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Αξιολογήθηκε από την τριμελή επιτροπή:

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This thesis is devoted to my friend and colleague Christos Spandonidis for all the help and advices that gave me, if it was not him I would not even start my master. Furthuremore, I would like to thank my supervisor for the interesting scientific topics that he taught me and introducing me to chaos, my professor from the Academy Ioannis Karatsompanis because he inspired me to deal with science. And at last but not least I thank my family for all the support all these years and my friend Alexandros for his advices.

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ABSTRACT

In all kind of industry there is a large interest in systems that can isolate energy. These systems are shown to be nonlinear because the linear systems remain constant in time and they lack of interest, they can be defined and analyzed with the theory of nonlinear dynamics and chaos. Systems that converse energy from one kind to another or that isolate energy can applications in many areas, one example is the damping of the energy from waves that crash to ships. In such systems a scientist may have an idea of what he wants to study, but eventually his conclusion will include much more phenomena such as energy transfer and perhaps modify his initial model. Our first try was to study a simplified, elementary model of the honeycomb structure but our preliminary study did not lead us to the desired conclusion, so we changed the model geometricaly. In that model, the new geometry added some extra nonlinearity. The examination of that model showed that there exist nonlinear phenomena. An interesting phenomenon that was observed from the experiment was the transformation of the vertical oscillation of the beam to the rotational of the mass and that the mechanical energy of the beam becomes mechanical of the mass.

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SUMMARY

In this thesis we started from a simplified honeycomb structure to find nonlinearities, but the theoritical investigation led us to a structure with different orientation. This orientation added some nonlinearities to the final model through its geometrical structure. After carring out some primary approximate calculations we found that there may hide an interesting phenomenon. Then we started to make the structure so as to take place the experiment and see how the system would naturaly behave. The examination of the experiment was done from studing the video frame by frame. Our goal was to isolate the energy from one system to another. Moreover we observed another phenomenon, the tranformation of the vertical oscillation of the beam to rotational of an eccentric mass which is the mechanism that transfers energy.

Part I
THEORETICAL BACKGROUND

1 BRIEF HISTORY OF DYNAMICS

A good start to understand the subject of this thesis is to have a clear and general view of the dynamics, this can be done with the presentation of the roots of dynamics. Dynamics is the theory that developed from the man's will to understant the world. As we will see none of the scientists that played a vital role to this theory were neither mathematicians nor phycicists. Our aim is not to present with detail everyones' subject and work, but give you a brief information.

The first person who delt with dynamics was Kepler in 1609 who published the three laws of planetary motion. Later in 1666 Isaak Newton after founded calculus he delt with the gravitational law, planetary motion and classical mechanics.

The third big step in dynamics and as we can say the foundier of dynamical systems was from H.J. Poincare. He studied the three-body problem and introduced the qualitative methods of studing the dynamical systems instead of quantitative with his geometric approach. In a nutshell we could say that instead of asking the exact position of a body in a particular moment, we could ask whether or not the system is stable. Then was the first moment that unofficially he talked about chaos, in which he noted that a deterministic system could exhibit aperiodic behaviour that depends sensitively on the initial conditions, which can lead in unpredictable behaviour.

In 1920 we have an interest in nonlinear dynamics that led in the development of lasers and radars. Later in 1963 with the help of computers, we have Lorenz, who experimented with his equations that led to chaotic motions and strange attractors.

The last breakthrough was from Ruelle and Takens who studied turbulence and chaos, May who connected chaos with logistic maps, Feigenbaum who proposed that different systems go chaotic in the same way, Winfree who introduced nonlinear dynamics in biology and last Mandelbrot who developed the fractals.

Our interest focuses in nonlinear dynamics because there rests interesting phenomena such as the attractors and generally there is a continuous alteration from one condition to another and a small alteration in the initial conditions can lead to a vast change in the solution which may not be unique as in the linear dynamics.

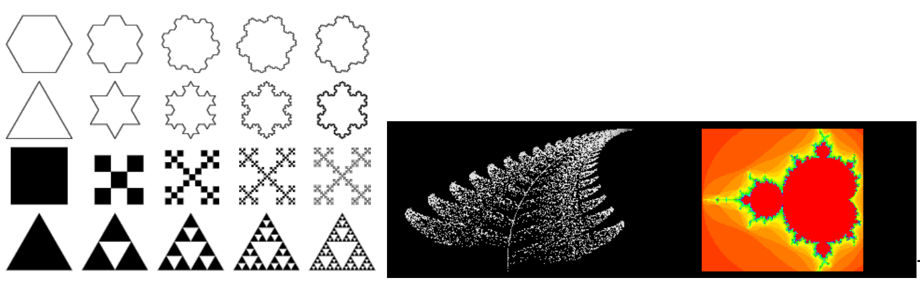
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Figure 1: *The most impressive subject of dynamical systems, the fractals. A fractal is an object that displays self similarity, not nessecarily of the same type, on all scales. In the first picture we can see characteristic examples of the L-systems, Koch's Snowflake and Pascal's Triangle, while in the second we can see the Mandelbrot's fractal.* [7]

2 DYNAMICAL SYSTEMS

A dynamical system is defined as a system that evolves/changes in time. For example consider a stopped car, it is regarded as a dynamical system that is in equillibrium, but if you assume of its engine that continiously combusts the gas and the pistons are moving, then it is a dynamical system that is reapeated in cycles.

Dynamical systems can be classified in two main categories: differential equations (continuous time) and iterated maps (discrete time). The following section explains in a few lines the most important mathematical tool to solve, describe or understand a dynamical system the differential equations. We will not expand in iterated maps because it is a special tool for chaos or where you can regard time as discrete [8].

2.1 DIFFERENTIAL EQUATIONS

Although differential equations (DE) is a subject of mathematics that is assumed to be well known for someone that wants to study dynamical systems, there is always a need to refresh the basics. So I would like to devote this little chapter to everyone that wants to refresh his basic mathematical knowledge and to everyone else that comes from a different field.

A differential equation is an equation that includes an unknown function and its derivatives. If the derivatives are partial or ordinary then the equation is of the same class- partial DE or ordinary DE (there will be only one independed variable). Class of the DE is the greatest derivative that appears in the DE. Solution of a DE with unknown function y and independed variable x is the function $y\left(x\right)$ that satisfies the equation [6].

$\frac{dy}{dx} = 3x + 2$	ODE, class 1
$\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial z^2} = 1$	PDE, class 2

Table 1: *The two main characteristic categories of the differential equations.*

The modeling of the experiment uses the second law of Newton to describe it and leads to a system of differential equations with equations the position in the 2D space of the masses and independed variable the time.

Another basic categorization of the DE is linearity. This characteristic is discribed later in chapter 3.

2.1.1 MATHEMATICA

In this section I think is the best to introduce the Mathematica's commands that are used to solve the DE.

First of all we have to define the equations or the system of equations that you want to be solved, then solve the equation/s. After those, the most common options that someone have in order to visualize the solution is to make the plot of the function against the time and the phase space (plot that has axes the functions in respect to time).

$$\text{system} = \{\text{here you write the equations}\};$$

$$\text{solution} = \text{NDSolve}[\{\text{system}, \text{initial\_conditions}\}, \{\text{functions that want to be solved}\}, \{\text{time range}\}];$$

$$\text{Plot}[\text{function1}/.\text{solution}, \{\text{time}\}]$$

$$\text{ParametricPlot}[\{\text{function1}, \text{function2}\}/.\text{solution}, \{\text{time}\}]$$

3 NONLINEAR DYNAMICS

Another classification of differential equations and also of dynamical systems is linearity and nonlinearity. The easiest way to classify them is the mathematical definition of the differential equation that describes the dynamical system. If the equation involves only linear terms then the phenomenon is linear, else if it contains nonlinear terms such as trigonometrical, exponential etc. functions, it is nonlinear. To have a better and more qualitative view of the above we can examine the equations in respect of their frequencies through their initial conditions.

In a linear system:

$$\ddot{x}(t) + \omega^2 x(t) = 0 \tag{1}$$

the frequency (ω) remains constant and independed of the initial conditions and this can be shown in the following phase space and the Fast Fourier Transform that gives us a constant frequency for every initial condition.

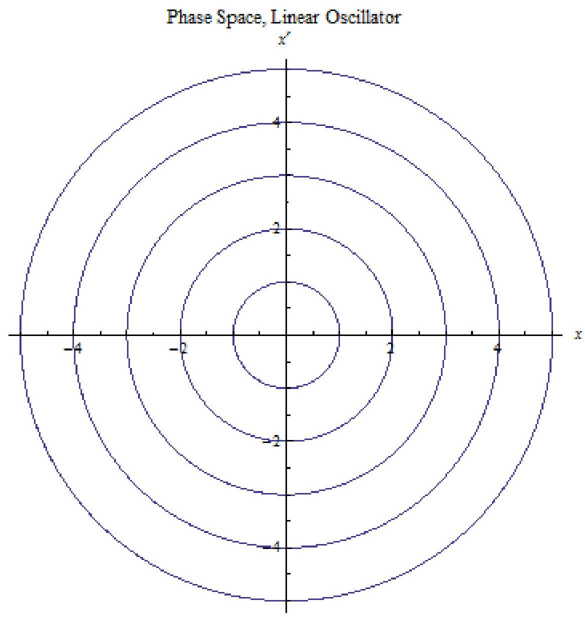

Figure 2: *Phase space of a linear oscillator with initial conditions:* $x(0) = 1, 2, 3, 4, 5$, $\dot{x}(0) = 0$ *and angular velocity:* $\omega = 1$.

Furthermore an another explanation that can be exluded from the equation 1 is the following: "Given a value to the angular velocity (or frequency $\omega = \frac{1}{f}$), then $x(0)$ and $\ddot{x}(0)$ must have an propotional values".

In a nonlinear system:

$$\ddot{x}(t) + ax(t)^3 = 0 \tag{2}$$

the frequency (a) does not remain constant and depends on the initial conditions. For the above instance, for $a = 1$, as $x(0)$ reaches from 1 to 0, so slowly the acceleration of the system changes $(\ddot{x}(t))$, and as $x(0)$ becomes greater than 1, so faster the acceleration increases. The coresponding phase space and Fast Fourier Transform are shown below.

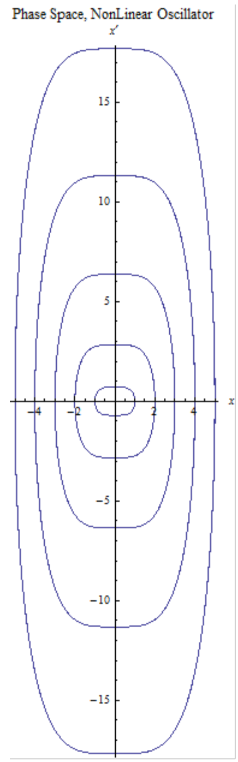

Figure 3: *Phase space of the nonlinear oscillator: 2 with initial conditions:* $x\,(0) = 1, 2, 3, 4, 5$, $\dot{x}\,(0) = 0$ *and constant* $a = 1$.

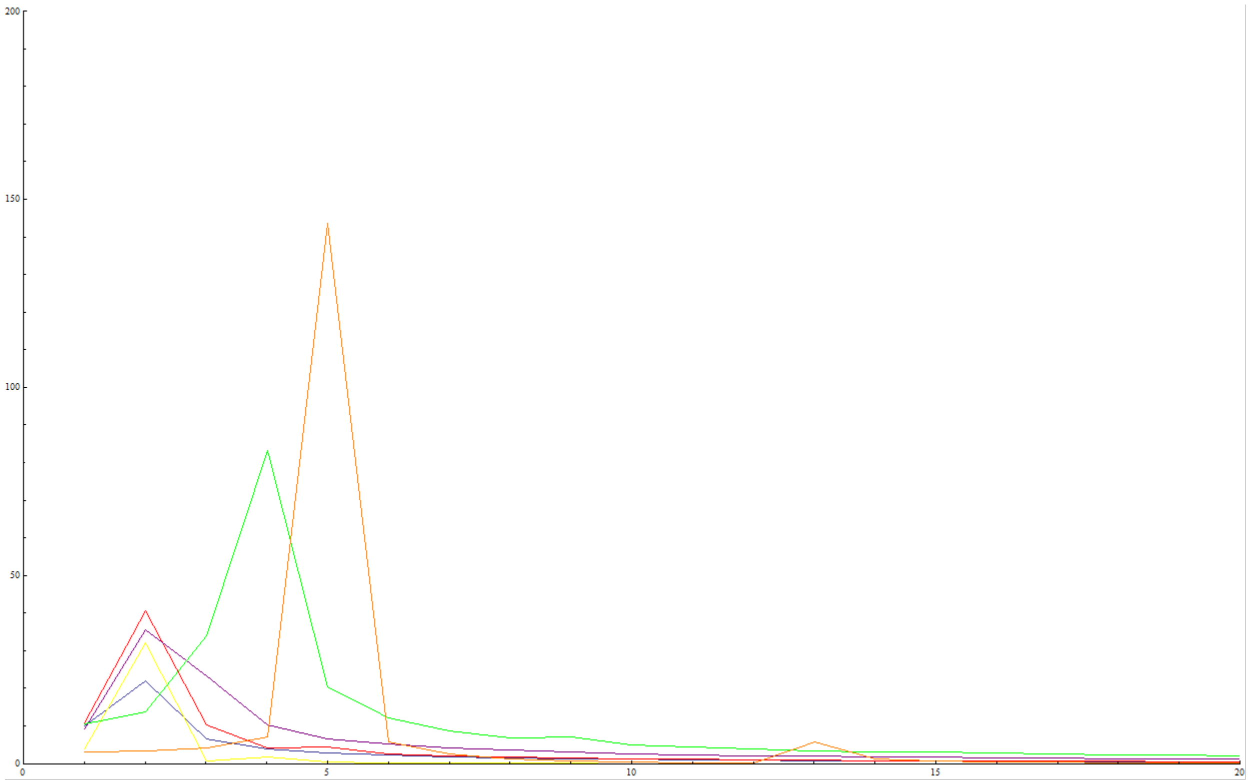

Figure 4: *Fast Fourier Transform for the nonlinear system 2 with initial conditions:* $x(0) = 0.5, 0.7, 0.9, 1, 2, 3$ *(blue, yellow, red, purple, green and orange respectively),* $\dot{x}(0) = 0$ *and constant* $a = 1$*. We notice that in the area of the initial conditions* $(0, 1)$ *the frequency remains constant, but for values greater than* 1 *the frequency becomes greater and greater and there appears to be nonlinearities with the presence of a second frequency that participates in the phenomenon. For* $x(0) = 3$*- orange- it becomes clear that a second frequency rises up, and as we diverge from* 1 *it contributes to the system. In contrast, in a linear system regardless of the initial conditions, you can observe only one frequency.*

4 CHAOS

In this section one could write hundrends of pages. I will limit it only to the very basics and important. The question that arises is what is chaos? Chaos is an aperiodic long term behavior in a deterministic system that exhibits sensitive dependence on initial conditions [8] and it is compination of randomness and structure [2]. Chaos theory focuses on nonlinear systems.

The term deterministic means that although we know the rules that a system relies on, e.g. it's equation, we cannot predict it's long term behaviour. The example that follows comes from cellular automata.

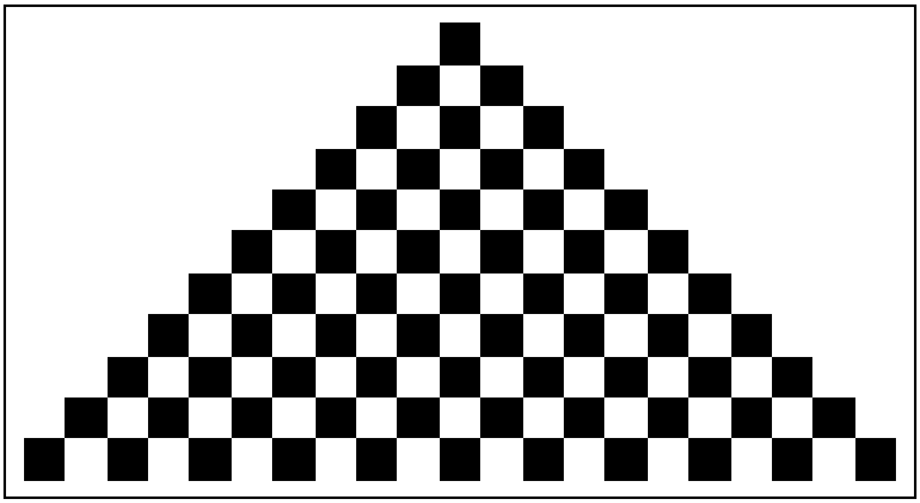

Figure 5: *Cellular Automaton rule 50, 10 steps of evolution. We know the rules of its evolution so it is a deterministic system. Whatever the initial conditions are, the evolution of the system is the same and we can predict that if we start with an empty array then no evolution at all happens, but if we have one or more black cells, the longterm evolution is the same. The rule of evolution is that a cell becomes white if both of its neighbours are white, otherwise it becomes black.*

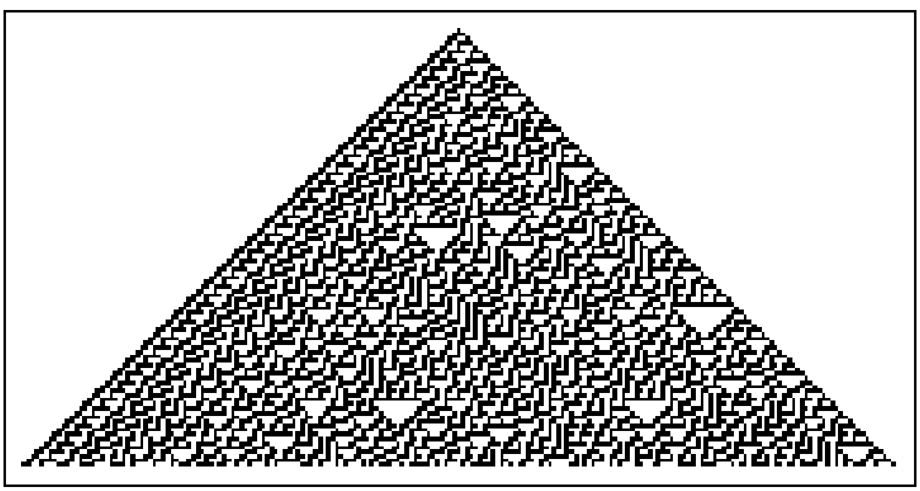

Figure 6: *Cellular Automaton rule 30, 100 steps of evolution. Again we deal with a deterministic system, we can predict the next step, but we cannot predict the long term behaviour of the system, whatever the initial conditions are. Thes rules of evolution are presented in the next table.*

111	110	101	100	011	010	001	000
0	0	0	1	1	1	1	0

Table 2: *The rules of evolution of the Cellular Automaton 30.*

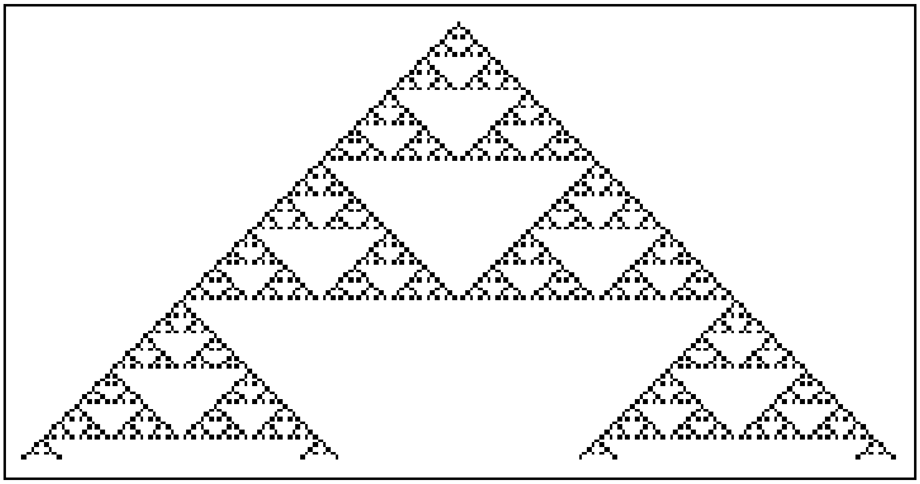

Figure 7: *Cellular Automaton rule 90, 100 steps of evolution. A cell becomes black when only one of the neighbours was black.*

In our experiment the system meets the requirements of being chaotic, it is deterministic in the sense that we know the rules that governs the system, it is sensitive to initial condition (a slight change in the initial displacement leads to different number of oscillations of the beam).

Another way for someone to understand what chaos theory deals with is through examples. I will give only two characteristic examples, if someone want to go deeper can find numerous.

4.1 COASTLINE OF BRITAIN

Benoit Mandelbrot wondered "How long is the coastline of Britain?". Someone could think that knows the answer from just looking a map. The problem is that the map is not so accurate and it is hard to measure so little distances. For now imagine that you have a really accurate map and you want to make the measures. Firstly you approximate Britain with a triangle, so you have a first approximation. Then you want to be more accurate, so you draw little "vortices" that correspond to the coves, so the perimeter becomes a little bit greater. As the details increase more and more, so the perimeter increases. Imagine that you add vortices that corresponds to the complicated shape of the beaches and other objects. As a result the perimeter extents to infinitum. That is a fractal .

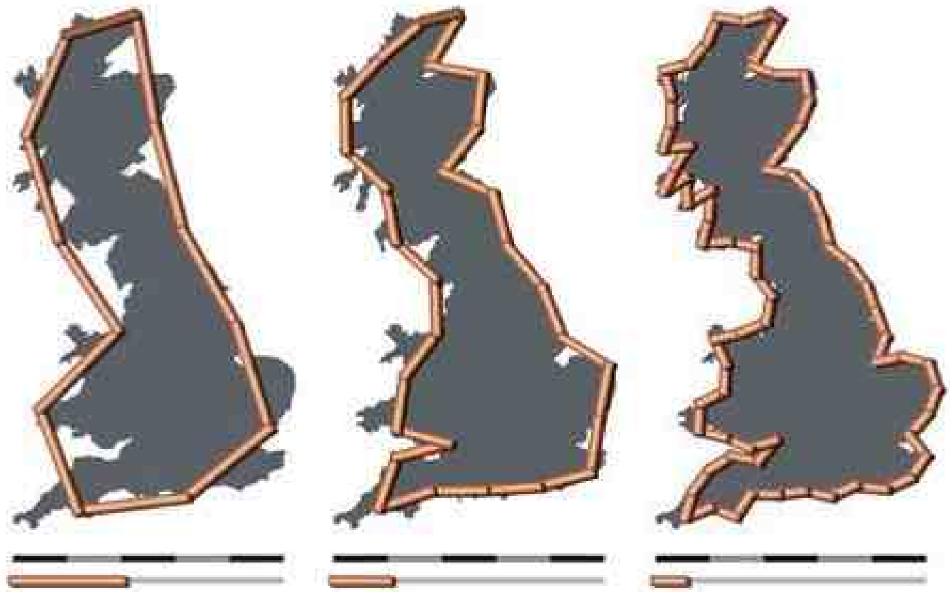

Figure 8: *A representation of the problem described above. At first we start with straight polygonal lines, but as we move up to more details we add "vortexes", so the perimeter continuously increases to infinum.*

4.2 CHAOTIC WATERWHEEL

In 1970 in MIT, Willem Malkus and Lou Howard invented a mechanical analog to the Lorenz equations, from which a simplest model will be described in this chapter. Imagine a waterwheel that in its perimeter has cups with bottom holes. Water is running to the waterwheel and the cups are filling. If the water flow is too slow then the cups are not filled enough to overcome the friction and they empty. If the flow rate increases then the wheel rotates in one direction, and if the flow icreases more, the rotation becomes chaotic. You can observe the wheel making 2 right turns, then 4 left turns etc in an unpredictable way.

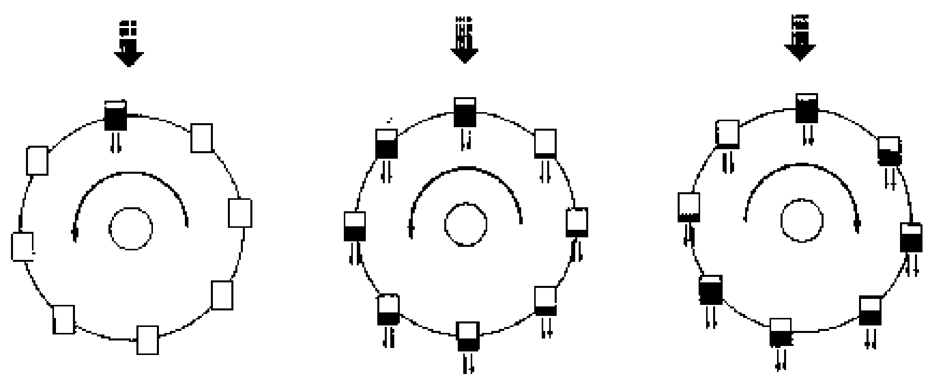

Figure 9: *The chaotic waterwheel. As we increase the water flow, so the system changes it's status from stationary, to equillibrium (constant rotation) and then to chaos (some turns right, some left etc.).*

At last I would like to mention Rudy Rucker's view of the world, who distinguishes chaos in everyday life and as a characteristic example of chaos patterns in Nature we can see the shells of the cones that appear in the next photo.[4]

Figure 10: *In this shell we see the "gnarly" patterns that came by the Nature's computations. Compare this picture with Figure 7, cellular automaton rule 30.*

In the experiment's section we make clear that in our approach we only care about the oscilaltions of the beam until the mass changes direction of rotation. This is the first stage of the phenomenon and this separation is made because it is the first phase of energy transition from one subsystem to another. In a further study we will correlate the oscillations that come after as it seems possible to contain behavior like the chaotic waterwheel.

5 PHASE SPACE

Nonlinear Differential Equations are hard to solve analytically, e.g.

$$\ddot{x} + x + \sin x = 0 \tag{3}$$

The method that is commonly used to solve that type of equations is the assumption $x \approx \sin x$ for small x. But in that way we focus only on a pretty small region of the phenomenon. The concept that introduced from Poincare is to study the evolution of a trajectory $(x, \dot{x})$. The idea is to start with a pair that serves as an initial condition, in this case the initial condition is the velocity and the position and then solve the equation with numerical methods. Then if we draw all the possible values that corresponds to the initial conditions (solutions of the DE) we can see that it is plotted a curve but the trajectories never intersect each other. For our example the phase space in different intitial conditions is shown below.

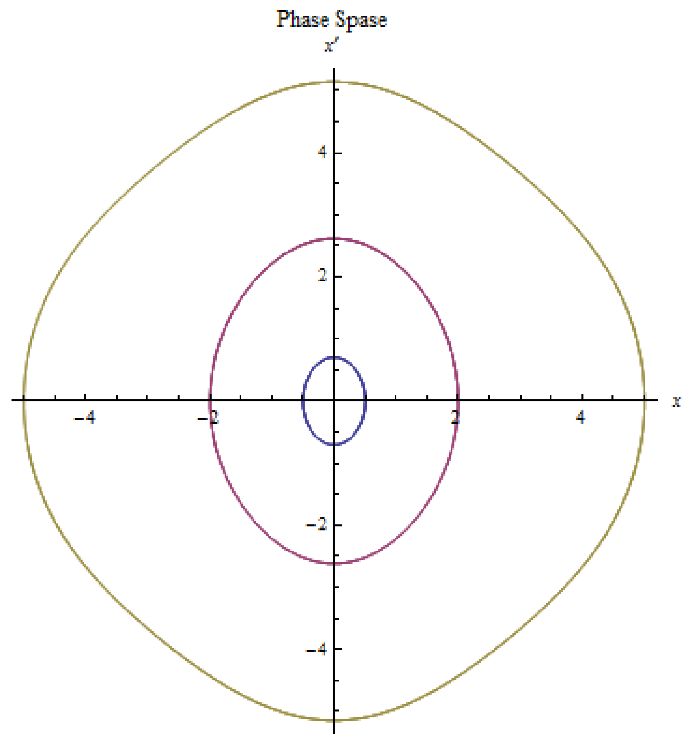

Figure 11: *Phase space of the equation 3 with initial conditions* 0.5 *(blue),* 0.2 *(purple) and* 5 *(yellow).*

6 FOURIER TRANSFORM

Fourier Tranform is a method that helps us find the frequency or frequencies of a system and how much every frequency contributes to the phenomenon. A Fourier diagram (Figure 4) is analyzed as follows, a pick in the horizontal axes represents the frequency whose height is its value, and the width of the curve near the pick is how much this frequency contributes to the system.

A brief explanation about Fourier Transform (FT) is that it analyzes a function in a sum of sinusoids. It can also deal with non-periodic phenomena and reveal periodicities. The definition of FT is:

$$f(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i \omega t} dt \tag{4}$$

From the definition we can conclude that FT is a transform from the time domain to the frequency domain.

6.1 DISCRETE FOURIER TRANSFORM

The Discrete Fourier Transform (DFT) is a generalization of FT in the case of discrete functions, the correlations that we have are the following: $f(t) \to f(t_k) \equiv f_k$ where $t_k = k\Delta$, which is the time interval. The definition is:

$$f(n) = \sum_{k=0}^{N-1} f_k e^{-2\pi i n k/N} \tag{5}$$

Until now nothing new happens, the definition is the same, except that we have transformed the integral to sum. The Descrete Fourier Transform is commonly used in the modeling of phenomena in

PCs, that all the calculations are descrete, through the principals of their operation. So a continuous mathematical operation is treated as descrete in a PC.

From the explanation of FT, we can additionaly say it can reveal periodicities as we can obtain them from each sine component. This is a very good tool because equations or time series of a system we cannot reveal the frequencies that play a vital role to the whole system. Let me give you an example here, consider a complex system, such as a car. We see that on this car, the windows of the front seats are vibrating continiously, so we put accelerometers (to the window) to exclude the time series of the vibration of the window. Till now we cannot figure out what really happens. After we apply the DFT and see that the window is vibrating with the frequency of the engine. So the problem is now defined, we should isolate the engine's vibration.

6.2 FAST FOURIER TRANSFORM

The Fast Fourier Transform (FFT) is an algorithm of computing the Descrete Fourier Transform fast, as you can imply from its name. In the examples that will follow we used the built-in Mathematica's code for computing the FFT as follows:

```
a1 = ListLinePlot[Abs[Fourier[Transpose[Table[y1[t]/.sol, {t, 0, 10, 0.001}]]]],
        PlotRange → {{0, 20}, {0, 300}}, AxesLabel → {ω, coef}, PlotStyle→Blue];
        a2 = ListLinePlot[Abs[Fourier[Transpose[Table[y2[t]/.sol, {t, 0, 10, 0.001}]]]],
        PlotRange → {{0, 20}, {0, 300}}, AxesLabel → {ω, coef}, PlotStyle → Red];
                                   Show[a1, a2, PlotLabel → Fast Fourier Transform]
```

7 LINEARIZATION

In order to simplify and linearize a nonlinear system we use the Taylor Series expansion. In a single variable model we have the following explanation:

Consider the function $f(x)$ with single variable x, we will linearize it in the equilibrium point $x = \bar{x}$, where $f(\bar{x}) = 0$. The Taylor Series expansion of $f(x)$ around the point $\bar{x}$ is given by:

$$f(x) = f(\bar{x}) + \frac{df}{dx}_{x=\bar{x}} (x - \bar{x}) + \frac{1}{2}\frac{d^2 f}{dx^2} |_{x=\bar{x}} (x - \bar{x})^2 + \frac{1}{3!}\frac{d^3 f}{dx^3} |_{x=\bar{x}} (x - \bar{x})^3 + ... \tag{6}$$

We ignore the higher order terms and since the $\bar{x}$ is the equilibrium point, we have:

$$f(x) = \dot{x} = \frac{df}{dx} |_{x=\bar{x}} (x - \bar{x}) = const\,(x - \bar{x})$$

In a multiviriable system of the form $\dot{x}_i = f_i(x_1, x_2, ..., x_n, u_{1,} u_2, ..., u_m)$ the equilibrium points are: $\bar{x}_1, \bar{x}_2, ..., \bar{x}_n, \bar{u}_{1,}\bar{u}_2, ..., \bar{u}_m$, so $f_i(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n, \bar{u}_{1,}\bar{u}_2, ..., \bar{u}_m) = 0$. Now the linearized model is:

$$f_i(x_1, x_2, ..., x_n, u_{1,} u_2, ..., u_m) \simeq \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} |_{x_j=\bar{x}_j} (x_j - \bar{x}_j) + \sum_{j=1}^{m} \frac{\partial f_i}{\partial u_j} |_{u_j=\bar{u}_j} (u_j - \bar{u}_j)$$

Part II
MODELING

8 INTRODUCTION

The purpose of this thesis was initialy to study a simpler case of composite materials with sandwich structure. After studing a simplified model we did not find any interesting phenomena and modified it. The models and the experiment that took place deals with beams and springs. We started by studying very simple models so as to understand the last model that we were about to use. Our aim was to see if there are phenomena such as transfering of energy.

All models have the same structure of analysis:

1. the differential equation,
2. the phase space,
3. Fast Fourier Transform and
4. Taylor linearization.

9 VERTICAL OSCILLATION OF A MASS WITH HORIZONTAL SPRINGS

The first and simplest case that we have to study to understand the motion of a mass coupled with springs is an arrengement that consists of a mass bounded to move in the vertical axes, coupled with two springs in the horizontal axes. The other ends of the springs are motionless. The whole system has one degree of freedom.

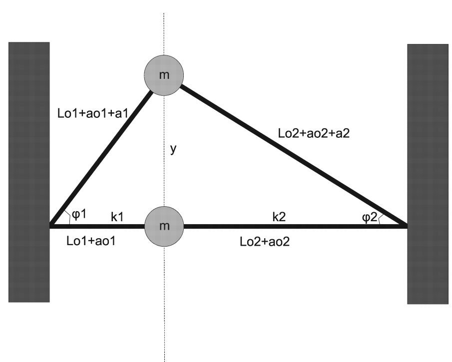

Figure 12: *This figure shows a mass attached with two springs or rubbers that has one degree of freedom and oscillates verticaly. The dimensions and springs'/rubbers' elasticity are shown above. The constants α_{01}, α_{02} are the initial displacements of the springs and α_1, $\alpha_2$is the elongation of each spring.*

The initial displacement means that in the horizontal plane where $y = 0$ the springs has not their neutral lenght, but are preloaded, either extended or contracted.

The equation of motion of the mass is the following:

$$m\ddot{y}(t) = -k_1 \left(y(t) - \frac{L_{01} y(t)}{\sqrt{(L_{01} + \alpha_{01})^2 + y(t)^2}} \right) - k_2 \left(y(t) - \frac{L_{02} y(t)}{\sqrt{(L_{02} + \alpha_{02})^2 + y(t)^2}} \right) \quad (7)$$

The change of the position of the mass in respect to time is the following:

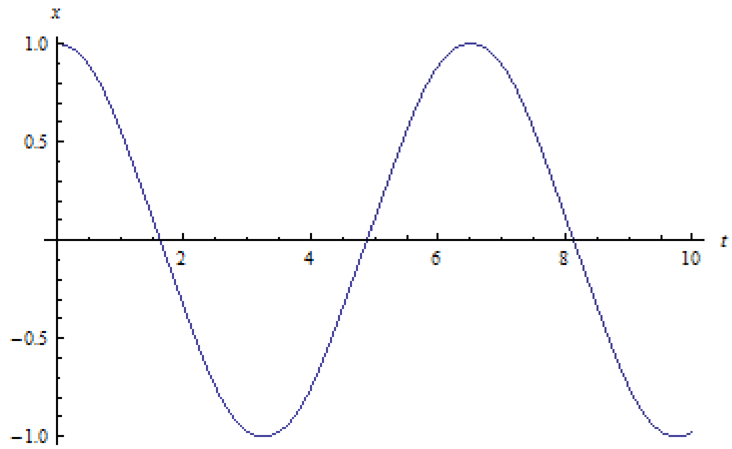

Figure 13: *The phenomenon is sinusoidal with constant period.*

Independently of the initial conditions, the period of the phenomenon remains constant, so the frequency remains constant as it is linear. We can confirm the above with the help of the Fast Fourier Transform (FFT), where we observe only one frequency.

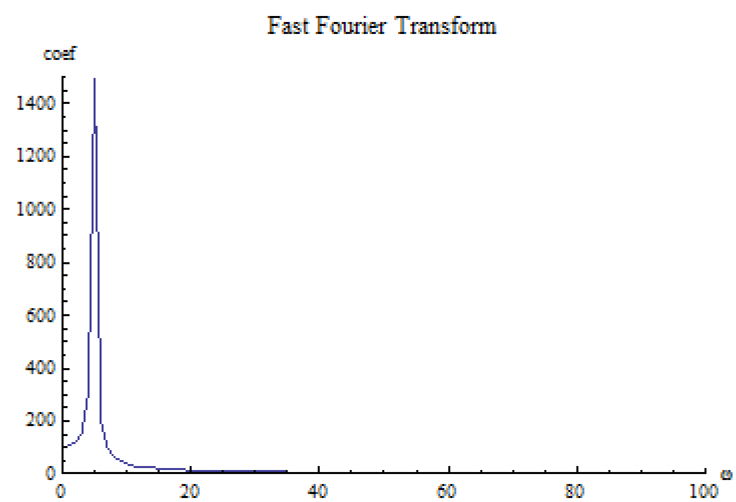

Figure 14: *The above profile of the Fast Fourier Transform remains constant, independed of the initial conditions.*

9.1 PHASE SPACE

The phase space constitutes of trajectories that have points $(y, \ddot{y})$. To plot the graph we solve the 7 for fixed y. As we see the equation of motion, we observe two constants: α_{01} and α_{02}, so the phase space changes as we change the value of these constants.

In our numerical experiment we let the values of the constants lay in the space $[-1, 1]$. Negative value means that we have a spring which is compressed and positive value means that it is streched.

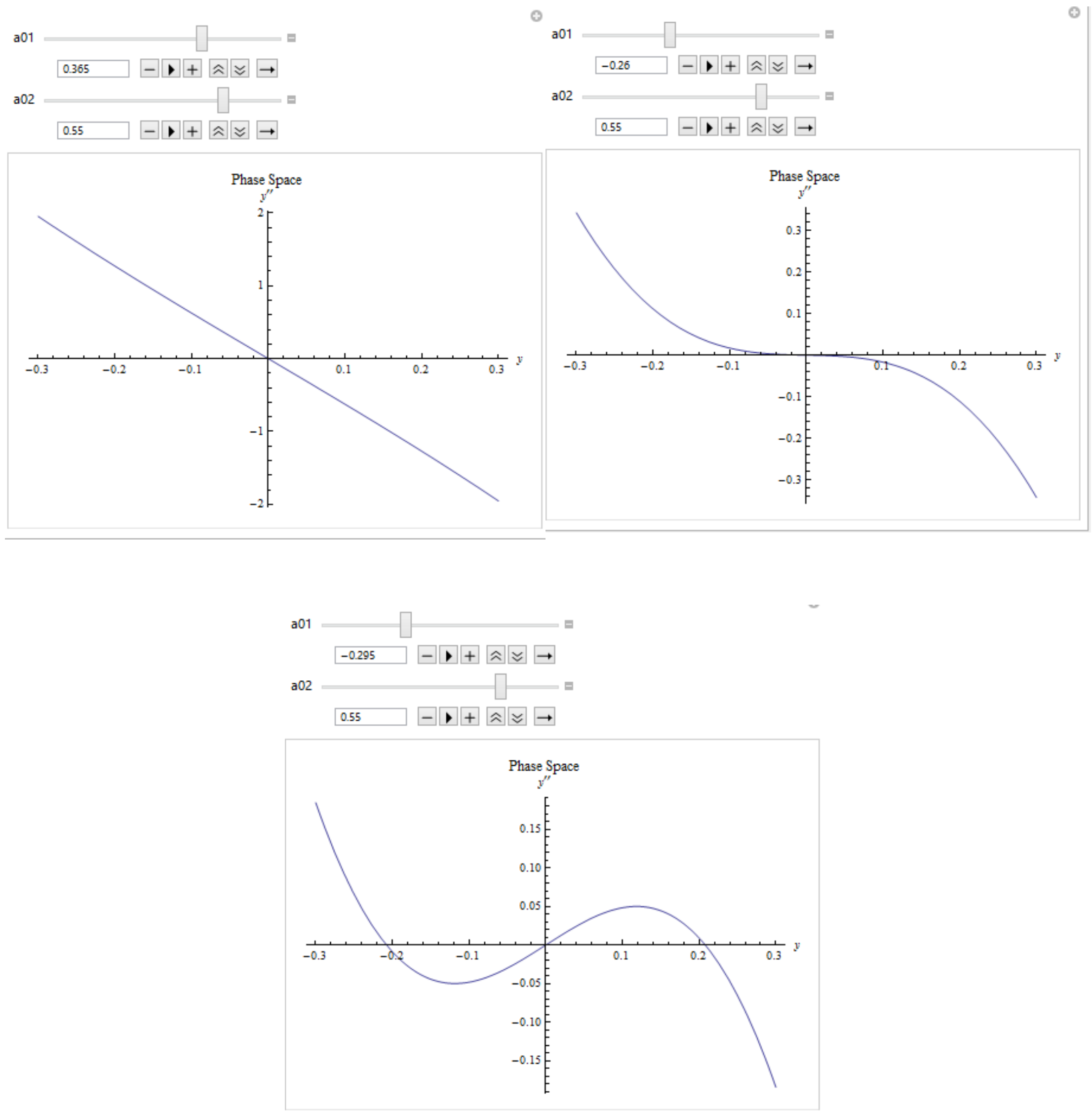

Figure 15: *The phase space of the system. The values of the parameters are shown in each picture.*

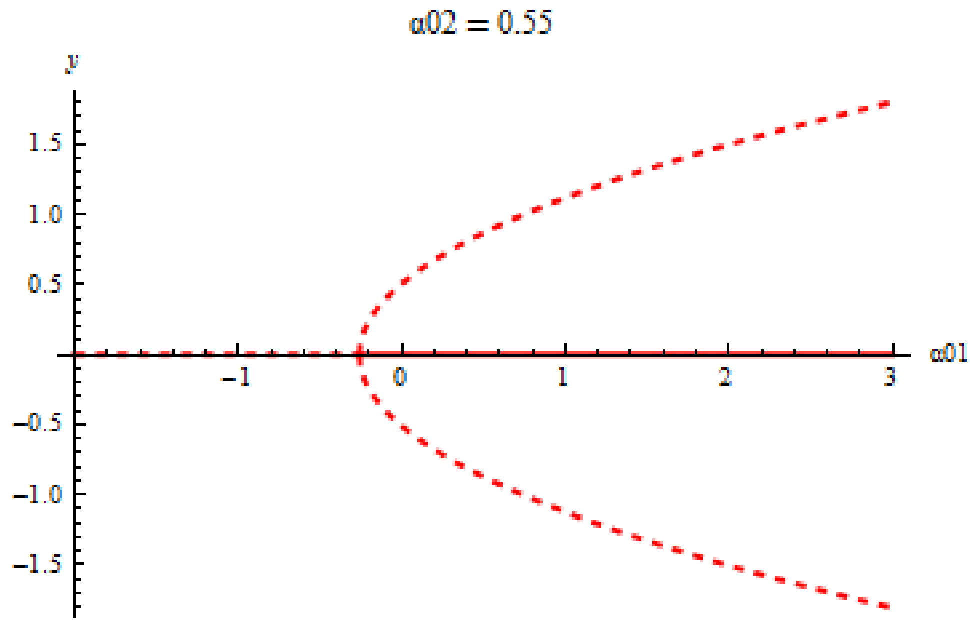

Figure 16: *Subcritical pitchfork bifurcation. For constant α_{02} ,and α_{01} changes values we observe that one unstable fixed point changes to 2 unstable fixed points and one stable. If we had a rubber instead of a spring we could not see that behavior because it would not be able to take negative values.*

9.2 LINEARIZATION

In this chapter is introduced the method that is used for linearizing through Taylor's expansion. In this example we linearize the equation 7 around $y = 0$. The purpose of this method is to find how the system behaves if we extinguish the nonlinear term. In some cases such as a linear spring, which is ideal, we can see how it could act if it was nonlinear by adding exponentials. The 7 is converted:

$$
\begin{aligned}
m\ddot{y}(t) = &-k_1\left(y(t) - \frac{L_{01}y(t)}{L_{01}+\alpha_{01}} + \frac{L_{01}y(t)^3}{2(L_{01}+\alpha_{01})^3} - \frac{3L_{01}y(t)^5}{8(L_{01}+\alpha_{01})^5}\right) \\
&-k_2\left(y(t) - \frac{L_{02}y(t)}{L_{02}+\alpha_{02}} + \frac{L_{02}y(t)^3}{2(L_{02}+\alpha_{02})^3} - \frac{3L_{02}y(t)^5}{8(L_{02}+\alpha_{02})^5}\right)
\end{aligned} \tag{8}
$$

If the preload is $\alpha_{01} = 0$ then the linear term is 0 and so remain only the nonlinear terms.

From the solution of the above equation we observe that the profile of the transpose versus time remains constant, in parallel the frequency remains constant, which is verified by the Fast Fourier Transform.

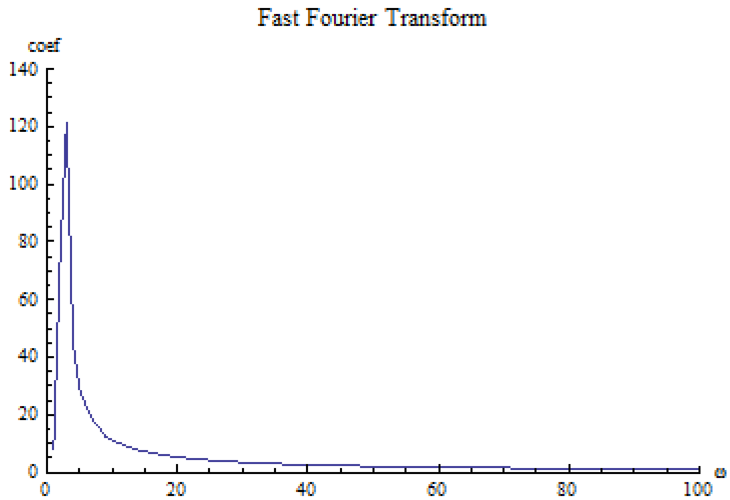

Figure 17: *The frequency remains constant. The only difference with the Figure 5 is the width, in other words the energy. It is inevitable because we changed the original equation.*

10 NONLINEAR OSCILLATOR

The next example is of a nonlinear oscillator. In simple words we have a mass that is coupled with 2 horizontal beams with 4 springs in "X" shape. One beam is motionless and the other is coupled with one vertical spring in a wall. The nonlinearity lies in the springs' positions, they have been installed with angle with each other.

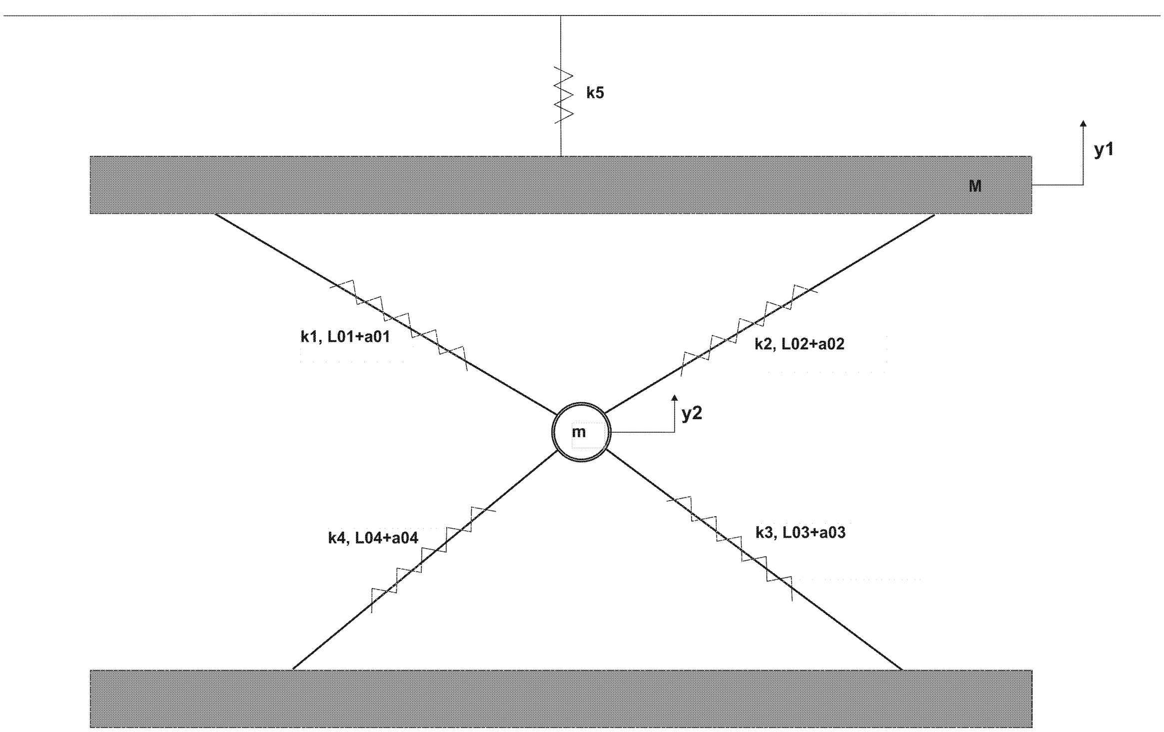

Figure 18: *Nonlinear oscillator system arrengement. The nonlinearity is geometrical.*

From the second law of Newton, in a random position we have:

$$m\ddot{y}_2(t) = F_{1y} + F_{2y} - F_{3y} - F_{4y} \tag{9}$$

$$m\ddot{y}_1(t) = -F_5 - F_{1y} - F_{2y} \tag{10}$$

The forces that are developed from the springs are:

$F_1 = k_1(\alpha_{01} - \alpha_1)$
$F_2 = k_2(\alpha_{02} - \alpha_2)$
$F_3 = k_3(\alpha_{03} - \alpha_3)$
$F_4 = k_4(\alpha_{04} - \alpha_4)$
$F_5 = k_5 y_1$

Table 3: *The forces that are implemented from the springs to the masses, where α_{0i} is the preload of each spring and $\alpha_i$is the deformation of the spring from its initial position.*

10.1 SPRINGS PRELOAD

We consider that the springs are not the same (different k, length), so the symmetrical middle of the beams is not the rest position of the mass. If we consider the middle of the beams as a reference position, then the springs will be deformated about $\alpha_{0i},\ i = 1, ..., 4$. This is the initial preload.

Next, we move the mass to y_2 and the beam to y_1. Then the springs will have a total deformation of $\alpha_{0i} + \alpha_i,\ i = 1, ..., 4$. The reason why we want to use the preload is because we aim to make small perturbances to our model. As you will see to the following computations, the preload is not explicitly contained.

In the following table are presented the geometrical characteristics of the model.

$\sin\phi_i = \frac{r_1 - y_2 + y_1}{L_{oi} + \alpha_{0i} - \alpha_i},\ i = 1, 2$	$\sin\phi_j = \frac{r_2 + y_2}{L_{0j} + \alpha_{0j} + \alpha_j}, j = 3, 4$
$(L_{oi} + \alpha_{0i} - \alpha_i)^2 = (r_1 - y_2 + y_1)^2 + c_i^2, i = 1, 2$	$(L_{0j} + \alpha_{0j} + \alpha_j)^2 = (r_2 + y_2)^2 + c_j^2, j = 3, 4$

Table 4: *Geometrical characteristics of the system, in the first row we have the sines and in the second the Pythagorean theorem.*

The system of the equations 9 and 10 after the substitution of the geometrical characteristics is:

$$\begin{aligned} m\ddot{y}_2(t) = k_1 &\left(r_1 - y_2(t) + y_1(t) - L_{01} \frac{r_1 - y_2(t) + y_1(t)}{\sqrt{(r_1 - y_2(t) + y_1(t))^2 + c_1^2}} \right) \\ &+ k_2 \left(r_1 - y_2(t) + y_1(t) - L_{02} \frac{r_1 - y_2(t) + y_1(t)}{\sqrt{(r_1 - y_2(t) + y_1(t))^2 + c_2^2}} \right) \\ &- k_3 \left(r_2 + y_2(t) - L_{03} \frac{r_2 + y_2(t)}{\sqrt{(r_2 + y_2(t))^2 + c_3^2}} \right) \\ &- k_4 \left(r_2 + y_2(t) - L_{04} \frac{r_2 + y_2(t)}{\sqrt{(r_2 + y_2(t))^2 + c_4^2}} \right) \end{aligned} \tag{11}$$

$$M\ddot{y}_1(t) = k_5 y_1(t) - k_1\left(r_1 - y_2(t) + y_1(t) - L_{01}\frac{r_1 - y_2(t) + y_1(t)}{\sqrt{(r_1 - y_2(t) + y_1(t))^2 + c_1^2}}\right)$$
$$- k_2\left(r_1 - y_2(t) + y_1(t) - L_{02}\frac{r_1 - y_2(t) + y_1(t)}{\sqrt{(r_1 - y_2(t) + y_1(t))^2 + c_2^2}}\right) \tag{12}$$

In the next figures are presented the diagrams that arise from the system of equations 11 and 12.

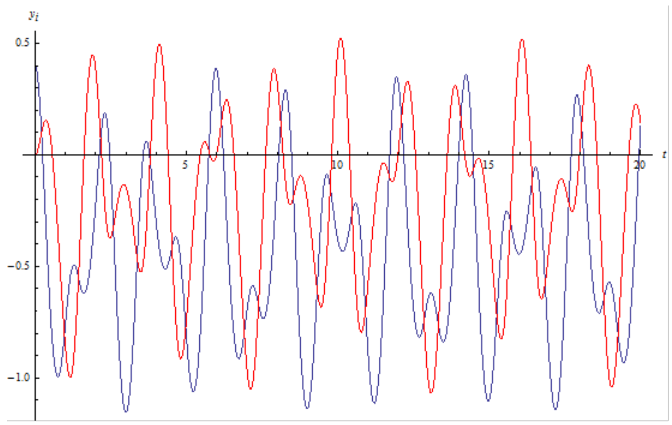

Figure 19: *Time evolution of* y_1 *(blue) and* y_2 *(red) for initial condition* $y_1(0) = 0.4$ *and* $y_2(0) = 0$.

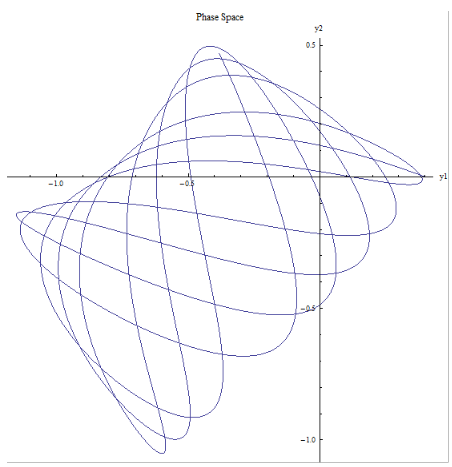

Figure 20: *Phase space for time* $t = 10$ *(time credits).*

At last the above model does not seem to have the desired behaviour, so it does not worth further study, neither theoritical nor experimental from our point of view. That is because the oscillation of the springs is bounded in only one plane and each two of the springs anihillate each other. Moreover, the preload is “lost” due to the Pythagorean theorem, so we cannot make little disturbances in the dynamics of the system. Our examinations does not conclude that this model is not suitable for further investigation.

11 VERTICAL OSCILLATION OF MASSES AND TRANSFER OF ENERGY

The last configuration that seems to have a more complex behavior is the following, because the springs are not in the same direction with the motion of the mass and the beams. We can model the beams as masses that are oscillating in the respective axes.

11.1 VERTICAL OSCILLATIONS

The model that will be studied is the one from the following figure.

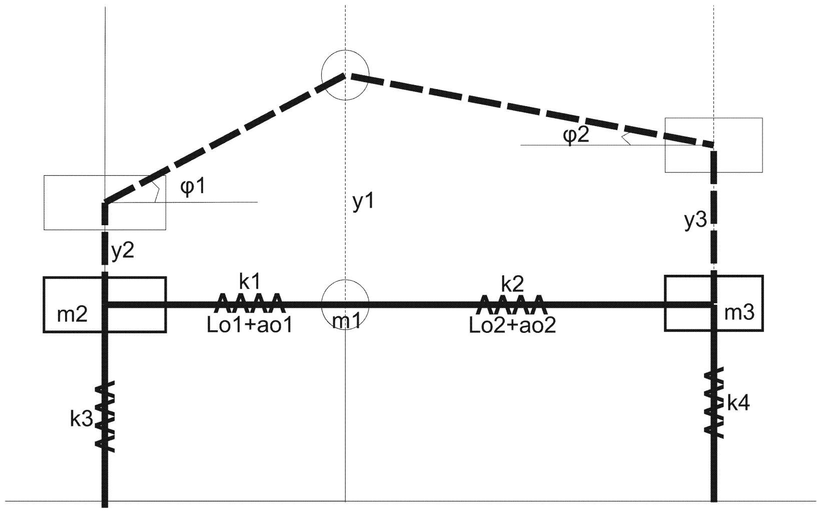

Figure 21: *In this figure we see the mass m_1 connected with springs or rubber to the masses m_2 and m_3. The masses m_2 and m_3 allong with the springs k_3 and k_4 can be thought as 2 beams that oscillate verticaly. Here is presented the starting and a random position of the system (dashed lines). The shift from the starting position to an another is measured by y_1, y_2 and y_3.*

In a random position the equations of motion extracted from the second law of Newton are:

$$\begin{aligned} m_1\ddot{y}_1(t) &= -F_{1y} - F_{2y} \\ m_2\ddot{y}_2(t) &= F_{1y} - F_3 \\ m_3\ddot{y}_3(t) &= F_{2y} - F_4 \end{aligned} \tag{13}$$

The forces that are applied to the masses for a random displacement of the masses are the following:

$F_1 = k_1(\alpha_{01} + \alpha_1)$
$F_2 = k_2(\alpha_{02} + \alpha_2)$
$F_3 = k_3(\alpha_{03} + \alpha_3)$
$F_4 = k_4(\alpha_{04} + \alpha_4)$

Table 5: *Forces from the springs.*

The spring 3 is displaced by y_2 in the vertical axes, the spring 4 by y_3, the spring 1 by $y_1 - y_2$ and the spring 2 by $y_1 - y_3$.

Only the springs number 1 and 2 have geometrical variations (angle) due to the springs' axes changes direction. As we see from a random position of the system a geometrical nonlinearity is introduced through the sinus terms that are analyzed below.

$\sin\phi_1 = \frac{y_1 - y_2}{L_{01}+\alpha_{01}+\alpha_1}$	$\sin\phi_2 = \frac{y_1 - y_3}{L_{02}+\alpha_{02}+\alpha_2}$
$L_{01}+\alpha_{01}+\alpha_1 = \sqrt{\left(L_{01}+\alpha_{01}\right)^2 + \left(y_1 - y_2\right)^2}$	$L_{02}+\alpha_{02}+\alpha_2 = \sqrt{\left(L_{02}+\alpha_{02}\right)^2 + \left(y_1 - y_3\right)^2}$

Table 6: *Geometrical characteristics of the springs 1 and 2.*

At last, the equations of motion of the system are:

$$\begin{aligned} m_1\ddot{y}_1(t) &= -k_1\left(y_1(t) - y_2(t) - \frac{L_{01}(y_1(t)-y_2(t))}{\sqrt{(L_{01}+\alpha_{01})^2+(y_1(t)-y_2(t))^2}}\right) \\ &\quad -k_2\left(y_1(t) - y_3(t) - \frac{L_{02}(y_1(t)-y_3(t))}{\sqrt{(L_{02}+\alpha_{02})^2+(y_1(t)-y_3(t))^2}}\right) \\ m_2\ddot{y}_2(t) &= k_1\left(y_1(t) - y_2(t) - \frac{L_{01}(y_1(t)-y_2(t))}{\sqrt{(L_{01}+\alpha_{01})^2+(y_1(t)-y_2(t))^2}}\right) - k_3 y_2(t) \\ m_3\ddot{y}_3(t) &= k_2\left(y_1(t) - y_3(t) - \frac{L_{02}(y_1(t)-y_3(t))}{\sqrt{(L_{02}+\alpha_{02})^2+(y_1(t)-y_3(t))^2}}\right) - k_4 y_3(t) \end{aligned} \tag{14}$$

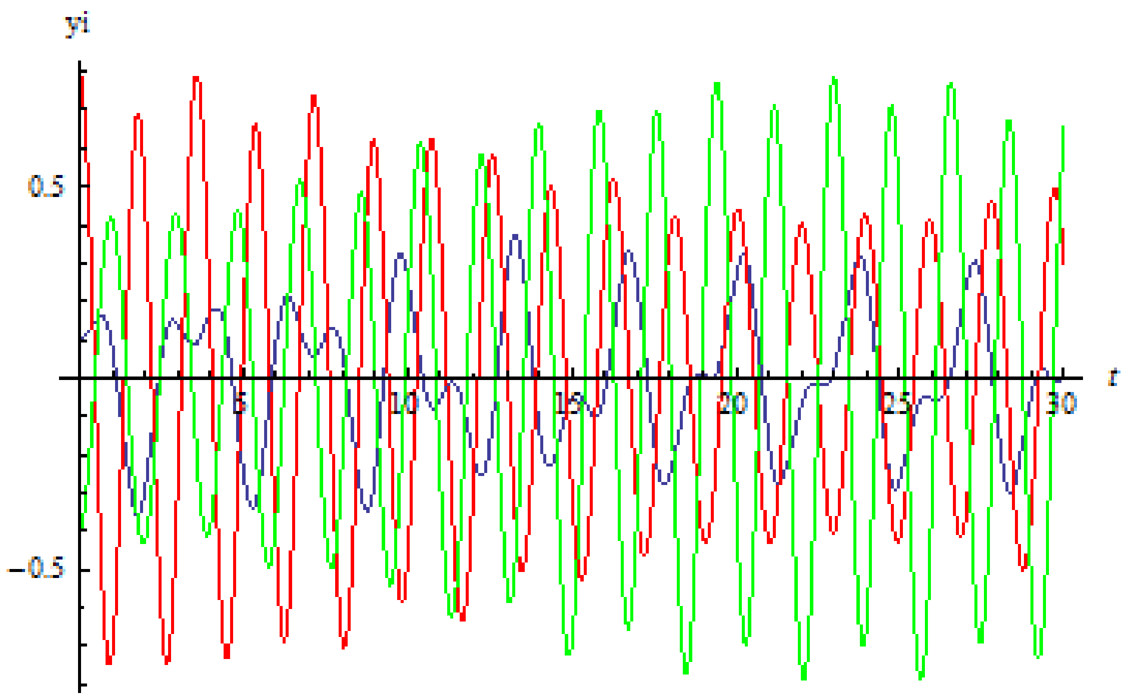

Figure 22: *The phase space of the system 14, y_1 is blue, y_2 is red and y_3 is green. The stiffness of the springs are $k_1 = k_2 = k_3 = k_4 = 10$.*

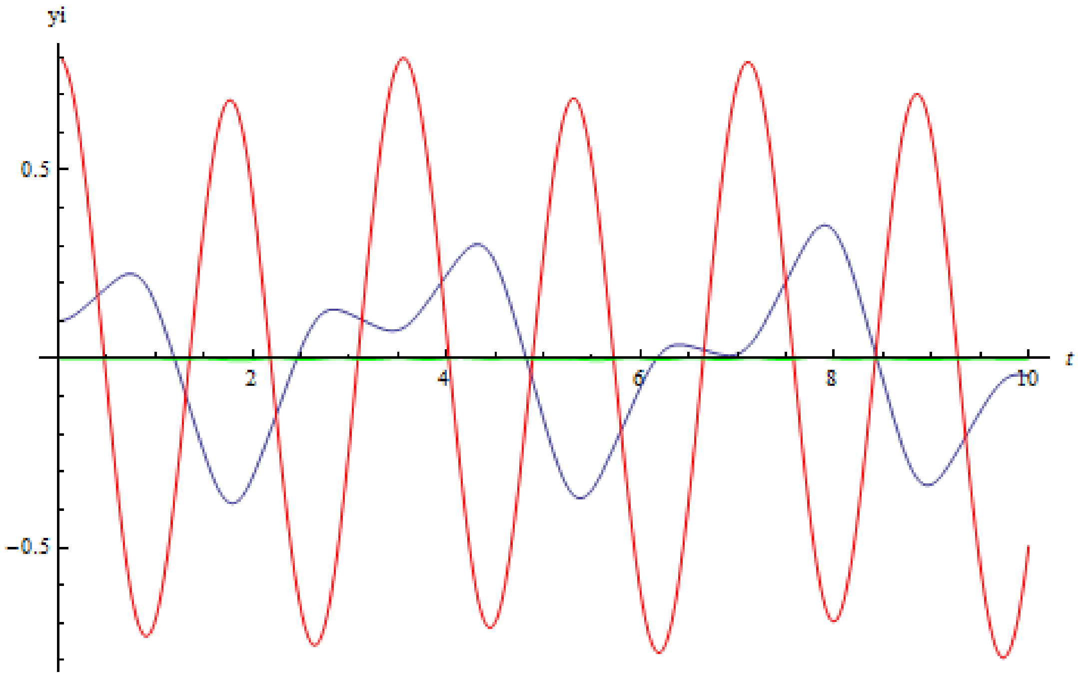

Figure 23: *The phase space of the system where the second beam is stiff and motionless. In the graph* y_1 *is blue and* y_2 *is red,* y_3 *is green and the stiffness of the springs are* $k_1 = k_2 = k_3 = 10$, $k_4 = 100$.

11.2 MODELING THE EXPERIMENT

In our experiment we had one beam that was free to oscillate vertically and the other one was stiff enough, not to make any oscillations. That means: $k_4 >> k_1$ and $y_3 = 0$. The new system is described by the following equations:

$$\begin{aligned} m_1 \ddot{y}_1(t) = &-k_1 \left(y_1(t) - y_2(t) - \frac{L_{01}(y_1(t)-y_2(t))}{\sqrt{(L_{01}+\alpha_{01})^2+(y_1(t)-y_2(t))^2}} \right) \\ &-k_2 \left(y_1(t) - \frac{L_{02} y_1(t)}{\sqrt{(L_{02}+\alpha_{02})^2+(y_1(t))^2}} \right) \\ m_2 \ddot{y}_2(t) = &\, k_1 \left(y_1(t) - y_2(t) - \frac{L_{01}(y_1(t)-y_2(t))}{\sqrt{(L_{01}+\alpha_{01})^2+(y_1(t)-y_2(t))^2}} \right) - k_3 y_2(t) \end{aligned} \tag{15}$$

The system has a new phase space that now makes it clear that we have transfer of energy from the beam to the mass. The beam's amplitude decreases as the mass's amplitude increases with time.

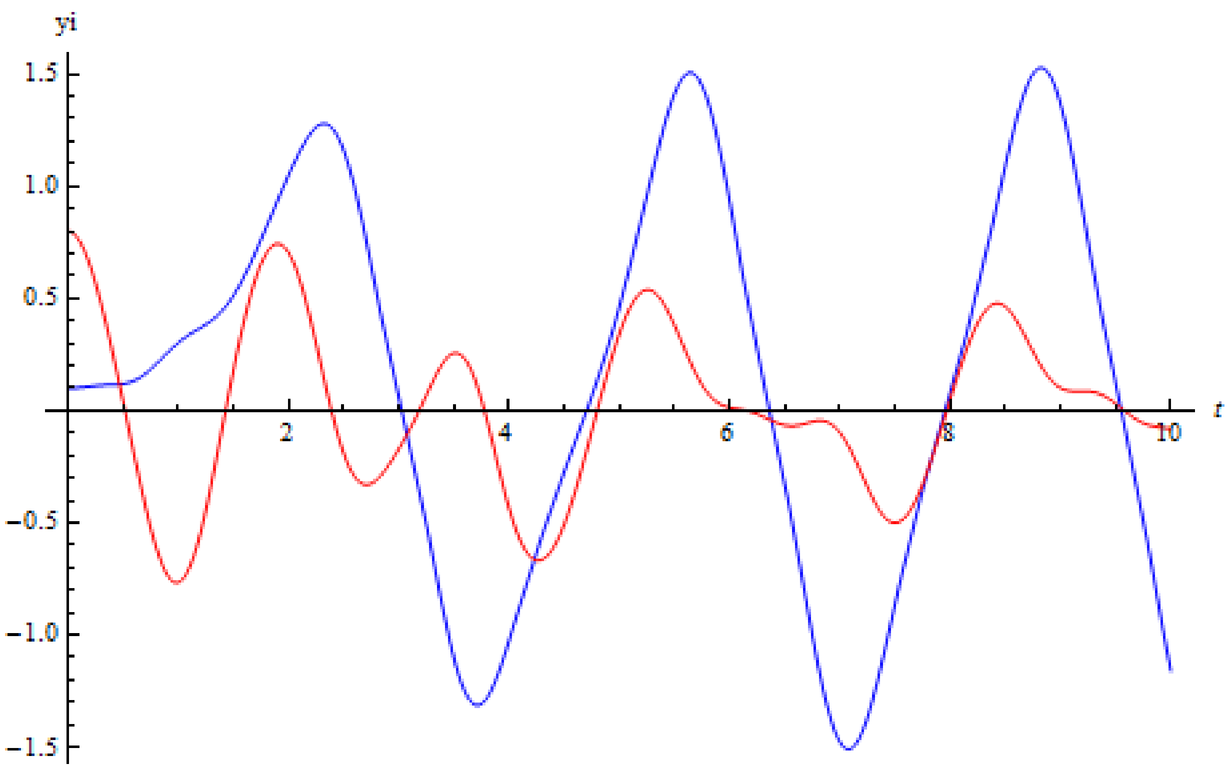

Figure 24: *The phase space of the system, y_1 is blue and y_2 is red. We have the same initial conditions as in the Figure 23. The beam starts to oscillate and gives a big amount of energy to the mass, that now oscillates with a much greater amplitude than before.*

In the Figure 24 we can observe that the amplitude of the beam is initialy large and the mass starts to absorbs energy from the beam as the time comes by. The oscillation of the beam slows down, till the enrgy flows back again and the phenomenon repeats, but now the amplitude of the beam is every time less than its previous status.

11.2.1 LINEARIZATION

After linearizing the system (15) with Taylor series we obtain the following system of equations:

$$
\begin{aligned}
m_1\ddot{y}_1(t) = k_1\left[-y_1+y_2+\tfrac{L_{01}(y_1-y_2)}{L_{01}+\alpha_{01}}+\tfrac{L_{01}\left(y_1^3+y_2^3\right)}{2(L_{01}+\alpha_{01})}\right]+k_2\left[-y_1+\tfrac{L_{02}y_1}{L_{02}+\alpha_{02}}-\tfrac{L_{02}y_1^3}{2(L_{02}+\alpha_{02})^3}\right] \\
m_2\ddot{y}_2(t) = k_1\left[-y_1+y_2+\tfrac{L_{01}(y_1-y_2)}{L_{01}+\alpha_{01}}-\tfrac{L_{01}\left(y_1^3-y_2^3\right)}{2(L_{01}+\alpha_{01})^3}\right]-k_3y_2
\end{aligned}
\tag{16}
$$

The nonlinear terms seem not to vanish after the Taylor's expansion.

Part III
EXPERIMENT

12 INTRODUCTION

In this part is going to take place the experiment. We are focused on finding arrangments that can isolate energy. In this thesis we are going to present a simple arrangement of two beams, one elastic (rubber) and one mass. The one beam is stiff. The whole arrangement can change significantly due to the changes of the mass, especially when it is eccentric.

Our experiment uses a mass that is eccentric, when it is not, nothing special happens. It consists of a metallic mass that is glued to the rubber so it is not able to slide across the rubber.

12.1 MODULUS OF ELASTICITY

An interesting property of a material is the elasticity, the ability to restore it's initial shape after distortion. A tool to measure this property is the Young's Modulus or Modulus of Elasticity which is used when the relationship between stress and strain is linear. It is described as

$$E = \frac{stress}{strain}\ (N/m^2)$$

, where:

$$stress = \frac{F}{A}\ (N/m^2)$$

and

$$strain = \frac{dL}{L}\ (m/m)$$

F is the force that is applied to the material, A is the area of the object, L is the initial dimension and dL is the elongation.

The mechanical characteristics of the polymers can be found in the same way as metals. By means you specify the initial dimension (L) and then you put loads and measure each dimension (dL) . Then you make a graph of the form found in Figure 25, the linear part is the elasticity. Polymers, and especially rubber are nonlinear and have a very low elasticity which varies with the stress. Furthermore, the elasticity changes with temperature. In our experiment the temperature remains constant and it equals to room temperature (approx.23^oC).

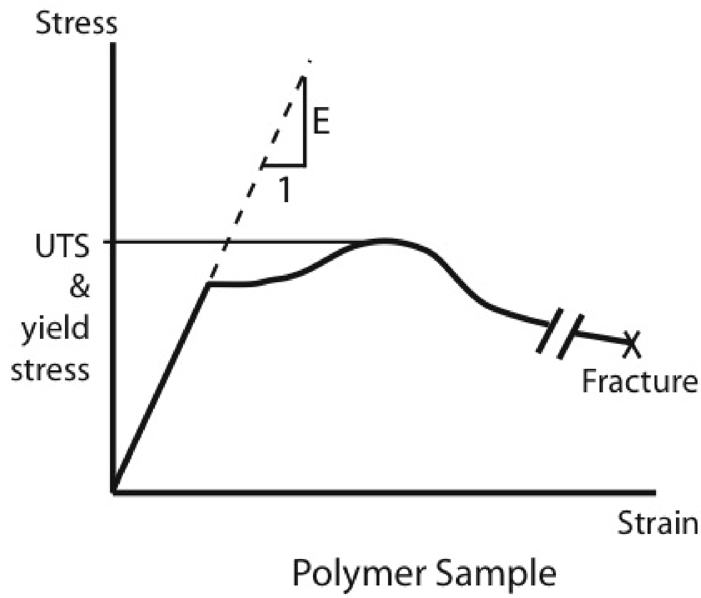

Figure 25: *Polymer tension graph.*

The area of the rubber can be found from the following geometrical term:

$$A = \pi \frac{d^2}{4} \tag{17}$$

where d is the diameter of the material. In our case the diameter and the area of the circle is shown Table 7 because the diameter decreases as the rubber elongates.

12.1.1 RUBBER'S ELASTICITY

To measure the rubber's elasticity we had to determine the elongation as we applied an increasing force. The experiment that took place was to fix the one point of the rubber and connect the other with a bottle as seen in Figure 26. In order not to measure the whole rubber each time that we apply a force, we marked two points. Adjacent to the rubber is placed a ruler so as to measure the shift of that point. As a force we added to the bottle $50\,mL$ of water in each time step. In total we had 22 time steps with the rest position.

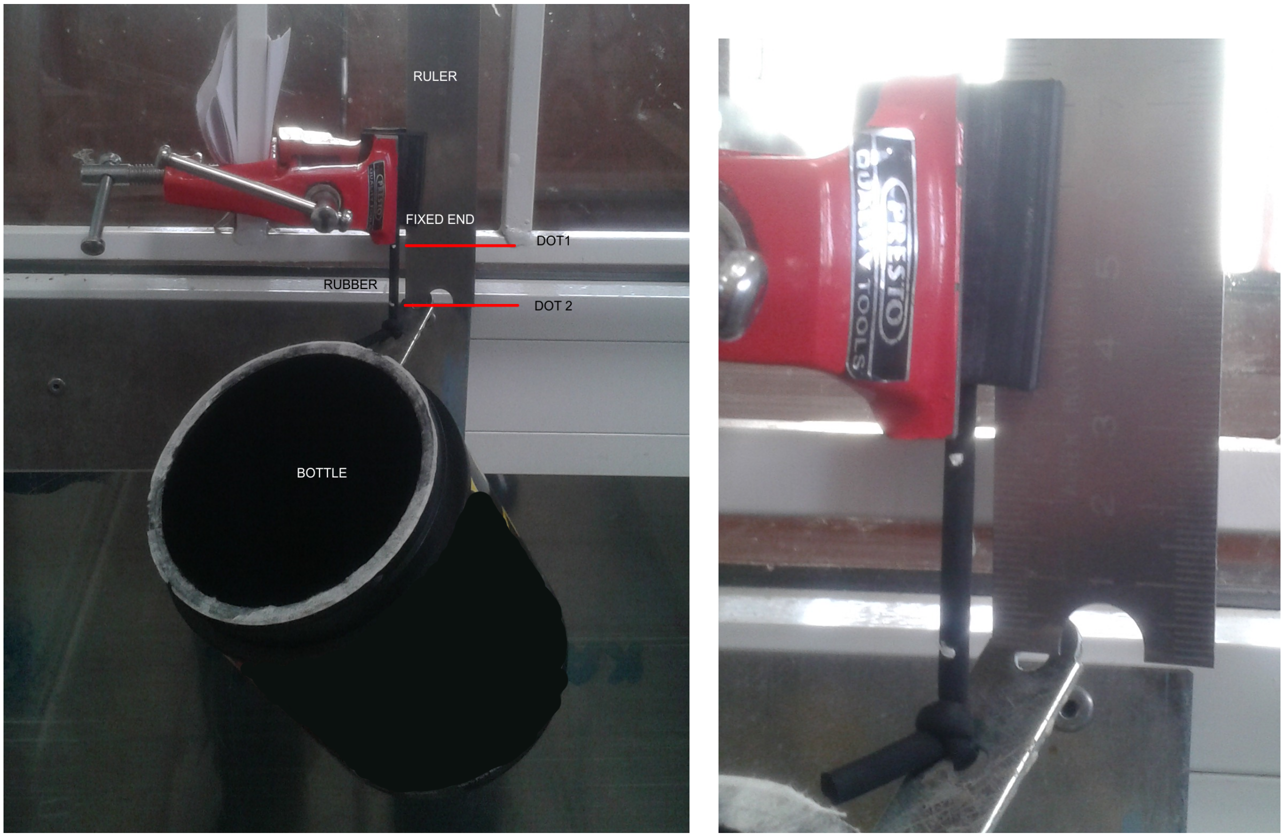

Figure 26: *The initial condition of the system. We see the ruler that is placed next to the rubber and the bottle where we fill it with water. The rubber's one end is fixed and across it are pointed two dots in order to measure the elongation. The ruler's* 0 *is always in the DOT* 2.

Figure 27: *A random position of the system. In the first figure is shown the whole system and in the second the measurement.*

For the rubber that was used in the experiment corresponds the following matrix. The matrix contains in the first column the weights/force (volume of water) that is applied to the rubber and it is measured in mL, in the second column is the elongation and in the third the area of the rubber that was measured with a micrometer.

Weight (mL)	Elongation-dL (cm)	Area $\left(cm^2\right)$
0	0	0.1256
50	0.1	0.1256
100	0.15	0.1256
150	1.25	0.1256
200	0.3	0.1256
250	0.15	0,1075
300	0.4	0.1075
350	0.25	0.1075
400	0.4	0.1075
450	0.25	0.1075
500	0.25	0.1075
550	0.35	0.1075
600	0.15	0.0962
650	0.15	0.0962
700	0.15	0.0962
750	0.2	0.0962
800	0.1	0.0962
850	0.05	0.0962
900	0.05	0.0962
950	0.2	0.0962
1000	0.05	0.0907
1050	0.03	0.0907

Table 7: In the first column are the weights that applied to the rubber and are measured in mL, in the second column is the elongation and in the third the area of the rubber.

In the next table are converted the weights/force from mL to $Newton$. The water's density is $1\,\frac{kg}{m^3}$ this equals to $1\,mL = 1\,gr = 9.8\,N$ and based on this is done the convertion. Furthermore the table contains the elongation, the strain and the stress.

Force (N)	Elongation (cm)	Strain	Stress (N/cm^2)
0	0	0	0
490	0.1	0.0714	3.9798
980	0.15	0.1000	7.9579
1470	1.25	0.7575	11.9369
1960	0.3	0.1034	15.9159
2450	0.15	0.0468	23.2519
2940	0.4	0.1194	27.9023
3430	0.25	0.0667	32.5527
3920	0.4	0.1000	37.2031
4410	0.25	0.0568	41.8535
4900	0.25	0.0537	46.5039
5390	0.35	0.0714	51.1543
5880	0.15	0.0285	62.3645
6370	0.15	0.0278	67.5616
6860	0.15	0.0270	72.7586
7350	0.2	0.0351	77.9557
7840	0.1	0.0169	83.1527
8330	0.05	0.0083	88.3498
8820	0.05	0.0082	93.5468
9310	0.2	0.0327	98.7439
9800	0.05	0.0079	110.1451
10290	0.03	0.004	115.6523

Table 8: *Sum up table of the elasticity's measurement of the rubber. The force that acts to the rubber is converted to N, the elongation is measured in cm. In the third column is the strain and in the last the stress.*

Now follows the graph that presents the relationship between stress and strain. In the horizontal axes is the strain and the vertical the stress.

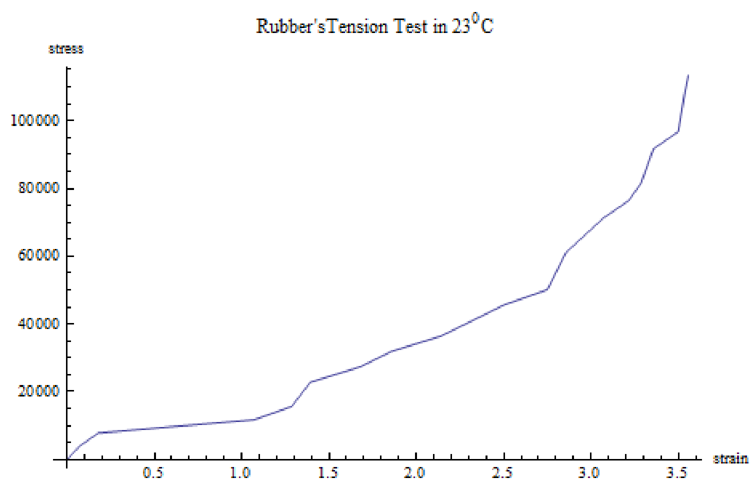

Figure 28: *The tension graph of the used polymer.*

13 EXPERIMENT

13.1 EXPERIMENT SPECIFICATIONS

As mentioned before the experiment consists of a horizontal metal beam, a rubber, a metal mass and a vertical metal beam. The beams are fixed to a working-table with clamps. In a fixed point in each beam we glued a metal sheet in "Gamma" (Γ) shape so as to attach the rubber. In order to change the rubber's stiffness we had to join it tighter. The mass is chosen to be a metallic clip because it has two metal grips that fold. This gives us the opportunity to change the state of the mass from neutral to eccentric. When the grips are fold, then the mass becomes eccentric.

13.1.1 DIMENSIONS

In this section we define the dimensions that the system had during the experiment. The beam's characteristics are shown in the next figure. The length from the point that the clamp was put till the attachement with the rubber was $104\,cm$ and till the free end is $117.8\,cm$

The distances between the elements are shown in the following figure.

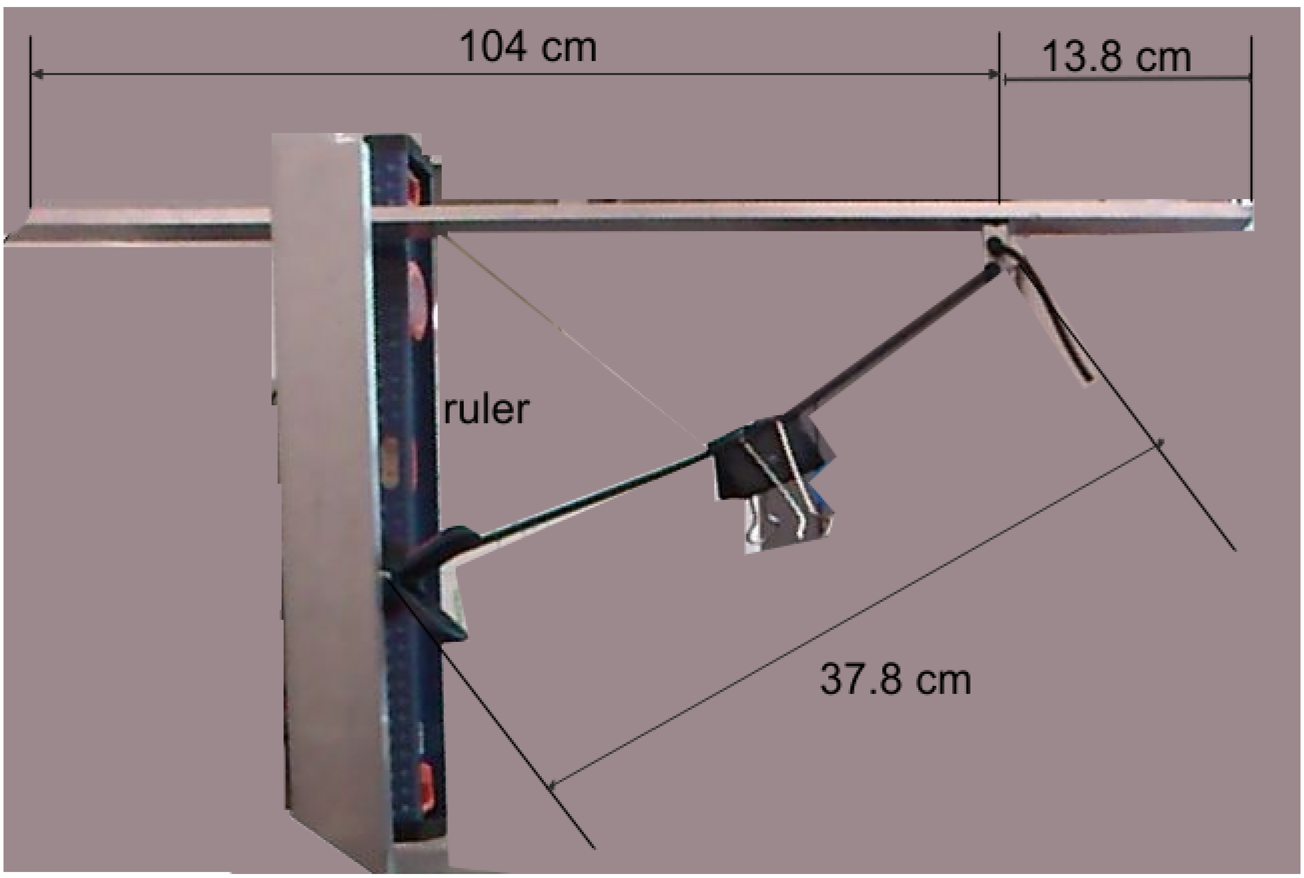

Figure 29: *The elements' distances. The beam has a total length* $117.8\,cm$.

14 PROCEDURE

The experiment that took place was to stimulate the beam and see how it oscillates as we modified the stiffness of the rubber. The stiffness of the rubber increases as you join it tighter. So we started by stimulating the free beam and then repeating the same stimulation as we increased the rubber's stiffness. The stimulation was an initial displacement (of the beam). The ruler that is next to the beam is to measure it's initial displacement.

Figure 30: *The coupled system.*

The free beam oscillates about 4 times more (in the sense of time) than the coupled one and the amplitute is significantly bigger. Moreover as the stiffness is low then the mass absorbs more energy than when the rubber is fully streched. The phenomenon that is observed is that the energy is transfered from the beam to the metallic mass and so the beam comes faster to rest. It is clear that the vertical oscillation of the beam transforms into three types: one vertical and two rotational oscillations of the mass. The mass now oscillates in the vertical axes, around the horizontal axes and around itself.

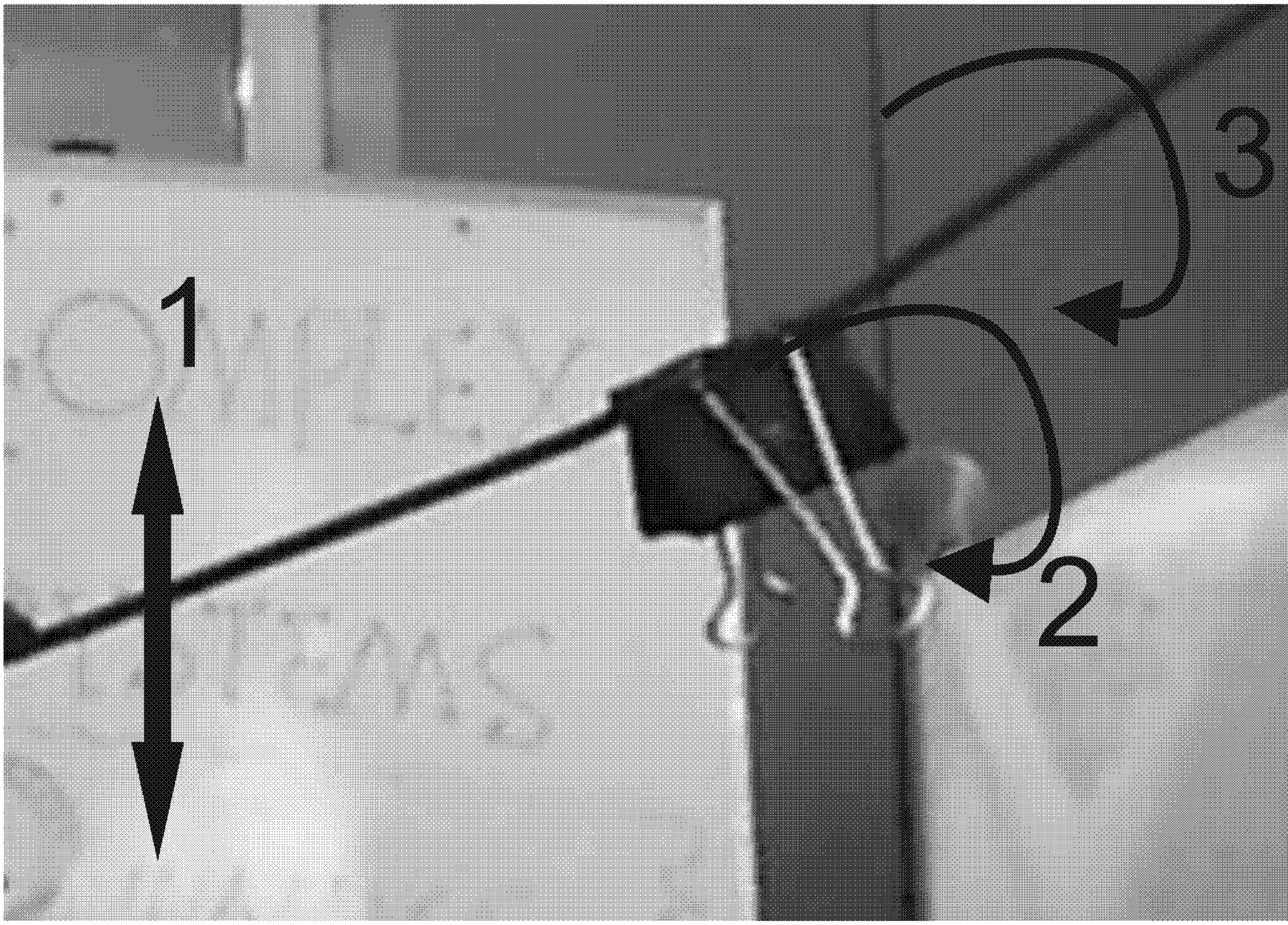

Figure 31: *The three types of oscillation that the mass do.*
1. vertical oscillation,
2. around itself,
3. around the rubber's axes.

The stimulation was an initial displacement of the horizontal beam. The stimulation was the same for every experiment that took place. It was measured with a ruler put in a fixed place adjacent to the beam.

What we observe is that if we stimulate the free beam, it starts to oscillate until it rests from its own, for about 2 minutes. Then in the coupled system the beam oscillates and so does the mass which vertical oscillation increases and starts two rotational oscillations, then the beam's oscillation starts to slow down. After some cycles the mass starts to change the direction of its rotation and then the energy flows back to the beam (till this point it is the important phenomenon). Now the beam after a momentary stasis it starts to increase its oscillation's amplitude but with a smaller maximum and the phenomenon repeats, the energy flows back to the mass and then after several seconds it comes to rest. We repeated the experiment several times and the total time that the beam came to rest depended on the change of the direction of oscillation of the mass.

The study of the phenomenon and the data that we exluded was with the help of the video that we captured because we had the chance to study it frame by frame.

If the rubber is in its neutral dimension or slightly streched, then the mass absorbs the energy of the beam very fast. If it is streched then it takes some time till it starts to rotate from the one side, then it changes side but with significantly less absorbsion than before. Yet the total amound of energy that is transfered from the beam to the mass is not negligible comparable to the free beam.

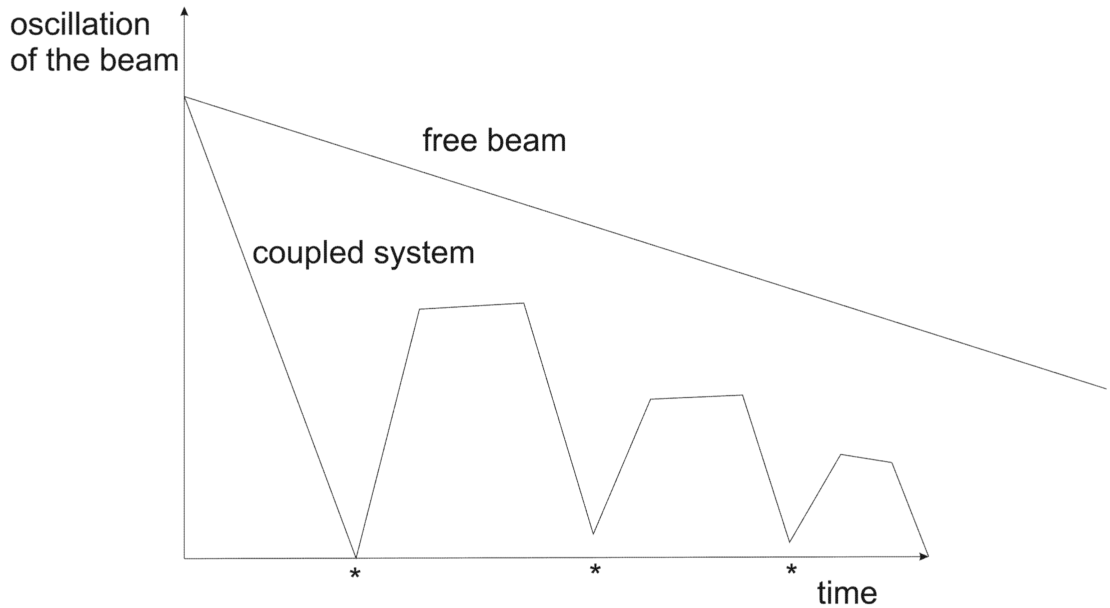

Figure 32: *A qualitative diagram that tries to sum up the whole experiment. The line "free beam" shows that the beam oscillates long after the coupled system came to rest. In the "coupled system" line the stars represent the change of oscillation of the mass. First we have a quick decrease in the oscillation of the beam as the mass absorbs much of the energy and it starts to rotate, then the mass changes rotation and the beam comes to rest. After that the beam starts to oscillate again but with a significantly lower amplitude, then the energy flows back to the mass again, and this phenomenon repeats till the system comes to rest.*

15 CONCLUSION

The experiment that took place was the last from some modifications that was made in the first model that we studied, after the mathematical modeling. The last system that was studied show us the phenomenon of transfer of energy from the beam to the mass. This energy was mechanical of the beam that transfered as mechanical of the mass. First, we stimulated the beam and it started to oscillate, then the mass after 2-3 vertical oscillations, due to its eccentricity rotated around itself and around the horizontal axes. Now it is clear that we have energy transition from one subsystem to another. When the mass' rotation changed side, the beam came to rest momentary, then the phenomenon continues back and forth untill it comes to rest. This last phenomenon is due to the the torsion of the rubber. In a further investigation someone could try to vanish this torsion so as not to flow back the energy back to the beam.

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