Complex Action Problem

Silver-Blaze Phenomenon in the relativistic Bose gas

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October 2014

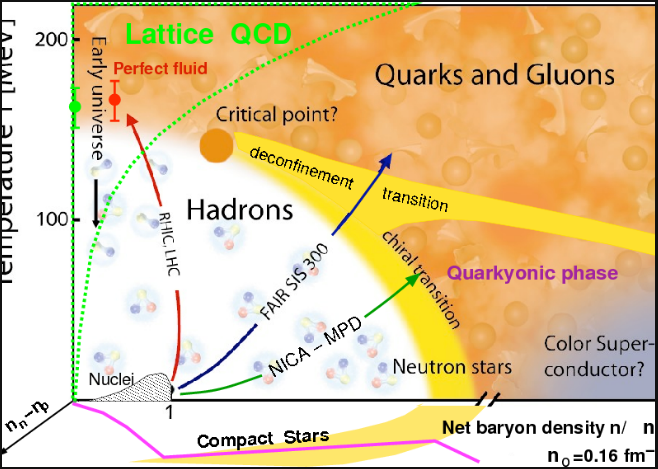
Outline

1. Complex Action Problem
   - Motivation
   - The Problem
   - Solutions (so far)
2. Stochastic quantization
   - Langevin equation
   - Fokker-Planck equation and distribution
   - Complex Langevin dynamics
3. Silver-Blaze phenomenon
   - Discrete Langevin dynamics
   - Relativistic Bose gas and simulations on a lattice
   - Summary

Outline

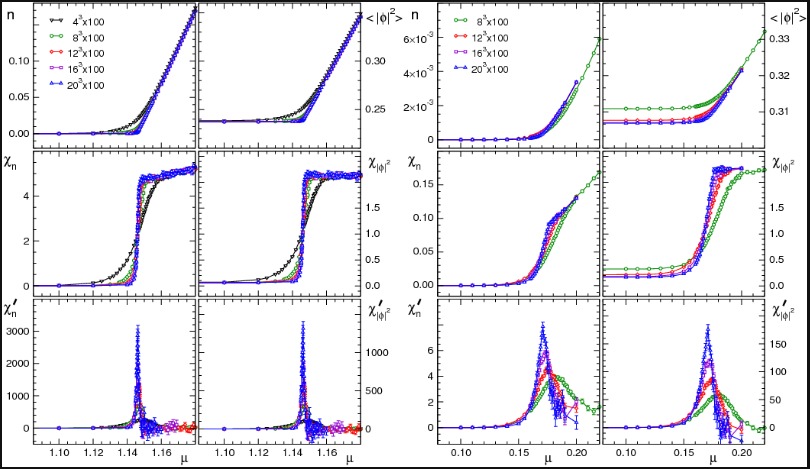
QCD phase diagram

Larry McLerran, arXiv:0906.2651v1 [hep-ph]



Silver-Blaze phenomenon

Christof Cattringer, Thomas Kloiber, arXiv:1206.2954v2 [hep-lat]



General Applications

Stochastic numerical calculations of integrals

- calculate expectation value integrals

$$\langle f \rangle_0 = \frac{\int_X f(x)\varrho(x)dx}{\int_X \varrho(x)dx}$$

by sampling configuration space via Monte Carlo

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- ϱ while natural is *not* always the best! (overlap problem)

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Complex weights

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- makes weight-sampling impossible

Re-weighting

Partial solution to the sign problem

- partial solution comes with re-weighting

$$\langle f \rangle = \frac{\dfrac{\int_X f(x)\varrho(x)e^{\imath\vartheta(x)}dx}{\int_X \varrho(x)dx}}{\dfrac{\int_X \varrho(x)e^{\imath\vartheta(x)}dx}{\int_X \varrho(x)dx}} = \frac{\langle f e^{\imath\vartheta}\rangle_0}{\langle e^{\imath\vartheta}\rangle_0}$$

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- using phase-quenched weights probability makes sense again

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The subset of X that contributes the most to the integral $\langle f \rangle$. Grows with sample size.

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- This is a general problem found in any weighting ϱ
- Re-weighting suffers from it too, though not as seriously as:

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The expectation value and relative error estimated by N independent measurements of $e^{\imath\vartheta}$ scale with the number of degrees of freedom $\dim X$ as

$$\langle e^{\imath\vartheta}\rangle_0 \propto e^{-\dim X} \text{ and } \frac{\Delta\langle e^{\imath\vartheta}\rangle_0}{\langle e^{\imath\vartheta}\rangle_0} \propto \frac{1}{\sqrt{N}} e^{\dim X}$$

meaning $N \propto e^{2\dim X}$ *at least* which is prohibitive.

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The presence of sign or phase factor in the integrand prevents thermalization (arrival at the important integration domain).

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Observables ((ground) expectation values)

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \mathcal{O}[\phi] \exp(-S[\phi]) \longleftarrow \prod_x \int d\phi_x \mathcal{O}[\phi] \exp(-S[\phi])$$

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- typical lattice simulation techniques fail

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G. Parisi, Y.-S. Wu, Sci. Sinica 24 (1981) 4S3

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Bonus! We get a configuration markovian chain in one package.

Langevin equation

Poul H. Damgaard, Helmuth Hüffel 152, Nos. 5 & 6 (1987) 227-398

(real) Langevin equation ($\phi \in \mathbb{R}$, $S \in \mathbb{R}$)

$$\frac{\partial}{\partial \tau}\phi(x,\tau) = K(\phi(x,\tau)) + \eta(x,\tau) \qquad \phi(x,\tau_0) = \phi_0(x)$$

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What's with Dirac 's bra-ket $\langle \_ | \_\_\_ | \_ \rangle$ symbol here?

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The Langevin process is actually a markovian one,

$$\langle\phi|\wp(\tau,\tau'')|\phi''\rangle = \int \mathcal{D}\phi\langle\phi|\wp(\tau,\tau')|\phi'\rangle\langle\phi'|\wp(\tau',\tau'')|\phi''\rangle$$

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The whole formulation develops on Langevin time τ as well as spacetime X. (extra degrees of freedom)

Fokker-Planck equation

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$\forall\tau$ in equilibrium (postulated)

$$\langle\mathcal{O}(\tau)\rangle = \int \mathcal{D}\eta\rho[\eta]\mathcal{O}[\phi(\tau)] = \int \mathcal{D}\phi\langle\phi|\wp(\tau,\tau_0)|\phi_0\rangle\mathcal{O}[\phi]$$

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$$\langle \phi | \wp(\tau, \tau_0) | \phi_0 \rangle = \langle \delta[\phi - \phi(\tau)] \rangle$$

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$$\langle\mathcal{O}(\tau)\rangle = \int \mathcal{D}\eta\rho[\eta]\mathcal{O}[\phi(\tau)] = \int \mathcal{D}\phi\langle\phi|\wp(\tau,\tau_0)|\phi_0\rangle\mathcal{O}[\phi]$$

$$\langle\phi|\wp(\tau,\tau_0)|\phi_0\rangle = \langle\delta[\phi-\phi(\tau)]\rangle$$

Fokker-Planck equation

$$\frac{\partial}{\partial\tau}\langle\phi|\wp(\tau,\tau_0)|\phi_0\rangle = \int_X d^{\dim X}x \frac{\delta}{\delta\phi(x)}\left(\frac{\delta}{\delta\phi(x)} + \frac{\delta}{\delta\phi(x)}S[\phi]\right)\langle\phi|\wp(\tau,\tau_0)|\phi_0\rangle$$

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Fokker-Planck equation with kernel $\mathcal{K}$

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Static (equilibrium) solution

Jean Zinn-Justin. International Series of Monographs on Physics 113.

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static solution to Fokker-Planck equation (exists!)

$$\langle\phi|\wp_\infty|\phi_0\rangle = Z^{-1}\exp(-S[\phi])$$

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Feynman path integral emergence

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$$\langle\mathcal{O}(\tau)\rangle = \rho_0^{-1}\int \mathcal{D}\eta\mathcal{O}[\phi(\tau)]\exp\left(-\frac{1}{4}\int_X d^{\dim X}x(\eta(x))^2\right) = \\ = Z^{-1}\int\mathcal{D}\phi\mathcal{O}[\phi]\exp(-S[\phi])$$

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- generate configurations via Langevin process in equilibrium
- calculate observables in ensemble with noise distribution
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  - we let the Langevin process do all the (markovian) work
- stochastic calculation matches that of path integral's!

Outline

Extension to complex Langevin

Stochastic quantization is solid in theory for $\phi \in \mathbb{R}$ and $S \in \mathbb{R}$.

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But first things first...

Complex index notation

$\forall\phi \in \mathbb{C}$, $\phi = \alpha^{-1}(\phi_0 + \imath\phi_1)$ where $\alpha > 0$ is a normalization.

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$$\bigcirc_{abc} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

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$\bullet_{abc} = \bigcirc_{bac}$

$\Diamond_{ab}\bigcirc_{bcd} = \bigcirc_{afe}\Diamond_{fd}\Diamond_{ec}$

$\phi \in \mathbb{C}$ and $S \in \mathbb{R}$

Langevin equation

$$\frac{\partial}{\partial \tau}\phi_a(x,\tau) = K_a(\phi(x,\tau)) + \eta_a(x,\tau)$$

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$$\alpha_{aa} \leq \delta_{aa} = \dim_{\mathbb{R}} \mathbb{C} = 2$$

$\phi \in \mathbb{C}$ and $S \in \mathbb{R}$

Fokker-Planck equation and equilibrium

Fokker-Planck hamiltonian

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The Feynman path integral is lost for the full theory!

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The Feynman path integral is lost for the full theory!

Unless of course $\alpha_{aa'} = \delta_{aa'}$, i.e. *full noise* is taken.

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We assume: $S = S_0 + \jmath S_1$

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A distinct from field's complex unity $\jmath$ plus no normalization.

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We assume: $S = S_0 + \jmath S_1$

Something's very wrong here!

$\phi \in \mathbb{C}$ and $S \in \mathbb{C}$

Langevin equation

$$\frac{\partial}{\partial \tau}\phi_a(x,\tau) = K_a(\phi(x,\tau)) + \eta_a(x,\tau)$$

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We assume: $S = S_0 + \jmath S_1$

We fix by extending real fields to complex: $\phi = \beta^{-1}(\phi_0 + \jmath\phi_1)$

$\phi \in \mathbb{C}$ and $S \in \mathbb{C}$

(modified) action

Technically, the original action non-writable in index form.

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Even the original imaginary part! The parameters actually spread out even in both parts of the new action S_a.

$\phi \in \mathbb{C}$ and $S \in \mathbb{C}$

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$\beta_{aa} < \delta_{aa}$ doesn't necessarily mean loss of information.

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The imaginary part ϕ_1 is auxiliary to start with.

$\phi \in \mathbb{C}$ and $S \in \mathbb{C}$

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$$\int_X d^{\dim X} x \frac{\delta}{\delta \phi_a(x)} \left(\beta_{aa'} \frac{\delta}{\delta \phi_{a'}(x)} + \bullet_{abc} \frac{\delta}{\delta \phi_b(x)} S_c[\phi] \right)$$

$\beta_{aa} < \delta_{aa}$ doesn't necessarily mean loss of information.

The imaginary part ϕ_1 is auxiliary to start with.

However S_0 is yet "unrecognizable".

$\phi \in \mathbb{C}$ and $S \in \mathbb{C}$

Fokker-Planck equation and equilibrium

Fokker-Planck hamiltonian

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equilibrium distribution

$$\wp_\infty[\phi] \propto \exp(-g[\phi])$$

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$$g[\phi] = g_a S_a[\phi]$$

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(bi)complex fields

$$\alpha_{aa} = \alpha^2 = 2$$

$$\beta_{aa} = \beta^2 = 1$$

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$$\sqrt{2}\phi = (\phi_{00} + \jmath\phi_{01}) + \imath(\phi_{10} + \jmath\phi_{11})$$

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Langevin dynamics test

- In general the Fokker-Planck distribution is away from the entropic factor $\exp(-S)$.

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- So is averaging over the Langevin ensemble still valid?
  - One way we can check (quickly): simulations!
  - There is also an interesting property regarding observables.

$\phi \in \mathbb{C} \otimes \mathbb{C}$ and $S \in \mathbb{C}$

observables

- Typical observables defined via specific parameters in action

$$\langle \mathcal{O} \rangle = \frac{\partial}{\partial \alpha} \log_e Z = \left\langle -\frac{\partial}{\partial \alpha} S \right\rangle$$

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- follows extension of observables $\mathcal{O} \longrightarrow \mathcal{O}_b$

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  - An interesting property: the auxiliary information is gone.
  - However, much like the action, $\langle \mathcal{O}_0 \rangle$ is not what we think: it contains data from the full original complex observable.

Outline

Discretization of Langevin time τ

continuous .... $\tau = n\epsilon$ .... discrete

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continuous	$\tau = n\epsilon$	discrete
$\phi(\tau)$		$\phi_n = \epsilon^{-1}\int_\tau^{\tau+\epsilon} d\tau \phi(\tau)$

Discretization of Langevin time τ

continuous .... $\tau = n\epsilon$ .... discrete

$$\frac{\partial}{\partial \tau}\phi(\tau) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \epsilon^{-1}(\phi_{n+1} - \phi_n)$$

Discretization of Langevin time τ

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$\int d\tau f(\tau)$		$\sum_n \epsilon f_n$

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ϵ is *not* a differential (time)!

Dynamic Langevin time step ϵ

standard drift average

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$$\log K = \sum_n \tau^{-1} \epsilon_n \log K_n$$

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Outline

Preliminaries

$$\bigcirc_{aa_1\ldots a_n} = \delta_{ab_0}\prod_{i=1}^{n-1}\bigcirc_{b_{i-1}a_ib_i}\delta_{b_{n-1}a_n}$$

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Relativistic Bose gas

action

$$\varkappa_{\dim X} = 2\dim X + m^2$$

$$\mathcal{L}_{d,x} = \frac{1}{2}\varkappa_{\dim X}\bigcirc_{def}\phi_{ge,x}\phi_{gf,x} + \frac{1}{4}\lambda\bigcirc_{defgh}\phi_{ie,x}\phi_{jf,x}\phi_{ig,x}\phi_{jh,x}$$
$$-\sum_{\alpha=1}^{\dim X}\cosh(\ell\mu\delta_{\alpha\,\dim\mathbb{L}})\delta_{de}\bigcirc_{efg}\delta_{hi}\phi_{hf,x}\phi_{ig,x+\hat{\alpha}}$$
$$-\sum_{\alpha=1}^{\dim X}\sinh(\ell\mu\delta_{\alpha\,\dim X})\varepsilon_{de}\bigcirc_{efg}\varepsilon_{hi}\phi_{hf,x}\phi_{ig,x+\hat{\alpha}}$$

Relativistic Bose gas

drift

$$\varkappa_{\dim X} = 2\dim X + m^2$$

$$K_{ab}(\phi_x) = -\varkappa_{\dim X}\phi_{ab,x} - \lambda\bigcirc_{bcde}\phi_{ac,x}\phi_{fd,x}\phi_{fe,x} + \sum_{\alpha=1}^{\dim X}\cosh(\mu\delta_{\alpha\dim X})\delta_{ac}\delta_{bd}(\phi_{cd,x+\hat{\alpha}} + \phi_{cd,x-\hat{\alpha}}) + \sum_{\alpha=1}^{\dim X}\sinh(\mu\delta_{\alpha\dim X})\varepsilon_{ac}\varepsilon_{bd}(\phi_{cd,x+\hat{\alpha}} - \phi_{cd,x-\hat{\alpha}})$$

Relativistic Bose gas

observables

$$n_{a,x} = -\frac{\partial}{\partial(\ell\mu)}\mathcal{L}_{a,x} =$$

$$= \sum_{\alpha=1}^{\dim X} \sinh(\ell\mu\delta_{\alpha \dim X})\delta_{ab}\bigcirc_{bcd}\delta_{ef}\phi_{ec,x}\phi_{fd,x+\hat{\alpha}}$$

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$$|\phi_x|_a^2 := \frac{\partial}{\partial((\ell m)^2)}\mathcal{L}_{a,x} = \frac{1}{2}\bigcirc_{abc}\phi_{db,x}\phi_{dc,x}$$

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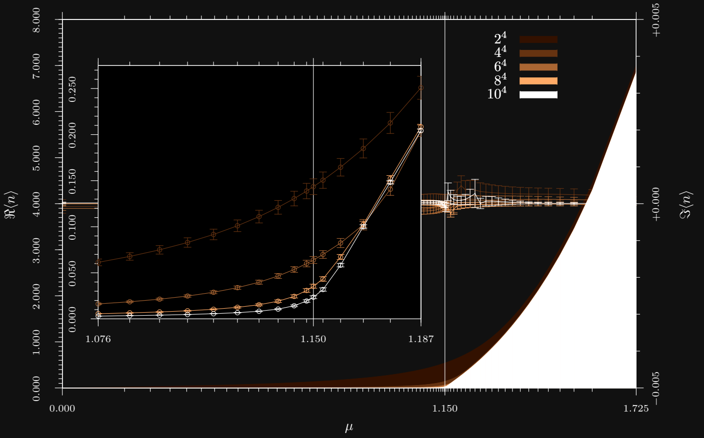
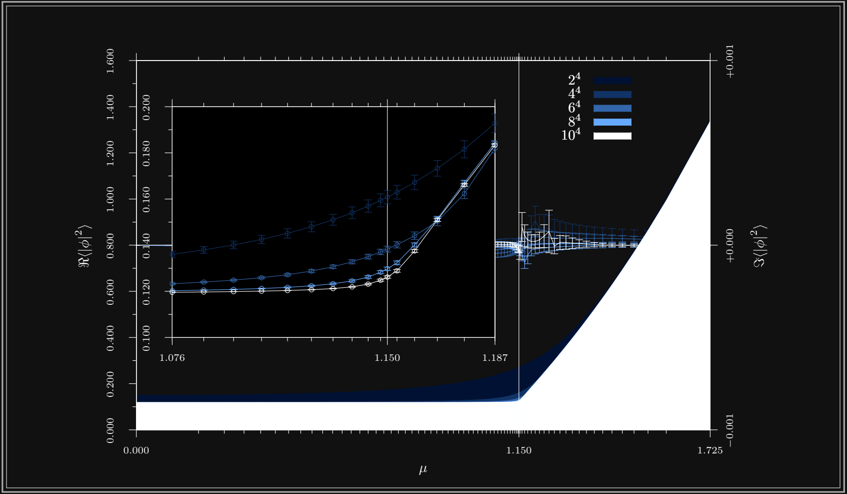
$\langle n \rangle$



$\langle|\phi|^2\rangle$



Outline

Conclusions

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  - when action is *complex*
    - stochastic quantization keeps probability interpretation

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    Christof Cattringer, Thomas Kloiber.
    arXiv:1206.2954v2 [hep-lat] 12 December 2012

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  - where does it fail and how/why may give guidelines towards modifying the method accordingly