National Technical University of Athens

School of Electrical and Computer Engineering

Division of Signals, Control and Robotics

Modeling and Study of a Debt Stabilization Dynamic Nash Game between a Financial and a Monetary Player in presence of Risk Premia

DIPLOMA THESIS

Tzanis Anevlavis

Supervisor : G. P. Papavassilopoulos

Professor NTUA

Athens, November 2015

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Περίληψη

Η παγκόσμια οικονομική κρίση των τελευταίων ετών έχει ως αποτέλεσμα την αύξηση του χρέους των κρατών σε πολλές χώρες της Ευρώπης, ιδιαίτερα στο μπλοκ του νότου, και έτσι η σταθεροποίηση του κόστους αυτού έχει μετατραπεί σε κύριο μέλημα στην ατζέντα πολλών χωρών. Σε αυτή τη Διπλωματική Εργασία επικεντρωνόμαστε στο γεγονός ότι οι αγορές ασκούν πίεση στα κράτη που φαίνονται ευάλωτα ως προς το μέγεθος του εκάστοτε χρέους, αλλά ως και την τάση του, ανοδική ή καθοδική. Αυτές οι πιέσεις αποτυπώνονται με τη μορφή των ασφαλίστρων κινδύνου. Εισάγουμε σε αυτή την εργασία έναν συντελεστή που αποτελεί ασφάλιστρο κινδύνου, βασισμένο στην τάση του χρέους, μαθηματικά την παράγωγο, δηλαδή στο αν το χρέος είναι σταθερό, ή έχει ανοδική ή καθοδική πορεία. Αυτός ο συντελεστής, εκτός από την πίεση που ασκούν οι αγορές στα κράτη δανειολήπτες, μπορεί και να αποτελέσει μια μορφή επιβράβευσης στις κυβερνήσεις που καταφέρνουν και μειώνουν το χρέος τους. Η εισαγωγή ενός τέτοιου συντελεστή εισαγάγει μια ισχυρή μη-γραμμικότητα στο μαθηματικό μοντέλο μας. Επιπρόσθετα, αναλογιζόμενοι την περίπτωση των κρατών-μελών της Ευρωπαΐκής Νομισματικής Ένωσης, εμφανίζεται η εξής ιδιαιτερότητα: Τα κράτη-μέλη της ένωσης έχουν μια κοινή νομισματική πολιτική που ασκείται κεντρικά από την Ε.Κ.Τ., σε αντίθεση με την οικονομική πολιτική που ασκείται από το Υπουργείο Οικονομικών της εκάστοτε χώρας. Αυτή είναι επίσης η περίπτωση και στις περισσότερες βιομηχανικές χώρες. Αυτή η ιδιαιτερότητα, μας παρακινεί να προσεγγίσουμε την κατάσταση ως ένα Μη-γραμμικό Δυναμικό Παίγνιο Ισορροπίας κατά Nash, μεταξύ δύο παικτών, της Κεντρικής Τράπεζας και της Κυβέρνησης-Υπουργείο Οικονομικών. Οι δύο αυτές αρχές είναι ανεξάρτητες και μπορεί να μη συνεργάζονται. Για το λόγο αυτό, αναλύουμε την περίπτωση του Open-Loop Μη-Συνεταιριστικού Παιγνίου καθώς και την περίπτωση του Συνεταιριστικού Παιγνίου. Επίσης ερευνούμε την περίπτωση του Πεπερασμένου Ορίζοντα, επιπροσθέτως του Παιγνίου Απείρου Μήκους, καθώς αποτελεί μια πιο πρακτική σκοπιά μιας που το γεγονός να είναι κάποιος από τους δύο παίκτες δεσμευμένος για πρακτικά τόσο μεγάλο χρονικό διάστημα, δεν έχει πρακτική εφαρμογή.

Λέξεις-κλειδιά: σταθεροποίηση χρέους, παίγνια δύο παικτών, μη-γραμμικά δυναμικά συστήματα, ισορροπία κατά Nash, ασφάλιστρα κινδύνου, άπειρος ορίζοντας, πεπερασμένος ορίζοντας, οικονομικά δυναμικά συστήματα

Abstract

On the grounds that the global financial crisis during recent years has resulted in a significant increment of the government debt, in many OECD countries, especially in the "south-block", government debt stabilization has taken a central stage in issues to be addressed. In this paper we focus on the fact that financial markets are adding pressures on countries that appear vulnerable when looking at the current levels of debt, as well as the current rate of change of the debt. This takes the form of requiring risk premia. The term we introduce, that depends on the rate of change of debt, represents apart from another form of pressure added by financial markets, can also be used to represent a measure of reward given by markets to governments that succeed in decreasing their debts. This term associated with the derivative of the governmental debt adds a strong nonlinearity to our mathematical model. In addition, when considering the Euro Area, an additional singularity arises: the members of the union are have a common monetary policy which is applied centrally by the E.C.B, in contrast with fiscal policy which is applied by each member country per se. This is also the case in most industrial countries, the size of fiscal deficits and the growth of monetary base are selected by two independent authorities. This suggests that we are facing a Two-Player Nonlinear Dynamic Nash Game under two modes of play, where the two authorities do or do not cooperate. Thus we analyze and solve under the Open-Loop Non-Cooperative mode of play and then under the Cooperative mode of play. We also investigate the finite time horizon, in addition to the infinite one, taking therefore a more practical approach as being bound in a policy for practically a really long time is sometimes of no application.

Keywords: debt stabilization, two-player games, nonlinear dynamic systems, dynamic nash game, risk premium, infinite horizon, finite horizon, economic dynamics

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On the grounds that the global financial crisis during recent years has resulted in a significant increment of the government debt, in many OECD countries, especially in the “south-block”, government debt stabilization has taken a central stage in issues to be addressed. In this paper we focus on the fact that financial markets are adding pressures on countries that appear vulnerable when looking at the current levels of debt, as well as the current rate of change of the debt. This takes the form of requiring risk premia. The term we introduce, that depends on the rate of change of debt, represents apart from another form of pressure added by financial markets, can also be used to represent a measure of reward given by markets to governments that succeed in decreasing their debts. This term associated with the derivative of the governmental debt adds a strong nonlinearity to our mathematical model. In addition, when considering the Euro Area, an additional singularity arises: the members of the union are have a common monetary policy which is applied centrally by the E.C.B, in contrast with fiscal policy which is applied by each member country per se. This is also the case in most industrial countries, the size of fiscal deficits and the growth of monetary base are selected by two independent authorities. This suggests that we are facing a Two-Player Nonlinear Dynamic Nash Game under two modes of play, where the two authorities do or do not cooperate. Thus we analyze and solve under the Open-Loop Non-Cooperative mode of play and then under the Cooperative mode of play. We also investigate the finite time horizon, in addition to the infinite one, taking therefore a more practical approach as being bound in a policy for practically a really long time is sometimes of no application.

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1 INTRODUCTION

One of the most significant effects of the global financial crisis during recent years, is the increase in the government debt. In many OECD countries, especially in the “south-block”, the ensuing economic slowdown, the fiscal balance deterioration, as well as the lack of much needed deep reforms are to explain the rigorous rise of debt. Thus, in most OECD countries, government debt stabilization has taken a central stage in issues to be addressed. Our approach focuses on the fact that financial markets are adding pressures on countries that appear vulnerable when looking at the current levels of debt, as well as the current rate of change of the debt. This takes the form of requiring risk premia on the aforementioned statistics. Furthermore, when considering the Euro Area, an additional singularity arises: the members of the union are have a common monetary policy which is applied centrally by the E.C.B, in contrast with fiscal policy which is applied by each member country per se. Hence, each member cannot finance budgetary deficits. Since, these two authorities are relatively independent and subject to different incentives and constraints, it is rational to assume they have different objectives. It must be mentioned also, that in most industrial countries, the size of fiscal deficits and the growth of monetary base are selected by two independent authorities. Thus, the general framework analyzed could be applied in any country working in that manner. Specifically, in the United States of America, the monetary policy is applied by the Federal Reserve Bank, while the fiscal policy is applied by the government. There is a fundamental difference however, between the U.S. (or countries operating in a similar way) and the countries of the OECD is that in the former both authorities have an eye on the economy and work together by having common interests, while in the latter the E.C.B. majorly takes into account statistics that are taken as average values from all the OECD countries. That is, E.C.B. has an eye on the overall economy of the union. This could result in partially neglecting countries with weak economies as long as there are countries with stronger economies that compensate for the former in the average. This paper examines how this interaction between the two authorities takes place. It investigates the government debt stabilization in the presence of endogenous risk premia from the point of the two independent authorities, fiscal and monetary. These two independent authorities are considered as two players in a dynamic game. In fact though, the financial players are more than one and all interact with the same monetary player. But in this simple approach we will not model this case, however this additional problem is encouraged as a future research proposal.

Similar approaches have already been made. G.Tabellini in [1] analyzed a dynamic game between fiscal and monetary authorities. In his simple approach the strategic interaction between these two authorities, takes the form of a game with a linear model and quadratic cost functions. One of his main findings was the benefit of cooperation, which states that when the two policymakers coordinate their efforts a smaller steady-state debt value is achieved and this is achieved more rapidly. Another one was that in all equilibria, the time path of public debt could be stable even if the real interest rate exceeds the rate of growth of real output. Also, decreasing the relative weight each player assigns to the debt stabilization assigns more burden to the other player for the adjustment. In a more recent approach from J.Enwerda, B.v Aarle et al. in [2], a similar model was used concerning the OECD countries. An endogenous risk premium term was introduced denoting the pressure added by financial markets. This risk premium term was dependent on the level of debt changing the game model to a non-linear one. Their main results are summed up to the fact that the differential game has at least one equilibrium point due to the risk premium term included, whereas in the Tabellini case there could be no equilibrium in some cases. In addition, as the risk premium term increases, the steady-state of debt decreases in both non-cooperative and cooperative modes of play. Also, they found out that the cooperative case is preferable only when the strength of risk premium term is not large, contradicting with the result of Tabellini. Furthermore, the presence of risk premium totally changes the dynamic game and the optimal strategies found by Tabellini, and in this case the reduction of the relative weight a player assign to the debt stabilization does no longer necessarily increase the steady-state of debt.

In this paper, we introduce another risk premium term which is dependent on the rate of change of debt. This term represents another form of pressure added by financial markets, but also it can be used to represent a measure of reward given by markets to governments that succeed in decreasing their debts. Hence, in contrast with the risk premium term dependent on the level of debt, this new one can be used as much as a penalty, as a reward. It is a suggestion that helps the governments to sustain and decrease their debt levels, either by forcing them adopt a stricter approach in order to not let the debt increase, or by helping the invigoration and development of economies that are already on a rising direction. For example, one can consider Japan, a country with a considerable amount of debt which is however sustainable and manageable. Japan also has a strong economy in overall. Thus, Japan is able to get favorable interest rates when getting loans. This example serves as a justification for our new risk premium term. Another distinction from the aforementioned works is that we also investigate the finite time horizon, as well as the infinite one. So this can be considered a more practical approach from this point of view, as the steady-state may need many years to be achieved and models of this type cannot account for all the unpredictable changes and events during that period. Although from an ethical point of view, one could say that there is no clear foreseeable future in the economy and this way the study of infinity horizon is justified, in our case being bound in a policy for practically a really long time is of no application and that justifies the need of investigation and importance of the finite horizon case as well.

Our findings were that in both Non-Cooperative Open-Loop Nash Equilibrium Game and Co-operative modes of play, the debt levels achieved are decreasing as the introduced risk premium term increases. In fact concerning the Non-Cooperative Open-Loop Nash Equilibrium case, we observed that the new risk premium term had a stronger effect than the risk premium term depending on the level of debt. However this inclusion did not change the maximum and minimum number of equilibrium points for the non-linear system, and thus there is always at least one and at maximum three equilibria. In addition, the effect of the risk premium term depending on the rate of change of debt, is evident even in short finite horizons and is still beneficial. It is remarkable how much the debt is able to decrease in the presence of this mechanism that works both as a penalty and reward. The rewarding nature is also being demonstrated by another finding. That is, when the risk premium term on the rate of change of debt increases the players become slightly less active. A justification of this is that when the debt is declining, the risk premium term b is acting as a reward by decreasing the real interest rate. This is helping the debt level to decrease and be contained more easily, allowing the players to relax slightly. Regarding now the Co-operative we observed similar qualitative behaviors, however there were significant improvements on the debt levels achieved comparing to the Non-Cooperative mode. Another interesting observation was that the results yielded were better when the one player was stronger than the other.

The outline of this paper is as follows. Section 2 sets up the model used, defines the variables and the constants and a brief justification of the values used is presented. Section 3 deals with the Non-Cooperative Open-Loop Nash Equilibrium case and the effect of the two risk premium terms is analyzed. Section 4 presents the Cooperative case, analyzing in addition the effect of the bargaining power each player has. Finally, in Section 5 we summarize the results produced and point out future research directions. Proofs of some theorems follow in the Appendix in the end of the paper.

2 MODEL DEFINITION AND PURPOSE

Our analytical framework is inspired by the debt stabilization game of Tabellini [1] and its extension by Engwerda, Van Aarle et al. [2]. In this paper we analyze government debt stabilization problems in the presence of risk premia associated with both the levels of debt and its rate of change. That is, we introduce extended risk premium terms that depend on the levels of debt and its rate of change. A justification for this is that not only the debt level should be taken under consideration, but also the way it is handled. That means, a government with sustainable debt - which is depicted with a negative rate of change (a decreasing debt) or with a zero rate of change (thus a steady debt and under control) - should be considered more reliable and hence rewarded with less pressure from financial markets. This extension adds a strong non-linearity to our problem. In that game our two players, monetary and fiscal policy makers, are engaged in a dynamic conflict of debt stabilization. Both authorities are assumed to have their own objective as well as an interest in government debt stabilization. Fiscal authorities try to reduce fiscal deficits. This can be achieved either by reducing government spending or by increasing taxes. In that way, the accumulation of debt is reduced. Monetary authorities on the other hand, are handling the monetary financing (money growth) and by increasing it, they also contribute in debt reduction.

The model derived for the debt stabilization game described consists of two players, one monetary authority (per say a central bank) responsible for applying the monetary policy, and a fiscal authority responsible for applying a country's fiscal policy. The two players are engaged in a dynamic conflict under the government budget constraint that depicts the accumulation of the debt and relates government debt, monetary financing, fiscal deficits as well as interest payments and risk premia required by financial markets:

$$\dot{d}(t) = r(t)d(t) + u_F(t) - u_M(t) \tag{1.a}$$

In this differential equation:
- d denotes the government debt as a percentage of the national output, note that negative value of debt denotes that government has obtained a claim on private sector assets
- u_F denotes the primary fiscal deficit as a percentage of the national output, note that f denotes deficit we can assume that negative values of f denote surplus for the fiscal authorities
- u_M denotes the monetary financing measured as a fraction of aggregate output
-r denotes the real interest rate adjusted for the rate of output growth

Both control laws belong to appropriate sets U_F and U_M respectively. In our case because the controls are scalar, these sets are subsets of $\mathbb{R}$. Therefore, without loss of generality we can assume that they are intervals of $\mathbb{R}$. Considering that u_F denotes the fiscal deficits, it is normal to assume that there is an upper bound that the authorities are willing to let the deficits grow to. Similarly, there is a lower bound for the fiscal deficits, e.g. there is a line to the taxes enforced in order to lower fiscal deficits. In similar fashion, there are upper and lower bounds for the money growth too. Thus, we could say that the sets are also compact, but we will not use that assumption.

It was stated that fiscal authorities have an objective of their own in addition to being interested in the government debt as well. This objective is to contain fiscal deficits around a neighborhood of a target fiscal deficit $\bar{f}$. We depict that in the following loss function that the player intends to minimize:

$$J_F = {}^1\!/_2 \int_0^T e^{-\theta t} \left(\left(u_F(t) - \bar{f} \right)^2 + \beta_F \left(d(t) - \overline{d_F} \right)^2 \right) dt \tag{1.b}$$

where $\overline{d_F}$ is the fiscal authorities' target for government debt and β_F is the weight assigned to this. It indicates the relative preference concerning debt stabilization of the fiscal authority.

Monetary authorities have also an objective of their own in addition to being interested in the government debt as well. This objective concerns money growth and is to contain it around a neighborhood of a target value $\bar{m}$. We depict that in the following loss function that the player intends to minimize:

$$J_M = {}^1\!/_2 \int_0^T e^{-\theta t} \left((u_M(t) - \bar{m})^2 + \beta_M \left(d(t) - \overline{d_M} \right)^2 \right) dt \tag{1.c}$$

where $\overline{d_M}$ denotes the monetary authorities' target for government debt and β_M is the weight assigned to this. It indicates the relative preference concerning debt stabilization of the monetary authority.

It is clear from these loss functions that the factors that determine the strategies of the two players are, the relative weights for debt stabilization, the target values for debt and each policy action and the respective initial conditions. The real goal of our players is to minimize their respective loss functions by stabilizing government debt at some steady state value, using policies that converge to steady states. The parameter θ denotes a discount factor.

Concerning our main extension with respect to the Tabellini model and the one proposed from Engwerda and Van Aarle, we have to analyze the real interest rate $r(t)$. In the Tabellini model it remained constant, thus leading in a linear first-order differential game with quadratic cost functions. In the Engwerda and Van Aarle model, inspired by the global crisis hitting the countries of the European South as well as Ireland and several other cases, an endogenous risk premia term is introduced depending on the level of government debt. The conclusions produced indicate that with the introduction of risk premium the game always has at least one equilibrium (and at most three) in contrast to the Tabellini model in which we could have no equilibria. Furthermore, equilibrium debt decreases in case the strength of the risk premium parameter increases, but its effect fades off when gone beyond a threshold value. Also, from a debt minimization point of view the cooperative case is produces better results than the non-cooperative only for values of risk premium parameter that are not too large. Else, non-cooperative steady-state of debt is lower than the cooperative, in contrast to the Tabellini case where cooperation produces always better results.

In this paper, as already discussed, we introduce another term of risk premium associated with the rate of change of debt, this modifies the real interest rates of the aforementioned papers:

$$r(t) = \bar{r} + a\, d(t) + b\, \dot{d}(t) \tag{1.d}$$

In our case, real interest rate consists of three terms:

- $\bar{r}$ denotes the difference between nominal interest rate and inflation, and we assume it to be constant in our approach

- a denotes the risk premium coefficient depending on the debt level that was introduced by Engwerda and Van Aarle

- b denotes the risk premium coefficient depending on the rate of change of debt

- both $a\ and\ b$ are positive real numbers

We believe it is justified that interest rate is dependent on the direction of the evolving debt. For example, a high debt (as a percent of national output) which decreases rapidly in the last years should not be necessarily connected with high interest rate, because it looks to be sustainable and that the situation is fully under control. Thus, the government could be "rewarded" with lower interest rate.

Considering the appropriate values for our risk premia parameters, to begin with, empirical studies confirm the dependence of sovereign bond risk premia on debt levels as much as the way it fluctuates – often described on an impact of a 1% change of debt to GDP ratio. De Grauwe and Ji [3] argue that, since the start of the sovereign debt crisis markets have been making errors in the direction of overestimating risks, while before crisis they also falsely tended to underestimate risks. They found evidence that a large part of the surge in the spreads of the periphery countries between 2010 and 2011 was disconnected from underlying increases in the debt-to-GDP ratios and current-account positions, and was the result of negative market sentiments, even panic, that became very strong starting at the end of 2010. That was interpreted in their empirical estimates as a value for a typically between 0.02 and 0.08, maybe even more as we're going deeper into crisis. In addition, Engen and Hubbard [4] conclude, according to their preferred metric, that increasing the ratio of debt to GDP by 1 percentage point will increase longterm real interest rates by 0.035 percentage point. Hence, they characterize their results as showing that the marginal effect of debt on long-term interest rates is small, but positive. However, the pressures for fiscal discipline, and thus debt accumulation discipline, coming from financial markets will be much stronger in the future than they had been before the crisis, as L.Schuknecht, J.von Hagen and G.Wolswijk [5] argue. Thus it is normal that values of a could be around and even greater the 0.10 threshold, while b could approach the value of 0.20.

In this paper we aim to analyze the debt stabilization game discussed, in presence of the risk premium parameters we just introduced. The model presented will be used and different types of equilibria will be considered. First we will consider non-cooperative Open-Loop Nash equilibria, and we try to solve the model under this mode of play. Then, we will consider cooperative Pareto equilibria, solve model under this mode and compare with the former case.

3 The Open-Loop Nash Equilibrium Case

In the first section we consider the game (1.a)-(1.d) under the Non-cooperative Open-Loop Nash Equilibrium case. That means, the players have no means to communicate with each other and have an open-loop information structure (that is, at time $t = 0$ both players have all information about the game and determine their actions, which they are obliged to apply for the whole planning horizon). Also the type of Nash Equilibrium means that our players are looking for a pair of strategies $({u_F}^*, {u_M}^*)$ such that:

$$\begin{gathered} \textit{For any other strategy within the set of admissible strategies} \\ \textit{for the pairs } ({u_F}^*, u_M) \textit{ and } (u_F, {u_M}^*) \textit{ we will have:} \\ J_F(u_F, {u_M}^*) \geq J_F({u_F}^*, {u_M}^*) \textit{ and } J_M({u_F}^*, u_M) \geq J_M({u_F}^*, {u_M}^*) \\ \textit{That means a player cannot achieve better results if he diverges} \\ \textit{from his equilibrium strategy, while the other one sticks to it.} \end{gathered}$$

Now that we have set the stage for this section, we attempt to solve the dynamic game under Non-cooperative Open-loop Nash Equilibrium. The Open-loop case can be seen as an optimal control problem for each player separately and can be solved using Pontryagin's Minimum Principle (A.W.Starr & Y.C.Ho [6]). Combining equations (1.a) and (1.d) we get the differential equation to act as a constraint to the minimization problem of our two players:

$$\dot{d}(t) = \frac{\bar{r}\, d(t) + a\, d(t)^2 + u_F(t) - u_M(t)}{1 - b\, d(t)} \tag{3.1}$$

<u>*Note 1:*</u> It is to be stated, that throughout the remainder of the paper we assume $1 - b\, d(t) \neq 0$. This is in accordance with reality, as b does normally take values around 0.20, thus giving a value of d around 5.0 in order for $1 - b\, d(t)$ to equal zero. This means governmental debt should be around 500% of the GDP of the country under consideration, and we do not aim to even get near that figure. Consequently, we can assume that $1 - b\, d(t) > 0$.

The players want to minimize their respective loss functions:

$$J_F = {}^{1}\!/_{2} \int_0^T e^{-\theta t} \left(\left(u_F(t) - \bar{f}\right)^2 + \beta_F \left(d(t) - \overline{d_F}\right)^2 \right) dt = \int_0^T h_F(t, d, u_F)\, dt \tag{3.2.a}$$

$$J_M = {}^{1}\!/_{2} \int_0^T e^{-\theta t} \left((u_M(t) - \bar{m})^2 + \beta_M \left(d(t) - \overline{d_M}\right)^2 \right) dt = \int_0^T h_M(t, d, u_M)\, dt \tag{3.2.b}$$

Assuming a solution exists we make use of Pontryagin's Minimum Principle and derive the necessary conditions that the Nash Equilibrium strategies should satisfy. To find Nash Equilibrium solutions we need to simultaneously solve two optimal control problems, where the optimal solution of the first enters as a parameter in the second problem, and vice versa.

It will be showed later on that the game always admits a Nash Equilibrium.

In order to solve the Open-Loop Nash Equilibrium problem we make use of Pontryagin's Minimum Principle:

- Pontryagin's Minimum Principle is applied for the fiscal authorities:
  - Hamiltonian: $H_F = \frac{1}{2} e^{-\theta t} \left(\left(u_F - \bar{f} \right)^2 + \beta_F \left(d - \overline{d_F} \right)^2 \right) + \lambda_F \, \frac{\bar{r}\, d + a\, d^2 + u_F - u_M}{1 - b\, d}$
  - $\dot{\lambda_F} = -\, {}^{\partial H_F}/_{\partial d}$, $\lambda_F(T) = 0$
  - ${u_F}^* = arg \min\limits_{u_F} \{H_F\}$

- Working in similar fashion for the monetary authorities:
  - Hamiltonian: $H_M = \frac{1}{2} e^{-\theta t} \left((u_M - \bar{m})^2 + \beta_M \left(d - \overline{d_M} \right)^2 \right) + \lambda_M \, \frac{\bar{r}\, d + a\, d^2 + u_F - u_M}{1 - b\, d}$
  - $\dot{\lambda_M} = -\, {}^{\partial H_M}/_{\partial d}$, $\lambda_M(T) = 0$
  - ${u_M}^* = arg \min\limits_{u_M} \{H_M\}$

However, the conditions from Pontryagin's Minimum Principle are necessary but not sufficient. This means that any solutions that provide the aforementioned equilibrium should satisfy them, but the solutions satisfying them are not guaranteed to be Nash Equilibrium, thus we name them candidate solutions. A rather restrictive yet sufficient condition, in order for the candidate solutions, satisfying the conditions of Pontryagin's Minimum Principle, to be a Nash Equilibrium, is given by the following theorem:

Theorem 3.1: The sets of strategies U_F, U_M are intervals of $\mathbb{R}$ and the Hamiltonians are convex in d. Then the conditions derived by using Pontryagin's Minimum Principle are also sufficient and thus the pair $({u_F}^*, {u_M}^*)$ satisfying them is a Nash Equilibrium.
Proof of the above theorem is provided in the Appendix (6.1).

The convexity of the strategy sets is given as they are intervals of $\mathbb{R}$, while the Hamiltonians are proved to be convex in our area of interest, $d \in (0, d_{init})$. A necessary condition for the Hamiltonians to be convex in d is that the area of interest is left of the value that makes the denominator of the system equal to zero, namely $1/b$. Given that for the values we mentioned in *Note 1(pg.18)*, $1/b$ is much greater than the initial value, and due to the fact that we intend to drive the system asymptotically near to the desired values, it is sufficient to consider the area $(0, d_{init})$ for the convexity of the Hamiltonians.

Hence, by *Theorem 3.1* from the conditions provided by Pontryagin's Minimum Principle we will obtain optimal policies. Then we use the following substitutions for computational ease:

Define: $\mu_F = e^{\theta t} \lambda_F$, $\mu_M = e^{\theta t} \lambda_M$, then introduce: $\mu = \mu_F + \mu_M$. This is a transformation of the costate variables we used for computational ease.

Considering the Finite Horizon problem we prove in the Appendix (6.1) the following theorem:

<u>Theorem 3.2:</u> If $(u_F{}^*, u_M{}^*)$ is a pair of Open-Loop Nash strategies for the game (3.1)-(3.2), there exist a trajectory for debt d^* and an associated costate variable μ^* that satisfy the set of non-linear differential equations:

$$\dot{d}^*(t) = \frac{1}{1-b\,d(t)}\left(\bar{r}\,d(t) + a\,d(t)^2 - \mu\frac{1}{1-b\,d(t)} + \bar{f} - \bar{m}\right) \quad (3.3.a)$$

$$\begin{aligned} &\dot{\mu}^*(t) \\ &= -\left(\beta_F\left(d(t) - \overline{d_F}\right) + \beta_M\left(d(t) - \overline{d_M}\right)\right) \\ &+ \mu\left(\theta - \frac{\left(\bar{r} + 2a\,d(t)\right)\left(1 - b\,d(t)\right) + b\left(\bar{r}\,d(t) + a\,d^2 + \bar{f} - \bar{m}\right)}{\left(1 - b\,d(t)\right)^2}\right) \\ &+ \mu^2\frac{\mathrm{b}}{\left(1 - b\,d(t)\right)^3} \end{aligned} \quad (3.3.\mathrm{b})$$

With $d^*(0) = d_0$. Also, we proved that the system of differential equations (3.3) admits no periodic solutions.

The corresponding expressions for the optimal control policies are:

$$u_F{}^*(t) = \bar{f} - e^{\theta t}\lambda_F{}^*(t)\frac{1}{1-b\,d(t)} = \bar{f} - \mu_F{}^*(t)\frac{1}{1-b\,d(t)} \quad (3.4.a)$$

$$u_M{}^*(t) = \bar{m} + e^{\theta t}\lambda_M{}^*(t)\frac{1}{1-b\,d(t)} = \bar{m} + \mu_M{}^*(t)\frac{1}{1-b\,d(t)} \quad (3.4.b)$$

Now let us assume that the finite horizon T is long enough so that the debt trajectory as well as the costate variables' trajectory enters a steady-state path. To find the values of the debt and of the costate variable in the steady-state we set the differential equations (3.3) simultaneously equal to zero. Denote these Non-Cooperative Open-Loop steady-state values of the trajectories by (d_e^{OL}, μ_e^{OL}). Therefore they should satisfy the following equations derived by setting the differential equations in (3.3) equal to zero:

$$\mu_e^{OL} = -a\,b\,(d_e^{OL})^3 + (a - b\,\bar{r})\,(d_e^{OL})^2 + \left(\bar{r} - b\left(\bar{f} - \bar{m}\right)\right)d_e^{OL} + \left(\bar{f} - \bar{m}\right)$$
(3.5)

And then d_e^{OL} as the solution of the following cubic polynomial:

$$-a\,(2a + b\,\theta)\,(d_e^{OL})^3 + \left(a\,\gamma_2 - \bar{r}(2a + b\,\theta)\right)(d_e^{OL})^2 + \left(\bar{r}\,\gamma_2 - \bar{u}(2a + b\,\theta) - (\beta_F + \beta_M)\right)d_e^{OL} + (\gamma_3 + \bar{u}\,\gamma_2) = 0 \quad (3.6)$$

Where: $\gamma_1 = 1 - b\,d_e^{OL}$, $\gamma_2 = \theta - \bar{r}$, $\gamma_3 = \beta_F\overline{d_F} + \beta_M\overline{d_M}$, $\bar{u} = \bar{f} - \bar{m}$

As we see the steady-state values are independent of the initial value of debt d_0.

Let us consider for a moment the Infinite Horizon problem. That is, the costs of the two authorities now are:

$$J_F = {}^1\!/_2 \int_0^\infty e^{-\theta t}\left(\left(u_F(t) - \bar{f}\right)^2 + \beta_F\left(d(t) - \overline{d_F}\right)^2\right) dt$$
$$J_M = {}^1\!/_2 \int_0^\infty e^{-\theta t}\left((u_M(t) - \bar{m})^2 + \beta_M\left(d(t) - \overline{d_M}\right)^2\right) dt$$

It should be noted though, that the control policies (3.4), do not imply necessarily the existence of solutions for the infinite time case. Papavassilopoulos and Olsder [9] discuss and exhibit a class of linear-quadratic Nash games, where while closed-loop no-memory strategies for any finite-time interval exist, for the infinite time horizon there might not exist, exist many solution or a unique one. Furthermore, the limit of the finite-time horizon solution does not necessarily have to converge to the infinite time horizon solution. In addition, Mageirou [10] had previously shown for the linear quadratic case, that the solution of the algebraic Riccati equation, which coincides with the limit of the solution of the dynamic Riccati equation when the time goes to infinity, determines the value of the infinite time horizon game. However, the strategies yielded for that particular solution of the algebraic Riccati equation are not necessarily in equilibrium.

Returning to our game and the Infinite Horizon case, one can easily observe that if we use again Pontryagin's Minimum Principle, the same expressions for control policies and costate functions are obtained. But, now there is a significant difference. The infinite horizon lacks transversality conditions, as the transversality conditions do not extend asymptotically. It is easy to prove that the equilibrium equations for the infinite horizon case are identical to the set of equations (3.3). This set of equations yields again the decoupled to equations (3.5) and (3.6). Therefore, if we assume that there exists a solution to equations (3.6), we obtain a steady-state equilibrium point. Then as Haurie claims in [13], the turnpike property makes the trajectories of the debt and of the costate variables attracted to those points. Note that for both finite and infinite horizon cases, it is natural that only real roots of (3.6) are considered and accepted as steady-state values, due to the fact that they represent debt value.

Therefore, one could conjecture that for the Finite Horizon case and for T long enough, equations (3.3) become equal to zero. Then there exists $\mathrm{T}' < \mathrm{T}$ such that for some t, $\mathrm{t} < \mathrm{T}'$, the debt trajectory of the Finite Horizon enters a steady-value path, in the sense that the value of debt remains constant. The debt trajectory in both the Finite and Infinite Horizon cases is attracted by the same steady-state value as the expressions of the optimal policies for both cases are identical and therefore the trajectory of the Finite Horizon can get in a neighborhood of the steady-state value of the Infinite Horizon. Then, for $\mathrm{T}' \leq \mathrm{t} \leq \mathrm{T}$, the trajectory of the Finite Horizon deviates from the aforementioned steady-state value as the optimal policies for the Finite Horizon deviate from those of the Infinite Horizon because the transversality conditions for the costate variables $\lambda_{\mathrm{i}}(\mathrm{T}) = 0, \mathrm{i} = \mathrm{F}, \mathrm{M}$ need to be met. In fact, Haurie in [13] comments that there is also a finite horizon turnpike property in the sense that on a long journey, the optimal trajectory should spend most of the time in the vicinity of the turnpike.

Although we will not address in detail the problem of proving such properties, benchmarks show that they might hold as well. We hence claim, that since for horizons long enough, the trajectory could enter the steady-path and stays on it for the most part of the debt trajectory except for some final time, it is important to study this steady-state path. Thus, during the Open-Loop Nash Equilibrium mode of play, we first focus on the steady-state of debt and on the effects which the additional risk premium (on the rate of change of debt) might have. Then we present as benchmark examples, various values for finite time horizon T and compare the results. We will elaborate later that the time needed to achieve the equilibrium might be too long to practically consider it a viable solution, however the steady-state presents properties that appear as well in smaller horizons, enough to provide a practical and viable solution. We will also present comparisons with the infinite horizon game. From an ethical point of view, one could say that there is no clear foreseeable future in the economy and this way the study of infinity horizon is justified. However, in our case binding in a policy for practically a really long time is of no application and thus the finite horizon case should be investigated as well.

Before proceeding with our model, we recall some results for the steady-state proven in literature, for the infinite horizon case:

3.1 The Tabellini Model: No Risk Premia

First, for reference we present the results of the Tabellini model. That is, in eq.(3.1) we set the parameters $a = 0 \; and \; b = 0$, and in (3.2) T goes to infinity. Also $\overline{d_F} = \overline{d_M} = 0$ was considered and the following conclusions had been derived:

1. This game has a unique set of admissible equilibrium actions that allow for a feedback synthesis.
2. This game has a unique set of admissible equilibrium actions if the policymakers are sufficiently impatient, i.e., if $\theta > 2\bar{r}$.
3. This game has an infinite number of admissible equilibrium actions if $\theta < 2\bar{r}$. However, all equilibrium actions yield the same closed-loop system. Furthermore, equilibrium actions converge to the same steady state.
4. When $b > 0$ the game has a unique steady state values of debt for every initial d_0, leading to the steady state values of: $d^e = \frac{\gamma_0}{b}$, $f^e = \frac{\beta_F(\bar{r}\overline{d_F}+\bar{f}-\bar{m})}{b}$, $m^e = \bar{m} - \frac{\beta_M}{\beta_F}(f^e - \bar{f})$.
5. Also if $b < 0$ and the players do not care much about the debt (that is $\theta > \bar{r}$ and β_F, β_M sufficiently small), then the game admits no Open-Loop Nash equilibrium unless $d_0 = d^e = \frac{\gamma_0}{b}$.

3.2 ENGWERDA AND VAN AARLE: RISK PREMIUM ON DEBT LEVEL

Engwerda and van Aarle extended the Tabellini model by introducing a parameter of risk premium depending on government debt level (in eq. (3.1) $a > 0 \; and \; b = 0$, and in (3.2) T goes to infinity). This extension produces at least one and at most three steady states of debt. They conclude to a third-order equation of steady state of debt, d^e:

$$-2a^2d^3 + a(\theta - 3\bar{r})d^2 + \gamma_1 d + \gamma_0 = 0$$

$$with \; \gamma_0 = \beta_F\overline{d_F} + \beta_M\overline{d_M} + (\bar{f} - \bar{m})(\theta - \bar{r}) \; and \; \gamma_1 = -(\bar{r}(\bar{r} + \theta) + \beta_F + \beta_M) - 2a(\bar{f} - \bar{m})$$

The discriminant is: $h(a) = 8{\gamma_1}^3 - 36a\gamma_1\gamma_0 s - 40a\gamma_0 s^3 - 108a^2{\gamma_0}^2 + {\gamma_1}^2 s^2, \quad s = \theta - 3\bar{r}$. Hence we obtain:

1. One steady state of debt if either $i) \; h(a) < 0 \; or \; ii) \; h(a) = 0 \; and \; a^2s^2 = 3\gamma_1$.
2. Two steady states (from which only one applies as an open loop equilibrium if $d_0 = d^e$) if $h(a) = 0 \; and \; a^2s^2 \neq 3\gamma_1$.
3. Three steady states (from which the middle one applies as an open loop equilibrium if $d_0 = d^e$), if $h(a) > 0$.
4. Of these steady states, at least one is a saddle-point, denoting the impact of including risk premia.
5. Also, they have extracted the conclusion that the steady-state value of debt decreases as the value of a becomes larger.

3.3 INTRODUCING: RISK PREMIUM ON DEBT RATE OF CHANGE:

First, throughout this section we assume that the horizon T chosen, is long enough so that equilibrium value for the debt is attained. For our main model consider again eq.(3.1) now with both $a > 0 \; and \; b > 0$. From equation (3.6) the equilibrium points are defined. This is a cubic polynomial and thus an analytical solution is available. It is natural that only real roots of (3.6) are considered and accepted, due to the fact that they represent debt value.

3.3.1 Effect on Number of Equilibrium Points and Qualitative Behavior

In the Appendix (Section 6.1.2-6.1.4) we prove the following:

- If $\Delta(a, b)$ is the discriminant of the cubic polynomial (3.6), as a function of the risk premium coefficients $a \; and \; b$, then the number of steady state values of debt given by (3.6) is subject to the following rule:
  - If $\Delta(a, b) > 0$, there are *3 distinct real roots* and hence, *3 candidate solutions* for the game.
  - If $\Delta(a, b) < 0$, there is *1 distinct real root and 2 complex conjugate roots*. Hence, *1 candidate solution* for the game.

- If $\Delta(a,b) = 0$, there are *at least 2 roots coincide and they are all real*. That means there are either a double real root and distinct single real root, or a triple real root. That is we either have 2 or 1 candidate solutions for the game respectively.
  $\left(\bar{r}\,\gamma_2 - \bar{u}(2a + b\,\theta) - (\beta_F + \beta_M)\right)\left(a\,\gamma_2 - \bar{r}(2a + b\,\theta)\right) + 9a(2a + b\,\theta)(\gamma_3 + \bar{u}\,\gamma_2) = 0$ (3.7)
  If (3.7) holds there is *1 triple real root*, else there is *1 double root and 1 distinct real root*.

➢ The qualitative behavior of the system near the equilibrium points should be explored. This is achieved via the Linearized System around each equilibrium point. Linearizing system (3.3) we get:

$$\begin{pmatrix} \dot{d(t)} \\ \dot{\mu(t)} \end{pmatrix} = \begin{pmatrix} -\frac{a}{b} - \frac{2\,b\,\mu_e^{OL}}{\gamma_1{}^3} + \frac{a+b\,\bar{r}+b^2\bar{u}}{b\,\gamma_1{}^2} & -\frac{1}{\gamma_1{}^2} \\ -(\beta_F + \beta_M) - \frac{2\,(a+b\,\bar{r}+b^2\,\bar{u})\,\mu_e^{OL}}{\gamma_1{}^3} + \frac{3(b\,\mu_e^{OL})^2}{\gamma_1{}^4} & \frac{a}{b} + \theta + \frac{2\,b\,\mu_e^{OL}}{\gamma_1{}^3} - \frac{a+b\,\bar{r}+b^2\bar{u}}{b\,\gamma_1{}^2} \end{pmatrix} \begin{pmatrix} d(t) \\ \mu(t) \end{pmatrix}$$

(3.8)

where (d_e^{OL}, μ_e^{OL}) is the equilibrium point under discussion.

The eigenvalues of the linearized system (3.7) are:

$$eig_{1,2} = \frac{(1 - bd_e^{OL})\,\theta\ \pm \sqrt{(1 - bd_e^{OL})^2\,\theta^2 - 4\,{}^{\partial\,g(d)}/_{\partial d}}}{2(1 - bd_e^{OL})} \qquad (3.9)$$

One can easily observe that the steady states will be one of the following:

- If $(1 - bd_e^{OL})^2\,\theta^2 - 4\,{}^{\partial\,g(d)}/_{\partial d} > 0\ and\ {}^{\partial\,g(d)}/_{\partial d} < 0$, then $eig_1 > 0$ and $eig_2 < 0$. Thus, the steady state is a saddle point.
- If $(1 - bd_e^{OL})^2\,\theta^2 - 4\,{}^{\partial\,g(d)}/_{\partial d} > 0\ and\ {}^{\partial\,g(d)}/_{\partial d} > 0$, then $eig_1 > 0$ and $eig_2 > 0$. Thus, the steady state is an unstable node.
- If $(1 - bd_e^{OL})^2\,\theta^2 - 4\,{}^{\partial\,g(d)}/_{\partial d} < 0$, then $eig_{1,2}$ are complex eigenvalues with positive real part. Thus, the steady state is an unstable focus.
- Last but not least, there is the case an eigenvalue becomes zero. That is, when $\Delta(a,b) = 0$ and (3.7) holds (a triple real root). As we have already argued the equilibrium point is also a root of the first derivative. It is then straightforward from (3.9) that an eigenvalue becomes zero while the other will be positive. This means the system matrix has a non-trivial null-space and has an equilibrium subspace than an equilibrium point. Due to the positive eigenvalue all trajectories diverge away from the equilibrium subspace.

- Combining the two previous bullet we conclude to the following proposition:
  Taking into account that the highest order coefficient $\delta_3 = -a(2a + b\,\theta)$ of (3.6) is always negative since $a, b, \theta > 0$, the sign of ${\partial\, g(d)}/{\partial d}$ near each equilibrium point can be determined. Thus:
  - If $\Delta(a,b) > 0$, there are *3 distinct real roots*. The first and the last are *saddle points*, while the middle one is *unstable node or focus*.
  - If $\Delta(a,b) < 0$, there is *1 distinct real root* and 2 complex conjugate roots. The derivative ${\partial\, g(d)}/{\partial d} < 0$ and the equilibrium point is a *saddle point*.
  - If $\Delta(a,b) = 0$ and (3.7) does not hold there is *1 double real root and 1 distinct single real root*, the double root is *unstable* while the single one is a *saddle point*.
  - If $\Delta(a,b) = 0$ and (3.7) does hold there is *1 triple real root* and we have an *equilibrium subspace* than an equilibrium point, and *all trajectories diverge away* from the equilibrium subspace.

Commenting on our analysis, the inclusion of another risk premium term again does give at least one steady state saddle point. Also, the maximum number of steady states remains the same, but of course now it varies with the values of $a\ and\ b$. Furthermore we have explored the case of a zero eigenvalue, and found that there exists an equilibrium subspace which acts as a solution of the dynamic game only when the initial value d_0 is already on that subspace. Else, the trajectory diverges away and no solution is obtained.

3.3.2 Effect on development of the Steady-State Value of Debt

It is now clear that risk premia coefficients $a\ and\ b$ can play major role in defining the equilibrium points. It is important then to see how the value of equilibrium of the system responds in variations to those coefficients and extract a qualitative conclusion, if possible.

In order to examine the sign of the derivative of steady state value debt w.r.t. $a\ and\ b$. Under the assumption that ${\partial\, g(d)}/{\partial d} \neq 0$ at the steady state d_e^{OL} (for example, not having a triple root as we've argued before) and that $d_e^{OL}(a)\ and\ d_e^{OL}(b)$ are continuous differentiable in their respective variables we obtain:

$$\partial\, d_e^{OL}(a)/\partial a = -\frac{\partial\, g/\partial d}{\partial\, g/\partial a} \qquad and \qquad \partial\, d_e^{OL}(b)/\partial b = -\frac{\partial\, g/\partial d}{\partial\, g/\partial b}$$

- $\partial\, g/\partial d = -3a(2a + b\theta){d_e^{OL}}^2 + 2(-3a\bar{r} + a\theta - b\bar{r}\theta)d_e^{OL} + \left(-\beta_F - \beta_M - \bar{u}(2a + b\theta) + \bar{r}(\theta - \bar{r})\right)$ (3.10)
- $\partial\, g/\partial a = -(4a + b\theta){d_e^{OL}}^3 + (\theta - 3\bar{r}){d_e^{OL}}^2 + (-2\bar{u})d_e^{OL}$ (3.11)
- $\partial\, g/\partial b = -a\,\theta {d_e^{OL}}^3 - \bar{r}\,\theta\, {d_e^{OL}}^2 - \bar{u}\,\theta\, d_e^{OL}$ (3.12)

Depending on the parameters we can derive the sign of the partial derivatives w.r.t. $a\ and\ b$. Furthermore as we analyzed in section 3.3.1 in case we have a unique steady state (a saddle point) the sign of ${\partial\, g(d)}/{\partial d}$ is negative. That is also the case for three distinct steady states, as two of them (the first and last are saddle points) have ${\partial\, g(d)}/{\partial d} < 0$.

Also, we assume that the steady-state of debt is positive. Thus for example, if $\theta < 3\,\bar{r}$ and $\bar{u} > 0 \Leftrightarrow \bar{f} - \bar{m} > 0$, then ${\partial\, g}/{\partial a} < 0$ and consequently: ${\partial\, d_e^{OL}(a)}/{\partial a} < 0$. This gives us the insight that debt decreases as the value of a gets larger. However, for another selection of parameters we could observe different behavior. For example, typically $\theta > 3\,\bar{r}$, thus we can observe the debt to increase for small values of a, but as it gets larger the "positive" effect of the second term in (3.11) is balanced by the other two "negative" terms and the derivative becomes negative again and the steady state value of debt declines again. This means we can except the value of debt to increase for small values of a and decrease again as a grows larger.

Considering now equation (3.12) the only non-negative term could be $-\bar{u}\,\theta\, d_e^{OL}$, when $\bar{u} < 0 \Leftrightarrow \bar{f} - \bar{m} < 0$. However, in most benchmarks is not large enough and the derivative remains negative. That means, the steady state value of debt will always decrease as b increases. The decaying of the steady state debt appears to be almost linear with the value of b.

As we will show in the next benchmarks, the effect of a to increase steady state value of debt for small values, is contained by the risk premium term on rate of change of debt. The higher the value of b is, the less this effect is visible.

3.3.3 Effect on the Steady-State Values of the Monetary and Fiscal Policies

The impact of risk premium terms is not restricted only on the steady-state value of debt, but as expected it affects the steady-state values of the optimal policies too. The steady-state values of the monetary and fiscal policies are computed in the Appendix by equations (6.1.6). We also provide them here for ease:

$$u_F{}^e = \bar{f} + \frac{\beta_F\left(d - \overline{d_F}\right)}{\bar{r} + 2a\,d - \theta(1 - b\,d)} \quad and \quad u_M{}^e = \bar{m} - \frac{\beta_M\left(d - \overline{d_M}\right)}{\bar{r} + 2a\,d - \theta(1 - b\,d)}$$

Where $u_F{}^e, u_M{}^e$ are the steady-state values of fiscal and monetary policies respectively. We can also assume that when in the steady-state, the debt has an equilibrium value too, thus it would be proper to write:

$$u_F{}^e = \bar{f} + \frac{\beta_F\left(d_e^{OL} - \overline{d_F}\right)}{\bar{r} + 2a\,d_e^{OL} - \theta(1 - b\,d_e^{OL})} \quad and \quad u_M{}^e = \bar{m} - \frac{\beta_M\left(d_e^{OL} - \overline{d_M}\right)}{\bar{r} + 2a\,d_e^{OL} - \theta(1 - b\,d_e^{OL})}$$

In order to derive assumptions on the effect of risk premia on the steady-state policies, we take the respective derivatives w.r.t. a:

$$\partial\, u_F^e(a)\Big/_{\partial a} = \frac{\beta_F \frac{\partial\, d_e^{OL}(a)}{\partial\, a}\left(\bar{r} + 2a\, d_e^{OL} - \theta(1 - b\, d_e^{OL})\right) - \beta_F\left(d_e^{OL} - \overline{d_F}\right)\left(2d_e^{OL} + 2a\frac{\partial\, d_e^{OL}(a)}{\partial\, a} + \theta b\frac{\partial\, d_e^{OL}(a)}{\partial\, a}\right)}{\left(\bar{r} + 2a\, d_e^{OL} - \theta(1 - b\, d_e^{OL})\right)^2}$$

$$\partial\, u_M^e(a)\Big/_{\partial a} = -\frac{\beta_M \frac{\partial\, d_e^{OL}(a)}{\partial\, a}\left(\bar{r} + 2a\, d_e^{OL} - \theta(1 - b\, d_e^{OL})\right) - \beta_M\left(d_e^{OL} - \overline{d_M}\right)\left(2d_e^{OL} + 2a\frac{\partial\, d_e^{OL}(a)}{\partial\, a} + \theta b\frac{\partial\, d_e^{OL}(a)}{\partial\, a}\right)}{\left(\bar{r} + 2a\, d_e^{OL} - \theta(1 - b\, d_e^{OL})\right)^2}$$

Next we differentiate w.r.t. b:

$$\partial\, u_F^e(b)\Big/_{\partial b} = \frac{\beta_F \frac{\partial\, d_e^{OL}(b)}{\partial\, b}\left(\bar{r} + 2a\, d_e^{OL} - \theta(1 - b\, d_e^{OL})\right) - \beta_F\left(d_e^{OL} - \overline{d_F}\right)\left(2a\frac{\partial\, d_e^{OL}(b)}{\partial\, b} + \theta d_e^{OL} + \theta b\frac{\partial\, d_e^{OL}(b)}{\partial\, b}\right)}{\left(\bar{r} + 2a\, d_e^{OL} - \theta(1 - b\, d_e^{OL})\right)^2}$$

$$\partial\, u_M^e(b)\Big/_{\partial b} = -\frac{\beta_M \frac{\partial\, d_e^{OL}(b)}{\partial\, b}\left(\bar{r} + 2a\, d_e^{OL} - \theta(1 - b\, d_e^{OL})\right) - \beta_M\left(d_e^{OL} - \overline{d_M}\right)\left(2a\frac{\partial\, d_e^{OL}(b)}{\partial\, b} + \theta d_e^{OL} + \theta b\frac{\partial\, d_e^{OL}(b)}{\partial\, b}\right)}{\left(\bar{r} + 2a\, d_e^{OL} - \theta(1 - b\, d_e^{OL})\right)^2}$$

One can see that if we assign same weights β_F, β_M and same target values for debt to the two authorities, we have $\frac{\partial\, u_F(a)}{\partial\, a} = -\frac{\partial\, u_M(a)}{\partial\, a}\ and\ \frac{\partial\, u_F(b)}{\partial\, b} = -\frac{\partial\, u_M(b)}{\partial\, b}$. A less restrictive, but more qualitative result would be to observe that the derivatives of the steady-state policies w.r.t. both risk premium terms have opposing signs. We are going to show in the benchmark example that follows that the derivative of the policies with respect to a is negative, however w.r.t. b it is positive. Keeping in mind that a negative value of fiscal deficits denotes a fiscal surplus, we can conclude that as the risk premium on the debt level increase, both the policies are being used more actively. In contrast as the risk premium term associated with the rate of change of the debt becomes greater, the system looks more independent and so the policies are less actively used. However, the impact of b is significantly less than the one of a.

3.3.4 Benchmark Examples

<u>*Note 2:*</u> We have already argued about the values of $a\ and\ b$ that they would be around 0.10 and 0.20 respectively given the current economic situation. Also, as stated in previous literature the weight values that the two authorities put on debt development, $\beta_F\ and\ \beta_M$, are of the same order of magnitude, that is 10^{-3}. The parameter θ, which denotes the degree in which the authorities care about the future development of the debt, will take two representative values: $\theta = 0.15$ when the players do care about the future debt development and $\theta = 0.75$ when they care less.

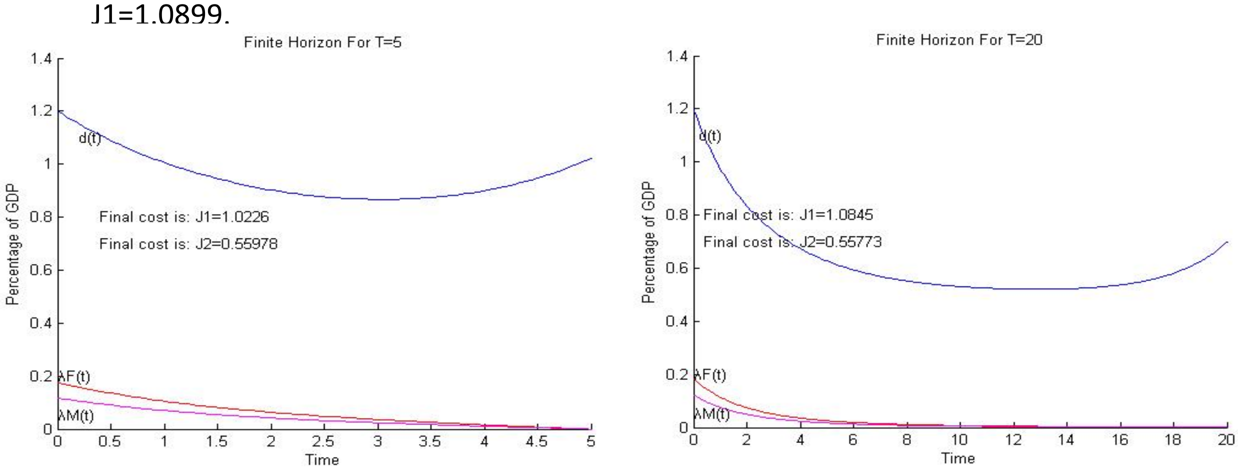
We choose the following parameters:

$$\beta_F = 0.06\ , \beta_M = 0.04\ , \overline{d_F} = \overline{d_M} = 0.5, \bar{r} = 0.03\ , \theta = 0.15$$

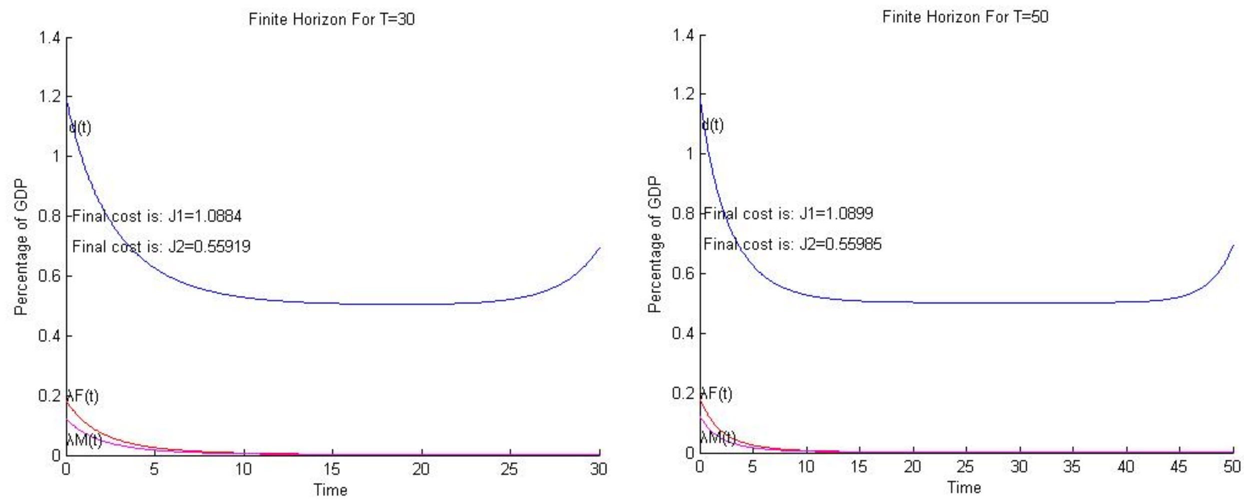
The justification of this choice is that in OECD countries for example, even in a non-cooperative mode there are some basic common lines such as the target of debt. Furthermore, the fiscal authorities of each country put greater weight to its governmental debt, than the ECB who should oversee the other country-members too. Last, in our first example the authorities are interested in the future development of the debt and do not play a hasty game. This justifies the value of θ. Last, the target values of the two authorities policies are: $\bar{f} = 0.01\ and\ \bar{m} = 0$, meaning that the ECB adopts a strict approach for the money growth, intending to be zero at steady state, while fiscal authorities care to more slowly reduce their deficits leaving a target value of 0.01. A stricter approach might mean that they would have to enforce higher taxes.

3.3.4.1 Trajectories of Debt, Costate Variables and Optimal Policies for the Finite Horizon

We begin our benchmarks by considering the finite horizon case for various values of T. We present the trajectories that the debt and the costate variables follow when solving the two point boundary value problem imposed by the use of Pontryagin's Minimum Principle. As we conjectured the debt trajectory is attracted by a steady-value, and either it reaches it or not, then it deviates as the transversality conditions have to be met. This is clear in Figures 3.1 to 3.4. In Fig. 3.2, where the horizon is T=20, there is a subtle suspicion of entering a steady-trajectory for a short-time, and for T=30 it is obvious that the trajectory enters a steady-path for a while. Then in Figure 3.4 for T=50 it is evident that the trajectory reaches the steady-path and remains in it except for some final time. In that time, as the transversality conditions have to be met, the control policies deviate from their steady values and therefore they affect the debt trajectory as well. Also the two costate variables, λ_F, λ_M, are always meeting their respective transversality condition at the end of the horizon. When referring to the respective costs at each horizon, of course the longer the horizon is the greater the cost for each player, however the increment is not that significant. On the grounds that the debt trajectory stays in a vicinity of a steady value for the most time of the horizon, then the accumulation that takes place for the cost is minute. For example when the horizon is T=20 the cost for the fiscal authorities is J1=1.0845, and for T=50 the cost is J1=1.0899.

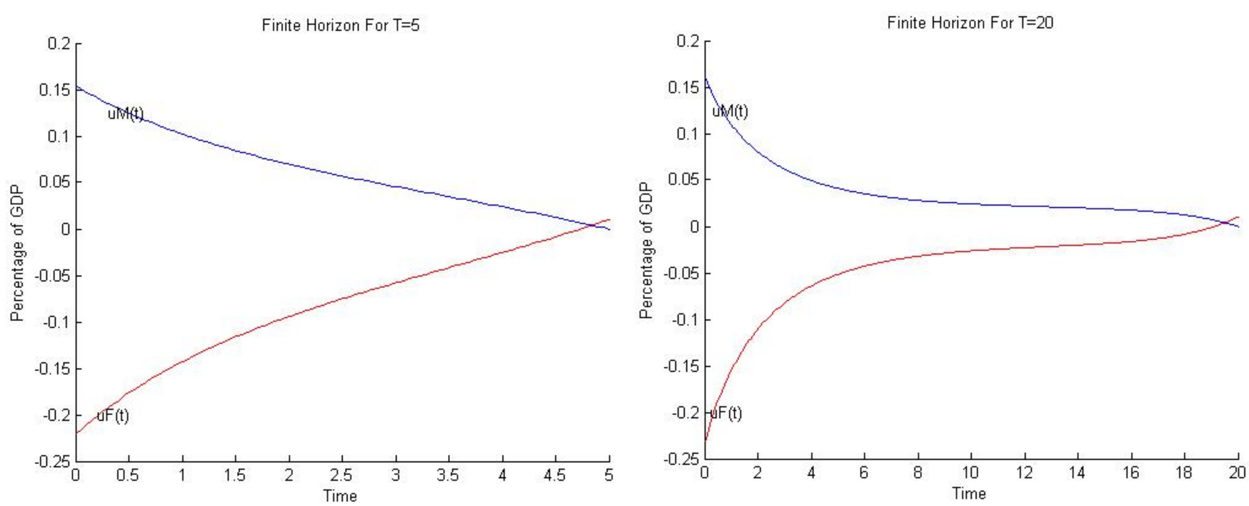


Figures 3.1 - 3.2: Finite Horizon Trajectories of Debt and Costate Variables or various values of finite horizons

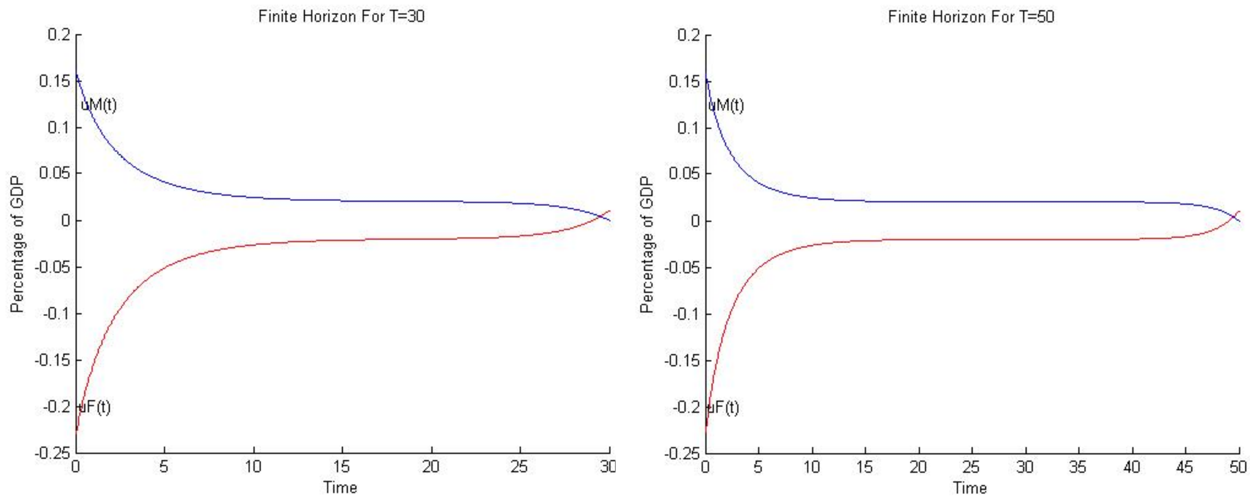


<u>Figures 3.3 - 3.4:</u> Finite Horizon Trajectories of Debt and Costate Variables or various values of finite horizons

Furthermore, the effect of deviating from the steady path in order to meet the transversality conditions, has an obvious effect on the optimal policies too. Recall the expressions derived from those policies, equations (3.4): $u_F{}^*(t) = \bar{f} - e^{\theta t}\lambda_F{}^*(t)\frac{1}{1-b\ d(t)}$, $u_M{}^*(t) = \bar{m} + e^{\theta t}\lambda_M{}^*(t)\frac{1}{1-b\ d(t)}$. When the two costate variables, λ_F, λ_M, meet their respective conditions $\lambda_F(T) = 0, \lambda_M(T) = 0$ then $u_F{}^*(T) = \bar{f},\ u_M{}^*(T) = \bar{m}$. This is exactly shown in Figures 3.5-3.8.



<u>Figures 3.5 - 3.6:</u> Finite Horizon Trajectories of Optimal Policies for various values of finite horizons

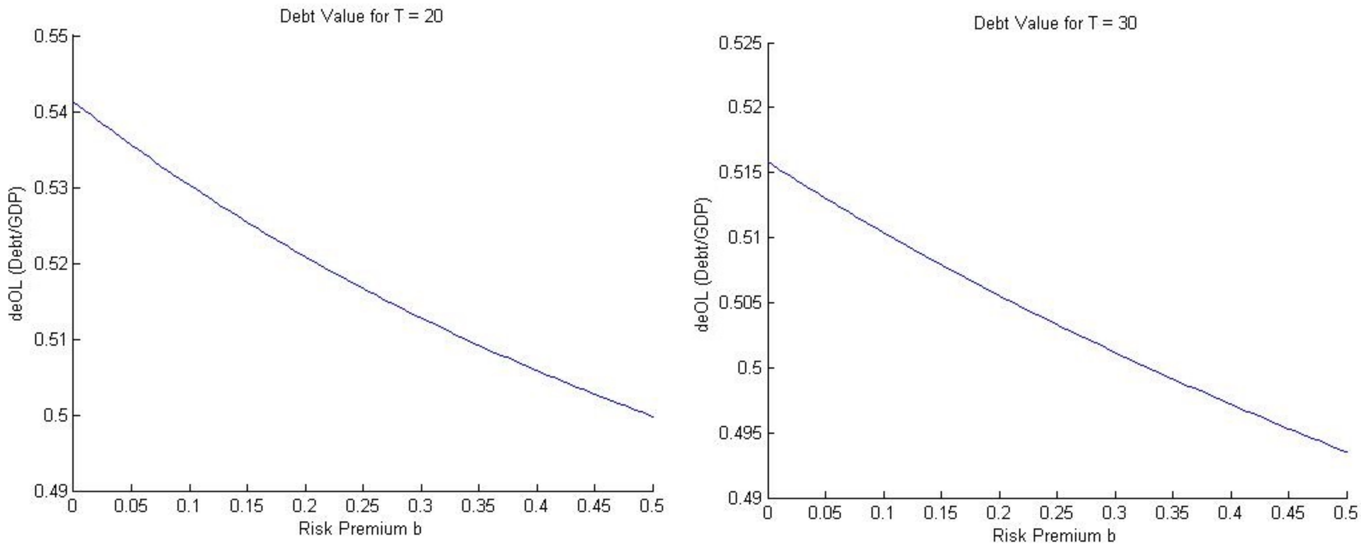


<u>Figures 3.7 - 3.8:</u> Finite Horizon Trajectories of Optimal Policies for various values of finite horizons

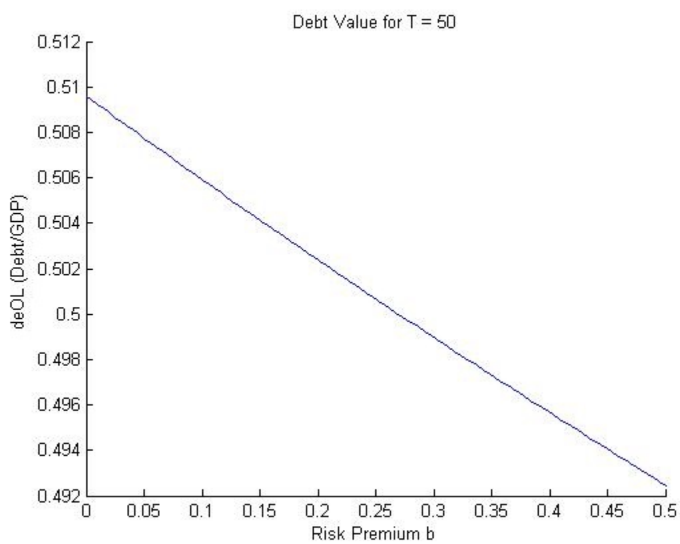
3.3.4.2 Effect of the Risk Premium Terms on Steady-Values of Debt

The Figures in the previous section were obtained for $a = 0.10\ and\ b = 0.20$. What is important to find out is how these in these plots, both steady part and final part, are affected by changes in the two risk premium terms. In other words, we will investigate the effect of risk premium terms in the finite horizon case. As we have already argued at the start of this section, and it is presented here, the steady part dominates for most of the time horizon. With this in mind, and also considering that practically this is the value in which the debt remains in a particular country for most of the time the policy is implemented, we first explore the effect of the risk premia in the steady-part.

Starting from a time horizon of T=20 we first explore the effect of varying the risk premium term b that depends on the rate of change of debt, while keeping a constant at 0.10. Figures 3.5 through 3.7, first of all show that the relation between the risk premium term b and the steady part of the debt trajectory is almost linear. As b increases the value the debt attains decreases too, independently of the horizon. This was also what we found out when analyzing theoretically the effect of the risk premium on the steady-state value of debt. However there are some differences that depend on the length of the horizon T. These differences are mainly attributed to how close to the steady path the trajectories tend. The longer the horizon is, the closer the trajectories tend to their respective steady values as it was also obvious from Figures 3.1-3.4 in the previous section. As the time horizon increases the behavior of these plots should be closer to the steady state trajectory. Indeed, T=50 is long enough and the plot is identical with the one from the steady state. One should keep in mind from these results, that no matter how close, in fact arbitrarily close, we can get to the steady state, the part that represents the steady part of our trajectory behaves always the same as b varies. We have an almost linear decrease in the steady value of debt as b increases.

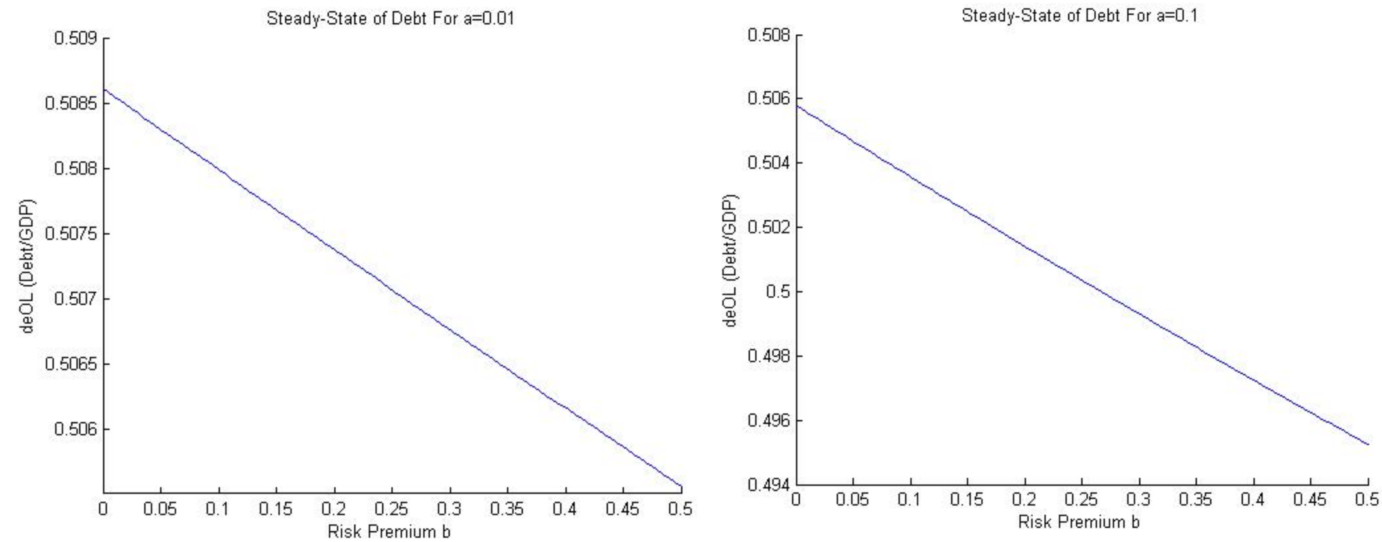


<u>Figures 3.9 - 3.10:</u> Effect of risk premium term b on steady part of the debt trajectory



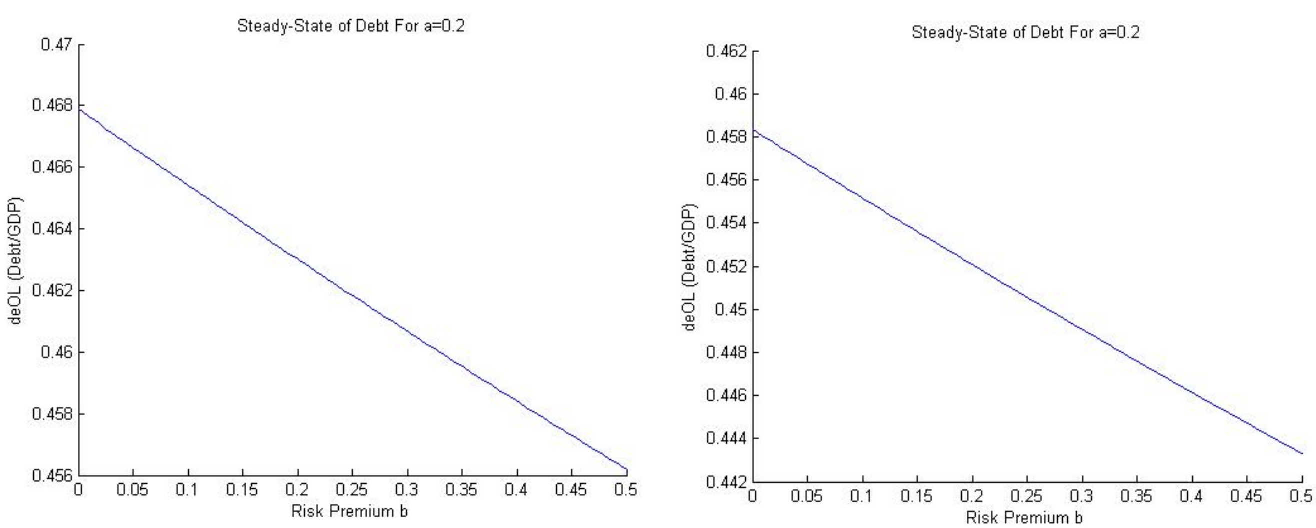
<u>Figures 3.11:</u> Effect of risk premium term b on steady part of the debt trajectory

Next we will investigate the infinite horizon case. That is the exact results obtained from the system of equations (3.3) or directly by solving (3.5) for a real valued solution. Figures 3.8 and 3.9 show two of the results we proposed theoretically in section 3.3.2. First, that the decline in the steady state value of debt, as b appears to be linear. Second, for small values of a we have higher steady state value of debt. As we can see, the plots are identical except being moved along the vertical axis. Another point to focus on, is that figure 3.9 is identical with 3.7 ($T = 50, a = 0.10$). This confirms the fact that T=50, is a long enough horizon and all the results deduced for steady-state in infinite time horizon, apply now to the steady part of the trajectory in finite time.



Figures 3.12 & 3.13: Effect of risk premium term b on steady state value of debt

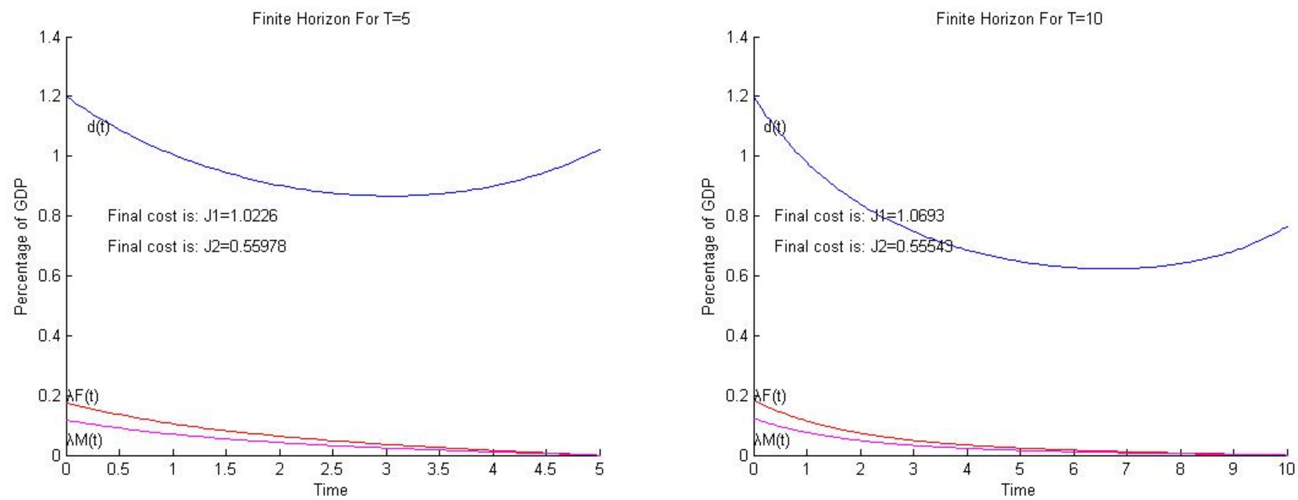
We continue to focus our analysis now on the steady-state for a while. In figures 3.10 and 3.11 we see how the sign of $\bar{f} - \bar{m}$ impacts on the steady state. As we expect, when the ECB is willing to let a higher money growth and fiscal authorities are stricter with their deficits, the steady state of value debt will be lower. Of course the relationship of the steady-state value of debt with the risk premium term b remains linear. Another observation to be made, is that as the value of risk premium a increases, in the second case, debt gets much lower than in the first. In other, words the slope is greater in the second case. Intuitively, we could say that when the main objective for each authority, namely the target values for money growth and fiscal deficit, are in the general direction of the decline of the debt value, the more active use of policies provoked by the greater risk premium terms yields lower steady-state debt.



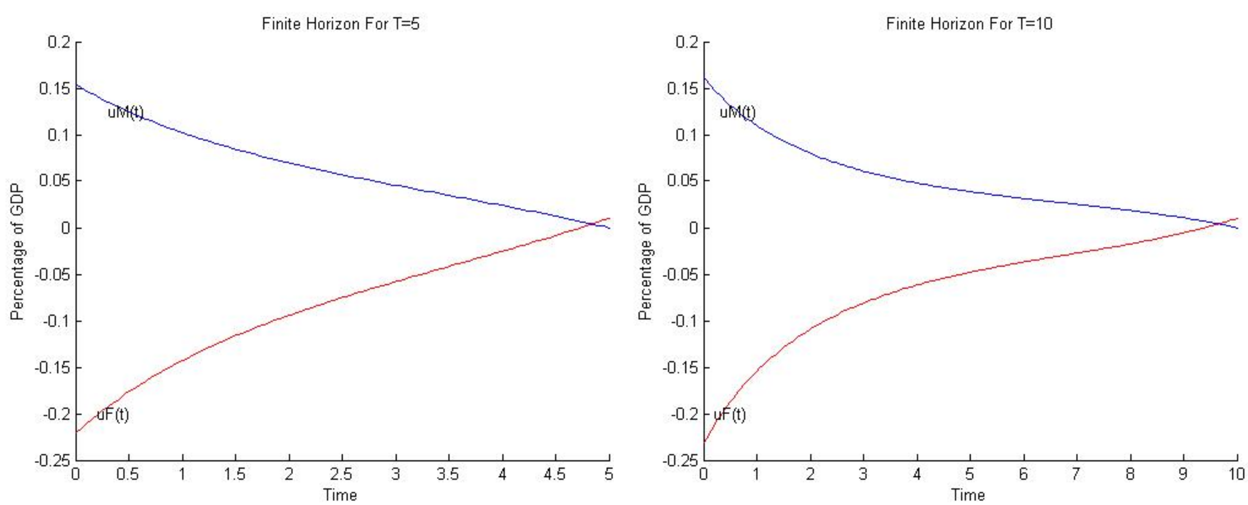
Figures 3.14 & 3.15: Steady state value of debt for $\bar{f} > \bar{m}(left)$ and $\bar{f} < \bar{m}(right)$

3.3.4.3 Effect of the Risk Premium on the Terminal Value Debt

However, as well as the steady-state analysis for long enough horizons, the analysis for shorter horizons is equivalently important. We've already argued that, depending on the global economic situation, the authorities might not prefer to bind themselves in such long-term agreements. Therefore we explore shorter horizons. We begin by presenting again the trajectories for two cases. First for a short horizon T=5 and second for an extended horizon of T=10. In both cases the debt neither reaches as low levels as in longer horizons, nor has a clearly steady part in the trajectory. One could say that the trajectory of debt here, is attracted by the steady-state but the horizon is too short, and it immediately deviates as well as the control policies, in order to meet the transversality conditions for the costate variable. To analyze the effect of the risk premium term depending on the rate of change of the debt in this case, we will examine what is the impact on the final value of debt. This is due to the fact that there is no steady-part long enough to justify domination throughout the horizon.

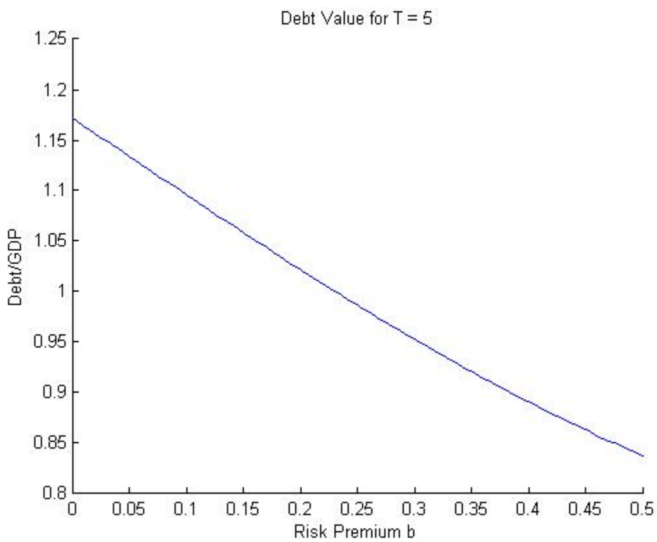


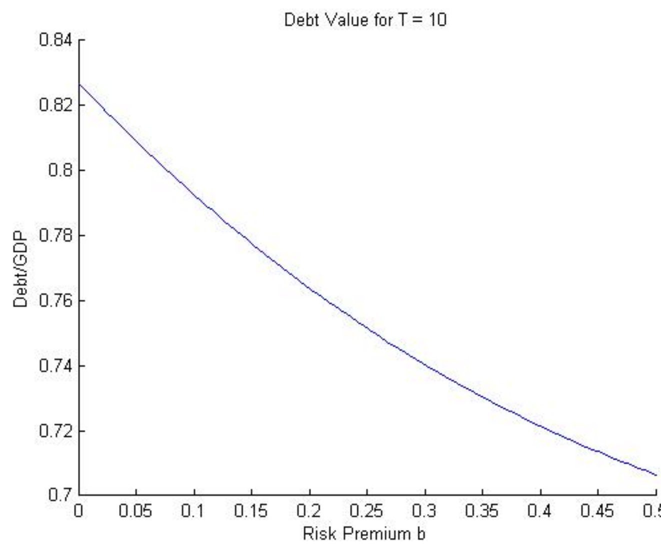
Figures 3.16 - 3.17: Finite Horizon Trajectories of Debt, Costate Variables for shorter finite horizons



Figures 3.18 - 3.19: Finite Horizon Trajectories of Optimal Policies for various values of finite horizons

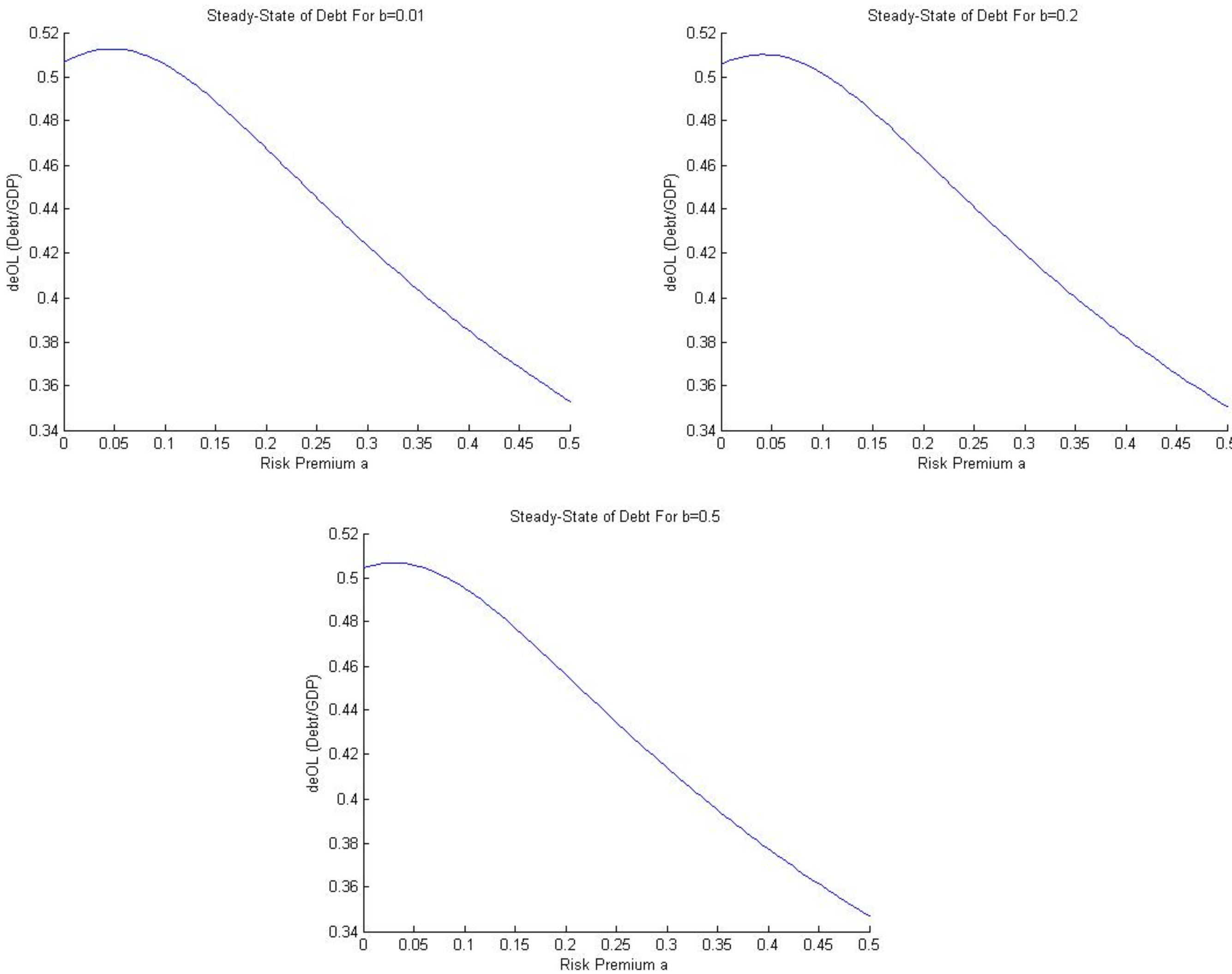
The risk premium term we introduced, even in shorter finite horizons, does decrease the level of debt as it increases. However, the values reached are not even near the steady state values or the ones achieved by longer horizons. An important feat though is how the debt is able to decrease, especially as b increases, in such short horizon. Keep also in mind that due to the transversality condition, which has to be satisfied, this is not the lowest value of debt throughout the horizon. This denotes the significance of the new term introduced, as it implements the pressure that the financial markets want to add, but also operates as a supporting mechanism for countries that are active and try to decrease their debt levels.





<u>Figures 3.20 - 3.21:</u> Effect of risk premium term b on the debt value at the end of the horizon

Another parameter to consider now is that of the risk premium term depending on the debt level. Engwerda, van Aarle et al [2] investigated this case as we have already mentioned, but now the dynamics of the system are changed and it would be appropriate to check again the effect of a on the steady state. From Figures 3.16-3.18, the first thing we have to mention is that the effect on the steady state value is no more linear as a increases, in contrast to [2]. We can observe easily the growth in steady state value of debt for small values of a, and then the decline that we described. Interest here focuses on the value of a that makes (3.10) equal to zero. It is there that the above curve obtains its maximum. One could argue that it's a value that should be avoided as it yields the maximum steady-state debt in the Open-Loop mode. This value can be obtained by solving the quadratic equation w.r.t a and choosing the positive solution. In addition, another fact we stated can be observed as well. As the value of b increases, this effect wares off by the dominance of the risk premium term depending on the rate of change of debt, on the term depending on the debt level.



<u>Figure 3.22, 3.23 & 3.24:</u> Effect of risk premium term a on steady state value of debt for various values of b

Next we present the changes as the two players become hasty, i.e. they do not care about the future development of the debt. That is for an increased value of $\theta, \theta = 0.75$. Theoretically we expect the steady state of debt to increase as θ increases. This is confirmed and depicted in Fig. 3.21. It is normal if we take a look at the cost functions (3.2). As θ increases the exponential term decreases faster. In fact it decreases too fast representing authorities that care significantly less about how the state of debt develops in the future. Also, we observe that the effect a has for small values is augmented in Fig. 3.21. This can be explained by equation (3.11). The second term containing $\theta - 3\bar{r}$ is much greater than the rest, thus dominating and resulting in a positive derivative w.r.t. a with relatively greater value than the respective values in the other cases. Thus, the spiking effect is enhanced as shown in Fig. 3.21.

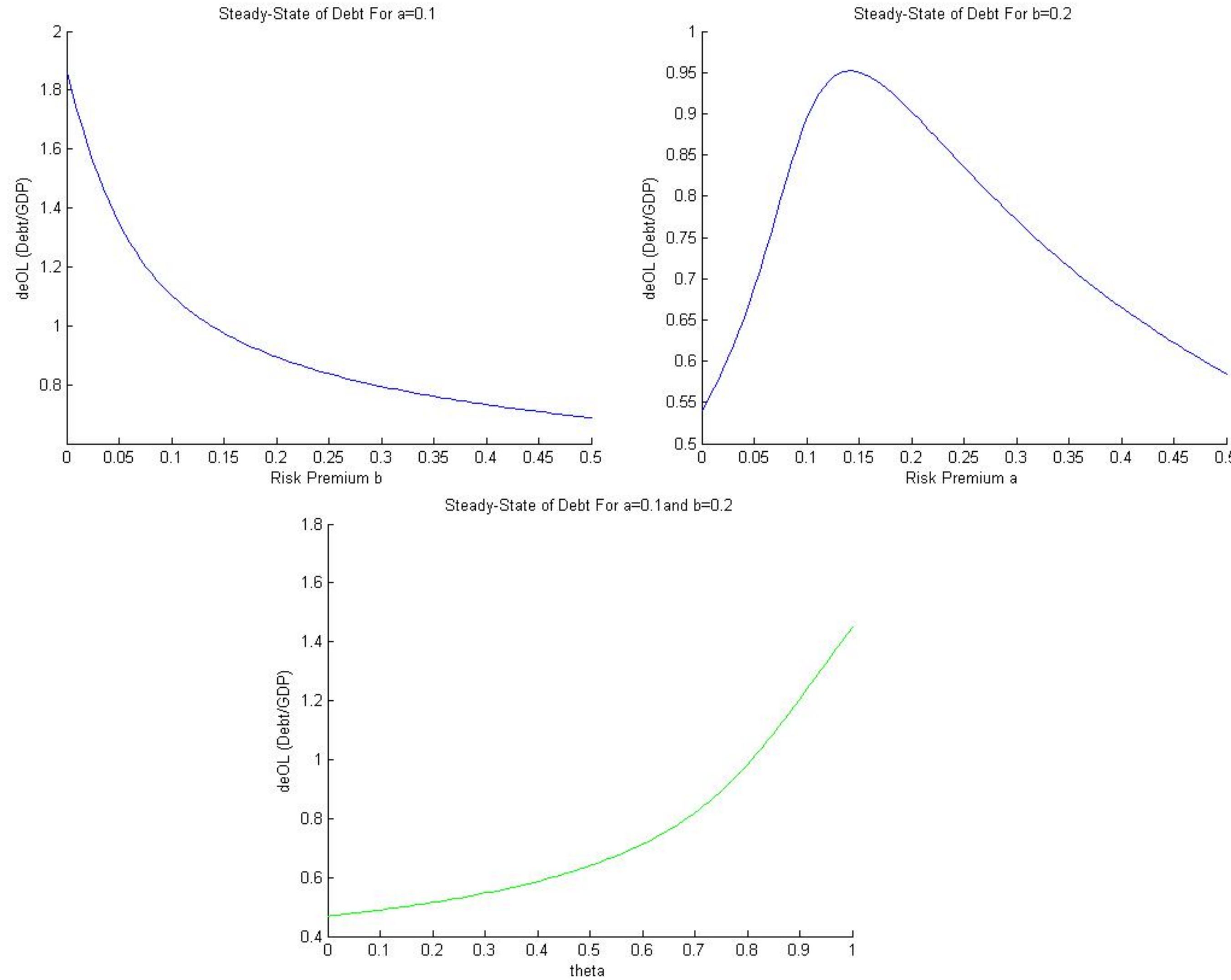


Figure 3.25, 3.26 & 3.27: Effect of parameter ϑ on steady state value of debt.
Left: How it affects the behavior as b varies
Right: How it affects the behavior as a varies
Bottom: Direct Effect for $a = 0.10$ *and* $b = 0.20$

3.3.4.4 Effect of the Risk Premium on the Control Policies

Last, we provide examples on the effect that risk premia have on the steady-state policies. In the previous section it was stated that the more the risk premium term on the level of debt increases, the more active the player gets, implying greater values for the respective policy. These behaviors are representative not only when the steady-state path is reached, but also for smaller horizons where the debt trajectory enters or approaches the steady state for a while. Target values are set to $\bar{f} = 0, \bar{m} = 0.01$ for this simulation. The results derived in the previous section (3.3.3) are depicted here. The increase of the term of risk premium depending on the level of debt results in more active policies from both players. Intuitively, this represents the fact that, when markets add more pressure to the countries by demanding great values of risk premium on the debt level, then the authorities are more alert and active trying to restrict the level of debt within acceptable standards. The more this risk premium term increases, the more the players have to lose by maintaining high debt levels, and thus they act to lower it (Fig. 3.22).

One the other hand, when the risk premium term on the rate of change of debt increases the players become slightly less active. A justification of this is that when the debt is declining, the risk premium term b is acting as a reward by decreasing the real interest rate. This is helping the debt level to decrease and be contained more easily, allowing the players to relax slightly (Fig. 3.23).

In significantly smaller horizons where we analyzed the final value for debt, it is straightforward from the equations derived for the optimal policies (3.4):

$$u_F^* = \bar{f} - \mu_F^* \frac{1}{1 - b\,d},\ u_M^* = \bar{m} + \mu_M^* \frac{1}{1 - b\,d}$$

At the end of the horizon, where $\mu_F(T) = \mu_M(T) = 0$, the policies reach their respective target values $\bar{f}, \bar{m}$.

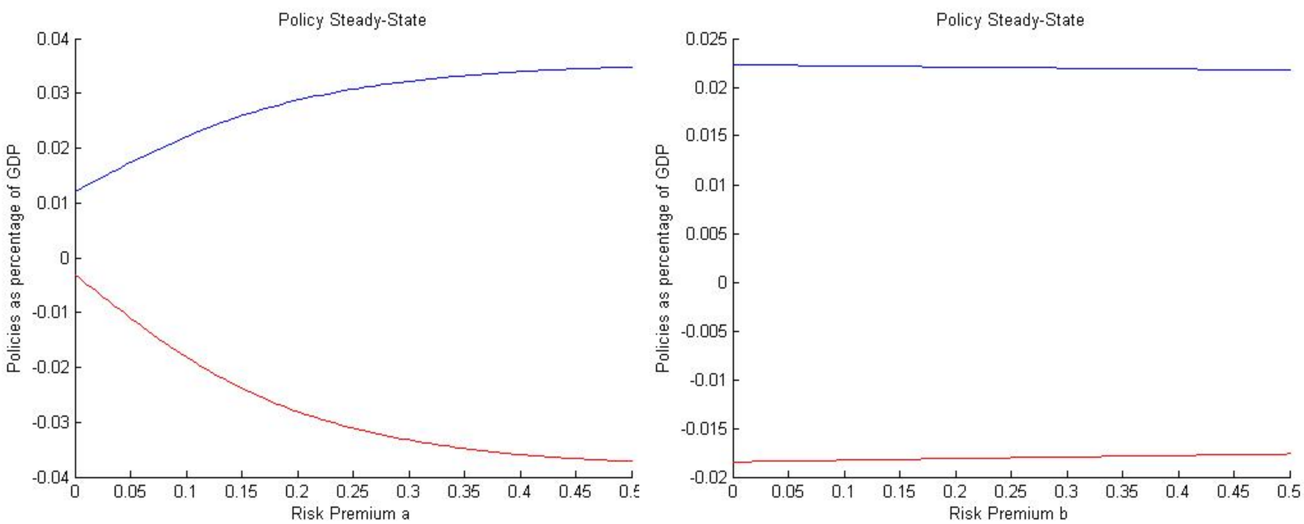


Figure 3.28, 3.29: Impact of risk premia on optimal policy steady-states.
Left: Impact of risk premium depending on debt level (a)
Right: Impact of risk premium depending on rate of change of debt (b)
Red represents fiscal deficits, Blue represents money growth

4 THE CO-OPERATIVE CASE

In this section we consider the game (1.a)-(1.d) under the Co-operative case. That means, the players communicate and seal agreements to achieve their goals. The cost of each player is not defined a priori, but after the cooperative talks take place. That means it is dependent on the bargaining strength of each player. In our case, the two-player game we denote the bargaining strength of the fiscal authorities with $\omega, 0 < \omega < 1$. Consequently the strength of monetary authorities is $(1-\omega)$. Thus, the following parameterized optimal control problem is at hand:

$$\min_{u_F,u_M} J, \quad J = \omega\, J_F \ + (1-\omega) J_M \tag{4.1}$$

$$= {}^1\!/_2 \int_0^T e^{-\theta t}\, \omega \left(\left(u_F(t) - \bar{f}\right)^2 + \beta_F \left(d(t) - \overline{d_F}\right)^2 \right) dt$$
$$+ {}^1\!/_2 \int_0^T e^{-\theta t}\, (1-\omega) \left((u_M(t) - \bar{m})^2 + \beta_M \left(d(t) - \overline{d_M}\right)^2 \right) dt$$

Subject to the government budget constraint discussed in the previous section:

$$\dot{d}(t) = \frac{\bar{r}\, d(t) + a\, d(t)^2 + u_F(t) - u_M(t)}{1 - b\, d(t)} \tag{4.2}$$

We are looking for Pareto efficient solutions. The equilibrium strategy $\hat{\gamma}$, chosen from the set of admissible strategies must satisfy:

$$\begin{gathered} If\ for\ any\ other\ strategy\ \gamma\ within\ the\ set\ of\ admissible\ strategies: \\ the\ set\ of\ inequalities\ J_i(\gamma) \le J_i(\hat{\gamma})\ , i = F, M\ with\ at\ least\ one \\ of\ them\ being\ strict, has\ no\ solution\ for\ any\ \gamma, then\ strategy \\ \hat{\gamma}\ is\ Pareto\ Efficient \end{gathered}$$

The dynamic game has now transformed into an optimal control problem from an analytical point of view. By solving the optimal control problem (4.1)-(4.2) for every $\omega \in (0,1)$ a curve of Pareto Efficient solutions is obtained. But which curve will be selected by the player, is defined by the choice of the parameter ω, i.e. the coordination of the system. Note that a Pareto solution is not unique. Since our game under the Co-operative case belongs to the class of optimal control problems, it can be solved using Pontryagin's Minimum Principle. As in the Non-Cooperative case we first prove that the conditions of Pontryagin's Minimum Principle are not only necessary but also sufficient for any $\omega \in (0,1)$ for which the candidate controls are obtained.

<u>Theorem 4.1:</u> The sets of strategies U_F, U_M are intervals of $\mathbb{R}$ and the Hamiltonians are convex in d. Then the conditions derived by using Pontryagin's Minimum Principle are also sufficient and thus the pair $({u_F}^*, {u_M}^*)$ satisfying them is a Nash Equilibrium.
Proof of the above theorem is provided in the Appendix (6.2).

- Applying Pontryagin's Minimum Principle we obtain:
  - Hamiltonian:
$$H = \frac{1}{2} e^{-\theta t} \omega \left(\left(u_F - \bar{f} \right)^2 + \beta_F \left(d - \overline{d_F} \right)^2 \right) + \frac{1}{2} e^{-\theta t} \, (1-\omega) \left((u_M(t) - \bar{m})^2 + \beta_M \left(d(t) - \overline{d_M} \right)^2 \right) + \lambda \, \frac{\bar{r}\, d + a\, d^2 + u_F - u_M}{1 - b\, d}$$
  - $\dot{\lambda} = - {}^{\partial H}/_{\partial d}, \lambda(T) = 0$
  - ${u_F}^* = arg \min_{u_F} \{H\}$
  - ${u_M}^* = arg \min_{u_M} \{H\}$
  - $\left({u_F}^* - \bar{f} \right) = -\frac{1-\omega}{\omega} ({u_M}^* - \bar{m})$

As in the previous section we define: $\mu = e^{\theta t} \lambda$. This is a transformation of the costate variable we use for computational ease. Using the results from Pontryagin's Minimum Principle we prove the following in the Appendix (6.2):

<u>Theorem 4.2:</u> If $({u_F}^*, {u_M}^*)$ is a set of Pareto Efficient strategies that are obtained for a choice of bargaining power $\omega \in (0,1)$ for the optimal control problem (4.1)-(4.2), there exist a trajectory for debt d^* and an associated costate variable μ^* that satisfy the set of nonlinear differential equations:

$$\dot{d}^*(t) = \frac{1}{1 - b\, d(t)} \left(\bar{r}\, d(t) + a\, d(t)^2 - \mu \frac{1}{\omega(1-\omega)(1 - b\, d(t))} + \bar{f} - \bar{m} \right) \tag{4.3.a}$$

$$\dot{\mu}^*(t) = -\left(\omega \beta_F \left(d(t) - \overline{d_F} \right) + (1-\omega) \beta_M \left(d(t) - \overline{d_M} \right) \right) + \mu \left(\theta - \frac{\left(\bar{r} + 2a\, d(t) \right)\left(1 - b\, d(t) \right) + b\left(\bar{r}\, d(t) + a\, d^2 + \bar{f} - \bar{m} \right)}{\left(1 - b\, d(t) \right)^2} \right) + \mu^2 \frac{\mathrm{b}}{\omega(1-\omega)\left(1 - b\, d(t) \right)^3} \tag{4.3.b}$$

And the optimal control laws derived: $${u_F}^* - {u_M}^* = \bar{f} - \bar{m} - \mu \, \frac{1}{\omega(1-\omega)(1 - b\, d)} \tag{4.4}$$

With $d^*(0) = d_0$. Also, the system of differential equations (4.3) admits no periodic solutions as we prove in the Appendix (section 6.2).

Again, now assume that the horizon is long enough so that equilibrium is reached. Then the Co-operative equilibrium points are (d_e^{CO}, μ_e^{CO}). These points satisfy the following equations:

$$\mu_e^{CO} = \omega(1-\omega) \left(-ab(d_e^{CO})^3 + (a - b\bar{r})(d_e^{CO})^2 + \left(\bar{r} - \, b\left(\bar{f} - \bar{m} \right) \right) \, d_e^{CO} + \left(\bar{f} - \bar{m} \right) \right) \tag{4.5}$$

Now, replacing μ_e^{CO} to the second equilibrium equation (4.3.b) we get:

$$\delta_3\ (d_e^{CO})^3 + \delta_2\ (d_e^{CO})^2 + \delta_1\ d_e^{CO} + \delta_0 = 0 \tag{4.6}$$

Defining: $\delta_3 = -\omega(1-\omega)a(2\alpha + b\theta), \delta_2 = \omega(1-\omega)\big(a\,\gamma_2 - \bar{r}(2\alpha + b\theta)\big)$,
$\delta_1 = \omega(1-\omega)(\bar{r}\,\gamma_2 - \bar{u}(2\alpha + b\theta) - (\omega\beta_F + (1-\omega)\beta_M)\ , \delta_0 = \omega(1-\omega)\bar{u}\,\gamma_2 + \omega\beta_F\overline{d_F} + (1-\omega)\beta_M\overline{d_M}$

Where: $\gamma_1 = 1 - b\ d_e^{OL}$, $\gamma_2 = \theta - \bar{r}$, $\gamma_3 = \beta_F\overline{d_F} + \beta_M\overline{d_M}$, $\bar{u} = \bar{f} - \bar{m}$

Equation (4.6), which defines the steady state value of debt, is a cubic equation. It is natural that only real roots of (4.6) are considered and accepted as solutions of the game, due to the fact that they represent debt value. It is easily observed that when we assign ${\beta_F}^{CO} = (1-\omega){\beta_F}^{OL}$ and ${\beta_M}^{CO} = \omega{\beta_M}^{OL}$ the cooperative equilibrium cubic polynomial (4.6) is the open-loop equilibrium cubic polynomial (3.5) multiplied by $\omega(1-\omega)$. This indicates that $d_e^{CO} = d_e^{OL}$ and equilibrium equation (4.5) gives $\mu_e^{CO} = \omega(1-\omega)\mu_e^{OL}$. Thus, the steady state value debt is the same in both non-cooperative Open-Loop and Cooperative case using the aforementioned weights assigned by the two authorities for the development of the debt.

Also, we speculate that the same conjecture we made in Non-Cooperative case, about how the steady-state attracts the trajectory of the state variables, will also apply here. In the same way we commented on, on the Non-Cooperative case, the equations from which steady-state value of debt is obtained is the same in the Finite Horizon and in the Infinite Horizon. Therefore, we hypothesize that there exists $\mathrm{T}' < \mathrm{T}$ such that for some t, $\mathrm{t} < \mathrm{T}'$, the debt trajectory of the Finite Horizon enters a steady-value path, in the sense that the value of debt remains constant. The debt trajectory in both the Finite and Infinite Horizon cases is attracted by the same steady-state value as the expressions of the optimal policies for both cases are identical and therefore the trajectory of the Finite Horizon can get in a neighborhood of the steady-state value of the Infinite Horizon. However again, for some final time the trajectory of the Finite Horizon deviates from the aforementioned steady-state value as the optimal policies for the Finite Horizon deviate from those of the Infinite Horizon because the transversality condition for the costate variable $\lambda(\mathrm{T}) = 0$ needs to be met.

Again, we focus on the steady state of debt and on the effects which the additional risk premium (on the rate of change of debt) might have. Before that we recall the aforementioned results from literature:

4.1 THE TABELLINI MODEL: NO RISK PREMIA

First, for reference we present the results of the Tabellini model. That is, in our control problem (4.1)-(4.2) we set the parameters $a = 0 \; and \; b = 0$. Also $\overline{d_F} = \overline{d_M} = 0$ was considered and the following conclusions had been derived:

1. The co-operative equilibrium has always higher speed of adjustment than the Nash equilibrium.
2. The co-operative equilibrium has always lower steady state value of debt than the Nash equilibrium.
3. There are unique equilibrium policies, for every initial debt, choice of parameters and bargaining power of the players. Furthermore, the deviations of the policy of each player from their respective targets, are proportional to each other (the constant of proportionality being ω).
4. Hence, in contrast with the non-cooperative case for every initial parameter the problem has a solution.

4.2 ENGWERDA AND VAN AARLE: RISK PREMIUM ON DEBT LEVEL

Engwerda and van Aarle extended the Tabellini model by introducing a parameter of risk premium depending on government debt level (in eq. (4.2) $a > 0 \; and \; b = 0$). Also, they confirm too that when ${\beta_F}^{CO} = (1-\omega){\beta_F}^{OL}$ and ${\beta_M}^{CO} = \omega {\beta_M}^{OL}$, we have the same steady state value of debt. And then, using these values the results for the steady states are the same:

$$-2a^2 d^3 + a(\theta - 3\bar{r})d^2 + \gamma_1 d + \gamma_0 = 0$$

$$with \; \gamma_0 = \beta_F \overline{d_F} + \beta_M \overline{d_M} + (\bar{f} - \bar{m})(\theta - \bar{r}) \; and \; \gamma_1 = (\bar{r}(\bar{r} - \theta) - \beta_F - \beta_M) - 2a(\bar{f} - \bar{m})$$

With a discriminant: $h(a) = 8{\gamma_1}^3 - 36a\gamma_1\gamma_0 s - 40a\gamma_0 s^3 - 108a^2{\gamma_0}^2 + {\gamma_1}^2 s^2, \quad s = \theta - 3\bar{r}$. Hence we have:

1. One steady state of debt if either $i) \; h(a) < 0 \; or \; ii) \; h(a) = 0 \; and \; a^2 s^2 = 3\gamma_1$.
2. Two steady states (from which only one applies as an open loop equilibrium if $d_0 = d^e$) if $h(a) = 0 \; and \; a^2 s^2 \neq 3\gamma_1$.
3. Three steady states (from which the middle one applies as an open loop equilibrium if $d_0 = d^e$), if $h(a) > 0$.
4. Of these steady states, at least one is a saddle-point, denoting the impact of including risk premia.

4.3 INTRODUCING: RISK PREMIUM ON DEBT RATE OF CHANGE:

It is clear that risk premia coefficients a and b can play major role defining the equilibrium points in the Co-operative case as well. It is important then to see how the equilibrium of the system responds in variations of those to coefficients and extract a qualitative conclusion, if possible. We will also explore the effect of the changing bargaining strength ω. The equilibrium points are defined by equation (4.6), a cubic polynomial and thus an analytical solution is available.

4.3.1 Effect on Number of Equilibrium Points and Qualitative Behavior

In the Appendix (Section 6.2.2-6.2.4) we prove the following:

- If $\Delta(a, b)$ is the discriminant of the cubic polynomial (4.6), as a function of the risk premium coefficients $a \; and \; b$, then the number of steady state values of debt given by (4.6) is subject to the following rule:

  - If $\Delta(a, b, \omega) > 0$, there are *3 distinct real roots* and hence, *3 candidate solutions* for the game.
  - If $\Delta(a, b, \omega) < 0$, there is *1 distinct real root and 2 complex conjugate roots*. Hence, *1 candidate solution* for the game.
  - If $\Delta(a, b, \omega) = 0$, there are *at least 2 roots coincide and they are all real*. That means there are either a double real root and distinct single real root, or a triple real root. That is we either have 2 or 1 candidate solutions for the game respectively.

$$\Big(\omega(1-\omega)\big(\bar{r}\,\gamma_2 - \bar{u}(2a + b\,\theta)\big) - (\omega\beta_F + (1-\omega)\beta_M)\Big)\big(a\,\gamma_2 - \bar{r}(2a + b\,\theta)\big)$$
$$+9a(2a + b\,\theta)\big(\omega\beta_F\overline{d_F} + (1-\omega)\beta_M\overline{d_M} + \omega(1-\omega)\bar{u}\,\gamma_2\big) = 0 \tag{4.7}$$

  If (4.7) holds there is *1 triple real root*, else there is *1 double root and 1 distinct real root*.

- The qualitative behavior of the system near the equilibrium points should be explored. This is achieved via the Linearized System around each equilibrium point. Linearizing system (4.3) we get:

$$\begin{pmatrix} \dot{d(t)} \\ \dot{\mu(t)} \end{pmatrix} = \begin{pmatrix} -\frac{a}{b} - \frac{2\,b\,\mu_e^{CO}}{\gamma_1{}^3(1-\omega)\omega} + \frac{(a+b\,\bar{r}+b^2\,\bar{u})}{b\,\gamma_1{}^2} & -\frac{1}{\gamma_1{}^2(1-\omega)\omega} \\ -(\omega\beta_F + (1-\omega)\beta_M) - \frac{2\,(a+b\,\bar{r}+b^2\,\bar{u})\,\mu_e^{CO}}{\gamma_1{}^3} + \frac{3(b\,\mu_e^{CO})^2}{\gamma_1{}^4(1-\omega)\omega} & \frac{a}{b} + \theta + \frac{2\,b\,\mu_e^{CO}}{\gamma_1{}^3} - \frac{(a+b\,\bar{r}+b^2\,\bar{u})}{b\,\gamma_1{}^2(1-\omega)\omega} \end{pmatrix} \begin{pmatrix} d(t) \\ \mu(t) \end{pmatrix} \tag{4.8}$$

  where (d_e^{CO}, μ_e^{CO}) is the equilibrium point under discussion.

The eigenvalues of the linearized system (4.8) are:

$$eig_{1,2} = \frac{(1-bd_e^{CO})^3\,\theta \pm \sqrt{\frac{(1-bd_e^{CO})^4}{\omega(1-\omega)}\left(\omega(1-\omega)(1-bd_e^{CO})^2\,\theta^2 - 4\,{}^{\partial\, g(d)}/_{\partial d}\right)}}{2(1-bd_e^{CO})^3} \tag{4.9}$$

It is pretty straightforward that $\frac{(1-bd_e^{CO})^4}{\omega(1-\omega)} > 0$. Hence, the sign of the quantity under the root and consequently the type of equilibria are subject to the following rules:

- If $\omega(1-\omega)(1-bd_e^{OL})^2\,\theta^2 - 4\,{}^{\partial\, g(d)}/_{\partial d} > 0$ and ${}^{\partial\, g(d)}/_{\partial d} < 0$, then $eig_1 > 0$ and $eig_2 < 0$. Thus, the steady state is a saddle point.

- If $\omega(1-\omega)(1-bd_e^{OL})^2\,\theta^2 - 4\,{}^{\partial\, g(d)}/_{\partial d} > 0$ and ${}^{\partial\, g(d)}/_{\partial d} > 0$, then $eig_1 > 0$ and $eig_2 > 0$. Thus, the steady state is an unstable node.

- If $\omega(1-\omega)(1-bd_e^{OL})^2\,\theta^2 - 4\,{}^{\partial\, g(d)}/_{\partial d} < 0$, then $eig_{1,2}$ are complex eigenvalues with positive real part. Thus, the steady state is an unstable focus.

- Last but not least, there an eigenvalue could become zero. That is, when $\Delta(a,b,\omega) = 0$ and (4.7) holds (a triple real root). As we have already argued the equilibrium point is also a root of the first derivative. It is then straightforward from (4.7) that an eigenvalue becomes zero while the other will be positive. This means the system matrix has a non-trivial null-space and has an equilibrium subspace than an equilibrium point. Due to the positive eigenvalue all trajectories diverge away from the equilibrium subspace.

➢ Combining the two previous bullets we conclude to the following proposition:

The highest order coefficient $\delta_3 = -\omega(1-\omega)a(2a + b\,\theta)$ of (4.6) is always negative since $a, b, \theta > 0$ and $\omega \in (0,1)$. Thus, depending on how many equilibrium points we have we can determine the sign of ${}^{\partial\, g(d)}/_{\partial d}$ near each equilibrium point:

- If $\Delta(a,b,\omega) > 0$, there are *3 distinct real roots*. The first and the last are *saddle points*, while the middle one is *unstable node or focus*.
- If $\Delta(a,b,\omega) < 0$, there is *1 distinct real root* and 2 complex conjugate roots. The derivative ${}^{\partial\, g(d)}/_{\partial d} < 0$ and the equilibrium point is a *saddle point*.
- If $\Delta(a,b,\omega) = 0$, and (4.7) does not hold, there is *1 double real root* and *1 distinct single real root*. The double root is *unstable* while the single one is a *saddle point*.

- If $\Delta(a,b,\omega)=0$, and (4.7) does hold there is *1 triple real root* and we have an *equilibrium subspace* than an equilibrium point, and *all trajectories diverge away* from the equilibrium subspace.

As we can see the qualitative behavior when playing under cooperative mode, does not change. However, the bargaining power of each authority can alter the number and type of equilibrium points, just like the risk premia coefficients, as it appears on both the discriminant of the equilibrium equation as well as in its partial derivative w.r.t. d.

4.3.2 Effect on development on Steady-State Value of Debt

It is now clear that risk premia coefficients $a\ and\ b$ can play major role in defining the equilibrium points. It is important then to see how the value of equilibrium of the system responds in variations to those coefficients and extract a qualitative conclusion, if possible.

We will want to examine the sign of the derivative of steady state value debt w.r.t. $a\ and\ b$. Under the assumption that ${}^{\partial\, g(d)}/_{\partial d} \neq 0$ at the steady state d_e^{OL} (for example, not having a triple root as we've argued before) and that $d_e^{OL}(a)\ and\ d_e^{OL}(b)$ are continuous differentiable in their respective variables we obtain:

$$ {}^{\partial\, d_e^{CO}(a)}/_{\partial a} = -\frac{{}^{\partial\, g}/_{\partial d}}{{}^{\partial\, g}/_{\partial a}} \qquad and \qquad {}^{\partial\, d_e^{CO}(b)}/_{\partial b} = -\frac{{}^{\partial\, g}/_{\partial d}}{{}^{\partial\, g}/_{\partial b}} $$

Simple calculations show:

- ${}^{\partial\, g}/_{\partial a}{}_{CO} = \omega(1-\omega)\, {}^{\partial\, g}/_{\partial a}{}_{OL}$ (4.10)
- ${}^{\partial\, g}/_{\partial b}{}_{CO} = \omega(1-\omega)\, {}^{\partial\, g}/_{\partial b}{}_{OL}$ (4.11)

Knowing that $\omega(1-\omega)$ is always positive for $\omega \in (0,1)$, exactly the same analysis as in section 3.3.2 applies here. We confirm this in the benchmarks that will follow.

In addition to the risk premia terms we will explore the effects of the coordination parameter ω of the system, i.e. the bargaining power of each player. Simple calculations on equation (4.6) show that:

$$ {}^{\partial\, d_e^{CO}}/_{\partial \omega} = -\frac{{}^{\partial\, g}/_{\partial d}}{{}^{\partial\, g}/_{\partial \omega}} $$

$$ {}^{\partial\, g}/_{\partial \omega} = -(1-2\omega)a(2a+b\theta)(d_e^{CO})^3 + (1-2\omega)\big(a\,\gamma_2 - \bar{r}(2\alpha+b\theta)\big)(d_e^{CO})^2 $$
$$ + \Big((1-2\omega)\left(\bar{r}\,\gamma_2 - \bar{u}(2\alpha+b\theta)\right) - \left(\beta_F - \beta_M\right)\Big)\, d_e^{CO} + (1-2\omega)\bar{u}\,\gamma_2 + \beta_F\overline{d_F} - \beta_M\overline{d_M} = 0 \qquad (4.12) $$

As we can see in general, the value of the coordination parameter ω plays a major role in the sign of (4.12). Also, the target values of each player, along with the respective weights act like an offset value in the derivative. Thus they affect too the way the debt is driven as ω varies. We will confirm numerically that for various set of parameters the steady state of debt goes from $\overline{d_M}$ (as ω starts near zero) to $\overline{d_F}$ (as ω approaches one), thus the slope with respect to ω, equivalently (4.12), is positive if $\overline{d_F} > \overline{d_M}$, and negative if $\overline{d_F} < \overline{d_M}$ and changes from positive to negative, as ω goes from zero to one, if $\overline{d_F} = \overline{d_M}$.

4.3.3 Benchmark Example

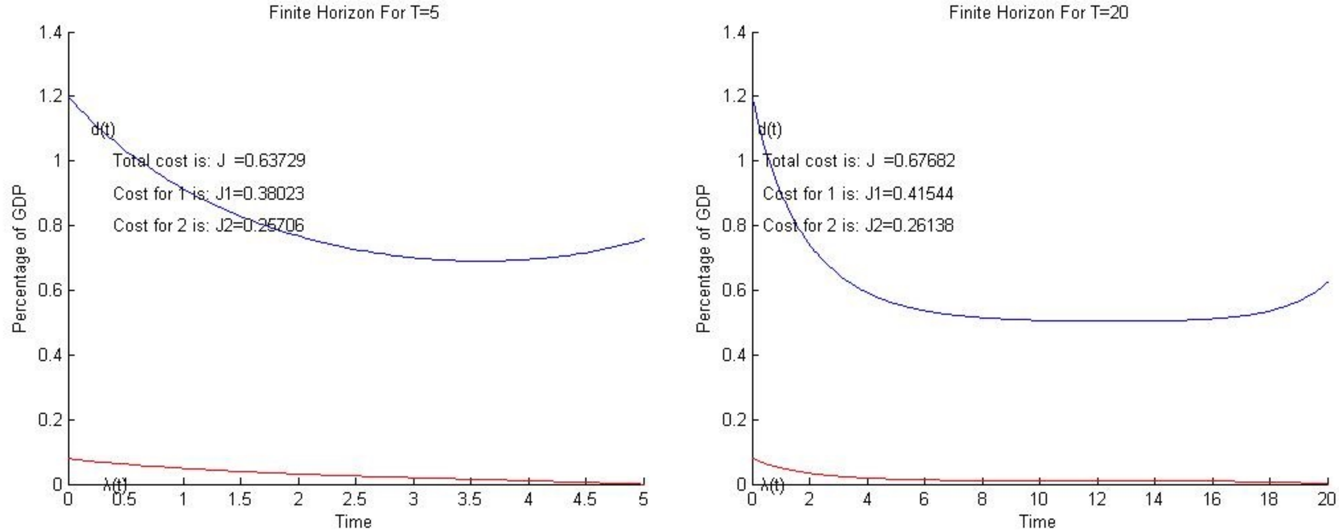
The parameters for our benchmarks are chosen again in accordance to *Note 2* (pg.27). That is the values of $a \; and \; b$ would be around 0.10 and 0.20 respectively given the current economic situation. Also, as stated in previous literature the weight values that the two authorities put on debt development, $\beta_F \; and \; \beta_M$, are of the same order of magnitude, that is 10^{-3}. The parameter θ, which denotes the degree in which the authorities care about the future development of the debt, will take two representative values: $\theta = 0.15$ when the players do care about the future debt development and $\theta = 0.75$ when they care less.

$$\beta_F = 0.06 \;\; , \beta_M = 0.04 \;\; , \overline{d_F} = \overline{d_M} = 0.5, \bar{r} = 0.03 \, , \theta = 0.15$$

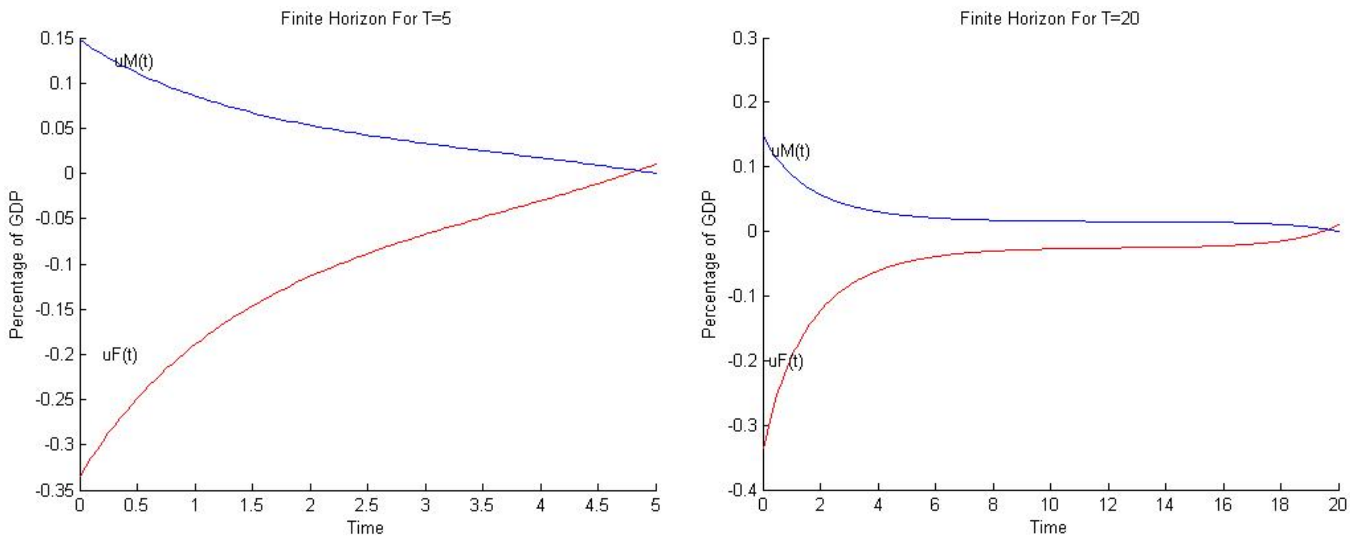
4.3.3.1 Trajectories of Debt, Costate Variables and Optimal Policies for the Finite Horizon

First of all, we present the trajectories for various values of the finite horizon T. We expect that the Cooperative case would yield better results than the Non-Cooperative. This expectation seems to be valid. Comparing Figures 4.1-4.4 with the respective ones from the Non-Cooperative case, Figures 3.1-3.4, it is observed that the cooperation yields two benefits that show up in these figures. First, the debt levels are lower for any given horizon. This decrease is significant as it is around 20% in the final debt value. Second, the trajectory enters the steady-state much faster. From T=20 (Fig.4.2) this is already evident, in fact the plot for T=20 in the Cooperative mode is quite similar to the one for T=30 in the Non-Cooperative mode. These plots were also obtained for $a = 0.10 \; and \; b = 0.20$, while the bargaining power was set to $\omega = 0.3$. The bargaining power is chosen in a way to depict the difference in power and influence of the monetary authority (ECB) and a particular fiscal authority, belonging to a country which could be part of the “south block”.

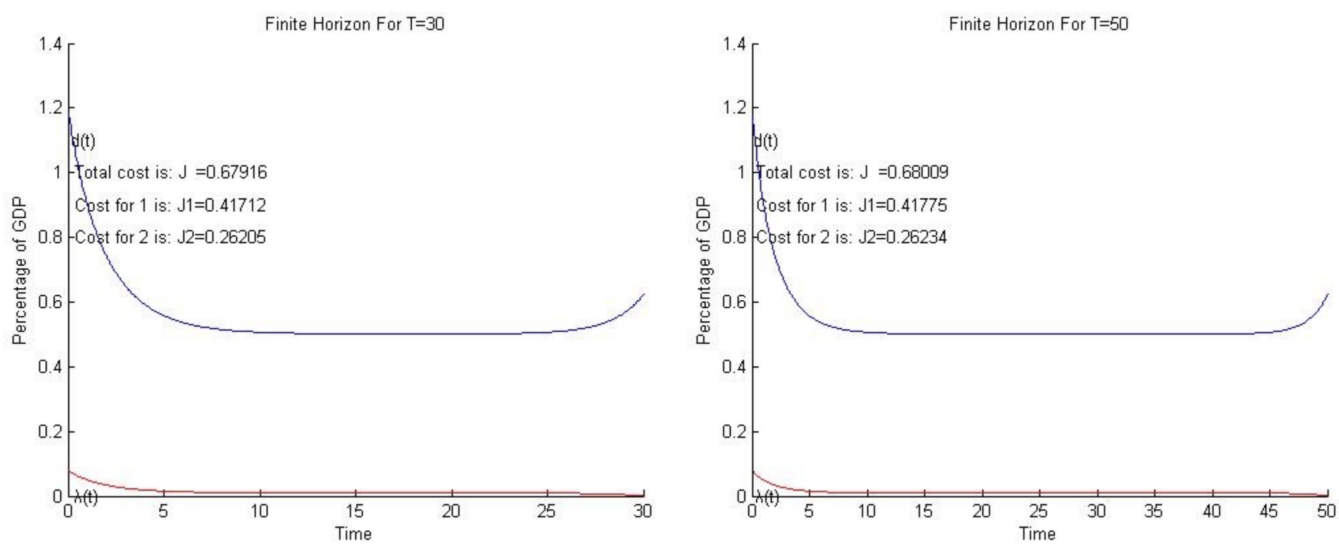
In addition, the cost of each player, compared to the ones in the Non-Cooperative case, is significantly less, namely more than 50% in the cases we present. For example, for T=5 in the Non-Cooperative case the costs were *J1=1.0226, J2=0.55978* while in the Cooperative case we obtain *J1=0.3802, J2=0.25706*. Such reduction in the players’ costs is apparent in all the other cases we present. Also, it is again evident that the accumulation of the cost as the horizon gets longer is significantly small, like in the Non-Cooperative case. This is due to the fact that the trajectory, as well as the controls, stay in the vicinity of a turnpike as long as possible. Therefore, both players do want to cooperate as they benefit from it (at least when ω=0.3, we will examine other cases as well on the next sections).



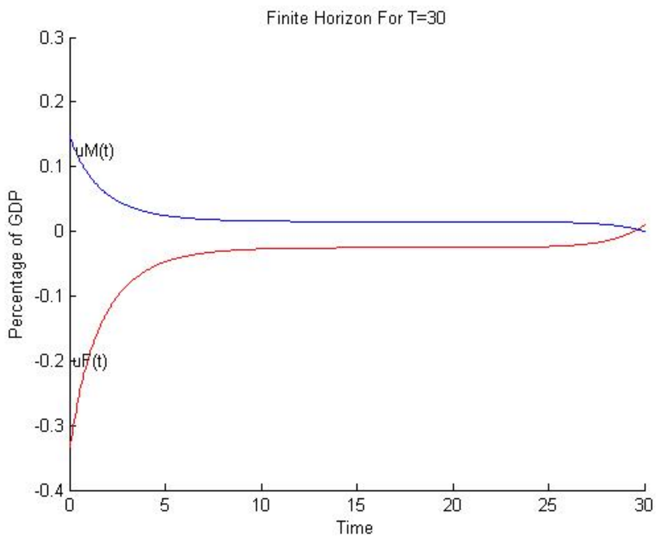
__Figures 4.1 - 4.2:__ Finite Horizon Trajectories of Debt, Costate Variables for finite horizons T=5 and T=20

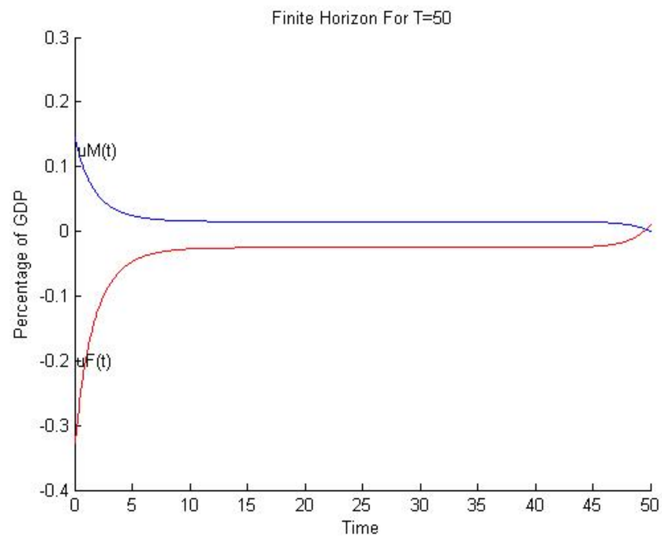


__Figures 4.3 - 4.4:__ Finite Horizon Optimal Policies for finite horizons T=5 and T=20



__Figures 4.5 - 4.6:__ Finite Horizon Trajectories of Debt, Costate Variables for finite horizons T=30 and T=50

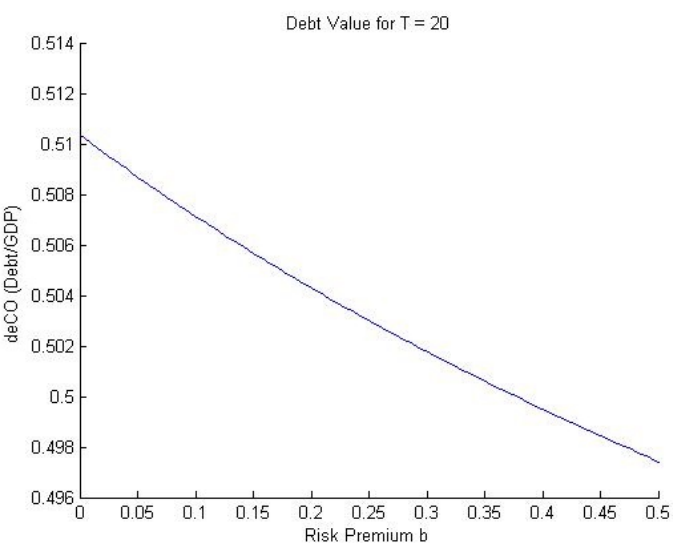


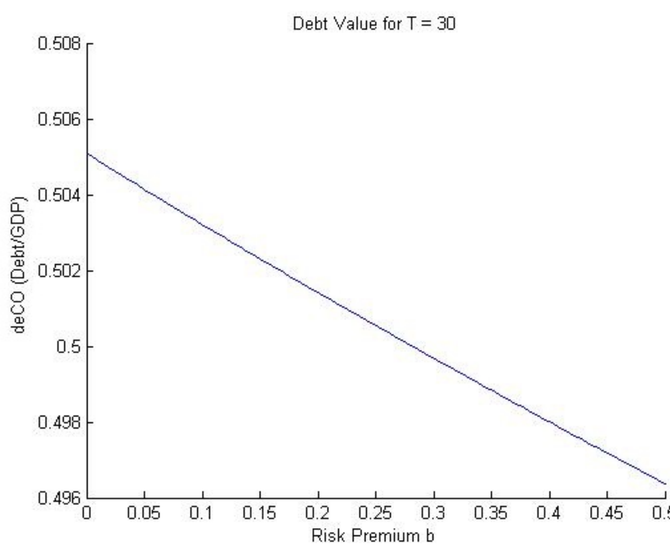


<u>Figures 4.7 - 4.8:</u> Finite Horizon Optimal Policies for finite horizons T=30 and T=50

4.3.3.2 Effect of the Risk Premium on Steady-Values of Debt

Again we will investigate how the varying values of risk premium term b effect the steady part and the final part of the debt trajectory. We expect, subject to our theoretical analysis, that the effect of the risk premium terms will have similar behavior to that of the Non-Cooperative case. Figures 4.9 through 4.11 show that the relation between the risk premium term b and the steady part of the debt trajectory is almost linear, as in the Non-Cooperative case and thus our results are validated. As b increases the value the debt attains decreases too, independently of the horizon. However now, it is clear that the value of debt achieved is significantly lower comparing to Figs 3.9-3.11. This indicates the benefits of the cooperating players. In similar way to the trajectories we observed in the previous section, the reduction in debt is again of considerable amount, namely around 30%.





<u>Figures 4.9 - 4.10:</u> Effect of risk premium term b on steady part of the debt trajectory

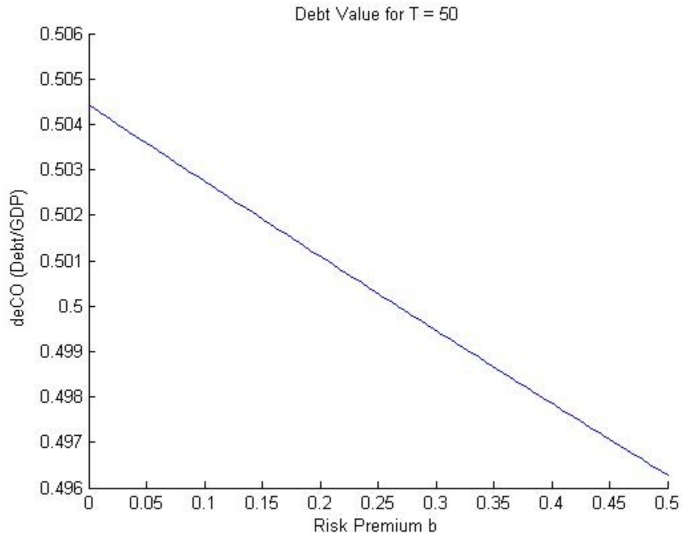
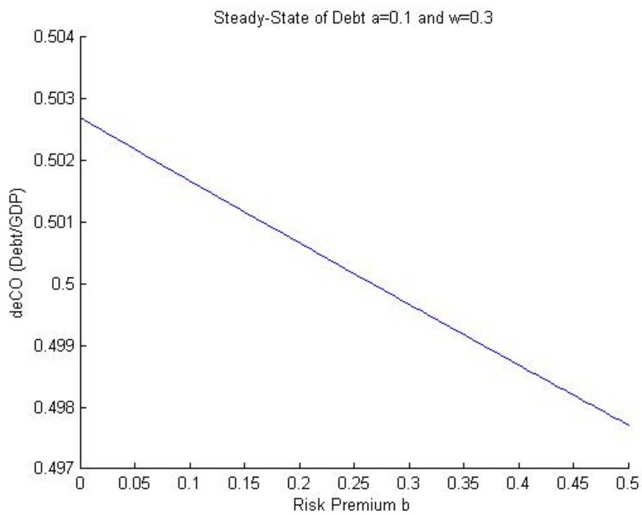
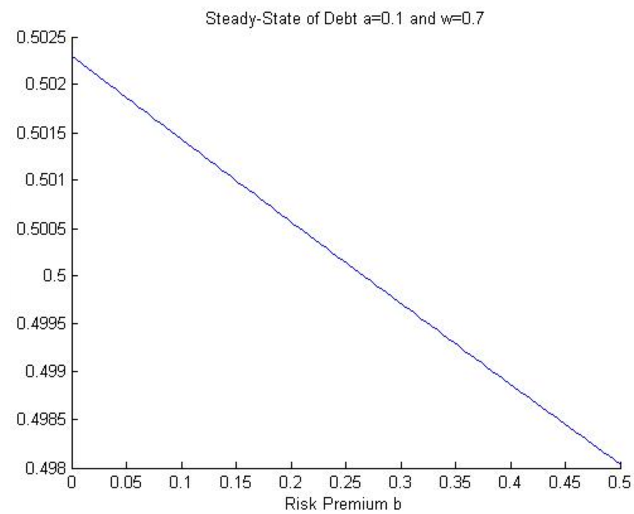


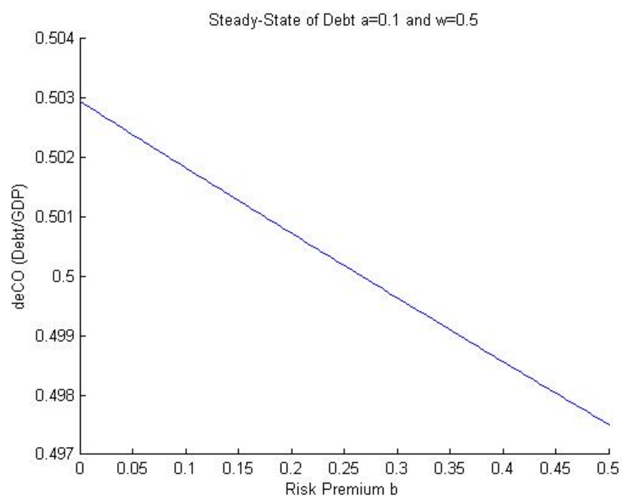
Figure 4.11: Effect of risk premium term b on steady part of the debt trajectory

In order to determine how close these results are to the steady-state we examine the infinite horizon case. That is the exact results obtained from the system of equations (4.3) or directly by solving (4.6) for a real valued solution. We begin by setting $\omega = 0.3$ denoting the superior bargaining power of the ECB. The target values for debt are now set $\overline{d_F} = 0.5 = \overline{d_M}$. We do so, in order to compare the results, for exactly the same parameters, during the non-cooperative Open-Loop Nash mode of play, and the Cooperative mode of play. Later on, we present how different target values can affect the results. Also, we will investigate the role of the bargaining power each player has. We will also explore various cases of the relative bargaining power between the two authorities. Namely we explore the cases of $\omega = 0.3,\ \omega = 0.5\ and\ \omega = 0$. Again it is important to compare the results yielded with the ones from the Non-Cooperative case. Taking a look to Figures 4.9-4.11 and recalling the respective plot from the Non-Cooperative case (Figure 3.9), one can see that for small values of b cooperation yields better results, i.e. smaller steady-state debt, than when the players are not cooperating. However, for larger values of b, non-cooperative play yields lower steady state value. We should state, that for the value of b around 0.20 (the reference value in crisis period) cooperation produces slightly better results, therefore it is suggested that the monetary authorities of the union should try and work together with the fiscal authorities of a country in benefit of both of them. Also, comparing Fig. 4.8 (where ω=0.3) with Figs 4.5-4.7 it is evident that for T=50 we have identical plots, meaning the trajectory is well within the steady-state, and also for T=30 the trajectory approaches really close to the steady-state with the difference being less than 1% of Debt/GDP. In contrast with Figures 3.5-3.7 of the Non-Cooperative case, it is clear that in the Cooperative mode of play we get closer to the steady-state in less time. This is evident from the fact that even for T=20 the debt levels achieved are 30% less and deviate from the steady-state values approximately 6%, whereas in the Non-Cooperative case this deviation was around 31% of Debt/GDP.





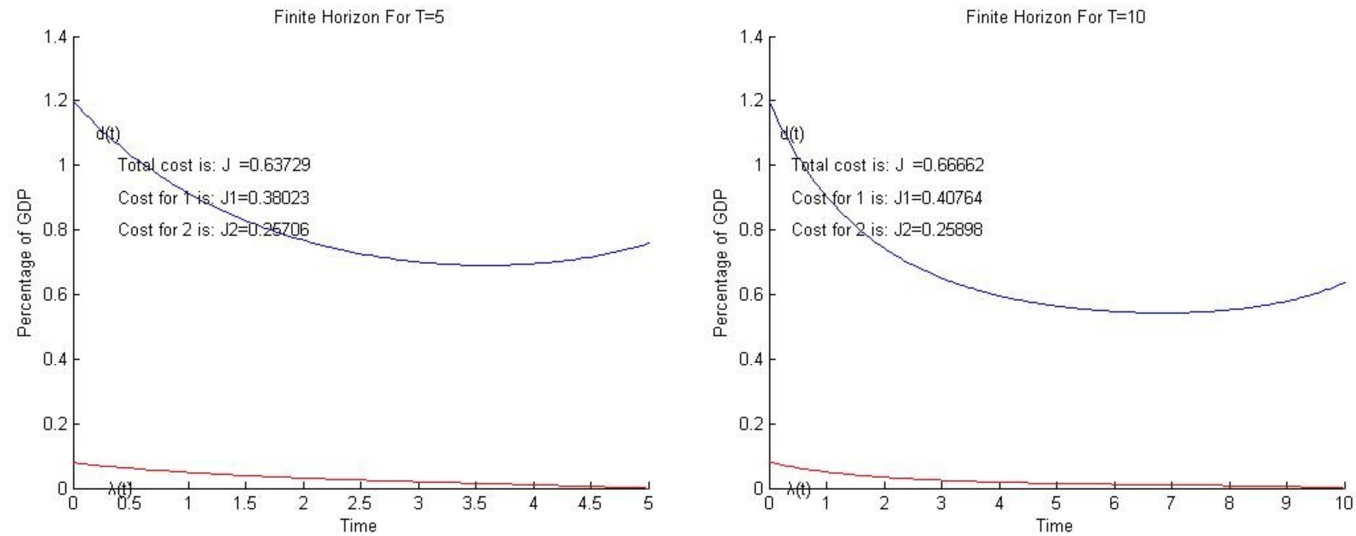
<u>Figures 4.12-4.13:</u> Effect of risk premium term b on steady state value of debt, same target for both players
Left: ω=0.3, Right: ω=0.7



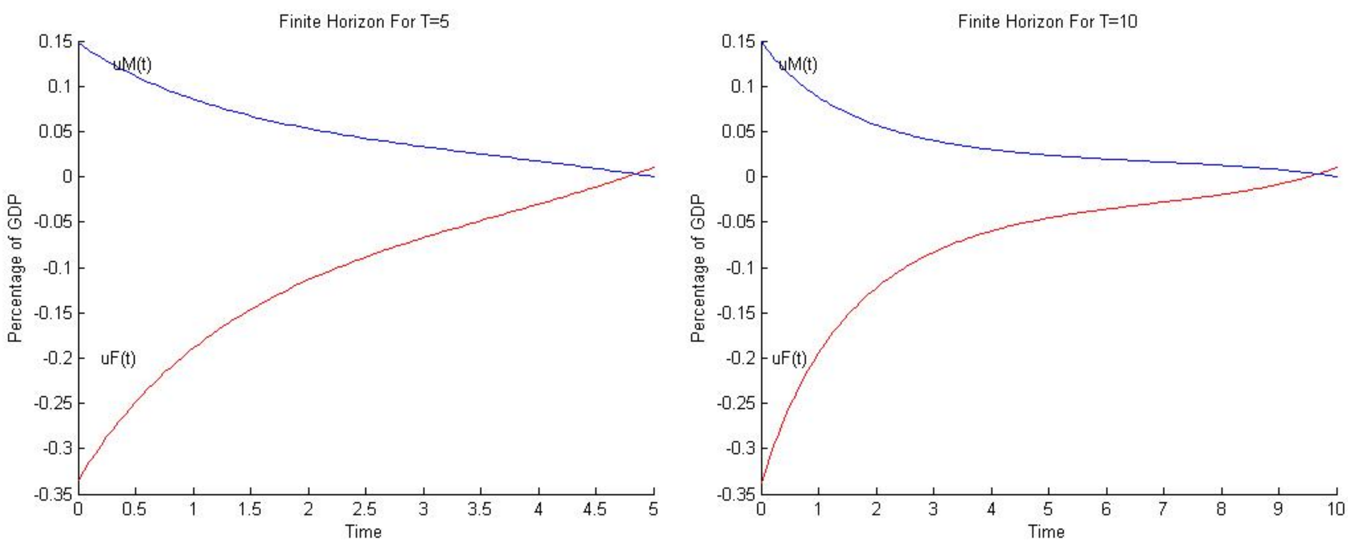
<u>Figures 4.14:</u> Effect of risk premium term b on steady state value of debt, same target for both players
ω=0.5

4.3.3.3 Effect of the Risk Premium on the Terminal Value Debt

It is also of equivalent significance to explore the effects of risk premia in shorter horizons when the steady-state is not approached as well as in longer horizons, and thus we cannot assume that the infinite case is representative in terms of behavior. We have already argued that, depending on the global economic situation, the authorities might not prefer to bind themselves in such long-term agreements. Therefore we explore shorter horizons. We begin by presenting again the trajectories for two cases. First for a short horizon T=5 and second for an extended horizon of T=10. Equivalently with the Non-Cooperative mode of play, the terminal value of debt in a short finite horizon neither reaches as low levels as in longer horizons, nor has a clearly steady part in the trajectory. Again the accumulation of the cost is small as when T=5 we obtain *Jtot=0.63729, J1=0.38023 and J2=0.25706*. When T=10 the corresponding values for cost are *Jtot=0.66662, J1=0.40764 and J2=0.25898*. Comparing these values to the respective ones from the Non-Cooperative case we verify once more that the two players would intend to cooperate as they both benefit.



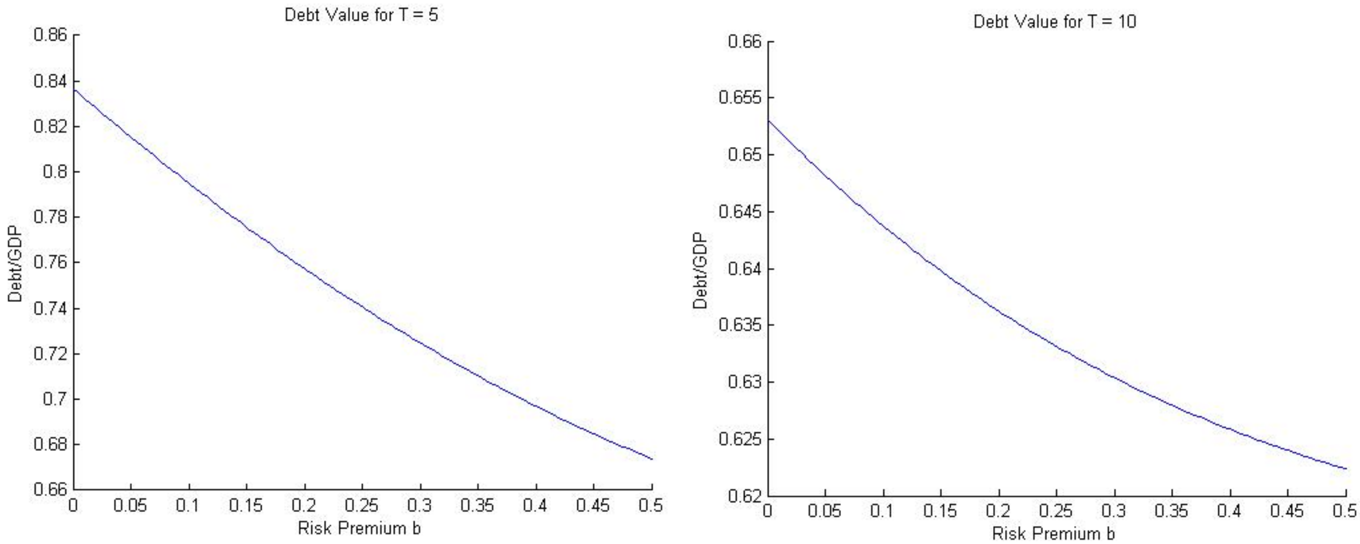
Figures 4.15 - 4.16: Finite Horizon Trajectories of Debt, Costate Variables for shorter finite horizons



Figures 4.17 - 4.18: Finite Horizon Optimal Policies for shorter finite horizons

To analyze the effect of the risk premium term depending on the rate of change of the debt in this case, we will examine what is the impact on the final value of debt as we did in the Non-Cooperative case. This is due to the fact that there is no steady-part long enough to justify domination throughout the horizon. It is clear even in this case, that the risk premium term we introduced does decrease the level of debt as it increases. However, the values reached are not even near the steady state values or the ones achieved by longer horizons.

Although, comparing them with the Non-Cooperative case, Figs 3.14-3.15 the debt level achieved is again significantly lower. This highlights once more the benefits of Cooperation in the finite case for shorter horizons, in an even more impressive way. Again this denotes the significance of the new term introduced, as it implements the pressure that the financial markets want to add, but also operates as a supporting mechanism for countries that are active and try to decrease their debt levels.

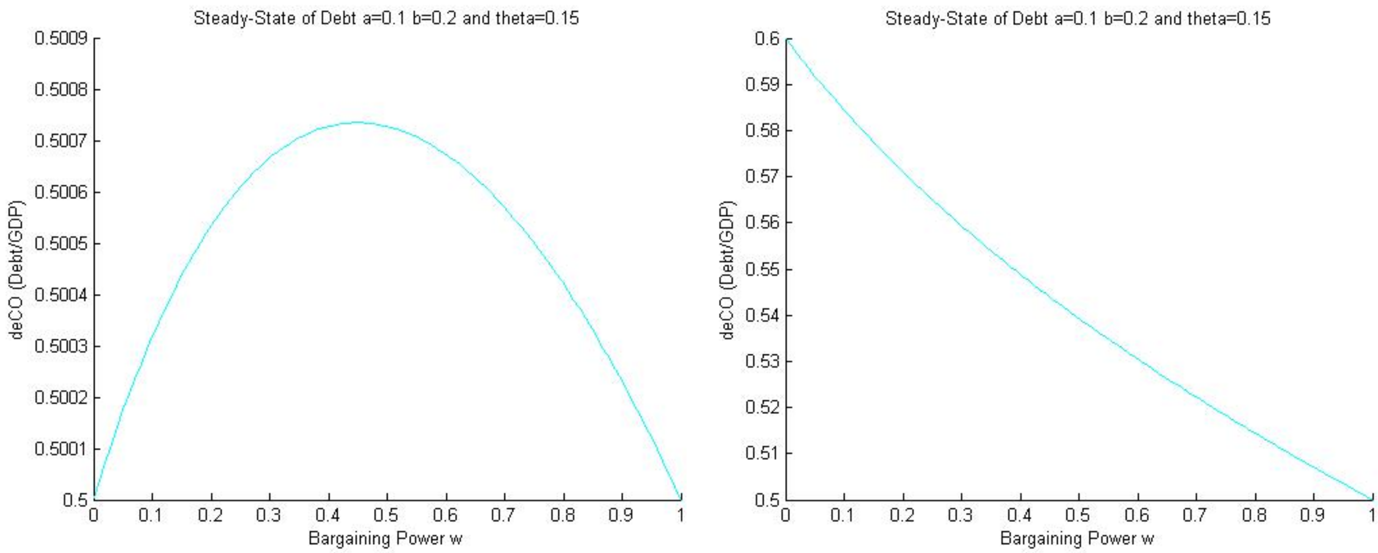


<u>Figures 4.19 - 4.20:</u> Effect of risk premium term b on the debt value at the end of the horizon

4.3.3.4 The impact of the Bargaining Strength

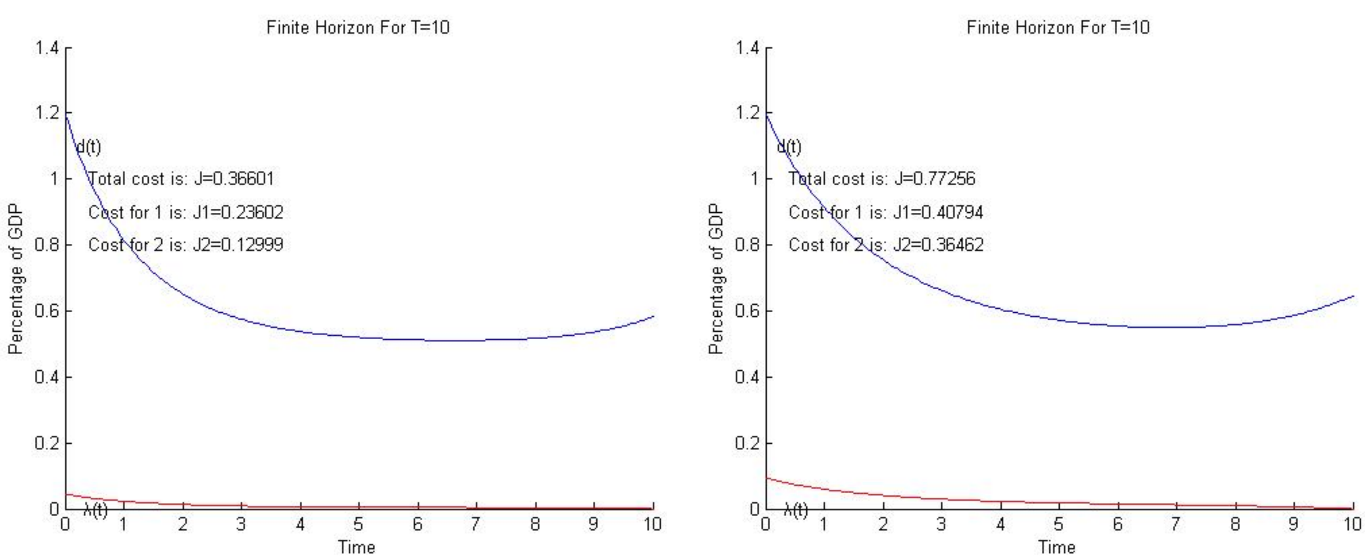
Before presenting some results for risk premium term a, we investigate the effect that the bargaining power of each player has. Namely, we select two values $\omega = 0.3$ and $\omega = 0.7$. The first value could correspond to an OECD country-member of the “south-block”. Their bargaining strength is relatively small when comparing to the ECB. In similar fashion, the second value could belong to a more powerful country-member with significant influence.

The first thing we would want to explore is how the bargaining power affects the steady state value of governmental debt. In the first figure we present the direct effect of the bargaining power of each player. If one first observes Figure 4.15, it can be seen that the more one of the two players is in control, the lower the steady state value debt is. However this should not trick us into thinking that dominance of a player gives always better results. The current graph has this particular form because the two players have the same target for government debt. Figure 4.2 confirms, as it was expected, that the more power has a player, the more the debt value tends to the value this particular player has as his target. In Figure 4.2 fiscal authorities have a target value for debt of 0.5, while the monetary authorities a target value of 0.6. As the bargaining power of the fiscal authorities approaches zero, the steady state of value debt approaches the target set by monetary authorities. The respective phenomenon takes place as ω approaches the value of 1.0.

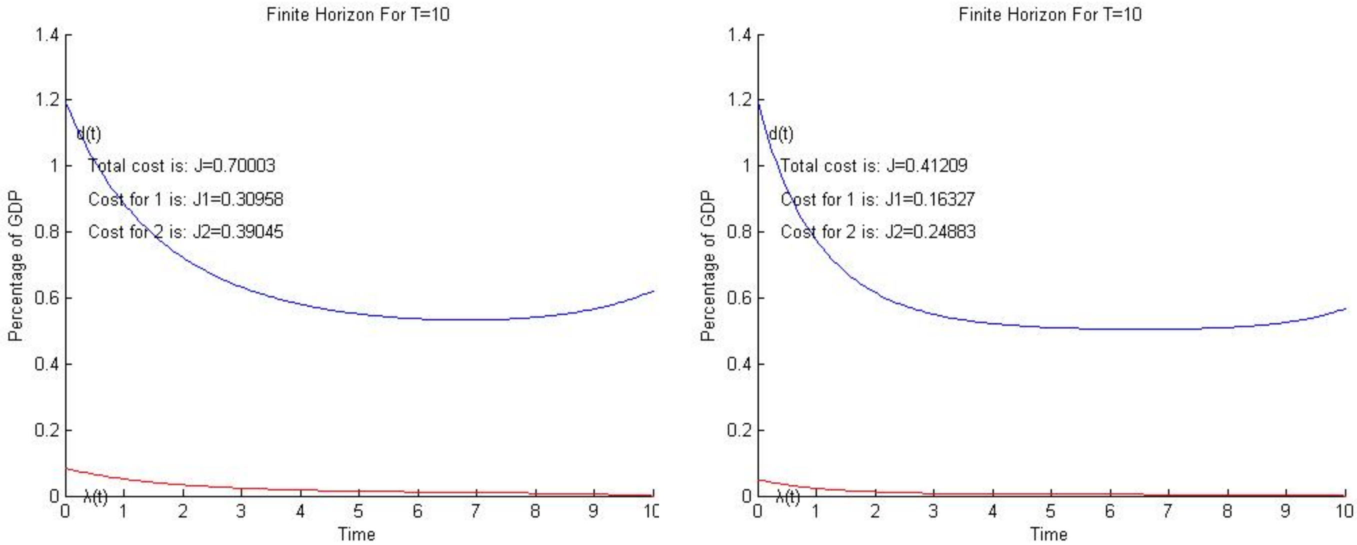


<u>Figure 4.21-4.22:</u> Effect of bargaining power ω on steady state value of debt
Left: Same debt target for both players
Right: Different debt target for each player

Now in order to investigate if the players still benefit from cooperation for various values of the bargaining strength we present the respective costs for some cases over a short finite horizon. Namely, we set T=10 and explore the cases of a really weak fiscal player (ω=0.1) and its counterpart for the monetary player (ω=0.9). Also we explore the cases for values in between of the aforementioned. As Figure 4.23 and its counterpart Figure 4.26 suggest, with the total dominance of one player both do minimize their costs, with the stronger one getting the lowest of the two. For each of the values used we can conclude that both players benefit from cooperation and in fact the stronger player gets the lower cost. Therefore we suggest that in the majority if not in any case the players have no reason not to cooperate.



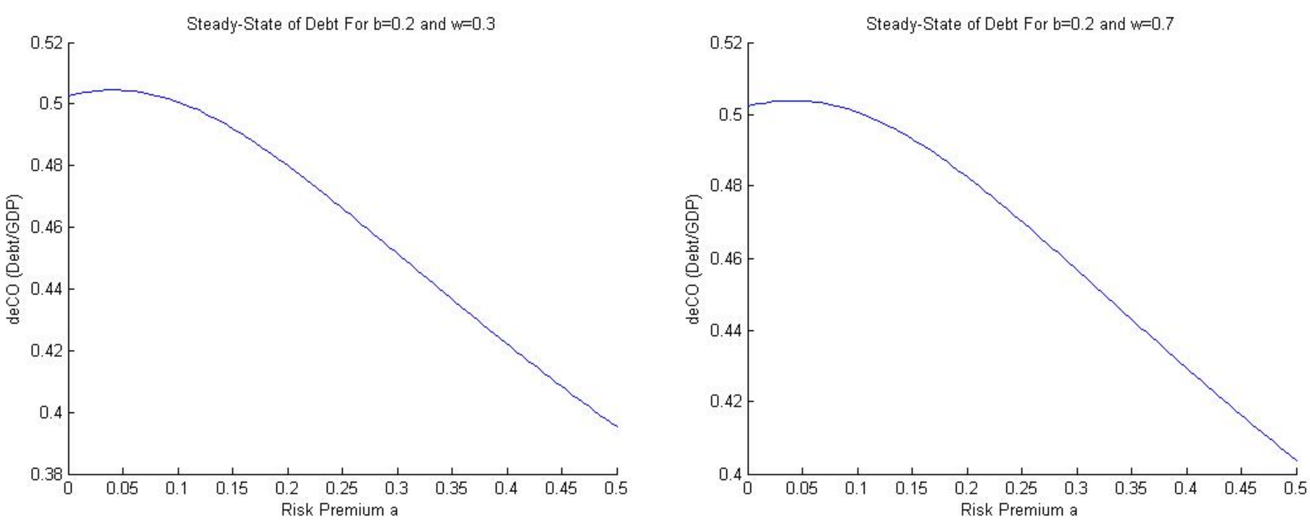
<u>Figure 4.23-4.24:</u> Effect of bargaining power ω on debt trajectory and the costs for T=10
Left: ω=0.1 , Right: ω=0.5



<u>Figure 4.25-4.26:</u> Effect of bargaining power ω on debt trajectory and the costs for T=10
Left: ω=0.7 , Right: ω=0.9

4.3.3.5 Effect of the Risk Premium Term $\boldsymbol{a}$

For completion we also present figures that support our theoretical analysis about how the steady state value of government debt behaves, in response to variations in the value of a. As we see in Figures 4.8 and 4.9, we have exactly the same behavior we analyzed. The variations in a look to have the same effect as in the Non-Cooperative case. Also, another benefit produced by the cooperation is that the spiking that was observed for small values of a in the Non-Cooperative case seems to have been soothed. Another thing that one also expects to be present again, is that for small values of a cooperation yields better results, i.e. smaller steady state debt, than when the players are not cooperating. However, for larger values of a, the risk premium non-linearity grows too strong making the non-cooperative play to yield smaller values for debt.



<u>Figure 4.27 & 4.28:</u> Effect of risk premium term a on steady state value of debt for various values of b
Left: ω=0.3 , Right: ω=0.7

5 Conclusion

In this paper we analyzed the impact of an endogenous risk premium on the rate of change of debt, in addition to risk premium on the level of debt. We examined the policies in a simple dynamic game between the fiscal and monetary authorities in a country. We considered both the Non-cooperative Nash Open-Loop and the Cooperative mode of play for theoretical results and benchmark examples. As intended we analyzed both Finite and Infinite Horizons (in the sense of achieving a steady-state). Furthermore we derived analytic expressions for the evolution of debt by a coordinate transformation in the Open-Loop mode and solved analytically the Cooperative mode. Unfortunately, it is not possible to track the non-cooperative policies analytically. However, we presented that there are cases where one can still compute these policies numerically. In addition to that, we solved numerically for representative values the Finite Horizon case. The results for the control strategies in both cases shows sufficient resemblance, indicated that the limit when time tends to infinity, of the Finite Horizon case could exist and attain the values of the Infinite Solution case.

We showed in both Non-Cooperative and Co-operative modes of play that in the finite horizon case, if the horizon T is long enough so that equilibrium equations are satisfied, then the trajectory enters the same equilibrium path of the infinite horizon, only to deviate from it in the final time. Due to this fact, it is important to study this steady-state path. Thus, during the Open-Loop Nash Equilibrium mode of play, we first focused on the steady state of debt and on the effects which the additional risk premium (on the rate of change of debt) might have, and then we investigated what happens in shorter horizons.

Concerning the equilibrium points of the game we showed that by including this second risk premium term the game always has at least one and at most three equilibria, thus not affecting the number of equilibrium points indicated by [2]. The number of equilibrium point is affected by the value of both risk premium terms. We derived conditions in order to obtain each case of equilibrium points, and proved that at least one of them is a saddle point. Another observation was that every cooperative equilibrium could be realized as a non-cooperative equilibrium for specific weight values concerning the development of the debt.

A significant result shown in our benchmark cases and examples is that the term of risk premium depending on the rate of change of debt has a greater effect on the steady state value of debt than the term depending on the level of debt. Specifically, the greater the value of risk premium is, the lower the steady state of debt becomes. This is justified if one thinks that there is a reward for managing to lower the debt value faster and the greater the value of risk premium is, the greater the reward. In similar fashion one could say that there is a heavy penalty for letting the debt increase instead of keeping it steady or making it decrease. Both the parameters that measure the strength of the risk premium mechanism, are of crucial importance in the debt stabilization game. Also, we observed that as b increases the value the debt attains decreases too, independently of the horizon. However we observe differences in depending on the length of the horizon, which are mainly attributed to how close to the steady path the trajectories tend, as the time horizon increases the behavior of these trajectories is closer to the steady state trajectory. Regarding the short finite horizons, it was evident that even here, the risk premium term we introduced does decrease the level of debt as it increases. However, the values reached were not even near the steady state values or the ones achieved by longer horizons. An important feat though was how much the debt is able to decrease, especially as b increases, in such short horizon.

Concerning the actions of each player we observed that the increase of the term of risk premium depending on the level of debt results in more active policies from both players. Intuitively, this represent the fact that, when markets add more pressure to the countries by demanding great values of risk premium on the debt level, then the authorities have to restrict it within acceptable standards. The more this risk premium term increases, the more the players have to lose by maintaining high debt levels, and thus they act to lower it. One the other hand, when the risk premium term on the rate of change of debt increases the players become slightly less active. A justification of this is that when the debt is declining, the risk premium term b is acting as a reward by decreasing the real interest rate. This is helping the debt level to decrease and be contained more easily, allowing the players to relax slightly.

In addition, we observed that the cooperation is beneficial, when the risk premia are not beyond a certain threshold value, as shown in our benchmark cases. This was the case for both short and long finite horizons, and for the infinite horizon as well. In every benchmark computed the cooperative mode produced better results. In fact the decrease was significant and also the trajectory entered the steady-state much faster. Again in the short finite horizon case the values reached were not even near the steady state values or the ones achieved by longer horizons. Although, comparing them with the Non-Cooperative case, once more the benefits of Cooperation in the finite case for shorter horizons were highlighted in an even more impressive way. Again this denotes the significance of the new term introduced, as it implements the pressure that the financial markets want to add, but also operates as a supporting mechanism for countries that are active and try to decrease their debt levels. An interesting observation that the results in case one of the players was superior to the other, were better than the case where the players shared equal bargaining strength. We are obliged to mention that for the empirical values that correspond with real life models cooperation does give better results and thus it suggested as the beneficial mode of play.

A topic for future research could be how this interaction affected by risk premium is depicted when having more than one country, e.g. more fiscal agents. A more interesting approach would be to introduce models taking into account the effect of the inflation and how it interacts with the risk premium terms added. This could better reflect the role of the markets, and could be an adequate representation when having more than one country or blocks of countries participating in the game. Our result could be also tested with a model consisting of many countries or blocks of countries and analyze the results in that case. Also, when it comes to the values of the parameters used, an interesting idea could be to use the theory of adaptive control and specifically system identification techniques, using past data to identify the parameter values.

6 APPENDIX

6.1 THE NASH OPEN-LOOP CASE:

Theorem 3.1 *(Proof)*:

Consider the Hamiltonians: $H_i = \frac{1}{2} e^{-\theta t}\left((u_i - \bar{\iota})^2 + \beta_i\left(d - \bar{d}_i\right)^2\right) + \lambda_i \frac{\bar{r}\, d + a\, d^2 + u_F - u_M}{1 - b\, d}$ for *i=F,M.*

Then as a function of d they are equivalent to:

$$H_F = A\, d^2 + \lambda_F\left(B\frac{d}{1 - b\, d} + C\frac{d^2}{1 - b\, d} + D\frac{1}{1 - b\, d}\right)$$

All these terms are convex in any interval lying left of the value that makes the denominator equal to zero, namely $1/b$. Restricting now ourselves to any such interval that contains d:

Let $u_i\,:\,[0, T] \to U_i\,, i = F, M$ be any measurable control function. Then

$$J_i(u) - J_i^*(u^*) = \int_0^T h_i(t, d, u_i)dt - \int_0^T h_i(t, d^*, u_i^*)dt$$

$$= \int_0^T \left(H_i(t, \lambda_i, d, u_i) - \lambda_i\, \dot{d}(t)\right)\, dt - \int_0^T \left(H_i(t, \lambda_i, d^*, u_i^*) - \lambda_i\, \dot{d}^*(t)\right)\, dt$$

Since u_i^* satisfies the optimality condition: $u_i{}^* = \bar{\iota} \mp e^{\theta t}\lambda_i{}^* \frac{1}{1 - b\, d}\;\;,\; i = F, M$

One has: $\qquad H_i(t, \lambda_i, d^*, u_i^*) = H_i(t, \lambda_i, d) = H_{min}\;,\; H(t, \lambda_i, d, u_i) \geq H_{min}$

Using these inequalities in the cost function expression we obtain:

$$J_i(u) - J_i^*(u^*) \geq \int_0^T \left(H_i(t, \lambda_i, d, u_i) - \lambda_i\, \dot{d}(t)\right)\, dt - \int_0^T \left(H_i(t, \lambda_i, d^*, u_i^*) - \lambda_i\, \dot{d}^*(t)\right)\, dt$$

If the map $d \to H(t, d, \lambda)$ is differentiable, the convexity assumption implies:

$$H_i(t, \lambda_i, d) - H_i(t, \lambda_i, d^*) \geq {}^{\partial H_i}\!/_{\partial d}\,(t, \lambda_i, d^*)\left(d(t) - d^*(t)\right) = -\dot{\lambda}_i\left(d(t) - d^*(t)\right)$$

Using now this inequality we finally obtain:

$$J_i(u) - J_i^*(u^*) \geq \int_0^T \left(-\dot{\lambda}_i\left(d(t) - d^*(t)\right) - \lambda_i\left(\dot{d}(t) - \dot{d}^*(t)\right)\right)\, dt =$$

$$= -\lambda_i(T)\left(d(T) - d^*(T)\right) - \lambda_i(0)\left(d(0) - d^*(0)\right) = 0$$

Given that:

$$d(0) = d^*(0) = d_0\,,\; \lambda_i(T) = 0\,, i = F, M\; as\; there\; is\; no\; terminal\; cost\; function.$$

6.1.1 Solution Using Pontryagin's Minimum Principle

Theorem 3.2: The Open-loop case can be seen as an optimal control problem for each player separately and can be solved using Pontryagin's Minimum Principle, when the pair of optimal controls $({u_F}^*, {u_M}^*)$ belongs to the set of admissible strategies. Combining equations (1.a) and (1.d) we get the differential equation to act as a constraint to the minimization problem of our two players:

$$\dot{d}(t) = \frac{\bar{r}\, d(t) + a\, d(t)^2 + u_F(t) - u_M(t)}{1 - b\, d(t)} \tag{6.1.1}$$

While the players want to minimize their respective loss functions:

$$J_F = {}^{1}\!/_{2} \int_0^T e^{-\theta t} \left(\left(u_F(t) - \bar{f}\right)^2 + \beta_F\left(d(t) - \overline{d_F}\right)^2 \right) dt \tag{6.1.2.a}$$

$$J_M = {}^{1}\!/_{2} \int_0^T e^{-\theta t} \left((u_M(t) - \bar{m})^2 + \beta_M\left(d(t) - \overline{d_M}\right)^2 \right) dt \tag{6.1.2.b}$$

- We apply Pontryagin's Minimum Principle for the fiscal authorities:
  - Hamiltonian: $H_F = \frac{1}{2} e^{-\theta t} \left(\left(u_F - \bar{f}\right)^2 + \beta_F\left(d - \overline{d_F}\right)^2 \right) + \lambda_F \frac{\bar{r}\, d + a\, d^2 + u_F - u_M}{1 - b\, d}$ (6.1.3.a)
  - $\dot{\lambda_F} = -{}^{\partial H_F}\!/_{\partial d}$, $\lambda_F(T) = 0$
  - ${u_F}^* = arg \min\limits_{u_F}\{H_F\}$

Differentiating the Hamiltonian (6.1.3.a) with respect to the state-vector we get:

$$\dot{\lambda_F} = -e^{-\theta t} \beta_F\left(d - \overline{d_F}\right) - \lambda_F \frac{(\bar{r} + 2a\, d)(1 - b\, d) + b(\bar{r}\, d + a\, d^2 + u_F - u_M)}{(1 - b\, d)^2}$$

We introduce $\mu_F = e^{\theta t} \lambda_F \Rightarrow \dot{\mu_F} = \theta\, \mu_F + e^{\theta t} \dot{\lambda}_F$

Yielding:

$$\dot{\mu_F} = -\beta_F\left(d - \overline{d_F}\right) + \mu_F \left(\theta - \frac{(\bar{r} + 2a\, d)(1 - b\, d) + b\left(\bar{r}\, d + a\, d^2 + u_F - u_M\right)}{(1 - b\, d)^2} \right) \tag{6.1.4.a}$$

The optimal control for the F - player is:

$${u_F}^* = \bar{f} - e^{\theta t} {\lambda_F}^* \frac{1}{1 - b\, d} = \bar{f} - {\mu_F}^* \frac{1}{1 - b\, d} \tag{6.1.5.a}$$

And the derivative with respect to time, of the optimal control is:

$$\dot{u_F}^* = \left(u_F - \bar{f}\right)\left(\theta - \frac{\bar{r} + 2a\, d}{1 - b\, d}\right) + \frac{\beta_F\left(d - \overline{d_F}\right)}{1 - b\, d}$$

$$\xRightarrow{\dot{u_F}^* = 0} {u_F}^* = \bar{f} + \frac{\beta_F(d - \overline{d_F})}{\bar{r} + 2a\, d - \theta(1 - b\, d)} \tag{6.1.6.a}$$

- Working in similar fashion for the monetary authorities:
    - Hamiltonian: $H_M = \frac{1}{2}e^{-\theta t}\left((u_M - \overline{m})^2 + \beta_M\left(d - \overline{d_M}\right)^2\right) + \lambda_M\ \frac{\bar{r}\ d + a\ d^2 + u_F - u_M}{1 - b\ d}$ (6.1.3.b)
    - $\dot{\lambda_M} = -\,{}^{\partial H_M}/_{\partial d}\ ,\ \lambda_M(T) = 0$
    - $u_M{}^* = arg\min_{u_M}\{H_M\}$

Differentiating the Hamiltonian (6.1.3.b) with respect to the state-vector we get:

$$\dot{\lambda_M} = -e^{-\theta t}\beta_M\left(d - \overline{d_M}\right) - \lambda_M\ \frac{(\bar{r} + 2a\ d)(1 - b\ d) + b(\bar{r}\ d + a\ d^2 + u_F - u_M)}{(1 - b\ d)^2}$$

We introduce $\mu_M = \ e^{\theta t}\lambda_F \Rightarrow \dot{\mu_M} = \theta\ \mu_M + e^{\theta t}\dot{\lambda}_M$

Yielding:

$$\dot{\mu_M} = -\beta_M\left(d - \overline{d_M}\right) + \mu_M\left(\theta - \frac{(\bar{r}+2a\ d)(1-b\ d)+b\left(\bar{r}\ d+a\ d^2+u_F-u_M\right)}{(1-b\ d)^2}\right) \quad (6.1.4.b)$$

The optimal control for the M - player is:

$$u_M{}^* = \overline{m} + e^{\theta t}\lambda_M{}^*\frac{1}{1-b\ d} = \ \overline{m} + \mu_M{}^*\frac{1}{1-b\ d} \quad (6.1.5.b)$$

And the derivative with respect to time, of the optimal control is:

$$\dot{u_M}^* = (u_M - \overline{m})\left(\theta - \frac{\bar{r} + 2a\ d}{1 - b\ d}\right) - \frac{\beta_M\left(d - \overline{d_M}\right)}{1 - b\ d}$$

$$\xRightarrow{\dot{u_M}^*=0} u_M{}^* = \overline{m} - \frac{\beta_M(d-\overline{d_M})}{\bar{r}+2a\ d-\theta(1-b\ d)} \quad (6.1.6.b)$$

Now we use the following substitution: $\mu = \mu_F + \mu_M$. Combined with the optimal control laws we derived we get: $u_F{}^* - u_M{}^* = \bar{f} - \overline{m} - \mu\ \frac{1}{1-b\ d}$

Replacing those we get the optimal trajectories:

$$\dot{d}^*(t) = \ \frac{1}{1-b\ d(t)}\left(\bar{r}\ d(t) + a\ d(t)^2 - \mu\frac{1}{1-b\ d(t)} + \bar{f} - \overline{m}\right) \quad (6.1.7.a)$$

$$\dot{\mu}^*(t) = -\left(\beta_F\left(d(t) - \overline{d_F}\right) + \beta_M\left(d(t) - \overline{d_M}\right)\right) + \ \mu\ \left(\theta - \frac{\left(\bar{r}+2a\ d(t)\right)\left(1-b\ d(t)\right)+b\left(\bar{r}\ d(t)+a\ d^2+\bar{f}-\overline{m}\right)}{\left(1-b\ d(t)\right)^2}\right) + \mu^2\frac{\mathrm{b}}{\left(1-b\ d(t)\right)^3} \quad (6.1.7.b)$$

Checking for periodic solutions we differentiate (6.1.7.a) w.r.t. d and (6.1.7.b) w.r.t. μ and adding them we get: $\frac{\partial(\dot{d}^*(t))}{\partial d} + \frac{\partial(\dot{\mu}^*(t))}{\partial \mu} = \theta$, so from Bendixson's Criterion we have no periodic solutions if $\theta \neq 0$.

6.1.2 Exploring Open-Loop Equilibrium Points

Now let's assume that the Open-Loop equilibrium points are (d_e^{OL}, μ_e^{OL}). To find those points we set equations (6.1.7) equal to zero:

$$\dot{d}^*(t) = 0 \Leftrightarrow \bar{r}\, d_e^{OL} + a\,(d_e^{OL})^2 - \mu_e^{OL}\frac{1}{1 - b\, d_e^{OL}} + \bar{f} - \bar{m} = 0$$

$$\Leftrightarrow -a\, b\,(d_e^{OL})^3 + (a - b\,\bar{r})\,(d_e^{OL})^2 + \left(\bar{r} - b\left(\bar{f} - \bar{m}\right)\right) d_e^{OL} + \left(\bar{f} - \bar{m} - \mu_e^{OL}\right) = 0$$

$$\Leftrightarrow \mu_e^{OL} = -a\, b\,(d_e^{OL})^3 + (a - b\,\bar{r})\,(d_e^{OL})^2 + \left(\bar{r} - b\left(\bar{f} - \bar{m}\right)\right) d_e^{OL} + \left(\bar{f} - \bar{m}\right) \qquad (6.1.8)$$

Now, replacing μ_e^{OL} to (6.1.8) we get d_e^{OL} as the solution of the following cubic polynomial:

$$-a\,(2a + b\,\theta)\,(d_e^{OL})^3 + \left(a\,\gamma_2 - \bar{r}(2a + b\,\theta)\right)(d_e^{OL})^2 + \left(\bar{r}\,\gamma_2 - \bar{u}(2a + b\,\theta) - (\beta_F + \beta_M)\right) d_e^{OL} + (\gamma_3 + \bar{u}\,\gamma_2) = 0 \qquad (6.1.9)$$

Where: $\gamma_1 = 1 - b\, d_e^{OL}$, $\gamma_2 = \theta - \bar{r}$, $\gamma_3 = \beta_F \overline{d_F} + \beta_M \overline{d_M}$, $\bar{u} = \bar{f} - \bar{m}$

Equation (6.1.9), which defines the steady state value of debt, is a cubic equation. It is natural that only real roots of (6.1.9) are considered and accepted, due to the fact that they represent debt value. The discriminant of the general cubic polynomial:

$$g(x) = \delta_3\, x^3 + \delta_2 x^2 + \delta_1 x + \delta_0$$

is $\Delta = 18\,\delta_3\,\delta_2\,\delta_1\delta_0 - 4{\delta_2}^3\delta_0 + (\delta_2\delta_1)^2 - 4\delta_3{\delta_1}^3 - 27(\delta_3\delta_0)^2$. Thus the roots of the polynomial in our case are determined as follows:

$$\delta_3 = -a(2a + b\,\theta)\,, \delta_2 = a\,\gamma_2 - \bar{r}(2a + b\,\theta)\,, \delta_1 = \bar{r}\,\gamma_2 - \bar{u}(2a + b\,\theta) - (\beta_F + \beta_M)\,, \delta_0 = (\gamma_3 + \bar{u}\,\gamma_2)$$

- If $\Delta(a, b) > 0$, there are 3 distinct real roots and hence, 3 candidate solutions for the game.
- If $\Delta(a, b) < 0$, there is 1 distinct real root and 2 complex conjugate roots. Hence, 1 candidate solution for the game.
- If $\Delta(a, b) = 0$, there are at least 2 roots coincide and they are all real. That means there are either a double real root and distinct single real root, or a triple real root. That is we either have 2 or 1 candidate solutions for the game respectively.

Since a triple root of a cubic polynomial has the property of being also a root of its first and second derivative we can derive the condition for a triple root to exist. Let d_{3e} be a triple root of $g(d)$:

$$\partial^2 g(d)/\partial d^2\,|_{d=d_{3e}} = 6\delta_3 d_{3e} + 2\delta_2 = 0 \Rightarrow d_{3e} = -\frac{\delta_2}{3\delta_3} \quad (6.1.10.a)$$

$$\partial\, g(d)/\partial d\,\,|_{d=d_{3e}} = 3\delta_3 {d_{3e}}^2 + 2\delta_2 d_{3e} + \delta_1 = 0 \xRightarrow{(6.1.10.a)} \delta_1 = \frac{{\delta_2}^2}{3\delta_3} \quad (6.1.10.b)$$

$$g(d)|_{d=d_{3e}} = 0 \Rightarrow \delta_3\, {d_{3e}}^3 + \delta_2 {d_{3e}}^2 + \delta_1 d_{3e} + \delta_0 = 0 \xRightarrow{(6.1.10.a),(6.1.10.b)} {\delta_2}^3 = 27\delta_0\,{\delta_3}^2 \; (6.1.10.c)$$

$$(6.1.10.b), (6.1.10.c) \Rightarrow \delta_1\delta_2 = 9\delta_0\delta_3 \; (6.1.10.d)$$

In our case (6.1.10.d) becomes:

$$\left(\bar{r}\,\gamma_2 - \bar{u}(2a + b\,\theta) - (\beta_F + \beta_M)\right)\left(a\,\gamma_2 - \bar{r}(2a + b\,\theta)\right) + 9a(2a + b\,\theta)(\gamma_3 + \bar{u}\,\gamma_2) = 0 \quad (6.1.11)$$

Thus, if $\Delta(a,b)=0$ and (6.1.11) holds there is 1 (triple) real root, and hence 1 candidate solution (and its value is: $d_{3e}=-\frac{\delta_2}{3\delta_3}=-\frac{a\,\gamma_2-\bar{r}(2a+b\,\theta)}{3a(2a+b\,\theta)}$). Else, if (6.1.11) does not hold there is 1 double real root and distinct single real root, thus 2 candidate solutions.

6.1.3 Qualitative Behavior of Equilibrium Points

The qualitative behavior of the system near the equilibrium points should be explored. This is achieved via the Linearized System around each equilibrium point. Linearizing system (3.3) we get:

$$\begin{pmatrix}\dot{d(t)}\\ \dot{\mu(t)}\end{pmatrix}=\begin{pmatrix}-\frac{a}{b}-\frac{2\,b\,\mu_e^{OL}}{\gamma_1{}^3}+\frac{a+b\,\bar{r}+b^2\bar{u}}{b\,\gamma_1{}^2} & -\frac{1}{\gamma_1{}^2}\\ -(\beta_F+\beta_M)-\frac{2\,(a+b\,\bar{r}+b^2\,\bar{u})\,\mu_e^{OL}}{\gamma_1{}^3}+\frac{3(b\,\mu_e^{OL})^2}{\gamma_1{}^4} & \frac{a}{b}+\theta+\frac{2\,b\,\mu_e^{OL}}{\gamma_1{}^3}-\frac{a+b\,\bar{r}+b^2\bar{u}}{b\,\gamma_1{}^2}\end{pmatrix}\begin{pmatrix}d(t)\\ \mu(t)\end{pmatrix} \tag{6.1.12}$$

where (d_e^{OL},μ_e^{OL}) is the equilibrium point under discussion.

The eigenvalues of the linearized system (6.1.12) are:

$$eig_{1,2}=\frac{(1-bd_e^{OL})\,\theta\;\pm\sqrt{\begin{matrix}(1-bd_e^{OL})^2\,\theta^2+4(bf+bm+\bar{u}(2a+b\theta)-\bar{r}\gamma_2)\\ +12a(2a+b\theta){d_e^{OL}}^2-8\big(a\gamma_2-(2a+b\theta)\big)d_e^{OL}\end{matrix}}}{2(1-bd_e^{OL})}$$

$$\Leftrightarrow eig_{1,2}=\frac{(1-bd_e^{OL})\,\theta\;\pm\sqrt{(1-bd_e^{OL})^2\,\theta^2-4\,{}^{\partial\,g(d)}/_{\partial d}}}{2(1-bd_e^{OL})} \tag{6.1.13}$$

One can easily observe that the steady states will be one of the following:

- If $(1-bd_e^{OL})^2\,\theta^2-4\,{}^{\partial\,g(d)}/_{\partial d}>0\ and\ {}^{\partial\,g(d)}/_{\partial d}<0$, then $eig_1>0\ and\ eig_2<0$. Thus, the steady state is a saddle point.
- If $(1-bd_e^{OL})^2\,\theta^2-4\,{}^{\partial\,g(d)}/_{\partial d}>0\ and\ {}^{\partial\,g(d)}/_{\partial d}>0$, then $eig_1>0\ and\ eig_2>0$. Thus, the steady state is an unstable node.
- If $(1-bd_e^{OL})^2\,\theta^2-4\,{}^{\partial\,g(d)}/_{\partial d}<0$, then $eig_{1,2}$ are complex eigenvalues with positive real part. Thus, the steady state is an unstable focus.
- Last but not least, there is the case an eigenvalue becomes zero. That is, when $\Delta(a,b)=0$ and (6.1.11) holds (a triple real root). As we have already argued the equilibrium point is also a root of the first derivative. It is then straightforward from (6.1.13) that an eigenvalue becomes zero while the other will be positive. This means the system matrix has a non-trivial null-space and has an equilibrium subspace than an equilibrium point. Due to the positive eigenvalue all trajectories diverge away from the equilibrium subspace.

6.2 THE CO-OPERATIVE CASE:

Theorem 4.1 *(Proof)*:

Consider the Hamiltonian:

$$H = \frac{1}{2}e^{-\theta t}\left(\omega\left(\left(u_F - \bar{f}\right)^2 + \beta_F\left(d - \overline{d_F}\right)^2\right) + (1-\omega)\left((u_M(t) - \bar{m})^2 + \beta_M\left(d(t) - \overline{d_M}\right)^2\right)\right)$$
$$+ \lambda\,\frac{\bar{r}\,d + a\,d^2 + u_F - u_M}{1 - b\,d}$$

Then as a function of d H is are equivalent to:

$$H = A\,d^2 + \lambda\left(B\frac{d}{1-b\,d} + C\frac{d^2}{1-b\,d} + D\frac{1}{1-b\,d}\right)$$

All these terms are convex in any interval lying left of the value that makes the denominator equal to zero, namely $1/b$. Restricting now ourselves to any such interval that contains d:

Let $u_i : [0,T] \to U_i\,, i = F, M$ be any measurable control function. Then

$$J(u_F,u_M) - J^*(u_F^*,u_M^*) = \int_0^T h(t,d,u_F,u_M)dt - \int_0^T h(t,d^*,u_F^*,u_M^*)dt$$
$$= \int_0^T \left(H(t,\lambda,d,u_F,u_M) - \lambda\,\dot{d}(t)\right)\,dt - \int_0^T \left(H(t,\lambda,d^*,u_F^*,u_M^*) - \lambda\,\dot{d}^*(t)\right)\,dt$$

Since u_F^*, u_M^* satisfy their optimality conditions:

$$u_F{}^* = \bar{f} - e^{\theta t}\lambda\frac{1}{\omega(1-b\,d)} \text{ and } u_M{}^* = \bar{m} + e^{\theta t}\lambda\frac{1}{(1-\omega)(1-b\,d)}$$

We obtain: $\quad H(t,\lambda,d^*,u_F^*,u_M^*) = H(t,\lambda,d) = H_{min}\;,\; H(t,\lambda,d,u_F,u_M) \geq H_{min}$

Using these inequalities in the cost function expression we obtain:

$$J(u_F,u_M) - J^*(u_F^*,u_M^*) \geq \int_0^T \left(H(t,\lambda,d,u_F,u_M) - \lambda\,\dot{d}(t)\right)\,dt - \int_0^T \left(H(t,\lambda,d^*,u_F^*,u_M^*) - \lambda\,\dot{d}^*(t)\right)\,dt$$

If the map $d \to H(t,d,\lambda)$ is differentiable, the convexity assumption implies:

$$H(t,\lambda,d) - H(t,\lambda,d^*) \geq {}^{\partial H}/_{\partial d}\,(t,\lambda,d^*)\,\left(d(t) - d^*(t)\right) = -\dot{\lambda}\left(d(t) - d^*(t)\right)$$

Using now this inequality we finally obtain:

$$J(u_F,u_M) - J^*(u_F^*,u_M^*) \geq \int_0^T \left(-\dot{\lambda}\left(d(t) - d^*(t)\right) - \lambda\left(\dot{d}(t) - \dot{d}^*(t)\right)\right)\,dt =$$
$$= -\lambda(T)\left(d(T) - d^*(T)\right) - \lambda(0)\left(d(0) - d^*(0)\right) = 0$$

Given that:

$$d(0) = d^*(0) = d_0\,,\; \lambda(T) = 0 \; as\ there\ is\ no\ terminal\ cost\ function.$$

6.2.1 Solution Using Pontryagin's Minimum Principle

Theorem 4.2: The dynamic game has now transformed into an optimal control problem from an analytical point of view. By solving the optimal control problem (4.1)-(4.2) for every $\omega \in (0,1)$ a curve of Pareto Efficient solutions is obtained. But which curve will be selected by the player, is defined by the choice of the parameter ω, i.e. the coordination of the system. Note that a Pareto solution is not unique. Since our game under the Co-operative case belongs to the class of optimal control problems, it can be solved using Pontrygin's Minimum Principle.

$$\min_{u_F,u_M} J, \quad J = \omega\, J_F \ + (1-\omega) J_M \tag{6.2.1}$$

$$= {}^1\!/_2 \int_0^T e^{-\theta t}\, \omega\, \left(\left(u_F(t) - \bar{f}\right)^2 + \beta_F\left(d(t) - \overline{d_F}\right)^2 \right) dt$$
$$+ {}^1\!/_2 \int_0^T e^{-\theta t}\, (1-\omega) \left((u_M(t) - \bar{m})^2 + \beta_M\left(d(t) - \overline{d_M}\right)^2 \right) dt$$

Subject to the government budget constraint discussed in the previous section:

$$\dot{d}(t) = \frac{\bar{r}\, d(t) + a\, d(t)^2 + u_F(t) - u_M(t)}{1 - b\, d(t)} \tag{6.2.2}$$

- Applying Pontryagin's Minimum Principle we obtain:
- Hamiltonian:

$$H = \frac{1}{2} e^{-\theta t} \omega \left(\left(u_F - \bar{f}\right)^2 + \beta_F\left(d - \overline{d_F}\right)^2 \right) \tag{6.2.3}$$
$$+ \frac{1}{2} e^{-\theta t}\, (1-\omega) \left((u_M(t) - \bar{m})^2 + \beta_M\left(d(t) - \overline{d_M}\right)^2 \right) + \lambda\, \frac{\bar{r}\, d + a\, d^2 + u_F - u_M}{1 - b\, d}$$

- $\dot{\lambda} = -{}^{\partial H}\!/_{\partial d}\,, \lambda(T) = 0$
- $u_F{}^* = arg \min_{u_F}\{H\}$
- $u_M{}^* = arg \min_{u_M}\{H\}$

Differentiating the Hamiltonian with respect to the state-vector we get:

$$\dot{\lambda} = -e^{-\theta t}\, \omega\, \beta_F\left(d - \overline{d_F}\right) - e^{-\theta t}\, (1-\omega)\beta_M\left(d(t) - \overline{d_M}\right)$$
$$- \lambda\, \frac{(\bar{r} + 2a\, d)(1 - b\, d) + b(\bar{r}\, d + a\, d^2 + u_F - u_M)}{(1 - b\, d)^2}$$

We introduce $\mu = e^{\theta t}\lambda \Rightarrow \dot{\mu} = \theta\,\mu + e^{\theta t}\dot{\lambda}$

Yielding:

$$\dot{\mu} = -\omega\,\beta_F\big(d - \overline{d_F}\big) - (1-\omega)\beta_M\big(d(t) - \overline{d_M}\big) + \mu\left(\theta - \frac{(\bar{r}+2a\,d)(1-b\,d)+b\big(\bar{r}\,d+a\,d^2+u_F-u_M\big)}{(1-b\,d)^2}\right) \quad (6.2.4)$$

The optimal control for the F - player is:

$$u_F{}^* - \bar{f} = -e^{\theta t}\lambda\frac{1}{\omega(1-b\,d)} = -\mu\frac{1}{\omega(1-b\,d)} \quad (6.2.5.a)$$

The optimal control for the M - player is:

$$u_M{}^* - \bar{m} = e^{\theta t}\lambda\frac{1}{(1-\omega)(1-b\,d)} = \mu\frac{1}{(1-\omega)(1-b\,d)} \quad (6.2.5.b)$$

Combining the equations above: $u_F{}^* - \bar{f} = -\frac{1-\omega}{\omega}(u_M{}^* - \bar{m})$ (6.2.6)

The optimal control laws we derived give: $u_F{}^* - u_M{}^* = \bar{f} - \bar{m} - \mu\,\frac{1}{\omega(1-\omega)(1-b\,d)}$

Replacing the optimal controls to equations (6.2.2) and (6.2.4) we obtain the optimal trajectories:

$$\dot{d}^*(t) = \frac{1}{1-b\,d(t)}\left(\bar{r}\,d(t) + a\,d(t)^2 - \mu\frac{1}{\omega(1-\omega)(1-b\,d(t))} + \bar{f} - \bar{m}\right) \quad (6.2.7.a)$$

$$\dot{\mu}^*(t) = -\Big(\omega\beta_F\big(d(t) - \overline{d_F}\big) + (1-\omega)\beta_M\big(d(t) - \overline{d_M}\big)\Big)$$
$$+\,\mu\left(\theta - \frac{\big(\bar{r}+2a\,d(t)\big)\big(1-b\,d(t)\big)+b\big(\bar{r}\,d(t)+a\,d^2+\bar{f}-\bar{m}\big)}{\big(1-b\,d(t)\big)^2}\right) + \mu^2\frac{\mathrm{b}}{\omega(1-\omega)\big(1-b\,d(t)\big)^3} \quad (6.2.7.b)$$

Checking for periodic solutions we differentiate (6.2.7.a) w.r.t. d and (6.2.7.b) w.r.t. μ and adding them we get: $\frac{\partial(\dot{d}^*(t))}{\partial d} + \frac{\partial(\dot{\mu}^*(t))}{\partial\mu} = \theta$, so from Bendixson's Criterion we have no periodic solutions if $\theta \neq 0$.

6.2.2 Exploring Open-Loop Equilibrium Points

Now let's assume that the Co-operative equilibrium points are (d_e^{CO}, μ_e^{CO}). To find those points we set equations (6.2.7) equal to zero:

$$\dot{d}(t) = 0 \Leftrightarrow \bar{r}\,d_e^{CO} + a\,(d_e^{CO})^2 - \mu_e^{CO}\frac{1}{\omega(1-\omega)(1-b\,d_e^{CO})} + \bar{f} - \bar{m} = 0$$

$$\Leftrightarrow \mu_e^{CO} = \omega(1-\omega)\Big(-ab(d_e^{CO})^3 + (a-b\bar{r})(d_e^{CO})^2 + \big(\bar{r} - b(\bar{f}-\bar{m})\big)\,d_e^{CO} + \big(\bar{f}-\bar{m}\big)\Big) = 0 \quad (6.2.8)$$

Now, replacing μ_e^{CO} to the second equilibrium equation we get:

$$\delta_3\,(d_e^{CO})^3 + \delta_2\,(d_e^{CO})^2 + \delta_1\,d_e^{CO} + \delta_0 = 0 \quad (6.2.9)$$

Defining: $\delta_3 = -\omega(1-\omega)a(2a+b\theta), \delta_2 = \omega(1-\omega)\big(a\,\gamma_2 - \bar{r}(2\alpha+b\theta)\big)$,
$\delta_1 = \omega(1-\omega)(\bar{r}\,\gamma_2 - \bar{u}(2\alpha+b\theta) - (\omega\beta_F + (1-\omega)\beta_M)\,, \delta_0 = \omega(1-\omega)\bar{u}\,\gamma_2 + \omega\beta_F\overline{d_F} + (1-\omega)\beta_M\overline{d_M}$

Where: $\gamma_1 = 1 - b\,d_e^{OL}$, $\gamma_2 = \theta - \bar{r}$, $\gamma_3 = \beta_F\overline{d_F} + \beta_M\overline{d_M}$, $\bar{u} = \bar{f} - \bar{m}$

Equation (6.2.9), which defines the steady state value of debt, is a cubic equation. It is natural that only real roots of (6.2.9) are considered and accepted, due to the fact that they represent debt value. The discriminant of the general cubic polynomial:

$$g(x) = \delta_3\, x^3 + \delta_2 x^2 + \delta_1 x + \delta_0$$

is $\Delta = 18\,\delta_3\,\delta_2\,\delta_1\delta_0 - 4{\delta_2}^3\delta_0 + (\delta_2\delta_1)^2 - 4\delta_3{\delta_1}^3 - 27(\delta_3\delta_0)^2$. Thus the roots of the polynomial in our case are determined as follows:

- If $\Delta(a,b,\omega) > 0$, there are 3 distinct real roots and hence, 3 candidate solutions for the game.
- If $\Delta(a,b,\omega) < 0$, there is 1 distinct real root and 2 complex conjugate roots. Hence, 1 candidate solution for the game.
- If $\Delta(a,b,\omega) = 0$, there are at least 2 roots coincide and they are all real. That means there are either a double real root and distinct single real root, or a triple real root. That is we either have 2 or 1 candidate solutions for the game respectively.

Since a triple root of a cubic polynomial has the property of being also a root of its first and second derivative we can derive the condition for a triple root to exist. Let d_{3e} be a triple root of $g(d)$:

$$\partial^2 g(d)/_{\partial d^2}\,|_{d=d_{3e}} = 6\delta_3 d_{3e} + 2\delta_2 = 0 \Rightarrow d_{3e} = -\frac{\delta_2}{3\delta_3} \quad (6.2.10.a)$$

$$\partial\, g(d)/_{\partial d}\,|_{d=d_{3e}} = 3\delta_3 {d_{3e}}^2 + 2\delta_2 d_{3e} + \delta_1 = 0 \xRightarrow{(6.1.10.a)} \delta_1 = \frac{{\delta_2}^2}{3\delta_3} \quad (6.2.10.b)$$

$$g(d)|_{d=d_{3e}} = 0 \Rightarrow \delta_3\, {d_{3e}}^3 + \delta_2 {d_{3e}}^2 + \delta_1 d_{3e} + \delta_0 = 0 \xRightarrow{(6.1.10.a),(6.1.10.b)} {\delta_2}^3 = 27\delta_0\, {\delta_3}^2 \quad (6.2.10.c)$$

$$(6.1.10.b), (6.1.10.c) \Rightarrow \delta_1\delta_2 = 9\delta_0\delta_3 \quad (6.1.10.d)$$

In our case (6.2.10.d) becomes:

$$\Big(\omega(1-\omega)\big(\bar{r}\,\gamma_2 - \bar{u}(2a + b\,\theta)\big) - (\omega\beta_F + (1-\omega)\beta_M)\Big)\big(a\,\gamma_2 - \bar{r}(2a + b\,\theta)\big)$$
$$+9a(2a + b\,\theta)\big(\omega\beta_F\overline{d_F} + (1-\omega)\beta_M\overline{d_M} + \omega(1-\omega)\bar{u}\,\gamma_2\big) = 0 \quad (6.2.11)$$

Thus, if $\Delta(a,b) = 0$ and (6.2.11) holds there is 1 (triple) real root, and hence 1 candidate solution (and its value is: $d_{3e} = -\frac{\delta_2}{3\delta_3} = -\frac{a\,\gamma_2 - \bar{r}(2a+b\,\theta)}{3a(2a+b\,\theta)}$). Else, if (6.2.11) does not hold there is 1 double real root and distinct single real root, thus 2 candidate solutions.

6.2.3 Qualitative Behavior of Equilibrium Points

The qualitative behavior of the system near the equilibrium points should be explored. This is achieved via the Linearized System around each equilibrium point. Linearizing system (4.3) we obtain:

$$\begin{pmatrix} \dot{d(t)} \\ \dot{\mu(t)} \end{pmatrix} = \begin{pmatrix} -\frac{a}{b} - \frac{2\,b\,\mu_e^{CO}}{{\gamma_1}^3(1-\omega)\omega} + \frac{(a+b\,\bar{r}+b^2\,\bar{u})}{b\,{\gamma_1}^2} & -\frac{1}{{\gamma_1}^2(1-\omega)\omega} \\ -(\omega\beta_F + (1-\omega)\beta_M) - \frac{2\,(a+b\,\bar{r}+b^2\,\bar{u})\,\mu_e^{CO}}{{\gamma_1}^3} + \frac{3(b\,\mu_e^{CO})^2}{{\gamma_1}^4(1-\omega)\omega} & \frac{a}{b} + \theta + \frac{2\,b\,\mu_e^{CO}}{{\gamma_1}^3} - \frac{(a+b\,\bar{r}+b^2\,\bar{u})}{b\,{\gamma_1}^2(1-\omega)\omega} \end{pmatrix} \begin{pmatrix} d(t) \\ \mu(t) \end{pmatrix} \quad (6.2.12)$$

where (d_e^{CO}, μ_e^{CO}) is the equilibrium point under discussion.

The eigenvalues of the linearized system (6.2.12) are:

$$eig_{1,2} = \frac{(1-bd_e^{CO})^3\,\theta \;\pm\; \sqrt{\frac{(1-bd_e^{CO})^4}{\omega(1-\omega)}\left(\omega(1-\omega)(1-bd_e^{CO})^2\,\theta^2 - 4\,{}^{\partial\, g(d)}\!/_{\partial d}\right)}}{2(1-bd_e^{CO})^3} \qquad (6.2.13)$$

It is pretty straightforward that $\frac{\left(1-bd_e^{CO}\right)^4}{\omega(1-\omega)} > 0$. Hence, the sign of the quantity under the root and consequently the type of equilibria are subject to the following rules:

- If $\omega(1-\omega)(1-bd_e^{OL})^2\,\theta^2 - 4\,{}^{\partial\, g(d)}\!/_{\partial d} > 0 \; and \; {}^{\partial\, g(d)}\!/_{\partial d} < 0$,
  then $eig_1 > 0 \; and \; eig_2 < 0$. Thus, the steady state is a saddle point.

- If $\omega(1-\omega)(1-bd_e^{OL})^2\,\theta^2 - 4\,{}^{\partial\, g(d)}\!/_{\partial d} > 0 \; and \; {}^{\partial\, g(d)}\!/_{\partial d} > 0$,
  then $eig_1 > 0 \; and \; eig_2 > 0$. Thus, the steady state is an unstable node.

- If $\omega(1-\omega)(1-bd_e^{OL})^2\,\theta^2 - 4\,{}^{\partial\, g(d)}\!/_{\partial d} < 0$, then $eig_{1,2}$ are complex eigenvalues with positive real part. Thus, the steady state is an unstable focus.

- Last but not least, there is the case an eigenvalue becomes zero. That is, when $\Delta(a,b,\omega) = 0$ and (6.2.11) holds (a triple real root). As we have already argued the equilibrium point is also a root of the first derivative. It is then straightforward from (6.2.13) that an eigenvalue becomes zero while the other will be positive. This means the system matrix has a non-trivial null-space and has an equilibrium subspace than an equilibrium point. Due to the positive eigenvalue all trajectories diverge away from the equilibrium subspace.

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