# NATIONAL TECHNICAL UNIVERSITY OF ATHENS



## SCHOOL OF NAVAL ARCHITECTURE AND MARINE ENGINEERING

Section of Naval and Marine Hydrodynamics

## Hydrodynamic interaction and power absorption efficiency of wave energy converters in the frequency domain, using the boundary element method

Diploma Thesis

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#### ABSTRACT

The increasing interest in renewable energy has motivated a large number of researchers to deal with such themes. Although solar and aeolic applications are already in production, ocean energy means and specifically wave energy conversion through oscillating bodies is under development. This delay mainly lies on the complexity of such systems as interaction effects occur throughout the domain around them and not only in their wake, which is the case for wind turbines. First, the linear hydrodynamic and mechanical problems are presented, followed by the proper numerical boundary element method formulation applied. The necessary discretization program developed is also described. The motions and power absorption of a wave energy converter is dependent on three main factors. First on the exciting forces acting on the body, second on the characteristics of the body which can be divided in the hydrodynamic and in the mass distribution ones and finally on external factors like the ones of the power take-off mechanism. This thesis deals with the interaction effects concerning the hydrodynamic problem, examined for a number of array cylinder configurations in terms of exciting forces, added masses and damping coefficients. Specifically, near trapping relating effects were found to influence significantly the exciting forces magnitude hence the sensitivity of results on changing parameters like the separating distance and the position of the cylinders is significant. On the mechanical problem, motions and power absorption in wave energy farms are examined in the case of heaving power absorption of fully movable bodies, with all six degrees of freedom modelled. As observed appropriate design leading to frequency coincidence of some phenomena can built constructive effects, greatly improving the total efficiency of wave energy converter configurations.

## TABLE OF CONTENTS

ACKNO	WLEDGEMENTS	i
ABSTRA	ACT	ii
TABLE (	OF CONTENTS	iii
LIST OF	SYMBOLS	v
LIST OF	ABBREVIATIONS	vi
LIST OF	FIGURES	. vii
INTROD	UCTION	xi
CHAPTE	ER 1: FORMULATION OF THE HYDRODYNAMIC PROBLEM	1
1.1	Euler equations and potential theory	1
1.2	Time and frequency domain formulations	2
1.3	Indirect boundary integral equations formulation	8
1.3.	1 Single layer distribution	8
1.3.2	2 Double layer distribution	9
1.3.3	3 Single layer, diffraction and radiation problems	. 10
1.4	Hybrid, indirect boundary integral equations formulation	. 12
CHAPTE	ER 2: FORMULATION OF THE OSCILLATING DYNAMIC PROBLEM	. 17
2.1	One body equations of motion	. 17
2.2	Any number of bodies equations of motion expansion	. 21
2.3	PTO modelling and quantification	. 24
2.3.	Equations of motion with PTO modelled	. 24
2.3.2	2 PTO calculation	. 25
CHAPTE	ER 3: NUMERICAL FORMULATION USING PLANE PANEL BEM	. 27
3.1	Influence matrices	. 27
3.2	Numerical formulation of the indirect, hybrid problem	. 31
3.3	Numerical calculation of exciting forces and hydrodynamic coefficients	. 35
CHAPTE	ER 4: DEVELOPMENT OF A GRID GENERATION PROGRAM	. 37
4.1	Transfinite interpolation	. 37
4.2	Free surface grid	. 38
4.2.1	1 Inside area	. 38
4.2.2	2 Outside area	. 42
4.2.3	3 Total free surface grid and special cases	. 45
4.3	Bodies grid	. 47
4.4	Matching boundary grid	. 48
4.5	Sea bed grid	. 49
4.6	Panel size specifications	. 50

CHAPTE	ER 5: COMPARISON BETWEEN BEM AND SEMI-ANALYTIC	53
5 1	Introduction	53
5.2	Pesults concerning one body	55
5.2	1 Scattering problem	54
5.2.1	2 Padiation problems	54
5 3	Posults concerning a 1x2 cylinder bodies array	54
5.21	Results concerning a Tx2 cylinder bodies array	50
5.3.1	Development	30
5.3.2	2 Radiation problems	38
J.4	Results concerning an 2x2 cylinder bodies array	62
CHAPIE	SR 6: HYDRODYNAMIC INTERACTION IN WEFS	63
6.1	Explanation of the obtained results	63
6.1.1	1 Excitation forces	63
6.1.2	2 Near trapping effect	64
6.1.3	Added mass and damping coefficients	65
6.2	Exciting forces acting on WECs	67
6.2.1	1 Results with respect to oscillating cylinder, one row array	67
6.2.2	2 Results with respect to oscillating cylinders, two rows array	76
6.2.3	3 Results with respect to incident wave angle, one row array	78
6.2.4	4 Results with respect to incident wave angle, two rows array	87
6.2.5	5 Combined configurations, near trapping indicating results	91
6.3	Interaction added masses and damping coefficients	93
CHAPTE	ER 7: HEAVING MOTIONS AND POWER ABSORPTION	107
7.1	Isolated body's motion and absorbed power	107
7.1.1	1 Cylinder characteristics	107
7.1.2	2 External damping tuning and spring value	109
7.2	Heaving mode motions	110
7.3	Power absorption and park effect	122
7.3.1	1 Park effect	122
7.3.2	2 Results with respect to separating distance and external spring value	122
7.3.3	3 Comparing results between different array configurations	135
OVERVIEW AND CONCLUSIONS		
BIBLIOGRAPHY		

## LIST OF SYMBOLS

D	:	fluid domain with $\mathbf{x}$ as the defining space variable.
$\partial D$	:	fluid domain's D boundary.
t	:	time variable.
ω	:	incident wave's frequency.
λ	:	incident wave's length.
β	:	incident wave's direction.
$k_0$	:	incident wave's number, real solution of the dispersion relation.
$\Phi(\mathbf{x},t)$	:	time dependent velocity potential in position $\mathbf{x}$ of the domain $D$ .
$\mathbf{v}(\mathbf{x},t)$	:	time dependent velocity in position $\mathbf{x}$ of the domain $D$ .
$\eta(\mathbf{x},t)$	:	time dependent wave surface elevation in position $\mathbf{x}$ .
А	:	wave amplitude.
$\xi_k^n(t)$	:	time dependent motion of $n$ body in the $k$ direction.
$\overset{{}_\circ}{\Phi}(\mathbf{x})$	:	complex amplitude of the velocity potential in position $\mathbf{x}$ of the domain $D$ .
$\mathring{\eta}(\mathbf{x})$	:	complex amplitude of the wave elevation in position $\mathbf{x}$ .
$\overset{{}_\circ}{\pmb{\sigma}}$	:	complex source strength.
$\overset{\circ}{\xi}{}^{n}_{k}$	:	complex amplitude of the motion of body $n$ body in the $k$ direction.
$A^{pq}_{ij}$	:	added mass of body $p$ in the $i$ direction due to the motion of body $q$ in the $j$
$B^{pq}_{ij}$	:	damping coefficient of body $p$ in the <i>i</i> direction due to the motion of body $q$ in the <i>j</i> direction.
${\mathop{\boldsymbol{\tilde{A}}^{pq}}}$	:	the 6x6 interaction matrix of added masses on body $p$ from body $q$ .
${\it B}^{{}^{pq}}$	:	the 6x6 interaction matrix of damping coefficients on body $p$ from body $q$ .
Ą	:	the global added mass matrix of the multi bodies problem.
${\substack B \\ \widetilde{\mathcal L}}$	:	the global damping coefficients matrix of the multi bodies problem.
$\overline{P_{_{ext,3,iso}}}(\omega)$	:	mean power absorbed by the heaving motion of an isolated body.
$\overline{P_{_{ext,3}}^{n}}(\omega)$	:	mean power absorbed by the heaving motion of $n$ body.
$\overline{P_{_{ext,3}}^{_{tot}}}(\omega)$	:	total mean power absorbed by the heaving motion of all bodies.
$q(\omega)$	:	q-factor, denoting the interaction efficiency of the bodies configuration.

## LIST OF ABBREVIATIONS

- PTO : Power Take-Off
- BEM : Boundary Element Method
- WEC : Wave Energy Converter
- WEF : Wave Energy Farm
- ORE : Ocean Renewable Energy

## LIST OF FIGURES

Figure 1-1 Hydrodynamic problem's geometry	2
Figure 1-2 Inclusion of point $\mathbf{x}_s$ , in domain D.	8
Figure 2-1 Free floating body	17
Figure 3-1 Quadrilateral source strength element. Picture taken from (Katz & Plotkin, 2	2001)
Figure 4-1 Application of transfinite interpolation	37
Figure 4-2 Basic "box"	38
Figure 4-3 Basic corner "box"	39
Figure 4-4 Slender body inside "box". No control of the panels, up. Control of the p	oints,
down	39
Figure 4-5 Bodies array without "gaps"	40
Figure 4-6 Bodies array with "gaps"	40
Figure 4-7 Bodies array with intermediate positioned column, without "gaps"	41
Figure 4-8 Bodies array with intermediate positioned column, with "gaps"	41
Figure 4-9 The previous array grid with an "empty box"	42
Figure 4-10 Outside part	43
Figure 4-11 Curved part between inside and outside parts	43
Figure 4-12 Outside part with panel split	44
Figure 4-13 Panel division	44
Figure 4-14 Example of a total free surface grid	45
Figure 4-15 Column array free surface grid	45
Figure 4-16 Line array free surface grid	46
Figure 4-17 One body free surface grid	46
Figure 4-18 Cylinder body grid	47
Figure 4-19 Ellipsoid body grid	47
Figure 4-20 Matching boundary grid without use of geometric progression	48
Figure 4-21 Matching boundary grid with use of geometric progression	48
Figure 4-22 Sea bed grid with panel division	49
Figure 4-23 Free surface panel dimensions	50
Figure 5-1 Hydrodynamic problem's geometry after the matching boundary definition	53
Figure 5-2 Numbering of bodies, inside an array	53
Figure 5-3 Isolated cylinder: Horizontal scattering force on isolated cylinder by G (1971)	arrett
Figure 5-4 Isolated cylinder: Vertical scattering force by Garrett (1971)	55
Figure 5-5 Isolated cylinder: Hydrodynamic coefficients in surge motion by Yeung (1980	)). 55
Figure 5-6 Isolated cylinder: Hydrodynamic coefficients in heave motion by Yeung (1980	0) 56
Figure 5-7 Isolated cylinder: Hydrodynamic coefficients in heave motion by Yeung (1980	0) 56
Figure 5-8 Isolated cylinder: Coupled hydrodynamic coefficients in pitch and heave m	otion
by Yeung (1980)	57
Figure 5-9 Isolated cylinder: Hydrodynamic coefficients in pitch motion by Yeung (1980	).57
Figure 5-10 Horizontal and vertical exciting forces, 1x2 array, cylinder 1 (continuous 1	lines:
freFLOW, dashed lines: Matsui and Tamaki 1981)	58
Figure 5-11 Horizontal and vertical scattering forces, 1x2 array, cylinder 2 (continuous 2)	lines:
freFLOW, dashed lines: Matsui and Tamaki 1981)	59
Figure 5-12 Hydrodynamic coefficients, 1x2 array, (continuous lines: freFLOW, dashed	lines:
Matsui and Tamaki 1981)	59
Figure 5-13 Interaction hydrodynamic coefficients, 1x2 array, (continuous lines: freFL	LOW,
dashed lines: Mavrakos 1991).	60

Figure 5-14 Interaction hydrodynamic coefficients, 1x2 array, (continuous lines:	freFLOW,
Gastieu files: Matsul and Tamaki 1981).	
Figure 5-15 Hydrodynamic coefficients, 1x2 array, (continuous lines: freFLOW, da	sned lines:
Matsul and Tamaki 1981).	
Figure 5-16 Interaction hydrodynamic coefficients, 1x2 array, (continuous lines:	freFLOW,
dashed lines: Mavrakos 1991).	
Figure 5-17 Interaction hydrodynamic coefficients, 2x2 array, (continuous lines:	freFLOW,
dashed lines: Mavrakos 1991)	62
Figure 5-18 Interaction hydrodynamic coefficients, 2x2 array, (continuous lines:	freFLOW,
dashed lines: Mavrakos 1991).	
Figure 6-1 Numbering of bodies inside an array	63
Figure 6-2 Field pattern of the radiated wave for an axisymmetric body moving	g in heave
(left), surging (right) motion, taken from (Babarit, 2015)	66
Figure 6-3 Exciting forces, 1x2 array, $\beta=0$ deg	67
Figure 6-4 Exciting forces, 1x2 array, β=45deg	
Figure 6-5 Exciting forces, 1x2 array, $\beta=90 \text{ deg}$	
Figure 6-6 Exciting forces, 1x3 array, $\beta=0$ deg.	
Figure 6-7 Exciting forces 1x3 array $B=45 \deg$	71
Figure 6-8 Exciting forces 1x3 array $\beta = 90 \text{ deg}$	
Figure 6.0 Exciting forces, $1xJ$ array, $\beta=0$ deg	
Figure 6.10 Exciting foreas $1x4$ array $\beta=45$ dag	
Figure 6-10 Exciting forces, 1x4 analy, $p=45$ deg.	
Figure 6-11 Exclude forces, 1x4 array, p=90 deg	
Figure 6-12 Exciting forces, $2x^2$ array, $p=0$ deg	
Figure 6-13 Exciting forces, $2x^2$ array, $\beta=45$ deg	
Figure 6-14 Exciting forces, 1x2 array, 1=5a	
Figure 6-15 Exciting forces, 1x2 array, 1=8a	
Figure 6-16 Exciting forces, 1x3 array, cylinder No1	80
Figure 6-17 Exciting forces, 1x3 array, cylinder No2	81
Figure 6-18 Exciting forces, 1x3 array, cylinder No3	82
Figure 6-19 Exciting forces, 1x4 array, cylinder No1	83
Figure 6-20 Exciting forces, 1x4 array, cylinder No2	
Figure 6-21 Exciting forces, 1x4 array, cylinder No3	85
Figure 6-22 Exciting forces, 1x4 array, cylinder No4	86
Figure 6-23 Exciting forces, 2x2 array, cylinder No1	87
Figure 6-24 Exciting forces, 2x2 array, cylinder No2	88
Figure 6-25 Exciting forces, 2x2 array, cylinder No3.	89
Figure 6-26 Exciting forces, 2x2 array, cylinder No4	
Figure 6-27 Exciting forces on the middle cylinder in row arrays of N bodies. $\beta=0$	deg. l=5a.
The magenta dashed line corresponds to $kl = \kappa \pi$ with $\kappa$ being odd number	and green
The magenta dashed line corresponds to $\kappa_0 t = \kappa t$ with $\kappa$ being out number	
dashed line corresponding to even number.	
Figure 6-28 Exciting forces of cylinder No1 in row arrays of N bodies, $\beta=0$ deg,	I=5a. The
magenta dashed line corresponds to $k_0 l = \kappa \pi$ with $\kappa$ being odd number and greater that the second sec	en dashed
line corresponding to even number.	
Figure 6-29 Added masses-damping coefficients. 1x2 array.	
Figure 6-30 Interaction added masses-damping coefficients, 1x2 array	
Figure 6-31 Added masses-damping coefficients 1x3 array 1=5a	95
Figure 6-32 Added masses-damping coefficients 1x3 array 1–8a	96
Figure 6-33 Interaction added masses damning coefficients 1v3 array 1-5a	
Figure 6-34 Interaction added masses-damping coefficients, 1x3 array 1-9a.	۲۶۹۲ ۵۵
Figure 6.25 Added masses domning coefficients, 1x4 error 1-5.	
Figure 0-55 Added masses-damping coefficients, 1x4 allay, 1–5a.	

Figure 6-36 Added masses-damping coefficients, 1x4 array, 1=8a.	100
Figure 6-3/ Interaction added masses-damping coefficients, 1x4 array, 1=5a	101
Figure 6-39 Added masses-damping coefficients, 2x2 array	102
Figure 6-40 Interaction added masses-damping coefficients, 2x2 array, 1=5a	103
Figure 6-41 Interaction added masses-damping coefficients, 2x2 array, 1=8a	105
Figure 7-1 Heaving motion for variable constant external damping. For constant spring va	alue,
for increasing damping value motion decreases. The damping value step is equal to 0.1 t	imes
the hydrodynamic damping in the resonance frequency of the isolated cylinder	107
for increasing damping value absorbed power increases. The damping value step is equ	alue, al to
0.1 the hydrodynamic damping in the resonance frequency of the isolated cylinder	108
Figure 7-3 Numbering of bodies inside an array (1)	110
Figure 7-4 Heaving motion, 1x2 array, $K_{33} = 0.0C_{33}$	112
Figure 7-5 Heaving motion, 1x2 array, $K_{33} = 0.5C_{33}$	113
Figure 7-6 Heaving motion, 1x3 array, $K_{33} = 0.0C_{33}$	114
Figure 7-7 Heaving motion, 1x3 array, $K_{33} = 0.5C_{33}$	115
Figure 7-8 Heaving motion, 1x4 array, $K_{33} = 0.0C_{33}$	116
Figure 7-9 Heaving motion, 1x4 array, $K_{33} = 0.5C_{33}$	117
Figure 7-10 Heaving motion, 2x2 array, $K_{33} = 0.0C_{33}$	118
Figure 7-11 Heaving motion, 2x2 array, $K_{33} = 0.5C_{33}$	119
Figure 7-12 Heaving motion, 1x5 and 2x2-3 arrays, $K_{33} = 0.0C_{33}$ , l=5a	120
Figure 7-13 Heaving motion, 1x5 and 2x2-3 arrays, $K_{33} = 0.5C_{33}$ , l=5a	121
Figure 7-14 Mean power absorbed per body, 1x2 array, $K_{33} = 0.0C_{33}$	123
Figure 7-15 Mean power absorbed per body, 1x2 array, $K_{33} = 0.5C_{33}$	124
Figure 7-16 Total mean power absorbed, 1x2 array	125
Figure 7-17 q-factor, 1x2 array.	125
Figure 7-18 Mean power absorbed per body, 1x3 array, $K_{33} = 0.0C_{33}$	126
Figure 7-19 Mean power absorbed per body, 1x3 array, $K_{33} = 0.5C_{33}$ .	127
Figure 7-20 Total mean power absorbed, 1x3 array	128
Figure 7-21 q-factor, 1x3 array.	128
Figure 7-22 Mean power absorbed per body, 1x4 array, $K_{33} = 0.0C_{33}$ .	129
Figure 7-23 Mean power absorbed per body, 1x4 array, $\kappa_{33} = 0.5C_{33}$ .	121
Figure 7-24 Total mean power absorbed, 1x4 array Figure 7-25 q-factor, 1x4 array	131
Figure 7-26 Mean power absorbed per body, 2x2 array, $K_{33} = 0.0C_{33}$	132
Figure 7-27 Mean power absorbed per body, 2x2 array, $K_{33} = 0.5C_{33}$	133
Figure 7-28 Total mean power absorbed, 2x2 array	134
Figure 7-29 q-factor, 2x2 array.	134
Figure 7-50 Numbering of bodies inside an array (2)	136
Figure 7-51 Wean absorbed power per body, 1x5 and 2x2-3 arrays, $K_{33} = 0.0C_{33}$ , 1=5a	13/
Figure 7-52 Mean absorbed power per body, 1x5 and 2x2-3 arrays, $K_{33} = 0.5C_{33}$ , 1=5a	138
Figure 7-55 Total mean power absorbed, 1x5 and 2x2-5 arrays, I=5a	139

Figure 7-34 q-factor, 1x5 and 2x2-3 arrays, 1=5a	139
Figure 7-35 Mean absorbed power per row, 1x4, 2x4 and 3x4 arrays, $K_{33} = 0.0C_{33}$ , 1=5a	140
Figure 7-36 Total mean power absorbed, 2x4 and 3x4 arrays, 1=5a.	141
Figure 7-37 q-factor, 2x4 and 3x4 arrays, 1=5a.	141
Figure 7-38 Mean absorbed power per row, 3x4 and 3x4-3-4 arrays, $K_{33} = 0.0C_{33}$ , l=5a	142
Figure 7-39 Total mean power absorbed, 3x4 and 3x4-3-4 arrays, 1=5a	143
Figure 7-40 q-factor, 3x4 and 3x4-3-4 arrays, l=5a.	143

#### **INTRODUCTION**

As the problem of climate change has become more and more significant in the long term mankind's energy production ability, the notion of renewable energy means was introduced. Even now there already exist a wide number of solar and aeolic energy applications and ocean energy ones are mainly still under development. As stated in (Chowdhury, et al., 2015), the term ocean renewable energy refers to a number of different forms such as tidal energy, thermal energy conversion, current power and wave energy. The most promising out of these is the wave energy since the local problems of the tidal energy are overcome, the thermal energy conversion has a low cost effectiveness and current power shows low energy density. Picture 1 showing the annual wave mean power around the globe indicates the large amount of power available however the applications are still limited. This is basically due to the complexity of the hydrodynamic interaction problems that take place and the optimization difficulty that arises. Furthermore, the neighboring operation of complex equipment with the water environment and the environmental issues that must be determined and overcome, as discussed in (Chowdhury, et al., 2015), make wave energy converters applications very challenging for engineers. Aim of this thesis is to approach the hydrodynamic interaction field and to figure the impact it has in the energy absorption efficiency. As presented in (Ilyas, et al., 2014), (Babarit, 2015) and (Day, et al., 2015) a number of different wave energy converters has been introduced. Specifically their performance comparison is presented in (Babarit, et al., 2011) The ones examined in this thesis are the six degrees of freedom heaving absorption cylindrical buoys positioned in a number of configurations.

In chapter 1, the linear hydrodynamic theory of a number of oscillating bodies in waves in the frequency domain is presented. It begins with the time domain non-linear formulation and continues with its evolution into an indirect boundary integral equation problem using complex representation. Finally the hybrid formulation is introduced in order to reduce computational cost and make possible the examination of multi body arrays.

In chapter 2, first the linear dynamic problem of a body oscillating under the excitation of waves is presented based on (Athanassoulis & Belibassakis, 2012) presentation. After that the expansion for any number of bodies, introducing the appropriate values and symbols, is presented. As stated previously all modes of motion have been modelled but the energy absorption is assumed to be performed only through the linear generator hydraulic power take off (PTO) mechanism of the heaving motion. Modelling of the PTO mechanism was based on (Ekstrom, et al., 2014) and (Li, et al., 2015). Comparison between linear and non-linear PTO mechanisms is presented in (Zhang, et al., 2014) and (Zhang & Yang, 2015) and novel PTO systems in (Xiao, et al., 2017).

In chapter 3 the numerical formulation of the hydrodynamic problem, using BEM, is introduced. After discretizing the boundary in a number of quadrilateral panels, the classic constant strength equations by Hess and Smith in (Katz & Plotkin, 2001) were used to calculate the induced velocity potential and normal derivative. Fortran 90 program freFLOW is based on this formulation and it was used in order to obtain all the hydrodynamic results presented. It is interesting to mention that techniques combining BEM with semi-analytic methods have been presented as in (Singh & Babarit, 2014).

In chapter 4, the developed program GAWEC is presented. This program produced all the necessary grids for the discretization of the boundary surfaces needed for the application of the BEM theory. The programming language was Matlab®2016a as there was no high performance need from a more efficient program language. The basic principle it was based on was the transfinite interpolation presented in (Gordon & Thiel, 1982) and (Dyken &

Floater, 2009). Total grid consisted of a number of structured grids. This program supports a large number of array configurations giving the ability for competitive ones to be compared. Before its development open source programs on the internet were sought but their interaction with other codes used was found problematic and the computational cost of the unstructured grids extruded could not be controlled.

In chapter 5, a number of BEM hydrodynamic results obtained was compared to semianalytic results. As stated in (Chowdhury, et al., 2015) in order to solve the boundary problem a number of methods is applied like the point absorption theory (PA), the multiple scattering scheme (MS) and plane wave theory (PW). The first assumes the bodies to be much smaller in comparison to the separating distance allowing the neglect of diffraction potential. The second introduced a scheme of iterations of multiple diffracted or radiated waves and the third one under the assumption of far positioned bodies uses plane waves to calculate the interactions. A comparison between those methods is presented in (Mavrakos & McIver, 1997). Some of those methods which refer to axisymmetric bodies were used by (Garrett, 1971), (Yeung, 1980), (Williams & Demirbilek, 1988), (Williams & Abul-Azm, 1989) and (Matsui & Tamaki, 1981). These are the sources of the hydrodynamic results comparing to. Similar results using the methods described above were also presented by (McIver & Evans, 1984), (McIver, 1984), (Kagemoto & Yue, 1986) and (Mavrakos & Koumoutsakos, 1987). All these results were produced at the 70's, 80's and 90's in order to support mainly the offshore industry with information about submersible column leg platforms of the oil industry and this explains the relatively small separating distances presented. Recently these semi analytic methods are used again in order to obtain results for large farms as in (Goteman, et al., 2015).

In chapter 6, a wide variety of BEM results obtained in this thesis is represented concerning array configurations of vertical circular cylinders. The aim of those results is to indicate the interaction phenomena observed in the calculation of the excitation forces and the hydrodynamic coefficients. In order to meet the wave energy converters arrays specifications, larger separating distances between the bodies were also considered than those examined for the comparisons of the previous chapter. Special reference was made to near trapping effect, introduced in (Evans & Porter, 1997), (Evans & Porter, 1999) and (Newman, 2001), which is the dominant factor affecting the results.

Last chapter of this thesis, chapter 7 was dedicated to results concerning body motions and energy absorption. This theme has risen great interest due to its complexity. In (Babarit, 2013) the park effect is presented which greatly affects the power production of a wave energy farm and therefore cost effectiveness. More specific observations in the constructive and destructive phenomena are presented in (Thomas, 2011) and (Chen, et al., 2016). Furthermore the idea of small clusters instead of large arrays is introduced in (Borgarino, et al., 2011). Apart from positioning, power absorption is highly dependent on the tuning of the PTO damping. A number of tuning strategies has been proposed in (Borgarino, et al., 2012), (Falnes, 2004) and (Wang, et al., 2016). In (Wang, et al., 2016), the tuning strategy proposed is also subject to motion constraints. Different approaches, like phase control with declutching and resonance control through PTO characteristics selection, are also presented in (Wahyudie, et al., 2016), (Song, et al., 2016) and (Cargo, et al., 2016).



Picture 1 Global annual mean wave power in kW/m, source (Chowdhury, et al., 2015).

#### **CHAPTER 1: FORMULATION OF THE HYDRODYNAMIC PROBLEM**

#### **1.1** Euler equations and potential theory

Describing the kinetic behavior of an incompressible fluid, Newton's second law of motion comes in the form of the **Navier-Stokes** equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \cdot \mathbf{v} = \mathbf{F} - \frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \Delta \cdot \mathbf{v}, \qquad \mathbf{x} \in D,$$
(1.1.1)

where

 $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$  is the domain velocity,

 $\rho$  is the fluid's density,

 $p = p(\mathbf{x}, t)$  is the domain's pressure,

 $\mu$  is the dynamic viscosity coefficient,

**F** are the global forces per mass unit.

These equations lead to the Euler ones for a non-viscous fluid,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \cdot \mathbf{v} = \mathbf{F} - \frac{1}{\rho} \nabla p, \qquad \mathbf{x} \in D.$$
(1.1.2)

Examining the kinematic part of the problem, assuming furthermore irrotational flow, the domain's velocity arises from the velocity potential  $\Phi = \Phi(\mathbf{x}, t)$  as

$$\mathbf{v} = \nabla \Phi, \qquad \mathbf{x} \in D. \tag{1.1.3}$$

Using this velocity expression, the integration of the Euler equations results to the **Bernoulli** equation

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left| \nabla \Phi \right|^2 + g z + \frac{p}{\rho} = const, \qquad (1.1.4)$$

which provides us with the hydrodynamic pressure

$$p_{DYN} = -\rho \; \frac{\partial \Phi}{\partial t} + \frac{1}{2} \rho \left| \nabla \Phi \right|^2. \tag{1.1.5}$$

Another necessary definition is that of the mass conservation law which is described by

$$\nabla \cdot \mathbf{v} = 0, \qquad \mathbf{x} \in D, \tag{1.1.6}$$

hence the mass conservation of an irrotational flow is represented by the Laplace equation

$$\Delta \Phi = 0, \quad \mathbf{x} \in D. \tag{1.1.7}$$

#### **1.2** Time and frequency domain formulations



Figure 1-1 Hydrodynamic problem's geometry

As stated in the previous paragraph, the velocity potential inside the fluid domain is represented using the Laplace equation. As for the boundary conditions, these are categorized in the following way.

• On the wetted surface of body *n*, the fluid's velocity must match the velocity of the rigid body (no insertion condition),

$$\frac{\partial \Phi}{\partial n} = \sum_{k=1}^{6} \frac{\partial \xi_k^n}{\partial t} n_k^n, \qquad \mathbf{x} \in \partial D_{B_n}.$$
(1.2.1)

• On the free surface, there exist two conditions. One kinematic condition which states that the velocity on the free surface is the same with the velocity of the fluid's surface particles,

$$\frac{D}{Dt}(z-\eta) = \frac{\partial\eta}{\partial t} + \frac{\partial\Phi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{\partial\eta}{\partial y} - \frac{\partial\Phi}{\partial z} = 0, \quad \mathbf{x} \in \partial D_F, \quad (1.2.2)$$

which after the linearization takes the form of

$$\frac{\partial \Phi}{\partial z} = \frac{\partial \eta}{\partial t}, \qquad \mathbf{x} \in \partial D_F : \mathbf{z} = \mathbf{0}.$$
(1.2.3)

Also one dynamic condition which states that the pressure on the surface must be constant,

$$\rho \frac{\partial \Phi}{\partial t} + \frac{1}{2} \rho \left| \nabla \Phi \right|^2 + \rho g \eta = 0, \qquad \mathbf{x} \in \partial D_F, \tag{1.2.4}$$

which after the linearization takes the form of

$$\frac{\partial \Phi}{\partial t} = g\eta, \qquad \mathbf{x} \in \partial D_F : \mathbf{z} = \mathbf{0}. \tag{1.2.5}$$

The combination of those conditions results in the following equation which acts on the mean position of the surface due to the linearization,

$$\frac{\partial^2 \Phi}{\partial t^2} \Phi + g \frac{\partial \Phi}{\partial z} = 0, \qquad \mathbf{x} \in \partial D_F : z = 0.$$
(1.2.6)

• On the sea bed the no insertion condition takes the form,

$$\frac{\partial \Phi}{\partial n} = 0, \quad \mathbf{x} \in \partial D_{\Pi}.$$
 (1.2.7)

• These boundary conditions must be fulfilled with another condition on the horizontal direction of the form,

$$\nabla \Phi = 0, \qquad r \to \infty. \tag{1.2.8}$$

In the frequency domain the system will be examined under the excitation of a regular wave of frequency  $\omega$ . Due to the linearity adopted, all values are regular too, of the same frequency. A much more comfortable way to work is using the complex form of the regular values. Euler's formula is denoted by

$$e^{j\omega t+\varphi} = \cos(j\omega t+\varphi) + j\sin(j\omega t+\varphi), \qquad (1.2.9)$$

hence

$$\operatorname{Re}\left(e^{j\omega t+\varphi}\right) = \cos(j\omega t+\varphi). \tag{1.2.10}$$

In this way for instance,

$$\Phi(\mathbf{x},t) = \Phi_{A}(\mathbf{x})\cos(\omega t + \varphi) = \operatorname{Re}\left(\Phi_{A}(\mathbf{x}) \cdot e^{j(\omega t + \varphi)}\right) \Longrightarrow$$
$$\Longrightarrow \operatorname{Re}\left(\Phi_{A}(\mathbf{x}) \cdot e^{j\varphi} \cdot e^{j\omega t}\right) = \operatorname{Re}\left(\overset{\circ}{\Phi}(\mathbf{x}) \cdot e^{j\omega t}\right), \quad \mathbf{x} \in D.$$
(1.2.11)

Similarly,

$$\eta(\mathbf{x},t) = \operatorname{Re}\left(\mathring{\eta}(\mathbf{x}) \cdot e^{j\omega t}\right), \quad \mathbf{x} \in \partial D_F,$$
(1.2.12)

$$\xi_k^n(t) = \operatorname{Re}\left(\mathring{\xi}_k^n \cdot e^{j\omega t}\right).$$
(1.2.13)

The use of complex variables also serves the transformation of the differential equations of motion into algebraic ones according to the following formula concerning time derivatives.

$$\frac{d}{dt}\cos(\omega t) = \operatorname{Re}(j\omega \cdot e^{j\omega t}), \qquad (1.2.14)$$

$$\frac{d^{2}}{dt^{2}}\cos(\omega t) = \operatorname{Re}\left(\left(j\omega\right)^{2} \cdot e^{j\omega t}\right)$$
(1.2.15)

The hydrodynamic problem can be summed then in the following equations for N bodies,

$$\Delta \mathring{\Phi}(\mathbf{x}) = \frac{\partial^2 \mathring{\Phi}}{\partial x^2} + \frac{\partial^2 \mathring{\Phi}}{\partial y^2} + \frac{\partial^2 \mathring{\Phi}}{\partial z^2} = 0, \quad \mathbf{x} \in D, \quad (1.2.16)$$

$$\frac{\partial \overset{\circ}{\Phi}}{\partial n} = \sum_{k=1}^{6} j \omega \cdot \overset{\circ}{\xi}_{k}^{1} \cdot n_{k}, \qquad \mathbf{x} \in \partial D_{B_{1}},$$
  

$$\vdots \qquad \vdots \qquad \vdots \qquad (1.2.17)$$

$$\frac{\partial \overset{\circ}{\Phi}}{\partial n} = \sum_{k=1}^{6} j \omega \cdot \overset{\circ}{\xi}_{k}^{N} \cdot n_{k}, \qquad \mathbf{x} \in \partial D_{B_{N}},$$

$$\frac{\omega^2}{g}\mathring{\Phi} - \frac{\partial\mathring{\Phi}}{\partial z} = 0, \qquad \mathbf{x} \in \partial \mathbf{D}_F : z = 0, \qquad (1.2.18)$$

$$\frac{\partial \Phi}{\partial n} = 0, \qquad \mathbf{x} \in \partial D_{\Pi}, \qquad (1.2.19)$$

In the frequency domain the wave's propagation is assumed to have fully expanded to an infinite distance from the body, so the horizontal boundary condition takes the form of the «Sommerfeld's radiation condition» for infinite water depth,

$$\left(\frac{\partial \mathring{\Phi}}{\partial r} + jk_0 \mathring{\Phi}\right) = O\left(\left(k_0 r\right)^{-3/2}\right), \quad k_0 r \to \infty,$$
(1.2.20)

where  $k_0$  is the real solution of the dispersion relation.

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The fact that the Laplace equation is a linear one, in addition to the linearized boundary conditions, allows us to assume the velocity potential  $\mathring{\Phi}(\mathbf{x})$  to be obtained from the superposition of potentials corresponding to simpler problems. Specifically the whole hydrodynamic problem can be split in the incident wave, diffraction and radiation problems,

$$\mathring{\Phi}(\mathbf{x}) = \mathring{\Phi}_{I}(\mathbf{x}) + \mathring{\Phi}_{D}(\mathbf{x}) + \sum_{n=1}^{N} \mathring{\Phi}_{R_{n}}(\mathbf{x}). \qquad (1.2.21)$$

The incident wave's potential is known from the bibliography as,

$$\mathring{\Phi}_{I}(\mathbf{x}) = \frac{jgA}{\omega} \cdot \frac{\cosh[k_{0}(z+h)]}{\cosh(k_{0}h)} \exp(-j\mathbf{k}_{0}\cdot\mathbf{R}), \qquad (1.2.22)$$

where

$$\mathbf{k}_0 = k_0 (\cos \beta \mathbf{i}_1 + \sin \beta \mathbf{i}_2),$$
$$\mathbf{R} = x \mathbf{i}_1 + y \mathbf{i}_2 + 0 \mathbf{i}_3,$$

 $\beta$  is the incident wave's direction.

The diffraction potential describes the fluid's potential resulting from the wave scattered by the surface of the grounded rigid bodies. The radiation ones on the other hand, are the result of body n oscillating on its k direction assuming zero further movements as

$$\dot{\xi}_{l}^{n} = 1,$$

$$\dot{\xi}_{l}^{n} = 0, \quad l = \{1, 2, ..., 6\} - \{k\},$$

$$\dot{\xi}_{l}^{m} = 0, \quad l = \{1, 2, ..., 6\}, \quad m = \{1, 2, ..., N\} - \{n\}.$$
(1.2.23)

In order to represent the various problems in a unique way, the incident wave potential, the diffraction potential and the radiation ones are written in the following form,

$$\mathring{\Phi}_{I}(\mathbf{x}) = j\omega A \mathring{\Phi}_{0}(\mathbf{x}), \qquad \mathbf{x} \in D, \qquad (1.2.24)$$

$$\overset{\circ}{\Phi}_{D}(\mathbf{x}) = j\omega A \overset{\circ}{\Phi}_{d}(\mathbf{x}), \qquad \mathbf{x} \in D, \qquad (1.2.25)$$

$$\overset{\circ}{\Phi}_{R_{1}}(\mathbf{x}) = \sum_{k} j\omega \overset{\circ}{\xi}_{k}^{1} \overset{\circ}{\Phi}_{k}^{1}, \qquad \mathbf{x} \in D, \qquad k = 1, 2, \dots, 6,$$
  
$$\vdots \qquad \vdots \qquad \vdots \qquad (1.2.26)$$

$$\mathring{\Phi}_{R_N}(\mathbf{x}) = \sum_k j\omega \mathring{\xi}_k^N \mathring{\Phi}_k^N, \qquad \mathbf{x} \in D,$$

so the total potential is given by,

$$\mathring{\Phi}(\mathbf{x}) = j\omega A\left(\mathring{\Phi}_0(\mathbf{x}) + \mathring{\Phi}_d(\mathbf{x})\right) + \sum_{n=1}^N \sum_{k=1}^6 j\omega \mathring{\xi}_k^n \mathring{\Phi}_k^n, \qquad \mathbf{x} \in D. \quad (1.2.27)$$

The diffraction potential problem for each direction is obtained by solving the following boundary problem,

$$\Delta \mathring{\Phi}_{d}(\mathbf{x}) = \frac{\partial^{2} \mathring{\Phi}_{d}}{\partial x^{2}} + \frac{\partial^{2} \mathring{\Phi}_{d}}{\partial y^{2}} + \frac{\partial^{2} \mathring{\Phi}_{d}}{\partial z^{2}} = 0, \qquad \mathbf{x} \in D,$$
(1.2.28)

$$\frac{\partial \mathring{\Phi}_{d}}{\partial n} = n_{d}^{1}, \qquad \mathbf{x} \in \partial D_{B_{1}},$$
  

$$\vdots \qquad \vdots \qquad (1.2.29)$$

$$\frac{\partial \overset{\circ}{\Phi}_d}{\partial n} = n_d^N, \qquad \mathbf{x} \in \partial D_{B_N},$$

$$\frac{\omega^2}{g}\mathring{\Phi}_d - \frac{\partial \mathring{\Phi}_d}{\partial z} = 0, \qquad \mathbf{x} \in \partial D_F : \mathbf{z} = 0, \qquad (1.2.30)$$

$$\frac{\partial \mathring{\Phi}_d}{\partial n} = 0, \qquad \mathbf{x} \in \partial D_{\Pi}, \qquad (1.2.31)$$

$$\left(\frac{\partial \mathring{\Phi}_d}{\partial r} + jk_0 \cdot \mathring{\Phi}_d\right) = 0\left(\left(k_0 \cdot r\right)^{-3/2}\right), \qquad k_0 r \to \infty, \qquad (1.2.32)$$

where

$$n = 1, 2, ..., N,$$
  $n_d^n = -\frac{\partial \overset{\circ}{\Phi}_0}{\partial x} n_1^n - \frac{\partial \overset{\circ}{\Phi}_0}{\partial y} n_2^n - \frac{\partial \overset{\circ}{\Phi}_0}{\partial z} n_3^n.$ 

The radiation problem of body n oscillating in the k direction is defined as,

$$\Delta \mathring{\Phi}_{k}^{n}(\mathbf{x}) = \frac{\partial^{2} \mathring{\Phi}_{k}^{n}}{\partial x^{2}} + \frac{\partial^{2} \mathring{\Phi}_{k}^{n}}{\partial y^{2}} + \frac{\partial^{2} \mathring{\Phi}_{k}^{n}}{\partial z^{2}} = 0, \quad \mathbf{x} \in D, \quad (1.2.33)$$

$$\frac{\partial \mathring{\Phi}_k^n}{\partial n} = n_k^n, \qquad \mathbf{x} \in \partial D_{B_n}, \qquad (1.2.34)$$

$$\frac{\partial \overset{\circ}{\Phi}_{k}^{n}}{\partial n} = 0, \qquad \mathbf{x} \in \partial D_{B_{m}} : m = \{1, 2, ..., N\} - \{n\}, \qquad (1.2.35)$$

$$\frac{\omega^2}{g} \mathring{\Phi}_k^n - \frac{\partial \mathring{\Phi}_k^n}{\partial z} = 0, \qquad \mathbf{x} \in \partial \mathbf{D}_F : \mathbf{z} = \mathbf{0}, \qquad (1.2.36)$$

$$\frac{\partial \mathring{\Phi}_{k}^{n}}{\partial n} = 0, \qquad \mathbf{x} \in \partial D_{\Pi}, \qquad (1.2.37)$$

$$\left(\frac{\partial \mathring{\Phi}_{k}^{n}}{\partial r} + jk_{0} \cdot \mathring{\Phi}_{k}^{n}\right) = 0\left(\left(k_{0} \cdot r\right)^{-3/2}\right), \qquad k_{0}r \to \infty.$$
(1.2.38)

As a result of the aforementioned representation, the potentials are functions of the following factors,

$$\mathring{\Phi}_{k}^{n}(\mathbf{x}) = \mathring{\Phi}_{k}^{n}(\mathbf{x};\partial D_{B_{n}},h,\omega), \qquad (1.2.39)$$

$$\overset{\circ}{\Phi}_{d}\left(\mathbf{x}\right) = \overset{\circ}{\Phi}_{d}\left(\mathbf{x};\partial D_{B_{n}},h,\omega,\beta\right), \qquad (1.2.40)$$

$$n = 1, 2, ..., N, k = 1, 2, ..., 6.$$

Assuming that the potentials are known, the hydrodynamic forces and moments on body p in the *i* direction are given using the linearized Bernoulli equation in the form,

$$F_{i}^{p} = \iint_{\partial D_{B_{p}}} p^{HYD} \mathbf{x}, t \ n_{i}^{p} dS_{B_{p}} = -\rho \iint_{\partial D_{B_{p}}} \dot{\Phi} n_{i}^{p} dS_{B_{p}},$$
  
 $i = 1, 2, ..., 6, \ p = 1, 2, ..., N.$ 
(1.2.41)

Using the complex representation and the previously obtained division of the total potential the hydrodynamic forces are given by the following components,

$$\overset{\circ}{F}_{i}^{p} = \overset{\circ}{X}_{0i}^{p} + \overset{\circ}{X}_{di}^{p} + \sum_{q=1}^{N} \sum_{j=1}^{6} \overset{\circ}{X}_{ij}^{pq}, \qquad (1.2.42)$$

where

$$\overset{\circ}{X}_{0i}^{p} = -(j\omega)^{2} \rho A \iint_{\partial D_{B_{p}}} \overset{\circ}{\Phi}_{0} n_{i}^{p} dS_{B_{p}}$$
(1.2.43)

are Froude-Krylov forces & moments on body p in direction i.

$$\overset{\circ}{X}_{di}^{p} = -(j\omega)^{2} \rho A \iint_{\partial D_{B_{p}}} \overset{\circ}{\Phi}_{d} n_{i}^{p} dS_{B_{p}}$$
(1.2.44)

are diffraction forces & moments on body p in direction i.

$$\overset{\circ}{X}_{ij}^{pq} = -(j\omega)^2 \rho \overset{\circ}{\xi}_{j}^{q} \iint_{\partial D_{B_p}} \overset{\circ}{\Phi}_{j}^{q} n_i^{p} dS_{B_p}$$
(1.2.45)

are radiation forces & moments on body p in direction i by the oscillation of body q in j direction.

Especially the radiation forces and moments can be formulated in the form,

$$\overset{\,\,{}_{}_{}}{X}_{ij}^{\,\,pq} = -(j\omega)^2 \, \overset{2}{\xi}_{j}^{\,q} \, \overset{\,\,{}_{}_{}_{}}{\Pi}_{ij}^{\,\,pq}, \qquad (1.2.46)$$

where

$$\mathring{\Pi}_{ij}^{pq} = \rho \iint_{\partial D_{B_p}} \mathring{\Phi}_j^q n_i^p dS_{B_p} = A_{ij}^{pq}(\omega) + \frac{1}{j\omega} B_{ij}^{pq}(\omega)$$
(1.2.47)

are the so called hydrodynamic coefficients. Specifically  $A_{ij}^{pq}$  and  $B_{ij}^{pq}$  denote the hydrodynamic masses and damping coefficients respectively.

#### 1.3 Indirect boundary integral equations formulation

#### 1.3.1 Single layer distribution

In this formulation the unknown potential is represented as the superposition of continuously distributed singular potentials (sources or dipoles) on the boundary of the fluid domain. If the singular potential is a simple (free space) source then the distribution is called a **single layer** distribution, while if the potential is a dipole it is called a **double-layer** distribution. It is a necessity for the formulation to stand the strengths to be bounded. Starting from the single layer distribution, free space source definition is given.

$$F(\mathbf{x}|\boldsymbol{\xi}) = \frac{1}{4\pi} \frac{1}{\|\mathbf{x}-\boldsymbol{\xi}\|} = \frac{1}{4\pi} \frac{1}{\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2}}, \ \mathbf{x} \in \mathbb{R}^3 - \{\boldsymbol{\xi}\}. \ (1.3.1)$$

The velocity potential  $\overset{\circ}{\Phi}(\mathbf{x})$  in a bounded domain *D* with boundary  $\partial D$  is assumed to be expressed as

$$\overset{\circ}{\Phi}(\mathbf{x}) = \iint_{\partial D} \overset{\circ}{\sigma}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{1}{\|\mathbf{x} - \boldsymbol{\xi}\|} dS_{\boldsymbol{\xi}}, \quad \mathbf{x} \in D,$$
(1.3.2)

where  $\sigma(\xi)$  is an appropriate continuous source-strength function defined over the boundary  $\partial D$ , to be determined. As known from the theory, the velocity inside the domain D is given by

$$\mathbf{v}(\mathbf{x}) = \nabla \overset{\circ}{\Phi}(\mathbf{x}) = \nabla \iint_{\partial D} \overset{\circ}{\sigma}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{1}{\|\mathbf{x} - \boldsymbol{\xi}\|} dS_{\boldsymbol{\xi}} =$$
$$= \iint_{\partial D} - \overset{\circ}{\sigma}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{\mathbf{x} - \boldsymbol{\xi}}{\|\mathbf{x} - \boldsymbol{\xi}\|^3} dS_{\boldsymbol{\xi}}, \ \mathbf{x} \in D.$$
(1.3.3)

The velocity component in the direction of any unit vector  $\mathbf{n}$  is given by

$$\frac{\partial \mathring{\Phi}(\mathbf{x})}{\partial \mathbf{n}} = \nabla \mathring{\Phi}(\mathbf{x}) \cdot \mathbf{n} = \iint_{\partial D} - \overset{\circ}{\sigma}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x} - \boldsymbol{\xi}) \cdot \mathbf{n}}{\|\mathbf{x} - \boldsymbol{\xi}\|^3} dS_{\boldsymbol{\xi}}, \ \mathbf{x} \in D.$$
(1.3.4)

Since the formulation must be fitted on boundary conditions, the potential value and its normal derivative must be found for a point  $\mathbf{x}_s$  which lies on the boundary. Since representations (1.3.2) and (1.3.4) become singular on the boundary points  $\mathbf{x}_s$ , a special treatment is needed.



Figure 1-2 Inclusion of point  $\mathbf{x}_{s}$ , in domain D.

More specific, as presented in (Power & Wrobel, 1995), the boundary point  $\mathbf{x}_s$  is included in the inner domain D by changing locally the boundary, using a small hemisphere of radius  $\mathcal{E}$  around that point. The values of  $\mathring{\Phi}(\mathbf{x}_s)$  and  $\partial \mathring{\Phi}(\mathbf{x}_s) / \partial \mathbf{n}(\mathbf{x}_s)$  are obtained by calculating the limits as  $\mathcal{E} \to 0$ . Specifically,

$$\overset{\circ}{\Phi}(\mathbf{x}_{s}) = \iint_{\partial D} \overset{\circ}{\sigma}(\xi) \frac{1}{4\pi} \frac{1}{\|\mathbf{x}_{s} - \xi\|} dS_{\xi}, \quad \mathbf{x}_{s} \in \partial D,$$
(1.3.5)

and

$$\frac{\partial \overset{\circ}{\Phi}(\mathbf{x}_{s})}{\partial \mathbf{n}(\mathbf{x}_{s})} = \frac{\overset{\circ}{\sigma}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \overset{\circ}{\sigma}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}}, \ \mathbf{x}_{s} \in \partial D.$$
(1.3.6)

Here, it must be denoted that these values correspond to the approach of the boundary  $\partial D$  from inside the domain D. If the approach takes place from the outside domain then,

$$\overset{\circ}{\Phi}(\mathbf{x}_{s})\Big|_{outside} = \iint_{\partial D} \overset{\circ}{\sigma}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{1}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|} dS_{\boldsymbol{\xi}}, \quad \mathbf{x}_{s} \in \partial D,$$
(1.3.7)

and

$$\frac{\partial \overset{\circ}{\Phi}(\mathbf{x}_{s})}{\partial \mathbf{n}(\mathbf{x}_{s})}\bigg|_{outside} = -\frac{\overset{\circ}{\sigma}(\mathbf{x}_{s})}{2} + \iint_{\partial D} -\overset{\circ}{\sigma}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}}, \ \mathbf{x}_{s} \in \partial D.$$
(1.3.8)

From the above equations it is clear that the velocity potential of a single layer distribution is continuous through the boundary surface, something that does not apply to the normal derivative. There is a jump in its value on the boundary, given by  $\overset{\circ}{\sigma}(\mathbf{x}_s)$ .

#### 1.3.2 Double layer distribution

As described previously, a double layer distribution corresponds to dipoles in the direction of the boundary. Such a dipole is denoted as,

$$M(\mathbf{x}|\boldsymbol{\xi}) = -\frac{1}{4\pi} \frac{(\mathbf{x}-\boldsymbol{\xi}) \cdot \mathbf{n}(\boldsymbol{\xi})}{\|\mathbf{x}-\boldsymbol{\xi}\|^{3}} = -\frac{1}{4\pi} \frac{(x_{1}-\xi_{1})n_{1}(\boldsymbol{\xi}) + (x_{2}-\xi_{2})n_{2}(\boldsymbol{\xi}) + (x_{3}-\xi_{3})n_{3}(\boldsymbol{\xi})}{\sqrt{(x_{1}-\xi_{1})^{2} + (x_{2}-\xi_{2})^{2} + (x_{3}-\xi_{3})^{2}}}, \ \mathbf{x} \in \mathbb{R}^{3} - \{\boldsymbol{\xi}\}.$$
(1.3.9)

The velocity potential is given by

$$\overset{\circ}{\Phi}(\mathbf{x}) = \iint_{\partial D} - \overset{\circ}{\mu}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x} - \boldsymbol{\xi}) \cdot \mathbf{n}(\boldsymbol{\xi})}{\|\mathbf{x} - \boldsymbol{\xi}\|^3} dS_{\boldsymbol{\xi}}, \ \mathbf{x} \in D.$$
(1.3.10)

Using the same methodology as for the single layer distribution, the value on point  $\mathbf{x}_s$  of the boundary  $\partial D$  is given by

$$\mathring{\Phi}(\mathbf{x}_{s}) = -\frac{\mathring{\mu}(\mathbf{x}_{s})}{2} + \iint_{\partial D} -\mathring{\mu}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\boldsymbol{\xi})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}}, \ \mathbf{x}_{s} \in \partial D.$$
(1.3.11)

If the approach of the boundary is from outside the domain D then,

$$\dot{\Phi}(\mathbf{x}_{s})\Big|_{outside} = \frac{\dot{\mu}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \dot{\mu}(\xi) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \xi) \cdot \mathbf{n}(\xi)}{\|\mathbf{x}_{s} - \xi\|^{3}} dS_{\xi}, \ \mathbf{x}_{s} \in \partial D, \quad (1.3.12)$$

hence there is also a value jump as in the case of the single layer normal derivative. As for the continuity of the normal derivative of the double layer representation approaching the boundary, even its existence is under examination. In (Power & Wrobel, 1995) it is stated that it exists not only if the distribution is continuous and bounded but also if certain continuity conditions apply. Assuming meeting the existence conditions for either inside-outside approaching limits, Lyapunov-Tauber theorem guarantees the existence and equality of the other.

$$\frac{\partial \overset{\circ}{\Phi}(\mathbf{x}_{s})}{\partial \mathbf{n}(\mathbf{x}_{s})} = \frac{\partial \overset{\circ}{\Phi}(\mathbf{x}_{s})}{\partial \mathbf{n}(\mathbf{x}_{s})} \bigg|_{outside}, \quad \mathbf{x}_{s} \in \partial D.$$
(1.3.13)

#### 1.3.3 Single layer, diffraction and radiation problems

Use of the mentioned single layer representation in the examined problem, requires the domain D to be bounded. As described, boundary  $\partial D$  consists of the free surface boundary  $\partial D_F$ , the bottom boundary  $\partial D_{\Pi}$  and the body boundaries  $\partial D_{B_n}$ . In order for it to close in the horizontal direction, a cylindrical surface is introduced of radius  $R_{\infty}$ ,  $(\partial D_{R_{\infty}})$  such that  $k_0 R_{\infty} \gg 1$ . Then the boundary conditions for the diffraction problem take the form,

$$\frac{\overset{\circ}{\sigma}_{d}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \overset{\circ}{\sigma}_{d}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}} = n_{d}^{1}(\mathbf{x}_{s}), \qquad \mathbf{x}_{s} \in \partial D_{B_{1}},$$
  
$$\vdots \qquad \vdots \qquad \vdots \qquad (1.3.14)$$

$$\frac{\overset{\circ}{\sigma_{d}}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \overset{\circ}{\sigma_{d}}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}} = n_{d}^{N}(\mathbf{x}_{s}), \qquad \mathbf{x}_{s} \in \partial D_{B_{N}},$$

$$\frac{\overset{\circ}{\sigma}_{d}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \overset{\circ}{\sigma}_{d}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}} = 0, \qquad \mathbf{x}_{s} \in \partial D_{\Pi}, (1.3.15)$$

$$\frac{\overset{\circ}{\sigma}_{d}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \overset{\circ}{\sigma}_{d}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}} - \frac{\omega^{2}}{g} \iint_{\partial D} \overset{\circ}{\sigma}_{d}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{1}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|} dS_{\boldsymbol{\xi}} = 0, \quad \mathbf{x}_{s} \in \partial D_{F},$$
(1.3.16)

$$\frac{\overset{\circ}{\sigma}_{d}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \overset{\circ}{\sigma}_{d}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}} + 
+ jk_{0} \iint_{\partial D} \overset{\circ}{\sigma}_{d}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{1}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|} dS_{\boldsymbol{\xi}} = 0\left(\left(k_{0} \cdot r\right)^{-3/2}\right), \begin{cases} \mathbf{x}_{s} \in \partial D_{R_{\infty}} \\ k_{0}R_{\infty} \gg 1 \end{cases},$$
(1.3.17)

where

$$n = 1, 2, ..., N,$$
  $n_d^n = -\frac{\partial \overset{\circ}{\Phi}_0}{\partial x} n_1^n - \frac{\partial \overset{\circ}{\Phi}_0}{\partial y} n_2^n - \frac{\partial \overset{\circ}{\Phi}_0}{\partial z} n_3^n.$ 

The boundary conditions for the radiation problems take the form,

$$\frac{\mathring{\sigma}_{k}^{n}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \mathring{\sigma}_{k}^{n}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}} = n_{k}^{n}(\mathbf{x}_{s}), \quad \mathbf{x}_{s} \in \partial D_{B_{n}}, (1.3.18)$$

$$\frac{\mathring{\sigma}_{k}^{n}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \mathring{\sigma}_{k}^{n}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}} = 0, \quad \mathbf{x}_{s} \in \partial D_{B_{m}}, (1.3.19)$$

$$\frac{\mathring{\sigma}_{k}^{n}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \mathring{\sigma}_{k}^{n}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}} = 0, \quad \mathbf{x}_{s} \in \partial D_{\Pi}, (1.3.20)$$

$$\frac{\mathring{\sigma}_{k}^{n}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \mathring{\sigma}_{k}^{n}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}} = 0, \quad \mathbf{x}_{s} \in \partial D_{\Pi}, (1.3.20)$$

$$\frac{\mathring{\sigma}_{k}^{n}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \mathring{\sigma}_{k}^{n}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}} - \left(\frac{\omega^{2}}{g} \iint_{\partial D} \mathring{\sigma}_{k}^{n}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{1}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|} dS_{\boldsymbol{\xi}} = 0, \quad \mathbf{x}_{s} \in \partial D_{F}, \quad (1.3.21)$$

$$\frac{\mathring{\sigma}_{k}^{n}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \mathring{\sigma}_{k}^{n}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}} + jk_{0} \iint_{\partial D} \mathring{\sigma}_{k}^{n}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{1}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|} dS_{\boldsymbol{\xi}} = 0\left(\left(k_{0} \cdot r\right)^{-3/2}\right), \begin{cases} \mathbf{x}_{s} \in \partial D_{R_{\infty}} \\ k_{0}R_{\infty} \gg 1 \end{cases},$$

$$(1.3.22)$$

where  $n = 1, 2, ..., N, m = \{1, 2, ..., N\} - \{n\}.$ 

#### 1.4 Hybrid, indirect boundary integral equations formulation

The set of equations representing the diffraction and radiation problems with indirect formulation are closed ones and therefore solvable. The difficulty arises from the fact that analytical solutions can be extracted only for some specific geometries. For arbitrary shaped bodies numerical schemes must be adopted to obtain solutions. Under numerical formulations distance  $R_{\infty}$  must be equal to a number of wave lengths in order for value  $k_0 R_{\infty}$  to increase sufficiently for the solution to converge. This approach however has a significant computer sources cost. In order to overcome the aforementioned problem, the total domain D is divided in the near field  $D_{R_*}$  and the far field  $D - D_{R_*}$  by a vertical cylindrical surface totally enclosing the bodies:  $\{x = (r, \theta, z) : r = R_*, 0 \le \theta \le 2\pi, -h \le z \le 0\}$ . The cost of this action is that the boundary condition in the horizontal direction no longer applies. The connection between the potential and its normal derivative is unknown and the system is not closed. In order for it to close a proper connection between the two values is sought. Specifically, the potential in the outer field is represented by its eigenfunction expansion in cylindrical coordinates  $x = (r, \theta, z)$  as,

$$\Phi^{*}(r,\theta,z) = \sum_{m=0}^{\infty} \frac{H_{m}^{(2)}(k_{0}r)}{H_{m}^{(2)}(k_{0}R_{*})} \cdot \left\{ \begin{array}{l} a_{0m}\cos m\theta + \\ + b_{0m}\sin m\theta \end{array} \right\} \cdot \frac{g_{0}(z)}{\left\| g_{0}(z) \right\|} + \\ + \sum_{n=1}^{\infty} \left( \sum_{m=0}^{\infty} \frac{K_{m}(k_{n}r)}{K_{m}(k_{n}R_{*})} \cdot \left\{ \begin{array}{l} a_{nm}\cos m\theta + \\ + b_{nm}\sin m\theta \end{array} \right\} \right) \cdot \frac{g_{n}(z)}{\left\| g_{n}(z) \right\|}, \end{array}$$
(1.4.1)

where

 $a_{nm}, b_{nm}$  are coefficients depending on the problem's geometry and the frequency  $\omega$ ,

 $H_m^{(2)}$  denotes the Hankel function of the second kind,

 $K_m$  denotes the modified Bessel function of the first kind,

 $g_0(z)$  is the propagating mode's vertical eigenfunction of the Sturm-Liouville problem,

 $g_{n}(z)$  are the evanescent mode's vertical eigenfunctions of the Sturm-Liouville problem,

$$\begin{cases} \frac{g_0(z)}{\|g_0(z)\|} = \frac{\cosh[k_0(z+h)]}{\sqrt{\frac{1}{2}\left(1 + \frac{\sinh 2k_0h}{2k_0h}\right)}}, & \text{non deep water case} \\ \frac{g_0(z)}{\|g_0(z)\|} = \frac{e^{k_0 z}}{\sqrt{\frac{1}{2}\frac{1}{k_0h}}}, & \text{deep water case} \\ \frac{g_n(z)}{\|g_n(z)\|} = \frac{\cos[k_n(z+h)]}{\sqrt{\frac{1}{2}\left(1 + \frac{\sin 2k_nh}{2k_nh}\right)}}, & n = 1, 2, \dots \end{cases}$$
(1.4.2)

 $k_0, k_n$  are the eigenvalues of the vertical Sturm-Liouville problem, which are solutions of equations,

$$\mu h = k_0 h \tanh(k_0 h)$$
 and  $-\mu h = k_n h \tanh(k_n h)$ ,  $n = 1, 2, ...$  (1.4.3)

After calculating the derivative of the special functions in the radial direction as

$$\frac{dH_m^{(2)}(k_0r)}{dR} = -k_0 H_{m+1}^{(2)}(k_0r) + \frac{m}{r} H_m^{(2)}(k_0r), \qquad (1.4.4)$$

$$\frac{dK_m(k_n r)}{dR} = -k_n K_{m+1}(k_n r) + \frac{m}{r} K_m(k_n r), \qquad (1.4.5)$$

the derivative of the outer domain potential in the radial direction is formed as,

$$\frac{\partial \Phi^{*}(r,\theta,z)}{\partial r} = \sum_{m=0}^{\infty} \left( \frac{-k_{0}H_{m+1}^{(2)}(k_{0}r) + \frac{m}{r}H_{m}^{(2)}(k_{0}r)}{H_{m}^{(2)}(k_{0}R_{*})} \right) \cdot \left\{ \begin{array}{l} a_{0m}\cos m\theta + \\ +b_{0m}\sin m\theta \end{array} \right\} \cdot \frac{g_{0}(z)}{\|g_{0}(z)\|} + \\ +\sum_{n=1}^{\infty} \left( \sum_{m=0}^{\infty} \left( \frac{-k_{n}K_{m+1}(k_{n}r) + \frac{m}{r}K_{m}(k_{n}r)}{K_{m}(k_{n}R_{*})} \right) \cdot \left\{ \begin{array}{l} a_{nm}\cos m\theta + \\ +b_{nm}\sin m\theta \end{array} \right\} \right) \cdot \frac{g_{n}(z)}{\|g_{n}(z)\|}. \end{array}$$
(1.4.6)

This potential representation refers not to a different potential than the one sought. It is only the form the potential takes from the boundary cylinder to the infinity and guarantees the compliance with the radiation condition. This means that both the potential and also its normal derivative on the boundary cylinder must be equal on either side  $(r \rightarrow R_* + 0 \text{ or } r \rightarrow R_* - 0)$ . This statement can be written as follows,

$$\lim_{r \to R_* \to 0} \Phi(r, \theta, z) = \lim_{r \to R_* \to 0} \Phi^*(r, \theta, z), \qquad (1.4.7)$$

$$\lim_{r \to R_* = 0} \frac{\partial \Phi(r, \theta, z)}{\partial r} = \lim_{r \to R_* + 0} \frac{\partial \Phi^*(r, \theta, z)}{\partial r}, \qquad (1.4.8)$$

$$\forall (\theta, z) \in [0, 2\pi) \times [-h, 0].$$

These conditions are referred to as matching conditions. The same way the cylindrical boundary is referred to as matching boundary  $\partial D_M$ . Appropriate handling of the outer domain representation, using these equations, enrich the problem with a closure condition involving the velocity potential and its normal derivative on the matching boundary  $\partial D_M$ . This handling requires use of the following functions orthogonality,

$$\{1, \cos m\theta, \sin m\theta\}.$$

and

$$\{g_n(z), n=0,1,2,...\}.$$

Specifically, first the value  $\cos m\theta \cdot \frac{g_0(z)}{\|g_0(z)\|}$  is multiplied on both sides of equation (1.4.1) and then follows integration on the domain  $[0, 2\pi) \times [-h, 0]$  giving the result,

$$\iint_{\partial D_M} \Phi^*(R_*, \theta, z) \cdot \cos m\theta \cdot \frac{g_0(z)}{\|g_0(z)\|} d\theta dz = \pi_m \cdot a_{0m}, \qquad (1.4.9)$$

where  $\pi_m = \begin{cases} 2\pi & \text{for } m = 0 \\ \pi & \text{for } m > 0 \end{cases}$ .

In a similar way using the value  $\sin m\theta \cdot \frac{g_0(z)}{\|g_0(z)\|}$ ,

$$\iint_{\partial D_M} \Phi^*(R_*, \theta, z) \cdot \sin m\theta \cdot \frac{g_0(z)}{\|g_0(z)\|} d\theta dz = \pi_m \cdot b_{0m}.$$
(1.4.10)

Following the same procedure for every coefficient  $a_{nm}$  and  $b_{nm}$  with n > 0, the result given is

$$\iint_{\partial D_M} \Phi^*(R_*, \theta, z) \cdot \cos m\theta \cdot \frac{g_n(z)}{\|g_n(z)\|} d\theta dz = \pi_m \cdot a_{nm}, \qquad (1.4.11)$$

$$\iint_{\partial D_M} \Phi^*(R_*,\theta,z) \cdot \sin m\theta \cdot \frac{g_n(z)}{\|g_n(z)\|} d\theta dz = \pi_m \cdot b_{nm}.$$
(1.4.12)

Introducing these relations in equation (1.4.6), the following representation is gained,

$$\frac{\partial \Phi^{*}(r,\theta,z)}{\partial r} = \sum_{m=0}^{\infty} \left\{ \frac{-k_{0}H_{m+1}^{(2)}(k_{0}r) + \frac{m}{r}H_{m}^{(2)}(k_{0}r)}{\pi_{m} \cdot H_{m}^{(2)}(k_{0}R_{*})} \right\} \cdot \frac{g_{0}(z)}{\|g_{0}(z)\|} \cdot \left\{ \left( \iint_{\partial D_{M}} \Phi^{*}(R_{*},\theta,z) \cdot \cos m\theta \cdot \frac{g_{0}(z)}{\|g_{0}(z)\|} d\theta dz \right) \cdot \cos m\theta + \left( \iint_{\partial D_{M}} \Phi^{*}(R_{*},\theta,z) \cdot \sin m\theta \cdot \frac{g_{0}(z)}{\|g_{0}(z)\|} d\theta dz \right) \cdot \sin m\theta \right\} + \\
+ \sum_{n=1}^{\infty} \left[ \sum_{m=0}^{\infty} \left( \frac{-k_{n}K_{m+1}(k_{n}r) + \frac{m}{r}K_{m}(k_{n}r)}{\pi_{m} \cdot K_{m}(k_{n}R_{*})} \right) \cdot \frac{g_{n}(z)}{\|g_{n}(z)\|} \cdot \\
\cdot \left\{ \left( \iint_{\partial D_{M}} \Phi^{*}(R_{*},\theta,z) \cdot \cos m\theta \cdot \frac{g_{n}(z)}{\|g_{n}(z)\|} d\theta dz \right) \cdot \cos m\theta + \left( \iint_{\partial D_{M}} \Phi^{*}(R_{*},\theta,z) \cdot \sin m\theta \cdot \frac{g_{n}(z)}{\|g_{n}(z)\|} d\theta dz \right) \cdot \sin m\theta \right\} \right].$$
(1.4.13)

This equation which is a Dirichlet to Neumann map (DtN) when applied for  $r = R_*$  becomes the boundary condition on the matching boundary.

As stated previously, the total domain D is divided in the near field  $D_{R_*}$  and the far field  $D - D_{R_*}$ . Adopting this definition, the solution sought refers to domain  $D_{R_*}$ . In order not to make the symbolism complex, the symbol D is used to describe the inner domain which is bounded by the free surface boundary  $\partial D_F$ , the bottom boundary  $\partial D_{\Pi}$ , the body boundaries

 $\partial D_{B_n}$  and the matching boundary  $\partial D_M \left( \partial D = \partial D_F \cup \partial D_\Pi \cup \sum_n \partial D_{B_n} \cup \partial D_M \right)$ . Then the diffraction problem boundary conditions are given by,

$$\frac{\overset{\circ}{\sigma_{d}}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \overset{\circ}{\sigma_{d}}(\xi) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \xi) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \xi\|^{3}} dS_{\xi} = n_{d}^{1}(\mathbf{x}_{s}), \qquad \mathbf{x}_{s} \in \partial D_{B_{1}},$$
  
$$\vdots \qquad \vdots \qquad \vdots \qquad (1.4.14)$$

$$\frac{\overset{\circ}{\sigma}_{d}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \overset{\circ}{\sigma}_{d}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}} = n_{d}^{N}(\mathbf{x}_{s}), \qquad \mathbf{x}_{s} \in \partial D_{B_{N}},$$

$$\frac{\overset{\circ}{\sigma}_{d}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \overset{\circ}{\sigma}_{d}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}} = 0, \qquad \mathbf{x}_{s} \in \partial D_{\Pi}, (1.4.15)$$

$$\frac{\overset{\circ}{\sigma}_{d}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \overset{\circ}{\sigma}_{d}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}} - \frac{\omega^{2}}{g} \iint_{\partial D} \overset{\circ}{\sigma}_{d}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{1}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|} dS_{\boldsymbol{\xi}} = 0, \quad \mathbf{x}_{s} \in \partial D_{F},$$
(1.4.16)

$$\frac{\overset{\circ}{\sigma}_{d}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \overset{\circ}{\sigma}_{d}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|^{3}} dS_{\boldsymbol{\xi}} = DtN\left(\iint_{\partial D} \overset{\circ}{\sigma}_{d}(\boldsymbol{\xi}) \frac{1}{4\pi} \frac{1}{\|\mathbf{x}_{s} - \boldsymbol{\xi}\|} dS_{\boldsymbol{\xi}}\right), \quad \mathbf{x}_{s} \in \partial D_{M},$$

$$(1.4.17)$$

where

$$n = 1, 2, ..., N,$$
  $n_d^n = -\frac{\partial \Phi_0}{\partial x} n_1^n - \frac{\partial \Phi_0}{\partial y} n_2^n - \frac{\partial \Phi_0}{\partial z} n_3^n.$ 

The radiation ones of body n oscillating in the k direction are,

$$\frac{\mathring{\sigma}_{k}^{n}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \mathring{\sigma}_{k}^{n}(\xi) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \xi) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \xi\|^{3}} dS_{\xi} = n_{k}^{n}(\mathbf{x}_{s}), \quad \mathbf{x}_{s} \in \partial D_{B_{n}}, (1.4.18)$$

$$\frac{\mathring{\sigma}_{k}^{n}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \mathring{\sigma}_{k}^{n}(\xi) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \xi) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \xi\|^{3}} dS_{\xi} = 0, \qquad \begin{cases} \mathbf{x}_{s} \in \partial D_{B_{m}}, (1.4.19) \\ m \neq n, \end{cases}, (1.4.19)$$

$$\frac{\mathring{\sigma}_{k}^{n}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \mathring{\sigma}_{k}^{n}(\xi) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \xi) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \xi\|^{3}} dS_{\xi} = 0, \qquad \mathbf{x}_{s} \in \partial D_{\Pi}, (1.4.20)$$

$$\frac{\mathring{\sigma}_{k}^{n}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \mathring{\sigma}_{k}^{n}(\xi) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \xi) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \xi\|^{3}} dS_{\xi} - \left( -\frac{\omega^{2}}{g} \iint_{\partial D} \mathring{\sigma}_{k}^{n}(\xi) \frac{1}{4\pi} \frac{\mathbf{n}}{\|\mathbf{x}_{s} - \xi\|^{3}} dS_{\xi} - \left( -\frac{\omega^{2}}{g} \iint_{\partial D} \mathring{\sigma}_{k}^{n}(\xi) \frac{1}{4\pi} \frac{\mathbf{n}}{\|\mathbf{x}_{s} - \xi\|^{3}} dS_{\xi} = 0, \qquad \mathbf{x}_{s} \in \partial D_{F}, \end{cases}$$

$$\frac{\mathring{\sigma}_{k}^{n}(\mathbf{x}_{s})}{2} + \iint_{\partial D} - \mathring{\sigma}_{k}^{n}(\xi) \frac{1}{4\pi} \frac{(\mathbf{x}_{s} - \xi) \cdot \mathbf{n}(\mathbf{x}_{s})}{\|\mathbf{x}_{s} - \xi\|^{3}} dS_{\xi} = 0, \qquad \mathbf{x}_{s} \in \partial D_{F}, \qquad (1.4.21)$$

$$= \operatorname{DtN}\left(\iint_{\delta D} \mathring{\sigma}_{k}^{n}(\xi) \frac{1}{4\pi} \frac{1}{\|\mathbf{x}_{s} - \xi\|} dS_{\xi}\right), \qquad \mathbf{x}_{s} \in \partial D_{M}, \qquad (1.4.22)$$

where

$$n = 1, 2, ..., N, m = \{1, 2, ..., N\} - \{n\}.$$

# CHAPTER 2: FORMULATION OF THE OSCILLATING DYNAMIC PROBLEM

#### 2.1 One body equations of motion



Figure 2-1 Free floating body

The problem of one rigid body moving in space-time is formulated using the translating and rotational momentum conservation theorem as,

$$\frac{d}{dt}\left(\int_{D_B}\rho_B\left(\mathbf{u}+\boldsymbol{\omega}\times\mathbf{r}\right)dD_B\right) = \mathbf{F},\tag{2.1.1}$$

$$\frac{d}{dt}\left(\int_{D_B} \rho_B \mathbf{r} \times (\mathbf{u} + \boldsymbol{\omega} \times \mathbf{r}) dD_B\right) + M \mathbf{u} \times \mathbf{u}_G = \mathbf{K}, \qquad (2.1.2)$$

where

$$M = \int_{D_B} \rho_B dD_B$$
 is the total body's mass,

**F** and **K** are the influence forces and moments respectively.

Using as a main argument the fact that the body's density remains constant in time derivations (Athanassoulis & Belibassakis, 2012) the selected reference system is a cartesian one with axes fixed on the body  $(Ox_1x_2x_3)$ . As a result equations (2.1.1) and (2.1.2) are transformed in,

$$\int_{D_B} \rho_B \frac{d}{dt} (\mathbf{u} + \mathbf{\omega} \times \mathbf{r}) dD_B = \mathbf{F}, \qquad (2.1.3)$$

$$\int_{D_B} \rho_B \frac{d}{dt} \left\{ \mathbf{r} \times \left( \mathbf{u} + \boldsymbol{\omega} \times \mathbf{r} \right) \right\} dD_B + M \mathbf{u} \times \mathbf{u}_G = \mathbf{K}.$$
(2.1.4)

The time derivatives shown inside the integrals are formed based on the body fixed system as,

$$\frac{d}{dt}(\mathbf{u}+\mathbf{\omega}\times\mathbf{r})=\partial_t\mathbf{u}-\mathbf{r}\times\partial_t\mathbf{\omega}+\mathbf{\omega}\times\mathbf{u}+\mathbf{\omega}(\mathbf{\omega}\times\mathbf{r})-\mathbf{r}(\mathbf{\omega}\times\mathbf{\omega}),\qquad(2.1.5)$$

$$\frac{d}{dt} \{ \mathbf{r} \times (\mathbf{u} + \boldsymbol{\omega} \times \mathbf{r}) \} = \mathbf{r} \times \partial_t \mathbf{u} + (\mathbf{r} \times \mathbf{r}) \partial_t \boldsymbol{\omega} - \mathbf{r} (\mathbf{r} \times \partial_t \boldsymbol{\omega}) + \mathbf{r} (\mathbf{u} \times \boldsymbol{\omega}) - \mathbf{u} (\mathbf{r} \times \boldsymbol{\omega}) + (\mathbf{r} \times \boldsymbol{\omega}) (\mathbf{r} \times \boldsymbol{\omega}),$$
(2.1.6)

where  $\partial_t$  refers to the time derivative based on the body fixed axes. Under the assumption of small amplitude oscillations the linearized form of the derivatives is,

$$\frac{d}{dt} (\mathbf{u} + \boldsymbol{\omega} \times \mathbf{r}) = \partial_t \mathbf{u} - \mathbf{r} \times \partial_t \boldsymbol{\omega}, \qquad (2.1.7)$$

$$\frac{d}{dt} \left\{ \mathbf{r} \times \left( \mathbf{u} + \boldsymbol{\omega} \times \mathbf{r} \right) \right\} = \mathbf{r} \times \partial_t \mathbf{u} + \left( \mathbf{r} \times \mathbf{r} \right) \partial_t \boldsymbol{\omega} - \mathbf{r} \left( \mathbf{r} \times \partial_t \boldsymbol{\omega} \right).$$
(2.1.8)

By inserting equations (2.1.7) and (2.1.8) in (2.1.3) and (2.1.4) respectively and furthermore by neglecting  $M\mathbf{u} \times \mathbf{u}_{G}$  as a second order factor, the result is,

$$\left(\int_{D_B} \rho_B dD_B\right) \partial_t \mathbf{u} - \left(\int_{D_B} \rho_B \mathbf{r} dD_B\right) \times \partial_t \boldsymbol{\omega} = \mathbf{F}, \qquad (2.1.9)$$

$$\left(\int_{D_{B}}\rho_{B}\mathbf{r}dD_{B}\right)\times\partial_{t}\mathbf{u}+\int_{D_{B}}\rho_{B}\left\{\left(\mathbf{r}\times\mathbf{r}\right)\partial_{t}\boldsymbol{\omega}-\mathbf{r}\left(\mathbf{r}\times\partial_{t}\boldsymbol{\omega}\right)\right\}dD_{B}=\mathbf{K},\qquad(2.1.10)$$

where

$$\int_{D_B} \rho_B dD_B = M \quad \text{is the total body's mass,}$$
(2.1.11)

$$\int_{D_B} \rho_B \mathbf{r} dD_B = \mathbf{J} = M \mathbf{R}_{\mathbf{G}}, \qquad (2.1.12)$$

$$\int_{D_{B}} \rho_{B} \left\{ \left( \mathbf{r} \times \mathbf{r} \right) \partial_{t} \boldsymbol{\omega} - \mathbf{r} \left( \mathbf{r} \times \partial_{t} \boldsymbol{\omega} \right) \right\} dD_{B} =$$

$$= \int_{D_{B}} \rho_{B} \left\{ \sum_{k=1}^{3} r_{k}^{2} \sum_{m=1}^{3} \partial_{t} \omega_{m} \hat{\mathbf{x}}_{m} - \sum_{k=1}^{3} r_{k} \hat{\mathbf{x}}_{k} \sum_{m=1}^{3} r_{m} \partial_{t} \omega_{m} \right\} dD_{B} = (2.1.13)$$

$$= \left( \sum_{k=1}^{3} I_{1k} \partial_{t} \omega_{k} \right) \hat{\mathbf{x}}_{1} + \left( \sum_{k=1}^{3} I_{2k} \partial_{t} \omega_{k} \right) \hat{\mathbf{x}}_{2} + \left( \sum_{k=1}^{3} I_{3k} \partial_{t} \omega_{k} \right) \hat{\mathbf{x}}_{3}$$

and

$$I_{kk} = \int_{D_B} \rho_B \left( \sum_{m=1}^3 r_m^2 - r_k^2 \right) dD_B, \qquad k = 1, 2, 3, \qquad (2.1.14)$$

$$I_{kk} = \int_{D_B} \rho_B r_k r_m dD_B, \qquad k \neq m, k = 1, 2, 3. \qquad (2.1.15)$$

After replacing (2.1.11)-(2.1.15) into (2.1.9) and (2.1.10)

$$\begin{aligned} M\dot{u}_{1} + & 0 + & 0 + & J_{3}\dot{\omega}_{2} - J_{2}\dot{\omega}_{3} = F_{1}, \\ 0 + M\dot{u}_{2} + & 0 - J_{3}\dot{\omega}_{1} + & 0 + J_{1}\dot{\omega}_{3} = F_{2}, \\ 0 + & 0 + M\dot{u}_{3} + J_{2}\dot{\omega}_{1} - J_{1}\dot{\omega}_{2} + & 0 = F_{3}, \\ 0 - J_{3}\dot{u}_{2} + J_{2}\dot{u}_{3} + I_{11}\dot{\omega}_{1} + I_{12}\dot{\omega}_{2} + I_{13}\dot{\omega}_{3} = K_{1}, \\ J_{3}\dot{u}_{1} + & 0 - J_{1}\dot{u}_{3} + I_{21}\dot{\omega}_{1} + I_{22}\dot{\omega}_{2} + I_{23}\dot{\omega}_{3} = K_{2}, \\ -J_{2}\dot{u}_{1} + J_{1}\dot{u}_{2} + & 0 + I_{31}\dot{\omega}_{1} + I_{32}\dot{\omega}_{2} + I_{33}\dot{\omega}_{3} = K_{3}. \end{aligned}$$

$$(2.1.16)$$

Under the assumption of linear theory, all the velocities u and  $\omega$  also the accelerations  $\dot{u}$  and  $\dot{\omega}$  based on the fixed on body axes can be linked to the motions based on an earth system as,

$$u_a = \dot{\xi}_a, \quad \omega_b = \dot{\xi}_b, \quad a = 1, 2, 3, \quad b = 4, 5, 6,$$
 (2.1.17)

$$\dot{u}_a = \ddot{\xi}_a, \quad \dot{\omega}_b = \ddot{\xi}_b, \quad a = 1, 2, 3, \quad b = 4, 5, 6.$$
 (2.1.18)

A matrix presentation for equations (2.1.16) using (2.1.17) and (2.1.18) is

$$\begin{pmatrix} M & 0 & 0 & 0 & J_{3} & J_{2} \\ 0 & M & 0 & J_{3} & 0 & J_{1} \\ 0 & 0 & M & J_{2} & J_{1} & 0 \\ 0 & J_{3} & J_{2} & I_{11} & I_{12} & I_{13} \\ J_{3} & 0 & J_{1} & I_{21} & I_{22} & I_{23} \\ -J_{2} & J_{1} & 0 & I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \ddot{\xi}_{1} \\ \ddot{\xi}_{2} \\ \ddot{\xi}_{3} \\ \ddot{\xi}_{4} \\ \ddot{\xi}_{5} \\ \ddot{\xi}_{6} \end{pmatrix} = \begin{pmatrix} F_{1} \\ F_{2} \\ F_{3} \\ K_{1} \\ K_{2} \\ K_{3} \end{pmatrix} = \begin{pmatrix} F_{1} \\ F_{2} \\ F_{3} \\ F_{4} \\ F_{5} \\ F_{6} \end{pmatrix}.$$
(2.1.19)

Giving attention to the right section of the above equations, special treatment was applied to the definition of the active forces. One basic component are the hydrostatic forces which act as reset forces. These forces are direct products of the motions with reverse sign as

$$F_{k,ST} = -\sum_{m=1}^{6} C_{km} \xi_m, \qquad (2.1.20)$$

where matrix C is given by

 $\begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$ ,  $\begin{vmatrix} x_1 & or & x_2 \\ are planes of symmetry. \end{vmatrix}$ 

Specifically the hydrostatic coefficients are denoted as:

$$C_{33} = \rho g S,$$
  

$$C_{34} = C_{43} = \rho g S_2,$$
  

$$C_{35} = C_{53} = -\rho g S_1,$$
  

$$C_{44} = Mg \mathbf{GB} + \rho g S_{22},$$
  

$$C_{45} = C_{54} = -\rho g S_{12},$$
  

$$C_{55} = Mg \mathbf{GB} + \rho g S_{11},$$
  
(2.1.22)

where B is the center of floatation.

The factors S are the waterline's surface area and its first and second moments as,

$$S_k = \int_{S} x_k dS, \qquad S_{km} = \int_{S} x_k x_m dS.$$
 (2.1.23)

Adding the hydrostatic forces denotation, equation (2.1.19) is formed as

or with alternative symbolism

$$M_{\tilde{z}}^{1} \cdot \xi_{\tilde{z}}^{1} + C_{\tilde{z}}^{1} \cdot \xi_{\tilde{z}}^{1} = F_{\tilde{z}}^{1}.$$
(2.1.25)
### 2.2 Any number of bodies equations of motion expansion

In the case of multiple bodies oscillating, motion of body n in the k direction is symbolized as  $\xi_k^n$ . Writing the modes of motion variables for all N bodies in a vector form

$$\begin{bmatrix} \xi_1^1, \xi_2^1, \xi_3^1, \xi_4^1, \xi_5^1, \xi_6^1, \dots, \xi_1^n, \xi_2^n, \xi_3^n, \xi_4^n, \xi_5^n, \xi_6^n, \dots, \xi_1^N, \xi_2^N, \xi_3^N, \xi_4^N, \xi_5^N, \xi_6^N \end{bmatrix}^T$$
(2.2.1)

and by inserting the symbols  $\xi^n$ ,  $\xi^n$ ,  $M^n$ ,  $F^n$ ,  $C^n$  for the local matrixes of body n, the total motion equations can be written in the next global matrix form

Introducing the complex form introduced in Chapter 1,

$$(j\omega)^{2} \begin{pmatrix} \mathcal{M}^{1} \dots \mathcal{Q}_{6x6} \dots \mathcal{Q}_{6x6} \\ \vdots & \vdots & \vdots \\ \mathcal{Q}_{6x6} \dots \mathcal{M}^{n} \dots \mathcal{Q}_{6x6} \\ \vdots & \vdots & \vdots \\ \mathcal{Q}_{6x6} \dots \mathcal{Q}_{6x6} \dots \mathcal{M}^{N} \end{pmatrix} \begin{pmatrix} \mathring{\xi}^{1} \\ \vdots \\ \mathring{\xi}^{n} \\ \vdots \\ \mathring{\xi}^{N} \\ \mathring{\xi}^{N} \end{pmatrix}^{+} \begin{pmatrix} \mathcal{Q}^{1} \dots \mathcal{Q}_{6x6} \dots \mathcal{Q}_{6x6} \\ \vdots & \vdots & \vdots \\ \mathcal{Q}_{6x6} \dots \mathcal{Q}_{6x6} \\ \vdots & \vdots & \vdots \\ \mathcal{Q}_{6x6} \dots \mathcal{Q}_{6x6} \dots \mathcal{Q}_{N} \end{pmatrix} \begin{pmatrix} \mathring{\xi}^{1} \\ \vdots \\ \mathring{\xi}^{n} \\ \vdots \\ \mathring{\xi}^{N} \\ \mathring{\xi}^{N} \end{pmatrix}^{-} = \begin{pmatrix} \mathring{F}^{1} \\ \vdots \\ \mathring{F}^{n} \\ \vdots \\ \mathring{\xi}^{N} \\ \mathring{\xi}^{N} \end{pmatrix}.$$
(2.2.3)

In order to analyze the hydrodynamic forces, it is necessary to introduce the interaction added mass and damping matrices between oscillating bodies,

$$\tilde{\mathcal{A}}^{pq} = \begin{pmatrix} A_{11}^{pq} & A_{12}^{pq} & A_{13}^{pq} & A_{14}^{pq} & A_{15}^{pq} & A_{16}^{pq} \\ A_{21}^{pq} & A_{22}^{pq} & A_{23}^{pq} & A_{24}^{pq} & A_{25}^{pq} & A_{26}^{pq} \\ A_{31}^{pq} & A_{32}^{pq} & A_{33}^{pq} & A_{34}^{pq} & A_{35}^{pq} & A_{36}^{pq} \\ A_{41}^{pq} & A_{42}^{pq} & A_{43}^{pq} & A_{44}^{pq} & A_{45}^{pq} & A_{46}^{pq} \\ A_{51}^{pq} & A_{52}^{pq} & A_{53}^{pq} & A_{54}^{pq} & A_{55}^{pq} & A_{56}^{pq} \\ A_{61}^{pq} & A_{62}^{pq} & A_{63}^{pq} & A_{64}^{pq} & A_{65}^{pq} & A_{66}^{pq} \end{pmatrix},$$

$$\mathcal{B}^{pq} = \begin{pmatrix} B_{11}^{pq} & B_{12}^{pq} & B_{13}^{pq} & B_{14}^{pq} & B_{15}^{pq} & B_{16}^{pq} \\ B_{21}^{pq} & B_{22}^{pq} & B_{23}^{pq} & B_{24}^{pq} & B_{25}^{pq} & B_{26}^{pq} \\ B_{31}^{pq} & B_{32}^{pq} & B_{33}^{pq} & B_{34}^{pq} & B_{35}^{pq} & B_{36}^{pq} \\ B_{41}^{pq} & B_{42}^{pq} & B_{43}^{pq} & B_{44}^{pq} & B_{45}^{pq} & B_{46}^{pq} \\ B_{51}^{pq} & B_{52}^{pq} & B_{53}^{pq} & B_{54}^{pq} & B_{55}^{pq} & B_{56}^{pq} \\ B_{61}^{pq} & B_{62}^{pq} & B_{63}^{pq} & B_{64}^{pq} & B_{65}^{pq} & B_{66}^{pq} \end{pmatrix}.$$

$$(2.2.5)$$

The global added mass and damping matrices are formed as

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} \mathbf{A}^{11} & \dots & \mathbf{A}^{1n} & \dots & \mathbf{A}^{1N} \\
\vdots & \vdots & \vdots & \vdots \\
\mathbf{A}^{n1} & \dots & \mathbf{A}^{nn} & \dots & \mathbf{A}^{nN} \\
\vdots & \vdots & \vdots & \vdots \\
\mathbf{A}^{N1} & \dots & \mathbf{A}^{Nn} & \dots & \mathbf{A}^{NN} \end{pmatrix}, \\
\end{aligned}$$

$$\mathbf{B} &= \begin{pmatrix} \mathbf{B}^{11} & \dots & \mathbf{B}^{1n} & \dots & \mathbf{B}^{1N} \\
\vdots & \vdots & \vdots & \vdots \\
\mathbf{B}^{n1} & \dots & \mathbf{B}^{nn} & \dots & \mathbf{B}^{nN} \\
\vdots & \vdots & \vdots & \vdots \\
\mathbf{B}^{N1} & \dots & \mathbf{B}^{Nn} & \dots & \mathbf{B}^{NN} \end{pmatrix}.$$
(2.2.6)

By analyzing the hydrodynamic forces, using equations (1.2.42), (1.2.46) and (1.2.47), the right part of equation (2.2.3) is formulated as

$$\begin{pmatrix} \mathring{F}^{1} \\ \vdots \\ \mathring{F}^{n} \\ \vdots \\ \mathring{F}^{N} \end{pmatrix} = \begin{pmatrix} \mathring{E}_{0}^{1} \\ \vdots \\ \mathring{E}_{0}^{n} \\ \vdots \\ \mathring{E}_{0}^{N} \end{pmatrix} + \begin{pmatrix} \mathring{E}_{d}^{1} \\ \vdots \\ \mathring{E}_{d}^{n} \\ \vdots \\ \mathring{E}_{d}^{N} \end{pmatrix} -$$

$$- (j\omega)^{2} \begin{cases} \begin{pmatrix} \mathring{A}^{11} & \dots & \mathring{A}^{1n} & \dots & \mathring{A}^{1N} \\ \vdots & \vdots & \vdots \\ \mathring{A}^{n1} & \dots & \mathring{A}^{nn} & \dots & \mathring{A}^{nN} \\ \vdots & \vdots & \vdots \\ \mathring{A}^{N1} & \dots & \mathring{A}^{Nn} & \dots & \mathring{A}^{NN} \end{pmatrix} + \frac{1}{j\omega} \begin{pmatrix} \mathring{B}^{11} & \dots & \mathring{B}^{1n} & \dots & \mathring{B}^{1N} \\ \mathring{B}^{n1} & \dots & \mathring{B}^{nn} & \dots & \mathring{B}^{nN} \\ \vdots & \vdots & \vdots & \vdots \\ \mathring{B}^{N1} & \dots & \mathring{B}^{Nn} & \dots & \mathring{B}^{NN} \end{pmatrix} \\ \end{pmatrix} \begin{pmatrix} \mathring{\xi}^{2} \\ \vdots \\ \mathring{\xi}^{2} \\ \vdots \\ \mathring{\xi}^{2} \\ \vdots \\ \mathring{\xi}^{2} \\ \vdots \\ \mathring{\xi}^{2} \\ \ddots \\ \mathring{\xi}^{2} \\ \vdots \\ \mathring{\xi}^{2} \\ \ddots \\ \ddots \\ \mathring{\xi}^{2} \\ \ddots \\ \mathring{\xi}^{2} \\ \ddots \\ \mathring{\xi}^{2} \\ \ddots \\ \ddots \\ \mathring{\xi}^{2} \\ \ddots \\ \ddots \\ \mathring{\xi}^{2} \\ \ddots \\ & \vdots \\ \mathring{\xi}^{2} \\ \ddots \\ & \vdots \\ \mathring{\xi}^{2} \\ \ddots \\ & \vdots \\ & \vdots \\ & & & & & \\ \end{pmatrix}$$

Equation (2.2.3) is then given by

$$(j\omega)^{2} \begin{cases} \left( \underbrace{M}_{2}^{1} \cdots \underbrace{Q}_{6x6} \cdots \underbrace{Q}_{6x6}_{0} \\ \vdots & \vdots & \vdots \\ \underbrace{Q}_{6x6} \cdots \underbrace{M}_{2}^{n} \cdots \underbrace{Q}_{6x6}_{0} \\ \vdots & \vdots & \vdots \\ \underbrace{Q}_{6x6} \cdots \underbrace{Q}_{6x6} \cdots \underbrace{M}_{2}^{N} \right)^{+} \begin{pmatrix} \underbrace{A}^{11} \cdots \underbrace{A}^{1n} \cdots \underbrace{A}^{1N}_{1} \\ \vdots & \vdots & \vdots \\ \underbrace{A}^{n1} \cdots \underbrace{A}^{nn} \cdots \underbrace{A}^{nN}_{1} \\ \vdots & \vdots & \vdots \\ \underbrace{A}^{n1} \cdots \underbrace{A}^{nn} \cdots \underbrace{A}^{nN}_{1} \\ \underbrace{A}^{nN} \cdots \underbrace{A}^{nN}_{1} \\ \underbrace{A}^{nN} \\ \vdots & \vdots & \vdots \\ \underbrace{B}^{n1} \cdots \underbrace{B}^{nn} \cdots \underbrace{B}^{nN}_{1} \\ \vdots \\ \underbrace{B}^{n1} \cdots \underbrace{B}^{nn} \cdots \underbrace{B}^{nN}_{1} \\ \vdots \\ \underbrace{B}^{n1} \cdots \underbrace{B}^{nn} \cdots \underbrace{B}^{nN}_{2} \\ \underbrace{A}^{nN}_{2} \\ \underbrace{A}^{n$$

or in a shorter form,

$$(j\omega)^{2} (\underline{M} + \underline{A}) \underline{\xi} + j\omega \,\underline{B} \underline{\xi} + \underline{C} \underline{\xi} = \underline{F}_{0} + \underline{F}_{d} . \qquad (2.2.10)$$

By solving this algebraic equation all the modes of motion of each body become available.

## 2.3 PTO modelling and quantification

### 2.3.1 Equations of motion with PTO modelled

There is a number of conceptual devices in the process of the wave energy absorption (Day, et al., 2015). In this thesis the heaving buoy farms are examined. The differentiation on the free floating bodies case by the heaving PTO mechanism is an additional damping and mooring force (Ekstrom, et al., 2014) and (Li, et al., 2015). Equation (2.2.10) is then changed to

$$(j\omega)^{2} (\underline{M} + \underline{A}) \underline{\xi} + j\omega \,\underline{B} \underline{\xi} + C \underline{\xi} = \underline{F}_{0} + \underline{F}_{d} + \underline{F}_{ext}, \qquad (2.3.1)$$

The PTO force which acts on body n is given in the regular time domain from

$$F_{ext,3}^{n}(\omega) = -B_{ext,33}^{nn} \dot{\xi}_{3}(\omega) - K_{33}\xi_{3}(\omega), \qquad (2.3.2)$$

and in the frequency domain from

$$\mathring{F}_{ext,3}^{n} = -j\omega B_{ext,33}^{nn} \mathring{\xi}_{3}^{n} - K_{33} \mathring{\xi}_{3}^{n}.$$
(2.3.3)

So in matrix form,

So for a total of N bodies,

Equations of motion (2.2.9) are then formed as,

$$\left\{ \left( j\omega \right)^{2} \left[ \begin{pmatrix} \mathcal{M}^{1} \dots \mathcal{Q}_{6x6} \dots \mathcal{Q}_{6x6} \\ \vdots & \vdots & \vdots \\ \mathcal{Q}_{6x6} \dots \mathcal{M}^{n} \dots \mathcal{Q}_{6x6} \\ \vdots & \vdots & \vdots \\ \mathcal{Q}_{6x6} \dots \mathcal{Q}_{6x6} \dots \mathcal{M}^{N} \end{pmatrix} + \begin{pmatrix} \mathcal{A}^{11} \dots \mathcal{A}^{1n} \dots \mathcal{A}^{1N} \\ \vdots & \vdots & \vdots \\ \mathcal{A}^{n1} \dots \mathcal{A}^{nm} \dots \mathcal{A}^{nN} \\ \vdots & \vdots & \vdots \\ \mathcal{A}^{N1} \dots \mathcal{A}^{Nn} \dots \mathcal{A}^{NN} \end{pmatrix} \right] +$$

$$+ j\omega \left[ \begin{pmatrix} \mathcal{B}^{11} & \dots & \mathcal{B}^{1n} & \dots & \mathcal{B}^{1N} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{B}^{n1} & \dots & \mathcal{B}^{nn} & \dots & \mathcal{B}^{nN} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{B}^{N1} & \dots & \mathcal{B}^{Nn} & \dots & \mathcal{B}^{NN} \end{pmatrix} + \begin{pmatrix} \mathcal{B}^{11}_{ext} & \dots & \mathcal{Q}_{6x6} & \dots & \mathcal{Q}_{6x6} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{Q}_{6x6} & \dots & \mathcal{Q}_{6x6} & \dots & \mathcal{Q}_{6x6} \\ \vdots & \vdots & \vdots \\ \mathcal{Q}_{6x6} & \dots & \mathcal{Q}_{6x6} & \dots & \mathcal{B}^{NN} \\ \end{pmatrix} \right] +$$

$$+ \begin{bmatrix} \begin{pmatrix} \mathcal{L}^{1} & \dots & \mathcal{Q}_{6x6} & \dots & \mathcal{Q}_{6x6} \\ \vdots & \vdots & \vdots \\ \mathcal{Q}_{6x6} & \dots & \mathcal{L}^{n} & \dots & \mathcal{Q}_{6x6} \\ \vdots & \vdots & \vdots \\ \mathcal{Q}_{6x6} & \dots & \mathcal{Q}_{6x6} & \dots & \mathcal{L}^{N} \end{bmatrix} + \begin{pmatrix} \mathcal{K}^{1} & \dots & \mathcal{Q}_{6x6} & \dots & \mathcal{Q}_{6x6} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{Q}_{6x6} & \dots & \mathcal{Q}_{6x6} & \dots & \mathcal{L}^{N} \end{bmatrix} \end{bmatrix} \begin{cases} \begin{pmatrix} \dot{\xi}^{1} \\ \vdots \\ \vdots \\ \dot{\xi}^{n} \\ \vdots \\ \dot{\xi}^{N} \end{pmatrix} = \begin{pmatrix} \dot{\mathcal{K}}_{0}^{1} \\ \vdots \\ \dot{\mathcal{K}}_{0}^{n} \\ \vdots \\ \dot{\mathcal{K}}_{0}^{N} \end{pmatrix} + \begin{pmatrix} \dot{\mathcal{K}}_{d}^{1} \\ \vdots \\ \dot{\mathcal{K}}_{d}^{n} \\ \vdots \\ \dot{\mathcal{K}}_{d}^{N} \end{pmatrix}.$$
(2.3.6)

## 2.3.2 PTO calculation

The energy absorbed by body n in the heaving mode of motion is given by,

$$P_{ext,3}^{n}(t \mid \omega) = F_{ext,3}^{n}(t \mid \omega) \cdot \dot{\xi}_{3}^{n}(t \mid \omega) = B_{ext,33}^{nn} \cdot \left(\dot{\xi}_{3}^{n}(t \mid \omega)\right)^{2}.$$
(2.3.7)

Since

$$\dot{\xi}_{3}^{n}(t \mid \omega) = \dot{\xi}_{3A}^{n}(\omega) \cdot \sin(\omega t + \varphi_{n}) = \dot{\xi}_{3A}^{n}(\omega) \cdot \cos(\omega t + \varphi_{n} + \frac{\pi}{2}), \qquad (2.3.8)$$

equation (2.3.7) leads to

$$P_{ext,3}^{n}(t \mid \omega) = B_{ext,33}^{nn} \cdot \dot{\xi}_{3A}^{2}(\omega) \cdot \cos^{2}(\omega t + \varphi_{n} + \frac{\pi}{2}).$$
(2.3.9)

Mean absorbed power is denoted as,

$$\overline{P_{ext,3}^{n}}(\omega) = B_{ext,33}^{nn} \cdot \dot{\xi}_{3A}^{2}(\omega) \cdot \overline{\cos^{2}(\omega t + \varphi_{n} + \frac{\pi}{2})} =$$

$$= B_{ext,33}^{nn} \cdot \dot{\xi}_{3A}^{2}(\omega) \cdot \frac{\overline{\cos(2\omega t + 2\varphi_{n} + \pi) + 1}}{2} = \frac{1}{2} B_{ext,33}^{nn} \cdot \dot{\xi}_{3A}^{2}(\omega).$$
(2.3.10)

Heaving velocity magnitude is given by

$$\dot{\xi}_{3A} = \omega \xi_{3A} = \left| j \omega \dot{\xi}_{3}^{n} \right|, \qquad (2.3.11)$$

therefore absorbed mean power is expressed as

$$\overline{P_{ext,3}^{n}}(\omega) = \frac{1}{2} B_{ext,33}^{nn} \cdot \omega^{2} \cdot |\dot{\xi}_{3}^{n}|^{2}.$$
(2.3.12)

The total mean power absorbed by N bodies is denoted as

$$\overline{P_{ext,3}^{tot}}(\omega) = \sum_{n=1}^{N} \overline{P_{ext,3}^{n}}(\omega)$$
(2.3.13)

#### **CHAPTER 3: NUMERICAL FORMULATION USING PLANE PANEL BEM**

#### 3.1 Influence matrices

In geometric first order approximation BEM, the boundary surfaces are discretized into a number of quadrilateral panels. In order to calculate the potential and its normal derivative value in a point P, which may lie inside the fluid domain or on its boundary, a sum of products must be calculated, each corresponding to a specific panel. As for the source distribution approximation, the simplest case is the zero-order one. In this case the strength is supposed to be constant, equal to the value on the center of the panel surface  $\mathbf{x}_c$ . The potential value on point  $\mathbf{x}_c$  is given by,

$$\overset{\circ}{\Phi}(\mathbf{x}_{c}) = \sum_{i=1}^{N_{TOT}} \overset{\circ}{\sigma}_{i} \cdot \iint_{\partial D_{i}} \frac{1}{4\pi \|\mathbf{x}_{c} - \boldsymbol{\xi}_{i}\|} dS_{\boldsymbol{\xi}_{i}}, \ \mathbf{x}_{c} \in \partial D_{p}, \ p = 1, 2, ..., N_{TOT}.$$
(3.1.1)

The same value on all panels can be written in matrix form as,

$$\boldsymbol{\Phi} = \mathbf{A}\mathbf{S} - \mathbf{G}_{N_{TOT} \times N_{TOT}} \cdot \boldsymbol{\sigma}_{N_{TOT} \times 1}.$$
(3.1.2)

Each line of  $\Phi$ ,  $\sigma$  corresponds to the potential value and source strength respectively on each panel and **AS** – **G** is the so called influence matrix. Each line of the influence matrix holds an integral of the sum in (3.1.1). These integrals are introduced for the quadrilateral, constant strength case by Hess and Smith in (Katz & Plotkin, 2001) as,

$$\begin{split} \iint_{\partial D_{i}} \frac{1}{4\pi \left\| \mathbf{x}_{c} - \mathbf{\xi}_{i} \right\|} dS_{\xi_{i}} &= \frac{1}{4\pi} \left\{ \left[ \frac{\left( x - x_{1} \right) \left( y_{2} - y_{1} \right) - \left( y - y_{1} \right) \left( x_{2} - x_{1} \right)}{d_{12}} \ln \frac{r_{1} + r_{2} + d_{12}}{r_{1} + r_{2} - d_{12}} + \right. \\ &+ \frac{\left( x - x_{2} \right) \left( y_{3} - y_{2} \right) - \left( y - y_{2} \right) \left( x_{3} - x_{2} \right)}{d_{23}} \ln \frac{r_{2} + r_{3} + d_{23}}{r_{2} + r_{3} - d_{23}} + \\ &+ \frac{\left( x - x_{3} \right) \left( y_{4} - y_{3} \right) - \left( y - y_{3} \right) \left( x_{4} - x_{3} \right)}{d_{34}} \ln \frac{r_{3} + r_{4} + d_{34}}{r_{3} + r_{4} - d_{34}} + \\ &+ \frac{\left( x - x_{4} \right) \left( y_{1} - y_{4} \right) - \left( y - y_{4} \right) \left( x_{1} - x_{4} \right)}{d_{41}} \ln \frac{r_{4} + r_{1} + d_{41}}{r_{4} + r_{1} - d_{41}} \right] - \\ &- \left| z \right| \left[ \tan^{-1} \left( \frac{m_{12}e_{1} - h_{1}}{zr_{1}} \right) - \tan^{-1} \left( \frac{m_{12}e_{2} - h_{2}}{zr_{2}} \right) + \\ &+ \tan^{-1} \left( \frac{m_{23}e_{2} - h_{2}}{zr_{2}} \right) - \tan^{-1} \left( \frac{m_{34}e_{4} - h_{4}}{zr_{4}} \right) + \\ &+ \tan^{-1} \left( \frac{m_{41}e_{4} - h_{4}}{zr_{4}} \right) - \tan^{-1} \left( \frac{m_{41}e_{1} - h_{1}}{zr_{1}} \right) \right] \right\}$$
(3.1.3)

and

$$d_{12} = \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2},$$
  

$$d_{23} = \sqrt{(x_3 - x_2)^2 - (y_3 - y_2)^2},$$
  

$$d_{34} = \sqrt{(x_4 - x_3)^2 - (y_4 - y_3)^2},$$
  

$$d_{41} = \sqrt{(x_1 - x_4)^2 - (y_1 - y_4)^2}$$
(3.1.4)

and

$$m_{12} = \frac{y_2 - y_1}{x_2 - x_1},$$

$$m_{23} = \frac{y_3 - y_2}{x_3 - x_2},$$

$$m_{34} = \frac{y_4 - y_3}{x_4 - x_3},$$

$$m_{41} = \frac{y_1 - y_4}{x_1 - x_4}$$
(3.1.5)

and

$$r_{k} = \sqrt{(x - x_{k})^{2} + (y - y_{k})^{2} + z^{2}},$$

$$e_{k} = (x - x_{k})^{2} + z^{2},$$

$$h_{k} = (x - x_{k})(y - y_{k}),$$

$$k = 1, 2, 3, 4.$$
(3.1.6)



Figure 3-1 Quadrilateral source strength element. Picture taken from (Katz & Plotkin, 2001)

It must be denoted that (x, y, z) are the pane's local coordinates of the calculation point which in this case is point  $\mathbf{x}_c$ . As for the velocity components in the local system of each panel, these are denoted as

$$u(\mathbf{x}_{c}) = \frac{\sigma}{4\pi} \left[ \frac{y_{2} - y_{1}}{d_{12}} \ln \frac{r_{1} + r_{2} + d_{12}}{r_{1} + r_{2} - d_{12}} + \frac{y_{3} - y_{2}}{d_{23}} \ln \frac{r_{2} + r_{3} + d_{23}}{r_{2} + r_{3} - d_{23}} + \frac{y_{4} - y_{3}}{d_{34}} \ln \frac{r_{3} + r_{4} + d_{34}}{r_{3} + r_{4} - d_{34}} + \frac{y_{1} - y_{4}}{d_{41}} \ln \frac{r_{4} + r_{1} + d_{41}}{r_{4} + r_{1} - d_{41}} \right],$$
(3.1.7)

$$v(\mathbf{x}_{c}) = \frac{\sigma}{4\pi} \left[ \frac{x_{1} - x_{2}}{d_{12}} \ln \frac{r_{1} + r_{2} + d_{12}}{r_{1} + r_{2} - d_{12}} + \frac{x_{2} - x_{3}}{d_{23}} \ln \frac{r_{2} + r_{3} + d_{23}}{r_{2} + r_{3} - d_{23}} + \frac{x_{3} - x_{4}}{d_{34}} \ln \frac{r_{3} + r_{4} + d_{34}}{r_{3} + r_{4} - d_{34}} + \frac{x_{4} - x_{1}}{d_{41}} \ln \frac{r_{4} + r_{1} + d_{41}}{r_{4} + r_{1} - d_{41}} \right],$$
(3.1.8)

$$w(\mathbf{x}_{c}) = -\frac{\sigma}{4\pi} \cdot \left[ \tan^{-1} \left( \frac{m_{12}e_{1} - h_{1}}{zr_{1}} \right) - \tan^{-1} \left( \frac{m_{12}e_{2} - h_{2}}{zr_{2}} \right) + \\ + \tan^{-1} \left( \frac{m_{23}e_{2} - h_{2}}{zr_{2}} \right) - \tan^{-1} \left( \frac{m_{23}e_{3} - h_{3}}{zr_{3}} \right) + \\ + \tan^{-1} \left( \frac{m_{34}e_{3} - h_{3}}{zr_{3}} \right) - \tan^{-1} \left( \frac{m_{34}e_{4} - h_{4}}{zr_{4}} \right) + \\ + \tan^{-1} \left( \frac{m_{41}e_{4} - h_{4}}{zr_{4}} \right) - \tan^{-1} \left( \frac{m_{41}e_{1} - h_{1}}{zr_{1}} \right) \right],$$
(3.1.9)

for  $\mathbf{x}_c \in \partial D_p$ ,  $p = 1, 2, ..., N_{TOT}$ .

If the calculation point is on the quadrilateral's surface, defining the self induced case, the normal velocity component on the panel takes the value

$$w(\mathbf{x}_c) = \frac{\sigma}{2}.$$
 (3.1.10)

If the calculation point is far from the panel the integral takes the form

$$\iint_{\partial D_{i}} \frac{1}{4\pi \left\| \mathbf{x}_{c} - \boldsymbol{\xi}_{i} \right\|} dS_{\boldsymbol{\xi}_{i}} = \frac{1}{4\pi} \cdot \frac{A_{i}}{\sqrt{\left(x - x_{0}\right)^{2} + \left(y - y_{0}\right)^{2} + z^{2}}}$$
(3.1.11)

where  $A_i$  corresponds to the *i* panel's surface area. The velocity components are given by,

$$u(\mathbf{x}_{c}) = \frac{1}{4\pi} \cdot \frac{\overset{\circ}{\sigma} A_{i}(x - x_{0})}{\left[\left(x - x_{0}\right)^{2} + \left(y - y_{0}\right)^{2} + z^{2}\right]^{3/2}},$$
(3.1.12)

$$v(\mathbf{x}_{c}) = \frac{1}{4\pi} \cdot \frac{\sigma A_{i}(y - y_{0})}{\left[\left(x - x_{0}\right)^{2} + \left(y - y_{0}\right)^{2} + z^{2}\right]^{3/2}},$$
(3.1.13)

$$w(\mathbf{x}_{c}) = \frac{1}{4\pi} \cdot \frac{\sigma^{\circ} A_{i} z}{\left[\left(x - x_{0}\right)^{2} + \left(y - y_{0}\right)^{2} + z^{2}\right]^{3/2}}.$$
 (3.1.14)

where  $x_0, y_0, z_0$  are the centroid coordinates of the influencing panel. After the transformation in the global system, the normal derivative of the potential in matrix form is,

$$\frac{\partial \mathbf{\Phi}}{\partial n} = \mathbf{AS}_{-\mathbf{th}} \mathbf{G}_{N_{TOT} \times N_{TOT}} \cdot \mathbf{\sigma}_{N_{TOT} \times 1}.$$
(3.1.15)

# 3.2 Numerical formulation of the indirect, hybrid problem

The hydrodynamic problem is then formed in the following form,

$$\mathbf{AS}_{N_{TOT} \times N_{TOT}} \cdot \mathbf{\sigma}_{N_{TOT} \times 1} = \mathbf{BS}_{N_{TOT} \times 1}.$$
(3.2.1)

Arrays  $\mathbf{AS}_{N_{TOT} \times N_{TOT}}$ ,  $\mathbf{BS}_{N_{TOT} \times 1}$  are filled according to the boundary conditions introduced in equations to and to as,

$$\frac{\partial \mathring{\Phi}_{k}^{n}}{\partial n} = n_{k}^{n} \mathbf{x} \in \partial D_{B_{n}} \rightarrow \mathbf{AS}(N_{1}:N_{2},1:N_{TOT}) = \mathbf{AS}_{-}\mathbf{thG}(N_{1}:N_{2},1:N_{TOT}),$$

$$\mathbf{BS}(N_{1}:N_{2},1) = \mathbf{n}_{j}, \quad j = 1,...,N_{B_{n}},$$

$$N_{1} = \sum_{z=1}^{z=n-1} N_{B_{z}} + 1, \quad N_{2} = \sum_{z=1}^{z=n-1} N_{B_{z}} + N_{B_{n}},$$
where  $N_{B_{n}}$ : body's *n* number of panels.
$$(3.2.2)$$

$$\frac{\partial \mathring{\Phi}_{k}^{n}}{\partial n} = 0 \quad \mathbf{x} \in \partial D_{B_{m}} : m = \{1, 2, ..., N\} - n \rightarrow \mathbf{AS}(N_{1} : N_{2}, 1 : N_{TOT}) = \mathbf{AS\_thG}(N_{1} : N_{2}, 1 : N_{TOT}),$$

$$\mathbf{BS}(N_{1} : N_{2}, 1) = \mathbf{n}_{j}, \quad j = 1, ..., N_{B_{n}},$$

$$N_{1} = 1, \quad N_{2} = \sum_{z=1}^{z=N} N_{B_{z}} \text{ excluding}$$

$$\text{lines from } \sum_{z=1}^{z=n-1} N_{B_{z}} + 1 \text{ to } \sum_{z=1}^{z=n-1} N_{B_{z}} + N_{B_{n}}.$$

$$(3.2.3)$$

$$\frac{\partial \mathring{\Phi}_{dork}^{n}}{\partial n} = 0 \quad \mathbf{x} \in \partial D_{\Pi} \quad \rightarrow$$

$$\mathbf{AS}(N_{1}:N_{2},1:N_{TOT}) = \mathbf{AS\_thG}(N_{1}:N_{2},1:N_{TOT}),$$

$$\mathbf{BS}(N_{1}:N_{2},1) = \mathbf{n}_{j}, \quad j = 1,...,N_{\Pi},$$

$$N_{1} = \sum_{z=1}^{z=N} N_{B_{z}} + 1, \quad N_{2} = \sum_{z=1}^{z=N} N_{B_{z}} + N_{\Pi}.$$
(3.2.4)

$$\frac{\partial \mathring{\Phi}_{dork}^{n}}{\partial z} - \frac{\omega^{2}}{g} \mathring{\Phi}_{dork}^{n} = 0 \quad \mathbf{x} \in \partial D_{F} : z = 0 \quad \rightarrow$$

$$\mathbf{AS}(N_{1}:N_{2},1:N_{TOT}) = \mathbf{AS\_thG} - \frac{\omega^{2}}{g} \mathbf{AS\_G} \quad (N_{1}:N_{2},1:N_{TOT}), \quad (3.2.5)$$

$$\mathbf{BS}(N_{1}:N_{2},1) = \mathbf{0},$$

$$N_{1} = \sum_{z=1}^{z=N} N_{B_{z}} + N_{\Pi} + 1, \quad N_{2} = \sum_{z=1}^{z=N} N_{B_{z}} + N_{\Pi} + N_{F}.$$

According to equation (1.4.13) the matching boundary condition is formed as,

$$\frac{\partial \Phi^*(r,\theta,z)}{\partial r} = \sum_{s=0}^{P_{max}} \left[ \sum_{m=0}^{M_{max}} SF_{ms} \cdot \frac{g_s(z)}{\|g_s(z)\|} \cdot \left( I_{1,ms} \cdot \cos m\theta + I_{2,ms} \cdot \sin m\theta \right) \right], \quad (3.2.6)$$

where

$$I_{1,ms} = \iint_{\partial D_{M}} \Phi^{*}(R_{*},\theta,z) \cdot \cos m\theta \cdot \frac{g_{s}(z)}{\|g_{s}(z)\|} d\theta dz, \quad \Rightarrow \qquad (3.2.7)$$

$$I_{2,ms} = \iint_{\partial D_{M}} \Phi^{*}(R_{*},\theta,z) \cdot \sin m\theta \cdot \frac{g_{s}(z)}{\|g_{s}(z)\|} d\theta dz,$$

$$\Rightarrow I_{1,ms} = \sum_{j=1}^{N_{M}} \Phi_{j}^{*} \cdot \iint_{\partial D_{M,j}} \cos m\theta \cdot \frac{g_{s}(z)}{\|g_{s}(z)\|} d\theta dz = \sum_{j=1}^{N_{M}} \Phi_{j}^{*} \cdot S_{1j,ms}, \qquad (3.2.8)$$

$$I_{2,ms} = \sum_{j=1}^{N_{M}} \Phi_{j}^{*} \cdot \iint_{\partial D_{M,j}} \sin m\theta \cdot \frac{g_{s}(z)}{\|g_{s}(z)\|} d\theta dz = \sum_{j=1}^{N_{M}} \Phi_{j}^{*} \cdot S_{2j,ms}, \qquad (3.2.9)$$

$$S_{1j,ms} = \int_{z_{1}}^{z_{2}} \int_{\theta_{1}}^{\theta_{2}} \sin m\theta d\theta \frac{g_{s}(z)}{\|g_{s}(z)\|} dz = -\frac{\sin m\theta}{m} \Big|_{\theta_{1}}^{\theta_{2}} \cdot IZ_{s}, \qquad (3.2.9)$$

$$\begin{cases} IZ_{0} = \frac{\sinh\left[k_{0}(z+h)\right]\Big|_{z_{1}}^{z_{2}}}{k_{0}\sqrt{\frac{1}{2}\left(1+\frac{\sinh 2k_{0}h}{2k_{0}h}\right)}}, & \text{non deep water case} \\ IZ_{0} = \frac{e^{k_{0}z}\Big|_{z_{1}}^{z_{2}}}{k_{0}\sqrt{\frac{1}{2}\frac{1}{k_{0}h}}}, & \text{deep water case} \\ IZ_{n} = \frac{\sin\left[k_{n}(z+h)\right]\Big|_{z_{1}}^{z_{2}}}{k_{n}\sqrt{\frac{1}{2}\left(1+\frac{\sin 2k_{n}h}{2k_{n}h}\right)}}, & n = 1, 2, \dots \end{cases}$$
(3.2.10)

$$SF_{m0} = \frac{-k_0 H_{m+1}^{(2)}(k_0 r) + \frac{m}{r} H_m^{(2)}(k_0 r)}{\pi_m \cdot H_m^{(2)}(k_0 R_*)},$$

$$SF_{mn} = \frac{-k_n K_{m+1}(k_n r) + \frac{m}{r} K_m(k_n r)}{\pi_m \cdot K_m(k_n R_*)}.$$
(3.2.11)

Equation (1.4.13) then is formulated as

$$\frac{\partial \Phi^*(r,\theta,z)}{\partial r} = \sum_{s=0}^{Pmax} \left[ \sum_{m=0}^{Mmax} SF_{ms} \cdot \frac{g_s(z)}{\|g_s(z)\|} \cdot \sum_{j=1}^{N_M} \left( \Phi_j^* \cdot \left( S_{1j,ms} \cdot \cos m\theta + S_{2j,ms} \cdot \sin m\theta \right) \right) \right]. \quad (3.2.12)$$

Using the representation of equations (3.1.1) and (3.1.15) for the potential and it's normal derivative the aforementioned equation referring to each matching boundary panel takes the form,

$$\mathbf{AS\_thG}(N_{1}:N_{2},1:N_{TOT})\cdot\mathbf{\sigma}_{N_{TOT}\times1} = \sum_{s=0}^{Pmax} \left[ \sum_{m=0}^{Mmax} SF_{ms} \cdot \frac{g_{s}(z)}{\|g_{s}(z)\|} \cdot \sum_{j=1}^{N_{M}} \left( \mathbf{AS\_G}(j,1:N_{TOT})\cdot\mathbf{\sigma}_{N_{TOT}\times1} \cdot \left( S_{1j,ms}\cdot\cos m\theta + S_{2j,ms}\cdot\sin m\theta \right) \right) \right] \Rightarrow$$
(3.2.13)  
$$\mathbf{AS\_thG}(N_{1}:N_{2},1:N_{TOT})\cdot\mathbf{\sigma}_{N_{TOT}\times1} = \sum_{s=0}^{Pmax} \left[ \sum_{m=0}^{Mmax} SF_{ms} \cdot \frac{g_{s}(z)}{\|g_{s}(z)\|} \cdot \sum_{j=1}^{N_{M}} \left( \mathbf{AS\_G}(j,1:N_{TOT}) \cdot \left( S_{1j,ms}\cdot\cos m\theta + S_{2j,ms}\cdot\sin m\theta \right) \right) \right] \cdot \mathbf{\sigma}_{N_{TOT}\times1} \Rightarrow$$
(3.2.14)

$$\Rightarrow \left\{ \mathbf{AS\_thG}(N_1:N_2,1:N_{TOT}) - \sum_{s=0}^{Pmax} \left[ \sum_{m=0}^{Mmax} SF_{ms} \cdot \frac{g_s(z)}{\|g_s(z)\|} \cdot \frac{\sum_{j=1}^{N_m} \left( \mathbf{AS\_G}(j,1:N_{TOT}) \cdot \left( S_{1j,ms} \cdot \cos m\theta + S_{2j,ms} \cdot \sin m\theta \right) \right) \right] \right\} \cdot \mathbf{\sigma}_{N_{TOT} \times 1} = 0.$$

$$(3.2.15)$$

So matrix AS and BS are filled with their final part as,

$$\mathbf{AS}(N_1:N_2,1:N_{TOT}) = \left\{ \mathbf{AS\_thG}(N_1:N_2,1:N_{TOT}) - \sum_{s=0}^{Pmax} \left[ \sum_{m=0}^{Mmax} SF_{ms} \cdot \frac{g_s(z)}{\|g_s(z)\|} \cdot \sum_{j=1}^{N_M} \left( \mathbf{AS\_G}(j,1:N_{TOT}) \cdot \left( S_{1j,ms} \cdot \cos m\theta + S_{2j,ms} \cdot \sin m\theta \right) \right) \right] \right\},$$
(3.2.16)

 $BS(N_1:N_2,1) = 0,$ 

$$N_1 = \sum_{z=1}^{z=N} N_{B_z} + N_{\Pi} + N_F + 1, \ N_2 = \sum_{z=1}^{z=N} N_{B_z} + N_{\Pi} + N_F + N_M$$

Calculation of matrices **AS** and **BS** leads to the source strength values of each problem, making the potentials and their normal derivatives available. Those potentials are functions of the following factors,

$$\mathring{\Phi}_{k}^{n}(\mathbf{x}) = \mathring{\Phi}_{k}(\mathbf{x}; \partial D_{B_{n}}, h, \omega, R_{*}, P_{max}, M_{max}, discretization), \qquad (3.2.17)$$

$$\overset{\circ}{\Phi}_{d}\left(\mathbf{x}\right) = \overset{\circ}{\Phi}_{d}\left(\mathbf{x};\partial D_{B_{n}},h,\omega,\beta,R_{*},P_{max},M_{max},discretization\right), \quad (3.2.18)$$

$$k = 1,2,...,6, \quad n = 1,2,...,N,$$

where *discretization* refers to the transferring of the continuous boundaries into their panel counterparts.

# 3.3 Numerical calculation of exciting forces and hydrodynamic coefficients

Numerical formulation of exciting forces based on equations (1.2.43) and (1.2.44), is as follows,

$$\overset{\circ}{X}_{0i}^{p} = -(j\omega)^{2} \rho A \iint_{\partial D_{B_{p}}} \overset{\circ}{\Phi}_{0} n_{i}^{p} dS_{B_{p}} = -(j\omega)^{2} \rho A \sum_{w=1}^{N_{B_{p}}} \iint_{\partial D_{B_{p,w}}} \overset{\circ}{\Phi}_{0} n_{i}^{p} dS_{B_{p}} = -(j\omega)^{2} \rho A \sum_{w=1}^{N_{B_{p}}} \overset{\circ}{\Phi}_{0}(\mathbf{x}_{c,w}) n_{i}^{p,w} S_{B_{p,w}},$$
(3.3.1)

$$\overset{\circ}{X}_{di}^{p} = -(j\omega)^{2} \rho A \iint_{\partial D_{B_{p}}} \overset{\circ}{\Phi}_{d} n_{i}^{p} dS_{B_{p}} = -(j\omega)^{2} \rho A \sum_{w=1}^{N_{B_{p}}} \iint_{\partial D_{B_{p,w}}} \overset{\circ}{\Phi}_{d} n_{i}^{p} dS_{B_{p}} = -(j\omega)^{2} \rho A \sum_{w=1}^{N_{B_{p}}} \overset{\circ}{\Phi}_{d} (\mathbf{x}_{c,w}) n_{i}^{p,w} S_{B_{p,w}}.$$
(3.3.2)

Radiation forces are also formulated in a similar way based on equation (1.2.47),

$$\overset{\circ}{\Pi}_{ij}^{pq} = \rho \iint_{\partial D_{B_p}} \overset{\circ}{\Phi}_{j}^{q} n_{i}^{p} dS_{B_p} = \rho \sum_{w=1}^{N_{B_p}} \iint_{\partial D_{B_{p,w}}} \overset{\circ}{\Phi}_{j}^{q} n_{i}^{p} dS_{B_p} =$$

$$= \rho \sum_{w=1}^{N_{B_p}} \overset{\circ}{\Phi}_{j}^{q} (\mathbf{x}_{c,w}) n_{i}^{p,w} S_{B_{p,w}}.$$

$$(3.3.3)$$

The added mass and damping coefficients are then given by

$$A_{ij}^{pq}(\omega) = \operatorname{Re}\left(\overset{\circ}{\Pi}_{ij}^{pq}\right),$$
  

$$B_{ij}^{pq}(\omega) = -\omega\operatorname{Im}\left(\overset{\circ}{\Pi}_{ij}^{pq}\right).$$
(3.3.4)

#### **CHAPTER 4: DEVELOPMENT OF A GRID GENERATION PROGRAM**

### 4.1 Transfinite interpolation

MATLAB®R2016a program GAWEC was developed in order to provide the necessary grid on the boundary surfaces, needed in order to apply the aforementioned hybrid numerical method. The boundary surfaces are divided into four types in accordance to the theory, body, free surface, sea bed and matching boundary type. Each of these types may consist of one or more parts numbered accordingly. The basic numerical tool used by this program is the transfinite interpolation in compliance to (Gordon & Thiel, 1982) and (Dyken & Floater, 2009) which was inspired from (Belibassakis, et al., 2016). Given the parametric four curves enclosing the planar surface  $\mathbf{c}_1(u)$ ,  $\mathbf{c}_2(v)$ ,  $\mathbf{c}_3(u)$ ,  $\mathbf{c}_4(v)$  any point inside this surface is given by,

$$\mathbf{S}(u,v) = (1-v)\mathbf{c}_{1}(u) + v\mathbf{c}_{3}(u) + (1-u)\mathbf{c}_{2}(v) + u\mathbf{c}_{4}(v) - \left[ (1-u)(1-v)\mathbf{P}_{1,2} + uv\mathbf{P}_{3,4} + u(1-v)\mathbf{P}_{1,4} + (1-u)v\mathbf{P}_{3,2} \right],$$
(4.1.1)

where  $P_{ab}$  is the point curves  $c_a$ ,  $c_b$  meet.

By discretizing the continuous parameters  $u, v \in [0,1]$  the nodes of the required structured grid are given.



Figure 4-1 Application of transfinite interpolation

# 4.2 Free surface grid

The free surface, which has a circular shape, is divided in two areas, the "inside" and "outside" one. The inside area includes the bodies whose waterline cuts the free surface, creating a number of holes.

## 4.2.1 Inside area

The basic idea applied in this area was to enclose each body in a rectangle of decided size which in program terms is called a "box". The number of panels in the azimuthal direction is controlled from the body's perimeter discretization selection and in the radial direction through the input selection. One fundamental requirement, for the proper function of the program, is the angle step to be a divisor of 90 degrees. It is not a necessity to also be a divisor of 45 degrees, as the program can handle an odd number of panels in a quadrant with proper distribution between the horizontal and vertical sides of the box.



Figure 4-2 Basic "box"

If the box is positioned on the corners of the inner area the box is rounded accordingly in order for the grid to be more homogeneous in terms of panel geometry and size.



Figure 4-3 Basic corner "box"

Another feature of the program is that the additional number of panels in two parallel sides of the box can be controlled in the case of slender bodies with high aspect ratio.



Figure 4-4 Slender body inside "box". No control of the panels, up. Control of the points, down.

The bodies are placed in an array system with rows and columns. The distances between rows and columns are controlled. If the size of the boxes and the distances are specific, some parts which are called "gaps" are created in order to close the inner surface.



Figure 4-5 Bodies array without "gaps"

Another feature of the program is that in whichever row it can be selected the bodies to be placed in a position which lies between the positions of the original columns. This position is controlled through a ratio of the original distance.



Figure 4-6 Bodies array with "gaps"



Figure 4-7 Bodies array with intermediate positioned column, without "gaps"



Figure 4-8 Bodies array with intermediate positioned column, with "gaps"

Also with the appropriate selection, each box can be left empty.



Figure 4-9 The previous array grid with an "empty box"

Concerning the number of panels in the gaps these are in part controlled directly and part from the number of panels in the boxes. The red ones, which in program terms are referred to as "vertical", are formed vertically from the number of panels in the vertical side of the box and horizontally through selection. The green ones, called "horizontal", have a vertical number of panels directly selected, and a horizontal in accordance with the number of panels of the horizontal box's side and the selection for the vertical gaps.

### 4.2.2 Outside area

This area's grid closes the free surface grid between the inner part and the outer circular limit. Basically it is formed in one part using the transfinite interpolation between the given boundary curves. The most complex boundary curve is the one that lies on the limit between the inside and outside areas. This curve is generally formed from four ellipsoidal curves and four lines if that is necessary. The number of panels and consequently the nodes on the curve parts are given by the ones of the inside part. The number of panels in the linear part is controlled independently.



Figure 4-10 Outside part

This boundary curve selection guarantees that there are no openings on the free surface grid.



Figure 4-11 Curved part between inside and outside parts

The number of nodes on the bottom curve determines the angle step of the outer part and on the matching boundary grid. A feature of the program is the ability to divide the outside area into multiple parts that have azimuthally a number of panels that is product of the initial number with the desirable values.



Figure 4-12 Outside part with panel split

The aforementioned panel division is done in such a way that no openings are created.



Figure 4-13 Panel division

**4.2.3** Total free surface grid and special cases The total free surface grid has the following general form.



Figure 4-14 Example of a total free surface grid

The program can also create either line or column array and single body grids.



Figure 4-15 Column array free surface grid



Figure 4-16 Line array free surface grid



Figure 4-17 One body free surface grid

# 4.3 Bodies grid

The available types of bodies are cylinders and ellipsoid solids. The necessary grids are available in one grid part for each body. The number of panels in the azimuthal direction are directly controlled. This also applies to the number of panels in the vertical direction and on the radial one in the bottom.



Figure 4-18 Cylinder body grid



Figure 4-19 Ellipsoid body grid

## 4.4 Matching boundary grid

The matching boundary is a cylindrical surface that encloses all the bodies. The simplest discretization available for this surface is to divide it into panels of constant height and angular width. This distribution is very wasteful in terms of computational memory for a lot of hydrodynamic problems. This comes as a result to the fact that a body without a big draught does not face any influence from panels far from it, in big depths. This defines no need for the grid to be dense on that depths. Another need is on the other hand, the panel's size distribution to be as homogeneous as possible. These requirements are met through the adoption of the geometric progression for the height of the panels. So these two options are available.



Figure 4-20 Matching boundary grid without use of geometric progression



Figure 4-21 Matching boundary grid with use of geometric progression

## 4.5 Sea bed grid

The construction of the sea bed grid follows the same method as the outside part of the free surface grid. By means of transfinite interpolation the starting point  $\mathbf{x}_0, \mathbf{y}_0$  is linked to the external boundary, a cycle with radius equal to that of the matching boundary's one. The number of panels in the azimuthal direction are controlled from the value calculated for the matching boundary. The option of panel division is also available for the sea bed grid.



Figure 4-22 Sea bed grid with panel division

# 4.6 Panel size specifications

Control of the number of panels, as mentioned in the previous paragraphs, leads to a specific dimensioning of them. In order to examine the arising dimensions, a number of distances is defined for the panels.

Free surface



Figure 4-23 Free surface panel dimensions

dbox <sub>rx</sub>	: horizontal distance in the radial direction
dbox <sub>ry</sub>	: vertical distance in the radial direction
dbox <sub>rd</sub>	: diagonal distance in the radial direction
dbox <sub>x</sub>	: horizontal distance on the box's perimeter
dboxy	: vertical distance on the box's perimeter
dout <sub>rx</sub>	: horizontal distance in the radial direction
dout <sub>ry</sub>	: vertical distance in the radial direction
dout <sub>rd</sub>	: diagonal distance in the radial direction
dout <sub>x</sub>	: horizontal distance on the inside area's perimeter
douty	: vertical distance on the inside area's perimeter
dgap <sub>vx</sub>	: horizontal distance of the vertical gaps
dgap <sub>v0x</sub>	: horizontal distance of the vertical gaps, in "intermediate" rows
dgap <sub>hy</sub>	: vertical distance of the horizontal gaps
dtnRst	: the distance on the matching boundary cycle

# Sea bed

: radial size of each panel

# Matching boundary

dh dz	<ul><li>vertical height of each panel, constant step</li><li>vertical height of each panel, geometric progression</li></ul>
Bodies	
drb <sub>1</sub> drb <sub>2</sub> dTb	<ul> <li>: horizontal distance on the body's bottom</li> <li>: vertical distance on the body's bottom</li> <li>: vertical height on the body's side</li> </ul>
Total grid	
dth <sub>b</sub> dth <sub>m</sub>	<ul><li>: the angle step on the creation of each body</li><li>: the angle step on the creation of the matching boundary</li></ul>

# CHAPTER 5: COMPARISON BETWEEN BEM AND SEMI-ANALYTIC HYDRODYNAMIC RESULTS

## 5.1 Introduction

The numerical scheme presented in chapter 3 is implemented in freFLOW (Manolas, 2015) which is a FORTRAN 90 program solving both the scattering, for a number of directions, and the radiation problems, for any number of bodies allowed by computer sources. Equipped with the provided grid nodes by GAWEC, freFLOW was used in order to obtain numerous results. In this chapter these results were compared with semi-analytic ones published by a number of authors who studied such problems, motivated by offshore industry interest. The problem geometry after the hybrid formulation adoption and the numbering of the bodies is represented in Figure 5-1 and Figure 5-2 respectively. It must be stated that this geometry naming is followed also in the following chapters.



Figure 5-1 Hydrodynamic problem's geometry after the matching boundary definition



Figure 5-2 Numbering of bodies, inside an array.

### 5.2 Results concerning one body

### 5.2.1 Scattering problem

The scattering problem of a vertical cylinder oscillating in waves using a semi-analytic method was examined in (Miles & Gilbert, 1968). Garrett found errors to that work and presented his results in (Garrett, 1971). The results obtained were compared to those results showing a great convergence, both for the horizontal (Figure 5-3) and the vertical forces (Figure 5-4). The cylinder dimension choices used were two, with the draught changing and the rest of the dimensions constant.



Figure 5-3 Isolated cylinder: Horizontal scattering force on isolated cylinder by Garrett (1971)

#### 5.2.2 Radiation problems

The results obtained for the radiation problems were compared to those published in (Yeung, 1980). Yeung used a semi-analytic method to obtain the hydrodynamic coefficients similar to that presented by Garrett. The program results showed perfect matching for a variety of cylinder dimensions. Specifically the hydrodynamic coefficients in surge (Figure 5-5) and heave (Figure 5-6 & Figure 5-7) motion both for a = 0.5 and a = 0.2 follow exactly the semi-analytic results in low frequencies and slightly underestimate the added mass in high frequencies. The coupled pitch-surge hydrodynamic results (Figure 5-8) and the pitch (Figure 5-9) ones matched exactly, especially in the d = 0.9 case.



Figure 5-4 Isolated cylinder: Vertical scattering force by Garrett (1971)



Figure 5-5 Isolated cylinder: Hydrodynamic coefficients in surge motion by Yeung (1980)



Figure 5-6 Isolated cylinder: Hydrodynamic coefficients in heave motion by Yeung (1980)



Figure 5-7 Isolated cylinder: Hydrodynamic coefficients in heave motion by Yeung (1980)


Figure 5-8 Isolated cylinder: Coupled hydrodynamic coefficients in pitch and heave motion by Yeung (1980)



Figure 5-9 Isolated cylinder: Hydrodynamic coefficients in pitch motion by Yeung (1980)

# 5.3 Results concerning a 1x2 cylinder bodies array

Although the isolated cylinder's response was examined and a number of results were available by the early 70s, the absence of an appropriate interaction theory led to the study of this subject by a large number of researchers. In the case of axisymmetric bodies, a number of semi-analytic theories was developed with respect to the poor computer sources of that period. Most of those simplified presentations used the superposition of the isolated cylinder's data with special theories.

#### 5.3.1 Scattering problem

The data used for comparisons were those presented in (Matsui & Tamaki, 1981). The horizontal and vertical forces of cylinder 1 (*Figure 5-10*) showed good convergence as did the ones for cylinder 2 (Figure 5-11).



Figure 5-10 Horizontal and vertical exciting forces, 1x2 array, cylinder 1 (continuous lines: freFLOW, dashed lines: Matsui and Tamaki 1981)

#### 5.3.2 Radiation problems

The results concerning the radiation problems were compared to those presented in (Matsui & Tamaki, 1981) and in (Mavrakos, 1991). The matching of the results obtained was satisfying with the exception of cylinder 1 self influenced case in surge motion (Figure 5-12) where 8-9% differences in the data peaks were reported.



Figure 5-11 Horizontal and vertical scattering forces, 1x2 array, cylinder 2 (continuous lines: freFLOW, dashed lines: Matsui and Tamaki 1981)



Figure 5-12 Hydrodynamic coefficients, 1x2 array, (continuous lines: freFLOW, dashed lines: Matsui and Tamaki 1981).



Figure 5-13 Interaction hydrodynamic coefficients, 1x2 array, (continuous lines: freFLOW, dashed lines: Mavrakos 1991).



Figure 5-14 Interaction hydrodynamic coefficients, 1x2 array, (continuous lines: freFLOW, dashed lines: Matsui and Tamaki 1981).



Figure 5-15 Hydrodynamic coefficients, 1x2 array, (continuous lines: freFLOW, dashed lines: Matsui and Tamaki 1981).



Figure 5-16 Interaction hydrodynamic coefficients, 1x2 array, (continuous lines: freFLOW, dashed lines: Mavrakos 1991).

#### 5.4 Results concerning an 2x2 cylinder bodies array

Mavrakos in (Mavrakos, 1991) presented a number of results for different configurations such as the 2x2 cylinder array. The comparison between the results obtained and the presented ones was held for the hydrodynamic interaction coefficients corresponding to each body's horizontal motion in cylinder No1's vertical motion. The results showed very good matching behavior with small differences in some sharp areas of the curves.



Figure 5-17 Interaction hydrodynamic coefficients, 2x2 array, (continuous lines: freFLOW, dashed lines: Mavrakos 1991).



Figure 5-18 Interaction hydrodynamic coefficients, 2x2 array, (continuous lines: freFLOW, dashed lines: Mavrakos 1991).

# **CHAPTER 6: HYDRODYNAMIC INTERACTION IN WEFs**

# 6.1 Explanation of the obtained results

As it will be stated with power terms in proceeding chapter, the interaction between WECs is of fundamental importance when arranging them in a WEF. The interaction in some cases can lead to higher exciting forces compared to the isolated body case and in others lower. In order not to make the results incomprehensible, the same dimensions were applied for all the circular cylinders examined. The configurations shown in Figure 6-1 which can be divided in one row and multiple row ones were used to obtain the sought interaction phenomena.

### 6.1.1 Excitation forces

The first results presented concern the two cylinder case. In the  $\beta=0^{\circ}$  case, cylinder No2 follows the results of the isolated one in terms of the horizontal and vertical force, but with a decrease in value occurring. On the other hand, the cylinder greatly affected as seen in Figure 6-3 is No1. Depending on the wave length of the incident wave, in some frequencies exciting forces can be higher or lower. This behavior is a classic result observed in multi-body interactions and appears due to the diffracted waves from the rear in a row cylinders to the ones in front of them. The dependence on wave length and not on specific frequencies can be confirmed from the repeated pattern which is shown between the results obtained for separated distance l=5a and l=8a. In the  $\beta=90^{\circ}$  case, the solution becomes symmetric for cylinders No1 and No2 and so the results are the same for both of them. In Figure 6-5 force on y-direction is almost equal to the isolated horizontal one and a small force is shown in the x-direction caused



Figure 6-1 Numbering of bodies inside an array.

by the side diffracted waves. In the vertical direction the solution is affected by the diffracted waves from each cylinder and wave length dependent peaks are observed. When the incident wave direction is  $\beta$ =45°, the force in x-direction follows the  $\beta$ =0° behavior while the force in y-direction the  $\beta$ =90° one. Not obvious to predict is the fact that the total resultant horizontal force for some wave lengths is significantly higher than in the isolated case. As for the vertical force, as seen in Figure 6-4, the aforementioned fluctuations occur, which are more intense in No1 cylinder than in No2. It is interesting though that peak values occur in higher frequencies, something directly linked to the incident wave angle as seen in Figure 6-14 and Figure 6-15. The one row configuration results of three and four cylinders agree with the observations for the two cylinder case with slight differences. The last cylinder of the row, No 3 or No 4 respectively, still is the one that experiences the mildest interaction effects when  $\beta=0^{\circ}$ . In Figure 6-6 and Figure 6-9 it is shown that the greatest, in terms of value, fluctuations are developed on the first cylinder of the row. This is due to the superposition of the diffracted waves from all the cylinders behind it. Similarly, the second cylinder is influenced from the ones behind it only and its peak values are the second highest. When  $\beta=90^{\circ}$  as seen in Figure 6-8 for the 1x3 configuration, No2 cylinder shows zero value x-direction force due to the symmetric diffracted waves from No1 and No3 ones. Also cylinders No1 and No2 have the same response as No3 and No4 respectively, in the 1x4 configuration presented in Figure 6-11 for the same reason. Differences in comparison to one row results can be observed in the fundamentally important for array configurations, 2x2 case. When the incident wave angle is  $\beta=0^{\circ}$ , cylinders behave similarly to the ones of the 1x2 configuration with the exception that the y-direction force has non zero value. This is a result of the diffracted waves from the opposite row cylinders. New observations are made in the  $\beta$ =45° case, as seen in Figure 6-13. Specifically, No2 cylinder although being the last cylinder a wave crest reaches, shows some sort of fluctuation both in the horizontal and vertical forces, probably due to the diffracted waves from cylinders No1 and No4. These cylinders present an increased, same, resultant horizontal force compared to the isolated case. The cylinder on the other hand that is greatly affected is cylinder No3 which for specific wave lengths shows high peaks.

# 6.1.2 Near trapping effect

As stated in the previous paragraph, for a certain configuration, there are some frequencies which correspond to wave lengths that the interaction effects are significant. Specifically when the separating distance between two bodies is a product of the half wave length,

$$k_0 l = \kappa \pi \Leftrightarrow \frac{l}{\lambda/2} = \kappa, \quad \kappa = 1, 2, 3, \dots,$$
 (6.1.1)

then the wave's crest diffracted from the body which lies behind, in terms of incident direction, reaches the first body simultaneously with one crest of the incident wave. The superposition of these two waves results in a peak either higher or lower to the isolated exciting force value. This phenomenon was extensively investigated in (Evans & Porter, 1997), (Evans & Porter, 1999) and (Newman, 2001) and is called near trapping. It is observed in row arrays of identical bodies separated by the same distance. The term "near" is used because firstly trapping refers to the resonance on infinite element row arrays or bodies in channels and secondly trapping is found when  $\kappa = 1$  and the rest values cannot be considered trapped too in the examined case. As stated, near trapping actually occurs in a wave length slightly lower than the values in (6.1.1). In order for this phenomenon to be presented in the examined cases a set of figures was prepared, focused on indicating it. First the 1x3

configuration under  $\beta=0^{\circ}$  is examined. In Figure 6-6 the horizontal and vertical forces are presented for separating distances l = 5a and l = 8a with special plotting of  $k_0 l$  value. On the horizontal force, as stated in the previous analysis, cylinder No3 shows a decreased but similar behavior to the isolated case and no peaks exist since no diffraction wave reaches this last cylinder in the row. Cylinder No2 which lies in front of cylinder No3 shows a behavior similar to that of the last cylinder but with the superposition of a sinusoidal with high peak when  $k_0 l / \pi$  reaches values such as 1, 2, 3 and low peak when this factor is equal to 1.5, 2.5, 3.5 and so on. Odd integer numbers refer to 180° difference in the phase of the two cylinders, and even ones to no difference. On the other hand non integer values like the ones denoted refer to the case the incident wave and the diffracted one reach the front cylinder with 180° difference in the phase resulting in zero influence and decrease in value. Cylinder No1 which is the first cylinder to face the incident wave, responds the way an isolated cylinder should with the superposition of two diffracted waves. The first wave comes from cylinder No2 and has the same impact as in cylinder No2 and No3 interaction. The second wave comes from cylinder No3 which is positioned in a 21 separating distance, which means that even when

 $k_0 l/\pi$  is equal to 1.5, 2.5, 3.5, ...,  $k_0 2l/\pi$  is equal to 3, 5, 7 and so on. This results in an increased value, even on such wave lengths. On the vertical force, the same observations apply but now the high peaks correspond to low ones and vice versa. In other configurations, 1x4 is presented in Figure 6-9 and 2x2 in Figure 6-12. When in a row the fourth cylinder and the whole configuration follows the observations for the three cylinder case with respect to the 31 separating distance between the first and the last cylinder. When four cylinders are positioned in a 2x2 configuration, the behavior of the front ones with respect to the incident wave direction is similar to that of cylinder No1, in a 1x2 configuration and of the ones behind them similar to that of cylinder No2. Deviations from this behavior are due to the diffracted waves from the second row cylinders and are more significant for the greater separating distance l = 8a. The observations that were presented and their appliance to any number of members row can be understood from Figure 6-27 and Figure 6-28. With an increasing number of bodies the forces increase since more diffracted waves exist and also the amplitude of the diffracted waves and their impact decreases with the increasing distance between two bodies. This is the reason for cylinder No1 to have the greatest high or low peak and also to have the maximum number of local peaks even if those can be considered of secondary order. For instance the interaction effect of cylinder No2 on No1 can be considered of secondary order and the interaction effect of cylinder No3 on No1 of third order.

#### 6.1.3 Added mass and damping coefficients

As stated in preceding paragraphs  $A_{ij}^{pq}$  and  $B_{ij}^{pq}$  correspond to added mass and damping coefficients respectively, of body p in the *i* direction due to the motion of body q in the *j* direction. Based on the radiation potentials whose result those coefficients constitute, the interaction effects can be divided in two types. The first one refers to the coefficients of a body which are the result of diffracted waves from neighboring stationary bodies, of the radiated wave by this body itself. The second one refers to coefficients of a body by another body's radiated wave. Furthermore, on interaction effects, a key role is played by the mode of motion that results in a radiated wave. As seen in Figure 6-2 surging waves have a significant impact in the direction introduced by the motion, on the contrary heaving waves are symmetric around the axis of the cylinder. In Figure 6-29 for the two body case, the coefficients concerning the potential induced by the body itself show significant fluctuations which follow a pattern similar to that of the exciting forces. Peak values are observed near wave lengths which follow the previously introduced equation (6.1.1). The fact that it is not obvious exactly which wave lengths result in those increased or decreased values is because the main dimension of the bodies are not relatively small compared to the separating distance between them. As a result wave crests can reach one body, in phase, in a different location on it, changing the result obtained. In Figure 6-30, the coefficients of cylinder No1 due to the motion of cylinder No2 are presented. The ones induced by surging mode of motion follow the general described behavior and the fluctuations occur around zero value, which means the coefficients take also negative values. The ones induced by heaving mode of motion on the other hand show a damping behavior which takes approximately zero value from  $k_a a = 1.8$ and on. In the 1x3 configuration the same behavior is observed but the difference lies on the fact that the cylinder influenced by two interactions and not only one is cylinder No2. This is indicated in Figure 6-31 and Figure 6-32 where cylinder No2 reaches a higher value than cylinder No1. Concerning the interaction coefficients, Figure 6-33 and Figure 6-34 show as expected that a coefficient value increases as the separating distance decreases and the number of high and low peaks depends on the number of half wave lengths that fit in the separating distance. This is also confirmed in the figures referring to 1x4 configuration. The last case considered is the 2x2 configuration, which is very indicative due to the positions of the bodies. Coefficients of the self induced case in Figure 6-39 reach a high peak for wave length  $k_0 l / \pi$ equal to 2, in the l = 5a separating distance while for l = 8a the values follow the usual pattern explained. As for the interaction ones in Figure 6-40 and Figure 6-41 they indicate the difference between surge and heave modes of motion. In the heaving case the interaction effects of cylinders No2 and No3 are the same in contrast to their effect when surging. Specifically when surging, No1 cylinder lies in the direction of the radiated wave from No2 cylinder and not in the direction of No3 one. Especially in the surging No3 cylinder occasion, cylinder No1 is positioned vertically to the radiated wave's direction hence the interaction effects are even lower than the ones in No4 cylinder's surging case, which lies in a greater separating distance.



Figure 6-2 Field pattern of the radiated wave for an axisymmetric body moving in heave (left), surging (right) motion, taken from (*Babarit*, 2015)

# 6.2 Exciting forces acting on WECs



# 6.2.1 Results with respect to oscillating cylinder, one row array

Figure 6-3 Exciting forces, 1x2 array,  $\beta$ =0 deg.



Figure 6-4 Exciting forces, 1x2 array,  $\beta$ =45deg.



Figure 6-5 Exciting forces, 1x2 array,  $\beta$ =90 deg.



Figure 6-6 Exciting forces, 1x3 array,  $\beta$ =0 deg.



Figure 6-7 Exciting forces, 1x3 array,  $\beta$ =45 deg.



Figure 6-8 Exciting forces, 1x3 array,  $\beta$ =90 deg.



Figure 6-9 Exciting forces, 1x4 array,  $\beta$ =0 deg.



Figure 6-10 Exciting forces, 1x4 array,  $\beta$ =45 deg.



Figure 6-11 Exciting forces, 1x4 array,  $\beta$ =90 deg.



Figure 6-12 Exciting forces, 2x2 array,  $\beta$ =0 deg.



Figure 6-13 Exciting forces, 2x2 array,  $\beta$ =45 deg.



6.2.3 Results with respect to incident wave angle, one row array

Figure 6-14 Exciting forces, 1x2 array, l=5a.



Figure 6-15 Exciting forces, 1x2 array, l=8a.



Figure 6-16 Exciting forces, 1x3 array, cylinder No1.



Figure 6-17 Exciting forces, 1x3 array, cylinder No2.



Figure 6-18 Exciting forces, 1x3 array, cylinder No3.



Figure 6-19 Exciting forces, 1x4 array, cylinder No1.



Figure 6-20 Exciting forces, 1x4 array, cylinder No2.



Figure 6-21 Exciting forces, 1x4 array, cylinder No3.



Figure 6-22 Exciting forces, 1x4 array, cylinder No4.



Figure 6-23 Exciting forces, 2x2 array, cylinder No1.



Figure 6-24 Exciting forces, 2x2 array, cylinder No2.



Figure 6-25 Exciting forces, 2x2 array, cylinder No3.



Figure 6-26 Exciting forces, 2x2 array, cylinder No4.



6.2.5 Combined configurations, near trapping indicating results

Figure 6-27 Exciting forces on the middle cylinder in row arrays of N bodies,  $\beta=0$  deg, l=5a. The magenta dashed line corresponds to  $k_0 l = \kappa \pi$  with  $\kappa$  being odd number and green dashed line corresponding to even number.



Figure 6-28 Exciting forces of cylinder No1 in row arrays of N bodies,  $\beta=0 \text{ deg}$ , l=5a. The magenta dashed line corresponds to  $k_0 l = \kappa \pi$  with  $\kappa$  being odd number and green dashed line corresponding to even number.


# 6.3 Interaction added masses and damping coefficients

Figure 6-29 Added masses-damping coefficients, 1x2 array.



Figure 6-30 Interaction added masses-damping coefficients, 1x2 array.



Figure 6-31 Added masses-damping coefficients, 1x3 array, l=5a.



Figure 6-32 Added masses-damping coefficients, 1x3 array, l=8a.



Figure 6-33 Interaction added masses-damping coefficients, 1x3 array, l=5a.



Figure 6-34 Interaction added masses-damping coefficients, 1x3 array, l=8a.



Figure 6-35 Added masses-damping coefficients, 1x4 array, l=5a.



Figure 6-36 Added masses-damping coefficients, 1x4 array, l=8a.



Figure 6-37 Interaction added masses-damping coefficients, 1x4 array, l=5a.



Figure 6-38 Interaction added masses-damping coefficients, 1x4 array, l=8a.



Figure 6-39 Added masses-damping coefficients, 2x2 array.



Figure 6-40 Interaction added masses-damping coefficients, 2x2 array, l=5a.



Figure 6-41 Interaction added masses-damping coefficients, 2x2 array, l=8a.

#### **CHAPTER 7: HEAVING MOTIONS AND POWER ABSORPTION**

## 7.1 Isolated body's motion and absorbed power

Obtaining hydrodynamic results for a number of WEC configurations, made it possible to also solve the mechanical problem introduced in chapter 2. In this way a MATLAB®R2016a program was developed, named freOSCIP after the hydrodynamic one, which was able to calculate both the motions of each body but also the absorbed power. Although this program, is able to model the external damping and spring in any mode of motion, it was used only for heaving motion.

## 7.1.1 Cylinder characteristics

As applied for the hydrodynamic results the same cylinder dimensions were used, a = 0.1, h = 1.0, d = 0.95 and the same assumptions were made. Due to lack of specific WEC design and mass matrices, cylinders were considered homogeneous, solid, non-hollow. Their characteristics are

Displacement

$$\Delta = \pi R^2 T \rho \tag{7.1.1}$$

where R is the cylinder's radius, T is its draught and  $\rho = 1025 kg / m^3$  is salt waters density.



Figure 7-1 Heaving motion for variable constant external damping. For constant spring value, for increasing damping value motion decreases. The damping value step is equal to 0.1 times the hydrodynamic damping in the resonance frequency of the isolated cylinder.



Figure 7-2 Absorbed power for variable constant external damping. For constant spring value, for increasing damping value absorbed power increases. The damping value step is equal to 0.1 the hydrodynamic damping in the resonance frequency of the isolated cylinder.

Moments of inertia

$$I_1 = I_2 = \frac{\Delta \cdot H^2}{12}$$

$$I_3 = \frac{\pi R^4 T \rho}{2}$$
(7.1.2)

where H is the cylinder's height, considered equal to 1.2T in this thesis.

Radius of gyration

$$i_{v} = \sqrt{\frac{I_{v}}{\Delta}}, v = 1, 2, 3, \Rightarrow$$

$$\Rightarrow \begin{cases} i_{1or2} = \sqrt{\frac{H^{2}}{12}} \\ i_{3} = \sqrt{\frac{R^{2}}{2}} \end{cases}$$
(7.1.3)

Metacentric height

$$GM = KM - KG = KB + BM - KG \tag{7.1.4}$$

where

$$KB = \frac{T/2}{2}$$

$$BM = \frac{\pi R^4/4}{\pi R^2 T} = \frac{R^2}{4T}$$

$$KG = \frac{H/2}{2}$$

$$(7.1.5)$$

#### 7.1.2 External damping tuning and spring value

In Figure 7-1 and Figure 7-2, heaving motion and absorbed power respectively of an isolated cylinder are presented for various values of external spring factor  $K_{33}$  and external damping  $B_{ext,33}$ . Although in most cases spring factor  $K_{33}$  depends on construction factors, the damping one is able to be tuned and a lot of interest has been shown on this issue as in (Falnes, 2004), (Borgarino, et al., 2012), (Cargo, et al., 2016) and (Wang, et al., 2016). In this thesis the aim was to search the interaction effects caused in selected configurations and not to find the optimal damping value for a specific configuration. The damping coefficient was held the same for every cylinder and tuned as stated in (Borgarino, et al., 2012) as equal to the hydrodynamic damping of an isolated cylinder at its resonant frequency. On the spring value, two cases were examined. One with zero spring value  $K_{33} = 0.0C_{33}$  and one equal to  $K_{33} = 0.5C_{33}$ .

## 7.2 Heaving mode motions

The heaving motion response of a body depends on three main factors. First on the exciting forces acting on the body, second on the characteristics of the body which can be divided in the hydrodynamic and in the mass distribution ones and finally on external factors like the PTO mechanism. As indicated in the previous chapter, the dominant phenomenon effecting the exciting forces and the hydrodynamic characteristics of one body is near trapping. On the other hand the characteristics of the body in combination with the external factors determine the resonant frequency where the motion is expected to reach the highest value. As shown in *Figure 7-1*, resonance frequency in the case of  $K_{33} = 0.0C_{33}$  is equal to  $k_0a \approx 0.8$  and in the case of  $K_{33} = 0.5C_{33}$  equal to  $k_0a \approx 1.5$ . The configurations examined are shown in *Figure 7-3*. Starting from the two cylinder configuration for incident wave angle equal to zero ( $\beta=0^\circ$ ), the interaction effects are similar to those observed for the exciting forces in *Figure 6-3*. As seen in *Figure 7-4* and *Figure 7-5* cylinder No2 has a response similar to that of the isolated body of a lower magnitude. On the contrary No1 cylinder's response shows fluctuations indicated



Figure 7-3 Numbering of bodies inside an array (1)

by the near trapping effect. Specifically, frequencies near the ones given from (6.1.1)correspond to low peak values and intermediate frequencies to high peak ones following the vertical exciting force behavior. When the incident wave angle is equal to  $\beta=90^{\circ}$ , due to the symmetry of the configuration, the response of both cylinders is the same. For  $K_{ii} = 0.0C_{ii}$  and separating distance l = 5a, cylinders reach a significant increase in maximum value of  $\simeq 15\%$ on the resonant frequency. An interesting observation is that in this configuration the intermediate near trapping frequency in which the vertical force increases coincides with the resonant one. This is not observed for l = 8a. For  $K_{33} = 0.5C_{33}$  similar behavior is shown with the exception of the heaving motion just reaching the one of the isolated body for l = 5a. The impact of this coincidence is more clear for  $\beta=90^{\circ}$  because of the contribution of body's hydrodynamic coefficients in the motion response. The difference between heaving and surging/swaying radiation waves as shown in Figure 6-2 results in small values for coefficients  $A_{32}^{12}$  and  $B_{32}^{12}$ . Sway motion being the dominant motion in the horizontal direction makes the effect of the heaving radiated waves and therefore near trapping significant. When  $\beta=0^{\circ}$  surging through coefficients  $A_{31}^{12}$  and  $B_{31}^{12}$  impacts on the vertical motion, not letting similar behavior to the  $\beta$ =90° one. In the intermediate incident wave angle  $\beta$ =45°, the results are similar to those for the zero angle case. Peak values are not as high and the motion of cylinder No2 is increased leading to higher values than the isolated one, especially for  $K_{ii} = 0.5C_{ii}$ . For this external spring value intermediate near trapping frequencies exist close to value  $k_0 a \approx 1.5$ , justifying this increase. These observations are generalized for the rest one row results presented. Like in the case of exciting forces, all the diffraction waves and the radiated ones create fluctuations of different order depending on the separating distance and the position in the row cylinders have. It is interesting to confirm that when the intermediate near trapping frequency coincides with the resonance one, the resulting response is again significantly high for the  $\beta$ =90° incident wave angle. This is increase in value is more significant as the position of one cylinder is near the middle of the row. In the 2x2 configuration the behavior of cylinders resembles that of the 1x2 configuration under  $\beta=90^{\circ}$ incident wave angle for the  $\beta=0^{\circ}$  and  $\beta=90^{\circ}$  cases. In the  $\beta=45^{\circ}$  case the results are highly fluctuating. No1 and No4 cylinders have the same response which shows high and low peaks in the intermediate near trapping frequencies as their distance from the others is equal to l. No2 and No3 cylinders on the other hand take their peak values on frequencies that correspond again to near trapping effect with respect that their separating distance is equal to  $l/\cos 45$ . No3 cylinder shows the highest value increase comparing to the isolated one. The last heaving motion results were about two different configurations carrying five cylinders each. The 1x5 configuration shows inferior behavior than the 2x2-3 one under zero incident wave angle as almost all cylinders oscillate with magnitude lower than the one of the isolated case. Only cylinder No1 shows slightly higher values in some frequencies. On the contrary most elements of the 2x2-3 configuration with the exception of cylinders No2 and No5 show higher magnitudes than in the isolated case. When the incident wave is equal to  $\beta=90^{\circ}$  the behavior reverses. The row configuration shows a high value peak at resonant frequency and the 2x2-3 configuration has a decreased performance. For incident wave angle equal to  $\beta=60^{\circ}$  the behavior of both configurations is more complicated and the best option is pointed out through the total energy absorption calculation.



Figure 7-4 Heaving motion, 1x2 array,  $K_{_{33}} = 0.0C_{_{33}}$ .



Figure 7-5 Heaving motion, 1x2 array,  $K_{_{33}} = 0.5C_{_{33}}$ .



Figure 7-6 Heaving motion, 1x3 array,  $K_{_{33}} = 0.0C_{_{33}}$ .



Figure 7-7 Heaving motion, 1x3 array,  $K_{_{33}} = 0.5C_{_{33}}$ .



Figure 7-8 Heaving motion, 1x4 array,  $K_{_{33}} = 0.0C_{_{33}}$ .



Figure 7-9 Heaving motion, 1x4 array,  $K_{_{33}} = 0.5C_{_{33}}$ .



Figure 7-10 Heaving motion, 2x2 array,  $K_{_{33}} = 0.0C_{_{33}}$ .



Figure 7-11 Heaving motion, 2x2 array,  $K_{_{33}} = 0.5C_{_{33}}$ .



Figure 7-12 Heaving motion, 1x5 and 2x2-3 arrays,  $K_{33} = 0.0C_{33}$ , l=5a.



Figure 7-13 Heaving motion, 1x5 and 2x2-3 arrays,  $K_{_{33}} = 0.5C_{_{33}}$ , l=5a

## 7.3 Power absorption and park effect

#### 7.3.1 Park effect

As expected and confirmed by a wide number of researchers as (Babarit, 2013), the operation of each individual WEC in a WEF is affected by the operation of the rest ones. This results in a total absorbed power different than the one absorbed by all WECs, assuming they oscillate isolated from each other. In this way the efficiency of the WEF is given by an introduced q-factor. It is denoted as the ratio of the absorbed power to the one absorbed by the isolated WECs,

$$q(\omega) = \frac{\sum_{n=1}^{N} \overline{P_{ext,3}^{n}}(\omega)}{N \cdot \overline{P_{ext,3,iso}}(\omega)} \frac{\overline{P_{ext,3}^{tot}}(\omega)}{N \cdot \overline{P_{ext,3,iso}}(\omega)}.$$
(7.1.6)

This ratio can be greater, less or of course even to unit. These values correspond to a constructive or destructive interaction between WECs.

### 7.3.2 Results with respect to separating distance and external spring value

The power absorbed by the heaving motion of a cylinder is given from equation (2.3.12). Since the damping coefficient is assumed constant for all results obtained, power absorption behavior is determined from the heaving motion one with respect to the quadratic connection. For this reason the total power absorbed is examined. Furthermore, attention must be paid on the normalization process of the results. Each power value is normalized with the maximum power absorbed by the isolated body having the same external factors. This means that although the results for  $K_{33} = 0.0C_{33}$  and  $K_{ii} = 0.5C_{ii}$  can be misunderstood as comparable, they are totally of different magnitude order as seen in Figure 7-2. The total power absorbed in the 1x2 configuration depends highly on the incident wave direction and on the cylinder external factors. In the  $K_{33} = 0.0C_{33}$  case for separating distance l = 5a, when the incident wave angle is  $\beta$ =90° both cylinders reach a higher value than in the isolated case resulting in maximum total power absorption. The zero angle case shows the second highest value and lower than the isolated case remains the  $\beta=45^{\circ}$  one near the resonant frequency. On the l=8aseparating distance, zero incident wave angle case is similar to the isolated one as does the  $\beta=45^{\circ}$ case around the resonant frequency. In the area after that frequency both  $\beta$ =45° and  $\beta$ =90° curves show a significant reserve in power absorption in contrast to the isolated one. When  $K_{ij} = 0.5C_{ij}$  the most power absorption rich case is the  $\beta$ =45° case no matter the separating distance. This is due to near trapping and resonance effects acting in close frequencies as in the zero incident wave angle for  $K_{ii} = 0.0C_{ii}$  case. In all other cases the absorption remains near the level of the isolated case with the exception of the  $\beta=90^{\circ}$  and l=8a case where the interaction is significantly destructive. In Figure 7-17 the described behavior is presented through the q-factor. What seems misleading is the high values observed, corresponding to low absorbed power differences mainly for frequencies greater than  $k_0 a \simeq 2.0$ . These observations are generalized for the rest of one row configurations. As for the 2x2 configuration,  $\beta = 0^{\circ}$  and  $\beta = 90^{\circ}$  cases give the same constructive result in the  $K_{33} = 0.0C_{33}$ , l = 5acase around resonant frequency. As for the rest cases the  $\beta$ =45° incident wave angle seems not only more widely distributed but when  $K_{33} = 0.5C_{33}$  also gives the highest power absorption values.



Figure 7-14 Mean power absorbed per body, 1x2 array,  $K_{_{33}} = 0.0C_{_{33}}$ .



Figure 7-15 Mean power absorbed per body, 1x2 array,  $K_{_{33}} = 0.5C_{_{33}}$ .



Figure 7-16 Total mean power absorbed, 1x2 array.



Figure 7-17 q-factor, 1x2 array.



Figure 7-18 Mean power absorbed per body, 1x3 array,  $K_{_{33}} = 0.0C_{_{33}}$ .



Figure 7-19 Mean power absorbed per body, 1x3 array,  $K_{_{33}} = 0.5C_{_{33}}$ .



Figure 7-20 Total mean power absorbed, 1x3 array.



Figure 7-21 q-factor, 1x3 array.


Figure 7-22 Mean power absorbed per body, 1x4 array,  $K_{_{33}} = 0.0C_{_{33}}$ .



Figure 7-23 Mean power absorbed per body, 1x4 array,  $K_{_{33}} = 0.5C_{_{33}}$ .



Figure 7-24 Total mean power absorbed, 1x4 array.



Figure 7-25 q-factor, 1x4 array.



Figure 7-26 Mean power absorbed per body, 2x2 array,  $K_{_{33}} = 0.0C_{_{33}}$ .



Figure 7-27 Mean power absorbed per body, 2x2 array,  $K_{_{33}} = 0.5C_{_{33}}$ .



Figure 7-28 Total mean power absorbed, 2x2 array.



Figure 7-29 q-factor, 2x2 array.

## 7.3.3 Comparing results between different array configurations

As stated before, the resulting output behavior prediction is very complicating due to the many parameters affecting it. Especially when the coincidence of near trapping effects and resonance ones take place. In the five cylinder configurations such complicated behavior can be observed. In Figure 7-33, for the 1x5 configuration, it is shown that for  $K_{33} = 0.0C_{33}$  the highest power absorption occurs for  $\beta = 90^{\circ}$  incident wave angle and when  $K_{33} = 0.5C_{33}$  for  $\beta$ =60°. This emphasizes the impact the resonance frequency has on the result, as in the first case intermediate near trapping frequency coincides with  $k_0 a \simeq 0.8$  when  $\beta = 90^\circ$  under the existence of some more parameters discussed in a previous paragraph and in the second one when  $\beta = 60^{\circ}$ . On the 2x2-3 configuration, for  $K_{33} = 0.0C_{33}$  the  $\beta = 90^{\circ}$  case shows the highest value despite the expectance for the  $\beta=0^{\circ}$  one and for  $K_{33}=0.5C_{33}$  the  $\beta=60^{\circ}$  case. The effect behind this unexpected behavior for  $K_{33} = 0.0C_{33}$  is the same as for the 1x5 configuration. The difference which causes the lower high peak value is that cylinders No1 and No2 are lying behind those three cylinders the wave meets first, decreasing their heaving response. Last observation on these configurations is that as expected,  $\beta=0^{\circ}$  cases are characterized as nonefficient in contrast to their output in the 2x2-3 configuration where for  $K_{33} = 0.5C_{33}$  the power absorption is above the isolated one. Respectively,  $\beta$ =90° cases can be assumed non efficient for the 2x2-3 configuration when no other phenomena take place. The next set of results presented from Figure 7-35 to Figure 7-37 concerns the impact of additional rows in the power absorbed. In all configurations, the separating distance is supposed constant equal to 1=5a as does the external spring factor ( $K_{33} = 0.0C_{33}$ ). In many sources as in (Babarit, 2013), it is advised to keep the number of rows with respect to the incident wave direction as low as possible. Returning to the results obtained for incident wave angle  $\beta=0^{\circ}$ , the isolated case seems to be more power rich with the one row configuration (1x4) following next. For this incident wave angle it is the 3x4 configuration that shows improved power absorption in contrast to the 2x4 configuration. A possible explanation is that the same phenomenon occurs as in the row configurations under  $\beta$ =90° incident wave angle, where an increase in elements leads to higher values. For  $\beta$ =45° the response of the multi-row configurations is more complicated and the total power absorbed must be examined. As expected the front rows show improved behavior. It is important to observe that the third row added in the 3x4 configuration has a significant low power absorption. The picture changes in the  $\beta$ =90° case where the resonance and near trapping effects cause the absorbed power of the row configurations to increase. Although the second row, in the 2x4 configuration, adds to the power absorbed by the front one about 40% in peak value compared to the 1x4 configuration, the total power absorbed by the two rows is about the same as twice the one absorbed by the 1x4 configuration. In the 3x4 configuration the front row (r = 3) output decreases compared to the 2x4 configuration (r = 2), as do the two other rows compared to the 1x4 configuration. Due to the fact that the compared configurations have a different number of cylinders, it is very useful to examine their total absorbed power efficiency through the q-factor. Examining Figure 7-37, makes it clear that the presence of additional rows in an array decreases the power efficiency of the configuration. As seen in the case examined appropriate design can lead into increased power absorption in multi-row configurations compared to the isolated bodies case. This is the reason only for incident wave angle  $\beta=0^{\circ}$  increasing number of rows corresponds to better results. The last results obtained concern the different impact two different middle row types, as introduced in Figure 7-30 for the 3x4 and 3x4-3-4 configurations, have on the total configurations. In Figure 7-38 for  $\beta=0^\circ$ , the constructive effect observed in the 3x4 configuration is not shown in the 3x4-3-4 one as the cylinders are no longer aligned. As

expected the  $\beta$ =60° behavior of the 3x4-3-4 configuration is improved compared to the 3x4 one, reaching the 2x4 configuration value level. Finally, when the incident wave angle is equal to  $\beta$ =90°, the behavior of both configurations is similar, with the 3x4 configuration being slightly superior, in terms of efficiency, to the 3x4-3-4 one.



Figure 7-30 Numbering of bodies inside an array (2)



Figure 7-31 Mean absorbed power per body, 1x5 and 2x2-3 arrays,  $K_{_{33}} = 0.0C_{_{33}}$ , l=5a.



Figure 7-32 Mean absorbed power per body, 1x5 and 2x2-3 arrays,  $K_{_{33}} = 0.5C_{_{33}}$ , 1=5a.



Figure 7-33 Total mean power absorbed, 1x5 and 2x2-3 arrays, l=5a.



Figure 7-34 q-factor, 1x5 and 2x2-3 arrays, l=5a.



Figure 7-35 Mean absorbed power per row, 1x4, 2x4 and 3x4 arrays,  $K_{_{33}} = 0.0C_{_{33}}$ , l=5a.



Figure 7-36 Total mean power absorbed, 2x4 and 3x4 arrays, l=5a.



Figure 7-37 q-factor, 2x4 and 3x4 arrays, l=5a.



Figure 7-38 Mean absorbed power per row, 3x4 and 3x4-3-4 arrays,  $K_{_{33}} = 0.0C_{_{33}}$ , l=5a.



Figure 7-39 Total mean power absorbed, 3x4 and 3x4-3-4 arrays, l=5a.



Figure 7-40 q-factor, 3x4 and 3x4-3-4 arrays, l=5a.

## **OVERVIEW AND CONCLUSIONS**

The main objective of this thesis was to examine the hydrodynamic interactions that take place in arrays of wave energy converters and the power efficiency achieved. In this direction, the frequency domain, BEM, hydrodynamic program freFLOW was used in order to obtain the exciting forces, added masses and damping coefficients of all array elements. In order to use this BEM code, a grid generation program (GAWEC) was developed, which supported us with all the necessary grids in the boundary discretization process. Furthermore, the requirement of obtaining motion and power absorption results led also to the development of another program (freOSCIP), able to solve the rigid body equations of motion with external PTO mechanisms modelled for all modes of motion. In this study the heaving power production from fully movable cylinders, with constant external damping, was examined for a variety of array configurations. The results obtained led to the following conclusions.

The dominant interaction phenomenon that determines the exciting forces, added masses and damping coefficients of the array elements is near trapping effect. Its dependence on the separating distance between two bodies relative to the wave length and the type of radiation wave, being a surging or a heaving one makes it complicated. When it comes to maximizing the heaving response of a body, separating distances as the ones obtained from the following equation are suggested.

$$k_0 l = \left(\kappa + \frac{1}{2}\right)\pi \iff \frac{l}{\lambda/2} = \kappa + \frac{1}{2}, \quad \kappa = 1, 2, 3, \dots$$

- The heaving motion response of a body depends on three main factors. First on the exciting forces acting on the body, second on the characteristics of the body which can be divided in the hydrodynamic and in the mass distribution ones and finally on external factors like the ones of PTO mechanism. The dominant phenomenon effecting the exciting forces and the hydrodynamic characteristics of one body is near trapping as stated before. On the other hand the characteristics of the body in combination with the external factors determine the resonant frequency where the motion is expected to reach the highest value. The results obtained lead us to the conclusion that coincidence of both these effects can result in highly constructive behavior close to the resonant area of frequencies, in which the highest power absorption values occur. Therefore the calculation of the resonant frequency of the isolated body is suggested first and then the separating distance can be determined appropriately.
- Constructive performance frequency areas are followed by areas in which efficiency drops in a relative way. This means that if the q-factor value is bigger than unity near to a frequency area, it will be smaller in another one. As shown through a number of results though the efficiency scale is highly dependent on the magnitude of the value to which it refers. This means that increased efficiency must be sought in power rich regions.
- As stated by many authors, the number of rows in an array must be kept as low as possible. This suggestion is confirmed by the results obtained since as the number of rows increased, the efficiency dropped. However since very long row arrays are not an option especially in geographically limited available areas, proper design respecting the second conclusion is shown that can lead in highly constructive behavior in comparison to the isolated cylinders case. Additionally, if the required by the wave spectrum bigger spread in wave directions is needed, it is shown that there are configurations that can serve such a demand.

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