



ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ
ΣΧΟΛΗ ΗΛΕΚΤΡΟΛΟΓΩΝ ΜΗΧΑΝΙΚΩΝ ΚΑΙ ΜΗΧΑΝΙΚΩΝ ΥΠΟΛΟΓΙΣΤΩΝ
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ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

Theory and Simulation of Interacting Particle Systems

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Αθήνα, Ιούλιος 2017



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Απαγορεύεται η αντιγραφή, αποθήκευση και διανομή της παρούσας εργασίας, εξ ολοκλήρου ή τμήματος αυτής, για εμπορικό σκοπό. Επιτρέπεται η ανατύπωση, αποθήκευση και διανομή για σκοπό μη κερδοσκοπικό, εκπαιδευτικής ή ερευνητικής φύσης, υπό την προϋπόθεση να αναφέρεται η πηγή προέλευσης και να διατηρείται το παρόν μήνυμα. Ερωτήματα που αφορούν τη χρήση της εργασίας για κερδοσκοπικό σκοπό πρέπει να απευθύνονται προς τον συγγραφέα. Οι απόψεις και τα συμπεράσματα που περιέχονται σε αυτό το έγγραφο εκφράζουν τον συγγραφέα και δεν πρέπει να ερμηνευθεί ότι αντιπροσωπεύουν τις επίσημες θέσεις του Εθνικού Μετσόβιου Πολυτεχνείου.

Ευχαριστίες

Αρχικά, θα ήθελα να ευχαριστήσω ιδιαίτερα τον κ. Μιχαήλ Λουλάκη για τη συνεχή βοήθεια και καθοδήγηση του τον τελευταίο χρόνο. Η διεκπαιρέωση της παρούσας εργασίας ήταν εξαιρετικά ενδιαφέρουσα και δημιουργική.

Επιπλέον, θα ήθελα να ευχαριστήσω τους συμφοιτητές μου, οι οποίοι έκαναν την εκπόνηση των σπουδών μου στο Εθνικό Μετσόβιο Πολυτεχνείο πολύ ευχάριστη και ιδιαίτερα παραγωγική.

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Περίληψη

Τα Συστήματα Αλληλεπιδρώντων Σωματιδίων είναι μοντέλα που συναντώνται σε πολλά φυσικά συστήματα. Ο κύριος σκοπός της μελέτης τους είναι η εξαγωγή της μακροσκοπικής συμπεριφοράς από τη μικροσκοπική δυναμική. Στο φάσμα αυτής της εργασίας θεωρούμε πρώτα ένα σύστημα σωματιδίων χωρίς αλληλεπιδράσεις και συνάγουμε το μακροσκοπικό του προφίλ χρησιμοποιώντας τεχνικές που επεκτείνονται και σε συστήματα με αλληλεπιδράσεις. Στη συνέχεια, χρησιμοποιώντας αυτές τις τεχνικές, μελετάμε δύο από τα πιο δημοφιλή μοντέλα αλληλεπιδρώντων σωματιδιακών συστημάτων: Απλή Διαδικασία Αποκλεισμού και Διαδικασία Μηδενικού Εύρους. Επιπλέον, συζητάμε για την πιθανότητα καθολικότητας της κατανομής Tracy-Widom που εμφανίζεται πολύ συχνά σε συστήματα με αλληλεπιδρώντα δομικά στοιχεία. Τέλος, παρουσιάζουμε τα αποτελέσματα που προέκυψαν από τις προσομοιώσεις Monte Carlo στα Συστήματα Αλληλεπιδρώντων Σωματιδίων που μελετήσαμε.

Λέξεις Κλειδιά

Markov διαδικασίες, συστήματα αλληλεπιδρώντων σωματιδίων, απλές διαδικασίες αποκλεισμού, διαδικασίες μηδενικού εύρους, μέθοδοι Monte Carlo

Abstract

Interacting Particle Systems are models encountered in many natural systems. The main purpose in their study is to deduce the macroscopic behavior from the microscopic dynamics. In the spectre of this thesis we consider first a particle system without interaction and deduce its macroscopic profile using techniques which are also applied in systems with interaction. Then, using these techniques, we study two of the most popular models of Interacting Particle Systems: Simple Exclusion Process and Zero Range Process. In addition, we discuss the possible universality of the Tracy-Widom distribution which very often emerges in systems with interacting components. Last but not least, we present results acquired by Monte Carlo simulations on our Interacting Particle Systems.

Keywords

Markov Processes, Interacting Particle Systems, Simple Exclusion Processes, Zero Range Processes, Monte Carlo Methods

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Chapter 1

Introduction

Interacting Particle Systems are models encountered in many natural systems, for which the local mechanics are very simple, but it might be considerably difficult to extract a global behavior. In most cases, the factors that contribute to this difficulty is the introduction of stochastic dynamics and interaction into them. Examples can be found in problems from natural sciences, such as reaction diffusion and gas particles systems, extending to problems from social sciences such as traffic flow, opinion dynamics and spread of epidemics.

The main purpose in the study of such Interacting Particle Systems is to deduce the macroscopic behavior, which is usually described by hydrodynamic equations, from the microscopic interaction, namely the underlying stochastics. The mathematical term associated to this is called the *scaling limits*. Scaling limits are of great interest in physics and in particular in mathematics. Furthermore, it usually interests us to find equilibrium states in our system, namely when the macroscopic profile does not change. These equilibriums are described mathematically by the establishment of *invariant distributions*.

1.1 Object of Thesis

On the one part, the goal of the current thesis is to study and apply mathematical techniques on the subject of Interacting Particle Systems and help us elucidate specific behaviors from a theoretical view. In this study we will consider the following models of Particle Systems:

- Independent Random Walks
- Simple Exclusion Process
- Zero Range Process

On the other part, we present Monte Carlo simulations on such models, which help us visualize their characteristics and approximate specific scaling limits for which there has not been an analytic result.

1.2 Following Chapters

In chapter 2 we present the basic concepts and "tools" from the field of Stochastic Processes which will assist us to establish concepts in Interacting Particle Systems.

In chapter 3 we will attempt to make the reader familiar with the mathematical techniques usually applied on Interacting Particle Systems.

In chapter 4 we will study Simple Exclusion Process and establish an invariant distribution. Moreover, we will give results on a certain variance of the process and discuss the emergence of a special distribution.

In chapter 5 we will study Zero Range Process, establish again an invariant distribution and discuss special properties of the model.

In chapter 6 we will present the results acquired by the Monte Carlo simulations on our Interacting Particle Systems.

Chapter 2

Theoretical Background

In this chapter we will introduce basic theorems and definitions of Stochastic Processes that will be used on a regular basis in the following chapters. Keep in mind that each Interacting Particle System is a continuous-time Markov process describing the collective behavior of stochastically interacting components. We use this section to state properties related to Markov processes which yield the background for our following investigation on interacting particle systems.

2.1 Markov Processes

Let us start with some basic notation. For a topological space X , we denote with $\mathcal{B}(X)$ the Borel σ -algebra generated by the open sets of X and equip X with this σ -algebra if not stated otherwise. $\mathcal{M}(X)$ denotes the space of Borel measures on X and $\mathcal{M}_1(X)$ is the subset of all probability measures.

A continuous time stochastic process $(\eta_t)_{t \geq 0}$ is a family of random variables η_t taking values in a compact metric space X , which is called the state space of the process. Let

$$D[0, \infty) = \{\eta : [0, \infty) \rightarrow X \text{ càdlàg}\}$$

be the set of right continuous functions with left limits (*càdlàg*), which is the canonical path space for a stochastic process on X . To define a reasonable measurable structure on $D[0, \infty)$, namely a suitable σ -algebra, let \mathcal{F} be the smallest σ -algebra on $D[0, \infty)$ such that all the mappings $\eta \mapsto \eta_s$ for $s \geq 0$ are measurable with respect to \mathcal{F} . That means that every path can be evaluated at arbitrary times s , namely

$$\{\eta_s \in A\} = \{\eta \mid \eta_s \in A\} \in \mathcal{F}$$

for all measurable subsets $A \in X$. If \mathcal{F}_t is the smallest σ -algebra on $D[0, \infty)$ relative to which all the mappings $\eta \mapsto \eta_s$ for $s \leq t$ are measurable, then

$(\mathcal{F}_t : t \geq 0)$ provides a natural filtration for the process. The filtered space $\{D[0, \infty), \mathcal{F}, (\mathcal{F}_t : t \geq 0)\}$ provides a generic choice for the probability space of a stochastic process which can be defined as a probability measure \mathbb{P} on $D[0, \infty)$.

Definition 1. A (time-homogeneous) *Markov process* on X is a collection $(\mathbb{P}^\zeta : \zeta \in X)$ of probability measures on $D[0, \infty)$ with the following properties:

1. $\mathbb{P}^\zeta(\eta \in D[0, \infty) : \eta_0 = \zeta) = 1$ for all $\zeta \in X$, namely \mathbb{P}^ζ is normalized on all paths with initial condition $\eta_0 = \zeta$.
2. The mapping $\zeta \mapsto \mathbb{P}^\zeta(A)$ is measurable for every $A \in \mathcal{F}$.
3. $\mathbb{P}^\zeta(\eta_{t+} \in A | \mathcal{F}_t) = \mathbb{P}^{\eta_t}(A)$ for all $\zeta \in X$, $A \in \mathcal{F}$ and $t > 0$. (Markov property)

2.2 Markov Chains

Let X now be a countable set. Then a Markov process $(\eta_t)_{t \geq 0}$ is called a *Markov chain* and it can be characterized by *transition rates* $c(\zeta, \zeta') \geq 0$, which have to be specified for all $\zeta, \zeta' \in X$. Often $c(\cdot, \cdot)$ is described as a matrix. For a given process $(\mathbb{P}^\zeta : \zeta \in X)$ the rates are defined via

$$\mathbb{P}^\zeta(\eta_t = \zeta') = c(\zeta, \zeta')t + o(t) \text{ as } t \searrow 0, \quad (2.1)$$

and represent probabilities per unit time.

At this point, we would like to give an intuitive understanding of the time evolution and the role of the transition rates in a process. Denote by

$$W_\zeta := \inf\{t \geq 0 : \eta_t \neq \zeta\}$$

the *holding time* in state ζ . The value of this time is related to the total exit rate out of state ζ ,

$$c_\zeta := \sum_{\zeta' \neq \zeta} c(\zeta, \zeta').$$

If $c_\zeta = 0$, ζ is called an *absorbing state* and $W_\zeta = \infty$.

Proposition 1. *If $c_\zeta > 0$, then $W_\zeta \sim \text{Exp}(c_\zeta)$ and $\mathbb{P}^\zeta(\eta_{W_\zeta} = \zeta') = c(\zeta, \zeta')/c_\zeta$.*

Proof. W_ζ has the Markov property

$$\mathbb{P}^\zeta(W_\zeta > s + t | W_\zeta > s) = \mathbb{P}^\zeta(W_\zeta > s + t | \eta_s = \zeta) = \mathbb{P}^\zeta(W_\zeta > t).$$

Therefore $\mathbb{P}^\zeta(W_\zeta > s + t) = \mathbb{P}^\zeta(W_\zeta > s)\mathbb{P}^\zeta(W_\zeta > t)$. This is the functional equation for an exponential and it suggests that

$$\mathbb{P}^\zeta(W_\zeta > t) = e^{-\lambda t} \text{ with initial condition } \mathbb{P}^\zeta(W_\zeta > 0) = 1.$$

For the parameter λ we will have that

$$\lambda = \left. \frac{d}{dt} \mathbb{P}^\zeta(W_\zeta > t) \right|_{t=0} = \lim_{t \searrow 0} \frac{\mathbb{P}^\zeta(W_\zeta > t) - 1}{t} = -c_\zeta,$$

since according to equation 2.1

$$\mathbb{P}^\zeta(W_\zeta > 0) = 1 - \mathbb{P}^\zeta(\eta_t \neq \zeta) + o(t) = 1 - c_\zeta t + o(t).$$

Now, conditioned on a jump occurring we get

$$\mathbb{P}^\zeta(\eta_t = \zeta' | W_\zeta < t) = \frac{\mathbb{P}^\zeta(\eta_t = \zeta')}{\mathbb{P}^\zeta(W_\zeta < t)} \rightarrow \frac{c(\zeta, \zeta')}{c_\zeta} \text{ as } t \searrow 0$$

by L' Hospital's rule. With the right-continuity of paths, this implies the second statement. \square

Remark 1. Let $\{W_1, \dots, W_n\}$ be a sequence of independent exponentials $W_i \sim \text{Exp}(\lambda_i)$. Regarding the distribution of $W = \min\{W_1, \dots, W_n\}$, we will have that

$$\begin{aligned} \mathbb{P}(W > t) &= \mathbb{P}(W_1 > t, \dots, W_n > t) = \\ &= \mathbb{P}(W_1 > t) \cdots \mathbb{P}(W_n > t) = e^{-\lambda_1 t} \cdots e^{-\lambda_n t} = e^{-(\lambda_1 + \dots + \lambda_n)t} \\ &\Leftrightarrow W \sim \text{Exp}\left(\sum_{i=1}^n \lambda_i\right). \end{aligned}$$

Keep in mind that what was presented in this section, will be very useful at the design of our simulations in chapter 6.

2.3 Feller Processes, Semigroups and Generators

Let X be a compact metric space and denote by

$$C(X) = \{f : X \rightarrow \mathbb{R} \text{ continuous}\}$$

the set of real-valued continuous functions, which is a Banach space with sup-norm $\|f\|_\infty = \sup_{\eta \in X} |f(\eta)|$. Functions f can be regarded as observables, and we are interested in their time evolution.

Definition 2. For a given process $(\eta_t)_{t \geq 0}$ on X , for each $t \geq 0$ we define the operator $S(t) : C(X) \rightarrow C(X)$ by

$$S(t)f(\eta) := \mathbb{E}^\eta f(\eta_t).$$

In general $f \in C(X)$ does not imply $S(t)f \in C(X)$, but all the processes we consider have this property and are called *Feller processes*.

Proposition 2. *Let η be a Feller process on X . Then the family $(S(t) : t \geq 0)$ is a **Markov semigroup**, namely*

1. $S(0) = Id$, (identity at $t = 0$)
2. $t \mapsto S(t)f$ is right-continuous for all $f \in C(X)$, (right-continuity)
3. $S(t+s)f = S(t)S(s)f$ for all $f \in C(X)$, $s, t \geq 0$, (Markov property)
4. $S(t)1 = 1$ for all $t \geq 0$, (conservation of probability)
5. $S(t)f \geq 0$ for all non-negative $f \in C(X)$. (positivity)

Proof. 1. $S(0)f(\eta) = \mathbb{E}^\eta(f(\eta_0)) = f(\eta)$ since $\eta_0 = \eta$ (def. 1(1)).

2. follows from right-continuity of η_t and continuity of f .

3. follows from the Markov property of η_t (def. 1(3))

$$\begin{aligned} S(t+s)f(\eta) &= \\ \mathbb{E}^\eta f(\eta_{t+s}) &= \mathbb{E}^\eta(\mathbb{E}(f(\eta_{t+s})|\mathcal{F}_t)) = \mathbb{E}^\eta(\mathbb{E}^{\eta_t}(f(\eta_{t+s}))) = \mathbb{E}^\eta((S(s)f)(\eta_t)) \\ &= S(t)S(s)f. \end{aligned}$$

4. $S(t)1 = \mathbb{E}^\eta(\mathbf{1}_{\eta_t}(X)) = 1$ since $\eta_t \in X$ for all $t \geq 0$ (conservation of probability).

5. is immediate by definition. □

The Markov semigroup $S(t)$ will appear frequently in our computations and theorems as it expresses in a natural way how a process will "behave" at a later time t . It determines the expected value of observables f on X at time t for a given Markov process η . Specification of all these expected values provides a full representation of η .

Let $\mathcal{P}(X)$ denote the set of all probability measures on X , with the topology of *weak convergence*:

$$\mu_n \rightarrow \mu \text{ if and only if } \int f d\mu_n \rightarrow \int f d\mu$$

for all $f \in C(X)$. Note in particular that with respect to this topology, $\mathcal{P}(X)$ is compact since X is compact. If $\mu \in \mathcal{P}(X)$ and $\{\mathbb{P}^\eta, \eta \in X\}$ is a Markov process, then the corresponding Markov process with initial distribution μ is a stochastic process η_t whose distribution is given by

$$\mathbb{P}^\mu := \int_X \mathbb{P}^\eta \mu(d\eta).$$

In view of this,

$$\mathbb{E}^\mu f(\eta_t) = \int_X S(t) f d\mu$$

for all $f \in C(X)$. This leads to the following definition.

Definition 3. For a process $(S(t) : t \geq 0)$ with initial distribution μ we denote by $\mu S(t) \in \mathcal{P}(X)$ the *distribution at time t* , which is uniquely determined by

$$\int_X f d[\mu S(t)] := \int_X S(t) f d\mu$$

for all $f \in C(X)$.

Now, since $(S(t) : t \geq 0)$ has the Markov property, in analogy with the proof of proposition 1 we expect that it has the form of an exponential generated by the linearization $S'(0)$, namely

$$S(t) = "e^{S'(0)t}" = Id + S'(0)t + o(t) \text{ with } S(0) = Id,$$

which is made precise in the following.

Definition 4. The generator $\mathcal{L} : \mathcal{D}_{\mathcal{L}} \rightarrow C(X)$ for the process $(S(t) : t \geq 0)$ is given by

$$\mathcal{L}f := \lim_{t \searrow 0} \frac{S(t)f - f}{t}$$

for $f \in \mathcal{D}_{\mathcal{L}}$, where the domain $\mathcal{D}_{\mathcal{L}} \subseteq C(X)$ is the set of functions for which the limit exists.

Note that, in general, $\mathcal{D}_{\mathcal{L}}$ is a proper subset of $C(X)$ for processes on infinite lattices, and this is in fact the case even for the simplest examples.

Proposition 3. \mathcal{L} as defined above is a **Markov Generator**, namely

1. $1 \in \mathcal{D}_{\mathcal{L}}$ and $\mathcal{L}1 = 0$, (*conservation of probability*)
2. for $f \in \mathcal{D}_{\mathcal{L}}$, $\lambda \geq 0$: $\min_{\zeta \in X} f(\zeta) \geq \min_{\zeta \in X} (f - \lambda \mathcal{L}f)(\zeta)$, (*positivity*)
3. $\mathcal{D}_{\mathcal{L}}$ is dense in $C(X)$ and the range $R(Id - \lambda \mathcal{L}) = C(X)$ for sufficiently small $\lambda > 0$.

The proof is rather technical and can be found in [12].

In general, for Markov chains with countable X and jump rates $c(\eta, \eta')$ the generator is given by

$$\mathcal{L}f(\eta) = \sum_{\eta' \in X} c(\eta, \eta') (f(\eta') - f(\eta))$$

which, using equation 2.1, follows for small $t \searrow 0$ from

$$\begin{aligned} S(t)f(\eta) &= \mathbb{E}^\eta(f(\eta_t)) = \sum_{\eta' \in X} \mathbb{P}^\eta(\eta_t = \eta')f(\eta') \\ &= \sum_{\eta' \neq \eta} c(\eta, \eta')f(\eta')t + f(\eta) \left(1 - \sum_{\eta' \neq \eta} c(\eta, \eta')t \right) + o(t) \end{aligned}$$

and the definition of \mathcal{L} .

Definition 5. For $X = \{0, 1\}^\Lambda$, where Λ is a countable lattice, $f \in C(X)$ is a *cylinder function* if there exists a finite subset $\Delta \subseteq \Lambda$ such that $f(\eta^x) = f(\eta)$ for all $x \in \Delta$, $\eta \in X$, namely f depends only on a finite set of coordinates of a configuration. We write $C_0(X) \subseteq C(X)$ for the set of all cylinder functions.

2.4 Invariant Measures

One of the main questions we need to address in the study of Interacting Particle Systems is the characterization of all invariant measures.

Definition 6. A measure μ is *invariant* or *stationary* if $\mu S(t) = \mu$. Equivalently,

$$\int_X S(t)f d\mu = \int_X f d\mu$$

or shorter $\mu(S(t)f) = \mu(f)$ for all $f \in C(X)$.

The set of all invariant measures of a process is denoted by \mathcal{I} . In addition, a measure μ is called *reversible* if $\mu(fS(t)g) = \mu(gS(t)f)$ for all $f, g \in C(X)$.

Taking $g = 1$ in the previous equation we see that every reversible measure is also stationary. Stationarity of μ implies that

$$\mathbb{P}^\mu(\eta_t \in A) = \mathbb{P}^\mu(\eta_{t+} \in A)$$

for all $t \geq 0$, $A \in \mathcal{F}$, namely if a state η follows a distribution with respect to μ , it will continue to do so after time t . Using $\eta_t \sim \mu$ as initial distribution, the definition of a stationary process can also be extended to negative times on the path space $D(-\infty, \infty)$.

Proposition 4. Consider a Feller process on a compact state space X with generator \mathcal{L} . Then

$$\mu \in \mathcal{I} \Leftrightarrow \mu(\mathcal{L}f) = 0$$

for all $f \in C_0(X)$, and similarly

$$\mu \text{ is reversible} \Leftrightarrow \mu(f\mathcal{L}g) = \mu(g\mathcal{L}f)$$

for all $f, g \in C_0(X)$.

Proof. Follows from the definitions of semigroup/generator and the fact that $\mu(f_n) \rightarrow \mu(f)$ if $\|f_n - f\|_\infty \rightarrow 0$ by continuity of f_n, f and compactness of X . \square

In particular, not every Markov chain has an invariant distribution. If X is finite there exists at least one invariant distribution, as a direct result of linear algebra (Perron-Frobenius theorem). For Interacting Particle Systems we have compact state spaces X , for which a similar result holds.

Theorem 1. *For every Feller process with compact state space X we have:*

1. \mathcal{I} is non-empty, compact and convex.
2. Suppose the weak limit $\mu = \lim_{t \rightarrow \infty} \pi S(t)$ exists for some initial distribution $\pi \in \mathcal{P}(X)$, namely

$$\pi S(t)(f) = \int_X S(t)f d\pi \rightarrow \mu(f)$$

for all $f \in C(X)$, then $\mu \in \mathcal{I}$.

For the proof see Theorem 1.9 in [6].

Definition 7. A Markov process $(\mathbb{P}^\eta : \eta \in X)$ is called *irreducible*, if for all $\eta, \eta' \in X$

$$\mathbb{P}^\eta(\eta_t = \eta') > 0$$

for some $t \geq 0$.

So an irreducible Markov process can sample the whole state space, and if X is countable this implies that it has at most one invariant distribution.

Definition 8. A Markov process with semigroup $(S(t) : t \geq 0)$ is *ergodic* if

1. $\mathcal{I} = \{\mu\}$ is a singleton, and (unique stationary measure)
2. and $\lim_{t \rightarrow \infty} \pi S(t) = \mu$ for all $\pi \in \mathcal{P}(X)$. (convergence to equilibrium)

Note that in an irreducible Markov process we can observe phase transitions, that is, mathematically speaking, a change between invariant distributions. Phase transitions are related to the breakdown of ergodicity in irreducible systems, in particular, non-uniqueness of invariant measures.

Proposition 5. *An irreducible Markov chain with finite state space X is ergodic.*

Proof. A result of linear algebra, in particular the Perron-Frobenius theorem: The finite matrix $c(\eta, \eta')$ has eigenvalue 0 with unique eigenvector μ . \square

Consequently, mathematically phase transitions occur only in infinite systems. Infinite systems are often studied as limits of finite systems, which show traces of a phase transition by divergence or non-analytic behavior of certain observables. In terms of applications, infinite systems are approximations or idealizations of large finite systems, so results have to be interpreted with "care".

Chapter 3

Independent Random Walks

In this chapter we want to investigate the system of indistinguishable particles following independent random walks. Our main goal is for the reader to get familiar with the main concepts and techniques used in the field of Interacting Particles System.

3.1 Model

Denote by \mathbb{Z}^d the d -dimensional integer lattice. For a positive integer L , denote by \mathbb{T}_L the torus with L points: $\mathbb{T}_L = \mathbb{Z}_L = \{0, 1, \dots, L - 1\}$ and let $\mathbb{T}_L^d = (\mathbb{T}_L)^d$. Here L represents the inverse of the distance between the points of \mathbb{T}_L^d , namely the particle sites, which are represented by x, y and z . By letting $L \rightarrow \infty$, the distance between particles will go to zero and so, we pass from microscopic to macroscopic.

We want to describe the evolution of the system, so let N denote the total number of particles and let x_1, x_2, \dots, x_N denote their initial positions. Also, because particles evolve as independent translation invariant discrete time random walks on the torus, we need to fix a translation invariant transition probability $p(x, y)$ on \mathbb{Z}^d , for which $p(x, y) = p(0, y - x) =: p(y - x)$ for some probability $p(\cdot)$ on \mathbb{Z}^d , called the *elementary transition probability* of the system. This probability expresses the stochastic characteristic of the random walk.

Let $p_t(x, y)$ represent the probability of being at time t on site y for a discrete time random walk with elementary transition probability $p(\cdot)$ starting from x . In addition, we have that $p_t(\cdot, \cdot)$ inherits the translation invariance property from $p(\cdot, \cdot)$, and so $p_t(x, y) = p_t(0, y - x) =: p_t(y - x)$.

It is time to describe the motion of each particle. Let us take N independent random walk variables $\{Z_t^1, Z_t^2, \dots, Z_t^N\}$ on \mathbb{Z}^d with elementary transition probability $p(\cdot)$ and initially at zero. As a result, the position of each particle i on the torus \mathbb{T}_L^d at time t will be

$$X_t^i = x_i + Z_t^i \text{ mod } L.$$

However, since particles are indistinguishable in our model, it does not interest us the exact position of each particle, but the number of particles on each site of the space. Specifically, the state space of the system, also called configuration space, is $\mathbb{N}^{\mathbb{T}_L^d}$. Configurations are denoted by η, ζ and ξ . Under this definition, if x is a site of \mathbb{T}_L^d , then $\eta(x)$ is the number of particles on this site for the configuration η . Therefore, if the particles are initially at x_1, x_2, \dots, x_N , then

$$\eta(x) = \sum_{i=1}^N \mathbb{1}\{x = x_i\}.$$

On the other hand, if we are given $(\eta(x) : x \in \mathbb{T}_L^d)$, we can first label the particles and then let them evolve according to the stochastic dynamics we have described.

Of course, we want the configuration at time t , which will be denoted by $\eta_t(x)$ and defined by

$$\eta_t(x) = \sum_{i=1}^N \mathbb{1}\{x = X_t^i\}.$$

Moreover, the process $(\eta_t)_{t \geq 0}$ inherits the Markov property from the random walks $\{X_t^i, 1 \leq i \leq N\}$ because all particles have the same elementary transition probability and they do not interact with each other.

3.2 Poisson Measure

Since the state space is finite and since the total number of particles is the unique quantity conserved by the dynamics of the system, for every positive integer N representing the total number of particles, there is only one invariant measure, as long as the support of the elementary transition probability $p(\cdot)$ generates \mathbb{Z}^d , namely the process is irreducible. The Poisson measures in our study will play a central role.

Recall that a Poisson distribution of parameter $\alpha > 0$ is the probability measure $\{p_{\alpha, k} = p_k, k > 1\}$ on \mathbb{N} given by

$$p_k = e^{-\alpha} \frac{\alpha^k}{k!}, \quad k \in \mathbb{N}$$

and its Laplace transform is equal to

$$e^{-\alpha} \sum_{k=0}^{\infty} e^{-\lambda k} \frac{\alpha^k}{k!} = e^{-\alpha} e^{\alpha e^{-\lambda}} = e^{\alpha(e^{-\lambda} - 1)},$$

for all $\lambda > 0$.

Definition 9. For a fixed positive function $\rho : \mathbb{T}_L^d \rightarrow \mathbb{R}_+$, we call Poisson measure on \mathbb{T}_L^d associated to the function ρ , a probability on the configuration space $\mathbb{N}^{\mathbb{T}_L^d}$, denoted by $\nu_{\rho(\cdot)}^L$, having the following two properties:

- under $\nu_{\rho(\cdot)}^L$ the random variables $(\eta(x) : x \in \mathbb{T}_L^d)$, representing the number of particles at each site, must be independent,
- for every fixed site $x \in \mathbb{T}_L^d$, $\eta(x)$ is distributed according to a Poisson distribution of parameter $\rho(x)$.

In the case where the function ρ is constant equal to α , we denote $\nu_{\rho(\cdot)}^L$, just by ν_α^L . We will, also, denote expectation with respect to a measure ν by E_ν .

The measure $\nu_{\rho(\cdot)}^L$ is characterized by its multidimensional Laplace transform:

$$E_{\nu_{\rho(\cdot)}^L} \left[e^{-\sum_{x \in \mathbb{T}_L^d} \lambda(x) \eta(x)} \right] = \prod_{x \in \mathbb{T}_L^d} e^{\rho(x)(e^{-\lambda(x)} - 1)} = e^{\sum_{x \in \mathbb{T}_L^d} \rho(x)(e^{-\lambda(x)} - 1)}$$

for all positive sequences $(\lambda(x) : x \in \mathbb{T}_L^d)$ [4].

Now, let us move on to establishing that the Poisson measures associated to constant functions are invariant for a system of independent random walks.

Proposition 6. *If particles are initially distributed according to a Poisson measure associated to a constant function equal to α then the distribution at time t is exactly the same Poisson measure. [11]*

Proof. Denote by $\mathbb{P}_{\nu_\alpha^L}$ the probability measure on the path space $\Omega_L = \mathbb{N}^{\mathbb{T}_L^d} \times \mathbb{N}^{\mathbb{T}_L^d} \times \dots$, namely the space of $(\eta_t)_{t \geq 0}$, induced by the independent random walk dynamics and the initial measure ν_α^L . Expectation with respect to $\mathbb{P}_{\nu_\alpha^L}$ is denoted by $\mathbb{E}_{\nu_\alpha^L}$. At this point, notice the difference between $E_{\nu_\alpha^L}$, which is the expectation with respect to the measure defined on $\mathbb{N}^{\mathbb{T}_L^d}$, and $\mathbb{E}_{\nu_\alpha^L}$, which is the expectation with respect to the measure defined on the path space Ω_L . It is easy to see that

$$\mathbb{E}_{\nu_\alpha^L}[F(\eta_0)] = E_{\nu_\alpha^L}[F(\eta)]$$

for all bounded continuous functions F on $\mathbb{N}^{\mathbb{T}_L^d}$.

Since the measure $\nu_{\rho(\cdot)}^L$ is characterized by its multidimensional Laplace transform, we will compute here the expectation

$$\mathbb{E}_{\nu_\alpha^L} \left[e^{-\sum_{x \in \mathbb{T}_L^d} \lambda(x) \eta_t(x)} \right]$$

for all positive sequences $(\lambda(x) : x \in \mathbb{T}_L^d)$. Furthermore, for a site $y \in \mathbb{T}_L^d$, we will denote by $X_t^{y,k}$ the position at time t of the k -th particle starting from y . In this way, the number of particles on site x at time t will be:

$$\eta_t(x) = \sum_{y \in \mathbb{T}_L^d} \sum_{k=1}^{\eta_0(y)} \mathbb{1}\{x = X_t^{y,k}\}.$$

And then, by inverting the order of summation, we will get that:

$$\sum_{x \in \mathbb{T}_L^d} \lambda(x) \eta_t(x) = \sum_{y \in \mathbb{T}_L^d} \sum_{k=1}^{\eta_0(y)} \lambda(X_t^{y,k}).$$

Since each particle evolves independently and the total number of particles at each site at time 0 is distributed according to a Poisson distribution of parameter α ,

$$\begin{aligned} \mathbb{E}_{\nu_\alpha^L} \left[e^{-\sum_{x \in \mathbb{T}_L^d} \lambda(x) \eta_t(x)} \right] &= \mathbb{E}_{\nu_\alpha^L} \left[e^{-\sum_{y \in \mathbb{T}_L^d} \sum_{k=1}^{\eta_0(y)} \lambda(X_t^{y,k})} \right] = \\ &= \prod_{y \in \mathbb{T}_L^d} \mathbb{E}_{\nu_\alpha^L} \left[e^{-\sum_{k=1}^{\eta_0(y)} \lambda(X_t^{y,k})} \right] = \\ &= \prod_{y \in \mathbb{T}_L^d} \int \nu_\alpha^L(d\eta) \left(E \left[e^{-\lambda(X_t^{y,1})} \right] \right)^{\eta_0(y)} = \\ &= \prod_{y \in \mathbb{T}_L^d} e^{\alpha \left(E \left[e^{-\lambda(X_t^{y,1})} \right] - 1 \right)} = \prod_{y \in \mathbb{T}_L^d} e^{\alpha \left(E \left[e^{-\lambda(y+X_t)} \right] - 1 \right)} \end{aligned}$$

where X_t is a random walk at time t on the torus \mathbb{T}_L^d starting from the origin and with transition probability $p_t^L(\cdot)$ defined by

$$p_t^L(x, y) = \sum_{z \in \mathbb{Z}^d} p_t(x, y + Lz)$$

for $x, y \in \mathbb{T}_L^d$. Since, by definition,

$$E \left[e^{-\lambda(y+X_t)} \right] = \sum_{x \in \mathbb{T}_L^d} p_t^L(x - y) e^{-\lambda(x)},$$

then, by inverting the order of summation and that $\sum_{y \in \mathbb{T}_L^d} p_t^L(x - y) = 1$, we obtain

$$\mathbb{E}_{\nu_\alpha^L} \left[e^{-\sum_{x \in \mathbb{T}_L^d} \lambda(x) \eta_t(x)} \right] = e^{\sum_{x \in \mathbb{T}_L^d} \alpha (e^{-\lambda(x)} - 1)}.$$

□

Remark 2. Since the total number of particles $\sum_{x \in \mathbb{T}_L^d} \eta(x)$ is conserved by the stochastic dynamics it might seem more natural to consider as reference probability measures the extremal invariant measures that are concentrated on the "hyper-planes" of all configurations with a fixed total number of particles. These measures are given by

$$\nu_{\mathbb{T}_L^d, N}^L(\cdot) := \nu_\alpha^L \left(\cdot \mid \sum_{x \in \mathbb{T}_L^d} \eta(x) = N \right).$$

We should, also, note that the Poisson distributions are such that their expectation is equal to

$$\sum_{k \geq 0} e^{-\alpha} \frac{\alpha^k}{k!} k = \alpha.$$

The Poisson measures are in this way naturally parametrized by the density of particles. In addition, by the weak law of large numbers,

$$\lim_{L \rightarrow \infty} \frac{1}{|\mathbb{T}_L^d|} \sum_{x \in \mathbb{T}_L^d} \eta(x) = \alpha$$

in probability with respect to ν_α^L . Consequently, imagine that the parameter α describes the mean density of particles in a "large" box.

In conclusion, for this section, we established in proposition 6 the existence of an one-parameter family of invariant measures indexed by the density of particles, which is the unique quantity conserved by the time evolution.

3.3 Local Equilibrium

We said before that one of our main goals is to deduce the macroscopic behavior of an Interacting Particles System. In this case, the passage from microscopic to macroscopic will be done by performing a limit in which the distance between the particle sites will go to zero. This is not difficult to formalize and it is a technique used in many areas of Mathematics.

If we imagine the discrete torus \mathbb{T}_L^d as embedded in a continuous torus $\mathbb{T}^d = [0, 1)^d$, namely taking the lattice \mathbb{T}^d with "vertices" at x/L , $x \in \mathbb{T}_L^d$, then the distances between molecules is $1/L$ and tends to zero as $L \rightarrow \infty$. In this way, for the inverse mapping, each macroscopic point u in \mathbb{T}^d is associated to a microscopic site $x = [uL]$ in \mathbb{T}_L^d . We should note that, here and below, for a d -dimensional real $r = (r_1, r_2, \dots, r_d)$, $[r]$ denotes the integer part of r : $[r] = ([r_1], [r_2], \dots, [r_d])$.

Now, we will start building the concept of a local equilibrium. Let $\rho_0 : \mathbb{T}^d \rightarrow \mathbb{R}_+$ be a function describing a density profile. Then, we distribute particles according to a Poisson measure with slowly varying parameter on \mathbb{T}^d , that is, for each positive L we fix the parameter of the Poisson distribution at site x to be equal to $\rho_0(x/L)$. This is one way to describe a local equilibrium and since this type of measure will appear frequently, we introduce the following terminology.

Definition 10. For each smooth function $\rho_0 : \mathbb{T}^d \rightarrow \mathbb{R}_+$, we represent by $\nu_{\rho_0(\cdot)}^L$ the measure on the state space $\mathbb{N}^{\mathbb{T}_L^d}$ having the following two properties. Under $\nu_{\rho_0(\cdot)}^L$ the variables $(\eta(x) : x \in \mathbb{T}_L^d)$ are independent and, for a site

$x \in \mathbb{T}_L^d$, $\eta(x)$ is distributed according to a Poisson distribution of parameter $\rho_0(x/L)$:

$$\nu_{\rho_0(\cdot)}^L\{\eta; \eta(x) = k\} = \nu_{\rho_0(x/L)}^L\{\eta; \eta(0) = k\}$$

for all x in \mathbb{T}_L^d and k in \mathbb{N} . [11]

Therefore, we have associated to each smooth profile $\rho_0 : \mathbb{T}^d \rightarrow \mathbb{R}_+$ and each positive integer L a Poisson measure on the torus \mathbb{T}_L^d .

Let us take it one step further and define a limit for the Poisson measures with slowly varying parameter. Notice that, as the parameter L increases to infinity, the discrete torus \mathbb{T}_L^d tends to the full lattice \mathbb{Z}^d . We can, also, define a Poisson measure on the space of configurations over \mathbb{Z}^d . For each $\alpha \geq 0$ we will denote by ν_α the probability on $\mathbb{N}^{\mathbb{T}^d}$ that makes the variables $(\eta(x) : x \in \mathbb{Z}^d)$ independent and under which, for every x in \mathbb{Z}^d , $\eta(x)$ is distributed according to a Poisson law of parameter α .

Now, with the definition of $\nu_{\rho_0(\cdot)}^L$, and since $\rho_0 : \mathbb{T}^d \rightarrow \mathbb{R}_+$ is assumed to be smooth, as $L \rightarrow \infty$ and we look very close to a point $u \in \mathbb{T}^d$, around $x = [Lu]$, we observe a Poisson measure of parameter almost constant and equal to $\rho_0(u)$. In fact, since the function $\rho_0(\cdot)$ is smooth, for every positive integer l and for every positive family of parameters $(\lambda(x) : |x| \geq l)$,

$$\lim_{L \rightarrow \infty} E_{\nu_{\rho_0(\cdot)}^L} \left[e^{-\sum_{|x| \geq l} \lambda(x) \eta([uL]+x)} \right] = E_{\nu_{\rho_0(u)}} \left[e^{-\sum_{|x| \geq l} \lambda(x) \eta(x)} \right]. \quad (3.1)$$

In this sense the sequence $\nu_{\rho_0(\cdot)}^L$ describes an example of local equilibrium. We should note here that, for $u = (u_1, u_2, \dots, u_d)$ in \mathbb{R}^d , $\|u\|$ stands for Euclidean norm of u and $|u|$ the max norm:

$$\|u\|^2 = \sum_{1 \leq i \leq d} u_i^2, \quad |u| = \max_{1 \leq i \leq d} u_i.$$

In the configuration space $\mathbb{N}^{\mathbb{T}_L^d}$ we denote by $(\tau_x : x \in \mathbb{T}_L^d)$ the group of translations, namely for a site x , $\tau_x \eta$ is the configuration that, at site y , has $\eta(x+y)$ particles:

$$(\tau_x \eta)(y) = \eta(x+y), \quad y \in \mathbb{T}_L^d.$$

Keep in mind that the action of the translation group extends in a natural way to the space of functions and to the space of probability measures on $\mathbb{N}^{\mathbb{T}_L^d}$. In fact, for a site x and a probability measure μ , $(\tau_x \mu)$ is the measure such that

$$\int f(\eta)(\tau_x \mu)(d\eta) = \int f(\tau_x \eta) \mu(d\eta)$$

for all bounded continuous f .

With this topological setting, equation 3.1 expresses that for all points $u \in \mathbb{T}^d$, the sequence $\tau_{[uL]} \nu_{\rho_0(\cdot)}^L$ converges weakly to the measure $\nu_{\rho_0(u)}^L$. Finally, we will present the following definition.

Definition 11. A sequence of probability measures $(\mu^L)_{L \geq 1}$ on $\mathbb{N}^{\mathbb{T}_L^d}$ is a **local equilibrium** of profile $\rho_0 : \mathbb{T}^d \rightarrow \mathbb{R}_+$ if

$$\lim_{L \rightarrow \infty} \tau_{[uL]} \mu^L = \nu_{\rho_0(u)}$$

for all continuity points u of $\rho_0(\cdot)$. [11]

3.4 Macroscopic Profile

3.4.1 Scaling Limits

In this section, we will address the matter of what will be the macroscopic profile of our system after time t . We assume that the initial state will follow a product measure with slowly varying parameter as defined in definition 10. We see that if we start from a Poisson measure with slowly varying parameter then

$$\begin{aligned} \mathbb{E}_{\nu_{\rho_0(\cdot)}^L} \left[e^{-\sum_{x \in \mathbb{T}_L^d} \lambda(x) \eta_t(x)} \right] &= \\ e^{\sum_{x \in \mathbb{T}_L^d} \rho_0(x/L) \sum_{y \in \mathbb{T}_L^d} p_t^L(y-x) (e^{-\lambda(y)} - 1)} &= \\ e^{\sum_{y \in \mathbb{T}_L^d} (e^{-\lambda(y)} - 1) \sum_{x \in \mathbb{T}_L^d} p_t^L(y-x) \rho_0(x/L)} &=: \\ e^{\sum_{y \in \mathbb{T}_L^d} (e^{-\lambda(y)} - 1) \psi_{L,t}(y)} & \end{aligned}$$

In the above equation [11] the first step is reached by repeating the same computations we did to prove proposition 6, and then by inverting the order of summation, we get that, at time t , we still have a Poisson measure with slowly varying parameter, which is now $\psi_{L,t}(\cdot)$ instead of $\rho_0(\cdot/L)$.

It is true that, up to this point, we have not really discussed much about $p_t(\cdot)$ and how it affects the system. We have only said that it makes $p_t(\cdot, \cdot)$ translation invariant and thus bistochastic: $\sum_x p_t(x, y) = 1$ for every y . Let us now see what happens when t is fixed and L increases to infinity. In this case $p_t(\cdot)$ is a function with essentially finite support, that is, for all $\epsilon > 0$, there exists $A = A(t, \epsilon) > 0$ so that

$$\sum_{x \leq |A|} p_t(x) \geq 1 - \epsilon.$$

From the explicit form of $\psi_{L,t}$, we have that for every continuity point u of ρ_0 ,

$$\lim_{L \rightarrow \infty} \psi_{L,t}([uL]) = \rho_0(u).$$

What the above equation tells us is that the profile remained unchanged. Even though time t have passed, it seems that the system did not have

enough time to evolve and this reflects the fact that at the macroscopic scale the particles did not move. Consider the following test: select a particle at the origin; since it evolves as a discrete time random walk, if X_t denotes its position at time t , for every $\epsilon > 0$, there exists $A = A(t, \epsilon) > 0$ such that $\mathbb{P}(|X_t| > A) \leq \epsilon$. Therefore, with probability close to 1, in the macroscopic scale, the test particle at time t is at distance of order L^{-1} from the origin.

In order to solve this problem, we need to distinguish between two different time scales, as we already have different space scales, \mathbb{T}^d and $L^{-1}\mathbb{T}_L^d$. Respectively, we need a microscopic time t and a macroscopic time which would be infinitely large with respect to t .

To introduce the macroscopic time scale, notice that the transition probabilities $p_t(\cdot)$ are equal to

$$p_t(x) = \sum_{k=0}^{\infty} e^{-t} \frac{t^k}{k!} p^{*k}(x),$$

where p^{*k} stands for the k -th convolution power of the elementary transition probability of each particle.

Assume that the elementary transition probability $p(\cdot)$ has finite expectation: $m := \sum xp(x) \in \mathbb{R}^d$. We say that the random walk is *asymmetric* if $m \neq 0$, that it is *mean-zero asymmetric* if $p(\cdot)$ is not symmetric but $m = 0$ and that it is *symmetric* if $p(\cdot)$ is symmetric. Recall that X_t stands for the position at time t of a discrete time random walk with transition probability $p(\cdot)$ and initially at the origin. By the law of large numbers for random walks, for all $\epsilon > 0$,

$$\lim_{L \rightarrow \infty} \sum_{x; |x/L - mt| \leq \epsilon} p_{tL}(x) = \lim_{L \rightarrow \infty} \mathbb{P} \left(\left| \frac{X_{tL}}{L} - mt \right| \leq \epsilon \right) = 1$$

In particular, from the explicit expression for $\psi_{L,tL}$ and since we assumed the initial profile to be smooth, we have that

$$\lim_{L \rightarrow \infty} \psi_{L,tL}([uL]) = \rho_0(u - mt) =: \rho(t, u)$$

for every u in \mathbb{T}^d .

Now you can see that with the new time scale, tL the profile did not remain unchanged. On the contrary, we observe a new macroscopic profile: the original one translated by mt . More precisely, in this macroscopic scale tL we observe a local equilibrium profile that has been translated by mt since $\psi_{L,tL}$ is itself slowly varying in the macroscopic scale.

3.4.2 Hydrodynamic Equation

Of course, the profile $\rho(t, u)$ satisfies the partial differential equation

$$\partial_t \rho + m \cdot \nabla \rho = 0$$

if $\nabla\rho$ denotes the gradient of ρ : $\nabla\rho = (\partial_{u_1}\rho, \partial_{u_2}\rho, \dots, \partial_{u_d}\rho)$.

We see that if we restrict ourselves to a particular class of initial measures, we are able to establish the existence of a time and space scales in which the particles density evolves according to the linear partial differential equation. Consequently, we have derived from the microscopic stochastic dynamics a macroscopic deterministic evolution for the unique conserved quantity.

An interacting particle system for which there exists a time and space macroscopic scales in which the conserved quantities evolve according to some partial differential equation is said to have a *hydrodynamic description*. Moreover, the partial differential equation is called the *hydrodynamic equation* associated to the system.

Proposition 7. *A system of particles evolving as independent asymmetric random walks with finite first moment on a d -dimensional torus has a hydrodynamic description. The evolution of the density profile is described by the solution of the differential equation*

$$\partial_t\rho + m \cdot \nabla\rho = 0.$$

However, when the random walk is not asymmetric and the expectation m vanishes, the solution of this differential equation is constant, which means that the profile didn't change in the time scale tL . This makes sense intuitively, as the system now is unbiased in direction. Still, if we consider a larger time scale, times of order L^2 , even when the mean displacement m vanishes, we can observe an interesting time evolution. [11]

Let $(S^L(t) : t \geq 0)$ be the semigroup associated to the Markov process $(\eta_t)_{t \geq 0}$. In Proposition 7, we have proved that there is a time renormalization θ_L such that

$$\lim_{L \rightarrow \infty} S^L(t\theta_L)\tau_{[uL]}\nu_{\rho_0(\cdot)}^L = \nu_{\rho(t,u)},$$

for all $t \geq 0$ and all continuity points u of $\rho(t, \cdot)$.

All in all, we have proved in this chapter the following results:

- Description of the equilibrium states of the system.
- Conservation of the local equilibrium in time evolution.
- Characterization at a later time of the new parameters describing the local equilibrium and derivation of a partial differential equation that determines how the parameters evolve in time.

3.5 Equivalence of Ensembles

It is true that, we have chose a class of invariant measures to describe the equilibrium states (the Poisson measures) when others would seem more

appropriate. Following remark 2 and the fact that we want to describe the equilibrium state associated to a given density on the torus \mathbb{T}^d , we would be led to study the behavior, as $L \rightarrow \infty$ and β is kept fixed, of

$$\nu_\alpha^L \left(\cdot \left| \sum_{x \in \mathbb{T}_L^d} \eta(x) = L^d \beta \right. \right).$$

Performing a simple computation presented below, we get that for each fixed positive integer r , and for all sequences (k_1, \dots, k_r) in \mathbb{N}^r and (x_1, \dots, x_r) in \mathbb{Z}^r ,

$$\begin{aligned} \lim_{L \rightarrow \infty} \nu_\alpha^L \left(\eta(x_1) = k_1, \dots, \eta(x_r) = k_r \left| \sum_{x \in \mathbb{T}_L^d} \eta(x) = L^d \beta \right. \right) \\ = \nu_\beta(\eta(x_1) = k_1, \dots, \eta(x_r) = k_r). \end{aligned}$$

Indeed, it is easy to check by computing the Laplacian transform that the addition of independent Poisson distributions, is still a Poisson distribution with parameter equal to the sum of the parameters. Consequently, the left hand side of the above formula is equal to

$$\begin{aligned} \frac{\alpha^{k_1 + \dots + k_r} e^{-L^d \alpha} ((L^d - r)\alpha)^{L^d \beta - (k_1 + \dots + k_r)}}{k_1! \dots k_r! (L^d \beta - (k_1 + \dots + k_r))!} \left(e^{-L^d \alpha} \frac{(L^d \alpha)^{L^d \beta}}{(L^d \beta)!} \right)^{-1} = \\ \frac{(L^d - r)^{L^d \beta - (k_1 + \dots + k_r)}}{k_1! \dots k_r! (L^d)^{L^d \beta}} (L^d \beta) (L^d \beta - 1) \dots (L^d \beta - (k_1 + \dots + k_r) + 1) \end{aligned}$$

which, as $L \rightarrow \infty$, converges to

$$\frac{\beta^{k_1 + \dots + k_r}}{k_1! \dots k_r!} e^{-r\beta}.$$

So, we see that the Poisson measures are "natural" for our system and at the same time computations are made much easier and the definition of local equilibrium is expressed in a very simple and elegant way in terms of these measures. In this sense, this fact is known to the physicists as the "equivalence of ensembles".

Chapter 4

Simple Exclusion Process

In this chapter we are interested in Simple Exclusion processes (SEP). This model was introduced in [16] and it is among the simplest and most widely studied interacting particle systems. The Simple Exclusion process, in contrast with the independent random walks studied in the previous chapter, allows at most one particle per site.

4.1 Model

First of all, the state space is $\{0, 1\}^{\mathbb{T}_L^d}$. In order to prevent the occurrence of more than one particle per site we introduce an exclusion rule that suppresses each jump to an already occupied site. In fact, we shall focus only on the simplest class of exclusion processes: systems where particles jump, whenever the jump is allowed, independently of the others and according to the same translation invariant elementary transition probability.

Definition 12 (Elementary jump probability). Let p be a finite range, translation invariant, irreducible transition probability on \mathbb{Z}^d :

$$p(x, y) = p(0, y - x) =: p(y - x)$$

for all pair (x, y) of d -dimensional integers and for some finite range probability measure $p(\cdot)$ on \mathbb{Z}^d :

$$\sum_{z \in \mathbb{Z}^d} p(z) = 1 \text{ and } p(x) = 0 \text{ for } |x| \text{ large enough. [11]}$$

The generator

$$(\mathcal{L}f)(\eta) := \sum_{x \in \mathbb{T}_L^d} \sum_{z \in \mathbb{T}_L^d} \eta(x)(1 - \eta(x + z))p^L(z)(f(\eta^{x, x+z}) - f(\eta)),$$

where $\eta^{x,y}$ is the configuration obtained from η letting a particle jump from x to y , namely

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y, \\ \eta(x) - 1 & \text{if } z = x, \\ \eta(y) + 1 & \text{if } z = y \end{cases} \quad \text{and } p^L(z) := \sum_{y \in \mathbb{Z}^d} p(z + yL),$$

defines a Markov process called simple exclusion process with elementary jump probability $p(\cdot)$. In the particular case where $p(z) = p(-z)$ we say that it is a symmetric simple exclusion process.

We believe that the interpretation is clear. Between 0 and dt each particle tries, independently from the others, to jump from x to $x+z$ with rate $p^L(z)$. The jump is suppressed if it leads to an already occupied site.

We remind here that a Markov process is said to be irreducible if it is possible to get to any state from any state. Furthermore, since the transition probability is assumed to be of finite range, there exists A_0 in \mathbb{N} such that $p(z) = 0$ for all sites outside the cube $[-A_0, A_0]^d$. In particular, $p^L(\cdot)$ and $p(\cdot)$ coincide provided $L \geq A_0$. For this reason, from now on we omit the superscript L in the elementary jump probability.

At this point, it might be worthwhile to justify the terminology. The rule that forbids jumps to occupied sites explains the term exclusion. Notice, on the other hand, that the rate at which a particle jumps from x to y depends on the configuration η only through the occupation variables $\eta(x)$ and $\eta(y)$. This last dependence on $\eta(x)$ and $\eta(y)$ reflects the exclusion rule. Finally, notice that the total number of particles is conserved by the dynamics.

4.2 Bernoulli Measure

We denote by $\nu_\alpha = \nu_\alpha^L$, for $0 \leq \alpha \leq 1$, the Bernoulli product measure of parameter α , that is, the product and translation invariant measure on $\{0, 1\}^{\mathbb{T}_L^d}$ with density α . In particular, under ν_α , the variables $(\eta(x) : x \in \mathbb{T}_L^d)$ are independent with marginals given by

$$\nu_\alpha\{\eta(x) = 1\} = \alpha = 1 - \nu_\alpha\{\eta(x) = 0\}.$$

Proposition 8. *The Bernoulli measures $\{\nu_\alpha : 0 \leq \alpha \leq 1\}$ are invariant for simple exclusion processes. In addition, with respect to each ν_α , exclusion processes with elementary jump probability $\check{p}(z) := p(-z)$ are adjoint to processes with elementary jump probability $p(z)$. In particular, symmetric simple exclusion processes are self-adjoint with respect to each ν_α . [11]*

Proof. It is easy to notice that by a simple change of variables

$$\int f(\eta^{0,z})g(\eta)\eta(0)(1 - \eta(z))\nu_\alpha(d\eta) = \int f(\eta)g(\eta^{0,z})\eta(z)(1 - \eta(0))\nu_\alpha(d\eta).$$

This identity, the fact that $1 = \sum_{z \in \mathbb{Z}^d} p(z) = \sum_{z \in \mathbb{Z}^d} p(-z)$ and a change in the order of summation prove the proposition. \square

In this case, the family of invariant measures ν_α is parametrized by the density, for

$$E_{\nu_\alpha}[\eta(0)] = \nu_\alpha\{\eta(0) = 1\} = \alpha.$$

Remark 3. Since the total number of particles is conserved by the dynamics, the measures

$$\nu_{L,N}(\cdot) := \nu_\alpha \left(\cdot \left| \sum_{x \in \mathbb{T}_L^d} \eta(x) = N \right. \right).$$

are invariant and it could have seemed more natural to consider them instead of the Bernoulli product measures ν_α . Nevertheless, a simple computation on binomials shows that for all finite subsets E of \mathbb{Z}^d , for all sequences $\{\epsilon_x : x \in E\}$ with values in $\{0, 1\}$ and for all $0 \leq \alpha \leq 1$,

$$\lim_{L \rightarrow \infty} \nu_\alpha \left\{ \eta(x) = \epsilon_x, x \in E \left| \sum_{y \in \mathbb{T}_L^d} \eta(y) = [\alpha_0 L^d] \right. \right\} = \nu_{\alpha_0} \{\eta(x) = \epsilon_x, x \in E\}$$

uniformly in α_0 . Consequently, the Bernoulli product measures are obtained as limits of the invariant measures $\nu_{L,N}$, as the total number of sites increases to infinity.

4.3 Asymmetric Simple Exclusion Process with step initial condition

One extensively studied variance of the simple exclusion process is the one-dimensional nearest neighbour asymmetric simple exclusion process with step initial condition. In this area, many significant results have been achieved by Tracy and Widom.

Let us consider the integer lattice \mathbb{Z} . In the case of step initial condition, particles will begin from the positive integers \mathbb{Z}_+ . As you know, a particle waits exponential time, then moves to the right with probability p if that site is unoccupied or to the left with probability $q = 1 - p$ if that site is unoccupied. If the site where it is about to jump is occupied, then it stays put.

The main quantity that will concern us in this section is the position of the m th particle from the left at time t , denoted by

$$x_m(t), \text{ with } x_m(0) = m.$$

Here we shall, also, assume that $p < q$, so there is a drift to the left, and establish results on the position of the m th particle and the current of particles. Now we will start by describing the results which were presented and proved in [21].

The authors in [17], derived a formula for the quantity that interests us, valid when p and q are non-zero. It is given in terms of the Fredholm determinant of a kernel $K(\xi, \xi')$ on C_R , a circle with center zero and large radius R described counterclockwise. The Fredholm determinant of a kernel K is the operator determinant $\det(I - \lambda K)$. It acts as an operator by

$$f(\xi) \rightarrow \int_{C_R} K(\xi, \xi') f(\xi') d\xi'$$

for all $\xi \in C_R$. We will use the following notation

$$\gamma = q - p, \quad \tau = q/p.$$

The kernel is

$$K(\xi, \xi') = q \frac{\xi'^x e^{\epsilon(\xi')t/\gamma}}{p + q\xi\xi' - \xi},$$

where

$$\epsilon(\xi) = p\xi^{-1} + q\xi - 1.$$

The first formula is the following

$$\mathbb{P}(x_m(t/\gamma) \leq x) = \int \frac{\det(I - \lambda K)}{\prod_{k=0}^{m-1} (1 - \lambda\tau^k)} \frac{d\lambda}{\lambda}. \quad (4.1)$$

The integral is taken over a contour enclosing the singularities of the integrand at $\lambda = 0$ and $\lambda = \tau - k$, $k = 0, \dots, m - 1$. It is easily derived from the above equation that

$$\mathbb{P}(x_1(t/\gamma) > x) = \det(I - K).$$

It is clear probabilistically that $\mathbb{P}(x_m(t) > x) = 0$ for all t when $x \geq m$, as for a particle to be to the right of its initial position, all particles would have to move simultaneously to the right, which surely has probability zero.

Although, the above formula required $p > 0$, the statement makes sense when $p = 0$. The process where $p = 0$ and the particles move only to the left is called Totally asymmetric simple exclusion process.

For the first asymptotic result, denote by \hat{K} the operator on $L^2(\mathbb{R})$ with kernel,

$$\hat{K}(z, z') = \frac{q}{\sqrt{2\pi}} e^{-(p^2+q^2)(z^2+z'^2)/4+pqzz'}.$$

Assume that $0 < p < q$. For fixed m the limit

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{x_m(t/\gamma) + t}{\gamma^{1/2}t^{1/2}} \leq s \right)$$

is equal to the integral 4.1 with K replaced by the operator $\hat{K}\mathcal{X}_{(-s, \infty)}$. It is easy again to derive the special case

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{x_1(t/\gamma) + t}{\gamma^{1/2}t^{1/2}} > -s \right) = \det(1 - \hat{K}\mathcal{X}_{(s, \infty)}).$$

This is an apparently new family of distribution functions, parametrized by p . When $p = 0$ the kernel has rank one and the determinant equals a standard normal distribution.

Furthermore, we will state the asymptotic result when m and x both go to infinity. We use the notation

$$\sigma = m/t, \quad c_1 = -1 + 2\sqrt{\sigma}, \quad c_2 = \sigma^{-1/6}(1 - \sqrt{\sigma})^{2/3}.$$

Theorem 2. *When $0 \leq p \leq q$ we have*

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{x_m(t/\gamma) - c_1 t}{c_2 t^{1/3}} \leq s \right) = F_2(s)$$

uniformly for σ in a compact subset of $(0, 1)$.

In the above theorem, the function $F_2(s)$ that arises asymptotically is the Tracy-Widom distribution. The Tracy-Widom distributions are a family of probability distributions that were described explicitly by Craig Tracy and Harold Widom [18, 19], and shown to govern the maximal eigenvalue of large random matrices.

The cumulative distribution function of the Tracy-Widom distribution can be given as an integral

$$F_2(s) = e^{\int_s^\infty (x-s)q^2(x)dx} \quad (4.2)$$

and q is the unique solution to the Painlevé II equation

$$q'' = sq + 2q^3, \quad \text{with boundary condition } q(s) \sim Ai(s) \text{ as } s \rightarrow \infty,$$

where Ai is the Airy function.

Let us now introduce the following quantity. As the particles are initially located at \mathbb{Z}_+ and we have assumed that $p < q$, then there will be on average a net flow of particles, or current, to the left. The *total current* \mathcal{I} at position $x \leq 0$ at time t ,

$$\mathcal{I}(x, t) := \text{number of particles with position } \leq x \text{ at time } t.$$

With step initial condition, it stands that, for $0 \leq c \leq \gamma$, the current \mathcal{I} satisfies the strong law [12],

$$\lim_{t \rightarrow \infty} \frac{\mathcal{I}([-ct], t)}{t} = \frac{1}{4}(\gamma - c)^2.$$

Now, we want to examine the behavior of the current fluctuations

$$\mathcal{I}(x, t) - \frac{1}{4}(\gamma - c)^2 t$$

for large x and t . It has been proved [10, 22] that to obtain a nontrivial limiting distribution the correct normalization of the fluctuations is cube root in t .

Theorem 3. For an asymmetric simple exclusion process with step initial condition we have, for $0 \leq v \leq 1$,

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{\mathcal{I}([-vt], t/\gamma) - \alpha_1 t}{\alpha_2 t^{1/3}} \leq s \right) = 1 - F_2(-s),$$

where $\alpha_1 = \frac{1}{4}(1-v)^2$ and $\alpha_2 = 2^{-4/3}(1-v^2)^{2/3}$. [22]

Proof. We are interested in the probability of the event,

$$\{\mathcal{I}(x, t) = m\} = \{x_m(t) \leq x, x_{m+1}(t) > x\}.$$

The sample space consists of the four disjoint events $\{x_m(t) \leq x, x_{m+1}(t) > x\}$, $\{x_m(t) \leq x, x_{m+1}(t) \leq x\}$, $\{x_m(t) > x, x_{m+1}(t) > x\}$, and $\{x_m(t) > x, x_{m+1}(t) \leq x\}$, and because of the exclusion property, we have

$$\begin{aligned} \{x_m(t) \leq x, x_{m+1}(t) \leq x\} &= \{x_{m+1}(t) \leq x\}, \\ \{x_m(t) > x, x_{m+1}(t) > x\} &= \{x_m(t) > x\}, \\ \{x_m(t) > x, x_{m+1}(t) \leq x\} &= \emptyset. \end{aligned}$$

These observations result to the intuitively obvious

$$\mathbb{P}(\mathcal{I}(x, t) = m) = \mathbb{P}(x_m(t) \leq x) - \mathbb{P}(x_{m+1}(t) \leq x).$$

And, since $\mathbb{P}(\mathcal{I}(x, t) = 0) = \mathbb{P}(x_1(t) > x)$, we have

$$\mathbb{P}(\mathcal{I}(x, t) \leq m) = 1 - \mathbb{P}(x_{m+1}(t) \leq x).$$

Therefore, since x and $x_{m+1}(t)$ are integers, the statement of the Theorem is equivalent to the statement that

$$\lim_{t \rightarrow \infty} \mathbb{P}(x_{m+1}(t/\gamma) \leq -vt) = F_2(s),$$

when $m = [\alpha_1 t - \alpha_2 s t^{1/3}]$. In particular, we shall show that

$$\lim_{t \rightarrow \infty} \mathbb{P}(x_m(t/\gamma) \leq -vt) = F_2(s),$$

when

$$m = \alpha_1 t - \alpha_2 s t^{1/3} + o(t^{1/3}).$$

Now, in order to obtain the last limit from Theorem 2, we determine σ so that

$$-vt = c_1 t + c_2 s t^{1/3}.$$

Thus, after some computations, we get that

$$\sigma = \left(\frac{1-v}{2} \right)^2 - s 2^{-4/3} (1-v^2)^{2/3} t^{-2/3} + o(t^{-2/3}).$$

Since this is exactly the statement that $m = \sigma t$ must satisfy, we see that the Theorem is established. \square

4.4 Universality of the Tracy-Widom Distribution

In this section, we will discuss about the universality that Tracy-Widom distribution has been found to exhibit lately. We will start by understanding how we ended up in such hypothesis.

It is a fact that Random Matrix Theory has found a huge number of applications ranging from statistical physics of disordered systems, quantum information, finance, telecommunication networks to number theory, combinatorics and integrable systems. Among the recent developments in Random Matrix Theory, the study of the largest eigenvalue λ_{max} of large random matrices has attracted particular attention. The first questions were related to the fluctuations of λ_{max} , belonging to the wider topic of extreme value statistics. Such extreme value questions arise naturally in the statistical physics of complex and disordered systems like interacting particle systems. In particular, the eigenvalues of a random matrix provide an interesting "toolset" to study extreme value statistics of strongly correlated random variables.

Biologist Robert May realized in 1972 a natural application of the statistics of λ_{max} , which is to provide a criterion of physical stability in dynamical systems such as ecosystems [14]. May considered a population of N distinct species and introduced strong pair-wise interactions between the species. May assumed that the interactions between pairs of species can be modeled by a random matrix J , of size $N \times N$, which is real and symmetric. A natural question is then: what is the probability, $P_{stable}(\alpha, N)$, where α represents the strength of interactions, that the system remains stable once the interactions are switched on? [14] After some computations, May derived that the system will remain stable, provided the eigenvalues λ_i of the random matrix J satisfy the inequality:

$$\alpha \lambda_i - 1 \leq 0,$$

for all $i = 1, \dots, N$. This is obviously equivalent to the statement that the largest eigenvalue $\lambda_{max} = \max_{1 \leq i \leq N} \lambda_i$ satisfies the inequality:

$$\lambda_{max} \leq \frac{1}{\alpha}.$$

Hence the probability that the system is stable gets naturally related to the cumulative distribution function of the largest eigenvalue λ_{max} .

The Tracy-Widom distribution was first established in 1992, two decades later, by Tracy and Widom [18, 19], who observed it by studying the same concept, namely the fluctuations of the largest eigenvalue λ_{max} of random matrices. Later in 1999, Baik, Deift and Johansson [2], discovered that the same statistical distribution also describes variations in sequences of shuffled integers - a completely unrelated mathematical abstraction. Specifically, let S_N be the group of all permutations of N numbers with uniform distribution and let $l_N(\pi)$ be the length of the longest increasing subsequence of $\pi \in S_N$.

Let χ be a random variable whose distribution function is F_2 (4.2). Then, as $N \rightarrow \infty$

$$\frac{l_N - 2\sqrt{N}}{N^{1/6}} \rightarrow \chi$$

in distribution.

Soon the distribution started to appear in models all over physics and mathematics. Systems of many interacting components kept producing the same statistical curve. This puzzling curve seemed to be the complex cousin of the familiar bell curve, or Gaussian distribution, which represents the natural variation of independent random variables. Like the Gaussian, the Tracy-Widom distribution exhibits universality, a mysterious phenomenon in which diverse microscopic effects give rise to the same collective behavior.

The Tracy-Widom distribution is an asymmetrical statistical bump which is steeper on the left side than the right. Suitably scaled, its summit sits at a telltale value: $\sqrt{2N}$, the square root of twice the number of variables in the systems that give rise to it.

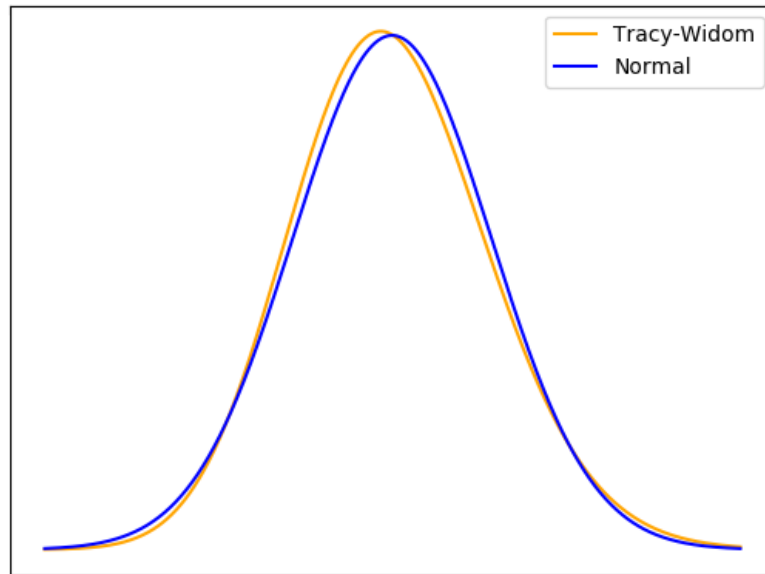


Figure 4.1: Probability density functions of Normal and Tracy-Widom distributions

When the Tracy-Widom distribution turned up in the integer sequences problem and other contexts that had nothing to do with random matrix theory, researchers began searching for the "hidden thread" tying all its manifestations together, just as mathematicians in the 18th and 19th centuries sought a theorem that would explain the ubiquity of the Normal distribution.

The Central Limit Theorem, which was finally made rigorous about a century ago, certifies that natural observations and other "uncorrelated" variables - meaning any of them can change without affecting the rest - will form a bell curve. By contrast, the Tracy-Widom curve appears to arise from variables that are strongly correlated, such as interacting species, stock prices and matrix eigenvalues. The feedback loop of mutual effects between correlated variables makes their collective behavior more complicated than that of uncorrelated variables.

While researchers have rigorously proved certain classes of random matrices in which the Tracy-Widom distribution universally holds, they have a looser handle on its manifestations in counting problems, random walk problems, growth models and beyond.

So far, there have been characterized three forms of the Tracy-Widom distribution: rescaled versions of one another that describe strongly correlated systems with different types of inherent randomness, namely different types of random matrix ensembles. Specifically, the three classes of $F_\beta(s)$ are indexed by

$$\beta = \begin{cases} 1, & \text{for Gaussian orthogonal ensemble} \\ 2, & \text{for Gaussian unitary ensemble} \\ 4, & \text{for Gaussian symplectic ensemble} \end{cases} .$$

But there could be many more than three, perhaps even an infinite number, of Tracy-Widom universality classes.

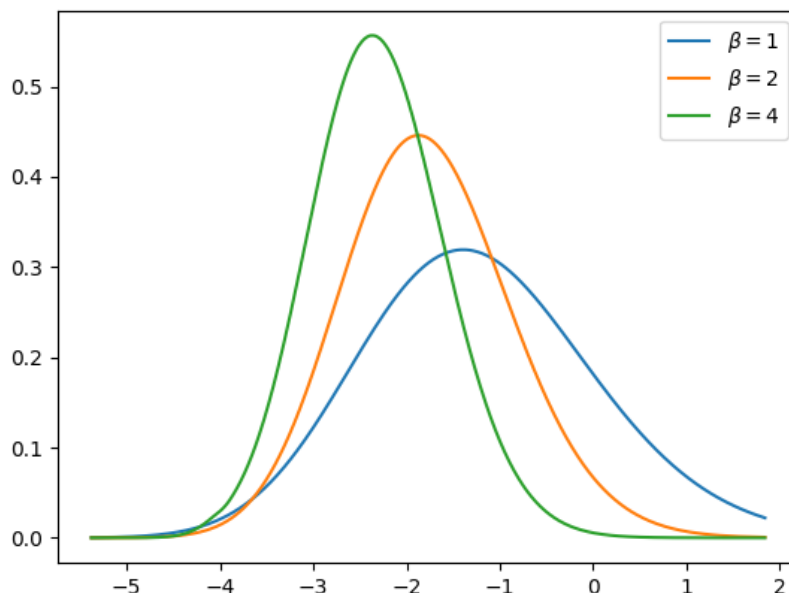


Figure 4.2: Classes of Tracy-Widom distribution

Lately it is being extensively discussed the fact that the asymmetric character of the distribution may represent some kind of universal phase transition [13]. In May's ecosystem model, for example, the critical point at $\sqrt{2N}$ separates a stable phase of weakly coupled species, whose populations can fluctuate individually without affecting the rest, from an unstable phase of strongly coupled species, in which fluctuations cascade through the ecosystem and throw it off balance. In general, systems in the Tracy-Widom universality class exhibit one phase in which all components act in concert, left tail, and another phase in which the components act alone, right tail.

Right now, many physicists and mathematicians are working in the field of seeking some universal law tied to the Tracy-Widom distribution. If such breakthrough were to be achieved, we would be able to interpret the macroscopic elements of systems with interacting components in a much more natural way.

Chapter 5

Zero Range Process

In this chapter we will study another widely known model of Interacting Particle Systems. It was also originally introduced as a simple example of an interacting Markov process in [16]. It is called Zero Range Process and its name originates from the fact that the particles will only interact with particles sitting on the same site.

5.1 Model

As in chapter 3, we will consider evaluations without restrictions on the total number of particles per site. The state space will therefore be $\mathbb{N}^{\mathbb{T}^d}$. The process is defined through a function $g : \mathbb{N} \rightarrow \mathbb{R}_+$ vanishing at zero, which represents the rate at which one particle leaves a site, and a translation invariant transition probability $p(\cdot, \cdot)$ on \mathbb{Z}^d . Its dynamics goes as follows. If there are k particles at a site x , independently of the number of particles on other sites, at rate $g(k)p(x, y)$ one of the particles at x jumps to y . In this way particles interact only with particles in the same site.

Definition 13. Let $g : \mathbb{N} \rightarrow \mathbb{R}_+$ be a function with $g(0) = 0$ and $p(\cdot, \cdot)$ be a finite range, irreducible, translation invariant transition probability. We assume that g is strictly positive on the set of positive integers and that it has bounded variation in the following sense:

$$g^* := \sup_{k \geq 0} |g(k+1) - g(k)| < \infty.$$

Now, let $Z : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the partition function defined by

$$Z(\phi) = \sum_{k \geq 0} \frac{\phi^k}{g!(k)}$$

and denote by ϕ_c the radius of convergence of Z . In the last formula $g!(k)$ stands for $\prod_{1 \leq j \leq k} g(j)$ and by convention $g!(0) = 1$. Furthermore, notice

that Z is analytic and strictly increasing in $[0, \phi_c)$. Assume that $Z(\cdot)$ increases to ∞ as ϕ converges to ϕ_c :

$$\lim_{\phi \rightarrow \phi_c} Z(\phi) = \infty. \quad (5.1)$$

This assumption is not necessary to define the process, but will always be required to prove following results. The generator

$$(\mathcal{L}f)(\eta) = \sum_{x \in \mathbb{T}_L^d} \sum_{z \in \mathbb{T}_L^d} p^L(z) g(\eta(x)) (f(\eta^{x, x+z}) - f(\eta))$$

defines a Markov process on $\mathbb{N}^{\mathbb{T}_L^d}$, called zero range process with parameters (g, p) . Also here, as in chapter 4, $\eta^{x,y}$ represents the configuration η where one particle jumped from x to y and $p^L(\cdot)$ represents the transition probability translated to the origin and restricted to the torus:

$$p^L(z) := p^L(0, z) = \sum_{y \in \mathbb{T}_L^d} p(0, z + yL)$$

for every d -dimensional integer z .

In zero range processes each particle jumps, independently of particles sitting at other sites, from x to y at rate

$$p^L(y - x) g(\eta(x)) \frac{1}{\eta(x)}.$$

In particular, if $g(k) = k$ for every $k \geq 0$, we obtain the superposition of independent random walks studied in chapter 3. On the other hand, the case $g(k) = \mathbf{1}\{k \geq 1\}$ models a system of queues with mean-one exponential random times of service. Moreover, we will study, later, the case where g is a decreasing function, and also perform experiments on it.

5.2 Invariant measures

We now turn our attention, as usual, to the characterization of invariant measures for the process. Since the zero range process is irreducible and the state space is finite, we have a unique invariant measure which we denote by $\mu^{N,L}$. We will refer to the measures $\mu^{N,L}$ as the canonical ensembles. They can be explicitly computed, but they can also be obtained by conditioning the grand-canonical ensembles, whose definition follows, on the total number of particles.

Definition 14. For each $0 \leq \phi < \phi_c$, let $\bar{\nu}_{\phi,g} = \bar{\nu}_{\phi,g}^L$ denote the product measure on $\mathbb{N}^{\mathbb{T}_L^d}$ with marginals given by

$$\bar{\nu}_{\phi,g}\{\eta, \eta(x) = k\} = \frac{1}{Z(\phi)} \frac{\phi^k}{g!(k)}$$

for each $k \geq 0$ and $x \in \mathbb{T}_L^d$.

Proposition 9. For each $0 \leq \phi < \phi_c$ the product measure $\bar{\nu}_{\phi,g}$ is invariant for the zero range process with parameters (g,p) . Moreover, the adjoint process with respect to any of the measures $\bar{\nu}_{\phi,g}$ is the zero range process with parameters (g,\check{p}) . In particular, if p is symmetric the process is self-adjoint. [11]

Proof. The proof relies on the same computations we did for Proposition 8 and on the following identity

$$g(k) \frac{\phi^k}{g!(k)} \frac{\phi^j}{g!(j)} = g(j+1) \frac{\phi^{k-1}}{g!(k-1)} \frac{\phi^{j+1}}{g!(j+1)}.$$

□

Also, because the function $g(\cdot)$ will always be fixed, to keep notation simple, we omit the dependence on g of the measure $\bar{\nu}_{\phi,g}$ and denote it simply by $\bar{\nu}_\phi$. And so,

$$\mu^{N,L}(\cdot) = \bar{\nu}_\phi \left(\cdot \left| \sum_{x \in \mathbb{T}_L^d} \eta(x) = N \right. \right).$$

Now, let $\rho(\phi)$ denote the expected value of the occupation variable, namely the density, under $\bar{\nu}_\phi$:

$$\rho(\phi) = E_{\bar{\nu}_\phi}[\eta(x)] = \frac{1}{Z(\phi)} \sum_{k \geq 0} k \frac{\phi^k}{g!(k)}. \quad (5.2)$$

The range of ρ is the interval $[0, \rho_c)$, with $\rho(0) = 0$ and

$$\rho_c = \lim_{\phi \rightarrow \phi_c} \rho(\phi)$$

the critical density. Also, equation 5.2 can easily be transformed into the following relation, which is usually seen in the concept of partition functions and will be often used later:

$$\rho(\phi) = \frac{Z'(\phi)\phi}{Z(\phi)} = \phi \partial_\phi \log Z(\phi). \quad (5.3)$$

Computing the first derivative of $\rho(\cdot)$ shows that it is strictly increasing.

Remark 4. A natural object of interest is to explore the behavior of these measures in the thermodynamic limit, namely as $N, L \rightarrow \infty$ in such a way that the average particle density N/L converges to a constant ρ . Well, in the subcritical case, when $\rho < \rho_c$, there exists a fugacity ϕ such that $\rho = \rho(\phi)$ and the standard equivalence of ensembles for independent random variables holds [11]. That is, the finite dimensional marginals of the canonical

ensembles $\mu^{N,L}$ converge to the grand-canonical ensemble corresponding to fugacity ϕ . The equivalence of ensembles for critical and supercritical densities, when $\rho \geq \rho_c$, was established in [7]. Using relative entropy methods the authors prove convergence of the finite dimensional marginals of $\mu^{N,L}$ to the grand-canonical ensemble at critical fugacity. Later in [1], the authors showed that in the thermodynamic limit the sites have joint distribution equal to the grand canonical measure at critical density, except one site which accommodates a macroscopically large number of particles.

Furthermore, we have often parametrized the invariant measures by the conserved quantity, which here is the density of particles. For that reason, we change variables in the definition of the invariant measures $\bar{\nu}_\phi$ as follows. For $\alpha \geq 0$, define the product measure ν_α by

$$\nu_\alpha(\cdot) = \bar{\nu}_{\Phi(\alpha)}(\cdot),$$

where $\Phi(\cdot)$ stands for

$$\Phi(\rho) = \begin{cases} \text{inverse of } \rho(\phi), & \text{for } \rho < \rho_c \\ \phi_c, & \text{for } \rho \geq \rho_c \end{cases}$$

In the next lemma we show that assumption 5.1 guarantees that the range of the function $\rho(\cdot)$ is all \mathbb{R}_+ . In this way, we obtained a family $\{\nu_\alpha : \alpha \geq 0\}$ of invariant measures parametrized by the density, since the expected value of the occupation variables $\eta(x)$ under ν_α is equal to α :

$$E_{\nu_\alpha}[\eta(x)] = E_{\bar{\nu}_{\Phi(\alpha)}}[\eta(x)] = \rho(\Phi(\alpha)) = \alpha$$

for every $\alpha \geq 0$. Moreover, a simple computation shows that the function $\Phi(\alpha)$ is the expected value of the jump rate $g(\eta(0))$ under the measure ν_α :

$$\Phi(\alpha) = E_{\nu_\alpha}[g(\eta(0))].$$

Lemma 1. *Recall that we denoted by ϕ_c the radius of convergence of the partition function Z .*

$$\rho_c = \lim_{\phi \rightarrow \phi_c} \rho(\phi) = \infty.$$

Furthermore, for each $0 \leq \phi < \phi_c$ the measure $\bar{\nu}_\phi$ has a finite exponential moment: there exists $\theta(\phi) > 0$ such that

$$E_{\bar{\nu}_\phi}[e^{\theta\eta(0)}] < \infty.$$

Proof. Now to prove this we consider separately two different cases. Assume first that Z is defined for all positive reals, namely the radius of convergence is infinite. Suppose, by contradiction, that the function ρ is bounded by some constant C_0 . Then from equation 5.3 it would follow

$$\partial_\phi \log Z(\phi) \leq C_0 \phi^{-1}.$$

Then, by integrating over ϕ we get that for every $\phi > 1$,

$$Z(\phi) \leq Z(1)\phi^{C_0}.$$

However, this is in contradiction with the fact that

$$Z(\phi) \geq \frac{\phi^k}{g!(k)}$$

for every integer k by the definition of Z .

Let us assume now, for the second part, that the radius of convergence is finite. Fix some positive $\phi_0 < \phi_c$. Since $Z(\cdot)$ is a smooth increasing function, for $\phi \geq \phi_0$,

$$\log Z(\phi) \leq \log Z(\phi_0) + \frac{1}{\phi_0} \int_{\phi_0}^{\phi} \psi \partial_{\psi} \log Z(\psi) d\psi.$$

Since, on the other hand,

$$\rho(\phi) = \phi \partial_{\phi} \log Z(\phi),$$

it follows that

$$\phi_0 \log \left(\frac{Z(\phi)}{Z(\phi_0)} \right) \leq \int_{\phi_0}^{\phi} \rho(\psi) d\psi.$$

Since the left hand side of this inequality, by assumption 5.1, increases to ∞ as $\phi \rightarrow \phi_c$, we obtain that

$$\lim_{\phi \rightarrow \phi_c} \int_{\phi_0}^{\phi} \rho(\psi) d\psi = \infty.$$

Since the function ρ is increasing the first statement of the lemma is proved.

Lastly, notice that

$$E_{\bar{\nu}_{\phi}}[e^{\theta \eta(0)}] = \frac{Z(\phi e^{\theta})}{Z(\phi)}.$$

Therefore, the second statement follows from assumption 5.1. \square

At this point we will give an example of zero range dynamics that does not possess an invariant product measure for each density $\rho \geq 0$. However, because of the previous lemma, the partition function $Z(\cdot)$ cannot satisfy the assumption 5.1.

Example 1. Consider a one-dimensional, nearest neighbor, symmetric zero range process, that is $p(-1) = p(1) = 1/2$, with jump rate $g(k) = (1 + k^{-1})^3$ for $k \geq 1$. Then, $\phi_c = 1$ and the partition function is

$$Z(\phi) = 1 + \sum_{k \geq 1} \frac{\phi^k}{(k+1)^3}$$

so that

$$\lim_{\phi \rightarrow 1} Z(\phi) = 1 + \sum_{k \geq 1} \frac{1}{(k+1)^3} < \infty.$$

Consider an invariant product measure ν . From proposition 4, since ν is invariant, we have that $\int \mathcal{L}\eta(x) d\nu = 0$ for every x . Denote by ϕ_x the expectation of $g(\eta(x))$ under ν : $\phi_x = E_\nu[g(\eta(x))]$. Since $\mathcal{L}\eta(x) = (1/2)(g(\eta(x+1)) + g(\eta(x-1)) - 2g(\eta(x)))$, the previous identity gives that $(\Delta_L \phi)_x = 0$, if Δ_L stands for the discrete Laplacian. This identity forces ϕ_x to be constant; say equal to ϕ .

On the other hand, for every $x \in \mathbb{T}_L^d$ and $\rho > 0$, $\int \mathcal{L}\mathbf{1}\{\eta(x) = a\} \nu(d\eta) = 0$. Since

$$\begin{aligned} \mathcal{L}\mathbf{1}\{\eta(x) = \rho\} &= -g(\alpha)\mathbf{1}\{\eta(x) = \rho\} \\ &\quad + (1/2)\mathbf{1}\{\eta(x) = \rho - 1\}(g(\eta(x+1)) + g(\eta(x-1))), \end{aligned}$$

since the measure ν is assumed to be product and since $E_\nu[g(\eta(x))] = \phi$ is constant, we have that

$$g(\alpha)\nu\{\eta, \eta(x) = \rho\} = \phi\nu\{\eta, \eta(x) = \rho - 1\}.$$

Furthermore, in this example, since $g(k) = (1 + k^{-1})^3$,

$$\rho(\phi) = \sum_{k \geq 1} k \frac{\phi^k}{(k+1)^3}$$

so that

$$\lim_{\phi \rightarrow 1} \rho(\phi) = \sum_{k \geq 1} \frac{k}{(k+1)^3} = \rho_c < \infty.$$

Consequently, for $\rho > \rho_c$, there is no invariant product measure with density ρ .

5.3 Relation to the Simple Exclusion Process

There exists an exact mapping from the one-dimensional zero-range process to the one-dimensional simple exclusion process. This is illustrated in the figure below. The idea is to consider the particles of the zero range process as the zeros (empty sites) of the exclusion process. Then the sites of the zero range process become the moving particles of the exclusion process. This is possible because of the preservation of the order of particles under the simple exclusion dynamics. Note that in the exclusion process we will have L particles hopping on a lattice of $L + N$ sites.

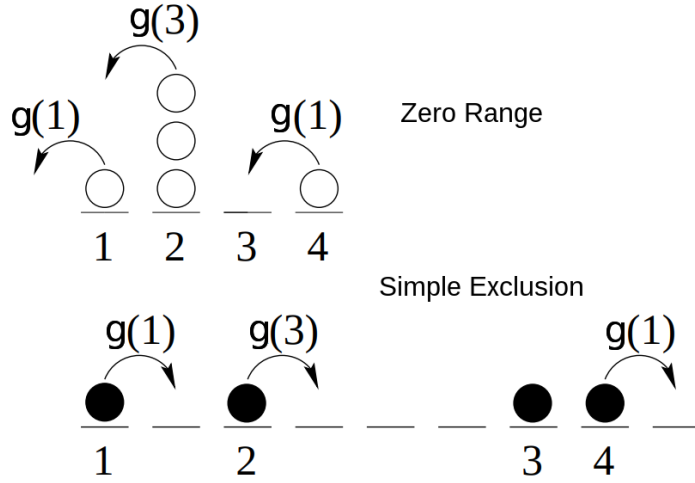


Figure 5.1: Equivalence of zero range process and simple exclusion process

An interesting feature of the mapping is that it converts a model where the local degree of freedom can take unbounded values (particle number in the zero range process) to a model where the local site variable is restricted to two values. On the other hand, a hopping rate $g(k)$ which is dependent on k corresponds to a hopping rate in the exclusion process which depends on the gap to the particle in front. Thus, the particles can "feel" each other's presence and it is possible to have a long-range interaction.

5.4 Supercritical Properties

Throughout this section we will see properties of zero range processes where $\rho_c < \infty$. As a result, there is a critical background density and excess particles condense on a non-extensive fraction of the volume. Precisely, if $\rho > \rho_c$ then

$$\lim_{\substack{N, L \rightarrow \infty \\ N/L \rightarrow \rho}} \mu^{N, L} \left(\max_{x \in \mathbb{T}_L^d} \eta(x) \geq (\rho - \rho_c)L \right) = 1.$$

While, in the subcritical case, the size of the largest component is of order $\log(L)$ [9]. If we were to get the picture of the system in the supercritical phase, we would distribute the bulk of the sites according to independent copies of ν_{ρ_c} and pile all the excess mass on a single randomly located site.

We will now study the case where $g(k)$ is decreasing in k , which then induces an effective attraction between particles. Specifically, the jump rates will be given by

$$g(k) = \begin{cases} 0, & \text{if } k = 0 \\ 1 + \frac{b}{k}, & \text{if } k \geq 1 \end{cases}$$

as studied in [3]. The author, also, observed for $b > 2$ that $\lim_{\phi \rightarrow \phi_c} Z(\phi)$ and ρ_c are both finite. With this choice of g , we also get

$$g!(k) = \frac{(1+b)_k}{k!} = \frac{\Gamma(b+k+1)}{\Gamma(b+1)k!} \sim \frac{k^b}{\Gamma(b+1)},$$

where $(\alpha)_k = \prod_{i=0}^{k-1} (\alpha+i)$ denotes the Pochhammer symbol and $\Gamma(\cdot)$ denotes the standard Gamma function. The grand canonical partition function is

$$Z(\phi) = \sum_{k=0}^{\infty} \frac{(1)_k (1)_k \phi^k}{(1+b)_k k!}$$

and its radius of convergence is $\phi_c = 1$.

At this point, we will analyze the grand-canonical single site measure $\bar{\nu}_\phi = \nu_\rho$ in the limit $\phi \rightarrow 1$, namely near the critical density ρ_c . For $\rho_c < \infty$ the limit $\bar{\nu}_1$ is well defined and it is the distribution of the non-condensed phase for super critical systems with $N/L = \rho > \rho_c$. As long as $\phi < 1$ the distribution $\bar{\nu}_\phi$ has exponential moments. For $\phi = 1$ the exponential tail of $\bar{\nu}_\phi$ disappears and the tail becomes proportional to $1/g!(k)$. These distributions have moments up to order $b-1$. Therefore, different scenarios are encountered as b is varied.

For $b \leq 1$ we get that

$$\begin{aligned} Z(\phi) &\rightarrow \infty, \\ \rho(\phi) &\rightarrow \rho_c = \infty, \end{aligned}$$

as $\phi \rightarrow 1$. For every density, the invariant distribution in the limit $L \rightarrow \infty$ is given by the grand-canonical measure $\bar{\nu}_\phi$. The probability to have a fixed number of particles on a given site vanishes with increasing density. Thus in the limit there is an infinite number of particles on every site with probability one, as it should be for homogeneous systems with $\rho \rightarrow \infty$.

For $1 < b \leq 2$, as we see a change of order, we get that

$$\begin{aligned} Z(\phi) &\rightarrow Z(1) = \frac{b}{b-1}, \\ \rho(\phi) &\rightarrow \rho_c = \infty, \end{aligned}$$

as $\phi \rightarrow 1$. In particular, $\rho_c = \infty$ and the stationary distribution is described by the grand-canonical ensemble for every density ρ .

However, the character of this distribution for large ρ differs from the previous case, where $b \leq 1$. Since $Z(1)$ is finite, $\bar{\nu}_1$ is well defined and there is a positive probability to have a fixed number of particles at a given site,

$$\begin{aligned} \bar{\nu}_1(0) &= \frac{1}{Z(1)} = \frac{b-1}{b}, \\ \bar{\nu}_1(k) &= \frac{1}{Z(1)g!(k)} \sim \Gamma(b)(b-1)k^{-b} \text{ for large } k. \end{aligned}$$

For example, the probability of an empty site, given by

$$\nu_\rho(0) = \frac{1}{Z(\Phi(\rho))},$$

decreases monotonically. Note that in case $b \leq 1$ the probability vanishes in the limit $\rho \rightarrow \infty$, while here, it reaches the non-zero value $\bar{\nu}_1(0) = (b-1)/b$. So no matter how large the density, the fraction of empty sites in a typical configuration is always greater than $(b-1)/b$.

For $b > 2$, besides the normalization also the first moment of the grand canonical distribution converges:

$$Z(\phi) \rightarrow Z(1) = \frac{b}{b-1},$$

$$\rho(\phi) \rightarrow \rho_c = \frac{1}{b-2},$$

as $\phi \rightarrow 1$. In addition, for $b > 3$ also the second moment σ^2 of the distribution $\bar{\nu}_1$ is finite, with

$$\sigma^2 = \frac{(b-1)^2}{(b-2)^2(b-3)},$$

and the number of particles satisfies the usual central limit theorem

$$\frac{\sum \eta(x) - \rho_c L}{\sigma \sqrt{L}} \sim \mathcal{N}(0, 1).$$

One more thing we should note is that the invariant distribution investigated so far carries no information on the dynamics of the condensation. A natural initial condition is to start with particles uniformly distributed at the supercritical density $\rho > \rho_c$. In the beginning the excess particles condense at a few random sites. Such a site containing many excess particles is called a cluster site. On the remaining sites, called bulk sites, the distribution relaxes to $\bar{\nu}_1$. With increasing time the larger clusters will gain particles at the expense of the smaller ones, causing some of the clusters to disappear. Eventually only a single cluster containing all excess particles survives, which is typical for the invariant distribution, as was discussed starting this section. We will observe the above image later in our simulations.

Chapter 6

Experimental Results

In this chapter we will show all the results we acquired from our experiments on the subject. We have simulated models of Interacting Particle Systems that we studied in the previous chapters and furthermore, we have tried to determine various scaling limits. In addition, we give an efficient method for simulating such Interacting Particle Systems.

6.1 Independent Random Walks

The dynamics of independent random walks were specified in section 3.1. Moreover, in section 3.2, it was proved that for a system of indistinguishable particles following random walks, there is a unique family of invariant measures, parametrized by the density of particles, called the Poisson measure.

6.1.1 Invariant Distribution

First, let us validate the result of the invariant distribution. We consider an one-dimensional torus with 1000 sites. Then, we distribute the particles initially according to the Poisson measure and we observe the distribution at later times of order N and N^2 . In the following figures, we see, for different particle densities, that this result, indeed, stands.

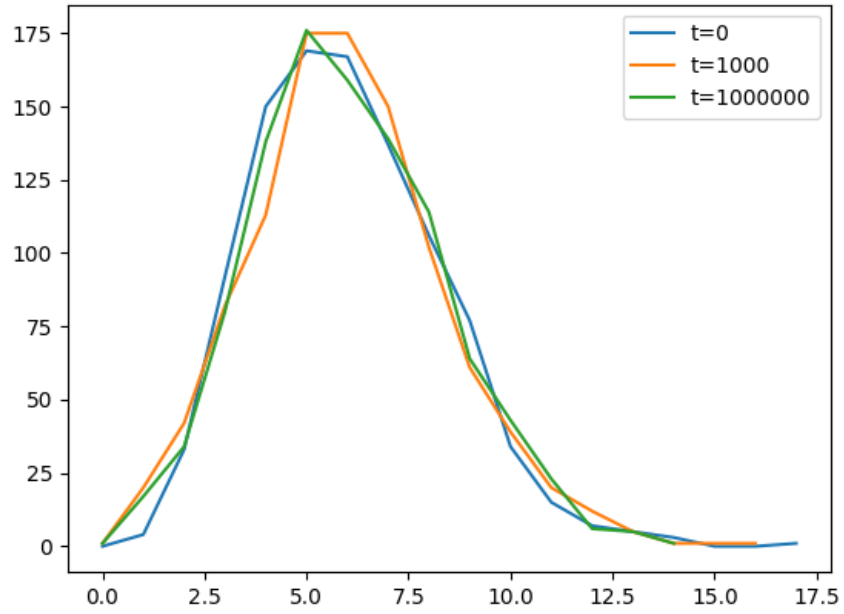


Figure 6.1: Distribution at time t of IRW on \mathbb{T}_{1000}^1 with $\alpha = 5$

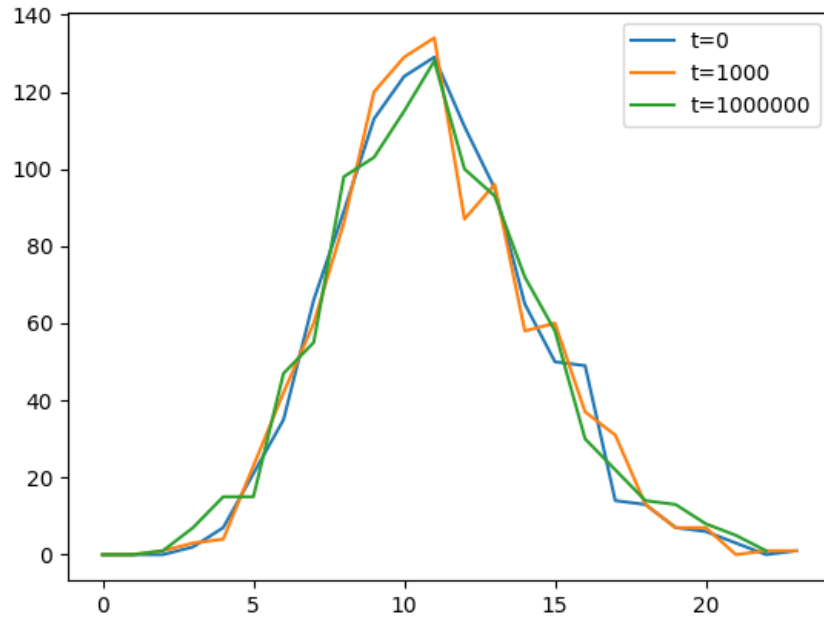


Figure 6.2: Distribution at time t of IRW on \mathbb{T}_{1000}^1 with $\alpha = 10$

6.1.2 2D Visualization

In addition, we wanted to visualize the evolution of a system of independent random walks in two dimensions. In order to achieve that, we have considered an image which represent the two-dimensional torus and we created a logarithmic color scale for the number of particles in each site. According to the normalized logarithmic value of the number of particles, the color is outputted in the following spectre:

$$\text{white} \rightarrow \text{yellow} \rightarrow \text{red} \rightarrow \text{black}. \quad (6.1)$$

In the images placed below, we see the evolution of the system, with total density 10, after some time, with the particles initially positioned at the center of the torus. In the first image, the elementary transition probability $\{p(x, y) : (x, y) \in \mathbb{T}_{100}^2\}$ is symmetric, when in the second is asymmetric with:

$$p(1, 0) = 0.3 \text{ (right)}, p(-1, 0) = 0.1 \text{ (left)},$$

$$p(0, 1) = 0.5 \text{ (up)}, p(0, -1) = 0.1 \text{ (down)}.$$

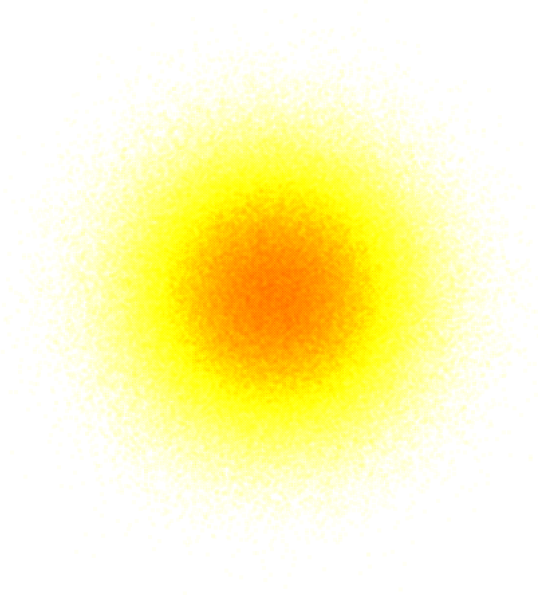


Figure 6.3: Symmetric IRW on \mathbb{T}_{100}^2

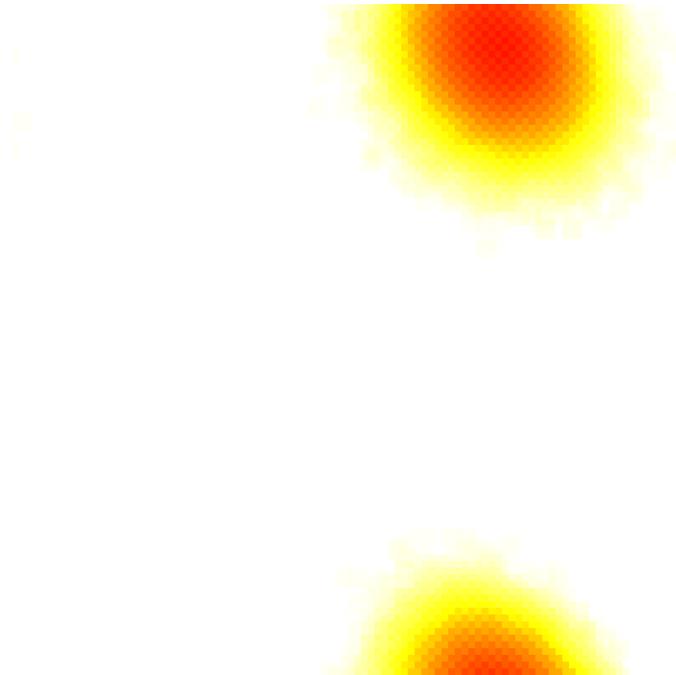


Figure 6.4: Asymmetric IRW on \mathbb{T}_{100}^2 at time $t_1 > 0$

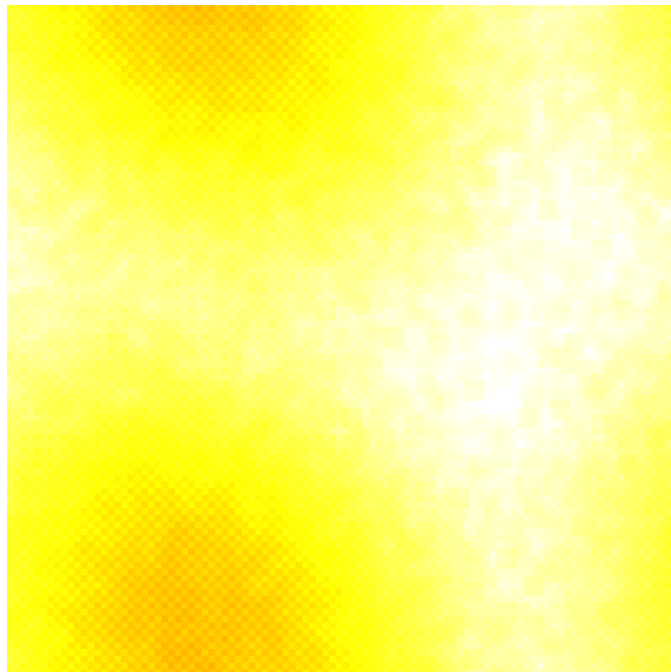


Figure 6.5: Asymmetric IRW on \mathbb{T}_{100}^2 at time $t_2 > t_1$

6.2 Continuous to Discrete Time Simulations

In this section we see an efficient method for simulating particle systems with interaction. This method was not needed for the previous model, as interaction does not actually exist. Throughout the previous chapters we define the time evolution of our systems to be continuous and that is something necessary in order to support other concepts like scaling limits. However, as you know, in a computer, the evolution must happen in discrete times.

In section 2.2 we said that a Markov chain is a Markov process $(\eta_t)_{t \geq 0}$ defined on a countable set and it is characterized by the transition rates $c(\zeta, \zeta') \geq 0$. Furthermore, we denoted by

$$W_\zeta = \inf\{t \geq 0 : \eta_t \neq \zeta\}$$

the holding time in state ζ , and by

$$c_\zeta = \sum_{\zeta' \neq \zeta} c(\zeta, \zeta').$$

the total exit rate out of state ζ . Obviously if $c_\zeta = 0$, then $W_\zeta = \infty$. After that, we proved proposition 1, which certifies that if $c_\zeta > 0$, then

$$W_\zeta \sim \text{Exp}(c_\zeta),$$

where $\text{Exp}(\lambda)$ denotes the exponential distribution with rate λ .

While simulating a system of particles with interaction on a torus \mathbb{T}_L^d , there is always a large loop, in which each iteration represents a transition, namely the jump of one particle. The choice of the particle is strongly connected with the holding time in its state. Note here that the transition rates for the zero range process are:

$$c(\eta, \eta^{x,y}) = g(\eta(x))p(y-x)\frac{1}{\eta(x)}.$$

With this notation, the steps of the simulation of zero range processes (similarly in SEP) are expressed by the following algorithm.

```

rates  $\leftarrow$  sum{ $g(\eta(x))$  for each  $x$  in  $\mathbb{T}_L^d$ }
time  $\leftarrow$  0
while true do
  choose a site  $x$  with probability  $g(\eta(x))/\textit{rates}$ 
  choose a site  $y$  with probability  $p(y-x)$ 
   $\eta(x) \leftarrow \eta(x) - 1$ 
   $\eta(y) \leftarrow \eta(y) + 1$ 
  time  $\leftarrow$  time +  $\text{Exp}(\textit{rates})$ 
  rates  $\leftarrow g(\eta(x)) + g(\eta(y)) - g(\eta(x) + 1) - g(\eta(y) - 1)$ 
end while

```

We believe that all the steps are pretty straight-forward. Remember remark 1, where we proved that

$$\min(\text{Exp}(\lambda_1), \dots, \text{Exp}(\lambda_n)) \sim \text{Exp}\left(\sum_{i=1}^n \lambda_i\right).$$

This justifies the increments in *time* variable.

6.3 Simple Exclusion Process

The dynamics of simple exclusion process were specified in section 4.1. Furthermore, in section 4.2, it was proved that for a system of interacting particles following simple exclusion process, there is a unique family of invariant measures, parametrized by the density of particles, called the Bernoulli measure.

Later in that chapter, in section 4.3, we presented the nearest neighbor asymmetric simple exclusion process with step initial condition. We will now perform experiments regarding the behavior of the distance $|X_1|$ that the marginal particle has covered on a given time.

First, we will establish the dependence of the mean value of $|X_1(t)|$ from the time t , for both totally asymmetric and non-totally asymmetric cases. We assume that

$$E(|X_1(t)|) \sim ct^\alpha,$$

which is expressed linearly

$$\log E(|X_1(t)|) \sim \alpha \log t + \log c.$$

For the totally asymmetric case we found that

$$E(|X_1(t)|) \sim t.$$

Now, remember the notation $\gamma = q - p$, where p is the probability that a particle jumps one place at the right and q that jumps one place at the left. When in the totally asymmetric case is $\gamma = 1$, we also consider the asymmetric case where $\gamma = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$. The result in this case was

$$E(|X_1(t)|) \sim \frac{t}{2} = \gamma t.$$

Finally, one could say that

$$E(|X_1(t/\gamma)|) \sim t$$

for every $\gamma \in (0, 1]$.

Next, we need to establish the order of the fluctuations around the mean value as the time increases. Following the same strategy as before we found for the totally asymmetric case that

$$|X_1(t)| - E(|X_1(t)|) \sim \mathcal{O}(\sqrt{t}).$$

Now regarding the fluctuations for the non-totally asymmetric version we found that

$$|X_1(t)| - E(|X_1(t)|) \sim \mathcal{O}(t^{0.6}).$$

And then, by looking at the following plot of the distribution of the translated and normalized value of $|X_1(t)|$ for the non-totally asymmetric case and the following table we notice an extraordinary similarity with the Tracy-Widom distribution.

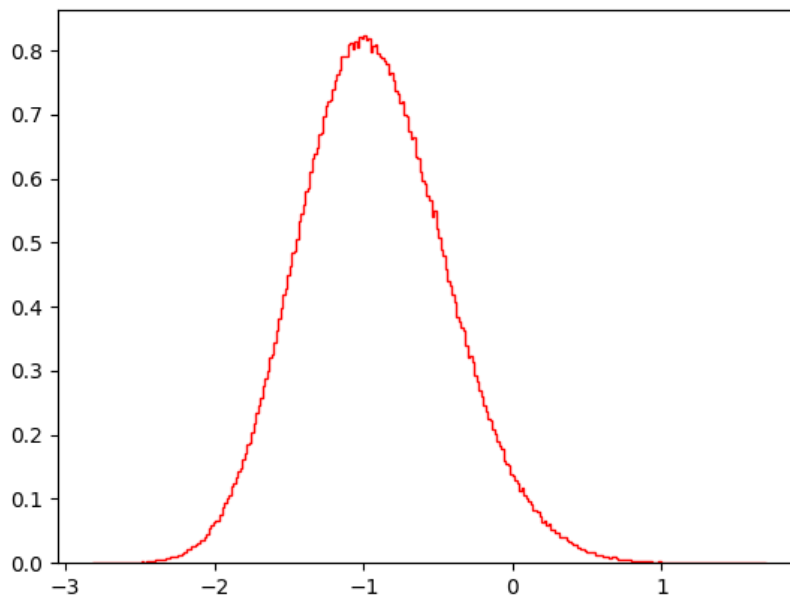


Figure 6.6: Distribution of $|X_1(t)|$ with $\gamma = 0.5$

Mean	Variance	Skewness	Excess kurtosis
-0.93205	0.241756	0.23716	0.0714276

Table 6.1: Characteristics of the distribution

6.4 Zero Range Process

Throughout this section we will perform simulations on Zero Range Processes, considering the Evans model [3], where

$$g(k) = \begin{cases} 0, & \text{if } k = 0 \\ 1 + \frac{b}{k}, & \text{if } k \geq 1 \end{cases}$$

In addition, we will consider only supercritical cases with density $\rho > \rho_c$ and $b > 2$ to ensure that $\rho_c < \infty$.

The following image represents a two-dimensional torus equipped with the normalized logarithmic color scale described in 6.1. In order to produce it, we have let a zero range process, with particles uniformly initially distributed, unfold for a considerable amount of time t . Then each site in the picture represents the maximum number of particles accommodated in that site for some time $t' \leq t$. With that information we can roughly deduce all the system evolution from time 0 in the following way.

As we described in section 5.4, starting from initial condition with particles uniformly distributed at the supercritical density $\rho > \rho_c$, the excess particles condense at a few random sites, called cluster sites. With increasing time the larger clusters will gain particles at the expense of the smaller ones, causing some of the clusters to disappear. In that way, if you choose a threshold density $\rho_{th} > \rho_c$ and filter the image with it, then you get the cluster sites for some arbitrary time. And if you increase that density, then you get fewer cluster sites which correspond to some later time.

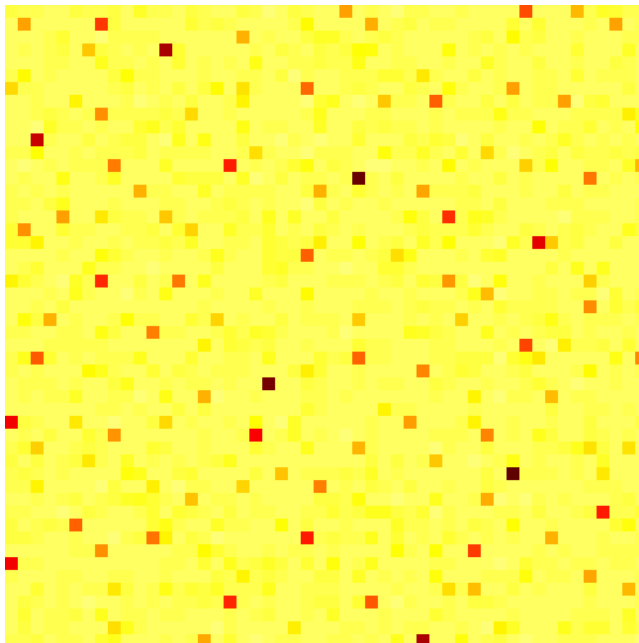


Figure 6.7: Zero Range Process with state $\zeta_t(x) = \max_{0 \leq t' \leq t} \eta_{t'}(x)$

6.4.1 Scaling Order

Now, we will attempt to determine the order of the time needed for the system to reach equilibrium. We will work in one dimension. Remember here that the equilibrium state contains one condensed site and all the others distributed according to ν_{ρ_c} . Consider the following three initial distributions:

1. $\eta(x) = \rho$, $x \in \mathbb{T}_L$
2. $\eta(0) = \rho L - 2\rho_c(L - 1)$ and $\eta(x) = 2\rho_c$, $x \in \mathbb{T}_L/\{0\}$
3. $\eta(0) = \rho L - 4\rho_c L/5$,
 $\eta(x) = 4\rho_c$, $x \in [2L/5, 3L/5)$, otherwise $\eta(x) = 0$

The last case is visualized as follows.

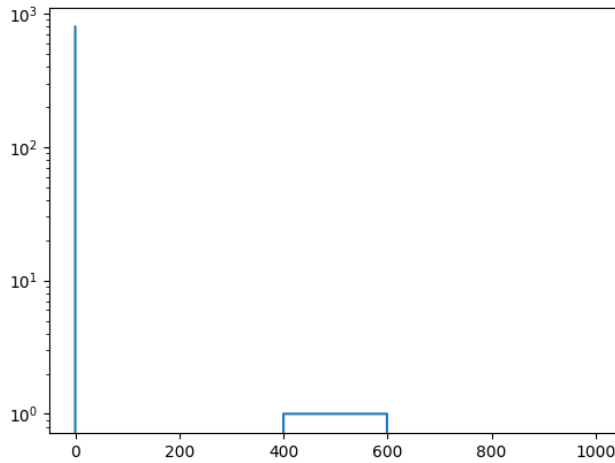


Figure 6.8: Example with $L = 1000$, $p = 1$, $p_c = 0.1$

Although the actual time for reaching equilibrium from these states is different, the order of the time is the same. After running simulations for two different cases of elementary jump probability we found the following results. In a totally asymmetric zero range process, namely with $p(1) = 1$, we have that

$$T_{eq} = \mathcal{O}(L^2).$$

When in a symmetric zero range process, namely with $p(-1) = p(1) = 1/2$, we have that,

$$T_{eq} = \mathcal{O}(L^3).$$

We would expect the time in the symmetric case to be of higher order as the particles diffuse without a drift.

6.4.2 Tagged Particles

Here we will assume the third initial distribution described above and we will study the dynamics of a tagged particle, whose location will be traced, in contrast with the others indistinguishable particles. Consider the following three classes of a tagged particle. If a particle is about to jump from the site x where the tagged particle is located,

- First class: then this particle will always be the tagged one,
- Random: then this particle will be the tagged one with probability $1/\eta(x)$,
- Second class: then this particle will be the tagged one if it is the last one at the site.

We are called here to determine the dependence of the distance of the tagged particle from its initial position from the time t . The tagged particle will be initially located at site $x = L/2$. We also scale the space by L and the time by L^2 [15]. The simulations gave us:

- First class: $X_{tag}(tL^{-2})/L = \mathcal{O}(1)$,
- Random: $X_{tag}(tL^{-2})/L = \mathcal{O}(\sqrt{t})$,
- Second class: $X_{tag}(tL^{-2})/L = \mathcal{O}(\sqrt{t})$.

We suspect that the first class tagged particles do not seem to depend on the time because of their tendency to reach fast the condensed site.

Chapter 7

Conclusion

Closing this diploma thesis, we would like to do a short review of the ideas and problems studied and sum up its contribution. Furthermore, we will describe, in a bit, some possible future lines of work on the subject.

7.1 Contribution

We started by establishing all the essential mathematical foundations, derived from the field of Stochastic Processes, in order to describe completely the notion of Interacting Particle Systems. This necessity has emerged due to the wide range of applications of Interacting Particle Systems in natural problems. The next step, was to define the simplest form of a particle system, namely the independent random walks, and then start asking questions about the underlying behaviors. These questions included the existence of invariant distributions in the system, the deduction of the macroscopic profile of the system in given time and space scales, defining these scales, etc. After that, it was time to turn our attention to a bit more complex Interacting Particle Systems, like the Simple Exclusion Process and the Zero Range Process. These models along with many popular variances of them are already widely studied by the researchers. In one of the variances of Simple Exclusion Process, the asymmetric one with step initial condition, the Tracy-Widom distribution appears. Occasioned by that, follows a discussion regarding the universality that this specific distribution seem to exhibit lately. In addition, while studying Zero Range Processes, we focused especially on a variance where attraction is introduced between the particles. Such model has started to be applied more and more often. Furthermore, the simulations which we performed gave us a very good intuitive understanding of our models and help us determine and validate several properties.

7.2 Future Work

As future lines of work on the field, we would suggest gaining a deeper understanding on the properties of Tracy-Widom distribution, as it seem to play a central role in concepts with interacting components. Moreover, it would be very advantageous to apply Variance Reduction Techniques on the simulations including tagged particles, as they could manifest considerably big fluctuations due to the nature of the quantity we want to approximate.

Συνοπτική Ελληνική Εκδοχή

Τα Συστήματα Αλληλεπιδρώντων Σωματιδίων είναι μοντέλα που συναντώνται σε πολλά φυσικά συστήματα, για τα οποία οι τοπικοί μηχανισμοί είναι πολύ απλοί, αλλά μπορεί να είναι πολύ δύσκολο να εξαχθεί μια καθολική συμπεριφορά. Στις περισσότερες περιπτώσεις, οι παράγοντες που συμβάλλουν στη δυσκολία αυτή είναι η εισαγωγή στοχαστικής δυναμικής και αλληλεπίδρασης σε αυτά. Παραδείγματα μπορούν να εντοπιστούν σε προβλήματα από τις φυσικές επιστήμες, όπως συστήματα διάχυσης της αντίδρασης και σωματίδια αερίων, που επεκτείνονται σε προβλήματα από τις κοινωνικές επιστήμες, όπως η ροή της κυκλοφορίας, η δυναμική της γνώσης και η εξάπλωση των επιδημιών.

Ο κύριος σκοπός της μελέτης τέτοιων Συστημάτων Αλληλεπιδρώντων Σωματιδίων είναι η εξαγωγή της μακροσκοπικής συμπεριφοράς, η οποία συνήθως περιγράφεται από υδροδυναμικές εξισώσεις, από τη μικροσκοπική αλληλεπίδραση, δηλαδή από τους υποκείμενους στοχαστικούς μηχανισμούς. Ο μαθηματικός όρος που συσχετίζεται με αυτό ονομάζεται *όρια κλιμάκωσης*. Τα όρια κλιμάκωσης έχουν μεγάλο ενδιαφέρον για τη φυσική και ιδιαίτερα για τα μαθηματικά. Επιπλέον, μας ενδιαφέρει συνήθως να βρούμε καταστάσεις ισορροπίας στο σύστημά μας, δηλαδή όταν το μακροσκοπικό προφίλ δεν αλλάζει. Αυτές οι ισορροπίες περιγράφονται μαθηματικά με τον χαρακτηρισμό *αναλλοίωτων κατανομών*.

Ανεξάρτητοι Τυχαίοι Περίπατοι

Σε αυτήν την ενότητα θέλουμε να διερευνήσουμε το σύστημα των μη διαχωρίσιμων σωματιδίων που εκτελούν ανεξάρτητους τυχαίους περιπάτους. Ο κύριος στόχος μας είναι ο αναγνώστης να εξοικειωθεί με βασικές έννοιες και τεχνικές που χρησιμοποιούνται στον τομέα των Συστημάτων Αλληλεπιδρώντων Σωματιδίων.

Μοντέλο

Σημειώστε με \mathbb{Z}^d το d -διάστατο πλέγμα ακέραιων αριθμών. Για ένα θετικό ακέραιο L , σημειώστε με \mathbb{T}_L τον τόμο με L σημεία: $\mathbb{T}_L = \mathbb{Z}_L = \{0, 1, \dots, L-1\}$ και $\mathbb{T}_L^d = (\mathbb{T}_L)^d$. Εδώ το L αντιπροσωπεύει το αντίστροφο της απόστασης

μεταξύ των σημείων \mathbb{T}_L^d , δηλαδή των περιοχών των σωματιδίων, οι οποίες αντιπροσωπεύονται από x , y και z . Αφήνοντας $L \rightarrow \infty$, η απόσταση μεταξύ των σωματιδίων θα πάει στο μηδέν και έτσι περνάμε από το μικροσκοπικό στο μακροσκοπικό πεδίο.

Θέλουμε να περιγράψουμε την εξέλιξη του συστήματος, οπότε αφήστε το N να δηλώσει τον συνολικό αριθμό των σωματιδίων και αφήστε τα x_1, x_2, \dots, x_N να δηλώσουν τις αρχικές τους θέσεις. Επίσης, επειδή τα σωματίδια εξελίσσονται ως ανεξάρτητοι τυχαίοι περίπατοι διακριτού χρόνου, πρέπει να ορίσουμε μια αναλλοίωτη πιθανότητα μετάβασης $p(x, y)$ στο \mathbb{Z}^d , για την οποία $p(x, y) = p(0, y - x) =: p(y - x)$ για κάποια πιθανότητα $p(\cdot)$ στο \mathbb{Z}^d , που ονομάζεται *στοιχειώδης μεταβατική πιθανότητα* του συστήματος. Αυτή η πιθανότητα εκφράζει το *στοχαστικό χαρακτηριστικό* του τυχαίου περιπάτου.

Έστω ότι το $p_t(x, y)$ αντιπροσωπεύει την πιθανότητα να είναι ένα σωματίδιο στο χώρο y στο χρόνο t για μια διακριτή χρονική τυχαία πορεία με στοιχειώδη πιθανότητα μετάβασης $p(\cdot)$ ξεκινώντας από το x . Επιπλέον, το $p_t(\cdot, \cdot)$ κληρονομεί την ιδιότητα μετάβασης από το $p(\cdot, \cdot)$ και έτσι $p_t(x, y) = p_t(0, y - x) =: p_t(y - x)$.

Είναι καιρός να περιγράψουμε την κίνηση κάθε σωματιδίου. Ας πάρουμε N ανεξάρτητες μεταβλητές τυχαίων περιπάτων $\{Z_t^1, Z_t^2, \dots, Z_t^N\}$ στο \mathbb{Z}^d με στοιχειώδη πιθανότητα μετάβασης $p(\cdot)$ και αρχικά στο μηδέν. Ως αποτέλεσμα, η θέση κάθε σωματιδίου i στον τόμο \mathbb{T}_L^d στο χρόνο t θα είναι

$$X_t^i = x_i + Z_t^i \text{ mod } L.$$

Ωστόσο, επειδή τα σωματίδια δεν διακρίνονται στο μοντέλο μας, δεν μας ενδιαφέρει η ακριβής θέση κάθε σωματιδίου, αλλά ο αριθμός των σωματιδίων σε κάθε θέση του χώρου. Συγκεκριμένα, ο χώρος κατάστασης του συστήματος, που ονομάζεται επίσης χώρος διαμόρφωσης, είναι $\mathbb{N}^{\mathbb{T}_L^d}$. Οι καταστάσεις σημειώνονται με η , ζ και ξ . Σύμφωνα με αυτόν τον ορισμό, εάν το x είναι χώρος στο \mathbb{T}_L^d , τότε $\eta(x)$ είναι ο αριθμός των σωματιδίων αυτού του χώρου για την κατάσταση η . Επομένως, αν τα σωματίδια είναι αρχικά στα x_1, x_2, \dots, x_N , τότε

$$\eta(x) = \sum_{i=1}^N \mathbf{1}\{x = x_i\}.$$

Από την άλλη πλευρά, εάν μας δώσουνε $(\eta(x) : x \in \mathbb{T}_L^d)$, μπορούμε πρώτα να επισημάνουμε τα σωματίδια και να τα αφήσουμε να εξελιχθούν σύμφωνα με την στοχαστική δυναμική που περιγράψαμε.

Φυσικά, θέλουμε την διαμόρφωση σε χρόνο t , η οποία θα συμβολίζεται με $\eta_t(x)$ και θα οριστεί από

$$\eta_t(x) = \sum_{i=1}^N \mathbf{1}\{x = X_t^i\}.$$

Επιπλέον, η διαδικασία $(\eta_t)_{t \geq 0}$ κληρονομεί την ιδιότητα Markov από τους τυχαίους περιπάτους $\{X_t^i, 1 \leq i \leq N\}$ επειδή όλα τα σωματίδια έχουν την ίδια στοιχειώδη πιθανότητα μετάβασης και δεν αλληλεπιδρούν μεταξύ τους.

Μέτρο Poisson

Δεδομένου ότι ο χώρος κατάστασης είναι πεπερασμένος και δεδομένου ότι ο συνολικός αριθμός των σωματιδίων είναι η μοναδική ποσότητα που διατηρείται από τη δυναμική του συστήματος, για κάθε θετικό ακέραιο N που αντιπροσωπεύει τον συνολικό αριθμό σωματιδίων, υπάρχει μόνο ένα αμετάβλητο μέτρο. Η υποστήριξη της στοιχειώδους πιθανότητας μετάβασης $p(\cdot)$ παράγει το \mathbb{Z}^d , δηλαδή η διαδικασία είναι μη υποβιβασίμη. Τα μέτρα της Poisson στη μελέτη μας θα παίξουν κεντρικό ρόλο.

Θυμηθείτε ότι η κατανομή Poisson με παράμετρο $\alpha > 0$ είναι το μέτρο πιθανότητας στον \mathbb{N} με

$$p_k = e^{-\alpha} \frac{\alpha^k}{k!}, \quad k \in \mathbb{N}$$

και ο μετασχηματισμός Laplace είναι ίσος με

$$e^{-\alpha} \sum_{k=0}^{\infty} e^{-\lambda k} \frac{\alpha^k}{k!} = e^{-\alpha} e^{\alpha e^{-\lambda}} = e^{\alpha(e^{-\lambda}-1)},$$

για κάθε $\lambda > 0$.

Ορίζουμε το παρακάτω μέτρο. Για μια θετική συνάρτηση $\rho : \mathbb{T}_L^d \rightarrow \mathbb{R}_+$, ονομάζουμε το Poisson μέτρο στο \mathbb{T}_L^d που σχετίζεται με τη συνάρτηση ρ , μια πιθανότητα στον χώρο διαμόρφωσης $\mathbb{N}^{\mathbb{T}_L^d}$, που υποδηλώνεται από $\nu_{\rho(\cdot)}^L$, έχοντας τις παρακάτω δύο ιδιότητες.

- κάτω από το $\nu_{\rho(\cdot)}^L$ οι τυχαίες μεταβλητές $(\eta(x) : x \in \mathbb{T}_L^d)$, που αντιπροσωπεύουν τον συνολικό αριθμό σωματιδίων, πρέπει να είναι ανεξάρτητες,
- για κάθε χώρο $x \in \mathbb{T}_L^d$, η $\eta(x)$ κατανέμεται σύμφωνα με μια κατανομή Poisson παραμέτρου $\rho(x)$.

Στην περίπτωση όπου η συνάρτηση ρ είναι σταθερή και ίση με α , υποδηλώνουμε το $\nu_{\rho(\cdot)}^L$, μόνο με ν_{α}^L . Επίσης, θα υποδηλώνουμε την αναμενόμενη τιμή σε σχέση με ένα μέτρο ν ως E_{ν} .

Το μέτρο $\nu_{\rho(\cdot)}^L$ χαρακτηρίζεται από τον πολυδιάστατο μετασχηματισμό Laplace:

$$E_{\nu_{\rho(\cdot)}^L} \left[e^{-\sum_{x \in \mathbb{T}_L^d} \lambda(x) \eta(x)} \right] = \prod_{x \in \mathbb{T}_L^d} e^{\rho(x)(e^{-\lambda(x)}-1)} = e^{\sum_{x \in \mathbb{T}_L^d} \rho(x)(e^{-\lambda(x)}-1)}$$

για κάθε θετική ακολουθία $(\lambda(x) : x \in \mathbb{T}_L^d)$ [4].

Τώρα, ας προχωρήσουμε στο να διαπιστώσουμε ότι τα μέτρα Poisson που σχετίζονται με σταθερές συναρτήσεις είναι αναλλοίωτα για ένα σύστημα ανεξάρτητων τυχαίων περιπάτων.

Πρόταση. Αν τα σωματίδια αρχικά κατανεμηθούν σύμφωνα με ένα μέτρο Poisson που σχετίζεται με μια σταθερή συνάρτηση ίση με α τότε η κατανομή στο χρόνο t είναι ακριβώς το ίδιο μέτρο Poisson. [11]

Απόδειξη. Ορίστε ως $\mathbb{P}_{\nu_\alpha^L}$ το μέτρο πιθανότητας στο χώρο μονοπατιών $\Omega_L = \mathbb{N}^{\mathbb{T}_L^d} \times \mathbb{N}^{\mathbb{T}_L^d} \times \dots$, δηλαδή ο χώρος της $(\eta_t)_{t \geq 0}$ που παράγεται από τις δυναμικές των ανεξάρτητων τυχαίων περιπάτων και το αρχικό μέτρο ν_α^L . Η αναμενόμενη τιμή ως προς το $\mathbb{P}_{\nu_\alpha^L}$ υποδηλώνεται από $\mathbb{E}_{\nu_\alpha^L}$. Σε αυτό το σημείο, παρατηρήστε τη διαφορά μεταξύ $\mathbb{E}_{\nu_\alpha^L}$, που είναι η αναμενόμενη τιμή σε σχέση με το μέτρο που ορίζεται στο $\mathbb{N}^{\mathbb{T}_L^d}$ και $\mathbb{E}_{\nu_\alpha^L}$, η οποία είναι η αναμενόμενη τιμή σε σχέση με το μέτρο που ορίζεται στο χώρο μονοπατιών Ω_L . Είναι εύκολο να δεις ότι

$$\mathbb{E}_{\nu_\alpha^L}[F(\eta_0)] = \mathbb{E}_{\nu_\alpha^L}[F(\eta)]$$

για κάθε φραγμένη συνεχή συνάρτηση F στο $\mathbb{N}^{\mathbb{T}_L^d}$.

Δεδομένου ότι το μέτρο $\nu_{\rho(\cdot)}^L$ χαρακτηρίζεται από τον πολυδιάστατο μετασχηματισμό Laplace, θα υπολογίσουμε εδώ την αναμενόμενη τιμή

$$\mathbb{E}_{\nu_\alpha^L} \left[e^{-\sum_{x \in \mathbb{T}_L^d} \lambda(x) \eta_t(x)} \right]$$

για κάθε θετική ακολουθία $(\lambda(x) : x \in \mathbb{T}_L^d)$. Επιπλέον, για έναν χώρο $y \in \mathbb{T}_L^d$, θα υποδείξουμε με $X_t^{y,k}$ τη θέση στο χρόνο t του k -οστού σωματιδίου ξεκινώντας από y . Με αυτό τον τρόπο, ο αριθμός σωματιδίων στον χώρο x στο χρόνο t θα είναι:

$$\eta_t(x) = \sum_{y \in \mathbb{T}_L^d} \sum_{k=1}^{\eta_0(y)} \mathbb{1}\{x = X_t^{y,k}\}.$$

Και τότε, αναστρέφοντας τη σειρά αθροίσεως, θα πάρουμε:

$$\sum_{x \in \mathbb{T}_L^d} \lambda(x) \eta_t(x) = \sum_{y \in \mathbb{T}_L^d} \sum_{k=1}^{\eta_0(y)} \lambda(X_t^{y,k}).$$

Δεδομένου ότι κάθε σωματίδιο εξελίσσεται ανεξάρτητα και ο συνολικός αριθμός σωματιδίων σε κάθε θέση στο χρόνο 0 κατανέμεται σύμφωνα με μια

κατανομή Poisson παραμέτρου α ,

$$\begin{aligned} \mathbb{E}_{\nu_\alpha^L} \left[e^{-\sum_{x \in \mathbb{T}_L^d} \lambda(x) \eta_t(x)} \right] &= \mathbb{E}_{\nu_\alpha^L} \left[e^{-\sum_{y \in \mathbb{T}_L^d} \sum_{k=1}^{\eta_0(y)} \lambda(X_t^{y,k})} \right] = \\ &= \prod_{y \in \mathbb{T}_L^d} \mathbb{E}_{\nu_\alpha^L} \left[e^{-\sum_{k=1}^{\eta_0(y)} \lambda(X_t^{y,k})} \right] = \\ &= \prod_{y \in \mathbb{T}_L^d} \int \nu_\alpha^L(d\eta) \left(E \left[e^{-\lambda(X_t^{y,1})} \right] \right)^{\eta_0(y)} = \\ &= \prod_{y \in \mathbb{T}_L^d} e^{\alpha \left(E \left[e^{-\lambda(X_t^{y,1})} \right] - 1 \right)} = \prod_{y \in \mathbb{T}_L^d} e^{\alpha \left(E \left[e^{-\lambda(y+X_t)} \right] - 1 \right)} \end{aligned}$$

Όπου X_t είναι ένας τυχαίος περίπατος στο χρόνο t στον τόμο \mathbb{T}_L^d ξεκινώντας από την αρχή και με πιθανότητα μετάβασης $p_t^L(\cdot)$ με

$$p_t^L(x, y) = \sum_{z \in \mathbb{Z}^d} p_t(x, y + Lz)$$

για $x, y \in \mathbb{T}_L^d$. Δεδομένου ότι, εξ ορισμού,

$$E \left[e^{-\lambda(y+X_t)} \right] = \sum_{x \in \mathbb{T}_L^d} p_t^L(x - y) e^{-\lambda(x)},$$

τότε, αντιστρέφοντας τη σειρά αθροίσεως και ότι $\sum_{y \in \mathbb{T}_L^d} p_t^L(x - y) = 1$, παίρνουμε

$$\mathbb{E}_{\nu_\alpha^L} \left[e^{-\sum_{x \in \mathbb{T}_L^d} \lambda(x) \eta_t(x)} \right] = e^{\sum_{x \in \mathbb{T}_L^d} \alpha (e^{-\lambda(x)} - 1)}. \square$$

Θα πρέπει επίσης να σημειώσουμε ότι οι κατανομές Poisson είναι τέτοιες ώστε η αναμενόμενη τιμή τους να ισούται με

$$\sum_{k \geq 0} e^{-\alpha} \frac{\alpha^k}{k!} k = \alpha.$$

Τα μέτρα Poisson με αυτόν τον τρόπο φυσιολογικά παραμετροποιούνται από την πυκνότητα των σωματιδίων. Επιπλέον, από τον νόμο των μεγάλων αριθμών,

$$\lim_{L \rightarrow \infty} \frac{1}{|\mathbb{T}_L^d|} \sum_{x \in \mathbb{T}_L^d} \eta(x) = \alpha$$

κατά πιθανότητα σε σχέση με ν_α^L . Συνεπώς, φανταστείτε ότι η παράμετρος α περιγράφει τη μέση πυκνότητα των σωματιδίων σε ένα "μεγάλο κουτί".

Συμπερασματικά, για αυτή την ενότητα, διαπιστώσαμε την ύπαρξη μιας οικογένειας αναλλοίωτων μέτρων μιας παραμέτρου που προσδιορίζει την πυκνότητα των σωματιδίων, η οποία είναι η μοναδική ποσότητα που διατηρείται από την εξέλιξη του χρόνου.

Απλή Διαδικασία Αποκλεισμού

Σε αυτήν την ενότητα μας ενδιαφέρουν οι διαδικασίες απλού αποκλεισμού. Αυτό το μοντέλο προτάθηκε στο [16] και είναι ένα από τα απλούστερα και πιο μελετημένα συστήματα αλληλεπιδρώντων σωματιδίων. Η διαδικασία απλού αποκλεισμού, σε αντίθεση με τους ανεξάρτητους τυχαίους περιπάτους που μελετήθηκαν στην προηγούμενη ενότητα, επιτρέπει το πολύ ένα σωματίδιο ανά χώρο.

Πρώτα απ' όλα, ο χώρος κατάστασης είναι $\{0, 1\}^{\mathbb{T}_L^d}$. Προκειμένου να αποφευχθεί η εμφάνιση περισσότερων από ένα σωματιδίων ανά τοποθεσία, εισάγουμε έναν κανόνα αποκλεισμού που καταστέλλει κάθε άλμα σε ήδη κατεχόμενο χώρο. Στην πραγματικότητα, θα επικεντρωθούμε μόνο στην απλούστερη κατηγορία διαδικασιών αποκλεισμού: συστήματα όπου τα σωματίδια πηδούν, όποτε επιτρέπεται το άλμα, ανεξάρτητα από τα άλλα και σύμφωνα με την ίδια στοιχειώδη μεταβατική πιθανότητα.

Ο γεννήτορας

$$(\mathcal{L}f)(\eta) := \sum_{x \in \mathbb{T}_L^d} \sum_{z \in \mathbb{T}_L^d} \eta(x)(1 - \eta(x+z))p^L(z)(f(\eta^{x,x+z}) - f(\eta)),$$

όπου $\eta^{x,y}$ είναι κατάσταση που παίρνουμε από την η αφήνοντας ένα σωματίδιο να μεταπηδήσει από το x στο y , δηλαδή

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{αν } z \neq x, y, \\ \eta(x) - 1 & \text{αν } z = x, \\ \eta(y) + 1 & \text{αν } z = y \end{cases} \quad \text{και } p^L(z) := \sum_{y \in \mathbb{Z}^d} p(z + yL),$$

ορίζει μια διαδικασία Markov που ονομάζεται απλή διαδικασία αποκλεισμού με στοιχειώδη πιθανότητα μεταβάσεων $p(\cdot)$. Στη συγκεκριμένη περίπτωση όπου $p(z) = p(-z)$ λέμε ότι είναι μια συμμετρική απλή διαδικασία αποκλεισμού.

Πιστεύουμε ότι η ερμηνεία είναι σαφής. Μεταξύ 0 και dt , κάθε σωματίδιο προσπαθεί, ανεξάρτητα από τα άλλα, να πηδήξει από το x στο $x+z$ με (εκθετικό) ρυθμό $p^L(z)$. Το άλμα καταστέλλεται εάν οδηγεί σε ήδη κατεχόμενο χώρο.

Υπενθυμίζουμε εδώ ότι μια διαδικασία Markov λέγεται ότι είναι μη υποβιβάσιμη αν είναι δυνατόν να φτάσουμε σε οποιαδήποτε κατάσταση από οποιαδήποτε κατάσταση. Επιπλέον, δεδομένου ότι η πιθανότητα μετάβασης θεωρείται ότι είναι πεπερασμένου εύρους, υπάρχει A_0 στο \mathbb{N} τέτοιο ώστε $p(z) = 0$ για όλες τις τοποθεσίες εκτός του κύβου $[-A_0, A_0]^d$. Συγκεκριμένα, τα $p^L(\cdot)$ και $p(\cdot)$ συμπίπτουν για $L \geq A_0$. Γι' αυτόν τον λόγο, από εδώ και πέρα θα παραλείψουμε τον δείκτη L στην στοιχειώδη πιθανότητα μετάβασης. Ακόμα, παρατηρούμε ότι ο ρυθμός με τον οποίο ένα σωματίδιο μεταπηδά από το x στο y εξαρτάται από τη διαμόρφωση η μόνο μέσω των μεταβλητών διαμόρφωσης $\eta(x)$ και $\eta(y)$. Αυτή η τελευταία εξάρτηση από τα $\eta(x)$ και $\eta(y)$ αντικατοπτρίζει τον κανόνα αποκλεισμού. Τέλος, παρατηρήστε ότι ο συνολικός αριθμός σωματιδίων διατηρείται από τη δυναμική.

Δηλώνουμε με $\nu_\alpha = \nu_\alpha^L$, για το $0 \leq \alpha \leq 1$, το μέτρο Bernoulli παραμέτρου α , δηλαδή το αναλλοίωτο μέτρο στο $\{0, 1\}^{\mathbb{T}_L^d}$ με πυκνότητα α . Συγκεκριμένα, κάτω από ν_α , οι μεταβλητές $(\eta(x) : x \in \mathbb{T}_L^d)$ είναι ανεξάρτητες με

$$\nu_\alpha\{\eta(x) = 1\} = \alpha = 1 - \nu_\alpha\{\eta(x) = 0\}.$$

Πρόταση. Τα μέτρα Bernoulli $\{\nu_\alpha : 0 \leq \alpha \leq 1\}$ είναι αμετάβλητα για απλές διαδικασίες αποκλεισμού. Επιπλέον, σε σχέση με κάθε ν_α , οι διεργασίες αποκλεισμού με πιθανότητα στοιχειώδους μετάβασης $\check{p}(z) := p(-z)$ είναι συμπληρωματικές με διαδικασίες με στοιχειώδη πιθανότητα πιθανότητας $p(z)$. Συγκεκριμένα, οι συμμετρικές απλές διαδικασίες αποκλεισμού είναι αυτοσυμπληρωματικές σε σχέση με κάθε ν_α . [11]

Απόδειξη. Είναι εύκολο να παρατηρήσετε ότι με μια απλή αλλαγή των μεταβλητών

$$\int f(\eta^{0,z})g(\eta)\eta(0)(1 - \eta(z))\nu_\alpha(d\eta) = \int f(\eta)g(\eta^{0,z})\eta(z)(1 - \eta(0))\nu_\alpha(d\eta).$$

Αυτή η ταυτότητα, το γεγονός ότι $1 = \sum_{z \in \mathbb{Z}^d} p(z) = \sum_{z \in \mathbb{Z}^d} p(-z)$ και μια αλλαγή στη σειρά αθροίσεως, αποδεικνύουν την πρόταση. \square

Σε αυτή την περίπτωση, η οικογένεια των αναλλοίωτων μέτρων ν_α παραμετροποιείται από την πυκνότητα, αφού

$$E_{\nu_\alpha}[\eta(0)] = \nu_\alpha\{\eta(0) = 1\} = \alpha.$$

Καθολικότητα κατανομής Tracy-Widom

Σε αυτή την ενότητα θα συζητήσουμε για την καθολικότητα που έχει διαπιστωθεί πρόσφατα ότι δείχνει η κατανομή Tracy-Widom.

Είναι γεγονός ότι η Θεωρία Τυχαίων Πινάκων έχει βρει έναν τεράστιο αριθμό εφαρμογών που κυμαίνονται από τη στατιστική φυσική μέχρι τη θεωρία αριθμών, τα συνδυαστικά και τα ολοκληρωτικά συστήματα. Μεταξύ των πρόσφατων εξελίξεων στη θεωρία τυχαίων πινάκων, η μελέτη της μεγαλύτερης ιδιοτιμής λ_{max} μεγάλων τυχαίων πινάκων έχει προσελκύσει ιδιαίτερη προσοχή. Οι πρώτες ερωτήσεις αφορούσαν τις διακυμάνσεις του λ_{max} , που ανήκουν στο ευρύτερο θέμα στατιστικών ακραίων τιμών.

Ο βιολόγος Robert May πραγματοποίησε το 1972 μια φυσική εφαρμογή των στατιστικών του λ_{max} , που είναι να παράσχει ένα κριτήριο της φυσικής σταθερότητας σε δυναμικά συστήματα όπως τα οικοσυστήματα [14]. Θεώρησε έναν πληθυσμό N διακριτών ειδών και εισήγαγε ισχυρές αλληλεπιδράσεις μεταξύ των ειδών. Υπέθεσε ότι οι αλληλεπιδράσεις μεταξύ ζευγών ειδών μπορούν να μοντελοποιηθούν από τυχαίο πίνακα J , μεγέθους $N \times N$. Ένα φυσικό ερώτημα είναι λοιπόν: ποια είναι η πιθανότητα $P_{stable}(\alpha, N)$, όπου το α αντιπροσωπεύει τη δύναμη των αλληλεπιδράσεων, ότι το σύστημα παραμένει σταθερό όταν ενεργοποιηθούν οι αλληλεπιδράσεις; [14] Μετά από μερικούς υπολογισμούς, ο

May απέδωσε ότι το σύστημα θα παραμείνει σταθερό, υπό την προϋπόθεση ότι οι ιδιοτιμές λ_i του τυχαίου πίνακα J ικανοποιούν την ανισότητα:

$$\alpha\lambda_i - 1 \leq 0,$$

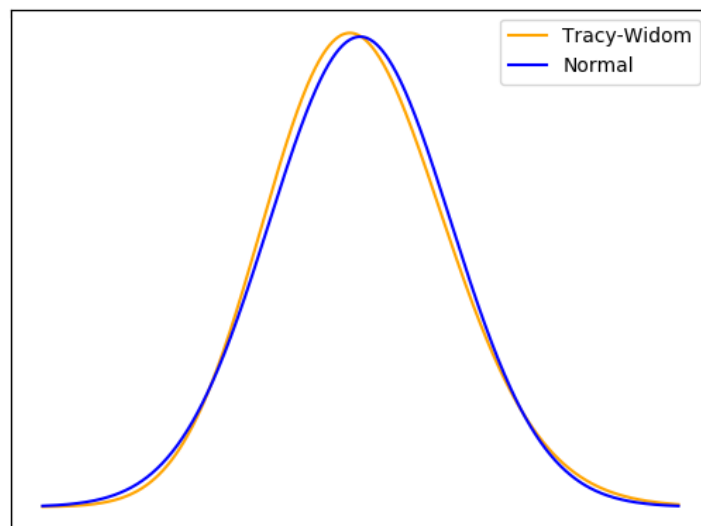
για όλα τα $i = 1, \dots, N$. Αυτό είναι προφανώς ισοδύναμο με τη δήλωση ότι η μεγαλύτερη ιδιοτιμή $\lambda_{max} = \max_{1 \leq i \leq N} \lambda_i$ ικανοποιεί την ανισότητα:

$$\lambda_{max} \leq \frac{1}{\alpha}.$$

Επομένως η πιθανότητα σταθερότητας του συστήματος σχετίζεται φυσικά με την κατανομή της μεγαλύτερης ιδιοτιμής λ_{max} .

Η κατανομή Tracy-Widom δημιουργήθηκε για πρώτη φορά το 1992, δύο δεκαετίες αργότερα, από τον Tracy και τον Widom [18, 19], οι οποίοι την παρατήρησαν μελετώντας την ίδια έννοια, δηλαδή τις διακυμάνσεις της μεγαλύτερης ιδιοτιμής λ_{max} τυχαίων πινάκων. Αργότερα το 1999, οι Baik, Deift, Johansson [2], ανακάλυψαν ότι η ίδια στατιστική κατανομή περιγράφει επίσης τις διακυμάνσεις σε ακολουθίες αναδιαταγμένων ακέραιων αριθμών - μια εντελώς ανεξάρτητη μαθηματική έννοια.

Σύντομα η στατιστική καμπύλη άρχισε να εμφανίζεται σε μοντέλα σε όλη τη φυσική και τα μαθηματικά. Αυτή η αινιγματική καμπύλη φαινόταν να είναι ο σύνθετος ξάδελφος της γνωστής καμπύλης καμπάνας, της Gaussian κατανομής, η οποία αντιπροσωπεύει τη φυσική διακύμανση των ανεξάρτητων τυχαίων μεταβλητών. Όπως και η Gaussian, η κατανομή Tracy-Widom παρουσιάζει καθολικότητα, ένα μυστηριώδες φαινόμενο στο οποίο διαφορετικά μικροσκοπικά γεγονότα δίνουν την ίδια συλλογική συμπεριφορά.



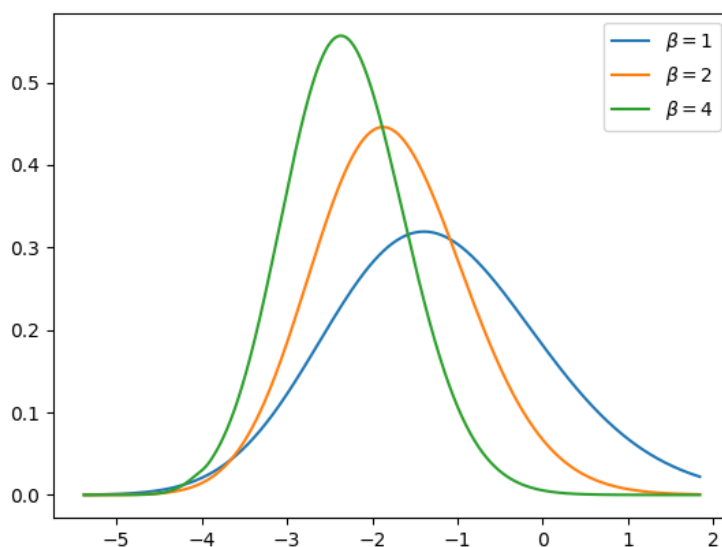
Συναρτήσεις πυκνότητας πιθανότητας Κανονικής και Tracy-Widom κατανομής

Όταν η κατανομή Tracy-Widom εμφανίστηκε στο πρόβλημα των ακέραιων ακολουθιών και σε άλλα πλαίσια που δεν είχαν καμία σχέση με τη θεωρία τυχαίων πινάκων, οι ερευνητές άρχισαν να ψάχνουν για το 'κρυμμένο νήμα' που συνδέει όλες τις πτυχές μαζί, όπως οι μαθηματικοί τον 18ο και 19ο αιώνα αναζητούσαν ένα θεώρημα που θα εξηγούσε την πανταχού παρούσα κανονική κατανομή.

Το κεντρικό οριακό θεώρημα, το οποίο τελικά έγινε αυστηρό πριν από περίπου έναν αιώνα, πιστοποιεί ότι οι φυσικές παρατηρήσεις και άλλες 'μη συσχετισμένες' μεταβλητές - που σημαίνει ότι οποιαδήποτε από αυτές μπορούν να αλλάξουν χωρίς να επηρεάσουν τις υπόλοιπες - θα αποτελούν καμπύλη καμπάνας. Αντίθετα, η καμπύλη Tracy-Widom φαίνεται να προκύπτει από μεταβλητές που συσχετίζονται έντονα, όπως η αλληλεπίδραση μεταξύ των ειδών, οι τιμές των μετοχών και οι ιδιοτιμές των πινάκων. Ο κύκλος ανάδρασης των αμοιβαίων αποτελεσμάτων μεταξύ των συσχετισμένων μεταβλητών καθιστά τη συλλογική συμπεριφορά τους πιο περίπλοκη από αυτή των μη συσχετισμένων μεταβλητών.

Ενώ οι ερευνητές έχουν αποδείξει αυστηρά ορισμένες κατηγορίες τυχαίων πινάκων στις οποίες επικρατεί γενικώς η κατανομή Tracy-Widom, έχουν περικυκλώσει λιγότερο τις εμφανίσεις της σε προβλήματα καταμέτρησης, προβλήματα τυχαίων περιπάτων, μοντέλα ανάπτυξης και άλλα.

Μέχρι στιγμής, έχουν χαρακτηριστεί τρεις μορφές διανομής Tracy-Widom. Είναι μετασχηματισμένες εκδοχές η μία της άλλης που περιγράφουν ισχυρά συσχετισμένα συστήματα με διαφορετικούς τύπους εγγενούς τυχειότητας, δηλαδή διαφορετικούς τύπους τυχαίων πινάκων. Συγκεκριμένα, οι τρεις κλάσεις του $F_\beta(s)$ σημειώνονται από $\beta = 1, 2, 4$. Ωστόσο, θα μπορούσαν να υπάρχουν περισσότερες από τρεις, ίσως ακόμη και ένας άπειρος αριθμός, τάξεων.



Κλάσεις κατανομής Tracy-Widom

Τον τελευταίο καιρό συζητείται εκτενώς το γεγονός ότι ο ασύμμετρος χαρακτήρας της κατανομής μπορεί να αντιπροσωπεύει κάποιο είδος καθολικής μετάβασης φάσης [13]. Στο μοντέλο του οικοσυστήματος του May, για παράδειγμα, το κρίσιμο σημείο στο $\sqrt{2N}$ χωρίζει μια σταθερή φάση των ασθενών συζευγμένων ειδών, των οποίων οι πληθυσμοί μπορούν να κυμαίνονται μεμονωμένα χωρίς να επηρεάζονται τα υπόλοιπα, από μια ασταθή φάση των ισχυρά συζευγμένων ειδών, όπου οι διακυμάνσεις 'ταξιδεύουν' ισχυρά μέσω του οικοσυστήματος και μπορούν να το πετάξουν εκτός ισορροπίας. Γενικά, τα συστήματα στην τάξη καθολικότητας Tracy-Widom παρουσιάζουν μία φάση στην οποία όλα τα συστατικά ενεργούν σε συνεννόηση, η αριστερή ουρά, και μια άλλη φάση στην οποία τα συστατικά ενεργούν μόνα τους, η δεξιά ουρά.

Αυτή τη στιγμή, πολλοί φυσικοί και μαθηματικοί εργάζονται στον τομέα της επιδίωξης κάποιου καθολικού νόμου που συνδέεται με την κατανομή Tracy-Widom. Εάν επιτευχθεί ένα τέτοιο κατόρθωμα, θα μπορούσαμε να ερμηνεύσουμε τα μακροσκοπικά χαρακτηριστικά των συστημάτων με αλληλεπιδρώντα συστατικά με πολύ πιο φυσικό τρόπο.

Διαδικασίες Μηδενικού Εύρους

Σε αυτήν την ενότητα θα μελετήσουμε ένα άλλο ευρέως γνωστό μοντέλο Συστημάτων Αλληλεπιδρόντων Σωματιδίων. Εισήχθη επίσης αρχικά ως ένα απλό παράδειγμα μιας διαδικασίας Markov στο [16]. Ονομάζεται Διαδικασία Μηδενικού Εύρους και το όνομά της προέρχεται από το γεγονός ότι τα σωματίδια αλληλεπιδρούν μόνο με σωματίδια που κάθονται στον ίδιο χώρο.

Μοντέλο

Ο χώρος κατάστασης είναι $\mathbb{N}^{\mathbb{Z}^d}$. Η διαδικασία καθορίζεται από τη συνάρτηση $g : \mathbb{N} \rightarrow \mathbb{R}_+$ με $g(0) = 0$, που αντιπροσωπεύει τον ρυθμό με τον οποίο ένα σωματίδιο φεύγει από μια θέση και μια πιθανότητα μετάβασης $p(\cdot, \cdot)$ στο \mathbb{Z}^d . Η δυναμική της πηγαινει ως εξής. Εάν υπάρχουν k σωματίδια σε μια θέση x , ανεξάρτητα από τον αριθμό των σωματιδίων σε άλλους χώρους, με ρυθμό $g(k)p(x, y)$ ένα από τα σωματίδια στο x μεταβαίνει στο y . Με αυτό τον τρόπο τα σωματίδια αλληλεπιδρούν μόνο με σωματίδια στην ίδια θέση. Υποθέτουμε επίσης ότι η g είναι αυστηρά θετική στο σύνολο των θετικών ακεραίων και ότι έχει φραγμένη διακύμανση, δηλαδή:

$$g^* := \sup_{k \geq 0} |g(k+1) - g(k)| < \infty.$$

Τώρα, έστω $Z : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ είναι η συνάρτηση διαμέρισης με

$$Z(\phi) = \sum_{k \geq 0} \frac{\phi^k}{g!(k)}$$

και έστω ϕ_c η ακτίνα σύγκλισης της Z . Στην τελευταία σχέση είναι $g!(k) = \prod_{1 \leq j \leq k} g(j)$ και $g!(0) = 1$. Επιπλέον, παρατηρήστε ότι η Z είναι αναλυτική και αυστηρά αύξουσα στο $[0, \phi_c)$. Υποθέτουμε ότι η $Z(\cdot)$ αυξάνει στο ∞ όταν το ϕ συγχλίνει στο ϕ_c :

$$\lim_{\phi \rightarrow \phi_c} Z(\phi) = \infty.$$

Αυτή η υπόθεση δεν είναι απαραίτητη για τον ορισμό της διαδικασίας, αλλά θα χρειάζεται για να αποδείξουμε αρκετά αποτελέσματα. Ως εκ τούτου, προτιμάται να συμπεριληφθεί στον ορισμό. Ο γεννήτορας

$$(\mathcal{L}f)(\eta) = \sum_{x \in \mathbb{T}_L^d} \sum_{z \in \mathbb{T}_L^d} p^L(z)g(\eta(x))(f(\eta^{x,x+z}) - f(\eta))$$

καθορίζει μια διαδικασία Markov στο $\mathbb{N}^{\mathbb{T}_L^d}$, που ονομάζεται διαδικασία μηδενικού εύρους με παραμέτρους (g, p) . Επίσης εδώ, όπως και σε προηγούμενη ενότητα, η $\eta^{x,y}$ αντιπροσωπεύει τη διαμόρφωση η όπου ένα σωματίδιο πήδηξε από το x στο y και η $p^L(\cdot)$ αντιπροσωπεύει την πιθανότητα μετάβασης που μετακινείται στην αρχή και περιορίζεται στον τόμο:

$$p^L(z) := p^L(0, z) = \sum_{y \in \mathbb{T}_L^d} p(0, z + yL)$$

για κάθε d -διάστατο ακέραιο z .

Σε διεργασίες μηδενικής εμβέλειας κάθε σωματίδιο πηδά, ανεξάρτητα από τα σωματίδια που κάθονται σε άλλες τοποθεσίες, από το x στο y με ρυθμό

$$p^L(y - x)g(\eta(x))\frac{1}{\eta(x)}.$$

Αναλλοίωτο Μέτρο

Στρέφουμε τώρα την προσοχή μας, ως συνήθως, στον χαρακτηρισμό των αναλλοίωτων μέτρων για τη διαδικασία. Δεδομένου ότι η διαδικασία μηδενικού εύρους είναι μη υποβιβάσιμη και ο χώρος κατάστασης είναι πεπερασμένος, έχουμε ένα μοναδικό αναλλοίωτο μέτρο που υποδηλώνουμε με $\mu^{N,L}$. Θα αναφερθούμε στα μέτρα $\mu^{N,L}$ ως κανονικά. Μπορούν να υπολογιστούν με σαφήνεια, αλλά μπορούν επίσης να ληφθούν και με την δέσμευση των μεγάλων κανονικών μέτρων, των οποίων ο ορισμός ακολουθεί, επί του συνολικού αριθμού των σωματιδίων.

Για κάθε $0 \leq \phi < \phi_c$, έστω $\bar{\nu}_{\phi,g} = \bar{\nu}_{\phi,g}^L$ το μεγάλο κανονικό μέτρο στο $\mathbb{N}^{\mathbb{T}_L^d}$ με

$$\bar{\nu}_{\phi,g}\{\eta, \eta(x) = k\} = \frac{1}{Z(\phi)} \frac{\phi^k}{g!(k)}$$

για κάθε $k \geq 0$ και $x \in \mathbb{T}_L^d$.

Πρόταση. Για κάθε $0 \leq \phi < \phi_c$ το μέτρο $\bar{\nu}_{\phi,g}$ είναι αναλλοίωτο για τη διαδικασία μηδενικού εύρους με τις παραμέτρους (g, p) . Επιπλέον, η συμπληρωματική διαδικασία σε σχέση με οποιοδήποτε από τα μέτρα $\bar{\nu}_{\phi,g}$ είναι η διαδικασία μηδενικού εύρους με τις παραμέτρους (g, \check{p}) . Ειδικότερα, εάν το p είναι συμμετρικό, η διαδικασία είναι αυτο-συμπληρωματική. [11]

Απόδειξη. Η απόδειξη βασίζεται στους ίδιους υπολογισμούς που κάναμε για τα μέτρα Bernoulli και στην παρακάτω ταυτότητα

$$g(k) \frac{\phi^k}{g!(k)} \frac{\phi^j}{g!(j)} = g(j+1) \frac{\phi^{k-1}}{g!(k-1)} \frac{\phi^{j+1}}{g!(j+1)}. \quad \square$$

Επίσης, επειδή η συνάρτηση $g(\cdot)$ θα έχει σταθερή περιγραφή, για να διατηρήσουμε απλούς τους συμβολισμούς, παραλείπουμε την εξάρτηση από το g του μέτρου $\bar{\nu}_{\phi,g}$ και το γράφουμε απλά με $\bar{\nu}_\phi$. Και έτσι,

$$\mu^{N,L}(\cdot) = \bar{\nu}_\phi \left(\cdot \left| \sum_{x \in \mathbb{T}_L^d} \eta(x) = N \right. \right).$$

Τώρα, συμβολίζουμε με $\rho(\phi)$ την αναμενόμενη τιμή της μεταβλητής διαμόρφωσης, δηλαδή της πυκνότητας, κάτω από $\bar{\nu}_\phi$:

$$\rho(\phi) = E_{\bar{\nu}_\phi}[\eta(x)] = \frac{1}{Z(\phi)} \sum_{k \geq 0} k \frac{\phi^k}{g!(k)}.$$

Το εύρος της ρ είναι το $[0, \rho_c)$, με $\rho(0) = 0$ και

$$\rho_c = \lim_{\phi \rightarrow \phi_c} \rho(\phi)$$

η κρίσιμη πυκνότητα. Επίσης από την παραπάνω εξίσωση εύκολα παίρνουμε ότι:

$$\rho(\phi) = \frac{Z'(\phi)\phi}{Z(\phi)} = \phi \partial_\phi \log Z(\phi).$$

Ο υπολογισμός της πρώτης παραγώγου της $\rho(\cdot)$ δείχνει ότι είναι αυστηρώς αύξουσα.

Σημείωση. Ένα φυσικό αντικείμενο ενδιαφέροντος είναι να διερευνήσουμε τη συμπεριφορά αυτών των μέτρων στο θερμοδυναμικό όριο, δηλαδή όταν $N, L \rightarrow \infty$ με τέτοιο τρόπο ώστε η μέση πυκνότητα σωματιδίων N/L να συγκλίνει σε ένα σταθερό ρ . Λοιπόν, στην υποκρίσιμη περίπτωση, όταν $\rho < \rho_c$, υπάρχει ϕ τέτοιο ώστε $\rho = \rho(\phi)$ και η τυπική ισοδυναμία των μέτρων για ανεξάρτητες τυχαίες μεταβλητές ισχύει κανονικά [11]. Δηλαδή, τα πεπερασμένα διαστασιακά περιθωριακά των κανονικών μέτρων $\mu^{N,L}$ συγκλίνουν στο μεγάλο κανονικό μέτρο που αντιστοιχεί στο ϕ . Η ισοδυναμία των μέτρων για κρίσιμες και υπερκρίσιμες πυκνότητες, όταν $\rho \geq \rho_c$, αποδείχθηκε στο [7]. Χρησιμοποιώντας μεθόδους σχετικής εντροπίας, οι συγγραφείς αποδεικνύουν τη σύγκλιση

των πεπερασμένων διαστασιακών περιθωρίων του $\mu^{N,L}$ στο μεγάλο κανονικό μέτρο στην κρίσιμη κατάσταση ϕ_c . Αργότερα στο [1], οι συγγραφείς έδειξαν ότι στο θερμοδυναμικό όριο οι χώροι έχουν κοινή κατανομή ίση με το μεγάλο κανονικό μέτρο σε κρίσιμη πυκνότητα, εκτός από μία θέση που φιλοξενεί έναν μακροσκοπικά μεγάλο αριθμό σωματιδίων.

Επιπλέον, έχουμε συχνά παραμετροποιήσει τα αναλλοίωτα μέτρα από την ποσότητα που διατηρείτε, η οποία εδώ είναι η πυκνότητα των σωματιδίων. Για το λόγο αυτό, αλλάζουμε μεταβλητές στον ορισμό των αμετάβλητων μέτρων $\bar{\nu}_\phi$ ως εξής. Για $\alpha \geq 0$, ορίστε το μέτρο ν_α ως

$$\nu_\alpha(\cdot) = \bar{\nu}_{\Phi(\alpha)}(\cdot),$$

όπου η $\Phi(\cdot)$ ορίζεται ως

$$\Phi(\rho) = \begin{cases} \text{αντίστροφη της } \rho(\phi), & \text{για } \rho < \rho_c \\ \phi_c, & \text{για } \rho \geq \rho_c \end{cases}$$

Στο επόμενο λήμμα φαίνεται ότι η υπόθεση που κάναμε για τη συνάρτηση διαμέρισης Z εγγυάται ότι το εύρος της συνάρτησης $\rho(\cdot)$ είναι όλο το \mathbb{R}_+ . Με αυτό τον τρόπο αποκτήσαμε μια οικογένεια $\{\nu_\alpha : \alpha \geq 0\}$ αναλλοίωτων μέτρων που παραμετροποιήθηκαν από την πυκνότητα, αφού η αναμενόμενη τιμή των μεταβλητών διαμόρφωσης $\eta(x)$ κάτω από ν_α ισούται με α :

$$E_{\nu_\alpha}[\eta(x)] = E_{\bar{\nu}_{\Phi(\alpha)}}[\eta(x)] = \rho(\Phi(\alpha)) = \alpha$$

για κάθε $\alpha \geq 0$. Ακόμα, με έναν απλό υπολογισμό βλέπουμε ότι η συνάρτηση $\Phi(\alpha)$ είναι η αναμενόμενη τιμή $g(\eta(0))$ κάτω από το μέτρο ν_α :

$$\Phi(\alpha) = E_{\nu_\alpha}[g(\eta(0))].$$

Λήμμα. Έχουμε σημειώσει ως ϕ_c την ακτίνα σύγκλισης της συνάρτησης διαμέρισης Z .

$$\rho_c = \lim_{\phi \rightarrow \phi_c} \rho(\phi) = \infty.$$

Επίσης, για κάθε $0 \leq \phi < \phi_c$ για το μέτρο $\bar{\nu}_\phi$, υπάρχει $\theta(\phi) > 0$ τέτοιο ώστε

$$E_{\bar{\nu}_\phi}[e^{\theta\eta(0)}] < \infty.$$

Προσομοίωση

Σε αυτή την ενότητα βλέπουμε μια αποτελεσματική μέθοδο για την προσομοίωση συστημάτων σωματιδίων με αλληλεπίδραση. Σε όλες τις προηγούμενες ενότητες ορίζουμε τη χρονική εξέλιξη των συστημάτων μας να είναι συνεχής και αυτό είναι κάτι απαραίτητο για να υποστηρίξουμε άλλες έννοιες όπως τα όρια κλιμάκωσης. Ωστόσο, όπως γνωρίζετε, σε έναν υπολογιστή, η εξέλιξη πρέπει να συμβαίνει σε διακριτούς χρόνους.

Ορίζουμε μια αλυσίδα Markov να είναι μια διαδικασία Markov ορισμένη σε ένα μετρήσιμο σύνολο και να χαρακτηρίζεται από τους ρυθμούς μετάβασης $c(\zeta, \zeta') \geq 0$. Επιπλέον, σημειώνουμε με

$$W_\zeta := \inf\{t \geq 0 : \eta_t \neq \zeta\}$$

τον χρόνο παραμονής σε μια κατάσταση ζ , και με

$$c_\zeta := \sum_{\zeta' \neq \zeta} c(\zeta, \zeta').$$

τους συνολικούς ρυθμούς εξόδου από την ζ . Προφανώς αν $c_\zeta = 0$, τότε $W_\zeta = \infty$. Ακόμα, ισχύει ότι αν $c_\zeta > 0$, τότε

$$W_\zeta \sim \text{Exp}(c_\zeta),$$

όπου με $\text{Exp}(\lambda)$ σημειώνουμε την εκθετική κατανομή με παράμετρο λ .

Όσο προσομοιώνουμε ένα σύστημα σωματιδίων με αλληλεπίδραση σε έναν τόμο \mathbb{T}_L^d , υπάρχει πάντα ένας μεγάλος βρόχος, στον οποίο κάθε επανάληψη αντιπροσωπεύει μια μετάβαση, δηλαδή το άλμα ενός σωματιδίου. Η επιλογή του σωματιδίου συνδέεται στενά με το χρόνο παραμονής στην κατάσταση του. Σημειώστε ότι οι μεταβατικοί ρυθμοί για τη διαδικασία μηδενικού εύρους είναι:

$$c(\eta, \eta^{x,y}) = g(\eta(x))p(y-x) \frac{1}{\eta(x)}.$$

Τα βήματα προσομοίωσης της διαδικασίας μηδενικού εύρους εκφράζονται από τον παρακάτω αλγόριθμο.

```

rates ← sum{g(η(x)) για κάθε x στο  $\mathbb{T}_L^d$ }
time ← 0
while true do
  επέλεξε ένα x με πιθανότητα g(η(x))/rates
  επέλεξε ένα y με πιθανότητα p(y-x)
  η(x) ← η(x) - 1
  η(y) ← η(y) + 1
  time ← time + Exp(rates)
  rates ← g(η(x)) + g(η(y)) - g(η(x) + 1) - g(η(y) - 1)
end while

```

Πιστεύουμε ότι όλα τα βήματα είναι αρκετά απλά. Σημειώνουμε βέβαια πως ισχύει ότι

$$\min(\text{Exp}(\lambda_1), \dots, \text{Exp}(\lambda_n)) \sim \text{Exp}\left(\sum_{i=1}^n \lambda_i\right).$$

Το παραπάνω αιτιολογεί τις αυξήσεις της μεταβλητής *times*.

Επίλογος

Τα τελευταία χρόνια έχει προκύψει έντονα η ανάγκη της μελέτης των Συστημάτων Αλληλεπιδρώντων Σωματιδίων, εξαιτίας του ευρέος φάσματος εφαρμογών σε φυσικά προβλήματα. Στην παρούσα εργασία, ξεκινήσαμε ορίζοντας την απλούστερη μορφή ενός συστήματος σωματιδίων, δηλαδή τους ανεξάρτητους τυχαίους περιπάτους, και στη συνέχεια αρχίσαμε να θέτουμε ερωτήματα σχετικά με τις υποκείμενες συμπεριφορές. Αυτές οι ερωτήσεις περιλάμβαναν την ύπαρξη αμετάβλητων κατανομών στο σύστημα, την εξαγωγή του μακροσκοπικού προφίλ του συστήματος σε δεδομένη χρονική και χωρική κλίμακα, την οριοθέτηση αυτών των κλιμάκων κλπ. Μετά από αυτό, ήρθε η ώρα να στρέψουμε την προσοχή μας σε λίγο πιο σύνθετα συστήματα αλληλεπιδρώντων σωματιδίων, όπως η απλή διαδικασία αποκλεισμού και η διαδικασία μηδενικού εύρους. Αυτά τα μοντέλα μαζί με πολλές δημοφιλείς διακυμάνσεις τους έχουν ήδη μελετηθεί ευρέως από τους ερευνητές. Σε μία από τις διακυμάνσεις της απλής διαδικασίας αποκλεισμού εμφανίζεται η κατανομή Tracy-Widom. Με την ευκαιρία αυτή, ξεκινήσαμε μια συζήτηση σχετικά με την καθολικότητα που αυτή η συγκεκριμένη διανομή φαίνεται να εκδηλώνει τελευταία. Επιπλέον μελετήσαμε τις διαδικασίες μηδενικού εύρους και καταλήξαμε στην αναλλοίωτη κατανομή τους.

Ως μελλοντικές γραμμές εργασίας στο πεδίο, προτείνουμε να αποκτηθεί μια βαθύτερη κατανόηση των ιδιοτήτων της κατανομής Tracy-Widom, καθώς φαίνεται να διαδραματίζει κεντρικό ρόλο σε έννοιες με αλληλεπιδρώντα συστατικά. Επιπλέον, θα ήταν πολύ αποδοτική η εφαρμογή τεχνικών μείωσης διασποράς σε προσομοιώσεις συστημάτων αλληλεπιδρώντων σωματιδίων.

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