ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ

ΣΧΟΛΗ ΜΗΧΑΝΟΛΟΓΩΝ ΜΗΧΑΝΙΚΩΝ

ΤΟΜΕΑΣ ΜΗΧΑΝΟΛΟΓΙΚΩΝ ΚΑΤΑΣΚΕΥΩΝ

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του

ΙΩΑΝΝΗ ΤΣΟΓΙΑ

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ΕΡΓΑΣΤΗΡΙΟ ΑΥΤΟΜΑΤΟΥ ΕΛΕΓΧΟΥ

Αθήνα, Ιούλιος 2016

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Ευχαριστίες

Ευχαριστώ πολύ τον κ. Κυριακόπουλο για την ανάθεση αυτής της διπλωματικής και για την ευκαιρία να συμμετέχω στην έρευνα του εργαστηρίου Αυτομάτου Ελέγχου. Ιδιαίτερες ευχαριστίες αποδίδω και στον επιστημονικό συνεργάτη δρ. Χαράλαμπο Μπεχλιούλη για την πολύτιμη βοήθειά του σε θέματα Μαθηματικών και Συστημάτων. Ακόμη, ευχαριστώ θερμότατα τους γονείς μου και την αδερφή μου για την αμέριστη συμπαράσταση και στήριξή τους στις εύκολες και τις δύσκολες στιγμές. Τέλος, ευχαριστώ εγκαρδίως όλους τους φίλους μου που μου στάθηκαν τα τελευταία χρόνια.

Περίληψη

Η ραγδαία αύξηση των ρομποτικών χειριστών που απαιτούνται για την περάτωση μιας εργασίας έχει καταστήσει επιτακτική ανάγκη τη δημιουργία λεπτομερών σχημάτων ελέγχου. Ωστόσο, όσο η προσοχή δίνεται στη δημιουργία αποτελεσματικών σχημάτων ελέγχου, τόσο απομακρύνεται από τη δημιουργία σωστών και λεπτομερών κινηματικών μοντέλων. Ένα κινηματικό μοντέλο αποτελεί βάση για τον έλεγχο καθώς προσφέρει γνώση επάνω στη συμπεριφορά του συστήματος και τις ιδιότητές του. Το ρομπότ KUKA youBot είναι ένας βραχίονας επάνω σε μια κινούμενη πλατφόρμα, αναπτυγμένο κατά κύριο λόγο για ερευνητικούς και εκπαιδευτικούς σκοπούς. Ο βραχίονας πέντε βαθμών ελευθερίας παρουσιάζει εξαιρετικό ενδιαφέρον καθώς το πρόβλημα της κινηματικής του -ιδιαιτέρως της αντίστροφης- περιπλέκεται. Επιπλέον, και η πλατφόρμα παντώς διεύθυνσης επάνω στην οποία προσαρμόζεται ο βραχίονας, αποτελεί αντικείμενο προς μελέτη στην περίπτωση που είναι αυτούσια ή ακόμη και στην περίπτωση που θα ήταν μη ολονομική.

Το αντικείμενο αυτής της διπλωματικής είναι η ανάπτυξη του κινηματικού μοντέλου για το KUKA youBot αρχικά ξεχωριστά για κάθε κομμάτι του ρομπότ και εν συνεχεία για ολόκληρο το σύστημα. Ταυτόχρονα, εξερευνούνται οι ιδιότητες του συστήματος με τη βοήθεια ποικίλλων μαθηματικών εργαλείων.

Στο πρώτο μέρος της διπλωματικής, αναλύεται πλήρως η κινηματική του βραχίονα και αναπτύσσονται κριτήρια για την ύπαρξη, την σωστότητα και τον αριθμό των λύσεων στο πρόβλημα της αντίστροφης κινηματικής του. Παράλληλα, γίνεται ανάλυση ιδιοτιμών του με τη μέθοδο SVD. Στο δεύτερο μέρος, διακρίνονται δυο περιπτώσεις για την πλατφόρμα: η μη ολονομική και η παντός κατεύθυνσης κίνησης. Τέλος, ακολουθούνται τα ίδια βήματα για το σύνθετο σύστημα στο τελευταίο κομμάτι και γίνεται μια ανάλυση για την προσέγγιση των επικίνδυνων σημείων (εσωτερικές ιδιομορφίες και ιδιομορφίες αναπαράστασης).

Λέξεις Κλειδιά:

Κινηματική, Αντίστροφη Κινηματική, Ιακωβιανή, Ιδιοτιμές, Singular Value Decomposition, Μη Ολονομική Πλατφόρμα, Πλατφόρμα Παντώς Διεύθυνσης, Κινούμενος Βραχίονας.

Abstract

The increasing number of robot manipulators that are needed to complete a task, has made the creation of detailed control schemes a vital need. However, as the attention is placed primarily in the efficiency and stability of these schemes, it is therefore shifted from the creation of correct and detailed kinematics models. Such models constitute the basis of any control scheme as they provide basic knowledge about the properties of the system. The KUKA youBot manipulator is a manipulator mounted on a mobile platform developed for research and educational purposes. The five (low)-DOF manipulator is an interesting case study because its kinematics -especially its inverse kinematics problem- is actually complicated. In addition, the omnidirectional platform the manipulator is mounted on, is an interesting case itself too whether as omnidirectional or even if it was considered non-holonomic.

The goal of this thesis is to develop a kinematic model for the KUKA youBot starting separately from its parts and afterwards for the system as a whole. Meanwhile, the properties of the system are explored with the help of various algebraic, geometric and linear algebra tools.

In the first part of the thesis, the kinematics of the manipulator is detailed including forward, inverse kinematics, criterions that determine the existence, the correctness and the number of the inverse kinematics problem and singularity analysis with the Singular value Decomposition (SVD). The second part of the thesis includes the analysis of the platform in two scenarios: the omnidirectional case and the non-holonomic case. Last but not least, the same steps are followed for the composite system followed by an analysis of how close can the values of the joint go to "dangerous" configurations (i.e. representational and internal singularities).

Keywords:

Kinematics, Inverse Kinematics, Jacobian, Singular Values, Singular Value Decomposition, Non-holonomic Platform, Omnidirectional Platform, Mobile Manipulator.

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Chapter 1

Introduction

The KUKA youBot is a mobile manipulator that was primarily developed for research and education. It is slowly becoming a standard platform in many laboratories and institutions around the world that focus on development and testing of new robotic technologies. Such a system, offers at low cost the capability to access and experiment basic robot functionalities.

An overview of the mobile manipulator To begin with, let us examine the KUKA youBot; Its two main parts are:

- an omnidirectional mobile platform that consists of the robot chassis, four mecanum wheels, motors, power and an onboard PC board. Users can either run programs on this board, or control it remotely from a computer.

- a five degree-of-freedom (DOF) robotic arm with a two-finger gripper. The arm can either be controlled by the onboard PC or without the mobile platform by using an own PC connected via Ethernet cable.

Of course, additional sensors can be mounted on the robot to facilitate the solution of various control problems.

Figure 1.1: KUKA youBot

A robotic platform like this offers a high potential of application. However, its design provides limited performances. A manipulator with less than six degrees of freedom (DOF), or so called a low-DOF manipulator, is not capable of positioning and orienting an object efficiently. However, for specific industrial applications such as welding, painting and loading/unloading, a low-DOF manipulator may be sufficient in theory. What is considered an advantage of a low-DOF manipulator compared to 6-DOF or redundant robots is that it has a simpler mechanical structure (i.e. less motors and links), a simpler controller, better stiffness and a lower cost. Thus, the exponential increase in usage is highly justified.

Despite the above, challenges arise when it comes to formulating and/or solving the Kinematics problem for a low-DOF mobile manipulator. As stated before, the positioning and orientation of an object by the manipulator is possible yet inefficient. Usually, one has to forgo the ability to rotate around an axis and properly design the arm in order to achieve arbitrary manipulation and orientation. The degrees of freedom of a system can be simplistically viewed as the minimum number of coordinates required to specify a configuration in space. Applying this definition, six variables are needed in our case. But the number of actuators on the arm -five to be exact- is not enough. This fact conveniently leads to the next point of interest; the arm of the manipulator is **trivially underactuated** since it has a lower number of actuators than degrees of freedom. Underactuation is a technical term used in robotics and control theory to describe mechanical systems that cannot be commanded to follow arbitrary trajectories in configuration space.

Furthermore, other mathematical intricacies appear due to the nature of the manipulator. For example, the matrices used to describe forward and differential kinematics are rectangular. That said, there is not a 1-to-1 mapping between the Cartesian space (workspace) and the joint space, making the velocity and singularity analysis of such manipulators a challenging problem that requires specific techniques to find (possibly multiple) solutions of complicated nonlinear and transcendental equations. And even then, a closed form is usually not obtainable.

The purpose of this thesis is to provide readers with a thorough kinematics analysis of the KUKA youBot with its base/platform while exploring its various intricacies. Correct kinematics analyses can provide guidance for the robot's motion control or the basis for its trajectory planning and even act as a reference for the next step optimization design of the robot. Therefore, the kinematics analysis is a solid foundation of any control problem. Moreover, this thesis has a methodological structure/feel, hence allowing any potential researcher to follow a train of thought so as to conduct kinematics analysis in general.

1.1 Related Work

The existing literature on robot kinematics and control, especially with regards to the papers/dissertations before 2005, focuses primarily on 6-DOF, redundant robots and mobile manipulators with a particular focus on non-holonomy. Examples include but are not limited to [PP00b] , [PP00a] , [TK01], [TK00] , [BL89], [Chu97], [MLS94].

However, the increased popularity of low-DOF manipulators has spawned quite the research activity over the past years. In particular, attention has been drawn to the inverse kinematics problem since it poses one of the most significant challenges. There mainly exist two strategies for inverse kinematics, which can be found in a multitude of classic robotics books ([SS96] , [SK08] , [SB01], [Pie68], [Cra05]). On the one hand, there exists the closed form solution. It takes advantage of the geometric and algebraic properties that the structure of the robot possesses to identify every single possible solution. On the other hand, there is the numerical solution. This one usually adopts an iterative method to find just one solution that stems from a set of starting values. Admittedly, there is a (usually high) difficulty to derive the former depending on the complexity of the system and often many algebraic/geometrical tricks and techniques are required but its speed more than compensates for that. In addition, there is always the danger of a numerical method failing to converge, making it unable to determine safely whether there is actually a solution. make the closed form solutions much more attractive. Therefore, a closed form solution is generally more attractive than the numerical approach, but it should be noted that the choice of method gratly depends on the system under examination. Instances of numerical methods include, but are not limited to the (modified) Newton-Raphson method, neural networks [OCAM01], genetic algorithms [Nea98], evolutionary approaches [KMA06], optimal search [RWS15] and combined/hybrid methods.

Furthermore, various specific cases of five (low)-DOF manipulators have been studied the past years. Some cases involve full kinematics analyses whereas others involve modeling on the kinematic level and control. In [XCGH05] the forward and inverse kinematics for an accessory for the family of Pioneer Mobile Robots (PArm, five-DOF manipulator) are derived. Afterwards, the arm is commanded to follow a specific trajectory while maintaining the orientation of one axis in the end-effector frame. In a similar fashion, Gan et al. in [GOHR05] , [GR03] proposed a complete model and analytical solution to the kinematics of the Pioneer2-arm of the same family.

In [LZLZ13] Liu et al. came up with an efficient inverse kinematics approach which features fast computing performance for a PUMA560 robot manipulator. Their solution included taking advantage of the geometric properties of orthogonal and block matrices. Manseur [MD92] solved the inverse kinematics of all five-DOF (5R) robot manipulators by using an one-dimensional iterative technique, which is similar to Newton-Raphson. A five-DOF elbow-type manipulator and a four-DOF Stanford type manipulator were analyzed in [FT03]. The inverse kinematics problem was solved for both with the help of reciprocal screws and a mapping was determined between the independent velocity components in the Cartesian space and the joint rates in the joint space.

Another example of a specific inverse kinematics technique can be found in [SCT11] where Sariyildiz et al. employed quaternion algebra, dualquaternion algebra, and exponential mapping methods (matrix algebra). In [CL14] a new path-planning algorithm was proposed based on inverse kinematics while satisfying the criterion of maximum mobility of the end effector. Also, a youBot mobile manipulator is studied in [ZLX15] where a hybrid algebraic and analytical geometric solution is proposed to solve its inverse kinematics.

Some cases of low-DOF manipulators with even more specialized applications were found in literature. In [LXW14] the kinematic model of such a manipulator is established followed by a workspace analysis and an inverse kinematics solution is derived. In order to solve the intricacies of the trajectory planning problem for a 5-DOF agricultural picking robot, Lu et al. in [LYGB10] analyzed its Denavit-Hartenberg parameters and established its kinematics model. In addition, an algorithm based on the comprehensive application of analytic and geometric method was also developed, and the inverse kinematics analytical solution was also obtained. In [Hua11], the kinematics model for a five-DOF cutting robot was established with the modified Denavit-Hartenberg method and the inverse kinematics was solved using inverse transformation method.

Moreover, Shen ([She10]) developed a kinematic model for a 5-DOF Carrying Manipulator, that is primarily used for teaching. After obtaining solutions for the forward and inverse kinematics, he analyzed the workspace of the manipulator with graphical solutions as a function of the working condition and other technological parameters. Based on the geometric model and the kinematics analysis of a 5-DOF manipulator used for rehabilitation purposes, Lu et al. ([LST07]) found accurate analytical solutions for all joint angles while offering reliance for the actual intellectual control of the position and speed of the rehabilitating robot.

1.2 Thesis Structure

The second chapter is essentially the derivation of the matrix that has an inherently vital role in the upcoming analysis: the homogeneous transformation that describes the forward kinematics for the manipulator. A short description of the Denavit-Hartenberg convention is also presented.

The third chapter, tackles the notorious problem of the inverse kinematics for the manipulator. After processing the matrices and solving the problem in a systematic way, two criterions are generated and proven so as to verify the existence, the correctness and the number of feasible solutions to the problem.

In chapter four, the geometric Jacobian of the manipulator is calculated in a systematic way. In addition, a singularity analysis is conducted and the Singular Value Decomposition is presented in order to assist in the analysis.

Chapter five is a short chapter that describes a significant property of the Singular Value Decomposition that is used to derive the inverse of the aforementioned Jacobian matrix.

The sixth chapter is an analysis of the mobile platform on which the arm will be mounted eventually. This analysis is split in two parts: the omnidirectional case and the non-holonomic case.

Last but not least, the combined system kinematics (forward, differential and the respective inverse) equations are presented and the issue of proximity to singular points is discussed.

Chapter 2

Forward Kinematics - Arm

Forward kinematics refers to the use of the kinematic equations of a robot to compute the position of the end-effector from specified values for the joint parameters. As such, obtaining the map that transforms the joint angles, for an open chain, revolute joint (5R) arm in our case, to the end-effector position in the Cartesian space, is primarily a "complex" geometric problem and an algebraic problem secondarily.

However, there are systematic approaches to this problem. In particular, one of the most popular methods in the field of robotics is the Denavit-Hartenberg (D-H) convention. An open chain manipulator, generally, is constituted by $n+1$ links connected by n joints. By default, the 0^{th} link is attached to the ground or to the manipulator base in our case. Since a joint connects two consecutive links, a homogeneous transformation could describe the position and orientation of a coordinate frame on a link with respect to the previous/next one. The purpose of the D-H convention is to facilitate the derivation of these homogeneous transformations and find the forward kinematics map recursively by matrix multiplication of these transformations. In particular, the D-H convention uses four parameters that completely specify the position of frame i with respect to frame i–1:

- a_i, the distance between the origins of the two coordinate frames O_i , $O_{i'}$
- d_i, the coordinate of O_i along z_{i-1},
- α_i, the angle between axes z_{i-1} and z_i about axis x_i to be taken positive when rotation is made counter-clockwise,
- θ_i, the angle between axes x_{i-1} and x_i about axis z_{i-1} to be taken positive when rotation is made counter-clockwise

Furthermore, the convention specifies some rules for the attachment of the frames. These can be found in any classic Robotics book.

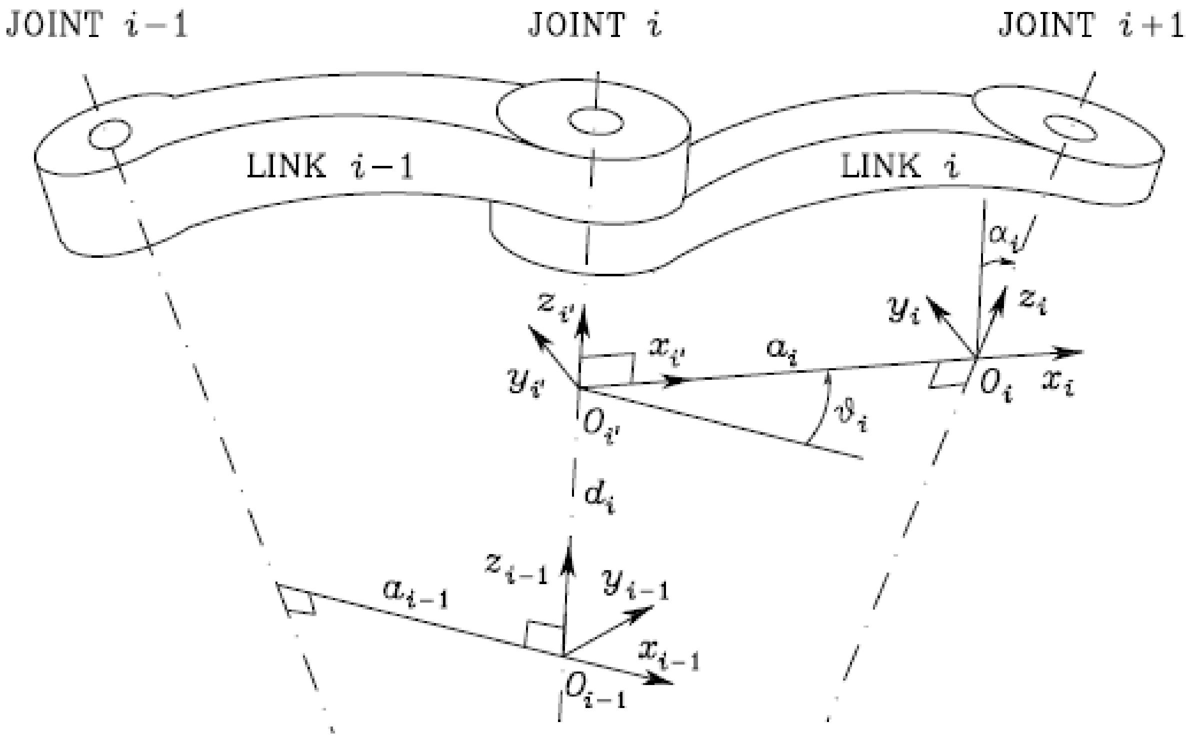


Figure 2.1: The classic Denavit-Hartenberg convention, frames and parameters.

In this thesis, in order to derive the forward kinematics map for the manipulator, the **modified Denavit-Hartenberg convention** -as explained in [Cra05]- was used. The modified D-H convention differs from the conventional one in the sense that the frame i is attached to the i^{th} link instead of the $(i+1)^{\text{th}}$ and its center is positioned on the i^{th} axis.

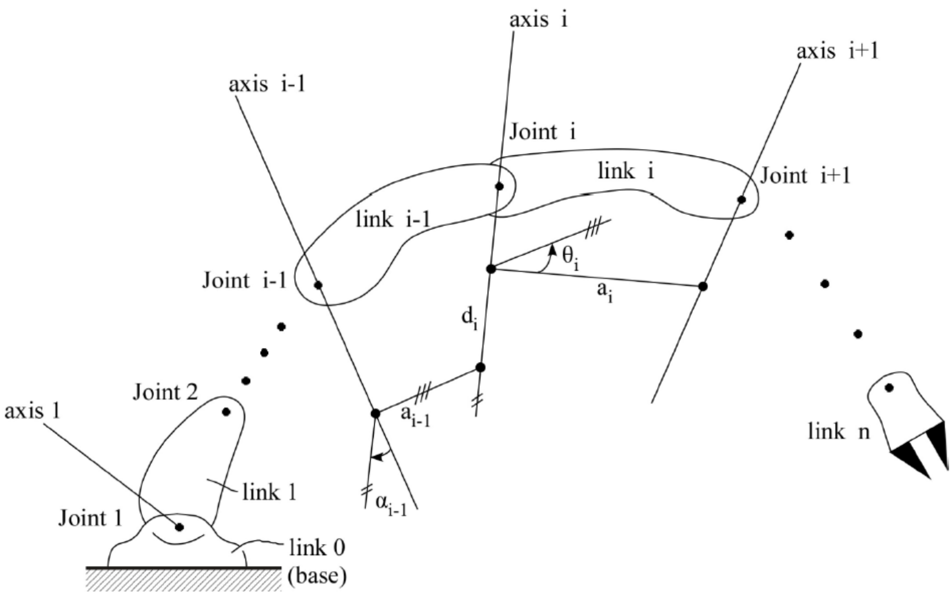


Figure 2.2: Axes and parameters according to the modified D-H convention.

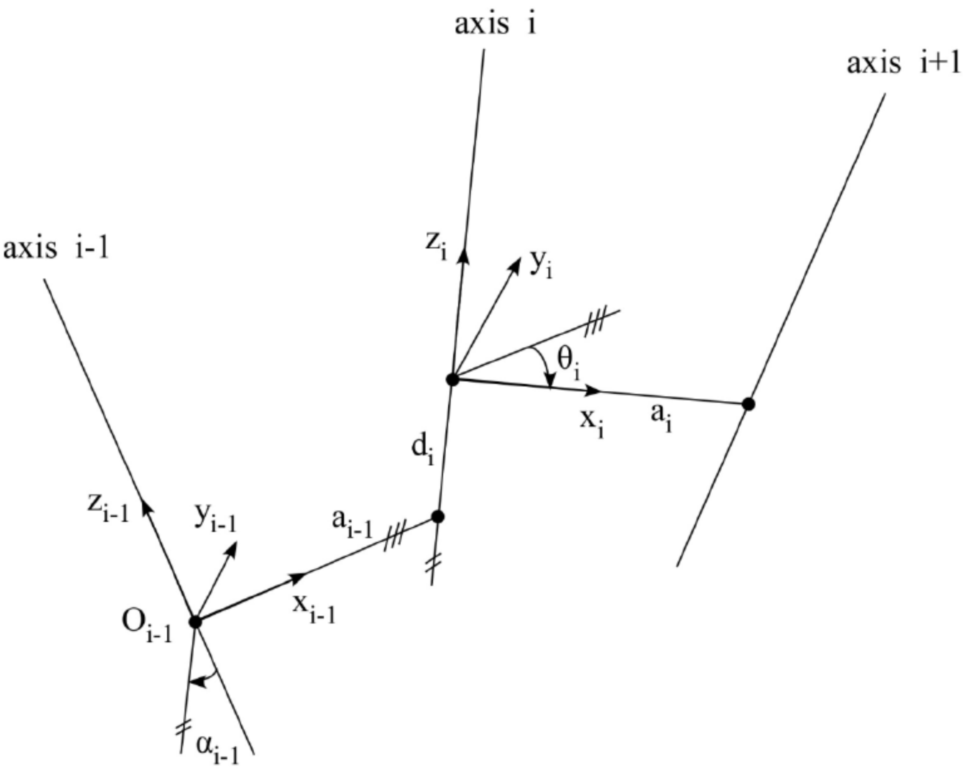


Figure 2.3: Frame attachment according to the modified D-H convention.

Following the D-H rules for frame attachment, the frames for the KUKA youBot are presented below. Note that the y-axes are not drawn, but they are selected so as to create right handed coordinate systems.

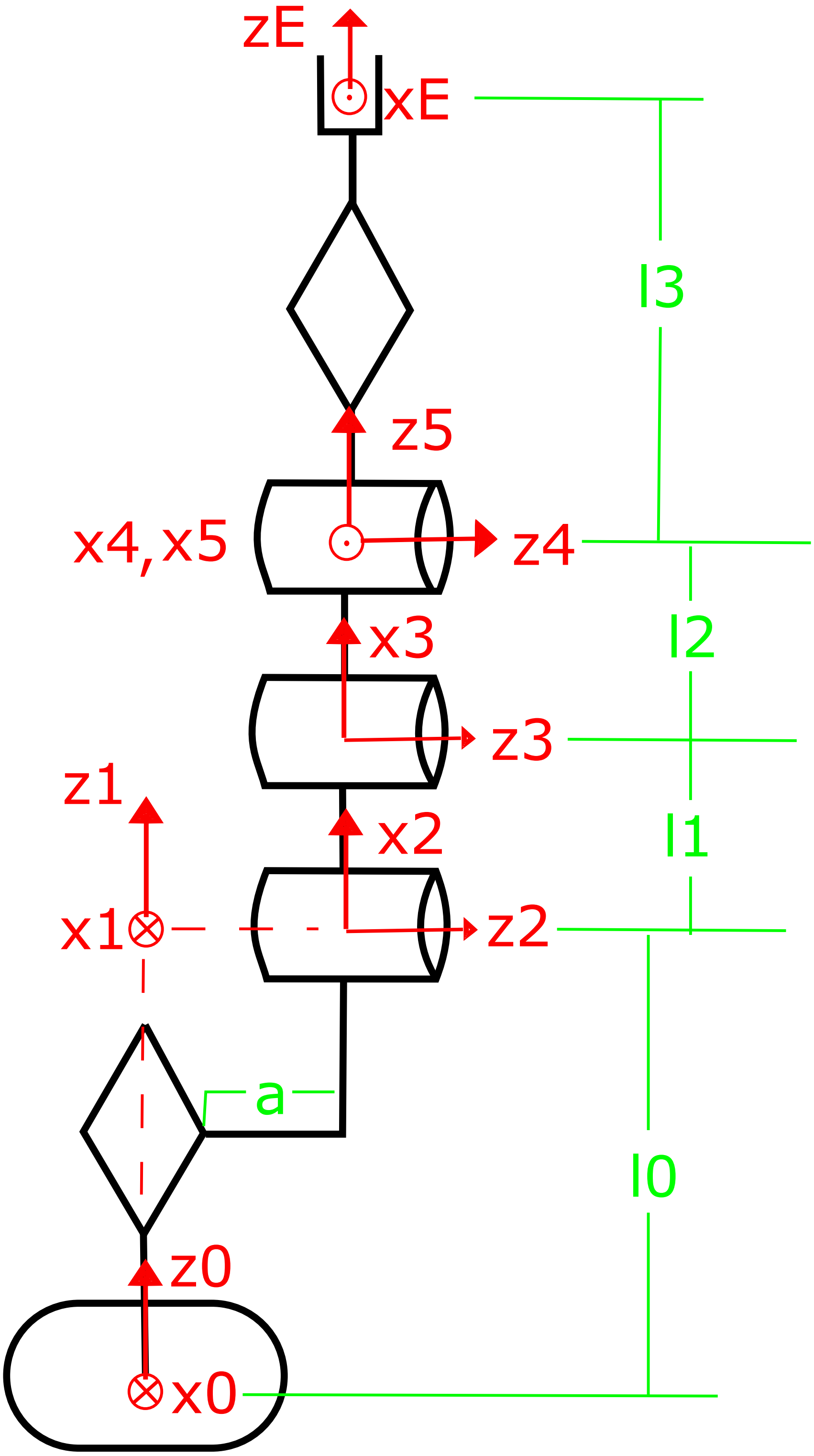


Figure 2.4: Frame attachment on the KUKA youBot.

The values of the D-H parameters are presented in the table below:

i	a_{i-1}	α_{i-1}	d_i	q_i
1	0	0	l_0	q_1
2	a	$\frac{\pi}{2}$	0	$q_2 + \frac{\pi}{2}$
3	l_1	0	0	q_3
4	l_2	0	0	$q_4 + \frac{\pi}{2}$
5	0	$\frac{\pi}{2}$	0	q_5
E	0	0	l_3	0

where a is the offset between the two vertical z-axes of the arm. According to the modified D-H convention, the homogeneous transformation from frame i to frame i-1 is effectively:

$$^{i-1}\mathbf{T}_i = \begin{bmatrix} cq_i & -sq_i & 0 & a_{i-1} \\ sq_ic\alpha_{i-1} & cq_ic\alpha_{i-1} & -s\alpha_{i-1} & -d_is\alpha_{i-1} \\ sq_is\alpha_{i-1} & cq_is\alpha_{i-1} & c\alpha_{i-1} & d_ic\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For the KUKA Youbot 5 D.O.F. manipulator, the end effector-to-base transformation is:

$$^0\mathbf{T}_E = {}^0\mathbf{T}_1{}^1\mathbf{T}_2{}^2\mathbf{T}_3{}^3\mathbf{T}_4{}^4\mathbf{T}_5{}^5\mathbf{T}_E \tag{2.1}$$

At this point, it should be noted that the last homogeneous transformation is just a translation. Furthermore, some adjustments were made to the frame attachments in order to easily manipulate the produced equations in the later portions of the analysis.

After substitution with the parameters from the table, we get the respective frame-to-frame homogeneous transformations:

$$^0\mathbf{T}_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^1\mathbf{T}_2 = \begin{bmatrix} -s_2 & -c_2 & 0 & a \\ 0 & 0 & -1 & 0 \\ c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^2\mathbf{T}_3 = \begin{bmatrix} c_3 & -s_3 & 0 & l_1 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
{}^{3}\mathbf{T}_4 = \begin{bmatrix} -s_4 & -c_4 & 0 & l_2 \\ c_4 & -s_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{4}\mathbf{T}_5 = \begin{bmatrix} c_5 & -s_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{5}\mathbf{T}_E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

And of course the end-effector-to-base transformation:

$$
{}^{0}\mathbf{T}_E = \begin{bmatrix} -c_1c_{234}c_5+s_1s_5 & c_1c_{234}s_5+c_5s_1 & -c_1s_{234} & c_1(a-l_1s_2-l_2s_{23}-l_3s_{234}) \\ -s_1c_{234}c_5-c_1s_5 & s_1c_{234}s_5-c_1c_5 & -s_1s_{234} & s_1(a-l_1s_2-l_2s_{23}-l_3s_{234}) \\ -s_{234}c_5 & s_{234}s_5 & c_{234} & l_0+l_1c_2+l_2c_{23}+l_3c_{234} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2}
$$

Some final notes on the kinematics. There are some methods that can check, even partially, the validity of the forward kinematics map. Firstly, as far as the orientation part of the homogeneous transformation, i.e. the upper left 3×3 block matrix, is concerned, the squared sum of its lines and columns MUST be equal to $+1$, since it is an orthogonal matrix. This property is equivalent to its determinant being $+1$. As far as the part with the lengths is concerned, we can substitute the joint variables with values that reflect a known position for the end-effector (e.g. home position, full stretch, etc.) and see if the map actually holds up. Both conditions are met for our transformation.

Chapter 3

Inverse Kinematics - Arm

The inverse kinematics problem, as stated in the related work section, consists of the determination of the joint variables corresponding to a given end-effector orientation and position. The solution to this problem is mandatory so as to transform the motion of the end-effector in the operational space, into the corresponding joint space motions.

In the previous chapter, the forward kinematics map was computed in a unique manner, meaning that once the joint variables are given, the position of the end-effector can be derived. On the other hand, the inverse kinematics problem is much more complex for the following reasons:

- The inverse kinematics equations are in general nonlinear, and thus it is not always possible to find a closed-form solution.
- Multiple or even infinite solutions may exist, e.g., in the case of a kinematically redundant manipulator
- There might be no admissible solutions, in view of the manipulator kinematic structure.

The existence of solutions is guaranteed only if the given end-effector position and orientation belong to the manipulator dexterous workspace whereas the problem of multiple solutions depends not only on the number of DOFs but also on the number of non-null DH parameters; in general, the greater the number of non-null parameters, the greater the number of admissible solutions. In addition, the existence of mechanical joint limits may eventually reduce further the number of admissible multiple solutions for the real structure.

In general, computation of closed-form solutions requires either algebraic intuition to find those significant equations containing the unknowns or geometric intuition to find those significant points on the structure with respect to which it is convenient to express position and/or orientation as a function of a reduced number of unknowns.

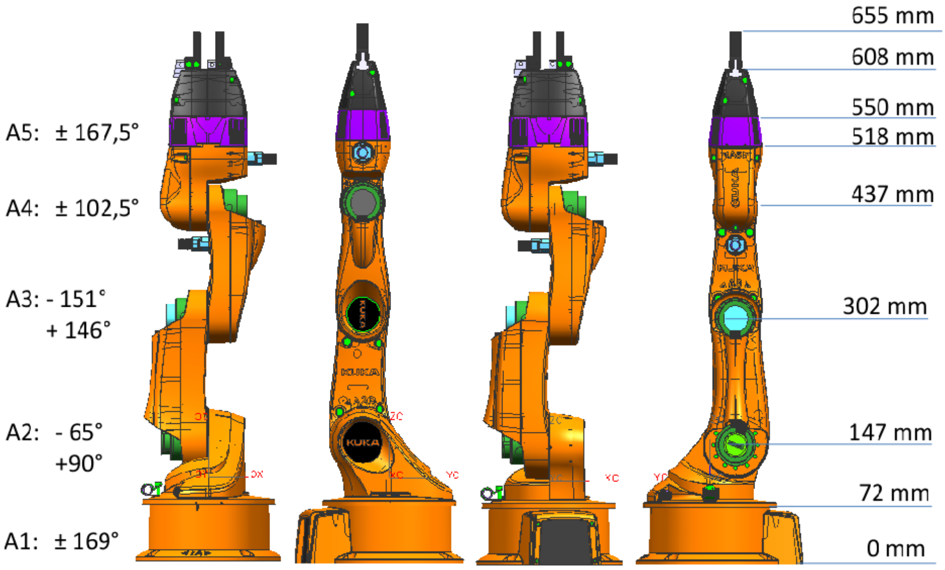


Figure 3.1: The KUKA youBot kinematic structure and its joint limits.

In this thesis, the solution for the inverse kinematics problem starts from the relationship (2.1) :

$$^{0}\mathbf{T}_E = {}^{0}\mathbf{T}_1{}^{1}\mathbf{T}_2{}^{2}\mathbf{T}_3{}^{3}\mathbf{T}_4{}^{4}\mathbf{T}_5{}^{5}\mathbf{T}_E$$

We identify that except for joint 1, every other joint is lying on the same plane, thus we decompose the above relationship as follows:

$$[{}^{0}\mathbf{T}_1]^{-1}\ {}^{0}\mathbf{T}_E = {}^{1}\mathbf{T}_2{}^{2}\mathbf{T}_3{}^{3}\mathbf{T}_4{}^{4}\mathbf{T}_5{}^{5}\mathbf{T}_E \tag{3.1}$$

where:

$$[{}^{0}\mathbf{T}_1]^{-1} = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & -l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{1}\mathbf{T}_2{}^{2}\mathbf{T}_3{}^{3}\mathbf{T}_4{}^{4}\mathbf{T}_5{}^{5}\mathbf{T}_E = \begin{bmatrix} -c_{234}c_5 & c_{234}s_5 & -s_{234} & a - l_1 s_2 - l_2 s_{23} - l_3 s_{234} \\ -s_5 & -c_5 & 0 & 0 \\ -c_5 s_{234} & s_5 s_{234} & c_{234} & l_1 c_2 + l_2 c_{23} + l_3 c_{234} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$^{0}\mathbf{T}_E = \begin{bmatrix} n'_x & o'_x & a'_x & p'_x \\ n'_y & o'_y & a'_y & p'_y \\ n'_z & o'_z & a'_z & p'_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The elements of ${}^{0}\mathbf{T}_E$ are completely known for this section. To be precise, (p'_x, p'_y, p'_z) represents the position, and $\left(\left\{n'_x, n'_y, n'_z\right\}, \left\{o'_x, o'_y, o'_z\right\}, \left\{a'_x, a'_y, a'_z\right\}\right)$ the orientation of the end-effector.

the elements of which are known. After computations, the equation (3.1) becomes:

$$\begin{bmatrix} c_1 n'_x + s_1 n'_y & c_1 o'_x + s_1 o'_y & c_1 a'_x + s_1 a'_y & c_1 p'_x + s_1 p'_y \\ -s_1 n'_x + c_1 n'_y & -s_1 o'_x + c_1 o'_y & -s_1 a'_x + c_1 a'_y & -s_1 p'_x + c_1 p'_y \\ n'_z & o'_z & a'_z & p'_z - l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -c_{234} c_5 & c_{234} s_5 & -s_{234} & a - l_1 s_2 - l_2 s_{23} - l_3 s_{234} \\ -s_5 & -c_5 & 0 & 0 \\ -c_5 s_{234} & s_5 s_{234} & c_{234} & l_1 c_2 + l_2 c_{23} + l_3 c_{234} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By equating the (2,4) elements of both matrices we get:

$$-p'_x s_1 + p'_y c_1 = 0$$

This equation has two solutions for q_1:

$$q_{1,1} = atan2(p'_x, p'_y) \text{ and } q_{1,2} = atan2(-p'_x, -p'_y) \tag{3.2}$$

where atan2 is the arctangent function with two arguments. The purpose of using two arguments instead of one is to gather information on the signs of the inputs in order to return the appropriate quadrant of the computed angle, which is not possible for the single-argument arctangent function. It also avoids the problems of division by zero.

By equating the (2,1) elements of both matrices and the (2,2) elements afterwards, we get the following two equations:

$$-s_5 = -s_1 n'_x + c_1 o'_x$$

$$-c_5 = -s_1 n'_y + c_1 o'_y$$

The only solution to this equation is $q_5 = atan2(s_1 n'_x - c_1 o'_x, s_1 n'_y - c_1 o'_y)$. However, since there are two solutions for q_1, two solutions are also generated for q_5 too, hence:

$$q_{5,i} = atan2(s_{1,i} n'_x - c_{1,i} o'_x, s_{1,i} n'_y - c_{1,i} o'_y) \ , \ i = 1,2 \tag{3.3}$$

Carrying on with the same procedure, we equate the (1,3) elements and the (3,3) ones, thus we get:

$$c_{234} = a'_z$$

$$-s_{234} = s_1 a'_y + c_1 a'_x$$

Similarly, two solutions are generated for the sum $q_2 + q_3 + q_4$:

$$(q_2 + q_3 + q_4)_i = atan2(a'_z, -s_{1,i} a'_y - c_{1,i} a'_x) \ , \ i = 1,2 \tag{3.4}$$

Before we continue, there are some pertinent questions to be answered. So far, it is assumed that the solution to the inverse kinematics problem actually exists. However, it is evident that the inverse kinematics for a 5-DOF manipulator will have no solution for some given positions and orientations of the end-effector. How can we know in which cases there will be no solution to the inverse kinematics? One way to figure it out would be to generate the workspace of the manipulator and find out if the given position and orientation belongs in said workspace. Although this is a more than viable solution, it is counter-intuitive to the algebraic/geometric approach that is usually followed.

That said, the approach would be to discover a geometric/algebraic property that holds when a solution to the inverse kinematics problem exists and is correct. Starting again from (2.1), we isolate the end-effector frame from the rest of the arm.

$$[{}^3\mathbf{T}_4]^{-1}\ [{}^2\mathbf{T}_3]^{-1}\ [{}^1\mathbf{T}_2]^{-1}\ [{}^1\mathbf{T}_0]^{-1}\ {}^0\mathbf{T}_E{}^0\mathbf{T}_E = {}^4\mathbf{T}_5{}^5\mathbf{T}_E \tag{3.5}$$

and after substitutions

$$\begin{bmatrix} *** & *** & *** & *** \\ -n'_z c_{234} + (n'_x c_1 + n'_y s_1) s_{234} & -o'_z c_{234} + (o'_x c_1 + o'_y s_1) s_{234} & *** & *** \\ *** & *** & -a'_y c_1 + a'_x s_1 & *** \end{bmatrix} =$$

$$\begin{bmatrix} *** & *** & *** & *** \\ 0 & 0 & *** & *** \\ *** & *** & 0 & *** \end{bmatrix}$$

The starred terms are of no particular importance. However, we can extract three paramount equations from the non-starred ones.

$$-n'_z c_{234} + (n'_x c_1 + n'_y s_1) s_{234} = 0$$
$$-o'_z c_{234} + (o'_x c_1 + o'_y s_1) s_{234} = 0$$
$$-a'_y c_1 + a'_x s_1 = 0$$

There is a physical meaning to the above equations, which becomes more apparent if they are rewritten in a different fashion:

$$\begin{bmatrix} n'_x\ n'_y\ n'_z \end{bmatrix} . [-c_1 c_{234}\ \ - s_1 s_{234}\ c_{234}]^T = 0$$
$$\begin{bmatrix} o'_x\ o'_y\ o'_z \end{bmatrix} . [-c_1 c_{234}\ \ - s_1 s_{234}\ c_{234}]^T = 0$$
$$\begin{bmatrix} a'_x\ a'_y\ a'_z \end{bmatrix} . [s_1\ \ - c_1\ 0]^T = 0$$

The physical meaning of the last equation is that the vector of the directional cosines that represents the z-axis of the end-effector frame, is perpendicular to the vector of the directional cosines that represents the y-axis of frame 1. Similarly, the first and second equations reflect the orthogonality between the directional cosine vectors that represent the x-axis and y-axis of the end-effector frame respectively and the directional cosine vector of the z-axis of the end-effector frame. The latter is evident from the third column of the forward kinematics map.

But isn't the orthogonality between the axes of the end-effector frame supposedly taken for granted? The answer is no in the case of inverse kinematics. More specifically, the starting point of the solution is a matrix that **possibly** represents the position and orientation of the end-effector. The orthogonality would hold if and only if, after substituting the joint angles $q_i,\ i = 1,2,3,4,5$ that were found by solving the inverse kinematics problem, the vector $[-c_1c_{234}\ \ -s_1s_{234}\ c_{234}]^T$ coincided with the directional cosine vector of the z-axis of the end-effector frame. This obviously may not be the case, since the given values of $n'_i, o'_i, a'_i,\ i = x, y, z$ may or may not produce valid/correct joint angles.

All three equations involve perpendicular mechanics between known and unknown parameters, but only the first two include all the joint angles that have been found already. Thus the focus will be shifted to these two. The criterions become:

$$-n'_x c_1 s_{234} - n'_y s_1 s_{234} + n'_z c_{234} = 0 \tag{3.6}$$

$$-o'_x c_1 s_{234} - o'_y s_1 s_{234} + o'_z c_{234} = 0 \tag{3.7}$$

Theorem 1: The inverse kinematics problem for the youBot has correct solutions, if and only if at least one of the solutions for q_1, $q_2+q_3+q_4$, q_5 given in (3.2), (3.4), and (3.3) satisfies both (3.6) and (3.7).
Proof of the necessity: If none of the equations is satisfied, then the given position and orientation of the manipulator cannot be satisfied by these solutions. The given position and orientation form the vectors $\left[n'_x\ n'_y\ n'_z\right]$ and $\left[o'_x\ o'_y\ o'_z\right]$. Then, substituting the joint angles found above in the vector $[-c_1c_{234}\ \ -s_1s_{234}\ c_{234}]$, should in theory - i.e. if the joint angles represent valid solutions- produce the vector of the directional cosines of the z-axis of the end-effector frame. But it was assumed that the solutions do not satisfy the equations, thus they do not represent valid solutions. Consequently, the supposed orthogonality of the vectors is violated.
Proof of the sufficiency: Let it be assumed that the equation (3.7) is satisfied. Then a similar proof should cover the case when (3.6) is satisfied too. If (3.7) is satisfied by any number of solutions $(q_1, q_2+q_3+q_4, q_5)_i$ $(i = 1,2$ in our case), then the solutions will satisfy (3.5), because of the orthogonality of the rotation transformation matrix for the orientation. That is, the desired position and orientation of the end-effector can be realized.

Lastly, we need to evaluate the components of the sum $q_2+q_3+q_4$ separately, for any values of the sum that were found above. As stated in the beginning, it is noticeable that the joints 2,3,4 lie on the same plane, called the manipulator plane from now on. The manipulator plane is presented below:
It can be easily determined that:

$$d_x = l_1c_2 + l_2c_{23}$$

$$d_z = l_1s_2 + l_2s_{23}$$

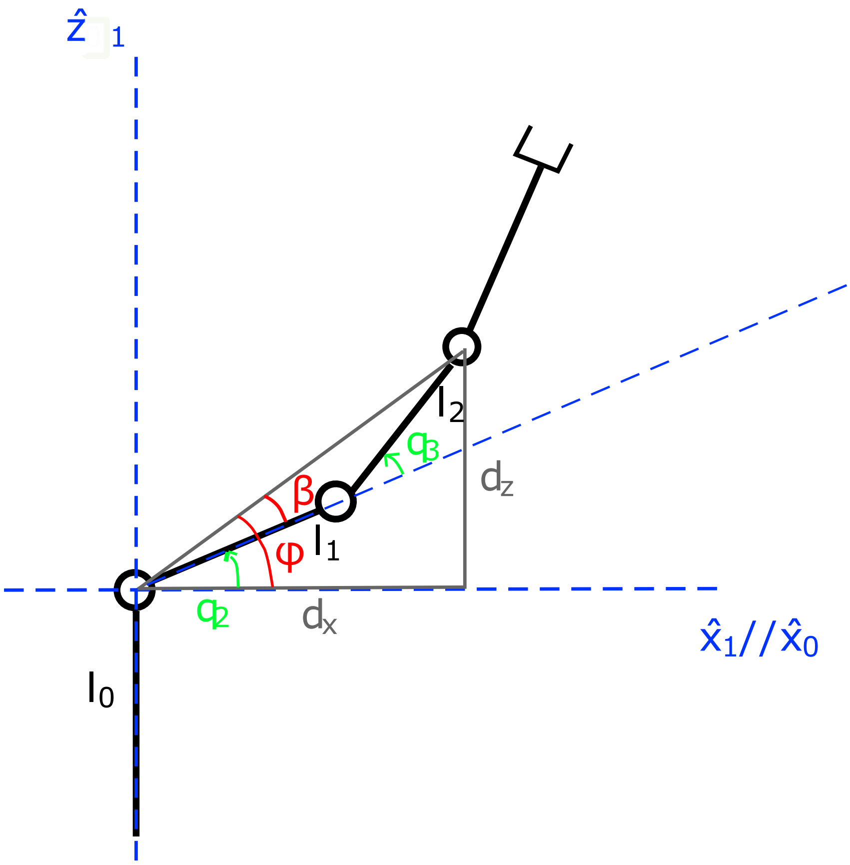


Figure 3.2: The KUKA youBot manipulator plane.

These quantities are unknown, so they must be expressed as a function of known quantities. By taking a closer look at p'_x and p'_z we see that:

$$p'_x = c_1(l_1c_2 + l_2c_{23} + l_3s_{234}) = c_1(d_x + l_3s_{234})$$

$$p'_z = l_0 + l_1s_2 + l_2s_{23} - l_3c_{234} = l_0 + d_z - l_3c_{234}$$

or:

$$d_x = \frac{p'_x}{c_1} - l_3s_{234}$$

$$d_z = p'_z - l_0 + l_3c_{234}$$

The angle q_3 can be calculated by applying the cosine law in the triangle that is formed by joints 2,3 and 4:

$$l_1^2 + l_2^2 - 2l_1l_2\cos(180 - q_3) = d_x^2 + d_z^2$$

and we eventually get:

$$c_3 = \frac{l_1^2 + l_2^2 - d_x^2 - d_z^2}{2l_1l_2} \tag{3.8}$$

This equation yields 2 symmetric solutions for q_3 one for elbow-up and one for elbow-down position. Furthermore, this equation introduces a mechanical constraint. In order for a valid solution to exist, $|c_3| \leq 1$ and after calculations:

$$(l_1 - l_2)^2 \leq d_x^2 + d_z^2 \leq (l_1 + l_2)^2 \tag{3.9}$$

which means that if this constraint is invalidated, there is no solution for the inverse kinematics.
Afterwards, the angles ϕ and β are calculated first in order to calculate q_2. We have:

$$\phi = atan2(d_x, d_z)$$

$$\cos\beta = \frac{d_x^2 + d_z^2 + l_1^2 - l_2^2}{2l_1\sqrt{d_x^2 + d_z^2}}$$

At this point we note that $0 \leq \beta \leq \pi$ due to it being an angle of a triangle, hence there is only one solution for its equation. Now, q_2 can be easily calculated:

$$q_2 = \begin{cases} \phi + \beta & \text{if } q_3 < 0 \\ \phi - \beta & \text{if } q_3 > 0 \end{cases} \tag{3.10}$$

Finally:

$$q_4 = (q_2 + q_3 + q_4) - q_2 - q_3 \tag{3.11}$$

The final question to be addressed is the following. Since in the beginning, two trees of solutions were discovered, when are both trees actually valid solutions? Again, we employ the geometrical properties of the associated vectors. The two trees of solutions, generate two pairs of vectors, $v_i = [-c_1c_{234} \ -s_1s_{234} \ c_{234}]_i^T$, $i = 1, 2$. These two vectors lie on the same plane, on the grounds that they are not identical. That plane is defined by its vertical vector which is: $v_v = v_i \times v_j$, $i \neq j$.

Theorem 2: If either $v_v \times \left[n'_x \ n'_y \ n'_z\right]^T = 0$ or $v_v \times \left[o'_x \ o'_y \ o'_z\right]^T = 0$ then multiple groups of solutions exist.

Proof: If any of the above cross products equals to zero, then v_v is parallel to either $\left[n'_x \ n'_y \ n'_z\right]^T$ or $\left[o'_x \ o'_y \ o'_z\right]^T$ respectively. Consequently, both solutions i, j are orthogonal to their respective vectors of directional cosines. And after taking into consideration the first theorem, these vectors would satisfy (3.6) or (3.7) making both trees valid solutions to the inverse kinematics problem.

In conclusion, the process of solving the inverse kinematics problem is summed up below:

1. Evaluate the joint angle q_1 from (3.2), check if joint limits are violated. Maximum two solutions are expected.

2. Evaluate $s_1 n'_x - c_1 o'_x$ and $s_1 n'_y - c_1 o'_y$ (for as many q_i as possible) and check if both quantities are ≤ 1 in absolute value since they represent s_5 and c_5 respectively. If they are not, then no valid solution exists since the given position/orientation matrix is invalid.

3. Evaluate q_5 from (3.3), check if joint limits are violated. Maximum two solutions are expected.

4. Evaluate $-s_{1,i} a'_y - c_{1,i} a'_x$ and check if it is ≤ 1 in absolute value along with $|a'_x| \leq 1$ as they represent s_{234} and c_{234} respectively. If they are not, then no valid solution exists since the given position/orientation matrix is invalid.

5. Evaluate $q_2 + q_3 + q_4$ from (3.4). Maximum two solutions are expected.

6. Check if the angles found above satisfy both (3.6) and (3.7) and apply the first theorem. If they are not satisfied, then there exist no valid inverse kinematics solutions.

7. Evaluate $d_x = \frac{p'_x}{c_1} - l_3 s_{234}$ and $d_z = p'_z - l_0 + l_3 c_{234}$.

8. Check if the inequality (3.9) holds. If it does not, then there are no solutions to the inverse kinematics problem due to violation of mechanical constraints.

9. Evaluate q_2, q_3, q_4 separately from (3.10),(3.8),(3.11). Check if joint limits are violated. Note that q_3 will yield two symmetric solutions for each tree stemming from the two solutions for q_1.

10. Apply the second theorem to check the number of valid trees of solutions. If its assumptions are validated, then both trees stemming from the two solutions for q_1 represent correct solutions. Else, choose only the valid tree.

Chapter 4

Differential Kinematics - Arm

4.1 Derivation of the Jacobian

In the previous chapter, direct and inverse kinematics equations establishing were derived, thus establishing the relationship between the joint variables and the end-effector position and orientation. In this chapter, differential kinematics is presented which gives the relationship between the joint velocities and the corresponding end-effector linear and angular velocities. This mapping is described by a matrix, termed **geometric Jacobian**, ${}^{0}\mathbf{J}_{Gm}$ which depends on the manipulator configuration:

$$^{0}\left[\mathbf{v}_E \ \omega_E\right]^T (6 \times 1) =^{0} \mathbf{J}_{Gm}(\mathbf{q_m})\dot{\mathbf{q_m}}$$

where $\mathbf{q_m}$ is the (5×1) vector of the KUKA youBot manipulator joint angles and $\mathbf{v}_E$, ω_E are the end-effector linear and angular velocities respectively. The superscript on the left side of matrices and vectors indicates that these matrices and vectors are expressed with respect to the frame with that specific number.
Alternatively, if the end-effector pose is expressed with reference to a minimal representation in the operational space (e.g. Euler angles), it is possible to compute the Jacobian matrix via differentiation of the direct kinematics function with respect to the joint variables. The resulting Jacobian is the **analytical Jacobian** and in general differs from the geometric one. There is a connection between the Jacobian matrices but this will be discussed later. The Jacobian constitutes one of the most important tools for manipulator characterization; in fact, it is useful for finding singularities, analyzing redundancy, determining inverse kinematics algorithms, describing the mapping between forces applied to the end-effector and resulting torques at the joints (statics) and much more.

To begin with, the process of finding the geometric Jacobian is presented below. For $i = 1, ..., 5$ the columns of the geometric Jacobian are given by this relationship:

$$^{0}\mathbf{J}_{Gm,i^{th}column} = \begin{bmatrix} \left({}^{0}\hat{\mathbf{z}}_i\right)^x \, {}^{0}\mathbf{p}^i_E \\ {}^{0}\hat{\mathbf{z}}_i \end{bmatrix}$$

where the notation $(\mathbf{k})^x$ is the skew-symmetric matrix operator, i.e:

$$(\mathbf{k})^x = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$

and:

$${}^0\hat{\mathbf{z}}_i = {}^0\mathbf{R}_i[0\ 0\ 1]^T$$

$${}^0\mathbf{p}_E^i = {}^0\mathbf{p}_E - {}^0\mathbf{p}_i$$

Essentially, ${}^0\hat{\mathbf{z}}_i$ is the third column of the rotation matrix from the frame of reference of joint i to the base of the manipulator and ${}^0\mathbf{p}_E^i$ is the difference between the fourth column of ${}^0\mathbf{T}_E$ and the fourth column of ${}^0\mathbf{T}_i$
For the sake of reference, all the vectors are listed:

$${}^0\hat{\mathbf{z}}_1 = [0\ 0\ 1]^T$$
$${}^0\hat{\mathbf{z}}_2 = {}^0\hat{\mathbf{z}}_3 = {}^0\hat{\mathbf{z}}_4 = [s_1 - c_1\ 0]^T$$
$${}^0\hat{\mathbf{z}}_5 = [-c_1 s_{234}\ \ - s_1 s_{234}\ c_{234}]^T$$

$${}^0\mathbf{p}_E^1 = [c_1(a - l_1 s_2 - l_2 s_{23} - l_3 s_{234})\ s_1(a - l_1 s_2 - l_2 s_{23} - l_3 s_{234})\ l_1 c_2 + l_2 c_{23} + l_3 c_{234}]^T$$

$${}^0\mathbf{p}_E^2 = [-c_1(l_1 s_2 + l_2 s_{23} + l_3 s_{234})\ \ - s_1(l_1 s_2 + l_2 s_{23} + l_3 s_{234})\ l_1 c_2 + l_2 c_{23} + l_3 c_{234}]^T$$

$${}^0\mathbf{p}_E^3 = [-c_1(l_2 s_{23} + l_3 s_{234})\ \ - s_1(l_2 s_{23} + l_3 s_{234})\ l_2 c_{23} + l_3 c_{234}]^T$$

$${}^0\mathbf{p}_E^4 = {}^0\mathbf{p}_E^5 = [-l_3 c_1 s_{234}\ \ - l_3 s_1 s_{234}\ l_3 c_{234}]^T$$

After substitution and calculations, the geometric Jacobian matrix of the manipulator w.r.t frame 0 is presented below:

$${}^0\mathbf{J}_{Gm} = \begin{bmatrix} s_1(-a+l_1s_2+l_2s_{23}+l_3s_{234}) & -c_1(l_1c_2+l_2c_{23}+l_3c_{234}) & -c_1(l_2c_{23}+l_3c_{234}) & -l_3c_1c_{234} & 0 \\ c_1(a-l_1s_2-l_2s_{23}-l_3s_{234}) & -s_1(l_1c_2+l_2c_{23}+l_3c_{234}) & -s_1(l_2c_{23}+l_3c_{234}) & -l_3s_1c_{234} & 0 \\ 0 & -l_1s_2-l_2s_{23}-l_3s_{234} & -l_2s_{23}-l_3s_{234} & -l_3s_{234} & 0 \\ 0 & s_1 & s_1 & s_1 & -c_1s_{234} \\ 0 & -c_1 & -c_1 & -c_1 & -s_1s_{234} \\ 1 & 0 & 0 & 0 & c_{234} \end{bmatrix} \tag{4.1}$$

One last note on the geometric Jacobian. In order to check if the result is correct, we have to take into account a basic property of the Jacobian matrix. Since it describes a differential mapping, any arbitrary small change in joint angles is mapped to a respective arbitrary small change in the position and orientation of the end-effector. I.e. :

$${}^0\delta\mathbf{x}_E ={}^0 \mathbf{J}_{Gm}\delta\mathbf{q}_m$$

Therefore, let two random (non-singular) configurations $\mathbf{q}_m, \mathbf{q}_n$ be selected with their difference being adequately small. These two configurations correspond to two end-effector position and orientation vectors $\mathbf{x}_{Em}, \mathbf{x}_{En}$ that can be found via the forward kinematics map. If the difference between these two vectors actually equals to the product of the Jacobian with the difference between the configuration vectors then the Jacobian matrix is correct.

4.2 Singularity Analysis

As stated before, the Jacobian in the differential kinematics equation of a manipulator defines a linear mapping between the vector of joint velocities and the vector of end-effector velocity. The Jacobian is, in general, a function of the configuration. Those configurations at which the Jacobian matrix is rank-deficient are defined as singularities. To find the singularities of a manipulator is of great interest for the following reasons:

- Singularities represent configurations at which the mobility of the structure is reduced. In other words, it is impossible to impose an arbitrary motion to the end-effector.
- When the manipulator is at a singularity, infinite solutions to the inverse kinematics problem may exist.
- In the neighborhood of a singularity, small velocities in the operational space may cause large velocities (and torques) in the joint space.

There are two classes of singularities:

- Boundary singularities that occur when the manipulator is either stretched or retracted. These singularities do not represent a true drawback, since they can be avoided by not driving the manipulator to the boundaries of its reachable workspace.
- Internal singularities that occur inside the reachable workspace and are generally caused by the alignment of two or more axes of motion, or else by the attainment of particular end-effector configurations. Unlike the above, these singularities constitute a serious problem, as they can be encountered anywhere in the reachable workspace when a path is given in the operational space.

Consequently, the configurations that lead to singularities of the matrix (??) need to be found. It is known that whenever a matrix loses its rank, its determinant becomes zero. However, since the Jacobian is rectangular and its determinant is not defined, a new methodology must be employed. That methodology takes advantage of the fact that when a rectangular matrix loses rank, all square sub-matrices of the same dimension as the lower dimension of the rectangular matrix also become singular. The rank-deficiency locus (i.e. every single possible combination of joint angles that lead to matrix rank deficiency) of the rectangular matrix is the intersection of the singularity loci of the square submatrices resulting from all possible combinations of rows of ${}^{0}\mathbf{J}_{Gm}(\mathbf{q}_m)$, meaning:

$$\mathbf{S} = \bigcap_i \mathbf{S}_{sq_i}, \ \ i = 1...n_{loci}$$

Unfortunately this method proves unwieldy as the number of square submatrices increases combinatorially with the number of degrees of freedom of the manipulator and the number of redundant degrees of freedom. Assuming that the Jacobian matrix $m \times n$ has more rows than columns $m > n$, which corresponds to an overdetermined system of equations, the number of the singularity loci of the square submatrices i is:

$$n_{loci} = \frac{m!}{n!(m-n)!}$$

and in our specific case equals to 6, a relatively small number. For matrices of greater dimensions the so called Singular Vector Algorithm ([PH01]) can be used. The main advantage of the Singular Vector Algorithm is that it can handle symbolically rectangular Jacobians of any row and column dimension. Although this algorithm was not deemed necessary for use in our case, the same results were produced after its implementation.

The methodology was implemented as follows; We construct the square matrices ${}^{0}\mathbf{J}_{Gm,sq_i}(\mathbf{q}_m), \ i = 1...6$ by removing each time one row out of the original ${}^{0}\mathbf{J}_{Gm}(\mathbf{q}_m)$. Afterwards, the rank-deficiency locus of each square submatrix is found by equating each determinant to zero:

$$\mathbf{S}_{sq_i} = \left\{\mathbf{q}_m^* | det\left[{}^{0}\mathbf{J}_{Gm,sq_i}(\mathbf{q}_m^*)\right] = 0\right\}, \ \ i = 1...6$$

More specifically:

- By removing row 1: $det\left[{}^{0}\mathbf{J}_{Gm,sq_1}(\mathbf{q}_m)\right] = -l_1 l_2 s_1 s_3 s_{234} \Rightarrow$ $\mathbf{S}_{sq_1} = \{q_1 = 0 \text{ or } q_3 = 0 \text{ or } q_2 + q_3 + q_4 = 0, \pi\}$
- By removing row 2: $det\left[{}^{0}\mathbf{J}_{Gm,sq_2}(\mathbf{q}_m)\right] = -l_1 l_2 c_1 s_3 s_{234} \Rightarrow$ $\mathbf{S}_{sq_2} = \left\{q_1 = \pm\frac{\pi}{2} \text{ or } q_3 = 0 \text{ or } q_2 + q_3 + q_4 = 0, \pi\right\}$
- By removing row 3: $det\left[{}^{0}\mathbf{J}_{Gm,sq_3}(\mathbf{q}_m\right] = 0$ identically
- By removing row 4: $det\left[{}^{0}\mathbf{J}_{Gm,sq_4}(\mathbf{q}_m)\right] = -l_1 l_2 c_1 c_{234} s_3 (l_1 s_2 + l_2 s_{23} + l_3 s_{234} - a) \Rightarrow$ $\mathbf{S}_{sq_4} = \left\{q_1 = \pm\frac{\pi}{2} \text{ or } q_3 = 0 \text{ or } q_2 + q_3 + q_4 = \pm\frac{\pi}{2} \text{ or } l_1 s_2 + l_2 s_{23} + l_3 s_{234} = a\right\}$
- By removing row 5: $det\left[{}^{0}\mathbf{J}_{Gm,sq_5}(\mathbf{q}_m)\right] = l_1 l_2 s_1 c_{234} s_3 (l_1 s_2 + l_2 s_{23} + l_3 s_{234} - a) \Rightarrow$ $\mathbf{S}_{sq_5} = \left\{q_1 = 0 \text{ or } q_3 = 0 \text{ or } q_2 + q_3 + q_4 = \pm\frac{\pi}{2} \text{ or } l_1 s_2 + l_2 s_{23} + l_3 s_{234} = a\right\}$
- By removing row 6: $det\left[{}^{0}\mathbf{J}_{Gm,sq_6}(\mathbf{q}_m)\right] = -l_1 l_2 s_3 s_{234} (l_1 s_2 + l_2 s_{23} + l_3 s_{234} - a) \Rightarrow$ $\mathbf{S}_{sq_6} = \{q_3 = 0 \text{ or } q_2 + q_3 + q_4 = 0, \pi \text{ or } l_1 s_2 + l_2 s_{23} + l_3 s_{234} = a\}$

After inspection and after removing loci that are subsets of others, the intersection of the above loci is found to be:

$$\mathbf{S}_{sq_i} = \{q_3 = 0;\ q_2 + q_3 + q_4 = 0, \pi \text{ and } l_1 s_2 - l_2 s_4 = a\}$$

Some notes about the above result:

1. All angles that were not compatible with the joint limits were discarded immediately.
2. The identity $det\,[{}^0\mathbf{J}_{Gm,sq_3}(\mathbf{q}_m)] = 0$ obviously means that any vector $\mathbf{q}_m$ validates it.
3. The first condition that is common for every single locus is $q_3 = 0$. For this value of q_3 the rank of the original Jacobian drops from 5 to 4.
4. The second condition is not as simple though. It can be easily identified that every single determinant has one more common term, other than s_3, which is s_{234}. If this term becomes zero, meaning $q_2 + q_3 + q_4 = 0, \pi$, then in order for an intersection to exist for all singularity loci of the sub-matrices, it must be $l_1 s_2 + l_2 s_{23} + l_3 s_{234} = l_1 s_2 + l_2 s_{23} = l_1 s_2 - l_2 s_4 = a$. Hence, the condition $q_2 + q_3 + q_4 = 0, \pi$ nulls the determinants of all sub-matrices but comes along with an extra mandatory condition: $l_1 s_2 - l_2 s_4 = a$. These values of q_2, q_3 and q_4 also reduce the rank of the original Jacobian from 5 to 4.
5. If $q_3 = 0$ and $q_2 + q_3 + q_4 = 0, \pi$ and $l_1 s_2 - l_2 s_4 = a$ then the rank of the original Jacobian becomes 3.

Consequently, the rank-deficiency locus of the original Jacobian is:

$$\mathbf{S} = \Bigg\{ q_3 = 0,\ \text{rank}({}^0\mathbf{J}_{Gm}) = 4;\ q_2 + q_3 + q_4 = 0, \pi \text{ and } l_1 s_2 - l_2 s_4 = a,\ \text{rank}({}^0\mathbf{J}_{Gm}) = 4;$$
$$\text{rank}({}^0\mathbf{J}_{Gm}) = 3, \text{ if all of the above apply.} \Bigg\} \tag{4.2}$$

4.3 The Singular Value Decomposition (SVD)

The above method or even the Singular Vector Algorithm sometimes has a general drawback. Since it is a symbolic method, when applied to floating point computations on computers, it can be unreliable. Often linear dependencies in a matrix are masked by measurement error. Thus, although computationally speaking, the columns of a matrix appear to be linearly independent, with perfect measurement the dependencies would have been detected. Or, put another way, it may be possible to make the columns of the matrix dependent by perturbing the entries by small amounts, on the same order as the measurement errors already present in the elements of the matrix. Therefore, numerical determination of rank requires a criterion for deciding when a value should be treated as zero, a practical choice which depends on both the matrix and the application. That said, in order to verify our results, a rank-revealing decomposition should be used instead. That way the focus is shifted on the singular values. Gaussian elimination is a classic method but its robustness is questionable. Another choice is the QR decomposition with pivoting (so-called rank-revealing QR factorization).

In our case, the alternative that will be examined is the Singular Value Decomposition (SVD). In the past few years, the SVD has been implemented by default in many mathematical packages. The increase in computational power -making its use more efficient- along with the attractive properties of this decomposition, make it an excellent candidate to assist in performing eigenvalue analysis in Robotics. By definition, if ${}^0\mathbf{J}_{Gm}$ is a real $m \times n$ matrix, the geometric Jacobian in this case, then there exist orthogonal matrices

$$\mathbf{U} = [\mathbf{u}_1, ..., \mathbf{u}_m] \in \Re^{m\times m} \text{ and } \mathbf{V} = [\mathbf{v}_1, ..., \mathbf{v}_n] \in \Re^{n\times n}$$

such that:

$$\mathbf{U}^T\, {}^0\mathbf{J}_{Gm}\mathbf{V} = \mathbf{\Sigma} \in \Re^{m\times n} \Leftrightarrow {}^0\mathbf{J}_{Gm} = \mathbf{U\Sigma V}^T \tag{4.3}$$

The matrix $\mathbf{\Sigma}$ has the "almost diagonal" form:

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \sigma_p & 0 \end{bmatrix}$$

where $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_p \geq 0$, $p = min\{m, n\} = n$ (in our case), the singular values of the Jacobian.
At this point, it is convenient to have the following notation for designating singular values:

$$\begin{aligned} \sigma_i({}^0\mathbf{J}_{Gm}) &= \text{ the ith largest singular value of } {}^0\mathbf{J}_{Gm} \\ \sigma_{max(min)}({}^0\mathbf{J}_{Gm}) &= \text{ the largest(smallest) singular value of } {}^0\mathbf{J}_{Gm} \\ \frac{\sigma_{max}}{\sigma_{min}} &= \text{ condition number of the Jacobian} \end{aligned}$$

The SVD reveals a great deal about the structure of ${}^0\mathbf{J}_{Gm}$. If we define r by:

$$\sigma_1 \geq ... \geq \sigma_r > \sigma_{r+1} = ... = \sigma_p = 0$$

then

$$\begin{aligned} rank({}^0\mathbf{J}_{Gm}) &= r \\ null({}^0\mathbf{J}_{Gm}) &= span\{\mathbf{v}_{r+1}, ..., \mathbf{v}_n\} \\ ran({}^0\mathbf{J}_{Gm}) &= span\{\mathbf{u}_1, ..., \mathbf{u}_r\} \\ {}^0\mathbf{J}_{Gm} &= \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T \end{aligned}$$

Another valuable property of the SVD is given by this theorem, also known as the **Eckhart-Young Theorem** (its proof can be found in [GVL83]); let the SVD of a matrix $\mathbf{A}$ be given. If $k < r = rank(\mathbf{A})$ and

$$\mathbf{A}_k = \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$$

then

$$min||\mathbf{A} - \mathbf{B}||_2 = ||\mathbf{A} - \mathbf{A}_k||_2 = \sigma_{k+1}$$

where $\mathbf{B}$ is a random matrix with $rank(\mathbf{B}) = k$.

The above theorem essentially states that when we substitute $\mathbf{A}$ with the geometric Jacobian, its smallest singular value is the 2-norm distance of ${}^0\mathbf{J}_{Gm}$ to the set of all rank-deficient matrices. Furthemore, when a matrix approaches rank deficiency, its smallest singular value similarly approaches zero.

A final note about the singular values: their numerical value is exactly the positive square root of the common (always greater than or equal to zero) eigenvalues of the square matrices $({}^0\mathbf{J}_{Gm})^{T\,0}\mathbf{J}_{Gm}$ and ${}^0\mathbf{J}_{Gm}({}^0\mathbf{J}_{Gm})^T$. This is a property that holds for any matrix based on results of the symmetric eigenvalue problem. Hence, sometimes, it could be easier to calculate these values initially and provide a semi-closed form (i.e. solutions of equations).

The computation of the singular values this way results in:

- Three eigenvalues which are the positive square roots of the roots of the cubic equation:

$$\alpha_0(q_3, q_4) + \alpha_1(q_3, q_4)(\lambda^2) + \alpha_2(q_3, q_4)(\lambda^2)^2 + \alpha(q_3, q_4)(\lambda^2)^3 = 0 \qquad (4.4)$$

  where,

  - $\alpha_0(q_3, q_4) = -2l_1^2 l_2^2 s_3$
  - $\alpha_1(q_3, q_4) = \alpha_{11} + \alpha_{12}c_3 + \alpha_{13}\cos(2q_3) + \alpha_{14}\left[c_4 - \cos(2q_3 + q_4)\right] + \alpha_{15}\cos(2q_4) + \alpha_{16}\cos(2q_3 + 2q_4)$
  - $\alpha_2(q_3, q_4) = -6 - 2l_1^2 - 4l_2^2 - 6l_3^2 - 4l_1l_2c_3 - 8l_2l_3c_4 - 4l_1l_3c_{34}$
  - $\alpha_3(q_3, q_4) = 2$

α_{11}	$4(l_1^2 + l_2^2) + l_1^2(l_2^2 + 2l_3^2) + 2l_2^2l_3^2$
α_{12}	$2l_1l_2(2 + l_3^2)$
α_{13}	$-l_1^2l_2^2$
α_{14}	$2l_1^2l_2l_3$
α_{15}	$-2l_2^2l_3^2$
α_{16}	$2l_1^2l_3^2$

- Two eigenvalues that are the positive square roots of the values obtained by the relationship:

$$4\lambda_{4,5}^2 = \beta_0 \pm \sqrt{\beta_0^2 - 8\beta_0 + 24 + 8\cos\left[2(q_2+q_3+q_4)\right]} \tag{4.5}$$

  where $\beta_0 = 4+l_1^2+l_2^2+l_3^2+2a^2-l_1^2\cos(2q_2)+2l_1l_2c_3-l_2^2\cos\left[2(q_2+q_3)\right]-2l_1l_2\cos(2q_2+q_3)+2l_2l_3c_4+2l_1l_3c_{34}-l_3^2\cos\left[2(q_2+q_3+q_4)\right]-2l_1l_3\cos\left[2q_2+q_3+q_4\right]-2l_2l_3\cos(2q_2+2q_3+q_4]-4l_1as_2-4l_2as_{23}-4l_3as_{234}$

Some observations regarding the eigenvalues follow:

1. The discriminant of the cubic equation of (3.4) is

$$\begin{aligned}\Delta(q_3,q_4) = {} & 36\alpha_0(q_3,q_4)\alpha_1(q_3,q_4)\alpha_2(q_3,q_4) - 4\alpha_2^3(q_3,q_4)\alpha_0(q_3,q_4) + \alpha_1^2(q_3,q_4)\alpha_2^2(q_3,q_4) \\ & -8\alpha_1^3(q_3,q_4) - 108\alpha_0^2(q_3,q_4)\end{aligned} \tag{4.6}$$

   As we can see from the plot below, the discriminant is always greater than zero, meaning that the cubic equation always gives three real distinct eigenvalues. A closed form of these eigenvalues as a function of joint variables can be obtained but that form is quite complicated.

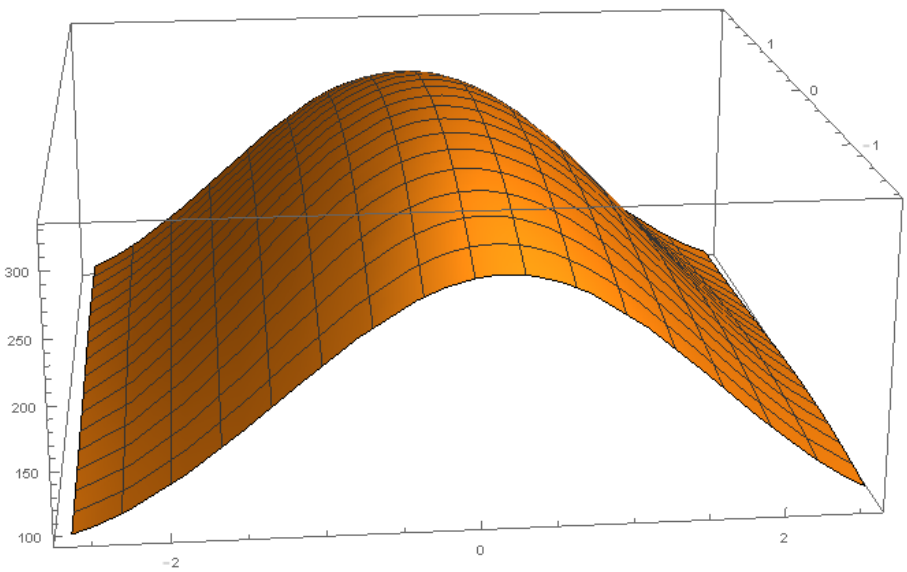


Figure 4.1: The discriminant of the cubic eigenvalue equation.

2. Assume that $q_3 = 0$. We remind that for this value of q_3 the rank of the Jacobian matrix drops from 5 to 4. According to the theorem derived from the SVD, one singular value will become zero. Indeed, if $q_3 = 0$, then $\alpha_0(q_3,q_4) = 0$ and thus one solution for equation (4.4) is zero. Furthermore, due to (4.4) being a function of only q_3, q_4 any changes in the other joint angles do not affect the corresponding eigenvalues. This leads to the next point.

3. Assume now that $q_2 + q_3 + q_4 = 0, \pi$ and $l_1 s_2 - l_2 s_4 = a$. For this combination of values the rank of the Jacobian also drops from 5 to 4. Since this rank deficiency depends on q_2, the eigenvalue(s) that becomes zero MUST be one from the equation that actually depends on q_2. That said, the singular value that becomes zero is the one from (4.5) with the minus sign.

4. Last but not least, if both conditions apply, two separate eigenvalues become zero. At this point, it should be stressed that the rest of the eigenvalues are always positive (for positive lengths obviously). Therefore, the minimum rank of the KUKA youBot Jacobian is 3.

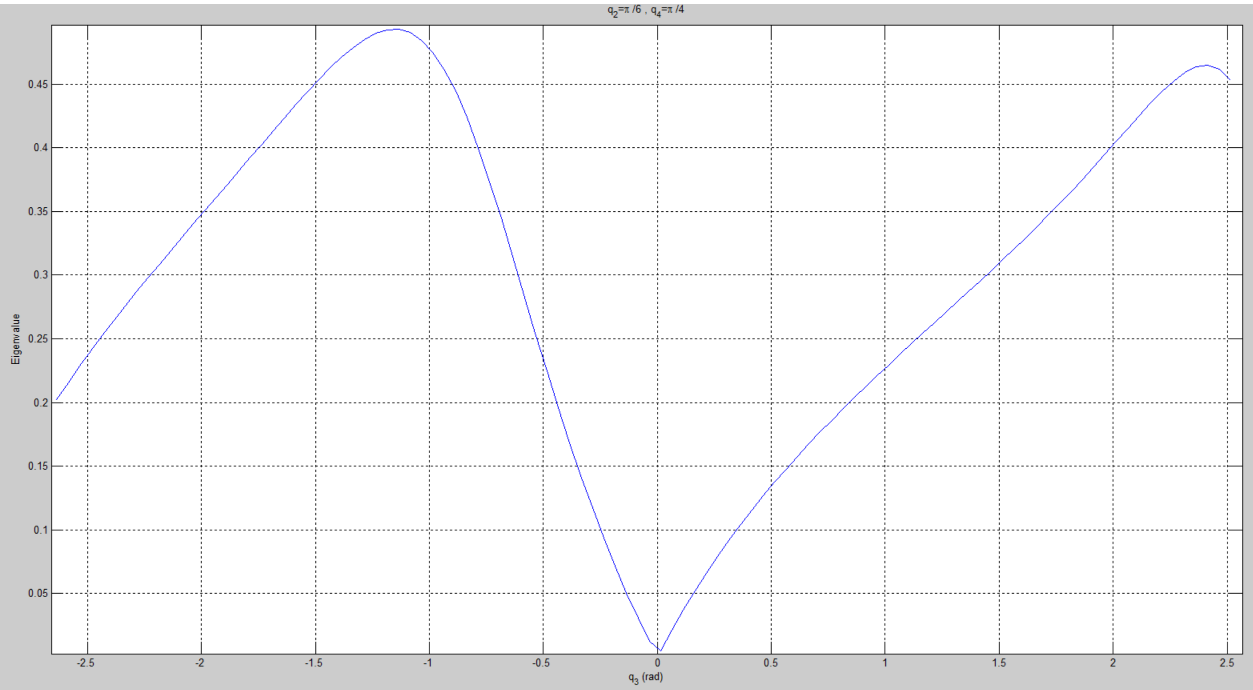


Figure 4.2: The change of the eigenvalue that determines the rank as a function of q_3 when q_2, q_4 are constant.

Chapter 5

Inverse Differential Kinematics - Arm

In this chapter, we derive the inverse differential kinematics model for the manipulator. The problem statement is: find a vector $\dot{\mathbf{q}}_m \in \Re^n$ such that ${}^0\mathbf{J}_{Gm}\dot{\mathbf{q}}_m = \dot{\mathbf{x}}_E$ where the Jacobian matrix ${}^0\mathbf{J}_{Gm} \in \Re^{m\times n}$ and the Cartesian velocity of the end-effector $\dot{\mathbf{x}}_E \in \Re^m$ are given. However, in our case, since $m > n$, the system is **overdetermined**. This means that the system has no exact solution since the end-effector velocities must be an element of $ran({}^0\mathbf{J}_{Gm})$, a proper subspace of $\Re^m$. Hence, the solution of the problem comes with the least squares minimization of $||{}^0\mathbf{J}_m\dot{\mathbf{q}}_m - \dot{\mathbf{x}}_E||_p$ for a suitable choice of p.

The above applies only when ${}^0\mathbf{J}_{Gm}$ is full rank. When ${}^0\mathbf{J}_{Gm}$ is rank deficient, there are an infinite number of solutions to the LS problem. That said, all cases must be examined accordingly. However, a property of the SVD is that it supplies the pseudoinverse of a matrix whether it is rank deficient or not.

Assuming that the SVD of ${}^0\mathbf{J}_{Gm}$ is ${}^0\mathbf{J}_{Gm} = \mathbf{U\Sigma V}^T$, then its pseudoinverse, ${}^0\mathbf{J}^\dagger_{Gm}$ is:

$$ {}^0\mathbf{J}^\dagger_{Gm} = \mathbf{V\Sigma}^\dagger\mathbf{U}^T \tag{5.1} $$

where

$$ \mathbf{\Sigma}^\dagger = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\sigma_3} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\sigma_p} & 0 \end{bmatrix} \in \Re^{n\times m} $$

If a singular value is exactly zero, meaning that $\text{rank}({}^0\mathbf{J}_{Gm}) = r < n$ then the corresponding σ_i is replaced by a zero in $\mathbf{\Sigma}^\dagger$. Since the values for which the singular values become zero are known, as explained in the previous chapter, predictions regarding when to replace them are possible.

Problems emerge when the singular values are very close to zero, though. Suppose that the SVD of ${}^0\mathbf{J}_{Gm}$ is given by (4.3). Then the minimizer with the smallest 2-norm of the least squares problem is given by the relationship:

$$\dot{\mathbf{q}}_{m,LS} = \sum_{i=1}^{r} \frac{\mathbf{u}_i^T \dot{\mathbf{x}}_E}{\sigma_i} \mathbf{v}_i =^0 \mathbf{J}_{Gm}^{\dagger} \dot{\mathbf{x}}_E \tag{5.2}$$

and the residual is given by the relationship:

$$\rho_{LS}^2 = ||^0\mathbf{J}_{Gm}\dot{\mathbf{q}}_{m,LS} - \dot{\mathbf{x}}_E||_2^2 = \sum_{i=r+1}^{m} (\mathbf{u}_i^T \dot{\mathbf{x}}_E)^2 \tag{5.3}$$

This residual is significant when the singular values are close to zero. Therefore, it can be used as another metric to determine the degree of rank deficiency of the Jacobian. The advantage of using the SVD instead of other LS methods is that the residual is quite smaller. In fact, in most other methods the order of magnitude of the residual is that of $||\dot{\mathbf{x}}_E||$. What is gained in accuracy during rank deficiency, is offset by the increase in computational power; the number of flops SVD near rank deficiency manifolds is approximately $4mn^2 + 8n^3$ whereas for other LS methods is half to a quarter of that number.

An extremely helpful quantity for the inversion is the condition number of the matrix, defined in previous chapter. A large condition number corresponds to a matrix whose inverse is very sensitive to relatively small perturbations in the matrix. Such a matrix is termed ill conditioned or poorly conditioned with respect to inversion. The importance of the condition number lies elsewhere though; any fractional change in the inverse can be ”condition number” times as large as the fractional change in the original with it counting as a bound.

Chapter 6

The Mobile Platform

As of now, a full kinematic analysis for the arm was performed. In order to proceed with the combined system analysis, a separate analysis for the mobile platform should be completed before.

To begin with, the coordinate system for the platform w.r.t the global-inertial frame is presented below.

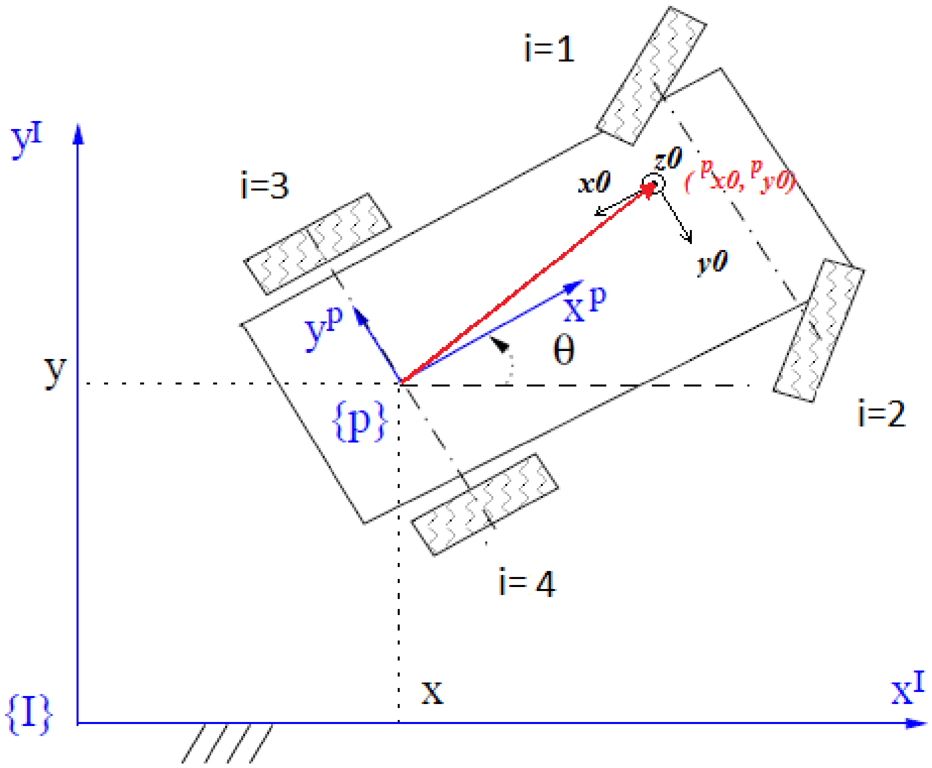


Figure 6.1: The KUKA youBot platform and the associated coordinate frames.

The notations 'p' and 'I' refer to the platform and global-inertial frame respectively. In addition, we assume that the origin of the platform frame coincides with the center of the platform to make the analysis easier (although it could be placed so as to coincide with any point on the platform, as shown in the figure above). We can easily determine that the rotation matrix from the manipulator base frame to the frame of the platform is:

$$ {}^{P}\mathbf{R}_0 = \mathbf{R}_{z_0 \equiv z_P}|_{\theta_z = 180^o} = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\theta_z = 180^o} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Furthermore, the rotation matrix from the coordinate system of the platform to the global-inertial coordinate system is:

$$ {}^{I}\mathbf{R}_P = \begin{bmatrix} \cos\theta_P & \sin\theta_P & 0 \\ -\sin\theta_P & \cos\theta_P & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Hence the rotation matrix from the manipulator base frame to the global-inertial frame is eventually:

$$ {}^{I}\mathbf{R}_0 = {}^{I}\mathbf{R}_P {}^{P}\mathbf{R}_0 = \begin{bmatrix} -\cos\theta_P & -\sin\theta_P & 0 \\ \sin\theta_P & -\cos\theta_P & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.1} $$

Moreover, in order to fully define the position of the manipulator base frame w.r.t. the global-inertial frame, we assume that:

- it is constant along the global-inertial z-axis, represented by the value ${}^{I}z_0$.
- it is given by a pair $({}^{P}x_0, {}^{P}y_0)$ w.r.t. the frame of the mobile platform

Having clarified the above, which will prove useful later on, we now analyze the kinematic structure of the platform. In general, the differential model of the platform is:

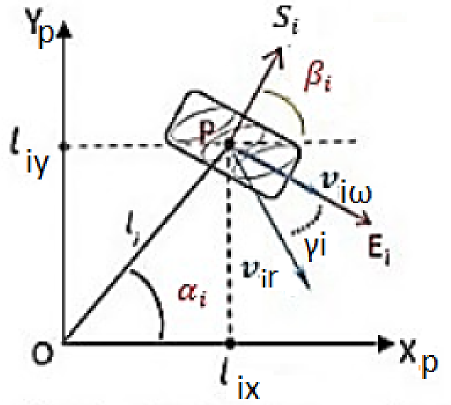
$$ \dot{\mathbf{q}}_P = \mathbf{J}_P \mathbf{u}_P \tag{6.2} $$

where $\mathbf{q}_P = [x_P \ y_P \ \theta_P]^T$ is the vector of the degrees of freedom of the platform, $\mathbf{u}_P$ is the vector of the control variables, i.e. the inputs and $\mathbf{J}_P$ a Jacobian-esque matrix for the platform that associates the two vectors which is in general a function of the DOFs of the platform.

6.1 The Omnidirectional Case

The youBot platform is equipped with Mecanum wheels. These conventional wheels have a series of rollers attached to their circumference. By alternating wheels with left and right-handed rollers, in such a way that each wheel applies force roughly at right angles to the wheelbase diagonal the wheel is on, the vehicle is stable and can be made to move in any direction and turn by varying the speed and direction of rotation of each wheel. Moving all four wheels in the same direction causes forward or backward movement, running the wheels on one side in the opposite direction to those on the other side causes rotation of the vehicle, and running the wheels on

(α′) A typical Mecanum wheel.



(β′) The coordinate system of the Mecanum wheel.

Figure 6.2: Mecanum Wheel

one diagonal in the opposite direction to those on the other diagonal causes sideways movement. Combinations of these wheel motions allow for vehicle motion in any direction with any vehicle rotation.

The modeling of an omnidirectional platform is based on the simple premise that the linear and angular velocity of the platform are tied to the rotational velocities of the Mecanum wheels. Naturally, there are geometric methods that can be applied so as to derive these relationships. These methods are based on evaluating the matrix $\mathbf{J}_P$ given a number of geometric quantities. The above figure, represents the coordinate system of the wheel $S_iP_iE_i$ w.r.t. the coordinate frame of the platform. Thus, the aforementioned quantities are defined as shown in the figure. Moreover, $v_{i\omega}$ is defined as the velocity vector corresponding to wheel i revolutions and v_{ir} as the velocity of the passive roller in the wheel i. Typical parameters for these angles, assuming that the wheels are completely identical with radius r, are presented in the table below:

i	α_i	β_i	γ_i	l_i	l_{ix}	l_{iy}
1	$\pi/4$	$\pi/2$	$-\pi/4$	l	l_x	l_y
2	$-\pi/4$	$-\pi/2$	$\pi/4$	l	l_x	l_y
3	$3\pi/4$	$\pi/2$	$\pi/4$	l	l_x	l_y
4	$-3\pi/4$	$-\pi/2$	$-\pi/4$	l	l_x	l_y

Figure 6.3: Typical parameters for identical wheels.

By employing simple velocity vector and geometric relationships or by substituting the above values in kinematic models (e.g. [TBG15]), we derive the kinematic equation:

$$\dot{\mathbf{q}}_P = \mathbf{J}_{P,o}\mathbf{u}_{P,o} = \frac{r}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -\frac{1}{l_x+l_y} & \frac{1}{l_x+l_y} & -\frac{1}{l_x+l_y} & \frac{1}{l_x+l_y} \end{bmatrix} [\omega_1 \ \omega_2 \ \omega_3 \ \omega_4 \ ]^T \tag{6.3}$$

6.2 The Non-holonomic Case

In this section, it is assumed that the platform is non-holonomic. By definition, in Robotics, a nonholonomic constraint is a constraint on velocity; there exist directions that the robot cannot achieve (e.g. rotate on the spot), but there exists no point in the reachable "workspace" that the robot cannot go to. In mathematical terms, the velocity constraint that is imposed on the platform is not integrable, i.e. the derivation of another constraint as a function of the position of the platform is impossible.

In practice, the omnidirectional platform facilitates the implementation of control laws. However, the non-holonomic case presents a particularly interesting case study and the purpose is to examine the implications of the non-holonomity on the kinematic structure of the mobile manipulator as a whole. The omnidirectional platform is converted to a non-holonomic by locking its back wheels. Subsequently, the motion of the origin of the mobile platform frame is considered now to be subject to the non-holonomic constraint of not being able to slide sideways along the y_p axis, or rotate in place.

To be precise, this non-holonomic constraint -by adopting the unicycle model- is given by the relationship:

$$\dot{x}_P \sin\theta_P - \dot{y}_P \cos\theta_P = 0$$

or in differential form:

$$dx_P \sin\theta_P - dy_P \cos\theta_P = 0$$

But the form that will prove more useful is the Pfaffian form:

$$\mathbf{A}(\mathbf{q}_P)\dot{\mathbf{q}}_P = 0 \tag{6.4}$$

where $\mathbf{q}_P = [x_P \ y_P \ \theta_P]^T$ the vector of the degrees of freedom of the platform and $\mathbf{A} = [\sin\theta_P \ -\cos\theta_P \ 0]$. The feasible trajectories of the platform lie in the tangent space of the co-distribution A defined by the above differential form. Assume that n is the number of d.o.f. of the platform and k is the number of the imposed non-holonomic constraints. Then, the above means that there exists a distribution G such that:

$$G = span\{\mathbf{g}_1,...,\mathbf{g}_{n-k}\} = A^{\perp}$$

We say that the distribution G annihilates the co-distribution A and thus, a control system is defined by the relationship:

$$\dot{\mathbf{q}}_P = \sum_{i=1}^{n-k} \mathbf{g}_i(\mathbf{q}_P)u_i \tag{6.5}$$

In our case $n = 3$, $k = 1$, hence we need to define two control inputs $u_1 = v_p$ the forward velocity of the platform and $u_2 = \omega_p$ the steering velocity of the platform. We can easily find the annihilator of the co-distribution A to be $G = [\cos\theta_P \ \ \sin\theta_P \ 1]$ and also define $g_1 = [\cos\theta_P \ \ \sin\theta_P \ 0]^T$ and $g_2 = [0 \ 0 \ 1]$ so as to be linearly independent. If we substitute in (6.5) we obtain eventually:

$$\dot{\mathbf{q}}_P = \begin{bmatrix} \cos\theta_P \\ \sin\theta_P \\ 0 \end{bmatrix} v_P + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega_P = \begin{bmatrix} \cos\theta_P & 0 \\ \sin\theta_P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_P \\ \omega_P \end{bmatrix} = \mathbf{J}_{P,nh}\mathbf{u}_{P,nh} \tag{6.6}$$

and this concludes the non-holonomic case.

Chapter 7

Combined System Kinematics

In the previous chapters the kinematic models were derived and analyzed for the manipulator and the mobile platform separately. In this chapter, the arm is considered mounted on the platform and the kinematic model (forward, differential and the inverse kinematics) is derived.

7.1 Forward Kinematics

To begin with, the homogeneous transformation from the end-effector to the global-inertial frame is given by the following relationship:

$$^{I}\mathbf{T}_E = \begin{bmatrix} ^{I}\mathbf{R}_0{}^{0}\mathbf{R}_E & \mathbf{r} + {}^{I}\mathbf{R}_0{}^{0}\mathbf{r}_E \\ 0\ 0\ 0 & 1 \end{bmatrix} \tag{7.1}$$

where:

- the matrix $^{I}\mathbf{R}_0$ was defined in the previous chapter.
- the vector $\mathbf{r} = \begin{bmatrix} x_P + {}^{P}x_0 & y_P + {}^{P}y_0 & {}^{I}z_0 \end{bmatrix}$ is comprised by values which were also defined in the previous chapter.
- the vector $^{0}\mathbf{r}_E$ is the first three elements of the fourth column of (2.2)

To be precise:

$$^{I}\mathbf{R}_E = {}^{I}\mathbf{R}_0{}^{0}\mathbf{R}_E = \begin{bmatrix} c_{234}c_5\cos(q_1-\theta_P)-s_5\sin(q_1-\theta_P) & -c_{234}\cos(q_1-\theta_P)s_5-c_5\sin(q_1-\theta_P) & \cos(q_1-\theta_P)s_{234} \\ \cos(q_1-\theta_P)s_5+c_{234}c_5\sin(q_1-\theta_P) & -c_{234}\sin(q_1-\theta_P)s_5+c_5\cos(q_1-\theta_P) & s_{234}\sin(q_1-\theta_P) \\ -s_{234}c_5 & s_{234}s_5 & c_{234} \end{bmatrix}$$

and:

$$\mathbf{r} + {}^{I}\mathbf{R}_0{}^{0}\mathbf{r}_E = \begin{bmatrix} x_P + {}^{P}x_0 - \cos(q_1-\theta_P)(a - l_1s_2 - l_2s_{23} - l_3s_{234}) \\ y_P + {}^{P}y_0 - \sin(q_1-\theta_P)(a - l_1s_2 - l_2s_{23} - l_3s_{234}) \\ {}^{I}z_0 + l_0 + l_1c_2 + l_2c_{23} + l_3c_{234} \end{bmatrix}$$

7.2 Differential Kinematics

Now, we have to derive the composite Jacobian $\mathbf{J}^*$. However, the geometric method that was applied for the manipulator in previous chapter is not applicable now. Thus, the time differentiation method, i.e. differentiating the relationship (7.1) w.r.t. time in order to derive relationships between end-effector and joint velocities/control variables, will be implemented in this case.

As usual, the Jacobian has two parts, one is associated with the linear velocities of the end-effector and the other with its angular velocities. We begin with the linear part, $\mathbf{J}_L^*$. If we differentiate (w.r.t. time) the first three elements of the fourth column of (7.1) we have:

$$\frac{d}{dt}\left(\mathbf{r} + {}^I\mathbf{R}_0{}^0\mathbf{r}_E\right) = \frac{d\mathbf{r}}{dt} + \frac{d({}^I\mathbf{R}_0)}{dt}{}^0\mathbf{r}_E + {}^I\mathbf{R}_0\frac{d({}^0\mathbf{r}_E)}{dt} \tag{7.2}$$

For the first term we have:

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left[x_P + {}^Px_0\ y_P + {}^Py_0\ {}^Iz_0\right]^T = [\dot{x}_P\ \dot{y}_P\ 0]^T$$

In the second term, the time derivative of a rotation matrix appears. As we know, for any rotation matrix $\mathbf{R}$, its time derivative is $\dot{\mathbf{R}} = (\boldsymbol{\omega})^x\mathbf{R}$ where $(\boldsymbol{\omega})^x$ is the skew symmetric matrix operator, as defined in Chapter 4, applied on the angular velocity of the rotating body. In our specific case $\boldsymbol{\omega}_P = \begin{bmatrix}0 & 0 & \dot{\theta}_P\end{bmatrix}^T$. Hence:

$$\frac{d({}^I\mathbf{R}_0)}{dt}{}^0\mathbf{r}_E = (\boldsymbol{\omega}_P)^x\ {}^I\mathbf{R}_0{}^0\mathbf{r}_E = \dot{\theta}_P\begin{bmatrix}-\sin\theta_P & \cos\theta_P & 0\\ -\cos\theta_P & \sin\theta_P & 0\\ 0 & 0 & 0\end{bmatrix}{}^0\mathbf{r}_E$$

Lastly, for the third term we have:

$${}^I\mathbf{R}_0\frac{d({}^0\mathbf{r}_E)}{dt} = {}^I\mathbf{R}_0\frac{\partial\ {}^0\mathbf{r}_E}{\partial\mathbf{q}_m}\frac{\partial\mathbf{q}_m}{\partial t} = {}^I\mathbf{R}_0\ {}^0\mathbf{J}_{Gm,L}\dot{\mathbf{q}}_m$$

where ${}^0\mathbf{J}_{Gm,L}$ is the part of the manipulator geometric Jacobian that corresponds to the linear velocities, i.e. the first three rows. Having said the above, (7.2) becomes:

$$\frac{d}{dt}\left(\mathbf{r} + {}^I\mathbf{R}_0{}^0\mathbf{r}_E\right) = [\dot{x}_P\ \dot{y}_P\ 0]^T + \dot{\theta}_P\begin{bmatrix}-\sin\theta_P & \cos\theta_P & 0\\ -\cos\theta_P & \sin\theta_P & 0\\ 0 & 0 & 0\end{bmatrix}{}^0\mathbf{r}_E + {}^I\mathbf{R}_0\ {}^0\mathbf{J}_{Gm,L}\dot{\mathbf{q}}_m \tag{7.3}$$

where:

$$\dot{\theta}_P\begin{bmatrix}-\sin\theta_P & \cos\theta_P & 0\\ -\cos\theta_P & \sin\theta_P & 0\\ 0 & 0 & 0\end{bmatrix}{}^0\mathbf{r}_E = \begin{bmatrix}\sin(q_1-\theta_P)(a-l_1s_2-l_2s_{23}-l_3s_{234})\\ -\cos(q_1+\theta_P)(a-l_1s_2-l_2s_{23}-l_3s_{234})\\ 0\end{bmatrix}\dot{\theta}_P$$

and:

$${}^I\mathbf{R}_0\ {}^0\mathbf{J}_{m,L} = \begin{bmatrix}\sin(q_1-\theta_P)(a-l_1s_2-l_2s_{23}-l_3s_{234}) & \cos(q_1-\theta_P)(l_1c_2+l_2c_{23}+l_3c_{234}) & \cos(q_1-\theta_P)(l_2c_{23}+l_3c_{234}) & l_3c_{234}\cos(q_1-\theta_P) & 0\\ -\cos(q_1-\theta_p)(a-l_1s_2-l_2s_{23}-l_3s_{234}) & \sin(q_1-\theta_p)(l_1c_2+l_2c_{23}+l_3c_{234}) & \sin(q_1-\theta_p)(l_2c_{23}+l_3c_{234}) & l_3c_{234}\sin(q_1-\theta_p) & 0\\ 0 & -l_1s_2-l_2s_{23}-l_3s_{234} & -l_2s_{23}-l_3s_{234} & -l_3s_{234} & 0\end{bmatrix}$$

The angular part $\mathbf{J}_A^*$ is a bit more difficult to calculate. For this particular purpose we employ a specific representation of the angular velocities of the end-effector. To be precise, we cannot construct this part of the Jacobian geometrically and the use of a representation other than the vector $[\omega_x\ \omega_y\ \omega_z]$ will enable us to obtain this part by differentiation w.r.t. time. As a first approach we use the Roll-Pitch-Yaw (RPY) Euler angles. If we begin from the complete rotation matrix ${}^I\mathbf{R}_E$ and its elements r_{ij} we have:

$$\begin{aligned} \alpha_E &= Atan2(r_{21}, r_{11}) = f_\alpha(\mathbf{q}) \\ \beta_E &= Atan2(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2}) = f_\beta(\mathbf{q}) \\ \gamma_E &= Atan2(r_{32}, r_{33}) = f_\gamma(\mathbf{q}) \end{aligned} \tag{7.4}$$

Note that the above solution for β_E is in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and all solutions degenerate when $\cos\beta_E = 0$. Then, only the sum or difference of α_E, γ_E can be calculated. Regardless, we assume for the time being that this is not the case and $\beta_E \neq \pm 90^o$. The angular part of the Jacobian is extracted then by the below relationships:

$$\begin{aligned} \dot{\alpha}_E &= \frac{\partial f_\alpha}{\partial \mathbf{q}}\dot{\mathbf{q}} = \frac{\partial f_\alpha}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial f_\alpha}{\partial q_n}\dot{q}_n \\ \dot{\beta}_E &= \frac{\partial f_\beta}{\partial \mathbf{q}}\dot{\mathbf{q}} = \frac{\partial f_\beta}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial f_\beta}{\partial q_n}\dot{q}_n \\ \dot{\gamma}_E &= \frac{\partial f_\gamma}{\partial \mathbf{q}}\dot{\mathbf{q}} = \frac{\partial f_\gamma}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial f_\gamma}{\partial q_n}\dot{q}_n \end{aligned}$$

where the partial derivatives constitute the elements of $\mathbf{J}_A^*$. That said, we have:

$$\dot{\alpha}_E = \dot{q}_1 + \frac{s_{234}s_5c_5}{1 - s_{234}^2c_5^2}\dot{q}_2 + \frac{s_{234}s_5c_5}{1 - s_{234}^2c_5^2}\dot{q}_3 + \frac{s_{234}s_5c_5}{1 - s_{234}^2c_5^2}\dot{q}_4 + \frac{c_{234}}{1 - s_{234}^2c_5^2}\dot{q}_5 - \dot{\theta}_P \tag{7.5}$$

$$\dot{\beta}_E = -\frac{c_{234}c_5}{\sqrt{1 - s_{234}^2c_5^2}}\dot{q}_2 - \frac{c_{234}c_5}{\sqrt{1 - s_{234}^2c_5^2}}\dot{q}_3 - \frac{c_{234}c_5}{\sqrt{1 - s_{234}^2c_5^2}}\dot{q}_4 + \frac{s_{234}s_5}{\sqrt{1 - s_{234}^2c_5^2}}\dot{q}_5 \tag{7.6}$$

$$\dot{\gamma}_E = \frac{s_5}{1 - s_{234}^2c_5^2}\dot{q}_2 + \frac{s_5}{1 - s_{234}^2c_5^2}\dot{q}_3 + \frac{s_5}{1 - s_{234}^2c_5^2}\dot{q}_4 + \frac{0.5c_5 \sin\left[2(q_2 + q_3 + q_4)\right]}{1 - s_{234}^2c_5^2}\dot{q}_5 \tag{7.7}$$

At this point, we have to examine two cases, the Omnidirectional case and the Non-holonomic case.

1. **Omnidirectional case**: In this case, the equation (6.3) in in effect. After substitutions are made so as to replace the DOFs of the platform with the control variables $\mathbf{u}_{P,o}$, the parts of the Jacobian become:

$$\mathbf{J}^*_{L,0} = \begin{bmatrix} {}^I\mathbf{R}_0 \ {}^0\mathbf{J}_{Gm,L}|_{row1} & \kappa_1 & \kappa_2 & \kappa_1 & \kappa_2 \\ {}^I\mathbf{R}_0 \ {}^0\mathbf{J}_{Gm,L}|_{row2} & -\kappa_3 & \kappa_3 & \kappa_4 & -\kappa_4 \\ {}^I\mathbf{R}_0 \ {}^0\mathbf{J}_{Gm,L}|_{row3} & 0 & 0 & 0 & 0 \end{bmatrix}$$

where:

$$\kappa_1 = \frac{r}{4} - \frac{r(a - l_1 s_2 - l_2 s_{23} - l_3 s_{234})}{4(l_x + l_y)} \sin(q_1 - \theta_p)$$

$$\kappa_2 = \frac{r}{4} + \frac{r(a - l_1 s_2 - l_2 s_{23} - l_3 s_{234})}{4(l_x + l_y)} \sin(q_1 - \theta_p)$$

$$\kappa_3 = \frac{r}{4} + \frac{r(a - l_1 s_2 - l_2 s_{23} - l_3 s_{234})}{4(l_x + l_y)} \cos(q_1 + \theta_p)$$

$$\kappa_4 = \frac{r}{4} - \frac{r(a - l_1 s_2 - l_2 s_{23} - l_3 s_{234})}{4(l_x + l_y)} \cos(q_1 + \theta_p)$$

$$\mathbf{J}^*_{A,o} = \begin{bmatrix} 1 & +\frac{s_{234}s_5c_5}{1-s_{234}^2c_5^2} & +\frac{s_{234}s_5c_5}{1-s_{234}^2c_5^2} & +\frac{s_{234}s_5c_5}{1-s_{234}^2c_5^2} & +\frac{c_{234}}{1-s_{234}^2c_5^2} & 0 & \kappa_0 & -\kappa_0 & \kappa_0 & -\kappa_0 \\ 0 & -\frac{c_{234}c_5}{\sqrt{1-s_{234}^2c_5^2}} & -\frac{c_{234}c_5}{\sqrt{1-s_{234}^2c_5^2}} & -\frac{c_{234}c_5}{\sqrt{1-s_{234}^2c_5^2}} & \frac{s_{234}s_5}{\sqrt{1-s_{234}^2c_5^2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{s_5}{1-s_{234}^2c_5^2} & \frac{s_5}{1-s_{234}^2c_5^2} & \frac{s_5}{1-s_{234}^2c_5^2} & \frac{0.5c_5\sin[2(q_2+q_3+q_4)]}{1-s_{234}^2c_5^2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\kappa_0 = \frac{r}{4(l_x+l_y)}$.

$$\mathbf{J}^*_o = \begin{bmatrix} \mathbf{J}^*_{L,o} \\ \mathbf{J}^*_{A,o} \end{bmatrix} \tag{7.8}$$

2. **Non-holonomic case**: Assuming that the mobile platform is now non-holonomic, i.e. the relationship (6.6) is in effect and the control variables are $\mathbf{u}_{P,nh}$, then, in the same vein as before the parts of the Jacobian become:

$$\mathbf{J}^*_{L,nh} = \begin{bmatrix} {}^I\mathbf{R}_0 \ {}^0\mathbf{J}_{Gm,L}|_{row1} & \cos\theta_P & \sin(q_1 - \theta_P)(a - l_1 s_2 - l_2 s_{23} - l_3 s_{234}) \\ {}^I\mathbf{R}_0 \ {}^0\mathbf{J}_{Gm,L}|_{row2} & \sin\theta_P & -\cos(q_1 + \theta_P)(a - l_1 s_2 - l_2 s_{23} - l_3 s_{234}) \\ {}^I\mathbf{R}_0 \ {}^0\mathbf{J}_{Gm,L}|_{row3} & 0 & 0 \end{bmatrix}$$

and:

$$\mathbf{J}^*_{A,nh} = \begin{bmatrix} 1 & +\frac{s_{234}s_5c_5}{1-s_{234}^2c_5^2} & +\frac{s_{234}s_5c_5}{1-s_{234}^2c_5^2} & +\frac{s_{234}s_5c_5}{1-s_{234}^2c_5^2} & +\frac{c_{234}}{1-s_{234}^2c_5^2} & 0 & -1 \\ 0 & -\frac{c_{234}c_5}{\sqrt{1-s_{234}^2c_5^2}} & -\frac{c_{234}c_5}{\sqrt{1-s_{234}^2c_5^2}} & -\frac{c_{234}c_5}{\sqrt{1-s_{234}^2c_5^2}} & \frac{s_{234}s_5}{\sqrt{1-s_{234}^2c_5^2}} & 0 & 0 \\ 0 & \frac{s_5}{1-s_{234}^2c_5^2} & \frac{s_5}{1-s_{234}^2c_5^2} & \frac{s_5}{1-s_{234}^2c_5^2} & \frac{0.5c_5\sin[2(q_2+q_3+q_4)]}{1-s_{234}^2c_5^2} & 0 & 0 \end{bmatrix}$$

$$\mathbf{J}^*_{nh} = \begin{bmatrix} \mathbf{J}^*_{L,nh} \\ \mathbf{J}^*_{A,nh} \end{bmatrix} \tag{7.9}$$

In these cases, the rank deficiency locus for both Jacobians can be found by utilizing the methods discussed in Chapter 4. What is worth noting at this point is that (obviously) the new maximum rank of these Jacobians is 6 and the old rank deficiency locus is a subset of the new one. Despite the fact that the values that reduce the rank of the manipulator Jacobian are intact, there are two major additions; a value because of the Euler angle representation that reduces the rank and a value $q_1(\theta_P)$ that reduces the rank due to the addition of the platform.

7.3 Inverse Kinematics

As far as the inverse kinematics is concerned, the algorithm follows a few extra steps compared to the one of the manipulator only. Assuming that the elements of the homogeneous transformation

$$
{}^{I}\mathbf{T}_E = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

are known now, we begin from the relationship:

$$
{}^{I}\mathbf{R}_0^T\left[i_j\right] = \begin{bmatrix} -n_x\cos\theta_P + n_y\sin\theta_P & -o_x\cos\theta_P + o_y\sin\theta_P & -a_x\cos\theta_P + a_y\sin\theta_P \\ -n_x\sin\theta_P - n_y\cos\theta_P & -o_x\sin\theta_P - o_y\cos\theta_P & -a_x\sin\theta_P - a_y\cos\theta_P \\ n_z & o_z & a_z \end{bmatrix}
$$

where $i = n, o, a$ and $j = x, y, z$. This relationship equals to the transformation ${}^{0}\mathbf{R}_E = {}^{0}\mathbf{R}_1{}^{1}\mathbf{R}_E$. If we multiply both sides of the above equations with ${}^{0}\mathbf{R}_1^T$ we get:

$$
\begin{bmatrix} -n_x\cos(q_1-\theta_P)-n_y\sin(q_1-\theta_P) & -o_x\cos(q_1-\theta_P)-o_y\sin(q_1-\theta_P) & -a_x\cos(q_1-\theta_P)-a_y\sin(q_1-\theta_P) \\ n_x\sin(q_1-\theta_P)-n_y\cos(q_1-\theta_P) & o_x\sin(q_1-\theta_P)-o_y\cos(q_1-\theta_P) & a_x\sin(q_1-\theta_P)-a_y\cos(q_1-\theta_P) \\ n_z & o_z & a_z \end{bmatrix} = \begin{bmatrix} -c_{234}c_5 & c_{234}s_5 & -s_{234} \\ -s_5 & -c_5 & 0 \\ -c_5s_{234} & s_5s_{234} & c_{234} \end{bmatrix} \quad (7.10)
$$

By equating the (2,3) elements of both matrices we obtain two values of $q_1 - \theta_P$. Afterwards, we can obtain two values of the sum $q_2 + q_3 + q_4$ easily from the elements $(1,3), (2,3), (3,3)$ and two values of q_5 from $(2,1), (2,2)$. Finally, in order to obtain separately the values of q_1 and θ_P we need to backtrack a step and view carefully the equation:

$$
{}^{I}\mathbf{R}_0^T\left[i_j\right] = {}^{0}\mathbf{R}_E
$$

The unknown variables of this equation, after substitution of the previously obtained values, are indeed q_1 and θ_P but separated. Hence, if we equate the (1,3) and (2,3) elements of both matrices we obtain the below system of equations:

$$\begin{aligned} a_x \cos\theta_P - a_y \sin\theta_P &= -c_1 s_{234} \\ a_x \sin\theta_P + a_y \cos\theta_P &= -s_1 s_{234} \end{aligned} \tag{7.11}$$

the solution of which is:

$$\theta_P = Atan2(-a_x c_1 s_{234} - a_y s_1 s_{234}, -a_x s_1 s_{234} + a_y c_1 s_{234} = \theta_P(q_1)$$

and thus, knowing the value of $q_1 - \theta_P$, it is possible to find both angles. The algorithm continues exactly in the same manner as the one for the manipulator. Furthermore, the criterions that are used in the case of the manipulator to determine the existence, correctness and quantity of the solutions to the inverse kinematics problem can still be applied since the orthogonality principle that they are based upon is still in effect.

7.4 The Proximity Issue

The last issue that has not been touched so far is the question of "how close can we go to the values of **q** that either reduce the rank of the Jacobian or produce representational singularities before the matrices "explode" ".

Let us begin with the representational singularities. When the method of representation is RPY Euler Angles, then our constraint is $c_5^2 s_{234}^2 \neq 1$. To be precise, $c_5 \neq \pm 1$ and $s_{234} \neq \pm 1$ or $q_5 \neq 0$ and $q_2 + q_3 + q_4 \neq \pm\frac{\pi}{2}$. As a preliminary analysis, we consider the function:

$$f(q_2, q_3, q_4, q_5) = 1 - \sin^2(q_2 + q_3 + q_4)\cos^2 q_5$$

After differentiation, we get:

$$\delta f = \sin 2q_5 \sin^2(q_2 + q_3 + q_4)\delta q_5 - \sin[2(q_2 + q_3 + q_4)]\cos^2 q_5(\delta q_2 + \delta q_3 + \delta q_4)$$

which essentially means that the rate of change for q_2, q_3, q_4 is the same, hence the sum can be considered as a single variable for the purpose of this analysis. The above function can be written as:

$$f(x, y) = 1 - \cos^2 x \sin^2 y$$

By substituting $x \to 0 + \delta x,\ y \to \frac{\pi}{2} + \delta y$ we get the variance of f around the point of interest:

$$\delta f = 1 - \cos^2 \delta x \cos^2 \delta y$$

and a plot of $\frac{1}{\delta f}$ (since δf is the denominator of some terms in the Jacobian) is presented below:

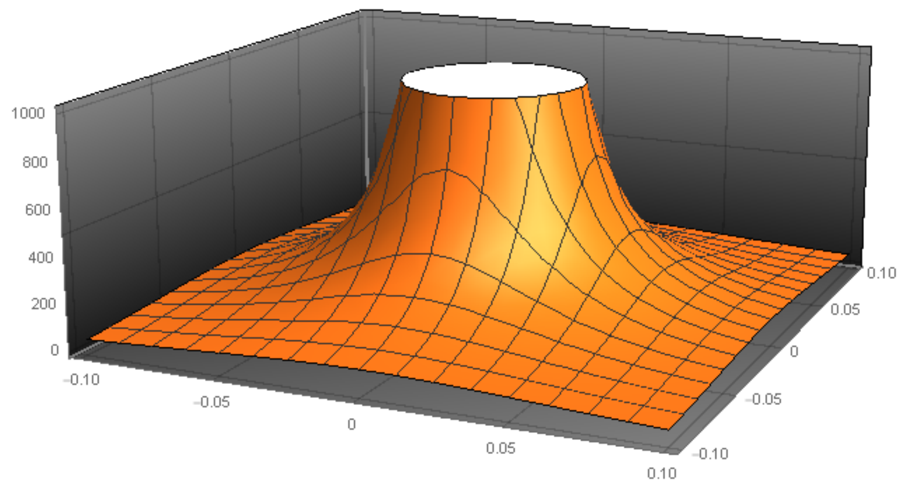


Figure 7.1: The change in joint angles for Euler angle representation.

As we see from the plot and after experimentation, the admissible values of the angles are $q_i < q_{i,danger} - 0.03$ and $q_i > q_{i,danger} + 0.03$. The above variances were considered successfully for the rank deficiency neighborhood too by examining the smallest singular value instead of δf. These values are subject to change though, depending on the sensitivity of e.g. the control algorithm that could be used for a task.

In vector algebra fashion, there is a method to determine what would be the minimum 2-norm perturbing matrix $\mathbf{\Delta}$ so that the matrix $(\mathbf{J}^* + \mathbf{\Delta})$ is rank deficient. Assuming that the SVD of the Jacobian exists, then $\mathbf{\Delta}$ is product of the smallest singular value of the composite Jacobian with the vector product of the two associated left and right singular vectors, i.e. :

$$\mathbf{\Delta} = -\sigma_{min} u_{min} v_{min}^T$$

Consequently, this matrix represents a change in the elements of the Jacobian. If each element is equated to the corresponding terms of Jacobian, the solution of this non-linera system respresents the desired change in joint angles.

Chapter 8

Conclusion and Future Directions

This thesis presented a systematic methodology so as to derive the complete kinematic model for a KUKA youBot mobile manipulator. We accomplished the following:

- Derive a kinematic model not only for the manipulator but also for the platform and for the combined system.
- Examine the cases of two types of platforms: ominidirectional and non-holonomic. In addition, two representations for the Jacobians were used for the system: the regular geometric Jacobian (manipulator) and the Euler angle representation (combined system).
- Analyze the models with the help of the Singular Value Decomposition and its properties while stressing its importance in Robotics. Among the results were the inverse differential kinematics model, the magnitude of change in the Jacobian and its inverse as a function of change in joint angles, a way to determine if any Jacobian matrix is close to rank deficiency via the singular values and the minimum values of joint angles as a perturbing matrix that lead to rank deficiency.
- Criterions that determine the existence, the correctness and the number of solutions for the inverse kinematic problem, which is also the focal point of the thesis.

Future Directions

- As far as improvements to the kinematic model are concerned, a particularly interesting case to examine is modeling with quaternions instead of Euler angle representation. The unique property of quaternions is that they are not subject to representational singularities. That said, it is expected that the minimum rank of that Jacobian would be greater than the one in our case. The price to pay for that would possibly be more complicated expressions.
- As far as next steps are concerned, the model could be expanded for two or more manipulators that handle an object. Do the same principles apply? What singular values are left intact then? Moreover, the possibility of the existence of

an alternative to the Singular Value Decomposition should be examined in case the model as a whole or just the equations that produce the eigenvalues are surprisingly complex.

- Explore the possibility of the minimum singular value of the Jacobian matrix acting as a metric in a control scheme. For instance, in navigation function control schemes, a manipulability measure, e.g. $\omega = \sqrt{\det(\mathbf{J})\det(\mathbf{J}^T)}$ is used so as to avoid the manifolds that correspond to rank deficiency. Perhaps, the properties of the minimum singular value could lead to it being a considered as a valid alternative.

Appendices

A. Additive Perturbation

Theorem Suppose $A \in \mathbb{C}^{m \times n}$ has full column rank $(= n)$. Then

$$\min_{\Delta \in \mathbb{C}^{m \times n}} \{\|\Delta\|_2 \mid A + \Delta \text{ has rank } < n\} = \sigma_n(A) \quad \text{(minimum singular value)}$$

Proof: Suppose $A + \Delta$ has rank $< n$. Then there exists $x \neq 0$ such that $\|x\|_2 = 1$ and

$$(A + \Delta)\, x = 0 \ .$$

Since $\Delta x = -Ax$,

$$\begin{aligned} \|\Delta x\|_2 &= \|Ax\|_2 \\ &\geq \sigma_n(A) \ . \end{aligned}$$

From the properties of induced norms we also know that

$$\|\Delta\|_2 \|x\|_2 \geq \|\Delta x\|_2.$$

Using the fact that $\|x\|_2 = 1$, we arrive at the following:

$$\begin{aligned} \|\Delta\|_2 &\geq \|\Delta x\|_2 \\ &\geq \sigma_n(A) \end{aligned} \qquad (1)$$

To complete the proof, we must show that the lower bound from Equation (1) can be achieved. Thus, we must construct a Δ so that $A + \Delta$ has rank $< n$ and $\|\Delta\|_2 = \sigma_n(A)$; such a Δ will be a minimizing solution. For this, choose

$$\Delta = -\sigma_n u_n v_n^{\mathsf{T}}$$

where u_n, v_n are the left and right singular vectors associated with the smallest singular value σ_n of A. Notice that $\|\Delta\|_2 = \sigma_n(A)$. This choice yields

$$\begin{aligned} (A + \Delta)\, v_n &= \sigma_n u_n - \sigma_n u_n v_n^* v_n \\ &= \sigma_n u_n - \sigma_n u_n \\ &= 0 \ . \end{aligned}$$

That is, $A + \Delta$ has rank $< n$. This completes the proof.

B. Condition Number

Taking differentials in the defining equation $A^{-1}A = I$, we find

$$d(A^{-1})\,A + A^{-1}\,dA = 0\ ,$$

where the order of the terms in each half of the sum is important, of course. (Rather than working with differentials, we could equivalently work with perturbations of the form $A+\epsilon P$, etc., where ϵ is vanishingly small, but this really amounts to the same thing.) Rearranging the preceding expression, we find

$$d(A^{-1}) = -A^{-1}\ dA\ A^{-1}$$

Taking norms, the result is

$$\|d(A^{-1})\| \leq \|A^{-1}\|^2\|dA\|$$

or equivalently

$$\frac{\|d(A^{-1})\|}{\|A^{-1}\|} \leq \|A\|\|A^{-1}\|\frac{\|dA\|}{\|A\|}$$

This derivation holds for any submultiplicative norm. The product $\|A\|\|A^{-1}\|$ is termed the *condition number* of A with respect to inversion (or simply the condition number of A) and denoted by $K(A)$:

$$K(A) = \|A\|\|A^{-1}\|$$

When we wish to specify which norm is being used, a subscript is attached to $K(A)$. Our earlier results on the SVD show, for example, that

$$K_2(A) = \sigma_{max}/\sigma_{min}$$

Bibliography

[AS09] Sunil K. Agrawal and Vivek Sangwan. Differentially flat designs of underactuated open-chain planar robots. *IEEE Transactions on Robotics*, 24(6):1445–1451, 2009.

[BHP11] R. Bischoff, U. Huggenberger, and E. Prassler. Kuka youbot- a mobile manipulator for research and education. *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA)*, pages 1–4, 2011.

[BL89] Jerome Barraquand and Jean-Claude Latombe. On non-holonomic mobile robots and optimal maneuvering. , 1989.

[Chu97] Jae H. Chung. Modeling and control of mobile manipulators. *AHMCT Research Report*, 1997.

[CL14] Hu Chen and Jangmyung Lee. Path planning of 5-dof manipulator based on maximum mobility. *International Journal of Precision Engineering and Manufacturing*, 15(1):45–52, 2014.

[Cra05] John J. Craig. *Introduction to Robotics. Mechanics and Control*. Pearson Prentice Hal, 2005.

[DBB+02] O. Diegel, A. Badve, G. Bright, J. Potgieter, and S. Tlale. Improved mecanum wheel design for omni-directional robots. 2002.

[DLIMO03] Alessandro De Luca, Stefano Iannitti, Raffaella Mattone, and Giuseppe Oriolo. Underactuated manipulators: Control properties and techniques. *Machine Intelligence and Robotic Control*, 2003.

[FT03] Y. Fang and L.W. Tsai. Inverse velocity and singularity analysis of low-dof serial manipulators. *Journal of Robotic Systems*, 20(4):177—188, 2003.

[GOHR05] J.Q. Gan, E. Oyama, H. Hu, and E.M. Rosales. A complete analytical solution to the inverse kinematics of the pioneer2 robotic arm. *Robotica*, 23:123–129, 2005.

[GR03] J.Q. Gan and E.M. Rosales. Forward and inverse kinematics models for a 5-dof pioneer2 robotic arm. *Technical report, University of Essex*, 2003.

[GVL83] Gene H. Golub and Charles F. Van Loan. *Matrix Computations*. 1983.

[Hua11] Shaoping Huang. Research on trajectory planning and motion control of 5-dof cutting robot. *Dissertation for the Master's Degree in Engineering, Harbin Institute of Technology*, 2011.

[Kal96] Dan Kalman. A singularly valuable decomposition: The svd of a matrix. *The College Mathematics Journal*, 27(1):2–23, 1996.

[KMA06] P. Kalra, P.B. Mahapatra, and D.K. Aggarwal. An evolutionary approach for solving the multimodal inverse kinematics problem of industrial robots. *Mech. Mach. Theory*, 41(10):1213–1229, 2006.

[KUK13] Kuka youbot: User manual, 2013.

[LC12] Zexiang Li and J.F. Canny. *Non-holonomic Motion Planning*. Springer, 2012.

[LSL98] J.P. Laumond, S. Sekhavat, and F. Lamiraux. *Guidelines in Nonholonomic Motion Planning for Mobile Robots*. 1998.

[LST07] Guangming Lu, Lining Sun, and Yuyong Tang. The invertible solution of kinematics about 5-dof upper limb rehabilitant robot. *Journal of Natural Science of Heilongjiang University*, 24(1):54–57, 2007.

[LXW14] Jinyan Lu, De Xu, and Peng Wang. A kinematics analysis for a 5-dof manipulator. *26th Chinese Control and Decision Conference (CCDC)*, 2014.

[LYGB10] Junyi Lu, Qinghua Yang, Feng Gao, and Guanjun Bao. Trajectory planning of a 5-dof agricultural picking robot. *Journal of Mechanical and Electrical Engineering*, 27(12):1–6, 2010.

[LZLZ13] H.S. Liu, W.N. Zhou, X.B. Lai, and S.Q. Zhu. An efficient inverse kinematics algorithm for a puma 560-structured robot manipulator. *Int. J. Adv. Robot. Syst.*, 10:236–240, 2013.

[MD92] R. Manseur and K.L. Doty. Fast inverse kinematics of five-revolute axis robot manipulators. *Mech. Mach. Theory*, 27(5):587–597, 1992.

[MLS94] Richard M. Murray, Zexiang Li, and Shankar S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. 1994.

[Nea98] A.C. Nearchou. Solving the inverse kinematics problem of redundant robots operating in complex environments via a modified genetic algorithm. *Mech. Mach. Theory*, 33(3):273–292, 1998.

[OCAM01] E. Oyama, N.Y. Chong, A. Agah, and T. Maeda. Inverse kinematics learning by modular architecture neural networks with performance prediction networks. *IEEE Int. Conf. Robot. Autom. (ICRA), Seoul, Korea*, 1:1006–1012, 2001.

[PH01] E. Papadopoulos and V. Hayward. The singular vector algorithm for the computation of rank-deficiency loci of rectangular jacobians. *Proceedings of IEEE/RSJ International Conference on Intelligent Robots and Systems*, 1:324–329, 2001.

[Pie68] D.I. Pieper. The kinematics of manipulators under computer control. *Ph.D. dissertation, Dept. Computer Science, Stanford Univ., Stanford*, CA USA, 1968.

[PP00a] Evangelos Papadopoulos and Ioannis Poulakakis. On motion planning of nonholonomic mobile robots. *Int. Symposium of Robotics*, pages 77–82, 2000.

[PP00b] Evangelos Papadopoulos and Ioannis Poulakakis. Planning and model-based control for mobile manipulators. *In proceedings of IEEE/RSJ International Conference on Intelligent Robots and Systems, Takamatsu, Japan*, pages 1810—1815, 2000.

[RWS15] Z.W. Ren, Z.H. Wang, and L.N. Sun. A hybrid biogeography-based optimization method for the inverse kinematics problem of an 8-dof redundant humanoid manipulator. *Front. Inf. Technol. Electron. Eng.*, 16(7):607–616, 2015.

[SB01] Stefano Stramigioli and Herman Bruyninckx. *Geometry and Screw Theory for Robotics*. 2001.

[SCT11] E. Sariyildiz, E. Cakiray, and H. Temeltas. A comparative study of three inverse kinematic methods of serial industrial robot manipulators in the screw theory framework. *Int. J. Adv. Robot. Syst.*, 8:9–24, 2011.

[She10] Xiaoqing Shen. Research on a 5-dof carrying manipulator for teaching. *Dissertation for the Master's Degree in Engineering, Suzhou University*, 2010.

[SK08] Bruno Siciliano and Oussama Khatib. *Handbook of Robotics*. Springer, 2008.

[Spi70] M. Spivak. *A Comprehensive Introduction to Differential Geometry*. 1970.

[SS96] Bruno Siciliano and Lorenzo Sciavicco. *Modelling and Control of Robot Manipulators*. Springer, 1996.

[Tan01] H. G. Tanner. Modeling of multiple mobile manipulators handling a common deformable object. *Dissertation for the Doctor of Philosophy title, National Technical University if Athens, Greece*, 2001.

[TBG15] H. Taheri, Q. Bing, and N. Ghaeminezhad. Kinematic model of a four mecanum wheeled mobile robot. *International Journal of Computer Applications*, 113(3), 2015.

[TBK04] P. Tang, R. Bhatt, and V. Krovi. Decentralized kinematic control of payload by a system of mobile manipulators. *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA)*, 3, 2004.

[TK00] Herbert G. Tanner and Kostas J. Kyriakopoulos. Nonholonomic motion planning for mobile manipulators. *IEEE International Conference on Robotics and Automation, 2000. Proceedings. ICRA '00*, 2:1233–1238, 2000.

[TK01] Herbert G. Tanner and Kostas J. Kyriakopoulos. Mobile manipulator modeling with kane's approach. *Robotica*, 19(6):675–690, September 2001.

[XCGH05] D. Xu, C.A.A. Calderon, J.Q. Gan, and H. Hu. An analysis of the inverse kinematics for a 5-dof manipulator. *Int. J. Autom. Comput.*, 2(2):114–124, 2005.

[ZLX15] Yaolun Zhang, Yangmin Li, and Xiao Xiao. A novel kinematics analysis for a 5-dof manipulator based on kuka youbot. *IEEE International Conference on Robotics and Biomimetics (ROBIO)*, 2015.