also available, through consideration of how the premiss of contraction was derived.

The translations we give also apply to non-normal derivations. Normalization can be achieved through translation to sequent calculus followed by cut elimination and translation back. The normal form thus obtained is not unique as cut elimination is not unique. Direct normalization through detour and permutation conversions, instead, will give strong normalization and uniqueness of normal form for natural deduction with general elimination rules.

3.2 Hidden convertibilities in natural deduction

Normal derivations with the usual natural deduction rules for conjunction and implication have a pleasant property: In each step of inference, the formula below is an immediate subformula of a formula above, or the other way around. With disjunction elimination, this simple subformula structure along all branches of a normal derivation tree is lost. But on the other hand, if the major premiss of an elimination step is concluded by disjunction elimination, the derivation converts into a more direct form. For example, if both steps are disjunction eliminations, we have

$$\dfrac{\dfrac{A \lor B \quad \begin{matrix}[A]^1\\ \vdots\\ C \lor D\end{matrix} \quad \begin{matrix}[B]^2\\ \vdots\\ C \lor D\end{matrix}}{C \lor D}\,{\scriptstyle \lor\mathrm{E},1,2} \quad \begin{matrix}[C]^3\\ \vdots\\ E\end{matrix} \quad \begin{matrix}[D]^4\\ \vdots\\ E\end{matrix}}{E}\,{\scriptstyle \lor\mathrm{E},3,4}$$

This derivation converts into

$$\dfrac{A \lor B \quad \dfrac{\begin{matrix}[A]^1\\ \vdots\\ C \lor D\end{matrix} \quad \begin{matrix}[C]^3\\ \vdots\\ E\end{matrix} \quad \begin{matrix}[D]^4\\ \vdots\\ E\end{matrix}}{E}\,{\scriptstyle \lor\mathrm{E},3,4} \quad \dfrac{\begin{matrix}[B]^2\\ \vdots\\ C \lor D\end{matrix} \quad \begin{matrix}[C]^3\\ \vdots\\ E\end{matrix} \quad \begin{matrix}[D]^4\\ \vdots\\ E\end{matrix}}{E}\,{\scriptstyle \lor\mathrm{E},3,4}}{E}\,{\scriptstyle \lor\mathrm{E},1,2}$$

If disjunction elimination is used to conclude a major premiss of conjunction or implication elimination, translations similar to the above apply. These permutation conversions were found by Prawitz in 1965. It is possible that the last step in the derivation of $C \lor D$ from A or B is $\lor$I. Elimination with major premiss $A \lor B$

separates the introduction of $C \vee D$ from an elimination of $C \vee D$. A permutation conversion can reveal such a "hidden" detour convertibility.

In terms of sequent calculus, where the rule corresponding to $\vee$E is the left disjunction rule L$\vee$, the first derivation is

$$\dfrac{\dfrac{A \Rightarrow C \vee D \qquad B \Rightarrow C \vee D}{A \vee B \Rightarrow C \vee D}\,\text{L}\vee \qquad \dfrac{C \Rightarrow E \qquad D \Rightarrow E}{C \vee D \Rightarrow E}\,\text{L}\vee}{A \vee B \Rightarrow E}\,\text{Cut}$$

The second derivation corresponds to

$$\dfrac{\dfrac{A \Rightarrow C \vee D \qquad \dfrac{C \Rightarrow E \qquad D \Rightarrow E}{C \vee D \Rightarrow E}\,\text{L}\vee}{A \Rightarrow E}\,\text{Cut} \qquad \dfrac{B \Rightarrow C \vee D \qquad \dfrac{C \Rightarrow E \qquad D \Rightarrow E}{C \vee D \Rightarrow E}\,\text{L}\vee}{B \Rightarrow E}\,\text{Cut}}{A \vee B \Rightarrow E}\,\text{L}\vee$$

Thus, the conversion of the natural deduction derivation into a more direct form corresponds to a step of cut elimination, where the cut is permuted with L$\vee$, to move it upwards in the derivation.

In Schroeder-Heister (1984), the following general conjunction elimination rule is presented,

$$\dfrac{A \wedge B \qquad \begin{matrix}[A, B] \\ \vdots \\ C\end{matrix}}{C}\,\wedge\text{E}$$

The standard rules come out as special cases when $C = A$ and $C = B$, respectively:

$$\dfrac{A \wedge B \qquad [A]}{A}\,\wedge\text{E}_{\text{R}} \qquad\qquad \dfrac{A \wedge B \qquad [B]}{B}\,\wedge\text{E}_{\text{L}}$$

In the other direction, leaving out the dummy discharged assumptions in these special cases, if C is derivable from A, B, we have

$$\begin{matrix}\dfrac{A \wedge B}{A} \qquad \dfrac{A \wedge B}{B} \\ \vdots \\ C\end{matrix}$$

But the structural properties of these two special elimination rules are quite different from those of the general elimination rule. To give an example, with the special rules we have the derivation

$$\dfrac{\dfrac{(A \wedge B) \wedge C}{A \wedge B}}{A}$$

With the general rule, this becomes the derivation

$$\dfrac{\dfrac{(A \wedge B) \wedge C \qquad [A \wedge B]^1}{A \wedge B}{\scriptstyle \wedge\mathrm{E},1} \qquad [A]^2}{A}{\scriptstyle \wedge\mathrm{E},2} \tag{3.1}$$

Here the major premiss of the second elimination is itself a conclusion of general conjunction elimination and a permutation conversion can be made:

$$\dfrac{(A \wedge B) \wedge C \qquad \dfrac{[A \wedge B]^1 \qquad [A]^2}{A}{\scriptstyle \wedge\mathrm{E},2}}{A}{\scriptstyle \wedge\mathrm{E},1} \tag{3.2}$$

Now the major premisses of both instances of the elimination rule have become assumptions.

Schroeder-Heister's general conjunction elimination rule corresponds to the left conjunction rule of sequent calculus, through the correspondence

$$\dfrac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}{\scriptstyle \mathrm{L}\wedge} \qquad \rightsquigarrow \qquad \dfrac{A \wedge B \qquad \begin{array}{c}[A, B], \Gamma \\ \vdots \\ C\end{array}}{C}{\scriptstyle \wedge\mathrm{E}}$$

Derivation (3.1) with the general elimination rule corresponds, in a way to be made exact below, to

$$\dfrac{\dfrac{\dfrac{A \wedge B \Rightarrow A \wedge B}{A \wedge B, C \Rightarrow A \wedge B}{\scriptstyle \mathrm{W}}}{(A \wedge B) \wedge C \Rightarrow A \wedge B}{\scriptstyle \mathrm{L}\wedge} \qquad \dfrac{\dfrac{A \Rightarrow A}{A, B \Rightarrow A}{\scriptstyle \mathrm{W}}}{A \wedge B \Rightarrow A}{\scriptstyle \mathrm{L}\wedge}}{(A \wedge B) \wedge C \Rightarrow A}{\scriptstyle \mathrm{Cut}}$$

Derivation (3.2) corresponds to

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, B \Rightarrow A}{\scriptstyle \mathrm{W}}}{A \wedge B \Rightarrow A}{\scriptstyle \mathrm{L}\wedge}}{A \wedge B, C \Rightarrow A}{\scriptstyle \mathrm{W}}}{(A \wedge B) \wedge C \Rightarrow A}{\scriptstyle \mathrm{L}\wedge}$$

It can be obtained from the first one by permuting the cut up twice, first with L$\vee$ and then with weakening in the left premiss. We observe that the elimination of cut corresponds to the conversion of major premisses of $\wedge$E rules into assumptions.

With the standard implication elimination rule, or modus ponens, we observe the same phenomenon: A derivation such as

$$\dfrac{\dfrac{A \to (B \to C) \qquad A}{B \to C} \qquad B}{C}$$

does not convert. But if in a sequent calculus derivation the last rule is L$\to$ and it is translated analogously to rules L$\vee$ and L$\wedge$ a general implication elimination rule is found:

$$\dfrac{\Gamma \Rightarrow A \qquad B, \Delta \Rightarrow C}{A \to B, \Gamma, \Delta \Rightarrow C}\,{\scriptstyle \mathrm{L}\to} \quad \rightsquigarrow \quad \dfrac{A \to B \qquad \begin{array}{c}\Gamma \\ \vdots \\ A\end{array} \qquad \begin{array}{c}[B], \Delta \\ \vdots \\ C\end{array}}{C}\,{\scriptstyle \to\mathrm{E}}$$

Again, we obtain the standard elimination rule as a special case, by setting $C = B$. In the other direction, if C is derivable from B, we have

$$\begin{array}{c}\dfrac{A \to B \qquad A}{B} \\ \vdots \\ C\end{array}$$

With the general rule, our example derivation is:

$$\dfrac{\dfrac{A \to (B \to C) \qquad A \qquad [B \to C]^1}{B \to C}\,{\scriptstyle \to\mathrm{E},1} \qquad B \qquad [C]^2}{C}\,{\scriptstyle \to\mathrm{E},2}$$

It converts into the derivation

$$\dfrac{A \to (B \to C) \qquad A \qquad \dfrac{[B \to C]^1 \qquad B \qquad [C]^2}{C}\,{\scriptstyle \to\mathrm{E},2}}{C}\,{\scriptstyle \to\mathrm{E},1}$$

Translations of these derivations into sequent calculus are: For the first, we have

$$\dfrac{\dfrac{A \Rightarrow A \qquad B \to C \Rightarrow B \to C}{A \to (B \to C),\, A \Rightarrow B \to C}\,{\scriptstyle \mathrm{L}\to} \qquad \dfrac{B \Rightarrow B \qquad C \Rightarrow C}{B \to C,\, B \to C}\,{\scriptstyle \mathrm{L}\to}}{A \to (B \to C),\, A,\, B \Rightarrow C}\,{\scriptstyle \mathrm{Cut}}$$

The second one gives instead the cut-free derivation

$$\dfrac{A \Rightarrow A \qquad \dfrac{B \Rightarrow B \qquad C \Rightarrow C}{B \to C,\, B \to C}\,{\scriptstyle L\to}}{A \to (B \to C),\, A,\, B \Rightarrow C}\,{\scriptstyle L\to}$$

There are altogether twelve cases of hidden convertibilities in natural deduction for propositional logic with special elimination rules.

For quantifiers, the standard elimination rules are

$$\dfrac{\forall x A}{A[t/x]}\,{\scriptstyle \forall E} \qquad\qquad \dfrac{\exists x A \qquad \begin{matrix}[A]\\ \vdots \\ C\end{matrix}}{C}\,{\scriptstyle \exists E}$$

where t is a term free for x in A and usual variable restrictions for $\exists$E apply. Similarly to the case of propositional logic, if the major premiss of an elimination step is derived by $\forall$E, the derivation does not convert. This brings out twelve new cases of hidden convertibilities, all eliminable by the use of the general elimination rule for the universal quantifier,

$$\dfrac{\forall x A \qquad \begin{matrix}[A[t/x]]\\ \vdots \\ C\end{matrix}}{C}\,{\scriptstyle \forall E}$$

This rule will permit a full normal form for derivations in intuitionistic first-order logic. The special elimination rule follows by setting $C = A[t/x]$. In the other direction, if C is derivable from $A[t/x]$, we have the derivation

$$\begin{matrix}\dfrac{\forall x A}{[A[t/x]]}\\ \vdots \\ C\end{matrix}$$

The detailed treatment of quantifiers brings no essential new aspects and is left to another occasion.

What has been said of conjunction and implication elimination extends to falsity elimination $\dfrac{\bot}{C}$. In full normal form, its major premiss $\bot$ is an assumption. Thus, in intuitionistic derivations in full normal form instances of rule $\bot$E are top inferences, followed by a derivation in minimal logic. A typical case of conversion is

$$\dfrac{\dfrac{A \to \bot \qquad A \qquad [\bot]^1}{\bot}{\scriptstyle \to\text{E},1}}{C}{\scriptstyle \bot\text{E}} \quad \rightsquigarrow \quad \dfrac{A \to \bot \qquad A \qquad \dfrac{[\bot]^1}{C}{\scriptstyle \to\text{E}}}{E}{\scriptstyle \to\text{E},1}$$

that cannot be done with the modus ponens rule.

3.3 A sequent calculus isomorphic to natural deduction

We shall introduce a sequent calculus, to be called *G0i*, corresponding precisely to natural deduction with logical introduction and general elimination rules.

G0i

Logical axiom: $A \Rightarrow A$

Logical Rules:

$$\frac{A,\, B,\, \Gamma \Rightarrow C}{A \wedge B,\, \Gamma \Rightarrow C}\,{\scriptstyle \text{L}\wedge} \qquad \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma,\, \Delta \Rightarrow A \wedge B}\,{\scriptstyle \text{R}\wedge}$$

$$\frac{A,\, \Gamma \Rightarrow C \qquad B, \Delta \Rightarrow C}{A \vee B,\, \Gamma,\, \Delta \Rightarrow C}\,{\scriptstyle \text{L}\vee} \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\,{\scriptstyle \text{R}\vee_\text{L}} \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\,{\scriptstyle \text{R}\vee_\text{R}}$$

$$\frac{\Gamma \Rightarrow A \qquad B,\, \Delta \Rightarrow C}{A \to B,\, \Gamma,\, \Delta \Rightarrow C}\,{\scriptstyle \text{L}\to} \qquad \frac{A,\, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\,{\scriptstyle \text{R}\to}$$

$$\frac{}{\bot \Rightarrow C}\,{\scriptstyle \text{L}\bot}$$

$$\frac{A[t/x],\, \Gamma \Rightarrow C}{\forall x A,\, \Gamma,\, \Rightarrow C}\,{\scriptstyle \text{L}\forall} \qquad \frac{\Gamma \Rightarrow A[y/x]}{\Gamma \Rightarrow \forall x A}\,{\scriptstyle \text{R}\forall}$$

$$\frac{A[y/x],\, \Gamma \Rightarrow C}{\exists x A,\, \Gamma,\, \Rightarrow C}\,{\scriptstyle \text{L}\exists} \qquad \frac{\Gamma \Rightarrow A[t/x]}{\Gamma \Rightarrow \exists x A}\,{\scriptstyle \text{R}\exists}$$

Rules of weakening and contraction:

$$\frac{\Gamma \Rightarrow C}{A,\, \Gamma \Rightarrow C}\,{\scriptstyle \text{W}} \qquad \frac{A,\, A,\, \Gamma \Rightarrow C}{A,\, \Gamma \Rightarrow C}\,{\scriptstyle \text{Ctr}}$$

The restriction in R$\forall$ and in L$\exists$ is that y does not occur free in the conclusion. The first axiom applies to arbitrary formulas. Therefore, in particular, it gives $\bot \Rightarrow \bot$ as an instance, where falsity $\bot$ is not an atomic formula but a logical constant of length 0. To emphasize that L$\bot$ is a logical rule, we have written it as a zero-premiss left rule. If it is left out, a sequent calculus for minimal logic is obtained.

Each rule has a context, a finite multiset of formulas designated by Γ, Δ in the above rules, active formulas designated by A and B, and a principal formula that is introduced by the rule in question. Corresponding to the treatment of assumptions in natural deduction, two-premiss rules have independent contexts, both collected in the antecedent of the conclusion.

The rule of cut,

$$\frac{\Gamma \Rightarrow A \qquad A, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\,\text{Cut}$$

is proved admissible in Chapter 4.

We shall give an inductive definition of a translation from cut-free derivations in *G0i* to natural deduction derivations with general elimination rules. It is sometimes thought that natural deduction is not able to express the rule of weakening. One defines instead derivability in natural deduction by: C is derivable from Γ if there is a derivation with open assumptions contained in Γ. Here we shall consider the more strict, formal derivability relation.

Definition 3.1. *A formula in a sequent calculus derivation is used if it is active in an antecedent in a rule.*

Rules that use a formula make it disappear from an antecedent, so these are the left rules and R$\to$. In natural deduction, use of formulas corresponds to the discharge of assumptions. The little numbers written next to the mnemonic symbol for the rule applied and on top of formulas indicate what is discharged and where.We shall adjust the translation from sequent calculus to natural deduction accordingly, by adding labels to used formulas. Labels on top of discharged formulas are called assumption labels and those next to the rules discharge labels.

Principle (Unique discharge) 3.2. *No two rules in a derivation must have the same discharge labels.*

The translation of a cut-free sequent calculus derivation of a sequent $\Gamma \Rightarrow C$ to natural deduction, where Γ has no unused weakening or contraction formulas, starts with the last step and works root-first step by step until it reaches axioms and instances of L$\bot$. In this process, it is crucial to keep track of how formulas in the antecedents turn into assumptions. To satisfy Principle 3.2, each rule that discharges assumptions must have fresh discharge labels. Below, in each case of translation, we write the result of the first step of translation with a rule in natural deduction notation, and the premisses from which the translation continues in sequent calculus notation. We also add square brackets and treat labelled and bracketed formulas in the same way as other formulas when continuing the translation. The natural deduction derivation comes out from the translation all finished:

If the last rule to be translated is logical, we have

$$\dfrac{\begin{array}{c}\vdots\\ A,\, B,\, \Gamma \Rightarrow C\end{array}}{A \wedge B,\, \Gamma \Rightarrow C}\,{\scriptstyle L\wedge} \quad \rightsquigarrow \quad \dfrac{A \wedge B \qquad \begin{array}{c}\vdots\\ [A]^1,\, [B]^2,\, \Gamma \Rightarrow C\end{array}}{C}\,{\scriptstyle \wedge E,1,2}$$

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow A\end{array} \qquad \begin{array}{c}\vdots\\ \Delta \Rightarrow B\end{array}}{\Gamma,\, \Delta \Rightarrow A \wedge B}\,{\scriptstyle R\wedge} \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow A\end{array} \qquad \begin{array}{c}\vdots\\ \Delta \Rightarrow B\end{array}}{A \wedge B}\,{\scriptstyle \wedge I}$$

$$\dfrac{\begin{array}{c}\vdots\\ A,\, \Gamma \Rightarrow C\end{array} \qquad \begin{array}{c}\vdots\\ B, \Delta \Rightarrow C\end{array}}{A \vee B,\, \Gamma,\, \Delta \Rightarrow C}\,{\scriptstyle L\vee} \quad \rightsquigarrow \quad \dfrac{A \vee B \qquad \begin{array}{c}\vdots\\ [A]^1,\, \Gamma \Rightarrow C\end{array} \qquad \begin{array}{c}\vdots\\ [B]^2,\, \Delta \Rightarrow C\end{array}}{C}\,{\scriptstyle \vee E,1,2}$$

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow A\end{array}}{\Gamma \Rightarrow A \vee B}\,{\scriptstyle R\vee_L} \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow A\end{array}}{A \vee B}\,{\scriptstyle \vee I_L} \qquad \dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow B\end{array}}{\Gamma \Rightarrow A \vee B}\,{\scriptstyle R\vee_R} \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow B\end{array}}{A \vee B}\,{\scriptstyle \vee I_R}$$

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow A\end{array} \qquad \begin{array}{c}\vdots\\ B,\, \Delta \Rightarrow C\end{array}}{A \rightarrow B,\, \Gamma,\, \Delta \Rightarrow C}\,{\scriptstyle L\rightarrow} \quad \rightsquigarrow \quad \dfrac{A \rightarrow B \qquad \begin{array}{c}\vdots\\ \Gamma \Rightarrow A\end{array} \qquad \begin{array}{c}\vdots\\ [B]^1,\, \Delta \Rightarrow C\end{array}}{C}\,{\scriptstyle \rightarrow E,1}$$

$$\dfrac{\begin{array}{c}\vdots\\ A,\, \Gamma \Rightarrow B\end{array}}{\Gamma \Rightarrow A \rightarrow B}\,{\scriptstyle R\rightarrow} \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\vdots\\ [A]^1,\, \Gamma \Rightarrow B\end{array}}{A \rightarrow B}\,{\scriptstyle \rightarrow I,1}$$

$$\dfrac{}{\bot \Rightarrow C}\,{\scriptstyle L\bot} \quad \rightsquigarrow \quad \dfrac{\bot}{C}\,{\scriptstyle \bot E}$$

If the last rule is weakening or contraction, we have

$$\cfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow C\end{array}}{[A]^n,\, \Gamma \Rightarrow C}\,\scriptstyle W \quad \rightsquigarrow \quad \begin{array}{c}\vdots\\ \Gamma \Rightarrow C\end{array} \qquad \cfrac{A,\, A,\, \Gamma \Rightarrow C}{[A]^n,\, \Gamma \Rightarrow C}\,\scriptstyle Ctr \quad \rightsquigarrow \quad \begin{array}{c}\vdots\\ [A]^n,\, [A]^n,\, \Gamma \Rightarrow C\end{array}$$

If the last rule is an axiom, we have

$$A \Rightarrow A \rightsquigarrow A$$

By the assumption of no unused weakening or contraction formulas, the translation can only reach weakening or contraction formulas indicated as discharged by square brackets. The topsequents of derivations are axioms or instances of $\bot \Rightarrow C$. If the translation arrives to these sequents and they do not have labels, their antecedents turn into open assumptions of the natural deduction derivation. When a formula is used, the translation produces formulas with labels and we can reach topsequents $[A]^n \Rightarrow A$ and $[\bot]^n \Rightarrow C$ with a label in the antecedent. These are translated into $[A]^n$ and $\cfrac{\bot}{C}\,\scriptstyle \bot E$, with discharged assumptions. Note that if a labelled formula gets decomposed further up in the derivation, the labelled formula itself becomes a major premiss of an elimination rule that has been assumed. The components, instead, do not inherit that label but only those indicated in the above translations. Two different labels must be used for assumptions A and B in rules $\vee$E and $\wedge$E. The translation produces derivations in which the major premisses of elimination rules always are (open or discharged) assumptions:

Definition 3.3. *A derivation in natural deduction is in full normal form if all major premisses of E-rules are assumptions.*

We shall refer to such derivations briefly as normal. Notice that $\bot$ in $\bot$E is counted as a major premiss of an E-rule.

Translation of derivations with cuts will be discussed in Section 3.4.

The translation is an algorithm that works its way up from the endsequent in a local way, reflecting the local character of sequent calculus rules. It produces syntactically correct derivation trees with discharges fully formalized.

The translation of applications of the rule of weakening into natural deduction may seem somewhat surprising, but it will lead to a useful insight about the nature of this rule. Natural deduction rules permit to discharge formulas that have not occurred in a derivation. Similarly, natural deduction rules permit to discharge any number of occurrences of an assumption, not just the occurrence indicated in the schematic rule.

Definition 3.4. *Rule* $\to I$ *and the elimination rules produce a vacuous (multiple) discharge whenever*

1. *In* $\to I$ *concluding* $A \to B$ *no occurrence (more than one occurrence) of assumption* A *was discharged.*
2. *In* $\wedge E$ *and* $\vee E$ *with major premisses* $A \wedge B$ *and* $A \vee B$*, no occurrence of* A *or* B *(more than one occurrence of* A *or* B*, or more than two if* $A = B$*), was discharged.*
3. *In* $\to E$ *with major premiss* $A \to B$ *no (more than one) occurrence of* B *was discharged.*

A weakening formula (resp. contraction formula) is a formula A introduced by weakening (contraction) in a derivation. There can be applications of weakening that have no correspondence in natural deduction: Whenever we have a derivation with weakening formulas that are not used, the endsequent is of the form $A, \Gamma \Rightarrow C$, with A an inactive weakening formula throughout.

The condition of no inactive weakening or contraction formulas in a sequent calculus derivation permits a correspondence with the formal derivability relation of natural deduction:

Theorem 3.5. *Given a derivation of* $\Gamma \Rightarrow C$ *with no inactive weakening or contraction formulas, there is a natural deduction derivation of* C *from* Γ *with each formula of* Γ *an open assumption.*

Απόδειξη. The proof is by induction on the height of derivation, using the translation from sequent calculus. If $\Gamma \Rightarrow C$ is an axiom or instance of L$\perp$, $\Gamma = C$ or $\Gamma = \perp$, and the translation gives the natural deduction derivations C and $\dfrac{\perp}{C}\,{\scriptstyle \perp E}$ with open assumptions C and $\perp$, respectively. If the last rule is L$\wedge$, we have $\Gamma = A \wedge B, \Gamma'$ and the translation gives

$$\dfrac{A \wedge B \qquad \begin{matrix} \vdots \\ [A]^1, [B]^2, \Gamma' \Rightarrow C \end{matrix}}{C}\;{\scriptstyle \wedge E,1,2}$$

If there are no inactive weakenings or contractions in the derivation of $A, B, \Gamma' \Rightarrow C$, there is by inductive hypothesis a natural deduction derivation of C from open assumptions A, B, Γ'. Now assume $A \wedge B$ and apply $\wedge$E to obtain a derivation of C from $A \wedge B$, Γ'.

If there is an inactive weakening or contraction formula in the derivation of $A,\, B,\, \Gamma' \;\Rightarrow\; C$ it is by assumption not in Γ' so it is $A or B$ or both. Deleting the weakenings and contractions with unused formulas we obtain a derivation of $A^m,\, B^n,\, \Gamma' \Rightarrow C$, with $m, n \geq 0$ copies of A and B, respectively. By the inductive hypothesis, there is a corresponding natural deduction derivation with open assumptions A^m, B^n, Γ'. Application of $\wedge$E now gives a derivation of C from $A \wedge B, \Gamma$. All the other cases of logical rules are dealt with similarly. The last step cannot be weakening or contraction by the assumption about no inactive weakening or contraction formulas. □

By the translation, the natural deduction derivation in Theorem 3.5 is normal. Later we show the converse result. Equivalence of derivability between sequent calculus and natural deduction only applies if unused weakenings and contractions are absent.

Theorem 3.6. *Given a derivation of $\Gamma \Rightarrow C$ with no inactive weakening or contraction formulas, if A is a weakening (contraction) formula in the derivation, then A is vacuously (multiply) discharged in the corresponding natural deduction derivation.*

Απόδειξη. Formula A can be used in left rules and R→ only. Applying the translation to natural deduction, A becomes a labelled formula in the antecedent, and translating further, it disappears when a weakening with A is translated, and is multiplied when a contraction on A is translated. □

Perhaps the simplest example is, with the corresponding natural deduction at right,

$$\cfrac{\cfrac{\cfrac{A \Rightarrow A}{A,\, B \Rightarrow A}\,\scriptstyle W}{A \wedge B \Rightarrow A}\,\scriptstyle L\wedge}{\Rightarrow A \wedge B \to A}\,\scriptstyle R\to \qquad\qquad \cfrac{\cfrac{[A \wedge B]^1 \qquad [A]^2}{A}\,\scriptstyle \wedge E,2,3}{A \wedge B \to A}\,\scriptstyle \to I,1$$

In the natural deduction derivation, B is vacuously discharged. The translation produces, as a trace of the weakening, the discharge label 3 to which no assumption label corresponds. An intermediate stage of the translation just before the disappearance of the weakening formula is

$$\cfrac{\cfrac{[A \wedge B]^1 \qquad \cfrac{A \Rightarrow A}{[A]^2,\, [B]^3 \Rightarrow A}\,\scriptstyle W}{A}\,\scriptstyle \wedge E,2,3}{A \wedge B \to A}\,\scriptstyle \to I,1$$

In Gentzen's original sequent calculus there were two left rules for conjunction:

$$\cfrac{A,\, \Gamma \Rightarrow C}{A \wedge B,\, \Gamma \Rightarrow C}\,\scriptstyle L\wedge_L \qquad\qquad \cfrac{B,\, \Gamma \Rightarrow C}{A \wedge B,\, \Gamma \Rightarrow C}\,\scriptstyle L\wedge_R$$

These left rules correspond to the standard elimination rules for conjunction, and the derivation of $A \land B \to A$ and its translation become

$$\cfrac{\cfrac{A}{A \land B \Rightarrow A}\,{\scriptstyle L\land_L}}{\Rightarrow A \land B \to A}\,{\scriptstyle R\to} \qquad\qquad \cfrac{\cfrac{[A \land B]^1}{A}\,{\scriptstyle \land E}}{A \land B \to A}\,{\scriptstyle \to I,1}$$

Weakening is hidden in Gentzen's left conjunction rules and vacuous discharge in the special conjunction elimination rules. It is not possible to state fully the meaning of weakening in terms of natural deduction without using the general elimination rules. The premiss of a contraction step can arise in essentially three ways: First, the duplication A, A comes from a rule with two premisses each having one occurrence of A. Second, A is the principal formula of a left rule and a premiss had A already in the antecedent. Third, weakening is applied to a premiss having A in the antecedent. The simplest example of a multiple discharge should be the derivation of $A \to A \land A$, given here both in *G0i* with a contraction and in translation to natural deduction with a double discharge:

$$\cfrac{\cfrac{\cfrac{A \Rightarrow A \qquad A \Rightarrow A}{A,\, A \Rightarrow A \land A}\,{\scriptstyle R\land}}{A \Rightarrow A \land A}\,{\scriptstyle Ctr}}{\Rightarrow A \to A \land A}\,{\scriptstyle R\to} \qquad\qquad \cfrac{\cfrac{[A]^1 \qquad [A]^1}{A \land A}\,{\scriptstyle \land I}}{A \to A \land A}\,{\scriptstyle \to I,1}$$

In Definition 3.4, the clause about more than two occurrences in $\land$E and $\lor$E in case of $A = B$, is exemplified by the derivation of $A \lor A \to A$:

$$\cfrac{\cfrac{A \Rightarrow A \qquad A \Rightarrow A}{A \lor A \Rightarrow A}\,{\scriptstyle L\lor}}{\Rightarrow A \lor A \to A}\,{\scriptstyle R\to} \qquad\qquad \cfrac{\cfrac{[A \lor A]^3 \qquad [A]^1 \qquad [A]^2}{A}\,{\scriptstyle \lor E,1,2}}{A \lor A \to A}\,{\scriptstyle \to I,3}$$

Here there is no contraction even if two occurrences of A are discharged at $\lor$E.

We now come to the translation from natural deduction to sequent calculus. It is essential to use multisets to see how natural deduction can keep track of contraction. This is no problem since it is well defined how many times open assumptions $A, B, C, \dots$ appear above any given formula in a derivation.

Translation from fully normal natural deduction derivations with unique discharge to the calculus *G0i* is defined inductively according to the last rule used:

1. *The last rule is $\land$I:*

$$\cfrac{\begin{matrix}\Gamma & & \Delta \\ \vdots & & \vdots \\ A & & B\end{matrix}}{A \land B}\,{\scriptstyle \land I} \quad\rightsquigarrow\quad \cfrac{\begin{matrix}\Gamma & & \Delta \\ \vdots & & \vdots \\ A & & B\end{matrix}}{\Gamma,\, \Delta \Rightarrow A \land B}\,{\scriptstyle R\land}$$

2. *The last rule is ∧E:* The natural deduction derivation, with m-fold discharge on A and n-fold on B, is

$$\dfrac{A \wedge B \qquad \begin{matrix}[A^m]^1, [B^n]^2, \Gamma \\ \vdots \\ C\end{matrix}}{C}\ {\scriptstyle \wedge\mathrm{E},1,2}$$

The translation is by cases according to values of m and n:
$m = 0, n = 0:$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Gamma \\ \vdots \\ C\end{matrix}}{A,\, \Gamma \Rightarrow C}\ {\scriptstyle \mathrm{W}}}{A,\, B,\, \Gamma \Rightarrow C}\ {\scriptstyle \mathrm{W}}}{A \wedge B,\, \Gamma \Rightarrow C}\ {\scriptstyle \mathrm{L}\wedge}$$

$m = 1, n = 1:$

$$\dfrac{\begin{matrix}A,\, B,\, \Gamma \\ \vdots \\ C\end{matrix}}{A \wedge B,\, \Gamma \Rightarrow C}\ {\scriptstyle \mathrm{L}\wedge}$$

Note that the discharge labels and brackets have been removed. The cases of $m = 1, n = 0$ and $m = 0, n = 1$ have one weakening step before the L∧ inference.
$m > 1, n = 0:$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}A^m,\, \Gamma \\ \vdots \\ C\end{matrix}}{A,\, \Gamma \Rightarrow C}\ {\scriptstyle \mathrm{Ctr}^m}}{A,\, B,\, \Gamma \Rightarrow C}\ {\scriptstyle \mathrm{W}}}{A \wedge B,\, \Gamma \Rightarrow C}\ {\scriptstyle \mathrm{L}\wedge}$$

Here Ctr^m indicates an $m - 1$ fold contraction, and discharges in m occurrences of assumption A have been removed. The rest of the cases for ∧E are similar.

3. *The last rule is ∨I:*

$$\dfrac{\begin{matrix}\Gamma \\ \vdots \\ A\end{matrix}}{A \vee B}\ {\scriptstyle \vee\mathrm{I_L}} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\Gamma \\ \vdots \\ A\end{matrix}}{\Gamma \Rightarrow A \vee B}\ {\scriptstyle \mathrm{R}\vee_\mathrm{L}} \qquad \dfrac{\begin{matrix}\Gamma \\ \vdots \\ B\end{matrix}}{A \vee B}\ {\scriptstyle \vee\mathrm{I_R}} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\Gamma \\ \vdots \\ B\end{matrix}}{\Gamma \Rightarrow A \vee B}\ {\scriptstyle \mathrm{R}\vee_\mathrm{R}}$$

4. *The last rule is* $\vee E$*:* The natural deduction derivation is

$$\dfrac{A \vee B \qquad \begin{matrix}[A^m]^1,\ \Gamma \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B^n]^2,\ \Delta \\ \vdots \\ C\end{matrix}}{C}\ {\scriptstyle \vee\mathrm{E},1,2}$$

and the translation is again by cases according to the values of m and n:
$m = 0, n = 0$:

$$\dfrac{\dfrac{\begin{matrix}\Gamma \\ \vdots \\ C\end{matrix}}{A,\ \Gamma \Rightarrow C}\ {\scriptstyle \mathrm{W}} \qquad \dfrac{\begin{matrix}\Delta \\ \vdots \\ C\end{matrix}}{B,\ \Delta \Rightarrow C}\ {\scriptstyle \mathrm{W}}}{A \vee B,\ \Gamma,\ \Delta \Rightarrow C}\ {\scriptstyle \mathrm{L}\vee}$$

$m = 1, n = 1$

$$\dfrac{\begin{matrix}A,\ \Gamma \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}B,\ \Delta \\ \vdots \\ C\end{matrix}}{A \vee B,\ \Gamma,\ \Delta \Rightarrow C}\ {\scriptstyle \mathrm{L}\vee}$$

Here again, the assumptions have been opened. The general case is $m > 1, n > 1$:

$$\dfrac{\dfrac{\begin{matrix}A^m,\ \Gamma \\ \vdots \\ C\end{matrix}}{A,\ \Gamma \Rightarrow C}\ {\scriptstyle \mathrm{Ctr}^m} \qquad \dfrac{\begin{matrix}B^n,\ \Delta \\ \vdots \\ C\end{matrix}}{B,\ \Delta \Rightarrow C}\ {\scriptstyle \mathrm{Ctr}^n}}{A \vee B,\ \Gamma,\ \Delta \Rightarrow C}\ {\scriptstyle \mathrm{L}\vee}$$

5. *The last rule is* $\rightarrow I$*:* The general case with $m > 1$ is translated by

$$\dfrac{\begin{matrix}[A^m], \Gamma \\ \vdots \\ B\end{matrix}}{A \rightarrow B}\ {\scriptstyle \rightarrow\mathrm{I},1} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\begin{matrix}A^m, \Gamma \\ \vdots \\ B\end{matrix}}{A,\ \Gamma \Rightarrow B}\ {\scriptstyle \mathrm{Ctr}^m}}{\Gamma \Rightarrow A \rightarrow B}\ {\scriptstyle \mathrm{R}\rightarrow}$$

Again assumptions have been opened. If $m = 0$, there is a weakening instead of contraction, if $m = 1$, there is just rule R$\rightarrow$.

6. *The last rule is →E:* The general case is translated as

$$\dfrac{A \to B \qquad \begin{matrix}\Gamma \\ \vdots \\ A\end{matrix} \qquad \begin{matrix}[B^n]^1, \Delta \\ \vdots \\ C\end{matrix}}{C}\,{\scriptstyle \to\mathrm{E},1} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\Gamma \\ \vdots \\ A\end{matrix} \qquad \dfrac{\begin{matrix}B^n, \Delta \\ \vdots \\ C\end{matrix}}{B,\, \Delta \Rightarrow C}\,{\scriptstyle \mathrm{Ctr}^m}}{A \to B,\, \Gamma,\, \Delta \Rightarrow C}\,{\scriptstyle \mathrm{L}\to}$$

The other cases are translated analogously to above.

7. *The last rule is ⊥E:*

$$\dfrac{\bot}{C}\,{\scriptstyle \bot\mathrm{E}} \quad \rightsquigarrow \quad \dfrac{}{\bot \Rightarrow C}\,{\scriptstyle \mathrm{L}\bot}$$

8. *The last formula is an assumption:*

$$A \quad \rightsquigarrow \quad A \Rightarrow A$$

Notice that in a fully normal derivation, the premiss of rule ⊥E is an assumption and nothing remains to be translated in step 7. If in 7 or 8 there are discharge labels and brackets they are removed.

Theorem 3.7. *Given a fully normal natural deduction derivation of C from open assumptions Γ, there is a derivation of $\Gamma \Rightarrow C$ in* G0i.

Απόδειξη. By the translation defined. □

There are no unused weakenings or contractions in the derivation of $\Gamma \Rightarrow C$. By the translation, we obtain the converse to theorem 3.6:

Theorem 3.8. *If A is vacuously (multiply) discharged in the derivation of C from open assumptions Γ, then A is a weakening (contraction) formula used in the derivation of $\Gamma \Rightarrow C$ in G0i.*

The usual explanation of contraction runs something like "if you can derive a formula using assumption A twice you can also derive it using A only once." But this is just a verbal statement of the rule of contraction. Logical rules of natural deduction that discharge assumptions vacuously or multiply are reproduced as weakenings or contractions plus a logical rule in sequent calculus, but the weakening and contraction rules in themselves have no proof-theoretical meaning, as was pointed out by Gentzen (1936, pp. 513-14) already.

By the translation of a derivation from natural deduction to sequent calculus, each formula in the former appears in the latter. We therefore have a somewhat surprising proof of the

Corollary (Subformula property) 3.9. *In a normal derivation of C from open assumptions Γ, each formula in the derivation is a subformula of Γ, C.*

The translations we have defined from natural deduction to sequent calculus and the other way around do not quite establish an isomorphism between the two: the order of logical rules is preserved, but it is possible to permute weakenings and contractions on a formula A as long as A remains inactive so that isomorphism obtains modulo such permutations. This, however, is a minor point that can be handled by doing weakening and contraction on formula A right before A is used.

Translation of non-normal derivations will be discussed in Section 3.5.

3.4 Derivation with cuts

We show that derivations with cuts can be translated into natural deduction if the cuts are of a suitable kind: the detour cuts and permutation cuts corresponding to cuts with the cut formula principal in both premisses and right premiss only, respectively. These are the principal cuts, the rest are nonprincipal cuts. Principal cuts correspond, in terms of natural deduction, to instances of rules of elimination in which the major premisses are not assumptions.

A sequent calculus derivation has an equivalent in natural deduction only if it has no unused weakening or contraction formulas. By this criterion, there is no correspondence in natural deduction for many of the nonprincipal cuts of sequent calculus. In particular, if the right premiss of cut has been derived by contraction, the contraction formula is not used in the derivation and there is no corresponding natural deduction derivation. This is precisely the problematic case that led Gentzen to use the rule of multicut. If cut and contraction are permuted, the right premiss of a cut becomes derived by another cut and there is likewise no translation.

In translating derivations with cuts, if the left premiss is an axiom the cut is deleted. There are three detour cuts and another twelve permutation cuts with left premiss derived by a logical rule to be translated. We also translate principal cuts on $\bot$ as well as cases where the left premiss has been derived by a structural rule, but derivations with other cases of cuts will not be translated. Translation of rules other than cut have been given in Section 3.3.

1. Detour cut on $A \wedge B$, and we have the deivation

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow A\end{matrix} \quad \begin{matrix}\vdots\\ \Delta \Rightarrow B\end{matrix}}{\Gamma, \Delta \Rightarrow A \wedge B}\,\scriptstyle{R\wedge} \qquad \dfrac{\begin{matrix}\vdots\\ A, B, \Theta \Rightarrow C\end{matrix}}{A \wedge B, \Theta \Rightarrow C}\,\scriptstyle{L\wedge}}{\Gamma, \Delta, \Theta \Rightarrow C}\,\scriptstyle{Cut}$$

The translation is:

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow A\end{array} \quad \begin{array}{c}\vdots\\ \Delta \Rightarrow B\end{array}}{A \wedge B}\,{\scriptstyle \wedge\mathrm{I}} \qquad \begin{array}{c}\vdots\\ [A]^1,\, [B]^2,\, \Theta \Rightarrow C\end{array}}{C}\,{\scriptstyle \wedge\mathrm{E},1,2}$$

Translation now continues from the premisses.

2.,3. Detour cuts on $A \vee B$ and $A \to B$. The translations are analogous to 1, with the left and right rules translated as in Section 3.3.

4. Permutation cut on $C \wedge D$ with left premiss derived by L$\wedge$:

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\ A,\, B,\, \Gamma \Rightarrow C \wedge D\end{array}}{A \wedge B,\, \Gamma \Rightarrow C \wedge D}\,{\scriptstyle \mathrm{L}\wedge} \qquad \cfrac{\begin{array}{c}\vdots\\ C,\, D,\, \Delta \Rightarrow E\end{array}}{C \wedge D,\, \Delta \Rightarrow E}\,{\scriptstyle \mathrm{L}\wedge}}{A \wedge B,\, \Gamma\, \Delta \Rightarrow E}\,{\scriptstyle \mathrm{Cut}}$$

The translation is

$$\cfrac{\cfrac{A \wedge B \qquad \begin{array}{c}\vdots\\ [A]^1,\, [B]^2,\, \Gamma \Rightarrow C \wedge D\end{array}}{C \wedge D}\,{\scriptstyle \wedge\mathrm{E},1,2} \qquad \begin{array}{c}\vdots\\ [C]^3,\, [D]^4,\, \Delta \Rightarrow E\end{array}}{E}\,{\scriptstyle \wedge\mathrm{E},3,4}$$

The rest of the permutation cuts with L$\wedge$, L$\vee$ and L$\to$ are translated analogously.

5. We also have permutation cuts on $\bot$E but no detour cuts since $\bot$ can never be principal in the left premiss. The derivation and its translation are, where L stands for a (one-premiss) left rule and E for an elimination,

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\ \Gamma' \Rightarrow \bot\end{array}}{\Gamma \Rightarrow \bot}\,{\scriptstyle \mathrm{L}} \qquad \cfrac{}{\bot \Rightarrow C}\,{\scriptstyle \mathrm{L}\bot}}{\Gamma \Rightarrow C}\,{\scriptstyle \mathrm{Cut}} \quad \rightsquigarrow \quad \cfrac{\cfrac{\begin{array}{c}\vdots\\ \Gamma' \Rightarrow \bot\end{array}}{\bot}\,{\scriptstyle \mathrm{E}}}{C}\,{\scriptstyle \bot\mathrm{E}}$$

6. 'Structural' cuts with left premiss derived by weakening, contraction or cut. For weakening and contraction the translation reaches, by the condition of no unused weakening or contraction formulas, a conclusion of cut of the form $[A^n],\, \Gamma,\, \Delta \Rightarrow C$. In the case of weakening, the left premiss of cut $A,\, \Gamma \Rightarrow B$ has been derived from $\Gamma \Rightarrow B$, in the case of contraction from $A,\, A,\, \Gamma \Rightarrow B$. The cuts are translated with left premiss replaced by $\Gamma \Rightarrow B$ and $[A^n],\, [A^n],\, \Gamma \Rightarrow B$, respectively.

For left premiss of cut derived by another cut the translation is modular and the upper cut is handled as before.

3.5 Non-normal derivations

We began in Section 3.2 with examples of non-normal natural deduction derivations corresponding to sequent calculus derivations with cuts. The latter are produced by translations defined inductively according to the last step. Derived major premisses are called conversion formulas. There are three cases of non-normality in which the major premiss of an elimination rule has been derived by the corresponding introduction rule:

1. The conversion formula has been derived by $\wedge$I and the derivation is

$$\dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}\qquad\begin{matrix}\Delta\\ \vdots\\ B\end{matrix}}{A\wedge B}{\scriptstyle\wedge\mathrm{I}}\qquad\begin{matrix}[A^m]^1,[B^n]^2,\Theta\\ \vdots\\ C\end{matrix}}{C}{\scriptstyle\wedge\mathrm{E},1,2}$$

The translation is by cases according to values of m and n. The general case is

$$\dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}\qquad\begin{matrix}\Delta\\ \vdots\\ B\end{matrix}}{\Gamma,\,\Delta\Rightarrow A\wedge B}{\scriptstyle\mathrm{R}\wedge}\qquad\dfrac{\dfrac{\begin{matrix}A^m,\,B^n,\,\Theta\\ \vdots\\ C\end{matrix}}{A,\,B,\,\Theta\Rightarrow C}{\scriptstyle\mathrm{Ctr}^m,\,\mathrm{Ctr}^n}}{A\wedge B,\,\Theta\Rightarrow C}{\scriptstyle\mathrm{L}\wedge}}{\Gamma,\,\Delta,\,\Theta\Rightarrow C}{\scriptstyle\mathrm{Cut}}$$

There is an $m+n-2$ fold contraction in case $m,n>1$.

2., 3. The conversion formula has been derived by $\vee$I or $\rightarrow$I and the translation is analogous.

When detour conversions are applied, the open assumptions in a derivation can change. For example, the derivation

$$\dfrac{\dfrac{A\qquad B}{A\wedge B}{\scriptstyle\wedge\mathrm{I}}\qquad[A]^1}{A}{\scriptstyle\wedge\mathrm{E},1}$$

converts into the derivation A. Translation gives

$$\dfrac{\dfrac{A\Rightarrow A\qquad B\Rightarrow B}{A,\,B\Rightarrow A\wedge B}{\scriptstyle\mathrm{R}\wedge}\qquad\dfrac{\dfrac{A\Rightarrow A}{A,\,B\Rightarrow A}{\scriptstyle\mathrm{W}}}{A\wedge B\Rightarrow A}{\scriptstyle\mathrm{L}\wedge}}{A,\,B\Rightarrow A}{\scriptstyle\mathrm{Cut}}$$

Cut elimination produces the derivation

$$\frac{A \Rightarrow A}{A,\, B \Rightarrow A}\,\text{W}$$

Deletion of the unused weakening gives the derivation $A \Rightarrow A$, corresponding to the result of the detour conversion.

Given a (cut-free) derivation of $\Gamma \Rightarrow C$, we can first delete the unused weakenings, then translate to natural deduction, and last add the unused weakening formulas of Γ to the natural deduction derivation by the above trick on formula B, to obtain a non-normal derivation of C from open assumptions Γ.

There is a good number of non-normalities with a permutation convertibility but we only show one typical case:

4. The conversion formula $C \wedge D$ has been derived by $\wedge$E from $A \wedge B$:

$$\dfrac{\dfrac{A \wedge B \qquad \begin{matrix}[A^m]^1,\, [B^n]^2,\, \Gamma \\ \vdots \\ C \wedge D\end{matrix}}{C \wedge D}\,{\scriptstyle \wedge\text{E},1,2} \qquad \begin{matrix}[C^k]^3, [D^l]^4, \Delta \\ \vdots \\ E\end{matrix}}{E}\,{\scriptstyle \wedge\text{E},3,4}$$

The translation is by cases according to values of m, n, k, l, with the general case

$$\dfrac{\dfrac{\dfrac{\begin{matrix}A^m,\, B^n,\, \Gamma \\ \vdots \\ C \wedge D\end{matrix}}{A,\, B,\, \Gamma \Rightarrow C \wedge D}\,{\scriptstyle \text{Ctr}^m, \text{Ctr}^n}}{A \wedge B,\, \Gamma \Rightarrow C \wedge D}\,{\scriptstyle \text{L}\wedge} \qquad \dfrac{\dfrac{\begin{matrix}C^k,\, D^l,\, \Delta \\ \vdots \\ C\end{matrix}}{C,\, D,\, \Delta \Rightarrow E}\,{\scriptstyle \text{Ctr}^k, \text{Ctr}^l}}{C \wedge D,\, \Delta \Rightarrow E}\,{\scriptstyle \text{L}\wedge}}{A \wedge B,\, \Gamma,\, \Delta \Rightarrow E}\,{\scriptstyle \text{Cut}}$$

If $A \wedge B$ in turn is a conversion formula, a cut on $A \wedge B$ is inserted after the rule L$\wedge$ that concludes the left premiss of the cut on $C \wedge D$.

Translations when $\vee$E and $\to$E have been used are analogous to the one for $\wedge$E. Translation when $\perp$E has been used is the converse to translation 5 in Section 3.4. If the major premiss in the derivation of a conversion formula is again a conversion formula, another cut is inserted.

Consider a typical principal cut, say, on $A \wedge B$:

$$\dfrac{\Gamma \Rightarrow A \wedge B \qquad \dfrac{\begin{matrix}\vdots \\ A,\, B,\, \Delta \Rightarrow C\end{matrix}}{A \wedge B, \Delta \Rightarrow C}\,{\scriptstyle \text{L}\wedge}}{\Gamma,\, \Delta \Rightarrow C}\,{\scriptstyle \text{Cut}}$$

We see that the cut is redundant, in the sense that its left premiss is an axiom, precisely when $A \wedge B$ is an assumption in the corresponding natural deduction derivation. In this case, the cut is not translated but deleted. We have, in general:

> A non-normal instance of a logical rule in natural deduction is represented in sequent calculus by the corresponding left rule and a cut.

Let us compare this explanation of cut to the presentation of cut as a combination of two lemmas $\Gamma \Rightarrow A$ and $A, \Delta \Rightarrow C$ into a theorem $\Gamma, \Delta \Rightarrow C$. Consider the derivation of C from assumptions A, Δ in natural deduction. Obviously A plays an essential role only if it is analyzed into components by an elimination rule, thus, A is a major premiss of that elimination rule. If not, it acts just as a parameter in the derivation. Our explanation of cut makes more precise the idea of cut as a combination of lemmas: In terms of sequent calculus, the cut formula has to be principal in a left rule in the derivation of $A, \Delta \Rightarrow C$.

Given a non-normal derivation, translation to sequent calculus, followed by cut elimination and translation back to natural deduction, will produce a normal derivation:

Theorem (Normalization) 3.10. *Given a natural deduction derivation of* C *from* Γ*, the derivation converts to a normal derivation of* C *from* Γ^* *where each formula in* Γ^* *is a formula in* Γ*.*

This process of normalization will not produce a unique result since cut elimination will not.

3.6 The structure of normal derivations

Theorem 3.10 gave a proof of normalization for intuitionistic natural deduction with general elimination rules, through a translation to sequent calculus, cut elimination and translation back to natural deduction. Strong normalization and uniqueness of normal form (modulo the choices in simplification convertibilities on disjunction, see below) for our system of natural deduction is given by Joachimski and Matthes (2001). Their proof uses a system of term assignment.

We consider three different types of non-normalities of a natural deduction derivation with general elimination rules that depend on how a major premiss of an elimination rule was derived. Then the subformula structure of normal derivations is detailed, with a direct proof of the subformula property.

1. **Detour conversions:** Gentzen's original notion of a normal derivation in natural deduction was that no conclusion of an introduction rule must be

the major premiss of an elimination rule. (This is seen from the example on implication in his 1934-35, Sec. II. §5.12.) Non-normal derivations are transformed into normal ones by detour conversions that delete each such pair of introduction and elimination rule instances. In a fully general form, a detour convertibility on the formula $A \wedge B$ obtains in a derivation whenever it has a part of the form

$$\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}\vdots\\ [A]\end{array} \quad \begin{array}{c}\vdots\\ [B]\end{array}}{A \wedge B}{\scriptstyle \wedge I} \qquad \begin{array}{c}[A^m]^1, [B^n]^2\\ \vdots\\ C\end{array}}{C}{\scriptstyle \wedge E1,2}\\
\vdots
\end{array}$$

Detour conversion on $A \wedge B$ gives, through simultaneous substitution, the modified derivation

$$\begin{array}{c}
\begin{array}{c}\vdots\\ A\end{array}, \overset{m\times}{\dots}, \begin{array}{c}\vdots\\ A\end{array} \qquad \begin{array}{c}\vdots\\ B\end{array}, \overset{n\times}{\dots}, \begin{array}{c}\vdots\\ B\end{array}\\
\vdots\\
C\\
\vdots
\end{array}$$

A detour convertibility on $A \vee B$ is quite analogous. For implication, the situation is more complicated since a vacuous or multiple discharge is possible also in the introduction of the conversion formula:

$$\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}[A^m]^1\\ \vdots\\ B\end{array}}{A \to B}{\scriptstyle \to,I,1} \qquad \begin{array}{c}\vdots\\ A\end{array} \qquad \begin{array}{c}[B^n]^2\\ \vdots\\ C\end{array}}{C}{\scriptstyle \to,E,2}\\
\vdots
\end{array}$$

Detour conversion on $A \to B$ gives the modified derivation

$$\begin{array}{c} \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\ A,\overset{m\times}{\cdots},A \qquad A,\overset{m\times}{\cdots},A \\ \vdots \qquad\quad \vdots \\ B,\overset{n\times}{\cdots},B \\ \vdots \\ C \\ \vdots \end{array}$$

In detour conversions, open assumptions typically get multiplied or deleted.

2. **Permutation conversions for general elimination rules:** There are four elimination rules which gives sixteen cases of permutation convertibilities, major premisses of elimination rules that are derived by another elimination rule. All of these act in a similar way on derivations and we only show one:

   A permutation convertibility on major premiss $C \wedge D$ derived by $\wedge$E on $A \wedge B$ obtains whenever a derivation has the part

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\vdots\\ A \wedge B\end{array} \qquad \begin{array}{c}[A^m, B^n]\\ \vdots\\ C \wedge D\end{array}}{C \wedge D}\,{\scriptstyle \wedge \mathrm{E}} \qquad \begin{array}{c}[C^k, D^l]\\ \vdots\\ E\end{array}}{E}\,{\scriptstyle \wedge \mathrm{E}} \\ \vdots \end{array}$$

   After the permutation conversion the part is

$$\begin{array}{c} \dfrac{\begin{array}{c}\vdots\\ A \wedge B\end{array} \qquad \dfrac{\begin{array}{c}[A^m, B^n]\\ \vdots\\ C \wedge D\end{array} \qquad \begin{array}{c}[C^k, D^l]\\ \vdots\\ E\end{array}}{E}\,{\scriptstyle \wedge \mathrm{E}}}{E}\,{\scriptstyle \wedge \mathrm{E}} \\ \vdots \end{array}$$

   The effect of the conversion is that the height of derivation of major premiss $C \wedge D$, as measured from the discharged assumptions A, B, is diminished by one.

3. **Simplification conversions:** Analogously to Section 1.2.3, we have

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A \wedge B\end{array} \quad \begin{array}{c}\Delta\\ \vdots\\ C\end{array}}{C}\,{\scriptstyle \wedge\mathrm{E}} \qquad \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A \to B\end{array} \quad \begin{array}{c}\Delta\\ \vdots\\ A\end{array} \quad \begin{array}{c}\Theta\\ \vdots\\ C\end{array}}{C}\,{\scriptstyle \to\mathrm{E}}$$

In both inferences, C is already concluded without the elimination rule, and simplification conversion extends to all elimination rules, quantifier rules included. The notion is captured by the

Definition 3.11. *A simplification convertibility in a derivation is an instance of an E-rule with no discharged assumptions, or an instance of* $\vee E$ *with no discharges of at least one disjunct.*

A simplification convertibility can prevent the normalization of a derivation, as is shown by the following:

$$\dfrac{\dfrac{\dfrac{[A]^1}{A \to A}\,{\scriptstyle \to\mathrm{I},1} \quad \dfrac{[B]^2}{B \to B}\,{\scriptstyle \to\mathrm{I},2}}{(A \to A) \wedge (B \to B)}\,{\scriptstyle \wedge\mathrm{I}} \qquad \dfrac{[C]^3}{C \to C}\,{\scriptstyle \to,3}}{C \to C}\,{\scriptstyle \wedge\mathrm{E}}$$

There is a detour convertibility but due to the vacuous discharge in $\wedge$E, the pieces of derivation do not fit together in the right way to remove it. Instead a simplification conversion into the derivation

$$\dfrac{[C]^3}{C \to C}\,{\scriptstyle \to\mathrm{I},3}$$

will remove the detour convertibility.

It is possible that in a simplification convertibility with $\vee$E, both auxiliary assumptions are vacuously discharged. In this case, there are two converted derivations of the conclusion.

Κεφάλαιο 4

A proof of Gentzen's Hauptsatz without multicut

Gentzen"s original proof of the Hauptsatz used a rule of multicut in the case that the right premiss of cut was derived by contraction. Cut elimination is here proved without multicut, by transforming suitably the derivation of the premiss of the contraction.

4.1 Introduction

Gentzen in his original 1934 proof of the Hauptsatz for sequent calculus, or cut elimination theorem, met the following problem: If the right premiss of cut is derived by contraction, the permutation of cut with contraction does not move the cut higher up in the derivation. Gentzen introduced the "mix" rule, or rule of multicut, that lets one eliminate $m \geq 1$ occurrences $A, \ldots, A = A^m$ of the cut formula in one step:

$$\frac{\Gamma \Rightarrow A \qquad A^m, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\,\text{Cut}^*$$

The reason for having to make recourse to this rule is as follows: Consider the derivation

$$\frac{\Gamma \Rightarrow A \qquad \dfrac{A, A, \Delta \Rightarrow C}{A, \Delta \Rightarrow C}\,\text{Ctr}}{\Gamma, \Delta \Rightarrow C}\,\text{Cut}$$

Permuting cut with contraction, we obtain

$$\dfrac{\dfrac{\Gamma \Rightarrow A \qquad \dfrac{\Gamma \Rightarrow A \qquad A, A, \Delta \Rightarrow C}{A, \Gamma, \Delta \Rightarrow C}\,\text{Cut}}{\Gamma, \Gamma, \Delta \Rightarrow C}\,\text{Cut}}{\Gamma, \Delta \Rightarrow C}\,\text{Ctr}$$

Here the lower cut is on the same formula A and has a sum of heights of derivations of the premisses not less than that in the first derivation. With multicut, instead, we transform the derivation into:

$$\frac{\Gamma \Rightarrow A \qquad A^2, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\text{Cut}^*$$

Here the height of derivation of the right premiss is diminished by one. A proof of multicut elimination can be given by induction on the length of the cut formula and a subinduction on cut-height, i.e., the sum of the derivation heights of the two premisses of cut. The proof consists of permuting multicut up with the rules used for deriving its premisses, until it reaches logical axioms and disappears(see, e.g., Takeuti 1975). A calculus with multicut is equivalent to a calculus with cut, in the sense that the same sequents are derivable. Ordinary cut is a special case of multicut, so that elimination follows from elimination of multicut.

We shall give proofs of the Hauptsatz without multicut for an intuitionistic single-succedent and a classical multi-succedent calculus. These can be considered standard calculi when contexts in rules with two premisses are treated as independent. In the problematic case of contraction, we make a more global proof transformation by cases on the derivation of the premiss of contraction.

4.2 Hauptsatz for the intuitionistic calculus

We will use the *G0i* system again. To prove the admissibility of cut, i.e., the derivability of its conclusion whenever the premisses are derivable, we define length $w(C)$ by $w(\bot) = 0, w(P) = 1$ for atoms P, and $w(A \circ B) = w(A) + w(B) + 1$ for conjunction, disjunction and implication. Height of a derivation is the greatest number of consecutive steps of inference in it. Cut-height is the sum of heights of the two premisses in an instance of the cut rule.

The proof of cut elimination is organized as follows: We first consider cases in which one premiss is an axiom of form $A \Rightarrow A$ or the left premiss is an axiom of form $\bot \Rightarrow C$, then cases with a premiss obtained by weakening. Next, reflecting normalization in natural deduction, we have cases where the cut formula is principal in both premisses and cases where the cut formula is principal in the right premiss only. Then we have the case that both premisses are derived by a logical rule and the cut formula is not principal in either rule instance. The last cases concern contraction. Cut elimination proceeds by first eliminating cuts that are not preceded by other cuts. The following lemma is needed:

Lemma 4.1. *(i) If $A \wedge B, \Gamma \Rightarrow C$ is derivable, also $A, B, \Gamma \Rightarrow C$ is derivable.*

(ii) If $A \vee B, \Gamma \Rightarrow C$ is derivable, also $A, \Gamma \Rightarrow C$ and $B, \Gamma \Rightarrow C$ are derivable.

(iii) If $A \vee B, \Gamma \Rightarrow C$ is derivable, also $B, \Gamma \Rightarrow C$ is derivable.

Απόδειξη. In each case, trace up from the endsequent the occurrence of the formula in question. If at some stage the formula is principal in contraction, trace up from the premiss both occurrences. In this way, a number of first occurrences of the formula is located. (i) If a first occurrence of $A \wedge B$ is obtained by weakening, weaken with A and with B and continue as before after the weakenings until either a derivation of $A, B, G \Rightarrow C$ is reached or a step found where a contraction on $A \wedge B$ was done in the first derivation. In the latter case, the transformed derivation will have A, A, B, B in place of $A \wedge B, A \wedge B$, and a contraction on A and on B is done and the derivation continued as before. If a first occurrence of $A \wedge B$ is obtained by an axiom $A \wedge B \Rightarrow A \wedge B$, the axiom is substituted by

$$\frac{A \Rightarrow A \qquad B \Rightarrow B}{A, B \Rightarrow A \wedge B}\text{R}\wedge$$

and the derivation continued as before. Otherwise, a first occurrence of $A \wedge B$ is obtained by L$\wedge$, and deletion of this rule will give a derivation of $A, B, \Gamma \Rightarrow C$ as before. For (ii), weakening and axiom are treated similarly to (i). Otherwise, the L$\vee$ rule introducing $A \vee B$ in the antecedent is

$$\frac{A, \Gamma'' \Rightarrow C' \qquad A, \Gamma''' \Rightarrow C'}{A \vee B, \Gamma' \Rightarrow C'}\text{L}\vee$$

where $\Gamma' = \Gamma'', \Gamma'''$. Repeated weakening of the premisses gives $A, \Gamma' \Rightarrow C'$ and $B, \Gamma' \Rightarrow C'$, and continuing as before derivations of $A, \Gamma \Rightarrow C$ and $B, \Gamma \Rightarrow C$ are obtained. (iii) is proved similarly to (ii). □

Note that the inversions do not need to preserve height of derivation.

Theorem 4.2. *The rule of cut*

$$\frac{\Gamma \Rightarrow D \qquad D, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\,\mathit{Cut}$$

is admissible in G0i.

Απόδειξη. The proof is an induction on (n, m) where n is the length of the cut formula and m the cut-height. Uppermost cuts (in the obvious sense) are eliminated first, so it is sufficient to consider a derivation with just one cut. For all cases, a transformation is given that either reduces the length of cut formula, or the cut-height while maintaining the cut formula.

1. One premissisof form $A \Rightarrow A$: The conclusion is equal to the other premiss and the cut is deleted. The left premiss is $\bot \Rightarrow A$: The conclusion of cut is obtained from the axiom $\bot \Rightarrow C$ by weakening.

2. A premiss is derived by weakening: If the first premiss is derived by weakening, cut and weakening are permuted to obtain a cut with lower cut-height, and similarly if the second premiss is derived by weakening and the cut formula is different from the weakening formula. If it is not, the conclusion $G, \Delta \Rightarrow C$ of cut is derived from the premiss $\Delta \Rightarrow C$ of the weakening by repeated weakening and no cut.

3. Cut formula D is principal in both premisses. There are three subcases each of which has a transformation of cut into cuts on shorter formulas:

3.1 $D = A \wedge B$, and the derivation

$$\dfrac{\dfrac{\Gamma \Rightarrow A \qquad \Delta \to B}{\Gamma, \Delta \Rightarrow A \wedge B}\,\text{R}\wedge \qquad \dfrac{A, B, \Theta \Rightarrow C}{A \wedge B, \Theta \Rightarrow C}\,\text{L0}\wedge}{\Gamma, \Delta, \Theta \Rightarrow C}\,\text{Cut}$$

is transformed into

$$\dfrac{\Delta \to B \qquad \dfrac{\Gamma \Rightarrow A \qquad A, B, \Theta \Rightarrow C}{B, \Gamma, \Theta \Rightarrow C}\,\text{Cut}}{\Gamma, \Delta, \Theta \Rightarrow C}\,\text{Cut}$$

Note that cut-height can increase in the transformation, but the cut formula is reduced.

3.2 $D = A \vee B$, and the derivation is

$$\dfrac{\dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\,R\wedge_L \qquad \dfrac{A, \Delta \Rightarrow C \qquad B, \Theta \Rightarrow C}{A \vee B, \Delta, \Theta \Rightarrow C}\,\text{L}\vee}{\Gamma, \Delta, \Theta \Rightarrow C}\,\text{Cut}$$

This is transformed into

$$\dfrac{\Gamma \Rightarrow A \qquad A, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\,\text{Cut}$$

that gives the conclusion by repeated weakening. Similarly if the second R$\vee$ rule was used. Length of cut formula is reduced.

3.3 $D = A \to B$, and the derivation is

$$\dfrac{\dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\,R\to \qquad \dfrac{\Delta \Rightarrow A \qquad B, \Theta \Rightarrow C}{A \to B, \Delta, \Theta \Rightarrow C}\,\text{L}\to}{\Gamma, \Delta, \Theta \Rightarrow C}\,\text{Cut}$$

This is transformed into

$$\dfrac{\Delta \Rightarrow A \qquad \dfrac{A, \Gamma \Rightarrow B \qquad B, \Theta \Rightarrow C}{A, \Gamma, \Theta \Rightarrow C}\,{\scriptstyle \text{Cut}}}{\Gamma, \Delta, \Theta \Rightarrow C}\,{\scriptstyle \text{Cut}}$$

4. Cut formula D is principal in the right premiss only. Counting $\perp\Rightarrow C$ as a zero-premiss left rule with principal formula $\perp$, we have four cases and in each, four subcases according to the left rule used for deriving the left premiss. In all cases, cut is permuted up in the derivation of the left premiss:

4.1 The left premiss has been derived by L$\wedge$, and we have $\Gamma = A \wedge B, \Gamma'$ and the derivation is

$$\dfrac{\dfrac{A, B, \Gamma' \Rightarrow D}{A \wedge B, \Gamma' \Rightarrow D}\,{\scriptstyle \text{L}\wedge} \qquad D, \Delta \Rightarrow C}{A \wedge B, \Gamma', \Delta \Rightarrow C}\,{\scriptstyle \text{Cut}}$$

It is transformed into

$$\dfrac{\dfrac{A, B, \Gamma' \Rightarrow D \qquad D, \Delta \Rightarrow C}{A, B, \Gamma', \Delta \Rightarrow C}\,{\scriptstyle \text{Cut}}}{A \wedge B, \Gamma', \Delta \Rightarrow C}\,{\scriptstyle \text{L}\wedge}$$

4.2 The left premiss is derived by L$\vee$. The transformation is analogous to case 4.1.

4.3 The left premiss is derived by L$\rightarrow$. As for case 4.2.

4.4 If the left premiss is $\perp\Rightarrow D$, we are back to case 1.

5. Cut formula is not principal in the right premiss: The derivation, with a two-premiss rule R and D in the antecedent of the left premiss of rule R, is

$$\dfrac{\Gamma \Rightarrow D \qquad \dfrac{D, \Delta' \Rightarrow C' \qquad \Delta'' \Rightarrow C''}{D, \Delta \Rightarrow C}\,{\scriptstyle \text{R}}}{\Gamma, \Delta \Rightarrow C}\,{\scriptstyle \text{Cut}}$$

This is transformed into

$$\dfrac{\dfrac{\Gamma \Rightarrow D \qquad D, \Delta' \Rightarrow C'}{\Gamma, \Delta' \Rightarrow C'}\,{\scriptstyle \text{Cut}} \qquad \Delta'' \Rightarrow C''}{\Gamma, \Delta \Rightarrow C}\,{\scriptstyle \text{R}}$$

Cut formula is the same but cut-height is reduced by at least one. Other cases of rules are variants of this one.

6. The left premiss is derived by contraction: The derivation

$$\dfrac{\dfrac{A, A, \Gamma \Rightarrow D}{A, \Gamma \Rightarrow D}\,{\scriptstyle \text{Ctr}} \qquad D, \Delta \Rightarrow C}{A, \Gamma, \Delta \Rightarrow C}\,{\scriptstyle \text{Cut}}$$

is transformed into

$$\dfrac{\dfrac{A, A, \Gamma \Rightarrow D \qquad D, \Delta \Rightarrow C}{A, A, \Gamma, \Delta \Rightarrow C}\,\text{Cut}}{A, \Gamma, \Delta \Rightarrow C}\,\text{Ctr}$$

7. The right premiss $D, \Delta \Rightarrow C$ is derived by contraction on D. If the left premiss has been derived by a left rule, cut is permuted up on the left premiss with cut-height diminished as in 4 above. Otherwise the derivation is transformed according to the derivation of the premiss of contraction.

7.1 If premiss of contraction is derived by weakening on D, both weakening and contraction are deleted.

7.2 If neither contraction formula occurrence is active in the rule deriving $D, D, \Delta \Rightarrow C$, the rule and contraction are permuted, except when copies of D come from different premisses: In this case cut is permuted up to the premisses and then the rule is applied, followed by contraction on Γ.

The cases remain in which the contraction formula is principal in the left premiss and in the premiss of contraction, or the latter has been derived by another contraction on D. We trace up the derivation of the right premiss of cut until the rule is not a contraction on D, and find a sequent with n copies of formula D in the antecedent. If D is not principal in the rule concluding this sequent, either 7.1 or 7.2 applies. If it is the former, deleting the weakening and contraction leaves $n-1$ copies of D. If it is the latter, we either permute the rule down through the $n-1$ contractions until it concludes the right premiss of cut, or copies of D come from two premisses. If more than one copy come from a premiss, they are contracted to one, and then cut is permuted up with two cuts of reduced cut-height as result.

If the contraction formula D is principal in the rule concluding the sequent with n copies of D, with $D = A \wedge B$ the rule is L$\wedge$, with $D = A \vee B$ the rule is L$\vee$, and with $D = A \rightarrow B$ the rule is L$\rightarrow$. The left premiss has been derived by a right rule corresponding to the left rule by which D was derived. The cases are:

7.3 If $D = A \wedge B$, we have the derivation

$$\dfrac{\dfrac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \wedge B}\,\text{R}\wedge \qquad \dfrac{\dfrac{A, B, (A \wedge B)^{n-1}, \Theta \Rightarrow C}{A \wedge B, (A \wedge B)^{n-1}, \Theta \Rightarrow C}\,\text{L}\wedge}{A \wedge B, \Theta \Rightarrow C}\,\text{Ctr}^*}{\Gamma, \Delta, \Theta \Rightarrow A \wedge B}\,\text{Cut}$$

Application of the transformation of lemma 4.1 will give a derivation of $A, B, \Theta \Rightarrow C$, and cut is permuted to cuts on shorter formulas as in 3.1 above.

7.4 Next consider the case of $D = A \vee B$, with the premiss of the first contraction on D derived by L$\vee$. This step is

$$\dfrac{A, \Delta' \Rightarrow C \qquad B, \Delta'' \Rightarrow C}{A \vee B, (A \vee B)^{n-1}, \Delta \Rightarrow C}\,{\scriptstyle \mathrm{L}\vee}$$

with $(A \vee B)^{n-1}, \Delta = \Delta', \Delta''$. As before, application of lemma 4.1 gives derivations of $A, \Delta \Rightarrow C$ and $B, \Delta \Rightarrow C$ where contractions have been done on A and B. If the left premiss of cut is derived by R$\vee_{\mathrm{L}}$, cut is permuted to a shorter formula as in case 3.2 above. If the left premiss is derived by the second rule R$\vee_{\mathrm{R}}$, a cut with B is done.

7.5 With $D = A \rightarrow B$ the contraction formula, the step concluding the premiss of the uppermost contraction is

$$\dfrac{\Delta \Rightarrow A \qquad B, \Theta' \Rightarrow C}{A \rightarrow B, (A \rightarrow B)^{n-1}, \Theta \Rightarrow C}\,{\scriptstyle \mathrm{L}\rightarrow}$$

Here $(A \rightarrow B)^{n-1}, \Theta = \Delta, \Theta'$. The $n-1$ copies of formula $A \rightarrow B$ are divided in Δ and Θ' with $\Delta = (A \rightarrow B)^k, \Lambda$ and $\Theta' = (A \rightarrow B)^l, \Theta''$ and $k + l = n - 1$. Each formula in Λ and in Θ'' is also in Θ. The derivation can now be written, with Ctr^n standing for an $n-1$ fold contraction, as

$$\dfrac{\dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}\,{\scriptstyle \mathrm{R}\rightarrow} \qquad \dfrac{\dfrac{(A \rightarrow B)^k, \Lambda \Rightarrow A \qquad B, (A \rightarrow B)^l, \Theta'' \Rightarrow C}{A \rightarrow B, (A \rightarrow B)^{n-1}, \Theta \Rightarrow C}\,{\scriptstyle \mathrm{L}\rightarrow}}{A \rightarrow B, \Theta \Rightarrow C}\,{\scriptstyle \mathrm{Ctr}^n}}{\Gamma, \Theta \Rightarrow C}\,{\scriptstyle \mathrm{Cut}}$$

The transformed derivation, with a $k-1$ fold contraction before the first cut, is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}\,{\scriptstyle \mathrm{R}\rightarrow} \qquad \dfrac{(A \rightarrow B)^k, \Lambda \Rightarrow A}{A \rightarrow B, \Lambda \Rightarrow A}\,{\scriptstyle \mathrm{Ctr}^k}}{\Gamma, \Lambda \Rightarrow A}\,{\scriptstyle \mathrm{Cut}} \qquad A, \Gamma \Rightarrow B}{\Gamma^2, \Lambda \Rightarrow B}\,{\scriptstyle \mathrm{Cut}} \qquad B, \Theta \Rightarrow C}{\Gamma^2, \Lambda, \Theta \Rightarrow C}\,{\scriptstyle \mathrm{Cut}}}{\Gamma, \Theta \Rightarrow C}\,{\scriptstyle \mathrm{Ctr}^*}$$

where $B, \Theta \Rightarrow C$ follows by lemma 4.1 from the right premiss of cut. Since $k \leq n-1$, the first cut has a reduced cut-height. The other two cuts are on shorter formulas, and finally the contractions in the end are justified by the fact that each formula of Λ is a formula of Θ. □

4.3 Hauptsatz for the classical calculus

The rules for the calculus, designated *G0c*, are as follows:

G0c

Logical axiom: $A \Rightarrow A$

Logical Rules:

$$\frac{A,\, B,\, \Gamma \Rightarrow \Delta}{A \wedge B,\, \Gamma \Rightarrow \Delta}\,\text{L}\wedge \qquad \frac{\Gamma \Rightarrow A,\, \Delta \qquad \Gamma' \Rightarrow B,\, \Delta'}{\Gamma,\, \Gamma' \Rightarrow A \wedge B,\, \Delta,\, \Delta'}\,\text{R}\wedge$$

$$\frac{A,\, \Gamma \Rightarrow \Delta \qquad B, \Gamma' \Rightarrow \Delta'}{A \vee B,\, \Gamma,\, \Gamma' \Rightarrow \Delta,\, \Delta'}\,\text{L}\vee \qquad \frac{\Gamma \Rightarrow A,\, B,\, \Delta}{\Gamma \Rightarrow A \vee B,\, \Delta}\,\text{R}\vee$$

$$\frac{\Gamma \Rightarrow A,\, \Delta \qquad B,\, \Gamma' \Rightarrow \Delta'}{A \to B,\, \Gamma,\, \Gamma' \Rightarrow \Delta,\, \Delta'}\,\text{L}\to \qquad \frac{A,\, \Gamma \Rightarrow B,\, \Delta}{\Gamma \Rightarrow A \to B,\, \Delta}\,\text{R}\to$$

$$\frac{}{\bot \Rightarrow \Delta}\,\text{L}\bot$$

$$\frac{A[t/x],\, \Gamma \Rightarrow \Delta}{\forall x A,\, \Gamma, \Rightarrow \Delta}\,\text{L}\forall \qquad \frac{\Gamma \Rightarrow A[y/x],\, \Delta}{\Gamma \Rightarrow \forall x A,\, \Delta}\,\text{R}\forall$$

$$\frac{A[y/x],\, \Gamma \Rightarrow \Delta}{\exists x A,\, \Gamma, \Rightarrow \Delta}\,\text{L}\exists \qquad \frac{\Gamma \Rightarrow A[t/x],\, \Delta}{\Gamma \Rightarrow \exists x A,\, \Delta}\,\text{R}\exists$$

Rules of weakening:

$$\frac{\Gamma \Rightarrow \Delta}{A,\, \Gamma \Rightarrow \Delta}\,\text{LW} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A,\, \Delta}\,\text{RW}$$

Rules of contraction:

$$\frac{A,\, A,\, \Gamma \Rightarrow \Delta}{A,\, \Gamma \Rightarrow \Delta}\,\text{LC} \qquad \frac{\Gamma \Rightarrow A,\, A,\, \Delta}{\Gamma \Rightarrow A,\, \Delta}\,\text{RC}$$

The restrictions in the L∃ and R∀ rules are that y must not occur free in the conclusion.

Lemma 4.3. *(i) If $A \wedge B, \Gamma \Rightarrow \Delta$ is derivable, also $A, B, \Gamma \Rightarrow \Delta$ is derivable.*

(ii) If $\Gamma \Rightarrow \Delta, A \wedge B$ is derivable, also $\Gamma \Rightarrow \Delta, A$ and $\Gamma \Rightarrow \Delta, B$ are derivable.

(iii) If $A \vee B, \Gamma \Rightarrow \Delta$ is derivable, also $A, \Gamma \Rightarrow \Delta$ and $B, \Gamma \Rightarrow \Delta$ are derivable.

(iv) If $\Gamma \Rightarrow \Delta, A \vee B$ is derivable, also $\Gamma \Rightarrow \Delta, A, B$ is derivable.

(v) If $A \to B, \Gamma \Rightarrow \Delta$ *is derivable, also* $B, \Gamma \Rightarrow \Delta$ *is derivable.*

(vi) If $\Gamma \Rightarrow \Delta, A \to B$ *is derivable, also* $A, \Gamma \Rightarrow \Delta, B$ *is derivable.*

Απόδειξη. (i) Similar to lemma 4.1. For (ii), if $A \wedge B$ is obtained by weakening, weaken first with A, then with B. If it is obtained as an axiom, conclude instead $A \wedge B \Rightarrow A$ from $A \Rightarrow A$ by weakening with B and L$\wedge$, and similarly for $A \wedge B \Rightarrow B$. If $A \wedge B$ is introduced by R$\wedge$, apply repeated weakening instead, dually to case (ii) of lemma 4.1. (iii) and (iv) are dual to previous. (v) If $A \to B$ in the antecedent is obtained by weakening, weaken with B on left instead. If $A \to B$ is obtained by an axiom $A \to B \Rightarrow A \to B$, conclude $B \Rightarrow A \to B$ from $B \Rightarrow B$ by left weakening with A followed by R$\to$. If $A \to B$ is obtained by L$\to$, proof is similar to (ii). (vi) If $A \to B$ in the succedent is obtained by right weakening, do left weakening with A and right weakening with B instead. If $A \to B$ is obtained by axiom $A \to B \Rightarrow A \to B$, conclude $A, A \to B \Rightarrow B$ from $A \Rightarrow A$ and $B \Rightarrow B$ by L$\to$ instead. If $A \to B$ is concluded by R$\to$, delete the rule. □

We note that the inversions of *G0c* are not height-preserving. Cut elimination is proved by arguments quite similar to those in theorem 4.2:

Theorem 4.4. *The rule of cut,*

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}$$

is admissible in G0c.

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