National Technical University of Athens

School of Applied Mathematical and Physical Sciences

Department of Mathematics

The Inverse Spectral Problem for the Reconstruction of the Refractive Index from the Interior Transmission Problem

PhD Thesis

of

Nikolaos Pallikarakis

Athens, April 2017

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Nikolaos Pallikarakis

Diploma in Applied Mathematical and Physical Sciences NTUA
MSc in Applied Mathematical Sciences NTUA

Thesis Committee: Gintides Drossos (supervisor)
Cakoni Fioralba
Kiriaki Kiriakie

The thesis is submitted to the School of Applied Mathematical and Physical Sciences of the National Technical University of Athens in fulfilment of the requirements for the degree of Doctor of Philosophy.

Examining Committee:

...	...	...
D. Gintides	F. Cakoni	K. Kiriaki
Associate Professor	Professor	Professor
NTUA	Rutgers University	NTUA
...	...	...
A. Charalambopoulos	E. H. Georgoulis	I. G. Stratis
Professor	Associate Professor	Professor
NTUA	NTUA	University of Athens
	...	
	N. Yannakakis	
	Assistant Professor	
	NTUA	

Athens, April 2017

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Abstract

The main object of this thesis is the investigation of the inverse transmission eigenvalue problem, that is the determination of the refractive index of an inhomogeneous medium from transmission eigenvalues. Using some known results for the case where the refractive index is a radially symmetric and C^2 function, we introduce the corresponding interior transmission eigenvalue problem for a discontinuous refractive index. We examine the asymptotic properties of the eigenfunctions for large values of the spectral parameter, and investigate their dependence upon the discontinuity. We prove that the discontinuous refractive index is uniquely determined from the knowledge of all transmission eigenvalues, with no restrictions on the position of the discontinuity.

Furthermore, we propose a numerical method to compute transmission eigenvalues. We adopt a Galerkin-type method which is based on the variational formulation of the problem. Using a proper operator representation we show convergence of the method. We define the inverse transmission problem and show that numerically the problem can be considered as an inverse quadratic eigenvalue problem. We investigate the case of a spherically symmetric and piecewise constant refractive index and show that a small number of eigenvalues is sufficient for the reconstructions. We also introduce a computational method based on a Newton-type algorithm for reconstructions of arbitrary piecewise constant index from transmission eigenvalues. We illustrate our method with several examples.

Particularly, our aim is to study the properties of the discontinuous transmission eigenvalue problem and examine how the presence of a discontinuity affect the inverse problem. Our work has been motivated by the inverse problem of recovering material properties of a medium with layers with applications in non-destructive testing and target identification.

Περιληψη

Σκοπός της παρούσας διδακτορικής διατριβής είναι η μελέτη του αντίστροφου εσωτερικού προβλήματος διαπερατότητας (inverse transmission eigenvalue problem), δηλαδή του προσδιορισμού του δείκτη διάθλασης ενός μη ομογενούς μέσου από τις ιδιοτιμές διαπερατότητας (transmission eiegenvalues). Βασιζόμενοι σε γνωστά αποτελέσματα για την περίπτωση όπου ο δείκτης διάθλασης είναι σφαιρικά συμμετρικός και C^2 συνάρτηση, διατυπώνουμε το αντίστροφο φασματικό πρόβλημα για τον δείκτη διάθλασης με ασυνέχειες. Διερευνάμε την ασυμπτωτική συμπεριφορά των ιδιοσυναρτήσεων για μεγάλες τιμές της φασματικής παραμέτρου, και μελετάμε την εξάρτησή τους από την ασυνέχεια. Αποδεικνύουμε ότι ο ασυνεχής δείκτης μπορεί να προσδιοριστεί μοναδικά από τη γνώση όλων των ιδιοτιμών διαπερατότητας, χωρίς περιορισμούς στη θέση της ασυνέχειας.

Στη συνέχεια, προτείνουμε μία αριθμητική μέθοδο για τον υπολογισμό των ιδιοτιμών διαπερατότητας. Υιοθετούμε μία μέθοδο τύπου Galerkin η οποία βασίζεται στη μεταβολική διατύπωση του προβλήματος. Με την κατάλληλη αναπαράσταση του προβλήματος ως ένα πρόβλημα τελεστών αποδεικνύουμε τη σύγκλιση της μεθόδου. Έπειτα, ορίζουμε το αντίστροφο πρόβλημα διαπερατότητας και δείχνουμε ότι το πρόβλημα μπορεί να θεωρηθεί ως ένα αντίστροφο τετραγωνικό πρόβλημα ιδιοτιμών (inverse quadratic eigenvalue problem). Μελετάμε την περίπτωση του σφαιρικά συμμετρικού και κατά τμήματα σταθερού δείκτη διάθλασης και δείχνουμε ότι ένας σχετικά μικρός αριθμός ιδιοτιμών επαρκεί για τις ανακατασκευές. Επίσης εισάγουμε έναν αλγόριθμο τύπου Newton για ανακατασκευές στη γενική περίπτωση του κατά τμήματα σταθερού δείκτη διάθλασης. Η μέθοδός μας συνοδεύεται από παραδείγματα.

Ανακεφαλαιώνοντας, σκοπός μας είναι να μελετήσουμε το ασυνεχές πρόβλημα διαπερατότητας εστιάζοντας στο κατά πόσο η ύπαρξη ασυνεχειών επιδρά στο αντίστροφο πρόβλημα. Κίνητρο για την εργασία μας είναι το αντίστροφο πρόβλημα της εύρεσης ιδιοτήτων υλικών με διαστρωματώσεις, το οποίο έχει εφαρμογή στο μη καταστροφικό έλεγχο και την αναγνώριση αντικειμένων.

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Contents

1

Introduction

In this introductory chapter we shall present the basic concepts of direct and inverse spectral problems and give some physical motivations for our investigations. Next, we introduce the reader to the theory of inverse scattering and pose a relevant eigenvalue problem, namely the interior transmission eigenvalue problem. Finally, we provide an outline of this thesis.

Contents

1.1 Direct and inverse spectral problems

In general, spectral problems are connected with the investigation of vibrations of several physical systems like strings or membranes, mechanical systems, atoms or molecules etc. The *direct spectral problem* consists on finding the frequencies and the mode shapes of the vibrations of a system for which all physical properties are known. The *inverse spectral problem* is to determine some properties of the system from a knowledge of its natural frequencies and modes of vibration.

From a mathematical perspective, inverse spectral problems concern the recovery of coefficients of a differential equation from the knowledge of the eigenvalues, i.e. the spectral values of the corresponding differential operator. The first study on the spectral properties of the differential operator

$$\ell y := -y'' + q(x)y = 0$$

was performed by Sturm and Liouville in 1836 and the corresponding inverse problem was firstly considered by Ambartsumyan in 1929, (for more information we refer to [26, 45, 79]).

Example 1.1.1. *(Sturm-Liouville eigenvalue problem, [63])*
We consider a string of length L and mass density $\rho = \rho(x) > 0,\ 0 \leq x \leq L$ which is fixed at the endpoints $x = 0$ and $x = L$. If an external force sets the string into motion, tones are produced due to vibrations. Let $u(x,t)$ be the transverse displacement at x and time t. It obeys the wave equation:

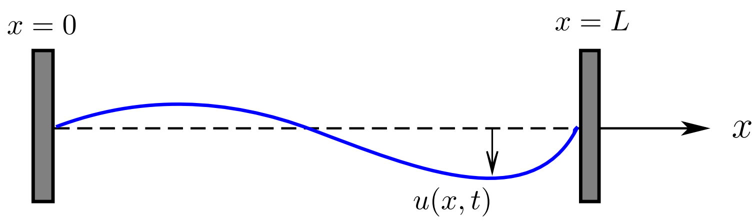

Figure 1.1: A vibrating string

$$\rho(x)\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}, \qquad 0 < x < L,\ t > 0$$

subject to boundary conditions

$$u(0,t) = u(L,t) = 0, \qquad t > 0.$$

A periodic vibration of the form

$$u(x,t) = y(x)[a \cos kt + b \sin kt]$$

with frequency k is called pure tone. Thus, u solves the boundary value problem if and only if y and k satisfy the Sturm-Liouville eigenvalue problem:

$$\begin{cases} y''(x) + k^2 \rho(x) y(x) = 0, & 0 < x < L \\ y(0) = y(L) = 0. \end{cases}$$

For the direct problem we assume that $\rho(x)$ is known and we want to compute the eigenfrequencies k and the corresponding eigenfunctions. The inverse problem is to recover the mass density $\rho(x)$ from a set of measured frequencies k. This is associated with the following uniqueness question: "Does a given set of frequencies correspond to a unique mass distribution of the string?". An answer to this question is not that simple and in general a single spectrum is insufficient to recover $\rho(x)$, [26].

Example 1.1.2. *(Potential Scattering, [26])*
We consider the scattering of a non-relativistic particle by a fixed force or the scattering of two particles by each other, and we use the time-dependent Schrödinger equation:

$$i \frac{\partial \Psi(t)}{\partial t} = \mathscr{H} \Psi(t),$$

where the Hamiltonian is given by the sum of kinetic and potential energy:

$$\mathscr{H} = -\Delta + V.$$

We restrict ourselves to the time-independent Schrödinger equation in one space dimension

$$-u'' + V(x)u = Eu,$$

where E corresponds to the energy in the particular mode of the state $u(x)$ which represents the amplitude at the point x. The quantity

$$\int_x^{x+\delta x} |u(x)|^2 \mathrm{d}x$$

represents the probability of finding the particle between x and $x + \delta x$. For the finite interval case, the values of constant E are allowed to form a discrete sequence $\{E_n\}$, which means that not all possible energy levels are possible. The inverse spectral problem is given the possible energy levels of the quantum mechanical system to reconstruct the unknown potential, and hence recover the qualitative characteristics of the underlying force.

There is an extensive literature on solving inverse spectral problems, for example in vibration of discrete systems with applications in engineering [50]. Most of the problems of this category, like the above examples, are based on the self-adjoint formulation of the problems. From a mathematical point of view this means that the corresponding differential operator is self-adjoint and hence has an infinite and discrete set of real eigenvalues. This fact is crucial in applications, since only real data (frequencies) can be measured.

1.2 Inverse scattering and the interior transmission problem

In this section, we present an introduction to the scattering theory for inhomogeneous media. The interior transmission problem appears in scattering theory as a special eigenvalue problem, associated with non-scattering waves. In the following, we show how transmission eigenvalues can be used for the inverse problem of determining the shape and the material properties of a scatterer.

1.2.1 The scattering problem

We consider the *scattering problem* of an inhomogeneous anisotropic medium of bounded and simply connected support $D \subset \mathbb{R}^d,\ d = 2, 3$, which is assumed to have a Lipschitz boundary ∂D. We present only the basic results of scattering theory that are necessary for our work and refer to the monographs [12, 15, 22, 30, 63] for more details and proofs. The physical characteristics of the medium are described by a bounded function $n \in L^\infty(D)$ and a matrix-valued function $A \in L^\infty(D, \mathbb{C}^{d\times d})$. We assume that A is symmetric such that $\overline{\xi}\mathrm{Im}(A(x))\xi \leq 0$ for all $\xi \in \mathbb{C}^d$ and $\mathrm{Re}(n(x)) > 0,\ \mathrm{Im}(n(x)) \geq 0$. The scattering problem for an incident wave u^i which satisfies the Helmholtz equation $\Delta u^i + k^2 u^i = 0,$ in $\mathbb{R}^d$ is the following:

Find the total field $u = u^i + u^s$ such that:

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^d \setminus \overline{D}, \tag{1.2.1}$$

$$\nabla \cdot A(x)\nabla u + k^2 n(x) u = 0 \qquad \text{in } D, \tag{1.2.2}$$

$$u^+ = u^- \qquad \text{on } \partial D, \tag{1.2.3}$$

$$\left(\frac{\partial u}{\partial \nu}\right)^+ = \left(\frac{\partial u}{\partial \nu_A}\right)^- \qquad \text{on } \partial D, \tag{1.2.4}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{1.2.5}$$

where ν is the outward normal vector to the boundary, $k > 0$ is the wave number, u^s is the scattered field and $r = |x|$. Condition (1.2.5) is the *Sommerfeld radiation condition* which guarantees that the scattered field is outgoing and is assumed to hold uniformly in $\hat{x} := x/|x|$. Moreover, with the notation $f^{\pm}$ we express the limits $f^{\pm} := \lim_{h\to 0} f(x \pm h\nu),\ h > 0$ and $x \in \partial D$ and also

$$\frac{\partial u}{\partial \nu_A} := \nu \cdot A(x)\nabla u, \qquad x \in \partial D.$$

It is well known that the scattering problem (1.2.1)-(1.2.5) has a unique solution in the space $C^2(\mathbb{R}^d)$. The application of a variational method [12] ensures the existence of a solution in the "larger" space $u \in H^1_{loc}(\mathbb{R}^d)$ where

$$H^1_{loc}(\mathbb{R}^d) = \left\{ u \in L^2(K) : \text{such that } \nabla u \in \left(L^2(K)\right)^d, \text{ for any ball } K = K(0,R) \right\}$$

provided that $\bar{\xi}\mathrm{Re}(A(x))\xi \geq |\xi|^2 > 0$ for all $\xi \in \mathbb{C}^d$ and almost everywhere in D.

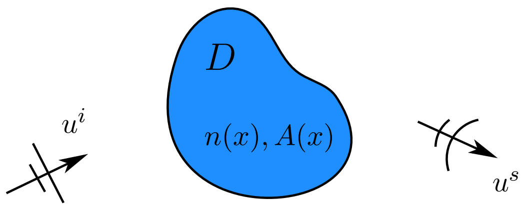

Figure 1.2: Scattering by an inhomogeneous medium

The *direct scattering problem in* $\mathbb{R}^3$ models the propagation of time harmonic acoustic waves of frequency ω in an inhomogeneous medium where $k = \omega/c_0$ is the wave number and the constant c_0 is the background sound speed. The *refractive index* is defined as:

$$n(x) := \frac{c_0^2}{c^2(x)}, \qquad x \in \mathbb{R}^3, \tag{1.2.6}$$

where $c(x)$ is the sound speed in the medium and $n(x) = 1,$ for $x \in \mathbb{R}^3 \setminus D$. A refractive index with an imaginary component corresponds to a medium with absorption.

The *direct scattering problem in* $\mathbb{R}^2$ models the scattering of time harmonic electromagnetic waves by an infinitely long cylinder with the corresponding (electric or magnetic) field being polarized parallel to the cylinder's axis. This model is the reduction of the three-dimensional vector problem for Maxwell's equations to the corresponding two-dimensional scalar problem. In this case, D is the cross section of this cylinder, for which A and n are related to the electric permittivity $\varepsilon(x)$ and magnetic permeability $\mu(x)$ respectively. The background medium is assumed to be non-conducting and homogeneous with constant electric permittivity ε_0 and magnetic permeability μ_0.

The wave number is $k = \omega\sqrt{\varepsilon_0\mu_0}$, where $\omega > 0$ is the frequency of the time harmonic electromagnetic wave.

Of main interest is the corresponding *inverse scattering problem*, that is the determination of the support and the physical properties of the inhomogeneous medium ($n(x)$ and $A(x)$) from measurements of the total field u away from the scatterer, for several incident fields u^i and wave numbers k. This inverse problem is non-linear and improperly posed in the sense that small perturbations in the measured data can cause large errors in the determination of the scatterer. Nevertheless, it has several applications in computer tomography, seismic and electromagnetic geophysical exploration, medical imaging and non-destructive testing of materials.

From now on, we restrict ourselves to the isotropic medium case i.e. $A = I$. The scattering problem (1.2.1)-(1.2.5) is simplified in the form:

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^d \setminus \overline{D}, \tag{1.2.7}$$

$$\Delta u + k^2 n(x) u = 0 \qquad \text{in } D, \tag{1.2.8}$$

$$u^+ = u^- \qquad \text{on } \partial D, \tag{1.2.9}$$

$$\left(\frac{\partial u}{\partial \nu}\right)^+ = \left(\frac{\partial u}{\partial \nu}\right)^- \qquad \text{on } \partial D, \tag{1.2.10}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0. \tag{1.2.11}$$

1.2.2 The interior transmission problem

We raise the following question:
"Are there any incident waves u^i such that the scattered field u^s is identically zero?"
The answer to this question leads to the *interior transmission eigenvalue problem.*

More precisely, we assume that the incident field is a plane wave given by $u^i(x) = \mathrm{e}^{ikx\cdot\hat{\theta}}$ where $\hat{\theta}$ is the unit vector parallel to the direction of propagation. Then, the corresponding scattered field satisfies the asymptotic formula [30]:

$$u^s(x) = \mathrm{e}^{ikr} r^{-\frac{d-1}{2}} u_\infty(\hat{x}) + \mathrm{O}\left(r^{-\frac{d+1}{2}}\right), \qquad \text{in } \mathbb{R}^d,\ d = 2, 3, \tag{1.2.12}$$

for $r = |x| \to \infty$ and uniformly in $\hat{x} = x/|x|$. We denote Ω the $(d-1)$-dimensional sphere $\Omega = \{x \in \mathbb{R}^d : \ |x| = 1\}$. The *far field pattern* $u_\infty : \Omega \to \mathbb{C}$ is defined as

$$u_\infty(\hat{x}) := \frac{k^2}{4\pi}\int_{\mathbb{R}^d} (n(y) - 1)\, \mathrm{e}^{-ik\hat{x}\cdot y} u(y) \mathrm{d}y \qquad \hat{x} \in \Omega. \tag{1.2.13}$$

and is an analytic function on Ω. The far field patterns define the *far field operator* $F : L^2(\Omega) \to L^2(\Omega)$

$$(Fg)(\hat{x}) := \int_{\Omega} u_{\infty}(\hat{x};\hat{\theta}) g(\hat{\theta}) \mathrm{d}s(\hat{\theta}) \qquad \hat{x} \in \Omega, \tag{1.2.14}$$

where $\hat{\theta}$ is the incident direction and $\hat{x}$ is the direction of observation. A special category of wave functions, the *Herglotz wave functions* are defined by:

$$v_g(x) := \int_{\Omega} \mathrm{e}^{ikx\cdot\hat{\theta}} g(\hat{\theta}) \mathrm{d}s(\hat{\theta}), \qquad g \in L^2(\Omega),\ x \in \mathbb{R}^d,\ d = 2,3. \tag{1.2.15}$$

Non-scattering waves are associated with the following eigenvalue problem:

Interior transmission eigenvalue problem:
Find $k > 0$ and $v, w \in C^2(D)$ such that:

$$\Delta w + k^2 n(x) w = 0 \quad \text{in } D, \tag{1.2.16}$$

$$\Delta v + k^2 v = 0 \quad \text{in } D, \tag{1.2.17}$$

$$w = v \quad \text{on } \partial D, \tag{1.2.18}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial D. \tag{1.2.19}$$

The values of k for which there exists a non-trivial solution (v, w) of (1.2.16)-(1.2.19) are the *interior transmission eigenvalues.*

This eigenvalue problem was firstly introduced by Kirsch [65], and was associated with the injectivity of the far field operator. A few years later, Colton and Monk in [37] used the interior transmission problem to solve the inverse scattering problem in inhomogeneous media. It will become clear later, that the interior transmission eigenvalue problem is a non-self-adjoint eigenvalue problem and is not covered by the spectral theory of elliptic differential operators. For the time being, we assume that transmission eigenvalues do exist and form a discrete set and we will discuss these issues in more detail afterwards.

In general, it is not possible to construct an incident wave that does not scatter. Although, from the density of Herglotz functions on the space of solutions to the Helmholtz equation in D [31], we infer that if k is a transmission eigenvalue corresponding to a non-trivial pair of solutions (v, w) then for $\varepsilon > 0$ there exists a Herglotz function v_{g_ε} which $\varepsilon-$approximates v and the corresponding scattered field is $\varepsilon-$small, in the $L^2(D)-$norm. We also note that if n is identically equal to 1, the interior transmission problem is degenerate. Indeed, in this case, every $k \in \mathbb{C}$ is a transmission eigenvalue and hence any incident field does not scatter.

The following theorem associates the injectivity of the far field operator with the interior transmission problem [30, theorem 8.9], and a similar result for the anisotropic case can be found in [12, theorem 6.2].

Theorem 1.2.1. *The far field operator $F : L^2(\Omega) \to L^2(\Omega)$ corresponding to the scattering problem* $(1.2.7) - (1.2.11)$ *is injective and has dense range if and only if k^2 is not a transmission eigenvalue of* $(1.2.16) - (1.2.19)$*, such that the function v of the corresponding non-trivial solution has the form of a Herglotz function* (1.2.15).

An application of the above theorem is for the inverse problem of determining the unknown shape of the scatterer via the *Linear Sampling Method*, which was introduced by Colton and Kirsch [28]. The aim of this method is to recover the support of the inhomogeneous medium from a knowledge of the far field pattern of the scattered wave. This method is based on solving a linear integral equation of the first kind and then using the solution as indicator function to reconstruct the boundary of the scatterer. More precisely, we assume that $u_\infty(\hat{x}, \hat{\theta})$ is known, which implies the knowledge of the far field operator F. We introduce the *far field equation:*

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z) \tag{1.2.20}$$

where $\Phi_\infty(\hat{x}, z)$ is the far field pattern corresponding to the *fundamental solution* $\Phi(x, z)$ of the Helmholtz equation:

$$\begin{aligned} \Phi(x,z) &:= \frac{\mathrm{e}^{ik|x-z|}}{4\pi|x-z|} && \text{in } \mathbb{R}^3 \\ \Phi(x,z) &:= \frac{i}{4} H_0^{(1)}(k|x-z|) && \text{in } \mathbb{R}^2 \end{aligned}$$

and $H_0^{(1)}$ is the Hankel function of order zero.

In order to solve the far field equation (1.2.20), we must introduce the interior transmission problem with non-homogeneous transmission conditions on the boundary:

Non-homogeneous interior transmission problem:

$$\Delta w + k^2 n(x) w = 0 \quad \text{in } D, \tag{1.2.21}$$

$$\Delta v + k^2 v = 0 \quad \text{in } D, \tag{1.2.22}$$

$$w - v = \Phi(\,\cdot\,, z) \quad \text{on } \partial D, \tag{1.2.23}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = \frac{\partial \Phi(\,\cdot\,, z)}{\partial \nu} \quad \text{on } \partial D \tag{1.2.24}$$

and we have the following theorem:

Theorem 1.2.2. *([30, theorem 10.22]) We assume that k is not a transmission eigenvalue. Given $\varepsilon > 0$, there exists an approximate solution to the far filed equation* (1.2.20)*, denoted with $g_{\varepsilon,z}$ (i.e. $||Fg_{\varepsilon,z} - \Phi_\infty(\,\cdot\,, z)||^2_{L^2} < \varepsilon$) such that*

a. if $z \in D$, then the corresponding Herglotz function $v_{g_{\varepsilon,z}}$ satisfies

$$\lim_{\varepsilon \to 0} ||v_{g_{\varepsilon,z}}|| = ||v_z|| \qquad \text{in } L^2(D)$$

where (w_z, v_z) is the unique solution of the non-homogeneous interior transmission problem

b. *if $z \notin D$, then*

$$\lim_{\varepsilon \to 0} ||v_{g_{\varepsilon,z}}|| = \infty \qquad \text{in } L^2(D).$$

The far field equation is not in general solvable since Φ_∞ is not in the range of F. Thus, we use regularization methods (e.g. Tikhonov regularization [30, 53]) to solve the far field equation for an approximate solution $g_{z,\alpha}$. The boundary is reconstructed by those points where $||g_{z,\alpha}||_{L^2}$ becomes large.

Since the interior transmission problem was introduced, the main research interest was focused on the discreteness of transmission eigenvalues [29, 81]. The problem of discreteness was important because the various sampling methods for the reconstruction of the support of an inhomogeneous medium fail at frequencies corresponding to transmission eigenvalues. For the sampling methods, transmission eigenvalues must be avoided and thus the fact that they form a discrete set was sufficient. After almost 20 years, it was suggested that transmission eigenvalues carry information about material properties of the scattering object [11, 17, 18]. Hence, the question of existence of real or complex eigenvalues became important to answer. The first existence result for the arbitrary (non-spherically symmetric) case was given by Päivärinta and Sylvester [77], for sufficiently large refractive index and for at least one eigenvalue. This paper was soon followed by several others [13, 20, 23, 64]. The problem of existence was completely answered by Cakoni, Gintides and Haddar [21], only under the restriction that $0 < n < 1$ or $n > 1$. We state here the theorem and refer to [21] for a detailed proof.

Theorem 1.2.3. *We denote by $n_* = \inf n$ and $n^* = \sup n$. We assume that $n \in L^\infty(D)$ satisfy either one of the following assumptions:*

a. $1 + \alpha \leq n_* \leq n(x) \leq n^* < \infty$

b. $0 < n_* \leq n(x) \leq n^* < 1 - \beta$

for some constants $\alpha > 0$ and $\beta > 0$. Then, there exist an infinite set of real transmission eigenvalues with $+\infty$ as the only accumulation point.

The proof also applies to inhomogeneous isotropic and anisotropic media for both Helmholtz and Maxwell's equations including the case of media with cavities.

Now, assuming that the shape of the scatterer is known, we consider the inverse problem of determining the unknown refractive index $n(x)$ from transmission eigenvalues. For this problem, in contrast with the sampling methods, we require the knowledge of transmission eigenvalues. We will refer to this problem as the *inverse spectral problem for transmission eigenvalues* or *inverse transmission eigenvalue problem.* This problem is motivated by the fact that real transmission eigenvalues can be measured by scattering data [16, theorem 3.2]:

Theorem 1.2.4. *Assume that k is a transmission eigenvalue, and for given $\varepsilon > 0$ let $g_{z,\varepsilon}$ be such that*

$$||Fg_{\varepsilon,z} - \Phi_\infty(\,\cdot\,,z)||^2_{L^2} \leq \varepsilon$$

with $v_{g_{\varepsilon,z}}$ being the corresponding Herglotz function. Then, for almost every $z \in D$, the norm $||v_{g_{\varepsilon,z}}||_{L^2}$ can not be bounded as $\varepsilon \to 0$.

Therefore, if D is known, and if we estimate a regularized solution of the far field equation for a specific interval of wave numbers, we can use the L^2-norm of this solution as an indicator for transmission eigenvalues in this interval. In practice, if we use a regularization method to solve the far field equation, we can plot the norm of the solution $||g_{z,\alpha}||_{L^2}$ against the wave number k and then sharp peaks should appear at the values of k which correspond to transmission eigenvalues.

As we mentioned above, transmission eigenvalues provide information about material properties of the scatterer. This result has also been investigated numerically [6, 18, 19, 48, 49, 84, 85], where eigenvalues are used to reconstruct the unknown refractive index. Hence, the inverse transmission eigenvalue problem has great importance for many applications. For example, transmission eigenvalues can be used for non-destructive testing of materials, to identify cracks or abnormalities inside media [11, 55, 56].

1.3 Outline of the thesis

In this thesis we investigate the inverse transmission eigenvalue problem, to determine an unknown refractive index from transmission eigenvalues. We examine this inverse problem both from the theoretical and the computational point of view.

In chapter 2, we consider the interior transmission problem for the special case of the spherically stratified inhomogeneous medium in $\mathbb{R}^3$. In this introductory chapter, we present some well-known results for the direct and inverse spectral problem when the refractive index is a C^2 function, which are necessary for our further investigations. We introduce spherical coordinates and reduce the problem into a boundary value problem where the spectral parameter appears in the boundary condition. Concerning the direct problem, we investigate the existence and discreteness of an infinite set of transmission eigenvalues. The approach is based on the asymptotic behaviour of the eigenfunctions for large values of the spectral parameter k. Next, we pose the corresponding inverse spectral problem and examine the necessary conditions under which transmission eigenvalues uniquely determine the refractive index. Finally, we present some results on existence and location of complex transmission eigenvalues which verify that the interior transmission problem is non-self-adjoint.

In chapter 3, we introduce the interior transmission problem for a radially symmetric and discontinuous refractive index. Following the theory of chapter 2, we examine the properties of the direct problem for a piecewise C^2 refractive index. We use separation of variables and derive a discontinuous boundary value problem. Using the formulation of the discontinuous Sturm-Liouville eigenvalue problem, we obtain the asymptotic formulas for the eigenfunctions which we prove that are functionally dependent on the characteristics of the discontinuity. Afterwards, we consider the inverse spectral problem for the corresponding discontinuous interior transmission problem. We prove that the special transmission eigenvalues corresponding to spherically symmetric eigenfunctions can recover some characteristics of the discontinuity. We assume next that the whole transmission spectrum is known and we prove that the refractive index is uniquely determined, with no restrictions on the position of the discontinuity. Uniqueness in this inverse problem demonstrates that transmission eigenvalues can be used to recover the material properties of a medium with layers. We conclude the chapter with the investigation of the asymptotic behaviour of the real eigenvalues for large values of the spectral parameter k. We give an example which shows that if the refractive index has a discontinuity then the transmission eigenvalues might not have an asymptotic expansion.

We propose a numerical method to solve the direct transmission eigenvalue problem in chapter 4. Our approach is based on the variational formulation of an equivalent fourth order eigenvalue problem. Firstly, we show that the interior transmission eigenvalue problem can be written as an eigenvalue problem for a compact and non-self-adjoint block operator. Afterwards, we introduce a Galerkin-type method in the Sobolev space H_0^2 and pose the corresponding discrete problem to compute transmission eigenvalues. We prove that the eigenvalues of the discrete problem converge to the corresponding original by using some abstract results for convergence in Banach spaces. Lastly, we present a review of the existing numerical methods for transmission eigenvalues.

In chapter 5, we face the inverse transmission eigenvalue problem numerically. Our goal is the reconstruction of a two-dimensional piecewise constant refractive index from a sufficiently small number of transmission eigenvalues. Firstly, we consider a minimization scheme for discs with constant or piecewise constant refractive index. We minimize the error between measured and computed eigenvalues. Using only few lower transmission eigenvalues we obtain the reconstructions. Furthermore, we give numerical evidence that both real and complex eigenvalues are useful for the reconstructions. Next, we propose a Newton-type algorithm for the reconstruction of an arbitrary piecewise constant index. This algorithm can be performed without having knowledge of the exact position of the eigenvalues in the spectrum. We give some numerical examples of reconstructions in multi-layered domains and compare the performance of our method with some other minimization methods. We finish the chapter with the application of the Newton method to non-spherically stratified domains.

Finally, in appendix, we consider a generalization of the interior transmission problem for a refractive index with a finite number of discontinuities and by induction we derive the asymptotic formula of the eigenfunctions. We also recall some results from entire function theory which are required for the inverse spectral problem.

Some of the work presented in this thesis is included in the following papers:

1. Gintides D. and Pallikarakis N. (2017), The inverse transmission eigenvalue problem for a discontinuous refractive index, *Inverse Problems* **33** 055006

2. Gintides D. and Pallikarakis N. (2013), A computational method for the inverse transmission eigenvalue problem, *Inverse Problems* **29** 104010

and was announced in the international conferences:

1. Gintides D. and Pallikarakis N. (2015), Uniqueness theorems for the inverse transmission eigenvalue problem with discontinuous refractive index, *International Conference on Modern Mathematical Methods in Science and Technology* **(M3ST)**, Kalamata-Greece

2. Gintides D. and Pallikarakis N. (2014), The inverse transmission eigenvalue problem for a discontinuous refractive index, 7^{th} *International Conference, Inverse Problems: Modeling and Simulation* **(IPMS)**, Fethiye-Turkey

2

The interior transmission problem for the spherically stratified medium with continuous refractive index

In this chapter we present all the well-known results for the interior transmission problem of a ball $B \subset \mathbb{R}^3$ where the corresponding refractive index is a continuous function, depending only on the radius. These results are necessary for the next chapters, because in the latter, we use similar techniques and methods. We discuss the conditions for the existence and discreteness of an infinite set of transmission eigenvalues. We also set the corresponding inverse spectral problem, and we present the conditions under transmission eigenvalues uniquely determine the refractive index. Furthermore, we present some recent results concerning existence and distribution of complex transmission eigenvalues.

Contents

2.1 Formulation of the problem

We examine the interior transmission problem corresponding to the acoustic scattering problem for an isotropic inhomogeneous medium B_a, where we assume that the medium is spherically stratified in $\mathbb{R}^3$, that is $B_a := \{x \in \mathbb{R}^3 : |x| < a\}$ and the refractive index $n(|x|) := n(r)$ is a function depending only on the radius. The *interior transmission eigenvalue problem* for this special case is defined as the following problem:

Find $k \in \mathbb{C}$ and a non-trivial solution $w, v \in L^2(B_a)$ such that $w - v \in H_0^2(B_a)$ satisfying:

$$\Delta w + k^2 n(r) w = 0 \quad \text{in } B_a, \tag{2.1.1}$$

$$\Delta v + k^2 v = 0 \quad \text{in } B_a, \tag{2.1.2}$$

$$w = v \quad \text{on } \partial B_a, \tag{2.1.3}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial B_a, \tag{2.1.4}$$

where the Sobolev space H_0^2 is defined as:

$$H_0^2(D) = \left\{ u \in H^2(D) : \text{ such that } u = 0 \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D \right\}.$$

As it will be shown in chapter 4, the above function spaces provide the appropriate setting for the weak form of this eigenvalue problem, which appears to be non-self-adjoint. In this chapter we consider only classical solutions, i.e. $w, v \in C^2(B_a) \cap C^1(\overline{B_a})$. Problem (2.1.1)-(2.1.4), corresponds to the simplest possible case for the interior transmission problem and a considerable amount of information is known about transmission eigenvalues.

We make the following assumptions for the refractive index (unless otherwise stated):

$$\mathrm{Im}(n(r)) = 0, \; n(r) > 0 \text{ and } n \in C^2[0, \infty), \tag{2.1.5}$$

and

$$n(r) = 1 \quad \text{for} \quad r \geq a, \tag{2.1.6}$$

where we assume as well that $n(r)$ is not identically equal to 1 for $r < a$. Condition (2.1.6) implies that there is no inhomogeneity for $r \geq a$ and since $n \in C^2$ the refractive index is smooth enough for $r = a$ (e.g. $n'(a) = 0$).

Following the theory of [30] developed for the spherically stratified dielectric medium (see also [37]), we introduce spherical coordinates (r, θ, ϕ) and using separation of variables we expand w and v in series of spherical harmonics

$$v(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} j_l(kr) Y_l^m(\theta, \phi)$$

$$w(x) = \frac{1}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b_{lm} y_l(r) Y_l^m(\theta, \phi),$$

where j_l is a spherical Bessel function. Moreover, assuming azimuthal symmetry, we look for a pair of solutions of (2.1.1)-(2.1.4) of the form

$$v(r, \theta) = a_l j_l(kr) P_l(\cos\theta), \tag{2.1.7}$$

$$w(r, \theta) = b_l \frac{y_l(r)}{r} P_l(\cos\theta), \tag{2.1.8}$$

where P_l is a Legendre's polynomial, a_l and b_l are constants and function y_l satisfies the following initial value problem:

$$y_l''(r) + \left(k^2 n(r) - \frac{l(l+1)}{r^2}\right) y_l(r) = 0 \tag{2.1.9}$$

for $r > 0$, with

$$\lim_{r\to 0} \left(\frac{y_l(r)}{r} - j_l(kr)\right) = 0. \tag{2.1.10}$$

We note that the ordinary differential equation (2.1.9) has a singular coefficient for $l \geq 1$. Initial condition (2.1.10) is chosen such that $y_l(r)$ behaves like $r j_l(kr)$ as $r \to 0$, for simplicity reasons, i.e.

$$\lim_{r\to 0} r^{-(l+1)} y_l(r) = \frac{\sqrt{\pi} k^l}{2^{l+1}\Gamma(l+3/2)}.$$

Now, (2.1.7)-(2.1.8) will be a non-trivial pair of solutions to the eigenvalue problem (2.1.1)-(2.1.4), if there exists a non-trivial solution of the following system:

$$b_l \frac{y_l(a)}{a} - a_l j_l(ak) = 0$$

$$b_l \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{y_l(r)}{r}\right)\bigg|_{r=a} - a_l \frac{\mathrm{d}}{\mathrm{d}r} j_l(kr)|_{r=a} = 0.$$

Thus, the transmission eigenvalue problem is written as

$$y_l''(r) + \left(k^2 n(r) - \frac{l(l+1)}{r^2}\right) y_l(r) = 0 \qquad \text{for} \;\; 0 < r < a, \tag{2.1.11}$$

$$\lim_{r\to 0} \left(\frac{y_l(r)}{r} - j_l(kr)\right) = 0, \tag{2.1.12}$$

$$D_l(k) := det \begin{pmatrix} \frac{y_l(r)}{r} & -j_l(kr) \\ \\ \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{y_l(r)}{r}\right) & -\frac{\mathrm{d}}{\mathrm{d}r}j_l(kr) \end{pmatrix} = 0 \qquad \text{for} \quad r = a. \tag{2.1.13}$$

The values of k for which there is a non-trivial solution of (2.1.11)-(2.1.13), are the transmission eigenvalues. Equivalently, we can deduce that k is a (real or complex) transmission eigenvalue if and only if $D_l(k) = 0$. We will refer to $\{D_l(k)\}_{l\geq 0}$ as *characteristic functions*.

Furthermore, we examine the special case of (2.1.1)-(2.1.4) where only spherically symmetric eigenfunctions are allowed, namely $l = 0$. For this special case, eigenfunctions are reduced to

$$v(r) = a_0 j_0(kr)$$

and

$$w(r) = b_0 \frac{y_0(r)}{r}.$$

Thus, the corresponding boundary value problem is equivalent with the following Sturm-Liouville-type eigenvalue problem with the spectral parameter appearing in the boundary condition at the right endpoint [2]:

$$y_0''(r) + k^2 n(r) y_0(r) = 0, \quad 0 < r < a \tag{2.1.14}$$

$$y_0(0) = 0, \tag{2.1.15}$$

$$D_0(k) = \frac{\sin ka}{k} y_0'(a) - \cos ka \; y_0(a) = 0 \tag{2.1.16}$$

where we used that $j_0(kr) = \sin kr/kr$. The eigenvalues of (2.1.14)-(2.1.16) are the transmission eigenvalues. We will refer to them as *special transmission eigenvalues*.

In the next subsections, we discuss existence, discreteness and uniqueness issues for transmission eigenvalues for both $l = 0$ and $l \geq 1$. These two eigenvalue problems have different formulations and their study requires various mathematical methods.

2.2 Existence and discreteness of transmission eigenvalues

To show the existence of an infinite and discrete set of (real) transmission eigenvalues such that (2.1.7)-(2.1.8) is a non-trivial solution of the interior transmission problem (2.1.1)-(2.1.4), we must study the asymptotic behaviour of y_l and D_l for large values of k. We examine both $l \geq 1$ and $l = 0$ cases.

2.2.1 The case $l \geq 1$

This problem was firstly considered by Colton and Päivärinta in [39]. Here we present the results from [30, section 9.4]. We reformulate (2.1.11) using the Liouville transformation

$$\xi(r) := \int_0^r \sqrt{n(t)}\mathrm{d}t, \qquad z(\xi) := n(r)^{1/4} y_l(r), \tag{2.2.17}$$

and we define the quantity

$$A := \int_0^a \sqrt{n(t)}\mathrm{d}t \tag{2.2.18}$$

which has a physical meaning as the *travel time* for a wave to move from $r = 0$ to $r = a$ in the wave scattering problem, [2, 57]. Also, we assume that $A \neq a$ which is satisfied if either $n(r) > 1$ or $0 < n(r) < 1$ for $0 \leq r < a$. We will show that this condition suffices for the existence of infinite number of transmission eigenvalues.

The problem is transformed in the following form:

$$\frac{\mathrm{d}^2 z(\xi)}{\mathrm{d}\xi^2} + \left(k^2 - \frac{l(l+1)}{\xi^2} - g(\xi)\right) z(\xi) = 0 \tag{2.2.19}$$

where

$$g(\xi) := \frac{l(l+1)}{r^2 n(r)} - \frac{l(l+1)}{\xi^2} + \frac{n''(r)}{4n(r)^2} - \frac{5}{16}\frac{n'(r)^2}{n(r)^3}. \tag{2.2.20}$$

Since $n(r) > 0$ for $r \geq 0$, the Liouville transformation is invertible, g is well defined and is a continuous function for $r > 0$. Moreover, (2.1.6) implies that

$$\int_0^1 \xi |g(\xi)| \mathrm{d}\xi < \infty \quad \text{and} \quad \int_1^\infty |g(\xi)| \mathrm{d}\xi < \infty \tag{2.2.21}$$

Equation (2.2.19) is a radial Schrödinger equation, singular at $\xi = 0$.

Following the analysis of [76, p. 436 – 437], for $\lambda > 0$ we define functions E_λ and M_λ by

$$E_\lambda(\xi) = \begin{cases} \left[-\frac{Y_\lambda(\xi)}{J_\lambda(\xi)}\right]^{1/2}, & 0 < \xi < \xi_\lambda \\ 1, & \xi_\lambda \leq \xi < \infty \end{cases} \tag{2.2.22}$$

$$M_\lambda(\xi) = \begin{cases} [2|Y_\lambda(\xi)|J_\lambda(\xi)]^{1/2}, & 0 < \xi < \xi_\lambda \\ [J_\lambda^2(\xi) + Y_\lambda^2(\xi)]^{1/2}, & \xi_\lambda \leq \xi < \infty \end{cases} \tag{2.2.23}$$

where $J_\lambda,\ Y_\lambda$ are Bessel and Neumann functions and ξ_λ is the smallest positive root of the equation $J_\lambda(\xi) + Y_\lambda(\xi) = 0$. We also introduce G_λ as:

$$G_\lambda(k,\xi) := \frac{\pi}{2}\int_0^\xi M_\lambda^2(kt)t|g(t)|\mathrm{d}t. \tag{2.2.24}$$

Now, the following theorem follows:

Theorem 2.2.1. *([30, theorem 9.9]) Let $k > 0$ and $l \geq -1/2$. Then (2.2.19) has a solution z which, as a function of ξ, is continuous in $[0,\infty)$, twice continuously differentiable in $(0,\infty)$, and is given by*

$$z(\xi) = \sqrt{\frac{\pi\xi}{2k}}\{J_\lambda(k\xi) + \varepsilon_l(k,\xi)\} \tag{2.2.25}$$

where

$$\lambda = l + \frac{1}{2}$$

and

$$|\varepsilon_l(k,\xi)| \leq \frac{M_\lambda(k\xi)}{E_\lambda(k\xi)}\{e^{G_\lambda(k,\xi)} - 1\}.$$

In order to obtain an asymptotic expansion for y_l, we need the following estimates:

$$Y_\lambda(x) = \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right) + \mathrm{O}\left(\frac{1}{x^{3/2}}\right), \quad x \to +\infty, \tag{2.2.26}$$

$$J_\lambda(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right) + \mathrm{O}\left(\frac{1}{x^{3/2}}\right), \quad x \to +\infty, \tag{2.2.27}$$

$$Y_\lambda(x) \sim -\frac{\Gamma(\lambda)}{\pi}\left(\frac{2}{x}\right)^\lambda, \quad x \to 0 \tag{2.2.28}$$

$$J_\lambda(x) \sim \left(\frac{x}{2}\right)^\lambda \frac{1}{\Gamma(\lambda+1)}, \quad x \to 0 \tag{2.2.29}$$

Using the above estimates, we also infer

$$M_\lambda(x) \sim \left(\frac{2}{\pi x}\right)^{1/2}, \quad x \to \infty$$

$$M_\lambda(x) \sim \left(\frac{2}{\pi\lambda}\right)^{1/2}, \quad x \to 0$$

Afterwards, we fix $\xi > 0$ and for k large, using (2.2.21) we get the following estimate:

$$|G_\lambda(k,\xi)| \leq c\left(\frac{\ln k}{k} + \frac{1}{k}\right) \tag{2.2.30}$$

for some positive constant c independent of k, where we used that

$$\frac{\pi}{2}\int_1^\infty M_\lambda^2(kt)t|g(t)|\mathrm{d}t \sim \frac{1}{k}\int_1^\infty |g(t)|\mathrm{d}t$$

and

$$\frac{\pi}{2}\int_0^1 M_\lambda^2(kt)t|g(t)|\mathrm{d}t \leq c_1\left(\int_0^{1/k} M_\lambda^2(kt)\mathrm{d}t + \int_{1/k}^1 M_\lambda^2(kt)\mathrm{d}t\right) \sim \left(\frac{1}{k}+\frac{1}{k}\int_{1/k}^1 \frac{\mathrm{d}t}{t}\right)$$

for a positive constant c_1 and $k \to \infty$. Hence, we conclude that

$$z(\xi) = \sqrt{\frac{\pi\xi}{2k}}J_\lambda(k\xi) + \mathrm{O}\left(\frac{\ln k}{k^2}\right) = \frac{1}{k}\cos\left(k\xi - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right) + \mathrm{O}\left(\frac{\ln k}{k^2}\right) \tag{2.2.31}$$

for fixed $\xi > 0$. The above estimate can be differentiated with respect to ξ, with the error being $\mathrm{O}(\ln k/k)$.

Afterwards, using the Liouville transformation $y_l(r) = n(r)^{-1/4}z(\xi)$, we get

$$y_l(r) = c\frac{1}{n(r)^{1/4}}\sqrt{\frac{\pi\xi}{2k}}J_\lambda(k\xi) + \mathrm{O}\left(\frac{\ln k}{k^2}\right)$$

where the constant c is chosen so that y_l satisfies the initial condition (2.1.10). From the asymptotics for $r \to 0$ we get that $c = 1/(n(0)^{l/2+1/4})$. Finally, we conclude that

$$y_l(r) = \frac{1}{kn(r)^{1/4}n(0)^{l/2+1/4}}\cos\left(k\xi - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right) + \mathrm{O}\left(\frac{\ln k}{k^2}\right) \tag{2.2.32}$$

The corresponding asymptotic formula for the determinants D_l follows from (2.1.13) and the expansions of the spherical Bessel functions:

$$j_l(kr) = \frac{1}{kr}\cos\left(kr - \frac{l\pi}{2} - \frac{\pi}{2}\right) + \mathrm{O}\left(\frac{1}{k^2}\right), \quad k \to \infty, \tag{2.2.33}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}j_l(kr) = \frac{1}{r}\sin\left(kr - \frac{l\pi}{2} + \frac{\pi}{2}\right) + \mathrm{O}\left(\frac{1}{k}\right), \quad k \to \infty. \tag{2.2.34}$$

Using the addition formula for the sine function, and keeping only the higher order terms, we derive the following estimate:

$$D_l(k) = \frac{1}{a^2kn(0)^{l/2+1/4}}\sin k\,(a - A) + \mathrm{O}\left(\frac{\ln k}{k^2}\right) \tag{2.2.35}$$

(the above estimate is also valid for $l = 0$). A sufficient condition for the existence of an infinite and discrete set of real roots of (2.1.13) is that the sine function of (2.2.35) has non-trivial argument, i.e. $A \neq a$. This is satisfied if either $n(r) > 1$ or $0 < n(r) < 1$ for $0 \leq r < a$.

Therefore, we have the following theorem [30, 39]:

Theorem 2.2.2. *Assume that* $\mathrm{Im}(n) = 0$ *and that* $n(x) = n(r)$ *is spherically stratified,* $n(r) = 1$ *for* $r \geq a$, $n(r) > 1$ *or* $0 < n(r) < 1$ *for* $0 \leq r < a$ *and, as a* *function of r, $n \in C^2$. Then there exists an infinite set of transmission eigenvalues for* (2.1.1) – (2.1.4).

2.2.2 The case $l = 0$

Now, we examine the special case where the eigenfunctions are depending only on the radius r which corresponds to the boundary value problem (2.1.14)-(2.1.16) and was introduced in [37], (see also [12, 22, 30, 40]).

Theorem 2.2.3. *Assume that $n \in C^2[0, a]$, $\mathrm{Im}(n(r)) = 0$ and either $n(a) \neq 1$ or $n(a) = 1$ and $A \neq a$. Then there exists an infinite discrete set of transmission eigenvalues for* (2.1.1) – (2.1.4) *with spherically symmetric eigenfunctions.*

Proof. We introduce the following auxiliary initial value problem:

$$y_0''(r) + k^2 n(r) y_0(r) = 0, \quad 0 < r < a \tag{2.2.36}$$

$$y_0(0) = 0, \quad y_0'(0) = 1 \tag{2.2.37}$$

Using again the Liouville transformation (2.2.17), the problem is transformed in the following form:

$$\frac{\mathrm{d}^2 z(\xi)}{\mathrm{d}\xi^2} + [k^2 - p(\xi)]z(\xi) = 0, \quad 0 < \xi < A \tag{2.2.38}$$

$$z(0) = 0, \quad \frac{\mathrm{d}z(0)}{\mathrm{d}\xi} = n(0)^{-1/4} \tag{2.2.39}$$

where

$$p(\xi) := \frac{n''(r)}{4n(r)^2} - \frac{5}{16}\frac{n'(r)^2}{n(r)^3} \tag{2.2.40}$$

and A is given by (2.2.18). We can rewrite the initial value problem (2.2.38)-(2.2.39) as a Volterra integral equation of the second kind [78]:

$$z(\xi) = \frac{\sin k\xi}{kn(0)^{1/4}} + \int_0^\xi \frac{\sin k(\xi - t)}{k} z(t)p(t)\mathrm{d}t. \tag{2.2.41}$$

If we apply the method of successive approximations to estimate the order of the integral part of (2.2.41) as function of k, we show that the solution of (2.2.38)-(2.2.39) satisfies

$$z(\xi) = \frac{\sin k\xi}{kn(0)^{1/4}} + \mathrm{O}\left(\frac{1}{k^2}\right) \qquad \text{and} \qquad \frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi} = \frac{\cos k\xi}{n(0)^{1/4}} + \mathrm{O}\left(\frac{1}{k}\right).$$

Finally, the Liouville transform implies:

$$y_0(r) = \frac{1}{k[n(r)n(0)]^{1/4}} \sin k\xi + \mathrm{O}\left(\frac{1}{k^2}\right) \tag{2.2.42}$$

and

$$y_0'(r) = \left[\frac{n(r)}{n(0)}\right]^{1/4} \cos k\xi + \mathrm{O}\left(\frac{1}{k}\right) \tag{2.2.43}$$

uniformly on $[0, a]$.

Before we proceed with the expansion of the determinant, we make the following hypothesis; we assume that either $n(a) \neq 1$ or both $n(a) = 1$ and $A \neq a$. We mention that if $n(a) \neq 1$ and $n(r) = 1$ for $r > a$, the refractive index is in $C^2[0, a]$ with a jump discontinuity across the boundary ∂B_a of the inhomogeneous ball. We show that there exists an infinite set of eigenvalues in both cases.

From the characteristic equation (2.1.16), and the estimates (2.2.42)-(2.2.43) we get:

$$D_0(k) = \frac{1}{a^2 k}\left[\left(\frac{n(a)}{n(0)}\right)^{1/4} \cos kA \sin ka - \frac{1}{(n(a)n(0))^{1/4}} \sin kA \cos ka\right] + \mathrm{O}\left(\frac{1}{k^2}\right) \tag{2.2.44}$$

or equivalently,

$$D_0(k) = \frac{1}{a^2 k}\{C \cos kA \sin ka - B \sin kA \cos ka\} + \mathrm{O}\left(\frac{1}{k^2}\right) \tag{2.2.45}$$

where

$$B = \frac{1}{(n(a)n(0))^{1/4}} \quad \text{and} \quad C = \left(\frac{n(a)}{n(0)}\right)^{1/4}. \tag{2.2.46}$$

If $n(a) = 1$, then $B = C$ and using the sine addition formula we conclude that

$$D_0(k) = \frac{1}{a^2 k n(0)^{1/4}} \sin k\,(a - A) + \mathrm{O}\left(\frac{1}{k^2}\right) \tag{2.2.47}$$

and thus, since $A \neq a$, there exist an infinite and discrete set of transmission eigenvalues. Moreover, if $n(a) \neq 1$ and if A is a rational number, then the first term in (2.2.45) is a periodic function taking both positive and negative values. This means that for k large enough, there exists infinitely many real transmission eigenvalues. This argument also holds for $A = a$. Finally, if A is irrational, then the first term in (2.2.45) is almost-periodic and takes positive and negative values as well [22, 40]. □

We note that the asymptotic formulas (2.2.35) and (2.2.47) have the same leading part, but (2.2.35) provides a sharper estimate on the error term.

2.3 Uniqueness for the inverse spectral problem

The most interesting and important issue related to the interior transmission problem is the corresponding inverse spectral problem, that is the determination of the refractive index from transmission eigenvalues. As we noted in the introduction, these eigenvalues carry information about the refractive index and are used in sampling type methods for the reconstruction of the support of an inhomogeneous medium. Moreover, an important question is under what conditions the transmission eigenvalues uniquely determine the corresponding refractive index. This inverse spectral problem is investigated only when the domain is a ball in $\mathbb{R}^3$ and $n(|x|) := n(r)$ is spherically stratified. The first problem that we examine is the general inverse spectral problem i.e. the determination of the refractive index from the knowledge of all transmission eigenvalues. Afterwards, we consider the inverse spectral problem for a subset of the spectrum that is the special transmission eigenvalues corresponding to spherically symmetric eigenfunctions.

2.3.1 Uniqueness in the general case

We consider the inverse spectral problem for (2.1.1) - (2.1.4) and we assume that the whole spectrum is known, including multiplicities of the eigenvalues. We present the ideas of Cakoni, Colton and Gintides in [14, theorem 2.1]. The proof is based on an equivalent integral representation of the solution of (2.1.9), asymptotic estimates for the determinants and a final reduction of the inverse problem to an equivalent moment type problem.

A basic tool for inverse Sturm-Liouville eigenvalue problems is the use of the Gelfand-Levitan-Marchenko transformation operator and the corresponding Goursat problem for the kernel of the operator. For more information about Goursat problems see [26, sec. 3.6] and [63, sec. 4.4 and 4.5]. Here, we use a modification of the integral representation of solutions of (2.1.1) in interior domains [33]. We define the contrast function $m(r) := 1 - n(r)$.

Theorem 2.3.1. *Assume that $n \in C^2[0,\infty)$, $\operatorname{Im}(n(r)) = 0$ and $m(r)$ does not change sign. If $n(0)$ is given then $n(r)$ is uniquely determined from a knowledge of all transmission eigenvalues and their multiplicity as a zero of $D_l(k)$.*

Proof. In the case where $n(r) \in C^2[0,\infty)$, following [33] we can represent $y_l(r)$ in the form:

$$y_l(r) = j_l(kr) + \int_0^r G(r,s,k) j_l(ks)\mathrm{d}s \qquad (2.3.48)$$

for

$$w(r,\theta) = b_l y_l(r) P_l(\cos\theta) \qquad \text{and} \qquad y_l'' + \frac{2}{r}y_l' + \left(k^2 n(r) - \frac{l(l+1)}{r^2}\right) y_l = 0$$

in (2.1.8) and (2.1.9) respectively. Obviously, the characteristic equation (2.1.13) is written as

$$D_l(k) = det \begin{pmatrix} y_l(r) & -j_l(kr) \\ \frac{\mathrm{d}}{\mathrm{d}r} y_l(r) & -\frac{\mathrm{d}}{\mathrm{d}r} j_l(kr) \end{pmatrix} = 0 \qquad \text{for} \quad r = a. \qquad (2.3.49)$$

If we substitute (2.3.48) into (2.1.1) and integrate by parts we find that the kernel $G(r,s,k)$ satisfies the following Goursat problem for $0 < s \leq r < a$

$$r^2 \left[\frac{\partial^2 G}{\partial r^2} + \frac{2}{r}\frac{\partial G}{\partial r} + k^2 n(r) G \right] = s^2 \left[\frac{\partial^2 G}{\partial s^2} + \frac{2}{s}\frac{\partial G}{\partial s} + k^2 G \right] \qquad (2.3.50)$$

$$G(r,r,k) = \frac{k^2}{2r} \int_0^r t m(t)\mathrm{d}t \qquad (2.3.51)$$

$$G(r,s,k) = \mathrm{O}\left((rs)^{1/2}\right) \qquad (2.3.52)$$

Equation (2.9) in [33] implies that $G(r,s,k)$ satisfies the following integral equation

$$\begin{aligned} G(r,s,k) = & -\frac{k^2}{2\sqrt{rs}} \int_0^{\sqrt{rs}} t(n(t)-1)\mathrm{d}t \\ & -\frac{k^2}{\sqrt{rs}} \int_1^{\sqrt{\frac{r}{s}}} \int_0^{\sqrt{rs}} t^2 \tau \left[n(t\tau) - \frac{1}{\tau^4} \right] G(t\tau, t/\tau, k)\mathrm{d}t\mathrm{d}\tau \qquad (2.3.53) \end{aligned}$$

where uniqueness of G is proved using Neumann series. Moreover, G is an entire function of k of exponential type, is even and satisfies

$$G(r,s,k) = \frac{k^2}{2\sqrt{rs}} \int_0^{\sqrt{rs}} t m(t)\mathrm{d}t \left(1 + \mathrm{O}(k^2)\right). \qquad (2.3.54)$$

(For more information on entire functions we refer to appendix A.4). Next, we assume that c_{2l+2} is the coefficient of k^{2l+2} in the Taylor expansion of $D_l(k)$. We can estimate this constant from the asymptotic formula of $D_l(k)$ for $k \to 0$. The spherical Bessel function satisfies

$$j_l(kr) = \frac{\sqrt{\pi}(kr)^l}{2^{l+1}\Gamma(l+3/2)}(1 + \mathrm{O}(k^2 r^2)), \quad k \to 0 \qquad (2.3.55)$$

and using (2.3.48) and (2.3.54) we get

$$y_l(r) = j_l(kr) + \frac{k^2}{2}\int_0^r \int_0^{\sqrt{rs}} \frac{tm(t)}{\sqrt{rs}} j_l(ks)\mathrm{d}t\mathrm{d}s + \mathrm{O}(k^{l+4}). \tag{2.3.56}$$

By substituting (2.3.55) and (2.3.56) in (2.3.49), after several calculations we arrive at

$$\begin{aligned} c_{2l+2}\left[\frac{2^{l+1}\Gamma(l+3/2)}{\sqrt{\pi}a^{(l-1)/2}}\right]^2 &= a\int_0^a \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{2\sqrt{rs}}\int_0^{\sqrt{rs}} tm(t)\mathrm{d}t\right)\bigg|_{r=a} s^l \mathrm{d}s \\ &\quad -l\int_0^a \frac{1}{2\sqrt{as}}\int_0^{\sqrt{as}} tm(t)\mathrm{d}t\ s^l\mathrm{d}s + \frac{a^l}{2}\int_0^a tm(t)\mathrm{d}t. \end{aligned} \tag{2.3.57}$$

Interchanging the orders of integration and using a change of variables, the previous relation is simplified in (see appendix A.1)

$$c_{2l+2} = \frac{\pi}{a^2(2^{l+1}\Gamma(l+3/2))^2}\int_0^a t^{2l+2}m(t)\mathrm{d}t. \tag{2.3.58}$$

Furthermore, we mention that j_l is even if l is even and odd if l is odd. Thus, since G is even and the product of two evens or two odds is even, we conclude that D_l is an even function of k. Moreover, since j_l and G are entire functions of k of exponential type, so is y_l and consequently D_l. From the asymptotic behaviour (2.2.35) for $k \to \infty$, we see that the order of D_l as a function of k is one, and hence by Hadamard's factorization theorem (appendix, theorem A.4.8)

$$D_l(k) = k^{2l+2}c_{2l+2}\prod_{n=1}^{\infty}\left(1-\frac{k^2}{k_{nl}^2}\right) \tag{2.3.59}$$

where c_{2l+2} is given by (2.3.58) provided that $m(r)$ does not change sign (in order $c_{2l+2} \neq 0$ for any l) and k_{nl} are the complex transmission eigenvalues in the right half plane.

Now, we assume that all transmission eigenvalues are known and hence

$$\frac{D_l(k)}{c_{2l+2}} = k^{2l+2}\prod_{n=1}^{\infty}\left(1-\frac{k^2}{k_{nl}^2}\right) \tag{2.3.60}$$

the left hand side of (2.3.60) is also known. Moreover, we define the auxiliary constants

$$\gamma_l := \frac{1}{c_{2l+2}n(0)^{l/2+1/4}} \neq 0 \tag{2.3.61}$$

and we can express the asymptotic formula (2.2.35) as

$$\frac{D_l(k)}{c_{2l+2}} = \frac{\gamma_l}{a^2k}\sin k(a-A) + \mathrm{O}\left(\frac{\ln k}{k^2}\right). \tag{2.3.62}$$

Now, we claim that since the left hand side of (2.3.62) is known, so is the constant γ_l. To prove this, we assume that two refractive indices n_1 and n_2, which satisfy the prerequisites of the theorem, have the same corresponding transmission eigenvalues. If we denote $D_{l_i},\ c_{2l+2_i},\ \gamma_{l_i},\ A_i$ for $i = 1, 2$ the characteristics of each problem, since

$$\frac{D_{l_1}(k)}{c_{2l+2_1}} = \frac{D_{l_2}(k)}{c_{2l+2_2}}$$

we conclude that

$$\gamma_{l_1} \sin k(a - A_1) = \gamma_{l_2} \sin k(a - A_2)$$

for k large enough. Using a linear dependence argument for the sine functions, we derive $A_1 = A_2$ and $\gamma_{l_1} = \gamma_{l_2}$.

Finally, from (2.3.58) we have

$$\int_0^a t^{2l+2} m(t)\mathrm{d}t = \frac{a^2(2^{l+1}\Gamma(l+3/2))^2}{n(0)^{l/2+1/4}\gamma_l \pi}. \tag{2.3.63}$$

If $n(0)$ is given, then the right hand side of (2.3.63) is known and the refractive index is uniquely determined by applying the Müntz's theorem [41] for the C^2 function $m(t)$. $\square$

Remark 2.3.2. *The Müntz's theorem is also used to prove the existence of a countably infinite number of transmission eigenvalues if $n \in C^2[0, a]$ and $n(r)$ is not identically equal to one ([38, theorem 2.1]).*

2.3.2 Uniqueness for special transmission eigenvalues

Afterwards, we are interested in the inverse spectral problem of determining the refractive index from the special transmission eigenvalues including multiplicities, i.e. the eigenvalues of (2.1.14)-(2.1.16). In the following we present the main results of this problem.

This problem was firstly examined by McLaughlin and Polyakov in [72], (see also [73, 74]). The refractive index is assumed to satisfy: $n \in C^1(\mathbb{R})$, $n'' \in L^2[0, a]$ and $\mathrm{Im}(n(r)) = 0$. Under these assumptions and given that $A \neq a$, the authors in [72, lemma 2] proved the following asymptotic relation for the real transmission eigenvalues

$$k_j^2 = \frac{j^2\pi^2}{(A-a)^2} + \mathrm{O}(1), \quad j \to +\infty. \tag{2.3.64}$$

Hence, the travel time A can be determined if the real eigenvalues of (2.1.14)-(2.1.16) are known. Moreover, authors gave a uniqueness result for the inverse spectral problem under restrictive assumptions on $n(r)$:

Theorem 2.3.3. *([72, theorem 3]) Let $n_1(r),\ n_2(r) \in C^1(\mathbb{R})$ be positive functions, such that $n_i''(r) \in L^2[0,a]$ and $n_i(r) = 1$ for $r \geq a$ $(i = 1, 2)$. Let $A_i = \int_0^a \sqrt{n_i(r)}\mathrm{d}r > a$ or $A_i < a$ for $i = 1, 2$ simultaneously. Let there exists $M > 0$ such that all real eigenvalues of $(2.1.14) - (2.1.16)$ for n_1 and n_2, that are greater of M coincide. Then we have that $A_1 = A_2$ and we can denote by A their common value.*

Suppose there is a common subsequence of eigenvalues of $(2.1.14) - (2.1.16)$, $k_j^2,\ j = 1, \ldots, \infty$ for n_1 and n_2 satisfying the following properties:

a. there exists $m_0 \in \mathbb{Z}^+$ such that $|k_j^2| < (m + 1/2)^2\pi^2/(A - a)^2$ for $j = 1, \ldots, m$ and $m \geq m_0$ and

b. for $j > m_0$ all k_j^2 are real and satisfy $|k_j^2| > (m_0 + 1/2)^2\pi^2/(A - a)^2$.

If also

1. $A \leq a/3$ or,

2. $A < a < 3A$ and $n_1(r) = n_2(r)$ for r satisfying $0 \leq \int_r^a \sqrt{n_i(r)}\mathrm{d}r \leq (3A - a)/2$ or,

3. $A > a \geq 0$ and $n_1(r) = n_2(r)$ for r satisfying $0 \leq \int_r^a \sqrt{n_i(r)}\mathrm{d}r \leq (A + a)/2$

then $n_1(r) = n_2(r)$.

Several years later, Aktosun, Gintides and Papanikolaou in [2] showed that the special transmission eigenvalues of (2.1.14)-(2.1.16) including multiplicities, uniquely determine $n(r)$ under the less restrictive condition $A < a$:

Theorem 2.3.4. *([2, theorem 3.2]) Assume that $n(r) \in C^1(0, a)$ such that $n''(r) \in L^2(0, 1)$, $\mathrm{Im}(n(r)) = 0$ and $A < a$. Then $n(r)$ is uniquely determined from the eigenvalues of $(2.1.14) - (2.1.16)$ and their multiplicities as zeros of $D_0(k)$.*

The proof is based in the reduction of the inverse problem to the classic inverse Sturm-Liouville problem, and it breaks down if $A > a$. Moreover, it is proved that if $A = a$, the eigenvalues together with the constant coefficient of the corresponding Hadamard's factorization of the determinant, uniquely determine $n(r)$. Recently, Buterin, Yang and Yurko in [10] proved that the spectrum alone (without the constant) does not determine $n(r)$ if $A = a$.

A shorter proof of theorem 2.3.4, was given by Colton and Leung in [35] (see also [12, section 9.4]) provided that $n \in C^2[0, a]$, $0 < n < 1$ and given that $n(0)$ is known. In the following, we present the main idea of the proof.

Theorem 2.3.5. *Assume that $n \in C^2[0,a],\ n(a)=1,\ n'(a)=0$ and $n(0)$ is given. Then if $0<n(r)<1$ for $r\in(0,a)$ the transmission eigenvalues (including multiplicity) uniquely determine $n(r)$.*

Proof. Using the theory of transformation operators ([12, section 9.2]), we can represent the solution of (2.2.38)-(2.2.39) in the following form

$$z(\xi) = \frac{1}{n(0)^{1/4}}\left[\frac{\sin k\xi}{k} + \int_0^\xi K(\xi,t)\frac{\sin kt}{k}\mathrm{d}t\right], \qquad 0\le\xi\le A \tag{2.3.65}$$

where $K(\xi,t)$ is the unique solution of

$$K_{\xi\xi} - K_{tt} - p(\xi)K = 0, \qquad 0<t<\xi<A$$

$$K(\xi,\xi) = \frac{1}{2}\int_0^\xi p(s)\mathrm{d}s, \qquad 0\le\xi\le A$$

$$K(\xi,0)=0, \qquad 0\le\xi\le A.$$

Existence of such a K can be proved using the method of successive approximations. Now, given that $n(a)=1$ and $n'(a)=0$ we infer that the solution of (2.2.36)-(2.2.37) satisfies:

$$y_0(a) = \frac{1}{n(0)^{1/4}}\left[\frac{\sin kA}{k} + \int_0^A K(A,t)\frac{\sin kt}{k}\mathrm{d}t\right], \tag{2.3.66}$$

$$y_0'(a) = \frac{1}{n(0)^{1/4}}\left[\cos kA + \frac{\sin kA}{2k}\int_0^A p(s)\mathrm{d}s + \int_0^A K_\xi(A,t)\frac{\sin kt}{k}\mathrm{d}t\right]. \tag{2.3.67}$$

We recall that the characteristic determinant satisfies

$$D_0(k) = \frac{1}{a^2 k n(0)^{1/4}}\sin k\,(a-A) + \mathrm{O}\left(\frac{1}{k^2}\right)$$

and from Hadamard's factorization (2.3.59) we have the representation

$$D_0(k) = c_0 k^2\prod_{n=1}^{\infty}\left(1-\frac{k^2}{k_{n0}^2}\right).$$

The determinant has a zero of order two at the origin, provided that $\int_0^a t^2 m(t)\mathrm{d}t \neq 0$ and c_0 is given by (2.3.58). Following similar arguments with theorem 2.3.1, if $n(0)$ and the special transmission eigenvalues $\{k_{n0}\}_{n=1}^{\infty}$ (including multiplicities) are known, then $c_0,\ A$ and consequently $D_0(k)$ are also known.

Moreover, since $D_0(k)$ satisfies

$$D_0(k) = \frac{\sin ka}{k}y_0'(a) - \cos ka\ y_0(a),$$

evaluating $D_0(k)$ at $k = m\pi/a,\ m = 1, 2, \dots$ gives

$$\frac{m\pi}{a} D_0\left(\frac{m\pi}{a}\right) = \frac{(-1)^{m+1}}{n(0)^{1/4}}\left[\sin\frac{m\pi A}{a} + \int_0^A K(A,t)\sin\frac{m\pi t}{a}\mathrm{d}t\right].$$

Now, since $\{\sin\frac{m\pi t}{a}\}_{m=1}^{\infty}$ forms a complete set in $L^2[0,A]$ if $A < a$ ([94, p. 97]) and using the fact that $n(0)$, A and D_0 are known we conclude that $K(A,t)$ is uniquely determined. We use similar arguments for $k = m\pi/A,\ m = 1, 2, \dots$. From (2.3.67) we get

$$\frac{m\pi}{A} D_0\left(\frac{m\pi}{A}\right) = -y_0(a)\frac{m\pi}{A}\cos\frac{m\pi a}{A} + \frac{\sin\frac{m\pi a}{A}}{n(0)^{1/4}}\left[(-1)^m + \frac{A}{m\pi}\int_0^A K_\xi(A,t)\sin\frac{m\pi t}{A}\mathrm{d}t\right].$$

From the completeness of $\{\sin\frac{m\pi t}{A}\}_{m=1}^{\infty}$ in $L^2[0,A]$ and since $y_0(a)$ is known, we infer that $K_\xi(A,t)$ is uniquely determined. From [80] (see also [63, section 4.7]) we conclude that the Cauchy data $K(A,t)$, $K_\xi(A,t)$ uniquely determine $p(\xi)$ for $0 \le \xi \le A$.

Now we assume that two different refractive indices n_1 and n_2, which satisfy the prerequisites of the theorem, have the same eigenvalues including multiplicities. Then, the above considerations imply that $p(\xi_1) = p(\xi_2)$ for fixed r where

$$\xi_i = \int_0^r \sqrt{n_i(t)}\mathrm{d}t, \qquad i = 1, 2.$$

From the definition of $p(\xi)$ in (2.2.40), since $n_1(a) = n_2(a) = 1$ and $n_1'(a) = n_2'(a) = 0$, we have that both $n_1^{1/4}(r(\xi_1))$ and $n_2^{1/4}(r(\xi_2))$ satisfy

$$\frac{\mathrm{d}^2 n_i^{1/4}}{\mathrm{d}\xi_i^2} - p(\xi_i)n_i^{1/4} = 0, \qquad 0 \le \xi_i \le A,$$

$$n_i^{1/4}(r(A)) = 1,$$
$$\frac{\mathrm{d}n_i^{1/4}(r(A))}{\mathrm{d}\xi_i} = 0.$$

for $i = 1, 2$. By the uniqueness theorem for initial value problem for ordinary differential equations we conclude that $n_1(\xi_1) = n_2(\xi_2)$ for each fixed r. Finally, taking into account that $r_i = r(\xi_i)$ satisfies

$$\frac{\mathrm{d}r_i}{\mathrm{d}\xi_i} = \frac{1}{\sqrt{n_i(\xi_i)}}, \qquad 0 \le \xi_i \le A,$$

$$r_i(0) = 0$$

for $i = 1, 2$, and using again the uniqueness for initial value problems it follows that $r(\xi_1) = r(\xi_2)$. This implies that $\xi_1 = \xi_2$ and hence $n_1 = n_2$.

□

Remark 2.3.6. *We note that the condition* $0 < n(r) < 1$ *that we used in the preceding theorem is somewhat stronger than* $0 < A < a$ *of theorem* 2.3.4.

For some further contributions in the inverse spectral problem for transmission eigenvalues we refer to [4, 9, 88, 91, 92].

2.4 Complex transmission eigenvalues

We have already mentioned that the interior transmission eigenvalue problem is non-self-adjoint and hence complex eigenvalues may exist. We note that complex eigenvalues cannot be detected using far field measurements or other methods up to now. Since only real eigenvalues can be determined from scattering data, we are interested in the existence of complex eigenvalues or the conditions under which there are no complex eigenvalues at all. We consider domains in $\mathbb{R}^2$ and $\mathbb{R}^3$ where both constant and variable refractive indices are inspected.

2.4.1 Complex eigenvalues in $\mathbb{R}^2$

A first existence result was given by Cakoni, Colton and Gintides in [14, section 2.2], for the simple case of the unit disc in $\mathbb{R}^2$ with constant refractive index $n \neq 1$, sufficiently small. The interior transmission problem (2.1.1) - (2.1.4), for spherically symmetric eigenfunctions in this case is written as:

$$w_{rr} + \frac{1}{r} w_r + k^2 n w = 0, \qquad 0 \le r \le 1, \tag{2.4.68}$$

$$v_{rr} + \frac{1}{r} v_r + k^2 v = 0, \qquad 0 \le r \le 1, \tag{2.4.69}$$

$$w(1) = v(1) \tag{2.4.70}$$

$$w_r(1) = v_r(1). \tag{2.4.71}$$

Theorem 2.4.1. *We consider the interior transmission problem* $(2.1.1)-(2.1.4)$ *where the domain is the unit disc in* $\mathbb{R}^2$*. If the refractive index is a constant* $n > 1$ *sufficiently small, there exist complex transmission eigenvalues.*

Proof. Let B_a be the unit disc in $\mathbb{R}^2$ and we also assume that the refractive index is a positive real constant. The solutions of the Helmholtz equations (2.1.1) and (2.1.2) are

$$w(r) = J_m(k\sqrt{n}r)\sin mr, \; w(r) = J_m(k\sqrt{n}r)\cos mr, \qquad m = 0, 1, \ldots$$

and

$$v(r) = J_m(kr)\sin mr,\ v(r) = J_m(kr)\cos mr, \qquad m = 0, 1, \ldots$$

respectively. We consider only spherically symmetric eigenfunctions that is $m = 0$. The corresponding characteristic determinant is

$$D_0(k) = det \begin{pmatrix} J_0(kr) & J_0(k\sqrt{n}r) \\ J_0'(kr) & J_0'(k\sqrt{n}r) \end{pmatrix} = 0, \quad \text{for } r = 1.$$

As a result, k is a transmission eigenvalue if and only if

$$D_0(k) = k(J_1(k)J_0(k\sqrt{n}) - \sqrt{n}J_0(k)J_1(k\sqrt{n}) = 0, \tag{2.4.72}$$

where we used that $J_0'(kr) = -kJ_1(kr)$. Now, if we consider D_0 as a function of two variables, $D_0(k, \sqrt{n})$, and differentiate with respect to $\sqrt{n}$ we compute

$$D_0'(k, \sqrt{n})|_{\sqrt{n}=1} = k(kJ_1(k)J_0'(k) - J_0(k)J_1(k) - kJ_0(k)J_1'(k))$$

Furthermore, using that $\frac{\mathrm{d}}{\mathrm{d}t}(tJ_1(t)) = tJ_0(t)$ we infer that

$$-kJ_0(k)J_1(k) - k^2J_0(k)J_1'(k) = -k^2J_0(k)\left(\frac{J_1(k)}{k} + J_1'(k)\right) = -k^2J_0^2(k)$$

and hence

$$D_0'(k, \sqrt{n})|_{\sqrt{n}=1} = -k^2(J_1^2(k) + J_0^2(k))$$

or equivalently

$$f(k) := \lim_{\sqrt{n}\to 1^+} \frac{D_0(k, \sqrt{n}) - D_0(k, 1)}{\sqrt{n} - 1} = -k^2(J_1^2(k) + J_0^2(k)). \tag{2.4.73}$$

Since $J_0(k)$ and $J_1(k)$ do not have any common roots, $f(k)$ is strictly negative for $k \neq 0$ and the only possible roots of $f(k)$ are complex. Since $f(k)$ is an even entire function of exponential type, from Hadamard's factorization theorem, it has an infinite number of complex roots. Furthermore, the convergence in (2.4.73) is uniform in compact subsets of $\mathbb{C}$ since the sequence in the right hand side of the limit is locally uniformly bounded in compact subsets (Montel's theorem, [32, p. 213]). Finally, by applying Hurwitz's theorem ([32, p. 213]), since $f(k)$ is a non-constant analytic function, for any root k_0 of $f(k)$ and for any $\varepsilon > 0$ small enough, we conclude that $D_0(k)$ has a (complex) root in the disc $|k - k_0| < \varepsilon$ for n sufficiently close to 1. This completes the proof. □

Later, Colton and Leung showed that complex eigenvalues in $\mathbb{R}^2$ exist for constant n not necessarily small.

Theorem 2.4.2. *([69, theorem 2.4]) Assume $n = n(r)$ is a positive constant not equal to* 1*. Then there exists an infinite number of complex eigenvalues of* (2.4.68) – (2.4.71).

2.4.2 Complex eigenvalues in $\mathbb{R}^3$

We present the main results on existence and distribution of complex eigenvalues for the unit ball of $\mathbb{R}^3$. If the refractive index is constant and the eigenfunctions are spherically symmetric, we have the following [69]:

Theorem 2.4.3. *If n is a positive integer not equal to one then all transmission eigenvalues corresponding to spherically symmetric eigenfunctions are real. On the other hand if n is a rational positive number $n = p/q$ such that either $q < p < 2q$ or $p < q < 2p$ then there exists an infinite number of complex eigenvalues.*

For the variable index case, in [69, theorem 4.2], authors proved the existence of complex eigenvalues under several restrictions on $n(r)$. Moreover, Colton and Leung examined the location of the complex eigenvalues on the complex plane :

Theorem 2.4.4. *([35, theorem 2.1]) Assume that $n(1) \neq 1$ and $n \in C^2[0,1]$. Then if complex eigenvalues exist, all of them lie in a strip parallel to the real axis.*

Recently, Colton, Leung and Meng in [36] examined the existence and distribution of complex eigenvalues in association with the values of the travel time $A = \int_0^1 \sqrt{n(t)}\mathrm{d}t$ and the values of $n(r)$ in the boundary, i.e. $n(1), n'(1)$ and $n''(1)$.

Theorem 2.4.5. *([36, theorem 4.1]) Suppose the refractive index $n \in C^2[0,1]$ with $n(1) = 1,\ n'(1) = 0$ and $A \neq 1$. Then under the extra assumption that $n''(1) \neq 0$ the entire function $D_0(k)$ has infinitely many non-real zeros and infinitely many real zeros.*

Also, in contrast to theorem 2.4.4, authors showed the following:

Theorem 2.4.6. *([36, theorem 4.2]) Suppose the refractive index $n \in C^2[0,1]$ with $n(1) = 1$ and $A \neq 1$. If either $n'(1)$ or $n''(1)$ is non-zero, the zeros of $D_0(k)$ do not lie inside a fixed strip parallel to the real axis.*

Finally, for the case $A = 1$ we have the next theorem

Theorem 2.4.7. *([36, theorem 5.1]) Let the refractive index $n \in C^2[0,1]$. Suppose $A = 1$ and $n(1) \neq 1$. Then there are at most finitely many complex transmission eigenvalues. However if both $A = 1$ and $n(1) = 1$, then it is possible to have only finitely many real eigenvalues.*

For the one-dimensional problem and some further results, we refer to [34, 86, 97].

3

The interior transmission problem for the spherically stratified medium with discontinuous refractive index

We formulate the interior transmission problem for a spherically symmetric and discontinuous refractive index and the corresponding auxiliary initial value problems [47]. We investigate the properties of solutions of the initial value problems and the asymptotic behaviour of the determinants. Next, we examine whether the eigenvalues corresponding to spherically symmetric eigenfunctions can determine some characteristics of the discontinuous refractive index. We show that the knowledge of all transmission eigenvalues including multiplicities uniquely determine the refractive index. Finally, we derive the asymptotic behaviour of real transmission eigenvalues in the form of an asymptotic inequality and present an example of a specific transmission eigenvalue problem, where the eigenvalues do not have an asymptotic expansion.

Contents

3.1 Formulation of the problem

We consider the interior transmission problem (2.1.1) - (2.1.4) for the unit ball of $\mathbb{R}^3$, i.e. $a = 1$, where the corresponding refractive index is discontinuous at a point $d \in (0, 1)$, see figure 3.1. We use similar methods with the continuous problem, but this procedure is more complicated. The refractive index is assumed to satisfy:

$$n(r) > 0,\ \mathrm{Im}(n(r)) = 0,\ n(r) = 1 \text{ for } r \geq 1, \text{ and } n'(1) = 0. \tag{3.1.1}$$

Also, $n(r)$ is C^2 in each $[0, d), (d, \infty)$ and the one-sided limits at d are finite. We introduce the following jump conditions:

$$n(d^+) = an(d^-) \tag{3.1.2}$$

$$n'(d^+) = a^{-1}n'(d^-) + bn(d^-) \tag{3.1.3}$$

where the constants a and b satisfy

$$a > 0,\ |a - 1| + |b| > 0 \tag{3.1.4}$$

(we note that the discontinuity size a, is different from the radius of the ball B_a). The assumption (3.1.4) ensures that n and/or n' is discontinuous at d.

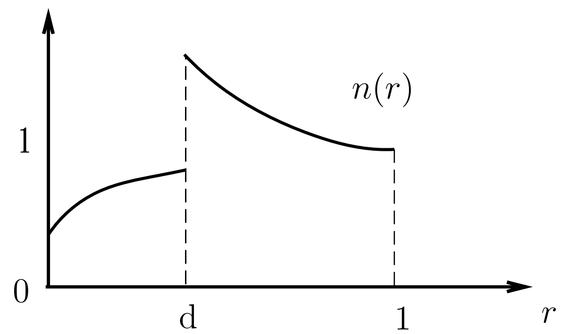

Figure 3.1: The discontinuous refractive index

Using separation of variables, we derive a boundary value problem equivalent to (2.1.11) - (2.1.13) for $y_l(r)$:

$$y_l''(r) + \left(k^2 n(r) - \frac{l(l+1)}{r^2}\right) y_l(r) = 0 \qquad \text{for } \ 0 < r < 1,\ r \neq d \tag{3.1.5}$$

$$\lim_{r \to 0} \left(\frac{y_l(r)}{r} - j_l(kr)\right) = 0, \tag{3.1.6}$$

$$D_l(k) := det \begin{pmatrix} \frac{y_l(r)}{r} & -j_l(kr) \\ \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{y_l(r)}{r}\right) & -\frac{\mathrm{d}}{\mathrm{d}r} j_l(kr) \end{pmatrix} = 0 \qquad \text{for} \quad r = 1, \tag{3.1.7}$$

and if we use the Liouville transformation (2.2.17), we arrive at the problem:

$$\frac{\mathrm{d}^2 z(\xi)}{\mathrm{d}\xi^2} + \left(k^2 - \frac{l(l+1)}{\xi^2} - g(\xi)\right) z(\xi) = 0 \qquad \text{for} \quad 0 < \xi < A \tag{3.1.8}$$

where

$$g(\xi) := \frac{l(l+1)}{r^2 n(r)} - \frac{l(l+1)}{\xi^2} + \frac{n''(r)}{4n(r)^2} - \frac{5}{16}\frac{n'(r)^2}{n(r)^3}.$$

The travel time is given by

$$A := \int_0^1 \sqrt{n(t)}\mathrm{d}t \tag{3.1.9}$$

and we also define the relevant travel time

$$\tilde{d} := \int_0^d \sqrt{n(t)}\mathrm{d}t. \tag{3.1.10}$$

We note that for the discontinuous problem, $g(\xi)$ is not defined at $\xi = \tilde{d}$. Since $n(r) > 0$ for $r \geq 0$, the Liouville transformation is invertible and g is well defined in each interval $(0, \tilde{d}), (\tilde{d}, \infty)$ and is a piecewise continuous function for $r > 0$. Thus, the Liouville transformation can be used only locally, on each interval $(0, d)$ and (d, ∞), [3]. Moreover, since $n(r) = 1$ for $r \geq 1$ we have

$$\int_0^A \xi|g(\xi)|\mathrm{d}\xi < \infty \quad \text{and} \quad \int_A^\infty |g(\xi)|\mathrm{d}\xi < \infty \tag{3.1.11}$$

The following lemma ensures that z is discontinuous at $\xi = \tilde{d}$ even though y_l is continuous at $r = d$.

Lemma 3.1.1. *The solution $z(\xi)$ of the transformed problem* (3.1.8)*, is discontinuous at $\xi = \tilde{d}$ and satisfies the jump conditions:*

$$z(\tilde{d}^+) = \tilde{a}\, z(\tilde{d}^-) \tag{3.1.12}$$

$$\frac{\mathrm{d}z(\tilde{d}^+)}{\mathrm{d}\xi} = \tilde{a}^{-1}\frac{\mathrm{d}z(\tilde{d}^-)}{\mathrm{d}\xi} + \tilde{b}\, z(\tilde{d}^-) \tag{3.1.13}$$

where:

$$\tilde{a} := a^{1/4} \tag{3.1.14}$$

$$\tilde{b} := \frac{1}{4}\left[\frac{n'(d^+)}{n(d^+)^{3/2}}\tilde{a} - \frac{n'(d^-)}{n(d^-)^{5/4} n(d^+)^{1/4}}\right]. \tag{3.1.15}$$

Proof. Since y_l is continuous at $r = d$ we have $y_l(d^+) = y_l(d^-)$ and from (2.2.17), $z(\tilde{d}^+)/n(d^+)^{1/4} = z(\tilde{d}^-)/n(d^-)^{1/4}$. Thus, $z(\tilde{d}^+) = z(\tilde{d}^-) n(d^+)^{1/4}/n(d^-)^{1/4}$ which implies that

$$z(\tilde{d}^+) = \tilde{a} z(\tilde{d}^-) \quad \text{where} \quad \tilde{a} = \frac{n(d^+)^{1/4}}{n(d^-)^{1/4}} = a^{1/4}. \tag{3.1.16}$$

Also, using the chain rule we have:

$$y_l'(r) = \frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi} n(r)^{1/4} - \frac{1}{4} z(\xi) \frac{n'(r)}{n(r)^{5/4}}. \tag{3.1.17}$$

From the continuity of y_l' at $r = d$ we have $y_l'(d^+) = y_l'(d^-)$ and from (3.1.17)

$$\frac{\mathrm{d}z(\tilde{d}^+)}{\mathrm{d}\xi} n(d^+)^{1/4} - \frac{1}{4} z(\tilde{d}^+) \frac{n'(d^+)}{n(d^+)^{5/4}} = \frac{\mathrm{d}z(\tilde{d}^-)}{\mathrm{d}\xi} n(d^-)^{1/4} - \frac{1}{4} z(\tilde{d}^-) \frac{n'(d^-)}{n(d^-)^{5/4}}$$

Therefore,

$$\frac{\mathrm{d}z(\tilde{d}^+)}{\mathrm{d}\xi} = \frac{\mathrm{d}z(\tilde{d}^-)}{\mathrm{d}\xi} \frac{n(d^-)^{1/4}}{n(d^+)^{1/4}} - \frac{1}{4} z(\tilde{d}^-) \frac{n'(d^-)}{n(d^+)^{1/4} n(d^-)^{5/4}} + \frac{1}{4} z(\tilde{d}^+) \frac{n'(d^+)}{n(d^+)^{1/4} n(d^+)^{5/4}}$$

Now, using (3.1.12) and (3.1.16) we conclude that

$$\frac{\mathrm{d}z(\tilde{d}^+)}{\mathrm{d}\xi} = \frac{\mathrm{d}z(\tilde{d}^-)}{\mathrm{d}\xi} \tilde{a}^{-1} + z(\tilde{d}^-) \left[\frac{1}{4} \frac{n'(d^+)}{n(d^+)^{3/2}} \tilde{a} - \frac{1}{4} \frac{n'(d^-)}{n(d^-)^{5/4} n(d^+)^{1/4}} \right].$$

□

Moreover, the discontinuity relation (3.1.4) implies the same assumptions for the constants $\tilde{a}$ and $\tilde{b}$ in the transformed problem.

Lemma 3.1.2. *The existence of a discontinuity in the refractive index, in the form* (3.1.4) *implies that*

$$|\tilde{a} - 1| + |\tilde{b}| > 0 \tag{3.1.18}$$

for the transformed problem.

Proof. From (3.1.3) and (3.1.15) we have

$$\tilde{b} = \frac{1}{4} \frac{a^{1/4}}{n(d^+)^{3/2}} \left(b n(d^-) + a^{-1} n'(d^-) \right) - \frac{1}{4} \frac{n'(d^-)}{n(d^-)^{5/4} n(d^+)^{1/4}}$$

and using (3.1.2) we conclude that

$$\tilde{b} = \frac{1}{4} b \frac{n(d^-)^{3/4}}{n(d^+)^{5/4}} + \frac{1}{4} n'(d^-) \left(\frac{1}{n(d^+)^{3/2} a^{3/4}} - \frac{1}{n(d^-)^{5/4} n(d^+)^{1/4}} \right) \tag{3.1.19}$$

Now, the last term in (3.1.19) gives

$$\frac{1}{n(d^+)^{3/2} a^{3/4}} - \frac{1}{n(d^-)^{5/4} n(d^+)^{1/4}} = \frac{n(d^-)^{3/4}}{n(d^+)^{3/2} n(d^+)^{3/4}} - \frac{1}{n(d^-)^{5/4} n(d^+)^{1/4}}$$

$$
\begin{aligned}
&= \frac{n(d^-)^{3/4}}{n(d^+)^{9/4}} - \frac{1}{n(d^-)^{5/4}n(d^+)^{1/4}} \\
&= \frac{n(d^-)^{8/4} - n(d^+)^{8/4}}{n(d^-)^{5/4}n(d^+)^{9/4}} \\
&= \frac{n(d^-)^2(1-a^2)}{n(d^-)^{5/4}n(d^+)^{9/4}} \\
&= \frac{n(d^-)^{3/4}}{n(d^+)^{9/4}}(1-a^2)
\end{aligned}
$$

and (3.1.19) becomes

$$\tilde{b} = \frac{1}{4}b\frac{n(d^-)^{3/4}}{n(d^+)^{5/4}} + \frac{1}{4}n'(d^-)\frac{n(d^-)^{3/4}}{n(d^+)^{9/4}}(1-a^2) \tag{3.1.20}$$

Let $|a-1|+|b|>0$, then $a \neq 1$ or $b \neq 0$. In the case that both $a \neq 1$ and $b \neq 0$ then from equations (3.1.16) and (3.1.20) we have that $|\tilde{a}-1|+|\tilde{b}|>0$. Now let $a=1$ and $b \neq 0$. Then (3.1.16) implies that $\tilde{a}=1$ and (3.1.20) implies that $\tilde{b} \neq 0$ and thus $|\tilde{a}-1|+|\tilde{b}|>0$. With the same arguments, if $b=0$ and $a \neq 1$ we conclude that $|\tilde{a}-1|+|\tilde{b}|>0$. □

3.2 Properties of the eigenfunctions and the determinants

We study now the properties of the solutions $y_l(r)$ and $z(\xi)$ for the discontinuous problem. Along similar lines with the continuous problem, we distinguish between two different cases. First, we consider the case $l=0$, which corresponds to the transmission eigenvalue problem with spherically symmetric eigenfunctions. Afterwards, we look for solutions depending both on r an θ, i.e. for $l \geq 1$, and we study their asymptotic behaviour for large values of k. Using these asymptotic formulas we derive the asymptotic expansion of the determinant (3.1.7), which is required for the inverse problem.

3.2.1 The case $l=0$

In the special case of (2.1.1)-(2.1.4), where only spherically symmetric eigenfunctions are allowed, the corresponding boundary value problem is equivalent with the Sturm-Liouville-type eigenvalue problem

$$y_0''(r) + k^2 n(r) y_0(r) = 0, \quad 0 < r < 1, \ r \neq d \tag{3.2.21}$$

$$y_0(0) = 0, \tag{3.2.22}$$

$$D_0(k) = \frac{\sin k}{k} y_0'(1) - \cos k \; y_0(1) = 0 \tag{3.2.23}$$

Let $y_0(r) = y_0(r;k)$ be the unique solution of the initial value problem:

$$y_0''(r) + k^2 n(r) y_0(r) = 0, \quad 0 < r < 1 \tag{3.2.24}$$

$$y_0(0) = 0, \quad y_0'(0) = 1 \tag{3.2.25}$$

Using the Liouville transformation (2.2.17), the initial value problem is transformed in the form (2.2.38)-(2.2.39), where now $p(\xi)$ is a piecewise continuous function. To study the properties of $y_0(r)$, we consider the solution $z(\xi)$ which satisfies the jump conditions (3.1.12)-(3.1.13). We follow the work of Hald [54] developed for the Sturm-Liouville eigenvalue problem, and we adapt it to the discontinuous transmission problem.

Proposition 3.2.1. *Function $z(\xi)$ satisfies the following Volterra integral equations:*

$$z(\xi) = \frac{\sin k\xi}{kn(0)^{1/4}} + \int_0^\xi p(t) \frac{\sin k(\xi - t)}{k} z(t) \mathrm{d}t, \; 0 \le \xi < \tilde{d} \tag{3.2.26}$$

$$\begin{aligned} z(\xi) = {} & \frac{1}{kn(0)^{1/4}} \left[\tilde{a} \sin k\tilde{d} \cos k(\xi - \tilde{d}) + \tilde{a}^{-1} \cos k\tilde{d} \sin k(\xi - \tilde{d}) + \frac{\tilde{b}}{k} \sin k\tilde{d} \sin k(\xi - \tilde{d}) \right] \\ & + \frac{1}{k} \int_0^{\tilde{d}} \left[\tilde{a} \sin k(\tilde{d} - t) \cos k(\xi - \tilde{d}) + \tilde{a}^{-1} \cos k(\tilde{d} - t) \sin k(\xi - \tilde{d}) \right. \\ & \left. + \frac{\tilde{b}}{k} \sin k(\tilde{d} - t) \sin k(\xi - \tilde{d}) \right] p(t) z(t) \mathrm{d}t + \int_{\tilde{d}}^\xi p(t) \frac{\sin k(\xi - t)}{k} z(t) \mathrm{d}t, \; \tilde{d} < \xi \le A \end{aligned} \tag{3.2.27}$$

Proof. For $\xi < \tilde{d}$ we have that z satisfies (3.2.26) since this case corresponds to the continuous problem. For $\xi > \tilde{d}$, following [54], we multiply both sides of equation (2.2.38) by a $G(\xi, t)$, integrate with respect to t from $\tilde{d}$ to ξ and use partial integration twice. This leads us to the representation:

$$\begin{aligned} & \frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi} G(\xi, \xi) - \frac{\mathrm{d}z(\tilde{d}^+)}{\mathrm{d}\xi} G(\xi, \tilde{d}) - z(\xi) G_t(\xi, \xi) + z(\tilde{d}^+) G_t(\xi, \tilde{d}) \\ & + \int_{\tilde{d}}^\xi z(t) \left(G_{tt} + k^2 G \right) \mathrm{d}t = \int_{\tilde{d}}^\xi p(t) G(\xi, t) z(t) \mathrm{d}t. \end{aligned}$$

By solving the differential equation $G_{tt} + k^2 G = 0$ together with the appropriate conditions $G(\xi, \xi) = 0$ and $G_t(\xi, \xi) = -1$ we conclude that:

$$z(\xi) = -z(\tilde{d}^+) G_t(\xi, \tilde{d}) + \frac{\mathrm{d}z(\tilde{d}^+)}{\mathrm{d}\xi} G(\xi, \tilde{d}) + \int_{\tilde{d}}^\xi p(t) G(\xi, t) z(t) \mathrm{d}t \tag{3.2.28}$$

where
$$G(\xi,t)=\frac{\sin k(\xi-t)}{k}.$$
Using the jump conditions (3.1.12)-(3.1.13) we get the Volterra integral equation (3.2.27). □

The previous proposition states that the solution z satisfies a Volterra integral equation for $\xi\neq\tilde{d}$, where the inhomogeneous term is discontinuous at $\xi=\tilde{d}$. Although, y_0 is continuous at $r=d$ and satisfies the Volterra equations (3.2.26)-(3.2.27) for $y_0(r)=z(\xi)n(r)^{-1/4}$.

In the sequel, we examine the dependence of the solution z upon the parameter k and the corresponding asymptotic behaviour for large values of k. We use the following lemma which can be found in [54, lemma 1].

Lemma 3.2.2. *Consider the integral equation*
$$u(x)-\int_a^x K(x,t)p(t)u(t)\mathrm{d}t=f(x)$$
where f and K are continuous and p is integrable. This equation has a unique solution u which is continuous and satisfies
$$|u(x)|\leq M(x)e^{L(x)\rho(x)},$$
where
$$M(x)=\max_{a\leq t\leq x}|f(t)|,\ \ L(x)=\max_{a\leq t\leq x}|K(x,t)|\ \textit{and}\ \rho(x)=\int_a^x|p(t)|\mathrm{d}t.$$

Proposition 3.2.3. *Let z be the solution of equations* (3.2.26) *and* (3.2.27) *on each interval respectively. Then, each component of z is an entire function of k^2 of order* 1/2.

Proof. Let $k=\sigma+i\tau$ and $|\tau|=|\mathrm{Im}(k)|:=\nu$. We rewrite (3.2.26) as
$$\mathrm{e}^{-\nu\xi}z(\xi)=\frac{\sin k\xi}{kn(0)^{1/4}}\mathrm{e}^{-\nu\xi}+\int_0^\xi p(t)\frac{\sin k(\xi-t)}{k}\mathrm{e}^{-\nu(\xi-t)}\mathrm{e}^{-\nu t}z(t)\mathrm{d}t$$
Moreover, the complex trigonometric functions satisfy the following inequalities:
$$|\cos k\xi|,|\sin k\xi|,\frac{|\sin k\xi|}{|k\xi|}\leq\mathrm{e}^{\nu\xi},\ \text{for}\ \xi\geq0.\tag{3.2.29}$$
Using these estimates, and by the notion of lemma 3.2.2 we have
$$M(\xi)=\max_{0\leq t\leq\xi}|f(t)|=\max_{0\leq t\leq\xi}\left|\frac{\sin kt}{kt}\frac{t}{n(0)^{1/4}}\mathrm{e}^{-\nu t}\right|\leq\frac{\xi}{|n(0)^{1/4}|},\tag{3.2.30}$$

$$L(\xi) = \max_{0\le t\le\xi} |K(\xi,t)| = \max_{0\le t\le\xi} \left| \frac{\sin k(\xi-t)}{k(\xi-t)} \mathrm{e}^{-\nu(\xi-t)}(\xi-t) \right| \le \xi$$

Therefore, from lemma 3.2.2 we have

$$|z(\xi)| \le \frac{\xi}{|n(0)^{1/4}|} \mathrm{e}^{\xi\rho(\xi)+\nu\xi} \tag{3.2.31}$$

Since f and K are entire functions of k, we have that z is an entire function too. Setting

$$m = \max\left\{ \frac{1}{|n(0)|^{1/4}}, \int_0^A |p(t)|\mathrm{d}t, |\tilde{b}| \right\} \tag{3.2.32}$$

we can write inequality (3.2.31) as

$$|z(\xi)| \le \xi m \mathrm{e}^{\xi m+\nu\xi} < mA\mathrm{e}^{\xi m+\nu\xi}, \text{ for } 0 \le \xi \le \tilde{d}. \tag{3.2.33}$$

Thus, z is an entire function of k^2, of order 1/2. Following similar ideas, we rewrite (3.2.27) as

$$\begin{aligned}
\mathrm{e}^{-\nu\xi}z(\xi) = &\frac{1}{n(0)^{1/4}} \left[\tilde{a}\frac{\sin k\tilde{d}}{k}\mathrm{e}^{-\nu\tilde{d}} \cos k(\xi-\tilde{d})\mathrm{e}^{-\nu(\xi-\tilde{d})} + \tilde{a}^{-1}\cos k\tilde{d}\mathrm{e}^{-\nu\tilde{d}}\frac{\sin k(\xi-\tilde{d})}{k}\mathrm{e}^{-\nu(\xi-\tilde{d})} \right. \\
&\left. + \tilde{b}\frac{\sin k\tilde{d}}{k}\mathrm{e}^{-\nu\tilde{d}}\frac{\sin k(\xi-\tilde{d})}{k}\mathrm{e}^{-\nu(\xi-\tilde{d})} \right] + \int_0^{\tilde{d}} \left[\tilde{a}\frac{\sin k(\tilde{d}-t)}{k}\mathrm{e}^{-\nu(\tilde{d}-t)}\cos k(\xi-\tilde{d}) \right. \\
&\mathrm{e}^{-\nu(\xi-\tilde{d})} + \tilde{a}^{-1}\cos k(\tilde{d}-t)\mathrm{e}^{-\nu(\tilde{d}-t)}\frac{\sin k(\xi-\tilde{d})}{k}\mathrm{e}^{-\nu(\xi-\tilde{d})} + \tilde{b}\frac{\sin k(\tilde{d}-t)}{k}\mathrm{e}^{-\nu(\tilde{d}-t)} \\
&\left. \frac{\sin k(\xi-\tilde{d})}{k}\mathrm{e}^{-\nu(\xi-\tilde{d})} \right] \mathrm{e}^{-\nu t}p(t)z(t)\mathrm{d}t + \int_{\tilde{d}}^{\xi} p(t)\frac{\sin k(\xi-t)}{k}\mathrm{e}^{-\nu(\xi-t)}\mathrm{e}^{-\nu t}z(t)\mathrm{d}t.
\end{aligned}$$

Since f and K are entire functions of k, we have that z is an entire function too. Now, (3.2.29) and the inequality (3.2.33) imply that

$$\begin{aligned}
|f(\xi)| \le &\frac{1}{|n(0)^{1/4}|} \left[\tilde{a}\tilde{d} + \tilde{a}^{-1}(\xi-\tilde{d}) + |\tilde{b}|(\xi-\tilde{d})\tilde{d} \right] \\
&+ \int_0^{\tilde{d}} \left[\tilde{a}(\tilde{d}-t) + \tilde{a}^{-1}(\xi-\tilde{d}) + |\tilde{b}|(\xi-\tilde{d})(\tilde{d}-t) \right] |p(t)|mA\mathrm{e}^{\tilde{d}\rho(\tilde{d})}\mathrm{d}t.
\end{aligned}$$

Moreover, considering that $\tilde{a}+\tilde{a}^{-1} > 1$ and using (3.2.32), after several calculations we have:

$$M(\xi) = \max_{\tilde{d}\le t\le\xi} |f(t)| < mA(1+mA)^2(\tilde{a}+\tilde{a}^{-1})\mathrm{e}^{\xi\rho(\tilde{d})}. \tag{3.2.34}$$

Also,

$$L(\xi) = \max_{\tilde{d}\le t\le\xi} |K(\xi,t)| = \max_{\tilde{d}\le t\le\xi} \left| \frac{\sin k(\xi-t)}{k(\xi-t)} \mathrm{e}^{-\nu(\xi-t)}(\xi-t) \right| < \xi$$

and from lemma 3.2.2 we conclude that

$$|z(\xi)| < mA(1+mA)^2(\tilde{a}+\tilde{a}^{-1})\mathrm{e}^{m\xi+\nu\xi}, \text{ for } \tilde{d} \le \xi \le A. \tag{3.2.35}$$

thus z is entire function of k^2 of order 1/2. $\square$

Corollary 3.2.4. *The derivative* $\mathrm{d}z/\mathrm{d}\xi$ *of* (3.2.26) – (3.2.27) *is entire function of* k^2 *of order* 1/2.

Corollary 3.2.5. *Functions* y_0 *and* y_0' *are entire in* k^2*, of order* 1/2

To derive the asymptotic expansions of the eigenfunctions of our problem we must study the behaviour of z and y_0 for large k. If we set $p = \tilde{b} = 0$ in (3.2.26) and (3.2.27), we get the leading part in the asymptotic inequalities.

Proposition 3.2.6. *Let z be the solution of* (3.2.26) – (3.2.27) *and* $\nu := |\mathrm{Im}(k)|$. *Then there exist some positive constants $\mathcal{C}$ and $\mathcal{D}$ such that:*

$$\left| z(\xi) - \frac{\sin k\xi}{kn(0)^{1/4}} \right| \leq \frac{1}{|k|^2} \mathcal{C} e^{\nu\xi} \tag{3.2.36}$$

$$\left| \frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi} - \frac{\cos k\xi}{n(0)^{1/4}} \right| \leq \frac{1}{|k|} \mathcal{C} e^{\nu\xi} \tag{3.2.37}$$

$$\left| z(\xi) - \frac{1}{kn(0)^{1/4}} \left[\tilde{a} \sin k\tilde{d} \cos k(\xi - \tilde{d}) + \tilde{a}^{-1} \cos k\tilde{d} \sin k(\xi - \tilde{d}) \right] \right| \leq \frac{1}{|k|^2} \mathcal{D} e^{\nu\xi} \tag{3.2.38}$$

$$\left| \frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi} - \frac{1}{n(0)^{1/4}} \left[-\tilde{a} \sin k\tilde{d} \sin k(\xi - \tilde{d}) + \tilde{a}^{-1} \cos k\tilde{d} \cos k(\xi - \tilde{d}) \right] \right| \leq \frac{1}{|k|} \mathcal{D} e^{\nu\xi} \tag{3.2.39}$$

where (3.2.36), (3.2.37) *are satisfied for* $0 \leq \xi \leq \tilde{d}$ *and* (3.2.38), (3.2.39) *for* $\tilde{d} \leq \xi \leq A$ *respectively.*

Proof. For large values of k, we can express the bounds of $z(\xi)$ in terms of $1/|k|$ instead of ξ. From the definition of the function $M(\xi)$ for the Volterra integral equation when $\xi \leq \tilde{d}$, we infer:

$$M(\xi) = \max_{0 \leq t \leq \xi} |f(t)| = \max_{0 \leq t \leq \xi} \left| \frac{\sin kt}{kn(0)^{1/4}} \mathrm{e}^{-\nu t} \right| \leq \frac{1}{|kn(0)^{1/4}|}$$

and using lemma 3.2.2 we have

$$|z(\xi)| \leq \frac{1}{|k|} m \mathrm{e}^{\tilde{d}m + \nu\xi}, \quad 0 \leq \xi \leq \tilde{d}. \tag{3.2.40}$$

From (3.2.26) we have

$$\mathrm{e}^{-\nu\xi} \left| z(\xi) - \frac{\sin k\xi}{kn(0)^{1/4}} \right| = \left| \int_0^\xi \mathrm{e}^{-\nu\xi} p(t) \frac{\sin k(\xi - t)}{k} z(t) \mathrm{d}t \right|$$

Now, using (3.2.40):

$$\mathrm{e}^{-\nu\xi}\left|z(\xi)-\frac{\sin k\xi}{kn(0)^{1/4}}\right|\leq\frac{1}{|k|^2}\int_0^{\xi}|p(t)|m\mathrm{e}^{\tilde{d}m}\mathrm{d}t$$

which implies

$$\left|z(\xi)-\frac{\sin k\xi}{kn(0)^{1/4}}\right|\leq\frac{1}{|k|^2}m^2\mathrm{e}^{\tilde{d}m}\mathrm{e}^{\nu\xi}:=\frac{1}{|k|^2}\mathcal{C}\mathrm{e}^{\nu\xi}.$$

Similarly, for $\mathrm{d}z(\xi)/\mathrm{d}\xi$ we have

$$\left|\frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi}-\frac{\cos k\xi}{n(0)^{1/4}}\right|\leq\frac{1}{|k|}m^2\mathrm{e}^{\tilde{d}m}\mathrm{e}^{\nu\xi}=\frac{1}{|k|}\mathcal{C}\mathrm{e}^{\nu\xi}.$$

With the same arguments, using again lemma 3.2.2 and the definition of $M(\xi)$ for the interval $\tilde{d}\leq t\leq\xi$, after several calculations we have

$$M(\xi)=\max_{\tilde{d}\leq t\leq\xi}|f(t)|<\frac{1}{|k|}m(1+mA)^2(\tilde{a}+\tilde{a}^{-1})\mathrm{e}^{\xi\rho(\tilde{d})}$$

and consequently the following estimate holds

$$|z(\xi)|<\frac{1}{|k|}m(1+mA)^2(\tilde{a}+\tilde{a}^{-1})\mathrm{e}^{mA+\nu\xi},\quad \tilde{d}\leq\xi\leq A. \tag{3.2.41}$$

For the asymptotic formula for $\xi\geq\tilde{d}$, we use relations (3.2.27) and (3.2.41):

$$\begin{aligned}
&\mathrm{e}^{-\nu\xi}\left|z(\xi)-\frac{1}{kn(0)^{1/4}}\left[\tilde{a}\sin k\tilde{d}\cos k(\xi-\tilde{d})+\tilde{a}^{-1}\cos k\tilde{d}\sin k(\xi-\tilde{d})+\frac{\tilde{b}}{k}\sin k\tilde{d}\right.\right.\\
&\left.\left.\sin k(\xi-\tilde{d})\right]\right|\leq\left|\frac{1}{k}\int_0^{\tilde{d}}\mathrm{e}^{-\nu\xi}\left[\tilde{a}\sin k(\tilde{d}-t)\cos k(\xi-\tilde{d})+\tilde{a}^{-1}\cos k(\tilde{d}-t)\sin k(\xi-\tilde{d})\right.\right.\\
&\qquad\left.\left.+\frac{\tilde{b}}{k}\sin k(\tilde{d}-t)\sin k(\xi-\tilde{d})\right]p(t)z(t)\mathrm{d}t\right|+\left|\int_{\tilde{d}}^{\xi}\mathrm{e}^{-\nu\xi}p(t)\frac{\sin k(\xi-t)}{k}z(t)\mathrm{d}t\right|\\
&\qquad\leq\frac{1}{|k|^2}\int_0^{\tilde{d}}[\tilde{a}+\tilde{a}^{-1}+mA]|p(t)|m\mathrm{e}^{\tilde{d}m}\mathrm{d}t\\
&\qquad+\frac{1}{|k|^2}\int_{\tilde{d}}^{\xi}m(1+mA)^2(\tilde{a}+\tilde{a}^{-1})\mathrm{e}^{mA}|p(t)|\mathrm{d}t
\end{aligned}$$

So we conclude that

$$\left|z(\xi)-\frac{1}{kn(0)^{1/4}}\left[\tilde{a}\sin k\tilde{d}\cos k(\xi-\tilde{d})+\tilde{a}^{-1}\cos k\tilde{d}\sin k(\xi-\tilde{d})+\frac{\tilde{b}}{k}\sin k\tilde{d}\sin k(\xi-\tilde{d})\right]\right|$$

$$<\frac{1}{|k|^2}m^2(\tilde{a}+\tilde{a}^{-1})(1+mA)(2+mA)\mathrm{e}^{mA}\mathrm{e}^{\nu\xi}$$

and we proved (3.2.38). In the same manner, we derive (3.2.39). □

Remark 3.2.7. *Similar estimates for the continuous problem are given in [2, proposition 2.2].*

Moreover, we study the properties of the characteristic function (3.2.23). From corollary 3.2.5 and since $\sin k/k$ and $\cos k$ are entire functions of k^2 of order 1/2 we conclude that $D_0(k)$ is entire in k^2 of order at most 1/2. Also, from (3.2.23), the integral equation (3.2.27), and using that $n(1) = 1$ and $n'(1) = 0$ we obtain the following representation for $D_0(k)$:

$$\begin{aligned}
D_0(k) = &\frac{1}{kn(0)^{1/4}}\left[\frac{\tilde{a}^2+1}{2\tilde{a}}\sin k(1-A) + \frac{1-\tilde{a}^2}{2\tilde{a}}\sin k(1-A+2\tilde{d})\right] \\
&+\frac{\tilde{b}}{2k^2n(0)^{1/4}}\left[\cos k(1-A) - \cos k(1-A+2\tilde{d})\right] \\
&+\frac{\sin k}{k}\int_0^{\tilde{d}}\left[-\tilde{a}\sin k(A-\tilde{d})\sin k(\tilde{d}-t) + \tilde{a}^{-1}\cos k(A-\tilde{d})\cos k(\tilde{d}-t)\right. \\
&\left.+\frac{\tilde{b}}{k}\cos k(A-\tilde{d})\sin k(\tilde{d}-t)\right]p(t)z(t)\mathrm{d}t - \frac{\cos k}{k}\int_0^{\tilde{d}}\left[\tilde{a}\cos k(A-\tilde{d})\sin k(\tilde{d}-t)\right. \\
&\left.+\tilde{a}^{-1}\sin k(A-\tilde{d})\cos k(\tilde{d}-t) + \frac{\tilde{b}}{k}\sin k(A-\tilde{d})\sin k(\tilde{d}-t)\right]p(t)z(t)\mathrm{d}t \\
&+\frac{\sin k}{k}\int_{\tilde{d}}^{A}\cos k(A-t)p(t)z(t)\mathrm{d}t - \frac{\cos k}{k}\int_{\tilde{d}}^{A}\sin k(A-t)p(t)z(t)\mathrm{d}t. \qquad (3.2.42)
\end{aligned}$$

Furthermore, we examine the asymptotic estimate for the characteristic function. We apply the inequalities of proposition 3.2.6 and the Liouville transformation $y_0(r) = z(\xi)n(r)^{-1/4}$ in (3.2.23) and we conclude that $D_0(k)$ satisfies the following asymptotic formula for $k \to \infty$ along the positive real axis:

$$D_0(k) = \frac{1}{kn(0)^{1/4}}\left[\frac{\tilde{a}^2+1}{2\tilde{a}}\sin k\,(1-A) + \frac{1-\tilde{a}^2}{2\tilde{a}}\sin k\left(1-A+2\tilde{d}\right)\right] + \mathrm{O}\left(\frac{1}{k^2}\right) \qquad (3.2.43)$$

Remark 3.2.8. *If we consider $\tilde{a}=1$ which corresponds to the continuous problem, the asymptotic estimate* (3.2.43) *is simplified to the corresponding one in* (2.2.47).

3.2.2 The case $l \geq 1$

In this part, we study problems (3.1.5) and (3.1.8) for $l \geq 1$. From theorem 2.2.1, we have that for $\xi < \tilde{d}$, $z(\xi)$ satisfies the following asymptotic formula for large values of

k:

$$z(\xi) = \sqrt{\frac{\pi\xi}{2k}} J_\lambda(k\xi) + \mathrm{O}\left(\frac{\ln k}{k^2}\right) \tag{3.2.44}$$

where $\lambda = l + 1/2$. This formula is developed for the continuous problem and remains valid since $n(r) \in C^2[0, d)$. For $\xi > \tilde{d}$ we have the following result:

Proposition 3.2.9. *Let $k > 0$ and $l \geq -1/2$. Then the solution of* (3.1.8) *is discontinuous at $\xi = \tilde{d}$, satisfies the jump conditions* (3.1.12) *and* (3.1.13), *and for $\xi > \tilde{d}$ and large k:*

$$\begin{aligned} z(\xi) = &\sqrt{\frac{\pi\xi}{2k}} J_\lambda(k\xi) \left[\frac{\pi}{2}\tilde{d}k \left(\tilde{a}^{-1} Y_\lambda(k\tilde{d}) J_{\lambda+1}(k\tilde{d}) - \tilde{a} J_\lambda(k\tilde{d}) Y_{\lambda+1}(k\tilde{d})\right)\right] \\ &+ \sqrt{\frac{\pi\xi}{2k}} Y_\lambda(k\xi) \left(\frac{\pi}{2}\tilde{d}k(\tilde{a} - \tilde{a}^{-1}) J_\lambda(k\tilde{d}) J_{\lambda+1}(k\tilde{d})\right) + \mathrm{O}\left(\frac{\ln k}{k^2}\right) \end{aligned} \tag{3.2.45}$$

where J_λ and Y_λ are Bessel and Neumann functions respectively.

Proof. Following [76, p. 449-451], z satisfies the integral equation

$$z(\xi) = \sqrt{\frac{\pi\xi}{2k}} J_\lambda(k\xi) + \frac{\pi}{2} \int_0^\xi \sqrt{t\xi} \left(Y_\lambda(k\xi) J_\lambda(kt) - J_\lambda(k\xi) Y_\lambda(kt)\right) g(t) z(t) \mathrm{d}t \tag{3.2.46}$$

for $0 < \xi < \tilde{d}$. To study the solution for $\xi > \tilde{d}$, we use the same technique as in proposition 3.2.1. We multiply both sides of (3.1.8) by a $G(\xi, t)$ and integrate with respect to t from $\tilde{d}$ to ξ. Integration by parts twice leads us to (3.2.28) where now we solve

$$G_{tt} + \left(k^2 - \frac{l(l+1)}{t^2}\right) G = 0, \quad G(\xi, \xi) = 0 \text{ and } G_t(\xi, \xi) = -1$$

and hence

$$G(\xi, t) = \sqrt{t\xi} \frac{\pi}{2} \left(Y_\lambda(k\xi) J_\lambda(kt) - J_\lambda(k\xi) Y_\lambda(kt)\right). \tag{3.2.47}$$

From (3.2.46) and the jump condition (3.1.12) we have:

$$z(\tilde{d}^+) = \tilde{a} z(\tilde{d}^-) = \tilde{a} \sqrt{\frac{\pi\tilde{d}}{2k}} J_\lambda(k\tilde{d}) + \tilde{a} \frac{\pi}{2} \int_0^{\tilde{d}} \sqrt{t\tilde{d}} \left(Y_\lambda(k\tilde{d}) J_\lambda(kt) - J_\lambda(k\tilde{d}) Y_\lambda(kt)\right) g(t) z(t) \mathrm{d}t.$$

Furthermore, from (3.2.46) and the jump condition (3.1.13) we have:

$$\begin{aligned} \frac{\mathrm{d}z(\tilde{d}^+)}{\mathrm{d}\xi} &= \tilde{a}^{-1} \frac{\mathrm{d}z(\tilde{d}^-)}{\mathrm{d}\xi} + \tilde{b} z(\tilde{d}^-) = \tilde{a}^{-1} \sqrt{\frac{\pi}{2k\tilde{d}}} \left((l+1) J_\lambda(k\tilde{d}) - k\tilde{d} J_{\lambda+1}(k\tilde{d})\right) \\ &+ \tilde{a}^{-1} \frac{\pi}{2\sqrt{\tilde{d}}} \int_0^{\tilde{d}} \sqrt{t} \left[-(l+1) J_\lambda(k\tilde{d}) Y_\lambda(kt) + k\tilde{d} J_{\lambda+1}(k\tilde{d}) Y_\lambda(kt)\right. \\ &\left. + J_\lambda(kt) \left((l+1) Y_\lambda(k\tilde{d}) - k\tilde{d} Y_{\lambda+1}(k\tilde{d})\right)\right] g(t) z(t) \mathrm{d}t + \tilde{b} \sqrt{\frac{\pi\tilde{d}}{2k}} J_\lambda(k\tilde{d}) \end{aligned}$$

$$+\tilde{b}\frac{\pi}{2}\int_0^{\tilde{d}}\sqrt{t\tilde{d}}\left(Y_\lambda(k\tilde{d})J_\lambda(kt)-J_\lambda(k\tilde{d})Y_\lambda(kt)\right)g(t)z(t)\mathrm{d}t$$

By applying the above representations in (3.2.28) and using the expressions for G and G_t we can write z as:

$$z(\xi)=\sqrt{\xi}Y_\lambda(k\xi)\mathcal{A}(k,\tilde{d})+\sqrt{\xi}J_\lambda(k\xi)\mathcal{B}(k,\tilde{d})+\sqrt{\xi}\,\varepsilon_l(k,\xi) \tag{3.2.48}$$

where

$$\varepsilon_l(k,\xi)=\int_{\tilde{d}}^{\xi}\frac{\pi}{2}\sqrt{t}\left(Y_\lambda(k\xi)J_\lambda(kt)-J_\lambda(k\xi)Y_\lambda(kt)\right)g(t)z(t)\mathrm{d}t, \tag{3.2.49}$$

$$\begin{aligned}\mathcal{A}(k,\tilde{d})=&\sqrt{\frac{\pi}{2k}}\frac{\pi}{2}\left[J_\lambda(k\tilde{d})(\tilde{a}^{-1}-\tilde{a})\left((l+1)J_\lambda(k\tilde{d})-k\tilde{d}J_{\lambda+1}(k\tilde{d})\right)+\tilde{b}\tilde{d}J_\lambda^2(k\tilde{d})\right]\\&+\int_0^{\tilde{d}}\frac{\pi^2}{4}\sqrt{t}\left\{\tilde{a}\left(Y_\lambda(k\tilde{d})J_\lambda(kt)-J_\lambda(k\tilde{d})Y_\lambda(kt)\right)\left(-(l+1)J_\lambda(k\tilde{d})+k\tilde{d}J_{\lambda+1}(k\tilde{d})\right)\right.\\&+\tilde{a}^{-1}J_\lambda(k\tilde{d})\left[Y_\lambda(kt)\left(-(l+1)J_\lambda(k\tilde{d})+k\tilde{d}J_{\lambda+1}(k\tilde{d})\right)+J_\lambda(kt)\left((l+1)Y_\lambda(k\tilde{d})\right.\right.\\&\left.\left.\left.-k\tilde{d}Y_{\lambda+1}(k\tilde{d})\right)\right]+\tilde{b}\tilde{d}J_\lambda(k\tilde{d})\left(Y_\lambda(k\tilde{d})J_\lambda(kt)-J_\lambda(k\tilde{d})Y_\lambda(kt)\right)\right\}g(t)z(t)\mathrm{d}t,\end{aligned} \tag{3.2.50}$$

$$\begin{aligned}\mathcal{B}(k,\tilde{d})=&\sqrt{\frac{\pi}{2k}}\frac{\pi}{2}\left[J_\lambda(k\tilde{d})\tilde{a}\left((l+1)Y_\lambda(k\tilde{d})-k\tilde{d}Y_{\lambda+1}(k\tilde{d})\right)\right.\\&\left.+Y_\lambda(k\tilde{d})\tilde{a}^{-1}\left(-(l+1)J_\lambda(k\tilde{d})+k\tilde{d}J_{\lambda+1}(k\tilde{d})\right)-\tilde{b}\tilde{d}J_\lambda(k\tilde{d})Y_\lambda(k\tilde{d})\right]\\&+\int_0^{\tilde{d}}\frac{\pi^2}{4}\sqrt{t}\left\{\tilde{a}\left(Y_\lambda(k\tilde{d})J_\lambda(kt)-J_\lambda(k\tilde{d})Y_\lambda(kt)\right)\left((l+1)Y_\lambda(k\tilde{d})-k\tilde{d}Y_{\lambda+1}(k\tilde{d})\right)\right.\\&+\tilde{a}^{-1}Y_\lambda(k\tilde{d})\left[Y_\lambda(kt)\left((l+1)J_\lambda(k\tilde{d})-k\tilde{d}J_{\lambda+1}(k\tilde{d})\right)-J_\lambda(kt)\left((l+1)Y_\lambda(k\tilde{d})\right.\right.\\&\left.\left.\left.-k\tilde{d}Y_{\lambda+1}(k\tilde{d})\right)\right]-\tilde{b}\tilde{d}Y_\lambda(k\tilde{d})\left(Y_\lambda(k\tilde{d})J_\lambda(kt)-J_\lambda(k\tilde{d})Y_\lambda(kt)\right)\right\}g(t)z(t)\mathrm{d}t.\end{aligned} \tag{3.2.51}$$

Now, applying the asymptotic formula (3.2.44) for z, we conclude that $\mathcal{A}$ and $\mathcal{B}$ satisfy the following estimates:

$$\mathcal{A}(k,\tilde{d})=\sqrt{\frac{\pi}{2k}}\frac{\pi}{2}k\tilde{d}(\tilde{a}-\tilde{a}^{-1})J_\lambda(k\tilde{d})J_{\lambda+1}(k\tilde{d})+\mathrm{O}\left(\frac{\ln k}{k^{3/2}}\right) \tag{3.2.52}$$

$$\mathcal{B}(k,\tilde{d})=\sqrt{\frac{\pi}{2k}}\frac{\pi}{2}k\tilde{d}\left(\tilde{a}^{-1}Y_\lambda(k\tilde{d})J_{\lambda+1}(k\tilde{d})-\tilde{a}J_\lambda(k\tilde{d})Y_{\lambda+1}(k\tilde{d})\right)+\mathrm{O}\left(\frac{\ln k}{k^{3/2}}\right) \tag{3.2.53}$$

The error term ε_l can be estimated using the method of successive approximations, following the arguments of section 2.2.1 of the continuous problem. We define functions

E_λ and M_λ by (2.2.22)-(2.2.23) and we also introduce $G_\lambda^{(1)}$ and $G_\lambda^{(2)}$:

$$G_\lambda^{(1)}(k,\xi) := \frac{\pi}{2}\int_{\tilde{d}}^{\xi} M_\lambda^2(kt)t|g(t)|\mathrm{d}t$$

$$G_\lambda^{(2)}(k,\xi) := \frac{\pi}{2}\int_{\tilde{d}}^{\xi} M_\lambda^2(kt)E_\lambda^2(kt)t|g(t)|\mathrm{d}t.$$

After several calculations, the integral equation (3.2.49) and the asymptotic formulas (3.2.52), (3.2.53) imply that:

$$|\varepsilon_l(k,\xi)| \le \frac{c}{\sqrt{k}}\frac{M_\lambda(k\xi)}{E_\lambda(k\xi)}\left\{\mathrm{e}^{G_\lambda^{(1)}(k,\xi)} + \mathrm{e}^{G_\lambda^{(2)}(k,\xi)} - 2\right\} \tag{3.2.54}$$

for some positive constant c and k large enough. Moreover, since $n(r) = 1$ for $r \ge 1$, from (3.1.11) we have:

$$\int_{\tilde{d}}^{A} \xi|g(\xi)|\mathrm{d}\xi < \infty, \quad \text{and} \quad \int_{A}^{\infty} |g(\xi)|\mathrm{d}\xi < \infty.$$

Thus, using (3.2.54) we conclude that $\varepsilon_l(k,\xi) = \mathrm{O}\left(1/k^2\right)$ for k large. Finally, the asymptotic formula (3.2.45) is derived by applying (3.2.52) and (3.2.53) to (3.2.48). $\square$

Moreover, from the previous proposition and the Liouville transformation (2.2.17) we infer the following result:

Lemma 3.2.10. *The solution of* (3.1.5) *and* (3.1.6) *for* $r > d$ *and large values of* k *satisfies the estimate:*

$$\begin{aligned} y_l(r) = &\frac{1}{kn(r)^{1/4}n(0)^{l/2+1/4}}\left[\frac{\tilde{a}+1}{2\tilde{a}}\sin\left(k\xi - \frac{l\pi}{2}\right)\right. \\ &\left.+\frac{1-\tilde{a}^2}{2\tilde{a}}\sin\left(k\xi + \frac{l\pi}{2} - 2k\tilde{d}\right)\right] + \mathrm{O}\left(\frac{\ln k}{k^2}\right) \end{aligned} \tag{3.2.55}$$

Proof. The asymptotic estimates for Neumann and Bessel functions are:

$$Y_\lambda(k\xi) = \sqrt{\frac{2}{k\pi\xi}}\sin\left(k\xi - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right) + \mathrm{O}\left(\frac{1}{k^{3/2}}\right), \quad k \to +\infty$$

$$J_\lambda(k\xi) = \sqrt{\frac{2}{k\pi\xi}}\cos\left(k\xi - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right) + \mathrm{O}\left(\frac{1}{k^{3/2}}\right), \quad k \to +\infty.$$

Using these estimates and (3.2.45) we conclude that

$$\begin{aligned} z(\xi) = &\frac{1}{k}\cos\left(k\xi - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right)\left[\tilde{a}\sin^2\left(k\tilde{d} - \frac{l\pi}{2}\right) + \tilde{a}^{-1}\cos^2\left(k\tilde{d} - \frac{l\pi}{2}\right)\right] \\ &+\frac{1}{k}\sin\left(k\xi - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right)\frac{\tilde{a}^{-1}-\tilde{a}}{2}\sin\left(2k\tilde{d} - l\pi\right) + \mathrm{O}\left(\frac{\ln k}{k^2}\right) \end{aligned}$$

Now, the Liouville transformation and the initial condition (3.1.6) imply (3.2.55). □

This expansion can be differentiated with respect to r with the error estimate being $\mathrm{O}(\ln k/k)$. Finally, by applying the asymptotic expansions of the spherical Bessel functions $j_l(kr)$, $j_l'(kr)$ in the characteristic equation (3.1.7), we derive the following formula:

$$\begin{aligned} D_l(k) = \frac{1}{kn(0)^{l/2+1/4}} & \left[\frac{\tilde{a}^2+1}{2\tilde{a}} \sin k\,(1-A) \right. \\ & \left. +(-1)^l \frac{1-\tilde{a}^2}{2\tilde{a}} \sin k \left(1 - A + 2\tilde{d}\right) \right] + \mathrm{O}\left(\frac{\ln k}{k^2}\right) \end{aligned} \tag{3.2.56}$$

Remark 3.2.11. *In the case where* $\tilde{a} = 1$*, equations* (3.2.45), (3.2.55) *and* (3.2.56) *are simplified to the corresponding ones in the subsection* 2.2.1.

In appendix A.3, we consider the interior transmission problem for a refractive index with a finite number of discontinuities and by induction we obtain an asymptotic formula, equivalent to (3.2.56).

3.3 Uniqueness for the inverse discontinuous problem

In this section we consider the inverse discontinuous transmission eigenvalue problem. In the previous chapter we proved the uniqueness in the inverse spectral problem for the continuous refractive index. We examine whether this is possible for a discontinuous index. Firstly, we examine the discontinuous inverse spectral problem of determining some characteristics of the refractive index from the special transmission eigenvalues corresponding to spherically symmetric eigenfunctions. We show next that if $n(0)$ is known then $n(r)$ is uniquely determined from all transmission eigenvalues including multiplicities.

The uniqueness question about this problem is important because a positive answer shows that methods based on transmission eigenvalues can be used for non-destructive methods for materials with layers. Potentially could be used for the numerical investigation of the inverse problem based on Newton-type algorithms using a piecewise constant approximation of the refractive index because it provides injectivity (see section 5.2 and [48]). There is a thorough literature concerning inverse spectral problems in Sturm-Liouville form in which the eigenfunctions have discontinuities in interior points (see for example [5, 45, 54, 89]). Most of the inverse problems in this category are based on the self-adjoint structure of the problems.

3.3.1 Uniqueness for the determination of discontinuity parameters

We investigate whether the unknown parameters of the discontinuity of the refractive index, like the position d or the size a and b, can be uniquely determined from the eigenvalues. We study the case $l=0$, where only spherically symmetric eigenfunctions are allowed. Since the characteristic function $D_0(k)$ is an entire function of k^2 of order at most $1/2$, it follows from Hadamard's factorization theorem that

$$D_0(k) = c_0 k^2 \prod_{n=1}^{\infty}\left(1 - \frac{k^2}{k_{n0}^2}\right) \tag{3.3.57}$$

where c_0 is a constant and k_{n0} are the zeros in the right half plane (in equivalence with (2.3.59)). We also define the auxiliary constant

$$\gamma_0 := \frac{1}{c_0 n(0)^{1/4}} \neq 0 \tag{3.3.58}$$

and we write the asymptotic formula (3.2.43) as

$$\frac{D_0(k)}{c_0} = \frac{\gamma_0}{k}\left[\frac{\tilde{a}^2+1}{2\tilde{a}}\sin k\,(1-A) + \frac{1-\tilde{a}^2}{2\tilde{a}}\sin k\left(1-A+2\tilde{d}\right)\right] + \mathrm{O}\left(\frac{1}{k^2}\right). \tag{3.3.59}$$

We assume that two transmission eigenvalue problems corresponding to the discontinuous indices $n_1(r)$ and $n_2(r)$ respectively, have the same special transmission eigenvalues. Let $D_{0_i}(k), c_{0_i}, \gamma_{0_i}, A_i, \tilde{d}_i,$ and $\tilde{a}_i$ characterize each problem for $i = 1, 2$. Since these two problems have the same eigenvalues, from (3.3.57) we have that $D_{0_1}/c_{0_1} = D_{0_2}/c_{0_2}$ for any $k \in \mathbb{C}$ and consequently from (3.3.59):

$$\frac{\gamma_{0_1}}{k}\left[\frac{\tilde{a}_1^2+1}{2\tilde{a}_1}\sin k\,(1-A_1) + \frac{1-\tilde{a}_1^2}{2\tilde{a}_1}\sin k\left(1-A_1+2\tilde{d}_1\right)\right] + \mathrm{O}\left(\frac{1}{k^2}\right)$$

$$= \frac{\gamma_{0_2}}{k}\left[\frac{\tilde{a}_2^2+1}{2\tilde{a}_2}\sin k\,(1-A_2) + \frac{1-\tilde{a}_2^2}{2\tilde{a}_2}\sin k\left(1-A_2+2\tilde{d}_2\right)\right] + \mathrm{O}\left(\frac{1}{k^2}\right). \tag{3.3.60}$$

for any $k > 0$ large enough. In theorem 2.3.1 we proved that A and c_0 are uniquely determined by the eigenvalues. The discontinuous problem requires more in-depth consideration.

Lemma 3.3.1. *We assume that two discontinuous refractive indices have the same special transmission eigenvalues. Then the corresponding travel times are also equal, provided that either $A_i > 1$, $i = 1, 2$ or $A_i < 1$, $i = 1, 2$.*

Proof. From the asymptotic equation (3.3.60) we conclude that

$$\gamma_{0_1}\frac{\tilde{a}_1^2+1}{2\tilde{a}_1}\sin k\,(1-A_1)-\gamma_{0_2}\frac{\tilde{a}_2^2+1}{2\tilde{a}_2}\sin k\,(1-A_2)$$
$$+\gamma_{0_1}\frac{1-\tilde{a}_1^2}{2\tilde{a}_1}\sin k\left(1-A_1+2\tilde{d}_1\right)-\gamma_{0_2}\frac{1-\tilde{a}_2^2}{2\tilde{a}_2}\sin k\left(1-A_2+2\tilde{d}_2\right)=0 \qquad (3.3.61)$$

for any $k>0$ large enough. We examine all the possible cases for which the sine functions of (3.3.61) might be linearly dependent. Since $A_1, A_2 > 1$ or $A_1, A_2 < 1$ and assuming that $A_1 \neq A_2$, we have that $\sin k(1-A_1)$ and $\sin k(1-A_2)$ are linearly independent functions.

First we assume that $A_1 \neq A_2$ and $\tilde{d}_1 \neq \tilde{d}_2$. So we have to examine all the cases for $\tilde{d}_1$ and $\tilde{d}_2$ respectively. In each case, we use (3.3.61) and the fact that γ_{0_1} and γ_{0_2} are never equal to zero. We always arrive at a contradiction as it shown in the table:

Case	Conclusion
1. $\tilde{d}_1 \neq A_1-1, \frac{A_1-A_2}{2}, \frac{A_1+A_2}{2}-1$ $\tilde{d}_2 \neq A_2-1, \frac{A_2-A_1}{2}, \frac{A_1+A_2}{2}-1$	$\gamma_{0_1}=0$ and $\gamma_{0_2}=0$
2. $\tilde{d}_1 = A_1-1,\ \tilde{d}_2 \neq A_2-1$	$\gamma_{0_1}\tilde{a}_1=0$ and $\gamma_{0_2}=0$
3. $\tilde{d}_1 = \frac{A_1-A_2}{2},\ \tilde{d}_2 \neq \frac{A_1+A_2}{2}-1$	$\gamma_{0_1}=0$ and $\gamma_{0_2}=0$
4. $\tilde{d}_1 = \frac{A_1+A_2}{2}-1,\ \tilde{d}_2 \neq \frac{A_2-A_1}{2}$	$\gamma_{0_1}=0$ and $\gamma_{0_2}=0$
5. $\tilde{d}_2 = A_2-1,\ \tilde{d}_1 \neq A_1-1$	$\gamma_{0_1}=0$ and $\gamma_{0_2}\tilde{a}_2=0$
6. $\tilde{d}_2 = \frac{A_2-A_1}{2},\ \tilde{d}_1 \neq \frac{A_1+A_2}{2}-1$	$\gamma_{0_1}=0$ and $\gamma_{0_2}=0$
7. $\tilde{d}_2 = \frac{A_1+A_2}{2}-1,\ \tilde{d}_1 \neq \frac{A_1-A_2}{2}$	$\gamma_{0_1}=0$ and $\gamma_{0_2}=0$
8. $\tilde{d}_1 = A_1-1,\ \tilde{d}_2 = A_2-1$	$\gamma_{0_1}\tilde{a}_1=0$ and $\gamma_{0_2}\tilde{a}_2=0$
9. $\tilde{d}_1 = \frac{A_1-A_2}{2},\ \tilde{d}_2 = \frac{A_1+A_2}{2}-1$	$\tilde{a}_1^2\tilde{a}_2^2=-1$
10. $\tilde{d}_1 = \frac{A_1+A_2}{2}-1,\ \tilde{d}_2 = \frac{A_2-A_1}{2}$	$\tilde{a}_1^2\tilde{a}_2^2=-1$

Cases $1-7$ are straightforward. For example, in case 2 we assume that $\tilde{d}_1 = A_1-1$ and $\tilde{d}_2 \neq A_2-1$. Hence, $\sin k(1-A_1)$ and $\sin k(1-A_1+2\tilde{d}_1)$ are linearly dependent because $\sin k(1-A_1+2\tilde{d}_1) = -\sin k(1-A_1)$. Moreover, $\sin k(1-A_2)$ and $\sin k(1-A_2+2\tilde{d}_2)$ are linearly independent. Then, equation (3.3.61) implies that:

$$\gamma_{0_1}\tilde{a}_1\sin k\,(1-A_1)-\gamma_{0_2}\frac{\tilde{a}_2^2+1}{2\tilde{a}_2}\sin k\,(1-A_2)-\gamma_{0_2}\frac{1-\tilde{a}_2^2}{2\tilde{a}_2}\sin k\left(1-A_2+2\tilde{d}_2\right)=0$$

for any k large enough, which is possible if and only if $\gamma_{0_1}\tilde{a}_1=0$ and $\gamma_{0_2}=0$.

For 8, equation (3.3.61) implies:

$$\gamma_{0_1}\left(\frac{\tilde{a}_1^2+1-(1-\tilde{a}_1^2)}{2\tilde{a}_1}\right)=0 \ \text{ and } \ \gamma_{0_2}\left(\frac{\tilde{a}_2^2+1-(1-\tilde{a}_2^2)}{2\tilde{a}_2}\right)=0,$$

and hence $\gamma_{0_1}\tilde{a}_1 = \gamma_{0_2}\tilde{a}_2 = 0$. In case 9, from (3.3.61) we get

$$\gamma_{0_1}\frac{\tilde{a}_1^2+1}{2\tilde{a}_1} + \gamma_{0_2}\frac{1-\tilde{a}_2^2}{2\tilde{a}_2} = 0 \;\text{ and }\; \gamma_{0_1}\frac{1-\tilde{a}_1^2}{2\tilde{a}_1} - \gamma_{0_2}\frac{\tilde{a}_2^2+1}{2\tilde{a}_2} = 0$$

and solving this system we conclude that $\tilde{a}_1^2\tilde{a}_2^2 = -1$. Finally, case 10 is treated in the same way. We also used that $A_1 - 1 \neq \frac{A_1-A_2}{2} \neq \frac{A_1+A_2}{2} - 1$ and $A_2 - 1 \neq \frac{A_2-A_1}{2} \neq \frac{A_1+A_2}{2} - 1$, due to $A_1, A_2 > 1$ or $A_1, A_2 < 1$.

Afterwards, we assume that $\tilde{d}_1 = \tilde{d}_2 = \tilde{d}$ and $A_1 \neq A_2$. Equation (3.3.61) becomes

$$\gamma_{0_1}\frac{\tilde{a}_1^2+1}{2\tilde{a}_1}\sin k\,(1-A_1) - \gamma_{0_2}\frac{\tilde{a}_2^2+1}{2\tilde{a}_2}\sin k\,(1-A_2)$$

$$+\,\gamma_{0_1}\frac{1-\tilde{a}_1^2}{2\tilde{a}_1}\sin k\left(1-A_1+2\tilde{d}\right) - \gamma_{0_2}\frac{1-\tilde{a}_2^2}{2\tilde{a}_2}\sin k\left(1-A_2+2\tilde{d}\right) = 0 \qquad (3.3.62)$$

With analogous ideas, we consider the following cases for $\tilde{d}$:

Case	Conclusion
1. $\tilde{d} \neq A_1 - 1, \frac{A_1-A_2}{2}, \frac{A_1+A_2}{2} - 1,$ $A_2 - 1, \frac{A_2-A_1}{2}$	$\gamma_{0_1} = 0$ and $\gamma_{0_2} = 0$
2. $\tilde{d} = A_1 - 1$	$\gamma_{0_1}\tilde{a}_1 = 0$ and $\gamma_{0_2} = 0$
3. $\tilde{d} = A_2 - 1$	$\gamma_{0_1} = 0$ and $\gamma_{0_2}\tilde{a}_2 = 0$
4. $\tilde{d} = \frac{A_1-A_2}{2}$	$\gamma_{0_1} = 0$ and $\gamma_{0_2} = 0$
5. $\tilde{d} = \frac{A_2-A_1}{2}$	$\gamma_{0_1} = 0$ and $\gamma_{0_2} = 0$
6. $\tilde{d} = \frac{A_1+A_2}{2} - 1$	$\tilde{a}_1^2 + \tilde{a}_2^2 = 0$

and we arrive at a contradiction. As a consequence we conclude that $A_1 = A_2$. □

The subsequent theorem states the relationship between the eigenvalues and the discontinuity of the transformed problem and is based on [54, lemma 6].

Theorem 3.3.2. *The constant $\tilde{a}$ and the discontinuity $\tilde{d}$ of the transformed problem are uniquely determined by the eigenvalues of* (3.2.21)−(3.2.23) *provided* $|\tilde{a}-1|+|\tilde{b}| > 0$ *in the cases:*

1. $0 < A < 1$ *and* $\tilde{d} \in (0, A)$
2. $A > 1$ *and* $\tilde{d} \in \left(0, \frac{A-1}{2}\right)$ *or* $\tilde{d} \in \left(\frac{A-1}{2}, A-1\right) \cup (A-1, A)$

Proof. If $\tilde{a} = 1$ and $\tilde{b} = 0$ then $a = 1$ and $b = 0$. In this case there is no discontinuity in the refractive index, so we insist that $|\tilde{a} - 1| + |\tilde{b}| > 0$.

We consider two problems with the same special transmission eigenvalues. From the previous lemma we have that $A_1 = A_2 = A$ and let $\gamma_{0_1}, \tilde{d}_1, \tilde{a}_1$ and $\gamma_{0_2}, \tilde{d}_2, \tilde{a}_2$ be the constants of each problem. As a result, from (3.3.61) we get

$$\left(\gamma_{0_1}\frac{\tilde{a}_1^2+1}{2\tilde{a}_1} - \gamma_{0_2}\frac{\tilde{a}_2^2+1}{2\tilde{a}_2}\right)\sin k\,(1-A) + \gamma_{0_1}\frac{1-\tilde{a}_1^2}{2\tilde{a}_1}\sin k\left(1-A+2\tilde{d}_1\right)$$
$$-\gamma_{0_2}\frac{1-\tilde{a}_2^2}{2\tilde{a}_2}\sin k\left(1-A+2\tilde{d}_2\right) = 0 \tag{3.3.63}$$

for any $k > 0$ large enough.

We distinguish between two different cases. First we assume that $\tilde{d}_1 = \tilde{d}_2 = \tilde{d}$ and then

$$\left(\gamma_{0_1}\frac{\tilde{a}_1^2+1}{2\tilde{a}_1} - \gamma_{0_2}\frac{\tilde{a}_2^2+1}{2\tilde{a}_2}\right)\sin k(1-A) + \left(\gamma_{0_1}\frac{1-\tilde{a}_1^2}{2\tilde{a}_1} - \gamma_{0_2}\frac{1-\tilde{a}_2^2}{2\tilde{a}_2}\right)\sin k\left(1-A+2\tilde{d}\right) = 0 \tag{3.3.64}$$

From the above equation, if $\tilde{d} \neq (A-1)/2, A-1$ the sine functions are linearly independent. Solving the system of the coefficients we conclude that $\tilde{a}_1 = \tilde{a}_2$ and $\gamma_{0_1} = \gamma_{0_2}$. Therefore we proved that $\tilde{a}_1 = \tilde{a}_2$ if $\tilde{d}_1 = \tilde{d}_2$.

In the second case we assume that $\tilde{d}_1 \neq \tilde{d}_2$ and we will show that this leads to a contradiction. From (3.3.63), if $\tilde{d}_1, \tilde{d}_2 \neq (A-1)/2,\ A-1$ and $\tilde{d}_1 + \tilde{d}_2 \neq A-1$, the sine functions are linearly independent. As a result we derive $\gamma_{0_1} = \gamma_{0_2}$ and $\tilde{a}_1 = \tilde{a}_2 = 1$.

In order to obtain a contradiction we will show that $\tilde{b}_1 = \tilde{b}_2 = 0$. Since $\tilde{a}_1 = 1,\ \tilde{a}_2 = 1$, we write equation (3.2.42) in the following form:

$$\begin{aligned}
D_0(k) &= \frac{\sin k(1-A)}{kn(0)^{1/4}} + \frac{\tilde{b}}{2k^2 n(0)^{1/4}}\cos k(1-A) - \frac{\tilde{b}}{2k^2 n(0)^{1/4}}\cos k(1-A+2\tilde{d}) \\
&+ \frac{\sin k}{k}\int_0^{\tilde{d}} \cos k(A-t)p(t)z(t)\mathrm{d}t + \frac{\sin k}{k^2}\tilde{b}\int_0^{\tilde{d}} \cos k(A-\tilde{d})\sin k(\tilde{d}-t)p(t)z(t)\mathrm{d}t \\
&- \frac{\cos k}{k}\int_0^{\tilde{d}} \sin k(A-t)p(t)z(t)\mathrm{d}t - \frac{\cos k}{k^2}\tilde{b}\int_0^{\tilde{d}} \sin k(A-\tilde{d})\sin k(\tilde{d}-t)p(t)z(t)\mathrm{d}t \\
&+ \frac{\sin k}{k}\int_{\tilde{d}}^{A} \cos k(A-t)p(t)z(t)\mathrm{d}t - \frac{\cos k}{k}\int_{\tilde{d}}^{A} \sin k(A-t)p(t)z(t)\mathrm{d}t
\end{aligned} \tag{3.3.65}$$

After several calculations we rewrite the previous equation as

$$\begin{aligned}
D_0(k) &= \frac{\sin k(1-A)}{kn(0)^{1/4}} + \frac{\tilde{b}}{2k^2 n(0)^{1/4}}\cos k(1-A) - \frac{\tilde{b}}{2k^2 n(0)^{1/4}}\cos k(1-A+2\tilde{d}) \\
&+ \frac{\sin k}{k^2 n(0)^{1/4}}\int_0^{A} \cos k(A-t)\sin kt\, p(t)\mathrm{d}t - \frac{\cos k}{k^2 n(0)^{1/4}}\int_0^{A} \sin k(A-t)\sin kt\, p(t)\mathrm{d}t \\
&+ \frac{E(k)}{k^2}
\end{aligned}$$

where

$$
\begin{aligned}
E(k) : = & \, k \sin k \int_0^{\tilde{d}} \cos k(A-t) p(t) \left[z(t) - \frac{\sin kt}{kn(0)^{1/4}} \right] \mathrm{d}t \\
& + k \sin k \int_{\tilde{d}}^{A} \cos k(A-t) p(t) \left[z(t) - \frac{\sin kt}{kn(0)^{1/4}} \right] \mathrm{d}t \\
& + \tilde{b} \sin k \int_0^{\tilde{d}} \cos k(A-\tilde{d}) \sin k(\tilde{d}-t) p(t) z(t) \mathrm{d}t \\
& - k \cos k \int_0^{\tilde{d}} \sin k(A-t) p(t) \left[z(t) - \frac{\sin kt}{kn(0)^{1/4}} \right] \mathrm{d}t \\
& - k \cos k \int_{\tilde{d}}^{A} \sin k(A-t) p(t) \left[z(t) - \frac{\sin kt}{kn(0)^{1/4}} \right] \mathrm{d}t \\
& - \tilde{b} \cos k \int_0^{\tilde{d}} \sin k(A-\tilde{d}) \sin k(\tilde{d}-t) p(t) z(t) \mathrm{d}t
\end{aligned}
$$

We observe that E is an even function of k and is real if k is real. Then, from propositions 3.2.3 and 3.2.6 it follows that

$$
\int_{-\infty}^{\infty} |E(x)|^2 \mathrm{d}x < \infty \quad \text{and} \quad |E(k)| \leq C \mathrm{e}^{(A+1)|\mathrm{Im}(k)|}, \; C > 0.
$$

Thus E satisfies the assumptions in the Paley-Wiener theorem (appendix, theorem A.4.12) and there exists $V \in L^2(0, A+1)$ such that

$$
E(k) = \int_0^{A+1} V(t) \cos kt \mathrm{d}t
$$

Moreover, using elementary trigonometric identities we have that:

$$
\begin{aligned}
& \frac{\sin k}{k^2 n(0)^{1/4}} \int_0^A \cos k(A-t) \sin kt p(t) \mathrm{d}t - \frac{\cos k}{k^2 n(0)^{1/4}} \int_0^A \sin k(A-t) \sin kt p(t) \mathrm{d}t \\
& = \frac{\cos k(1-A)}{2k^2 n(0)^{1/4}} \int_0^A p(t) \mathrm{d}t - \frac{\cos k}{2k^2 n(0)^{1/4}} \int_0^A \cos k(A-2t) p(t) \mathrm{d}t \\
& - \frac{\sin k}{2k^2 n(0)^{1/4}} \int_0^A \sin k(A-2t) p(t) \mathrm{d}t
\end{aligned}
$$

and using a change of variables we can rewrite

$$
\int_0^A \sin k(A-2t) p(t) \mathrm{d}t = \int_0^A W(t) \sin kt \mathrm{d}t
$$

and

$$
\int_0^A \cos k(A-2t) p(t) \mathrm{d}t = \int_0^A U(t) \cos kt \mathrm{d}t
$$

where

$$
U(t) := \frac{1}{2} \left[p\left(\frac{A+t}{2} \right) - p\left(\frac{A-t}{2} \right) \right], \; W(t) := -\frac{1}{2} \left[p\left(\frac{A+t}{2} \right) + p\left(\frac{A-t}{2} \right) \right]
$$

Finally, equation (3.3.65) takes the form

$$
\begin{aligned}
D_0(k) = & \frac{\sin k(1-A)}{kn(0)^{1/4}} + \frac{\tilde{b}}{2k^2 n(0)^{1/4}}\cos k(1-A) - \frac{\tilde{b}}{2k^2 n(0)^{1/4}}\cos k(1-A+2\tilde{d}) \\
& + \frac{\cos k(1-A)}{2k^2 n(0)^{1/4}}\int_0^A p(t)\mathrm{d}t - \frac{\cos k}{2k^2 n(0)^{1/4}}\int_0^A U(t)\cos kt\mathrm{d}t \\
& - \frac{\sin k}{2k^2 n(0)^{1/4}}\int_0^A W(t)\sin kt\mathrm{d}t + \frac{1}{k^2}\int_0^{A+1} V(t)\cos kt\mathrm{d}t
\end{aligned}
\tag{3.3.66}
$$

Considering that $\gamma_{0_1} = \gamma_{0_2}$ and $D_{0_1}/c_{0_1} = D_{0_2}/c_{0_2}$, equation (3.3.58) implies that:

$$
D_{0_1}(k) = \frac{n_2(0)^{1/4}}{n_1(0)^{1/4}} D_{0_2}(k).
$$

Using this equation and the representation (3.3.66) we have:

$$
\begin{aligned}
& \frac{\cos k(1-A)}{2k^2 n_1(0)^{1/4}}(\tilde{b}_1 - \tilde{b}_2) + \frac{\cos k(1-A)}{2k^2 n_1(0)^{1/4}}\left[\int_0^A p_1(t)\mathrm{d}t - \int_0^A p_2(t)\mathrm{d}t\right] - \frac{\cos k(1-A+2\tilde{d}_1)}{2k^2 n_1(0)^{1/4}}\tilde{b}_1 \\
& \quad + \frac{\cos k(1-A+2\tilde{d}_2)}{2k^2 n_1(0)^{1/4}}\tilde{b}_2 = \frac{\sin k}{2k^2 n_1(0)^{1/4}}\int_0^A (W_1(t) - W_2(t))\sin kt\mathrm{d}t \\
& \qquad + \frac{\cos k}{2k^2 n_1(0)^{1/4}}\int_0^A (U_1(t) - U_2(t))\cos kt\mathrm{d}t \\
& \qquad + \frac{1}{k^2}\int_0^{A+1}\left(\frac{n_2(0)^{1/4}}{n_1(0)^{1/4}}V_2(t) - V_1(t)\right)\cos kt\mathrm{d}t
\end{aligned}
\tag{3.3.67}
$$

If we multiply (3.3.67) by $2n_1(0)^{1/4}k^2\cos k(1-A+2\tilde{d}_1)T^{-1}$ and integrate with respect to k from τ to T for some $\tau > 0$, in order propositions 3.2.3 and 3.2.6 to be valid, we have

$$
\begin{aligned}
& (\tilde{b}_1 - \tilde{b}_2)\mathrm{O}\left(\frac{1}{T}\right) + \left[\int_0^A p_1(t)\mathrm{d}t - \int_0^A p_2(t)\mathrm{d}t\right]\mathrm{O}\left(\frac{1}{T}\right) - \tilde{b}_1\left(\frac{1}{2} + \mathrm{O}\left(\frac{1}{T}\right)\right) + \tilde{b}_2\mathrm{O}\left(\frac{1}{T}\right) \\
& = \int_0^A \left[(W_1(t) - W_2(t))\int_\tau^T \frac{1}{T}\sin k\cos k(1-A+2\tilde{d}_1)\sin kt\mathrm{d}k\right]\mathrm{d}t \\
& \quad + \int_0^A \left[(U_1(t) - U_2(t))\int_\tau^T \frac{1}{T}\cos k\cos k(1-A+2\tilde{d}_1)\cos kt\mathrm{d}k\right]\mathrm{d}t \\
& \quad + 2\int_0^{A+1}\left[(n_2(0)^{1/4}V_2(t) - n_1(0)^{1/4}V_1(t))\int_\tau^T \frac{1}{T}\cos k(1-A+2\tilde{d}_1)\cos kt\mathrm{d}k\right]\mathrm{d}t
\end{aligned}
\tag{3.3.68}
$$

where we used Fubini's theorem to interchange the order of integration in the right hand side of the equation. We can write the right hand side of (3.3.68) in the following form

$$
\int_0^A (W_1(t) - W_2(t))f_T\mathrm{d}t + \int_0^A (U_1(t) - U_2(t))h_T\mathrm{d}t
$$

$$+2\int_0^{A+1}(n_2(0)^{1/4}V_2(t)-n_1(0)^{1/4}V_1(t))g_T\mathrm{d}t$$

where $|f_T|,|g_T|,|h_T|\leq 1$ and $f_T,g_T,h_T \overset{T\to\infty}{\longrightarrow} 0$ almost everywhere. Letting $T\to\infty$ in (3.3.68) and using Lebesgue's dominated convergence theorem we conclude that $\tilde{b}_1=0$ provided that restrictions $\tilde{d}_1\neq (A-1)/2,\ A-1$ and $\tilde{d}_1+\tilde{d}_2\neq A-1$ are satisfied.

With similar arguments, if we multiply (3.3.67) by $2n_1(0)^{1/4}k^2\cos k(1-A+2\tilde{d}_2)T^{-1}$ and integrate with respect to k from τ to T we conclude that $\tilde{b}_2=0$ provided that $\tilde{d}_2\neq (A-1)/2, A-1$. Since $|\tilde{a}_i-1|+|\tilde{b}_i|>0$ for $i=1,2$, we have arrived to a contradiction and hence $\tilde{d}_1=\tilde{d}_2$.

Now, we need to determine the appropriate subintervals of $(0,A)$ where the uniqueness result is valid, taking into account the restrictions on the position of the discontinuity. We distinguish between the following cases:

1. In the first case we assume that $0<A<1$. Obviously, all restrictions are satisfied. So, in this case $\tilde{d}\in(0,A)$.

2. For the second case we assume that $A>1$. In this case we need to consider all restrictions so:

$$\tilde{d}\in\left(0,\ \frac{A-1}{2}\right)\quad\text{or}\quad \tilde{d}\in\left(\frac{A-1}{2},\ A-1\right)\cup\left(A-1,\ A\right).$$

□

Remark 3.3.3. *The previous theorem states that if two discontinuous transmission eigenvalue problems have the same special eigenvalues, then the discontinuities of the corresponding transformed problems must occur at the same point and the size of the jump in the indices in* (3.1.2)*, is also the same.*

Remark 3.3.4. *We mention that a problem with a discontinuity such that* $\tilde{d}\neq (A-1)/2, A-1$*, cannot have the same special transmission eigenvalues with a problem without discontinuities. Assuming that they do have the same eigenvalues then* $D_{0_1}/c_{0_1}=D_{0_2}/c_{0_2}$ *where* D_{0_1} *satisfies* (2.2.47)*:*

$$D_{0_1}(k)=\frac{1}{kn_1(0)^{1/4}}\sin k(1-A_1)+\mathrm{O}\left(\frac{1}{k^2}\right)$$

and D_{0_2} *satisfies* (3.2.43)*. Following the proof of lemma* 3.3.1 *and theorem* 3.3.2 *we conclude that* $\tilde{a}=1$ *and* $\tilde{b}=0$ *and this contradicts our assumption. Although, we cannot derive a contradiction if* $\tilde{d}=(A-1)/2$ *or* $\tilde{d}=A-1$*. This case will be examined in the following uniqueness theorem.*

3.3.2 The uniqueness theorem

We complete our study with a proof of the uniqueness for the inverse problem. In the previous section we proved that transmission eigenvalues corresponding to spherically symmetric eigenfunctions can recover only the jumps of the transformed problem. As a result, we assume now that the whole spectrum is known, including multiplicities. The proof is based on Müntz's theorem (see also theorem 2.3.1).

In the case where $n(r) \in C^2[0,1]$, we can represent $y_l(r)$ in the form (2.3.48):

$$y_l(r) = j_l(kr) + \int_0^r G(r,s,k) j_l(ks) \mathrm{d}s$$

and the kernel $G(r,s,k)$ satisfies the Goursat problem (2.3.50)-(2.3.52).

Now we assume that the refractive index is piecewise C^2 and satisfies the jump conditions (3.1.1)-(3.1.4). In this case, the representation (2.3.53) implies that $G(r,s,k)$ is continuous in $0 \le s \le r \le 1$. Also, differentiating by parts we conclude that $G_r(r,s,k)$ and $G_s(r,s,k)$ are discontinuous in $\sqrt{rs} = d$ and $G_{rr}(r,s,k)$, $G_{ss}(r,s,k)$ are discontinuous in $r = d$ and $\sqrt{rs} = d$, (see appendix A.2). Thus, equation (2.3.48) is well defined for the discontinuous problem, and y_l satisfies the continuity conditions $y_l(d^+) = y_l(d^-)$, $y_l'(d^+) = y_l'(d^-)$.

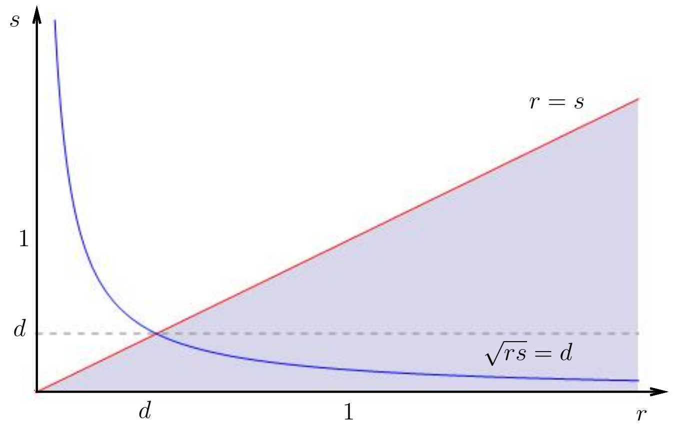

Figure 3.2: The Goursat problem for discontinuous $n(r)$

Following analogous ideas with theorem 2.3.1, from the asymptotics of y_l and D_l for $k \to 0$, we derive the identity

$$c_{2l+2} = \frac{\pi}{(2^{l+1}\Gamma(l+3/2))^2} \int_0^1 t^{2l+2} m(t) \mathrm{d}t. \tag{3.3.69}$$

where c_{2l+2} is the coefficient of k^{2l+2} in the Taylor expansion of $D_l(k)$. We observe that constant c_{2l+2} is functionally dependent on $n(r)$ and therefore with the discontinuity constants a, b and d. The determinant $D_l(k)$ is an entire function of k of order one, and hence by Hadamard's factorization theorem,

$$D_l(k) = k^{2l+2} c_{2l+2} \prod_{n=1}^{\infty} \left(1 - \frac{k^2}{k_{nl}^2}\right) \tag{3.3.70}$$

where c_{2l+2} is given by (3.3.69) provided that $m(r) = 1 - n(r)$ does not change sign and k_{nl} are the complex transmission eigenvalues in the right half plane.

We define the auxiliary constants

$$\gamma_l := \frac{1}{c_{2l+2}\, n(0)^{l/2+1/4}} \neq 0 \tag{3.3.71}$$

and in the next proposition we examine whether γ_l can be uniquely determined. This result is straightforward in the continuous problem but the discontinuous case is more complicated.

Proposition 3.3.5. *We assume that two refractive indices have the same transmission eigenvalues including multiplicities and either $A_i > 1, i = 1, 2$ or $A_i < 1, i = 1, 2$. Then $A_1 = A_2 = A$ and the corresponding constants γ_{l_1} and γ_{l_2} are also equal, provided that $\tilde{d}_1, \tilde{d}_2 \neq (A-1)/2,\ A-1$. In all other cases $\gamma_{l_1} = \tilde{c}\gamma_{l_2}$ where $\tilde{c}$ is a constant depending on $\tilde{a}_1$ and $\tilde{a}_2$.*

Proof. Let $\gamma_{l_i}, \tilde{d}_i,$ and $\tilde{a}_i$ characterize each problem for $i = 1, 2$. We follow similar arguments with the first part of the proof of theorem 3.3.2. Since the two problems have the same eigenvalues, from lemma 3.3.1 we infer directly that $A_1 = A_2 = A$. Also, from (3.2.56) and (3.3.70) we have:

$$\left(\gamma_{l_1} \frac{\tilde{a}_1^2 + 1}{2\tilde{a}_1} - \gamma_{l_2} \frac{\tilde{a}_2^2 + 1}{2\tilde{a}_2}\right) \sin k\,(1 - A) + \gamma_{l_1} (-1)^l \frac{1 - \tilde{a}_1^2}{2\tilde{a}_1} \sin k \left(1 - A + 2\tilde{d}_1\right)$$

$$- \gamma_{l_2} (-1)^l \frac{1 - \tilde{a}_2^2}{2\tilde{a}_2} \sin k \left(1 - A + 2\tilde{d}_2\right) = 0 \tag{3.3.72}$$

for any $k > 0$ large enough. With analogous ideas with lemma 3.3.1, we examine all the possible cases for which the sine functions of (3.3.72) are linearly dependent and we use the fact that γ_{l_1} and γ_{l_2} are never equal to zero. When $\tilde{d}_1 = \tilde{d}_2 = \tilde{d} \neq (A-1)/2,\ A-1$, the sine functions $\sin k(1 - A)$ and $\sin k(1 - A + 2\tilde{d})$ are linearly independent and we conclude that $\gamma_{l_1} = \gamma_{l_2}$ and $\tilde{a}_1 = \tilde{a}_2$. Now we consider the case where $\tilde{d}_1 + \tilde{d}_2 = A - 1$. Thus, $\sin k(1 - A + 2\tilde{d}_1)$ and $\sin k(1 - A + 2\tilde{d}_2)$ are linearly dependent and from (3.3.72):

$$\gamma_{l_1} \frac{\tilde{a}_1^2 + 1}{2\tilde{a}_1} - \gamma_{l_2} \frac{\tilde{a}_2^2 + 1}{2\tilde{a}_2} = 0 \quad \text{and} \quad \gamma_{l_1} \frac{1 - \tilde{a}_1^2}{2\tilde{a}_1} + \gamma_{l_2} \frac{1 - \tilde{a}_2^2}{2\tilde{a}_2} = 0.$$

Solving the system we get $\gamma_{l_1} = \gamma_{l_2}$ and $\tilde{a}_1 = 1/\tilde{a}_2$.

Now we investigate all cases where γ_l is not uniquely determined. Firstly we assume that $\tilde{d}_1 = \tilde{d}_2 = A - 1$. In this case all sine functions degenerate to $\sin k(1-A)$ and from (3.3.72) we get:

$$\gamma_{l_1} \frac{\tilde{a}_1^2 + 1 - (-1)^l(1-\tilde{a}_1^2)}{2\tilde{a}_1} = \gamma_{l_2} \frac{\tilde{a}_2^2 + 1 - (-1)^l(1-\tilde{a}_2^2)}{2\tilde{a}_2}.$$

So, for l even we conclude that $\gamma_{l_1} = \gamma_{l_2}\tilde{a}_2/\tilde{a}_1$ and for l odd, $\gamma_{l_1} = \gamma_{l_2}\tilde{a}_1/\tilde{a}_2$. Next we take $\tilde{d}_1 = \tilde{d}_2 = (A-1)/2$ and from (3.3.72) we deduce

$$\gamma_{l_1} = \gamma_{l_2} \frac{\tilde{a}_1\tilde{a}_2^2 + \tilde{a}_1}{\tilde{a}_1^2\tilde{a}_2 + \tilde{a}_2}.$$

Afterwards, we investigate all the other possible cases, with $\tilde{d}_1 \neq \tilde{d}_2$. We use the linear independence of the sine functions, and solving the system of the coefficients we get the results. In some cases we distinguish between even and odd values of l.

Case	Conclusion
1. $\tilde{d}_1 = A-1, \tilde{d}_2 \neq \frac{A-1}{2}$	$\gamma_{l_1} = \gamma_{l_2}/\tilde{a}_1$, for $l =$ even
	$\gamma_{l_1} = \tilde{a}_1\gamma_{l_2}$, for $l =$ odd
2. $\tilde{d}_2 = A-1, \tilde{d}_1 \neq \frac{A-1}{2}$	$\gamma_{l_1} = \tilde{a}_2\gamma_{l_2}$, for $l =$ even
	$\gamma_{l_1} = \gamma_{l_2}/\tilde{a}_2$, for $l =$ odd
3. $\tilde{d}_1 = A-1, \tilde{d}_2 = \frac{A-1}{2}$	$\gamma_{l_1} = \gamma_{l_2}\frac{\tilde{a}_2^2+1}{2\tilde{a}_1\tilde{a}_2}$, for $l =$ even
	$\gamma_{l_1} = \gamma_{l_2}\frac{\tilde{a}_1\tilde{a}_2^2+\tilde{a}_1}{2\tilde{a}_2}$, for $l =$ odd
4. $\tilde{d}_2 = A-1, \tilde{d}_1 = \frac{A-1}{2}$	$\gamma_{l_1} = \gamma_{l_2}\frac{2\tilde{a}_1\tilde{a}_2}{1+\tilde{a}_1^2}$, for $l =$ even
	$\gamma_{l_1} = \gamma_{l_2}\frac{2\tilde{a}_1}{\tilde{a}_1^2\tilde{a}_2+\tilde{a}_2}$, for $l =$ odd
5. $\tilde{d}_1 = \frac{A-1}{2}, \tilde{d}_2 \neq A-1$	$\gamma_{l_1} = \gamma_{l_2}\frac{2\tilde{a}_1}{1+\tilde{a}_1^2}$
6. $\tilde{d}_2 = \frac{A-1}{2}, \tilde{d}_1 \neq A-1$	$\gamma_{l_1} = \gamma_{l_2}\frac{\tilde{a}_2^2+1}{2\tilde{a}_2}$

□

The subsequent theorem establishes uniqueness for the inverse problem whether the index is a continuous function or not, with no restrictions on the position of the discontinuity.

Theorem 3.3.6. *We consider the transmission problem* (2.1.1) – (2.1.4) *for the unit ball of* $\mathbb{R}^3$*, where* $n(r)$ *is a* C^2 *or a piecewise* C^2 *function that satisfies* (3.1.1) – (3.1.4) *and* $m(r)$ *does not change sign. Then if* $n(0)$ *is known,* $n(r)$ *is uniquely determined from the knowledge of all transmission eigenvalues including multiplicities.*

Proof. We assume that the same transmission eigenvalues are corresponding to two different refractive indices $n_1(r)$ and $n_2(r)$ with $n_1(0) = n_2(0) = n(0)$. From equations (3.3.69) and (3.3.71) we derive the following moment type relations:

$$\int_0^1 t^{2l+2} m_i(t) \mathrm{d}t = \frac{(2^{l+1}\Gamma(l+3/2))^2}{n(0)^{l/2+1/4}\gamma_{l_i}\pi}, \; i = 1,2 \tag{3.3.73}$$

for the corresponding contrast functions $m_i(r), \; i = 1,2$. Since $m_i(r), \; i = 1,2$ do not change sign, either $A > 1$ or $A < 1$. We mention that from lemma 3.3.1, $A_1 = A_2 = A$. According to the previous proposition we consider several cases. If $\tilde{d}_1 = \tilde{d}_2 \neq (A-1)/2, \; A-1$ then $\gamma_{l_1} = \gamma_{l_2}$. Using the Müntz's theorem [41] for the piecewise C^2 function $m(t)$ in equation (3.3.73) we conclude that $n_1(r) = n_2(r)$. Next we assume that $\tilde{d}_1 = \tilde{d}_2 = (A-1)/2$. From proposition 3.3.5 we know that $\gamma_{l_1} = \gamma_{l_2}\frac{\tilde{a}_1\tilde{a}_2^2+\tilde{a}_1}{\tilde{a}_1^2\tilde{a}_2+\tilde{a}_2}$ and (3.3.73) implies that

$$(1 - n_1(r)) = \frac{\tilde{a}_1^2\tilde{a}_2 + \tilde{a}_2}{\tilde{a}_1\tilde{a}_2^2 + \tilde{a}_1}(1 - n_2(r)).$$

Taking account that $n_1(0) = n_2(0)$, we conclude that $n_1(r) = n_2(r)$. If $\tilde{d}_1 = \tilde{d}_2 = A-1$ we deal with the even and odd l's separately. For l even, $\gamma_{l_1} = \gamma_{l_2}\tilde{a}_2/\tilde{a}_1$. We can use the Müntz's theorem in (3.3.73) for the exponents with even l's, and derive that

$$(1 - n_1(r)) = \frac{\tilde{a}_1}{\tilde{a}_2}(1 - n_2(r)).$$

Using again that $n_1(0) = n_2(0)$ we attain uniqueness. For l odd we can use the same arguments. In the same way, for all other cases of proposition 3.3.5 where $\tilde{d}_1 \neq \tilde{d}_2$, we prove that $n_1(r) = n_2(r)$ and hence we arrive at a contradiction.

Finally, we examine two problems where the first corresponds to a continuous index and the second has a discontinuity and we assume that they have the same transmission eigenvalues. Then $D_{l_1}(k)/c_{2l+2_1} = D_{l_2}(k)/c_{2l+2_2}$ where $D_{l_1}(k)$ has the following asymptotic behaviour from (2.2.35):

$$D_{l_1}(k) = \frac{1}{kn(0)^{l/2+1/4}} \sin k(1 - A_1) + \mathrm{O}\left(\frac{\ln k}{k^2}\right)$$

Following the ideas of lemma 3.3.1 and proposition 3.3.5 and applying the Müntz's theorem in (3.3.73) we arrive at a contradiction, even if $\tilde{d} = (A-1)/2$ or $\tilde{d} = A-1$. This completes the proof. □

Remark 3.3.7. *We mention that the position and the size of the discontinuity of the refractive index are uniquely determined from the transmission eigenvalues.*

Remark 3.3.8. *We believe that this uniqueness result can be extended to a piecewise C^2 refractive index with more discontinuities (see appendix A.3). A similar technique can be used but the procedure will be much more complex and is a worthwhile future* *project. For the inverse Sturm-Liouville problem with two discontinuities we refer to [89]. Furthermore, the above considerations can be applied to other discontinuous eigenvalue problems that can be reduced to one-dimensional ones (e.g. Schrödinger equation with discontinuous potential).*

3.4 Asymptotics of transmission eigenvalues

In the sequel, we present some results concerning the asymptotic behaviour of transmission eigenvalues for the discontinuous problem. As we have already mentioned in section 2.3.2, the real transmission eigenvalues corresponding to spherically symmetric eigenfunctions for $n(r) \in C^2$ satisfy the estimate (2.3.64):

$$k_m^2 = \frac{m^2\pi^2}{(A-1)^2} + \mathrm{O}(1), \tag{3.4.74}$$

(where we consider the unit ball of $\mathbb{R}^3$). This relation is obtained from a sharper representation [72], which contains a Fourier coefficient for $p(t)$

$$k_m^2 = \frac{m^2\pi^2}{(1-A)^2} + \frac{1}{1-A}\int_0^1 p(t)\mathrm{d}t - \frac{1}{1-A}\int_0^1 p(t)\cos\frac{2\pi m t}{1-A}\mathrm{d}t + \mathrm{O}\left(\frac{1}{m}\right)$$

for m sufficiently large and $p(t)$ given by (2.2.40). In [72, lemma 2], authors proved that there exists a unique real transmission eigenvalue in each interval

$$I_m(C_1, C_2) = \left(\frac{\pi m}{|1-A|}\left(1 - C_1/m\right), \frac{\pi m}{|1-A|}\left(1 + C_2/m\right)\right)$$

for some appropriate small positive constants C_1, C_2. We examine whether this is possible for the discontinuous problem.

Proposition 3.4.1. *We consider the interior transmission problem* (3.2.21) – (3.2.23) *for a piecewise C^2 refractive index satisfying the conditions* (3.1.1) – (3.1.4). *Then there exists at least one real eigenvalue k_m in each interval*

$$I_m = \left(-\frac{\pi}{2|1-A|} + \frac{\pi m}{|1-A|}, \frac{\pi m}{|1-A|} + \frac{\pi}{2|1-A|}\right),$$

for m sufficiently large.

Proof. The characteristic determinant of the discontinuous problem satisfies the asymptotic formula (3.2.43):

$$D_0(k) = \frac{1}{kn(0)^{1/4}}\left[\frac{\tilde{a}^2+1}{2\tilde{a}}\sin k\,(1-A) + \frac{1-\tilde{a}^2}{2\tilde{a}}\sin k\left(1-A+2\tilde{d}\right)\right] + \mathrm{O}\left(\frac{1}{k^2}\right)$$

and thus there exists a positive constant C such that

$$\left|D_0(k) - \frac{1}{kn(0)^{1/4}}\left[\frac{\tilde{a}^2+1}{2\tilde{a}}\sin k\,(1-A) + \frac{1-\tilde{a}^2}{2\tilde{a}}\sin k\left(1-A+2\tilde{d}\right)\right]\right| \leq \frac{C}{k^2}. \tag{3.4.75}$$

If we set $k_m^+ := \frac{\pi m}{|1-A|} + \frac{\pi}{2|1-A|}$ and calculate $D_0(k)$ at k_m^+ we have

$$\begin{aligned} D_0(k_m^+) &\geq \frac{\tilde{a}^2+1}{k_m^+ 2\tilde{a}n(0)^{1/4}}\left[\sin\frac{1-A}{|1-A|}\left(m\pi+\frac{\pi}{2}\right) + \frac{1-\tilde{a}^2}{\tilde{a}^2+1}\sin k_m^+(1-A+2\tilde{d})\right] - \frac{C}{{k_m^+}^2} \\ &= \frac{\tilde{a}^2+1}{k_m^+ 2\tilde{a}n(0)^{1/4}}\left[(-1)^m\mathrm{sgn}(1-A) + \frac{1-\tilde{a}^2}{\tilde{a}^2+1}\sin k_m^+(1-A+2\tilde{d})\right] - \frac{C}{{k_m^+}^2}. \end{aligned}$$

Now since $-1 < \frac{1-\tilde{a}^2}{\tilde{a}^2+1} < 1$, if m is even and $A<1$ or m is odd and $A>1$, we conclude that

$$D_0(k_m^+) \geq \left(1 - \frac{1-\tilde{a}^2}{\tilde{a}^2+1}\right)\frac{\tilde{a}^2+1}{k_m^+ 2\tilde{a}n(0)^{1/4}} - \frac{C}{{k_m^+}^2} > 0 \tag{3.4.76}$$

for m large enough.

Next, we set $k_m^- := \frac{\pi m}{|1-A|} - \frac{\pi}{2|1-A|}$ and calculate $D_0(k)$ at k_m^-. From (3.4.75) we have

$$D_0(k_m^-) \leq \frac{\tilde{a}^2+1}{k_m^- 2\tilde{a}n(0)^{1/4}}\left[(-1)^{m+1}\mathrm{sgn}(1-A) + \frac{1-\tilde{a}^2}{\tilde{a}^2+1}\sin k_m^-(1-A+2\tilde{d})\right] + \frac{C}{{k_m^-}^2}$$

so if m is even and $A<1$ or m is odd and $A>1$ we conclude that

$$D_0(k_m^-) \leq \left(\frac{1-\tilde{a}^2}{\tilde{a}^2+1} - 1\right)\frac{\tilde{a}^2+1}{k_m^+ - \tilde{a}n(0)^{1/4}} + \frac{C}{{k_m^-}^2} < 0 \tag{3.4.77}$$

for m large enough. Inequalities (3.4.76) and (3.4.77) imply that $D_0(k)$ attains at least one root at I_m for m even and $A<1$ or for m odd and $A>1$.

With analogous ideas for m even and $A>1$ or for m odd and $A<1$ we conclude that $D_0(k)$ has at least one root in each I_m for m large enough and $A \neq 1$. □

The previous proposition does not ascertain the existence of only one eigenvalue in each interval I_m. Uniqueness for the continuous problem is established by proving that $\mathrm{d}D_0(k)/\mathrm{d}k$ is never zero in each $I_m(C_1, C_2)$. This argument breaks down in our case. Furthermore, in [54, lemma 4], it is shown that every eigenvalue of the discontinuous Sturm-Liouville problem lies in an interval of the form $J_m = |k_m - m| < 1/2$, as an application of the Rouché's theorem [82]. The proof is based on showing that the corresponding characteristic function and $\sin k\pi$ have the same number of roots in each interval J_m. We do not expect this result to remain valid in our case since $D_0(k)$ may has complex roots.

In the following example we show that the existence of a discontinuity implies that we may have more than one root in each I_m.

Example 3.4.2. *We introduce the following Halm-type refractive index (see [36, example 2])*

$$n_1(r) := \frac{(0.9)^2}{(0.9+(1-r)^2)^2} \tag{3.4.78}$$

which satisfies $n_1(r) \in C^2[0,1]$, $n_1(1) = 1$ *and* $n_1'(1) = 0$. *Moreover, its Liouville transform is constant,* $p(\xi) \simeq -1.11$ *and the corresponding travel time is* $A = \int_0^1 (n_1(t))^{1/2}\mathrm{d}t \simeq 0.77$.

We also consider the refractive index $n_2(r)$ *which is discontinuous at* $r = d = 1/2$

$$n_2(r) := \begin{cases} n_c \simeq 0.88, & 0 < r < 1/2 \\ \frac{(0.08)^2}{(0.08+(1-r)^2)^2}, & 1/2 < r < 1 \end{cases} \tag{3.4.79}$$

where the constant value n_c *is chosen so that* n_1 *and* n_2 *have equal travel times i.e.*

$$\int_0^1 (n_1(t))^{1/2}\mathrm{d}t = \int_0^{1/2} (n_c)^{1/2}\mathrm{d}t + \int_{1/2}^1 \left(\frac{(0.08)^2}{(0.08+(1-t)^2)^2}\right)^{1/2} \mathrm{d}t.$$

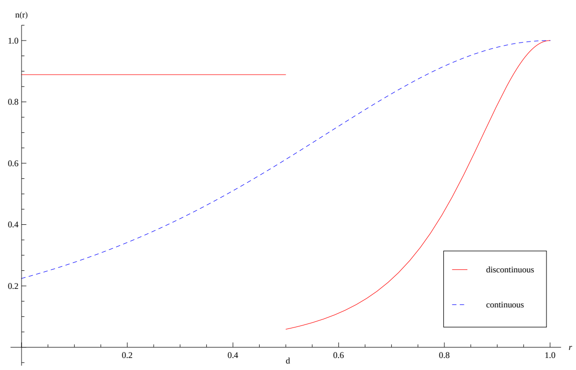

Figure 3.3: The refractive indices $n_1(r)$ and $n_2(r)$

From the asymptotic formulas (2.2.47) *and* (3.2.43) *for the determinants we have*

$$D_{0_1}(k) = \frac{1}{kn_1(0)^{1/4}} \sin k\,(1-A) + \mathrm{O}\left(\frac{1}{k^2}\right) \tag{3.4.80}$$

and

$$D_{0_2}(k) = \frac{1}{kn_c^{1/4}} \left[\frac{\tilde{a}^2+1}{2\tilde{a}} \sin k\,(1-A) + \frac{1-\tilde{a}^2}{2\tilde{a}} \sin k\left(1-A+2\tilde{d}\right)\right] + \mathrm{O}\left(\frac{1}{k^2}\right) \tag{3.4.81}$$

where $\tilde{d} = \int_0^{1/2}(n_c)^{1/2}\mathrm{d}t \simeq 0.47$ *and* $\tilde{a} = (n_2(1/2)/n_c)^{1/4} \simeq 0.51$*. Obviously,* $D_{0_1}(k)$ *has only one root in each interval* I_m *but this does not hold for* $D_{0_2}(k)$*, as it is demonstrated in figure* 3.4*.*

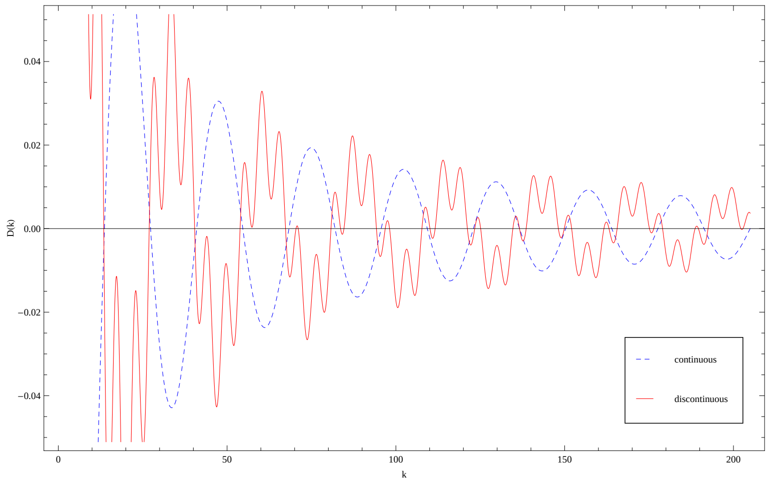

Figure 3.4: Distribution of real eigenvalues for $D_{0_1}(k)$ and $D_{0_2}(k)$

Using root finding software, we compute the real roots of the leading terms of (3.4.80) *and* (3.4.81) *in the interval* $[0, 15\pi/|1 - A|]$*. As we can see, for* $m \geq 5$*,* $D_{0_2}(k)$ *has*

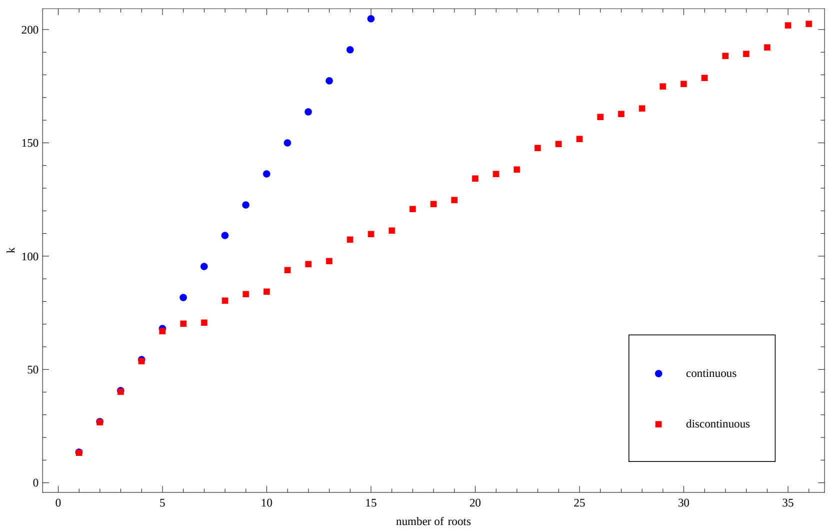

Figure 3.5: Number of roots for the continuous and discontinuous refractive index

more than one root in each I_m *and hence an asymptotic formula of the form* (3.4.74) *cannot be used for the discontinuous refractive index. As a result, we conclude that transmission eigenvalues corresponding to a discontinuous index might not have an analytic asymptotic expansion. Nevertheless, they distribute at infinity with a specific way described by proposition* 3.4.1*, with* $+\infty$ *the only accumulation point.*

4

Numerical methods for the direct transmission eigenvalue problem

In this chapter we present a numerical method for the direct transmission eigenvalue problem [48]. Firstly, we show that the interior transmission eigenvalue problem is non-self-adjoint provided that the contrast $n-1$ does not change sign. We investigate the discrete version of the direct problem using the standard variational formulation for elliptic problems. We also introduce a Galerkin-type method in the Sobolev space $H_0^2(D)$ to compute transmission eigenvalues. Moreover, using a proper operator representation of the problem we show convergence of the method. Finally, we discuss some recent results on numerical methods for transmission eigenvalues.

Contents

4.1 The discrete transmission eigenvalue problem

We consider the problem of computing transmission eigenvalues corresponding to a known refractive index $n(x)$. The first numerical study of this problem was considered in [38]. Since then, the research in the area of numerical methods for transmission eigenvalues has been developed and is still ongoing. In the following, we propose a simple computational method, based on an equivalent fourth order eigenvalue problem.

4.1.1 Operator representation of the eigenvalue problem

Let D be a bounded and simply connected subregion of $\mathbb{R}^2$ with piecewise smooth boundary ∂D and $n \in L^\infty(D)$. We assume that $n(x)$ and and $1/|n(x)-1| > 0$ are bounded positive real valued functions defined in D. The interior transmission eigenvalue problem (1.2.16)-(1.2.19) can be transformed into a fourth order equation for $u := w - v \ \in H_0^2(D)$

$$(\Delta + k^2 n)\frac{1}{n-1}(\Delta + k^2)u = 0, \quad \text{in } D \tag{4.1.1}$$

together with the homogeneous boundary conditions

$$u = 0 \ \text{ and } \ \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial D \tag{4.1.2}$$

where ν denotes the outward normal vector to ∂D (see [22, 81]). Indeed, since $u = w - v$, we have that u satisfies:

$$(\Delta + k^2)u = k^2(1-n)w \tag{4.1.3}$$

and w satisfies (1.2.16) in D. To eliminate w from (4.1.3) we divide by $n-1$ and apply the $(\Delta + k^2 n)$ operator. Thus we arrive at (4.1.1). The results of this section can be applied to the same equations in $\mathbb{R}^3$ as well. For simplicity we assume that $n(x) - 1 \geq \delta > 0$ almost everywhere, although the following theory also holds true for $n(x)$ strictly less than 1.

In the variational form, the problem (4.1.1) – (4.1.2) is equivalent to find a function $u \in H_0^2(D)$ such that:

$$\int_D \frac{1}{n-1}(\Delta u + k^2 u)(\Delta\overline{\phi} + k^2 n\overline{\phi})\mathrm{d}x = 0, \quad \forall\, \phi \in H_0^2(D). \tag{4.1.4}$$

We define the following bounded sesquilinear forms on $H_0^2(D) \times H_0^2(D)$:

$$a(u,\phi) := \Big(\frac{1}{n-1}\Delta u, \Delta\phi\Big)_D$$

$$a_1(u,\phi) := -\Big(\frac{n}{n-1}\Delta u, \phi\Big)_D - \Big(\frac{1}{n-1}u, \Delta\phi\Big)_D$$

$$a_2(u,\phi) := \Big(\frac{n}{n-1}u, \phi\Big)_D$$

$\forall u, \phi \in H_0^2(D)$, where $(\,\cdot\,,\,\cdot\,)_D$ denotes the $L^2(D)$ inner product. The problem (4.1.4) is written in the subsequent form:

$$a(u,\phi) - k^2 a_1(u,\phi) + k^4 a_2(u,\phi) = 0, \quad \forall\ \phi \in H_0^2(D).$$

For $n(x) - 1 \geq \delta > 0$, $a(u,\phi)$ is coercive i.e.

$$a(u,u) \geq c||u||^2_{H_0^2(D)}$$

for some positive constant c. This result is obtained using the fact that the $H_0^2(D)$ norm of a function is equivalent to the $L^2(D)$ norm of its Laplacian, as a consequence of the Poincaré inequality. The coercivity of $a(u,\phi)$ implies that $k = 0$ is not an eigenvalue of this problem. Furthermore, $a_2(u,\phi)$ is non-negative.

Using the Riesz representation theorem we define the following bounded linear operators $T : H_0^2(D) \to H_0^2(D),\ T_1 : H_0^2(D) \to H_0^2(D),\ T_2 : H_0^2(D) \to H_0^2(D)$ by:

$$a(u,\phi) = (Tu,\phi)_{H^2(D)}$$

$$a_1(u,\phi) = (T_1u,\phi)_{H^2(D)}$$

$$a_2(u,\phi) = (T_2u,\phi)_{H^2(D)}$$

Setting $k^2 := \tau$ we can write (4.1.4) as a quadratic pencil operator problem:

$$Tu - \tau T_1 u + \tau^2 T_2 u = 0 \tag{4.1.5}$$

Since $a(u,\phi), a_1(u,\phi)$ and $a_2(u,\phi)$ are hermitian, operators T, T_1 and T_2 are self-adjoint. By definition, T is positive definite. Also, from the compact embedding of $H_0^2(D)$ in $L^2(D)$, we have that T_1, T_2 are compact operators. As a result of the coercivity of T we have that T^{-1} is bounded, and the transmission eigenvalue problem can be written equivalently in the following form:

$$u - \tau K_1 u + \tau^2 K_2 u = 0 \tag{4.1.6}$$

where K_1 and K_2 are self-adjoint, compact operators and K_2 is non-negative, defined as [22]:

$$K_1 = T^{-1/2}T_1T^{-1/2} \quad \text{and} \quad K_2 = T^{-1/2}T_2T^{-1/2}. \tag{4.1.7}$$

We use for simplicity in (4.1.6) the symbol u instead of $T^{1/2}u$ following similar notation in the literature. Note that if U is a separable Hilbert space and A is a bounded, positive definite and self-adjoint operator on U, we define the operators $A^{\pm 1/2} := \int_0^\infty \lambda^{\pm 1/2} \mathrm{d}E_\lambda$ where $\mathrm{d}E_\lambda$ is the spectral measure associated with A. It is well known that $A^{\pm 1/2}$ are also bounded, positive definite and self-adjoint operators on U, $A^{-1/2}A^{1/2} = I$ and $A^{1/2}A^{1/2} = A$.

Now, the transmission problem can be written in operator form as the linear eigenvalue problem:

$$(\mathbb{K} - \xi \mathbb{I})U = 0, \quad U = (u, \tau K_2^{1/2} u)^\top, \ \xi := \frac{1}{\tau} \tag{4.1.8}$$

for the non-self-adjoint compact operator $\mathbb{K} : H_0^2(D) \times H_0^2(D) \to H_0^2(D) \times H_0^2(D)$, given by:

$$\mathbb{K} := \begin{pmatrix} K_1 & -K_2^{1/2} \\ K_2^{1/2} & 0 \end{pmatrix}$$

We note here that ξ is well posed since $\tau = k^2 = 0$ is not a transmission eigenvalue. The above form is completely equivalent to the quadratic form (4.1.6). Moreover, this expression of $\mathbb{K}$ declares that the transmission eigenvalue problem is non-self-adjoint. As a result, from here one can see that the set of transmission eigenvalues is at most discrete with $+\infty$ as the only one accumulation point.

4.1.2 A Galerkin-type method for the direct problem

We adopted a simple Galerkin-type method, based on the weak formulation of the problem. The main difficulty of our approximation method is that the problem under consideration is a non-self-adjoint eigenvalue problem.

We assume that $\{\phi_i\}_{i=1}^\infty$ is a set of eigenfunctions of the problem:

$$L\phi_i = \mu_i \phi_i \ \text{ in } D \tag{4.1.9}$$

$$\phi_i = 0, \ \frac{\partial \phi_i}{\partial \nu} = 0 \ \text{ on } \partial D \tag{4.1.10}$$

where L is a fourth order elliptic operator. We can use any elliptic operator L. In our work we adopt the Bilaplacian operator for which the eigenpairs can be easily computed and the eigenfunctions form a Hilbert basis in $H_0^2(D)$. For fourth order eigenvalue problems there are many approximation methods e.g. the Weinstein-Aronjszajn method [52], based on similar simple Hilbert basis.

Assume now that $\{\phi_i\}_{i=1}^\infty$ is such a set and any transmission eigenfunction u_k can be expanded in this system as:

$$u_k = \sum_{i=1}^{\infty} c_i \phi_i. \tag{4.1.11}$$

which is a convergent series in $H_0^2(D)$. We approximate u_k by:

$$u_k^{(N)} = \sum_{i=1}^{(N)} c_i \phi_i. \tag{4.1.12}$$

We enter $u_k^{(N)}$ in (4.1.4) and we use as test functions the eigenfunctions $\phi_i,\ i = 1, \dots, N$. So, the approximate non-linear eigenvalue problem is written in matrix form as:

$$[A^{(N)} - (k^{(N)})^2 B^{(N)} + (k^{(N)})^4 C^{(N)}]\mathbf{c} = 0, \tag{4.1.13}$$

where

$$\begin{aligned} A^{(N)} &:= \int_D \frac{1}{n(x)-1} \Delta\phi_i \Delta\overline{\phi_j} \mathrm{d}x \\ B^{(N)} &:= -\left(\int_D \frac{n(x)}{n(x)-1} \Delta\phi_i \overline{\phi_j} \mathrm{d}x + \int_D \frac{1}{n(x)-1} \phi_i \Delta\overline{\phi_j} \mathrm{d}x \right) \\ C^{(N)} &:= \int_D \frac{n(x)}{n(x)-1} \phi_i \overline{\phi_j} \mathrm{d}x \end{aligned} \tag{4.1.14}$$

are $N \times N$ matrices and $\mathbf{c} = (c_1, c_2, \dots, c_N)^\top,\ \ i, j = 1, \dots, N$.

Equation (4.1.13) is a typical *quadratic eigenvalue problem* [87] and it is completely analogous to the operator eigenvalue problem (4.1.5). More precisely $A^{(N)}, B^{(N)}, C^{(N)}$ are self-adjoint and $A^{(N)}, C^{(N)}$ are positive definite. These properties of the matrices follow directly from the properties of the corresponding operators in (4.1.5). Now we can transform (4.1.13) into a linear eigenvalue problem for a block matrix:

$$(\mathbb{K}^{(N)} - \xi^{(N)} \mathbb{I}^{(N)}) U^{(N)} = 0, \tag{4.1.15}$$

where

$$U^{(N)} = (\mathbf{c}, \tau^{(N)} (K_2^{(N)})^{1/2} \mathbf{c})^\top,\ \xi^{(N)} := \frac{1}{\tau^{(N)}},$$

the block matrix is defined as

$$\mathbb{K}^{(N)} := \begin{pmatrix} K_1^{(N)} & -(K_2^{(N)})^{1/2} \\ (K_2^{(N)})^{1/2} & 0 \end{pmatrix} \tag{4.1.16}$$

and

$$K_1^{(N)} := (A^{(N)})^{-1/2} B^{(N)} (A^{(N)})^{-1/2} \tag{4.1.17}$$

$$K_2^{(N)} := (A^{(N)})^{-1/2} C^{(N)} (A^{(N)})^{-1/2} \tag{4.1.18}$$

We underline that any positive semi-definite matrix has a unique positive semi-definite square root [67].

In the sequel, we study the existence and convergence of the eigenvalues of the discrete problem. The following proposition shows the existence of eigenvalues of (4.1.13).

Proposition 4.1.1. *Assume $n(x) > 1$ for $x \in \overline{D}$. Then there exist $2N$ eigenvalues of problem* (4.1.13).

Proof. This result comes directly from the linearized problem (4.1.15), which is a standard eigenvalue problem for a square $2N \times 2N$ matrix. Thus there exist $2N$ eigenvalues of (4.1.13). □

We can deduce an existence result for the case $0 < n(x) < 1$, after trivial changes in the definition of the sesquilinear forms (see also [38]).

In order to prove that the eigenvalues of the discrete problem converge to the corresponding eigenvalues of the original problem we have to prove that $\tau^{(N)} \to \tau$ (or equivalently $\xi^{(N)} \to \xi$). But, the main difficulty of our problem is that it is non-self-adjoint and we can not apply convergence results for compact and self-adjoint eigenvalue problems. To avoid this difficulty we use some abstract results for convergence in Banach spaces, [8, 43] where the main tools are based on compactness arguments and convergence behaviour of isolated eigenvalues.

Let X be a complex Banach space with $|| \; ||$ norm and $\{X_n\}$ be a sequence of finite dimensional subspaces of X parameterized by n, which will be identified with the dimension. We introduce the following framework, [43]: $\Pi_n : X \to X$ are linear projectors with range X_n, $A : X \to X$ is a linear bounded operator, the linear operators $B_n : X \to X$, with range in X_n, are supposed to approximate A. A_n is then defined as the restriction of B_n to X_n. B_n will be called the Galerkin approximations of A if $B_n = \Pi_n A$ and then $A_n := \Pi_n A|_{X_n} : X_n \to X_n$.

Following [8], we introduce the necessary framework for convergence of compact operators. Let H be a complex Banach space and $\mathbb{K} : H \to H$ a linear compact operator. We assume that μ is a non-zero eigenvalue of $\mathbb{K}$ with algebraic multiplicity m and that $\mathbb{K}^{(N)} : H \to H$ is a sequence of linear operators which converge in $\mathbb{K}$ in norm as $N \to \infty$. Let Γ be a circle in the complex plane centered at μ which lies in the resolvent set, which is defined as:

$$\rho(\mathbb{K}) = \{\lambda \in \mathbb{C} : (\mathbb{K} - \lambda)^{-1} \text{ is linear and bounded}\},$$

and no other eigenvalues of $\mathbb{K}$ are contained in Γ. Then, it is proved in [8] that for N sufficient large, $\Gamma \subset \rho(\mathbb{K}^{(N)})$ and counting according to algebraic multiplicity there are m eigenvalues of $\mathbb{K}^{(N)}$ in Γ. If we denote these eigenvalues by $\mu_{1,N}, \mu_{2,N}, \dots, \mu_{m,N}$ and if Γ' is another circle centered at μ with an arbitrarily small radius, we have that

$\mu_{1,N}, \mu_{2,N}, \ldots, \mu_{m,N}$ are all contained in Γ' for N sufficiently large. So we see that $\lim_{N\to\infty} \mu_{j,N} = \mu \; for \; j = 1, \cdots, m$.

Now, we can prove the following theorem, which gives us the desired convergence result:

Theorem 4.1.2. *Eigenvalues of the linear matrix problem* (4.1.15) *converge to the corresponding eigenvalues of the operator problem* (4.1.8) *for* $N \to \infty$.

Proof. We define as $X := H_0^2(D)$, as $X_N := H_0^2(D)_{,N}$ an $N-$dimensional subspace of $H_0^2(D)$ and the linear orthogonal projectors $\Pi_N : H_0^2(D) \to H_0^2(D)$ with range $H_0^2(D)_{,N}$. We introduce the linear bounded operators:

$$T^{(N)} := \Pi_N T|_{X_N} : X_N \to X_N$$

$$T_1^{(N)} := \Pi_N T_1|_{X_N} : X_N \to X_N$$

$$T_2^{(N)} := \Pi_N T_2|_{X_N} : X_N \to X_N.$$

The matrices given by (4.1.14) expressing our Galerkin approximation correspond to the linear operators defined above. Also we define

$$K_1^{(N)} : = (T^{(N)})^{-1/2} T_1^{(N)} (T^{(N)})^{-1/2}, \tag{4.1.19}$$
$$K_2^{(N)} : = (T^{(N)})^{-1/2} T_2^{(N)} (T^{(N)})^{-1/2} \tag{4.1.20}$$

with corresponding matrices defined in (4.1.17)-(4.1.18). Let $H := H_0^2(D) \times H_0^2(D)$ then $\mathbb{K} : H \to H$ is compact. Moreover, we set

$$\mathbb{K}^{(N)} := \begin{pmatrix} K_1^{(N)} & -(K_2^{(N)})^{1/2} \\ (K_2^{(N)})^{1/2} & 0 \end{pmatrix}$$

with corresponding block matrix defined in (4.1.16). $\mathbb{K}^{(N)}$ represents a sequence of operators which approximate operator $\mathbb{K}$ and have the same eigenvalues with the block matrices (4.1.16). Therefore, we can prove that the eigenvalues of $\mathbb{K}^{(N)}$ converge to those of $\mathbb{K}$ if we infer that $\mathbb{K}^{(N)} \to \mathbb{K}$ in norm as $N \to \infty$. From the definition of the operator norm for a block operator we have:

$$||\mathbb{K} - \mathbb{K}^{(N)}|| \leq ||K_1 - K_1^{(N)}|| + 2||K_2^{1/2} - (K_2^{(N)})^{1/2}||.$$

From the form of K_1 and $K_1^{(N)}$ as products of operators given by (4.1.7) and (4.1.19) respectively, we have:

$$\begin{aligned} ||K_1 - K_1^{(N)}|| =& ||T^{-1/2} T_1 T^{-1/2} - (T^{(N)})^{-1/2} T_1^{(N)} (T^{(N)})^{-1/2}|| \\ \leq& ||(T^{-1/2} - (T^{(N)})^{-1/2}) T_1 T^{-1/2}|| + ||(T^{(N)})^{-1/2}|| ||T_1 (T^{-1/2} - (T^{(N)})^{-1/2})|| \\ &+ ||T_1 - T_1^{(N)}|| ||(T^{(N)})^{-1/2}||^2 \end{aligned}$$

The sequence $(T^{(N)})^{-1/2}$ is uniformly bounded and pointwise convergent, T_1 is compact and self-adjoint and $T^{-1/2}$ is self-adjoint. From the result that multiplying by a compact operator on the right converts a pointwise convergent sequence of bounded operators into a norm convergent one [51, p. 108], we infer that

$$||(T^{-1/2} - (T^{(N)})^{-1/2})T_1T^{-1/2}|| \to 0$$

and

$$||(T^{(N)})^{-1/2}||||T_1(T^{-1/2} - (T^{(N)})^{-1/2})|| = ||(T^{(N)})^{-1/2}||||(T^{-1/2} - (T^{(N)})^{-1/2})T_1|| \to 0.$$

Since T_1 is compact and $(T^{(N)})^{-1/2}$ is uniformly bounded we deduce directly that

$$||(T_1 - T_1^{(N)})||||(T^{(N)})^{-1/2}||^2 \to 0.$$

So, we conclude $||K_1 - K_1^{(N)}|| \to 0$. Using the same arguments for the operators $K_2^{1/2}$ and $(K_2^{(N)})^{1/2}$, having a similar structure with K_1 and $K_1^{(N)}$ respectively, we deduce that $||K_2^{1/2} - (K_2^{(N)})^{1/2}|| \to 0$. So, the desired convergence result follows. □

4.2 Numerical methods for transmission eigenvalues

The problem of computing transmission eigenvalues has attracted many researchers since 2010 and the main effort has been focused on developing numerical methods for accurate computations. This problem is very challenging since its neither linear nor self-adjoint (as it is shown in the previous section). As a result, all numerical methods for the problem have been established very recently.

Most of the existing numerical methods are developed for the case of transmission eigenvalues corresponding to scattering for inhomogeneous media (i.e. scalar Helmholtz equation). In [38] Colton, Monk and Sun firstly proposed three finite element methods namely Argyris, continuous and mixed, to compute transmission eigenvalues for several domains. The first numerical method which was supported by rigorous convergence analysis, was introduced by Sun in [83]. After that, most of the numerical methods for transmission eigenvalues are based on finite element methods, where convergence issues have also been considered (see [25, 27, 58, 61, 62, 71, 90, 93]). In [46], Geng et al. used the discontinuous Galerkin method with C^0 Lagrange elements to solve the interior transmission problem, which is easier in implementation than classical finite element methods. An and Shen have developed spectral-Galerkin and spectral-element methods in [6] and [7] respectively. Moreover, some integral-based methods are considered in [24, 60, 66, 68, 95]. The vector case, that is computation of transmission eigenvalues for Maxwell's equations, is examined in [59, 75, 85, 96].

The computational method that we establish has several advantages. Since it is based in the equivalent fourth order formulation, the zero eigenvalue is naturally eliminated. Moreover, due to the easily constructed Hilbert basis, it is simple in implementation. Also, we can calculate as many real and complex eigenvalues as we need at the same time. Of course, our method is not that accurate compared to other methods, but it can be useful for the numerical solution of the inverse problem which is the subject of the following chapter.

5

Numerical methods for the inverse transmission eigenvalue problem

In this chapter we present numerical results for the inverse transmission eigenvalue problem [48]. We use the Galerkin method, as described in the previous chapter, and pose the inverse eigenvalue problem for a piecewise constant index of refraction $n(x)$. We first investigate the case of a spherically stratified medium in $\mathbb{R}^2$. We also propose a Newton scheme for the general piecewise constant index problem. In all cases, since there does not exist a general uniqueness theory for the inverse spectral problem, we assume that we have situations where all the original and known transmission eigenvalues are well separated and the computed eigenvalues are very close to the corresponding original eigenvalues. We mention that in the next subsections we examine examples of problems where this assumption has also been checked analytically.

Contents

5.1 The inverse problem for spherically stratified domains

We consider the inverse problem of reconstructing the unknown refractive index from transmission eigenvalues. We examine cases where the domain is spherically stratified in $\mathbb{R}^2$ (where the term *spherically stratified* can be used for domains in $\mathbb{R}^2$ and $\mathbb{R}^3$ without loss of generality). The following computational methods can be applied for domains in $\mathbb{R}^3$ as well. The numerical solution of the inverse transmission eigenvalue problem has been investigated very recently. Some works concerning the reconstruction of the refractive index using the lowest eigenvalue are considered in [6, 18, 19, 49, 84, 85]. In these works, authors are focused on using only the first real transmission eigenvalue and a Faber-Krahn-type inequality [40] to provide a lower bound for the refractive index.

5.1.1 Circular domain

Let D be a circular domain of radius r with constant refractive index $n(x) = n$. It is well known that we can recover n from the knowledge of only the first transmission eigenvalue, [14]:

Theorem 5.1.1. *The constant index of refraction n is uniquely determined from the knowledge of the corresponding smallest transmission eigenvalue $k_{1,n} > 0$ provided that it is known a priori that either $n > 1$ or $0 < n < 1$.*

We use this result to reconstruct the refractive index for unit disks. We can analytically compute the eigenvalues and eigenfunctions of the clamped plate (4.1.9)-(4.1.10), and the transmission eigenvalues. The first transmission eigenvalue is the lowest positive value for which [30, 40]:

$$det \begin{pmatrix} J_m(kr) & J_m(k\sqrt{n}r) \\ J'_m(kr) & J'_m(k\sqrt{n}r) \end{pmatrix} = 0, \quad m = 0, 1, \dots \tag{5.1.1}$$

where J_m are the Bessel functions of the first kind. This relation can be derived from separation of variables for the Helmholtz equations (2.1.1), (2.1.2) defined in the ball of $\mathbb{R}^2$ with radius r. Using root finding software we can compute the lowest real transmission eigenvalues k_0 from (5.1.1). The distribution of the determinants as functions of k for a unit disc with $n = 6$ is shown in figure 5.1.

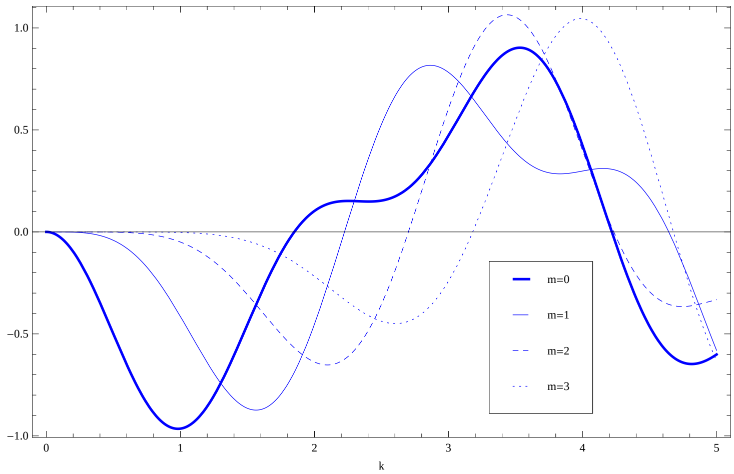

Figure 5.1: Distribution of real transmission eigenvalues for $n = 6$ and $m = 0, 1, 2, 3$ respectively

Now we construct the basis $\{\phi_i\}_{i=1}^{(N)}$ for the problem $(4.1.9) - (4.1.10)$ with $L = \Delta\Delta$. In polar coordinates the eigenfunctions u_i for one μ are linear combinations of

$$J_i(\mu r)\cos i\theta,\ \ J_i(\mu r)\sin i\theta,\ \ I_i(\mu r)\cos i\theta,\ \ I_i(\mu r)\sin i\theta,$$

where I_i are hyperbolic Bessel functions. The eigenvalues can be computed from the relation:

$$det\begin{pmatrix} J_i(kr) & J_i'(kr) \\ I_i(kr) & I_i'(kr) \end{pmatrix} = 0, \quad i = 0, 1, \dots \tag{5.1.2}$$

We constructed a basis with 20 eigenfunctions, $\{\phi_i\}_{i=1}^{20}$ and computed the 20×20 matrices $A^{(N)}, B^{(N)}, C^{(N)}$ for $r = 1$. We used the Matlab function `polyeig` with step 0.01 to solve the direct quadratic eigenvalue problem (4.1.13) for different values of n in the intervals $[0.01, 0.5]$ and $[2, 21]$, and the results are shown in table 5.1.

Table 5.1: Reconstructions for constant index of refraction

original n	first eigenvalue k_0	error $\lvert k_0 - k_0^{(N)}\rvert$	estimated n^*
1/10	4.0992	6.05×10^{-4}	0.10
1/3	7.20401	0.0060	0.33
3	4.15924	0.0027	3.01
6	1.84972	5.81×10^{-4}	6.00
10	1.29630	1.91×10^{-4}	10.00
12	1.16612	1.53×10^{-4}	12.00
20	0.88154	9.47×10^{-5}	20.00

Using only the lowest transmission eigenvalue and a few eigenfunctions of the clamped plate we can estimate n very well by minimizing $|k_0^{(N)} - k_0|$ by simply considering

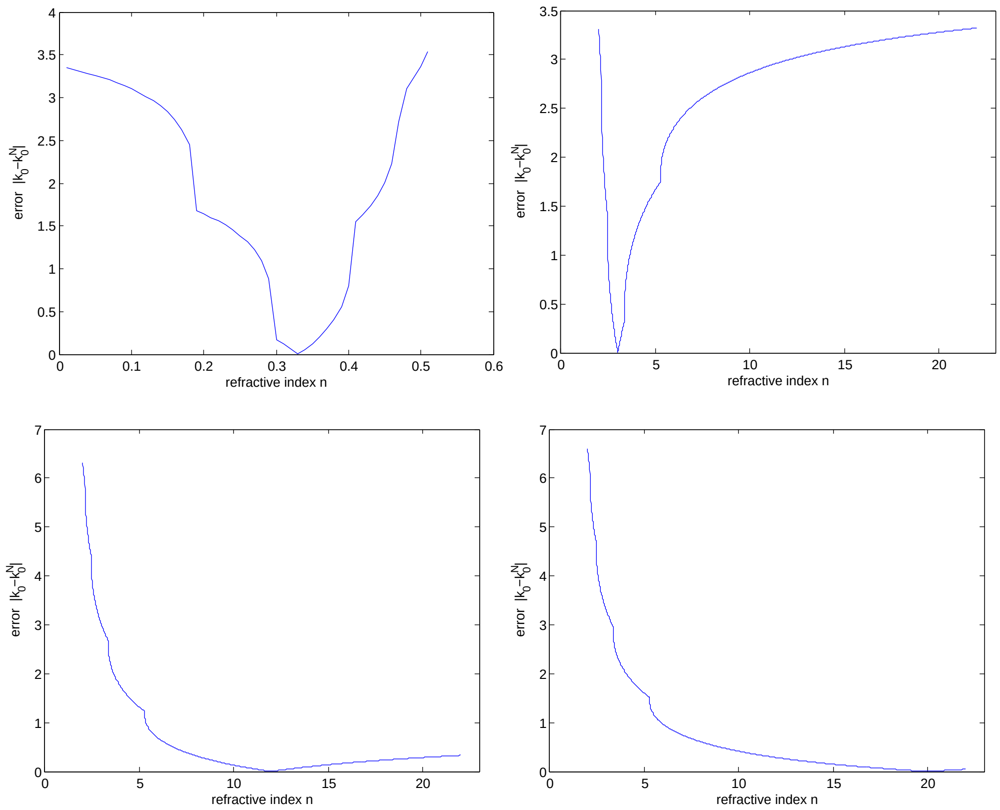

Figure 5.2: Plots of the $|k_0 - k_0^{(N)}|$ versus n for original $n = 1/3,\ n = 3,\ n = 12$ and $n = 20$ respectively

$k_0^{(N)} = k_0^{(N)}(n)$. Some plots of the error $|k_0^{(N)} - k_0|$ versus the index n are shown in figure 5.2. We see that the error is minimized for estimated index very close to the original one which corresponds to the analytically known first transmission eigenvalue. We also tested the method adding a small error to the transmission eigenvalue and the reconstructions where also accurate.

In [49], authors develop a numerical method to estimate an unknown constant refractive index from the first transmission eigenvalue. In this method, when the domain is a disk with constant index the reconstructions are accurate but for larger values of n the approximation of the lowest transmission eigenvalue is not precise enough ([49, figure 2]). With our method, we observe that the as n becomes larger the error $|k_0^{(N)} - k_0|$ becomes smaller and the reconstructions are accurate as it is shown in table 5.1. Moreover, to demonstrate the above considerations we plot the error versus the values of n for $n \in [2, 50]$ and the result is shown in figure 5.3.

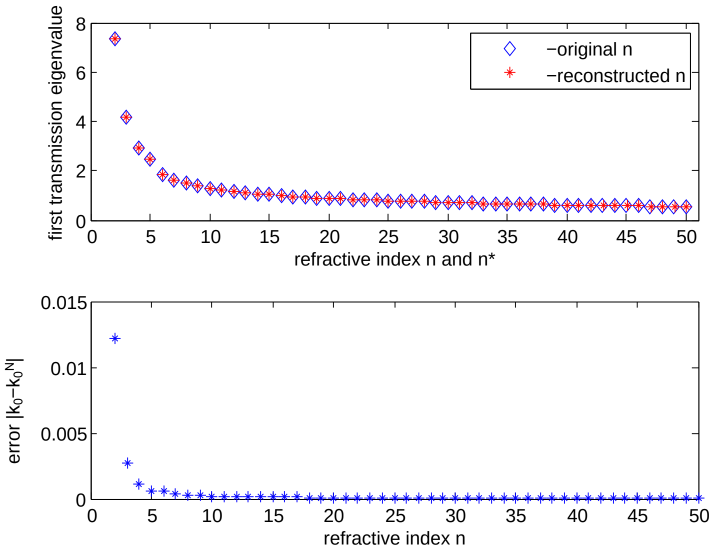

Figure 5.3: Reconstructions of a constant refractive index in a disc and the corresponding relative error

5.1.2 Domain with two layers

We choose D to be a disc of radius $r = R$, but we now assume that the refractive index is a piecewise constant function with different values at each layer:

$$n(r) = \begin{cases} n_1, & 0 < r < r_1 \\ n_2, & r_1 < r < R \end{cases}$$

The original transmission eigenvalues are analytically computed from the equation:

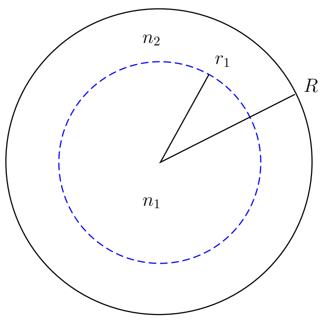

Figure 5.4: A disc with two layers

$$det \begin{pmatrix} J_m(kR) & 0 & J_m(k\sqrt{n_2}R) & N_m(k\sqrt{n_2}R) \\ J'_m(kr)|_{r=R} & 0 & J'_m(k\sqrt{n_2}r)|_{r=R} & N'_m(k\sqrt{n_2}r)|_{r=R} \\ 0 & J_m(k\sqrt{n_1}r_1) & J_m(k\sqrt{n_2}r_1) & N_m(k\sqrt{n_2}r_1) \\ 0 & J'_m(k\sqrt{n_1}r)|_{r=r_1} & J'_m(k\sqrt{n_2}r)|_{r=r_1} & N'_m(k\sqrt{n_2}r)|_{r=r_1} \end{pmatrix} = 0 \qquad (5.1.3)$$

for $m = 0, 1, \dots,$ where N_m are Neumann functions and r_1 is the inner radius. This relation is completely analogous with (5.1.1). The eigenfunctions of the Helmholtz equations (2.1.1), (2.1.2) are linear combinations of Bessel and Neumann functions since the domain is spherically stratified. Using root finding software we compute both real and complex transmission eigenvalues.

The main idea of the inverse problem is the following. We assume that in all cases we know the position of each eigenvalue in the spectrum and we minimize the error $\sum_{i=1}^{l} |k_i^{(N)} - k_i|$ considering that the computed eigenvalues are functions of (n_1, n_2, r_1). We examine two different problems; first having the knowledge of the lowest real transmission eigenvalues and afterwards using both real and complex transmission eigenvalues.

5.1.2.1 Reconstructions using only real transmission eigenvalues

We have reconstructions using as test data the first 4 real transmission eigenvalues for examples in the range $0.1 \le r_1 \le 1$ and $5 \le n_1, n_2 \le 20$. In this case, the refractive index is not close enough to $n(x) = 1$. From numerical computations we have that complex eigenvalues with small modulus do not appear and consequently we can use as first eigenvalues only real eigenvalues.

We used the same basis $\{\phi_i\}_{i=1}^{20}$ as in constant index case and we computed the $A^{(N)}, B^{(N)}, C^{(N)}$ matrices for $0.1 \le r_1 \le 1$ with step 0.1. We solved the direct problem using Matlab function `polyeig` for $5 \le n_1, n_2 \le 20$ with step 0.1 and constructed a database of eigenvalues $k^{(N)}$ for all possible combinations between n_1, n_2 and r_1. The algorithm for the inverse problem is based on minimizing the error between the lower real transmission eigenvalues and the computed eigenvalues. Minimizing the error $\sum_{i=1}^{l} |k_i - k_i^{(N)}|^2$ for $l = 4$, we reconstruct $n(x)$ with relatively good accuracy as we can see in table 5.2. We mention here that we used the inner radius as an unknown parameter. So, the method can cover cases where the exact size of the layer is not known. We note that some reconstructions in table 5.2 are more accurate than those in [48, table 2], since we used a larger basis for the Galerkin method.

There are cases where the reconstruction of the unknown index is not successful. For example the index $(19, 5, 0.7)$ is reconstructed as $(9, 15.3, 0.4)$. This problem is circumvented if we use more transmission eigenvalues. Indeed, if we use as test data the first 8 transmission eigenvalues instead of 4, the index is reconstructed as $(19, 5.9, 0.7)$ (see figure 5.5). We notice that the non-successful reconstructions correspond to refractive indices with relatively large jump between the layers. To validate this result, we consider the inverse problem for indices with $r_1 = 0.7$ and values $n_1, n_2 \in [5, 19]$ such that $n_1 + n_2 = 24$ (for example $(5, 19)$, $(6, 18), \dots$). We solved the inverse problem using 4 and 8 eigenvalues respectively. The results are demonstrated in figure

Table 5.2: Reconstructions for piecewise constant index of refraction using only real transmission eigenvalues

(n_1, n_2, r_1)	analytic t.e. $\{k_0, k_1, k_2, k_3\}$	approximate $\{k_0^*, k_1^*, k_2^*, k_3^*\}$	(n_1^*, n_2^*, r_1^*)
(5 14 0.1)	1.0916, 1.4019, 1.7279, 2.0480	1.0915, 1.4028, 1.7290, 2.0495	(7.9 14 0.1)
(16 8 0.2)	1.3121, 1.8406, 2.3058, 2.7321	1.3108, 1.8538, 2.2972, 2.7175	(17.2 8.1 0.2)
(18 5 0.3)	1.3787, 1.8635, 2.6059, 3.3562	1.4699, 1.8619, 2.5485, 3.3044	(14 5 0.4)
(7 17 0.4)	1.1847, 1.4027, 1.6352, 1.8894	1.1743, 1.4186, 1.6368, 1.8814	(7.6 17.3 0.4)
(13 5 0.5)	1.4957, 1.7336, 2.1694, 2.6974	1.4654, 1.7104, 2.1691, 2.7361	(14 5 0.5)
(5 8 0.6)	1.7889, 2.2483, 2.6654, 3.0329	1.7612, 2.2483, 2.6766, 3.0430	(5.0 8.4 0.6)
(10 8 0.7)	1.3716, 1.7304, 2.1086, 2.4934	1.3723, 1.7309, 2.1098, 2.4955	(10.0 8.0 0.7)
(13 11 0.8)	1.1487, 1.4884, 1.8248, 2.1547	1.1472, 1.4872, 1.8242, 2.1550	(13.0 11.1 0.8)
(6 13 0.9)	1.5885, 2.0239, 2.5120, 3.0009	1.5849, 2.0260, 2.5124, 3.0014	(5.9 15.4 0.9)

5.5. Nevertheless, the numerical method we presented above can be useful in many applications such as non-destructive testing of materials because it can completely estimate the unknown index using only a few transmission eigenvalues.

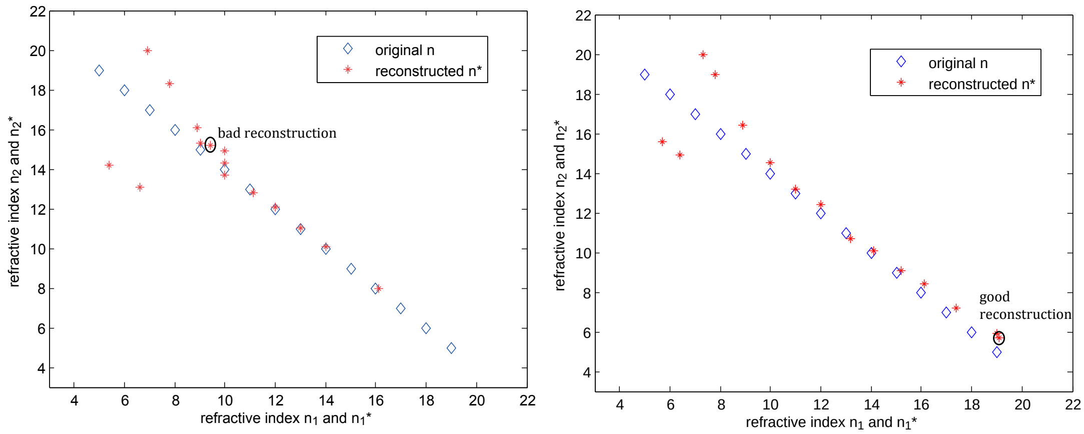

Figure 5.5: Reconstructions for $r_1 = 0.7$ using 4 eigenvalues (left) and 8 eigenvalues (right)

5.1.2.2 Reconstructions using both real and complex transmission eigenvalues

In the case when $n(x)$ is closer to 1, more complex transmission eigenvalues arise as it is shown in theorem 2.4.1. For more information about complex transmission eigenvalues we refer to section 2.4. Note that since $n(x)$ has to be real, the complex eigenvalues must appear in complex conjugate pairs. Using root finding software and contour plots we compute the eigenvalues from the equations derived from separation of variables (5.1.3). The transmission eigenvalues with the lower modulus for the refractive index $(5, 2, 0.1)$ are shown in figure 5.6.

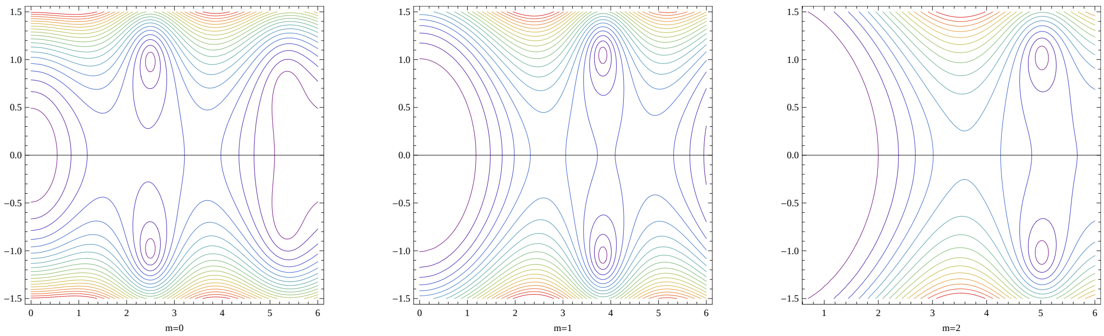

Figure 5.6: Contour plots of the determinants for $n_1 = 5,\ n_2 = 2$ and $r_1 = 0.1$

Table 5.3: Reconstructions for piecewise constant index of refraction using both real and complex transmission eigenvalues

(n_1, n_2, r_1)	analytic t.e. $\{k_0, k_1, k_2, k_3\}$	approximate $\{k_0^*, k_1^*, k_2^*, k_3^*\}$	(n_1^*, n_2^*, r_1^*)
(5 2 0.1)	2.4965+0.9904i, 3.8331+1.0549i, 5.0181+1.0370i, 5.3422+0.6662i	2.4450+0.9839i, 3.8243+1.0543i, 5.0163+1.0480i, 5.2558+0.6727i	(4.5 2.0 0.2)
(4 3 0.3)	2.3183+0.74235i, 3.7319+0.4273i, 3.8728, 4.0985	2.3135+0.7357i, 3.7259+0.4588i, 3.8397, 4.1180	(4.2 3.0 0.3)
(2 4 0.5)	2.4317+0.6964i, 3.5768+0.5709i, 3.9082, 4.1101	2.3847+0.7374i, 3.5806+0.5381i, 3.9076, 4.1165	(2.4 3.5 0.4)
(5 3 0.6)	2.2966+0.7428i, 2.8845, 3.1834, 3.2932	2.2827+0.7408i, 2.8600, 3.1933, 3.2845	(5.1 3.0 0.6)
(2 4 0.7)	2.4257+0.6292i, 3.7545+0.6136i, 4.9642+0.4386i, 5.0402	2.3969+0.6496i, 3.6879+0.6052i, 4.9051+0.4229i, 5.0442	(2.1 3.9 0.7)
(3 6 0.8)	2.2202+0.4778i, 3.0247, 3.6685, 3.7963	2.2063+0.5026i, 3.0612, 3.6779, 3.8045	(3.0 6.0 0.8)
(6 2 0.9)	2.0212, 2.4177, 2.7382+0.4470i, 2.9295	2.0344, 2.4250, 2.7187+0.4648i, 2.9778	(6.0 2.6 0.9)

We used the same basis $\{\phi_i\}_{i=1}^{20}$ and solved the inverse problem for $2 \le n_1, n_2 \le 6$ with step 0.1 and $0.1 \le r_1 \le 1$ with step 0.1. Minimizing the error

$$\sum_{i=1}^{l} \left(|\mathrm{Re}k_i - \mathrm{Re}k_i^{(N)}|^2 + |\mathrm{Im}k_i - \mathrm{Im}k_i^{(N)}|^2 \right)$$

we reconstruct the index $n(x)$.

As we can see in table 5.3, we have reconstructions of $n(x)$ with good accuracy in the case when the index is close to 1. Note that in this case, if we use the first 4 real transmission eigenvalues the reconstructions are not satisfying enough. For example the index $(5, 2, 0.1)$ is reconstructed as $(2.5, 2, 0.3)$. That is the reason why we treated this case individually. This result is important because from this numerical method we can see that both real and complex eigenvalues carry information about the refractive index.

5.2 A Newton-type method for the inverse problem

We now propose an algorithm for estimating the unknown index of refraction for domains with two or more layers. The main advantage of this method, beside the fact that it can be used for more general domains, is that we don't have to pair the analytically known eigenvalues with the numerically estimated correctly according to their position on the spectrum, in the complex plane.

We suppose that $D \subset \mathbb{R}^2$ is a spherically stratified domain with $k-$layers such that $D = \cup_{i=1}^{k} D_i$ and $\{\partial D_i\}_{i=1}^{k}$ are concentric circles.

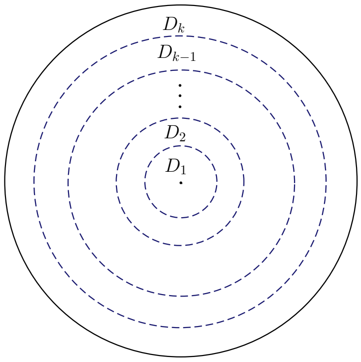

Figure 5.7: A disc with k-layers

The unknown piecewise constant index of refraction $n(x)$ is given by:

$$n(x) = \begin{cases} n_1, & x \in D_1 \\ \vdots & \\ n_k, & x \in D_k \end{cases}$$

We assume that $n(x) > 1$. The transmission eigenvalues are the zeros of the determinant of a $2k \times 2k$ matrix, analogous to (5.1.3), which can be obtained using separation of variables.

We can solve the corresponding inverse quadratic eigenvalue problem (4.1.13), using a Newton method [44]. We can write the $N \times N$ matrices $A^{(N)}, B^{(N)}, C^{(N)}$ in the following form:

$$A^{(N)} = \sum_{l=1}^{k} \frac{1}{n_l - 1} A_l \tag{5.2.4}$$

$$B^{(N)} = \sum_{l=1}^{k} \left(\frac{n_l}{n_l - 1} B_l^{(1)} + \frac{1}{n_l - 1} B_l^{(2)} \right) \tag{5.2.5}$$

$$C^{(N)} = \sum_{l=1}^{k} \frac{n_l}{n_l - 1} C_l \tag{5.2.6}$$

where:
$A_l = \int_{D_l} \Delta\phi_i \Delta\overline{\phi_j} \mathrm{dx}$, $\mathrm{B}_\mathrm{l}^{(1)} = \int_{\mathrm{D_l}} \Delta\phi_\mathrm{i}\overline{\phi_\mathrm{j}} \mathrm{dx}$, $\mathrm{B}_\mathrm{l}^{(2)} = \int_{\mathrm{D_l}} \phi_\mathrm{i}\Delta\overline{\phi_\mathrm{j}} \mathrm{dx}$ and $\mathrm{C_l} = \int_{\mathrm{D_l}} \phi_\mathrm{i}\overline{\phi_\mathrm{j}} \mathrm{dx}$, for $i, j = 1, \cdots, N$. From the analysis of the previous chapter we have that $\{A_l\}_{l=1}^k, \{(B_l^{(1)} + B_l^{(2)})\}_{l=1}^k, \{C_l\}_{l=1}^k$ are symmetric and $\{A_l\}_{l=1}^k$ are positive definite. Also if we set $a_l := 1/(n_l - 1)$ then (5.2.4)-(5.2.6) can be written as:

$$A^{(N)} = \sum_{l=1}^{k} a_l A_l \tag{5.2.7}$$

$$B^{(N)} = \sum_{l=1}^{k} B_l^{(1)} + \sum_{l=1}^{k} a_l (B_l^{(1)} + B_l^{(2)}) \tag{5.2.8}$$

$$C^{(N)} = \sum_{l=1}^{k} C_l + \sum_{l=1}^{k} a_l C_l \tag{5.2.9}$$

Now the inverse problem has the following form:
given a set of transmission eigenvalues $S = \{\mu_i\}_{i=1}^k$, *find scalars* $\{a_l\}_{l=1}^k$ *which are such that the pencil* $P(\lambda) = \lambda^4 C^{(N)} + \lambda^2 B^{(N)} + A^{(N)}$ *has spectrum* $\sigma(A^{(N)}, B^{(N)}, C^{(N)}) = S$.

We used a modification of an algorithm designed for higher degree matrix polynomial inverse eigenvalue problems in [44], adapted for the special case of our quadratic problem. We denote $a = (a_1, a_2, \cdots, a_k)$ the set of the unknown coefficients. The main idea of this Newton-based iterative method is solving the non-linear system $f(a) := (f_1(a), \cdots, f_k(a))^\top = (0, \cdots, 0)^\top$ where:

$$f_i(a) = det\left[\mu_i^4 \left(\sum_{l=1}^{k} C_l + \sum_{l=1}^{k} a_l C_l\right) + \mu_i^2 \left(\sum_{l=1}^{k} B_l^{(1)} + \sum_{l=1}^{k} a_l (B_l^{(1)} + B_l^{(2)})\right) + \sum_{l=1}^{k} a_l A_l\right]$$

rather than minimizing a cost functional like

$$g(a) = \sum_{i=1}^{k} |\lambda_i(a) - \mu_i|$$

which was our first approach for the problem.

With the previous method we had to pair $\lambda_i(a)$ with μ_i correctly in each iterative step. With the new method this problem is circumvented. This result is crucial in applications because we can estimate the unknown $n(x)$ using eigenvalues for which we do not know the exact position in the spectrum of the transmission problem. This could happen if our eigenvalues were derived from scattering data in a specific interval of wave numbers.

When the number of layers is equal to the number of the eigenvalues, the inverse problem is reduced to the special case of solving a system of non-linear equations.

When k is less than the number of eigenvalues (and thus number of equations), we have an overdetermined system of equations and the inverse problem corresponds to an unconstrained minimization problem [42]. In this case, the subject of minimization is the functional:

$$\min_{a \in \mathbb{R}^k} g(a) := \frac{1}{2} f(a)^\top f(a) = \frac{1}{2} \sum_{i=1}^{N} f_i(a)^2$$

where the residual function $f : \mathbb{R}^k \to \mathbb{R}^N$ is non-linear at a. The corresponding iterative method is a Gauss-Newton method, which has the advantage that second derivatives which have a large computational cost are omitted. The rate of convergence of the Gauss-Newton method depends on the relative non-linearity and the norm size of the residual at the minimization point ([42, theorem 10.2.1]).

5.2.1 The algorithm

In the following, we describe the main steps of the Newton-type iteration method.

The Newton method
Input

- the set of $\{A_l\}_{l=1}^k$, $\{B_l\}_{l=1}^k$, $\{C_l\}_{l=1}^k$ of $N \times N$ matrices.
- an initial estimate of $a^{(0)} = (a_1^{(0)}, a_2^{(0)}, \cdots, a_k^{(0)})$ of the unknown $\{a_l\}_{l=1}^k$ set.
- the set $S = \{\mu_i\}_{i=1}^k$ of transmission eigenvalues.

Output
At convergence the method produces a vector of the unknown $\{a_l\}$ which are such $\sigma(A^{(N)}, B^{(N)}, C^{(N)}) = S$.

The Iteration

1. Choose a starting value $a^{(0)}$ for the vector of the unknown coefficients
2. for $m = 0, 1, \cdots$
   a) compute the Jacobian $J(a^{(m)})$ and the function $f(a^{(m)})$ by the algorithm below
   b) solve the system: $J(a^{(m)})\xi^{(m)} = -f(a^{(m)})$ for $\xi^{(m)}$
   c) compute the new estimate of the coefficients vector $a^{(m+1)} = \xi^{(m)} + a^{(m)}$

d) stop if $||\xi^{(m)}||$ is sufficiently small

end loop (2) ($m-$loop).

Computing the Jacobian

Input

- the set of $\{A_l\}_{l=1}^k$, $\{B_l\}_{l=1}^k$, $\{C_l\}_{l=1}^k$ of $N \times N$ matrices.
- the $k-$vector of $a^{(m)}$ of the coefficients resulting to the m^{th} iteration step of the Newton method.
- the set $S = \{\mu_i\}_{i=1}^k$ of transmission eigenvalues.

Output

- the function $f(a^{(m)})$
- the Jacobian matrix $J(a^{(m)})$ with il^{th} component $\partial f_i(a^{(m)})/\partial a_l$

The algorithm

1. for each $i = 1, 2, \cdots, k$
   a) compute the $N \times N$ matrix
   $H = \mu_i^4 \left(\sum_{l=1}^k C_l + \sum_{l=1}^k a_l C_l\right) + \mu_i^2 \left(\sum_{l=1}^k B_l^{(1)} + \sum_{l=1}^k a_l (B_l^{(1)} + B_l^{(2)})\right)$
   $+ \sum_{l=1}^k a_l A_l$

   b) use LU or QR factorization to triangularize H and then compute $f(a^{(m)})$ as the product of the diagonal elements of the triangle

   c) for each $l = 1, 2, \cdots, k$
      i. compute the $N \times N$ matrix:
      $D = \mu_i^4 \Big(\sum_{j=1}^k C_j + \sum_{j=1, j\neq l}^k a_j C_j \Big) + \mu_i^2 \Big(\sum_{j=1}^k B_j^{(1)} + \sum_{j=1, j\neq l}^k a_j (B_j^{(1)} +$
      $B_j^{(2)}) \Big) + \sum_{j=1, j\neq l}^k a_j A_j$

      ii. use the QZ (Generalized Schur decomposition) algorithm to find matrices Q and R with determinant unity which simultaneously triangularize the pair $A_l + \mu_i^2(B_l^{(1)} + B_l^{(2)}) + \mu_i^4 C_l, D$

      iii. denote as $(\alpha_i, \beta_i)_{i=1}^N$ the pairs of the diagonal elements of the QZ triangular matrices

      iv. determine the number of non-zero α_i

v. relabel (α_i, β_i) so that $\alpha_{r+1} = \alpha_{r+2} = \cdots = \alpha_N = 0$

vi. set:
$\left[J(a^{(m)})\right]_{il} = \left(\prod_{i=r+1}^{N} \beta_i\right) \sum_{i=1}^{r} \alpha_i \prod_{j=1,j\neq i}^{N} (a_l^{(m)} \alpha_j + \beta_j)$

end loop (c) (l-loop)

end loop (1) (i-loop).

Remark 5.2.1. *Note that this algorithm requires the same number k of known transmission eigenvalues and layers. In the case where the number of layers is less that the amount of transmission eigenvalues, the Jacobian matrix of step (c)-vi is not square and then the system (2)-b can be solved using the generalized inverse of the Jacobian matrix, or using a Tikhonov-type inversion method.*

5.2.2 Numerical examples

In the following examples we test the algorithm in domains with two or more layers, using as input data a set of transmission eigenvalues. We also compare our method with other minimization routines of Matlab and Mathematica. All computations are performed on an regular intel core i5 laptop.

5.2.2.1 Domain with two layers

We have tested the algorithm for the simple case of spherically stratified domain with two layers where the position of the discontinuity is known. We used a set of 8×8 square matrices A, B, C in the form (5.2.7), (5.2.8) and (5.2.9). The set S of the 16 real and complex eigenvalues was computed numerically from the pencil $P(\lambda) = \lambda^4 C + \lambda^2 B + A$, using Matlab function `polyeig`.

Example 5.2.2. *We consider a unit disc with two layers and piecewise constant refractive index with values* 12 *and* 6 *respectively. The position of the discontinuity is at* $r = 0.8$*. The results are shown in the following table:*

(n_1, n_2)	initial guess	tol	reconstruction	steps	res. norm
$(12, 6)$	$(9, 9)$	10^{-6}	$(12.000, 5.999)$	6	10^8

We note that the norm of the residual vector is large since the determinant of $\lambda^4 C + \lambda^2 B + A$ *is highly non-linear at* λ*. The norm becomes remarkably smaller if we* *use arbitrary precision rather than the default double precision floating point for the computations (see for example [1]).*

We also estimate the rate of convergence of our Gauss-Newton method by computing the slope of the line that fits the error at step $n+1$ and n respectively.

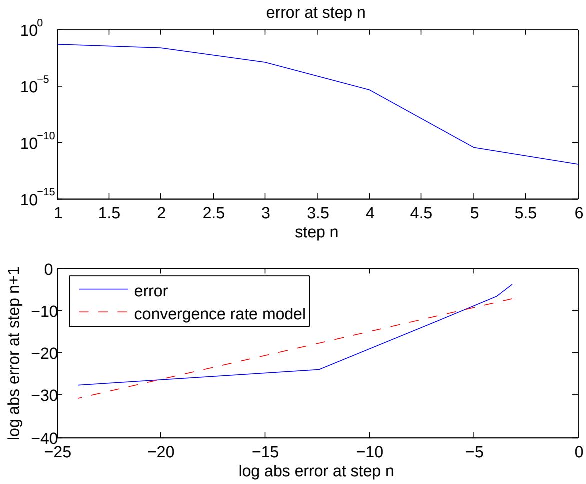

Figure 5.8: Rate of convergence for example 5.2.2

The slope of the red line in figure 5.8 *indicates that the algorithm converges with rate* 1.13*.*

Example 5.2.3. *We consider a unit disc with two layers and piecewise constant refractive index with values* $\sqrt{2}$ *and* 12.31 *respectively. The position of the discontinuity is at* $r = 0.2$*. The results are shown in the following table:*

(n_1, n_2)	initial guess	tol	reconstruction	steps	res. norm
$(\sqrt{2}, 12.31)$	$(8, 8)$	10^{-6}	$(1.4142, 12.310)$	7	10^7

The slope of the red line in figure 5.9 *indicates that the algorithm converges with rate* 1.04*.*

Analytically computed transmission eigenvalues:
We also tested the algorithm using as input data the first 6 *analytically computed transmission eigenvalues from separation of variables* (5.1.3)*. For the minimization problem, we used a set of* 20×20 *square matrices. If we use smaller matrices (i.e. a smaller basis for the Galerkin method) the reconstructions are not satisfying. This is a direct consequence of the convergence theorem* 4.1.2 *of the matrix problem. The results are shown in the following table.*

(n_1, n_2)	initial guess	tol	reconstruction	steps	res. norm
$(12, 6)$	$(9, 9)$	10^{-6}	$(11.922, 6.524)$	25	10^{33}
$(\sqrt{2}, 12.31)$	$(8, 8)$	10^{-6}	$(3.189, 12.598)$	15	10^{31}

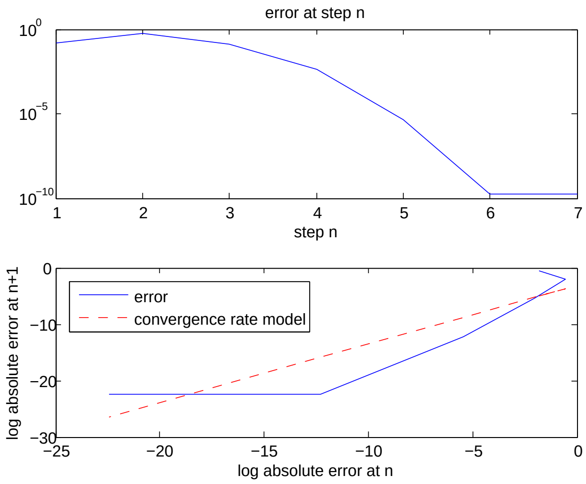

Figure 5.9: Rate of convergence for example 5.2.3

Other minimization routines:

We compare our minimization method with the Matlab `lsqnonlin` *and Mathematica* `FindMinimum` *routines. We used* 8×8 *matrices and the corresponding* 16 *real and complex eigenvalues.*

Table 5.4: Reconstructions for example 5.2.2

method	initial guess	reconstruction	steps	elapsed time
Gauss-Newton	$(9, 9)$	$(12.000, 5.999)$	6	0.41 sec
`lsqnonlin`	$(9, 9)$	$(11.999, 6.000)$	139	21.04 sec
`FindMinimum`	$(9, 9)$	$(12.000, 6.000)$	8	2.41 sec

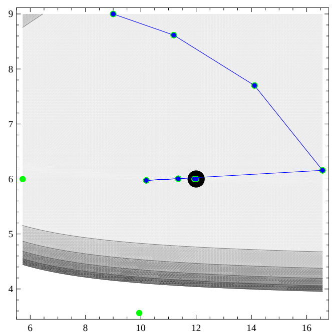

Figure 5.10: Minimization with `FindMinimum` for example 5.2.2

Table 5.5: Reconstructions for example 5.2.3

method	initial guess	reconstruction	steps	elapsed time
Gauss-Newton	$(8, 8)$	$(1.414, 12.310)$	7	0.39 sec
`lsqnonlin`	$(1.5, 8)$	$(1.414, 12.309)$	9	1.45 sec
`FindMinimum`	$(1.5, 8)$	$(1.414, 12.310)$	10	2.17 sec

We note that for example 5.2.3*, the other methods did not converge when we used as initial guess the values* $(8, 8)$*.*

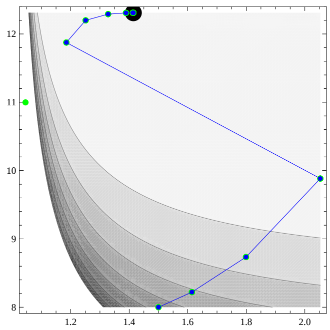

Figure 5.11: Minimization with `FindMinimum` for example 5.2.3

5.2.2.2 Domain with more than two layers

Next, we implement the algorithm for domains with more than two layers. We consider a unit disc with five layers with width 0.2 and we compute the corresponding matrices using (5.2.7), (5.2.8) and (5.2.9). We used 8×8 matrices and the corresponding 16 real and complex eigenvalues. For the inverse problem, we do not assume that the position of each layer is a priori known. This method can be useful in applications such as testing of materials, where the inner structure of the domain is not known.

Example 5.2.4. *We consider a unit disc with five layers and piecewise constant refractive index with values* $(15.2, 5.3, 19.2, 18, 8.3)$ *at each layer respectively. The results are shown in the following table:*

$(n_1, n_2, n_3, n_4, n_5)$	initial guess	tol	reconstruction	steps
$(15.2, 5.3, 19.2, 18, 8.3)$	$(10, 10, 10, 10, 10)$	10^{-6}	$(15.200, 5.299, 19.200, 17.999, 8.300)$	6

Example 5.2.5. *We consider a unit disc with two layers and piecewise constant refractive index with values* 5 *and* 8 *where the position of the discontinuity is at* $r = 0.6$*. We tested the algorithm assuming that the unknown index has at most five layers. The results are shown in the following table:*

$(n_1, n_2, n_3, n_4, n_5)$	initial guess	tol	reconstruction	steps
$(5, 5, 5, 8, 8)$	$(6.5, 6.5, 6.5, 6.5, 6.5)$	10^{-6}	$(4.999, 5.000, 4.999, 8.000, 8.000)$	7

We see that we can recover the position of the discontinuity as well.

We also tested the algorithm for the above piecewise constant refractive index, using as input data the first 6 *analytically computed transmission eigenvalues and a set of* 20×20 *matrices. The reconstruction is shown in the next table:*

$(n_1, n_2, n_3, n_4, n_5)$	initial guess	tol	reconstruction	steps
$(5, 5, 5, 8, 8)$	$(6.5, 6.5, 6.5, 6.5, 6.5)$	10^{-6}	$(5.279, 4.721, 5.408, 7.915, 8.038)$	12

From this example we confirm that we can recover the qualitative properties of the unknown refractive index using original transmission eigenvalues as well.

5.2.3 Generalization of the method for non-spherically stratified domains

We can apply the previous method to any bounded and simply connected domain $D \subset \mathbb{R}^2$, $D = \cup_{i=1}^m D_i$, $D_i \cap D_j = \emptyset, i \neq j$, where the index of refraction is piecewise constant:

$$n(x) = \begin{cases} n_1, & x \in D_1 \\ \vdots \\ n_m, & x \in D_m \end{cases}$$

The final scheme and the algorithm is the same, the only difference is that the integrals in matrices given in (5.2.4),(5.2.5) and (5.2.6), are defined on the corresponding domains D_i, $i = 1, \ldots, m$. We assume that the exact geometry of subdomains is known.

Example 5.2.6. *We give an example of a disc with two inner elliptical layers with equations* $(2x)^2 + (5y/4)^2 = 1$ *and* $(4x)^2 + (5y/3)^2 = 1$.

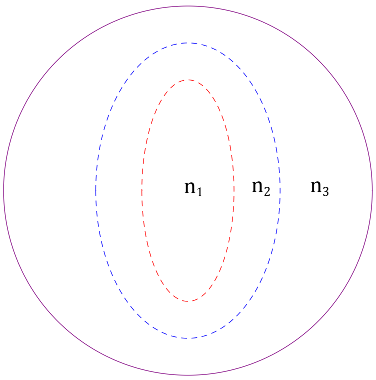

Figure 5.12: A disk with two elliptical layers

We tested the algorithm using 8×8 *matrices and* 16 *eigenvalues and the reconstruction is given in the following table:*

(n_1, n_2, n_3)	initial guess	tol	reconstruction	steps
$(12.2, 2.3, 8.7)$	$(6, 6, 6)$	10^{-6}	$(12.199, 2.299, 8.700)$	7

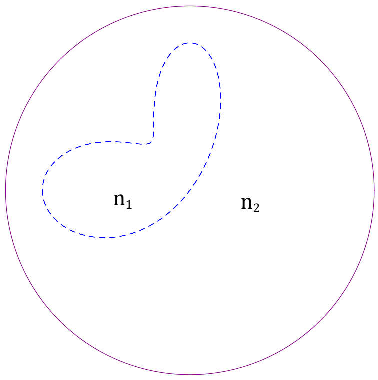

Figure 5.13: A disc with a peanut-shaped inclusion

Example 5.2.7. *We consider a unit disc with a peanut-shaped inclusion with equation* $4(\sin^7\theta - \cos^3\theta)/5$ *in polar coordinates.*

We used 8×8 *matrices and* 16 *eigenvalues and the reconstruction is shown in the following table:*

(n_1, n_2)	initial guess	tol	reconstruction	steps
$(6.2, 18.1)$	$(12, 12)$	10^{-6}	$(6.199, 18.100)$	6

Remark 5.2.8. *The Newton method we used in the above examples provides accurate reconstructions for simple domains with piecewise constant refractive index. The application of this method to more complex domains with arbitrary geometry and the corresponding error and stability analysis is a worthwhile future task.*

A

Appendix

Contents

A.1 Representation of c_{2l+2}

From equation (2.3.57) we have

$$c_{2l+2}\left[\frac{2^{l+1}\Gamma(l+3/2)}{\sqrt{\pi}a^{(l-1)/2}}\right]^2 = a\int_0^a \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{2\sqrt{rs}}\int_0^{\sqrt{rs}} tm(t)\mathrm{d}t\right)\bigg|_{r=a} s^l\mathrm{d}s$$
$$-l\int_0^a \frac{1}{2\sqrt{as}}\int_0^{\sqrt{as}} tm(t)\mathrm{d}t\, s^l\mathrm{d}s + \frac{a^l}{2}\int_0^a tm(t)\mathrm{d}t \quad \text{(A.1.1)}$$

We compute:

$$a\int_0^a \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{2\sqrt{rs}}\int_0^{\sqrt{rs}} tm(t)\mathrm{d}t\right)\bigg|_{r=a} s^l\mathrm{d}s \;= a\int_0^a \frac{sm(\sqrt{as})}{4\sqrt{as}}s^l\mathrm{d}s$$
$$-a\int_0^a \frac{s}{4(as)^{3/2}}\int_0^{\sqrt{as}} tm(t)\mathrm{d}t s^l\mathrm{d}s \quad \text{(A.1.2)}$$

We make the change of variables $y=\sqrt{as},\ \mathrm{d}s = 2y/a\,\mathrm{d}y$ and $0<y<a$ and hence:

$$a\int_0^a \frac{sm(\sqrt{as})}{4\sqrt{as}}s^l\mathrm{d}s = a\int_0^a \frac{m(y)y^2}{4ay}\frac{y^{2l}}{a^l}\frac{2y}{a}\mathrm{d}y = \int_0^a \frac{y^{2l+2}m(y)}{2a^{l+1}}\mathrm{d}y \quad \text{(A.1.3)}$$

Furthermore, the domain of integration for the double integral in the right hand side of (A.1.2) is:

$$D = \{(t,s):\ 0<t<\sqrt{as}\ \text{ and }\ 0<s<a\} = \{(t,s):\ 0<t<a\ \text{ and }\ t^2/a<s<a\}$$

So, if we interchange the orders of integration we compute

$$-a\int_0^a \frac{s}{4(as)^{3/2}}\int_0^{\sqrt{as}} tm(t)\mathrm{d}t\, s^l\mathrm{d}s = -a\int_{t^2/a}^a\int_0^a \frac{s}{4(as)^{3/2}}tm(t)s^l\mathrm{d}t\mathrm{d}s$$
$$= -a\int_0^a\int_{t^2/a}^a \frac{s}{4(as)^{3/2}}tm(t)s^l\mathrm{d}s\mathrm{d}t$$
$$= -a\int_0^a \frac{tm(t)}{4a^{3/2}}\int_{t^2/a}^a s^{l-1/2}\mathrm{d}s\mathrm{d}t$$
$$= -a\int_0^a \frac{tm(t)}{4a^{3/2}}\left(\frac{a^{l+1/2}}{l+1/2} - \frac{t^{2(l+1/2)}}{a^{l+1/2}(l+1/2)}\right)\mathrm{d}t$$
$$= \frac{-a^l}{2(2l+1)}\int_0^a tm(t)\mathrm{d}t + \frac{1}{2(2l+1)a^{l+1}}\int_0^a t^{2l+2}m(t)\mathrm{d}t$$
$$\text{(A.1.4)}$$

A change of the orders of integration for the second term of equation (A.1.1) implies:

$$
\begin{aligned}
-l\int_0^a \frac{1}{2\sqrt{as}}\int_0^{\sqrt{as}} tm(t)\mathrm{d}t\, s^l \mathrm{d}s &= -l\int_0^a \frac{tm(t)}{2\sqrt{a}}\int_{t^2/a}^a s^{l-1/2}\mathrm{d}s\mathrm{d}t \\
&= -l\int_0^a \frac{tm(t)}{2\sqrt{a}}\left(\frac{a^{l+1/2}}{l+1/2}-\frac{t^{2(l+1/2)}}{a^{l+1/2}(l+1/2)}\right)\mathrm{d}t \\
&= \frac{-la^l}{2l+1}\int_0^a tm(t)\mathrm{d}t + \frac{l}{(2l+1)a^{l+1}}\int_0^a t^{2l+2}m(t)\mathrm{d}t
\end{aligned}
\tag{A.1.5}
$$

Thus, by substituting (A.1.3)-(A.1.5) into (A.1.1) we conclude

$$
\begin{aligned}
c_{2l+2} =& \frac{\pi a^{l-1}}{\left(2^{l+1}\Gamma(l+3/2)\right)^2}\left[\frac{1}{2a^{l+1}}\int_0^a t^{2l+2}m(t)\mathrm{d}t - \frac{a^l}{2(2l+1)}\int_0^a tm(t)\mathrm{d}t\right. \\
&+\frac{1}{2(2l+1)a^{l+1}}\int_0^a t^{2l+2}m(t)\mathrm{d}t - \frac{la^l}{2l+1}\int_0^a tm(t)\mathrm{d}t + \frac{l}{(2l+1)a^{l+1}}\int_0^a t^{2l+2}m(t)\mathrm{d}t \\
&\left.+\frac{a^l}{2}\int_0^a tm(t)\mathrm{d}t\right] \\
=& \frac{\pi a^{l-1}}{\left(2^{l+1}\Gamma(l+3/2)\right)^2}\left[\left(\frac{1}{2a^{l+1}}+\frac{1}{2(2l+1)a^{l+1}}+\frac{l}{(2l+1)a^{l+1}}\right)\int_0^a t^{2l+2}m(t)\mathrm{d}t\right. \\
&\left.+\left(-\frac{a^l}{2(2l+1)}-\frac{la^l}{2l+1}+\frac{a^l}{2}\right)\int_0^a tm(t)\mathrm{d}t\right] \\
=& \frac{\pi a^{l-1}}{\left(2^{l+1}\Gamma(l+3/2)\right)^2}\left[\frac{1}{a^{l+1}}\int_0^a t^{2l+2}m(t)\mathrm{d}t\right] \\
=& \frac{\pi}{a^2\left(2^{l+1}\Gamma(l+3/2)\right)^2}\int_0^a t^{2l+2}m(t)\mathrm{d}t
\end{aligned}
$$

end representation (2.3.58) is proved.

A.2 The discontinuous Goursat problem

From the representation (2.3.53), we have:

$$G(r,s,k) = -\frac{k^2}{2\sqrt{rs}}\int_0^{\sqrt{rs}} t(n(t)-1)\mathrm{d}t \\ -\frac{k^2}{\sqrt{rs}}\int_1^{\sqrt{r/s}}\int_0^{\sqrt{rs}} t^2\tau\left[n(t\tau)-\frac{1}{\tau^4}\right]G\left(t\tau,\frac{t}{\tau},k\right)\mathrm{d}t\mathrm{d}\tau \qquad \text{(A.2.6)}$$

where G is the unique solution of the Goursat problem (2.3.50)-(2.3.52), for $0 < s \leq r < 1$.

Now, we assume that the refractive index $n(r)$ is discontinuous at $d \in (0,1)$, and satisfies the jump conditions (3.1.2)-(3.1.4). We examine how the discontinuity of the refractive index affect the Goursat problem.

If we differentiate (A.2.6) with respect of r, we get:

$$G_r(r,s) = -\frac{k^2\sqrt{s}}{4\sqrt{r}}\left(n(\sqrt{rs})-1\right) + \frac{k^2}{4\sqrt{s}r^{3/2}}\int_0^{\sqrt{rs}} t\,(n(t)-1)\,\mathrm{d}t \\ -\frac{k^2 s}{2}\int_1^{\sqrt{r/s}}\tau\left[n(\tau\sqrt{rs})-\frac{1}{\tau^4}\right]G\left(\tau\sqrt{rs},\frac{\sqrt{rs}}{\tau}\right)\mathrm{d}\tau \\ -\frac{k^2}{2\sqrt{r}s^{3/2}}\int_0^{\sqrt{rs}} t^2\left[n\left(\sqrt{\frac{r}{s}}t\right)-\frac{s^2}{r^2}\right]G\left(\sqrt{\frac{r}{s}}t,\sqrt{\frac{s}{r}}t\right)\mathrm{d}t \\ +\frac{k^2}{2\sqrt{s}r^{3/2}}\int_1^{\sqrt{r/s}}\int_0^{\sqrt{rs}}\tau t^2\left[n(t\tau)-\frac{1}{\tau^4}\right]G\left(t\tau,\frac{t}{\tau}\right)\mathrm{d}t\mathrm{d}\tau. \qquad \text{(A.2.7)}$$

Now, since $n(r)$ is discontinuous at $r = d$, using the jump condition (3.1.2) and (A.2.7) we conclude that:

$$G_r(r,s)|_{\sqrt{rs}=d^+} = G_r(r,s)|_{\sqrt{rs}=d^-} - \frac{k^2\sqrt{s}}{4\sqrt{r}}(a-1)n(d^-)$$

and hence $G_r(r,s)$ is discontinuous at $\sqrt{rs} = d$.

With the same arguments, we differentiate (A.2.6) with respect of s and we have:

$$G_s(r,s) = -\frac{k^2\sqrt{r}}{4\sqrt{s}}\left(n(\sqrt{rs})-1\right) + \frac{k^2}{4\sqrt{r}s^{3/2}}\int_0^{\sqrt{rs}} t\,(n(t)-1)\,\mathrm{d}t \\ -\frac{k^2 r}{2}\int_1^{\sqrt{r/s}}\tau\left[n(\tau\sqrt{rs})-\frac{1}{\tau^4}\right]G\left(\tau\sqrt{rs},\frac{\sqrt{rs}}{\tau}\right)\mathrm{d}\tau \\ +\frac{k^2\sqrt{r}}{2s^{5/2}}\int_0^{\sqrt{rs}} t^2\left[n\left(\sqrt{\frac{r}{s}}t\right)-\frac{s^2}{r^2}\right]G\left(\sqrt{\frac{r}{s}}t,\sqrt{\frac{s}{r}}t\right)\mathrm{d}t$$

$$+\frac{k^2}{2\sqrt{r}s^{3/2}}\int_1^{\sqrt{r/s}}\int_0^{\sqrt{rs}}\tau t^2\left[n(t\tau)-\frac{1}{\tau^4}\right]G\left(t\tau,\frac{t}{\tau}\right)\mathrm{d}t\mathrm{d}\tau. \quad \text{(A.2.8)}$$

Using again the jump condition (3.1.2), we obtain the following discontinuity relation for $G_s(r,s)$, at $\sqrt{rs}=d$:

$$G_s(r,s)|_{\sqrt{rs}=d^+}=G_s(r,s)|_{\sqrt{rs}=d^-}-\frac{k^2\sqrt{r}}{4\sqrt{s}}(a-1)n(d^-).$$

For the second order derivatives, we differentiate (A.2.6) twice with respect of r and compute:

$$\begin{aligned}
G_{rr}(r,s) \;=\; & \frac{k^2\sqrt{s}}{4r^{3/2}}\left(n(\sqrt{rs})-1\right)-\frac{k^2}{2}G(r,s)\left(n(r)-\frac{s^2}{r^2}\right)-\frac{k^2 s}{8r}n'(\sqrt{rs})\\
& -\frac{3k^2}{8\sqrt{s}r^{5/2}}\int_0^{\sqrt{rs}}t(n(t)-1)\mathrm{d}t\\
& +\frac{k^2 s}{2r}\int_1^{\sqrt{r/s}}\tau\left[n(\tau\sqrt{rs})-\frac{1}{\tau^4}\right]G\left(\tau\sqrt{rs},\frac{\sqrt{rs}}{\tau}\right)\mathrm{d}\tau\\
& +\frac{3k^2}{4(rs)^{3/2}}\int_0^{\sqrt{rs}}t^2\left[n\left(\sqrt{\frac{r}{s}}t\right)-\frac{s^2}{r^2}\right]G\left(\sqrt{\frac{r}{s}}t,\sqrt{\frac{s}{r}}t\right)\mathrm{d}t\\
& -\frac{3k^2}{4\sqrt{s}r^{5/2}}\int_1^{\sqrt{r/s}}\int_0^{\sqrt{rs}}\tau t^2\left[n(t\tau)-\frac{1}{\tau^4}\right]G\left(t\tau,\frac{t}{\tau}\right)\mathrm{d}t\mathrm{d}\tau\\
& -\frac{k^2 s}{4r}\int_1^{\sqrt{\frac{r}{s}}}\frac{1}{\tau^4}\left\{\sqrt{rs}\left(\tau^4 n(\tau\sqrt{rs})-1\right)\left[\tau^2 G_r\left(\tau\sqrt{rs},\frac{\sqrt{rs}}{\tau}\right)+G_s\left(\tau\sqrt{rs},\frac{\sqrt{rs}}{\tau}\right)\right]\right.\\
& \left.+\tau G\left(\tau\sqrt{rs},\frac{\sqrt{rs}}{\tau}\right)\left[\tau^4\left(\tau\sqrt{rs}n'(\tau\sqrt{rs})+n(\tau\sqrt{rs})\right)-1\right]\right\}\mathrm{d}\tau\\
& -\frac{k^2}{4r^4 s^2}\int_0^{\sqrt{rs}}t^2\left\{t\left(s^2-r^2 n\left(\sqrt{\frac{r}{s}}t\right)\right)\left[sG_s\left(\sqrt{\frac{r}{s}}t,\sqrt{\frac{s}{r}}t\right)-rG_r\left(\sqrt{\frac{r}{s}}t,\sqrt{\frac{s}{r}}t\right)\right]\right.\\
& \left.+G\left(\sqrt{\frac{r}{s}}t,\sqrt{\frac{s}{r}}t\right)\left(r^3 t n'\left(\sqrt{\frac{r}{s}}t\right)+r^2\sqrt{rs}\,n\left(\sqrt{\frac{r}{s}}t\right)+3s^3\sqrt{\frac{r}{s}}\right)\right\}\mathrm{d}t \qquad \text{(A.2.9)}
\end{aligned}$$

Since both n and n' appear in the above equation, we use conditions (3.1.2) and (3.1.3) as well. We have the following discontinuity condition for G_{rr}, along the curve $\sqrt{rs}=d$:

$$G_{rr}(r,s)|_{\sqrt{rs}=d^+}=G_{rr}(r,s)|_{\sqrt{rs}=d^-}-\frac{k^2 s}{8r}(a^{-1}-1)n'(d^-)+\frac{k^2}{4}n(d^-)\left(\frac{\sqrt{s}}{r^{3/2}}(a-1)-\frac{s}{2r}b\right)$$

and

$$G_{rr}(r,s)|_{r=d^+}=G_{rr}(r,s)|_{r=d^-}-\frac{k^2}{2}(a-1)n(d^-)G(d^-,s)$$

for $r=d$.

With analogous ideas, we differentiate (A.2.6) twice with respect of s and arrive at:

$$\begin{aligned}
G_{ss}(r,s) \;=&\; \frac{k^2\sqrt{r}}{4s^{3/2}}\Big(n(\sqrt{rs})-1\Big)+\frac{k^2r^2}{2s^2}G(r,s)\left(n(r)-\frac{s^2}{r^2}\right)-\frac{k^2r}{8s}n'(\sqrt{rs})\\
&-\frac{3k^2}{8\sqrt{r}s^{5/2}}\int_0^{\sqrt{rs}} t(n(t)-1)\mathrm{d}t\\
&+\frac{k^2r}{2s}\int_1^{\sqrt{r/s}}\tau\left[n(\tau\sqrt{rs})-\frac{1}{\tau^4}\right]G\left(\tau\sqrt{rs},\frac{\sqrt{rs}}{\tau}\right)\mathrm{d}\tau\\
&-\frac{5k^2\sqrt{r}}{4s^{7/2}}\int_0^{\sqrt{rs}} t^2\left[n\left(\sqrt{\frac{r}{s}}t\right)-\frac{s^2}{r^2}\right]G\left(\sqrt{\frac{r}{s}}t,\sqrt{\frac{s}{r}}t\right)\mathrm{d}t\\
&-\frac{3k^2}{4\sqrt{r}s^{5/2}}\int_1^{\sqrt{r/s}}\int_0^{\sqrt{rs}}\tau t^2\left[n(t\tau)-\frac{1}{\tau^4}\right]G\left(t\tau,\frac{t}{\tau}\right)\mathrm{d}t\mathrm{d}\tau\\
&-\frac{k^2r}{4s}\int_1^{\sqrt{\frac{r}{s}}}\frac{1}{\tau^4}\left\{\sqrt{rs}\left(\tau^4n(\tau\sqrt{rs})-1\right)\left[\tau^2G_r\left(\tau\sqrt{rs},\frac{\sqrt{rs}}{\tau}\right)+G_s\left(\tau\sqrt{rs},\frac{\sqrt{rs}}{\tau}\right)\right]\right.\\
&\left.+\tau G\left(\tau\sqrt{rs},\frac{\sqrt{rs}}{\tau}\right)\left[\tau^4\left(\tau\sqrt{rs}n'(\tau\sqrt{rs})+n(\tau\sqrt{rs})\right)-1\right]\right\}\mathrm{d}\tau\\
&-\frac{k^2}{4s^4r^2}\int_0^{\sqrt{rs}} t^2\left\{t\left(s^2-r^2n\left(\sqrt{\frac{r}{s}}t\right)\right)\left[sG_s\left(\sqrt{\frac{r}{s}}t,\sqrt{\frac{s}{r}}t\right)-rG_r\left(\sqrt{\frac{r}{s}}t,\sqrt{\frac{s}{r}}t\right)\right]\right.\\
&\left.+G\left(\sqrt{\frac{r}{s}}t,\sqrt{\frac{s}{r}}t\right)\left(r^3tn'\left(\sqrt{\frac{r}{s}}t\right)+r^2\sqrt{rs}n\left(\sqrt{\frac{r}{s}}t\right)+3s^3\sqrt{\frac{r}{s}}\right)\right\}\mathrm{d}t \qquad \text{(A.2.10)}
\end{aligned}$$

Finally, by applying the jump conditions in (A.2.10) we conclude that:

$$G_{ss}(r,s)|_{\sqrt{rs}=d^+} = G_{ss}(r,s)|_{\sqrt{rs}=d^-}-\frac{k^2r}{8s}(a^{-1}-1)n'(d^-)+\frac{k^2}{4}n(d^-)\left(\frac{\sqrt{r}}{s^{3/2}}(a-1)-\frac{r}{2s}b\right)$$

for $\sqrt{rs}=d$ and

$$G_{ss}(r,s)|_{r=d^+} = G_{ss}(r,s)|_{r=d^-}+\frac{k^2d^2}{2s^2}(a-1)n(d^-)G(d^-,s)$$

for $r=d$.

A.3 Refractive index with a finite number of discontinuities

We investigate the interior transmission problem for a spherically symmetric and piecewise C^2 refractive index which has a finite number of discontinuities and we extend the results of sections 3.1 and 3.2.2. We assume that the refractive index is discontinuous at the points $\{d_i\}_{i=1}^m,\ m \in \mathbb{N}$ and satisfies:

$$n(r) > 0,\ \mathrm{Im}(n(r)) = 0,\ n(r) = 1 \text{ for } r \geq 1,\ \text{ and } n'(1) = 0. \tag{A.3.11}$$

Also, $n(r)$ is C^2 in each $[0, d_1), (d_1, d_2), \ldots, (d_m, \infty)$ and the one-sided limits at d_i

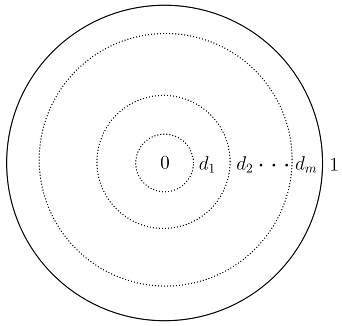

Figure A.1: The cross section of a spherical medium with m-discontinuities

are finite. We introduce the following jump conditions:

$$n(d_i^+) = a_i n(d_i^-) \tag{A.3.12}$$
$$n'(d_i^+) = a_i^{-1} n'(d_i^-) + b_i n(d_i^-) \tag{A.3.13}$$

where

$$a_i > 0,\ |a_i - 1| + |b_i| > 0 \text{ for } i = 1, \ldots, m \tag{A.3.14}$$

The interior transmission problem is equivalent with the boundary value problem (3.1.5)-(3.1.7), for the piecewise C^2 refractive index. We use the Liouville transformation (2.2.17) and define the relevant travel times

$$\tilde{d}_i := \int_0^{d_i} \sqrt{n(t)}\mathrm{d}t,\ i = 1, \ldots, m \tag{A.3.15}$$

The differential equation is transformed in the following form:

$$\frac{\mathrm{d}^2 z(\xi)}{\mathrm{d}\xi^2} + \left(k^2 - \frac{l(l+1)}{\xi^2} - g(\xi)\right) z(\xi) = 0 \tag{A.3.16}$$

where $g(\xi)$ is defined for $\xi \in \mathbb{R}^+ \setminus \{\tilde{d}_1, \ldots, \tilde{d}_m\}$. Since $n(r) > 0$ for $r \geq 0$, the Liouville transformation is invertible and g is well defined in each interval $(0, \tilde{d}_1), \ldots, (\tilde{d}_m, \infty)$ and is a piecewise continuous function for $r > 0$. Thus, the Liouville transformation can be used only locally, on each interval $(0, d_1), \ldots, (d_m, \infty)$.

The following lemma ensures that z is discontinuous at $\xi = \tilde{d}_i$ even though y_l is continuous at $r = d_i$, for each $i = 1, \ldots, m$.

Lemma A.3.1. *The solution $z(\xi)$ of the transformed problem (A.3.16), is discontinuous at $\xi = \tilde{d}_i$ and satisfies the jump conditions:*

$$z(\tilde{d}_i^+) = \tilde{a}_i z(\tilde{d}_i^-) \tag{A.3.17}$$

$$\frac{\mathrm{d}z(\tilde{d}_i^+)}{\mathrm{d}\xi} = \tilde{a}_i^{-1} \frac{\mathrm{d}z(\tilde{d}_i^-)}{\mathrm{d}\xi} + \tilde{b}_i z(\tilde{d}_i^-) \tag{A.3.18}$$

where:

$$\tilde{a}_i := a_i^{1/4} \tag{A.3.19}$$

$$\tilde{b}_i := \frac{1}{4}\left[\frac{n'(d_i^+)}{n(d_i^+)^{3/2}}\tilde{a}_i - \frac{n'(d_i^-)}{n(d_i^-)^{5/4} n(d_i^+)^{1/4}}\right] \tag{A.3.20}$$

for $i = 1, \ldots, m$.

Our purpose is to estimate the asymptotic expansions of the eigenfunctions and the determinants for large values of k. We follow the ideas of the simple case with one discontinuity, but the general case is much more complex. We define the following quantities:

$$A(k, \tilde{d}_i) = \frac{\pi}{2} k\tilde{d}_i(\tilde{a}_i - \tilde{a}_i^{-1}) J_\lambda(k\tilde{d}_i) J_{\lambda+1}(k\tilde{d}_i) \tag{A.3.21}$$

$$B(k, \tilde{d}_i) = \frac{\pi}{2} k\tilde{d}_i \left(\tilde{a}_i^{-1} Y_\lambda(k\tilde{d}_i) J_{\lambda+1}(k\tilde{d}_i) - \tilde{a}_i J_\lambda(k\tilde{d}_i) Y_{\lambda+1}(k\tilde{d}_i)\right) \tag{A.3.22}$$

$$C(k, \tilde{d}_i) = \frac{\pi}{2} k\tilde{d}_i(\tilde{a}_i^{-1} - \tilde{a}_i) Y_\lambda(k\tilde{d}_i) Y_{\lambda+1}(k\tilde{d}_i) \tag{A.3.23}$$

$$D(k, \tilde{d}_i) = \frac{\pi}{2} k\tilde{d}_i \left(\tilde{a}_i Y_\lambda(k\tilde{d}_i) J_{\lambda+1}(k\tilde{d}_i) - \tilde{a}_i^{-1} J_\lambda(k\tilde{d}_i) Y_{\lambda+1}(k\tilde{d}_i)\right) \tag{A.3.24}$$

Moreover, for $m \geq 2$ we define $P_m(k, \tilde{d}_i)$ and $Q_m(k, \tilde{d}_i)$ as

$$\begin{aligned} P_m &:= P_{m-1} D(k, \tilde{d}_m) + Q_{m-1} A(k, \tilde{d}_m) \\ Q_m &:= P_{m-1} C(k, \tilde{d}_m) + Q_{m-1} B(k, \tilde{d}_m) \end{aligned} \tag{A.3.25}$$

where

$$P_1 := A(k, \tilde{d}_1) \text{ and } Q_1 := B(k, \tilde{d}_1).$$

Therefore, we write the asymptotic formula (3.2.45)

$$z(\xi) = \sqrt{\frac{\pi\xi}{2k}} J_\lambda(k\xi) \left(\frac{\pi}{2}\tilde{d}_1 k[\tilde{a}_1^{-1} Y_\lambda(k\tilde{d}_1) J_{\lambda+1}(k\tilde{d}_1) - \tilde{a}_1 J_\lambda(k\tilde{d}_1) Y_{\lambda+1}(k\tilde{d}_1)]\right)$$

$$+\sqrt{\frac{\pi\xi}{2k}}Y_\lambda(k\xi)\left(\frac{\pi}{2}\tilde{d}_1k(\tilde{a}_1-\tilde{a}_1^{-1})J_\lambda(k\tilde{d}_1)J_{\lambda+1}(k\tilde{d}_1)\right)+O\left(\frac{\ln k}{k^2}\right)$$

as

$$z(\xi)=\sqrt{\frac{\pi\xi}{2k}}Y_\lambda(k\xi)P_1+\sqrt{\frac{\pi\xi}{2k}}J_\lambda(k\xi)Q_1+O\left(\frac{\ln k}{k^2}\right)$$

Proposition A.3.2. *The solution of (A.3.16), for large values of k and $\xi > \tilde{d}_m$, satisfies the estimate*

$$z(\xi)=\sqrt{\frac{\pi\xi}{2k}}Y_\lambda(k\xi)P_m+\sqrt{\frac{\pi\xi}{2k}}J_\lambda(k\xi)Q_m+O\left(\frac{\ln k}{k^2}\right) \tag{A.3.26}$$

Proof. For $m=1$ the result is obvious. We assume that (A.3.26) is satisfied for m and we will prove that is also satisfied for $m+1$ by induction. Indeed, let

$$z(\xi)=\sqrt{\frac{\pi\xi}{2k}}Y_\lambda(k\xi)P_m+\sqrt{\frac{\pi\xi}{2k}}J_\lambda(k\xi)Q_m+O\left(\frac{\ln k}{k^2}\right). \tag{A.3.27}$$

Along similar lines with proposition 3.2.9, for $\xi > \tilde{d}_{m+1}$

$$z(\xi)=-z(\tilde{d}_{m+1}^+)G_t(\xi,\tilde{d}_{m+1})+\frac{\mathrm{d}z(\tilde{d}_{m+1}^+)}{\mathrm{d}\xi}G(\xi,\tilde{d}_{m+1})+\int_{\tilde{d}_{m+1}}^{\xi}g(t)G(\xi,t)z(t)\mathrm{d}t.$$

Using the representation (A.3.27) for m and the jump conditions (A.3.17)-(A.3.18), after several calculations we conclude that

$$\begin{aligned}
z(\xi)=&\sqrt{\frac{\pi\xi}{2k}}Y_\lambda(k\xi)Q_m\frac{\pi}{2}k\tilde{d}_{m+1}(\tilde{a}_{m+1}-\tilde{a}_{m+1}^{-1})J_\lambda(k\tilde{d}_{m+1})J_\lambda(k\tilde{d}_{m+1})\\
&+\sqrt{\frac{\pi\xi}{2k}}Y_\lambda(k\xi)P_m\frac{\pi}{2}k\tilde{d}_{m+1}\left(\tilde{a}_{m+1}Y_\lambda(k\tilde{d}_{m+1})J_{\lambda+1}(k\tilde{d}_{m+1})-\tilde{a}_{m+1}^{-1}J_\lambda(k\tilde{d}_{m+1})Y_{\lambda+1}(k\tilde{d}_{m+1})\right)\\
&+\sqrt{\frac{\pi\xi}{2k}}J_\lambda(k\xi)Q_m\frac{\pi}{2}k\tilde{d}_{m+1}\left(\tilde{a}_{m+1}^{-1}Y_\lambda(k\tilde{d}_{m+1})J_{\lambda+1}(k\tilde{d}_{m+1})-\tilde{a}_{m+1}J_\lambda(k\tilde{d}_{m+1})Y_{\lambda+1}(k\tilde{d}_{m+1})\right)\\
&+\sqrt{\frac{\pi\xi}{2k}}J_\lambda(k\xi)P_m\frac{\pi}{2}k\tilde{d}_{m+1}(\tilde{a}_{m+1}^{-1}-\tilde{a}_{m+1})Y_\lambda(k\tilde{d}_{m+1})Y_{\lambda+1}(k\tilde{d}_{m+1})+O\left(\frac{\ln k}{k^2}\right),\quad \xi>\tilde{d}_{m+1}
\end{aligned}$$

which can be written as:

$$\begin{aligned}
z(\xi)=&\sqrt{\frac{\pi\xi}{2k}}Y_\lambda(k\xi)\left(P_mD(k,\tilde{d}_{m+1})+Q_mA(k,\tilde{d}_{m+1})\right)\\
&+\sqrt{\frac{\pi\xi}{2k}}J_\lambda(k\xi)\left(P_mC(k,\tilde{d}_{m+1})+Q_mB(k,\tilde{d}_{m+1})\right)+O\left(\frac{\ln k}{k^2}\right).
\end{aligned}$$

Thus, for $\xi > \tilde{d}_{m+1}$

$$z(\xi) = \sqrt{\frac{\pi\xi}{2k}} Y_\lambda(k\xi) P_{m+1} + \sqrt{\frac{\pi\xi}{2k}} J_\lambda(k\xi) Q_{m+1} + O\left(\frac{\ln k}{k^2}\right)$$

and the result follows. □

Furthermore, using the asymptotic expansions of Bessel and Neumann functions for large values of k we derive:

$$\begin{aligned}
A(k,\tilde{d}_i) = C(k,\tilde{d}_i) &= \frac{\tilde{a}_i^{-1} - \tilde{a}_i}{2} \sin\left(2k\tilde{d}_i - l\pi\right) + O\left(\frac{1}{k}\right)\\
B(k,\tilde{d}_i) &= \tilde{a}_i^{-1}\cos^2\left(k\tilde{d}_i - \frac{l\pi}{2}\right) + \tilde{a}_i \sin^2\left(k\tilde{d}_i - \frac{l\pi}{2}\right) + O\left(\frac{1}{k}\right)\\
D(k,\tilde{d}_i) &= \tilde{a}_i \cos^2\left(k\tilde{d}_i - \frac{l\pi}{2}\right) + \tilde{a}_i^{-1}\sin^2\left(k\tilde{d}_i - \frac{l\pi}{2}\right) + O\left(\frac{1}{k}\right)
\end{aligned}$$

Using these expansions, the general asymptotic form of z and the Liouville transform we can express y_l in a closed asymptotic form. We define the following sets:

$$I = \{1,\dots,m\} \quad \text{and} \quad I_s = \{J \subset I : \#J = s\} \tag{A.3.28}$$

where I_s is the set of all subsets of I with cardinality s. Also, for any $J \in I_s$ we define the ordered set

$$\tilde{J} = \langle i_j \rangle_{1\le j\le s},\ i_j < i_{j+1} \text{ for any } i_j \in J. \tag{A.3.29}$$

Finally, for large values of k and $r > d_m$:

$$\begin{aligned}
y_l(r) = &\frac{1}{kn(r)^{1/4} n(0)^{l/2+1/4} 2^m \prod_{i=1}^m \tilde{a}_i} \Bigg[\sum_{s=0}^{m} \sum_{J\in I_s} \Bigg\{ \prod_{i\in J} (\tilde{a}_i^2 - 1) \prod_{i\in I\setminus J} (\tilde{a}_i^2+1)\\
&\sin\left(\sum_{i_j\in\tilde{J}} (-1)^{j+1} 2k\tilde{d}_{i_j} - \frac{l\pi}{2} + (-1)^s k\xi\right)\Bigg\}\Bigg] + O\left(\frac{\ln k}{k^2}\right)
\end{aligned} \tag{A.3.30}$$

where, for $J = \{\emptyset\}$ we take $\prod_{i\in J}(\tilde{a}_i^2 - 1) = 1$ and $\sum_{i_j\in\tilde{J}}(-1)^{j+1}2k\tilde{d}_{i_j} = 0$ and for $I \setminus J = \{\emptyset\}$ we take $\prod_{i\in I\setminus J}(\tilde{a}_i^2+1) = 1$.

To make clear this rather complicated form, consider $s = 0$. Then $I_s, J = \{\emptyset\}$ and $I = I \setminus J = \{1,\dots,m\}$. Thus the corresponding term in the sum is:

$$(\tilde{a}_1^2+1)\cdots(\tilde{a}_m^2+1)\sin\left(k\xi - \frac{l\pi}{2}\right).$$

For $s = 1$, $I_s = \{\{1\},\dots,\{m\}\}$ and $J = \{1\},\dots,\{m\}$. Now the corresponding terms in the sum are:

$$(\tilde{a}_1^2 - 1)(\tilde{a}_2^2 + 1)\cdots(\tilde{a}_m^2 + 1)\sin\left(2k\tilde{d}_1 - \frac{l\pi}{2} - k\xi\right),$$
$$\vdots$$
$$(\tilde{a}_1^2 + 1)\cdots(\tilde{a}_{m-1}^2 + 1)(\tilde{a}_m^2 - 1)\sin\left(2k\tilde{d}_m - \frac{l\pi}{2} - k\xi\right).$$

For $s \geq 2$, J consists of s-sets, and for the corresponding sums in the sine functions we consider the ordered sets $\tilde{J}$. Finally for $s = m$, $I_s = J = \{1, \ldots, m\}$ and $I \setminus J = \{\emptyset\}$. Therefore, the corresponding term is:

$$(\tilde{a}_1^2 - 1)\cdots(\tilde{a}_m^2 - 1)\sin\left(2k\tilde{d}_1 - 2k\tilde{d}_2 + \cdots + (-1)^{m+1}2k\tilde{d}_m - \frac{l\pi}{2} + (-1)^m k\xi\right).$$

Moreover, by applying equation (A.3.30) and the asymptotic expansions for the spherical Bessel functions to the characteristic equation (3.1.7), we obtain the corresponding asymptotic formula:

$$\begin{aligned} d_l(k) = \frac{1}{kn(0)^{l/2+1/4}2^m\prod_{i=1}^m \tilde{a}_i} & \left[\sum_{s=0}^m \left(\sum_{\substack{J\in I_s\\ s=odd}} \left\{-\prod_{i\in J}(\tilde{a}_i^2 - 1)\prod_{i\in I\setminus J}(\tilde{a}_i^2 + 1)\right.\right.\right. \\ & \sin\left(k - kA + \sum_{i_j\in\tilde{J}}(-1)^{j+1}2k\tilde{d}_{i_j} - l\pi\right)\Bigg\} + \sum_{\substack{J\in I_s\\ s=even}}\Bigg\{\prod_{i\in J}(\tilde{a}_i^2 - 1)\prod_{i\in I\setminus J}(\tilde{a}_i^2 + 1) \\ & \left.\left.\left.\sin\left(k - kA + \sum_{i_j\in\tilde{J}}(-1)^j 2k\tilde{d}_{i_j}\right)\right\}\right)\right] + O\left(\frac{\ln k}{k^2}\right) \end{aligned} \tag{A.3.31}$$

Example A.3.3. *In the case where $m = 2$, the corresponding asymptotic expansion for y_l is:*

$$\begin{aligned} y_l(r) = \frac{1}{kn(r)^{1/4}n(0)^{l/2+1/4}4\tilde{a}_1\tilde{a}_2} & \left[(\tilde{a}_1^2 + 1)(\tilde{a}_2^2 + 1)\sin\left(k\xi - \frac{l\pi}{2}\right) + (\tilde{a}_1^2 - 1)(\tilde{a}_2^2 + 1)\right. \\ & \sin\left(2k\tilde{d}_1 - \frac{l\pi}{2} - k\xi\right) + (\tilde{a}_1^2 + 1)(\tilde{a}_2^2 - 1)\sin\left(2k\tilde{d}_2 - \frac{l\pi}{2} - k\xi\right) \\ & \left.+(\tilde{a}_1^2 - 1)(\tilde{a}_2^2 - 1)\sin\left(2k\tilde{d}_1 - 2k\tilde{d}_2 - \frac{l\pi}{2} + k\xi\right)\right] + O\left(\frac{\ln k}{k^2}\right), \quad r > d_2 \end{aligned}$$

and the determinant satisfies:

$$\begin{aligned} d_l(k) = \frac{1}{kn(0)^{l/2+1/4}4\tilde{a}_1\tilde{a}_2} & \Big[(\tilde{a}_1^2 + 1)(\tilde{a}_2^2 + 1)\sin(k - kA) - (\tilde{a}_1^2 - 1)(\tilde{a}_2^2 + 1) \\ & \sin\left(k - kA + 2k\tilde{d}_1 - l\pi\right) - (\tilde{a}_1^2 + 1)(\tilde{a}_2^2 - 1)\sin\left(k - kA + 2k\tilde{d}_2 - l\pi\right) \end{aligned}$$

$$+(\tilde{a}_1^2-1)(\tilde{a}_2^2-1)\sin\left(k-kA-2k\tilde{d}_1+2k\tilde{d}_2\right)\Big]+O\left(\frac{\ln k}{k^2}\right).$$

A.4 Entire functions

We recall some basic results from entire function theory that are required for the investigation of the inverse spectral problem for transmission eigenvalues. We present results from [12, 70, 94] and we refer therein for further details and proofs.

Definition A.4.1. *A function of a complex variable z that is analytic in the whole complex plane, i.e. a function represented by a power series:*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad \lim_{n\to\infty} \sqrt[n]{|c_n|} = 0 \tag{A.4.32}$$

is called an entire function.

Typical examples of entire functions are polynomials and the exponential function. We define the maximum modulus

$$M(r) := \max_{|z|=r} |f(z)|. \tag{A.4.33}$$

The relationship between the distribution of an entire function's roots and its growth is the main subject of the theory of entire functions. In order to classify entire functions according to their growth we introduce the following concepts:

Definition A.4.2. *The order ρ of an entire function $f(z)$ is defined as*

$$\rho := \overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r}. \tag{A.4.34}$$

or equivalently:

Definition A.4.3. *The order ρ of an entire function $f(z)$ is the greatest lower bound of all $\rho > 0$ such that*

$$|f(z)| \leq Ae^{B|z|^{\rho}}, \quad \textit{for all} \ \ z \in \mathbb{C}$$

and for some positive constants A, B.

Definition A.4.4. *An entire function $f(z)$ of finite order ρ has type τ, where*

$$\tau := \overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^{\rho}}. \tag{A.4.35}$$

Definition A.4.5. *An entire function of order $\rho = 1$ and finite type is said to be of exponential type.*

We illustrate these definitions with the following examples:

Example A.4.6. *The function* $f(z) = \sin Az$ *is of order* $\rho = 1$ *and type* $\tau = |A|$, *(exponential type* $|A|$*).*

Indeed, since $|f(z)| = |e^{iAz} - e^{-iAz}|/|2i| \le e^{|A||z|}$, *we have that* $M(r) = e^{|A|r}$. *Calculating the limits (A.4.34) and (A.4.35) we conclude that* $\rho = 1$ *and* $\tau = |A|$.

Example A.4.7. *The function* $f(z) = \sin \sqrt{z}/\sqrt{z}$ *is of order* $\rho = 1/2$ *and type* $\tau = 1$.

From the inequality

$$\left|\frac{\sin\sqrt{z}}{\sqrt{z}}\right| \le \int_0^1 |\cos\sqrt{z}t \;\mathrm{d}t| \le \int_0^1 \cosh|\mathrm{Im}(\sqrt{z})|t \;\mathrm{d}t \le e^{|\mathrm{Im}(\sqrt{z})|} \le e^{|z|^{1/2}}$$

we have that $M(r) = e^{r^{1/2}}$ *and hence we get* $\rho = 1/2$ *and* $\tau = 1$.

An important result on entire functions of finite order is the *Hadamard's factorization theorem.* The main idea is to relate the growth of an entire function with the number of zeros it possesses. Before we state the theorem, we introduce the *Weierstrass prime factors*:

$$G(u,p) := \begin{cases} 1-u, & p = 0 \\ (1-u)\exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^p}{p}\right), & p > 0. \end{cases} \tag{A.4.36}$$

Theorem A.4.8. *An entire function of finite order* ρ *can be represented in the form:*

$$f(z) = z^m e^{P_q(z)} \prod_{n=1}^{\infty} G\left(\frac{z}{z_n}, p\right) \tag{A.4.37}$$

where $z_1, z_2, \dots$ *are the non-zero roots of* $f(z)$, $p \le \rho$, $P_q(z)$ *is a polynomial of* z *with degree* $q \le \rho$ *and* m *is the multiplicity of the root at the origin. The constant* p *is the smallest positive integer such that* $\sum_{n=1}^{\infty} |z_n|^{-p-1}$ *converges.*

We refer to [70] for the detailed proof. The integer $g = \max(p, q)$ is called the *genus* of the entire function f. If the order is not an integer, then g is equal with the integer part of ρ, $g = [\rho]$. Else, $g = \rho$ or $g = \rho - 1$.

Example A.4.9. *An entire function of order zero or order* $\rho < 1$ *has the following representation:*

$$f(z) = Cz^m \prod_{n=1}^{\omega} \left(1 - \frac{z}{z_n}\right),$$

where $\omega \le \infty$ *and* $\sum_{n=1}^{\omega} 1/|z_n| < \infty$.

Example A.4.10. *An entire function of genus one has the following representation:*

$$f(z) = Cz^m e^{az+b} \prod_{n=1}^{\omega} \left(1 - \frac{z}{z_n}\right) e^{z/z_n},$$

for some constants C, a, b.

Example A.4.11. *The entire function* $\sin \pi\sqrt{z}/\pi\sqrt{z}$ *has the representation:*

$$\frac{\sin \pi\sqrt{z}}{\pi\sqrt{z}} = \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(1 - \frac{\sqrt{z}}{n}\right) e^{\sqrt{z}/n}.$$

As we showed in example A.4.7, $f(z)$ *is entire of order* $\rho = 1/2$ *and has only real roots at* $z_n = n^2,\ n \in \mathbb{N}^*$. *Hence, by Hadamard's theorem*

$$\frac{\sin \pi\sqrt{z}}{\pi\sqrt{z}} = C \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2}\right).$$

Since $f(0) = 1$, *we conclude that* $C = 1$. *Furthermore, by substituting* z^2 *instead of* z *we have:*

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \tag{A.4.38}$$

and we claim that the product can be written as

$$\prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}.$$

Indeed, from the above product we have:

$$\ldots \left(1 - \frac{z}{-2}\right) e^{-z/2} (1+z)\, e^{-z} (1-z)\, e^{z} \left(1 - \frac{z}{2}\right) e^{z/2} \ldots$$

or equivalently

$$\cdots (1+z)(1-z)\left(1+\frac{z}{2}\right)\left(1-\frac{z}{2}\right)\cdots = (1-z^2)\left(1-\frac{z^2}{2^2}\right)\cdots$$

and thus, we arrive at (A.4.38).

The final topic concerns the *Paley-Wiener theorem.* We consider a square integrable function $\phi \in L^2[-A, A]$ and we define the function

$$f(z) = \int_{-A}^{A} \phi(t) \mathrm{e}^{izt} \mathrm{d}t.$$

Then, it is easily verified that $f(z)$ is entire of exponential type and belongs to L^2 on the real axis. The Paley-Wiener theorem states that every entire function of exponential type is obtained in this way.

Theorem A.4.12. *Let* $f(z)$ *be an entire function of exponential type at most* A *such that*

$$\int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x < \infty.$$

Then there exists $\phi \in L^2[-A, A]$ *such that*

$$f(z) = \int_{-A}^{A} \phi(t) e^{izt} \mathrm{d}t.$$

A detailed proof can be found in [94].

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