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Lyapunov-Schmidt Reduction Methods For Solving PDE's

ΖΑΧΑΡΕΝΑΚΗΣ ΔΗΜΗΤΡΙΟΣ
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Περίληψη

Μελετάμε τη μέθοδο Lyapunov-Schmidt Reduction που ξεκίνησε από την πρωτοποριακή δουλειά των Floer A., Weinstein A. [35] και μάλιστα εξελίχθηκε μέχρι που λύθηκε η εικασία του De Giorgi για $N = 9$ διαστάσεις. Αναφέρουμε τη σχέση της μεθόδου με τις εξίσωσεις Allen-Cahn, NLS και τα minimal surfaces, constant mean curvature surfaces αντίστοιχα.

Ιστορικά, υπάρχει πλούσιο υλικό από το έργο των De Giorgi [16], Del Pino, Kowalczyk, Wei, Felmer, Pacard [25],[27],[28],[29], [30],[31],[32],[33],[34], [42], Ambrosetti, Malchiodi [3],[4],[5],[6],[7],[8],[9],[10],[11], [40] Modica [41], Berestycki [15] και πολλών άλλων.

Abstract

We study the Lyapunov-Schmidt Reduction Method, that started from the pioneering work by Floer and Weinstein [35] and evolved untill the De Giorgi conjecture was solved for $N = 9$ dimensions. The relation of this method with the Allen-Cahn , NLS equations and minimal surfaces, constant mean curvature surfaces respectively, is also mentioned.

Historically, there is a vast amount of work by De Giorgi [16], Del Pino, Kowalczyk, Wei, Felmer, Pacard [25],[27],[28],[29], [30],[31],[32],[33],[34],[42], Ambrosetti, Malchiodi [3],[4],[5],[6],[7],[8],[9],[10],[11],[40] Modica [41], Berestycki [15] and many other.

Contents

Πρόλογος

Χωριζουμε την εργασια σε 3 κυρια μέρη.

Το Κεφάλαιο 1, που περιέχει μια εισαγωγή στη Γ - σύγκλιση με κάποια ιστορικά στοιχεία αλλά και αρκετούς ορισμούς και θεωρήματα, βασικές έννοιες της θεωρίας αυτής, όπως επίσης και τη σύνδεση με φαινόμενα αλλαγής φάσης.

Το Κεφαλαιο 2, στο οποιο κάνουμε μια εισαγωγή για την εξίσωση Allen-Cahn, όπου έχουμε transition layers, δείχνουμε ύπαρξη λύσης για κάθε ϵ, μέσω μεθόδων μεταβολών και έπειτα μελετάμε τη συμπεριφορά όταν $\epsilon \to 0$ εφαρμόζοντας τη μέθοδο Lyapunov-Schmidt Reduction.

Το Κεφάλαιο 3, στο οποίο κάνουμε μια εισαγωγή για την εξίσωση NLS, όπου έχουμε spikes, τονίζοντας τη πρωτοποριακή δουλειά των Floer A., Weinstein A. [35] που κατασκεύασαν μια θετική λύση μέσω της μεθόδου Lyapunov-Schmidt Reduction.

Τέλος, στο παράρτημα κάνουμε μια ανασκόπηση της Lyapunov-Schmidt μεθόδου για ελλειπτικές εξισώσεις και το πώς πηγε ιστορικα από την αναγωγή πεπερασμένης διάστασης σε άπειρης ώστε να λύσει την εικασία του De Giorgi. Επίσης, τονίζουμε το γεγονός ότι η μέθοδος επεκτείνεται και σε παραβολικά προβλήματα.

Θα ήθελα να ευχαριστήσω τον επιβλέποντα καθηγητή Ρόθο Βασίλειο αυτής της διπλωματικής εργασίας, για την υπομονή και κατανόησή του. Ευχαριστώ, ακόμα το καθηγητή Σούρδη Χρήστο για τη βοήθεια και συνεισφορά του στη συγγραφή της, όπως επίσης και τη τριμελή επιτροπή.

Chapter 1

Γ - Convergence

1.1 Introduction

The theory of Γ-convergence, from the time of its inception by Ennio De Giorgi in the the 1970's, has become a powerful tool in a variational framework. It emanated from previous notions of convergence related mainly to elliptic operators as G-convergence or H-convergence or to convex functionals as Mosco convergence. The last forty years have seen an increasing interest for variational convergences. But why a variational convergence?

In many mathematical problems, may they come from the world of Physics, applications to problems in Partial Differential Equations (phase transitions, singular perturbations, boundary value problems in wildly perturbed domains, and nonsmooth analysis) or abstract mathematical questions, some parameter ϵ appears (small or large, of geometric or constitutive origin, coming from an approximation process or a discretization argument, at times more than a single parameter) which makes those problems increasingly complex or more and more degenerate. Nevertheless, as we vary this ϵ-parameter, it is often possible to anticipate some "limit" behaviour, and assume that we may substitute the complex, degenerate problems we started with, with a new one, simpler and with a more comprehensible behaviour, possibly of a completely different type, where the parameters have disappeared, or appear in a more handy way.

Therefore it plays a central role for its compactness properties and for the large number of results concerning Γ-limits of integral functionals. It also, provides a indespensible tool for studying global and local minimizers. An essential matter in the definition of Γ-convergence is examining the behaviour of a family of global minimum problems (minimum values and minimizers) of a sequence f_ϵ, in an abstract notation

$$min\{f_\epsilon(x) : x \in X_\epsilon\} \tag{1.1.1}$$

by the computation of an "effective" minimum problem

$$min\{f(x) : x \in X\} \tag{1.1.2}$$

involving the (properly defined) Γ-limit of this sequence.

Even though the definition of such a limit is local (in that in defining its value at a point x we only take into account sequences converging to x), its computation in general does not describe the behaviour of local minimizers of f_ϵ (i.e., points x_ϵ which are absolute minimizers of the restriction of f_ϵ to a small neighbourhood of the point x_ϵ itself).

In Figure 1, we can see a possible situation, in a simplified picture, where f_ϵ has many local minimizers. However, after the Γ-convergence procedure some or all minimizers are "integrated out" (note that this happens even when the oscillations depth does not vanish). When we have an isolated local minimizer x of the Γ-limit, we may track the behaviour of local minimizers as absolute minimizers of f_ϵ restricted to a fixed neighbourhood of x and conclude the existence of local minimizers for f_ϵ close to x. Kohn and Sternberg used this general principle and found the existence of local minimizers of the Allen-Cahn equation by considering a surface that locally minimizes its area (minimal surface).

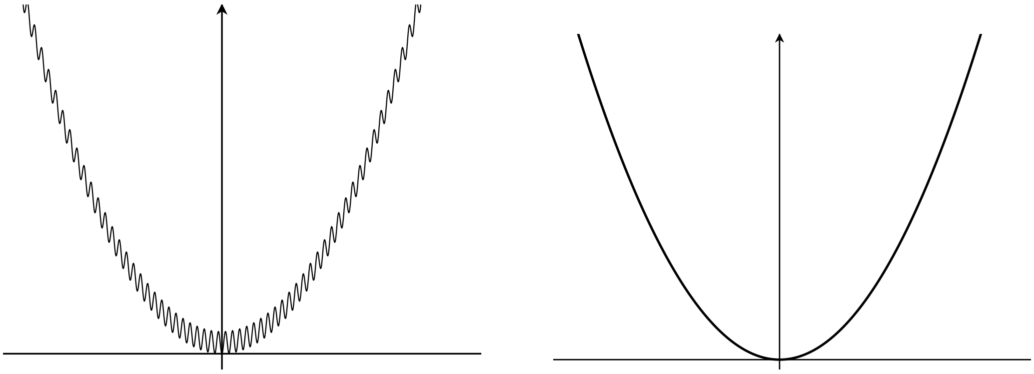

Figure 1.1

1.2 Some definitions of Γ-convergence

Some properties

1. Γ-limits are stable under continuous perturbations. This means that once a Γ-limit is computed we do not have to redo all computations if "lower-order terms" are added. Conversely, we can always remove such terms to simplify calculations.

2. Under suitable conditions Γ-convergence implies convergence of minimum values and minimizers. Note that some minimizers of the Γ-limit may not be limit of minimizers, so that Γ-convergence may be interpreted as a choice criterion.

3. The computation of Γ-limits can be separated into computing lower and upper bounds. The first involving lower-semicontinuity inequalities, the second the construction of suitable approximating sequences of functions. In order to better handle these operations Γ-Iower and upper limits are introduced.

4. The natural setting of Γ-convergence are lower semicontinuous functions. In particular Γ-upper and lower limits are lower semicontinuous functions, and the operation of Γ-limit does not change if functionals are replaced by their lower semicontinuous envelopes (which, in turn, are usually easier to handle).

5. The choice of the convergence with respect to which computing the Γ-limit is essential. Since the arguments of Γ-convergence rely on compactness issues, it is usually more convenient to use weaker topologies, which explains why spaces of "weakly-differentiable functions" are preferred.

Definition 1.2.1. *Let $f : X \to [-\infty, +\infty]$, where X is a metric space equipped with the distance d. We define the* ***lower limit*** *of f at x as*

$$\begin{aligned}\lim_{y\to x} inf f(y) &= inf\left\{\lim_{\epsilon} inf f(x_\epsilon) : x_\epsilon \in X, x_\epsilon \to x\right\}\\ &= inf\left\{\lim_{\epsilon} f(x_\epsilon) : x_\epsilon \in X, x_\epsilon \to x,\ \exists \lim_{\epsilon} f(x_\epsilon)\right\}\end{aligned}$$

We define the ***upper limit*** *of f at x as*

$$\begin{aligned}\lim_{y\to x} sup f(y) &= sup\left\{\lim_{\epsilon} sup f(x_\epsilon) : x_\epsilon \in X, x_\epsilon \to x\right\}\\ &= sup\left\{\lim_{\epsilon} f(x_\epsilon) : x_\epsilon \in X, x_\epsilon \to x,\ \exists \lim_{\epsilon} f(x_\epsilon)\right\}\end{aligned}$$

By taking $x_\epsilon = x$ we always get $\lim\limits_{y\to x} inf f(y) \le f(x)$. It can also be checked that

$$\begin{aligned}&\lim_{y\to x} inf\,(-f(y)) = -\lim_{y\to x} sup f(y)\\ &\lim_{y\to x} inf\,(f(y) + g(y)) \ge \lim_{y\to x} inf f(y) + \lim_{y\to x} inf g(y)\\ &\lim_{y\to x} inf\,(f(y) + g(y)) \le \lim_{y\to x} sup f(y) + \lim_{y\to x} inf g(y)\end{aligned}$$

Definition 1.2.2. *A function $f : X \to \overline{\mathbb{R}}$ is called (sequentially) lower semicontinuous (l.s.c. for short) at x, if for every sequence (x_ϵ) converging to x we have*

$$f(x) \le \lim_{\epsilon} inf f(x_\epsilon)$$

Remark 1.1. *(i) If f and g are l.s.c. at x, then so is $f + g$*

(ii) If $f = \mathbb{X}_E$ is the characteristic function of the set E, then f is l.s.c. if and only if E is open.

(iii) A function $f : X \to \overline{\mathbb{R}}$ is called upper semicontinuous if $-f$ is l.s.c. Then $f = \mathbb{X}_E$ is upper semicontinuous if and only if E is closed.

Definition 1.2.3. *We say that the function f is a* ***lower bound*** *for the sequence (f_ϵ) if for all $x \in X$ we have*

$$f(x) \le \lim_{\epsilon\to 0} inf f_\epsilon(x_\epsilon),\ \forall x_\epsilon \to x$$

Definition 1.2.4. *We say that the function f is an* ***upper bound*** *for the sequence (f_ϵ) if for all $x \in X$ we have that there exists a $x_\epsilon \to x$ such that*

$$f(x) \ge \lim_{\epsilon\to 0} sup f_\epsilon(x_\epsilon)$$

Definition 1.2.5. *We say that f is the Γ-limit for the sequence (f_ϵ) if it is both lower and upper bound. If both bounds hold at a point x, then we say that f is the Γ-limit at x and we write*

$$f(x) = \Gamma - \lim_{\epsilon\to 0} f_\epsilon(x)$$

In this notation f_ϵ Γ*-converges to* f *if and only if* $f(x) = \Gamma - \lim\limits_{\epsilon\to 0} f_\epsilon(x)$ *at all* $x \in X$.

Remark 1.2. *If* f *is a lower bound then requiring that upper bound holds is equivalent to any of the following*

(i) there exists a $x_\epsilon \to x$ *such that*

$$f(x) = \lim_{\epsilon\to 0} f_\epsilon(x_\epsilon)$$

(ii) $\forall \eta > 0$ *there exists* $x_\epsilon \to x$ *such that* $f(x) + \eta \geq \lim\limits_{\epsilon\to 0} sup f_\epsilon(x_\epsilon)$

A sequence satisfying the first one is called recovery sequence. The second one is called approximate limsup inequality.

Definition 1.2.6. *We define*

$$\Gamma - \lim_{\epsilon\to 0} inf f_\epsilon(x) = inf \left\{ \lim_{\epsilon\to 0} inf f_\epsilon(x_\epsilon) : x_\epsilon \to x \right\}$$
$$\Gamma - \lim_{\epsilon\to 0} sup f_\epsilon(x) = inf \left\{ \lim_{\epsilon\to 0} sup f_\epsilon(x_\epsilon) : x_\epsilon \to x \right\}$$

as the lower and upper Γ*-limits respectively.*

Remark 1.3. *(i) The* Γ*-limit exists at a point* x *if and only if*

$$\Gamma - \lim_{\epsilon\to 0} inf f_\epsilon(x) = \Gamma - \lim_{\epsilon\to 0} sup f_\epsilon(x)$$

(ii) Comparing with the trivial sequence $x_\epsilon = x$*, we obtain*

$$\Gamma - \lim_{\epsilon\to 0} inf f_\epsilon(x) \leq \lim_{\epsilon\to 0} inf f_\epsilon(x)$$
$$\Gamma - \lim_{\epsilon\to 0} sup f_\epsilon(x) \leq \lim_{\epsilon\to 0} sup f_\epsilon(x)$$

Proposition 1.2.1. *An important property of* Γ*-convergence is its stability under continuous perturbations: Let* f_ϵ Γ*-converge to* f *and* g_ϵ *converge continuously to* g *(i.e.,* $g_\epsilon(x_\epsilon) \to g(x)$ *if* $x_\epsilon \to x$*). Then* $f_\epsilon + g_\epsilon \to f + g$.

Note that this proposition applies to $g_\epsilon = g$ if g is continuous, but is in general false for $g_\epsilon = g$ even if g is lower semicontinuous.

1.3 Gradient theory of phase transitions

We consider a fluid, under isothermal conditions, confined to a container which occupies a bounded, open region $\Omega \subset \mathbb{R}^n$. If we denote the concentration of the fluid with a function $u : \Omega \to [0,1]$, then the classical problem of determining the equilibrium configurations of the fluid is to minimize a suitable energy depending on u under a mass constraint:

$$\min\left\{E(u) : u : \Omega \to [0,1], \ \int_\Omega u dx = m\right\} \tag{1.3.1}$$

where m is the total mass of the fluid in Ω and the energy, if there are no other contributions, is given by the functional

$$E(u) = \int_\Omega W(u(x))dx$$

The energy density, per unit volume, $W : (0,\infty) \to \mathbb{R}$ is a non-convex function, of the density u, given by the Van der Waals Cabn Hilliard theory, whose graph is of the form as in Fig. 1.2.

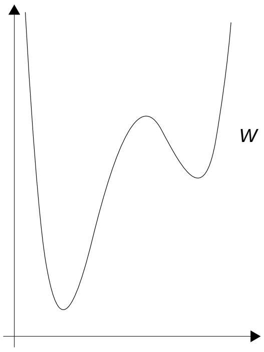

Figure 1.2

In order to understand the properties of minimizers, we may add an affine change of variable to W, replacing W by $W(u) + c_1 u + c_2$. The minimum problem remains unchanged, since it amounts to add the fixed quantity

$$\int_\Omega (c_1 u + c_2)\, dx = mc_1 + c_2 |\Omega|$$

to $E(u)$. It is customary to choose c_1 and c_2 so that the new energy density, which we still denote by W, is continuous, non-negative, capable of supporting two phases, and attains the minimum value of zero at exactly two points α and β ($\alpha < \beta$), as in Fig. 1.3

Definition 1.3.1. *We will call such a potential double-well, also called bi-stable and balanced when*

$$W(x) > 0 \ if \ x \neq \beta \ or \ x \neq \alpha, \ W(\alpha) = W(\beta) = 0, \ W''(\alpha) > 0, \ W''(\beta) > 0$$

If this is allowed by the mass constraint, minimizers of the original problem (1.3.1) will be simply given by all functions u which take only the values α and β and still satisfy the constraint $\int_\Omega u dx = m$. The two values α and β of the density u correspond to a stable, two-phase configuration of the fluid and form a partition of Ω. Note that minimizing the original problem does not provide any information about the interface between the two phases, which may be irregular or even dense in Ω. In particular, there is no way to recover the physically reasonable criterion that among these minimizers some special

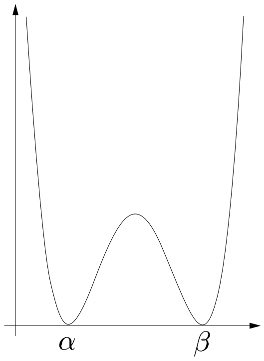

Figure 1.3

configuration are preferred, instead, and precisely those with minimal interface between the phases. This minimal-interface criterion is interpreted as a consequence of higher-order terms: in order to prevent the appearance of irregular interfaces, based on the the van der Waals-Cahn-Hilliard theory we perturb energy functionals (singular perturbation) by a gradient term of u, which may be interpreted as giving a (small) surface tension between the phases. The mathematical problem is then to study the asymptotic behaviour, as $\epsilon \to 0^+$, of the solutions u_ϵ of the minimization problems

$$\min\left\{\int_\Omega \left(\epsilon^2 |Du|^2 + W(u)\right) dx : \int_\Omega u dx = m\right\} \tag{1.3.2}$$

where ϵ^2 is a small, positive parameter, which accounts for surface energy between phases and we also require some more regularity on u.

It is also proved that u_ϵ converges to a function, which takes only the values α and β and for which the interface between the sets $\{u = \alpha\}$ and $\{u = \beta\}$ has minimal area. The solutions u_ϵ of this problem have the form

$$u_\epsilon(x) \approx u(x) + u_1\left(\frac{dist(x, S)}{\epsilon}\right),$$

where $u : \Omega \to \{\alpha, \beta\}$ is a phase-transition function with minimal interface S in Ω, and $u_1 : \mathbb{R} \to \mathbb{R}$ is a function with limit 0 at infinity, which gives the optimal profile between the phases at $\epsilon > 0$.

Fig. 1.4 picture a minimizer u_ϵ corresponding to a minimal u with a minimal (linear) interface between

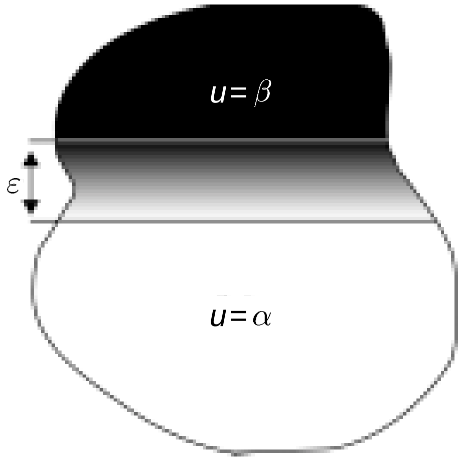

Figure 1.4

the phases. This is a natural ansatz and is proved rigorously by a Γ-convergence arguments. We can picture this behaviour in the one-dimensional case, where, then, u is simply a function with a single discontinuity point. In Fig.1.5 are represented functions u_ϵ for various values of ϵ.

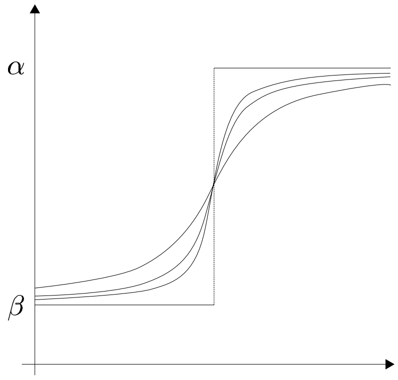

Figure 1.5

The behaviour of u_ϵ cannot be read out directly by examining small-energy functions for problem (1.3.2), but may be more easily deduced if that problem is rewritten as

$$\min\left\{\int_\Omega\left(\epsilon\,|Du|^2+\frac{W(u)}{\epsilon}\right)dx:\int_\Omega u\,dx=m\right\} \tag{1.3.3}$$

in order for each piece in the integral to have the same relative size as $\epsilon\to 0$ for minimizing sequences.

The qualitative effect of the first term is to penalize the spatial inhomogeneity of u, while the effect of the second term is that u tends to get closer to α or β. It can be seen that problem (1.3.3) is well approximated as ϵ gets small by a minimal interface problem:

$$\min\left\{Per\left(\{u=\alpha\},\Omega\right):u:\Omega\to\{\alpha,\beta\},\int_\Omega u\,dx=m\right\} \tag{1.3.4}$$

where $Per(A,\Omega)$ denotes the properly defined perimeter of A in Ω.

We refer to [2],[18],[19],[41] for further details in Γ-convergence and phase transitions and we proceed to the next Chapter applying this theory to Allen-Cahn Equation.

Chapter 2

Allen-Cahn Equation

2.1 Introduction

We study the semilinear elliptic problem

$$\Delta u + f(u) = 0 \quad in\ \mathbb{R}^n \tag{2.1.1}$$

where $f(x) = -W'(x)$ and W is a " double-well potential " , as seen in Definition 1.3.1. A typical example of such a double well potential is given by

$$W(x) = \frac{1}{4}\left(1 - x^2\right)^2 \ where\ f(x) = \left(1 - x^2\right)x$$

Equation (2.1.1) is a prototype for the continuous modeling of phase transition phenomena, for example when two materials try to coexist in a domain Ω of a " binary mixture " , water in solid phase (+1) and water in liquid phase(-1), while minimizing their interaction which is proportional to the $(n-1)$ - dimensional volume of the interface $\{x \in \Omega \mid u(x) = 0\}$.

Let $\Lambda \subset \mathbb{R}^n$ be an open, connected, bounded subset of Ω with $\partial\Lambda$ minimally smooth (smooth and with small perimeter). The configuration above can be described as a function

$$u_\epsilon(x) \approx \begin{cases} +1, & in\ \Lambda \\ -1, & in\ \Omega \setminus \Lambda \end{cases}$$

where $\epsilon > 0$ is a small parameter. This function has a sharp transition between these values across a narrow layer, called the interface, of width roughly $O(\epsilon)$.

Definition 2.1.1. *(Domain with minimally smooth boundary)*

An open set $\Omega \subset \mathbb{R}^n$ $(n = 2, 3, \ldots)$ is said to be a domain with minimally smooth boundary if there exist $\epsilon > 0$, $N \in \mathbb{N}$, $M > 0$ and a sequence $\{U_i\}_{i\in\mathbb{N}}$ of open subsets of $\mathbb{R}^n$ such that

1. *for any $x \in \partial\Omega$, $B(x, \epsilon) \subset U_i$ holds for some $i \in \mathbb{N}$*
2. *no point in $\mathbb{R}^n$ belongs to more than N of the U_i*
3. *for any $i \in \mathbb{N}$, there exists a special Lipschitz domain Ω_i, whose Lipschitz bound is not more than M, such that $U_i \cap \Omega = U_i \cap \Omega_i$*

We will consider the case in which the container isn't homogeneous so that distinct costs are paid for parts of the interface in different locations. Then the Allen-Cahn energy in the bounded domain $\Omega \subseteq \mathbb{R}^n$ is

$$J_\epsilon(u) = \int_\Omega \left[\frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{4\epsilon}\left(1-u^2\right)^2 \right] a(x)dx$$

where $a(x)$ is smooth and we make the assumption that there exists β, γ such that $0 < \gamma \leq a(x) \leq \beta$, $\forall\ x \in \mathbb{R}$. The system has variational structure as solutions are critical points of the Euler-Lagrange functional. We refer to [25],[27],[28],[40],[42] for further motivation and references on the subject.

2.2 Critical points of $J_\epsilon(u)$

We recall In order to find the critical points of $J_\epsilon(u)$ we vary u and try to calculate the value of the functional on a new position in $W^{1,2}(\Omega)$ which corresponds to the function $u_\epsilon + t\varphi$, where $\varphi \in C_0^\infty(\Omega)$ and $t > 0$ is small real number.

$$J_\epsilon(u_\epsilon + t\varphi) = \int_\Omega \left(\epsilon \frac{|\nabla\left(u_\epsilon + t\varphi\right)|^2}{2} + \frac{1}{\epsilon}\int_\Omega \frac{\left(1-\left(u_\epsilon + t\varphi^2\right)\right)^2}{4} \right) a(x)dx \tag{2.2.1}$$

So,

$$J_\epsilon(u_\epsilon + t\varphi) - J_\epsilon(u_\epsilon) = \left(\epsilon \frac{2\nabla u_\epsilon \nabla\left(t\varphi\right) + |\nabla\left(t\varphi\right)|^2}{2} + \frac{1}{\epsilon}\int_\Omega u_\epsilon\left(-1+u_\epsilon^2\right)t\varphi \right) a(x)dx$$

where we excluded the higher order terms of t ($O(t^2)$), and we now have

$$\begin{aligned} \frac{\partial}{\partial t}\left[J_\epsilon(u_\epsilon + t\varphi)\right]_{t=0} &= 0 = DJ_\epsilon(u_\epsilon;\varphi) = \\ &= \lim_{t\to 0}\frac{J_\epsilon(u_\epsilon + t\varphi) - J_\epsilon(u_\epsilon)}{t} = \epsilon\int_\Omega\left(\nabla u_\epsilon \nabla\varphi\right)\ a(x)dx - \frac{1}{\epsilon}\int_\Omega (1-u_\epsilon^2)u_\epsilon\varphi\ a(x)dx \end{aligned} \tag{2.2.2}$$

where the notation $DJ_\epsilon(u_\epsilon;\varphi)$ implies that we look at the "infinitesmal" variation of the functional at the position u along the direction φ (vanishes on the boundary). Thus, at a critical point of J, it holds that $DJ_\epsilon(u_\epsilon;\varphi) = 0$. Otherwise, u is called regular.

We recall in 1-dimension that if X is a Banach space and the functional $J \in C^1(X,\mathbb{R})$, then a critical point of J is an element $x \in X$ such that $F'(x) = 0$. We say that J achieves its minimum, whenever there exists $x_0 \in X$ such that

$$J(x_0) = \inf_{x\in X} J(x)$$

Then x_0 is a critical point of J.

For a set $M \subset W^{1,2}(\Omega)$, a point $u \in M$ is an absolute minimizer for J on M if $\forall v \in M$ there holds $J(v) \geq J(u)$. A solution $u \in C^2(\mathbb{R}^n)$ is called globally minimizing if

$$J(u;\Omega) \leq J(u+\varphi;\Omega)$$

for every smooth, bounded domain $\Omega \subset \mathbb{R}^n$ and $\forall\varphi \in C_0^\infty(\Omega)$.

Integrating (2.2.2) by parts and if $u_\epsilon \in C^2(\Omega)$, we obtain

$$\int_\Omega \left(-\epsilon \nabla \cdot (a \nabla u_\epsilon) + \frac{a}{\epsilon} W'(u_\epsilon)\right) \varphi dx = 0, \;\; \forall \varphi \in C_0^\infty(\Omega) \tag{2.2.3}$$

Thus, the Euler-Lagrange equation is the weighted Allen Cahn equation in Ω

$$-\epsilon \nabla \cdot (a \nabla u_\epsilon) - \frac{a}{\epsilon}\left(1 - u^2\right) u = 0, \;\; in\ \Omega \tag{2.2.4}$$

If $\Omega = \mathbb{R}$, we obtain

$$\epsilon^2 u'' + \epsilon^2 u' \frac{a'}{a} + \left(1 - u^2\right) u = 0, \;\; in\ \mathbb{R} \tag{2.2.5}$$

Multiplying (2.2.5) against u' and integrating by parts we obtain

$$\int_{-\infty}^{+\infty} \frac{d}{dx}\left(\epsilon \frac{u'}{2} - \frac{\left(1-u^2\right)^2}{4\epsilon}\right) + \epsilon \int_{-\infty}^{+\infty} \frac{a'}{a} u'^2 = 0 \tag{2.2.6}$$

Assuming that $u(\infty) = 1, \;\; u(-\infty) = -1, \;\; u'(\pm\infty) = 0$, we obtain

$$\epsilon \int_{-\infty}^{+\infty} \frac{a'}{a} u'^2 = 0$$

Observation 2.2.1. *If a is monotone and $a' \neq 0$, then there are no solutions. We need the existence (if $a' \neq 0$) of local maximum or local minimum of a. Given a local maximum or local minimum x_0 of a non-degenerate ($a''(x_0) \neq 0$), a solution to (2.2.5) exists, with transition layer.*

Letting $a = 1, \;\; \epsilon = 1, \;\; u(x) \approx w(t)$ in (2.2.5), we obtain the limit fast system

$$w'' + \left(1 - w^2\right) w = 0, \;\; w(+\infty) = 1, \;\; w(-\infty) = -1 \tag{2.2.7}$$

The solution

$$w(t) = tanh\left(\frac{t}{\sqrt{2}}\right)$$

is unique up to translations, vanishes at $t = 0$ and tends to $+1$ at $+\infty$ and -1 at $-\infty$. This solution is called the "heteroclinic solution".

Indeed, if

$$w'' + f(w) = 0, \;\; in\ \mathbb{R}, \;\; w(+\infty) = 1, \;\; w(-\infty) = -1 \tag{2.2.8}$$

where $f(w) = -W'(w)$, the heteroclinic solution exists and defined uniquely up to a constant translation $a \in \mathbb{R}$, by the identity

$$\int_0^{w(t)} \frac{ds}{\sqrt{2W(s)}} = t - a,$$

which follows from the fact that

$$w'^2 - 2W(w) = E,$$

where E is constant and $w(+\infty) = 1,\ w(-\infty) = -1$ if an only if $E = 0$.

We fix in what follows the unique w for which

$$\int_{\mathbb{R}} tw'^2(t)dt = 0$$

In general, w approaches its limits at exponential rates,

$$w(t) \to \pm 1,\ as\ t \to \pm\infty$$

and $w(0) = 0$.

Changing variables for $x = x_0 + \epsilon(t + h)$, with $x_0 \in \mathbb{R}$ and $h \in \mathbb{R}$, we set

$$\begin{aligned}
v(t) &= u(x) = u(x_0 + \epsilon(t + h)) \Rightarrow \\
\dot{v}(t) &= \epsilon u'(x_0 + \epsilon(t + h)) \Rightarrow \\
\ddot{v}(t) &= \epsilon^2 u''(x_0 + \epsilon(t + h))
\end{aligned}$$

and substituting in (2.2.5) we obtain

$$\begin{aligned}
&\epsilon^2 u''(x_0 + \epsilon(t + h)) + \epsilon^2 u'(x_0 + \epsilon(t + h)) \frac{a'}{a}(x_0 + \epsilon(t + h)) + \left(1 - v^2(t)\right) v(t) = 0 \Rightarrow \\
&\ddot{v}(t) + \epsilon\dot{v}(t)\frac{a'}{a}(x_0 + \epsilon(t + h)) + \left(1 - v^2(t)\right) v(t) = 0, \quad v(\pm\infty) = \pm 1 \qquad (2.2.9)
\end{aligned}$$

A natural way is to find approximations first and then to look for genuine solutions as small perturbations of approximations. Letting $\epsilon = 0$, we observe that we obtain (2.2.7), and we look for a solution $v(t) = w(t) + \varphi$, where φ is a small error in ϵ. We make the following assumptions,

1. There exists β, γ such that $0 < \gamma \leq a(x) \leq \beta,\ \forall\ x\ \in\ \mathbb{R}$
2. $||a'||_{L^\infty(\mathbb{R})}, ||a''||_{L^\infty(\mathbb{R})} < +\infty$
3. x_0 is a non-degenerate critical point of $a\,(a'(x_0) = 0, a''(x_0) \neq 0)$.

2.3 Lyapunov-Schmidt Reduction Method

2.3.1 The Linear Projected Problem

Theorem 2.3.1. *$\forall\epsilon > 0$ sufficiently small, there exists a solution $v = v_\epsilon$ to (2.2.9) for some $h = h_\epsilon$, where $|h_\epsilon| \leq C\epsilon$ and $v_\epsilon(t) = w(t) + \varphi_\epsilon(t)$ and*

$$||\varphi_\epsilon|| \leq C\epsilon$$

Proof. Substituting $v(t) = w(t) + \varphi(t)$ in (2.2.9) we obtain

$$\begin{aligned}
&w'' + \varphi'' + \epsilon\frac{a'}{a}(x_0 + \epsilon(t + h))(w' + \varphi') + \left(1 - (w(t) + \varphi(t))^2\right)(w(t) + \varphi(t)) = 0 \Rightarrow \\
&w''(t) + \varphi''(t) + \epsilon\frac{a'}{a}(x_0 + \epsilon(t + h))w' + \epsilon\frac{a'}{a}(x_0 + \epsilon(t + h))\varphi' \\
&\qquad + f(w + \varphi) - f(w) - f'(w)\varphi + f(w) + f'(w)\varphi = 0,\ \varphi(+\infty) = \varphi(-\infty) = 0
\end{aligned}$$

where $f(v) = (1 - v^2)v$. Considering that $w'' + f(w) = 0$ from (2.2.8), we write the above in the following way

$$\varphi'' + f'(w)\varphi + E + B(\varphi) + N(\varphi) = 0, \;\; \varphi(+\infty) = \varphi(-\infty) = 0$$

where

$$\begin{cases} E & = \epsilon \frac{a'}{a} \left(x_0 + \epsilon \left(t + h\right)\right) w' \\ B(\varphi) & = \epsilon \frac{a'}{a} \left(x_0 + \epsilon \left(t + h\right)\right) \varphi' \\ N(\varphi) & = f(w + \varphi) - f(w) - f'(w)\varphi = -3w\varphi^2 - \varphi^3 \end{cases} \tag{2.3.1}$$

We consider the problem

$$L(\varphi) = \varphi'' + f'(w)\varphi = -g(t), \;\; \varphi \in L^\infty(\mathbb{R}) \tag{2.3.2}$$

In order to solve (2.3.2), we try to invert the linear operator L so that we can rephrase the problem as a fixed point problem.

Let $g \in L^\infty(\mathbb{R})$ and multiply the above equation against w' we get

$$\begin{aligned} \int_{-\infty}^{+\infty} \left(w''' + f'(w)w'\right)\varphi + \int_{-\infty}^{+\infty} gw' = 0 \Rightarrow \\ \int_{-\infty}^{+\infty} gw' = 0 \end{aligned} \tag{2.3.3}$$

So a necessary and sufficient condition in order to have a solution is that g in (2.3.3) is orthogonal to the kernel. Indeed, if we write

$$\begin{aligned} \varphi &= w'\Psi \Rightarrow \\ \varphi' &= w'\Psi' + w''\Psi \Rightarrow \\ \varphi'' &= w'''\Psi + 2w''\Psi' + w'\Psi'' \end{aligned}$$

then (2.3.2) becomes

$$w'''\Psi + 2w''\Psi' + w'\Psi'' + f'(w)w'\Psi + g = 0$$

and multiplying by w' we have

$$\begin{aligned} 2w''w'\Psi' + w'^2\Psi'' &= -gw' \Rightarrow \\ \left(w'^2\Psi'\right)' &= -gw' \Rightarrow \\ w'^2\Psi'(t) &= -\int_{-\infty}^{+\infty} g(s)w'(s)ds \end{aligned}$$

Then

$$\Psi(t) = -\int_0^t \frac{d\tau}{w'^2(\tau)} \int_{-\infty}^{\tau} g(s)w'(s)ds$$

and

$$\varphi(t) = -w'(t) \int_0^t \frac{d\tau}{w'^2(\tau)} \int_{-\infty}^{\tau} g(s) w'(s) ds$$

□

where $w'(t) \approx 2\sqrt{2} e^{-\sqrt{2}\,|t|}$.

Lemma 2.3.1. *If $\int_{-\infty}^{+\infty} gw' = 0$, then we have the following estimate*

$$\|\varphi\|_\infty \leq C \|g\|_\infty$$

If $t > 0$,

$$|\varphi(t)| \leq |w'(t)| \int_0^t \frac{C}{e^{-2\sqrt{2}\tau}} \left| \int_\tau^{+\infty} gw' ds \right| d\tau \leq C \|g\|_\infty e^{-\sqrt{2}t} \int_0^t e^{\sqrt{2}\tau} d\tau \leq C \|g\|_\infty .$$

If $t < 0$, a similar estimate yields, so

$$|\varphi(t)| \leq C \|g\|_\infty$$

2.3.2 The Nonlinear Projected Problem

Lemma 2.3.2. *Given $g \in L^\infty(\mathbb{R})$, there exists a unique $C = C(g) = \frac{\int_{-\infty}^{+\infty} gw'}{\int_{-\infty}^{+\infty} w'^2}$ and $\varphi \in L^\infty(\mathbb{R})$ with $\varphi(0) = 0$ such that*

$$\varphi'' + f'(w)\varphi + (g - cw') = 0, \ \in \mathbb{R} \tag{2.3.4}$$

has a solution, which defines an operator $\varphi = T[g]$ with

$$\|T[g]\|_\infty \leq C \|g\|_\infty$$

In fact, if $\hat{T}[\hat{g}]$ is the solution find in the previous step then $\varphi = \hat{T}[g - C(g)w']$ solves (2.3.4) and

$$\|\varphi\|_\infty \leq C \|g\|_\infty + |C(g)|\, C \leq C \|g\|_\infty$$

Proof. Rather than solving the problem directly, we consider a projected version of it

$$L(\varphi) = \varphi'' + f'(w)\varphi = -E - B(\varphi) - N(\varphi) + Cw', \ \ \varphi \in L^\infty(\mathbb{R})$$

where

$$C = \frac{\int_{\mathbb{R}} [E + B(\varphi) + N(\varphi)]\, w'}{\int_{\mathbb{R}} w'^2}$$

Step 1: Given the parameter function h, we find a solution $\varphi = \Phi(h)$ to the problem. We assume $|h| \leq 1$, and we write in fixed point form

$$\varphi = T\left[E + B(\varphi) + N(\varphi)\right] = M[\varphi]$$

Remark: Given the relations in (2.3.1), we obtain

$$\begin{aligned}||E||_\infty &\leq C\epsilon^2 \\ ||B(\varphi)||_\infty &\leq C\epsilon \left\|\varphi'\right\|_\infty \\ ||N(\varphi)||_\infty &\leq C\left(\left\|\varphi^2\right\|_\infty + \left\|\varphi^3\right\|_\infty\right)\end{aligned}$$

with C uniform on $|h| \leq 1$.

$$||M||_\infty + \left\|\frac{d}{dt}M\right\|_\infty \leq C\left(||E||_\infty + ||B(\varphi)||_\infty + ||N(\varphi)||_\infty\right) \leq C\left(\epsilon^2 + \epsilon\left\|\varphi'\right\|_\infty + \left\|\varphi^2\right\|_\infty + \left\|\varphi^3\right\|_\infty\right)$$

If $||\varphi'||_\infty + ||\varphi||_\infty \leq M\epsilon^2$, we have

$$||M||_\infty + \left\|\frac{d}{dt}M\right\|_\infty \leq C^*\epsilon^2$$

We define the space $X = \left\{\varphi \in C^1(\mathbb{R}) : ||\varphi'||_\infty + ||\varphi||_\infty \leq M\epsilon^2\right\}$. Let us observe that $M(X) \subset X$ and

$$||M(\varphi_1) - M(\varphi_2)||_\infty + \left\|\frac{d}{dt}\left(M(\varphi_1) - M(\varphi_2)\right)\right\|_\infty \leq C\epsilon\left(||\varphi_1 - \varphi_2||_\infty + \left\|\varphi_1' - \varphi_2'\right\|_\infty\right)$$

So if ϵ is small, M is a contraction mapping which implies that there exists a unique $\varphi \in X$ such that $\varphi = M[\varphi] = \Phi(h)$. □

Step 2: We need to find h such that $C = 0$ in for $\varphi = \Phi(h)$.

$$C = 0 \Leftrightarrow \alpha_\epsilon(h) = \int_{\mathbb{R}} \left[E + B(\varphi) + N(\varphi)\right] w' = 0$$

If we call $\psi(x) = \frac{a'}{a}(x)$, then

$$\psi\left(x_0 + \epsilon\left(t + h\right)\right) = \psi\left(x_0\right) + \psi'\left(x_0\right)\epsilon\left(t + h\right) + \int_0^1 (1 - s)\,\psi''\left(x_0 + s\epsilon\left(t + h\right)\right)\epsilon^2\left(t + h\right)^2 ds$$

We want $\psi'' \in L^\infty(\mathbb{R})$, so $a''' \in L^\infty(\mathbb{R})$. The first term of the integral gives

$$\int E_h w' = \epsilon^2\psi'\left(x_0\right)\int\left(t + h\right)w'^2(t) + \epsilon^3\int_{\mathbb{R}}\left(\int_0^1 (1 - s)\,\psi''\left(x_0 + s\epsilon\left(t + h\right)\right)ds\right)\left(t + h\right)^2 w'(t)dt$$

Given,

$$\begin{cases} \int_{\mathbb{R}} tw'^2(t) = 0 \\ \left|\int_{\mathbb{R}}\left(B(\varphi) + N(\varphi)\right)w'\right| \leq C\left(\epsilon\,||\Phi(h)||_{C^1} - ||\Phi(h)||_{L^\infty}\right) \leq C\epsilon^3 \end{cases}$$

we conclude that

$$\alpha_\epsilon(h) = \psi'(x_0)\epsilon^2\left(h + O(\epsilon)\right)$$

The term inside the parenthesis changes sign, so $\exists h_\epsilon$: $|h_\epsilon| \leq M\epsilon$ such that $\alpha_\epsilon(h) = 0$, which means $C = 0$.

Observation 2.3.1.

$$\bar{L}(\varphi) = \varphi'' + (1-3w^2)\varphi + \epsilon\psi + \frac{1}{2}f''(w+s\varphi)\varphi + O(\epsilon^2)e^{-2\sqrt{2}|t|} = 0,\ |t| > R$$

We consider $t > R$ *and* $\frac{1}{2}f''(w+s\varphi)\varphi = O(\epsilon^2)$. *Using* $\hat{\varphi} = \epsilon e^{-|t|} + \delta e^{|t|}$ *and maximum principle, we obtain* $\varphi \leq \epsilon e^{-|t|}$, *as* $\delta \to 0$.

Lemma 2.3.3. *Given the bilinear form of the operator* $L(\varphi) = \varphi'' + f'(w)\varphi,\ \varphi \in H^2(\mathbb{R})$

$$B(\varphi,\varphi) = -\int_{\mathbb{R}} L(\varphi)\varphi = \int_{\mathbb{R}} \varphi'^2 - f'(w)\varphi^2,\ \varphi \in H^1(\mathbb{R})$$

Then,

$$B(\varphi,\varphi) \geq 0,\ \forall\varphi \in H^1(\mathbb{R}) \text{ and } B(\varphi,\varphi) = 0 \ \Leftrightarrow\ \varphi = cw'(t)$$

In fact $J''(w)[\varphi,\varphi] = B(\varphi,\varphi)$.

Proof. We write again $\varphi = w'\Psi,\ \varphi \in C_0^\infty(\mathbb{R}), \Psi \in C_0^\infty(\mathbb{R})$. Then $L(\varphi) = L(w'\Psi) = 2w''\Psi' + w'\Psi'' = \frac{1}{w'}\left[2w''w'\Psi' + w'^2\Psi''\right] = \frac{1}{w'}\left(w'^2\Psi'\right)'$ and

$$B(\varphi,\varphi) = -\int \frac{1}{w'}\left(w'^2\Psi'\right)' w'\Psi = -\left(w'^2\Psi'\Psi\ \Big|_{-\infty}^{\infty}\right) + \int_{\mathbb{R}} w'^2\Psi'^2,\ \forall\varphi \in C_0^\infty(\mathbb{R})$$

Same is valid $\forall\varphi \in H^1(\mathbb{R})$. We have,

$$B(\varphi,\varphi) = \int_{\mathbb{R}} \left|\varphi'\right|^2 - f'(w)\varphi^2 = \int_{\mathbb{R}} w'^2\ \left|\Psi'\right|^2 \geq 0$$

and also $B(\varphi,\varphi) = 0 \Leftrightarrow \Psi' = 0$, which means that $\varphi = cw'$. □

We now give a *spectral gap estimate*:

Corollary 2.3.1. *There exists* $\gamma > 0$ *such that if* $\varphi \in H^1(\mathbb{R})$ *and* $\int_{\mathbb{R}} \varphi w' = 0$ *then*

$$B(\varphi,\varphi) \geq \gamma \int_{\mathbb{R}} \varphi^2$$

Proof. If not, there exists φ_n such that $0 \leq B(\varphi_n,\varphi_n) < \frac{1}{n}\int_{\mathbb{R}} \varphi_n^2$. W.l.o.g, we normalize $\int_{\mathbb{R}} \varphi_n^2 = 1$, and using the Rellich-Kondrachov Theorem implies that

$$\varphi_n \rightharpoonup \varphi \in H^1(\mathbb{R})$$

and $\varphi_n \to \varphi$ uniformly $\in L^2$, so

$$0 = \lim_{n\to\infty} \int_{\mathbb{R}} \varphi_n w' = \int_{\mathbb{R}} \varphi w'$$

Also,

$$B(\varphi_n,\varphi_n) = \int \left|\varphi_n'\right|^2 + 2\int \varphi_n^2 - 3\int \left(1-w^2\right)\varphi_n^2 \to 0$$

and $B(\varphi_n, \varphi_n) \to B(\varphi, \varphi)$, so $B(\varphi, \varphi) = 0$ and $\int_{\mathbb{R}} \varphi w' = 0$, which means $\varphi = 0$. But

$$2 \leq 3 \int \left(1 - w^2\right) \varphi_n^2 + o(1)$$

which implies that $2 \leq 3 \int \left(1 - w^2\right) \varphi^2$. This means that $\varphi \neq 0$, so we have a contradiction. □

Observation 2.3.2. *If we choose* $\delta = \frac{\gamma}{2\|f'\|_\infty}$, *then*

$$\int \varphi'^2 - (1 + \delta) f'(w) \varphi^2 \geq 0.$$

This implies that

$$B(\varphi, \varphi) \geq \alpha \int \varphi'^2.$$

Chapter 3

Nonlinear Schrödinger Equation (NLS)

3.1 Introduction

We study semiclassical states of nonlinear Schrödinger equations

$$\epsilon i\Psi_t + \epsilon^2\Delta\Psi - W(x)\Psi + |\Psi|^{p-1}\Psi = 0 \tag{3.1.1}$$

where i is the imaginary unit and $W(x)$ given potential that may exhibit vanishing and singularity while allowing decays and unboundedness at infinity. We are also interested in spike type standing waves concentrating at the singularities of the potentials.

Equation (3.1.1) arises in may fields of physics, in particular when we describe Bose-Einstein condensates and the propagation of light in some nonlinear optical materials (see the introduction and references in [21]). We already know that $\int_{\mathbb{R}^N}|\Psi|^2 = \text{constant}$. In this section, we are concerned with standing waves of the nonlinear Schrödinger equation for small $\epsilon > 0$. These standing wave solutions are refereed as semiclassical states and have the form $\Psi(x,t) = e^{-iEt}u(x)$, where $u(x)$ is a real-valued function and E is the energy of the wave. In what follows, we shall only consider positive, finite energy solutions of (3.1.1)

In order to obtain a bound state we require $u \in W^{1,2}(\mathbb{R}^N)$. Replacing the ansatz $\Psi(x,t)$ into (3.1.1), we obtain

$$\epsilon Eu + \epsilon^2\Delta u - Wu + u^p = 0$$

where $\Delta = \sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}$ stands for the Laplace operator.

With a simple rescaling, choosing $E = \frac{\lambda}{\epsilon}$ and defining $V(x) = (W(x) - \lambda)$, we obtain

$$\begin{cases} \epsilon^2\Delta u - V(x)u + u^p = 0 \\ u > 0, u \in W^{1,2}(\mathbb{R}^N),\ \lim_{|x|\to\infty} u(x) = 0 \end{cases} \tag{3.1.2}$$

Its structure has variational form and solutions can be found as critical points of the following Euler-Lagrange functional $J_\epsilon(u) : W^{1,2}(\mathbb{R}^N) \to R$

$$J_\epsilon(u) = \int_{\mathbb{R}^N}\left(\frac{\epsilon^2|\nabla u|^2}{2} + \frac{V(x)u^2}{2}\right) - \frac{1}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}, \quad u \in W^{1,2}(\mathbb{R}^N)$$

In recent years many authors have tried to understand solutions structure of (3.1.2) as $\epsilon \to 0$. A characteristic feature is that the semiclassical bound states exhibit concentration behaviors as $\epsilon \to 0$, in the sense that, out of a certain concentration set, the function $u_\epsilon(x)$ decays uniformly to zero as $\epsilon \to 0$. When this concentration set is a single point these solutions are usually called spikes.

Floer and Weinstein [35] investigated the special case where $N = 1$ and $p = 3$. Assuming that V is globally bounded potential having a non-degenerate critical point, say $x = 0$, and $\inf_{x\in\mathbb{R}^N} V(x) > 0$, they constructed a positive solution u_ϵ of (3.1.2) for small $\epsilon > 0$ via the Lyapunov-Schmidt Reduction. They proved that the solution concentrates around the critical point of V, i.e most of the mass of u_ϵ is contained in a neighbourhood of 0 that shrinks to a single point as $\epsilon \to 0$. Their results were generalized by Y.-G. Oh [36],[37] to the higher-dimensional case with $1 < p < \frac{N+2}{N-2}$ and were obtained multi-peak solutions concentrating near several non-degenerate critical points of V. We refer the reader to the (still incomplete) list of papers [3],[4],[5],[6],[7],[8],[9],[10],[11],[14],[22],[23],[29],[30],[31],[32],[47],[48].

We also refer to P. L. Lions [39], Y. Li [38], Bahri and P. L. Lions [12] as well as to their bibliographies for other works involving variational methods to treat the existence of standing waves for nonlinear Schrödinger equations.

3.2 Lyapunov-Schmidt Reduction Method

We study first **the case of dimension** 1:

$$\begin{cases} \epsilon^2 u'' - V(x)u + u^p = 0, \ x \in \mathbb{R}, \ p > 1 \\ u(x) > 0, \ \lim_{|x|\to\infty} u(x) = 0 \end{cases} \tag{3.2.1}$$

where $\inf_{x\in\mathbb{R}} V(x) > 0$ and we assume $V \geq \gamma > 0$, $V, V', V'', V''' \in L^\infty$ and $V \in C^3(\mathbb{R})$.

Rescaling the above equation to fast variable $t = \epsilon x$, we obtain

$$w'' - w + w^p = 0, \ w > 0, \ w(\pm\infty) = 0, \ p > 1$$

There exists a homoclinic solution

$$w(t) = \frac{C_p}{\cosh\left(\frac{p-1}{2}t\right)^{\frac{2}{p-1}}}, \quad C_p = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}}$$

and $w(t) \approx 2^{2/(p-1)} C_p e^{-|t|}$, as $t \to \infty$.

Observation 3.2.1. *Given x_0 we can assume $V(x_0) = 1$. We write*

$$u(x) = \lambda^{\frac{2}{p-1}} v\left(\lambda x_0 + (1-\lambda)x_0\right)$$

and we obtain from (3.2.1)

$$\epsilon^2 v''(y) - \hat{V}(y)v + v^p = 0$$

where $y = \lambda x_0 + (1-\lambda)x_0$ and $\hat{V}(y) = V\left(\frac{y-(1-\lambda)x_0}{\lambda}\right)$. If we choose $\lambda = \sqrt{V(x_0)}$ then $\hat{V}(x_0) = 1$.

Theorem 3.2.1. *We assume $V(x_0) = 1, V'(x_0) = 0, V''(x_0) \neq 0$. Then there exists a solution to (3.2.1) with* *the form*

$$u_\epsilon(x) \approx w\left(\frac{x - x_0}{\epsilon}\right)$$

hence a solution that concentrates at x_0. We say that a solution u_ϵ of (3.2.1) concentrates at x_0 (as $\epsilon \to 0$) provided

$$\forall \delta > 0, \exists \epsilon_0 > 0, R > 0 : u_\epsilon(x) \leq \delta, \forall \, |x - x_0| \geq \epsilon R, \;\; \epsilon < \epsilon_0$$

This kind of solutions are called spike layers or simply spikes. From the physical point of view, spikes are important because they show that (focusing) NLS of the type (3.2.1) are not dispersive but the energy is localized in packets.

Remark 3.1. *If u_ϵ is a solution of (3.2.1) with minimal energy concentrating at x_0, then x_0 is a global minimum of V. Moreover, any solution concentrating at some x_0 has a unique maximum which converges to x_0. This justifies the name spikes given to these solutions.*

Following the same procedure, as in Section 2.3, we change variables $x = x_0 + \epsilon\,(t + h)$, with $x_0 \in \mathbb{R}$ and $h \in \mathbb{R}$, we set

$$v(t) = u(x) = u\left(x_0 + \epsilon\,(t + h)\right)$$

and substituting in (3.2.1) we obtain

$$\begin{aligned}
&\epsilon^2 u''\left(x_0 + \epsilon\,(t + h)\right) - V\left(x_0 + \epsilon\,(t + h)\right) u\left(x_0 + \epsilon\,(t + h)\right) + u^p\left(x_0 + \epsilon\,(t + h)\right) = 0 \Rightarrow \\
&\ddot{v}(t) - V\left(x_0 + \epsilon\,(t + h)\right) v + v^p = 0 \qquad (3.2.2)
\end{aligned}$$

We look for a solution $v(t) = w(t) + \varphi$, where φ is a small error in ϵ.

Then,

$$\begin{aligned}
&\varphi'' + w'' - V\left(x_0 + \epsilon\,(t + h)\right) w - V\left(x_0 + \epsilon\,(t + h)\right) \varphi + (w + \varphi)^p + p w^{p-1}\varphi - p w^{p-1}\varphi = 0 \Rightarrow \\
&\varphi'' - \varphi + p w^{p-1}\varphi + w'' - w - \left[V\left(x_0 + \epsilon\,(t + h)\right) - V(x_0)\right] w - \left[V\left(x_0 + \epsilon\,(t + h)\right) - V(x_0)\right] \varphi + (w + \varphi)^p - p w^{p-1}\varphi = 0 \Rightarrow \\
&\varphi'' - \varphi + p w^{p-1}\varphi - \left[V\left(x_0 + \epsilon\,(t + h)\right) - V(x_0)\right] \varphi + (w + \varphi)^p - w^p - p w^{p-1}\varphi - \left[V\left(x_0 + \epsilon\,(t + h)\right) - V(x_0)\right] w = 0 \\
&\qquad (3.2.3)
\end{aligned}$$

where we used the fact that $w'' - w + w^p = 0$ and $V(x_0) = 1$. Now, we write the above in the following way,

$$\varphi'' - \varphi + p w^{p-1}\varphi = E + N(\varphi) + B(\varphi), \;\; \varphi(\pm\infty) = 0 \qquad (3.2.4)$$

where

$$\begin{cases}
E & = \left[V\left(x_0 + \epsilon\,(t + h)\right) - V(x_0)\right] w \\
B(\varphi) & = \left[V\left(x_0 + \epsilon\,(t + h)\right) - V(x_0)\right] \varphi \\
N(\varphi) & = (w + \varphi)^p - w^p - p w^{p-1}\varphi = f(w + \varphi) - f(w) - f'(w)\varphi = -3w\varphi^2 - \varphi^3
\end{cases} \qquad (3.2.5)$$

where we used the fact that $V'(x_0) = 0$ and $f(v) = v^p - v$.

Observation 3.2.2. *In order to have a solution, V' needs to change sign and $V \neq 0$. Consider $V'(x) \geq 0$. Multiplying equation (3.2.2) by u' and integrating by parts, we obtain that $\int_{\mathbb{R}} \dot{v}\frac{u^2}{2} = 0$. This implies that $u = 0$.*

3.2.1 The Linear Projected Problem

We consider the problem

$$L(\varphi) = \varphi'' - \varphi + pw^{p-1}\varphi = g(t), \;\; \varphi \in L^{\infty}(\mathbb{R}) \tag{3.2.6}$$

and want to know when it is solvable. Let $g \in L^{\infty}(\mathbb{R})$ and multiply the above equation against w' we get

$$\begin{aligned} \int_{-\infty}^{+\infty} \left(w''' - w' + pw^{p-1}w'\right)\varphi = \int_{-\infty}^{+\infty} gw' \Rightarrow \\ \int_{-\infty}^{+\infty} gw' = 0 \end{aligned} \tag{3.2.7}$$

because $w'' - w + w^p = 0$. So g in (3.2.7) is orthogonal to the kernel. If we write

$$\begin{aligned} \varphi &= w'\Psi \Rightarrow \\ \varphi' &= w'\Psi' + w''\Psi \Rightarrow \\ \varphi'' &= w'''\Psi + 2w''\Psi' + w'\Psi'' \end{aligned}$$

multiplying operator $L(\varphi)$ by w' we obtain

$$\begin{aligned} & w'''\Psi w' + 2w''\Psi' w' + w'^2\Psi'' - w'^2\Psi + pw^{p-1}w'\Psi = gw' \Rightarrow \\ & 2w''\Psi' w' + w'^2\Psi'' = gw' \Rightarrow \\ & \left(w'^2\Psi'\right)' = gw', \;\; for\ t \;\neq 0 \end{aligned} \tag{3.2.8}$$

Then, for $t < 0$

$$\Psi(t) = \int_t^{-1} \frac{d\tau}{w'^2(\tau)} \int_{-\infty}^{\tau} g(s)w'(s)ds$$

and

$$\varphi(t) = w'(t) \int_t^{-1} \frac{d\tau}{w'^2(\tau)} \int_{-\infty}^{\tau} g(s)w'(s)ds$$

and for $t > 0$

$$\Psi(t) = \int_1^{t} \frac{d\tau}{w'^2(\tau)} \int_{\tau}^{\infty} g(s)w'(s)ds$$

$$\varphi(t) = w'(t) \int_1^{t} \frac{d\tau}{w'^2(\tau)} \int_{\tau}^{\infty} g(s)w'(s)ds$$

Lemma 3.2.1. *If $\int_{-\infty}^{+\infty} gw' = 0$, then we have the following estimate*

$$||\varphi||_{\infty} \leq C\, ||g||_{\infty}$$

If $t > 0$,

$$|\varphi(t)| \leq |w'(t)| \int_0^t \frac{C}{e^{-2\sqrt{2}\tau}} \left| \int_\tau^{+\infty} gw'ds \right| d\tau \leq C\,\|g\|_\infty e^{-\sqrt{2}t} \int_0^t e^{\sqrt{2}\tau} d\tau \leq C\,\|g\|_\infty .$$

If $t < 0$, *a similar estimate yields, so*

$$|\varphi(t)| \leq C\,\|g\|_\infty$$

There exists a unique solution with $\varphi(O^-) = \varphi(O^+) = 0$

$$\varphi(O^-) = \lim_{t\to O^-} \frac{-\int_{-1}^t \frac{d\tau}{w'^2(\tau)} \int_{-\infty}^\tau g(s)w'(s)ds}{\frac{1}{w'(t)}} = \lim_{t\to O^-} \frac{-\frac{1}{w'(t)^2}\int_{-\infty}^t gw'}{-\frac{1}{w'(t)^2}w''(t)} = \frac{\int_{-\infty}^0 gw'}{w''(0)} = 0$$

$$\varphi(O^+) = \lim_{t\to O^+} \frac{\int_1^t \frac{d\tau}{w'^2(\tau)} \int_\tau^\infty g(s)w'(s)ds}{\frac{1}{w'(t)}} = \frac{\int_0^\infty gw'}{w''(0)} = 0$$

3.2.2 The Nonlinear Projected Problem

Lemma 3.2.2. *Given* $g \in L^\infty(\mathbb{R})$, *there exists a unique* $C = C(g) = \frac{\int_{-\infty}^{+\infty} gw'}{\int_{-\infty}^{+\infty} w'^2}$ *and* $\varphi \in L^\infty(\mathbb{R})$ *with* $\varphi(0) = 0$ *such that*

$$\varphi'' - \varphi + pw^{p-1}\varphi + (g - cw') = 0,\ \varphi \in L^\infty(\mathbb{R}) \tag{3.2.9}$$

has a solution, which defines an operator $\varphi = T[g]$ *with*

$$\|T[g]\|_\infty \leq C\,\|g\|_\infty$$

In fact if $\hat{T}[\hat{g}]$ *is the solution find in the previous step then* $\varphi = \hat{T}[g - C(g)w']$ *solves and*

$$\|\varphi\|_\infty \leq C\,\|g\|_\infty + |C(g)|\,C \leq C\,\|g\|_\infty$$

Proof. We consider a projected version of the problem

$$L(\varphi) = \varphi'' - \varphi + pw^{p-1}\varphi = -E - B(\varphi) - N(\varphi) - Cw',\ \ \varphi \in L^\infty(\mathbb{R})$$

where

$$C = \frac{\int_{\mathbb{R}} [E + B(\varphi) + N(\varphi)]\, w'}{\int_{\mathbb{R}} w'^2}$$

Step 1: Given the parameter function h, we find a solution $\varphi = \Phi(h)$ to the problem. We assume $|h| \leq 1$, and we write in fixed point form

$$\varphi = T\,[E + B(\varphi) + N(\varphi)] = M[\varphi]$$

Remark: Given the relations in (2.3.1), we obtain

$$\begin{aligned} ||E||_\infty &\le C\epsilon^2 \\ ||B(\varphi)||_\infty &\le C\epsilon^2\, ||\varphi||_\infty \\ ||N(\varphi)||_\infty &\le C\left(\left\|\varphi^2\right\|_\infty + \left\|\varphi^3\right\|_\infty\right) \end{aligned}$$

with C uniform on $|h| \le 1$.

We define the space $X = \left\{\varphi \in C^0(\mathbb{R}) : ||\varphi||_\infty \le M\epsilon^2\right\}$. Let us observe that $M(X) \subset X$ and

$$||M(\varphi_1) - M(\varphi_2)||_\infty \le C\epsilon\left(||\varphi_1 - \varphi_2||_\infty\right)$$

So if ϵ is small, M is a contraction mapping, which implies that there exists a unique $\varphi \in X$ such that $\varphi = M[\varphi] = \Phi(h)$. □

Step 2: We need to find h, such that $C = 0$ in for $\varphi = \Phi(h)$.

$$C = 0 \Leftrightarrow C_h = \int_{\mathbb{R}} [E + B(\varphi) + N(\varphi)]\, w' = 0$$

The first term of the integral gives

$$\begin{aligned} \int E_h w' &= \int_{-\infty}^{\infty} [V(x_0 + \epsilon(t+h)) - V(x_0)]\, ww'dt \\ &= -\epsilon \int_{-\infty}^{\infty} V'(x_0 + \epsilon(t+h))\, \frac{w^2}{2} dt \end{aligned}$$

where we integrated by parts and used the fact that the approximation $w(t)$ is zero at infinity. We now use Taylor expansion for $V'(x_0 + \epsilon(t+h))$

$$V'(x_0 + \epsilon(t+h)) = V'(x_0) + V''(x_0)(\epsilon t + \epsilon h) + \frac{V'''(\xi)}{2}\epsilon^2 (t+h)^2$$

Uusing the fact that $\int_{-\infty}^{\infty} w^2 V''(x_0)\epsilon t = 0$ and $V'(x_0) = 0$, we obtain

$$-\epsilon \int_{-\infty}^{\infty} \left[V''(x_0)\,\epsilon h + \frac{V'''(\xi)}{2}\epsilon^2 (t+h)^2\right] \frac{w^2}{2} dt = 0$$

so

$$C_h = V''(x_0)\,\epsilon(h + O(\epsilon^2))$$

Thus, the reduced problem is a smooth function and $\exists h_\epsilon$: $|h_\epsilon| \le 1$ such that $C_h = 0$, which means $C = 0$.

Appendix A

Brief review of Lyapunov-Schmidt history and further remarks

As we mentioned Allen-Cahn and NLS equations have attracted the interest of many mathematicians and the existence of positive solutions under various assumptions has been proved using different methods. As the problem has generated an impressive amount of publications, it is impossible to give a comprehensive list of references here, but we will try to list as much as we can.

The formulation of a Lyapunov-Schmidt type procedure was first introduced by Floer and Weinstein in [35] who investigated the one-dimensional case. It uses in an essential way the non-degeneracy of the critical point of the potential V, so that one can address the natural question whether alternative arguments may be used to extend their result to a degenerate setting, that is whether solutions concentrating around possibly degenerate critical points of the potential can be obtained. Many authors have subsequently extended this result to higher dimensions to the construction of solutions exhibiting high concentration around one or more points of space under various assumptions on the potential and nonlinearity. Specifically, Oh's [36; 37] result led to so-called multi-bump standing waves which reduces the original problem to a finite dimensional one.

The Lyapunov-Schmidt reduction was then combined with variational arguments by Ambrosetti [3; 7; 8; 10] and [38] for multibump solutions. On the other hand, Rabinowitz [44] was the first in dealing with the question from a global variational point of view, then mainly relayed by del Pino and Felmer [29; 30; 29; 32]. A difficulty faced with variational characterizations of critical values, is that they do not always allow easily to localize properties of associated critical points, especially if they do not enjoy a minimizing or least-energy character. On the other hand this is an advantage of the implicit-function Lyapunov-Schmidt type approach, which discovers the solutions around a small neighborhood of a well chosen first approximation. However, this approach relies heavily on non-degeneracy properties of the linearized problem around this first approximation, thus this reduction procedure is possible only with very fine information on the the limiting equation. In a number of interesting problems exhibiting point concentration this type of information is simply not available, and could be very hard to be obtained even for simplest possible nonlinearities. The need is then created of finding ways of localizing without linearizing.

In [43], Pacard and Ritoré started from a minimal hypersurface Σ in a compact Riemannian manifold M and, under suitable assumptions, they showed that it can be achieved as the limit as $\epsilon \to 0$ of nodal sets (that is 0-level sets) of solutions u_ϵ of the rescaled Allen-Cahn equation. These solutions u_ϵ were constructed with techniques such as fixed point theorems and the Lyapunov-Schmidt reduction, and are not necessarily minimizers. Despite several results lead to think that, in some sense, the nodal sets of the solutions to the Allen-Cahn equation resemble minimal surfaces, there are also solutions for which the nodal set is far from being minimal. For instance, Agudelo, Del Pino and Wei constructed axially symmetric solutions $u = u(|x'|, x_3)$ in $\mathbb{R}^3$ such that the components of the nodal set, for $|x'|$ large enough, look like a catenoid (see [1]).

The Lyapunov-Schmidt reduction was also applied to the non compact case, to construct entire solutions to the Allen-Cahn equation in $\mathbb{R}^9$ that are monotone in one variable but not one-dimensional, since their nodal set resembles the Bombieri-De Giorgi-Giusti graph, that is a minimal graph over $\mathbb{R}^8$ that is not affine (see [16; 26]). The close connection between minimal surfaces and the entire solutions of Allen-Cahn equation led De Giorgi to formulate a celebrated conjecture on the Allen-Cahn equation, that asserts that,

DE GIORGI'S CONJECTURE. *Let u be a bounded solution of the Allen-Cahn equation such that* $\partial_{x_N} u > 0$. *Then the level sets* $\{u = \lambda\}$ *are all hyperplanes, at least for dimension* $N \leq 8$

- True for $N = 2$, Ghoussoub and Gui (1998)
- True for $N = 3$ Ambrosio and Cabré (1999)
- True for $4 \leq N \leq 8$, Savin (2009), thesis (2003), if in addition

$$\lim_{x_N \to \pm\infty} u\,(x', x_N) = \pm 1, \ \forall \ x' \in \mathbb{R}^{N-1}$$

The monotonicity of u implies that the scaled functions (see Section 2.2) are, in a suitable sense, local minimizers of the functional. Moreover, the level sets of u are all graphs. In this setting, De Giorgi's conjecture is a natural, parallel statement to Bernstein's theorem for minimal graphs, which in its most general form, due to Simons [35], states that any minimal hypersurface in $\mathbb{R}^N$, which is also a graph of a function of $N - 1$ variables, must be a hyperplane if $N \leq 8$.

Bernstein"s problem (by Fleming, 1962). *Is it true that all entire minimal graphs are hyperplanes, namely any entire solution F of the minimal surface equation*

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0, \ \textit{in}\ \mathbb{R}^{N-1}$$

must be a linear affine function?

This claim is true for $N \leq 8$.

- Bernstein (1917), Fleming (1962), $N = 3$
- De Giorgi (1965), $N = 4$
- Almgren (1966), $N = 5$
- Simons (1968), $N = 6, 7, 8$

Strikingly, Bombieri, De Giorgi and Giusti (1969) proved that this fact is false in dimension $N \geq 9$

After the famous poincare conjecture and Grigori Perelman's proof, the Hamilton-Ricci flow theory enjoyed a lot of attention. Recent results by P. Daskalopoulos, M. del Pino, N. Sesum [24] in geometric flows, also use the Lyapunov-Schmidt reduction techniques in the parabolic setting in order to construct new ancient solutions to the Yamabe flow.

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