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Το μοντέλο Sachdev-Ye-Kitaev

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ
ΤΟΥ
ΔΗΜΟΣΘΕΝΗ ΘΕΟΦΙΛΟΠΟΥΛΟΥ

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"I have noticed that even people who claim everything is predetermined and that we can do nothing to change it, look before they cross the road."
Stephen Hawking, 1942-2018

Ευχαριστίες

Θα ήθελα να ευχαριστήσω κατ'αρχήν τον επιβλέποντα καθηγητή μου κ. Νίκο Ήργε για την ανάθεση ενός τόσο ενδιαφέροντος και μοντέρνου θέματος αλλά και για την αμέριστη και συνεχή στήριξη και συμπαράστασή του καθ'όλη την διάρκεια εκπόνησης της εργασίας μου. Επίσης, θα ήθελα να ευχαριστήσω τους καθηγητές κ. Κουτσούμπα και κ. Κεχαγιά ως μέλη της τριμελούς επιτροπής αλλά και τους υπόλοιπους καθηγητές μου στο μεταπτυχιακό πρόγραμμα.

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Abstract

The subjects of this Master thesis is to study the Sachdev-Ye-Kitaev (SYK) model, focusing on the Conformal Field Theory aspect, viewing it through the AdS/CFT correspondence. This particular model presents some very interesting features and thus attracted a lot of attention. It is now solved and there is ongoing research for its bulk dual.

To begin, the two-point and four-point functions are presented and from the latter the operators of the model are deduced. These Green's functions are studied in the IR limit where conformal symmetry is emergent. Then, the 6-point and 8-point functions are presented along with their features that make the model solvable.

Moreover, the effective action of the model and its symmetries are discussed along with certain reparametrizations and their physical meaning as Nambu-Goldstone modes. After presenting their Schwarzian action and a brief introduction to the quantum butterfly effect, the chaotic behaviour of the model is discussed. Finally, a brief discussion concerning the bulk dual and SYK-like models is presented. In the appendices, an introduction to conformal symmetry and the OPE expansion is given and also technical computations are included.

Περίληψη

Σκοπός της μεταπτυχιακής αυτής εργασίας είναι η μελέτη του μοντέλου SachdevYe-Kitaev (SYK), εστιάζοντας ιδιαίτερα στην πλευρά της Σύμμορφης Θεωρίας Πεδίου εφ'όσον το μοντέλο ιδωθεί μέσα από το πρίσμα της AdS/CFT. Το συγκεκριμένο μοντέλο παρουσιάζει μερικά ιδιαίτερα ενδιαφέροντα χαρακτηριστικά και γι'αυτό έχει τραβήξει την προσοχή της αντίστοιχης επιστημονικής κοινότητας.

Αρχικά, παρουσιάζονται οι συναρτήσεις δύο και τεσσάρων σημείων και από την τελευταία συνάγονται οι τελεστές του μοντέλου. Αυτές οι συναρτήσεις Green μελετώνται στο υπέρυθρο όριο, όπου αναδύεται η σύμμορφη συμμετρία. Στη συνέχεια, παρουσιάζονται οι συναρτήσεις έξι και οκτώ σημείων μαζί με τα χαρακτηριστικά τους, τα οποία καθιστούν το μοντέλο επιλύσιμο.

Επιπρόσθετα, συζητιούνται η ενεργός (effective) δράση του μοντέλου μαζί με τις συμμετρίες της παράλληλα με τις επαναπαραμετροποιήσεις του μοντέλου και την φυσική ερμηνεία τους ως Nambu-Goldstone τρόπους (modes). Αφού παρουσιαστεί η Schwarzian δράση τους και δοθεί μια σύντομη εισαγωγή στην κβαντομηχανική εκδοχή του κβαντικού χάους (ή κβαντικό φαινόμενο της πεταλούδας), συζητιέται η χαοτική συμπεριφορά του μοντέλου. Τέλος, γίνεται μια σύντομη συζήτηση που αφορά το βαρυτικό αντίστοιχο του μοντέλου και τα διάφορα παρόμοια μοντέλα που έχουν προταθεί. Στα παραρτήματα, εκτός από μια εισαγωγή στη σύμμορφη θεωρία πεδίου και στη μέθοδο OPE αυτής και στις σπινοριακές αναπαραστάσεις σε N διαστάσεις, βρίκονται και κάποιοι τεχνικοί υπολογισμοί της εργασίας.

Εισαγωγή

Προέλευση και κίνητρα για την μελέτη του SYK

Το μοντέλο Sachdev-Ye-Kitaev προτάθηκε πρόσφατα από τον A. Kitaev [1]. Αποτελεί ένα κβαντομηχανικό μοντέλο που απαρτίζεται από αλληλεπιδράσεις τετάρτης τάξης μεταξύ N Majorana φερμιονίων. Η Χαμιλτονιανή του δίνεται από

$$H = \frac{1}{4!}\sum_{j,k,l,m}^{N} J_{jklm} x_j x_k x_l x_m. \tag{1}$$

Μπορεί να ιδωθεί ως μια παραλλαγή ενός μοντέλου που προτάθηκε από τους Sachdev και Ye [15], το οποίο εισήχθη για να περιγράψει έναν μαγνήτη Heisenberg με τυχαίες αλληλεπιδράσεις. Η Χαμιλτονιανή του μοντέλου του Sachdev-Ye είναι

$$H = \frac{1}{\sqrt{NM}}\sum_{i>j}^{N} J_{ij}\mathbf{S}_i\mathbf{S}_j, \tag{2}$$

όπου οι τελεστές $\mathbf{S}_i, \mathbf{S}_j$ ικανοποιούν την άλγεβρα $su(M)$ ενώ οι συντελεστές J_{ij} είναι τυχαίοι αριθμοί διαλεγμένοι από μια κανονική κατανομή και δηλώνουν την ισχύ των αλληλεπιδράσεων. Το μοντέλο SY είναι επιλύσιμο στο όριο όπου $N \to \infty, M \to \infty$ και για πρώτη φορά συζητήθηκε στα πλαίσια της ολογραφίας στο [31], όπου ο Sachdev ανέδειξε την στενή αντιστοιχία μεταξύ ολογραφικών μετάλλων κοντά σε φορτισμένες AdS μελανές οπές και της κλασματοποιημένης (fractionalized) υγρής φάσης Fermi του πλεγματικού μοντέλου του Anderson.

Το μοντέλο SYK καθίσταται επιλύσιμο στο όριο $N \to \infty$, όπου τα διαγράμματα Feynman αποτελούνται από melonic διαγράμματα για την συνάρτηση δύο σημείων και από ladder διαγράμματα για την συνάρτηση τεσσάρων σημείων. Συνεπώς, είναι δυνατόν να εξαχθούν σχετικά απλές εκφράσεις για τις εξισώσεις Schwinger-Dyson των συναρτήσεων δύο και τεσσάρων σημείων.

Στις χαμηλές ενέργειες (υπέρυθρο όριο), οι συναρτήσεις δύο σημείων είναι επιλύσιμες και παρουσιάζουν σύμμορφη συμμετρία, η οποία όμως σπάει σε υψηλότερες ενέργειες. Η συμμετρία αυτή αλλά και το σπάσιμό της εμφανίζονται και στις συνάρτησεις τεσσάρων σημείων.

Επιπλέον, μελετώντας τις out-of-time-order συναρτήσεις συσχέτισης έχει αποδειχθεί ότι ο εκθέτης Lyapunov που χαρακτηρίζει τη χαοτική συμπεριφορά του συστήματος παίρνει τη μέγιστη τιμή του, $\lambda_L = \frac{2\pi}{\beta}$. Σύμφωνα με το [9] εικάζεται ότι αυτός είναι και ο μέγιστος επιτρεπόμενος εκθέτης Lyapunov για ένα μεγάλο αριθμό κβαντικών συστημάτων, όπως τα συστήματα με μεγάλο N των οποίων το μοντέλο SYK είναι ένα παράδειγμα . Το ίδιο όριο είναι αυτό που συναντάει κανείς σε μια μελανή οπή σε βαρυτικές θεωρίες [11].

Η μέγιστη χαοτική συμπεριφορά, η επιλυσιμότητα και οι ομοιότητες μεταξύ των συναρτήσεων δύο και τεσσάρων σημείων μίας μελανής οπής Schwarzschild 1+1 διαστάσεων και εκείνων του μοντέλου SYK, οδήγησε τον Kitaev να προτείνει αυτό το μοντέλο ως ένα ολογραφικό αντίστοιχο μιας τέτοιας μελανής οπής η οποία ασυμπτωτικά προσεγγίζει μια AdS μετρική [1].

Η προτεινόμενη δυαδικότητα μεταξύ του μοντέλου SYK και των μαύρων οπών είναι ένα παράδειγμα της AdS/CFT αντιστοιχίας, μια εικασία δυισμού μεταξύ ισχυρά συζευγμένων θεωρίων βαθμίδας και βαρυτικών θεωριών σε AdS χωρόχρονο. Η εικασία αυτή πρωτοεμφανίστηκε στην πρωτοποριακή δημοσίευση "The Large N Limit of Superconformal Field Theories and Supergravity" από τον J. Maldacena [18]. O βασικός ισχυρισμός της AdS/CFT αντιστοιχίας είναι ότι οι συναρτήσεις-γεννήτριες των ισχυρών συζευγμένων θεωριών βαθμίδας και των βαρυτικών θεωριών σε χωρόχρονους AdS είναι ίσες [29]. Αν και η AdS / CFT αντιστοιχία δεν έχει αποδειχθεί ρητά, επιτρέπει τη μελέτη των ισχυρά συζευγμένων θεωριών βαθμίδας εξετάζοντας ασθενώς συνδεδεμένες θεωρίες βαρύτητας στον AdS χωρόχρονο και αντίστροφα. Αυτό είναι χρήσιμο διότι οι διαταρακτικές μέθοδοι δεν λειτουργούν σε ισχυρά συζευγμένα συστήματα λόγω της έλλειψης μιας μικρής παραμέτρου γύρω από την οποία θα εφαρμοστεί η διαταραχή.

Ως εργαλείο διερεύνησης μαύρων οπών, το μοντέλο SYK παρουσιάζει μερικά προβλήματα. Ένα από αυτά είναι ότι τα πραγματικά κβαντικά συστήματα δεν έχουν τυχαίες αλληλεπιδράσεις στις οποίες είναι δύσκολο να αποδοθεί κβαντική συμπεριφορά. Επομένως δεν είναι άμεσα σαφές εάν το μοντέλο SYK μπορεί να χρησιμοποιηθεί για τη διερεύνηση των ιδιοτήτων των μαύρων οπών. Πρόσφατα, προτάθηκε [4, 5] ένα τανυστικό μοντέλο χωρίς αυτές τις τυχαίες αλληλεπιδράσεις, το οποίο όμως μοιράζεται τα σημαντικότερα χαρακτηριστικά του μοντέλου SYK.

Μια τελευταία παρατήρηση είναι ότι το μοντέλο SYK δεν μπορεί να θεωρηθεί ως παράδειγμα της AdS_2/CFT_1, γιατί όπως θα δούμε η σύμμορφη συμμετρία είναι αυθόρμητα αλλά και ρητά σπασμένη. O κατάλληλος τρόπος για να μελετήσει κανείς το μοντέλο είναι στα πλαίσια της $nAdS_2/nCFT_1$, όπου το n σημαίνει "σχεδόν (nearly)". Στη συνέχεια, το βαρυτικό αντίστοιχο θεωρείται ότι είναι ένα μοντέλο $nAdS_2$ το οποίο παρουσιάζει το ίδιο μοτίβο σπασίματος της σύμμορφης συμμετρίας [12, 23, 24].

Σχεδιάγραμμα της εργασίας αυτής

Η μεταπτυχιακή αυτή εργασία εστιάζει στην πλευρά που σχετίζεται με την σύμμορφη θεωρία πεδίου (CFT) του μοντέλου SYK. Στο **κεφάλαιο δύο**, εξάγουμε τις εξισώσεις κίνησης του μοντέλου SYK και εξάγουμε τον έλευθερο διαδότη. Στη συνέχεια μελετάμε την συνάρτηση Green δύο σημείων, χρησιμοποιώντας βασικές τεχνικές των διαταρακτικών θεωρίων, δείχνουμε ότι στο όριο $N \to \infty$ τα μοναδικά διαγράμματα που συνεισφέρουν είναι αυτά που έχουν melonic μορφή. Εκμεταλλευόμενοι το χαρακτηριστικό αυτό εξάγουμε τις εξισώσεις Schwinger-Dyson και στην συνέχεια τις μελετάμε στο όριο των χαμηλών ενεργειών. Σ' αυτό το όριο αποδεικνύουμε την αναδύουσα σύμμορφη συμμετρία και στη συνέχεια διαλέγοντας μια δοκιμαστική συνάρτηση (ansatz) για τον διαδότη, μελετάμε πως η συμμετρία αυτή σπάει αυθόρμητα.

Στο **κεφάλαιο τρία**, συνεχίζουμε την μελέτη μας εστιάζοντας στις συναρτήσεις Green τεσσάρων σημείων και στα ladder διαγράμματα τα οποία είναι και τα μοναδικά που συνεισφέρουν. Δείχνουμε ότι τα διαγράμματα αυτά ικανοποιούν αναδρομικές σχέσεις με κύριο χαρακτηριστικό τον τελεστή kernel ο οποίος δρα ως γεννήτορας τους. Έπειτα, χρησιμοποιώντας την σύμμορφη συμμετρία στο όριο των χαμηλών ενεργειών υπολογίζουμε όλες τις απαραίτητες ποσότητες (όπως οι ιδιοσυναρτήσεις του τελεστή Casimir, οι ιδιοτιμές του kernel και διάφορα εσωτερικά γινόμενα) οι οποίες μας οδηγούν σε μια τελική έκφραση για την συνάρτηση τεσσάρων σημείων. Στη συνέχεια αναλύουμε αυτήν την έκφραση, μέσω θεωρητικών εργαλείων που προέρχονται από τις σύμμορφες θεωρίες πεδίου και εξάγουμε διάφορες χρήσιμες ποσότητες όπως τις διαστάσεις διαφόρων τελεστών, τους συντελεστές OPE και τα conformal blocks. Είναι σημαντικό να επισημάνουμε ότι κατά την διάρκεια όλης αυτής της ανάλυσης έχουμε σκόπιμα αγνοήσει τις συνεισφορές οι οποίες οδηγούν σε αποκλίσεις. Φυσικά κάτι τέτοιο δεν είναι αποδεκτό αφού τότε η συνάρτηση τεσσάρων σημείων στερείται φυσικού νοήματος και στο τέλος του κεφαλαίου αυτού, κάνουμε μια σύντομη συζήτηση γι αυτό βασιζόμενοι στο [2].

Στο **κεφάλαιο τέσσερα**, παρουσιάζουμε και μελετάμε τις συναρτήσεις έξι και οκτώ σημείων ή ισοδύναμα τις συναρτήσεις τριών και τεσσάρων σημείων κατάλληλων διγραμμικών τελεστών αντίστοιχα. Για την συνάρτηση έξι σημείων, κατηγοριοποιούμε τα διαγράμματα που συνεισφέρουν σε contact και planar και παρουσιάζουμε τη συνεισφορά του καθενός. Όσον αφορά τη συνάρτηση οκτώ σημείων, κάνουμε μια σύντομη συζήτηση για το πως μπορεί κανείς να παράγει όλες τις συναρτήσεις 2n-σημείων με το να "κόψει" το melonic διάγραμμα κενού όσες φορές χρειαστεί. Στη συνέχεια, παραθέτουμε τα διαγράμματα που συνεισφέρουν στη συνάρτηση οκτώ σημείων μαζί με το ιδιαίτερα σημαντικό γεγονός ότι είναι απόλυτα καθορισμένα από ποσοτήτες που έχουμε υπολογίσει ήδη στις συναρτήσεις δύο, τεσσάρων και έξι σημείων. Βασιζόμενοι σ' αυτό το γεγονός συζητάμε πως κάτι τέτοιο υπονοεί ουσιαστικά την πλήρη επιλυσιμότητα (με την έννοια του υπολογισμού κάθε συνάρτησης 2n-σημείων) του μοντέλου SYK. Σχετικά με το θέμα αυτού του κεφαλαίου, περισσότερες τεχνικές λεπτομέρειες μπορούν να βρεθούν στο [12].

Στο **κεφάλαιο πέντε**, μελετάμε την ενεργό δράση του μοντέλου SYK η οποία εξάγεται αφού εκτελέσουμε την μέθοδο annealed disorder, η οποία πρακτικά είναι η άμεση ολοκλήρωση ως προς τις τυχαίες μεταβλητές. Στη συνέχεια αφού ολοκληρώσουμε ως προς τα φερμιόνια και εισάγουμε τα διγραμμικά πεδία G, Σ, βρίσκουμε ότι το μοντέλο παρουσιάζει κλασσική συμπεριφορά στο όριο για μεγάλα N και η ενεργός του δράση δίνει τις ίδιες εξισώσεις Schwinger-Dyson με αυτές που βρήκαμε στο δεύτερο κεφάλαιο. Ακολούθως, αποδεικνύουμε τη συμμετρία $\mathcal{O}(N)$ της δράσης του μοντέλου SYK και εξάγουμε το διατηρούμενο ρεύμα μόνο για την ελεύθερη θεωρία αφού αν συμπεριλάβουμε τον όρο αλληλεπίδρασης τότε έχουμε μια μη-τοπική δράση για την οποία δεν ισχύει το θεώρημα της Noether.

Στο **κεφάλαιο έξι**, μελετάμε τις διακυμάνσεις των διγραμμικών πεδίων γύρω από το σαγματικό σημείο τους ή αλλοιώς τις επαναπαραμετροποιήσεις τους. Πριν από αυτό, αποδεικνύουμε ρητά την σύμμορφη συμμετρία της δράσης στο όριο των χαμηλών ενεργειών και θέτουμε υπο συζήτηση το λόγο για τον οποίο αυτή η συμμετρία σπάει ρητά μακριά από αυτό το όριο. Μετά, θεωρώντας τις διακυμάνσεις των διγραμμικών πεδίων, φτάνουμε σε μια δράση της οποίας οι μηδενικοί τρόποι (modes) είναι ουσιαστικά εκείνες οι ιδιοσυναρτήσεις του kernel που έχουν ιδιοτιμή ένα. Αυτές είναι και οι ιδιοσυναρτήσεις που οδηγούν σε μια αποκλίνουσα και συνεπώς στερούμενη φυσικού νοήματος συνάρτηση τεσσάρων σημείων, όπως έχουμε ήδη προαναφέρει. Συζητούμε το φυσικό νόημα αυτών των μηδενικών τρόπων ως Nambu-Goldstone μποζόνια. Η ερμηνεία αυτή βασίζεται στο γεγονός ότι αυτοί οι τρόποι είναι ουσιαστικά παραμετροποιήσεις του διαδότη μακρία από το όριο που η θεωρία έχει σύμμορφη συμμετρία, συνεπώς προκύπτουν από το αυθόρμητο σπάσιμο αυτής. Τέλος, παραθέτοντας κάποια ποιοτικά χαρακτηριστικά που περιμένουμε να έχει η δράση αυτών των παραμετροποιήσεων καταλήγουμε στην τελική μορφή της η οποία είναι αυτή της παραγώγου Schwarzian.

Στο **κεφάλαιο επτά**, παρουσιάζουμε μια ανασκόπηση των βασικών εννοιών του κβαντικού χάους, του scrambling της πληροφορίας και της εικασίας για την ύπαρξη ενός παγκόσμιου (universal) φράγματος στην τιμή του εκθετικού του Lyapunov. Έχοντας αναφέρει όλα τα βασικά εργαλεία που θα χρειαστούμε στην περαιτέρω μας ανάλυση, μελατάμε τις out-of-time-order συναρτήσεις συσχέτισης του μοντέλου SYK. Καταλήγουμε σε ένα από τα πιο σημαντικά στοιχεία του μοντέλου: την μέγιστη χαοτική συμπεριφορά του.

Στο **κεφάλαιο οκτώ**, παρουσιάζουμε εν συντομία μερικές ενναλακτικές εκδοχές του μοντέλου SYK συνοδευόμενες από τα πλεονεκτήματα τους αλλά και τα μειονεκτήματα τους. Με αυτό τον τρόπο ο αναγνώστης παίρνει μια πρώτη ιδέα για την πιθανή κατεύθυνση της μελλοντικής έρευνας γύρω από το μοντέλο SYK. Στη συνέχεια, γίνεται μια περιληπτική συζήτηση που αφορά το βαρυτικό αντίστοιχο του μοντέλου και στο τέλος κλείνουμε το κύριο μέρος της μεταπτυχιακής εργασίας παραθέτοντας κάποια συμπεράσματα που αφορούν το υπό μελέτη μοντέλο και τα οποία έχουμε εξάγει κατά τη διάρκεια εκπόνησης της εργασίας αυτής.

Στα **Παραρτήματα**, έχουμε συμπεριλάβει διάφορα κεφάλαια που αφορούν στοιχεία που θεωρούμε ότι δεν είχαν θέση στο κυρίως μέρος της εργασίας αλλά είναι απαραίτητα και συνεισφέρουν σε έναν από τους βασικού στόχους μιας μεταπτυχιακής εργασίας, την πληρότητα της. Συγκεκριμένα, στο **Παράρτημα A**, δίνουμε μια αυτοτελή εισαγωγή στις σύμμορφες θεωρίες πεδίου. Φυσικά, τα περιεχόμενα της ενότητας αυτής μπορούν βρεθούν σε εξαιρετικά βιβλία και διαλέξεις, από τα οποία μερικά παρατίθενται στο τέλος. Στα **Παραρτήματα B,C**, βρίσκονται κάποιο τεχνικοί υπολογισμοί οι οποίοι θα αποπροσανατόλιζαν τον αναγνώστη έαν παρατίθονταν στο κύριο μέρος της εργασίας. Τέλος, στο **Παράρτημα D**, δίνεται μια εισαγωγή στις σπινοριακές αναπαραστάσεις σε N διαστάσεις μαζί με την μεθοδολογία για το πως υπολογίζονται αυτές για το μοντέλο SYK.

Contents

Chapter 1

Introduction

1.1 Origin and Motivation

The Sachdev-Ye-Kitaev model was recently proposed by A. Kitaev [1]. It a quantum mechanical model with quartic all-to-all interactions between N Majorana fermions. Its Hamiltonian is

$$H = \frac{1}{4!} \sum_{j,k,l,m}^{N} J_{jklm} x_j x_k x_l x_m. \tag{1.1}$$

It can be thought as variant of a model proposed by Sachdev and Ye [15], which was introduced to describe a Heisenberg magnet with random all-to-all interactions. The Hamiltonian of the Sachdev-Ye (SY) is given by

$$H = \frac{1}{\sqrt{NM}} \sum_{i>j}^{N} J_{ij} \hat{\mathbf{S}}_i \hat{\mathbf{S}}_j, \tag{1.2}$$

where the operators $\hat{\mathbf{S}}_i, \hat{\mathbf{S}}_j$ obey the $su(M)$ algebra and the coefficients J_{ij} are random numbers chosen from a normal distribution and denote the strength of the interactions. The SY model is solvable at $N \to \infty, M \to \infty$ and was first discussed in connection with holographic correspondence in [31], where Sachdev showed a close correspondence between holographic metals near charged AdS black holes and the fractionalised Fermi liquid phase of the lattice Anderson model.

The SYK model becomes solvable in the $N \to \infty$ limit, where the Feynman diagrams consist of melonic diagrams for the two-point function and ladder diagrams for the four-point function. Consequently, it is possible to derive relatively simple expressions for the Schwinger-Dyson equations of the two-point and the four-point functions.

At low energies (IR limit), the two-point functions are solvable and exhibit conformal symmetry, which, however, is broken at higher energies.

The conformal symmetry and its breaking at higher energies is also present at the four-point function.

Furthermore, by studying the out-of-order-time correlators it has been shown that the Lyapunov exponent that characterizes the chaotic behaviour of the system takes its maximum value, $\lambda_L = \frac{2\pi}{\beta}$. It has be shown that this is the maximal allowed Lyapunov exponent for a large number of quantum systems, such as the large N systems of which the SYK model is an example [9]. The same bound is saturated by a black hole in Einstein gravity [11].

The maximally chaotic behaviour, the exact solvability and the similarities between the two- and four-point functions of a 1+1 dimensional Schwarzschild black hole and those of the SYK model, led Kitaev to propose this model as a holographic dual of a Schwarzschild black hole in $1 + 1$ dimensional spacetime that is asymptotically AdS [1].

The proposed duality between the SYK model and black holes is an example of the AdS/CFT correspondence, a conjectured duality between strongly coupled gauge theories and gravitational theories on the AdS spacetime. The conjecture was introduced in the groundbreaking paper "The Large N Limit of Superconformal field theories and supergravity" by Maldacena [18]. The main claim of AdS/CFT correspondence is that the generating functionals of strongly coupled gauge theories and gravitational theories on the AdS spacetime are equal [29]. Although the AdS/CFT correspondence has not been proven rigorously, it allows the study of strongly coupled gauge theories by considering weakly-coupled gravitational theories on the AdS spacetime and vice versa. This is useful because the perturbation methods do not work with strongly-coupled systems due to the lack of a small expansion parameter.

As a tool of investigating black holes, the SYK model has a few problems. One of the main problems is that real quantum systems do not have random interactions that are averaged over a probability distribution. Therefore it is not immediately clear if the SYK model can be used to investigate subtler properties of black holes. Recently, a tensor model without these random interactions has been proposed [4,5], which shares the most important features of the SYK model without having random interactions.

One final remark, is that the SYK model can not be thought as and example of AdS_2/CFT_1 as as we will see the conformal symmetry is spontaneously and explicitly broken. The appropriate way to study the model is to consider it as a $nAdS_2/nCFT_1$, where n stand for nearly. Then the bulk dual is thought to be a $nAdS_2$ model which exhibits the same symmetry breaking pattern [12, 23, 24].

1.2 Outline of this thesis

The focus of this Master thesis is the CFT aspect of the SYK model. In **chapter two**, we study the two-point function and show, using standard perturbation techniques, that in the large N limit the only contribution to the two-point function comes from melonic diagrams. Taking advantage of this property we derive the Schwinger-Dyson equations and then study the IR limit of these equations. We prove the emergent conformal symmetry of the S-D equations and its spontaneous breaking by the chosen ansatz.

In **chapter three**, we study the four-point function and the ladder diagrams, which are the only ones that contribute to the four-point function. Then, by using conformal symmetry in the low energy limit, we compute all the relevant quantities (such as the eigenfunctions/eigenvalues of the Casimir and the kernel) that lead us to a final expression for the four-point function. Throughout this chapter as we will mention we avoid the $h = 2$ contribution that leads to divergences, but in the end we present a short discussion/review of this particular contribution based on [2].

In **chapter four**, we present and discuss the six- and eight-point function, or equivalently the bilinear three- and four-point function respectively. For the six-point function we categorize the contributing diagrams to contact and planar diagrams and present the contribution of each one. Considering the eight-point function, we make a short discussion about how one can produce all the relevant diagrams for all 2n-point functions by cutting the vacuum melon diagram. Then, we present the contributing eight-point diagrams along with their interesting fact that they are completely determined by quantities already computed in the two-, four- and six-point functions. Based on this result, we discuss how such a fact implies the full solvability of the SYK model. More technical details for the topic presented in this chapter can be found in [13].

In **chapter five**, we study the effective action of the model which is derived by performing the annealed disorder method which practically means that one directly averages the partition function. Then, by integrating out the fermion fields and introducing the bilocal field G, Σ we find out the the model becomes classical in large N and the effective action gives the already derived S-D equations. Then, we prove the $\mathcal{O}(N)$ symmetry of the action of the model and derive the conserved current only for the free action since the interaction term is bilocal and thus the Noether theorem is not valid.

In **chapter six**, we study the fluctuations of the bilocal fields or their reparametrizations. Before that, we prove rigorously the emergent conformal symmetry of the action in the IR limit and discuss the reason it is explicitly broken away from this limit. Then by considering the fluctuations of the bilocal fields we end up with an action that its zero modes turn to be eigenfunctions of the kernel with eigenvalue one. These are exactly the eigenfunctions that lead to a divergent four-point function. We discuss the physical interpretation of these zero modes as Nambu-Goldstone modes. Then, by imposing some qualitative argument we find that the action of these reparemetrizations is the Schwarzian derivative.

In **chapter seven**, we present a review of the basic concepts of quantum chaos, the scrambling of information and the conjecture about a universal bound on the Lyapunov exponent. Equipped with these tools, we study the out-of-time-order correlators of the SYK model, only to find out one of its most important features: its maximally chaotic behaviour.

In **chapter eight**, we shortly present some intriguing variants of the SYK model along with their advantages and problems. Then, a short discussion is made about the active research for the bulk dual for the model under study and finally we end the main part of this master thesis by providing some conclusion about the hallmark features and the problems of the SYK model.

In **appendix A**, we review the basic aspects of conformal field theory as its features are heavily used throughout this work. Although, the context presented in this appendix can be found in many excellent textbooks and lecture notes, this appendix helps towards one of the main goals of this thesis, its completeness. In **appendices B,C**, we have included some technical computations that would disorient the reader if they were included in the main part. Finally, in **appendix D**, we present a short discussion about spinor representations in various dimensions and how we can find the appropriate representation for the SYK model.

Chapter 2

Basic aspects of the Sachdev-Ye-Kitaev model

2.1 Introduction to the SYK model

The SYK model is a quantum mechanical model in 0+1 dimensions that consists of a Hamiltonian with random all-to-all interactions between N Majorana fermions[1]. The Hamiltonian is given by

$$H = \frac{1}{4!} \sum_{j,k,l,m}^{N} J_{jklm} x_j x_k x_l x_m. \tag{2.1}$$

The Majorana fermions x_i obey the anticommutation relations $\{x_i, x_j\} = 2\delta_{ij}$, from which we can deduce that these operators are dimensionless and that $x_i^2 = 1$. Regarding the variable J_{jklm}, it is randomly drawn from a normal/Gaussian distribution and it is time independent. This means that we study a model with quenched disorder. These variables have dimension of energy.

We can show that J_{jklm} is completeley antisymmetric because

$$\begin{aligned} J_{jklm} x_j x_k x_l x_m &= \frac{1}{2}(J_{jklm} x_j x_k x_l x_m + J_{kjlm} x_k x_j x_l x_m) \\ &= \frac{1}{2}(J_{jklm} x_j x_k x_l x_m - J_{kjlm} x_j x_k x_l x_m). \end{aligned}$$

and for this sum to be different than zero we must have $J_{jklm} = -J_{kjlm}$. This property of antisymmetry holds for all possible pairs of indices. It is important to mention that these variables vanish if 2 or more indices are the same.

[1]For more details, see Appendix D

The J_{jklm} are drawn from a distribution with

$$P(J_{jklm}) = \sqrt{\frac{N^3}{12\pi J^2}} exp\big(- \frac{N^3 J^2_{jklm}}{12 J^2}\big), \tag{2.2}$$

where $J^2_{jklm} = J_{jklm} J_{jklm}$ and $J^2 = \sum_{j,k,l,m}^{N} J_{jklm} J_{jklm}$. To compute the average $\overline{J_{jklm}}$, we will integrate over the probability distribution (with no summation over the indices):

$$\begin{aligned}\overline{J_{jklm}} &= \int_{-\infty}^{\infty} d(J_{jklm}) J_{jklm} P(J_{jklm}) = \\ &= \sqrt{\frac{N^3}{12\pi J^2}} \int_{-\infty}^{\infty} d(J_{jklm}) J_{jklm} exp\big(- \frac{N^3 J^2_{jklm}}{12 J^2}\big) = 0,\end{aligned}$$

as an odd function integrated on a symmetric interval. Moreover, we compute

$$\begin{aligned}\overline{J^2_{jklm}} &= \int_{-\infty}^{\infty} d(J_{jklm}) J^2_{jklm} P(J_{jklm}) = \\ &= \sqrt{\frac{N^3}{12\pi J^2}} \int_{-\infty}^{\infty} d(J_{jklm}) J^2_{jklm} exp\big(- \frac{N^3 J^2_{jklm}}{12 J^2}\big) = \frac{3! J^2}{N^3},\end{aligned}$$

where we have used the Gaussian integral $\int_{-\infty}^{\infty} dx x^2 e^{-ax^2} = \sqrt{\frac{\pi}{4a^3}}$.
So, we have

$$\begin{aligned}\overline{J_{jklm}} &= 0, \qquad (2.3)\\ \overline{J^2_{jklm}} &= \frac{3! J^2}{N^3}. \qquad (2.4)\end{aligned}$$

The Lagrangian (in Euclidean space) of this model is

$$L = \frac{1}{2} x_j \frac{dx_j}{d\tau} - H. \tag{2.5}$$

Using the Euler-Lagrange equations and the anticommutation relations of the fermions we have:

$$\begin{aligned}&\frac{dL}{dx_j} - \frac{d}{d\tau}\big(\frac{dL}{d\dot{x_j}}\big) = 0 \\ &\Leftrightarrow \frac{1}{2}\dot{x_j} - \frac{1}{3!} J_{jklm} x_k x_l x_m - (-\frac{1}{2}\dot{x_j}) = 0 \\ &\dot{x_j} = \frac{1}{3!} \sum_{klm} J_{jklm} x_k x_l x_m. \qquad (2.6)\end{aligned}$$

2.2 Two-point function

The two-point Green function in Euclidean space is defined as:

$$G_{ij}(\tau) \equiv < Tx_i(\tau)x_j(0) > \equiv < x_i(\tau)x_j(0) > \theta(\tau) - < x_i(0)x_j(\tau) > \theta(-\tau). \tag{2.7}$$

An important quantity that we will use from now on, is the normalized trace of the above two-point function. We will denote it as:

$$G_0(\tau) = \frac{1}{N}\sum_{i=1}^{N} G_{0,ii}. \tag{2.8}$$

The generating functional for the free Majorana fermion is given by (where J_j is a source):

$$\begin{aligned} Z_0[J] &= \int Dx_1 ... Dx_N e^{-\int d\tau\left(\frac{1}{2}x_j \frac{dx_j}{d\tau} + J_j x_j\right)} \\ &= N e^{\frac{1}{2}\int d\tau d\tau' J_k(\tau)\Delta(\tau-\tau')J_k(\tau')}. \end{aligned} \tag{2.9}$$

The propagator $\Delta(\tau - \tau')$ satisfies the equation

$$\frac{d}{d\tau}\Delta(\tau - \tau') = \delta(\tau - \tau'). \tag{2.10}$$

This equation is solved by the function $\mathrm{sgn}(\tau) = 2\theta(\tau) - 1$, so we find that

$$\Delta(\tau - \tau') = \frac{1}{2}\mathrm{sgn}(\tau - \tau').$$

Now, we are able to compute the two-point function using the equation

$$G_{0,ij}(\tau) = \frac{\delta_{ij}}{Z[0]}\frac{\delta}{\delta J_i(\tau)}\frac{\delta}{\delta J_j(0)} Z_0[J]|_{J=0}. \tag{2.11}$$

We have

$$G_{0,ij}(\tau) = \frac{\delta_{ij}}{Z[0]}\frac{\delta}{\delta J_i(\tau)}\left[\frac{1}{2}\int d\tau' \Delta(-\tau')J_j(\tau') + \frac{1}{2}\int d\tau J_j(\tau)\Delta(\tau)\right]$$

$$\times\, e^{\frac{1}{2}\int d\tau d\tau' J_k(\tau)\Delta(\tau-\tau')J_k(\tau')}.$$

Differentiating once more and setting $J = 0$ we get

$$G_{0,ij}(\tau) = \Delta(\tau)\delta_{ij} = \frac{1}{2}\mathrm{sgn}(\tau)\delta_{ij}, \quad G_0(\tau) = \frac{1}{N}\sum_{i=1}^{N} G_{0,ii}(\tau) = \frac{1}{2}\mathrm{sgn}(\tau).$$

By Fourier transformation we can calculate $G_0(\omega)$ with $\omega = \frac{(2n+1)\pi}{\beta}$ where ω are the Matsubara frequencies and β the inverse temperature. Thus, for $i = j$ we have:

$$\begin{aligned} G_0(\omega) &= \int_0^\beta d\tau e^{i\omega\tau} G_0(\tau) = \frac{1}{2}\int_0^\beta d\tau e^{i\omega\tau}\mathrm{sgn}(\tau) = \\ &= \frac{1}{2}\int_0^\beta d\tau e^{i\omega\tau} = -\frac{1}{2}\frac{i}{\omega}(e^{i\omega\beta}-1) = -\frac{1}{i\omega}. \end{aligned} \tag{2.12}$$

As expected due to the fermionic nature of the theory, the Green's function is odd and antiperiodic (the period being β). Summarizing, we have two important results:

$$G_0(\tau) = \frac{1}{2}\mathrm{sgn}(\tau), \tag{2.13}$$

$$G_0(\omega) = -\frac{1}{i\omega} = \frac{i}{\omega}. \tag{2.14}$$

2.3 Two-point function of the full interacting theory for large N

We will now compute the full two-point function of the model using perturbation theory and as we will see due to the large N limit and the disorder average, we will end up with some simple diagrams.
We have:

$$\begin{aligned} &< T(x_a(\tau_1)x_b(\tau_2) >= \int Dx_i e^{-S} x_a(\tau_1)x_b(\tau_2) = \\ &= \int Dx_i e^{-\int L_0 d\tau} x_a(\tau_1)x_b(\tau_2)\Big(1 - \int d\tau L_{int} + \frac{1}{2}(\int d\tau L_{\text{int}})^2 + ...\Big), \end{aligned} \tag{2.15}$$

where $L_{int} = \frac{1}{4!}\sum_{j,k,l,m}^N J_{jklm}x_jx_kx_lx_m$. We will now compute each term separately. The first term is the free two-point function as we have showed in the previous section:

$$\int Dx_i e^{-\int L_0 d\tau} x_a(\tau_1)x_b(\tau_2) = \frac{1}{2}\delta_{ab}\mathrm{sgn}(\tau_1-\tau_2). \tag{2.16}$$

The second term is:

$$\int Dx_i e^{-\int L_0 d\tau}\sum_{j,k,l,m}^N \overline{J_{jklm}}\frac{1}{4!}\int d\tau x_j(\tau)x_k(\tau)x_l(\tau)x_m(\tau)x_a(\tau_1)x_b(\tau_2) = 0, \tag{2.17}$$

because as we have shown $\overline{J_{jklm}} = 0$. So, in first order we do not have any contribution to the two-point function.
Next, we will compute the second order term:

$$\frac{1}{2}\frac{1}{4!4!}\int Dx_i e^{-\int L_0 d\tau} x_a(\tau_1)x_b(\tau_2) \sum_{j,k,l,m,n,p,q,r}^{N} \overline{J_{jklm}J_{npqr}} \int d\tau \\ \int d\tau' x_j(\tau)x_k(\tau)x_l(\tau)x_m(\tau)x_n(\tau')x_p(\tau')x_q(\tau')x_r(\tau'). \tag{2.18}$$

We will now use Wick's theorem and the fact that every contraction gives $G_{0,ij}(\tau-\tau')\delta_{ij}$. It is also important to avoid contractions that will result in giving the same indices in J. For simplicity, we will denote the contraction showing only the indices of each operator.
Thus, we have:

$$abjklmnpqr$$

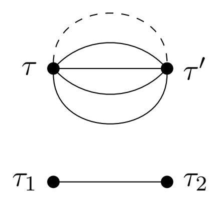
The first possible contraction is:

$$abjkllnpqr \tag{2.19}$$

We end up with the expression:

$$\frac{1}{2}\frac{4!}{4!4!}\int d\tau d\tau' \sum_{jklmnpqr} \overline{J_{jklm}J_{npqr}} \delta_{ab}\delta_{jn}\delta_{kp}\delta_{lq}\delta_{mr} G_0(\tau_1-\tau_2)[G_0(\tau-\tau')]^4. \tag{2.20}$$

This expression gives us the following Feynman diagram, in which after dividing by Z_0 the vacuum bubble will be eliminated.



We now move to a different contraction:

$$abjklmnpqr \tag{2.21}$$

The expression is:

$$\frac{1}{2}\frac{S}{4!4!}\int d\tau d\tau' \sum_{jklmnpqr} \overline{J_{jklm}J_{npqr}} G_{0,aj}\delta_{aj}(\tau_1-\tau)G_{0,bn}(\tau'-\tau_2)\delta_{bn}$$

$$\times G_{0,kp}(\tau-\tau')\delta_{kp}G_{0,lq}(\tau-\tau')\delta_{lq}G_{0,mr}(\tau-\tau')\delta_{mr},$$

where S is the symmetry factor (in this diagram $S = 4\cdot4\cdot4\cdot3\cdot2\cdot1\cdot3\cdot2\cdot1$).

To compute above expression, we will impose the rules that come from the delta functions and in the end we will compute the disorder average. Using

$$\sum_{klmpqr} \delta_{kp}\delta_{lq}\delta_{mr} = N^3,$$

we end up with the expression

$$\frac{1}{2}\frac{S}{4!4!}N^3 \int d\tau d\tau' \overline{J_{aklm}J_{bklm}} G_{0,aa}(\tau_1-\tau)G_{0,bb}(\tau'-\tau_2)[G_0(\tau-\tau')]^3. \quad (2.22)$$

We must now compute the disorder average. Using the results (2.3),(2.4) we have:

$$\overline{J_{aklm}J_{bklm}} = \begin{cases} \overline{J_{aklm}} \times \overline{J_{bklm}} = 0 & \text{if } a \neq b \\ \overline{J^2_{aklm}} = \frac{3!J^2}{N^3} & \text{if } a = b \end{cases}$$

Finally, we get

$$\frac{3!J^2}{2}\frac{S}{4!4!}\int d\tau d\tau' G_{0,aa}(\tau_1-\tau)G_{0,bb}(\tau'-\tau_2)[G_0(\tau-\tau')]^3, \quad (2.23)$$

which corresponds to the watermelon diagram:

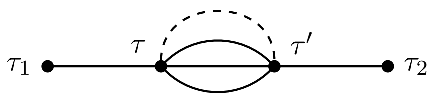
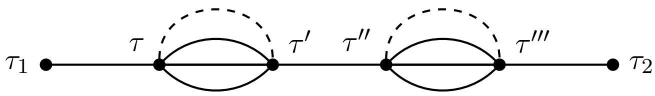


Figure 2.1: The watermelon diagram

The next non-vanishing contribution comes from $\frac{1}{4!}\Big(\int d\tau L_{int}\Big)^4$, we have the diagram



The above diagram corresponds to the following expression:

$$\frac{S}{(4!)^5}\int d\tau d\tau' d\tau'' d\tau''' \sum_{jklmnpqr}\sum_{cdefghsw} \overline{J_{jklm}J_{Npqr}J_{cdef}J_{ghsw}}$$
$$\times G_{0,aj}(t_1-\tau')\delta_{aj}[G_0(\tau-\tau')]^3\delta_{kp}\delta_{lq}\delta_{mr}G_{0,nc}(\tau'-\tau'')\delta_{nc}$$
$$\times [G_0(\tau''-\tau''')]^3\delta_{dh}\delta_{es}\delta_{fw}\delta_{bg}G_{0,bg}(\tau'''-\tau_2).$$

Again, from the summation of the delta functions we get

$$\sum_{jklmnpqr}\sum_{cdefghsw} \delta_{kp}\delta_{lq}\delta_{mr}\delta_{dh}\delta_{es}\delta_{fw} = N^6.$$

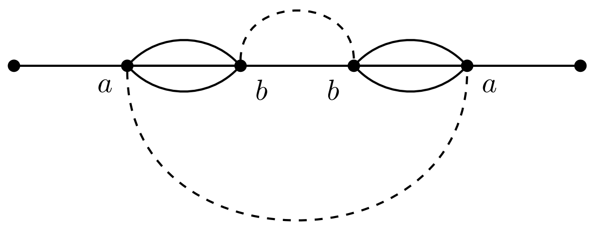
Thus, (ignoring the symmetry factor and the constants from perturbation theory) we have

$$N^6 \int d\tau d\tau' d\tau'' d\tau''' \overline{J_{aklm} J_{cklm} J_{cdef} J_{bdef}} G_0(\tau_1 - \tau')[G_0(\tau - \tau')]^3 G_0(\tau' - \tau'')$$
$$\times [G_0(\tau'' - \tau''')]^3 G_0(\tau''' - \tau_2).$$

In order to obtain a non-vanishing result we must set impose now the condition $c = b = a$. Then, the above expression becomes

$$\frac{(3!)^2 J^4 N^6}{N^6} \int d\tau d\tau' d\tau'' d\tau''' G_0(t_1 - \tau')[G_0(\tau - \tau')]^3 G_0(\tau' - \tau'')$$
$$\times [G_0(\tau'' - \tau''')]^3 G_0(\tau''' - \tau_2).$$

Again, we see that in the large N limit, this diagram contributes to the full two-point function. From the calculations of the previous two diagrams we can deduce the rule that every internal propagator that is included between the 2 vertices that we compute the disorder average contributes a factor N to the diagram. That means that in 2.1 the three internal propagators between the vertices τ, τ' give a N^3 contribution. Keeping this rule in mind, we examine the following diagram:



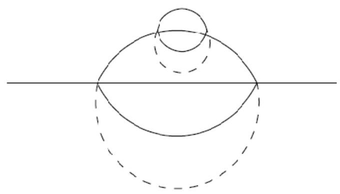
As we can see, it is the same diagram as before except that the disorder averages are computed in a different way. doing the same procedure as before we get:

$$\frac{(3!)^2 J^4}{N^6} \sum_{i,j,k,b} \delta_{ii}\delta_{jj}\delta_{kk}\delta_{ii}\delta_{bb} \int d\tau d\tau' d\tau'' d\tau''' G_0(t_1 - \tau')[G_0(\tau - \tau')]^3 G_0(\tau' - \tau'')$$
$$\times [G_0(\tau'' - \tau''')]^3 G_0(\tau''' - \tau_2),$$

where the indices i, j, k correspond to the internal propagators that connect the vertices a, b and the index b corresponds to the internal propagator that connects the vertices b, b. It is crucial to mention that the disorder average from vertex a to the other vertex a impose the condition that the internal propagators from the vertices $a \to b, b \to a$ must be the same. Because of this, we get a factor N^4 instead of N^6 from the summation of the delta functions. Overall, we get:

$$\frac{(3!)^2 J^4 N^4}{N^6} \sum_{i,j,k,b} \delta_{ii}\delta_{jj}\delta_{kk}\delta_{ii}\delta_{bb} \int d\tau d\tau' d\tau'' d\tau''' G_0(t_1 - \tau')[G_0(\tau - \tau')]^3 G_0(\tau' - \tau'') \\ \times [G_0(\tau'' - \tau''')]^3 G_0(\tau''' - \tau_2), \tag{2.24}$$

This diagram contributes as $\frac{1}{N^2}$ and in the large N limit it vanishes. We can conclude that the only diagram that contribute to the two-point function in this limit are the ones with the structure of the watermelon diagram. We demonstrate another diagram:



Now, inspired by 2.1 we define the self energy

$$\Sigma(\tau_1, \tau_2) = J^2 G(\tau_1, \tau_2)^3. \tag{2.25}$$

Then, we have the Dyson equation for the full two-point function:

$$\begin{aligned} G &= G_0 + G_0 \Sigma G + G_0 \Sigma G_0 \Sigma G_0 + ... \\ &= G_0[1 + \Sigma G_0 + \Sigma G_0 \Sigma G_0 + ...] \\ &= G_0[1 - \Sigma G_0]^{-1}, \end{aligned} \tag{2.26}$$

where in the last line we have resummed a geometric series. This can also be written as

$$G^{-1} = G_0^{-1} - \Sigma, \tag{2.27}$$

and in frequency space we have:

$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega). \tag{2.28}$$

2.4 Strong coupling limit

As we have seen, the parameter J has dimensions of energy so it is a relevant coupling. This means that in the IR limit it is strong and we can't use perturbation theory. The dimensionless coupling of the theory is βJ. So the strong coupling limit (at low energies) is defined by $J >> \omega >> \beta^{-1}$. In this limit, we can ignore the first term in the right hand side of equation (2.27). Thus , we have $G \times \Sigma = -1$. Using Fourier transformation we have the following equation to solve

$$\int d\tau' G(\tau, \tau')\Sigma(\tau', \tau'') = -\delta(\tau - \tau''). \tag{2.29}$$

One very interesting consequence of the IR limit is the emergent conformal symmetry of the above equation. In one dimension, every smooth transformation is a conformal transformation so we have $Conf(R^1) \cong Diff(R^1)$. Suppose we make the parametrization $\tau = f(\tau)$. We suppose the following equation:

$$J^2 \int d\tau' G(f(\tau), f(\tau'))[G(f(\tau'), f(\tau''))]^3 = -\delta(f(\tau) - f(\tau'')). \tag{2.30}$$

We want to show that if and only if the two-point function transforms as a conformal two-point function, that means

$$G(\tau, \tau') = |f'(\tau), f'(\tau')|^{\Delta} G(f(\tau), f(\tau')). \tag{2.31}$$

where $\Delta = \frac{1}{4}$, then equation (2.29) is invariant under the reparametrization $\tau = f(\tau)$.We have (for generality, we replace the interaction appearing in (2.30) with general q interactions between the fermions):

$$\begin{aligned}
&J^2 \int d\tau' G(\tau, \tau')[G(\tau', \tau'')]^{q-1} \\
&= J^2 \int d\tau' |f'(\tau), f'(\tau')|^{\Delta} |f'(\tau'), f'(\tau'')|^{\Delta(q-1)} G(f(\tau), f(\tau'))[G(f(\tau'), f(\tau'')]^{q-1} \\
&= J^2 \int d\tau' f'(\tau') \cdot \left[\frac{f'(\tau)}{f'(\tau'')}\right]^{q-1} f'(\tau'') G(f(\tau), f(\tau'))[G(f(\tau'), f(\tau'')]^{q-1}.
\end{aligned}$$

We now change the integration variable: $\tilde{f} = f(\tau'), d\tilde{f} = f'(\tau')d\tau'$. Thus,we get:

$$\begin{aligned}
&J^2 \int d\tau' G(\tau, \tau')[G(\tau', \tau'')]^{q-1} \\
&= J^2 \int d\tilde{f} G(f(\tau), f(\tau'))[G(f(\tau'), f(\tau'')]^{q-1} \left[\frac{f'(\tau)}{f'(\tau'')}\right]^{q-1} f'(\tau'').
\end{aligned} \tag{2.32}$$

Using equation (2.29) we finally get:

$$\begin{aligned} & J^2 \int d\tau' G(\tau),\tau')[G(\tau',\tau'']^{q-1} \\ & = -f(\tau'')\delta(f(\tau)-f(\tau'')) = -\delta(\tau-\tau''), \end{aligned} \tag{2.33}$$

To derive this, we have used the identity $\frac{1}{|f'(x_0)|}\delta(x-x_0) = \delta(f(x)-f(x_0))$. Thus, we have proved the emergent conformal symmetry of the propagator in the strong coupling limit. It is believed that this a necessary and sufficient condition for the entire model to be a conformal field theory in this limit.

We now give a simple example that justifies this consequence. Suppose $f(\tau) = a\tau$. Using the transformation rules, we have

$$G(\tau,\tau') = \sqrt{a}G(f(\tau), f(\tau').$$

Plugging this result to equation (1.26) we get

$$J^2 \int a d\tau' G(f(\tau), f(\tau'))G(f(\tau'), f(\tau'') = -\frac{1}{a}\delta(\tau-\tau'),$$

which is the correct transformation of the equation (2.30).

Taking under consideration the emergent conformal symmetry, we use an ansatz of the form:

$$G(\tau) = \frac{b}{|\tau|^{2\Delta}}\mathrm{sgn}(\tau). \tag{2.34}$$

By inserting this expression in (1.26) we will determine the constant b. We have:

$$J^2 b^4 \int d\tau' \frac{\mathrm{sgn}(\tau-\tau')}{|\tau-\tau'|^{2\Delta}} \frac{\mathrm{sgn}(\tau-\tau'')}{|\tau-\tau''|^{6\Delta}}. \tag{2.35}$$

We now use the Fourier transformation:

$$\frac{\mathrm{sgn}(\tau)}{|\tau|^{2\Delta}} = \int \frac{d\omega}{2\pi} e^{-i\omega\tau}|\omega|^{2\Delta-1} i 2^{1-2\Delta}\sqrt{\pi}\frac{\Gamma(1-\Delta)}{\Gamma(\frac{1}{2}+\Delta)}\mathrm{sgn}(\omega).$$

Thus, (2.35) becomes (for $\Delta = \frac{1}{4}$):

$$-J^2 b^4 \pi \int d\tau' \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi} e^{-i\omega\tau} e^{i\omega'\tau''} e^{i\tau'(\omega-\omega')}|\omega|^{-\frac{1}{2}}|\omega'|^{\frac{1}{2}} \frac{\Gamma(1-\frac{1}{4})}{\Gamma(\frac{1}{2}+\frac{1}{4})}\frac{\Gamma(1-\frac{3}{4})}{\Gamma(\frac{1}{2}+\frac{3}{4})}.$$

From the properties of Gamma function we get $\Gamma(\frac{5}{4}) = \frac{\Gamma(\frac{1}{4})}{4}$. Using

$$\delta(\omega-\omega') = \int \frac{d\tau'}{2\pi} e^{i\tau'(\omega-\omega')},$$

the equation (2.29) for the given ansatz becomes:

$$-4J^2b^4\pi \int \frac{d\omega}{2\pi} e^{i\omega(\tau-\tau')} = -\delta(\tau-\tau'). \tag{2.36}$$

We now use the Fourier transformation

$$\delta(\tau-\tau'') = \int \frac{d\omega}{2\pi} e^{-i\omega(\tau-\tau'')}.$$

Finally, we find that:

$$b = \left(\frac{1}{4J^2\pi}\right)^{\frac{1}{4}}.$$

For general q, we have

$$J^2b^q\pi = \left(\frac{1}{2}-\frac{1}{q}\right)\tan\left(\frac{\pi}{q}\right).$$

The full two-point function (in the strong coupling limit and zero temperature) is given by:

$$G(\tau) = \left(\frac{1}{4J^2\pi}\right)^{\frac{1}{4}} \frac{\mathrm{sgn}(\tau)}{|\tau|^{2\Delta}}. \tag{2.37}$$

To get the finite temperature version of the full two-point function, we use $f(\tau) = \tan\frac{\pi\tau}{\beta}$ as a reparametrization. Using the transformation properties (2.31) we get:

$$G_\beta(\tau-0) = \left(\frac{\pi}{\beta}\right)^{\frac{1}{2}} \left(\frac{1}{4J^2\pi}\right)^{\frac{1}{4}} \left(\frac{1}{\cos^2\frac{\pi\tau}{\beta}}\right)^{\frac{1}{4}} \frac{1}{\sqrt{\tan\frac{\pi\tau}{\beta}}}.$$

The full two-point function for a finite temperature is:

$$G_\beta(\tau) = \frac{\pi^{\frac{1}{4}}}{\sqrt{2J\beta}} \frac{1}{\sqrt{\sin\frac{\pi\tau}{\beta}}} \mathrm{sgn}\tau. \tag{2.38}$$

2.5 Spontaneous symmetry breaking

It is very important to notice that the ansatz (2.37) we have chosen causes the spontaneous breaking of conformal symmetry down to SL(2,R). The SL(2,R) is defined by Moebious transformations of the form

$$\tau \to f(\tau) = \frac{a\tau+b}{c\tau+d}, \quad ad-bc=1. \tag{2.39}$$

The derivative of this transformation is:

$$f'(\tau) = \frac{1}{(c\tau+d)^2}.$$

We will prove now the invariance of $G(\tau, \tau')$ under (2.39). For simplicity, we take $\tau' = 0$. Using the appropriate transformation rule we have:

$$\begin{aligned} G(\tau, 0) = |f'(\tau), f'(0)|^{\Delta} G(f(\tau), f(0)) &= \frac{1}{|c\tau + d|^{2\Delta}} \cdot \frac{1}{|d|^{2\Delta}} \frac{\text{sgn}(\tau)}{|\frac{a\tau+b}{c\tau+d} - \frac{b}{d}|^{2\Delta}} \\ &= \frac{1}{|c\tau + d|^{2\Delta}} \cdot \frac{1}{|d|^{2\Delta}} \frac{\text{sgn}(\tau)|d(c\tau + d)|^{2\Delta}}{|\tau(ad - bc)|^{2\Delta}} = \frac{\text{sgn}(\tau)}{|\tau|^{2\Delta}}. \end{aligned} \tag{2.40}$$

But if we apply a transformation $f(\tau) \in Conf(R^1)$ but $f(\tau) \notin SL(2, R)$ we can see that our ansatz is not invariant. For example we take $f(\tau) = a\tau^2$. Using the transformation rule we have:

$$G(\tau, \tau') = |\tau|^2 |\tau'|^2 \cdot \frac{\text{sgn}(a\tau^2 - a\tau'^2)}{|\tau^2 - \tau'^2|^{2\Delta}} \neq \frac{\text{sgn}(\tau - \tau')}{|\tau - \tau'|^{2\Delta}}. \tag{2.41}$$

With this two derivations we have proven the spontaneous breaking of the conformal symmetry which is a landmark feature of the SYK model.

Chapter 3

Four-point function

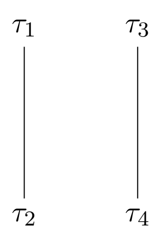
In this chapter we will study the four-point function which lead to some interesting results. As we have seen, in any correlation function the average over the disorder $J_{i_1,..,i_q}$ will give zero unless the indices are equal in pairs. Keeping this in mind the most general four-point function is

$$\langle x_i(\tau_1)x_i(\tau_2)x_j(\tau_3)x_j(\tau_4)\rangle. \tag{3.1}$$

Now we consider averaging over i, j indices. Thus, the averaged correlator is:

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\langle T(x_i(\tau_1)x_i(\tau_2)x_j(\tau_3)x_j(\tau_4))\rangle = G(\tau_{12})G(\tau_{34}) + \frac{1}{N}\mathcal{F}(\tau_1,..,\tau_4) \tag{3.2}$$

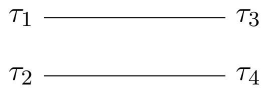
where $\frac{1}{N^2}$ is put for normalization purposes. In the right hand side,we have the disconnected piece after the contraction of the full propagators plus a power series in $1/N$. The disconnected diagram



gives a contribution of N^2 due to the summation over i, j and together with the normalization factor $1/N^2$, it gives a zeroth order contribution in powers of N.

3.1 Ladder diagrams

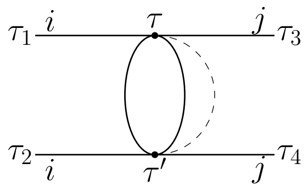
We will now analyse the first ladder diagrams that are needed to compute $\mathcal{F}$. These are diagrams with n rungs. Denoting a diagram with n rungs as $\mathcal{F}_n$, we have $\mathcal{F} = \sum_n \mathcal{F}_n$. The first diagram that contributes to the four-point function is



and its expression is given by the product of the propagators

$$\mathcal{F}_0(\tau_1..\tau_4) = -G(\tau_{13})G(\tau_{24}) + G(\tau_{14})G(\tau_{23}), \tag{3.3}$$

where the second term gives the same diagram but with $(\tau_3 \leftrightarrow \tau_4)$ and a relative minus sign due to the anticommutation relations that the fermions obey. The propagators set $i = j$ in the sum (3.2) so this gives an N contribution. Combined with the factor $1/N^2$, this diagram gives a $1/N$ contribution. Moving to the next diagram we have one rung. The diagram is the following

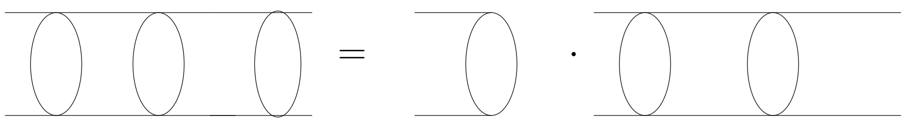


and the expression is given by:

$$\begin{aligned}\mathcal{F}_1 = J^2(q-1)\int d\tau d\tau' \Big[& G(\tau_1-\tau)G(\tau_2-\tau')G(\tau-\tau')^{q-2} \\ & \times G(\tau-\tau_3)G(\tau'-\tau_4) - (\tau_3 \leftrightarrow \tau_4)\Big],\end{aligned} \tag{3.4}$$

where we have integrated over the locations of the ends of the rung. This diagram also contributes $1/N$. That is because we get a factor $1/N^3$ from the disorder average and we also have a factor of N^2 from the internal propagators and a factor N^2 from the sum over the indices i, j. Thus a factor N divided by N^2 (the normalization factor) gives us at the end a contribution of $1/N$. Another way to check this is by ignoring the sum of i,j indices and the normalization factor and focus on the rung. Then we get a factor $1/N^{q-1}$ from the disorder average and a factor of N^{q-2} from the sum of the internal $(q-2)$ indices of the internal propagators.

The factor $(q-1)$ comes from the choice of which of the lines coming out of the interaction vertex should be contracted into a rung and which should continue on as the side rail. In our case, where $q = 4$ we get a factor 3. We will use now the fact that every ladder diagram is produced by multiplication by a kernel K as we can see in the following figure:



As we can see we can write a ladder diagram as:

$$\mathcal{F}_{n+1}(\tau_1,\tau_2,\tau_3,\tau_4) = \int d\tau d\tau' K(\tau_1,\tau_2;\tau,\tau')\mathcal{F}_n(\tau,\tau';\tau_3,\tau_4), \tag{3.5}$$

with the kernel being

$$K(\tau_1,\tau_2;\tau_3,\tau_4) \equiv -J^2(q-1)G(\tau_{13})G(\tau_{24})G(\tau_{34})^{q-2}. \tag{3.6}$$

We can think of the integral transform in the previous equation as matrix multiplication with the first 2 arguments of the kernel forming one index and the last 2 arguments forming the index that gets integrated. This way, the sum of all ladder diagrams is a geometric series.

$$\begin{aligned}\mathcal{F} = \sum_{n=0}^{\infty}\mathcal{F}_n = \sum_{n=0}^{\infty} K^n\mathcal{F}_0 = \frac{1}{1-K}\mathcal{F}_0 \\ = \sum_h \Psi_h(\chi)\frac{1}{1-k_c(h)}\frac{\langle\Psi_h(\chi),\mathcal{F}_0\rangle}{\langle\Psi_h(\chi),\Psi_h(\chi)\rangle},\end{aligned} \tag{3.7}$$

where $\Psi_h(\chi)$ are the eigenfunctions of the Casimir operator and k_c the eigenvalues of the kernel. To compute this sum we must first understand how to diagonalize K. As we can see by the way we have defined the kernel, it is not symmetric in $(\tau_1,\tau_2) \leftrightarrow (\tau_3,\tau_4)$ because:

$$K(\tau_3,\tau_4,\tau_1,\tau_2) = -J^2(q-1)G(\tau_1,\tau_3)G(\tau_2,\tau_4)G(\tau_1,\tau_2)^{q-2}. \tag{3.8}$$

However we can define the symmetric version as follows:

$$\begin{aligned}\tilde{K}(\tau_1,\tau_2;\tau_3,\tau_4) &\equiv |G(\tau_{12})|^{\frac{q-2}{2}}K(\tau_1,\tau_2;\tau_3,\tau_4)|G(\tau_{34})|^{\frac{2-q}{2}} \\ &= -J^2(q-1)|G(\tau_{12})|^{\frac{q-2}{2}}G(\tau_{12})G(\tau_{24})|G(\tau_{34})|^{\frac{q-2}{2}}.\end{aligned} \tag{3.9}$$

Now that we have shown how to symmetrize K, we can deduce that K has a complete set of eigenvectors. We will consider this kernel to act on the space of antisymmetric functions of two arguments.

3.2 Using conformal symmetry

What we have said is true for any value of the coupling βJ. Now we will consider the conformal limit $\beta J \gg 1$. We can now use the expressions we have found for $G_c(\tau)$. Using these expressions we can see that the kernel has no J dependence. Indeed:

$$K \propto J^2\left(\frac{1}{J^2}\right)^{1/q}\left(\frac{1}{J^2}\right)^{1/q}\left(\frac{1}{J^2}\right)^{q-2/q} \propto J^2\frac{1}{J^2}. \tag{3.10}$$

To find the explicit expressions for the kernel we substitute $G_c(\tau)$. We have:

$$K_c(\tau_1, \tau_2; \tau_3, \tau_4) = -\frac{1}{\alpha_0} \frac{\text{sgn}(\tau_{13})\text{sgn}(\tau_{24})\text{sgn}(\tau_{34})}{|\tau_{13}|^{2\Delta}|\tau_{24}|^{2\Delta}|\tau_{34}|^{2-4\Delta}}, \tag{3.11}$$

where

$$\alpha_0 = \frac{1}{(q-1)J^2 b^q} = \frac{2\pi q}{(q-1)(q-2)\tan\left(\frac{\pi}{q}\right)}. \tag{3.12}$$

Using this expression we can compute some of the correlators in the conformal limit but we have to be careful because some eigenfunctions of K can have the eigenvalue $K_c = 1$ and then the geometric series (3.7) will diverge. The crucial property to diagonalize the kernel is the use of conformal invariance. We will use the following generators:

$$\hat{D} = -\tau\partial_\tau - \Delta, \quad \hat{P} = \partial_\tau, \quad \hat{K} = \tau^2\partial_\tau + 2\tau\Delta\,. \tag{3.13}$$

We will now derive their commutation relations: For example,

$$\begin{aligned}
[\hat{D}, \hat{K}] &= (-\tau\partial_\tau - \Delta)(\tau^2\partial_\tau + 2\tau\Delta) - (\tau^2\partial_\tau + 2\tau\Delta)(-\tau\partial_\tau - \Delta) \\
&= (-2\tau^2\partial_\tau - \tau^3\partial_\tau^2 - \hat{K}\Delta - 2\Delta\tau) - (-\tau^2\partial_\tau - \tau^3\partial_\tau^2 - \hat{K}\Delta - 2\tau^2\partial_\tau \\
&= -\tau^2\partial_\tau - 2\Delta\tau = -\hat{K}.
\end{aligned}$$

Finally, we can see that the generators obey the following commutation relations:

$$[\hat{D}, \hat{P}] = \hat{P}, \quad [\hat{D}, \hat{K}] = -\hat{K}, \quad [\hat{P}, \hat{K}] = -2\hat{D}\,. \tag{3.14}$$

These generators commute with the kernel K_c up to total derivatives with respect to τ_3, τ_4. That means:

$$(\hat{D}_1 + \hat{D}_2)K_c(\tau_1, \tau_2; \tau_3, \tau_4) = K_c(\tau_1, \tau_2; \tau_3, \tau_4)(\hat{D}_3 + \hat{D}_4)\,. \tag{3.15}$$

The same holds for the $\hat{P}$ and $\hat{K}$ generators. We will confirm this relation for the $\hat{P}$ generators. We have:

$$(\hat{P}_1 + \hat{P}_2)K_c(\tau_1, \tau_2; \tau_3, \tau_4) = K_c(\tau_1, \tau_2; \tau_3, \tau_4)(\hat{P}_3 + \hat{P}_4)\,. \tag{3.16}$$

The left hand side can be written as $\int d\tau_1 d\tau_2 (\frac{dK_c}{d\tau_1} + \frac{dK_c}{d\tau_2})u_a(\tau_1, \tau_2)$, where u_a are the eigenfunctions of the kernel. The right hand side is:

$$\begin{aligned}
&\int d\tau_3 d\tau_4 K_c(\tau_1, \tau_2; \tau_3, \tau_4)\hat{P_{34}}u_a(\tau_3, \tau_4) = \int d\tau_3 d\tau_4 \hat{P_{34}}(K_c \cdot u_a) - \\
&\int d\tau_3 d\tau_4 (\hat{P_{34}}K_c)u_a = \frac{d}{d\tau_3}(K_c \cdot u_a)|_{-\infty}^{\infty} + \frac{d}{d\tau_4}(K_c \cdot u_a)|_{-\infty}^{\infty} - \\
&\int d\tau_3 d\tau_4 (\hat{P_{34}}K_c)u_a(\tau_3, \tau_4) = \frac{d}{d\tau_3}(K_c \cdot u_a)|_{-\infty}^{\infty} + \frac{d}{d\tau_4}(K_c \cdot u_a)|_{-\infty}^{\infty} - \\
&\int d\tau_1 d\tau_2 (\hat{P_{12}}K_c)u_a(\tau_1, \tau_2).
\end{aligned} \tag{3.17}$$

As we can see we get the left hand side plus some total derivatives. We should be careful about dropping the boundary terms. The commutation we have shown means that the generators of the conformal algebra take the solutions of $Ku_a = g(a)u_a$ to new solutions with the same eigenvalue.

We find conformal symmetry to be useful in two ways.

- First, as we have seen $\mathcal{F}_0$ is a product of two-point Green functions which transform in a conformal way. That means that also $\mathcal{F}_0$ transforms as a conformal four-point function. That implies that we can write the ladder diagrams $\mathcal{F}_n$ as simple powers times a function of the invariant cross ratio
$$\chi = \frac{\tau_{12}\tau_{34}}{\tau_{13}\tau_{24}}.$$
This will allow us to represent the kernel in the space of function of a single cross ratio $K_c(\chi; \tilde{\chi})$ instead of function of $\tau_1, \tau_2, \tau_3, \tau_4$.

- Second, it implies that the kernel commutes with the Casimir operator C_{1+2} defined as:
$$C_{1+2} = (\hat{D}_1 + \hat{D}_2)^2 - \frac{1}{2}(\hat{K}_1 + \hat{K}_2)(\hat{P}_1 + \hat{P}_2) - \frac{1}{2}(\hat{P}_1 + \hat{P}_2)(\hat{K}_1 + \hat{K}_2)\,. \tag{3.18}$$
Using the commutation relations of the generators and the fact that two generators acting on different times commute we get the expression for the Casimir:
$$C_{1+2} = 2(\Delta^2 - \Delta) - \hat{K}_1\hat{P}_2 - \hat{P}_1\hat{K}_2 + 2\hat{D}_1\hat{D}_2\,. \tag{3.19}$$
The Casimir is a differential operator with eigenfunctions given by simple powers times function $\Psi_h(\chi)$. The fact that the kernel commutes with the Casimir operator tells us that the eigenfunctions $\Psi_h(\chi)$ of the Casimir are also the exact eigenfunctions of the kernel $K_c(\chi; \tilde{\chi})$.

3.2.1 The four-point function as a function of cross ratios

In the strong coupling limit, taking advantage of the emergent conformal symmetry, we deduce that the ladder diagrams $\mathcal{F}_n$ will transform as conformal four-point functions:

$$\mathcal{F}_n(\tau_1, \tau_2, \tau_3, \tau_4) = G_c(\tau_{12})G_c(\tau_{34})\mathcal{F}_n(\chi)\,. \tag{3.20}$$

We use the antisymmetry under $\tau_1 \leftrightarrow \tau_2$,$\tau_3 \leftrightarrow \tau_4$,the symmetry under $(\tau_1, \tau_2) \leftrightarrow (\tau_3, \tau_4)$ and SL(2) transformations and we can set $\tau_1 = 0, \tau_3 = 1, \tau_4 = \infty$ and $\tau_2 > 0$. The cross ratio now becomes $\chi = \tau_2 > 0$. Using the time-ordering in the definition of the four-point averaged correlator we have the following cases:

$$\begin{cases} +\langle x_j(\infty)x_j(1)x_i(\chi)x_i(0)\rangle & \text{if } 0<\chi<1 \\ -\langle x_j(\infty)x_i(\chi)x_j(1)x_i(0)\rangle & \text{if } 1<\chi<\infty \end{cases}$$

In the region $\chi > 1$, the correlation function has an extra symmetry. To see this, we use the following nonstandard map

$$\frac{\tau-2}{\tau}=\tan\frac{\theta}{2}.$$

The operators for $\tau = 0, 1, \infty$ get sent to the points $-\pi, -\frac{\pi}{2}, \frac{\pi}{2}$. For $\tau_2 = \chi$ the operator get sent to some coordinate θ. The symmetry $\theta \to -\theta$ of the circle translates to $\chi \to \frac{\chi}{\chi-1}$. This can be checked as following:

$$\frac{\frac{\tau_2}{\tau_2-1}-2}{\frac{\tau_2}{\tau_2-1}}=\frac{\frac{\tau_2-2\tau_2+2}{\tau_2-1}}{\frac{\tau_2}{\tau_2-1}}=\frac{-\tau_2+2}{\tau_2}=\tan\frac{-\theta}{2}\,, \tag{3.21}$$

which is the correct transformation. As a consequence of this symmetry, in the region where $\chi > 1$ we must have $\mathcal{F}(\chi) = \mathcal{F}(\frac{\chi}{\chi-1})$. Moreover using this symmetry we have $\mathcal{F}(1) = \mathcal{F}(\infty)$. This maps the interval $1 < \chi < 2$ to the range $2 < \chi < \infty$ with fixed point at $\chi = 2$. This implies that the full $\mathcal{F}(\chi)$ can be fully determined once we know the function in $0 < \chi < 2$, and that the derivative of $\mathcal{F}$ must vanish at $\chi = 2$. This can be checked by differentiating $\mathcal{F}(\chi) = \mathcal{F}(\frac{\chi}{\chi-1})$. We get $\mathcal{F}'(2) = -\mathcal{F}'(2)$, so the derivative must vanish.
Another advantage is that ladder kernel can be written as

$$\mathcal{F}_{n+1}(\chi)=\int_0^2\frac{d\tilde{\chi}}{\chi^2}K_c(\chi,\tilde{\chi})\mathcal{F}_n(\chi)\,, \tag{3.22}$$

where $K_c(\chi, \tilde{\chi})$ is the symmetric kernel in terms of hypergeometric function as we have shown in chapter 3 of the appendices.
In the following section we will compute the eigenfunctions of the Casimir operator, the eigenvalues of the kernel and the necessary inner products for equation (3.7).

3.2.2 Eigenfunctions of the Casimir operator

We will now compute a complete set of eigenfunctions of the Casimir C_{1+2} with the required properties. We will derive now how the Casimir operator acts on functions of the cross-ratio. The explicit calculation can be found on the appendices. We find the following relationship:

$$C_{1+2}\frac{1}{|\tau_{12}|^{2\Delta}}f(\chi)=\frac{1}{|\tau_{12}|^{2\Delta}}\mathcal{C}f(\chi)\,, \tag{3.23}$$

with $\mathcal{C} \equiv \chi^2(1-\chi)\partial_\chi^2 - \chi^2\partial_\chi$. Writing the eigenvalues as $h(h-1)$, we have to solve the equation $\mathcal{C}f = h(h-1)f$. The general solution is a linear combination of

$$\chi^h {}_2F_1(h,h,2h,\chi), \qquad \chi^{1-h} {}_2F_1(1-h,1-h,2-2h,\chi)\,. \tag{3.24}$$

We now need to select a from this set a complete basis for functions that satisfy $f'(2)=0$. Moreover these functions should be normalizable with respect to the inner product

$$\langle g,f\rangle = \int_0^2 \frac{d\chi}{\chi^2} g^*(\chi)f(\chi)\,. \tag{3.25}$$

We will use this product to make C hermitian. As we know, the eigenfunctions of a hermitian operator are complete so we will use this fact and try to make the boundary terms that will occur vanish. The hermiticity condition is:

$$\begin{aligned} 0 = \langle g,Cf\rangle - \langle Cg,f\rangle &= \int_0^2 d\chi \left[g^*(1-\chi)f'' - g^*f' - (g^{*\prime\prime}(1-\chi)f - g^{*\prime}f)\right] \\ &= \left[g^*(1-\chi)f' - g^{*\prime}(1-\chi)f\right]' = \left[g^*(1-\chi)f' - g^{*\prime}(1-\chi)f\right]\Big|_0^2\,. \end{aligned} \tag{3.26}$$

As we have seen $f'(2)=0$, thus the boundary term at $\chi=2$ vanishes. The boundary term also vanishes at $\chi=0$ if we impose that $f\to 0$ faster than $\chi^{1/2}$. The eigenfunctions (3.24) at $\chi=1$ have logarithmic divergence as they can be approximated by $f \approx A + B\log(1-\chi)$ for $\chi\to 1^-$ and $f \approx C + D\log(\chi-1)$ for $\chi \to 1^+$. Consequently, for this boundary term to vanish we must impose that $A=C, B=D$. In other words, the eigenfunctions must approach χ from $\chi\to 1^+$ and $\chi \to 1^-$ in the same way. We will now determine the correct set of eigenfunction with the properties we mentioned. First, in the region $\chi>1$ we impose the condition $f'(2)=0$. Using special hypergeometric identities and a convenient normalization factor we get:

$$\Psi_h = \frac{\Gamma(1/2-h/2)\Gamma(h/2)}{\sqrt{\pi}} {}_2F_1(h/2,h/2,h,\frac{(2-\chi)^2}{\chi^2}) \quad \chi>1\,. \tag{3.27}$$

In the region $\chi<1$ we have the linear combination:

$$\Psi_h = A\frac{\Gamma(h)^2}{\Gamma(2h)}\chi^h {}_2F_1(h,h,2h,\chi) + B\frac{\Gamma(1-h)^2}{\Gamma(2-2h)}\chi^{1-h} {}_2F_1(1-h,1-h,2-2h,\chi)\,. \tag{3.28}$$

As we expected from the equation of the Casimir operator, in both region we have $\Psi_h = \Psi_{1-h}$. We can now expand these eigenfunctions for $\chi = 1+\epsilon$

and $\chi = 1 - \epsilon$ to study the behaviour around $\chi = 1$ and determine the factors A, B. We get:

$$A(h) = \frac{1}{\tan\frac{\pi h}{2}} \frac{\tan\pi h}{2} \quad B = A(1-h) = -\tan\frac{\pi h}{2}\frac{\tan\pi h}{2}\,. \tag{3.29}$$

The last condition we have to impose is that the eigenfunctions must vanish as fast as $x^{1/2}$ as χ approaches 0. We have two types of solutions based on the fact that $h > 1/2$.

1. $h = 2n, n = 1, 2, 3...$ .For these values $A = 1$ and B vanishes.

2.$h = \frac{1}{2} + is$ Together with these two sets the eigenfunctions given by (3.27),(3.28) in the related regions form a complete basis of normalizable function with the conditions we have imposed.

3.2.3 Eigenvalues of the kernel

Having now computed the eigenfunctions of the Casimir operator, we proceed to the computation of its eigenvalues. As we have shown, the Casimir operator commutes with the kernel K_c (3.15), so the Casimir eigenvalues are also eigenvalues of the kernel. The most intuitive way to compute them is to solve the equation $K_c(\chi, \tilde{\chi})\Psi_h(\tilde{\chi}) = k_c(h)\Psi(\tilde{\chi})$, where we have avoided the integration for simplicity. However, there is a simpler way to get the answer. We think about the Casimir as acting on two times C_{1+2}. We then have the equation $C_{1+2}\Psi_h = k_c(h)\Psi_h$. The eigenfunctions of the operator turn to be of the form of a conformal three-point function of two fermions and an operator of dimension h. Written as linear combination we have:

$$\begin{aligned}
&\Psi_h(\tau_1, \tau_2) = \int d\tau_o g(\tau_o) f(\tau_1, \tau_2), \\
&\text{where} \quad f_h^{\tau_0}(\tau_1, \tau_2) = \frac{\text{sgn}(\tau_1 - \tau_2)}{|\tau_1 - \tau_0|^h |\tau_2 - \tau_0|^h |\tau_1 - \tau_2|^{2\Delta - h}}.
\end{aligned} \tag{3.30}$$

To determine the eigenvalues we have to solve the equation:

$$k_c(h) f_h^{\tau_0}(\tau_1, \tau_2) = \int d\tau d\tau' K_c(\tau_1, \tau_2; \tau, \tau') \frac{\text{sgn}(\tau - \tau')}{|\tau - \tau_0|^h |\tau' - \tau_0|^h |\tau - \tau'|^{2\Delta - h}}\,. \tag{3.31}$$

Using SL(2,R) symmetry we can move τ_0 around also set $\tau_1 = 1, \tau_2 = 0$. We, then set τ_0 to approach infinity. Thus:

$$\begin{aligned}
k_c(h) &= \int d\tau d\tau' K_c(1, 0; \tau, \tau') \frac{\text{sgn}(\tau - \tau')}{|\tau - \tau'|^{2\Delta - h}} \\
&= -\frac{1}{\alpha_0} \int d\tau d\tau' \frac{\text{sgn}(1 - \tau)\text{sgn}(-\tau')\text{sgn}(\tau - \tau')}{|\tau - \tau'|^{2 - 2\Delta - h} |\tau'|^{2\Delta} |1 - \tau|^{2\Delta}}\,.
\end{aligned} \tag{3.32}$$

The straightforward way to evaluate this integral is by dividing the integration regions according to the sign functions. But there is a quicker and more elegant way. We make use of:

$$\frac{\mathrm{sgn}(\tau)}{|\tau|^{\alpha}} = \int \frac{d\omega}{2\pi} e^{-i\omega\tau} c(\alpha) |\omega|^{\alpha-1} \mathrm{sgn}(\omega), \quad c(\alpha) = 2i2^{-\alpha}\sqrt{\pi}\frac{\Gamma(1-\frac{\alpha}{2})}{\Gamma(\frac{1}{2}+\frac{\alpha}{2})}. \tag{3.33}$$

By using this identity we can write the factor $1/|\tau-\tau'|^{2-2\Delta-h}$ as a Fourier transformation. We get:

$$\int \frac{d\omega}{2\pi} |\omega|^{2-2\Delta-h-1} C(2-2\Delta-h)\mathrm{sgn}(\omega) * \int d\tau e^{-i\omega\tau} \frac{\mathrm{sgn}(1-\tau)}{|1-\tau|^{2\Delta}}$$
$$* \int d\tau' e^{i\omega\tau'} \frac{(-1)\cdot \mathrm{sgn}(\tau')}{|-\tau'|^{2\Delta}},$$

where we also have used $\mathrm{sgn}(-\tau') = -\mathrm{sgn}(\tau')$. Changing integration variables, the second integral can be written as:

$$\int d\tau e^{-i\omega\tau} \frac{\mathrm{sgn}(1-\tau)}{|1-\tau|^{2\Delta}} = \int du e^{-i\omega(1-u)} \frac{\mathrm{sgn}(u)}{|u|^{2\Delta}}.$$

Thus we get 2 similar integrals (up to a minus sign) of the form of (3.33). Thus we use this formula 2 more times and we finally find:

$$k_c = \frac{1}{\alpha_0} \frac{c(2-2\Delta-h)}{c(2\Delta-h)} [c(2\Delta)]^2 \,. \tag{3.34}$$

Substituting α_0 and the coefficients c from (3.33) we get:

$$k_c = -\frac{(q-1)(q-2)}{2q} \tan\left(\frac{\pi}{q}\right) \frac{[\Gamma(1-\frac{1}{q})]^2}{[\Gamma(\frac{1}{2}+\frac{1}{q})]^2} \frac{\Gamma(\frac{1}{q}+\frac{h}{2})}{\Gamma(\frac{3}{2}-\frac{1}{q}-\frac{h}{2})} \frac{\Gamma(\frac{1}{2}+\frac{1}{q}-\frac{h}{2})}{\Gamma(1-\frac{1}{q}+\frac{h}{2})}. \tag{3.35}$$

We can express the tangent function in terms of Gamma functions using the following:

$$\sin(\pi z) = \frac{\pi}{\Gamma(z)\Gamma(1-z)} \quad \tan\left(\frac{\pi}{q}\right) = \frac{\sin\left(\frac{\pi}{q}\right)}{\sin\left(\frac{\pi}{2}-\frac{\pi}{q}\right)}.$$

Thus,

$$\tan\left(\frac{\pi}{q}\right) = \frac{\Gamma(\frac{1}{2}+\frac{1}{q})\Gamma(\frac{1}{2}-\frac{1}{q})}{\Gamma(1-\frac{1}{q})\Gamma\left(\frac{1}{q}\right)}$$

Then we can use the identity $z\Gamma(z) = \Gamma(z+1)$ and we can write

$$\left(\frac{1}{2}-\frac{1}{q}\right)\Gamma\left(\frac{1}{2}-\frac{1}{q}\right) = \Gamma\left(\frac{3}{2}-\frac{1}{q}\right) \quad .$$

Substituting the above in (3.35), we finally get the eigenvalues of the kernel:

$$k_c(h) = -(q-1)\frac{\Gamma(1-\frac{1}{q})\Gamma\left(\frac{3}{2}-\frac{1}{q}\right)}{\Gamma(\frac{1}{2}+\frac{1}{q})\Gamma\left(\frac{1}{q}\right)}\frac{\Gamma(\frac{1}{q}+\frac{h}{2})}{\Gamma(\frac{3}{2}-\frac{1}{q}-\frac{h}{2})}\frac{\Gamma(\frac{1}{2}+\frac{1}{q}-\frac{h}{2})}{\Gamma(1-\frac{1}{q}+\frac{h}{2})}. \tag{3.36}$$

The eigenvalues are real for $h = \frac{1}{2} + is$ and $h = 2n$. Moreover, they satisfy $k_c(h) = k_c(1-h)$ as expected. We will now compute them for q=4. We get:

$$k_c(h) = -3\frac{\Gamma(5/4)}{\Gamma(1/4)}\frac{\Gamma\left(\frac{1}{4}+\frac{h}{2}\right)\Gamma\left(\frac{3}{4}-\frac{h}{2}\right)}{\Gamma\left(\frac{5}{4}-\frac{h}{2}\right)\Gamma\left(\frac{3}{4}+\frac{h}{2}\right)}$$

We can write

$$\Gamma(5/4) = \frac{1}{4}\Gamma(1/4) \quad \Gamma\left(\frac{5}{4}-\frac{h}{2}\right) = \left(\frac{1}{4}-\frac{h}{2}\right)\Gamma\left(\frac{1}{4}-\frac{h}{2}\right)$$

and then use $\sin(\pi z) = \frac{\pi}{\Gamma(z)\Gamma(1-z)}$. We arrive at:

$$k_c(h) = -\frac{3}{2}\frac{\tan\frac{\pi(h-1/2)}{2}}{(h-1/2)}$$

We summarize some simple cases:

$$k_c(h) = -\frac{3}{2}\frac{\tan\frac{\pi(h-1/2)}{2}}{(h-1/2)} \qquad q=4 \tag{3.37}$$

$$k_c(h) = \frac{2}{h(h-1)} \qquad q=\infty \tag{3.38}$$

$$k_c(h) = -1 \qquad q=2 \tag{3.39}$$

For the cases $q = 4, q = \infty$, we get the important result $k_c(2) = 1$.

3.2.4 Relevant inner products

In this section, we will compute all the relevant inner products that are present in the four-point function. To start, we will compute the norm of the eigenfunctions $\langle \Psi_h, \Psi_{h'} \rangle$. For the continuum case $h = \frac{1}{2} + is$ we expect that this inner product will be proportional to $\delta(s - s')$. Such a contribution can only come from small χ, where we can use the expansion:

$${}_2F_1(a, b, c, \chi) \approx 1 + \frac{ab\chi}{c} + \mathcal{O}(\chi^2). \tag{3.40}$$

Thus, in the small χ region we can replace the hypergeometric functions by one. The inner product becomes:

$$\langle \Psi_h, \Psi_{h'} \rangle = \int_0^\epsilon \frac{d\chi}{\chi^2}\Psi_h^* \cdot \Psi_{h'} = \int_0^\epsilon \frac{d\chi}{\chi^2}\left[A(h)\frac{\Gamma(h)^2}{\Gamma(2h)}\chi^h + B(h)\frac{\Gamma(1-h)^2}{\Gamma(2-2h)}\chi^{1-h}\right]^* \cdot \left[A(h')\frac{\Gamma(h')^2}{\Gamma(2h')}\chi^{h'} + B(h)\frac{\Gamma(2h')^2}{\Gamma(2-2h')}\chi^{1-h'}\right] \tag{3.41}$$

Now we will keep only the finite terms when $s \to s'$ as the other terms will have no contribution since eigenfunctions with different h must be orthogonal. Thus:

$$\begin{aligned}\langle \Psi_h, \Psi_{h'} \rangle \sim \int_0^{\epsilon} \frac{d\chi}{\chi^2} \Big[& A(h')A(h^*) \frac{\Gamma(h')^2}{\Gamma(2h')} \frac{\Gamma(h^*)^2}{\Gamma(2h^*)} \chi^{h^*+h} \\ & + B(h')B(h^*) \frac{\Gamma(1-h^*)^2}{\Gamma(2-2h^*)} \frac{\Gamma(1-h')^2}{\Gamma(2-2h')} \chi^{1-h^*+1-h'} \Big] \, .\end{aligned} \tag{3.42}$$

Now we will work out the coefficients of χ. It is easy to show that $h^* = \frac{1}{2} - is = 1 - (\frac{1}{2} + is) = 1 - h$. This means that(in the limit $s \to s'$) we have

$$A(h^*) \frac{\Gamma(h^*)^2}{\Gamma(2h^*)} = A(1-h) \frac{\Gamma(1-h)^2}{\Gamma(2-2h)} = B(h) \frac{\Gamma(1-h)^2}{\Gamma(2-2h)} \, . \tag{3.43}$$

Applying the same relation to the other coefficient, we see that they are the same as expected. In detail:

$$\begin{aligned} & A(h)B(h) \frac{\Gamma(h)^2}{\Gamma(2h)} \frac{\Gamma(1-h)^2}{\Gamma(2-2h)} = -\frac{\tan^2\pi h}{4} \frac{\Gamma(h)^2 \sqrt[\pi]{2}^{1-2h}}{\Gamma(h)\Gamma(\frac{1}{2}+h)} \frac{\Gamma(1-h)^2 \sqrt{\pi} 2^{-1+2h}}{\Gamma(1-h)\Gamma(\frac{3}{2}-h)} \\ & = -\frac{\tan^2\pi h}{4} \frac{\pi^2}{\sin\pi h} \frac{1}{\Gamma(\frac{1}{2}+h)\Gamma(\frac{3}{2}-h)} \\ & = -\frac{\tan^2\pi h}{4} \frac{\pi}{\sin\pi h} \frac{1}{(h-\frac{1}{2})\Gamma(\frac{3}{2}-h)\Gamma(\frac{1}{2}-h)} = -\frac{\pi\tan^2\pi \mathrm{h}}{4} \frac{\pi}{\sin\pi h} \frac{\sin\pi(h-1/2)}{\pi} \\ & = \frac{\pi\tan^2\pi \mathrm{h}}{4} \frac{\cos\pi h}{\sin\pi h} \frac{1}{(h-1/2)} = \frac{\pi\tan\pi h}{4h-2} \, . \end{aligned} \tag{3.44}$$

The inner product becomes:

$$\langle \Psi_h, \Psi_{h'} \rangle \sim \frac{\pi\tan\pi h}{4h-2} \int_0^{\epsilon} \frac{d\chi}{\chi} \left(\chi^{i(s-s')} + \chi^{-i(s-s')} \right) \, . \tag{3.45}$$

Making the change of variables $u = \log(\chi)$, we have:

$$\langle \Psi_h, \Psi_{h'} \rangle \sim \frac{\pi\tan\pi h}{4h-2} \int_{-\infty}^{\log(\epsilon)} du \left(e^{iu(s-s')} + e^{-iu(s-s')} \right) \, . \tag{3.46}$$

Taking the limit $\epsilon \to 1$, and changing in the second term the integration variable to $-u$ we end up with:

$$\langle \Psi_h, \Psi_{h'} \rangle \sim \frac{\pi\tan\pi h}{4h-2} \int_{-\infty}^{+\infty} du e^{iu(s-s')} \sim \frac{\pi\tan\pi h}{4h-2} 2\pi\delta(s-s') \, . \tag{3.47}$$

Maybe while taking the limit $\epsilon \to 1$ we would have anticipated more finite contributions because we cant set ${}_2F_1(a,b,c,\chi) \approx 1$ for large χ, but this is not the case since based on the requirement of orthogonality the only strong divergence comes from the point where $\chi = 0$.

Now we will compute the same inner product but for $h = 2n$. Using the definition of the Legendre Q function, we can write $\Psi_h(\chi) = 2Re[Q_{h-1}(y)]$, where $y = (2-\chi)/\chi$. Substituting this expression to the inner product we have:

$$\langle \Psi_h, \Psi_{h'} \rangle = 2\int_0^\infty dy Re[Q_h(y)] Re[Q_{h'}(y)] = \frac{\delta_{hh'}\pi^2}{4h-2} \,. \tag{3.48}$$

The last inner product we need to compute is $\langle \Psi_h, \mathcal{F}_0 \rangle$. First we are going to use the definition of $\mathcal{F}_0$ to express it as a function of cross ratios:

$$\begin{aligned}
\mathcal{F}_0(\chi) &= \frac{\mathcal{F}_0(\tau_1,\tau_2,\tau_3,\tau_4}{G(\tau_1,\tau_2)G(\tau_3,\tau_4)} = -\frac{G(\tau_1,\tau_3)G(\tau_2,\tau_4}{G(\tau_1,\tau_2)G(\tau_3,\tau_4)} + \frac{G(\tau_1,\tau_4)G(\tau_2,\tau_3)}{G(\tau_1,\tau_2)G(\tau_3,\tau_4)} \\
&= -\mathrm{sgn}\left(\frac{\tau_{12}\tau_{34}}{\tau_{13}\tau_{24}}\right)\sqrt{\left|\frac{\tau_{12}\tau_{34}}{\tau_{13}\tau_{24}}\right|} \\
&+ \mathrm{sgn}\left(\frac{\tau_{12}\tau_{34}}{\tau_{14}\tau_{23}}\right)\sqrt{\left|\frac{\tau_{12}\tau_{34}}{\tau_{14}\tau_{23}}\right|} \\
&= -\mathrm{sgn}(\chi)\sqrt{|\chi|} + \mathrm{sgn}\left(\frac{\chi}{1-\chi}\right)\sqrt{\left|\frac{\chi}{1-\chi}\right|} .
\end{aligned} \tag{3.49}$$

The overall sign of this expression depends on the value of χ. We have (for general q, we replace the square root by 2Δ):

$$\mathcal{F}_0(\chi) = \begin{cases} -\chi^{2\Delta} + \left(\frac{\chi}{1-\chi}\right)^{2\Delta} & \text{if } 0 < \chi < 1 \\ -\chi^{2\Delta} - \left(\frac{\chi}{1-\chi}\right)^{2\Delta} & \text{if } \chi > 1 \end{cases}$$

Now to calculate the inner product $\langle \Psi_h, \mathcal{F}_\prime \rangle$ we use the following symmetry $\mathcal{F}(\chi) = \mathcal{F}(\frac{\chi}{\chi-1})$ for all χ. This symmetry comes from the integral representation:

$$\Psi_h(\chi) = \frac{1}{2}\int_{-\infty}^{\infty} dy \frac{|\chi|^h}{|y|^h|\chi-y|^h|1-y|^{1-h}} \,. \tag{3.50}$$

Moreover we can see from the expression of $\mathcal{F}_0(\chi)$, that it is antisymmetric under $\chi \to \frac{\chi}{1-\chi}$ for $0 < \chi < 1$ and symmetric under $\chi \to \frac{\chi}{\chi-1}$ for $\chi > 1$. The inner product becomes:

$$\begin{aligned}
\langle \Psi_h, \mathcal{F}_0 \rangle &= \int_0^2 \frac{d\chi}{\chi^2}\Psi_h^*(\chi)\mathcal{F}_0(\chi) = \int_0^2 \frac{d\chi}{\chi^2}\Psi_h^*(\chi)\mathrm{sgn}(\chi)|\chi|^{2\Delta} \\
&+ \int_0^1 \frac{d\chi}{\chi^2}\Psi_h^*(\chi)\mathrm{sgn}\left(\frac{\chi}{1-\chi}\right)\left|\frac{\chi}{1-\chi}\right|^{2\Delta} + \int_1^2 \frac{d\chi}{\chi^2}\Psi_h^*(\chi)\mathrm{sgn}\left(\frac{\chi}{\chi-1}\right)\left|\frac{\chi}{\chi-1}\right|^{2\Delta}
\end{aligned}$$

In the second term we can change variables $\chi \to \frac{\chi}{1-\chi}$ and use the relevant symmetries and we get:

$$\int_0^1 \frac{d\chi}{\chi^2}\Psi_h^*(\chi)\mathrm{sgn}\left(\frac{\chi}{1-\chi}\right)\left|\frac{\chi}{1-\chi}\right|^{2\Delta} = \int_{-\infty}^0 \frac{d\chi}{\chi^2}\Psi_h^*(\chi)\mathrm{sgn}(\chi)|\chi|^{2\Delta}\,. \quad (3.51)$$

The third term, using the appropriate symmetries and changing variables $\chi \to \frac{\chi}{\chi-1}$ becomes:

$$\int_1^2 \frac{d\chi}{\chi^2}\Psi_h^*(\chi)\mathrm{sgn}\left(\frac{\chi}{\chi-1}\right)\left|\frac{\chi}{\chi-1}\right|^{2\Delta} = \int_2^\infty \frac{d\chi}{\chi^2}\Psi_h^*(\chi)\mathrm{sgn}(\chi)|\chi|^{2\Delta}\,. \quad (3.52)$$

Thus, we get:

$$\langle\Psi_h, \mathcal{F}_0\rangle = \int_{-\infty}^\infty \frac{d\chi}{\chi^2}\Psi_h^*(\chi)\mathrm{sgn}(\chi)|\chi|^{2\Delta} = \frac{1}{2}\int_{-\infty}^\infty d\chi dy \frac{\mathrm{sgn}(\chi)}{|\chi|^{2-h-2\Delta}|y|^h|\chi-y|^h|1-y|^{1-h}} \quad (3.53)$$

This integral is similar to (3.32). We can make the integration variable change $y = \frac{1}{\tau}, \chi = \frac{1}{\tau\tau'}$. We then have:

$$\int d\tau \frac{\mathrm{sgn}(\tau)}{|\tau|^{2\Delta}|1-\tau|^{1-h}} \cdot \int d\tau' \frac{\mathrm{sgn}(\tau')}{|\tau'|^{2\Delta}|1-\tau'|^{1-h}}\,. \quad (3.54)$$

Now, we change the integration variables as $\tau = \tau_1 - 1, \tau' = \frac{\tau_1\tau_2}{\tau_2-1}$ in (3.32)/a_0 and get the above expression. In the end our inner product becomes:

$$\langle\Psi_h, \mathcal{F}_0\rangle = \frac{a_0}{2}\mathrm{k_c}(h)\,. \quad (3.55)$$

Another way of solving the previous integral is to divide the regions of integration and use the Euler beta function.

3.2.5 Summing all ladder diagrams

Now that we have computed all relevant quantities we can use (3.7) to give an expression for the four-point function. The desired function will be an integral over continuous values of h and a sum over their discrete values. Thus, we have:

$$\begin{aligned}\mathcal{F}(\chi) &= \sum_h \Psi_h(\chi)\frac{1}{1-k_c(h)}\frac{\langle\Psi_h(\chi), \mathcal{F}_0\rangle}{\langle\Psi_h(\chi), \Psi_h(\chi)\rangle} \\ &= a_0\int_0^\infty \frac{ds}{2\pi}\frac{2h-1}{\pi \mathrm{tan}\pi \mathrm{h}}\frac{k_c(h)}{1-k_c(h)}\Psi_h(\chi) + a_0\sum_{n=1}^\infty \frac{2h-1}{\pi^2}\frac{k_c(h)}{1-k_c(h)}\Psi_h(\chi)|_{h=2n}\,.\end{aligned} \quad (3.56)$$

As we have seen before (3.7) for $k_c(2) = 1$ the relevant term ($n = 1$) in the sum diverges. This causes a serious problem as the four-point function should be finite. To avoid this problem we have to treat the divergence outside the conformal limit. Later, we will briefly discuss this contribution. In the following we will focus in the $h \neq 2$ eigenfunctions. Thus, we have the expression:

$$\frac{\mathcal{F}_{h\neq 2}}{a_0} = \int_0^\infty \frac{ds}{2\pi} \frac{2h-1}{\pi \tan \pi \mathrm{h}} \frac{k_c(h)}{1-k_c(h)} \Psi_h(\chi) + \sum_{n=2}^{\infty} \frac{2h-1}{\pi^2} \frac{k_c(h)}{1-k_c(h)} \Psi_h(\chi)|_{h=2n} . \tag{3.57}$$

To simplify the expression we use the identity:

$$\frac{2}{\tan \pi h} = \frac{1}{\tan \frac{\pi h}{2}} - \frac{1}{\tan \frac{\pi(1-h)}{2}} . \tag{3.58}$$

Thus, the integral becomes:

$$\int_0^\infty \frac{ds}{2\pi} \frac{h-1/2}{\pi \tan \frac{\pi h}{2}} \frac{k_c(h)}{1-k_c(h)} \Psi_h(\chi) - \int_0^\infty \frac{ds}{2\pi} \frac{h-1/2}{\tan \frac{\pi(1-h)}{2}} \frac{k_c(h)}{1-k_c(h)} \Psi_h(\chi) . \tag{3.59}$$

We focus now on the second integral. First, we change the integration measure from $s \to -s$ so we extend the region to all values of s. In the second step we use the symmetry of the eigenvalues and eigenfunction under $h \to 1-h$. In detail:

$$\begin{aligned} -\int_0^\infty \frac{ds}{2\pi} \frac{h-1/2}{\tan \frac{\pi(1-h)}{2}} \frac{k_c(h)}{1-k_c(h)} \Psi_h(\chi) &\overset{s\to -s}{=} -\int_{-\infty}^0 \frac{ds}{2\pi} \frac{h-1/2}{\tan \frac{\pi(1-h)}{2}} \frac{k_c(h)}{1-k_c(h)} \Psi_h(\chi) \\ \overset{1-h\to h}{=} + \int_{-\infty}^0 \frac{ds}{2\pi} \frac{h-1/2}{\tan \frac{\pi(h)}{2}} \frac{k_c(1-h)}{1-k_c(1-h)} \Psi_{1-h}(\chi) &= + \int_{-\infty}^0 \frac{ds}{2\pi} \frac{h-1/2}{\tan \frac{\pi(h)}{2}} \frac{k_c(h)}{1-k_c(h)} \Psi_h(\chi) . \end{aligned} \tag{3.60}$$

In the end, the integral we have is:

$$\int_{-\infty}^{+\infty} \frac{ds}{2\pi} \frac{h-1/2}{\tan \frac{\pi(h)}{2}} \frac{k_c(h)}{1-k_c(h)} \Psi_h(\chi) . \tag{3.61}$$

Now we can manipulate the sum over $h = 2n$ using the residue theorem and we can write is as a sum over the residues of the poles of $1/\tan \pi h$. We compute:

$$\begin{aligned} \mathrm{Res}(\frac{1}{\tan \pi h}, h = 2n) &= \lim_{h\to h_0 = 2n} \left((h-h_0) \frac{1}{\tan \pi h} \right) \\ &= lim_{h\to h_0=2n} \frac{\cos(\pi h/2) - (h-h_0) \cancelto{0}{\sin(\pi h/2)}}{\frac{\pi}{2} \cos(\pi h/2} = \frac{2}{\pi} . \end{aligned} \tag{3.62}$$

Of course

$$\sum_{n=2} \frac{2h-1}{\pi^2}|_{h=2n} = \sum_{n=2} \frac{2}{\pi} \frac{(h-1/2)}{\pi}|_{h=2n} = \sum_{n=2} Res \left(\frac{(h-1/2)}{\pi \tan \pi h} \right)$$

The expression for the

$$\frac{\mathcal{F}_{h\neq 2}}{a_0} = \int_{-\infty}^{+\infty} \frac{ds}{2\pi} \frac{h-1/2}{\tan(\frac{\pi h}{2})} \frac{k_c(h)}{1-k_c(h)} \Psi_h(\chi) + \sum_{n=2} Res \left(\frac{(h-1/2)}{\pi \tan(\frac{\pi h}{2})} \frac{k_c(h)}{1-k_c(h)} \Psi_h(\chi) \right)_{h=2n} . \tag{3.63}$$

This formula can be thought as a single contour integral over the complex plane h:

$$\frac{1}{2\pi i} \int_{\mathcal{C}} dh = \int_{-\infty}^{+\infty} \frac{ds}{2\pi} + \sum_{n=2}^{\infty} \text{Res}_{h=2n} . \tag{3.64}$$

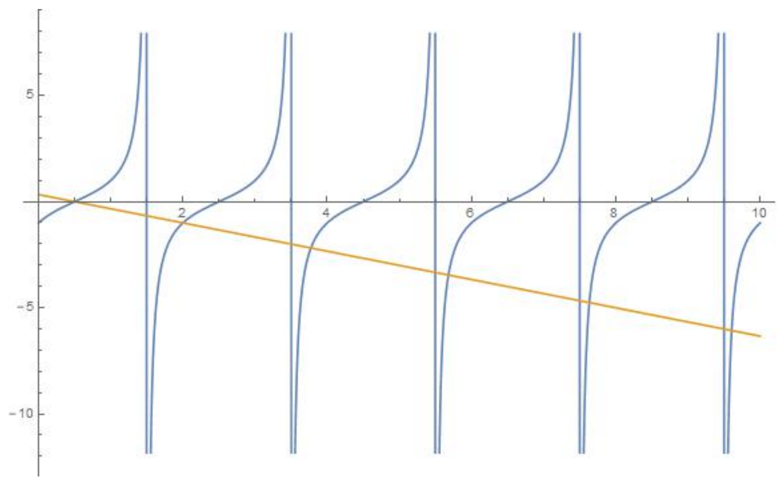
Examining the expression for Ψ_h (in both regions) we see that it has poles on $h = 1 + 2n$. But these poles are canceled by the zeros that come from $1/\tan(\frac{\pi h}{2})$ at these particular values. Therefore we are left only with the poles at $h = 2n$. We can now deform the contour to annihilate the s axis (vertical line) together with the explicit residues around the poles $h = 2n$. This comes with the cost of picking up poles from the equation:

$$\frac{k_c(h)}{1-k_c(h)} . \tag{3.65}$$

This equation can be solved graphically. For example, for q=4 we have:

$$k_c(h) = -\frac{3}{2} \frac{\tan \frac{\pi(h-1/2)}{2}}{(h-1/2)} = 1 . \tag{3.66}$$

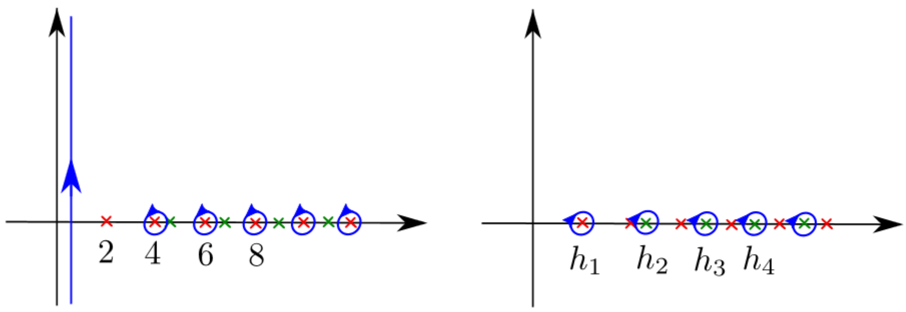
The solution are demonstrated graphically in the next figure.



We can see that the solutions h_m are left to 2n. It has been shown that for q=4 and m», they behave like

$$h_m = 2\Delta + 1 + 2m + \frac{3}{2\pi m}\,. \tag{3.67}$$

Before getting into more details about how we deform the contour we demonstrate the whole procedure below.



The way this deformation it is done is by analysing $\mathcal{F}$ in two separate regions.

- In the region $\chi > 1$, we can push the contour rightward to infinity by picking the poles of $k_c(h_m) = 1$. The four-point function becomes:

$$\mathcal{F}_{h\neq 2} = -a_0 \sum_{h_m} \text{Res}\left(\frac{(h-1/2)}{\pi \tan(\frac{\pi h}{2})}\frac{k_c(h)}{1-k_c(h)}\Psi_h(\chi)\right)_{h_m}, \quad x > 1, \tag{3.68}$$

  where the minus sign comes from the fact that we have close the pole clockwise.

- In the region, $0 < \chi < 1$, we cannot push the contour to large positive h. The reason that ${}_2F_1(1-h, 1-h, 2-2h, \chi)$ must have positive arguments. Thus, we must use the symmetries of the integrand under $h \to (1-h)$. In the end we get another copy of the A term of (3.28) and replacing in the sum $\frac{\Gamma(h)^2}{\Gamma(2h)}\chi^h_2 F_1(h, h, 2h, \chi)$ we end up:

$$\begin{aligned}\frac{\mathcal{F}_{h\neq 2}}{a_0} &= \int_{-\infty}^{+\infty} \frac{ds}{2\pi}\frac{h-1/2}{\tan(\frac{\pi h}{2})}\frac{k_c(h)}{1-k_c(h)}\frac{\Gamma(h)^2}{\Gamma(2h)}\chi^h{}_2F_1(h, h, 2h, \chi)\\ &+ \sum_{n=2} \text{Res}\left(\frac{(h-1/2)}{\pi \tan(\frac{\pi h}{2})}\frac{k_c(h)}{1-k_c(h)}\frac{\Gamma(h)^2}{\Gamma(2h)}\chi^h{}_2F_1(h, h, 2h, \chi)\right)_{h=2n}.\end{aligned} \tag{3.69}$$

Now we can push the vertical axis to the right and pick up poles at h_m, just like we did in the first region. The four-point function becomes:

$$\mathcal{F}_{h\neq 2} = N \sum_{m=1} c_m^2 [\chi_2^h F_1(h,h,2h,\chi)], \quad \chi < 1 . \tag{3.70}$$

where

$$c_m^2 = -\frac{a_0}{N} \frac{(h_m - 1/2)}{\pi \tan(\frac{\pi h}{2})} \frac{1}{-k_c'(h_m)} \frac{\Gamma(h_m)^2}{\Gamma(2h_m)} . \tag{3.71}$$

The N term will prove useful when we want to relate $\mathcal{F}$ with the four-point function given in (3.2). The expression (3.70) is the expected OPE expansion with the the quantity in brackets being the conformal blocks.

3.2.6 OPE expansion and operators of the model

Now that we have derived the four-point function, we will study the OPE of short time limit. The main expression that we will use is (3.70). Before that we will state the OPE of two Majorana fermions:

$$\frac{1}{N} \sum_i x_i(\tau_1) x_i(\tau_2) = \frac{1}{\sqrt{N}} \sum_n c_n \mathcal{C}_n(\tau_{12}, \partial_{\tau_2}) \mathcal{O}_n , \tag{3.72}$$

where c_n are the OPE coefficients (3.71), $\mathcal{O}_n$are O(N) invariant bilinear primary operators of the form $\mathcal{O}_n \simeq \sum_n x_i \partial_\tau^{2n+1} x_i$ with dimensions h_n and $\mathcal{C}_n(\tau_{12}, \partial_{\tau_2})$ is a function that is fully determined by conformal invariance and includes the contribution of the descendants of $\mathcal{O}_n$. In correspondence with (A.119) we have

$$\mathcal{C}_n((\tau_{12}), \partial_{\tau_2}) = G(\tau_{12}) |\tau_{12}|^{h_n} \left(1 + \frac{1}{2}\tau_{12}\partial_{\tau_2} + ...\right) . \tag{3.73}$$

Now we will focus on the four-point function. When taking the short time limit $|\tau_{12}| << 1$ and consequently $\chi \to 0$, after replacing the hypergeometric function of (3.70) by one, we have:

$$\mathcal{F}_{h\neq 2}(\tau_1, \tau_2, \tau_3, \tau_4) = G(\tau_{12}) G(\tau_{34}) \sum_{m=1} c_n^2 \left| \frac{\tau_{12}\tau_{34}}{\tau_{23}\tau_{24}} \right|^{h_n} \quad |\tau_{12}| << 1 . \tag{3.74}$$

In addition, if $|\tau_{34}| << 1$ we have(after replacing τ_3 with τ_4 in τ_{23}):

$$\mathcal{F}_{h\neq 2}(\tau_1, \tau_2, \tau_3, \tau_4) = G(\tau_{12}) G(\tau_{34}) \sum_{m=1} c_n^2 \frac{|\tau_{12}|^{h_n} |\tau_{34}|^{h_n}}{|\tau_{24}|^{h_n}} \quad |\tau_{12}| << 1, |\tau_{34}| << 1 . \tag{3.75}$$

Using (3.72) twice for the four-point function in the small time limit we get:

$$\mathcal{F}_c = \sum_{n,m} c_n c_m \mathcal{C}_n(\tau_{12}, \partial_{\tau_2}) \mathcal{C}_n(\tau_{34}, \partial_{\tau_4}) \langle \mathcal{O}_n(\tau_2) \mathcal{O}_m(\tau_4) \rangle \, . \tag{3.76}$$

Comparing this expression with (3.75), we get the two-point function of the bilinears:

$$\langle \mathcal{O}_n(\tau_2) \mathcal{O}_m(\tau_4) \rangle = \frac{\delta_{n,m}}{|\tau_{24}|^{2h_n}} \, . \tag{3.77}$$

If we write and the sum the complete series of $\mathcal{C}_n(\tau_{12}, \partial_{\tau_2}), \mathcal{C}_n(\tau_{34}, \partial_{\tau_4})$, we will reproduce the expression (3.70) of the four-point function.

Now, although studying the bulk dual of the SYK model in not included in the purposes of this thesis, we will make a short comment using the AdS/CFT dictionary. As we have found from the four-point function, there is a tower of $\mathcal{O}(N)$ invariant, bilinear primary operators with dimensions h_n. In the bulk dual of the theory this translates to massive field ϕ_n with mass $m_n^2 = h_n(h_n - 1)$. These fields are described by:

$$\int d^2x \sqrt{g} \sum_n \left(\frac{1}{2} (\partial \phi_n)^2 + \frac{1}{2} m_n^2 \phi_n^2 \right) \, . \tag{3.78}$$

3.2.7 The $h = 2$ contribution

Up until now we have avoided the contribution from the $h = 2$ eigenfunction of the Casimir operator as it causes divergences. To include them in the four-point function and get a finite answer we have to treat these eigenfunctions outside the conformal limit by applying perturbation theory in the non-conformal correction to the kernel. This correction is a product of the non-conformal correction δG to the correlators. The small parameter is the inverse coupling, $1/\beta J$. This analysis is too technical that it exceeds the scope of this thesis and here we will only give a review of the calculations. Full technical details can be found in [2].

The starting point of the analysis is that since we consider non-conformal corrections to the kernel, the perturbation δK will break the conformal symmetry. As a consequence, the line and finite temperature can't be treated equivalently and the study has to be done in the circle. To do this we use angular coordinates $\theta = 2\pi\tau/\beta$ with $0 \leq \theta < 2\pi$. Now the kernel is given by doing he transformation $\tau_i = \tan\left(\frac{\theta_i}{2}\right)$. Next, we find the SL(2,R) generators that commute with the kernel on the thermal circle. They are

$$P = e^{-i\theta} \left[\partial_\theta - i/2 \right] , \quad K = -e^{-i\theta} \left[\partial_\theta + i/2 \right] , \quad D = i\partial_\theta \, . \tag{3.79}$$

Then we can find the eigenfunctions of the corresponding Casimir with $h =$

2. These turn to be:

$$\Psi_{2,n} = \frac{3}{2\pi^2|n|(n^2-1)} \frac{e^{-in\frac{\theta_1+\theta_2}{2}}}{\sin\left(\frac{\theta_1-\theta_2}{2}\right)} \left(\frac{\sin\left(2\frac{\theta_1-\theta_2}{2}\right)}{\tan\left(2\frac{\theta_1-\theta_2}{2}\right)} - n\cos\left(2\frac{\theta_1-\theta_2}{2}\right) \right) . \tag{3.80}$$

With the use of these eigenfunctions and the kernel perturbation δK, we can find the correction to the eigenvalue $k(2,n)$. It turns to be:

$$k(2,n) = 1 + k_c'(2)\frac{q\alpha|n|}{\beta J} + \mathcal{O}(1/(\beta J)^2)\,, \tag{3.81}$$

where $\alpha \approx 0.1872$. Adding all these results, we arrive at the leading contribution to the four-point function:

$$\frac{\delta\mathcal{F}(\theta_1,..\theta_4}{G(\theta_{12})G(\theta_{34})} = \frac{6\alpha_0}{\pi^2\alpha_K}\beta J \sum_{|n|\geq 2} \frac{e^{in(y'-y)}}{n^2(n^2-1)} \left[\frac{\sin\frac{nx}{2}}{\tan\frac{x}{2}} - n\cos\frac{nx}{2}\right] \left[\frac{\sin\frac{nx'}{2}}{\tan\frac{x'}{2}} - n\cos\frac{nx'}{2}\right] , \tag{3.82}$$

with

$$x = \theta_{12}, \quad x' = \theta_{34}, \quad y = \frac{\theta_1+\theta_2}{2}, \quad y' = \frac{\theta_3+\theta_4}{2},$$

and $\alpha_K = -qk_c'(2)\alpha_G$. This contribution is very large compared to the $h \neq 2$ contributions that we have found. The reason is the term βJ. Moreover, this contribution is not conformal invariant as it is not a function of the cross ratio. Finally, to be more consistent we have to also include the corrections δG and the corrections of the term $1/(1-k(2,n))$. To include all these correction we have to find the second order corrections of the eigenvalues. This is a very difficult task and up until now only conjectures have been made [2].

Chapter 4

Higher-point correlation functions

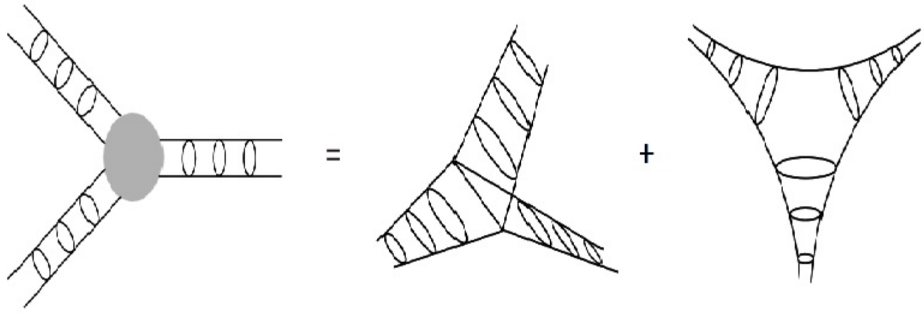
Up until now, the two-point and four-point functions have been calculated. In this chapter we will review the process for calculating the six-point, the eight-point and eventually all higher-point correlation functions of the SYK model. This chapter is heavily based on [13].

4.1 Bilinear three-point function

The six-point function of the model can be equivalently viewed as a three-point function of the bilinear O(N) invariant fermion primaries $\mathcal{O}_{h_i}$, where h_i denotes the dimension of the operators. The six-point function of the Majorana fermions is:

$$\frac{1}{N^3}\sum_{i,j,l=1}^{N}\langle x_i(\tau_1)x_i(\tau_2)x_j(\tau_3)x_j(\tau_4)x_l(\tau_5)x_l(\tau_6) = ... + \frac{1}{N^2}\mathcal{S}(\tau_1,..,\tau_6) + ... \,, \tag{4.1}$$

with $\mathcal{S}$ the lowest order in $1/N$ with fully connected diagrams. As shown below, there are 2 classes of diagrams that contribute: the contact diagrams (left) and the planar diagrams (right).



Denoting the contribution from contact as $\mathcal{S}_1$ and the contribution from planar as $\mathcal{S}_2$, we can write: $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$. Using the OPE formalism, the three-point function of the bilinears is:

$$\langle \mathcal{O}_1(\tau_1)\mathcal{O}_2(\tau_2)\mathcal{O}_3(\tau_3)\rangle = \frac{1}{\sqrt{N}} \frac{c_{123}}{|\tau_{12}|^{h_1+h_2-h_3}|\tau_{13}|^{h_1+h_3-h_2}|\tau_{24}|^{h_4+h_2-h_1}}\,, \tag{4.2}$$

with $c_{123} = c_{123}^{(1)} + c_{123}^{(2)}$ as it has contributions from contact and planar diagrams respectively.

4.1.1 Contact diagrams

As shown in the above figure, the contact diagrams are composed of three four-point functions glued with two interaction vertices connected by $q-3$ propagators. Their expression is:

$$\mathcal{S}_1 = (q-1)(q-2)J^2 \int d\tau_a d\tau_b G(\tau_{ab})^{q-3} \mathcal{F}(\tau_1,\tau_2,\tau_a,\tau_b)\mathcal{F}(\tau_3,\tau_4,\tau_a,\tau_b)\mathcal{F}(\tau_5,\tau_6,\tau_a,\tau_b).$$

It would be more useful to write the conformal blocks of the four-point function in terms of the operator $\mathcal{C}_n(\tau_{12},\partial_2)$ as in (3.73). Thus, we write:

$$\mathcal{F}(\tau_1,\tau_2,\tau_a,\tau_b) = \sum_n c_n \mathcal{C}_n(\tau_{12},\partial_2)\langle \mathcal{O}_n(\tau_2)x(\tau_a)x(\tau_b)\rangle\,, \tag{4.3}$$

where the three-point function is given by (3.30). Using this formula the contribution from the contact diagrams becomes:

$$\sum_{n_1,n_2,n_3}\prod_{i=1}^{3} c_{n_i}\mathcal{C}_{n_i}(\tau_{2i-1,2i},\partial_{2i})\langle \mathcal{O}_{n_1}(\tau_2)\mathcal{O}_{n_2}(\tau_4)\mathcal{O}_{n_3}(\tau_6)\rangle\,, \tag{4.4}$$

where

$$\langle \mathcal{O}_{n_1}(\tau_1)\mathcal{O}_{n_2}(\tau_2)\mathcal{O}_{n_3}(\tau_3)\rangle = (q-1)(q-2)J^2 \int d\tau_a d\tau_b G(\tau_{ab})^{q-3}\prod_{i=1}^{3}\langle \mathcal{O}_{n_i}(\tau_i)x(\tau_a)x(\tau_b)\rangle.$$

Now, if we replace the three-point functions of the bilinears given by (4.2) we get:

$$\langle \mathcal{O}_{n_1}(\tau_1)\mathcal{O}_{n_2}(\tau_2)\mathcal{O}_{n_3}(\tau_3)\rangle = c_{n_1}c_{n_2}c_{n_3}(q-1)(q-2)b^q I_{123}^{(1)}(\tau_1,\tau_2,\tau_3)\,, \tag{4.5}$$

with

$$I_{123}^{(1)}(\tau_1,\tau_2,\tau_3) = \int d\tau_a d\tau_b \frac{|\tau_{ab}|^{h_1+h_2+h_3-2}}{|\tau_{1a}|^{h_1}|\tau_{1b}|^{h_1}|\tau_{2a}|^{h_2}|\tau_{2b}|^{h_2}|\tau_{3a}|^{h_3}|\tau_{4a}|^{h_4}}\,. \tag{4.6}$$

We will now present the final result avoiding technical details as they are out of scope of this thesis. In the end, we end up with

$$c_{123}^{(1)} = c_1 c_2 c_3 \mathcal{I}_{123}^{(1)} \,, \tag{4.7}$$

where $\mathcal{I}_{123}^{(1)}$ is the coefficient of $I_{123}^{(1)}(\tau_1, \tau_2, \tau_3)$ written as a conformal three-point function

$$I_{123}^{(1)}(\tau_1, \tau_2, \tau_3) = \frac{\mathcal{I}_{123}^{(1)}}{|\tau_{12}|^{h_1+h_2-h_3}|\tau_{13}|^{h_1+h_3-h_2}|\tau_{24}|^{h_4+h_2-h_1}} .$$

The expression that is found for $\mathcal{I}_{123}^{(1)}$ is:

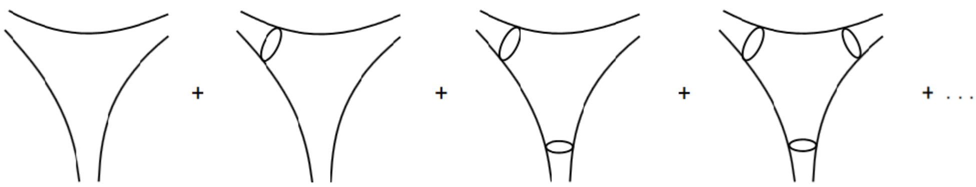
$$\begin{aligned}\mathcal{I}_{123}^{(1)} = & \frac{\sqrt{\pi}2^{h_1+h_2+h_3-1}\Gamma(1-h_1)\Gamma(1-h_2)\Gamma(1-h_3)}{\Gamma(\frac{2-h_1-h_2+h_3}{2})\Gamma(\frac{2-h_1-h_3+h_2}{2})}[\rho(h_1, h_2, h_3) \\ & + \rho(h_2, h_3, h_1) + \rho(h_3, h_1, h_2)] \,,\end{aligned} \tag{4.8}$$

where

$$\rho(h_1, h_2, h_3) = \frac{\Gamma(\frac{h_2+h_3-h_1}{2})}{\Gamma(\frac{2-h_2+h_1+h_3}{2})\Gamma(\frac{2-h_1-h_3+h_2}{2})}\left(1 + \frac{\sin(\pi h_2)}{\sin(\pi h_3) - \sin(\pi h_1 + \pi h_2)}\right) .$$

4.1.2 Planar diagrams

We will now present the contribution of the contact diagrams. As before they contain three four-point functions glued together in a smooth way. Based on



we can write the expression:

$$\begin{aligned}\mathcal{S}_2 = \int d\tau_a d\tau_{\bar{a}} d\tau_b d\tau_{\bar{b}} d\tau_c d\tau_{\bar{c}} \mathcal{F}(\tau_1, \tau_2, \tau_a, \tau_{\bar{b}}) D(\tau_{b\bar{b}}) \mathcal{F}(\tau_3, \tau_4, \tau_c, \tau_{\bar{a}} D(\tau_{\bar{a}a}) \\ \times \mathcal{F}(\tau_5, \tau_6, \tau_b, \tau_{\bar{c}}) D(\tau_{\bar{c}c}) \,,\end{aligned} \tag{4.9}$$

where $D(\tau_{\bar{a}a})$ is the inverse propagator that satisfies:

$$\int d\tau_0 D(\tau_{10}) G(\tau_{02}) = \delta(\tau_{12}) \,. \tag{4.10}$$

It is inserted to avoid the over-counting of the external propagators. In the IR limit $D(\tau) = -\Sigma(\tau) = -J^2 G(\tau)^{q-1}$. This expression comes from the Schwinger-Dyson equations (2.29). Now, following the same procedure as before and substituting the fermion four-point function by (4.3), we arrive at:

$$\langle \mathcal{O}_1(\tau_1)\mathcal{O}_2(\tau_2)\mathcal{O}_3(\tau_3)\rangle = \int d\tau_a d\tau_{\bar{a}} d\tau_b d\tau_{\bar{b}} d\tau_c d\tau_{\bar{c}} \langle \mathcal{O}_1(\tau_1)x(\tau_a)x(\tau_{\bar{b}})\rangle D(\tau_{b\bar{b}}) \\ \cdot \langle \mathcal{O}_2(\tau_2)x(\tau_c)x(\tau_{\bar{a}})\rangle D(\tau_{\bar{a}a})\langle \mathcal{O}_3(\tau_3)x(\tau_b)x(\tau_{\bar{c}})\rangle D(\tau_{\bar{c}c})\,. \tag{4.11}$$

Skipping the technical details as we did in the contact diagrams we arrive at the expression:

$$\langle \mathcal{O}_1(\tau_1)\mathcal{O}_2(\tau_2)\mathcal{O}_3(\tau_3)\rangle = c_1 c_2 c_3 \xi(h_1)\xi(h_2)\xi(h_3) I^{(2)}_{123}(\tau_1,\tau_2,\tau_3)\,, \tag{4.12}$$

where

$$\xi(h) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{2\Delta+1}{2})}{\Gamma(1-\Delta)} \frac{\Gamma(\frac{1-h}{2})}{\Gamma(\frac{h}{2})} \frac{\Gamma(\frac{2-2\Delta+h}{2})}{\Gamma(\frac{1+2\Delta-h}{2})}\,. \tag{4.13}$$

It is finally found that:

$$c^{(2)}_{123} = c_1 c_2 c_3 \xi(h_1)\xi(h_2)\xi(h_3)\mathcal{I}^{(2)}_{123}\,, \tag{4.14}$$

where $\mathcal{I}^{(2)}_{123}$ is once again the coefficient of $I^{(2)}_{123}(\tau_1,\tau_2,\tau_3)$ written as a conformal three-point function. It is a sum of four generalized hypergeometric function and its explicit expression can be found in [13].

In summary, the coefficient of the three-point function is a sum over the coefficients of the contact and the planar diagrams. We can write it as $c_{123} = c^{(1)}_{123} + c^{(2)}_{123}$ and in more convenient form as

$$c_{123} = c_1 c_2 c_3 \mathcal{I}_{123}\,. \tag{4.15}$$

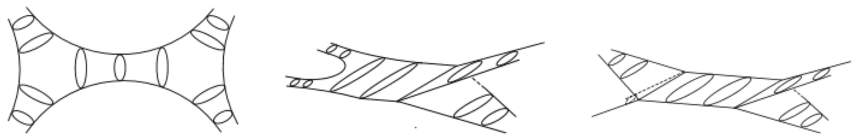
We can read the following contributions. The coefficients $c_1c_2c_3$ are the ones that appear when by the OPE formalism two fermions x_i are turned into $\mathcal{O}_i$ as it can been seen from (3.72). It reflects the sum of the ladder diagrams. This particular sum is the one that determines the dimensions h_i of $\mathcal{O}_i$. The second contribution, encoded in $\mathcal{I}_{123}$ is the effect of gluing together the ladder diagrams. One can say the coefficients c_{123} are universal since are determined by an integral with the only parameters being the dimension of the fermions Δ and the dimension h_i of $\mathcal{O}_i$.

4.2 Bilinear four-point function

4.2.1 Cutting melons

It is noted in [4], that in any large N theory the Feynman diagrams that will contribute to a 2p-point function are found by drawing all the diagrams that contribute to the vacuum energy and then considering all cuts of the propagators.

That means that a cut gives a diagrams that contributes to the two-point function (for example the watermelon diagram 2.1). Then a cut of the melon gives the ladder diagram that contributes to the four-point function. Now there are two ways to cut a propagator. One may either cut the propagator along a rail or cut a melon that is along a rung. The first option will give the six-point planar diagram while the second option will give a contact six-point diagram. To find the diagrams of the eight-point function, there are four possible cuts. Regarding the planar diagram, a cut of a melon along a rail gives a planar diagram contribution to the eight-point function while a cut of a melon along a rung will give a mixed planar/contact diagram (second diagram in the figure below). Considering the six-point contact diagram, a cut along the rail will also give a mixed planar/contact diagram while a cut along the rung will give a contact/contact contribution to the eight-point function (third diagram in the figure above).



The same procedure can be used to find the diagrams contributing to higher-point functions.

4.2.2 Summing the eight-point diagrams

Now that we have outlined the diagrams that will contribute to the eight-point function, we will list the necessary diagrams that have to be added. First, we will denote as $\mathcal{E}_s(\tau_1, .., \tau_8)$ the following diagram and $\langle \mathcal{O}_1(\tau_1)...\mathcal{O}_4(\tau_4)\rangle_s^0$ its contribution to the bilinear four-point function

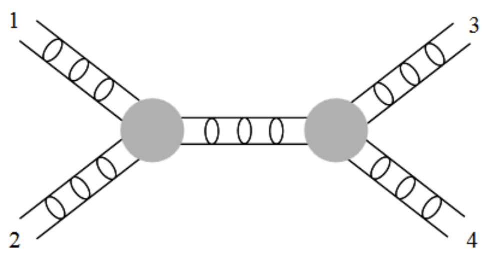
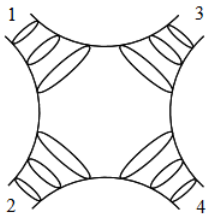


Figure 4.1: Eight-point function

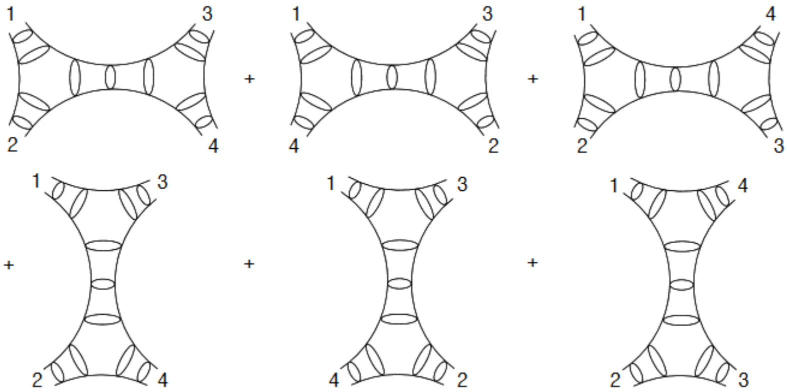
Moreover, we denote $\mathcal{E}_S^0$ and $\langle \mathcal{O}_1(\tau_1)...\mathcal{O}_4(\tau_4)\rangle^0$ the planar diagrams with no exchanged melons such as



Thus, the four-point bilinear function is given by:

$$\begin{aligned}\langle \mathcal{O}_1(\tau_1)...\mathcal{O}_4(\tau_4)\rangle = (\langle \mathcal{O}_1(\tau_1)...\mathcal{O}_4(\tau_4)\rangle_s + (2 \leftrightarrow 3) + (3 \leftrightarrow 4)) \\ - \frac{1}{2}\big(\langle \mathcal{O}_1(\tau_1)...\mathcal{O}_4(\tau_4)\rangle_s^0 + (2 \leftrightarrow 3) + (3 \leftrightarrow 4)\big).\end{aligned} \tag{4.16}$$

To justify this equation we demonstrate some of the planar diagrams that contribute to the eight-point functions. These are



The first line of the above figure are three different channels. The second line shows the same diagrams although now the exchanged melons are in the other direction. If there are no exchanged melons then the two lines demonstrate the same diagrams and a factor 1/2 must be included to avoid double counting. That justifies the second line of (4.16). The first line of (4.16) accounts for the sum of all the 6 diagrams. Finally, just as we did for the four-point function where it is antisymmetric under interchange of the first two or last fermions, $\mathcal{E}_s(\tau_1, .., \tau_8)$ correspond to the sum of the first and last diagrams of the first line of the above figure.

4.2.3 Computing $\langle \mathcal{O}_1(\tau_1)...\mathcal{O}_4(\tau_4)\rangle_s$

We will present the computation for the eight-point function as it is found in [13]. It follows the same procedure done for the computation of the six-point function. The eight-point function can be written:

$$\frac{1}{N^4}\sum_{i_1,i_2,..,i_4}\langle x_{i_1}(\tau_1)x_{i_1}(\tau_2)x_{i_2}(\tau_3)x_{i_2}(\tau_4)x_{i_3}(\tau_5)x_{i_3}(\tau_6)x_{i_4}(\tau_7)x_{i_4}(\tau_8)\rangle = ...$$
$$+\frac{1}{N^3}\mathcal{E}(\tau_1,..,\tau_8)+...\,, \tag{4.17}$$

with $\mathcal{E}(\tau_1,..,\tau_8)$ the lowest order term that contains connected diagrams.

We will focus on the contribution $\mathcal{E}_s(\tau_1,..,\tau_8)$. From the corresponding figure 4.1, we see that it consists of two six-point functions glued together. The details of the interactions can be encoded in the piece $\mathcal{S}_{core}$ (shaded circle), which is attached to the three external four-point functions. The general expression of the six-point functions is:

$$\begin{aligned}\mathcal{S}(\tau_1,...\tau_6) = \int d\tau_{a_1}...d\tau_{a_6}\Big[&\mathcal{F}(\tau_1,\tau_2,\tau_{a_1},\tau_{a_2})\mathcal{F}(\tau_3,\tau_4,\tau_{a_3},\tau_{a_4})\\ &\cdot\mathcal{F}(\tau_5,\tau_6,\tau_{a_5},\tau_{a_6})\mathcal{S}_{core}(\tau_{a_1},...,\tau_{a_6})\Big]\,.\end{aligned} \tag{4.18}$$

With the use of this expression and the fact that $\mathcal{E}_s(\tau_1,..,\tau_8)$ consists of two six-point functions glued together we get:

$$\begin{aligned}&\mathcal{E}_s(\tau_1,..,\tau_8) = \int d\tau_{a_1}...d\tau_{a_8}d\tau_{b_1}...d\tau_{b_8}\mathcal{S}_{core}(\tau_{a_1},...,\tau_{a_6},\tau_{b_1},\tau_{b_2})\mathcal{S}_{core}(\tau_{a_1},...,\tau_{a_6},\tau_{b_3},\tau_{b_4})\\ &\mathcal{F}(\tau_1,\tau_2,\tau_{a_1},\tau_{a_2})\mathcal{F}(\tau_3,\tau_4,\tau_{a_3},\tau_{a_4})\mathcal{F}(\tau_{b_1},...,\tau_{b_4})\mathcal{F}(\tau_5,\tau_6,\tau_{a_5},\tau_{a_6})\mathcal{F}(\tau_7,\tau_8,\tau_{a_7},\tau_{a_8})\,.\end{aligned} \tag{4.19}$$

Using now (4.3) we get:

$$\begin{aligned}&\langle\mathcal{O}_1(\tau_1)...\mathcal{O}_4(\tau_4)\rangle_s = \int d\tau_{a_1}...d\tau_{a_8}d\tau_{b_1}...d\tau_{b_8}\mathcal{S}_{core}(\tau_{a_1},...,\tau_{a_6},\tau_{b_1},\tau_{b_2})\mathcal{S}_{core}(\tau_{a_1},...,\tau_{a_6},\tau_{b_3},\tau_{b_4})\\ &\cdot\langle\mathcal{O}_1(\tau_1)x(\tau_{a_1})x(\tau_{a_2})\rangle\langle\mathcal{O}_2(\tau_2)x(\tau_{a_3})x(\tau_{a_4})\rangle\mathcal{F}(\tau_{b_1},...,\tau_{b_4})\\ &\cdot\langle\mathcal{O}_3(\tau_3)(\tau_3)x(\tau_{a_5})x(\tau_{a_6})\rangle\langle\mathcal{O}_2(\tau_4)x(\tau_{a_7})x(\tau_{a_8})\rangle\,.\end{aligned} \tag{4.20}$$

To evaluate this challenging integral, we use the representation

$$\frac{\mathcal{F}_{h\neq 2}}{a_0} = \int_{\mathcal{C}}\frac{dh}{2\pi i}\frac{h-1/2}{\tan(\frac{\pi h}{2})}\frac{k_c(h)}{1-k_c(h)}\Psi_h(\chi) = \int_{\mathcal{C}}\frac{dh}{2\pi i}\rho(h)\Psi_h(\chi)\,. \tag{4.21}$$

We will now state the final result avoiding the technical details as it is its form that we are interested in. In the end, we get:

$$\langle\mathcal{O}_1(\tau_1)...\mathcal{O}_4(\tau_4)\rangle_s = \int_{\mathcal{C}}\frac{dh}{2\pi i}\frac{\rho(h)}{c_h^2}\frac{\Gamma(h)^2}{\Gamma(2h)}c_{12h}c_{34h}\mathcal{F}^h_{1234}(\chi)\,, \tag{4.22}$$

where $\mathcal{F}^h_{1234}(\chi)$ is the conformal block of operators $\mathcal{O}_1, ..., \mathcal{O}_4$ and the exchanged operator $\mathcal{O}_h$. It is given by the expression:

$$\mathcal{F}^h_{1234}(\chi) = \left|\frac{\tau_{24}}{\tau_{14}}\right|^{h_{12}} \left|\frac{\tau_{14}}{\tau_{13}}\right|^{h_{34}} \frac{1}{|\tau_{12}|^{h_1+h_2}|\tau_{34}|^{h_3+h_4}} \chi^h {}_2F_1(h-h_{12}, h+h_{34}, 2h, \chi)\,. \tag{4.23}$$

Recalling the form $c_{123} = c_1 c_2 c_3 \mathcal{I}_{123}$, which separates c_i that come from the sum of ladders of the four-point function and $\mathcal{I}_{123}$ which denotes effect of gluing the ladder to get the six-point function, we can write:

$$\langle \mathcal{O}_1(\tau_1)...\mathcal{O}_4(\tau_4)\rangle_s = c_1 c_2 c_3 c_4 \int_{\mathcal{C}} \frac{dh}{2\pi i}\rho(h)\frac{\Gamma(h)^2}{\Gamma(2h)}\mathcal{I}_{12h}\mathcal{I}_{34h}\mathcal{F}^h_{1234}(\chi)\,. \tag{4.24}$$

4.3 Solving SYK

It is time to recap what we have presented up until now as we have all the ingredients required to fully solve the SYK model. First of all, from the analysis of the two-point function we got the fermion dimensions $\Delta = 1/q$ using the conformal symmetry in the IR limit. Moving to the four-point function, and studying the ladder diagrams we arrive at the expression for $\rho(h)$, see (4.21). For the six-point function, we presented an expression with all the information encoded in $c_{123} = c_1 c_2 c_3 \mathcal{I}_{123}$. The coeeficients c_i, which are functions depending on $\rho(h)$, were computed in the analysis of the four-point function, see (3.71). One step further, we presented the eight-point function. The final expression that is found in (4.24) can be fully determined only from knowing $\Delta, \rho(h), \mathcal{I}_{12h}$. These are the exact expressions computed from the two, four and six-point functions. Nothing new was added in the computation. Moreover, it is proved in [13] that all the information about the higher-point function is encoded in these three parameters. In summary, once one solves the two,four and six-point functions is then able to solve the SYK model. It is also stated in [13] that this result is also applicable to any theory in which higher-point functions are built from four-point functions.

Chapter 5

Effective action

So far, we have studied our model and derived its basic properties by analyzing the 2, 4 and higher-point functions. In this chapter we will use the path integral representation and apart from getting the same Schwinger-Dyson equations we will also see clearer some other interesting features of the model like its classical behaviour in large N. This chapter is based mainly on [2, 33].

5.1 Annealed disorder

The partition function as a path integral is:

$$Z(J_{ijkl}) = \int Dx_i exp\left[-\int d\tau\left(\frac{1}{2}\sum_i x_i\partial_\tau x_i + \frac{1}{4!}\sum_{i,j,k,l}^{N} J_{ijkl}x_ix_jx_kx_l\right)\right]. \tag{5.1}$$

Now we want to do the average over the disorder J_{ijkl}. There are 2 physical ways to do this and they are equivalent up to order $1/N$.

- The annealed disorder method when you average directly the partition function $\langle Z\rangle_J$ while you treat J_{ijkl} as a microscopic variable.
- The replica trick. In this method you average the free energy $\langle logZ\rangle_J$. This method is more complicated but it bears more physical relevance in condensed matter theory when you want to describe lattice errors in crystals.

We will use the annealed disorder method for simplicity. To average the partition function we to do the Gaussian expectation values. Practically that means that we have to solve the integral:

$$\langle Z\rangle_J = \int dJ_{ijkl} exp\left(-\frac{\sum J_{ijkl}^2}{2\frac{3!J^2}{N^3}}\right)\cdot Z(J_{ijkl}). \tag{5.2}$$

This a Gaussian integral and we will use the formula:

$$\int dx e^{ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}.$$

As usually the constant factor will be absorbed into the integration measure. We have to be careful to get the correct numerical factor in the exponent. We have to include a factor of 4! as for any i, j, k, l since there exist 4! terms as an effect of the anticommutation rules of the fermions. This term can be ignored if we write $\frac{1}{4!}\sum_{i,j,k,l}^{N} = \sum_{1\leq i<j<k<l\leq N}$. Thus we have:

$$\langle Z\rangle_J \sim \int Dx_i exp\left(-\int d\tau \frac{1}{2}\sum_i x_i\partial_\tau x_i + \frac{1}{2}\frac{J^2}{4N^3}\int d\tau d\tau' \left[\sum_i x_i(\tau)x_i(\tau')\right]^4\right),$$

where we have used

$$\sum_{1\leq i<j<k<l\leq N}(x_ix_jx_kx_l)(\tau)(x_ix_jx_kx_l)(\tau') = \frac{1}{4!}\left[\sum_i x_i(\tau)x_i(\tau')\right]^4.$$

Our next task is to integrate out the fermions. To do this we first introduce the bilocal field:

$$\tilde{G}(\tau_1,\tau_2) = \frac{1}{N}\sum_{i=1}^{N} x_i(\tau_1)x_i(\tau_2)\,. \tag{5.3}$$

We can introduce this bilocal field in the path integral by inserting 1 in the following way:

$$\begin{aligned} 1 &= \int D\tilde{G}\delta\left(N\tilde{G}(\tau_1,\tau_2) - \frac{1}{N}\sum_{i=1}^{N} x_i(\tau_1)x_i(\tau_2)\right) \\ &\sim \int D\tilde{G}\tilde{\Sigma} exp\left(-\frac{N}{2}\int\int d\tau d\tau' \tilde{\Sigma}(\tilde{G} - \frac{1}{N}\sum x_i(\tau)x_i(\tau'))\right), \end{aligned} \tag{5.4}$$

where $\tilde{\Sigma}(\tau_1,\tau_2)$ plays the role of a Lagrange multiplier. Inserting this expression in the partition function we get:

$$\begin{aligned} \langle Z\rangle_J \sim &\int Dx_i D\tilde{G}D\tilde{\Sigma} exp[-\int d\tau \frac{1}{2}\sum_i x_i\partial_\tau x_i \\ &-\frac{1}{2}\int\int d\tau d\tau' N\tilde{\Sigma}(\tilde{G} - \frac{1}{N}\sum x_i(\tau)x_i(\tau')) + \frac{J^2N}{2\cdot 4}\int\int d\tau d\tau' [\tilde{G}(\tau_1,\tau_2)]^4\,. \end{aligned} \tag{5.5}$$

Exploiting this trick we have managed to obtain an exponential that is bilinear in the fermion fields x_i. We use again the Gaussian integral

$$\int dx e^{-\frac{1}{2}xAx} = \sqrt{detA}.$$

We arrive at:

$$\begin{aligned}\langle Z\rangle_J &\sim \int D\tilde{G}D\tilde{\Sigma}[det(\partial_\tau - \tilde{\Sigma})]^{\frac{N}{2}} exp\left(-\frac{N}{2}\int\int(\tilde{\Sigma}\tilde{G} - \frac{1}{4}J^2\tilde{G}^4)\right) \\ &= \int D\tilde{G}D\tilde{\Sigma}e^{-NI[\tilde{G}\tilde{\Sigma}]}\,,\end{aligned} \tag{5.6}$$

with

$$I[\tilde{G},\tilde{\Sigma}] = -\frac{1}{2}logdet(\partial_\tau - \tilde{\Sigma}) + \int\int d\tau d\tau'(\tilde{\Sigma}\tilde{G} - \frac{1}{4}J^2\tilde{G}^4)\,. \tag{5.7}$$

Now, we can clearly see that N plays the role of $\hbar^{-1}$ and in the large N limit the model becomes classical. Extremizing this action with respect to $\tilde{\Sigma}, \tilde{G}$ gives us the classical equations of motion:

$$\frac{\delta I}{\delta\tilde{G}} = 0 \Leftrightarrow \tilde{\Sigma}^2 = J^2\tilde{G} \tag{5.8}$$

$$\frac{\delta I}{\delta\tilde{\Sigma}} = 0 \Leftrightarrow [\partial_\tau - \tilde{\Sigma}]^{-1} = \tilde{G}, \tag{5.9}$$

where we have used the identity $\mathrm{logdet}M = \mathrm{Trlog}M$. These are exactly the Schwinger-Dyson equations that we have found in the second chapter.

5.2 $\mathcal{O}(N)$ symmetry

In this section we will prove the $\mathcal{O}(N)$ symmetry of the SYK model. We will also derive the conserved current for the free action while for the full action we will show the absence of such current as an effect of the non-locality of the action.

5.2.1 Free action

The action for a free Majorana particle is given by:

$$S = \frac{1}{2}\int d\tau x_i(\tau)\dot{x}^i(\tau)\,. \tag{5.10}$$

Now we apply a O(N) transformation $x_i \to O^i_{\ j}x^j$. Its infinitesimal form is:

$$\delta x^i = \xi_a(T^a)^i_{\ j}x^j + \mathcal{O}(\xi^2)\,, \tag{5.11}$$

with (T^a) denoting the generators and ξ_a the constant (for now) parameter of the transformation. The generators are antisymmetric. This is easily proven:

$$O^TO = 1 \Leftrightarrow e^{aT_a^T}e^{aT_a} = 1 \Leftrightarrow T_a = -T_a^T\,. \tag{5.12}$$

Varying the action under this transformation we get:

$$\begin{aligned}\delta S &= \frac{1}{2}\int d\tau\left(\xi_a (T^a)_i^j x_j \dot{x}^i + x_i \frac{d}{d\tau}(\xi_a (T^a)_j^i x^j)\right)\\ &= \frac{1}{2}\int d\tau\left(\xi_a (T^a)_i^j x_j \dot{x}^i - \frac{1}{2}\int d\tau \xi_a (T^a)_i^j x_j \dot{x}^i\right) = 0\,,\end{aligned} \tag{5.13}$$

where in the last step we have interchanged the indices i, j and used the antisymmetry of the generators. Now we promote our transformation to be a local one. That means that now ξ has dependence on time. Varying once more the action:

$$\begin{aligned}\delta S &= \frac{1}{2}\int d\tau\left(\xi_a(\tau)(T^a)_i^j x_j \dot{x}^i + x_i \frac{d}{d\tau}(\xi_a(\tau)(T^a)_j^i x^j)\right)\\ &= \frac{1}{2}\int d\tau(\partial_\tau \xi_a(\tau)) \cdot (x_i (T^a)_j^i x^j)\\ &= \frac{1}{2}\frac{d}{d\tau}\left(\int d\tau \xi_a(\tau) x_i (T^a)_j^i x^j\right) - \frac{1}{2}\int d\tau \xi_a(\tau)\frac{d}{d\tau}(x_i (T^a)_j^i x^j)\,.\end{aligned} \tag{5.14}$$

The first term is 0 as a boundary term and from the second, requiring that the variation of the action is 0 with respect to this particular transformation, we can obtain the the conserved current:

$$j^a(\tau) = \frac{1}{2} x_i(\tau)(T^a)_j^i x^j(\tau)\,. \tag{5.15}$$

This Noether current can also be derived by the standard formula:

$$j_\mu = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)}\frac{\delta \phi_n}{\delta \alpha}\,. \tag{5.16}$$

In our case, we have:

$$j^a = \frac{\partial(\frac{1}{2}x_i(\tau)\dot{x}^i(\tau))}{d(\dot{x}^i(\tau))}\frac{\delta x^i}{\delta \xi} = \frac{1}{2}x_i(\tau)(T^a)_j^i x^j(\tau)\,. \tag{5.17}$$

5.2.2 Full action

We now consider the full action of the model as given by:

$$\begin{aligned}Z = \int Dx_i dJ_{ijkl} exp\left(-\frac{\sum J_{ijkl}^2}{2\frac{3!J^2}{N^3}}\right) exp\Big[-\int d\tau\Big(\frac{1}{2}\sum_i x_i \partial_\tau x_i\\ + \frac{1}{4!}\sum_{i,j,k,l}^N J_{ijkl} x_i x_j x_k x_l\Big)\Big]\end{aligned}$$

We will now again apply a O(N) transformation but we have to be careful. As we have seen J_{ijkl} has no dynamics and thus they cannot be treated as a quantum mechanic variable. This means that we do not know how these parameters transform and such a transformation will not yield any a conserved current. To overcome this obstacle we again use annealed disorder to integrate them out and then we will apply the desired transformation. After integration, we have the already computed result:

$$\langle Z\rangle_J \sim \int Dx_i exp\left(-\int d\tau \frac{1}{2}\sum_i x_i\partial_\tau x_i + \frac{1}{2}\frac{J^2}{4N^3}\int d\tau d\tau' \left[\sum_i x_i(\tau)x_i(\tau')\right]^4\right).$$

The first term of the exponent is the free action and will add nothing new to our calculations as we have already seen that it is symmetric under O(N) transformations. Thus, we will ignore this term. Now we vary the remaining part of the action:

$$\begin{aligned}\delta S &= -\frac{1}{2}\frac{J^2}{16N^3}\int d\tau d\tau' [\xi_a (T^a)^k_j x_k(\tau)x^j(\tau') + \xi_a (T^a)^j_k x_k(\tau)x^j(\tau')]^3 \\ &= -\frac{1}{2}\frac{J^2}{16N^3}\int d\tau d\tau' [\xi_a (T^a)^k_j x_k(\tau)x^j(\tau') - \xi_a (T^a)^k_j x_k(\tau)x^j(\tau')]^3 = 0\,.\end{aligned} \tag{5.18}$$

Thus, we find that the SYK model has O(N) symmetry. Now we promote the transformation to be a local one in expectation to find a conserved current. Doing the same manipulations as in the free theory we arrive at:

$$\begin{aligned}\delta S = &\frac{1}{2}\int d\tau (\partial_\tau \xi_a(\tau))\cdot (x_i (T^a)^i{}_j x^j) \\ &- \frac{1}{2}\frac{J^2}{16N^3}\int d\tau d\tau' (\xi_a(\tau)-\xi_a(\tau'))^3 (x_k(\tau)x^j(\tau')(T^a)^k_j)^3\,.\end{aligned} \tag{5.19}$$

The first term is of course the one we get from varying the free action. The second term is the one that comes from the interaction. Unfortunately, we can not write this term as $\dot{\xi}_a j^a$ so we can by partial integration arrive at a conserved current. This seems catastrophic as we have an action that has a continuous symmetry but there is no Noether current associated. But the crucial point is to observe that Noether theorem is only valid for local action while our action is clearly bilocal. In the end, Noether theorem is not violated.

Chapter 6

Theory of reparametrizations

Now that that we have derived the effective action of the SYK model, we are going to use it to expose some interesting properties of the model that are not clear from the diagrammatic treatment of its two and four-point functions. First, we are going to examine the emergent conformal symmetry of this action in the IR limit. Then, we will introduce some fluctuations of the bilocal fields and obtain their effective action. This action will have zero modes that are connected to fluctuations of the conformal propagator. After discussing the physical interpretation of these modes we will derive the action of this reparametrizations of the conformal propagator.

6.1 Conformal symmetry of the action

As we have seen, there is an emergent conformal symmetry of the Schwinger Dyson equations in the IR limit. Now, we will examine if this symmetry also appears in the effective action. In the IR limit we can ignore the kinetic term ∂_τ since it has dimensions of energy. Thus our action becomes (for general q interactions and $\Delta = 1/q$):

$$I[\tilde{G},\tilde{\Sigma}] = -\frac{1}{2} logdet(-\tilde{\Sigma}) + \int\int d\tau_1 d\tau_2 [\tilde{\Sigma}(\tau_1,\tau_2)\tilde{G}(\tau_1,\tau_2) - \frac{J^2}{q}\tilde{G}(\tau_1,\tau_2)^q].$$

We take the reparametrizations

$$\tau_1 \to f(\tau_1), \quad \tau_2 \to f(\tau_2).$$

The transformation rules for the fields are:

$$\tilde{G}(\tau_1,\tau_2) = |f'(\tau_1), f'(\tau_2)|^{\Delta}\tilde{G}(f(\tau_1), f(\tau_2)) \tag{6.1}$$

$$\tilde{\Sigma}(\tau_1,\tau_2) = |f'(\tau_1), f'(\tau_2)|^{\Delta(q-1)}\tilde{\Sigma}(f(\tau_1), f(\tau_2)) \tag{6.2}$$

The first term is easy to see that it is invariant under reparametrizations. Explicitly we have:

$$
\begin{aligned}
\mathrm{logdet}(-\tilde{\Sigma}) \to & \ \mathrm{logdet}(-|f'(\tau_1), f'(\tau_2)|^{\Delta(q-1)}\tilde{\Sigma}(f(\tau_1), f(\tau_2))) \\
& = \log[|f'(\tau_1), f'(\tau_2)|^{\Delta(q-1)} det(-\tilde{\Sigma}(f(\tau_1), f(\tau_2)))] \\
& = \log[|f'(\tau_1), f'(\tau_2)|^{\Delta(q-1)}|f'(\tau_1), f'(\tau_2)|^{-\Delta(q-1)}\det(-\tilde{\Sigma}(\tau_1,\tau_2))] \\
& = \mathrm{logdet}(-\tilde{\Sigma}(\tau_1,\tau_2))\,.
\end{aligned}
\tag{6.3}
$$

Moving on to the integral we have:

$$
\begin{aligned}
& \int |f'(\tau_1)||f'(\tau_2)| d\tau_1 d\tau_2 \left\{ \tilde{G}(f(\tau_1), f(\tau_2))\tilde{\Sigma}(f(\tau_1), f(\tau_2)) - \frac{J^2}{q}\tilde{G}(f(\tau_1), f(\tau_2))^q \right\} \\
= & \int |f'(\tau_1)||f'(\tau_2)| d\tau_1 d\tau_2 \Big\{ |f'(\tau_1), f'(\tau_2)|^{-\Delta}|f'(\tau_1), f'(\tau_2)|^{-\Delta(q-1)} \\
& \cdot \tilde{\Sigma}(\tau_1,\tau_2)\tilde{G}(\tau_1,\tau_2) - \frac{J^2}{q}|f'(\tau_1), f'(\tau_2)|^{-q\Delta}\tilde{G}(\tau_1,\tau_2)^q] \Big\}.
\end{aligned}
$$

All the derivatives cancel and thus our action is conformally invariant in IR limit. Until now, we have seen that the ansatz of the conformal propagator spontaneously breaks the conformal symmetry down to SL(2,R). The kinetic term, which we ignored in this limit, is responsible for explicitly breaking the conformal symmetry away from the strong coupling limit.

6.2 Fluctuations

The classical saddle point solutions of Schwinger-Dyson equations will be denoted by G, Σ. We will now try to derive an action of fluctuations around these solutions. Fluctuations are of the form:

$$
\tilde{G} = G + |G|^{\frac{2-q}{2}} g \tag{6.4}
$$

$$
\tilde{\Sigma} = \Sigma + |G|^{\frac{-2+q}{2}} \sigma \tag{6.5}
$$

The reason we chose this form for the fluctuations will become obvious later. We will plug this in the effective action and we will keep only second order terms in g,σ. The linear terms will yield zero as they are expansion around a saddle point. Also we can ignore terms that are independent of g, σ because they wont be integrated and can be thought as constants. We have:

$$
\begin{aligned}
I = & -\frac{1}{2}\mathrm{logdet}[\partial_\tau - (\Sigma + |G|^{\frac{-2+q}{2}}\sigma)] + \frac{1}{2}\int d\tau_1 d\tau_2 \Big\{ (\Sigma + |G|^{\frac{-2+q}{2}}\sigma)(G + |G|^{\frac{2-q}{2}} g) \\
& - \frac{J^2}{q}(G + |G|^{\frac{2-q}{2}} g) \Big\}\,.
\end{aligned}
\tag{6.6}
$$

We will now treat each term separately:

- For the first term we will use $\log(b+ax) = \log b + \frac{ax}{b} - \frac{a^2x^2}{2b^2} + O(x^3)$ and the identity $\log\det A = \mathrm{Tr}\log A$. Also we use the saddle point equation $\partial_\tau - \Sigma = G^{-1}$. Expanding this term and keeping only second order we have:

  $$-\frac{1}{2}Tr(-\frac{1}{2}\sigma(\tau_1,\tau_2)|G(\tau_1,\tau_2)|^{\frac{q-2}{2}}G(\tau_1,\tau_3)G(\tau_2,\tau_4)|G(\tau_3,\tau_4)|^{\frac{q-2}{2}}\sigma(\tau_3,\tau_4)).$$

  This expression can be written using the symmetric kernel we have defined $\tilde{K}(\tau_1,\tau_2;\tau_3,\tau_4) = -J^2(q-1)|G(\tau_{12})|^{\frac{q-2}{2}}G(\tau_{12})G(\tau_{24})|G(\tau_{34})|^{\frac{q-2}{2}}$ as:

  $$-\frac{1}{4J^2(q-1)}\int d\tau_1 d\tau_2 d\tau_3 d\tau_4[\sigma(\tau_1,\tau_2)\tilde{K}(\tau_1,\tau_2;\tau_3,\tau_4)\sigma(\tau_3,\tau_4)]\,. \quad (6.7)$$

- From the second term the only non trivial contribution is :

  $$G^q(1+G^{-1}|G|^{\frac{2-q}{2}}g)^q \approx G^q((1+q(q-1)G^{-2}|G|^{2-q}g^2) \approx (\mathrm{sgn}G)^{q-2}\cdot g(\tau_1,\tau_2),$$

  where we have used the definition of sgn function $x = \mathrm{sgn}(x)\cdot|x|$. But in our model $q = 2n$ so $q-2 \in \mathbb{Z}$. That means that $(\mathrm{sgn}G)^{q-2} = 1$.

Then the second term becomes:

$$\frac{1}{2}\int d\tau_1 d\tau_2(\sigma(\tau_1,\tau_2)g(\tau_1,\tau_2) - \frac{1}{2}J^2(q-1)g^2(\tau_1,\tau_2))\,. \quad (6.8)$$

To keep our final expression for the partition function readable, from now one we will use the shorthand notation:

$$\langle\sigma|\tilde{K}|\sigma\rangle = \int d\tau_1 d\tau_2 d\tau_3 d\tau_4[\sigma(\tau_1,\tau_2)\tilde{K}(\tau_1,\tau_2;\tau_3,\tau_4)\sigma(\tau_3,\tau_4)],$$
$$\langle\sigma|g\rangle = \int d\tau_1 d\tau_2 \sigma(\tau_1,\tau_2)g(\tau_1,\tau_2).$$

Concerning the integration measure we have $d\tilde{\Sigma}d\tilde{G} = |G|^{\frac{2-q}{2}}|G|^{\frac{-2+q}{2}}d\sigma dg = d\sigma dg$. Our action becomes:

$$\frac{I}{N} = -\frac{1}{4J^2(q-1)}\langle\sigma|\tilde{K}|\sigma\rangle + \frac{1}{2}\langle\sigma|g\rangle - \frac{1}{4}J^2(q-1)\langle g|g\rangle\,. \quad (6.9)$$

Inserting this action to our partition function we get:

$$\langle Z\rangle_J \sim \int D\tilde{g}D\tilde{\sigma}\exp N\cdot\{-\frac{1}{4J^2(q-1)}\langle\sigma|\tilde{K}|\sigma\rangle + \frac{1}{2}\langle\sigma|g\rangle - \frac{1}{4}J^2(q-1)\langle g|g\rangle\}\,. \quad (6.10)$$

Using the Gaussian formula

$$\int dx e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}+c},$$

we can integrate out σ to get an action for the fluctuations g of the bilocal field $\tilde{G}$. We end up with the action:

$$\frac{I(g)}{N} = \frac{J^2(q-1)}{4}\langle g|\tilde{K}^{-1} - 1|g\rangle \,. \tag{6.11}$$

It is important to mention that the above expression is valid for all energies.

6.2.1 Conformal limit and Nambu-Goldstone modes

Now we once again study the action in the conformal limit. We can use the already known expressions for the conformal propagator and the symmetric kernel. The action (6.11) can yield zero when g is an eigenfunction of the symmetric kernel with eigenvalue 1. We will now prove that there exist such eigenfunctions using the Schwinger Dyson equations.
As we have proved the Schwinger-Dyson equations are invariant under conformal transformations. That means that if we take a transformation $\tau \to \tau + \epsilon(\tau)$ and G_c is a solution, then also $G_c + \delta_\epsilon G_c$ is also a solution. To find the explicit form of $\delta_\epsilon G_c$ we use the transformation rule for the propagator for $f(\tau) = \tau + \epsilon(\tau)$ and expand in powers of ϵ. In detail:

$$\begin{aligned}
&G_c(\tau,\tau') = |f(\tau)f(\tau')|^{\Delta} G_c(f(\tau), f(\tau')) \\
&= \big(1 + \Delta\epsilon'(\tau) + \epsilon'(\tau')\big)\big(G_c(\tau,\tau') + \epsilon(\tau)\partial_\tau G_c(\tau,\tau') + \epsilon(\tau')\partial_{\tau'} G_c(\tau,\tau')\big) \\
&= G_c(\tau,\tau') + \big(\Delta\epsilon'(\tau) + \epsilon'(\tau') + \epsilon(\tau)\partial_\tau + \epsilon(\tau')\partial_{\tau'}\big)G_c(\tau,\tau').
\end{aligned}$$

Thus, we have:

$$\delta_\epsilon G_c(\tau,\tau') = (\Delta\epsilon'(\tau) + \epsilon'(\tau') + \epsilon(\tau)\partial_\tau + \epsilon(\tau')\partial_{\tau'})G_c(\tau,\tau') \,. \tag{6.12}$$

Now we can plug this transformation in the Schwinger-Dyson equation:

$$\begin{aligned}
&\int d\tau' G(\tau,\tau')\Sigma(\tau',\tau'') = -\delta(\tau-\tau'') \\
&= \int d\tau' \,\big(G_c(\tau,\tau') + \delta_\epsilon G_c(\tau,\tau')\big)\,\big(\Sigma_c(\tau,\tau') + \delta_\epsilon\Sigma_c(\tau,\tau')\big) \\
&= \int d\tau' \,\big(G_c(\tau,\tau')\cdot\delta_\epsilon\Sigma_c(\tau,\tau') + \Sigma_c(\tau,\tau')\cdot\delta_\epsilon G_c(\tau,\tau')\big) = 0\,.
\end{aligned} \tag{6.13}$$

For simplicity, we can write:

$$\delta_\epsilon G_c * \Sigma_c + G_c\delta_\epsilon * \Sigma_c = 0\,, \tag{6.14}$$

where integration is implied. In the IR limit we can use: $\Sigma = G_c^{-1}$ and also the definition $\Sigma_c = J^2 G_c^{q-1}$. We can now multiply (actually we involute) from the right with G_c. Using also the chain rule $\delta_\epsilon\Sigma(G_c) = \delta_G\Sigma\cdot\delta_\epsilon G_c$ we have:

$$\delta_\epsilon G_c + G_c * (J^2(q-1)G_c^{q-2}\delta_\epsilon G_c) * G_c = 0\,. \tag{6.15}$$

Using the expression for the kernel:

$$K(\tau_3, \tau_4, \tau_1, \tau_2) = -J^2(q-1)G(\tau_1, \tau_3)G(\tau_2, \tau_4)G(\tau_1, \tau_2)^{q-2}\,,$$

our equation becomes:

$$(1 - K_c)\delta_\epsilon G_c = 0\,. \tag{6.16}$$

We see that $\delta_\epsilon G_c$ are eigenfunctions of the kernel with eigenvalue 1. We can rewrite this expression in a different way to contain the symmetric kernel. Using our definition (and once again avoid writing the integrals):

$$\begin{aligned} 1 - K_c &= 1 - \tilde{K}_c |G(\tau_{12})|^{\frac{2-q}{2}} |G(\tau_{34})|^{\frac{-2+q}{2}} \\ &= |G(\tau_{34})|^{\frac{2-q}{2}} - \tilde{K}_c |G(\tau_{12})|^{\frac{2-q}{2}} = (1 - \tilde{K})|G|^{\frac{2-q}{2}}\,. \end{aligned} \tag{6.17}$$

This is a shorthand for:

$$\int d\tau_1 d\tau_2 \tilde{K}_c(\tau_1, \tau_2; \tau_3, \tau_4)|G(\tau_{12})|^{\frac{2-q}{2}} \delta_\epsilon G_c(\tau_{12}) = |G(\tau_{34})|^{\frac{2-q}{2}} \delta_\epsilon G_c(\tau_{34})\,. \tag{6.18}$$

We just have proven that there exist eigenfunctions of $\tilde{K}$ that have eigenvalue 1. These as we have said above are the ones that make the action 0. Moreover, we have shown that these eigenfunctions are the reparametrizations of the conformal propagator. The physical interpretation is that these zero modes are associated with the spontaneous symmetry breaking of the conformal symmetry of the action down to SL(2,R) by the solution G_c. Thus these zero modes can be viewed as the associated Nambu-Goldstone modes.

6.3 Action of the reparametrizations

We want to find the action for finite reparametrizations $\tau \to f(\tau)$ when they are included in the original action. Here we will follow an intuitive argument inspired by effective field theory. We are searching for an expression of lowest order in derivatives that is invariant under SL(2,R) transformations. We will work at zero temperature. We can state that we want an action that satisfies the following:

- If $f \in SL(2,R) \leftrightarrow f(\tau) = \frac{a\tau+b}{c\tau+d} \to S[f] = 0$,
- If $f \notin SL(2,R) \to S[f]$ must be invariant under $f \to \frac{af+b}{cf+d}$.

The first statement follows from the fact that G_c is invariant under SL(2,R) transformations and hence $\delta_{SL(2,R)} G_c$ must yield zero. The second statement follows from the fact that at zero temperature G_c is invariant under SL(2,R) transformations and therefore the reparametrization action must have an exact symmetry under these transformations. Putting these statements in another way we are searching for combination of derivatives of a SL(2,R) transformation F that reduces to exactly the same combination of derivatives in f. We find the first and second derivative of F:

$$\begin{aligned}F(\tau) &= \frac{af(\tau)+b}{cf(\tau)+d}\\ F' &= \frac{f'}{(cf(\tau)+d)^2}\\ F'' &= \frac{f''}{(cf(\tau)+d)^2} - \frac{2c(f')^2}{(cf(\tau)+d)^3}\end{aligned}$$

In both these derivative we have a common term $1/(cf(\tau)+d)^2$. Thus we consider:

$$\frac{F''}{F'} = \frac{f''}{f'} - \frac{2cf}{cf(\tau)+d}$$

In the above expression the second term is exactly what we want but the third term needs to be get rid of. We differentiate once more:

$$F''' = \frac{f'''}{(cf(\tau)+d)^2} - \frac{6cf'f''}{(cf(\tau)+d)^3} + \frac{6c^2(f')^3}{(cf(\tau)+d)^4}\,. \tag{6.19}$$

Dividing by F' we get:

$$\frac{F'''}{F'} = \frac{f'''}{f'} - \frac{6cf''}{(cf(\tau)+d)} + \frac{4c^2(f')^2}{(cf(\tau)+d)^2}\,. \tag{6.20}$$

This term is similar to $\frac{F''}{F'}$. Thus we can square $\frac{F''}{F'}$ and we get:

$$\left(\frac{F''}{F'}\right)^2 = \left(\frac{f''}{f'}\right)^2 - \frac{4cf''}{(cf+d)} + \frac{4c^2(f')^2}{(cf+d)^2}\,. \tag{6.21}$$

We can make the following combination:

$$\frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \equiv \{f,\tau\}\,. \tag{6.22}$$

This expression is called the Schwartzian Derivative. It is an operator that is invariant under SL(2,R) transformation. We are going to prove the above statement. Suppose

$$f(\tau) = \frac{a\tau+b}{c\tau+d} \in SL(2,R).$$

Then we can define $u(\tau) = f'(\tau^{-1/2}) = c\tau + d$. Then:

$$\begin{aligned}u'' = 0 &\leftrightarrow -\frac{3}{4}(f')^{-5/2}f'' - \frac{1}{2}(f')^{-3/2}f''' = 0\\ &\leftrightarrow -\frac{1}{2}(f')^{-1/2}\left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right) = 0\\ &\leftrightarrow -\frac{1}{2}(f')^{-1/2}\{f,\tau\} = 0\\ &\leftrightarrow \{f,\tau\} = 0\,.\end{aligned} \tag{6.23}$$

In the end, we have found an expression that yields 0 if the reparametrization is a linear fractional transformation and moreover it is invariant under a SL(2,R) transformation of the reparametrization. The action can be written as:

$$\frac{I}{N} = \frac{-c}{J} \int d\tau \{f, \tau\} \, . \tag{6.24}$$

The constant c is found to be (for large q):

$$c = \frac{1}{4q^2}$$

We can go now to finite temperature (we map the line to a circle) by $f(\tau) = \exp(\frac{2\pi i \tau}{\beta})$ or $f(\tau) = \tan(\frac{\pi\tau}{\beta})$. The action becomes:

$$\frac{I_\beta}{N} = \frac{-c}{J} \int_0^\beta \{\exp(\frac{2\pi i \tau}{\beta}), \tau\} = \frac{-c}{J} \int_0^\beta \left(\frac{2\pi^2}{\beta^2}\right) = \frac{-c}{J} \frac{2\pi^2}{\beta} \, . \tag{6.25}$$

Chapter 7

Chaotic behaviour of SYK model

So far we have treated the SYK model in zero temperature ($\beta \to \infty$) and have only given the related expressions in finite temperature. In this chapter we are going to study the chaotic behaviour of the SYK model in finite temperature. First we are going to give a short introduction on the relative new field of chaotic behaviour of quantum systems and then we are going to study the chaotic behaviour of the SYK model.

7.1 Chaotic behaviour in Quantum Systems

The chaotic behaviour (or else strong chaos or the butterfly effect) of a general quantum system can be characterized using a time-separated commutator $[W(t), V(0)]$ between two Hermitian operators W(t) and V(t). This commutator measures the effect of a perturbation by V(0) on the measurements of $W(t)$ at a later time t and vice versa.One possible measure of such effects is

$$C(t) = -\langle [W(t), V(0)]^2 \rangle = Z^{-1} Tr[e^{\beta H} [W(t), V(0)]^2] \,, \tag{7.1}$$

where $Z = Tr[e^{-\beta H}]$. A quantum definition of strong chaos is that

$$C(t) \sim 2\langle W(t)W(t)\rangle \langle V(0)V(0)\rangle \,, \tag{7.2}$$

for large t, regardless the choice of W, V with the restriction that they have zero thermal one-point function.
To see how the commutator $C(t)$ has a connection with chaos, we can choose $W(t) = q(t)$ and $V(t) = p(t)$. In the semi classical limit the commutator becomes a Poisson bracket:

$$[W(t), V(0)] \to i\hbar \{q(t), p(0)\} = i\hbar \frac{\partial q(t)}{\partial q(0)} \,. \tag{7.3}$$

We see that the commutator describes the way the position of the system depends on its initial position or the measure of the divergence between nearby trajectories. In classical theory, such a a divergence is characterized by the Lyapunov exponent λ_L. This quantity is defined by:

$$|\delta q(t)| \sim e^{\lambda_L t} |\delta q(0)| \,. \tag{7.4}$$

If we take the limit $|\delta q(t)| \to 0$, we have:

$$\lim_{|\delta q(0)| \to 0} \frac{|\delta q(t)|}{|\delta q(0)|} = \frac{\partial q(t)}{\partial q(0)} \sim e^{\lambda_L t} \,. \tag{7.5}$$

We see that, at least in the semi classical approximation, the commutator $[W(t), V(0)]$ has a direct connection with chaotic behaviour. From a purely quantum mechanical point of view we can view $C(t)$ as a measure of the growth of the operator $W(t)$ expressed as a sum of products of basis operators. For example in an Ising system (in 1-d) these operators are the Pauli matrices. Evaluating C(t) we find that:

$$C(t) = 2\langle W^2(t) V^2(0) \rangle - 2\langle W(t) V(0) W(t) V(0) \rangle \,. \tag{7.6}$$

At large t, the first term converges to a constant value while the second term (the out-of-time order correlator) vanishes exponentially. This means that the increase of C(t) is caused by the decrease of the OTO correlator. From this, follows the quantum definition of strong chaos (7.2).
Now, we will present another intuitive way that connects the out-of-time-correlator of 7.6 with quantum chaos. Suppose we have a quantum state $|\psi\rangle$.

- First, we are going to act on the state with the perturbation V. Then we evolve our state at time t, we act with W and then we evolve back to $t = 0$. This procedure is:

$$|\psi\rangle \to |\psi_1\rangle = U(0,t) W U(t,0) V |\psi\rangle = W(t) V(0) |\psi\rangle \,. \tag{7.7}$$

- Now we do the reversal procedure. We evolve our state to t, we act with W, then we evolve back at $t = 0$ and then we act with V. This is translated to:

$$|\psi\rangle \to |\psi_2\rangle = V U(0,t) W U(t,0) |\psi\rangle = V(0) W(t) |\psi\rangle \,. \tag{7.8}$$

Now we take the inner product $\langle \psi_2 | \psi_1 \rangle$. This gives the OTO correlator we are interested in. Moreover, this inner product is, in some sense, a measure of the similarity between these two states. Thus, the exponential vanishing of this correlator means that after some time these two states are nothing alike. These 2 completely different states is a characteristic of the butterfly effect.

7.2 Scrambling of information

Up until now we haven't given any formal definition of quantum chaos but intuitively we have tried to connect the growth of 7.6 with concepts of classical chaos. There is no clear evidence that this commutator is in agreement with previous work done in the field of quantum chaos. Bearing this in mind, the main purpose of the work done with OTOC is to search for connections with scrambling of information either from the black hole perspective or by the quantum information one. Scrambling of information can be thought as how fast information about a specific part of the system spreads to the rest of it.

A nice and relative simple model, proposed in [11] to study such effect is an ensemble of qubits [1] interacting with an Ising system. Before presenting the qualitative results of this work, we will recall some useful quantities.

First, we know that the Hilbert space of a composite quantum system can be written as the tensor product of the Hilbert space of each subsystem: $\mathcal{H} = \mathcal{H_A} \otimes \mathcal{H_B}$.... Suppose we have only two subsystems A,B in a state $|\Psi\rangle$. To compute the density matrix of one subsystem we trace out the degree of freedoms of the other system. For example, $\rho^A = Tr_B |\Psi\rangle \langle\Psi|$. If our system is in a mixed (entangled) state then the density matrices of the subsystems are also mixed states. These states can be thought as ensembles with the density matrix given by $\rho = \sum_k p_k |\psi_k\rangle \langle\psi_k|$. One famous ensemble is the canonical:$\rho = e^{-\beta H}$. Another way to see if the two subsystems are entangled is to calculate the von Neumann entropy of a subsystem: $S_A = -tr_B(-\rho^A \log \rho^A)$. This entropy is greater than zero if and only if $|\langle\rangle\, \Psi\rangle$ is entangled.

Now, suppose we have mixed state but we want to construct a pure one. This is done by the thermofield double formalism. The trick is to treat our mixed state ρ as a pure state in larger system. To do this, we consider two identical copies of a subsystem. Then we consider the thermofield double state

$$|TFD\rangle = \frac{1}{Z^{1/2}} \sum_n e^{-\beta E_n/2} |n\rangle_L |n\rangle_R \,. \tag{7.9}$$

The total density matrix is $\rho = |TFD\rangle \langle TFD|$, as expected from a pure state. Then the density matrix for the subsystem L is given by: $\rho_L = \mathrm{tr}_R \rho = e^{-\beta H_1}$. The thermofield double also plays a crucial role in black hole physics/information theory and in AdS/CFT [9, 11, 12].

Another essential quantity for the analysis that we will present below is the notion of mutual information which is a measure of correlation between subsystems. For two subsystems A, B it is defined as: $I = S_A + S_B - S_{A\cup B}$, with S the von Neumann entropy.

[1]Particles with 1/2 spin that are frozen in sites and interact among each other because of their spin

Now that we have recalled all the neccesary quantities we will expose the qualitative results of [11]. In this work, they prepare a thermofield double state with 10 qubits on the left subsystem and 10 on the right. The Ising Hamiltonian is written in a way that is not integrable. Then the fifth qubit of the left system is perturbed by σ_z in the past (at t_w and at $t = 0$ the state is:

$$\left|\Psi'\right\rangle = e^{-iH_L t_w} \sigma_z e^{+iH_L t_w} \left|\Psi\right\rangle .$$

Since σ_z does not commute with the Hamiltonian, the perturbation alters the state of the system. Then mutual information is calculated for the first two cubits of the left and the same qubits on the right. The qubit that was perturbed was not included in the calculation. The plot is shown below:

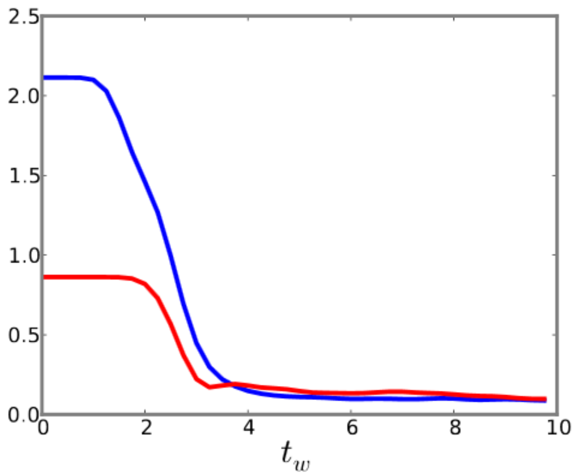


Figure 7.1: Mutual information (blue line) and spin-spin correlation as a function of the time of the perturbation.

As we see, the mutual information remains constant for some time and then suddenly it drops to zero. The constant part is due to the time that the perturbation on the fifth qubit needs to propagate to the qubits under study. But as $I \to 0$, we cant learn anything from the qubits of the R subsystem as the correlations of the thermofield double are destroyed and information gets scrambled. Moreover, we see that the scrambling effects do not happen instantly but it depends on how fast the perturbation propagates. The scrambling time should depend on the size of the system as in a system with more qubits, the propagation will be slower. In [9] it is stated that the scrambling time is $t_* \sim \log N$, with N the number of qubits. The logarithm here is another similarity to the classical chaos as we can see from (7.5):

$$\lambda_L = \lim_{t\to+\infty} \lim_{|q_0|\to 0} \left(\frac{1}{t} \log \frac{|\delta q(t)|}{|\delta q(0)|} \right) . \tag{7.10}$$

Another interesting fact that we can see from the behaviour of this model is the similarity with the butterfly effect as in both cases a small effect changed drastically the correlation between different parts of the system. Moreover, the vanishing mutual information bears resemblance with the vanishing OTO correlators.

7.3 A bound on chaos

For many years, the black hole information paradox has been a subject of interest. Although, we are not going to get into details here, this problem arises when one tries to study the quantum nature of a black hole. The unitarity of quantum mechanics does not allow loss of information but scrambling is allowed. Black holes then scramble information in such a way that it becomes inaccessible.

A lot of work, such as [11], has focused on quantifying the scrambling of information inside a black hole, mainly by the notion of mutual information. Using holography, it has been found [11] that mutual information is an exponentially decreasing function $\sim e^{2\pi t_w/b}/S$ of the perturbation time t_w. As we know, S increases with the size of the system (although in black holes $S \sim N^2$). Along with the exponential decay of the mutual information we get the logarithmic law for the scrambling time. Thus, mutual information reaches zero when

$$t_* \sim \frac{\beta}{2\pi} \log S \,. \tag{7.11}$$

Recalling that this exponential is the qualitative analogue of the Lyapunov exponent, we see that in black holes

$$\lambda_L = \frac{2\pi}{\beta} \,. \tag{7.12}$$

Therefore, we can write $t_* \sim \frac{\ln S}{\lambda_L}$. This means that a larger system (larger S) scrambles slower, while the higher the Lyapunov exponent the smaller is the scrambling time.

As far the size of the system everything is pretty straightforward. But regarding the Lyapunov exponent as a measure of chaos there isn't much information. Why the black hole value $\lambda_L = \frac{2\pi}{\beta}$ is important has been recently answered. By using mathematical arguments, concerning the OTOC function $F(t) = \langle W(t)V(0)W(t)V(0)\rangle$ as analytic in the complex time contour, a universal bound has been established [9]. The universal bound is found to be

$$\lambda_L \leq \frac{2\pi}{\beta} \,. \tag{7.13}$$

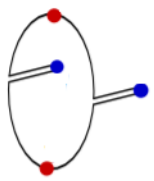
This result along with the expression 7.12, promotes black holes as the systems where scrambling is the fastest. Following the AdS/CFT conjecture, for a large N system to have a bulk dual it must certainly saturate this bound. It also speculated [9] that this saturation is enough for such a system to have a bulk dual at least at near horizon region.

7.4 Chaotic behaviour of the SYK model

Having introduced the theoretical tools to study the chaotic behaviour of a quantum system, now we are going to derive the OTO correlators for the SYK model. The analysis presented in this section is mainly based on [2]. For our case, we will consider an OTO correlator in real time with the fermions separated by a quarter of the thermal circle:

$$F(t_1,t_2) = Tr[yx_i(t_1)yx_j(0)yx_i(t_2)y_j(0)], \quad y \equiv \rho(\beta)^{1/4}\,. \tag{7.14}$$

The $1/N$ contribution of F is determined once again by a set of ladder diagrams that live on the thermal circle and a pair of real time folds for the operators $x_i(t_1)x_i(t_2)$. Pictorially the time contour we are interested in is:



where the blue dots denote the real time folds. As t_1, t_2 become large, these folds grow and in the end the growth of the $1/N$ part of F is determined only by the real time part of the contour. Working on real time now, we have to define the retarded propagators. First, we find the expression for the propagator in Lorentzian time by setting $\tau = it$ in (2.34). Since the correlator we have computed is not analytic at $\tau = 0$, we have to determine if we are doing the analytic continuation of $\tau > 0$ or $\tau < 0$. For $\tau > 0$ we have:

$$\begin{aligned}\langle x(t)x(0)\rangle = G_c(it+\epsilon) =& b\frac{1}{(it+\epsilon)^{2\Delta}} \\ &= b\frac{1}{(-1)^{\Delta}(t-i\epsilon)^{2\Delta}} = b\frac{e^{i\Delta\pi}}{(t-i\epsilon)^{2\Delta}}\,.\end{aligned} \tag{7.15}$$

But we are interested to find the correlators at finite temperature. This means we have to do the same but in the expression:

$$G_\beta(\tau) = \frac{\pi^{\frac{1}{4}}}{\sqrt{2J\beta}}\frac{1}{\sqrt{sin\frac{\pi\tau}{\beta}}}\text{sgn}\tau\,. \tag{7.16}$$

The retarded propagator is defined as:

$$G_{c,R} = \langle x(t)x(0) + x(0)x(t)\rangle\theta(t)\,. \tag{7.17}$$

For finite temperature β, using simple algebra and standard trigonometric identities we get:

$$G_{c,R} = 2b\mathrm{cos}(\pi\Delta)\left[\frac{\pi}{\beta\mathrm{sinh}\frac{\pi t}{\beta}}\right]^{2\Delta}\theta(t)\,. \tag{7.18}$$

To help us analyse this ladder diagram, we define the retarded kernel:

$$K_R(t_1,t_2,t_3,t_4) = J^2(q-1)G_R(t_{13}G_R(t_{24})G_{lr}(t_{34})^{q-2}\,. \tag{7.19}$$

G_{lr} is the Wightman correlator, where the two operators of (2.38)are separated, apart from the real time separation, by a half the thermal circle. It's expression is given by

$$G_{lr}(t) = \langle x(i\tau + \pi\beta/2)x(0)\rangle = b\left[\frac{\pi}{\beta\mathrm{cosh}\frac{\pi t}{\beta}}\right]^{2\Delta}. \tag{7.20}$$

According to [2], the asymptotic growth of F is determined by the condition that adding one rung to the ladder does not change the value of the ladder. This means that F must be an eigenvector of the retarded kernel with eigenvalue one. Thus it must satisfy the integral equation:

$$F(t_1,t_2) = \int dt_3 dt_4 K_R(t_1,t_2,t_3,t_4)F(t_3,t_4)\,. \tag{7.21}$$

To solve this equation an exponentially growing ansatz is used:

$$F(t_3,t_4) = f(t_3-t_4)e^{\lambda_L(t_3+t_4)/2}\,. \tag{7.22}$$

The motivation for choosing such an ansatz is that for $t_3 = t_4 = t$, F should be proportional to the Lyapunov exponent, as we have mentioned in (7.1). In other words,

$$F(t,t) = Tr[yx_i(t)yx_j(0)yx_i(t)y_j(0)] \sim e^{\lambda_L t}\,. \tag{7.23}$$

The integral equation becomes:

$$\begin{aligned} F(t_1,t_2) = 4J^2(q-1)b^q\mathrm{cos}^2(\pi\Delta)\pi^2\int_{-\infty}^{t_1}dt_3\int_{-\infty}^{t_2}dt_4\left(\frac{1}{\mathrm{cosh}\frac{\pi t_{34}}{\beta}}\right)^{2-4\Delta} \\ \cdot\left(\frac{1}{\beta\mathrm{sinh}\frac{\pi t_{24}}{\beta}}\right)^{2\Delta}\left(\frac{1}{\beta\mathrm{sinh}\frac{\pi t_{13}}{\beta}}\right)^{2\Delta} f(t_3-t_4)e^{\lambda_L(t_3+t_4)/2}\,. \end{aligned} \tag{7.24}$$

Solving this equation [2], one finds:

$$F(t_1,t_2) = \frac{e^{-h\pi/\beta(t_1+t_2)}}{\left[\cosh\frac{\pi t_{12}}{\beta}\right]^{2\Delta-h}}, \quad k_R(h) = \frac{\Gamma(3-2/q)\Gamma(2/q-h)}{\Gamma(1+2/q)\Gamma(2-2/q-h)}\,. \tag{7.25}$$

As we have said, we are interested in $k_R = 1$. The only solution to this is $h = -1$ as once can see. For $h = -1$ and $t_1 = t_2 = t$ we have:

$$F(t,t) = e^{\frac{2\pi}{\beta}t}\,. \tag{7.26}$$

From this expression and our expectation from theory one can recognize that $\lambda_L = \frac{2\pi}{\beta}$. This means that the SYK model saturates the universal bound value for the Lyapunov exponent and according to previously stated conjectures, it should have a bulk dual.

Chapter 8

Final remarks

In this final chapter, we are going to present some of the most interesting varieties of the SYK model along with their important features. Moreover, a short discussion concerning the bulk dual of the SYK model will be made and in the end we will close the main part this master thesis by a section with conclusions.

8.1 SYK-like models

Along with the research concerning the SYK model, there have been proposed interesting models that try to address and encounter its main problems. There has been a research on supersymmetric SYK models [14]. Similarly to the SYK model the full superconformal symmetry is spontaneously and explicitly broken and there exists also SuperSchwarzian action. The important difference between the supersymmetric model and SYK is that in the supersymmetric model the Gaussian variables and not independent.

There is a SYK tensor model studied in [4], [5]. Its main advantage is the lack of the disorder average (since its lack of dynamics an AdS/CFT interpretation obscure). The SYK-tensor model reproduces the diagrammatic structure of the original SYK. On the other hand there is no effective action describing the model and as a consequence there is no Schwarzian action. The lack of these features is concerning as they consist of basic features of the SYK model.

There is also a generalized SYK model proposed [6]. This particular model consists of f flavours of fermions, each occupying $N\alpha$ sites and appearing with a $q\alpha$ order in the interaction. It has been shown that in this model there is always a dimension-two operator in the spectrum of the bilinear singlet operators. This implies that there is a conformal symmetry breaking and maximal chaos in the infrared four-point function.

Another interest generalization of the SYK model is the 2d QFT analog [7]. The action of this model consists of an unconventional kinetic term imposed so that the fermion field is dimensionless. As a consequence, the coupling of the interaction is relevant. This makes the model strongly coupled in IR as the original SYK and in conformal limit the SD equations are diffeomorphism invariant.

8.2 Search for bulk dual

There is active research on describing the physics of the SYK model as a two dimensional gravitational theory. The fact that the action of the SYK model looks like a classical system for the bilocal fields allow us to think of a field theory defined on two dimensional space. Moreover, in $nAdS_2$ gravity we have the same pattern of symmetry breaking. Reading the propagating modes, we get an infinite tower of dimensions but their shift is of order one and not $1/N$. Thus, we can't view these states as a two particle state in a weakly interacting bulk dual.

There is another interesting approach [8], [13], [17] that takes advantage of the solvability of the model in large N. The action is constructed by massive scalar fields (due to AdS/CFT correspondence) from the bilinear $\mathcal{O}(N)$ operators up to order $1/N$. The masses of the fields are related to the dimensions of the primary operators of the SYK model. Then, the coefficients of the interactions of the bulk dual are fixed to match the SYK correlation functions. But there should be a string-like interpretation of the bulk dual that hasn't been found. As proposed in [13], the research to understand the bulk should start with the correlators of large dimension operators, thus the interaction between very massive bulk fields.

8.3 Conclusions and discussion

In this thesis we have studied the SYK model. Along the way we have encountered many interesting features of the model. Also in the beginning of this chapter we have given an overview of various SYK- like model that there is active research and the current state of the bulk dual of the model.

To conclude this thesis we now list the main reasons that made the SYK model to attract so much attention:

- The hallmark feature of the model is the emergent conformal symmetry at strong coupling. This symmetry is spontaneously broken down to SL(2,R). Moreover, this symmetry is also explicitly broken when we consider the model away from the IR limit. Due to this breaking, the reparametrization modes can be thought as Goldstone modes as they acquire a non-zero action. This symmetry breaking pattern is identical to the one found in Jackiw-Teitelboim gravity.[1] Taking this breaking pattern under consideration, the model can be thought as a $nCFT_1/nAdS_2$ example.

- The action of the reparametrization modes is found to be a Schwarzian. The same action can be found in boundary dynamics of the spacetime in Jackiw-Teitelboim gravity.
- In the large N limit the model becomes classical and therefore solvable. The equation that need to be solved involve the bilocal fields G, Σ. These fields can be thought as 'living' in two dimensions.
- The model has $\mathcal{O}(N)$ symmetry.
- The model saturates the Lyapunov exponent and thus, it should have a bulk dual.
- The model has been completely solved as even eight-point functions have been calculated and the scheme for their calculation can be expanded to higher-point functions.

[1] As the gravity dual goes beyond the scope of this thesis, there will be given relevant references for the interested reader at the end.

Appendices

Appendix A

Conformal Field Theory

A.1 Conformal transformations

A conformal field theory is a theory that is left invariant under conformal transformations. In a d-dimensional space-time we define the conformal transformations as the transformations that leave invariant the metric $g_{\mu\nu}$ up to a local scale factor.

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \,. \tag{A.1}$$

For $\Lambda(x) = 1$, we have isometries. In the flat space where $g_{\mu\nu} = \eta_{\mu\nu}$, the group of isometries is the Poincare group, a subgroup of the conformal group. The case where $\Lambda(x) =$ const. corresponds to scale transformations/dilatations. These transformations preserve the angle between intersecting curves.

An infinitesimal coordinate transformation is expressed as :

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \,, \tag{A.2}$$

where $\epsilon^\mu(x)$ is very small. The metric is a (0,2) tensor and under coordinate transformations transforms as:

$$g'_{\mu\nu}(x') = \frac{\partial x^a}{\partial x'^\mu} \frac{\partial x^b}{\partial x'^\nu} g_{ab} \,. \tag{A.3}$$

Under the previous infinitesimal transformation, we have:

$$\begin{aligned} g'_{\mu\nu} &= (\delta^a{}_\mu - \partial_\mu \epsilon^a)(\delta^b{}_\nu - \partial_\nu \epsilon^b) g_{ab} \\ &= g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) + \mathcal{O}(\epsilon^2) \,. \end{aligned} \tag{A.4}$$

With $\Lambda(x) \simeq 1 - f(x)$, we get

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = f(x) g_{\mu\nu} \,. \tag{A.5}$$

By taking the trace of the previous expression:

$$f(x) = \frac{2}{d}\partial_\rho \epsilon^\rho \, . \tag{A.6}$$

For our purposes, we take $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, 1, ..., 1)$. The treatment will be identical to Minkowski spacetime too. Once we take the derivative ∂_ρ of (A.5), we have:

$$(\partial_\rho \partial_\mu \epsilon_\nu + \partial_\rho \partial_\nu \epsilon_\mu) = (\partial_\rho f(x)) g_{\mu\nu} \, . \tag{A.7}$$

Permuting $(\mu \leftrightarrow \rho)$ and $(\nu \leftrightarrow \rho)$, we get

$$2\partial_\mu \partial_\nu \epsilon_\rho = \eta_{\mu\rho}\partial_\nu f + \eta_{\nu\rho}\partial_\mu f - \eta_{\mu\nu}\partial_\rho f \, . \tag{A.8}$$

Multiplying by $\eta^{\mu\nu}$ we get:

$$\begin{aligned} 2\eta^{\mu\nu}\partial_\mu \partial_\nu \epsilon_\rho &= \eta^{\mu\nu}\eta_{\mu\rho}\partial_\nu f + \eta^{\mu\nu}\eta_{\nu\rho}\partial_\mu f - \eta^{\mu\nu}\eta_{\mu\nu}\partial_\rho f \\ \Leftrightarrow 2\partial^2 \epsilon_\rho &= (2-d)\partial_\rho f \, . \end{aligned} \tag{A.9}$$

Differentiating with ∂_ν and using ∂^2 of (A.5), we end up with:

$$(2-d)\partial_\mu \partial_\nu f = \eta_{\mu\nu}\partial^2 f \rightarrow (d-1)\partial^2 f = 0 \, . \tag{A.10}$$

For the case $d = 1$, we have no restrictions in f and we will study later this particular case. The $d = 2$ is special and we will not study this case in this thesis. For $d > 2$,the above equation implies that $\partial_\mu \partial_\nu$ so f is at most linear in x^μ.

$$f = A + B_\mu x^\mu \, , \tag{A.11}$$

with A, B constants. So at the level of ϵ, this implies that ϵ is at most quadratic in coordinates.

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \, , \tag{A.12}$$

with $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. Plugging this in (A.5) we get:

$$\partial_\mu(a_\nu + b_{\mu\kappa}x^\kappa + c_{\mu\kappa\rho}x^\kappa x^\rho) + \partial_\nu(a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho) = \frac{2}{d}\partial_\lambda \epsilon_\mu \eta^{\mu\lambda}.$$

Comparing the term of the right and left side of the equation we get:

- a_μ has no constraints and corresponds to infinitesimal translations.
- $b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu}$ with $m_{\mu\nu} = -m_{\nu\mu}$. The trace part corresponds to an infinitesimal scale transformation ($\alpha = \frac{1}{d} b^\lambda{}_\lambda$ and $m_{\mu\nu}$ corresponds to an infinitesimal Lorentz rotation.
- $c_{\mu\rho\nu} = \eta_{\mu\rho} b_\nu + \eta_{\mu\nu} b_\rho - \eta_{\nu\rho} b_\mu$ with $b_\mu = \frac{1}{d} c^\kappa{}_{\kappa\mu}$ a constant vector.

The last transformation is called Special Conformal Transformation (SCT) and acts on coordinates as:

$$x^\mu \to x'^\mu = x^\mu + 2(x \cdot b) x^\mu - b^\mu x^2 \,. \tag{A.13}$$

The finite versions of the above transformations are:

- Translations: $x'^\mu = x^\mu + \alpha^\mu$.
- Dilatations: $x'^\mu = a x^\mu$.
- Rotations: $x'^\mu = M^\mu \nu x^\nu$.
- SCT: $x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - b \cdot x + b^2 x^2}$.

If we introduce the inversion transformation I such that:

$$I : x^\mu \to x'^\mu = \frac{x^\mu}{x^2} \,, \tag{A.14}$$

with $I^2 = 1$ we can see that the SCT is equivalent with performing an inversion followed by a translation and then another inversion.

Finally, counting the generators in d dimensions we have: d generators from translations, d generators from dilatations, $\frac{d(d-1)}{2}$ generators from rotations and d generators from SCT. So, we have $\frac{(d+1)(d+2)}{2}$ generators.

A.2 Conformal group

The conformal transformations posses the structure of a group. The composition of conformal transformations yields another conformal transformation and every conformal transformation has an inverse that is a conformal transformation too. We will now construct a representation of conformal generators that act on fields/functions.

Given a conformal transformation $x \to x' = x'(x)$ we define the action on fields $\Phi(x)$ as

$$\Phi(x') \equiv \Phi(x) \,. \tag{A.15}$$

We can always write an infinitesimal coordinate transformation as:

$$x'^{\mu} = x^{\mu} + \omega_{\alpha} \frac{\delta x^{\mu}}{\delta \omega_{\alpha}} \,, \tag{A.16}$$

where ω_{α} is very small. We define the generators G_{α} of such transformation as:

$$\delta_{\omega}\Phi(x) = \Phi'(x) - \Phi(x) = -i\omega_{\alpha} G_{\alpha} \Phi(x) \,. \tag{A.17}$$

Taylor expanding $\Phi'(x)$ we have:

$$\Phi'(x) = \Phi(x - \omega_{\alpha} \frac{\delta x^{\mu}}{\delta \omega_{\alpha}}) \simeq \Phi(x) - \omega_{\alpha} \frac{\delta x^{\mu}}{\delta \omega_{\alpha}} \partial_{\mu} \Phi(x) \,. \tag{A.18}$$

In the end we have:

$$iG_{\alpha}\Phi(x) = \frac{\delta x^{\mu}}{\delta \omega_{\alpha}} \partial_{\mu} \Phi(x) \,. \tag{A.19}$$

We focus now on the conformal transformations. For translations, $x'^{\mu} = x^{\mu} + \omega^{\nu} \delta^{\mu}{}_{\nu}$. Thus, $\frac{\delta x^{\mu}}{\delta \omega_{\alpha}} = \delta^{\mu}{}_{\nu}$. Putting this in the equation (1.19), we find that the the generator for the translations is:

$$P_{\nu} = -i\partial_{\nu} \,. \tag{A.20}$$

An infinitesimal Lorentz transformation can be written as:

$$\begin{aligned} x'^{\mu} &= x^{\mu} + \omega^{\mu}{}_{\nu} x^{\nu} = x^{\mu} + \omega_{\rho\nu} \eta^{\rho\mu} x^{\nu} \\ &= x^{\mu} + \frac{1}{2}(\omega_{\rho\nu} \eta^{\rho\mu} x^{\nu} + \omega_{\nu\rho} \eta^{\nu\mu} x^{\rho}) \,, \end{aligned} \tag{A.21}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Using this relation we find that

$$\frac{\delta x^{\mu}}{\delta \omega_{\rho\nu}} = \frac{1}{2}(\eta^{\rho\mu} x^{\nu} - \eta^{\nu\mu} x^{\rho}).$$

The generator that corresponds to Lorentz transformation is given by the expression:

$$iG_{\rho\nu}\Phi(x) = \frac{1}{2}(\eta^{\rho\mu} x^{\nu} - \eta^{\nu\mu} x^{\rho}) \partial_{\mu} = \frac{1}{2}(x^{\nu} \partial_{\rho} - x^{\rho} \partial_{\nu}) \,. \tag{A.22}$$

Thus, we get the familiar expression:

$$L_{\mu\nu} = i(x_{\mu} \partial_{\nu} - x_{\nu} \partial_{\mu}) \,. \tag{A.23}$$

For a dilatation, we write the infinitesimal transformation as:

$$x'^{\mu} = x^{\mu} + \alpha x^{\mu} = x^{\mu} + \alpha \eta^{\mu\nu} x_{\nu} \,. \tag{A.24}$$

We can easily see that the generator of dilatations is given by:

$$D = -ix^{\mu} \partial_{\mu} \,. \tag{A.25}$$

We summarize the generators of the conformal algebra as differential operators acting on functions:

- Translations: $P_\mu = -i\partial_\mu$.
- Rotations: $L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$.
- Dilatations: $D = -ix^\mu\partial_\mu$.
- Special CT: $K_\mu - i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$.

We can now obtain the conformal algebra:

$$
\begin{aligned}
&[D, P_\mu] = iP_\mu. \\
&[D, K_\mu] = -iK_\mu. \\
&[K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - L_{\mu\nu}). \\
&[L_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu). \\
&[L_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu). \\
&[L_{\mu\nu}, L_{\rho\sigma}] = -i(L_{\mu\rho}\eta_{\nu\sigma} - L_{\mu\sigma}\eta_{\nu\rho} - L_{\nu\rho}\eta_{\mu\sigma} + L_{\nu\sigma}\eta_{\mu\rho}). \\
&[D, L_{\mu\nu}] = 0. \\
&[P_\mu, P_\nu] = 0. \\
&[K_\mu, K_\nu] = 0. \\
&[D, D] = 0
\end{aligned}
$$

We define the above generators:

$$J_{\mu\nu} = L_{\mu\nu}, J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu).$$

$$J_{-1,0} = D, J_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu).$$

and $J_{a,b} = -J_{b,a}$ with $a, b \in \{-1, 0, 1.., d\}$. The above generators satisfy the following algebra:

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac})\,, \tag{A.26}$$

with $\eta_{ab} = \text{diag}(-1, 1, ..., 1)$.

The above relations show the isomorphism between the conformal group in d dimensions and the $SO(d+1, 1)$ which has $\frac{(d+1)(d+2)}{2}$ too. It is important to notice that the Poincare group together with translations form a subgroup of the full conformal group. This means that a theory invariant under rotations, translations and dilatations is not necessarily invariant under the special conformal transformations.

A.2.1 Conformal transformations in $d = 1$ dimensions

In one dimension there is no definition of angles so we are left with the transformation. The only transformations we can have are:

- Translations: $\tau' = \tau + a$
- Dilatations: $\tau' = b\tau$
- Special conformal transformations: $\tau' = a + b \cdot \tau + c \cdot \tau^2$

Moreover, we can see from equation (A.10) we can conclude that in one dimension there is no constrain on f,so we can say that every smooth transformation is conformal. In terms of group theory, $Conf(R^1) \cong Diff(R^1)$.

A.3 Action on operators

Symmetries in quantum field theory are realized as operators acting on the Hilbert space (Schrodinger picture) or on local operators (Heisenberg picture). We follow the second view and we have a multicomponent operator $\phi_\alpha(x)$. In this picture the spacetime dependence is given by:

$$\phi_\alpha(x) = e^{-iPx}\phi_\alpha(0)e^{+iPx}\,. \tag{A.27}$$

We now take the derivative:

$$\begin{aligned} \partial_\mu\phi_\alpha(x) &= e^{-iPx}(-iP_\mu\phi_\alpha(0) + \phi_\alpha(0)iP_\mu)e^{iPx} \\ &= -iP_\mu\phi_\alpha(x) + \phi_\alpha(x)iP_\mu \\ &= -i[P_\mu, \phi_\alpha(x)]\,. \end{aligned} \tag{A.28}$$

From this we obtain the action of the generator on the operator:

$$[P_\mu, \phi_\alpha(x)] = i\partial_\mu \phi_\alpha(x)\,. \tag{A.29}$$

We will find now the the action of the remaining generators on our operator. First, we will focus in the stability group, the group that leaves the origin invariant. In case of the conformal group, it is spanned by Lorentz rotations, dilatations and special conformal transformations. We define the actions of these operators at the origin:

$$[D, \phi_\alpha(0)] = i\Delta\phi_\alpha(0). \tag{A.30}$$

$$[L_{\mu\nu}, \phi_\alpha(0)] = i(S_{\mu\nu})^\beta_\alpha \phi_\beta(0). \tag{A.31}$$

$$[K_\mu, \phi_\alpha(0)] = 0. \tag{A.32}$$

where Δ is the scaling dimension and $S_{\mu\nu}$ is a spin associated matrix which is zero for scalar fields. The above transformations are the definition for a primary operator of scaling dimension Δ. A primary operator is an operator that is annihilated by a special conformal transformation at the origin.

With the use of (A.27) combined with the conformal algebra we are able to derive the action the action of the conformal generators on $\phi_\alpha(x)$. We will also use the Haussdorff formula:

$$e^{-A}Be^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + ...$$

For the dilatation operators we get:

$$\begin{aligned}
[D, \phi_\alpha(x)] &= De^{-iPx}\phi_\alpha(0)e^{iPx} - e^{-iPx}\phi_\alpha(0)e^{iPx}D \\
&= e^{-iPx}e^{iPx}De^{-iPx}\phi_\alpha(0)e^{iPx} - e^{-iPx}\phi_\alpha(0)e^{iPx}De^{-iPx}e^{iPx} \\
&= e^{-iPx}\hat{D}\phi_\alpha(0)e^{iPx} - e^{-iPx}\phi_\alpha(0)\hat{D}e^{iPx} \\
&= e^{-iPx}[\hat{D}, \phi_\alpha(0)]e^{iPx}\,,
\end{aligned} \tag{A.33}$$

where we have defined $\hat{D} = e^{iPx}De^{-iPx}$. Using the Haussdorff formula we find:

$$\hat{D} = D + ix^\mu[P_\mu, D] + ... = D + x^\mu P_\mu\,. \tag{A.34}$$

Thus, we obtain:

$$[D, \phi_\alpha(x)] = i(\Delta + x^\mu\partial_\mu)\phi_\alpha(x)\,. \tag{A.35}$$

Moving to the generator of rotations we get:

$$[L_{\mu\nu}, \phi_\alpha(x)] = e^{-iPx}[\hat{L_{\mu\nu}}, \phi_\alpha(0)]e^{iPx}\,. \tag{A.36}$$

Again, using the Haussdorff formula and the conformal algebra

$$\begin{aligned} \hat{L_{\mu\nu}} &= e^{iPx} L_{\mu\nu} e^{-iPx} = L_{\mu\nu} + [L_{\mu\nu}, -iP_\rho x^\rho] \\ &= L_{\mu\nu} - (x^\rho \eta_{\mu\rho} P_\nu - x^\rho \eta_{\nu\rho} P_\mu) \, . \end{aligned} \tag{A.37}$$

Finally, we get:

$$[L_{\mu\nu}, \phi_\alpha(x)] = -i(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi_\alpha(x) + i(S_{\mu\nu})_{\alpha\beta}\phi_\beta(x) \, . \tag{A.38}$$

Following the same procedure for the generator of special conformal transformations we find:

$$[K_\mu, \phi_\alpha(x)] = 2ix_\mu \Delta \phi_\alpha(x) + i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)\phi_\alpha(x) + 2ix^\rho (S_{\rho\mu})_{\alpha\beta}\phi_\beta(x) \, . \tag{A.39}$$

For a primary scalar field $\Phi(x)$ of scaling dimension Δ we find

$$\Phi(x') \to \Phi'(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\Delta/d} \Phi(x) \, . \tag{A.40}$$

The Jacobian of the transformation is given by:

$$\left| \frac{\partial x'}{\partial x} \right| = \Lambda(x)^{-d/2} \, . \tag{A.41}$$

Thus, we have:

$$\Phi'(x') = \Lambda(x)^{\Delta/2} \Phi(x) \, . \tag{A.42}$$

A.4 Consequences of conformal invariance

A.4.1 Classical symmetries in quantum field theory

In this section we will study continuous transformations in quantum field theory. We consider a general action:

$$S = \int d^d x \mathcal{L}(\phi, \partial_\mu \phi) \, . \tag{A.43}$$

Under a general transformation,

$$\begin{aligned} & x \to x' = x'(x). \\ & \phi \to \phi'(x') = \mathcal{F}(\phi(x)). \end{aligned}$$

the action becomes:

$$\begin{aligned} S[\phi'] &= \int d^d x \mathcal{L}(\phi', \partial_\mu \phi') \\ &= \int d^d x' \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x')) \\ &= \int d^d x' \mathcal{L}(\mathcal{F}(\phi(x)), \partial'_\mu \mathcal{F}(\phi(x))) \\ &= \int d^d x \left| \frac{\partial x'}{\partial x} \right| \mathcal{L}(\mathcal{F}(\phi(x)), \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \mathcal{F}(\phi(x))) \, . \end{aligned} \tag{A.44}$$

Now we must specify the conditions under which the actions is invariant $S[\phi'] = S[\phi]$ for a given transformation. Then we can say that the transformation is a symmetry of the theory at classical level. For start, we consider translations:

$$
\begin{aligned}
x'^{\mu} &= x^{\mu} + a^{\mu}. \\
\phi'(x+a) &= \phi(x)
\end{aligned}
$$

For those transformations, we have $\frac{\partial x^{\nu}}{\partial x'^{\mu}} = \delta^{\mu}_{\nu}$, $J = 1$, $\mathcal{F} = I_d$. It is trivial to see that the action is invariant unless it depends explicitly on x.

Next, we consider Lorentz transformations:

$$
x^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}. \tag{A.45}
$$

$$
\phi'(\Lambda x) = L_{\Lambda}\phi(x). \tag{A.46}
$$

where $\Lambda^{\mu}{}_{\nu}$ satisfy $\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$ and the matrices L_{Λ} form a representation of the Lorentz group. The Jacobian of these transformations is $J = \left|\frac{\partial x}{\partial x'}\right| = 1$.

Thus, the action becomes:

$$
S[\phi'] = \int d^d x \mathcal{L}(L_{\Lambda}\phi, \Lambda^{-1} \cdot \partial(L_{\Lambda}\phi)) \,. \tag{A.47}
$$

Suppose ϕ is a scalar field. Then $L_{\Lambda} = 1$ and the action is invariant provided that the derivatives ∂_{μ} are properly contracted.

Now, we analyse scale transformations

$$
\begin{aligned}
x' &= \lambda x \to J = \left|\frac{\partial x}{\partial x'}\right| = \lambda^{-d}. \\
\phi'(\lambda x) &= \lambda^{-\Delta}\phi(x).
\end{aligned}
$$

The transformed action becomes:

$$
S[\phi'] = \lambda^{d} \int d^d x \mathcal{L}(\lambda^{-\Delta}\phi, \lambda^{-1-\Delta}\partial_{\mu}\phi) \,. \tag{A.48}
$$

For example, the action of a free scalar field is

$$
S[\phi] = \int d^d x \partial_{\mu}\phi\partial^{\mu}\phi \,. \tag{A.49}
$$

It transforms as:

$$
\begin{aligned}
S[\phi'] &= \lambda^{d} \int d^d x (\lambda^{-1-\Delta}\partial_{\mu}\phi)(\lambda^{-1-\Delta}\partial^{\mu}\phi) \\
&= \lambda^{d-2-2\Delta} S[\phi] \,.
\end{aligned} \tag{A.50}
$$

For the theory to be invariant under the scale transformation, $\Delta = \frac{d}{2} - 1$. We can also add an interaction of the form

$$S[\phi]_{int} = \int d^d x \phi^n,$$

but for the theory to be invariant we must have $n = \frac{d}{\Delta} = \frac{2d}{d-2}$. So we can add a ϕ^3 interaction in 6 dimensions or a ϕ^4 interaction in four dimensions.

A.4.2 Implications for the stress-energy tensor

For our discussion, we will consider infinitesimal transformations:

$$x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a}, \tag{A.51}$$

$$\phi'(x') = \phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}. \tag{A.52}$$

We consider now the change of the action under these transformations. The Jacobian changes as follows:

$$\frac{\partial x'^\nu}{\partial x^\mu} = \delta^\nu_\mu + \partial_\mu \omega_a \frac{\delta x^\nu}{\delta \omega_a}. \tag{A.53}$$

Using the identity $\det(1 + A) \approx 1 + \mathrm{Tr} A$, we obtain:

$$\left| \frac{\partial \mathbf{x}'}{\partial \mathbf{x}} \right| = 1 + \partial_\mu \omega_a \frac{\delta x^\mu}{\delta \omega_a}. \tag{A.54}$$

Thus, the transformed action becomes:

$$\begin{aligned} S' = &\int d^d x \Big(1 + \partial_\mu \omega_a \frac{\delta x^\mu}{\delta \omega_a}\Big) \\ &\times \mathcal{L}\Big(\phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}, [\delta^\nu_\mu + \partial_\mu \omega_a \frac{\delta x^\nu}{\delta \omega_a}](\partial_\nu \phi(x) + \partial_\nu \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a})\Big). \end{aligned} \tag{A.55}$$

We can now expand the Lagrangian and keep only terms of first derivatives of ω_a. We get:

$$\delta S = -\int d^d x j^\mu_a \omega_a, \tag{A.56}$$

with the current associated to the infinitesimal transformation:

$$j^\mu_a = \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}\Big) \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \mathcal{F}}{\delta \omega_a}. \tag{A.57}$$

The Noether theorem states that to every continuous symmetry of the action, we may associate a current that is classically conserved. Assuming now that the transformation (A.51) is symmetry of the action, that means that it leaves the action invariant, and the fields satisfy the equations of motion then $\delta S = 0$ for every variation ω_a. We integrate by parts (A.56) and arrive at the conservation equation:

$$\partial_\mu j^\mu_a = 0. \tag{A.58}$$

We can then define a conserved charges:

$$Q_a = \int d^{d-1}x j^0_a. \tag{A.59}$$

Then,

$$\partial_t Q_a = \int d^{d-1}\partial_t x j^0_a = \int d^{d-1}\partial_i x j^i_a = 0. \tag{A.60}$$

In the last step, we have assumed that the fields and thus j^i_a vanish sufficiently fast at infinity. We can always redefine the given current as

$$j^\mu_a \to j^\mu_a + \partial_\nu B^{\mu\nu}, \tag{A.61}$$

with $B^{\mu\nu} = -B^{\nu\mu}$. It is is to see that the redefined current is also conserved as:

$$\partial_\mu j^\mu_a + \partial_\mu \partial_\nu B^{\mu\nu} = \partial_\mu j^\mu_a = 0, \tag{A.62}$$

as the term $\partial_\mu \partial_\nu B^{\mu\nu}$ is the product of derivatives, which are symmetric under $\mu \leftrightarrow \nu$, with the antisymmetric $B^{\mu\nu}$ and consequently it vanishes.

Energy-momentum tensor

Now we consider an infinitesimal translation $x^\mu \to x^\mu + \epsilon^\mu$. Then we have

$$\frac{\delta x^\mu}{\delta x^\nu} = \delta^\mu_\nu, \qquad \frac{\delta \mathcal{F}}{\delta \epsilon^\nu} = 0.$$

Using these relations, we get:

$$T^{\mu\nu}_c = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \eta^{\mu\nu} \mathcal{L}, \tag{A.63}$$

where $T^{\mu\nu}_c$ is the canonical energy momentum tensor. The conservation law becomes:

$$\partial_\mu T^{\mu\nu}_c = 0. \tag{A.64}$$

The conserved charge is the momentum:

$$P^\nu = \int d^{d-1}x T^{0\nu}_c. \tag{A.65}$$

For example, the energy is:

$$P^0 = \int d^{d-1}x T^{00}_c = \int d^{d-1}x \Big(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} \Big) = \int d^{d-1}x \mathcal{H}. \tag{A.66}$$

It is found out that when the theory is Poincare invariant, we redefine a new symmetric tensor.

$$T_c^{\mu\nu} \to T^{\mu\nu}, \tag{A.67}$$

with $T^{\mu\nu} = T^{\nu\mu}$. This new tensor is called the Belifante tensor. It is based on the freedom we have to redefine the energy momentum tensor as:

$$T^{\mu\nu} = T_c^{\mu\nu} + \partial_\rho B^{\rho\mu\nu}, \tag{A.68}$$

with $B^{\rho\mu\nu} = -B^{\mu\rho\nu}$. Of course this redefinition does not violate the conservation law. The tensor $B^{\rho\mu\nu}$ is then constructed in such a way that the antisymmetric part of the redefined $T^{\mu\nu}$ vanishes. For more details, see [27].

We consider now an infinitesimal diffeomorphism:

$$x'^\mu = x^\mu + \epsilon^\mu(x). \tag{A.69}$$

From (A.56), we have:

$$\delta S = -\int d^dx T^{\mu\nu}\partial_\mu\epsilon_\nu = -\frac{1}{2}\int d^dx T^{\mu\nu}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu). \tag{A.70}$$

Conformal invariance and stress-energy tensor

As we will now see the scale and the conformal symmetry has important implications for the stress-energy tensor. Suppose we have translation and Poincare invariance and we impose now conformal invariance. We have shown that for an infinitesimal conformal transformation:

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \eta_{\mu\nu} f(x). \tag{A.71}$$

Thus, equation (A.70) becomes:

$$\delta S = -\int d^dx T^\mu_\mu f(x). \tag{A.72}$$

For a scale invariant action, we have $\delta S = 0$ and $f(x) = a$. Thus, for scale invariant theories the stress-energy tensor is traceless, $T^\mu_\mu = 0$. For special conformal transformation, the function $f(x)$ is not arbitrary. But if the stress-energy tensor is traceless, $T^\mu_\mu = 0$ then we have once again $\delta S = 0$. Then, the theory is conformally invariant.

A conformal field theory must have a conserved and symmetric stress energy tensor $(\partial_\mu T^{\mu\nu} = 0, T^{\mu\nu} = T^{\nu\mu})$ that is also traceless$T^\mu_\mu = 0$.

The free boson

As an example of the above, we will study the Euclidean action for a free boson. The action is

$$S[\phi] = \frac{1}{2} \int d^d x \partial_\mu \phi \partial^\mu \phi. \tag{A.73}$$

From equation (A.63), we obtain the stress-energy tensor of the theory:

$$T^{\mu\nu} = -\frac{1}{2} \eta^{\mu\nu} \partial_\kappa \phi \partial^\kappa \phi + \partial^\nu \phi \partial^\mu \phi. \tag{A.74}$$

Of course, the tensor is already symmetric. It also conserved when we use the e.o.m $\partial_\mu \partial^\mu \phi = 0$. The trace is:

$$\begin{aligned} T^\mu_\mu = \eta^{\mu\nu} T^{\mu\nu} &= -\frac{1}{2} d \partial_\kappa \phi \partial^\kappa \phi + \partial_\kappa \phi \partial^\kappa \phi \\ &= -\frac{1}{2}(d-2) \partial_\kappa \phi \partial^\kappa \phi. \end{aligned} \tag{A.75}$$

We see that in $d = 2$, the trace vanishes and thus the theory is conformally invariant.

A.4.3 Quantum conformal symmetry: implications for correlators

As we have seen when an action is invariant under some symmetry that means that the theory has a classical symmetry. Now we will study the quantum theory. The natural object that we are interested in are the correlation functions:

$$\langle \phi(x_1) ... \phi(x_n) \rangle = \frac{1}{Z} \int [d\phi] \phi(x_1) ... \phi(x_n) e^{-S[\phi]}. \tag{A.76}$$

We will assume that the action and the integration measure are invariant. The fact that the integration measure is invariant is highly non-trivial for scale transformation because QFT come with a scale (UV-cutoff).

Provided the action has a symmetry we can show that:

$$\langle \phi(x_1') ... \phi(x_n') \rangle = \langle \phi'(x_1') ... \phi'(x_n') \rangle = \langle \mathcal{F}(\phi(x_1)) ... \mathcal{F}(\phi(x_n)) \rangle. \tag{A.77}$$

Indeed under a transformation

$$\begin{aligned} \int [d\phi] \phi(x_1') ... \phi(x_n') e^{-S[\phi]} &= \int [d\phi'] \phi'(x_1) ... \phi'(x_n) e^{-S[\phi']} \\ &= \int [d\phi] \mathcal{F}(\phi(x_1)) ... \mathcal{F}(\phi(x_n)) e^{-S[\phi]}. \end{aligned} \tag{A.78}$$

In the last line we have assumed the invariance of the action and the integration measure.

For example, for translations we get:

$$\langle \phi(x_1+\vec{a})...\phi(x_2+\vec{a})\rangle = \langle \phi(x_1)...\phi(x_n)\rangle \,, \tag{A.79}$$

and we see that the correlators, in a theory with translational invariance, are only functions of relative positions. Now we consider Lorentz transformations acting on scalar operators. From equation (A.77) we have

$$\langle \phi(\Lambda^{\mu}{}_{\nu}x_1^{\nu})...\phi(\Lambda^{\mu}{}_{\nu}x_n^{\nu})\rangle = \langle \phi(x_1)...\phi(x_n)\rangle \,. \tag{A.80}$$

A.4.4 Conformal invariance constraint on correlators

We assume now that our theory , apart from the Poincare invariance, has also full conformal invariance. We will focus on correlators of primary scalar fields. Under a conformal transformation $x \to x'$ equation (A.77) implies

$$\langle \phi(x_1')...\phi(x_n')\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{-\Delta_1/d} .... \left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{-\Delta_n/d} \langle \phi(x_1)...\phi(x_n)\rangle \,, \tag{A.81}$$

or equivalently

$$\langle \phi(x_1)...\phi(x_n)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d} .... \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\Delta_n/d} \langle \phi(x_1)...\phi(x_n)\rangle \,. \tag{A.82}$$

We will now consider the implications of these results in two and four-point functions.

Two-point function

Taking under consideration Poincare invariance (translations and rotations) the two-point function is

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1-x_2|) \,. \tag{A.83}$$

Scale transformations $x \to x' = \lambda x$ imply

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle \,. \tag{A.84}$$

We can deduce that $f(x) = \lambda^{\Delta_1+\Delta_2} f(\lambda x)$. Then the two-point function becomes:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}} \,. \tag{A.85}$$

Now we consider special conformal transformations. They satisfy

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1-2b\cdot x+b^2x^2)^d} \,. \tag{A.86}$$

Introducing $\gamma_i = 1 - 2b \cdot x_i + b^2 x_i^2$ we have,

$$\left|x'_i - x'_j\right| = \frac{|x_i - x_j|}{\gamma_i^{1/2}\gamma_j^{1/2}}. \tag{A.87}$$

Then special conformal transformations imply that:

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \frac{C_{12}}{|x'_1 - x'_2|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \frac{(\gamma_1\gamma_2)^{\frac{\Delta_1+\Delta_2}{2}}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}$$

Since γ_1 and γ_2 are independent, the above equation can be satisfied only if $\Delta_1 = \Delta_2$. That means that primary fields are correlated only when they have the same scaling dimension Δ. Introducing the notation $x_{12} = x_1 - x_2$, we arrive at a very important result:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \begin{cases} 0 & \text{if } \Delta_1 \neq \Delta_2 \\ \frac{C_{12}}{|x_{12}|^{2\Delta}} & \text{if } \Delta_1 = \Delta_2 = \Delta \end{cases}$$

Three-point function

We consider three-point functions. In a theory with Poinacare invariance they can be written:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\rangle = \frac{C_{123}}{|x_{12}|^a |x_{23}|^b |x_{13}|^c}. \tag{A.88}$$

Following the same arguments as before, we can see that scale invariance now implies:

$$a + b + c = \Delta_1 + \Delta_2 + \Delta_3. \tag{A.89}$$

This condition does not completely fix a, b, c and we must impose invariance under special conformal transformations too. The requirement for invariance under these transformations is:

$$\frac{C_{123}}{|x_{12}|^a |x_{23}|^b |x_{13}|^c} = \frac{(\gamma_1\gamma_2)^{a/2}(\gamma_2\gamma_3)^{b/2}(\gamma_1\gamma_3)^{c/2}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_3}\gamma_3^{\Delta_3}} \frac{C_{123}}{|x_{12}|^a |x_{23}|^b |x_{13}|^c}. \tag{A.90}$$

Hence, we get:

$$a + c = 2\Delta_1, \tag{A.91}$$
$$a + b = 2\Delta_2, \tag{A.92}$$
$$b + c = 2\Delta_3. \tag{A.93}$$

The solutions for this system are:

$$a = \Delta_1 + \Delta_2 - \Delta_3, \tag{A.94}$$
$$b = \Delta_2 + \Delta_3 - \Delta_1, \tag{A.95}$$
$$c = \Delta_3 + \Delta_1 - \Delta_2. \tag{A.96}$$

Finally, the three-point function is:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{13}|^{\Delta_3+\Delta_1-\Delta_2}}. \quad \text{(A.97)}$$

As we have no more freedom to normalize C_{ijk}, after normalizing $C_{ij} = 1$, we can deduce that C_{ijk} has non-trivial physical meaning.

Four-point function

We move now to the four-point functions which play a crucial role in the model we study. As we have seen

$$x'_{ij} = \frac{x_{ij}^2}{\gamma_i\gamma_j}\,. \quad \text{(A.98)}$$

For four points x_1, x_2, x_3, x_4 we can construct the following cross ratios that are left invariant under conformal transformations:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \quad , \quad \upsilon = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \quad \text{(A.99)}$$

That said:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{G(u,\upsilon)}{\prod_{i<j}\left|x_{ij}^2\right|^{\delta_{ij}}}\,, \quad \text{(A.100)}$$

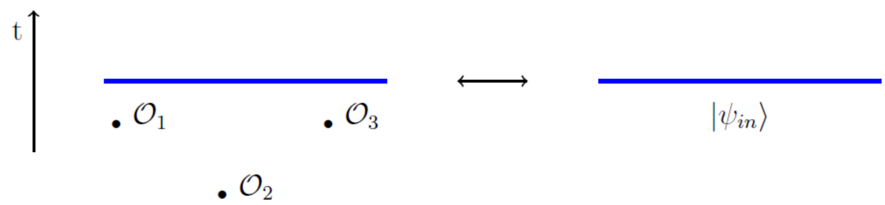
with $\sum_{i\neq j}\delta_{ij} = \Delta_i$,which is the constrain due to scale transformations. $F(u,\upsilon)$ can be any function of the cross ratios.

A.5 Radial quantization and the OPE

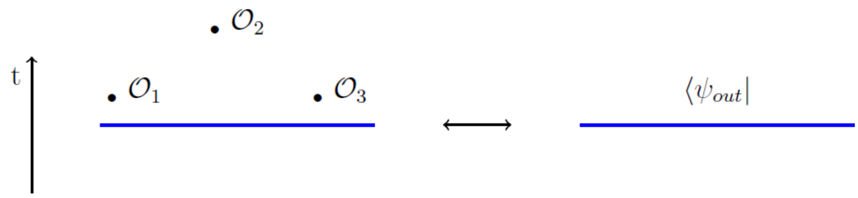
A.5.1 Radial quantization

In this section we will study a parallel view of the correlation functions, from the point of view of Hilbert space and quantum mechanical evolution. In quantum field theory we can foliate the Minkowskian space-time by surfaces of equal time. Then, the space-time is composed by the union of infinite equal time surfaces.

The in states of a Hilbert space can be created by inserting operators in the past of the surface:



The out states can be created by inserting operators in the future of the surface:



The correlator among these operators is given by the following inner product

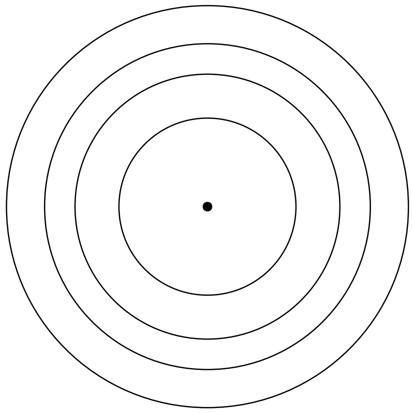
$$\langle\psi_{out}|\psi_{in}\rangle = (\langle 0|O_1O_3O_2)|(O_2O_1O_3|0\rangle).$$

These states live in a different leaf (different time), so we need to evolve them with the operator $U = e^{-iP_0\Delta t}$ and the correlator is:

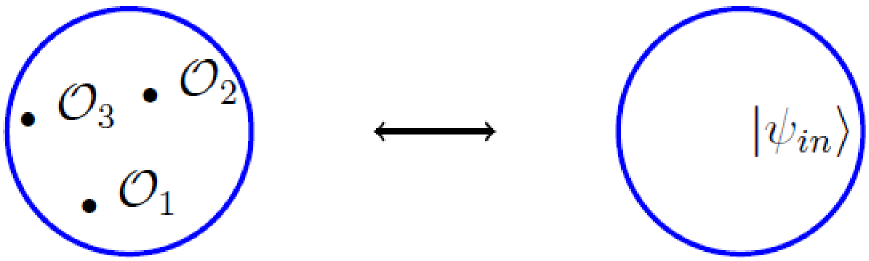
$$\langle\psi_{out}|U|\psi_{in}\rangle.$$

The operator P_0 commutes with the generators P^μ so we can characterize the states living on the surfaces by their momenta $P^\mu|k\rangle = k^\mu|k\rangle$.

We now consider a conformal field theory in Euclidean space. It is more convenient to foliate the space by spheres S^{d-1} with the origin at the center.



Now the in and out states are created by inserting operators inside and out side of the sphere. For example:



The operator that translates from one sphere to another one of different radius is the dilatation operator, $U = e^{i\Delta t}$, with $t = \log r$. To justify this, lets look at a metric in $\mathbb{R}^d$ in spherical coordinates. We have:

$$ds^2 = dr^2 + r^2 d\Omega_{d-1} = r^2[\frac{dr^2}{r^2} + \Omega_{d-1}]. \tag{A.101}$$

If we set $t = \log r$, we get:

$$\frac{dr^2}{r^2} + \Omega_{d-1} = dt^2 + \Omega_{d-1}. \tag{A.102}$$

This is a metric on $\mathbb{R} \times S^{d-1}$. Consider now that we are studying a CFT on $\mathbb{R}^d$. Under such rescale the metric should be invariant. Thus, studying a CFT in $\mathbb{R} \times S^{d-1}$ is equivalent. This map takes circles of constant radius in $\mathbb{R}^d$ to constant t slices on $\mathbb{R} \times S^{d-1}$. Thus, the dilatation operator in $\mathbb{R}^d$, which maps circles to circles with different radius, corresponds to time translations on $\mathbb{R} \times S^{d-1}$. It acts just like a Hamiltonian. This discussion justifies the argument we made above.

In the sphere, states are now classified by their scaling dimension and their $SO(d)$ spin. We have:

$$D\,|\Delta\rangle = i\Delta\,|\Delta\rangle\,. \tag{A.103}$$

$$L_{\mu,\nu}\,|\Delta, l\rangle_a = i(S_{\mu,\nu})_a^b\,|\Delta, l\rangle_b\,. \tag{A.104}$$

A.5.2 State/operator correspondence

We haven seen that by inserting operators inside the sphere, we generate states that live on the sphere. To see how this works, we are going to give some examples:

- First, the vacuum state $|0\rangle$ corresponds to no insertion and has zero dilatation eigenvalue. Moreover, it is annihilated by all operators K, L, P.
- Suppose we insert a primary operator $\phi_\Delta(0)$ at the origin. We get a state $|\Delta\rangle = \phi_\Delta(0)\,|0\rangle$. We will now find its eigenvalue of the dilatation operator. We expect it to be Δ. We have:

$$\begin{aligned} D\,|\Delta\rangle = D\phi_\Delta(0)\,|0\rangle &= [D, \phi_\Delta(0)]\,|0\rangle + \phi_\Delta(0)D\,|0\rangle \\ &= [D, \phi_\Delta(0)]\,|0\rangle = i\Delta\,|\Delta\rangle\,, \end{aligned} \tag{A.105}$$

  as it was expected. Moreover, as we inserted a primary operator, we expect to have created a primary state. That means that the state should be annihilated by $\hat{K}$. Let's see:

$$K\,|\Delta\rangle = K\phi_\Delta(0)\,|0\rangle = [K, \phi_\Delta(0)]\,|0\rangle + \phi_\Delta(0)K\,|0\rangle = 0\,|\Delta\rangle\,. \tag{A.106}$$

  Indeed it is.
- Suppose we insert a primary operator $\phi_\Delta(x)$ but not at the origin. We then get the state $|\psi\rangle = \phi_\Delta(x)\,|0\rangle$. We will show that this state in not an eigenvalues of the dilatation operator. We have:

$$\begin{aligned} |\psi\rangle &= \phi_\Delta(x)\,|0\rangle = e^{-iPx}\phi_\Delta(0)e^{+iPx}\,|0\rangle = e^{-iPx}\phi_\Delta(0)(1 + ixP + ...)\,|0\rangle \\ &= e^{-iPx}\phi_\Delta(0)\,|0\rangle = e^{-iPx}\,|\Delta\rangle = \sum_n \frac{1}{n!}(-iPx)^n\,|\Delta\rangle = |\Delta\rangle - ixP\,|\Delta\rangle + ... \end{aligned} \tag{A.107}$$

  Indeed, the state we created is a superposition of states with different scaling dimensions/energies. Moreover, we will show that the eigenvalues of $P_\mu\,|\Delta\rangle$ under the dilatation operator is $|\Delta + 1\rangle$. That bares a close resemblance to the quantum harmonic oscillator and the creation operator. Let's justify it:

$$D(P_\mu\,|\Delta\rangle) = ([D, P_\mu] + P_\mu D)\,|\Delta\rangle = (iP_\mu + i\Delta P_\mu)\,|\Delta\rangle = i(\Delta + 1)\,|\Delta\rangle\,. \tag{A.108}$$

Schematically this can be illustrated as:

$$|\Delta\rangle \xrightarrow{P} |\Delta+1\rangle \xrightarrow{P} |\Delta+2\rangle \ldots \tag{A.109}$$

It is also easy to show that the state $P_\mu |\Delta\rangle$ is not a primary state since it is not annihilated by K_μ. We have:

$$K_\nu(P_\mu |\Delta\rangle) = [K_\nu, P_\mu] |\Delta\rangle + P_\mu K_\nu |\Delta\rangle = [K_\nu, P_\mu] |\Delta\rangle \neq 0 \tag{A.110}$$

We can also check that while P_μ raises the dimension, K_μ lowers it. We have:

$$\begin{aligned} D(K_\mu |\Delta+1\rangle) &= [D, K_\mu] |\Delta+1\rangle + K_\mu D |\Delta+1\rangle \\ &= (-i + i(\Delta+1)) K_\mu |\Delta+1\rangle = i\Delta(K_\mu |\Delta+1\rangle). \end{aligned} \tag{A.111}$$

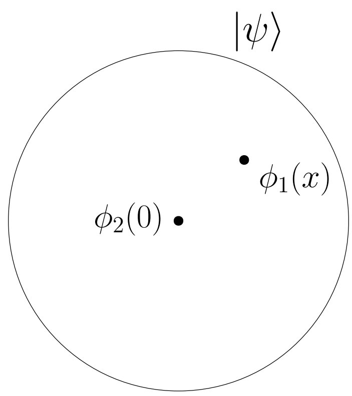
We showed that the state $K_\mu |\Delta+1\rangle$ has dimension Δ. That means that the operator K_μ lowers the dimensions. This is similar to the annihilation operator in quantum harmonic oscillator. This fact allows us to justify the existence of a primary state/operators as an axiom since the dimensions should be bounded from below.

Through these examples we can justify the state/operators correspondence. It says that a state that has dimensions Δ and it is annihilated by K_μ corresponds to the insertion of a local primary operator at the origin. Furthermore, each eigenstate of the dilatation operator is either a primary or a descendant or even a linear combination of those.

A.5.3 The OPE in CFT

The operator product expansion (OPE) is used in QFT to write a product of two operators that are close to each other, as a product of local operators at the middle point. In CFTs we will see that the OPE acquires powerful properties thanks to the radial quantization.

Let's consider the insertion of two operators inside a sphere:



They generate the state:

$$|\psi\rangle = \phi_1(x)\phi_2(0)\,|0\rangle\,. \tag{A.112}$$

Then we can expand this state in a basis of eigenstates of the dilatation operator $|E_n\rangle$. Thus,

$$|\psi\rangle = \sum_n c_n(x)\,|E_n\rangle \tag{A.113}$$

But as we have seen through the state/operator correspondence, each $|E_n\rangle$ is linear combination of primaries and their derivatives/descendants. We can write:

$$\phi_1(x)\phi_2(0)\,|0\rangle = \sum_{primaries} C_\Delta(x,\partial)\phi_\Delta(0)\,|0\rangle\,. \tag{A.114}$$

This expression has algebraic origin since we have expanded a state in a complete basis. Practically, this means that in contrary with QFT the operators don't have to be close, just inside the sphere. Now, we analyse the functions $C_\Delta(x,\partial)$. For simplicity, we consider only one primary field $\phi_\Delta(0)$.

$$\phi_1(x)\phi_2(0)\,|0\rangle = \frac{\text{const.}}{|x|^k}(\phi_\Delta(0)+...)\,|0\rangle\,, \tag{A.115}$$

where the dots stand for descendants and other primaries. We act with the dilatation on the L.H.S of (A.115). We get:

$$\begin{aligned} D\phi_1(x)\phi_2(0)\,|0\rangle &= i(\Delta + x^\mu\partial_\mu\phi_1(x)\phi_2(0)\,|0\rangle + i\Delta_2\phi_1(x)\phi_2(0)\,|0\rangle \\ &= i(\Delta_1+\Delta_2-k)\frac{\text{const.}}{|x|^k}(\phi_\Delta(0)+...)\,|0\rangle\,, \end{aligned} \tag{A.116}$$

where in the last line we have used the R.H.S of (A.115). Now, we act with the dilatation operator on the R.H.S of (A.115).

$$D\frac{\text{const.}}{|x|^k}(\phi_\Delta(0)+...)\,|0\rangle = i\Delta\frac{\text{const.}}{|x|^k}(\phi_\Delta(0)+...)\,|0\rangle\,. \tag{A.117}$$

Comparing the previous equations we arrive at:

$$k = \Delta_1+\Delta_2-\Delta. \tag{A.118}$$

Next, we focus on the descendant contribution, that is the next order term of $C_\Delta(x,\partial)$. We have:

$$\phi_1(x)\phi_2(0)\,|0\rangle = \frac{\text{const.}}{|x|^{\Delta_1+\Delta_2-\Delta}}(\phi_\Delta(0)+cx^\mu\partial_\mu\phi_\Delta(0)+...)\,|0\rangle\,. \tag{A.119}$$

As before we use the conformal symmetry to fix the constant c. Now, we will act with K_μ on both sides of (A.119). This will allow us to use the definition of the primary operator, that is the annihilation by K_μ. Moreover, we will use the transformation rule for a scalar field:

$$[K_\mu, \phi_\alpha(x)] = 2ix_\mu \Delta \phi_\alpha(x) + i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)\phi_\alpha(x). \tag{A.120}$$

Acting on the LHS of (A.119). We get:

$$\begin{aligned} K_\mu \phi_1(x)\phi_2(0) |0\rangle &= 2ix_\mu \Delta_1 \phi_\alpha(x) + i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu))\phi_1(x)\phi_2(0) |0\rangle \\ &= ix_\mu(\Delta_1 + \Delta - \Delta_2)\frac{\text{const.}}{|x|^{\Delta_1+\Delta_2-\Delta}}(\phi_\Delta(0) + ...) |0\rangle . \end{aligned} \tag{A.121}$$

Acting on the RHS of (A.119) and using $[K_\mu, P_\nu]\phi_\Delta(0) = K_\mu P_\nu \phi_\Delta(0) = 2i\eta_{\mu\nu}\Delta\phi_\Delta(0)$ and $-i[P_\mu, \phi_\alpha(x)] = \partial_\mu \phi_\alpha(x)$, we arrive at:

$$\begin{aligned} &K_\mu \left(\frac{\text{const.}}{|x|^{\Delta_1+\Delta_2-\Delta}}(\phi_\Delta(0) + cx^\mu \partial_\mu \phi_\Delta(0) + ...) |0\rangle \right) \\ &= \frac{\text{const.}}{|x|^{\Delta_1+\Delta_2-\Delta}}(2ic\Delta x_\mu \phi_\Delta(0) + ...). \end{aligned} \tag{A.122}$$

Comparing again the two sides of the equation, we fix the constant c:

$$c = \frac{\Delta_1 + \Delta_2 - \Delta}{2\Delta}. \tag{A.123}$$

Once again, the conformal invariance fixes the constant. Following this procedure, we can deduce that conformal invariance fully fixes the function $C_\Delta(x, \partial)$, up to an overall factor $\mathcal{C}_{12\Delta}$. It is important to notice that the function $C_\Delta(x, \partial)$ has dependence only on the scaling dimensions of the inserted fields and the dimension of the primary.

We consider now a three-point function of primaries and we take the OPE of the first two operators. We have:

$$\langle \phi_1(x)\phi_2(0)\phi_\Delta(z)\rangle = \sum_{\text{primaries}\Delta'} C_{12\Delta'} C_{\Delta'} \langle \phi_{\Delta'}(y)\phi_\Delta\rangle|_{y=0}. \tag{A.124}$$

Considering the two-point function of the above equation, we have seen that for the two primaries to be correlated they must have the same dimensions. Moreover, we assume we have contribution from only one primary. Thus, it must have dimensions Δ. The three-point function then, becomes:

$$\langle \phi_1(x)\phi_2(0)\phi_\Delta(z)\rangle = C_{12\Delta} C_\Delta \langle \phi_\Delta(y)\phi_\Delta\rangle|_{y=0}. \tag{A.125}$$

In the previous section, we have found the expressions for a conformal invariant two and three-point function. Thus we can substitute in the above expression. To determine $C_\Delta(x, \partial)$, we have to expands in powers of x the three-point function on the L.H.S of A.125. The coefficients $C_{12\Delta}$ are the same that appear in (A.97) and are called OPE coefficients.

A.5.4 Conformal blocks

Equipped with the function $C_\Delta(x, \partial)$, with successive application of the OPE, we can fix the building blocks of any correlation function. Let's see how this is done for the four-point function. We apply the OPE two times:

$$
\begin{aligned}
\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle &= \sum_\Delta c_{12\Delta} C_\Delta(x_{12}, \partial_y)\langle \phi_\Delta(y)\phi(x_3)\phi(x_4)\rangle \\
&= \sum_\Delta c_{12\Delta} c_{34\Delta}\Big[C_\Delta(x_{12}, \partial_y) C_\Delta(x_{34}, \partial_z)\langle \phi_\Delta(y)\phi_\Delta(z)\rangle\Big].
\end{aligned} \tag{A.126}
$$

Since, $C_\Delta(x_{12}, \partial_y), C_\Delta(x_{34}, \partial_z)$ together with the two-point function are fixed by conformal invariance, the whole quantity in the brackets is fixed. Recalling, (A.100), we can define:

$$
\Big[C_\Delta(x_{12}, \partial_y) C_\Delta(x_{34}, \partial_z)\langle \phi_\Delta(y)\phi_\Delta(z)\rangle\Big] = \frac{G_{\Delta,l}(u,v)}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}} = \mathcal{F}^\Delta_{1234}. \tag{A.127}
$$

The functions $G_{\Delta,l}(u,v)$ are called conformal blocks and they only depend on the dimensions of the primaries, their spin l and the dimension Δ of the operator appearing during the OPE. In the end, the four-point function can be written as:

$$
\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_\Delta c_{12\Delta} c_{34\Delta} \mathcal{F}^\Delta_{1234}. \tag{A.128}
$$

We have arrived at a remarkable conclusion.

In a conformal field theory, the dimensions of the primaries along with the OPE coefficients and the structure of the OPE is enough to write any correlation function.

Appendix B

The kernel as a function of cross ratios

As we have seen, the kernel gives the $(n+1)$-ladder diagram in terms of the n-ladder diagram.

$$\frac{\text{sgn}(\tau_{12})\text{sgn}(\tau_{34})}{|\tau_{12}|^{2\Delta}|\tau_{34}|^{2\Delta}}\mathcal{F}_{n+1}(\chi) = -\frac{1}{\alpha_0}\int d\tau_\alpha d\tau_b \frac{\text{sgn}(\tau_{1\alpha})\text{sgn}(\tau_{2b})}{|\tau_{2b}|^{2\Delta}|\tau_{1\alpha}|^{2\Delta}|\tau_{\alpha b}|^{2-4\Delta}} \cdot \frac{\text{sgn}(\tau_{\alpha b})\text{sgn}(\tau_{34})}{|\tau_{\alpha b}|^{2\Delta}|\tau_{34}|^{2\Delta}}\mathcal{F}_n(\tilde{\chi})$$

We will use conformal symmetry to turn this into a one-dimensional integral equation. As we have seen the cross ratios are:

$$\chi = \frac{\tau_{12}\tau_{34}}{\tau_{13}\tau 24} \qquad \tilde{\chi} = \frac{\tau_{ab}\tau_{34}}{\tau_{a3}\tau_{b4}}.$$

Using the conformal symmetry and we take $\tau_1 = 0, \tau_3 = 1, \tau_4 = \infty$. Thus we get $\chi = \tau_2$ and $\tilde{\chi} = \frac{\tau_{ab}}{\tau_a - 1}$. We now replace the τ_b integration variable by $\tilde{\chi}$. That means that $d\tau_b = -d\tilde{\chi}(\tau_a - 1)$ so the measure becomes $d\tau_a d\tau_b = d\tau_a d\tilde{\chi}(1-\tau_a)$. From the definition of $\tilde{\chi}$, we get $\tau_b = \tau_a - \tilde{\chi}(\tau_a - 1)$. For simplicity we will write $\tau_a = \tau$ and we will use the property that the sgn function is odd. We now substitute all the above to the initial integral. On the right hand size (ignoring the integration measure) we have the following two fractions:

- $$\begin{aligned}\frac{\text{sgn}(\tau_{1\alpha})\text{sgn}(\tau_{2b})}{|\tau_{2b}|^{2\Delta}|\tau_{1\alpha}|^{2\Delta}|\tau_{\alpha b}|^{2-4\Delta}} &= \frac{\text{sgn}(-\tau_a)\text{sgn}(\chi - \tau_b)}{|\chi-\tau_b|^{2\Delta}|-\tau_a|^{2\Delta}|\tau_a - \tau_b|^{2-4\Delta}} \\ &= \frac{-\text{sgn}(\tau)\text{sgn}(\chi - \tau + \tilde{\chi}(\tau-1))}{|-\tau|^{2\Delta}|\chi - \tau + \tilde{\chi}(\tau-1)|^{2\Delta}|\tilde{\chi}(\tau-1)|^{2-4\Delta}} \\ &= \frac{-\text{sgn}(\tau)\text{sgn}(\chi-\tilde{\chi})\text{sgn}(1-\frac{\tau(1-\tilde{\chi})}{\chi-\tilde{\chi}})}{|\tau|^{2\Delta}|\chi-\tilde{\chi}|^{2\Delta}|\text{sgn}(1-\frac{\tau(1-\tilde{\chi})}{\chi-\tilde{\chi}})|^{2\Delta}|\tilde{\chi}(\tau-1)|^{2-4\Delta}}\end{aligned}$$

-
$$\frac{\text{sgn}(\tau_{\alpha b})\text{sgn}(\tau_{34})}{|\tau_{\alpha b}|^{2\Delta}|\tau_{34}|^{2\Delta}} = \frac{\text{sgn}(\tau_{34})\text{sgn}(\tilde{\chi}(\tau-1))}{|\tilde{\chi}(\tau-1)|^{2\Delta}|\tau_{34}|^{2\Delta}}$$

Together with the transformed measure we multiply these two terms and divide their product by $\frac{\text{sgn}(\tau_{12})\text{sgn}(\tau_{34})}{|\tau_{12}|^{2\Delta}|\tau_{34}|^{2\Delta}}$. In the end we get:

$$\mathcal{F}_{n+1}(\chi) = \frac{1}{\alpha_0}\int \frac{d\tilde{\chi}^2}{|\tilde{\chi}|}\left(\frac{|\chi||\tilde{\chi}|}{|\chi-\tilde{\chi}|}\right)^{2\Delta} \text{sgn}(\chi\tilde{\chi})m(\chi,\tilde{\chi})\mathcal{F}_n(\tilde{\chi})\,, \tag{B.1}$$

where

$$m(\chi,\tilde{\chi}) = \text{sgn}(\chi-\tilde{\chi})\int_{-\infty}^{\infty} d\tau \frac{\text{sgn}(\tau)\text{sgn}(1-\tau)\text{sgn}(1-\frac{\tau(1-\tilde{\chi})}{\chi-\tilde{\chi}})}{|\tau|^{2\Delta}|1-\tau|^{1-2\Delta}|1-\frac{\tau(1-\tilde{\chi})}{\chi-\tilde{\chi}}|^{2\Delta}}\,. \tag{B.2}$$

Appendix C

Casimir operator acting on cross ratio

In this chapter we are going to derive the way the Casimir operator acts on function of the cross ratio. Explicitly we will end in:

$$C_{1+2}\frac{1}{|\tau_{12}|^{2\Delta}}f(\chi) = \frac{1}{|\tau_{12}|^{2\Delta}}\mathcal{C}f(\chi)\,, \tag{C.1}$$

with $\mathcal{C} \equiv \chi^2(1-\chi)\partial_\chi^2 - \chi^2\partial_\chi$. The Casimir operator is given by:

$$C_{1+2} = 2(\Delta^2 - \Delta) - K_1P_2 - P_1K_2 + 2D_1D_2,$$

where

$$D = -\tau\frac{\partial}{\partial\tau}, \quad P = \frac{\partial}{\partial\tau}, \quad K = \tau^2\frac{\partial}{\partial\tau} + 2\tau\Delta.$$

To derive the desired relationship we are going to use the following:

$$\frac{d\chi}{d\tau_1} = \frac{\tau_{34}\tau_{23}}{\tau_{24}\tau_{13}^2}, \quad \frac{d\chi}{d\tau_2} = \frac{\tau_{34}\tau_{41}}{\tau_{13}\tau_{24}^2}, \quad \frac{d}{d\chi}\left(\frac{1}{|\chi-a|^b}\right) = b(a-\chi)|\chi-a|^{-b-2}.$$

First we are going to act with P_2 on $\frac{1}{|\tau_{12}|^{2\Delta}}f(\chi)$. We get:

$$P_2\left(\frac{1}{|\tau_{12}|}f(\chi)\right) = \left(2\Delta\tau_{12}\frac{1}{|\tau_{12}|^{2\Delta+2}}\right)f(\chi) + \left(\frac{1}{|\tau_{12}|^{2\Delta}}\cdot\frac{\tau_{34}\tau_{41}}{\tau_{13}\tau_{24}^2}\right)\partial_\chi f(\chi)$$

Then we act on the above result with K_1. We arrive at:

$$\begin{aligned}
K_1P_2\left(\frac{1}{|\tau_{12}|^{2\Delta}}f(\chi)\right) &= \frac{1}{|\tau_{12}|^{2\Delta}}f(\chi)\bigg(2\Delta\tau_1^2\frac{1}{|\tau_{12}|^2} - 4\Delta^2\tau_1^2\frac{\tau_{12}^2}{|\tau_{12}|^4} \\
&- 4\Delta\tau_{12}^2\frac{\tau_1^2}{|\tau_{12}|^4} + 4\tau_1\Delta^2\frac{\tau_{12}}{|\tau_{12}|^2}\bigg) + \frac{1}{|\tau_{12}|^{2\Delta}}\partial_\chi f(\chi)\bigg(2\Delta\tau_{12}\frac{\tau_1^2}{|\tau_{12}|^2}\frac{\tau_{34}\tau_{23}}{\tau_{24}\tau_{13}^2} \\
&- 2\Delta\frac{\tau_{12}}{|\tau_{12}|^2}\frac{\tau_{34}\tau_{41}\tau_1^2}{\tau_{13}\tau_{24}^2} + \tau_1^2\frac{\tau_{34}^2}{\tau_{24}^2\tau_{13}^2} + 2\Delta\tau_1\frac{\tau_{34}\tau_{41}}{\tau_{13}\tau_{24}^2}\bigg) \\
&+ \frac{1}{|\tau_{12}|^{2\Delta}}\partial_\chi^2 f(\chi)\left(\tau_1^2\frac{\tau_{34}^2\tau_{41}\tau_{23}}{\tau_{24}^3\tau_{13}^3}\right)
\end{aligned}$$

Moving on to the next term we first act with K_2.We get:

$$K_2\left(\frac{1}{|\tau_{12}|^{2\Delta}}f(\chi)\right)=2\Delta\tau_{12}\tau_2^2\frac{1}{|\tau_{12}|^2}\frac{1}{|\tau_{12}|^{2\Delta}}f(\chi)$$
$$+\tau_2^2\frac{\tau_{34}\tau_{41}}{\tau_{13}\tau_{24}^2}\frac{1}{|\tau_{12}|^{2\Delta}}\partial_\chi f(\chi)+2\tau_2\Delta\frac{1}{|\tau_{12}|^{2\Delta}}f(\chi)$$

Then, we act with P_1. Finally this term gives:

$$P_1K_2\left(\frac{1}{|\tau_{12}|^{2\Delta}}f(\chi)\right)=\frac{1}{|\tau_{12}|^{2\Delta}}f(\chi)\bigg(2\Delta\tau_2^2\frac{1}{|\tau_{12}|^2}-4\Delta^2\tau_2^2\frac{\tau_{12}^2}{|\tau_{12}|^4}$$
$$-4\Delta\tau_{12}^2\frac{\tau_2^2}{|\tau_{12}|^4}-4\tau_2\Delta^2\frac{\tau_{12}}{|\tau_{12}|^2}\bigg)+\frac{1}{|\tau_{12}|^{2\Delta}}\partial_\chi f(\chi)\bigg(2\Delta\tau_{12}\frac{\tau_2^2}{|\tau_{12}|^2}\frac{\tau_{34}\tau_{23}}{\tau_{24}\tau_{13}^2}$$
$$-2\Delta\frac{\tau_{12}}{|\tau_{12}|^2}\frac{\tau_{34}\tau_{41}\tau_2^2}{\tau_{13}\tau_{24}^2}+\tau_2^2\frac{\tau_{34}^2}{\tau_{24}^2\tau_{13}^2}+2\Delta\tau_2\frac{\tau_{34}\tau_{23}}{\tau_{24}\tau_{13}^2}\bigg)$$
$$+\frac{1}{|\tau_{12}|^{2\Delta}}\partial_\chi^2 f(\chi)\left(\tau_2^2\frac{\tau_{34}^2\tau_{41}\tau_{23}}{\tau_{24}^3\tau_{13}^3}\right)$$

Now, we will compute the final term $2D_1D_2$ following the same procedure. At first we have:

$$D_2\left(\frac{1}{|\tau_{12}|^{2\Delta}}f(\chi)\right)=\frac{1}{|\tau_{12}|^{2\Delta}}f(\chi)\left(-\Delta-2\Delta\tau_2\frac{\tau_{12}}{|\tau_{12}|^2}\right)$$
$$+\frac{1}{|\tau_{12}|^{2\Delta}}\partial_\chi f(\chi)\left(-\tau_2\frac{\tau_{34}\tau_{41}}{\tau_{13}\tau_{24}^2}\right)$$

Then, we act with $2D_1$. To make our final computations clearer we write the expression that occurs by acting with $2(\Delta^2-\Delta)-K_1P_2-P_1K_2+2D_1D_2$ in terms of $f,\partial f,\partial^2$. The terms proportional to $\frac{1}{|\tau_{12}|^{2\Delta}}f(\chi)$ are:

$$2(\Delta^2-\Delta)-\left(2\Delta\tau_1^2\frac{1}{|\tau_{12}|^2}-4\Delta^2\tau_1^2\frac{\tau_{12}^2}{|\tau_{12}|^4}-4\Delta\tau_{12}^2\frac{\tau_1^2}{|\tau_{12}|^4}+4\tau_1\Delta^2\frac{\tau_{12}}{|\tau_{12}|^2}\right)$$
$$-\left(2\Delta\tau_2^2\frac{1}{|\tau_{12}|^2}-4\Delta^2\tau_2^2\frac{\tau_{12}^2}{|\tau_{12}|^4}-4\Delta\tau_{12}^2\frac{\tau_2^2}{|\tau_{12}|^4}-4\tau_2\Delta^2\frac{\tau_{12}}{|\tau_{12}|^2}\right)$$
$$+\bigg(4\Delta\frac{\tau_1\tau_2}{|\tau_{12}|^2}-8\Delta\frac{\tau_1\tau_2\tau_{12}^2}{|\tau_{12}|^4}-4\Delta^2\frac{\tau_1\tau_{12}}{|\tau_{12}|^2}-8\Delta^2\frac{\tau_1\tau_2\tau_{12}^2}{|\tau_{12}|^4}$$
$$+2\Delta^2+4\Delta^2\frac{\tau_2\tau_{12}}{|\tau_{12}|^2}\bigg)$$

The last line is the outcome of the action of $2D_1D_2$. Collecting the terms with the same denominator and using trivial identities, we get:

$$2\Delta^2+4\Delta^2-8\Delta^2+2\Delta^2=0$$

Now, the terms proportional to $\frac{1}{|\tau_{12}|^{2\Delta}}\partial_\chi f(\chi)$ are:

$$
\begin{aligned}
&-\left(2\Delta\tau_{12}\frac{\tau_1^2}{|\tau_{12}|^2}\frac{\tau_{34}\tau_{23}}{\tau_{24}\tau_{13}^2} - 2\Delta\frac{\tau_{12}}{|\tau_{12}|^2}\frac{\tau_{34}\tau_{41}\tau_1^2}{\tau_{13}\tau_{24}^2} + \tau_1^2\frac{\tau_{34}^2}{\tau_{24}^2\tau_{13}^2} + 2\Delta\tau_1\frac{\tau_{34}\tau_{41}}{\tau_{13}\tau_{24}^2}\right)\\
&-\left(2\Delta\tau_{12}\frac{\tau_2^2}{|\tau_{12}|^2}\frac{\tau_{34}\tau_{23}}{\tau_{24}\tau_{13}^2} - 2\Delta\frac{\tau_{12}}{|\tau_{12}|^2}\frac{\tau_{34}\tau_{41}\tau_2^2}{\tau_{13}\tau_{24}^2} + \tau_2^2\frac{\tau_{34}^2}{\tau_{24}^2\tau_{13}^2} + 2\Delta\tau_2\frac{\tau_{34}\tau_{23}}{\tau_{24}\tau_{13}^2}\right)\\
&+\left(\frac{2\Delta\tau_1\tau_{34}\tau_{23}}{\tau_{13}^2\tau_{24}} + \frac{4\tau_1\tau_2\tau_{34}\tau_{23}\tau_{12}}{\tau_{13}^2\tau_{24}|\tau_{12}|^2} - \frac{4\tau_1\tau_2\tau_{34}\tau_{41}\tau_{12}}{\tau_{13}\tau_{24}^2|\tau_{12}|^2} + \frac{2\tau_1\tau_2\tau_{34}^2}{\tau_{13}^2\tau_{24}^2}\right.\\
&\left.+\frac{2\Delta\tau_2\tau_{34}\tau_{41}}{\tau_{13}\tau_{24}^2}\right)
\end{aligned}
$$

The last line is again the outcome of the action of $2D_1D_2$. Using the definition of the cross ratio $\chi = \frac{\tau_{12}\tau_{34}}{\tau_{13}\tau_{24}}$ which appears in every term of the above expression we get:

$$
\begin{aligned}
&-\cancel{\frac{(\tau_1-\tau_2)^2}{\tau_{12}^2}}\chi^2 + \frac{\Delta\chi}{|\tau_{12}|^2\tau_{13}}\left(-2\tau_1^2\tau_{23} - 2\tau_2^2\tau_{23} + 4\tau_1\tau_2\tau_{23}\right)\\
&+\frac{\Delta\chi}{\tau_{12}\tau_{24}}\Big(-2\tau_1\tau_{41} + 2\tau_2\tau_{41}\Big) - \frac{\Delta\chi}{|\tau_{12}|^2\tau_{24}}\left(-2\tau_1^2\tau_{41} - 2\tau_2^2\tau_{41} + 4\tau_1\tau_2\tau_{41}\right)\\
&+\frac{\Delta\chi}{\tau_{12}\tau_{13}}\Big(-2\tau_2\tau_{23} + \tau_1\tau_{23}\Big)
\end{aligned}
$$

Working out the expression at the parentheses we arrive at:

$$
-\chi^2 - \cancel{\frac{2\Delta\chi\tau_{23}\tau_{12}^2}{|\tau_{12}|^2\tau_{13}}} + \cancel{\frac{2\Delta\chi\tau_{41}\tau_{12}^2}{|\tau_{12}|^2\tau_{24}}} + \cancel{\frac{2\Delta\chi\tau_{23}\tau_{12}}{\tau_{12}\tau_{13}}} - \cancel{\frac{2\Delta\chi\tau_{41}\tau_{12}}{\tau_{12}\tau_{24}}} = -\chi^2
$$

Finally, we are left with the terms that are proportional to $\frac{1}{|\tau_{12}|^{2\Delta}}\partial_\chi^2 f(\chi)$. These are:

$$
\begin{aligned}
&-\tau_1^2\frac{\tau_{34}^2\tau_{41}\tau_{23}}{\tau_{24}^3\tau_{13}^3} - \tau_2^2\frac{\tau_{34}^2\tau_{41}\tau_{23}}{\tau_{24}^3\tau_{13}^3} + 2\frac{\tau_1\tau_2\tau_{34}^2\tau_{23}\tau_{41}}{\tau_{13}^3\tau_{24}^3}\\
&= -\frac{\tau_1^2\chi^2\tau_{41}\tau_{23}}{\tau_{24}\tau_{13}\tau_{12}^2} - \frac{\tau_2^2\chi^2\tau_{41}\tau_{23}}{\tau_{24}\tau_{13}\tau_{12}^2} + \frac{2\tau_1\tau_2\chi^2\tau_{23}\tau_{41}}{\tau_{24}\tau_{13}\tau_{12}^2}\\
&= -\frac{\tau_1^2\chi^3\tau_{41}\tau_{23}}{\tau_{34}\tau_{12}^3} - \frac{\tau_2^2\chi^3\tau_{41}\tau_{23}}{\tau_{34}\tau_{12}^3} + \frac{2\tau_1\tau_2\chi^3\tau_{23}\tau_{41}}{\tau_{34}\tau_{12}^3}\\
&= -\frac{\chi^3\tau_{41}\tau_{23}}{\tau_{12}\tau_{34}},
\end{aligned}
$$

but

$$
\frac{\tau_{41}\tau_{23}}{\tau_{12}\tau_{34}} = \frac{\tau_4\tau_2 - \tau_4\tau_3 - \tau_1\tau_2 + \tau_1\tau_3}{\tau_1\tau_3 - \tau_1\tau_4 - \tau_2\tau_3 + \tau_2\tau_4}.
$$

Moreover, computing $1 - 1/\chi$ we find that:

$$
\begin{aligned}
-\frac{1}{\chi} = \frac{\tau_{12}\tau_{34} - \tau_{13}\tau_{24}}{\tau_{12}\tau_{34}} &= \frac{\tau_1\tau_3 - \cancel{\tau_1\tau_4} - \cancel{\tau_2\tau_3} + \tau_2\tau_4}{\tau_1\tau_3 - \tau_1\tau_4 - \tau_2\tau_3 + \tau_2\tau_4} \\
&= -\frac{-\tau_1\tau_2 + \cancel{\tau_1\tau_4} + \cancel{\tau_3\tau_2} - \tau_3\tau_4}{\tau_1\tau_3 - \tau_1\tau_4 - \tau_2\tau_3 + \tau_2\tau_4} = \frac{\tau_{41}\tau_{23}}{\tau_{12}\tau_{34}}.
\end{aligned}
$$

Thus,

$$
\frac{1}{|\tau_{12}|^{2\Delta}}\partial_\chi^2 f(\chi)\left(-\chi^3 \cdot (1 - \frac{1}{\chi})\right) = \frac{1}{|\tau_{12}|^{2\Delta}} \cdot \chi^2(1-\chi)\,. \qquad \text{(C.2)}
$$

Adding together all the terms corresponding to to $f, \partial f, \partial^2$ we arrive at the desired expression:

$$
C_{1+2}\frac{1}{|\tau_{12}|^{2\Delta}} f(\chi) = \frac{1}{|\tau_{12}|^{2\Delta}}\mathcal{C} f(\chi),
$$

with $\mathcal{C} \equiv \chi^2(1-\chi)\partial_\chi^2 - \chi^2\partial_\chi$.

Appendix D

Spinor representations in various dimensions

As we have said the Hamiltonian of the SYK model is given by

$$H = \frac{1}{4!} \sum_{j,k,l,m}^{N} J_{jklm} x_j x_k x_l x_m, \tag{D.1}$$

while the Majorana fermions x_i obey the anticommutation relations

$$\{x_i, x_j\} = 2\delta_{ij}. \tag{D.2}$$

These fermions are simple matrices, which we are going to build now. These matrices will satisfy the Clifford algebra $\{x_i, x_j\} = 2\delta_{ij}$, and they will be representations of the orthogonal group, since we are working on Euclidean space, for general dimension N. Moreover, the fact that we are dealing with Majorana fermions implies that we are going to look for Hermitian representations, ${x_i}^\dagger = x_i$. We will restrict our discussion for even dimensions, $N = 2K$.

We can define the following raising and lowering operators:

$$c_i = \frac{1}{2}(x_{2i} - ix_{2i+1}), \quad c_i^\dagger = \frac{1}{2}(x_{2i} + ix_{2i+1}). \quad i = 1, ...K - 1. \tag{D.3}$$

It is trivial to see that they obey the following anticommutation relations:

$$\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0, \quad \{c_i, c_j^\dagger\} = \delta_{ij}. \tag{D.4}$$

These are exactly the fermion anticommutation relations. We now assume that there exists a state, $|0\rangle$ which is annihilated by c_i and thus, we can build our basis by acting with $c_i^\dagger$ on the vacuum. We must be careful and keep in mind that our basis will be composed by the states created by acting will all the possible ways with $c_i^\dagger$, at most once since $(c_i^\dagger)^2 = 0$. Thus, our basis consists of states like

$$(c_1^\dagger)^n....(c_K^\dagger)^n |0\rangle , \quad n = 0, 1. \tag{D.5}$$

The number of the basis states and consequently the dimension of the Hilbert space will be 2^K, corresponding to whether a state is occupied or not. For example, suppose we have $N = 4 \to K = 2$. We have the following states:

$$(c_1^\dagger) |0\rangle \propto |1\rangle , \tag{D.6}$$
$$(c_2^\dagger) |0\rangle \propto |2\rangle , \tag{D.7}$$
$$(c_1^\dagger)(c_2^\dagger) |0\rangle \propto |12\rangle , \tag{D.8}$$
$$|0\rangle . \tag{D.9}$$

As expected, we have 4 orthogonal states that form our basis. For $N = 2$, we have the following 2×2 matrices

$$x_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{D.10}$$

It is trivial to check that these 2 matrices satisfy (D.2). We will now give a recursion relation for the representation matrices

$$x_i^K = x_i^{K-1} \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = 1, 2, ..., N-2 \tag{D.11}$$
$$x_{N-1}^K = \mathcal{I}_{2^{K-1}} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{D.12}$$
$$x_N^K = \mathcal{I}_{2^{K-1}} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{D.13}$$

where $\mathcal{I}_d$ is the $d \times d$ identity matrix. From (D.11), we can see that x_i^K are $2^K \times 2^K$ matrices. For more details on this subject see [32]

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