ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ
ΣΧΟΛΗ ΗΛΕΚΤΡΟΛΟΓΩΝ ΜΗΧΑΝΙΚΩΝ
ΚΑΙ ΜΗΧΑΝΙΚΩΝ ΥΠΟΛΟΓΙΣΤΩΝ
ΤΟΜΕΑΣ ΤΕΧΝΟΛΟΓΙΑΣ ΠΛΗΡΟΦΟΡΙΚΗΣ & ΥΠΟΛΟΓΙΣΤΩΝ

Παίγνια Διαμόρφωσης Άποψης σε Κοινωνικά Δίκτυα

ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

Ευστράτιος-Παντελεήμων Χ. Σκουλάκης

Επιβλέπων : Δημήτρης Φωτάκης
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Αθήνα, Οκτώβριος 2014

Εθνικο Μετσοβιο Πολυτεχνειο
Σχολη Ηλεκτρολογων Μηχανικων
και Μηχανικων Υπολογιστων
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Ευχαριστίες

Η πραγματοποίηση της παρούσας διπλωματικής εργασίας ως έχει δεν θα είχε καταστεί δυνατή χωρίς τη συμβολή των παρακάτω προσώπων. Οφείλω να ευχαριστήσω τον επιβλέποντα μου Δημήτρη Φωτάκη για την ουσιαστική καθοδήγηση που προσέφερε κατά την συγγραφική διαδικασία, αλλά και γιατί είναι αυτός που με γνώρισε με τον κόσμο της Θεωρητικής Πληροφορικής. Ευχαριστώ επίσης τους καθηγητές Στάθη Ζάχο και Άρη Παγουρτζή για τις πολύτιμες γνώσεις του και συμβουλές που μου έχουν πρόσφερει όλα αυτά τα χρόνια, καθώς και όλα τα μέλη του Εργαστηρίου Λογικής και Επιστήμης Υπολογιστών. Τέλος, η πραγματοποίηση αυτής της διπλωματικής σημαδοτοδοτεί την ολοκλήρωση των σπουδών μου και την έξοδο μου από τον φοιτητικό κόσμο. Θέλω να ευχαριστήσω την οικογένεια μου για την συμπαράσταση της καθόλη την διάρκεια των σπιουδών μου, αλλά και τους φίλους μου για άλα αυτά τα υπέροχα χρόνια.

Περίληψη

Στην παρούσα διπλωματική εργασία ασχολούμαστε με το θέμα της διαμόρφωσης άποψης στα κοινωνικά δίκτυα. Σήμερα τα κοινωνικά δίκτυα παίζουν καθοριστικό ρόλο στην σημερινή κοινωνία καθώς επηρεάζουν ολοένα και περισσότερο τις κοινωνικές και οικονομικές δραστηριότητες. Καθώς τα μέλη ενός κοινωνικού δικτύου αλλάζουν τις απόψεις τους μέσω της αλληλεπίδρασης μεταξύ τους, είναι πολύ σημαντικό να γνωρίζουμε αν τα μέλη του δικτύου θα υιοθετήσουν τελικά μία συγκεκριμένη άποψη καθώς και ποια άποψη θα είναι αυτή. Σε αυτή την διπλωματική ποικίλα μοντέλα διαμόρφωσης άποψης εχουν μελετηθεί όπως:
το μοντέλο DeGroot, Kleiberg-Bindel καθώς και το Decentralized Opinion model.
Τέλος παρουσιάζονται μερικά βασικά αποτελεσμάτα για μοντέλα μη σταθερών γραφημάτων όπως: το K-NN και το Hegelsman-Krause model.

Λέξεις Κλειδιά: Αλγοριθμική Θεωρία Παιγνίων, Κοινωνικά Δίκτυα, Δυναμική Διαμόρφωση Άποψης

Abstract

In this thesis we deal with opinion dynamics in social networks. Social Networks play a major role in today's life since they affect most of the economic and social activities. Since members of a society change their opinions while intefering with other people, it is very important knowing whether they will finally adopt a specific opinion and which opinion it will be. In this thesis various models for opinion dynamics such as DeGroot, Kleiberg-Bindel model and Decentralized opinion dynamics are examined. Finally, some basic results are presented for non-steady graph models such as the K-NN model and the Hegelsman-Krause model.

Keywords: Algorithmic game theory, Social Networks, Opinion dynamics

Contents

Introduction

The unterstanding of human behavior was always a major study field in various sciences. Psychologists, Sociologists and Political Scientists were always interested in how humans form their opinions and consequently their behaviour. Although biology has taught us that human characteristics such as height or colour are imprinted on us by our genes, opinions or beliefs have nothing to do with genes. So a major question arises: Where do the opinions come from?

Today we are quite confident that the way that we form our beliefs depends on the experiences that we get from our birth to our death. Apparently, different individuals have very different experiences, something that explains why there exist such vast differences in human's behavior around the world. The causes that lead a certain individual to adopt a certain opinion on a specific subject are various and very complicated. For example economic welfare, education ,religion and cultural backround play a major role in someone's beliefs. All these factors are very heteregenous, but they all something in common: They are all trasmitted by the interaction of people with other people. Thus, society plays an very important role in the opinion formation. Notice that it is very likely that someone supports the football team that his friends support rather than another one. Psycology has reavealed the huge impact that a social network has to its members. This impact has at the same time beneficiary and negative results for the welfare of the society. On the one hand, if a member of the society has no idea if it will rain tomorrow, it is very likely that he will ask one of his friends. As a result, society permits the diffusion of information and knowledge that helps people form opinions and beliefs for various subjects. On the other hand, this impact can affect people's *rationality*. Racism is an example of this case: There are no scientific evidence that Afroamerican are inferior, but many white men adopt this belief influenced by their social enviroment.

The evolution of computer science and modern fields of mathematics such us statistics or game theory provide us tools to model and study how people form their opinions in a modern and productive way. The invasion of this quite different field to sociology started with Condorcet's jury theorem in his 1785 work Essay on the Application of Analysis to the Probability of Majority Decisions. Latter in 1907 the British scientist Francis Galton asked around 800 villagers in Plymouth to guess the weight of an ox (none of them was an expert), suprisingly the mean value of the values that the villagers had reported was very close to the actual weight of the ox (the actual weight was 1198 pounds and the mean value was 1197). As a result, Galton claimed that the collective intelligence of a group of people is much more than the knowledge that each of them has, something that is known today as wisdom of crowds.

These previous works inspired many scientists in the last century to study social networks in a more consistent and formal way. In 1965 American statistician Morris H. DeGroot proposed a model according to which the opinions in a social network are formed. This model is known as DeGroot model and it will be studied latter in an exhaustive way, but the main idea is that each individual trusts some other individuals who affect his opinion. DeGoot has represented the social network as a graph $G(V, E)$ at which the nodes stand for the members of the social network and the edges stand for the trust between them. After DeGroot model many other models have emerged trying to capture the way that the members of a social network form their opinions. Although these models present differences, they follow a quite common framework: There exists a graph $G(V, E)$ representing the social network and there are cost functions that are defined over the underlying graph and the expressed opinions of the nodes. The cost functions stands for the cost that disagreement causes to each individual. As a result, opinion formation in a social network can be viewed as a game and many of the knowledge on the game theory is very useful in this area. As we have previously said, the personal cost for disagreement for a specific model, is explicitly defined in it and thus there exists a implicit updating rule for every member of the society. The latter is very reasonable since the opinions in a social network are not stable but they change over time. For every model that we will examine, we will be mainly interested in computing equilibrium points (Nash Equilibrium in other words) which are stable over the updating rules. Apart from that, we investigate mechanisms that lead the agents adopt a specific opinion. Something that is really important and this is why this field is frequently referred as Opinion Dynamics. Finally notions from game theory such as Price of Anarchy(PoA) or Price of Stability(PoS) are studied for this class of games. Other very interesting but more complicating questions are the social influence of certains agents and maximizing or minimizing this influence.

Chapter 1

Kleiberg-Bindel Model

1.1 Introduction

In this section we will see a quite simple model proposed by Bindel,Kleinberg and Oren [4]. In this model we have a weighted undirected graph $G(V, E)$ representing a social network. Each node $i \in V$ represents who an agent has an opinion $x_i \in [0, 1]$ and an internal opinion $s_i \in [0, 1]$ which remains constant over time. The weight $w_{ij} = w_{ji} \geqslant 0$ of the edge (i, j) represents the influnce between the nodes i and j and each node has also a weight $w_i \geqslant 0$ which is the stuborness to his initial opinion. We also assume that for every connected component of the graph $G(V, E)$ there exists at least one agent $i \in V$ with $w_i > 0$. For convenience, we consider that if the nodes i and j are not linked with an edge then $w_{ij} = w_{ji} = 0$.

More pecisely, let $\vec{x}$ be our strategy profile. Then, each agent i has a personal cost:

$$c_i(\vec{x}) = \sum_{j \in V} w_{ij}(x_i - x_j)^2 + w_i(x_i - s_i)^2$$

As a result given a strategy profile $\vec{x}$, node i in order to minimize its pesonal cost updates its personal opinion as follows:

$$x_i = \frac{\sum_{j\in V} w_{ij}x_{ij} + w_i s_i}{\sum_{j\in V} w_{ij} + w_i}$$

Apparently disagreement in a society is always something that provokes a cost. Now we would like to quantify this cost and we define the social cost function.

$$\begin{aligned} SC(\vec{x}) &= \sum_{i\in V} c_i \\ &= \sum_{i\in V}(\sum_{j\in V} w_{ij}(x_i - x_j)^2 + w_i(x_i - si)^2) \\ &= 2\sum_{i<j} w_{ij}(x_i - x_j)^2 + \sum_{i\in V} w_i(x_i - s_i)^2 \end{aligned}$$

Now, some very natural questions arise. Will the society reach at a consesus? If not, will there be a state where nobody wants to deviate or the agents will always change their opinions?

In the next section we will try to shed light on these questions.

1.2 Nash Equilibrium

In the previous section we have seen that all the agents change their opinions in their effort to minimize their personal cost. A very natural question is *Will this ever stop*?. In other words, is there a state in which all the players are satisfied and noone wants to change his opinion? Apparently, we are asking whether there exists a pure Nash Equilibrium in this game, a question that is not trivial at all.

Let the function $\Phi(x) = \sum_{i<j} w_{ij}(x_i - x_j)^2 + \sum_{i \in V} w_i(x_i - s_i)^2$ and $N(i)$ be all the neighbours of the agent i. We can easily see that:

$$\begin{aligned}
\Phi(x_i, x_{-i}) - \Phi(x_i{'}, x_{-i}) &= \sum_{i<j} w_{ij}(x_i - x_j)^2 + \sum_{i \in V} w_i(x_i - s_i)^2 \\
&\quad - \sum_{i<j} w_{ij}(x_i{'} - x_j)^2 + \sum_{i \in V} w_i(x_i{'} - s_i)^2 \\
&= \sum_{j \in N(i)} w_{ij}(x_i - x_j)^2 + w_i(x_i - s_i)^2 \\
&\quad + \sum_{j \in N(i)} w_{ij}(x_i{'} - x_j)^2 + w_i(x_i{'} - s_i)^2 \\
&= C_i(x_i, x_{-i}) - C(x_i{'}, x_{-i})
\end{aligned}$$

Observation 1. *If $\Phi(x^*)$ is a minimum of $\Phi(x)$. Then x^* is a Nash Equilibrium.*

Proof. Let $\Phi(x^*)$ be a minimum. Thus, $\Phi(x_i{}^*, x_{-i}{}^*) - \Phi(x_i{'}, x_{-i}{}^*) = C_i(x_i{}^*, x_{-i}{}^*) - C_i(x_i{'}, x_{-i}{}^*) < 0$

$$\implies \forall i \in V : C_i(x_i{}^*, x_{-i}{}^*) < C_i(x_i{'}, x_{-i}{}^*)$$

Thus, x^* is a Nash Equilibrium. □

Generally $\Phi(x)$ is called potential function and the games that have a potential function are called potential games [17]. Now, we can answer some of the previous questions. $\Phi(x)$ is a continous function which is also bounded. Thus, there exists a $x^* \in [0,1]^n$ which is a global minimum of the potential function and consequently is a Nash Equilibrium [1, 6]. We can also observe that $\Phi(x)$ is a strictly convex function. As a result, there are no local minimums and there exists a unique global minimum which is also the unique Nash Equilibrium of the game. The next question is how we can compute this unique equilibrium? From the above it easy to see that we just need to find the $x^* \in [0,1]^n$ at which the potential function $\Phi(x)$ is minimized. We will use standard optimization theory to compute x^*.

We define the matrices $A \in \mathrm{R}^{\mathrm{n}\times\mathrm{n}}, B \in \mathrm{R}^{\mathrm{n}\times 1}, C \in \mathrm{R}$ as follows:

$$A_{ij} = \begin{cases} \sum_{i\neq j} w_{ij} + w_i & \text{if } i = j \\ -w_{ij} & \text{if } i \neq j \end{cases}$$

$B_i = w_i \cdot s_i$ and $C = \sum_{i\in V} w_i \cdot s_i^2$

Thus,

$$\Phi(x) = x^T \cdot A \cdot x - 2 \cdot B \cdot x + C$$

Observation 2. *Let x^* be The Nash Equilibrium then $x^* = A^{-1} \cdot B$*

Proof. If x^* is the Nash Equilibrium then x^* minimizes $\Phi(x)$. As a result, $\nabla\Phi(x^*) = 0 \Longrightarrow 2 \cdot A \cdot x - 2 \cdot B = 0 \Longrightarrow x^* = A^{-1} \cdot B$ □

1.3 Local Smoothness and Price of Anarchy

In the previous section we have seen that our game has a unique Nash Equilibrium. Let $x^* \in [0,1]$ be the N.E., then x^* minimizes the potential function $\Phi(x) = \sum_{i<j} w_{ij}(x_i - x_j)^2 + \sum_i w_i(x_i - s_i)^2$. In this section we will investigate how bad is the social cost of the Nash Equilibrium. We remind that the social cost function $SC(x) = 2\sum_{i<j} w_{ij}(x_i - x_j)^2 + \sum_i w_i(x_i - s_i)^2 \neq \Phi(x)$. As a result, a Nash Equilibrium is never the optimal solution. Now will introduce the notion of Price

of Anarchy(PoA) which is a quantifier of how bad a N.E. can be in respect to the optimal solution [14].

Definition 1. *Let a game G with social cost function $SC(x)$ and I an instance of the game. Let also $N(I)$ be the set of all Nash Equilibrium of instance I then:*

$$PoA = \max_{\forall I, \forall x \in N(I)} \frac{SC(x)}{SC(y)}$$

where $SC(y)$ is the minimum of the cost function.

The Price of Anarchy measures how eficiency of the system degrades due to selfish behavior of its agents. From the definition the Price of Anarchy follows that for every N.E. $x^* : \frac{SC(x^*)}{SC(y^*)} \leqslant PoA$, where y^* is the optimal solution.
In this section we will prove that $PoA = \frac{9}{8}$ for this game, which is something good because the social cost of the optimal solution is very close to the optimal [4, 3]. Before proving that $PoA = \frac{9}{8}$ we will give an easy upper bound to the PoA.

Observation 3. *The $PoA \leqslant 2$*

Proof. Let x^* be a N.E. and y^* is the optimal solution. It is easy to see verify that $\forall x \in [0, 1] : \Phi(x) \leqslant SC(x) \leqslant 2 \cdot \Phi(x)$ We also know that x^* is the minimizer of $\Phi(x)$ and y is the minimizer of $SC(x)$.

$$PoA = \frac{SC(x^*)}{SC(y^*)} \leqslant \frac{2 \cdot \Phi(x^*)}{\Phi(y^*)} \leqslant 2 \cdot \frac{\Phi(y^*)}{\Phi(y^*)} = 2 \Longrightarrow$$

$$PoA \leqslant 2$$

□

Local Smoothness

We have already found an upper bound to the PoA, but this is not enough. We would like whether to prove that this bound is tight or to find a lower upper bound. In order to do this, we will use the local smoothess technique that we will explain latter [3]. Now, we will give some definitions that are necessary to continue. Until now, we have seen the notion of Nash Equilibrium. Now, we will define two more general notions of equilibrium, the mixed Nash Equilibrium and the correlated Equilibrium.

Definition 2. *A mixed N.E. is the equilibrium at which each agent has picked a distribution σ_i over expressed strategies, so that we have:*

$$E_{x_{-i}\sim\sigma_{-i}}[C_i(x_i, x_{-i})] \leqslant E_{x_{-i}\sim\sigma_{-i}}[C_i(x, x_{-i})]$$

where σ_{-i} denotes the joint distribution of others agents strategies.
Respectively, mixed $PoA = \frac{E_{x\sim\sigma}[SC(x)]}{SC(Opt)}$.

Definition 3. *A correlated equilibrium σ is a distribution such that for each player i and each opinion x_i in the support of σ,*

$$E_{x_{-i}\sim\sigma_{-i}|x_i}[C_i(x_i, x_{-i})] \leqslant E_{x_{-i}\sim\sigma_{-i}}[C_i(x, x_{-i})]$$

where $\sigma_{-i}|x_i$ denotes the distribution σ conditioned on x_i.
Respectively, correlated $PoA = \frac{E_{x\sim\sigma}[SC(x)]}{SC(Opt)}$

From the defitions of Nash Equilibrium , mixed Nash Equilibrium and correlated Equilibrium follows that a Nash Equilibrium is a mixed Nash Equilibrium and a mixed Nash Equilibrium is a correlated Equilibrium. As a result, an upper bound at the correlated PoA is an upper bound to the PoA.

We will use the local smoothness thechnique to give an upper bound to the correlated PoA [18].

Let a fixed strategy profile o, $\lambda > 0$ and $\mu < 1$ such that:

$$\sum_i [C_i(x_i, x_{-i}) + (o_i - x_i)\frac{d}{d_{xi}} C_i(x_i, x_{-i})] \leqslant \lambda \cdot SC(o) + \mu \cdot SC(x) \ (1)$$

Theorem 1. *Let σ be a correlated Equilibrium. If equation (1) holds for any profile x with respect to a fixed profile o. Then, $\frac{E_{x\sim\sigma}[SC(x)]}{SC(o)} \leqslant \frac{\lambda}{1-\mu}$. In particular, when o denotes the optimal profile, the correlated $PoA \leqslant \frac{\lambda}{1-\mu}$.*

Before things start getting too complicated we will focus on what we have seen and that will be used to find an upper bound to the PoA. From the above theorem we know that:

Let $\lambda > 0, \mu < 1$ such that:

$\forall x \in [0,1]: \ \sum_i [C_i(x_i, x_{-i}) + (y_i - x_i)\frac{d}{d_{xi}} C_i(x_i, x_{-i})] \leqslant \lambda \cdot SC(y) + \mu \cdot SC(x)$ (2)

where y is the the optimal profile. Then,

$$PoA \leqslant \frac{\lambda}{1-\mu}$$

Now, we are starting to get some intuition on how we proceed. At first, we will find a set $A \subseteq R^2$ such that: $\forall(\lambda, \mu) \in A \implies$ *equation* (2) *holds* and then we will find $inf\{\frac{\lambda}{1-\mu}, (\lambda, \mu) \in A\}$.

Of course our goal is to find an upper bound lower than 2 that we have already found.

Let $A_1, A_2 \subseteq \mathrm{R}^2$

- $A_1 := \{(\lambda, \mu) : \forall x, y \geqslant 0,\ f(x) + \frac{(y-x)}{2} \cdot f'(x) \leqslant \lambda \cdot f(y) + \mu \cdot f(x)\}$

- $A_2 := \{(\lambda, \mu) : \forall x, y \geqslant 0,\ g(x) + (y-x) \cdot g'(x) \leqslant \lambda \cdot g(y) + \mu \cdot g(x)\}$

where $f(x) = g(x) = x^2$

We will show that $A_1 \cap A_2$ is the $A \subseteq R^2$ that we are searching.

Observation 4. $\forall (\lambda, \mu) \in A_1 \cap A_2 \implies$ *equation* (2) *holds.*

Proof. Let $f_{ij}(x) = w_{ij} \cdot f(x)$ and $g_i(x) = w_i \cdot g(x)$, where $f(x) = g(x) = x^2$ then:

$C_i(x_i, x_{-i}) = \sum_{i \neq j} f_{ij}(x_i - x_j) + g_i(x_i - s_i) \implies$

$$\begin{aligned}\sum_i [C_i(x_i, x_{-i}) + (y_i - x_i)\frac{d}{d_{xi}} C_i(x_i, x_{-i})] \quad &= \quad \sum_i \sum_{i \neq j} f_{ij}(x_i - x_j) + \sum_i g_i(x_i - s_i) \\ &\quad + \sum_i (y_i - x_i) \sum_{i \neq j} f'_{ij}(x_i - x_j) \\ &\quad + \sum_i (y_i - x_i) g'_i(x_i - s_i)\end{aligned}$$

$= \sum_{i<j} [2 \cdot f_{ij}(x_i - x_j) + (y_i - x_i) \cdot f'_{ij}(x_i - x_j) + (y_j - x_j) \cdot f'_{ij}(x_j - x_i)]$

$+ \sum_i [g_i(x_i - s_i) + (y_i - x_i) \cdot g'_i(x_i - s_i)]$

$= \sum_{i<j} [2 \cdot f_{ij}(x_i - x_j) + (y_i - y_j) \cdot f'_{ij}(x_i - x_j) - (x_i - x_j) \cdot f'_{ij}(x_i - x_j)]$

$+ \sum_i [g_i(x_i - s_i) + (y_i - x_i) \cdot g'_i(x_i - s_i)]$

$= \sum_{i<j} 2 \cdot [f_{ij}(x_i - x_j) + \frac{(y_i - y_j)}{2} \cdot f'_{ij}(x_i - x_j) - \frac{(x_i - x_j)}{2} \cdot f'_{ij}(x_i - x_j)]$

$+ \sum_i [g_i(x_i - s_i) + ((y_i - s_i) - (x_i - s_i)) \cdot g'_i(x_i - s_i)]$ (3)

Now let $(\lambda, \mu) \in A_1 \cap A_2 \implies$

$f(x_i - x_j) + \frac{(y_i - y_j) - (x_i - x_j)}{2} \cdot f'(x_i - x_j) \leqslant \lambda \cdot f(y_i - y_j) + \mu \cdot f(x_i - x_j) \implies$

$w_{ij} \cdot f(x_i - x_j) + w_{ij} \cdot \frac{(y_i - y_j) - (x_i - x_j)}{2} \cdot f'(x_i - x_j) \leqslant \lambda \cdot w_{ij} \cdot f(y_i - y_j) + \mu \cdot w_{ij} \cdot f(x_i - x_j) \implies$

$f_{ij}(x_i-x_j)+\frac{(y_i-y_j)}{2}\cdot f'_{ij}(x_i-x_j)-\frac{(x_i-x_j)}{2}\cdot f'_{ij}(x_i-x_j) \leqslant \lambda\cdot f_{ij}(y_i-y_j)+\mu\cdot f_{ij}(x_i-x_j) \Longrightarrow$

$\sum_{i<j} 2\cdot [f_{ij}(x_i-x_j)+\frac{(y_i-y_j)}{2}\cdot f'_{ij}(x_i-x_j)-\frac{(x_i-x_j)}{2}\cdot f'_{ij}(x_i-x_j)] \leqslant$

$\lambda\cdot\sum_{i<j} 2\cdot f_{ij}(yi-yj)+\mu\cdot\sum_{i<j} 2\cdot f_{ij}(x_i-x_j)$ (4)

Respectively,

$g(x_i-s_i)+((y_i-s_i)-(x_i-s_i))\cdot g'(x_i-s_i) \leqslant \lambda\cdot g(y_i-s_i)+\mu\cdot g(x_i-s_i) \Longrightarrow$

$w_i\cdot g(x_i-s_i)+w_i\cdot((y_i-s_i)-(x_i-s_i))\cdot g'(x_i-s_i) \leqslant \lambda\cdot w_i\cdot g(y_i-s_i)+\mu\cdot w_i\cdot g(x_i-s_i) \Longrightarrow$

$g_i(x_i-s_i)+((y_i-s_i)-(x_i-s_i))\cdot g'_i(x_i-s_i) \leqslant \lambda\cdot g_i(y_i-s_i)+\mu\cdot g_i(x_i-s_i) \Longrightarrow$

$\sum_i[g_i(x_i-s_i)+((y_i-s_i)-(x_i-s_i))\cdot g'_i(x_i-s_i)] \leqslant \lambda\cdot\sum_i g_i(y_i-s_i)+\mu\cdot\sum_i g_i(x_i-s_i)$ (5)

$(3),(4),(5) \Longrightarrow$

$\sum_i[C_i(x_i,x_{-i})+(y_i-x_i)\frac{d}{d_{xi}}C_i(x_i,x_{-i})] \leqslant \lambda\cdot\sum_{i<j} 2\cdot f_{ij}(yi-yj)$

$+\mu\cdot\sum_{i<j} 2\cdot f_{ij}(xi-xj)+\lambda\cdot\sum_i g_i(y_i-s_i)+\mu\cdot\sum_i g_i(x_i-s_i) = \lambda\cdot SC(y)+\mu\cdot SC(x)$

As a result,

$\forall(\lambda,\mu)\in A_1\cap A_2 \Longrightarrow \sum_i[C_i(x_i,x_{-i})+(y_i-x_i)\frac{d}{d_{xi}}C_i(x_i,x_{-i})] \leqslant \lambda\cdot SC(y)+\mu\cdot SC(x)$ □

Because of the previous observation $PoA \leqslant \inf\{\frac{\lambda}{1-\mu},(\lambda,\mu)\in A_1\cap A_2\}$. As we have said before we will find $A_1\cap A_2$ and then we will find $inf\{\frac{\lambda}{1-\mu},(\lambda,\mu)\in A_1\cap A_2\}$.

Observation 5. $inf\{\frac{\lambda}{1-\mu},(\lambda,\mu)\in A_1\cap A_2\} \leqslant 2$

Proof. It is easy to verify that $\forall x,y \geqslant 0$:

$$\begin{cases} x^2+\frac{(y-x)}{2}\cdot 2\cdot x \leqslant y^2+\frac{x^2}{2} \\ x^2+\frac{(y-x)}{2}\cdot 2\cdot x \leqslant y^2+\frac{x^2}{2} \end{cases}$$

Then, $(1,\frac{1}{2})\in A_1\cap A_2 \Longrightarrow inf\{\frac{\lambda}{1-\mu},(\lambda,\mu)\in A_1\cap A_2\} \leqslant 2$ □

The last observation shows as that the local smoothness technique will produce an upper bound for PoA lower than 2, that we have previously very easily found Something that encourages us to continue.

Now, it's time to find a closed from for the set $A_1 \cap A_2$. We will find closed forms for A_1, A_2 respectively and then we will find their intersection.

Finding A_1

It is easy to see that $A_1 := \{(\lambda, \mu) : \ \lambda > 0, \mu < 1\} \cap A_1'$

,where $A_1' := \{(\lambda, \mu) : \ \forall x, y \geqslant 0 : \ f(x) + \frac{(y-x)}{2} \cdot f'(x) \leqslant \lambda \cdot f(y) + \mu \cdot f(x)\} \equiv \{(\lambda, \mu) : \ \forall x, y \geqslant 0 : \ x \cdot y \leqslant \lambda \cdot y^2 + \mu \cdot x^2\}$.

We have described $A_1' \subseteq R^2$ in a very simple way and the next observation produces a closed form for the set.

Observation 6. $(\lambda, \mu) \in A_1' \iff \{\lambda \geqslant \frac{1}{4 \cdot \mu}, \lambda > 0, \mu > 0\}$

Proof. $\Longrightarrow$

$\forall x, y \geqslant 0 : \ x \cdot y \leqslant \lambda \cdot y^2 + \mu \cdot x^2 \Longrightarrow$

$\forall x, y > 0 : \ x \cdot y \leqslant \lambda \cdot y^2 + \mu \cdot x^2 \Longrightarrow$

$\forall x, y > 0 : \ \frac{y}{x} \leqslant \lambda \cdot (\frac{y}{x})^2 + \mu \Longrightarrow$

$\forall a > 0 : \ \lambda \cdot a^2 - a + \mu \geqslant 0 \Longrightarrow$

$\{\Delta \leqslant 0, \lambda > 0\} \Longrightarrow$

$\{1 - 4 \cdot \lambda \cdot \mu \leqslant 0, \lambda \geqslant 0\} \Longrightarrow \{4 \cdot \lambda \cdot \mu \geqslant 1, \lambda \geqslant 0\} \Longrightarrow \{\lambda \geqslant \frac{1}{4 \cdot \mu}, \lambda > 0, \mu > 0\}$

$\Longleftarrow$

$\{\lambda \geqslant \frac{1}{4 \cdot \mu}, \lambda > 0, \mu > 0\} \Longrightarrow$

$\{1 - 4 \cdot \lambda \cdot \mu \geqslant 0, \lambda > 0, \mu > 0\} \Longrightarrow$

$\forall a : \lambda \cdot a^2 - a + \mu \geqslant 0 \Longrightarrow$

$\forall x > 0, y \geqslant 0 : \; x \cdot y \leqslant \lambda \cdot y^2 + \mu \cdot x^2$

For $x = 0$: $\lambda \cdot y^2 \geqslant 0$ because $\lambda > 0$ $\square$

Since the previous observation provides us a closed form for A_1', we can easily find that $A_1 := \{(\lambda, \mu) \in R^2 : \lambda \geqslant \frac{1}{4 \cdot \mu}, \lambda > 0, 0 < \mu < 1\}$. We will procceed with finding an closed form for the $A_2 \subseteq R^2$.

Finding A_2

As before we can easily see that: $A_2 := \{(\lambda, \mu) : \; \lambda > 0, \mu < 1\} \cap A_2'$,where $A_2' := \{(\lambda, \mu) : \; \forall x, y \geqslant 0 : \; g(x) + (y - x) \cdot g'(x) \leqslant \lambda \cdot g(y) + \mu \cdot g(x)\} \equiv \{(\lambda, \mu) : \; \forall x, y \geqslant 0 : \; 2 \cdot x \cdot y \leqslant \lambda \cdot y^2 + (\mu + 1)x^2$. We will follow the same path as before.

Observation 7. $(\lambda, \mu) \in A_2' \iff \{\lambda \geqslant \frac{1}{\mu+1}, \lambda > 0, \mu > -1\}$

Proof. $\Longrightarrow$

$\forall x, y \geqslant 0 : \; 2 \cdot x \cdot y \leqslant \lambda \cdot y^2 + (\mu + 1) \cdot x^2 \Longrightarrow$

$\forall x, y > 0 : \; 2 \cdot x \cdot y \leqslant \lambda \cdot y^2 + (\mu + 1) \cdot x^2 \Longrightarrow$

$\forall x, y > 0 : \; 2 \cdot \frac{y}{x} \leqslant \lambda \cdot (\frac{y}{x})^2 + (\mu + 1) \Longrightarrow$

$\forall a > 0 : \; \lambda \cdot a^2 - 2 \cdot a + (\mu + 1) \geqslant 0 \Longrightarrow$

$\{\Delta \leqslant 0, \lambda > 0\} \Longrightarrow$

$\{4 - 4 \cdot \lambda \cdot (\mu + 1) \leqslant 0, \lambda \geqslant 0\} \Longrightarrow$

$\{\lambda \cdot (\mu + 1) \geqslant 1, \lambda \geqslant 0\} \Longrightarrow$

$\{\lambda \geqslant \frac{1}{\mu+1}, \lambda > 0, \mu > -1\}$

$\Longleftarrow$

$\{\lambda \geqslant \frac{1}{\mu+1}, \lambda > 0, \mu > -1\} \Longrightarrow$

$\{4 - 4 \cdot \lambda \cdot \mu > -1, \lambda > 0\} \Longrightarrow$

$\forall a : \lambda \cdot a^2 - 2 \cdot a + (\mu + 1) \geqslant 0 \Longrightarrow$

$\forall x > 0, y \geqslant 0 :\ 2 \cdot x \cdot y \leqslant \lambda \cdot y^2 + (\mu + 1) \cdot x^2$

For $x = 0$: $\lambda \cdot y^2 \geqslant 0$ because $\lambda > 0$ $\square$

As a result, $A_2 := \{(\lambda, \mu) \in R^2 : \lambda \geqslant \frac{1}{\mu+1}, \lambda > 0, -1 < \mu < 1\}$.

Now, we can have a closed form for $A_1 \cap A_2$:

$A_1 := \{(\lambda, \mu) \in R^2 : \lambda \geqslant \frac{1}{4 \cdot \mu}, \lambda > 0, 0 < \mu < 1\}$

$A_2 := \{(\lambda, \mu) \in R^2 : \lambda \geqslant \frac{1}{\mu+1}, \lambda > 0, \mu > -1\}$

The intersection of the previous two sets is:

$$A_1 \cap A_2 := \{(\lambda, \mu) \in R^2 : \lambda > 0, 0 < \mu < 1, \lambda \geqslant \begin{cases} \frac{1}{4 \cdot \mu} & \text{if } 0 < \mu \leqslant \frac{1}{3} \\ \frac{1}{\mu+1} & \text{if } \frac{1}{3} < \mu < 1 \end{cases} \}$$

We last closed form for $A_1 \cap A_2$ is a very useful tool in order to find the $inf\{\frac{\lambda}{1-\mu}$:

$(\lambda, \mu) \in A_1 \cap A_2\}$. The following theo

$$A_1 := \{(\lambda, \mu) : \ \forall x, y \geqslant 0 : \ x \cdot y \leqslant \lambda \cdot y^2 + \mu \cdot x^2 \ , \lambda > 0, \mu < 1\}$$

$$\equiv \{(\lambda, \mu) \in R^2 : \lambda \geqslant \frac{1}{4 \cdot \mu}, \lambda > 0, 0 < \mu < 1\}$$

rem uses this closed form to prove that $inf\{\frac{\lambda}{1-\mu} : (\lambda, \mu) \in A_1 \cap A_2\} = \frac{9}{8}$.

Theorem 2. $inf\{\frac{\lambda}{1-\mu} : (\lambda, \mu) \in A_1 \cap A_2\} = \frac{9}{8}$

Proof. Let the lines $h_k(\mu) = -k \cdot \mu + k, k > 1$. Let $(\lambda, \mu) \in h_k$ then $\lambda = -k \cdot \mu + k \Longrightarrow$ $\forall (\lambda, \mu) \in h_k : \ \frac{\lambda}{1-\mu} = k$.

We can easily see that $(\frac{3}{4}, \frac{1}{3}) \in A_1 \cap A_2$ and that $h_{\frac{9}{8}}(\frac{1}{3}) = \frac{3}{4}$.

As a result, $inf\{\frac{\lambda}{1-\mu} : (\lambda, \mu) \in A_1 \cap A_2\} \leqslant \frac{9}{8}$.

Now, we have only to prove that there is no other point $(\lambda^*, \mu^*) \in A_1 \cap A_2$ such that $\frac{\lambda}{1-\mu} = k^* < \frac{9}{8}$. Let (λ^*, μ^*) such that $\frac{\lambda}{1-\mu} = k^* < \frac{9}{8}$ then $(\lambda^*, \mu^*) \in h^*_k$. We can also observe that $(k^*, 0) \in h_{k^*}$, $(\frac{9}{8}, 0) \in h_{\frac{9}{8}}$ and that $h_{k^*}, h_{\frac{9}{8}}$ intersect at $(0, 1)$. Since $k^* < \frac{9}{8}$, then $\forall \mu \in (0, 1) : \ h_{k^*}(\mu) < h_{\frac{9}{8}}(\mu)$

It is trivial that: $h_{\frac{9}{8}}(\mu) = -\frac{9}{8} \cdot \mu + \frac{9}{8} \leqslant \begin{cases} \frac{1}{4 \cdot \mu} & \text{if } 0 < \mu \leqslant \frac{1}{3} \\ \frac{1}{\mu+1} & \text{if } \frac{1}{3} < \mu < 1 \end{cases}$

Thus, there is no such point in $A_1 \cap A_2$.

∎

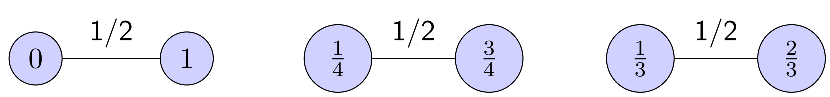
Until now, we have proven that the $PoA \leqslant \frac{9}{8}$ but we have no idea whether this upper bound is tight or not. We will give an instance of our game at which the $\frac{SC(x^*)}{SC(y)} = \frac{9}{8}$, where x^*, y are the Nash Equilibrium and the optimal profile respectively. By this we will prove that $PoA = \frac{9}{8}$ [4, 3] and we will complete this section.

Let the graph with the nodes n_1, n_2 at which $s_1 = 0, s_2 = 1, w_{12} = w_{21} = \frac{1}{2}$ and $w_1 = w_2 = 1$.

- Nash Equilibruim: $\Longrightarrow \frac{d\Phi}{dx_1} = \frac{d\Phi}{dx_2} = 0 \Longrightarrow \{x_1 = \frac{x_2}{3}, x_2 = \frac{x_1+2}{3}\} \Longrightarrow x^* = (\frac{1}{4}, \frac{3}{4})$
- Optimal profile: $\Longrightarrow \frac{dSC(x)}{dx_1} = \frac{dSC(x)}{dx_2} = 0 \Longrightarrow \{x_1 = \frac{x_2}{2}, x_2 = \frac{x_1+1}{2}\} \Longrightarrow$
  $y^* = (\frac{1}{3}, \frac{2}{3})$

Then,

$SC(x^*) = \frac{3}{8}$ and $SC(y^*) = \frac{3}{9} \Longrightarrow \frac{SC(x^*)}{SC(y^*)} = \frac{9}{8}$



Initial Opinions *Nash Equilibrium* *Optimal Profile*

Now, we can be sure that $PoA = \frac{9}{8}$, something that is really encouraging because any Nash Equilibrium may be not an optimal solution but it is really close it.

1.4 Sequential Best Response Dynamics

Before we start this section, it will be useful to remind what we save seen in the previous sections. We already know that our model always has a unique Nash Equilibrium, which can be very easily computed by a matrix multiplication. It is very positive that the Nash Equilibrium can be computed in polynomial time, but this implies that there exist an *external authority* that computes the Equilibrium and then if we want stability in our system this *authority* must force each of the agents to adopt a specific opinion. Apparently, it not is always possible or desirable to force the agents to adopt an opinion. As a result, it will be very important if there were a mechanism according to which the agents play and finally the system would end up to the Nash Equilibrium. At this section, we will present such a mechanism and we will prove that the system always converges to a Nash Equilibrium. Now we will give the definition of the mechanism according to which the agents play. .

Definition 4. *Sequential Best Response Dynamics*

Let a random permutation π *of the* n *agents.Let* x^t *be the opinion vector, such that* x_i^t *is the opinion of the* i*-th agent in* π *at time step* t*. Let* s_π *be the vector of the initial opinions of the agents according to the permutation* π*. Then,*

$$x^0 = s_\pi \text{ and } x_i^t = \begin{cases} x_i^{t-1} & \text{if } t \neq i \bmod n \\ \frac{\sum_{i \neq j} w_{ij} \cdot x_j^{t-1} + w_i \cdot s_i}{\sum_{i \neq j} w_{ij} + w_i} & \text{if } t = i \bmod n \end{cases}$$

Now let's try to describe the above mechanism, there are rounds, each of which consists of n time steps. At each time step, only one agent plays and the i-th player in the permutation is the i-th that is permitted to play in the round. When the game starts each agent adopts his internal opinion, something that is reasonable because he has no information for the opinions of the others. We also assume that the player that is permitted to play at each time step, plays his best response. Consequently, agent i at time t adopts the opinion $\frac{\sum_{i \neq j} w_{ij} \cdot x_j(t-1) + w_i \cdot s_i}{\sum_{i \neq j} w_{ij} + w_i}$, which is the best response according to the situation at time $t-1$.

Before proving that if the agents play according to this mechanism, they always converge to a Nash Equilibrium, we will give some intuition on it. We have already seen in the first section that the unique Nash Equilibrium of our model is the unique minimizer of the potential function $\Phi(x)$ and vice versa. We also know that $\Phi(x_i, x{-}i) - \Phi(x'_i, x{-}i) = C_i(x_i, x{-}i) - C_i(x'_i, x{-}i)$ (7). According to the above mechanism at each time step, the player that is permitted to play, plays his best response. Thus, his personal cost is reduced and according to the equation (7), at each time step the potential function is reduced. Let Φ^t denotes the value of the potential function at time step t. Apparently, $\Phi^0 = \Phi(s_\pi)$. As a result, we know that the potential function has a unique global minimum, is bounded from above and at each time step Φ is reduced, things that make us be confident about the convergence.

Theorem 3. *There is not $a \in R$ such that: $\lim_{t\to\infty} \Phi^t = a > \Phi_{min}$, where Φ_{min} is the global minimum of the potential function.*

Proof. Let's assume that there exist $a \in R$ such that $\lim_{t\to\infty} \Phi^t = a > \Phi_{min}$. Then, $\lim_{t\to\infty} x^t \neq x^*$, where x^* is the Nash Equilibrium. This is true because let $\lim_{t\to\infty} x^t = x^* \implies \lim_{t\to\infty} \Phi^t = \Phi_{min}$, which is opposite to our assumption. Now, we will prove that $\Phi^{t+n} < \Phi^t$, where n is the number of the agents.

From equation (7) we know that $\Phi^{t+1} \leqslant \Phi^t \implies \Phi^{t+n} \leqslant \Phi^t$. Now, let $t_0 \in N$ such that $\Phi^{t_0+n} = \Phi^{t_0}$, which means that all the player had the chance to decrease their personal cost and noone of them did, thus x^{t_0} is a Nash Equilibrium. This means, that $\Phi^{t_0} = \Phi_{min}$ which contradicts our assumption.

$$\implies \forall t \in N: \ \Phi^{t+n} < \Phi^t \ (8)$$

Now, let $\epsilon(t) = \Phi^t - a$. Then, $\forall t \in N : \epsilon(t) \geqslant 0$ because let t_0 such that $\epsilon(t_0) < 0 \Longrightarrow$ $\Phi_0^t < a \Longrightarrow \lim_{t\to\infty} \Phi^t < a$, which is a contradiction.

Because of equation (8) : $\forall t \in N :\ \epsilon(t+n) < \epsilon(t)$ and $\epsilon(t) \geqslant 0$

$$\Longrightarrow \lim_{t\to\infty} \epsilon(t) = 0 \ (9)$$

Now, it easy to see that $\Phi^t - \Phi^{t+1} \leqslant \epsilon(t)$ because let $t_0 \in N$ such that $\Phi^{t_0} - \Phi^{t_0+1} > \epsilon(t_0) \Longrightarrow \Phi^{t_0+1} < a \Longrightarrow \lim_{t\to\infty} \Phi^t < a$, which contradicts our assumption.Thus,

$$\forall t \in N :\ 0 \leqslant \Phi^t - \Phi^{t+1} \leqslant \epsilon(t) \ (10)$$

$$(9), (10) \Longrightarrow \lim_{t\to\infty} (\Phi^t - \Phi^{t+1}) = 0 \ (11)$$

We now define the sequences $\forall i \in \{1, n\} : \alpha_k^i = C_i(x^{n\cdot k+i}) - C_i(x^{n\cdot k+i+1})$.

It is easy to verify that $\alpha_k^i \geqslant 0$. We will show that $\forall i \in \{1, n\} : \lim_{k\to\infty} \alpha_k^i = 0$.

Because of equation (11):

$\forall \delta > 0, \exists t_0$ such that:

$\forall t \geqslant t_0 :\ \Phi^t - \Phi^{t+1} < \delta$.

Let $t_1 = t_0 + n - (t_0 \ mod \ n) + \ i = n \cdot k_1 + i \geqslant t_0$. Then,

$\forall k \geqslant k_1 : n \cdot k + i \geqslant n \cdot k_1 + i \geqslant t_0 \Longrightarrow \Phi^{n\cdot k+i} - \Phi^{n\cdot k+i+1} < \delta \Longrightarrow$

$C_i(x^{n\cdot k+i}) - C_i(x^{n\cdot k+i+1}) < \delta \Longrightarrow \alpha_k^i < \delta \Longrightarrow$

$\forall \delta > 0, \exists k_1$ such that $\forall k \geqslant k_1 : \alpha_k^i < \delta$. We also know that $\forall k : \alpha_k^i \geqslant 0$. Thus,

$$\forall i \in \{1, n\} : \lim_{k\to\infty} {\alpha_k}^i = 0$$

$$\begin{aligned}\lim_{k\to\infty}\alpha_k^i &= \lim_{k\to\infty}(C_i^{n\cdot+i} - C_i^{n\cdot+i}) \\ &= (\sum_{i\neq j} w_{ij} + w_i)^2 \lim_{k\to\infty}(x_i^{n\cdot k+i+1} - \frac{\sum_{i\neq j} w_{ij}\cdot x_j^{n\cdot k+i} + w_i \cdot s_i}{\sum_{i\neq j} w_{ij} + w_i})^2 = 0\end{aligned}$$

$\Longrightarrow \forall i \in \{1, n\} : \lim_{k\to\infty} x_i^{n\cdot k+i+1} = \lim_{k\to\infty} \frac{\sum_{i\neq j}(w_{ij}\cdot x_j^{n\cdot k+i} + w_i\cdot s_i)}{\sum_{i\neq j} w_{ij}+w_i}$

$\Longrightarrow \forall i \in \{1, n\} :\ \ \lim_{t\to\infty} x_i^t = \lim_{t\to\infty} \frac{\sum_{i\neq j}(w_{ij}\cdot x_j^t + w_i\cdot s_i)}{\sum_{i\neq j} w_{ij}+w_i} \Longrightarrow$

$$\lim_{t\to\infty} x^t = x^*$$

But, this is impossible because if $\lim_{t\to\infty} x^t = x^*$ then $\lim_{t\to\infty} \Phi^t = \Phi_{min}$. $\square$

Because of the previous theorem it is very easy to prove the convergence to the Sequential Best Response Mechanism. As we have seen before $\Phi(x)$ is a strictly convex function which is bounded from above. We also know that at each time step the potential function deceases. Thus, $\exists a \in R$ such that $\lim_{t\to\infty} \Phi^t = a$, but the above theorem claims that this cannot be possible if $a > \Phi_{min}$. Thus, $\lim_{t\to\infty} \Phi^t = \Phi_{min}$. Because $\Phi(x)$ is a continous function if $\lim_{t\to\infty} \Phi^t = \Phi_{min} \Longrightarrow \lim_{t\to\infty} = x^*$, where x^* is a Nash Equilibrium.

Now, we can be sure that if the agents play according to the mechanism that we described it is certain that they will converge to a Nash Equilibrium. This also provides us an alternate method to compute the Nash Equilibrium. We just let the players play till the reach the equilibrium.(Of course we don't know whether this will happen after a polynomial number of steps). An easy observation is that in the proof of the convergence we haven't used that at the first time step $x^0 = s_\pi$. This means that at the first time step x^0 can be a random vector with values in $[0, 1]$ without any effect to the convergence.

1.5 Parallel Best Response Dynamics

In the previous section we have seen a mechanism at which one agent play at each time step making always his best response. In this section, we will decribe a similar but quite different mechanism, at which at each time step all the agents play their best response [10]. We will call this mechanism Parallel Best Response Mechanism. Obviously, we would like to know that if the agents play according to Parallel Best Response Mechanism, they will converge to a Nash Equilibrium and we will prove this latter in this section. Before we start analyzing this mechanism, we will give a more formal definition.

Definition 5. *Parallel Best Response Dynamics*

Let a random permutation π of the n agents.Let x^t be the opinion vector, such that x_i^t is the opinion of the i-th agent in π at time step t. Let s_π be the vector of the inital opinions of the agents according to the permutation π. Then,

$$x_i^t = \begin{cases} s_{\pi_i} & \textit{if } t = 0 \\ \frac{\sum_{i \neq j} w_{ij} \cdot x_j^{t-1} + w_i \cdot s_i}{\sum_{i \neq j} w_{ij} + w_i} & \textit{if } t > 0 \end{cases}$$

As we have said before this model may be similar to the Sequential Best Response, but they have some major differences that make this case more difficult to prove that according to this mechanism the system converges to a Nash Equilibrium. In the previous section, we knew that at each time step the potential function $\Phi(x)$ decreased, by using this property we proved the convergence. Unfornunately, this property is not valid in this case and we have to find another aproach. As we have said many times before the problem of convergence to a Nash Equilibrium is equivalent with the problem of minimizing the potential function $\Phi(x)$. As we already know we want to minimize $\Phi(x), x \in [0,1]^n$, which is a constrained optimization problem, but we already know that there is a unique global minimum $x^* \in [0,1]^n$ and no local minimums, something that simplifies our problem because we can use unconstrained optimization techniques like gradient method. Before we prove the convergence we will make an introduction to the gradient descent methods [1].

Assume that we want to minimize the function $f(x)$. Many gradient methods are specified in the form: $x^{k+1} = x^k - a^k \cdot D^k \cdot \nabla f(x^k)$. , where D^k is a positive definite symmetric matrix. According to the a^k, D^k that we select we have different gradient methods that differ at the convergence and at the convergence rate. There also many conditions for various gradient methods that can garantee us the convergence of the method. Some of the most popular methods are:

- Steepest Descent: $D^k = I$,$k = 1, 2, \cdots$ and I is the $n \times n$ identity matrix
- Newton's Method: $D^k = (\nabla^2 f(x^k))^{-1})$, $k = 1, 2, \cdots$ and provided $\nabla^2 f(x^k)$ is positive definite.

- Diagonially Scaled Steepest Descent:

$$D^k = \begin{pmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{pmatrix}$$

where $d_i^k = (\frac{d^2 f(x^k)}{(dx_i)^2})^{-1}$ and $d_k^i > 0$.

Observation 8. *The Parallel Best Response Mechanism is the Diagonially Scaled Steepest Descent with $a^k = 1$ applied to the potential function $\Phi(x)$.*

Proof. Let's take the the Diagonially Scaled Steepest Descent with $a^k = 1$ applied to the potential function $\Phi(x)$. Then, $x^{k+1} = x^k - D^k \cdot \nabla\Phi(x^k)$ (12)
,where $D_{ij}^k = \begin{cases} 0 & \text{if } i \neq j \\ (\frac{d^2\Phi(x^k)}{(dx_i)^2})^{-1} & \text{if } i = j \end{cases}$
$\frac{d\Phi(x^k)}{dx_i} = 2(\sum_{i\neq j} w_{ij} + w_i) \cdot x_i - 2 \cdot (\sum_{i\neq j} w_{ij} \cdot x_j^{k-1} + w_i \cdot s_i) \Longrightarrow$
$(\frac{d^2\Phi(x^k)}{(dx_i)^2})^{-1} = \frac{1}{2(\sum_{i\neq j} w_{ij} + w_i)}$
(12) $\Longrightarrow \forall i \in \{1, n\}: \ x_i^t = \frac{\sum_{i\neq j} w_{ij} \cdot x_j^{t-1} + w_i \cdot s_i}{\sum_{i\neq j} w_{ij} + w_i}$.
Thus, the Parallel Best Response Mechanism is a Diagonially Scaled Steepest Descent applied to the potential function $\Phi(x)$. □

The above observation is very important because we have reduced the problem of proving that our mechanism converges to proving that a gradient method converges. Before proving that this gradient method converges to the global minimum we will prove another theorem which we will use in our final proof.

Observation 9. *Let the method $x^{k+1} = A \cdot x^k$, where A symmetric. $\lim_{k\to\infty} x^k = 0$ if and only if $-1 < \lambda(A) < 1$, for all the eigenvalues $\lambda(A)$ of matrix A.*

Proof. It is easy to see that $\lim_{k\to\infty} x^k = 0 \iff \lim_{k\to\infty} ||x^k|| = 0$ So ,we will proof that $\lim_{k\to\infty} ||x^k|| = 0 \iff -1 < \lambda(A) < 1$, where $\lambda(A)$ is the eigenvalues of matrix A.

$\Longrightarrow$

$\lim_{k\to\infty} ||x^k|| = 0$ then $\forall\delta > 0 : \exists k_0$ such that $\forall k \geqslant k_0 :\ ||x^k|| < \delta$.
Then, $\forall ||x^0|| \in R :\ \exists k_0$ such that $\forall k \geqslant k_0$: $||x^k|| < ||x^0|| \Longrightarrow ||A^k \cdot x^0|| < ||x^0|| \Longrightarrow ||A^k \cdot x^0||^2 < ||x^0||^2 \Longrightarrow (x^0)^T \cdot A^{2\cdot k} \cdot x^0 < (x^0)^T \cdot x^0 \Longrightarrow \forall x^0 :\ (x^0)^T \cdot (I - A^{2\cdot k}) \cdot x^0 > 0$.
Then matrix $(I - A^{2\cdot k})$ is positive definite$\Longrightarrow 1 - \lambda^{2\cdot k}(A) > 0$ [15, 19] $\Longrightarrow -1 < \lambda(A) < 1$, for all the eigenvalues $\lambda(A)$ of matrix A..

$\Longleftarrow$

Let $-1 < \lambda(A) < 1$,for all the eigenvalues $\lambda(A)$ of matrix A. Then,
$-1 < \lambda_{max}(A) < 1$. $\lim_{k\to\infty} ||x^k||^2 = \lim_{k\to\infty} ||A^k \cdot x^0||^2 = \lim_{k\to\infty}(x^0)^T \cdot A^{2\cdot k} \cdot x^0 \leqslant \lim_{k\to\infty} ||\lambda_{max}(A)^{2\cdot k}|| \cdot ||x^0||^2 = 0$,($A^2$ is positive definite matrix) $\Longrightarrow$
$\lim_{k\to\infty} ||x^k|| = 0$. □

Theorem 4. *Let $f(x) = \frac{1}{2} \cdot x^T \cdot Q \cdot x - b \cdot x$, where Q is a positive definite matrix. Let the method $x^{k+1} = x^k - a \cdot D \cdot \nabla f(x^k)$, where D is a positive definite matrix and $a > 0$. The method converges to $x^* = Q^{-1} \cdot b$ if and only if $a \in (0, \frac{2}{L})$, where L is the maximum eigenvalue of the matrix $D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}$.*

Proof. Let $x^{k+1} = x^k - a \cdot D \cdot \nabla f(x^k)$ (13).
(13) can be written as :
$x^{k+1} - x^* = (I - a \cdot D \cdot Q)(x^k - x^*) \iff$

$$
\begin{aligned}
D^{-\frac{1}{2}} \cdot x^{k+1} - D^{-\frac{1}{2}} \cdot x^* &= D^{-\frac{1}{2}} \cdot (I - a \cdot D \cdot Q)(x^k - x^*) \\
&= (D^{-\frac{1}{2}} - a \cdot D^{\frac{1}{2}} \cdot Q) \cdot (x^k - x^*) \\
&= (D^{-\frac{1}{2}} - a \cdot D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}} \cdot D^{-\frac{1}{2}}) \cdot (x^k - x^*) \\
&= (I - a \cdot D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}) \cdot (D^{-\frac{1}{2}} \cdot x^k - D^{-\frac{1}{2}} \cdot x^*)
\end{aligned}
$$

Let $y^k = x^k$ and $y^* = x^*$. Then,
$y^{k+1} - y^* = (I - a \cdot D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}) \cdot (y^k - y^*)$.
Now, it is easy to see that $x^{k+1} = x^k - a \cdot D \cdot \nabla f(x^k)$ converges to $Q^{-1} \cdot b$ if and only if $y^{k+1} - y^* = (I - a \cdot D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}) \cdot (y^k - y^*)$ converges to 0
Let $u^{k+1} = y^{k+1} - y^* \Longrightarrow u^k = y^k - y^*$ and $A = I - a \cdot D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}$. Then, according to the previous observation the method converges if and only if $\forall \lambda(A)$ eigenvalue of matrix A: $-1 < \lambda(A) < 1$.
We will show that $\forall \lambda(A) : -1 < \lambda(A) < 1 \Longleftrightarrow -1 < 1 - a \cdot L < 1$, where L is the maximum eigenvalue of the matrix $D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}$.

$\Longrightarrow$

$\forall \lambda(A) : \quad -1 < \lambda(A) < 1 \Longrightarrow -1 < 1 - a \cdot L < 1.$

$\Longleftarrow$

$-1 < 1 - a \cdot L < 1 \Longrightarrow \forall \lambda(D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}) : -1 < 1 - \lambda(D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}})$. But, the matrix $D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}$ is positive definite. As a result, $\forall \lambda(D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}) : \lambda(D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}) > 0 \Longrightarrow \forall \lambda(D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}) : -1 < 1 - \lambda(D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}) < 1$. Consequently, the method converges if and only if $-1 < 1 - a \cdot L < 1 \Longleftrightarrow a \in (0, \frac{2}{L})$. □

Now, we have the necessary tools to prove that the Parallel Best Response Mechanism converges to a Nash Equilibrium. Let the matrices $Q_{n\times n}, b_{n\times 1}$. As we have seen in the first section $\Phi(x) = \frac{1}{2} \cdot x^T \cdot Q \cdot x^T - b \cdot x + c$, where:

$$Q_{ij} = \begin{cases} 2 \cdot (\sum_{i\neq j} w_{ij} + w_i) & \text{if } i = j \\ -2 \cdot w_{ij} & \text{if } i \neq j \end{cases}$$

$b_i = 2 \cdot w_i \cdot s_i$ and $c = \sum_{i \in V} w_i \cdot s_i^2$.

Minimizing $\Phi(x)$ is equivalent as minimizing $\frac{1}{2} \cdot x^T \cdot Q \cdot x^T - b \cdot x$. So, without loss of generality we can set $\Phi(x) = \frac{1}{2} \cdot x^T \cdot Q \cdot x^T - b \cdot x$. Now, we can easily verify that the Parallel Best Response Mechanism is equivalent to the method $x^{k+1} = x^k - a \cdot D \cdot \nabla\Phi(x^k)$ if $a = 1$ and $D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$, $d_i = (\frac{d^2\Phi(x^k)}{(dx_i)^2})^{-1} = \frac{1}{2\cdot(\sum_{i\neq j} w_{ij} + w_i})$.

Because of the previous theorem we just need to show that: $0 < \lambda_{max}(D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}) < 2$

Now, we will give a more clear form to the matrix $D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}$

$D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}} = I - W$ where: $W_{ij} = \begin{cases} 0 & \text{if } i = j \\ \frac{w_{ij}}{\sqrt{\sum_{k\neq i} w_{ik} + w_i} \cdot \sqrt{\sum_{l \neq j} w_{jl} + w_j}} & \text{if } i \neq j \end{cases}$

$$\Longrightarrow \lambda_{max}(D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}) = 1 - \lambda_{min}(W)$$

It is easy to see that the matrix $D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}$ is positive definite. Thus, its eigenvalues are positive. Then, in order to complete our proof, we just need to show that $\lambda_{min}(W) > -1$.

$$\begin{aligned} \text{Let } \Phi'(x) &= -\sum_{i<j} w_{ij} \cdot (x_i + x_j)^2 - \sum_i w_i \cdot x_i^2 \\ &= \frac{1}{2} \cdot x^T \cdot Q' \cdot x, \text{where } Q'_{ij} = \begin{cases} -Q_{ij} & \text{if } i = j \\ Q_{ij} & \text{if } i \neq j \end{cases} \end{aligned} \tag{1.1}$$

We have already assumed that for any connected component of our social network, there exists $w_i > 0$. (This is the minimum condition in order to differ from De Groot model). Because of this assumption $\forall x \in R^n : \Phi'(x) < 0 \Longleftrightarrow x^T \cdot Q' \cdot x < 0 \Longrightarrow Q'$ is a negative definite matrix . It is also easy to verify that since $D^{\frac{1}{2}}$ is a positive definite matrix and Q' is a negative definite matrix. Then, $D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}$ is a negative definite matrix. Notice that $D^{\frac{1}{2}} \cdot Q' \cdot D^{\frac{1}{2}} = -I - W$ and consequently $\forall \lambda(W) : 1 - \lambda(W) < 0 \Longrightarrow \lambda_{min}(W) > -1$ (14).
(14) $\Longrightarrow -\lambda_{min}(W) < 1 \Longrightarrow 1 - \lambda_{min}(W) < 2 \Longrightarrow 0 < \lambda_{max}(D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}) < 2$
Because $D^{\frac{1}{2}} \cdot Q \cdot D^{\frac{1}{2}}$ is a positive definite matrix.
So we have proven that if all agents play at each round simultaneously it is certain that they will end up to the unique Nash Equilibrium of the system.

Chapter 2

De Groot model

2.1 Introduction

DeGroot model is one of the first models trying to descibe how the members of a social network form an opinion according to a specific matter, we consider that the opinion is a real number in $[0, 1]$ [8, 13]. This opinion may denote the probability voting for a specific party or whether it will rain a lot this year. The social network is represented by a weighted directed graph $G(V, E)$ at which each node represents a node in the social network and the weight w_{ij} of each edge in the graph represents the trust that node i has to node j(if the egde $(ij) \notin E$ we can consider $w_{ij} = 0$). There can also exist self loops w_{ii} in the graph that represent the *stubborness* of each node. Without loss of generality we consider that $\forall i \in V : \sum_{(ij)\in E} w_{ij} = 1$. At first all the nodes have an initial opinion which is denoted by the vector $x(0)$ and then at each time step $x(t + 1) = T \cdot x(t)$,where T is the updating matrix of the model. The underlying graph $G(V, E)$ and the updating matrix T have an 1-by-1 relation.

$$T_{ij} = \begin{cases} w_{ij} & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E \end{cases}$$

The question that arise is: Will the agents converge to a specific opinion?

2.2 Markov chains and random walks

In this section, we will see some basic theory of Markov chains and random walks that we will help to continue our analysis to the De Groot model[16]. Let a weighted directed graph $G(V, E)$ such that $\forall i \in V : \sum_{(ij)\in E} w_{ij} = 1$. A *random walk* on G is the following process starting, which occurs in the sequence of discrete steps: Starting at a vertex v_0, at t=1 we select with probalility w_{ij}, one of the edges adjacent to v_0 and we traverse it. In the next vertex, we repeat the previous process.

Definition 6. *A finite discrete time Makov chain is a random walk on weighted directed graph* $G(V, E)$ *such that* $\forall i \in V : \sum_{(ij)\in E} w_{ij} = 1$.

Now, let the random variables $X_0, X_1, ... \in$ V such that

$$\Pr[X_t = v_i] = \Pr[\text{ the Markov chain be at vertex } v_i \text{ at time step t}]$$

Then, $\Pr[X_t = v_j | X_{t-1} = v_i] = w_{ij} \implies \Pr[X_t = v_j] = \sum_{(ij)\in E} w_{ij} \cdot \Pr[X_{t-1} = v_i]$. Now, let the vector $\pi^t \in \mathrm{R}_{\mathrm{n}\times 1}$ such that $\pi_i(t) = \Pr[X_t = v_i]$ and the matrix $A_{n\times n}$ such that $A_{ij} = w_{ij}$. It is easy to see that:

$$\pi^T(t) = \pi^T(t-1) \cdot A$$

A is called the *transition matrix* and also notice that for a given graph $G(V, E)$ $A = T$ where T is the matrix in the updating rule of De Groot model.

Definition 7. *A stationary distribution π^* is a stochastic vector such that* $(\pi^*)^T = (\pi^*)^T \cdot A$*.*

As we have seen, $\pi^T(t) = \pi^T(t-1) \cdot A$. So, if such a π^* exists and there exists t_0 such that $\pi(t_0) = \pi^*$. Then, $\forall t \geqslant t_0 : \pi(t) = \pi^*$. As a result, if we start a Markov chain with distribution over the vertices π^* or at some time step the distribution become π^*. Then, this distribution will hold forever. Someone can also observe that π^* depends only at the matrix A and not at the initial distribution $\pi(0)$. Now, a very important question arise. Under what conditions there exists a stationary distribution at a given Markov chain? Before we continue, we will give some definition that are necessary in order to continue.

Definition 8. *A Markov chain is irreducible if and only if its underlying graph is strongly connected.*

Let the underlyig graph G of a given Markov chain and $l_1, l_2, ..., l_k$ are the lengths of all directed cycles in graph G. *Periodicity* of the Markov chain is the $gcd(l_1, l_2, ..., l_k)$.

Definition 9. *A Markov chain is aperiodic if and only if its periodicity is 1.*

Now we will give the fundamental theorem of the Markov chains:

Theorem 5. *Any finite, irreducible and aperiodic Markov chain with transition matrix A:*

- *has a unique stationary distribution π^*.*

- *for any initial distribution* $\pi(0)$*:* $\lim_{t\to\infty}(\pi^T(0)\cdot A^t) = \pi^*$.

This above theorem tells us something really interesting. Let a Markov chain which is finite, irreducible and aperiodic. If we let the chain procced for a long time, then, we will have a distribution π^* over the vetrices that is independent of the initial distribution $\pi(0)$.

2.3 Strongly connected case

In this section we will study a special case of De Groot at which the undelying graph of the updating matrix T is strongly connected. A directed graph is strongly connected if and only if $\forall(u,v)\in V$ there exists a directed path from u to v. Notice that in this case there are no opinion leaders($i\in V\colon\, T_{ii}<1$). We present this case because it is simple and it will help us to analyze the general case. As we will see consesus in the society is possible and also notice that if there were two or more opinion leaders ($T_{ii}=1$), consesus in the society wouldn't be possible. As it is already said the major questions that we are interested in are whether consesus in the society is possible and whether there exists a time step at which all the agents adopt a specific opinion. We will see that in this case the agents either adopt the same opinion or the don't adopt any opinion at all. Apparently, we would like to know the conditions under which the agents end up to consesus, how fast do they converge to the conseus and of course a way to compute this consesus.

2.3.1 Convergence to consesus

From the previous section we already know that the updating matrix T can be viewed as a trasition matrix of a Markov chain. From the fundamental theorem of Markov chains, we know that if a Markov chain is irreducible and aperiodic. Then, for any initial distribution π_0, $\pi_0^T \cdot \lim_{t\to\infty} T^t = \pi^*$, where π^* is the stationary distribution. Without abuse of notation we will say that an updating matrix T is strongly connected and aperiodic if the underlyig graph is strongly connected and aperiodic. From now on we will consider the updating matrix T as strongly connected without mentioning it.

Observation 10. *If the updating matrix T is aperiodic then T^t converges and $\lim_{t\to\infty} T^t = 1^n \cdot (\pi^*)^T$, where 1^n is the vector with 1.*

Proof. Since T is strongly connected and aperiodic, the Markov chain with transition martix T is irreducible and aperiodic. From Markov chains theory we know that $\lim_{t\to\infty} \pi_0^T \cdot T^t = (\pi^*)^T$, for every initial distribution π_0. Now, let the distribution u_i such that $u_i(i) = 1$ and 0 otherwise. Let also $T^t = \begin{pmatrix} T_1^t \\ T_2^t \\ \vdots \\ T_n^t \end{pmatrix}$.

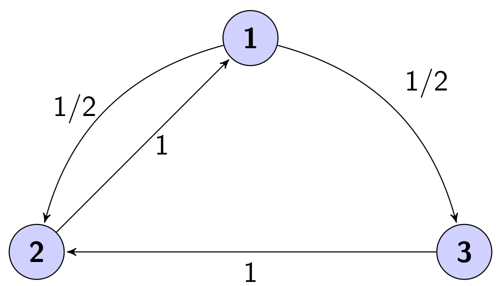
Then,

$\forall i \in V : \lim_{t\to\infty} u_i^T \cdot T^t = \pi^* \Longrightarrow \forall i \in V : T_i^t = \pi^* \Longrightarrow \lim_{t\to\infty} T^t = 1^n \cdot \pi^*$ □

Now, we will give an example of this case. Let the social network with the updating matrix

$$T = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

the graph of this matrix is illustrated in the following figure.



$$x(1) = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1(0) \\ x_2(0 \\ x_3(0 \end{pmatrix}, \quad x(2) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1(0) \\ x_2(0 \\ x_3(0 \end{pmatrix}$$

. It is easy to see that this matrix is strongly connected and aperiodic and that

$$T^{\infty} = \begin{pmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

Now, let $x(\infty) = \lim_{t\to\infty} T^t \cdot x(0) = \begin{pmatrix} \frac{2}{5} \cdot x_1(0) + \frac{2}{5} \cdot x_2(0) + \frac{1}{5} \cdot x_3(0) \\ \frac{2}{5} \cdot x_1(0) + \frac{2}{5} \cdot x_2(0) + \frac{1}{5} \cdot x_3(0) \\ \frac{2}{5} \cdot x_1(0) + \frac{2}{5} \cdot x_2(0) + \frac{1}{5} \cdot x_3(0) \end{pmatrix}$

The above example shows us not only a strongly connected and aperiodic matrix that converges, but also that a the 3-member social network that used this updating matrix, finally adopted the same opinion.

Observation 11. *Let a social network using DeGroot model with updating matrix T, which is strongly connected and aperiodic. Then, $\forall i \in V : x_i(\infty) = \sum_{i \in V} \pi_i^* \cdot x_i(0)$, where $(\pi^*)^T \cdot T = (\pi^*)^T$.*

Proof. From the previous observation we know that: $\lim_{t\to\infty} T^t = 1^n \cdot (\pi^*)^T \implies x(\infty) = \lim_{t\to\infty} T^t \cdot x(0) = 1^n \cdot (\pi^*)^T \implies \forall i \in V : x_i(\infty) = \sum_{i\in V} \pi_i^* \cdot x_i(0)$ □

Observations 10 and 11 are very important because they give the sufficient conditions for convergence and consesus and at the same time it give us a closed form for the opinion that the agents eventually adopt. But this not enough, we would like to fully characterize the convergence in the case of a strongly connected updating matrix T, which means that we want to find necessary and sufficient conditions. What we will see later in this section is that, aperiodicity is also necessary condition in this case. From now on we will focus in proving that if matrix T is not aperiodic then it cannot be convergent.

Let a strongly connected updating matrix T, with periodicity $d \geqslant 2$ that converges. Thus, $\lim_{t\to\infty} T^t = T^*$. Apparently, $\forall x(0) \in \mathrm{R}^n : \lim_{t\to\infty} T^t \cdot x(0) = T^* \cdot x(0)$. What we will do is that we will use the periodicity of T to construct an instance $x(0)$ such that $T^t \cdot x(0)$ doesn't converge. Before doing this we will give an observation that we will help us with this construction.

Observation 12. *Let a stongly connected matrix updating matrix T, which has periodicity $d \geqslant 2$. Now let $n^* \in V$ and $A_0, A_1, ..., A_{d-1} \subseteq V$ such that $n \in A_i$ if and only if length(path(n^*, n)) $\equiv i$ mod d. Then, $A_0, A_1, ..., A_{d-1}$ is a partition of V.*

Proof. Since the T is strongly connected then it is certain that $\forall n \in V$ there exist a path form n^* to n. Thus, $\forall n \in V : \exists i \in \{0, d-1\}$ such that $n \in A_i$. Now, we just have to show that there if $n \in A_i$ then $n \notin A_j$, $\forall j \neq i$.

By definition $gcd(C_1, C_2, ..., C_k) = d \geqslant 2$ where $C_1, C_2, ..., C_k$ are the directed cycles of the graph. Let a path $p_{n_1 \to n_2}$ from n_1 to n_2 such that length$(p_{n_1 \to n_2}) \equiv i \bmod d$. Then, there exists a path $p'_{n_2 \to n_1}$ form n_2 to n_1(T is stongly connected) such that length$(p'_{n_1 \to n_2}) \equiv (d-i) \bmod d$. This holds because let the sequence of nodes $P_{n_1 \to n_1} = (p_{n_1 \to n_2}, p'_{n_2 \to n_1})$ (notice that $P_{n_1 \to n_1}$ is not a cycle because it is possible that some nodes are repeated). Then, if we *shrink* all the cycles contained into $P_{n_1 \to n_1}$ we will get $P'_{n_1 \to n_1}$,which is obviously a cycle $\Longrightarrow$ length$(P'_{n_1 \to n_1}) = a' \cdot d$. But, the lengths of all the cycles that we removed from $P_{n_1 \to n_1}$ are also multiples of d. As a result, length$(P_{n_1 \to n_1}) = a \cdot d \Longrightarrow$ length$(p'_{n_2 \to n_1}) \equiv (d-i) \bmod d$.

Now let $n \in A_i, A_j, i > j$ then there exists a path $p_{1(n^* \to n)}$ with length$(p_{1(n^* \to n)}) \equiv i \bmod d$ and consequently a path $p'_{1(n \to n^*)}$ with length$(p'_{1(n^* \to n)}) \equiv (d-i) \bmod d$. There exists also a path $p_{2(n^* \to n)}$ with length$(p_{2(n^* \to n)}) \equiv j \bmod d$, as also a path $p'_{2(n \to n^*)}$ with length$(p'_{2(n^* \to n)}) \equiv (d-j) \bmod d$. Then $P_{n^* \to n^*} = (p_{1(n^* \to n)}, p'_{2(n \to n^*)})$, which has length $a \cdot d + (i - j)$. With the same argument as above, if we remove all the cycles of $P_{n^* \to n^*}$ we will get a cycle a cycle $P'_{n^* \to n^*}$, with length$(P'_{n^* \to n^*}) \equiv (i-j) \bmod d$. Something that contradicts the periodicity d of the matrix T. As a result, we have proven that $A_0, A_1, ..., A_{d-1}$ is a partition of V. □

Now, we are ready to give the proof that the periodicity is a necessary condition for convergence.

Observation 13. *Let T be a strongly connected updating matrix. If T has periodicity*

$d \geqslant 2$ then T doesn't converge.

Proof. Let $x(t) = T \cdot x(t-1) = T^t \cdot x(0)$. As we have already discussed we just need to find a $x(0) \in \mathrm{R}^n$ so as $x(t)$ doesn't converge. Let a $n^* \in V$ and $A_0, A_1, ..., A_{d-1}$ a partition of V, as was defined in the previous observation. Notice that $n^* \in A_0$ and that $\forall i \in V : T_{ii} = 0$ since T has periodicity $d \geqslant 2$. Let $x_i(0) = \begin{cases} \frac{1}{|A_0|} & \text{if } i \in A_0 \\ 0 & \text{otherwise} \end{cases}$.

We can imagine that $x_i(t)$ is a quantity of "money" that node i has at time step t. Then at $t+1$ node i tranfers all of its money to its neighbours proportionally to T_{ij}. Observe that since $T_{ii} = 0$ then noone of the nodes keeps money for himself and that the total amount of money in the network remains always 1. Since each node belongs to a unique A_i and all nodes of A_i give all their money to the nodes of A_{i+1}. We can see that the total amount of money is tranferred at time step $t \equiv (i+1) \bmod d$ from A_i to A_{i+1}. All nodes in A_i remain moneyless until $t = i \bmod d$ at which A_{i-1} gives to A_i. Now let $\lim_{t\to\infty} x(t) = x^*$. Then $\lim_{t\to\infty} x_{A_i}(t) = x^*_{A_i}$, where $x_{A_i}(t)$ is the vector denoting the money each node in A_i has at time step t. As we have explained

$$x_{A_i}(t) \begin{cases} \neq 0 & \text{if } t \equiv i \bmod d \\ = 0 & \text{otherwise} \end{cases}$$

But in this case $x_{A_i}(t)$ can only converge to 0: $\forall i \in \{0, d-1\} : \lim_{t\to\infty} x_{A_i}(t) = 0 \Longrightarrow \lim_{t\to\infty} x(t) = 0$ something that is impossible because $\forall t : \sum_{\forall i \in V} x_i(t) = 1$. Thus, our assumption that $x(t)$ converges doesn't hold. Since we can find such a $x(0)$,such that $T^t \cdot x(0)$ doesn't converge, for every strongly connected updating matrix T with periodicity $d \geqslant 2$. Then, if T is not aperiodic then it doesn't converge. □

Finally, we can express our final theorem that fully describes the case of the strongly connected matrix T.

Theorem 6. *A strongly connected updating T converges if and only if T is aperiodic.*

Proof. The proof of this Theorem is a direct implementation of observation 11 and observation 13. □

The last Theorem also show us that the DeGroot model in this case converges if and only if the updating matrix T is aperiodic.

2.3.2 Convergence rate to Consensus

We just have seen that if the updating matrix T is strongly and connected and aperiodic, then the social network converges to a consesus. Apparently, we would like to know how much time do the society members need in order to reach consesus? This is the question that we will try to answer in next lines. From now one, we will consider matrix T as strongly connected and aperiodic.

Since T is a stochastic matrix, strongly connected and aperiodic then from Perron-Frobenius Theorem it follows that [19]: $1 = \lambda_1 > |\lambda_2| \geqslant ... \geqslant |\lambda_n|$, where λ_i are the eigenvalues of the matrix T. According to eigendecomposition

$$T = U^{-1} \cdot \Lambda \cdot U$$

, where Λ is the diagonal matrix with entries the eigenvalues of T and U is the matrix of corresponding row eigenvectors. It follows that:

$$T^t_{ij} = \pi^*_j + \sum_{k \geqslant 2} \lambda^t_k \cdot U^{-1}_{ik} \cdot U_{kj} \Longrightarrow$$

$$x_i(t) = \sum_{j \in V} T^t \cdot x_j(0) + \sum_{j \in V} x_j(0) \cdot \sum_{k \geqslant 2} \lambda_k^t \cdot U_{ik}^{-1} \cdot U_{kj} \Longrightarrow$$

$$\begin{aligned}
|x_i(t) - x_i(\infty)| &= |\sum_{j \in V} x_j(0) \cdot \sum_{k \geqslant 2} \lambda_k^t \cdot U_{ik}^{-1} \cdot U_{kj}| \\
&\leqslant \sum_{j \in V} x_j(0) \cdot |\sum_{k \geqslant 2} \lambda_k^t \cdot U_{ik}^{-1} \cdot U_{kj}| \\
&\leqslant \sum_{j \in V} x_j(0) \cdot |\lambda_2|^t \cdot |\sum_{k \geqslant 2} \cdot U_{ik}^{-1} \cdot U_{kj}| \\
&\leqslant C' \cdot \sum_{j \in V} x_j(0) \cdot |\lambda_2|^t \\
&\leqslant C \cdot |\lambda_2|^t
\end{aligned}$$

Thus, the convergence rate is $|\lambda_2|$, noticy that the λ_2 may be complex number, but we know that $|\lambda_2| < 1$.

2.3.3 Social Influence

Let us consider the following scenario: We have to two different parties the *democrats* and the *republicans*. Each of the voters has to vote either the *democrats* or the *republicans*. Let also $x_i(t) \in [0, 1]$ denotes the probalility agent i votes for the *republicans* if the elections took place at time step t. We also consider that the voters updates their probabilities at each time step according to the DeGroot model, with an updating matrix T that is strongly connected and aperiodic. From the previous analysis we already know that if the voters have enough time to update their probabilities before the elections then:

$$\text{E}[\# \text{ votes for the republican}] = n \cdot \sum_{i \in V} x_i(0) \cdot \pi_i^*$$

,where $(\pi^*)^T \cdot T = (\pi^*)^T$. Apparently, *republicans* want to maximize $n \cdot \sum_{i \in V} x_i(0) \cdot \pi_i^*$ and the *democrats* want to minimize it. Now, consider the following problem: The republicans have the chance to bride a voter so $x_i(0) = 1$. The problem is which voter is the best to bride?

Since $\forall i \in V : x_i(\infty) = \sum_{i \in V} \pi_i^* \cdot x_i(0)$ the influence in the consesus of each node is π_i^*. Apparently, in order to maximize the $n \cdot \sum_{i \in V} x_i(0) \cdot \pi_i^*$ we just need the voter that maximizes $(1 - x_i(0)) \cdot \pi_i^*$. Also notice that $(\pi^*)^T \cdot T = (\pi^*)^T$, thus $(\pi^*)^T$ is the left unit eigenvector, which can be computed at $O(n^3)$. More generally the vector π^* not only gives as a way to compute the consesus, but also its entries describe the influence of each node at the final opinion. Trivially, the most influential node is the one, whose entry in π^* is the maximum. Thus, we have found an easy way to measure *centrality* in a social network.

2.4 The Stubborn case

In the previous section, we have examined a special case of the DeGroot model, a case at which the undelying graph of the updating matrix T was strongly connected. We have managed to show that aperiodicity of T is necessary and suficient conditions for convergence and that convergence and consesus are equivalent. Apart from that, it was shown that the consesus opinion is a linear combination of the initial opinions of the nodes, providing also a efficient way to compute the influence of each of them at the final common opinion. Although all these are really positive, it is easy for someone to understand that the assumption that the undelying graph is strongly connected is really strong. To understand that, imagine that if only one of the agents were stubborn($T_{ii} = 1$) the graph wouldn't be strongly connected and our previous analysis collapse. The reason that we have analyzed so much this special case is that it will help us to deal with the general case. In this section we introduce stubborness to our analysis. At first we will see another special case which is somehow the complement of the previous case. Then, combining the results of these two special cases it will be really easy to understand the DeGroot model in all its generality.

2.4.1 Opinion leaders

Now in this special case we will not demand that the updating matrix is strongly connected. On the contrary our assumption is that there are two types of nodes the *leaders* and the *followers* [20]. In a more formal way the node $i \in L(\text{Leaders}) \iff T_{ii} = 1$ and $i \in F(\text{Followers}) \iff T_{ii} < 1$. We also assume that $\forall i \in F$ there is at least one node $j \in F$ such that there is directed path from node i to node j. From now one we will consider that the updating matrix T fullfils the two previous properties. Until now, we can notice two interesting facts 1) the updating matrix T is aperiodic 2) there are no closed and strongly connected groups of *followers*. Rember that a group of nodes $C \subseteq V$ is a closed if and only if $\forall i, j :\ i \in C, j \notin C$ then $(i, j) \notin E$. Another interesting observation is that consesus is not possible in this case because the *leaders* never change their initial opinion. Although consensus is not possible, we would like to know if each node adopts a specific opinion. This leads us to two questions: Is there an opinion vector x^* sucn that $x^* = T \cdot x^*$? If there is such an opinion vector. Will the agents converge to x^*? if they constantly update their opinions according to the update martix T.

Let $x_F(t), x_L(t)$ denote the opinion vector of the *followers* and the *leaders* respectively. Without loss of generality $x^T(t) = (x_F(t), x_L(t))^T$ and $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & \mathrm{I} \end{pmatrix}$ where T_{11} are the weights between the $followers$, T_{12} are the weights that followers pose to the *leaders* and $\mathrm{I} = \mathrm{I}_{|L|\times|L|}$.

Observation 14. $x^* = \begin{pmatrix} (\mathrm{I} - T_{11})^{-1} \cdot T_{12} \\ \mathrm{I} \end{pmatrix} \cdot x_L(0)$

Proof. By definition $x^* = T \cdot x^* \iff \begin{cases} x_F^* = T_{11} \cdot x_F^* + T_{12} \cdot x_L^* \\ x_L^* = x_L(0) \end{cases}$

The matrix $\mathrm{I} - T_{11}$ is substochastic and thus it is reversible. As a result:

$$x^* = \begin{pmatrix} (\mathrm{I} - T_{11})^{-1} \cdot T_{12} \\ \mathrm{I} \end{pmatrix} \cdot x_L(0)$$

□

Until now, we have proven that there exists an opinion vector x^*, which is stable. We can understand that this is not enough we would like to know whether the nodes will reach to x^* if they follow the DeGroot model. What we will prove is that $\lim_{t\to\infty} T^t = \begin{pmatrix} 0 & (\mathrm{I} - T_{11})^{-1} \cdot T_{12} \cdot \\ 0 & \mathrm{I} \end{pmatrix}$,which is implies that $\lim_{t\to\infty} x(t) = x^*$. In order to do this we will use some Markov chain theory, which we have already seen.

Observation 15. *The matrix* T_{11}^t *converges and* $\lim_{t\to\infty} T_{11}^t = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

Proof. Let the Markov chain with transition matrix the matrix T. In terms of Makov chain theory the *leaders* are called absorbing states because once the chain reaches such a state, it remains there. Now, it is easy to see that for non absorbing states it holds that:

$$p_F^T(t) = p_F^T(t-1) \cdot T_{11} = p_F^T(0) \cdot T_{11}^t$$

,where $p_F^T(t)$ is the probalility vector of the non absorbing states.

Now from our assumption in the beginning of this section, every $follower$ is connected with a directed path to at least one *leader*. We have already observed that this implies that there are no closed and strongly connected groups of $followers$. The interpretation of this to the Markov chain theory implies that if the Markov chain starts at non absorbing state it is certain that it will end up to an absorbing state. As a result:

$$\forall i \in \{1, |F|\} : \lim_{t\to\infty} u_i^T \cdot T_{11}^t = \lim_{t\to\infty} p_F^T(t) = 0, \text{ where } u_{i_j} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

$$\Longrightarrow \lim_{t\to\infty} T_{11}^t = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

□

Now let's return to the updating matrix T. As we have seen $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & \mathrm{I} \end{pmatrix}$.

It is easy to see that $T^t = \begin{pmatrix} T_{11}^t & K(t) \\ 0 & \mathrm{I} \end{pmatrix}$ and $K(t+1) = T_{11}^t \cdot T_{12} + K(t) = \sum_{i=0}^{t} T_{11}^i \cdot T_{12}$

Now we have just to prove that the matrix $K(t)$ converges.

Observation 16. *The matrix $K(t)$ converges and* $\lim_{t\to\infty} K(t) = (I - T_{11})^{-1} \cdot T_{12}$

Proof. Firstly we will prove that $\sum_{i=0}^{\infty} T_{11}^{t}$ converges.

$I - T_{11}^{t+1} = \sum_{i=0}^{t} T_{11}^{i} - T_{11} \cdot \sum_{i=0}^{t} T_{11}^{i} = (I - T_{11}) \cdot \sum_{i=0}^{t} T_{11}^{t} \implies$

$(I - T_{11}) \cdot \lim_{t\to\infty} \sum_{i=0}^{t} T_{11}^{t} = \lim_{t\to\infty}(I - T_{11}^{t+1}) = I$, which holds because of the Observation 14. $\implies \lim_{t\to\infty} \sum_{i=0}^{t} T_{11}^{t} = (I - T_{11})^{-1}$ (We have already proven that $(I - T_{11})$ is a reversible matrix). As a result:

$$\lim_{t\to\infty} K(t) = (I - T_{11})^{-1} \cdot T_{12}$$

□

Using the previous two observation we can see that

$$\lim_{t\to\infty} T^{t} = \begin{pmatrix} 0 & (I - T_{11})^{-1} \cdot T_{12} \\ 0 & \mathrm{I} \end{pmatrix}$$

As a result we have proven that in this case that each of nodes converges to a specific opinion.

2.4.2 DeGroot model(general case)

Having the previous two specific cases in mind, we are ready to handle the general case of *DeGroot*. We already know that consesus is not generally possible, but we would like to know if each of the nodes converge to a specific opinion. Having already obtained some intuition on the model, we will give a theorem that fully describes convergence in the general case [11].

Theorem 7. *The updating matrix T converges if and only if each closed and strongly connected subset of the nodes is aperiodic.*

Proof. $\Longrightarrow$ It is quite simple to prove this direction of the Theorem. Since T is convergent then every subset of the nodes is aperiodic. A closed and strongly connected group can be viewed as a separate social network, which is strongly connected. We have proven that since this social network is strongly connnected and convergent then it is aperiodic.

$\Longleftarrow$ Let an update matrix T and $B_1, B_2, ..., B_k$ are the closed and strongly connected groups of nodes in the social network. Notice that a $leader(T_{ii} = 1)$ is also a closed and strongly connected group. If $\forall i \in \{1, k\} : B_i$ is aperiodic, then the nodes in B_i will reach a consensus, which is a linear combination of their initial opinions $x_i(0)$, let this opinion be $x_{\{B_i\}}(0)$. Without loss of generality

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \text{,where } T_{22} = \begin{pmatrix} T_{B_1} & 0 & \cdots & 0 \\ 0 & T_{B_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{B_k} \end{pmatrix}$$

Also notice that $\forall i \in \{1, k\} : \lim_{t\to\infty} T^t_{B_i} = T^*_{B_i}$ and consequently $\lim_{t\to\infty} T^t_{22} = T^*_{22}$. Now, let's *shrink* all the closed and strongly connected groups to single stubborn nodes($leaders$). Thus, we have another social network with updating matrix T', which is the case described in (1.4.1), since there are $leaders$ and there doesn't exist a closed and strongly connected group of $followers$. Using the previous notation F denotes the set of the followers, L' denotes the set of the leaders and $L \equiv B_1 \cup \cdots \cup B_k$. Observe that :

$$\begin{pmatrix} x_F(t) \\ x_L(t) \end{pmatrix} = T \cdot \begin{pmatrix} x_F(t-1) \\ x_L(t-1) \end{pmatrix} \text{ and } \begin{pmatrix} x_F(t) \\ x'_L(t) \end{pmatrix} = T' \cdot \begin{pmatrix} x_F(t-1) \\ x'_L(t-1) \end{pmatrix}$$

$$\text{,where } T' = \begin{pmatrix} T_{11} & T'_{12} \\ 0 & \mathrm{I} \end{pmatrix}$$

Let $x(0) \in [0,1]^n$ we construct a $x'(0) \in [0,1]^n$ as follows: $x'_F(0) = x_F(0)$ and $x'_{Li}(0) = x_{\{B_i\}}(0)$ ($x_{\{B_i\}}(0)$ is defined above). From (1.4.1) we know that:

$$\begin{aligned} \lim_{t\to\infty} x_F(t) &= (\mathrm{I} - T_{11})^{-1} \cdot T'_{12} \cdot x'_L(0) \\ &= (\mathrm{I} - T_{11})^{-1} \cdot T_{12} \cdot T^*_{22} \cdot x_L(0) \end{aligned}$$

We also know that: $\lim_{t\to\infty} x_L(t) = \lim_{t\to\infty} T^t_{22} \cdot x_L(0) = T^*_{22} \cdot x_L(0)$. As a result:

$$\forall x(0) \in [0,1]^n : \ \lim_{t\to\infty} x(t) = \lim_{t\to\infty} T^t \cdot x(0) = \begin{pmatrix} 0 & (\mathrm{I} - T_{11})^{-1} \cdot T_{12} \cdot T^*_{22} \\ 0 & T^*_{22} \end{pmatrix} \cdot x(0) \Longrightarrow$$

$$\lim_{t\to\infty} T^t = \begin{pmatrix} 0 & (\mathrm{I} - T_{11})^{-1} \cdot T_{12} \cdot T^*_{22} \\ 0 & T^*_{22} \end{pmatrix}$$

So finally we have proven that if each closed and strongly connected group of nodes is aperiodic, then T^t converges. □

As a result we have found necessary and sufficient conditions for the convergence of the DeGroot model. Notice that we can check in polynomial time whether all the closed group of the updating matrix are aperiodic. Thus, it is easy to know whether an updating matrix converges or not. We can alsoo see that The convergence rate to the opinion vector x^* is as in the previous case $\lambda_2(T)$ and the proof remains the same.

Chapter 3

Decentralized Opinion model

3.1 Introduction

In the previous section, we have seen the Kleiberg-Bindel model in which each agent has an internal opinion $s_i \in [0, 1]$ and can adopt an opinion $x_i \in [0, 1]$. It is easy to argue that although it is possible each agent has an internal opinion $s_i \in [0, 1]$, it is not always possible to adopt an opinion $x_i \in [0, 1]$. Elections is a very good example of this case, because each voter can have an internal opinion $s_i \in [0, 1]$, denoting that voter i is a fan of the republicans with degree s_i and $1 - s_i$ fan of the democrats. However at the elections he cannot split his vote a s_i and $1 - s_i$, he has to vote only for one of them. Decentralized opinion dynamics model proposed by Ferroli,Goldberg and Ventre tries to capture this case [9]. More formally in this model each agent i has an internal opinion $s_i \in [0, 1]$, there is an undirected graph $G(V, E)$ representig the structure of the social network. Each agent i can adopt an opinion $x_i \in \{0, 1\}$

and the cost for choosing opinion x_i is:

$$C_i(x) = (x_i - s_i)^2 + \sum_{(i,j)\in E} (x_i - x_j)^2$$

Obviously each agent wants to minimize its cost. We give some defintions that help us simplify our notation. Let N_{xi} be all the neighbours of agent i which have the same opinion with agent i. Let $N_{\overline{xi}}$ be all the neighbours of agent i which have the oposite opinion of player i and $\overline{x_i}$ be the opposite of player 's i strategy. It's trivial that:

$$C_i(x) = (x_i - b_i)^2 + N_{\overline{xi}}$$

Respectively we define the social cost function :

$$SC(x) = \sum_{i\in V} C_i(x) = \sum_{i\in V} (x_i - b_i)^2 + 2D(x)$$

where $D(x)$ is the number of discording edges in opinion vector x.

Now we have a complete view of this model, which can be viewed as a *rounded* case of the Kleiberg-Bindel model. In the latter, we have seen that the $PoA = \frac{9}{8}$, something very positive since all the N.E. are very close to the optimal solution. Unfortunately, this doesn't hold in this model and in the next theorem is proved that the $PoA = \infty$.

Theorem 8. *the Price of Anarchy (PoA) of this game is ∞.*

Proof. Let our instance be a ring graph such that

$$\forall i: \ s_i = 0$$

It is trivial that the optimal strategy profile is that all agents adopt 0. Now consider the strategy profile in which all agents play 1. This opinion vector is a Nash

Equilibrium because each agent i has personal cost $C_i(x) = 1$ and if he changes his strategy to 0 his personal cost will be 2 (there are 2 neighbours that play 1). Let this strategy profile be $y \implies SC(y) = n$, where n is the number of the players. Apparently, the optimal opinion profile is that $\forall i \in V: \ x_i = 0 \implies OPT = 0$. Thus, $PoA \geqslant \frac{SC(y)}{OPT} = \infty \implies PoA = \infty$ $\square$

Apart from knowing the PoA of this model, we are also interested in finding an efficient way to compute the equilibrium of this model. In the previous chapter, it was proved that the Kleiberg-Bindel model has a potential function and in order to find the Nash Equilibrium, we had just to find the minimum of the potential function. In this model, there is also a function $\Phi(x)$ such that $\Phi(x_i, x{-}i) - \Phi(\overline{x_i}, x_{-i}) = C_i(x_i, x{-}i) - C_i(\overline{x_i}, x_{-i})$, which means that $\Phi(x)$ is a potential function of this game.

Observation 17. *The decentralized opinion formation game is a potential game with potential function*

$$\Phi(x) = \sum_{i \in V} (x_i - s_i)^2 + D(x)$$

Proof. Let $\Phi(x) = \sum_{i \in V} (x_i - s_i)^2 + D(x) \implies$

$$\begin{aligned}
\Phi(x_i, x_{-i}) - \Phi(\overline{x_i}, x_{-i}) &= (x_i - s_i)^2 + D(x_i, x_{-i}) - (\overline{x_i} - s_i)^2 - D(\overline{x_i}, x_{-i}) \\
&= (x_i - s_i)^2 - (\overline{x_i} - s_i)^2 + N_{\overline{x_i}} + D_{-i}(x_i, x_{-i}) - N_{x_i} - D{-}i(\overline{x_i}, x_{-i}) \\
&= (x_i - s_i)^2 + N_{\overline{x_i}} - (\overline{x_i} - s_i)^2 - N_{x_i} \\
&= C_i(x_i, x_{-i}) - C_i(\overline{x_i}, x_{-i})
\end{aligned}$$

,where $D_{-i}(x_i, x_{-i})$ is the number of discording edges apart from those that belong to node i. Thus $\Phi(x) = C_i(x_i, x_{-i}) - C_i(\overline{x_i}, x_{-i})$ $\square$

Now, it easy to understand that an opinion profile $x \in \{0,1\}$ is a N.E. if and only if x is a local or global minimum of $\Phi(x)$. In our case the potential function is defined over all the the possible opinion vectors $x \in \{0,1\}^n$. Since $|\{x : x \in \{0,1\}^n\}| = 2^n$ it is certain that $\Phi(x)$ has at least one minimum. As a result, the existence of the potential function implies the existence of a N.E. of this model.

Notice that when a node i updates its opinion in order to reduce its personal cost the value of the potential function is reduced. This property holds for every game having a potential function, but this model has an additional property which is very important. When an agent i reduces his personal cost not only the potential function, but also the social cost fuction $SC(x)$ reduces. This property is proved in the next observation.

Observation 18. *If* $C_i(x_i, x_{-i}) > C_i(\overline{x_i}, x_{-i})$ *then* $SC(x_i, x_{-i}) > SC(\overline{x_i}, x_{-i})$

Proof. At first we prove that if $C_i(x_i, x_{-i}) > C_i(\overline{x_i}, x_{-i})$ then $N_{xi} \leqslant N_{\overline{xi}}$.

Let $N_{xi} > N_{\overline{xi}} \implies N_{xi} \geqslant N_{\overline{xi}} + 1 \geqslant (x_i - s_i)^2 + N_{\overline{xi}} \implies (\overline{x_i} - s_i)^2 + N_{xi} \geqslant (x_i - s_i)^2 + N_{\overline{xi}} \implies C_i(\overline{x_i}, x_{-i}) \geqslant C_i(x_i, x_{-i})$.

So if $C_i(x_i, x_{-i}) > C_i(\overline{x_i}, x_{-i})$ then $N_{xi} \leqslant N_{\overline{xi}}$. As a result there are two case:

- $N_{xi} = N_{\overline{xi}}$

$$C_i(x_i, x_{-i}) > C_i(\overline{x_i}, x_{-i}) \implies (x_i - s_i)^2 > (\overline{x_i} - s_i)^2 \implies$$

$$(x_i - s_i)^2 + \sum_{j \neq i}(x_j - s_j)^2 + 2 \cdot D_{-i}(\overline{x_i}, x_{-i}) + 2N_{\overline{xi}} >$$
$$(\overline{x_i} - s_i)^2 + \sum_{j \neq i}(x_j - s_j)^2 + 2 \cdot D_{-i}(x_i, x_{-i}) + 2N_{xi}$$
$$\implies SC(x_i, x_{-i}) > SC(\overline{x_i}, x_{-i})$$

- $N_{\overline{xi}} > N_{xi} \Longrightarrow$

$$N_{\overline{xi}} \geqslant N_{xi} + 1 \Longrightarrow$$

$$2 \cdot D(x_i, x_{-i}) \geqslant 2 \cdot D(\overline{x_i}, x_{-i}) + 2$$

  It is also easy to verify that

$$(x_i - s_i)^2 - (\overline{x_i} - s_i)^2 \geqslant -1$$

  Adding the two previous inequalities we have

$$(x_i - s_i)^2 + 2 \cdot D(x_i, x_{-i}) \geqslant (\overline{x_i} - s_i)^2 + 2 \cdot D(\overline{x_i}, x_{-i}) + 1 \Longrightarrow$$

$$SC(x_i, x_{-i}) > SC(\overline{x_i}, x_{-i})$$

Consequently, if $C_i(x_i, x_{-i}) > C_i(\overline{x_i}, x_{-i})$ then $SC(x_i, x_{-i}) > SC(\overline{x_i}, x_{-i})$ □

Previously, we proved that the Price of Anarchy (PoA) of the game is ∞, which means that there is a N.E. that has a great social cost in respect to the optimal solution. Let us introduce the notion of the Price of Stability (PoS). The Price of Stability is the fraction of the social cost of the N.E. with the smallest social cost and the optimal social cost of the game. ($PoS = \frac{\text{SC(Nash Equilibrium wih the smallest cost)}}{OPT}$). It is easy to see that for every game $PoS \geqslant 1$ and $PoA \geqslant PoS$. Although in this model the $PoA = \infty$, the $PoS = 1$ which means that the optimal opinion profile is a N.E. (notice that this doesn't hold in the Kleiberg-Bindel model).

Observation 19. *the Price of Stability (PoS) = 1*

Proof. Because of the previous observation it is easy to see that if $SC(x_i, x_{-i}) \leqslant SC(\overline{x_i}, x_{-i})$ then $C_i(x_i, x_{-i}) \leqslant C_i(\overline{x_i}, x_{-i})$. Let y be the optimal opinion profile. By definition $\forall i$: $SC(y_i, y_{-i}) \leqslant SC(\overline{y_i}, y_{-i}) \Longrightarrow \forall i$: $C_i(y_i, y_{-i}) \leqslant C_i(\overline{y_i}, y_{-i})$ so the optimal profile is also a Nash Equilibrium. Thus, $PoS = 1$. □

3.2 Best Response Dynamics

From the the previous section, we already know that the Decentralized Opinion model always admits a Nash Equilibrium. As before we would like to know whether there exists a mechanism that garantees the convergence of the agents at a Nash Equilibrium. In this section, we will examine the Best Response mechanism that garantees the convergence of the agents to a Nash Equilibrium after a polynomial number of steps.

We assume that at each time step only one player plays a move and this move is always his best response. At first we prove that from any initial state the agents converge to a Nash Equilibrium at a finite number steps.

Observation 20. *From any initial state x_0 the Best Response Dynamics will reach a N.E. after a finite number of steps.*

Proof. Consider a directed $\overline{G}(\overline{V}, \overline{E})$ in which each node $x \in \overline{V}$ corresponds to an opinion vector $x \in \{0,1\}^n$. Obviously there are 2^n nodes. There are acres only from node (x_i, x_{-i}) to node $(\overline{x_i}, x_{-i})$ if and only if $\Phi(x_i, x_{-i}) > \Phi(\overline{x_i}, x_{-i})$. Apparently there are no cycles in $\overline{G}$ because let a cycle $y_1, y_2, ..., y_k$. By definition $\Phi(y_1) > \Phi(y_2) > ... > \Phi(y_k) > \Phi(y_1)$ which is impossible. As a result $\overline{x}$ is a DAG and there are nodes that are *sinks*. All the sinks in this graph correspond to N.E. because the are node acres starting from them, which by definition means that $\forall i \in V : \Phi(x_i, x_{-i}) \leqslant \Phi(\overline{x_i}, x_{-i})$. The initial state x_0 of our system is either a sink(N.E.) or a normal node. A best response move from agent i at state x corresponds to traversing the acre $\overrightarrow{((x_i, x_{-i}), (\overline{x_i}, x_{-i}))}$. Because there are no cycles in the graph and the number of nodes is finite, after a finite number of best response moves we will end up at a sink, which is a N.E. $\square$

Theorem 9. *The Best Response Dynamics converges to a N.E. after a polynomial number of steps.*

Proof. Let an opinion vector $x = (x_i, x_{-i})$ and an agent i with $s_i \in (\frac{1}{2}, 1)$. It is easy to verify that agent's i best response is:

$$\begin{cases} \alpha \in \{0,1\} & , N_\alpha > N_{\overline{\alpha}} \\ 1 & , N_{xi} = N_{\overline{xi}} \end{cases}$$

As a result agent i plays the same moves as if s_i were $\frac{3}{4}$.

Respectively if $s_i \in (0, \frac{1}{2})$, agent's i best response is:

$$\begin{cases} \alpha \in \{0,1\} & , N_\alpha > N_{\overline{\alpha}} \\ 0 & , N_{xi} = N_{\overline{xi}} \end{cases}$$

As a result agent i plays the same moves as if $b_i = \frac{1}{4}$.

Now assume that we round each $s_i \in (0, \frac{1}{2})$ to $\frac{1}{4}$ and $s_i \in (\frac{1}{2}, 0)$ to $\frac{3}{4}$. We can now see in both *rounded* and *unrounded* casse agent i take the exact same decision. As a result, the state graphs of the both cases are the same because not only the states are the same, but also the acres that connect them.

Let the longest directed path in the state graph $\overline{G}$ of the game from a node x_0 to a sink node x_1 with length P_{max}. We can observe that P_{max} is the maximum number of times steps needed to reach a N.E from any initial state, which is the same for both *rounded* and *unrounded* case.

Let' s take the *rounded* case:

$$\Phi(x_0) - \Phi(x_1) \geqslant P_{max} \cdot \Delta\Phi min$$

$$\Phi(x) = \sum_{i \in v} (x_i - b_i)^2 + D(x) \leqslant n^2 + n \Longrightarrow$$

$$P_{max} \cdot \Delta\Phi min \leqslant n^2 + n$$

$$\Delta\Phi min = \frac{1}{16} \Longrightarrow$$

$$P_{max} \leqslant 16(n^2 + n)$$

So the maximum path length from any node to a leaf is polynomial as a result the number of steps of best-response dynamics to a N.E. is also poynomial. □

3.3 Bounding the PoA by Best Response

We have already seen that the PoA is unbounded for this game. In this section we will describe a best reponse mechanism that leads to Nash Equilibrium with bounded Price of Anarchy.

More precisely: Let a random sequence of the nodes $p = (p_1, p_2, ..., p_n)$ at each time step only one agents plays according to this sequence. We assume that all agents always play their best response move and at the first time step each agent i plays:

$$\begin{cases} 0 & , s_i \leqslant \frac{1}{2} \\ 1 & , s_i > \frac{1}{2} \end{cases}$$

According to observation 20 the agents will converge always to a Nash Equilibrium since at each time step the potential function is reduced. Let $x_p \in \{0,1\}^n$ be the

Nash Equilibrium at which the agents converge if they play according to the above mechanism with permutation p. We will bound the above the $SC(x_p)$ in respect to the OPT and n, which is the number of agents.

Theorem 10. *Let x_p be the N.E. in which the agents converge using the previous mechanism with permutation p. Then,* $\frac{SC(x_p)}{OPT} \leqslant 8 \cdot n - 6$

Proof. Let S^t be the state of the system at time step t. Notice that $S^0 = (s_{p1}, s_{p2}, ..., s_{pn})$. In observation 18 it is proven that when an agents plays his best response move the social cost is reduced. As a result: $SC(S^{t+1}) < SC(S^t) \Longrightarrow \frac{SC(S^t)}{OPT} \leqslant \frac{SC(S^0)}{OPT} \Longrightarrow$

$$\frac{SC(x_p)}{OPT} \leqslant \frac{SC(S^0)}{OPT}$$

Now we have just to bound the $\frac{SC(S^0)}{Opt}$.. Before continuing let us give some definitions:

- A_0 be the set of players i that play 0 in the optimal solution and $b_i \leqslant \frac{1}{2}$
- A_1 be the set of players i that play 1 in the optimal solution and $b_i \leqslant \frac{1}{2}$
- B_0 be the set of players i that play 0 in the optimal solution and $b_i > \frac{1}{2}$
- B_1 be the set of players i that play 1 in the optimal solution and $b_i > \frac{1}{2}$

we also define:

- $D(x)$ be the set of the discording edges in the strategy profile x.
- $(A \cdot B)$ be the number of edges (i, j) : $i \in A$ *and* $j \in B$

$$
\begin{aligned}
\frac{SC(S^0)}{SC(OPT)} &= \frac{\sum_{i\in A_0} b_i^2 + \sum_{i\in A_1} b_i^2 + \sum_{i\in B_0}(1-b_i)^2 + \sum_{i\in B_1}(1-b_i)^2 + 2|D(S^0)|}{\sum_{i\in A_0} b_i^2 + \sum_{i\in A_1}(1-b_i)^2 + \sum_{i\in B_0} b_i^2 + \sum_{i\in B_1}(1-b_i)^2 + 2|D(OPT)|} \\
&\leqslant 1 + \frac{\sum_{i\in A_1} b_i^2 + \sum_{i\in B_0}(1-b_i)^2 + 2(A_0\cdot B_0) + 2(A_1\cdot B_1)}{\sum_{i\in A_1}(1-b_i)^2 + \sum_{i\in B_0} b_i^2 + 2(A_0\cdot A_1) + 2(B_0\cdot B_1)} \\
&\leqslant 1 + \frac{\frac{|A_1|}{4} + \frac{|B_0|}{4} + 2(A_0\cdot B_0) + 2(A_1\cdot B_1)}{\frac{|A_1|}{4} + \frac{|B_0|}{4}} \\
&\leqslant 2 + 8\cdot\frac{|A_0||B_0| + |A_1||B_1|}{|A+1| + |B_0|} \\
&\leqslant 2 + 8\cdot(|A_0| + |B_1|) \\
&\leqslant 2 + 8\cdot(n-1) \\
&\leqslant 8\cdot n - 6
\end{aligned}
$$

without loss of generality we assume $|A_0| + |B_1| \geqslant 1$, otherwise $|A_0| = |B_1| = 0$ and $\frac{SC(S^0)}{SC(Opt)} = 1$. So finally we have proven that $\frac{SC(S^0)}{OPT} \leqslant 8\cdot n - 6 \Longrightarrow$

$$\frac{SC(x_p)}{OPT} \leqslant 8\cdot n - 6$$

□

The last theorem tells us that although the PoA of this model is unbounded, if agents play reasonably the will avoid ending up to the really "*bad*″ equilibriums.

3.4 Computing Nash Equilibriums

The problem of computing an equilibrium for a game with two or more players is PPAD-complete, which means that the existence of an algorithm that solves the problem polynomially in its generality is quite unlikely [7]. However in our case, we have already found a polynomial algorithm that computes a Nash Equilibrium for our problem. We just let the players play according to the mechanism that we described in the previous section and they will reach a N.E. at $O(n^2)$ steps. Notice that if we want find the optimal opinion vector, which is also a N.E. the above algorithm fails. Because we have no idea whether the N.E. that the players converge is the optimal. In this section we provide an algorithm for the optimal opinion profile and a similar algorithm for computing equilibriums at which two specific nodes adopt opposite opinions. Notice that in this model there are at least two equilibriums. This holds because if an opinion vector is a N.E. it is easy to prove that the opinion profile $\overline{x^*}$ which is the dual complement of x^* is also a N.E.

3.4.1 Computing the optimal Nash Equilibrium

Now we provide the algorithm for the optimal Nash Equilibrium. We show that the problem can be reduced to an equivalent max-flow/min-cut problem to a specially constructed directed graph.

Let the underlying graph $G(V, E)$ of the social network. We construct a directed graph $G'(V', E')$ in the following way:

1. $\forall i$: $i \in V$ we construct a node $i' \in V'$, we also add two additional nodes 0 and 1.

2. $\forall (i, j) \in E$ we construct an edge $\overrightarrow{(i, j)} \in E'$ and an edge $\overrightarrow{(j, i)} \in E'$ both with capacity 2.

3. $\forall i'$: $i' \in V'$: we add an edge $\overrightarrow{(0, i')}$ with capacity $(1 - b_i^2)$ and an edge $\overrightarrow{(i', 1)}$ with capacity b_i^2

Observation 21. *Let a max-flow/min-cut from node 0 to node 1 in G'. If the nodes that belong in the same set with the node 0 adopt 0 and the nodes that belong in the same set with the node 1 adopt 1, then we have the optimal equilibrium.*

Proof. Let the $OPT < min - cut$ from 0 to 1. Let the sets A, B be the sets of agents that play 0 and 1 respectively in the optimal equilibrium and S the set of edges between the nodes of A and B. Apparently:

$$OPT = \sum_{i \in A} b_i^2 + \sum_{i \in B} (1 - b_i)^2 + 2|S|$$

The optimal equilibrium defines a cut in G' with the sets $A' = A \cup \{0\}$ and $B' = B \cup \{1\}$. By definition of G' the weight of this cut is $\sum_{i \in A'} b_i^2 + \sum_{i \in B'} (1 - b_i)^2 + 2|S|$ because for each edge $(i, j) \in E$ there is an edge $\overrightarrow{(i, j)} \in E'$ with capacity 2. Consequently $OPT \geqslant min - cut(G')$ (1).

Reversively, the min-cut of G' defines a solution y for our game where agent i plays 0 if he belongs in the same set with node 0 and 1 if he belongs in the same set with the node 1. Thus, $min - cut(G') = SC(y) \leqslant OPT$ (2)

$$(1), (2) \Longrightarrow SC(y) = OPT$$

As a result, $OPT =$ max-flow from node o to node 1 □

So we have a polynomial algorithm for computing the optimal equilibrium.

3.4.2 Computing an Equilibrium in which two specific agents play 0 and 1

Before closing this chapter we will provide an algorithm for the following problem: Let the underlying graph $G(V, E)$ of a social network whose nodes play according the

Decentralized opinion model. We want to find a N.E. such that nodes i_1, i_2 play 0 and 1 respectively. The general idea is the same as before, we will reduce this problem to min-cut problem in an appropriate directed graph $G'(V', E')$. As we described before we construct from a graph $G(V, E)$ a directed graph $G'(V', E')$ as follows:

1. For each $i \in V$ we add an edge $\overrightarrow{(i_1, i)} \in E'$ and $\overrightarrow{(i, i_2)} \in E'$ with capacity $c(\overrightarrow{(i_1, i)}) = (1 - b_i)^2$, $c(\overrightarrow{(i, i_2)}) = b_i^2$ if $(i_1, i) \notin E$ and $(i, i_2) \notin E$

2. if $(i_1, i) \in E$ and $(i, i_2) \notin E$ then we add an edge $\overrightarrow{(i_1, i)} \in E'$ and $\overrightarrow{(i, i_2)} \in E'$ with $c(\overrightarrow{(i_1, i)}) = 2 + (1 - b_i)^2$ and $c(\overrightarrow{(i, i_2)}) = b_i^2$

3. if $(i_1, i) \notin E$ and $(i_1, i) \in E$ then we add an edge $\overrightarrow{(i_1, i)} \in E'$ and $\overrightarrow{(i, i_2)} \in E'$ with $c(\overrightarrow{(i_1, i)}) = (1 - b_i)^2$ and $c(\overrightarrow{(i, i_2)} = 2 + b_i^2$

4. if $(i_1, i) \in E$ and $(i_1, i) \in E$ then we add an edge $\overrightarrow{(i_1, i)} \in E'$ and $\overrightarrow{(i, i_2)} \in E'$ with $c(\overrightarrow{(i_1, i)}) = 2 + (1 - b_i)^2$ and $c(\overrightarrow{(i, i_2)}) = 2 + b_i^2$

5. for each egde $(i, j) \in E$ with $i, j \neq i_1, i_2$ we add the edges $\overrightarrow{(i, j)}, \overrightarrow{(j, i)} \in E'$ with $c(\overrightarrow{(i, j)}) = c(\overrightarrow{(j, i)}) = 2$

Observation 22. *The max-flow/min-cut from node i_1 to node i_2 in the directed graph $G'(V', E')$ is also a Nash Equilibrium if players in the same set with i_1 play 0 and those in the same set with the i_2 play 1.*

Proof. Let $A, B \subset V$ is the min-cut from node i_1 to i_2 such that $i_1 \in A$ and $i_2 \in B$. Let that player i with strategy $x_i = 0$ wants to deviate. So $c_i(x_i, x_{-i}) > c_i(\overline{x_i}, x_{-i})$ $\implies SC(x_i, x_{-i}) > SC(\overline{x_i}, x_{-i})$ (1). It is trivial that $SC(x_i, x_{-i}) = b_{i1}^2 + (1 - b_{i2})^2 +$

$W(A,B)$ (where $W(A,B)$ is the weight of the cut) and $SC(\overline{x_i}, x_{-i}) = b_{i1}^2 + (1-b_{i2})^2 + W(A \setminus \{i\}, B \cup \{i\})$

$$(1) \implies W(A \setminus \{i\}, B \cup \{i\}) \; > \; W(A,B)$$

which is by definition invalid. As a result none of the agents wants to deviate and we have a Nash Equilibrium. $\square$

Chapter 4

Non Steady Graph Models

In the previous chapters we have seen various models, simulating the way that the members of a social network form their opinions. The general framework of the previous model was common, each agent had a cost function whose analytic form was invariant to the time and to the opinions of the other agents. In other words, the network was represented as a graph whose edges and weights were always the same. Although in same cases this aproach may be sufficient, experience shows us that this is not always true. In many cases people tend to trust more those who have similar opinion with them and generally the trust between two agents is not something that is always constant. In this chapter we will introduce some models, that try to descibe this behavior of the society. Unfortunately, these models are much complex to analyze since in these cases the cost functions have no or very complicated analytic forms. The very general framework is the same , we will consider that there is set of nodes(V) representig the members of the society and that the node i has an opinion at time t: $x_i(t) \in [0, 1]$ and as in the previous chapters we will be interested in equilibrium points, convergence time and price of anarchy(PoA) of these models.

4.1 Hegselmann-Krause model

The Hegselmann-Krause model tries to capture the tension of the agents to trust only those that have similar opinions [12]. Let V denotes the set of the agents. Each agent i has a initial opinion $x_i(0) \in [0,1]$ and the updating rule of the model is:

$$x_i(t+1) = \frac{\sum_{j:|x_i-x_j|<d} x_j}{\sum_{j:|x_i-x_j|<d} 1}$$

,where $d \in (0,1]$ is constant parameter of the model. Let another instance of the HK-model at which $x'_i(0) = d \cdot x_i(0)$ and $d' = 1$. Then, $\forall t : x_i(t) = \frac{1}{d} \cdot x'_i(t)$. So without loss of generality we will study the case at which $x_i(0) \geqslant 0$ and the updating rule is:

$$x_i(t+1) = \frac{\sum_{j:|x_i-x_j|<1} x_j}{\sum_{j:|x_i-x_j|<1} 1}$$

Notice that the edges of the underlying graph of the network change at each time step. The first question that we would ask is whether the agents will converge to specific opinion if they update their opinions according to this model [5]. Before proving this we will give a very important property of this model.

Observation 23. *If $x_i(0) \leqslant x_j(0)$. Then, $\forall t : x_i(t) \leqslant x_j(t)$ if $i \leqslant j$.*

Proof. Let $i, j \in V$ such that $x_i(0) \leqslant x_j(0)$. We have just to prove that if $x_i(t) \leqslant x_j(t)$ then $x_i(t+1) \leqslant x_j(t+1)$. We assume that $x_i(t) \leqslant x_j(t)$.

Let $N_i(t) \equiv \{$the nodes that are connected to node i and not to node j at time step t$\}$, $N_j(t) \equiv \{$the nodes that are connected to node j and not to node i at time step t$\}$ and

$N_{ij}(t) \equiv \{$the nodes that are connected to node i and to node j at time step t$\}$

Since $x_i(t) \leqslant x_j(t)$ then $\forall(k1, k_2, k_3) \in N_i \times N_{ij} \times N_j : x_{k_1}(t) \leqslant x_{k_2}(t) \leqslant x_{k_3}(t) \Longrightarrow$

$$\begin{cases} x_i(t+1) = \frac{N_i \cdot \overline{x}_{N_i}(t) + N_{ij} \cdot \overline{x}_{N_{ij}}(t)}{|N_i(t)| + |N_{ij}(t)|} \leqslant \overline{x}_{N_{ij}(t)} \\ x_j(t+1) = \frac{N_j \cdot \overline{x}_{N_j}(t) + N_{ij} \cdot \overline{x}_{N_{ij}}(t)}{|N_j(t)| + |N_{ij}(t)|} \geqslant \overline{x}_{N_{ij}(t)} \end{cases} \Longrightarrow x_i(t+1) \leqslant x_j(t+1)$$

By induction it is proved that $\forall t : x_i(t) \leqslant x_j(t)$

□

From now on, without loss of generality we will consider that the nodes are ordered according to the ording $x_1(0) \leqslant x_2(0) \leqslant ... \leqslant x_n(0)$. Notice that if there exists t_0 such that $x_{i+1}(t_0) - x_i(t_0) \geqslant 1$, then $\forall t \geqslant t_0 : x_{i+1}(t) - x_i(t) \geqslant 1$ since the value x_i will not increase and the value x_{i+1} will not decrease. Before trying to prove convergence we would like to know how the equilibrium looks like. Let $x_1^*, x_2^*, ..., x_k^*$ such that $\forall i, j \in \{1, k\} : |x_i^* - x_j^*| \geqslant 1$. It is easy to verify that if all the agents adopt one of the above opinions, then they will be in an equilibrium. Also notice that the above property has to hold for every equilibrium. Now, we are ready to prove the convergence of this model.

Theorem 11. *Let the agents update their opinions according to the HK-model. Then* $\forall i \in V : x_i(t)$ *converges to* x_i^* *in a finite number of steps and* $\forall i \neq j : x_i^* = x_j^*$ *or* $|x_i^* - x_j^*| \geqslant 1$

Proof. Since node 1 has only *right* neighbors then $x_1(t) \leqslant x_1(t+1)$. Respectively node n has only $left$ neighbors and then $x_n(t) \geqslant x_n(t+1)$. Thus, $\forall t : x_1(0) \leqslant x_1(t) \leqslant x_n(t) \leqslant x_n(0)$. As a result, $x_1(t) \leqslant x_1(t+1)$ and $x_1(t) \leqslant x_n(0)$ then $\exists x_1^*$ such that

$\lim_{t\to\infty} x_1(t) = x_1^*$.

Now let p the largest index such that $x_p(t)$ converges to x_1^*. We will show that there exists t_0 such that $x_{p+1}(t_0) - x_p(t_0) \geqslant 1$. Let $\forall t : x_{p+1} - x_p < 1$ then node $p+1$ is always a neighbor of node p. Since $p+1$ doen't converge to x_1 then there exists δ_0 such that $\forall t : x_{p+1}(t) - x_1^* > \delta_0$. On the other hand since $\forall i \in \{1, p\} : x_i(t)$ converges to x_1^* then there exists t_0 such that $\forall t \geqslant t_0 : x_i(t) - x_1^* \leqslant \epsilon_0 = \frac{p}{p+1} \cdot \delta_0$. From the definition of HK-model $x_p(t_0+1) = \frac{\sum_{i=1}^{p} x_i}{p+1} + \frac{1}{p+1} \cdot x_{p+1} > \frac{p}{p+1} \cdot x_1^* - \frac{p}{p+1} \cdot \epsilon_0 + \frac{1}{p+1} \cdot x_1 + \frac{1}{p+1} \cdot \delta_0 = \epsilon_0$. Something that contradicts that $x_p - x_1^* \leqslant \epsilon_0$. Now the nodes $\{p+1, ..., n\}$ are decomposed from the nodes $\{1, ..., p\}$ and as a result we can repeat the same argument to prove that every node i converges to x_i^* and that $\forall i \neq j : x_i^* = x_j^*$ or $|x_i^* - x_j^*| \geqslant 1$. Now we have to prove that the convergence will occur in a finite number of steps. Without loss of generality we will consider that all the nodes $\{1, ..., n\}$ converge to x^*. Then there exist t_0 such that $\forall t \geqslant t_0 : x^* - x_1(t_0) < \frac{1}{2}$ and $x_n(t_0) - x^* < \frac{1}{2}$. Adding the two previous inequalities $x_n(t_0) - x_1(t_0) < 1$, thus at time step t_0 every node has neighbors all the other nodes and consequently all of them compute the same value x^*. As a result the convergence occurs in a finite number of steps. □

The convergence to a limiting value at finite time, is important. But as always we would like to know how fast will the system converge. We will prove that this happens at $O(n^3)$ steps [2]. Before proving that we will give some definitions and observations that are necessary in order to continue.

Since now we have not demanded that two different nodes start with a different initial value. Observe that in this case these two nodes are actually one, that has twice as influence as the others. Also see that if two or more nodes start with different initial values, but at some time step they adopt the same opinion, then they will never be separated. Now we will introduce the notion of the weight of the node i at time step t, $w_i(t) = |\{j : x_j(t) = x_i(t)\}|$. As a result, from now on when two nodes adopt the same opinion, we will consider that they form one node whose weight is increased by one. Also notice, that the weight of a node never decreases and that $\sum_{i=1}^{k} w_i(t) = n$,where k is the number of nodes at time step t and n is the initial number of nodes. From now on, $N_i(t)$ will denote the number of neighbors node i has at time step t (we have already used this notation denoting something similar but different).

Observation 24. *Let $l(t)$ denotes the most left node (the minimum indexed node) that has not converged to its limiting value at time t. Then, at time $t+2$ the node $l(t)$ has increased its weight(1) or has converged(2) or has moved to the right at least $\frac{1}{n^2}$(3).*

Proof. Since $l(t)$ is the minimum indexed node that has not already converged, it is certain that $l(t)$ has no left neighbors and it has at least one right neighbor(r). Let $N_{l(t)}(t) = N_r(t)$, then at time $t+1 : x_r(t+1) = x_{l(t)}(t+1)$ which means that $l(t)$ has increased its weight. Now let $N_{l(t)}(t) \neq N_r(t)$ then there exist a node s such that $x_s(t) > x_r(t)$ and $x_s(t) - x_{l(t)}(t) \geqslant 1$. As a result:

$$x_r(t+1) \geqslant \frac{(N_r(t)-1) \cdot x_{l(t)}(t) + x_s(t)}{N_r(t)} \geqslant x_{l(t)}(t) + \frac{1}{N_r(t)} \geqslant x_{l(t)}(t) + \frac{1}{n} \Longrightarrow$$

$$x_r(t+1) \geqslant x_{l(t)}(t) + \frac{1}{n}$$

We already know that since $l(t)$ has no left neighbors is tis certain that $x_{l(t)}(t+1) \geqslant x_{l(t)}(t)$. Notice that at time $t+1$ it is possible that $x_r(t+1) - x_{l(t)}(t+1) \geqslant 1$ then the node $l(t)$ has neither right nor left neighbors which means that it has converged to its limiting value. On the other hand if $x_r(t+1) - x_{l(t)}(t+1) < 1$ then $l(t+1) = l(t)$ and r is the smallest right neighbor of $l(t+1)$. Thus,

$$\begin{aligned} x_{l(t)}(t+2) &\geqslant \frac{w_{l(t+1)}(t+1) \cdot x_{l(t+1)} + (N_{l(t+1)}(t+1) - w_{l(t+1)}(t+1)) \cdot x_r(t+1)}{N_{l(t+1)}(t+1)} \\ &\geqslant x_{l(t)}(t) + \frac{N_{l(t+1)}(t+1) - w_{l(t+1)}(t+1)}{N_{l(t+1)}(t+1)} \cdot \frac{1}{n} \\ &\geqslant x_{l(t)}(t) + \frac{1}{n^2} \end{aligned}$$

□

The last observation will lead us to proving that the system converges to $O(n^3)$ steps. Without loss of generality we can assume that $x_n(0) - x_1(0) \leqslant 1$, because otherwise the system can be decomposed into independently evolving subsystems. Notice that $l(t)$ can increase its weight at most n times. Let t_1 be the number of time steps at which $l(t)$ increased its weight, then $t_1 \leqslant 2 \cdot n$. Respectively, $l(t)$ can't converge to its limiting value more that n times. As a result, $t_2 \leqslant 2 \cdot n$, where t_2 is the number of time steps at which $l(t)$ converged to its limiting value. Finally, let t_3 be the number of times steps that the case (3) of the above observation occurs. Since $x_n(0) - x_1(0) \leqslant n \Longrightarrow x_n(t) - l(t) \leqslant n - \frac{t}{2 \cdot n^2}$. As a result, $t_3 \leqslant 2 \cdot n^3$. Consequently, at time step $t > 4 \cdot n + 2 \cdot n^3$ the system has converged.

4.2 K-NN model

In this section we present another non steady graph model, the K-NN model. Following the general framework we have the set V, $|V| = n$ of nodes representig the agents. Each node $i \in V$, has an internal opinion $s_i \in [0, 1]$, which never changes. Given an opinion vector $x = (x_i, x_{-i})$, each node i forms directed edges to the K closest nodes to s_i(the nodes with the smallest $|z_j - s_i|$). This implies that the underlying graph of the network is not steady, but it is a function of the opinion vector. The personal cost of each node $i \in V$ at the opinion vector x is:

$$C_i(x_i, x_{-i}) = \sum_{j \in S_x(i)} (x_j - x_i)^2 + \rho \cdot K \cdot (x_i - s_i)^2$$

,where $S_x(i) \subseteq V$ denotes the K closest nodes to s_i at opinion vector x. As a result, the opinion x'_i that the agent i adopts is:

$$x'_i = \frac{1}{1+\rho} \cdot (\frac{\sum_{j \in S_x(i)} x_j}{K} + \rho \cdot s_i)$$

The social cost function is respectively:

$$SC(x) = \sum_{i \in V} C_i(x_i, x_{-i}) = \sum_{i \in V} (\sum_{j \in S_x(i)} (x_j - x_i)^2 + \rho \cdot K \cdot (x_i - s_i)^2)$$

As before, we want to know whether this model has an equilibrium point and an efficient way to find it. Unfortunately, the K-NN model doesn't always have a N.E. something that is proved in the following observation.

Observation 25. *Let the instance of the K-NN model:* $s_1 = 0, s_2 = \frac{1}{2}, s_3 = 1, K = 1$ *and* $\rho = 1$. *This instance has no N.E.*

Proof. Let a N.E. $x^* = (a, b, c)$ exists, where a, b, c are the opinions of the nodes 1,2,3 respectively. Let $a = b$ then $a = \frac{a}{2}$ and $a = \frac{a+\frac{1}{2}}{2}$, which is impossible. With the same arguments it can be proved that $a \neq b, b \neq c$ and $c \neq a$.

Let $S_{x^*}(c) = a$ then $1-a < 1-b \implies b < a < c$, since $c = \frac{1+a}{2}$. Then, $S_{x^*}(a) = b (b < c)$ and $a = \frac{b}{2} > b$, which is impossible. As a result, $S_{x^*}(c) = b \implies a < b < \frac{1+b}{2} = c \implies a < b < c$.

Because of the previous relation: $\begin{cases} a = \frac{b}{2} \\ c = \frac{1+b}{2} \end{cases}$, now there are two cases:

- $S_{x^*}(b) = a \implies b = \frac{\frac{1}{2}+a}{\frac{1}{2}} = \frac{\frac{1}{2}+\frac{b}{2}}{\frac{1}{2}} \implies b = \frac{1}{3} \implies x^* = (\frac{1}{6}, \frac{1}{3}, \frac{2}{3})$, but in this case $S_{x^*}(b) = c$ because $\frac{1}{2} - \frac{1}{6} > \frac{2}{3} - \frac{1}{2}$.

- $S_{x^*}(b) = c \implies b = \frac{\frac{1}{2}+c}{\frac{1}{2}} = \frac{\frac{1}{2}+\frac{1+b}{2}}{\frac{1}{2}} \implies b = \frac{2}{3} \implies x^* = (\frac{1}{3}, \frac{2}{3}, \frac{5}{6})$, but now $S_{x^*}(b) = a$ since $\frac{1}{2} - \frac{1}{3} < \frac{5}{6} - \frac{1}{2}$

Consequently, in this game there is no N.E. □

The above observation provides us an instance at which there is not a N.E., which implies that on the contrary of the models that we have already seen, the K-NN model doesn't always have an equilibrium point. However, we would like to know the PoA at the instances that have a N.E. We will see that if $\rho = 1 + \epsilon, \epsilon > 0$ then $PoA \leqslant \frac{(7+\epsilon)\cdot(2+\epsilon)}{\epsilon\cdot(1+\epsilon)}$. On the other hand, when $\rho < 1$ then $PoA \geqslant \frac{1}{\rho^2}$ which means that as ρ reduces the PoA is unbounded.
We start with the case at which $\rho > 1$. Firstly, we will prove that $SC(s) \leqslant \frac{\rho+6}{\rho} \cdot OPT$, where $s = (s_1, s_2, ..., s_n)$ is the internal opinion vector and OPT is the minimum of the social cost function. We give two observations that will help us proving this.

Observation 26. *Let the opinion vector x and $A_x(i) = |j : i \in S_x(j)|$, is the number of the nodes that have i as neighbor. Let $s = (s_1, s_2, ..., s_n)$ is the internal opinion vector then $\forall i \in V : A_s(i) \leqslant 2 \cdot K$.*

Proof. Without loss of generality $s_1 < s_2 < ... < s_n$. Then it is certain that $i \notin S_s(i + K + 1)$, since there exist K nodes between i and $i + K + 1$. Respectively, $i \notin S_s(i - K - 1)$. As a result, $A_s(i) \leqslant 2 \cdot K$. □

Observation 27. *Let* $s = (s_1, s_2, ..., s_n)$ *and the internal opinion vector and* $o = (o_1, o_2, ..., o_n)$ *the optimal opinion profile such that* $SC(o) = OPT$*. Then,*

$$\sum_{i \in V}(\sum_{j \in S_0(i)} (o_j - o_i)^2 + \frac{\rho}{3} \cdot (o_j - s_j)^2 + \frac{\rho}{3} \cdot (o_i - s_i)^2) \geqslant \frac{\rho}{\rho + 6} \cdot \sum_{i \in V} \sum_{j \in S_s(i)} (s_i - s_j)^2$$

,where $SC(s) = \sum_{i \in V} \sum_{j \in S_s(i)} (s_i - s_j)^2$.

Proof. With basic algebra it is easy to prove that:

$$\forall \rho \geqslant 0 : \quad r^2 + \frac{\rho}{3} \cdot a^2 + \frac{\rho}{3} \cdot b^2 \geqslant \frac{\rho}{\rho + 6} \cdot (a + b + r)^2$$

Setting $r = |o_j - o_i|, a = |o_j - s_j|$ and $b = |o_i - s_i| \Longrightarrow$

$$\begin{aligned}(o_j - o_i)^2 + \frac{\rho}{3} \cdot (o_j - s_j)^2 + \frac{\rho}{3} \cdot (o_i - s_i)^2 \quad &\geqslant \quad (|o_j - o_i| + |o_j - s_j| + |o_i - s_i|)^2 \\ &\geqslant \quad \frac{\rho}{\rho + 6} \cdot (s_i - s_j)^2 \ (1)\end{aligned}$$

Since by the triangular inequality: $|s_i - s_j| \leqslant |o_j - o_i| + |o_j - s_j| + |o_i - s_i|$.

Adding the last inequality(1) for every $j \in S_o(i)$, we get:

$$\sum_{j \in S_0(i)} (o_j - o_i)^2 + \frac{\rho}{3} \cdot (o_j - s_j)^2 + \frac{\rho}{3} \cdot (o_i - s_i)^2 \geqslant \frac{\rho}{\rho + 6} \cdot \sum_{j \in S_o(i)} (s_i - s_j)^2 \ (2)$$

Notice that $S_s(i)$ are the K closest s_j to $s_i \Longrightarrow$

$$\sum_{j \in S_s(i)} (s_i - s_j)^2 \leqslant \sum_{j \in S_o(i)} (s_i - s_j)^2 \ (3)$$

$$\begin{aligned}(2),(3) \Longrightarrow \ &\sum_{i \in V}(\sum_{j \in S_0(i)} (o_j - o_i)^2 + \frac{\rho}{3} \cdot (o_j - s_j)^2 + \frac{\rho}{3} \cdot (o_i - s_i)^2) \geqslant \frac{\rho}{\rho + 6} \cdot \sum_{i \in V} \sum_{j \in S_s(i)} (s_i - s_j)^2 \\ \Longrightarrow &\sum_{i \in V}(\sum_{j \in S_0(i)} (o_j - o_i)^2 + \frac{\rho}{3} \cdot (o_j - s_j)^2 + \frac{\rho}{3} \cdot (o_i - s_i)^2) \geqslant \frac{\rho}{\rho + 6} \cdot SC(s)\end{aligned}$$

$\square$

Now, we are ready to continue with our proof. It is easy to notice that:

$$\sum_{i \in V}(\sum_{j \in S_0(i)} (o_j - o_i)^2 + \frac{\rho}{3} \cdot (o_j - s_j)^2 + \frac{\rho}{3} \cdot (o_i - s_i)^2) =$$

$$\sum_{i \in V} \sum_{j \in S_0(i)} (o_j - o_i)^2 + \frac{\rho \cdot K}{3} \cdot \sum_{i \in V}(o_i - s_i)^2 + \frac{\rho}{3} \cdot \sum_{i \in V} A_o(i) \cdot (o_i - s_i)^2$$

Because of observation 20: $A_o(i) \leqslant 2 \cdot K \Longrightarrow$

$$\sum_{i \in V} \sum_{j \in S_0(i)} (o_j - o_i)^2 + \frac{\rho \cdot K}{3} \cdot \sum_{i \in V}(o_i - s_i)^2 + \frac{\rho}{3} \cdot \sum_{i \in V} A_o(i) \cdot (o_i - s_i)^2 \leqslant$$

$$\sum_{i \in V} \sum_{j \in S_0(i)} (o_j - o_i)^2 + \frac{\rho \cdot K}{3} \cdot \sum_{i \in V}(o_i - s_i)^2 + \frac{\rho}{3} \cdot 2 \cdot K \cdot \sum_{i \in V} \cdot (o_i - s_i)^2 =$$

$$\sum_{i \in V} \sum_{j \in S_0(i)} (o_j - o_i)^2 + \rho \cdot K \cdot \sum_{i \in V}(o_i - s_i)^2 = OPT \Longrightarrow$$

$$\sum_{i \in V}(\sum_{j \in S_0(i)} (o_j - o_i)^2 + \frac{\rho}{3} \cdot (o_j - s_j)^2 + \frac{\rho}{3} \cdot (o_i - s_i)^2) \leqslant OPT \ (4)$$

Because of observation 21 and the inequality (4) $\Longrightarrow SC(s) \leqslant \frac{\rho+6}{\rho} \cdot OPT$ (5). We use the last inequality to bound the PoA, when $\rho = 1 + \epsilon, \epsilon > 0 (\rho > 1)$. The local smoothness technique gives us the upper bound to the PoA.

Theorem 12. *For $\rho = 1 + \epsilon, \epsilon > 0$ the PoA of the game is at most $\frac{(7+\epsilon)\cdot(2+\epsilon)}{\epsilon\cdot(1+\epsilon)}$*

Proof. From inequality (5) we know that $SC(s) \leqslant \frac{\rho+6}{\rho} \cdot OPT = \frac{7+\epsilon}{1+\epsilon} \cdot OPT$.

Let $\lambda > 0$ and $\mu < 1$ such that:

$$\forall x: \ \sum_{i \in V} C_i(x_i, x_{-i}) + (s_i - x_i) \cdot \frac{d}{dx_i} C_i(x_i, x_{-i}) \leqslant \lambda \cdot SC(s) + \mu \cdot SC(x) \ (6)$$

Then $PoA \leqslant \frac{7+\epsilon}{1+\epsilon} \cdot \frac{\lambda}{1-\mu}$. We will set $\mu = 0$ and we will find a $\lambda > 0$ such that the relation (6) holds. In this case the $PoA \leqslant \frac{7+\epsilon}{1+\epsilon} \cdot \lambda$.

From now on we will find a $\lambda > 0$ such that:

$$\forall x: \sum_{i\in V}(\sum_{j\in S_x(i)}(x_i-x_j)^2+\rho\cdot K(x_i-s_i))+2\cdot(s_i-x_i)\cdot\sum_{i\in V}(\rho\cdot K(x_i-s_i)+\sum_{j\in S_x(i)}(x_i-x_j))$$

$$\leqslant \lambda\cdot\sum_{i\in V}\sum_{j\in S_s(i)}(s_i-s_j)^2 \ (7)$$

Notice that $(x_i-x_j)^2+(x_i-s_i)^2+2\cdot(s_i-x_i)(x_i-x_j) = (s_i-x_j)^2$ and then inequality (7) can be transformed to the following inequality:

$$\sum_{i\in V}\sum_{j\in S_x(i)}(s_i-x_j)^2 \leqslant \sum_{i\in V}((\rho+1)\cdot K\cdot(s_i-x_i)^2+\lambda\cdot\sum_{j\in S_s(i)}(s_j-s_i)^2)$$

Also notice that $\sum_{j\in S_x(i)}(s_i-x_j)^2 \leqslant \sum_{j\in S_s(i)}(s_i-x_j)^2$ because $S_x(i)$ are the K closest x_j to s_i. As a result it sufficient to find a $\lambda > 0$ such that:

$$\forall x: \sum_{i\in V}\sum_{j\in S_s(i)}(s_i-x_j)^2 \leqslant \sum_{i\in V}((\rho+1)\cdot K\cdot(s_i-x_i)^2+\lambda\cdot\sum_{j\in S_s(i)}(s_j-s_i)^2)$$

It is easy to prove that for every a, b, d $(a+b)^2 \leqslant (d^2+1)\cdot a^2+(\frac{1}{d^2}+1)\cdot b^2$. Setting $a=(s_i-s_j), b=(s_j-x_j)$ and $d^2=\frac{(\rho-1)}{2}$, we get:

$$\sum_{j\in S_s(i)}(s_i-x_j)^2 \leqslant \sum_{j\in S_s(i)}((1+\frac{2}{\rho-1})\cdot(s_j-s_i)^2+\frac{\rho+1}{2}\cdot(s_j-x_j)^2) \Longrightarrow$$

$$\forall x: \sum_{i\in V}\sum_{j\in S_s(i)}(s_i-x_j)^2 \leqslant \sum_{i\in V}((\rho+1)\cdot K\cdot(s_i-x_i)^2)+(1+\frac{2}{\rho-1})\sum_{j\in S_s(i)}(s_j-s_i)^2)$$

As a result we have found $\lambda = 1+\frac{2}{\rho-1} > 0$,$\mu = 0$ such that the relation (6) holds. Thus, $PoA \leqslant \frac{7+\epsilon}{1+\epsilon}\cdot\lambda = \frac{(7+\epsilon)\cdot(2+\epsilon)}{\epsilon\cdot(1+\epsilon)}$. □

The last theorem tells us that in the instances with $\rho > 1$ in which an N.E. exists, the $PoA \leqslant \frac{(7+\epsilon)\cdot(2+\epsilon)}{\epsilon\cdot(1+\epsilon)}$. Now it's time to deal with the instances at which $\rho < 1$. As we have already claimed the PoA in this case is at least $\frac{1}{\rho^2}$. Let the following instance of our game:

Observation 28. *Let the insance of the K-NN model, at which K=1, $s_0 = s_1 = 0$,$s_5 = s_6 = 1$ and $s_3 = x, s_4 = 1 - x$, $x < \frac{1}{2}$. The following game has PoA $> \frac{1}{\rho^2}$.*

Proof. Let x the opinion vector at which $x_0 = x_1 = 0, x_5 = x_6 = 1$ and the nodes 3 and 4 point to each other. Let,

$$\begin{cases} x_1 = \frac{1}{\rho+1} \cdot (x_2 + \rho \cdot x) \\ x_2 = \frac{1}{\rho+1} \cdot (x_1 + \rho \cdot (1-x)) \end{cases} \Longrightarrow x_1 = \frac{1+\rho \cdot x}{\rho+2} \text{ and } x_2 = \frac{1+\rho\cdot(1-x)}{\rho+2}$$

Notice that since nodes 3 and 4 point to each other:

$$\begin{cases} \frac{1+\rho\cdot x}{\rho+2} \geqslant \frac{1+\rho\cdot(1-x)}{\rho+2} - x \\ 1 - \frac{1+\rho\cdot(1-x)}{\rho+2} \geqslant \frac{1+\rho\cdot(1-x)}{\rho+2} - x \end{cases} \iff \frac{\rho\cdot\delta}{\rho+2} \leqslant x \text{ (8), where } \delta = |s_3 - s_4| = 1 - 2\cdot x.$$

If inequality (8) holds then: $SC(x) = (x_3 - x_4)^2 + \rho\cdot(x_3 - s_3)^2 + \rho\cdot(x_4 - s_4)^2 = \frac{\delta^2\cdot\rho}{\rho+2}$, then, $OPT \leqslant SC(x) \leqslant \frac{\delta^2\cdot\rho}{\rho+2} \leqslant \frac{2\cdot\rho\cdot\delta^2}{\rho+1} \Longrightarrow OPT \leqslant \frac{2\cdot\rho\cdot\delta^2}{\rho+1}$

Now let the x^* the opinion vector at which $x^*_0 = x^*_1 = 0$,$x^*_5 = x^*_6 = 1$ and node 3 points to 0 and node 4 points to 5. As a result, $x^*_3 = \frac{\rho\cdot x}{\rho+1}$ and $x^*_4 = 1 - \frac{\rho\cdot x}{\rho+1}$.

Then: $$\begin{cases} x \leqslant 1 - \frac{\rho\cdot x}{\rho+1} - x \\ 1 - (1-x) \leqslant 1 - x - \frac{\rho\cdot x}{\rho+1} \end{cases} \iff \frac{\rho\cdot x}{\rho+1} \leqslant \delta \text{ (9)}$$

If we set $x = \frac{\rho+1}{\rho}\cdot\delta$ then both relation (8) and (9) hold. It is also easy to see that: $SC(x^*) = \frac{2 \cdot x^2 \cdot \rho}{1+\rho}$ and that x^* is a N.E. $\Longrightarrow$

$$PoA \geqslant \frac{SC(x^*)}{OPT} \geqslant \frac{SC(x^*)}{SC(x)} \geqslant (\frac{x}{\delta})^2 \geqslant (1 + \frac{1}{\rho})^2 > (\frac{1}{\rho})^2$$

$\square$

The above observation provide us an instance of the K-NN model at which the $PoA > \frac{1}{\rho^2}$. In the case that $\rho \geqslant 1$ the above instance doesn't have something to say since by definition $PoA \geqslant 1$. On the contrary, if $\rho < 1$ then the PoA is unbounded as ρ reduces. Now,we have complete our analysis concernig the Nash Equilibrium and the PoA of this model, However there are still open questions. We would like to whether the exists an polynomial time algorithm that computes the Nash Equilibrium of this game as also a mechanism tha leads the agents to a N.E. [13]

Bibliography

[1] D. P. Bertsekas. *Nonlinear Programming*. Athena Scientific, Belmont, MA, 1999.

[2] A. Bhattacharyya, M. Braverman, B. Chazelle, and H. L. Nguyen. On the convergence of the hegselmann-krause system. *CoRR*, abs/1211.1909, 2012.

[3] K. Bhawalkar, S. Gollapudi, and K. Munagala. Coevolutionary opinion formation games. In *Proceedings of the Forty-fifth Annual ACM Symposium on Theory of Computing*, STOC '13, pages 41–50, New York, NY, USA, 2013. ACM.

[4] D. Bindel, J. M. Kleinberg, and S. Oren. How bad is forming your own opinion? *CoRR*, abs/1203.2973, 2012.

[5] V. D. Blondel, J. M. Hendrickx, and J. N. Tsitsiklis. On krause's consensus formation model with state-dependent connectivity. *CoRR*, abs/0807.2028, 2008.

[6] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, New York, NY, USA, 2004.

[7] C. Daskalakis, P. W. Goldberg, and C. H. Papadimitriou. The complexity of computing a nash equilibrium. pages 71–78. ACM Press, 2006.

[8] M. DeGroot. Reaching a consesus. *Journal of the American Statistical Association*, 1974.

[9] D. Ferraioli, P. W. Goldberg, and C. Ventre. Decentralized dynamics for finite opinion games. *CoRR*, abs/1311.1610, 2013.

[10] J. Ghaderi and R. Srikant. Opinion dynamics in social networks: A local interaction game with stubborn agents. *CoRR*, abs/1208.5076, 2012.

[11] B. Golub and M. O. Jackson. Nave learning in social networks: Convergence, influence, and the wisdom of crowds, 2007.

[12] R. Hegselmann and U. Krause. Opinion dynamics and bounded confidence: Models, analysis and simulation. *Journal of Artificial Societies and Social Simulation*, 5:1–24, 2002.

[13] M. O. Jackson. *Social and Economic Networks*. Princeton University Press, Princeton, NJ, USA, 2008.

[14] E. Koutsoupias and C. Papadimitriou. Worst-case equilibria. In *IN PROCEEDINGS OF THE 16TH ANNUAL SYMPOSIUM ON THEORETICAL ASPECTS OF COMPUTER SCIENCE*, pages 404–413, 1999.

[15] C. D. Meyer, editor. *Matrix Analysis and Applied Linear Algebra*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2000.

[16] R. Motwani and P. Raghavan. *Randomized Algorithms*. Cambridge University Press, New York, NY, USA, 1995.

[17] R. W. Rosenthal. A class of games possessing pure-strategy nash equilibria. *International Journal of Game Theory*, 2:65–67, 1973.

[18] T. Roughgarden and F. Schoppmann. Local smoothness and the price of anarchy in atomic splittable congestion games. In *Proceedings of the Twenty-second Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '11, pages 255–267. SIAM, 2011.

[19] G. Strang. *Introduction to Linear Algebra*. Wellesley-Cambridge Press, 2003.

[20] E. Yildiz, A. Ozdaglar, D. Acemoglu, A. Saberi, and A. Scaglione. Binary opinion dynamics with stubborn agents. *ACM Trans. Econ. Comput.*, 1(4):19:1–19:30, Dec. 2013.