NATIONAL TECHNICAL UNIVERSITY OF ATHENS
POSTGRADUATE COMPUTATIONAL MECHANICS

STOCHASTIC SOLUTION FOR NON LINEAR PROBLEM WITH POLYNOMIAL CHAOS METHOD

Master Thesis

Dimitris Karachalios

As. Prof. **Vissarion Papadopoulos**

Athens

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Abstract

In the deterministic case, the solution of a nonlinear problem can easily approached by iterative methods. In the case of, that we have to deal with a stochastic nonlinear problem the domain does not however, have a physical meaning that permits a sensible discretization. In this context the abstract Hilbert space foundation of the finite element method becomes useful as it can be extended to deal with random functions. In this thesis we intended to solve one nonlinear problem with a random variable. In a more applied sense, it is sought to describe random processes in such a manner that they can be implemented in a finite element formulation of the physical problem.

I would like also to express my gratitude to As. Professor Vissarion Papadopoulo for the assignment of this thesis and Dr. Dimitri Giovani for all explanations and programming implementations.

1 Introduction

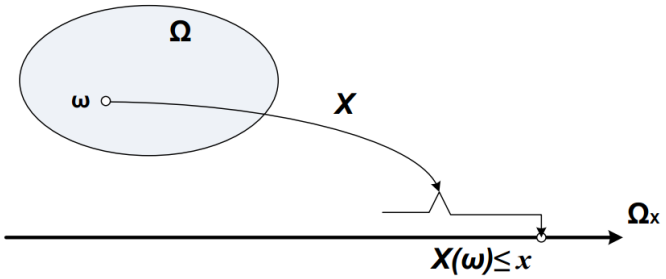
1.1 Definitions in Stochastic Measurement

1.1.1 Random Variables

Definition
A **random variable** $X(\omega)$, $\omega \in \Omega$ is a function defined on a sample space Ω such that for every real number x there exists the probability $P[\omega : X(\omega) \leq x]$. In more simple terms a random variable is a rule that assigns a numerical value to each possible outcome of a probabilistic experiment. Examples:

• The age of a randomly selected student here today.

• The values of the Young's modulus



Random variables can be:

1. **Discrete**, that is, taking any of a specified finite or countable list of values.
2. **Continuous**, taking any numerical value in an interval or collection of intervals.

Rule of thumb: before the experiment is run, if you can determine/list all possible values of the random variable, it is a discrete random variable, else it is a continuous random variable.

1.1.2 Cumulative distribution function (CDF)

The probability distribution or **cumulative** distribution function (CDF) $FX(x)$ describes the **probability** that a random variable X takes on a value less than or equal to a number x.

$$F_x(x) = P\left[X \leq x\right] = P\left\{\omega \in \Omega, -\infty \leq X(\omega) \leq x\right\}$$

Properties:

- $\lim_{x\to\infty} F_x(x) = 1$
- If $a \leq b$ then $F_x(a) \leq F_x(b)$.
- For any positive **(** $\varepsilon > 0$ **)**
- $\lim_{\varepsilon\to 0} F_x(a+\varepsilon) = \lim_{a\to\infty} F_x(a^+)$

- $F_x(x)$ is a continuous from the right

- $P\left[a \le x \le b\right] = F_x(b) - F_x(a)$

1.1.3 Probability density function (PDF)

The probability density function (PDF) *fX*(*x*) is defined to be that function when **integrated** yields the **CDF**:

$$F_x(x) = \int_{-\infty}^{x} f_x(x)dx \Rightarrow f_x(x) = \frac{dF_x(x)}{dx}$$

Properties:

- $f_x(x) \ge 0$
- $\int_{-\infty}^{+\infty} f_x(\xi)d\xi = 1$
- $\int_{-\infty}^{x} f_x(\xi)d\xi = F_x(x)$

A probability density function can be **unimodal and/or multimodal**. The **mode** is the value that appears most often in a set of data.

1.1.4 Moments of random Variables

- $m_n(x) = E\left[X^n\right] = \int_{-\infty}^{+\infty} x^n f_x(x)dx$

- $\mu_x = m_1 = E\left[X\right] = mean$

- $m_2 = E\left[X^2\right] = mean\ square$

Central Moments

- $K_n(x) = E\left[\left(X - \mu_x\right)^n\right] = \int_{-\infty}^{+\infty} \left(x - \mu_x\right)^n f_x(x)dx$

- $\sigma_x^2 = K_2(x) = E\left[\left(X - \mu_x\right)^2\right] = Variance$

- $\sigma_x = \sqrt{Variance} = Std$ **(Standard Deviation)**

- $Var(x) = E\left[X^2\right] - E\left[X\right]^2 = E\left[X^2\right] - \mu_x^2$

Coefficient of Variation

$$C.O.V = \frac{\sigma_x}{\mu_x}$$

Coefficient of Skewness

Skewness is a measure of the asymmetry of the probability distribution of a random variable about its mean:

$$\gamma_1 = \frac{K_3}{\sigma_x^3}$$

Coefficient of Kyrtosis

$$\gamma_2 = \frac{K_4}{\sigma_x^4}$$

It is often compared to a **standard value** of 3. The value of 3 is chosen only because it is the value of the **kurtosis coefficient of the Gaussian distribution**. A $(\gamma_2 - 3) > 0$, implies a slim sharp peak in the neighborhood of a mode in a unimodal distribution (sharper than the Gaussian distribution having the same σ_x), while a negate $(\gamma_2 - 3) < 0$ implies as a rule, a flattened peak (flatter than the Gaussian distribution having the same σ_x).

1.1.5 Functions of random variables

Mapping

Consider the problem where we have two random variables *X* and *Y* related as *Y* = *g*(*X*). We need to compute the cumulative distribution of *Y* in terms of the cumulative distribution of *X*. The easiest case for transformations of continuous random variables is the case of *g* to be 1–1.

• We first consider the case of *g* increasing on the range of the random variable *X*. In this case, g^{-1} is also an increasing function.

$$F_Y(y) = P\{Y \le y\} = P\{g(X) \le y\} = P\{X \le g^{-1}(y)\} = F_X\left(g^{-1}(y)\right)$$

• Consider the case of *g* decreasing on the range of the random variable *X*. function.

$$F_Y(y) = P\{Y \le y\} = P\{g(X) \le y\} = P\{X \ge g^{-1}(y)\} = 1 - F_X\left(g^{-1}(y)\right)$$

The probability density function is then obtained as:

$$f_Y(y) = f_x\left(g^{-1}(y)\right)\left|\frac{d}{dy}\left(g^{-1}(y)\right)\right|$$

Moments of functions of random variables

- $mean = E\left[G(X)\right] = \int_{-\infty}^{+\infty} G(X)\, f_X(x)dx$
- $Variance = E\left[\left(G(X) - E\left[G(X)\right]\right)^2\right] = \int_{-\infty}^{+\infty} \left(G(X) - E\left[G(X)\right]\right)^2 f_X(x)dx$

Jointly distributed variables

Joint probability distribution function

$$F_{XY}(x,y) = P\left[\left(X \le x\right) \cap (Y \le y)\right]$$

Properties:

- $F_{XY}(-\infty,-\infty) = 0$
- $F_{XY}(+\infty,+\infty) = 1$
- $F_{XY}(-\infty,y) = F_{XY}(x,-\infty) = 0$

Joint probability density function

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(\xi_X,\xi_Y)d\xi_X d\xi_Y$$

If the random variables *X* and *Y* are statistically **independent** (the occurrence of one does not affect the probability of the other) **if and only if**

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

Moments of Jointly distributed random variables

$$\mu'_{\nu\nu'} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^{\nu} y^{\nu'} f_{XY}(x,y)dxdy$$

- $\mu'_{00} = 1$
- $\mu'_{10} = \mu_X$
- $\mu'_{01} = \mu_Y$

Central Moments of jointly distributed random variables

$$\mu'_{\nu\nu'} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(x - E\left[X\right]\right)^2 \cdot \left(y - E\left[Y\right]\right)^2 f_{XY}(x,y)dxdy$$

- $\mu_{00} = 1$
- $\mu_{10} = \mu_{01} = 0$
- $\mu_{20} = Var(x)$
- $\mu_{02} = Var(y)$

Covariance

$$\mu_{11} = K_{XY} = E\left[\left(X-\mu_X\right)\left(Y-\mu_Y\right)\right] = \int_{-\infty}^{y}\int_{-\infty}^{x}\left(\xi_X-\mu_X\right)\left(\xi_Y-\mu_Y\right) f_{XY}(\xi_X,\xi_Y)d\xi_X d\xi_Y$$
$$= E\left[XY\right]-\mu_X\mu_Y$$

If the random variables are **uncorrelated**

$$K_{XY}=0 \Leftrightarrow E\left[XY\right]=\mu_X\mu_Y$$

$$\rho_{XY}=\frac{K_{XY}}{\sigma_X\sigma_Y}$$

Properties:

- $\left|\rho_{XY}\right| \le 1 \Leftrightarrow \left|K_{XY}\right| \le \sigma_X\sigma_y$
- If $\rho_{XY}=\pm 1$ then the random variables *X* and *Y* are fully correlated, i.e.: $Y=aX+b$.
- If $\rho_{XY}=0$ then the random variables *X* and *Y* are uncorrelated.

Important:

IF *X*, *Y* : Independent $\Rightarrow$ *X*, *Y* : Uncorrelated
X, *Y* : Independent : **IF** *X*, *Y* : Uncorrelated

1.1.6 Gaussian (Normal) random variable

A random variable *X* is considered to be normally distributed ($X \sim N(\mu, \sigma 2)$) if its probability density function is of the form:

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < +\infty$$

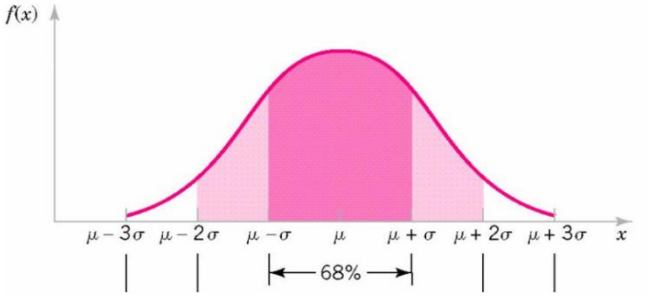
where μ is its mean value and σ the standard deviation.

This probability density is often called the Laplacian density by the French [after the French mathematician the Marquis Pierre Simon De Laplace (1749-1827)] and the Gaussian density by the Germans [after the German mathematician Johann Friedrich Carl Gauss (1777-1855)]. Partly for historical reasons, probability theorists and statisticians commonly use normal, while physicists and engineers often use Gaussian. We shall use all these names interchangeably. If $\mu = 0$ and $\sigma = 1$ the random variable is the standard Gaussian random variable, denoted as $Z \sim N(0, 1)$ and the $\varphi(x)$ is the

standardized pdf. The corresponding probability distribution function $\Phi(x)$ is obtained by integration of $\varphi(x)$.

$$\Phi(x) = \int_{-\infty}^{x} \Phi(\xi) d\xi = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\xi-\mu)^2}{2\sigma^2}} d\xi$$

$$P(\mu - \sigma < X < \mu + \sigma) = 0.6827$$
$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9545$$
$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$



1-1 Normal Distribution with quartiles

Using the following change of variable

$$\zeta = \frac{\xi - \mu}{\sigma}\ ,\ z = \frac{x - \mu}{\sigma}$$

we obtain the cumulative distribution function of the **standard Gaussian** random variable Z

$$\Phi_Z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{\zeta^2}{2}} d\zeta$$

Central Limit Theorem

I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the Law of Frequency of Error. The law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement, amidst the wildest confusion. The huger the mob, and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of Unreason. Whenever a large sample of chaotic elements are taken in hand and marshaled in the order of their magnitude, an unsuspected and most beautiful form of regularity proves to have been latent all along.–Sir Francis Dalton (1889). Let Xn denote a sequence of independent random variables with mean value μ and finite variance σ^2. The Central Limit Theorem states that the arithmetic mean as n → +∞ will be approximately normally distributed, regardless of the underlying distribution,

$$\frac{1}{\sqrt{n}}\left[\left(\sum_{i=1}^{n} X_i\right)-\mu\right] \sim \mathrm{N}\left(0,\sigma^2\right) \quad for \quad n\to\infty$$

Because of its generality, this theorem is often used to simplify calculations involving finite sums of non- Gaussian random variables. However, attention is seldom paid to the convergence rate of the Central Limit Theorem. Consequently, the Central Limit Theorem, as a finite-sample distributional approximation, is only guaranteed to hold near the mode of the Gaussian, with huge numbers of observations needed to specify the tail behavior.

1.1.7 Gaussian random vectors

When working with all these random variables together, we will often find it convenient to put them in a vector $X=\left[X_1, X_2, ..., X_n\right]^T$. We call the resulting vector a random vector (more formally, a random vector is a mapping from Ω to $\mathbb{R}^n$. It should be clear that random vectors are simply an alternative notation for dealing with *n* random variables, so the notions of joint pdf and cdf will apply to random vectors as well. Consider an arbitrary function $g(\mathbf{X})$: $g(X):\mathbb{R}^n \to \mathbb{R}$ of the random vector **X**. The mathematical expectation of this function can be defined as:

$$E\left[g(X)\right]=\int_{R^n} g(x_1,x_2,...,x_n)\, f_{X_1,X_2,...,X_N}(x_1,x_2,...,x_n)dx_1dx_2...dx_n$$

where $\int_{R^n}$ is *n* consecutive integrations from $-\infty$ to ∞. If *g* is a function from , then the value of *g* is the element-wise expected values of the output vector. The covariance matrix **C** of the random vector is the *nxn* matrix whose entries are given b $\left[C_{ij}\right]_{nxn}=Cov\left[X_i,X_j\right]=E\left[\left(X_i-\mu_{X_i}\right)\left(X_j-\mu_{X_j}\right)\right]$. The covariance matrix has a number of useful properties like being positive semi-definite and symmetric. The mean value vector μ_X of a random vector **X** is a vector containing the mean value of each component random variable

$$\mu_X=\left[\mu_{X_1},\mu_{X_2},...,\mu_{X_n}\right]^T$$

The multivariate Gaussian distribution.

One particularly important example of a probability distribution over random vectors X is called the multivariate Gaussian or multivariate normal distribution. A random vector $X \in \mathbb{R}^n$ is said to have a multivariate normal (or Gaussian) distribution with mean $m \in \mathbb{R}^n$ and covariance matrix C.

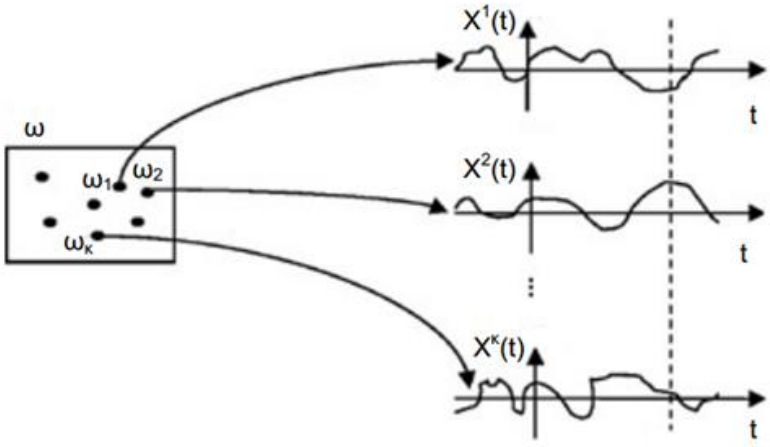
$$f_X(x) = f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n; \mu, C) = \frac{1}{(2\pi)^{\frac{n}{2}} C^{\frac{1}{2}}} \exp\left(-\frac{1}{2} (X - \mu)^T C^{-1} (X - \mu) \right)$$

We written this as $X \sim (\mu, C)$.

1.2 Definitions of Stochastic Processes

• Random variable $X(\omega)$: Mapping from a random outcome ω to a real number.

• Random (stochastic) process $X(t,\omega)$: Mapping from a random outcome ω to a *function*. A random process can be a function of time and/or space.
For a given outcome $\omega = \omega_k$ the function $X(t,\omega k)$ is called a sample function.
Therefore, a stochastic process is a collection of functions (or a family of functions) generated by the random outcome ω.



The parameter *t* of the random process $X(t,\omega)$ can be either:

1. **Discrete**: Discrete-time random process

2. **Continuous**: Continuous-time random process

At any given time *t* the random process $X(t,\omega)$ represents a **random variable** and as the time changes the random process generates different random variables

$$X(t_1,\omega k), X(t_2,\omega k),...$$

1.2.1 Probability functions of random processes

Since a random process $X(\mathbf{t}, \omega)$ is a random variable at any given time *t*, the process generates a sequence of random variables at a series of discrete time instances $t_1, t_2, ..., t_k$:

$$X_1 = X(t_1,\zeta), X_2 = X(t_2,\zeta),..., X_k = X(t_k,\zeta)$$

Therefore we can define the joint probability functions (CDF,PDF) of these random variables:

1. Joint Cumulative distribution function (cdf)

$$F_{X_1,X_2,...,X_k}(x_1,x_2,...,x_k) = P\left[X_1 \le x_1, X_2 \le x_2,..., X_k \le x_k\right]$$

2. Joint Probability density function (pdf)

$$f_{X_1,X_2,...,X_k}(x_1,x_2,...,x_k)$$

1.2.2 Moments of random process

Let $f_{X(t)}(x)$ be the pdf of a random variable $X(t)$. We can define the kth moment as:

$$E\left[\left(X(t)\right)^k\right]=\int_{-\infty}^{+\infty}x^k f_{x(t)}(x)dx$$

A special case of the kth moment is the **mean**:

$$\mu_X(t)=E\left[X(t)\right]=\int_{-\infty}^{+\infty}xf_{x(t)}(x)dx$$

If $f_{X(t)}(x)$ be the pdf of a random variable $X(t)$ then the kth central moment is defined as:

$$E\left[\left(X(t)-\mu_X(t)\right)^k\right]=\int_{-\infty}^{+\infty}(x-\mu_X(t))^k f_{x(t)}(x)dx$$

A special case of the kth central moment is the **Variance**:

$$Var_X(t)=E\left[\left(X(t)-\mu_X(t)\right)^2\right]=\int_{-\infty}^{+\infty}(x-\mu_X(t))^2 f_{x(t)}(x)dx$$

1.2.3 Joint moments of a random process

Two sample times $X(t_1), X(t_2)$ of the random process $X(t)$ represent two random variable with a joint pdf $f_{X(t_1),X(t_2)}(x,y)$

<u>Autocorrelation function</u>

The autocorrelation function $R_X(t_1,t_2)$ of the random process is defined as:

$$R_X(t_1,t_2)=E\left[X(t_1)X(t_2)\right]=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}x_1x_2 f_{X(t_1),X(t_2)}(x_1,x_2)dx_1dx_2$$

Properties:

- Symmetry: $R_X(t_1,t_2)=R_X(t_2,t_1)$
- Cauchy – Schwarz inequality: $R_x^2(t_1,t_2)\le R_X(t_1,t_1)R_X(t_2,t_2)$
- Non-negative definite: $\sum_{j=1}^{n}\sum_{k=1}^{n}R_X(t_j-t_k)g(t_j)g(t_k)\ge 0$
- $\lim_{\tau\to 0}R_X(\tau)=\sigma_X^2$ and $\lim_{\tau\to\infty}R_X(\tau)=0$.

The **cross-correlation function** of the random process is defines as:

$$R_{XY}(t_1,t_2) = E\left[X(t_1)Y(t_2)\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xyf_{X(t_1),X(t_2)}(x,y)dxdy$$

The **autocovariance function** of the random process is defined as:

$$C_X(t_1,t_2) = E\left[\left\{X(t_1)-\mu_X(t_1)\right\}\left\{X(t_2)-\mu_X(t_2)\right\}\right]$$

The value of $C(t,t)$ on the diagonal $t_1 = t_2 = t$ equals the variance of $X(t)$ at the time t.

$$Var\left(X(t)\right) = C_X(t,t) = E\left[\left\{X(t)-\mu_X(t)\right\}^2\right]$$

The Cross-covariance function $C_X(t_1,t_2) = R_X(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$

Autocorrelation coefficient:

$$\rho_X(t_1,t_2) = \frac{C_X(t_1,t_2)}{\sqrt{C_X(t_1,t_1)}\sqrt{C_X(t_2,t_2)}}$$

Several models of autocorrelation coefficient functions for homogeneous random fields have been proposed. Common models are:

$$\rho_X(\tau) = e^{-\frac{|\tau|}{b}}$$

$$\rho_X(\tau) = e^{-\left(\frac{|\tau|}{b}\right)^2}$$

The parameter *b* is the correlation length of the respective correlation models. A small correlation length signifies fast reduction of the correlation coefficient as the distance τ increases and thus a high variability in the random field realization. Conversely, large correlation lengths correspond to slowly varying realizations. Moreover, the limit case of an infinite correlation length can be modeled by one random variable. **Note:** The mean and correlation function of a stochastic process provide a partial characterization of the process, referred to as **second-moment characterization**. It is clear that stochastic processes with the same second-moment characteristics can have very different sample properties.

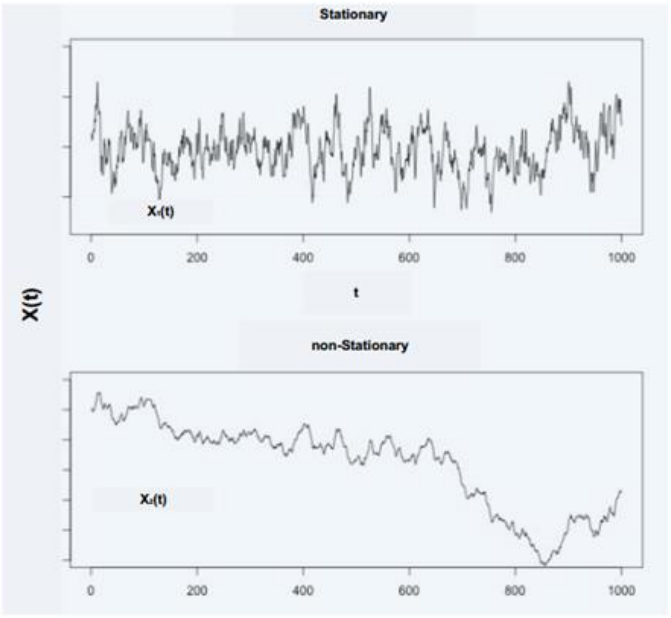
1.2.4 Stationary random process

Stationary random process is a stochastic process whose joint probability distribution **does not change when shifted in time**. Consequently, parameters such as the mean and variance, if they are present, also do not change over time and do not follow any trends.

$$\mu_X(t) \to \mu_x \ , \ Var(X(t)) \to Var(X) \ , \ R_X(t_1,t_2) \to R_X(\tau)$$
$$C_X(t_1,t_2) \to C_X(\tau) \ , \ \ \rho_X(t_1,t_2) \to \rho_X(\tau) \ , \ \tau = t_2 - t_1$$

For a stationary random process, the bounds of $R_X(\tau)$ and $C_X(\tau)$ is:

- $\left|R_X(\tau)\right| \le R(0) \ , \ \ R_X(0) = E\left[X(t)^2\right] = const$
- $C_X(\tau) \le \sigma_X^2 = Variance$



1-2 Stationary and non – Stationary stochastic process

1.2.5 Ergodicity of a stochastic process

The study of **Ergodic Theory** was first introduced in classical statistical mechanics and kinetic theory in order, for example, to relate the average properties of a particular system of molecules in a closed space to the ensemble behavior of all molecules at any given time. For the theory of random processes to be useful, we need to estimate the mean value and autocorrelation function by measurements. This estimation is done by using a large number of sample functions or realizations of the random process.

Enseble Averages(averages over the entire number of sample functions).The *interchangeability* of ensemble and time averages has considerable appeal in practice because, when statistical averages of a stochastic process need to be computed, what is generally available is not a representative collection of sample functions but rather certain pieces of sample functions or a long, single observation of one sample function.

$$E\left[X(t_1)\right] = \mu_X(t_1)$$
$$E\left[X(t_1)X(t_2)\right] = R_X(t_1,t_2)$$

Unfortunately, we usually have only a very small number of sample functions of the random process. One naturally asks if certain statistical averages of a stochastic process can be determined from appropriate time averages associated with a single sample function. If a random process is stationary, SOMETIMES it is possible to estimate the mean and autocorrelation function from just ONE sufficiently LONG sample function. A stationary random process is called ERGODIC if its second-order information can be obtained from a single realization $X(t)$ of the field. That is, the ensemble averages equal the corresponding time averages with probability one.

Ergodicity and stationarity

• Wide-sense stationary: Mean and Autocorrelation is constant over time

• Strictly stationary: All statistics is constant over time

Weak form of ergodicity: The complete statistics is often difficult to estimate so we are often only interested in:

1. Ergodicity in the Mean

2. Ergodicity in the Autocorrelation

1.2.6 Spectral density function

The theories of Fourier series and integrals cannot be applied directly to random signals. This is because the periodic requirement for the Fourier series is not met which rules out the Fourier series. Moreover, if we consider a random signal to continue over infinite time, neither the real or imaginary part of the Fourier transform converges to a steady value which is why it is not possible to use the concept of Fourier integrals: Instead, we introduce the spectral density, which has no convergence problems.

The power spectrum of a random process X(t) describes how the variance of the data $x(t)$ is distributed over the frequency components into which they may be decomposed. A theorem due to S. BOCHNER asserts that every non-negative definite function has a nonnegative Fourier.

Consider a stationary ergodic random process $x(t)$ which is assumed to have started at $t = -\infty$ and continue until $t = +\infty$. Such a signal is not periodic and it is thus impossible to define its Fourier transform. It is possible, however, to determine the Fourier transform of a signal which is equal to $x(t)$ over the interval $-\frac{T}{2} \le t \le \frac{T}{2}$ and zero at all other times.

The spectral density or power density of $x(t)$ is now defined as:

$$S(\omega) = \lim_{T\to\infty}\left(\frac{1}{\pi T} \left| A_T(\omega) \right|^2 \right)$$

so that

$$\sigma_X^2 = \int_0^\infty S(\omega) d\omega$$

The power density indicates how the harmonic content of $x(t)$ is spread over the frequency domain. The amount of $x^2(t)$ associated with a narrow frequency band $\Delta\omega$ is equal to $S(\omega)\Delta\omega$. Different realizations of a stationary ergodic random process have a common $S(\omega)$. The power density $S(\omega)$ is thus a constant, non-random, statistical parameter of the random process $x(t)$. If a signal has a spectrum that is uniform (constant) over the whole frequency domain, the spectrum is said to be white and the signal is referred to as white noise.

1.3 The Mathematical Model

1.3.1 Hilbert Spaces

The class of problems dealt with in this study is not of the conventional engineering kind in that it involves concepts of a rather abstract and mathematical nature. It is both necessary and instructive to introduce at this point the mathematical concepts which are used in thesis. The Hilbert space of functions (Oden, 1979) defined over a domain **D**, with values on the real line $\mathbb{R}$, is denoted by **H**. Let (Ω, Ψ, P) denote a probability space. Let x be an element of **D** and **θ** be an element of **Ω.** Then, the space of functions mapping Ω onto the real line is denoted by **Θ**. Each map $\Omega \to \mathbb{R}$ defines a random variable. Elements of **H** and **Θ** are denoted by roman Greek letters respectively. Capital letters are used to denote algebraic structures and spaces as well as operators defined on these spaces, with Greek letters referring again to those operators defined on spaces of random functions.

1.3.2 Inner product over Hilbert Spaces

The inner products over **H** and over **Θ** are defined using the Lebesgue measure and the probability measure respectively. That is, for any two elements $h_i(x)$ and $h_j(x)$ in **H,** their inner product defined as

$$(h_i(x), h_j(x)) = \int_D h_i(x) h_j(x) dx \tag{1.7}$$

The domain **D** represents the physical space over which the problem is defined. Similarly, given any two elements $\alpha(\theta)$ and $\beta(\theta)$ in **Θ,** their inner product is defined as

$$(\alpha(\theta), \beta(\theta)) = \int_\Omega \alpha(\theta)\beta(\theta) dP \tag{1.8}$$

where dP is a probability measure. Under very general conditions, the integral in equation (1.8) is equivalent to the average of the integrand with respect to the probability measure dP, so that

$$(\alpha(\theta), \beta(\theta)) = \alpha(\theta)\beta(\theta) \tag{1.9}$$

Any two elements of the Hilbert spaces defined above are said to be orthogonal if their inner product vanishes. A random process may then be described as a function defined on the product space $D \times \Omega$. Viewed from this perspective a random process can be regarded as a curve in either of **H** or **Θ.**

1.3.3 Mathematical Representation of the General Problem

The physical model under consideration involves a medium whose properties exhibit random spatial fluctuations and which is subjected to a random external excitation. The mathematical representation of this problem involves an operator equation

$$\Lambda(x, \theta)u(x, \theta) = f(x, \theta) \tag{1.10}$$

where $\Lambda(x, \theta)$ is some operator defined on $\mathrm{H} \times \Theta$. In other words, **Λ** is a differential operator with coefficients exhibiting random fluctuations with respect to one or more of independent variables. The aim then is to solve for the response $u(x, \theta)$ as a function of both arguments. With no loss of generality, **Λ** is assumed to be a differential operator, whose random coefficients are restricted to being second order random processes. This is not a severe restriction for practical problems, since most physically measurable processes are of the second order type. Then, each one of these coefficients $a_k(x, \theta)$ can be decomposed into a purely deterministic component and a purely random component in the form

$$a_k(x, \theta) = \bar{a}_k(x) + a_k(x, \theta) \tag{1.11}$$

where $\bar{a}_k(x)$ is equal to the mathematical expectation of the process $a_k(x, \theta)$, and $a_k(x, \theta)$ is a zero mean random process, having the same covariance function as the process $a_k(x, \theta)$. Equation (1.10) can be written as

$$\left[L(x) + \Pi(x, \theta)\right]u(x, \theta) = f(x, \theta) \tag{1.12}$$

where $L(x)$ is a deterministic differential operator and $\Pi(x, \theta)$ is a differential operator whose coefficients are zero-mean random processes. Before a solution to equation (1.12) is sought, it is essential to clarify what is meant by such a solution.

As was mentioned above, **Θ** denotes the Hilbert space of functions defined on the σ-field of events generated by the physical problem, with range the interval [0,1]. In other words, if the possible realizations of the random process were numbered continuously on the interval [0,1], and these numbers were assigned to the variable θ, then, for a fixed $\theta = \theta^* \in [0,1]$, the process $a_k(x, \theta^*)$ is a deterministic function of x, a realization of the process. From observing a finite number of realizations of this process, distribution theory can be used to construct the distribution of the process along the θ-dimension. For a given x, $a_k(x, \theta)$ is a random variable with such a

distribution. Obviously, for a complete description of the process, the joint distribution at all $x \in D$ is required. However, if the process is assumed to be Gaussian, all the finite dimensional distribution functions are also Gaussian. Clearly, the usually limited number of observed realizations of a random process cannot, in general, suggest any definite distribution. However, invoking the central limit theorem, the Gaussian distribution appears to be the most likely candidate for many physical applications.

Once the coefficients in the operator equation (1.10) have been defined through their probability distribution functions, the main question remains as to what is meant by a solution to the problem. Obviously, the discussion and comments just made concerning the coefficient processes apply to the solution process as well. A conceptual modification, through, can be introduced. Specifically, a quite general form of the solution process can be expressed as

$$u = g\left[a_k(x,\theta),\, f(x,\theta), x\right] \tag{1.13}$$

where $g[.]$ is some linear functional of its arguments. Clearly, a complete description of the response would involve the prescription of its joint distribution with the various processes appearing in equation (1.13). This information could form the basis for a rational reliability and risk assessment. However, given the infinite dimensional structure of the random process appearing in equation (1.13), such a task seems to exceed the capability of currently used methods. A finite – dimensional description of the processes involved is required if the solution is to proceed in a computational setting. Given the abstract nature of the functional spaces over which random processes are defined, a finite dimensional representation cannot be achieved through partitioning of these spaces as it usually done with the deterministic finite element analysis. Alternatively, an abstraction of the discretization process can be introduced which is mathematically equivalent to a discretization with respect to a spectral measure. Indeed, a spectral representation is introduced in the text which permits the algebraic manipulation of random processes through that of an equivalent discrete set of random variables.

2 Representation of Stochastic Processes

2.1 The Mathematical Model

2.2 General Theory of Random Processes

A continuous random process is formally defined as an indexed set of random variables, the index belonging to some continuous uncountable set (Ghanem & Spanos, 2003). Then, the process can be approximated as closely as desired be restricting the index to a set dense in the indexing set. Stretching mathematical rigor further, a random process can be represented by its values at a discrete set of points in its domain of definition. It is then clear how the foregoing discussion for random variables can be extended to the case of random processes. In the context of the finite element analysis of a given system, the random processes involved are substituted for by random variables that are so chosen as to coincide with some local average of the process over each element. It is expected, however, that the result would depend to a notable extend on the averaging method used. Local averages are usually of two kinds. These are weighted average over each a subset of the indexing set, and the collocation average over each such subset whereby the process over the subset is replaced by its value at some point in the subset. It is clear that, in general, the first approach smoothes the random process, whereas the second approach introduces additional irregularities. This suggests that the two approaches provide lower and upper bounds, respectively, for the random behavior of the process. In the limit, as the size of each subset becomes vanishingly small, the representation resulting from the two approaches should converge to the exact process. It is obvious that a relatively large number of random variables is required to represent a random process in this fashion. It is noted that local averaging parallels point wise approximations of deterministic functions. The size of the individual subsets used for the averaging process does in general depend on the frequency content of the process. That is, the broader the frequency content of the process, the smaller the region over which the process shows a definite pattern, and thus the smaller the size of the necessary subset to meet a certain precision criterion. This problem with local averaging is particularly crucial in the context of the finite element analysis of structures with curved and irregular geometry. The finite element analysis of these systems usually requires recourse to curved elements and non-uniform spacing of the nodal points. The shape and size of the finite element mesh is generally dictated by the stress distribution within the structure. This stress distribution is usually independent of that of the uncertainty of the random parameters involved. This fact necessitates either the use of an independent mesh for the simulation, or the use of a mesh size such that both stresses and material properties are adequately and consistently represented. In either case, the dimension of the problem, as reflected by the number of random variables used to represent the underlying processes, is quite large. The associated computational problem is, in general, of prohibitive dimensions.

Alternatively to the heuristic argument associated with the local averaging approach, a rigorous exposition of the basic concepts of the theory of representation for random processes can be formulated (Parsen, 1959). This theory is a quite rich and mature mathematical subject. The development of the theory parallels that of the modern theory of random processes, and has had its origin in the need for more sophisticated models in applied statistics. Most of the related results have been derived for the class of second order processes. Perhaps the most important results is the spectral

representation of random processes (Gel'fand and Vilenkin), which, in its most general form, can be stated as

$$\omega(x,\theta) = \int g(x) d\mu(\theta) \tag{2.1}$$

where $\omega(x,\theta)$ is a stochastic process whose covariance function $C_{\omega\omega}(x_1, x_2)$ admits of the decomposition

$$C_{\omega\omega}(x_1, x_2) = \int g(x_1) g(x_2) < d\mu_1(\theta) d\mu_2(\theta) > \tag{2.2}$$

In equation (2.1), $g(x)$ is a deterministic function. Further, $d\mu(\theta)$ is an orthogonal set function, also termed orthogonal stochastic measure, defined on the σ-field Ψ of random events. An important specialization of the spectral decomposition occurs if the process $\omega(x,\theta)$ is wide sense stationary. In this case, equation (2.1) can be shown to reduce to the Wiener-Khintchine relation (Yaglom, 1962) and the following equations hold

$$\omega(x,\theta) = \int_{-\infty}^{+\infty} e^{ix^T\omega} d\mu(\omega,\theta) \tag{2.3}$$

and

$$C_{\omega\omega}(x_1, x_2) = \int_{-\infty}^{+\infty} e^{i(x_1 - x_2)^T\omega} S(\omega) d\omega^T \tag{2.4}$$

Here, the symbol T denotes vector transposition, S(ω) is the usual spectral density of the stationary process, and ω is the wave number vector.

The preceding representations have had a strong impact on the subsequent development of the theory of random processes. However, their applications have been restricted to randomly excited deterministic systems. This is largely attributed to the fact ha all of these representations involve differentials of random functions, and are therefore set in an infinite dimensional space, not readily amenable to computational algorithms. An alternative formulation of the spectral representation, and one which is extensively used in the sequel, is the Karhunen – Loeve expansion whereby a random process $\omega(x,\theta)$ can be expanded in terms of a denumerable set of orthogonal random variables in the form

$$\omega(x,\theta) = \sum_{i=1}^{\infty} \mu_i(\theta) g_i(x) \tag{2.5}$$

where $\{\mu_i(\theta)\}$ is a set of orthogonal random variables and $g_i(x)$ are deterministic functions, which can be related to the covariance kernel of $\omega(x,\theta)$. Note that since equation (2.5) constitutes a representation of the random process in terms of a denumerable set of random variables, it may be regarded as an abstract discretization of the random process. Further, it is important to note that this equation, can be viewed

as a representation of the process $\omega(x,\theta)$ as a curve in the Hilbert space spanned by the set $\{\xi_i(\theta)\}$. The random process $\omega(x,\theta)$ is expressed as a direct sum of orthogonal projections in this Hilbert space hereby the magnitudes of the projections on successive basis vectors are proportional to the corresponding eigenvalues of the covariance function associated with the process $\omega(x,\theta)$.

The representations discussed thus far can be thought of as linear operators or filters acting on processes with independent increments (Doob, 1953). Interestingly, these concepts can be generalized to allow for the representation of nonlinear functional of the orthogonal stochastic measures $d\mu(\theta)$. The theory of non linear functional was developed by Volterra (1913). He generalized the Taylor expansion of functions to the case of functionals. It was Wiener, however, who first applied Volterra's ideas to stochastic analysis, and developed what is now known as the Homogeneous Chaos. Based on Wiener's work, Cameron and Martin (1947) developed the Fourier – Hermite expansion, which is a Fourier – type expansion for nonlinear functionals. Again, it was (Wiener, 1958) who first applied the new theory to problems involving random phenomena, using the ideas he had developed on Differential Spaces (1923) and Homogeneous Chaos (1938).

2.2.1 Derivation

One of the major difficulties associated with the numerical incorporation of random processes in finite element analyses, is the necessity to deal with abstract measure spaces that have limited physical intuitive support. The major conceptual difficulty from the viewpoint of the class of problems considered herein involves the treatment of functions defined on these abstract spaces, namely random variables defined on the σ-field of random events. The most widely used method, the Monte Carlo simulation, consists of sampling these functions at randomly chosen elements of this σ-field, in a random, collocation – like, scheme. Obviously, a quite large number of points needs to be sampled if a good approximation is to be achieved. A theoretically more sound and more appealing approach would be to expand these functions in a Fourier – type series as

$$\omega(x,\theta) = \sum_{n=0}^{\infty} \sqrt{\lambda_n}\,\xi_n(\theta)\, f_n(x) \tag{2.6}$$

where $\{\xi_n(\theta)\}$ is a set of random variables to be determined, λ_n is some constant, and $\{f_n(x)\}$ is an orthogonal set of deterministic functions. This is exactly the Karhunen Loeve expansion achieves. The expansion was derived independently by a number of investigators (Karhunen, 1947; Loeve 1948; Ka cans Siegert, 1947). Let $\omega(x,\theta)$ be a random process, function of the position vector x defined over the domain D, with θ belonging to the space of random events Ω. Let $\bar{\omega}(x)$ denote the expected value of $\omega(x,\theta)$ over all possible realizations of the process, and $C(x_1,x_2)$ denote its covariance function, it is bounded, symmetric and positive definite. Thus, it has the spectral decomposition (Hilbert, 1953)

$$C(x_1,x_2) = \sum_{n=0}^{\infty} \lambda_n\, f_n(x_1)\, f_n(x_2) \tag{2.7}$$

where λ_n and $f_n(x)$ are the eigenvalues and the eigenvector of the covariance kernel respectively. That is, they are the solution to the integral equation

$$\int_D C(x_1, x_2) f_n(x) dx_1 = \lambda_n f_n(x_2) \tag{2.8}$$

Due to the symmetry and the positive definiteness of the covariance kernel (Loeve, 1977), its eigenfunctions are orthogonal and form a complete set. They can be normalized according to the following criterion

$$\int_D f_n(x) f_m(x) dx = \delta_{nm} \tag{2.9}$$

where δ_{nm} is the Kronecker delta. Clearly, $\omega(x,\theta)$ can be written as

$$\omega(x,\theta) = \overline{\omega}(x) + \alpha(x,\theta) \tag{2.10}$$

where $\alpha(x,\theta)$ is a process with zero mean and covariance function $C(x_1, x_2)$. The process $\alpha(x,\theta)$ can be expanded in terms of the eigenfunctions $f_n(x)$ as

$$a(x,\theta) = \sum_{n=0}^{\infty} \xi_n(\theta) \sqrt{\lambda_n} f_n(x) \tag{2.11}$$

Second order properties of the random variables ξ_n can be determined by multiplying both sides of equation (2.11) by $\alpha(x_2,\theta)$ and taking the expectation on both sides. Specifically, it is found that

$$C(x_1, x_2) = < a(x_1,\theta) a(x_2,\theta) > = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} < \xi_n(\theta) \xi_m(\theta) > \sqrt{\lambda_n \lambda_m} f_n(x_1) f_m(x_2) \tag{2.12}$$

Then, multiplying both sides of equation (2.12) by $f_k(x_2)$, integrating over the domain D, and making use of the orthogonality of the eigenfunctions, yields

$$\int_D C(x_1, x_2) f_k(x_2) dx_2 = \lambda_k f_k(x_1) = \sum_{n=0}^{\infty} < \xi_n(\theta) \xi_k(\theta) > \sqrt{\lambda_n \lambda_k} f_n(x_1) \tag{2.13}$$

Multiplying once more by $f_l(x_1)$ and integrating over D, gives

$$\lambda_k \int_D f_k(x_1) f_l(x_1) dx_1 = \sum_{n=0}^{\infty} E < \xi_n(\theta) \xi_k(\theta) > \sqrt{\lambda_n \lambda_k} \delta_{nl} \tag{2.14}$$

Then, using equation (2.15) leads to

$$\lambda_k \delta_{kl} = \sqrt{\lambda_k \lambda_l} < \xi_k(\theta) \xi_l(\theta) > \tag{2.15}$$

Equation (2.15) can be rearranged to give

$$< \xi_k(\theta) \xi_l(\theta) > = \delta_{kl} \tag{2.16}$$

Thus, the random process $\omega(x,\theta)$ can be written as

$$\omega(x,\theta) = \bar{\omega}(x) + \sum_{n=0}^{\infty} \xi_n(\theta)\sqrt{\lambda_n}\, f_n(x) \tag{2.17}$$

where,

$$< \xi_n(\theta) >= 0 \quad , \quad < \xi_n(\theta)\xi_m(\theta) >= \delta_{nm} \tag{2.18}$$

and $\lambda_n, f_n(x)$ are solution to equation (2.8). Truncating the series in equation (2.17) at the M^{th} term , gives

$$\omega(x,\theta) = \bar{\omega}(x) + \sum_{n=0}^{M} \xi_n(\theta)\sqrt{\lambda_n}\, f_n(x) \tag{2.19}$$

An explicit expressions for $\xi_n(\theta)$ can be obtained by multiplying equation (2.11) by $f_n(x)$ and integrating over the domain D. That is,

$$\xi_n(\theta) = \frac{1}{\sqrt{\lambda_n}} \int_D \alpha(x,\theta) f_n(x) dx \tag{2.20}$$

Viewed from a Reproducing Kernel Hilbert Space (RKHS) point of view (Aronszajn, 1950; Parzen, 1959), either of equations (2.11) or (2.20), is an expression for the congruence that maps the Hilbert space spanned by the functions $f_n(x)$ to the Hilbert space spanned by the set of random variables $\{\xi_n(\theta)\}$. It is this congruence along with the covariance function of the process that determines uniquely the random process $\omega(x,\theta)$. Observed the similarity of equations (2.11) and (2.20) with equations (2.7) and (2.8), respectively. Indeed, it can be shown (Parzen 1959) that if a function f can be represented in terms of linear operations on the family $\{C(\bullet, x_2), x_2 \in Z\}$, then f belongs to the (RKHS) corresponding to the kernel $C(x_1, x_2)$, and of an orthogonal family spanning this RKHS by means of the same linear operations used to represent f in terms of $\{C(\bullet, x_2), x_2 \in Z\}$. Another point of practical importance is that the expansion given by equation (2.19) can be used in a numerical simulation scheme to obtain numerical realizations of the random process. It is optimal in the Fourier sense, as it minimizes the mean square error resulting from truncation after a finite number of terms. The expansion is used extensively in the fields of detection, estimation, pattern recognition and image processing as an efficient tool to store random processes (Devijver and Kittler, 1982). It is worth noting at this point that the Karhunen – Loeve exoansion was independently derived in connection with stochastic turbulence problems (Lumley, 1970). In that context, the associated eigenfunctions can be identified with the characteristic eddies of the turbulence field.

It is well known from functional analysis that the steeper a bilinear form decays to zero as a function of one of its arguments, the more terms are needed in its spectral representation in order to reach a preset accuracy. Noting that the Fourier transform operator is a spectral representation, it may be concluded that the faster the autocorrelation function tends to zero, the broader is the corresponding spectral

density and the greater the number of requisite terms to represent the underlying random process by the Karhunem – Loeve expansion.

For the spectral case of a random process possessing a rational spectrum, the integral eigenvalue problem can be replaced by an equivalent differential equation that is easier to solve (Van Trees, 1968). In the same context, it is reminded that a necessary and sufficient condition for a process to have a finite dimensional Markov realization is that its spectrum be rational (Soize, 1986). It may seem that any complete set of functions can be used in-lieu of the eigenfunctions of the covariance kernel in the expansion (2.11). However, it will now be shown that the Karhunen – Loeve expansion as described above, has some desirable properties that make it a preferable choice for some of the present approach.

2.2.2 Properties

Error Minimizing Property

The generalized coordinate system defined by the eigenfunctions of the covariance kernel is optimal in the sense that the mean – square error resulting from a finite representation of the process $\omega(x,\theta)$ is minimized. This property can be proved as follows. Given a complete orthogonal set of functions $h_n(x)$, the process $\omega(x,\theta)$ can be approximated in a convergent series of the form

$$\omega(x,\theta) = \sum_{n=0}^{\infty} \lambda_n \xi_n(\theta) h_n(x) \tag{2.21}$$

Truncating equation (2.21) at the Mth term results in an error e_M equal to

$$e_M = \sum_{n=M+1}^{\infty} \lambda_n \xi_n(\theta) h_n(x) \tag{2.22}$$

Multiplying equation (2.21) by $h_m(x)$ and integrating throughout gives

$$\xi_m(\theta) = \frac{1}{\lambda_m} \int_D \omega(x,\theta) h_m(x) dx \tag{2.23}$$

where use is made of the orthogonality property of the set $h_n(x)$. Substituting equation (2.23) for $\xi_m(\theta)$ back into equation (2.22), the mean – square error e_M^2 can be written as

$$\begin{aligned} e_M^2 &= \left[\sum_{m=M+1}^{\infty} \sum_{n=M+1}^{\infty} h_m(x) h_n(x) \int_D \int_D < S(x_1,\theta) S(x_2,\theta) > h_m(x_1) h_n(x_2) dx_1 dx_2 \right] \\ &= \sum_{m=M+1}^{\infty} \sum_{n=M+1}^{\infty} h_m(x) h_n(x) \int_D \int_D R_{\omega\omega}(x_1,x_2) h_m(x_1) h_n(x_2) dx_1 dx_2 \end{aligned} \tag{2.24}$$

Integrating equation (2.24) over D and using the orthogonality of the set $\{h_i(x)\}$ yields

$$F\left[h_k(x)\right] = \sum_{m=M+1}^{\infty} \int_D \int_D R_{\omega\omega}(x_1, x_2) h_m(x_1) h_m(x_2) dx_1 dx_2 - \lambda_m \left[\int_D h_m(x) h_m(x) dx - 1 \right] \quad (2.26)$$

Differentiating equation (2.26) with respect to $h_i(x)$ and setting the result equal to zero, gives

$$\frac{\partial F\left[h_m(x)\right]}{\partial h_i(x)} = \int_D \left[\int_D R_{\omega\omega}(x_1, x_2) h_i(x_1) dx_1 - \lambda_i h_i(x_2) \right] dx_2 = 0 \quad (2.27)$$

which is satisfied when

$$\int_D R_{\omega\omega}(x_1, x_2) h_i(x_2) dx_2 = \lambda_i h_i(x_1) \quad (2.28)$$

Uniqueness of the Expansion

The random variables appearing in an expansion of the kind given by equation (2.10) are orthogonal if and only if the orthogonal functions $\{f_n(x)\}$ and the constants $\{\lambda_n\}$ are respectively the eigenfunctions and the eigenvalues of the covariance kernel as given by equation (2.8). The "if" part is an immediate consequence of equation (2.11). To show the "only if" part, equation (2.12) can be used with $<\xi_n(\theta)\xi_m(\theta)>=\delta_{nm}$ to obtain

$$C(x_1, x_2) = \sum_{n=0}^{\infty} \lambda_n f_n(x_1) f_n(x_2) \quad (2.29)$$

Multiplying both sides by $f_m(x_2)$ and integrating over D gives

$$\int_D C(x_1, x_2) f_m(x_2) dx_2 = \sum_{n=0}^{\infty} \lambda_n f_n(x_1) \delta_{nm} = \lambda_m f_m(x_1) \quad (2.30)$$

In the context of this last theorem, it is interesting to note that some investigators (e.g. Lawrence, 1987) have used an expansion of the kind given by equation (2.11) with orthogonal random variables and orthogonal deterministic functions that do not satisfy equation (2.8). It is obvious that such an expansion cannot form a basis for the representation of random processes.

Expansion of Gaussian Processes

Let $\omega(x,\theta)$ be a Gaussian process with covariance function $C(x_1, x_2)$. Then $\omega(x,\theta)$ has the Karhunen – Loeve decomposition given by equation (2.17) with the random variables $\xi_i(\theta)$ forming a Gaussian vector. That is, any subset of $\xi_i(\theta)$ is jointly Gaussian. Since these random variables are uncorrelated, their Gaussian property impliew their independence. Some important consequences derive from this property. Specifically,

$$< \xi_1(\theta), ..., \xi_{2n+1}(\theta) >= 0 \tag{2.31}$$

and

$$< \xi_1(\theta), ..., \xi_{2n}(\theta) >= \sum \prod < \xi_i(\theta) \xi_j(\theta) > \tag{2.32}$$

where the summation extends over all the partitions of the set $\{\xi_i(\theta)\}_{i=1}^{2n}$ into sets of two elements, and the product is over all such sets in a given partition. Furthermore, it can be shown (Loeve, 1977) that for Gaussian processes, the Karhunen – Loeve expansion is almost surely convergent.

Other Properties

In addition to the mean – square error minimizing property, the Karhunen – Loeve expansion has some additional desirable properties. Of these, the minimum representation entropy property is worth mentioning. These properties, however, are of no relevance to the present study and will not be discussed any further. A detailed study of the properties of the Karhunen – Loeve expansion is given by Devijver and kittler (1982).

2.2.3 Solution of the Integral Equation

Preliminary Remarks

The usefulness of the Karhunen – Loeve expansion hinges on the ability to solve the integral equation of the form

$$\int_D C(x_1, x_2) f(x_2) dx_2 = \lambda f(x_1) \tag{2.33}$$

where $C(x_1, x_2)$ is an autocovariance function. Equation (2.33) is a homogeneous Fredholm integral equation of the second kind. The theory underlying this kind of equations has been extensively investigated and is well documented in a number of monographs (Mikhlin, 1957). Beaing an autocovariance function the kernel $C(x_1, x_2)$ is bounded, symmetric, and positive definite. This fact simplifies the ensuing analysis considerably in that it guarantees a number of properties for the eigenfunctions and the eigenvalues that are solution to equation (2.33). Specifically,

1. The set $f_i(x)$ of eigenfunctions is orthogonal and complete.
2. For each eigenvalue λ_k, there correspond at most a finite number of linearly independent eigenfunctions.
3. There are at most a countably infinite set of eigenvalues.
4. The eigenvalues are all positive real numbers.
5. The kernel $C(x_1, x_2)$ admits of the following uniformly convergent expansion

$$C(x_1, x_2) = \sum_{k=1}^{\infty} \lambda_k f_k(x_1) f_k(x_2) \tag{2.34}$$

The Karhunen – Loeve expansion of a process was derived based on the preceding analytical properties of its covariance function. These properties are independent of the stochastic nature of the process involved, which allows the expansion to be applied to a wide range of processes including non-stationary and multidimensional processes.

Numerical Solution

In this section, a Galerkin type procedure is described for the solution of the Fredholm equation (2.33). Let $h_i(x)$ be a complete set of functions in the Hilbert space H. Each eigenfunction of the kernel $C(x_1, x_2)$ may be represented as

$$f_k(x) = \sum_{i=1}^{N} d_i^{(k)} h_i(x) \tag{2.35}$$

with an error e_N resulting from truncation the summation after the N[th] term. This error is equal to the difference between the left hand side and the right hand side of equation (2.33). Substituting equation (2.74) into equation (2.33) yields the following expression for the error

$$e_N = \sum_{i=1}^{N} d_i^{(k)} \left[\int_D C(x_1, x_2) h_i(x_2) dx_2 - \lambda_n h_i(x_1) \right] \tag{2.36}$$

Requiring the error to be orthogonal to the approximating space yields equations of the following form,

$$(e_N, h_j(x)) = 0 \ , \quad j = 1, ..., N \tag{2.37}$$

Equivalently,

$$\sum_{i=1}^{N} d_i^{(k)} \left[\int_D \left[\int_D C(x_1, x_2) h_i(x_2) dx_2 \right] h_j(x_1) dx_1 - \lambda_n \int_D h_i(x) h_j(x) dx_1 \right] = 0 \tag{2.38}$$

Denoting

$$C_{ij} = \int_D \int_D C(x_1, x_2) h_i(x_2) dx_2 h_j(x_1) dx_1 dx_2 \tag{2.39}$$

$$B_{ij} = \int_D h_i(x) h_j(x) dx \tag{2.40}$$

$$D_{ij} = d_i^{(k)} \tag{2.41}$$

and

$$\Lambda_{ij} = \delta_{ij} \lambda_i \tag{2.41}$$

Equation (2.38) becomes

$$CD = \Lambda BD \tag{2.42}$$

where C,B and D are three N-dimensional matrices whose elements are given by equations (2.39)-(2.40). Equation (2.42) represents a generalized algebraic eigenvalue problem which may be solved for the matrix D and the eigenvalues λ_k. Back substituting into equation (2.35) yields the eigenfunctions of the covariance kernel. The preceding procedure can be implemented using piecewise polynomials as the basis for the expansion. With this choice of basis functions, the columns of the matrix D becomes the eigenvectors computed at the respective nodal points of the induced mesh, and the ijth element of the matrix C becomes the weighed correlation between the process at nodes i and j. Both matrices C and B are symmetric positive definite, a fact that substantially simplifies the numerical solution. Obviously, the eigenvectors and eigenvalues computed based on the above scheme provide convergent estimates to the true values. Certain properties of these particular estimates make this scheme computationally attractive. Specifically, the Galerkin scheme described above can be shown to be equivalent to variations treatments of the problem. This property ensures that the computed eigenvalues are a lower bound of the correspondingly numbered exact eigenvalues. This implies that the convergence of each eigenvalue is monotonic in N. Further, note that the accuracy in estimating the eigenvalues is better than that achieved for the eigenfunctions (Delves and Mohamed, 1985).

2.3 Homogeneous Chaos

2.3.1 Preliminary Remarks

It is clear from the preceding discussion that the implementation of the Karhunen – Loeve expansion requires knowing the covariance function of the process being expanded. As far as the system under consideration is concerned, this implies that the expansion can be used for the random coefficient in the operator equation. However, it cannot be implemented for the solution process, since its covariance function and therefore the corresponding eigenfunctions are not known. An alternative expansion is clearly needed which circumvents this problem. Such an expansion could involve a basis of known random functions with deterministic coefficient to be found by minimizing some norm of the error resulting from a finite representation. This could be construed as similar to the Fourier series solution of deterministic differential equations, whereby the series coefficients are determined so as to satisfy some optimality criterion. To clarify this important idea further, a useful modification of the problem suggested by equation $u = g\left[a_k(x,\theta), f(x,\theta), x\right]$ is noted. So we can note,

$$u = h\left[\xi_i(\theta), x\right] \tag{2.43}$$

where $h[\cdot]$ is a nonlinear functional of its arguments. In equation (2.43) the random processes involved, have all been replaced by their corresponding Karhunen – Loeve representations. It is clear now that what is required is a nonlinear expansion of $h[\cdot]$ in terms of the set of random variables $\xi_i(\theta)$. If the processes defining the operator are Gaussian, this set is a sampled derivative of the Wiener process (Doop, 1953). In this case, equation (2.43) involves functionals of the Brownian motion. This is exactly

what the concept of Homogeneous Chaos provides. This concept was first introduced by Wiener (1938) and consists of an extension of the then obscure Volterra's work on the generalization of Taylor series to functionals (Volterra, 1913). Wiener's contributions were the result of his investigations of nonlinear functionals of the Brownian motion. Based on Wienere's ideas, Cameron and Martin (1947) contracted an orthogonal basis for nonlinear functionals in terms of Fourier – Hermite functionals. Wiener's Homogeneous Chaos was subsequently refined by Ito (1951) into what is known as the "Multiple Wiener Integral". About the same time that this analytical and measure – theoretic development of the theory was being pursued, Murray and Von – Neumann (1936) were establishing the parallel algenraic structure of rings of operators. Wiener's theory was further developed through research efforts that led to a series of reports (Bose, 1956; George, 1959). Numerical implementation of the basic ideas as well as convergence properties were addressed in these reports. This theory has drawn the interest of investigators in the fields of communication (Yasui, 1979), neuro-science (Palm and Poggio, 1977), engineering mechanics (Jahedi and Ahmadi, 1983), statistical phusics (Imamura et.al, 1965) and mathematics (Hida and Ikeda, 1965; McKean, 1973; Huang and Cambanis, 1978, 1979; Kallianpur, 1980; Engels, 1982). In particular, Yasui (1979), Engels (1982) and Kallianpur (1980) have attempted to develop a unified treatment of the Volterra series, the Wiener series, the Cameron – Martin expansion and the Ito approach. They have concluded that the last three approaches are equivalent and that they are superior to the Volterra series in terms of their convergence properties and their applicability. In the same manner that the Homogeneous Chaos can be viewed as an orthogonal development for nonlinear functionals with Gaussian measure, the Discrete Chaos (Wintner and Wiener, 1943; Hida and Ikedia, 1965; Ogura, 1972) is an orthogonal development with respect to the Poisson measure. Extensions to general measures have been investigated by Segall and Kailath (1976).

2.3.2 Definition and Properties

Let $\{\xi_i(\theta)\}_{i=1}^{\infty}$ be a set of orthogonal Gaussian random variables. Consider the space $\hat{\Gamma}_p$ of all polynomials in $\{\xi_i(\theta)\}_{i=1}^{\infty}$ of degree not exceeding p. Let Γ_p represent the set of all polynomials in $\hat{\Gamma}_p$ orthogonal to $\hat{\Gamma}_{p-1}$. Finally, let $\overline{\Gamma}_p$ be the space spanned by Γ_p. Then, the subspace $\overline{\Gamma}_p$ of $\boldsymbol{\Theta}$ is called the pth Homogeneous Chaos, and Γ_p is called the polynomial Chaos of order p. Based on the above definitions, the Polynomial Chaoses of any order p consist of all orthogonal polynomials of order p involving any combination of the random variables $\{\xi_i(\theta)\}_{i=1}^{\infty}$. It is clear, then, that the number of Polynomial Chaoses of order p, which involve a specific random variable out of the set $\{\xi_i(\theta)\}_{i=1}^{\infty}$ increases with p. This fact plays an important role in connection with the finite dimensional Polynomial Chaoses to be introduced in the sequel. Furthermore, since random variables are themselves functions, it becomes clear that Polynomial Chaoses are functions of functions and are therefore functionals. The set of Polynomial Chaoses is a linear subspace of the space of square – integrable (L_2) random variables Θ, and is a ring with respect to the functional multiplication $\Gamma_p\Gamma_l(\omega) = \Gamma_p(\omega)\Gamma_l(\omega)$. In this context, square integrability must be construed to be

with respect to the probability measure defining the random variables. Denoting the Hilbert space spanned by the set $\{\xi_i(\theta)\}_{i=1}^{\infty}$ by $\Theta(\xi)$, the resulting ring is denoted by $\Phi_{\Theta(\xi)}$, and is called the ring of functions generated by $\Theta(\xi)$. Then, it can be shown that under some general conditions, the ring $\Phi_{\Theta(\xi)}$ is dense the space Θ (Kakutani, 1961). This means that any square – integrable random function ($\Omega \to \mathbb{R}$) can be approximated as closely as desired by elements from $\Phi_{\Theta(\xi)}$. Thus, any element $\mu(\theta)$ from the space Θ admits the following representation,

$$\mu(\theta) = \sum_{p \geq 0} \sum_{n_1 + \ldots n_r = p} \sum_{p_1, \ldots, p_r} a_{p_1 \ldots p_r}^{n_1 \ldots n_r} \Gamma_p(\xi_{p_1}(\theta), \ldots, \xi_{p_r}(\theta)) \tag{2.44}$$

where $\Gamma_p(.)$ is the Polynomial Chaos of order p. The superscript n_i refers to the number of occurrences of $\xi_{p_i}(\theta)$ in the argument list for $\Gamma_p(.)$. Also, the double subscript provides for the possibility of repeated arguments in the argument list of the Polynomial Chaoses, thus preserving the generality of the representation given by equation (2.44). Briefly stated, the Polynomial Chaos appearing in equation (2.44) involves r distinct random variables out of the set $\{\xi_i(\theta)\}_{i=1}^{\infty}$, with the k^{th} tandom variable $\xi_k(\theta)$ having multiplicity n_k, and such that the total number of random variables involved is equal to the order p of the Polynomial Chaos. The Polynomial Chaoses of any order will be assumed to be symmetrical with respect to their arguments. Such a symmetrization is always possible. Indeed, a symmetrical polynomial can be obtained from a non – symmetrical one by taking the average of the polynomial aver all permutations of its arguments. The form of the coefficients appearing in equations (2.44) can then simplified, resulting in the following expanded expression for the representation of random variables,

$$\begin{aligned} \mu(\theta) = a_0 \Gamma_0 &+ \sum_{i_1=1}^{\infty} a_{i_1} \Gamma_1(\xi_{i_1}(\theta)) \\ &+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} a_{i_1 i_2} \Gamma_2(\xi_{i_1}(\theta), \xi_{i_2}(\theta)) \\ &+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3} \Gamma_3(\xi_{i_1}(\theta), \xi_{i_2}(\theta), \xi_{i_3}(\theta)) \\ &+ \ldots\ldots \end{aligned} \tag{2.45}$$

where $\Gamma_p(.)$ are successive Polynomial Chaoses of their arguments, the expansion being convergent in the mean – square sense. The upper limits on the summations in equation (2.45) reflect the symmetry of the Polynomial Chaoses with respect to their arguments, as discussed above. The Polynomial Chaoses of order greater than one have mean zero. Polynomials of different order are orthogonal to each other; so are same order polynomials with different argument list. At times in the ensuing developments, it will prove notationally expedient to rewrite equation (2.45) in the form

$$\mu(\theta) = \sum_{j=0}^{\infty} \hat{a}_j \Psi_j[\xi(\theta)] \tag{2.46}$$

where there is a one to one (1-1) correspondence between the functionals $\Psi[.]$ and $\Gamma[.]$, and also between the coefficient $\hat{a}_j$ and $\hat{a}_{i_1 \ldots i_r}$ appearing in equation (2.45). Implicit in equation (2.45) is the assumption that the expansion (2.45) is carried out in the order indicated by that equation. In other words, the contribution of polynomials of lower order is accounted for first. Up to now, and throughout the previous theoretical development, the symbol θ has been used to emphasize the random character of the quantities involved. It is felt that, although somewhat cumbersome, this notation underlines the fact that a random variable is a function defined over the space of events of which θ is an element. Having noted this, the symbol θ will be deleted in the ensuing development whenever the random nature of a certain quantity is obvious from the context. As defined above, each Polynomial Chaos is a function of the infinite set $\{\xi_i\}$, and is therefore an infinite dimensional polynomial. In a computational setting, however, this infinite set has to be replaced by a finite one. In view of that, it seems logical to introduce the concept of a finite dimensional Polynomial Chaos. Specifically, the n-dimensional Polynomial Chaos of order p is the subset of the polynomial Chaos of order p, as defined above, which is a function of only n of the uncorrelated random variables ξ_i. As $n \to \infty$, the Polynomial Chaos as defined previously is recovered. Obviously, the convergence properties of a representation based on the n-dimensional Polynomial Chaoses depend on n as well as on the choice of the subset $\{\xi_{\lambda_i}\}_{i=1}^{n}$ out of the infinite set. In the ensuing analysis, this choice will be based on the Karhunen – Loeve expansion of an appropriate random process. Since the finite dimensional Polynomial Chaos is a subset of the (infinite-dimensional) Polynomial Chaos, the same symbol will be used for both, with the dimension being specified. Note that for this case, the infinite upper limit on the dimension of the Polynomial involved. For clarity, the two- dimensional counterpart of equation (2.45) is rewritten, in a fully expanded form as

$$\begin{aligned}\mu(\theta) = {} & a_0\Gamma_0 + a_1\Gamma_1(\xi_i) + a_2\Gamma_1(\xi_2) + \\ & + a_{11}\Gamma_2(\xi_1,\xi_1) + a_{12}\Gamma_2(\xi_2,\xi_1) + a_{22}\Gamma_2(\xi_2,\xi_2) \\ & + a_{111}\Gamma_3(\xi_1,\xi_1,\xi_1) + a_{211}\Gamma_3(\xi_2,\xi_1,\xi_1) + a_{221}\Gamma_3(\xi_2,\xi_2,\xi_1) + a_{222}\Gamma_3(\xi_2,\xi_2,\xi_2) \\ & + \ldots\end{aligned} \tag{2.47}$$

In view of this last equation, it becomes clear that, except for a different indexing convention, the functionals $\Psi[.]$ and $\Gamma[.]$ are identical. In this regard, equation (2.47) can be recast in terms of $\Psi_j[.]$ as follows

$$\mu(\theta) = \hat{a}_0\Psi_0 + \hat{a}_1\Psi_1 + \hat{a}_2\Psi_2 + \ldots \tag{2.48}$$

From which the correspondence between $\Psi[.]$ and $\Gamma[.]$ is evident. For example, the term $a_{211}\Gamma_3(\xi_2,\xi_1,\xi_1)$ of equation (2.47) is identified with the term $\hat{a}_7\Psi_7$ of equation (2.48).

2.3.3 Construction of the Polynomial Chaos

A direct approach to construct the successive Polynomial Chaoses is to start with the set of homogeneous polynomials in ant to proceed, through a sequence of orthogonalization procedures. The zeroth order polynomial is a constant and it can be chosen to be 1. That is

$$\Gamma_0 = 1 \tag{2.49}$$

The first order polynomial has to be chosen so that it is orthogonal to all zeroth order polynomials. Since the set $\{\xi_i\}$ consists of zero – mean elements, the orthogonality condition implies

$$\Gamma_1(\xi_i) = \xi_i \tag{2.50}$$

The second order Polynomial Chaos consists of second order polynomials in $\{\xi_i\}$ that are orthogonal to both constants and first order polynomials. Formally, a second order polynomial can be written as

$$\Gamma_2(\xi_{i_1}, \xi_{i_2}) = a_0 + a_{i_1}\xi_{i_1} + a_{i_2}\xi_{i_2} + a_{i_1}a_{i_2}\xi_{i_1}\xi_{i_2} \tag{2.51}$$

where the constants are so chosen as to satisfy the orthogonality conditions. The second of these requires that

$$< \Gamma_2(\xi_{i_1}, \xi_{i_2})\xi_{i_3} >= 0 \tag{2.52}$$

This result in the following equation

$$a_{i_1}\delta_{i_1 i_3} + a_{i_2}\delta_{i_2 i_3} = 0 \tag{2.53}$$

Allowing i_3 to be equal to i_1 and i_2 successively, permits the evaluation of the coefficients a_{i_1} and a_{i_2} as

$$a_{i_1} = 0, a_{i_2} = 0 \tag{2.54}$$

The first orthogonality condition results in

$$a_0 + a_{i_1 i_2}\delta_{i_1 i_2} = 0 \tag{2.55}$$

Equation (2.55) can be normalized by requiring that

$$a_{i_1 i_2} = 1 \tag{2.56}$$

This leads to

$$a_0 = -\delta_{i_1 i_2} \tag{2.57}$$

Thus, the second Polynomial Chaos can be expressed as

$$\Gamma_2(\xi_{i_1},\xi_{i_2}) = \xi_{i_1}\xi_{i_2} - \delta_{i_1 i_2} \tag{2.58}$$

In a similar manner, the third order Polynomial Chaos has the general form

$$\begin{aligned}\Gamma_3(\xi_{i_1},\xi_{i_2},\xi_{i_3}) = a_0 + a_{i_1}\xi_{i_1} + a_{i_2}\xi_{i_2} + a_{i_3}\xi_{i_3} + a_{i_1 i_2}\xi_{i_1}\xi_{i_2} + \\ + a_{i_1 i_3}\xi_{i_1}\xi_{i_3} + a_{i_2 i_3}\xi_{i_2}\xi_{i_3} + a_{i_1 i_2 i_3}\xi_{i_1}\xi_{i_2}\xi_{i_3}\end{aligned} \tag{2.59}$$

With conditions of being orthogonal to all constants, first order polynomials and second order polynomials. The first of these conditions implies that

$$<\Gamma_3(\xi_{i_1},\xi_{i_2},\xi_{i_3})>=0 \tag{2.60}$$

That is

(2.61)

The second condition implies that

$$<\Gamma_3(\xi_{i_1},\xi_{i_2},\xi_{i_3})>=0 \tag{2.62}$$

which leads to

$$a_{i_1}\delta_{i_1 i_4} + a_{i_2}\delta_{i_2 i_4} + a_{i_3}\delta_{i_3 i_4} + a_{i_1 i_2 i_3} <\xi_{i_1}\xi_{i_2}\xi_{i_3}\xi_{i_4}>=0 \tag{2.63}$$

The last orthogonality condition is equivalent to

$$<\Gamma_3(\xi_{i_1},\xi_{i_2},\xi_{i_3})\xi_{i_4}\xi_{i_5}>=0 \tag{2.64}$$

which results in

$$a_0\delta_{i_4 i_5}a_{i_1 i_2} <\xi_{i_1}\xi_{i_2}\xi_{i_4}\xi_{i_5}> + a_{i_1 i_3} <\xi_{i_1}\xi_{i_3}\xi_{i_4}\xi_{i_5}> + a_{i_2 i_3} <\xi_{i_2}\xi_{i_3}\xi_{i_4}\xi_{i_5}>=0 \tag{2.65}$$

The above equations can be normalized by requiring that

$$a_{i_1 i_2 i_3} = 1 \tag{2.66}$$

Then equation (2.63) becomes

(2.67)

Due to the Gaussian property of the set $\{\xi_i\}$, the following equation holds

$$<\xi_{i_1}\xi_{i_2}\xi_{i_3}\xi_{i_4}>=\delta_{i_1 i_2}\delta_{i_3 i_4} + \delta_{i_1 i_3}\delta_{i_2 i_4} + \delta_{i_1 i_2}\delta_{i_2 i_3} \tag{2.67}$$

Substituting for the expectations in equations (2.67) and (2.65) yields

$$a_{i_1}\delta_{i_1 i_4} + a_{i_2}\delta_{i_2 i_4} + a_{i_3}\delta_{i_3 i_4} + \\ + \delta_{i_1 i_2}\delta_{i_3 i_4} + \delta_{i_1 i_3}\delta_{i_2 i_4} + \delta_{i_1 i_4}\delta_{i_2 i_3} = 0 \tag{2.68}$$

and

$$a_0\delta_{i_4 i_5} + a_{i_1 i_2}\left[\delta_{i_1 i_2}\delta_{i_4 i_5} + \delta_{i_1 i_4}\delta_{i_2 i_5} + \delta_{i_1 i_5}\delta_{i_2 i_4}\right] \\ + a_{i_1 i_3}\left[\delta_{i_1 i_3}\delta_{i_4 i_5} + \delta_{i_1 i_4}\delta_{i_3 i_5} + \delta_{i_1 i_5}\delta_{i_3 i_4}\right] \\ + a_{i_2 i_3}\left[\delta_{i_2 i_3}\delta_{i_4 i_5} + \delta_{i_2 i_4}\delta_{i_3 i_5} + \delta_{i_2 i_5}\delta_{i_3 i_4}\right] = 0 \tag{2.69}$$

Substituting for α_0 from equation (2.63), equation (2.69) can be written as

$$a_{i_1 i_2}\left[\delta_{i_1 i_4}\delta_{i_2 i_5} + \delta_{i_1 i_5}\delta_{i_2 i_4}\right] + a_{i_1 i_3}\left[\delta_{i_1 i_4}\delta_{i_3 i_5} + \delta_{i_1 i_5}\delta_{i_3 i_4}\right] \\ + a_{i_2 i_3}\left[\delta_{i_2 i_4}\delta_{i_3 i_5} + \delta_{i_2 i_5}\delta_{i_3 i_4}\right] = 0 \tag{2.70}$$

From equation (2.70), the coefficient $a_{i_1 i_2}, a_{i_1 i_3}, a_{i_2 i_3}$ can be evaluated as

$$a_{i_1 i_2} = 0 \\ a_{i_1 i_3} = 0 \\ a_{i_2 i_3} = 0 \tag{2.71}$$

Using equation (2.69) again , it is found that

$$a_0 = 0 \tag{2.72}$$

Equation (2.68) can be written as

$$\delta_{i_1 i_4}(a_{i_1} + \delta_{i_2 i_3}) + \delta_{i_1 i_4}(a_{i_1} + \delta_{i_2 i_3}) + \delta_{i_1 i_4}(a_{i_1} + \delta_{i_2 i_3}) = 0 \tag{2.73}$$

From which the coefficient $a_{i_1}, a_{i_2}, a_{i_3}$ are found to be

$$a_{i_1} = -\delta_{i_2 i_3} \\ a_{i_2} = -\delta_{i_1 i_3} \\ a_{i_3} = -\delta_{i_1 i_2} \tag{2.74}$$

The third order Polynomial Chaos can then be written as

$$\Gamma_3(\xi_{i_1}, \xi_{i_2}, \xi_{i_3}) = \xi_{i_1}\xi_{i_2}\xi_{i_3} - \xi_{i_1}\delta_{i_2 i_3} - \xi_{i_2}\delta_{i_1 i_3} - \xi_{i_3}\delta_{i_1 i_2} \tag{2.75}$$

After laborious algebraic manipulations, the fourth order Polynomial Chaos can be expressed as

$$\begin{aligned}\Gamma_4(\xi_{i_1},\xi_{i_2},\xi_{i_3},\xi_{i_4}) = {} & \xi_{i_1}\xi_{i_2}\xi_{i_3}\xi_{i_4} - \xi_{i_1}\xi_{i_2}\delta_{i_3i_4} - \xi_{i_1}\xi_{i_3}\delta_{i_2i_4} - \xi_{i_1}\xi_{i_4}\delta_{i_2i_3} \\ & - \xi_{i_2}\xi_{i_3}\delta_{i_1i_4} - \xi_{i_2}\xi_{i_4}\delta_{i_1i_3} - \xi_{i_3}\xi_{i_4}\delta_{i_1i_2} \\ & - \delta_{i_1i_2}\delta_{i_3i_4} - \delta_{i_1i_3}\delta_{i_2i_4} - \delta_{i_1i_4}\delta_{i_2i_3}\end{aligned} \tag{2.76}$$

It is readily seen that, in general, the n[th] order Polynomial Chaos can be written as

$$\Gamma_p(\xi_{i_1},...,\xi_{i_n}) = \begin{cases} \sum\limits_{\substack{r=n \\ r\,even}} (-1)^r \sum\limits_{\pi(i_1,...,i_n)} \prod\limits_{k=1}^{r} \xi_{i_k} < \prod\limits_{l=r+1}^{n} \xi_{i_l} >, \; n \; even \\ \sum\limits_{\substack{r=n \\ r\,even}} (-1)^{r-1} \sum\limits_{\pi(i_1,...,i_n)} \prod\limits_{k=1}^{r} \xi_{i_k} < \prod\limits_{l=r+1}^{n} \xi_{i_l} >, \; n \; odd \end{cases} \tag{2.77}$$

where $\pi(.)$ denotes a permutation of its arguments, and the summation is over all such permutations such that the sets $\left\{\xi_{i_1},...,\xi_{i_r}\right\}$ is modified by the permutation.
Thus, the Polynomial Chaos of order n can be obtained as

$$\Gamma_n(\xi_{i_1},...,\xi_{i_n}) = (-1)^n \frac{\partial^n}{\partial\xi_{i_1}...\partial\xi_{i_n}} e^{-\frac{1}{2}\xi^T\xi} \tag{2.78}$$

Equation (2.78) can be readily evaluated symbolically at the following tables for one dimensional Polynomial Chaoses up to the fourth order along with the values of their variances. The term Ψ_i in this table refers to the quantity appearing in equation (2.46).

j	*p , Order of the Homogeneous Chaos*	*j[th] Polynomial Chaos* Ψ_j	$<\Psi_j^2>$
0	$p=0$	1	1
1	$p=1$	ξ_1	1
2	$p=2$	ξ_1^2-1	2
3	$p=3$	$\xi_1^3-3\xi_1$	6
4	$p=4$	$\xi_1^4-6\xi_1^2+3$	24

Table 2.1: One – Dimensional Polynomial Chaoses and their Variance; n=1

3 Newton - Raphson method

3.1 Method in general

Newton's method is a general procedure that can be applied in many diverse situations. When specialized to the problem of locating a zero of real-valued function of a real variable, it is often called Newton-Raphson iteration. In general, Newton's method is faster than the bisection method and Fixed-Point iteration since its convergence is quadratic rather than linear. Once the quadratic becomes effective, that is, the values of Newton's method sequence are sufficiently close to the root, the convergence is so rapid that only a few more values are needed. Unfortunately, the method is not guaranteed always to convergence. Newton's method is often combined with other slower method in a hybrid method that is numerically globally convergence. Suppose that we have a function f whose zeros are to be determined numerically. Let r be a zero of $f(x)$ and let x be an approximation to r. If f'' exists and is continuous, then by Taylor's Theorem,

$$0 = f(r) = f(x+h) = f(x) + hf'(x) + o(h^2) \tag{3.1}$$

where $h = r - x$. If h is small (that is, x is near r), then it is reasonable to ignore the $o(h^2)$ term and solve the remaining equation for h. If we do this, the result is $h = -f(x)/f'(x)$. If x is an approximation to r, then $x - f(x)/f'(x)$ should be r. Newton's method begins with an estimate x_0 of r and then defines inductively

(3.2)

3.2 Newton's Algorithm

$$\begin{aligned} x_0 &= \text{initial guess} \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for } n = 0, 1, 2, \cdots \end{aligned} \tag{3.3}$$

Before examining the theoretical basis for Newton's method, let's give a graphical interpretation of it. From the description already given, we can say that Newton's method involves linearizing the function. That is, f was replaced by a linear function. The usual way of doing this is to replace f by the first two terms in the Taylor series. Thus, if

$$f(x) = f(c) + f'(c)(x-c) + \frac{1}{2!} f''(c)(x-c)^2 + \cdots \tag{3.4}$$

Then the linearization (at c) produces the linear function

$$l(x) = f(c) + f'(c)(x - c) \tag{3.5}$$

Note that l is a good approximation to f in the vicinity of c, and in fact we have $l(c) = f(c)$ and $l'(c) = f'(c)$. Thus, the linear function has the same value and the same slope as fit the point c. So in Newton's method we are constructing the target line to f at a point near r, and finding where the target line intersects the x-axis.

3.3 Error Analysis (Quadratic convergence of Newton's method)

Now we analyze the errors in Newton's method. By errors, we mean the quantities $e_n = x_n - x$. (We are not considering round-off errors) Let us assume that f'' is continuous and r is a simple zero of f so that $f(r) = 0 \neq f'(r)$. From the definition of the Newton iteration, we have

$$\begin{aligned} e_{n+1} &= x_{n+1} - r = x_n - \frac{f(x_n)}{f'(x_n)} - r \\ &= e_n - \frac{f(x_n)}{f'(x_n)} = \frac{e_n f'(x_n) - f(x_n)}{f'(x_n)} = \frac{e_n f'(x_n) - (f(x_n) - f(r))}{f'(x_n)}. \end{aligned} \tag{3.6}$$

By Taylor's Theorem, we have

$$f(x_n) - f(r) = f'(x_n)(x_n - r) + \frac{1}{2} e_n^{\,2} f''(\xi_n) \tag{3.7}$$

$$e_{n+1} \approx \frac{\frac{1}{2} f(\xi_n) e_n^{\,2}}{f'(r)} \approx \frac{\frac{1}{2} f''(r)}{f'(r)} e_n^{\,2} = c e_n^{\,2} \tag{3.8}$$

Suppose that $c \approx 1$ and $e_n \approx 10^{-4}$. Then by Equation (3.8), we have $e_{n+1} \approx 10^{-8}$ and $l_{n+1} \approx 10^{-16}$. We are impressed that only a few additional iterations are needed to obtain more than machine precision.

This equation tells us that e_{n+1} is roughly a constant times $e_n^{\,2}$. This desirable state of affairs is called quadratic convergence. It accounts for the apparent doubling of precision with each iteration of Newton's method in many applications.

3.3.1 Definition (Quadratically convergent):

A iterative method is quadratically convergent if

$$M = \lim_{n \to \infty} \frac{e_{n+1}}{e_n^2} < \infty$$

We have yet to establish the convergence of the method. By Equation (3.8), the idea of the proof is simple: If e_n is small and if $\frac{1}{2} f''(\xi_n)/f'(x_n)$ is not too large, then e_{n+1} will be smaller than e_n. Define a quantity $c(\xi)$ dependent on δ by

$$c(\delta)=\frac{1}{2}\max_{|x-r|\le\delta}|f''(x)|\Big/\min_{|x-r|\le\delta}|f'(x)| \qquad (\delta>0) \tag{3.9}$$

We select small enough δ to be sure that the denominator in Equation (3.9) is positive, and then if necessary we decrease δ so that $\delta c(\delta)<1$. This is possible because as δ converges to 0, $c(\xi)$ converges to $\left|\frac{1}{2} f''(r)/f'(r)\right|$, and so $\delta c(\delta)$ converges to 0. Having fixed δ, set $\rho=\delta c(\delta)$. Suppose that we start the Newton iteration with a point x_0 satisfying $|x_0-r|\le\delta$. Then $|e_0|\le\delta$ and $|\xi_0-r|\le\delta$. Hence, by the definition of $c(\delta)$, we have

(3.10)

Therefore, Equation (3.8) yields

$$|x_1-r|=|e_1|\le e_0{}^2 c(\delta)=|e_0||e_0|c(\delta)\le|e_0|\delta c(\delta)\le|e_0|\rho<|e_0|\le\delta \tag{3.11}$$

This shows that the next point x_1, also lies within δ units of r. Hence, the argument can be repeated, with the results

$$|e_1|\le\rho|e_0|\ ,\quad |e_2|\le\rho|e_1|\le\rho^2|e_0| \quad\text{and}\quad |e_3|\le\rho|e_2|\le\rho^3|e_0| \tag{3.12}$$

In general, we have

$$|e_n|\le\rho^n|e_0| \tag{3.13}$$

Since $0\le\rho<1$, we have $\lim_{n\to\infty}\rho^n=0$, and so $\lim_{n\to\infty}e_n=0$. Summarizing, we obtain the following theorem on Newton's method.

3.4 Theorem (Theorem on Newton's method)

Let f be twice continuously differentiable and $f(r)=0$. If $f'(r)\neq0$, then Newton's method is locally and quadratic ally convergent to r and satisfies

$$|x_{n+1}-r|\le c|x_n-r|^2 \qquad (n=0) \tag{3.14}$$

In some situations Newton's method can be guaranteed to converge from an arbitrary starting point. We give one such theorem as a sample.

3.4.1 Theorem (Theorem on Newton's method for a convex function)

If f belongs to $C^2(R)$, is increasing, convex and has a zero, then the zero is unique, and the Newton's method will converge to it from any starting point.

3.4.2 Theorem (Linear convergence of Newton's Method)

Assume that the $(m+1)$-time continuously differentiable function f on $[a,b]$ has a multiplicity m root at r. Then Newton's Method is locally and linearly convergent to r. Newton's Method, like Fixed-Point Method, may not converge to a root.

4 Spectral Galerkin Method - Newton- Raphson (SGM-NR)

4.1 SGM-NR Method

At this paragraph is presented the method SGM-NR (Spectral Galerkin Method – Newton - Raphson) for a stochastic non linear problem. Let k is a random variable (scalar), so the dimension is one (m=1) and f is a non linear function of variable Δ. The right hand side of the equation is a constant number.

$$kf(\Delta) = R \text{ , with } k \sim N\left(\mu, \sigma^2\right) \tag{4.1}$$

An equivalent problem to (4.1) is:

$$J(\Delta) = kf(\Delta) - R = 0 \tag{4.2}$$

Let Δ_0 was an initial (displacement) guess, and then Δ can be expressed as:

$$\Delta = \Delta_0 + \delta\Delta \tag{4.3}$$

With first order Taylor expansion at eq. (4.2) we can create the following expression:

$$J(\Delta) = J(\Delta_0 + \delta\Delta) \overset{Taylor}{=} J(\Delta_0) + \delta\Delta \cdot \left.\frac{dJ(\Delta)}{d\Delta}\right|_{\Delta=\Delta_0} \tag{4.4}$$

Then the equation (4.4) can be in the form:

$$\left.\frac{dJ(\Delta)}{d\Delta}\right|_{\Delta=\Delta_0} \cdot \delta\Delta = -J(\Delta_0) \tag{4.5}$$

In order to solve the stochastic problem we can re-write equation (4.5) in the form:

$$\left.\frac{dJ(\Delta,k)}{d\Delta}\right|_{\Delta=\Delta_0} \cdot \delta\Delta(k) = -J(\Delta_0,k) \tag{4.6}$$

Further, we can note from the following equations that:

$$\left.\frac{dJ(\Delta,k)}{d\Delta}\right|_{\Delta=\Delta_0} = \left.\frac{d\left(kf(\Delta)-R\right)}{d\Delta}\right|_{\Delta=\Delta_0} = k\left.\frac{d\left(f(\Delta)\right)}{d\Delta}\right|_{\Delta=\Delta_0} \tag{4.7}$$

and

From equations (4.6) and (4.7) we can write equation (4.5) in the form:

$$k\left(\left.\frac{d\left(f(\Delta)\right)}{d\Delta}\right|_{\Delta=\Delta_0} \cdot \delta\Delta\right) = -J(\Delta_0,k) \tag{4.10}$$

Equation (4.10) it's the equivalent problem with (4.1) if we assume that k is a random variable is distributed by Gaussian field $k \sim N\left(\mu, \sigma^2\right)$ with dimension 1. At this step it is obvious from the equation (4.10) that we have to deal with a stochastic equation because k is a random variable. Further it is important to notice that Δ is depended from variable k and that's has as result that Δ is a random variable with respect to k. For that reason we can note that $\delta\Delta = \delta\Delta(k)$. So,

$$k \cdot f'(\Delta_0) \cdot \delta\Delta(k) = R - kf(\Delta_0) \tag{4.12}$$

Now we can write the quantity $\delta\Delta(k)$ with the polynomial chaos expansion form (2.46). If we choose the order of polynomial chaos expansion to be p and Ψ_i are the Hermite orthogonal up to p degree then the expansion with truncated over $P = \begin{pmatrix} p+1 \\ p \end{pmatrix} = \frac{(p+1)!}{p!1!} = p+1$ is:

$$\delta\Delta(k) = \sum_{i=0}^{P-1} d_i \Psi_i(k) \text{ , where } d_i, i = 0,...,P-1 \text{ are unknown constants} \tag{4.13}$$

Then equation (4.12) with (4.13) can be expressed as

$$k \cdot f'(\Delta_0) \cdot \sum_{i=0}^{P-1} d_i \Psi_i(k) = R - kf(\Delta_0) \tag{4.14}$$

Further, if we multiply equation (4.14) with Ψ_j , $j = 0,...,P-1$ where Ψ_j are Hermite polynomials, equation (4.14) can take the form:

$$k \cdot f'(\Delta_0) \cdot \sum_{i=0}^{P-1} d_i \Psi_i(k) \Psi_j(k) = \left(R - kf(\Delta_0)\right) \Psi_j(k) \tag{4.15}$$

As a property of Newton – Raphson convergence is that $f'(\Delta_0) \neq 0$.
So

$$k \cdot \sum_{i=0}^{P-1} d_i \Psi_i(k) \Psi_j(k) = \frac{R - kf(\Delta_0)}{f'(\Delta_0)} \cdot \Psi_j(k), \quad j = 0,...,P-1 \tag{4.16}$$

Summarized the above results of equations (4.16) and (4.17), equation (4.15) can be expressed as:

$$\sum_{i=0}^{P-1} d_i k \Psi_i(k) \Psi_j(k) = \frac{R - kf(\Delta_0)}{f'(\Delta_0)} \Psi_j(k) \text{ , for } j = 0,...,P-1 \tag{4.17}$$

and with the estimations, equation (4.18) can be expressed as:

$$\sum_{i=0}^{P-1} d_i \left\langle k \Psi_i(k) \Psi_j(k) \right\rangle = \left\langle B(k) \Psi_j(k) \right\rangle \text{ , for } j = 0,...,P-1 \tag{4.18}$$

Finally with the notation

$$K = K_{ij} = \left\langle k \cdot \Psi_i(k) \cdot \Psi_j(k) \right\rangle$$
$$B = B_j = \left\langle \frac{R - kf(\Delta_0)}{f'(\Delta_0)} \cdot \Psi_j(k) \right\rangle \tag{4.19}$$

the augmented linear system can expressed as:

$$Kd = B$$

$$\begin{aligned} \dim[K] &= (p+1)\times(p+1) \\ \dim[d] &= (p+1)\times(1) \\ \dim[B] &= (p+1)\times(1) \end{aligned} \tag{4.20}$$

More analytically

$$K = \begin{pmatrix} \langle k\Psi_0\Psi_0 \rangle & \langle k\Psi_0\Psi_1 \rangle & \dots & \langle k\Psi_0\Psi_P \rangle \\ \langle k\Psi_1\Psi_0 \rangle & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \langle k\Psi_P\Psi_0 \rangle & \cdots & \cdots & \langle k\Psi_P\Psi_P \rangle \end{pmatrix}, \; d = \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_P \end{pmatrix}, \; B = \begin{pmatrix} \langle B_0 \rangle \\ \langle B_1 \rangle \\ \vdots \\ \langle B_P \rangle \end{pmatrix}$$

As we assume before, k is a random variable distributed as $k \sim N(\mu, \sigma^2)$. If we standardized variable k we can exact:

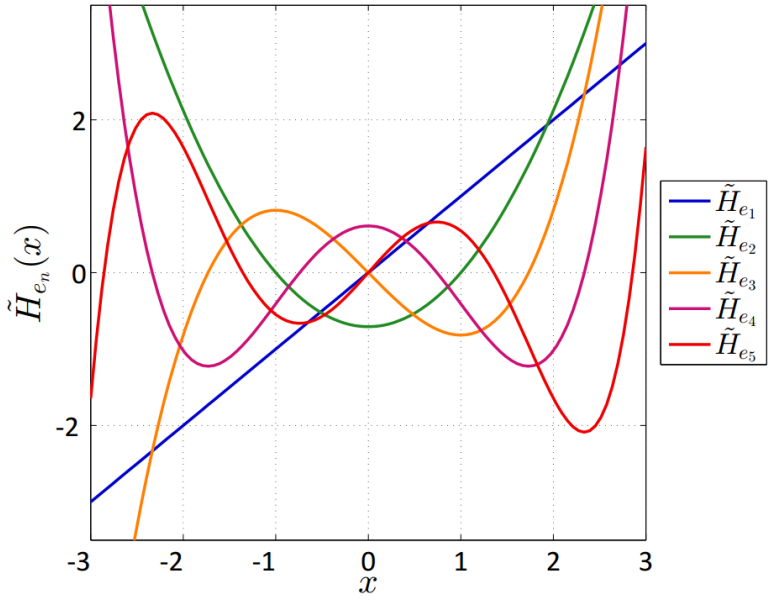
$$\xi = \frac{k-\mu}{\sigma} \sim N(0,1)$$
$$k = \sigma \cdot \xi + \mu \tag{4.21}$$

Hermite polynomials are (Xiu & Karniadakis, 2014)

$$H_{-1}(\xi) = H_0(\xi) = 1$$
$$H_{n+1}(\xi) = \xi \cdot H_n(\xi) - nH_{n-1}(\xi) \tag{4.22}$$
$$\|H_n\|^2 = n!$$

So, the first Polynomials are

j	p , *Order of the Homogeneous Chaos*	j^{th} *Polynomial Chaos* Ψ_j	$<\Psi_j^2>$
0	$p=0$	1	1
1	$p=1$	ξ	1
2	$p=2$	ξ^2-1	2
3	$p=3$	$\xi^3-3\xi$	6
4	$p=4$	$\xi^4-6\xi^2+3$	24



4-1 Hermite Polynomials

For the inner functional product we have the following properties:

$$\begin{aligned}
&\bullet \left\langle \Psi_i(\xi), \Psi_j(\xi) \right\rangle = \int_{-\infty}^{+\infty} \Psi_i(\xi)\Psi_j(\xi)\frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi \\
&\bullet \left\langle \Psi_0(\xi) \right\rangle = 1 \\
&\bullet \left\langle \Psi_i(\xi) \right\rangle = 0 \ , \quad for\ i = 1,\ldots, p \\
&\bullet \left\langle \Psi_i(\xi), \Psi_j(\xi) \right\rangle = a_{ij}\delta_{ij} \ , \quad for\ i, j = 1,\ldots, p
\end{aligned} \tag{4.23}$$

4.1.1 K Matrix

$$K = K_{ij} = \left\langle k \cdot \Psi_i(k) \cdot \Psi_j(k) \right\rangle \tag{4.24}$$

From the equations (4.21), (4.22) and (4.23) we have the following:

- For the K_{00} element we have:

$$\mathrm{K}_{00} = \left\langle k\Psi_0\Psi_0 \right\rangle = \left\langle (\sigma\xi + \mu)\cdot 1 \cdot 1 \right\rangle = \int_{-\infty}^{+\infty} (\sigma\xi + \mu) \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi$$

$$= \sigma \int_{-\infty}^{+\infty} \xi \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi + \mu \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi = \mu$$

- For the K_{01} element we have:

$$\mathrm{K}_{01} = \left\langle k\Psi_0\Psi_1 \right\rangle = \left\langle (\sigma\xi + \mu)\cdot 1 \cdot \xi \right\rangle = \int_{-\infty}^{+\infty} (\sigma\xi^2 + \mu\xi) \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi$$

$$= \sigma \int_{-\infty}^{+\infty} \xi^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi + \mu \int_{-\infty}^{+\infty} \xi \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi = \sigma$$

- For the K_{02} element we have:

$$\mathrm{K}_{02} = \left\langle k\Psi_0\Psi_2 \right\rangle = \left\langle (\sigma\xi + \mu)\cdot 1 \cdot (\xi^2 - 1) \right\rangle = \int_{-\infty}^{+\infty} (\sigma\xi^3 + \mu\xi^2 - \sigma\xi - \mu) \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi$$

$$= \sigma \int_{-\infty}^{+\infty} (\xi^3 - \xi) \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi + \mu \int_{-\infty}^{+\infty} (\xi^2 - 1) \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi = 0$$

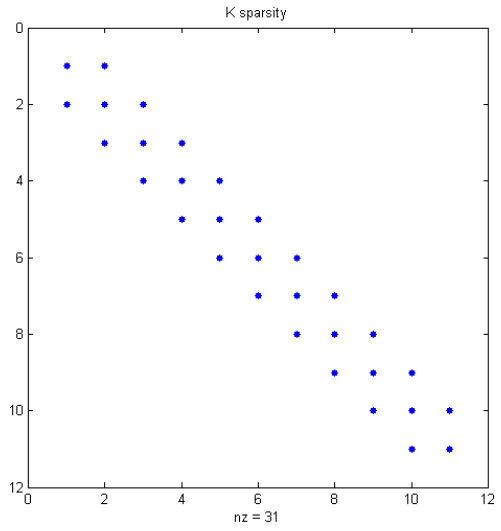
- For the K_{12} element we have:

$$\mathrm{K}_{12} = \left\langle k\Psi_1\Psi_2 \right\rangle = \left\langle (\sigma\xi + \mu)\cdot \xi \cdot (\xi^2 - 1) \right\rangle = \int_{-\infty}^{+\infty} (\sigma\xi^4 + \mu\xi^3 - \sigma\xi^2 - \mu\xi) \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi$$

$$= \sigma \int_{-\infty}^{+\infty} (\xi^4 - \xi^2) \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi + \mu \int_{-\infty}^{+\infty} (\xi^3 - \xi) \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi = 2\sigma$$

K is also a symmetric matrix so we can have the following sparsity (Stavroulakis, Giovanis, Papadrakakis, & Papadopoulos, 2014):

$$\mathrm{K}=\begin{pmatrix} \mu & \sigma & 0 & & & & \\ \sigma & \mu & 2\sigma & 0 & & 0 & \\ 0 & 2\sigma & 2\mu & 6\sigma & 0 & & \\ & 0 & 6\sigma & 6\mu & 24\sigma & \ddots & \\ & & 0 & 24\sigma & 24\mu & \ddots & 0 \\ & 0 & & \ddots & \ddots & \ddots & n!\sigma \\ & & & & 0 & n!\sigma & n!\mu \end{pmatrix}$$



4-2 K sparsity

4.1.2 B Matrix

$$B=B_j=\left\langle \frac{R-kf(\Delta_0)}{f'(\Delta_0)}\cdot\Psi_j(k)\right\rangle=\frac{R}{f'(\Delta_0)}\cdot\left\langle\Psi_j(k)\right\rangle-\frac{f(\Delta_0)}{f'(\Delta_0)}\left\langle k\cdot\Psi_j(k)\right\rangle \qquad (4.25)$$

- For the B_0 element we have:

$$B_0 = \frac{R}{f'(\Delta_0)} \cdot \langle \Psi_0(k) \rangle - \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \langle k \cdot \Psi_0(k) \rangle =$$

$$= \frac{R}{f'(\Delta_0)} \cdot 1 - \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \langle (\sigma\xi + \mu) \cdot 1 \rangle =$$

$$= \frac{R}{f'(\Delta_0)} - \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \int_{-\infty}^{+\infty} (\sigma\xi + \mu) \cdot \left(\frac{1}{\sqrt{2\pi}} \right) \cdot e^{-\frac{\xi^2}{2}} d\xi =$$

$$= \frac{R}{f'(\Delta_0)} - \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \sigma \cdot \int_{-\infty}^{+\infty} \xi \cdot \left(\frac{1}{\sqrt{2\pi}} \right) \cdot e^{-\frac{\xi^2}{2}} d\xi - \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \mu \cdot \int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}} \right) \cdot e^{-\frac{\xi^2}{2}} d\xi =$$

$$= \frac{R}{f'(\Delta_0)} - 0 - \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \mu = \frac{R}{f'(\Delta_0)} - \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \mu$$

- For the B_1 element we have:

$$B_1 = \frac{R}{f'(\Delta_0)} \cdot \langle \Psi_1(k) \rangle - \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \langle k \cdot \Psi_1(k) \rangle =$$

$$= 0 - \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \langle (\sigma\xi + \mu) \cdot \xi \rangle =$$

$$= -\frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \int_{-\infty}^{+\infty} (\sigma\xi + \mu) \cdot \xi \cdot \left(\frac{1}{\sqrt{2\pi}} \right) \cdot e^{-\frac{\xi^2}{2}} d\xi =$$

$$= -\frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \sigma \cdot \int_{-\infty}^{+\infty} \xi^2 \cdot \left(\frac{1}{\sqrt{2\pi}} \right) \cdot e^{-\frac{\xi^2}{2}} d\xi - \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \mu \cdot \int_{-\infty}^{+\infty} \xi \cdot \left(\frac{1}{\sqrt{2\pi}} \right) \cdot e^{-\frac{\xi^2}{2}} d\xi =$$

$$= -\frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \sigma \cdot 1 = -\frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \sigma$$

- For the B_2 element we have:

$$B_2 = \frac{R}{f'(\Delta_0)} \cdot \langle \Psi_2(k) \rangle - \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \langle k \cdot \Psi_2(k) \rangle =$$

$$= 0 - \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \langle (\sigma\xi + \mu) \cdot (\xi^2 - 1) \rangle =$$

$$= -\frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \int_{-\infty}^{+\infty} (\sigma\xi + \mu) \cdot (\xi^2 - 1) \cdot \left(\frac{1}{\sqrt{2\pi}} \right) \cdot e^{-\frac{\xi^2}{2}} d\xi =$$

$$= -\frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \sigma \cdot \int_{-\infty}^{+\infty} \xi^3 \cdot \left(\frac{1}{\sqrt{2\pi}} \right) \cdot e^{-\frac{\xi^2}{2}} d\xi +$$

$$+ \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \sigma \cdot \int_{-\infty}^{+\infty} \xi \cdot \left(\frac{1}{\sqrt{2\pi}} \right) \cdot e^{-\frac{\xi^2}{2}} d\xi$$

$$- \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \mu \cdot \int_{-\infty}^{+\infty} \xi^2 \cdot \left(\frac{1}{\sqrt{2\pi}} \right) \cdot e^{-\frac{\xi^2}{2}} d\xi$$

$$+ \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \mu \cdot \int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}} \right) \cdot e^{-\frac{\xi^2}{2}} d\xi =$$

$$= 0$$

- For the B_i with $i = 2,..., p+1$ we have $B_i = 0$.

Proof

$$B_n = \left\langle \frac{R - kf(\Delta_0)}{f'(\Delta_0)} \cdot \Psi_n(k) \right\rangle = \frac{R}{f'(\Delta_0)} \cdot \langle \Psi_n(k) \rangle - \frac{f(\Delta_0)}{f'(\Delta_0)} \langle k \cdot \Psi_n(k) \rangle$$

$$= \frac{R}{f'(\Delta_0)} \cdot \langle \Psi_n(k) \rangle - \frac{f(\Delta_0)}{f'(\Delta_0)} \langle k \cdot \Psi_n(k) \rangle \overset{n>1}{=} \frac{R}{f'(\Delta_0)} \cdot 0 - \frac{f(\Delta_0)}{f'(\Delta_0)} \langle (\sigma\xi + \mu) \cdot \Psi_n(k) \rangle =$$

$$= -\frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \sigma \cdot \langle \xi \cdot \Psi_n(k) \rangle - \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \mu \cdot \langle \Psi_n(k) \rangle =$$

$$= -\frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \sigma \cdot \langle \Psi_1 \cdot \Psi_n(k) \rangle - \frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \mu \cdot 0 =$$

$$\overset{n>1}{=} -\frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \sigma \cdot \langle \Psi_1 \cdot \Psi_n(k) \rangle = -\frac{f(\Delta_0)}{f'(\Delta_0)} \cdot \sigma \cdot 0 = 0$$

■

4.1.3 Statistics measurements of SGM

- **Mean of SGM**

The solution of the linear system (4.20) is $\delta\Delta(k) = \sum_{i=0}^{P} d_i \Psi_i(k)$

$$\begin{aligned} Mean(\delta\Delta) &= E\left(\sum_{i=0}^{p} d_i \Psi_\iota\right) = E\left(d_0\Psi_0 + d_1\Psi_1 + ... + d_p\Psi_p\right) = \\ &= d_0 \mathrm{E}(\Psi_0) + d_1 \mathrm{E}(\Psi_1) + ... + d_p \mathrm{E}(\Psi_p) = \\ &\overset{\Psi_0=1}{=} d_0 \mathrm{E}(1) + d_1 \mathrm{E}(\Psi_1) + ... + d_p \mathrm{E}(\Psi_p) = \\ &= d_0 + d_1 \mathrm{E}(\Psi_1) + ... + d_p \mathrm{E}(\Psi_p) \overset{E(\Psi_i)=0,\ for\ i=1,...,p}{=} \\ &= d_0 \end{aligned} \tag{4.26}$$

- **Variance (Standard Deviation)**

$$\begin{aligned} Variance(\delta\Delta) &= Var\left(\sum_{i=0}^{p} d_i \Psi_\iota\right) = Var\left(d_0\Psi_0 + d_1\Psi_1 + ... + d_p\Psi_p\right) = \\ &= d_0^2 Var(\Psi_0) + d_1^2 Var(\Psi_1) + ... + d_p^2 Var(\Psi_p) = \\ &\overset{\Psi_0=1}{=} d_0^2 Var(1) + d_1^2 Var(\Psi_1) + ... + d_p^2 Var(\Psi_p) = \\ &= d_0 \cdot 0 + d_1^2 Var(\Psi_1) + ... + d_p^2 Var(\Psi_p) = \sum_{i=1}^{p} d_i^2 Var(\Psi_\iota) \end{aligned} \tag{4.27}$$

So if we use the identity $Var(x) = E\left(x^2\right) - E^2(x)$ we can express (4.27) as:

$$\begin{aligned} Variance(\delta\Delta) &= \sum_{i=1}^{p} d_i^2 \left(E\left(\Psi_i^2\right) - E^2\left(\Psi_i\right)\right) \overset{E(\Psi_i)=0,\ i=1,...,p}{=} \\ &= \sum_{i=1}^{p} d_i^2 E\left(\Psi_i^2\right) = d_1^2 \cdot 1! + d_2^2 \cdot 2! + ... + d_p^2 \cdot p! \end{aligned} \tag{4.28}$$

Summary:

- $\overline{\delta\Delta} = d_0$
- $std(\delta\Delta) = \sqrt{d_1 + 2!d_2 + ... + n!d_n}$

(4.29)

And for the Δ we have:

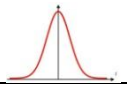
- $\overline{\Delta} = \Delta_0 + d_0$

(4.30)

- $std(\Delta) = \sqrt{d_1 + 2!d_2 + ... + n!d_n}$ is the same because $Var(\Delta_0) = 0$

4.1.4 Algorithm SGM – Newton – Raphson

For the solution of the problem $kf(\Delta) = R$, with $k \sim N\left(\mu, \sigma^2\right)$ we have the next algorithm:

Start	SGM-NR
Step 1	Input the Dimension of the variable k (m) and the order of Polynomial Chaos (p) **(next)**
Step 2	Specify R (right hand side) and generate one random k from $N\left(\mu, \sigma^2\right)$ **(next)**
Step 3	Calculate Polynomial Chaos parameters (Basis, norms etc) **(next)**
Step 4	Newton – Raphson (Specify: tolerance) **(next)**
Step 5	Specify the initial (seed) Δ_0 **(next)**
Step 6	Solve the linear system (4.20) and calculate $\delta\Delta$ if $\|\delta\Delta\| \le Tolerance$ go to (**step 7)** else $\Delta_0 \leftarrow \Delta_0 + \delta\Delta$ and go to **(step 5)**
Step 7	If Newton – Raphson Converged, then Calculate the statistics $Mean(\Delta)$ and $std = \sqrt{Var(\Delta)}$ then you have pdf 
End	**SGM-NR**

4.2 Monte Carlo Simulation (MCS)

4.2.1 Monte Carlo Method

A **Monte Carlo method** is a technique that involves using random numbers and probability to solve problems. The term Monte Carlo Method was coined by S. Ulam and Nicholas Metropolis in reference to games of chance, a popular attraction in Monte Carlo, Monaco (Hoffman, 1998; Metropolis and Ulam, 1949).

Computer simulation has to do with using computer models to imitate real life or *make predictions*. When you create a model with a spreadsheet, you have a certain number of *input parameters* and a few equations that use those inputs to give you a set of *outputs* (or *response* variables). This type of model is usually **deterministic**, meaning that you get the same results no matter how many times you re-calculate.

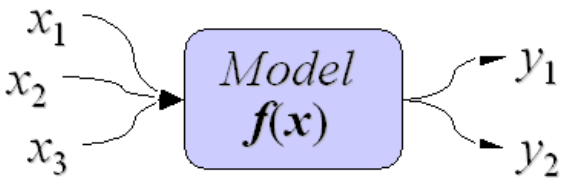


Figure 1: A parametric deterministic model maps a set of input variables to a set of output variables.

Monte Carlo simulation is a method for *iteratively* evaluating a deterministic model using sets of random numbers as inputs. This method is often used when the model is complex, nonlinear, or involves more than just a couple uncertain parameters. A simulation can typically involve *over 10,000 evaluations* of the model, a task which in the past was only practical using super computers.

By using random inputs, you are essentially turning the deterministic model into a stochastic model.

We can use simple *uniform random numbers* as the inputs to the model. However, a uniform distribution is not the only way to represent uncertainty. Before describing the steps of the general MC simulation in detail, a little word about *uncertainty propagation*:

The Monte Carlo method is just one of many methods for analyzing uncertainty propagation, where the goal is to determine how *random variation, lack of knowledge*, or *error* affects the *sensitivity*, *performance*, or *reliability* of the system that is being modeled. Monte Carlo simulation is categorized as a sampling method because the inputs are randomly generated from *probability distributions* to simulate the process of sampling from an actual *population*. So, we try to choose a distribution for the inputs that most closely *matches data we already have*, or best represents our *current state of knowledge*. The data generated from the simulation can be represented as probability distributions (or histograms) or converted to *error bars*, *reliability predictions*, *tolerance zones*, and *confidence intervals*. (See Figure 2).

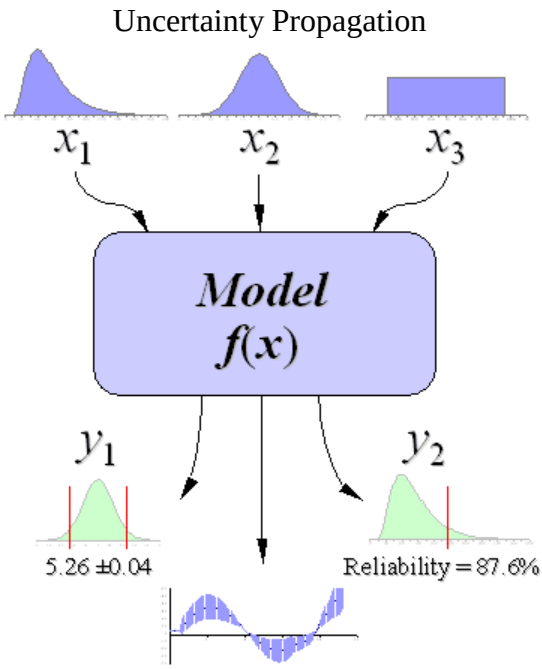


Figure 2: Schematic showing the principal of stochastic uncertainty propagation.

(The basic principle behind Monte Carlo simulation.)

The steps in Monte Carlo simulation corresponding to the uncertainty propagation shown in Figure 2 are fairly simple, and can be easily implemented for models. All we need to do is follow the **five simple steps** listed below:

Step 1: Create a parametric model, $y = f(x_1, x_2, ..., x_q)$.

Step 2: Generate a set of random inputs, $x_{i1}, x_{i2}, ..., x_{iq}$.

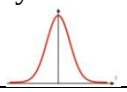
Step 3: Evaluate the model and store the results as y_i.

Step 4: Repeat steps 2 and 3 for $i = 1$ to n.

Step 5: Analyze the results using histograms, summary statistics, confidence intervals, etc.

4.2.2 Algorithm Monte Carlo – Newton – Raphson

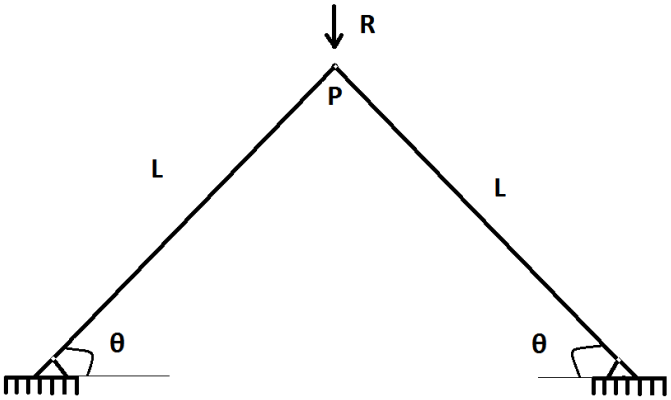
For the solution of the problem $kf(\Delta) = R$, with $k \sim N\left(\mu, \sigma^2\right)$ we have the next algorithm:

Start	**MCS-NR**
Step 1	Specify R (right hand side) and generate one random k from $N\left(\mu, \sigma^2\right)$ **(next)**
Step 2	Newton – Raphson (Specify: tolerance) **(next)**
Step 3	Specify the initial (seed) Δ_0 **(next)**
Step 4	Solve the linear system (4.10) and calculate $\delta\Delta$ if $\|\delta\Delta\| \le Tolerance$ go to (**step 5)** else $\Delta_0 \leftarrow \Delta_0 + \delta\Delta$ and go to **(step 3)**
Step 5	If Newton – Raphson Converged, then Calculate the statistics $Mean(\Delta)$ and $std = \sqrt{Var(\Delta)}$ Then you have pdf 
End	**MCS-NR**

5 Example - Results

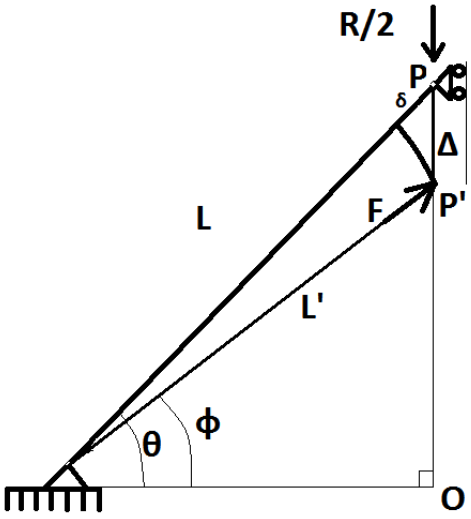
5.1 Definition of the problem

At the following truss we must find the displacement of the point P under loading R.



5-1 **Symmetric Truss with two bars under loading R**

It is obvious that we have a symmetric problem so we can make the calculation on the next equivalent geometrical scheme:



5-2 Displacement of point P

$$F \sin\varphi = \frac{R}{2} \tag{5.1}$$

From cosine rule

$$L' = \sqrt{L^2 - 2L\Delta\sin\theta + \Delta^2} \tag{5.2}$$

If we note $\delta = L - L'$ we have

$$\delta = L - \sqrt{L^2 - 2L\Delta\sin\theta + \Delta^2} \tag{5.3}$$

$$\sin\varphi = \frac{L\sin\theta - \Delta}{L - \delta} \tag{5.4}$$

Now, from equations (5.1),(5.3) and (5.4) we have the following expression:

$$F\frac{L\sin\theta - \Delta}{L - \delta} = \frac{R}{2} \tag{5.5}$$

For k=EA where E: Young's Module and area A we have

$$F = k\delta = k\frac{(L - L')}{L} \text{ , where k=EA} \tag{5.6}$$

$$2k\frac{(L - L')}{L}\cdot\frac{L\sin\theta - \Delta}{L'} = R \tag{5.7}$$

Further,

$$2k\left(-1 + \frac{L}{L'}\right)\cdot\left(\sin\theta - \frac{\Delta}{L}\right) = R \tag{5.8}$$

From the equation (5.2) equation (5.8) can be expressed as:

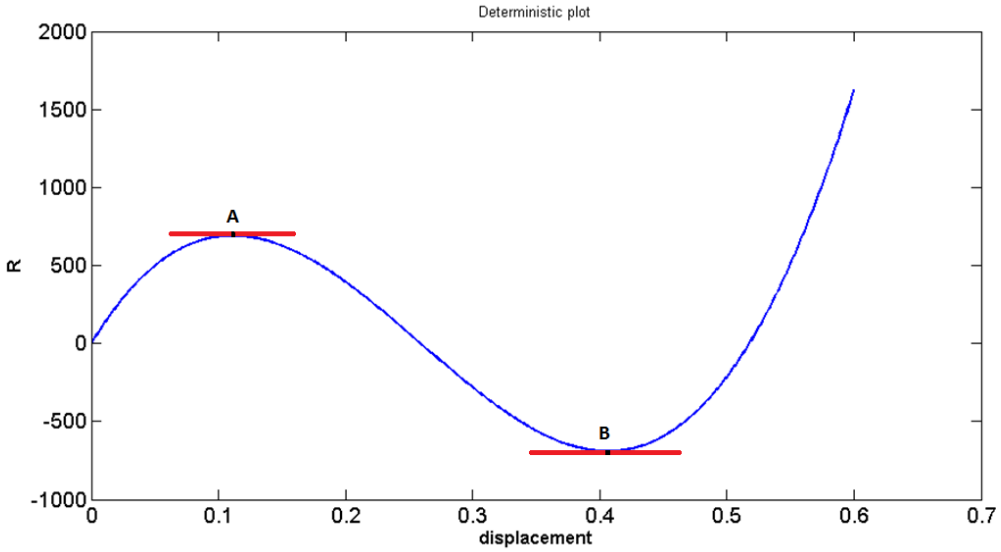
$$2k\left(-1 + \frac{L}{\sqrt{L^2 - 2L\Delta\sin\theta + \Delta^2}}\right)\cdot\left(\sin\theta - \frac{\Delta}{L}\right) = R \tag{5.9}$$

And finally

$$2k\left(-1 + \frac{1}{\left[1 - 2\frac{\Delta}{L}\sin\theta + \left(\frac{\Delta}{L}\right)^2\right]^{1/2}}\right)\cdot\left(\sin\theta - \frac{\Delta}{L}\right) = R \tag{5.10}$$

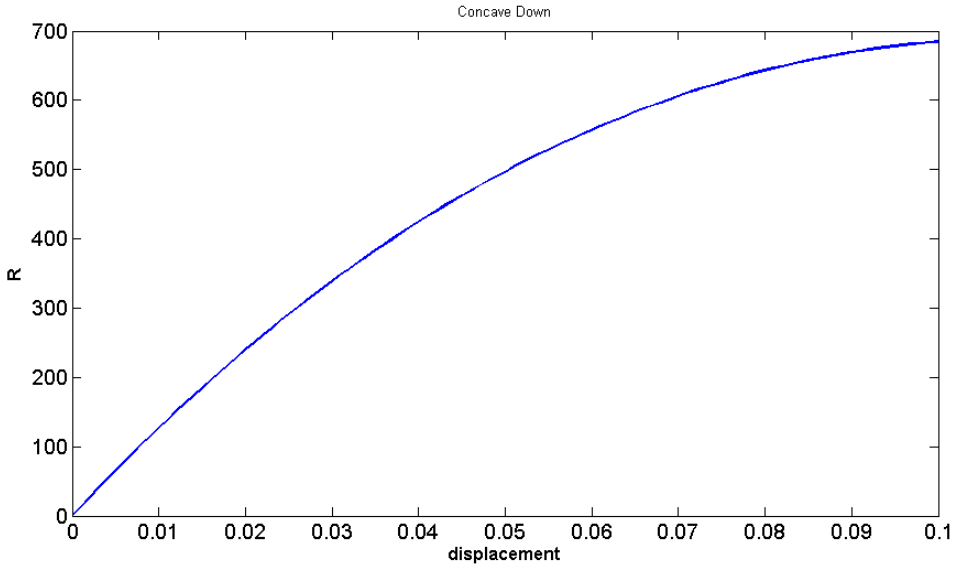
5.2 Deterministic Solution

Let $k = 10^5, (\mathrm{k} = EA = 10^5 \cdot 1)$ with length of each bar is $L = 1 un.length$ and $\theta = 15^0$.



5-3 Solution of deterministic problem for all displacements and stationary points A and B

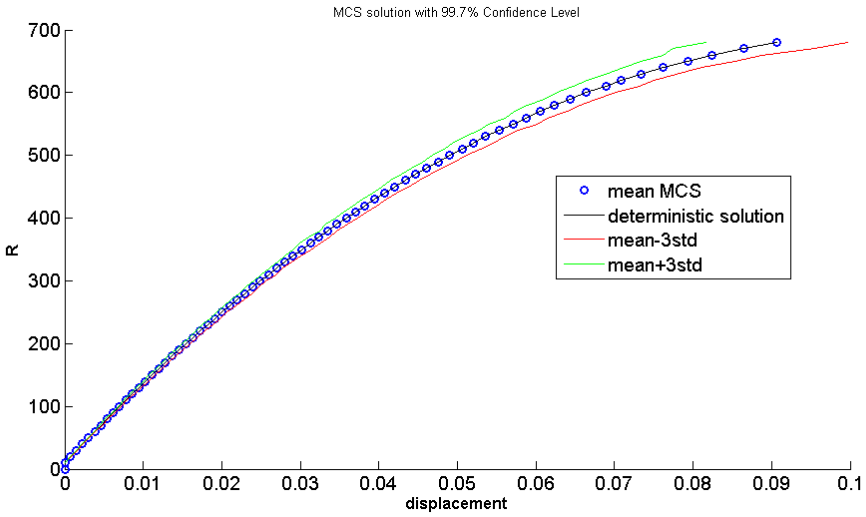
At this part, it is important to note the stationary points A and B of the diagram (5-3) because at those points Newton – Raphson it is unstable as the bar of the truss. In order to pass through this points, we must use path techniques (Arc Length algorithms) but isn't the purpose of this thesis. For that reason we can deal with the first concave down part to confirm the method of this thesis.



5-4 Concave down part of deterministic solution

5.2.1 Stochastic Solution with Monte Carlo Simulation (MCS)

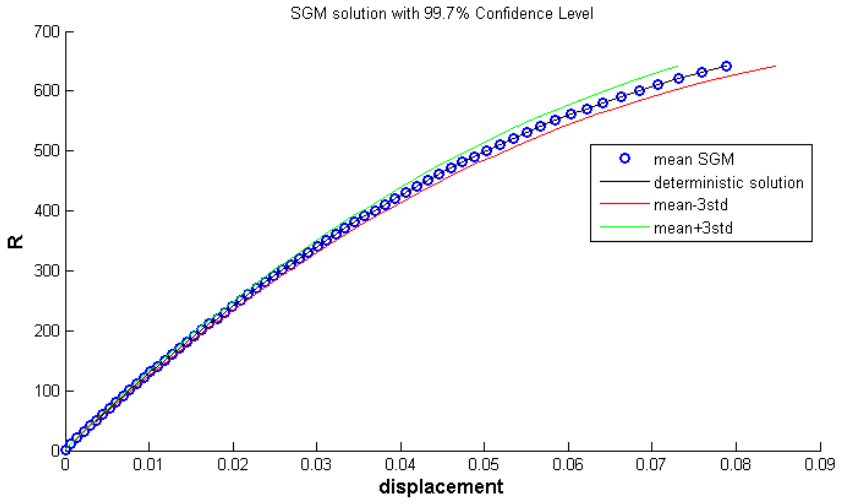
For the problem (5.10) with $k \sim N(10^5, 10^3)$, $L = 1unit$ and $\theta = 15^0$ after the application of the algorithm (MCS-NR) as it proposed in the paragraph (4.4.2) for $R \in [0, 650]$ and for each R we estimate the pdf for all displacements Δ with 1000 realizations. The following diagram is the solution of stochastic problem.



5-5 Monte Carlo Simulation - NR with $k \sim N\left(10^5, 10^3\right)$

5.2.2 Solution of Stochastic problem with SGM

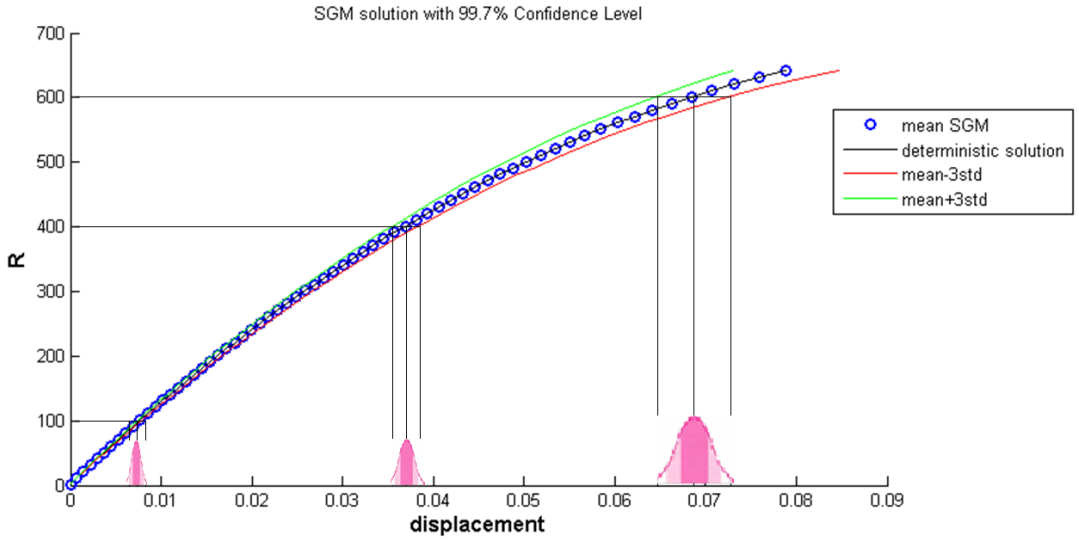
At this paragraph we solve the problem (5.10) again with $k \sim N(10^5, 10^3)$, $L = 1unit$ and $\theta = 15^0$ now with the application of the algorithm (SGM-NR) as it proposed in the paragraph (4.4.1) for $R \in [0, 650]$ and for each R we estimate the pdf for all displacements Δ without any realizations. The following diagram is the solution of stochastic problem and as we can see it is similar with (5-5) diagram.



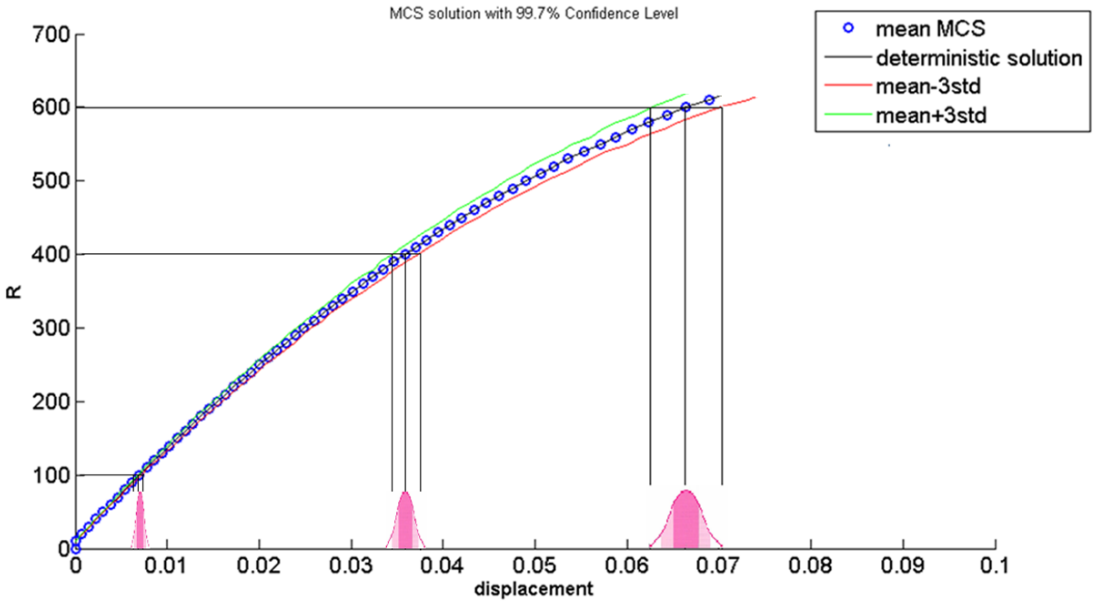
5-6 Spectral Galerkin Method -NR Simulation with $k \sim N\left(10^5, 10^3\right)$

5.3 Comparison of two methods

At this paragraph we introduce the way that we can compare SGM-NR method with reliability. The following diagrams represent for the specific loadings R=100, R=400 and R=600 the probability distribution function (pdf) and give the idea for the reliable comparison test at the following (5.3.1) – (5.3.3) sections.

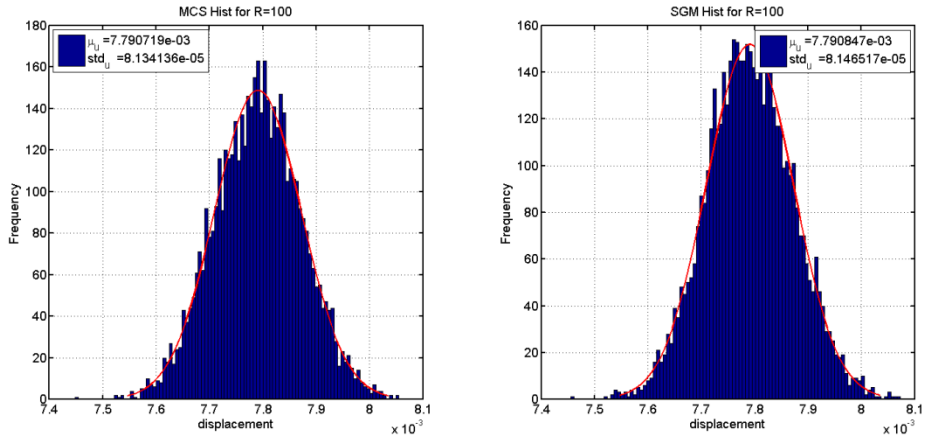


5-7 at this diagram for loading R=100, R=400 and R=600
Is the estimation for the pdf with the SGM-NR method



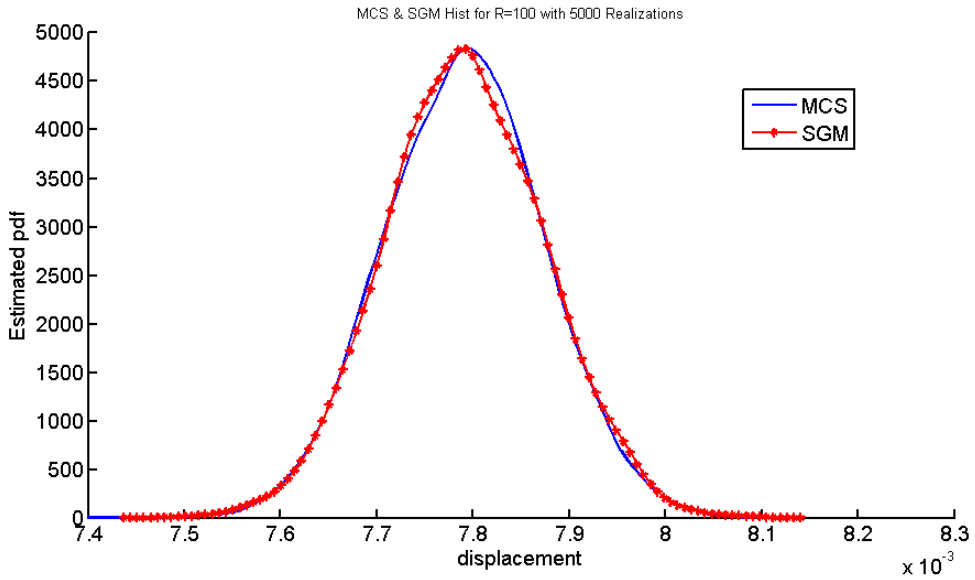
5-8 At this diagram for loading R=100, R=400 and R=600
Is the estimation for the pdf with the MCS-NR method.

5.3.1 R=100 with 5000 Realizations and p=2 polynomial Chaos



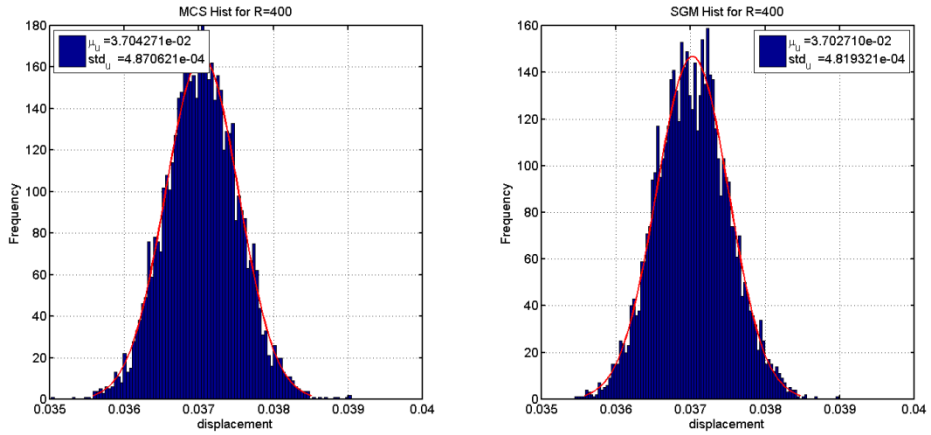
5-9 Left histogram is the pdf estimation with MCS-NR and the right is the pdf estimation with SGM-NR with R=100

R=100 (5000 Realizations)	Mean	Standard Deviation
MCS	$7.790719 \cdot 10^{-3}$	$8.134136 \cdot 10^{-5}$
SGM (p=2)	$7.790847 \cdot 10^{-3}$	$8.146517 \cdot 10^{-5}$
Absolute Error	$1.28 \cdot 10^{-7}$ **O(-7)**	$1.2381 \cdot 10^{-7}$ **O(-7)**



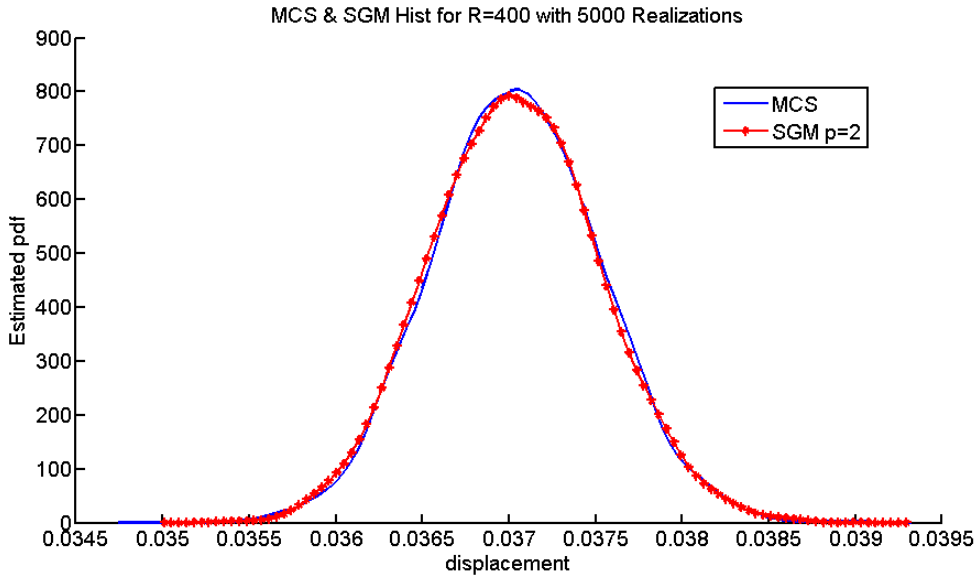
5-10 both kernel density plots for R=100

5.3.2 R=400 with 5000 Realizations and p=2 polynomial Chaos



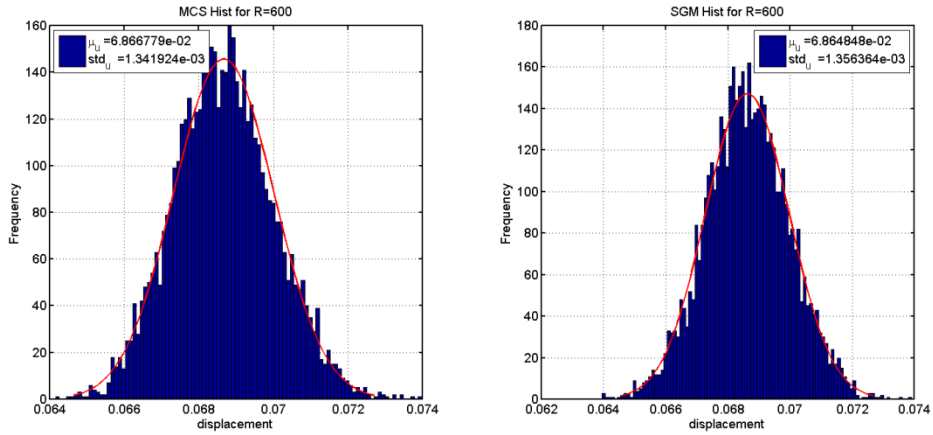
5-11 Left histogram is the pdf estimation with MCS-NR and the right is the pdf estimation with SGM-NR with R=400

R=400 (5000 Realizations)	Mean	Standard Deviation
MCS	$3.704271 \cdot 10^{-2}$	$4.870621 \cdot 10^{-4}$
SGM (p=2)	$3.702710 \cdot 10^{-2}$	$4.819321 \cdot 10^{-4}$
Absolute Error	$1.1561 \cdot 10^{-5}$ **O(-5)**	$5.13 \cdot 10^{-6}$ **O(-6)**



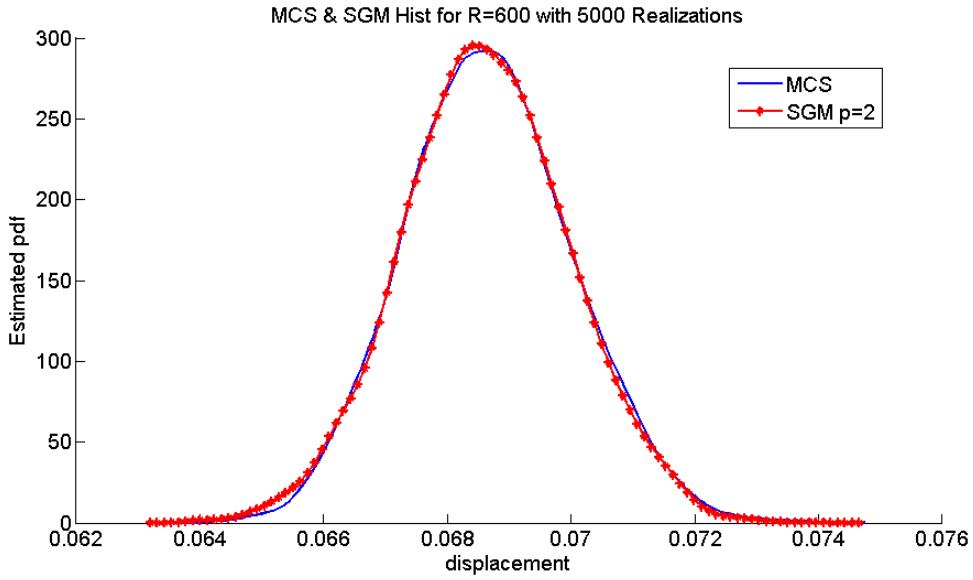
5-12 both kernel density plots for R=400

5.3.3 R=600 with 5000 Realizations and p=2 polynomial Chaos



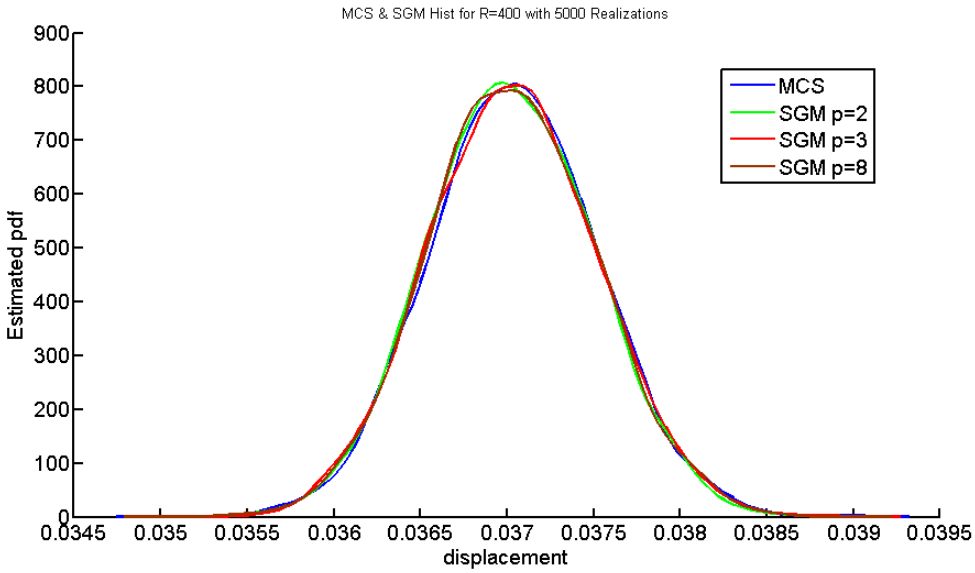
5-13 Left histogram is the pdf estimation with MCS-NR and the right is the pdf estimation with SGM-NR with R=600

R=600 (5000 Realizations)	Mean	Standard Deviation
MCS	$6.866779 \cdot 10^{-2}$	$1.341924 \cdot 10^{-3}$
SGM (p=2)	$6.864848 \cdot 10^{-2}$	$1.356364 \cdot 10^{-3}$
Absolute Error	$3.287 \cdot 10^{-5}$ **O(-5)**	$1.444 \cdot 10^{-5}$ **O(-5)**



5-14 both kernel density plots for R=600

5.3.4 Parametric Analysis



5-5 At this multi kernel density plots we can see the convergence of high orders of Polynomial Chaos Expansion method

R=400 (5000 Realizations)	Mean	Standard Deviation
MCS	$3.704271 \cdot 10^{-2}$	$4.870621 \cdot 10^{-4}$
SGM (p=2)	$3.702710 \cdot 10^{-2}$	$4.819321 \cdot 10^{-4}$
SGM (p=4)	0,0370271041200518	0,000481932178769749
SGM (p=8)	0,0370271041200518	0,000481932178769775

6 References

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