Διπλωματικη Εργασια- Diploma Thesis

Το Φαινόμενο CASIMIR

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Στη μνήμη του καθηγητή Ι. Μπάκα

Ευχαριστίες

Η παρούσα διπλωματική εργασία εκπονήθηκε στον Τομέα Φυσικής της Σχολής Εφαρμοσμένων Μαθηματικών και Φυσικών Επιστημών (ΣΕΜΦΕ) σε συνεργασία με τη Σχολή Πολιτικών Μηχανικών του Εθνικού Μετσοβίου Πολυτεχνείου (ΕΜΠ). Η δυνατότητα αυτή μου δόθηκε στο πλαίσιο εκπόνησης Διατμηματικών Διπλωματικών που προσφέρει το ΕΜΠ δίνοντας την ευκαιρία και ελευθερία στο φοιτητή να διευρύνει τα ενδιαφέροντά του και να εφαρμόσει τις αποκτηθείσες γνώσεις και σε άλλους παρεμφερείς τομείς τονίζοντας την διεπιστημονικότητα των αντικειμένων.

Την επιθυμία μου να εκπονήσω μια διπλωματική στο Τομέα της Φυσικής εκπλήρωσε ο εκλιπών καθηγητής Μπάκας. Ο αείμνηστος Μπάκας μου προσέφερε το πρώτο έναυσμα για να ακολουθήσω τα ενδιαφέροντά μου. Μου πρόσφερε, επίσης, παρακαταθήκη ανεκτίμητης αξίας για τη μετέπειτα πορεία μου. Κατά τη διάρκεια της εκπόνησης της διπλωματικής, η προθυμία του για βοήθεια, εφοδιασμένη με ένα σπάνιο ενδιαφέρον, ήταν για μένα στήριξη και πηγή πνευματικής δύναμης. Αυτή η Διπλωματική Εργασία, ειλικρινώς, από καρδιάς και εξ' ολοκλήρου αφιερώνεται στη μνήμη του.

Επιπλέον, θα ήθελα να ευχαριστήσω τον καθηγητή Γεώργιο Κουτσούμπα του Τομέα Φυσικής της ΣΕΜΦΕ που ήταν πρόθυμος να με δεχτεί για τη συνέχιση της Διπλωματικής Εργασίας. Αν και η διπλωματική είχε σχεδόν ολοκληρωθεί, ο κ. Κουτσούμπας μου έδωσε μια σημαντική ώθηση στο να καταλάβω καλύτερα το μαθηματικό υπόβαθρο του θέματος.

Επίσης, ευχαριστώ την καθηγήτρια Μαθηματικών της ΣΕΜΦΕ και της Σχολής Πολιτικών Μηχανικών, κ. Κυριακή Κυριάκη, με την οποία οι συνεχείς συζητήσεις μας πάνω σε θέματα Επιστήμης και επαγγελματικού προσανατολισμού ήταν ιδιαίτερης σημασίας.

Τέλος, θα ήταν μεγάλη παράληψη αν δεν ευχαριστήσω τους δικούς μου ανθρώπους. Ευχαριστώ τους φίλους μου για όλη τη στήριξη και όλες τις εποικοδομητικές μας συζητήσεις όλα αυτά τα χρόνια. Ευχαριστώ, επίσης, τους γονείς μου για την αγάπη τους, την στήριξη, την υπομονή και τη συγκαταβατικότητά τους, τους ευχαριστώ επειδή σε αυτούς οφείλω τα πάντα.

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Περίληψη

Η παρούσα διπλωματική εργασία αποτελεί μια μελέτη του Φαινομένου Casimir. Το φαινόμενο αυτό είναι ένα κβαντικό φαινόμενο που συναντάται σε πολλές περιοχές της Φυσικής αλλά και σε διάφορες τεχνολογικές εφαρμογές προσφέροντας έτσι μια μακροσκοπική εκδήλωση του κβαντομηχανικού μικρόκοσμου.

Στο κεφάλαιο 1, παρουσιάζεται η έννοια της ενέργειας μηδενικού σημείου (zero-point energy) και πώς αυτή αναδεικνύεται μέσα από τη κβαντική περιγραφή ενός πεδίου. Η διαταραχές της ενέργειας αυτής λόγω της παρουσίας κάποιων συνόρων (συνοριακών συνθηκών) στο χώρο οδηγεί στην εμφάνιση δυνάμεων επι των συνόρων (δυνάμεις Casimir) που τείνουν να φέρουν το σύστημα στην κατάσταση με την ελάχιστη δυνατή ενέργεια.

Στο κεφάλαιο 2, γίνεται μια λεπτομερής μελέτη του ηλεκτρομαγνητικού (Η/Μ) φαινομένου Casimir για το κλασικό παράδειγμα των δύο ιδανικά αγώγιμων επιπέδων πλακών. Υπολογίζεται η δύναμη Casimir μέσω της άθροισης των ενεργειών των κανονικών τρόπων ταλάντωσης του Η/Μ πεδίου. Έπειτα, υπολογίζονται οι θερμικές δυνάμεις που προέρχονται από τη παρουσία Η/Μ ακτινοβολίας και οι οποίες δρούν επί των επιπέδων, με σκοπό τη σύγκριση μεταξύ αυτών των δύο δυνάμεων. Σε θερμοκρασίες δωματίου και για μικρές αποστάσεις των πλακών, κατά μια πρώτη προσέγγιση, εξετάζεται ο βαθμός συμμετοχής των δυνάμεων Casimir οδηγώντας έτσι σε ένα συμπέρασμα για το αν πρέπει να αμεληθούν κατά τη κατασκευή μικρο-μηχανοηλεκτρονικών συστημάτων (microelectromechanical systems-MEMS).

Εν συνεχεία, στο κεφάλαιο 3, προσεγγίζεται το ίδιο πρόβλημα με τη μέθοδο των δυαδικών συναρτήσεων Green. Η προσέγγιση αυτή είναι πιο περίπλοκη για το παράδειγμα των επιπέδων πλακών απ' ότι η προηγούμενη, ωστόσο, είναι ιδιαίτερα χρήσιμη σε διατάξεις με διαφορετική γεωμετρία, π.χ. σφαιρική, κυλινδρική, ελλειψοειδή, κτλ.

Στο κεφάλαιο 4, υπολογίζεται η Η/Μ δύναμη Casimir για έναν τέλεια αγώγιμο σφαιρικό φλοιό μέσω δυαδικών συναρτήσεων Green. Στο κεφάλαιο αυτό, χρησιμοποιούνται μαθηματικά εργαλεία και ειδικές συναρτήσεις τα οποία παρατείθενται, όσο αναλυτικότερα ήταν δυνατόν, στα παραρτήματα της εργασίας.

Κλείνοντας, στο κεφάλαιο 5, παρουσιάζεται το φαινόμενο Casimir, για τις ίδιες γεωμετρίες (επίπεδη και σφαιρική) για τη περίπτωση ενός φερμιονικού πεδίου.

Μέρος I

Κεφάλαιο 1

Εισαγωγή

1.1. Ιστορική Αναδρομή

Το 1940, ο J.T.G. Overbeek στα εργαστήρια Phillips εξέτασε την αλληλεπίδραση μεταξύ σωματιδίων κολλοειδών συστημάτων και διαπίστωσε ότι οι αλληλεπιδράσεις Van der Walls δεν περιγράφουν με ακρίβεια τα πειραματικά αποτελέσματα. Έδειξε ότι η αλληλεπίδραση μεταξύ των σωματιδίων φθίνει ως r^{-7} και όχι ως r^{-6} όπως έλεγε η υπάρχουσα θεωρία. Αυτό τον οδήγησε στο να κάνει την υπόθεση οτι η απόκλιση οφείλεται στην πεπερασμένη ταχύτητα του φωτός (retardation).

Τα αποτελέσματα του Overbeek οδήγησαν τους H.B.G Casimir και D. Polder το 1948 να επανεξετάσουν θεωρητικά τις δυνάμεις Van der Walls. Λαμβάνοντας υπ΄ όψη στους υπολογισμούς τους την πεπερασμένη ταχύτητα του φωτός, επιβεβαίωσαν ακριβώς την υπόθεση του Overbeek.

Η απλότητα των υπολογισμών τους, ενθουσίασε τον Casimir και τον ώθησε να ψάξει για μια βαθύτερη φυσική εξήγηση, μιας και οι δυνάμεις Van der Walls είναι φαινομενολογικές. Έτσι, την ίδια χρονιά, 1948, δημοσιεύει ένα άρθρο με τίτλο "On the attraction between two perfectly conducting plates" όπου δείχνει οτι στα ίδια αποτελέσματα μπορεί να καταλείξει κάποιος υπολογίζοντας τις διακυμάνσεις την ενέργειας του κενού. Στο άρθρο αυτό, θεώρησε δύο ουδέτερες ιδανικά αγώγιμες πλάκες, και υπέδειξε την ύπαρξη μιας ελκτικής δύναμης:

$$\boxed{f(L) = -\frac{\pi^2 \hbar c}{240 L^4}}. \tag{1.1}$$

1.2. Ενέργεια Μηδενικού Σημείου/ Κενού Χώρου

Η ενέργεια μηδενικού σημείου είναι η χαμηλότερη ενέργεια που μπορεί να έχει ένα σύστημα.

Για ένα **κλασικό σύστημα** αυτή η ενέργεια είναι μηδέν, και επιτυγχάνεται σε θερμοκρασία $T = 0^o K$.

Αντιθέτως, για ένα **κβαντικό σύστημα** αυτό δεν μπορεί να επιτευχθεί αλλά υπάρχει μια αναπόφευκτη ενέργεια μηδενικού σημείου.

1.3. Ηλεκτρομαγνητική (Η/Μ) Ενέργεια Μηδενικού Σημείου

Κλασικά, ένας χώρος γεμάτος Η/Μ ακτινοβολία έχει ενεργειακή πυκνότητα:

$$u = \frac{1}{8\pi}(\boldsymbol{E}^2 + \boldsymbol{B}^2), \tag{1.2}$$

όπου $\boldsymbol{E}$ & $\boldsymbol{B}$ είναι το ηλεκτρικό και μαγνητικό πεδίο αντίστοιχα. Στον κενό χώρο έχουμε φυσικά $E = 0 = B$, οπότε οδηγούμαστε σε $\underline{u = 0}$.

Κβαντομηχανικά, στην περιγραφή του Η/Μ πεδίου αντιστοιχούμε τα $\boldsymbol{E}$ & $\boldsymbol{B}$ σε τελεστές οι οποίοι ακολουθούν την ακόλουθη σχέση μετάθεσης:

$$[\boldsymbol{E}_i(\boldsymbol{r}), \boldsymbol{B}_j(\boldsymbol{r})] = i\hbar\delta_{ij}\delta^3(\boldsymbol{r} - \boldsymbol{r}'). \tag{1.3}$$

Επειδή αυτή, για το ίδιο σημείο του χώρου, δεν μηδενίζεται οδηγεί σε μια σχέση αβεβαιότητας μεταξύ αυτών των μεγεθών πράγμα που σημαίνει ότι αυτά τα δύο δεν μπορούν να μετρηθούν ταυτόχρονα με απόλυτη ακρίβεια (σε κάθε σημείο του χώρου) και έτσι δεν μπορούν να είναι και τα δύο ταυτόχρονα μηδέν. Αυτό σημαίνει ότι η ενεργειακή πυκνότητα (αν δει κανένας τον κλασικό ορισμό της) δεν μπορεί να πάρει την τιμή μηδέν σε κανένα σημείο του χώρου.

Πράγματι, κάτι αυτό ακριβώς συμβαίνει. Στη κβαντομηχανική περιγραφή του πεδίου η Χαμηλτονιανή δίνεται από τη σχέση:

$$H_j(\boldsymbol{k}) = \hbar\omega(k)\Big(a_j^\dagger(\boldsymbol{k})a_j(\boldsymbol{k}) + \frac{1}{2}\Big), \tag{1.4}$$

όπου $j = 1, 2$ είναι για τις δύο ανεξάρτητες πολώσεις και οι τελεστές $a_j(\boldsymbol{k}, t)$ και $a_j^\dagger(\boldsymbol{k}, t)$ υπακούουν την μεταθετική σχέση:

$$[a_j(\boldsymbol{k}), a_{j'}^\dagger(\boldsymbol{k}')] = \delta_{jj'}\delta(\boldsymbol{k} - \boldsymbol{k}'). \tag{1.5}$$

Η ολικη Χαμηλτονιανή στο ελεύθερο πεδίο, λαμβάνοντας υπ΄ όψη όλους τους τρόπους ταλάντωσης, είναι προφανώς:

$$H = \sum_j \int_{\boldsymbol{k}} H_j(\boldsymbol{k})\, d^3k. \tag{1.6}$$

Επομένως, για να πάρουμε την ενέργεια μηδενικού σημείου εκτελούμε τη πράξη:

$$\mathcal{E}_f = \langle 0|\, H\, |0\rangle = \sum_j \int_{\boldsymbol{k}} \frac{\hbar ck}{2}\, d^3k = 2 \cdot \frac{\hbar c}{2} \int_0^\infty k \frac{1}{8} 4\pi k^2\, dk, \tag{1.7}$$

όπου χρησιμοποιήσαμε σφαιρικές συντεταγμένες για την ολοκλήρωση στο χώρο (k_x, k_y, k_z).

Αυτό το ολοκλήρωμα μας δίνει για την ενέργεια μηδενικού σημείου άπειρη τιμή γεγονός το οποίο φαίνεται να μην έχει κάποιο φυσικό νόημα.

Ωστόσο, αν με κάποιον τρόπο διαταραχθεί ο κενός χώρος εφαρμόζοντας κάποιες συνοριακές συνθήκες (π.χ. υλικά σύνορα) το σύστημα μπορεί να οδηγηθεί σε μια νέα κατάσταση με διαφορετική ενέργεια μηδενικού σημείου, μεγαλύτερη ή μικρότερη, αλλά εν γένει πάλι άπειρη. Το φυσικό σύστημα 'προτιμά' να ελαχιστοποιήσει την ενέργειά του και η ενεργειακή διαφορά μεταξύ των δύο καταστάσεων οδηγεί στην εμφάνηση δυνάμεων μεταξύ των συνόρων οι οποίες τείνουν να φέρουν το σύσημα στη χαμηλότερη δυνατη ενέργεια (μηδενικού σημείου).

1.4. Φερμιονική Ενέργεια Μηδενικού Σημείου

Όπως παρουσιάζεται και στο κύριο Παράρτημα η Χαμηλτονιανή που περιγράφει ένα φερμιονικό πεδίο δίνεται από τη σχέση:

$$\mathcal{H} = \sum_{k,s} \Big(a_{\boldsymbol{k}}^{s\dagger} a_{\boldsymbol{k}}^{s} - b_{\boldsymbol{k}}^{s\dagger} b_{\boldsymbol{k}}^{s} \Big) \omega_k, \tag{1.8}$$

όπου οι εκθετικοί δείκτες $s = 1, 2$ υποδηλώνουν τις δύο προβολές του σπίν ($(+)$ & $(-)$) πάνω στην κατεύθυνση της γραμμικής ορμής $\boldsymbol{k}$.

Εφόσον το πεδίο είναι φερμιονικό, οι συντελεστές ακολουθούν τις αντιμεταθετικές σχέσεις:

$$\{a_{\boldsymbol{k}}^{s}, a_{\boldsymbol{k}'}^{s'\dagger}\} = \delta_{\boldsymbol{k}\boldsymbol{k}'} \delta_{ss'} \tag{1.9α'}$$

$$\{b_{\boldsymbol{k}}^{s}, b_{\boldsymbol{k}'}^{s'\dagger}\} = \delta_{\boldsymbol{k}\boldsymbol{k}'} \delta_{ss'}, \tag{1.9β'}$$

και επομένως η Χαμηλτονιανή γράφεται ως:

$$H = \sum_{k,s} \Big(a_{\boldsymbol{k}}^{s\dagger} a_{\boldsymbol{k}}^{s} + b_{\boldsymbol{k}}^{s} b_{\boldsymbol{k}}^{s\dagger} \Big) \omega_k + \sum_{k} \omega_k. \tag{1.10}$$

Ο δεύτερος όρος της εξίσωσης αυτής είναι η ενέργεια του κενού χώρου. Αν και είναι άπειρη, μπορεί να οδηγήσει σε *Φερμιονικές Δυνάμεις Casimir* με τον ίδιο ακριβώς τρόπο όπως και στην περίπτωση του Η/Μ κενού χώρου.

Κεφάλαιο 2

Η/Μ Δύναμη Casimir Ανάμεσα σε Ιδανικά Αγώγιμες Παράλληλες Πλάκες

Θεωρούμε δύο τέλεια αγώγιμες επίπεδες πλάκες που απέχουν απόσταση L εμβαδού $\mathcal{A} = L_y \times L_z$. Στην περίπτωση αυτή υπάρχουν ενδογενείς στάσιμα κύματα τα οποία έχουν συχνότητες $\omega = c\sqrt{\frac{\pi^2 n^2}{L^2} + k_y^2 + k_z^2}$ (αναπόφευκτη) θεμελιώδη ενέργεια $\hbar\omega/2$ έκαστο.

Αθροίζοντας τις ενέργειες όλων των κανονικών τρόπων ταλάντωσης των Η/Μ κυμάτων μέσα στη κοιλότητα λαμβάνουμε:

$$\mathcal{E}_0(L) = \frac{\hbar c}{2}\frac{L_y L_z}{\pi^2}\sum_{n=0}^{\infty}{}' \int_{k_y,k_z}\sqrt{\frac{\pi^2 n^2}{L^2} + k_y^2 + k_z^2} dk_y dk_z, \tag{2.1}$$

η οποία είναι η έκφραση για την ενέργεια μηδενικού σημείου παρουσία των πλακών. Ο τόνος στο άθροισμα δηλώνει οτι πρέπει για $n = 0$ να προστεθεί ένα παράγοντας $1/2$ καθώς η έκφραση αυτή είναι και για τις δύο πολώσεις των Η/Μ κυμάτων ενώ για $n = 0$ υπάρχει μόνο μία.

Το ολοκλήρωμα αυτό αποκλίνει, ωστόσο, πρέπει να λάβουμε υπ'όψη και την ενέργεια μηδενικού σημείου του κενού χώρου απουσία πλακών, δηλαδη την περίπτωση όπου η απόσταση των πλάκών είναι στο όριο $L \to \infty$. Παίρνοντας αυτό το όριο για την ανωτέρω έκφραση, το άθροισμα πάνω στα n μετατρέπεται σε ολοκλήρωση επειδή η απόσταση μεταξύ δύο διαδοχικών k_x γίνεται απειροστή και το n μπορεί να ληφθεί ως συνεχής παράμετρος. Επίσης, κάνοντας την αλλαγή μεταβλητών $dn = \frac{L}{\pi}dk$, η ενέργεια μηδενικού σημείου του κενού χώρου (απουσία πλακών) δίνεται από τη σχέση:

$$\mathcal{E}(\infty) = \lim_{L\to\infty}\mathcal{E}(L) = \frac{L}{\pi}\frac{\hbar c\mathcal{A}}{2\pi}\int_0^{\infty}\int_0^{\infty}\sqrt{k_x^2 + k_\rho^2}\; k_\rho \, dk_x dk_\rho, \tag{2.2}$$

το οποίο είναι και αυτό άπειρο. Όμως, όπως είπαμε, η διαφορά των ενεργειών των δύο αυτών καταστάσεων είναι αυτή που έχει σημασία.

Επομένως, η δυναμική ενέργεια της διάταξης δίνεται από τη διαφορά των ενεργειών (2.1) και (2.2), δηλαδή $\tilde{\mathcal{E}}_0(L) = \mathcal{E}(L) - \mathcal{E}(\infty)$ το οποίο οδηγεί στη σχέση:

$$\tilde{\mathcal{E}}_0(L) = \frac{\hbar c \mathcal{A}}{2\pi}\left({\sum_n}' \int_0^\infty \sqrt{\frac{\pi^2 n^2}{L^2} + k_\rho^2}\ k_\rho\, dk_\rho - \frac{L}{\pi}\int_0^\infty\int_0^\infty \sqrt{k_x^2 + k_\rho^2}\ k_\rho\, dk_x dk_\rho \right). \tag{2.3}$$

Παρ΄ όλο που περιμένουμε η έκφραση (2.3) να οδηγήσει σε πεπερασμένο αποτέλεσμα, στην πραγματικότητα δίνει και αυτή άπειρο. Παρ΄ όλ΄ αυτά, υπάρχει μια φυσική αναγκαιότητα για να άρουμε αυτή την απειρία. Οι τρόποι ταλάντωσης υψηλών συχνοτήτων αντιστοιχούν σε φωτόνια πολύ υψηλών ενεργειών τα οποία μπορούν να διεισδύσουν μέσα στον αγωγό και να τον διαπεράσουν, με συνέπεια να μη συμμετέχουν στην επιθυμητή δυναμική ενέργεια της διάταξης. Για να αποκλίσουμε, λοιπόν, αυτές τις υψηλές συχνότητες εισάγουμε μια cut-off συνάρτηση $f(k)$ τέτοια ώστε $f(k) = 1$ όταν $k < k_c$ και $f(k) = 0$ όταν $k > k_c$, όπου η τιμή k_c μπορεί να εκλεχθεί η τιμή για μήκος κύματος ίσο με την ακτίνα του $Bohr$. Τελικά, η επιλογή της συνάρτησης αυτής δεν επηρεάζει το τελικό αποτέλεσμα. Έτσι η διαφορά των ενεργειών γίνεται:

$$\mathcal{E}_{reg}(L) = \frac{\hbar c \mathcal{A}}{2\pi} {\sum_n}' \int_0^\infty \sqrt{\frac{\pi^2 n^2}{L^2} + k_\rho^2}\ f\left(\sqrt{\frac{\pi^2 n^2}{L^2} + k_\rho^2}\right)\frac{1}{2} dk_\rho^2 \tag{2.4α΄}$$

$$\mathcal{E}_{reg}(\infty) = \frac{L}{\pi}\frac{\hbar c \mathcal{A}}{2\pi}\int_0^\infty\int_0^\infty \sqrt{k_x^2 + k_\rho^2}\, f\left(\sqrt{k_x^2 + k_\rho^2}\right)\frac{1}{2}\ dk_x dk_\rho^2 \tag{2.4β΄}$$

$$\tilde{\mathcal{E}}_0(L) = \mathcal{E}_{reg}(L) - \mathcal{E}_{reg}(\infty) \tag{2.4γ΄}$$

όπου θέσαμε $k_\rho dk_\rho = \frac{1}{2} dk_\rho^2$ και ο δείκτης reg υποδηλώνει ότι αυτή είναι η ενέργεια μετά την κανονικοποίηση (ή τακτοποίηση (regularization)).

Τελικά, εκτελώντας τις πράξεις καταλήγουμε στο αποτέλεσμα (δες κύριο Παράρτημα):

$$\tilde{\mathcal{E}}_0(L) = -\frac{\pi^2 \hbar c \mathcal{A}}{720 L^3}. \tag{2.5}$$

Ορίζοντας την δύναμη Casimir με το σύμβολο $\tilde{\mathcal{X}}_0(L)$[1] και τη δύναμη ανα μονάδα επιφανείας ως $\tilde{\mathcal{P}}_C$ έχουμε:

$$\tilde{\mathcal{P}}_C(L) = \frac{1}{A}\tilde{\mathcal{X}}_0(L) = -\frac{1}{A}\frac{\partial \tilde{\mathcal{E}}_0}{\partial L} = -\frac{\pi^2 \hbar c}{240 L^4}, \tag{2.6}$$

η οποία είναι η πίεση Casimir μεταξύ των δύο πλακών.

[1]Ο δείκτης 0 υποδηλώνει οτι ο χώρος ειναι κενός, δηλαδή απουσία Η/Μ ακτινοβολίας.

2.1. Η/Μ Ελεύθερη Ενέργεια σε Πεπερασμένη Θερμοκρασία

Σε αυτό τμήμα θα μελετήσουμε τις θερμικές δυνάμεις που ασκούνται στα τοιχώματα της κοιλότητας εαν αυτή εμβαπτιστεί μέσα σε Η/Μ ακτινοβολία. Η περίπτωση αυτή μας ενδιαφέρει για να συγκρίνουμε τις θερμικές δυνάμεις με τις δυνάμεις Casimir στο όριο των χαμηλών θερμοκρασιών και αποστάσεων L. Αυτό το όριο είναι σημαντικό για τη κατασκευή μικρο-μηχανοηλεκτορνικών συστημάτων (microelectromechanical systems-MEMS) τα οποία, προφανώς, είναι μικρών διαστάσεων και κατασκευάζονται σε θερμοκρασία δωματίου. Θέλουμε να δούμε αν μπορούμε να αμελήσουμε τις δυνάμεις Casimir ή όχι.

Θεωρούμε λοιπόν οτι μέσα στη κοιλότητα υπάρχει Η/Μ ακτινοβολία όλων των επιτρεπόμενων συχνοτήτων ω σε θερμοκρασία T. Η ελεύθερη ενέργεια, από κάθε κανονικό τρόπο ταλάντωσης συχνότητας ω, είναι:

$$\begin{aligned} f(\omega) &= -\frac{1}{\beta}\ln\sum_{N=0}^{\infty} e^{-\beta E_N} = -\frac{1}{\beta}\ln\sum_{N=0}^{\infty} e^{-\beta\hbar\omega(N+\frac{1}{2})} \\ &= -\frac{1}{\beta}\ln e^{-\beta\frac{1}{2}\hbar\omega} - \frac{1}{\beta}\ln\sum_{N=1}^{\infty} e^{-N\beta\hbar\omega} \\ &= \frac{1}{2}\hbar\omega + \frac{1}{\beta}\ln\Big(1-e^{-\beta\hbar\omega}\Big), \end{aligned} \tag{2.7}$$

όπου $\beta = 1/k_B T$, k_B είναι η σταθερά του Boltzmann. Στην ανωτέρω σχέση αντικαταστήσαμε $\sum_{N=1}^{\infty} e^{-N\beta\hbar\omega} = 1 - e^{-\beta\hbar\omega}$ αφού πρόκειται για το άθροισμα όλων των όρων της ακολουθίας $\{\alpha_\nu\}$ με $\alpha_0 = 1$ και κοινό λόγο $r = e^{-\beta\hbar\omega}$ με $|r| < 1$.

Ο πρώτος όρος είναι η συνεισφορά του κενού χώρου καθώς ο δεύτερος είναι η θερμική συνεισφορά η οποία θα οδηγήσει στις θερμικές δυνάμεις. Γράφουμε τον δεύτερο όρο ως: $f_T(\omega) = \beta^{-1}\phi(\beta\hbar\omega) = \beta^{-1}\ln\Big(1-e^{-\beta\hbar\omega}\Big)$.

Οπότε, αθροίζοντας για όλα τα επιτρεπόμενα ω, βρίσκουμε τη θερμική συνεισφορά της Η/Μ ακτινοβολίας στην ελεύθερη ενέργεια η οποία είναι:

$$\mathcal{F}_T = 2\frac{\mathcal{A}}{\pi^2}\frac{\pi}{4}{\sum_n}' \int_{\boldsymbol{k}} f_T\left(c\sqrt{\frac{\pi^2 n^2}{L^2} + k_\rho^2}\,\right) dk_\rho^2. \tag{2.8}$$

Κάνοντας αλλαγή μεταβλητών από το k στο ω παίρνουμε την έκφραση:

$$\mathcal{F}_T = 2\frac{\mathcal{A}}{2\pi\beta}c^{-2}{\sum_n}' \int_{\frac{\pi n c}{L}}^{\infty} \phi(\beta\hbar\omega)\,\omega d\omega. \tag{2.9}$$

Σε αυτό το σημείο πρέπει να αφαιρέσουμε την ελεύθερη ενέργεια απουσία των πλακών. Αυτό θα προκύψει παίρνοντας και εδώ το όριο $L \to \infty$, το οποίο οδηγεί στη σχέση:

$$\mathcal{F}_T(\infty) = -\frac{\mathcal{A}L}{\beta}\frac{1}{(\beta\hbar c)^3}\frac{\pi^2}{45}. \qquad (2.10)$$

Η συνολική λοιπόν θερμική δύναμη δίνεται από τη σχέση (θέτουμε την αδιάστατη μεταβλητή $\beta\hbar\omega = \alpha$):

$$F_T(L) = -\frac{\partial(E_T(L) - E_T(\infty))}{\partial L}$$
$$= (2)\cdot\frac{\mathcal{A}}{2\pi\beta L}\frac{1}{(\beta\hbar c)^2}\sum_n n^2\alpha^2\phi(n\alpha) - \frac{\mathcal{A}}{\beta}\frac{1}{(\beta\hbar c)^3}\frac{\pi^2}{45}. \qquad (2.11)$$

Στην εξίσωση (2.11) εμφανίζεται η μεταβλητή α η οποία είναι αντιστρόφως ανάλογη της θερμοκρασίας T και της απόστασης μεταξύ των πλακών L και επομένως είναι αυτή που μας ενδιαφέρει για το όριο των χαμηλών θερμοκρασιών και των μικρών αποστάσεων. Έτσι, μπορούμε να πάρουμε, ταυτόχρονα, ένα όριο για χαμηλές θερμοκρασίες και μικρές αποστάσεις για να ελέγξουμε σε αυτό το όριο το μέγεθος της θερμικής συνεισφοράς στη δύναμη. Έτσι, απαιτώντας στην (2.11) το $\alpha \gg 1$, οδηγούμαστε στη σχέση:

$$f_T(L) = -\frac{1}{\beta}\frac{1}{(\beta\hbar c)^3}\frac{\pi^2}{45} + \frac{1}{\beta}\frac{\pi}{L^3}e^{-\alpha} + \mathcal{O}\Big(e^{-2\alpha}\Big) \qquad (2.12)$$

,
όπου διαιρέσαμε με το $\mathcal{A}$ για να έχουμε τη δύναμη ανα μονάδα επιφανείας. Στην εξίσωση (2.12) ο κυρίαρχος όρος για $\alpha \gg 1$ είναι μόνο ο πρώτος καθώς οι υπόλοιποι φθίνουν εκθετικά.

Οπότε, είμαστε τώρα σε θέση να συγκρίνουμε τις δυνάμεις Casimir με τις θερμικές δυνάμεις παίρνοντας τον λόγο $\gamma = \frac{f_T}{f_0} \approx \frac{f_T^\infty}{f_0} = \frac{16\pi^4}{3\alpha^4}$, όπου πήραμε μόνο τον κυρίαρχο όρο από την θερμική συνεισφορά (εξ. (2.12)). Βλέπουμε ότι ο λόγος αυτός εξαρτάται μόνο από τη σημαντική παράμετρο α όπως προαναφέραμε[2].

Για θερμοκρασία δωματίου $T = 300^o K$ και για απόσταση $L = 500nm$ η οποία είναι μια εύλογη απόσταση για μια (ηλεκτρομηχανική) νανοδιάταξη, παίρνουμε $\alpha \simeq 48$. Έτσι, καταλήγουμε στον λόγο $\gamma \simeq 0.98 \times 10^{-4}$. Επομένως, καταλήγουμε στο εξής αξιόλογο συμπέρασμα:

Η συνεισφορά στην ολική δύναμη λόγω των Η/Μ διακυμάνσεων του κενού χώρου ειναι πολύ μεγαλύτερη από αυτήν που προέρχεται από τη θερμική ακτινοβολία, ακόμα και σε θερμοκρασία δωματίου⇒ 1) Η δύναμη Casimir μπορεί να μετρηθεί σε πειράματα και 2) η συνεισφορά των δυνάμεων αυτών πρέπει να ληφθεί υπόψιν στην κατασκευή MEMS.

[2] Αυτή η παράμετρος του προβλήματος δεν μπορεί να εξαρτάται και από κάτι άλλο αφού μόνο τα L και T είναι τα αυθαίρετα μεγέθη του προβλήματός μας.

Κεφάλαιο 3

Η/Μ Φαινόμενο Casimir σε Τέλεια Σφαιρική Κοιλότητα

Σε αυτό το κεφάλαιο θα μελετήσουμε μια τέλεια σφαιρική αγώγιμη μπάλα ακτίνας a, στον κενό χώρο. Σε αυτή τη περίπτωση, λόγω της σφαιρικής γεωμετρίας, εισάγονται κάποια επιπλέον μαθηματικά εργαλεία, τα οποία παρουσιάζονται συνοπτικά παρακάτω. Για μια πιο αναλυτική παρουσίαση μπορεί ο αναγνώστης να ανατρέξει στα Παραρτήματα του κυρίου Παραρτήματος.

Αρχικά, γράφουμε τις εξισώσεις Maxwell σε δυαδική μορφή. Υποθέτουμε τρεις διπολικές κατανομές $\boldsymbol{P}_j(\boldsymbol{r},t), \;\; j=(x_1,x_2,x_3)$. Οπότε οι εξισώσεις του Maxwell μπορούν να γραφούν στη μορφή[1]:

$$\boldsymbol{\nabla}\times\boldsymbol{E}_j = i\omega\boldsymbol{H}_j \tag{3.1α′}$$
$$\boldsymbol{\nabla}\times\boldsymbol{H}_j = -i\omega(4\pi\boldsymbol{P}_j+\boldsymbol{E}_j) \tag{3.1β′}$$
$$\boldsymbol{\nabla}\cdot(\boldsymbol{E}_j+4\pi\boldsymbol{P}_j) = 0 \tag{3.1γ′}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{H}_j = 0. \tag{3.1δ′}$$

όπου $\boldsymbol{P}_j$ είναι η διπολική κατανομή κατά τη κατεύθυνση j και $\boldsymbol{E}_j$, $\boldsymbol{H}_j$ είναι η ένταση του ηλεκτρικού και μαγνητικού πεδίου, αντίστοιχα, που δημιουργούνται από αυτή τη κατανομή.

Τοποθετόντας ένα μοναδιαίο διάνυσμα $\hat{x}_j$ στα δεξιά και των δύο μελών αυτών των εξισώσεων και αθροίζοντας τα τρια σετ των εξισώσεων για κάθε j, λαμβάνουμε τις εξισώσεις Maxwell σε δυαδική μορφή η οποία είναι:

$$\boldsymbol{\nabla}\times\overline{\boldsymbol{E}} = i\omega\overline{\boldsymbol{H}} \tag{3.2α′}$$
$$\boldsymbol{\nabla}\times\overline{\boldsymbol{H}} = -i\omega(4\pi\overline{\boldsymbol{P}}+\overline{\boldsymbol{E}}) \tag{3.2β′}$$
$$\boldsymbol{\nabla}\cdot(\overline{\boldsymbol{E}}+4\pi\overline{\boldsymbol{P}}) = 0 \tag{3.2γ′}$$
$$\boldsymbol{\nabla}\cdot\overline{\boldsymbol{H}} = 0, \tag{3.2δ′}$$

όπου εξ′ ορισμού έχουμε:

[1] Εδώ, χρησιμοποιούμε μονάδες Gauss και θέτουμε $\hbar=c=1$ τα οποία θα αποκαταστήσουμε στο τελικό αποτέλεσμα.

$$\overline{\boldsymbol{E}} = \sum_j \boldsymbol{E}_j \hat{x}_j = \sum_{ij} E_{ij} \hat{x}_i \hat{x}_j \tag{3.3α′}$$

$$\overline{\boldsymbol{H}} = \sum_j \boldsymbol{H}_j \hat{x}_j = \sum_{ij} H_{ij} \hat{x}_i \hat{x}_j \tag{3.3β′}$$

$$\overline{\boldsymbol{P}} = \sum_j \boldsymbol{P}_j \hat{x}_j = \sum_{ij} P_{ij} \hat{x}_i \hat{x}_j . \tag{3.3γ′}$$

με E_{ij} να είναι το πεδίο που παράγεται κατά τον άξονα $\hat{x}_i$ εξάιτίας της κατανομής P_j.

Αντί να λύσουμε τις (3.3), λύνουμε τις:

$$\begin{aligned}
&\boldsymbol{\nabla} \times \overline{\boldsymbol{\Gamma}} = i\omega \overline{\boldsymbol{\Phi}} \\
&\boldsymbol{\nabla} \times \overline{\boldsymbol{\Phi}} = -i\omega\left(4\pi\overline{\mathbf{1}}\delta(\boldsymbol{r}-\boldsymbol{r'}) + \overline{\boldsymbol{\Gamma}}\right) \\
&\boldsymbol{\nabla} \cdot \left(\overline{\boldsymbol{\Gamma}} + 4\pi\overline{\mathbf{1}}\delta(\boldsymbol{r}-\boldsymbol{r'})\right) = 0 \\
&\boldsymbol{\nabla} \cdot \overline{\boldsymbol{\Phi}} = 0,
\end{aligned} \tag{3.4}$$

όπου $\overline{\boldsymbol{\Gamma}}$ και $\overline{\boldsymbol{\Phi}}$ είναι οι **δυαδικές συναρτήσεις Green.**

Γενικά, οι συναρτήσεις Green δίνουν το πεδίο που παράγεται σε κάποιο σημείο $\boldsymbol{r}$ εξάιτίας μια συγκεντρωμένης πηγής στο σημείο $\boldsymbol{r}'$. Εδώ, που η πηγή έχει και κάποια κατεύθυνση κατά τον j άξονα, χρησιμοποιούμε τις δυαδικές συναρτήσεις Green οι οποίες μας δίνουν το πεδίο σε κάποιο τυχόν σημείο $\boldsymbol{r}$ εξ′ αιτίας μια συγκεντρωμένης πηγής στο σημείο $\boldsymbol{r}'$ κατά τη τυχούσα κατεύθυνση.

Η μέθοδος του Green μας λέει ότι αν η κατανομή είναι γνωστή, τότε τα $\boldsymbol{E}$ & $\boldsymbol{B}$ προκύπτουν από τις σχέσεις:

$$\begin{aligned}
\boldsymbol{E}(\boldsymbol{r},t) &= \int \overline{\boldsymbol{\Gamma}}(\boldsymbol{r},\boldsymbol{r'},\tau) \cdot \boldsymbol{P}(\boldsymbol{r}',t')\, dv' \\
\boldsymbol{H}(\boldsymbol{r},t) &= \int \overline{\boldsymbol{\Phi}}(\boldsymbol{r},\boldsymbol{r'},\tau) \cdot \boldsymbol{P}(\boldsymbol{r}',t')\, dv', \quad \tau = t - t'.
\end{aligned} \tag{3.5}$$

<u>Επιπροσθέτως</u>, χρησιμοποιούνται οι διανυσματικές σφαιρικές αρμονικές οι οποίες αποτελούν διανυσματικές συναρτήσεις οι οποίες μπορούν να χρησιμοποιηθούν ως βάση ανάπτυξης των $\boldsymbol{E}$ & $\boldsymbol{H}$. Αυτές ορίζονται ως εξής:

$$\boldsymbol{X}_{lm} = \frac{1}{\sqrt{l(l+1)}} \boldsymbol{L} Y_{lm}(\theta,\phi), \tag{3.6}$$

όπου $\boldsymbol{L}$ είναι ο τελεστης της στροφορμής που δίνεται από τη σχέση:

$$\boldsymbol{L} = -i\boldsymbol{r} \times \boldsymbol{\nabla}.$$

Οι διανυσματικές σφαιρικές αρμονικές αποτελούν μια ορθοκανονική βάση, με σχέσεις ορθογωνιότητας:

$$\int d\Omega \boldsymbol{X}^*_{l'm'} \cdot \boldsymbol{X}_{lm} = \delta_{ll'}\delta_{mm'}$$

και σχέση πληρότητας:

$$\sum_{m=-l}^{l} |\boldsymbol{X}_{lm}(\theta,\phi)|^2 = \frac{2l+1}{4\pi}.$$

Έτσι, αναπτύσσουμε το ηλεκτρικό και μαγνητικό πεδίο ως εξής:

$$\begin{aligned}
\overline{\boldsymbol{\Gamma}} &= \sum_{l,m}\Big(\boldsymbol{f}_l(r,\boldsymbol{r}')\boldsymbol{X}_{lm}(\theta,\phi) + \frac{i}{\omega}\boldsymbol{\nabla}\times\boldsymbol{g}_l(r,\boldsymbol{r}')\boldsymbol{X}_{lm}(\theta,\phi)\Big)\\
\overline{\boldsymbol{\Phi}} &= \sum_{l,m}\Big(\tilde{\boldsymbol{g}}_l(r,\boldsymbol{r}')\boldsymbol{X}_{lm}(\theta,\phi) - \frac{i}{\omega}\boldsymbol{\nabla}\times\tilde{\boldsymbol{f}}_l(r,\boldsymbol{r}')\boldsymbol{X}_{lm}(\theta,\phi)\Big). \qquad (3.7)
\end{aligned}$$

Οι συντελεστές $\boldsymbol{g}_l$ και $\boldsymbol{f}_l$ ικανοποιούν τις κατωτέρω σχέσεις (για λεπτομερέστερα βλέπε κύριο παράρτημα):

$$\begin{aligned}
(D_l+\omega^2)\tilde{\boldsymbol{g}}_l &= 4\pi i\omega\int d\Omega''\boldsymbol{X}^*_{lm}(\Omega'')\cdot[\boldsymbol{\nabla}''\times\overline{\mathbf{1}}\delta(\boldsymbol{r}''-\boldsymbol{r}')]\\
(D_l+\omega^2)\boldsymbol{f}_l &= -4\pi\int d\Omega''\boldsymbol{X}^*_{lm}(\Omega'')\cdot\boldsymbol{\nabla}''\times[\boldsymbol{\nabla}''\times\overline{\mathbf{1}}\delta(\boldsymbol{r}''-\boldsymbol{r}')] \qquad (3.8)
\end{aligned}$$

με

$$\begin{aligned}
\tilde{\boldsymbol{f}}_l(r,\boldsymbol{r}') &= \boldsymbol{f}_l - \frac{4\pi}{r^2}\delta(r-r')\boldsymbol{X}_{lm}(\Omega')\\
\tilde{\boldsymbol{g}}_l(r,\boldsymbol{r}') &= \boldsymbol{g}_l, \qquad (3.9)
\end{aligned}$$

όπου ο διαφορικός τελεστής D_l είναι:

$$D_l = \frac{1}{r}\frac{d^2}{dr^2}r - \frac{l(l+1)}{r^2}.$$

Σε αυτό το σημείο μπορούμε να εκφράσουμε τις $\boldsymbol{f}_l(r,\boldsymbol{r}')$ και $\boldsymbol{g}_l(r,\boldsymbol{r}')$ συναρτήσει βαθμωτών συναρτήσεων Green, $F_l(r,r')$ και $G_l(r,r')$ αντίστοιχα, οι οποίες θα ικανοποιούν τις σχέσεις:

$$(D_l+\omega^2)\Delta_l(r,r') = -\frac{1}{r^2}\delta(r-r'),. \qquad (3.10)$$

με γενική λύση της μορφής:

$$\Delta_l(r,r') = \begin{cases} A_1 j_l(kr), & 0 \le r < r' \\ A_2 h_l^{(1)}(kr) + B_2 j_l(kr), & r' < r \le a, \end{cases}$$

για το εσωτερικό της σφαίρας η οποία ικανοποιεί τη συνθήκη ότι στο κέντρο της σφαίρας δεν είναι ανώμαλη. Για το κενό η αντίστοιχη λύση είναι:

$$\Delta_l^0(r,r') = ikj_l(kr_<)h_l^{(1)}(kr_>).$$

Έτσι, λύνοντας την (3.10) παρουσία και απουσία του σφαιρικού φλοιού και εφαρμόζοντας τις κατάλληλες συνοριακές συνθήκες παίρνουμε τις λύσεις:

$$r,r' < a : \left\{ \begin{array}{c} \tilde{G}_l \\ \tilde{F}_l \end{array} \right\} = -A_{G,F}\, ikj_l(kr)j_l(kr') \tag{3.11α'}$$

$$r,r' > a : \left\{ \begin{array}{c} \tilde{G}_l \\ \tilde{F}_l \end{array} \right\} = -B_{G,F}\, ikh_l^{(1)}(kr)h_l^{(1)}(kr'). \tag{3.11β'}$$

όπου:

$$\left\{ \begin{array}{c} \tilde{G}_l \\ \tilde{F}_l \end{array} \right\} = \left\{ \begin{array}{c} G_l \\ F_l \end{array} \right\} - \underbrace{\left\{ \begin{array}{c} G_l^0 \\ F_l^0 \end{array} \right\}}_{\text{κενό}}.$$

Οι συντελεστές $A_{G,F}$ και $B_{G,F}$ βρίσκονται εφαρμόζοντας τις συνοριακές σχέσεις πάνω στην αγώγιμη σφαίρα οι οποίες οδηγούν στις:

$$F_l(a,r') = 0 \tag{3.12α'}$$

$$\frac{d}{dr}[rG_l(r,r')]_{r=a} = 0. \tag{3.12β'}$$

και υπολογίζονται:

$$A_F = h_l^{(1)}(ka)/j_l(ka) = B_F^{-1}, \tag{3.13}$$

και:

$$A_G = [kah_l^{(1)}(ka)]'/[kaj_l(ka)]' = B_G^{-1}, \tag{3.14}$$

όπου ο τόνος υποδηλώνει παραγώγιση ως προς ka.

Επομένως, η δυαδική συνάρτηση Green για το ηλεκτρικό πεδίο, μπορεί να γραφεί ως:

$$\tilde{\overline{\boldsymbol{\Gamma}}}(\boldsymbol{r},\boldsymbol{r}',\omega)=4\pi\sum_{lm}\Big\{\omega^2\tilde{F}_l(r,r')\boldsymbol{X}_{lm}(\Omega)\boldsymbol{X}^*_{lm}(\Omega')$$
$$-\boldsymbol{\nabla}\times[\tilde{G}_l(r,r')\boldsymbol{X}_{lm}(\Omega)\boldsymbol{X}^*_{lm}(\Omega')]\times\boldsymbol{\nabla}'\Big\}$$
$$+4\pi\frac{1}{r^2}\delta(r-r')\sum_{lm}\boldsymbol{X}_{lm}(\Omega)\boldsymbol{X}^*_{lm}(\Omega')-4\pi\overline{\mathbf{1}}\delta(r-r'). \tag{3.15}$$

Η Η/Μ ενεργειακή πυκνότητα γράφεται ως:

$$u=\frac{1}{8\pi}\Big(\boldsymbol{E}^2+\boldsymbol{H}^2\Big). \tag{3.16}$$

Μπορούμε να βρούμε τις αναμενόμενες τιμές για το κενό υπολογίζοντας τις σχέσεις:

$$\Big\langle E_j(\boldsymbol{r})E_k(\boldsymbol{r}')\Big\rangle_0=\frac{1}{i}\Gamma_{jk}(\boldsymbol{r},\boldsymbol{r}') \tag{3.17α'}$$
$$\Big\langle H_j(\boldsymbol{r})H_k(\boldsymbol{r}')\Big\rangle_0=\Big(-\frac{1}{i}\frac{1}{\omega^2}\boldsymbol{\nabla}\times\overline{\boldsymbol{\Gamma}}(\boldsymbol{r},\boldsymbol{r}')\times\overleftarrow{\boldsymbol{\nabla}}\Big)_{jk}$$
$$=\epsilon_{jlm}\epsilon_{knp}\frac{1}{i}\frac{1}{\omega^2}\nabla_l\nabla'_n\Gamma_{mp}(\boldsymbol{r},\boldsymbol{r}',\omega), \tag{3.17β'}$$

όπου ο τελεστής $\overleftarrow{\boldsymbol{\nabla}}=({}_x\partial,{}_y\partial,{}_z\partial)$ δρα από τα αριστερά. Αντικαθιστώντας, και ολοκληρώνοντας για όλες τις συχνότητες και σε όλο το χώρο, μπορεί να δειχθεί ότι η ενέργεια παίρνει τη μορφή[2]:

$$E=\int(d\boldsymbol{r})\frac{1}{2i}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\sum_{lm}\Bigg\{k^2[\tilde{F}_l(r,r')+\tilde{G}_l(r,r')]\boldsymbol{X}_{lm}(\Omega)\cdot\boldsymbol{X}^*_{lm}(\Omega')$$
$$-\boldsymbol{\nabla}\times\Big[\boldsymbol{X}_{lm}(\Omega)\cdot[\tilde{F}_l(r,r')+\tilde{G}_l(r,r')]\cdot\boldsymbol{X}^*_{lm}(\Omega')\Big]\times\boldsymbol{\nabla}'\Bigg\}\Bigg|_{\boldsymbol{r}=\boldsymbol{r}'}. \tag{3.18}$$

Αντικαθιστώντας τα $\tilde{F}_l$ και $\tilde{G}_l$ μπορούμε να καταλήξουμε στην εξής έκφραση (δες κύριο Παράρτημα):

$$E=\frac{i}{2a}\sum_l(2l+1)\int_{-\infty}^{\infty}\frac{d(\omega a)}{2\pi}e^{-i\omega\tau}z\Bigg\{\frac{(zj_l)'}{zj_l}+\frac{(zj_l)''}{(zj_l)'}+\frac{(zh_l^{(1)})'}{zh_l^{(1)}}+\frac{(zh_l^{(1)})''}{zh_l^{(1)})'}\Bigg\}, \tag{3.19}$$

όπου $z=ka$ και η παραγώγιση γίνεται ως προς το z.

Το ολοκλήρωμα αυτό δεν μπορεί να λυθεί αναλυτικά. Ακολουθώντας προσεγγιστικές μεθόθους μπορεί να βρεθεί ότι η ενέργεια Casimir είναι:

[2] Για αναφορές σε άρθρα και κεφάλαια βιβλίων ο αναγνώστης μπορεί να ανατρέξει στο κείμενο του κύριου Παραρτήματος.

$$E = 0.092353 \frac{\hbar c}{2a} \tag{3.20}$$

όπου αποκαταστήσαμε τα $\hbar$, c.

Επομένως, μπορεί να υπολογιστεί και η τάση (πίεση) Casimir ως εξής:

$$f(a) = -\frac{1}{4\pi a^2}\frac{\partial E}{\partial a} = \frac{0.092353\hbar c}{8\pi a^4}.$$

Η τάση αυτή είναι απωστικής φύσης και θέλει να διογκώσει τη σφαίρα, να την αποσυνθέσει.

Συμπεράσματα

- Η παρουσία συνόρων στον κενό χώρο οδηγεί στην ανάπτυξη δυνάμεων (Casimir)
- Οι δυνάμεις Casimir είναι απολύτως σημαντικές στην κατασκευή μικροδιατάξεων
- Εξαρτώνται από τη γεωμετρία των συνόρων:
  - Επίπεδες πλάκες: **ελκτική**
  - Σφαιρική κοιλότητα: **απωστική**

Παρατήρηση: Οι δυνάμεις Casimir αναπτύσσονται σε οποιοδήποτε πεδίο, π.χ. φερμιονικό, κτλ...

Σημείωση: Στο κύριο Παράρτημα παρουσιάζεται και η περίπτωση του Φερμιονικού πεδίου, όπου υπολογίζεται η φερμιονική πίεση (τάση) Casimir για την περίπτωση των επίπεδων πλακών και της σφαιρικής κοιλότητας, με τα εξής αποτελέσματα:

- για τις παράλληλες επίπεδες πλάκες:

$$f_F = -\frac{7\pi^2}{1920a^4}, \tag{3.21}$$

- για την τέλεια σφαιρική κοιλότητα:

$$T_{rr}\Big|_a = \frac{0.0204}{4\pi a^4}. \tag{3.22}$$

Μέρος II

Chapter 1

Introduction

1.1. A Brief History

In the 1940s, J.T.G. Overbeek at the Phillips Laboratory carried out experiments on suspensions of quartz powder used in manufacturing lamps and cathode ray tubes. His results seemed to diverge from the existing theory of interparticle interaction of colloid systems that he had developed with E.J.W. Verwey. He pointed out that the interaction might fall off more rapidly at relatively large interparticle distances (distances comparable to the wavelength corresponding to the atomic frequencies). He finally suggested that the divergence was due to retardation, the fact that it takes time for light to travel between two points because of its finite velocity.

Overbeek's observations and suggestion to include the retardation of light prompted H.B.G. Casimir and D. Polder [6] to reconsider van der Walls interaction. They concluded that, indeed, the interaction between particles falls off as r^{-7} rather than r^{-6} for large intermolecular separations, r, due to retardation.

The simplicity of their work, astonished Casimir who took a second look to the results in order to find a deeper understanding. The result of his research was that the same results can be reproduced manipulating changes of the, so called, zero-point energy of the field.

In 1948, he presents a paper titled “On the attraction between two perfectly conducting plates” where he associated the effect with the electromagnetic zero-point energy [5]. The paper indicated that two neutral parallel perfect conductors attract each other while classically there should be no force exerted. Casimir's attractive force (per unit area) does not depend on the nature of the plates or their charges and it seems to be universal:

$$\boxed{f(L) = -\frac{\pi^2 \hbar c}{240 L^4}}. \tag{1.1}$$

1.2. Zero-point energy

Zero-point energy is the energy of a system at temperature T=0^{o}K (absolute zero). Classically, in this limit all motion ceases, however, in a quantum mechanical system this is not the case. There is an inevitable non-zero energy even at absolute zero temperature coming as a consequence of Heisenberg's uncertainty principle.

The first appearance of the concept dates back in 1925 when Planck suggested the "second theory" of the blackbody radiation and proposed a modified expression of his original formula, given by:

$$U = \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1} + \frac{1}{2}\hbar\omega, \tag{1.2}$$

for the average energy of an oscillator of frequency ω in equilibrium with radiation of temperature T. While classical physics demand no motion and thus no energy when $T \to 0$ equation (1.2) leads to the finite result of $\frac{1}{2}\hbar\omega$.

The remaining finite result seems of no physical meaning, a fact that troubled Planck. Einstein and Stern, however, noted that the zero-point energy is a necessity in order for (1.2) to produce the classical formula $U = kT$ in the classical limit $kT \gg \hbar\omega$. In addition, strong evidence of its existence comes from the numerous experiments that have been made confirming zero-point fluctuations.

1.3. Electromagnetic zero-point energy

A distribution of electric (and magnetic) charges in space produces electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$ respectively. Moreover, as function of the electromagnetic field the Hamiltonian (energy) density of this field (in Gaussian CGS units) is expressed as:

$$\mathcal{H} = \frac{1}{8\pi}(\boldsymbol{E}^2 + \boldsymbol{B}^2), \tag{1.3}$$

and the overall field Hamiltonian (energy) is the integral of the energy density over all space:

$$H = \int \mathcal{H}\, d^3r. \tag{1.4}$$

In the sequel the classical and quantum versions of the electromagnetic field in a vacuum (absence of charges) are being developed in order to find the electromagnetic zero-point energy making use of (1.4).

1.3.1. The Classical Electromagnetic Field

The electric and magnetic fields obey Maxwell's equations which, in the absence of charges, are written as follows:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0 \tag{1.5a}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \tag{1.5b}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{1.5c}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}. \tag{1.5d}$$

One can decouple differential equations (1.5) and get the wave equations for $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{1.6a}$$

$$\nabla^2 \boldsymbol{B} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2}. \tag{1.6b}$$

For a single mode $\omega(\boldsymbol{k}) = ck$, $\boldsymbol{E}$ and $\boldsymbol{B}$ can be separated into temporal and spatial functions, $a(t)$ and $\boldsymbol{E}_o(\boldsymbol{r})$ respectively (we work only with the electric vector field and apply the same steps for the magnetic). General expressions of the electric and magnetic fields are the superposition of the mode functions over all modes. Plugging the single mode function back into eqs.(1.6) requires that:

$$a_{\boldsymbol{k}}(t) = a_{\boldsymbol{k}}(0)e^{-i\omega t}, \tag{1.7}$$

where $a_{\boldsymbol{k}}(0)$ is to be determined from initial conditions, while $\boldsymbol{E}_o(\boldsymbol{r})$ satisfy the Helmholtz equation, which is:

$$\nabla^2 \boldsymbol{E}_o + k^2 \boldsymbol{E}_o = \boldsymbol{0}. \tag{1.8}$$

Solution of (1.8) is in general complex and, by convention, $\boldsymbol{E}$ is the real part of the product $a_{\boldsymbol{k}}(t)\boldsymbol{E}_o$ given by:

$$\boldsymbol{E}(\boldsymbol{r}, t) = a^*_{\boldsymbol{k}}(t)\boldsymbol{E}^*_o(\boldsymbol{r}) + a_{\boldsymbol{k}}(t)\boldsymbol{E}_o(\boldsymbol{r}). \tag{1.9}$$

Plugging (1.9) into (1.5b) yields the magnetic field:

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{i}{k}(a^*_{\boldsymbol{k}}(t)\boldsymbol{\nabla} \times \boldsymbol{E}^*_o(\boldsymbol{r}) + a_{\boldsymbol{k}}(t)\boldsymbol{\nabla} \times \boldsymbol{E}_o(\boldsymbol{r})). \tag{1.10}$$

Solutions (1.9) and (1.10) imply that the electromagnetic field undergo harmonic oscillations at each point of the field. Wave equations (1.6) have the trivial solution $\boldsymbol{E}$, $\boldsymbol{B} = \mathbf{0}$ for the case that no electromagnetic wave exists. Therefore, the overall zero-point energy, evaluated from (1.4), for the classical electromagnetic field is zero.

Moving on, plugging the electric (1.9) and magnetic field (1.10) into (1.4) and making use of (1.5a), (1.8) one can show that the Hamiltonian of the field for a single mode is:

$$H_k = \frac{\hbar\omega}{2}(a^*_{\boldsymbol{k}} a_{\boldsymbol{k}} + a_{\boldsymbol{k}} a^*_{\boldsymbol{k}}), \tag{1.11}$$

where we made use of the normalization:

$$\int |\boldsymbol{E}_o(\boldsymbol{r})|^2 \, d^3r = 2\pi\hbar\omega, \tag{1.12}$$

without any loss of generality.

Again, (1.11) gives zero energy for each mode if the initial condition is $a_{\boldsymbol{k}}(0) = 0$ which is the case of empty space (without any electromagnetic wave).

1.3.2. The Quantized Electromagnetic Field

In the quantum description of the field states are given by vectors $|\psi\rangle$ in a Hilbert space rather than by values of the electromagnetic field. Moreover, all observable quantities, like Hamiltonian and the fields, are operators. In this spirit, $a_{\boldsymbol{k}}$ and $a^*_{\boldsymbol{k}}$ are promoted to the creation and annihilation operators $a_{\boldsymbol{k}}$ and $a^\dagger_{\boldsymbol{k}}$. They satisfy the commutation relation:

$$[a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}}] = a_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} - a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} = 1, \tag{1.13}$$

and given the dependence of $\boldsymbol{E}$ and $\boldsymbol{B}$ on $a_{\boldsymbol{k}}$ and $a^\dagger_{\boldsymbol{k}}$ the field operators $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{B}(\boldsymbol{r})$ satisfy the following commutation relation:

$$[\boldsymbol{E}_i(\boldsymbol{r}), \boldsymbol{B}_j(\boldsymbol{r})] = i\hbar\delta_{ij}\delta^3(\boldsymbol{r} - \boldsymbol{r}'). \tag{1.14}$$

Equation (1.14) gives rise to an uncertainty relation between the electric and magnetic fields meaning that the electric and magnetic fields can no longer be simoultaneously measured at the same point without uncertainty and therefore they cannot be simultaneously zero, leading, through (1.4), to an inevitable non-vanishing zero-point energy.

Given (1.13), the *Hamiltonian of a single mode* (1.11) is written as:

$$H_k = \hbar\omega\left(a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{1}{2}\right). \tag{1.15}$$

For a single mode, the eigenvalues of the Hamiltonian (1.15) are $E_N = \hbar\omega\left(N + \frac{1}{2}\right)$ where N=0,1,2,... denotes the degree of excitation of the mode $\boldsymbol{k}$ which physically is realized as the number of photons in this mode. The states of a mode $\boldsymbol{k}$ are denoted as $|n_{\boldsymbol{k}}\rangle$ where n indicates the number of photons within the state with a wavevector $\boldsymbol{k}$.

If that is true, it is clear that each mode has a lowest non-vanishing energy, $\frac{1}{2}\hbar\omega$, even when there is no measurable photon in the field, $N = 0$. In free space, all possible wave vectors $\boldsymbol{k}$ are available for the field. Equation (1.8) implies that each mode has two linearly independent, but physically identical, solutions normal to $\boldsymbol{k}$; the two independent polarizations of the wave. So, the Hamiltonian (1.15) in its complete form reads:

$$H_j(\boldsymbol{k}) = \hbar\omega(k)\left(a^\dagger_j(\boldsymbol{k}) a_j(\boldsymbol{k}) + \frac{1}{2}\right), \tag{1.16}$$

where $j = 1, 2$ are the two independent polarizations and operators $a_j(\boldsymbol{k}, t)$ and $a^\dagger_j(\boldsymbol{k}, t)$ obey the commutation relation:

$$[a_j(\boldsymbol{k}), a^\dagger_{j'}(\boldsymbol{k}')] = \delta_{jj'}\delta(\boldsymbol{k} - \boldsymbol{k}'). \tag{1.17}$$

The overall Hamiltonian in free space accounting for all modes is simply the sum of the individual mode Hamiltonians:

$$H = \sum_j \int_{\boldsymbol{k}} H_j(\boldsymbol{k})\, d^3k. \tag{1.18}$$

In order to get the vacuum energy, we simply set $n_{\boldsymbol{k},j} = 0$ for all $\boldsymbol{k}$ and $j = 1, 2$. Thus, we get:

$$\mathcal{E}_f = \langle 0|\, H\, |0\rangle = \sum_j \int_{\boldsymbol{k}} \frac{\hbar c k}{2}\, d^3k = 2 \cdot \frac{\hbar c}{2} \int_0^\infty k \frac{1}{8} 4\pi k^2\, dk, \tag{1.19}$$

where we used spherical coordinates for the integration on the (k_x, k_y, k_z) space.

This final equation states that the zero-point energy (or vacuum energy) is *infinite* which is of no physical meaning.

Nonetheless, if one disturb free space applying some boundary conditions (e.g. material boundaries, particles etc.) the system is driven to a state with different electromagnetic zero-point energy, greater or lower but (in general) also infinite. The physical system "prefers" to minimize its energy and the difference between the energies of the two states is realized as forces between the boundaries which tend to move them towards a different orientation where the system will have the lowest possible (zero-point) energy.

1.4. Fermionic zero point energy

In field theory, the electromagnetic field is said to be a part of a family of fields called *bosonic fields*. That is because its corresponding particle, the *photon*, is a boson since it carries an integer spin.

In this section, we wish to describe the zero-point energy of a *fermionic field*, that is a field whose quanta are particleswith a half-integer spin.

Particularly, we will study the so called *Dirac field* whose quanta are particles with a spin $s = 1/2$.

1.4.1. The "Classical" Dirac field

Dirac's equation can be written in the form

$$(\gamma^\mu \hat{p}_\mu - m)\psi = 0 \tag{1.20}$$

where
γ^μ are *Dirac's (or gamma) matrices*,
$\hat{p}_\mu$ the 4-momentum operator and
$\psi = \psi(x,t)$ the wavefunction which has 4 components, *ie*:

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 . \end{bmatrix} \tag{1.21}$$

The first two components of the wavefunction refer to the field's particle having two possible states: one with an up and one with a down spin, respectively, while the rest two components are for the field's antiparticle.

A usual representation of gamma matrices is

$$\gamma^0 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}, \quad \boldsymbol{\gamma} = \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix} \tag{1.22}$$

where $\boldsymbol{\sigma}$ denotes the known 2×2 *Pauli spin matrices.*

From expansion of (1.20) we have

$$\begin{aligned} (\gamma^0 \hat{p}_0 - \boldsymbol{\gamma} \cdot \boldsymbol{p} - m)\psi = 0 \Rightarrow \\ \gamma^0 \gamma^0 i\partial_t \psi - \gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{p}\psi - m\gamma^0 \psi = 0 \Rightarrow \\ i\partial_t \psi = (\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m)\psi \Rightarrow \\ i\partial_t \psi = \hat{H}\psi \end{aligned} \tag{1.23}$$

which have an identical form with Schrodinger's equation with the difference that here the Hamiltonian is given by:

$$H = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m, \qquad \boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}, \quad \beta = \gamma^0. \tag{1.24}$$

In addition, it is important to mention the relations that the matrices $\boldsymbol{a}$ and β (and thus γ^0 and $\boldsymbol{\gamma}$) obey[1]:

$$\{a_i, a_j\} = 2\delta_{ij} \tag{1.25a}$$
$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{1.25b}$$
$$\{a_i, \beta\} = 0 \tag{1.25c}$$
$$\beta^2 = 1, \tag{1.25d}$$

in which the *Minkowski metric* appears:

$$g_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}. \tag{1.26}$$

Now, we are interested in finding the solutions of the Dirac field. Given (1.23), we can write the solution in the form:

$$\psi(\boldsymbol{r}, t) = \psi(\boldsymbol{r}) e^{-iEt}, \tag{1.27}$$

where $\psi(\boldsymbol{r})$ satisfies the time-independent Dirac's equation:

$$\hat{H}\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}). \tag{1.28}$$

Since $[\hat{H}, \hat{\boldsymbol{p}}] = 0$ the eigenstates of the Hamiltonian will also be eigenstates of the momentum, $\hat{\boldsymbol{p}}$, and thus, they will have the form of plane waves, *ie*:

$$\psi(\boldsymbol{r}) = u(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{1.29}$$

and because of (1.28) we have:

$$H(\boldsymbol{k})u \equiv (\boldsymbol{\alpha} \cdot \boldsymbol{k} + \beta m)u = Eu. \tag{1.30}$$

[1]These relations can also be shown using the necessity that $\hat{H}^2 = \boldsymbol{p}^2 + m^2$.

The eigenvalues of (1.30) are given by:

$$E = +\omega_k \quad , \quad E = -\omega_k, \tag{1.31}$$

where $\omega_k = \sqrt{\boldsymbol{k}^2 + m^2}$.

It can easily be shown (e.g. [2]), that the general solution of (1.30) is written as:

$$\psi(\boldsymbol{r}) = \int \frac{d^3k}{(2\pi)^3} \sum_{s=1,2} \left[a^s_{\boldsymbol{k}} \; u^s_{\boldsymbol{k}} \; e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + b^s_{\boldsymbol{k}} \; v^s_{\boldsymbol{k}} \; e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \right], \tag{1.32}$$

in which

$u^s_{\boldsymbol{k}}$ and $v^s_{\boldsymbol{k}}$ are the (independent) solutions of (1.30) for the positives and the negatives eigenvalues respectively and

the superscript $s = 1, 2$ denotes the two possible projections of the spin ($1 \leftrightarrow +$ and $2 \leftrightarrow -$) onto the direction of the linear momentum $\boldsymbol{k}$ called *helicity*.

In our case though, we are interested for the vacuum, that is the field with no particles. Therefore, the wavefunction in this situation is clearly:

$$\psi(\boldsymbol{r}) = 0, \tag{1.33}$$

which is indeed a solution of (1.28). So, the total expected energy will be:

$$E = \int (\psi^\dagger H \psi) \; d^3r = 0. \tag{1.34}$$

Nonetheless, this is not correct as (1.30) leads to negative energies and thus this theory is problematic leading to wrong solutions. In order to overcome the theory's problems, Paul Dirac proceeded a step further quantizing the field itself leading to an extreme clarification.

1.4.2. Quantized Dirac field

The quantization of the field is to represent the field $\psi(\boldsymbol{r})$ as an operator and, consequently, the coefficients $a^s_{\boldsymbol{k}}$ and $b^s_{\boldsymbol{k}}$. The latter, alongside with their conjugates, will play the role of creation and annihilation operators of the field's particle and its associated antiparticle.

It is easy to show that the field Hamiltonian is given by the same form as the expected energy value of the state ψ, *ie*:

$$\mathcal{H} = \int (\psi^\dagger H \psi) \; d^3r. \tag{1.35}$$

Thus, owing to (1.32) we obtain:

$$\mathcal{H} = \sum_{k,s} \left(a_{\boldsymbol{k}}^{s\dagger} a_{\boldsymbol{k}}^{s} - b_{\boldsymbol{k}}^{s\dagger} b_{\boldsymbol{k}}^{s} \right) \omega_k, \tag{1.36}$$

in which the sum on $\boldsymbol{k}$ may be replaced with an integral for a free-boundary space; though, we write it this way for simplicity.

Since the Dirac field is a fermionic one, coefficients will obey anti-commutation relations, that is:

$$\{a_{\boldsymbol{k}}^{s}, a_{\boldsymbol{k}'}^{s'\dagger}\} = \delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{ss'} \tag{1.37a}$$

$$\{b_{\boldsymbol{k}}^{s}, b_{\boldsymbol{k}'}^{s'\dagger}\} = \delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{ss'}, \tag{1.37b}$$

and hence (1.36) is written as:

$$\mathcal{H} = \sum_{k,s} \left(a_{\boldsymbol{k}}^{s\dagger} a_{\boldsymbol{k}}^{s} + b_{\boldsymbol{k}}^{s} b_{\boldsymbol{k}}^{s\dagger} \right) \omega_k + \sum_{k} \omega_k. \tag{1.38}$$

It is the second term which denotes the vacuum energy of the field. Although it is infinite, it can give rise to *(Fermionic) Casimir forces* in the exactly same way as the electomagnetic vacuum energy.

Chapter 2

E/M Casimir Force Between Perfectly Conducting Parallel Plates

Consider two identical perfectly conducting plates, parallel to the yz plane, with an area $\mathcal{A} = L_y \times L_z$ and separated by a distance L from each other along the Ox axis. Since the plates are perfect conductors, the electric field must be normal to the surface, while the magnetic field must be tangential. Let $\hat{n}$ be the normal unit vector to the planes going outwards, $\boldsymbol{E}$ the electric field and $\boldsymbol{B}$ the magnetic field. Then, the boundary conditions may be expressed in the following way:

$$\hat{n} \times \boldsymbol{E} = \mathbf{0} \tag{2.1a}$$
$$\hat{n} \cdot \boldsymbol{B} = \mathbf{0}. \tag{2.1b}$$

The vacuum, into this new state, has a different zero-point energy, from that of the vacuum for the infinite space, and the difference in the energy of the two states is the Casimir energy, referred to as the potential energy of the configuration. This finite energy gives rise to an attractive Casimir force between the plates that falls off as L^{-4}.

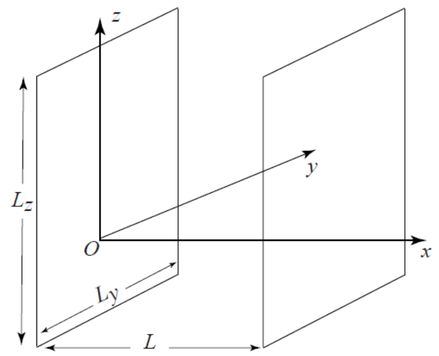

Figure 2.1: Parallel perfectly conducting plates.

2.1. Dimensional Analysis

To start with, we make use of the power of dimensional analysis. It helps us to find, from the very beginning, the form that the Casimir pressure (force per unit area) must have.

The way of working is the following: A quantum pressure due to electromagnetic fluctuations should indeed depend on $\hbar$, c and of course on the characteristic length of the problem L. To get pressure units $P \propto Nm^{-2} = Jm^{-3}$, with $\hbar \propto Js$, $c \propto ms^{-1}$ and $L \propto m$ one has to find the one and only combination that gives the correct units, and that is:

$$P_{Casimir} \propto \frac{\hbar c}{L^4}. \tag{2.2}$$

2.2. Casimir Force via Mode summation Method

Firstly, the total ground state energy is calculated for all modes and a finite plate separation L. The potential energy (Casimir energy) of the configuration is given by the difference between the ground state energy for finite L and the ground state energy for $L \to \infty$. This potential energy still diverges, however, there is a physical demand to introduce a cutoff function at large values of the wave vector k leading to a finite result and finally the Casimir force between the plates.

Boundary conditions (2.1) give rise to stationary modes that are described by wave vectors $(k_x, \boldsymbol{k})$, where $\boldsymbol{k} = (k_y, k_z)$ is the random wave vector parallel to the plates. In the longitudinal direction, boundary conditions give the discrete series: $k_x = \dfrac{\pi n}{L}$, where $n \in \mathbb{N}$, while along the plates k_y, k_z can take on any non-negative values as $L_{y,z} \to \infty$. So, the $(n, \boldsymbol{k}) = (n, k_y, k_z)$ eigenmode oscillates with a frequency:

$$\omega = \omega(n, k_y, k_z) = ck = c\sqrt{\frac{\pi^2 n^2}{L^2} + k_y^2 + k_z^2}. \tag{2.3}$$

The total zero-point energy of the cavity equals the sum $\mathcal{E}$ of the zero-point energies of all the eigenmodes:

$$\mathcal{E}(L) = 2 \cdot \frac{1}{2}\hbar \frac{L_y L_z}{\pi^2} \sum_n{}' \int_{\boldsymbol{k}} \omega(n, k_y, k_z) dk_y dk_z \tag{2.4}$$

In (2.4) the factor of 2 comes from the two different polarization modes that have the same wavenumber while the prime in the sum denotes that this factor must be excluded for the mode with $n = 0$ because then there is only one polarization; equally a factor of $\frac{1}{2}$ must be included. Moreover, the factor of $\frac{L_y L_z}{\pi^2}$ comes from the integral on k_y and k_z because their original form is $n_y \pi / L_y$ and $n_z \pi / L_z$; as $L_{y,z} \gg L$, $n_{y,z}$ are the continuous variables.

We may now, make use of cylindrical coordinates in the first quartile of (k_y, k_z)-space defining $k_\rho^2 = k_y^2 + k_z^2$. Doing so, equation (2.4) is written as:

$$\begin{aligned}\mathcal{E}(L) &= 2 \cdot \frac{1}{2}\hbar\frac{\mathcal{A}}{\pi^2}\sum_n{}' \int_0^\infty c \cdot \sqrt{\frac{\pi^2 n^2}{L^2} + k_\rho^2}\ \left(\frac{1}{4}2\pi k_\rho\right)dk_\rho \\ &= \frac{\hbar c\mathcal{A}}{2\pi}\sum_n{}' \int_0^\infty \sqrt{\frac{\pi^2 n^2}{L^2} + k_\rho^2}\ k_\rho\, dk_\rho \end{aligned} \tag{2.5}$$

Equation (2.5) gives the zero-point energy of the infinite parallel plates separated by a distance L.

Now, we turn to the case in which the plates are separated by a large distance $L \to \infty$. The sum on n becomes an integral over n because the spacing between the values of k_x becomes infinitesimally small and n can be treated as a continuous variable. After the change of variables $dn = \frac{L}{\pi}dk$, the zero-point energy of free space is:

$$\mathcal{E}(\infty) = \lim_{L\to\infty} \mathcal{E}(L) = \frac{L}{\pi}\frac{\hbar c\mathcal{A}}{2\pi}\int_0^\infty \int_0^\infty \sqrt{k_x^2 + k_\rho^2}\ k_\rho\, dk_x dk_\rho. \tag{2.6}$$

Thus, the potential energy of the configuration is given by the difference between (2.5) and (2.6), $\tilde{\mathcal{E}}_0(L) = \mathcal{E}(L) - \mathcal{E}(\infty)$, and it reads as:

$$\tilde{\mathcal{E}}_0(L) = \frac{\hbar c\mathcal{A}}{2\pi}\left(\sum_n{}' \int_0^\infty \sqrt{\frac{\pi^2 n^2}{L^2} + k_\rho^2}\ k_\rho\, dk_\rho - \frac{L}{\pi}\int_0^\infty \int_0^\infty \sqrt{k_x^2 + k_\rho^2}\ k_\rho\, dk_x dk_\rho\right). \tag{2.7}$$

At this point, (2.7) should give the finite difference between the energies, however all equations (2.5) & (2.6) and even (2.7) diverge. We can overcome this problem by making use of a physical necessity: for modes with high frequency the boundaries may not be perfectly conducting; high energy photons, $E = \hbar\omega$, probe into the atomic scale of the plates and realize that they are not perfect conductors at this scale. To take that into account we introduce a cut-off function $f(k)$ such that $f(k) = 1$ when $k \ll k_c$ and $f(k) = 0$ when $k \gg k_c$, where the cut-off value k_c may be chosen to correspond to a wavelength equal to Bohr's radius. It turns out that the selection of the cut-off function is irrelevant for the final result. Using this, equations(2.5)-(2.7) become:

$$\mathcal{E}_{reg}(L) = \frac{\hbar c \mathcal{A}}{2\pi} \sum_n{}' \int_0^\infty \sqrt{\frac{\pi^2 n^2}{L^2} + k_\rho^2} \; f\Big(\sqrt{\frac{\pi^2 n^2}{L^2} + k_\rho^2}\Big) \frac{1}{2} dk_\rho^2 \tag{2.8a}$$

$$\mathcal{E}_{reg}(\infty) = \frac{L}{\pi} \frac{\hbar c \mathcal{A}}{2\pi} \int_0^\infty \int_0^\infty \sqrt{k_x^2 + k_\rho^2}\, f\Big(\sqrt{k_x^2 + k_\rho^2}\Big) \frac{1}{2}\, dk_x dk_\rho^2 \tag{2.8b}$$

$$\tilde{\mathcal{E}}_0(L) = \mathcal{E}_{reg}(L) - \mathcal{E}_{reg}(\infty) \tag{2.8c}$$

where we have set $k_\rho dk_\rho = \frac{1}{2} dk_\rho^2$ for later convenience and the subscript *reg* notes that these are the energies after the regularization.

To calculate (2.8c) it is convenient to define new variables as follows: $\frac{\pi\zeta}{L} = k_x$ or $\zeta = \frac{k_x L}{\pi}$ and $k_\rho^2 = \frac{\pi^2 \chi}{L^2}$ or $\chi = \frac{L^2 k_\rho^2}{\pi^2}$ with their corresponding differentials being $d\zeta = \frac{L}{\pi} dk_x$ and $d\chi = \frac{L^2}{\pi^2} dk_\rho^2$. Therefore, (2.8c) is written as

$$\begin{aligned} \tilde{\mathcal{E}}_0(L) = \frac{\pi^2 \hbar c \mathcal{A}}{4L^3} \Bigg(& \sum_{\zeta \in \mathbb{N}}{}' \int_0^\infty \sqrt{\zeta^2 + \chi} f\Big(\frac{\pi}{L}\sqrt{\zeta^2 + \chi}\Big) \; d\chi \\ & - \int_0^\infty \int_0^\infty \sqrt{\zeta^2 + \chi}\, f\Big(\frac{\pi}{L}\sqrt{\zeta^2 + \chi}\Big)\; d\chi \; d\zeta \Bigg) \\ = \frac{\pi^2 \hbar c \mathcal{A}}{4L^3} \Bigg(& \sum_{N=0}^\infty{}' g(N) \; - \int_0^\infty g(\zeta)\; d\zeta \Bigg) \end{aligned} \tag{2.9}$$

where $g(\zeta) = \int_0^\infty \sqrt{\zeta^2 + \chi}\; f\Big(\frac{\pi}{L}\sqrt{\zeta^2 + \chi}\Big)\; d\chi$ is the inner integral with respect to χ. This kind of differences can be carries out using the *Euler-Maclaurin* formula which reads:

$$\sum_{N=0}^\infty{}' g(N) \; - \int_0^\infty g(\zeta)\; d\zeta = -\frac{1}{12} g'(0) + \frac{1}{6!} g'''(0) + \mathcal{O}(g^{(5)}(0)), \tag{2.10}$$

which involves all odd order derivatives of a function g which decays sufficiently fast as ζ tends to infinity. Indeed, $g(x \gg x_c) = 0$ as well as all its derivatives, because of the cut-off function. Taking derivatives with respect to ζ, one can show that $g'(0) = 0$, $g'''(0) = -4$, thus we finally get the potential energy (or Casimir energy) of the configuration of the two plates:

$$\tilde{\mathcal{E}}_0(L) = -\frac{\pi^2 \hbar c \mathcal{A}}{720 L^3}. \tag{2.11}$$

We may denote the Casimir force as $\tilde{\mathcal{X}}_0(L)$[1] and the Casimir force per unit area (Casimir pressure) as $\tilde{\mathcal{P}}_C$ and thus write:

[1]The subscript 0 in this section is to denote that the space is empty of any radiation.

$$\tilde{\mathcal{P}}_C(L) = \frac{1}{A}\tilde{\mathcal{X}}_0(L) = -\frac{1}{A}\frac{\partial \tilde{\mathcal{E}}_0}{\partial L} = -\frac{\pi^2 \hbar c}{240 L^4}, \tag{2.12}$$

which has exactly the same form with the result in (2.2). All of what we did from the beginning of this section was just to find the factor of $-\pi^2/240$. Again here, the result is independent of the material of the plates; for the derivation of the force the plates participated as geometrical boundaries that quantized the electromagnetic field without invoking their material properties. This is the advantage of Casimir's idea to manipulate with the changes of the zero-point energy, regarding only the state of the field and considering the material boundaries as mathematical-geometrical boundary conditions, instead of calculating the interaction between the molecules of the two plates.

2.3. Electromagnetic free energy at finite temperature

Although Casimir forces result from changes in the zero-point energy which is fascinating, it is also of great importance to investigate what happens if the cavity is filled with radiation. That is because the micromechanical systems used in engineering, where Casimir forces should be taken into consideration, are all at finite temperature. Hence, we wish to compare the zero-point with the thermal effect to see if one of them is much smaller so that it can be neglected.

The idea of this section is the following: In a material-free space filled with radiation of temperature T there is a thermal free energy $\mathcal{F}_T(\infty)$, while when space is disturbed putting the two parallel plates there is a change in the thermal free energy which becomes $\mathcal{F}_T(L)$. The change $\tilde{\mathcal{F}}_T = \mathcal{F}_T(L) - \mathcal{F}_T(\infty)$ give rise to thermal forces applied on the plates.

The thermal free energy $\mathcal{F}_T$ turns out that depends on a quantity $\alpha = \beta\pi\hbar/L$. Therefore taking the limit, $\alpha \gg 1$, enables us to seek for the magnidute of thermal forces compared to the ones coming from zero-point fluctuations, at low temperature or, equivalently, for short separations between the plates [?].

2.3.1. Vacuum and thermal contribution

When one says about radiation, we can think of it as a bunch of travelling photons. When the cavity is in thermal equilibrium at temperature T, is filled with photons which obey the blackbody statistics; *ie* their distribution over the frequency spectrum follows the blackbody's characteristic curve. Each electromagnetic mode of frequency ω given by (2.3) has an associated Hamiltonian operator given by (1.15) with the corresponding eigenvalues $E_N = \hbar\omega\left(N + \frac{1}{2}\right)$ where $N \in \mathbb{N}$ represents the number of photons in the mode.

The *free energy* of a single mode at a finite temperature T has the following form:

$$
\begin{aligned}
f(\omega) &= -\frac{1}{\beta}\ln\sum_{N=0}^{\infty} e^{-\beta E_N} = -\frac{1}{\beta}\ln\sum_{N=0}^{\infty} e^{-\beta\hbar\omega(N+\frac{1}{2})} \\
&= -\frac{1}{\beta}\ln e^{-\beta\frac{1}{2}\hbar\omega} - \frac{1}{\beta}\ln\sum_{N=1}^{\infty} e^{-N\beta\hbar\omega} \\
&= \frac{1}{2}\hbar\omega + \frac{1}{\beta}\ln\Big(1-e^{-\beta\hbar\omega}\Big),
\end{aligned} \tag{2.13}
$$

where $\beta = 1/k_BT$, k_B being Boltzmann's constant. In (2.13), we substituded $\sum_{N=1}^{\infty} e^{-N\beta\hbar\omega} = 1 - e^{-\beta\hbar\omega}$ since it is the infinite geometric series of the sequence $\{a_\nu\}$ with $a_o = 1$ and common ratio $r = e^{-\beta\hbar\omega}$ with $|r| < 1$.

We rewrite this equation as:

$$
f(\omega) = \epsilon_o(\omega) + f_T(\omega), \tag{2.14}
$$

with

$$
\epsilon_o(\omega) = \frac{1}{2}\hbar\omega\ ,\ f_T(\omega) = \beta^{-1}\phi(\beta\hbar\omega) = \beta^{-1}\ln\Big(1-e^{-\beta\hbar\omega}\Big), \tag{2.15}
$$

just to highlight that $\epsilon_o(\omega)$ and $f_T(\omega)$ are the zero-point and thermal contributions to the mode's free energy, respectively. We may set $\beta\hbar\omega = x$, hence we write:

$$
f_T(\omega) = \beta^{-1}\phi(x) \tag{2.16}
$$

$$
= \beta^{-1}\ln\Big(1-e^{-x}\Big) \le 0. \tag{2.17}
$$

Hence, the overall *thermal part* contribution of the electromagnetic free energy between the plates is:

$$
\mathcal{F}_T = 2\frac{\mathcal{A}}{\pi^2}\sum_n{}' \int_{\boldsymbol{k}} f_T\Bigg(c\sqrt{\frac{\pi^2n^2}{L^2}+k_y^2+k_z^2}\,\Bigg)\,dk_y dk_z, \tag{2.18}
$$

where, again, the factor of 2 comes from the two indepented states of polarisation while the factor of $\mathcal{A}/\pi^2$ from the differentials dk_y, dk_z.

It can be shown that the sum (2.18) is convergent, in contrast with the vacuum situation. Once again, we may use cylindrical coordinates; $k_y^2 + k_z^2 = k_\rho^2$ with $dk_xdk_z = \frac{1}{4}2\pi k_\rho dk_\rho = \frac{1}{4}\pi dk_\rho^2$, and obtain:

$$\mathcal{F}_T = 2\frac{\mathcal{A}}{\pi^2}\frac{\pi}{4}\sum_n{}' \int_{\boldsymbol{k}} f_T\left(c\sqrt{\frac{\pi^2 n^2}{L^2} + k_\rho^2}\,\right) dk_\rho^2. \tag{2.19}$$

Here, we make a change of variables setting:

$$\begin{cases} c^2k^2 & = c^2\sqrt{\frac{\pi^2n^2}{L^2} + k_\rho^2} = \omega^2 \\ c^2dk_\rho & = 2\omega d\omega \\ k_\rho = 0 & \Rightarrow \omega = \frac{\pi n c}{L} \\ k_\rho \to \infty & \Rightarrow \omega \to \infty, \end{cases}$$

and therefore, (2.19) is written as:

$$\begin{aligned} \mathcal{F}_T &= 2\frac{\mathcal{A}}{\pi^2}\frac{\pi}{4}2c^{-2}\sum_n{}' \int_{\boldsymbol{k}} f_T(\omega)\,\omega d\omega \\ &= 2\frac{\mathcal{A}}{2\pi\beta}c^{-2}\sum_n{}' \int_{\frac{\pi n c}{L}}^{\infty} \phi(\beta\hbar\omega)\,\omega d\omega. \end{aligned} \tag{2.20}$$

We may here introduce the integral:

$$\psi(u) = \int_u^\infty \phi(x) x dx \tag{2.21}$$

Thus, in terms of the dimensionless variable $x = \beta\hbar\omega$ and of $\psi(u)$, the free energy $\mathcal{F}_T(L)$ is obtained as a simple convergent series:

$$\mathcal{F}_T(L) = 2\frac{\mathcal{A}}{2\pi\beta}\frac{1}{(\beta\hbar c)^2}\sum_n{}'\psi(n\alpha), \quad \alpha = \beta\pi\hbar c/L. \tag{2.22}$$

2.3.2. Material-free space; the limit $L \to \infty$

In the limit $L \to \infty$ we have that $\alpha = \beta\pi\hbar c/L \ll 1$, so the series in (2.22) is given by the integral:

$$\mathcal{F}_T(L) = 2\frac{\mathcal{A}}{2\pi\beta}\frac{1}{(\beta\hbar c)^2}\frac{1}{\alpha}\int_0^\infty \psi(u)du, \tag{2.23}$$

in which:

$$\begin{aligned}\int_0^\infty \psi(u)du &= \int_0^\infty x^2\phi(x)dx = \int_0^\infty x^2 \ln\Big(1-e^{-x}\Big)dx \\ &= \frac{1}{3}\Big[x^3\ln\Big(1-e^{-x}\Big)\Big]\Big|_0^\infty - \frac{1}{3}\int_0^\infty \frac{x^3e^{-x}}{1-e^{-x}}dx \\ &= -\frac{1}{3}\int_0^\infty \frac{x^3}{e^x-1}dx. \end{aligned} \tag{2.24}$$

This integral can be evaluated using the $\zeta(u)$ and $\Gamma(u)$ functions. They obey they obey the following relation:

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx, \tag{2.25}$$

and thus, the result for (2.24) is:

$$\int_0^\infty \psi(u)du = -\frac{1}{3}\zeta(4)\Gamma(4) = -2\zeta(4) = -\frac{\pi^4}{45}.$$

Hence, the free energy in the unbounded limit is the following finite result:

$$\mathcal{F}_T(\infty) = -\frac{\mathcal{A}L}{\beta}\frac{1}{(\beta\hbar c)^3}\frac{\pi^2}{45}. \tag{2.26}$$

Now, we are in position to ask for the subtracted free energy given by the formula:

$$\tilde{\mathcal{F}}_T = \mathcal{F}_T(L) - \mathcal{F}_T(\infty). \tag{2.27}$$

Owing to (2.22) and (2.26), has the following form:

$$\tilde{\mathcal{F}}_T(L) = 2\frac{\mathcal{A}}{2\pi\beta}\frac{1}{(\beta\hbar c)^2}\Bigg[\sum_n{}' \psi(n\alpha) + \frac{2}{\alpha}\zeta(4)\Bigg]. \tag{2.28}$$

Finally, we may express the thermal force excerted on the plates, as usual, writting:

$$\tilde{\mathcal{X}}_T(L) = -\frac{\partial\tilde{\mathcal{F}}_T(L)}{\partial L} = \frac{\beta\pi\hbar c}{L^2}\frac{\partial\mathcal{F}_T}{\partial\alpha}. \tag{2.29}$$

2.4. A different point of view

In this section, we present a slightly different way of thinking on the forces acting on the plates. Until now, the forces appeared due to changes in zero and thermal free energies owing to the appearance of the two plates. These changes were a function of the distance L so they led to the appearance of a perpendicular to the plates forces, eqs (2.12) and (2.29). Nonetheless, there is an another, maybe more intuitive, way of thinking upon their presence.

Zero-point and thermal free energies for a finite distance L between the plates, (2.8a) and (2.22), are calculated taking into account only whatever happens within the cavity, the quantization of the electromagnetic fields, thus, they can be seen as the free energy inside the cavity. When the distance L varies, this free energy changes and that leads to a force acting on the plate from the inside; these forces come from the pressure of the enclosed zero-point and thermal radiation contributions. Equivalently, we may write:

$$\mathcal{X}_0(L) = -\frac{\partial \mathcal{E}_0(L)}{\partial L} \tag{2.30a}$$

$$\mathcal{X}_T(L) = -\frac{\partial \mathcal{F}_T(L)}{\partial L}. \tag{2.30b}$$

On the other hand, outside the cavity there is an infinite space and the free energy there is the same as $\mathcal{F}(L \to \infty)$. Hence, equations (2.8b) and (2.26) can be seen as the zero and thermal energies of the exterior space, respectively. Again, the changes due to variation of distance lead to a force acting on the plate from the outside; these forces come from the pressure of the external zero-point and thermal radiation. Equivalently, we may write:

$$\mathcal{X}_0(\infty) = -\frac{\partial \mathcal{E}_0(\infty)}{\partial L} \tag{2.31a}$$

$$\mathcal{X}_T(\infty) = -\frac{\partial \mathcal{F}_T(\infty)}{\partial L}. \tag{2.31b}$$

Owing to (2.22) and (2.26) and using definitions (2.30b) and (2.31b) we have:

$$\mathcal{X}_T(L)\frac{1}{\mathcal{A}} = -2\frac{1}{2\pi\beta L}\frac{1}{(\beta\hbar c)^2}\sum_n{}' n^2\alpha^2\phi(n\alpha) \tag{2.32a}$$

and

$$\mathcal{X}_T(\infty)\frac{1}{\mathcal{A}} = \frac{1}{\beta}\frac{1}{(\beta\hbar c)^3}\frac{\pi^2}{45}, \tag{2.32b}$$

which are the recovery of the black-body radiation pressures in a finite and an infinite geometry, respectively.

The total zero-point and thermal forces are given by the formulas:

$$\begin{aligned}\tilde{\mathcal{X}}_{0,T} &\equiv -\frac{\partial \tilde{\mathcal{F}}_{0,T}(L)}{\partial L} \\ &= -\Big(\frac{\partial \mathcal{F}_{0,T}(L)}{\partial L} - \frac{\partial \mathcal{F}_{0,T}(\infty)}{\partial L}\Big) \\ &= \mathcal{X}_{0,T}(L) - \mathcal{X}_{0,T}(\infty),\end{aligned} \tag{2.33}$$

where the subscripts $\{0, T\}$ denote the zero-point and thermal contributions.

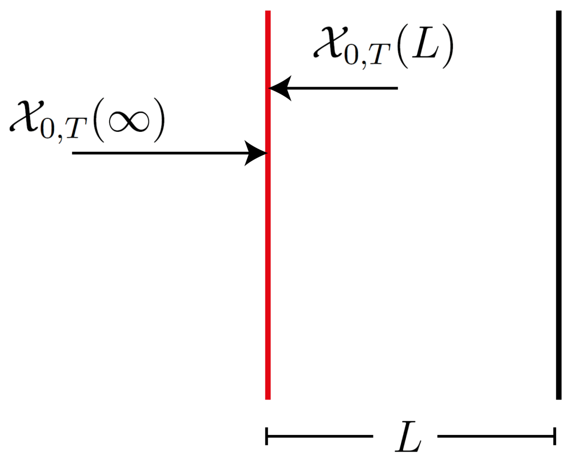

Figure 2.2: The zero-point and thermal forces acting on the plates. The exterior forces are bigger in magnitude than the interior ones leading to a total attractive force.

2.5. Low temperature or short distance expansion

The thermal forces (2.32) can be expressed in the limit of low-temperatures or equally, since they are functions of the variable $\alpha = \beta\pi\hbar c/L$, in the limit of short distances . This limit is important for applications in electromechanical nanodivices, in which the temperature is around room's temperature and the distances are extremely small. While (2.32b) remains as it is, the series (2.32a), in the limit $\alpha = \beta\pi\hbar c/L \gg 1$, has the following expansion [9]:

$$\mathcal{X}_T(L)\frac{1}{\mathcal{A}} = \frac{1}{\beta}\frac{\pi}{L^3}\left[e^{-\alpha} + \mathcal{O}\left(e^{-2\alpha}\right)\right], \quad \alpha \gg 1, \tag{2.34}$$

where we took into account that $\phi(n\alpha) = \ln(1 - e^{-n\alpha})$ behaves as $-e^{-n\alpha}$ for $\alpha \gg 1$; $n = 1$ giving the leading exponential. Therefore, the resulting thermal force at this limit is the attractive force [9]:

$$\tilde{\mathcal{X}}_T(L)\frac{1}{\mathcal{A}} = \left(\mathcal{X}_T(L) - \mathcal{X}_T(\infty)\right)\frac{1}{\mathcal{A}} = -\frac{1}{\beta}\frac{1}{(\beta\hbar c)^3}\frac{\pi^4}{45} + \frac{1}{\beta}\frac{\pi}{L^3}\left[e^{-\alpha} + \mathcal{O}\left(e^{-2\alpha}\right)\right], \quad (2.35)$$

in which minus sign (-) means that the exterior force is bigger.

2.6. Comparison of zero-point and thermal effects

As already mentioned, we are interested to compare zero-point and thermal effects at short distances. The usual length in microelectromechanical systems (MEMS) is around 500 nm. If we suppose that these microdevices operate at room temperature $T = 300$ K, then $\alpha \approx 48$. In order make the comparison, we may introduce the ratio $\gamma = \tilde{\mathcal{X}}_T/\tilde{\mathcal{X}}_0$. With $\alpha \approx 48$, the interior thermal pressure contributes only as $\mathcal{X}_T(L) \simeq 0.104 \cdot e^{-48}$ Thus, the ratio γ is given by $\gamma \simeq -\dfrac{\mathcal{X}_T(\infty)}{\tilde{\mathcal{X}}_0(L)} = \dfrac{16\pi^4}{3\alpha^4} =\approx 0.98 \times 10^{-4}$, which is fairly small.

So, we may reach to the conclution that even at room temperature, $\tilde{\mathcal{X}}_0 \gg \tilde{\mathcal{X}}_T$ meaning that the zero-point effect greatly overcome the thermal effects. This have two important consequences: first, Casimir vacuum effects can be detected in experiments and second, Casimir forces may play an important role in any micromechanical system.

2.7. Total free energy and total force [9]

Let us now write down the *total free energy* and *total for* for the plate orientation.

The total free energy is, of course, the sum of the vacuum (zero-point) and thermal free energy contributions and that is:

$$F = \tilde{\mathcal{E}}_0 + \tilde{\mathcal{F}}_T. \quad (2.36)$$

Following, the total force acting on the plates is the sum of the vacuum and thermal forces and is perpendicular to its surface, that is:

$$X = \tilde{\mathcal{X}}_0 + \tilde{\mathcal{X}}_T \quad \perp S. \quad (2.37)$$

Equally, the total force is given by the spatial derivative of the total free energy as:

$$X = -\frac{\partial F}{\partial L}. \tag{2.38}$$

Using equations (2.11) and (2.28), we have:

$$F = \mathcal{A}\frac{\pi^2\hbar c}{L^3}\left[-\frac{1}{720} + \mathcal{G}(\alpha)\right] \tag{2.39a}$$

and

$$\mathcal{G}(\alpha) = \frac{1}{\alpha^3}\left[{\sum_n}'\psi(n\alpha) + \frac{2}{\alpha}\zeta(4)\right]. \tag{2.39b}$$

Hence, we can calculate the total force (2.38) using the final free energy formulas given in (2.39).

2.7.1. Low-temperature expansion

Although we saw that in this limit the thermal contributions are of low importance, let us mention, , for the shake of completeness, the formula for the low-temperature expansion for total force.

For $\alpha \gg 1$ the function $\phi(n\alpha)$ behaves as $-e^{n\alpha}$, so the function $\psi(n\alpha)$ difined in (2.21) gives: $n = 0, \psi(0) = -\zeta(3); n = 1, \psi(\alpha) = -(\alpha+1)\left[e^{-\alpha} + \mathcal{O}\left(e^{-2\alpha}\right)\right]$ while for $n > 1$ there is a minor contribution.

Thus, finally we get the following formula:

$$F = \mathcal{A}\frac{\pi^2\hbar c}{L^3}\left\{-\frac{1}{720} + \frac{1}{\alpha^3}\left[-\frac{1}{2}\zeta(3) + \frac{2}{\alpha}\zeta(4) - (\alpha+1)[e^{-\alpha} + \mathcal{O}\left(e^{-2\alpha}\right)]\right]\right\}. \tag{2.40}$$

2.7.2. High-temperature expansion

In order to evaluate this expansion, we make use of the so called duality formula between low and high temperatures. This mathematical "trick" states the following:

$$\begin{aligned} & \alpha^2 G(a) = \alpha'^2 G(\alpha'), \quad \alpha\alpha' = (2\pi)^2 \\ \textit{or} \quad & G(\alpha) = \frac{\alpha'^2}{\alpha^2} G\left((2\pi)^2/\alpha\right) = \frac{(\alpha\alpha')^2}{\alpha^4} G\left((2\pi)^2/\alpha\right) \\ \textit{or} \quad & G(\alpha) = \left(\frac{2\pi}{\alpha}\right)^4 G\left((2\pi)^2/\alpha\right). \end{aligned} \tag{2.41}$$

Consequently, for the high-temperature limit we have:

$$F(L, \alpha) = \left(\frac{2\pi}{\alpha}\right)^4 F\left(L, (2\pi)^2/\alpha\right) \tag{2.42}$$

Substituting (2.40) and making the appopriate calculations we may write:

$$F = -\mathcal{A}\frac{\zeta(3)}{8\pi\beta L^2} + \mathcal{O}\Big(\beta^{-2}e^{-4\pi^2/\alpha}\Big), \quad \alpha \ll 1. \tag{2.43}$$

In the following we attemp to use a different way to calculate the zero-point forces via the *dyadic Green's function method.* This approach may seem more complicated but it is usefull when one treats zero-point fluctuations in more complex boundaries such as cylinders, spheres, wedges, et cetera.

Chapter 3

Dyadic Green's function Method

In this section, we introduce the dyadic Green's function method, for the case of the EM Casimir force between the parallel plates [3].

In general, a Green function $G(\boldsymbol{r}, \boldsymbol{r}')$ is the value of the field at position $\boldsymbol{r}$ produced by a source located at a position $\boldsymbol{r}'$ (for further discussion see Appendix A). In a charge-free space filled with electromagnetic waves we make use of the Green's function method untroducing as a source an infinitesimal current oscillating dipole at a position $\mathbf{r}'$.

3.1. Dyadics in Electromagnetic Theory [3, 4]

Let us first derive Maxwell's equations for dyadic Green's functions $\overline{\boldsymbol{\Gamma}}$ and $\overline{\boldsymbol{\Phi}}$. Manipulations of dyadic functions are introduced fully in Appendix B and the reader is strongly recommended to start his reading from there.

First, we may write Maxwell's equation in a dyadic form. We consider three sets of harmonically oscillating fields with the same frequency produced by three distinct, harmonically oscillating, dipole distributions $\boldsymbol{P}_j(\boldsymbol{r}, t)$ with $j = (x_1, x_2, x_3)$. Then, Maxwell's equations for these fields are written as[1]:

$$\boldsymbol{\nabla} \times \boldsymbol{E}_j = i\omega \boldsymbol{H}_j \tag{3.1a}$$
$$\boldsymbol{\nabla} \times \boldsymbol{H}_j = -i\omega(4\pi \boldsymbol{P}_j + \boldsymbol{E}_j) \tag{3.1b}$$
$$\boldsymbol{\nabla} \cdot (\boldsymbol{E}_j + 4\pi \boldsymbol{P}_j) = 0 \tag{3.1c}$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{H}_j = 0. \tag{3.1d}$$

By juxtaposing a unit vector $\hat{x}_j$ at the posterior position of these equations and summing the three sets of equations with respect to j, we obtain Maxwell's equations in dyadic form, that is

[1]Here we use Gaussian units and set $\hbar = c = 1$ which will be retrieved in the final result.

$$\nabla \times \overline{\boldsymbol{E}} = i\omega \overline{\boldsymbol{H}} \tag{3.2a}$$
$$\nabla \times \overline{\boldsymbol{H}} = -i\omega(4\pi\overline{\boldsymbol{P}} + \overline{\boldsymbol{E}}) \tag{3.2b}$$
$$\nabla \cdot (\overline{\boldsymbol{E}} + 4\pi\overline{\boldsymbol{P}}) = 0 \tag{3.2c}$$
$$\nabla \cdot \overline{\boldsymbol{H}} = 0, \tag{3.2d}$$

where by definition:

$$\overline{\boldsymbol{E}} = \sum_j \boldsymbol{E}_j \hat{x}_j = \sum_{ij} E_{ij} \hat{x}_i \hat{x}_j \tag{3.3a}$$
$$\overline{\boldsymbol{H}} = \sum_j \boldsymbol{H}_j \hat{x}_j = \sum_{ij} H_{ij} \hat{x}_i \hat{x}_j \tag{3.3b}$$
$$\overline{\boldsymbol{P}} = \sum_j \boldsymbol{P}_j \hat{x}_j = \sum_{ij} P_{ij} \hat{x}_i \hat{x}_j. \tag{3.3c}$$

E_{ij} is the field produced along the $\hat{x}_i$ direction due to the P_j distribution.

Now, we are in position to introduce dyadic Green's funtions $\overline{\boldsymbol{\Gamma}}(\boldsymbol{r}, \boldsymbol{r'}, \tau)$ and $\overline{\boldsymbol{\Phi}}(\boldsymbol{r}, \boldsymbol{r'}, \tau)$ with $\tau = t - t'$, for the electric and magnetic field values, respectively. These functions give the field that is produced at point $\boldsymbol{r}$ by a concentrated current at point $\mathbf{r}'$. The dyadic function may be represented as a 3×3 matrix having as elements $\Gamma_{ij}(\boldsymbol{r}, \boldsymbol{r'})$, $i, j = x_1, x_2, x_3$. The $\Gamma_{ij}(\boldsymbol{r}, \boldsymbol{r}')$ component represents the field value at the point $\mathbf{r}$ in the $\hat{\imath}$ direction due to a concentrated, infinitesimal, current at point $\mathbf{r}'$ along the $\hat{\jmath}$ direction.

The infinitesimal current is an oscillating dipole located at $\boldsymbol{r'}$, *ie* $\overline{\boldsymbol{P}}(\boldsymbol{r}, t) = \overline{\mathbf{1}} p(t)\delta(\boldsymbol{r} - \boldsymbol{r'})$, where $p(t)$, as already said, is an harmonically oscillating function and $\overline{\mathbf{1}}$ is the unit dyad defined by:

$$\overline{\mathbf{1}} = \hat{x}_1\hat{x}_1 + \hat{x}_2\hat{x}_2 + \hat{x}_3\hat{x}_3. \tag{3.4}$$

Inserting (3.4) into (3.2), we obtain the associated *dyadic Green's differential equations*. These are the following[2]:

$$\nabla \times \overline{\boldsymbol{\Gamma}} = i\omega \overline{\boldsymbol{\Phi}} \tag{3.5a}$$
$$\nabla \times \overline{\boldsymbol{\Phi}} = -i\omega \left(4\pi \overline{\mathbf{1}} \delta(\boldsymbol{r} - \boldsymbol{r'}) + \overline{\boldsymbol{\Gamma}} \right) \tag{3.5b}$$
$$\nabla \cdot \left(\overline{\boldsymbol{\Gamma}} + 4\pi \overline{\mathbf{1}} \delta(\boldsymbol{r} - r') \right) = 0 \tag{3.5c}$$
$$\nabla \cdot \overline{\boldsymbol{\Phi}} = 0. \tag{3.5d}$$

[2] In (3.5) and in the following text appear quantities as $\nabla \times \left[\overline{\mathbf{1}}\delta(\boldsymbol{r} - \boldsymbol{r'})\right]$. These are differentiations of dyadics which are developed in Appendix B. For example here $\nabla \times \left[\overline{\mathbf{1}}\delta(\boldsymbol{r} - \boldsymbol{r'})\right] = \left[\nabla \delta(\boldsymbol{r} - \boldsymbol{r'})\right] \times \overline{\mathbf{1}}$.

where we have used the Fourier transforms in frequency of $\overline{\mathbf{\Gamma}}$, $\overline{\mathbf{\Phi}}$ and $\overline{\mathbf{P}}$:

$$\overline{\mathbf{\Gamma}}(\boldsymbol{r}, \boldsymbol{r}'; t, t') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \overline{\mathbf{\Gamma}}(\boldsymbol{r}, \boldsymbol{r'}, \omega). \tag{3.6}$$

Thus, following Green's function method, if we solve (3.5), given the real distribution $\boldsymbol{P}(\boldsymbol{r}, t)$, we may find the solutions to (3.1), *ie* the electric and magnetic fields, through the formulas[3,4]:

$$\boldsymbol{E}(\boldsymbol{r}, t) = \int \overline{\mathbf{\Gamma}}(\boldsymbol{r}, \boldsymbol{r'}, \tau) \cdot \boldsymbol{P}(\boldsymbol{r}', t')\, dv' \tag{3.7a}$$

$$\boldsymbol{H}(\boldsymbol{r}, t) = \int \overline{\mathbf{\Phi}}(\boldsymbol{r}, \boldsymbol{r'}, \tau) \cdot \boldsymbol{P}(\boldsymbol{r}', t')\, dv'. \tag{3.7b}$$

Hence, we can find the total energy of the space using the formula (1.4) and consequently derive the potential energy and the force (per unit area) of a given configuration.

Alternatively, we can, straightforwardly, derive the force per unit area through Maxwell's electromagnetic stress tensor:

$$T_{ij} = \frac{1}{4\pi}\Big(\frac{1}{2}\delta ij \boldsymbol{E}^2 - E_i E_j + \frac{1}{2}\delta_{ij}\boldsymbol{B}^2 - B_i B_j\Big). \tag{3.8}$$

Let us now take it a step further. Here, we see that the magnetic Green dyad, $\overline{\mathbf{\Phi}}$, is solenoidal, eq.(3.5d), while the electric one is not, eq.(3.5c). Though, we can set:

$$\overline{\mathbf{\Gamma}}' = \overline{\mathbf{\Gamma}} + 4\pi \overline{\mathbf{1}} \delta(\boldsymbol{r} - \boldsymbol{r'}), \tag{3.9}$$

such that $\boldsymbol{\nabla} \cdot \overline{\mathbf{\Gamma}}' = 0$.

The system of equations (3.5) is valid in the absence of dielectrics, as we wished. Equations (3.5a) and (3.5b) can be easily converted into a second-ordered set of equations. For the latter, we take the curl on both sides and we obtain:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \overline{\mathbf{\Phi}}) = -i\omega\Big(4\pi \boldsymbol{\nabla} \times \overline{\mathbf{1}}\delta(\boldsymbol{r} - \boldsymbol{r'}) + \boldsymbol{\nabla} \times \overline{\mathbf{\Gamma}}\Big) \overset{(3.5a)}{\Longrightarrow}$$

$$\boldsymbol{\nabla}(\cancelto{0}{\boldsymbol{\nabla} \cdot \overline{\mathbf{\Phi}}}) - \nabla^2 \overline{\mathbf{\Phi}} = -i\omega\bigg(i\omega\overline{\mathbf{\Phi}} + 4\pi \boldsymbol{\nabla} \times \Big[\overline{\mathbf{1}}\delta(\boldsymbol{r} - \boldsymbol{r'})\Big]\bigg) \Rightarrow$$

[3]We have used here $\boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{B}/\mu$ instead of $\boldsymbol{B}$ because we have introduced the oscillating dipole. However, the final Maxwell's equations for $\overline{\mathbf{\Gamma}}$ and $\overline{\mathbf{\Phi}}$ will still be valid in the absence of dielectrics.

[4]Again, take a look at Appendix B for the formulas considering the dot product between a dyadic and a vector

$$(\nabla^2+\omega^2)\overline{\mathbf{\Phi}} = 4\pi i\omega \boldsymbol{\nabla}\times\Big[\overline{\mathbf{1}}\delta(\boldsymbol{r}-\boldsymbol{r'})\Big]. \tag{3.10}$$

Similarly, for the former, eq.(3.5a), making use of (3.9) we obtain:

$$\boldsymbol{\nabla}\times\overline{\mathbf{\Gamma}}' = i\omega\overline{\mathbf{\Phi}} + 4\pi\boldsymbol{\nabla}\times\Big[\overline{\mathbf{1}}\delta(\boldsymbol{r}-\boldsymbol{r'})\Big] \overset{\boldsymbol{\nabla}\times}{\Longrightarrow} \tag{3.11}$$

$$-\nabla^2\overline{\mathbf{\Gamma}}' = i\omega\boldsymbol{\nabla}\times\overline{\mathbf{\Phi}} + 4\pi\boldsymbol{\nabla}\times\Bigg[\boldsymbol{\nabla}\times\Big[\overline{\mathbf{1}}\delta(\boldsymbol{r}-\boldsymbol{r'})\Big]\Bigg] \overset{(3.5b)}{\Longrightarrow}$$

$$-\nabla^2\overline{\mathbf{\Gamma}}' = i\omega\Bigg[i\omega\Big(\overline{\mathbf{\Gamma}} + 4\pi\overline{\mathbf{1}}\delta(\boldsymbol{r}-\boldsymbol{r'})\Big)\Bigg] + 4\pi\boldsymbol{\nabla}\times\Bigg[\boldsymbol{\nabla}\times\Big[\overline{\mathbf{1}}\delta(\boldsymbol{r}-\boldsymbol{r'})\Big]\Bigg] \overset{(3.9)}{\Longrightarrow}$$

$$(\nabla^2+\omega^2)\overline{\mathbf{\Gamma}}' = -4\pi\boldsymbol{\nabla}\times\Bigg[\boldsymbol{\nabla}\times\Big[\overline{\mathbf{1}}\delta(\boldsymbol{r}-\boldsymbol{r'})\Big]\Bigg]. \tag{3.12}$$

The system of equations (3.10) and (3.12) are quite general and, as we said, valid in the absence of dielectrics.

3.2. The Casimir Force between Parallel Plates [4]

Let $(x_1, x_2, x_3) = (x, y, z)$ with the plates being perpendicular to the x-axis as shown in figure 2.1. We specialize to the case of parallel plates by introducing the transverse Fourier transform:

$$\overline{\mathbf{\Gamma}}'(\boldsymbol{r},\boldsymbol{r}',\omega) = \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\boldsymbol{k}_\perp\cdot(\boldsymbol{r}_\perp-\boldsymbol{r}'_\perp)}\overline{\boldsymbol{g}}(\boldsymbol{r},\boldsymbol{r}',\boldsymbol{k}_\parallel,\omega) \tag{3.13}$$

and

$$\delta(\boldsymbol{r}-\boldsymbol{r}') = \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\boldsymbol{k}_\perp\cdot(\boldsymbol{r}_\perp-\boldsymbol{r}'_\perp)}\delta(x-x') \tag{3.14}$$

where $\boldsymbol{k}_\perp = (k_y, k_z)$ are the wave-vectors transverse to the plates' direction. In (3.13) $\overline{\boldsymbol{g}}$, the reduced dyadic Green's function, have as components the scalar functions $g_{ij}(x, x'; \boldsymbol{k}_\parallel, \omega)$, and it is written as:

$$\overline{\boldsymbol{g}} = \sum_{ij} g_{ij}\hat{x}_i\hat{x}_j,$$

or as a tensor:

$$\overline{\boldsymbol{g}} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}.$$

These scalar functions, g_{ij}, denote the electric field of the mode $(\boldsymbol{k}_{\parallel}, \omega)$ at position $\boldsymbol{r}$ which lies along the $\hat{x}_i$ axis produced by an oscillating infinitesimal dipole at $\boldsymbol{r}'$ which lies along the $\hat{x}_j$ axis.

From (3.12) using (3.13) we can find the Cartesian components g_{ij}. Also, we have to recognize that the components of the operator $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \overline{\mathbf{1}})$ are:

$$[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \overline{\mathbf{1}})]_{ij} = [(\boldsymbol{\nabla}\boldsymbol{\nabla} - \nabla^2)\overline{\mathbf{1}}\ ]_{ij}$$

$$= \partial_i \partial_j - \delta_{ij}\partial^2, \tag{3.15}$$

where $\partial^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$. Therefore, in terms of the Fourier transform, the differential equations for the components are:

$$\left(\frac{\partial^2}{\partial x^2} - k_\perp^2 + \omega^2\right) g_{xx} = -4\pi k_\perp^2 \delta(x - x') \tag{3.16a}$$

$$\left(\frac{\partial^2}{\partial x^2} - k_\perp^2 + \omega^2\right) g_{yy} = 4\pi \left(- k_z^2 + \frac{\partial^2}{\partial x^2}\right) \delta(x - x') \tag{3.16b}$$

$$\left(\frac{\partial^2}{\partial x^2} - k_\perp^2 + \omega^2\right) g_{zz} = 4\pi \left(- k_y^2 + \frac{\partial^2}{\partial x^2}\right) \delta(x - x') \tag{3.16c}$$

$$\left(\frac{\partial^2}{\partial x^2} - k_\perp^2 + \omega^2\right) g_{xy} = -4\pi i k_y \frac{\partial}{\partial x} \delta(x - x') \tag{3.16d}$$

$$\left(\frac{\partial^2}{\partial x^2} - k_\perp^2 + \omega^2\right) g_{xz} = -4\pi i k_z \frac{\partial}{\partial x} \delta(x - x') \tag{3.16e}$$

$$\left(\frac{\partial^2}{\partial x^2} - k_\perp^2 + \omega^2\right) g_{yz} = 4\pi k_y k_z \delta(x - x'). \tag{3.16f}$$

We solve these equations subject to the boundary condition (2.1a) that the transverse components of the electric field vanish on the conducting surfaces; that for the electric dyadic applies as:

$$\hat{n} \times \overline{\boldsymbol{\Gamma}}'\Big|_{z=0,L} = 0, \tag{3.17}$$

where $\hat{n}$ is the normal to the surface going outwards. This boundary condition leads to the condition $g_{ij}(0) = 0 = g_{ij}(L)$ for any x or y component of $\overline{\boldsymbol{\Gamma}}'$.

To find g_{yz} is particularly simple. From (3.16f) into the regions $(0, x')$, (x', L) and setting $k_\perp^2 - \omega^2 = -\lambda^2$ the solutions are:

$$g_{yz}(x, x') = \begin{cases} A \sin \lambda x, & 0 \leq x < x' \leq L \\ B \sin \lambda (x - L), & 0 \leq x' < x \leq L, \end{cases} \tag{3.18}$$

which make use of the boundary conditions.

According to (3.16f), g_{yz} is continuous at $x = x'$ but its derivative has a discontinuity, that is:

$$A \sin \lambda x' - B \sin \lambda (x' - L) = 0 \tag{3.19a}$$

$$k_x A \cos \lambda x' - k_x B \cos \lambda (x' - L) = -4\pi k_y k_z, \tag{3.19b}$$

which has as solutions for A and B the following:

$$A = \frac{4\pi k_y k_z}{\lambda} \frac{\sin \lambda (x' - L)}{\sin \lambda L} \tag{3.20a}$$

$$B = \frac{4\pi k_y k_z}{\lambda} \frac{\sin \lambda x'}{\sin \lambda L}. \tag{3.20b}$$

Thus, we may simplify the solution as:

$$g_{yz}(x, x') = g_{zy}(x, x') = \frac{4\pi k_y k_z}{\lambda \sin \lambda L} \sin \lambda x_< \sin \lambda (x_> - L)$$

$$= \frac{4\pi k_y k_z}{\lambda \sin \lambda L} (ss), \tag{3.21}$$

where we have set $(ss) = \sin \lambda x_< \sin \lambda (x_> - L)$ for simplicity; $x_>$ $(x_<)$ is the greater (lesser) of x and x'.

To find g_{yy} we set $g_{yy} = g'_{yy} + 4\pi \delta(x - x')$ and hence (3.16b) becomes:

$$\left(\frac{\partial^2}{\partial x^2} + \lambda^2 \right) g'_{yy} = 4\pi (k_y^2 - \omega^2) \delta(x - x'), \tag{3.22}$$

and thus, following the same procedure as before, we finally get:

$$g'_{yy} = \frac{4\pi (k_y^2 - \omega^2)}{\lambda \sin \lambda L} (ss). \tag{3.23}$$

Similarly, for g_{zz} we have:

$$g'_{zz} = \frac{4\pi(k_z^2 - \omega^2)}{\lambda \sin \lambda L}(ss). \tag{3.24}$$

Components g_{yx}, g_{zx}, obeying (3.16d) and (3.16e), have the property that they are discontinuous at $x = x'$ while, apart from a δ-function, their derivatives are continuous. In addition with the vanishing condition at $z = \{0, L\}$ we have:

$$g_{yx} = \frac{4\pi i k_y}{\sin \lambda L}(sc) \tag{3.25a}$$

$$g_{zx} = \frac{4\pi i k_z}{\sin \lambda L}(sc), \tag{3.25b}$$

where we have set:

$$(sc) = \begin{cases} \sin \lambda x \cos \lambda(x' - L), & 0 \leq x < x' \leq L \\ \cos \lambda x' \sin \lambda(x - L), & 0 \leq x' < x \leq L. \end{cases}$$

Finally, for the normal components, g_{xx}, g_{xy} and g_{xz} the boundary condition comes from the solenoidal nature of $\overline{\mathbf{\Gamma}}'$. While all components depend on the x variable, this condition implies that:

$$\frac{\partial}{\partial x} g_{xx} = 0 \tag{3.26a}$$
$$\frac{\partial}{\partial x} g_{xy} = 0 \tag{3.26b}$$
$$\frac{\partial}{\partial x} g_{xz} = 0. \tag{3.26c}$$

For the g_{xx} component this boundary condition leads to the formula:

$$g_{xx}(x, x') = \begin{cases} A \cos \lambda x, & 0 \leq x < x' \leq L \\ B \cos \lambda(x - L), & 0 \leq x' < x \leq L, \end{cases} \tag{3.27}$$

while the matching conditions fix the integration coefficients A and B. Finally, after simple calculations similar to the ones given above, the solution is given by:

$$g_{xx} = -\frac{4\pi(k_y^2 + k_z^2)}{\lambda \sin \lambda L}(cc) = -\frac{4\pi k_\perp^2}{\lambda \sin \lambda L}(cc) \tag{3.28}$$

where $(cc) = \cos \lambda x_< \cos \lambda(x_> - L)$.

We finally conclude finding g_{xy} and g_{xz}, subjected to boundary conditions (3.26), which obey the same matching conditions as g_{yx} and g_{zx}, and thus we take:

$$g_{xy} = -\frac{4\pi i k_y}{\sin \lambda L}(cs) \tag{3.29a}$$
$$g_{xz} = -\frac{4\pi i k_z}{\sin \lambda L}(cs) \tag{3.29b}$$

where, again, we have:

$$(cs) = \begin{cases} \cos \lambda x \sin \lambda(x' - L), & 0 \leq x < x' \leq L \\ \sin \lambda x' \cos \lambda(x - L), & 0 \leq x' < x \leq L. \end{cases}$$

We easily see that $g_{xy}(x, x') = g_{yx}(x', x)$ and $g_{xz}(x, x') = g_{zx}(x', x)$ which reflects that $\Gamma_{ij}(\boldsymbol{r}, \boldsymbol{r'}) = \Gamma_{ji}(\boldsymbol{r'}, \boldsymbol{r})$.

At this point, after we have determined $\overline{\boldsymbol{\Gamma}}'$, we can find the force per unit area acting on the plates using the normal-normal component (xx-component) of Maxwell's stress tensor. Owing to (3.8), the xx-component of T_{ij} reads:

$$T_{xx} = \frac{1}{4\pi}\Big(\frac{1}{2}\boldsymbol{E}^2 + \frac{1}{2}\boldsymbol{B}^2 - E_x^2 - B_x^2\Big). \tag{3.30}$$

However, in order to find the Casimir force, a force coming from the vacuum, we have to calculate *vacuum expectation value* of T_{xx}. Let us denoted as $\left\langle T_{xx} \right\rangle_0$. We may find the vacuum expectation values of each term in (3.30) given by the following formulas:

$$\left\langle E_j(\boldsymbol{r}) E_k(\boldsymbol{r'}) \right\rangle_0 = \frac{1}{i}\Gamma_{jk}(\boldsymbol{r}, \boldsymbol{r'}) \tag{3.31a}$$
$$\begin{aligned} \left\langle H_j(\boldsymbol{r}) H_k(\boldsymbol{r'}) \right\rangle_0 &= \Big(-\frac{1}{i}\frac{1}{\omega^2}\boldsymbol{\nabla} \times \overline{\boldsymbol{\Gamma}}(\boldsymbol{r}, \boldsymbol{r'}) \times \overleftarrow{\boldsymbol{\nabla}}\Big)_{jk} \\ &= \epsilon_{jlm}\epsilon_{knp}\frac{1}{i}\frac{1}{\omega^2}\nabla_l \nabla'_n \Gamma_{mp}(\boldsymbol{r}, \boldsymbol{r'}, \omega), \end{aligned} \tag{3.31b}$$

where $\overleftarrow{\boldsymbol{\nabla}} = ({}_x\partial, {}_y\partial, {}_z\partial)$ that acts from the right.

From equation (3.30) in follows that:

$$\left\langle T_{xx} \right\rangle_0 = \frac{1}{8\pi}\Bigg(-\left\langle E_x^2 \right\rangle_0 - \left\langle B_x^2 \right\rangle_0 + \left\langle E_y^2 \right\rangle_0 + \left\langle E_z^2 \right\rangle_0 + \left\langle B_y^2 \right\rangle_0 + \left\langle B_z^2 \right\rangle_0\Bigg). \tag{3.32}$$

Now, we wish to evaluate each term of (3.32). Owing to (3.31), in terms of the Fourier transforms of the electric components Γ_{ij}, we may write:

$$\begin{aligned}\left\langle B_x^2\right\rangle &= \frac{1}{i\omega^2}\Big(\epsilon_{xyz}\epsilon_{xyz}\nabla_y\nabla'_y\Gamma_{zz} + \epsilon_{xzy}\epsilon_{xzy}\nabla_z\nabla'_z\Gamma_{yy} \\ &\qquad +\epsilon_{xyz}\epsilon_{xzy}\nabla_y\nabla'_z\Gamma_{zy} + \epsilon_{xzy}\epsilon_{xyz}\nabla_z\nabla'_y\Gamma_{yz}\Big) \\ &= \frac{1}{i\omega^2}\Big(k_y^2 g_{zz} + k_z^2 g_{yy} - k_y k_z(g_{yz} + g_{zy})\Big) \qquad (3.33)\end{aligned}$$

and similarly for the rest:

$$\left\langle E_x^2\right\rangle = \frac{1}{i}g_{xx} \qquad (3.34a)$$

$$\left\langle E_y^2\right\rangle = \frac{1}{i}g_{yy} \qquad (3.34b)$$

$$\left\langle E_z^2\right\rangle = \frac{1}{i}g_{zz} \qquad (3.34c)$$

$$\left\langle B_y^2\right\rangle = \frac{1}{i\omega^2}\Big(k_z^2 g_{xx} + \frac{\partial}{\partial x}\frac{\partial}{\partial x'}g_{zz} + ik_z\frac{\partial}{\partial x}g_{zx} - ik_z\frac{\partial}{\partial x'}g_{xz}\Big) \qquad (3.34d)$$

$$\left\langle B_z^2\right\rangle = \frac{1}{i\omega^2}\Big(\frac{\partial}{\partial x}\frac{\partial}{\partial x'}g_{yy} + k_y^2 g_{xx} + ik_y\frac{\partial}{\partial x}g_{yx} - ik_y\frac{\partial}{\partial x'}g_{xy}\Big). \qquad (3.34e)$$

These are the electric and magnetic expectation values for a single mode ω.

Here, we introduce the normal-normal stress t_{xx} owing to the mode with frequency ω. The from of t_{xx} is the same as (3.32) with the difference that $\left\langle E_i\right\rangle_0$ and $\left\langle B_j\right\rangle_0$ are the vacuum expectation values of a single mode. Thus, we have:

$$\begin{aligned}\left\langle t_{xx}\right\rangle = -\frac{1}{8\pi i\omega^2}\Big[&\omega^2 g_{xx} - \omega^2 g_{yy} - \omega^2 g_{zz} + \\ &k_y^2 g_{zz} + k_z^2 g_{yy} - k_y k_z(g_{yz} + g_{zy}) - \\ &- k_z^2 g_{xx} - \frac{\partial}{\partial x}\frac{\partial}{\partial x'}g_{zz} - ik_z\frac{\partial}{\partial x}g_{zx} + ik_z\frac{\partial}{\partial x'}g_{xz} - \\ &- \frac{\partial}{\partial x}\frac{\partial}{\partial x'}g_{yy} - k_y^2 g_{xx} - ik_y\frac{\partial}{\partial x}g_{yx} + ik_y\frac{\partial}{\partial x'}g_{xy}\Big]\end{aligned}$$

$$\begin{aligned}= \frac{1}{8\pi i\omega^2}\Big[&-(\omega^2 - k_\perp^2)g_{xx} + (\omega^2 - k_z^2)g_{yy} + (\omega^2 - k_y^2)g_{zz} \\ &-ik_y\Big(\frac{\partial}{\partial x'}g_{xy} - \frac{\partial}{\partial x}g_{yx}\Big) - ik_z\Big(\frac{\partial}{\partial x'}g_{xz} - \frac{\partial}{\partial x}g_{zx}\Big) \\ &+k_y k_z(g_{yz} + g_{zy}) + \frac{\partial}{\partial x}\frac{\partial}{\partial x'}(g_{yy} + g_{zz})\Big] \qquad (3.35)\end{aligned}$$

When the appropriate Green's functions are inserted into the above, enormous simplification occurs on the surface, and we are left with:

$$\left\langle t_{xx} \right\rangle\Big|_{z=0,L} = i\lambda \cot \lambda L. \tag{3.36}$$

Now, we can integrate over the transverse wave-vectors and the frequency and get the force per unit area. The latter integral is best done by performing a compex frequency rotation as:

$$\omega \to i\zeta, \qquad \lambda \to i\sqrt{\zeta^2 + k_\perp^2} = i\kappa. \tag{3.37}$$

So, (3.36) becomes:

$$\left\langle t_{xx} \right\rangle\Big|_{z=0,L} = -\kappa \coth(\kappa L).$$

Thus, the force per unit area is given by:

$$\mathcal{X}_0 = \int \frac{d^2k_\perp}{(2\pi)^2} \int \frac{d\zeta}{2\pi} \left\langle t_{xx} \right\rangle\Big|_{z=0,L} \tag{3.38}$$

$$= -\int \frac{d^2k_\perp}{(2\pi)^2} \int \frac{d\zeta}{2\pi} \kappa \coth \kappa L. \tag{3.39}$$

This integral does not exist. We expected that since the force given by (3.39) is the zero-point pressure from the inside environment. We must include the pressure from the external space.

Outside the cavity there are zero-point electromagnetic travelling waves. Thus, in order to find the stress on the $x = L$ plate, we must solve (3.16) for $x, x' > L$ with two boundary conditions: one at $x = L$ where the same boundary conditions for g_{ij} apply, because of the plate's perfect conductor behaviour, and one in the limit $x \to \infty$ where all the components behave as $\sim e^{i\lambda x}$. Calculations similar to the ones given above lead to the following solutions:

$$g_{yz} = g_{zy} = -\frac{4\pi k_y k_z}{\lambda} \sin \lambda(x_< - L) e^{i\lambda(x_> - L)} \tag{3.40a}$$

$$g'_{yy} = \frac{4\pi(\omega^2 - k_y^2)}{\lambda} \sin \lambda(x_< - L) e^{i\lambda(x_> - L)} \tag{3.40b}$$

$$g'_{zz} = \frac{4\pi(\omega^2 - k_z^2)}{\lambda} \sin \lambda(x_< - L) e^{i\lambda(x_> - L)} \tag{3.40c}$$

$$g_{yx} = 4\pi k_y \begin{cases} \sin \lambda(x - L) e^{i\lambda(x' - L)}, & x < x' \\ -i \cos \lambda(x' - L) e^{i\lambda(x - L)}, & x > x' \end{cases} \tag{3.40d}$$

$$g_{zx} = 4\pi k_z \begin{cases} \sin\lambda(x-L)e^{i\lambda(x'-L)}, & x < x' \\ -i\cos\lambda(x'-L)e^{i\lambda(x-L)}, & x > x' \end{cases} \tag{3.40e}$$

$$g_{xx} = \frac{4\pi i k_\perp^2}{\lambda}\cos\lambda(x_< - L)e^{i\lambda(x_> - L)} \tag{3.40f}$$

$$g_{xy} = 4\pi k_y \begin{cases} -i\cos\lambda(x-L)e^{i\lambda(x'-L)}, & x < x' \\ \sin\lambda(x'-L)e^{i\lambda(x-L)}, & x > x' \end{cases} \tag{3.40g}$$

$$g_{xz} = 4\pi k_z \begin{cases} -i\cos\lambda(x-L)e^{i\lambda(x'-L)}, & x < x' \\ \sin\lambda(x'-L)e^{i\lambda(x-L)}, & x > x'. \end{cases} \tag{3.40h}$$

Plugging these into (3.35), after trivial calculations, we obtain the result:

$$\left\langle t_{xx} \right\rangle\Big|_{x=x'=L} = \lambda. \tag{3.41}$$

So, from the difference between (3.36) and (3.41) we get the net force per unit area on the conducting surface

$$\tilde{\mathcal{X}} = -\int \frac{d^2k_\perp}{(2\pi)^2}\int \frac{d\zeta}{2\pi}\kappa(\coth\kappa L - 1) \tag{3.42}$$

Using polar coordinates we have

$$\begin{aligned} \tilde{\mathcal{X}} &= -\frac{4\pi}{(2\pi)^3}\int_0^\infty \kappa^2 d\kappa \frac{2\kappa}{e^{2\kappa L}-1} \\ &= -\frac{8\pi}{(2\pi)^3(2L)^3 2L}\int_0^\infty \frac{(2\kappa L)^3}{e^{2\kappa L}-1}d(2\kappa L) \\ &= -\frac{8\pi}{8\pi\pi^2(2L)^4}\Gamma(4)\zeta(4) \\ &= -\frac{1}{\pi^2 2^4 L^4}\cdot 6\cdot\frac{\pi^4}{90} = \frac{\pi^2}{240L^4}, \end{aligned} \tag{3.43}$$

which, after recovering $\hbar$ and c constants, is written as:

$$\tilde{\mathcal{X}} = -\frac{\pi^2\hbar c}{240L^4}, \tag{3.44}$$

.

Equation (3.44) is indeed the well known result for the case of parallel conducting plates.

Chapter 4

Electromagnetic Casimir Effect with Perfect Spherical Boundaries

In this chapter we calculate the Casimir stress on a perfectly conducting uncharged sphere of a radius a using the Green's function method. Furthermore, the temperature dependence is briefly discussed.

Historically, Casimir thought that such forces sould play the role of stresses in stabilizing the a semiclassical model of the electron. However, it turns out that, in this case, the Casimir force is *repulsive* opposed to Casimir's intuition.

More specifically, we can take as a model of the electron a perfectly conducting shell of radius a carrying the total charge e. Then, the Coulomb repulsion must be balanced by an attractive force; that is the force that Casimir postulated to be risen out of vacuum flactuations. In this case, the effective energy of the model would be

$$E = \frac{e^2}{2a} - \frac{Z}{a}\hbar c \tag{4.1}$$

where Z is a factor in Casimir's energy.

For the electron's stability we must have $E = 0$ which implies

$$Z = \frac{1}{2}\frac{e^2}{\hbar c} = \frac{1}{2}\alpha \tag{4.2}$$

in which $\alpha = e^2/\hbar c$ is the fine-structure constant.

Unfortunately, a decade later, Tim Boyer calculated that $Z = -0.04618$ so the Casimir force in this case is *repulsive* and hence Casimir's idea was wrong. Therefore, the above illustation is nothing more than an easy semiclassical calculation of the fine-structure constant.

4.1. Casimir Energy via dyadic Green's function Method [4, 7]

We start by presenting the formulation of the approprate eigenmodes (*vector spherical harmonics*) for the spherical geometry which will be the base of the electric and magnetic Green's dyadics which, however, still obey the formalism of Chapter 3.

In this chapter we make use of mathematical tools, commonly used in a spherical coordinate system, which are fully developed in Appendix C. The electromagnetic modes can be expanded by vector spherical harmonics defined by:

$$\boldsymbol{X}_{lm} = \frac{1}{\sqrt{l(l+1)}}\boldsymbol{L}Y_{lm}(\theta,\phi), \tag{4.3}$$

where $\boldsymbol{L}$ is the angular momentum operator:

$$\boldsymbol{L} = -i\boldsymbol{r}\times\boldsymbol{\nabla}. \tag{4.4}$$

The vector spherical harmonics satisfy the orthonormality condition:

$$\int d\Omega \boldsymbol{X}^*_{l'm'}\cdot\boldsymbol{X}_{lm} = \delta_{ll'}\delta_{mm'} \tag{4.5}$$

as well as the sum rule:

$$\sum_{m=-l}^{l}|\boldsymbol{X}_{lm}(\theta,\phi)|^2 = \frac{2l+1}{4\pi}. \tag{4.6}$$

Additionally, it will be useful to demonstrate the following orthogonality statements [7]:

$$\int d\Omega[f(r')\boldsymbol{X}_{l'm'}]^*\cdot[g(r)\boldsymbol{X}_{lm}] = f^*(r')g(r)\delta_{ll'}\delta_{mm'} \tag{4.7a}$$

$$\int d\Omega[f(r')\boldsymbol{X}_{l'm'}]^*\cdot[\boldsymbol{\nabla}\times g(r)\boldsymbol{X}_{lm}] = 0 \tag{4.7b}$$

$$\int d\Omega[\boldsymbol{\nabla}\times f(r')\boldsymbol{X}_{l'm'}]^*\cdot[\boldsymbol{\nabla}\times g(r)\boldsymbol{X}_{lm}] =$$
$$= \frac{1}{rr'}\Big[\frac{d}{dr'}(r'f^*(r'))\frac{d}{dr}(rg(r)) + l(l+1)f^*(r')g(r)\Big]\delta_{ll'}\delta_{mm'}. \tag{4.7c}$$

Furthermore, for an arbitary vector function, $\boldsymbol{V}(\boldsymbol{r})$, the following indentities hold [7]:

$$\frac{1}{r^2}\left[\frac{d}{dr}r + l(l+1)\right]\int d\Omega \boldsymbol{X}^*_{lm}\cdot \boldsymbol{V}(\boldsymbol{r}) = \int d\Omega[\boldsymbol{\nabla}\times \boldsymbol{X}^*_{lm}]\cdot \boldsymbol{\nabla}\times \boldsymbol{V}(\boldsymbol{r}) \tag{4.8a}$$

$$\begin{aligned} D_l \int d\Omega \boldsymbol{X}^*_{lm}\cdot \boldsymbol{V}(\boldsymbol{r}) &= \int d\Omega \boldsymbol{X}^*_{lm}\cdot [\nabla^2 \boldsymbol{V}(\boldsymbol{r})] \\ &= -\int d\Omega \boldsymbol{X}^*_{lm}\cdot [\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times \boldsymbol{V}(\boldsymbol{r}))], \end{aligned} \tag{4.8b}$$

where the radial differential operator occuring here is:

$$D_l = \frac{1}{r}\frac{d^2}{dr^2}r - \frac{l(l+1)}{r^2}. \tag{4.9}$$

Finally, it can be also shown [1] that the (divergenceless) dyadics $\overline{\boldsymbol{\Gamma}}'$ and $\overline{\boldsymbol{\Phi}}$ can be expanded in terms of the vector spherical harmonics as:

$$\overline{\boldsymbol{\Gamma}}' = \sum_{l,m}\Big(\boldsymbol{f}_l(r,\boldsymbol{r}')\boldsymbol{X}_{lm}(\theta,\phi) + \frac{i}{\omega}\boldsymbol{\nabla}\times \boldsymbol{g}_l(r,\boldsymbol{r}')\boldsymbol{X}_{lm}(\theta,\phi)\Big) \tag{4.10a}$$

$$\overline{\boldsymbol{\Phi}} = \sum_{l,m}\Big(\tilde{\boldsymbol{g}}_l(r,\boldsymbol{r}')\boldsymbol{X}_{lm}(\theta,\phi) - \frac{i}{\omega}\boldsymbol{\nabla}\times \tilde{\boldsymbol{f}}_l(r,\boldsymbol{r}')\boldsymbol{X}_{lm}(\theta,\phi)\Big), \tag{4.10b}$$

in which the radial coefficient functions are the unknown quantities to be found.

Substituting into (3.5b), for $\overline{\boldsymbol{\Gamma}}'$ and $\overline{\boldsymbol{\Phi}}$, we have:

$$\begin{aligned} \sum_{l,m}\Big[\boldsymbol{\nabla}\times \tilde{\boldsymbol{g}}_l \boldsymbol{X}_{lm} - \frac{i}{\omega}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times \tilde{\boldsymbol{f}}_l \boldsymbol{X}_{lm})\Big] &= -i\omega\sum_{l,m}\Big[\boldsymbol{f}_l \boldsymbol{X}_{lm} + \frac{i}{\omega}\boldsymbol{\nabla}\times \boldsymbol{g}_l \boldsymbol{X}_{lm}\Big] \overset{(4.8b)}{\Longrightarrow} \\ \sum_{l,m}\Big[\boldsymbol{\nabla}\times \tilde{\boldsymbol{g}}_l \boldsymbol{X}_{lm} + \frac{i}{\omega}D_l \tilde{\boldsymbol{f}}_l \boldsymbol{X}_{lm})\Big] &= \sum_{l,m}\Big[-i\omega \boldsymbol{f}_l \boldsymbol{X}_{lm} + \boldsymbol{\nabla}\times \boldsymbol{g}_l \boldsymbol{X}_{lm}\Big], \end{aligned} \tag{4.11a}$$

leading to:

$$D_l \tilde{\boldsymbol{f}}_l = -\omega^2 \boldsymbol{f}_l \tag{4.12a}$$

$$\tilde{\boldsymbol{g}}_l = \boldsymbol{g}_l. \tag{4.12b}$$

Also, equation (3.11) gives:

$$\sum_{l,m}\Big[\boldsymbol{\nabla}\times \boldsymbol{f}_l \boldsymbol{X}_{lm} - \frac{i}{\omega}D_l \boldsymbol{g}_l \boldsymbol{X}_{lm})\Big] - \sum_{l,m}\Big[-i\omega \tilde{\boldsymbol{g}}_l \boldsymbol{X}_{lm} + \boldsymbol{\nabla}\times \tilde{\boldsymbol{f}}_l \boldsymbol{X}_{lm}\Big] = 4\pi \boldsymbol{\nabla}\times \overline{\boldsymbol{1}}\delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{4.13}$$

thus, using orthonormality relations and the identity (4.8a), we are led to:

$$(D_l + \omega^2)\tilde{\boldsymbol{g}}_l = 4\pi i\omega \int d\Omega'' \boldsymbol{X}^*_{lm}(\Omega'') \cdot [\boldsymbol{\nabla}'' \times \overline{\mathbf{1}}\delta(\boldsymbol{r}'' - \boldsymbol{r}')], \tag{4.14}$$

and:

$$\begin{aligned}\int d\Omega''[\boldsymbol{\nabla} \times \boldsymbol{X}^*_{lm}] \cdot [\boldsymbol{\nabla} \times \boldsymbol{f}_l \boldsymbol{X}_{lm}] = \int d\Omega''[\boldsymbol{\nabla} \times \boldsymbol{X}^*_{lm}] \cdot [\boldsymbol{\nabla} \times \tilde{\boldsymbol{f}}_l \boldsymbol{X}_{lm}] \\ +4\pi \int d\Omega''[\boldsymbol{\nabla} \times \boldsymbol{X}^*_{lm}(\Omega'')] \cdot [\boldsymbol{\nabla}'' \times \overline{\mathbf{1}}\delta(\boldsymbol{r}'' - \boldsymbol{r}')] \Rightarrow \\ \int d\Omega'' \boldsymbol{X}^*_{lm} \cdot \boldsymbol{f}_l \boldsymbol{X}_{lm} = \int d\Omega'' \boldsymbol{X}^*_{lm} \cdot \tilde{\boldsymbol{f}}_l \boldsymbol{X}_{lm} \\ +4\pi \int d\Omega'' \boldsymbol{X}^*_{lm}(\Omega'') \cdot \overline{\mathbf{1}}\delta(\boldsymbol{r}'' - \boldsymbol{r}') \Rightarrow\end{aligned}$$

$$\tilde{\boldsymbol{f}}_l = \boldsymbol{f}_l - \frac{4\pi}{r^2}\delta(r - r')\boldsymbol{X}_{lm}(\Omega'), \tag{4.15}$$

in which the vector $\boldsymbol{r}''$ has length r and we have also used Dirac's function, $\delta(\boldsymbol{r}-\boldsymbol{r}')$, expressed in spherical coordinates defined as:

$$\delta(\boldsymbol{r} - \boldsymbol{r}') = \frac{1}{r^2 \sin\theta}\delta(r - r')\delta(\theta - \theta')\delta(\phi - \phi'). \tag{4.16}$$

In the same spirit, an equation similar to (4.14) for the coefficient function $f_l(r)$ is obtained after manipulation of (3.12), using (4.8a) and (4.8b), *ie*:

$$(D_l + \omega^2)\boldsymbol{f}_l = -4\pi \int d\Omega'' \boldsymbol{X}^*_{lm}(\Omega'') \cdot \boldsymbol{\nabla}'' \times [\boldsymbol{\nabla}'' \times \overline{\mathbf{1}}\delta(\boldsymbol{r}'' - \boldsymbol{r}')]. \tag{4.17}$$

The set of scalar equations (4.14) and (4.17) can be solved in terms of Green's functions for the differential operator D_l:

$$(D_l + \omega^2)\Delta_l(r, r') = -\frac{1}{r^2}\delta(r - r'), \tag{4.18}$$

which satisfy appropriate boundary conditions descussed below. We denote the scalar Green's functions for $\boldsymbol{f}_l$ and $\boldsymbol{g}_l$ by F_l and G_l, respectively.

The solutions to equations (4.12)-(4.17) are

$$\boldsymbol{g}_l(r) = \tilde{\boldsymbol{g}}_l(r) = -4\pi i\omega \boldsymbol{\nabla}' \times [G_l(r, r')\boldsymbol{X}^*_{lm}(\Omega')] \tag{4.19a}$$

$$\boldsymbol{f}_l(r) = 4\pi\omega^2 F_l(r, r')\boldsymbol{X}^*_{lm}(\Omega') + 4\pi\frac{1}{r^2}\delta(r - r')\boldsymbol{X}^*_{lm}(\Omega') \tag{4.19b}$$

$$\tilde{\boldsymbol{f}}_l(r) = 4\pi\omega^2 F_l(r, r')\boldsymbol{X}^*_{lm}(\Omega'), \tag{4.19c}$$

hence, the problem is now shifted on solving (4.18).

Solutions to the homogeneous equation analogous to (4.18) are spherical Bessel functions. The boundary conditions are that the solutions must be finite as $r \to 0$ and consist of outgoing spherical waves as $r \to \infty$ (see Appendix D). Additionally, maching conditions at $r = r'$ for the Δ_l Green function must be applied.

In the domain $[0, a]$ we have:

$$\Delta_l(r, r') = \begin{cases} A_1 j_l(kr), & 0 \leq r < r' \\ A_2 h_l^{(1)}(kr) + B_2 j_l(kr), & r' < r \leq a, \end{cases} \tag{4.20}$$

where only j_l appears in the first form because the solution must be finite in the origin, as mentioned. The second solution to the equation:

$$n_l = \sqrt{\frac{\pi}{2x}} N_{l+1/2}(x), \tag{4.21}$$

where N_ν is the Neumann function, is singular at $x = 0$. Moreover, the second form in (4.20) consist the general solution of (4.18).

In sequence, the scalar function Δ_l must be continuous at $r = r'$ which leads to:

$$A_1 j_l(kr') = A_2 h_l^{(1)}(kr') + B_2 j_l(kr'), \tag{4.22}$$

as well as, discontinuous at $r = r'$ according to (4.18) which gives:

$$\begin{aligned}
\frac{d}{d(kr)} \Delta_l(r, r') \Big|_{kr'_-}^{kr'_+} &= -\frac{k}{(kr')^2} \Rightarrow \\
A_2 h_l'^{(1)}(kr') + B_2 j_l'(kr') - A_1 j_l'(kr') &= -\frac{k}{(kr')^2} \overset{(4.22)}{\Rightarrow} \\
A_2 h_l'^{(1)}(kr') + B_2 j_l'(kr') - A_2 \frac{h_l^{(1)}(kr')}{j_l(kr')} j_l'(kr') - B_2 j_l'(kr') &= -\frac{k}{(kr')^2} \Rightarrow \\
A_2 \frac{h_l'^{(1)}(kr') j_l(kr') - h_l^{(1)}(kr') j_l'(kr')}{j_l(kr')} &= -\frac{k}{(kr')^2} \Rightarrow \\
A_2 \frac{W(j_l, h_l^{(1)})}{j_l(kr')} = -\frac{k}{(kr')^2} &\Rightarrow
\end{aligned}$$

$$A_2 = ikj_l(kr'), \tag{4.23}$$

where the prime denotes differentiation with respect to kr.

Hence, from (4.22) :

$$
\begin{aligned}
A_1 &= ikh_l^{(1)}(kr') + B_2(kr') \Rightarrow \\
A_1 &= ik\Big[h_l^{(1)}(kr') - Aj_l(kr')\Big],
\end{aligned}
\tag{4.24}
$$

where we have set $B_2 = B_2(kr') = -ikAj_l(kr')$, A is the constant to be determined and $k = |\omega|$.

Thus, we end up with the solution which is:

$$
\Delta_l(r, r') = \begin{cases} ikj_l(kr)\Big[h_l^{(1)}(kr') - Aj_l(kr')\Big], & 0 \leq r < r' \\ ikj_l(kr')h_l^{(1)}(kr) - ikAj_l(kr')j_l(kr), & r' < r \leq a, \end{cases}
\tag{4.25}
$$

or in a more simplified manner:

$$
\Delta_l(r, r') = ikj_l(kr_<)\Big[h_l^{(1)}(kr_>) - Aj_l(kr_>)\Big].
\tag{4.26}
$$

Similarly, the solution in the domain $[a, +\infty)$ is:

$$
\Delta_l(r, r') = ik\Big[j_l(kr_<) - Bh_l^{(1)}(kr_<)\Big]h_l^{(1)}(kr_>).
\tag{4.27}
$$

At this point, we imply the condition imposed by the perfectly conducting sphere that the tangential components of the electric field on the surface must be zero. Owing to (3.7), this condition can be expressed as follows:

$$
\hat{n} \times \overline{\boldsymbol{\Gamma}}(\boldsymbol{r}, \boldsymbol{r'})\Big|_{|\boldsymbol{r}|=a} = \boldsymbol{0},
\tag{4.28}
$$

for $|\boldsymbol{r'}| \neq a$.

Now, equation (4.10a), owing to (4.19), leads to the dyadic Green's function $\overline{\boldsymbol{\Gamma}}$ written in terms of the scalar Green's functions F_l and G_l as follows:

$$
\begin{aligned}
\overline{\boldsymbol{\Gamma}}(\boldsymbol{r}, \boldsymbol{r}', \omega) = 4\pi \sum_{lm} \Big\{ &\omega^2 F_l(r, r') \boldsymbol{X}_{lm}(\Omega) \boldsymbol{X}^*_{lm}(\Omega') \\
&- \boldsymbol{\nabla} \times [G_l(r, r') \boldsymbol{X}_{lm}(\Omega) \boldsymbol{X}^*_{lm}(\Omega')] \times \boldsymbol{\nabla}' \Big\} \\
&+ 4\pi \frac{1}{r^2} \delta(r - r') \sum_{lm} \boldsymbol{X}_{lm}(\Omega) \boldsymbol{X}^*_{lm}(\Omega') - 4\pi \overline{\boldsymbol{1}} \delta(r - r').
\end{aligned}
\tag{4.29}
$$

It can be shown [7] that the boundary condition implies the following for F_l and G_l:

$$F_l(a, r') = 0 \tag{4.30a}$$

$$\frac{d}{dr}[rG_l(r, r')]_{r=a} = 0, \tag{4.30b}$$

so that F is the TE, and G is the TM, Green's function[1].

Therefore, after simple calculations, the coefficients A and B in (4.26) and (4.27), respectiveley, are found to be:

for F_l:

$$A_F = h_l^{(1)}(ka)/j_l(ka) = B_F^{-1}, \tag{4.31}$$

and for G_l:

$$A_G = [kah_l^{(1)}(ka)]'/[kaj_l(ka)]' = B_G^{-1}, \tag{4.32}$$

where prime denotes differentiation with respect to ka.

We may write these scalar Green's function with reference to the vacuum Green's function, G_l^0, *ie*:

$$\left\{ \begin{array}{c} G_l \\ F_l \end{array} \right\} = G_l^0 + \left\{ \begin{array}{c} \tilde{G}_l \\ \tilde{F}_l \end{array} \right\}, \tag{4.33}$$

where:

$$G_l^0 = ikj_l(kr_<)h_l^{(1)}(kr_>), \tag{4.34}$$

which is the solution in empty space. In the interior and the exterior of the sphere respectively:

$$r, r' < a : \left\{ \begin{array}{c} \tilde{G}_l \\ \tilde{F}_l \end{array} \right\} = -A_{G,F}ikj_l(kr)j_l(kr') \tag{4.35a}$$

$$r, r' > a : \left\{ \begin{array}{c} \tilde{G}_l \\ \tilde{F}_l \end{array} \right\} = -B_{G,F}ikh_l^{(1)}(kr)h_l^{(1)}(kr'). \tag{4.35b}$$

Finally, we have found F_l and G_l and hence the coefficient functions $\boldsymbol{f}_l$ and $\boldsymbol{g}_l$ are now known.

[1] *Transverse electric (TE) modes:* no electric field in the direction of propagation. These are sometimes called H modes because there is only a magnetic field along the direction of propagation (H is the conventional symbol for magnetic field).
Transverse magnetic (TM) modes: no magnetic field in the direction of propagation. These are sometimes called E modes because there is only an electric field along the direction of propagation.

After all the mathematical formulation for finding $\overline{\mathbf{\Gamma}}$, we are now in position to derive an expression for the total Casimir energy of a conducting sphere of radius a. The electromagnetic energy density is:

$$u(x) = \frac{1}{8\pi}[\boldsymbol{E}(x) \cdot \boldsymbol{E}(x') + \boldsymbol{H}(x) \cdot \boldsymbol{H}(x')]_{x \to x'}, \tag{4.36}$$

We seek the change in the energy from the vacuum value of zero-point energy when the conducting sphere is introduced, thus, instead of $\overline{\mathbf{\Gamma}}$ we use the subtracted Green dyadic:

$$\tilde{\overline{\mathbf{\Gamma}}} = \overline{\mathbf{\Gamma}} - \overline{\mathbf{\Gamma}}^0, \tag{4.37}$$

which has exactly the same stucture as $\overline{\mathbf{\Gamma}}$, (4.29), with the substitutions:

$$\left\{ \begin{array}{c} G_l \\ F_l \end{array} \right\} \to \left\{ \begin{array}{c} \tilde{G}_l \\ \tilde{F}_l \end{array} \right\}. \tag{4.38}$$

Considering (3.31) and the fact that in the limit $x \to x'$ the delta functions in (4.29) do not contribute, one can find the following formula for the energy of the system:

$$\begin{aligned} E = \int (d\boldsymbol{r}) \frac{1}{2i} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \sum_{lm} \Bigg\{ & k^2 [\tilde{F}_l(r,r') + \tilde{G}_l(r,r')] \boldsymbol{X}_{lm}(\Omega) \cdot \boldsymbol{X}^*_{lm}(\Omega') \\ & - \boldsymbol{\nabla} \times \Big[\boldsymbol{X}_{lm}(\Omega) \cdot [\tilde{F}_l(r,r') + \tilde{G}_l(r,r')] \cdot \boldsymbol{X}^*_{lm}(\Omega') \Big] \times \boldsymbol{\nabla}' \Bigg\} \Bigg|_{\boldsymbol{r}=\boldsymbol{r}'} . \end{aligned} \tag{4.39}$$

We want to evaluate the result when $x = x'$. However, we put the two spatial points to coincide while leave the temporal seperation, $t' - t = \tau$, which will be set to zero in the end of our calculation and therefore serves as a kind of regulator.

First, we perform the angular integrations using (4.7a) and (4.7c). Before the calculation, it is convenient to rewrite the right-hand side of (4.7c) when $r = r'$ and f and g are spherical Bessel functions of order l as:

$$\begin{aligned} & \frac{1}{r^2}\Big[\frac{d}{dr}(rf)\frac{d}{dr}(rg) + l(l+1)fg\Big]\delta_{ll'}\delta_{mm'} \\ & = k^2 \Big\{ fg + \frac{1}{k^2 r^2} \frac{d}{dr}\Big[rf \frac{d}{dr}(rg) \Big] \Big\} \delta_{ll'}\delta_{mm'}. \end{aligned} \tag{4.40}$$

So,for $f = 1$, and $g = \tilde{F}_l(r,r') + \tilde{G}_l(r,r')$, and performing the angular integration, noting that there are $2l+1$ m values for each $l : (-l, -l+1, ..., l-1, l)$, we have:

$$E = \frac{1}{2i}\sum_l (2l+1)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\int_0^{\infty} r^2 dr\Bigg\{k^2\Big(\tilde{F}_l(r,r') + \tilde{G}_l(r,r')\Big)$$
$$+ k^2\Big(\tilde{F}_l(r,r') + \tilde{G}_l(r,r')\Big) + \frac{1}{r^2}\frac{d}{dr}r\left[\frac{d}{dr'}r'\Big(\tilde{F}_l(r,r') + \tilde{G}_l(r,r')\Big)\right]\Bigg\}\Bigg|_{r=r'} \quad (4.41)$$

Substituting $\tilde{F}_l$ and $\tilde{G}_l$ we evaluate:

$$E = -\sum_l (2l+1)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}k^3\Bigg\{\int_0^a r^2 dr(A_G + A_F)[j_l(kr)]^2$$
$$+ \int_a^{\infty} r^2 dr(B_G + B_F)[h_l^{(1)}(kr)]^2\Bigg\}$$
$$= -\sum_l (2l+1)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\Bigg[(A_G + A_F)I_1 + (B_G + B_F)I_2\Bigg]$$
$$= -\sum_l (2l+1)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}(A_G + A_F)\left[I_1 + \frac{j_l(zj_l)'}{h_l^{(1)}(zh_l^{(1)})'}I_2\right], \quad (4.42)$$

where:

$$I_1 = \int_0^a z^2[j_l(z)]^2 dz \quad (4.43a)$$
$$I_2 = \int_a^{\infty} z^2[h_l^{(1)}(z)]^2 dz, \qquad z = kr, \quad (4.43b)$$

and the integral with the derivative term in (4.41) equals zero:

$$\int_0^{\infty} r^2 dr\frac{1}{r^2}\frac{d}{dr}r\left[\frac{d}{dr'}r'\Big(\tilde{F}_l(r,r') + \tilde{G}_l(r,r')\Big)\right] =$$
$$= \int_0^{\infty} d\Bigg\{r\Big[\frac{d}{dr'}r'\Big(\tilde{F}_l(r,r') + \tilde{G}_l(r,r')\Big)\Big]\Bigg\} =$$
$$= r\Big[\frac{d}{dr'}r'\Big(\tilde{F}_l(r,r') + \tilde{G}_l(r,r')\Big)\Big]\Bigg|_0^{\infty} = 0. \quad (4.44)$$

In order to carry these integrations we denote the following identity that holds for any spherical Bessel function, that is:

$$\begin{aligned}\int z^2 dz f_l^2 &= \frac{1}{2} z^3 (f_l^2 - f_{l-1} f_{l+1}) \\ &= \frac{1}{2} z \Big\{ [(z f_l)']^2 - f_l (z f_l)' + (z^2 - l(l+1)) f_l^2 \Big\} \\ &= \frac{1}{2} z \Big\{ [(z f_l)']^2 - f_l (z f_l)' + z (z f_l)'' f_l \Big\}\end{aligned} \tag{4.45}$$

where we have used the recurrence relations:

$$(2l+1) f_l = z (f_{l-1} + f_{l+1}) \tag{4.46a}$$

$$(2l+1) f_l' = l f_{l-1} - (l+1) f_{l+1}. \tag{4.46b}$$

Thus, after explicit calculations, we are left with:

$$E = \frac{i}{2a} \sum_l (2l+1) \int_{-\infty}^{\infty} \frac{d(\omega a)}{2\pi} e^{-i\omega\tau} z \left\{ \frac{(zj_l)'}{zj_l} + \frac{(zj_l)''}{(zj_l)'} + \frac{(zh_l^{(1)})'}{zh_l^{(1)}} + \frac{(zh_l^{(1)})''}{zh_l^{(1)})'} \right\}, \tag{4.47}$$

with $z = ka$.

It is convenient to perform an Euclidean rotation (*complex frequency rotation*):

$$\begin{cases} \omega \to i k_4 \\ k \to i |k_4| \\ t - t' = \tau \to i \tau_4, \end{cases} \tag{4.48}$$

and particularly set:

$$\begin{cases} \tau / a \to i \delta \\ ka \to iy \\ |y| = x. \end{cases} \tag{4.49}$$

So, instead of (4.47) we have:

$$\begin{aligned} E &= -\frac{1}{2\pi a} \sum_{l=1}^{\infty} (2l+1) \frac{1}{2} \int_{-\infty}^{\infty} dy e^{i\delta y} x \left[\frac{s_l'}{s_l} + \frac{s_l''}{s_l'} + \frac{e_l'}{e_l} + \frac{e_l''}{e_l'} \right] \\ &= -\frac{1}{2\pi a} \sum_{l=1}^{\infty} (2l+1) \frac{1}{2} \int_{-\infty}^{\infty} dy e^{i\delta y} x \frac{d}{dx} \ln \Big(1 - \lambda_l^2 \Big), \end{aligned} \tag{4.50}$$

where $\lambda_l = [s_l(x)e_l(x)]'$ and s_l, e_l are the Ricatti-Bessel functions of a complex argument $z = ix$:

$$s_l(x) = \sqrt{\frac{\pi x}{2}} I_{l+1/2}(x) \tag{4.51a}$$

$$e_l(x) = \sqrt{\frac{2x}{\pi}} K_{l+1/2}(x), \tag{4.51b}$$

where $I_{l+1/2}$ and $K_{l+1/2}$ are modified Bessel functions:

$$I_\nu(x) = e^{-\nu\pi i/2} J_\nu(ix) \tag{4.52a}$$

$$K_\nu(x) = \frac{\pi}{2} i e^{\nu\pi i/2} H_\nu^{(1)}(ix). \tag{4.52b}$$

The second line in (4.50) is obtained using that:

$$W(e_l, s_l) = 1. \tag{4.53}$$

We, finally, have reached our purpose, that is a final formula for Casimir's energy of a conducting ball (4.50). Now, we wish to calculate this integral but, unfortunatelly, it cannot be evaluated analytically.

However, a rapidly convergent evaluation can be achieved by using the uniform asymptotic expansions for the Bessel functions:

$$s_l \sim \frac{1}{2} \frac{z^{1/2}}{(1+z^2)^{1/4}} e^{\nu\eta} \left[1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right] \tag{4.54a}$$

$$e_l \sim \frac{z^{1/2}}{(1+z^2)^{1/4}} e^{-\nu\eta} \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k u_k(t)}{\nu^k} \right], \quad l \to \infty, \tag{4.54b}$$

where:

$$x = \nu z, \quad \nu = l + 1/2, \quad t = (1+z^2)^{-1/2}, \quad \eta = t^{-1} + \ln \frac{z}{1+t^{-1}}, \tag{4.55}$$

and $u_k(t)$ are polynomials of t and of order $3k$, the first few of which are

$$u_1(t) = \frac{1}{24}(3t - 5t^3) \tag{4.56a}$$

$$u_2(t) = \frac{1}{1152}(81t^2 - 462t^4 + 285t^6) \tag{4.56b}$$

$$u_3(t) = \frac{1}{414720}(30375t^3 - 369603t^5 + 765765t^7 - 425425t^9) \tag{4.56c}$$

$$\begin{aligned} u_4(t) = \frac{1}{39813120}(&4465125t^4 - 94121676t^6 + 349922430t^8 \\ &- 446185740t^{10} + 185910725t^{12}). \end{aligned} \tag{4.56d}$$

We find that:

$$s_l(x)e_l(x) \sim \frac{z}{2}\frac{1}{(1+z^2)^{1/2}}, \tag{4.57}$$

as well as:

$$(2\nu)^2 \ln\left(1 - \lambda_l^2(x)\right) \sim -\frac{1}{(1+z^2)^3}. \tag{4.58}$$

Moreover, we may write (4.50) as:

$$E = -\frac{1}{2a}\sum_{l=1}^{\infty} J(l,\delta), \tag{4.59}$$

and consequently, for $\delta = 0$, we obtain:

$$\begin{aligned} J(l,0) &= \frac{1}{\pi}(2l+1)\frac{1}{2}\int_{-\infty}^{\infty} dx \;\; x\frac{d}{dx}\ln\Big(1 - \lambda_l^2(x)\Big) \\ &\sim -\frac{1}{\pi}\int_{-\infty}^{\infty} dz \;\; z\frac{d}{dz}\lambda_l^2(x) \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty} dz \;\; z\frac{d}{dz}\left(\frac{d}{dz}\frac{z/2}{(1+z^2)^{1/2}}\right)^2 \\ &= \frac{1}{\pi}\int_0^{\infty}\frac{3z^2}{(1+z^2)^4}dz = \frac{3}{\pi}\cdot\frac{\pi}{32} = \frac{3}{32}, \qquad l \to \infty. \end{aligned} \tag{4.60}$$

Now, we wish to find a finite sum (4.59), thus, we must keep $\delta \neq 0$ until the end of our calculation. We may , add and subtract the leading approximation to the logarithm, leading to:

$$J(l,\delta) = \frac{1}{2\pi}\int_0^\infty dz \quad z\frac{d}{dz}\left[(2l+1)^2 \ln(1-\lambda_l^2) + \frac{1}{(1+z^2)^3}\right] + S_l(\delta)$$

$$= R_l + S_l(\delta), \tag{4.61}$$

so that the first integral has a significant contribution only for small values of l and we can ignore the exponential cutoff. Additionally, the extra term, $S_l(\delta)$:

$$S_l(\delta) = -\frac{1}{4\pi}\int_{-\infty}^{\infty} z\ dz\ e^{i\delta\nu z}\frac{d}{dz}\frac{1}{(1+z^2)^3}, \tag{4.62}$$

has a significant contribution only for large l where the exponential must be in position.

To evaluate $\sum_{l=1}^{\infty} S_l(\delta)$ we break the sum as follows:

$$\sum_{l=1}^{\infty} = \sum_{l=1}^{L} + \sum_{l=L+1}^{\infty}. \tag{4.63}$$

For the first summation, where $1 \ll L \ll 1/\delta$, we may set $\delta = 0$ and get:

$$\sum_{l=1}^{L} S_l(\delta) = L\frac{3}{32}, \tag{4.64}$$

while for the second summation we can write:

$$\begin{aligned}\sum_{l=L+1}^{\infty} S_l(\delta) &\longrightarrow \int_{L+1}^{\infty} dl\ S_l(\delta) + \frac{1}{2}S_{L+1}(\delta=0)\\ &= \int_{L+1}^{\infty} dl\ S_l(\delta) + \frac{1}{2}\cdot\frac{3}{32},\end{aligned} \tag{4.65}$$

because for $l = L+1 \gg 1$ the integral results the asymptotic value (4.60).

To find the integral in (4.65), we first calculate $S_l(\delta)$ by means of contour integral. From (4.62) we have:

$$S_l(\delta) = -\frac{1}{4\pi}\int_{-\infty}^{\infty} dz\ e^{i\delta\nu z}\frac{6z^2}{(1+z^2)^4} = -\frac{1}{4\pi}\int_{-\infty}^{\infty} dz \frac{\frac{e^{i\delta\nu z}6z^2}{(z+i)^4}}{(z-i)^4}$$

$$= -\frac{1}{4\pi}\int_C dz \frac{f(z)}{(z-i)^4} = -\frac{1}{4\pi}\cdot\frac{2\pi i}{3!}f^{(3)}(i)$$

$$= \frac{3}{32}\Big[1 - \delta\nu - \frac{1}{2}(\delta\nu)^3\Big]e^{-\delta\nu}, \tag{4.66}$$

and performing the l integration:

$$\begin{aligned}\int_{L+1}^{\infty} dl\ S_l(\delta) &= \frac{3}{32}\Big[1 - \delta\frac{d}{d\delta} + \frac{1}{3}\delta^3\frac{d^3}{d\delta^3}\Big]\int_{L+3/2}^{\infty} d\nu\ e^{-\delta\nu} \\ &= -\frac{3}{32}\Big(L + \frac{3}{2}\Big), \quad \delta \to 0. \end{aligned} \tag{4.67}$$

Thereby, obtaining the sum:

$$\sum_{l=1}^{\infty} S_l(\delta) = L\frac{3}{32} - \frac{3}{32}\Big(L + \frac{3}{2}\Big) + \frac{1}{2}\cdot\frac{3}{32} = -\frac{3}{32}. \tag{4.68}$$

Consequently, Casimir energy can be written as:

$$E = \frac{1}{2a}\left(\frac{3}{32} + \sum_{l=1}^{\infty} R_l\right), \tag{4.69}$$

where, integrating R_l given in (4.61) by parts, we have:

$$R_l = -\frac{1}{2\pi}\int_0^{\infty} dz\left[(2l+1)^2 \ln\left(1 - \lambda_l^2\right) + \frac{1}{(1+z^2)^3}\right]. \tag{4.70}$$

The sum on l of the R_l integral can be evaluated numerically and changes the result by less than 2%. Using the numerically integrated results for $l = 1, 2, 3, 4$ and the asymptotic result we obtain:

$$E = \frac{0.092353}{2a}, \tag{4.71}$$

or retrieving the $\hbar$ and c constants:

$$E = 0.092353\frac{\hbar c}{2a} \tag{4.72}$$

for the Casimir energy of a spherical conducting shell.

Finally, the force per unit area is given by:

$$\tilde{\mathcal{P}} = \tilde{\mathcal{X}}\frac{1}{\mathcal{A}} = -\frac{1}{4\pi a^2}\frac{\partial E}{\partial a}. \tag{4.73}$$

4.2. Temperature Dependence [4]

Balian and Duplantier have found, using the *multiple scattering expansion* approach, the *free energy* for the electromagnetic Casimir effect of a sphere in both low and high temperature limits, which is

$$F(a,T) \sim \frac{0.04618}{a} - (\pi a)^3 \frac{(kT)^4}{15}, \quad kT \ll 1/a \tag{4.74a}$$

$$F(a,T) \sim -\frac{kT}{4}(\ln kTa + 0.769) - \frac{1}{3840 kTa^2}, \quad kT \gg 1/a. \tag{4.74b}$$

The corresponding radial force acting on the spherical shell is repulsive and owing to (4.73) is given by:

$$\tilde{\mathcal{X}}(a,T) \sim \frac{0.04618}{a^2} + \pi^3 a^2 \frac{(kT)^4}{5}, \quad kT \ll 1/a \tag{4.75a}$$

$$\tilde{\mathcal{X}}(a,T) \sim \frac{kT}{4a} - \frac{1}{1920 kTa^3}, \quad kT \gg 1/a. \tag{4.75b}$$

Chapter 5

Fermionic Casimir Effect

We are interested in evaluating Carimir forces arising from flactuations of a massless Dirac fermionic field. Reading (1.20) and setting $m = 0$ we get:

$$i\gamma^\mu \partial_\mu \psi = 0. \tag{5.1}$$

Equation (5.1) accompanied with the appropriate boundary conditions will lead us to Fermionic Casimir forces applied on the boundaries. These calculations, especially in the spherical geometry, have important applications in hardronic physics as we will see in the next chapter. Yet, until then, we must define the problem, that is, except the governing equation (5.1) we have to describe the appropriate boundary conditions.

5.1. Boundary Conditions [4]

The appropriate boundary conditions come from the physical neccesity that there should be no particle current, $\boldsymbol{j} = \psi^\dagger \gamma^0 \boldsymbol{\gamma} \psi = \psi^\dagger \boldsymbol{\alpha} \psi$, through the surfaces, *ie*:

$$\begin{aligned} \hat{\boldsymbol{n}} \cdot \boldsymbol{j} = 0 \Rightarrow \\ \psi^\dagger \gamma^0 \hat{\boldsymbol{n}} \cdot \boldsymbol{\gamma} \psi = 0 \Rightarrow \\ \bar{\psi} \hat{\boldsymbol{n}} \cdot \boldsymbol{\gamma} \psi = 0, \end{aligned} \tag{5.2}$$

where:
$\hat{\boldsymbol{n}}$ is the outward unit vector normal to the surface and:
we have set $\psi^\dagger \gamma^0 = \bar{\psi}$.

To find a boundary condition including only ψ let us take:

$$
\begin{aligned}
(i\hat{\boldsymbol{n}}\cdot\boldsymbol{\gamma})^2 &= -(\hat{\boldsymbol{n}}\cdot\boldsymbol{\gamma})(\hat{\boldsymbol{n}}\cdot\boldsymbol{\gamma}) = -n_\mu\gamma^\mu\, n_\nu\gamma^\nu \\
&= -\frac{1}{2}n_\mu n_\nu\{\gamma^\mu,\gamma^\nu\} = -g^{\mu\nu}n_\mu n_\nu \\
&= -n^\nu n_\nu = -[(n^0)^2 - \hat{\boldsymbol{n}}^2] = 1.
\end{aligned} \tag{5.3}
$$

In this way, the quantity $i\hat{\boldsymbol{n}}\cdot\boldsymbol{\gamma}$ must have eigenvalues ±1. Let us assume that on the surface is:

$$
\begin{aligned}
i\hat{\boldsymbol{n}}\cdot\boldsymbol{\gamma}\psi &= -\psi \Rightarrow \\
(1 + i\hat{\boldsymbol{n}}\cdot\boldsymbol{\gamma})\psi &= 0.
\end{aligned} \tag{5.4}
$$

Then, it is easy to show that (5.4) for ψ is equivalent with (5.2). Indeed, we have:

$$
i\hat{\boldsymbol{n}}\cdot\boldsymbol{j} = i\bar{\psi}\hat{\boldsymbol{n}}\cdot\boldsymbol{\gamma}\psi \overset{(5.4)}{=} -\bar{\psi}\psi \qquad \text{on the surface.} \tag{5.5}
$$

On the other hand, (5.4) can be written as:

$$
\begin{aligned}
i\hat{\boldsymbol{n}}\cdot\gamma^0\boldsymbol{\gamma}\psi &= -\gamma^0\psi \Rightarrow \\
(i\hat{\boldsymbol{n}}\cdot\gamma^0\boldsymbol{\gamma}\psi)^\dagger &= -(\gamma^0\psi)^\dagger \Rightarrow \\
i\psi^\dagger(\gamma^0\boldsymbol{\gamma})^\dagger\cdot\hat{\boldsymbol{n}} &= \gamma^{0\dagger}\psi^\dagger \Rightarrow \\
i\gamma^0\psi^\dagger\boldsymbol{\gamma}\cdot\hat{\boldsymbol{n}} &= \gamma^0\psi^\dagger \Rightarrow \\
i\bar{\psi}\boldsymbol{\gamma}\cdot\hat{\boldsymbol{n}} &= \bar{\psi},
\end{aligned} \tag{5.6}
$$

so, once again we have:

$$
i\hat{\boldsymbol{n}}\cdot\boldsymbol{j} = i\bar{\psi}\hat{\boldsymbol{n}}\cdot\boldsymbol{\gamma}\psi \overset{(5.6)}{=} \bar{\psi}\psi \qquad \text{on the surface.} \tag{5.7}
$$

Hence, considering (5.5) and (5.7) we finally obtain that $\hat{\boldsymbol{n}}\cdot\boldsymbol{j} = 0$ on the sureface as required by (5.2). In other words, (5.2) and (5.4) are equivalent and the latter will be the problem's boundary condition including only the field's value ψ, as we wished.

5.2. Parallel plates (Mode summation Method)[4]

Let us assume here the orientation of the parallel plates to be vertical to the z-axis. It follows that boundary condition (5.4) becomes:

$$(1 \mp i\gamma^3)\psi = 0, \tag{5.8}$$

at $z = 0$ and $z = a$, respectively. At the following we will choose a representation of the Dirac matrices in which $i\gamma_5$ is diagonal:

$$i\gamma_5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.9}$$

while:

$$\gamma^0 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \tag{5.10}$$

from which the explicit form of the other gamma matrices follow from:

$$\boldsymbol{\gamma} = i\gamma^0\gamma_5\boldsymbol{\sigma}, \tag{5.11}$$

in which the matrix $\boldsymbol{\sigma}$ denote the *Pauli matrices* which, in this situtation, are represented as:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{5.12}$$

In this section, in order to get the Fermionic Casimir forces we are going to use a summing of modes method. For this case, we introduce a Fourier transform in time and the transverse spatial direction:

$$\psi(x) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{d^k}{(2\pi)^2} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \psi(z; \boldsymbol{k}, \omega). \tag{5.13}$$

On that accound, for a massless fermion and for a single mode, (5.1) becomes; in the coordinate system in which the parallel to the plates $\boldsymbol{k}$ vector lies along the x-axis:

$$\begin{aligned} \gamma^0\partial_0\psi + \gamma^1\partial_1\psi + \gamma^2\partial_2\psi + \gamma^3\partial_3\psi = 0 \Rightarrow \\ \gamma^0(-i\omega)\begin{bmatrix} u_\pm \\ v_\pm \end{bmatrix} + ik\gamma^0 i\gamma_5\sigma^1 \begin{bmatrix} u_\pm \\ v_\pm \end{bmatrix} + \gamma^0\frac{\partial}{\partial z} i\gamma_5\sigma^3 \begin{bmatrix} u_\pm \\ v_\pm \end{bmatrix} = 0 \Rightarrow \\ -i\omega \begin{bmatrix} u_\pm \\ v_\pm \end{bmatrix} + iki\gamma_5 \begin{bmatrix} v_\pm \\ u_\pm \end{bmatrix} + \frac{\partial}{\partial z} i\gamma_5 \begin{bmatrix} u_\pm \\ -v_\pm \end{bmatrix} = 0 \Rightarrow \end{aligned}$$

$$-\omega \begin{bmatrix} u_{\pm} \\ v_{\pm} \end{bmatrix} + k \begin{bmatrix} \pm v_{\pm} \\ \pm u_{\pm} \end{bmatrix} - i\frac{\partial}{\partial z} \begin{bmatrix} \pm u_{\pm} \\ \mp v_{\pm} \end{bmatrix} = 0 \Rightarrow$$

$$\Big(- \omega + \mp i\frac{\partial}{\partial z}\Big) u_{\pm} \pm k v_{\pm} = 0 \tag{5.14a}$$

$$\pm k u_{\pm} + \Big(- \omega + \pm i\frac{\partial}{\partial z}\Big) v_{\pm} = 0, \tag{5.14b}$$

where the subscripts denote the eigenvalues of $i\gamma_5$ and u and v are the eigenvectors of the σ^3 operator with eingenvalue $+1$ or -1, respectively.

The boundary conditions (5.8) are written as:

$$\begin{bmatrix} u_{\pm} \\ v_{\pm} \end{bmatrix} \mp i\gamma^0 i\gamma_5 \sigma^3 \begin{bmatrix} u_{\pm} \\ v_{\pm} \end{bmatrix} = 0 \Rightarrow$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mp \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} u_{\pm} \\ v_{\pm} \end{bmatrix} = 0 \Rightarrow$$

$$\begin{bmatrix} 1 & \mp 1 \\ \pm 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{\pm} \\ v_{\pm} \end{bmatrix} = 0 \Rightarrow$$

$$u_+ + u_- \Big|_{z=0} = 0 \tag{5.16a}$$

$$v_+ - v_- \Big|_{z=0} = 0 \tag{5.16b}$$

$$u_+ - u_- \Big|_{z=a} = 0 \tag{5.16c}$$

$$v_+ + v_- \Big|_{z=a} = 0. \tag{5.16d}$$

Next, to solve the problem, we decouple (5.14) and we note that each component satisfies:

$$\Big(\frac{\partial^2}{\partial^2 z} + \lambda^2 \Big) \psi = 0, \tag{5.17}$$

where $\lambda^2 = \omega^2 - k^2$. Therefore, each component, subjected to boundary conditions, can be written as follows:

$$u_+ + u_- = A \sin \lambda z \tag{5.18a}$$

$$v_+ - v_- = B \sin \lambda z \tag{5.18b}$$

$$u_+ - u_- = C \sin \lambda (z - a) \tag{5.18c}$$

$$v_+ + v_- = D \sin \lambda (z - a). \tag{5.18d}$$

Inserting these into (5.14) we get:

$$\begin{aligned} &-\omega(u_+ + u_-) - i\frac{\partial}{\partial z}(u_+ - u_-) + k(v_+ - v_-) = 0 \Rightarrow \\ &(kB - \omega A)\sin\lambda a - i\lambda C = 0 \qquad \text{and} \qquad \cos\lambda a = 0, \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} &-\omega(v_+ - v_-) + i\frac{\partial}{\partial z}(v_+ - v_-) + k(u_+ - u_-) = 0 \Rightarrow \\ &(kA - \omega B)\sin\lambda a + i\lambda D = 0 \qquad \text{and} \qquad \cos\lambda a = 0. \end{aligned} \tag{5.20}$$

Thus, overall, we have a condition on λ:

$$\cos\lambda a = 0 \Rightarrow \lambda a = \left(n + \frac{1}{2}\right)\pi,$$

and two independent solutions for the coeffiecients:

$$\begin{cases} A \neq 0 \\ B = 0 \\ C = \frac{i\omega}{\lambda}(-1)^n A \\ D = \frac{ik}{\lambda}(-1)^n A, \end{cases} \tag{5.21}$$

and

$$\begin{cases} A = 0 \\ B \neq 0 \\ C = \frac{k}{i\lambda}(-1)^n B \\ D = \frac{\omega}{i\lambda}(-1)^n B. \end{cases} \tag{5.22}$$

Ergo, the fermionic zero-point energy of the configuration is:

$$\begin{aligned} u &= -2 \cdot \frac{1}{2}\sum_n \int_k \frac{d^2k}{(2\pi)^2}\omega_n(k) \\ &= -2 \cdot \frac{1}{2}\sum_n \int_k \frac{d^2k}{(2\pi)^2}\sqrt{k^2+\lambda^2} \\ &= -\sum_n \int_k \frac{d^2k}{(2\pi)^2}\sqrt{k^2 + \frac{(n+1/2)^2\pi^2}{a^2}}. \end{aligned} \tag{5.23}$$

The factor of 2 refers to the two spin modes of the fermion.

To evaluate (5.23) we employ the Schwinger proper-time representation for the square root:

$$u = -\sum_n \int_k \frac{d^2k}{(2\pi)^2} \frac{1}{\Gamma\Big(-\frac{1}{2}\Big)} \int_0^\infty \frac{dt}{t} t^{-1/2} e^{-t(k^2+(n+1/2)^2\pi^2/a^2)}. \tag{5.24}$$

We next carry out the Gaussian integration over k:

$$u = \frac{1}{2\sqrt{\pi}} \frac{1}{4\pi} \sum_n \int_0^\infty \frac{dt}{t} t^{-1/2-1} e^{-t(n+1/2)^2\pi^2/a^2}. \tag{5.25}$$

Then, we use the Euler representation for the gamma funtion:

$$\int_0^\infty dt \ \ t^{z-1} e^{-t\lambda} = \Gamma(z) \frac{1}{\lambda^z}, \tag{5.26}$$

and, for $z = -3/2$, we finally get:

$$\begin{aligned} u &= \frac{1}{8\pi^{3/2}} \Gamma\Big(-\frac{3}{2}\Big) \sum_{n=0}^\infty \frac{(n+1/2)^3\pi^3}{a^3} \\ &= \frac{1}{8\pi^{3/2}} \cdot \frac{4\sqrt{\pi}}{3} \cdot \frac{\pi^3}{a^3} \cdot \Big(-\frac{7}{8}\Big)\zeta(-3) \\ &= -\frac{\pi^2}{6a^3} \frac{7}{8} \frac{1}{120} \Rightarrow \end{aligned}$$

$$u = -\frac{7\pi^2}{5760 a^3}, \tag{5.27}$$

And, for the Casimir force we get:

$$f_F = -\frac{7\pi^2}{1920 a^4}. \tag{5.28}$$

5.3. Spherical shell (Green's function Method)[4, 8]

5.3.1. Green's Function

In this case, the Fermion Green's function satisfies the equation:

$$\left(\gamma\frac{1}{i}\partial\right)\mathcal{G}(x,x') = \delta(x,x'), \tag{5.29}$$

where $x = (x^0, x^1, x^2, x^3)$ is the space-time component.

Once again, equation (5.29) is to be solved subject to the linear boundary condition:

$$(1 + i\hat{\boldsymbol{n}}\cdot\boldsymbol{\gamma})\mathcal{G}(x,x')\Big|_S = 0, \tag{5.30}$$

where $\hat{\boldsymbol{n}}$ is the outward normal unit vector to the ball.

To solve the problem we introduce a time Fourier transform:

$$\mathcal{G}(x,x') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G(\boldsymbol{r},\boldsymbol{r}';\omega). \tag{5.31}$$

Here, we adopt the same representation for the gamma matrices used before in Sec. 5.2. Owing to (5.11) and (5.29), G satisfies:

$$(-\omega + \gamma_5\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})G(\boldsymbol{r},\boldsymbol{r}') = \gamma^0\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{5.32}$$

In this representation, and introducing the representation for the G matrcices given above, that is:

$$G = \begin{bmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{bmatrix}, \tag{5.33}$$

equation (5.32) reads:

$$(-\omega \pm i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})G_{\mp\mp}(\boldsymbol{r},\boldsymbol{r}') = 0 \tag{5.34a}$$
$$(-\omega \pm i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})G_{\mp\pm}(\boldsymbol{r},\boldsymbol{r}') = \pm\delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{5.34b}$$

where the subscripts of the components of G correspond to the eigenvalues of $i\gamma_5$.

Additionally, it is convenient to make an angular momentum decomposition. For the massless spin-1/2 particle, the eigenstates of the total angular momentum:

$$\boldsymbol{J} = \boldsymbol{L} + \frac{1}{2}\boldsymbol{\sigma}, \tag{5.35}$$

are given by (take a look in Appendix C.2.1):

$$Z_{JM}^{l=J\pm 1/2}(\Omega) = \left(\frac{l+\frac{1}{2}\mp M}{2l+1}\right)^{1/2} Y_{l\,M-1/2}(\Omega)\,|+\rangle$$
$$\mp\left(\frac{l+\frac{1}{2}\pm M}{2l+1}\right)^{1/2} Y_{l\,M+1/2}(\Omega)\,|-\rangle\,, \tag{5.36}$$

which wave the property

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}} Z_{JM}^{l=J\pm 1/2}(\Omega) = Z_{JM}^{l=J\mp 1/2}(\Omega), \tag{5.37}$$

that is, the *radial spin operator*, $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}$, interchanges the spin.

Representing $|+\rangle = (1\;0)^T$ and $|-\rangle = (0\;1)^T$ one obtains:

$$Z_{JM}^{l=J\mp 1/2}(\Omega) = \frac{1}{\sqrt{2l+1}}\begin{bmatrix}\sqrt{l+\frac{1}{2}\mp M}\;Y_{l\,M-1/2}(\Omega)\\ \sqrt{l+\frac{1}{2}\pm M}\;Y_{l\,M+1/2}(\Omega)\end{bmatrix}. \tag{5.38}$$

In the two-dimensional spin space spanned by $Z_{JM}^{l=J\pm 1/2}(\Omega)$, the operator in (5.34) is represented as:

$$(-\omega\pm i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}) = \begin{bmatrix}-\omega & \pm\frac{i}{r}\left(\frac{\partial}{\partial r}r - J - \frac{1}{2}\right)\\ \pm\frac{i}{r}\left(\frac{\partial}{\partial r}r + J + \frac{1}{2}\right) & -\omega\end{bmatrix}. \tag{5.39}$$

We now expand $G_{ab}(\boldsymbol{r},\boldsymbol{r}')$, $\;a,b=\pm$, in terms of these eigenstates and we get:

$$G_{ab}(\boldsymbol{r},\boldsymbol{r}') = \sum_{J\,M}\left[f_J^{ab}(r,\boldsymbol{r}')Z_{JM}^{l=J+1/2}(\Omega) + g_J^{ab}(r,\boldsymbol{r}')Z_{JM}^{l=J-1/2}(\Omega)\right]. \tag{5.40}$$

Using the orthonormality property:

$$\int d\Omega\; Z_{JM}^{l}(\Omega)Z_{J'M'}^{l'*}(\Omega) = \delta_{JJ'}\delta_{MM'}\delta_{ll'}, \tag{5.41}$$

and equations (5.34), we take the component equation:

$$(-\omega\mp\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\sum_{J\,M}\left[f_J^{\pm\pm}(r,\boldsymbol{r}')Z_{JM}^{l=J+1/2}(\Omega) + g_J^{\pm\pm}(r,\boldsymbol{r}')Z_{JM}^{l=J-1/2}(\Omega)\right] = 0 \Rightarrow$$
$$-\omega\sum_{J\,M}\left[f_J^{\pm\pm}(r,\boldsymbol{r}')Z_{JM}^{l=J+1/2}(\Omega) + g_J^{\pm\pm}(r,\boldsymbol{r}')Z_{JM}^{l=J-1/2}(\Omega)\right]$$
$$\mp\frac{i}{r}\left(\frac{\partial}{\partial r}r - J - \frac{1}{2}\right)\sum_{J\,M}\left[f_J^{\pm\pm}(r,\boldsymbol{r}')Z_{JM}^{l=J-1/2}(\Omega) + g_J^{\pm\pm}(r,\boldsymbol{r}')Z_{JM}^{l=J+1/2}(\Omega)\right] = 0 \Rightarrow$$

$$-\omega f_J^{\pm\pm} \mp \frac{i}{r}\left(\frac{\partial}{\partial r}r - J - \frac{1}{2}\right)g_J^{\pm\pm} = 0 \quad (5.42a)$$

and similarly

$$\mp \frac{i}{r}\left(\frac{\partial}{\partial r}r - J - \frac{1}{2}\right)f_J^{\pm\pm} - \omega g_J^{\pm\pm} = 0. \quad (5.42b)$$

Following the same proccedure for the $G_{\pm\mp}$ components we get:

$$-\omega f_J^{\pm\mp} \mp \frac{i}{r}\left(\frac{\partial}{\partial r}r - J - \frac{1}{2}\right)g_J^{\pm\mp} = \mp\frac{i}{r^2}\delta(r-r')Z_{JM}^{l=J+1/2*}(\Omega') \quad (5.43a)$$

$$\mp\frac{i}{r}\left(\frac{\partial}{\partial r}r - J - \frac{1}{2}\right)f_J^{\pm\mp} - \omega g_J^{\pm\mp} = \mp\frac{i}{r^2}\delta(r-r')Z_{JM}^{l=J-1/2*}(\Omega'). \quad (5.43b)$$

The system of equations (5.43) can be solved in terms of scalar Green's functions satisfying:

$$\left(\frac{1}{r}\frac{d^2}{dr^2}r - \frac{l(l+1)}{r^2} + \omega^2\right)\Delta_l(r,r') = -\frac{1}{r^2}\delta(r-r') \quad (5.44)$$

which for $r, r' < a$ has a solution:

$$\Delta_l = ikj_l(kr_<)[h_l(kr_>) - A_l j_l(kr_>)]. \quad (5.45)$$

That is valid having set:

$$f_J^{\pm\mp} = \frac{1}{r'}\left(\frac{\partial}{\partial r'}r' + J + \frac{1}{2}\right)F_{J+1/2}(r,r')Z_{JM}^{l=J-1/2*}(\Omega')$$
$$\mp i\omega F_{J+1/2}(r,r')Z_{JM}^{l=J+1/2*}(\Omega') \quad (5.46a)$$

$$g_J^{\pm\mp} = \frac{1}{r'}\left(\frac{\partial}{\partial r'}r' - J - \frac{1}{2}\right)G_{J-1/2}(r,r')Z_{JM}^{l=J+1/2*}(\Omega')$$
$$\mp i\omega G_{J-1/2}(r,r')Z_{JM}^{l=J-1/2*}(\Omega'), \quad (5.46b)$$

where G_l and F_l are related by the relation:

$$G_{J-1/2}(r,r') = \frac{1}{\omega^2}\frac{1}{r}\left(\frac{\partial}{\partial r}r + J + \frac{1}{2}\right)\frac{1}{r'}\left(\frac{\partial}{\partial r'}r' + J + \frac{1}{2}\right)F_{J+1/2}(r,r'). \quad (5.47)$$

Then, $G_{\pm\mp}(\boldsymbol{r},\boldsymbol{r}')$ in (5.40) is written as:

$$\begin{aligned} G_{\pm\mp}(\boldsymbol{r},\boldsymbol{r}') = \sum_{JM} \Bigg[& \frac{1}{r'}\left(\frac{\partial}{\partial r'}r' + J + \frac{1}{2}\right) F_{J+1/2}(r,r') Z_{JM}^{l=J+1/2}(\Omega) Z_{JM}^{l=J-1/2}(\Omega')^* \\ & + \frac{1}{r'}\left(\frac{\partial}{\partial r'}r' - J - \frac{1}{2}\right) G_{J-1/2}(r,r') Z_{JM}^{l=J-1/2}(\Omega) Z_{JM}^{l=J+1/2}(\Omega')^* \\ & \mp i\omega F_{J+1/2}(r,r') Z_{JM}^{l=J+1/2}(\Omega) Z_{JM}^{l=J+1/2}(\Omega')^* \\ & \mp i\omega G_{J-1/2}(r,r') Z_{JM}^{l=J-1/2}(\Omega) Z_{JM}^{l=J-1/2}(\Omega')^* \Bigg]. \end{aligned} \tag{5.48}$$

It follows from (5.45) and (5.47), as well as the fact that:

$$\left(\frac{d}{dr}r \pm l\right) j_l(kr) = \pm kr j_{l\mp1}(kr), \tag{5.49}$$

the following coefficient relation:

$$A^G_{J-1/2} = A^F_{J+1/2}. \tag{5.50}$$

We now switch our attention to the homogeneous system of equations (5.42). This system of equations is subjected to the antisymmetry constrain for the Green's function, that is:

$$[\gamma^0 \mathcal{G}(x,x')]^T = -\gamma^0 \mathcal{G}(x,x'), \tag{5.51}$$

which leads to the form:

$$\begin{aligned} G_{\pm\pm}(\boldsymbol{r},\boldsymbol{r}') = \sum_{JM} \Bigg[& a_J\, j_{J+1/2}(kr) j_{J+1/2}(kr') Z_{JM}^{l=J+1/2}(\Omega) Z_{JM}^{l=J+1/2}(\Omega')^* \\ & + b_J\, j_{J-1/2}(kr) j_{J-1/2}(kr') Z_{JM}^{l=J-1/2}(\Omega) Z_{JM}^{l=J-1/2}(\Omega')^* \\ & \pm c_J\, j_{J-1/2}(kr) j_{J+1/2}(kr') Z_{JM}^{l=J-1/2}(\Omega) Z_{JM}^{l=J+1/2}(\Omega')^* \\ & \pm c_J\, j_{J+1/2}(kr) j_{J-1/2}(kr') Z_{JM}^{l=J+1/2}(\Omega) Z_{JM}^{l=J-1/2}(\Omega')^* \Bigg]. \end{aligned} \tag{5.52}$$

Having found the form of all the components of G we now try to define the coefficients by applying boundary condition (5.30). That reads:

$$\begin{bmatrix} 1 & -\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}} \\ -\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}} & 1 \end{bmatrix} \cdot \begin{bmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{bmatrix} \Bigg|_{|\boldsymbol{r}|=a} = 0. \tag{5.53}$$

Using (5.37), the boundary condition impies:

$$A^F_{J+1/2} = \frac{h_{J+1/2}(ka)j_{J+1/2}(ka) - h_{J-1/2}(ka)j_{J-1/2}(ka)}{[j_{J+1/2}(ka)]^2 - [j_{J-1/2}(ka)]^2} \tag{5.54a}$$

and

$$-a_J = b_J = -i\frac{k}{\omega}c_J = \frac{1/a^2}{[j_{J+1/2}(ka)]^2 - [j_{J-1/2}(ka)]^2}, \tag{5.54b}$$

where in (5.54b) we used the property:

$$h_{l-1}(x)j_l(x) - h_l(x)j_{l-1}(x) = \frac{i}{x^2}. \tag{5.55}$$

Therefore, we have found Dirac Green's function in a closed form. However, we are interested in the reduced Green's function, that is:

$$\tilde{G} = G - G^{(0)}, \tag{5.56}$$

where the $G^{(0)}$ is the free Dirac Green's function that can easily be obtained once we recognize that:

$$\Delta^{(0)}_l(r,r') = ikj_l(kr_<)h_l(kr_>), \quad r,r' < a. \tag{5.57}$$

and that $a^{(0)}_J = b^{(0)}_J = c^{(0)}_J = 0$ in the limit $a \to \infty$.

Finally, using a matrix representation for the two-dimensional spin space spanned by $Z^{l=J\pm1/2}_{JM}$ we get:

$$\tilde{G}_{\pm\mp} = -ik\sum_J A^F_{J+1/2}\begin{bmatrix} \mp i\omega j_{J+1/2}(kr)j_{J+1/2}(kr') & kj_{J+1/2}(kr)j_{J-1/2}(kr') \\ -kj_{J-1/2}(kr)j_{J+1/2}(kr') & \mp i\omega j_{J-1/2}(kr)j_{J-1/2}(kr') \end{bmatrix} \tag{5.58a}$$

and

$$\tilde{G}_{\pm\pm} = -ik\sum_J \frac{1/k^2a^2}{[j_{J+1/2}(ka)]^2 - [j_{J-1/2}(ka)]^2}\begin{bmatrix} -ikj_{J+1/2}(kr)j_{J+1/2}(kr') & \mp\omega j_{J+1/2}(kr)j_{J-1/2}(kr') \\ \mp\omega j_{J-1/2}(kr)j_{J+1/2}(kr') & ikj_{J-1/2}(kr)j_{J-1/2}(kr') \end{bmatrix}. \tag{5.58b}$$

Similarly, one can find the Green's functions for $a \le r,r' < \infty$. We proceed finding the energy due to the interior modes and, then, the total energy due to both interior and exterior modes comes as an immediate generalization.

5.3.2. Casimir Stress

The fermionic stress tensor is:

$$T^{\mu\nu} = \frac{1}{2}\overline{\psi}\gamma^0\frac{1}{2}\left(\gamma^\mu\frac{1}{i}\partial^\nu + \gamma^\nu\frac{1}{i}\partial^\mu\right)\psi + g^{\mu\nu}\mathcal{L}, \tag{5.59}$$

in which $\mathcal{L}$ is the fermionic Lagrange function.

Following, we find the contribution from wuantum fluctuations by making the replacement:

$$i\psi(x)\overline{\psi}(x') \rightarrow \mathcal{G}(x,x'), \tag{5.60}$$

which leads to the radial stress:

$$T_{rr} = \frac{1}{2}\frac{\partial}{\partial r}\operatorname{tr}\boldsymbol{\gamma}\cdot\hat{\boldsymbol{r}}\mathcal{G}(x,x')\bigg|_{x'\to x}, \tag{5.61}$$

where we ignore the δ-terms coming from the Lagrangian.

In the same representation for the gamma matrices we get:

$$\begin{aligned}T_{rr} &= \frac{i}{2}\int\frac{d\omega}{2\pi}e^{-i\omega\tau}\frac{\partial}{\partial r}\operatorname{tr}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}(G_{-+}+G_{+-})\\ &\overset{(5.37)\&(5.48)}{=\!=\!=} i\int\frac{d\omega}{2\pi}e^{-i\omega\tau}\frac{\partial}{\partial r}\sum_{JM}\operatorname{tr}\left[\frac{1}{r'}\left(\frac{\partial}{\partial r'}r' + J + \frac{1}{2}\right)F_{J+1/2}(r,r')Z_{JM}^{l=J-1/2}(\Omega)Z_{JM}^{l=J-1/2}(\Omega')^*\right.\\ &\qquad\left.+\frac{1}{r'}\left(\frac{\partial}{\partial r'}r' - J - \frac{1}{2}\right)G_{J-1/2}(r,r')Z_{JM}^{l=J+1/2}(\Omega)Z_{JM}^{l=J+1/2}(\Omega')^*\right]\bigg|_{\boldsymbol{r}'=\boldsymbol{r}}.\end{aligned} \tag{5.62}$$

Applying the completeness relation for the harmonics:

$$\sum_{M=-J}^{J}\operatorname{tr}Z_{JM}^{J\pm1/2}(\Omega)\,Z_{JM}^{J\pm1/2}(\Omega)^* = \frac{2J+1}{4\pi}, \tag{5.63}$$

we obtain

$$\begin{aligned}T_{rr} = i\frac{\partial}{\partial r}\sum_{J=1/2}^{\infty}\int\frac{d\omega}{2\pi}e^{-i\omega\tau}\frac{2J+1}{4\pi}&\left[\frac{1}{r'}\left(\frac{\partial}{\partial r'}r' + J + \frac{1}{2}\right)F_{J+1/2}(r,r')\right.\\ &\left.+\frac{1}{r'}\left(\frac{\partial}{\partial r'}r' - J - \frac{1}{2}\right)G_{J-1/2}(r,r')\right]\bigg|_{r'=r}.\end{aligned} \tag{5.64}$$

At this point, we have to substract the vacuum part, given by eq.(5.56), or equally making use of $\tilde{G}_{\pm\mp}$ istead of $G_{\pm\mp}$ and hence we are driven to:

$$T_{rr}\Big|_{r=a} = \sum_{J=1/2}^{\infty} \frac{2J+1}{4\pi} \int \frac{d\omega}{2\pi} e^{-i\omega\tau} k^3 A^F_{J+1/2} \frac{\partial}{\partial(kr)} \Big(j_{J+1/2}(kr) j_{J-1/2}(kr') + j_{J-1/2}(kr) j_{J+1/2}(kr') \Big)\Big|_{r=a},$$

which finally is written as:

$$\begin{aligned} T_{rr}\Big|_{r=a} &= \sum_{J=1/2}^{\infty} \frac{2J+1}{4\pi} \int \frac{d\omega}{2\pi} e^{-i\omega\tau} k^3 \frac{h_{J+1/2}(ka) j_{J+1/2}(ka) - h_{J-1/2}(ka) j_{J-1/2}(ka)}{[j_{J+1/2}(ka)]^2 - [j_{J-1/2}(ka)]^2} \\ &\qquad \times [j'_{J+1/2}(ka) j_{J-1/2}(ka) - j'_{J-1/2}(ka) j_{J+1/2}(ka)] \\ &= \sum_{l=0}^{\infty} \frac{2(l+1)}{4\pi} \int \frac{d\omega}{2\pi} e^{-i\omega\tau} k^3 \frac{h_{l+1}(ka) j_{l+1}(ka) - h_l(ka) j_l(ka)}{[j_{l+1}(ka)]^2 - [j_l(ka)]^2} \\ &\qquad \times [j'_{l+1}(ka) j_l(ka) - j'_l(ka) j_{l+1}(ka)]. \end{aligned} \tag{5.65}$$

At this point, again, we have an oscillating term under integration, thus it is convenient to make a Euclidean rotation:

$$\begin{cases} \omega \to i\omega \\ k \to ik \\ \tau \to i\tau. \end{cases} \tag{5.66}$$

as well as using $x = ka \to ix$, $y = \omega a \to iy$ and $\delta = \tau/a$.

Herein, the radial stress component transforms to:

$$\begin{aligned} T_{rr}\Big|_a &= \sum_{l=0}^{\infty} \frac{2(l+1)}{8\pi^2} \int_{-\infty}^{\infty} i\frac{dy}{a} e^{iy\delta} \frac{i^3 x^3}{a^3} \frac{h_{l+1}(ix) j_{l+1}(ix) - h_l(ix) j_l(ix)}{[j_{l+1}(ix)]^2 - [j_l(ix)]^2} \\ &\qquad \times [j'_{l+1}(ix) j_l(ix) - j'_l(ix) j_{l+1}(ix)] \\ &= \sum_{l=0}^{\infty} \frac{2(l+1)}{8\pi^2 a^4} \int_{-\infty}^{\infty} dy\, e^{iy\delta} x^3 \frac{h_{l+1}(ix) j_{l+1}(ix) - h_l(ix) j_l(ix)}{[j_{l+1}(ix)]^2 - [j_l(ix)]^2} \\ &\qquad \times [j'_{l+1}(ix) j_l(ix) - j'_l(ix) j_{l+1}(ix)]. \end{aligned} \tag{5.67}$$

Here, we introduce the modified spherical Bessel functions:

$$s_n(x) = i^{-n} x j_n(ix) \tag{5.68a}$$

$$e_n(x) = -i^n x h_n(ix), \tag{5.68b}$$

alognside with:

$$
\begin{aligned}
s'_n(x) = i^{-n+1}xj'_n(ix) + i^{-n}j_n(ix) \Rightarrow \\
xj'_n(x) = i^{n-1}s'_n(x) - i^{n-1}\frac{s_n(x)}{x}.
\end{aligned}
\tag{5.69}
$$

Therefore, we obtain:

$$
\begin{aligned}
T_{rr} &= \sum_{l=0}^{\infty} \frac{2(l+1)}{8\pi^2 a^4} \int_{-\infty}^{\infty} dy\; e^{iy\delta} x \frac{xh_{l+1}(ix)xj_{l+1}(ix) - xh_l(ix)xj_l(ix)}{[xj_{l+1}(ix)]^2 - [xj_l(ix)]^2} \\
&\qquad \times [xj'_{l+1}(ix)xj_l(ix) - xj'_l(ix)xj_{l+1}(ix)] \Rightarrow \\
&= \sum_{l=0}^{\infty} \frac{2(l+1)}{8\pi^2 a^4} \int_{-\infty}^{\infty} dy\; e^{iy\delta} x \frac{-i^{-l-1}e_{l+1}(x)i^{l+1}s_{l+1}(x) + i^{-l}e_l(x)i^l s_l(x)}{i^{2l}i^2[s_{l+1}(x)]^2 - i^{2l}[s_l(x)]^2} \\
&\qquad \times [(i^l s'_{l+1}(x) - i^l\frac{s_{l+1}(x)}{x})i^l s_l(x) - (i^{l-1}s'_l(x) - i^{l-1}\frac{s_l(x)}{x})i^{l+1}s_{l+1}(x)] \\
&= \sum_{l=0}^{\infty} \frac{2(l+1)}{8\pi^2 a^4} \int_{-\infty}^{\infty} dy\; e^{iy\delta} x \frac{-e_{l+1}(x)s_{l+1}(x) + e_l(x)s_l(x)}{-\cancel{i^{2l}}\left([s_{l+1}(x)]^2 + [s_l(x)]^2\right)} \\
&\qquad \times \cancel{i^{2l}}[s'_{l+1}(x)s_l(x) - \cancel{\frac{s_{l+1}(x)s_l(x)}{x}} - s'_l(x)s_{l+1}(x) + \cancel{\frac{s_l(x)s_{l+1}(x)}{x}}] \\
&= \sum_{l=0}^{\infty} \frac{2(l+1)}{8\pi^2 a^4} \int_{-\infty}^{\infty} dy\; e^{iy\delta} x \frac{e_{l+1}(x)s_{l+1}(x) - e_l(x)s_l(x)}{[s_{l+1}(x)]^2 + [s_l(x)]^2}[s'_{l+1}(x)s_l(x) - s'_l(x)s_{l+1}(x)].
\end{aligned}
\tag{5.70}
$$

It can be shown, after a further manipulation, that the above equation can equally be written as:

$$
T_{rr} = \sum_{l=0}^{\infty} \frac{2(l+1)}{8\pi^2 a^4} \int_0^{\infty} dx\; x\cos\delta x \left[\frac{d}{dx}\ln\left([s_{l+1}(x)]^2 + [s_l(x)]^2\right) - 2(s'_{l+1}(x)e_l(x) + s'_l(x)e_{l+1}(x)\right], \tag{5.71}
$$

where we consider the limit $\delta \to 0$.

Until here, we considered only the internal modes of the problem. However, there exist external modes which contribute to the shell's stress. The inclusion of the exterior modes proceeds similarly, and we obtain the generalization of eq. (5.71):

$$
T_{rr} = \frac{2}{8\pi^2 a^4} \sum_{l=0}^{\infty}(l+1) \int_0^{\infty} dx\; x\cos\delta x \frac{d}{dx}\ln\left[\left(s_{l+1}(x)]^2 + [s_l(x)]^2\right)\left(e_{l+1}(x)]^2 + [e_l(x)]^2\right)\right]. \tag{5.72}
$$

This expression may be numerically evalutated through use of uniform asymptotic approximations, as we have already done in Chapter 4. Doing so, we find for the radial Casimir stress:

$$T_{rr}\Big|_a = \frac{0.0204}{4\pi a^4}, \tag{5.73}$$

and the Casimir energy reads as:

$$T_{rr}\Big|_a = -\frac{1}{4\pi a^2}\frac{\partial}{\partial a}E \Rightarrow$$

$$E = -\int 4\pi a^2 \frac{0.0204}{4\pi a^4} da = -\int \frac{0.0204}{a^2} da \Rightarrow$$

$$E = \frac{0.0204}{a}. \tag{5.74}$$

So, in this Chapter we illustrated that there exist fermionic fluctuations and we found their zero-point energy for the cases of parallel plates and spherical shell. The important conclution is that **for the case of parallel plates the fermionic Casimir force is attractive while for the case of the spherical shell is repulsive, *ie* the same image as for the EM Casimir forces.**

Appendices

Appendix A

Green's Functions

A.1. The General Idea

Consider an ordinary differential equation

$$\mathcal{L}y(x) = f(x) \tag{A.1}$$

where $\mathcal{L}$ is a linear differential operator, and $f(x)$ is a given function.

Generally, the solution of this equation is not easily found, except in some particular situations which are discussed in a relative undergraduate course. However, there is a more general way of treating it, and that is through the use of Green's functions.

Instead of (A.1) we try to solve the following equation

$$\mathcal{L}G(x, x') = \delta(x - x') \tag{A.2}$$

where $\delta(x - x')$ is Dirac's δ-function

$$\delta(x - x') = \begin{cases} 0, & x \neq x' \\ \infty, & x = x'. \end{cases}$$

The function $G(x, x')$ is called Green's function of the $\mathcal{L}$ differential operator. Once $G(x, x')$ is found, the solution of (A.1) is given by the formula

$$y(x) = \xi(x) + \int_{-\infty}^{\infty} G(x, x') f(x') dx'. \tag{A.3}$$

where $\xi(x)$ is the general solution to the homogeneous equation

$$\mathcal{L}y(x) = 0. \tag{A.4}$$

That is the mathematical framework to define the Green's function. However, in order to gain a physical understanding, let us try to re-define the Green function in the following, more intuitive, way.

Imagine that we have a system evolving in time, and that its governing equation is (A.1). Then, x represents time and $y(x)$ the state of the system at time x. The non-homogeneous part, $f(x)$, is the governing "force" at time $x > 0$ which changes the system's state. Then, for given initial conditions, say $\{y(0) = y_0, y'(0) = y'_0, \dots\}$, the solution $y(x)$ represents the state of the system in any specific time $x > 0$.

Now, instead of solving (A.1) we prefer to solve the differential equation (A.2) whose physical meaning is the following:

It is the governing differential equation of the system when an ideal impulse $\delta(x - x')$ is applied at time $x = x'$. The solution $G(x, x')$ is the state of the system at time x because of the impulse at x'.

Then, for the general solution we think that:

The state at x due to an impulse at x' is $G(x, x')$. However, in reality, at x' we have the $f(x')$ instead of $\delta(x - x')$. So the solution due to $f(x')$ is $G(x, x')f(x')$; the solution due to $\delta(x - x')$ times the "real force" $f(x')$. Finally, considering that $f(x')$ occurs for all x', we have to sum over all times $x' > 0$.

Example 1

Take the operator $\mathcal{L} \equiv f_0(t)\frac{d^2}{dt^2} + f_1(t)\frac{d}{dt} + f_2(t)$, with $t \in \mathcal{D} : f_0(t) \neq 0, \;\; \mathcal{D} \subset \mathbb{R}$ and $\{f_i \in C(\mathcal{D}), \;\; i = 0, 1, 2\}$.

Let $x_1(t)$ and $x_2(t)$ be two linearly independent solutions to the homogeneous equation

$$\mathcal{L}x(t) = 0.$$

We wish to solve the non-homogeneous equation

$$\mathcal{L}x(t) = f(t).$$

For this reason, we search the Green's function, $G(t, t')$, obeying the associated diff. equation

$$\mathcal{L}G(t, t') = f_0(t)\frac{d^2G(t, t')}{dt^2} + f_1(t)\frac{dG(t, t')}{dt} + f_2(t)G(t, t') = \delta(t - t') \qquad \text{(A.5)}$$

For $t \neq t'$ (A.5) is just the homogeneous equation with the known solutions x_1 and x_2.

Thus, we may write

$$G(t,t') = \begin{cases} a_1 x_1(t) + a_2 x_2(t), & t > t' \\ b_1 x_1(t) + b_2 x_2(t), & t < t'. \end{cases} \tag{A.6}$$

Now, we may determine constants a_1, a_2, b_1 and b_2 applying matching conditions for $G(t,t')$ at $t = t'$. At $t = t'$ $G(t,t')$ must be continuous; if it were not, dG/dt would contain a δ-function which means that d^2G/dt^2 would contain a $\dot{\delta}$-function which is not the case reading (A.5). On the other hand, dG/dt is discontinuous at $t = t'$.

We integrate both sides of (A.5) from $t = t' - \epsilon$ to $t = t' + \epsilon$ and we obtain

$$\int_{t'-\epsilon}^{t'+\epsilon} f_0(t) \frac{d^2G(t,t')}{dt^2} dt + \int_{t'-\epsilon}^{t'+\epsilon} f_1(t) \frac{dG(t,t')}{dt} dt + \int_{t'-\epsilon}^{t'+\epsilon} f_2(t) G(t,t') = 1.$$

For $\epsilon \to 0$ the functions f_0, f_1 and f_2 vary negligibly in the region of integration, so we can replace them by their values at $t = t'$, that is

$$f_0(t') \int_{t'-\epsilon}^{t'+\epsilon} \frac{d^2G(t,t')}{dt^2} dt + f_1(t') \int_{t'-\epsilon}^{t'+\epsilon} \frac{dG(t,t')}{dt} dt + f_2(t') \int_{t'-\epsilon}^{t'+\epsilon} G(t,t') = 1.$$

The last integral on the left of this equation vanishes for arbitrarily small ϵ, since $G(t,t')$ is continuous at $t = t'$. So we have

$$f_0(t') \left\{ \left[\frac{dG(t,t')}{dt} \right]_{t=t'+\epsilon} - \left[\frac{dG(t,t')}{dt} \right]_{t=t'-\epsilon} \right\} + f_1(t') [G(t'+\epsilon, t') - G(t'-\epsilon, t')] = 1.$$

Again, $G(t,t')$ is continuous at $t = t'$, so the second term vanishes and we are left with

$$\left[\frac{dG(t,t')}{dt} \right]_{t=t'+\epsilon} - \left[\frac{dG(t,t')}{dt} \right]_{t=t'-\epsilon} = \frac{1}{f_0(t')}. \tag{A.7}$$

The continuity of $G(t,t')$ at $t = t'$ and equation (A.7) are the matching conditions.

Applying them on $G(t,t')$ we get the following two relations for the coefficients

$$\left. \begin{aligned} a_1 x_1(t') + a_2 x_2(t') &= b_1 x_1(t') + b_2 x_2(t') \\ a_1 \dot{x}_1(t') + a_2 \dot{x}_2(t') - [b_1 \dot{x}_1(t') + b_2 \dot{x}_2(t')] &= \frac{1}{f_0(t')} \end{aligned} \right\} \Rightarrow$$

$$\begin{cases} (a_1 - b_1)x_1(t') + (a_2 - b_2)x_2(t') = 0 \\ \\ (a_1 - b_1)\dot{x}_1(t') + (a_2 - b_2)\dot{x}_2(t') = \dfrac{1}{f_0(t')}. \end{cases}$$

The solutions for $(a_1 - b_1)$ and $(a_2 - b_2)$ are given by Cramer's rule

$$a_1 - b_1 = -\frac{x_2(t')}{f_0(t')W(t')}, \qquad a_2 - b_2 = \frac{x_1(t')}{f_0(t')W(t')} \tag{A.8}$$

where $W(t') = x_1(t')\dot{x}_2(t') - \dot{x}_1(t')x_2(t')$ is the *Wronskian* of the functions x_1 and x_2 evaluated at $t = t'$.

We may substitute a_1 and a_2 into (A.6) and get

$$G(t,t') = \begin{cases} b_1 x_1(t) + b_2 x_2(t) - \dfrac{x_1(t)x_2(t') - x_2(t)x_1(t')x_2(t)}{f_0(t')W(t')}, & t > t' \\ \\ b_1 x_1(t) + b_2 x_2(t), \quad t < t'. \end{cases}$$

The two remaining constants, b_1 and b_2, are chosen so as to satisfy the appropriate boundary conditions. The final form of the Green's function is strongly dependent on the type of boundary conditions.

For initial conditions, $\{x_0, \dot{x}_0\}$ we may require $b_1 = 0 = b_2$, so the Green's function becomes

$$G(t,t') = -\theta(t - t')\frac{x_1(t)x_2(t') - x_2(t)x_1(t')x_2(t)}{f_0(t')W(t')} \tag{A.9}$$

and the solution is

$$x(t) = Ax_1(t) + Bx_2(t) + \int_{t_0}^{t} G(t,t')f(t')dt'. \tag{A.10}$$

A.2. Eigenfunction Expansion

Consider a linear ordinary differential equation

$$\mathcal{L}y(x) = f(x) \tag{A.11}$$

where $\mathcal{L}$ is a linear differential operator and f is a given function.

Suppose that the operator $\mathcal{L}$ possesses a complete, orthonormal set of eigenfunctions $\{\phi_n(x)\}$, so that

$$\mathcal{L}\psi_n(x) = \lambda_n \phi_n(x). \tag{A.12}$$

So, the solution of (A.11) can be written as a linear combination of the eigenfunction, that is

$$y(x) = \sum_{n=1}^{\infty} a_n \phi_n(x). \tag{A.13}$$

We have

$$\begin{gathered}\mathcal{L}y(x) = f(x) \Rightarrow \\ \mathcal{L}\sum_{n=1}^{\infty} a_n \phi_n(x) = \sum_{n=1}^{\infty} a_n \mathcal{L}\phi_n(x) = \sum_{n=1}^{\infty} a_n \lambda_n \phi_n(x) = f(x).\end{gathered} \tag{A.14}$$

We may conduct the inner product with $\phi_n(x)$, thus

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \lambda_n (\phi_m, \phi_n) &= (\phi_m, f) \Rightarrow \\ a_m \lambda_m &= (\phi_m, f) \Rightarrow \\ a_n &= \frac{1}{\lambda_n}(\phi_n, f).\end{aligned} \tag{A.15}$$

Thus, the solution is written as

$$y(x) = \sum_{n=1}^{\infty} \frac{(\phi_n, f)}{\lambda_n} \phi_n(x). \tag{A.16}$$

If some $\lambda_n = 0$, then the solution exists only if the function f has no ϕ_n component, ie $(\phi_n, f) = 0$. Let us suppose that $\lambda_n \neq 0$.

Equation (A.16) is

$$
\begin{aligned}
y(x) &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \phi_n(x)(\phi_n, f) \\
&= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \phi_n(x) \int \psi_n^*(x') f(x') dx' \\
&= \int \sum_{n=1}^{\infty} \frac{\phi_n(x)\psi_n^*(x')}{\lambda_n} f(x') dx' = \int G(x, x') f(x') dx' \qquad \text{(A.17)}
\end{aligned}
$$

where

$$
G(x, x') = \sum_{n=1}^{\infty} \frac{\phi_n(x)\psi_n^*(x')}{\lambda_n}. \qquad \text{(A.18)}
$$

Thus we obtain the Green's function in terms of an infinite eigenfunction expansion.

Until here we have treated Green's function in one dimension. Let us now turn to the case of a three dimensional space.

A.3. Green's Function in Three Dimensions

Consider a partial differential equation

$$
\mathcal{M} y(\boldsymbol{r}) = f(\boldsymbol{r}) \qquad \text{(A.19)}
$$

where $f(\boldsymbol{r})$ is a given function and $\mathcal{M}$ is a linear, partial differential operator. Green's funtion obeys the equation

$$
\mathcal{M} G(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}') \qquad \text{(A.20)}
$$

where $\delta(\boldsymbol{r} - \boldsymbol{r}')$ is the three dimensional Dirac function.

Once the Green's function is found, the general solution to the initial equation takes the form

$$
y(\boldsymbol{r}) = \xi(\boldsymbol{r}) + \int G(\boldsymbol{r}, \boldsymbol{r}') f(\boldsymbol{r}') d^3\boldsymbol{r}' \qquad \text{(A.21)}
$$

where $\xi(\boldsymbol{r})$ is the general solution of the homogeneous equation.

That is the definition of Green's function in a three dimensional space. However, in many problems, one may not follow exactly these steps in order to solve the differential equation but, instead, follow some mathematical technique. Let us make an example.

Example 2

Let us take $\mathcal{M} = \nabla^2 + \lambda$. We wish to find the Green's function and to solve the equation

$$\mathcal{M}y(\boldsymbol{r}) = (\nabla^2 + \lambda)y(\boldsymbol{r}) = F(\boldsymbol{r}). \tag{A.22}$$

We may solve this equation making use of the Fourier transformation supposing that both $y(\boldsymbol{r})$ and $F(\boldsymbol{r})$ have a Fourier transform, that is, they go to zero for large values of $|\boldsymbol{r}|$.

Assume

$$\hat{y}(\boldsymbol{k}) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} y(\boldsymbol{r}) d^3\boldsymbol{r} \tag{A.23a}$$

$$\hat{F}(\boldsymbol{k}) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} F(\boldsymbol{r}) d^3\boldsymbol{r}. \tag{A.23b}$$

Taking the Fourier transform of both sides of (A.22) we have

$$\frac{1}{(2\pi)^{3/2}} \int e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \nabla^2 y(\boldsymbol{r}) d^3\boldsymbol{r} + \lambda \hat{y}(\boldsymbol{k}) = \hat{F}(\boldsymbol{k}). \tag{A.24}$$

At this point, we may use Green's theorem to express the integral in terms of $\hat{y}(\boldsymbol{k})$. The theorem states that

$$\int_V (F\nabla^2 G - G\nabla^2 F) d^3\boldsymbol{r} = \int_S (G\boldsymbol{\nabla} F - F\boldsymbol{\nabla} G) \cdot \boldsymbol{n}\, dS$$

where S is the surface enclosing the volume V and $\boldsymbol{n}$ is the unit vector normal to S, pointing outwards. Applying this, the integral appearing in (A.24) is written

$$\frac{1}{(2\pi)^{3/2}} \int_V e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \nabla^2 y(\boldsymbol{r}) d^3\boldsymbol{r} = \frac{1}{(2\pi)^{3/2}} \int_V y(\boldsymbol{r}) \nabla^2 e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d^3\boldsymbol{r}$$
$$+ \frac{1}{(2\pi)^{3/2}} \int_S [e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \nabla y(\boldsymbol{r}) d^3\boldsymbol{r} - y(\boldsymbol{r}) \nabla e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}] \cdot \boldsymbol{n}\, dS \tag{A.25}$$

We integrate over all space, so we may take for the surface a sphere of radius R and take the limit $R \to \infty$. In this case, $\boldsymbol{n}$ is a unit vector in the radial direction. So, the surface integral becomes

$$\frac{1}{(2\pi)^{3/2}}\int_S [e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\nabla y(\boldsymbol{r})d^3\boldsymbol{r} - y(\boldsymbol{r})\nabla e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}]\cdot \boldsymbol{n}\, dS$$

$$= \frac{1}{(2\pi)^{3/2}}\lim_{R\to\infty} R\left\{\int_S \left[e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\frac{d}{dr}y(\boldsymbol{r})d^3\boldsymbol{r} - y(\boldsymbol{r})\frac{d}{dr}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right]d\Omega\right\}\Bigg|_{r=R} \tag{A.26}$$

where $d\Omega = \sin\theta\, d\theta d\phi$. Since we suppose that $y(\boldsymbol{r})$ tends to zero sufficiently rapidly as $|\boldsymbol{r}| \to \infty$, the surface term vanishes and we are left with

$$\frac{1}{(2\pi)^{3/2}}\int_V e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\nabla^2 y(\boldsymbol{r})d^3\boldsymbol{r} = \frac{1}{(2\pi)^{3/2}}\int_V y(\boldsymbol{r})\nabla^2 e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}d^3 = -k^2\hat{y}(\boldsymbol{k}).$$

Consequently, equation (A.24) becomes

$$(-k^2+\lambda)\hat{y}(\boldsymbol{k}) = \hat{F}(\boldsymbol{k}). \tag{A.27}$$

Here, we distinguish two cases: (i)$\lambda < 0$ and (ii)$\lambda \geq 0$.

(i) We write $\underline{\lambda = -\kappa^2 < 0}$. Then, since $(-k^2 - \kappa^2)$ never vanishes, we may write

$$\hat{y}(\boldsymbol{k}) = -\frac{\hat{F}(\boldsymbol{k})}{k^2+\kappa^2} \tag{A.28}$$

and by the inverse Fourier operation we obtain

$$y(\boldsymbol{r}) = \xi(\boldsymbol{r}) - \frac{1}{(2\pi)^{3/2}}\int \frac{\hat{F}(\boldsymbol{k})}{k^2+\kappa^2}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}d^3\boldsymbol{k} \tag{A.29}$$

where $\xi(\boldsymbol{r})$ is an arbitrary linear combination of the associated homogeneous equation.

Using (A.23b) we get

$$\begin{aligned} y(\boldsymbol{r}) &= \xi(\boldsymbol{r}) - \frac{1}{(2\pi)^3}\int d^3\boldsymbol{k}\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2+\kappa^2}\int d^3\boldsymbol{r}\; e^{i\boldsymbol{k}\cdot\boldsymbol{r}'}F(\boldsymbol{r}') \\ &= \xi(\boldsymbol{r}) - \frac{1}{(2\pi)^3}\int\left(\int \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{k^2+\kappa^2}d^3\boldsymbol{k}\right)F(\boldsymbol{r}')\; d^3\boldsymbol{r}' \\ &= \xi(\boldsymbol{r}) + \int G(\boldsymbol{r},\boldsymbol{r}')F(\boldsymbol{r}')d^3\boldsymbol{r}' \end{aligned}$$

where

$$G(\boldsymbol{r},\boldsymbol{r}') = -\frac{1}{(2\pi)^3}\int \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{k^2+\kappa^2}d^3\boldsymbol{k}. \tag{A.30}$$

is the Green's function.

It is known from residue theory of complex functions that

$$\int_{-\infty}^{\infty} R(x)e^{iax}\,dx = 2\pi i\sum_{y>0}\mathrm{Res}[R(z)e^{iaz}] \tag{A.31}$$

so, for brevity, we let the reader see that the integral over $\boldsymbol{k}$-space in (A.30) leads to

$$G(\boldsymbol{r},\boldsymbol{r}') = -\frac{1}{4\pi}\frac{e^{-\kappa|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{A.32}$$

Thus, we have found Green's function indirectly, using a Fourier transform technique, rather than solving (A.20). Now, to find the general solution, we need to determine $\xi(\boldsymbol{r})$, which obeys

$$\mathcal{M}y(\boldsymbol{r}) = (\nabla^2-\kappa^2)y(\boldsymbol{r}) = 0$$

which is easily solved leading to

$$\xi(\boldsymbol{r}) = e^{\kappa_1 x}e^{\kappa_2 y}e^{\kappa_3 z}, \qquad \kappa^2 = \kappa_1^2+\kappa_2^2+\kappa_3^2.$$

However, in the most physical problems we want $y(\boldsymbol{r})$ to remain bounded as $|\boldsymbol{r}|$ becomes very large, so all these solutions are not acceptable while they are not bounded, except in the directions in which $\kappa_1 x$, $\kappa_2 y$ and $\kappa_3 z$ are negative. Therefore, the final result is written as

$$y(\boldsymbol{r}) = -\frac{1}{4\pi}\int \frac{e^{-\kappa|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}F(\boldsymbol{r}')d^3\boldsymbol{r}'. \tag{A.33}$$

(ii) For $\underline{\lambda \geq 0}$ we see that the factor in (A.27) has a root at $k = \pm\lambda$, so we cannot invert it as it stands. We can avoid this difficulty by setting $\sqrt{\lambda}$ to be a complex number with a positive real part and an arbitrary small imaginary part, so that

$$\lambda = (q\pm i\epsilon)^2, \qquad \epsilon > 0. \tag{A.34}$$

After all calculations we may let ϵ tend to zero and hope to obtain a well-behaved result. Thus, (A.27) is

$$\hat{y}_{\pm}(\boldsymbol{k}) = -\frac{\hat{F}(\boldsymbol{k})}{k^2 - (q \pm \epsilon)^2} \tag{A.35}$$

and following the same procedure as before we obtain

$$y_{\pm}(\boldsymbol{r}) = \xi(\boldsymbol{r}) + \int G_{\pm}(\boldsymbol{r}, \boldsymbol{r'}) F(\boldsymbol{r'}) d^3\boldsymbol{r'} \tag{A.36}$$

where

$$G_{\pm}(\boldsymbol{r}, \boldsymbol{r'}) = -\frac{1}{(2\pi)^3} \int \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r'})}}{k^2 - (q \pm i\epsilon)^2} d^3\boldsymbol{k}. \tag{A.37}$$

We proceed evaluating the integral, starting with the angular integration

$$G_{\pm}(\boldsymbol{r}, \boldsymbol{r'}) = -\frac{1}{(2\pi)^3} 2\pi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r'}|\cos\theta}}{k^2 - (q \pm i\epsilon)^2}$$

Setting $\cos\theta = t \Rightarrow -\sin\theta d\theta = dt$, with $\{\theta \to 0 \Rightarrow t \to 1, \quad \theta \to \pi \Rightarrow t \to -1\}$ we get

$$\begin{aligned}
G_{\pm}(\boldsymbol{r}, \boldsymbol{r'}) &= -\frac{1}{(2\pi)^3} 2\pi \int_{-1}^{1} dt \int_0^{\infty} k^2 dk \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r'}|t}}{k^2 - (q \pm i\epsilon)^2} \\
&= -\frac{1}{(2\pi)^3} \frac{2\pi}{i|\boldsymbol{r}-\boldsymbol{r'}|} \int_0^{\infty} k dk \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r'}|} - e^{-ik|\boldsymbol{r}-\boldsymbol{r'}|}}{(k - q \mp i\epsilon)(k + q \pm i\epsilon)} \Rightarrow \\
&= -\frac{1}{(2\pi)^3} \frac{2\pi}{i|\boldsymbol{r}-\boldsymbol{r'}|} \frac{1}{2} \int_{-\infty}^{\infty} k dk \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r'}|} - e^{-ik|\boldsymbol{r}-\boldsymbol{r'}|}}{(k - q \mp i\epsilon)(k + q \pm i\epsilon)} \Rightarrow \\
&= -\frac{1}{8\pi^2 i|\boldsymbol{r}-\boldsymbol{r'}|} \int_{-\infty}^{\infty} k dk \left[\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r'}|}}{(k - q \mp i\epsilon)(k + q \pm i\epsilon)} - \frac{e^{-ik|\boldsymbol{r}-\boldsymbol{r'}|}}{(k - q \mp i\epsilon)(k + q \pm i\epsilon)} \right].
\end{aligned}$$

Consider the first integral

$$I_{\pm} = \int_{-\infty}^{\infty} k \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r'}|}}{(k - q \mp i\epsilon)(k + q \pm i\epsilon)} dk. \tag{A.38}$$

This integral has the form

$$I_{\pm} = \int_{-\infty}^{\infty} R(k) e^{iak} dk, \qquad a = |\boldsymbol{r}-\boldsymbol{r'}| \tag{A.39}$$

hence, according to (A.31), we have

$$
\begin{aligned}
I_\pm &= 2\pi i \sum_{y>0} \mathrm{Res}[R(k)e^{iak}] \\
&= 2\pi i \sum_{y>0} \mathrm{Res}\left[\frac{ke^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{(k-q\mp i\epsilon)(k+q\pm i\epsilon)}\right] \\
&= 2\pi i \frac{(\pm q + i\epsilon)e^{i(\pm q+i\epsilon)|\boldsymbol{r}-\boldsymbol{r}'|}}{2(\pm q+i\epsilon)} = \pi i e^{\pm iq|\boldsymbol{r}-\boldsymbol{r}'|}e^{-\epsilon|\boldsymbol{r}-\boldsymbol{r}'|}. \qquad \text{(A.40)}
\end{aligned}
$$

Similarly, the other integral, with the substitution $k \to -k$ is exactly the opposite of (A.40), that is

$$
\int_{-\infty}^{\infty} k \frac{e^{-ik|\boldsymbol{r}-\boldsymbol{r}'|}}{(k-q\mp i\epsilon)(k+q\pm i\epsilon)} dk = -\pi i e^{\pm iq|\boldsymbol{r}-\boldsymbol{r}'|}e^{-\epsilon|\boldsymbol{r}-\boldsymbol{r}'|}.
$$

Thus, the Green's function takes the form

$$
G_\pm(\boldsymbol{r},\boldsymbol{r}') = -\frac{1}{4\pi}\frac{e^{\pm iq|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}e^{-\epsilon|\boldsymbol{r}-\boldsymbol{r}'|}. \qquad \text{(A.41)}
$$

Therefore, equation (A.36) becomes

$$
y_\pm(\boldsymbol{r}) = \xi(\boldsymbol{r}) - \frac{1}{4\pi}\int \frac{e^{i(\pm q+i\epsilon)|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}F(\boldsymbol{r}')d^3\boldsymbol{r}'. \qquad \text{(A.42)}
$$

Notice that the choice of positive or negative imaginary part in (A.34) has a profound effect on the final solution. This choice depends on the desired behaviour of the solution at large distances, $|\boldsymbol{r}| \to \infty$. Now, to end up, we have to find $\xi(\boldsymbol{r})$, the general solution to the homogeneous equation

$$
(\nabla^2 + q^2)y(\boldsymbol{r}) = 0.
$$

The solution is $Ce^{i\boldsymbol{q}\cdot\boldsymbol{r}}$, where $\boldsymbol{q}$ is an arbitrary vector. So, the final solution is

$$
y(\boldsymbol{r}) = Ce^{i\boldsymbol{q}\cdot\boldsymbol{r}} - \frac{1}{4\pi}\int \frac{e^{i(\pm q+i\epsilon)|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}F(\boldsymbol{r}')d^3\boldsymbol{r}' \qquad \text{(A.43)}
$$

in which the constant of integration C and the direction of $\boldsymbol{q}$ are determined by the initial conditions of the problem.

A.4. Time-dependent Green's Functions

At this section, we concentrate to equations where the time evolution is of great importance. In physical problems we wish to know the time evolution of a system given some initial data. We shall see the general problems whose governing differential equations are of first and second order with respect the time variable.

A.4.1. First order

Consider the differential equation

$$Hy(\boldsymbol{r},t) + \frac{\partial}{\partial t}y(\boldsymbol{r},t) = 0 \tag{A.44}$$

where t is the time variable and H is a time-independent linear operator. Also, suppose that the H operator posses a complete set of eigenfunctions $\{\phi_n(\boldsymbol{r})\}$

$$H\phi_n(\boldsymbol{r}) = \lambda_n\phi(\boldsymbol{r}). \tag{A.45}$$

With some initial condition $y(\boldsymbol{r},t') =$ known, it is easily shown that the solution to (A.44) is a linear combination of the time-evolved eigenfuntions

$$y(\boldsymbol{r},t) = \sum_n A_n(t')\phi_n(\boldsymbol{r})e^{-\lambda_n(t-t')}. \tag{A.46}$$

At time $t = t'$ we have

$$\begin{aligned} y(\boldsymbol{r},t') &= \sum_n A_n(t')\phi_n(\boldsymbol{r}) \Rightarrow \\ A(t') &= \int \phi_n^*(\boldsymbol{r})y(\boldsymbol{r},t')d^3\boldsymbol{r}. \end{aligned}$$

Thus

$$\begin{aligned} y(\boldsymbol{r},t) &= \sum_n \int \phi_n^*(\boldsymbol{r}')y(\boldsymbol{r}',t')d^3\boldsymbol{r} \ \ \phi_n(\boldsymbol{r})e^{-\lambda_n(t-t')} \\ &= \int \sum_n \phi_n^*(\boldsymbol{r}')\phi_n(\boldsymbol{r})e^{-\lambda_n(t-t')}y(\boldsymbol{r}',t')d^3\boldsymbol{r} \\ &= \int G_1(\boldsymbol{r},\boldsymbol{r}',t,t')y(\boldsymbol{r}',t') \end{aligned} \tag{A.47}$$

where

$$G_1(\boldsymbol{r},\boldsymbol{r}',t,t') = \sum_n \phi_n^*(\boldsymbol{r}')\phi_n(\boldsymbol{r})e^{-\lambda_n(t-t')}. \tag{A.48}$$

Equation (A.47) states that $G_1(\boldsymbol{r}, \boldsymbol{r}', t, t')$ is the *Kernel* that *propagates* the function y in time from t' to $t > t'$. For this reason, G_1 is often referred to as a *propagator*.

Note that G_1 does not obey the equation

$$\left(H + \frac{\partial}{\partial t}\right) G(\boldsymbol{r}, \boldsymbol{r}', t, t') = \delta^3(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t') \tag{A.49}$$

but, instead, it obeys

$$\left(H + \frac{\partial}{\partial t}\right) G_1(\boldsymbol{r}, \boldsymbol{r}', t, t') = 0. \tag{A.50}$$

That is why we have put the subscript 1 on the G-function. In the following we treat the problem involving (A.49) and it will be clear that G_1, defined here, still plays an important role.

Suppose that we have the differential equation

$$Hy(\boldsymbol{r}, t) + \frac{\partial}{\partial t} y(\boldsymbol{r}, t) = F(\boldsymbol{r}, t). \tag{A.51}$$

where $F(\boldsymbol{r}, t)$ is a given ”source” function.

The associated Green's function, G, satisfies (A.49). However, G satisfies (A.50) for $\boldsymbol{r} \neq \boldsymbol{r}'$ and $t \neq t'$, thus, G is equal with G_1 everywhere except this point, $(\boldsymbol{r}', t')$.

Here, we introduce the *Heaviside Equation*

$$\theta(t - t') = \begin{cases} 0, & t < t' \\ 1, & t > t' \end{cases}$$

and claim that $G(\boldsymbol{r}, \boldsymbol{r}', t, t') = \theta(t - t')G_1(\boldsymbol{r}, \boldsymbol{r}', t, t')$ is the solution of (A.49). Let us show that this is indeed correct.

$$\begin{aligned}\left(H + \frac{\partial}{\partial t}\right) G(\boldsymbol{r}, \boldsymbol{r}', t, t') &= H[\theta(t - t')G_1(\boldsymbol{r}, \boldsymbol{r}', t, t')] + \frac{\partial}{\partial t}[\theta(t - t')G_1(\boldsymbol{r}, \boldsymbol{r}', t, t')] = \\ &= \theta(t - t')HG_1(\boldsymbol{r}, \boldsymbol{r}', t, t') + G_1(\boldsymbol{r}, \boldsymbol{r}', t, t')\frac{\partial}{\partial t}\theta(t - t') \\ &+ \theta(t - t')\frac{\partial}{\partial t}G_1(\boldsymbol{r}, \boldsymbol{r}', t, t') \\ &= \theta(t - t')\left(H + \frac{\partial}{\partial t}\right)G_1(\boldsymbol{r}, \boldsymbol{r}', t, t') + G_1(\boldsymbol{r}, \boldsymbol{r}', t, t')\frac{\partial}{\partial t}\theta(t - t')\end{aligned}$$

Using equation (A.50) and the θ-function identity

$$\frac{\partial}{\partial t}\theta(t - t') = \delta(t - t') \tag{A.52}$$

we are left with

$$\left(H + \frac{\partial}{\partial t}\right) G(\boldsymbol{r}, \boldsymbol{r}', t, t') = G_1(\boldsymbol{r}, \boldsymbol{r}', t, t')\delta(t - t').$$

The $\delta(t-t')$ equation is zero except at $t = t'$ so we are interested in $G_1(\boldsymbol{r}, \boldsymbol{r}', t, t')$ at $t = t'$. But from equation (A.48) it follows the identity that

$$\lim_{t \to t'} G_1(\boldsymbol{r}, \boldsymbol{r}', t, t') = \sum_n \phi_n^*(\boldsymbol{r}')\phi_n(\boldsymbol{r}) = \delta^3(\boldsymbol{r} - \boldsymbol{r}') \tag{A.53}$$

using the *completeness relation* for the complete set of eigenfunctions $\{\phi_n(\boldsymbol{r})\}$.

So, the final result is

$$\left(H + \frac{\partial}{\partial t}\right) G(\boldsymbol{r}, \boldsymbol{r}', t, t') = \delta^3(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t'). \tag{A.54}$$

Thus, we have reaches the conclusion that $G(\boldsymbol{r}, \boldsymbol{r}', t, t') = \theta(t - t')G_1(\boldsymbol{r}, \boldsymbol{r}', t, t')$ is indeed the solution to (A.49). We could say that, the propagator is a Green's function, subject to the boundary condition $G(\boldsymbol{r}, \boldsymbol{r}', t < t', t') = 0$.

So, the general solution to (A.51) is

$$y(\boldsymbol{r}, t) = y_0(\boldsymbol{r}, t) + \int d^3\boldsymbol{r}' \int_{-\infty}^{t} dt' G_1(\boldsymbol{r}, \boldsymbol{r}', t, t')F(\boldsymbol{r}', t') \tag{A.55}$$

where $y_0(\boldsymbol{r}, t)$ is the solution to the homogeneous equation.

A.4.2. Second Order

Suppose that we have the differential equation

$$Hy(\boldsymbol{r}, t) + \frac{\partial^2}{\partial t^2} y(\boldsymbol{r}, t) = 0 \tag{A.56}$$

where H is a time-independent linear operator that possesses a complete set of eigenfunctions

$$H\phi_n(\boldsymbol{r}) = \lambda_n \phi(\boldsymbol{r}).$$

Additionally, we have two initial conditions, say $y(\boldsymbol{r}, 0),\ \dot{y}(\boldsymbol{r}, 0) =$ known.

Again, it is easily shown that the general solution of (A.56) is given by

$$y(\boldsymbol{r}, t) = \sum_n \left(a_n e^{i\sqrt{\lambda}t} + b_n e^{-i\sqrt{\lambda}t}\right)\phi_n(\boldsymbol{r}). \tag{A.57}$$

Initial conditions lead to

$$
\begin{aligned}
y(\boldsymbol{r},0) &= \sum_n (a_n + b_n)\phi(\boldsymbol{r}) \\
\dot{y}(\boldsymbol{r},0) &= i\sum_n \sqrt{\lambda_n}(a_n - b_n)\phi(\boldsymbol{r})
\end{aligned}
$$

from which follows that

$$
\begin{aligned}
a_n + b_n &= \int \phi^*(\boldsymbol{r})y(\boldsymbol{r},0)d^3\boldsymbol{r} \\
a_n - b_n &= -\frac{i}{\sqrt{\lambda_n}}\int \phi^*(\boldsymbol{r})\dot{y}(\boldsymbol{r},0)d^3\boldsymbol{r}.
\end{aligned}
$$

so each coefficient is

$$
\begin{aligned}
a_n &= \frac{1}{2}\left[\int \phi^*(\boldsymbol{r})y(\boldsymbol{r},0)d^3\boldsymbol{r} - \frac{i}{\sqrt{\lambda_n}}\int \phi^*(\boldsymbol{r})\dot{y}(\boldsymbol{r},0)d^3\boldsymbol{r}\right] \\
b_n &= \frac{1}{2}\left[\int \phi^*(\boldsymbol{r})y(\boldsymbol{r},0)d^3\boldsymbol{r} + \frac{i}{\sqrt{\lambda_n}}\int \phi^*(\boldsymbol{r})\dot{y}(\boldsymbol{r},0)d^3\boldsymbol{r}\right].
\end{aligned}
$$

Substituting these into (A.57) we get

$$
\begin{aligned}
y(\boldsymbol{r},t) = &\int \sum_n \cos\left(\sqrt{\lambda_n}t\right)\phi(\boldsymbol{r})\phi^*(\boldsymbol{r'})y(\boldsymbol{r'},0)d^3\boldsymbol{r'} + \\
&+ \int \sum_n \frac{\sin\left(\sqrt{\lambda_n}t\right)}{\sqrt{\lambda_n}}\phi(\boldsymbol{r})\phi^*(\boldsymbol{r'})\dot{y}(\boldsymbol{r'},0)d^3\boldsymbol{r'} = \\
&= \int G_2(\boldsymbol{r},\boldsymbol{r'},t)y(\boldsymbol{r},0)d^3\boldsymbol{r'} + \int \tilde{G}_2(\boldsymbol{r},\boldsymbol{r'},t)\dot{y}(\boldsymbol{r'},0)d^3\boldsymbol{r'} \qquad \text{(A.61)}
\end{aligned}
$$

in which we define two Green's functions as

$$
G_2(\boldsymbol{r},\boldsymbol{r'},t) = \sum_n \cos\left(\sqrt{\lambda_n}t\right)\phi(\boldsymbol{r})\phi^*(\boldsymbol{r'})y(\boldsymbol{r'},0) \qquad \text{(A.62a)}
$$

$$
\tilde{G}_2(\boldsymbol{r},\boldsymbol{r'},t) = \sum_n \frac{\sin\left(\sqrt{\lambda_n}t\right)}{\sqrt{\lambda_n}}\phi(\boldsymbol{r})\phi^*(\boldsymbol{r'})\dot{y}(\boldsymbol{r'},0). \qquad \text{(A.62b)}
$$

Note that

$$G_2(\boldsymbol{r}, \boldsymbol{r}', t) = \frac{\partial}{\partial t}\tilde{G}_2(\boldsymbol{r}, \boldsymbol{r}', t)$$

and that initial conditions are satysfied by (A.61) since

$$G_2(\boldsymbol{r}, \boldsymbol{r}', 0) = \sum_n \phi(\boldsymbol{r})\phi^*(\boldsymbol{r}') = \delta^3(\boldsymbol{r} - \boldsymbol{r}'). \tag{A.63}$$

Now, we switch our attention to the case where there is a "source" function, *ie* with the inhomogeneous differential equation

$$Hy(\boldsymbol{r}, t) + \frac{\partial^2}{\partial t^2}y(\boldsymbol{r}, t) = F(\boldsymbol{r}, t) \tag{A.64}$$

where $F(\boldsymbol{r}, t)$ is given.

We search for the Green's function G such that

$$\left(H + \frac{\partial^2}{\partial t^2}\right)G = \delta^3(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t'). \tag{A.65}$$

Letting $\mathcal{G}$ stand for either G_2 or $\tilde{G}_2$ we have that

$$\left(H + \frac{\partial^2}{\partial t^2}\right)\mathcal{G} = 0. \tag{A.66}$$

Again, we claim that the solution is $G(\boldsymbol{r}, \boldsymbol{r}', t, t') = \theta(t - t')\mathcal{G}(\boldsymbol{r}, \boldsymbol{r}', t, t')$ and following the same procedure as for the first order Green's function case, we let the reader to show that the result for G is

$$G(\boldsymbol{r}, \boldsymbol{r}', t, t') = \theta(t - t')\tilde{G}_2(\boldsymbol{r}, \boldsymbol{r}', t, t'). \tag{A.67}$$

Therefore, the general solution to (A.64) is

$$y(\boldsymbol{r}, t) = y_0(\boldsymbol{r}, t) + \int d^3\boldsymbol{r} \int_{-\infty}^{t} dt\tilde{G}_2(\boldsymbol{r}, \boldsymbol{r}', t, t')F(\boldsymbol{r}', t') \tag{A.68}$$

where $y_0(\boldsymbol{r}, t)$ is a solution of the homogeneous equation.

Appendix B

Dyadic Analysis

B.1. Definition of a Dyadic Function

A *scalar function* (or *scalar field*) is a mapping $f : \mathcal{D} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$. In other words, it corresponds to each point of a n-dimensional space a scalar value (eg. temperature function).

A *vector function* (or *vector field*) is a mapping $\boldsymbol{F} : \mathcal{D} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$. In other words, it corresponds to each point of a n-dimensional space a n-dimensional vector (eg. velocity field).

In a 3-dimensional Cartesian space, with $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ being the orthonormal Cartesian vector base, the vector function is written as

$$\boldsymbol{F}(\boldsymbol{r}) = (F_1, F_2, F_3) = F_1(\boldsymbol{r})\hat{x}_1 + F_2(\boldsymbol{r})\hat{x}_2 + F_3(\boldsymbol{r})\hat{x}_3 = \sum_{i=1}^{3} F_i \hat{x}_i$$

where F_i with $i = 1, 2, 3$, are the scalar components of the vector and $\hat{x}_i$ are the unit vectors in the direction of $\boldsymbol{x}_i$.

Now, we consider three distinct vector functions $\boldsymbol{F}_j(\boldsymbol{r})$ denoted by

$$\boldsymbol{F}_j(\boldsymbol{r}) = \sum_{i=1}^{3} F_{ij} \hat{x}_i. \tag{B.1}$$

In other words, we define three distinct vectors at each point of space, $\boldsymbol{r}$. We then, may define the dyadic function (or *dyadic field*, or just *dyadic*) as

$$\overline{\boldsymbol{F}}(\boldsymbol{r}) = \sum_{j=1}^{3} \boldsymbol{F}_j(\boldsymbol{r}) \hat{x}_j \tag{B.2}$$

where $\boldsymbol{F}_j$ with $j = 1, 2, 3$, are the three vector components of the dyadic $\overline{\boldsymbol{F}}$.

By substituting (B.1) into (B.2) we get

$$\overline{\boldsymbol{F}}(\boldsymbol{r}) = \sum_{i=1}^{3} \sum_{j=1}^{3} F_{ij} \hat{x}_i \hat{x}_j \tag{B.3}$$

where F_{ij} with $i, j = 1, 2, 3$, are the nine scalar components of the dyadic $\overline{\boldsymbol{F}}$ and the doublet $\hat{x}_i\hat{x}_j$ are the nine unit dyadics, being formed by a pair of unit vectors. It holds that the unit dyadic vectors are not commutative, that is

$$\hat{x}_i\hat{x}_j \neq \hat{x}_j\hat{x}_i.$$

The 3×3 dyadic's scalar components F_{ij} can be arranged in a marix form, denoted by

$$[F_{ij}] = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix}. \tag{B.4}$$

In this case the dyadic is reffered as a *tensor of rank 2*.

B.2. Dyadic Algebra

B.2.1. The transpose of a dyadic

The *transpose* of a dyadic is defined by

$$(\overline{\boldsymbol{F}})^T = \sum_{j=1}^{3} \hat{x}_j \boldsymbol{F}_j = \sum_{i,j} F_{ij}\hat{x}_j\hat{x}_i = \sum_{i,j} F_{ji}\hat{x}_i\hat{x}_j. \tag{B.5}$$

Comparing (B.5) with (B.3) we see that the scalar components F_{ij} in $\overline{\boldsymbol{F}}$ have been replaced by F_{ji} in $(\overline{\boldsymbol{F}})^T$.

When the scalar components of a dyadic are symmetrical, such that

$$F_{ij} = F_{ji} \quad or \quad (\overline{\boldsymbol{F}})^T = \overline{\boldsymbol{F}}$$

the dyadic is called *symmetric dyadic* and the corresponding tensor, *symmetric tensor*.

On the other hand, when the components of a dyadic are antisymmetrical, such that

$$F_{ij} = -F_{ji} \quad or \quad (\overline{\boldsymbol{F}})^T = -\overline{\boldsymbol{F}}$$

the dyadic is called *antisymmetric dyadic* and the corresponding tensor, *antisymmetric tensor*.

One special case of a symmetric dyadic is when $F_{ii} = 1$ and $F_{ij} = 0$ with $i \neq j$. This dyadic is called *idem factor*, is denoted by $\overline{\boldsymbol{I}}$ and is defined by

$$\overline{\boldsymbol{I}} = \sum_{i=1}^{3} \hat{x}_i \hat{x}_i. \tag{B.6}$$

B.2.2. Scalar and vector product properties

We can define two *scalar products* between a vector $\boldsymbol{a}$ and a dyadic $\overline{\boldsymbol{F}}$. Firstly, the *anterior scalar product* defined by

$$\boldsymbol{a} \cdot \overline{\boldsymbol{F}} = \sum_{j=1}^{3} (\boldsymbol{a} \cdot \boldsymbol{F}_j) \hat{x}_j = \sum_i \sum_j a_i F_{ij} \hat{x}_j$$

which is a vector. Secondly, we define the *posterior scalar product* as

$$\overline{\boldsymbol{F}} \cdot \boldsymbol{a} = \sum_j \boldsymbol{F}_j (\hat{x}_j \cdot \boldsymbol{a}) = \sum_i \sum_j a_j F_{ij} \hat{x}_i = \sum_i \sum_j a_i F_{ji} \hat{x}_j$$

which is also a vector. In general, these two scalar products are not equivalent unless the dyadic is symmetric. Thus, in general, we have

$$\boldsymbol{a} \cdot \left(\overline{\boldsymbol{F}}\right)^T = \overline{\boldsymbol{F}} \cdot \boldsymbol{a}. \tag{B.7}$$

If $\overline{\boldsymbol{F}}$ is symmetrical, then

$$\boldsymbol{a} \cdot \overline{\boldsymbol{F}}_S = \overline{\boldsymbol{F}}_S \cdot \boldsymbol{a}.$$

If $\overline{\boldsymbol{F}}$ is antisymmetrical, then

$$\boldsymbol{a} \cdot \overline{\boldsymbol{F}}_{AS} = -\overline{\boldsymbol{F}}_{AS} \cdot \boldsymbol{a}.$$

If $\overline{\boldsymbol{F}} = \overline{\boldsymbol{I}}$, then

$$\boldsymbol{a} \cdot \overline{\boldsymbol{I}} = \overline{\boldsymbol{I}} \cdot \boldsymbol{a} = \boldsymbol{a}.$$

In addition, we can define two *vector products* between a vector $\boldsymbol{a}$ and a dyadic $\overline{\boldsymbol{F}}$. The *anterior vector product* defined by

$$\boldsymbol{a} \times \overline{\boldsymbol{F}} = \sum_{j=1}^{3} (\boldsymbol{a} \times \boldsymbol{F}_j)\hat{x}_j = \sum_{i,j,k} F_{ij} a_k (\hat{x}_k \times \hat{x}_i)\hat{x}_j$$

and the *posterior vector product* defined by

$$\overline{\boldsymbol{F}} \times \boldsymbol{a} = \sum_{j} \boldsymbol{F}_j (\hat{x}_j \times \boldsymbol{a}) = \sum_{i,j,k} F_{ij} a_k \hat{x}_i (\hat{x}_j \times \hat{x}_k)$$

which both are dyadics.

B.2.3. Mixed products

In vector analysis we have mixed product identities, between three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, which are

1. *Scalar mixed product*:

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}).$$

2. *Vector mixed product*:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}. \tag{B.8}$$

These identities can be generalized to include dyadics. For the scalar mixed product, we rewrite the equalities as

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = -\boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{c}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}).$$

and then, we make the substitution $\boldsymbol{c} = \overline{\boldsymbol{c}} \cdot \boldsymbol{d}$ where $\overline{\boldsymbol{c}}$ is a dyadic and $\boldsymbol{d}$ is an arbitrary vector. So, we take

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \overline{\boldsymbol{c}}) \cdot \boldsymbol{d} = -\boldsymbol{b} \cdot (\boldsymbol{a} \times \overline{\boldsymbol{c}}) \cdot \boldsymbol{d} = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \overline{\boldsymbol{c}} \cdot \boldsymbol{d}.$$

and since $\boldsymbol{d}$ is arbitrary we obtain

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \overline{\boldsymbol{c}}) = -\boldsymbol{b} \cdot (\boldsymbol{a} \times \overline{\boldsymbol{c}}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \overline{\boldsymbol{c}} \tag{B.9}$$

which is the desired identity for the scalar mixed product between two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, and a dyadic, $\overline{\boldsymbol{c}}$.

Similarly, one can show the ”dyadic version” for the vector mixed product, which is written as

$$\boldsymbol{a} \times (\boldsymbol{b} \times \overline{\boldsymbol{c}}) = \boldsymbol{b}(\boldsymbol{a} \cdot \overline{\boldsymbol{c}}) - (\boldsymbol{a} \cdot \boldsymbol{b})\overline{\boldsymbol{c}}. \tag{B.10}$$

B.3. Differentiation of a Dyadic Function

B.3.1. Divergence (div) & Circulation (curl)

We define the *divergence of a dyadic function* $\overline{\boldsymbol{F}}$, denoted by $\boldsymbol{\nabla} \cdot \overline{\boldsymbol{F}}$, as follows

$$\boldsymbol{\nabla} \cdot \overline{\boldsymbol{F}} = \sum_j (\boldsymbol{\nabla} \cdot \boldsymbol{F}_j)\hat{x}_j = \sum_i \sum_j \frac{\partial F_{ij}}{\partial x_i}\hat{x}_j \tag{B.11}$$

which is a vector function.

Now, the dyadic may be constructed by the idem factor and a scalar function as

$$\overline{\boldsymbol{F}} = f\overline{\boldsymbol{I}}$$

In this case we have

$$\boldsymbol{\nabla} \cdot \overline{\boldsymbol{F}} = \boldsymbol{\nabla} \cdot (f\overline{\boldsymbol{I}}) = \sum_j (\boldsymbol{\nabla} \cdot f\hat{x}_i)\hat{x}_i = \sum_i \frac{\partial f}{\partial x_i}\hat{x}_i = \boldsymbol{\nabla} f \tag{B.12}$$

Similarly, the *curl of a dyadic function*, denoted by $\boldsymbol{\nabla} \times \overline{\boldsymbol{F}}$, is given by

$$\boldsymbol{\nabla} \times \overline{\boldsymbol{F}} = \sum_j (\boldsymbol{\nabla} \times \boldsymbol{F}_j)\hat{x}_j \tag{B.13}$$

which is a dyadic funtion.

We may use that

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \sum_i \boldsymbol{\nabla} \times (F_i\hat{x}_i) = \sum_i \boldsymbol{\nabla} F_i \times \hat{x}_i$$

where $\boldsymbol{F}$ is a vector, and rewrite (B.13) in the form

$$\boldsymbol{\nabla} \times \overline{\boldsymbol{F}} = \sum_i \sum_j (\boldsymbol{\nabla} F_{ij} \times \hat{x}_i)\hat{x}_j. \tag{B.14}$$

which is a dyadic.

Again, if the dyadic is written as

$$\overline{\boldsymbol{F}} = f\overline{\boldsymbol{I}}$$

we have

$$\boldsymbol{\nabla} \times \overline{\boldsymbol{F}} = \boldsymbol{\nabla} \times (f\overline{\boldsymbol{I}}) = \sum_j \boldsymbol{\nabla} \times (f\hat{x}_i)\hat{x}_i = \sum_i (\boldsymbol{\nabla} f \times \hat{x}_i)\hat{x}_i = \boldsymbol{\nabla} f \times \overline{\boldsymbol{I}} \qquad \text{(B.15)}$$

which is also a dyadic.

B.3.2. Gradient of a Vector

In dyadic analysis is allowed to define the *gradient of a vector function*, denoted by $\boldsymbol{\nabla F}$, where $\boldsymbol{F}$ is a vector function. We write

$$\boldsymbol{\nabla F} = \sum_j (\boldsymbol{\nabla} F_j)\hat{x}_j = \sum_i \sum_j \frac{\partial F_j}{\partial x_i}\hat{x}_i\hat{x}_j \qquad \text{(B.16)}$$

which is dyadic.

We can see that (B.16) recovers the divergence of the vector function $\boldsymbol{F}$ if we put the scalar product, $\cdot$, between vectors on both sides.

B.3.3. Differentiation Identities

For any vector function $\boldsymbol{a}$, any scalar function f and any dyadic function $\overline{\boldsymbol{b}}$ the following identities hold:

1. $\boldsymbol{\nabla} \cdot (f\overline{\boldsymbol{b}}) = f(\boldsymbol{\nabla} \cdot \overline{\boldsymbol{b}}) + (\boldsymbol{\nabla} f) \cdot \overline{\boldsymbol{b}}$
2. $\boldsymbol{\nabla} \times (f\overline{\boldsymbol{b}}) = f(\boldsymbol{\nabla} \times \overline{\boldsymbol{b}}) + (\boldsymbol{\nabla} f) \times \overline{\boldsymbol{b}}$
3. $\boldsymbol{\nabla} \cdot (\boldsymbol{a} \times \overline{\boldsymbol{b}}) = (\boldsymbol{\nabla} \times \boldsymbol{a}) \cdot \overline{\boldsymbol{b}} - \boldsymbol{a} \cdot (\boldsymbol{\nabla} \times \overline{\boldsymbol{b}})$
4. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \overline{\boldsymbol{b}}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \overline{\boldsymbol{b}})$
5. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \overline{\boldsymbol{b}}) = 0$

B.4. Dyadic Integral Theorems

We can illustrate here the alternative version of *vector Green's theorems* involving dyadic functions. These dyadic integral theorems are derived by replacing the vector functions with dyadic functions.

Firstly, we may write down the vector Green's theorems. In order to do so, we introduce the Gauss theorem which states that

$$\oint_S \boldsymbol{F} \cdot \hat{n}\, dS = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{F}\, dV \tag{B.17}$$

where V is the enclosed space by the surface S and $\hat{n}$ is the normal unit vector to S going outwards.

By letting $\boldsymbol{F} = \boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b})$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors, we have

$$\begin{cases} \boldsymbol{\nabla} \cdot F = \boldsymbol{\nabla} \cdot [\boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b})] = (\boldsymbol{\nabla} \times \boldsymbol{b}) \cdot (\boldsymbol{\nabla} \times \boldsymbol{a}) - \boldsymbol{a} \cdot [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{b})] \\ \\ \hat{n} \cdot \boldsymbol{F} = \hat{n} \cdot [\boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b})]. \end{cases}$$

By substituting these into (B.17) we get

$$\int_V \Big\{ (\boldsymbol{\nabla} \times \boldsymbol{b}) \cdot (\boldsymbol{\nabla} \times \boldsymbol{a}) - \boldsymbol{a} \cdot [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{b})] \Big\}\, dV = \oint_S \hat{n} \cdot [\boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b})]\, dS. \tag{B.18}$$

Equation (B.18) is reffered as *first vector Green's theorem.*

Additionally, we may now let $\boldsymbol{F} = \boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b}) - \boldsymbol{b} \times (\boldsymbol{\nabla} \times \boldsymbol{a})$. By substituting $\boldsymbol{\nabla} \cdot \boldsymbol{F}$ and $\hat{n} \cdot \boldsymbol{F}$ into (B.17) we get the *second vector Green's theorem* which is

$$\int_V \Big\{ \boldsymbol{b} \cdot [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a})] - \boldsymbol{a} \cdot [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{b})] \Big\}\, dV = \oint_S \hat{n} \cdot [\boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b}) - \boldsymbol{b} \times (\boldsymbol{\nabla} \times \boldsymbol{a})]\, dS. \tag{B.19}$$

Thus, we are now in position to introduce the dyadic Green's theorems.

1. **First vector-dyadic Green's theorem**

   Here, we replace, in (B.18), one of the two vectors with a dyadic. To do this, we rewrite (B.18) as follows

   $$\int_V \Big\{ (\boldsymbol{\nabla} \times \boldsymbol{a}) \cdot (\boldsymbol{\nabla} \times \boldsymbol{b}) - \boldsymbol{a} \cdot [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{b})] \Big\}\, dV = \oint_S \hat{n} \cdot [\boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b})]\, dS \tag{B.20}$$

   where we have purposely placed $(\boldsymbol{\nabla} \times \boldsymbol{b})$ in the posterior position in the left hand side of the integral equation.
   Concider three distict vectors $\boldsymbol{b}_j$ with $j = 1, 2, 3$, such that the obey (B.20). By putting $\hat{x}_j$ at the posterior position of each term in (B.20) and each of the three equations and summing them we obtain

$$\int_V \left\{(\boldsymbol{\nabla}\times\boldsymbol{a})\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{b}})-\boldsymbol{a}\cdot[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\overline{\boldsymbol{b}})]\right\} dV = \oint_S \hat{n}\cdot[\boldsymbol{a}\times(\boldsymbol{\nabla}\times\overline{\boldsymbol{b}})]\; dS \quad \text{(B.21)}$$

where $\overline{\boldsymbol{b}}$ is the dyadic function. By definition we have that

$$\boldsymbol{\nabla}\times\overline{\boldsymbol{b}} = \sum_j (\boldsymbol{\nabla}\times\boldsymbol{b}_j)\hat{x}_j$$

and

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\overline{\boldsymbol{b}}) = \sum_j [\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{b}_j)]\hat{x}_j.$$

Equation (B.21) is known as *first vector-dyadic Green's theorem of type A*. By interchanging $\boldsymbol{a}$ and $\overline{\boldsymbol{b}}$ we obtain the *first vector-dyadics Green's theorem of type B* which is

$$\int_V \left\{(\boldsymbol{\nabla}\times\boldsymbol{a})\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{b}})-[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{a})]\cdot\overline{\boldsymbol{b}}\right\} dV = -\oint_S \hat{n}\cdot[(\boldsymbol{\nabla}\times\boldsymbol{a})\times\overline{\boldsymbol{b}})]\; dS. \quad \text{(B.22)}$$

2. Second vector-dyadic Green's theorem

The second vector-dyadic Green's theorem comes from equation (B.19) by substituting the vector $\boldsymbol{b}$ with the dyadic $\overline{\boldsymbol{b}}$ as we did before. Equally, this theorem can be produced by substracting (B.21) from (B.22). Either way, we get

$$\int_V \left\{\boldsymbol{a}\cdot[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\overline{\boldsymbol{b}})]-[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{a})]\cdot\overline{\boldsymbol{b}}\right\} dV = -\oint_S \hat{n}\cdot[\boldsymbol{a}\times(\boldsymbol{\nabla}\times\overline{\boldsymbol{b}})+(\boldsymbol{\nabla}\times\boldsymbol{a})\times\overline{\boldsymbol{b}})]\; dS. \quad \text{(B.23)}$$

3. First dyadic-dyadic Green's theorem

Using identiy (B.7) and (B.9) we can rewrite (B.21) as

$$\int_V \left\{\left[(\boldsymbol{\nabla}\times\overline{\boldsymbol{b}})\right]^T\cdot(\boldsymbol{\nabla}\times\boldsymbol{a})-\left[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\overline{\boldsymbol{b}})\right]^T\cdot\boldsymbol{a}\right\} dV = \oint_S \left[\boldsymbol{\nabla}\times\overline{\boldsymbol{b}}\right]^T\cdot(\hat{n}\times\boldsymbol{a})\; dS$$

in order to place the vectors including the vector $\boldsymbol{a}$ in the posterior position. Now, we convert the vector $\boldsymbol{a}$ into a dyadic $\overline{\boldsymbol{a}}$ as we did in first vector-dyadic Green's theorem and we obtain the *first dyadic-dyadic Green's theorem of type A* given by the formula

$$\int_V \left\{ \left[(\boldsymbol{\nabla} \times \overline{\boldsymbol{b}})\right]^T \cdot (\boldsymbol{\nabla} \times \overline{\boldsymbol{a}}) - \left[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \overline{\boldsymbol{b}})\right]^T \cdot \overline{\boldsymbol{a}} \right\} dV = \oint_S \left[\boldsymbol{\nabla} \times \overline{\boldsymbol{b}}\right]^T \cdot (\hat{n} \times \overline{\boldsymbol{a}})\, dS. \tag{B.24}$$

By interchanging $\overline{\boldsymbol{a}}$ and $\overline{\boldsymbol{b}}$ we obtain the *first dyadic-dyadic Green's theorem of type B*, which is

$$\int_V \left\{ \left[(\boldsymbol{\nabla} \times \overline{\boldsymbol{b}})\right]^T \cdot (\boldsymbol{\nabla} \times \overline{\boldsymbol{a}}) - \left[\overline{\boldsymbol{b}}\right]^T \cdot \left[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \overline{\boldsymbol{a}})\right] \right\} dV = \oint_S \left[\overline{\boldsymbol{b}}\right]^T \cdot \left[\hat{n} \times (\boldsymbol{\nabla} \times \overline{\boldsymbol{a}})\right] dS. \tag{B.25}$$

4. Second dyadic-dyadic Green's theorem

Substracting (B.24) from (B.25), we get the *second dyadic-dyadic Green's theorem* given by the formula

$$\int_V \left\{ \left[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \overline{\boldsymbol{b}})\right]^T \cdot \overline{\boldsymbol{a}} - [\overline{\boldsymbol{b}}]^T \cdot \left[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \overline{\boldsymbol{a}})\right] \right\} dV$$

$$= - \oint_S \left\{ \left[\boldsymbol{\nabla} \times \overline{\boldsymbol{b}}\right]^T \cdot (\hat{n} \times \overline{\boldsymbol{a}}) - [\overline{\boldsymbol{b}}]^T \cdot \left[\hat{n} \times (\boldsymbol{\nabla} \times \overline{\boldsymbol{a}})\right] \right\} dS. \tag{B.26}$$

Appendix C

Spherical Harmonics

C.1. (Scalar) Spherical Harmonics

Spherical Harmonics are functions, Y, defined of the surface of a sphere, $Y : S^2 \to \mathbb{R}$. They form a complete set of orthogonal functions on the sphere and they represent the normal modes of oscillation of a spherical surface. Thus, they may be used to express functions defined on the surface of a sphere.

Let us now evaluate their mathematical form.

Suppose we have the *Laplace Equation*

$$\nabla^2 f(\boldsymbol{r}) = 0$$

where $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ and is reffered as *Laplacian*.

Every function that satisfies this equation is said to be an *Harmonic Function*. Thus, one may could guess that the Spherical Harmonics are (a part of) the solutions to the Laplace Equation expressed in spherical coordinates.

Laplace Equation expressed in spherical coordinates has the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} = 0.$$

with $r \in (0,\infty)$, $\theta \in [0,\pi]$ and $\phi[0,2\pi)$.

Let us try to solve the differential equation applying *separation of variables*, $f(r,\theta,\phi) = R(r)Y(\theta,\phi)$. Doing so, we are led to two differential equations

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) = \lambda R(r) \tag{C.1a}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -\lambda Y. \tag{C.1b}$$

The functions $Y = Y(\theta,\phi)$ are functions of θ, ϕ only, and are the desired *Spherical Harmonics*.

We now apply separation of variables again, by setting $Y(\theta,\phi) = \Theta(\theta)\Phi(\phi)$, and we get

$$\frac{d^2\Phi(\phi)}{d\phi^2} = -m^2\Phi \tag{C.2a}$$

$$\lambda\Theta(\theta)\sin^2\theta + \sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) = m^2\Theta(\theta). \tag{C.2b}$$

Solution to (C.2a) is, of course, $e^{im\phi}$. However, Φ must be a periodic function with a period $T = 2\pi$. So, we have

$$e^{im(2\pi+\phi)} = e^{im\phi} \Rightarrow e^{i2\pi m} = 1 \Rightarrow 2\pi m = 2\pi n \Rightarrow m \in \mathbb{Z}.$$

Solution to (C.2b): We set $t := \cos\theta$, $\{\theta \to 0 \Rightarrow t \to 1, \quad \theta \to \pi \Rightarrow t \to -1\}$, thus the differential equation takes the form

$$\frac{d}{dt}\left[(1-t^2)\frac{d\Theta(t)}{dt}\right] + \left(\lambda - \frac{m^2}{1-t^2}\right)\Theta(t) = 0 \tag{C.3}$$

which is a family of differential equations, one for each $m \in \mathbb{Z}$.

Equation (C.3) is known as *General Legendre Equation*, named after the French mathematician Adrien-Marie Legendre (1752-1833).

For $m = 0$ we take the *Legendre Equation*

$$\frac{d}{dt}\left[(1-t^2)\frac{d\Theta(t)}{dt}\right] + \lambda\Theta(t) = 0. \tag{C.4}$$

It is easily shown that the unknown constant λ must be the product two consecutive integers $l(l+1)$.

Equation (C.4), for $\lambda = l(l+1)$, possesses as solutions polynomials of rank l, known as *Legendre polynomials*, which may be expressed through the Rodrigues' formula as

$$P_l(t) = \frac{1}{2!l!}\frac{d^l}{dt^l}\left[(t^2-1)^l\right]. \tag{C.5}$$

These polynomials form an orthogonal base

$$\int_{-1}^{1} P_l(t)P_{l'}(t)\,dt = \frac{2}{2l+1}\delta_{ll'}.$$

For $m \neq 0$ solutions to (C.3) are the *associated Legendre functions* which are produced by the Legendre's polynomials through the formula

$$P_l^m(t) := (1-t^2)^{m/2}\frac{d^m}{dt^m}\Big(P_l(t)\Big), \quad m \leq l \tag{C.6}$$

with $\lambda = l(l+1)$ and $m \in \mathbb{Z}^+ = 0, 1, 2, \ldots, (l-1), l$.

Associated Legendre functions form an orthogonal base

$$\int_{-1}^{1} P_l^m(t)P_{l'}^{m'}(t)\ dt = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}\delta_{mm'}.$$

for $l = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots, l$.

Combining the $\Theta(\theta)$, $\Phi(\phi)$ solutions we can obtain the normalized Spherical Harmonics $Y_l^m(\theta, \phi)$

$$Y_m^l(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}(-1)^m P_l^m(\cos\theta)e^{im\phi}, \quad l \geq m \geq 0. \tag{C.7}$$

We expand the definition for negative integers of m as

$$Y_l^m(\theta,\phi) = (-1)^m\Bigg(Y_l^{-m}(\theta,\phi)\Bigg)^*$$

Therefore, the set of functions

$$\{Y_l^m(\theta,\phi),\ \ l = 0, 1, 2, 3, \ldots;\ \ -l \leq m \leq l,\ \ m \in \mathbb{Z}\}$$

is the complete, orthonormal set of spherical harmonics, with the orthonormality relation

$$\int Y_l^m(\theta,\phi)Y_{l'}^{m'*}(\theta,\phi)d\Omega = \delta_{ll'}\delta_{mm'}.$$

and the closure relation

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_l^m(\theta,\phi)Y_l^{m*}(\theta',\phi') = \delta(\Omega-\Omega') = \frac{1}{\sin\theta}\delta(\theta-\theta')\delta(\phi-\phi').$$

Therefore, on the unit sphere, any square-integrable function can be expanded as a linear combination of these function, as

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=l} f_l^m Y_l^m(\theta,\phi). \tag{C.8}$$

C.1.1. Connection with Orbital Angular Momentum

In quantum mechanics the orbital angular momentum is represented by a vector operator, $\hat{\boldsymbol{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$, given by

$$\hat{\boldsymbol{L}} = -i\boldsymbol{r} \times \boldsymbol{\nabla}$$

where we have set $\hbar = 1$.

Its components, $\hat{L}_i, i = x, y, z$, follow the commutation relations

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_i jk\hat{L}_k$$

that is, they do not commute.

Furthermore, we have that

$$[\hat{\boldsymbol{L}}^2, \hat{L}_i] = 0, \quad i = x, y, z.$$

This means that from the four physical quantities $L_x, L_y, L_z, \boldsymbol{L}^2$ we can simultaneously measure only the square of the angular momentum's magnitude, $\boldsymbol{L}^2$, and one of its components, say L_z. Mathematically speaking, their associated operators posses common eigenstates-eigenfunctions.

We may now express the operators $\hat{\boldsymbol{L}}^2$ and $\hat{L}_z$ in spherical coordinates.

For $\hat{\boldsymbol{L}}^2$ we have

$$\begin{aligned}\hat{\boldsymbol{L}}^2 &= -r^2\nabla^2 + \left(r\frac{\partial}{\partial r} + 1\right)r\frac{\partial}{\partial r} \\ &= -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.\end{aligned} \tag{C.9}$$

Now, taking a closer look at equation (C.1b) one can see that this equation is nothing more than

$$\hat{\boldsymbol{L}}^2 Y_l^m(\theta, \phi) = l(l+1)Y_l^m(\theta, \phi) \tag{C.10}$$

where we have set $\lambda = l(l+1)$ and we have put as subscript and upperscript the $l, m \in \mathbb{Z} : -l \leq m \leq l$, denoting the correct associated spherical harmonic.

Therefore, $Y_l^m(\theta, \phi)$ are eigenfunctions of the orbital angular momentum operator $\hat{\boldsymbol{L}}^2$ with their corresponding eigenvalues being $l(l+1)$.

For$\hat{L}_z$ one can see that

$$\begin{aligned}\hat{L}_z Y^l_m(\theta,\phi) &= -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)Y^l_m(\theta,\phi)\\ &= -i\frac{\partial}{\partial\phi}Y^l_m(\theta,\phi) = -i(im)Y^l_m(\theta,\phi) = mY^l_m(\theta,\phi) \end{aligned} \tag{C.11}$$

so, $Y^l_m(\theta,\phi)$ are also eigenfunctions of the $\hat{L}_z$ operator with their corresponding eigenvalues being m.

Hence, we have found that $Y^l_m(\theta,\phi)$ are the common eigenstates of the simultaneously measurable quantities $\boldsymbol{L}^2,\ L_z$.

In the following section we treat the situation where, except of the orbital angular momentum, there is a spin angular momentum in presence.

C.2. Tensor Spherical Harmonics

Consider a system with an orbital angular momentum $\boldsymbol{L}$ and a spin angular momentum $\boldsymbol{S}$. Then, the total angular momentum of the system is denoted by $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$. Both orbital and spin angular momenta posses their own eigenstates, and in Dirac's notation that is

$$\begin{cases}\hat{\boldsymbol{L}}^2\,|lm_l\rangle = l(l+1)\,|lm_l\rangle\\ \hat{L}_z\,|lm_l\rangle = m_l\,|lm_l\rangle\end{cases} \tag{C.12a}$$

$$\begin{cases}\hat{\boldsymbol{S}}^2\,|sm_s\rangle = s(s+1)\,|sm_s\rangle\\ \hat{S}_z\,|sm_s\rangle = m_s\,|sm_s\rangle\,.\end{cases} \tag{C.12b}$$

where $l \in \mathbb{Z}^+$, $s \in \frac{1}{2}\mathbb{Z}^+$ and $-l, s \leq m_{l,s} \leq l, s$ with an integer step.[1] One sees that for fixed l, the $\{|lm_l\rangle\}$ basis consists of $(2l+1)$ independent eigenstates. The same stands for the $\{|sm_s\rangle\}$ basis.

Note: The eigenstates $|lm_l\rangle$ and $|sm_s\rangle$ are two different *irreducible* representations of the same group, SO(3), known as *rotation group*.

All possible eigenstates of the total angular momentum can be given by the direct product of these two bases, $|lm_l\rangle \otimes |sm_s\rangle := |lm_l; sm_s\rangle$. See that for fixed l and s the (*uncoupled*) direct product basis has $(2l+1)(2s+1)$ elements, and thus, the total angular momentum states exist in a $(2l+1)(2s+1)$-dimensional vector space.

[1] $\frac{1}{2}\mathbb{Z}^+$ denotes all the non-negative integers and all positive half integers.

However, the total angular momentum operator, as an ordinary angular momentum operator, satisfies the same commutation relations as $\boldsymbol{L}$ and $\boldsymbol{S}$

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad i,j,k = \{x,y,z\}.$$

Hence, it do possesses its own *coupled* eigenstates, $|jm;ls\rangle := |jm\rangle$, as any angular momentum operator, and that is

$$\begin{cases} \hat{\boldsymbol{J}}\,|jm\rangle = j(j+1)\,|jm\rangle \\ \hat{J}_z\,|jm\rangle = m\,|jm\rangle\,. \end{cases} \tag{C.13}$$

where $|l-s| \leq j \leq l+s$ and $m \in \{-J,\ldots,J-1,J\}$. Of course, the number of the elements of the *coupled* basis $\{|jm\rangle\}$ must be equal with $(2l+1)(2s+1)$; l, s fixed.

Note: The combination of the two angular momenta; *ie* the combination of the two *irreducible* representations of SO(3), $\{|lm_l\rangle\}$ and $\{|sm_s\rangle\}$, leads to a different representation, $\{|jm\rangle\}$ of the same group, but this time a reducible one; for each j we have one $(2j+1)$-dimensional proper sub-representation of SO(3).

Now, one can express the total angular momentum basis $\{|jm\rangle\}$ (*coupled basis*) in terms of the direct product basis $\{|lm_l;sm_s\rangle\}$ (*uncoupled basis*)[2]

$$|jm;ls\rangle := |jm\rangle = \sum_{m_l=-l}^{l}\sum_{m_s=-s}^{s} a_{m_l m_s}\,|lm_l;sm_s\rangle \tag{C.14}$$

where the expansion coefficients are given by

$$a_{m_l m_s} = \langle lm_l;sm_s|jm\rangle\,.$$

which are known as *Clebsch–Gordan coefficients*, named after the German mathematicians Alfred Clebsch (1833-1872) and Paul Gordan (1837-1912).

Applying the operator

$$\hat{J}_z = \hat{L}_z \otimes 1 + 1 \otimes \hat{S}_z \tag{C.15}$$

on both sides of (C.14), we get

$$m\,|jm\rangle = \sum_{m_l=-l}^{l}\sum_{m_s=-s}^{s} a_{m_l m_s}(m_l+m_s)\,|lm_l;sm_s\rangle \tag{C.16}$$

and acting from the left with $\langle lm_l;sm_s|$, we get

[2]Since all operators are Hermitean, $\{|lm_l\rangle\}$, $\{|sm_s\rangle\}$ and $\{|jm\rangle\}$ are all complete bases.

$$
\begin{aligned}
m \langle lm_l'; sm_s' | jm \rangle &= \sum_{m_l=-l}^{l} \sum_{m_s=-s}^{s} a_{m_l m_s}(m_l + m_s) \langle lm_l'; sm_s' | lm_l; sm_s \rangle \\
&= a_{m_l' m_s'}(m_l' + m_s') = (m_l' + m_s') \langle lm_l'; sm_s' | jm \rangle \Rightarrow \\
[m - (m_l + m_s)] \langle lm_l; sm_s | jm \rangle = 0. & \qquad \text{(C.17)}
\end{aligned}
$$

Thus, for $m \neq m_l + m_s$ we have $a_{m_l m_s} = \langle lm_l; sm_s | jm \rangle = 0$ which means that Clebsch-Gordon coefficients exist only for $m = m_l + m_s$.

Let us now return to the coordinate representation.

As already said, $Y_l^{m_l}(\theta, \phi)$ are simultaneous eigenstates of $\boldsymbol{L}^2$ and L_z (see equations (C.10) and (C.11)). We may similarly define $\chi_s^{m_s}$ to be simultaneous eigenstates of $\boldsymbol{S}^2$ and S_z

$$
\begin{cases}
\hat{\boldsymbol{S}}^2 \chi_s^{m_s} = s(s+1)\chi_s^{m_s} \\
\hat{S}_z \chi_s^{m_s} = m_s \chi_s^{m_s}.
\end{cases}
\qquad \text{(C.18)}
$$

Then, we denote the direct product basis in the coordinate representation as $Y_l^{m_l} \chi_{sm_s}$.

Additionally, as we have mentioned, the total angular momentum basis consists of simultaneous eigenstates of $\boldsymbol{J}^2$ and J_z alongside with $\boldsymbol{L}^2$ and $\boldsymbol{S}^2$. These eigenstates may be called *tensor spherical harmonics* and symbolized by $\mathcal{Y}_{jm}^{ls}$. By definition, they satisfy

$$
\begin{cases}
\hat{\boldsymbol{J}}^2 \mathcal{Y}_{jm}^{ls}(\theta, \phi) = j(j+1)\mathcal{Y}_{jm}^{ls}(\theta, \phi) \\
\hat{J}_z \mathcal{Y}_{jm}^{ls}(\theta, \phi) = m\mathcal{Y}_{jm}^{ls}(\theta, \phi)
\end{cases}
$$

$$
\begin{aligned}
\hat{\boldsymbol{L}}^2 \mathcal{Y}_{jm}^{ls}(\theta, \phi) &= l(l+1)\mathcal{Y}_{jm}^{ls}(\theta, \phi) \\
\hat{\boldsymbol{S}}^2 \mathcal{Y}_{jm}^{ls}(\theta, \phi) &= s(s+1)\mathcal{Y}_{jm}^{ls}(\theta, \phi).
\end{aligned}
$$

So, according to (C.14), we may express the total angular momentum eigenstates, $\mathcal{Y}_{jm}^{lm}$, in terms of the direct product basis, $Y_l^{m_l} \chi_s^{m_s}$

$$
\mathcal{Y}_{jm}^{ls}(\theta, \phi) = \sum_{m_l=-l}^{l} \sum_{m_s=-s}^{s} \langle lm_l; sm_s | jm \rangle \, Y_l^{m_l}(\theta, \phi) \chi_s^{m_s}
$$

Since Clebsch-Gordon coefficients vanish for all $m \neq m_l + m_s$ the above formula becomes

$$\mathcal{Y}^{ls}_{jm}(\theta,\phi) = \sum_{m_s=-s}^{s} \langle l, m-m_s; sm_s|jm\rangle\, Y_l^{m-m_s}(\theta,\phi)\chi_s^{m_s} \tag{C.20}$$

letting us with a single summation over m_s.

Now, we shall continue with two special cases: one for spin $s = 1/2$ and one for spin $s = 1$ where arise some special tensor spherical harmonics, known as *spin spherical harmonics* and *vector spherical harmonics*.

C.2.1. Spin Spherical Harmonics (s=1/2)

For $s = 1/2$ the possible values of j are $j = l \pm \frac{1}{2}$; l =fixed, where, of course, for $l = 0$ the total angular momentum is just the spin $j = s = \frac{1}{2}$. For each value of j we have $m_s = \pm\frac{1}{2}$. The corresponding Clebsch Gordon coefficients are given in Table C.1.

j	$m_s = \frac{1}{2}$	$m_s = -\frac{1}{2}$
$\ell + \frac{1}{2}$	$\left(\frac{\ell+m+\frac{1}{2}}{2\ell+1}\right)^{1/2}$	$\left(\frac{\ell-m+\frac{1}{2}}{2\ell+1}\right)^{1/2}$
$\ell - \frac{1}{2}$	$-\left(\frac{\ell-m+\frac{1}{2}}{2\ell+1}\right)^{1/2}$	$\left(\frac{\ell+m+\frac{1}{2}}{2\ell+1}\right)^{1/2}$

Table C.1: Clebsch-Gordon coefficients for $s = 1/2$.

According to equation (C.20) we have

$$\mathcal{Y}^{l\frac{1}{2}}_{j=l\pm\frac{1}{2},m}(\theta,\phi) = \pm\left(\frac{l\pm m+\frac{1}{2}}{2l+1}\right)^{1/2} Y_l^{m-\frac{1}{2}}(\theta,\phi)\chi_{\frac{1}{2}}^{\frac{1}{2}} + \left(\frac{l\mp m+\frac{1}{2}}{2l+1}\right)^{1/2} Y_l^{m+\frac{1}{2}}(\theta,\phi)\chi_{\frac{1}{2}}^{-\frac{1}{2}}$$

Furthermore, we may represent $\chi_{\frac{1}{2}}^{\frac{1}{2}} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\chi_{\frac{1}{2}}^{-\frac{1}{2}} = \begin{bmatrix}0\\1\end{bmatrix}$, thus, the spin spherical harmonics are formulated as

$$\mathcal{Y}^{l\frac{1}{2}}_{l\pm\frac{1}{2},m}(\theta,\phi) = \frac{1}{\sqrt{2l+1}}\begin{bmatrix}\pm\sqrt{l\pm m+\frac{1}{2}}\; Y_l^{m-\frac{1}{2}}(\theta,\phi)\\ \sqrt{l\mp m+\frac{1}{2}}\; Y_l^{m+\frac{1}{2}}(\theta,\phi)\end{bmatrix}. \tag{C.21}$$

For $l = 0$ there is only one spin spherical harmonic, for $j = 1/2$

$$\mathcal{Y}^{0\frac{1}{2}}_{j=\frac{1}{2},m}(\theta,\phi) = \begin{bmatrix}\sqrt{m+\frac{1}{2}}\; Y_0^{m-\frac{1}{2}}(\theta,\phi)\\ \sqrt{\frac{1}{2}-m}\; Y_0^{m+\frac{1}{2}}(\theta,\phi)\end{bmatrix}. \tag{C.22}$$

These are two-dimensional vectors; the square of the first component represents the probability for the particle to be at "position" (θ, ϕ) with spin up, $m_s = \frac{1}{2}$, while the square of the second component with spin down, $m_s = -\frac{1}{2}$.

Thus, for the $l = 0$ case, if $m = \frac{1}{2}$ the second component in (C.22) is zero since $m = m_s = \frac{1}{2}$ and there is no probability to find the particle (anywhere in space) with a down spin. The same stands for $m = -\frac{1}{2}$.

C.2.2. Vector Spherical Harmonics (s=1)

For $s = 1$ the possible values of j are $j = l \pm 1$; l =fixed, where, of course, for $l = 0$ the total angular momentum has $j = 1$. For each value of j we have $(2 \cdot 1 + 1)$ different states for $m_s = -1, 0, +1$. Thus, we have three linearly independent *vector spherical harmonics* $\{\mathcal{Y}^{l1}_{j=l\pm1},\ \mathcal{Y}^{l1}_{j=l}\}$ forming a basis in $\mathbb{R}^3$. Let us write them in terms of the (scalar) spherical harmonics Y_l^m whose mathematical expression is known.

We expand each vector spherical harmonic in terms of the direct product eigenfunctions, $Y_L^{m-m_s}\chi_1^{m_s}$, representing $\chi_1^1 = [1\ 0\ 0]^T$, $\chi_1^0 = [0\ 1\ 0]^T$ and $\chi_1^{-1} = [0\ 0\ 1]^T$. Given the corresponding Clebsch-Gordon coefficients in Table C.2 the expansion of $\mathcal{Y}^{l1}_{j=l,m}$ is

j	$m_s = 1$	$m_s = 0$	$m_s = -1$
$\ell + 1$	$\left[\frac{(\ell+m)(\ell+m+1)}{(2\ell+1)(2\ell+2)}\right]^{1/2}$	$\left[\frac{(\ell-m+1)(\ell+m+1)}{(\ell+1)(2\ell+1)}\right]^{1/2}$	$\left[\frac{(\ell-m)(\ell-m+1)}{(2\ell+1)(2\ell+2)}\right]^{1/2}$
ℓ	$-\left[\frac{(\ell-m+1)(\ell+m)}{2\ell(\ell+1)}\right]^{1/2}$	$\frac{m}{\sqrt{\ell(\ell+1)}}$	$\left[\frac{(\ell-m)(\ell+m+1)}{2\ell(\ell+1)}\right]^{1/2}$
$\ell - 1$	$\left[\frac{(\ell-m)(\ell-m+1)}{2\ell(2\ell+1)}\right]^{1/2}$	$-\left[\frac{(\ell-m)(\ell+m)}{\ell(2\ell+1)}\right]^{1/2}$	$\left[\frac{(\ell+m)(\ell+m+1)}{2\ell(2\ell+1)}\right]^{1/2}$

Table C.2: Clebsch-Gordon coefficients for $s = 1$.

$$\mathcal{Y}^{l1}_{j=l,m} = \begin{bmatrix} -\left[\frac{(l-m+1)(l+m)}{2l(l+1)}\right]^{1/2} Y_l^{m-1}(\theta,\phi) \\ \frac{m}{\sqrt{l(l+1)}} Y_l^m(\theta,\phi) \\ \left[\frac{(l-m)(l+m+1)}{2l(l+1)}\right]^{1/2} Y_l^{m+1}(\theta,\phi) \end{bmatrix}. \tag{C.23}$$

The other two vectors $\mathcal{Y}^{l1}_{j=l\pm1,m}$ are similarly expressed, using the corresponding Glebsch-Gordan coefficients. For $l = 0$, only $\mathcal{Y}^{l1}_{j=1,m}$ survives.

We could leave this discussion here, however, there is a more elegant and compact formulation than the one given above.

We choose to work in a Cartesian basis where the $\chi_1^{m_s}$ are eigenvectors of $\mathcal{S}_3$, which is the third element of the spin-1 spin matrices given by $(\mathcal{S}_k)_{ij} = -i\epsilon_{ijk}$. Particularly,

$$\mathcal{S}_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathcal{S}_3 \chi_1^{m_s} = m_s \chi_1^{m_s}.$$

Searching for these eigenvectors we find

$$\chi_1^{\pm 1} = \begin{bmatrix} \mp 1 \\ -i \\ 0 \end{bmatrix}, \qquad \chi_1^{0} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, expansion (C.20) leads to

$$\mathcal{Y}^{l1}_{j=l,m}(\theta,\phi) = \frac{1}{2\sqrt{l(l+1)}} \begin{bmatrix} [(l-m+1)(l+m]^{1/2} Y_l^{m-1}(\theta,\phi) + [(l+m+1)(l-m)]^{1/2} Y_l^{m+1}(\theta,\phi) \\ i[(l-m+1)(l+m]^{1/2} Y_l^{m-1}(\theta,\phi) - i[(l+m+1)(l-m)]^{1/2} Y_l^{m+1}(\theta,\phi) \\ 2m Y_l^m(\theta,\phi) \end{bmatrix}.$$

This is a Cartesian vector in $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ basis. It is convenient, though, to express it in terms of spherical coordinates $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$

$$\begin{aligned} \hat{\boldsymbol{x}} &= \hat{\boldsymbol{r}} \sin\theta\cos\phi + \hat{\boldsymbol{\theta}} \cos\theta\cos\phi - \hat{\boldsymbol{\phi}} \sin\phi \\ \hat{\boldsymbol{y}} &= \hat{\boldsymbol{r}} \sin\theta\sin\phi + \hat{\boldsymbol{\theta}} \cos\theta\sin\phi + \hat{\boldsymbol{\phi}} \cos\phi \\ \hat{\boldsymbol{z}} &= \hat{\boldsymbol{r}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta. \end{aligned} \tag{C.24}$$

After calculations, it can be shown that $\mathcal{Y}^{l1}_{j=l,m}$ expressed in spherical coordinates is given by the formula

$$\mathcal{Y}^{l1}_{j=l,m} = \frac{i}{\sqrt{l(l+1}} \left[\frac{\hat{\boldsymbol{\theta}}}{\sin\theta} \frac{\partial}{\partial\phi} Y_l^m(\theta,\phi) - \hat{\boldsymbol{\phi}} \frac{\partial}{\partial\theta} Y_l^m(\theta,\phi) \right]. \tag{C.25}$$

in which one may recognize the appearence of the differential operator

$$\hat{\boldsymbol{L}} = -i\boldsymbol{r} \times \boldsymbol{\nabla} = i\left(\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\boldsymbol{\phi}}\frac{\partial}{\partial\theta}\right)$$

which is the angular momentum operator expressed in spherical coordinates. Hence, (C.25) is just

$$\mathcal{Y}^{l1}_{j=l,m} = \frac{1}{\sqrt{l(l+1}}\hat{\boldsymbol{L}}Y_l^m(\theta,\phi), \quad l \neq 0.$$

So, we have found the first vector spherical harmonic in terms of just $Y_l^m(\theta,\phi)$. We may use the notation

$$\boldsymbol{X}_{lm}(\theta,\phi) := \mathcal{Y}^{l1}_{j=l,m} = \frac{1}{\sqrt{l(l+1}}\hat{\boldsymbol{L}}Y_{lm}(\theta,\phi). \tag{C.26}$$

In the same fashion, one can derive the other two vector spherical harmonics

$$\boldsymbol{X}_{l-1,m} := \mathcal{Y}^{l1}_{j=l-1,m}(\theta,\phi) = \frac{-1}{\sqrt{(j+1)(2j+1)}}\Big[l\hat{\boldsymbol{n}} + r\boldsymbol{\nabla}\Big]Y_{l-1,m}(\theta,\phi), \quad l \neq 0 \tag{C.27}$$

and

$$\boldsymbol{X}_{l+1,m} := \mathcal{Y}^{l1}_{j=l+1,m}(\theta,\phi) = \frac{1}{\sqrt{j(2j+1)}}\Big[(l+1)\hat{\boldsymbol{n}} + r\boldsymbol{\nabla}\Big]Y_{l+1,m}(\theta,\phi). \tag{C.28}$$

where $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$. So, finally, we can choose as the three independent normalized vector spherical harmonics the following elements

$$\left\{-\frac{ir}{\sqrt{j(j+1)}}\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}Y_{jm}(\theta,\phi), \quad \frac{r}{\sqrt{j(j+1)}}\boldsymbol{\nabla}Y_{jm}(\theta,\phi), \quad \hat{\boldsymbol{n}}Y_{jm}(\theta,\phi)\right\}. \tag{C.29}$$

Note that $\hat{\boldsymbol{n}}Y_{jm}(\theta,\phi)$ is longitudinal (*ie* $\parallel \hat{\boldsymbol{n}}$), $\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}Y_{jm}(\theta,\phi) = \partial_r Y_{jm}(\theta,\phi) = 0$ and $\hat{\boldsymbol{n}} \cdot (\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}Y_{jm}(\theta,\phi) = 0$. Hence, the first two vector spherical harmonics are transverse (*ie* $\perp \hat{\boldsymbol{n}}$).

Closing, it is important to say that only the first vector spherical harmonic is an eigenstate of $\boldsymbol{L}^2$.

Appendix D

Spherical Bessel's Functions

Let us define the linear differential equation, known as *Helmholtz Equation*

$$(\nabla^2 + k^2)f(\boldsymbol{r}) = 0.$$

In spherical coordinates this equation is written as

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + k^2 f\right] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} = 0.$$

which is the same with the Laplace Equation, displayed in Appendix C, with an extra term $k^2 f$. If we set $f(\boldsymbol{r}) = \phi(r)Y(\theta, \phi)$ and use equation (C.10), we obtain

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + k^2 f\right]\phi_l(r)Y_l^m(\theta,\phi) - \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\phi_l(r)Y_l^m(\theta,\phi) = 0. \Rightarrow$$

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + k^2 f\right]\phi_l(r)Y_l^m(\theta,\phi) - \frac{l(l+1)}{r^2}\phi_l(r)Y_l^m(\theta,\phi) = 0 \Rightarrow$$

$$\frac{d^2}{dr^2}\phi_l(r) + \frac{2}{r}\frac{d}{dr}\phi_l(r) + \left[k^2 - \frac{l(l+1)}{r^2}\right]\phi_l(r) = 0. \tag{D.1}$$

Employing the change of variable $x = kr$, equation (D.1) becomes

$$\frac{d^2}{dx^2}\phi_l(x) + \frac{2}{x}\frac{d}{dx}\phi_l(x) + \left[1 - \frac{l(l+1)}{x^2}\right]\phi_l(x) = 0. \tag{D.2}$$

For $x \to 0$ a solution to (D.2) can tend to either x^l or $x^{-(l+1)}$. The solution which behaves like x^l near $x = 0$ is called *spherical Bessel function* of order l, and the solution which behaves like $x^{-(l+1)}$ is called *spherical Neumann function*.

In bibliography, they both are mentioned as spherical Bessel functions. These functions are the two linearly independent solutions of (D.2). They can be expressed in the form of complex integral representations with

$$j_l(x) = \frac{1}{2\pi}\frac{(-2)^l l!}{x^{l+1}} \oint_{C_j} \frac{e^{-ixz}}{(z+1)^{l+1}(z-1)^{l+1}} \tag{D.3}$$

being the spherical Bessel function, when C_j is any contour enclosing the poles at $z = \pm 1$ traced out in a counterclockwise direction, and

$$n_l(x) = \frac{1}{2\pi i}\frac{(-2)^l l!}{x^{l+1}} \oint_{C_n} \frac{e^{-ixz}}{(z+1)^{l+1}(z-1)^{l+1}} \tag{D.4}$$

being the spherival Neumann function, but here we take two contours, the one about $z = -1$ being traced out counterclockwise and the other about $z = +1$ being traced out clockwise.

Alternatively, they can be given by the *Rayleigh's formula*

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x} \tag{D.5}$$

$$n_l(x) = -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x}. \tag{D.6}$$

We also introduce the *spherical Hankel functions* of the first an second kind

$$h_l^{(1)}(x) := j_l(x) + i n_l(x)$$
$$h_l^{(2)}(x) := j_l(x) - i n_l(x)$$

which are also solutions to (D.2) and can equally be chosen as the basis of solutions.

Except spherical Bessel's functions, one can may use the so called *Modified Spherical Bessel's Functions*, given by

$$i_l(x) := i^l j_l(ix) \tag{D.7a}$$

$$k_l(x) := -i^l h_l^{(1)}(ix) \tag{D.7b}$$

Sometimes in bibliography, they are given by the following alternative form

$$i_l(x) := i^l x j_l(ix) \tag{D.8a}$$

$$k_l(x) := -i^l x h_l^{(1)}(ix). \tag{D.8b}$$

D.1. Radial Green's Functions

Consider we have the non-homogeneous version of (D.2), that is

$$\frac{d^2}{dx^2}\phi_l(x) + \frac{2}{x}\frac{d}{dx}\phi_l(x) + \left[1 - \frac{l(l+1)}{x^2}\right]\phi_l(x) = f(x), \quad x = kr. \tag{D.9}$$

We wish to solve this equation in terms of Green's functions. According to the Example of Appendix A, for $f_0(x) = 1$, $f_1(x) = 2/x$ and $f_2(x) = 1 - l(l+1)/x^2$, the *radial* Green's function will has the form

$$G(x, x') = \begin{cases} b_1 j_l(x) + b_2 n_l(x) - \dfrac{j_l(x)n_l(x') - j_l(x')n_l(x)}{W(x')}, & x > x' \\ b_1 j_l(x) + b_2 n_l(x), \quad x < x'. \end{cases} \tag{D.10}$$

It is easily shown that $W(x') = j_(x)n_l'(x) - j_l(x)n_l'(x) = 1/x^2$. Thus we can rewrite

$$G(x, x') = \begin{cases} b_1 j_l(x) + b_2 n_l(x) - x^2[j_l(x)n_l(x') - j_l(x')n_l(x)], & x > x' \\ b_1 j_l(x) + b_2 n_l(x), \quad x < x'. \end{cases} \tag{D.11}$$

At this point, we can determine b_1 and b_2 applying boundary conditions. For instance, we may want the solutions to be non-singular at the origin, so for $x < x'$, so $g(x, x')$ must have no singularity at $x = 0$. However, $n_l(x)$ behaves like $(1/x)^{l+1}$ near $x = 0$, so we must exclude this term from the solution setting $b_2 = 0$.

In addition, for large r, we may want the solution to have a specific asymptotic behavior. For example, we may require to behave like e^{ix}/x, that is, a damping outgoing wave. But, one can see that this is the asymptotic form of the Hankel function $h_l^{(1)}(x)$. So, we may choose the remaining b_1 so as to ensure this behavior for $r > r'$. For this purpose, we take

$$b_1 = -ix'^2 h_l^{(1)}(x').$$

Then, for $x > x'$ we have

$$g_l(x, x') = -ix'^2 h_l^{(1)}(x) j_l(x')$$

and for $x < x'$

$$g_l(x, x') = -ix'^2 j_l(x) h_l^{(1)}(x').$$

If we use the symbolism $x_> = \max(x, x')$ and $x_< = \min(x, x')$ we may write the results in the following compact form

$$g_l(x, x') = -ix'^2 j_l(x_<) h_l^{(1)}(x_>). \tag{D.12}$$

Thus, the general solution to (D.9) is given by the formula

$$\phi_l(x) = A j_l(x) + \int_0^\infty g_l(x, x') f(x') dx' \tag{D.13}$$

where we have included the one and only solution to the homogeneous equation, $j_l(x)$, which is regular at the origin $x = 0$.

Βιβλιογραφία

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