NATIONAL TECHINICAL UNIVERSITY OF ATHENS
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Modelling & Analysis of Elastic Plates through Variational Principles

Diploma Thesis

by

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“There two possible outcomes:
If the result confirms the hypothesis, then you’ve made a measurement.
If the result is contrary to the hypothesis, then you’ve made a discovery”
Enrico Fermi (1901-1954)

Dedicated to my father.

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Summary in English

The present thesis is the result of our investigation over the mathematical description of two "Plate Theories" and the final comparison of the "results" of the two different models of the plates.

Consequently, the present dissertation is divided into three parts, with the first part (Part A) dealing with the Classical Plate Theory of the Plate (or Kirchhoff's Plate Theory - CPT) developed though the Variational Principles, while the second part (Part B) investigates a more accurate kinematic model for the plates, the so called Third-Order Plate Theory (or Levinson's Plate Theory - TSDT). The last is also examined though Variational Principles. These two parts are the main line of work of this dissertation. However, there is also a third shorter part (Part C), in which the dispersion relations of the wave propagation through a free (from external loads) and one-directional infinite medium are derived for each model (Classical and Third-Order). After that work, there are illustrated the dispersion curves of each model.

Although the structure and rationality of the indexes of each one of the two first parts (Part A and B) does not differ substantially, we decided to set two of shorter duration introductory sections in the first pages of each part, because there are substantial differences as for the initial assumptions, the modelling and consequently the governing equations of motion and the boundary conditions of the plate. However for the sake of completeness, we set the Preface, which is usually found on the first pages of the thesis. Further, we create a global section regarding the References combined with this diploma thesis. The section of the references is as usual located on the end of the dissertation. Also due to the fact that the most of the references include elements of various theories of plates in order to gain comparable results between each other, there is no reason to distinguish them into different section at the end of each part. Moreover inside the sections of each one of the three main parts, there are pointed out specific chapters or sections of these references inside brackets where regarded appropriate. Note also that the references considered as determinant and important to our work on a particular part, can be found additionally in its introductory section.

The first part (Part A) is divided into eight sections. The first one is an introductory section presenting the basic definitions of the quantities and the notation used in the sequel. However, the very importance of this section is the initial assumptions of the CPT and the way the last are inserted on the kinematic model of the plate. Subsequently, follows the second section, in which the geometric configuration and the externally applied loads on the (external) surface of the plate are prescribed. Note that the choice of the aforementioned must be compatible with the initial assumptions of the modelling, described on the first section. The third section combines all the above descriptions for the model and then the kinematics of the thin plate are produced. The far most important section of the Part A is the fourth, because though the Variational Principles the governing equations of motion of the model of the CPT are constructed. These equations are derived for two different cases. The first case is for an orthotropic but in-plane anisotropic material and the second is for an orthotropic but in-plane isotropic material. In the context of the fifth section, the boundary conditions of the model of CPT are derived, again with the aid of Variational Principles. The equations of motion and boundary conditions, derived on the fourth and fifth section respectively, are written in terms of some (appropriate for the problem) thickness-integrated quantities. This fact inserts some confusion on the study and the application of those relations with physical interpretation. Thus, the sections six and seven aim to give the governing equations of motion and the boundary conditions of the plate explicitly in terms of the displacement field of the model. Last but not least, on the section eight is defined the functional space in which the total mathematical formulation of our problem is developed.

As for the second part (Part B) of this diploma thesis, it has similar structure with the Part A composed of eight sections. However, the initial assumptions of the TSDT and consequently its resulting equations of motion and boundary conditions differ substantially. Of course, there are common terms (namely exist both on the CPT and TSDT) inside the aforementioned relations, which are highlighted when regarded useful, due to the fact that the TSDT is essentially an "extension in accuracy" of the modelling in comparison with the CPT.

The last part (Part C) is subdivided into three sections. On the first section is analyzed the process of deriving the dispersion relation of CPT. On the second section is prescribed the way of producing the dispersion relation of the TSDT. The previous are performed in accordance with the assumption of one-directional wave propagation along the infinite (along the same dimension) medium (plate). On the third section, we just compare and comment on the results of the two previous sections.

The present work is supplemented with two appendices, namely the Appendix A and B (located at the end of this dissertation), where additional information and proofs about the results of the parts A, B and C are presented. On the Appendix A is analyzed the transformation from the Cartesian to a curvilinear coordinate system used to derive the full set of boundary conditions demanded for each one of the two models. Finally on the Appendix B, there are the governing equations of motion and the dispersion relations as for the First-Order Shear Deformable Plate Theory (FSDT), in order to compare them with the corresponding equations motion and dispersion curves of the models of CPT and TSDT.

Σύνοψη στα Ελληνικά (Summary in Greek)

Η εν λόγω διπλωματική εργασία είναι το αποκύημα μίας διερεύνησης ως προς το μαθηματικό φορμαλισμό δύο «Θεωρητικών Προσεγγίσεων για Μοντέλα Πλακών» και η τελική σύγκριση των «αποτελεσμάτων» των δύο διαφορετικών μοντέλων.

Συνεπώς, η εν λόγω διατριβή χωρίζεται σε τρία μέρη, εκ των οποίων το πρώτο (Μέρος Α`) διαπραγματεύεται την «Κλασσική Θεωρία Πλακών» (ή την Θεωρία Πλακών του Kirchhoff) που αναπτύσσεται μέσω των Μεταβολικών Αρχών, ενώ το δεύτερο μέρος (Μέρος Β`) διερευνά ένα πιο ακριβές κινηματικό μοντέλο για πλάκες, αποκαλούμενο ως «Η Τρίτης τάξης Θεωρία Πλακών» (ή Θεωρία Πλακών του Levinson). Η τελευταία εξετάζεται επίσης μέσω των Μεταβολικών Αρχών. Αυτά τα δύο μέρη είναι η κύρια γραμμή δουλειάς αυτής της διατριβής. Ωστόσο υπάρχει και ένα τρίτο μέρος (Μέρος Γ`), μικρότερης έκτασης, στο οποίο δίδονται οι σχέσεις διασποράς της κυματικής διάδοσης μέσω ενός ελεύθερου (από εξωτερικά φορτία) και μονοδιάστατα άπειρου μέσου, για κάθε μοντέλο (Κλασσική και Τρίτης τάξης Θεωρία Πλακών). Έπειτα αναπαρίστανται τα διαγράμματα των σχέσεων διασποράς για κάθε μοντέλο.

Αν και η δομή και η λογική των περιεχομένων των δύο πρώτων μερών (Μέρος Α` και Β`) δε διαφέρουν ουσιαστικά, αποφασίζουμε να θέσουμε εισαγωγικές ενότητες, μικρότερης έκτασης, στις πρώτες σελίδες κάθε μέρους, διότι υπάρχουν ουσιώδεις διαφορές στις αρχικές υποθέσεις, στη μοντελοποίηση και κατά συνέπεια στις εξισώσεις κίνησης και στις συνοριακές συνθήκες της πλάκας. Ωστόσο, χάριν πληρότητας, παραθέτουμε τον Πρόλογο, ο οποίος συνηθίζεται να βρίσκεται στις πρώτες σελίδες των εργασιών. Περαιτέρω, δημιουργούμε ένα από κοινού για όλα τα μέρη κεφάλαιο για τις Αναφορές που εμπλέκονται στην παρούσα διατριβή. Οι αναφορές συνήθως τοποθετούνται στο τέλος της διατριβής. Ακόμη εξαιτίας του γεγονότος ότι η περισσότερες αναφορές περιέχουν στοιχεία από ποικίλα μοντέλα πλακών προκειμένου να εξάγουν συγκρίσιμα αποτελέσματα μεταξύ αυτών, δεν υπάρχει λόγος να διαχωρίσουμε τις αναφορές σε διαφορετικά εδάφια στο τέλος κάθε μέρους. Τονίζεται επίσης ότι οι αναφορές που θεωρούνται καθοριστικές και σημαντικές για τη δουλείας μας σε ένα συγκεκριμένο μέρος, παρατίθενται και στο εισαγωγικό του εδάφιο.

Το πρώτο μέρος (Μέρος Α`) χωρίζεται σε οκτώ ενότητες. Η πρώτη είναι ένα εισαγωγικό κεφάλαιο όπου παρουσιάζονται οι βασικοί ορισμοί των μεγεθών και ο συμβολισμός που θα χρησιμοποιηθεί στην συνέχεια. Ωστόσο, η μεγάλη σημασία του εν λόγω κεφαλαίου έγκειται στις αρχικές υποθέσεις της Κλασσικής Θεωρίας Πλακών και στον τρόπο μέσω του οποίου εισέρχονται στο κινηματικό μοντέλο της πλάκας. Στη συνέχεια, ακολουθεί το δεύτερο κεφάλαιο, στο οποίο προδιαγράφονται η γεωμετρική διαμόρφωση και τα εξωτερικώς επιβαλλόμενα φορτία στην επιφάνεια της πλάκας. Τονίζεται ότι η επιλογή των προαναφερθέντων θα πρέπει να είναι συμβατή με τις αρχικές υποθέσεις της μοντελοποίησης που περιγράφεται στο πρώτο εδάφιο. Στο τρίτο κεφάλαιο συνδυάζονται όλες οι παραπάνω περιγραφές για το μοντέλο και τότε παράγεται η κινηματική της λεπτής πλάκας. Το μακράν σημαντικότερο κεφάλαιο του πρώτου μέρους είναι το τέταρτο, διότι μέσω των Μεταβολικών Αρχών δομούνται οι εξισώσεις κίνησης του μοντέλου της Κλασσικής Θεωρίας Πλακών. Αυτές οι εξισώσεις παράγονται για δύο διαφορετικές περιπτώσεις. Η πρώτη είναι για ένα ορθοτροπικό αλλά ανισοτροπικό στο οριζόντιο επίπεδο υλικό και η δεύτερη για ένα ορθοτροπικό αλλά ισοτροπικό στο οριζόντιο επίπεδο υλικό. Στα πλαίσια του πέμπτου κεφαλαίου, οι συνοριακές συνθήκες του εν λόγω μοντέλου της πλάκας, εξάγονται επίσης με τη βοήθεια των Μεταβολικών Αρχών. Οι εξισώσεις κίνησης και οι συνοριακές συνθήκες που παράγονται στο τέταρτο και πέμπτο κεφάλαιο αντιστοίχως, γράφονται μέσω (κατάλληλων για το πρόβλημα) ποσοτήτων που προκύπτουν από την ολοκλήρωση κατάλληλων μεγεθών (τάσεων) κατά το πάχος της πλάκας. Το παραπάνω εισάγει κάποια σύγχυση στη μελέτη και την εφαρμογή αυτών των σχέσεων με φυσική ερμηνεία. Εν τέλει, στο κεφάλαιο οκτώ ορίζεται ο συναρτησιακός χώρος μέσα στον οποίο αναπτύσσεται ο συνολικός μαθηματικός φορμαλισμός του προβλήματος που μελετάμε.

Όσον φορά το δεύτερο μέρος (Μέρος Β`) αυτής της διπλωματικής εργασίας, έχει παρόμοια δομή με το πρώτο μέρος και αποτελείται από οκτώ εδάφια. Ωστόσο οι αρχικές υποθέσεις της Τρίτης τάξης Θεωρίας Πλακών και κατά συνέπεια οι καταληκτικές εξισώσεις κίνησης και συνοριακές συνθήκες διαφέρουν ουσιωδώς. Υπάρχουν βέβαια κοινοί όροι (δηλαδή που εμφανίζονται και στα δύο μοντέλα) μέσα στις προαναφερθείσες σχέσεις, οι οποίοι τονίζονται όπου κρίνεται χρήσιμο, εξαιτίας του γεγονότος ότι η Τρίτης τάξης Θεωρίας πλακών είναι ουσιαστικά μία «επέκταση στην ακρίβεια» της μοντελοποίησης σε σύγκριση με την Κλασσική Θεωρία Πλακών.

Το τελευταίο μέρος (Μέρος Γ`) υποδιαιρείται σε τρείς ενότητες. Στην πρώτη αναλύεται η διαδικασία εξαγωγής των εξισώσεων διασποράς της Κλασσικής Θεωρίας Πλακών. Στη δεύτερη ενότητα προδιαγράφεται ο τρόπος παραγωγής της εξίσωσης διασποράς για το τριτο-τάξιο μοντέλο της πλάκας. Τα παραπάνω γίνονται σε συμφωνία με την υπόθεση της μονοδιάστατης κυματικής διαταραχής κατά μήκος ενός άπειρου (κατά την ίδια διάσταση) μέσου (πλάκας). Στο τρίτο κεφάλαιο, συγκρίνουμε απλά και σχολιάζουμε τα αποτελέσματα των προηγούμενων δύο.

Η εν λόγω διατριβή συμπληρώνεται και από δύο παραρτήματα, δηλαδή το Παράρτημα Α` και Β` (που βρίσκονται στο τέλος αυτής της διατριβής), όπου παρουσιάζονται επιπρόσθετη πληροφορία και αποδείξεις σχετικά με τα αποτελέσματα των Μερών Α`, Β` και Γ`. Στο Παράρτημα Α` αναλύεται ο μετασχηματισμός από το Καρτεσιανό στο επικαμπύλιο σύστημα συντεταγμένων, το οποίο χρησιμοποιείται για να παράγουμε το σύνολο των συνοριακών συνθηκών που απαιτούνται για καθένα μοντέλο. Εν τέλει στο Παράρτημα Β, υπάρχουν οι εξισώσεις κίνησης και οι σχέσεις διασποράς για την Πρώτης τάξης Θεωρία Πλακών, προκειμένου να συγκριθούν με τις αντίστοιχες της Κλασσικής και της Τρίτης τάξης Θεωρίας Πλακών.

Introduction (Preface)

[References: **1.** *Reddy J.N. (2007), "Theory and Analysis of Elastic Plates and Shells"/ Chapter 1.1*, **2.** *Love A.E.H. (1994), "A Treatise on the Mathematical Theory", 4th edition New York Dover/ Introduction*, **3.** *Timoshenko S., Young D.H. Weaver W. (1974), "Vibration Problems in Engineering", 4th edition John Wiley & Sons/ Chapter 5*, **4.** *Babuska I. and Li L. (1991), "Hierarchic Modeling of Plates", Journal on Computer & Structures, Pergamon Press*, **5.** *Szilard R., Dr. –Ing. P.E. (2004), "Theory and Applications of Plate Analysis- Classical Numerical and Engineering Methods", John Wiley & Sons*,
6. https://en.wikipedia.org/wiki/Mechanician].

Generally speaking, plates are straight, plane two-dimensional structural components whose one dimension called the thickness of the plate, is much smaller than the other dimensions. Geometrically, they are bound either by straight or curved lateral boundary. As exactly their counterparts, the beams, they are not only used as structural components but can also form complete structures. These structure can be Slab Bridge, floating oil extraction platforms, barrage and seawalls, or even seawave energy harvesting systems, the last of which are one of the most popular devices nowadays due to the matters concerning energy efficiency and renewable sources of energy. Statically plates have free, simply supported and fixed boundary conditions, including elastic supports and elastic restrains, or, in some cases, even points support [Figure 1]. The static and dynamic loads carried by plates are predominantly perpendicular to the horizontal faces of the plate. These external loads are carried by internal bending and torsional moments as well as by transverse shear forces.

Because of the fact that the loading-carrying action of plates resembles to a certain extent that of beams, plates can be approximated by gridworks beams. Such an approximation, however, arbitrarily breaks the continuity of the structure and usually leads to incorrect results unless the actual two-dimensional behavior of plate is taken correctly into account.

The two-dimensional structural action of plates results in lighter structures and consequently gives more economical assets. Furthermore, numerous structural configuration require partial or even complete enclosure that can easily be accomplished by plates without the use of additional covering, resulting in further savings in materials and labor cost for the erection of the total structure. As a direct consequence, plates and plate-type structures have gained special importance and remarkably widespread, engineering applications in recent years. A large number of structural components in engineering structures are floor and foundation slabs, lock-gates, thin retaining walls and more specifically as for the naval architecture and marine engineering structures, we come up against decks of ships, longitudinal and transverse bulkheads, double bottom, hatches, and parts of the superstructures of ships and so on. Further, plates are also indispensable in aerospace industries. The wings and a large part of the fuselage of an aircraft, for example, consists of a slightly curved plate skin with an array of stiffened ribs. Plates are also frequently parts of machineries and other mechanical plate devices.

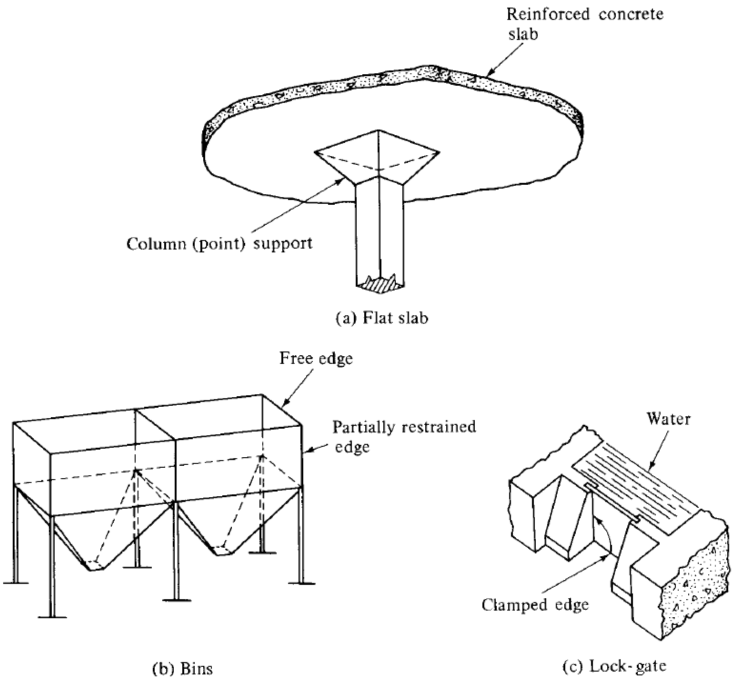

Figure 1: Static Loads

The majority of the plate structures is analyzed by applying the governing equations of the Theory of Elasticity. However, the “exact” solutions of the various governing differential equations of plate “theories” (namely, approximation of the real behavior of the plate) can only be obtained for special boundary and loading conditions. Note that in most cases, the various energy methods can yield quite usable analytical solutions for most practical problems. Almost all the numerical methods are based on the discretization of the plate continuum, but these methods are not going to occupy us on the context of this quotation, because the usage of the Calculus of Variations and the arguments of the Variational Principles are going to give us adequately exact equations of motion and boundary conditions for the kinematic models of the plate (the presentation of whom is the main part of this dissertation). Although the equations of motion and boundary conditions are given in a generalized form as will be shown on the last sections of the Pat A and B, the interested reader could specify and modify them according to the shape of the plate and the kind of boundary conditions under consideration.

In all structural analysis the engineer is forced, due to the complexity of any real structure, to replace the structure by a simplified analysis model equipped only with those important parameters that mostly influence its static or dynamic response to loads. In plate analysis such idealizations concern mainly, the geometry of the plate and its supports, the behavior of the material used and also the type of loads and their way of application.

To proceed to the modelling of the motion of the plate with the specific chosen characteristics referred above, we have to think about the most efficient way to handle it. Thus, regarding the plate as a three-dimensional continuum is a highly impractical approach since it would create almost unbeatable mathematical difficulties. Even if the solution could be easily found, we have to confront with unfairly painful amount of calculations. Consequently, we distinguish the plate into four different categories with inherently different structural behavior and different governing differential equations of their mathematical modelling. The four plate-types

might be categorized, to same extent, using their slenderness ratio (ratio of thickness to an in-plane dimension, breadth or length) h/L.
Although, the boundaries between these individual plate types are somewhat fuzzy, we can attempt to subdivide plates into the following major categories:

- **Membranes** ($h/L \leq 0.02$), are very thin plates without flexural rigidity, carrying loads by axial and central shear forces. This load – carrying action can be can be approximated by the stresses only, because of their extreme thickness, their moment resistance is of negligible order.
- **Thin Plates** ($h/L = 0.02 \div 0.1$), are thin plates with flexural rigidity, carrying loads two dimensionally, mostly by internal (bending and torsional) moments and transverse shear forces. These loading condition is similar to those of beams.
- **Moderately Thick Plates** ($h/L = 0.1 \div 0.2$), are in many respects similar to the thin plates, with the notable exception that the effects of transverse shear forces on the normal stress components are also taken into account.
- **Thick Plates** ($h/L \geq 0.2$), have an internal stress condition that resembles to that of three-dimensional continua (3D Elasticity).

The above cases are illustrated on the Figure 2.

Taking advantage of the above subdivision of the different type of plates, we are going to analyze two different mathematical models or "theories" of plates. The first one is the usually applied on engineering problems, Classical Plate Theory (Kirchhoff's Plate Theory), presented on the Part A of this thesis. This model concerns the case of Thin Plate (above). The second is a higher-order plate theory, found on the literature as the Third-Order Shear Deformation Plate Theory or Levinson's Plate Theory, which is presented on the Part B of this dissertation. This model concerns the case of Moderately Thick Plates (above). Finally, on the Part C on this thesis there are some applications concerning the dispersion relations and curves of the two kinematic model for wave propagation through infinite medium.

For completeness reasons, note that the case of Thick Plate (generally three-dimensional orthotropic material) will not occupy us here, because there is no practical use of extremely thick plates in engineering applications, in which the thickness dimension along the vertical axis to the flat surfaces of the plate influences considerably the in-plane motion of the plate.

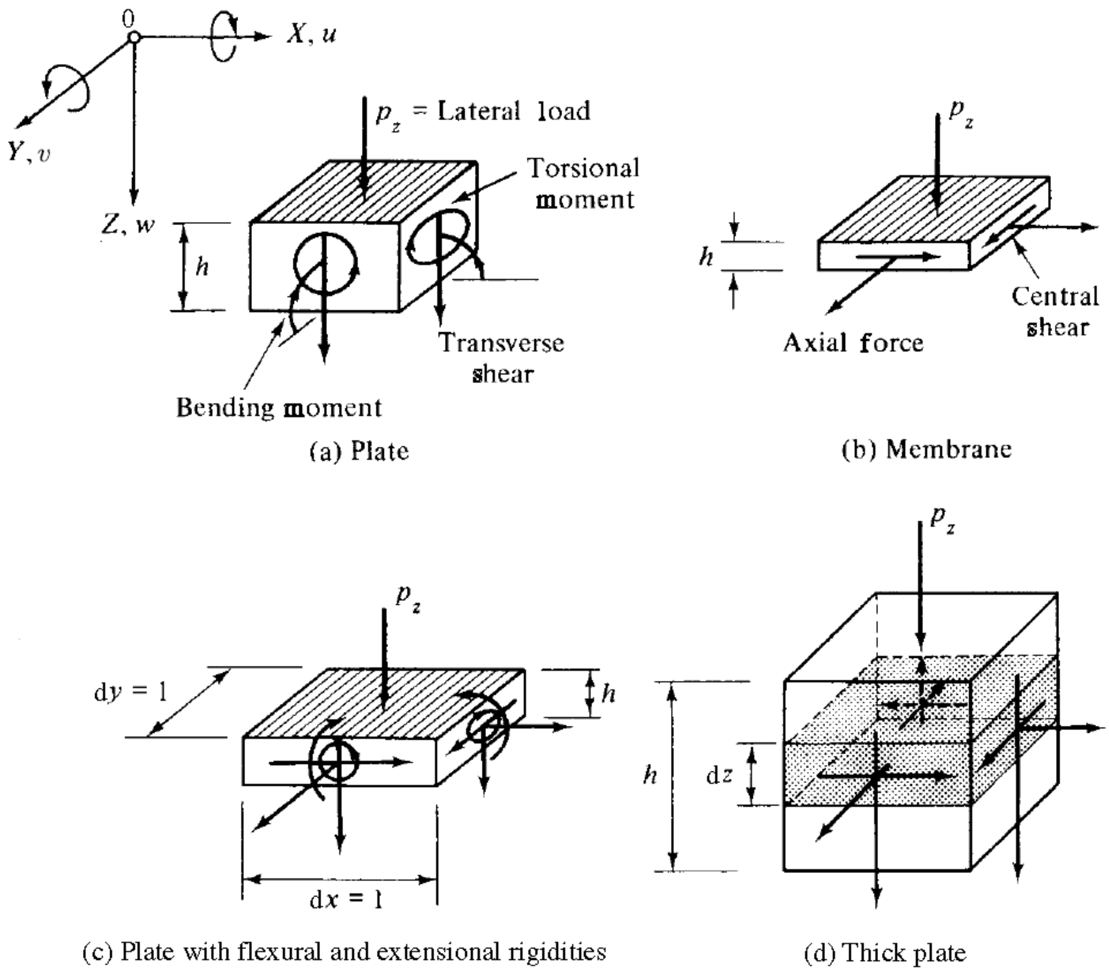

Figure 2: Main categories of the plates

A few words about the Comparison of the Kinematic Models:
The classical plate theory and the first-order shear deformation theory are the simplest equivalent single-layer theories, and they adequately describe the kinematic behavior of most laminates. Higher-order theories, such the third-order shear deformation theory, can represent the motion of the plate better, may not demand shear correction factors and also can yield more accurate interlaminar stress distributions. However, the disadvantage is that they involve higher-order stress resultants that are difficult to interpret physically and require considerably more computational effort. Therefore, such theories should be used only when necessary.

In principle, it is possible to expand the displacement field in terms of the thickness coordinate up to any desired degree. However, the higher than third-order plate theories are misleading and unusual, due to the algebraic complexity and computational effort involved with these theories in return for marginal gain in accuracy. The reason for expanding the displacements up to the cubic term in the thickness coordinate is to appear quadratic variation of the transverse shear strains and transverse shear stress through each layer of the laminated composite plates. This result avoids the need for shear correction coefficients used in the first-order theory, fact the insert a relative fault in the approximation due to the demanding experience used to predict the appropriate values of this factor and to calibrate adequately the model of the plate.

There is a great variety of papers on the third-order theories and their applications (some of them exist on the References of this dissertation). Although many of them seem to differ from each other on the surface and as a consequence on the boundary conditions deriving from the analysis of the kinematic model invoking the arguments of the Variational Principles.

PART A:

CLASSICAL THEORY OF PLATES

1. Introduction

1.1. Basic (general) Definition of a plate

Generally speaking, a plate is a structural element with planform dimensions (e.g. length, breadth) that are large compared to its thickness and is subjected to loads that cause bending deformation in conjunction with stretching. Usually a plate is regarded as thin plate, when its thickness is ten times smaller than the smallest in-plane dimension. As shown in Figure 1, h/L or $h/B \leq 0.1$ [*Reddy J. N.* (2007), "*Theory and Analysis of Elastic Plates and Shells"/ Chapter 3.1* and *Onate E.* (*2013*), *"Structural Analysis with the Finite Element Method. Linear Statics: Beams, Plates and Shells"/ Chapter 6.1*].

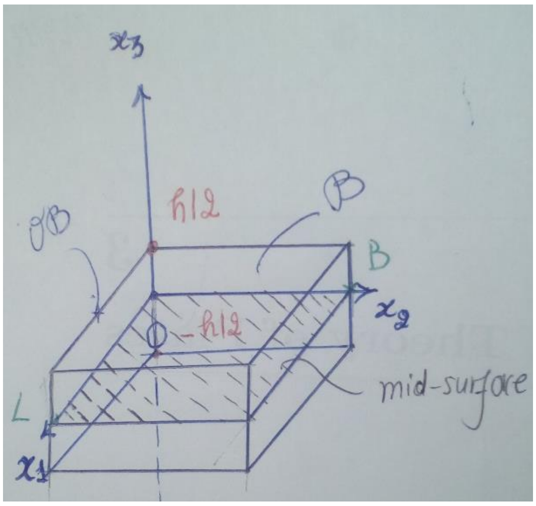

Figure 1: A usual rectangular plate or thin structural element

Because of the very small thickness, there is no reason to model those problems with 3D Elasticity. The simpler equations of 2D Elasticity, are sufficient in order to analyze the strains and stresses upon plates.
In addition we assume that the reference system of axes (here: the Cartesian coordinate system), namely the origin of the axes, is located on the middle plane of the plate. This plane will be usually called mid-surface on the next sections and it is regarded as the reference plane for deriving the kinematic equations of the plate. Also we assume that the mid-surface is equidistant from the upper and lower surface of the plate, which means that each point upon the plate is described by zero vertical coordinate $x_3 = 0$.

As for the material of the plate under consideration, we are going to model two kind of structural material separately on the below corresponding sections. The first one the orthotropic but in-plane anisotropic plate and the second one is the orthotropic again but in-plane isotropic plate. Note that the generally isotropic plate will not occupy us on the context of this quotation due to the ratio of its vertical and in-plane dimensions, as justified on the following sections.

1.2. Important Assumptions of the classical plate theory (CPT) (Kirchhoff's plate theory)

1.2.1. Straight lines perpendicular to the mid-surface (i.e. transverse normals) before deformation remain straight after deformation. This assumption can be called, <u>the straightness assumption.</u>

1.2.2. The transverse normals do not experience elongation. In view of the small thickness of the plate, the vertical movement of any point of the plate is identical to that of the point of the middle surface or alternatively the points along a normal to the middle plane have the same vertical displacement. This assumption is simply, the inextensibility assumption.

1.2.3. The transverse lines (normals) rotate such that they remain perpendicular to the middle surface after deformation. This hypothesis is called shortly, the normality assumption. The normality assumption is also found on the literature as the normal orthogonality condition. Note that this assumption (condition) only holds for thin plates (thickness/average in-plane dimensions ratio: h/L or $h/B \leq 0.05$). For moderately thick ($0.05 \leq h/L \leq 0.1$) and very thick ($h/L \geq 0.1$) plates the distortion of the normal during deformation increases.

1.3. Consequences of Kirchhoff' s Assumptions

1.3.1. As for the straightness assumption, the rate of change of the planar dimensions of the plate is small in comparison with the rate of change of the vertical dimension (thickness). As a consequence, the derivatives $\frac{\partial u}{\partial x_1}, \frac{\partial v}{\partial x_2}, \frac{\partial u}{\partial x_2}, \frac{\partial v}{\partial x_1}$ are regarded small in comparison with $\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}$. So the first two are neglected to the following calculations. The essential point of this assumption is that the section before and after deformation remains linear, whereas on other higher-order plate theories we come up i.e. cubic curves (sections). The symbols of the virtual displacements u , v , w are apparent below:

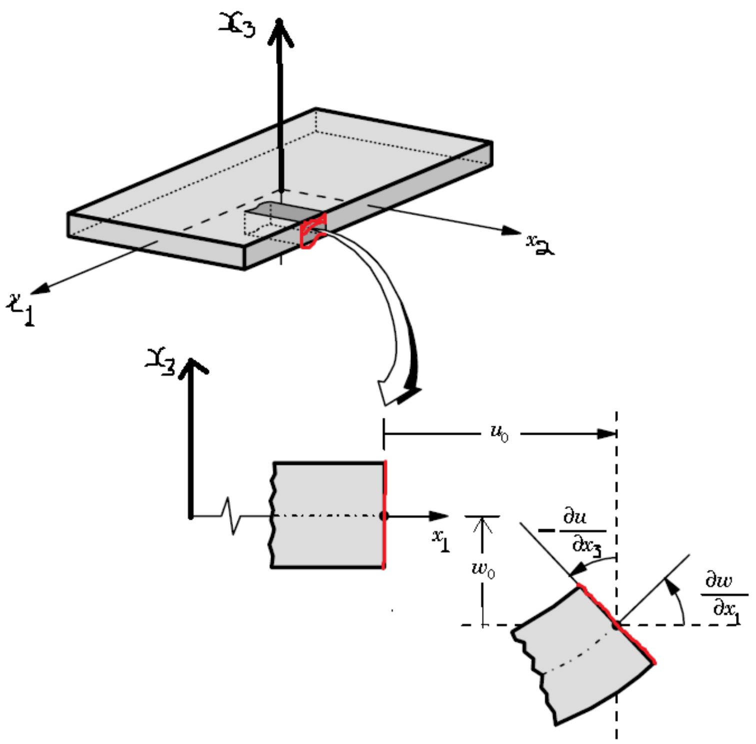

Figure 2: The straightness assumption of the cross-section of the plate during its deformation.

As seems on the Figure 2, the cross-section before and after deformation of the plate is illustrated with the straight red line.

1.3.2. As for the inextensibility assumption (section 1.2.2.), we note that the structures are usually composed of stiff materials. Consequently, the transverse deformation-displacement is independent of the vertical coordinate x_3. This assumption is conceptually the inextensibility of the cross section.

Thus, $\dfrac{\partial w}{\partial x_3} = 0 \Rightarrow w(\boldsymbol{x};t) = w(x_1, x_2;t) = w_0(x_1, x_2;t)$.

1.3.3. The important point of the normality assumption (section 1.2.3.), is that the transverse shear strains are zero, so that

$$\gamma_{13} = 2e_{13} = \frac{\partial u}{\partial x_3} + \frac{\partial w}{\partial x_1} = 0 \tag{1}$$

and

$$\gamma_{23} = 2e_{23} = \frac{\partial v}{\partial x_3} + \frac{\partial w}{\partial x_2} = 0 \tag{2}$$

According to the normality assumption, the transverse lines (sections) rotate remaining perpendicular to the mid- surface after the deformation of the plate.
Keeping the above in mind and taking a look at the Figure 3 (which represents a cross-section of the plate on $x_1 x_3$-plane), we note that the tangent line to the mid-surface of the deformed plate with the horizontal line of the undeformed mid-surface (or simply the x_1-axis) define an angle ϑ_1 so that,

$$\vartheta_1 \cong \tan\vartheta_1 = \frac{\partial w}{\partial x_1}$$

The first equality is due to the smallness of the rotations of mid-surface and transverse normals (essential assumption of our model of CPT).
Respectively, the vertical line on the undeformed mid-surface with the vertical line on the tangent to the deformed mid-surface [at each point (u, w)], create an angle ϑ_2 so that,

$$\vartheta_2 \cong \tan\vartheta_2 = -\frac{\partial u}{\partial x_3}$$

These angles (ϑ_1 and ϑ_2) have their sides per two between them vertical, as seem on the Figure 3 with the two continuous and other two dashed lines. Thus, as a geometric consequence the angles have to be equal each other. Namely,

$$\vartheta_1 = \vartheta_2 \Rightarrow \frac{\partial w}{\partial x_1} = -\frac{\partial u}{\partial x_3} \Rightarrow \frac{\partial w}{\partial x_1} + \frac{\partial u}{\partial x_3} = 0 \xrightarrow{Eq.(1)} \gamma_{13} = 2e_{13} = 0$$

Similarly, is proved the Eq. (2),

$$\frac{\partial w}{\partial x_1} = -\frac{\partial u}{\partial x_3} \Rightarrow \frac{\partial w}{\partial x_1} + \frac{\partial u}{\partial x_3} = 0 \xrightarrow{Eq.(2)} \gamma_{23} = 2e_{23} = 0$$

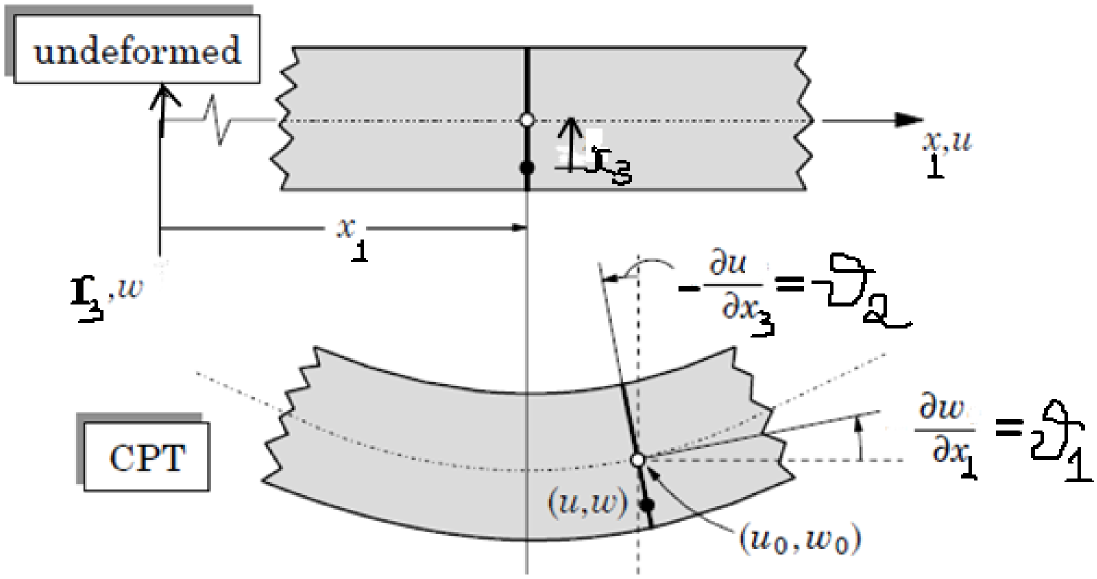

Figure 3: Consequences of the normality assumption during the deformation of the plate

Understanding the physical meaning of the above relationships, in the following paragraphs we are going to study their application on the kinematic model of the CPT.

2. Geometric configuration and boundary conditions

The shape of plate considered herein is the one of a homogeneous cylinder having a basis of arbitrary (smooth) shape, and height (thickness) h, much smaller than the in-plane dimensions. The domain occupied by the plate (the cylinder) is denoted by B. The total boundary of the plate is denoted by ∂B, and consists of lateral boundary (surface) $\partial B^{(lat)}$, and the two flat faces $\partial B^{(f)} = \partial B^{(uf)} \cup \partial B^{(lf)}$. One of these two flat surfaces is conventionally called the upper face, $\partial B^{(uf)}$, and the other is called the lower face, $\partial B^{(lf)}$. That is

$$\partial B = \partial B^{(lat)} \cup \partial B^{(uf)} \cup \partial B^{(lf)}.$$

Another (different) subdivision of the total boundary ∂B is also useful for our analysis, according to the boundary conditions applied to the various parts of it. Thus, we denote by ∂B_T the parts of the total boundary surface tractions (stresses) are prescribed, and ∂B_u the parts of the total boundary on which the displacements are given. Of course,

$$\partial B = \partial B_T \bigcup \partial B_u .$$

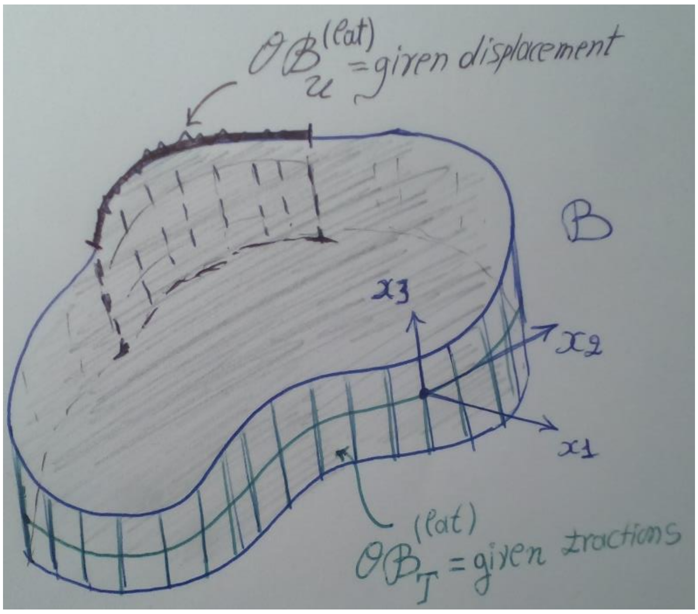

Figure 4: Geometry and loading conditions of the plate.

In addition to the above conventions, we remark that the total boundary of the prescribed surface tractions (∂B_T) includes the boundaries $\partial B^{(f)} = \partial B^{(uf)} \cup \partial B^{(lf)}$ and part of the lateral boundary denoted by $\partial B_T^{(lat)}$.

Similarly, the total boundary of the given displacements (∂B_u) includes the boundaries $\partial B^{(f)} = \partial B^{(uf)} \cup \partial B^{(lf)}$ and part of the lateral boundary denoted by $\partial B_u^{(lat)}$.

Now as for the loads set to the formulation of the problem CPT, we consider the following. First, we assume that on the top or/ and bottom of the plate there is a normal load distribution $q(x_1, x_2)$. There is no matter if the algebraic sum of the vertical load q is on the positive or

negative direction of the x_3-axis. Also it is incompatible to the model of CPT to consider this load on the lateral surface, because it has negligible influence on the boundary conditions due to the smallness of the thickness of the plate. For this reason the aforementioned load q will not be treated in curvilinear integrals (as seem on next sections) as the residual external loads (tractions), but is going to appear on the volume integrals of the variational equations and as a consequence on the equations of motion of the vibrating plate. Thus, the assumption of the thin plate results to the fact that the load q is applied on the mid-surface Ω of the plate [or the plane $(x_1, x_2, 0)$]. Let Ω be the common projection of the upper and lower faces of the plate on its mid-surface. The last is surrounded by the curve Γ, which is the projection of the (vertical) lateral boundary of the mid-surface.
Respectively to the above notation, let $d\omega = dx_1 dx_2$ be an infinitesimal element of the domain Ω and $d\gamma$ an infinitesimal arc of the curve Γ.

Second, at the edge of the plate, we have surface-distributed loads (surface tractions), whose components are going to be analyzed below.

Generally we have, $\boldsymbol{T}(\boldsymbol{x};t) = \boldsymbol{T}_0(\boldsymbol{x})$, where $\boldsymbol{x} \in \partial B_T$ and

$$\boldsymbol{T}_0(\boldsymbol{x}) = \hat{T}_1(x_1, x_2, x_3)\,\boldsymbol{e}_{x_1} + \hat{T}_2(x_1, x_2, x_3)\,\boldsymbol{e}_{x_2} + \hat{T}_3(x_1, x_2, x_3)\,\boldsymbol{e}_{x_3}$$

Note that we consider here the surface tractions independent of the time variable.

Now separating the components of the surface tractions along the three axis of the Cartesian coordinate system, we get

$$\hat{T}_1(x_1, x_2, x_3) = a_{T0}(x_1, x_2) + a_{T1}(x_1, x_2)\,x_3 = a_{T0}(\gamma) + a_{T1}(\gamma)\,x_3 \tag{1}$$

$$\hat{T}_2(x_1, x_2, x_3) = b_{T0}(x_1, x_2) + b_{T1}(x_1, x_2)\,x_3 = b_{T0}(\gamma) + b_{T1}(\gamma)\,x_3 \tag{2}$$

$$\hat{T}_3(x_1, x_2, x_3) = c_{T0}(x_1, x_2) = c_{T0}(\gamma) \tag{3}$$

The above configuration of the surface tractions, including the arbitrary but appropriate functions a_{T0}, a_{T1}, b_{T0}, b_{T1} and c_{T0} is compatible with the initial assumptions of the kinematic model of the CPT. Due to the inextensibility assumption (section 1.2.2) and the smallness of the thickness of the plate, there is no dependence of the tractions along the x_3-axis $(\hat{T}_3)$ from the x_3 spatial variable. To express the above differently, our model cannot carry shear strains apart from those existing parallel to $x_1 x_2$-plane.

The in-plane tractions ($\hat{T}_1$ and $\hat{T}_2$) are linearly dependent from the x_3 variable, fact that is consistent with the normality assumption (section 1.2.3), which restricts each cross-section of the plate to remain normal to the mid-surface during the deformation.

Highlight also that the notation of the zero sub index is essential for the first part of the right-hand side of the Eqs. (1), (2) and (3), because we want to show the dependence of the a_{T0}, b_{T0}, c_{T0} functions from the curve Γ (γ-arc around the curve) of the lateral boundary, on which the x_3-variable is zero, whereas the functions a_{T1} and b_{T1} notated by unit sub index declare the linear dependence of the surface tractions $\hat{T}_1$, $\hat{T}_2$ from the vertical spatial variable.

As for the specific parts of the boundary, where displacements are prescribed, we assume the following boundary conditions. These kinematic boundary conditions are alternatively called essential conditions of the problem, because they are considered as a priori constraint affecting the space of the admissible functions and variations of the problem of CPT.

Generally the form of the given displacements is,

$$\boldsymbol{u}(\boldsymbol{x};t) = \boldsymbol{u}_0(\boldsymbol{x}) = \text{given} , \qquad \text{where } \boldsymbol{x} \in \partial B_u \quad \text{and}$$

$$\boldsymbol{u}_0(\boldsymbol{x}) = u_1(x_1, x_2, x_3)\, \boldsymbol{e}_{x_1} + u_2(x_1, x_2, x_3)\, \boldsymbol{e}_{x_2} + u_3(x_1, x_2, x_3)\, \boldsymbol{e}_{x_3}$$

Note that we consider here the above displacements independent of the time variable.

Now separating the components of the displacement field on the boundary along the three axes of the Cartesian coordinate system, we get

$$u_1(x_1, x_2, x_3) = a_{0u}(x_1, x_2) + a_{1u}(x_1, x_2)\, x_3 = a_{0u}(\gamma) + a_{1u}(\gamma)\, x_3 , \tag{4}$$

$$u_2(x_1, x_2, x_3) = b_{0u}(x_1, x_2) + b_{1u}(x_1, x_2)\, x_3 = b_{0u}(\gamma) + b_{1u}(\gamma)\, x_3 \tag{5}$$

$$u_3(x_1, x_2, x_3) = c_{0u}(x_1, x_2) = c_{0u}(\gamma) \tag{6}$$

The notation follows the same rationality as this of the surface tractions, expressed above. The only difference is the form of the functions a_{0u}, a_{1u}, b_{0u}, b_{1u}, c_{0u}. However they must be compatible with the “nature” of our problem, as exactly the above functions a_{T0}, a_{T1}, b_{T0}, b_{T1} and c_{T0} are.

Further the rightness and compatibility of the above form of the essential conditions, is verified by the initial assumptions of the modelling of the problem of CPT, and specifically by the normality and the straightness assumptions of the sections 1.2.3 and 1.2.1 respectively.

In conclusion all the aforementioned boundary and loading conditions, leads to the fact that parallel to the mid-surface (in-plane motion) there are two contributions. The first are stretching actions due to loads at the edge of the plate which act parallel to the mid-surface of the plate. The second contribution is attributed to bending.

3. Kinematics of Thin Plates

The **in-plane displacements** (due to the total loads acting on the plate) can be approximated by a **few terms of the Taylor expansion** around each point $(x_1, x_2, 0)$ of the mid-surface, with respect to $x_3 \in [-h/2, h/2]$. We choose to expand Taylor with respect to x_3- axis (namely along the smallest dimension, -thickness of the plate), since Taylor's expansions (polynomials) are adequate approximations only in a small region $(-h/2, h/2)$ around the central points $(x_1, x_2, 0)$. Thus, the form of the $u, v-$components of the displacement is assumed of the form:

$$u(x_1,x_2,x_3;t) = u(x_1,x_2,0;t) + \frac{(x_3-0)}{1!}\frac{\partial u(x_1,x_2,0;t)}{\partial x_3} + \frac{(x_3-0)^2}{2!}\frac{\partial^2 u(x_1,x_2,0;t)}{\partial^2 x_3} + $$
$$+ \frac{x_3-0}{3!}^3 \frac{\partial^3 u \;\; x_1,x_2,0;t}{\partial^3 x_3} + \cdots \tag{1}$$

$$v(x_1,x_2,x_3;t) = v(x_1,x_2,0;t) + \frac{(x_3-0)}{1!}\frac{\partial v(x_1,x_2,0;t)}{\partial x_3} + \frac{(x_3-0)^2}{2!}\frac{\partial^2 v(x_1,x_2,0;t)}{\partial^2 x_3} + $$
$$+ \frac{(x_3-0)^3}{3!}\frac{\partial^3 v(x_1,x_2,0;t)}{\partial^3 x_3} + \cdots \tag{2}$$

At this point, using the normality assumption (1.2.3.), we have:

$$\gamma_{13} = \frac{\partial u}{\partial x_3} + \frac{\partial w}{\partial x_1} = 0 \Rightarrow \frac{\partial u}{\partial x_3} = -\frac{\partial w}{\partial x_1} \tag{3}$$
$$\gamma_{23} = \frac{\partial v}{\partial x_3} + \frac{\partial w}{\partial x_2} = 0 \Rightarrow \frac{\partial v}{\partial x_3} = -\frac{\partial w}{\partial x_2} \tag{4}$$

Note that the Eqs. (3) and (4) coincidence with the Eqs. (1) and (2) of the section 1.3.3, which have been proved above.

According to the small-strain assumption of the CPT, the higher order derivatives of the relationships (1) and (2) are neglected.

Thus, neglecting second- and higher-order terms in the expansions (1) and (2), and taking into account Eqs. (3) and (4), we obtain:

$$u(x_1,x_2,x_3;t) = u(x_1,x_2,0;t) - x_3\frac{\partial w(x_1,x_2,0;t)}{\partial x_1}$$
$$v(x_1,x_2,x_3;t) = v(x_1,x_2,0;t) - x_3\frac{\partial w(x_1,x_2,0;t)}{\partial x_2}$$

Also the elimination of the higher-order terms in these equations is essential and compatible with the basic assumptions of our model and specifically with the straightness assumption

(1.2.1). This hypothesis points out that the in plane displacements u, v are linearly dependent from the x_3 variable and they cannot be expressed from higher-order terms, including x_3^2, x_3^3 and so on and so forth.

Adopting the notation

$$u(x_1,x_2,0;t)=u_0(x_1,x_2;t), \qquad v(x_1,x_2,0;t)=v_0(x_1,x_2;t),$$
$$\text{and} \qquad w(x_1,x_2,0;t)=w_0(x_1,x_2;t),$$

we obtain the following model of the displacement field:

$$u(x_1,x_2,x_3;t) = u_0(x_1,x_2;t) - x_3\frac{\partial w_0(x_1,x_2;t)}{\partial x_1} \tag{5}$$

$$v(x_1,x_2,x_3;t) = v_0(x_1,x_2;t) - x_3\frac{\partial w_0(x_1,x_2;t)}{\partial x_2} \tag{6}$$

$$w(x_1,x_2,x_3;t) = w_0(x_1,x_2;t) \tag{7}$$

We notice that the Eq. (7) is compatible with the inextensibility assumption (1.2.2), which declares that each point of the plate is subjected to the same vertical displacement w_0.

On the basis of Eqs. (5) – (7), we conclude that the displacement field (u, v, w) is fully described in terms of deformation of the mid-surface (u_0, v_0, w_0). As far as the strain field, we have:

$$e_{11}=e_{uu}=\frac{\partial u(x_1,x_2,x_3;t)}{\partial x_1} = \frac{\partial u_0(x_1,x_2;t)}{\partial x_1}-x_3\frac{\partial^2 w_0(x_1,x_2;t)}{\partial^2 x_1} \tag{8}$$

$$e_{22}=e_{vv}=\frac{\partial v(x_1,x_2,x_3;t)}{\partial x_2} = \frac{\partial v_0(x_1,x_2;t)}{\partial x_2}-x_3\frac{\partial^2 w_0(x_1,x_2;t)}{\partial^2 x_2} \tag{9}$$

$$e_{33}=e_{ww}=\frac{\partial w(x_1,x_2,x_3;t)}{\partial x_3}=0 \quad \text{the above assumption}$$

$$e_{12}=e_{21}=e_{uv}=e_{vu} = \frac{1}{2}\left(\frac{\partial u}{\partial x_2}+\frac{\partial v}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial u_0}{\partial x_2}-x_3\frac{\partial^2 w_0}{\partial x_2 \partial x_1}+\frac{\partial v_0}{\partial x_1}-x_3\frac{\partial^2 w_0}{\partial x_1 \partial x_2}\right) =$$
$$= \frac{1}{2}\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}\right) - x_3\frac{\partial^2 w_0}{\partial x_1 \partial x_2}$$

thus,

$$2e_{12}=\gamma_{12}=\frac{\partial u}{\partial x_2}+\frac{\partial v}{\partial x_1}=\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}\right)-2x_3\frac{\partial^2 w_0}{\partial x_1 \partial x_2}. \tag{10}$$

$$e_{32}=e_{23}=e_{wv}=e_{vw}=\frac{1}{2}\left(\frac{\partial w}{\partial x_2}+\frac{\partial v}{\partial x_3}\right) = \frac{1}{2}\left(\cancel{\frac{\partial w_0}{\partial x_2}}+\cancel{\frac{\partial v_0}{\partial x_3}}-\cancel{\frac{\partial w_0}{\partial x_2}}-x_3\cancel{\frac{\partial^2 w}{\partial x_3 \partial x_2}}\right)=0$$

$$e_{13}=e_{31}=e_{uw}=e_{wu}=\frac{1}{2}\left(\frac{\partial w}{\partial x_1}+\frac{\partial u}{\partial x_3}\right) = \frac{1}{2}\left(\cancel{\frac{\partial w_0}{\partial x_1}}+\cancel{\frac{\partial u_0}{\partial x_3}}-\cancel{\frac{\partial w_0}{\partial x_1}}-x_3\cancel{\frac{\partial^2 w_0}{\partial x_3 \partial x_1}}\right)=0$$

Generally refer that on the literature, the model of CPT does not taking for granted the non-zero in plane displacements (u_0, v_0). However, in case of the existence of an in-plane external loading condition or the assumption of heterogeneous material of the plate, the in-plane displacements and consequently in-plane strains are nonzero. Thus, due to the assumptions of the kinematic model presented on the previous sections 1 and 2, our choice of displacement and strain field is rational.

4. Equations of Motion- Variational Principles

Now we are going to produce the differential equation of motion of the plate and its boundary conditions, replacing the expressions of the displacement field to the variational equation and using the Hamilton's Principle in Elastodynamics [*Athanassoulis G.A. (2016)*, *Hamilton's Principle in Elastodynamics*, NTUA Lecture Notes of *Functional Analysis*].

We formulate the Elastodynamic Lagrangian function in a constraint form, which means we impose as a priori constraint the condition $u_i(\boldsymbol{x};t) = \hat{u}_i(\boldsymbol{x};t) = given, \boldsymbol{x} \in \partial B_u$ (essential condition):

$$\mathrm{L}\left[\boldsymbol{u}(\cdot;t)\right] = \iiint_B \left[K(\dot{\boldsymbol{u}}) - U(\boldsymbol{e})\right] dV + \iint_{\partial B_T} \hat{T}_i u_i \, dS$$

Next, we have to define the action functional, corresponding to the above Lagrangian function:

$$S\left[\boldsymbol{u}(\cdot,\cdot)\right] = \int_{t_1}^{t_2} \mathrm{L}\left[\boldsymbol{u}(\cdot;t)\right] dt$$

In order to find the differential equations of the CPT, we have to find the stationary points of the action functional (Hamilton's Principle):

$$\delta S\left[\boldsymbol{u};\delta\boldsymbol{u}\right] = \delta \int_{t_1}^{t_2} \mathrm{L}\left[\boldsymbol{u}(\cdot;t)\right] dt = 0 , \qquad \forall\, \delta\boldsymbol{u} \in \left\{ \begin{matrix} \text{space of admissible} \\ \text{variations} \end{matrix} \right\} \quad \Leftrightarrow$$

$$\delta \int_{t_1}^{t_2} \iiint_B \left[K(\dot{\boldsymbol{u}}) - U(\boldsymbol{e})\right] dV\, dt + \delta \int_{t_1}^{t_2} \iint_{\partial B_T} \hat{T}_i u_i \, dS\, dt = 0 , \quad \forall\, \delta\boldsymbol{u} \in \left\{ \begin{matrix} \text{space of admissible} \\ \text{variations} \end{matrix} \right\}$$

$$\int_{t_1}^{t_2} \iiint_B \delta K(\dot{\boldsymbol{u}}) dV\, dt - \int_{t_1}^{t_2} \iiint_B \delta U(\boldsymbol{e}) dV\, dt + \int_{t_1}^{t_2} \iint_{\partial B_T} \hat{T}_i\, \delta u_i\, dS\, dt = 0, \qquad (1)$$

$$\forall\, \delta\boldsymbol{u} \in \left\{\text{space of admissible variations}\right\}.$$

Now we calculate separately the terms of the above variational equation:

4.1. Variation of the Kinetic-Energy Part

The calculation of the kinetic-energy part of the action functional is standard. Integrating by parts the time integral, we find:

$$\delta J_K = \int_{t_1}^{t_2} \iiint_B \delta K(\dot{\boldsymbol{u}}) dV\, dt = \int_{t_1}^{t_2} \iiint_B \frac{1}{2} \rho\, \dot{u}_i\, \delta \dot{u}_i\, dV\, dt =$$

$$= -\int_{t_1}^{t_2} \iiint_B \rho\, \ddot{u}_i\, \delta u_i\, dV\, dt,$$

Thus, $\delta J_K = -\int_{t_1}^{t_2}\iiint_B \rho\ \left(\ddot{u}_1\delta u_1 + \ddot{u}_2\delta u_2 + \ddot{u}_3\delta u_3\right)\ dV\,dt =$

$$= -\int_{t_1}^{t_2}\iiint_B \rho\ \left(\ddot{u}\,\delta u + \ddot{v}\,\delta v + \ddot{w}\,\delta w\right)\ dV\,dt\ , \tag{2}$$

where $\ddot{u} = \ddot{u}_0 - x_3\dfrac{\partial \ddot{w}_0}{\partial x_1},\quad \ddot{v} = \ddot{v}_0 - x_3\dfrac{\partial \ddot{w}_0}{\partial x_2}$ (3)

$$\delta u = \delta u_0 - x_3\frac{\partial \delta w_0}{\partial x_1},\quad \delta v = \delta v_0 - x_3\frac{\partial \delta w_0}{\partial x_2}. \tag{4}$$

From Eqs. (2), (3) and (4), the variation of the kinetic part takes the form:

$$\delta J_K = -\int_{t_1}^{t_2}\iiint_B \rho\left\{\begin{array}{l}\left(\ddot{u}_0 - x_3\dfrac{\partial \ddot{w}_0}{\partial x_1}\right)\left(\delta u_0 - x_3\dfrac{\partial \delta w_0}{\partial x_1}\right) + \\ \qquad + \left(\ddot{v}_0 - x_3\dfrac{\partial \ddot{w}_0}{\partial x_2}\right)\left(\delta v_0 - x_3\dfrac{\partial \delta w_0}{\partial x_2}\right) + \ddot{w}_0\,\delta w_0\end{array}\right\} dVdt =$$

$$= -\int_{t_1}^{t_2}\iiint_B \rho\left\{\begin{array}{l}\ddot{u}_0\delta u_0 - x_3\,\delta u_0\dfrac{\partial \ddot{w}_0}{\partial x_1} - x_3\,\ddot{u}_0\dfrac{\partial \delta w_0}{\partial x_1} + x_3^2\dfrac{\partial \ddot{w}_0}{\partial x_1}\dfrac{\partial \delta w_0}{\partial x_1} + \\ + \ddot{v}_0\delta v_0 - x_3\,\delta v_0\dfrac{\partial \ddot{w}_0}{\partial x_2} - x_3\,\ddot{v}_0\dfrac{\partial \delta w_0}{\partial x_2} + x_3^2\dfrac{\partial \ddot{w}_0}{\partial x_2}\dfrac{\partial \delta w_0}{\partial x_2} + \ddot{w}_0\,\delta w_0\end{array}\right\} dV\,dt =$$

$$= -\int_{t_1}^{t_2}\iiint_B \rho\left\{\begin{array}{l}\left(\ddot{u}_0\delta u_0 + \ddot{v}_0\delta v_0 + \ddot{w}_0\,\delta w_0\right) + x_3^2\left(\dfrac{\partial \ddot{w}_0}{\partial x_1}\dfrac{\partial \delta w_0}{\partial x_1} + \dfrac{\partial \ddot{w}_0}{\partial x_2}\dfrac{\partial \delta w_0}{\partial x_2}\right) - \\ - x_3\left(\ddot{u}_0\dfrac{\partial \delta w_0}{\partial x_1} + \delta u_0\dfrac{\partial \ddot{w}_0}{\partial x_1} + \ddot{v}_0\dfrac{\partial \delta w_0}{\partial x_2} + \delta v_0\dfrac{\partial \ddot{w}_0}{\partial x_2}\right)\end{array}\right\} dV\,dt$$

$$\begin{aligned}\delta J_K = &-\int_{t_1}^{t_2}\iiint_B \rho\left(\ddot{u}_0 - x_3\frac{\partial \ddot{w}_0}{\partial x_1}\right)\delta u_0\ dV\,dt - \int_{t_1}^{t_2}\iiint_B \rho\left(\ddot{v}_0 - x_3\frac{\partial \ddot{w}_0}{\partial x_2}\right)\delta v_0\ dV\,dt - \\ &-\int_{t_1}^{t_2}\iiint_B \rho\ddot{w}_0\delta w_0 dVdt - \int_{t_1}^{t_2}\iiint_B \rho\left(x_3^2\frac{\partial \ddot{w}_0}{\partial x_1} - x_3\ddot{u}_0\right)\frac{\partial \delta w_0}{\partial x_1}\ dV\,dt - \\ &-\int_{t_1}^{t_2}\iiint_B \rho\left(x_3^2\frac{\partial \ddot{w}_0}{\partial x_2} - x_3\ddot{v}_0\right)\frac{\partial \delta w_0}{\partial x_2} dVdt\end{aligned} \tag{5}$$

In Eq. (5) there are not only the variations δu_0, δv_0, δw_0, but also the variation of the first spatial derivatives (x_1, x_2-derivatives) of δw_0. To eliminate the later we perform an **integration by parts with respect to the corresponding spatial variables**. These integrations by parts will generate boundary terms, which will contribute to the construction of the appropriate boundary conditions of the CPT. For further simplification, we neglect for the present calculations the time integral. Thus,

$$\delta I_{\delta w_0, x_1} = \iiint_B \rho\left(x_3^2\frac{\partial \ddot{w}_0}{\partial x_1} - x_3\ddot{u}_0\right)\frac{\partial \delta w_0}{\partial x_1}dV =$$

$$= \iint_{\partial B^{(lat)}} \rho\left(x_3^2\frac{\partial \ddot{w}_0}{\partial x_1} - x_3\ddot{u}_0\right)n_{x_1}\delta w_0\, dS - \iiint_B \rho\left(x_3^2\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1} - x_3\frac{\partial \ddot{u}_0}{\partial x_1}\right)\delta w_0 dV, \qquad (6a)$$

$$\delta I_{\delta w_0, x_2} = \iiint_B \rho\left(x_3^2\frac{\partial \ddot{w}_0}{\partial x_2} - x_3\ddot{v}_0\right)\frac{\partial \delta w_0}{\partial x_2}dV =$$

$$= \iint_{\partial B^{(lat)}} \rho\left(x_3^2\frac{\partial \ddot{w}_0}{\partial x_2} - x_3\ddot{v}_0\right)n_{x_2}\delta w_0\, dS - \iiint_B \rho\left(x_3^2\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2} - x_3\frac{\partial \ddot{v}_0}{\partial x_2}\right)\delta w_0 dV. \qquad (6b)$$

Consequently, substituting Eqs. (6a) and (6b) in Eq. (5), we get

$$\delta J_K = -\int_{t_1}^{t_2}\iiint_B \rho\left(\ddot{u}_0 - x_3\frac{\partial \ddot{w}_0}{\partial x_1}\right)\delta u_0\, dV\,dt - \int_{t_1}^{t_2}\iiint_B \rho\left(\ddot{v}_0 - x_3\frac{\partial \ddot{w}_0}{\partial x_2}\right)\delta v_0\, dV\,dt -$$

$$-\int_{t_1}^{t_2}\iiint_B \rho\ddot{w}_0\delta w_0 dVdt - \int_{t_1}^{t_2}\iint_{\partial B^{(lat)}}\left[\rho\left(x_3^2\frac{\partial \ddot{w}_0}{\partial x_1} - x_3\ddot{u}_0\right)n_{x_1} + \rho\left(x_3^2\frac{\partial \ddot{w}_0}{\partial x_2} - x_3\ddot{v}_0\right)n_{x_2}\right]\delta w_0\, dS\,dt +$$

$$+\int_{t_1}^{t_2}\iiint_B\left[\rho x_3^2\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1} + \frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right) - \rho x_3\left(\frac{\partial \ddot{u}_0}{\partial x_1} + \frac{\partial \ddot{v}_0}{\partial x_2}\right)\right]\delta w_0\, dV\,dt =$$

$$= -\int_{t_1}^{t_2}\iiint_B \rho\left(\ddot{u}_0 - x_3\frac{\partial \ddot{w}_0}{\partial x_1}\right)\delta u_0\, dV\,dt - \int_{t_1}^{t_2}\iiint_B \rho\left(\ddot{v}_0 - x_3\frac{\partial \ddot{w}_0}{\partial x_2}\right)\delta v_0\, dV\,dt -$$

$$-\int_{t_1}^{t_2}\iiint_B\left[\rho\ddot{w}_0 - \rho x_3^2\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1} + \frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right) + \rho x_3\left(\frac{\partial \ddot{u}_0}{\partial x_1} + \frac{\partial \ddot{v}_0}{\partial x_2}\right)\right]\delta w_0\, dVdt -$$

$$-\int_{t_1}^{t_2}\iint_{\partial B^{(lat)}}\left[\rho\left(x_3^2\frac{\partial \ddot{w}_0}{\partial x_1} - x_3\ddot{u}_0\right)n_{x_1} + \rho\left(x_3^2\frac{\partial \ddot{w}_0}{\partial x_2} - x_3\ddot{v}_0\right)n_{x_2}\right]\delta w_0\, dS\,dt \qquad (6c)$$

Eq. (6c) can be simplified, by observing that the x_3-dependence of all integrands is explicit, and thus the vertical integration can be performed explicitly. To this end, it is convenient to define the "mass-moment" quantities:

$$I_i = \int_{-h/2}^{h/2} \rho x_3^i \, dx_3, \quad i = 0, 1, 2, ..., 6 .$$

Note that all odd-order I_i's are zero. More precisely, we have

$$I_1 = I_3 = I_5 = 0,$$

$$I_0 = \int_{-h/2}^{h/2} \rho \, dx_3 = \rho h, \qquad I_2 = \int_{-h/2}^{h/2} \rho x_3^2 \, dx_3 = \rho \frac{h^3}{12},$$

$$I_4 = \int_{-h/2}^{h/2} \rho x_3^4 \, dx_3 = \rho \frac{h^5}{80}, \qquad I_6 = \int_{-h/2}^{h/2} \rho x_3^6 \, dx_3 = \rho \frac{h^7}{448}.$$

To treat the volume-integral terms (appearing in the first and second row of the right-most side of Eq. (6)), we decompose them as follows:

$$\iiint_B (\bullet) \, dV = \iint_\Omega d\omega \int_{-h/2}^{h/2} (\bullet) \, dx_3 , \qquad d\omega = dx_1 dx_2 ,$$

where Ω is the common projection of the upper and lower faces of the plate on the mid-surface. Similarly, to treat the terms in the last row of the right-most side of Eq. (6), we have to decompose the lateral surface integral as follows:

$$\iint_{\partial B^{(lat)}} (\bullet) \, dS = \int_{-h/2}^{h/2} \oint_\Gamma (\bullet) \, d\gamma \, dx_3 ,$$

where Γ is the curve defined by the projection of the (vertical) lateral boundary on the mid-surface.

Substituting the above decomposed integrals to the Eq. (6c), we have the following,

$$\delta J_K = -\int_{t_1}^{t_2} \iint_\Omega \int_{-h/2}^{h/2} \rho \left(\ddot{u}_0 - x_3 \frac{\partial \ddot{w}_0}{\partial x_1} \right) \delta u_0 \, dx_3 \, d\omega \, dt - \int_{t_1}^{t_2} \iint_\Omega \int_{-h/2}^{h/2} \rho \left(\ddot{v}_0 - x_3 \frac{\partial \ddot{w}_0}{\partial x_2} \right) \delta v_0 \, dx_3 \, d\omega \, dt -$$

$$- \int_{t_1}^{t_2} \iint_\Omega \int_{-h/2}^{h/2} \left[\rho \ddot{w}_0 - \rho x_3^2 \left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1} + \frac{\partial^2 \ddot{w}_0}{\partial^2 x_2} \right) + \rho x_3 \left(\frac{\partial \ddot{u}_0}{\partial x_1} + \frac{\partial \ddot{v}_0}{\partial x_2} \right) \right] \delta w_0 \, dx_3 \, d\omega \, dt -$$

$$- \int_{t_1}^{t_2} \oint_\Gamma \int_{-h/2}^{h/2} \left[\rho \left(x_3^2 \frac{\partial \ddot{w}_0}{\partial x_1} - x_3 \ddot{u}_0 \right) n_{x_1} + \rho \left(x_3^2 \frac{\partial \ddot{w}_0}{\partial x_2} - x_3 \ddot{v}_0 \right) n_{x_2} \right] \delta w_0 \, dx_3 \, d\gamma \, dt \Longrightarrow$$

$$\delta J_K = -\int_{t_1}^{t_2}\iint_{\Omega} I_0 \ddot{u}_0 \,\delta u_0 \, d\omega \, dt - \int_{t_1}^{t_2}\iint_{\Omega} I_0 \ddot{v}_0 \,\delta v_0 \, d\omega \, dt -$$

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left[I_0 \ddot{w}_0 - I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1} + \frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right)\right]\delta w_0 \, d\omega \, dt - \qquad \textbf{(6`)}$$

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left[I_2 \frac{\partial \ddot{w}_0}{\partial x_1} n_{x_1} + I_2 \frac{\partial \ddot{w}_0}{\partial x_2} n_{x_2}\right]\delta w_0 \, d\gamma \, dt$$

4.2. Stress-Strain Relations and Elastic Potential Energy

According to “*Athanassoulis G.A. (2016), Hamilton’s Principle in Elastodynamics, NTUA Lecture Notes of Functional Analysis*” and “*Athanassoulis G.A. (2017), Elastic potential energy – Energy function, NTUA Lecture Notes of Functional Analysis*”, we have the general form of the elastic potential energy of the problem,

$U(\boldsymbol{e}) = \frac{1}{2}\sigma_{ij}(\boldsymbol{e})e_{ij} = \frac{1}{2}C_{ijkl}\,e_{ij}\,e_{kl}$, where the strains e_{kl} are expressed in terms of the displacement field as $e_{ij} = \frac{1}{2}\; u_{i,j} + u_{j,i}$ and $C_{ijkl} = C_{ijkl}(\boldsymbol{x})$ are the stiffness coefficients (material properties).

As for the variation of the elastic potential energy, we derive the following:

$$\delta U(\boldsymbol{e}) = \frac{1}{2}\delta\sigma_{ij}\,e_{ij} + \frac{1}{2}\sigma_{ij}\,\delta e_{ij} \xrightarrow{\text{Hooke's Law 1st term}}$$

$$\delta U(\boldsymbol{e}) = \frac{1}{2}C_{ijkl}\,\delta e_{kl}\,e_{ij} + \frac{1}{2}\sigma_{ij}\,\delta e_{ij} \xrightarrow[\text{stiffness coefficients}]{\text{Minor symmetry of matrix of}}$$

$$\delta U(\boldsymbol{e}) = \frac{1}{2}C_{klij}\,e_{ij}\,\delta e_{kl} + \frac{1}{2}\sigma_{ij}\,\delta e_{ij} \xrightarrow{\text{Hooke's Law 1st term}}$$

$$\delta U(\boldsymbol{e}) = \frac{1}{2}\sigma_{kl}\,\delta e_{kl} + \frac{1}{2}\sigma_{ij}\,\delta e_{ij} \xrightarrow{\text{dummy indexes}} \delta U(\boldsymbol{e}) = \sigma_{ij}\,\delta e_{ij}$$

Consequently,

$$U(\boldsymbol{e}) = \frac{1}{2}\;\; \sigma_{uu}e_{uu} + \sigma_{vv}e_{vv} + \sigma_{ww}e_{ww} + \sigma_{vw}\gamma_{vw} + \sigma_{wu}\gamma_{wu} + \sigma_{uv}\gamma_{uv} \;\; =$$

$$= \frac{1}{2}\;\; \sigma_{11}e_{11} + \sigma_{22}e_{22} + \sigma_{33}e_{33} + \sigma_{23}\gamma_{23} + \sigma_{31}\gamma_{31} + \sigma_{12}\gamma_{12} \;\; = \text{Voigt Notation}$$

$$= \frac{1}{2}\;\; \sigma_1 e_1 + \sigma_2 e_2 + \sigma_3 e_3 + \sigma_4\gamma_4 + \sigma_5\gamma_5 + \sigma_6\gamma_6 \qquad (1)$$

$$\delta U(\boldsymbol{e}) = \sigma_1\delta e_1 + \sigma_2\delta e_2 + \sigma_3\delta e_3 + \sigma_4\delta\gamma_4 + \sigma_5\delta\gamma_5 + \sigma_6\delta\gamma_6 \qquad (2)$$

According to the last paragraph, we observe that some of the terms of the elastic potential energy, are equal to zero. So we derive,

$$U(\boldsymbol{e}) = \frac{1}{2}\;\; \sigma_1 e_1 + \sigma_2 e_2 + \sigma_6\gamma_6 \qquad (3)$$

From (1) and the proof of (2) we have,

$$\boxed{\delta U(\boldsymbol{e}) = \sigma_1\cdot\delta e_1 + \sigma_2\cdot\delta e_2 + \sigma_6\cdot\delta\gamma_6} \qquad (4)$$

From (5) of the section 3: $\delta e_1 = \delta e_{11} = \delta e_{uu} = \dfrac{\partial\delta u_0(x_1,x_2;t)}{\partial x_1} - x_3\dfrac{\partial^2\delta w_0(x_1,x_2;t)}{\partial^2 x_1}$

From (6) of the section 3: $\delta e_2 = \delta e_{22} = \delta e_{vv} = \dfrac{\partial\delta v_0(x_1,x_2;t)}{\partial x_2} - x_3\dfrac{\partial^2\delta w_0(x_1,x_2;t)}{\partial^2 x_2}$

From (7) of the section 3: $\delta\gamma_6 = \delta\gamma_{12} = \delta\gamma_{uv} = \left(\dfrac{\partial \delta u_0}{\partial x_2} + \dfrac{\partial \delta v_0}{\partial x_1}\right) - 2x_3 \dfrac{\partial^2 \delta w_0}{\partial x_1 \partial x_2}$

4.2.1. Orthotropic, in-plane anisotropic material. Stress- Strain Relations

A wide range of engineering materials, including certain piezoelectric materials and fiber-reinforced composites (i.e. laminated plates composed of multiple orthotropic layers), are orthotropic. By definition an orthotropic material has at least two orthogonal planes of symmetry, where material properties are independent of the direction within each plane. Such materials require nine (9) independent variables (i.e. constants) in their constitutive matrices. In contrast, a material without any planes of symmetry is fully anisotropic and requires at least twenty-one (21) elastic constants (due to the symmetry of the constitutive matrices), whereas a material with an infinite number of symmetry planes (i.e. every plane is a plane of symmetry), is isotropic and requires only two elastic constants (Lame's constants) [*"An Introduction to Continuum Mechanics", Chapter 6 (2013), J.N. Reddy and "Theory and Analysis of Elastic Plates and Shells", Chapter 3 (2007), J.N. Reddy*].

By convention, the nine elastic constants in orthotropic constitutive equations are comprised of three Young's modulus of elasticity (E_1, E_2, E_3), three Poisson's ratios (v_{23}, v_{31}, v_{12}) or (v_4, v_5, v_6) and three shear moduli (G_{23}, G_{31} ,G_{12}) or (G_4, G_5 ,G_6).

According to the process followed on the Lecture Notes "Stress-Strain Relations: Hooke's Law-Orthotropic Materials (First-Principle Approach)", G.A. Athanassoulis (2016), the three-dimensional compliance matrix takes the form,

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{E_1} & -\frac{v_{21}}{E_2} & -\frac{v_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{v_{12}}{E_1} & \frac{1}{E_2} & -\frac{v_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{v_{31}}{E_1} & -\frac{v_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & & \\ 0 & 0 & 0 & & \frac{1}{G_{31}} & \\ 0 & 0 & 0 & & & \frac{1}{G_{12}} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix}$$

However, in the case of CPT, we keep only the stress-strain relations which represent the two-dimensional constitutive equations. Thus,

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{E_1} & -\frac{v_{21}}{E_2} \\ -\frac{v_{12}}{E_1} & \frac{1}{E_2} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \quad \text{and} \quad \gamma_6 = \frac{1}{G_6}\sigma_6$$

Note that, in orthotropic materials there is no interaction between the normal stresses σ_1, σ_2 and shear strain $\gamma_6 = 2e_6 = 2e_{12}$. Further, the symmetry of the compliance coefficients leads directly to the following Symmetry Relations for Poisson rations:

$$\frac{v_{12}}{E_1} = \frac{v_{21}}{E_2}$$

Now looking forward on the reduced stiffness matrix C , which is the following,

$$C = \begin{pmatrix} \frac{1}{E_1} & -\frac{v_{21}}{E_2} \\ -\frac{v_{12}}{E_1} & \frac{1}{E_2} \end{pmatrix} \qquad (1)$$

we perform Gaussian elimination (also known as row reduction) in order to express the stresses in terms of strains [https://en.wikipedia.org/wiki/Gaussian_elimination], as seem below,

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{E_1} & -\frac{v_{21}}{E_2} \\ -\frac{v_{12}}{E_1} & \frac{1}{E_2} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \quad \text{or} \quad \begin{Bmatrix} e_1 = \frac{1}{E_1}\sigma_1 - \frac{v_{21}}{E_2}\sigma_2 \\ e_2 = -\frac{v_{12}}{E_1}\sigma_1 + \frac{1}{E_2}\sigma_2 \end{Bmatrix}$$

and regarding the system of equations, with unknown quantities the stresses σ_1, σ_2 ,

$$\begin{Bmatrix} \frac{1}{E_1}\sigma_1 - \frac{v_{21}}{E_2}\sigma_2 = e_1 \\ -\frac{v_{12}}{E_1}\sigma_1 + \frac{1}{E_2}\sigma_2 = e_2 \end{Bmatrix} \sim \left[\begin{array}{cc|c} \frac{1}{E_1} & -\frac{v_{21}}{E_2} & e_1 \\ -\frac{v_{12}}{E_1} & \frac{1}{E_2} & e_2 \end{array}\right] \xrightarrow{L_2 + v_{12} L_1 \to L_2}$$

$$\left[\begin{array}{cc|c} \frac{1}{E_1} & -\frac{v_{21}}{E_2} & e_1 \\ 0 & \frac{1-v_{12}v_{21}}{E_2} & e_2 + v_{12} e_1 \end{array}\right] \xrightarrow{L_1 + \frac{v_{21}}{1-v_{12}v_{21}} L_1 \to L_2}$$

$$\left[\begin{array}{cc|c} \frac{1}{E_1} & 0 & \frac{1}{1-v_{12}v_{21}} e_1 + \frac{v_{21}}{1-v_{12}v_{21}} e_2 \\ 0 & \frac{1-v_{12}v_{21}}{E_2} & e_2 + v_{12} e_1 \end{array}\right] \xrightarrow[L_2 \times \frac{E_2}{1-v_{12}v_{21}} \to L_2]{L_1 \times E_1 \to L_1} \quad \text{echelon or triangular form}$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{E_1}{1-v_{12}v_{21}} e_1 + \frac{v_{21}E_1}{1-v_{12}v_{21}} e_2 \\ 0 & 1 & \frac{E_2}{1-v_{12}v_{21}} \left(e_2 + v_{12} e_1\right) \end{array}\right] \sim \quad [\text{identity matrix}]$$

$$\left\{\begin{aligned}\sigma_1 &= \frac{E_1}{1-v_{12}v_{21}}\,e_1+\frac{v_{21}E_1}{1-v_{12}v_{21}}\,e_2\\ \sigma_2 &= \frac{v_{12}E_2}{1-v_{12}v_{21}}\,e_1+\frac{E_2}{1-v_{12}v_{21}}\,e_2\end{aligned}\right\} \text{ or in matrix form}$$

$$\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\dfrac{E_1}{1-v_{12}v_{21}} & \dfrac{v_{21}E_1}{1-v_{12}v_{21}}\\ \dfrac{v_{12}E_2}{1-v_{12}v_{21}} & \dfrac{E_2}{1-v_{12}v_{21}}\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix}$$

These results of elastic coefficients $C_{11} = \dfrac{E_1}{1-v_{12}v_{21}}$, $C_{22} = \dfrac{E_2}{1-v_{12}v_{21}}$ and

$C_{12} = C_{21} = \dfrac{v_{21}E_1}{1-v_{12}v_{21}}$ due to symmetry of matrix C , are verified according to the remarks of the Lecture Notes “Stress-Strain Relations: Hooke’s Law-Orthotropic Materials (First-Principle Approach)”, G.A. Athanassoulis (2016). Noticing the coefficients C_{11}, C_{22}, C_{12}, C_{21} and calculating the quantity $\Delta = \dfrac{1-v_{12}v_{21}}{E_1E_2E_3}$ and after substituting it on the corresponding relations (11) of the Lecture Notes, we get,

$$C_{11} = \frac{1-\cancel{v_{23}v_{32}}}{E_2E_3\Delta} = \frac{1}{\cancel{E_2E_3}\dfrac{1-v_{12}v_{21}}{E_1\,\cancel{E_2E_3}}} = \frac{E_1}{1-v_{12}v_{21}} \tag{2}$$

$$C_{22} = \frac{1-\cancel{v_{13}v_{31}}}{E_1E_3\Delta} = \frac{1}{\cancel{E_1E_3}\dfrac{1-v_{12}v_{21}}{\cancel{E_1}\,E_2\,\cancel{E_3}}} = \frac{E_2}{1-v_{12}v_{21}} \tag{3}$$

$$C_{12} = C_{21} = \frac{v_{21}+\cancel{v_{23}v_{31}}}{E_2E_3\Delta} = \frac{v_{21}}{\cancel{E_2E_3}\dfrac{1-v_{12}v_{21}}{E_1\,\cancel{E_2E_3}}} = \frac{v_{21}E_1}{1-v_{12}v_{21}} \tag{4}$$

The last relationships verify our results of the stress – strain relationships of <u>an orthotropic but in-plane anisotropic</u> material, which are finally,

$$\sigma_{11} = \frac{E_1}{1-v_{12}v_{21}}\,e_{11}+\frac{v_{21}E_1}{1-v_{12}v_{21}}\,e_{22} \tag{5}$$

$$\sigma_{22} = \frac{v_{12}E_2}{1-v_{12}v_{21}}\,e_{11}+\frac{E_2}{1-v_{12}v_{21}}\,e_{22} \tag{6}$$

$$\sigma_{12} = \sigma_6 = G_{12}\gamma_{12} = G_6\gamma_6 \tag{7}$$

In addition, we can express the above stresses in terms of displacements, substituting the Eqs. (8), (9) and (10) of the section 3 into the Eqs. (5), (6) and (7), as seem below,

$$\sigma_{11} = \frac{E_1}{1-v_{12}v_{21}}\left(\frac{\partial u_0}{\partial x_1} - x_3\frac{\partial^2 w_0}{\partial^2 x_1}\right) + \frac{v_{21}E_1}{1-v_{12}v_{21}}\left(\frac{\partial v_0}{\partial x_2} - x_3\frac{\partial^2 w_0}{\partial^2 x_2}\right) \tag{8}$$

$$\sigma_{22} = \frac{v_{12}E_2}{1-v_{12}v_{21}}\left(\frac{\partial u_0}{\partial x_1} - x_3\frac{\partial^2 w_0}{\partial^2 x_1}\right) + \frac{E_2}{1-v_{12}v_{21}}\left(\frac{\partial v_0}{\partial x_2} - x_3\frac{\partial^2 w_0}{\partial^2 x_2}\right) \tag{9}$$

$$\sigma_6 = \sigma_{12} = \sigma_{21} = G_6\gamma_6 = G_6\left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1} - 2x_3\frac{\partial^2 w_0}{\partial x_1 \partial x_2}\right) \tag{10}$$

4.2.2. Orthotropic, in-plane isotropic material. Stress - Strain Relations

The simplest way to derive the stress – strain relations of an orthotropic but in-plane isotropic material, is to notice and elaborate appropriately the stress – strain relations of the orthotropic in-plane anisotropic plate. Thus, remembering the matrix of stiffness coefficients (1) of the section 4.2.1 and the fact that the modulus of elasticity and the elastic coefficients of an elastically isotropic solid body are constant regardless of the rotation of the Cartesian system [*Lecture Notes*, *"Stress-Strain Relations: Some Formal Considerations on Hooke's Law applied to Isotropic Materials"* , *G.A. Athanassoulis 2015*], we have:

From Eq. (2) of the section 4.2.1, $C_{11} = \dfrac{E}{1-v^2}$

From Eq. (3) of the section 4.2.1, $C_{22} = \dfrac{E}{1-v^2}$

From Eq. (4) of the section 4.2.1, $C_{12} = C_{21} = \dfrac{vE}{1-v^2}$

Consequently, the matrix of the Eqs. (1) in the section 4.2.1., is converted to,

$$\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\dfrac{E}{1-v^2} & \dfrac{vE}{1-v^2}\\ \dfrac{vE}{1-v^2} & \dfrac{E}{1-v^2}\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix} \Leftrightarrow \begin{Bmatrix}\sigma_1 = \dfrac{E}{1-v^2}e_1 + \dfrac{vE}{1-v^2}e_2\\ \sigma_2 = \dfrac{vE}{1-v^2}e_1 + \dfrac{E}{1-v^2}e_2\end{Bmatrix} \Rightarrow \left\{\begin{array}{l}\text{For the specific}\\ \text{model of CPT}\end{array}\right.$$

$$\begin{Bmatrix}\sigma_1 = \dfrac{E}{1-v^2}\left(\dfrac{\partial u_0}{\partial x_1} - x_3\dfrac{\partial^2 w_0}{\partial^2 x_1}\right) + \dfrac{vE}{1-v^2}\left(\dfrac{\partial v_0}{\partial x_2} - x_3\dfrac{\partial^2 w_0}{\partial^2 x_2}\right)\\ \sigma_2 = \dfrac{vE}{1-v^2}\left(\dfrac{\partial u_0}{\partial x_1} - x_3\dfrac{\partial^2 w_0}{\partial^2 x_1}\right) + \dfrac{E}{1-v^2}\left(\dfrac{\partial v_0}{\partial x_2} - x_3\dfrac{\partial^2 w_0}{\partial^2 x_2}\right)\end{Bmatrix} \Leftrightarrow$$

$$\begin{Bmatrix}\sigma_1 = \dfrac{E}{1-v^2}\dfrac{\partial u_0}{\partial x_1} - x_3\dfrac{E}{1-v^2}\dfrac{\partial^2 w_0}{\partial^2 x_1} + \dfrac{vE}{1-v^2}\dfrac{\partial v_0}{\partial x_2} - x_3\dfrac{vE}{1-v^2}\dfrac{\partial^2 w_0}{\partial^2 x_2}\\ \sigma_2 = \dfrac{vE}{1-v^2}\dfrac{\partial u_0}{\partial x_1} - x_3\dfrac{vE}{1-v^2}\dfrac{\partial^2 w_0}{\partial^2 x_1} + \dfrac{E}{1-v^2}\dfrac{\partial v_0}{\partial x_2} - x_3\dfrac{E}{1-v^2}\dfrac{\partial^2 w_0}{\partial^2 x_2}\end{Bmatrix} \tag{5}$$

and the in-plane stress – strain relationships are already apparent.

As for the shear stress – strain relation, we have obviously

$$\sigma_6 = G_6 \gamma_6 = G \gamma_6 = G \left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1} - 2x_3 \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \right) \tag{5`}$$

To avoid confusion and possible doubt about the selection of the appropriate stress-strain relations, we remark that the plate cannot be regarded isotropic with respect to all its directions, because the plate is in-plane isotropic as aforementioned above. Consequently, the stress-strain relations are not obtained from the general form of Hooke's Law for an isotropic, homogeneous material (continuum) and the Theorem of the most general form of a 4th order tensor of the book *"Continuum Mechanics"* [*Chandrasekharaiah & Debnath (1994), Section 2.6*] is not valid in our case. Thus, the Poisson's ratio v of the previous relations v is an identity of the material referred to its planar directions, namely x_1 and x_2-axis. This effect is quantified by,

$$v_{12} = -\frac{e_{22}}{e_{11}} = v_{21} = -\frac{e_{11}}{e_{22}} = v$$

In conclusion, is assumed that linear theory is applied and that the material of the plate is homogenous and isotropic with regard to directions in the $x_1 x_2$-plane [*E. Reissner* (*1963*) *"On the derivation of boundary conditions for the plate theory", MIT, page 179*]. The previous ascertainment means that the Poisson's ratio and the modulus of elasticity coincidence on the directions x_1 and x_2 of the material, but differ from the corresponding on the direction of its x_3-axis (vertical direction). Also, as seems on the section 4.2.1 in the context of the CPT, we keep only the stress-strain relations which represent the two-dimensional constitutive equations, so that the Poisson's ration which relates the in-plane with the vertical strains and the modulus of elasticity and shear modulus on the direction of x_3-axis are eliminated and as a consequence they will not occupy us on this quotation. Thus,

$$E_1 = E_2 = E \qquad \text{and} \qquad G_{12} = G_{21} = G_6 = = G$$

In addition to the above and for completeness gift, we refer the so called Lame's constants λ and μ, which are related to the modulus of elasticity E, the shear modulus of elasticity G and the Poisson's ratio v, as seems from the following relations [*Some Formal Considerations on Hooke's Law applied to Isotropic Materials", G.A. Athanassoulis (2015)* and *"An Introduction to Continuum Mechanics", J.N. Reddy (2013), 2nd edition CUP*].

$$\mu = G \quad , \quad G = \frac{E}{2(1+v)} \quad \text{and} \quad \lambda = \frac{vE}{(1-2v)(1+v)}.$$

4.3. Variation of the Elastic Potential Energy

Due to the proof of the previous section (4.2), the only variations appearing on the variation of the elastic potential energy are the variations of the strains. Note that the variations of stresses are not appear explicitly on the following relation (or the Eq. (4) of the section 4.2), because they are a priori included in the variation of the elastic potential energy. This fact is declared on the section 4.2 by the use of the Hooke's Law in conjunction with the identity of symmetry of matrix composed of the stiffness coefficients as well as the contrivances of index notation.

At this point, we keep in mind the Eqs. (8) – (10) of the section 4.2.1 or the Eqs. (6) - (8) of the section 4.2.2 and we are not going to replace the last, in order to avoid difficult and time-consuming calculations. Thus, the relationship (4) is converted to,

$$\delta U(\boldsymbol{e})=\sigma_1\left(\frac{\partial \delta u_0}{\partial x_1}-x_3\frac{\partial^2\delta w_0}{\partial^2 x_1}\right)+\sigma_2\left(\frac{\partial \delta v_0}{\partial x_2}-x_3\frac{\partial^2\delta w_0}{\partial^2 x_2}\right)+\sigma_6\left[\left(\frac{\partial \delta u_0}{\partial x_2}+\frac{\partial \delta v_0}{\partial x_1}\right)-2x_3\frac{\partial^2\delta w_0}{\partial x_1\partial x_2}\right]$$

Finally replacing (4) to the expression of the variation of the elastic potential part, we derive the last expression:

$$\delta J_U=\int_{t_1}^{t_2}\iiint_B \delta U(\boldsymbol{e})\,dV\,dt=\int_{t_1}^{t_2}\iiint_B \sigma_1\delta e_1+\sigma_2\delta e_2+\sigma_6\delta e_6\;dVdt=$$

$$=\int_{t_1}^{t_2}\iiint_B \sigma_1\frac{\partial\delta u_0}{\partial x_1}dVdt-\int_{t_1}^{t_2}\iiint_B x_3\sigma_1\frac{\partial^2\delta w_0}{\partial^2 x_1}dVdt+\int_{t_1}^{t_2}\iiint_B \sigma_2\frac{\partial\delta v_0}{\partial x_2}dVdt-$$

$$-\int_{t_1}^{t_2}\iiint_B x_3\sigma_2\frac{\partial^2\delta w_0}{\partial^2 x_2}dVdt+\int_{t_1}^{t_2}\iiint_B \sigma_6\frac{\partial\delta u_0}{\partial x_2}dVdt+\int_{t_1}^{t_2}\iiint_B \sigma_6\frac{\partial\delta v_0}{\partial x_1}dVdt-$$

$$-\int_{t_1}^{t_2}\iiint_B\left(2\sigma_6 x_3\frac{\partial^2\delta w_0}{\partial x_1\partial x_2}\right)dVdt \tag{1}$$

On the last equation (1) appear not only the variations δu_0, δv_0, δw_0 but also their x_1, x_2 – derivatives. To eliminate the later, we perform one or two (by case) **integration(s) by parts with respect to the variable $\boldsymbol{x}$**. This integrations by parts will generate boundary terms kinematic and dynamic, which will contribute to the construction of the appropriate boundary conditions of the CPT. For further simplification we neglect for the present calculations the time integral. We have also to be careful about the integration(s) by parts with respect to spatial variable, because the boundary terms of the following relations are related with the natural boundary conditions of the problem (or dynamic boundary conditions of the elastic continuum).

$$\delta I_{\delta u_0,x_1}=\iiint_B \sigma_1\frac{\partial\delta u_0}{\partial x_1}dV=$$

$$=\iint_{\partial B^{(lat)}}\sigma_1 n_{x_1}\delta u_0\,dS-\iiint_B\frac{\partial\sigma_1}{\partial x_1}\delta u_0\,dV \tag{2a}$$

$$\delta I_{\delta w_0,x_1x_1} = \iiint\limits_B x_3\,\sigma_1 \frac{\partial^2 \delta w_0}{\partial x_1{}^2} dV =$$

$$= \iint\limits_{\partial B^{(lat)}} x_3\,\sigma_1\, n_{x_1} \frac{\partial \delta w_0}{\partial x_1} dS - \iiint\limits_B x_3 \frac{\partial \sigma_1}{\partial x_1} \frac{\partial \delta w_0}{\partial x_1} dV =$$

$$= \iint\limits_{\partial B^{(lat)}} x_3\,\sigma_1\, n_{x_1} \frac{\partial \delta w_0}{\partial x_1} dS - \iint\limits_{\partial B^{(lat)}} x_3 \frac{\partial \sigma_1}{\partial x_1} n_{x_1} \delta w_0\, dS + \iiint\limits_B x_3 \frac{\partial^2 \sigma_1}{\partial^2 x_1} \delta w_0\, dV \tag{2b}$$

$$\delta I_{\delta v_0,x_2} = \iiint\limits_B \sigma_2 \frac{\partial \delta v_0}{\partial x_2} dV =$$

$$= \iint\limits_{\partial B^{(lat)}} \sigma_2\, n_{x_2} \delta v_0\, dS - \iiint\limits_B \frac{\partial \sigma_2}{\partial x_2} \delta v_0\, dV \tag{2c}$$

$$\delta I_{\delta w_0,x_2x_2} = \iiint\limits_B \sigma_2\, x_3 \frac{\partial^2 \delta w_0}{\partial x_2{}^2} dV =$$

$$= \iint\limits_{\partial B^{(lat)}} \sigma_2\, x_3\, n_{x_2} \frac{\partial \delta w_0}{\partial x_2} dS - \iiint\limits_B \frac{\partial \sigma_2}{\partial x_2} x_3 \frac{\partial \delta w_0}{\partial x_2} dV =$$

$$= \iint\limits_{\partial B^{(lat)}} \sigma_2\, x_3\, n_{x_2} \frac{\partial \delta w_0}{\partial x_2} dS - \iint\limits_{\partial B^{(lat)}} \frac{\partial \sigma_2}{\partial x_2} x_3\, n_{x_2} \delta w_0\, dS + \iiint\limits_B \frac{\partial^2 \sigma_2}{\partial^2 x_2} x_3\, \delta w_0\, dV \tag{2d}$$

$$\delta I_{\delta u_0,x_2} = \iiint\limits_B \sigma_6 \frac{\partial \delta u_0}{\partial x_2} dV =$$

$$= \iint\limits_{\partial B^{(lat)}} \sigma_6\, n_{x_2} \delta u_0\, dS - \iiint\limits_B \frac{\partial \sigma_6}{\partial x_2} \delta u_0\, dV \tag{2e}$$

$$\delta I_{\delta v_0,x_1} = \iiint\limits_B \sigma_6 \frac{\partial \delta v_0}{\partial x_1} dV =$$

$$= \iint\limits_{\partial B^{(lat)}} \sigma_6\, n_{x_1} \delta v_0\, dS - \iiint\limits_B \frac{\partial \sigma_6}{\partial x_1} \delta v_0\, dV \tag{2f}$$

In order to derive the final terms of the following volume integral $\delta I_{\delta w_0,x_1x_2}$, we follow a different path.

$$\delta I_{\delta w_0,x_1x_2} = \iiint_B 2\sigma_6 x_3 \frac{\partial^2 \delta w_0}{\partial x_1 \partial x_2} dV =$$

$$= \underbrace{\iiint_B \sigma_6 x_3 \frac{\partial^2 \delta w_0}{\partial x_1 \partial x_2} dV}_{J_1} + \underbrace{\iiint_B \sigma_6 x_3 \frac{\partial^2 \delta w_0}{\partial x_1 \partial x_2} dV}_{J_2} \equiv J_1 + J_2 \qquad (2g)$$

As for the first term J_1, we perform integrations by parts firstly according to x_1 and secondly according to x_2 variable, while as for the second term J_2, we perform integrations by parts initially according to x_2 and subsequently according to x_1 spatial variable. This concept is adopted to the following calculations, because we desire to derive boundary conditions with a "symmetric" formulation between the terms (of the variation of the action functional) with the same variations $(\delta u_0, \delta v_0, \delta w_0)$.

Thus,

$$J_1 = \iiint_B \sigma_6 x_3 \frac{\partial^2 \delta w_0}{\partial x_1 \partial x_2} dV = \iint_{\partial B^{(lat)}} \sigma_6 x_3 n_{x_1} \frac{\partial \delta w_0}{\partial x_2} dS - \iiint_B x_3 \frac{\partial \sigma_6}{\partial x_1} \frac{\partial \delta w_0}{\partial x_2} dV =$$

$$= \iint_{\partial B^{(lat)}} \sigma_6 x_3 n_{x_1} \frac{\partial \delta w_0}{\partial x_2} dS - \iint_{\partial B^{(lat)}} \frac{\partial \sigma_6}{\partial x_1} x_3 n_{x_2} \delta w_0 \, dS + \iiint_B \frac{\partial^2 \sigma_6}{\partial x_1 \partial x_2} x_3 \delta w_0 \, dV$$

and

$$J_2 = \iiint_B \sigma_6 x_3 \frac{\partial^2 \delta w_0}{\partial x_1 \partial x_2} dV = \iint_{\partial B^{(lat)}} \sigma_6 x_3 n_{x_2} \frac{\partial \delta w_0}{\partial x_1} dS - \iiint_B x_3 \frac{\partial \sigma_6}{\partial x_2} \frac{\partial \delta w_0}{\partial x_1} dV =$$

$$= \iint_{\partial B^{(lat)}} \sigma_6 x_3 n_{x_2} \frac{\partial \delta w_0}{\partial x_1} dS - \iint_{\partial B^{(lat)}} \frac{\partial \sigma_6}{\partial x_2} x_3 n_{x_1} \delta w_0 \, dS + \iiint_B \frac{\partial^2 \sigma_6}{\partial x_2 \partial x_1} x_3 \delta w_0 \, dV$$

Consequently, the equation (15g) is converted to,

$$\delta I_{\delta w_0,x_1x_2} = \iint_{\partial B^{(lat)}} \sigma_6 x_3 n_{x_1} \frac{\partial \delta w_0}{\partial x_2} dS + \iint_{\partial B^{(lat)}} \sigma_6 x_3 n_{x_2} \frac{\partial \delta w_0}{\partial x_1} dS -$$

$$- \iint_{\partial B^{(lat)}} \left\{ \frac{\partial \sigma_6}{\partial x_1} x_3 n_{x_2} + \frac{\partial \sigma_6}{\partial x_2} x_3 n_{x_1} \right\} \delta w_0 \, dS + \iiint_B 2 \frac{\partial^2 \sigma_6}{\partial x_1 \partial x_2} x_3 \delta w_0 \, dV \qquad (2g)$$

Now using (2a) - (2g), the equation (1) is converted to,

$$\delta J_U = \int_{t_1}^{t_2}\iiint_B \delta U(\boldsymbol{e})\,dV\,dt = \int_{t_1}^{t_2}\iiint_B \left(\sigma_1\delta e_1+\sigma_2\delta e_2+\sigma_6\delta e_6\right) dV\,dt =$$

$$= \int_{t_1}^{t_2}\iint_{\partial B^{(lat)}} (\sigma_1 n_{x_1}+\sigma_6 n_{x_2})\delta u_0\, dS\,dt + \int_{t_1}^{t_2}\iint_{\partial B^{(lat)}} (\sigma_1 x_3 n_{x_1}+\sigma_6 x_3 n_{x_2})\frac{\partial \delta w_0}{\partial x_1}\,dS\,dt -$$

$$-\int_{t_1}^{t_2}\iint_{\partial B^{(lat)}}\left(x_3\left\{\frac{\partial\sigma_1}{\partial x_1}+\frac{\partial\sigma_6}{\partial x_2}\right\}n_{x_1}+x_3\left\{\frac{\partial\sigma_2}{\partial x_2}+\frac{\partial\sigma_6}{\partial x_1}\right\}n_{x_2}\right)\delta w_0\,dS\,dt +$$

$$+\int_{t_1}^{t_2}\iint_{\partial B^{(lat)}} (\sigma_2 n_{x_2}+\sigma_6 n_{x_1})\delta v_0\, dS\,dt + \int_{t_1}^{t_2}\iint_{\partial B^{(lat)}} (\sigma_2 x_3 n_{x_2}+\sigma_6 x_3 n_{x_1})\frac{\partial \delta w_0}{\partial x_2}\,dS\,dt -$$

$$-\int_{t_1}^{t_2}\iiint_B\left(\frac{\partial\sigma_1}{\partial x_1}+\frac{\partial\sigma_6}{\partial x_2}\right)\delta u_0\,dV\,dt - \int_{t_1}^{t_2}\iiint_B\left(\frac{\partial\sigma_2}{\partial x_2}+\frac{\partial\sigma_6}{\partial x_1}\right)\delta v_0\,dV\,dt +$$

$$+\int_{t_1}^{t_2}\iiint_B\left(2\frac{\partial^2\sigma_6}{\partial x_1\partial x_2}x_3+\frac{\partial^2\sigma_1}{\partial^2 x_1}x_3+\frac{\partial^2\sigma_2}{\partial^2 x_2}x_3\right)\delta w_0\,dV\,dt =$$

$$= \int_{t_1}^{t_2}\oint_\Gamma\int_{-h/2}^{h/2} (\sigma_1 n_{x_1}+\sigma_6 n_{x_2})\delta u_0\, dx_3\,d\gamma\,dt + \int_{t_1}^{t_2}\oint_\Gamma\int_{-h/2}^{h/2} (\sigma_1 x_3 n_{x_1}+\sigma_6 x_3 n_{x_2})\frac{\partial \delta w_0}{\partial x_1}\,dx_3\,d\gamma\,dt -$$

$$-\int_{t_1}^{t_2}\oint_\Gamma\int_{-h/2}^{h/2}\left(x_3\left\{\frac{\partial\sigma_1}{\partial x_1}+\frac{\partial\sigma_6}{\partial x_2}\right\}n_{x_1}+x_3\left\{\frac{\partial\sigma_2}{\partial x_2}+\frac{\partial\sigma_6}{\partial x_1}\right\}n_{x_2}\right)\delta w_0\,dx_3\,d\gamma\,dt +$$

$$+\int_{t_1}^{t_2}\oint_\Gamma\int_{-h/2}^{h/2} (\sigma_2 n_{x_2}+\sigma_6 n_{x_1})\delta v_0\, dx_3\,d\gamma\,dt + \int_{t_1}^{t_2}\oint_\Gamma\int_{-h/2}^{h/2} (\sigma_2 x_3 n_{x_2}+\sigma_6 x_3 n_{x_1})\frac{\partial \delta w_0}{\partial x_2}\,dx_3\,d\gamma\,dt -$$

$$-\int_{t_1}^{t_2}\iint_\Omega\int_{-h/2}^{h/2}\left(\frac{\partial\sigma_1}{\partial x_1}+\frac{\partial\sigma_6}{\partial x_2}\right)\delta u_0\,dx_3\,dS\,dt - \int_{t_1}^{t_2}\iint_\Omega\int_{-h/2}^{h/2}\left(\frac{\partial\sigma_2}{\partial x_2}+\frac{\partial\sigma_6}{\partial x_1}\right)\delta v_0\,dx_3\,dS\,dt +$$

$$+\int_{t_1}^{t_2}\iint_\Omega\int_{-h/2}^{h/2}\left(2\frac{\partial^2\sigma_6}{\partial x_1\partial x_2}x_3+\frac{\partial^2\sigma_1}{\partial^2 x_1}x_3+\frac{\partial^2\sigma_2}{\partial^2 x_2}x_3\right)\delta w_0\,dx_3\,dS\,dt \quad (3)$$

In the Eq. (3) is followed the same process (as those of the kinetic part) of decomposition of the volume and surface integrals. Thus, the above relation takes a new form including

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} = \begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} dx_3 \quad , \quad \begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} x_3 \, dx_3 \qquad (4)$$

Thus, to calculate the variation of the elastic potential energy part, we have to define additionally these stress resultants as,

the thickness-integrated forces $(N_{11}, N_{22}, N_{12}) = (N_1, N_2, N_6)$

and the thickness-integrated moments $(M_{11}, M_{22}, M_{12}) = (M_1, M_2, M_6)$,

which are called alternatively as **stress resultants**.

At this point is important to clarify that the above stress resultants are nothing more than "abbreviations" of the stress field of the material. By this way, we gather together the components of stress field, which are expressed in terms of the displacement field (u_0, v_0, w_0), namely the unknowns, as seems from the Eqs. (8) - (10) of the section 4.2.1 or the Eqs. (5) and (8) of the section 4.2.2.
Substituting the Eqs. (8) - (10) of the section 4.2.1 or the Eqs. (5) and (8) of the section 4.2.2, into the Eq. (4) of the stress resultants, we can express the thickness-integrated forces and moments directly in terms of the displacements (u_0, v_0, w_0).
Thus, the last aforementioned relations are going to appear on next sections in order to derive easier the equations of motion and the boundary conditions of the plate (in terms of the displacement field). Complementarily note that the total number of the resulting scalar equations of the problem must be the same with the number of unknowns so that our problem has a unique solution. In that case the number of unknowns is three. Consequently, we expect to derive three equations from the variational principle, including the unknowns (u_0, v_0, w_0), and finally solve a 3x3 system.

Thus, the equation (3) is converted to:

$$\begin{aligned}
\delta J_U = & \int_{t_1}^{t_2}\oint_{\Gamma} N_1 n_{x_1} + N_6 n_{x_2} \; \delta u_0 \, d\gamma \, dt + \int_{t_1}^{t_2}\oint_{\Gamma} M_1 n_{x_1} + M_6 n_{x_2} \; \frac{\partial \delta w_0}{\partial x_1} \, d\gamma \, dt - \\
& - \int_{t_1}^{t_2}\oint_{\Gamma} \left(\left\{ \frac{\partial M_1}{\partial x_1} + \frac{\partial M_6}{\partial x_2} \right\} n_{x_1} + \left\{ \frac{\partial M_2}{\partial x_2} + \frac{\partial M_6}{\partial x_1} \right\} n_{x_2} \right) \delta w_0 \, d\gamma \, dt + \\
& + \int_{t_1}^{t_2}\oint_{\Gamma} N_2 n_{x_2} + N_6 n_{x_1} \; \delta v_0 \, d\gamma \, dt + \int_{t_1}^{t_2}\oint_{\Gamma} M_2 n_{x_2} + M_6 n_{x_1} \; \frac{\partial \delta w_0}{\partial x_2} \, d\gamma \, dt - \\
& - \int_{t_1}^{t_2}\iint_{\Omega} \left(\frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} \right) \delta u_0 \, d\omega \, dt - \int_{t_1}^{t_2}\iint_{\Omega} \left(\frac{\partial N_2}{\partial x_2} + \frac{\partial N_6}{\partial x_1} \right) \delta v_0 \, d\omega \, dt + \\
& + \int_{t_1}^{t_2}\iint_{\Omega} \left(2\frac{\partial^2 M_6}{\partial x_1 \partial x_2} + \frac{\partial^2 M_1}{\partial^2 x_1} + \frac{\partial^2 M_2}{\partial^2 x_2} \right) \delta w_0 \, d\omega \, dt \qquad \mathbf{(3`)}
\end{aligned}$$

4.4. Virtual Work of the Externally Applied Loads

As for the last terms of the variational equation, we have to calculate the **variations of the externally applied forces**, apart from Kinetic and Elastic Potential Energy.
Within the Classical Plate Theory, we assume that there is a normal distributed external load at the top or/ and bottom surface $\partial B^{(f)}$ of the plate (surface force/ traction at $x_3 = h/2$ or/ and $x_3 = -h/2$) $q(x_1, x_2; t)$. At this point we clarify that the normal distributed external load q is regarded as the algebraic sum between the load at the top and the bottom of the external boundary of the plate ($q = q_{top} + q_{bottom}$), as exactly shown on the reference, *M. Amabili (2004), "Nonlinear vibrations of rectangular plates with different boundary conditions: theory and experiments", Italy, Journal of Computers & Structures* on pp. 2589.
Also it is necessary to quantify the **virtual work of the traction field** at the edge of the plate. This work is related to the virtual displacements δu_1, δu_2 and δu_3, which are the displacements on the direction of x_1-axis, x_2-axis and x_3-axis respectively.
Thus, the variation of the functional of the external surface traction, due to the surface distributed load (surface tractions) at the adjacent surface and the horizontally distributed vertical load q (as illustrated on the following figure, **Figure 5**), is:

$$\delta J_T = \int_{t_1}^{t_2} \iint_{\partial B_T} \hat{T}_i \, \delta u_i \, dS \, dt + \int_{t_1}^{t_2} \iint_{\Omega} q \, \delta w_0 \, d\omega \, dt \Rightarrow$$

$$\delta J_T = \underbrace{\int_{t_1}^{t_2} \iint_{\partial B_T^{(lat)}} \left(\hat{T}_1 \delta u_1 + \hat{T}_2 \delta u_2 + \hat{T}_3 \delta u_3 \right) dS \, dt}_{\delta J_T^{(lat)}} +$$

$$+ \underbrace{\int_{t_1}^{t_2} \iint_{\partial B^{(uf)}} \hat{T}_3 \delta u_3 \, dS \, dt}_{\delta J_T^{(uf)}} + \underbrace{\int_{t_1}^{t_2} \iint_{\partial B^{(lf)}} \hat{T}_3 \delta u_3 \, dS \, dt}_{\delta J_T^{(lf)}} + \int_{t_1}^{t_2} \iint_{\Omega} q \, \delta w_0 \, d\omega \, dt .$$

We examine separately the three integrals of the lateral, upper and bottom surface. To simplify their expressions, we neglect the time integration at this moment.
Due to the basic assumptions of the model of CPT, the second and the third term of the above variation are eliminated, because of the normality assumption (section 1.2.3), which gives zero shear strains $\gamma_{23} = 2e_{23} = 0$ and $\gamma_{13} = 2e_{13} = 0$ all over the plate. This fact is also obvious from the form of the surface tractions which are prescribed on the section 2 [Eqs. (1) and (2)].
Thus,

$$\delta J_T^{(uf)} = \delta J_T^{(lf)} = 0 \qquad \text{, on the flat surfaces}$$

By this way the only term that remains to be analyzed, on the above variation of the externally applied loads is the first one.

$$\delta J_T = \int_{t_1}^{t_2} \iint_{\partial B_T^{(lat)}} \hat{T}_1 \delta u_1 + \hat{T}_2 \delta u_2 + \hat{T}_3 \delta u_3 \;\; dS \; dt \; + \int_{t_1}^{t_2} \iint_{\Omega} q \, \delta w_0 \, d\omega \; dt \qquad (1)$$

At this point we procced to further study of the Eq. (1) and neglecting again the time integral to simplify the calculations, we get

$$\delta J_T^{(lat)} = \iint_{\partial B_T^{(lat)}} \hat{T}_1 \delta u_1 + \hat{T}_2 \delta u_2 + \hat{T}_3 \delta u_3 \;\; dS =$$

$$= \iint_{\partial B_T^{(lat)}} \hat{T}_1 \delta u_1 \, dS + \iint_{\partial B_T^{(lat)}} \hat{T}_2 \delta u_2 \, dS + \iint_{\partial B_T^{(lat)}} \hat{T}_3 \delta u_3 \, dS =$$

$$= \iint_{\partial B_T^{(lat)}} \hat{T}_1 \delta u \, dS + \iint_{\partial B_T^{(lat)}} \hat{T}_2 \delta v \, dS + \iint_{\partial B_T^{(lat)}} \hat{T}_3 \delta w \, dS \qquad (2)$$

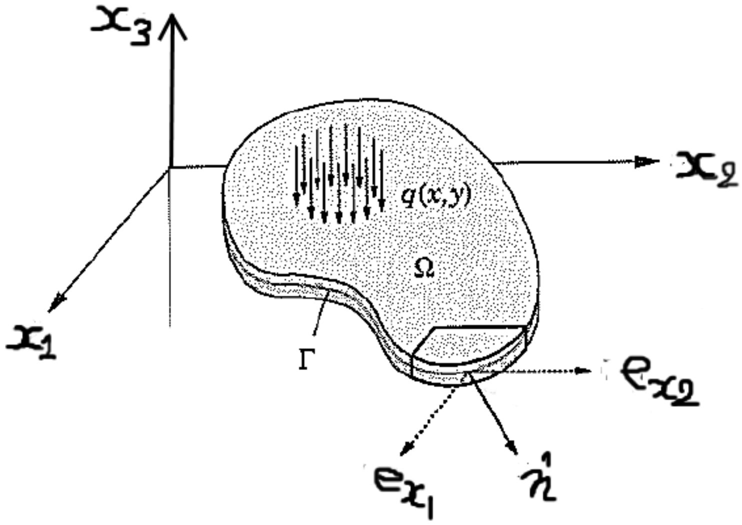

Figure 5: Externally applied and horizontally distributed vertical load

Now we assume that we have given surface tractions $\hat{T}_1$, $\hat{T}_2$ and $\hat{T}_3$ at the specific parts of the lateral boundary $\partial B^{(lat)}$ $(\partial B_T^{(lat)})$ and using the Eqs. (5), (6) and (7) of the section 3, we derive the following,

$$\delta J_T^{(lat)} = \iint_{\partial B^{(lat)}} \hat{T}_1 \left(\delta u_0 - x_3 \frac{\partial \delta w_0}{\partial x_1} \right) dS \; dt + \iint_{\partial B^{(lat)}} \hat{T}_2 \left(\delta v_0 - x_3 \frac{\partial \delta w_0}{\partial x_2} \right) dS \; dt + \iint_{\partial B^{(lat)}} \hat{T}_3 \delta w_0 \, dS =$$

$$= \iint_{\partial B^{(lat)}} \hat{T}_1 \delta u_0 \, dS \; dt - \iint_{\partial B^{(lat)}} \hat{T}_1 x_3 \frac{\partial \delta w_0}{\partial x_1} dS + \iint_{\partial B^{(lat)}} \hat{T}_2 \delta v_0 \, dS \; -$$

$$- \iint_{\partial B^{(lat)}} \hat{T}_2 \, x_3 \frac{\partial \delta w_0}{\partial x_2} \, dS \; + \iint_{\partial B^{(lat)}} \hat{T}_3 \, \delta w_0 \, dS \qquad (3)$$

Now as for the form of the given surface tractions $\hat{T}_1$, $\hat{T}_2$ and $\hat{T}_3$, we recall the configurations of the section 2. We notice also that the deformation at the edge of the plate during its motion in conjunction with the externally applied loads must be linearly dependent from the x_3-variable. This fact is justified due to the same dependence of the in-plane displacements (u, v) from the x_3-variable.

Further, the quantities which multiply the variations δu_0, δv_0 and δw_0 inside the integrals of the Eq. (3), are expected to match with the respective terms of the variation of the Elastic-Potential Energy part (section 4.3). The last contains boundary terms (surface integrals) similarly linear dependent of the x_3-variable.

Taking all the aforementioned into account, we present here for convenience again the form of the given surface tractions prescribed on the section 2 by the Eqs. (1), (2), (3),

$$\hat{T}_1 = a_{T0}(x_1, x_2) + a_{T1}(x_1, x_2)\, x_3 = a_{T0}(\gamma) + a_{T1}(\gamma)\, x_3 \qquad (4a)$$

$$\hat{T}_2 = b_{T0}(x_1, x_2) + b_{T1}(x_1, x_2)\, x_3 = b_{T0}(\gamma) + b_{T1}(\gamma)\, x_3 \qquad (4b)$$

$$\hat{T}_3 = c_{T0}(x_1, x_2) = c_{T0}(\gamma) \qquad (4c)$$

However, it is essential to note that the above form of the surface traction field is a simplified approximation of the real values of the surface tractions at each point upon the edge of the plate as to the Cartesian coordinate system (because on the curvilinear one, these tractions are going to take a different form as will show on the section 5). Certainly, this approximation is enough accurate in the context of our problem of CPT (and compatible with our model), because of the thin plate which permits only small variations on the values of the tractions along its thickness.

Furthermore, we substitute the Eqs. (4a) - (4c) into the Eq. (3) and after that we use the mass-moment quantities and the process of decomposition of the surface integrals. Thus, the Eq. (3) is modified as follows,

$$\delta J_T^{(lat)} = \iint_{\partial B^{(lat)}} a_{T0} + a_{T1} x_3 \;\; \delta u_0 \, dS - \iint_{\partial B^{(lat)}} a_{T0} + a_{T1} x_3 \;\; x_3 \frac{\partial \delta w_0}{\partial x_1} dS \; +$$

$$+ \iint_{\partial B^{(lat)}} b_{T0} + b_{T1} x_3 \;\; \delta v_0 \, dS - \iint_{\partial B^{(lat)}} b_{T0} + b_{T1} x_3 \;\; x_3 \frac{\partial \delta w_0}{\partial x_2} dS \; + \iint_{\partial B^{(lat)}} c_{T0} \, \delta w_0 \, dS \; =$$

$$= \iint_{\partial B^{(lat)}} a_{T0} \, \delta u_0 + a_{T1} x_3 \, \delta u_0 \;\; dS - \iint_{\partial B^{(lat)}} \left(a_{T0} \, x_3 \frac{\partial \delta w_0}{\partial x_1} + a_{T1} x_3^2 \frac{\partial \delta w_0}{\partial x_1} \right) dS \; +$$

$$+ \iint_{\partial B^{(lat)}} b_{T0} \, \delta v_0 + b_{T1} x_3 \, \delta v_0 \;\; dS - \iint_{\partial B^{(lat)}} \left(b_{T0} \, x_3 \frac{\partial \delta w_0}{\partial x_2} + b_{T1} x_3^2 \frac{\partial \delta w_0}{\partial x_2} \right) dS \; +$$

$$+\iint_{\partial B^{(lat)}} c_{T0}\,\delta w_0\, dS =$$

$$= \oint_{\Gamma}\int_{-h/2}^{h/2} a_{T0}\,\delta u_0 + a_{T1} x_3\,\delta u_0 \;\; dx_3\, d\gamma\, dt - \oint_{\Gamma}\int_{-h/2}^{h/2}\left(a_{T0}\, x_3 \frac{\partial \delta w_0}{\partial x_1} + a_{T1} x_3^2 \frac{\partial \delta w_0}{\partial x_1}\right) dx_3\, d\gamma \; +$$

$$+ \oint_{\Gamma}\int_{-h/2}^{h/2} b_{T0}\,\delta v_0 + b_{T1} x_3\,\delta v_0 \;\; dx_3\, d\gamma - \oint_{\Gamma}\int_{-h/2}^{h/2}\left(b_{T0}\, x_3 \frac{\partial \delta w_0}{\partial x_2} + b_{T1} x_3^2 \frac{\partial \delta w_0}{\partial x_2}\right) dx_3\, d\gamma \; +$$

$$+ \oint_{\Gamma}\int_{-h/2}^{h/2} c_{T0}\,\delta w_0\, dx_3\, d\gamma =$$

$$= \oint_{\Gamma}\left(a_{T0}\frac{I_0}{\rho}\delta u_0 + a_{T1}\frac{\cancel{I_1}}{\rho}\delta u_0\right) d\gamma - \oint_{\Gamma}\left(a_{T0}\frac{\cancel{I_1}}{\rho}\frac{\partial \delta w_0}{\partial x_1} + a_{T1}\frac{I_2}{\rho}\frac{\partial \delta w_0}{\partial x_1}\right) d\gamma \; +$$

$$+ \oint_{\Gamma}\left(b_{T0}\frac{I_0}{\rho}\delta v_0 + b_{T1}\frac{\cancel{I_1}}{\rho}\delta v_0\right) d\gamma - \oint_{\Gamma}\left(b_{T0}\frac{\cancel{I_1}}{\rho}\frac{\partial \delta w_0}{\partial x_2} + b_{T1}\frac{I_2}{\rho}\frac{\partial \delta w_0}{\partial x_2}\right) d\gamma \; +$$

$$+ \oint_{\Gamma} c_{T0}\frac{I_0}{\rho}\delta w_0\, d\gamma =$$

$$= \oint_{\Gamma} a_{T0}\frac{I_0}{\rho}\delta u_0\, d\gamma - \oint_{\Gamma} a_{T1}\frac{I_2}{\rho}\frac{\partial \delta w_0}{\partial x_1} d\gamma + \oint_{\Gamma} b_{T0}\frac{I_0}{\rho}\delta v_0\, d\gamma \; -$$

$$- \oint_{\Gamma} b_{T1}\frac{I_2}{\rho}\frac{\partial \delta w_0}{\partial x_2} d\gamma + \oint_{\Gamma} c_{T0}\frac{I_0}{\rho}\delta w_0\, d\gamma \tag{5}$$

Finally, substituting the Eq. (5) into the Eq. (1), we get

$$\delta J_T = \int_{t_1}^{t_2}\oint_{\Gamma} a_{T0}\frac{I_0}{\rho}\delta u_0\, d\gamma\, dt - \int_{t_1}^{t_2}\oint_{\Gamma} a_{T1}\frac{I_2}{\rho}\frac{\partial \delta w_0}{\partial x_1} d\gamma\, dt + \int_{t_1}^{t_2}\oint_{\Gamma} b_{T0}\frac{I_0}{\rho}\delta v_0\, d\gamma\, dt \; -$$

$$- \int_{t_1}^{t_2}\oint_{\Gamma} b_{T1}\frac{I_2}{\rho}\frac{\partial \delta w_0}{\partial x_2} d\gamma dt + \int_{t_1}^{t_2}\oint_{\Gamma} c_{T0}\frac{I_0}{\rho}\delta w_0\, d\gamma\, dt + \int_{t_1}^{t_2}\iint_{\Omega} q\,\delta w_0\, d\omega\, dt \tag{6}$$

4.5. The variational equation of the CPT

Now we are able to substitute the results of separate parts, Eq. **(6`**) of the section 4.1., **(3`)** of the section 4.3. and **(6)** of the section 4.4. into the variational equation **(1)** of the section 4. The next step is to gather separately the different terms according to the kind of their variations e.g. δu_0, δv_0, δw_0. By this way it is easier to extract the equations of motion and the boundary conditions of the model of Classical Theory of Plates.
To facilitate the calculations and substitutions, we repeat the equation (1) of the section 4:

$$\int_{t_1}^{t_2}\iiint_B \delta K(\dot{\boldsymbol{u}})dV\,dt-\int_{t_1}^{t_2}\iiint_B \delta U(\boldsymbol{e})dV\,dt+\int_{t_1}^{t_2}\iint_{\partial B_T}\hat{T}_i\,\delta u_i\,dS\,dt=0 \qquad (1)$$

Then the equation (1), is converted to:

$$-\int_{t_1}^{t_2}\iint_\Omega I_0\ddot{u}_0\,\delta u_0\,d\omega\,dt-\int_{t_1}^{t_2}\iint_\Omega I_0\ddot{v}_0\,\delta v_0\,d\omega\,dt-$$

$$-\int_{t_1}^{t_2}\iint_\Omega\left[I_0\ddot{w}_0-I_2\left(\frac{\partial^2\ddot{w}_0}{\partial^2x_1}+\frac{\partial^2\ddot{w}_0}{\partial^2x_2}\right)\right]\delta w_0\,d\omega\,dt-$$

$$-\int_{t_1}^{t_2}\oint_\Gamma\left[I_2\frac{\partial\ddot{w}_0}{\partial x_1}n_{x_1}+I_2\frac{\partial\ddot{w}_0}{\partial x_2}n_{x_2}\right]\delta w_0\,d\gamma\,dt-$$

$$-\int_{t_1}^{t_2}\oint_\Gamma(N_1n_{x_1}+N_6n_{x_2})\delta u_0\,d\gamma\,dt-\int_{t_1}^{t_2}\oint_\Gamma(N_2n_{x_2}+N_6n_{x_1})\delta v_0\,d\gamma\,dt+$$

$$+\int_{t_1}^{t_2}\oint_\Gamma\left(\left\{\frac{\partial M_1}{\partial x_1}+\frac{\partial M_6}{\partial x_2}\right\}n_{x_1}+\left\{\frac{\partial M_2}{\partial x_2}+\frac{\partial M_6}{\partial x_1}\right\}n_{x_2}\right)\delta w_0\,d\gamma\,dt-$$

$$-\int_{t_1}^{t_2}\oint_\Gamma(M_1n_{x_1}+M_6n_{x_2})\frac{\partial\delta w_0}{\partial x_1}d\gamma\,dt-\int_{t_1}^{t_2}\oint_\Gamma(M_2n_{x_2}+M_6n_{x_1})\frac{\partial\delta w_0}{\partial x_2}d\gamma\,dt+$$

$$+\int_{t_1}^{t_2}\iint_\Omega\left(\frac{\partial N_1}{\partial x_1}+\frac{\partial N_6}{\partial x_2}\right)\delta u_0\,d\omega\,dt+\int_{t_1}^{t_2}\iint_\Omega\left(\frac{\partial N_6}{\partial x_1}+\frac{\partial N_2}{\partial x_2}\right)\delta v_0\,d\omega\,dt+$$

$$+\int_{t_1}^{t_2}\iint_\Omega\left(2\frac{\partial^2M_6}{\partial x_1\partial x_2}+\frac{\partial^2M_1}{\partial^2x_1}+\frac{\partial^2M_2}{\partial^2x_2}\right)\delta w_0\,d\omega\,dt+$$

$$+\int_{t_1}^{t_2}\oint_\Gamma a_{T0}\frac{I_0}{\rho}\delta u_0\,d\gamma\,dt-\int_{t_1}^{t_2}\oint_\Gamma a_{T1}\frac{I_2}{\rho}\frac{\partial\delta w_0}{\partial x_1}d\gamma\,dt+\int_{t_1}^{t_2}\oint_\Gamma b_{T0}\frac{I_0}{\rho}\delta v_0\,d\gamma\,dt-$$

$$- \int_{t_1}^{t_2} \oint_{\Gamma} b_{T1} \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial x_2} d\gamma dt + \int_{t_1}^{t_2} \oint_{\Gamma} c_{T0} \frac{I_0}{\rho} \delta w_0 \, d\gamma \, dt + \int_{t_1}^{t_2} \iint_{\Omega} q \, \delta w_0 \, d\omega \, dt \quad (2)$$

For further simplification, we gather together the terms with surface and curvilinear integrals, taking care of the kind of variation (δu_0, δv_0 and δw_0) of each term. Thus, the final form of the variational equation of the problem of the CPT is the following. Note that the Eq. (3) below is exactly the same as the variational Eq. (1) of the section 4.

$$-\int_{t_1}^{t_2} \iint_{\Omega} \left\{ I_0 \ddot{u}_0 - \frac{\partial N_{11}}{\partial x_1} - \frac{\partial N_{12}}{\partial x_2} \right\} \delta u_0 \, d\omega \, dt - \int_{t_1}^{t_2} \iint_{\Omega} \left\{ I_0 \ddot{v}_0 - \frac{\partial N_{12}}{\partial x_1} - \frac{\partial N_{22}}{\partial x_2} \right\} \delta v_0 \, d\omega \, dt -$$

$$-\int_{t_1}^{t_2} \iint_{\Omega} \left\{ I_0 \ddot{w}_0 - I_2 \left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1} + \frac{\partial^2 \ddot{w}_0}{\partial^2 x_2} \right) - 2 \frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} - \frac{\partial^2 M_{11}}{\partial^2 x_1} - \frac{\partial^2 M_{22}}{\partial^2 x_2} - q \right\} \delta w_0 \, d\omega \, dt -$$

$$-\int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \left(I_2 \frac{\partial \ddot{w}_0}{\partial x_1} - \frac{\partial M_{11}}{\partial x_1} - \frac{\partial M_{12}}{\partial x_2} \right) n_{x_1} + \left(I_2 \frac{\partial \ddot{w}_0}{\partial x_2} - \frac{\partial M_{22}}{\partial x_2} - \frac{\partial M_{12}}{\partial x_1} \right) n_{x_2} - c_{T0} \frac{I_0}{\rho} \right\} \delta w_0 \, d\gamma \, dt -$$

$$-\int_{t_1}^{t_2} \oint_{\Gamma} \left\{ N_{11} n_{x_1} + N_{12} n_{x_2} - a_{T0} \frac{I_0}{\rho} \right\} \delta u_0 \, d\gamma \, dt - \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ M_{11} n_{x_1} + M_{12} n_{x_2} + a_{T1} \frac{I_2}{\rho} \right\} \frac{\partial \delta w_0}{\partial x_1} \, d\gamma \, dt -$$

$$-\int_{t_1}^{t_2} \oint_{\Gamma} \left\{ N_{22} n_{x_2} + N_{12} n_{x_1} - b_{T0} \frac{I_0}{\rho} \right\} \delta v_0 \, d\gamma \, dt - \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ M_{22} n_{x_2} + M_{12} n_{x_1} + b_{T1} \frac{I_2}{\rho} \right\} \frac{\partial \delta w_0}{\partial x_2} \, d\gamma \, dt = 0 \quad \textbf{(3)}$$

4.6 Equations of Motion of the CPT in terms of thickness integrated forces and moments

Now in order to obtain the equations of motion of the model of CPT, we assume that each term (under surface or curvilinear integrals) of the Eq. (3) is continuous function of x_1 and x_2. These terms are multiplied with the variations δu_0, δv_0 and δw_0 or the spatial derivatives of the last (δw_0) variation. At this point, using the standard arguments of the calculus of variations [*"Calculus of Variations", I. M. Gelfand and S. V. Fomin, Lemma 1, p.9/Sec.3/Chap.1 and Lemma, p.22/Sec.5/Chap.1*], we derive the **three equations of motion of the plate**.

Accordingly, we first assume that, $\delta u_0 = \delta v_0 = \delta w_0 = \partial\delta w_0/\partial x_2 = \partial\delta w_0/\partial x_1 = 0$ on the boundary $(\boldsymbol{x}\in\Gamma)$, where t is arbitrary. The previous means that the variations and their spatial derivatives are not vary upon the boundary of the plate and obviously we have given displacements. Then (3) reduces to just,

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left\{I_0\ddot{u}_0-\frac{\partial N_1}{\partial x_1}-\frac{\partial N_6}{\partial x_2}\right\}\delta u_0\,d\omega\,dt-\int_{t_1}^{t_2}\iint_{\Omega}\left\{I_0\ddot{v}_0-\frac{\partial N_6}{\partial x_1}-\frac{\partial N_2}{\partial x_2}\right\}\delta v_0\,d\omega\,dt-$$

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left\{I_0\ddot{w}_0-I_2\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}\right)-2\frac{\partial^2 M_6}{\partial x_1\partial x_2}-\frac{\partial^2 M_1}{\partial^2 x_1}-\frac{\partial^2 M_2}{\partial^2 x_2}-q\right\}\delta w_0\,d\omega\,dt=0 \quad (3`)$$

Subsequently, we assume that $\delta v_0 = \delta w_0 = 0$ on the domain Ω (inside the body of the plate). Thus,

$$\int_{t_1}^{t_2}\iint_{\Omega}\left[I_0\ddot{u}_0-\frac{\partial N_1}{\partial x_1}-\frac{\partial N_6}{\partial x_2}\right]\delta u_0(\boldsymbol{x},t)\,d\omega\,dt = 0, \qquad \forall\ \delta u_0(\boldsymbol{x},t),$$

and using the arbitrariness of the variation δu_0 inside the $\Omega\times[t_1,t_2]$, we find the first equation of motion of the plate,

$$I_0\ddot{u}_0-\frac{\partial N_{11}}{\partial x_1}-\frac{\partial N_{12}}{\partial x_2} = 0. \qquad \textbf{(4)}$$

for $\boldsymbol{x}\in\Omega$ and $\forall\ t\in[t_1,\ t_2]$.

Next, we remove the restriction $\delta v_0 = 0$ on the domain Ω and taking into account the equation (4) which eliminates the first surface integral of (3`). Thus,

$$\int_{t_1}^{t_2}\iint_{\Omega}\left[I_0\ddot{v}_0-\frac{\partial N_6}{\partial x_1}-\frac{\partial N_2}{\partial x_2}\right]\delta v_0(\boldsymbol{x},t)\,d\omega\,dt = 0, \qquad \forall\ \delta v_0(\boldsymbol{x},t),$$

And using the arbitrariness of the variation δv_0 inside the space $\Omega\times[t_1,t_2]$, we result to the second equation of motion of the plate,

$$I_0\ddot{v}_0 - \frac{\partial N_{22}}{\partial x_2} - \frac{\partial N_{12}}{\partial x_1} = 0\,. \qquad \textbf{(5)}$$

for $\boldsymbol{x}\in\Omega$ and $\forall\ t\in[t_1,\ t_2]$.

Further, removing the restriction $\delta w_0 = 0$ on the surface Ω and taking into account the two previous equations (4) and (5), the result of the equation is (3`) is,

$$\int_{t_1}^{t_2}\iint_{\Omega}\left[I_0\ddot{w}_0 - I_2\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}\right) - 2\frac{\partial^2 M_6}{\partial x_1\partial x_2} - \frac{\partial^2 M_1}{\partial^2 x_1} - \frac{\partial^2 M_2}{\partial^2 x_2} - q\right]\delta w_0(\boldsymbol{x},t)\,d\omega\,dt = 0,$$

$$\forall\ \delta w_0(\boldsymbol{x},t)$$

Regarding also the arbitrariness of the variation δw_0 inside the $\Omega\times[t_1,t_2]$, we extract the third equation of motion of the plate,

$$I_0\ddot{w}_0 - I_2\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}\right) - 2\frac{\partial^2 M_{12}}{\partial x_1\partial x_2} - \frac{\partial^2 M_{11}}{\partial^2 x_1} - \frac{\partial^2 M_{22}}{\partial^2 x_2} = q \qquad \textbf{(6)}$$

Let it be noted that Eqs. (4), (5) are identical with the respective results (3.4.15), (3.4.16) of the book J.N. Reddy (2007) on the page 105. As for the comparison of the equation (6) and (3.4.17), we note that there is a difference on one term of the (3.4.17), which is not appear on the equation (6). This difference is easily justified because of another strain field, which is adopted for the development of the CPT according to J.N. Reddy and takes into account the geometric nonlinearities i.e. small strains but moderate rotations of transverse normal of the mid-surface $(10^o\text{-}15^o)$ [*J.N. Reddy (2007) "Theory and Analysis of Elastic Plates and Shells", page 98 – 99/ Chapter 3*].

It is essential to note that, the above system of three equations (4), (5), (6) is solvable, as will be seen explicitly on the section 6, because the number of unknown quantities is three. This is a fact due to the definition of the thickness-integrated forces and moments [Eqs. (4) of the section 4.3], which can be expressed directly in terms of the unknowns of the system, namely the displacement field (u_0, v_0, w_0), as will be shown on the section 6 again.

5. Boundary Conditions of CPT in terms of thickness -integrated forces and moments

Inspection of the previous Eq. (3) of the section 4.5 indicates that the quantities with a variation in the boundary integrals are the **primary variables** u_0, v_0, w_0, $\partial w_0 / \partial x_1$, $\partial w_0 / \partial x_2$ and their specification constitutes the **geometric** or **kinematic** (essential) boundary conditions. The mathematical expressions inside the brackets of the integrated quantities, which are coefficients of the varied quantities, are termed the **secondary** variables, and their specification gives the **dynamic** (natural) boundary conditions. Therefore, there are primary and secondary variables of the plate with faces parallel to $(x_1, x_2) -$ plane.

However, on this step we must not hustle to conclude about the final results of the boundary conditions of the problem, because at first glance [Variational Equation (3) of the section 4.5] the number of the boundary terms does not give the desirable number of boundary conditions. In other words, if the equations of motion are expressed in terms of displacements (as will be shown on the section 6), they would contain second-order spatial derivatives of u_0, v_0 and fourth-order spatial derivatives of w_0. This implies that there should be only four essential and four natural boundary conditions, whereas from the Variational Equation (1) below we note five essential and five natural boundary conditions. This fact is incompatible with our problem of CPT and must be corrected by specific treatments (section 5.2).

5.1. Variational boundary terms in Cartesian coordinates

Now, we isolate the curvilinear integrals in the Eq. (3) of the section 4.5, in order to illustrate better the aforementioned boundary terms.

Initially we remove the restrictions $\delta u_0 = \delta v_0 = \delta w_0 = \partial \delta w_0 / \partial x_2 = \partial \delta w_0 / \partial x_1 = 0$ from the boundary. Then, the action functional of the Eq. (3) becomes,

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\left(I_2\frac{\partial \ddot{w}_0}{\partial x_1}-\frac{\partial M_1}{\partial x_1}-\frac{\partial M_6}{\partial x_2}\right)n_{x_1}+\left(I_2\frac{\partial \ddot{w}_0}{\partial x_2}-\frac{\partial M_2}{\partial x_2}-\frac{\partial M_6}{\partial x_1}\right)n_{x_2}-c_{T0}\frac{I_0}{\rho}\right\}\delta w_0\, d\gamma\, dt\, -$$
$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{N_1 n_{x_1}+N_6 n_{x_2}-a_{T0}\frac{I_0}{\rho}\right\}\delta u_0\, d\gamma\, dt-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{M_1 n_{x_1}+M_6 n_{x_2}+a_{T1}\frac{I_2}{\rho}\right\}\frac{\partial \delta w_0}{\partial x_1}\, d\gamma\, dt\, -$$
$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{N_2 n_{x_2}+N_6 n_{x_1}-b_{T0}\frac{I_0}{\rho}\right\}\delta v_0\, d\gamma\, dt-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{M_2 n_{x_2}+M_6 n_{x_1}+b_{T1}\frac{I_2}{\rho}\right\}\frac{\partial \delta w_0}{\partial x_2}\, d\gamma\, dt = 0 \quad (1)$$

We are thinking exactly with the same rationality as on the paragraph 4.5 but at this moment to derive the boundary terms of the problem. Subsequently, on the section 5.3 we are going to extract the boundary conditions, which are independent from each other and finally compatible with our problem.

Additionally, note that the above process of deriving the equations of motion and the following boundary conditions is explained thoroughly on the Lecture Notes of Functional Analysis, *G.A. Athanassoulis* (*2016*) *"Necessary Conditions of Extremum of Functional"* and *"A further study of the Variational Problem as for integral type functional"*, as well as on the book of *Gelfand I.M.*, *Fomin S.V.* (*1963*), *"Calculus of Variations"*.

5.2. Transformation of the boundary conditions to a curvilinear boundary system

[References: *J.N. Reddy "Theory and Analysis of Elastic Plates and Shells"*, Chapter 1.4/ 3.5 and the *Lecture Notes, G.A. Athanassoulis* (*2016*), *"Invariances and transformation of physical quantities under rotations of the reference system"*].

Recalling the Variational Equation (1) of the section 5.1, which are examined so far as the possible boundary conditions, we remark the following.

On the one hand, the number of the possible geometrical and natural boundary conditions is five and five respectively. However, in general the geometrical and natural boundary conditions cannot take place concurrently, which means that the total number of the possible boundary conditions is five.

On the other hand, the total order of the partial differential equations of motion of the elliptic problem of the CPT [Eq. (9) of the section 6.1 or Eq. (9) of the section 6.2] is four, since these equations include fourth-order spatial derivatives of the displacement w_0. However, on the context of this section we are going to occupy with the equations of motion of the plate found on the section 4.6 and the possible boundary conditions of the section 5.1, which are expressed in terms of thickness-integrated forces and moments. The only difference between the equations of motion and the relations of the sections 4.6 / 5.1 and those of the sections 6.1 / 6.2, is that the first include implicitly the displacement field, whereas the second are expressed explicitly from the displacement field $\boldsymbol{u} = (u_0, v_0, w_0)$.

Consequently, we conclude that there is not the right number of primary (geometric) or secondary (natural) variables associated with the equations of motion governing the bending and stretching of the plate.

To remedy the aforementioned inconsistency, we will proceed to appropriate techniques, which are going to be developed on the following sections as well as on the Appendix A. Note that our struggle is to diminish the number of the boundary conditions from five to four.

First, we transform the appropriate boundary expressions in terms of the displacements, forces and moments over the edge of the plate (and specifically the arbitrary curve Γ surrounding the mid-surface of the plate). For this purpose the Cartesian orthogonal coordinate system (x_1, x_2, x_3) is transformed to a local coordinate system (n, s, z), which "follows" the shape of the arbitrary curve Γ on the lateral surface of the plate. The expression "follows", denotes that the coordinate system (n, s, z) moves on the curve Γ, so that the n-axis be normal to the lateral boundary (with a unit normal $\hat{\boldsymbol{n}}$) and s-axis be tangential to the same curve (with a unit tangential vector $\hat{\boldsymbol{s}}$). These vectors projected on the Cartesian coordinate system (x_1, x_2, x_3), are expressed as

$$\hat{\boldsymbol{n}} = n_{x_1} \boldsymbol{e}_{x_1} + n_{x_2} \boldsymbol{e}_{x_2} \tag{1}$$

$$\hat{\boldsymbol{s}} = s_{x_1} \boldsymbol{e}_{x_1} + s_{x_2} \boldsymbol{e}_{x_2} \tag{2}$$

Further, we suppose that the unit normal $\hat{\boldsymbol{n}}$ is oriented at an angle ϑ clockwise from the positive x_1-axis, then its direction cosines are $n_{x_1} = \cos\vartheta$ and $n_{x_2} = \sin\vartheta$. Similarly, the direction cosines of the vector $\hat{\boldsymbol{s}}$ are $s_{x_1} = -n_{x_2} = -\sin\vartheta$ and $s_{x_2} = n_{x_1} = \cos\vartheta$ (for additional remark and explanations, see APPENDIX A).

For this reason the aforementioned, local to the edge of the plate, coordinate system (n, s, z) is denoted as the **curvilinear coordinate system** and its components the **curvilinear coordinates** respectively. As for the choice of the previous designation (name) of the transformed coordinate system under rotation around the vertical axis x_3 or z, it is essential and meaningful because the direction of the in-plane axes (n, s) is directly adjusted to the shape of the lateral surface of the plate under consideration.

Further, as known the lateral surface (edge) of the plate is prescribed by the curve Γ surrounding the mid-surface of the plate and having arbitrary shape as exactly the form of its lateral boundary.

Thus, it is rational (reasonable) to describe the coordinates of the transformed system as **curvilinear-dependent**, since these coordinates enable us to model the problems of plates with curved boundaries of arbitrary shape.

After appropriate algebraic calculations and proofs referred extensively to the APPENDIX A, the final relations connecting the quantities of the Cartesian coordinate system to those of the curvilinear-dependent are presented below.
As for the displacement field of the problem of the CPT and its variation, we have

$$u_0 = n_{x_1} u_{0n} + n_{x_2} u_{0s} \quad (3) \quad \text{and} \quad \delta u_0 = n_{x_1} \delta u_{0n} + n_{x_2} \delta u_{0s} \quad (3`)$$

$$v_0 = -n_{x_2} u_{0n} + n_{x_1} u_{0s} \quad (4) \quad \text{and} \quad \delta v_0 = -n_{x_2} \delta u_{0n} + n_{x_1} \delta u_{0s} \quad (4`)$$

$$w_0 = 1 w_0 \quad (5) \quad \text{and} \quad \delta w_0 = 1 \delta w_0 \quad (5`)$$

Note that the vertical displacement and its variation remains the same during the transformation, since we occupy with the planar rotation of the $x_1 x_2$-plane around the vertical axis x_3.

As for the derivatives of the δw_0, we get

$$\frac{\partial \delta w_0}{\partial x_1} = n_{x_1} \frac{\partial \delta w_0}{\partial n} + n_{x_2} \frac{\partial \delta w_0}{\partial s} \quad (6)$$

$$\frac{\partial \delta w_0}{\partial x_2} = -n_{x_2} \frac{\partial \delta w_0}{\partial n} + n_{x_1} \frac{\partial \delta w_0}{\partial s} \quad (7)$$

Finally the relations which transform the stress field from the Cartesian coordinate system to the curvilinear one, are

$$\sigma_{11} = n_{x_1}^2 \sigma_{nn} + 2 n_{x_1} n_{x_2} \sigma_{ns} + n_{x_2}^2 \sigma_{ss} \quad (8)$$

$$\sigma_{22} = n_{x_2}^2 \sigma_{nn} - 2 n_{x_1} n_{x_2} \sigma_{ns} + n_{x_1}^2 \sigma_{ss} \quad (9)$$

$$\sigma_{12} = \sigma_{21} = n_{x_1} n_{x_2} (\sigma_{ss} - \sigma_{nn}) + (n_{x_1}^2 - n_{x_2}^2) \sigma_{ns} \quad (10)$$

By the definition of the thickness-integrated forces and moments, we have similarly with the transformation of the stress field the following relations,

$$N_{11} = n_{x_1}^2 N_{nn} + 2n_{x_1} n_{x_2} N_{ns} + n_{x_2}^2 N_{ss} \tag{11}$$

$$N_{22} = n_{x_2}^2 N_{nn} - 2n_{x_1} n_{x_2} N_{ns} + n_{x_1}^2 N_{ss} \tag{12}$$

$$N_{12} = N_{21} = n_{x_1} n_{x_2} (N_{ss} - N_{nn}) + (n_{x_1}^2 - n_{x_2}^2) N_{ns} \tag{13}$$

and

$$M_{11} = n_{x_1}^2 M_{nn} + 2n_{x_1} n_{x_2} M_{ns} + n_{x_2}^2 M_{ss} \tag{14}$$

$$M_{22} = n_{x_2}^2 M_{nn} - 2n_{x_1} n_{x_2} M_{ns} + n_{x_1}^2 M_{ss} \tag{15}$$

$$M_{12} = M_{21} = n_{x_1} n_{x_2} (M_{ss} - M_{nn}) + (n_{x_1}^2 - n_{x_2}^2) M_{ns} \tag{16}$$

Also before we proceed to the transformation of the boundary terms from the Cartesian to the curvilinear coordinate system, we have to present the same transformation law of the functions $a_{T0}(x_1, x_2)$, $b_{T0}(x_1, x_2)$, $a_{T1}(x_1, x_2)$, $b_{T1}(x_1, x_2)$, $c_{T0}(x_1, x_2)$, which describe the form of the given surface tractions (shown on the section 4.4). Thus, according to the transformation law (T0) and (T1) of the APPENDIX A, we get the following relations

$$a_{T0} = n_{x_1} \cdot a_{T0n} + n_{x_2} \cdot a_{T0s} \quad \text{(17a)} \quad \text{and} \quad a_{T1} = n_{x_1} \cdot a_{T1n} + n_{x_2} \cdot a_{T1s} \quad \text{(17b)}$$

$$b_{T0} = -n_{x_2} \cdot a_{T0n} + n_{x_1} \cdot a_{T0s} \quad \text{(17c)} \quad \text{and} \quad b_{T1} = -n_{x_2} \cdot a_{T1n} + n_{x_1} \cdot a_{T1s} \quad \text{(17d)}$$

$$c_{T0} = 1 \cdot c_{T0} \quad \text{(17e)}$$

where the function of the right-hand side of the above relations (17a) - (17e) have unique independent argument s, which counts the length of the curve Γ. Thus, as explained thoroughly by the mathematic definition of the curve on the APPENDIX A, the functions in the curvilinear system are written as,

$$a_{T0n} = a_{T0n}(s), \quad a_{T0s} = a_{T0s}(s), \quad a_{T1n} = a_{T1n}(s),$$
$$a_{T1s} = a_{T1s}(s) \quad \text{and} \quad c_{T0} = c_{T0}(s).$$

In view of the above relations, it is apparent that we have to occupy with the surface integrals (boundary terms) of the variational equation 3 of the section 4.5, namely the following

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{N_{11}n_{x_1}+N_{12}n_{x_2}-a_{T0}\frac{I_0}{\rho}\right\}\delta u_0\,d\gamma\,dt-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{N_{22}n_{x_2}+N_{12}n_{x_1}-b_{T0}\frac{I_0}{\rho}\right\}\delta v_0\,d\gamma\,dt-$$

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\left(I_2\frac{\partial\ddot{w}_0}{\partial x_1}-\frac{\partial M_1}{\partial x_1}-\frac{\partial M_6}{\partial x_2}\right)n_{x_1}+\left(I_2\frac{\partial\ddot{w}_0}{\partial x_2}-\frac{\partial M_2}{\partial x_2}-\frac{\partial M_6}{\partial x_1}\right)n_{x_2}-c_{T0}\frac{I_0}{\rho}\right\}\delta w_0\,d\gamma\,dt-$$

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{M_{11}n_{x_1}+M_{12}n_{x_2}+a_{T1}\frac{I_2}{\rho}\right\}\frac{\partial\delta w_0}{\partial x_1}\,d\gamma\,dt-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{M_{22}n_{x_2}+M_{12}n_{x_1}+b_{T1}\frac{I_2}{\rho}\right\}\frac{\partial\delta w_0}{\partial x_2}\,d\gamma\,dt$$

At this point, it is regarded important due to the nature of the problem of CPT and also convenient for the calculations, to separate the above equation into three parts and after that to perform calculations, integrations and so on. The first part includes the in-plane variations δu_0 and δv_0 (first row of the above expression), the second one includes the vertical variation δw_0 (second line), whereas the third part is related to the spatial derivatives of the vertical variation $\partial\delta w_0/\partial x_1$ and $\partial\delta w_0/\partial x_2$ (third line), that is the variation of the derivatives of δw_0 along the boundary curve Γ .

The following disjunction and grouping of the boundary terms of the just previous variational equation, is on purpose. First, the in-plane variations give surely the two of the boundary conditions of the problem of CPT, either they are expressed in terms of the Cartesian or curvilinear coordinates. Second in contrast to the in-plane variations, the boundary conditions associated with the spatial derivatives of the vertical variation $\partial\delta w_0/\partial x_1$ and $\partial\delta w_0/\partial x_2$, have to be merged and finally produce one boundary condition. The last satisfies our demand of totally four boundary conditions for the problem of CPT.

However, the second part including the variation δw_0 is correlated to the third one, as will be shown on the next paragraphs of this section (5.2).This fact takes place due to the performance of by parts integrations to the transformed quantities of the variations $\partial\delta w_0/\partial x_1$ and $\partial\delta w_0/\partial x_2$, which will give boundary terms explicitly related to the variation δw_0.

5.2.1. In-plane boundary conditions in the curvilinear boundary coordinates

To simplify the process of transformation of these terms from the Cartesian coordinate system to the local one, we neglect once again the time integration and further now the curvilinear integration.

Subsequently, taking apart each boundary condition multiplied with a different component of the variation of the displacement field and using the Eqs. (3`), (4`), (5`), (11)-(13), (17a) and (17c) we get the below.

$$\left\{N_{11} n_{x_1} + N_{12} n_{x_2} - a_{T0}\frac{I_0}{\rho}\right\}\delta u_0 =$$

$$= \left\{\begin{array}{c} (n_{x_1}^2 N_{nn} + 2 n_{x_1} n_{x_2} N_{ns} + n_{x_2}^2 N_{ss}) n_{x_1} + \\ +(n_{x_1} n_{x_2} (N_{ss} - N_{nn}) + (n_{x_1}^2 - n_{x_2}^2) N_{ns}) n_{x_2} - \\ -(n_{x_1} a_{T0n} + n_{x_2} a_{T0s})\dfrac{I_0}{\rho} \end{array}\right\}\{\delta u_{0n} n_{x_1} + \delta u_{0s} n_{x_2}\} =$$

$$= \left\{\begin{array}{c} (n_{x_1}^3 - n_{x_1} n_{x_2}^2) N_{nn} + 2 n_{x_2}^2 n_{x_1} N_{ss} + \\ + (3 n_{x_1}^2 n_{x_2} - n_{x_2}^3) N_{ns} - (n_{x_1} a_{T0n} + n_{x_2} a_{T0s})\dfrac{I_0}{\rho} \end{array}\right\}\{\delta u_{0n} n_{x_1} + \delta u_{0s} n_{x_2}\} =$$

$$\begin{aligned} &= (n_{x_1}^4 - n_{x_1}^2 n_{x_2}^2) N_{nn}\delta u_{0n} + 2 n_{x_2}^2 n_{x_1}^2 N_{ss}\delta u_{0n} + (3 n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3) N_{ns}\delta u_{0n} + \\ &+ (n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3) N_{nn}\delta u_{0s} + 2 n_{x_2}^3 n_{x_1} N_{ss}\delta u_{0s} + (3 n_{x_1}^2 n_{x_2}^2 - n_{x_2}^4) N_{ns}\delta u_{0s} - \\ &- (n_{x_1}^2 a_{T0n} + n_{x_1} n_{x_2} a_{T0s})\frac{I_0}{\rho}\delta u_{0n} - (n_{x_1} n_{x_2} a_{T0n} + n_{x_2}^2 a_{T0s})\frac{I_0}{\rho}\delta u_{0s} \end{aligned} \tag{19}$$

and

$$\left\{N_{22} n_{x_2} + N_{12} n_{x_1} - b_{T0}\frac{I_0}{\rho}\right\}\delta v_0 =$$

$$= \left\{\begin{array}{c} n_{x_2}^3 N_{nn} - 2 n_{x_1} n_{x_2}^2 N_{ns} + n_{x_1}^2 n_{x_2} N_{ss} + \\ + n_{x_1}^2 n_{x_2} N_{ss} - n_{x_1}^2 n_{x_2} N_{nn} + (n_{x_1}^3 - n_{x_2}^2 n_{x_1}) N_{ns} - \\ -(-n_{x_2} a_{T0n} + n_{x_1} a_{T0s})\dfrac{I_0}{\rho} \end{array}\right\}\{n_{x_1}\delta u_{0s} - n_{x_2}\delta u_{0n}\} =$$

$$= \left\{\begin{array}{c} (n_{x_2}^3 - n_{x_1}^2 n_{x_2}) N_{nn} + 2 n_{x_1}^2 n_{x_2} N_{ss} + \\ + (n_{x_1}^3 - 3 n_{x_1} n_{x_2}^2) N_{ns} - (n_{x_1} a_{T0s} - n_{x_2} a_{T0n})\dfrac{I_0}{\rho} \end{array}\right\}\{n_{x_1}\delta u_{0s} - n_{x_2}\delta u_{0n}\} =$$

$$= - \; n_{x_2}^4 - n_{x_1}^2 n_{x_2}^2 \; N_{nn}\delta u_{0n} - 2\, n_{x_1}^2 n_{x_2}^2 N_{ss}\delta u_{0n} + \; 3n_{x_1} n_{x_2}^3 - n_{x_1}^3 n_{x_2} \; N_{ns}\delta u_{0n} +$$
$$+(n_{x_2}^3 n_{x_1} - n_{x_1}^3 n_{x_2}) N_{nn}\delta u_{0s} + 2\, n_{x_1}^3 n_{x_2} N_{ss}\delta u_{0s} - (3n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4) N_{ns}\delta u_{0s} +$$
$$+ \; n_{x_2}^2 a_{T0n} + n_{x_1} n_{x_2} a_{T0s} \; \frac{I_0}{\rho}\delta u_{0n} - \; -n_{x_2} n_{x_1} a_{T0n} + n_{x_1}^2 a_{T0s} \; \frac{I_0}{\rho}\delta u_{0s} \qquad (20)$$

Adding the Eqs. (19) and (20), we have

$$\left\{ N_{11} n_{x_1} + N_{12} n_{x_2} - a_{T0}\frac{I_0}{\rho} \right\}\delta u_0 + \left\{ N_{22} n_{x_2} + N_{12} n_{x_1} - b_{T0}\frac{I_0}{\rho} \right\}\delta v_0 =$$
$$= (n_{x_1}^4 - n_{x_1}^2 n_{x_2}^2 - n_{x_2}^4 + n_{x_1}^2 n_{x_2}^2)\, N_{nn}\delta u_{0n} + 2\, n_{x_2}^2 n_{x_1}^2 \, N_{ss}\delta u_{0n} - 2\, n_{x_1}^2 n_{x_2}^2 \, N_{ss}\delta u_{0n} +$$
$$+ (3\, n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 - n_{x_1}^3 n_{x_2} + 3n_{x_1} n_{x_2}^3) N_{ns}\delta u_{0n} +$$
$$+ (n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 + n_{x_2}^3 n_{x_1} - n_{x_1}^3 n_{x_2}) N_{nn}\delta u_{0s} +$$
$$+ (2\, n_{x_2}^3 n_{x_1} + 2\, n_{x_1}^3 n_{x_2}) N_{ss}\delta u_{0s} + (3\, n_{x_1}^2 n_{x_2}^2 - n_{x_2}^4 + n_{x_1}^4 - 3n_{x_1}^2 n_{x_2}^2) N_{ns}\delta u_{0s} +$$
$$+ (-\, n_{x_2}^2 a_{T0n} + n_{x_1} n_{x_2} a_{T0s} - n_{x_1}^2 a_{T0n} - n_{x_1} n_{x_2} a_{T0s}) \frac{I_0}{\rho}\delta u_{0n} -$$
$$- (n_{x_1} n_{x_2} a_{T0n} + n_{x_2}^2 a_{T0s} - n_{x_1} n_{x_2} a_{T0n} + n_{x_1}^2 a_{T0s}) \frac{I_0}{\rho}\delta u_{0s} =$$

$$= (n_{x_1}^4 - n_{x_1}^2 n_{x_2}^2 - n_{x_2}^4 + n_{x_1}^2 n_{x_2}^2)\, N_{nn}\delta u_{0n} + \cancel{2\, n_{x_2}^2 n_{x_1}^2 \, N_{ss}\delta u_{0n}} - \cancel{2\, n_{x_1}^2 n_{x_2}^2 \, N_{ss}\delta u_{0n}} +$$
$$+ (2\, n_{x_1}^3 n_{x_2} + 2n_{x_1} n_{x_2}^3) N_{ns}\delta u_{0n} +$$
$$+ (n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 + n_{x_2}^3 n_{x_1} - n_{x_1}^3 n_{x_2}) N_{nn}\delta u_{0s} +$$
$$+ (2\, n_{x_2}^3 n_{x_1} + 2\, n_{x_1}^3 n_{x_2})\, N_{ss}\delta u_{0s} + (3\, n_{x_1}^2 n_{x_2}^2 - n_{x_2}^4 + n_{x_1}^4 - 3n_{x_1}^2 n_{x_2}^2)\, N_{ns}\delta u_{0s} +$$
$$+ (-\, n_{x_2}^2 a_{T0n} + n_{x_1} n_{x_2} a_{T0s} - n_{x_1}^2 a_{T0n} - n_{x_1} n_{x_2} a_{T0s}) \frac{I_0}{\rho}\delta u_{0n} -$$
$$- (n_{x_1} n_{x_2} a_{T0n} + n_{x_2}^2 a_{T0s} - n_{x_1} n_{x_2} a_{T0n} + n_{x_1}^2 a_{T0s}) \frac{I_0}{\rho}\delta u_{0s} =$$

[eliminating the zero terms and grouping together those with the same variation]

$$= \; n_{x_1}^4 - n_{x_2}^4 \; N_{nn}\delta u_{0n} + \; 2\, n_{x_1}^3 n_{x_2} + 2n_{x_1} n_{x_2}^3 \; N_{ns}\delta u_{0n} + \; -n_{x_2}^2 - n_{x_1}^2 \; a_{T0n}\frac{I_0}{\rho}\delta u_{0n} +$$
$$+ \; 2\, n_{x_2}^3 n_{x_1} + 2\, n_{x_1}^3 n_{x_2} \; N_{ss}\delta u_{0s} + \; n_{x_1}^4 - n_{x_2}^4 \; N_{ns}\delta u_{0s} - \; n_{x_2}^2 + n_{x_1}^2 \; a_{T0s}\frac{I_0}{\rho}\delta u_{0s} =$$

$$= \left\{ \; n_{x_1}^4 - n_{x_2}^4 \; N_{nn} + \; 2\, n_{x_1}^3 n_{x_2} + 2n_{x_1} n_{x_2}^3 \; N_{ns} - \; n_{x_2}^2 + n_{x_1}^2 \; a_{T0n}\frac{I_0}{\rho} \right\}\delta u_{0n} +$$
$$+ \left\{ \; n_{x_1}^4 - n_{x_2}^4 \; N_{ns} + \; 2\, n_{x_2}^3 n_{x_1} + 2\, n_{x_1}^3 n_{x_2} \; N_{ss} - \; n_{x_2}^2 + n_{x_1}^2 \; a_{T0s}\frac{I_0}{\rho} \right\}\delta u_{0s} =$$

$$= \left\{ n_{x_1}^2 - n_{x_2}^2 \;\; n_{x_1}^2 + n_{x_2}^2 \;\; N_{nn} + 2\, n_{x_1} n_{x_2} \;\; n_{x_1}^2 + n_{x_2}^2 \;\; N_{ns} - \;\; n_{x_2}^2 + n_{x_1}^2 \;\; a_{T0n} \frac{I_0}{\rho} \right\} \delta u_{0n} +$$

$$+ \left\{ n_{x_1}^2 - n_{x_2}^2 \;\; n_{x_1}^2 + n_{x_2}^2 \;\; N_{ns} + 2\, n_{x_2} n_{x_1} \;\; n_{x_2}^2 + n_{x_1}^2 \;\; N_{ss} - \;\; n_{x_2}^2 + n_{x_1}^2 \;\; a_{T0s} \frac{I_0}{\rho} \right\} \delta u_{0s}$$

However the above relation could be simplified further, by remembering that the normal vector to the lateral edge of the plate $\hat{\boldsymbol{n}}$ is unit. Thus, its meter is equal to the unit so that

$$\sqrt{n_{x_1}^2 + n_{x_2}^2} = n_{x_1}^2 + n_{x_2}^2 = 1.$$

Consequently, the final form of the above result is,

$$\left\{ N_{11} n_{x_1} + N_{12} n_{x_2} - a_{T0} \frac{I_0}{\rho} \right\} \delta u_0 + \left\{ N_{22} n_{x_2} + N_{12} n_{x_1} - b_{T0} \frac{I_0}{\rho} \right\} \delta v_0 =$$

$$= \left\{ n_{x_1}^2 - n_{x_2}^2 \;\; N_{nn} + 2\, n_{x_1} n_{x_2} \;\; N_{ns} - a_{T0n} \frac{I_0}{\rho} \right\} \delta u_{0n} +$$

$$+ \left\{ n_{x_1}^2 - n_{x_2}^2 \;\; N_{ns} + 2\, n_{x_2} n_{x_1} \;\; N_{ss} - a_{T0s} \frac{I_0}{\rho} \right\} \delta u_{0s} \qquad \textbf{(21)}$$

Note that the previous simplification of the results is going to be applied similarly on the next sections, namely 5.2.2 and 5.2.3, because we come up the same quantity ($n_{x_1}^2 + n_{x_2}^2$) inside the following relations.

On the basis of the above result, we notice that the functions $a_{T0n}(s)$ and $a_{T0s}(s)$ are not correlated inside the brackets multiplied with the variations δu_{0n} and δu_{0s} respectively. This fact is expected because the variations normal and tangent to the lateral surface of the plate, the δu_{0n} and δu_{0s} respectively, are independent essential conditions and also the quantities by which they are multiplied are two independent natural boundary conditions of the problem of CPT, as seems below

$$n_{x_1}^2 - n_{x_2}^2 \;\; N_{nn} + 2\, n_{x_1} n_{x_2} \;\; N_{ns} = a_{T0n} \frac{I_0}{\rho} \qquad (21a)$$

$$n_{x_1}^2 - n_{x_2}^2 \;\; N_{ns} + 2\, n_{x_2} n_{x_1} \;\; N_{ss} = a_{T0s} \frac{I_0}{\rho} \qquad (21b)$$

5.2.2. Transformation of the boundary conditions associated with $\partial_1 \delta w_0 , \partial_2 \delta w_0$ to the curvilinear coordinate system

To simplify the process of transformation of these terms from the Cartesian coordinate system to the curvilinear one, we neglect once again the time integration and further now the curvilinear integration.
Subsequently, taking apart each boundary condition multiplied with a different component of the variation of the displacement field and using the Eqs. (6), (7), (14)-(16), (17b) and (17d), we get the following results.

First, as for the first part involving the x_1-spatial derivative of the variation δw_0

$$\left\{ M_{11} n_{x_1} + M_{12} n_{x_2} + (n_{x_1} a_{T1n} + n_{x_2} a_{T1s}) \frac{I_2}{\rho} \right\} \frac{\partial \delta w_0}{\partial x_1} =$$

$$= \left\{ \begin{array}{l} (n_{x_1}^2 M_{nn} + 2 n_{x_1} n_{x_2} M_{ns} + n_{x_2}^2 M_{ss}) n_{x_1} + \\ + (n_{x_1} n_{x_2} (M_{ss} - M_{nn}) + (n_{x_1}^2 - n_{x_2}^2) M_{ns}) n_{x_2} + \\ + (n_{x_1} a_{T1n} + n_{x_2} a_{T1s}) \frac{I_2}{\rho} \end{array} \right\} \left\{ n_{x_1} \frac{\partial \delta w_0}{\partial n} - n_{x_2} \frac{\partial \delta w_0}{\partial s} \right\} =$$

$$= \left\{ \begin{array}{l} n_{x_1}^3 M_{nn} + 2 n_{x_1}^2 n_{x_2} M_{ns} + n_{x_2}^2 n_{x_1} M_{ss} + \\ + n_{x_1} n_{x_2}^2 (M_{ss} - M_{nn}) + (n_{x_1}^2 n_{x_2} - n_{x_2}^3) M_{ns} + \\ + (n_{x_1} a_{T1n} + n_{x_2} a_{T1s}) \frac{I_2}{\rho} \end{array} \right\} \left\{ n_{x_1} \frac{\partial \delta w_0}{\partial n} - n_{x_2} \frac{\partial \delta w_0}{\partial s} \right\} =$$

[Grouping together the terms with the same thickness-integrated moment multiplied with the same curvilinear derivative of the δw_0 and also separating the unknown terms from the given]

$$\begin{aligned} &= (n_{x_1}^4 - n_{x_1}^2 n_{x_2}^2) M_{nn} \frac{\partial \delta w_0}{\partial n} + 2 n_{x_1}^2 n_{x_2}^2 M_{ss} \frac{\partial \delta w_0}{\partial n} + \\ &+ (3 n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3) M_{ns} \frac{\partial \delta w_0}{\partial n} - (n_{x_1}^3 n_{x_2} + n_{x_1} n_{x_2}^3) M_{nn} \frac{\partial \delta w_0}{\partial s} - \\ &- 2 n_{x_2}^3 n_{x_1} M_{ss} \frac{\partial \delta w_0}{\partial s} - (n_{x_2}^4 + 3 n_{x_1}^2 n_{x_2}^2) M_{ns} \frac{\partial \delta w_0}{\partial s} + \\ &+ (n_{x_1}^2 a_{T1n} + n_{x_2} n_{x_1} a_{T1s}) \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial n} - (n_{x_1} n_{x_2} a_{T1n} + n_{x_2}^2 a_{T1s}) \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial s} \end{aligned} \tag{22}$$

Subsequently, as for the second part related to the x_2-spatial derivatives of the variation δw_0

$$\left\{M_{22}\, n_{x_2} + M_{12}\, n_{x_1} + (-n_{x_2}\, a_{T1n} + n_{x_1}\, a_{T1s})\frac{I_2}{\rho}\right\}\frac{\partial \delta w_0}{\partial x_2} =$$

$$= \left\{ \begin{array}{c} (n_{x_2}^2 M_{nn} - 2 n_{x_1} n_{x_2} M_{ns} + n_{x_1}^2 M_{ss}) n_{x_2} + \\ + (n_{x_1} n_{x_2} (M_{ss} - M_{nn}) + (n_{x_1}^2 - n_{x_2}^2) M_{ns}) n_{x_1} + \\ + (-n_{x_2}\, a_{T1n} + n_{x_1}\, a_{T1s})\dfrac{I_2}{\rho} \end{array} \right\} \left\{ n_{x_2} \frac{\partial \delta w_0}{\partial n} + n_{x_1} \frac{\partial \delta w_0}{\partial s} \right\} =$$

$$= \left\{ \begin{array}{c} n_{x_2}^3 M_{nn} - 2 n_{x_1} n_{x_2}^2 M_{ns} + n_{x_1}^2 n_{x_2} M_{ss} + \\ + n_{x_1}^2 n_{x_2} (M_{ss} - M_{nn}) + (n_{x_1}^3 - n_{x_2}^2 n_{x_1}) M_{ns} + \\ + (-n_{x_2}\, a_{T1n} + n_{x_1}\, a_{T1s})\dfrac{I_2}{\rho} \end{array} \right\} \left\{ n_{x_2} \frac{\partial \delta w_0}{\partial n} + n_{x_1} \frac{\partial \delta w_0}{\partial s} \right\} =$$

[Similarly to the previous, we gather the terms with the same thickness-integrated moment multiplied with the same curvilinear derivative of the δw_0 and also separating the unknown terms from the given]

$$\begin{aligned} &= n_{x_2}^4 - n_{x_1}^2 n_{x_2}^2 \; M_{nn} \frac{\partial \delta w_0}{\partial n} + 2\, n_{x_1}^2 n_{x_2}^2 \, M_{ss} \frac{\partial \delta w_0}{\partial n} + \\ &+ n_{x_1}^3 n_{x_2} - 3 n_{x_1} n_{x_2}^3 \; M_{ns} \frac{\partial \delta w_0}{\partial n} + n_{x_2}^3 n_{x_1} - n_{x_1}^3 n_{x_2} \; M_{nn} \frac{\partial \delta w_0}{\partial s} + \\ &+ 2\, n_{x_1}^3 n_{x_2} M_{ss} \frac{\partial \delta w_0}{\partial s} + n_{x_1}^4 - 3 n_{x_1}^2 n_{x_2}^2 \; M_{ns} \frac{\partial \delta w_0}{\partial s} + \\ &+ n_{x_1} n_{x_2} a_{T1s} - n_{x_2}^2 a_{T1n} \; \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial n} + n_{x_1}^2 a_{T1s} - n_{x_2} n_{x_1} a_{T1n} \; \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial s} \end{aligned} \qquad (23)$$

Adding now the Eqs. (22) and (23), we have

$$\left\{M_{11} n_{x_1} + M_{12} n_{x_2} + a_{T1} \frac{I_2}{\rho}\right\} \frac{\partial \delta w_0}{\partial x_1} + \left\{M_{22} n_{x_2} + M_{12} n_{x_1} + b_{T1} \frac{I_2}{\rho}\right\} \frac{\partial \delta w_0}{\partial x_2} =$$

$$= \; n_{x_1}^4 - n_{x_1}^2 n_{x_2}^2 + n_{x_2}^4 - n_{x_1}^2 n_{x_2}^2 \;\; M_{nn} \frac{\partial \delta w_0}{\partial n} + 4\, n_{x_1}^2 n_{x_2}^2 \;\; M_{ss} \frac{\partial \delta w_0}{\partial n} +$$

$$+ \;\; 3 n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 + n_{x_1}^3 n_{x_2} - 3 n_{x_1} n_{x_2}^3 \;\; M_{ns} \frac{\partial \delta w_0}{\partial n} -$$

$$- \; (n_{x_1}^3 n_{x_2} + n_{x_1} n_{x_2}^3 - n_{x_2}^3 n_{x_1} + n_{x_1}^3 n_{x_2}) M_{nn} \frac{\partial \delta w_0}{\partial s} -$$

$$- \, (n_{x_2}^4 + 3 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4 \; + 3 n_{x_1}^2 n_{x_2}^2) M_{ns} \frac{\partial \delta w_0}{\partial s} +$$

$$+ \; (2\, n_{x_1}^3 n_{x_2} - 2\, n_{x_2}^3 n_{x_1}) M_{ss} \frac{\partial \delta w_0}{\partial s} +$$

$$+ \; (n_{x_1}^2 a_{T1n} \; + n_{x_2} n_{x_1} a_{T1s}) \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial n} \; - \; (n_{x_1} n_{x_2} a_{T1n} \; + n_{x_2}^2 a_{T1s}) \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial s} +$$

$$+ \; (n_{x_1} n_{x_2} a_{T1s} \; - \; n_{x_2}^2 a_{T1n}) \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial n} + (n_{x_1}^2 a_{T1s} \; - \; n_{x_2} n_{x_1} a_{T1n}) \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial s} =$$

[Grouping together again the terms with the same the thickness-integrated quantities of the same curvilinear direction and of the same derivative of w_0 on the curvilinear coordinate system]

$$= \; n_{x_1}^4 - 2\, n_{x_1}^2 n_{x_2}^2 + n_{x_2}^4 \;\; M_{nn} \frac{\partial \delta w_0}{\partial n} + 4\, n_{x_1}^2 n_{x_2}^2 \;\; M_{ss} \frac{\partial \delta w_0}{\partial n} +$$

$$+ \;\; 4 n_{x_1}^3 n_{x_2} - 4 n_{x_1} n_{x_2}^3 \;\; M_{ns} \frac{\partial \delta w_0}{\partial n} - (2\, n_{x_1}^3 n_{x_2} + \cancel{n_{x_1} n_{x_2}^3} - \cancel{n_{x_2}^3 n_{x_1}}) M_{nn} \frac{\partial \delta w_0}{\partial s} -$$

$$- \; (n_{x_2}^4 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4) M_{ns} \frac{\partial \delta w_0}{\partial s} + (2\, n_{x_1}^3 n_{x_2} - 2\, n_{x_2}^3 n_{x_1}) M_{ss} \frac{\partial \delta w_0}{\partial s} +$$

$$+ \; (n_{x_1}^2 a_{T1n} \; + n_{x_2} n_{x_1} a_{T1s} \; + n_{x_1} n_{x_2} a_{T1s} \; - n_{x_2}^2 a_{T1n}) \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial n} -$$

$$- \; (n_{x_1} n_{x_2} a_{T1n} \; + n_{x_2}^2 a_{T1s} \; - n_{x_1}^2 a_{T1s} \; + n_{x_2} n_{x_1} a_{T1n}) \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial s} =$$

[Simplifying the terms inside the brackets concerning the direction cosines n_{x_1}, n_{x_2}]

$$= \;\; n_{x_1}^4 - 2\, n_{x_1}^2\, n_{x_2}^2 + n_{x_2}^4 \;\; M_{nn} \frac{\partial \delta w_0}{\partial n} \; + \; 4\, n_{x_1}^2\, n_{x_2}^2 \;\; M_{ss} \frac{\partial \delta w_0}{\partial n} \; +$$

$$+ \;\; 4 n_{x_1}^3\, n_{x_2} - \; 4 n_{x_1}\, n_{x_2}^3 \;\; M_{ns} \frac{\partial \delta w_0}{\partial n} \; - \; 2\, n_{x_1}^3\, n_{x_2} \, M_{nn} \frac{\partial \delta w_0}{\partial s} \; -$$

$$- \;\; n_{x_2}^4 + 6 n_{x_1}^2\, n_{x_2}^2 - n_{x_1}^4 \;\; M_{ns} \frac{\partial \delta w_0}{\partial s} \; + \; 2\, n_{x_1}^3\, n_{x_2} - \; 2\, n_{x_2}^3 n_{x_1} \;\; M_{ss} \frac{\partial \delta w_0}{\partial s} \; +$$

$$+ \;\; n_{x_1}^2\, a_{T1n} \; + \; 2\, n_{x_2}\, n_{x_1}\, a_{T1s} \; - \; n_{x_2}^2\, a_{T1n} \;\; \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial n} \; -$$

$$- \;\; (n_{x_2}^2\, a_{T1s} \; + \; 2\, n_{x_1}\, n_{x_2}\, a_{T1n} \; - \; n_{x_1}^2\, a_{T1s}) \, \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial s} \; =$$

[Finally we create smarter and shorter forms of the direction cosines inside the brackets and we conclude to the final curvilinear boundary terms]

$$= \;\; n_{x_1}^2 - n_{x_2}^2 \;\; {}^2 \, M_{nn} \frac{\partial \delta w_0}{\partial n} \; + \; 4\, n_{x_1}^2\, n_{x_2}^2 \;\; M_{ss} \frac{\partial \delta w_0}{\partial n} \; +$$

$$+ \; 4 n_{x_1}\, n_{x_2} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; M_{ns} \frac{\partial \delta w_0}{\partial n} \; - \; 2\, n_{x_1}^3\, n_{x_2} \, M_{nn} \frac{\partial \delta w_0}{\partial s} \; -$$

$$- \;\; n_{x_2}^4 + 6 n_{x_1}^2\, n_{x_2}^2 - n_{x_1}^4 \;\; M_{ns} \frac{\partial \delta w_0}{\partial s} \; + \; 2\, n_{x_1}\, n_{x_2} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; M_{ss} \frac{\partial \delta w_0}{\partial s} \; + \qquad \textbf{(24)}$$

$$+ \; ((n_{x_1}^2 \; - \; n_{x_2}^2)\, a_{T1n} \; + \; 2\, n_{x_2}\, n_{x_1}\, a_{T1s}) \, \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial n} \; -$$

$$- \; ((n_{x_2}^2 \; - \; n_{x_1}^2)\, a_{T1s} \; + \; 2\, n_{x_1}\, n_{x_2}\, a_{T1n}) \, \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial s}$$

On the basis of the above result, remark that the function $a_{T1n}(n, s)$ is related both with the variation $\partial \delta w_0 / \partial n$ as well as with the variation $\partial \delta w_0 / \partial s$. Exactly the same is valid for the function $a_{T1s}(n, s)$, which similarly is related both to the variation $\partial \delta w_0 / \partial n$ and $\partial \delta w_0 / \partial s$.

This configuration is expected because the derivatives of the vertical displacement as well as the corresponding natural conditions [terms inside the brackets of Eq. (24)], are correlated and finally they are going to give one boundary condition (as will show on the following sections).

5.2.3. Derivation of the boundary terms associated with the variation δw_0

Now, taking apart the second line of the boundary terms of the variational equation, this is repeated on the last paragraph of the section 5.2, we have

$$\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\underbrace{\left(I_2\frac{\partial \ddot{w}_0}{\partial x_1}-\frac{\partial M_{11}}{\partial x_1}-\frac{\partial M_{12}}{\partial x_2}\right)n_{x_1}}_{I_{\delta w_0,n_{x_1}}}+\underbrace{\left(I_2\frac{\partial \ddot{w}_0}{\partial x_2}-\frac{\partial M_{22}}{\partial x_2}-\frac{\partial M_{12}}{\partial x_1}\right)n_{x_2}}_{I_{\delta w_0,n_{x_2}}}-c_{T0}(x_1,x_2)\frac{I_0}{\rho}\right\}\delta w_0\,d\gamma\,dt$$

Subsequently, taking apart each term of the above expression multiplied with the direction cosines (in order to simplify the calculations) and using the Eqs. (6), (7), (14)-(16) and (17e), we get the following results. In addition, note that for the sake of convenience we neglect the time and curvilinear integrals.

However, due to the complication of the calculations, we present below the derivation of same terms separately and after that we substitute the results into the $I_{\delta w_0,n_{x_1}}$ and $I_{\delta w_0,n_{x_2}}$.

Thus,

$$\frac{\partial \ddot{w}_0}{\partial x_1}=n_{x_1}\frac{\partial \ddot{w}_0}{\partial n}+n_{x_2}\frac{\partial \ddot{w}_0}{\partial s} \quad \textbf{(25a)} \quad \text{and} \quad \frac{\partial \ddot{w}_0}{\partial x_2}=-n_{x_2}\frac{\partial \ddot{w}_0}{\partial n}+n_{x_1}\frac{\partial \ddot{w}_0}{\partial s} \quad \textbf{(25b)}$$

$$\begin{aligned}\frac{\partial M_{11}}{\partial x_1}&=n_{x_1}\frac{\partial M_{11}}{\partial n}+n_{x_2}\frac{\partial M_{11}}{\partial s}=\\&=n_{x_1}\frac{\partial}{\partial n}\,n_{x_1}^2M_{nn}+2n_{x_1}n_{x_2}M_{ns}+n_{x_2}^2M_{ss}\;+\\&+n_{x_2}\frac{\partial}{\partial s}\,n_{x_1}^2M_{nn}+2n_{x_1}n_{x_2}M_{ns}+n_{x_2}^2M_{ss}\;=\\&=n_{x_1}^3M_{nn,n}+2n_{x_1}^2n_{x_2}M_{ns,n}+n_{x_1}n_{x_2}^2M_{ss,n}\;+\\&+n_{x_2}n_{x_1}^2M_{nn,s}+2n_{x_1}n_{x_2}^2M_{ns,s}+n_{x_2}^3M_{ss,s}\end{aligned} \quad \textbf{(25c)}$$

$$\begin{aligned}\frac{\partial M_{12}}{\partial x_2}&=-n_{x_2}\frac{\partial M_{12}}{\partial n}+n_{x_1}\frac{\partial M_{12}}{\partial s}=\\&=-n_{x_2}\frac{\partial}{\partial n}\,n_{x_1}n_{x_2}\;M_{ss}-M_{nn}\;+\;n_{x_1}^2-n_{x_2}^2\;M_{ns}\;+\\&+n_{x_1}\frac{\partial}{\partial s}\,n_{x_1}n_{x_2}\;M_{ss}-M_{nn}\;+\;n_{x_1}^2-n_{x_2}^2\;M_{ns}\;=\\&=-n_{x_1}n_{x_2}^2\;M_{ss,n}-M_{nn,n}\;-\;n_{x_1}^2n_{x_2}-n_{x_2}^3\;M_{ns,n}\;+\\&+n_{x_1}^2n_{x_2}\;M_{ss,s}-M_{nn,s}\;+\;n_{x_1}^3-n_{x_1}n_{x_2}^2\;M_{ns,s}\end{aligned} \quad \textbf{(25d)}$$

$$\frac{\partial M_{22}}{\partial x_2} = -n_{x_2}\frac{\partial M_{22}}{\partial n} + n_{x_1}\frac{\partial M_{22}}{\partial s} =$$
$$= -n_{x_2}\frac{\partial}{\partial n}\; n_{x_2}^2 M_{nn} - 2n_{x_1}n_{x_2}M_{ns} + n_{x_1}^2 M_{ss} \;+$$
$$+\, n_{x_1}\frac{\partial}{\partial s}\; n_{x_2}^2 M_{nn} - 2n_{x_1}n_{x_2}M_{ns} + n_{x_1}^2 M_{ss} \;=$$
$$= -n_{x_2}^3 M_{nn,n} + 2n_{x_1}n_{x_2}^2 M_{ns,n} - n_{x_2}n_{x_1}^2 M_{ss,n} \;+$$
$$+\, n_{x_1}n_{x_2}^2 M_{nn,s} - 2n_{x_1}^2 n_{x_2}M_{ns,s} + n_{x_1}^3 M_{ss,s} \qquad \textbf{(25e)}$$

$$\frac{\partial M_{12}}{\partial x_1} = n_{x_1}\frac{\partial M_{12}}{\partial n} + n_{x_2}\frac{\partial M_{12}}{\partial s} =$$
$$= n_{x_1}\frac{\partial}{\partial n}\; n_{x_1}n_{x_2}\; M_{ss} - M_{nn} \;+\; n_{x_1}^2 - n_{x_2}^2 \; M_{ns} \;+$$
$$+\, n_{x_2}\frac{\partial}{\partial s}\; n_{x_1}n_{x_2}\; M_{ss} - M_{nn} \;+\; n_{x_1}^2 - n_{x_2}^2 \; M_{ns} \;=$$
$$= n_{x_1}^2 n_{x_2}\; M_{ss,n} - M_{nn,n} \;+\; n_{x_1}^3 - n_{x_1}n_{x_2}^2 \; M_{ns,n} \;+$$
$$+\, n_{x_1}n_{x_2}^2\; M_{ss,s} - M_{nn,s} \;+\; n_{x_2}n_{x_1}^2 - n_{x_2}^3 \; M_{ns,s} \qquad \textbf{(25f)}$$

And now substituting the Eqs. (25a) - (25f) into the expressions $I_{\delta w_0,\, n_{x_1}}$ and $I_{\delta w_0,\, n_{x_2}}$,

$$I_{\delta w_0,\, n_{x_1}} = I_2\left\{ n_{x_1}^2 \frac{\partial \ddot{w}_0}{\partial n} + n_{x_1}n_{x_2}\frac{\partial \ddot{w}_0}{\partial s}\right\} -$$
$$-\left\{\begin{array}{l} n_{x_1}^4 M_{nn,n} + 2n_{x_1}^3 n_{x_2} M_{ns,n} + n_{x_1}^2 n_{x_2}^2 M_{ss,n} \;+ \\ +\, n_{x_2}n_{x_1}^3 M_{nn,s} + 2n_{x_1}^2 n_{x_2}^2 M_{ns,s} + n_{x_2}^3 n_{x_1} M_{ss,s} \;- \\ -\, n_{x_1}^2 n_{x_2}^2\; M_{ss,n} - M_{nn,n} \;-\; (n_{x_1}^3 n_{x_2} - n_{x_1}n_{x_2}^3)\, M_{ns,n} \;+ \\ +\, n_{x_1}^3 n_{x_2}\; M_{ss,s} - M_{nn,s} \;+\; (n_{x_1}^4 - n_{x_1}^2 n_{x_2}^2)\, M_{ns,s} \end{array}\right\} =$$
$$= I_2\left\{ n_{x_1}^2 \frac{\partial \ddot{w}_0}{\partial n} + n_{x_1}n_{x_2}\frac{\partial \ddot{w}_0}{\partial s}\right\} +$$
$$+\left\{\begin{array}{l} -n_{x_1}^4 M_{nn,n} - 2n_{x_1}^3 n_{x_2} M_{ns,n} - n_{x_1}^2 n_{x_2}^2 M_{ss,n} \;- \\ -\, n_{x_2}n_{x_1}^3 M_{nn,s} - 2n_{x_1}^2 n_{x_2}^2 M_{ns,s} - n_{x_2}^3 n_{x_1} M_{ss,s} \;+ \\ +\, n_{x_1}^2 n_{x_2}^2\; M_{ss,n} - n_{x_1}^2 n_{x_2}^2 M_{nn,n} + (n_{x_1}^3 n_{x_2} - n_{x_1}n_{x_2}^3)\, M_{ns,n} \;- \\ -\, n_{x_1}^3 n_{x_2}\; M_{ss,s} + n_{x_1}^3 n_{x_2} M_{nn,s} - (n_{x_1}^4 - n_{x_1}^2 n_{x_2}^2)\, M_{ns,s} \end{array}\right\} =$$

$$= I_2\left\{n_{x_1}^2 \frac{\partial \ddot{w}_0}{\partial n} + n_{x_1} n_{x_2}\frac{\partial \ddot{w}_0}{\partial s}\right\} +$$

$$+\left\{\begin{array}{l} -n_{x_1}^4 M_{nn,n} - 2n_{x_1}^3 n_{x_2} M_{ns,n} - \cancel{n_{x_1}^2 n_{x_2}^2 M_{ss,n}} - \\ - \cancel{n_{x_2} n_{x_1}^3 M_{nn,s}} - 2n_{x_1}^2 n_{x_2}^2 M_{ns,s} - n_{x_2}^3 n_{x_1} M_{ss,s} + \\ + \cancel{n_{x_1}^2 n_{x_2}^2 M_{ss,n}} - n_{x_1}^2 n_{x_2}^2 M_{nn,n} + (n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3) M_{ns,n} - \\ - n_{x_1}^3 n_{x_2} M_{ss,s} + \cancel{n_{x_1}^3 n_{x_2} M_{nn,s}} - (n_{x_1}^4 - n_{x_1}^2 n_{x_2}^2) M_{ns,s} \end{array}\right\} =$$

$$= I_2\left\{n_{x_1}^2 \frac{\partial \ddot{w}_0}{\partial n} + n_{x_1} n_{x_2}\frac{\partial \ddot{w}_0}{\partial s}\right\} -$$

$$- \; n_{x_1}^4 + n_{x_1}^2 n_{x_2}^2 \; M_{nn,n} - \; n_{x_2}^3 n_{x_1} + n_{x_1}^3 n_{x_2} \; M_{ss,s} -$$

$$- \; n_{x_1} n_{x_2}^3 + n_{x_1}^3 n_{x_2} \; M_{ns,n} - \; n_{x_1}^4 + n_{x_1}^2 n_{x_2}^2 \; M_{ns,s}$$

and

$$I_{\delta w_0, n_{x_2}} = I_2\left\{- n_{x_2}^2 \frac{\partial \ddot{w}_0}{\partial n} + n_{x_1} n_{x_2}\frac{\partial \ddot{w}_0}{\partial s}\right\} -$$

$$- \left\{\begin{array}{l} - n_{x_2}^4 M_{nn,n} + 2n_{x_1} n_{x_2}^3 M_{ns,n} - n_{x_2}^2 n_{x_1}^2 M_{ss,n} + \\ + n_{x_1} n_{x_2}^3 M_{nn,s} - 2n_{x_1}^2 n_{x_2}^2 M_{ns,s} + n_{x_1}^3 n_{x_2} M_{ss,s} \end{array}\right\} -$$

$$- \left\{\begin{array}{l} n_{x_1}^2 n_{x_2}^2 \; M_{ss,n} - M_{nn,n} + \; n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 \; M_{ns,n} + \\ + n_{x_1} n_{x_2}^3 (M_{ss,s} - M_{nn,s}) + (n_{x_2}^2 n_{x_1}^2 - n_{x_2}^4) M_{ns,s} \end{array}\right\} =$$

$$= I_2\left\{- n_{x_2}^2 \frac{\partial \ddot{w}_0}{\partial n} + n_{x_1} n_{x_2}\frac{\partial \ddot{w}_0}{\partial s}\right\} +$$

$$+ n_{x_2}^4 M_{nn,n} - 2n_{x_1} n_{x_2}^3 M_{ns,n} + \cancel{n_{x_2}^2 n_{x_1}^2 M_{ss,n}} +$$

$$- \cancel{n_{x_1} n_{x_2}^3 M_{nn,s}} + 2n_{x_1}^2 n_{x_2}^2 M_{ns,s} - n_{x_1}^3 n_{x_2} M_{ss,s} -$$

$$- \cancel{n_{x_1}^2 n_{x_2}^2 M_{ss,n}} + n_{x_1}^2 n_{x_2}^2 M_{nn,n} - (n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3) M_{ns,n} -$$

$$- n_{x_1} n_{x_2}^3 M_{ss,s} + \cancel{n_{x_1} n_{x_2}^3 M_{nn,s}} - (n_{x_2}^2 n_{x_1}^2 - n_{x_2}^4) M_{ns,s} =$$

$$= I_2\left\{- n_{x_2}^2 \frac{\partial \ddot{w}_0}{\partial n} + n_{x_1} n_{x_2}\frac{\partial \ddot{w}_0}{\partial s}\right\} +$$

$$+ \; n_{x_2}^4 + n_{x_1}^2 n_{x_2}^2 \; M_{nn,n} - \; n_{x_1}^3 n_{x_2} + n_{x_1} n_{x_2}^3 \; M_{ns,n}$$

$$+ \; n_{x_1}^2 n_{x_2}^2 + n_{x_2}^4 \; M_{ns,s} - \; n_{x_1}^3 n_{x_2} + n_{x_1} n_{x_2}^3 \; M_{ss,s}$$

Finally the terms inside the brackets of the initial expression of the present paragraph, take the following form

$$
\begin{aligned}
& I_{\delta w_0, n_{x_1}} + I_{\delta w_0, n_{x_2}} - c_{T0}(s)\frac{I_0}{\rho} = \\
&= I_2\left\{ n_{x_1}^2 \frac{\partial \ddot{w}_0}{\partial n} + n_{x_1} n_{x_2} \frac{\partial \ddot{w}_0}{\partial s} - n_{x_2}^2 \frac{\partial \ddot{w}_0}{\partial n} + n_{x_1} n_{x_2} \frac{\partial \ddot{w}_0}{\partial s} \right\} - c_{T0}(s)\frac{I_0}{\rho} - \\
&- (n_{x_1}^4 + \cancel{n_{x_1}^2 n_{x_2}^2} - n_{x_2}^4 - \cancel{n_{x_1}^2 n_{x_2}^2}) M_{nn,n} - (n_{x_2}^3 n_{x_1} + n_{x_1}^3 n_{x_2} + n_{x_1}^3 n_{x_2} + n_{x_1} n_{x_2}^3) M_{ss,s} - \\
&- (n_{x_1} n_{x_2}^3 + n_{x_1}^3 n_{x_2} + n_{x_1}^3 n_{x_2} + n_{x_1} n_{x_2}^3)\, M_{ns,n} - (n_{x_1}^4 + \cancel{n_{x_1}^2 n_{x_2}^2} - \cancel{n_{x_1}^2 n_{x_2}^2} - n_{x_2}^4) M_{ns,s} = \\
&\qquad = I_2\left\{ n_{x_1}^2 - n_{x_2}^2 \frac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_1} n_{x_2} \frac{\partial \ddot{w}_0}{\partial s} \right\} - c_{T0}(s)\frac{I_0}{\rho} - \\
&\qquad - \; n_{x_1}^4 - n_{x_2}^4 \; M_{nn,n} - 2\, n_{x_1}^3 n_{x_2} + 2\, n_{x_1} n_{x_2}^3 \; M_{ss,s} - \\
&\qquad - \; 2\, n_{x_1} n_{x_2}^3 + 2\, n_{x_1}^3 n_{x_2} \; M_{ns,n} - \; n_{x_1}^4 - n_{x_2}^4 \; M_{ns,s} = \\
&\qquad = I_2\left\{ n_{x_1}^2 - n_{x_2}^2 \frac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_1} n_{x_2} \frac{\partial \ddot{w}_0}{\partial s} \right\} - c_{T0}(s)\frac{I_0}{\rho} - \\
&\qquad - \; n_{x_1}^2 - n_{x_2}^2 \; n_{x_1}^2 + n_{x_2}^2 \; M_{nn,n} - 2\, n_{x_1} n_{x_2} \; n_{x_1}^2 + n_{x_2}^2 \; M_{ss,s} - \\
&\qquad - \; 2\, n_{x_1} n_{x_2} \; n_{x_2}^2 + n_{x_1}^2 \; M_{ns,n} - \; n_{x_1}^2 - n_{x_2}^2 \; n_{x_1}^2 + n_{x_2}^2 \; M_{ns,s} = \\
&\qquad = I_2\left\{ n_{x_1}^2 - n_{x_2}^2 \frac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_1} n_{x_2} \frac{\partial \ddot{w}_0}{\partial s} \right\} - c_{T0}(s)\frac{I_0}{\rho} - \\
&\qquad\quad - \; n_{x_1}^2 - n_{x_2}^2 \; M_{nn,n} - 2\, n_{x_1} n_{x_2} M_{ss,s} - \\
&\qquad\quad - \; 2\, n_{x_1} n_{x_2} \; M_{ns,n} - \; n_{x_1}^2 - n_{x_2}^2 \; M_{ns,s}
\end{aligned}
$$

or

$$
\begin{aligned}
& I_{\delta w_0, n_{x_1}} + I_{\delta w_0, n_{x_2}} - c_{T0}(s)\frac{I_0}{\rho} = \\
&= I_2\left\{ n_{x_1}^2 - n_{x_2}^2 \frac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_1} n_{x_2} \frac{\partial \ddot{w}_0}{\partial s} \right\} - c_{T0}(s)\frac{I_0}{\rho} - \\
&- \; n_{x_1}^2 - n_{x_2}^2 \; M_{nn,n} + M_{ns,s} \; - 2\, n_{x_1} n_{x_2} \; M_{ss,s} + M_{ns,n}
\end{aligned}
\tag{26}
$$

At this moment, we ready to compose the final results of the sections 5.2.1, 5.2.2 and 5.2.3 in order to extract the total number of the boundary conditions of the problem of the CPT. This is going to be presented explicitly on the next section.

5.3. The full set of boundary conditions of the problem of CPT

Finally, after reorganizing the curvilinear integrals of the Eq. (1) of the section 5 including different variations on a more convenient way, we substitute the Eqs. **(21)**, **(24)** and **(26)** into the Eq. (1) of the section 5 and we get

$$
+\int_{t_1}^{t_2}\oint_{\Gamma}\left(\left\{N_{11}n_{x_1}+N_{12}n_{x_2}-a_{T0}\frac{I_0}{\rho}\right\}\delta u_0+\left\{N_{22}n_{x_2}+N_{12}n_{x_1}-b_{T0}\frac{I_0}{\rho}\right\}\delta v_0\right)d\gamma\, dt\ +
$$

$$
+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\left(I_2\frac{\partial\ddot{w}_0}{\partial x_1}-\frac{\partial M_{11}}{\partial x_1}-\frac{\partial M_{12}}{\partial x_2}\right)n_{x_1}+\left(I_2\frac{\partial\ddot{w}_0}{\partial x_2}-\frac{\partial M_{22}}{\partial x_2}-\frac{\partial M_{12}}{\partial x_1}\right)n_{x_2}-c_{T0}\frac{I_0}{\rho}\right\}\delta w_0\, d\gamma\, dt\ +
$$

$$
+\int_{t_1}^{t_2}\oint_{\Gamma}\left(\left\{M_{11}n_{x_1}+M_{12}n_{x_2}+a_{T1}\frac{I_2}{\rho}\right\}\frac{\partial\delta w_0}{\partial x_1}+\left\{M_{22}n_{x_2}+M_{12}n_{x_1}+b_{T1}\frac{I_2}{\rho}\right\}\frac{\partial\delta w_0}{\partial x_2}\right)d\gamma\, dt\ =0\Rightarrow
$$

$$
+\int_{t_1}^{t_2}\oint_{\Gamma}\left(\begin{array}{l}\left\{\ n_{x_1}^2-n_{x_2}^2\ \ N_{nn}+2\,n_{x_1}n_{x_2}\ N_{ns}-a_{T0n}\dfrac{I_0}{\rho}\right\}\delta u_{0n}\ +\\ +\left\{\ n_{x_1}^2-n_{x_2}^2\ \ N_{ns}+2\,n_{x_2}n_{x_1}\ N_{ss}-a_{T0s}\dfrac{I_0}{\rho}\right\}\delta u_{0s}\end{array}\right)ds\, dt\ +
$$

$$
+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}I_2\left\{(n_{x_1}^2-n_{x_2}^2)\dfrac{\partial\ddot{w}_0}{\partial n}+2\,n_{x_1}n_{x_2}\dfrac{\partial\ddot{w}_0}{\partial s}\right\}-c_{T0}(s)\dfrac{I_0}{\rho}-\\ -(n_{x_1}^2-n_{x_2}^2)(M_{nn,n}+M_{ns,s})-2\,n_{x_1}n_{x_2}(M_{ss,s}+M_{ns,n})\end{array}\right\}\delta w_0\, ds\, dt\ +
$$

$$
+\int_{t_1}^{t_2}\oint_{\Gamma}\left(\begin{array}{c}(n_{x_1}^2-n_{x_2}^2)^2M_{nn}\dfrac{\partial\delta w_0}{\partial n}+4\,n_{x_1}^2n_{x_2}^2\ M_{ss}\dfrac{\partial\delta w_0}{\partial n}+\\ +\ 4n_{x_1}n_{x_2}(n_{x_1}^2-n_{x_2}^2)M_{ns}\dfrac{\partial\delta w_0}{\partial n}-2\,n_{x_1}^3n_{x_2}\ M_{nn}\dfrac{\partial\delta w_0}{\partial s}-\\ -\ (n_{x_2}^4+6n_{x_1}^2n_{x_2}^2-n_{x_1}^4)M_{ns}\dfrac{\partial\delta w_0}{\partial s}+2\,n_{x_1}n_{x_2}(n_{x_1}^2-n_{x_2}^2)M_{ss}\dfrac{\partial\delta w_0}{\partial s}+\\ +\ ((n_{x_1}^2-n_{x_2}^2)a_{T1n}+2\,n_{x_2}n_{x_1}\ a_{T1s})\dfrac{I_2}{\rho}\dfrac{\partial\delta w_0}{\partial n}-\\ -\ ((n_{x_2}^2-n_{x_1}^2)a_{T1s}+2\,n_{x_1}n_{x_2}\ a_{T1n})\dfrac{I_2}{\rho}\dfrac{\partial\delta w_0}{\partial s}\end{array}\right)ds\, dt\ =0\Rightarrow
$$

$$
\begin{aligned}
&+\int_{t_1}^{t_2}\oint_{\Gamma}\left(\begin{aligned}&\left\{(n_{x_1}^2-n_{x_2}^2)N_{nn} + 2\,n_{x_1}n_{x_2}\,N_{ns} - a_{T0n}\frac{I_0}{\rho}\right\}\delta u_{0n} + \\ &+\left\{(n_{x_1}^2-n_{x_2}^2)N_{ns} + 2\,n_{x_2}n_{x_1}\,N_{ss} - a_{T0s}\frac{I_0}{\rho}\right\}\delta u_{0s}\end{aligned}\right) ds\,dt\; + \\
&+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{aligned}&I_2\left\{(n_{x_1}^2 - n_{x_2}^2)\frac{\partial \ddot{w}_0}{\partial n} + 2\,n_{x_1}n_{x_2}\frac{\partial \ddot{w}_0}{\partial s}\right\} - c_{T0}(s)\frac{I_0}{\rho} - \\ &-(n_{x_1}^2 - n_{x_2}^2)(M_{nn,n}+M_{ns,s}) - 2\,n_{x_1}n_{x_2}(M_{ss,s}+M_{ns,n})\end{aligned}\right\}\delta w_0\, ds\,dt\; + \\
&+\int_{t_1}^{t_2}\oint_{\Gamma}(n_{x_1}^2 - n_{x_2}^2)^2 M_{nn}\frac{\partial \delta w_0}{\partial n}\,ds\,dt + \int_{t_1}^{t_2}\oint_{\Gamma}4\,n_{x_1}^2 n_{x_2}^2\; M_{ss}\frac{\partial \delta w_0}{\partial n}\,ds\,dt\; + \\
&+\int_{t_1}^{t_2}\oint_{\Gamma}4n_{x_1}n_{x_2}(n_{x_1}^2 - n_{x_2}^2)M_{ns}\frac{\partial \delta w_0}{\partial n}\,ds\,dt - \int_{t_1}^{t_2}\oint_{\Gamma}2\,n_{x_1}^3 n_{x_2}\,M_{nn}\frac{\partial \delta w_0}{\partial s}\,ds\,dt\; - \\
&-\int_{t_1}^{t_2}\oint_{\Gamma}(n_{x_2}^4 + 6n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4)M_{ns}\frac{\partial \delta w_0}{\partial s}\,ds\,dt\; + \\
&+\int_{t_1}^{t_2}\oint_{\Gamma}2\,n_{x_1}n_{x_2}(n_{x_1}^2 - n_{x_2}^2)M_{ss}\frac{\partial \delta w_0}{\partial s}\,ds\,dt\; + \\
&+\int_{t_1}^{t_2}\oint_{\Gamma}((n_{x_1}^2 - n_{x_2}^2)\,a_{T1n} + 2\,n_{x_2}n_{x_1}\,a_{T1s})\frac{I_2}{\rho}\frac{\partial \delta w_0}{\partial n}\,ds\,dt\; - \\
&-\int_{t_1}^{t_2}\oint_{\Gamma}((n_{x_2}^2 - n_{x_1}^2)\,a_{T1s} + 2\,n_{x_1}n_{x_2}\,a_{T1n})\frac{I_2}{\rho}\frac{\partial \delta w_0}{\partial s}\,ds\,dt = 0
\end{aligned}
\tag{1}
$$

Now separating the integrals of the terms including boundary conditions with the variations $\partial\delta w_0/\partial n$ and $\partial\delta w_0/\partial s$ and also neglecting the time integration for reason of simplification, we have

$$I_{M_{nn,n}} = \oint_{\Gamma} n_{x_1}^2 - n_{x_2}^2 \;{}^2\; M_{nn}\frac{\partial \delta w_0}{\partial n}\,ds \tag{1a}$$

$$I_{M_{ss,n}} = \oint_{\Gamma} 4\,n_{x_1}^2 n_{x_2}^2\; M_{ss}\frac{\partial \delta w_0}{\partial n}\,ds \tag{1b}$$

$$I_{M_{ns,n}} = \oint_{\Gamma} 4n_{x_1}n_{x_2}\; n_{x_1}^2 - n_{x_2}^2\; M_{ns}\frac{\partial \delta w_0}{\partial n}\,ds \tag{1c}$$

$$I_{M_{nn,s}} = \oint_{\Gamma} 2\, n_{x_1}^3 n_{x_2} M_{nn} \frac{\partial \delta w_0}{\partial s} ds = \Big[2\, n_{x_1}^3 n_{x_2}\; M_{nn} \delta w_0 \Big]_{\Gamma} - \oint_{\Gamma} \frac{\partial}{\partial s}\; 2\, n_{x_1}^3 n_{x_2}\; M_{nn}\; \delta w_0\; ds$$

where the last integral of the right-hand side of the above relation is written as,

$$\oint_{\Gamma} \frac{\partial}{\partial s}\; 2\, n_{x_1}^3 n_{x_2}\; M_{nn}\; \delta w_0\; ds = \oint_{\Gamma} \left\{ \frac{\partial}{\partial s}\; 2\, n_{x_1}^3 n_{x_2}\;\; M_{nn} + 2\, n_{x_1}^3 n_{x_2} \frac{\partial M_{nn}}{\partial s} \right\} \delta w_0\; ds =$$

$$= \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \frac{\partial}{\partial s}\; 2\, n_{x_1}^3 n_{x_2}\;\; M_{nn} \right\} \delta w_0\; ds\, dt + \int_{t_1}^{t_2} \oint_{\Gamma} 2\, n_{x_1}^3 n_{x_2} M_{nn,s}\;\; \delta w_0\; ds\, dt$$

Consequently, we get

$$\begin{aligned} I_{M_{nn,s}} &= \Big[2\, n_{x_1}^3 n_{x_2}\; M_{nn} \delta w_0 \Big]_{\Gamma} - \oint_{\Gamma} \frac{\partial}{\partial s}\; 2\, n_{x_1}^3 n_{x_2}\; M_{nn}\; \delta w_0\; ds = \\ &= \Big[2\, n_{x_1}^3 n_{x_2}\; M_{nn} \delta w_0 \Big]_{\Gamma} - \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \frac{\partial}{\partial s}\; 2\, n_{x_1}^3 n_{x_2}\;\; M_{nn} \right\} \delta w_0\; ds\, dt \; - \\ &\qquad - \int_{t_1}^{t_2} \oint_{\Gamma} 2\, n_{x_1}^3 n_{x_2} M_{nn,s}\;\; \delta w_0\; ds\, dt \end{aligned} \tag{1d}$$

$$\begin{aligned} I_{M_{ns,s}} &= \oint_{\Gamma} n_{x_2}^4 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4\;\; M_{ns} \frac{\partial \delta w_0}{\partial s} ds = \\ &= \Big[n_{x_2}^4 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4\;\; M_{ns} \delta w_0 \Big]_{\Gamma} - \oint_{\Gamma} \frac{\partial}{\partial s}\;\; n_{x_2}^4 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4\;\; M_{ns}\;\; \delta w_0\; ds \end{aligned}$$

where the last integral of the right-hand side of the above relation is written as,

$$\begin{aligned} \oint_{\Gamma} \frac{\partial}{\partial s}\;\; n_{x_2}^4 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4\;\; M_{ns}\;\; \delta w_0\; ds &= \oint_{\Gamma} \left\{ \frac{\partial}{\partial s}\; n_{x_2}^4 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4\;\; M_{ns} \right\} \delta w_0\; ds \; + \\ &+ \oint_{\Gamma} \left\{ n_{x_2}^4 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4\;\; \frac{\partial M_{ns}}{\partial s} \right\} \delta w_0\; ds \end{aligned}$$

Consequently, we get

$$I_{M_{ns,s}} = \left[n_{x_2}^4 + 6n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4 \;\; M_{ns}\,\delta w_0 \right]_\Gamma - \oint_\Gamma \left\{ \frac{\partial}{\partial s} \;\; n_{x_2}^4 + 6n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4 \;\; M_{ns} \right\} \delta w_0 \, ds \; -$$

$$- \oint_\Gamma \;\; n_{x_2}^4 + 6n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4 \;\; M_{ns,s} \;\; \delta w_0 \, ds \qquad \text{(1e)}$$

$$I_{M_{ss,s}} = \oint_\Gamma 2\, n_{x_1} n_{x_2} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; M_{ss} \frac{\partial \delta w_0}{\partial s} ds = \left[2\, n_{x_1} n_{x_2} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; M_{ss}\,\delta w_0 \right]_\Gamma -$$

$$- \oint_\Gamma \frac{\partial}{\partial s} \;\; 2\, n_{x_1} n_{x_2} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; M_{ss} \;\; \delta w_0 \, ds$$

where the last integral of the right-hand side of the above relation is written as,

$$\oint_\Gamma \frac{\partial}{\partial s} \;\; 2\, n_{x_1} n_{x_2} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; M_{ss} \;\; \delta w_0 \, ds =$$

$$= \oint_\Gamma \frac{\partial}{\partial s} \;\; 2\, n_{x_1} n_{x_2} \;\; n_{x_1}^2 - n_{x_2}^2 \;\;\; M_{ss}\,\delta w_0 \, ds + \oint_\Gamma 2\, n_{x_1} n_{x_2} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; \frac{\partial M_{ss}}{\partial s} \delta w_0 \, ds$$

Consequently, we have

$$I_{M_{ss,s}} = \left[2\, n_{x_1} n_{x_2} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; M_{ss}\,\delta w_0 \right]_\Gamma - \oint_\Gamma \frac{\partial}{\partial s} \;\; 2\, n_{x_1} n_{x_2} \;\; n_{x_1}^2 - n_{x_2}^2 \;\;\; M_{ss}\,\delta w_0 \, ds \; -$$

$$- \oint_\Gamma 2\, n_{x_1} n_{x_2} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; M_{ss,s}\,\delta w_0 \, ds \qquad \text{(1f)}$$

$$I_{\delta w_{0,n}} = \oint_\Gamma \;\; n_{x_1}^2 - n_{x_2}^2 \;\; a_{T1n} \; + \; 2\, n_{x_2} n_{x_1} \, a_{T1s} \;\; \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial n} ds =$$

$$= \frac{I_2}{\rho} \oint_\Gamma \;\; n_{x_1}^2 - n_{x_2}^2 \;\; a_{T1n} \; + \; 2\, n_{x_2} n_{x_1} \, a_{T1s} \;\;\; \frac{\partial \delta w_0}{\partial n} ds \qquad \text{(1g)}$$

$$I_{\delta w_{0,s}} = \int_{t_1}^{t_2} \oint_\Gamma \;\; n_{x_2}^2 - n_{x_1}^2 \;\; a_{T1s} \; + \; 2\, n_{x_1} n_{x_2} \, a_{T1n} \;\; \frac{I_2}{\rho} \frac{\partial \delta w_0}{\partial s} ds\, dt =$$

$$= \frac{I_2}{\rho} \oint_\Gamma \;\; n_{x_2}^2 - n_{x_1}^2 \;\; a_{T1s} \; + \; 2\, n_{x_1} n_{x_2} \, a_{T1n} \;\;\; \frac{\partial \delta w_0}{\partial s} ds =$$

$$= \frac{I_2}{\rho} \left[\;\; n_{x_2}^2 - n_{x_1}^2 \;\; a_{T1s} \; + \; 2\, n_{x_1} n_{x_2} \, a_{T1n} \;\;\; \delta w_0 \right]_\Gamma -$$

$$- \frac{I_2}{\rho} \oint_\Gamma \frac{\partial}{\partial s} \;\;\; n_{x_2}^2 - n_{x_1}^2 \;\; a_{T1s} \; + \; 2\, n_{x_1} n_{x_2} \, a_{T1n} \;\;\; \delta w_0 \, ds \qquad \text{(1h)}$$

The previous choice to perform by parts integrations only along the curve Γ, namely following the arc s through the tangential derivatives $\partial\delta w_0/\partial s$, is intentional because it gives the desirable boundary terms and concurrently by this way the number of the total boundary conditions of the problem is reduced to the desirable. The last is attained by eliminating the derivative $\partial\delta w_0/\partial s$ from the boundary terms, as seems below.

Note also that the direction cosines n_{x_1} and n_{x_2} are s-dependent as exactly the functions a_{T0n}, a_{T0s}, a_{T1n}, a_{T1s} and c_{T0} (see section 5.2 and APPENDIX A). For this reason, the direction cosines $n_{x_1} = n_{x_1}(s)$ and $n_{x_2} = n_{x_2}(s)$ are under the derivation inside the integral terms, which remains after the by parts integrations [Eqs. (1a) - (1h)].

Subsequently, we are going to examine two cases. The first one is when the end points of the closed curve Γ coincide or when the terms inside the brackets $\cdot\ _\Gamma$ are equal to zero, namely,

$$M_{nn} = M_{ns} = M_{ss} = n_{x_2}^2 - n_{x_1}^2 \; a_{T1s} + 2\, n_{x_1} n_{x_2}\, a_{T1n} = 0\,. \tag{2}$$

Then the first terms of the right-hand side of the relations (1d), (1e), (1f), (1h) are eliminated,

$$\begin{aligned}\left[2\, n_{x_1}^3 n_{x_2}\; M_{nn}\,\delta w_0\right]_\Gamma &= \left[\; n_{x_2}^4 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4 \;\; M_{ns}\,\delta w_0\right]_\Gamma = \\ = \left[2\, n_{x_1} n_{x_2}\;\; n_{x_1}^2 - n_{x_2}^2 \;\; M_{ss}\,\delta w_0\right]_\Gamma &= \\ = \left[\;\; n_{x_2}^2 - n_{x_1}^2 \;\; a_{T1s} + 2\, n_{x_1} n_{x_2}\, a_{T1n} \;\; \delta w_0\right]_\Gamma &= 0 \end{aligned} \tag{3}$$

As for the second case, the terms of the right-hand side of the relations (1d), (1e), (1f), (1h) are non-zero and we have to take them into account inside the boundary conditions. However, the last demands the choice of specific parts of the lateral boundary on which the displacements (essential boundary conditions) or the quantities inside the intercalations of the Eqs. (3) (natural boundary conditions) will be prescribed. As we will show below, in case of given surface tractions along specific parts of the edge of the plate, the above terms (3) could not be taken equal to zero and we have to treat them properly.

Now, as for the first aforementioned case,

$$\begin{aligned} I_{M_{nn,s}} = \cancel{\left[2\, n_{x_1}^3 n_{x_2}\, M_{nn}\,\delta w_0\right]_\Gamma} &- \oint_\Gamma \frac{\partial}{\partial s}\{2\, n_{x_1}^3 n_{x_2}\}\, M_{nn}\,\delta w_0\, ds \, - \\ &- \int_{t_1}^{t_2}\oint_\Gamma 2\, n_{x_1}^3 n_{x_2} M_{nn,s}\,\delta w_0\, ds\, dt = \\ = -\oint_\Gamma \frac{\partial}{\partial s}\{2\, n_{x_1}^3 n_{x_2}\}\, M_{nn}\,\delta w_0\, ds &- \oint_\Gamma 2\, n_{x_1}^3 n_{x_2} M_{nn,s}\,\delta w_0\, ds \end{aligned}$$

$$I_{M_{ns,s}} = \cancel{\left[n_{x_2}^4 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4 \; M_{ns} \, \delta w_0 \right]_{\Gamma}} - \oint_{\Gamma} \frac{\partial}{\partial s} \{ n_{x_2}^4 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4 \} M_{ns} \, \delta w_0 \, ds -$$

$$- \oint_{\Gamma} (n_{x_2}^4 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4) M_{ns,s} \, \delta w_0 \, ds =$$

$$= - \oint_{\Gamma} \frac{\partial}{\partial s} \{ n_{x_2}^4 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4 \} M_{ns} \, \delta w_0 \, ds - \oint_{\Gamma} (n_{x_2}^4 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4) M_{ns,s} \, \delta w_0 \, ds$$

$$I_{M_{ss,s}} = \cancel{\left[2 \, n_{x_1} n_{x_2} \; n_{x_1}^2 - n_{x_2}^2 \; M_{ss} \, \delta w_0 \right]_{\Gamma}} - \oint_{\Gamma} \frac{\partial}{\partial s} \{ 2 \, n_{x_1} n_{x_2} (n_{x_1}^2 - n_{x_2}^2) \} M_{ss} \, \delta w_0 \, ds -$$

$$- \oint_{\Gamma} 2 \, n_{x_1} n_{x_2} (n_{x_1}^2 - n_{x_2}^2) M_{ss,s} \, \delta w_0 \, ds =$$

$$= - \oint_{\Gamma} \frac{\partial}{\partial s} \{ 2 \, n_{x_1} n_{x_2} (n_{x_1}^2 - n_{x_2}^2) \} M_{ss} \, \delta w_0 \, ds - \oint_{\Gamma} 2 \, n_{x_1} n_{x_2} (n_{x_1}^2 - n_{x_2}^2) M_{ss,s} \, \delta w_0 \, ds$$

$$I_{\delta w_{0,s}} = \frac{I_2}{\rho} \cancel{\left[((n_{x_2}^2 - n_{x_1}^2) \, a_{T1s} + 2 \, n_{x_1} n_{x_2} \, a_{T1n}) \delta w_0 \right]_{\Gamma}} -$$

$$- \frac{I_2}{\rho} \oint_{\Gamma} \frac{\partial}{\partial s} \{ (n_{x_2}^2 - n_{x_1}^2) \, a_{T1s} + 2 \, n_{x_1} n_{x_2} \, a_{T1n} \} \, \delta w_0 \, ds =$$

$$= - \frac{I_2}{\rho} \oint_{\Gamma} \frac{\partial}{\partial s} \{ (n_{x_2}^2 - n_{x_1}^2) \, a_{T1s} + 2 \, n_{x_1} n_{x_2} \, a_{T1n} \} \, \delta w_0 \, ds$$

In the last, the relations of $I_{M_{nn,n}}$, $I_{M_{ss,n}}$, $I_{M_{ns,n}}$, $I_{M_{nn,s}}$, $I_{M_{ns,s}}$, $I_{M_{ss,s}}$, $I_{\delta w_{0,n}}$ and $I_{\delta w_{0,s}}$ revised with the previous results are posed on the Eq. (1), referred above and we extract the following,

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\left(\begin{array}{l}\left\{\ n_{x_1}^2-n_{x_2}^2\ \ N_{nn}\ +\ 2\,n_{x_1}n_{x_2}\ N_{ns}\ -\ a_{T0n}\dfrac{I_0}{\rho}\right\}\delta u_{0n}\ +\\ +\left\{\ n_{x_1}^2-n_{x_2}^2\ \ N_{ns}\ +\ 2\,n_{x_2}n_{x_1}\ N_{ss}\ -\ a_{T0s}\dfrac{I_0}{\rho}\right\}\delta u_{0s}\end{array}\right)ds\,dt\ +$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}I_2\left\{(n_{x_1}^2\ -\ n_{x_2}^2)\dfrac{\partial\ddot{w}_0}{\partial n}\ +\ 2\,n_{x_1}n_{x_2}\dfrac{\partial\ddot{w}_0}{\partial s}\right\}\ -\ c_{T0}(s)\dfrac{I_0}{\rho}\ -\\ -\ (n_{x_1}^2\ -\ n_{x_2}^2)(M_{nn,n}+M_{ns,s})\ -\ 2\,n_{x_1}n_{x_2}\,(M_{ss,s}+M_{ns,n})\end{array}\right\}\delta w_0\ ds\,dt\ +$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}(n_{x_1}^2-n_{x_2}^2)^2M_{nn}\frac{\partial\delta w_0}{\partial n}\,ds\,dt\ +\int_{t_1}^{t_2}\oint_{\Gamma}4\,n_{x_1}^2\,n_{x_2}^2\ M_{ss}\frac{\partial\delta w_0}{\partial n}\,ds\,dt\ +$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}4n_{x_1}n_{x_2}(n_{x_1}^2-n_{x_2}^2)M_{ns}\frac{\partial\delta w_0}{\partial n}\,ds\,dt\ +$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\frac{\partial}{\partial s}\{2\,n_{x_1}^3\,n_{x_2}\}\ M_{nn}\right\}\delta w_0\ ds\,dt\ +\int_{t_1}^{t_2}\oint_{\Gamma}2\,n_{x_1}^3\,n_{x_2}M_{nn,s}\,\delta w_0\ ds\,dt\ +$$

$$+\int_{t_1}^{t_2}\frac{\partial}{\partial s}\{n_{x_2}^4+6n_{x_1}^2\,n_{x_2}^2-n_{x_1}^4\}M_{ns}\ dt\ +\int_{t_1}^{t_2}\oint_{\Gamma}(n_{x_2}^4+6n_{x_1}^2\,n_{x_2}^2-n_{x_1}^4)M_{ns,s}\ \delta w_0\ ds\,dt\ -$$

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\frac{\partial}{\partial s}\{2\,n_{x_1}n_{x_2}(n_{x_1}^2-n_{x_2}^2)\}M_{ss}\,\delta w_0\ ds\,dt\ -\int_{t_1}^{t_2}\oint_{\Gamma}2\,n_{x_1}n_{x_2}(n_{x_1}^2-n_{x_2}^2)M_{ss,s}\,\delta w_0\ ds\,dt\ +$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}((n_{x_1}^2-n_{x_2}^2)\,a_{T1n}\ +\ 2\,n_{x_2}n_{x_1}\ a_{T1s})\ \frac{I_2}{\rho}\ \frac{\partial\delta w_0}{\partial n}\,ds\,dt\ +$$

$$+\int_{t_1}^{t_2}\frac{I_2}{\rho}\oint_{\Gamma}\frac{\partial}{\partial s}\{(n_{x_2}^2-n_{x_1}^2)\,a_{T1s}\ +\ 2\,n_{x_1}n_{x_2}\ a_{T1n}\}\,\delta w_0\ ds\,dt\ =\ 0 \qquad \Rightarrow$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{ n_{x_1}^2 - n_{x_2}^2 \;\; N_{nn} + 2\, n_{x_1} n_{x_2} \; N_{ns} - a_{T0n}\frac{I_0}{\rho}\right\}\delta u_{0n}\, ds\, dt \; +$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{ n_{x_1}^2 - n_{x_2}^2 \;\; N_{ns} + 2\, n_{x_2} n_{x_1} \; N_{ss} - a_{T0s}\frac{I_0}{\rho}\right\}\delta u_{0s}\, ds\, dt \; +$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l} I_2\left\{(n_{x_1}^2 - n_{x_2}^2)\dfrac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_1} n_{x_2}\dfrac{\partial \ddot{w}_0}{\partial s}\right\} - c_{T0}(s)\dfrac{I_0}{\rho} - \\ - (n_{x_1}^2 - n_{x_2}^2) M_{nn,n} - 2\, n_{x_1} n_{x_2} M_{ns,n} + 2(n_{x_2}^2 + 3 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2) M_{ns,s} + \\ + 2\dfrac{\partial}{\partial s}\{n_{x_1}^3 n_{x_2}\}\, M_{nn} + 2\, n_{x_1}^3 n_{x_2} M_{nn,s} + \dfrac{\partial}{\partial s}\{n_{x_2}^2 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2\} M_{ns} - \\ - 2\dfrac{\partial}{\partial s}\{n_{x_1} n_{x_2}(n_{x_1}^2 - n_{x_2}^2)\} M_{ss} - 4\, n_{x_2} n_{x_1}^3 M_{ss,s} + \\ + \dfrac{I_2}{\rho}\dfrac{\partial}{\partial s}\{(n_{x_2}^2 - n_{x_1}^2)\, a_{T1s} + 2\, n_{x_1} n_{x_2}\, a_{T1n}\} \end{array}\right\}\delta w_0\, ds\, dt \; +$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}(n_{x_1}^2 - n_{x_2}^2)^2 M_{nn} + 4\, n_{x_1}^2 n_{x_2}^2 \; M_{ss} + 4 n_{x_1} n_{x_2}(n_{x_1}^2 - n_{x_2}^2) M_{ns} + \\ + ((n_{x_1}^2 - n_{x_2}^2)\, a_{T1n} + 2\, n_{x_2} n_{x_1}\, a_{T1s})\dfrac{I_2}{\rho}\end{array}\right\}\frac{\partial \delta w_0}{\partial n}\, ds\, dt = 0 \quad \mathbf{(4)}$$

As seems from the last version of the variational equation, the total number of the boundary conditions is <u>four</u> natural boundary conditions with primary variables u_{0n}, u_{0s}, w_0, $\partial \delta w_0 / \partial n$, which correspond to four essential boundary conditions respectively. Thus, to derive the essential and natural boundary conditions we follow the process explained below.

Now we invoke the fundamental arguments of the Calculus of Variations in order to extract the boundary conditions from the last version of the Variational Equation (4) including only the boundary terms. The following process is presented extensively on the Lecture Notes of Functional Analysis, *G.A. Athanassoulis* (*2016*) *"Necessary Conditions of Extremum of Functional"* and *"A further study of the Variational Problem as for integral type functional"*, as well as on the book of *Gelfand I.M., Fomin S.V.* (*1963*), *"Calculus of Variations"*.

First, we assume that δu_{0n} is arbitrary on the curve Γ, for arbitrary interval $[t_1, t_2]$ and keep the restrictions $\delta u_{0s} = \partial \delta w_0 / \partial n = \delta w_0 = 0$. Thus, the last equation is converted to,

$$\int_{t_1}^{t_2}\oint_{\Gamma}\left\{ n_{x_1}^2 - n_{x_2}^2 \;\; N_{nn} + 2\, n_{x_1} n_{x_2} \; N_{ns} - a_{T0n}\frac{I_0}{\rho}\right\}\delta u_{0n}(s;t)\, ds\, dt = 0,$$

$$\forall \quad \delta u_{0n}(s;t)$$

and using the arbitrariness of the variation δu_{0n} on the curvilinear domain $\Gamma \times [t_1, t_2]$, we find the <u>first natural boundary condition</u> of the problem of CPT,

$$n_{x_1}^2 - n_{x_2}^2 \quad N_{nn} + 2\, n_{x_1} n_{x_2} \; N_{ns} = a_{T0n} \frac{I_0}{\rho} \tag{4a}$$

Now, due to the above Eq. (4a) or boundary condition, the Variational Equation (4) is converted to the Eq. (4\`),

$$+ \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ n_{x_1}^2 - n_{x_2}^2 \quad N_{ns} + 2\, n_{x_2} n_{x_1} \; N_{ss} - a_{T0s} \frac{I_0}{\rho} \right\} \delta u_{0s} \, ds\, dt \; +$$

$$+ \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \begin{array}{c} I_2 \left\{ n_{x_1}^2 - n_{x_2}^2 \; \frac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_1} n_{x_2} \frac{\partial \ddot{w}_0}{\partial s} \right\} - c_{T0}(s) \frac{I_0}{\rho} - \\ - (n_{x_1}^2 - n_{x_2}^2) M_{nn,n} - 2\, n_{x_1} n_{x_2} M_{ns,n} + 2\,(n_{x_2}^2 + 3 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2) M_{ns,s} + \\ + 2 \frac{\partial}{\partial s} \{ n_{x_1}^3 n_{x_2} \} M_{nn} + 2\, n_{x_1}^3 n_{x_2} M_{nn,s} + \frac{\partial}{\partial s} \{ n_{x_2}^2 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2 \} M_{ns} - \\ - 2 \frac{\partial}{\partial s} \{ n_{x_1} n_{x_2} (n_{x_1}^2 - n_{x_2}^2) \} M_{ss} - 4\, n_{x_2} n_{x_1}^3 M_{ss,s} + \\ + \frac{I_2}{\rho} \frac{\partial}{\partial s} \{ (n_{x_2}^2 - n_{x_1}^2)\, a_{T1s} + 2\, n_{x_1} n_{x_2} \, a_{T1n} \} \end{array} \right\} \delta w_0 \, ds\, dt \; +$$

$$+ \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \begin{array}{c} n_{x_1}^2 - n_{x_2}^2 \;{}^2\, M_{nn} + 4\, n_{x_1}^2 n_{x_2}^2 \; M_{ss} + 4 n_{x_1} n_{x_2} \; n_{x_1}^2 - n_{x_2}^2 \; M_{ns} + \\ + \quad n_{x_1}^2 - n_{x_2}^2 \; a_{T1n} + 2\, n_{x_2} n_{x_1} \, a_{T1s} \; \frac{I_2}{\rho} \end{array} \right\} \frac{\partial \delta w_0}{\partial n} \, ds\, dt = 0 \tag{4\`}$$

Removing the restriction $\delta u_{0s} = 0$, assuming the arbitrariness of the function δu_{0s} and of the interval $[t_1, t_2]$ and taking into account the restrictions $\partial \delta w_0 / \partial n = \delta w_0 = 0$, we derive the following,

$$\int_{t_1}^{t_2} \oint_{\Gamma} \left\{ n_{x_1}^2 - n_{x_2}^2 \quad N_{ns} + 2\, n_{x_2} n_{x_1} \; N_{ss} - a_{T0s} \frac{I_0}{\rho} \right\} \delta u_{0s}(s;t) \, ds\, dt = 0,$$

$$\forall \;\; \delta u_{0s}(s;t)$$

and using the arbitrariness of the variation δu_{0s} on the $\Gamma \times [t_1, t_2]$, we get the second natural boundary condition of the problem of CPT,

$$n_{x_1}^2 - n_{x_2}^2 \quad N_{ns} + 2\, n_{x_2} n_{x_1} \; N_{ss} = a_{T0s} \frac{I_0}{\rho} \tag{4b}$$

Taking into account the boundary equations (4a) and (4b), the Eq. **(4\`)** remains just with the terms,

$$
+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{c}
I_2\left\{ (n_{x_1}^2 - n_{x_2}^2)\dfrac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_1} n_{x_2}\dfrac{\partial \ddot{w}_0}{\partial s}\right\} - c_{T0}(s)\dfrac{I_0}{\rho} - \\
-\,(n_{x_1}^2 - n_{x_2}^2)\, M_{nn,n} - 2\, n_{x_1} n_{x_2} M_{ns,n} + 2\,(n_{x_2}^2 + 3 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2)\, M_{ns,s} + \\
+\, 2\dfrac{\partial}{\partial s}\{n_{x_1}^3 n_{x_2}\}\, M_{nn} + 2\, n_{x_1}^3 n_{x_2} M_{nn,s} + \dfrac{\partial}{\partial s}\{n_{x_2}^2 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2\} M_{ns} - \\
-\, 2\dfrac{\partial}{\partial s}\, n_{x_1} n_{x_2}(n_{x_1}^2 - n_{x_2}^2)\, M_{ss} - 4\, n_{x_2} n_{x_1}^3 M_{ss,s} + \\
+\, \dfrac{I_2}{\rho}\dfrac{\partial}{\partial s}\left\{ (n_{x_2}^2 - n_{x_1}^2)\, a_{T1s} + 2\, n_{x_1} n_{x_2}\, a_{T1n} \right\}
\end{array}\right\}\delta w_0\, ds\, dt\, +
$$

$$
+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{c}
(n_{x_1}^2 - n_{x_2}^2)^2 M_{nn} + 4\, n_{x_1}^2 n_{x_2}^2\, M_{ss} + 4 n_{x_1} n_{x_2}\,(n_{x_1}^2 - n_{x_2}^2)\, M_{ns} + \\
+\, (n_{x_1}^2 - n_{x_2}^2)\, a_{T1n} + 2\, n_{x_2} n_{x_1}\, a_{T1s}\, \dfrac{I_2}{\rho}
\end{array}\right\}\frac{\partial \delta w_0}{\partial n}\, ds\, dt = 0 \qquad \textbf{(4``)}
$$

Removing the restriction $\delta w_0 = 0$, assuming the arbitrariness of the function δw_0 and of the interval $[t_1, t_2]$ and taking into account the restriction $\partial \delta w_0 / \partial n = 0$, we have

$$
\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{c}
I_2\left\{ (n_{x_1}^2 - n_{x_2}^2)\dfrac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_1} n_{x_2}\dfrac{\partial \ddot{w}_0}{\partial s}\right\} - c_{T0}(s)\dfrac{I_0}{\rho} - \\
-\,(n_{x_1}^2 - n_{x_2}^2) M_{nn,n} - 2\, n_{x_1} n_{x_2} M_{ns,n} + 2\,(n_{x_2}^2 + 3 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2)\, M_{ns,s} + \\
+\, 2\dfrac{\partial}{\partial s}\{n_{x_1}^3 n_{x_2}\}\, M_{nn} + 2\, n_{x_1}^3 n_{x_2} M_{nn,s} + \dfrac{\partial}{\partial s}\{n_{x_2}^2 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2\} M_{ns} - \\
-\, 2\dfrac{\partial}{\partial s}\{n_{x_1} n_{x_2}(n_{x_1}^2 - n_{x_2}^2)\} M_{ss} - 4\, n_{x_2} n_{x_1}^3 M_{ss,s} + \\
+\, \dfrac{I_2}{\rho}\dfrac{\partial}{\partial s}\{ (n_{x_2}^2 - n_{x_1}^2)\, a_{T1s} + 2\, n_{x_1} n_{x_2}\, a_{T1n} \}
\end{array}\right\}\delta w_0(s;t)\, ds\, dt = 0,
$$

$$
\forall \;\; \delta w_0(s;t)
$$

and using the arbitrariness of the variation δw_0 on the $\Gamma \times [t_1, t_2]$, we find the <u>third natural boundary condition</u> of the problem of CPT,

$$\begin{aligned}
& I_2 \left\{ n_{x_1}^2 - n_{x_2}^2 \frac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_1} n_{x_2} \frac{\partial \ddot{w}_0}{\partial s} \right\} - c_{T0}(s) \frac{I_0}{\rho} - \\
& - \; n_{x_1}^2 - n_{x_2}^2 \; M_{nn,n} - 2\, n_{x_1} n_{x_2} M_{ns,n} + 2\; n_{x_2}^2 + 3 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2 \; M_{ns,s} + \\
& + 2 \frac{\partial}{\partial s} \; n_{x_1}^3 n_{x_2} \; M_{nn} + 2\, n_{x_1}^3 n_{x_2} M_{nn,s} + \frac{\partial}{\partial s} \; n_{x_2}^2 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2 \; M_{ns} - \\
& - 2 \frac{\partial}{\partial s} \; n_{x_1} n_{x_2} \; n_{x_1}^2 - n_{x_2}^2 \quad M_{ss} - 4\; n_{x_2} n_{x_1}^3 M_{ss,s} + \\
& \qquad + \frac{I_2}{\rho} \frac{\partial}{\partial s} \{ (n_{x_2}^2 - n_{x_1}^2) a_{T1s} + 2\, n_{x_1} n_{x_2} \; a_{T1n} \} = 0
\end{aligned} \tag{4c}$$

Finally, taking into account the Eqs. (4a), (4b) and (4c), the Eq. (4``) remains just with the following terms,

$$\int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \begin{aligned} & n_{x_1}^2 - n_{x_2}^2 \;{}^2 M_{nn} + 4\, n_{x_1}^2 n_{x_2}^2 \; M_{ss} + 4 n_{x_1} n_{x_2} \; n_{x_1}^2 - n_{x_2}^2 \; M_{ns} + \\ & \qquad + \; n_{x_1}^2 - n_{x_2}^2 \; a_{T1n} + 2\, n_{x_2} n_{x_1} a_{T1s} \; \frac{I_2}{\rho} \end{aligned} \right\} \frac{\partial \delta w_0}{\partial n} \, ds\, dt = 0 \tag{4```}$$

Removing the restriction $\partial \delta w_0 / \partial n = 0$, assuming the arbitrariness of the variation $\partial \delta w_0 / \partial n$ and of the interval $[t_1, t_2]$ we derive from the Eq. (4```),

$$\int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \begin{aligned} & n_{x_1}^2 - n_{x_2}^2 \;{}^2 M_{nn} + 4\, n_{x_1}^2 n_{x_2}^2 \; M_{ss} + 4 n_{x_1} n_{x_2} \; n_{x_1}^2 - n_{x_2}^2 \; M_{ns} + \\ & \qquad + \; n_{x_1}^2 - n_{x_2}^2 \; a_{T1n} + 2\, n_{x_2} n_{x_1} a_{T1s} \; \frac{I_2}{\rho} \end{aligned} \right\} \frac{\partial \delta w_0(s;t)}{\partial n} \, ds\, dt = 0,$$

$$\forall \; \frac{\partial \delta w_0(s;t)}{\partial n}$$

and using the arbitrariness of the variation $\partial \delta w_0 / \partial n$ on the $\Gamma \times [t_1, t_2]$, we find the <u>fourth natural boundary condition</u> of the problem of CPT,

$$\begin{aligned}
& n_{x_1}^2 - n_{x_2}^2 \;{}^2 M_{nn} + 4\, n_{x_1}^2 n_{x_2}^2 \; M_{ss} + 4 n_{x_1} n_{x_2} \; n_{x_1}^2 - n_{x_2}^2 \; M_{ns} = \\
& \quad = \; n_{x_2}^2 - n_{x_1}^2 \; a_{T1n} - 2\, n_{x_2} n_{x_1} a_{T1s} \; \frac{I_2}{\rho}
\end{aligned} \tag{4d}$$

To compare easier the form of each boundary condition and to elaborate their results, we gather them below

$$n_{x_1}^2 - n_{x_2}^2 \;\; N_{nn} + 2\, n_{x_1} n_{x_2} \; N_{ns} = a_{T0n} \frac{I_0}{\rho} \tag{4a}$$

$$n_{x_1}^2 - n_{x_2}^2 \;\; N_{ns} + 2\, n_{x_2} n_{x_1} \; N_{ss} = a_{T0s} \frac{I_0}{\rho} \tag{4b}$$

$$I_2\left\{ n_{x_1}^2 - n_{x_2}^2 \frac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_1} n_{x_2} \frac{\partial \ddot{w}_0}{\partial s}\right\} - c_{T0}(s)\frac{I_0}{\rho} -$$
$$- (n_{x_1}^2 - n_{x_2}^2) M_{nn,n} - 2\, n_{x_1} n_{x_2} M_{ns,n} + 2(n_{x_2}^2 + 3n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2) M_{ns,s} +$$
$$+ 2\frac{\partial}{\partial s}\{n_{x_1}^3 n_{x_2}\} M_{nn} + 2\, n_{x_1}^3 n_{x_2} M_{nn,s} + \frac{\partial}{\partial s}\{n_{x_2}^2 + 6n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2\} M_{ns} - \tag{4c}$$
$$- 2\frac{\partial}{\partial s}\{n_{x_1} n_{x_2}(n_{x_1}^2 - n_{x_2}^2)\} M_{ss} - 4\, n_{x_2} n_{x_1}^3 M_{ss,s} =$$
$$= \frac{I_2}{\rho}\frac{\partial}{\partial s}\{(n_{x_1}^2 - n_{x_2}^2)\, a_{T1s} - 2\, n_{x_1} n_{x_2}\, a_{T1n}\}$$

$$(n_{x_1}^2 - n_{x_2}^2)^2 M_{nn} + 4\, n_{x_1}^2 n_{x_2}^2 M_{ss} + 4 n_{x_1} n_{x_2} (n_{x_1}^2 - n_{x_2}^2) M_{ns} =$$
$$= ((n_{x_2}^2 - n_{x_1}^2)\, a_{T1n} - 2\, n_{x_2} n_{x_1}\, a_{T1s})\frac{I_2}{\rho} \tag{4d}$$

Although regarding the second case, where

$$\left[2\, n_{x_1}^3 n_{x_2} M_{nn}\,\delta w_0\right]_\Gamma \neq 0 \tag{5a}$$
$$\left[n_{x_2}^4 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4 \; M_{ns}\,\delta w_0\right]_\Gamma \neq 0 \tag{5b}$$
$$\left[2\, n_{x_1} n_{x_2} \; n_{x_1}^2 - n_{x_2}^2 \; M_{ss}\,\delta w_0\right]_\Gamma \neq 0 \tag{5c}$$
$$\left[\; n_{x_2}^2 - n_{x_1}^2 \; a_{T1s} + 2\, n_{x_1} n_{x_2}\, a_{T1n} \quad \delta w_0\right]_\Gamma \neq 0 \tag{5d}$$

we are not mean to give extensionally the boundary conditions in the context of this dissertation, we let a short comment in order to show that after specification of the boundary conditions upon the lateral boundary of the plate these additional terms could give interesting result in various applications.

Thus, due to the terms (5a)-(5d) we are going to derive a different form of the the Eq. (1), which includes the terms with given values of thickness-integrated quantities and δw_0 upon the curve Γ of the edge of the plate.

Substituting the Eqs. (1a)- (1h) into the Variational Equation (1) [as done on the previous case], we get the same results for the terms inside the brackets which multiply each variation except for the boundary term of the variation δw_0.

The last occurs because there are additional prescribed terms upon the curve Γ, which are described by the expressions (5a)-(5d).
Thus, the last form of the variational equation (4) is invariable as for the previous integrals, but differs on the last integral as seems below

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\; n_{x_1}^2 - n_{x_2}^2 \;\; N_{nn} + 2\, n_{x_1} n_{x_2}\; N_{ns} - a_{T0n}\frac{I_0}{\rho}\right\}\delta u_{0n}\; ds\; dt\; +$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\; n_{x_1}^2 - n_{x_2}^2 \;\; N_{ns} + 2\, n_{x_2} n_{x_1}\; N_{ss} - a_{T0s}\frac{I_0}{\rho}\right\}\delta u_{0s}\; ds\; dt\; +$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{c} I_2\left\{(n_{x_1}^2 - n_{x_2}^2)\dfrac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_1} n_{x_2}\dfrac{\partial \ddot{w}_0}{\partial s}\right\} - c_{T0}(s)\dfrac{I_0}{\rho} - \\ -\,(n_{x_1}^2 - n_{x_2}^2)M_{nn,n} - 2\, n_{x_1} n_{x_2} M_{ns,n} + 2(n_{x_2}^2 + 3n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2)M_{ns,s} + \\ +\, 2\dfrac{\partial}{\partial s}\{n_{x_1}^3 n_{x_2}\}M_{nn} + 2\, n_{x_1}^3 n_{x_2} M_{nn,s} + \dfrac{\partial}{\partial s}\{n_{x_2}^2 + 6n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2\}M_{ns} - \\ -\, 2\dfrac{\partial}{\partial s}\{n_{x_1} n_{x_2}(n_{x_1}^2 - n_{x_2}^2)\}M_{ss} - 4\, n_{x_2} n_{x_1}^3 M_{ss,s} + \\ +\, \dfrac{I_2}{\rho}\dfrac{\partial}{\partial s}\{(n_{x_2}^2 - n_{x_1}^2)\, a_{T1s} + 2\, n_{x_1} n_{x_2}\, a_{T1n}\} \end{array}\right\}\delta w_0\; ds\; dt\; +$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{c} (n_{x_1}^2 - n_{x_2}^2)^2 M_{nn} + 4\, n_{x_1}^2 n_{x_2}^2\, M_{ss} + 4 n_{x_1} n_{x_2}\,(n_{x_1}^2 - n_{x_2}^2) M_{ns} + \\ +\, ((n_{x_1}^2 - n_{x_2}^2)\, a_{T1n} + 2\, n_{x_2} n_{x_1}\, a_{T1s})\dfrac{I_2}{\rho} \end{array}\right\}\frac{\partial \delta w_0}{\partial n}\; ds\; dt\; -$$

$$-\int_{t_1}^{t_2}\Big[\, 2\, n_{x_1}^3 n_{x_2}\; M_{nn}\,\delta w_0\Big]_{\Gamma} dt - \int_{t_1}^{t_2}\Big[(n_{x_2}^4 + 6n_{x_1}^2 n_{x_2}^2 - n_{x_1}^4)M_{ns}\,\delta w_0\Big]_{\Gamma} dt\; +$$

$$+\int_{t_1}^{t_2}\Big[\, 2\, n_{x_1} n_{x_2}(n_{x_1}^2 - n_{x_2}^2)M_{ss}\,\delta w_0\Big]_{\Gamma} dt - \int_{t_1}^{t_2}\frac{I_2}{\rho}\Big[((n_{x_2}^2 - n_{x_1}^2)\, a_{T1s} + 2\, n_{x_1} n_{x_2}\, a_{T1n})\,\delta w_0\Big]_{\Gamma} dt = 0 \;\Rightarrow$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{(n_{x_1}^2-n_{x_2}^2)\,N_{nn}+2\,n_{x_1}n_{x_2}\,N_{ns}-a_{T0n}\frac{I_0}{\rho}\right\}\delta u_{0n}\,ds\,dt\;+$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{(n_{x_1}^2-n_{x_2}^2)N_{ns}+2\,n_{x_2}n_{x_1}\,N_{ss}-a_{T0s}\frac{I_0}{\rho}\right\}\delta u_{0s}\,ds\,dt\;+$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\left[\begin{array}{c} I_2\left\{(n_{x_1}^2-n_{x_2}^2)\dfrac{\partial \ddot{w}_0}{\partial n}+2\,n_{x_1}n_{x_2}\dfrac{\partial \ddot{w}_0}{\partial s}\right\}-c_{T0}(s)\dfrac{I_0}{\rho}\;- \\ -(n_{x_1}^2-n_{x_2}^2)M_{nn,n}-2\,n_{x_1}n_{x_2}M_{ns,n}+2\,(n_{x_2}^2+3n_{x_1}^2n_{x_2}^2-n_{x_1}^2)M_{ns,s}\;+ \\ +\,2\dfrac{\partial}{\partial s}\{n_{x_1}^3n_{x_2}\}\,M_{nn}+2\,n_{x_1}^3n_{x_2}M_{nn,s}+\dfrac{\partial}{\partial s}\{n_{x_2}^2+6n_{x_1}^2n_{x_2}^2-n_{x_1}^2\}M_{ns}\;- \\ -\,2\dfrac{\partial}{\partial s}\{n_{x_1}n_{x_2}(n_{x_1}^2-n_{x_2}^2)\}M_{ss}-4\,n_{x_2}n_{x_1}^3M_{ss,s}\;+ \\ +\,\dfrac{I_2}{\rho}\dfrac{\partial}{\partial s}\{(n_{x_2}^2-n_{x_1}^2)\,a_{T1s}+2\,n_{x_1}n_{x_2}\,a_{T1n}\} \end{array}\right]\delta w_0\,ds\,dt\;+$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{c}(n_{x_1}^2-n_{x_2}^2)^2M_{nn}+4\,n_{x_1}^2n_{x_2}^2\,M_{ss}+4n_{x_1}n_{x_2}(n_{x_1}^2-n_{x_2}^2)M_{ns}\;+ \\ +\,((n_{x_1}^2-n_{x_2}^2)\,a_{T1n}+2\,n_{x_2}n_{x_1}\,a_{T1s})\dfrac{I_2}{\rho}\end{array}\right\}\frac{\partial\delta w_0}{\partial n}\,ds\,dt\;-$$

$$-\int_{t_1}^{t_2}\left\{\begin{array}{c}\left[2\,n_{x_1}^3n_{x_2}\,M_{nn}\,\delta w_0\right]_{\Gamma}+\left[n_{x_2}^4+6n_{x_1}^2n_{x_2}^2-n_{x_1}^4M_{ns}\,\delta w_0\right]_{\Gamma}\;- \\ -\left[2\,n_{x_1}n_{x_2}(n_{x_1}^2-n_{x_2}^2)M_{ss}\,\delta w_0\right]_{\Gamma}+\dfrac{I_2}{\rho}\left[((n_{x_2}^2-n_{x_1}^2)\,a_{T1s}+2\,n_{x_1}n_{x_2}\,a_{T1n})\,\delta w_0\right]_{\Gamma}\end{array}\right\}dt=0$$

At this point, each reader could decide about the kind of boundary conditions on the curve Γ. This means that has to choose between the free, simply supported or clamped boundary conditions as well as the two points upon the curve Γ, which define the specific natural or essential boundary conditions. However, as we referred above these cases will not occupy us further here.

In conclusion, we manage to remedy the initially unbalanced system of equations and unknown quantities. This balance ensures that the problem of CPT resulting from the conservation principles, the constitutive equations and the physically meaningful boundary conditions, is well – posed in the sense that the solution exists and it is unique.

6. Equations of motion of the CPT in terms of displacements

As we have aforementioned on the conclusion of the section 4.3, it is time to use the relations of the stress resultants [Eqs. (4) of the section 4.3] and to substitute into them the relations of stresses in terms of the displacements [Eqs. (8) - (10) of the section 4.2.1 or Eqs. (5) and (5`) of the section 4.2.2], meaning to express the stress resultants similarly in terms of the displacement field. Thus, we get the following relationships as seems on the sections 6.1 and 6.2 in case of an orthotropic in-plane anisotropic and orthotropic in-plane isotropic plate respectively.

6.1. Equations of motion of the CPT in terms of displacements for an orthotropic, in-plane anisotropic material

The thickness-integrated forces of the Eq. (4) of the section 4.3 are converted to the below, due to the Eqs. (8) - (10) of the section 4.2.1,

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} = \begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \dfrac{E_1}{1-v_{12}v_{21}} \left(\dfrac{\partial u_0}{\partial x_1} - x_3 \dfrac{\partial^2 w_0}{\partial^2 x_1} \right) + \dfrac{v_{21}E_1}{1-v_{12}v_{21}} \left(\dfrac{\partial v_0}{\partial x_2} - x_3 \dfrac{\partial^2 w_0}{\partial^2 x_2} \right) \\ \dfrac{v_{12}E_2}{1-v_{12}v_{21}} \left(\dfrac{\partial u_0}{\partial x_1} - x_3 \dfrac{\partial^2 w_0}{\partial^2 x_1} \right) + \dfrac{E_2}{1-v_{12}v_{21}} \left(\dfrac{\partial v_0}{\partial x_2} - x_3 \dfrac{\partial^2 w_0}{\partial^2 x_2} \right) \\ G_6 \left(\dfrac{\partial u_0}{\partial x_2} + \dfrac{\partial v_0}{\partial x_1} - 2x_3 \dfrac{\partial^2 w_0}{\partial x_1 \partial x_2} \right) \end{Bmatrix} dx_3 =$$

$$= \begin{Bmatrix} \dfrac{E_1}{1-v_{12}v_{21}} \left(\dfrac{\partial u_0}{\partial x_1} \dfrac{I_0}{\rho} - \dfrac{\partial^2 w_0}{\partial^2 x_1} \dfrac{\cancel{I_1}}{\rho} \right) + \dfrac{v_{21}E_1}{1-v_{12}v_{21}} \left(\dfrac{\partial v_0}{\partial x_2} \dfrac{I_0}{\rho} - \dfrac{\partial^2 w_0}{\partial^2 x_2} \dfrac{\cancel{I_1}}{\rho} \right) \\ \dfrac{v_{12}E_2}{1-v_{12}v_{21}} \left(\dfrac{\partial u_0}{\partial x_1} \dfrac{I_0}{\rho} - \dfrac{\partial^2 w_0}{\partial^2 x_1} \dfrac{\cancel{I_1}}{\rho} \right) + \dfrac{E_2}{1-v_{12}v_{21}} \left(\dfrac{\partial v_0}{\partial x_2} \dfrac{I_0}{\rho} - \dfrac{\partial^2 w_0}{\partial^2 x_2} \dfrac{\cancel{I_1}}{\rho} \right) \\ G_6 \left(\dfrac{\partial u_0}{\partial x_2} + \dfrac{\partial v_0}{\partial x_1} \right) \dfrac{I_0}{\rho} - 2 \dfrac{\cancel{I_1}}{\rho} G_6 \dfrac{\partial^2 w_0}{\partial x_1 \partial x_2} \end{Bmatrix} =$$

$$= \begin{Bmatrix} \dfrac{E_1}{1-v_{12}v_{21}} \dfrac{\partial u_0}{\partial x_1} \dfrac{I_0}{\rho} + \dfrac{v_{21}E_1}{1-v_{12}v_{21}} \dfrac{\partial v_0}{\partial x_2} \dfrac{I_0}{\rho} \\ \dfrac{v_{12}E_2}{1-v_{12}v_{21}} \dfrac{\partial u_0}{\partial x_1} \dfrac{I_0}{\rho} + \dfrac{E_2}{1-v_{12}v_{21}} \dfrac{\partial v_0}{\partial x_2} \dfrac{I_0}{\rho} \\ G_6 \left(\dfrac{\partial u_0}{\partial x_2} + \dfrac{\partial v_0}{\partial x_1} \right) \dfrac{I_0}{\rho} \end{Bmatrix}$$

And writing the above thickness-integrated forces separately, we get

$$N_1 = \frac{E_1}{1-\nu_{12}\nu_{21}}\frac{\partial u_0}{\partial x_1}\frac{I_0}{\rho} + \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}\frac{\partial v_0}{\partial x_2}\frac{I_0}{\rho} \tag{1}$$

$$N_2 = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}\frac{\partial u_0}{\partial x_1}\frac{I_0}{\rho} + \frac{E_2}{1-\nu_{12}\nu_{21}}\frac{\partial v_0}{\partial x_2}\frac{I_0}{\rho} \tag{2}$$

$$N_6 = G_6\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}\right)\frac{I_0}{\rho} \tag{3}$$

The thickness-integrated moments of the Eq. (4) of the section 4.3 are converted to the below, due to the Eqs. (8) - (10) of the section 4.2.1,

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}}\left(\frac{\partial u_0}{\partial x_1} - x_3\frac{\partial^2 w_0}{\partial^2 x_1}\right) + \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}\left(\frac{\partial v_0}{\partial x_2} - x_3\frac{\partial^2 w_0}{\partial^2 x_2}\right) \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}\left(\frac{\partial u_0}{\partial x_1} - x_3\frac{\partial^2 w_0}{\partial^2 x_1}\right) + \frac{E_2}{1-\nu_{12}\nu_{21}}\left(\frac{\partial v_0}{\partial x_2} - x_3\frac{\partial^2 w_0}{\partial^2 x_2}\right) \\ G_6\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1} - 2x_3\frac{\partial^2 w_0}{\partial x_1 \partial x_2}\right) \end{Bmatrix} x_3\, dx_3 =$$

$$= \int_{-h/2}^{h/2} \begin{Bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}}\left(\frac{\partial u_0}{\partial x_1}\, x_3 - x_3^2\frac{\partial^2 w_0}{\partial^2 x_1}\right) + \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}\left(\frac{\partial v_0}{\partial x_2}\, x_3 - x_3^2\frac{\partial^2 w_0}{\partial^2 x_2}\right) \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}\left(\frac{\partial u_0}{\partial x_1}\, x_3 - x_3^2\frac{\partial^2 w_0}{\partial^2 x_1}\right) + \frac{E_2}{1-\nu_{12}\nu_{21}}\left(\frac{\partial v_0}{\partial x_2}\, x_3 - x_3^2\frac{\partial^2 w_0}{\partial^2 x_2}\right) \\ G_6\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}\right) x_3 - 2G_6\, x_3^2\frac{\partial^2 w_0}{\partial x_1 \partial x_2} \end{Bmatrix} dx_3 =$$

$$= \begin{Bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}}\left(\frac{\partial u_0}{\partial x_1}\frac{\cancel{I_1}}{\rho} - \frac{\partial^2 w_0}{\partial^2 x_1}\frac{I_2}{\rho}\right) + \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}\left(\frac{\partial v_0}{\partial x_2}\frac{\cancel{I_1}}{\rho} - \frac{\partial^2 w_0}{\partial^2 x_2}\frac{I_2}{\rho}\right) \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}\left(\frac{\partial u_0}{\partial x_1}\frac{\cancel{I_1}}{\rho} - \frac{\partial^2 w_0}{\partial^2 x_1}\frac{I_2}{\rho}\right) + \frac{E_2}{1-\nu_{12}\nu_{21}}\left(\frac{\partial v_0}{\partial x_2}\frac{\cancel{I_1}}{\rho} - \frac{\partial^2 w_0}{\partial^2 x_2}\frac{I_2}{\rho}\right) \\ G_6\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}\right)\frac{\cancel{I_1}}{\rho} - 2\,G_6\frac{I_2}{\rho}\frac{\partial^2 w_0}{\partial x_1 \partial x_2} \end{Bmatrix} =$$

$$= \left\{ \begin{array}{c} -\dfrac{E_1}{1-\nu_{12}\nu_{21}}\dfrac{\partial^2 w_0}{\partial^2 x_1}\dfrac{I_2}{\rho} - \dfrac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}\dfrac{\partial^2 w_0}{\partial^2 x_2}\dfrac{I_2}{\rho} \\ -\dfrac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}\dfrac{\partial^2 w_0}{\partial^2 x_1}\dfrac{I_2}{\rho} - \dfrac{E_2}{1-\nu_{12}\nu_{21}}\dfrac{\partial^2 w_0}{\partial^2 x_2}\dfrac{I_2}{\rho} \\ -2G_6\dfrac{I_2}{\rho}\dfrac{\partial^2 w_0}{\partial x_1 \partial x_2} \end{array} \right\}$$

And writing the above thickness-integrated moments separately, we have

$$M_1 = -\frac{E_1}{1-\nu_{12}\nu_{21}}\frac{\partial^2 w_0}{\partial^2 x_1}\frac{I_2}{\rho} - \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}\frac{\partial^2 w_0}{\partial^2 x_2}\frac{I_2}{\rho} \tag{4}$$

$$M_2 = -\frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}\frac{\partial^2 w_0}{\partial^2 x_1}\frac{I_2}{\rho} - \frac{E_2}{1-\nu_{12}\nu_{21}}\frac{\partial^2 w_0}{\partial^2 x_2}\frac{I_2}{\rho} \tag{5}$$

$$M_6 = -2G_6\frac{I_2}{\rho}\frac{\partial^2 w_0}{\partial x_1 \partial x_2} \tag{6}$$

From the Eq. (4) of the section 4.5,

$$I_0\ddot{u}_0 - \frac{\partial N_1}{\partial x_1} - \frac{\partial N_6}{\partial x_2} = 0 \Rightarrow$$

$$I_0\ddot{u}_0 - \frac{\partial}{\partial x_1}\left[\frac{E_1}{1-\nu_{12}\nu_{21}}\frac{\partial u_0}{\partial x_1}\frac{I_0}{\rho} + \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}\frac{\partial v_0}{\partial x_2}\frac{I_0}{\rho}\right] - \frac{\partial}{\partial x_2}\left[G_6\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}\right)\frac{I_0}{\rho}\right] = 0 \Rightarrow$$

$$\cancel{I_0}\ddot{u}_0 - \left[\frac{E_1}{1-\nu_{12}\nu_{21}}\frac{\partial^2 u_0}{\partial^2 x_1}\frac{\cancel{I_0}}{\rho} + \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}\frac{\partial^2 v_0}{\partial x_1 \partial x_2}\frac{\cancel{I_0}}{\rho}\right] - \left[G_6\left(\frac{\partial^2 u_0}{\partial^2 x_2}+\frac{\partial^2 v_0}{\partial x_2 \partial x_1}\right)\frac{\cancel{I_0}}{\rho}\right] = 0 \Rightarrow$$

$$\rho\ddot{u}_0 - \frac{E_1}{1-\nu_{12}\nu_{21}}\frac{\partial^2 u_0}{\partial^2 x_1} - \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}\frac{\partial^2 v_0}{\partial x_1 \partial x_2} - G_6\left(\frac{\partial^2 u_0}{\partial^2 x_2}+\frac{\partial^2 v_0}{\partial x_2 \partial x_1}\right) = 0 \Rightarrow$$

$$\rho\ddot{u}_0 - \frac{E_1}{1-\nu_{12}\nu_{21}}\frac{\partial^2 u_0}{\partial^2 x_1} - \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}\frac{\partial^2 v_0}{\partial x_1 \partial x_2} - G_6\left(\frac{\partial^2 u_0}{\partial^2 x_2}+\frac{\partial^2 v_0}{\partial x_2 \partial x_1}\right) = 0 \tag{7}$$

Note that the above Eq. (7), can be found on the book of J.N. Reddy (2007), "Theory and Analysis of Elastic Plates and Shells", on the page 118 of the section 3.8. Thus, the corresponding equation of motion of the book is exactly the following,

$$A_{11}\left(\frac{\partial^2 u_0}{\partial^2 x_1}+\frac{\partial w_0}{\partial x_1}\frac{\partial^2 w_0}{\partial^2 x_1}\right) + A_{12}\left(\frac{\partial^2 v_0}{\partial x_1 \partial x_2}+\frac{\partial w_0}{\partial x_2}\frac{\partial^2 w_0}{\partial x_1 \partial x_2}\right) +$$

$$+A_{66}\left(\frac{\partial^2 u_0}{\partial^2 x_2}+\frac{\partial^2 v_0}{\partial x_1 \partial x_2}+\frac{\partial^2 w_0}{\partial x_1 \partial x_2}\frac{\partial w_0}{\partial x_2}+\frac{\partial w_0}{\partial x_1}\frac{\partial^2 w_0}{\partial^2 x_2}\right)-\left(\frac{\partial N_{11}^T}{\partial x_1}+\frac{\partial N_{12}^T}{\partial x_2}\right)=I_0\ddot{u}_0 \qquad (3.8.3)$$

Subsequently, by using the expressions of the extensional stiffness coefficients A_{ij} of the page 112 (Eqs 3.6.13). Also, comparing our result, namely Eq. (7), to our source, we notice that there are some additional second derivatives of w_0 because Reddy regards another strain field which takes into account the geometric nonlinearities i.e. small strains but moderate rotations of transverse normal of the mid-surface $(10^o\text{-}15^o)$ [*J.N. Reddy (2007) "Theory and Analysis of Elastic Plates and Shells", page 98 – 99/ Chapter 3*]. Finally, the relation (3.8.3) is converted to,

$$\frac{E_1 h}{1-v_{12}v_{21}}\frac{\partial^2 u_0}{\partial^2 x_1}+\frac{v_{21}E_1 h}{1-v_{12}v_{21}}\frac{\partial^2 v_0}{\partial x_1 \partial x_2}+G_{12}h\left(\frac{\partial^2 u_0}{\partial^2 x_2}+\frac{\partial^2 v_0}{\partial x_1 \partial x_2}\right)=I_0\ddot{u}_0$$

The last expression coincidence exactly with our result, namely the Eq. **(7)**.

From the Eq. (5) of the section 4.5,

$$I_0\ddot{v}_0-\frac{\partial N_6}{\partial x_1}-\frac{\partial N_2}{\partial x_2}=0 \Rightarrow$$

$$I_0\ddot{v}_0-\frac{\partial}{\partial x_1}\left[G_6\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}\right)\frac{I_0}{\rho}\right]-\frac{\partial}{\partial x_2}\left[\frac{v_{12}E_2}{1-v_{12}v_{21}}\frac{\partial u_0}{\partial x_1}\frac{I_0}{\rho}+\frac{E_2}{1-v_{12}v_{21}}\frac{\partial v_0}{\partial x_2}\frac{I_0}{\rho}\right]=0\Rightarrow$$

$$\cancel{I_0}\ddot{v}_0-G_6\left(\frac{\partial^2 u_0}{\partial x_1 \partial x_2}+\frac{\partial^2 v_0}{\partial^2 x_1}\right)\frac{\cancel{I_0}}{\rho}-\frac{v_{12}E_2}{1-v_{12}v_{21}}\frac{\partial^2 u_0}{\partial x_2 \partial x_1}\frac{\cancel{I_0}}{\rho}-\frac{E_2}{1-v_{12}v_{21}}\frac{\partial^2 v_0}{\partial^2 x_2}\frac{\cancel{I_0}}{\rho}=0\Rightarrow$$

$$\rho\ddot{v}_0-G_6\left(\frac{\partial^2 u_0}{\partial x_1 \partial x_2}+\frac{\partial^2 v_0}{\partial^2 x_1}\right)-\frac{v_{12}E_2}{1-v_{12}v_{21}}\frac{\partial^2 u_0}{\partial x_2 \partial x_1}-\frac{E_2}{1-v_{12}v_{21}}\frac{\partial^2 v_0}{\partial^2 x_2}=0 \qquad \mathbf{(8)}$$

Similarly to the previous note, the Eq. **(8)** is coincident to the Eq. (3.8.4) of the same book. Using the same justification, we convert the Eq. (3.8.4) to the following form, which is exactly our result.

$$I_0\ddot{v}_0-G_6 h\left(\frac{\partial^2 u_0}{\partial x_1 \partial x_2}+\frac{\partial^2 v_0}{\partial^2 x_1}\right)-\frac{v_{12}E_2 h}{1-v_{12}v_{21}}\frac{\partial^2 u_0}{\partial x_2 \partial x_1}-\frac{E_2 h}{1-v_{12}v_{21}}\frac{\partial^2 v_0}{\partial^2 x_2}=0 \qquad (3.8.4)$$

From the Eq. (6) of the section 4.5,

$$I_0\ddot{w}_0-I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right)-2\frac{\partial^2 M_6}{\partial x_1 \partial x_2}-\frac{\partial^2 M_1}{\partial^2 x_1}-\frac{\partial^2 M_2}{\partial^2 x_2}-q=0\Rightarrow$$

$$I_0\ddot{w}_0-I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right)+4\frac{\partial^2}{\partial x_1 \partial x_2}\left(G_6\frac{I_2}{\rho}\frac{\partial^2 w_0}{\partial x_1 \partial x_2}\right)+$$

$$+\frac{\partial^2}{\partial^2 x_1}\left(\frac{E_1}{1-\nu_{12}\nu_{21}}\frac{\partial^2 w_0}{\partial^2 x_1}\frac{I_2}{\rho}-\frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}\frac{\partial^2 w_0}{\partial^2 x_2}\frac{I_2}{\rho}\right)+$$
$$+\frac{\partial^2}{\partial^2 x_2}\left(\frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}\frac{\partial^2 w_0}{\partial^2 x_1}\frac{I_2}{\rho}-\frac{E_2}{1-\nu_{12}\nu_{21}}\frac{\partial^2 w_0}{\partial^2 x_2}\frac{I_2}{\rho}\right)=q\Rightarrow$$

$$I_0\ddot{w}_0-I_2\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}\right)+4G_6\frac{I_2}{\rho}\frac{\partial^4 w_0}{\partial^2 x_1\partial^2 x_2}+$$
$$+\frac{E_1}{1-\nu_{12}\nu_{21}}\frac{\partial^4 w_0}{\partial^4 x_1}\frac{I_2}{\rho}-\frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}\frac{\partial^4 w_0}{\partial^2 x_1\partial^2 x_2}\frac{I_2}{\rho}+$$
$$+\frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}\frac{\partial^4 w_0}{\partial^2 x_2\partial^2 x_1}\frac{I_2}{\rho}-\frac{E_2}{1-\nu_{12}\nu_{21}}\frac{\partial^4 w_0}{\partial^4 x_2}\frac{I_2}{\rho}=q\Rightarrow$$

$$I_0\ddot{w}_0-I_2\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}\right)+$$
$$+\left(4\frac{G_6 h^3}{12}+\frac{\nu_{12}E_2 h^3}{12(1-\nu_{12}\nu_{21})}+\frac{\nu_{21}E_1 h^3}{12(1-\nu_{12}\nu_{21})}\right)\frac{\partial^4 w_0}{\partial^2 x_1\partial^2 x_2}+$$
$$+\frac{E_1 h^3}{12(1-\nu_{12}\nu_{21})}\frac{\partial^4 w_0}{\partial^4 x_1}-\frac{E_2 h^3}{12(1-\nu_{12}\nu_{21})}\frac{\partial^4 w_0}{\partial^4 x_2}=q\Rightarrow$$

$$\rho h\ddot{w}_0-\frac{\rho h^3}{12}\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}\right)+$$
$$+\left(4\frac{G_6 h^3}{12}+\underbrace{\frac{\overbrace{\nu_{12}E_2}^{\nu_{12}E_2=\nu_{21}E_1}h^3}{12(1-\nu_{12}\nu_{21})}}_{D_{12}}+\frac{\nu_{21}E_1 h^3}{\underbrace{12(1-\nu_{12}\nu_{21})}_{D_{12}=\nu_{21}D_{11}}}\right)\frac{\partial^4 w_0}{\partial^2 x_1\partial^2 x_2}+$$
$$+\frac{E_1 h^3}{12(1-\nu_{12}\nu_{21})}\frac{\partial^4 w_0}{\partial^4 x_1}-\frac{E_2 h^3}{12(1-\nu_{12}\nu_{21})}\frac{\partial^4 w_0}{\partial^4 x_2}=q \qquad \textbf{(9)}$$

On the Eq. (9), we use the expressions of the bending stiffness coefficients of the page 112 of the aforementioned book of J.N. Reddy, which are presented below by the Eqs. (10) for convenience, in order to compare easier our resulting equation with the Eq. (3.8.5) of the page 118 of the same book.

$$D_{11}=\frac{E_1 h^3}{12(1-\nu_{12}\nu_{21})},\qquad D_{22}=\frac{E_2 h^3}{12(1-\nu_{12}\nu_{21})},$$
$$D_{12}=\frac{\nu_{21}E_1 h^3}{12(1-\nu_{12}\nu_{21})},\qquad D_{66}=\frac{G_{12}h^3}{12} \qquad (10)$$

Similarly with the previous equations of motion, we are not take into account thermal effects and the elastic foundation, because on frames of CPT, we have made no assumptions relative to thermal loads. Additionally the term including residual quantities due a different strain field, defined on the page 98-99, is also eliminated.

Note that the Eq. (3.8.5), is exactly the same, as this which is found on the Chapter 9.1 on the page 331 of the book of J.N. Reddy (2007), "Theory and Analysis of Elastic Plates and Shells". This relation is presented on the [Eqs. (9.1.1) and (9.1.2)].

Finally the Eq. (3.8.5) is converted to the following,

$$-D_{11}\frac{\partial^4 w_0}{\partial^4 x_1} - 2\,D_{12} + D_{66}\frac{\partial^4 w_0}{\partial^2 x_1 \partial^2 x_2} - D_{22}\frac{\partial^4 w_0}{\partial^4 x_2} - \cancel{k\,w_0} - \cancel{\left(\frac{\partial^2 M_{11}^T}{\partial^2 x_1} + 2\frac{\partial^2 M_{12}^T}{\partial x_1 \partial x_2} + \frac{\partial^2 M_{22}^T}{\partial^2 x_2}\right)} + \cancel{N(u_0, v_0, w_0)} + q = I_0\,\ddot{w}_0 - I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1} + \frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right) \Rightarrow$$

$$-D_{11}\frac{\partial^4 w_0}{\partial^4 x_1} - 2\,D_{12} + D_{66}\frac{\partial^4 w_0}{\partial^2 x_1 \partial^2 x_2} - D_{22}\frac{\partial^4 w_0}{\partial^4 x_2} + q = I_0\,\ddot{w}_0 - I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1} + \frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right) \qquad (3.8.5)$$

which is exactly our result , namely the Eq. **(9)**.

6.2. Equations of motion of the CPT in terms of displacements for an orthotropic, in-plane isotropic material

The thickness-integrated forces of the Eq. (4) of the section 4.3 are converted to the below, due to the Eqs. (5) and (5`) of the section 4.2.2,

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} = \begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \frac{E}{1-v^2}\frac{\partial u_0}{\partial x_1} - x_3 \frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{vE}{1-v^2}\frac{\partial v_0}{\partial x_2} - x_3 \frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 x_2} \\ \frac{vE}{1-v^2}\frac{\partial u_0}{\partial x_1} - x_3 \frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{E}{1-v^2}\frac{\partial v_0}{\partial x_2} - x_3 \frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 x_2} \\ G\left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1}\right) - 2x_3 G \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \end{Bmatrix} dx_3 =$$

$$= \begin{Bmatrix} \frac{E I_0}{\rho(1-v^2)}\frac{\partial u_0}{\partial x_1} - \frac{E \cancel{I_1}}{\rho(1-v^2)}\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{v I_0 E}{\rho(1-v^2)}\frac{\partial v_0}{\partial x_2} - \frac{v \cancel{I_1} E}{\rho(1-v^2)}\frac{\partial^2 w_0}{\partial^2 x_2} \\ \frac{v I_0 E}{\rho(1-v^2)}\frac{\partial u_0}{\partial x_1} - \frac{v \cancel{I_1} E}{\rho(1-v^2)}\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{I_0 E}{\rho(1-v^2)}\frac{\partial v_0}{\partial x_2} - \frac{\cancel{I_1} E}{\rho(1-v^2)}\frac{\partial^2 w_0}{\partial^2 x_2} \\ G\frac{I_0}{\rho}\left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1}\right) - 2\frac{\cancel{I_1}}{\rho} G \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \end{Bmatrix} =$$

$$= \begin{Bmatrix} \frac{E I_0}{\rho(1-v^2)}\frac{\partial u_0}{\partial x_1} + \frac{v I_0 E}{\rho(1-v^2)}\frac{\partial v_0}{\partial x_2} \\ \frac{v I_0 E}{\rho(1-v^2)}\frac{\partial u_0}{\partial x_1} + \frac{I_0 E}{\rho(1-v^2)}\frac{\partial v_0}{\partial x_2} \\ G\frac{I_0}{\rho}\left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1}\right) \end{Bmatrix}$$

And expressing the above thickness-integrated forces separately, we have

$$N_1 = \frac{E I_0}{\rho(1-v^2)}\frac{\partial u_0}{\partial x_1} + \frac{v I_0 E}{\rho(1-v^2)}\frac{\partial v_0}{\partial x_2} \tag{1}$$

$$N_2 = \frac{v I_0 E}{\rho(1-v^2)}\frac{\partial u_0}{\partial x_1} + \frac{I_0 E}{\rho(1-v^2)}\frac{\partial v_0}{\partial x_2} \tag{2}$$

$$N_6 = G\frac{I_0}{\rho}\left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1}\right) \tag{3}$$

The thickness-integrated moments of the Eq. (4) of the section 4.3 are converted to the below, due to the Eqs. (5) and (5`) of the section 4.2.2,

$$\left\{\begin{matrix} M_1 \\ M_2 \\ M_6 \end{matrix}\right\} = \left\{\begin{matrix} M_{11} \\ M_{22} \\ M_{12} \end{matrix}\right\} = \int_{-h/2}^{h/2} \left\{\begin{matrix} \frac{E}{1-v^2}\frac{\partial u_0}{\partial x_1} - x_3\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{vE}{1-v^2}\frac{\partial v_0}{\partial x_2} - x_3\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 x_2} \\ \frac{vE}{1-v^2}\frac{\partial u_0}{\partial x_1} - x_3\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{E}{1-v^2}\frac{\partial v_0}{\partial x_2} - x_3\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 x_2} \\ G\left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1}\right) - 2x_3 G\frac{\partial^2 w_0}{\partial x_1 \partial x_2} \end{matrix}\right\} x_3\, dx_3 =$$

$$= \int_{-h/2}^{h/2} \left\{\begin{matrix} \frac{E}{1-v^2}\, x_3 \frac{\partial u_0}{\partial x_1} - x_3^2\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{vE}{1-v^2}\, x_3\frac{\partial v_0}{\partial x_2} - x_3^2\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 x_2} \\ \frac{vE}{1-v^2}\, x_3\frac{\partial u_0}{\partial x_1} - x_3^2\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{E}{1-v^2}\, x_3\frac{\partial v_0}{\partial x_2} - x_3^2\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 x_2} \\ G\, x_3\left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1}\right) - 2x_3^2 G\frac{\partial^2 w_0}{\partial x_1 \partial x_2} \end{matrix}\right\} dx_3 =$$

$$= \left\{\begin{matrix} \frac{E\cancel{I_1}}{\rho(1-v^2)}\frac{\partial u_0}{\partial x_1} - \frac{EI_2}{\rho(1-v^2)}\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{v\cancel{I_1}E}{\rho(1-v^2)}\frac{\partial v_0}{\partial x_2} - \frac{vI_2E}{\rho(1-v^2)}\frac{\partial^2 w_0}{\partial^2 x_2} \\ \frac{v\cancel{I_1}E}{\rho(1-v^2)}\frac{\partial u_0}{\partial x_1} - \frac{vI_2E}{\rho(1-v^2)}\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{E\cancel{I_1}}{\rho(1-v^2)}\frac{\partial v_0}{\partial x_2} - \frac{EI_2}{\rho(1-v^2)}\frac{\partial^2 w_0}{\partial^2 x_2} \\ G\frac{\cancel{I_1}}{\rho}\left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1}\right) - 2\frac{I_2}{\rho}G\frac{\partial^2 w_0}{\partial x_1 \partial x_2} \end{matrix}\right\} =$$

$$= \left\{\begin{matrix} -\frac{EI_2}{\rho(1-v^2)}\frac{\partial^2 w_0}{\partial^2 x_1} - \frac{vI_2E}{\rho(1-v^2)}\frac{\partial^2 w_0}{\partial^2 x_2} \\ -\frac{vI_2E}{\rho(1-v^2)}\frac{\partial^2 w_0}{\partial^2 x_1} - \frac{EI_2}{\rho(1-v^2)}\frac{\partial^2 w_0}{\partial^2 x_2} \\ -\frac{I_2}{\rho}\frac{E}{1+v}\frac{\partial^2 w_0}{\partial x_1 \partial x_2} \end{matrix}\right\}$$

And writing the above thickness-integrated moments separately, we get

$$M_1 = -\frac{EI_2}{\rho(1-v^2)}\frac{\partial^2 w_0}{\partial^2 x_1} - \frac{vI_2E}{\rho(1-v^2)}\frac{\partial^2 w_0}{\partial^2 x_2} \tag{4}$$

$$M_2 = -\frac{vI_2E}{\rho(1-v^2)}\frac{\partial^2 w_0}{\partial^2 x_1} - \frac{EI_2}{\rho(1-v^2)}\frac{\partial^2 w_0}{\partial^2 x_2} \tag{5}$$

$$M_6 = -2\,\frac{I_2}{\rho}\,G\,\frac{\partial^2 w_0}{\partial x_1 \partial x_2} \tag{6}$$

Now to derive the equations of motion, in terms of the displacement field for an isotropic material, we recall the Eqs. (4) – (6) of the section 4.5 and substitute the Eqs. (1) – (6). By this way, we get the following:

From (4) of the section 4.5.,

$$I_0\ddot{u}_0 - \frac{\partial N_1}{\partial x_1} - \frac{\partial N_6}{\partial x_2} = 0 \Rightarrow$$

$$I_0\ddot{u}_0 - \frac{\partial}{\partial x_1}\left(\frac{E\,I_0}{\rho(1-v^2)}\frac{\partial u_0}{\partial x_1} + \frac{v\,I_0\,E}{\rho(1-v^2)}\frac{\partial v_0}{\partial x_2}\right) - \frac{G\,I_0}{\rho}\frac{\partial}{\partial x_2}\left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1}\right) = 0 \Rightarrow$$

$$\cancel{I_0}\,\ddot{u}_0 - \frac{E\,\cancel{I_0}}{\rho(1-v^2)}\frac{\partial}{\partial x_1}\left(\frac{\partial u_0}{\partial x_1} + v\,\frac{\partial v_0}{\partial x_2}\right) - \frac{G\,\cancel{I_0}}{\rho}\frac{\partial}{\partial x_2}\left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1}\right) = 0 \Rightarrow$$

$$\rho\ddot{u}_0 - \frac{E}{(1-v^2)}\left(\frac{\partial^2 u_0}{\partial^2 x_1} + v\,\frac{\partial^2 v_0}{\partial x_1 \partial x_2}\right) - \frac{E}{2(1+v)}\left(\frac{\partial^2 u_0}{\partial^2 x_2} + \frac{\partial^2 v_0}{\partial x_2 \partial x_1}\right) = 0 \tag{7}$$

Let it be highlighted that the Eq. (7) is practically identical with the respective result (3.8.3) of the book J.N. Reddy (2007) on the page 118. Comparing our result, namely Eq. (7), to our source, we notice that there are some additional second derivatives of w_0 because Reddy regards another strain field which takes into account the geometric nonlinearities i.e. small strains but moderate rotations of transverse normal of the mid-surface $(10^o\text{-}15^o)$ [*J.N. Reddy (2007) "Theory and Analysis of Elastic Plates and Shells", page 98 – 99/ Chapter 3*]. The relation (3.8.3) is shown below,

$$A_{11}\left(\frac{\partial^2 u_0}{\partial^2 x_1} + \frac{\partial w_0}{\partial x_1}\frac{\partial^2 w_0}{\partial^2 x_1}\right) + A_{12}\left(\frac{\partial^2 v_0}{\partial x_1 \partial x_2} + \frac{\partial w_0}{\partial x_2}\frac{\partial^2 w_0}{\partial x_1 \partial x_2}\right) +$$

$$+A_{66}\left(\frac{\partial^2 u_0}{\partial^2 x_2} + \frac{\partial^2 v_0}{\partial x_1 \partial x_2} + \frac{\partial^2 w_0}{\partial x_1 \partial x_2}\frac{\partial w_0}{\partial x_2} + \frac{\partial w_0}{\partial x_1}\frac{\partial^2 w_0}{\partial^2 x_2}\right) - \left(\frac{\partial N_{11}^T}{\partial x_1} + \frac{\partial N_{12}^T}{\partial x_2}\right) = I_0\ddot{u}_0 \tag{3.8.3}$$

In addition, the general "behavior" of the plate indicates that it must be regarded as orthotropic. Consequently, we take into account the extensional stiffness coefficients of the page 111-112 of the reference book, which are given below,

$$A_{11} = \frac{E_1 h}{1-v_{12}v_{21}}\,, \qquad A_{12} = v_{21}A_{11}\,,$$

$$A_{22} = \frac{E_2}{E_1}A_{11}\,, \qquad A_{66} = G_{12}\,h$$

However, the in-plane "behavior" (or alternatively the stress-strain relations between the in-plane displacements u_0, v_0) of the plate could be described as isotropic, as justified in the section 4.2.2. Thus, the previous coefficients could be written as,

$$A_{11} = \frac{E\,h}{1-v^2}\,, \qquad A_{12} = \frac{v\,E\,h}{1-v^2}\,,$$
$$A_{22} = \frac{E\,h}{1-v^2}\,, \qquad A_{66} = G\,h$$

Also, in the frames of the particular problem of the CPT, we assume that there are no thermal effect, so it is obvious that,

$$\frac{\partial N_{11}^T}{\partial x_1} = \frac{\partial N_{12}^T}{\partial x_2} = 0$$

Summing up the above notifications, the relation (3.8.3) is converted to the following,

$$A_{11}\left(\frac{\partial^2 u_0}{\partial^2 x_1} + \cancel{\frac{\partial w_0}{\partial x_1}\frac{\partial^2 w_0}{\partial^2 x_1}}\right) + A_{12}\left(\frac{\partial^2 v_0}{\partial x_1 \partial x_2} + \cancel{\frac{\partial w_0}{\partial x_2}\frac{\partial^2 w_0}{\partial x_1 \partial x_2}}\right) +$$
$$+A_{66}\left(\frac{\partial^2 u_0}{\partial^2 x_2} + \frac{\partial^2 v_0}{\partial x_1 \partial x_2} + \cancel{\frac{\partial^2 w_0}{\partial x_1 \partial x_2}\frac{\partial w_0}{\partial x_2}} + \cancel{\frac{\partial w_0}{\partial x_1}\frac{\partial^2 w_0}{\partial^2 x_2}}\right) - \left(\cancel{\frac{\partial N_{11}^T}{\partial x_1} + \frac{\partial N_{12}^T}{\partial x_2}}\right) = I_0\,\ddot{u}_0 \;\Rightarrow$$
$$\frac{E\,h}{1-v^2}\frac{\partial^2 u_0}{\partial^2 x_1} + \frac{v\,E\,h}{1-v^2}\frac{\partial^2 v_0}{\partial x_1 \partial x_2} + G\,h\left(\frac{\partial^2 u_0}{\partial^2 x_2} + \frac{\partial^2 v_0}{\partial x_1 \partial x_2}\right) = I_0\,\ddot{u}_0 \;\Rightarrow$$
$$\frac{E\,\cancel{h}}{1-v^2}\frac{\partial^2 u_0}{\partial^2 x_1} + \frac{v\,E\,\cancel{h}}{1-v^2}\frac{\partial^2 v_0}{\partial x_1 \partial x_2} + \frac{E}{2(1+v)}\cancel{h}\left(\frac{\partial^2 u_0}{\partial^2 x_2} + \frac{\partial^2 v_0}{\partial x_1 \partial x_2}\right) = \rho\,\cancel{h}\,\ddot{u}_0 \;\Rightarrow$$
$$\frac{E}{1-v^2}\left(\frac{\partial^2 u_0}{\partial^2 x_1} + v\frac{\partial^2 v_0}{\partial x_1 \partial x_2}\right) + \frac{E}{2(1+v)}\left(\frac{\partial^2 u_0}{\partial^2 x_2} + \frac{\partial^2 v_0}{\partial x_1 \partial x_2}\right) = \rho\,\ddot{u}_0$$

The last equation is identical to the Eq. (7).

From (5) of the section 4.5,

$$I_0\,\ddot{v}_0 - \frac{\partial N_6}{\partial x_1} - \frac{\partial N_2}{\partial x_2} = 0 \;\Rightarrow$$

$$\cancel{I_0}\,\ddot{v}_0 - G\,\frac{\cancel{I_0}}{\rho}\frac{\partial}{\partial x_1}\left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{v\,\cancel{I_0}\,E}{\rho(1-v^2)}\frac{\partial u_0}{\partial x_1} + \frac{\cancel{I_0}\,E}{\rho(1-v^2)}\frac{\partial v_0}{\partial x_2}\right) = 0 \;\Rightarrow$$

$$\ddot{v}_0 - \frac{E}{2\,\rho(1+v)}\left(\frac{\partial^2 u_0}{\partial x_1 \partial x_2} + \frac{\partial^2 v_0}{\partial^2 x_1}\right) - \frac{E}{\rho(1-v^2)}\left(v\,\frac{\partial^2 u_0}{\partial x_2 \partial x_1} + \frac{\partial^2 v_0}{\partial^2 x_2}\right) = 0 \;\Rightarrow$$

$$\rho\,\ddot{v}_0 - \frac{E}{2(1+v)}\left(\frac{\partial^2 v_0}{\partial^2 x_1} + \frac{\partial^2 u_0}{\partial x_1 \partial x_2}\right) - \frac{E}{1-v^2}\left(\frac{\partial^2 v_0}{\partial^2 x_2} + v\,\frac{\partial^2 u_0}{\partial x_2 \partial x_1}\right) = 0 \qquad (8)$$

On a similar way we are going to show that the relationship (8) is identical to the Eq. (3.8.4) of the book of J.N. Reddy (2007) “Theory and Analysis of Elastic Plates and Shells”, located on the Chapter 3 (page 118). However, we convert the equation (3.8.4) to the appropriate form for our problem, as we done above. Thus,

$$A_{66}\left(\frac{\partial^2 u_0}{\partial x_1 \partial x_2}+\frac{\partial^2 v_0}{\partial^2 x_1}+\frac{\partial w_0}{\partial x_2}\frac{\partial^2 w_0}{\partial^2 x_1}+\frac{\partial w_0}{\partial x_1}\frac{\partial^2 w_0}{\partial^2 x_2}\right)+A_{12}\left(\frac{\partial^2 u_0}{\partial x_1 \partial x_2}+\frac{\partial w_0}{\partial x_1}\frac{\partial^2 w_0}{\partial x_1 \partial x_2}\right)+$$

$$+A_{22}\left(\frac{\partial^2 v_0}{\partial^2 x_2}+\frac{\partial w_0}{\partial x_2}\frac{\partial^2 w_0}{\partial^2 x_2}\right)-\left(\frac{\partial N_{12}^T}{\partial x_1}+\frac{\partial N_{22}^T}{\partial x_2}\right) = I_0 \ddot{v}_0 \Rightarrow$$

$$A_{66}\left(\frac{\partial^2 u_0}{\partial x_1 \partial x_2}+\frac{\partial^2 v_0}{\partial^2 x_1}+\cancel{\frac{\partial w_0}{\partial x_2}\frac{\partial^2 w_0}{\partial^2 x_1}}+\cancel{\frac{\partial w_0}{\partial x_1}\frac{\partial^2 w_0}{\partial^2 x_2}}\right)+A_{12}\left(\frac{\partial^2 u_0}{\partial x_1 \partial x_2}+\cancel{\frac{\partial w_0}{\partial x_1}\frac{\partial^2 w_0}{\partial x_1 \partial x_2}}\right)+$$

$$+A_{22}\left(\frac{\partial^2 v_0}{\partial^2 x_2}+\cancel{\frac{\partial w_0}{\partial x_2}\frac{\partial^2 w_0}{\partial^2 x_2}}\right)-\left(\cancel{\frac{\partial N_{12}^T}{\partial x_1}+\frac{\partial N_{22}^T}{\partial x_2}}\right) = I_0 \ddot{v}_0 \Rightarrow$$

$$G\,h\left(\frac{\partial^2 u_0}{\partial x_1 \partial x_2}+\frac{\partial^2 v_0}{\partial^2 x_1}\right) + \frac{v E h}{1-v^2}\frac{\partial^2 u_0}{\partial x_1 \partial x_2} + \frac{E h}{1-v^2}\frac{\partial^2 v_0}{\partial^2 x_2} = I_0 \ddot{v}_0 \Rightarrow$$

$$G\,\cancel{h}\left(\frac{\partial^2 u_0}{\partial x_1 \partial x_2}+\frac{\partial^2 v_0}{\partial^2 x_1}\right) + \frac{v E \cancel{h}}{1-v^2}\frac{\partial^2 u_0}{\partial x_1 \partial x_2} + \frac{E \cancel{h}}{1-v^2}\frac{\partial^2 v_0}{\partial^2 x_2} = \rho \cancel{h} \ddot{v}_0 \Rightarrow$$

$$\frac{E}{2(1+v)}\left(\frac{\partial^2 v_0}{\partial^2 x_1}+\frac{\partial^2 u_0}{\partial x_1 \partial x_2}\right) + \frac{E}{1-v^2}\left(\frac{\partial^2 v_0}{\partial^2 x_2}+v\frac{\partial^2 u_0}{\partial x_1 \partial x_2}\right) = \rho \ddot{v}_0$$

The above equation is identical to the Eq. (8).

From (6) of the section 4.5,

$$I_0 \ddot{w}_0 - I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right) - 2\frac{\partial^2 M_6}{\partial x_1 \partial x_2} - \frac{\partial^2 M_1}{\partial^2 x_1} - \frac{\partial^2 M_2}{\partial^2 x_2} - q = 0 \Rightarrow$$

$$I_0 \ddot{w}_0 - I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right) + 2\frac{I_2}{\rho}\frac{E}{1+v}\frac{\partial^4 w_0}{\partial^2 x_1 \partial^2 x_2} + \frac{E I_2}{\rho(1-v^2)}\left(\frac{\partial^4 w_0}{\partial^4 x_1} + v\frac{\partial^4 w_0}{\partial^2 x_1 \partial^2 x_2}\right) +$$

$$+ \frac{E I_2}{\rho(1-v^2)}\left(v\frac{\partial^4 w_0}{\partial^2 x_2 \partial^2 x_1} + \frac{\partial^4 w_0}{\partial^4 x_2}\right) = q \Rightarrow$$

$$I_0 \ddot{w}_0 - I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right) + \frac{E I_2}{\rho(1-v^2)}\left(\frac{\partial^4 w_0}{\partial^4 x_1}+\frac{\partial^4 w_0}{\partial^4 x_2}\right)+$$

$$+\left(2\frac{I_2}{\rho}\frac{E}{1+v} + \frac{E I_2}{\rho(1-v^2)}v + \frac{E I_2}{\rho(1-v^2)}v\right)\frac{\partial^4 w_0}{\partial^2 x_1 \partial^2 x_2} = q \Rightarrow$$

$$I_0 \ddot{w}_0 - I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right) + \frac{E I_2}{\rho(1-v^2)}\left(\frac{\partial^4 w_0}{\partial^4 x_1}+\frac{\partial^4 w_0}{\partial^4 x_2}\right)+$$

$$+\left(2\frac{I_2}{\rho}\frac{E}{1+v} + 2v\frac{E I_2}{\rho(1-v^2)}\right)\frac{\partial^4 w_0}{\partial^2 x_1 \partial^2 x_2} = q \Rightarrow$$

$$I_0\ddot{w}_0 - I_2\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}\right) + \frac{EI_2}{\rho(1-v^2)}\left(\frac{\partial^4 w_0}{\partial^4 x_1}+\frac{\partial^4 w_0}{\partial^4 x_2}\right)+$$

$$+\frac{2EI_2}{\rho}\left(\frac{1}{1+v}+\frac{v}{1-v^2}\right)\frac{\partial^4 w_0}{\partial^2 x_1\partial^2 x_2} = q \Rightarrow$$

$$I_0\ddot{w}_0 - I_2\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}\right) + \frac{EI_2}{\rho(1-v^2)}\left(\frac{\partial^4 w_0}{\partial^4 x_1}+\frac{\partial^4 w_0}{\partial^4 x_2}\right)+\frac{2EI_2}{\rho}\frac{1}{1-v^2}\frac{\partial^4 w_0}{\partial^2 x_1\partial^2 x_2} = q \Rightarrow$$

$$\rho h\ddot{w}_0 - \frac{\rho h^3}{12}\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}\right) + \frac{E}{\not{\rho}(1-v^2)}\frac{\not{\rho}h^3}{12}\left(\frac{\partial^4 w_0}{\partial^4 x_1}+\frac{\partial^4 w_0}{\partial^4 x_2}\right)+$$

$$+\frac{2E}{\not{\rho}}\frac{\not{\rho}h^3}{12}\frac{1}{1-v^2}\frac{\partial^4 w_0}{\partial^2 x_1\partial^2 x_2} = q \Rightarrow$$

$$\rho h\ddot{w}_0 - \frac{\rho h^3}{12}\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}\right) + \frac{E}{(1-v^2)}\frac{h^3}{12}\left(\frac{\partial^4 w_0}{\partial^4 x_1}+\frac{\partial^4 w_0}{\partial^4 x_2}\right)+\frac{h^3}{12}\frac{2E}{1-v^2}\frac{\partial^4 w_0}{\partial^2 x_1\partial^2 x_2} = q$$

or

$$\rho h\ddot{w}_0 - \frac{\rho h^3}{12}\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}\right) + \frac{E}{(1-v^2)}\frac{h^3}{12}\left(\frac{\partial^4 w_0}{\partial^4 x_1}+2\frac{\partial^4 w_0}{\partial^2 x_1\partial^2 x_2}+\frac{\partial^4 w_0}{\partial^4 x_2}\right) = q$$

or

$$\rho h\ddot{w}_0 - \frac{\rho h^3}{12}\Delta\ddot{w}_0 + \frac{Eh^3}{12}\frac{1}{1-v^2}\Delta^2 w_0 = q \qquad (9)$$

The last equation of motion of the vibrating plate [Eq. (9)], there is on the resources; https://en.wikipedia.org/wiki/Kirchhoff%E2%80%93Love_plate_theory as well as on the book of Reddy J.N. (2007) “Theory and Analysis of Elastic Plates and Shells”/ Chapter 3/ page 118.

However, there are some terms on the Eq. (3.8.5) of the page 118 of the book “Theory and Analysis of Elastic Plates and Shells”, which differ from our result, namely the Eq. (9). This is justified as follows.

Comparing Eq. (9), to our source, we notice that there are some additional terms due to the use of another strain field which takes into account the geometric nonlinearities i.e. small strains but moderate rotations of transverse normal of the mid-surface $(10^o\text{-}15^o)$ [*J.N. Reddy (2007) “Theory and Analysis of Elastic Plates and Shells”, page 98 – 99/ Chapter 3*]. These terms are gathered together as seems on the Eq. (3.4.14) of the page 104 of the book,

$$N(u_0,v_0,w_0) = \frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial w_0}{\partial x_1}+N_{12}\frac{\partial w_0}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial w_0}{\partial x_1}+N_{22}\frac{\partial w_0}{\partial x_2}\right).$$

To make the comparison easier, we present the Eq. (3.8.5) below,

$$-D_{11}\frac{\partial^4 w_0}{\partial^4 x_1} - 2\ D_{12}+2D_{66}\frac{\partial^4 w_0}{\partial^2 x_1\partial^2 x_2} - D_{22}\frac{\partial^4 w_0}{\partial^4 x_2} - kw_0 + N(u_0,v_0,w_0) -$$

$$-\left(\frac{\partial^2 M_{11}^T}{\partial^2 x_1}+2\frac{\partial^2 M_{12}^T}{\partial x_1 \partial x_2}+\frac{\partial^2 M_{22}^T}{\partial^2 x_2}\right) = I_0 \ddot{w}_0 - I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right) - q \qquad (3.8.5)$$

In addition as we referred above, the general "behavior" of the plate indicates that it must be regarded as orthotropic. Consequently, we take into account the bending stiffness coefficients of the page 111-112 of the reference book, which are given below,

$$D_{11} = \frac{E_1 h^3}{12(1-v_{12}v_{21})}, \qquad D_{12} = v_{21} D_{11},$$

$$D_{22} = \frac{E_2}{E_1} D_{11}, \qquad D_{66} = \frac{G_{12} h^3}{12}$$

However, the in-plane "behavior" (or alternatively the stress-strain relations between the in-plane displacements u_0, v_0) of the plate could be described as isotropic, as justified on the section 4.2.2. Thus, the previous coefficients could be written as,

$$D_{11} = \frac{E h^3}{12(1-v^2)}, \qquad D_{12} = \frac{v E h^3}{12(1-v^2)},$$

$$D_{22} = \frac{E h^3}{12(1-v^2)}, \qquad D_{66} = \frac{G h^3}{12}$$

Also, in the frames of the particular problem of the CPT, we assume that there are not at all thermal effect, so it is obvious that,

$$\frac{\partial^2 M_{11}^T}{\partial^2 x_1} = \frac{\partial^2 M_{12}^T}{\partial x_1 \partial x_2} = \frac{\partial^2 M_{22}^T}{\partial^2 x_2} = 0 \text{ and there is not elastic foundation, } k = 0.$$

Summing up the above notifications, the relation (3.8.5) is converted to the following,

$$-D_{11}\frac{\partial^4 w_0}{\partial^4 x_1} - 2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w_0}{\partial^2 x_1 \partial^2 x_2} - D_{22}\frac{\partial^4 w_0}{\partial^4 x_2} - \cancel{k w_0} + \cancel{N(u_0, v_0, w_0)} -$$

$$-\cancel{\left(\frac{\partial^2 M_{11}^T}{\partial^2 x_1}+2\frac{\partial^2 M_{12}^T}{\partial x_1 \partial x_2}+\frac{\partial^2 M_{22}^T}{\partial^2 x_2}\right)} = I_0 \ddot{w}_0 - I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right) - q \Rightarrow$$

$$-D_{11}\frac{\partial^4 w_0}{\partial^4 x_1} - 2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w_0}{\partial^2 x_1 \partial^2 x_2} - D_{22}\frac{\partial^4 w_0}{\partial^4 x_2} = I_0 \ddot{w}_0 - I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right) - q \Rightarrow$$

$$-\frac{E h^3}{12(1-v^2)}\frac{\partial^4 w_0}{\partial^4 x_1} - 2\left(\frac{v E h^3}{12(1-v^2)}+2\frac{G h^3}{12}\right)\frac{\partial^4 w_0}{\partial^2 x_1 \partial^2 x_2} -$$

$$-\frac{E h^3}{12(1-v^2)}\frac{\partial^4 w_0}{\partial^4 x_2} = I_0 \ddot{w}_0 - I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right) - q \Rightarrow$$

$$\rho h \ddot{w}_0 - \frac{\rho h^3}{12}\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}+\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right) + \frac{E h^3}{12(1-v^2)}\left(\frac{\partial^4 w_0}{\partial^4 x_1}+\frac{\partial^4 w_0}{\partial^4 x_2}\right) + \frac{E h^3}{12}\frac{2}{(1-v^2)}\frac{\partial^4 w_0}{\partial^2 x_1 \partial^2 x_2} = q$$

The last equation is exactly our result, namely Eq. (9).

7. Boundary Conditions of the CPT in terms of displacements

Following the same process, as exactly on the section 6, where we derive the equations of motion in terms of displacements, but at this moment to derive the boundary conditions in terms of displacements for both cases of the material of the plate.

As we have aforementioned on the conclusion of the section 4.3, we use the relations of the stress resultants [Eqs. (4) of the section 4.3] but now transformed to the curvilinear coordinate system on which the boundary conditions are derived, as shown below.

$$\begin{Bmatrix} N_{nn} \\ N_{ss} \\ N_{ns} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{nn} \\ \sigma_{ss} \\ \sigma_{ns} \end{Bmatrix} dz \qquad \begin{Bmatrix} M_{nn} \\ M_{ss} \\ M_{ns} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{nn} \\ \sigma_{ss} \\ \sigma_{ns} \end{Bmatrix} z\, dz \tag{1}$$

Further, substitute into them the relations of stresses in terms of the displacements [Eqs. (8) - (10) of the section 4.2.1 or Eqs. (5) and (5`) of the section 4.2.2] similarly in the curvilinear form, meaning to express the stress resultants in terms of the displacement field, which is defined for the coordinate system (n, s, z). Note that we deserve to proceed to the analogous stress-strain relations from the Cartesian to the curvilinear coordinate system, because we study an orthotropic material regarding the vertical dimension.

Thus, we get the following form of the boundary conditions of the problem of CPT, as seems on the sections 7.1 and 7.2 in case of an orthotropic in-plane anisotropic and orthotropic in-plane isotropic plate respectively.

For the sake of convenience, we present again the full set of the boundary conditions of the CPT in terms of thickness-integrated forces and moments of the section 5.3.

$$n_{x_1}^2 - n_{x_2}^2 \; N_{nn} + 2\, n_{x_1} n_{x_2} \; N_{ns} = a_{T0n} \frac{I_0}{\rho} \tag{1a}$$

$$n_{x_1}^2 - n_{x_2}^2 \; N_{ns} + 2\, n_{x_2} n_{x_1} \; N_{ss} = a_{T0s} \frac{I_0}{\rho} \tag{1b}$$

$$\begin{aligned} & I_2 \left\{ n_{x_1}^2 - n_{x_2}^2 \frac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_1} n_{x_2} \frac{\partial \ddot{w}_0}{\partial s} \right\} - c_{T0}(s) \frac{I_0}{\rho} - \\ & - n_{x_1}^2 - n_{x_2}^2 \; M_{nn,n} - 2\, n_{x_1} n_{x_2} M_{ns,n} + 2\; n_{x_2}^2 + 3 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2 \; M_{ns,s} + \\ & + 2 \frac{\partial}{\partial s} \; n_{x_1}^3 n_{x_2} \; M_{nn} + 2\, n_{x_1}^3 n_{x_2} M_{nn,s} + \frac{\partial}{\partial s} \; n_{x_2}^2 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2 \; M_{ns} - \\ & - 2 \frac{\partial}{\partial s} \; n_{x_1} n_{x_2} \; n_{x_1}^2 - n_{x_2}^2 \; M_{ss} - 4\; n_{x_2} n_{x_1}^3 M_{ss,s} = \\ & \qquad = \frac{I_2}{\rho} \frac{\partial}{\partial s} \{ (n_{x_1}^2 - n_{x_2}^2)\, a_{T1s} - 2\, n_{x_1} n_{x_2}\, a_{T1n} \} \end{aligned} \tag{1c}$$

$$\left(n_{x_1}^2 - n_{x_2}^2\right)^2 M_{nn} + 4\, n_{x_1}^2 n_{x_2}^2\, M_{ss} + 4 n_{x_1} n_{x_2} \left(n_{x_1}^2 - n_{x_2}^2\right) M_{ns} =$$
$$= \left(n_{x_2}^2 - n_{x_1}^2\right) a_{T1n} - 2\, n_{x_2} n_{x_1}\, a_{T1s} \quad \frac{I_2}{\rho} \tag{1d}$$

7.1. The Boundary Conditions of the CPT in terms of displacements for an orthotropic, in-plane anisotropic material

As for the natural boundary conditions, we follow the same process to get the equations of motions in terms of displacements, but now using the Eqs. (4a), (4b), (4c), (4d) of the section 5.3.

First, taking into account the Eqs. (8) - (10) of the section 4.2.1, the substitute into them the analogous components of the displacement field (u_{0n}, u_{0s}, w_0) in order to derive the stress field (σ_{nn}, σ_{ss}, σ_{ns}) applied on the curvilinear coordinate system.

$$\sigma_{nn} = \frac{E_n}{1-\nu_{ns}\nu_{sn}}\left(\frac{\partial u_{0n}}{\partial n} - z\frac{\partial^2 w_0}{\partial^2 n}\right) + \frac{\nu_{sn}E_n}{1-\nu_{ns}\nu_{sn}}\left(\frac{\partial u_{0s}}{\partial s} - z\frac{\partial^2 w_0}{\partial^2 s}\right) \tag{2a}$$

$$\sigma_{ss} = \frac{\nu_{ns}E_s}{1-\nu_{ns}\nu_{sn}}\left(\frac{\partial u_{0n}}{\partial n} - z\frac{\partial^2 w_0}{\partial^2 n}\right) + \frac{E_s}{1-\nu_{ns}\nu_{sn}}\left(\frac{\partial u_{0s}}{\partial s} - z\frac{\partial^2 w_0}{\partial^2 s}\right) \tag{2b}$$

$$\sigma_{ns} = \sigma_{sn} = G_{ns}\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} - 2z\frac{\partial^2 w_0}{\partial n\,\partial s}\right) \tag{2c}$$

where, (E_n, E_s) the modulus of elasticity on the directions n, s respectively and G_{ns} the shear modulus of elasticity. In addition, the Poisson's ratio ν_{ns} or ν_{sn} is an identity of the material referred to its planar directions, namely n and s-axis, defined as

$$\nu_{ns} = -\frac{e_{ss}}{e_{nn}} \quad \text{and} \quad \nu_{sn} = -\frac{e_{nn}}{e_{ss}}.$$

Substituting the Eqs. (2a) - (2c) into the Eqs. (1), we get

$$N_{nn} = \int_{-h/2}^{h/2} \sigma_{nn}\, dz =$$
$$= \int_{-h/2}^{h/2} \frac{E_n}{1-\nu_{ns}\nu_{sn}}\left(\frac{\partial u_{0n}}{\partial n} - z\frac{\partial^2 w_0}{\partial^2 n}\right) dz + \int_{-h/2}^{h/2} \frac{\nu_{sn}E_n}{1-\nu_{ns}\nu_{sn}}\left(\frac{\partial u_{0s}}{\partial s} - z\frac{\partial^2 w_0}{\partial^2 s}\right) dz =$$
$$= \frac{E_n}{\rho(1-\nu_{ns}\nu_{sn})}\left(\frac{\partial u_{0n}}{\partial n} I_0 - \cancel{I_1}\frac{\partial^2 w_0}{\partial^2 n}\right) + \frac{\nu_{sn}E_n}{\rho(1-\nu_{ns}\nu_{sn})}\left(\frac{\partial u_{0s}}{\partial s} I_0 - \cancel{I_1}\frac{\partial^2 w_0}{\partial^2 s}\right) =$$
$$= \frac{E_n}{\rho(1-\nu_{ns}\nu_{sn})}\frac{\partial u_{0n}}{\partial n} I_0 + \frac{\nu_{sn}E_n}{\rho(1-\nu_{ns}\nu_{sn})}\frac{\partial u_{0s}}{\partial s} I_0 \tag{3a}$$

$$N_{ss} = \int_{-h/2}^{h/2} \sigma_{ss}\, dz =$$

$$= \frac{\nu_{ns} E_s}{1-\nu_{ns}\nu_{sn}} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0n}}{\partial n} - z \frac{\partial^2 w_0}{\partial^2 n} \right) dz + \frac{E_s}{1-\nu_{ns}\nu_{sn}} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0s}}{\partial s} - z \frac{\partial^2 w_0}{\partial^2 s} \right) dz =$$

$$= \frac{\nu_{ns} E_s}{\rho(1-\nu_{ns}\nu_{sn})} \left(\frac{\partial u_{0n}}{\partial n} I_0 - \cancel{I_1} \frac{\partial^2 w_0}{\partial^2 n} \right) + \frac{E_s}{\rho(1-\nu_{ns}\nu_{sn})} \left(\frac{\partial u_{0s}}{\partial s} I_0 - \cancel{I_1} \frac{\partial^2 w_0}{\partial^2 s} \right) =$$

$$= \frac{\nu_{ns} E_s}{\rho(1-\nu_{ns}\nu_{sn})} \frac{\partial u_{0n}}{\partial n} I_0 + \frac{E_s}{\rho(1-\nu_{ns}\nu_{sn})} \frac{\partial u_{0s}}{\partial s} I_0 \qquad \text{(3b)}$$

$$N_{ns} = \int_{-h/2}^{h/2} \sigma_{ns}\, dz = G_{ns} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} - 2z \frac{\partial^2 w_0}{\partial n \partial s} \right) dz =$$

$$= \frac{G_{ns}}{\rho} \left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} \right) I_0 - 2 \frac{G_{ns}}{\rho} \cancel{I_1} \frac{\partial^2 w_0}{\partial n \partial s} = G_{ns} \left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} \right) \frac{I_0}{\rho} \qquad \text{(3c)}$$

$$M_{nn} = \int_{-h/2}^{h/2} \sigma_{nn}\, z\, dz =$$

$$= \int_{-h/2}^{h/2} \left[\frac{E_n}{1-\nu_{ns}\nu_{sn}} \left(\frac{\partial u_{0n}}{\partial n} z - z^2 \frac{\partial^2 w_0}{\partial^2 n} \right) + \frac{\nu_{sn} E_n}{1-\nu_{ns}\nu_{sn}} \left(\frac{\partial u_{0s}}{\partial s} z - z^2 \frac{\partial^2 w_0}{\partial^2 s} \right) \right] dz =$$

$$= \frac{E_n}{1-\nu_{ns}\nu_{sn}} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0n}}{\partial n} z - z^2 \frac{\partial^2 w_0}{\partial^2 n} \right) dz + \frac{\nu_{sn} E_n}{1-\nu_{ns}\nu_{sn}} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0s}}{\partial s} z - z^2 \frac{\partial^2 w_0}{\partial^2 s} \right) dz =$$

$$= \frac{E_n}{\rho(1-\nu_{ns}\nu_{sn})} \left(\frac{\partial u_{0n}}{\partial n} \cancel{I_1} - I_2 \frac{\partial^2 w_0}{\partial^2 n} \right) + \frac{\nu_{sn} E_n}{\rho(1-\nu_{ns}\nu_{sn})} \left(\frac{\partial u_{0s}}{\partial s} \cancel{I_1} - I_2 \frac{\partial^2 w_0}{\partial^2 s} \right) =$$

$$= -\frac{E_n}{\rho(1-\nu_{ns}\nu_{sn})} I_2 \frac{\partial^2 w_0}{\partial^2 n} - \frac{\nu_{sn} E_n}{\rho(1-\nu_{ns}\nu_{sn})} I_2 \frac{\partial^2 w_0}{\partial^2 s} \qquad \text{(3d)}$$

$$M_{ss} = \int_{-h/2}^{h/2} \sigma_{ss}\, z\, dz =$$

$$= \int_{-h/2}^{h/2} \left[\frac{\nu_{ns} E_s}{1-\nu_{ns}\nu_{sn}} \left(\frac{\partial u_{0n}}{\partial n} z - z^2 \frac{\partial^2 w_0}{\partial^2 n} \right) + \frac{E_s}{1-\nu_{ns}\nu_{sn}} \left(\frac{\partial u_{0s}}{\partial s} z - z^2 \frac{\partial^2 w_0}{\partial^2 s} \right) \right] dz =$$

$$= \frac{\nu_{ns} E_s}{1-\nu_{ns}\nu_{sn}} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0n}}{\partial n} z - z^2 \frac{\partial^2 w_0}{\partial^2 n} \right) dz + \frac{E_s}{1-\nu_{ns}\nu_{sn}} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0s}}{\partial s} z - z^2 \frac{\partial^2 w_0}{\partial^2 s} \right) dz =$$

$$= \frac{\nu_{ns} E_s}{\rho(1-\nu_{ns}\nu_{sn})}\left(\frac{\partial u_{0n}}{\partial n}\cancel{I_1} - I_2\frac{\partial^2 w_0}{\partial^2 n}\right) + \frac{E_s}{\rho(1-\nu_{ns}\nu_{sn})}\left(\frac{\partial u_{0s}}{\partial s}\cancel{I_1} - I_2\frac{\partial^2 w_0}{\partial^2 s}\right) =$$

$$= -\frac{\nu_{ns} E_s}{\rho(1-\nu_{ns}\nu_{sn})} I_2 \frac{\partial^2 w_0}{\partial^2 n} - \frac{E_s}{\rho(1-\nu_{ns}\nu_{sn})} I_2 \frac{\partial^2 w_0}{\partial^2 s} \tag{3e}$$

$$M_{ns} = \int_{-h/2}^{h/2} \sigma_{ns} z\,dz = G_{ns}\int_{-h/2}^{h/2}\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} - 2z\frac{\partial^2 w_0}{\partial n\,\partial s}\right) z\,dz =$$

$$= G_{ns}\int_{-h/2}^{h/2}\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n}\right) z\,dz - \int_{-h/2}^{h/2} 2\,G_{ns} z^2 \frac{\partial^2 w_0}{\partial n\,\partial s} dz =$$

$$= G_{ns}\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n}\right)\frac{\cancel{I_1}}{\rho} - 2\,G_{ns}\frac{I_2}{\rho}\frac{\partial^2 w_0}{\partial n\,\partial s} = -2\,G_{ns}\frac{I_2}{\rho}\frac{\partial^2 w_0}{\partial n\,\partial s} \tag{3f}$$

Note that on the above calculations the z-dependence of all the integrands is explicit as exactly the x_3-dependence on the Cartesian coordinate system, because as referred above the x_3, z axes are parallel during the transformation. Thus, the vertical integration can be performed explicitly and the "mass-moments" quantities are defined as those of the section 4.1. Consequently,

$$I_i = \int_{-h/2}^{h/2} \rho\, x_3^i\, dx_3 = \int_{-h/2}^{h/2} \rho\, z^i\, dz, \quad i = 0, 1, 2, ..., 6 \qquad \text{where,}$$

$$I_1 = I_3 = I_5 = 0 \qquad \text{and}$$

$$I_0 = \int_{-h/2}^{h/2} \rho\, dz = \rho h, \qquad I_2 = \int_{-h/2}^{h/2} \rho\, z^2\, dz = \rho\frac{h^3}{12},$$

$$I_4 = \int_{-h/2}^{h/2} \rho\, z^4\, dz = \rho\frac{h^5}{80}, \qquad I_6 = \int_{-h/2}^{h/2} \rho\, z^6\, dz = \rho\frac{h^7}{448}.$$

And finally we are ready to set the results of the Eqs. (3a) and (3c) into the first natural boundary condition (1a),

$$n_{x_1}^2 - n_{x_2}^2 \left(\frac{E_n}{\cancel{\rho}(1-\nu_{ns}\nu_{sn})}\frac{\partial u_{0n}}{\partial n}\cancel{I_0} + \frac{\nu_{sn} E_n}{\cancel{\rho}(1-\nu_{ns}\nu_{sn})}\frac{\partial u_{0s}}{\partial s}\cancel{I_0}\right) +$$

$$+ 2\,n_{x_1} n_{x_2}\, G_{ns}\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n}\right)\frac{\cancel{I_0}}{\cancel{\rho}} = a_{T0n}\frac{\cancel{I_0}}{\cancel{\rho}} \quad \Rightarrow$$

$$n_{x_1}^2 - n_{x_2}^2 \;\; \frac{E_n}{(1-\nu_{ns}\nu_{sn})}\left(\frac{\partial u_{0n}}{\partial n} + \nu_{sn}\frac{\partial u_{0s}}{\partial s}\right) + 2\, n_{x_1} n_{x_2}\, G_{ns}\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n}\right) = a_{T0n} \qquad \text{(4a)}$$

As for the second natural boundary condition, taking into account the results of the Eqs. (3b) and (3c), we get

$$n_{x_1}^2 - n_{x_2}^2 \;\; G_{ns}\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n}\right)\cancel{\frac{I_0}{\rho}} \; +$$

$$+\; 2\, n_{x_2} n_{x_1}\left(\frac{\nu_{ns} E_s}{\cancel{\rho}(1-\nu_{ns}\nu_{sn})}\frac{\partial u_{0n}}{\partial n}\cancel{I_0} + \frac{E_s}{\cancel{\rho}(1-\nu_{ns}\nu_{sn})}\frac{\partial u_{0s}}{\partial s}\cancel{I_0}\right) = a_{T0s}\cancel{\frac{I_0}{\rho}} \;\Rightarrow$$

$$n_{x_1}^2 - n_{x_2}^2 \;\; G_{ns}\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n}\right) + 2\, n_{x_2} n_{x_1}\frac{E_s}{(1-\nu_{ns}\nu_{sn})}\left(\nu_{ns}\frac{\partial u_{0n}}{\partial n} + \frac{\partial u_{0s}}{\partial s}\right) = a_{T0s} \qquad \text{(4b)}$$

Further, substituting the appropriate from the above relations

$$I_2\left\{ n_{x_1}^2 - n_{x_2}^2 \frac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_1} n_{x_2}\frac{\partial \ddot{w}_0}{\partial s}\right\} - c_{T0}(s)\frac{I_0}{\rho} \; +$$

$$+\; n_{x_1}^2 - n_{x_2}^2 \;\frac{E_n I_2}{\rho(1-\nu_{ns}\nu_{sn})}\frac{\partial}{\partial n}\left\{\frac{\partial^2 w_0}{\partial^2 n} + \nu_{sn}\frac{\partial^2 w_0}{\partial^2 s}\right\} +$$

$$+\; 4\, n_{x_1} n_{x_2}\frac{G_{ns} I_2}{\rho}\frac{\partial}{\partial n}\left\{\frac{\partial^2 w_0}{\partial n \partial s}\right\} - 4\; n_{x_2}^2 + 3 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2 \;\frac{G_{ns} I_2}{\rho}\frac{\partial}{\partial s}\left\{\frac{\partial^2 w_0}{\partial n \partial s}\right\} -$$

$$-\; 2\frac{\partial}{\partial s}\{n_{x_1}^3 n_{x_2}\}\frac{E_n I_2}{\rho(1-\nu_{ns}\nu_{sn})}\left(\frac{\partial^2 w_0}{\partial^2 n} + \nu_{sn}\frac{\partial^2 w_0}{\partial^2 s}\right) - 2\, n_{x_1}^3 n_{x_2}\frac{E_n I_2}{\rho(1-\nu_{ns}\nu_{sn})}\frac{\partial}{\partial s}\left\{\frac{\partial^2 w_0}{\partial^2 n} + \nu_{sn}\frac{\partial^2 w_0}{\partial^2 s}\right\} +$$

$$+\; \frac{\partial}{\partial s}\{n_{x_2}^2 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2\} M_{ns} - 2\frac{\partial}{\partial s}\{n_{x_1} n_{x_2}(n_{x_1}^2 - n_{x_2}^2)\} M_{ss} \; +$$

$$+\; 4\, n_{x_2} n_{x_1}^3 \frac{E_s I_2}{\rho(1-\nu_{ns}\nu_{sn})}\frac{\partial}{\partial s}\left\{\nu_{ns}\frac{\partial^2 w_0}{\partial^2 n} + \frac{\partial^2 w_0}{\partial^2 s}\right\} = \frac{I_2}{\rho}\frac{\partial}{\partial s}\{(n_{x_1}^2 - n_{x_2}^2)\, a_{T1s} - 2\, n_{x_1} n_{x_2}\, a_{T1n}\} \qquad \Rightarrow$$

(and after performing the derivations we get the final form of the boundary condition)

$$
\begin{aligned}
& I_2 \left\{ n_{x_1}^2 - n_{x_2}^2 \frac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_1} n_{x_2} \frac{\partial \ddot{w}_0}{\partial s} \right\} + \\
& + n_{x_1}^2 - n_{x_2}^2 \frac{E_n I_2}{\rho(1-\nu_{ns}\nu_{sn})} \left\{ \frac{\partial^3 w_0}{\partial^3 n} + \nu_{sn} \frac{\partial^3 w_0}{\partial n \partial^2 s} \right\} + \\
& + 4\, n_{x_1} n_{x_2} \frac{G_{ns} I_2}{\rho} \frac{\partial^3 w_0}{\partial^2 n \partial s} - 4\, n_{x_2}^2 + 3 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2 \frac{G_{ns} I_2}{\rho} \frac{\partial^3 w_0}{\partial n \partial^2 s} - \\
& - 2 \frac{\partial}{\partial s} \{ n_{x_1}^3 n_{x_2} \} \frac{E_n I_2}{\rho(1-\nu_{ns}\nu_{sn})} \left(\frac{\partial^2 w_0}{\partial^2 n} + \nu_{sn} \frac{\partial^2 w_0}{\partial^2 s} \right) - \\
& - 2\, n_{x_1}^3 n_{x_2} \frac{E_n I_2}{\rho(1-\nu_{ns}\nu_{sn})} \left\{ \frac{\partial^3 w_0}{\partial s \partial^2 n} + \nu_{sn} \frac{\partial^3 w_0}{\partial^3 s} \right\} + \\
& + \frac{\partial}{\partial s} \{ n_{x_2}^2 + 6 n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2 \} M_{ns} - 2 \frac{\partial}{\partial s} \{ n_{x_1} n_{x_2} (n_{x_1}^2 - n_{x_2}^2) \} M_{ss} + \\
& + 4\, n_{x_2} n_{x_1}^3 \frac{E_s I_2}{\rho(1-\nu_{ns}\nu_{sn})} \left\{ \nu_{ns} \frac{\partial^3 w_0}{\partial s \partial^2 n} + \frac{\partial^3 w_0}{\partial^2 s} \right\} = \\
& = \frac{I_2}{\rho} \frac{\partial}{\partial s} \{ (n_{x_1}^2 - n_{x_2}^2) a_{T1s} - 2\, n_{x_1} n_{x_2} a_{T1n} \} + c_{T0}(s) \frac{I_0}{\rho}
\end{aligned}
\tag{4c}
$$

Last but not least, the substitution of the relations (3d), (3e) and (3f) into the Eqs. (1d) results to the following form of the fourth boundary condition,

$$
\begin{aligned}
& n_{x_1}^2 - n_{x_2}^2 {}^2 \frac{E_n}{1-\nu_{ns}\nu_{sn}} \left(\frac{\partial^2 w_0}{\partial^2 n} + \nu_{sn} \frac{\partial^2 w_0}{\partial^2 s} \right) + \\
& + 4\, n_{x_1}^2 n_{x_2}^2 \frac{E_s}{1-\nu_{ns}\nu_{sn}} \left(\nu_{ns} \frac{\partial^2 w_0}{\partial^2 n} + \frac{\partial^2 w_0}{\partial^2 s} \right) + \\
& + 8\, n_{x_1} n_{x_2}\, n_{x_1}^2 - n_{x_2}^2\, G_{ns} \frac{\partial^2 w_0}{\partial n \partial s} = n_{x_1}^2 - n_{x_2}^2\, a_{T1n} + 2\, n_{x_2} n_{x_1} a_{T1s}
\end{aligned}
\tag{4d}
$$

7.2. The boundary conditions of the problem of CPT in terms of displacements for an orthotropic, in-plane isotropic material

In the context of the section, we follow exactly the same process as shown on the previous section 7.1 the only difference is the form of the displacement field.

Thus, taking into account the Eqs. (5) and (5`) of the section 4.2.2, the components of the displacement field (σ_{nn}, σ_{ss}, σ_{ns}) for an in-plane isotropic plate, are

$$\sigma_{nn} = \frac{E}{1-v^2}\frac{\partial u_{0n}}{\partial n} - z\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 n} + \frac{vE}{1-v^2}\frac{\partial u_{0s}}{\partial s} - z\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 s} \tag{1a}$$

$$\sigma_{ss} = \frac{vE}{1-v^2}\frac{\partial u_{0n}}{\partial n} - z\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 n} + \frac{E}{1-v^2}\frac{\partial u_{0s}}{\partial s} - z\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 s} \tag{1b}$$

$$\sigma_{ns} = \sigma_{sn} = G\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} - 2z\frac{\partial^2 w_0}{\partial n\,\partial s}\right) \tag{1c}$$

Subsequently, we set the Eqs. (18)- (20) to the Eqs. (13) of the section 7.3,

$$N_{nn} = \int_{-h/2}^{h/2}\left(\frac{E}{1-v^2}\frac{\partial u_{0n}}{\partial n} - z\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 n} + \frac{vE}{1-v^2}\frac{\partial u_{0s}}{\partial s} - z\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 s}\right)dz =$$
$$= \frac{E}{1-v^2}\frac{\partial u_{0n}}{\partial n}\frac{I_0}{\rho} - \frac{\cancel{I_1}}{\rho}\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 n} + \frac{vE}{1-v^2}\frac{\partial u_{0s}}{\partial s}\frac{I_0}{\rho} - \frac{\cancel{I_1}}{\rho}\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 s} =$$
$$= \frac{E}{1-v^2}\frac{\partial u_{0n}}{\partial n}\frac{I_0}{\rho} + \frac{vE}{1-v^2}\frac{\partial u_{0s}}{\partial s}\frac{I_0}{\rho} \tag{2a}$$

$$N_{ss} = \int_{-h/2}^{h/2}\left(\frac{vE}{1-v^2}\frac{\partial u_{0n}}{\partial n} - z\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 n} + \frac{E}{1-v^2}\frac{\partial u_{0s}}{\partial s} - z\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 s}\right)dz =$$
$$= \int_{-h/2}^{h/2}\left(\frac{vE}{1-v^2}\frac{\partial u_{0n}}{\partial n}\frac{I_0}{\rho} - \frac{I_1}{\rho}\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 n} + \frac{E}{1-v^2}\frac{\partial u_{0s}}{\partial s}\frac{I_0}{\rho} - \frac{I_1}{\rho}\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 s}\right)dz =$$
$$= \frac{vE}{1-v^2}\frac{\partial u_{0n}}{\partial n}\frac{I_0}{\rho} - \frac{\cancel{I_1}}{\rho}\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 n} + \frac{E}{1-v^2}\frac{\partial u_{0s}}{\partial s}\frac{I_0}{\rho} - \frac{\cancel{I_1}}{\rho}\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 s} =$$
$$= \frac{vE}{1-v^2}\frac{\partial u_{0n}}{\partial n}\frac{I_0}{\rho} + \frac{E}{1-v^2}\frac{\partial u_{0s}}{\partial s}\frac{I_0}{\rho} \tag{2b}$$

$$N_{ns} = \int_{-h/2}^{h/2} G\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} - 2z\frac{\partial^2 w_0}{\partial n \partial s}\right)dz = G\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n}\right)\frac{I_0}{\rho} - 2G\frac{\cancel{I_1}}{\rho}\frac{\partial^2 w_0}{\partial n \partial s} =$$

$$= G\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n}\right)\frac{I_0}{\rho} \tag{2c}$$

$$M_{nn} = \int_{-h/2}^{h/2}\left(\frac{E}{1-v^2}\frac{\partial u_{0n}}{\partial n}z - z^2\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 n} + \frac{vE}{1-v^2}\frac{\partial u_{0s}}{\partial s}z - z^2\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 s}\right)dz =$$

$$= \frac{E}{1-v^2}\frac{\partial u_{0n}}{\partial n}\frac{\cancel{I_1}}{\rho} - \frac{I_2}{\rho}\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 n} + \frac{vE}{1-v^2}\frac{\partial u_{0s}}{\partial s}\frac{\cancel{I_1}}{\rho} - \frac{I_2}{\rho}\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 s} =$$

$$= -\frac{I_2}{\rho}\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 n} - \frac{I_2}{\rho}\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 s} \tag{2d}$$

$$M_{ss} = \int_{-h/2}^{h/2}\left(\frac{vE}{1-v^2}\frac{\partial u_{0n}}{\partial n}z - z^2\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 n} + \frac{E}{1-v^2}\frac{\partial u_{0s}}{\partial s}z - z^2\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 s}\right)dz =$$

$$= \frac{vE}{1-v^2}\frac{\partial u_{0n}}{\partial n}\frac{\cancel{I_1}}{\rho} - \frac{I_2}{\rho}\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 n} + \frac{E}{1-v^2}\frac{\partial u_{0s}}{\partial s}\frac{\cancel{I_1}}{\rho} - \frac{I_2}{\rho}\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 s} =$$

$$= -\frac{I_2}{\rho}\frac{vE}{1-v^2}\frac{\partial^2 w_0}{\partial^2 n} - \frac{I_2}{\rho}\frac{E}{1-v^2}\frac{\partial^2 w_0}{\partial^2 s} \tag{2e}$$

$$M_{ns} = G\int_{-h/2}^{h/2}\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n}z - 2z^2\frac{\partial^2 w_0}{\partial n \partial s}\right)dz = \left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n}\right)\frac{I_1}{\rho} - 2G\frac{I_2}{\rho}\frac{\partial^2 w_0}{\partial n \partial s} =$$

$$= -2G\frac{I_2}{\rho}\frac{\partial^2 w_0}{\partial n \partial s} \tag{2f}$$

Note that the above relations of the stress-resultants (2a)-(2f), are also found on a similar form on the paper of BO Haggblad and Klaus-Jurgen Bathe (1990), "Specifications of Boundary Conditions for Reissner/ Mindlin Plate Bending Finite Elements".

To the end of the above, we substitute the Eqs. (2a)-(2f) into the Eqs. (4a), (4b), (4c) and (4d) of the section 5.3 or Eqs. (1a), (1b), (1c), (1d) of the section 7, namely the boundary conditions and we derive the following results.

The first boundary condition is converted to,

$$n_{x_1}^2 - n_{x_2}^2\left(\frac{E}{1-v^2}\frac{\partial u_{0n}}{\partial n}\frac{I_0}{\rho} + \frac{vE}{1-v^2}\frac{\partial u_{0s}}{\partial s}\frac{I_0}{\rho}\right) + 2n_{x_1}n_{x_2}G\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n}\right)\frac{I_0}{\rho} = a_{T0n}\frac{I_0}{\rho} \Rightarrow$$

$$n_{x_1}^2 - n_{x_2}^2\frac{E}{1-v^2}\left(\frac{\partial u_{0n}}{\partial n} + v\frac{\partial u_{0s}}{\partial s}\right) + 2n_{x_1}n_{x_2}G\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n}\right) = a_{T0n} \tag{3a}$$

The second natural boundary condition, due to the Eqs. (2b) and (2c)

$$n_{x_1}^2 - n_{x_2}^2 \;\; G\left(\frac{\partial u_{0n}}{\partial s}+\frac{\partial u_{0s}}{\partial n}\right)\frac{I_0}{\rho} + 2\, n_{x_2} n_{x_1}\left(\frac{\nu E}{1-\nu^2}\frac{\partial u_{0n}}{\partial n}\frac{I_0}{\rho}+\frac{E}{1-\nu^2}\frac{\partial u_{0s}}{\partial s}\frac{I_0}{\rho}\right) = a_{T0s}\frac{I_0}{\rho} \Rightarrow$$

$$n_{x_1}^2 - n_{x_2}^2 \;\; G\left(\frac{\partial u_{0n}}{\partial s}+\frac{\partial u_{0s}}{\partial n}\right) + 2\, n_{x_2} n_{x_1}\frac{E}{1-\nu^2}\left(\nu\frac{\partial u_{0n}}{\partial n}+\frac{\partial u_{0s}}{\partial s}\right) = a_{T0s} \qquad (3b)$$

As for the third boundary condition, regarding the appropriate terms from the Eqs. (3c), (2e), (2f),

$$I_2\left\{ n_{x_1}^2 - n_{x_2}^2 \frac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_1} n_{x_2}\frac{\partial \ddot{w}_0}{\partial s}\right\} - c_{T0}(s)\frac{I_0}{\rho} +$$

$$+\; n_{x_1}^2 - n_{x_2}^2 \frac{EI_2}{\rho(1-\nu^2)}\frac{\partial}{\partial n}\left\{\frac{\partial^2 w_0}{\partial^2 n}+\nu\frac{\partial^2 w_0}{\partial^2 s}\right\} + 4\frac{GI_2}{\rho} n_{x_1} n_{x_2}\frac{\partial}{\partial n}\left\{\frac{\partial^2 w_0}{\partial n\partial s}\right\} -$$

$$-\; 4\;\; n_{x_2}^2 + 3n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2 \;\; \frac{GI_2}{\rho}\frac{\partial}{\partial s}\left\{\frac{\partial^2 w_0}{\partial n\partial s}\right\} -$$

$$-\; 2\frac{\partial}{\partial s}\{n_{x_1}^3 n_{x_2}\}\frac{EI_2}{\rho(1-\nu^2)}\left(\frac{\partial^2 w_0}{\partial^2 n}+\nu\frac{\partial^2 w_0}{\partial^2 s}\right) - 2\, n_{x_1}^3 n_{x_2}\frac{EI_2}{\rho(1-\nu^2)}\frac{\partial}{\partial s}\left\{\frac{\partial^2 w_0}{\partial^2 n}+\nu\frac{\partial^2 w_0}{\partial^2 s}\right\} +$$

$$+\; \frac{\partial}{\partial s}(n_{x_2}^2 + 6n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2) M_{ns} \; - 2\frac{\partial}{\partial s}\{n_{x_1} n_{x_2}(n_{x_1}^2 - n_{x_2}^2)\} M_{ss} \; +$$

$$+\; 4\; n_{x_2} n_{x_1}^3 \frac{EI_2}{\rho(1-\nu^2)}\frac{\partial}{\partial s}\left\{\nu\frac{\partial^2 w_0}{\partial^2 n}+\frac{\partial^2 w_0}{\partial^2 s}\right\} = \frac{I_2}{\rho}\frac{\partial}{\partial s}\left\{(n_{x_1}^2 - n_{x_2}^2)\, a_{T1s} - 2\, n_{x_1} n_{x_2}\, a_{T1n}\right\} \Rightarrow$$

$$I_2\left\{ n_{x_1}^2 - n_{x_2}^2 \frac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_1} n_{x_2}\frac{\partial \ddot{w}_0}{\partial s}\right\} +$$

$$+\; n_{x_1}^2 - n_{x_2}^2 \frac{EI_2}{\rho(1-\nu^2)}\left\{\frac{\partial^3 w_0}{\partial^3 n}+\nu\frac{\partial^3 w_0}{\partial n\partial^2 s}\right\} + 4\frac{GI_2}{\rho} n_{x_1} n_{x_2}\left\{\frac{\partial^3 w_0}{\partial^2 n\partial s}\right\} -$$

$$-\; 4\;\; n_{x_2}^2 + 3n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2 \;\; \frac{GI_2}{\rho}\left\{\frac{\partial^3 w_0}{\partial n\partial^2 s}\right\} -$$

$$-\; 2\frac{\partial}{\partial s}\{n_{x_1}^3 n_{x_2}\}\frac{EI_2}{\rho(1-\nu^2)}\left(\frac{\partial^2 w_0}{\partial^2 n}+\nu\frac{\partial^2 w_0}{\partial^2 s}\right) - 2\, n_{x_1}^3 n_{x_2}\frac{EI_2}{\rho(1-\nu^2)}\left\{\frac{\partial^3 w_0}{\partial s\partial^2 n}+\nu\frac{\partial^3 w_0}{\partial^3 s}\right\} + \qquad (3c)$$

$$+\; \frac{\partial}{\partial s}\{n_{x_2}^2 + 6n_{x_1}^2 n_{x_2}^2 - n_{x_1}^2\} M_{ns} \; - 2\frac{\partial}{\partial s}\{n_{x_1} n_{x_2}(n_{x_1}^2 - n_{x_2}^2)\} M_{ss} \; +$$

$$+\; 4\; n_{x_2} n_{x_1}^3 \frac{EI_2}{\rho(1-\nu^2)}\left\{\nu\frac{\partial^3 w_0}{\partial s\partial^2 n}+\frac{\partial^3 w_0}{\partial^3 s}\right\} =$$

$$= \frac{I_2}{\rho}\frac{\partial}{\partial s}\{(n_{x_1}^2 - n_{x_2}^2)\, a_{T1s} \; - 2\, n_{x_1} n_{x_2}\, a_{T1n}\} + c_{T0}(s)\frac{I_0}{\rho}$$

Finally, taking into account the Eqs. (2c), (2d), (2e) the Eq. (4d) is converted to

$$
\begin{aligned}
&-\left(n_{x_1}^2-n_{x_2}^2\right)^2\frac{EI_2}{\rho(1-\nu^2)}\left(\frac{\partial^2 w_0}{\partial^2 n}+\nu\frac{\partial^2 w_0}{\partial^2 s}\right)-4\,n_{x_1}^2 n_{x_2}^2\frac{EI_2}{\rho(1-\nu^2)}\left(\nu\frac{\partial^2 w_0}{\partial^2 n}+\frac{\partial^2 w_0}{\partial^2 s}\right)-\\
&-8n_{x_1}n_{x_2}\left(n_{x_1}^2-n_{x_2}^2\right)G\frac{I_2}{\rho}\frac{\partial^2 w_0}{\partial n\partial s}=\left[\left(n_{x_2}^2-n_{x_1}^2\right)a_{T1n}-2\,n_{x_2}n_{x_1}a_{T1s}\right]\frac{I_2}{\rho}\Rightarrow\\
&\left(n_{x_1}^2-n_{x_2}^2\right)^2\frac{E}{1-\nu^2}\left(\frac{\partial^2 w_0}{\partial^2 n}+\nu\frac{\partial^2 w_0}{\partial^2 s}\right)+4\,n_{x_1}^2 n_{x_2}^2\frac{E}{1-\nu^2}\left(\nu\frac{\partial^2 w_0}{\partial^2 n}+\frac{\partial^2 w_0}{\partial^2 s}\right)+\\
&+8n_{x_1}n_{x_2}\left(n_{x_1}^2-n_{x_2}^2\right)G\frac{\partial^2 w_0}{\partial n\partial s}=\left(n_{x_1}^2-n_{x_2}^2\right)a_{T1n}+2\,n_{x_2}n_{x_1}a_{T1s}
\end{aligned}
\tag{3d}
$$

In conclusion, we have managed to remedy the aforementioned inconsistency of the system of equations and unknown quantities, gathering finally the four natural boundary conditions (4a)-(4d) of the section 7.1 or (3a)-(3d) of the section 7.2 to solve the system of fourth-order partial differential equations (7)-(9) of the section 6.1 or (7)-(9) of the section 6.2 respectively.

8. Conclusions

8.1. Functional Spaces

In conclusion, it's meaningful to refer and verify the functional space in which the action functional of the Hamilton's Principle is defined.

We note that the equations of motion (1) - (3) of the section 6, are expressed in terms of the displacement (u_0, v_0, w_0) and they contain second- order derivatives of u_0, v_0, t and fourth-order spatial derivatives of w_0. Consequently, the CPT is said to be eight – order plate theory, because the total spatial differential order of the equations of motion, is eight [*Reddy J.N. 2007, "Theory and Analysis of Elastic Plates and Shells", Chap.3 & Mitchell Griffiths 1980, "The Finite Difference Method in Partial Differential Equations" J. Wiley & Sons, Chap.2*]. Thus, the functional space in which the displacement field is defined has to include up to fourth – order spatial and second – order time derivatives.

As for the boundary of the domain of virtual displacements, the equations (4`) – (8`) of the sections 6.1 and 6.2 highlight the need of a boundary equipped with at least third – order spatial derivatives (because of the existence of third - order derivative of w_0).

Consequently, inside the volume $B \in R^3$, must be defined at least the fourth spatial derivatives of $\boldsymbol{u}$ (C^4- continuity) and its second-time derivatives (C^2- continuity). This means the existence and the continuity of the fourth spatial and second time derivatives of $\boldsymbol{u}$.

Upon the boundary ∂B, which encloses the space B, we demand the existence and continuity up to the third spatial derivatives of $\boldsymbol{u}$ (C^3- continuity).
Thus, the action functional $S = S[\boldsymbol{u}(\cdot,\cdot)]$ is defined on the space of admissible functions $C^2 \ [t_1,t_2] \rightarrow Y$, where Y is the functional space $Y = \ \boldsymbol{u} \in C^4(B) \cap C^3(\bar{B})$,
while the admissible variations $\delta\boldsymbol{u}$ belong to the space $C^2 \ [t_1,t_2] \rightarrow A$, where A is a functional space $A = \ \delta\boldsymbol{u} \in C^4(B) \cap C^3(\bar{B}) : \delta\boldsymbol{u}(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \partial B$.

In addition to the above, note that $\bar{B} = B \cup \partial B$, is the reference domain $\bar{B}$, which consists of the open set B [interior of B or $cl(B)$] and its boundary ∂B.

PART B:

THE THIRD-ORDER SHEAR DEFORMATION THEORY OF PLATES

1. Introduction

1.1. Definition of the meaning of « Plate » in the context of the specific quotation.

Consider now again a plate of planar dimensions a and b, length and breadth respectively and thickness h (or height or depth). The previous presented model of CPT, give satisfactory results in case of thin plate. Consequently, it must be the need of a better and more accurate plate theory in order to model the kinematics of a **thick** plate, namely moderately thick plate ($0.05 \leq h/L \leq 0.1$) and very thick plates ($h/L \geq 0.1$) [*Reddy J.N. (2007), "Theory and Analysis of Elastic Plates and Shells, Chap. 10.1.1*], because the CPT underpredicts the deflections of a thick plate. To overcome this problem, we formulate higher-order plate theories from which the most popular are the first-order shear deformation plate theory (FSDT), also called as Reissner/Mindlin Plate Theory and the third-order shear deformation plate theory (TSDT), also known as Levinson's Plate Theory.

On the context of this quotation, we choose the TSDT to describe the kinematics of the thick plate, because the expansion of the displacements up to the cubic term in the thickness coordinate (as shown on the next sections) is to gain quadratic variation of the transverse shear strains and stresses through the plate thickness. By this way is approximated the exact distribution of the transverse shear stresses to the thickness of the plate and consequently is avoided the need for shear correction coefficients used in the FSDT [*Reddy J.N. (2007), "Theory and Analysis of Elastic Plates and Shells", Chapter 10.1.3*]. The exact and the assumed distribution of stresses is illustrated on the following figure.

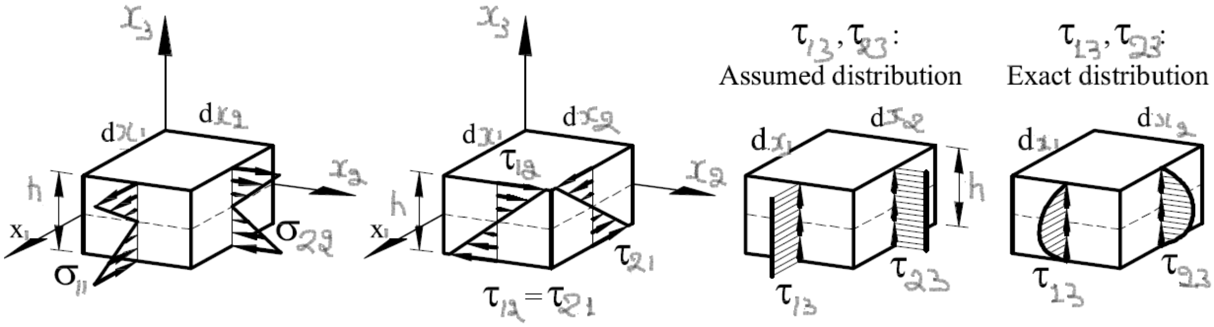

Figure 1: Shear correction factor and stresses along the edge of the plate

Complementary, note that the Reissner/ Mindlin plate theory is used sometimes for thin plates situations. However, this model suffer from the so called "shear –locking" defect, which influence the numerical solution of the problem. This defect is corrected by specific methods, which are not going to occupy us on the context of this text. [*Onãte E. (2013), "Structural Analysis with the Finite Element Method. Linear Statics: Beams, Plates and Shells", Chapter 6.1, 6.4*].

In order to express and develop the relations of the displacement field of the plate, we assume that the reference system of axes (here: the Cartesian coordinate system $Ox_1x_2x_3$ as seems on the **Figure 2** below), namely the origin of axes, is located on the middle plane of the plate. This plane will be called as mid-surface on the next sections and it is regarded as the reference plane for deriving the kinematic equations of the plate. Also we assume that the mid-surface is equidistant from the upper and lower surface of the plate, which means that each point upon the plate is described by zero vertical coordinate $x_3 = 0$.

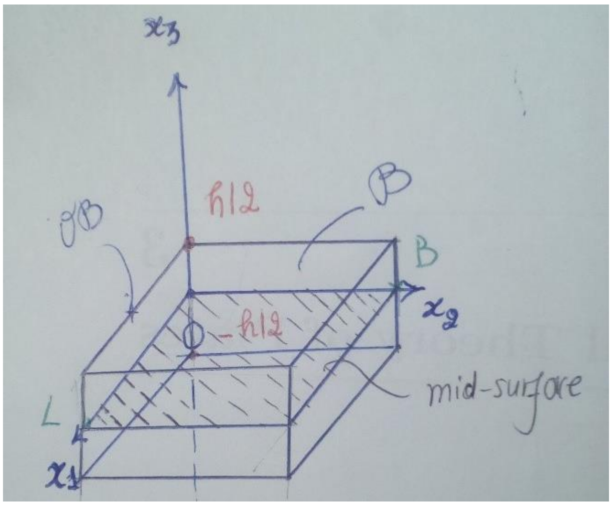

Figure 2: Cartesian coordinates and general notation as for the plate

As for the material of the plate under consideration, we are going to model two kind of structural materials separately on the following appropriate sections. The first one is the orthotropic but in-plane anisotropic plate and the second one is the orthotropic but in-plane isotropic plate. Note that the generally isotropic plate, namely with the same material properties on each direction, has no practical use on the structures and structural elements, because the stiffness and the material properties of an element (here: the plate) are purposely created different form one to another direction and proportional to the loading condition on which the plate is submitted.

1.2. Important Assumptions of the TSDT

On this quotation the assumptions of the straightness and normality of the transverse normal after deformation are neglected [*Reddy (2007), "Theory and Analysis of Elastic Plates and Shells", Chap. 10.3*].

1.2.1. The straightness assumption

In contrast to the CPT, the TSDT relaxes the kinematic hypothesis of CPT by removing the straightness assumption, i.e. straight normal to the middle plane before deformation may become cubic curves after deformation.

1.2.2. The inextensibility assumption

Although a fundamental assumption, which is conserved unchanged from the CPT, is the inextensibility of the cross section of the plate. Consequently the transverse normals do not experience elongation. In view of the small thickness of the plate, the vertical movement of any point of the plate is identical to that of the point of middle surface [*Reddy 2007, "Theory and Analysis of Elastic Plates and Shells, Chap. 3.2*].

1.2.3. The normality assumption

In addition, the TSDT relaxes the kinematic hypothesis by removing the normality restriction (as well as the FSDT or Reissner- Mindlin Plate Theory do generally) and allowing for arbitrary but constant rotation of transverse normals in comparison with the permanently transverse normals of the CPT.

The consequences on the mathematical modelling of the problem of TSDT of the three above assumptions ("no normality", "no straightness" and "inextensibility") are analyzed on the below paragraph 1.3.

1.3. Consequences of Levinson's Assumptions

Initially, these consequences are illustrated more schematically on the following figure. Shown also the main differences between the various kinds of plate theories and the way passing from the simplest (CPT) to a more complicated mathematical modeling of plates (TSDT). Thus, Figure 3 illustrates the following,

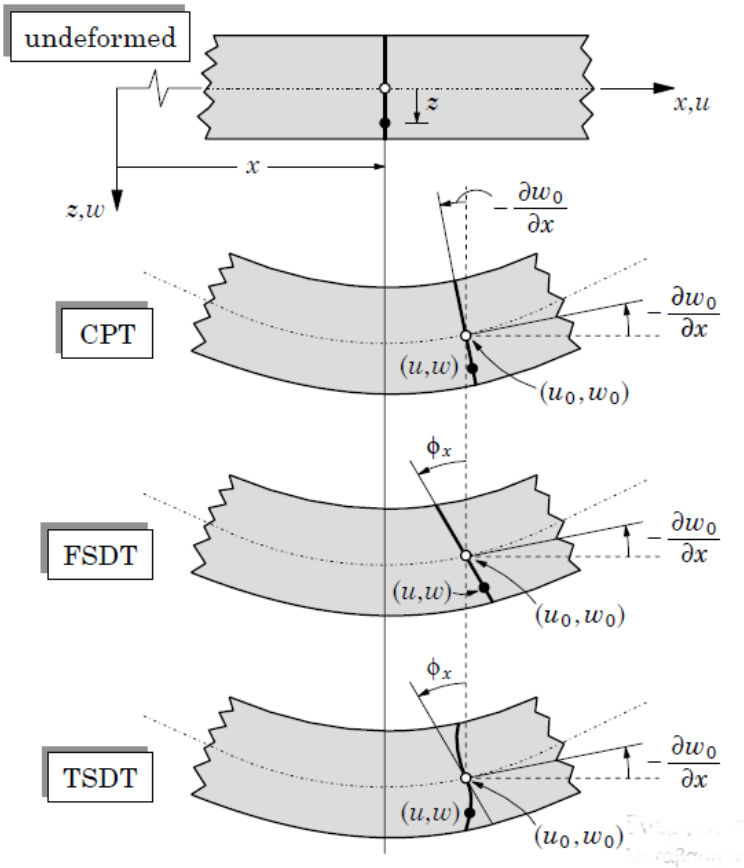

Figure 3: Deformation of the section of a plate according to the CPT, first-order deformation theory and third-order plate theories

1.3.1. The straightness of the cross-section

The lack of straightness assumption of the transverse normals after deformation, means that we are able to expand the displacements (u, v, w) as cubic functions along the thickness coordinate. Thus, as will see on the sequel, we are entitled to keep up to third-order terms on the Taylor expansion.

1.3.2. The inextensibility of the cross-section

As for the inextensibility of the cross section, we have to refer that the structures are usually composed of stiff materials. Consequently, the transverse displacement is independent of the vertical coordinate. This assumption is conceptually the inextensibility of the cross section.
Thus,

$$\frac{\partial w}{\partial x_3} = 0 \Rightarrow w(\boldsymbol{x};t) = w(x_1,x_2;t) = w_0(x_1,x_2;t) \tag{1}$$

This assumption remains valid even for a thick plate, because the slenderness ratio of the plate h/L is relatively small by the definition of the plate structure. Besides, this qualification is the essential difference of a plate from another structural element, such as solid cubes and cylinders.

1.3.3. The normality of the cross-section

The absence of the normality assumption of the transverse normals after deformation, implies that the rate of change of planar dimensions (u_0, v_0) of the plate are different from the rate of change of its vertical dimension (w_0), as seem on the figure below. The respective figure is valid for the $x_2 x_3$-section.

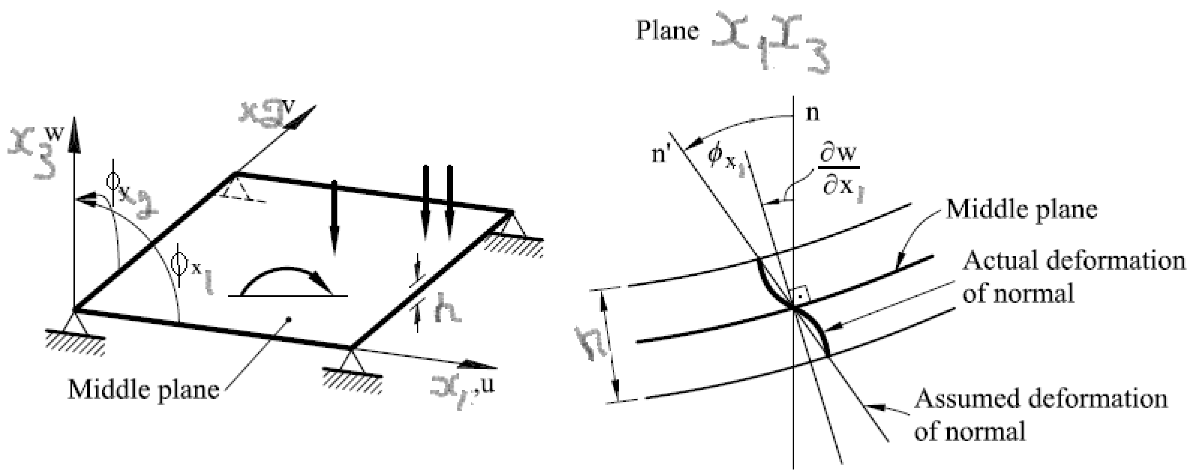

Figure 4: The lack of normality assumption on the $x_1 x_3$-section of the plate.

The obvious remarks of the above are that, $\partial u_0 / \partial x_3 \neq \partial w_0 / \partial x_1$ and $\partial v_0 / \partial x_3 \neq \partial w_0 / \partial x_2$. In addition, on the context of the TSDT (or FSDT) $\dfrac{\partial u_0}{\partial x_3} \approx \phi_x$ and $\dfrac{\partial v_0}{\partial x_3} \approx \phi_y$, where ϕ_x and ϕ_y are the real slopes of the transverse normals of the plate. In addition to the previous and as a direct consequence $\partial w_0 / \partial x_1 \neq \phi_x$ and $\partial w_0 / \partial x_2 \neq \phi_y$. These quantities ($\phi_x$, ϕ_y) are going to appear independent of the displacement field (u_0, v_0, w_0) on the equations of motion and the boundary conditions of the plate.

1.4. Voigt Notation

On the context of this quotation, it is followed the Voigt–Kelvin notation or contracted notation, in order to express more smartly the components of stresses and strains. The two-subscript components C_{ij} of the matrix of stiffness coefficients, are obtained from C_{ijkl} by the following change of subscripts:
Especially for the diagonal elements (trace) of the matrix of elastic coefficients, we derive,

$(11 \text{ or } x_1 x_1) \rightarrow 1$

$(22 \text{ or } x_2 x_2) \rightarrow 2$

$(33 \text{ or } x_3 x_3) \rightarrow 3$

Also regarding that the minor symmetries of the stiffness matrix are valid, we have the following notation for the off-diagonal elements,

$(23 \text{ or } x_2 x_3 \text{ or } 32 \text{ or } x_3 x_2) \rightarrow 4$

$(13 \text{ or } x_1 x_3 \text{ or } 31 \text{ or } x_3 x_1) \rightarrow 5$

$(12 \text{ or } x_1 x_2 \text{ or } 21 \text{ or } x_2 x_1) \rightarrow 6$

It is easily seen that the symmetries of stress and strain tensors lead to the following symmetries of stiffness (and compliance) coefficients:

$$C_{ijkl} = C_{klij} = C_{ijlk} = C_{jikl}$$

$$S_{ijkl} = S_{klij} = S_{ijlk} = S_{jikl}$$

The enormous importance of the minor symmetries of stiffness / compliance coefficients is that they reduce the number of independent coefficients from $3^4 = 81$ to $6^2 = 36$. As a consequence, the generalized Hooke's Law is simplified further [*G.A. Athanassoulis* (*2016*), *Lecture Notes of Functional Analysis, "Elastic Potential Energy- Energy function"*].
By this notation the generalized Hooke's Law relates the six components of stresses to the six components of strains, as seem below,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{Bmatrix}$$

[*J.N. Reddy 2007, "Theory and Analysis of Elastic Plates and Shells", Chap. 1.3.6, pp.28*].

Thus,

$$\sigma_1 = \sigma_{11} = \sigma_{x_1 x_1}$$

$$\sigma_2 = \sigma_{22} = \sigma_{x_2 x_2}$$

$$\sigma_3 = \sigma_{33} = \sigma_{x_3 x_3}$$

$$\sigma_4 = \sigma_{23} = \sigma_{32} = \sigma_{x_2 x_3} = \sigma_{x_3 x_2}$$

$$\sigma_5 = \sigma_{13} = \sigma_{31} = \sigma_{x_1 x_3} = \sigma_{x_3 x_1}$$

$$\sigma_6 = \sigma_{12} = \sigma_{21} = \sigma_{x_2 x_1} = \sigma_{x_1 x_2}$$

$$\gamma_4 = 2e_4 = 2e_{23} = 2e_{32} = 2e_{x_2 x_3} = 2e_{x_3 x_2}$$

$$\gamma_5 = 2e_5 = 2e_{31} = 2e_{13} = 2e_{x_3 x_1} = 2e_{x_1 x_3}$$
$$\gamma_6 = 2e_6 = 2e_{12} = 2e_{21} = 2e_{x_1 x_2} = 2e_{x_2 x_1}$$

2. Geometric Configuration and Loading Conditions

The shape of the plate considered herein is the one of homogeneous cylinder having a basis of arbitrary (smooth) shape and height (thickness) h, with slenderness ratio $0.05 \leq h/L \leq 0.1$ for moderately thick and $h/L \geq 0.1$ for very thick plates. The domain occupied by the plate (the cylinder) is denoted by B. The total boundary of the plate is denoted by ∂B, and consists of the lateral boundary (surface) $\partial B^{(lat)}$, and the two flat faces $\partial B^{(f)} = \partial B^{(uf)} \cup \partial B^{(lf)}$. One of these two flat faces is conventionally called the upper face, $\partial B^{(uf)}$, and the other is called the lower face, $\partial B^{(lf)}$. That is

$$\partial B = \partial B^{(lat)} \cup \partial B^{(uf)} \cup \partial B^{(lf)}.$$

Another (different) subdivision of the total boundary ∂B is also useful for our analysis, according to the boundary conditions applied to the various parts of it. Thus, we denote by ∂B_T the parts of the total boundary surface tractions (stresses) are prescribed, and ∂B_u the parts of the total boundary on which the displacements are given. Of course, we can define

$$\partial B = \partial B_T \cup \partial B_u.$$

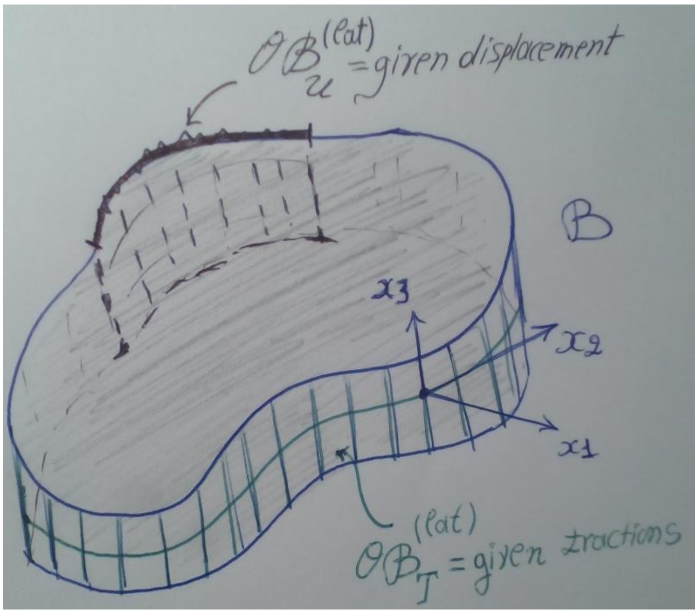

Figure 1: Geometry and loading conditions of the plate.

In addition to the above conventions, note the total boundary of the prescribed surface tractions (∂B_T) includes the boundaries $\partial B^{(f)} = \partial B^{(uf)} \cup \partial B^{(lf)}$ and part of the lateral boundary denoted by $\partial B_T^{(lat)}$.

Similarly, the total boundary of given displacements (∂B_u) includes the boundaries $\partial B^{(f)} = \partial B^{(uf)} \cup \partial B^{(lf)}$ and part of the lateral boundary $\partial B_u^{(lat)}$.

Now as for the loads set to the formulation of the problem of TSDT, we consider the following. First we assume that on the top or/ and bottom surface of the plate there is a normal load distribution $q(x_1, x_2)$. Obviously, this vertical load is horizontally distributed. Also there is no matter if the algebraic sum of the vertical load q is on the positive or negative direction of

the x_3-axis. Also due to the large thickness of the plate, we have to take into account the contribution of the load q on the boundary conditions except from its influence on the equations of motion of the plate (as seem on the model of CPT). Consequently, the load q will be treated in curvilinear integrals (as done with the residual external loads-tractions), from which we derive the boundary conditions, as well as in volume integrals of the variational equations, from which we derive the equations of motion of the model of TSDT. These kinds of integrals and its physical meaning will be presented on next sections. Thus, the assumption of the thick plate results to the fact that the load q is applied on the mid-surface Ω of the plate [or the plane $(x_1, x_2, 0)$]. Let Ω be a common projection of the upper and lower faces of the plate on its mid-surface. The last is surrounded by the curve Γ, which is the projection of the (vertical) lateral boundary of the mid-surface.
Respectively to the above notation, let $d\omega = dx_1 dx_2$ be an infinitesimal element of the domain Ω and $d\gamma$ an infinitesimal arc of the curve Γ.

Second, at the edge of the plate, we have surface distributed loads (surface tractions), whose components are going to be analyzed below.

These surface-distributed loads on the context of this study are chosen to depend on the plate thickness h in such a way that the limit problem ($h \to 0$), namely the CPT, will produce solutions and boundary conditions that are neither infinite nor zero. This is manage by letting the q_S above and the surface tractions below, be proportional to h^3 [*BO HÄGGBLAD*, *KLAUS- JÜRGEN BATHE (1990), "Specifications of Boundary Conditions for Reissner/ Mindlin Plate Bending Finite Elements", Journal*].

Generally we have, $\boldsymbol{T}(\boldsymbol{x};t) = \boldsymbol{T}_0(\boldsymbol{x})$, where $\boldsymbol{x} \in \partial B_T$ and

$$\boldsymbol{T}_0(\boldsymbol{x}) = \hat{T}_1(x_1, x_2, x_3)\, \boldsymbol{e}_{x_1} + \hat{T}_2(x_1, x_2, x_3)\, \boldsymbol{e}_{x_2} + \hat{T}_3(x_1, x_2, x_3)\, \boldsymbol{e}_{x_3}$$

Note that we consider here the surface tractions independent of the time variable.

Now taking apart from each other the components of the surface tractions along the three axes of the Cartesian coordinate system, we get

$$\begin{aligned} \hat{T}_1(x_1, x_2, x_3) &= a_{T0}(x_1, x_2) + a_{T1}(x_1, x_2)\, x_3 + a_{T3}(x_1, x_2)\, x_3^3 = \\ &= a_{T0}(\gamma) + a_{T1}(\gamma)\, x_3 + a_{T3}(\gamma)\, x_3^3 \end{aligned} \tag{1}$$

$$\begin{aligned} \hat{T}_2(x_1, x_2, x_3) &= b_{T0}(x_1, x_2) + b_{T1}(x_1, x_2)\, x_3 + b_{T3}(x_1, x_2)\, x_3^3 = \\ &= b_{T0}(\gamma) + b_{T1}(\gamma)\, x_3 + b_{T3}(\gamma)\, x_3^3 \end{aligned} \tag{2}$$

$$\begin{aligned} \hat{T}_3(x_1, x_2, x_3) &= c_{T0}(x_1, x_2) + c_{T1}(x_1, x_2)\, x_3 + c_{T3}(x_1, x_2)\, x_3^3 = \\ &= c_{T0}(\gamma) + c_{T1}(\gamma)\, x_3 + c_{T3}(\gamma)\, x_3^3 \end{aligned} \tag{3}$$

The above configuration of the surface tractions, including the arbitrary but appropriate functions a_{T0}, a_{T1}, a_{T3}, b_{T0}, b_{T1}, b_{T3}, c_{T0}, c_{T1} and c_{T3} are compatible with the initial assumptions of the kinematic model of the TSDT. Remark that one of the essential differences of this model of TSDT in comparison with the model of the CPT, is the existence of shear deformations $\gamma_{13} = 2e_{13}$ and $\gamma_{23} = 2e_{23}$ except for the in-plane shear deformation $\gamma_{12} = 2e_{12}$ (as shown on the following section 3). For this reason the vertical surface traction ($\hat{T}_3$) is chosen to be explicitly dependent to x_3 variable.

However, due to the in extensibility assumption (section 1.2.2) these tractions along the x_3-axis ($\hat{T}_3$) as well as the load q_S must be equal to zero upon the top and bottom surfaces of the plate at $x_3 = \pm h/2$.

The in-plane tractions ($\hat{T}_1$ and $\hat{T}_2$) are dependent up to x_3^3 from the thickness of the plate, fact that is consistent with the lack of normality and straightness assumption (section 1.2.1 and 1.2.3) in order to follow the deformed cross-section of the plate.

Highlight also that the notation of the notation of the zero sub index is on purpose because for the first part of the right-hand side of the Eqs. (1), (2) and (3), because we want to show the dependence of the a_{T0}, b_{T0}, c_{T0} functions from the curve Γ (γ-arc around the curve) of the lateral boundary, on which the x_3-variable is zero. In addition, the functions a_{T1}, b_{T1}, c_{T1} and a_{T3}, b_{T3}, c_{T3} notated by unit sub index declare the linear and cubic (respectively) dependence of the these surface tractions from the vertical spatial variable.

As for the specific parts of the boundary, where displacements are prescribed, we assume the following boundary conditions. These kinematic boundary conditions are alternatively called essential conditions of the problem, because they are considered as a priori constraint affecting the space of the admissible functions and variations of the problem of TSDT.

Generally the form of the given displacements is,

$$\boldsymbol{u}(\boldsymbol{x};t) = \boldsymbol{u}_0(\boldsymbol{x}) = given, \qquad \text{where } \boldsymbol{x} \in \partial B_u \text{ and}$$

$$\boldsymbol{u}_0(\boldsymbol{x}) = u_1(x_1, x_2, x_3)\,\boldsymbol{e}_{x_1} + u_2(x_1, x_2, x_3)\,\boldsymbol{e}_{x_2} + u_3(x_1, x_2, x_3)\,\boldsymbol{e}_{x_3}$$

Also note that we consider here the above displacements independent of the time variable. Now separating the components of the displacement field on the boundary along the three axes of the Cartesian coordinate system, we have

$$\begin{aligned} u_1(x_1, x_2, x_3) &= a_{0u}(x_1, x_2) + a_{1u}(x_1, x_2)\,x_3 + a_{3u}(x_1, x_2)\,x_3^3 = \\ &= a_{0u}(\gamma) + a_{1u}(\gamma)\,x_3 + a_{3u}(\gamma)\,x_3^3 \end{aligned} \tag{4}$$

$$\begin{aligned} u_2(x_1, x_2, x_3) &= b_{0u}(x_1, x_2) + b_{1u}(x_1, x_2)\,x_3 + b_{3u}(x_1, x_2)\,x_3^3 = \\ &= b_{0u}(\gamma) + b_{1u}(\gamma)\,x_3 + b_{3u}(\gamma)\,x_3^3 \end{aligned} \tag{5}$$

$$\begin{aligned} u_3(x_1, x_2, x_3) &= c_{0u}(x_1, x_2) + c_{1u}(x_1, x_2)\,x_3 + c_{3u}(x_1, x_2)\,x_3^3 = \\ &= c_{0u}(\gamma) + c_{1u}(\gamma)\,x_3 + c_{3u}(\gamma)\,x_3^3 \end{aligned} \tag{6}$$

The above notation follows the same rationality as this of the surface tractions, expressed above. The only difference is the form of the functions a_{0u}, a_{1u}, a_{3u}, b_{0u}, b_{1u}, b_{3u}. However they must be compatible with the "nature" of the problem, as exactly the above functions a_{T0}, b_{T0}, c_{T0}, a_{T1}, b_{T1}, c_{T1} and a_{T3}, b_{T3}, c_{T3} are.

Further the rightness and compatibility of the above form of the essential conditions, is verified by the initial assumptions of the modelling of the problem of TSDT, and especially by the normality and the straightness assumptions of the sections 1.2.3 and 1.2.1 respectively.

In conclusion all the aforementioned boundary and loading conditions, lead to the fact that parallel to the mid-surface (in-plane motion) there are three contributions. The first are the stretching actions due to loads at the edge of the plate which act parallel to the mid-surface of the plate. The second contribution is the bending of the plate due to the x_3 terms and the third one is the distortion of the cross-section of the plate due to the x_3^3 terms.

3. Kinematics of Thick plates

The **in-plane displacements** (due to the total loads acting on the plate) can be approximated by a **few terms of the Taylor expansion** around each point (x_1, x_2, 0) of the mid-surface, with respect to $x_3 \in [-h/2, h/2]$. We choose to expand Taylor with respect to x_3-axis (namely along the smallest dimension, -thickness of the plate), since Taylor's expansions (polynomials) are adequate approximations only in a small region $(-h/2, h/2)$ around the central points (x_1, x_2, 0). Thus, the form of the u, v-components of the displacement is assumed of the form:

$$u(x_1,x_2,x_3;t) = u(x_1,x_2,0;t)+\frac{(x_3-0)}{1!}\frac{\partial u(x_1,x_2,0;t)}{\partial x_3}+\frac{(x_3-0)^2}{2!}\frac{\partial^2 u(x_1,x_2,0;t)}{\partial^2 x_3}+ + \frac{(x_3-0)^3}{3!}\frac{\partial^3 u(x_1,x_2,0;t)}{\partial^3 x_3}+\cdots \tag{1}$$

$$v(x_1,x_2,x_3;t)=v(x_1,x_2,0;t)+\frac{(x_3-0)}{1!}\frac{\partial v(x_1,x_2,0;t)}{\partial x_3}+\frac{(x_3-0)^2}{2!}\frac{\partial^2 v(x_1,x_2,0;t)}{\partial^2 x_3}+ + \frac{(x_3-0)^3}{3!}\frac{\partial^3 v(x_1,x_2,0;t)}{\partial^3 x_3}+\cdots \tag{2}$$

Here maintaining until third-order terms and adopting the notation

$$u(x_1,x_2,0;t)=u_0(x_1,x_2;t), \qquad v(x_1,x_2,0;t)=v_0(x_1,x_2;t) \quad \text{and}$$

$$w(x_1,x_2,0;t)=w_0(x_1,x_2;t),$$

we can write more simply the above relations as:

$$u(x_1,x_2,x_3;t) = u_0(x_1,x_2;t)+x_3 u_{01}(x_1,x_2;t)+ +x_3^2 u_{02}(x_1,x_2;t)+x_3^3 u_{03}(x_1,x_2;t) \tag{3}$$

$$v(x_1,x_2,x_3;t) = v_0(x_1,x_2;t)+x_3 v_{01}(x_1,x_2;t)+ +x_3^2 v_{02}(x_1,x_2;t)+x_3^3 v_{03}(x_1,x_2;t) \tag{4}$$

Further we can modify some of the above terms, taking into account the Figures 3 and 4 of the section 1.3. Concerning u_1, v_1, we observe that,

$$u_1(x_1,x_2;t) = \frac{\partial u(x_1,x_2,x_3=0;t)}{\partial x_3} = \frac{\partial u_0}{\partial x_3} = \tan\varphi_x(x_1,x_2;t) \approx \varphi_x(x_1,x_2;t) \tag{5}$$

and

$$v_1(x_1,x_2;t) = \frac{\partial v(x_1,x_2,x_3=0;t)}{\partial x_3} = \frac{\partial v_0}{\partial x_3} = \tan\varphi_y(x_1,x_2;t) \approx \varphi_y(x_1,x_2;t) \tag{6}$$

because of the smallness of the angles ϕ_x and ϕ_y, which notate the rotations about the y and x-axis respectively [*Bo Häggblad, Klaus-Jürgen Bathe (1990), "Specifications of Boundary Conditions for Reissner/ Mindlin Plate Bending Finite Elements", International Journal for Numerical Methods in Engineering*].

Now substituting (5) and (6) on (3) and (4) respectively, we derive the following Eqs for the displacement field of the model of TSDT,

$$u(x_1,x_2,x_3;t) = u_0(x_1,x_2;t)+x_3\phi_x(x_1,x_2;t)+ \\ +x_3^2u_{02}(x_1,x_2;t)+x_3^3u_{03}(x_1,x_2;t) \tag{7}$$

$$v(x_1,x_2,x_3;t) = v_0(x_1,x_2;t)+x_3\phi_y(x_1,x_2;t)+ \\ +x_3^2v_{02}(x_1,x_2;t)+x_3^3v_{03}(x_1,x_2;t) \tag{8}$$

$$w(x_1,x_2,0;t)=w_0(x_1,x_2;t) \tag{9}$$

At this point it's meaningful to make a specific assumption, which is related to the upper and bottom surfaces of the plate at $x_3 = \pm h/2$.

For any loading condition acting on the plate purely in the vertical direction, the shear stresses σ_{13} and σ_{23} on the top and the bottom faces, $x_3 = \pm h/2$ of the plate should be zero [*G.A. Athanassoulis, Lecture Notes-Functional Analysis, "A variational approach to the third- order Bickford - Reddy Beam Theory"*].

$$\sigma_{13}\left(x_1,x_2,\pm\frac{h}{2}\right) = \sigma_{23}\left(x_1,x_2,\pm\frac{h}{2}\right) = 0 \Rightarrow \\ \sigma_5\left(x_1,x_2,\pm\frac{h}{2}\right) = \sigma_4\left(x_1,x_2,\pm\frac{h}{2}\right) = 0 \tag{10}$$

Thus, this choice of displacement field is expected to satisfy the following stress-free boundary conditions on the bottom and the top faces of the plate [*Reddy J.N. (2007), "Theory and Analysis of Elastic Plates and Shells", Chap. 10.3.2.*].

Consequently, we have the following kinematic boundary conditions,

$$e_{13}\left(x_1,x_2,\pm\frac{h}{2}\right) = e_{23}\left(x_1,x_2,\pm\frac{h}{2}\right) = 0 \Rightarrow \\ e_5\left(x_1,x_2,\pm\frac{h}{2}\right) = e_4\left(x_1,x_2,\pm\frac{h}{2}\right) = 0 \tag{11}$$

Now substituting the equations (7) and (8) on the known relations of strain-displacements,

$$2e_{13}=\gamma_{13}=\frac{\partial u}{\partial x_3}+\frac{\partial w}{\partial x_1}=\phi_x(x_1,x_2;t)+ \\ +2x_3u_2(x_1,x_2;t)+3x_3^2u_3(x_1,x_2;t)+\frac{\partial w_0}{\partial x_1} \tag{12}$$

$$2e_{23}=\gamma_{23}=\frac{\partial v}{\partial x_3}+\frac{\partial w}{\partial x_2}=\phi_y(x_1,x_2;t)+$$

$$+2x_3 v_2(x_1,x_2;t)+3x_3^2 v_3(x_1,x_2;t)+\frac{\partial w_0}{\partial x_2} \tag{13}$$

and using the Eq. (10), we derive easily the four following equations as seem subsequently:

- Substituting $x_3 = h/2$ on the Eq. (11) we have,

$$\phi_x(x_1,x_2;t)+\not{2}\frac{h}{\not{2}}u_2(x_1,x_2;t)+3\frac{h^2}{4}u_3(x_1,x_2;t)+\frac{\partial w_0}{\partial x_1} = 0 \tag{14a}$$

- Substituting $x_3 = -h/2$ on the Eq. (11) we have,

$$\phi_x(x_1,x_2;t)-\not{2}\frac{h}{\not{2}}u_2(x_1,x_2;t)+3\frac{h^2}{4}u_3(x_1,x_2;t)+\frac{\partial w_0}{\partial x_1} = 0 \tag{14b}$$

- Substituting $x_3 = h/2$ on the Eq. (12) we have,

$$\phi_y(x_1,x_2;t)+\not{2}\frac{h}{\not{2}}v_2(x_1,x_2;t)+3\frac{h^2}{4}v_3(x_1,x_2;t)+\frac{\partial w_0}{\partial x_2} = 0 \tag{14c}$$

- Substituting $x_3 = -h/2$ on the Eq. (12) we have,

$$\phi_y(x_1,x_2;t)-\not{2}\frac{h}{\not{2}}v_2(x_1,x_2;t)+3\frac{h^2}{4}v_3(x_1,x_2;t)+\frac{\partial w_0}{\partial x_2} = 0 \tag{14d}$$

Subtracting the Eq. (13a) from (13b), we find

$$2hu_2(x_1,x_2;t) = 0 \Rightarrow u_2(x_1,x_2;t) = 0 \tag{15a}$$

Subtracting the Eq. (13c) from (13d), we find

$$2hv_2(x_1,x_2;t) = 0 \Rightarrow v_2(x_1,x_2;t) = 0 \tag{15b}$$

Adding the Eqs. (13a) and (13b), we have

$$u_3(x_1,x_2;t) = -\frac{4}{3h^2}\phi_x(x_1,x_2;t)-\frac{4}{3h^2}\frac{\partial w_0}{\partial x_1} = -\frac{4}{3h^2}\left(\phi_x+\frac{\partial w_0}{\partial x_1}\right) \tag{16a}$$

Adding equations (13c) and (13d), we have

$$v_3(x_1,x_2;t) = -\frac{4}{3h^2}\phi_y(x_1,x_2;t)-\frac{4}{3h^2}\frac{\partial w_0}{\partial x_2} = -\frac{4}{3h^2}\left(\phi_y+\frac{\partial w_0}{\partial x_2}\right) \tag{16b}$$

To sum up, the final displacement field of the kinematic model of the third-order plate theory or TSDT, is expressed as:

$$u(x_1,x_2,x_3;t) = u_0(x_1,x_2;t)+x_3\phi_x(x_1,x_2;t)-x_3^3\frac{4}{3h^2}\left(\phi_x+\frac{\partial w_0}{\partial x_1}\right) \tag{17a}$$

$$v(x_1,x_2,x_3;t) = v_0(x_1,x_2;t) + x_3\phi_y(x_1,x_2;t) - x_3^3\frac{4}{3h^2}\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right) \tag{17b}$$

$$w(x_1,x_2,x_3;t) = w_0(x_1,x_2;t) \tag{17c}$$

At this point, we note that the displacement field (u, v, w) is fully described in terms of deformation of the mid- surface $(u_0,\ v_0,\ w_0)$ and the slopes of the transverse normal at $x_3 = 0$ namely $(\phi_x,\ \phi_y)$.

Thus, after calculations and grouping together separately the terms with the same virtual displacements $(u_0, v_0, w_0, \phi_x, \phi_y)$, we derive the following relations of the strains which are associated with the above displacement field (16a) - (16c).

$$e_{11} = \frac{\partial u}{\partial x_1} = \frac{\partial u_0}{\partial x_1} + x_3\frac{\partial \phi_x}{\partial x_1} - x_3^3\frac{4}{3h^2}\left(\frac{\partial \phi_x}{\partial x_1} + \frac{\partial^2 w_0}{\partial^2 x_1}\right) =$$

$$= \frac{\partial u_0}{\partial x_1} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \phi_x}{\partial x_1} - x_3^3\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_1} \tag{18a}$$

$$e_{22} = \frac{\partial v}{\partial x_2} = \frac{\partial v_0}{\partial x_2} + x_3\frac{\partial \phi_y}{\partial x_2} - x_3^3\frac{4}{3h^2}\left(\frac{\partial \phi_y}{\partial x_2} + \frac{\partial^2 w_0}{\partial^2 x_2}\right) =$$

$$= \frac{\partial v_0}{\partial x_2} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \phi_y}{\partial x_2} - x_3^3\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_2} \tag{18b}$$

$$e_{33} = \frac{\partial w}{\partial x_3} = \frac{\partial w_0}{\partial x_3} = 0 \tag{18c}$$

[As it was proved at the Eq. (1) of the assumptions of the TSDT, section 1.3.2]

$$2e_{23} = \gamma_{23} = \gamma_4 = \frac{\partial v}{\partial x_3} + \frac{\partial w}{\partial x_2} = \phi_y - \frac{4x_3^2}{h^2}\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right) + \frac{\partial w_0}{\partial x_2} =$$

$$= \left(1-\frac{4x_3^2}{h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right) \tag{18d}$$

$$2e_{13} = \gamma_{13} = \gamma_5 = \frac{\partial u}{\partial x_3} + \frac{\partial w}{\partial x_1} = \phi_x - \frac{4x_3^2}{h^2}\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right) + \frac{\partial w_0}{\partial x_1} =$$

$$= \left(1-\frac{4x_3^2}{h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right) \tag{18e}$$

$$2e_{12} = \gamma_{12} = \gamma_6 = \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} = \frac{\partial u_0}{\partial x_2} + x_3 \frac{\partial \phi_x}{\partial x_2} - x_3^3 \frac{4}{3h^2}\left(\frac{\partial \phi_x}{\partial x_2} + \frac{\partial^2 w_0}{\partial x_2 \partial x_1}\right) +$$

$$+ \frac{\partial v_0}{\partial x_1} + x_3 \frac{\partial \phi_y}{\partial x_1} - x_3^3 \frac{4}{3h^2}\left(\frac{\partial \phi_y}{\partial x_1} + \frac{\partial^2 w_0}{\partial x_2 \partial x_1}\right) =$$

$$= \frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1} + x_3\left(1 - \frac{4x_3^2}{3h^2}\right)\frac{\partial \phi_x}{\partial x_2} + x_3\left(1 - \frac{4x_3^2}{3h^2}\right)\frac{\partial \phi_y}{\partial x_1} - \frac{8x_3^3}{3h^2}\frac{\partial^2 w_0}{\partial x_2 \partial x_1} \tag{18f}$$

Note that the previous Eqs. (18a) -(18f) are found invariable on the book of Wang C.M., Reddy J.N., Lee K.H. (2000) “Shear Deformable Beams and Plates- Relations with Classical Solutions”, and specifically on the Chapter 6.4.

4. Equations of Motion - Variational Principles.

Now we are going to produce the differential equations of motion of the plate and its boundary conditions, replacing the expressions of the displacement field to the variational equation and using the Hamilton's Principle in Elastodynamics [*Athanassoulis G.A. (2016), Hamilton's Principle in Elastodynamics, NTUA Lecture Notes of Functional Analysis*].

We formulate the Elastodynamic Lagrangian function in a constraint form, which means we impose as a priori constraint the condition $u_i(\boldsymbol{x};t) = \hat{u}_i(\boldsymbol{x};t) = given \;,\; \boldsymbol{x} \in \partial B_u$ (essential condition):

$$\mathrm{L}\; \boldsymbol{u}(\cdot;t) = \iiint_B K(\dot{\boldsymbol{u}}) - U(\boldsymbol{e}) \; dV + \iint_{\partial B_T} \hat{T}_i u_i \, dS$$

Next, we have to define the action functional, corresponding to the above Lagrangian function:

$$S \;\; \boldsymbol{u}(\cdot,\cdot) = \int_{t_1}^{t_2} \mathrm{L}\;\; \boldsymbol{u}(\cdot;t) \;\; dt$$

In order to find the differential equations of the TSDT, we have to find the stationary points of the action functional (Hamilton's Principle):

$$\delta S \;\; \boldsymbol{u}\,;\delta\boldsymbol{u} = \delta \int_{t_1}^{t_2} \mathrm{L}\;\; \boldsymbol{u}(\cdot;t) \;\; dt = 0, \qquad \forall \;\; \delta\boldsymbol{u} \in \begin{Bmatrix} \text{space of admissible} \\ \text{variations} \end{Bmatrix} \Leftrightarrow$$

$$\delta \int_{t_1}^{t_2} \iiint_B K(\dot{\boldsymbol{u}}) - U(\boldsymbol{e}) \;\; dV \; dt + \delta \int_{t_1}^{t_2} \iint_{\partial B_T} \hat{T}_i u_i \, dS \; dt = 0, \forall \;\; \delta\boldsymbol{u} \in \begin{Bmatrix} \text{space of admissible} \\ \text{variations} \end{Bmatrix}$$

$$\int_{t_1}^{t_2} \iiint_B \delta K(\dot{\boldsymbol{u}}) \, dV \; dt - \int_{t_1}^{t_2} \iiint_B \delta U(\boldsymbol{e}) \, dV \; dt + \int_{t_1}^{t_2} \iint_{\partial B_T} \hat{T}_i \, \delta u_i \; dS \; dt = 0 \,, \qquad (1)$$

$$\forall \;\; \delta\boldsymbol{u} \in \;\; \text{space of admissible variations}$$

Now we calculate separately the terms of the above variational equation (1) :

4.1. Variation of the Kinetic-Energy Part

The calculation of the kinetic-energy part of the action functional is standard. Integrating by parts the time integral, we find:

$$\delta J_K = \int_{t_1}^{t_2} \iiint_B \delta K(\dot{\boldsymbol{u}})\, dV\, dt = \int_{t_1}^{t_2} \iiint_B \frac{1}{2} \rho \dot{u}_i\, \delta \dot{u}_i\, dV\, dt =$$

$$= -\int_{t_1}^{t_2} \iiint_B \rho \ddot{u}_i\, \delta u_i\, dV\, dt,$$

Thus, $$\delta J_K = -\int_{t_1}^{t_2} \iiint_B \rho \left(\ddot{u}_1 \delta u_1 + \ddot{u}_2 \delta u_2 + \ddot{u}_3 \delta u_3 \right) dV\, dt =$$

$$= -\int_{t_1}^{t_2} \iiint_B \rho \left(\ddot{u}\, \delta u + \ddot{v}\, \delta v + \ddot{w}\, \delta w \right) dV\, dt \tag{2}$$

where the double dots represent the time derivatives of the variations and using the relations (16a)-(16c) of the section 3,

$$\ddot{u}(x_1, x_2, x_3; t) = \ddot{u}_0(x_1, x_2; t) + x_3 \ddot{\phi}_x(x_1, x_2; t) - x_3^3 \frac{4}{3h^2} \left(\ddot{\phi}_x + \frac{\partial \ddot{w}_0}{\partial x_1} \right) \tag{3a}$$

$$\ddot{v}(x_1, x_2, x_3; t) = \ddot{v}_0(x_1, x_2; t) + x_3 \ddot{\phi}_y(x_1, x_2; t) - x_3^3 \frac{4}{3h^2} \left(\ddot{\phi}_y + \frac{\partial \ddot{w}_0}{\partial x_2} \right) \tag{3b}$$

$$\ddot{w}(x_1, x_2, x_3; t) = \ddot{w}_0(x_1, x_2; t) \tag{3c}$$

And their variations are,

$$\delta u(x_1, x_2, x_3; t) = \delta u_0(x_1, x_2; t) + x_3 \delta \phi_x(x_1, x_2; t) - x_3^3 \frac{4}{3h^2} \left(\delta \phi_x + \frac{\partial \delta w_0}{\partial x_1} \right) \tag{4a}$$

$$\delta v(x_1, x_2, x_3; t) = \delta v_0(x_1, x_2; t) + x_3 \delta \phi_y(x_1, x_2; t) - x_3^3 \frac{4}{3h^2} \left(\delta \phi_y + \frac{\partial \delta w_0}{\partial x_2} \right) \tag{4b}$$

$$\delta w(x_1, x_2, x_3; t) = \delta w_0(x_1, x_2; t) \tag{4c}$$

Before proceed to the appropriate substitutions, it’s meaningful to calculate firstly the products inside the brackets under the volume integral of the Eq. (2), in order to avoid confusing calculations.
Thus after careful handling,

$$\ddot{u}\,\delta u = \left[\ddot{u}_0 + x_3\ddot{\phi}_x - x_3^3\frac{4}{3h^2}\left(\ddot{\phi}_x + \frac{\partial \ddot{w}_0}{\partial x_1}\right)\right]\left[\delta u_0 + x_3\,\delta\phi_x - x_3^3\frac{4}{3h^2}\left(\delta\phi_x + \frac{\partial \delta w_0}{\partial x_1}\right)\right] =$$

$$= \left[\ddot{u}_0 + \left(x_3 - x_3^3\frac{4}{3h^2}\right)\ddot{\phi}_x - x_3^3\frac{4}{3h^2}\frac{\partial \ddot{w}_0}{\partial x_1}\right]\delta u_0 +$$

$$+\left[\ddot{u}_0\left(x_3 - x_3^3\frac{4}{3h^2}\right) + \left(x_3^2 - x_3^4\frac{8}{3h^2} + x_3^6\frac{16}{9h^4}\right)\ddot{\phi}_x + \frac{4}{3h^2}\left(x_3^6\frac{4}{3h^2} - x_3^4\right)\frac{\partial \ddot{w}_0}{\partial x_1}\right]\delta\phi_x +$$

$$+\frac{4}{3h^2}\left[-x_3^3\ddot{u}_0 + \left(x_3^6\frac{4}{3h^2} - x_3^4\right)\ddot{\phi}_x + x_3^6\frac{4}{3h^2}\frac{\partial \ddot{w}_0}{\partial x_1}\right]\frac{\partial \delta w_0}{\partial x_1}$$

$$\ddot{v}\,\delta v = \left(\ddot{v}_0 + x_3\ddot{\phi}_y - x_3^3\frac{4}{3h^2}\left(\ddot{\phi}_y + \frac{\partial \ddot{w}_0}{\partial x_2}\right)\right)\left(\delta v_0 + x_3\,\delta\phi_y - x_3^3\frac{4}{3h^2}\left(\delta\phi_y + \frac{\partial \delta w_0}{\partial x_2}\right)\right) =$$

$$= \left[\ddot{v}_0 + \left(x_3 - x_3^3\frac{4}{3h^2}\right)\ddot{\phi}_y - x_3^3\frac{4}{3h^2}\frac{\partial \ddot{w}_0}{\partial x_2}\right]\delta v_0 +$$

$$+\left[\ddot{v}_0\left(x_3 - x_3^3\frac{4}{3h^2}\right) + \left(x_3^2 - x_3^4\frac{8}{3h^2} + x_3^6\frac{16}{9h^4}\right)\ddot{\phi}_y + \frac{4}{3h^2}\left(\frac{4}{3h^2}x_3^6 - x_3^4\right)\frac{\partial \ddot{w}_0}{\partial x_2}\right]\delta\phi_y +$$

$$+\frac{4}{3h^2}\left[-x_3^3\ddot{v}_0 + \left(x_3^6\frac{4}{3h^2} - x_3^4\right)\ddot{\phi}_y + x_3^6\frac{4}{3h^2}\frac{\partial \ddot{w}_0}{\partial x_2}\right]\frac{\partial \delta w_0}{\partial x_2}$$

Now substituting the above results into the eq. (2), we extract a more extensive form of the kinetic-energy part of the action functional.

$$\delta J_K = -\int_{t_1}^{t_2}\iiint_B \rho\,\left(\ddot{u}\,\delta u + \ddot{v}\,\delta v + \ddot{w}\,\delta w\right)\,dV\,dt =$$

$$= -\int_{t_1}^{t_2}\iiint_B \rho\left[\ddot{u}_0 + \left(x_3 - x_3^3\frac{4}{3h^2}\right)\ddot{\phi}_x - x_3^3\frac{4}{3h^2}\frac{\partial \ddot{w}_0}{\partial x_1}\right]\delta u_0\,dV\,dt\; -$$

$$-\int_{t_1}^{t_2}\iiint_B \rho\left[\begin{array}{c}\ddot{u}_0\left(x_3 - x_3^3\frac{4}{3h^2}\right) + \left(x_3^2 - x_3^4\frac{8}{3h^2} + x_3^6\frac{16}{9h^4}\right)\ddot{\phi}_x\; + \\ +\frac{4}{3h^2}\left(x_3^6\frac{4}{3h^2} - x_3^4\right)\frac{\partial \ddot{w}_0}{\partial x_1}\end{array}\right]\delta\phi_x\,dV\,dt\; -$$

$$-\int_{t_1}^{t_2}\iiint_B \rho\frac{4}{3h^2}\left[-x_3^3\ddot{u}_0 + \left(x_3^6\frac{4}{3h^2} - x_3^4\right)\ddot{\phi}_x + x_3^6\frac{4}{3h^2}\frac{\partial \ddot{w}_0}{\partial x_1}\right]\frac{\partial \delta w_0}{\partial x_1}\,dV\,dt\; -$$

$$-\int_{t_1}^{t_2}\iiint_B \rho\left[\ddot{v}_0+\left(x_3-x_3^3\frac{4}{3h^2}\right)\ddot{\phi}_y-x_3^3\frac{4}{3h^2}\frac{\partial\ddot{w}_0}{\partial x_2}\right]\delta v_0\, dV\, dt\ -$$

$$-\int_{t_1}^{t_2}\iiint_B \rho\left[\begin{array}{c}\ddot{v}_0\left(x_3-x_3^3\frac{4}{3h^2}\right)+\left(x_3^2-x_3^4\frac{8}{3h^2}+x_3^6\frac{16}{9h^4}\right)\ddot{\phi}_y+\\ +\frac{4}{3h^2}\left(\frac{4}{3h^2}x_3^6-x_3^4\right)\frac{\partial\ddot{w}_0}{\partial x_2}\end{array}\right]\delta\phi_y\, dV\, dt\ -$$

$$-\int_{t_1}^{t_2}\iiint_B \rho\frac{4}{3h^2}\left[-x_3^3\ddot{v}_0+\left(x_3^6\frac{4}{3h^2}-x_3^4\right)\ddot{\phi}_y+x_3^6\frac{4}{3h^2}\frac{\partial\ddot{w}_0}{\partial x_2}\right]\frac{\partial\delta w_0}{\partial x_2}\, dV\, dt\ -$$

$$-\int_{t_1}^{t_2}\iiint_B \rho\,\ddot{w}_0\,\delta w_0\, dV\, dt \tag{5}$$

In Eq. (5) there are not only the variations δu_0, δv_0, δw_0, but also the first spatial derivatives, here the x_1, x_2-derivatives of δw_0. To eliminate the later we perform an **integration by parts with respect to the corresponding spatial variables**. These integrations by parts will generate boundary terms, which will contribute to the construction of the appropriate boundary conditions of the TSDT. For further simplification we neglect for the present calculations the time integral. Thus,

$$\delta I_{\delta w_0,x_1}=\iiint_B \rho\frac{4}{3h^2}\left[-x_3^3\ddot{u}_0+\left(x_3^6\frac{4}{3h^2}-x_3^4\right)\ddot{\phi}_x+x_3^6\frac{4}{3h^2}\frac{\partial\ddot{w}_0}{\partial x_1}\right]\frac{\partial\delta w_0}{\partial x_1}dV\ =$$

$$=\iint_{\partial B^{(lat)}} \rho\frac{4}{3h^2}\left[-x_3^3\ddot{u}_0+\left(x_3^6\frac{4}{3h^2}-x_3^4\right)\ddot{\phi}_x+x_3^6\frac{4}{3h^2}\frac{\partial\ddot{w}_0}{\partial x_1}\right]n_{x_1}\,\delta w_0\, dS\ -$$

$$-\iiint_B \rho\frac{4}{3h^2}\left[-x_3^3\frac{\partial\ddot{u}_0}{\partial x_1}+\left(x_3^6\frac{4}{3h^2}-x_3^4\right)\frac{\partial\ddot{\phi}_x}{\partial x_1}+x_3^6\frac{4}{3h^2}\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}\right]\delta w_0\, dV \tag{6}$$

$$\delta I_{\delta w_0,x_2}\ =\iiint_B \rho\frac{4}{3h^2}\left[-x_3^3\ddot{v}_0+\left(x_3^6\frac{4}{3h^2}-x_3^4\right)\ddot{\phi}_y\ +\ x_3^6\frac{4}{3h^2}\frac{\partial\ddot{w}_0}{\partial x_2}\right]\frac{\partial\delta w_0}{\partial x_2}dV=$$

$$=\iint_{\partial B^{(lat)}} \rho\frac{4}{3h^2}\left[-x_3^3\ddot{v}_0+\left(x_3^6\frac{4}{3h^2}-x_3^4\right)\ddot{\phi}_y+x_3^6\frac{4}{3h^2}\frac{\partial\ddot{w}_0}{\partial x_2}\right]n_{x_2}\,\delta w_0\, dS\ -$$

$$-\iiint_B \rho\frac{4}{3h^2}\left[-x_3^3\frac{\partial\ddot{v}_0}{\partial x_2}+\left(x_3^6\frac{4}{3h^2}-x_3^4\right)\frac{\partial\ddot{\phi}_y}{\partial x_2}+x_3^6\frac{4}{3h^2}\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}\right]\delta w_0\, dV \tag{7}$$

Consequently, substituting the Eqs. (6) and (7) into (5), we get

$$\delta J_K = -\int_{t_1}^{t_2}\iiint_B \rho\left(\ddot{u}\,\delta u + \ddot{v}\,\delta v + \ddot{w}\,\delta w\right) dV\,dt =$$

$$= -\int_{t_1}^{t_2}\iiint_B \rho\left[\ddot{u}_0 + \left(x_3 - x_3^3\frac{4}{3h^2}\right)\ddot{\phi}_x - x_3^3\frac{4}{3h^2}\frac{\partial \ddot{w}_0}{\partial x_1}\right]\delta u_0\, dV\,dt\; -$$

$$-\int_{t_1}^{t_2}\iiint_B \rho\left[\begin{array}{c}\ddot{u}_0\left(x_3 - x_3^3\frac{4}{3h^2}\right) + \left(x_3^2 - x_3^4\frac{8}{3h^2} + x_3^6\frac{16}{9h^4}\right)\ddot{\phi}_x\; + \\ + \frac{4}{3h^2}\left(x_3^6\frac{4}{3h^2} - x_3^4\right)\frac{\partial \ddot{w}_0}{\partial x_1}\end{array}\right]\delta\phi_x\, dV\,dt\; -$$

$$-\int_{t_1}^{t_2}\iint_{\partial B^{(lat)}} \rho\frac{4}{3h^2}\left[-x_3^3\ddot{u}_0 + \left(x_3^6\frac{4}{3h^2} - x_3^4\right)\ddot{\phi}_x + x_3^6\frac{4}{3h^2}\frac{\partial \ddot{w}_0}{\partial x_1}\right] n_{x_1}\,\delta w_0\, dS\,dt\; +$$

$$+\int_{t_1}^{t_2}\iiint_B \rho\frac{4}{3h^2}\left[-x_3^3\frac{\partial \ddot{u}_0}{\partial x_1} + \left(x_3^6\frac{4}{3h^2} - x_3^4\right)\frac{\partial \ddot{\phi}_x}{\partial x_1} + x_3^6\frac{4}{3h^2}\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}\right]\delta w_0\, dV\,dt\; -$$

$$-\int_{t_1}^{t_2}\iiint_B \rho\left[\ddot{v}_0 + \left(x_3 - x_3^3\frac{4}{3h^2}\right)\ddot{\phi}_y - x_3^3\frac{4}{3h^2}\frac{\partial \ddot{w}_0}{\partial x_2}\right]\delta v_0\, dV\,dt\; -$$

$$-\int_{t_1}^{t_2}\iiint_B \rho\left[\begin{array}{c}\ddot{v}_0\left(x_3 - x_3^3\frac{4}{3h^2}\right) + \left(x_3^2 - x_3^4\frac{8}{3h^2} + x_3^6\frac{16}{9h^4}\right)\ddot{\phi}_y\; + \\ + \frac{4}{3h^2}\left(\frac{4}{3h^2}x_3^6 - x_3^4\right)\frac{\partial \ddot{w}_0}{\partial x_2}\end{array}\right]\delta\phi_y\, dV\,dt\; -$$

$$-\int_{t_1}^{t_2}\iint_{\partial B^{(lat)}} \rho\frac{4}{3h^2}\left[-x_3^3\ddot{v}_0 + \left(x_3^6\frac{4}{3h^2} - x_3^4\right)\ddot{\phi}_y + x_3^6\frac{4}{3h^2}\frac{\partial \ddot{w}_0}{\partial x_2}\right] n_{x_2}\,\delta w_0\, dS\,dt\; +$$

$$+\int_{t_1}^{t_2}\iiint_B \rho\frac{4}{3h^2}\left[-x_3^3\frac{\partial \ddot{v}_0}{\partial x_2} + \left(x_3^6\frac{4}{3h^2} - x_3^4\right)\frac{\partial \ddot{\phi}_y}{\partial x_2} + x_3^6\frac{4}{3h^2}\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right]\delta w_0\, dV\,dt\; -$$

$$-\int_{t_1}^{t_2}\iiint_B \rho\,\ddot{w}_0\,\delta w_0\, dV\,dt =$$

$$
\begin{aligned}
&= -\int_{t_1}^{t_2}\iiint_B \rho\left[\ddot{u}_0+\left(x_3-x_3^3\frac{4}{3h^2}\right)\ddot{\phi}_x - x_3^3\frac{4}{3h^2}\frac{\partial \ddot{w}_0}{\partial x_1}\right]\delta u_0\, dV\, dt \; - \\
&-\int_{t_1}^{t_2}\iiint_B \rho\left[\begin{array}{c}\ddot{u}_0\left(x_3-x_3^3\frac{4}{3h^2}\right)+\left(x_3^2-x_3^4\frac{8}{3h^2}+x_3^6\frac{16}{9h^4}\right)\ddot{\phi}_x \; + \\ +\frac{4}{3h^2}\left(x_3^6\frac{4}{3h^2}-x_3^4\right)\frac{\partial \ddot{w}_0}{\partial x_1}\end{array}\right]\delta\phi_x\, dV\, dt \; - \\
&-\int_{t_1}^{t_2}\iint_{\partial B^{(lat)}} \rho\frac{4}{3h^2}\left\{\begin{array}{c}\left[-x_3^3\ddot{u}_0+\left(x_3^6\frac{4}{3h^2}-x_3^4\right)\ddot{\phi}_x + x_3^6\frac{4}{3h^2}\frac{\partial \ddot{w}_0}{\partial x_1}\right]\boldsymbol{n}_{x_1} + \\ +\left[-x_3^3\ddot{v}_0+\left(x_3^6\frac{4}{3h^2}-x_3^4\right)\ddot{\phi}_y + x_3^6\frac{4}{3h^2}\frac{\partial \ddot{w}_0}{\partial x_2}\right]\boldsymbol{n}_{x_2}\end{array}\right\}\delta w_0\, dS\, dt \; + \\
&+\int_{t_1}^{t_2}\iiint_B \rho\frac{4}{3h^2}\left[\begin{array}{c}-x_3^3\left(\frac{\partial \ddot{u}_0}{\partial x_1}+\frac{\partial \ddot{v}_0}{\partial x_2}\right)+\left(x_3^6\frac{4}{3h^2}-x_3^4\right)\left(\frac{\partial \ddot{\phi}_y}{\partial x_2}+\frac{\partial \ddot{\phi}_x}{\partial x_1}\right)+ \\ +x_3^6\frac{4}{3h^2}\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}+\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}\right)-\frac{3h^2}{4}\ddot{w}_0\end{array}\right]\delta w_0\, dV\, dt \; - \\
&-\int_{t_1}^{t_2}\iiint_B \rho\left[\ddot{v}_0+\left(x_3-x_3^3\frac{4}{3h^2}\right)\ddot{\phi}_y - x_3^3\frac{4}{3h^2}\frac{\partial \ddot{w}_0}{\partial x_2}\right]\delta v_0\, dV\, dt \; - \\
&-\int_{t_1}^{t_2}\iiint_B \rho\left[\begin{array}{c}\ddot{v}_0\left(x_3-x_3^3\frac{4}{3h^2}\right)+\left(x_3^2-x_3^4\frac{8}{3h^2}+x_3^6\frac{16}{9h^4}\right)\ddot{\phi}_y \; + \\ +\frac{4}{3h^2}\left(\frac{4}{3h^2}x_3^6-x_3^4\right)\frac{\partial \ddot{w}_0}{\partial x_2}\end{array}\right]\delta\phi_y\, dV\, dt
\end{aligned}
\qquad (8)
$$

Eq. (8) can be simplified, by observing that the x_3-dependence of all integrands is explicit, and thus the vertical integration can be performed explicitly. To this end, it is convenient to define the "mass-moment" quantities:

$$I_i = \int_{-h/2}^{h/2} \rho x_3^i\, dx_3, \quad i = 0, 1, 2, ..., 6$$

Note that odd-order "mass-moment" quantities are zero. More precisely, we get

$$I_1 = I_3 = I_5 = 0$$

$$I_0 = \int_{-h/2}^{h/2} \rho\, dx_3 = \rho h, \qquad\qquad I_2 = \int_{-h/2}^{h/2} \rho x_3^2\, dx_3 = \rho\frac{h^3}{12},$$

$$I_4 = \int_{-h/2}^{h/2} \rho x_3^4 \, dx_3 = \rho \frac{h^5}{80}, \qquad I_6 = \int_{-h/2}^{h/2} \rho x_3^6 \, dx_3 = \rho \frac{h^7}{448}.$$

To treat the volume-integral terms (appearing in the first, second, fourth, fifth and sixth row of the right-most side of Eq. (8)), we decompose them as follows:

$$\iiint_B (\bullet) \, dV = \iint_\Omega d\omega \int_{-h/2}^{h/2} (\bullet) \, dx_3, \qquad d\omega = dx_1 dx_2$$

where Ω is the common projection of the upper and lower faces of the plate on the mid-surface. Similarly, to treat the terms in the third row of the right-most side of the Eq. (8), we have to decompose the lateral surface integral as follows:

$$\iint_{\partial B^{(lat)}} (\bullet) \, dS = \int_{-h/2}^{h/2} \oint_\Gamma (\bullet) \, d\gamma \, dx_3$$

where Γ is the curve defined by the projection of the (vertical) lateral boundary on the mid-surface.

Substituting the above decomposed integrals to the Eq. (8), we have the following,

$$\begin{aligned}
\delta J_K = & - \int_{t_1}^{t_2} \iint_\Omega \int_{-h/2}^{h/2} \rho \left[\ddot{u}_0 + \left(x_3 - x_3^3 \frac{4}{3h^2} \right) \ddot{\phi}_x - x_3^3 \frac{4}{3h^2} \frac{\partial \ddot{w}_0}{\partial x_1} \right] \delta u_0 \, dx_3 \, d\omega \, dt \; - \\
& - \int_{t_1}^{t_2} \iint_\Omega \int_{-h/2}^{h/2} \rho \left[\begin{array}{l} \ddot{u}_0 \left(x_3 - x_3^3 \frac{4}{3h^2} \right) + \left(x_3^2 - x_3^4 \frac{8}{3h^2} + x_3^6 \frac{16}{9h^4} \right) \ddot{\phi}_x - \\ \quad - \frac{4}{3h^2} \left(x_3^4 - x_3^6 \frac{4}{3h^2} \right) \frac{\partial \ddot{w}_0}{\partial x_1} \end{array} \right] \delta \phi_x \, dx_3 \, d\omega \, dt \; - \\
& - \int_{t_1}^{t_2} \oint_\Gamma \int_{-h/2}^{h/2} \rho \frac{4}{3h^2} \left\{ \begin{array}{l} \left[- x_3^3 \ddot{u}_0 + \left(x_3^6 \frac{4}{3h^2} - x_3^4 \right) \ddot{\phi}_x + x_3^6 \frac{4}{3h^2} \frac{\partial \ddot{w}_0}{\partial x_1} \right] \boldsymbol{n}_{x_1} \; + \\ + \left[- x_3^3 \ddot{v}_0 + \left(x_3^6 \frac{4}{3h^2} - x_3^4 \right) \ddot{\phi}_y + x_3^6 \frac{4}{3h^2} \frac{\partial \ddot{w}_0}{\partial x_2} \right] \boldsymbol{n}_{x_2} \end{array} \right\} \delta w_0 \, dx_3 \, d\gamma \, dt \; + \\
& + \int_{t_1}^{t_2} \iint_\Omega \int_{-h/2}^{h/2} \rho \frac{4}{3h^2} \left[\begin{array}{l} - x_3^3 \left(\frac{\partial \ddot{u}_0}{\partial x_1} + \frac{\partial \ddot{v}_0}{\partial x_2} \right) + \left(x_3^6 \frac{4}{3h^2} - x_3^4 \right) \left(\frac{\partial \ddot{\phi}_y}{\partial x_2} + \frac{\partial \ddot{\phi}_x}{\partial x_1} \right) + \\ \quad + x_3^6 \frac{4}{3h^2} \left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2} + \frac{\partial^2 \ddot{w}_0}{\partial^2 x_1} \right) - \frac{3h^2}{4} \ddot{w}_0 \end{array} \right] \delta w_0 \, dx_3 \, d\omega \, dt \; - \\
& - \int_{t_1}^{t_2} \iint_\Omega \int_{-h/2}^{h/2} \rho \left[\ddot{v}_0 + \left(x_3 - x_3^3 \frac{4}{3h^2} \right) \ddot{\phi}_y - x_3^3 \frac{4}{3h^2} \frac{\partial \ddot{w}_0}{\partial x_2} \right] \delta v_0 \, dx_3 \, d\omega \, dt \; -
\end{aligned}$$

$$-\int_{t_1}^{t_2}\iint_{\Omega}\int_{-h/2}^{h/2}\rho\left[\begin{array}{c}\ddot{v}_0\left(x_3-x_3^3\frac{4}{3h^2}\right)+\left(x_3^2-x_3^4\frac{8}{3h^2}+x_3^6\frac{16}{9h^4}\right)\ddot{\phi}_y+\\ +\frac{4}{3h^2}\left(\frac{4}{3h^2}x_3^6-x_3^4\right)\frac{\partial\ddot{w}_0}{\partial x_2}\end{array}\right]\delta\phi_y\,dx_3\,d\omega\,dt=$$

$$=-\int_{t_1}^{t_2}\iint_{\Omega}\left[\ddot{u}_0 I_0+\left(\cancel{I_1}-\cancel{I_3}\frac{4}{3h^2}\right)\ddot{\phi}_x-\cancel{I_3}\frac{4}{3h^2}\frac{\partial\ddot{w}_0}{\partial x_1}\right]\delta u_0\,d\omega\,dt-$$

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left[\begin{array}{c}\ddot{u}_0\left(\cancel{I_1}-\cancel{I}_3\frac{4}{3h^2}\right)+\left(I_2-I_4\frac{8}{3h^2}+I_6\frac{16}{9h^4}\right)\ddot{\phi}_x-\\ -\frac{4}{3h^2}\left(I_4-I_6\frac{4}{3h^2}\right)\frac{\partial\ddot{w}_0}{\partial x_1}\end{array}\right]\delta\phi_x\,d\omega\,dt-$$

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\frac{4}{3h^2}\left\{\begin{array}{c}\left[-\cancel{I_3}\ddot{u}_0+\left(I_6\frac{4}{3h^2}-I_4\right)\ddot{\phi}_x+I_6\frac{4}{3h^2}\frac{\partial\ddot{w}_0}{\partial x_1}\right]\boldsymbol{n}_{x_1}+\\ +\left[-\cancel{I_3}\ddot{v}_0+\left(I_6\frac{4}{3h^2}-x_3^4\right)\ddot{\phi}_y+I_6\frac{4}{3h^2}\frac{\partial\ddot{w}_0}{\partial x_2}\right]\boldsymbol{n}_{x_2}\end{array}\right\}\delta w_0\,d\gamma\,dt+$$

$$+\int_{t_1}^{t_2}\iint_{\Omega}\frac{4}{3h^2}\left[\begin{array}{c}-\cancel{I_3}\left(\frac{\partial\ddot{u}_0}{\partial x_1}+\frac{\partial\ddot{v}_0}{\partial x_2}\right)+\left(I_6\frac{4}{3h^2}-I_4\right)\left(\frac{\partial\ddot{\phi}_y}{\partial x_2}+\frac{\partial\ddot{\phi}_x}{\partial x_1}\right)+\\ +I_6\frac{4}{3h^2}\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}\right)-\frac{3h^2}{4}I_0\ddot{w}_0\end{array}\right]\delta w_0\,d\omega\,dt-$$

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left[I_0\ddot{v}_0+\left(\cancel{I_1}-\cancel{I_3}\frac{4}{3h^2}\right)\ddot{\phi}_y-\cancel{I_3}\frac{4}{3h^2}\frac{\partial\ddot{w}_0}{\partial x_2}\right]\delta v_0\,d\omega\,dt-$$

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left[\begin{array}{c}\ddot{v}_0\left(\cancel{I_1}-\cancel{I_3}\frac{4}{3h^2}\right)+\left(I_2-I_4\frac{8}{3h^2}+I_6\frac{16}{9h^4}\right)\ddot{\phi}_y+\\ +\frac{4}{3h^2}\left(\frac{4}{3h^2}I_6-I_4\right)\frac{\partial\ddot{w}_0}{\partial x_2}\end{array}\right]\delta\phi_y\,d\omega\,dt=$$

$$= -\int_{t_1}^{t_2}\iint_{\Omega} \ddot{u}_0 I_0 \delta u_0 \, d\omega \, dt - \int_{t_1}^{t_2}\iint_{\Omega} I_0 \ddot{v}_0 \delta v_0 \, d\omega \, dt -$$

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left[I_0 \ddot{w}_0 - \frac{4}{3h^2}\left(I_6\frac{4}{3h^2} - I_4\right)\left(\frac{\partial \ddot{\phi}_y}{\partial x_2} + \frac{\partial \ddot{\phi}_x}{\partial x_1}\right) - I_6 \frac{16}{9h^4}\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2} + \frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}\right)\right]\delta w_0 \, d\omega \, dt -$$

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left[\left(I_2 - I_4\frac{8}{3h^2} + I_6\frac{16}{9h^4}\right)\ddot{\phi}_x + \frac{4}{3h^2}\left(I_6\frac{4}{3h^2} - I_4\right)\frac{\partial \ddot{w}_0}{\partial x_1}\right]\delta \phi_x \, d\omega \, dt -$$

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left[\left(I_2 - I_4\frac{8}{3h^2} + I_6\frac{16}{9h^4}\right)\ddot{\phi}_y + \frac{4}{3h^2}\left(\frac{4}{3h^2}I_6 - I_4\right)\frac{\partial \ddot{w}_0}{\partial x_2}\right]\delta \phi_y \, d\omega \, dt -$$

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\frac{4}{3h^2}\left\{\begin{array}{l}\left[\left(I_6\frac{4}{3h^2} - I_4\right)\ddot{\phi}_x + I_6\frac{4}{3h^2}\frac{\partial \ddot{w}_0}{\partial x_1}\right]\boldsymbol{n}_{x_1} + \\ \quad + \left[\left(I_6\frac{4}{3h^2} - I_4\right)\ddot{\phi}_y + I_6\frac{4}{3h^2}\frac{\partial \ddot{w}_0}{\partial x_2}\right]\boldsymbol{n}_{x_2}\end{array}\right\}\delta w_0 \, d\gamma \, dt \quad \Rightarrow$$

$$\delta J_K = -\int_{t_1}^{t_2}\iint_{\Omega}\left\{\begin{array}{c}\ddot{u}_0 I_0 \, \delta u_0 + I_0 \ddot{v}_0 \, \delta v_0 + \\ \left[I_0 \ddot{w}_0 - \frac{4}{3h^2}\left(I_6\frac{4}{3h^2} - I_4\right)\left(\frac{\partial \ddot{\phi}_y}{\partial x_2} + \frac{\partial \ddot{\phi}_x}{\partial x_1}\right) - I_6\frac{16}{9h^4}\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2} + \frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}\right)\right]\delta w_0 + \\ + \left[\left(I_2 - I_4\frac{8}{3h^2} + I_6\frac{16}{9h^4}\right)\ddot{\phi}_x + \frac{4}{3h^2}\left(I_6\frac{4}{3h^2} - I_4\right)\frac{\partial \ddot{w}_0}{\partial x_1}\right]\delta \phi_x + \\ + \left[\left(I_2 - I_4\frac{8}{3h^2} + I_6\frac{16}{9h^4}\right)\ddot{\phi}_y + \frac{4}{3h^2}\left(\frac{4}{3h^2}I_6 - I_4\right)\frac{\partial \ddot{w}_0}{\partial x_2}\right]\delta \phi_y\end{array}\right\} d\omega \, dt -$$

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}\left[\left(I_6\frac{16}{9h^4} - \frac{4}{3h^2}I_4\right)\ddot{\phi}_x + I_6\frac{16}{9h^4}\frac{\partial \ddot{w}_0}{\partial x_1}\right]\boldsymbol{n}_{x_1} + \\ \quad + \left[\left(I_6\frac{16}{9h^4} - \frac{4}{3h^2}I_4\right)\ddot{\phi}_y + I_6\frac{16}{9h^4}\frac{\partial \ddot{w}_0}{\partial x_2}\right]\boldsymbol{n}_{x_2}\end{array}\right\}\delta w_0 \, d\gamma \, dt \qquad \textbf{(9)}$$

4.2. Stress – Strain Relations and Elastic Potential Energy

According to "*Athanassoulis G.A. (2016), Hamilton's Principle in Elastodynamics, NTUA Lecture Notes of Functional Analysis*" and "*Athanassoulis G.A. (2017), Elastic potential energy – Energy function, NTUA Lecture Notes of Functional Analysis*", we have the general form of the elastic potential energy of the problem,

$U(\boldsymbol{e}) = \frac{1}{2}\sigma_{ij}(\boldsymbol{e})e_{ij} = \frac{1}{2}C_{ijkl}\,e_{ij}\,e_{kl}$, where the strains e_{kl} e_{kl} are expressed in terms of the displacement field as $e_{ij} = \frac{1}{2}\, u_{i,j}+u_{j,i}$ and $C_{ijkl} = C_{ijkl}(\boldsymbol{x})$ are the stiffness coefficients (material properties).

As for the variation of the elastic potential energy, we derive the following:

$$\delta U(\boldsymbol{e}) = \frac{1}{2}\delta\sigma_{ij}\,e_{ij} + \frac{1}{2}\sigma_{ij}\,\delta e_{ij} \xrightarrow{\text{Hooke's Law 1st term}}$$

$$\delta U(\boldsymbol{e}) = \frac{1}{2}C_{ijkl}\,\delta e_{kl}\,e_{ij} + \frac{1}{2}\sigma_{ij}\,\delta e_{ij} \xrightarrow[\text{stiffness coefficients}]{\text{Minor symmetry of matrix of}}$$

$$\delta U(\boldsymbol{e}) = \frac{1}{2}C_{klij}\,e_{ij}\,\delta e_{kl} + \frac{1}{2}\sigma_{ij}\,\delta e_{ij} \xrightarrow{\text{Hooke's Law 1st term}}$$

$$\delta U(\boldsymbol{e}) = \frac{1}{2}\sigma_{kl}\,\delta e_{kl} + \frac{1}{2}\sigma_{ij}\,\delta e_{ij} \xrightarrow{\text{dummy indexes}} \delta U(\boldsymbol{e}) = \sigma_{ij}\,\delta e_{ij}$$

Consequently,

$$U(\boldsymbol{e}) = \frac{1}{2}\;\sigma_{x_1x_1}e_{x_1x_1} + \sigma_{x_2x_2}e_{x_2x_2} + \sigma_{x_3x_3}e_{x_3x_3} + \sigma_{x_2x_3}\gamma_{x_2x_3} + \sigma_{x_3x_1}\gamma_{x_3x_1} + \sigma_{x_1x_2}\gamma_{x_1x_2} =$$

$$= \frac{1}{2}\;\sigma_{11}e_{11} + \sigma_{22}e_{22} + \sigma_{33}e_{33} + \sigma_{23}\gamma_{23} + \sigma_{31}\gamma_{31} + \sigma_{12}\gamma_{12} \;=\; \text{Voigt Notation}$$

$$= \frac{1}{2}\;\sigma_1 e_1 + \sigma_2 e_2 + \sigma_3 e_3 + \sigma_4\gamma_4 + \sigma_5\gamma_5 + \sigma_6\gamma_6 \tag{1}$$

$$\delta U(\boldsymbol{e}) = \frac{1}{2}\;\sigma_1\delta e_1 + \sigma_2\delta e_2 + \sigma_3\delta e_3 + \sigma_4\delta\gamma_4 + \sigma_5\delta\gamma_5 + \sigma_6\delta\gamma_6 \tag{2}$$

According to the last paragraph, we observe that some of the terms of the elastic potential energy, are equal to zero. So we derive,

$$U(\boldsymbol{e}) = \frac{1}{2}\;\sigma_1\delta e_1 + \sigma_2\delta e_2 + \sigma_4\delta\gamma_4 + \sigma_5\delta\gamma_5 + \sigma_6\delta\gamma_6 \tag{3}$$

From (1) and the proof of (2) we have,

$$\boxed{\delta U(\boldsymbol{e}) = \sigma_1\cdot\delta e_1 + \sigma_2\cdot\delta e_2 + \sigma_4\cdot\delta\gamma_4 + \sigma_5\cdot\delta\gamma_5 + \sigma_6\cdot\delta\gamma_6} \tag{4}$$

From the Eq. (17a) of the section 3:

$$\delta e_1 = \delta e_{11} = \frac{\partial\delta u_0}{\partial x_1} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial\delta\phi_x}{\partial x_1} - x_3^3\,\frac{4}{3h^2}\,\frac{\partial^2\delta w_0}{\partial^2 x_1} \tag{5a}$$

From the Eq. (17b) of the section 3:

$$\delta e_2 = \delta e_{22} = \frac{\partial \delta v_0}{\partial x_2} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \delta \phi_y}{\partial x_2} - x_3^3\frac{4}{3h^2}\frac{\partial^2 \delta w_0}{\partial^2 x_2} \tag{5b}$$

From the Eq. (17d) of the section 3:

$$\delta\gamma_4 = \delta\gamma_{23} = \left(1-\frac{4x_3^2}{h^2}\right)\left(\delta\phi_y + \frac{\partial \delta w_0}{\partial x_2}\right) \tag{5c}$$

From the Eq. (17e) of the section 3:

$$\delta\gamma_5 = \delta\gamma_{13} = \left(1-\frac{4x_3^2}{h^2}\right)\left(\delta\phi_x + \frac{\partial \delta w_0}{\partial x_1}\right) \tag{5d}$$

From the Eq. (17f) of the section 3:

$$\delta\gamma_6 = \delta\gamma_{12} = \frac{\partial \delta u_0}{\partial x_2} + \frac{\partial \delta v_0}{\partial x_1} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \delta \phi_x}{\partial x_2} + $$

$$+ x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \delta \phi_y}{\partial x_1} - \frac{8x_3^3}{3h^2}\frac{\partial^2 \delta w_0}{\partial x_2 \partial x_1} \tag{5e}$$

4.2.1. Orthotropic, in-plane anisotropic material. Stress – Strain Relations

A wide range of engineering materials, including certain piezoelectric materials and fiber-reinforced composites (i.e. laminated plates composed of multiple orthotropic layers), are orthotropic. By definition an orthotropic material has at least two orthogonal planes of symmetry, where material properties are independent of the direction within each plane. Such materials require nine (9) independent variables (i.e. constants) in their constitutive matrices. In contrast, a material without any planes of symmetry is fully anisotropic and requires at least twenty-one (21) elastic constants (due to the symmetry of the constitutive matrices), whereas a material with an infinite number of symmetry planes (i.e. every plane is a plane of symmetry), is isotropic and requires only two elastic constants (Lame's constants) [*"An Introduction to Continuum Mechanics", Chapter 6 (2013), J.N. Reddy and "Theory and Analysis of Elastic Plates and Shells", Chapter 3 (2007), J.N. Reddy*].

By convention, the nine elastic constants in orthotropic constitutive equations are comprised of three Young's modulus of elasticity (E_1, E_2, E_3), three Poisson's ratios (v_{23}, v_{31}, v_{12}) or (v_4, v_5, v_6) and three shear moduli (G_{23}, G_{31}, G_{12}) or (G_4, G_5, G_6).

According to the process followed on the Lecture Notes "Stress-Strain Relations: Hooke's Law-Orthotropic Materials (First-Principle Approach)", G.A. Athanassoulis (2016), the three-dimensional compliance matrix takes the form,

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & & \\ 0 & 0 & 0 & & \frac{1}{G_{31}} & \\ 0 & 0 & 0 & & & \frac{1}{G_{12}} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix}$$

However, in the case of the TSDT, we retain only the below stress-strain relations. Thus, we get

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \text{ and}$$

$$\gamma_4 = \frac{1}{G_4}\sigma_4 , \qquad \gamma_5 = \frac{1}{G_5}\sigma_5 , \qquad \gamma_6 = \frac{1}{G_6}\sigma_6$$

Note that, in orthotropic materials there is no interaction between the normal stresses σ_1, σ_2 and shear strains $\gamma_4 = 2e_4 = 2e_{23}$, $\gamma_5 = 2e_5 = 2e_{13}$, $\gamma_6 = 2e_6 = 2e_{12}$. Further, the symmetry of the compliance coefficients leads directly to the following Symmetry for Poisson ratios:

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}$$

Now following the same path, as exactly seem on the problem of CPT, we derive the below stress-strain relationships for the case of an orthotropic material,

$$\sigma_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} e_{11} + \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} e_{22} \tag{1}$$

$$\sigma_{22} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} e_{11} + \frac{E_2}{1-\nu_{12}\nu_{21}} e_{22} \tag{2}$$

$$\sigma_{23} = \sigma_4 = G_{23}\gamma_{23} = G_4\gamma_4 \tag{3}$$

$$\sigma_{13} = \sigma_5 = G_{13}\gamma_{13} = G_5\gamma_5 \tag{4}$$

$$\sigma_{12} = \sigma_6 = G_{12}\gamma_{12} = G_6\gamma_{21} \tag{5}$$

In the sequel, we express the strains of the above Eqs. (1) - (5) in terms of displacements, substituting the Eqs. (17a), (17b), (17d), (17e) and (17f) of the section 3 on the (1) - (5). Finally,

$$\sigma_{11} = \frac{E_1}{1-v_{12}v_{21}}\left(\frac{\partial u_0}{\partial x_1} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \phi_x}{\partial x_1} - x_3^3\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_1}\right) + \\ + \frac{v_{21}E_1}{1-v_{12}v_{21}}\left(\frac{\partial v_0}{\partial x_2} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \phi_y}{\partial x_2} - x_3^3\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_2}\right) \tag{1`}$$

$$\sigma_{22} = \frac{v_{12}E_2}{1-v_{12}v_{21}}\left(\frac{\partial u_0}{\partial x_1} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \phi_x}{\partial x_1} - x_3^3\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_1}\right) + \\ + \frac{E_2}{1-v_{12}v_{21}}\left(\frac{\partial v_0}{\partial x_2} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \phi_y}{\partial x_2} - x_3^3\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_2}\right) \tag{2`}$$

$$\sigma_4 = \sigma_{23} = \sigma_{32} = = G_4\gamma_4 = G_4\left(1-\frac{4x_3^2}{h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right) \tag{3`}$$

$$\sigma_5 = \sigma_{13} = \sigma_{31} = G_5\gamma_5 = G_5\left(1-\frac{4x_3^2}{h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right) \tag{4`}$$

$$\sigma_6 = \sigma_{12} = \sigma_{21} = G_6\gamma_6 = \\ = G_6\left[\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \phi_x}{\partial x_2} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \phi_y}{\partial x_1} - \frac{8x_3^3}{3h^2}\frac{\partial^2 w_0}{\partial x_2 \partial x_1}\right] \tag{5`}$$

Note also that the higher-order plate theories are very adequate for studying composite laminate materials for which shear deformation effects are important [Oñate E., "Structural Analysis with the FEM. Linear Statics: Volume 2, Beams, Plates and Shells"]

4.2.2. Orthotropic, in-plane isotropic material. Stress – Strain Relations

The simplest way to derive the stress-strain relations of an orthotropic but in-plane isotropic material, is to notice and elaborate appropriately the stress – strain relations of the orthotropic but in-plane anisotropic plate. As exactly shown on the respective sections of the problem of the CPT, we derive by the same way the following,

$$\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\frac{E}{1-v^2} & \frac{vE}{1-v^2}\\ \frac{vE}{1-v^2} & \frac{E}{1-v^2}\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix} \Leftrightarrow \begin{Bmatrix}\sigma_1 = \frac{E}{1-v^2}e_1 + \frac{vE}{1-v^2}e_2\\ \sigma_2 = \frac{vE}{1-v^2}e_1 + \frac{E}{1-v^2}e_2\end{Bmatrix} \Rightarrow \begin{cases}\text{For the specific model of TSDT}\\ \text{we get the Eqs. (1) \& (2) below}\end{cases}$$

Thus, the in-plane stress-strain relationships are already apparent.

As for the shear stress-strain relations, we have $\sigma_4 = G\gamma_4$, $\sigma_5 = G\gamma_5$, $\sigma_6 = G\gamma_6$.

Thus, the stresses are easily obtained from the generalized Hooke's Law:

$$\sigma_{11} = \frac{E}{1-v^2}\, e_{11} + v\, e_{22} =$$

$$= \frac{E}{1-v^2}\left\{\frac{\partial u_0}{\partial x_1} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \phi_x}{\partial x_1} - x_3^3\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_1} + v\left[\frac{\partial v_0}{\partial x_2} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \phi_y}{\partial x_2} - x_3^3\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_2}\right]\right\} =$$

$$= \frac{E}{1-v^2}\left\{\left(\frac{\partial u_0}{\partial x_1} + v\frac{\partial v_0}{\partial x_2}\right) + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\left(\frac{\partial \phi_x}{\partial x_1} + v\frac{\partial \phi_y}{\partial x_2}\right) - x_3^3\frac{4}{3h^2}\left(\frac{\partial^2 w_0}{\partial^2 x_1} + v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right\} \qquad (1)$$

$$\sigma_{22} = \frac{E}{1-v^2}\, v\, e_{11} + e_{22} =$$

$$= \frac{E}{1-v^2}\left\{v\left[\frac{\partial u_0}{\partial x_1} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \phi_x}{\partial x_1} - x_3^3\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_1}\right] + \frac{\partial v_0}{\partial x_2} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \phi_y}{\partial x_2} - x_3^3\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_2}\right\} =$$

$$= \frac{E}{1-v^2}\left\{\left(v\frac{\partial u_0}{\partial x_1} + \frac{\partial v_0}{\partial x_2}\right) + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\left(v\frac{\partial \phi_x}{\partial x_1} + \frac{\partial \phi_y}{\partial x_2}\right) - x_3^3\frac{4}{3h^2}\left(v\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{\partial^2 w_0}{\partial^2 x_2}\right)\right\} \qquad (2)$$

$$\sigma_{23} = \sigma_4 = G\gamma_4 = G\left(1-\frac{4x_3^2}{h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right) \qquad (3)$$

$$\sigma_{13} = \sigma_5 = G\gamma_5 = G\left(1-\frac{4x_3^2}{h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right) \qquad (4)$$

$$\sigma_{12} = \sigma_6 = G\gamma_6 =$$

$$= G\left\{\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \phi_x}{\partial x_2} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \phi_y}{\partial x_1} - \frac{8x_3^3}{3h^2}\frac{\partial^2 w_0}{\partial x_2\,\partial x_1}\right\}$$

$$= G\left\{\left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1}\right) + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\left(\frac{\partial \phi_x}{\partial x_2} + \frac{\partial \phi_y}{\partial x_1}\right) - \frac{8x_3^3}{3h^2}\frac{\partial^2 w_0}{\partial x_2\,\partial x_1}\right\} \qquad (5)$$

In conclusion, note that the same remark have made on the closure of the respective section 4.2.2 of the CPT (Part A), are valid here and used in the model of TSDT developed on this part (Part B).

4.3. Variation of the Elastic Potential Part – U (strain energy)

Due to the proof of the previous section 4.2, the only variations appearing on the variation of the elastic potential energy are the variations of strains and not those of stresses. The variations of stresses are not appear explicitly on the following relation (or the Eq. 4 of the section 4.2), because they are a priori included in the variation of the elastic potential energy. This fact is declared on the section 4.2 by the use of the Hooke's Law in conjunction with the identity of symmetry of matrix composed of the stiffness coefficients as well as the contrivances of the index notation.

Now keeping in mind the Eqs. (1`) – (5`) of the section 4.2.1 or Eqs. (1) - (5) of the section 4.2.2 and we are not going to replace the last, in order to avoid difficult and time-consuming calculations. Thus, the Eq. (4) of the section 4.2 using (17a) - (17f) is converted to,

$$\begin{aligned}
\delta U(\boldsymbol{e}) = {} & \sigma_{11}\left[\frac{\partial \delta u_0}{\partial x_1} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \delta\phi_x}{\partial x_1} - x_3^3\frac{4}{3h^2}\frac{\partial^2 \delta w_0}{\partial^2 x_1}\right] + \\
& + \sigma_{22}\left[\frac{\partial \delta v_0}{\partial x_2} + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \delta\phi_y}{\partial x_2} - x_3^3\frac{4}{3h^2}\frac{\partial^2 \delta w_0}{\partial^2 x_2}\right] + \\
& + \sigma_{23}\left(1-\frac{4x_3^2}{h^2}\right)\left(\delta\phi_y + \frac{\partial \delta w_0}{\partial x_2}\right) + \sigma_{13}\left(1-\frac{4x_3^2}{h^2}\right)\left(\delta\phi_x + \frac{\partial \delta w_0}{\partial x_1}\right) + \\
& + \sigma_{12}\left[\left(\frac{\partial \delta u_0}{\partial x_2} + \frac{\partial \delta v_0}{\partial x_1}\right) + x_3\left(1-\frac{4x_3^2}{3h^2}\right)\left(\frac{\partial \delta\phi_x}{\partial x_2} + \frac{\partial \delta\phi_y}{\partial x_1}\right) - \frac{8x_3^3}{3h^2}\frac{\partial^2 \delta w_0}{\partial x_2 \partial x_1}\right] \qquad (1)
\end{aligned}$$

Finally substituting Eq. (1) to the expression of the variation of the elastic potential part of the variation of action functional (1) of the section 3, we derive the last expression:

$$\begin{aligned}
\delta J_U = {} & \int_{t_1}^{t_2}\iiint_B \delta U(\boldsymbol{e})\, dV\, dt = \int_{t_1}^{t_2}\iiint_B \left(\begin{matrix}\sigma_1\delta e_1 + \sigma_2\delta e_2 + \sigma_4\delta\gamma_4 + \\ + \sigma_5\delta\gamma_5 + \sigma_6\delta\gamma_6\end{matrix}\right) dV dt = \\
= {} & \int_{t_1}^{t_2}\iiint_B \left\{\sigma_1\frac{\partial \delta u_0}{\partial x_1}\right\} dV\, dt + \int_{t_1}^{t_2}\iiint_B \left\{\sigma_1 x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \delta\phi_x}{\partial x_1}\right\} dV\, dt - \\
& - \int_{t_1}^{t_2}\iiint_B \left\{\sigma_1 x_3^3\frac{4}{3h^2}\frac{\partial^2 \delta w_0}{\partial^2 x_1}\right\} dV\, dt + \int_{t_1}^{t_2}\iiint_B \left\{\sigma_2\frac{\partial \delta v_0}{\partial x_2}\right\} dV\, dt + \\
& + \int_{t_1}^{t_2}\iiint_B \left\{\sigma_2 x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial \delta\phi_y}{\partial x_2}\right\} dV\, dt - \int_{t_1}^{t_2}\iiint_B \left\{\sigma_2 x_3^3\frac{4}{3h^2}\frac{\partial^2 \delta w_0}{\partial^2 x_2}\right\} dV\, dt +
\end{aligned}$$

$$+\int_{t_1}^{t_2}\iiint_B\left\{\sigma_4\left(1-\frac{4x_3^2}{h^2}\right)\delta\phi_y\right\}dV\,dt+\int_{t_1}^{t_2}\iiint_B\left\{\sigma_4\left(1-\frac{4x_3^2}{h^2}\right)\frac{\partial\delta w_0}{\partial x_2}\right\}dV\,dt+$$

$$+\int_{t_1}^{t_2}\iiint_B\left\{\sigma_5\left(1-\frac{4x_3^2}{h^2}\right)\delta\phi_x\right\}dV\,dt+\int_{t_1}^{t_2}\iiint_B\left\{\sigma_5\left(1-\frac{4x_3^2}{h^2}\right)\frac{\partial\delta w_0}{\partial x_1}\right\}dV\,dt+$$

$$+\int_{t_1}^{t_2}\iiint_B\left\{\sigma_6\frac{\partial\delta u_0}{\partial x_2}\right\}dV\,dt+\int_{t_1}^{t_2}\iiint_B\left\{\sigma_6\frac{\partial\delta v_0}{\partial x_1}\right\}dV\,dt+$$

$$+\int_{t_1}^{t_2}\iiint_B\left\{\sigma_6 x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial\delta\phi_x}{\partial x_2}\right\}dV\,dt+\int_{t_1}^{t_2}\iiint_B\left\{\sigma_6 x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial\delta\phi_y}{\partial x_1}\right\}dV\,dt-$$

$$-\int_{t_1}^{t_2}\iiint_B\left\{\sigma_6\frac{8x_3^3}{3h^2}\frac{\partial^2\delta w_0}{\partial x_2\,\partial x_1}\right\}dV\,dt \tag{2}$$

In the last Eq. (2) appear only the variations $\delta\phi_x$ and $\delta\phi_y$, but the most terms (integrals) include first or second derivatives of the variations (δu_0, δv_0, δw_0, $\delta\phi_x$, $\delta\phi_y$). Apparently, to eliminate the later we perform (by case) **integration(s) by parts with respect to the spatial variable** x. These integrations by parts will generate boundary terms kinematic and dynamic, which will contribute to the construction of the appropriate boundary conditions of the TSDT. For further simplification we neglect for the present calculations the time integral. We have also to pay attention to the integration(s) by parts with respect to the spatial variable, because the boundary terms of the following relations are related with the natural boundary conditions of the problem (or dynamic boundary conditions of the elastic continuum). Thus,

$$\delta I_{\delta u_0,x_1}=\iiint_B\sigma_1\frac{\partial\delta u_0}{\partial x_1}dV=\iint_{\partial B^{(lat)}}\sigma_1\delta u_0 n_{x_1}\,dS-\iiint_B\frac{\partial\sigma_1}{\partial x_1}\delta u_0\,dV \tag{3a}$$

$$\delta I_{\delta\phi_x,x_1}=\iiint_B\sigma_1 x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial\delta\phi_x}{\partial x_1}dV=$$

$$=\iint_{\partial B^{(lat)}}\sigma_1 x_3\left(1-\frac{4x_3^2}{3h^2}\right)\delta\phi_x n_{x_1}\,dS-\iiint_B\frac{\partial\sigma_1}{\partial x_1}x_3\left(1-\frac{4x_3^2}{3h^2}\right)\delta\phi_x\,dV \tag{3b}$$

$$\delta I_{\delta w_0,x_1,x_1}=\iiint_B\sigma_1 x_3^3\frac{4}{3h^2}\frac{\partial^2\delta w_0}{\partial^2 x_1}dV=$$

$$=\iint_{\partial B^{(lat)}}\sigma_1 x_3^3\frac{4}{3h^2}\frac{\partial\delta w_0}{\partial x_1}n_{x_1}\,dS-\iiint_B\frac{\partial\sigma_1}{\partial x_1}x_3^3\frac{4}{3h^2}\frac{\partial\delta w_0}{\partial x_1}dV=$$

$$= \iint_{\partial B^{(lat)}} \sigma_1 x_3^3 \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_1} n_{x_1} \, dS \; -$$

$$- \iint_{\partial B^{(lat)}} \frac{\partial \sigma_1}{\partial x_1} x_3^3 \frac{4}{3h^2} \delta w_0 \, n_{x_1} \, dS \; + \iiint_B \frac{\partial^2 \sigma_1}{\partial^2 x_1} x_3^3 \frac{4}{3h^2} \delta w_0 \, dV \qquad (3c)$$

$$\delta I_{\delta v_0, x_2} \; = \iiint_B \sigma_2 \frac{\partial \delta v_0}{\partial x_2} dV = \iint_{\partial B^{(lat)}} \sigma_2 \, \delta v_0 \, n_{x_2} \, dS \; - \iiint_B \frac{\partial \sigma_2}{\partial x_2} \delta v_0 \, dV \qquad (3d)$$

$$\delta I_{\delta \phi_y, x_2} \; = \iiint_B \sigma_2 x_3 \left(1 - \frac{4x_3^2}{3h^2} \right) \frac{\partial \delta \phi_y}{\partial x_2} dV \; =$$

$$= \iint_{\partial B^{(lat)}} \sigma_2 \, x_3 \left(1 - \frac{4x_3^2}{3h^2} \right) \delta \phi_y \, n_{x_2} \, dS \; - \iiint_B \frac{\partial \sigma_2}{\partial x_2} x_3 \left(1 - \frac{4x_3^2}{3h^2} \right) \delta \phi_y \, dV \qquad (3e)$$

$$\delta I_{\delta w_0, x_2, x_2} \; = \iiint_B \sigma_2 x_3^3 \frac{4}{3h^2} \frac{\partial^2 \delta w_0}{\partial^2 x_2} dV \; =$$

$$= \iint_{\partial B^{(lat)}} \sigma_2 \, x_3^3 \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_2} n_{x_2} \, dS \; - \iiint_B \frac{\partial \sigma_2}{\partial x_2} x_3^3 \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_2} dV \; =$$

$$= \iint_{\partial B^{(lat)}} \sigma_2 \, x_3^3 \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_2} n_{x_2} \, dS \; -$$

$$- \iint_{\partial B^{(lat)}} \frac{\partial \sigma_2}{\partial x_2} x_3^3 \frac{4}{3h^2} \delta w_0 \, n_{x_2} \, dS \; + \iiint_B \frac{\partial^2 \sigma_2}{\partial^2 x_2} x_3^3 \frac{4}{3h^2} \delta w_0 \, dV \qquad (3f)$$

$$\delta I_{\delta w_0, x_2} \; = \iiint_B \sigma_4 \left(1 - \frac{4x_3^2}{h^2} \right) \frac{\partial \delta w_0}{\partial x_2} dV =$$

$$= \iint_{\partial B^{(lat)}} \sigma_4 \left(1 - \frac{4x_3^2}{h^2} \right) \delta w_0 \, n_{x_2} \, dS - \iiint_B \frac{\partial \sigma_4}{\partial x_2} \left(1 - \frac{4x_3^2}{h^2} \right) \delta w_0 \, dV \qquad (3g)$$

$$\delta I_{\delta w_0, x_1} \; = \iiint_B \sigma_5 \left(1 - \frac{4x_3^2}{h^2} \right) \frac{\partial \delta w_0}{\partial x_1} dV =$$

$$= \iint_{\partial B^{(lat)}} \sigma_5 \left(1 - \frac{4x_3^2}{h^2} \right) \delta w_0 \, n_{x_1} \, dS - \iiint_B \frac{\partial \sigma_5}{\partial x_1} \left(1 - \frac{4x_3^2}{h^2} \right) \delta w_0 \, dV \qquad (3h)$$

$$\delta I_{\delta u_0, x_2} \; = \iiint_B \sigma_6 \frac{\partial \delta u_0}{\partial x_2} dV \; = \iint_{\partial B^{(lat)}} \sigma_6 \, \delta u_0 \, n_{x_2} \, dS \; - \iiint_B \frac{\partial \sigma_6}{\partial x_2} \delta u_0 \, dV \qquad (3i)$$

$$\delta I_{\delta v_0, x_1} = \iiint_B \sigma_6 \frac{\partial \delta v_0}{\partial x_1} dV = \iint_{\partial B^{(lat)}} \sigma_6 \, \delta v_0 \, n_{x_1} \, dS - \iiint_B \frac{\partial \sigma_6}{\partial x_1} \delta v_0 \, dV \quad (3j)$$

$$\delta I_{\delta \phi_x, x_2} = \iiint_B \sigma_6 \, x_3 \left(1 - \frac{4x_3^2}{3h^2} \right) \frac{\partial \delta \phi_x}{\partial x_2} dV =$$

$$= \iint_{\partial B^{(lat)}} \sigma_6 \, x_3 \left(1 - \frac{4x_3^2}{3h^2} \right) \delta \phi_x \, n_{x_2} \, dS - \iiint_B \frac{\partial \sigma_6}{\partial x_2} x_3 \left(1 - \frac{4x_3^2}{3h^2} \right) \delta \phi_x \, dV \quad (3k)$$

$$\delta I_{\delta \phi_y, x_1} = \iiint_B \sigma_6 \, x_3 \left(1 - \frac{4x_3^2}{3h^2} \right) \frac{\partial \delta \phi_y}{\partial x_1} dV =$$

$$= \iint_{\partial B^{(lat)}} \sigma_6 \, x_3 \left(1 - \frac{4x_3^2}{3h^2} \right) \delta \phi_y \, n_{x_1} \, dS - \iiint_B \frac{\partial \sigma_6}{\partial x_1} x_3 \left(1 - \frac{4x_3^2}{3h^2} \right) \delta \phi_y \, dV \quad (3l)$$

$$\delta I_{\delta w_0, x_1, x_2} = \iiint_B \sigma_6 \frac{8x_3^3}{3h^2} \frac{\partial^2 \delta w_0}{\partial x_2 \, \partial x_1} dV \quad (3m)$$

As for the different handling of the integral (3m), where going to present it thoroughly below.

Now following the same process (as those of the kinetic part) of decomposition of the volume and surface integrals, the above relations take a new form including additionally the known stress resultants (bending moments and shear stresses), the **higher-order stress resultants**.

$$\begin{Bmatrix} N_1 \\ M_1 \\ P_1 \end{Bmatrix} = \begin{Bmatrix} N_{11} \\ M_{11} \\ P_{11} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{11} \begin{Bmatrix} 1 \\ x_3 \\ x_3^3 \end{Bmatrix} dx_3 = \int_{-h/2}^{h/2} \sigma_1 \begin{Bmatrix} 1 \\ x_3 \\ x_3^3 \end{Bmatrix} dx_3 \ ,$$

$$\begin{Bmatrix} N_2 \\ M_2 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} N_{22} \\ M_{22} \\ P_{22} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{22} \begin{Bmatrix} 1 \\ x_3 \\ x_3^3 \end{Bmatrix} dx_3 = \int_{-h/2}^{h/2} \sigma_2 \begin{Bmatrix} 1 \\ x_3 \\ x_3^3 \end{Bmatrix} dx_3 \ ,$$

$$\begin{Bmatrix} N_6 \\ M_6 \\ P_6 \end{Bmatrix} = \begin{Bmatrix} N_{12} \\ M_{12} \\ P_{12} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{12} \begin{Bmatrix} 1 \\ x_3 \\ x_3^3 \end{Bmatrix} dx_3 = \int_{-h/2}^{h/2} \sigma_6 \begin{Bmatrix} 1 \\ x_3 \\ x_3^3 \end{Bmatrix} dx_3$$

$$\begin{Bmatrix} Q_4 \\ R_4 \end{Bmatrix} = \begin{Bmatrix} Q_{23} \\ R_{23} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{23} \begin{Bmatrix} 1 \\ x_3^2 \end{Bmatrix} dx_3 = \int_{-h/2}^{h/2} \sigma_4 \begin{Bmatrix} 1 \\ x_3^2 \end{Bmatrix} dx_3$$

$$\begin{Bmatrix} Q_5 \\ R_5 \end{Bmatrix} = \begin{Bmatrix} Q_{13} \\ R_{13} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{13} \begin{Bmatrix} 1 \\ x_3^2 \end{Bmatrix} dx_3 = \int_{-h/2}^{h/2} \sigma_5 \begin{Bmatrix} 1 \\ x_3^2 \end{Bmatrix} dx_3$$

Thus, to calculate the variation of the elastic potential part, we have to define additionally the stress resultants as,

the thickness-integrated forces $(N_{11}, N_{22}, N_{12}) = (N_1, N_2, N_6)$,

the thickness-integrated moments $(M_{11}, M_{22}, M_{12}) = (M_1, M_2, M_6)$,

the thickness-integrated higher moments $(P_{11}, P_{22}, P_{12}) = (P_1, P_2, P_6)$ and $(R_{23}, R_{13}) = (R_4, R_5)$,

and finally the **shear forces** $(Q_{23}, Q_{13}) = (Q_4, Q_5)$,

which are called alternatively as **stress resultants**.

The aforementioned higher-order moments are mathematically similar to the conventional moments. They represent the internal actions between the parts of the thick plate and they are reckoned across its whole thickness [*A.E.H. Love (1944), "A Treatise on the Mathematical Theory of Elasticity"*].

It is essential also to clear that the above stress resultants (as those of the CPT) are nothing more than "abbreviations" of the stress field of the material. By this way, we gather together the components of the stress field, which are expressed in terms of the displacement field (u_0, v_0, w_0, ϕ_x, ϕ_y), namely the unknowns, as shown on the Eqs. (1`) - (5`) of the section 4.2.1 and (1) – (5) of the section 4.2.2.

Consequently, substituting the Eqs. (1`) - (5`) of the section 4.2.1 and (1) – (5) of the section 4.2.2, into the relations of stress resultants, we can express the thickness-integrated moments, forces, higher moments and shear forces in terms of displacement field (u_0, v_0, w_0, ϕ_x, ϕ_y).

Thus, the previously referred relations are going to be presented on the appropriate following sections in order to derive easier the equations of motion and the boundary conditions of the plate (in terms of displacement field). Further remark that the total number of the resulting scalar equations of the problem should be the same with the number of unknowns so that our problem has a unique solution. In our case of the model of TSDT, the number of unknowns is five. Consequently, we expect to derive five equations from the variational principle, including the unknowns (u_0, v_0, w_0, ϕ_x, ϕ_y), and finally solve the 5x5 system.

Now in order to derive the final terms of the previous relations (3a) - (3m), is followed the same process (as those of the kinetic part) of the decomposition of the volume and surface integrals. Thus, the above relations take the now form (shown below) including the previously defined stress resultants.

$$\delta I_{\delta u_0, x_1} = \int_{-h/2}^{h/2} \oint_{\Gamma} \sigma_1 \delta u_0 n_{x_1} d\gamma\, dx_3 - \int_{-h/2}^{h/2} \iint_{\Omega} \frac{\partial \sigma_1}{\partial x_1} \delta u_0\, d\omega\, dx_3 =$$

$$= \oint_{\Gamma} N_1 \delta u_0 n_{x_1} d\gamma - \iint_{\Omega} \frac{\partial N_1}{\partial x_1} \delta u_0 \, d\omega \tag{4a}$$

$$\delta I_{\delta\phi_x, x_1} = \oint_{\Gamma} \int_{-h/2}^{h/2} \sigma_1 x_3 \left(1 - \frac{4x_3^2}{3h^2}\right) \delta\phi_x n_{x_1} dx_3 \, d\gamma - \iint_{\Omega} \int_{-h/2}^{h/2} \frac{\partial \sigma_1}{\partial x_1} x_3 \left(1 - \frac{4x_3^2}{3h^2}\right) \delta\phi_x \, dx_3 \, d\omega =$$

$$= \oint_{\Gamma} \left(M_1 - \frac{4}{3h^2} P_1\right) \delta\phi_x n_{x_1} d\gamma - \iint_{\Omega} \left(\frac{\partial M_1}{\partial x_1} - \frac{\partial P_1}{\partial x_1} \frac{4}{3h^2}\right) \delta\phi_x \, d\omega \tag{4b}$$

$$\delta I_{\delta w_0, x_1, x_1} = \oint_{\Gamma} \int_{-h/2}^{h/2} \sigma_1 x_3^3 \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_1} n_{x_1} dx_3 \, d\gamma - \oint_{\Gamma} \int_{-h/2}^{h/2} \frac{\partial \sigma_1}{\partial x_1} x_3^3 \frac{4}{3h^2} \delta w_0 n_{x_1} dx_3 \, d\gamma +$$

$$+ \iint_{\Omega} \int_{-h/2}^{h/2} \frac{\partial^2 \sigma_1}{\partial^2 x_1} x_3^3 \frac{4}{3h^2} \delta w_0 \, dx_3 \, d\omega =$$

$$= \oint_{\Gamma} P_1 \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_1} n_{x_1} d\gamma - \oint_{\Gamma} \frac{\partial P_1}{\partial x_1} \frac{4}{3h^2} \delta w_0 n_{x_1} d\gamma + \iint_{\Omega} \frac{\partial^2 P_1}{\partial^2 x_1} \frac{4}{3h^2} \delta w_0 \, d\omega \tag{4c}$$

$$\delta I_{\delta v_0, x_2} = \iint_{\partial B^{(lat)}} \sigma_2 \delta v_0 n_{x_2} dS - \iiint_{B} \frac{\partial \sigma_2}{\partial x_2} \delta v_0 \, dV =$$

$$= \oint_{\Gamma} \int_{-h/2}^{h/2} \sigma_2 \delta v_0 n_{x_2} dx_3 \, d\gamma - \iint_{\Omega} \frac{\partial}{\partial x_2} \int_{-h/2}^{h/2} \sigma_2 \delta v_0 \, dx_3 \, d\omega =$$

$$= \oint_{\Gamma} N_2 \delta v_0 n_{x_2} d\gamma - \iint_{\Omega} \frac{\partial N_2}{\partial x_2} \delta v_0 \, d\omega \tag{4d}$$

$$\delta I_{\delta\phi_y, x_2} = \oint_{\Gamma} \int_{-h/2}^{h/2} \sigma_2 x_3 \left(1 - \frac{4x_3^2}{3h^2}\right) \delta\phi_y n_{x_2} dx_3 \, d\gamma - \iint_{\Omega} \int_{-h/2}^{h/2} \frac{\partial \sigma_2}{\partial x_2} x_3 \left(1 - \frac{4x_3^2}{3h^2}\right) \delta\phi_y \, dx_3 \, d\omega =$$

$$= \oint_{\Gamma} \left(M_2 - \frac{4}{3h^2} P_2\right) \delta\phi_y n_{x_2} d\gamma - \iint_{\Omega} \left(\frac{\partial M_2}{\partial x_2} - \frac{\partial P_2}{\partial x_2} \frac{4}{3h^2}\right) \delta\phi_y \, d\omega \tag{4e}$$

$$\delta I_{\delta w_0, x_2, x_2} = \oint_{\Gamma} P_2 \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_2} n_{x_2} d\gamma - \oint_{\Gamma} \frac{\partial P_2}{\partial x_2} \frac{4}{3h^2} \delta w_0 n_{x_2} d\gamma +$$

$$+ \iint_{\Omega} \frac{\partial^2 P_2}{\partial^2 x_2} \frac{4}{3h^2} \delta w_0 \, d\omega \tag{4f}$$

[calculations similar to the Eq. (4c)]

$$\delta I_{\delta w_0, x_2} = \iint\limits_{\partial B^{(lat)}} \sigma_4 \left(1 - \frac{4x_3^2}{h^2}\right) \delta w_0 \, n_{x_2} \, dS - \iiint\limits_{B} \frac{\partial \sigma_4}{\partial x_2} \left(1 - \frac{4x_3^2}{h^2}\right) \delta w_0 \, dV =$$

$$= \oint\limits_{\Gamma} \int\limits_{-h/2}^{h/2} \sigma_4 \left(1 - \frac{4x_3^2}{h^2}\right) \delta w_0 \, n_{x_2} \, dx_3 \, d\gamma - \iint\limits_{\Omega} \int\limits_{-h/2}^{h/2} \frac{\partial \sigma_4}{\partial x_2} \left(1 - \frac{4x_3^2}{h^2}\right) \delta w_0 \, dx_3 \, d\omega =$$

$$= \oint\limits_{\Gamma} \left(Q_4 - R_4 \frac{4}{h^2} \right) \delta w_0 \, n_{x_2} \, d\gamma - \iint\limits_{\Omega} \left(\frac{\partial Q_4}{\partial x_2} - \frac{\partial R_4}{\partial x_2} \frac{4}{h^2} \right) \delta w_0 \, d\omega \qquad (4g)$$

$$\delta I_{\delta w_0, x_1} = \iint\limits_{\partial B^{(lat)}} \sigma_5 \left(1 - \frac{4x_3^2}{h^2}\right) \delta w_0 \, n_{x_1} \, dS - \iiint\limits_{B} \frac{\partial \sigma_5}{\partial x_1} \left(1 - \frac{4x_3^2}{h^2}\right) \delta w_0 \, dV =$$

$$= \oint\limits_{\Gamma} \int\limits_{-h/2}^{h/2} \sigma_5 \left(1 - \frac{4x_3^2}{h^2}\right) \delta w_0 \, n_{x_1} \, dx_3 \, d\gamma - \iint\limits_{\Omega} \int\limits_{-h/2}^{h/2} \frac{\partial \sigma_5}{\partial x_1} \left(1 - \frac{4x_3^2}{h^2}\right) \delta w_0 \, dx_3 \, d\omega =$$

$$= \oint\limits_{\Gamma} \left(Q_5 - R_5 \frac{4}{h^2} \right) \delta w_0 \, n_{x_1} \, d\gamma - \iint\limits_{\Omega} \left(\frac{\partial Q_5}{\partial x_1} - \frac{\partial R_5}{\partial x_1} \frac{4}{h^2} \right) \delta w_0 \, d\omega \qquad (4h)$$

$$\delta I_{\delta u_0, x_2} = \iint\limits_{\partial B^{(lat)}} \sigma_6 \, \delta u_0 \, n_{x_2} \, dS - \iiint\limits_{B} \frac{\partial \sigma_6}{\partial x_2} \delta u_0 \, dV =$$

$$= \oint\limits_{\Gamma} \int\limits_{-h/2}^{h/2} \sigma_6 \, \delta u_0 \, n_{x_2} \, dx_3 \, d\gamma - \iint\limits_{\Omega} \int\limits_{-h/2}^{h/2} \frac{\partial \sigma_6}{\partial x_2} \delta u_0 \, dx_3 \, d\omega =$$

$$= \oint\limits_{\Gamma} N_6 \, \delta u_0 \, n_{x_2} \, d\gamma - \iint\limits_{\Omega} \frac{\partial N_6}{\partial x_2} \delta u_0 \, d\omega \qquad (4i)$$

$$\delta I_{\delta v_0, x_1} = \iint\limits_{\partial B^{(lat)}} \sigma_6 \, \delta v_0 \, n_{x_1} \, dS - \iiint\limits_{B} \frac{\partial \sigma_6}{\partial x_1} \delta v_0 \, dV =$$

$$= \oint\limits_{\Gamma} \int\limits_{-h/2}^{h/2} \sigma_6 \, \delta v_0 \, n_{x_1} \, dx_3 \, d\gamma - \iint\limits_{\Omega} \int\limits_{-h/2}^{h/2} \frac{\partial \sigma_6}{\partial x_1} \delta v_0 \, dx_3 \, d\omega =$$

$$= \oint\limits_{\Gamma} N_6 \, \delta v_0 \, n_{x_1} \, d\gamma - \iint\limits_{\Omega} \frac{\partial N_6}{\partial x_1} \delta v_0 \, d\omega \qquad (4j)$$

$$\delta I_{\delta\phi_x,x_2} = \iint\limits_{\partial B^{(lat)}} \sigma_6 x_3\left(1-\frac{4x_3^2}{3h^2}\right)\delta\phi_x n_{x_2}\, dS - \iiint\limits_B \frac{\partial\sigma_6}{\partial x_2} x_3\left(1-\frac{4x_3^2}{3h^2}\right)\delta\phi_x\, dV =$$

$$= \oint\limits_\Gamma \int\limits_{-h/2}^{h/2}\left(\sigma_6 x_3 - \sigma_6\frac{4x_3^3}{3h^2}\right)\delta\phi_x n_{x_2}\, dx_3\, d\gamma - \iint\limits_\Omega \int\limits_{-h/2}^{h/2}\left(\frac{\partial\sigma_6}{\partial x_2}x_3 - \frac{\partial\sigma_6}{\partial x_2}\frac{4x_3^3}{3h^2}\right)\delta\phi_x\, dx_3\, d\omega =$$

$$= \oint\limits_\Gamma \int\limits_{-h/2}^{h/2}\left(\sigma_6 x_3 - \sigma_6\frac{4x_3^3}{3h^2}\right)dx_3\,\delta\phi_x n_{x_2}\, d\gamma - \iint\limits_\Omega \frac{\partial}{\partial x_2}\int\limits_{-h/2}^{h/2}\left(\sigma_6 x_3 - \sigma_6\frac{4x_3^3}{3h^2}\right)dx_3\,\delta\phi_x\, d\omega =$$

$$= \oint\limits_\Gamma \left(M_6 - P_6\frac{4}{3h^2}\right)\delta\phi_x n_{x_2}\, d\gamma - \iint\limits_\Omega\left(\frac{\partial M_6}{\partial x_2} - \frac{\partial P_6}{\partial x_2}\frac{4}{3h^2}\right)\delta\phi_x\, d\omega \qquad \text{(4k)}$$

$$\delta I_{\delta\phi_y,x_1} = \iint\limits_{\partial B^{(lat)}} \sigma_6 x_3\left(1-\frac{4x_3^2}{3h^2}\right)\delta\phi_y n_{x_1}\, dS - \iiint\limits_B \frac{\partial\sigma_6}{\partial x_1} x_3\left(1-\frac{4x_3^2}{3h^2}\right)\delta\phi_y\, dV =$$

$$= \oint\limits_\Gamma \int\limits_{-h/2}^{h/2}\left(\sigma_6 x_3 - \sigma_6\frac{4x_3^3}{3h^2}\right)dx_3\,\delta\phi_y n_{x_1}\, d\gamma - \iint\limits_\Omega \frac{\partial}{\partial x_1}\int\limits_{-h/2}^{h/2}\left(\sigma_6 x_3 - \sigma_6\frac{4x_3^3}{3h^2}\right)dx_3\,\delta\phi_y\, d\omega =$$

$$= \oint\limits_\Gamma \left(M_6 - P_6\frac{4}{3h^2}\right)\delta\phi_y n_{x_1}\, d\gamma - \iint\limits_\Omega\left(\frac{\partial M_6}{\partial x_1} - \frac{\partial P_6}{\partial x_1}\frac{4}{3h^2}\right)\delta\phi_y\, d\omega \qquad \text{(4l)}$$

In order to derive the final terms of the below volume integral $\delta I_{\delta w_0,x_1,x_2}$, we follow a different path.

$$\delta I_{\delta w_0,x_1,x_2} = \iiint\limits_B \sigma_6\frac{8x_3^3}{3h^2}\frac{\partial^2\delta w_0}{\partial x_2\,\partial x_1}dV =$$

$$= \underbrace{\iiint\limits_B \sigma_6\frac{4x_3^3}{3h^2}\frac{\partial^2\delta w_0}{\partial x_2\,\partial x_1}dV}_{J_1} + \underbrace{\iiint\limits_B \sigma_6\frac{4x_3^3}{3h^2}\frac{\partial^2\delta w_0}{\partial x_2\,\partial x_1}dV}_{J_2} = J_1 + J_2 \qquad \text{(4m)}$$

As for the first term J_1, we perform integrations by parts firstly according to x_1 and second according to x_2 variable, while as for the second term J_2, we perform integrations by parts initially according to x_2 and subsequently according to x_1 spatial variable. This concept is adopted to the following calculations, because we desire to derive boundary conditions with a “symmetric” formulation between the terms (of the variation of the action functional) with the same variations $(\delta u_0, \delta v_0, \delta w_0)$.

Thus,

$$J_1 = \iiint_B \sigma_6 \frac{4x_3^3}{3h^2}\frac{\partial^2 \delta w_0}{\partial x_2 \partial x_1} dV = \iint_{\partial B^{(lat)}} \sigma_6 \frac{4x_3^3}{3h^2}\frac{\partial \delta w_0}{\partial x_2} n_{x_1}\, dS - \iiint_B \frac{\partial \sigma_6}{\partial x_1}\frac{4x_3^3}{3h^2}\frac{\partial \delta w_0}{\partial x_2} dV =$$

$$= \iint_{\partial B^{(lat)}} \sigma_6 \frac{4x_3^3}{3h^2}\frac{\partial \delta w_0}{\partial x_2} n_{x_1}\, dS - \iint_{\partial B^{(lat)}} \frac{\partial \sigma_6}{\partial x_1}\frac{4x_3^3}{3h^2}\delta w_0\, n_{x_2}\, dS + \iiint_B \frac{\partial^2 \sigma_6}{\partial x_2 \partial x_1}\frac{4x_3^3}{3h^2}\delta w_0\, dV =$$

$$= \oint_\Gamma \int_{-h/2}^{h/2} \sigma_6 \frac{4x_3^3}{3h^2}\frac{\partial \delta w_0}{\partial x_2} n_{x_1}\, dx_3\, d\gamma - \oint_\Gamma \int_{-h/2}^{h/2} \frac{\partial \sigma_6}{\partial x_1}\frac{4x_3^3}{3h^2}\delta w_0\, n_{x_2}\, dx_3\, d\gamma +$$

$$+ \iint_\Omega \int_{-h/2}^{h/2} \frac{\partial^2 \sigma_6}{\partial x_2 \partial x_1}\frac{4x_3^3}{3h^2}\delta w_0\, dx_3\, d\omega =$$

$$= \oint_\Gamma P_6 \frac{4}{3h^2}\frac{\partial \delta w_0}{\partial x_2} n_{x_1}\, d\gamma - \oint_\Gamma \frac{\partial P_6}{\partial x_1}\frac{4}{3h^2}\delta w_0\, n_{x_2}\, d\gamma + \iint_\Omega \frac{\partial^2 P_6}{\partial x_2 \partial x_1}\frac{4}{3h^2}\delta w_0\, d\omega$$

and

$$J_2 = \iiint_B \sigma_6 \frac{4x_3^3}{3h^2}\frac{\partial^2 \delta w_0}{\partial x_2 \partial x_1} dV = \iint_{\partial B^{(lat)}} \sigma_6 \frac{4x_3^3}{3h^2}\frac{\partial \delta w_0}{\partial x_1} n_{x_2}\, dS - \iiint_B \frac{\partial \sigma_6}{\partial x_2}\frac{4x_3^3}{3h^2}\frac{\partial \delta w_0}{\partial x_1} dV =$$

$$= \iint_{\partial B^{(lat)}} \sigma_6 \frac{4x_3^3}{3h^2}\frac{\partial \delta w_0}{\partial x_1} n_{x_2}\, dS - \iint_{\partial B^{(lat)}} \frac{\partial \sigma_6}{\partial x_2}\frac{4x_3^3}{3h^2}\delta w_0\, n_{x_1}\, dS + \iiint_B \frac{\partial^2 \sigma_6}{\partial x_1 \partial x_2}\frac{4x_3^3}{3h^2}\delta w_0\, dV =$$

$$= \oint_\Gamma \int_{-h/2}^{h/2} \sigma_6 \frac{4x_3^3}{3h^2}\frac{\partial \delta w_0}{\partial x_1} n_{x_2}\, dx_3\, d\gamma - \oint_\Gamma \int_{-h/2}^{h/2} \frac{\partial \sigma_6}{\partial x_2}\frac{4x_3^3}{3h^2}\delta w_0\, n_{x_1}\, dx_3\, d\gamma +$$

$$+ \iint_\Omega \int_{-h/2}^{h/2} \frac{\partial^2 \sigma_6}{\partial x_1 \partial x_2}\frac{4x_3^3}{3h^2}\delta w_0\, dx_3\, d\omega =$$

$$= \oint_\Gamma P_6 \frac{4}{3h^2}\frac{\partial \delta w_0}{\partial x_1} n_{x_2}\, d\gamma - \oint_\Gamma \frac{\partial P_6}{\partial x_2}\frac{4}{3h^2}\delta w_0\, n_{x_1}\, d\gamma + \iint_\Omega \frac{\partial^2 P_6}{\partial x_1 \partial x_2}\frac{4}{3h^2}\delta w_0\, d\omega$$

Consequently, the equation (4m) is converted to,

$$\delta I_{\delta w_0, x_1, x_2} = \oint_\Gamma P_6 \frac{4}{3h^2}\left(\frac{\partial \delta w_0}{\partial x_2} n_{x_1} + \frac{\partial \delta w_0}{\partial x_1} n_{x_2}\right) d\gamma - \oint_\Gamma \frac{4}{3h^2}\left(\frac{\partial P_6}{\partial x_1} n_{x_2} + \frac{\partial P_6}{\partial x_2} n_{x_1}\right)\delta w_0\, d\gamma +$$

$$+ \iint_\Omega \frac{\partial^2 P_6}{\partial x_2 \partial x_1}\frac{8}{3h^2}\delta w_0\, d\omega \qquad (4m)$$

Now substituting (4a) - (4m) into the equation (2), we derive the following,

$$\delta J_U = \int_{t_1}^{t_2} \iiint_B \delta U(\boldsymbol{e})\, dV\, dt = \int_{t_1}^{t_2} \oint_\Gamma N_1\, \delta u_0\, n_{x_1}\, d\gamma\, dt - \int_{t_1}^{t_2} \iint_\Omega \frac{\partial N_1}{\partial x_1} \delta u_0\, d\omega\, dt +$$

$$+\int_{t_1}^{t_2} \oint_\Gamma \left(M_1 - \frac{4}{3h^2} P_1 \right) \delta\phi_x\, n_{x_1}\, d\gamma\, dt - \int_{t_1}^{t_2} \iint_\Omega \left(\frac{\partial M_1}{\partial x_1} - \frac{4}{3h^2} \frac{\partial P_1}{\partial x_1} \right) \delta\phi_x\, d\omega\, dt -$$

$$-\oint_\Gamma P_1 \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_1} n_{x_1}\, d\gamma + \oint_\Gamma \frac{\partial P_1}{\partial x_1} \frac{4}{3h^2} \delta w_0\, n_{x_1}\, d\gamma - \iint_\Omega \frac{\partial^2 P_1}{\partial^2 x_1} \frac{4}{3h^2} \delta w_0\, d\omega +$$

$$+\int_{t_1}^{t_2} \oint_\Gamma N_2\, \delta v_0\, n_{x_2}\, d\gamma\, dt - \int_{t_1}^{t_2} \iint_\Omega \frac{\partial N_2}{\partial x_2} \delta v_0\, d\omega\, dt + \int_{t_1}^{t_2} \oint_\Gamma \left(M_2 - \frac{4}{3h^2} P_2 \right) \delta\phi_y\, n_{x_2}\, d\gamma\, dt -$$

$$-\int_{t_1}^{t_2} \iint_\Omega \left(\frac{\partial M_2}{\partial x_2} - \frac{4}{3h^2} \frac{\partial P_2}{\partial x_2} \right) \delta\phi_y\, d\omega\, dt - \oint_\Gamma P_2 \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_2} n_{x_2}\, d\gamma + \oint_\Gamma \frac{\partial P_2}{\partial x_2} \frac{4}{3h^2} \delta w_0\, n_{x_2}\, d\gamma -$$

$$-\iint_\Omega \frac{\partial^2 P_2}{\partial^2 x_2} \frac{4}{3h^2} \delta w_0\, d\omega - \int_{t_1}^{t_2} \iint_\Omega \left(\frac{\partial Q_4}{\partial x_2} - \frac{4}{h^2} \frac{\partial R_4}{\partial x_2} \right) \delta w_0\, d\omega dt + \int_{t_1}^{t_2} \oint_\Gamma \left(Q_4 - R_4 \frac{4}{h^2} \right) \delta w_0\, n_{x_2}\, d\gamma\, dt +$$

$$+\int_{t_1}^{t_2} \iint_\Omega \left(Q_5 - \frac{4}{h^2} R_5 \right) \delta\phi_x\, d\omega dt + \int_{t_1}^{t_2} \iint_\Omega \left(Q_4 - \frac{4}{h^2} R_4 \right) \delta\phi_y\, d\omega dt +$$

$$+\int_{t_1}^{t_2} \oint_\Gamma \left(Q_5 - R_5 \frac{4}{h^2} \right) \delta w_0\, n_{x_1}\, d\gamma\, dt - \int_{t_1}^{t_2} \iint_\Omega \left(\frac{\partial Q_5}{\partial x_1} - \frac{\partial R_5}{\partial x_1} \frac{4}{h^2} \right) \delta w_0\, d\omega\, dt +$$

$$+\int_{t_1}^{t_2} \oint_\Gamma N_6\, \delta u_0\, n_{x_2}\, d\gamma\, dt - \int_{t_1}^{t_2} \iint_\Omega \frac{\partial N_6}{\partial x_2} \delta u_0\, d\omega\, dt + \int_{t_1}^{t_2} \oint_\Gamma N_6\, \delta v_0\, n_{x_1}\, d\gamma\, dt -$$

$$-\int_{t_1}^{t_2} \iint_\Omega \frac{\partial N_6}{\partial x_1} \delta v_0\, d\omega\, dt + \int_{t_1}^{t_2} \oint_\Gamma \left(M_6 - P_6 \frac{4}{3h^2} \right) \delta\phi_x\, n_{x_2}\, d\gamma\, dt -$$

$$-\int_{t_1}^{t_2} \iint_\Omega \left(\frac{\partial M_6}{\partial x_2} - \frac{\partial P_6}{\partial x_2} \frac{4}{3h^2} \right) \delta\phi_x\, d\omega\, dt + \int_{t_1}^{t_2} \oint_\Gamma \left(M_6 - P_6 \frac{4}{3h^2} \right) \delta\phi_y\, n_{x_1}\, d\gamma\, dt -$$

$$-\int_{t_1}^{t_2} \iint_\Omega \left(\frac{\partial M_6}{\partial x_1} - \frac{\partial P_6}{\partial x_1} \frac{4}{3h^2} \right) \delta\phi_y\, d\omega dt - \int_{t_1}^{t_2} \oint_\Gamma P_6 \frac{4}{3h^2} \left(\frac{\partial \delta w_0}{\partial x_2} n_{x_1} + \frac{\partial \delta w_0}{\partial x_1} n_{x_2} \right) d\gamma\, dt +$$

$$+\int_{t_1}^{t_2} \oint_\Gamma \frac{4}{3h^2} \left(\frac{\partial P_6}{\partial x_1} n_{x_2} + \frac{\partial P_6}{\partial x_2} n_{x_1} \right) \delta w_0\, d\gamma\, dt - \int_{t_1}^{t_2} \iint_\Omega \frac{\partial^2 P_6}{\partial x_2 \partial x_1} \frac{8}{3h^2} \delta w_0\, d\omega\, dt \;\Rightarrow$$

$$
\delta J_U = \int_{t_1}^{t_2} \oint_{\Gamma} \left\{
\begin{aligned}
& N_1 n_{x_1} + N_6 n_{x_2} \;\; \delta u_0 + \;\; N_2 n_{x_1} + N_6 n_{x_2} \;\; \delta v_0 + \\
& + \left[\begin{aligned} & \left(M_1 - \frac{4}{3h^2} P_1 \right) n_{x_1} + \\ & \qquad + \left(M_6 - P_6 \frac{4}{3h^2} \right) n_{x_2} \end{aligned} \right] \delta \phi_x + \left[\begin{aligned} & \left(M_2 - \frac{4}{3h^2} P_2 \right) n_{x_2} + \\ & \qquad + \left(M_6 - P_6 \frac{4}{3h^2} \right) n_{x_1} \end{aligned} \right] \delta \phi_y + \\
& + \left[\begin{aligned} & \left(Q_5 - R_5 \frac{4}{h^2} + \frac{4}{3h^2} \frac{\partial P_6}{\partial x_2} + \frac{4}{3h^2} \frac{\partial P_1}{\partial x_1} \right) n_{x_1} + \\ & \qquad + \left(Q_4 - R_4 \frac{4}{h^2} + \frac{4}{3h^2} \frac{\partial P_6}{\partial x_1} + \frac{4}{3h^2} \frac{\partial P_2}{\partial x_2} \right) n_{x_2} \end{aligned} \right] \delta w_0 - \\
& - \left(P_2 \frac{4}{3h^2} n_{x_2} + P_6 \frac{4}{3h^2} \, n_{x_1} \right) \frac{\partial \delta w_0}{\partial x_2} - \left(P_1 \frac{4}{3h^2} n_{x_1} + P_6 \frac{4}{3h^2} \, n_{x_2} \right) \frac{\partial \delta w_0}{\partial x_1}
\end{aligned}
\right\} d\gamma \, dt +
$$

$$
+ \int_{t_1}^{t_2} \iint_{\Omega} \left\{
\begin{aligned}
& - \left(\frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} \right) \delta u_0 - \left(\frac{\partial N_2}{\partial x_2} + \frac{\partial N_6}{\partial x_1} \right) \delta v_0 + \\
& + \left(Q_5 - \frac{4}{h^2} R_5 - \frac{\partial M_1}{\partial x_1} - \frac{\partial M_6}{\partial x_2} + \frac{\partial P_6}{\partial x_2} \frac{4}{3h^2} + \frac{\partial P_1}{\partial x_1} \frac{4}{3h^2} \right) \delta \phi_x + \\
& + \left(Q_4 - \frac{4}{h^2} R_4 - \frac{\partial M_2}{\partial x_2} - \frac{\partial M_6}{\partial x_1} + \frac{\partial P_6}{\partial x_1} \frac{4}{3h^2} + \frac{\partial P_2}{\partial x_2} \frac{4}{3h^2} \right) \delta \phi_y - \\
& - \left(\begin{aligned} & \frac{\partial Q_4}{\partial x_2} - \frac{\partial R_4}{\partial x_2} \frac{4}{h^2} + \frac{\partial Q_5}{\partial x_1} - \frac{\partial R_5}{\partial x_1} \frac{4}{h^2} + \\ & \qquad + \frac{\partial^2 P_1}{\partial^2 x_1} \frac{4}{3h^2} + \frac{\partial^2 P_6}{\partial x_2 \partial x_1} \frac{8}{3h^2} + \frac{\partial^2 P_2}{\partial^2 x_2} \frac{4}{3h^2} \end{aligned} \right) \delta w_0
\end{aligned}
\right\} d\omega \, dt \qquad \textbf{(5)}
$$

4.4. Virtual Work of the Externally Applied Loads

As for the last terms of the variational equation, we have to calculate the **variations of the externally applied forces**, apart from the Kinetic and Elastic Potential Energy.

Within the Transverse Shear Deformation Theory, we assume that there is a normal distributed external load at the top or/and bottom surface $\partial B^{(f)}$ of the plate (surface force/ traction at $x_3 = h/2$ or/ and $x_3 = -h/2$) $q(x_1, x_2; t)$. At this point we clarify that the normal distributed external load q is regarded as the algebraic sum between the load at the top and the bottom of the external boundary of the plate ($q = q_{top} + q_{bottom}$).

Also it is necessary to quantify the **virtual work of the traction field** at the edge of the plate. This work is related to the virtual displacements δu_1 ,δu_2, δu_3, δu_4, δu_5, from which the first three are the displacements on the direction of x_1-axis, x_2-axis, x_3-axis respectively and the last two are derivatives of the slopes of the in-plane displacements to the vertical x_3-axis. The displacements δu_4 and δu_5, could be analyzed further as illustrated on the figure below in order to show explicitly their physical meaning.

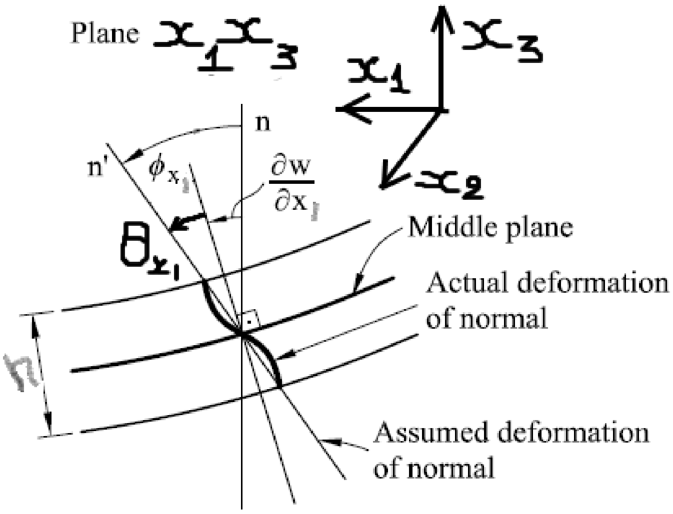

Figure 4.4.1: Slopes of the displacement field on the $x_1 x_3$-plane (similarly for the $x_2 x_3$-plane).

Thus, $\delta u_4 = \delta u_{23} = \delta\phi_y = \dfrac{\partial \delta w_0}{\partial x_2} + \theta_{x_2}$ and $\delta u_5 = \delta u_{13} = \delta\phi_x = \dfrac{\partial \delta w_0}{\partial x_1} + \theta_{x_1}$.

Note that, the above relations are going to appear again on the appropriate sections, where we mean to derive the boundary conditions of the problem of TSDT, for reason explained on the respective section 5.

Now, the variation of the functional of the external surface traction, due to the surface distributed load (surface tractions) at the adjacent surface and the horizontally distributed vertical load q (as illustrated on the following figure), is:

$$\delta J_T = \int_{t_1}^{t_2} \iint_{\partial B_T} \hat{\mathrm{T}}_i \, \delta u_i \, dS \, dt + \int_{t_1}^{t_2} \iint_{\Omega} q \, \delta w_0 \, d\omega \, dt \Rightarrow$$

$$\delta J_T = \underbrace{\int_{t_1}^{t_2}\iint_{\partial B_T^{(lat)}} \left(\hat{T}_1\delta u_1 + \hat{T}_2\delta u_2 + \hat{T}_3\delta u_3 + \hat{T}_4\delta u_4 + \hat{T}_5\delta u_5\right) dS\ dt}_{\delta J_T^{(lat)}} +$$

$$+ \underbrace{\int_{t_1}^{t_2}\iint_{\partial B^{(uf)}} \left(\hat{T}_3\delta u_3 + \hat{T}_4\delta u_4 + \hat{T}_5\delta u_5\right) dS\ dt}_{\delta J_T^{(uf)}} + \underbrace{\int_{t_1}^{t_2}\iint_{\partial B^{(lf)}} \left(\hat{T}_3\delta u_3 + \hat{T}_4\delta u_4 + \hat{T}_5\delta u_5\right) dS\ dt}_{\delta J_T^{(lf)}} +$$

$$+ \int_{t_1}^{t_2}\iint_{\Omega} q\,\delta w_0\, d\omega\, dt \qquad (1)$$

Note that on the section 2 we have described only the three components of the surface tractions $\hat{T}_1$, $\hat{T}_2$, $\hat{T}_3$ on the directions of the three axes on the Cartesian coordinate system. Consequently, the "shear" surface tractions $\hat{T}_4$, $\hat{T}_5$ can be expressed in terms of the given "normal" surface tractions $\hat{T}_1$, $\hat{T}_2$, $\hat{T}_3$. Through the parallelogram law, we get

$$\hat{T}_4 = \hat{T}_{23} = a_4\hat{T}_2 + b_4\hat{T}_3, \qquad (2a) \qquad \text{where } a_4, b_4 = \text{constants}$$

$$\hat{T}_5 = \hat{T}_{13} = a_5\hat{T}_1 + b_5\hat{T}_3, \qquad (2b) \qquad \text{where } a_5, b_5 = \text{constants}$$

We examine separately the three integrals of the lateral, upper and bottom surface. To simplify their expression, we neglect the time integration at this moment.
Due to the stress-free boundary conditions on the bottom and top faces of the plate, we have $\gamma_4 = 2e_4 = 0$ and $\gamma_5 = 2e_5 = 0$ on the bottom (lower) and top (upper) faces. This fact is also compatible with the form of the surface tractions which are prescribed on the section 2. Thus,

$$\delta J_T^{(uf)} = \delta J_T^{(lf)} = 0 \qquad \text{,on the flat surfaces}$$

By this way the only term that remains to be analyzed, on the variation of the externally applied loads is the first one.

$$\delta J_T^{(lat)} = \iint_{\partial B_T^{(lat)}} \left(\hat{T}_1\delta u_1 + \hat{T}_2\delta u_2 + \hat{T}_3\delta u_3 + \hat{T}_4\delta u_4 + \hat{T}_5\delta u_5\right) dS =$$

$$= \iint_{\partial B_T^{(lat)}} \hat{T}_1\delta u_1\, dS + \iint_{\partial B_T^{(lat)}} \hat{T}_2\delta u_2\, dS + \iint_{\partial B_T^{(lat)}} \hat{T}_3\delta u_3\, dS + \iint_{\partial B_T^{(lat)}} \hat{T}_4\delta u_4\, dS + \iint_{\partial B_T^{(lat)}} \hat{T}_5\delta u_5\, dS =$$

$$= \iint_{\partial B_T^{(lat)}} \hat{T}_1\delta u\, dS + \iint_{\partial B_T^{(lat)}} \hat{T}_2\delta v\, dS + \iint_{\partial B_T^{(lat)}} \hat{T}_3\delta w\, dS + \iint_{\partial B_T^{(lat)}} \hat{T}_4\delta\phi_y\, dS + \iint_{\partial B_T^{(lat)}} \hat{T}_5\delta\phi_x\, dS \qquad (2)$$

Now we assume that we have given surface tractions $\hat{T}_1, \hat{T}_2, \hat{T}_3$ at the specific parts of the lateral boundary $\partial B^{(lat)}(\partial B_T^{(lat)})$ and using the Eqs. (17a) - (17c) of the section 3 and the above Eqs. (2a) and (2b), we derive the following,

$$
\begin{aligned}
\delta J_T^{(lat)} = & \iint_{\partial B_T^{(lat)}} \hat{T}_1 \left[\delta u_0 + x_3 \delta\phi_x - x_3^3 \frac{4}{3h^2}\left(\delta\phi_x + \frac{\partial \delta w_0}{\partial x_1} \right) \right] dS + \\
& + \iint_{\partial B_T^{(lat)}} \hat{T}_2 \left[\delta v_0 + x_3 \delta\phi_y - x_3^3 \frac{4}{3h^2}\left(\delta\phi_y + \frac{\partial \delta w_0}{\partial x_2} \right) \right] dS + \\
+ & \iint_{\partial B_T^{(lat)}} \hat{T}_3 \delta w_0 \, dS + \iint_{\partial B_T^{(lat)}} \left[a_4 \hat{T}_2 + b_4 \hat{T}_3 \right] \delta\phi_y \, dS + \iint_{\partial B_T^{(lat)}} \left[a_5 \hat{T}_1 + b_5 \hat{T}_3 \right] \delta\phi_x \, dS = \\
= & \iint_{\partial B_T^{(lat)}} \hat{T}_1 \delta u_0 \, dS + \iint_{\partial B_T^{(lat)}} \hat{T}_2 \delta v_0 \, dS + \iint_{\partial B_T^{(lat)}} \hat{T}_3 \delta w_0 \, dS + \\
& + \iint_{\partial B_T^{(lat)}} \left(\hat{T}_1 x_3 - \hat{T}_1 x_3^3 \frac{4}{3h^2} + a_5 \hat{T}_1 + b_5 \hat{T}_3 \right) \delta\phi_x \, dS + \\
& + \iint_{\partial B_T^{(lat)}} \left(\hat{T}_2 x_3 - \hat{T}_2 x_3^3 \frac{4}{3h^2} + a_4 \hat{T}_2 + b_4 \hat{T}_3 \right) \delta\phi_y \, dS - \\
& - \iint_{\partial B_T^{(lat)}} \hat{T}_1 x_3^3 \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_1} dS - \iint_{\partial B_T^{(lat)}} \hat{T}_2 x_3^3 \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_2} dS
\end{aligned}
\tag{3}
$$

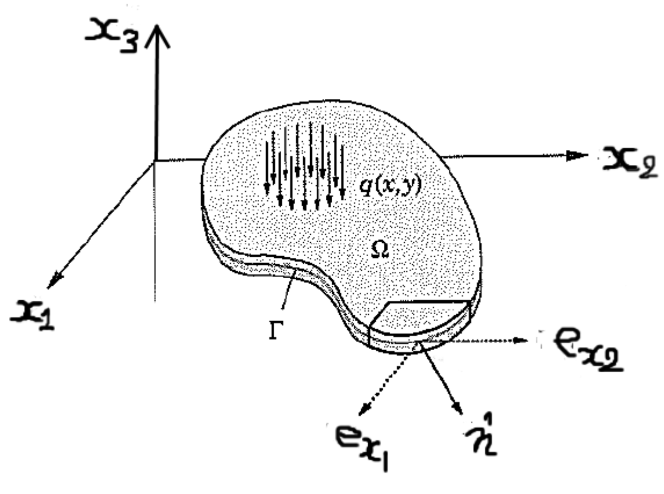

Figure 4.4.2: Externally applied and horizontally distributed vertical load.

Now as for the form of the given surface tractions $\hat{T}_1, \hat{T}_2, \hat{T}_3$, we recall the configurations of the section 2. Remark also that the deformation at the edge of the plate during its motion in conjunction with the externally can be cubic dependent from the x_3-variable. This fact is rational for our model, because we have the same dependence of the in-plane displacements (u, v) from the x_3-variable.

Further note that, the quantities which multiply the variations δu_0, δv_0, δw_0, $\delta\phi_x$, $\delta\phi_y$ inside the surface integrals of the Eq. (3), are expected to match with the respective terms of the variation of the Elastic-Potential Energy part (section 4.3). The last referred contains boundary terms (surface integrals) similarly cubic dependent of the x_3-variable.

Taking all the aforementioned into account, we present here for convenience again the form of the given surface tractions prescribed on the section 2 by the Eqs. (1) - (3).

$$\hat{T}_1(x_1, x_2, x_3) = a_{T0}(x_1, x_2) + a_{T1}(x_1, x_2)\, x_3 + a_{T3}(x_1, x_2)\, x_3^3 = \\ = a_{T0}(\gamma) + a_{T1}(\gamma)\, x_3 + a_{T3}(\gamma)\, x_3^3 \tag{1}$$

$$\hat{T}_2(x_1, x_2, x_3) = b_{T0}(x_1, x_2) + b_{T1}(x_1, x_2)\, x_3 + b_{T3}(x_1, x_2)\, x_3^3 = \\ = b_{T0}(\gamma) + b_{T1}(\gamma)\, x_3 + b_{T3}(\gamma)\, x_3^3 \tag{2}$$

$$\hat{T}_3(x_1, x_2, x_3) = c_{T0}(x_1, x_2) + c_{T1}(x_1, x_2)\, x_3 + c_{T3}(x_1, x_2)\, x_3^3 = \\ = c_{T0}(\gamma) + c_{T1}(\gamma)\, x_3 + c_{T3}(\gamma)\, x_3^3 \tag{3}$$

In addition, it is essential to note that the above form of the surface traction field is a simplified approximation of the real values of the surface tractions at each point upon the edge of the plate. Certainly, this approximation is enough accurate in the context of our problem of TSDT (and compatible with our model), because the thick plate can be deformed under the influence of tractions with larger amplitude (size) along its thickness in comparison with the thin plate of the CPT.

Subsequently, we substitute the previous Eqs. (1) - (3) into the Eq. (3) and after that we use the mass-moment quantities and the process of decomposition of the surface integrals. Thus, the Eq. (3) is modified as follows,

$$\delta J_T^{(lat)} = \iint_{\partial B_T^{(lat)}} a_{T0} + a_{T1}x_3 + a_{T3}x_3^3 \; \delta u_0 \, dS + \iint_{\partial B_T^{(lat)}} b_{T0} + b_{T1}x_3 + b_{T3}x_3^3 \; \delta v_0 \, dS +$$

$$+ \iint_{\partial B_T^{(lat)}} c_{T0} + c_{T1}x_3 + c_{T3}x_3^3 \; \delta w_0 \, dS +$$

$$+ \iint_{\partial B_T^{(lat)}} \begin{pmatrix} a_{T0}x_3 + a_{T1}x_3^2 + a_{T3}x_3^4 - a_{T0}x_3^3 + a_{T1}x_3^4 + a_{T3}x_3^6 \; \dfrac{4}{3h^2} + \\ + a_5 \; a_{T0} + a_{T1}x_3 + a_{T3}x_3^3 \; + b_5 \; c_{T0} + c_{T1}x_3 + c_{T3}x_3^3 \end{pmatrix} \delta\phi_x \, dS +$$

$$+ \iint_{\partial B_T^{(lat)}} \begin{pmatrix} (b_{T0}x_3 + b_{T1}x_3^2 + b_{T3}x_3^4) - (b_{T0}x_3^3 + b_{T1}x_3^4 + b_{T3}x_3^6) \dfrac{4}{3h^2} + \\ + a_4(b_{T0} + b_{T1}x_3 + b_{T3}x_3^3) + b_4(c_{T0} + c_{T1}x_3 + c_{T3}x_3^3) \end{pmatrix} \delta\phi_y \, dS -$$

$$- \iint_{\partial B_T^{(lat)}} (a_{T0}x_3^3 + a_{T1}x_3^4 + a_{T3}x_3^6) \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_1} dS -$$

$$- \iint_{\partial B_T^{(lat)}} (b_{T0}x_3^3 + b_{T1}x_3^4 + b_{T3}x_3^6) \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_2} dS \;\; \Rightarrow$$

$$
\begin{aligned}
\delta J_T^{(lat)} = {} & \oint_\Gamma \left(\frac{I_0}{\rho} a_{T0} + a_{T1}\frac{\cancel{I_1}}{\rho} + a_{T3}\frac{\cancel{I_3}}{\rho} \right) \delta u_0 \, d\gamma + \oint_\Gamma \left(\frac{I_0}{\rho} b_{T0} + b_{T1}\frac{\cancel{I_1}}{\rho} + b_{T3}\frac{\cancel{I_3}}{\rho} \right) \delta v_0 \, d\gamma + \\
& + \oint_\Gamma \left(\frac{I_0}{\rho} c_{T0} + c_{T1}\frac{\cancel{I_1}}{\rho} + c_{T3}\frac{\cancel{I_3}}{\rho} \right) \delta w_0 \, d\gamma + \\
& + \oint_\Gamma \left(\begin{aligned} & \left(a_{T0}\frac{\cancel{I_1}}{\rho} + a_{T1}\frac{I_2}{\rho} + a_{T3}\frac{I_4}{\rho} \right) - \left(a_{T0}\frac{\cancel{I_3}}{\rho} + a_{T1}\frac{I_4}{\rho} + a_{T3}\frac{I_6}{\rho} \right) \frac{4}{3h^2} + \\ & + a_5 \left(a_{T0}\frac{I_0}{\rho} + a_{T1}\frac{\cancel{I_1}}{\rho} + a_{T3}\frac{\cancel{I_3}}{\rho} \right) + b_5 \left(c_{T0}\frac{I_0}{\rho} + c_{T1}\frac{\cancel{I_1}}{\rho} + c_{T3}\frac{\cancel{I_3}}{\rho} \right) \end{aligned} \right) \delta\phi_x \, d\gamma + \\
& + \oint_\Gamma \left(\begin{aligned} & \left(b_{T0}\frac{\cancel{I_1}}{\rho} + b_{T1}\frac{I_2}{\rho} + b_{T3}\frac{I_4}{\rho} \right) - \left(b_{T0}\frac{\cancel{I_3}}{\rho} + b_{T1}\frac{I_4}{\rho} + b_{T3}\frac{I_6}{\rho} \right) \frac{4}{3h^2} + \\ & + a_4 \left(b_{T0}\frac{I_0}{\rho} + b_{T1}\frac{\cancel{I_1}}{\rho} + b_{T3}\frac{\cancel{I_3}}{\rho} \right) + b_4 \left(c_{T0}\frac{I_0}{\rho} + c_{T1}\frac{\cancel{I_1}}{\rho} + c_{T3}\frac{\cancel{I_3}}{\rho} \right) \end{aligned} \right) \delta\phi_y \, d\gamma - \\
& - \oint_\Gamma \left(a_{T0}\frac{\cancel{I_3}}{\rho} + a_{T1}\frac{I_4}{\rho} + a_{T3}\frac{I_6}{\rho} \right) \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_1} \, d\gamma - \\
& - \oint_\Gamma \left(b_{T0}\frac{\cancel{I_3}}{\rho} + b_{T1}\frac{I_4}{\rho} + b_{T3}\frac{I_6}{\rho} \right) \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_2} \, d\gamma \quad \Rightarrow
\end{aligned}
$$

$$
\begin{aligned}
\delta J_T^{(lat)} = {} & \oint_\Gamma \frac{I_0}{\rho} a_{T0} \, \delta u_0 \, d\gamma + \oint_\Gamma \frac{I_0}{\rho} b_{T0} \, \delta v_0 \, d\gamma + \oint_\Gamma \frac{I_0}{\rho} c_{T0} \, \delta w_0 \, d\gamma + \\
& + \oint_\Gamma \left(\begin{aligned} & a_{T1}\frac{I_2}{\rho} + a_{T3}\frac{I_4}{\rho} - a_{T1}\frac{I_4}{\rho}\frac{4}{3h^2} - a_{T3}\frac{I_6}{\rho}\frac{4}{3h^2} + \\ & + a_5 \, a_{T0}\frac{I_0}{\rho} + b_5 \, c_{T0}\frac{I_0}{\rho} \end{aligned} \right) \delta\phi_x \, d\gamma + \\
& + \oint_\Gamma \left(\begin{aligned} & b_{T1}\frac{I_2}{\rho} + b_{T3}\frac{I_4}{\rho} - b_{T1}\frac{I_4}{\rho}\frac{4}{3h^2} - b_{T3}\frac{I_6}{\rho}\frac{4}{3h^2} + \\ & + a_4 \, b_{T0}\frac{I_0}{\rho} + b_4 \, c_{T0}\frac{I_0}{\rho} \end{aligned} \right) \delta\phi_y \, d\gamma - \\
& - \oint_\Gamma \left(a_{T1}\frac{I_4}{\rho} + a_{T3}\frac{I_6}{\rho} \right) \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_1} \, d\gamma - \oint_\Gamma \left(b_{T1}\frac{I_4}{\rho} + b_{T3}\frac{I_6}{\rho} \right) \frac{4}{3h^2} \frac{\partial \delta w_0}{\partial x_2} \, d\gamma
\end{aligned} \tag{4}
$$

Finally, substituting the Eq. (4) into the Eq. (1), we get

$$\delta J_T = \oint_\Gamma \frac{I_0}{\rho} a_{T0}\, \delta u_0 \, d\gamma + \oint_\Gamma \frac{I_0}{\rho} b_{T0}\, \delta v_0 \, d\gamma + \oint_\Gamma \frac{I_0}{\rho} c_{T0}\, \delta w_0 \, d\gamma +$$

$$+ \oint_\Gamma \left(\begin{matrix} a_{T1}\dfrac{I_2}{\rho} + a_{T3}\dfrac{I_4}{\rho} - a_{T1}\dfrac{I_4}{\rho}\dfrac{4}{3h^2} - a_{T3}\dfrac{I_6}{\rho}\dfrac{4}{3h^2} + \\ + a_5\, a_{T0}\dfrac{I_0}{\rho} + b_5\, c_{T0}\dfrac{I_0}{\rho} \end{matrix} \right) \delta\phi_x \, d\gamma +$$

$$+ \oint_\Gamma \left(\begin{matrix} b_{T1}\dfrac{I_2}{\rho} + b_{T3}\dfrac{I_4}{\rho} - b_{T1}\dfrac{I_4}{\rho}\dfrac{4}{3h^2} - b_{T3}\dfrac{I_6}{\rho}\dfrac{4}{3h^2} + \\ + a_4\, b_{T0}\dfrac{I_0}{\rho} + b_4\, c_{T0}\dfrac{I_0}{\rho} \end{matrix} \right) \delta\phi_y \, d\gamma -$$

$$- \oint_\Gamma \left(a_{T1}\frac{I_4}{\rho} + a_{T3}\frac{I_6}{\rho} \right) \frac{4}{3h^2}\frac{\partial \delta w_0}{\partial x_1} d\gamma - \oint_\Gamma \left(b_{T1}\frac{I_4}{\rho} + b_{T3}\frac{I_6}{\rho} \right) \frac{4}{3h^2}\frac{\partial \delta w_0}{\partial x_2} d\gamma +$$

$$+ \int_{t_1}^{t_2} \iint_\Omega q\, \delta w_0 \, d\omega \, dt \qquad \textbf{(5)}$$

4.5. The variational equation of the TSDT

Now we are able to substitute the results of separate parts, **(9)** of the section 4.1, **(5)** of the section 4.3 and **(5)** of the section 4.4 into the variational equation **(1)** of the section 4. The next step is to gather separately the different terms according to the kind of their variations e.g. δu_0, δv_0, δw_0, $\delta\phi_x$, $\delta\phi_y$. By this way it is easier to extract the equations of motion and the boundary conditions of the model of the TSDT.
To facilitate the calculations and substitutions, we repeat the equation (1) of the section 4:

$$\int_{t_1}^{t_2}\iiint_B \delta K(\dot{\boldsymbol{u}})dV\,dt-\int_{t_1}^{t_2}\iiint_B \delta U(\boldsymbol{e})dV\,dt+\int_{t_1}^{t_2}\iint_{\partial B_T}\hat{T}_i\,\delta u_i\,dS\,dt=0 \tag{1}$$

Then the equation (1), is converted to:

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left\{\begin{aligned}&\left[\ddot{u}_0 I_0-\frac{\partial N_1}{\partial x_1}-\frac{\partial N_6}{\partial x_2}\right]\delta u_0+\left[I_0\ddot{v}_0-\frac{\partial N_2}{\partial x_2}-\frac{\partial N_6}{\partial x_1}\right]\delta v_0+\\
&\left[\begin{aligned}&I_0\ddot{w}_0-\frac{4}{3h^2}\left(I_6\frac{4}{3h^2}-I_4\right)\left(\frac{\partial\ddot{\phi}_y}{\partial x_2}+\frac{\partial\ddot{\phi}_x}{\partial x_1}\right)-I_6\frac{16}{9h^4}\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}\right)-\\
&-\frac{\partial Q_4}{\partial x_2}+\frac{\partial R_4}{\partial x_2}\frac{4}{h^2}-\frac{\partial Q_5}{\partial x_1}+\frac{\partial R_5}{\partial x_1}\frac{4}{h^2}-\\
&-\frac{4}{3h^2}\left(\frac{\partial^2 P_1}{\partial^2 x_1}+\frac{2\,\partial^2 P_6}{\partial x_2\,\partial x_1}+\frac{\partial^2 P_2}{\partial^2 x_2}\right)-q\end{aligned}\right]\delta w_0+\\
&+\left[\begin{aligned}&\left(I_2-I_4\frac{8}{3h^2}+I_6\frac{16}{9h^4}\right)\ddot{\phi}_x+\frac{4}{3h^2}\left(I_6\frac{4}{3h^2}-I_4\right)\frac{\partial\ddot{w}_0}{\partial x_1}+\\
&+Q_5-\frac{4}{h^2}R_5-\frac{\partial M_1}{\partial x_1}-\frac{\partial M_6}{\partial x_2}+\frac{\partial P_6}{\partial x_2}\frac{4}{3h^2}+\frac{\partial P_1}{\partial x_1}\frac{4}{3h^2}\end{aligned}\right]\delta\phi_x+\\
&+\left[\begin{aligned}&\left(I_2-I_4\frac{8}{3h^2}+I_6\frac{16}{9h^4}\right)\ddot{\phi}_y+\frac{4}{3h^2}\left(\frac{4}{3h^2}I_6-I_4\right)\frac{\partial\ddot{w}_0}{\partial x_2}+\\
&+Q_4-\frac{4}{h^2}R_4-\frac{\partial M_2}{\partial x_2}-\frac{\partial M_6}{\partial x_1}+\frac{\partial P_6}{\partial x_1}\frac{4}{3h^2}+\frac{\partial P_2}{\partial x_2}\frac{4}{3h^2}\end{aligned}\right]\delta\phi_y\end{aligned}\right\}d\omega\,dt-$$

$$
-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{N_1 n_{x_1}+N_6 n_{x_2}-\frac{I_0}{\rho}\,a_{T0}\right\}\delta u_0\, d\gamma\, dt\;-\;\int_{t_1}^{t_2}\oint_{\Gamma}\left\{N_2 n_{x_1}+N_6 n_{x_2}-\frac{I_0}{\rho}\,b_{T0}\right\}\delta v_0\, d\gamma\, dt\;-
$$

$$
-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}\left(M_1-\dfrac{4}{3h^2}P_1\right)n_{x_1}+\left(M_6-P_6\dfrac{4}{3h^2}\right)n_{x_2}-\\ -\left(\begin{array}{c}a_{T1}\dfrac{I_2}{\rho}+a_{T3}\dfrac{I_4}{\rho}-a_{T1}\dfrac{I_4}{\rho}\dfrac{4}{3h^2}-a_{T3}\dfrac{I_6}{\rho}\dfrac{4}{3h^2}+\\ +a_5\,a_{T0}\dfrac{I_0}{\rho}+b_5\,c_{T0}\dfrac{I_0}{\rho}\end{array}\right)\end{array}\right\}\delta\phi_x\, d\gamma\, dt\;-
$$

$$
-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}\left(M_2-\dfrac{4}{3h^2}P_2\right)n_{x_2}+\left(M_6-P_6\dfrac{4}{3h^2}\right)n_{x_1}-\\ -\left(\begin{array}{c}b_{T1}\dfrac{I_2}{\rho}+b_{T3}\dfrac{I_4}{\rho}-b_{T1}\dfrac{I_4}{\rho}\dfrac{4}{3h^2}-b_{T3}\dfrac{I_6}{\rho}\dfrac{4}{3h^2}+\\ +a_4\,b_{T0}\dfrac{I_0}{\rho}+b_4\,c_{T0}\dfrac{I_0}{\rho}\end{array}\right)\end{array}\right\}\delta\phi_y\, d\gamma\, dt\;-
$$

$$
-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}\left[\begin{array}{l}\left(I_6\dfrac{16}{9h^4}-\dfrac{4}{3h^2}I_4\right)\ddot{\phi}_x+I_6\dfrac{16}{9h^4}\dfrac{\partial\ddot{w}_0}{\partial x_1}+\\ +Q_5-R_5\dfrac{4}{h^2}+\dfrac{4}{3h^2}\dfrac{\partial P_6}{\partial x_2}+\dfrac{4}{3h^2}\dfrac{\partial P_1}{\partial x_1}\end{array}\right]\boldsymbol{n}_{x_1}+\\ +\left[\begin{array}{l}\left(I_6\dfrac{16}{9h^4}-\dfrac{4}{3h^2}I_4\right)\ddot{\phi}_y+I_6\dfrac{16}{9h^4}\dfrac{\partial\ddot{w}_0}{\partial x_2}+\\ +Q_4-R_4\dfrac{4}{h^2}+\dfrac{4}{3h^2}\dfrac{\partial P_6}{\partial x_1}+\dfrac{4}{3h^2}\dfrac{\partial P_2}{\partial x_2}\end{array}\right]\boldsymbol{n}_{x_2}-\dfrac{I_0}{\rho}c_{T0}\end{array}\right\}\delta w_0\, d\gamma\, dt\;+
$$

$$
+\int_{t_1}^{t_2}\oint_{\Gamma}\frac{4}{3h^2}\left(\left\{P_1\,n_{x_1}+P_6\,n_{x_2}-\left(a_{T1}\frac{I_4}{\rho}+a_{T3}\frac{I_6}{\rho}\right)\right\}\frac{\partial\delta w_0}{\partial x_1}\right)d\gamma\, dt\;+
$$

$$
+\int_{t_1}^{t_2}\oint_{\Gamma}\frac{4}{3h^2}\left(\left\{P_2\,n_{x_2}+P_6\,n_{x_1}-\left(b_{T1}\frac{I_4}{\rho}+b_{T3}\frac{I_6}{\rho}\right)\right\}\frac{\partial\delta w_0}{\partial x_2}\right)d\gamma\, dt\;=\;0\Rightarrow
$$

For further simplification, we gather together the terms with surface and curvilinear integrals, taking care of the kind of variation (δu_0, δv_0, δw_0, $\delta\phi_x$ and $\delta\phi_y$) of each term. Thus, the final form of the variational equation of the problem of the TSDT is the following. Note that the Eq. (2) below is exactly the same as the variational Eq. (1) of the section 4.

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left\{\left[\ddot{u}_0 I_0-\frac{\partial N_1}{\partial x_1}-\frac{\partial N_6}{\partial x_2}\right]\delta u_0\right\}d\omega\, dt\;-\int_{t_1}^{t_2}\iint_{\Omega}\left[I_0\ddot{v}_0-\frac{\partial N_2}{\partial x_2}-\frac{\partial N_6}{\partial x_1}\right]\delta v_0 d\omega\, dt\;-$$

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left[\begin{array}{c} I_0\ddot{w}_0-\frac{4}{3h^2}\left(I_6\frac{4}{3h^2}-I_4\right)\left(\frac{\partial\ddot{\phi}_y}{\partial x_2}+\frac{\partial\ddot{\phi}_x}{\partial x_1}\right)-I_6\frac{16}{9h^4}\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}\right)- \\ -\frac{\partial Q_4}{\partial x_2}+\frac{\partial R_4}{\partial x_2}\frac{4}{h^2}-\frac{\partial Q_5}{\partial x_1}+\frac{\partial R_5}{\partial x_1}\frac{4}{h^2}- \\ -\frac{4}{3h^2}\left(\frac{\partial^2 P_1}{\partial^2 x_1}+\frac{2\,\partial^2 P_6}{\partial x_2\,\partial x_1}+\frac{\partial^2 P_2}{\partial^2 x_2}\right)-q \end{array}\right]\delta w_0 d\omega\, dt\;-$$

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left[\begin{array}{c}\left(I_2-I_4\frac{8}{3h^2}+I_6\frac{16}{9h^4}\right)\ddot{\phi}_x+\frac{4}{3h^2}\left(I_6\frac{4}{3h^2}-I_4\right)\frac{\partial\ddot{w}_0}{\partial x_1}+ \\ +Q_5-\frac{4}{h^2}R_5-\frac{\partial M_1}{\partial x_1}-\frac{\partial M_6}{\partial x_2}+\frac{\partial P_6}{\partial x_2}\frac{4}{3h^2}+\frac{\partial P_1}{\partial x_1}\frac{4}{3h^2}\end{array}\right]\delta\phi_x\, d\omega\, dt\;-$$

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left[\begin{array}{c}\left(I_2-I_4\frac{8}{3h^2}+I_6\frac{16}{9h^4}\right)\ddot{\phi}_y+\frac{4}{3h^2}\left(\frac{4}{3h^2}I_6-I_4\right)\frac{\partial\ddot{w}_0}{\partial x_2}+ \\ +Q_4-\frac{4}{h^2}R_4-\frac{\partial M_2}{\partial x_2}-\frac{\partial M_6}{\partial x_1}+\frac{\partial P_6}{\partial x_1}\frac{4}{3h^2}+\frac{\partial P_2}{\partial x_2}\frac{4}{3h^2}\end{array}\right]\delta\phi_y\, d\omega\, dt\;-$$

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{N_1 n_{x_1}+N_6 n_{x_2}-\frac{I_0}{\rho}a_{T0}\right\}\delta u_0\, d\gamma\, dt\;-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{N_2 n_{x_1}+N_6 n_{x_2}-\frac{I_0}{\rho}b_{T0}\right\}\delta v_0\, d\gamma\, dt\;-$$

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}\left(M_1-\frac{4}{3h^2}P_1\right)n_{x_1}+\left(M_6-P_6\frac{4}{3h^2}\right)n_{x_2}- \\ -\left(\begin{array}{c} a_{T1}\frac{I_2}{\rho}+a_{T3}\frac{I_4}{\rho}-a_{T1}\frac{I_4}{\rho}\frac{4}{3h^2}-a_{T3}\frac{I_6}{\rho}\frac{4}{3h^2}+ \\ +a_5\, a_{T0}\frac{I_0}{\rho}+b_5\, c_{T0}\frac{I_0}{\rho}\end{array}\right)\end{array}\right\}\delta\phi_x\, d\gamma\, dt\;-$$

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left[\begin{array}{l}\left(M_2-\frac{4}{3h^2}P_2\right)n_{x_2}+\left(M_6-P_6\frac{4}{3h^2}\right)n_{x_1}- \\ -\left(\begin{array}{c} b_{T1}\frac{I_2}{\rho}+b_{T3}\frac{I_4}{\rho}-b_{T1}\frac{I_4}{\rho}\frac{4}{3h^2}-b_{T3}\frac{I_6}{\rho}\frac{4}{3h^2}+ \\ +a_4\, b_{T0}\frac{I_0}{\rho}+b_4\, c_{T0}\frac{I_0}{\rho}\end{array}\right)\end{array}\right]\delta\phi_y\, d\gamma\, dt\;-$$

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{bmatrix}\left(I_6\frac{16}{9h^4}-\frac{4}{3h^2}I_4\right)\ddot{\phi}_x+I_6\frac{16}{9h^4}\frac{\partial\ddot{w}_0}{\partial x_1}+ \\ +Q_5-R_5\frac{4}{h^2}+\frac{4}{3h^2}\frac{\partial P_6}{\partial x_2}+\frac{4}{3h^2}\frac{\partial P_1}{\partial x_1}\end{bmatrix}\boldsymbol{n}_{x_1}+ \\ +\begin{bmatrix}\left(I_6\frac{16}{9h^4}-\frac{4}{3h^2}I_4\right)\ddot{\phi}_y+I_6\frac{16}{9h^4}\frac{\partial\ddot{w}_0}{\partial x_2}+ \\ +Q_4-R_4\frac{4}{h^2}+\frac{4}{3h^2}\frac{\partial P_6}{\partial x_1}+\frac{4}{3h^2}\frac{\partial P_2}{\partial x_2}\end{bmatrix}\boldsymbol{n}_{x_2}-\frac{I_0}{\rho}c_{T0}\right\}\delta w_0\,d\gamma\,dt+$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\frac{4}{3h^2}\left(\left\{P_1\,n_{x_1}+P_6\,n_{x_2}-\left(a_{T1}\frac{I_4}{\rho}+a_{T3}\frac{I_6}{\rho}\right)\right\}\frac{\partial\delta w_0}{\partial x_1}\right)d\gamma\,dt+$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\frac{4}{3h^2}\left(\left\{P_2\,n_{x_2}+P_6\,n_{x_1}-\left(b_{T1}\frac{I_4}{\rho}+b_{T3}\frac{I_6}{\rho}\right)\right\}\frac{\partial\delta w_0}{\partial x_2}\right)d\gamma\,dt=0 \qquad \textbf{(2)}$$

4.6. Equations of Motion of the TSDT in terms of thickness integrated forces and moments

Now in order to obtain the equations of motion of the TSDT, we assume that each term (under surface or curvilinear integral) of the Eq. (2) is continuous function of x_1 and x_2. These terms are multiplied with the variations δu_0, δv_0, δw_0, $\delta\phi_x$ and $\delta\phi_y$ or the spatial derivatives of δw_0. At this point, using the standard arguments of the calculus of variations [*"Calculus of Variations", I. M. Gelfand and S. V. Fomin, Lemma 1, p.9/Sec.3/Chap.1 and Lemma, p.22/Sec.5/Chap.1* and *"Introduction to the Calculus of Variations", Sagan 1969, p.54 Lemma 2.4*], we derive the **five equations of motion of the plate**.

Accordingly, we first assume that, $\delta u_0=\delta v_0=\delta w_0=\partial\delta w_0/\partial x_2=\partial\delta w_0/\partial x_1=\delta\phi_x=\delta\phi_y=0$ on the boundary ($\boldsymbol{x}\in\Gamma$), where t is arbitrary. The previous means that the variations and their spatial derivatives are not vary upon the boundary of the plate and obviously we have given displacements ["*Calculus of Variations", I. M. Gelfand and S. V. Fomin, Chap.7, and especially paragraph 36.4*]. Then (2) reduces to just,

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left\{\left[\ddot{u}_0 I_0-\frac{\partial N_1}{\partial x_1}-\frac{\partial N_6}{\partial x_2}\right]\delta u_0\right\}d\omega\,dt-\int_{t_1}^{t_2}\iint_{\Omega}\left[I_0\ddot{v}_0-\frac{\partial N_2}{\partial x_2}-\frac{\partial N_6}{\partial x_1}\right]\delta v_0 d\omega\,dt-$$

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left[\begin{array}{l} I_0\ddot{w}_0-\frac{4}{3h^2}\left(I_6\frac{4}{3h^2}-I_4\right)\left(\frac{\partial\ddot{\phi}_y}{\partial x_2}+\frac{\partial\ddot{\phi}_x}{\partial x_1}\right)-I_6\frac{16}{9h^4}\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}\right)- \\ \quad -\frac{\partial Q_4}{\partial x_2}+\frac{\partial R_4}{\partial x_2}\frac{4}{h^2}-\frac{\partial Q_5}{\partial x_1}+\frac{\partial R_5}{\partial x_1}\frac{4}{h^2}- \\ \qquad -\frac{4}{3h^2}\left(\frac{\partial^2 P_1}{\partial^2 x_1}+\frac{2\,\partial^2 P_6}{\partial x_2\,\partial x_1}+\frac{\partial^2 P_2}{\partial^2 x_2}\right)-q \end{array}\right]\delta w_0 d\omega\,dt-$$

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left[\begin{array}{l}\left(I_2-I_4\frac{8}{3h^2}+I_6\frac{16}{9h^4}\right)\ddot{\phi}_x+\frac{4}{3h^2}\left(I_6\frac{4}{3h^2}-I_4\right)\frac{\partial\ddot{w}_0}{\partial x_1}+ \\ \quad +Q_5-\frac{4}{h^2}R_5-\frac{\partial M_1}{\partial x_1}-\frac{\partial M_6}{\partial x_2}+\frac{\partial P_6}{\partial x_2}\frac{4}{3h^2}+\frac{\partial P_1}{\partial x_1}\frac{4}{3h^2}\end{array}\right]\delta\phi_x\,d\omega\,dt-$$

$$-\int_{t_1}^{t_2}\iint_{\Omega}\left[\begin{array}{l}\left(I_2-I_4\frac{8}{3h^2}+I_6\frac{16}{9h^4}\right)\ddot{\phi}_y+\frac{4}{3h^2}\left(\frac{4}{3h^2}I_6-I_4\right)\frac{\partial\ddot{w}_0}{\partial x_2}+ \\ \quad +Q_4-\frac{4}{h^2}R_4-\frac{\partial M_2}{\partial x_2}-\frac{\partial M_6}{\partial x_1}+\frac{\partial P_6}{\partial x_1}\frac{4}{3h^2}+\frac{\partial P_2}{\partial x_2}\frac{4}{3h^2}\end{array}\right]\delta\phi_y\,d\omega\,dt=0 \qquad (2`)$$

Subsequently, we assume that $\delta v_0 = \delta w_0 = \delta\phi_x = \delta\phi_y = 0$ on the domain Ω (inside the body of the plate). Thus,

$$\int_{t_1}^{t_2}\iint_{\Omega}\left\{\left[\ddot{u}_0 I_0-\frac{\partial N_1}{\partial x_1}-\frac{\partial N_6}{\partial x_2}\right]\delta u_0(\boldsymbol{x};t)\right\}d\omega\,dt=0, \qquad \forall\ \delta u_0(\boldsymbol{x};t)$$

and using the arbitrariness of the variation δu_0 inside the $\Omega\times[t_1,t_2]$, we find the first equation of motion of the plate,

$$\ddot{u}_0 I_0-\frac{\partial N_{11}}{\partial x_1}-\frac{\partial N_{12}}{\partial x_2}=0 \qquad \textbf{(3)}$$

for $\boldsymbol{x}\in\Omega$ and $\forall\ t\in[t_1,\ t_2]$.

Next, we remove the restriction $\delta v_0 = 0$ on the domain $\Omega\times[t_1,\ t_2]$ and taking into account the equation (3), which eliminates the first surface integral (2), we derive

$$\int_{t_1}^{t_2}\iint_{\Omega}\left[I_0\ddot{v}_0-\frac{\partial N_2}{\partial x_2}-\frac{\partial N_6}{\partial x_1}\right]\delta v_0(\boldsymbol{x};t)\,d\omega\,dt=0, \qquad \forall\ \delta v_0(\boldsymbol{x};t)$$

And using the arbitrariness of the variation δv_0 inside the space $\Omega\times[t_1,\ t_2]$, we result to the second equation of motion of the plate,

$$I_0 \ddot{v}_0 - \frac{\partial N_{22}}{\partial x_2} - \frac{\partial N_{12}}{\partial x_1} = 0 \qquad \textbf{(4)}$$

for $\boldsymbol{x} \in \Omega$ and $\forall\ t \in [t_1, t_2]$.

Further, removing the restriction $\delta w_0 = 0$ on the surface Ω and taking into account the two previous Eqs. (3) and (4), the result of the Eq. (2`) is,

$$\int_{t_1}^{t_2} \iint_{\Omega} \left[\begin{array}{l} I_0 \ddot{w}_0 - \dfrac{4}{3h^2}\left(I_6 \dfrac{4}{3h^2} - I_4\right)\left(\dfrac{\partial \ddot{\phi}_y}{\partial x_2} + \dfrac{\partial \ddot{\phi}_x}{\partial x_1}\right) - I_6 \dfrac{16}{9h^4}\left(\dfrac{\partial^2 \ddot{w}_0}{\partial^2 x_2} + \dfrac{\partial^2 \ddot{w}_0}{\partial^2 x_1}\right) - \\ - \dfrac{\partial Q_4}{\partial x_2} + \dfrac{\partial R_4}{\partial x_2}\dfrac{4}{h^2} - \dfrac{\partial Q_5}{\partial x_1} + \dfrac{\partial R_5}{\partial x_1}\dfrac{4}{h^2} - \dfrac{4}{3h^2}\left(\dfrac{\partial^2 P_1}{\partial^2 x_1} + \dfrac{2\,\partial^2 P_6}{\partial x_2\,\partial x_1} + \dfrac{\partial^2 P_2}{\partial^2 x_2}\right) - q \end{array} \right] \delta w_0(\boldsymbol{x};t)\, d\omega\, dt = 0$$

$$,\forall \quad \delta w_0(\boldsymbol{x};t)$$

Regarding also the arbitrariness of variation δw_0 inside the $\Omega \times [t_1,\ t_2]$, we extract the third equation of motion of the plate,

$$I_0 \ddot{w}_0 - \frac{4}{3h^2}\left(I_6 \frac{4}{3h^2} - I_4\right)\left(\frac{\partial \ddot{\phi}_y}{\partial x_2} + \frac{\partial \ddot{\phi}_x}{\partial x_1}\right) - I_6 \frac{16}{9h^4}\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2} + \frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}\right) -$$
$$- \frac{\partial Q_{23}}{\partial x_2} + \frac{\partial R_{23}}{\partial x_2}\frac{4}{h^2} - \frac{\partial Q_{13}}{\partial x_1} + \frac{\partial R_{13}}{\partial x_1}\frac{4}{h^2} - \frac{4}{3h^2}\left(\frac{\partial^2 P_{11}}{\partial^2 x_1} + \frac{2\,\partial^2 P_{12}}{\partial x_2\,\partial x_1} + \frac{\partial^2 P_{22}}{\partial^2 x_2}\right) = q \qquad \textbf{(5)}$$

Now, remove the restriction $\delta\phi_x = 0$ from the domain Ω and taking into account the three previous Eqs. (3), (4), (5), the result of the Eq. (2`) is the following,

$$\int_{t_1}^{t_2} \iint_{\Omega} \left[\begin{array}{l} \left(I_2 - I_4 \dfrac{8}{3h^2} + I_6 \dfrac{16}{9h^4}\right)\ddot{\phi}_x + \dfrac{4}{3h^2}\left(I_6 \dfrac{4}{3h^2} - I_4\right)\dfrac{\partial \ddot{w}_0}{\partial x_1} + \\ + Q_5 - \dfrac{4}{h^2} R_5 - \dfrac{\partial M_1}{\partial x_1} - \dfrac{\partial M_6}{\partial x_2} + \dfrac{\partial P_6}{\partial x_2}\dfrac{4}{3h^2} + \dfrac{\partial P_1}{\partial x_1}\dfrac{4}{3h^2} \end{array} \right] \delta\phi_x(\boldsymbol{x};t)\, d\omega\, dt = 0$$

$$,\forall \quad \delta\phi_x(\boldsymbol{x};t)$$

Regarding also the arbitrariness of the variation $\delta\phi_x$ inside the $\Omega \times [t_1,\ t_2]$, we extract the fourth equation of motion of the plate,

$$\left(I_2 - I_4 \frac{8}{3h^2} + I_6 \frac{16}{9h^4}\right)\ddot{\phi}_x + \frac{4}{3h^2}\left(I_6 \frac{4}{3h^2} - I_4\right)\frac{\partial \ddot{w}_0}{\partial x_1} +$$
$$+ Q_{13} - \frac{4}{h^2} R_{13} - \frac{\partial M_{11}}{\partial x_1} - \frac{\partial M_{12}}{\partial x_2} + \frac{\partial P_{12}}{\partial x_2}\frac{4}{3h^2} + \frac{\partial P_{11}}{\partial x_1}\frac{4}{3h^2} = 0 \qquad \textbf{(6)}$$

And finally (as for the equations of motion) eliminating the restriction $\delta\phi_y = 0$ from the Ω and recalling the previous four Eqs. (3), (4), (5) and (6), the result of the Eq. (2`) is,

$$\int_{t_1}^{t_2}\iint_{\Omega}\left[\begin{array}{l}\left(I_2-I_4\frac{8}{3h^2}+I_6\frac{16}{9h^4}\right)\ddot{\phi}_y+\frac{4}{3h^2}\left(\frac{4}{3h^2}I_6-I_4\right)\frac{\partial\ddot{w}_0}{\partial x_2}+\\ +Q_4-\frac{4}{h^2}R_4-\frac{\partial M_2}{\partial x_2}-\frac{\partial M_6}{\partial x_1}+\frac{\partial P_6}{\partial x_1}\frac{4}{3h^2}+\frac{\partial P_2}{\partial x_2}\frac{4}{3h^2}\end{array}\right]\delta\phi_y(\boldsymbol{x};t)\,d\omega\,dt=0$$

$$,\forall\ \delta\phi_y(\boldsymbol{x};t)$$

Regarding also the arbitrariness of variation $\delta\phi_y$ inside the $\Omega\times[t_1,\ t_2]$, we extract the fifth equation of motion of the plate,

$$\left(I_2-I_4\frac{8}{3h^2}+I_6\frac{16}{9h^4}\right)\ddot{\phi}_y+\frac{4}{3h^2}\left(\frac{4}{3h^2}I_6-I_4\right)\frac{\partial\ddot{w}_0}{\partial x_2}+$$
$$+Q_{23}-\frac{4}{h^2}R_{23}-\frac{\partial M_{22}}{\partial x_2}-\frac{\partial M_{12}}{\partial x_1}+\frac{\partial P_{12}}{\partial x_1}\frac{4}{3h^2}+\frac{\partial P_{22}}{\partial x_2}\frac{4}{3h^2}=0 \qquad \textbf{(7)}$$

Let it be noted that Eqs. **(3)** - **(7)** are identical with the respective results (10.3.14), (10.3.15), (10.3.16), (10.3.17), (10.3.18) of the book of the *J.N. Reddy (2007), "Theory and Analysis of Elastic Plates and Shells"*, chapter 10 on the page 381.

It is essential to note that, the above system of the five equations (3), (4), (5), (6) and (7) is solvable, as will be proved on the section 6, because the number of unknown quantities is five. This is a fact due to the definition of the thickness-integrated forces and moments [Eqs. (4) of the section 4.3], which can be expressed directly in terms of the unknowns of the system, namely the displacement field $(u_0, v_0, w_0, \phi_x, \phi_y)$, as will be shown on the section 6 again.

5. Boundary Conditions of TSDT in terms of thickness-integrated forces and moments

Inspection of the above Eq. (2) of the section 4.5 indicates that the quantities with a variation in the boundary integrals are the **primary variables** $u_0, v_0, w_0, \partial w_0 / \partial x_1, \partial w_0 / \partial x_2, \phi_x, \phi_y$ and their specification constitutes the **geometric** or **kinematic** (essential) boundary conditions. The mathematical expressions inside the brackets of the integrated quantities, which are coefficients of the varied quantities, are termed the **secondary variables**, and their specification gives the **dynamic** (natural) boundary conditions. Therefore, there are primary and secondary variables of the plate with edges parallel to $x_1 x_2$-coordinates.

On this step similarly with the process followed on the respective section of the Part A, we must think a bit more about the final results of the boundary conditions of the problem, due to the unbalance between to possible boundary conditions (after noticing the boundary terms of the Variational Equation (2) of the section 4.5) and the desirable number of boundary conditions. To set it differently, if the equations of motion of the model of TSDT are expressed in terms of displacements (as will be shown on the section 6), they maximum order derivatives that appear on them are the second-order spatial derivatives of the in-plane displacements u_0, v_0, the third-order spatial derivatives of the slopes displacements ϕ_x, ϕ_y and the **fourth-order spatial** derivative of the vertical displacement w_0. The above imply that the total number of boundary conditions must be **four essential** and **four natural** boundary conditions, whereas from the Variational Equation (1) below we note **seven** essential and **seven** natural boundary conditions. The last is incompatible with our model of TSDT and must be treated appropriately as will be shown on the section 5.2.

5.1. Variational boundary terms in Cartesian coordinates

Now isolating the curvilinear integrals of the Eq. (2) of the section 4.5, in order to illustrate clearer the aforementioned boundary terms.

Initially we remove the restrictions $\delta u_0 = \delta v_0 = \delta w_0 = \partial \delta w_0 / \partial x_1 = \partial \delta w_0 / \partial x_2 = \delta \phi_x = \delta \phi_y = 0$ from the boundary. We deserve to isolate these integrals, because the existence of the Eqs. (3) - (7) of the section 4.6. Thus, the action functional (2) becomes,

$$
- \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ N_1 n_{x_1} + N_6 n_{x_2} - \frac{I_0}{\rho} a_{T0} \right\} \delta u_0 \, d\gamma \, dt - \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ N_2 n_{x_1} + N_6 n_{x_2} - \frac{I_0}{\rho} b_{T0} \right\} \delta v_0 \, d\gamma \, dt -
$$

$$
- \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \begin{array}{l} \left(M_1 - \frac{4}{3h^2} P_1 \right) n_{x_1} + \left(M_6 - P_6 \frac{4}{3h^2} \right) n_{x_2} - \\ - \left(\begin{array}{c} a_{T1} \frac{I_2}{\rho} + a_{T3} \frac{I_4}{\rho} - a_{T1} \frac{I_4}{\rho} \frac{4}{3h^2} - a_{T3} \frac{I_6}{\rho} \frac{4}{3h^2} + \\ + a_5 \, a_{T0} \frac{I_0}{\rho} + b_5 \, c_{T0} \frac{I_0}{\rho} \end{array} \right) \end{array} \right\} \delta \phi_x \, d\gamma \, dt -
$$

$$
\begin{aligned}
&-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}\left(M_2-\dfrac{4}{3h^2}P_2\right)n_{x_2}+\left(M_6-P_6\dfrac{4}{3h^2}\right)n_{x_1}-\\
-\left(\begin{array}{l}b_{T1}\dfrac{I_2}{\rho}+b_{T3}\dfrac{I_4}{\rho}-b_{T1}\dfrac{I_4}{\rho}\dfrac{4}{3h^2}-b_{T3}\dfrac{I_6}{\rho}\dfrac{4}{3h^2}+\\
+a_4 b_{T0}\dfrac{I_0}{\rho}+b_4 c_{T0}\dfrac{I_0}{\rho}\end{array}\right)\end{array}\right\}\delta\phi_y\,d\gamma\,dt-\\
&-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}\left[\begin{array}{l}\left(I_6\dfrac{16}{9h^4}-\dfrac{4}{3h^2}I_4\right)\ddot{\phi}_x+I_6\dfrac{16}{9h^4}\dfrac{\partial\ddot{w}_0}{\partial x_1}+\\
+Q_5-R_5\dfrac{4}{h^2}+\dfrac{4}{3h^2}\dfrac{\partial P_6}{\partial x_2}+\dfrac{4}{3h^2}\dfrac{\partial P_1}{\partial x_1}\end{array}\right]\boldsymbol{n}_{x_1}+\\
+\left[\begin{array}{l}\left(I_6\dfrac{16}{9h^4}-\dfrac{4}{3h^2}I_4\right)\ddot{\phi}_y+I_6\dfrac{16}{9h^4}\dfrac{\partial\ddot{w}_0}{\partial x_2}+\\
+Q_4-R_4\dfrac{4}{h^2}+\dfrac{4}{3h^2}\dfrac{\partial P_6}{\partial x_1}+\dfrac{4}{3h^2}\dfrac{\partial P_2}{\partial x_2}\end{array}\right]\boldsymbol{n}_{x_2}-\dfrac{I_0}{\rho}c_{T0}\end{array}\right\}\delta w_0\,d\gamma\,dt+\\
&+\int_{t_1}^{t_2}\oint_{\Gamma}\frac{4}{3h^2}\left(\left\{P_1 n_{x_1}+P_6 n_{x_2}-\left(a_{T1}\frac{I_4}{\rho}+a_{T3}\frac{I_6}{\rho}\right)\right\}\frac{\partial\delta w_0}{\partial x_1}\right)d\gamma\,dt+\\
&+\int_{t_1}^{t_2}\oint_{\Gamma}\frac{4}{3h^2}\left(\left\{P_2 n_{x_2}+P_6 n_{x_1}-\left(b_{T1}\frac{I_4}{\rho}+b_{T3}\frac{I_6}{\rho}\right)\right\}\frac{\partial\delta w_0}{\partial x_2}\right)d\gamma\,dt=0
\end{aligned}
\tag{1}
$$

We are thinking exactly with the same rationality as on the section 4.5 but at this time to derive the boundary terms of the problem. Subsequently, on the section 5.3 we are going to extract the boundary conditions, which are independent from each other and also compatible with our problem.

Note that the above process of deriving the equations of motion and the following boundary conditions is explained thoroughly on the Lecture Notes of Functional Analysis, *G.A. Athanassoulis* (*2016*) *"Necessary Conditions of Extremum of Functional"* and *"A further study of the Variational Problem as for integral type functional"*, as well as on the book of *Gelfand I.M., Fomin S.V.* (*1963*), *"Calculus of Variations"*.

5.2. Transformation of the boundary conditions to a curvilinear boundary system

[References: *J.N. Reddy "Theory and Analysis of Elastic Plates and Shells"*, Chapter 1.4/ 3.5/ 10.3 and the *Lecture Notes, G.A. Athanassoulis* (*2016*), *"Invariances and transformation of physical quantities under rotations of the reference system"* and *Wang C.M., Reddy J.N., Lee K.H. (2000), "Shear Deformable Beams and Plates-Relations with Classical Solutions"*, Chapter 7.4].

Thinking about the way of combining and grouping together the variations of the displacement field of the problem of TSDT and as a consequence the boundary terms related to each one of them and also taking into account the process followed on the respective sections for the problem of CPT (section 5.2 of Part A), we note that the corresponding displacements after a potential transformation to a curvilinear boundary system will be,

Cartesian Coordinates	Curvilinear Coordinates
u_0	u_{0n}
v_0	u_{0s}
w_0	w_0
$\partial w_0 / \partial x_1$ and $\partial w_0 / \partial x_2$	w_0 and $\partial w_0 / \partial n$
ϕ_x	ϕ_n
ϕ_y	ϕ_s

However, the higher order of the spatial derivatives of the equations of motion of the TSDT is **four** and the total number of essential or natural boundary conditions seems to be **six** and **six** respectively, fact that is not auxiliary to solve the system of differential equations of motion. For this reason, we take another path in order to conclude to a balance between the unknowns and the equations.

From the Eqs. (14a) - (14e) of the section 6.1 and 6.2, we note that the first two equations of motion, namely the Eqs. (14a) and (14b) are coupled between each other but decoupled form the residual three equations of motion, namely Eqs. (14c), (14d) and (14e). Also the previously referred Eqs. (14c), (14d) and (14e) are coupled between them. To express the previous differently the components of the displacement field (u_0, v_0) exist only on the Eqs. (14a) and (14b) of the sections 6.1 or 6.2, whereas the components (w_0, ϕ_x, ϕ_y) exist only on the Eqs. (14c), (14d) and (14e). The last means that the **shear** deformation (motion) of the plate is decoupled from its **bending** deformation (motion) and due to this ascertainment, we are entitled to study each one of these two "motions" separately and independently.

Actually, as will be shown on the following sections, on the one hand the shear motion has **two** essential and **two** natural boundary conditions related to the displacements u_{0n} and u_{0s}, fact that is compatible with the **second-order** differential equations (14a) and (14b). On the other hand the bending motion has **four** essential and **four** natural boundary conditions associated to the displacements w_0, $\partial w_0 / \partial n$, ϕ_n and ϕ_s, which is also compatible with the **fourth-order** differential equations (14c), (14d), (14e).

In addition, all the remarks about the transformation laws existing on the section 5.2 of the Part A and on the APPENDIX A are exactly the same and are used self-esteem on the following sections. For the sake of convenience, we repeat the most important transformation law to this section and after that we formulate some more transformation laws about the excess stress components of the displacement field of the TSDT (referred on the section 4.2) and about the higher-order thickness-integrated moments (referred on the section 4.2). Thus,

$u_0 = n_{x_1} u_{0n} + n_{x_2} u_{0s}$ (1) and $\delta u_0 = n_{x_1} \delta u_{0n} + n_{x_2} \delta u_{0s}$ (1`)

$v_0 = -n_{x_2} u_{0n} + n_{x_1} u_{0s}$ (2) and $\delta v_0 = -n_{x_2} \delta u_{0n} + n_{x_1} \delta u_{0s}$ (2`)

$w_0 = 1 w_0$ (3) and $\delta w_0 = 1 \delta w_0$ (3`)

But here we have two more components of the displacement field, which are converted to the curvilinear system by the same transformation law, since the nature of the transformation is identical to that performed in the context of the CPT. To remember, is a planar rotation around the vertical axis to the flat surfaces of the plate. For this reason, the x_3, z-axes remain parallel. Consequently,

$\phi_x = n_{x_1} \phi_n + n_{x_2} \phi_s$ (4) and $\delta\phi_x = n_{x_1} \delta\phi_n + n_{x_2} \delta\phi_s$ (4`)

$\phi_y = -n_{x_1} \phi_n + n_{x_2} \phi_s$ (5) and $\delta\phi_y = -n_{x_1} \delta\phi_n + n_{x_2} \delta\phi_s$ (5`).

As for the derivatives of the δw_0, we get

$$\frac{\partial \delta w_0}{\partial x_1} = n_{x_1} \frac{\partial \delta w_0}{\partial n} + n_{x_2} \frac{\partial \delta w_0}{\partial s} \quad (6)$$

$$\frac{\partial \delta w_0}{\partial x_2} = -n_{x_2} \frac{\partial \delta w_0}{\partial n} + n_{x_1} \frac{\partial \delta w_0}{\partial s} \quad (7)$$

Finally the relations which transform the stress field from the Cartesian coordinate system to the curvilinear one, are

$$\sigma_{11} = n_{x_1}^2 \sigma_{nn} + 2 n_{x_1} n_{x_2} \sigma_{ns} + n_{x_2}^2 \sigma_{ss} \quad (8)$$

$$\sigma_{22} = n_{x_2}^2 \sigma_{nn} - 2 n_{x_1} n_{x_2} \sigma_{ns} + n_{x_1}^2 \sigma_{ss} \quad (9)$$

$$\sigma_{12} = \sigma_{21} = n_{x_1} n_{x_2} \; \sigma_{ss} - \sigma_{nn} + n_{x_1}^2 - n_{x_2}^2 \; \sigma_{ns} \quad (10)$$

As for the two more stress components of the TSDT, taking into account the above relation (10) we get

$$\sigma_{13} = -\sigma_{31} = n_{x_1} n_{x_3} \; \sigma_{zz} - \sigma_{nn} + n_{x_1}^2 - n_{x_3}^2 \; \sigma_{nz} = -n_{x_1} \sigma_{nn} + n_{x_1}^2 - 1 \; \sigma_{nz} \quad (11)$$

$$\sigma_{23} = -\sigma_{32} = n_{x_2} n_{x_3} \; \sigma_{zz} - \sigma_{ss} + n_{x_2}^2 - n_{x_3}^2 \; \sigma_{nz} = -n_{x_2} \sigma_{ss} + n_{x_1}^2 - 1 \; \sigma_{sz} \quad (12)$$

The final results of the last two shear stresses are derived due to the initial assumption of the in extensibility of the cross section, $\sigma_{zz} = 0$ and due to the planar rotation of the coordinate system around the vertical axis, the direction cosine as for the vertical axis is $n_{x_3} = \cos 0 = 1$. It is also interesting that the shear stresses are transformed with opposite signs.

By the definition of the thickness-integrated forces and moments, we have similarly with the transformation of the stress field the following relations,

$$N_{11} = n_{x_1}^2 N_{nn} + 2n_{x_1} n_{x_2} N_{ns} + n_{x_2}^2 N_{ss} \tag{13}$$

$$N_{22} = n_{x_2}^2 N_{nn} - 2n_{x_1} n_{x_2} N_{ns} + n_{x_1}^2 N_{ss} \tag{14}$$

$$N_{12} = N_{21} = n_{x_1} n_{x_2} \; N_{ss} - N_{nn} \; + \; n_{x_1}^2 - n_{x_2}^2 \; N_{ns} \tag{15}$$

and

$$M_{11} = n_{x_1}^2 M_{nn} + 2n_{x_1} n_{x_2} M_{ns} + n_{x_2}^2 M_{ss} \tag{16}$$

$$M_{22} = n_{x_2}^2 M_{nn} - 2n_{x_1} n_{x_2} M_{ns} + n_{x_1}^2 M_{ss} \tag{17}$$

$$M_{12} = M_{21} = n_{x_1} n_{x_2} \; M_{ss} - M_{nn} \; + \; n_{x_1}^2 - n_{x_2}^2 \; M_{ns} \tag{18}$$

And also as for the higher-order moment appearing on the model of TSDT,

$$P_{11} = n_{x_1}^2 P_{nn} + 2n_{x_1} n_{x_2} P_{ns} + n_{x_2}^2 P_{ss} \tag{16`}$$

$$P_{22} = n_{x_2}^2 P_{nn} - 2n_{x_1} n_{x_2} P_{ns} + n_{x_1}^2 P_{ss} \tag{17`}$$

$$P_{12} = P_{21} = n_{x_1} n_{x_2} \; P_{ss} - P_{nn} \; + \; n_{x_1}^2 - n_{x_2}^2 \; P_{ns} \tag{18`}$$

Additionally, with similar way with that of the transformation law of the Eqs. (11) and (12), we get

$$Q_{13} = -Q_{31} = n_{x_1} n_{x_3} \; Q_{zz} - Q_{nn} \; + \; n_{x_1}^2 - n_{x_3}^2 \; Q_{nz} = -n_{x_1} Q_{nn} + \; n_{x_1}^2 - 1 \; Q_{nz} \tag{19}$$

$$Q_{23} = -Q_{32} = n_{x_2} n_{x_3} \; Q_{zz} - Q_{ss} \; + \; n_{x_2}^2 - n_{x_3}^2 \; Q_{nz} = -n_{x_2} Q_{ss} + \; n_{x_1}^2 - 1 \; Q_{sz} \tag{20}$$

Similarly, are transformed the higher order moments R_{13} and R_{23},

$$R_{13} = -R_{31} = n_{x_1} n_{x_3} \; R_{zz} - R_{nn} \; + \; n_{x_1}^2 - n_{x_3}^2 \; R_{nz} = -n_{x_1} R_{nn} + \; n_{x_1}^2 - 1 \; R_{nz} \tag{19`}$$

$$R_{23} = -R_{32} = n_{x_2} n_{x_3} \; R_{zz} - R_{ss} \; + \; n_{x_2}^2 - n_{x_3}^2 \; R_{nz} = -n_{x_2} R_{ss} + \; n_{x_1}^2 - 1 \; R_{sz} \tag{20`}$$

Before we proceed to the transformation of the boundary terms from the Cartesian to the curvilinear coordinate system, we have to present the same transformation law of the functions $a_{T0}(x_1, x_2)$, $b_{T0}(x_1, x_2)$, $c_{T0}(x_1, x_2)$, $a_{T1}(x_1, x_2)$, $b_{T1}(x_1, x_2)$, $c_{T1}(x_1, x_2)$, $a_{T3}(x_1, x_2)$, $b_{T3}(x_1, x_2)$, $c_{T3}(x_1, x_2)$, $a_4(x_1, x_2)$, $b_4(x_1, x_2)$, $a_5(x_1, x_2)$ and $b_5(x_1, x_2)$ which describe the form of the given surface tractions (shown on the section 4.4). Thus, according to the transformation law (T0) and (T1) of the APPENDIX A, we get the following relations

$$a_{T0} = n_{x_1} \cdot a_{T0n} + n_{x_2} \cdot a_{T0s} \quad (21a) \quad \text{and} \quad a_{T1} = n_{x_1} \cdot a_{T1n} + n_{x_2} \cdot a_{T1s} \quad (21b)$$

$$a_{T3} = n_{x_1} \cdot a_{T3n} + n_{x_2} \cdot a_{T3s} \quad (21c)$$

$$b_{T0} = -n_{x_2} \cdot a_{T0n} + n_{x_1} \cdot a_{T0s} \quad (21d) \quad \text{and} \quad b_{T1} = -n_{x_2} \cdot a_{T1n} + n_{x_1} \cdot a_{T1s} \quad (21e)$$

$$b_{T3} = -n_{x_2} \cdot a_{T3n} + n_{x_1} \cdot a_{T3s} \quad (21f)$$

$$c_{T0} = 1 \cdot c_{T0} \quad (21g) \quad \text{and} \quad c_{T1} = 1 \cdot c_{T1} \quad (21h)$$

$$c_{T3} = 1 \cdot c_{T3} \quad (21i)$$

However the functions c_{T1} and c_{T3} have been eliminated from the Variational Equations due to analysis of the section 4.4, and there not going to appear further after their definition on the surface traction field of the section 2. Further, we have

$$a_4 = n_{x_1} a_{4n} + n_{x_2} a_{4s} \quad (21j) \quad \text{and} \quad a_5 = n_{x_1} a_{5n} + n_{x_2} a_{5s} \quad (21k)$$

$$b_4 = -n_{x_1} b_{4n} + n_{x_2} b_{4s} \quad (21l) \quad \text{and} \quad b_5 = -n_{x_1} b_{5n} + n_{x_2} b_{5s} \quad (21m)$$

In the view of the above relation, it is obvious that we have to occupy with the surface integrals (boundary terms) of the variational equation 2 of the section 4.5, namely the below

$$- \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ N_1 n_{x_1} + N_6 n_{x_2} - \frac{I_0}{\rho} a_{T0} \right\} \delta u_0 \, d\gamma \, dt - \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ N_2 n_{x_1} + N_6 n_{x_2} - \frac{I_0}{\rho} b_{T0} \right\} \delta v_0 \, d\gamma \, dt -$$

$$- \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \begin{array}{l} \left(M_1 - \frac{4}{3h^2} P_1 \right) n_{x_1} + \left(M_6 - P_6 \frac{4}{3h^2} \right) n_{x_2} - \\ - \left(\begin{array}{c} a_{T1} \frac{I_2}{\rho} + a_{T3} \frac{I_4}{\rho} - a_{T1} \frac{I_4}{\rho} \frac{4}{3h^2} - a_{T3} \frac{I_6}{\rho} \frac{4}{3h^2} + \\ + a_5 \, a_{T0} \frac{I_0}{\rho} + b_5 \, c_{T0} \frac{I_0}{\rho} \end{array} \right) \end{array} \right\} \delta \phi_x \, d\gamma \, dt -$$

$$- \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \begin{array}{l} \left(M_2 - \frac{4}{3h^2} P_2 \right) n_{x_2} + \left(M_6 - P_6 \frac{4}{3h^2} \right) n_{x_1} - \\ - \left(\begin{array}{c} b_{T1} \frac{I_2}{\rho} + b_{T3} \frac{I_4}{\rho} - b_{T1} \frac{I_4}{\rho} \frac{4}{3h^2} - b_{T3} \frac{I_6}{\rho} \frac{4}{3h^2} + \\ + a_4 \, b_{T0} \frac{I_0}{\rho} + b_4 \, c_{T0} \frac{I_0}{\rho} \end{array} \right) \end{array} \right\} \delta \phi_y \, d\gamma \, dt -$$

$$- \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \begin{array}{l} \left[\begin{array}{l} \left(I_6 \frac{16}{9h^4} - \frac{4}{3h^2} I_4 \right) \ddot{\phi}_x + I_6 \frac{16}{9h^4} \frac{\partial \ddot{w}_0}{\partial x_1} + \\ + Q_5 - R_5 \frac{4}{h^2} + \frac{4}{3h^2} \frac{\partial P_6}{\partial x_2} + \frac{4}{3h^2} \frac{\partial P_1}{\partial x_1} \end{array} \right] n_{x_1} + \\ + \left[\begin{array}{l} \left(I_6 \frac{16}{9h^4} - \frac{4}{3h^2} I_4 \right) \ddot{\phi}_y + I_6 \frac{16}{9h^4} \frac{\partial \ddot{w}_0}{\partial x_2} + \\ + Q_4 - R_4 \frac{4}{h^2} + \frac{4}{3h^2} \frac{\partial P_6}{\partial x_1} + \frac{4}{3h^2} \frac{\partial P_2}{\partial x_2} \end{array} \right] n_{x_2} - \frac{I_0}{\rho} c_{T0} \end{array} \right\} \delta w_0 \, d\gamma \, dt +$$

$$+ \int_{t_1}^{t_2} \oint_{\Gamma} \frac{4}{3h^2} \left(\left\{ P_1 \, n_{x_1} + P_6 \, n_{x_2} - \left(a_{T1} \frac{I_4}{\rho} + a_{T3} \frac{I_6}{\rho} \right) \right\} \frac{\partial \delta w_0}{\partial x_1} \right) d\gamma \, dt +$$

$$+ \int_{t_1}^{t_2} \oint_{\Gamma} \frac{4}{3h^2} \left(\left\{ P_2 \, n_{x_2} + P_6 \, n_{x_1} - \left(b_{T1} \frac{I_4}{\rho} + b_{T3} \frac{I_6}{\rho} \right) \right\} \frac{\partial \delta w_0}{\partial x_2} \right) d\gamma \, dt = 0$$

As exactly on the corresponding section of the Part A, it is essential to separate the above equation into four parts and after that to perform calculations. The first parts will include the in-plane variations δu_0 and δv_0 (first row of the above expression), the second is associated with the derivatives of the vertical variation $\partial \delta w_0 / \partial x_1$ and $\partial \delta w_0 / \partial x_2$ (fifth and sixth row), whereas the third one is related to the vertical variation δw_0 (fourth row). Further, we have one more part in which we manage the terms related to the slopes $\delta \phi_x$ and $\delta \phi_y$ (second and third row).

As for the aforementioned disjunction of the above expression including all the boundary terms, we procced to the following separate sections.

5.2.1. Transformation of the in-plane boundary conditions to the curvilinear boundary system

For simplification reasons the of transformation of these terms from the Cartesian coordinate system to the local one, we neglect once again the time integration and further now the curvilinear integration.
Subsequently, taking apart each boundary condition multiplied with a different component of the variation of the displacement field and using the Eqs. (1`), (2`), (3`), (13), (14), (15) of the section 5.2 we get the following. Note that the transformation into curvilinear coordinates is exactly the same with that of the in-plane boundary conditions of the CPT (section 5.2.1 Part A). Thus, we will not continue to thorough calculations.

$$\begin{aligned}
&\left\{N_{11} n_{x_1} + N_{12} n_{x_2} - \frac{I_0}{\rho} a_{T0}\right\}\delta u_0 = \\
&= n^4_{x_1} - n^2_{x_1} n^2_{x_2} \; N_{nn}\,\delta u_{0n} + 2\, n^2_{x_2} n^2_{x_1} \; N_{ss}\,\delta u_{0n} + (3\, n^3_{x_1} n_{x_2} - n_{x_1} n^3_{x_2})\, N_{ns}\,\delta u_{0n} + \\
&+ n^3_{x_1} n_{x_2} - n_{x_1} n^3_{x_2} \; N_{nn}\,\delta u_{0s} + 2\, n^3_{x_2} n_{x_1} \; N_{ss}\,\delta u_{0s} + (3\, n^2_{x_1} n^2_{x_2} - n^4_{x_2})\, N_{ns}\,\delta u_{0s} - \\
&- n^2_{x_1} a_{T0n} + n_{x_1} n_{x_2} a_{T0s} \; \frac{I_0}{\rho}\,\delta u_{0n} - (n_{x_1} n_{x_2} a_{T0n} + n^2_{x_2} a_{T0s})\frac{I_0}{\rho}\,\delta u_{0s}
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
&\left\{N_{22} n_{x_1} + N_{12} n_{x_2} - \frac{I_0}{\rho} b_{T0}\right\}\delta v_0 = \\
&= -\; n^4_{x_2} - n^2_{x_1} n^2_{x_2} \; N_{nn}\,\delta u_{0n} - 2\, n^2_{x_1} n^2_{x_2} \; N_{ss}\,\delta u_{0n} + (3 n_{x_1} n^3_{x_2} - n^3_{x_1} n_{x_2}) N_{ns}\,\delta u_{0n} + \\
&+ n^3_{x_2} n_{x_1} - n^3_{x_1} n_{x_2} \; N_{nn}\,\delta u_{0s} + 2\, n^3_{x_1} n_{x_2} \; N_{ss}\,\delta u_{0s} - (3 n^2_{x_1} n^2_{x_2} - n^4_{x_1})\, N_{ns}\,\delta u_{0s} + \\
&+ n^2_{x_2} a_{T0n} + n_{x_1} n_{x_2} a_{T0s} \; \frac{I_0}{\rho}\,\delta u_{0n} - (-n_{x_2} n_{x_1} a_{T0n} + n^2_{x_1} a_{T0s})\frac{I_0}{\rho}\,\delta u_{0s}
\end{aligned} \tag{23}$$

Summing up the Eqs. (22) and (23), we derive

$$\begin{aligned}
&\left\{N_{11} n_{x_1} + N_{12} n_{x_2} - a_{T0}\frac{I_0}{\rho}\right\}\delta u_0 + \left\{N_{22} n_{x_2} + N_{12} n_{x_1} - b_{T0}\frac{I_0}{\rho}\right\}\delta v_0 = \\
&\qquad = \left\{ n^2_{x_1} - n^2_{x_2} \; N_{nn} + 2\, n_{x_1} n_{x_2} \; N_{ns} - a_{T0n}\frac{I_0}{\rho}\right\}\delta u_{0n} + \\
&\qquad + \left\{ n^2_{x_1} - n^2_{x_2} \; N_{ns} + 2\, n_{x_2} n_{x_1} \; N_{ss} - a_{T0s}\frac{I_0}{\rho}\right\}\delta u_{0s}
\end{aligned} \tag{23`}$$

The remarks and the explanations done on the respective section 5.2.1 of the Part A are valid here too. Finally, we extract the two independent and in-plane natural boundary conditions of the problem of TSDT using the same arguments of the Calculus of Variations applied on the Part A in order to derive the corresponding in-plane natural boundary conditions of the CPT. Thus, we get the following

$$n_{x_1}^2 - n_{x_2}^2 \quad N_{nn} + 2\, n_{x_1} n_{x_2} \, N_{ns} = a_{T0n} \frac{I_0}{\rho} \tag{24a}$$

$$n_{x_1}^2 - n_{x_2}^2 \quad N_{ns} + 2\, n_{x_2} n_{x_1} \, N_{ss} = a_{T0s} \frac{I_0}{\rho} \tag{24b}$$

Remark once again, that the boundary conditions- Eqs. (24a) and (24b) are identically same with those of the in-plane natural boundary conditions of the model of the CPT. This fact was expected and rational because the additional assumptions made in the context of the TSDT influences main differences of the displacement field along the thickness of the plate than those along the horizontal dimensions of the plate. Consequently, we expect to find remarkable differences on the residual boundary conditions presented on the next sections 5.2.2, 5.2.3 and 5.2.4.

Note also that we have so far managed to remedy the unbalance between the number of boundary conditions and the order of the 2x2 system of the partial differential equations (14a) and (14b) of the following sections 6.1. or 6.2.

5.2.2. Transformation of the boundary conditions associated with $\partial_1 \delta w_0$, $\partial_2 \delta w_0$ to the curvilinear system

To simplify the process of transformation of these terms from the Cartesian coordinate system to the curvilinear one, we neglect once again the time integration and further now the curvilinear integration.

Subsequently, taking apart each boundary condition multiplied with a different component of the variation of the displacement field and using the Eqs. (6), (7), (16), (17), (18) and (21b), (21c), (21de), (21f) of the section 5.2 we get the following results.

Note also that the reason why we examine the boundary terms related to the x_1, x_2-spatial derivatives of the vertical displacement w_0, is that after the transformation of these two boundary terms from the Cartesian to the curvilinear coordinate system and due to the Eqs. (6) and (7) of the section 5.2, each Cartesian derivative of the w_0 gives derivatives both on the in-plane curvilinear derivatives along n and s-axis.

As for the first part associated to the x_1-spatial derivative of the variation δw_0,

$$\oint_\Gamma \frac{4}{3h^2}\left\{P_{11}\, n_{x_1} + P_{12}\, n_{x_2} - \left(a_{T1}\frac{I_4}{\rho} + a_{T3}\frac{I_6}{\rho}\right)\right\}\frac{\partial \delta w_0}{\partial x_1}\, d\gamma =$$

$$= \frac{4}{3h^2}\oint_\Gamma \left\{\begin{array}{l} \{n_{x_1}^2 P_{nn} + 2n_{x_1} n_{x_2} P_{ns} + n_{x_2}^2 P_{ss}\}\, n_{x_1} + \\ + \{n_{x_1} n_{x_2} (P_{ss} - P_{nn}) + (n_{x_1}^2 - n_{x_2}^2) P_{ns}\}\, n_{x_2} - \\ - (n_{x_1}\cdot a_{T1n} + n_{x_2}\cdot a_{T1s})\dfrac{I_4}{\rho} - (n_{x_1}\cdot a_{T3n} + n_{x_2}\cdot a_{T3s})\dfrac{I_6}{\rho} \end{array}\right\}\left\{n_{x_1}\frac{\partial \delta w_0}{\partial n} + n_{x_2}\frac{\partial \delta w_0}{\partial s}\right\} ds =$$

$$= \frac{4}{3h^2}\oint_\Gamma \left\{\begin{array}{l} n_{x_1}^3 P_{nn} + 2n_{x_1}^2 n_{x_2} P_{ns} + n_{x_2}^2 n_{x_1} P_{ss} + \\ + \{n_{x_1} n_{x_2}^2 (P_{ss} - P_{nn}) + (n_{x_1}^2 n_{x_2} - n_{x_2}^3) P_{ns}\} - \\ - (n_{x_1}\cdot a_{T1n} + n_{x_2}\cdot a_{T1s})\dfrac{I_4}{\rho} - (n_{x_1}\cdot a_{T3n} + n_{x_2}\cdot a_{T3s})\dfrac{I_6}{\rho} \end{array}\right\}\left\{n_{x_1}\frac{\partial \delta w_0}{\partial n} + n_{x_2}\frac{\partial \delta w_0}{\partial s}\right\} ds =$$

$$= \frac{4}{3h^2}\oint_\Gamma \left\{\begin{array}{l} n_{x_1}^4 P_{nn} + 2n_{x_1}^3 n_{x_2} P_{ns} + n_{x_2}^2 n_{x_1}^2 P_{ss}\dfrac{\partial \delta w_0}{\partial n} + \\ + \{n_{x_1}^2 n_{x_2}^2 P_{ss} - n_{x_1}^2 n_{x_2}^2 P_{nn} + (n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3) P_{ns}\}\dfrac{\partial \delta w_0}{\partial n} - \\ - (n_{x_1}\cdot a_{T1n} + n_{x_2}\cdot a_{T1s})\dfrac{I_4}{\rho} n_{x_1}\dfrac{\partial \delta w_0}{\partial n} - (n_{x_1}\cdot a_{T3n} + n_{x_2}\cdot a_{T3s})\dfrac{I_6}{\rho} n_{x_1}\dfrac{\partial \delta w_0}{\partial n} + \\ + \{n_{x_1}^3 n_{x_2} P_{nn} + 2n_{x_1}^2 n_{x_2}^2 P_{ns} + n_{x_2}^3 n_{x_1} P_{ss}\}\dfrac{\partial \delta w_0}{\partial s} + \\ + \{n_{x_1} n_{x_2}^3 P_{ss} - n_{x_1} n_{x_2}^3 P_{nn} + (n_{x_1}^2 n_{x_2}^2 - n_{x_2}^4) P_{ns}\}\dfrac{\partial \delta w_0}{\partial s} - \\ - (n_{x_1} n_{x_2}\cdot a_{T1n} + n_{x_2}^2\cdot a_{T1s})\dfrac{I_4}{\rho}\dfrac{\partial \delta w_0}{\partial s} - (n_{x_1} n_{x_2}\cdot a_{T3n} + n_{x_2}^2\cdot a_{T3s})\dfrac{I_6}{\rho}\dfrac{\partial \delta w_0}{\partial s} \end{array}\right\} ds =$$

[And now grouping together the terms with the same the thickness-integrated quantities of the same curvilinear direction and of the same derivative of w_0 on the curvilinear coordinate system, after highlighting each one of them with a different color]

$$= \frac{4}{3h^2}\oint_\Gamma \left\{ \begin{array}{l} n_{x_1}^4 \, P_{nn} \frac{\partial \delta w_0}{\partial n} + 2 n_{x_1}^3 \, n_{x_2} \, P_{ns} \frac{\partial \delta w_0}{\partial n} + n_{x_2}^2 \, n_{x_1}^2 \, P_{ss} \frac{\partial \delta w_0}{\partial n} + \\ + n_{x_1}^2 \, n_{x_2}^2 \, P_{ss} \frac{\partial \delta w_0}{\partial n} - n_{x_1}^2 \, n_{x_2}^2 \, P_{nn} \frac{\partial \delta w_0}{\partial n} + (n_{x_1}^3 \, n_{x_2} - n_{x_1} n_{x_2}^3) \, P_{ns} \frac{\partial \delta w_0}{\partial n} - \\ - (n_{x_1} \cdot a_{T1n} + n_{x_2} \cdot a_{T1s}) \frac{I_4}{\rho} n_{x_1} \frac{\partial \delta w_0}{\partial n} - (n_{x_1} \cdot a_{T3n} + n_{x_2} \cdot a_{T3s}) \frac{I_6}{\rho} n_{x_1} \frac{\partial \delta w_0}{\partial n} + \\ + n_{x_1}^3 \, n_{x_2} \, P_{nn} \frac{\partial \delta w_0}{\partial s} + 2 n_{x_1}^2 \, n_{x_2}^2 \, P_{ns} \frac{\partial \delta w_0}{\partial s} + n_{x_2}^3 \, n_{x_1} \, P_{ss} \frac{\partial \delta w_0}{\partial s} + \\ + n_{x_1} \, n_{x_2}^3 \, P_{ss} \frac{\partial \delta w_0}{\partial s} - n_{x_1} \, n_{x_2}^3 \, P_{nn} \frac{\partial \delta w_0}{\partial s} + (n_{x_1}^2 \, n_{x_2}^2 - n_{x_2}^4) \, P_{ns} \frac{\partial \delta w_0}{\partial s} - \\ - (n_{x_1} n_{x_2} \cdot a_{T1n} + n_{x_2}^2 \cdot a_{T1s}) \frac{I_4}{\rho} \frac{\partial \delta w_0}{\partial s} - (n_{x_1} n_{x_2} \cdot a_{T3n} + n_{x_2}^2 \cdot a_{T3s}) \frac{I_6}{\rho} \frac{\partial \delta w_0}{\partial s} \end{array} \right\} ds =$$

$$= \frac{4}{3h^2}\oint_\Gamma \left\{ \begin{array}{l} (n_{x_1}^4 - n_{x_1}^2 \, n_{x_2}^2) \, P_{nn} \frac{\partial \delta w_0}{\partial n} + 2 \, n_{x_2}^2 \, n_{x_1}^2 \, P_{ss} \frac{\partial \delta w_0}{\partial n} + \\ + (3 n_{x_1}^3 \, n_{x_2} - n_{x_1} n_{x_2}^3) \, P_{ns} \frac{\partial \delta w_0}{\partial n} - \\ - \left\{ (n_{x_1}^2 \cdot a_{T1n} + n_{x_2} n_{x_1} \cdot a_{T1s}) \frac{I_4}{\rho} + (n_{x_1}^2 \cdot a_{T3n} + n_{x_2} n_{x_1} \cdot a_{T3s}) \frac{I_6}{\rho} \right\} \frac{\partial \delta w_0}{\partial n} + \\ + (n_{x_1}^3 \, n_{x_2} - n_{x_1} n_{x_2}^3) \, P_{nn} \frac{\partial \delta w_0}{\partial s} + (n_{x_2}^3 \, n_{x_1} + n_{x_1} n_{x_2}^3) \, P_{ss} \frac{\partial \delta w_0}{\partial s} + \\ + (3 n_{x_1}^2 \, n_{x_2}^2 - n_{x_2}^4) \, P_{ns} \frac{\partial \delta w_0}{\partial s} - \\ - \left\{ (n_{x_1} n_{x_2} \cdot a_{T1n} + n_{x_2}^2 \cdot a_{T1s}) \frac{I_4}{\rho} + (n_{x_1} n_{x_2} \cdot a_{T3n} + n_{x_2}^2 \cdot a_{T3s}) \frac{I_6}{\rho} \right\} \frac{\partial \delta w_0}{\partial s} \end{array} \right\} ds \qquad (25a)$$

As for the second part related to the x_2-spatial derivative of the variation δw_0,

$$\oint_\Gamma \frac{4}{3h^2} \left\{ P_2 \, n_{x_2} + P_6 \, n_{x_1} - \left(b_{T1} \frac{I_4}{\rho} + b_{T3} \frac{I_6}{\rho} \right) \right\} \frac{\partial \delta w_0}{\partial x_2} d\gamma =$$

$$= \frac{4}{3h^2}\oint_\Gamma \left\{ \begin{array}{c} n_{x_2}^2 \, P_{nn} - 2 n_{x_1} n_{x_2} P_{ns} + n_{x_1}^2 \, P_{ss} \quad n_{x_2} + \\ + \; n_{x_1} n_{x_2} \quad P_{ss} - P_{nn} \; + \; n_{x_1}^2 - n_{x_2}^2 \quad P_{ns} \quad n_{x_1} - \\ - \; n_{x_1} \cdot a_{T1s} - n_{x_2} \cdot a_{T1n} \; \frac{I_4}{\rho} - \; n_{x_1} \cdot a_{T3s} - n_{x_2} \cdot a_{T3n} \; \frac{I_6}{\rho} \end{array} \right\} \left\{ n_{x_1} \frac{\partial \delta w_0}{\partial s} - n_{x_2} \frac{\partial \delta w_0}{\partial n} \right\} ds =$$

$$= \frac{4}{3h^2}\oint_{\Gamma}\left\{\begin{array}{l} n_{x_2}^3 P_{nn} - 2n_{x_1} n_{x_2}^2 P_{ns} + n_{x_1}^2 n_{x_2} P_{ss} + \\ + n_{x_1}^2 n_{x_2} P_{ss} - n_{x_1}^2 n_{x_2} P_{nn} + (n_{x_1}^3 - n_{x_2}^2 n_{x_1}) P_{ns} - \\ -(n_{x_1}\cdot a_{T1s} - n_{x_2}\cdot a_{T1n})\frac{I_4}{\rho} - (n_{x_1}\cdot a_{T3s} - n_{x_2}\cdot a_{T3n})\frac{I_6}{\rho} \end{array}\right\}\left\{n_{x_1}\frac{\partial \delta w_0}{\partial s} - n_{x_2}\frac{\partial \delta w_0}{\partial n}\right\} ds =$$

$$= \frac{4}{3h^2}\oint_{\Gamma}\left\{\begin{array}{l} n_{x_2}^3 n_{x_1} P_{nn}\frac{\partial \delta w_0}{\partial s} - 2n_{x_1}^2 n_{x_2}^2 P_{ns}\frac{\partial \delta w_0}{\partial s} + n_{x_1}^3 n_{x_2} P_{ss}\frac{\partial \delta w_0}{\partial s} + \\ + n_{x_1}^3 n_{x_2} P_{ss}\frac{\partial \delta w_0}{\partial s} - n_{x_1}^3 n_{x_2} P_{nn}\frac{\partial \delta w_0}{\partial s} + (n_{x_1}^4 - n_{x_2}^2 n_{x_1}^2) P_{ns}\frac{\partial \delta w_0}{\partial s} - \\ -(n_{x_1}^2\cdot a_{T1s} - n_{x_2} n_{x_1}\cdot a_{T1n})\frac{I_4}{\rho}\frac{\partial \delta w_0}{\partial s} - (n_{x_1}^2\cdot a_{T3s} - n_{x_1} n_{x_2}\cdot a_{T3n})\frac{I_6}{\rho}\frac{\partial \delta w_0}{\partial s} - \\ -n_{x_2}^3 n_{x_2} P_{nn}\frac{\partial \delta w_0}{\partial n} + 2n_{x_1} n_{x_2}^3 P_{ns}\frac{\partial \delta w_0}{\partial n} - n_{x_1}^2 n_{x_2}^2 P_{ss}\frac{\partial \delta w_0}{\partial n} - \\ - n_{x_1}^2 n_{x_2}^2 P_{ss}\frac{\partial \delta w_0}{\partial n} + n_{x_1}^2 n_{x_2}^2 P_{nn}\frac{\partial \delta w_0}{\partial n} - (n_{x_1}^3 n_{x_2} - n_{x_2}^3 n_{x_1}) P_{ns}\frac{\partial \delta w_0}{\partial n} + \\ +(n_{x_1} n_{x_2}\cdot a_{T1s} - n_{x_2}^2\cdot a_{T1n})\frac{I_4}{\rho}\frac{\partial \delta w_0}{\partial n} + (n_{x_1} n_{x_2}\cdot a_{T3s} - n_{x_2}^2\cdot a_{T3n})\frac{I_6}{\rho}\frac{\partial \delta w_0}{\partial n} \end{array}\right\} ds =$$

[And now again by the same way followed on the first part, we gather together the terms with the same the thickness-integrated quantities of the same curvilinear direction and of the same derivative of w_0 on the curvilinear coordinate system, after highlighting each one of them with a different color]

$$= \frac{4}{3h^2}\oint_{\Gamma}\left\{\begin{array}{l} n_{x_2}^3 n_{x_1} P_{nn}\frac{\partial \delta w_0}{\partial s} - 2n_{x_1}^2 n_{x_2}^2 P_{ns}\frac{\partial \delta w_0}{\partial s} + n_{x_1}^3 n_{x_2} P_{ss}\frac{\partial \delta w_0}{\partial s} + \\ + n_{x_1}^3 n_{x_2} P_{ss}\frac{\partial \delta w_0}{\partial s} - n_{x_1}^3 n_{x_2} P_{nn}\frac{\partial \delta w_0}{\partial s} + (n_{x_1}^4 - n_{x_2}^2 n_{x_1}^2) P_{ns}\frac{\partial \delta w_0}{\partial s} - \\ -(n_{x_1}^2\cdot a_{T1s} - n_{x_2} n_{x_1}\cdot a_{T1n})\frac{I_4}{\rho}\frac{\partial \delta w_0}{\partial s} - (n_{x_1}^2\cdot a_{T3s} - n_{x_1} n_{x_2}\cdot a_{T3n})\frac{I_6}{\rho}\frac{\partial \delta w_0}{\partial s} - \\ -n_{x_2}^3 n_{x_2} P_{nn}\frac{\partial \delta w_0}{\partial n} + 2n_{x_1} n_{x_2}^3 P_{ns}\frac{\partial \delta w_0}{\partial n} - n_{x_1}^2 n_{x_2}^2 P_{ss}\frac{\partial \delta w_0}{\partial n} - \\ - n_{x_1}^2 n_{x_2}^2 P_{ss}\frac{\partial \delta w_0}{\partial n} + n_{x_1}^2 n_{x_2}^2 P_{nn}\frac{\partial \delta w_0}{\partial n} - (n_{x_1}^3 n_{x_2} - n_{x_2}^3 n_{x_1}) P_{ns}\frac{\partial \delta w_0}{\partial n} + \\ +(n_{x_1} n_{x_2}\cdot a_{T1s} - n_{x_2}^2\cdot a_{T1n})\frac{I_4}{\rho}\frac{\partial \delta w_0}{\partial n} + (n_{x_1} n_{x_2}\cdot a_{T3s} - n_{x_2}^2\cdot a_{T3n})\frac{I_6}{\rho}\frac{\partial \delta w_0}{\partial n} \end{array}\right\} ds =$$

$$= \frac{4}{3h^2}\oint_{\Gamma}\left\{\begin{array}{l} n_{x_2}^3 n_{x_1} - n_{x_1}^3 n_{x_2} \quad P_{nn}\dfrac{\partial \delta w_0}{\partial s} \quad + 2 n_{x_1}^3 n_{x_2} P_{ss}\dfrac{\partial \delta w_0}{\partial s} + \\ + \; n_{x_1}^4 - 3 n_{x_1}^2 n_{x_2}^2 \quad P_{ns}\dfrac{\partial \delta w_0}{\partial s} - \\ -\left\{ n_{x_1}^2 \cdot a_{T1s} - n_{x_2} n_{x_1} \cdot a_{T1n} \dfrac{I_4}{\rho} + n_{x_1}^2 \cdot a_{T3s} - n_{x_1} n_{x_2} \cdot a_{T3n} \dfrac{I_6}{\rho}\right\}\dfrac{\partial \delta w_0}{\partial s} + \\ + (n_{x_1}^2 n_{x_2}^2 - n_{x_2}^4) P_{nn}\dfrac{\partial \delta w_0}{\partial n} - 2 n_{x_1}^2 n_{x_2}^2 P_{ss}\dfrac{\partial \delta w_0}{\partial n} - \\ - (n_{x_1}^3 n_{x_2} + n_{x_1} n_{x_2}^3) P_{ns}\dfrac{\partial \delta w_0}{\partial n} + \\ + \left\{(n_{x_1} n_{x_2} \cdot a_{T1s} - n_{x_2}^2 \cdot a_{T1n})\dfrac{I_4}{\rho} + (n_{x_1} n_{x_2} \cdot a_{T3s} - n_{x_2}^2 \cdot a_{T3n})\dfrac{I_6}{\rho}\right\}\dfrac{\partial \delta w_0}{\partial n} \end{array}\right\} ds \qquad (25b)$$

Adding the Eqs. (25a) and (25b), we have

$$\oint_{\Gamma}\frac{4}{3h^2}\left\{P_{11}\, n_{x_1} + P_{12}\, n_{x_2} - \left(a_{T1}\frac{I_4}{\rho} + a_{T3}\frac{I_6}{\rho}\right)\right\}\frac{\partial \delta w_0}{\partial x_1} d\gamma \; +$$

$$+ \oint_{\Gamma}\frac{4}{3h^2}\left\{P_2\, n_{x_2} + P_6\, n_{x_1} - \left(b_{T1}\frac{I_4}{\rho} + b_{T3}\frac{I_6}{\rho}\right)\right\}\frac{\partial \delta w_0}{\partial x_2} d\gamma \quad =$$

$$= \frac{4}{3h^2}\oint_{\Gamma}\left\{\begin{array}{l} n_{x_1}^4 - n_{x_1}^2 n_{x_2}^2 + n_{x_1}^2 n_{x_2}^2 - n_{x_2}^4 \quad P_{nn}\dfrac{\partial \delta w_0}{\partial n} + \; 2 n_{x_2}^2 n_{x_1}^2 - 2 n_{x_1}^2 n_{x_2}^2 \quad P_{ss}\dfrac{\partial \delta w_0}{\partial n} + \\ \qquad + \; 2 n_{x_1}^3 n_{x_2} - 2 n_{x_1} n_{x_2}^3 \quad P_{ns}\dfrac{\partial \delta w_0}{\partial n} - \\ \qquad -\left\{\begin{array}{l} n_{x_1}^2 \cdot a_{T1n} + n_{x_2} n_{x_1} \cdot a_{T1s} - n_{x_2} n_{x_1} \cdot a_{T1s} - n_{x_2}^2 \cdot a_{T1n} \dfrac{I_4}{\rho} + \\ + \; n_{x_1}^2 \cdot a_{T3n} + n_{x_2} n_{x_1} \cdot a_{T3s} - n_{x_1} n_{x_2} \cdot a_{T3s} + n_{x_2}^2 \cdot a_{T3n} \dfrac{I_6}{\rho} \end{array}\right\}\dfrac{\partial \delta w_0}{\partial n} + \\ + (n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 + n_{x_2}^3 n_{x_1} - n_{x_1}^3 n_{x_2}) P_{nn}\dfrac{\partial \delta w_0}{\partial s} + (2 n_{x_2}^3 n_{x_1} + 2 n_{x_1}^3 n_{x_2}) P_{ss}\dfrac{\partial \delta w_0}{\partial s} + \\ \qquad + (3 n_{x_1}^2 n_{x_2}^2 - n_{x_2}^4 + n_{x_1}^4 - 3 n_{x_1}^2 n_{x_2}^2) P_{ns}\dfrac{\partial \delta w_0}{\partial s} - \\ \qquad -\left\{\begin{array}{l} (n_{x_1} n_{x_2} \cdot a_{T1n} + n_{x_2}^2 \cdot a_{T1s} + n_{x_1}^2 \cdot a_{T1s} - n_{x_2} n_{x_1} \cdot a_{T1n})\dfrac{I_4}{\rho} + \\ + (n_{x_1} n_{x_2} \cdot a_{T3n} + n_{x_2}^2 \cdot a_{T3s} + n_{x_1}^2 \cdot a_{T3s} - n_{x_1} n_{x_2} \cdot a_{T3n})\dfrac{I_6}{\rho} \end{array}\right\}\dfrac{\partial \delta w_0}{\partial s} \end{array}\right\} ds =$$

[And after simplification of the above sum, highlighting the terms including the direction co-sines which are eliminated inside the brackets and illustrated for the sake of convenience by orange color]

$$= \frac{4}{3h^2}\oint_{\Gamma}\left\{\begin{array}{l} n_{x_1}^4 - n_{x_2}^4 \;\; P_{nn}\dfrac{\partial \delta w_0}{\partial n} + \;\; 2n_{x_1}^3\, n_{x_2} - 2n_{x_1} n_{x_2}^3 \;\; P_{ns}\dfrac{\partial \delta w_0}{\partial n} - \\ -\left\{ n_{x_1}^2 \cdot a_{T1n} - n_{x_2}^2 \cdot a_{T1n} \;\; \dfrac{I_4}{\rho} + \;\; n_{x_1}^2 \cdot a_{T3n} + n_{x_2}^2 \cdot a_{T3n} \;\; \dfrac{I_6}{\rho}\right\}\dfrac{\partial \delta w_0}{\partial n} + \\ + \;\; 2n_{x_2}^3\, n_{x_1} + 2n_{x_1}^3\, n_{x_2} \;\; P_{ss}\dfrac{\partial \delta w_0}{\partial s} + (n_{x_1}^4 - n_{x_2}^4)\, P_{ns}\dfrac{\partial \delta w_0}{\partial s} - \\ -\left\{ n_{x_2}^2 \cdot a_{T1s} + n_{x_1}^2 \cdot a_{T1s} \;\; \dfrac{I_4}{\rho} + (n_{x_2}^2 \cdot a_{T3s} + n_{x_1}^2 \cdot a_{T3s})\dfrac{I_6}{\rho}\right\}\dfrac{\partial \delta w_0}{\partial s} \end{array}\right\} ds \qquad \textbf{(26)}$$

And finally expressing aggregately the previous transformation of the aforementioned boundary terms, we have

$$\oint_{\Gamma}\frac{4}{3h^2}\left\{P_{11}\, n_{x_1} + P_{12}\, n_{x_2} - \left(a_{T1}\frac{I_4}{\rho} + a_{T3}\frac{I_6}{\rho}\right)\right\}\frac{\partial \delta w_0}{\partial x_1}\, d\gamma \; +$$

$$+ \oint_{\Gamma}\frac{4}{3h^2}\left\{P_{2}\, n_{x_2} + P_{6}\, n_{x_1} - \left(b_{T1}\frac{I_4}{\rho} + b_{T3}\frac{I_6}{\rho}\right)\right\}\frac{\partial \delta w_0}{\partial x_2}\, d\gamma =$$

$$= \frac{4}{3h^2}\oint_{\Gamma}\left\{\begin{array}{c} n_{x_1}^2 - n_{x_2}^2 \;\; P_{nn}\dfrac{\partial \delta w_0}{\partial n} + 2n_{x_2} n_{x_1} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; P_{ns}\dfrac{\partial \delta w_0}{\partial n} + \\ + 2n_{x_2} n_{x_1}\, P_{ss}\dfrac{\partial \delta w_0}{\partial s} + (n_{x_1}^2 - n_{x_2}^2)\, P_{ns}\dfrac{\partial \delta w_0}{\partial s} - \\ -\left\{a_{T1n}(n_{x_1}^2 - n_{x_2}^2)\dfrac{I_4}{\rho} + a_{T3n}\dfrac{I_6}{\rho}\right\}\dfrac{\partial \delta w_0}{\partial n} - \left\{a_{T1s}\dfrac{I_4}{\rho} + a_{T3s}\dfrac{I_6}{\rho}\right\}\dfrac{\partial \delta w_0}{\partial s} \end{array}\right\} ds$$

The right-hand side of the last equation is simplified further due to the meter of normal unit $\hat{\boldsymbol{n}}$ where $n_{x_1}^2 + n_{x_2}^2 = 1$.

On the basis of the above result, remark that the functions $a_{T1n}(n,s)$, $a_{T3n}(n,s)$ are related both with the variation $\partial\delta w_0 / \partial n$ and are decoupled from the variation $\partial\delta w_0 / \partial s$. Exactly the analogous configuration is valid for the functions $a_{T1s}(n,s)$, $a_{T3s}(n,s)$ which are involved only on the variation $\partial\delta w_0 / \partial s$ and not at all on the $\partial\delta w_0 / \partial n$.

This configuration could be interpreted from the "nature" influence of the initially assumed surface traction $\hat{T}_1$ of the section 2, inserted in the model of TSDT.

However, we will not hustle to derive established conclusions about the above, due to the fact that the above boundary terms are going to be connected with further boundary terms deriving from those of the variation δw_0 (as will show on the following sections).

5.2.3. Transformation of the boundary conditions associated with δw_0 to the curvilinear boundary system

Now, taking apart the fourth row of the boundary terms of the variational equation repeated on the last paragraph of the section 5.2, we have

$$\oint_{\Gamma} \left\{ \begin{array}{l} \left[\underbrace{\left(I_6 \frac{16}{9h^4} - \frac{4}{3h^2} I_4\right)\ddot{\phi}_x + I_6 \frac{16}{9h^4}\frac{\partial \ddot{w}_0}{\partial x_1} + Q_5 - R_5 \frac{4}{h^2} + \frac{4}{3h^2}\frac{\partial P_6}{\partial x_2} + \frac{4}{3h^2}\frac{\partial P_1}{\partial x_1}}_{I_{\delta w_0, n_{x_1}}} \right] n_{x_1} + \\ + \left[\underbrace{\left(I_6 \frac{16}{9h^4} - \frac{4}{3h^2} I_4\right)\ddot{\phi}_y + I_6 \frac{16}{9h^4}\frac{\partial \ddot{w}_0}{\partial x_2} + Q_4 - R_4 \frac{4}{h^2} + \frac{4}{3h^2}\frac{\partial P_6}{\partial x_1} + \frac{4}{3h^2}\frac{\partial P_2}{\partial x_2}}_{I_{\delta w_0, n_{x_2}}} \right] n_{x_2} - \frac{I_0}{\rho} c_{T0} \end{array} \right\} \delta w_0 \, d\gamma$$

Subsequently, taking apart each term of the above expression multiplied with the direction cosines (in order to simplify the calculations) and using the Eqs. (4), (5), (6), (7), (16)-(18) and (19), (20), (19`), (20`), (21g) of the section 5.2, we get the following results. In addition, note that for the sake of convenience we neglect the time and curvilinear integrals.

However, due to the complication of the calculations, we present below the derivation of same terms separately and after that we substitute the results into the $I_{\delta w_0, n_{x_1}}$ and $I_{\delta w_0, n_{x_2}}$.

Thus, as for the transformation laws of the time-derivatives of the displacement field

$$\frac{\partial \ddot{w}_0}{\partial x_1} = n_{x_1}\frac{\partial \ddot{w}_0}{\partial n} + n_{x_2}\frac{\partial \ddot{w}_0}{\partial s} \quad \textbf{(27a)} \quad \text{and} \quad \frac{\partial \ddot{w}_0}{\partial x_2} = -n_{x_2}\frac{\partial \ddot{w}_0}{\partial n} + n_{x_1}\frac{\partial \ddot{w}_0}{\partial s} \quad \textbf{(27b)}$$

$$\ddot{\phi}_x = n_{x_1}\ddot{\phi}_n + n_{x_2}\ddot{\phi}_s \quad \textbf{(27d)} \quad \text{and} \quad \ddot{\phi}_y = -n_{x_1}\ddot{\phi}_n + n_{x_2}\ddot{\phi}_s \quad \textbf{(27d)}$$

As for the spatial-derivatives of the thickness-integrated and higher-order moments

$$\begin{aligned} \frac{\partial P_1}{\partial x_1} &= n_{x_1}\frac{\partial P_1}{\partial n} + n_{x_2}\frac{\partial P_1}{\partial s} = \\ &= n_{x_1}\frac{\partial}{\partial n}\left(n_{x_1}^2 P_{nn} + 2n_{x_1}n_{x_2}P_{ns} + n_{x_2}^2 P_{ss}\right) + n_{x_2}\frac{\partial}{\partial s}\left(n_{x_1}^2 P_{nn} + 2n_{x_1}n_{x_2}P_{ns} + n_{x_2}^2 P_{ss}\right) = \\ &= \frac{\partial}{\partial n}\left(n_{x_1}^3 P_{nn} + 2n_{x_1}^2 n_{x_2} P_{ns} + n_{x_1}n_{x_2}^2 P_{ss}\right) + \frac{\partial}{\partial s}\left(n_{x_2}n_{x_1}^2 P_{nn} + 2n_{x_1}n_{x_2}^2 P_{ns} + n_{x_2}^3 P_{ss}\right) = \\ &= \left\{n_{x_1}^3 \frac{\partial P_{nn}}{\partial n} + 2n_{x_1}^2 n_{x_2}\frac{\partial P_{ns}}{\partial n} + n_{x_1}n_{x_2}^2\frac{\partial P_{ss}}{\partial n}\right\} + \left\{n_{x_2}n_{x_1}^2\frac{\partial P_{nn}}{\partial s} + 2n_{x_1}n_{x_2}^2\frac{\partial P_{ns}}{\partial s} + n_{x_2}^3\frac{\partial P_{ss}}{\partial s}\right\} \end{aligned} \quad \textbf{(27e)}$$

$$
\begin{aligned}
\frac{\partial P_6}{\partial x_2} &= -n_{x_2}\frac{\partial P_6}{\partial n} + n_{x_1}\frac{\partial P_6}{\partial s} = \\
&= -n_{x_2}\frac{\partial}{\partial n}\left[n_{x_1}n_{x_2}\left(P_{ss}-P_{nn}\right) + \left(n_{x_1}^2-n_{x_2}^2\right)P_{ns}\right] + n_{x_1}\frac{\partial}{\partial s}\left[n_{x_1}n_{x_2}\left(P_{ss}-P_{nn}\right) + \left(n_{x_1}^2-n_{x_2}^2\right)P_{ns}\right] = \\
&= -\frac{\partial}{\partial n}\left[n_{x_1}n_{x_2}^2\left(P_{ss}-P_{nn}\right) + \left(n_{x_2}n_{x_1}^2-n_{x_2}^3\right)P_{ns}\right] + \frac{\partial}{\partial s}\left[n_{x_1}^2 n_{x_2}\left(P_{ss}-P_{nn}\right) + \left(n_{x_1}^3-n_{x_1}n_{x_2}^2\right)P_{ns}\right] = \\
&= -\left\{n_{x_1}n_{x_2}^2\left(\frac{\partial P_{ss}}{\partial n}-\frac{\partial P_{nn}}{\partial n}\right) + (n_{x_2}n_{x_1}^2-n_{x_2}^3)\frac{\partial P_{ns}}{\partial n}\right\} + \left\{n_{x_1}^2 n_{x_2}\left(\frac{\partial P_{ss}}{\partial s}-\frac{\partial P_{nn}}{\partial s}\right) + (n_{x_1}^3-n_{x_1}n_{x_2}^2)\frac{\partial P_{ns}}{\partial s}\right\}
\end{aligned}
\qquad \textbf{(27f)}
$$

$$
\begin{aligned}
\frac{\partial P_2}{\partial x_2} &= -n_{x_2}\frac{\partial P_2}{\partial n} + n_{x_1}\frac{\partial P_2}{\partial s} = \\
&= -n_{x_2}\frac{\partial}{\partial n}\left(n_{x_2}^2 P_{nn} - 2n_{x_1}n_{x_2}P_{ns} + n_{x_1}^2 P_{ss}\right) + n_{x_1}\frac{\partial}{\partial s}\left(n_{x_2}^2 P_{nn} - 2n_{x_1}n_{x_2}P_{ns} + n_{x_1}^2 P_{ss}\right) = \\
&= -\frac{\partial}{\partial n}\left(n_{x_2}^3 P_{nn} - 2n_{x_1}n_{x_2}^2 P_{ns} + n_{x_2}n_{x_1}^2 P_{ss}\right) + \frac{\partial}{\partial s}\left(n_{x_1}n_{x_2}^2 P_{nn} - 2n_{x_1}^2 n_{x_2}P_{ns} + n_{x_1}^3 P_{ss}\right) = \\
&= -\left\{n_{x_2}^3\frac{\partial P_{nn}}{\partial n} - 2n_{x_1}n_{x_2}^2\frac{\partial P_{ns}}{\partial n} + n_{x_2}n_{x_1}^2\frac{\partial P_{ss}}{\partial n}\right\} + \left\{n_{x_1}n_{x_2}^2\frac{\partial P_{nn}}{\partial s} - 2n_{x_1}^2 n_{x_2}\frac{\partial P_{ns}}{\partial s} + n_{x_1}^3\frac{\partial P_{ss}}{\partial s}\right\}
\end{aligned}
\qquad \textbf{(27g)}
$$

$$
\begin{aligned}
\frac{\partial P_6}{\partial x_1} &= n_{x_1}\frac{\partial P_6}{\partial n} + n_{x_2}\frac{\partial P_6}{\partial s} = \\
&= n_{x_1}\frac{\partial}{\partial n}\left[n_{x_1}n_{x_2}\left(P_{ss}-P_{nn}\right) + \left(n_{x_1}^2-n_{x_2}^2\right)P_{ns}\right] + n_{x_2}\frac{\partial}{\partial s}\left[n_{x_1}n_{x_2}\left(P_{ss}-P_{nn}\right) + \left(n_{x_1}^2-n_{x_2}^2\right)P_{ns}\right] = \\
&= \frac{\partial}{\partial n}\left[n_{x_1}^2 n_{x_2}\left(P_{ss}-P_{nn}\right) + \left(n_{x_1}^3-n_{x_1}n_{x_2}^2\right)P_{ns}\right] + \frac{\partial}{\partial s}\left[n_{x_1}n_{x_2}^2\left(P_{ss}-P_{nn}\right) + \left(n_{x_2}n_{x_1}^2-n_{x_2}^3\right)P_{ns}\right] = \\
&= \left\{n_{x_1}^2 n_{x_2}\left(\frac{\partial P_{ss}}{\partial n}-\frac{\partial P_{nn}}{\partial n}\right) + (n_{x_1}^3-n_{x_1}n_{x_2}^2)\frac{\partial P_{ns}}{\partial n}\right\} + \left\{n_{x_1}n_{x_2}^2\left(\frac{\partial P_{ss}}{\partial s}-\frac{\partial P_{nn}}{\partial s}\right) + (n_{x_2}n_{x_1}^2-n_{x_2}^3)\frac{\partial P_{ns}}{\partial s}\right\}
\end{aligned}
\qquad \textbf{(27h)}
$$

And now substituting the Eqs. (27a) - (27h) into the expressions $I_{\delta w_0, n_{x_1}}$ and $I_{\delta w_0, n_{x_2}}$,

$$
\begin{aligned}
I_{\delta w_0, n_{x_1}} &= \left(I_6 \frac{16}{9h^4} - \frac{4}{3h^2} I_4\right)\ddot{\phi}_x + I_6 \frac{16}{9h^4}\frac{\partial \ddot{w}_0}{\partial x_1} + Q_5 - R_5 \frac{4}{h^2} + \frac{4}{3h^2}\frac{\partial P_6}{\partial x_2} + \frac{4}{3h^2}\frac{\partial P_1}{\partial x_1} = \\
&= \left(I_6 \frac{16}{9h^4} - \frac{4}{3h^2} I_4\right) n_{x_1}\ddot{\phi}_n + n_{x_2}\ddot{\phi}_s + I_6 \frac{16}{9h^4}\left(n_{x_1}\frac{\partial \ddot{w}_0}{\partial n} + n_{x_2}\frac{\partial \ddot{w}_0}{\partial s}\right) - \\
&- n_{x_1} Q_{nn} + (n_{x_1}^2 - 1) Q_{nz} + n_{x_1} R_{nn}\frac{4}{h^2} - (n_{x_1}^2 - 1) R_{nz}\frac{4}{h^2} - \\
&- \frac{4}{3h^2}\left\{n_{x_1} n_{x_2}^2 \left(\frac{\partial P_{ss}}{\partial n} - \frac{\partial P_{nn}}{\partial n}\right) + (n_{x_2} n_{x_1}^2 - n_{x_2}^3)\frac{\partial P_{ns}}{\partial n}\right\} + \\
&+ \frac{4}{3h^2}\left\{n_{x_1}^2 n_{x_2}\left(\frac{\partial P_{ss}}{\partial s} - \frac{\partial P_{nn}}{\partial s}\right) + (n_{x_1}^3 - n_{x_1} n_{x_2}^2)\frac{\partial P_{ns}}{\partial s}\right\} + \\
&+ \frac{4}{3h^2}\left\{n_{x_1}^3 \frac{\partial P_{nn}}{\partial n} + 2 n_{x_1}^2 n_{x_2}\frac{\partial P_{ns}}{\partial n} + n_{x_1} n_{x_2}^2 \frac{\partial P_{ss}}{\partial n}\right\} + \\
&+ \frac{4}{3h^2}\left\{n_{x_2} n_{x_1}^2 \frac{\partial P_{nn}}{\partial s} + 2 n_{x_1} n_{x_2}^2 \frac{\partial P_{ns}}{\partial s} + n_{x_2}^3 \frac{\partial P_{ss}}{\partial s}\right\} =
\end{aligned}
$$

[Taking apart the Q and R moments from the P higher-order moments and after that performing calculations, gives the following result]

$$
\begin{aligned}
&= \left(I_6 \frac{16}{9h^4} - \frac{4}{3h^2} I_4\right) n_{x_1}\ddot{\phi}_n + n_{x_2}\ddot{\phi}_s + I_6 \frac{16}{9h^4}\left(n_{x_1}\frac{\partial \ddot{w}_0}{\partial n} + n_{x_2}\frac{\partial \ddot{w}_0}{\partial s}\right) - \\
&- n_{x_1} Q_{nn} + (n_{x_1}^2 - 1) Q_{nz} + n_{x_1} R_{nn}\frac{4}{h^2} - (n_{x_1}^2 - 1) R_{nz}\frac{4}{h^2} + \\
&+ \frac{4}{3h^2}\left\{\begin{array}{l}
n_{x_1} n_{x_2}^2 \dfrac{\partial P_{nn}}{\partial n} - n_{x_1} n_{x_2}^2 \dfrac{\partial P_{ss}}{\partial n} + (n_{x_2}^3 - n_{x_2} n_{x_1}^2)\dfrac{\partial P_{ns}}{\partial n} + \\
+ n_{x_1}^2 n_{x_2}\dfrac{\partial P_{ss}}{\partial s} - n_{x_1}^2 n_{x_2}\dfrac{\partial P_{nn}}{\partial s} + (n_{x_1}^3 - n_{x_1} n_{x_2}^2)\dfrac{\partial P_{ns}}{\partial s} + \\
+ n_{x_1}^3 \dfrac{\partial P_{nn}}{\partial n} + 2 n_{x_1}^2 n_{x_2}\dfrac{\partial P_{ns}}{\partial n} + n_{x_1} n_{x_2}^2 \dfrac{\partial P_{ss}}{\partial n} + \\
+ n_{x_2} n_{x_1}^2 \dfrac{\partial P_{nn}}{\partial s} + 2 n_{x_1} n_{x_2}^2 \dfrac{\partial P_{ns}}{\partial s} + n_{x_2}^3 \dfrac{\partial P_{ss}}{\partial s}
\end{array}\right\} =
\end{aligned}
$$

[Subsequently, we sum up separately the terms with the same thickness-integrated quantities or its derivatives and the derivatives of the displacements. To handle this grouping easier, we set different shades analogous to the “kind” of each term]

$$= \left(I_6 \frac{16}{9h^4} - \frac{4}{3h^2} I_4\right) n_{x_1} \ddot{\phi}_n + n_{x_2} \ddot{\phi}_s + I_6 \frac{16}{9h^4}\left(n_{x_1} \frac{\partial \ddot{w}_0}{\partial n} + n_{x_2} \frac{\partial \ddot{w}_0}{\partial s}\right) -$$

$$- n_{x_1} Q_{nn} + (n_{x_1}^2 - 1)\, Q_{nz} + n_{x_1} R_{nn} \frac{4}{h^2} - (n_{x_1}^2 - 1) R_{nz} \frac{4}{h^2} +$$

$$+ \frac{4}{3h^2} \left\{ \begin{array}{l} n_{x_1} n_{x_2}^2 \frac{\partial P_{nn}}{\partial n} - n_{x_1} n_{x_2}^2 \frac{\partial P_{ss}}{\partial n} + (n_{x_2}^3 - n_{x_2} n_{x_1}^2) \frac{\partial P_{ns}}{\partial n} + \\ + n_{x_1}^2 n_{x_2} \frac{\partial P_{ss}}{\partial s} - n_{x_1}^2 n_{x_2} \frac{\partial P_{nn}}{\partial s} + (n_{x_1}^3 - n_{x_1} n_{x_2}^2) \frac{\partial P_{ns}}{\partial s} + \\ + n_{x_1}^3 \frac{\partial P_{nn}}{\partial n} + 2 n_{x_1}^2 n_{x_2} \frac{\partial P_{ns}}{\partial n} + n_{x_1} n_{x_2}^2 \frac{\partial P_{ss}}{\partial n} + \\ + n_{x_2} n_{x_1}^2 \frac{\partial P_{nn}}{\partial s} + 2 n_{x_1} n_{x_2}^2 \frac{\partial P_{ns}}{\partial s} + n_{x_2}^3 \frac{\partial P_{ss}}{\partial s} \end{array} \right\}$$

Before extracting the final result, we elaborate individually the colorful terms, because the first two rows of the above expression do not need further work. Thus,

$$\frac{4}{3h^2} \left\{ \begin{array}{l} n_{x_1} n_{x_2}^2 \frac{\partial P_{nn}}{\partial n} + n_{x_1}^3 \frac{\partial P_{nn}}{\partial n} + n_{x_1} n_{x_2}^2 \frac{\partial P_{ss}}{\partial n} - n_{x_1} n_{x_2}^2 \frac{\partial P_{ss}}{\partial n} + \\ \qquad + (n_{x_2}^3 - n_{x_2} n_{x_1}^2) \frac{\partial P_{ns}}{\partial n} + 2 n_{x_1}^2 n_{x_2} \frac{\partial P_{ns}}{\partial n} + \\ + n_{x_1}^2 n_{x_2} \frac{\partial P_{ss}}{\partial s} + n_{x_2}^3 \frac{\partial P_{ss}}{\partial s} + (n_{x_1}^3 - n_{x_1} n_{x_2}^2) \frac{\partial P_{ns}}{\partial s} + 2 n_{x_1} n_{x_2}^2 \frac{\partial P_{ns}}{\partial s} \\ \qquad + n_{x_2} n_{x_1}^2 \frac{\partial P_{nn}}{\partial s} - n_{x_1}^2 n_{x_2} \frac{\partial P_{nn}}{\partial s} \end{array} \right\} =$$

$$= \frac{4}{3h^2} \left\{ \begin{array}{l} (n_{x_1} n_{x_2}^2 + n_{x_1}^3) \frac{\partial P_{nn}}{\partial n} + (n_{x_1} n_{x_2}^2 - n_{x_1} n_{x_2}^2) \frac{\partial P_{ss}}{\partial n} + \\ + (n_{x_2}^3 + n_{x_1}^2 n_{x_2}) \frac{\partial P_{ns}}{\partial n} + (n_{x_2} n_{x_1}^2 - n_{x_1}^2 n_{x_2}) \frac{\partial P_{nn}}{\partial s} + \\ + (n_{x_1}^2 n_{x_2} + n_{x_2}^3) \frac{\partial P_{ss}}{\partial s} + (n_{x_1}^3 + n_{x_1} n_{x_2}^2) \frac{\partial P_{ns}}{\partial s} \end{array} \right\} =$$

[Eliminating the appropriate combinations of direction cosines inside the brackets of the previous expression, we get the following]

$$= \frac{4}{3h^2} \left\{ \begin{array}{l} n_{x_1} n_{x_2}^2 + n_{x_1}^3 \quad \frac{\partial P_{nn}}{\partial n} + \quad n_{x_2}^3 + n_{x_1}^2 n_{x_2} \quad \frac{\partial P_{ns}}{\partial n} \\ + \quad n_{x_1}^2 n_{x_2} + n_{x_2}^3 \quad \frac{\partial P_{ss}}{\partial s} + \quad n_{x_1}^3 + n_{x_1} n_{x_2}^2 \quad \frac{\partial P_{ns}}{\partial s} \end{array} \right\} =$$

$$= \frac{4}{3h^2} \left\{ \begin{array}{l} n_{x_1} \quad n_{x_2}^2 + n_{x_1}^2 \quad \frac{\partial P_{nn}}{\partial n} + n_{x_2} \quad n_{x_2}^2 + n_{x_1}^2 \quad \frac{\partial P_{ns}}{\partial n} \\ + n_{x_2} \quad n_{x_1}^2 + n_{x_2}^2 \quad \frac{\partial P_{ss}}{\partial s} + n_{x_1} \quad n_{x_1}^2 + n_{x_2}^2 \quad \frac{\partial P_{ns}}{\partial s} \end{array} \right\} =$$

[And due to the unit normal $\hat{\boldsymbol{n}}$ on the lateral surface of the plate, $n_{x_1}^2 + n_{x_2}^2 = 1$]

$$= \frac{4}{3h^2}\left\{ n_{x_1}\frac{\partial P_{nn}}{\partial n} + n_{x_2}\frac{\partial P_{ns}}{\partial n} + n_{x_2}\frac{\partial P_{ss}}{\partial s} + n_{x_1}\frac{\partial P_{ns}}{\partial s}\right\}$$

Consequently, we get the result

$$I_{\delta w_0, n_{x_1}} = \left(I_6\frac{16}{9h^4} - \frac{4}{3h^2}I_4\right) n_{x_1}\ddot{\phi}_n + n_{x_2}\ddot{\phi}_s + I_6\frac{16}{9h^4}\left(n_{x_1}\frac{\partial \ddot{w}_0}{\partial n} + n_{x_2}\frac{\partial \ddot{w}_0}{\partial s}\right) -$$
$$- n_{x_1}Q_{nn} + n_{x_1}^2 - 1\; Q_{nz} + n_{x_1}R_{nn}\frac{4}{h^2} - n_{x_1}^2 - 1\; R_{nz}\frac{4}{h^2} +$$
$$+ \frac{4}{3h^2}\left\{ n_{x_1}\frac{\partial P_{nn}}{\partial n} + n_{x_2}\frac{\partial P_{ns}}{\partial n} + n_{x_2}\frac{\partial P_{ss}}{\partial s} + n_{x_1}\frac{\partial P_{ns}}{\partial s}\right\}$$

and

$$I_{\delta w_0, n_{x_2}} = \left(I_6\frac{16}{9h^4} - \frac{4}{3h^2}I_4\right) n_{x_2}\ddot{\phi}_s - n_{x_1}\ddot{\phi}_n + I_6\frac{16}{9h^4}\left(n_{x_1}\frac{\partial \ddot{w}_0}{\partial s} - n_{x_2}\frac{\partial \ddot{w}_0}{\partial n}\right) +$$
$$+ n_{x_1}^2 - 1\; Q_{sz} - n_{x_2}Q_{ss} + n_{x_2}R_{ss}\frac{4}{h^2} - n_{x_1}^2 - 1\; R_{sz}\frac{4}{h^2} +$$
$$+ \frac{4}{3h^2}\left\{ n_{x_1}^2 n_{x_2}\left(\frac{\partial P_{ss}}{\partial n} - \frac{\partial P_{nn}}{\partial n}\right) + n_{x_1}^3 - n_{x_1}n_{x_2}^2\; \frac{\partial P_{ns}}{\partial n}\right\} +$$
$$+ \frac{4}{3h^2}\left\{ n_{x_1}n_{x_2}^2\left(\frac{\partial P_{ss}}{\partial s} - \frac{\partial P_{nn}}{\partial s}\right) + n_{x_2}n_{x_1}^2 - n_{x_2}^3\; \frac{\partial P_{ns}}{\partial s}\right\} +$$
$$+ \frac{4}{3h^2}\left\{ -n_{x_2}^3\frac{\partial P_{nn}}{\partial n} + 2n_{x_1}n_{x_2}^2\frac{\partial P_{ns}}{\partial n} - n_{x_2}n_{x_1}^2\frac{\partial P_{ss}}{\partial n}\right\} +$$
$$+ \frac{4}{3h^2}\left\{ n_{x_1}n_{x_2}^2\frac{\partial P_{nn}}{\partial s} - 2n_{x_1}^2 n_{x_2}\frac{\partial P_{ns}}{\partial s} + n_{x_1}^3\frac{\partial P_{ss}}{\partial s}\right\} =$$

[Taking apart the Q and R moments from the P higher-order moments and after that performing calculations, gives the following result]

$$
=\left(I_6\frac{16}{9h^4}-\frac{4}{3h^2}I_4\right)\left(n_{x_2}\ddot{\phi}_s-n_{x_1}\ddot{\phi}_n\right)+I_6\frac{16}{9h^4}\left(n_{x_1}\frac{\partial\ddot{w}_0}{\partial s}-n_{x_2}\frac{\partial\ddot{w}_0}{\partial n}\right)+
$$

$$
+\left[\left(n_{x_1}^2-1\right)Q_{sz}-n_{x_2}Q_{ss}+n_{x_2}R_{ss}\frac{4}{h^2}-\left(n_{x_1}^2-1\right)R_{sz}\frac{4}{h^2}\right]+
$$

$$
+\frac{4}{3h^2}\left\{\begin{array}{l}
n_{x_1}^2n_{x_2}\left(\frac{\partial P_{ss}}{\partial n}-\frac{\partial P_{nn}}{\partial n}\right)+\left(n_{x_1}^3-n_{x_1}n_{x_2}^2\right)\frac{\partial P_{ns}}{\partial n}+\\
+n_{x_1}n_{x_2}^2\left(\frac{\partial P_{ss}}{\partial s}-\frac{\partial P_{nn}}{\partial s}\right)+\left(n_{x_2}n_{x_1}^2-n_{x_2}^3\right)\frac{\partial P_{ns}}{\partial s}-\\
-n_{x_2}^3\frac{\partial P_{nn}}{\partial n}+2n_{x_1}n_{x_2}^2\frac{\partial P_{ns}}{\partial n}-n_{x_2}n_{x_1}^2\frac{\partial P_{ss}}{\partial n}+\\
+n_{x_1}n_{x_2}^2\frac{\partial P_{nn}}{\partial s}-2n_{x_1}^2n_{x_2}\frac{\partial P_{ns}}{\partial s}+n_{x_1}^3\frac{\partial P_{ss}}{\partial s}
\end{array}\right\}=
$$

[Subsequently, we sum up separately the terms with the same thickness-integrated quantities or its derivatives and the derivatives of the displacements. To handle this grouping easier, we set different shades analogous to the “kind” of each term]

$$
=\left(I_6\frac{16}{9h^4}-\frac{4}{3h^2}I_4\right)\left(n_{x_2}\ddot{\phi}_s-n_{x_1}\ddot{\phi}_n\right)+I_6\frac{16}{9h^4}\left(n_{x_1}\frac{\partial\ddot{w}_0}{\partial s}-n_{x_2}\frac{\partial\ddot{w}_0}{\partial n}\right)+
$$

$$
+\left[\left(n_{x_1}^2-1\right)Q_{sz}-n_{x_2}Q_{ss}+n_{x_2}R_{ss}\frac{4}{h^2}-\left(n_{x_1}^2-1\right)R_{sz}\frac{4}{h^2}\right]+
$$

$$
+\frac{4}{3h^2}\left\{\begin{array}{l}
n_{x_1}^2n_{x_2}\frac{\partial P_{ss}}{\partial n}-n_{x_1}^2n_{x_2}\frac{\partial P_{nn}}{\partial n}+\left(n_{x_1}^3-n_{x_1}n_{x_2}^2\right)\frac{\partial P_{ns}}{\partial n}+\\
+n_{x_1}n_{x_2}^2\frac{\partial P_{ss}}{\partial s}-n_{x_1}n_{x_2}^2\frac{\partial P_{nn}}{\partial s}+\left(n_{x_2}n_{x_1}^2-n_{x_2}^3\right)\frac{\partial P_{ns}}{\partial s}-\\
-n_{x_2}^3\frac{\partial P_{nn}}{\partial n}+2n_{x_1}n_{x_2}^2\frac{\partial P_{ns}}{\partial n}-n_{x_2}n_{x_1}^2\frac{\partial P_{ss}}{\partial n}+\\
+n_{x_1}n_{x_2}^2\frac{\partial P_{nn}}{\partial s}-2n_{x_1}^2n_{x_2}\frac{\partial P_{ns}}{\partial s}+n_{x_1}^3\frac{\partial P_{ss}}{\partial s}
\end{array}\right\}
$$

Before extracting the final result, we elaborate individually the colorful terms, because the first two rows of the above expression do not need further work. Thus,

$$
\frac{4}{3h^2}\left\{\begin{array}{l}
n_{x_1}^2n_{x_2}\frac{\partial P_{ss}}{\partial n}-n_{x_2}n_{x_1}^2\frac{\partial P_{ss}}{\partial n}+\left(n_{x_1}^3-n_{x_1}n_{x_2}^2\right)\frac{\partial P_{ns}}{\partial n}+2n_{x_1}n_{x_2}^2\frac{\partial P_{ns}}{\partial n}+\\
+n_{x_1}n_{x_2}^2\frac{\partial P_{ss}}{\partial s}+n_{x_1}^3\frac{\partial P_{ss}}{\partial s}+n_{x_1}n_{x_2}^2\frac{\partial P_{nn}}{\partial s}-n_{x_1}n_{x_2}^2\frac{\partial P_{nn}}{\partial s}+\\
+\left(n_{x_2}n_{x_1}^2-n_{x_2}^3\right)\frac{\partial P_{ns}}{\partial s}-2n_{x_1}^2n_{x_2}\frac{\partial P_{ns}}{\partial s}-n_{x_2}^3\frac{\partial P_{nn}}{\partial n}-n_{x_1}^2n_{x_2}\frac{\partial P_{nn}}{\partial n}
\end{array}\right\}=
$$

$$
=\frac{4}{3h^2}\left\{\begin{array}{l}
n_{x_1}^2 n_{x_2}-n_{x_2} n_{x_1}^2 \dfrac{\partial P_{ss}}{\partial n}+\; n_{x_1}^3+n_{x_1} n_{x_2}^2 \dfrac{\partial P_{ns}}{\partial n}+\\
+\; n_{x_1} n_{x_2}^2+n_{x_1}^3 \dfrac{\partial P_{ss}}{\partial s}+\; n_{x_1} n_{x_2}^2-n_{x_1} n_{x_2}^2 \dfrac{\partial P_{nn}}{\partial s}-\\
-\; n_{x_2}^3+n_{x_1}^2 n_{x_2} \dfrac{\partial P_{ns}}{\partial s}-\; n_{x_2}^3+n_{x_1}^2 n_{x_2} \dfrac{\partial P_{nn}}{\partial n}
\end{array}\right\}=
$$

[Eliminating the appropriate combinations of direction cosines inside the brackets of the previous expression, we get the following]

$$
=\frac{4}{3h^2}\left\{n_{x_1}\; n_{x_1}^2+n_{x_2}^2 \frac{\partial P_{ns}}{\partial n}+n_{x_1}\; n_{x_2}^2+n_{x_1}^2 \frac{\partial P_{ss}}{\partial s}-n_{x_2}\; n_{x_2}^2+n_{x_1}^2 \frac{\partial P_{ns}}{\partial s}-n_{x_2}\; n_{x_2}^2+n_{x_1}^2 \frac{\partial P_{nn}}{\partial n}\right\}=
$$

$$
=\frac{4}{3h^2}\left\{n_{x_1} \frac{\partial P_{ns}}{\partial n}+n_{x_1} \frac{\partial P_{ss}}{\partial s}-n_{x_2} \frac{\partial P_{ns}}{\partial s}-n_{x_2} \frac{\partial P_{nn}}{\partial n}\right\}
$$

Subsequently, we get the result

$$
\boxed{\begin{array}{l}
I_{\delta w_0, n_{x_2}}=\left(I_6 \dfrac{16}{9h^4}-\dfrac{4}{3h^2} I_4\right) n_{x_2} \ddot{\phi}_s-n_{x_1} \ddot{\phi}_n+I_6 \dfrac{16}{9h^4}\left(n_{x_1} \dfrac{\partial \ddot{w}_0}{\partial s}-n_{x_2} \dfrac{\partial \ddot{w}_0}{\partial n}\right)+\\
\qquad+\; n_{x_1}^2-1\; Q_{sz}-n_{x_2} Q_{ss}+n_{x_2} R_{ss} \dfrac{4}{h^2}-\; n_{x_1}^2-1\; R_{sz} \dfrac{4}{h^2}+\\
\qquad+\dfrac{4}{3h^2}\left\{n_{x_1} \dfrac{\partial P_{ns}}{\partial n}+n_{x_1} \dfrac{\partial P_{ss}}{\partial s}-n_{x_2} \dfrac{\partial P_{ns}}{\partial s}-n_{x_2} \dfrac{\partial P_{nn}}{\partial n}\right\}
\end{array}}
$$

Finally the terms inside the brackets of the initial expression of the present section, take the following form

$$
I_{\delta w_0, n_{x_1}} \cdot n_{x_1}+I_{\delta w_0, n_{x_2}} \cdot n_{x_2}-c_{T0}(s) \frac{I_0}{\rho}=
$$

$$
\begin{array}{l}
=\left(I_6 \dfrac{16}{9h^4}-\dfrac{4}{3h^2} I_4\right) n_{x_1}^2 \ddot{\phi}_n+\; n_{x_2} n_{x_1} \ddot{\phi}_s+I_6 \dfrac{16}{9h^4}\left(n_{x_1}^2 \dfrac{\partial \ddot{w}_0}{\partial n}+n_{x_2} n_{x_1} \dfrac{\partial \ddot{w}_0}{\partial s}\right)-\\
-\; n_{x_1}^2 Q_{nn}+n_{x_1}\; n_{x_1}^2-1\; Q_{nz}+n_{x_1}^2 R_{nn} \dfrac{4}{h^2}-n_{x_1}\; n_{x_1}^2-1\; R_{nz} \dfrac{4}{h^2}-c_{T0} \dfrac{I_0}{\rho}+\\
\qquad+\dfrac{4}{3h^2}\left\{n_{x_1}^2 \dfrac{\partial P_{nn}}{\partial n}+n_{x_1} n_{x_2} \dfrac{\partial P_{ns}}{\partial n}+n_{x_1} n_{x_2} \dfrac{\partial P_{ss}}{\partial s}+n_{x_1}^2 \dfrac{\partial P_{ns}}{\partial s}\right\}+\\
+\left(I_6 \dfrac{16}{9h^4}-\dfrac{4}{3h^2} I_4\right)(n_{x_2}^2 \ddot{\phi}_s-n_{x_2} n_{x_1} \ddot{\phi}_n)+I_6 \dfrac{16}{9h^4}\left(n_{x_2} n_{x_1} \dfrac{\partial \ddot{w}_0}{\partial s}-n_{x_2}^2 \dfrac{\partial \ddot{w}_0}{\partial n}\right)+\\
\qquad+\; n_{x_2}(n_{x_1}^2-1)\, Q_{sz}-n_{x_2}^2 Q_{ss}+n_{x_2}^2 R_{ss} \dfrac{4}{h^2}-n_{x_2}(n_{x_1}^2-1) R_{sz} \dfrac{4}{h^2}+\\
\qquad+\dfrac{4}{3h^2}\left\{n_{x_2} n_{x_1} \dfrac{\partial P_{ns}}{\partial n}+n_{x_2} n_{x_1} \dfrac{\partial P_{ss}}{\partial s}-n_{x_2}^2 \dfrac{\partial P_{ns}}{\partial s}-n_{x_2}^2 \dfrac{\partial P_{nn}}{\partial n}\right\} \quad \Rightarrow
\end{array}
$$

And taking apart the terms with common coefficients, displacements or higher-order thickness-integrated quantities, we perform calculations and we get

$$
\begin{aligned}
& I_{\delta w_0, n_{x_1}} \cdot n_{x_1} + I_{\delta w_0, n_{x_2}} \cdot n_{x_2} - c_{T0}(s)\frac{I_0}{\rho} = \\
&= \left(I_6 \frac{16}{9h^4} - \frac{4}{3h^2} I_4 \right) \; n_{x_1} \; n_{x_1} - n_{x_2} \; \ddot{\phi}_n + n_{x_2} \; n_{x_1} + n_{x_2} \; \ddot{\phi}_s \; + \\
& + I_6 \frac{16}{9h^4} \left\{ \; n_{x_1}^2 - n_{x_2}^2 \; \frac{\partial \ddot{w}_0}{\partial n} + 2 \, n_{x_2} \, n_{x_1} \frac{\partial \ddot{w}_0}{\partial s} \right\} - c_{T0} \frac{I_0}{\rho} + \\
& + \frac{4}{3h^2} \left\{ \; n_{x_1}^2 - n_{x_2}^2 \; \left(\frac{\partial P_{nn}}{\partial n} + \frac{\partial P_{ns}}{\partial s} \right) + 2 \, n_{x_2} \, n_{x_1} \left(\frac{\partial P_{ns}}{\partial n} + \frac{\partial P_{ss}}{\partial s} \right) \right\} + \\
& + n_{x_2} (n_{x_1}^2 - 1) Q_{sz} - n_{x_2}^2 \, Q_{ss} + n_{x_2}^2 \, R_{ss} \frac{4}{h^2} - n_{x_2} (n_{x_1}^2 - 1) \, R_{sz} \frac{4}{h^2} + \\
& + n_{x_1} (n_{x_1}^2 - 1) \, Q_{nz} - n_{x_1}^2 \, Q_{nn} + n_{x_1}^2 \, R_{nn} \frac{4}{h^2} - n_{x_1} (n_{x_1}^2 - 1) \, R_{nz} \frac{4}{h^2}
\end{aligned}
\qquad \textbf{(28)}
$$

which is the analogous context inside the brackets of the curvilinear integral (presented on the first page of this section) but now expressed on the curvilinear coordinate system.

5.2.4. Transformation of the boundary conditions associated with the slopes $\delta\phi_x$, $\delta\phi_y$ to the curvilinear system

In the context on the present section, we handle terms associated with the slopes of the deformed cross section of the plate from the vertical one before the deformation (as explained on the initial assumptions on the section 1 of the Part B).

Point out that the reason for choosing to examine the boundary terms related to the $\delta\phi_x$, $\delta\phi_y$ w_0, is that after the transformation of these two boundary terms from the Cartesian to the curvilinear coordinate system and due to the Eqs. (4), (5) or (4`), (5`) of the section 5.2, each displacement ϕ_x, ϕ_y produces both of the in-plane curvilinear displacements ϕ_n and ϕ_s.

To avoid confusing calculations and faults, we examine separately each one of the boundary terms of the Variational Equation (referred on the section 5.2), which is related to the $\delta\phi_x$ and $\delta\phi_y$ respectively.

As for the first boundary term, we substitute into it the Eqs. (4`), (16), (16`), (8), (18`), (21a), (21b), (21c), (21g), (21k), (21m) of the section 5.2

$$\oint_{\Gamma}\left\{\begin{array}{l}\left(M_{11}-\frac{4}{3h^2}P_{11}\right)n_{x_1}+\left(M_{12}-P_{12}\frac{4}{3h^2}\right)n_{x_2}\;-\\ -\left(\begin{array}{c}a_{T1}\frac{I_2}{\rho}+a_{T3}\frac{I_4}{\rho}-a_{T1}\frac{I_4}{\rho}\frac{4}{3h^2}-a_{T3}\frac{I_6}{\rho}\frac{4}{3h^2}+\\ +a_5\,a_{T0}\frac{I_0}{\rho}+b_5\,c_{T0}\frac{I_0}{\rho}\end{array}\right)\end{array}\right\}\delta\phi_x\,d\gamma=$$

$$=\oint_{\Gamma}\left\{\begin{array}{l}\left\{\begin{array}{l}n_{x_1}^2 M_{nn}+2n_{x_1}n_{x_2}M_{ns}+n_{x_2}^2 M_{ss}\;-\\ -\frac{4}{3h^2}\;n_{x_1}^2 P_{nn}+2n_{x_1}n_{x_2}P_{ns}+n_{x_2}^2 P_{ss}\end{array}\right\}n_{x_1}\;+\\ +\left\{\begin{array}{l}n_{x_1}n_{x_2}\;M_{ss}-M_{nn}+n_{x_1}^2-n_{x_2}^2\;M_{ns}\;-\\ -\frac{4}{3h^2}\;n_{x_1}n_{x_2}\;P_{ss}-P_{nn}+n_{x_1}^2-n_{x_2}^2\;P_{ns}\end{array}\right\}n_{x_2}\;-\\ -\;n_{x_1}\cdot a_{T1n}+n_{x_2}\cdot a_{T1s}\;\frac{I_2}{\rho}-(n_{x_1}\cdot a_{T3n}+n_{x_2}\cdot a_{T3s})\frac{I_4}{\rho}\;+\\ +(n_{x_1}\cdot a_{T1n}+n_{x_2}\cdot a_{T1s})\frac{I_4}{\rho}\frac{4}{3h^2}+(n_{x_1}\cdot a_{T3n}+n_{x_2}\cdot a_{T3s})\frac{I_6}{\rho}\frac{4}{3h^2}\;-\\ -(n_{x_1}\,a_{5n}+n_{x_2}\,a_{5s})\,(n_{x_1}\cdot a_{T0n}+n_{x_2}\cdot a_{T0s})\frac{I_0}{\rho}\;-\\ \qquad -(n_{x_2}\,b_{5s}-n_{x_1}\,b_{5n})\,c_{T0}\,\frac{I_0}{\rho}\end{array}\right\}\{n_{x_1}\;\delta\phi_n+n_{x_2}\;\delta\phi_s\}\,ds\;=$$

$$
= \oint_{\Gamma} \left\{ \begin{array}{l}
\left\{ \begin{array}{l} n_{x_1}^3 M_{nn} + 2 n_{x_1}^2 n_{x_2} M_{ns} + n_{x_2}^2 n_{x_1} M_{ss} - \\ - \dfrac{4}{3h^2} n_{x_1}^3 P_{nn} + 2 n_{x_1}^2 n_{x_2} P_{ns} + n_{x_2}^2 n_{x_1} P_{ss} \end{array} \right\} + \\
+ \left\{ \begin{array}{l} n_{x_1} n_{x_2}^2 \;\; M_{ss} - M_{nn} \;\; + \;\; n_{x_1}^2 n_{x_2} - n_{x_2}^3 \;\; M_{ns} - \\ - \dfrac{4}{3h^2} n_{x_1} n_{x_2}^2 \;\; P_{ss} - P_{nn} \;\; + \;\; n_{x_1}^2 n_{x_2} - n_{x_2}^3 \;\; P_{ns} \end{array} \right\} - \\
- \;\; n_{x_1} \cdot a_{T1n} + n_{x_2} \cdot a_{T1s} \;\; \dfrac{I_2}{\rho} - (n_{x_1} \cdot a_{T3n} + n_{x_2} \cdot a_{T3s}) \dfrac{I_4}{\rho} + \\
+ (n_{x_1} \cdot a_{T1n} + n_{x_2} \cdot a_{T1s}) \dfrac{I_4}{\rho} \dfrac{4}{3h^2} + (n_{x_1} \cdot a_{T3n} + n_{x_2} \cdot a_{T3s}) \dfrac{I_6}{\rho} \dfrac{4}{3h^2} - \\
- (n_{x_1} a_{5n} + n_{x_2} a_{5s}) (n_{x_1} \cdot a_{T0n} + n_{x_2} \cdot a_{T0s}) \dfrac{I_0}{\rho} - \\
\qquad - (n_{x_2} b_{5s} - n_{x_1} b_{5n}) c_{T0} \dfrac{I_0}{\rho}
\end{array} \right\} \{ n_{x_1} \delta\phi_n + n_{x_2} \delta\phi_s \} \, ds
$$

At this point, we choose to perform the calculations separately for the boundary terms multiplied with the variations $\delta\phi_n$ and $\delta\phi_s$ in order to overcome the confusing and misleading calculations. This step is on purpose thinking than the corresponding boundary terms resulting from the variation $\delta\phi_y$ are going to be combined with those of the previous expression. Namely, the coefficients of $\delta\phi_n$ (or $\delta\phi_s$) deriving form the variation $\delta\phi_x$ are added to those deriving from the $\delta\phi_y$. Thus,

$$
I_{\phi_n}^{\phi_x} =
$$

$$
= \oint_{\Gamma} \left\{ \begin{array}{l}
\quad n_{x_1}^4 M_{nn} + 2 n_{x_1}^3 n_{x_2} M_{ns} + n_{x_2}^2 n_{x_1}^2 M_{ss} - \dfrac{4}{3h^2} n_{x_1}^4 P_{nn} + 2 n_{x_1}^3 n_{x_2} P_{ns} + n_{x_2}^2 n_{x_1}^2 P_{ss} + \\
+ \;\; n_{x_1}^2 n_{x_2}^2 \;\; M_{ss} - M_{nn} \;\; + \;\; n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 \;\; M_{ns} - \dfrac{4}{3h^2} n_{x_1}^2 n_{x_2}^2 \;\; P_{ss} - P_{nn} \;\; + \;\; n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 \;\; P_{ns} - \\
\qquad - \;\; n_{x_1}^2 \cdot a_{T1n} + n_{x_2} n_{x_1} \cdot a_{T1s} \;\; \dfrac{I_2}{\rho} - \;\; n_{x_1}^2 \cdot a_{T3n} + n_{x_1} n_{x_2} \cdot a_{T3s} \;\; \dfrac{I_4}{\rho} + \\
\quad + (n_{x_1}^2 \cdot a_{T1n} + n_{x_1} n_{x_2} \cdot a_{T1s}) \dfrac{I_4}{\rho} \dfrac{4}{3h^2} + (n_{x_1}^2 \cdot a_{T3n} + n_{x_1} n_{x_2} \cdot a_{T3s}) \dfrac{I_6}{\rho} \dfrac{4}{3h^2} - \\
\quad - (n_{x_1}^2 a_{5n} + n_{x_1} n_{x_2} a_{5s}) (n_{x_1} \cdot a_{T0n} + n_{x_2} \cdot a_{T0s}) \dfrac{I_0}{\rho} - (n_{x_1} n_{x_2} b_{5s} - n_{x_1}^2 b_{5n}) c_{T0} \dfrac{I_0}{\rho}
\end{array} \right\} \delta\phi_n \, ds
$$

$$I_{\phi_s}^{\phi_x} =$$

$$= \oint_{\Gamma} \left\{ \begin{array}{l} n_{x_1}^3 n_{x_2} M_{nn} + 2 n_{x_1}^2 n_{x_2}^2 M_{ns} + n_{x_2}^3 n_{x_1} M_{ss} - \frac{4}{3h^2} n_{x_1}^3 n_{x_2} P_{nn} + 2 n_{x_1}^2 n_{x_2}^2 P_{ns} + n_{x_2}^3 n_{x_1} P_{ss} + \\ + n_{x_1} n_{x_2}^3 \; M_{ss} - M_{nn} \; + \; n_{x_1}^2 n_{x_2}^2 - n_{x_2}^4 \; M_{ns} - \frac{4}{3h^2} n_{x_1} n_{x_2}^3 \; P_{ss} - P_{nn} \; + \; n_{x_1}^2 n_{x_2}^2 - n_{x_2}^4 \; P_{ns} - \\ \qquad - \; n_{x_1} n_{x_2} \cdot a_{T1n} + n_{x_2}^2 \cdot a_{T1s} \; \frac{I_2}{\rho} - (n_{x_1} n_{x_2} \cdot a_{T3n} + n_{x_2}^2 \cdot a_{T3s}) \frac{I_4}{\rho} + \\ \qquad + (n_{x_1} n_{x_2} \cdot a_{T1n} + n_{x_2}^2 \cdot a_{T1s}) \frac{I_4}{\rho} \frac{4}{3h^2} + (n_{x_2} n_{x_1} \cdot a_{T3n} + n_{x_2}^2 \cdot a_{T3s}) \frac{I_6}{\rho} \frac{4}{3h^2} - \\ - (n_{x_1} n_{x_2} a_{5n} + n_{x_2}^2 a_{5s}) (n_{x_2} \cdot a_{T0n} + n_{x_2} \cdot a_{T0s}) \frac{I_0}{\rho} - (n_{x_2}^2 b_{5s} - n_{x_1} n_{x_2} b_{5n}) c_{T0} \frac{I_0}{\rho} \end{array} \right\} \delta\phi_s \, ds$$

As for the second boundary term, we substitute into it the Eqs. (5\`), (17), (17\`), (18), (18\`), (21e), (21f), (21g), (21l) of the section 5.2

$$\oint_{\Gamma} \left\{ \begin{array}{l} \left(M_{22} - \frac{4}{3h^2} P_{22} \right) n_{x_2} + \left(M_{12} - P_{12} \frac{4}{3h^2} \right) n_{x_1} - \\ - \left(\begin{array}{c} b_{T1} \frac{I_2}{\rho} + b_{T3} \frac{I_4}{\rho} - b_{T1} \frac{I_4}{\rho} \frac{4}{3h^2} - b_{T3} \frac{I_6}{\rho} \frac{4}{3h^2} + \\ + a_4 b_{T0} \frac{I_0}{\rho} + b_4 c_{T0} \frac{I_0}{\rho} \end{array} \right) \end{array} \right\} \delta\phi_y \, d\gamma =$$

$$= \oint_{\Gamma} \left\{ \begin{array}{l} \left\{ \begin{array}{l} n_{x_2}^3 M_{nn} - 2 n_{x_1} n_{x_2}^2 M_{ns} + n_{x_1}^2 n_{x_2} M_{ss} - \\ - \frac{4}{3h^2} n_{x_2}^3 P_{nn} - 2 n_{x_1} n_{x_2}^2 P_{ns} + n_{x_1}^2 n_{x_2} P_{ss} \end{array} \right\} + \\ \quad + \left\{ \begin{array}{l} n_{x_1}^2 n_{x_2} \; M_{ss} - M_{nn} \; + \; n_{x_1}^3 - n_{x_1} n_{x_2}^2 \; M_{ns} - \\ - \frac{4}{3h^2} n_{x_1}^2 n_{x_2} \; P_{ss} - P_{nn} \; + \; n_{x_1}^3 - n_{x_1} n_{x_2}^2 \; P_{ns} \end{array} \right\} + \\ + \; n_{x_2} \cdot a_{T1n} - n_{x_1} \cdot a_{T1s} \; \frac{I_2}{\rho} + (n_{x_2} \cdot a_{T3n} - n_{x_1} \cdot a_{T3s}) \frac{I_4}{\rho} + \\ + (n_{x_1} \cdot a_{T1s} - n_{x_2} \cdot a_{T1n}) \frac{I_4}{\rho} \frac{4}{3h^2} + (n_{x_1} \cdot a_{T3s} - n_{x_2} \cdot a_{T3n}) \frac{I_6}{\rho} \frac{4}{3h^2} + \\ + (n_{x_1} a_{4n} + n_{x_2} a_{4s}) (n_{x_2} \cdot a_{T0n} - n_{x_1} \cdot a_{T0s}) \frac{I_0}{\rho} + (n_{x_1} b_{4n} - n_{x_2} b_{4s}) c_{T0} \frac{I_0}{\rho} \end{array} \right\} \{ n_{x_2} \delta\phi_s - n_{x_1} \delta\phi_n \} \, ds$$

Similarly to the previous step, we divide the above result to two parts according to the variations $\delta\phi_n$ and $\delta\phi_s$

$$I_{\phi_n}^{\phi_y} =$$

$$= \oint_{\Gamma} \left\{ \begin{array}{l} - n_{x_2}^3 n_{x_1} M_{nn} + 2 n_{x_1}^2 n_{x_2}^2 M_{ns} - n_{x_1}^3 n_{x_2} M_{ss} + \dfrac{4}{3h^2} n_{x_2}^3 n_{x_1} P_{nn} - 2 n_{x_1}^2 n_{x_2}^2 P_{ns} + n_{x_1}^3 n_{x_2} P_{ss} - \\ - n_{x_1}^3 n_{x_2} \; M_{ss} - M_{nn} \; - \; n_{x_1}^4 - n_{x_1}^2 n_{x_2}^2 \; M_{ns} + \dfrac{4}{3h^2} n_{x_1}^3 n_{x_2} \; P_{ss} - P_{nn} \; - \; n_{x_1}^4 - n_{x_1}^2 n_{x_2}^2 \; P_{ns} - \\ \qquad - n_{x_1} n_{x_2} \cdot a_{T1n} - n_{x_1}^2 \cdot a_{T1s} \dfrac{I_2}{\rho} - (n_{x_1} n_{x_2} \cdot a_{T3n} - n_{x_1}^2 \cdot a_{T3s}) \dfrac{I_4}{\rho} - \\ \quad - (n_{x_1}^2 \cdot a_{T1s} - n_{x_1} n_{x_2} \cdot a_{T1n}) \dfrac{I_4}{\rho} \dfrac{4}{3h^2} - (n_{x_1}^2 \cdot a_{T3s} - n_{x_1} n_{x_2} \cdot a_{T3n}) \dfrac{I_6}{\rho} \dfrac{4}{3h^2} - \\ - (n_{x_1}^2 a_{4n} + n_{x_1} n_{x_2} a_{4s}) (n_{x_2} \cdot a_{T0n} - n_{x_1} \cdot a_{T0s}) \dfrac{I_0}{\rho} - (n_{x_1}^2 b_{4n} - n_{x_1} n_{x_2} b_{4s}) c_{T0} \dfrac{I_0}{\rho} \end{array} \right\} \delta\phi_n \, ds$$

and

$$I_{\phi_s}^{\phi_y} =$$

$$= \oint_{\Gamma} \left\{ \begin{array}{l} n_{x_2}^4 M_{nn} - 2 n_{x_1} n_{x_2}^3 M_{ns} + n_{x_1}^2 n_{x_2}^2 M_{ss} - \dfrac{4}{3h^2} n_{x_2}^4 P_{nn} - 2 n_{x_1} n_{x_2}^3 P_{ns} + n_{x_1}^2 n_{x_2}^2 P_{ss} + \\ + n_{x_1}^2 n_{x_2}^2 \; M_{ss} - M_{nn} \; + \; n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 \; M_{ns} - \dfrac{4}{3h^2} n_{x_1}^2 n_{x_2}^2 \; P_{ss} - P_{nn} \; + \; n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 \; P_{ns} + \\ \qquad + n_{x_2}^2 \cdot a_{T1n} - n_{x_1} n_{x_2} \cdot a_{T1s} \dfrac{I_2}{\rho} + (n_{x_2}^2 \cdot a_{T3n} - n_{x_1} n_{x_2} \cdot a_{T3s}) \dfrac{I_4}{\rho} + \\ \qquad + (n_{x_1} n_{x_2} \cdot a_{T1s} - n_{x_2}^2 \cdot a_{T1n}) \dfrac{I_4}{\rho} \dfrac{4}{3h^2} + (n_{x_1} n_{x_2} \cdot a_{T3s} - n_{x_2}^2 \cdot a_{T3n}) \dfrac{I_6}{\rho} \dfrac{4}{3h^2} + \\ \quad + (n_{x_1} n_{x_2} a_{4n} + n_{x_2}^2 a_{4s}) (n_{x_2} \cdot a_{T0n} - n_{x_1} \cdot a_{T0s}) \dfrac{I_0}{\rho} + (n_{x_2} n_{x_1} b_{4n} - n_{x_2}^2 b_{4s}) c_{T0} \dfrac{I_0}{\rho} \end{array} \right\} \delta\phi_s \, ds$$

Subsequently, the total expression of the boundary terms associated to the $\delta\phi_n$ on the Variational Equation of the section 5.2 is given by the sum

$$I_{\phi_n} = I_{\phi_n}^{\phi_x} + I_{\phi_n}^{\phi_y} \tag{1}$$

And similarly the total expression of the boundary terms associated to the $\delta\phi_s$ on the Variational Equation of the section 5.2 is given by the sum

$$I_{\phi_s} = I_{\phi_s}^{\phi_x} + I_{\phi_s}^{\phi_y} . \tag{2}$$

However, we perform calculations inside the brackets of the integrals of the above expressions and after that we extract the whole boundary terms on the curvilinear coordinate system which are related to the variations $\delta\phi_n$ and $\delta\phi_s$.

As for the Eqs. (1), namely the boundary terms related to the $\delta\phi_n$ variation

$$n_{x_1}^4 M_{nn} - n_{x_1}^2 n_{x_2}^2 M_{nn} - n_{x_2}^3 n_{x_1} M_{nn} + n_{x_1}^3 n_{x_2} M_{nn} +$$

$$+ 2n_{x_1}^3 n_{x_2} M_{ns} + n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 M_{ns} - n_{x_1}^4 - n_{x_1}^2 n_{x_2}^2 M_{ns} + 2n_{x_1}^2 n_{x_2}^2 M_{ns} +$$

$$+ n_{x_2}^2 n_{x_1}^2 M_{ss} + n_{x_1}^2 n_{x_2}^2 M_{ss} - n_{x_1}^3 n_{x_2} M_{ss} - n_{x_1}^3 n_{x_2} M_{ss} -$$

$$- \frac{4}{3h^2}(n_{x_1}^4 P_{nn} + 2n_{x_1}^3 n_{x_2} P_{ns} + n_{x_2}^2 n_{x_1}^2 P_{ss}) - \frac{4}{3h^2}(n_{x_1}^2 n_{x_2}^2 P_{ss} - n_{x_1}^2 n_{x_2}^2 P_{nn} + (n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3) P_{ns}) +$$

$$+ \frac{4}{3h^2}(n_{x_2}^3 n_{x_1} P_{nn} - 2n_{x_1}^2 n_{x_2}^2 P_{ns} + n_{x_1}^3 n_{x_2} P_{ss}) + \frac{4}{3h^2}(n_{x_1}^3 n_{x_2} P_{ss} - n_{x_1}^3 n_{x_2} P_{nn} - (n_{x_1}^4 - n_{x_1}^2 n_{x_2}^2) P_{ns}) -$$

$$- (n_{x_1}^2 \cdot a_{T1n} + n_{x_2} n_{x_1} \cdot a_{T1s}) \frac{I_2}{\rho} - (n_{x_1}^2 \cdot a_{T3n} + n_{x_1} n_{x_2} \cdot a_{T3s}) \frac{I_4}{\rho} +$$

$$+ (n_{x_1}^2 \cdot a_{T1n} + n_{x_1} n_{x_2} \cdot a_{T1s}) \frac{I_4}{\rho} \frac{4}{3h^2} + (n_{x_1}^2 \cdot a_{T3n} + n_{x_1} n_{x_2} \cdot a_{T3s}) \frac{I_6}{\rho} \frac{4}{3h^2} -$$

$$- (n_{x_1}^2 a_{5n} + n_{x_1} n_{x_2} a_{5s}) (n_{x_1} \cdot a_{T0n} + n_{x_2} \cdot a_{T0s}) \frac{I_0}{\rho} - (n_{x_1} n_{x_2} b_{5s} - n_{x_1}^2 b_{5n}) c_{T0} \frac{I_0}{\rho} -$$

$$- (n_{x_1} n_{x_2} \cdot a_{T1n} - n_{x_1}^2 \cdot a_{T1s}) \frac{I_2}{\rho} - (n_{x_1} n_{x_2} \cdot a_{T3n} - n_{x_1}^2 \cdot a_{T3s}) \frac{I_4}{\rho} -$$

$$- (n_{x_1}^2 \cdot a_{T1s} - n_{x_1} n_{x_2} \cdot a_{T1n}) \frac{I_4}{\rho} \frac{4}{3h^2} - (n_{x_1}^2 \cdot a_{T3s} - n_{x_1} n_{x_2} \cdot a_{T3n}) \frac{I_6}{\rho} \frac{4}{3h^2} -$$

$$- (n_{x_1}^2 a_{4n} + n_{x_1} n_{x_2} a_{4s}) (n_{x_2} \cdot a_{T0n} - n_{x_1} \cdot a_{T0s}) \frac{I_0}{\rho} - (n_{x_1}^2 b_{4n} - n_{x_1} n_{x_2} b_{4s}) c_{T0} \frac{I_0}{\rho} =$$

[And grouping together the terms with common unknown quantities and given functions]

$$= n_{x_1} n_{x_1} + n_{x_2} {}^2 n_{x_1} - n_{x_2} M_{nn} + 2n_{x_1}^2 n_{x_2} n_{x_2} - n_{x_1} M_{ss} +$$

$$+ 3n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 - n_{x_1}^4 + 3n_{x_1}^2 n_{x_2}^2 M_{ns} -$$

$$- \frac{4}{3h^2} n_{x_1}^4 - n_{x_1}^2 n_{x_2}^2 P_{nn} + 2n_{x_2}^2 n_{x_1}^2 P_{ss} + 3n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 P_{ns} +$$

$$+ \frac{4}{3h^2} n_{x_2}^3 n_{x_1} - n_{x_1}^3 n_{x_2} P_{nn} + 2n_{x_1}^3 n_{x_2} P_{ss} - n_{x_1}^4 - 3n_{x_1}^2 n_{x_2}^2 P_{ns} -$$

$$- (n_{x_1}^2 \cdot a_{T1n} + n_{x_2} n_{x_1} \cdot a_{T1s} + n_{x_1} n_{x_2} \cdot a_{T1n} - n_{x_1}^2 \cdot a_{T1s}) \frac{I_2}{\rho} -$$

$$- (n_{x_1}^2 \cdot a_{T3n} + n_{x_1} n_{x_2} \cdot a_{T3s} + n_{x_1} n_{x_2} \cdot a_{T3n} - n_{x_1}^2 \cdot a_{T3s}) \frac{I_4}{\rho} +$$

$$+ (n_{x_1}^2 \cdot a_{T1n} + n_{x_1} n_{x_2} \cdot a_{T1s} - n_{x_1}^2 \cdot a_{T1s} + n_{x_1} n_{x_2} \cdot a_{T1n}) \frac{I_4}{\rho} \frac{4}{3h^2} +$$

$$+ (n_{x_1}^2 \cdot a_{T3n} + n_{x_1} n_{x_2} \cdot a_{T3s} - n_{x_1}^2 \cdot a_{T3s} + n_{x_1} n_{x_2} \cdot a_{T3n}) \frac{I_6}{\rho} \frac{4}{3h^2} -$$

$$- (n_{x_1}^2 a_{5n} + n_{x_1} n_{x_2} a_{5s}) (n_{x_1} \cdot a_{T0n} + n_{x_2} \cdot a_{T0s}) \frac{I_0}{\rho} - (n_{x_1} n_{x_2} b_{5s} - n_{x_1}^2 b_{5n}) c_{T0} \frac{I_0}{\rho} -$$

$$- (n_{x_1}^2 a_{4n} + n_{x_1} n_{x_2} a_{4s}) (n_{x_2} \cdot a_{T0n} - n_{x_1} \cdot a_{T0s}) \frac{I_0}{\rho} - (n_{x_1}^2 b_{4n} - n_{x_1} n_{x_2} b_{4s}) c_{T0} \frac{I_0}{\rho} =$$

$$= n_{x_1} \;\; n_{x_1} + n_{x_2} \;\;{}^{2}\;\; n_{x_1} - n_{x_2} \;\; M_{nn} \; + \; 2 n_{x_1}^{2} \, n_{x_2} \;\; n_{x_2} - n_{x_1} \;\; M_{ss} \; +$$

$$+ \; 3 n_{x_1}^{3} \, n_{x_2} \; - n_{x_1} n_{x_2}^{3} - n_{x_1}^{4} + \; 3 n_{x_1}^{2} \, n_{x_2}^{2} \;\; M_{ns} \; +$$

$$+ \frac{4}{3h^{2}} \begin{pmatrix} -n_{x_1} \;\; n_{x_1} + n_{x_2} \;\;{}^{2}\;\; n_{x_1} - n_{x_2} \;\; P_{nn} + 2 n_{x_1}^{2} \, n_{x_2} \;\; n_{x_1} - n_{x_2} \;\; P_{ss} - \\ - \;\; n_{x_1}^{4} - 3 n_{x_1}^{2} \, n_{x_2}^{2} - 3 n_{x_1}^{3} \, n_{x_2} + n_{x_1} \, n_{x_2}^{3} \;\; P_{ns} \end{pmatrix} -$$

$$- \; (n_{x_1}^{2} \cdot a_{T1n} \; + \; n_{x_2} \, n_{x_1} \cdot a_{T1s} + n_{x_1} \, n_{x_2} \cdot a_{T1n} - n_{x_1}^{2} \cdot a_{T1s}) \, \frac{I_2}{\rho} \; -$$

$$- \; (n_{x_1}^{2} \cdot a_{T3n} \; + \; n_{x_1} \, n_{x_2} \cdot a_{T3s} + n_{x_1} \, n_{x_2} \cdot a_{T3n} - n_{x_1}^{2} \cdot a_{T3s}) \, \frac{I_4}{\rho} \; +$$

$$+ \; (n_{x_1}^{2} \cdot a_{T1n} \; + \; n_{x_1} \, n_{x_2} \cdot a_{T1s} - n_{x_1}^{2} \cdot a_{T1s} + n_{x_1} \, n_{x_2} \cdot a_{T1n}) \, \frac{I_4}{\rho} \, \frac{4}{3h^{2}} \; +$$

$$+ \; (n_{x_1}^{2} \cdot a_{T3n} \; + \; n_{x_1} \, n_{x_2} \cdot a_{T3s} - n_{x_1}^{2} \cdot a_{T3s} + n_{x_1} \, n_{x_2} \cdot a_{T3n}) \, \frac{I_6}{\rho} \, \frac{4}{3h^{2}} \; -$$

$$- \; (n_{x_1}^{2} \, a_{5n} + \; n_{x_1} \, n_{x_2} \, a_{5s}) \, (n_{x_1} \cdot a_{T0n} \; + n_{x_2} \cdot a_{T0s}) \frac{I_0}{\rho} \; - \; (n_{x_1} \, n_{x_2} \, b_{5s} - n_{x_1}^{2} \, b_{5n}) \, c_{T0} \, \frac{I_0}{\rho} \; -$$

$$- \; (n_{x_1}^{2} \, a_{4n} + n_{x_1} \, n_{x_2} \, a_{4s}) \, (n_{x_2} \cdot a_{T0n} - n_{x_1} \cdot a_{T0s}) \, \frac{I_0}{\rho} \; - \; (n_{x_1}^{2} \, b_{4n} - n_{x_1} \, n_{x_2} \, b_{4s}) \, c_{T0} \, \frac{I_0}{\rho}$$

As for the Eqs. (2), namely the boundary terms related to the $\delta \phi_s$ variation

$$n_{x_1}^3 n_{x_2} M_{nn} - n_{x_1} n_{x_2}^3 M_{nn} + n_{x_2}^4 M_{nn} - n_{x_1}^2 n_{x_2}^2 M_{nn} +$$

$$+ 2 n_{x_1}^2 n_{x_2}^2 M_{ns} + \left(n_{x_1}^2 n_{x_2}^2 - n_{x_2}^4 \right) M_{ns} + \left(n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 \right) M_{ns} - 2 n_{x_1} n_{x_2}^3 M_{ns} +$$

$$+ n_{x_2}^3 n_{x_1} M_{ss} + n_{x_1} n_{x_2}^3 M_{ss} + n_{x_1}^2 n_{x_2}^2 M_{ss} + n_{x_1}^2 n_{x_2}^2 M_{ss} -$$

$$- \frac{4}{3h^2} (n_{x_1}^3 n_{x_2} P_{nn} + 2 n_{x_1}^2 n_{x_2}^2 P_{ns} + n_{x_2}^3 n_{x_1} P_{ss} + n_{x_2}^4 P_{nn} - 2 n_{x_1} n_{x_2}^3 P_{ns} + n_{x_1}^2 n_{x_2}^2 P_{ss}) -$$

$$- \frac{4}{3h^2} (n_{x_1} n_{x_2}^3 P_{ss} - n_{x_1} n_{x_2}^3 P_{nn} + (n_{x_1}^2 n_{x_2}^2 - n_{x_2}^4 + n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3) P_{ns} + n_{x_1}^2 n_{x_2}^2 P_{ss} - n_{x_1}^2 n_{x_2}^2 P_{nn}) -$$

$$- (n_{x_1} n_{x_2} \cdot a_{T1n} + n_{x_2}^2 \cdot a_{T1s}) \frac{I_2}{\rho} - (n_{x_1} n_{x_2} \cdot a_{T3n} + n_{x_2}^2 \cdot a_{T3s}) \frac{I_4}{\rho} +$$

$$+ (n_{x_1} n_{x_2} \cdot a_{T1n} + n_{x_2}^2 \cdot a_{T1s}) \frac{I_4}{\rho} \frac{4}{3h^2} + (n_{x_2} n_{x_1} \cdot a_{T3n} + n_{x_2}^2 \cdot a_{T3s}) \frac{I_6}{\rho} \frac{4}{3h^2} -$$

$$- (n_{x_1} n_{x_2} a_{5n} + n_{x_2}^2 a_{5s}) (n_{x_2} \cdot a_{T0n} + n_{x_2} \cdot a_{T0s}) \frac{I_0}{\rho} - (n_{x_2}^2 b_{5s} - n_{x_1} n_{x_2} b_{5n}) c_{T0} \frac{I_0}{\rho} +$$

$$+ (n_{x_2}^2 \cdot a_{T1n} - n_{x_1} n_{x_2} \cdot a_{T1s}) \frac{I_2}{\rho} + (n_{x_2}^2 \cdot a_{T3n} - n_{x_1} n_{x_2} \cdot a_{T3s}) \frac{I_4}{\rho} +$$

$$+ (n_{x_1} n_{x_2} \cdot a_{T1s} - n_{x_2}^2 \cdot a_{T1n}) \frac{I_4}{\rho} \frac{4}{3h^2} + (n_{x_1} n_{x_2} \cdot a_{T3s} - n_{x_2}^2 \cdot a_{T3n}) \frac{I_6}{\rho} \frac{4}{3h^2} +$$

$$+ (n_{x_1} n_{x_2} a_{4n} + n_{x_2}^2 a_{4s}) (n_{x_2} \cdot a_{T0n} - n_{x_1} \cdot a_{T0s}) \frac{I_0}{\rho} + (n_{x_2} n_{x_1} b_{4n} - n_{x_2}^2 b_{4s}) c_{T0} \frac{I_0}{\rho} =$$

[And grouping together the terms with common unknown quantities and given functions]

$$= n_{x_2} \; n_{x_1}^2 - n_{x_2}^2 \quad n_{x_1} + n_{x_2} \quad M_{nn} + 2 n_{x_2}^2 \, n_{x_1} \quad n_{x_2} + n_{x_1} \quad M_{ss} +$$

$$+ \; 3 n_{x_1}^2 \, n_{x_2}^2 - n_{x_2}^4 + n_{x_1}^3 \, n_{x_2} - 3 n_{x_1} \, n_{x_2}^3 \quad M_{ns} -$$

$$- \frac{4}{3h^2} \left(\begin{array}{c} n_{x_1}^3 \, n_{x_2} + n_{x_2}^4 - n_{x_1} \, n_{x_2}^3 - n_{x_1}^2 \, n_{x_2}^2 \quad P_{nn} + 2 n_{x_2}^2 \, n_{x_1} \quad n_{x_2} + n_{x_1} \quad P_{ss} + \\ + \; -n_{x_2}^4 + n_{x_1}^3 \, n_{x_2} + 3 n_{x_1}^2 \, n_{x_2}^2 - 3 n_{x_1} \, n_{x_2}^3 \quad P_{ns} \end{array} \right) -$$

$$- \left(n_{x_1} \, n_{x_2} \cdot a_{T1n} + n_{x_2}^2 \cdot a_{T1s} - n_{x_2}^2 \cdot a_{T1n} + n_{x_1} \, n_{x_2} \cdot a_{T1s}\right) \frac{I_2}{\rho} -$$

$$- \left(n_{x_1} \, n_{x_2} \cdot a_{T3n} + n_{x_2}^2 \cdot a_{T3s} - n_{x_2}^2 \cdot a_{T3n} + n_{x_1} \, n_{x_2} \cdot a_{T3s}\right) \frac{I_4}{\rho} +$$

$$+ \left(n_{x_1} \, n_{x_2} \cdot a_{T1n} + n_{x_2}^2 \cdot a_{T1s} + n_{x_1} \, n_{x_2} \cdot a_{T1s} - n_{x_2}^2 \cdot a_{T1n}\right) \frac{I_4}{\rho} \frac{4}{3h^2} +$$

$$+ \left(n_{x_2} \, n_{x_1} \cdot a_{T3n} + n_{x_2}^2 \cdot a_{T3s} + n_{x_1} \, n_{x_2} \cdot a_{T3s} - n_{x_2}^2 \cdot a_{T3n}\right) \frac{I_6}{\rho} \frac{4}{3h^2} -$$

$$- \left(n_{x_1} \, n_{x_2} \, a_{5n} + n_{x_2}^2 \, a_{5s}\right) \left(n_{x_2} \cdot a_{T0n} + n_{x_2} \cdot a_{T0s}\right) \frac{I_0}{\rho} - \left(n_{x_2}^2 \, b_{5s} - n_{x_1} \, n_{x_2} \, b_{5n}\right) c_{T0} \frac{I_0}{\rho} +$$

$$+ \left(n_{x_1} \, n_{x_2} \, a_{4n} + n_{x_2}^2 \, a_{4s}\right) \left(n_{x_2} \cdot a_{T0n} - n_{x_1} \cdot a_{T0s}\right) \frac{I_0}{\rho} + \left(n_{x_2} \, n_{x_1} \, b_{4n} - n_{x_2}^2 \, b_{4s}\right) c_{T0} \frac{I_0}{\rho}$$

Note that the expressions derived above are not expected to “give” additional boundary terms to the other boundary parts of the Variations Equation (section 5.2). Consequently, these boundary terms either are expressed to the Cartesian or to the curvilinear coordinate system are just two alternative forms of the same physical meaning and they express two independent natural boundary conditions given below.

Exactly the same configuration took place for the in-plane boundary conditions of the problem of shear deformation (section 5.2.1).

Furthermore to derive a shorter and more comprehensive form of the final boundary conditions, we notate the given terms, which include functions used to describe the given surface tractions (defined on the section 2 of the Part B) with smarter symbols after moving term to the right-hand side of the boundary conditions (deriving as known from the Variational Equations and by the application of the fundamental argument of the Calculus of Variations as will be explained thoroughly on the next section 5.3). Thus, we regard the following

$$T_{\delta\phi_n} = n_{x_1} \; n_{x_1} + n_{x_2} \cdot a_{T1n} + n_{x_1} \; n_{x_2} - n_{x_1} \cdot a_{T1s} \; \frac{I_2}{\rho} +$$

$$+ \; n_{x_1} \; n_{x_1} + n_{x_2} \cdot a_{T3n} + n_{x_1} \; n_{x_2} - n_{x_1} \cdot a_{T3s} \; \frac{I_4}{\rho} -$$

$$- \; n_{x_1} \; n_{x_1} + n_{x_2} \cdot a_{T1n} + n_{x_1} \; n_{x_2} - n_{x_1} \cdot a_{T1s} \; \frac{I_4}{\rho} \frac{4}{3h^2} -$$

$$- \; n_{x_1} \; n_{x_1} + n_{x_2} \cdot a_{T3n} + n_{x_1} \; n_{x_2} - n_{x_1} \cdot a_{T3s} \; \frac{I_6}{\rho} \frac{4}{3h^2} +$$

$$+ \; n_{x_1}^2 a_{5n} + n_{x_1} n_{x_2} a_{5s} \;\; n_{x_1} \cdot a_{T0n} + n_{x_2} \cdot a_{T0s} \; \frac{I_0}{\rho} + (n_{x_1} n_{x_2} b_{5s} - n_{x_1}^2 b_{5n}) \, c_{T0} \frac{I_0}{\rho} +$$

$$+ \; (n_{x_1}^2 a_{4n} + n_{x_1} n_{x_2} a_{4s}) \, (n_{x_2} \cdot a_{T0n} - n_{x_1} \cdot a_{T0s}) \frac{I_0}{\rho} + (n_{x_1}^2 b_{4n} - n_{x_1} n_{x_2} b_{4s}) \, c_{T0} \frac{I_0}{\rho}$$

and

$$T_{\delta\phi_s} = n_{x_1} \; n_{x_1} + n_{x_2} \cdot a_{T1n} + n_{x_1} \; n_{x_2} - n_{x_1} \cdot a_{T1s} \; \frac{I_2}{\rho} +$$

$$+ \; n_{x_1} \; n_{x_1} + n_{x_2} \cdot a_{T3n} + n_{x_1} \; n_{x_2} - n_{x_1} \cdot a_{T3s} \; \frac{I_4}{\rho} -$$

$$- \; n_{x_1} \; n_{x_1} + n_{x_2} \cdot a_{T1n} + n_{x_1} \; n_{x_2} - n_{x_1} \cdot a_{T1s} \; \frac{I_4}{\rho} \frac{4}{3h^2} -$$

$$- \; n_{x_1} \; n_{x_1} + n_{x_2} \cdot a_{T3n} + n_{x_1} \; n_{x_2} - n_{x_1} \cdot a_{T3s} \; \frac{I_6}{\rho} \frac{4}{3h^2} +$$

$$+ \; n_{x_1}^2 a_{5n} + n_{x_1} n_{x_2} a_{5s} \;\; n_{x_1} \cdot a_{T0n} + n_{x_2} \cdot a_{T0s} \; \frac{I_0}{\rho} + (n_{x_1} n_{x_2} b_{5s} - n_{x_1}^2 b_{5n}) \, c_{T0} \frac{I_0}{\rho} +$$

$$+ \; (n_{x_1}^2 a_{4n} + n_{x_1} n_{x_2} a_{4s}) \, (n_{x_2} \cdot a_{T0n} - n_{x_1} \cdot a_{T0s}) \frac{I_0}{\rho} + (n_{x_1}^2 b_{4n} - n_{x_1} n_{x_2} b_{4s}) \, c_{T0} \frac{I_0}{\rho}$$

Thus, as for the natural boundary condition with is multiplied with the variation $\delta\phi_n$ in the Variational Equation, we have finally

$$\begin{aligned} & n_{x_1} \; n_{x_1} - n_{x_2} \;\; n_{x_1} + n_{x_2} \;^2 M_{nn} + 2 n_{x_1}^2 n_{x_2} \; n_{x_2} - n_{x_1} \;\; M_{ss} + \\ & + 3 n_{x_1}^3 n_{x_2} \; - n_{x_1} n_{x_2}^3 - n_{x_1}^4 + 3 n_{x_1}^2 n_{x_2}^2 \;\; M_{ns} + \\ & + \frac{4}{3h^2} \left(\begin{matrix} -n_{x_1} \; n_{x_1} + n_{x_2} \;^2 \; n_{x_1} - n_{x_2} \; P_{nn} + 2 n_{x_1}^2 n_{x_2} \; n_{x_1} - n_{x_2} \;\; P_{ss} - \\ - \; n_{x_1}^4 + n_{x_1} n_{x_2}^3 - 3 n_{x_1}^2 n_{x_2}^2 - 3 n_{x_1}^3 n_{x_2} \;\; P_{ns} \end{matrix} \right) = T_{\delta\varphi_n} \;\; \text{data} \end{aligned} \qquad (3a)$$

And as for the natural boundary condition with is multiplied with the variation $\delta\phi_s$ in the Variational Equation, we have finally

$$
\begin{aligned}
& n_{x_2}\; n_{x_1} - n_{x_2}\;\; n_{x_1} + n_{x_2}\;\; {}^{2} M_{nn} \;+\; 2\, n_{x_2}^{2}\, n_{x_1}\;\; n_{x_2} + n_{x_1}\;\; M_{ss} \;+ \\
& +\;\; 3\, n_{x_1}^{2}\, n_{x_2}^{2} - n_{x_2}^{4} + n_{x_1}^{3}\, n_{x_2} - 3 n_{x_1}\, n_{x_2}^{3}\;\; M_{ns} \;- \qquad\qquad (3b) \\
& -\frac{4}{3h^{2}}\left(\begin{aligned} & n_{x_2}\;\; n_{x_1} - n_{x_2}\;\; {}^{2}\;\; n_{x_1} + n_{x_2}\;\; P_{nn} \;+\; 2\, n_{x_2}^{2}\, n_{x_1}\;\; n_{x_2} + n_{x_1}\;\; P_{ss} \;- \\ & \qquad -\;\; n_{x_2}^{4} - n_{x_1}^{3}\, n_{x_2} -\; 3 n_{x_1}^{2}\, n_{x_2}^{2} + 3 n_{x_1}\, n_{x_2}^{3}\;\; P_{ns} \end{aligned}\right) = T_{\delta\varphi_s}\;\; \text{data}
\end{aligned}
$$

Remind that the process of derivation of the above natural boundary conditions (3a) and (3b) is going to be explained extensively on the subsequent section 5.3, although is exactly the same present on the Part A in order to gain the natural boundary conditions of the model of CPT.

5.3. The full set of boundary conditions of the flexural response (bending) of the TSDT

On this section, we gather all the above results of the sections 5.2.2, 5.2.3 and 5.2.4 and substitute them into the curvilinear integrals of the Variational Equation of the section 5.2, after reorganizing the boundary terms by an appropriate way as seem below.

Remember also that in the context of the problem of TSDT, the shear (membrane) stresses (or strains) are decoupled from those of flexural (bending) stresses (or strains), as said on the section 5.2. For this reason the in-pane natural boundary conditions [Eqs. (24a), (24b) of the section 5.2.1] will not occupy us further from now on and we are going to extract here only those related the flexural response of the plate.

For the sake of completeness, we repeat the boundary terms of the Variational Equation of the section 5.2, expressed on the Cartesian coordinate system.

$$
\begin{aligned}
&\int_{t_1}^{t_2}\oint_{\Gamma} \frac{4}{3h^2}\left(\left\{P_1\, n_{x_1} + P_6\, n_{x_2} - \left(a_{T1}\frac{I_4}{\rho} + a_{T3}\frac{I_6}{\rho}\right)\right\}\frac{\partial \delta w_0}{\partial x_1}\right) d\gamma\, dt \; + \\
&+\int_{t_1}^{t_2}\oint_{\Gamma} \frac{4}{3h^2}\left(\left\{P_2\, n_{x_2} + P_6\, n_{x_1} - \left(b_{T1}\frac{I_4}{\rho} + b_{T3}\frac{I_6}{\rho}\right)\right\}\frac{\partial \delta w_0}{\partial x_2}\right) d\gamma\, dt \; - \\
&-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}
\left[\begin{array}{l}\left(I_6\dfrac{16}{9h^4} - \dfrac{4}{3h^2}I_4\right)\ddot{\phi}_x + I_6\dfrac{16}{9h^4}\dfrac{\partial \ddot{w}_0}{\partial x_1} \; + \\ \quad + Q_5 - R_5\dfrac{4}{h^2} + \dfrac{4}{3h^2}\dfrac{\partial P_6}{\partial x_2} + \dfrac{4}{3h^2}\dfrac{\partial P_1}{\partial x_1}\end{array}\right]\boldsymbol{n}_{x_1} \; + \\
+\left[\begin{array}{l}\left(I_6\dfrac{16}{9h^4} - \dfrac{4}{3h^2}I_4\right)\ddot{\phi}_y + I_6\dfrac{16}{9h^4}\dfrac{\partial \ddot{w}_0}{\partial x_2} \; + \\ + Q_4 - R_4\dfrac{4}{h^2} + \dfrac{4}{3h^2}\dfrac{\partial P_6}{\partial x_1} + \dfrac{4}{3h^2}\dfrac{\partial P_2}{\partial x_2}\end{array}\right]\boldsymbol{n}_{x_2} - \dfrac{I_0}{\rho}c_{T0}
\end{array}\right\}\delta w_0\, d\gamma\, dt \; - \\
&-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}
\left(M_1 - \dfrac{4}{3h^2}P_1\right)n_{x_1} + \left(M_6 - P_6\dfrac{4}{3h^2}\right)n_{x_2} \; - \\
-\left(\begin{array}{l} a_{T1}\dfrac{I_2}{\rho} + a_{T3}\dfrac{I_4}{\rho} - a_{T1}\dfrac{I_4}{\rho}\dfrac{4}{3h^2} - a_{T3}\dfrac{I_6}{\rho}\dfrac{4}{3h^2} \; + \\ \quad + a_5\, a_{T0}\dfrac{I_0}{\rho} + b_5\, c_{T0}\dfrac{I_0}{\rho}\end{array}\right)
\end{array}\right\}\delta\phi_x\, d\gamma\, dt \; - \\
&-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}
\left(M_2 - \dfrac{4}{3h^2}P_2\right)n_{x_2} + \left(M_6 - P_6\dfrac{4}{3h^2}\right)n_{x_1} \; - \\
-\left(\begin{array}{l} b_{T1}\dfrac{I_2}{\rho} + b_{T3}\dfrac{I_4}{\rho} - b_{T1}\dfrac{I_4}{\rho}\dfrac{4}{3h^2} - b_{T3}\dfrac{I_6}{\rho}\dfrac{4}{3h^2} \; + \\ \quad + a_4\, b_{T0}\dfrac{I_0}{\rho} + b_4\, c_{T0}\dfrac{I_0}{\rho}\end{array}\right)
\end{array}\right\}\delta\phi_y\, d\gamma\, dt = 0 \quad \Rightarrow
\end{aligned}
$$

$$
\begin{aligned}
&\frac{4}{3h^2}\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}
n_{x_1}^2 - n_{x_2}^2 \; P_{nn}\dfrac{\partial\delta w_0}{\partial n} + 2 n_{x_2} n_{x_1} \; n_{x_1}^2 - n_{x_2}^2 \; P_{ns}\dfrac{\partial\delta w_0}{\partial n} + \\
+ 2 n_{x_2} n_{x_1} P_{ss}\dfrac{\partial\delta w_0}{\partial s} + n_{x_1}^2 - n_{x_2}^2 \; P_{ns}\dfrac{\partial\delta w_0}{\partial s} - \\
- \left\{a_{T1n}\; n_{x_1}^2 - n_{x_2}^2 \; \dfrac{I_4}{\rho} + a_{T3n}\dfrac{I_6}{\rho}\right\}\dfrac{\partial\delta w_0}{\partial n} - \left\{a_{T1s}\dfrac{I_4}{\rho} + a_{T3s}\dfrac{I_6}{\rho}\right\}\dfrac{\partial\delta w_0}{\partial s}
\end{array}\right\} ds\, dt \; - \\
&- \int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}
\left(I_6\dfrac{16}{9h^4} - \dfrac{4}{3h^2}I_4\right)\{n_{x_1}(n_{x_1}-n_{x_2})\ddot{\phi}_n + n_{x_2}(n_{x_1}+n_{x_2})\ddot{\phi}_s\} + \\
+ I_6\dfrac{16}{9h^4}\left\{(n_{x_1}^2 - n_{x_2}^2)\dfrac{\partial\ddot{w}_0}{\partial n} + 2 n_{x_2} n_{x_1}\dfrac{\partial\ddot{w}_0}{\partial s}\right\} - c_{T0}\dfrac{I_0}{\rho} + \\
+ \dfrac{4}{3h^2}\left\{(n_{x_1}^2 - n_{x_2}^2)\left(\dfrac{\partial P_{nn}}{\partial n} + \dfrac{\partial P_{ns}}{\partial s}\right) + 2 n_{x_2} n_{x_1}\left(\dfrac{\partial P_{ns}}{\partial n} + \dfrac{\partial P_{ss}}{\partial s}\right)\right\} + \\
+ n_{x_2}(n_{x_1}^2 - 1) Q_{sz} - n_{x_2}^2 Q_{ss} + n_{x_2}^2 R_{ss}\dfrac{4}{h^2} - n_{x_2}(n_{x_1}^2 - 1) R_{sz}\dfrac{4}{h^2} + \\
+ n_{x_1}(n_{x_1}^2 - 1) Q_{nz} - n_{x_1}^2 Q_{nn} + n_{x_1}^2 R_{nn}\dfrac{4}{h^2} - n_{x_1}(n_{x_1}^2 - 1) R_{nz}\dfrac{4}{h^2}
\end{array}\right\}\delta w_0\, ds\, dt \; - \\
&- \int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}
n_{x_1}(n_{x_1}-n_{x_2})(n_{x_1}+n_{x_2})^2 M_{nn} + 2 n_{x_1}^2 n_{x_2}(n_{x_2}-n_{x_1}) M_{ss} + \\
+ (3 n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 - n_{x_1}^4 + 3 n_{x_1}^2 n_{x_2}^2) M_{ns} + \\
+ \dfrac{4}{3h^2}\left(\begin{array}{l}
- n_{x_1}(n_{x_1}+n_{x_2})^2(n_{x_1}-n_{x_2}) P_{nn} + 2 n_{x_1}^2 n_{x_2}(n_{x_1}-n_{x_2}) P_{ss} - \\
- (n_{x_1}^4 + n_{x_1} n_{x_2}^3 - 3 n_{x_1}^2 n_{x_2}^2 - 3 n_{x_1}^3 n_{x_2}) P_{ns}
\end{array}\right) - T_{\delta\varphi_n}
\end{array}\right\}\delta\phi_n\, ds\, dt \; - \qquad (1) \\
&- \int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}
n_{x_2}(n_{x_1}-n_{x_2})(n_{x_1}+n_{x_2})^2 M_{nn} + 2 n_{x_2}^2 n_{x_1}(n_{x_2}+n_{x_1}) M_{ss} + \\
+ (3 n_{x_1}^2 n_{x_2}^2 - n_{x_2}^4 + n_{x_1}^3 n_{x_2} - 3 n_{x_1} n_{x_2}^3) M_{ns} - \\
- \dfrac{4}{3h^2}\left(\begin{array}{l}
n_{x_2}(n_{x_1}-n_{x_2})^2(n_{x_1}+n_{x_2}) P_{nn} + 2 n_{x_2}^2 n_{x_1}(n_{x_1}+n_{x_2}) P_{ss} - \\
- (n_{x_2}^4 - n_{x_1}^3 n_{x_2} - 3 n_{x_1}^2 n_{x_2}^2 + 3 n_{x_1} n_{x_2}^3) P_{ns}
\end{array}\right) - T_{\delta\varphi_s}
\end{array}\right\}\delta\phi_s\, ds\, dt = 0
\end{aligned}
$$

At this moment we take apart the boundary terms multiplied with the variations δw_0, $\partial\delta w_0/\partial n$ and $\partial\delta w_0/\partial s$ in order to perform further calculations. There is no reason to elaborate the boundary terms of the variations $\delta\phi_n$ and $\delta\phi_s$ furthermore, because it is obvious then that the last will not contribute to the terms of the variations δw_0, $\partial\delta w_0/\partial n$ and $\partial\delta w_0/\partial s$.

Thus, separating the curvilinear integral which includes n,s-derivatives on the variation δw_0 and neglecting for the present the residual boundary terms, we have

$$\frac{4}{3h^2}\int_{t_1}^{t_2}\oint_{\Gamma} n_{x_1}^2 - n_{x_2}^2 \;\; P_{nn}\,\frac{\partial \delta w_0}{\partial n}\, ds\, dt + \frac{4}{3h^2}\int_{t_1}^{t_2}\oint_{\Gamma} 2\, n_{x_2}\, n_{x_1}\;\; n_{x_1}^2 - n_{x_2}^2 \;\; P_{ns}\,\frac{\partial \delta w_0}{\partial n}\, ds\, dt \;+$$

$$+\frac{4}{3h^2}\int_{t_1}^{t_2}\oint_{\Gamma} 2\, n_{x_2}\, n_{x_1}\, P_{ss}\,\frac{\partial \delta w_0}{\partial s}\, d\gamma\, dt + \frac{4}{3h^2}\int_{t_1}^{t_2}\oint_{\Gamma} n_{x_1}^2 - n_{x_2}^2 \;\; P_{ns}\,\frac{\partial \delta w_0}{\partial s}\, ds\, dt \;-$$

$$-\frac{4}{3h^2}\int_{t_1}^{t_2}\oint_{\Gamma}\left\{ a_{T1n}(n_{x_1}^2 - n_{x_2}^2)\,\frac{I_4}{\rho} + a_{T3n}\,\frac{I_6}{\rho}\right\}\frac{\partial \delta w_0}{\partial n}\, ds\, dt \;-$$

$$-\frac{4}{3h^2}\int_{t_1}^{t_2}\oint_{\Gamma}\left\{ a_{T1s}\,\frac{I_4}{\rho} + a_{T3s}\,\frac{I_6}{\rho}\right\}\frac{\partial \delta w_0}{\partial s}\, ds\, dt \;-$$

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l} \left(I_6\,\dfrac{16}{9h^4} - \dfrac{4}{3h^2}I_4\right)\{ n_{x_1}(n_{x_1} - n_{x_2})\,\ddot{\phi}_n + n_{x_2}\,(n_{x_1} + n_{x_2})\,\ddot{\phi}_s \} \;+ \\ + I_6\,\dfrac{16}{9h^4}\left\{ (n_{x_1}^2 - n_{x_2}^2)\dfrac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_2}\, n_{x_1}\,\dfrac{\partial \ddot{w}_0}{\partial s}\right\} - c_{T0}\,\dfrac{I_0}{\rho} \;+ \\ + \dfrac{4}{3h^2}\left\{ (n_{x_1}^2 - n_{x_2}^2)\left(\dfrac{\partial P_{nn}}{\partial n} + \dfrac{\partial P_{ns}}{\partial s}\right) + 2\, n_{x_2}\, n_{x_1}\left(\dfrac{\partial P_{ns}}{\partial n} + \dfrac{\partial P_{ss}}{\partial s}\right)\right\} \;+ \\ + n_{x_2}(n_{x_1}^2 - 1)\, Q_{sz} - n_{x_2}^2\, Q_{ss} + n_{x_2}^2\, R_{ss}\,\dfrac{4}{h^2} - n_{x_2}(n_{x_1}^2 - 1)\, R_{sz}\,\dfrac{4}{h^2} \;+ \\ + n_{x_1}(n_{x_1}^2 - 1)\, Q_{nz} - n_{x_1}^2\, Q_{nn} + n_{x_1}^2\, R_{nn}\,\dfrac{4}{h^2} - n_{x_1}(n_{x_1}^2 - 1)\, R_{nz}\,\dfrac{4}{h^2} \end{array}\right\}\delta w_0\, ds\, dt = 0 \qquad (2)$$

For simplification reasons, we neglect the time integration and we note the following boundary terms with spatial derivatives on the variations in order to perform by parts integrations to those with s-derivative on the variation δw_0.

$$I_{P_{nn,n}} = \oint_{\Gamma} n_{x_1}^2 - n_{x_2}^2 \;\; P_{nn}\,\frac{\partial \delta w_0}{\partial n}\, ds \qquad (3a)$$

$$I_{P_{ns,n}} = \oint_{\Gamma} 2\, n_{x_2}\, n_{x_1}\;\; n_{x_1}^2 - n_{x_2}^2 \;\; P_{ns}\,\frac{\partial \delta w_0}{\partial n}\, ds \qquad (3b)$$

$$\begin{aligned} I_{P_{ss,s}} &= \oint_{\Gamma} 2\, n_{x_2}\, n_{x_1}\, P_{ss}\,\frac{\partial \delta w_0}{\partial s}\, ds = \\ &= \left[2\, n_{x_2}\, n_{x_1} P_{ss}\;\delta w_0\right]_{\Gamma} - \oint_{\Gamma}\frac{\partial}{\partial s}\; 2\, n_{x_2}\, n_{x_1}\, P_{ss}\;\;\delta w_0\, ds = \\ &= \left[2\, n_{x_2}\, n_{x_1} P_{ss}\;\delta w_0\right]_{\Gamma} - \oint_{\Gamma}\frac{\partial}{\partial s}\; 2\, n_{x_2}\, n_{x_1}\;\; P_{ss}\,\delta w_0\, ds - \oint_{\Gamma} 2\, n_{x_2}\, n_{x_1} P_{ss,s}\;\delta w_0\, ds \end{aligned} \qquad (3c)$$

$$
\begin{aligned}
I_{P_{ns,s}} &= \oint_{\Gamma} \left(n_{x_1}^2 - n_{x_2}^2\right) P_{ns} \frac{\partial \delta w_0}{\partial s}\, ds = \\
&= \left[\left(n_{x_1}^2 - n_{x_2}^2\right) P_{ns}\, \delta w_0 \right]_{\Gamma} - \oint_{\Gamma} \frac{\partial}{\partial s} \left(\left(n_{x_1}^2 - n_{x_2}^2\right) P_{ns} \right) \delta w_0\, ds = \\
&= \left[\left(n_{x_1}^2 - n_{x_2}^2\right) P_{ns}\, \delta w_0 \right]_{\Gamma} - \oint_{\Gamma} \frac{\partial}{\partial s} \left(n_{x_1}^2 - n_{x_2}^2\right) P_{ns}\, \delta w_0\, ds - \oint_{\Gamma} \left(n_{x_1}^2 - n_{x_2}^2\right) P_{ns,s}\, \delta w_0\, ds
\end{aligned}
\tag{3d}
$$

$$
I_{\delta w_0,n} = \oint_{\Gamma} \left\{ a_{T1n} \left(n_{x_1}^2 - n_{x_2}^2\right) \frac{I_4}{\rho} + a_{T3n} \frac{I_6}{\rho} \right\} \frac{\partial \delta w_0}{\partial n}\, ds \tag{3e}
$$

$$
\begin{aligned}
I_{\delta w_0,s} &= \oint_{\Gamma} \left\{ a_{T1s} \frac{I_4}{\rho} + a_{T3s} \frac{I_6}{\rho} \right\} \frac{\partial \delta w_0}{\partial s}\, ds = \\
&= \left[\left\{ a_{T1s} \frac{I_4}{\rho} + a_{T3s} \frac{I_6}{\rho} \right\} \delta w_0 \right]_{\Gamma} - \oint_{\Gamma} \frac{\partial}{\partial s} \left\{ a_{T1s} \frac{I_4}{\rho} + a_{T3s} \frac{I_6}{\rho} \right\} \delta w_0\, ds = \\
&= \left[\left\{ a_{T1s} \frac{I_4}{\rho} + a_{T3s} \frac{I_6}{\rho} \right\} \delta w_0 \right]_{\Gamma} - \oint_{\Gamma} \left\{ \frac{\partial a_{T1s}}{\partial s} \frac{I_4}{\rho} + \frac{\partial a_{T3s}}{\partial s} \frac{I_6}{\rho} \right\} \delta w_0\, ds
\end{aligned}
\tag{3f}
$$

The previous choice to perform by parts integrations only along the curve Γ, namely following the arc s through the tangential derivatives $\partial \delta w_0 / \partial s$, is intentional because it gives the desirable boundary terms and concurrently by this way the number of the total boundary conditions of the problem is reduced to the desirable. The last is attained by eliminating the derivative $\partial \delta w_0 / \partial s$ from the boundary terms, as seems below.

We assume here that the end points of the closed curve Γ coincide or when terms inside the brackets $[\cdot]_{\Gamma}$ are equal to zero, namely,

$$
P_{ss} = P_{ns} = a_{T1s} \frac{I_4}{\rho} + a_{T3s} \frac{I_6}{\rho} = 0 \tag{4}
$$

Then the first terms of the right-hand side of the relations (3c), (3d) and (3f) are eliminated,

$$
\left[2 n_{x_2} n_{x_1} P_{ss}\, \delta w_0 \right]_{\Gamma} = \left[\left(n_{x_1}^2 - n_{x_2}^2\right) P_{ns}\, \delta w_0 \right]_{\Gamma} = \left[\left\{ a_{T1s} \frac{I_4}{\rho} + a_{T3s} \frac{I_6}{\rho} \right\} \delta w_0 \right]_{\Gamma} = 0 \tag{5}
$$

Thus,

$$
I_{P_{ss,s}} = - \oint_{\Gamma} \frac{\partial}{\partial s} \left(2 n_{x_2} n_{x_1}\right) P_{ss}\, \delta w_0\, ds - \oint_{\Gamma} 2 n_{x_2} n_{x_1} P_{ss,s}\, \delta w_0\, ds \tag{6}
$$

$$
I_{P_{ns,s}} = - \oint_{\Gamma} \frac{\partial}{\partial s} \left(n_{x_1}^2 - n_{x_2}^2\right) P_{ns}\, \delta w_0\, ds - \oint_{\Gamma} \left(n_{x_1}^2 - n_{x_2}^2\right) P_{ns,s}\, \delta w_0\, ds \tag{7}
$$

$$I_{\delta w_0,s} = -\oint_{\Gamma}\left\{\frac{\partial a_{T1s}}{\partial s}\frac{I_4}{\rho}+\frac{\partial a_{T3s}}{\partial s}\frac{I_6}{\rho}\right\}\delta w_0\,ds \tag{8}$$

Finally, substituting the relations (3a), (3b), (6), (7), (3e), (8) into the Eqs. (2), we get

$$\begin{aligned}
&\frac{4}{3h^2}\int_{t_1}^{t_2}\oint_{\Gamma} n_{x_1}^2 - n_{x_2}^2\;\; P_{nn}\,\frac{\partial\delta w_0}{\partial n}\,ds\,dt + \frac{4}{3h^2}\int_{t_1}^{t_2}\oint_{\Gamma} 2\,n_{x_2}\,n_{x_1}\;\; n_{x_1}^2 - n_{x_2}^2\;\; P_{ns}\,\frac{\partial\delta w_0}{\partial n}\,ds\,dt \;+\\
&-\int_{t_1}^{t_2}\frac{4}{3h^2}\oint_{\Gamma}\frac{\partial}{\partial s}\;2\,n_{x_2}\,n_{x_1}\;\; P_{ss}\;\delta w_0\,ds\,dt - \int_{t_1}^{t_2}\frac{4}{3h^2}\oint_{\Gamma} 2\,n_{x_2}\,n_{x_1} P_{ss,s}\;\delta w_0\,ds\,dt \;-\\
&-\int_{t_1}^{t_2}\frac{4}{3h^2}\oint_{\Gamma}\frac{\partial}{\partial s}\;\; n_{x_1}^2 - n_{x_2}^2\;\; P_{ns}\;\delta w_0\,ds\,dt - \int_{t_1}^{t_2}\frac{4}{3h^2}\oint_{\Gamma}\;\; n_{x_1}^2 - n_{x_2}^2\;\; P_{ns,s}\;\delta w_0\,ds\,dt \;-\\
&-\int_{t_1}^{t_2}\frac{4}{3h^2}\oint_{\Gamma}\left\{a_{T1n}(n_{x_1}^2 - n_{x_2}^2)\frac{I_4}{\rho} + a_{T3n}\frac{I_6}{\rho}\right\}\frac{\partial\delta w_0}{\partial n}\,ds\,dt + \int_{t_1}^{t_2}\frac{4}{3h^2}\oint_{\Gamma}\left\{\frac{\partial a_{T1s}}{\partial s}\frac{I_4}{\rho}+\frac{\partial a_{T3s}}{\partial s}\frac{I_6}{\rho}\right\}\delta w_0\,ds\,dt \;-
\end{aligned}$$

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}
\left(I_6\dfrac{16}{9h^4} - \dfrac{4}{3h^2}I_4\right)\{n_{x_1}(n_{x_1}-n_{x_2})\,\ddot{\phi}_n + n_{x_2}\,(n_{x_1}+n_{x_2})\,\ddot{\phi}_s\} \;+\\
\quad + I_6\dfrac{16}{9h^4}\left\{(n_{x_1}^2 - n_{x_2}^2)\dfrac{\partial\ddot{w}_0}{\partial n} + 2\,n_{x_2}\,n_{x_1}\dfrac{\partial\ddot{w}_0}{\partial s}\right\} - c_{T0}\dfrac{I_0}{\rho} \;+\\
+\;\dfrac{4}{3h^2}\left\{(n_{x_1}^2 - n_{x_2}^2)\left(\dfrac{\partial P_{nn}}{\partial n} + \dfrac{\partial P_{ns}}{\partial s}\right) + 2\,n_{x_2}\,n_{x_1}\left(\dfrac{\partial P_{ns}}{\partial n} + \dfrac{\partial P_{ss}}{\partial s}\right)\right\} \;+\\
+\; n_{x_2}(n_{x_1}^2 - 1)\,Q_{sz} - n_{x_2}^2\,Q_{ss} + n_{x_2}^2\,R_{ss}\dfrac{4}{h^2} - n_{x_2}(n_{x_1}^2 - 1)\,R_{sz}\dfrac{4}{h^2} \;+\\
+\; n_{x_1}(n_{x_1}^2 - 1)\,Q_{nz} - n_{x_1}^2\,Q_{nn} + n_{x_1}^2\,R_{nn}\dfrac{4}{h^2} - n_{x_1}(n_{x_1}^2 - 1)R_{nz}\dfrac{4}{h^2}
\end{array}\right\}\delta w_0\,ds\,dt = 0 \quad\Rightarrow$$

[Gathering separately the boundary terms associated to the variations δw_0, $\partial\delta w_0/\partial n$ and also writing the derivatives of the thickness-integrated quantities in a more abbreviate form, we get the final result of the boundary terms]

$$\int_{t_1}^{t_2} \frac{4}{3h^2} \oint_{\Gamma} \left[n_{x_1}^2 - n_{x_2}^2 \;\; P_{nn} + 2\, n_{x_2} n_{x_1} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; P_{ns} - \left\{ a_{T1n} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; \frac{I_4}{\rho} + a_{T3n} \frac{I_6}{\rho} \right\} \right] \frac{\partial \delta w_0}{\partial n}\, ds\, dt \; -$$

$$- \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \begin{array}{l} \left(I_6 \frac{16}{9h^4} - \frac{4}{3h^2} I_4 \right) \;\; n_{x_1} \;\; n_{x_1} - n_{x_2} \;\; \ddot{\phi}_n + \; n_{x_2} \;\; n_{x_1} + n_{x_2} \;\; \ddot{\phi}_s \;\; + \\ \quad + I_6 \frac{16}{9h^4} \left\{ (n_{x_1}^2 - n_{x_2}^2) \frac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_2}\, n_{x_1} \frac{\partial \ddot{w}_0}{\partial s} \right\} + \\ + \frac{4}{3h^2} \{ (n_{x_1}^2 - n_{x_2}^2)(P_{nn,n} + P_{ns,s}) + 2\, n_{x_2}\, n_{x_1} (P_{ns,n} + P_{ss,s}) \} + \\ + \frac{4}{3h^2} \frac{\partial}{\partial s} \{ 2\, n_{x_2}\, n_{x_1} \} P_{ss} + \frac{8}{3h^2} n_{x_2} n_{x_1} P_{ss,s} + \\ + \frac{4}{3h^2} \frac{\partial}{\partial s} \{ n_{x_1}^2 - n_{x_2}^2 \} P_{ns} + \frac{4}{3h^2} (n_{x_1}^2 - n_{x_2}^2)\, P_{ns,s} \\ + \; n_{x_2} (n_{x_1}^2 - 1)\, Q_{sz} - n_{x_2}^2\, Q_{ss} + n_{x_2}^2\, R_{ss} \frac{4}{h^2} - n_{x_2} (n_{x_1}^2 - 1)\, R_{sz} \frac{4}{h^2} + \\ + \; n_{x_1} (n_{x_1}^2 - 1)\, Q_{nz} - n_{x_1}^2\, Q_{nn} + n_{x_1}^2\, R_{nn} \frac{4}{h^2} - n_{x_1} (n_{x_1}^2 - 1) R_{nz} \frac{4}{h^2} - \\ \qquad - \frac{4}{3h^2} \left\{ \frac{\partial a_{T1s}}{\partial s} \frac{I_4}{\rho} + \frac{\partial a_{T3s}}{\partial s} \frac{I_6}{\rho} \right\} - c_{T0} \frac{I_0}{\rho} \end{array} \right\} \delta w_0 \, ds\, dt = 0 \qquad \Rightarrow$$

$$\int_{t_1}^{t_2} \frac{4}{3h^2} \oint_{\Gamma} \left[n_{x_1}^2 - n_{x_2}^2 \;\; P_{nn} + 2\, n_{x_2} n_{x_1} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; P_{ns} - \left\{ a_{T1n} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; \frac{I_4}{\rho} + a_{T3n} \frac{I_6}{\rho} \right\} \right] \frac{\partial \delta w_0}{\partial n}\, ds\, dt \; -$$

$$- \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \begin{array}{l} \qquad \left(I_6 \frac{16}{9h^4} - \frac{4}{3h^2} I_4 \right) \;\; n_{x_1} \;\; n_{x_1} - n_{x_2} \;\; \ddot{\phi}_n + \; n_{x_2} \;\; n_{x_1} + n_{x_2} \;\; \ddot{\phi}_s \;\; + \\ \qquad\quad + I_6 \frac{16}{9h^4} \left\{ \;\; n_{x_1}^2 - n_{x_2}^2 \;\; \frac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_2}\, n_{x_1} \frac{\partial \ddot{w}_0}{\partial s} \right\} + \\ + \frac{4}{3h^2} \{ (n_{x_1}^2 - n_{x_2}^2) P_{nn,n} + 2 (n_{x_1}^2 - n_{x_2}^2) P_{ns,s} + 2\, n_{x_2}\, n_{x_1}\, P_{ns,n} + 4\, n_{x_2}\, n_{x_1}\, P_{ss,s} \} + \\ \qquad + \frac{4}{3h^2} \frac{\partial}{\partial s} \{ 2\, n_{x_2}\, n_{x_1} \} P_{ss} + \frac{4}{3h^2} \frac{\partial}{\partial s} \{ n_{x_1}^2 - n_{x_2}^2 \} P_{ns} + \\ \quad + \; n_{x_2} (n_{x_1}^2 - 1)\, Q_{sz} - n_{x_2}^2\, Q_{ss} + n_{x_2}^2\, R_{ss} \frac{4}{h^2} - n_{x_2} (n_{x_1}^2 - 1)\, R_{sz} \frac{4}{h^2} + \\ \quad + \; n_{x_1} (n_{x_1}^2 - 1)\, Q_{nz} - n_{x_1}^2\, Q_{nn} + n_{x_1}^2\, R_{nn} \frac{4}{h^2} - n_{x_1} (n_{x_1}^2 - 1)\, R_{nz} \frac{4}{h^2} - \\ \qquad\qquad - \frac{4}{3h^2} \left\{ \frac{\partial a_{T1s}}{\partial s} \frac{I_4}{\rho} + \frac{\partial a_{T3s}}{\partial s} \frac{I_6}{\rho} \right\} - c_{T0} \frac{I_0}{\rho} \end{array} \right\} \delta w_0 \, ds\, dt = 0$$

After this demanding separation, we appear again the boundary terms related to the variations $\delta\,\phi_n$ and $\delta\,\phi_s$ in order to illustrate the total Variational Equation [Eq. (1)] in the final form on curvilinear coordinates.

$$\int_{t_1}^{t_2}\frac{4}{3h^2}\oint_{\Gamma}\left[n_{x_1}^2 - n_{x_2}^2 \ P_{nn} + 2\ n_{x_2} n_{x_1}\ n_{x_1}^2 - n_{x_2}^2\ P_{ns} - \left\{ a_{T1n}\ n_{x_1}^2 - n_{x_2}^2\ \frac{I_4}{\rho} + a_{T3n}\frac{I_6}{\rho}\right\}\right]\frac{\partial\delta w_0}{\partial n}\,ds\,dt \ -$$

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{c} \left(I_6\dfrac{16}{9h^4}-\dfrac{4}{3h^2}I_4\right)\ n_{x_1}\ n_{x_1} - n_{x_2}\ \ddot{\phi}_n + n_{x_2}\ n_{x_1} + n_{x_2}\ \ddot{\phi}_s \ + \\ + I_6\dfrac{16}{9h^4}\left\{ n_{x_1}^2 - n_{x_2}^2\ \dfrac{\partial\ddot{w}_0}{\partial n} + 2\,n_{x_2}\,n_{x_1}\dfrac{\partial\ddot{w}_0}{\partial s}\right\} + \\ + \dfrac{4}{3h^2}\{(n_{x_1}^2 - n_{x_2}^2)P_{nn,n} + 2\,(n_{x_1}^2 - n_{x_2}^2)P_{ns,s} + 2\,n_{x_2}\,n_{x_1}\,P_{ns,n} + 4\,n_{x_2}\,n_{x_1}\,P_{ss,s}\} + \\ + \dfrac{4}{3h^2}\dfrac{\partial}{\partial s}\{2\,n_{x_2}\,n_{x_1}\}P_{ss} + \dfrac{4}{3h^2}\dfrac{\partial}{\partial s}\{n_{x_1}^2 - n_{x_2}^2\}P_{ns} \ + \\ + \ n_{x_2}(n_{x_1}^2 - 1)\ Q_{sz} - \ n_{x_2}^2\ Q_{ss} + \ n_{x_2}^2\ R_{ss}\dfrac{4}{h^2} - \ n_{x_2}(n_{x_1}^2 - 1)\ R_{sz}\dfrac{4}{h^2} \ + \\ + \ n_{x_1}(n_{x_1}^2 - 1)\ Q_{nz} - \ n_{x_1}^2\ Q_{nn} + \ n_{x_1}^2\ R_{nn}\dfrac{4}{h^2} - \ n_{x_1}(n_{x_1}^2 - 1)R_{nz}\dfrac{4}{h^2} \ - \\ - \dfrac{4}{3h^2}\left\{\dfrac{\partial a_{T1s}}{\partial s}\dfrac{I_4}{\rho} + \dfrac{\partial a_{T3s}}{\partial s}\dfrac{I_6}{\rho}\right\} - \ c_{T0}\dfrac{I_0}{\rho} \end{array}\right\}\delta w_0\,ds\,dt \ -$$

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{c} n_{x_1}(n_{x_1} - n_{x_2})(n_{x_1} + n_{x_2})^2\ M_{nn} + 2\,n_{x_1}^2\ n_{x_2}(n_{x_2} - n_{x_1})\ M_{ss} \ + \\ + \ (3\,n_{x_1}^3\ n_{x_2} - n_{x_1}\ n_{x_2}^3 - n_{x_1}^4 + 3n_{x_1}^2\ n_{x_2}^2)\ M_{ns} \ + \\ + \dfrac{4}{3h^2}\left(\begin{array}{c} -n_{x_1}(n_{x_1} - n_{x_2})(n_{x_1} + n_{x_2})^2\ P_{nn} + 2\,n_{x_1}^2\ n_{x_2}(n_{x_1} - n_{x_2})\ P_{ss} \ - \\ - \ (n_{x_1}^4 + n_{x_1}\ n_{x_2}^3 - 3\,n_{x_1}^2\ n_{x_2}^2 - 3\,n_{x_1}^3\ n_{x_2})\ P_{ns} \end{array}\right) - \ T_{\delta\varphi_n} \end{array}\right\}\delta\phi_n\,ds\,dt \ -$$

$$-\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{c} n_{x_2}(n_{x_1} - n_{x_2})(n_{x_1} + n_{x_2})^2\ M_{nn} + 2\,n_{x_2}^2\ n_{x_1}(n_{x_2} + n_{x_1})\ M_{ss} \ + \\ + \ (3\,n_{x_1}^2\ n_{x_2}^2 - n_{x_2}^4 + n_{x_1}^3\ n_{x_2} - 3n_{x_1}\ n_{x_2}^3)\ M_{ns} \ - \\ - \dfrac{4}{3h^2}\left(\begin{array}{c} n_{x_2}(n_{x_1} - n_{x_2})^2(n_{x_1} + n_{x_2})\ P_{nn} + 2\,n_{x_2}^2\ n_{x_1}(n_{x_1} + n_{x_2})P_{ss} \ - \\ - \ (n_{x_2}^4 - n_{x_1}^3\ n_{x_2} - 3n_{x_1}^2\ n_{x_2}^2 + 3n_{x_1}\ n_{x_2}^3)\ P_{ns} \end{array}\right) - \ T_{\delta\varphi_s} \end{array}\right\}\delta\phi_s\,ds\,dt = 0$$

As seems from the last version of the variational equation, the total number of the boundary conditions is **four** natural boundary conditions with primary variables $\partial w_0/\partial n$, w_0, ϕ_n and ϕ_s. Each of the previous corresponds to four essential boundary conditions. Thus, to derive the essential and natural boundary conditions we follow the process explained below.

Now we invoke the fundamental arguments of the Calculus of Variations in order to extract the boundary conditions from the last version of the Variational Equation including only the boundary terms. The following process is presented extensively on the Lecture Notes of Functional Analysis, *G.A. Athanassoulis* (*2016*) *"Necessary Conditions of Extremum of Functional"* and *"A further study of the Variational Problem as for integral type functional"*, as well as on the book of *Gelfand I.M., Fomin S.V.* (*1963*), *"Calculus of Variations"*.
First, we assume that $\partial w_0/\partial n$ is arbitrary om the curve Γ, for arbitrary interval $[t_1, t_2]$ and keep the restrictions $\delta w_0 = \delta\phi_n = \delta\phi_s = 0$. Thus, the last equation is converted to

$$\int_{t_1}^{t_2} \frac{4}{3h^2} \oint_{\Gamma} \left[n_{x_1}^2 - n_{x_2}^2 \;\; P_{nn} + 2\; n_{x_2} n_{x_1} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; P_{ns} - \left\{ a_{T1n} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; \frac{I_4}{\rho} + a_{T3n} \frac{I_6}{\rho} \right\} \right] \frac{\partial \delta w_0(s;t)}{\partial n} \, ds \, dt = 0$$

$$\forall \;\; \frac{\partial \delta w_0(s;t)}{\partial n}$$

and using the arbitrariness of the variation $\partial \delta w_0 / \partial n$ on the curvilinear domain $\Gamma \times [t_1, t_2]$, we find the <u>first natural boundary condition</u> as for the **flexural** response of the problem of TSDT,

$$n_{x_1}^2 - n_{x_2}^2 \;\; P_{nn} + 2\; n_{x_2} n_{x_1} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; P_{ns} = a_{T1n} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; \frac{I_4}{\rho} + a_{T3n} \frac{I_6}{\rho} \qquad (9a)$$

Now, due to the Eq. (9a), the Variational Equation is diminished to the below

$$\int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \begin{array}{c} \left(I_6 \frac{16}{9h^4} - \frac{4}{3h^2} I_4 \right) \;\; n_{x_1} \;\; n_{x_1} - n_{x_2} \;\; \ddot{\phi}_n + n_{x_2} \;\; n_{x_1} + n_{x_2} \;\; \ddot{\phi}_s \;\; + \\ + I_6 \frac{16}{9h^4} \left\{ n_{x_1}^2 - n_{x_2}^2 \;\; \frac{\partial \ddot{w}_0}{\partial n} + 2\, n_{x_2} n_{x_1} \frac{\partial \ddot{w}_0}{\partial s} \right\} + \\ + \frac{4}{3h^2} \;\; n_{x_1}^2 - n_{x_2}^2 \;\; P_{nn,n} + 2 \;\; n_{x_1}^2 - n_{x_2}^2 \;\; P_{ns,s} + 2\, n_{x_2} n_{x_1} P_{ns,n} + 4\, n_{x_2} n_{x_1} P_{ss,s} \;\; + \\ + \frac{4}{3h^2} \frac{\partial}{\partial s} \, 2\, n_{x_2} n_{x_1} \;\; P_{ss} + \frac{4}{3h^2} \frac{\partial}{\partial s} \{ n_{x_1}^2 - n_{x_2}^2 \} P_{ns} \;\; + \\ + \; n_{x_2} (n_{x_1}^2 - 1) \; Q_{sz} - n_{x_2}^2 \, Q_{ss} + n_{x_2}^2 \; R_{ss} \frac{4}{h^2} - n_{x_2} (n_{x_1}^2 - 1) \; R_{sz} \frac{4}{h^2} \;\; + \\ + \; n_{x_1} (n_{x_1}^2 - 1) \; Q_{nz} - n_{x_1}^2 \, Q_{nn} + n_{x_1}^2 \; R_{nn} \frac{4}{h^2} - n_{x_1} (n_{x_1}^2 - 1) \; R_{nz} \frac{4}{h^2} \;\; - \\ - \frac{4}{3h^2} \left\{ \frac{\partial a_{T1s}}{\partial s} \frac{I_4}{\rho} + \frac{\partial a_{T3s}}{\partial s} \frac{I_6}{\rho} \right\} - c_{T0} \frac{I_0}{\rho} \end{array} \right\} \delta w_0 \; ds \, dt \;\; +$$

$$+ \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \begin{array}{c} n_{x_1} \;\; n_{x_1} - n_{x_2} \;\; n_{x_1} + n_{x_2} \;\; {}^2 \, M_{nn} + 2\, n_{x_1}^2 \, n_{x_2} \;\; n_{x_2} - n_{x_1} \;\; M_{ss} \;\; + \\ + \; 3\, n_{x_1}^3 \, n_{x_2} - n_{x_1} n_{x_2}^3 - n_{x_1}^4 + 3 n_{x_1}^2 \, n_{x_2}^2 \;\; M_{ns} \;\; + \\ + \frac{4}{3h^2} \left(\begin{array}{c} - n_{x_1} \;\; n_{x_1} + n_{x_2} \;\; {}^2 \;\; n_{x_1} - n_{x_2} \;\; P_{nn} + 2\, n_{x_1}^2 \, n_{x_2} \;\; n_{x_1} - n_{x_2} \;\; P_{ss} \;\; - \\ - \;\; n_{x_1}^4 + n_{x_1} n_{x_2}^3 - 3\, n_{x_1}^2 \, n_{x_2}^2 - 3\, n_{x_1}^3 \, n_{x_2} \;\; P_{ns} \end{array} \right) - T_{\delta \varphi_n} \end{array} \right\} \delta \phi_n \; ds \, dt \;\; +$$

$$+ \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \begin{array}{c} n_{x_2} \;\; n_{x_1} - n_{x_2} \;\; n_{x_1} + n_{x_2} \;\; {}^2 \, M_{nn} + 2\, n_{x_2}^2 \, n_{x_1} \;\; n_{x_2} + n_{x_1} \;\; M_{ss} \;\; + \\ + \; 3\, n_{x_1}^2 \, n_{x_2}^2 - n_{x_2}^4 + n_{x_1}^3 \, n_{x_2} - 3 n_{x_1} \, n_{x_2}^3 \;\; M_{ns} \;\; - \\ - \frac{4}{3h^2} \left(\begin{array}{c} n_{x_2} \;\; n_{x_1} - n_{x_2} \;\; {}^2 \;\; n_{x_1} + n_{x_2} \;\; P_{nn} + 2\, n_{x_2}^2 \, n_{x_1} \;\; n_{x_2} + n_{x_1} \;\; P_{ss} \;\; - \\ - \;\; n_{x_2}^4 - n_{x_1}^3 \, n_{x_2} - 3 n_{x_1}^2 \, n_{x_2}^2 + 3 n_{x_1} \, n_{x_2}^3 \;\; P_{ns} \end{array} \right) - T_{\delta \varphi_s} \end{array} \right\} \delta \phi_s \; ds \, dt = 0$$

Removing the restriction $\delta w_0 = 0$, assuming the arbitrariness of the function δw_0 and of the interval $[t_1, t_2]$ and taking into account the restrictions $\delta\phi_n = \delta\phi_s = 0$, we derive the following

$$\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l}\left(I_6\frac{16}{9h^4}-\frac{4}{3h^2}I_4\right)\left[\left(n_{x_1}\ n_{x_1} - n_{x_2}\right)\ddot{\phi}_n + \left(n_{x_2}\ n_{x_1} + n_{x_2}\right)\ddot{\phi}_s\right] + \\ \quad + I_6\frac{16}{9h^4}\left\{\left(n_{x_1}^2 - n_{x_2}^2\right)\frac{\partial \ddot{w}_0}{\partial n} + 2\,n_{x_2}\,n_{x_1}\frac{\partial \ddot{w}_0}{\partial s}\right\} + \\ + \frac{4}{3h^2}\left[\left(n_{x_1}^2 - n_{x_2}^2\right)P_{nn,n} + 2\left(n_{x_1}^2 - n_{x_2}^2\right)P_{ns,s} + 2\,n_{x_2}\,n_{x_1}P_{ns,n} + 4\,n_{x_2}\,n_{x_1}P_{ss,s}\right] + \\ \quad + \frac{4}{3h^2}\frac{\partial}{\partial s}\left(2\,n_{x_2}\,n_{x_1}P_{ss}\right) + \frac{4}{3h^2}\frac{\partial}{\partial s}\{n_{x_1}^2 - n_{x_2}^2\}P_{ns} + \\ \quad + n_{x_2}(n_{x_1}^2 - 1)\,Q_{sz} - n_{x_2}^2\,Q_{ss} + n_{x_2}^2\,R_{ss}\frac{4}{h^2} - n_{x_2}(n_{x_1}^2-1)\,R_{sz}\frac{4}{h^2} + \\ \quad + n_{x_1}(n_{x_1}^2 - 1)\,Q_{nz} - n_{x_1}^2\,Q_{nn} + n_{x_1}^2\,R_{nn}\frac{4}{h^2} - n_{x_1}(n_{x_1}^2-1)\,R_{nz}\frac{4}{h^2} - \\ \quad\quad - \frac{4}{3h^2}\left\{\frac{\partial a_{T1s}}{\partial s}\frac{I_4}{\rho} + \frac{\partial a_{T3s}}{\partial s}\frac{I_6}{\rho}\right\} - c_{T0}\frac{I_0}{\rho}\end{array}\right\}\delta w_0(s;t)\,ds\,dt = 0$$

$$\forall\ \ \delta w_0(s;t)$$

and using the arbitrariness of the variation δw_0 on the $\Gamma\times[t_1, t_2]$, we get the <u>second natural</u> boundary condition as for the flexural response of the problem of TSDT,

$$\begin{array}{l}\left(I_6\frac{16}{9h^4}-\frac{4}{3h^2}I_4\right)\left[\left(n_{x_1}\ n_{x_1} - n_{x_2}\right)\ddot{\phi}_n + \left(n_{x_2}\ n_{x_1} + n_{x_2}\right)\ddot{\phi}_s\right] + \\ \quad + I_6\frac{16}{9h^4}\left\{\left(n_{x_1}^2 - n_{x_2}^2\right)\frac{\partial \ddot{w}_0}{\partial n} + 2\,n_{x_2}\,n_{x_1}\frac{\partial \ddot{w}_0}{\partial s}\right\} + \\ + \frac{4}{3h^2}\left[\left(n_{x_1}^2 - n_{x_2}^2\right)P_{nn,n} + 2\left(n_{x_1}^2 - n_{x_2}^2\right)P_{ns,s} + 2\,n_{x_2}\,n_{x_1}P_{ns,n} + 4\,n_{x_2}\,n_{x_1}P_{ss,s}\right] + \\ \quad + \frac{4}{3h^2}\frac{\partial}{\partial s}\left(2\,n_{x_2}\,n_{x_1}P_{ss}\right) + \frac{4}{3h^2}\frac{\partial}{\partial s}\{n_{x_1}^2 - n_{x_2}^2\}P_{ns} + \\ \quad + n_{x_2}(n_{x_1}^2 - 1)\,Q_{sz} - n_{x_2}^2\,Q_{ss} + n_{x_2}^2\,R_{ss}\frac{4}{h^2} - n_{x_2}(n_{x_1}^2-1)\,R_{sz}\frac{4}{h^2} + \\ \quad + n_{x_1}(n_{x_1}^2 - 1)\,Q_{nz} - n_{x_1}^2\,Q_{nn} + n_{x_1}^2\,R_{nn}\frac{4}{h^2} - n_{x_1}(n_{x_1}^2-1)\,R_{nz}\frac{4}{h^2} = \\ \quad\quad = \frac{4}{3h^2}\left\{\frac{\partial a_{T1s}}{\partial s}\frac{I_4}{\rho} + \frac{\partial a_{T3s}}{\partial s}\frac{I_6}{\rho}\right\} + c_{T0}\frac{I_0}{\rho}\end{array} \tag{9b}$$

Taking into account the Eqs. (9a) and (9b), the Variational Equation remains just with the terms,

$$\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l} n_{x_1}\left(n_{x_1}-n_{x_2}\right)\left(n_{x_1}+n_{x_2}\right)^2 M_{nn} + 2n_{x_1}^2 n_{x_2}\left(n_{x_2}-n_{x_1}\right) M_{ss} + \\ \quad + \left(3n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 - n_{x_1}^4 + 3n_{x_1}^2 n_{x_2}^2\right) M_{ns} + \\ + \dfrac{4}{3h^2}\left(\begin{array}{l} -n_{x_1}\left(n_{x_1}+n_{x_2}\right)^2\left(n_{x_1}-n_{x_2}\right) P_{nn} + 2n_{x_1}^2 n_{x_2}\left(n_{x_1}-n_{x_2}\right) P_{ss} - \\ \quad - \left(n_{x_1}^4 + n_{x_1} n_{x_2}^3 - 3n_{x_1}^2 n_{x_2}^2 - 3n_{x_1}^3 n_{x_2}\right) P_{ns} \end{array}\right) - T_{\delta\varphi_n} \end{array}\right\}\delta\phi_n \, ds\, dt \, +$$

$$+\int_{t_1}^{t_2}\oint_{\Gamma}\left[\begin{array}{l} n_{x_2}\left(n_{x_1}-n_{x_2}\right)\left(n_{x_1}+n_{x_2}\right)^2 M_{nn} + 2n_{x_2}^2 n_{x_1}\left(n_{x_2}+n_{x_1}\right) M_{ss} + \\ \quad + \left(3n_{x_1}^2 n_{x_2}^2 - n_{x_2}^4 + n_{x_1}^3 n_{x_2} - 3n_{x_1} n_{x_2}^3\right) M_{ns} - \\ - \dfrac{4}{3h^2}\left(\begin{array}{l} n_{x_2}\left(n_{x_1}-n_{x_2}\right)^2\left(n_{x_1}+n_{x_2}\right) P_{nn} + 2n_{x_2}^2 n_{x_1}\left(n_{x_2}+n_{x_1}\right) P_{ss} - \\ \quad - \left(n_{x_2}^4 - n_{x_1}^3 n_{x_2} - 3n_{x_1}^2 n_{x_2}^2 + 3n_{x_1} n_{x_2}^3\right) P_{ns} \end{array}\right) - T_{\delta\varphi_s} \end{array}\right]\delta\phi_s \, ds\, dt = 0$$

Removing the restriction $\delta\phi_n = 0$, assuming the arbitrariness of the function $\delta\phi_n$ and of the interval $[t_1, t_2]$ and taking into account the restriction $\delta\phi_s = 0$, we have

$$\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l} n_{x_1}\left(n_{x_1}-n_{x_2}\right)\left(n_{x_1}+n_{x_2}\right)^2 M_{nn} + 2n_{x_1}^2 n_{x_2}\left(n_{x_1}-n_{x_2}\right) M_{ss} + \\ \quad + \left(3n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 - n_{x_1}^4 + 3n_{x_1}^2 n_{x_2}^2\right) M_{ns} + \\ + \dfrac{4}{3h^2}\left(\begin{array}{l} -n_{x_1}\left(n_{x_1}+n_{x_2}\right)^2\left(n_{x_1}-n_{x_2}\right) P_{nn} + 2n_{x_1}^2 n_{x_2}\left(n_{x_1}-n_{x_2}\right) P_{ss} - \\ \quad - \left(n_{x_1}^4 + n_{x_1} n_{x_2}^3 - 3n_{x_1}^2 n_{x_2}^2 - 3n_{x_1}^3 n_{x_2}\right) P_{ns} \end{array}\right) - T_{\delta\varphi_n} \end{array}\right\}\delta\phi_n(s;t)\, ds\, dt = 0 \qquad \forall \;\; \delta\phi_n(s;t)$$

and using the arbitrariness of the variation $\delta\phi_n$ on the $\Gamma\times[t_1, t_2]$, we find the third natural boundary condition as for the flexural response of the problem of TSDT,

$$\begin{array}{l} n_{x_1}\left(n_{x_1}-n_{x_2}\right)\left(n_{x_1}+n_{x_2}\right)^2 M_{nn} + 2n_{x_1}^2 n_{x_2}\left(n_{x_2}-n_{x_1}\right) M_{ss} + \\ \quad + \left(3n_{x_1}^3 n_{x_2} - n_{x_1} n_{x_2}^3 - n_{x_1}^4 + 3n_{x_1}^2 n_{x_2}^2\right) M_{ns} + \\ + \dfrac{4}{3h^2}\left(\begin{array}{l} -n_{x_1}\left(n_{x_1}+n_{x_2}\right)^2\left(n_{x_1}-n_{x_2}\right) P_{nn} + 2n_{x_1}^2 n_{x_2}\left(n_{x_1}-n_{x_2}\right) P_{ss} - \\ \quad - \left(n_{x_1}^4 + n_{x_1} n_{x_2}^3 - 3n_{x_1}^2 n_{x_2}^2 - 3n_{x_1}^3 n_{x_2}\right) P_{ns} \end{array}\right) = T_{\delta\varphi_n} \end{array} \qquad (9c)$$

where $T_{\delta\varphi_n}$ defined on the section 5.2.4 and gives the total data terms of the above boundary condition.

In conclusion, taking into account the Eqs. (9a), (9b) and (9c), the Variational Equation remains just with the following term

$$\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l} n_{x_2}\; n_{x_1} - n_{x_2}\quad n_{x_1} + n_{x_2}\quad ^{2} M_{nn} + 2n_{x_2}^2\, n_{x_1}\quad n_{x_2} + n_{x_1}\quad M_{ss} + \\ + \; 3n_{x_1}^2\, n_{x_2}^2 - n_{x_2}^4 + n_{x_1}^3\, n_{x_2} - 3n_{x_1}\, n_{x_2}^3\quad M_{ns} - \\ -\dfrac{4}{3h^2}\left(\begin{array}{l} n_{x_2}\; n_{x_1} - n_{x_2}\quad ^{2}\; n_{x_1} + n_{x_2}\quad P_{nn} + 2n_{x_2}^2\, n_{x_1}\quad n_{x_2} + n_{x_1}\quad P_{ss} - \\ -\; n_{x_2}^4 - n_{x_1}^3\, n_{x_2} - 3n_{x_1}^2\, n_{x_2}^2 + 3n_{x_1}\, n_{x_2}^3\quad P_{ns} \end{array}\right) - T_{\delta\varphi_s} \end{array}\right\}\delta\phi_s \; ds\, dt = 0$$

Removing the restriction $\delta\phi_s = 0$, assuming the arbitrariness of the variation $\delta\phi_s$ and of the interval $[t_1, t_2]$, we derive from the above (last) from of the variational equation

$$\int_{t_1}^{t_2}\oint_{\Gamma}\left\{\begin{array}{l} n_{x_2}\; n_{x_1} - n_{x_2}\quad n_{x_1} + n_{x_2}\quad ^{2} M_{nn} + 2n_{x_2}^2\, n_{x_1}\quad n_{x_2} + n_{x_1}\quad M_{ss} + \\ + \; 3n_{x_1}^2\, n_{x_2}^2 - n_{x_2}^4 + n_{x_1}^3\, n_{x_2} - 3n_{x_1}\, n_{x_2}^3\quad M_{ns} - \\ -\dfrac{4}{3h^2}\left(\begin{array}{l} n_{x_2}\; n_{x_1} - n_{x_2}\quad ^{2}\; n_{x_1} + n_{x_2}\quad P_{nn} + 2n_{x_2}^2\, n_{x_1}\quad n_{x_2} + n_{x_1}\quad P_{ss} - \\ -\; n_{x_2}^4 - n_{x_1}^3\, n_{x_2} - 3n_{x_1}^2\, n_{x_2}^2 + 3n_{x_1}\, n_{x_2}^3\quad P_{ns} \end{array}\right) - T_{\delta\varphi_s} \end{array}\right\}\delta\phi_s \; ds\, dt = 0$$

$$\forall \quad \delta\phi_s$$

and using the arbitrariness of the variation $\delta\phi_s$ on the $\Gamma \times [t_1, t_2]$, we find the <u>fourth natural boundary condition</u> as for the flexural response of the problem of TSDT,

$$\begin{array}{l} n_{x_2}\; n_{x_1} - n_{x_2}\quad n_{x_1} + n_{x_2}\quad ^{2} M_{nn} + 2n_{x_2}^2\, n_{x_1}\quad n_{x_2} + n_{x_1}\quad M_{ss} + \\ \quad + \; 3n_{x_1}^2\, n_{x_2}^2 - n_{x_2}^4 + n_{x_1}^3\, n_{x_2} - 3n_{x_1}\, n_{x_2}^3\quad M_{ns} - \\ -\dfrac{4}{3h^2}\left(\begin{array}{l} n_{x_2}\; n_{x_1} - n_{x_2}\quad ^{2}\; n_{x_1} + n_{x_2}\quad P_{nn} + 2n_{x_2}^2\, n_{x_1}\quad n_{x_2} + n_{x_1}\quad P_{ss} - \\ -\; n_{x_2}^4 - n_{x_1}^3\, n_{x_2} - 3n_{x_1}^2\, n_{x_2}^2 + 3n_{x_1}\, n_{x_2}^3\quad P_{ns} \end{array}\right) = T_{\delta\varphi_s} \end{array} \tag{9d}$$

where $T_{\delta\varphi_s}$ defined on the section 5.2.4 and gives the total data terms of the above boundary condition.

To compare easier the form of each natural boundary condition of the flexural response of model TSDT, we repeat them below

$$(n_{x_1}^2 - n_{x_2}^2)\, P_{nn} + 2\, n_{x_2} n_{x_1} (n_{x_1}^2 - n_{x_2}^2)\, P_{ns} = a_{T1n}(n_{x_1}^2 - n_{x_2}^2)\frac{I_4}{\rho} + a_{T3n}\frac{I_6}{\rho} \qquad \textbf{(9a)}$$

$$\left(I_6\frac{16}{9h^4}-\frac{4}{3h^2}I_4\right)\{n_{x_1}(n_{x_1}-n_{x_2})\ddot{\phi}_n+n_{x_2}(n_{x_1}+n_{x_2})\ddot{\phi}_s\}+$$
$$+I_6\frac{16}{9h^4}\left\{(n_{x_1}^2-n_{x_2}^2)\frac{\partial\ddot{w}_0}{\partial n}+2n_{x_2}n_{x_1}\frac{\partial\ddot{w}_0}{\partial s}\right\}+$$
$$+\frac{4}{3h^2}\{(n_{x_1}^2-n_{x_2}^2)P_{nn,n}+2(n_{x_1}^2-n_{x_2}^2)P_{ns,s}+2n_{x_2}n_{x_1}P_{ns,n}+4n_{x_2}n_{x_1}P_{ss,s}\}+$$
$$+\frac{4}{3h^2}\frac{\partial}{\partial s}\{2n_{x_2}n_{x_1}\}P_{ss}+\frac{4}{3h^2}\frac{\partial}{\partial s}\{n_{x_1}^2-n_{x_2}^2\}P_{ns}+ \qquad \textbf{(9b)}$$
$$+n_{x_2}(n_{x_1}^2-1)Q_{sz}-n_{x_2}^2Q_{ss}+n_{x_2}^2R_{ss}\frac{4}{h^2}-n_{x_2}(n_{x_1}^2-1)R_{sz}\frac{4}{h^2}+$$
$$+n_{x_1}(n_{x_1}^2-1)Q_{nz}-n_{x_1}^2Q_{nn}+n_{x_1}^2R_{nn}\frac{4}{h^2}-n_{x_1}(n_{x_1}^2-1)R_{nz}\frac{4}{h^2}=$$
$$=\frac{4}{3h^2}\left\{\frac{\partial a_{T1s}}{\partial s}\frac{I_4}{\rho}+\frac{\partial a_{T3s}}{\partial s}\frac{I_6}{\rho}\right\}+c_{T0}\frac{I_0}{\rho}$$

$$n_{x_1}(n_{x_1}-n_{x_2})(n_{x_1}+n_{x_2})^2M_{nn}+2n_{x_1}^2n_{x_2}\;n_{x_2}-n_{x_1}\;M_{ss}+$$
$$+(3n_{x_1}^3n_{x_2}-n_{x_1}n_{x_2}^3-n_{x_1}^4+3n_{x_1}^2n_{x_2}^2)M_{ns}+ \qquad \textbf{(9c)}$$
$$+\frac{4}{3h^2}\begin{pmatrix}-n_{x_1}(n_{x_1}-n_{x_2})(n_{x_1}+n_{x_2})^2P_{nn}+2n_{x_1}^2n_{x_2}(n_{x_1}-n_{x_2})P_{ss}-\\ -(n_{x_1}^4+n_{x_1}n_{x_2}^3-3n_{x_1}^2n_{x_2}^2-3n_{x_1}^3n_{x_2})P_{ns}\end{pmatrix}=T_{\delta\varphi_n}$$

$$n_{x_2}(n_{x_1}-n_{x_2})(n_{x_1}+n_{x_2})^2M_{nn}+2n_{x_2}^2n_{x_1}(n_{x_2}+n_{x_1})M_{ss}+$$
$$+(3n_{x_1}^2n_{x_2}^2-n_{x_2}^4+n_{x_1}^3n_{x_2}-3n_{x_1}n_{x_2}^3)M_{ns}- \qquad \textbf{(9d)}$$
$$-\frac{4}{3h^2}\begin{pmatrix}n_{x_2}(n_{x_1}-n_{x_2})^2(n_{x_1}+n_{x_2})P_{nn}+2n_{x_2}^2n_{x_1}(n_{x_1}+n_{x_2})P_{ss}-\\ -(n_{x_2}^4-n_{x_1}^3n_{x_2}-3n_{x_1}^2n_{x_2}^2+3n_{x_1}n_{x_2}^3)P_{ns}\end{pmatrix}=T_{\delta\varphi_s}$$

6. Equations of motion of the TSDT in terms of displacements

As we have aforementioned on the closure of the section 4.3, it is time to use the relations of the stress resultants (section 4.3) and to substitute into them the relations of stress in terms of displacements [Eqs. (1`) - (5`) of the section 4.2.1 or Eqs. (1) - (5) of the section 4.2.2], in order to express the stress resultants in terms of the displacement field of the problem of TSDT. Thus, we get the following relations as seems on the sections 6.1 and 6.2 in case of an orthotropic but in-plane anisotropic and in case of an orthotropic but in-plane isotropic material respectively.

6.1. Equations of motion of the TSDT in terms of displacements for an orthotropic, in-plane anisotropic material

The thickness-integrated forces of the aforementioned relations of the section 4.3 are converted to the below, due to the Eqs. (1`) – (5`) of the section 4.2.1,

$$\begin{Bmatrix} N_1 \\ M_1 \\ P_1 \end{Bmatrix} = \begin{Bmatrix} N_{11} \\ M_{11} \\ P_{11} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{11} \begin{Bmatrix} 1 \\ x_3 \\ x_3^3 \end{Bmatrix} dx_3 =$$

$$= \int_{-h/2}^{h/2} \begin{Bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} \left(\frac{\partial u_0}{\partial x_1} + x_3 \left(1 - \frac{4x_3^2}{3h^2}\right) \frac{\partial \phi_x}{\partial x_1} - x_3^3 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_1} \right) + \\ + \frac{\nu_{21} E_1}{1-\nu_{12}\nu_{21}} \left(\frac{\partial v_0}{\partial x_2} + x_3 \left(1 - \frac{4x_3^2}{3h^2}\right) \frac{\partial \phi_y}{\partial x_2} - x_3^3 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_2} \right) \end{Bmatrix} \begin{Bmatrix} 1 \\ x_3 \\ x_3^3 \end{Bmatrix} dx_3 \Leftrightarrow$$

$$N_{11} = \int_{-h/2}^{h/2} \begin{Bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} \left(\frac{\partial u_0}{\partial x_1} + \left(x_3 - \frac{4x_3^3}{3h^2}\right) \frac{\partial \phi_x}{\partial x_1} - x_3^3 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_1} \right) + \\ + \frac{\nu_{21} E_1}{1-\nu_{12}\nu_{21}} \left(\frac{\partial v_0}{\partial x_2} + \left(x_3 - \frac{4x_3^3}{3h^2}\right) \frac{\partial \phi_y}{\partial x_2} - x_3^3 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_2} \right) \end{Bmatrix} dx_3 =$$

$$= \frac{E_1}{1-\nu_{12}\nu_{21}} \left[\frac{\partial u_0}{\partial x_1} \frac{I_0}{\rho} + \left(\frac{I_1}{\rho} - \frac{4 I_3}{3\rho h^2} \right) \frac{\partial \phi_x}{\partial x_1} - \frac{I_3}{\rho} \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_1} \right] +$$

$$+ \frac{\nu_{21} E_1}{1-\nu_{12}\nu_{21}} \left[\frac{\partial v_0}{\partial x_2} \frac{I_0}{\rho} + \left(\frac{I_1}{\rho} - \frac{4 I_3}{3\rho h^2} \right) \frac{\partial \phi_y}{\partial x_2} - \frac{I_3}{\rho} \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_2} \right] =$$

$$= \frac{E_1}{1-\nu_{12}\nu_{21}} \frac{\partial u_0}{\partial x_1} \frac{I_0}{\rho} + \frac{\nu_{21} E_1}{1-\nu_{12}\nu_{21}} \frac{\partial v_0}{\partial x_2} \frac{I_0}{\rho} \tag{1}$$

$$
\begin{aligned}
M_{11} &= \int_{-h/2}^{h/2} \left\{ \begin{aligned} &\frac{E_1}{1-\nu_{12}\nu_{21}} \left(\frac{\partial u_0}{\partial x_1} x_3 + \left(x_3^2 - \frac{4x_3^4}{3h^2} \right) \frac{\partial \phi_x}{\partial x_1} - x_3^4 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_1} \right) + \\ &+ \frac{\nu_{21} E_1}{1-\nu_{12}\nu_{21}} \left(\frac{\partial v_0}{\partial x_2} x_3 + \left(x_3^2 - \frac{4x_3^4}{3h^2} \right) \frac{\partial \phi_y}{\partial x_2} - x_3^4 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_2} \right) \end{aligned} \right\} dx_3 = \\
&= \frac{E_1}{1-\nu_{12}\nu_{21}} \left[\frac{\partial u_0}{\partial x_1} \frac{\cancel{I_1}}{\rho} + \left(\frac{I_2}{\rho} - \frac{4 I_4}{3\rho h^2} \right) \frac{\partial \phi_x}{\partial x_1} - \frac{I_4}{\rho} \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_1} \right] + \\
&+ \frac{\nu_{21} E_1}{1-\nu_{12}\nu_{21}} \left[\frac{\partial v_0}{\partial x_2} \frac{\cancel{I_1}}{\rho} + \left(\frac{I_2}{\rho} - \frac{4 I_4}{3\rho h^2} \right) \frac{\partial \phi_y}{\partial x_2} - \frac{I_4}{\rho} \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_2} \right] = \\
&= \frac{E_1}{1-\nu_{12}\nu_{21}} \left[\left(\frac{I_2}{\rho} - \frac{4 I_4}{3\rho h^2} \right) \frac{\partial \phi_x}{\partial x_1} - \frac{I_4}{\rho} \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_1} \right] + \\
&+ \frac{\nu_{21} E_1}{1-\nu_{12}\nu_{21}} \left[\left(\frac{I_2}{\rho} - \frac{4 I_4}{3\rho h^2} \right) \frac{\partial \phi_y}{\partial x_2} - \frac{I_4}{\rho} \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_2} \right]
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
P_{11} &= \int_{-h/2}^{h/2} \left\{ \begin{aligned} &\frac{E_1}{1-\nu_{12}\nu_{21}} \left(\frac{\partial u_0}{\partial x_1} x_3^3 + \left(x_3^4 - \frac{4x_3^2}{3h^2} \right) \frac{\partial \phi_x}{\partial x_1} - x_3^6 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_1} \right) + \\ &+ \frac{\nu_{21} E_1}{1-\nu_{12}\nu_{21}} \left(\frac{\partial v_0}{\partial x_2} x_3^3 + \left(x_3^4 - \frac{4x_3^2}{3h^2} \right) \frac{\partial \phi_y}{\partial x_2} - x_3^6 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_2} \right) \end{aligned} \right\} dx_3 = \\
&= \frac{E_1}{1-\nu_{12}\nu_{21}} \left[\frac{\partial u_0}{\partial x_1} \frac{\cancel{I_3}}{\rho} + \left(\frac{I_4}{\rho} - \frac{4 I_2}{3\rho h^2} \right) \frac{\partial \phi_x}{\partial x_1} - \frac{I_6}{\rho} \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_1} \right] + \\
&+ \frac{\nu_{21} E_1}{1-\nu_{12}\nu_{21}} \left[\frac{\partial v_0}{\partial x_2} \frac{\cancel{I_3}}{\rho} + \left(\frac{I_4}{\rho} - \frac{4 I_2}{3\rho h^2} \right) \frac{\partial \phi_y}{\partial x_2} - \frac{I_6}{\rho} \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_2} \right] = \\
&= \frac{E_1}{1-\nu_{12}\nu_{21}} \left[\left(\frac{I_4}{\rho} - \frac{4 I_2}{3\rho h^2} \right) \frac{\partial \phi_x}{\partial x_1} - \frac{I_6}{\rho} \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_1} \right] + \\
&+ \frac{\nu_{21} E_1}{1-\nu_{12}\nu_{21}} \left[\left(\frac{I_4}{\rho} - \frac{4 I_2}{3\rho h^2} \right) \frac{\partial \phi_y}{\partial x_2} - \frac{I_6}{\rho} \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_2} \right]
\end{aligned}
\tag{3}
$$

Further,

$$\begin{Bmatrix} N_2 \\ M_2 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} N_{22} \\ M_{22} \\ P_{22} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{22} \begin{Bmatrix} 1 \\ x_3 \\ x_3^3 \end{Bmatrix} dx_3 =$$

$$= \int_{-h/2}^{h/2} \begin{bmatrix} \frac{\nu_{12} E_2}{1-\nu_{12}\nu_{21}} \left(\frac{\partial u_0}{\partial x_1} + x_3 \left(1 - \frac{4x_3^2}{3h^2}\right) \frac{\partial \phi_x}{\partial x_1} - x_3^3 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_1} \right) + \\ + \frac{E_2}{1-\nu_{12}\nu_{21}} \left(\frac{\partial v_0}{\partial x_2} + x_3 \left(1 - \frac{4x_3^2}{3h^2}\right) \frac{\partial \phi_y}{\partial x_2} - x_3^3 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_2} \right) \end{bmatrix} \begin{Bmatrix} 1 \\ x_3 \\ x_3^3 \end{Bmatrix} dx_3 \Leftrightarrow$$

$$N_{22} = \int_{-h/2}^{h/2} \begin{Bmatrix} \frac{\nu_{12} E_2}{1-\nu_{12}\nu_{21}} \left(\frac{\partial u_0}{\partial x_1} + x_3 \left(1 - \frac{4x_3^2}{3h^2}\right) \frac{\partial \phi_x}{\partial x_1} - x_3^3 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_1} \right) + \\ + \frac{E_2}{1-\nu_{12}\nu_{21}} \left(\frac{\partial v_0}{\partial x_2} + x_3 \left(1 - \frac{4x_3^2}{3h^2}\right) \frac{\partial \phi_y}{\partial x_2} - x_3^3 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_2} \right) \end{Bmatrix} dx_3 =$$

$$= \frac{\nu_{12} E_2}{1-\nu_{12}\nu_{21}} \left[\frac{\partial u_0}{\partial x_1} \frac{I_0}{\rho} + \left(\frac{\cancel{I_1}}{\rho} - \frac{4\cancel{I_3}}{3\rho h^2} \right) \frac{\partial \phi_x}{\partial x_1} - \frac{\cancel{I_3}}{\rho} \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_1} \right] +$$

$$+ \frac{E_2}{1-\nu_{12}\nu_{21}} \left[\frac{\partial v_0}{\partial x_2} \frac{I_0}{\rho} + \left(\frac{\cancel{I_1}}{\rho} - \frac{4\cancel{I_3}}{3\rho h^2} \right) \frac{\partial \phi_y}{\partial x_2} - \frac{\cancel{I_3}}{\rho} \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_2} \right] =$$

$$= \frac{\nu_{12} E_2}{1-\nu_{12}\nu_{21}} \frac{\partial u_0}{\partial x_1} \frac{I_0}{\rho} + \frac{E_2}{1-\nu_{12}\nu_{21}} \frac{\partial v_0}{\partial x_2} \frac{I_0}{\rho} \tag{4}$$

$$M_{22} = \int_{-h/2}^{h/2} \begin{Bmatrix} \frac{\nu_{12} E_2}{1-\nu_{12}\nu_{21}} \left(\frac{\partial u_0}{\partial x_1} x_3 + \left(x_3^2 - \frac{4x_3^4}{3h^2}\right) \frac{\partial \phi_x}{\partial x_1} - x_3^4 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_1} \right) + \\ + \frac{E_2}{1-\nu_{12}\nu_{21}} \left(\frac{\partial v_0}{\partial x_2} x_3 + \left(x_3^2 - \frac{4x_3^4}{3h^2}\right) \frac{\partial \phi_y}{\partial x_2} - x_3^4 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_2} \right) \end{Bmatrix} dx_3 =$$

$$= \frac{\nu_{12} E_2}{1-\nu_{12}\nu_{21}} \left(\frac{\partial u_0}{\partial x_1} \frac{\cancel{I_1}}{\rho} + \left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2} \right) \frac{\partial \phi_x}{\partial x_1} - \frac{I_4}{\rho} \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_1} \right) +$$

$$+ \frac{E_2}{1-\nu_{12}\nu_{21}} \left(\frac{\partial v_0}{\partial x_2} \frac{\cancel{I_1}}{\rho} + \left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2} \right) \frac{\partial \phi_y}{\partial x_2} - \frac{I_4}{\rho} \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 x_2} \right) =$$

$$= \frac{v_{12}E_2}{1-v_{12}v_{21}}\left[\left(\frac{I_2}{\rho}-\frac{4I_4}{3\rho h^2}\right)\frac{\partial\phi_x}{\partial x_1}-\frac{I_4}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_1}\right]+ \\ +\frac{E_2}{1-v_{12}v_{21}}\left[\left(\frac{I_2}{\rho}-\frac{4I_4}{3\rho h^2}\right)\frac{\partial\phi_y}{\partial x_2}-\frac{I_4}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_2}\right] \quad (5)$$

$$P_{22} = \int_{-h/2}^{h/2}\left\{\begin{matrix}\frac{v_{12}E_2}{1-v_{12}v_{21}}\left(\frac{\partial u_0}{\partial x_1}x_3^3+\left(x_3^4-\frac{4x_3^6}{3h^2}\right)\frac{\partial\phi_x}{\partial x_1}-x_3^6\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_1}\right)+ \\ +\frac{E_2}{1-v_{12}v_{21}}\left(\frac{\partial v_0}{\partial x_2}x_3^3+\left(x_3^4-\frac{4x_3^6}{3h^2}\right)\frac{\partial\phi_y}{\partial x_2}-x_3^6\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_2}\right)\end{matrix}\right\}dx_3 =$$

$$= \frac{v_{12}E_2}{1-v_{12}v_{21}}\left(\frac{\partial u_0}{\partial x_1}\frac{\cancel{I_3}}{\rho}+\left(\frac{I_4}{\rho}-\frac{4I_6}{3\rho h^2}\right)\frac{\partial\phi_x}{\partial x_1}-\frac{I_6}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_1}\right)+ \\ +\frac{E_2}{1-v_{12}v_{21}}\left(\frac{\partial v_0}{\partial x_2}\frac{\cancel{I_3}}{\rho}+\left(\frac{I_4}{\rho}-\frac{4I_6}{3\rho h^2}\right)\frac{\partial\phi_y}{\partial x_2}-\frac{I_6}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_2}\right)=$$

$$= \frac{v_{12}E_2}{1-v_{12}v_{21}}\left[\left(\frac{I_4}{\rho}-\frac{4I_6}{3\rho h^2}\right)\frac{\partial\phi_x}{\partial x_1}-\frac{I_6}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_1}\right]+ \\ +\frac{E_2}{1-v_{12}v_{21}}\left[\left(\frac{I_4}{\rho}-\frac{4I_6}{3\rho h^2}\right)\frac{\partial\phi_y}{\partial x_2}-\frac{I_6}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_2}\right] \quad (6)$$

In addition,

$$\begin{Bmatrix}N_6\\M_6\\P_6\end{Bmatrix}=\begin{Bmatrix}N_{12}\\M_{12}\\P_{12}\end{Bmatrix}=\int_{-h/2}^{h/2}\sigma_{12}\begin{Bmatrix}1\\x_3\\x_3^3\end{Bmatrix}dx_3 =$$

$$= \int_{-h/2}^{h/2}G_6\left\{\begin{matrix}\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}+x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial\phi_x}{\partial x_2}+ \\ +x_3\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial\phi_y}{\partial x_1}-\frac{8x_3^3}{3h^2}\frac{\partial^2 w_0}{\partial x_2\partial x_1}\end{matrix}\right\}\begin{Bmatrix}1\\x_3\\x_3^3\end{Bmatrix}dx_3 \Leftrightarrow$$

$$N_{12} = \int_{-h/2}^{h/2}G_6\left\{\begin{matrix}\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}+\left(x_3-\frac{4x_3^3}{3h^2}\right)\left(\frac{\partial\phi_x}{\partial x_2}+\frac{\partial\phi_y}{\partial x_1}\right)- \\ -\frac{8x_3^3}{3h^2}\frac{\partial^2 w_0}{\partial x_2\partial x_1}\end{matrix}\right\}dx_3 =$$

$$= G_6 \left\{ \begin{array}{c} \left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1} \right) \frac{I_0}{\rho} + \left(\frac{\cancel{I_1}}{\rho} - \frac{4\cancel{I_3}}{3\rho h^2} \right) \left(\frac{\partial \phi_x}{\partial x_2} + \frac{\partial \phi_y}{\partial x_1} \right) - \\ - \frac{8\cancel{I_3}}{3\rho h^2} \frac{\partial^2 w_0}{\partial x_2 \partial x_1} \end{array} \right\} = G_6 \left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1} \right) \frac{I_0}{\rho} \tag{7}$$

$$M_{12} = \int_{-h/2}^{h/2} G_6 \left\{ \begin{array}{c} \left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1} \right) x_3 + \left(x_3^2 - \frac{4x_3^4}{3h^2} \right) \left(\frac{\partial \phi_x}{\partial x_2} + \frac{\partial \phi_y}{\partial x_1} \right) - \\ - \frac{8x_3^4}{3h^2} \frac{\partial^2 w_0}{\partial x_2 \partial x_1} \end{array} \right\} dx_3 =$$

$$= G_6 \left\{ \begin{array}{c} \left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1} \right) \frac{\cancel{I_1}}{\rho} + \left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2} \right) \left(\frac{\partial \phi_x}{\partial x_2} + \frac{\partial \phi_y}{\partial x_1} \right) - \\ - \frac{8I_4}{3\rho h^2} \frac{\partial^2 w_0}{\partial x_2 \partial x_1} \end{array} \right\} =$$

$$= G_6 \left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2} \right) \left(\frac{\partial \phi_x}{\partial x_2} + \frac{\partial \phi_y}{\partial x_1} \right) - \frac{8I_4 G_6}{3\rho h^2} \frac{\partial^2 w_0}{\partial x_2 \partial x_1} \tag{8}$$

$$P_{12} = \int_{-h/2}^{h/2} G_6 \left\{ \begin{array}{c} \left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1} \right) x_3^3 + \left(x_3^4 - \frac{4x_3^6}{3h^2} \right) \left(\frac{\partial \phi_x}{\partial x_2} + \frac{\partial \phi_y}{\partial x_1} \right) - \\ - \frac{8x_3^6}{3h^2} \frac{\partial^2 w_0}{\partial x_2 \partial x_1} \end{array} \right\} dx_3 =$$

$$= G_6 \left\{ \begin{array}{c} \left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1} \right) \frac{\cancel{I_3}}{\rho} + \left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2} \right) \left(\frac{\partial \phi_x}{\partial x_2} + \frac{\partial \phi_y}{\partial x_1} \right) - \\ - \frac{8I_6}{3\rho h^2} \frac{\partial^2 w_0}{\partial x_2 \partial x_1} \end{array} \right\} =$$

$$= G_6 \left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2} \right) \left(\frac{\partial \phi_x}{\partial x_2} + \frac{\partial \phi_y}{\partial x_1} \right) - \frac{8G_6 I_6}{3\rho h^2} \frac{\partial^2 w_0}{\partial x_2 \partial x_1} \tag{9}$$

And also,

$$\begin{Bmatrix} Q_4 \\ R_4 \end{Bmatrix} = \begin{Bmatrix} Q_{23} \\ R_{23} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{23} \begin{Bmatrix} 1 \\ x_3^2 \end{Bmatrix} dx_3 = \int_{-h/2}^{h/2} G_4 \left(1 - \frac{4x_3^2}{h^2} \right) \left(\phi_y + \frac{\partial w_0}{\partial x_2} \right) \begin{Bmatrix} 1 \\ x_3^2 \end{Bmatrix} dx_3 \Leftrightarrow$$

$$Q_4 = \int_{-h/2}^{h/2} G_4 \left(1 - \frac{4x_3^2}{h^2} \right) \left(\phi_y + \frac{\partial w_0}{\partial x_2} \right) dx_3 = G_4 \left(\frac{I_0}{\rho} - \frac{4I_2}{\rho h^2} \right) \left(\phi_y + \frac{\partial w_0}{\partial x_2} \right) \tag{10}$$

$$R_4 = \int_{-h/2}^{h/2} G_4\left(x_3^2 - \frac{4x_3^4}{h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right)dx_3 = G_4\left(I_2 - \frac{4I_4}{\rho h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right) \quad (11)$$

Finally as for the thickness-integrated quantities, we get additionally

$$\begin{Bmatrix} Q_5 \\ R_5 \end{Bmatrix} = \begin{Bmatrix} Q_{13} \\ R_{13} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{13}\begin{Bmatrix} 1 \\ x_3^2 \end{Bmatrix}dx_3 = \int_{-h/2}^{h/2}\left\{G_5\left(1 - \frac{4x_3^2}{h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right)\right\}\begin{Bmatrix} 1 \\ x_3^2 \end{Bmatrix}dx_3 \Leftrightarrow$$

$$Q_5 = \int_{-h/2}^{h/2} G_5\left(1 - \frac{4x_3^2}{h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right)dx_3 = G_5\left(\frac{I_0}{\rho} - \frac{4I_2}{\rho h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right) \quad (12)$$

$$R_5 = \int_{-h/2}^{h/2} G_5\left(x_3^2 - \frac{4x_3^4}{h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right)dx_3 = G_5\left(\frac{I_2}{\rho} - \frac{4I_4}{\rho h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right) \quad (13)$$

Now, we have already prepare the path to express the equations of motion of the plate exclusively in terms of the displacement field of the problem of TSDT.

Substituting the Eqs. (1) and (7) into the Eq. **(3)** of the section 4.6 and by substituting

$$\ddot{u}_0 I_0 - \frac{\partial N_{11}}{\partial x_1} - \frac{\partial N_{12}}{\partial x_2} = 0 \Rightarrow \ddot{u}_0 I_0 - \frac{\partial}{\partial x_1}\left[\frac{E_1}{1-\nu_{12}\nu_{21}}\frac{\partial u_0}{\partial x_1}\frac{I_0}{\rho} + \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}\frac{\partial v_0}{\partial x_2}\frac{I_0}{\rho}\right] -$$

$$- \frac{\partial}{\partial x_2}\left[G_6\left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1}\right)\frac{I_0}{\rho}\right] = 0 \Rightarrow$$

$$\ddot{u}_0 I_0 - \frac{E_1}{1-\nu_{12}\nu_{21}}\frac{\partial^2 u_0}{\partial^2 x_1}\frac{I_0}{\rho} - \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}\frac{\partial^2 v_0}{\partial x_1 \partial x_2}\frac{I_0}{\rho} - G_6\left(\frac{\partial^2 u_0}{\partial^2 x_2} + \frac{\partial^2 v_0}{\partial x_2 \partial x_1}\right)\frac{I_0}{\rho} = 0 \Rightarrow$$

$$\rho\ddot{u}_0 - \frac{E_1}{1-\nu_{12}\nu_{21}}\frac{\partial^2 u_0}{\partial^2 x_1} - \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}\frac{\partial^2 v_0}{\partial x_1 \partial x_2} - G_6\left(\frac{\partial^2 u_0}{\partial^2 x_2} + \frac{\partial^2 v_0}{\partial x_2 \partial x_1}\right) = 0 \quad (14a)$$

Substituting the Eqs. (4) and (7) into the Eq. **(4)** of the section 4.6,

$$I_0 \ddot{v}_0 - \frac{\partial N_{22}}{\partial x_2} - \frac{\partial N_{12}}{\partial x_1} = 0 \Rightarrow I_0 \ddot{v}_0 - \frac{\partial}{\partial x_2}\left[\frac{v_{12} E_2}{1-v_{12} v_{21}} \frac{\partial u_0}{\partial x_1} \frac{I_0}{\rho} + \frac{E_2}{1-v_{12} v_{21}} \frac{\partial v_0}{\partial x_2} \frac{I_0}{\rho}\right] -$$

$$- \frac{\partial}{\partial x_1}\left[G_6 \left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1}\right) \frac{I_0}{\rho}\right] = 0 \Rightarrow$$

$$I_0 \ddot{v}_0 - \frac{v_{12} E_2}{1-v_{12} v_{21}} \frac{\partial^2 u_0}{\partial x_2 \partial x_1} \frac{I_0}{\rho} - \frac{E_2}{1-v_{12} v_{21}} \frac{\partial^2 v_0}{\partial^2 x_2} \frac{I_0}{\rho} - G_6 \left(\frac{\partial^2 u_0}{\partial x_1 \partial x_2} + \frac{\partial^2 v_0}{\partial^2 x_1}\right) \frac{I_0}{\rho} = 0 \Rightarrow$$

$$\rho \ddot{v}_0 - \frac{v_{12} E_2}{1-v_{12} v_{21}} \frac{\partial^2 u_0}{\partial x_2 \partial x_1} - \frac{E_2}{1-v_{12} v_{21}} \frac{\partial^2 v_0}{\partial^2 x_2} - G_6 \left(\frac{\partial^2 u_0}{\partial x_1 \partial x_2} + \frac{\partial^2 v_0}{\partial^2 x_1}\right) = 0 \qquad (14b)$$

Further, we replace the appropriate terms of the Eq. (5) of the section 4.6 with the Eqs. (3), (6), (9), (10), (11), (12) and (13),

$$I_0 \ddot{w}_0 - \frac{4}{3h^2}\left(I_6 \frac{4}{3h^2} - I_4\right)\left(\frac{\partial \ddot{\phi}_y}{\partial x_2} + \frac{\partial \ddot{\phi}_x}{\partial x_1}\right) - I_6 \frac{16}{9h^4}\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2} + \frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}\right) -$$

$$- \frac{\partial Q_{23}}{\partial x_2} + \frac{\partial R_{23}}{\partial x_2} \frac{4}{h^2} - \frac{\partial Q_{13}}{\partial x_1} + \frac{\partial R_{13}}{\partial x_1} \frac{4}{h^2} - \frac{4}{3h^2}\left(\frac{\partial^2 P_{11}}{\partial^2 x_1} + \frac{2\,\partial^2 P_{12}}{\partial x_2 \partial x_1} + \frac{\partial^2 P_{22}}{\partial^2 x_2}\right) = q \Rightarrow$$

$$I_0 \ddot{w}_0 - \frac{4}{3h^2}\left(I_6 \frac{4}{3h^2} - I_4\right)\left(\frac{\partial \ddot{\phi}_y}{\partial x_2} + \frac{\partial \ddot{\phi}_x}{\partial x_1}\right) - I_6 \frac{16}{9h^4}\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_2} + \frac{\partial^2 \ddot{w}_0}{\partial^2 x_1}\right) -$$

$$- \frac{\partial}{\partial x_2}\left[G_4 \left(\frac{I_0}{\rho} - \frac{4 I_2}{\rho h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right)\right] + \frac{4}{h^2}\frac{\partial}{\partial x_2}\left[G_4 \left(I_2 - \frac{4 I_4}{\rho h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right)\right] -$$

$$- \frac{\partial}{\partial x_1}\left[G_5 \left(\frac{I_0}{\rho} - \frac{4 I_2}{\rho h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right)\right] + \frac{4}{h^2}\frac{\partial}{\partial x_1}\left[G_5 \left(\frac{I_2}{\rho} - \frac{4 I_4}{\rho h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right)\right] -$$

$$- \frac{4}{3h^2}\frac{\partial^2}{\partial^2 x_1}\left[\begin{array}{l} \frac{E_1}{1-v_{12} v_{21}}\left[\left(\frac{I_4}{\rho} - \frac{4 I_2}{3\rho h^2}\right)\frac{\partial \phi_x}{\partial x_1} - \frac{I_6}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_1}\right] + \\ + \frac{v_{21} E_1}{1-v_{12} v_{21}}\left[\left(\frac{I_4}{\rho} - \frac{4 I_2}{3\rho h^2}\right)\frac{\partial \phi_y}{\partial x_2} - \frac{I_6}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_2}\right] \end{array}\right] -$$

$$- \frac{4}{3h^2}\frac{2\,\partial^2}{\partial x_2 \partial x_1}\left[G_6 \left(\frac{I_4}{\rho} - \frac{4 I_6}{3\rho h^2}\right)\left(\frac{\partial \phi_x}{\partial x_2} + \frac{\partial \phi_y}{\partial x_1}\right) - \frac{8 G_6 I_6}{3\rho h^2}\frac{\partial^2 w_0}{\partial x_2 \partial x_1}\right] -$$

$$- \frac{4}{3h^2}\frac{\partial^2}{\partial^2 x_2}\left[\begin{array}{l} \frac{v_{12} E_2}{1-v_{12} v_{21}}\left[\left(\frac{I_4}{\rho} - \frac{4 I_6}{3\rho h^2}\right)\frac{\partial \phi_x}{\partial x_1} - \frac{I_6}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_1}\right] + \\ + \frac{E_2}{1-v_{12} v_{21}}\left[\left(\frac{I_4}{\rho} - \frac{4 I_6}{3\rho h^2}\right)\frac{\partial \phi_y}{\partial x_2} - \frac{I_6}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_2}\right] \end{array}\right] = q \Rightarrow$$

$$I_0\ddot{w}_0 - \frac{4}{3h^2}\left(I_6\frac{4}{3h^2}-I_4\right)\left(\frac{\partial\ddot{\phi}_y}{\partial x_2}+\frac{\partial\ddot{\phi}_x}{\partial x_1}\right) - I_6\frac{16}{9h^4}\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}\right) -$$

$$-\left[G_4\left(\frac{I_0}{\rho}-\frac{4I_2}{\rho h^2}\right)\left(\frac{\partial\phi_y}{\partial x_2}+\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right] + \frac{4}{h^2}\left[G_4\left(\frac{I_2}{\rho}-\frac{4I_4}{\rho h^2}\right)\left(\frac{\partial\phi_y}{\partial x_2}+\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right] -$$

$$-\left[G_5\left(\frac{I_0}{\rho}-\frac{4I_2}{\rho h^2}\right)\left(\frac{\partial\phi_x}{\partial x_1}+\frac{\partial^2 w_0}{\partial^2 x_1}\right)\right] + \frac{4}{h^2}\left[G_5\left(\frac{I_2}{\rho}-\frac{4I_4}{\rho h^2}\right)\left(\frac{\partial\phi_x}{\partial x_1}+\frac{\partial^2 w_0}{\partial^2 x_1}\right)\right] -$$

$$-\frac{4}{3h^2}\left[\begin{array}{l}\dfrac{E_1}{1-v_{12}v_{21}}\left[\left(\dfrac{I_4}{\rho}-\dfrac{4I_2}{3\rho h^2}\right)\dfrac{\partial^3\phi_x}{\partial^3 x_1}-\dfrac{I_6}{\rho}\dfrac{4}{3h^2}\dfrac{\partial^4 w_0}{\partial^4 x_1}\right]+\\ +\dfrac{v_{21}E_1}{1-v_{12}v_{21}}\left[\left(\dfrac{I_4}{\rho}-\dfrac{4I_2}{3\rho h^2}\right)\dfrac{\partial^3\phi_y}{\partial^2 x_1\partial x_2}-\dfrac{I_6}{\rho}\dfrac{4}{3h^2}\dfrac{\partial^4 w_0}{\partial^4 x_2}\right]\end{array}\right] -$$

$$-\frac{8}{3h^2}\left[G_6\left(\frac{I_4}{\rho}-\frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial^3\phi_x}{\partial^2 x_2\partial x_1}+\frac{\partial^3\phi_y}{\partial x_2\partial^2 x_1}\right)-\frac{8G_6I_6}{3\rho h^2}\frac{\partial^4 w_0}{\partial^2 x_2\partial^2 x_1}\right] -$$

$$-\frac{4}{3h^2}\left[\begin{array}{l}\dfrac{v_{12}E_2}{1-v_{12}v_{21}}\left[\left(\dfrac{I_4}{\rho}-\dfrac{4I_6}{3\rho h^2}\right)\dfrac{\partial^3\phi_x}{\partial^2 x_2\partial x_1}-\dfrac{I_6}{\rho}\dfrac{4}{3h^2}\dfrac{\partial^4 w_0}{\partial^2 x_2\partial^2 x_1}\right]+\\ +\dfrac{E_2}{1-v_{12}v_{21}}\left[\left(\dfrac{I_4}{\rho}-\dfrac{4I_6}{3\rho h^2}\right)\dfrac{\partial^3\phi_y}{\partial^3 x_2}-\dfrac{I_6}{\rho}\dfrac{4}{3h^2}\dfrac{\partial^4 w_0}{\partial^4 x_2}\right]\end{array}\right] = q \Rightarrow$$

$$I_0\ddot{w}_0 - \frac{4}{3h^2}\left(I_6\frac{4}{3h^2}-I_4\right)\left(\frac{\partial\ddot{\phi}_y}{\partial x_2}+\frac{\partial\ddot{\phi}_x}{\partial x_1}\right) - I_6\frac{16}{9h^4}\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_2}+\frac{\partial^2\ddot{w}_0}{\partial^2 x_1}\right) -$$

$$-\frac{G_4}{\rho}\left(I_0-\frac{4I_2}{h^2}\right)\left(\frac{\partial\phi_y}{\partial x_2}+\frac{\partial^2 w_0}{\partial^2 x_2}\right)+\frac{4G_4}{\rho h^2}\left(I_2-\frac{4I_4}{h^2}\right)\left(\frac{\partial\phi_y}{\partial x_2}+\frac{\partial^2 w_0}{\partial^2 x_2}\right) -$$

$$-\frac{G_5}{\rho}\left(I_0-\frac{4I_2}{h^2}\right)\left(\frac{\partial\phi_x}{\partial x_1}+\frac{\partial^2 w_0}{\partial^2 x_1}\right)+\frac{4G_5}{\rho h^2}\left(I_2-\frac{4I_4}{h^2}\right)\left(\frac{\partial\phi_x}{\partial x_1}+\frac{\partial^2 w_0}{\partial^2 x_1}\right) -$$

$$-\frac{4}{3\rho h^2}\frac{E_1}{1-v_{12}v_{21}}\left(\left(I_4-\frac{4I_6}{3h^2}\right)\frac{\partial^3\phi_x}{\partial^3 x_1}-I_6\frac{4}{3h^2}\frac{\partial^4 w_0}{\partial^4 x_1}\right) -$$

$$-\frac{4}{3\rho h^2}\frac{v_{21}E_1}{1-v_{12}v_{21}}\left(\left(I_4-\frac{4I_6}{3h^2}\right)\frac{\partial^3\phi_y}{\partial^2 x_1\partial x_2}-I_6\frac{4}{3h^2}\frac{\partial^4 w_0}{\partial^2 x_1\partial^2 x_2}\right) -$$

$$-\frac{8G_6}{3\rho h^2}\left[\left(I_4-\frac{4I_6}{3h^2}\right)\left(\frac{\partial^3\phi_x}{\partial^2 x_2\partial x_1}+\frac{\partial^3\phi_y}{\partial^2 x_1\partial x_2}\right)-\frac{8I_6}{3h^2}\frac{\partial^4 w_0}{\partial^2 x_2\partial^2 x_1}\right] -$$

$$-\frac{4}{3\rho h^2}\left[\begin{aligned}&\frac{v_{12}E_2}{1-v_{12}v_{21}}\left(\left(I_4-\frac{4I_6}{3h^2}\right)\frac{\partial^3\phi_x}{\partial^2x_2\partial x_1}-\frac{4I_6}{3h^2}\frac{\partial^4 w_0}{\partial^2x_2\partial^2x_1}\right)+\\&+\frac{E_2}{1-v_{12}v_{21}}\left(\left(I_4-\frac{4I_6}{3h^2}\right)\frac{\partial^3\phi_y}{\partial^3x_2}-\frac{4I_6}{3h^2}\frac{\partial^4 w_0}{\partial^4x_2}\right)\end{aligned}\right]=q \qquad (14c)$$

Subsequently, substituting the Eqs. (2), (8), (9), (12) and (13) into the Eq. **(6)** of the section 4.6, we derive the fourth equation of motion of the TSDT in terms of displacements,

$$\left(I_2-I_4\frac{8}{3h^2}+I_6\frac{16}{9h^4}\right)\ddot{\phi}_x+\frac{4}{3h^2}\left(I_6\frac{4}{3h^2}-I_4\right)\frac{\partial\ddot{w}_0}{\partial x_1}+$$

$$+Q_{13}-\frac{4}{h^2}R_{13}-\frac{\partial M_{11}}{\partial x_1}-\frac{\partial M_{12}}{\partial x_2}+\frac{\partial P_{12}}{\partial x_2}\frac{4}{3h^2}+\frac{\partial P_{11}}{\partial x_1}\frac{4}{3h^2}=0\Rightarrow$$

$$\left(I_2-I_4\frac{8}{3h^2}+I_6\frac{16}{9h^4}\right)\ddot{\phi}_x+\frac{4}{3h^2}\left(I_6\frac{4}{3h^2}-I_4\right)\frac{\partial\ddot{w}_0}{\partial x_1}+$$

$$+G_5\left(\frac{I_0}{\rho}-\frac{4I_2}{\rho h^2}\right)\left(\phi_x+\frac{\partial w_0}{\partial x_1}\right)-\frac{4}{h^2}G_5\left(\frac{I_2}{\rho}-\frac{4I_4}{\rho h^2}\right)\left(\phi_x+\frac{\partial w_0}{\partial x_1}\right)-$$

$$-\frac{\partial}{\partial x_1}\left\{\begin{aligned}&\frac{E_1}{1-v_{12}v_{21}}\left[\left(\frac{I_2}{\rho}-\frac{4I_4}{3\rho h^2}\right)\frac{\partial\phi_x}{\partial x_1}-\frac{I_4}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2x_1}\right]+\\&+\frac{v_{21}E_1}{1-v_{12}v_{21}}\left[\left(\frac{I_2}{\rho}-\frac{4I_4}{3\rho h^2}\right)\frac{\partial\phi_y}{\partial x_2}-\frac{I_4}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2x_2}\right]\end{aligned}\right\}-$$

$$-\frac{\partial}{\partial x_2}\left\{G_6\left(\frac{I_2}{\rho}-\frac{4I_4}{3\rho h^2}\right)\left(\frac{\partial\phi_x}{\partial x_2}+\frac{\partial\phi_y}{\partial x_1}\right)-\frac{8I_4G_6}{3\rho h^2}\frac{\partial^2 w_0}{\partial x_2\partial x_1}\right\}+$$

$$+\frac{4}{3h^2}\frac{\partial}{\partial x_2}\left\{G_6\left(\frac{I_4}{\rho}-\frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial\phi_x}{\partial x_2}+\frac{\partial\phi_y}{\partial x_1}\right)-\frac{8G_6I_6}{3\rho h^2}\frac{\partial^2 w_0}{\partial x_2\partial x_1}\right\}+$$

$$+\frac{4}{3h^2}\frac{\partial}{\partial x_1}\left\{\begin{aligned}&\frac{E_1}{1-v_{12}v_{21}}\left[\left(\frac{I_4}{\rho}-\frac{4I_2}{3\rho h^2}\right)\frac{\partial\phi_x}{\partial x_1}-\frac{I_6}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2x_1}\right]+\\&+\frac{v_{21}E_1}{1-v_{12}v_{21}}\left[\left(\frac{I_4}{\rho}-\frac{4I_2}{3\rho h^2}\right)\frac{\partial\phi_y}{\partial x_2}-\frac{I_6}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2x_2}\right]\end{aligned}\right\}=0\Rightarrow$$

$$\left(I_2 - I_4\frac{8}{3h^2} + I_6\frac{16}{9h^4}\right)\ddot{\phi}_x + \frac{4}{3h^2}\left(I_6\frac{4}{3h^2} - I_4\right)\frac{\partial \ddot{w}_0}{\partial x_1} +$$
$$+ G_5\left(\frac{I_0}{\rho} - \frac{4I_2}{\rho h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right) - \frac{4}{h^2}G_5\left(\frac{I_2}{\rho} - \frac{4I_4}{\rho h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right) -$$
$$- \left\{ \begin{array}{l} \dfrac{E_1}{1-v_{12}v_{21}}\left[\left(\dfrac{I_2}{\rho} - \dfrac{4I_4}{3\rho h^2}\right)\dfrac{\partial^2\phi_x}{\partial^2 x_1} - \dfrac{I_4}{\rho}\dfrac{4}{3h^2}\dfrac{\partial^3 w_0}{\partial^3 x_1}\right] + \\ + \dfrac{v_{21}E_1}{1-v_{12}v_{21}}\left[\left(\dfrac{I_2}{\rho} - \dfrac{4I_4}{3\rho h^2}\right)\dfrac{\partial^2\phi_y}{\partial x_1\partial x_2} - \dfrac{I_4}{\rho}\dfrac{4}{3h^2}\dfrac{\partial^2 w_0}{\partial x_1\partial^2 x_2}\right] \end{array} \right\} -$$
$$- \left\{G_6\left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2}\right)\left(\frac{\partial^2\phi_x}{\partial^2 x_2} + \frac{\partial^2\phi_y}{\partial x_2\partial x_1}\right) - \frac{8I_4G_6}{3\rho h^2}\frac{\partial^3 w_0}{\partial^2 x_2\partial x_1}\right\} +$$
$$+ \frac{4}{3h^2}\left\{G_6\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial^2\phi_x}{\partial^2 x_2} + \frac{\partial^2\phi_y}{\partial x_2\partial x_1}\right) - \frac{8G_6I_6}{3\rho h^2}\frac{\partial^3 w_0}{\partial^2 x_2\partial x_1}\right\} +$$
$$+ \frac{4}{3h^2}\left\{ \begin{array}{l} \dfrac{E_1}{1-v_{12}v_{21}}\left[\left(\dfrac{I_4}{\rho} - \dfrac{4I_2}{3\rho h^2}\right)\dfrac{\partial^2\phi_x}{\partial^2 x_1} - \dfrac{I_6}{\rho}\dfrac{4}{3h^2}\dfrac{\partial^3 w_0}{\partial^3 x_1}\right] + \\ + \dfrac{v_{21}E_1}{1-v_{12}v_{21}}\left[\left(\dfrac{I_4}{\rho} - \dfrac{4I_2}{3\rho h^2}\right)\dfrac{\partial^2\phi_y}{\partial x_1\partial x_2} - \dfrac{I_6}{\rho}\dfrac{4}{3h^2}\dfrac{\partial^3 w_0}{\partial x_1\partial^2 x_2}\right] \end{array} \right\} = 0 \qquad (14d)$$

Finally, substituting the Eqs. (5), (8), (9), (10) and (11) into the Eq. (7) of the section 4.6, we derive the last equation of motion of the TSDT in terms of displacements,

$$\left(I_2 - I_4\frac{8}{3h^2} + I_6\frac{16}{9h^4}\right)\ddot{\phi}_y + \frac{4}{3h^2}\left(\frac{4}{3h^2}I_6 - I_4\right)\frac{\partial \ddot{w}_0}{\partial x_2} +$$
$$+ Q_{23} - \frac{4}{h^2}R_{23} - \frac{\partial M_{22}}{\partial x_2} - \frac{\partial M_{12}}{\partial x_1} + \frac{\partial P_{12}}{\partial x_1}\frac{4}{3h^2} + \frac{\partial P_{22}}{\partial x_2}\frac{4}{3h^2} = 0 \Rightarrow$$

$$\left(I_2 - I_4\frac{8}{3h^2} + I_6\frac{16}{9h^4}\right)\ddot{\phi}_y + \frac{4}{3h^2}\left(\frac{4}{3h^2}I_6 - I_4\right)\frac{\partial \ddot{w}_0}{\partial x_2} +$$

$$+ G_4\left(\frac{I_0}{\rho} - \frac{4I_2}{\rho h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right) - \frac{4}{h^2}G_4\left(I_2 - \frac{4I_4}{\rho h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right) -$$

$$- \frac{\partial}{\partial x_2}\left\{\begin{array}{l} \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}\left[\left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2}\right)\frac{\partial \phi_x}{\partial x_1} - \frac{I_4}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_1}\right] + \\ + \frac{E_2}{1-\nu_{12}\nu_{21}}\left[\left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2}\right)\frac{\partial \phi_y}{\partial x_2} - \frac{I_4}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_2}\right] \end{array}\right\} -$$

$$- \frac{\partial}{\partial x_1}\left\{G_6\left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2}\right)\left(\frac{\partial \phi_x}{\partial x_2} + \frac{\partial \phi_y}{\partial x_1}\right) - \frac{8I_4 G_6}{3\rho h^2}\frac{\partial^2 w_0}{\partial x_2 \partial x_1}\right\} +$$

$$+ \frac{4}{3h^2}\frac{\partial}{\partial x_1}\left\{G_6\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial \phi_x}{\partial x_2} + \frac{\partial \phi_y}{\partial x_1}\right) - \frac{8G_6 I_6}{3\rho h^2}\frac{\partial^2 w_0}{\partial x_2 \partial x_1}\right\} +$$

$$+ \frac{4}{3h^2}\frac{\partial}{\partial x_2}\left\{\begin{array}{l} \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}\left[\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\frac{\partial \phi_x}{\partial x_1} - \frac{I_6}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_1}\right] + \\ + \frac{E_2}{1-\nu_{12}\nu_{21}}\left[\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\frac{\partial \phi_y}{\partial x_2} - \frac{I_6}{\rho}\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 x_2}\right] \end{array}\right\} = 0 \Rightarrow$$

$$\left(I_2 - I_4\frac{8}{3h^2} + I_6\frac{16}{9h^4}\right)\ddot{\phi}_y + \frac{4}{3h^2}\left(\frac{4}{3h^2}I_6 - I_4\right)\frac{\partial \ddot{w}_0}{\partial x_2} +$$

$$+ G_4\left(\frac{I_0}{\rho} - \frac{4I_2}{\rho h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right) - \frac{4}{h^2}G_4\left(I_2 - \frac{4I_4}{\rho h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right) -$$

$$- \left\{\begin{array}{l} \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}\left[\left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2}\right)\frac{\partial^2 \phi_x}{\partial x_2 \partial x_1} - \frac{I_4}{\rho}\frac{4}{3h^2}\frac{\partial^3 w_0}{\partial x_2 \partial^2 x_1}\right] + \\ + \frac{E_2}{1-\nu_{12}\nu_{21}}\left[\left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2}\right)\frac{\partial^2 \phi_y}{\partial^2 x_2} - \frac{I_4}{\rho}\frac{4}{3h^2}\frac{\partial^3 w_0}{\partial^3 x_2}\right] \end{array}\right\} -$$

$$- \left\{G_6\left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2}\right)\left(\frac{\partial^2 \phi_x}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi_y}{\partial^2 x_1}\right) - \frac{8I_4 G_6}{3\rho h^2}\frac{\partial^3 w_0}{\partial x_2 \partial^2 x_1}\right\} +$$

$$+\frac{4}{3h^2}\left\{G_6\left(\frac{I_4}{\rho}-\frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial^2\phi_x}{\partial x_1\partial x_2}+\frac{\partial^2\phi_y}{\partial^2 x_1}\right)-\frac{8G_6 I_6}{3\rho h^2}\frac{\partial^3 w_0}{\partial x_2\partial^2 x_1}\right\}+$$

$$+\frac{4}{3h^2}\left\{\begin{aligned}&\frac{v_{12}E_2}{1-v_{12}v_{21}}\left[\left(\frac{I_4}{\rho}-\frac{4I_6}{3\rho h^2}\right)\frac{\partial^2\phi_x}{\partial x_2\partial x_1}-\frac{I_6}{\rho}\frac{4}{3h^2}\frac{\partial^3 w_0}{\partial x_2\partial^2 x_1}\right]+\\&+\frac{E_2}{1-v_{12}v_{21}}\left[\left(\frac{I_4}{\rho}-\frac{4I_6}{3\rho h^2}\right)\frac{\partial^2\phi_y}{\partial^2 x_2}-\frac{I_6}{\rho}\frac{4}{3h^2}\frac{\partial^3 w_0}{\partial^3 x_2}\right]\end{aligned}\right\}=0 \qquad (14e)$$

Let it be noted that the Eqs. (14a) - (14e) are practically identical with the respective results of the book of J.N. Reddy (2004), "Mechanics of Laminated Composite Plates and Shells-Theory and Analysis" and are especially found on the Chapter 11.

6.2. Equations of motion of the TSDT in terms of displacements for an orthotropic, in-plane isotropic material

The thickness-integrated forces and moments, referred on the section 4.3 are converted to the below, due to the Eqs. (1) – (5) of the section 4.2.2,

$$\begin{Bmatrix}N_1\\M_1\\P_1\end{Bmatrix}=\begin{Bmatrix}N_{11}\\M_{11}\\P_{11}\end{Bmatrix}=\int_{-h/2}^{h/2}\sigma_{11}\begin{Bmatrix}1\\x_3\\x_3^3\end{Bmatrix}dx_3=$$

$$=\int_{-h/2}^{h/2}\frac{E}{1-v^2}\left\{\begin{aligned}&\left(\frac{\partial u_0}{\partial x_1}+v\frac{\partial v_0}{\partial x_2}\right)+x_3\left(1-\frac{4x_3^2}{3h^2}\right)\left(\frac{\partial\phi_x}{\partial x_1}+v\frac{\partial\phi_y}{\partial x_2}\right)-\\&-x_3^3\frac{4}{3h^2}\left(\frac{\partial^2 w_0}{\partial^2 x_1}+v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\end{aligned}\right\}\begin{Bmatrix}1\\x_3\\x_3^3\end{Bmatrix}dx_3\Leftrightarrow$$

$$N_{11}=\int_{-h/2}^{h/2}\frac{E}{1-v^2}\left\{\begin{aligned}&\left(\frac{\partial u_0}{\partial x_1}+v\frac{\partial v_0}{\partial x_2}\right)+x_3\left(1-\frac{4x_3^2}{3h^2}\right)\left(\frac{\partial\phi_x}{\partial x_1}+v\frac{\partial\phi_y}{\partial x_2}\right)-\\&-x_3^3\frac{4}{3h^2}\left(\frac{\partial^2 w_0}{\partial^2 x_1}+v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\end{aligned}\right\}dx_3=$$

$$=\frac{E}{1-v^2}\int_{-h/2}^{h/2}\left\{\left(\frac{\partial u_0}{\partial x_1}+v\frac{\partial v_0}{\partial x_2}\right)+\left(x_3-\frac{4x_3^3}{3h^2}\right)\left(\frac{\partial\phi_x}{\partial x_1}+v\frac{\partial\phi_y}{\partial x_2}\right)-x_3^3\frac{4}{3h^2}\left(\frac{\partial^2 w_0}{\partial^2 x_1}+v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right\}dx_3=$$

$$=\frac{E}{1-v^2}\left[\left(\frac{\partial u_0}{\partial x_1}+v\frac{\partial v_0}{\partial x_2}\right)\frac{I_0}{\rho}+\left(\frac{\cancel{I_1}}{\rho}-\frac{4\cancel{I_3}}{3\rho h^2}\right)\left(\frac{\partial\phi_x}{\partial x_1}+v\frac{\partial\phi_y}{\partial x_2}\right)-\frac{4}{3h^2}\left(\frac{\partial^2 w_0}{\partial^2 x_1}+v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\frac{\cancel{I_3}}{\rho}\right]=$$

$$=\frac{E}{1-v^2}\left(\frac{\partial u_0}{\partial x_1}+v\frac{\partial v_0}{\partial x_2}\right)\frac{I_0}{\rho} \qquad (1)$$

$$M_{11} = \int_{-h/2}^{h/2} \frac{E}{1-v^2} \left\{ \begin{array}{c} \left(\frac{\partial u_0}{\partial x_1} + v \frac{\partial v_0}{\partial x_2} \right) + x_3 \left(1 - \frac{4x_3^2}{3h^2} \right) \left(\frac{\partial \phi_x}{\partial x_1} + v \frac{\partial \phi_y}{\partial x_2} \right) - \\ - x_3^3 \frac{4}{3h^2} \left(\frac{\partial^2 w_0}{\partial^2 x_1} + v \frac{\partial^2 w_0}{\partial^2 x_2} \right) \end{array} \right\} x_3 \, dx_3 =$$

$$= \int_{-h/2}^{h/2} \frac{E}{1-v^2} \left\{ \begin{array}{c} \left(\frac{\partial u_0}{\partial x_1} + v \frac{\partial v_0}{\partial x_2} \right) x_3 + x_3^2 \left(1 - \frac{4x_3^2}{3h^2} \right) \left(\frac{\partial \phi_x}{\partial x_1} + v \frac{\partial \phi_y}{\partial x_2} \right) - \\ - x_3^4 \frac{4}{3h^2} \left(\frac{\partial^2 w_0}{\partial^2 x_1} + v \frac{\partial^2 w_0}{\partial^2 x_2} \right) \end{array} \right\} dx_3 =$$

$$= \frac{E}{1-v^2} \int_{-h/2}^{h/2} \left\{ \left(\frac{\partial u_0}{\partial x_1} + v \frac{\partial v_0}{\partial x_2} \right) x_3 + \left(x_3^2 - \frac{4x_3^4}{3h^2} \right) \left(\frac{\partial \phi_x}{\partial x_1} + v \frac{\partial \phi_y}{\partial x_2} \right) - x_3^4 \frac{4}{3h^2} \left(\frac{\partial^2 w_0}{\partial^2 x_1} + v \frac{\partial^2 w_0}{\partial^2 x_2} \right) \right\} dx_3 =$$

$$= \frac{E}{1-v^2} \left[\left(\frac{\partial u_0}{\partial x_1} + v \frac{\partial v_0}{\partial x_2} \right) \frac{\not{I_1}}{\rho} + \left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2} \right) \left(\frac{\partial \phi_x}{\partial x_1} + v \frac{\partial \phi_y}{\partial x_2} \right) - \frac{4}{3h^2} \left(\frac{\partial^2 w_0}{\partial^2 x_1} + v \frac{\partial^2 w_0}{\partial^2 x_2} \right) \frac{I_4}{\rho} \right] =$$

$$= \frac{E}{1-v^2} \left[\left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2} \right) \left(\frac{\partial \phi_x}{\partial x_1} + v \frac{\partial \phi_y}{\partial x_2} \right) - \frac{4}{3h^2} \frac{I_4}{\rho} \left(\frac{\partial^2 w_0}{\partial^2 x_1} + v \frac{\partial^2 w_0}{\partial^2 x_2} \right) \right] \tag{2}$$

$$P_{11} = \int_{-h/2}^{h/2} \frac{E}{1-v^2} \left\{ \begin{array}{c} \left(\frac{\partial u_0}{\partial x_1} + v \frac{\partial v_0}{\partial x_2} \right) + x_3 \left(1 - \frac{4x_3^2}{3h^2} \right) \left(\frac{\partial \phi_x}{\partial x_1} + v \frac{\partial \phi_y}{\partial x_2} \right) - \\ - x_3^3 \frac{4}{3h^2} \left(\frac{\partial^2 w_0}{\partial^2 x_1} + v \frac{\partial^2 w_0}{\partial^2 x_2} \right) \end{array} \right\} x_3^3 \, dx_3 =$$

$$= \frac{E}{1-v^2} \int_{-h/2}^{h/2} \left\{ \left(\frac{\partial u_0}{\partial x_1} + v \frac{\partial v_0}{\partial x_2} \right) x_3^3 + \left(x_3^4 - \frac{4x_3^6}{3h^2} \right) \left(\frac{\partial \phi_x}{\partial x_1} + v \frac{\partial \phi_y}{\partial x_2} \right) - x_3^6 \frac{4}{3h^2} \left(\frac{\partial^2 w_0}{\partial^2 x_1} + v \frac{\partial^2 w_0}{\partial^2 x_2} \right) \right\} dx_3 =$$

$$= \frac{E}{1-v^2} \left[\left(\frac{\partial u_0}{\partial x_1} + v \frac{\partial v_0}{\partial x_2} \right) \frac{\not{I_3}}{\rho} + \left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2} \right) \left(\frac{\partial \phi_x}{\partial x_1} + v \frac{\partial \phi_y}{\partial x_2} \right) - \frac{I_6}{\rho} \frac{4}{3h^2} \left(\frac{\partial^2 w_0}{\partial^2 x_1} + v \frac{\partial^2 w_0}{\partial^2 x_2} \right) \right] =$$

$$= \frac{E}{1-v^2} \left[\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2} \right) \left(\frac{\partial \phi_x}{\partial x_1} + v \frac{\partial \phi_y}{\partial x_2} \right) - \frac{4}{3h^2} \frac{I_6}{\rho} \left(\frac{\partial^2 w_0}{\partial^2 x_1} + v \frac{\partial^2 w_0}{\partial^2 x_2} \right) \right] \tag{3}$$

Further,

$$\begin{Bmatrix} N_2 \\ M_2 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} N_{22} \\ M_{22} \\ P_{22} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{22} \begin{Bmatrix} 1 \\ x_3 \\ x_3^3 \end{Bmatrix} dx_3 =$$

$$= \int_{-h/2}^{h/2} \frac{E}{1-v^2} \left\{ \begin{aligned} &\left(v\frac{\partial u_0}{\partial x_1} + \frac{\partial v_0}{\partial x_2} \right) + x_3 \left(1 - \frac{4x_3^2}{3h^2} \right) \left(v\frac{\partial \phi_x}{\partial x_1} + \frac{\partial \phi_y}{\partial x_2} \right) - \\ &- x_3^3 \frac{4}{3h^2} \left(v\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{\partial^2 w_0}{\partial^2 x_2} \right) \end{aligned} \right\} \begin{Bmatrix} 1 \\ x_3 \\ x_3^3 \end{Bmatrix} dx_3 \Leftrightarrow$$

$$N_{22} = \frac{E}{1-v^2} \int_{-h/2}^{h/2} \left\{ \left(v\frac{\partial u_0}{\partial x_1} + \frac{\partial v_0}{\partial x_2} \right) + x_3 \left(1 - \frac{4x_3^2}{3h^2} \right) \left(v\frac{\partial \phi_x}{\partial x_1} + \frac{\partial \phi_y}{\partial x_2} \right) - x_3^3 \frac{4}{3h^2} \left(v\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{\partial^2 w_0}{\partial^2 x_2} \right) \right\} dx_3 =$$

$$= \frac{E}{1-v^2} \left[\left(v\frac{\partial u_0}{\partial x_1} + \frac{\partial v_0}{\partial x_2} \right) \frac{I_0}{\rho} + \left(\frac{\cancel{I_1}}{\rho} - \frac{4\cancel{I_3}}{3\rho h^2} \right) \left(v\frac{\partial \phi_x}{\partial x_1} + \frac{\partial \phi_y}{\partial x_2} \right) - \frac{4}{3h^2} \left(v\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{\partial^2 w_0}{\partial^2 x_2} \right) \frac{\cancel{I_3}}{\rho} \right] =$$

$$= \frac{E}{1-v^2} \left(v\frac{\partial u_0}{\partial x_1} + \frac{\partial v_0}{\partial x_2} \right) \frac{I_0}{\rho} \tag{4}$$

$$M_{22} = \int_{-h/2}^{h/2} \frac{E}{1-v^2} \left\{ \begin{aligned} &\left(v\frac{\partial u_0}{\partial x_1} + \frac{\partial v_0}{\partial x_2} \right) + x_3 \left(1 - \frac{4x_3^2}{3h^2} \right) \left(v\frac{\partial \phi_x}{\partial x_1} + \frac{\partial \phi_y}{\partial x_2} \right) - \\ &- x_3^3 \frac{4}{3h^2} \left(v\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{\partial^2 w_0}{\partial^2 x_2} \right) \end{aligned} \right\} x_3 \, dx_3 =$$

$$= \frac{E}{1-v^2} \int_{-h/2}^{h/2} \left\{ \left(v\frac{\partial u_0}{\partial x_1} + \frac{\partial v_0}{\partial x_2} \right) x_3 + \left(x_3^2 - \frac{4x_3^4}{3h^2} \right) \left(v\frac{\partial \phi_x}{\partial x_1} + \frac{\partial \phi_y}{\partial x_2} \right) - x_3^4 \frac{4}{3h^2} \left(v\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{\partial^2 w_0}{\partial^2 x_2} \right) \right\} dx_3 =$$

$$= \frac{E}{1-v^2} \left[\left(v\frac{\partial u_0}{\partial x_1} + \frac{\partial v_0}{\partial x_2} \right) \frac{\cancel{I_1}}{\rho} + \left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2} \right) \left(v\frac{\partial \phi_x}{\partial x_1} + \frac{\partial \phi_y}{\partial x_2} \right) - \frac{I_4}{\rho} \frac{4}{3h^2} \left(v\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{\partial^2 w_0}{\partial^2 x_2} \right) \right] =$$

$$= \frac{E}{1-v^2} \left[\left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2} \right) \left(v\frac{\partial \phi_x}{\partial x_1} + \frac{\partial \phi_y}{\partial x_2} \right) - \frac{I_4}{\rho} \frac{4}{3h^2} \left(v\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{\partial^2 w_0}{\partial^2 x_2} \right) \right] \tag{5}$$

$$P_{22} = \int_{-h/2}^{h/2} \frac{E}{1-v^2} \left\{ \begin{aligned} &\left(v\frac{\partial u_0}{\partial x_1} + \frac{\partial v_0}{\partial x_2} \right) + x_3 \left(1 - \frac{4x_3^2}{3h^2} \right) \left(v\frac{\partial \phi_x}{\partial x_1} + \frac{\partial \phi_y}{\partial x_2} \right) - \\ &- x_3^3 \frac{4}{3h^2} \left(v\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{\partial^2 w_0}{\partial^2 x_2} \right) \end{aligned} \right\} x_3^3 \, dx_3 =$$

$$= \frac{E}{1-v^2}\int_{-h/2}^{h/2}\left\{\left(v\frac{\partial u_0}{\partial x_1}+\frac{\partial v_0}{\partial x_2}\right)x_3^3+\left(x_3^4-\frac{4x_3^6}{3h^2}\right)\left(v\frac{\partial \phi_x}{\partial x_1}+\frac{\partial \phi_y}{\partial x_2}\right)-x_3^6\frac{4}{3h^2}\left(v\frac{\partial^2 w_0}{\partial^2 x_1}+\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right\}dx_3 =$$

$$= \frac{E}{1-v^2}\left[\left(v\frac{\partial u_0}{\partial x_1}+\frac{\partial v_0}{\partial x_2}\right)\frac{\cancel{I_3}}{\rho}+\left(\frac{I_4}{\rho}-\frac{4I_6}{3\rho h^2}\right)\left(v\frac{\partial \phi_x}{\partial x_1}+\frac{\partial \phi_y}{\partial x_2}\right)-\frac{4}{3h^2}\frac{I_6}{\rho}\left(v\frac{\partial^2 w_0}{\partial^2 x_1}+\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right] =$$

$$= \frac{E}{1-v^2}\left[\left(\frac{I_4}{\rho}-\frac{4I_6}{3\rho h^2}\right)\left(v\frac{\partial \phi_x}{\partial x_1}+\frac{\partial \phi_y}{\partial x_2}\right)-\frac{4}{3h^2}\frac{I_6}{\rho}\left(v\frac{\partial^2 w_0}{\partial^2 x_1}+\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right] \qquad (6)$$

In addition,

$$\begin{Bmatrix} N_6 \\ M_6 \\ P_6 \end{Bmatrix} = \begin{Bmatrix} N_{12} \\ M_{12} \\ P_{12} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{12}\begin{Bmatrix} 1 \\ x_3 \\ x_3^3 \end{Bmatrix} dx_3 =$$

$$= \int_{-h/2}^{h/2} G \left\{\begin{matrix}\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}\right)+x_3\left(1-\frac{4x_3^2}{3h^2}\right)\left(\frac{\partial \phi_x}{\partial x_2}+\frac{\partial \phi_y}{\partial x_1}\right)- \\ -\frac{8x_3^3}{3h^2}\frac{\partial^2 w_0}{\partial x_2\,\partial x_1}\end{matrix}\right\}\begin{Bmatrix} 1 \\ x_3 \\ x_3^3 \end{Bmatrix} dx_3 \Leftrightarrow$$

$$N_{12} = \int_{-h/2}^{h/2} G \left\{\begin{matrix}\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}\right)+x_3\left(1-\frac{4x_3^2}{3h^2}\right)\left(\frac{\partial \phi_x}{\partial x_2}+\frac{\partial \phi_y}{\partial x_1}\right)- \\ -\frac{8x_3^3}{3h^2}\frac{\partial^2 w_0}{\partial x_2\,\partial x_1}\end{matrix}\right\} dx_3 =$$

$$= G\left\{\begin{matrix}\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}\right)\frac{I_0}{\rho}+\left(\frac{\cancel{I_1}}{\rho}-\frac{4\cancel{I_3}}{3\rho h^2}\right)\left(\frac{\partial \phi_x}{\partial x_2}+\frac{\partial \phi_y}{\partial x_1}\right)- \\ -\frac{8\cancel{I_3}}{3\rho h^2}\frac{\partial^2 w_0}{\partial x_2\,\partial x_1}\end{matrix}\right\} = G\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}\right)\frac{I_0}{\rho} \qquad (7)$$

$$M_{12} = \int_{-h/2}^{h/2} G \left\{\begin{matrix}\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}\right)x_3+x_3^2\left(1-\frac{4x_3^2}{3h^2}\right)\left(\frac{\partial \phi_x}{\partial x_2}+\frac{\partial \phi_y}{\partial x_1}\right)- \\ -\frac{8x_3^4}{3h^2}\frac{\partial^2 w_0}{\partial x_2\,\partial x_1}\end{matrix}\right\} dx_3 =$$

$$= G\left\{\begin{matrix}\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}\right)\frac{\cancel{I_3}}{\rho}+\left(\frac{I_2}{\rho}-\frac{4I_4}{3\rho h^2}\right)\left(\frac{\partial \phi_x}{\partial x_2}+\frac{\partial \phi_y}{\partial x_1}\right)- \\ -\frac{8I_4}{3\rho h^2}\frac{\partial^2 w_0}{\partial x_2\,\partial x_1}\end{matrix}\right\} =$$

$$= G\left(\frac{I_2}{\rho}-\frac{4I_4}{3\rho h^2}\right)\left(\frac{\partial\phi_x}{\partial x_2}+\frac{\partial\phi_y}{\partial x_1}\right) - G\,\frac{8I_4}{3\rho h^2}\frac{\partial^2 w_0}{\partial x_2\,\partial x_1} \tag{8}$$

$$P_{12} = \int_{-h/2}^{h/2} G\left\{\begin{aligned}&\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}\right)x_3^3 + x_3^4\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial\phi_x}{\partial x_2} + \\ &+ x_3^4\left(1-\frac{4x_3^2}{3h^2}\right)\frac{\partial\phi_y}{\partial x_1} - \frac{8x_3^6}{3h^2}\frac{\partial^2 w_0}{\partial x_2\,\partial x_1}\end{aligned}\right\}dx_3 =$$

$$= G\left\{\begin{aligned}&\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial v_0}{\partial x_1}\right)\frac{\not{I_3}}{\rho} + \left(\frac{I_4}{\rho}-\frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial\phi_x}{\partial x_2}+\frac{\partial\phi_y}{\partial x_1}\right) - \\ &- \frac{8I_6}{3\rho h^2}\frac{\partial^2 w_0}{\partial x_2\,\partial x_1}\end{aligned}\right\} =$$

$$= G\left(\frac{I_4}{\rho}-\frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial\phi_x}{\partial x_2}+\frac{\partial\phi_y}{\partial x_1}\right) - G\,\frac{8I_6}{3\rho h^2}\frac{\partial^2 w_0}{\partial x_2\,\partial x_1} \tag{9}$$

And also,

$$\begin{Bmatrix}Q_4\\R_4\end{Bmatrix} = \begin{Bmatrix}Q_{23}\\R_{23}\end{Bmatrix} = \int_{-h/2}^{h/2}\sigma_{23}\begin{Bmatrix}1\\x_3^2\end{Bmatrix}dx_3 = \int_{-h/2}^{h/2}\left\{G\left(1-\frac{4x_3^2}{h^2}\right)\left(\phi_y+\frac{\partial w_0}{\partial x_2}\right)\right\}\begin{Bmatrix}1\\x_3^2\end{Bmatrix}dx_3 \Leftrightarrow$$

$$Q_{23} = \int_{-h/2}^{h/2} G\left(1-\frac{4x_3^2}{h^2}\right)\left(\phi_y+\frac{\partial w_0}{\partial x_2}\right)dx_3 = \frac{G}{\rho}\left(I_0-\frac{4I_2}{h^2}\right)\left(\phi_y+\frac{\partial w_0}{\partial x_2}\right) \tag{10}$$

$$R_{23} = \int_{-h/2}^{h/2} G\left(x_3^2-\frac{4x_3^4}{h^2}\right)\left(\phi_y+\frac{\partial w_0}{\partial x_2}\right)dx_3 = \frac{G}{\rho}\left(I_2-\frac{4I_4}{h^2}\right)\left(\phi_y+\frac{\partial w_0}{\partial x_2}\right) \tag{11}$$

Finally as for the thickness-integrated quantities, we get additionally

$$\begin{Bmatrix}Q_5\\R_5\end{Bmatrix} = \begin{Bmatrix}Q_{13}\\R_{13}\end{Bmatrix} = \int_{-h/2}^{h/2}\sigma_{13}\begin{Bmatrix}1\\x_3^2\end{Bmatrix}dx_3 = \int_{-h/2}^{h/2}\left\{G\left(1-\frac{4x_3^2}{h^2}\right)\left(\phi_x+\frac{\partial w_0}{\partial x_1}\right)\right\}\begin{Bmatrix}1\\x_3^2\end{Bmatrix}dx_3 \Leftrightarrow$$

$$Q_{13} = \int_{-h/2}^{h/2}\left\{G\left(1-\frac{4x_3^2}{h^2}\right)\left(\phi_x+\frac{\partial w_0}{\partial x_1}\right)\right\}dx_3 = \frac{G}{\rho}\left(I_0-\frac{4I_2}{h^2}\right)\left(\phi_x+\frac{\partial w_0}{\partial x_1}\right) \tag{12}$$

$$R_{13} = \int_{-h/2}^{h/2}\left\{G\left(x_3^2-\frac{4x_3^4}{h^2}\right)\left(\phi_x+\frac{\partial w_0}{\partial x_1}\right)\right\}dx_3 = \frac{G}{\rho}\left(I_2-\frac{4I_4}{h^2}\right)\left(\phi_x+\frac{\partial w_0}{\partial x_1}\right) \tag{13}$$

Note that the above expressions of the thickness-integrated quantities are also exist on the book of C.M. Wang, J.N. Reddy, K.H. Lee "Shear Deformable Beams & Plates-Relations with Classical Solutions" and especially on the chapter 6.4.2. Although the appearance of the Eqs. (6.4.11a)- (6.4.11j) of the book differ a little from our Eqs. (1) - (13), the last are verified that after appropriate substitutions coincide to the expressions of the reference. For instance, we present the proof of similarity of the above Eq. (2) and the Eq. (6.4.11a) of the reference.

From the Eq. (2), taking into account the moment quantities $I_0 = \rho h$, $I_2 = \rho h^3/12$, $I_4 = \rho h^5/80$, $I_6 = \rho h^7/448$ and the flexural rigidity $D = Eh^3/12(1-v^2)$, we get

$$M_{11} = \frac{E}{1-v^2}\left[\left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2}\right)\left(\frac{\partial\phi_x}{\partial x_1} + v\frac{\partial\phi_y}{\partial x_2}\right) - \frac{4}{3h^2}\frac{I_4}{\rho}\left(\frac{\partial^2 w_0}{\partial^2 x_1} + v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right] =$$

$$= \frac{E}{1-v^2}\left[\left(\frac{1}{\cancel{\rho}}\frac{\cancel{\rho}h^3}{12} - \frac{4}{3\cancel{\rho}\cancel{h^2}}\frac{\cancel{\rho}h^{\cancel{5}3}}{80}\right)\left(\frac{\partial\phi_x}{\partial x_1} + v\frac{\partial\phi_y}{\partial x_2}\right) - \frac{4}{3\cancel{h^2}}\frac{1}{\cancel{\rho}}\frac{\cancel{\rho}h^{\cancel{5}3}}{80}\left(\frac{\partial^2 w_0}{\partial^2 x_1} + v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right] =$$

$$= \frac{E}{1-v^2}\left[\left(\frac{h^3}{12} - \frac{\cancel{4}}{3}\frac{h^3}{\cancel{80}20}\right)\left(\frac{\partial\phi_x}{\partial x_1} + v\frac{\partial\phi_y}{\partial x_2}\right) - \frac{\cancel{4}}{3}\frac{h^3}{\cancel{80}20}\left(\frac{\partial^2 w_0}{\partial^2 x_1} + v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right] =$$

$$= \frac{Eh^3}{12(1-v^2)}\left[\left(1-\frac{1}{5}\right)\left(\frac{\partial\phi_x}{\partial x_1} + v\frac{\partial\phi_y}{\partial x_2}\right) - \frac{1}{5}\left(\frac{\partial^2 w_0}{\partial^2 x_1} + v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right] =$$

$$= \frac{4D}{5}\left(\frac{\partial\phi_x}{\partial x_1} + v\frac{\partial\phi_y}{\partial x_2}\right) - \frac{D}{5}\left(\frac{\partial^2 w_0}{\partial^2 x_1} + v\frac{\partial^2 w_0}{\partial^2 x_2}\right) \qquad \text{[reference book]}$$

For a further example, we prove the similarity of the above Eq. (3) and the Eq. (6.4.11b) of the reference,

$$P_{11} = \frac{E}{1-v^2}\left[\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial\phi_x}{\partial x_1} + v\frac{\partial\phi_y}{\partial x_2}\right) - \frac{4}{3h^2}\frac{I_6}{\rho}\left(\frac{\partial^2 w_0}{\partial^2 x_1} + v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right] =$$

$$= \frac{E}{1-v^2}\left[\left(\frac{1}{\cancel{\rho}}\frac{\cancel{\rho}\, h^5}{80} - \frac{4}{3\cancel{\rho}\,\cancel{h^2}}\frac{\cancel{\rho}\, h^{\cancel{7}5}}{448}\right)\left(\frac{\partial \phi_x}{\partial x_1} + v\frac{\partial \phi_y}{\partial x_2}\right) - \frac{4}{3\cancel{h^2}}\frac{1}{\cancel{\rho}}\frac{\cancel{\rho}\, h^{\cancel{7}5}}{448}\left(\frac{\partial^2 w_0}{\partial^2 x_1} + v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right] =$$

$$= \frac{E}{1-v^2}\left[\left(\frac{h^5}{80} - \frac{4}{3}\frac{h^5}{448}\right)\left(\frac{\partial \phi_x}{\partial x_1} + v\frac{\partial \phi_y}{\partial x_2}\right) - \frac{4}{3}\frac{h^5}{448}\left(\frac{\partial^2 w_0}{\partial^2 x_1} + v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right] =$$

$$= \frac{E\,h^3}{1-v^2}\left[\left(\frac{h^2}{80} - \frac{\cancel{4}}{3}\frac{h^2}{\cancel{448}112}\right)\left(\frac{\partial \phi_x}{\partial x_1} + v\frac{\partial \phi_y}{\partial x_2}\right) - \frac{\cancel{4}}{3}\frac{h^2}{\cancel{448}112}\left(\frac{\partial^2 w_0}{\partial^2 x_1} + v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right] =$$

$$= \frac{E\,h^3}{12(1-v^2)}\left[\left(\frac{h^2 12}{80} - \frac{12}{3}\frac{h^2}{112}\right)\left(\frac{\partial \phi_x}{\partial x_1} + v\frac{\partial \phi_y}{\partial x_2}\right) - \frac{12}{3}\frac{h^2}{112}\left(\frac{\partial^2 w_0}{\partial^2 x_1} + v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right] =$$

$$= \frac{E\,h^3}{12(1-v^2)}\left[\left(\frac{3h^2}{20} - \frac{h^2}{28}\right)\left(\frac{\partial \phi_x}{\partial x_1} + v\frac{\partial \phi_y}{\partial x_2}\right) - \frac{h^2}{28}\left(\frac{\partial^2 w_0}{\partial^2 x_1} + v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right] =$$

$$= \frac{D}{4}\left[\frac{16h^2}{35}\left(\frac{\partial \phi_x}{\partial x_1} + v\frac{\partial \phi_y}{\partial x_2}\right) - \frac{h^2}{7}\left(\frac{\partial^2 w_0}{\partial^2 x_1} + v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right] =$$

$$= \left[\frac{4h^2 D}{35}\left(\frac{\partial \phi_x}{\partial x_1} + v\frac{\partial \phi_y}{\partial x_2}\right) - \frac{h^2 D}{28}\left(\frac{\partial^2 w_0}{\partial^2 x_1} + v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right] \qquad \text{[reference book]}$$

Now following the same way as exactly on the previous section, regarding the orthotropic but in-plane anisotropic material, we are going to derive the equations of motion in terms of displacements for the orthotropic but in-plane isotropic material of the problem of TSDT,

Substituting the Eqs. (1) and (7) into the Eq. (3) of the section 4.6, we get

$$\ddot{u}_0 I_0 - \frac{\partial N_{11}}{\partial x_1} - \frac{\partial N_{12}}{\partial x_2} = 0 \Rightarrow$$

$$\ddot{u}_0 I_0 - \frac{\partial}{\partial x_1}\left\{\frac{E}{1-v^2}\left(\frac{\partial u_0}{\partial x_1} + v\frac{\partial v_0}{\partial x_2}\right)\frac{I_0}{\rho}\right\} - \frac{\partial}{\partial x_2}\left\{G\left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1}\right)\frac{I_0}{\rho}\right\} = 0 \Rightarrow$$

$$\ddot{u}_0 - \frac{E}{\rho(1-v^2)}\left(\frac{\partial^2 u_0}{\partial^2 x_1} + v\frac{\partial^2 v_0}{\partial x_1\,\partial x_2}\right) - \left(\frac{\partial^2 u_0}{\partial^2 x_2} + \frac{\partial^2 v_0}{\partial x_2\,\partial x_1}\right)\frac{G}{\rho} = 0 \Rightarrow$$

$$\rho\,\ddot{u}_0 - \frac{E}{(1-v^2)}\left(\frac{\partial^2 u_0}{\partial^2 x_1} + v\frac{\partial^2 v_0}{\partial x_1\,\partial x_2}\right) - \frac{E}{2(1+v)}\left(\frac{\partial^2 u_0}{\partial^2 x_2} + \frac{\partial^2 v_0}{\partial x_2\,\partial x_1}\right) = 0 \qquad (14a)$$

From the Eq. (4) of the section 4.6 in conjunction with the Eqs. (4) and (7), we have

$$I_0\,\ddot{v}_0 - \frac{\partial N_{22}}{\partial x_2} - \frac{\partial N_{12}}{\partial x_1} = 0 \Rightarrow$$

$$I_0\,\ddot{v}_0 - \frac{\partial}{\partial x_2}\left\{\frac{E}{1-v^2}\left(v\frac{\partial u_0}{\partial x_1} + \frac{\partial v_0}{\partial x_2}\right)\frac{I_0}{\rho}\right\} - \frac{\partial}{\partial x_1}\left\{G\left(\frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1}\right)\frac{I_0}{\rho}\right\} = 0 \Rightarrow$$

$$\rho\,\ddot{v}_0 - \frac{E}{(1-v^2)}\left(v\frac{\partial^2 u_0}{\partial x_2\,\partial x_1} + \frac{\partial^2 v_0}{\partial^2 x_2}\right) - \frac{E}{2(1+v)}\left(\frac{\partial^2 u_0}{\partial x_1\,\partial x_2} + \frac{\partial^2 v_0}{\partial^2 x_1}\right) = 0 \qquad (14b)$$

It is essential to note that the above two governing Eqs. (14a) and (14b) are identical to the respective governing Eqs. (7) and (8) of the analogous section 6.2 of the CPT (Part A). This fact was expected because as for the in-plane motions the two models (CPT and TSDT) have negligible differences.

From the Eq. **(5)** of the section 4.6 in conjunction with the Eqs. (3), (6), (9), (10), (11), we derive the third equation of motion in terms of displacements,

$$I_0\,\ddot{w}_0 - \frac{4}{3h^2}\left(I_6\frac{4}{3h^2} - I_4\right)\left(\frac{\partial\ddot{\phi}_y}{\partial x_2} + \frac{\partial\ddot{\phi}_x}{\partial x_1}\right) - I_6\frac{16}{9h^4}\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_2} + \frac{\partial^2\ddot{w}_0}{\partial^2 x_1}\right) -$$
$$-\frac{\partial Q_{23}}{\partial x_2} + \frac{\partial R_{23}}{\partial x_2}\frac{4}{h^2} - \frac{\partial Q_{13}}{\partial x_1} + \frac{\partial R_{13}}{\partial x_1}\frac{4}{h^2} - \frac{4}{3h^2}\left(\frac{\partial^2 P_{11}}{\partial^2 x_1} + \frac{2\,\partial^2 P_{12}}{\partial x_2\,\partial x_1} + \frac{\partial^2 P_{22}}{\partial^2 x_2}\right) = q \Rightarrow$$

$$I_0\,\ddot{w}_0 - \frac{4}{3h^2}\left(I_6\frac{4}{3h^2} - I_4\right)\left(\frac{\partial\ddot{\phi}_y}{\partial x_2} + \frac{\partial\ddot{\phi}_x}{\partial x_1}\right) - I_6\frac{16}{9h^4}\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_2} + \frac{\partial^2\ddot{w}_0}{\partial^2 x_1}\right) -$$
$$-\frac{\partial}{\partial x_2}\left\{\frac{G}{\rho}\left(I_0 - \frac{4I_2}{h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right)\right\} + \frac{4}{h^2}\frac{\partial}{\partial x_2}\left\{\frac{G}{\rho}\left(I_2 - \frac{4I_4}{h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right)\right\} -$$
$$-\frac{\partial}{\partial x_1}\left\{\frac{G}{\rho}\left(I_0 - \frac{4I_2}{h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right)\right\} + \frac{4}{h^2}\frac{\partial}{\partial x_1}\left\{\frac{G}{\rho}\left(I_2 - \frac{4I_4}{h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right)\right\} -$$
$$-\frac{4}{3h^2}\left(\begin{aligned}&\frac{\partial^2}{\partial^2 x_1}\left\{\frac{E}{1-v^2}\left[\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial\phi_x}{\partial x_1} + v\frac{\partial\phi_y}{\partial x_2}\right) - \frac{4}{3h^2}\frac{I_6}{\rho}\left(\frac{\partial^2 w_0}{\partial^2 x_1} + v\frac{\partial^2 w_0}{\partial^2 x_2}\right)\right]\right\} +\\ &+\frac{2\,\partial^2}{\partial x_2\,\partial x_1}\left\{G\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial\phi_x}{\partial x_2} + \frac{\partial\phi_y}{\partial x_1}\right) - G\frac{8I_6}{3\rho h^2}\frac{\partial^2 w_0}{\partial x_2\,\partial x_1}\right\} +\\ &+\frac{\partial^2}{\partial^2 x_2}\left\{\frac{E}{1-v^2}\left[\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\left(v\frac{\partial\phi_x}{\partial x_1} + \frac{\partial\phi_y}{\partial x_2}\right) - \frac{4}{3h^2}\frac{I_6}{\rho}\left(v\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{\partial^2 w_0}{\partial^2 x_2}\right)\right]\right\}\end{aligned}\right) = q \Rightarrow$$

$$I_0\ddot{w}_0 - \frac{4}{3h^2}\left(I_6\frac{4}{3h^2} - I_4\right)\left(\frac{\partial\ddot{\phi}_y}{\partial x_2} + \frac{\partial\ddot{\phi}_x}{\partial x_1}\right) - I_6\frac{16}{9h^4}\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_2} + \frac{\partial^2\ddot{w}_0}{\partial^2 x_1}\right) -$$

$$-\frac{G}{\rho}\left(I_0 - \frac{4I_2}{h^2}\right)\left(\frac{\partial\phi_y}{\partial x_2} + \frac{\partial^2 w_0}{\partial^2 x_2}\right) + \frac{4}{h^2}\frac{G}{\rho}\left(I_2 - \frac{4I_4}{h^2}\right)\left(\frac{\partial\phi_y}{\partial x_2} + \frac{\partial^2 w_0}{\partial^2 x_2}\right) -$$

$$-\frac{G}{\rho}\left(I_0 - \frac{4I_2}{h^2}\right)\left(\frac{\partial\phi_x}{\partial x_1} + \frac{\partial^2 w_0}{\partial^2 x_1}\right) + \frac{4}{h^2}\frac{G}{\rho}\left(I_2 - \frac{4I_4}{h^2}\right)\left(\frac{\partial\phi_x}{\partial x_1} + \frac{\partial^2 w_0}{\partial^2 x_1}\right) -$$

$$-\frac{4}{3h^2}\frac{E}{1-v^2}\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial^3\phi_x}{\partial^3 x_1} + v\frac{\partial^3\phi_y}{\partial^2 x_1\partial x_2}\right) + \frac{16}{9h^4}\frac{E}{1-v^2}\frac{I_6}{\rho}\left(\frac{\partial^4 w_0}{\partial^4 x_1} + v\frac{\partial^4 w_0}{\partial^2 x_1\partial^2 x_2}\right) +$$

$$-\frac{8G}{3h^2}\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial^3\phi_x}{\partial^2 x_2\partial x_1} + \frac{\partial^3\phi_y}{\partial x_2\partial^2 x_1}\right) + 2\frac{4G}{3h^2}\frac{8I_6}{3\rho h^2}\frac{\partial^4 w_0}{\partial^2 x_2\partial^2 x_1} +$$

$$-\frac{4}{3h^2}\frac{E}{1-v^2}\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\left(v\frac{\partial^3\phi_x}{\partial^2 x_2\partial x_1} + \frac{\partial^3\phi_y}{\partial^3 x_2}\right) + \frac{16}{9h^4}\frac{E}{1-v^2}\frac{I_6}{\rho}\left(v\frac{\partial^4 w_0}{\partial^2 x_2\partial^2 x_1} + \frac{\partial^4 w_0}{\partial^4 x_2}\right) = q \Rightarrow$$

$$I_0\ddot{w}_0 + \frac{4}{3h^2}\left(I_4 - \frac{4I_6}{3h^2}\right)\left(\frac{\partial\ddot{\phi}_y}{\partial x_2} + \frac{\partial\ddot{\phi}_x}{\partial x_1}\right) - \frac{16I_6}{9h^4}\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_2} + \frac{\partial^2\ddot{w}_0}{\partial^2 x_1}\right) -$$

$$-\frac{G}{\rho}\left(I_0 - \frac{8I_2}{h^2} + \frac{16I_4}{h^4}\right)\left(\frac{\partial\phi_y}{\partial x_2} + \frac{\partial^2 w_0}{\partial^2 x_2}\right) - \frac{G}{\rho}\left(I_0 - \frac{8I_2}{h^2} + \frac{16I_4}{h^4}\right)\left(\frac{\partial\phi_x}{\partial x_1} + \frac{\partial^2 w_0}{\partial^2 x_1}\right) -$$

$$-\frac{4}{3h^2}\frac{E}{\rho(1-v^2)}\left(I_4 - \frac{4I_6}{3h^2}\right)\left(\frac{\partial^3\phi_x}{\partial^3 x_1} + v\frac{\partial^3\phi_y}{\partial^2 x_1\partial x_2}\right) + \frac{16I_6}{9h^4}\frac{E}{\rho(1-v^2)}\left(\frac{\partial^4 w_0}{\partial^4 x_1} + v\frac{\partial^4 w_0}{\partial^2 x_1\partial^2 x_2}\right) +$$

$$-\frac{8G}{3\rho h^2}\left(I_4 - \frac{4I_6}{3h^2}\right)\left(\frac{\partial^3\phi_x}{\partial^2 x_2\partial x_1} + \frac{\partial^3\phi_y}{\partial x_2\partial^2 x_1}\right) + 2\frac{8G}{3h^2}\frac{4I_6}{3\rho h^2}\frac{\partial^4 w_0}{\partial^2 x_2\partial^2 x_1} +$$

$$-\frac{4}{3h^2}\frac{E}{\rho(1-v^2)}\left(I_4 - \frac{4I_6}{3h^2}\right)\left(v\frac{\partial^3\phi_x}{\partial^2 x_2\partial x_1} + \frac{\partial^3\phi_y}{\partial^3 x_2}\right) + \frac{16I_6}{9h^4}\frac{E}{\rho(1-v^2)}\left(v\frac{\partial^4 w_0}{\partial^2 x_2\partial^2 x_1} + \frac{\partial^4 w_0}{\partial^4 x_2}\right) = q \Rightarrow$$

$$I_0\ddot{w}_0 + \frac{4}{3h^2}\left(I_4 - \frac{4I_6}{3h^2}\right)\left(\frac{\partial\ddot{\phi}_y}{\partial x_2} + \frac{\partial\ddot{\phi}_x}{\partial x_1}\right) - \frac{16I_6}{9h^4}\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_2} + \frac{\partial^2\ddot{w}_0}{\partial^2 x_1}\right) -$$

$$-\frac{G}{\rho}\left(I_0 - \frac{8I_2}{h^2} + \frac{16I_4}{h^4}\right)\left(\frac{\partial\phi_y}{\partial x_2} + \frac{\partial^2 w_0}{\partial^2 x_2}\right) - \frac{G}{\rho}\left(I_0 - \frac{8I_2}{h^2} + \frac{16I_4}{h^4}\right)\left(\frac{\partial\phi_x}{\partial x_1} + \frac{\partial^2 w_0}{\partial^2 x_1}\right) -$$

$$-\frac{4}{3\rho h^2}\frac{2G}{1-v}\left(I_4 - \frac{4I_6}{3h^2}\right)\left(\frac{\partial^3\phi_x}{\partial^3 x_1} + v\frac{\partial^3\phi_y}{\partial^2 x_1\partial x_2}\right) + \frac{16I_6}{9\rho h^4}\frac{2G}{1-v}\left(\frac{\partial^4 w_0}{\partial^4 x_1} + v\frac{\partial^4 w_0}{\partial^2 x_1\partial^2 x_2}\right) +$$

$$-\frac{8G}{3\rho h^2}\left(I_4 - \frac{4I_6}{3h^2}\right)\left(\frac{\partial^3\phi_x}{\partial^2 x_2\partial x_1} + \frac{\partial^3\phi_y}{\partial x_2\partial^2 x_1}\right) + \frac{8G}{3h^2}\frac{4I_6}{3\rho h^2}\frac{2\partial^4 w_0}{\partial^2 x_2\partial^2 x_1} +$$

$$-\frac{4}{3\rho h^2}\frac{2G}{1-v}\left(I_4 - \frac{4I_6}{3h^2}\right)\left(v\frac{\partial^3\phi_x}{\partial^2 x_2\partial x_1} + \frac{\partial^3\phi_y}{\partial^3 x_2}\right) + \frac{16I_6}{9\rho h^4}\frac{2G}{1-v}\left(v\frac{\partial^4 w_0}{\partial^2 x_2\partial^2 x_1} + \frac{\partial^4 w_0}{\partial^4 x_2}\right) = q \Rightarrow$$

$$I_0\ddot{w}_0 + \frac{4}{3h^2}\left(I_4 - \frac{4I_6}{3h^2}\right)\left(\frac{\partial\ddot{\phi}_y}{\partial x_2} + \frac{\partial\ddot{\phi}_x}{\partial x_1}\right) - \frac{16I_6}{9h^4}\left(\frac{\partial^2\ddot{w}_0}{\partial^2 x_2} + \frac{\partial^2\ddot{w}_0}{\partial^2 x_1}\right) -$$
$$-\frac{G}{\rho}\left(I_0 - \frac{8I_2}{h^2} + \frac{16I_4}{h^4}\right)\left(\frac{\partial\phi_y}{\partial x_2} + \frac{\partial\phi_x}{\partial x_1} + \frac{\partial^2 w_0}{\partial^2 x_2} + \frac{\partial^2 w_0}{\partial^2 x_1}\right) -$$
$$-\frac{8G}{3\rho h^2}\frac{1}{1-v}\left(I_4 - \frac{4I_6}{3h^2}\right)\left(\frac{\partial^3\phi_x}{\partial^3 x_1} + \frac{\partial^3\phi_y}{\partial^3 x_2} + \frac{\partial^3\phi_y}{\partial^2 x_1\partial x_2} + \frac{\partial^3\phi_x}{\partial^2 x_2\partial x_1}\right) +$$
$$+\frac{32I_6G}{9\rho h^4}\frac{1}{1-v}\left(\frac{\partial^4 w_0}{\partial^4 x_1} + 2\frac{\partial^4 w_0}{\partial^2 x_2\partial^2 x_1} + \frac{\partial^4 w_0}{\partial^4 x_2}\right) = q$$

or

$$I_0\ddot{w}_0 + \frac{4}{3h^2}\left(I_4 - \frac{4I_6}{3h^2}\right)\left(\frac{\partial\ddot{\phi}_y}{\partial x_2} + \frac{\partial\ddot{\phi}_x}{\partial x_1}\right) - \frac{16I_6}{9h^4}\Delta\ddot{w}_0 -$$
$$-\frac{G}{\rho}\left(I_0 - \frac{8I_2}{h^2} + \frac{16I_4}{h^4}\right)\left(\frac{\partial\phi_y}{\partial x_2} + \frac{\partial\phi_x}{\partial x_1} + \Delta w_0\right) -$$
$$-\frac{8G}{3\rho h^2}\frac{1}{1-v}\left(I_4 - \frac{4I_6}{3h^2}\right)\left(\frac{\partial^3\phi_x}{\partial^3 x_1} + \frac{\partial^3\phi_y}{\partial^3 x_2} + \frac{\partial^3\phi_y}{\partial^2 x_1\partial x_2} + \frac{\partial^3\phi_x}{\partial^2 x_2\partial x_1}\right) +$$
$$+\frac{32I_6G}{9\rho h^4}\frac{1}{1-v}\Delta^2 w_0 = q \qquad (14c)$$

At this moment it is convenient to compare the form of the third governing equation of motion of the TSDT model with the corresponding of the CPT model. Thus, recalling the Eq. (9) of the section 6.2 of the Part A and finding the common terms of those equations inside the above Eq. (14c), we get

$$I_0\ddot{w}_0 + \frac{4}{3h^2}\left(I_4 - \frac{4I_6}{3h^2}\right)\left(\frac{\partial\ddot{\phi}_y}{\partial x_2} + \frac{\partial\ddot{\phi}_x}{\partial x_1}\right) - \frac{1}{28}\frac{\rho h^3}{12}\Delta\ddot{w}_0 -$$
$$-\frac{G}{\rho}\left(I_0 - \frac{8I_2}{h^2} + \frac{16I_4}{h^4}\right)\left(\frac{\partial\phi_y}{\partial x_2} + \frac{\partial\phi_x}{\partial x_1} + \Delta w_0\right) -$$
$$-\frac{8G}{3\rho h^2}\frac{1}{1-v}\left(I_4 - \frac{4I_6}{3h^2}\right)\left(\frac{\partial^3\phi_x}{\partial^3 x_1} + \frac{\partial^3\phi_y}{\partial^3 x_2} + \frac{\partial^3\phi_y}{\partial^2 x_1\partial x_2} + \frac{\partial^3\phi_x}{\partial^2 x_2\partial x_1}\right) +$$
$$+\frac{1}{21}\frac{Eh^3}{12}\frac{1}{1-v^2}\Delta^2 w_0 = q$$

Consequently, we notice that the red colored terms are exactly the same consisting the Eq. (9) of the section 6.2 of the Part A, namely the last (third) equation of motion of the model of CPT.

Substituting the Eqs. (2), (8), (9), (12) and (13) into the Eq. **(6)** of the section 4.6,

$$\left(I_2 - I_4\frac{8}{3h^2} + I_6\frac{16}{9h^4}\right)\ddot{\phi}_x + \frac{4}{3h^2}\left(I_6\frac{4}{3h^2} - I_4\right)\frac{\partial \ddot{w}_0}{\partial x_1} + $$

$$+ Q_{13} - \frac{4}{h^2}R_{13} - \frac{\partial M_{11}}{\partial x_1} - \frac{\partial M_{12}}{\partial x_2} + \frac{\partial P_{12}}{\partial x_2}\frac{4}{3h^2} + \frac{\partial P_{11}}{\partial x_1}\frac{4}{3h^2} = 0 \Rightarrow$$

$$\left(I_2 - I_4\frac{8}{3h^2} + I_6\frac{16}{9h^4}\right)\ddot{\phi}_x + \frac{4}{3h^2}\left(I_6\frac{4}{3h^2} - I_4\right)\frac{\partial \ddot{w}_0}{\partial x_1} + $$

$$+ \frac{G}{\rho}\left(I_0 - \frac{8I_2}{h^2} + \frac{16I_4}{h^4}\right)\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right) -$$

$$- \left\{\frac{E}{1-v^2}\left[\left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2}\right)\left(\frac{\partial^2\phi_x}{\partial^2 x_1} + v\frac{\partial^2\phi_y}{\partial x_1\partial x_2}\right) - \frac{4}{3h^2}\frac{I_4}{\rho}\left(\frac{\partial^3 w_0}{\partial^3 x_1} + v\frac{\partial^3 w_0}{\partial x_1\partial^2 x_2}\right)\right]\right\} -$$

$$- \left\{G\left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2}\right)\left(\frac{\partial^2\phi_x}{\partial^2 x_2} + \frac{\partial^2\phi_y}{\partial x_2\partial x_1}\right) - G\frac{8I_4}{3\rho h^2}\frac{\partial^3 w_0}{\partial^2 x_2\partial x_1}\right\} +$$

$$+ \frac{4}{3h^2}\left\{G\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial^2\phi_x}{\partial^2 x_2} + \frac{\partial^2\phi_y}{\partial x_2\partial x_1}\right) - G\frac{8I_6}{3\rho h^2}\frac{\partial^3 w_0}{\partial^2 x_2\partial x_1}\right\} +$$

$$+ \frac{4}{3h^2}\left\{\frac{E}{1-v^2}\left[\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial^2\phi_x}{\partial^2 x_1} + v\frac{\partial^2\phi_y}{\partial x_1\partial x_2}\right) - \frac{4}{3h^2}\frac{I_6}{\rho}\left(\frac{\partial^3 w_0}{\partial^3 x_1} + v\frac{\partial^3 w_0}{\partial x_1\partial^2 x_2}\right)\right]\right\} = 0 \Rightarrow$$

$$\left(I_2 - I_4\frac{8}{3h^2} + I_6\frac{16}{9h^4}\right)\ddot{\phi}_x + \frac{4}{3h^2}\left(I_6\frac{4}{3h^2} - I_4\right)\frac{\partial \ddot{w}_0}{\partial x_1} + \frac{G}{\rho}\left(I_0 - \frac{8I_2}{h^2} + \frac{16I_4}{h^4}\right)\left(\phi_x + \frac{\partial w_0}{\partial x_1}\right) -$$

$$- \frac{E}{1-v^2}\left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2}\right)\left(\frac{\partial^2\phi_x}{\partial^2 x_1} + v\frac{\partial^2\phi_y}{\partial x_1\partial x_2}\right) + \frac{4}{3h^2}\frac{E}{1-v^2}\frac{I_4}{\rho}\left(\frac{\partial^3 w_0}{\partial^3 x_1} + v\frac{\partial^3 w_0}{\partial x_1\partial^2 x_2}\right) -$$

$$- G\left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2}\right)\left(\frac{\partial^2\phi_x}{\partial^2 x_2} + \frac{\partial^2\phi_y}{\partial x_2\partial x_1}\right) + G\frac{8I_4}{3\rho h^2}\frac{\partial^3 w_0}{\partial^2 x_2\partial x_1} +$$

$$+ \frac{4}{3h^2}G\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial^2\phi_x}{\partial^2 x_2} + \frac{\partial^2\phi_y}{\partial x_2\partial x_1}\right) - \frac{4}{3h^2}G\frac{8I_6}{3\rho h^2}\frac{\partial^3 w_0}{\partial^2 x_2\partial x_1} +$$

$$+ \frac{E}{1-v^2}\frac{4}{3h^2}\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial^2\phi_x}{\partial^2 x_1} + v\frac{\partial^2\phi_y}{\partial x_1\partial x_2}\right) - \frac{E}{1-v^2}\frac{16}{9h^4}\frac{I_6}{\rho}\left(\frac{\partial^3 w_0}{\partial^3 x_1} + v\frac{\partial^3 w_0}{\partial x_1\partial^2 x_2}\right) = 0 \Rightarrow$$

$$\left(I_2-\frac{8I_4}{3h^2}+\frac{16I_6}{9h^4}\right)\ddot{\phi}_x-\frac{4}{3h^2}\left(I_4-\frac{4I_6}{3h^2}\right)\frac{\partial\ddot{w}_0}{\partial x_1}+\frac{G}{\rho}\left(I_0-\frac{8I_2}{h^2}+\frac{16I_4}{h^4}\right)\left(\phi_x+\frac{\partial w_0}{\partial x_1}\right)+$$
$$+\frac{4}{3h^2}\frac{E}{\rho(1-\nu^2)}\left(I_4-\frac{4I_6}{3h^2}\right)\left(\frac{\partial^3 w_0}{\partial^3 x_1}+\nu\frac{\partial^3 w_0}{\partial x_1\partial^2 x_2}\right)+\frac{8G}{3\rho h^2}\left(I_4-\frac{4I_6}{3h^2}\right)\frac{\partial^3 w_0}{\partial^2 x_2\partial x_1}-$$
$$-\frac{G}{\rho}\left(I_2-\frac{8I_4}{3h^2}+\frac{16I_6}{9h^4}\right)\left(\frac{\partial^2\phi_x}{\partial^2 x_2}+\frac{\partial^2\phi_y}{\partial x_2\partial x_1}\right)-\frac{E}{\rho(1-\nu^2)}\left(I_2-\frac{8I_4}{3h^2}+\frac{16I_6}{9h^4}\right)\left(\frac{\partial^2\phi_x}{\partial^2 x_1}+\nu\frac{\partial^2\phi_y}{\partial x_1\partial x_2}\right)=0\Rightarrow$$

$$\left(I_2-\frac{8I_4}{3h^2}+\frac{16I_6}{9h^4}\right)\ddot{\phi}_x-\frac{4}{3h^2}\left(I_4-\frac{4I_6}{3h^2}\right)\frac{\partial\ddot{w}_0}{\partial x_1}+$$
$$+\frac{G}{\rho}\left(I_0-\frac{8I_2}{h^2}+\frac{16I_4}{h^4}\right)\left(\phi_x+\frac{\partial w_0}{\partial x_1}\right)+$$
$$+\frac{8G}{3\rho h^2}\left(I_4-\frac{4I_6}{3h^2}\right)\left(\frac{1}{1-\nu}\frac{\partial^3 w_0}{\partial^3 x_1}+\frac{\nu}{1-\nu}\frac{\partial^3 w_0}{\partial x_1\partial^2 x_2}\right)+$$
$$+\frac{8G}{3\rho h^2}\left(I_4-\frac{4I_6}{3h^2}\right)\frac{\partial^3 w_0}{\partial^2 x_2\partial x_1}-\frac{G}{\rho}\left(I_2-\frac{8I_4}{3h^2}+\frac{16I_6}{9h^4}\right)\left(\frac{\partial^2\phi_x}{\partial^2 x_2}+\frac{\partial^2\phi_y}{\partial x_2\partial x_1}\right)-$$
$$-\frac{G}{\rho}\left(I_2-\frac{8I_4}{3h^2}+\frac{16I_6}{9h^4}\right)\left(\frac{2}{1-\nu}\frac{\partial^2\phi_x}{\partial^2 x_1}+\frac{2\nu}{1-\nu}\frac{\partial^2\phi_y}{\partial x_1\partial x_2}\right)=0\Rightarrow$$

$$\left(I_2-\frac{8I_4}{3h^2}+\frac{16I_6}{9h^4}\right)\ddot{\phi}_x-\frac{4}{3h^2}\left(I_4-\frac{4I_6}{3h^2}\right)\frac{\partial\ddot{w}_0}{\partial x_1}+$$
$$+\frac{G}{\rho}\left(I_0-\frac{8I_2}{h^2}+\frac{16I_4}{h^4}\right)\left(\phi_x+\frac{\partial w_0}{\partial x_1}\right)+$$
$$+\frac{8G}{3\rho h^2}\left(I_4-\frac{4I_6}{3h^2}\right)\frac{1}{1-\nu}\left(\frac{\partial^3 w_0}{\partial^3 x_1}+\frac{\partial^3 w_0}{\partial x_1\partial^2 x_2}\right)- \qquad (14d)$$
$$-\frac{G}{\rho}\left(I_2-\frac{8I_4}{3h^2}+\frac{16I_6}{9h^4}\right)\left(\frac{\partial^2\phi_x}{\partial^2 x_2}+\frac{2}{1-\nu}\frac{\partial^2\phi_x}{\partial^2 x_1}+\frac{1+\nu}{1-\nu}\frac{\partial^2\phi_y}{\partial x_1\partial x_2}\right)=0$$

From the Eq. (7) of the section 4.5,

$$\left(I_2-I_4\frac{8}{3h^2}+I_6\frac{16}{9h^4}\right)\ddot{\phi}_y+\frac{4}{3h^2}\left(\frac{4}{3h^2}I_6-I_4\right)\frac{\partial\ddot{w}_0}{\partial x_2}+$$
$$+Q_{23}-\frac{4}{h^2}R_{23}-\frac{\partial M_{22}}{\partial x_2}-\frac{\partial M_{12}}{\partial x_1}+\frac{4}{3h^2}\frac{\partial P_{12}}{\partial x_1}+\frac{4}{3h^2}\frac{\partial P_{22}}{\partial x_2}=0\Rightarrow$$

$$\left(I_2 - I_4\frac{8}{3h^2} + I_6\frac{16}{9h^4}\right)\ddot{\phi}_y + \frac{4}{3h^2}\left(\frac{4}{3h^2}I_6 - I_4\right)\frac{\partial \ddot{w}_0}{\partial x_2} +$$
$$+\frac{G}{\rho}\left(I_0 - \frac{4I_2}{h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right) - \frac{4}{h^2}\frac{G}{\rho}\left(I_2 - \frac{4I_4}{h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right) -$$
$$-\frac{\partial}{\partial x_2}\left\{\frac{E}{1-v^2}\left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2}\right)\left(v\frac{\partial \phi_x}{\partial x_1} + \frac{\partial \phi_y}{\partial x_2}\right) - \frac{E}{1-v^2}\frac{I_4}{\rho}\frac{4}{3h^2}\left(v\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{\partial^2 w_0}{\partial^2 x_2}\right)\right\} -$$
$$-\frac{\partial}{\partial x_1}\left\{G\left(\frac{I_2}{\rho} - \frac{4I_4}{3\rho h^2}\right)\left(\frac{\partial \phi_x}{\partial x_2} + \frac{\partial \phi_y}{\partial x_1}\right) - G\frac{8I_4}{3\rho h^2}\frac{\partial^2 w_0}{\partial x_2 \partial x_1}\right\} +$$
$$+\frac{4}{3h^2}\frac{\partial}{\partial x_1}\left\{G\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\left(\frac{\partial \phi_x}{\partial x_2} + \frac{\partial \phi_y}{\partial x_1}\right) - G\frac{8I_6}{3\rho h^2}\frac{\partial^2 w_0}{\partial x_2 \partial x_1}\right\} +$$
$$+\frac{4}{3h^2}\frac{\partial}{\partial x_2}\left\{\frac{E}{1-v^2}\left(\frac{I_4}{\rho} - \frac{4I_6}{3\rho h^2}\right)\left(v\frac{\partial \phi_x}{\partial x_1} + \frac{\partial \phi_y}{\partial x_2}\right) - \frac{4}{3h^2}\frac{E}{1-v^2}\frac{I_6}{\rho}\left(v\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{\partial^2 w_0}{\partial^2 x_2}\right)\right\} = 0 \Rightarrow$$

$$\left(I_2 - \frac{8I_4}{3h^2} + \frac{16I_6}{9h^4}\right)\ddot{\phi}_y + \frac{4}{3h^2}\left(\frac{4}{3h^2}I_6 - I_4\right)\frac{\partial \ddot{w}_0}{\partial x_2} + \frac{G}{\rho}\left(I_0 - \frac{8I_2}{h^2} + \frac{16I_4}{h^4}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right) -$$
$$-\frac{E}{\rho(1-v^2)}\left(I_2 - \frac{4I_4}{3h^2}\right)\left(v\frac{\partial^2 \phi_x}{\partial x_2 \partial x_1} + \frac{\partial^2 \phi_y}{\partial^2 x_2}\right) + \frac{E}{\rho(1-v^2)}\frac{4I_4}{3h^2}\left(v\frac{\partial^3 w_0}{\partial x_2 \partial^2 x_1} + \frac{\partial^3 w_0}{\partial^3 x_2}\right) -$$
$$-\frac{G}{\rho}\left(I_2 - \frac{4I_4}{3h^2}\right)\left(\frac{\partial^2 \phi_x}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi_y}{\partial^2 x_1}\right) + G\frac{8I_4}{3\rho h^2}\frac{\partial^3 w_0}{\partial x_2 \partial^2 x_1} +$$
$$+\frac{G}{\rho}\left(\frac{4I_4}{3h^2} - \frac{16I_6}{9h^4}\right)\left(\frac{\partial^2 \phi_x}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi_y}{\partial^2 x_1}\right) - G\frac{32I_6}{9\rho h^4}\frac{\partial^3 w_0}{\partial x_2 \partial^2 x_1} +$$
$$+\frac{E}{\rho(1-v^2)}\left(\frac{4I_4}{3h^2} - \frac{16I_6}{9h^4}\right)\left(v\frac{\partial^2 \phi_x}{\partial x_2 \partial x_1} + \frac{\partial^2 \phi_y}{\partial^2 x_2}\right) - \frac{E}{\rho(1-v^2)}\frac{16I_6}{9h^4}\left(v\frac{\partial^3 w_0}{\partial x_2 \partial^2 x_1} + \frac{\partial^3 w_0}{\partial^3 x_2}\right) = 0 \Rightarrow$$

$$\left(I_2 - \frac{8I_4}{3h^2} + \frac{16I_6}{9h^4}\right)\ddot{\phi}_y - \frac{4}{3h^2}\left(I_4 - \frac{4I_6}{3h^2}\right)\frac{\partial \ddot{w}_0}{\partial x_2} + \frac{G}{\rho}\left(I_0 - \frac{8I_2}{h^2} + \frac{16I_4}{h^4}\right)\left(\phi_y + \frac{\partial w_0}{\partial x_2}\right) +$$
$$+\frac{4}{3h^2}\frac{E}{\rho(1-v^2)}\left(I_4 - \frac{4I_6}{3h^2}\right)\left(v\frac{\partial^3 w_0}{\partial x_2 \partial^2 x_1} + \frac{\partial^3 w_0}{\partial^3 x_2}\right) + \frac{8G}{3\rho h^2}\left(I_4 - \frac{4I_6}{3h^2}\right)\frac{\partial^3 w_0}{\partial x_2 \partial^2 x_1} -$$
$$-\frac{G}{\rho}\left(I_2 - \frac{8I_4}{3h^2} + \frac{16I_6}{9h^4}\right)\left(\frac{\partial^2 \phi_x}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi_y}{\partial^2 x_1}\right) - \frac{E}{\rho(1-v^2)}\left(I_2 - \frac{8I_4}{3h^2} + \frac{16I_6}{9h^4}\right)\left(v\frac{\partial^2 \phi_x}{\partial x_2 \partial x_1} + \frac{\partial^2 \phi_y}{\partial^2 x_2}\right) = 0 \Rightarrow$$

or

$$\left(I_2-\frac{8I_4}{3h^2}+\frac{16I_6}{9h^4}\right)\ddot{\phi}_y-\frac{4}{3h^2}\left(I_4-\frac{4I_6}{3h^2}\right)\frac{\partial\ddot{w}_0}{\partial x_2}+$$
$$+\frac{G}{\rho}\left(I_0-\frac{8I_2}{h^2}+\frac{16I_4}{h^4}\right)\left(\phi_y+\frac{\partial w_0}{\partial x_2}\right)+$$
$$+\frac{8G}{3\rho h^2}\left(I_4-\frac{4I_6}{3h^2}\right)\left(\frac{v}{1-v}\frac{\partial^3 w_0}{\partial x_2\partial^2 x_1}+\frac{1}{1-v}\frac{\partial^3 w_0}{\partial^3 x_2}+\frac{\partial^3 w_0}{\partial x_2\partial^2 x_1}\right)-$$
$$-\frac{G}{\rho}\left(I_2-\frac{8I_4}{3h^2}+\frac{16I_6}{9h^4}\right)\left(\frac{\partial^2\phi_x}{\partial x_1\partial x_2}+\frac{\partial^2\phi_y}{\partial^2 x_1}\right)-$$
$$-\frac{2G}{\rho}\left(I_2-\frac{8I_4}{3h^2}+\frac{16I_6}{9h^4}\right)\left(\frac{v}{1-v}\frac{\partial^2\phi_x}{\partial x_2\partial x_1}+\frac{1}{1-v}\frac{\partial^2\phi_y}{\partial^2 x_2}\right)=0\Rightarrow$$

$$\left(I_2-\frac{8I_4}{3h^2}+\frac{16I_6}{9h^4}\right)\ddot{\phi}_y-\frac{4}{3h^2}\left(I_4-\frac{4I_6}{3h^2}\right)\frac{\partial\ddot{w}_0}{\partial x_2}+$$
$$+\frac{G}{\rho}\left(I_0-\frac{8I_2}{h^2}+\frac{16I_4}{h^4}\right)\left(\phi_y+\frac{\partial w_0}{\partial x_2}\right)+$$
$$+\frac{8G}{3\rho h^2}\left(I_4-\frac{4I_6}{3h^2}\right)\frac{1}{1-v}\left(\frac{\partial^3 w_0}{\partial x_2\partial^2 x_1}+\frac{\partial^3 w_0}{\partial^3 x_2}\right)-\qquad(14e)$$
$$-\frac{G}{\rho}\left(I_2-\frac{8I_4}{3h^2}+\frac{16I_6}{9h^4}\right)\left(\frac{\partial^2\phi_y}{\partial^2 x_1}+\frac{1+v}{1-v}\frac{\partial^2\phi_x}{\partial x_2\partial x_1}+\frac{2}{1-v}\frac{\partial^2\phi_y}{\partial^2 x_2}\right)=0$$

Let it be noted that the Eqs. (14a) - (14e) are practically identical with the respective results of the book of J.N. Reddy (2004), "Mechanics of Laminated Composite Plates and Shells-Theory and Analysis" and are especially found on the Chapter 11.

7. Boundary Conditions of the TSDT in terms of displacements

Following the same process, as exactly on the previous section 6, where we derive the equations of motion in terms of displacements, but at this moment to derive the boundary conditions in terms of displacements for both cases of the material of the plate.

As we have aforementioned on the conclusion of the section 4.3, we use the relations of the stress resultants of the section 4.3 and substitute into them the relations of stresses in terms of the displacements [Eqs. (1`) - (5`) of the section 4.2.1 or Eqs. (1) - (5) of the section 4.2.2], meaning to express the stress resultants similarly in terms of displacement field of the problem of TSDT. Consequently, we get the following relations as seems in the sequel on the sections 7.1 and 7.2 in case of an orthotropic in-plane anisotropic and orthotropic in-plane isotropic material respectively.

As we have aforementioned on the conclusion of the section 4.3, we use the relations of the stress resultants but now transformed to the curvilinear coordinate system on which the boundary conditions are derived, as shown below.

$$\begin{Bmatrix} N_{nn} \\ N_{ss} \\ N_{ns} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{nn} \\ \sigma_{ss} \\ \sigma_{ns} \end{Bmatrix} dz \,, \qquad \begin{Bmatrix} M_{nn} \\ M_{ss} \\ M_{ns} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{nn} \\ \sigma_{ss} \\ \sigma_{ns} \end{Bmatrix} z\, dz \,,$$

$$\begin{Bmatrix} P_{nn} \\ P_{ss} \\ P_{ns} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{nn} \\ \sigma_{ss} \\ \sigma_{ns} \end{Bmatrix} z^3\, dz \,,$$

$$\begin{Bmatrix} Q_{sz} \\ R_{sz} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{sz} \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} dz \,, \qquad \begin{Bmatrix} Q_{nz} \\ R_{nz} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{nz} \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} dz$$

The rationality of deriving the boundary conditions in terms of displacement of the model of TSDT, is corresponding to those followed on the section 7 of the Part A for the model of CPT. However due to the much more and difficult calculations demanded here, we are not going to proceed to final results for the boundary conditions, but we describe accurately the path should be followed in order to their the final forms.

Thus, substituting into the above thickness-integrated quantities the stress-strain relations, given on the section 4.2.1 and 4.2.2 for an orthotropic but in-plane anisotropic and an orthotropic but in-plane isotropic material respectively, we manage to express the above stress resultants explicitly in terms of displacement field (of the curvilinear coordinate system).

After that we can get the following form of the boundary conditions of the problem of TSDT, as seems on the sections 7.1 and 7.2 in case of an orthotropic in-plane anisotropic and orthotropic in-plane isotropic plate respectively.

Note again for convenience reasons that the boundary conditions in terms of thickness-integrated quantities, which will occupy us on this section are the Eqs. (24a) and (24b) of the section 5.2.1 and the Eqs. (9a) - (9d) referred on the section 5.3.

7.1. Boundary Conditions of the TSDT in terms of displacements for an orthotropic, in-plane anisotropic material

As for the natural boundary conditions, we follow the same process to get the equations of motions in terms of displacements, but now using the Eqs. (24a) and (24b) of the section 5.2.1 and the Eqs. (9a) - (9d) referred on the section 5.3.

First, taking into account the Eqs. (1`) - (5`) of the section 4.2.1, the substitute into them the analogous components of the displacement field (u_{0n}, u_{0s}, w_0, ϕ_n, ϕ_s) in order to derive the stress field (σ_{nn}, σ_{ss}, σ_{ns}, σ_{nz}, σ_{sz}) applied on the curvilinear coordinate system.

$$\sigma_{nn} = \frac{E_n}{1-\nu_{ns}\nu_{sn}}\left(\frac{\partial u_{0n}}{\partial n} + z\left(1-\frac{4z^2}{3h^2}\right)\frac{\partial \phi_n}{\partial n} - z^3\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 n}\right) + \frac{\nu_{sn}E_n}{1-\nu_{ns}\nu_{sn}}\left(\frac{\partial u_{0s}}{\partial s} + z\left(1-\frac{4z^2}{3h^2}\right)\frac{\partial \phi_s}{\partial s} - z^3\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 s}\right) \quad (1a)$$

$$\sigma_{ss} = \frac{\nu_{ns}E_s}{1-\nu_{ns}\nu_{sn}}\left(\frac{\partial u_{0n}}{\partial n} + z\left(1-\frac{4z^2}{3h^2}\right)\frac{\partial \phi_n}{\partial s} - z^3\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 s}\right) + \frac{E_s}{1-\nu_{ns}\nu_{sn}}\left(\frac{\partial u_{0s}}{\partial s} + z\left(1-\frac{4z^2}{3h^2}\right)\frac{\partial \phi_s}{\partial s} - z^3\frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 s}\right) \quad (1b)$$

$$\sigma_{sz} = \sigma_{zs} = G_{sz}\left(1-\frac{4z^2}{h^2}\right)\left(\phi_s + \frac{\partial w_0}{\partial s}\right) \quad (1c)$$

$$\sigma_{nz} = \sigma_{zn} = G_{nz}\left(1-\frac{4z^2}{h^2}\right)\left(\phi_n + \frac{\partial w_0}{\partial n}\right) \quad (1d)$$

$$\sigma_{ns} = \sigma_{sn} = G_{ns}\left[\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} + z\left(1-\frac{4z^2}{3h^2}\right)\frac{\partial \phi_n}{\partial s} + z\left(1-\frac{4z^2}{3h^2}\right)\frac{\partial \phi_s}{\partial n} - \frac{8z^3}{3h^2}\frac{\partial^2 w_0}{\partial s\,\partial n}\right] \quad (1e)$$

where, (E_n, E_s) the modulus of elasticity on the directions n, s respectively and G_{ns}, G_{nz}, G_{sz} the shear modulus of elasticity. In addition, the Poisson's ratio ν_{ns} or ν_{sn} is an identity of the material referred to its planar directions, namely n and s-axis, defined as

$$\nu_{ns} = -\frac{e_{ss}}{e_{nn}} \quad \text{and} \quad \nu_{sn} = -\frac{e_{nn}}{e_{ss}}.$$

The same are valid for the other Poisson's ratio ν_{nz} and ν_{sz} on the corresponding directions.

Substituting the Eqs. (1a)- (1e) into the above relations of the stress resultants,

$$N_{nn} = \int_{-h/2}^{h/2} \sigma_{nn}\, dz =$$

$$= \frac{E_n}{1-\nu_{ns}\nu_{sn}} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0n}}{\partial n} + z\left(1-\frac{4z^2}{3h^2}\right)\frac{\partial \phi_n}{\partial n} - z^3 \frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 n} \right) dz +$$

$$+ \frac{\nu_{sn} E_n}{1-\nu_{ns}\nu_{sn}} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0s}}{\partial s} + z\left(1-\frac{4z^2}{3h^2}\right)\frac{\partial \phi_s}{\partial s} - z^3 \frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 s} \right) dz =$$

$$= \frac{E_n}{\rho(1-\nu_{ns}\nu_{sn})} \left(\frac{\partial u_{0n}}{\partial n} I_0 + \left(I_1 - \frac{4I_3}{3h^2}\right)\frac{\partial \phi_n}{\partial n} - I_3 \frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 n} \right) +$$

$$+ \frac{\nu_{sn} E_n}{\rho(1-\nu_{ns}\nu_{sn})} \left(\frac{\partial u_{0s}}{\partial s} I_0 + \left(I_1 - \frac{4I_3}{3h^2}\right)\frac{\partial \phi_s}{\partial s} - I_3 \frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 s} \right) =$$

$$= \frac{E_n I_0}{\rho(1-\nu_{ns}\nu_{sn})}\frac{\partial u_{0n}}{\partial n} + \frac{\nu_{sn} E_n I_0}{\rho(1-\nu_{ns}\nu_{sn})}\frac{\partial u_{0s}}{\partial s} = \frac{E_n I_0}{\rho(1-\nu_{ns}\nu_{sn})}\left(\frac{\partial u_{0n}}{\partial n} + \nu_{sn}\frac{\partial u_{0s}}{\partial s} \right) \quad (2a)$$

The last result is identically same with this of the model of CPT [section 7.1, Part A, Eq. (3a)]. Similarly, the following result is the same with the corresponding Eq. (3b) of the section 7.1, Part A.

$$N_{ss} = \int_{-h/2}^{h/2} \sigma_{ss}\, dz = \frac{\nu_{ns} E_s I_0}{\rho(1-\nu_{ns}\nu_{sn})}\frac{\partial u_{0n}}{\partial n} + \frac{E_s I_0}{\rho(1-\nu_{ns}\nu_{sn})}\frac{\partial u_{0s}}{\partial s} =$$

$$= \frac{E_s I_0}{\rho(1-\nu_{ns}\nu_{sn})}\left(\nu_{ns}\frac{\partial u_{0n}}{\partial n} + \frac{\partial u_{0s}}{\partial s} \right) \quad (2b)$$

And also the below is identical to the Eq. (3c) of the section 7.1 of the problem of CPT.

$$N_{ns} = \int_{-h/2}^{h/2} \sigma_{ns}\, dz = G_{ns}\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} \right)\frac{I_0}{\rho} \quad (2c)$$

The above similarities of the stress resultants [Eqs. (2a), (2b) and (2c)] were expected due to the same shear deformation of the mid-surface of the plate in the context of each model, namely the CPT and the TSDT. Remark also that the Eqs. (3a), (3b), (3c) lead to the same boundary conditions as for the problem of shear deformation, as has already been explained on the section 5.2. Thus, we deserve to present directly the two boundary conditions of the problem of shear deformation of the model of TSDT which are an alternative form of the Eqs. (24a) and (24b) of the section 5.2.1, specialized here for an orthotropic but in-plane anisotropic material.

$$n_{x_1}^2 - n_{x_2}^2 \; \frac{E_n}{(1-\nu_{ns}\nu_{sn})}\left(\frac{\partial u_{0n}}{\partial n} + \nu_{sn}\frac{\partial u_{0s}}{\partial s} \right) + 2\, n_{x_1} n_{x_2}\; G_{ns}\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} \right) = a_{T0n} \quad (3a)$$

$$n_{x_1}^2 - n_{x_2}^2 \quad G_{ns}\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n}\right) + 2\, n_{x_2}\, n_{x_1} \frac{E_s}{(1-\nu_{ns}\nu_{sn})}\left(\nu_{ns}\frac{\partial u_{0n}}{\partial n} + \frac{\partial u_{0s}}{\partial s}\right) = a_{T0s} \quad (3b)$$

Now, as for the flexural response of the model of TSDT we calculate the residual stress resultants,

$$M_{nn} = \int_{-h/2}^{h/2} \sigma_{nn}\, z\, dz =$$

$$= \frac{E_n}{1-\nu_{ns}\nu_{sn}} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0n}}{\partial n} z + z^2\left(1-\frac{4z^2}{3h^2}\right)\frac{\partial \phi_n}{\partial n} - z^4 \frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 n}\right) dz \, +$$

$$+ \frac{\nu_{sn} E_n}{1-\nu_{ns}\nu_{sn}} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0s}}{\partial s} z + z^2\left(1-\frac{4z^2}{3h^2}\right)\frac{\partial \phi_s}{\partial s} - z^4 \frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 s}\right) dz =$$

$$= \frac{E_n}{\rho(1-\nu_{ns}\nu_{sn})}\left(\frac{\partial u_{0n}}{\partial n} \not{I_1} + \left(I_2 - \frac{4I_4}{3h^2}\right)\frac{\partial \phi_n}{\partial n} - I_4 \frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 n}\right) +$$

$$+ \frac{\nu_{sn} E_n}{\rho(1-\nu_{ns}\nu_{sn})}\left(\frac{\partial u_{0s}}{\partial s} \not{I_1} + \left(I_2 - \frac{4I_4}{3h^2}\right)\frac{\partial \phi_s}{\partial s} - I_4 \frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 s}\right) =$$

$$= \frac{E_n}{\rho(1-\nu_{ns}\nu_{sn})}\left(\left(I_2 - \frac{4I_4}{3h^2}\right)\frac{\partial \phi_n}{\partial n} - I_4 \frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 n}\right) + \frac{\nu_{sn} E_n}{\rho(1-\nu_{ns}\nu_{sn})}\left(\left(I_2 - \frac{4I_4}{3h^2}\right)\frac{\partial \phi_s}{\partial s} - I_4 \frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 s}\right) =$$

$$= \frac{E_n}{\rho(1-\nu_{ns}\nu_{sn})}\left(\left(I_2 - \frac{4I_4}{3h^2}\right)\left(\frac{\partial \phi_n}{\partial n} + \nu_{sn}\frac{\partial \phi_s}{\partial s}\right) - I_4 \frac{4}{3h^2}\left(\frac{\partial^2 w_0}{\partial^2 n} + \nu_{sn}\frac{\partial^2 w_0}{\partial^2 s}\right)\right) \quad (4a)$$

$$M_{ss} = \int_{-h/2}^{h/2} \sigma_{ss}\, z\, dz =$$

$$= \frac{\nu_{ns} E_s}{1-\nu_{ns}\nu_{sn}} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0n}}{\partial n} z + z^2\left(1-\frac{4z^2}{3h^2}\right)\frac{\partial \phi_n}{\partial s} - z^4 \frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 s}\right) dz \, +$$

$$+ \frac{E_s}{1-\nu_{ns}\nu_{sn}} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0s}}{\partial s} z + z^2\left(1-\frac{4z^2}{3h^2}\right)\frac{\partial \phi_s}{\partial s} - z^4 \frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 s}\right) dz =$$

$$= \frac{\nu_{ns} E_s}{\rho\,(1-\nu_{ns}\nu_{sn})}\left(\frac{\partial u_{0n}}{\partial n} I_0 + \left(I_1 - \frac{4I_3}{3h^2}\right)\frac{\partial \phi_n}{\partial s} - I_3 \frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 s}\right) +$$

$$+ \frac{E_s}{\rho\,(1-\nu_{ns}\nu_{sn})}\left(\frac{\partial u_{0s}}{\partial s} I_0 + \left(I_1 - \frac{4I_3}{3h^2}\right)\frac{\partial \phi_s}{\partial s} - I_3 \frac{4}{3h^2}\frac{\partial^2 w_0}{\partial^2 s}\right) =$$

$$= \frac{E_s\, I_0}{\rho\,(1-\nu_{ns}\nu_{sn})}\left(\nu_{ns}\frac{\partial u_{0n}}{\partial n} + \frac{\partial u_{0s}}{\partial s}\right) \quad (4b)$$

$$M_{ns} = \int_{-h/2}^{h/2} \sigma_{ns} z\, dz =$$

$$= G_{ns} \int_{-h/2}^{h/2} \left[\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} \right) z + z^2 \left(1 - \frac{4z^2}{3h^2} \right) \frac{\partial \phi_n}{\partial s} + z^2 \left(1 - \frac{4z^2}{3h^2} \right) \frac{\partial \phi_s}{\partial n} - \frac{8z^4}{3h^2} \frac{\partial^2 w_0}{\partial s \partial n} \right] dz =$$

$$= \frac{G_{ns}}{\rho} \left[\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} \right) I_1 + \left(I_2 - \frac{4I_4}{3h^2} \right) \frac{\partial \phi_n}{\partial s} + \left(I_2 - \frac{4I_4}{3h^2} \right) \frac{\partial \phi_s}{\partial n} - \frac{8I_4}{3h^2} \frac{\partial^2 w_0}{\partial s \partial n} \right] =$$

$$= \frac{G_{ns}}{\rho} \left[\left(I_2 - \frac{4I_4}{3h^2} \right) \left(\frac{\partial \phi_n}{\partial s} + \frac{\partial \phi_s}{\partial n} \right) - \frac{8I_4}{3h^2} \frac{\partial^2 w_0}{\partial s \partial n} \right] \tag{4c}$$

Note that on the above calculations the z-dependence of all the integrands is explicit as exactly the x_3-dependence on the Cartesian coordinate system, because as referred above the x_3, z axes are parallel during the transformation. Thus, the vertical integration can be performed explicitly and the “mass-moments” quantities are defined as those of the section 4.1.

Consequently, the terms eliminated from the above quantities were due to the following inertia’s

$$I_i = \int_{-h/2}^{h/2} \rho x_3^i dx_3 = \int_{-h/2}^{h/2} \rho z^i dz, \quad i = 0, 1, 2, ..., 6 \qquad \text{where,}$$

$$I_1 = I_3 = I_5 = 0 \qquad \text{and}$$

$$I_0 = \int_{-h/2}^{h/2} \rho dz = \rho h, \qquad I_2 = \int_{-h/2}^{h/2} \rho z^2 dz = \rho \frac{h^3}{12},$$

$$I_4 = \int_{-h/2}^{h/2} \rho z^4 dz = \rho \frac{h^5}{80}, \qquad I_6 = \int_{-h/2}^{h/2} \rho z^6 dz = \rho \frac{h^7}{448}$$

Further, as for the higher-order thickness –integrated quantities

$$P_{nn} = \int_{-h/2}^{h/2} \sigma_{nn} z^3 \, dz =$$

$$= \frac{E_n}{1-\nu_{ns}\nu_{sn}} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0n}}{\partial n} z^3 + z^4 \left(1 - \frac{4z^2}{3h^2}\right) \frac{\partial \phi_n}{\partial n} - z^4 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 n} \right) dz +$$

$$+ \frac{\nu_{sn} E_n}{1-\nu_{ns}\nu_{sn}} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0s}}{\partial s} z^3 + z^4 \left(1 - \frac{4z^2}{3h^2}\right) \frac{\partial \phi_s}{\partial s} - z^4 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 s} \right) dz =$$

$$= \frac{E_n}{\rho(1-\nu_{ns}\nu_{sn})} \left(\frac{\partial u_{0n}}{\partial n} I_3 + \left(I_4 - \frac{4I_6}{3h^2}\right) \frac{\partial \phi_n}{\partial n} - I_4 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 n} \right) +$$

$$+ \frac{\nu_{sn} E_n}{\rho(1-\nu_{ns}\nu_{sn})} \left(\frac{\partial u_{0s}}{\partial s} I_3 + \left(I_4 - \frac{4I_6}{3h^2}\right) \frac{\partial \phi_s}{\partial s} - I_4 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 s} \right) =$$

$$= \frac{E_n}{\rho(1-\nu_{ns}\nu_{sn})} \left(\left(I_4 - \frac{4I_6}{3h^2}\right) \left(\frac{\partial \phi_n}{\partial n} + \nu_{sn} \frac{\partial \phi_s}{\partial s} \right) - I_4 \frac{4}{3h^2} \left(\frac{\partial^2 w_0}{\partial^2 n} + \nu_{sn} \frac{\partial^2 w_0}{\partial^2 s} \right) \right) \tag{4d}$$

$$P_{ss} = \int_{-h/2}^{h/2} \sigma_{ss} z^3 \, dz =$$

$$= \frac{\nu_{ns} E_s}{1-\nu_{ns}\nu_{sn}} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0n}}{\partial n} z^3 + z^4 \left(1 - \frac{4z^2}{3h^2}\right) \frac{\partial \phi_n}{\partial s} - z^6 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 s} \right) dz +$$

$$+ \frac{E_s}{1-\nu_{ns}\nu_{sn}} \int_{-h/2}^{h/2} \left(\frac{\partial u_{0s}}{\partial s} z^3 + z^4 \left(1 - \frac{4z^2}{3h^2}\right) \frac{\partial \phi_s}{\partial s} - z^6 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 s} \right) dz =$$

$$= \frac{\nu_{ns} E_s}{\rho(1-\nu_{ns}\nu_{sn})} \left(\frac{\partial u_{0n}}{\partial n} I_3 + \left(I_4 - \frac{4I_6}{3h^2}\right) \frac{\partial \phi_n}{\partial s} - I_6 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 s} \right) +$$

$$+ \frac{E_s}{\rho(1-\nu_{ns}\nu_{sn})} \left(\frac{\partial u_{0s}}{\partial s} I_3 + \left(I_4 - \frac{4I_6}{3h^2}\right) \frac{\partial \phi_s}{\partial s} - I_6 \frac{4}{3h^2} \frac{\partial^2 w_0}{\partial^2 s} \right) =$$

$$= \frac{E_s}{\rho(1-\nu_{ns}\nu_{sn})} \left(\left(I_4 - \frac{4I_6}{3h^2}\right) \left(\frac{\partial \phi_s}{\partial s} + \nu_{ns} \frac{\partial \phi_n}{\partial s} \right) - I_6 \frac{4}{3h^2} \left(\frac{\partial^2 w_0}{\partial^2 s} + \nu_{ns} \frac{\partial^2 w_0}{\partial^2 s} \right) \right) \tag{4e}$$

$$P_{ns} = \int_{-h/2}^{h/2} \sigma_{ns} z^3 dz =$$

$$= G_{ns} \int_{-h/2}^{h/2} \left[\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} \right) z^3 + z^4 \left(1 - \frac{4z^2}{3h^2} \right) \left(\frac{\partial \phi_n}{\partial s} + \frac{\partial \phi_s}{\partial n} \right) - \frac{8z^6}{3h^2} \frac{\partial^2 w_0}{\partial s \partial n} \right] dz =$$

$$= \frac{G_{ns}}{\rho} \left[\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} \right) I_3 + \left(I_4 - \frac{4I_6}{3h^2} \right) \left(\frac{\partial \phi_n}{\partial s} + \frac{\partial \phi_s}{\partial n} \right) - \frac{8I_6}{3h^2} \frac{\partial^2 w_0}{\partial s \partial n} \right] =$$

$$= \frac{G_{ns}}{\rho} \left[\left(I_4 - \frac{4I_6}{3h^2} \right) \left(\frac{\partial \phi_n}{\partial s} + \frac{\partial \phi_s}{\partial n} \right) - \frac{8I_6}{3h^2} \frac{\partial^2 w_0}{\partial s \partial n} \right] \tag{4f}$$

$$Q_{sz} = \int_{-h/2}^{h/2} \sigma_{sz} dz = G_{sz} \int_{-h/2}^{h/2} \left(1 - \frac{4z^2}{h^2} \right) \left(\phi_s + \frac{\partial w_0}{\partial s} \right) dz = \frac{G_{sz}}{\rho} \left(I_0 - \frac{4I_2}{h^2} \right) \left(\phi_s + \frac{\partial w_0}{\partial s} \right) \tag{4g}$$

$$Q_{nz} = \int_{-h/2}^{h/2} \sigma_{nz} dz = G_{nz} \int_{-h/2}^{h/2} \left(1 - \frac{4z^2}{h^2} \right) \left(\phi_n + \frac{\partial w_0}{\partial n} \right) dz = \frac{G_{nz}}{\rho} \left(I_0 - \frac{4I_2}{h^2} \right) \left(\phi_n + \frac{\partial w_0}{\partial n} \right) \tag{4h}$$

$$R_{sz} = \int_{-h/2}^{h/2} \sigma_{sz} z^2 dz = G_{sz} \int_{-h/2}^{h/2} \left(z^2 - \frac{4z^4}{h^2} \right) \left(\phi_s + \frac{\partial w_0}{\partial s} \right) dz = \frac{G_{sz}}{\rho} \left(I_2 - \frac{4I_4}{h^2} \right) \left(\phi_s + \frac{\partial w_0}{\partial s} \right) \tag{4i}$$

$$R_{nz} = \int_{-h/2}^{h/2} \sigma_{nz} z^2 dz = G_{nz} \int_{-h/2}^{h/2} \left(z^2 - \frac{4z^4}{h^2} \right) \left(\phi_n + \frac{\partial w_0}{\partial n} \right) dz = \frac{G_{nz}}{\rho} \left(I_2 - \frac{4I_4}{h^2} \right) \left(\phi_n + \frac{\partial w_0}{\partial n} \right) \tag{4j}$$

Finally, through the substitution of the Eqs. (4a)- (4j) into the boundary conditions- Eqs. (9a) - (9d) referred on the section 5.3, we could get an alternative form of these conditions expressed in terms of the displacements for the problem of TSDT and specialized for an orthotropic but in-plane anisotropic plate.

Subsequently, we mean to examine the same boundary conditions but at this time for the case of an orthotropic, in-plane isotropic material.

7.2. Boundary Conditions of the TSDT in terms of displacement for an isotropic, in-plane isotropic material

On this section, we follow the same path as shown on the section 7.1, but for a stress (and consequently displacement) field.

Thus, taking into account the Eqs. (5) and (5`) of the section 4.2.2, the components of the displacement field (σ_{nn}, σ_{ss}, σ_{ns}, σ_{nz}, σ_{sz}) for an in-plane isotropic plate, are

$$\sigma_{nn} = \frac{E}{1-v^2}\left\{\left(\frac{\partial u_{0n}}{\partial n} + v\frac{\partial u_{0s}}{\partial s}\right) + z\left(1-\frac{4z^2}{3h^2}\right)\left(\frac{\partial \phi_n}{\partial n} + v\frac{\partial \phi_s}{\partial s}\right) - z^3\frac{4}{3h^2}\left(\frac{\partial^2 w_0}{\partial^2 n} + v\frac{\partial^2 w_0}{\partial^2 s}\right)\right\} \quad (1a)$$

$$\sigma_{ss} = \frac{E}{1-v^2}\left\{\left(v\frac{\partial u_{0n}}{\partial n} + \frac{\partial u_{0s}}{\partial s}\right) + z\left(1-\frac{4z^2}{3h^2}\right)\left(v\frac{\partial \phi_n}{\partial n} + \frac{\partial \phi_s}{\partial s}\right) - z^3\frac{4}{3h^2}\left(v\frac{\partial^2 w_0}{\partial^2 n} + \frac{\partial^2 w_0}{\partial^2 s}\right)\right\} \quad (1b)$$

$$\sigma_{sz} = G\left(1-\frac{4z^2}{h^2}\right)\left(\phi_s + \frac{\partial w_0}{\partial s}\right) \quad (1c)$$

$$\sigma_{nz} = G\left(1-\frac{4z^2}{h^2}\right)\left(\phi_n + \frac{\partial w_0}{\partial n}\right) \quad (1d)$$

$$\sigma_{ns} = G\left\{\left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n}\right) + z\left(1-\frac{4z^2}{3h^2}\right)\left(\frac{\partial \phi_n}{\partial s} + \frac{\partial \phi_s}{\partial n}\right) - \frac{8z^3}{3h^2}\frac{\partial^2 w_0}{\partial s\,\partial n}\right\} \quad (1e)$$

Thus, substituting the Eqs. (1a) – (1e) into the stress resultant presented on the beginning of the section 7, we get

$$N_{nn} = \int_{-h/2}^{h/2} \sigma_{nn}\,dz =$$

$$= \frac{E}{1-v^2}\int_{-h/2}^{h/2}\left\{\left(\frac{\partial u_{0n}}{\partial n} + v\frac{\partial u_{0s}}{\partial s}\right) + z\left(1-\frac{4z^2}{3h^2}\right)\left(\frac{\partial \phi_n}{\partial n} + v\frac{\partial \phi_s}{\partial s}\right) - z^3\frac{4}{3h^2}\left(\frac{\partial^2 w_0}{\partial^2 n} + v\frac{\partial^2 w_0}{\partial^2 s}\right)\right\}dz =$$

$$= \frac{E}{\rho(1-v^2)}\left[\left(\frac{\partial u_{0n}}{\partial n} + v\frac{\partial u_{0s}}{\partial s}\right)I_0 + \left(I_1 - \frac{4I_3}{3h^2}\right)\left(\frac{\partial \phi_n}{\partial n} + v\frac{\partial \phi_s}{\partial s}\right) - I_3\frac{4}{3h^2}\left(\frac{\partial^2 w_0}{\partial^2 n} + v\frac{\partial^2 w_0}{\partial^2 s}\right)\right] =$$

$$= \frac{E\,I_0}{\rho(1-v^2)}\left(\frac{\partial u_{0n}}{\partial n} + v\frac{\partial u_{0s}}{\partial s}\right) \quad (2a)$$

Similarly to the previous section 7.1, the in-plane stress resultants N_{nn}, N_{ss} and N_{ns} for the model of the TSDT are the same with the corresponding of the CPT [section 7.2 Part A]. Thus, we give directly the already known results for the other two N_{ss} and N_{ns}

$$N_{ss} = \int_{-h/2}^{h/2} \sigma_{ss}\, dz = = \frac{E\, I_0}{\rho(1-v^2)} \left(v \frac{\partial u_{0n}}{\partial n} + \frac{\partial u_{0s}}{\partial s} \right) \tag{2b}$$

$$N_{ns} = G \left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} \right) \frac{I_0}{\rho} \tag{2c}$$

Thus, as before the in-plane natural boundary conditions for the shear deformation problem of the model TSDT with in-plane isotropic material are identical to the corresponding of the model of CPT,

$$n_{x_1}^2 - n_{x_2}^2 \;\; \frac{E}{1-v^2} \left(\frac{\partial u_{0n}}{\partial n} + v \frac{\partial u_{0s}}{\partial s} \right) + 2\, n_{x_1} n_{x_2} \; G \left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} \right) = a_{T0n} \tag{3a}$$

$$n_{x_1}^2 - n_{x_2}^2 \;\; G \left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} \right) + 2\, n_{x_2} n_{x_1} \; \frac{E}{1-v^2} \left(v \frac{\partial u_{0n}}{\partial n} + \frac{\partial u_{0s}}{\partial s} \right) = a_{T0s} \tag{3b}$$

Now, as for the moments and higher-order thickness-integrated quantities of our case which are involved on the four natural boundary conditions of the flexural response of the model of TSDT, we get

$$M_{nn} = \int_{-h/2}^{h/2} \sigma_{nn}\, z\, dz =$$

$$= \frac{E}{1-v^2} \int_{-h/2}^{h/2} \left\{ \left(\frac{\partial u_{0n}}{\partial n} + v \frac{\partial u_{0s}}{\partial s} \right) z + z^2 \left(1 - \frac{4z^2}{3h^2} \right) \left(\frac{\partial \phi_n}{\partial n} + v \frac{\partial \phi_s}{\partial s} \right) - z^4 \frac{4}{3h^2} \left(\frac{\partial^2 w_0}{\partial^2 n} + v \frac{\partial^2 w_0}{\partial^2 s} \right) \right\} dz =$$

$$= \frac{E}{\rho(1-v^2)} \left\{ \left(\frac{\partial u_{0n}}{\partial n} + v \frac{\partial u_{0s}}{\partial s} \right) I_1 + \left(I_2 - \frac{4I_4}{3h^2} \right) \left(\frac{\partial \phi_n}{\partial n} + v \frac{\partial \phi_s}{\partial s} \right) - I_4 \frac{4}{3h^2} \left(\frac{\partial^2 w_0}{\partial^2 n} + v \frac{\partial^2 w_0}{\partial^2 s} \right) \right\} =$$

$$= \frac{E}{\rho(1-v^2)} \left\{ \left(I_2 - \frac{4I_4}{3h^2} \right) \left(\frac{\partial \phi_n}{\partial n} + v \frac{\partial \phi_s}{\partial s} \right) - \frac{4I_4}{3h^2} \left(\frac{\partial^2 w_0}{\partial^2 n} + v \frac{\partial^2 w_0}{\partial^2 s} \right) \right\} \tag{4a}$$

$$M_{ss} = \int_{-h/2}^{h/2} \sigma_{ss}\, z\, dz =$$

$$= \frac{E}{1-v^2} \int_{-h/2}^{h/2} \left\{ \left(v \frac{\partial u_{0n}}{\partial n} + \frac{\partial u_{0s}}{\partial s} \right) z + z^2 \left(1 - \frac{4z^2}{3h^2} \right) \left(v \frac{\partial \phi_n}{\partial n} + \frac{\partial \phi_s}{\partial s} \right) - z^4 \frac{4}{3h^2} \left(v \frac{\partial^2 w_0}{\partial^2 n} + \frac{\partial^2 w_0}{\partial^2 s} \right) \right\} dz =$$

$$= \frac{E}{\rho(1-v^2)} \left\{ \left(v \frac{\partial u_{0n}}{\partial n} + \frac{\partial u_{0s}}{\partial s} \right) I_1 + \left(I_2 - \frac{4I_4}{3h^2} \right) \left(v \frac{\partial \phi_n}{\partial n} + \frac{\partial \phi_s}{\partial s} \right) - I_4 \frac{4}{3h^2} \left(v \frac{\partial^2 w_0}{\partial^2 n} + \frac{\partial^2 w_0}{\partial^2 s} \right) \right\} =$$

$$= \frac{E}{\rho(1-v^2)} \left\{ \left(I_2 - \frac{4I_4}{3h^2} \right) \left(v \frac{\partial \phi_n}{\partial n} + \frac{\partial \phi_s}{\partial s} \right) - \frac{4I_4}{3h^2} \left(v \frac{\partial^2 w_0}{\partial^2 n} + \frac{\partial^2 w_0}{\partial^2 s} \right) \right\} \tag{4b}$$

$$M_{ns} = \int_{-h/2}^{h/2} \sigma_{ns} z \, dz =$$

$$= G\int_{-h/2}^{h/2}\left\{\left(\frac{\partial u_{0n}}{\partial s}+\frac{\partial u_{0s}}{\partial n}\right)z + z^2\left(1-\frac{4z^2}{3h^2}\right)\left(\frac{\partial \phi_n}{\partial s}+\frac{\partial \phi_s}{\partial n}\right)-\frac{8z^4}{3h^2}\frac{\partial^2 w_0}{\partial s\,\partial n}\right\}dz =$$

$$= \frac{G}{\rho}\left\{\left(\frac{\partial u_{0n}}{\partial s}+\frac{\partial u_{0s}}{\partial n}\right)I_1 + \left(I_2-\frac{4I_4}{3h^2}\right)\left(\frac{\partial \phi_n}{\partial s}+\frac{\partial \phi_s}{\partial n}\right)-\frac{8I_4}{3h^2}\frac{\partial^2 w_0}{\partial s\,\partial n}\right\} =$$

$$= \frac{G}{\rho}\left\{\left(I_2-\frac{4I_4}{3h^2}\right)\left(\frac{\partial \phi_n}{\partial s}+\frac{\partial \phi_s}{\partial n}\right)-\frac{8I_4}{3h^2}\frac{\partial^2 w_0}{\partial s\,\partial n}\right\} \tag{4c}$$

Note that after comparing the above expressions with the respective of the CPT, we main difference is the additional terms including the variations of the slopes of the deformed cross sections of the plate, ϕ_x and ϕ_y.

$$P_{nn} = \int_{-h/2}^{h/2} \sigma_{nn} z^3 \, dz =$$

$$= \frac{E}{1-v^2}\int_{-h/2}^{h/2}\left\{\left(\frac{\partial u_{0n}}{\partial n}+v\frac{\partial u_{0s}}{\partial s}\right)z^3 + z^4\left(1-\frac{4z^2}{3h^2}\right)\left(\frac{\partial \phi_n}{\partial n}+v\frac{\partial \phi_s}{\partial s}\right)-z^6\frac{4}{3h^2}\left(\frac{\partial^2 w_0}{\partial^2 n}+v\frac{\partial^2 w_0}{\partial^2 s}\right)\right\}dz =$$

$$= \frac{E}{\rho(1-v^2)}\left\{\left(\frac{\partial u_{0n}}{\partial n}+v\frac{\partial u_{0s}}{\partial s}\right)I_3 + \left(I_4-\frac{4I_6}{3h^2}\right)\left(\frac{\partial \phi_n}{\partial n}+v\frac{\partial \phi_s}{\partial s}\right)-\frac{4I_6}{3h^2}\left(\frac{\partial^2 w_0}{\partial^2 n}+v\frac{\partial^2 w_0}{\partial^2 s}\right)\right\} =$$

$$= \frac{E}{\rho(1-v^2)}\left\{\left(I_4-\frac{4I_6}{3h^2}\right)\left(\frac{\partial \phi_n}{\partial n}+v\frac{\partial \phi_s}{\partial s}\right)-\frac{4I_6}{3h^2}\left(\frac{\partial^2 w_0}{\partial^2 n}+v\frac{\partial^2 w_0}{\partial^2 s}\right)\right\} \tag{4d}$$

$$P_{ss} = \int_{-h/2}^{h/2} \sigma_{ss} z^3 \, dz =$$

$$= \frac{E}{1-v^2}\int_{-h/2}^{h/2}\left\{\left(v\frac{\partial u_{0n}}{\partial n}+\frac{\partial u_{0s}}{\partial s}\right)z^3 + z^4\left(1-\frac{4z^2}{3h^2}\right)\left(v\frac{\partial \phi_n}{\partial n}+\frac{\partial \phi_s}{\partial s}\right)-z^6\frac{4}{3h^2}\left(v\frac{\partial^2 w_0}{\partial^2 n}+\frac{\partial^2 w_0}{\partial^2 s}\right)\right\}dz =$$

$$= \frac{E}{\rho(1-v^2)}\left\{\left(v\frac{\partial u_{0n}}{\partial n}+\frac{\partial u_{0s}}{\partial s}\right)I_3 + \left(I_4-\frac{4I_6}{3h^2}\right)\left(v\frac{\partial \phi_n}{\partial n}+\frac{\partial \phi_s}{\partial s}\right)-\frac{4I_6}{3h^2}\left(v\frac{\partial^2 w_0}{\partial^2 n}+\frac{\partial^2 w_0}{\partial^2 s}\right)\right\} =$$

$$= \frac{E}{\rho(1-v^2)}\left\{\left(I_4-\frac{4I_6}{3h^2}\right)\left(v\frac{\partial \phi_n}{\partial n}+\frac{\partial \phi_s}{\partial s}\right)-\frac{4I_6}{3h^2}\left(v\frac{\partial^2 w_0}{\partial^2 n}+\frac{\partial^2 w_0}{\partial^2 s}\right)\right\} \tag{4e}$$

$$P_{ns} = \int_{-h/2}^{h/2} \sigma_{ns} z^3 \, dz =$$

$$= G \int_{-h/2}^{h/2} \left\{ \left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} \right) z^3 + z^4 \left(1 - \frac{4z^2}{3h^2} \right) \left(\frac{\partial \phi_n}{\partial s} + \frac{\partial \phi_s}{\partial n} \right) - \frac{8z^6}{3h^2} \frac{\partial^2 w_0}{\partial s \partial n} \right\} dz =$$

$$= \frac{G}{\rho} \left\{ \left(\frac{\partial u_{0n}}{\partial s} + \frac{\partial u_{0s}}{\partial n} \right) I_3 + \left(I_4 - \frac{4I_6}{3h^2} \right) \left(\frac{\partial \phi_n}{\partial s} + \frac{\partial \phi_s}{\partial n} \right) - \frac{8I_6}{3h^2} \frac{\partial^2 w_0}{\partial s \partial n} \right\} =$$

$$= \frac{G}{\rho} \left\{ \left(I_4 - \frac{4I_6}{3h^2} \right) \left(\frac{\partial \phi_n}{\partial s} + \frac{\partial \phi_s}{\partial n} \right) - \frac{8I_6}{3h^2} \frac{\partial^2 w_0}{\partial s \partial n} \right\} \tag{4f}$$

$$Q_{sz} = \int_{-h/2}^{h/2} \sigma_{sz} \, dz = G \int_{-h/2}^{h/2} \left(1 - \frac{4z^2}{h^2} \right) \left(\phi_s + \frac{\partial w_0}{\partial s} \right) dz = \frac{G}{\rho} \left(I_0 - \frac{4I_2}{h^2} \right) \left(\phi_s + \frac{\partial w_0}{\partial s} \right) \tag{4g}$$

$$Q_{nz} = \int_{-h/2}^{h/2} \sigma_{nz} \, dz = G \int_{-h/2}^{h/2} \left(1 - \frac{4z^2}{h^2} \right) \left(\phi_n + \frac{\partial w_0}{\partial n} \right) dz = \frac{G}{\rho} \left(I_0 - \frac{4I_2}{h^2} \right) \left(\phi_n + \frac{\partial w_0}{\partial n} \right) \tag{4h}$$

$$R_{sz} = \int_{-h/2}^{h/2} \sigma_{sz} z^2 \, dz = G \int_{-h/2}^{h/2} \left(1 - \frac{4z^2}{h^2} \right) \left(\phi_s + \frac{\partial w_0}{\partial s} \right) z^2 \, dz = \frac{G}{\rho} \left(I_2 - \frac{4I_4}{h^2} \right) \left(\phi_s + \frac{\partial w_0}{\partial s} \right) \tag{4i}$$

$$R_{nz} = \int_{-h/2}^{h/2} \sigma_{nz} z^2 dz = G \int_{-h/2}^{h/2} \left(1 - \frac{4z^2}{h^2} \right) \left(\phi_n + \frac{\partial w_0}{\partial n} \right) z^2 dz = \frac{G}{\rho} \left(I_2 - \frac{4I_4}{h^2} \right) \left(\phi_n + \frac{\partial w_0}{\partial n} \right) \tag{4j}$$

Finally, through the substitution of the Eqs. (4a)- (4j) into the boundary conditions- Eqs. (9a) - (9d) referred on the section 5.3, we could get another form of these conditions expressed in terms of the displacements for the problem of TSDT and specialized for an orthotropic but in-plane isotropic plate.

8. Conclusions

8.1. Functional Spaces

In conclusion, we mean to define the functional space in which the action functional of the Hamilton's Principle is located.

We remark that the equations of motion (1) – (5) of the section 6.1 or 6.2, are expressed in terms of the displacement field (u_0, v_0, w_0, ϕ_x, ϕ_y) and they 2nd order derivatives of u_0, v_0 and t, 4th order spatial derivatives of w_0 and 3rd order spatial derivatives with respect to ϕ_x and ϕ_y. Consequently, the functional space in which the TSDT problem takes place, has to include up to 4th order spatial derivatives and up to 2nd order time derivatives.

As for the boundary of the domain of virtual displacements, the Eqs. (6) - (12) of the section 6.1 or (6) – (12) of the section 6.2 highlight the need of a boundary equipped with at least 3rd order spatial derivatives (because of the existence of 3rd order derivative of w_0).

Consequently, inside the volume $B \in \mathbb{R}^3$, must be defined at least the 4th spatial derivative of the displacement field (C^4- continuity) and its second-time derivative (C^2- continuity). This means the existence and continuity of the fourth spatial and second time derivatives of the displacement field.'
Upon the boundary ∂B, which encloses the space B, we demand the existence and continuity up to the third spatial derivatives of $\boldsymbol{u}$ (C^3- continuity).
Thus, the action functional $S = S[\boldsymbol{u}(\cdot,\cdot)]$ is defined on the space of admissible functions $C^2\ [t_1,t_2] \rightarrow Y$, where Y is the functional space $Y = \quad \boldsymbol{u} \in C^4(B) \cap C^3(\bar{B})$,

while the admissible variations $\delta\boldsymbol{u}$ belong to the space $C^2\ [t_1,t_2] \rightarrow A$, where A is a functional space $A = \quad \delta\boldsymbol{u} \in C^4(B) \cap C^3(\bar{B}) : \delta\boldsymbol{u}(\boldsymbol{x}) = 0 \text{ for } \boldsymbol{x} \in \partial B$.

In addition to the above, note that $\bar{B} = B \cup \partial B$, is the reference domain $\bar{B}$, which consists of the open set B [interior of B or $cl(B)$] and its boundary ∂B.

8.2. Conjunction between the CPT and the TSDT (or generally the shear deformation plate theories) and the Beam Theories

[References: **1.** *C.M. Wang, J.N. Reddy, K.H. Lee* (*2000*), *"Shear Deformable Beams and Plates - Relations with Classical Solutions", Chapter 7 and Chapter 12*, **2.** *K.J. Bathe, F. Brezzi* (*1894*) *"On the convergence of a four-node plate bending element based on Reissner/ Mindlin plate theory and mixed interpolation", in J.R. Whitmann(ed.), Proc. MAFELAP Conference, Brunel University*, **3.** *Bo Haggblag, Klaus-Jurgen Bathe* (*1990*), *"Specifications of Boundary Conditions for Reissner/ Mindlin Plate Bending Finite Elements", International Journal for Numerical Methods in Engineering*, and especially page 985-986, **4.** *P.G. Carliet, P.Destuyner* (*1979*), *"Approximation of the three-dimensional models by two-dimensional models in plate theory", in R. Glowinski* (*ed.*), *Energy Methods in Finite Element Analysis, Wiley New York, pp. 33-34*].

Generally speaking, the TSDT or the FSDT are substantial improvements in the description of the physical behavior of the plate structures in comparison with the CPT.

In addition, note that the assumptions and the kinematic model of the "plate theories" are corresponding with the known "beam theories" and at this point we compare them for the sake of completeness. Thus, the CPT is the corresponding so called Euler-Bernoulli Beam Theory. Also the FSDT is analogous to the Timoshenko Beam Theory, whereas the TSDT occupying us on the previous sections is respective to the Reddy-Bickford Beam Theory.

PART C:

DISPERSION CURVES

&

COMPARISON OF THE KINEMATIC MODELS

[Main References: **1.** *Graff F. Karl (1975), "Wave Motion in Elastic Solids", Chapter 4.2, 5.1, 8* , **2.** *Diploma Thesis by Feruza Abdukadirovna Amirkulova (2011), Dispersion Relations for Elastic Waves in Plates and Rods*, **3.** *Papathanasiou_Belibassakis (2014) Hydroelastic analysis of VLFS based on a consistent coupled-mode system and FEM, Technical Paper*, **4.** *Liew K.M., Wang C.M., Xiang Y., Kitipornchai S. (1998), "Vibration of Mindlin Plates - Programming the p-Version Ritz Method"*].

1. Wave propagation through infinite medium for the CPT

Recalling the third equation of motion of the Classical Plate Theory, which is related to the vertical motion (vibration) of the plate, we are going to remove the horizontally distributed external load q, in order to study the homogeneous problem of the vibrating plate.

$$\rho\, h\, \ddot{w}_0 - \frac{\rho h^3}{12}\left(\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1} + \frac{\partial^2 \ddot{w}_0}{\partial^2 x_2}\right) + \frac{E}{(1-v^2)}\frac{h^3}{12}\left(\frac{\partial^4 w_0}{\partial^4 x_1} + 2\frac{\partial^4 w_0}{\partial^2 x_1 \partial^2 x_2} + \frac{\partial^4 w_0}{\partial^4 x_2}\right) = 0$$

or

$$\rho h \ddot{w}_0 - \frac{\rho h^3}{12}\Delta \ddot{w}_0 + \frac{E h^3}{12}\frac{1}{(1-v^2)}\Delta^2 w_0 = 0 \qquad (1)$$

To simplify the notation on the following calculations, we write the deflection of the plate in the x_3 direction $w = w_0$ and also the direction of the wave propagation is along the $x_1 = x$ - axis. Further, regarding the symbols which appear on the section 6.2 of the CPT concerning the orthotropic but in-plane isotropic material (plate), we have

$$I_0 = \rho h, \qquad I_2 = \frac{\rho h^3}{12} \quad \text{and} \quad D = \frac{E h^3}{12(1-v^2)}$$

By this way the initial form of the Eq. (1) becomes,

$$\rho\, h\, \ddot{w}_0 - \frac{\rho h^3}{12}\frac{\partial^2 \ddot{w}_0}{\partial^2 x_1} + \frac{E}{(1-v^2)}\frac{h^3}{12}\frac{\partial^4 w_0}{\partial^4 x_1} = 0$$

or

$$I_0 \ddot{w}_0 - I_2 \Delta \ddot{w}_0 + D \Delta^2 w_0 = 0 \qquad (1`)$$

where the Laplace and Biharmonic Operators are expressed only by the x_1-spatial derivatives of the displacement w_0.

Let now assume that the vertical displacement of the plate w is harmonic dependent from the time and the spatial variable x with an amplitude A. Then we are going to study "under what conditions can waves of the type

$$w = A e^{i(kx-\omega t)} \qquad (2)$$

exist in the plate? "
Subsequently, substituting the Eq. (2) into the governing equation of motion (1`) we have the following relation after a few calculations,

$$I_0(-i\omega)^2 A i\,exp(k\,x-\omega\,t)\;-\;I_2(i\,k)^2(-i\omega)^2 A i\,exp(k\,x-\omega\,t)\;+$$
$$+\;D(i\,k)^4 A i\,exp(k\,x-\omega\,t)\;=\;0\quad\Rightarrow$$
$$I_0(-i\omega)^2\,\cancel{A i\,exp(k\,x-\omega\,t)}\;-\;I_2(i\,k)^2(-i\omega)^2\,\cancel{A i\,exp(k\,x-\omega\,t)}\;+$$
$$+\;D(i\,k)^4\,\cancel{A i\,exp(k\,x-\omega\,t)}\;=\;0\quad\Rightarrow$$
$$-\,I_0\,\omega^2\;-\;I_2\,k^2\omega^2\;+\;D\,k^4\;=\;0\quad\Rightarrow$$
$$I_0\,\omega^2\;+\;I_2\,k^2\omega^2\;-\;D\,k^4\;=\;0 \qquad \textbf{(3)}$$

Thus, by the Eq. (3) we manage to express the angular frequency ω of the plate in terms of the parameter k. The last is called the wavenumber of the wave propagation through the body of the plate and is usually defined as, $k\;=\;2\pi\,/\,\lambda$, where λ is the wavelength of the wave propagation. Also the wave velocity is defined as $c\;=\;\omega\,/\,\lambda$.

Now is essential to proceed to a non-dimensional form of the above Eq. (3), in order to plot the graph of the relation between the frequency ω and the wavenumber k under the same scale. Thus, by the method of non-dimensional analysis and normalization [*J. David Logan, "Applied Mathematics" (1997) by John Wiley & Sons/ A. Papaioannou, "Fluid Mechanics" (2002) Korali editions*] we choose the following way to convert the dimensional quantities ω, k to the non-dimensional $\tilde{\omega}, \tilde{k}$.

Let the non-dimensional angular frequency be $\quad \tilde{\omega} = \dfrac{\omega}{\Omega}$.

Also, let the non-dimensional wavenumber be $\quad \tilde{k} = \dfrac{k}{K}$.

Substituting the dimensional angular frequency and wavenumber, $\omega = \tilde{\omega}\,\Omega$ and $k = \tilde{k}\,K$ respectively, into the Eq. (3) we get

$$I_0\;\tilde{\omega}\,\Omega\;^2\;+\;I_2\;\tilde{k}\,K\;^2\;\tilde{\omega}\,\Omega\;^2\;-\;D\;\tilde{k}\,K\;^4\;=\;0\;\Rightarrow$$

$$I_0\,\tilde{\omega}^2\,\Omega^2\;+\;I_2\,\tilde{k}^2\,K^2\,\tilde{\omega}^2\,\Omega^2\;-\;D\,\tilde{k}^4\,K^4\;=\;0\;\xrightarrow{\div I_0\,\Omega^2}$$

$$\tilde{\omega}^2\;+\;\frac{I_2\,K^2}{I_0}\,\tilde{k}^2\tilde{\omega}^2\;-\;\frac{D\,K^4}{I_0\,\Omega^2}\,\tilde{k}^4\;=\;0 \qquad (4)$$

Note that the above Eq. (4) has two degrees of freedom and as a consequence the way to convert it to a non-dimensional form is unique. Thus, we set

$$\frac{I_2\,K^2}{I_0}=1\;\Rightarrow\;K=\sqrt{\frac{I_0}{I_2}}\;\text{ or }\;K=\sqrt{\frac{12}{h^2}} \qquad (5a)$$

and
$$\frac{D\,K^4}{I_0\,\Omega^2}=1\;\Rightarrow\;\Omega=K^2\sqrt{\frac{D}{I_0}}=\frac{I_0}{I_2}\sqrt{\frac{D}{I_0}} \qquad (5b)$$

or

$$\Omega = \frac{12}{h^2}\sqrt{\frac{E\,h^3}{12\,(1-v^2)\,\rho\,h}} = \frac{12}{h}\sqrt{\frac{E}{12\,(1-v^2)\,\rho}}$$

Finally substituting the relations (5a) and (5b) into the Eq. (4), we derive the following

$$\tilde{\omega}^2 + \tilde{k}^2\tilde{\omega}^2 - \tilde{k}^4 = 0 \quad \Rightarrow \quad \tilde{\omega}^2 = \frac{\tilde{k}^4}{1+\tilde{k}^2} \quad \Rightarrow \quad \tilde{\omega} = \pm\frac{\tilde{k}^2}{\sqrt{1+\tilde{k}^2}} \quad \Leftrightarrow$$

$$\Leftrightarrow \left(\tilde{\omega} = \frac{\tilde{k}^2}{\sqrt{1+\tilde{k}^2}} \quad or \quad \tilde{\omega} = -\frac{\tilde{k}^2}{\sqrt{1+\tilde{k}^2}} \right)$$

where the negative frequency has no physical interpretation and consequently is rejected. Finally, we get the dispersion relation (red curve)

$$\tilde{\omega} = \frac{\tilde{k}^2}{\sqrt{1+\tilde{k}^2}} \qquad \textbf{(6)}$$

The last expression [Eq. **(6)**], is the dispersion relation between the dimensionless angular frequency $\tilde{\omega}$ and the dimensionless wavenumber $\tilde{k}$ of the plate, in the context of the problem of wave propagation through an infinite plate of the Kirchhoff's model and is illustrated on the figure below. On the first of the following figures as for the CPT, is illustrated the comparison between the initial regarded Kirchhoff's Plate Theory and the subsequently corrected Kirchhoff's Plate Theory. The second one is the above Eq. **(6)**, which is supplied with the rotary inertia term on the governing equation of motion. This term was proposed by the Lord Rayleigh and gives by far better results as for the physical interpretation of the "Kirchhoff's Plate Theory". Thus, the blue curve of the Figure 1 is the initial regarded Kirchhoff's Plate Theory, which derives from the governing equation (1) without the rotary inertia term, as seems below

$$\rho\,h\,\ddot{w}_0 + \frac{E\,h^3}{12}\,\frac{1}{(1-v^2)}\,\Delta^2 w_0 = 0 \qquad (7)$$

or $$I_0\,\ddot{w}_0 + D\,\Delta^2 w_0 = 0 \qquad (7`)$$

and regarding the same harmonic functions of the vertical displacement w_0 , we get

$$I_0\,\omega^2 - D\,k^4 = 0 \qquad \textbf{(8)}$$

Also we consider the same non-dimensionalization of the Eq. (8) in order to compare its form with those of the Eq. **(3)**. Thus, we get

$$I_0\;\tilde{\omega}\,\Omega^{\;2} - D\;\tilde{k}\,K^{\;4} = 0 \;\Rightarrow\; I_0\,\Omega^2\,\tilde{\omega}^2 - D\,K^4\,\tilde{k}^4 = 0 \;\Rightarrow$$

$$I_0\left(\frac{I_0}{I_2}\right)^2\frac{D}{I_0}\,\tilde{\omega}^2 - D\left(\frac{I_0}{I_2}\right)^2\tilde{k}^4 = 0 \;\Rightarrow\; \tilde{\omega}^2 = \tilde{k}^4 \;\Rightarrow\; \tilde{\omega} = \pm\tilde{k}^2 \;\Leftrightarrow$$

$$\tilde{\omega} = \tilde{k}^2 \quad or \quad \tilde{\omega} = -\,\tilde{k}^2$$

where the negative frequency has no physical interpretation and consequently is rejected. Finally, we get the dispersion relation (blue curve)

$$\tilde{\omega} = \tilde{k}^2 \qquad \textbf{(9)}$$

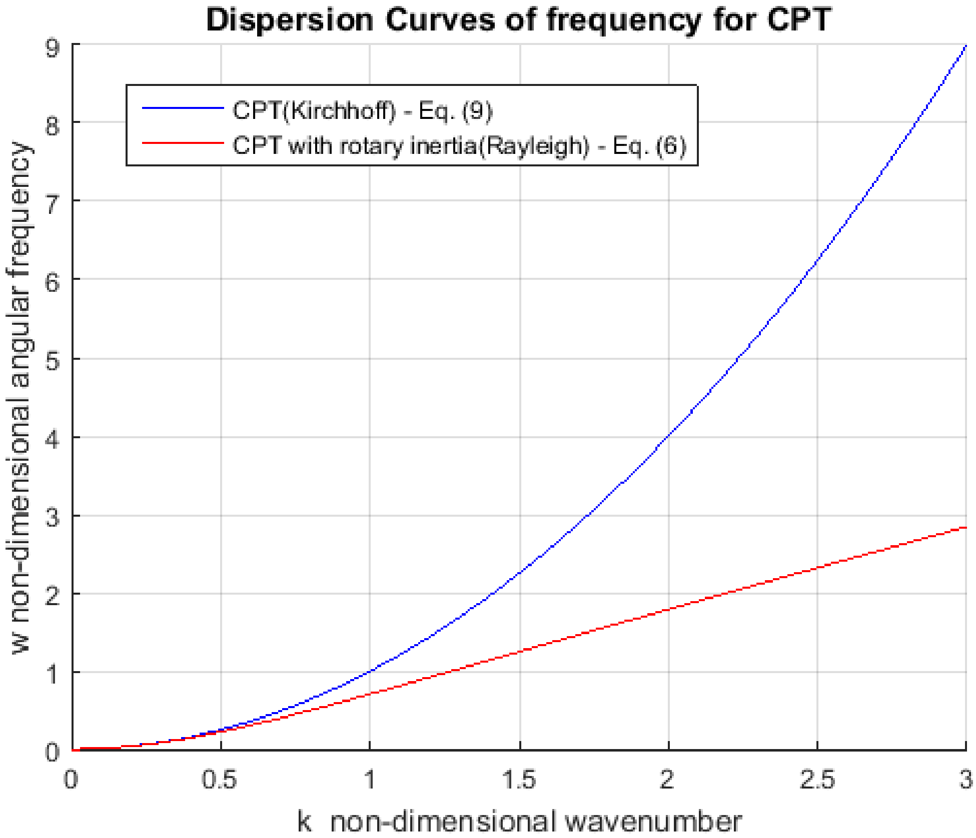

Figure 1: Dispersion Curves of frequency-wavenumber as for the Classical Kirchhoff's Plate Theory.

Subsequently, there are illustrated the dispersion curves of the non-dimensional phase velocity of the wave propagation and the non-dimensional wavenumber. Further, it is shown the relation between the non-dimensional group velocity of the wave propagation and the non-dimensional wavenumber. However, on the last section 3 of this part the comparison of the models of the plates is related to the CPT with the rotary inertia term, because it is more essential to compare our results of the higher-order plate theories with the "optimized" CPT with the rotary inertia term. Thus, by definition the non-dimensional phase and group velocities of the CPT (Rayleigh) are given respectively by the relations

$$\tilde{c}_p = \frac{\tilde{\omega}}{\tilde{k}} = \frac{\tilde{k}}{\sqrt{1+\tilde{k}^2}} \qquad (10)$$

and $$\tilde{c}_g = \frac{d\tilde{\omega}}{d\tilde{k}} = \frac{\tilde{k}^3 + 2\tilde{k}}{\sqrt{1+\tilde{k}^2}^{\,3}} \qquad (11)$$

and the non-dimensional phase and group velocities of the CPT without rotary inertia term are the following,

$$c_p = \frac{\tilde{\omega}}{\tilde{k}} = \tilde{k} \qquad (12) \qquad \text{and} \qquad c_g = \frac{d\tilde{\omega}}{d\tilde{k}} = 2\tilde{k} \qquad (13)$$

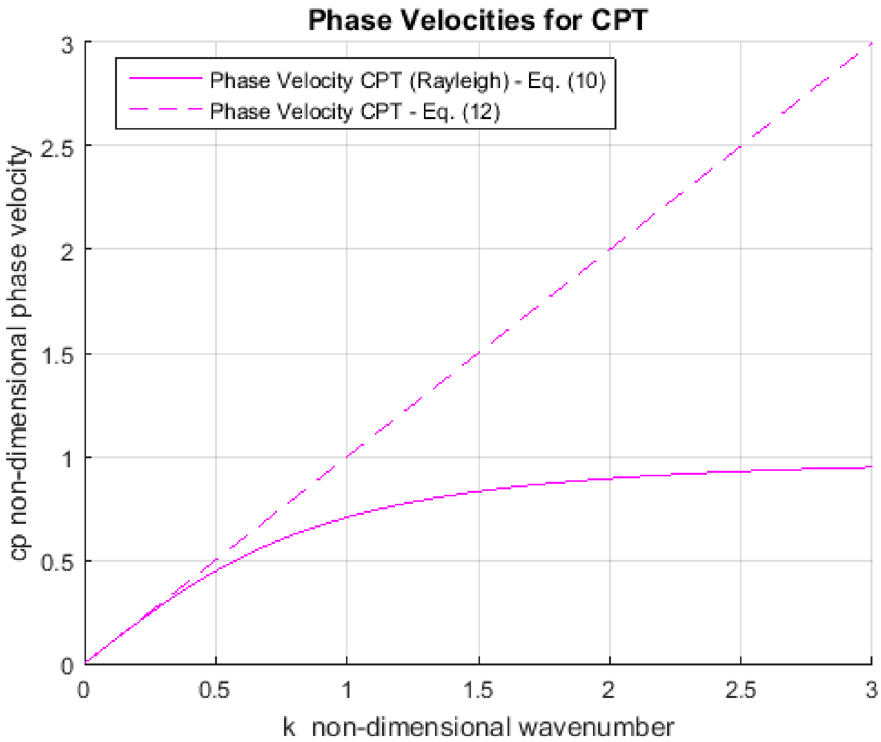

Figure 2: Dispersion Curves of phase velocity-wavenumber for the Classical (Kirchhoff's) Plate Theory.

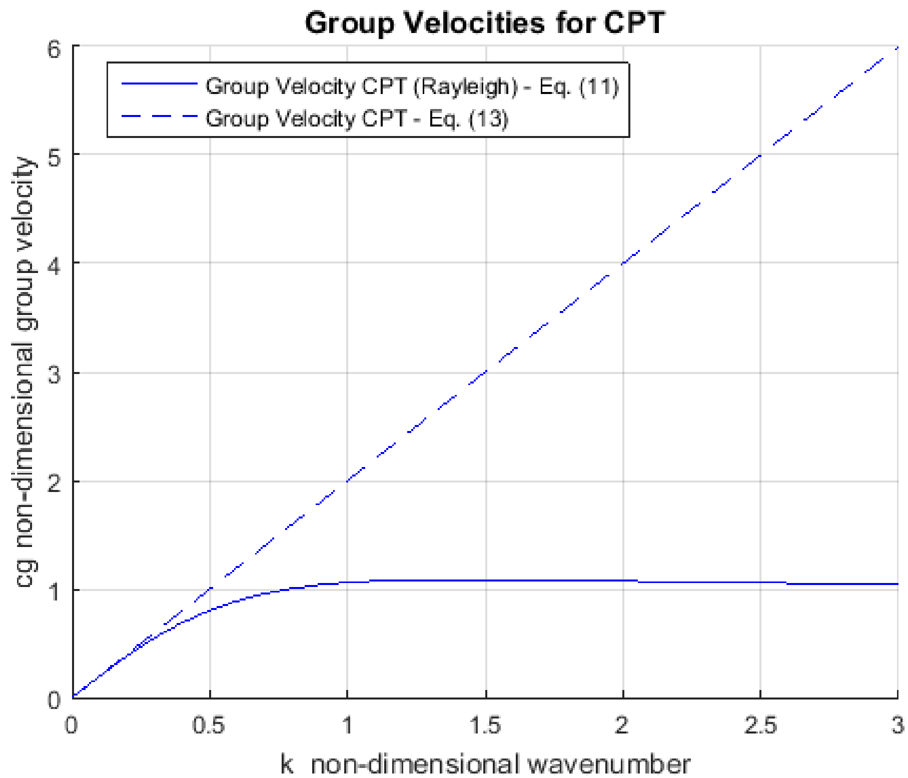

Figure 3: Dispersion Curves of group velocity-wavenumber for the Classical Plate Theory.

And finally we gather the curves of the two precious figures in order to compare more efficiently the results of the CPT with and without rotary inertia.

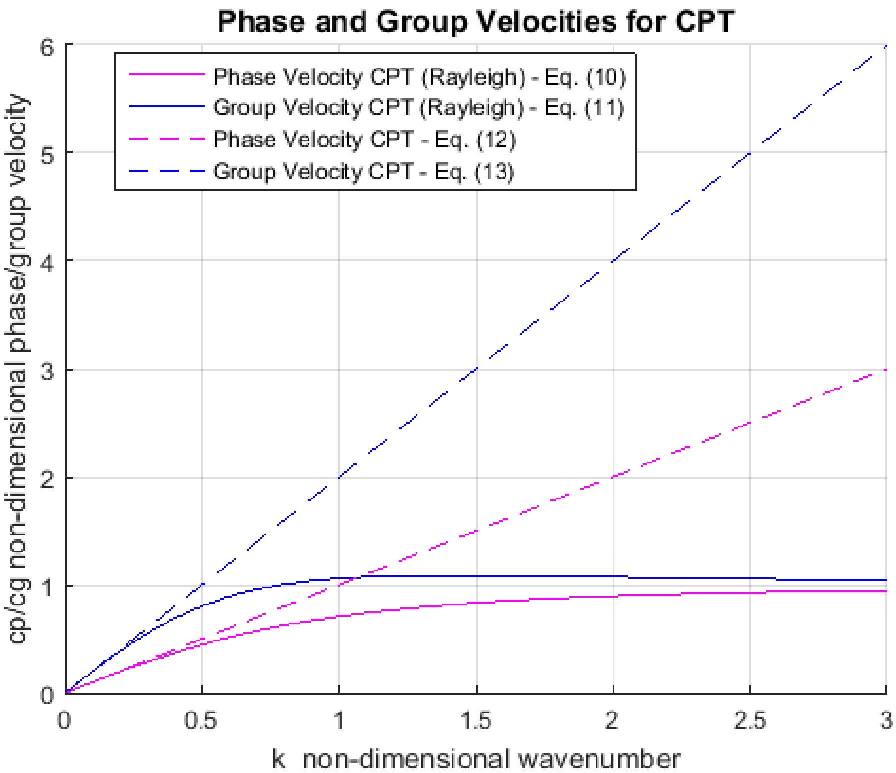

Figure 4: Comparison of the dispersion curves of phase and group velocity-wavenumber for the CPT.

Further, notice that the behavior of the phase and group velocities for the CPT with rotary inertia is bounded for larger and infinite values of wavenumber. To illustrate this fact better, we plot again only the phase and group velocities of the CPT with rotary inertia for a wider range of wavenumber. It is regarded adequate to make the choice for $\tilde{k} \in [0,10]$. Thus, we get the following figure

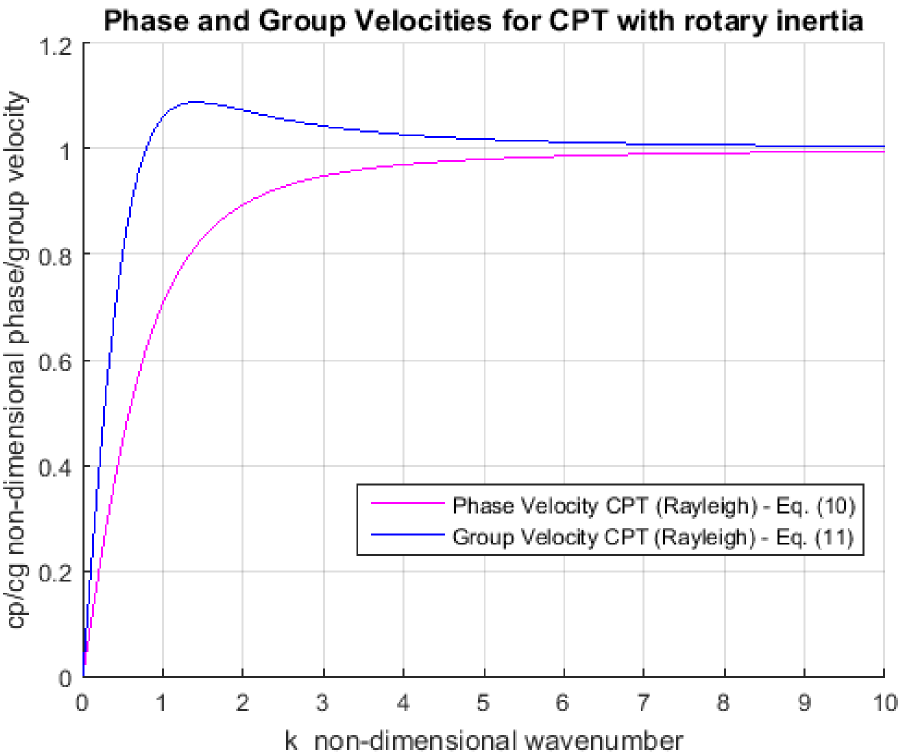

Figure 5: Dispersion Curves of phase/group velocity-wavenumber for the CPT with rotary inertia (Rayleigh).

2. Wave propagation through infinite medium for the TSDT

As for the model of the Third-Order Shear Deformation Theory, we get the three last equations of motion (of the total five) because these include the vertical variation w_0. These are the Eqs. (14c), (14d), (14e) of the section 6.2 of the Part B of this dissertation.
Subsequently, we are going to express in a shorter form the aforementioned equations in order to simplify the calculations. Thus, at a first glance some terms in the equations can be gather together and substitute by a single symbol as seems below.

$$c_4 = \frac{4}{3h^2}, \qquad c_2 = 3c_4 = \frac{4}{h^2}$$

$$\begin{cases} J_i = I_i - c_4 I_{i+2} \\ K_i = I_i - 2c_{i+2} I_{i+2} + c_{i+2}^2 I_{i+4} \\ A_i = I_i - c_{i+2} I_{i+2} \end{cases}, \qquad \text{where for } i = 0, 2, 4$$

$$\begin{cases} J_0 = I_0 - c_4 I_2 \\ K_0 = I_0 - 2c_2 I_2 + c_2^2 I_4 \\ A_0 = I_0 - c_2 I_2 \end{cases}, \qquad \begin{cases} J_2 = I_2 - c_4 I_4 \\ K_2 = I_2 - 2c_4 I_4 + c_4^2 I_6 \\ A_2 = I_2 - c_4 I_4 \end{cases},$$

$$\begin{cases} J_4 = I_4 - c_4 I_6 \\ K_4 = I_4 - 2c_6 I_6 + c_6^2 I_8 \\ A_4 = I_4 - c_6 I_6 \end{cases}$$

Substituting the necessary results of the above into the Eq. (14c) of the section 6.2 (PART B) and neglecting the external load in the right-hand side of the same equation, we get

$$\begin{aligned} & I_0 \ddot{w}_0 + c_4 J_4 \left(\frac{\partial \ddot{\phi}_y}{\partial x_2} + \frac{\partial \ddot{\phi}_x}{\partial x_1} \right) - c_4^2 I_6 \Delta \ddot{w}_0 - \\ & - \frac{G}{\rho} K_0 \left(\frac{\partial \phi_y}{\partial x_2} + \frac{\partial \phi_x}{\partial x_1} + \Delta w_0 \right) - \\ & - c_4 J_4 \frac{2G}{\rho} \frac{1}{1-v} \left(\frac{\partial^3 \phi_x}{\partial^3 x_1} + \frac{\partial^3 \phi_y}{\partial^3 x_2} + \frac{\partial^3 \phi_y}{\partial^2 x_1 \partial x_2} + \frac{\partial^3 \phi_x}{\partial^2 x_2 \partial x_1} \right) + \\ & + 2 c_4^2 \frac{I_6 G}{\rho} \frac{1}{1-v} \Delta^2 w_0 = q \end{aligned} \qquad \textbf{(1)}$$

Following the same path for the Eq. (14d) of the section 6.2 (PART B), we get

$$K_2\ddot{\phi}_x - c_4 J_4 \frac{\partial \ddot{w}_0}{\partial x_1} + \frac{G}{\rho} K_0 \left(\phi_x + \frac{\partial w_0}{\partial x_1} \right) + \\ + \frac{2G}{\rho(1-v)} c_4 J_4 \left(\frac{\partial^3 w_0}{\partial^3 x_1} + \frac{\partial^3 w_0}{\partial x_1 \partial^2 x_2} \right) - \\ - \frac{G}{\rho} K_2 \left(\frac{\partial^2 \phi_x}{\partial^2 x_2} + \frac{1+v}{1-v} \frac{\partial^2 \phi_y}{\partial x_1 \partial x_2} + \frac{2}{1-v} \frac{\partial^2 \phi_x}{\partial^2 x_1} \right) = 0 \quad \text{(2)}$$

And finally, as for the Eq. (14e),

$$K_2 \ddot{\phi}_y - c_4 J_4 \frac{\partial \ddot{w}_0}{\partial x_2} + \frac{G}{\rho} K_0 \left(\phi_y + \frac{\partial w_0}{\partial x_2} \right) + \\ + \frac{2G}{\rho} c_4 J_4 \left(\frac{v}{1-v} \frac{\partial^3 w_0}{\partial x_2 \partial^2 x_1} + \frac{1}{1-v} \frac{\partial^3 w_0}{\partial^3 x_2} \right) + \frac{2G}{\rho} c_4 J_4 \frac{\partial^3 w_0}{\partial x_2 \partial^2 x_1} - \\ - \frac{G}{\rho} K_2 \left(\frac{\partial^2 \phi_x}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi_y}{\partial^2 x_1} \right) - \frac{G}{\rho} K_2 \left(\frac{2v}{1-v} \frac{\partial^2 \phi_x}{\partial x_2 \partial x_1} + \frac{2}{1-v} \frac{\partial^2 \phi_y}{\partial^2 x_2} \right) = 0 \Rightarrow$$

$$K_2 \ddot{\phi}_y - c_4 J_4 \frac{\partial \ddot{w}_0}{\partial x_2} + \frac{G}{\rho} K_0 \left(\phi_y + \frac{\partial w_0}{\partial x_2} \right) + \\ + \frac{2G}{\rho(1-v)} c_4 J_4 \left(\frac{\partial^3 w_0}{\partial x_2 \partial^2 x_1} + \frac{\partial^3 w_0}{\partial^3 x_2} \right) - \\ - \frac{G}{\rho} K_2 \left(\frac{\partial^2 \phi_y}{\partial^2 x_1} + \frac{1+v}{1-v} \frac{\partial^2 \phi_x}{\partial x_2 \partial x_1} + \frac{2}{1-v} \frac{\partial^2 \phi_y}{\partial^2 x_2} \right) = 0 \quad \text{(3)}$$

Thus, we have managed to create a more abbreviated form of the last three governing equations of motion of the model of TSDT.

Since there are three degrees of freedom (w_0, ϕ_x, ϕ_y), this set of equations describes three wave modes. However, the three above equations of motion can be concentrated to a single equation in which the three degrees of freedom will be decoupled and consequently we are able to produce the dispersion relation of the model of TSDT. The previously described way is usual when we have a system of differential equations and we mean to couple the total number of them to one single equation. However, at this moment is regarded more efficient and smart to proceed with an alternative approach of extracting the dispersion relation, because the decoupling of the three partial differential equations in going to appear directly due to the initial assumption of the one-dimensional wave propagation along the x_1-axis of the plate, which demands the elimination of the x_2-spatial derivatives as shown on the consequence.

As for this alternative approach, we consider the equations **(1)** with $q = 0$ on the right-hand side, **(2)** and **(3)** directly and we assume solutions of the form

$$w_0 = w(x_1, x_2\,; t) = B_1\, e^{i(k\,\boldsymbol{n}\cdot\boldsymbol{r} - \omega t)} \qquad \text{(a)}$$

$$\phi_x = B_2\, e^{i(k\,\boldsymbol{n}\cdot\boldsymbol{r} - \omega t)} \qquad \text{(b)}$$

$$\phi_y = B_3\, e^{i(k\,\boldsymbol{n}\cdot\boldsymbol{r} - \omega t)} \qquad \text{(c)}$$

Thus, the three degrees of freedom of the governing equations **(1)**, **(2)** and **(3)** are harmonic functions of the time and space in the context of the problem of TSDT.

Note that the wave propagates dispersively along the horizontal planes of the plate. Further, on the above Eqs. (a), (b), (c) the vector $\boldsymbol{r}$ denotes the position vector to a point on the plane of the wave and $\boldsymbol{n}$ is the unit normal vector to the plane of wave ,as shown on the Figure 4.2 of the book of Karl F. Graff, "Wave Motion in Elastic Solids", pp. 215 presented below.

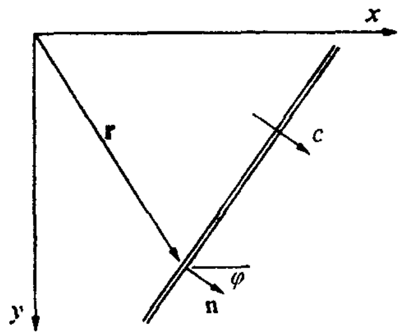

Fig. 4.2. A propagating plane disturbance in two dimensions.

In the context of this problem we study the in-plane wave propagation inside an in-plane isotropic media. Thus, considering a one-dimensional wave propagation along the x- axis of the isotropic material, we have only the x-dependence of the harmonic wave.

Consequently, the propagating two-dimensional plane disturbance of the above figure becomes one-dimensional by rotating the vector $\boldsymbol{r}$ until it coincide with the x-axis. For such a disturbance, each particle along the line ("plane") defined by $\boldsymbol{n}\cdot\boldsymbol{r} - \omega t = \text{constant}$, where $\omega = c\,\lambda$ (as defined above) has the same displacement as exactly its neighbor particles. On a next step, expressing the inter product of the vectors,

$$\boldsymbol{n} = l\,\boldsymbol{i} + m\,\boldsymbol{j} = \cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j} \xrightarrow[l=1]{1D} \boldsymbol{n} = \boldsymbol{i}$$

and $\qquad \boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} \xrightarrow{1D} \boldsymbol{r} = x\,\boldsymbol{i}\,.$

we finally get a simple product of the first components of each vector,

$\boldsymbol{n}\cdot\boldsymbol{r} = x = x_1$ and substitute it into the Eqs. (a), (b), (c)

$$w = B_1\, e^{i(k\,x_1 - \omega t)} \qquad \text{(a`)}$$

$$\phi_x = B_2\, e^{i(k\,x_1 - \omega t)} \qquad \text{(b`)}$$

$$\phi_y = B_3\, e^{i(k\,x_1 - \omega t)} \qquad \text{(c`)}$$

Remind here that k is the wavenumber of the propagation, ω is the angular frequency and c is the phase velocity of the wave propagation.

Further, to reduce the size and the complexity of calculations after the substitution of the Eqs. (a\`), (b\`), (c\`) into the Eqs we procced to some calculations of separate terms existing on the aforementioned governing equations. Is also obvious that we have zero x_2 (or y) - spatial derivatives, which leads to the elimination of the respective terms of the Eqs. **(1)**, **(2)**, **(3)**. Consequently, the final form of the governing equations of the TSDT through the one-dimensional wave propagation and without externally applied loads (free surface) are the following

$$I_0 \ddot{w}_0 + c_4 J_4 \frac{\partial \ddot{\phi}_x}{\partial x_1} - c_4^2 I_6 \frac{\partial^2 \ddot{w}_0}{\partial^2 x_1} - \frac{G}{\rho} K_0 \left(\frac{\partial \phi_x}{\partial x_1} + \frac{\partial^2 w_0}{\partial^2 x_1} \right) - \\ - c_4 J_4 \frac{2G}{\rho} \frac{1}{1-v} \frac{\partial^3 \phi_x}{\partial^3 x_1} + 2 c_4^2 \frac{I_6 G}{\rho} \frac{1}{1-v} \frac{\partial^4 w_0}{\partial^4 x_1} = 0 \qquad \textbf{(6a)}$$

$$K_2 \ddot{\phi}_x - c_4 J_4 \frac{\partial \ddot{w}_0}{\partial x_1} + \frac{G}{\rho} K_0 \left(\phi_x + \frac{\partial w_0}{\partial x_1} \right) + \\ + \frac{2G}{\rho(1-v)} c_4 J_4 \frac{\partial^3 w_0}{\partial^3 x_1} - K_2 \frac{2G}{\rho(1-v)} \frac{\partial^2 \phi_x}{\partial^2 x_1} = 0 \qquad \textbf{(6b)}$$

$$K_2 \ddot{\phi}_y + \frac{G}{\rho} K_0 \phi_y - \frac{G}{\rho} K_2 \frac{\partial^2 \phi_y}{\partial^2 x_1} = 0 \qquad \textbf{(6c)}$$

Note that the above Eqs. (6a), (6b) and (6c) are decoupled. Namely the first two Eqs. are coupled and both of them includes two degrees of freedom w_0 and ϕ_x, whereas the third one is decoupled from the aforementioned equations because it includes only the ϕ_y variable. This fact was expected due to our initial assumption of the infinite plate along one direction, here the x_1-direction. This consideration leads to the elimination of the lateral (to the direction of the wave propagation) distortion of the plate, namely $\phi_y = 0$ and finally we treat it as a beam or so called plate strip (since the length of the plate is regarded here very large in comparison with its breadth). Thus, we are going to occupy with a 2x2 system and through the two degrees of freedom w_0 and ϕ_x and the Eqs. (6a) and (6b), we are going to extract the dispersion curve of the problem.

Subsequently, substituting the above calculated derivatives into the Eqs. (6a) and (6b) we get the following results.

First, from the Eq. **(6a)**

$$- B_1 I_0 \omega^2 - i B_2 c_4 J_4 \omega^2 k - B_1 c_4^2 I_6 \omega^2 k^2 - \\ - i B_2 \frac{G}{\rho} K_0 k + B_1 \frac{G}{\rho} K_0 k^2 + \\ + i B_2 c_4 J_4 \frac{2G}{\rho} \frac{1}{1-v} k^3 + B_1 2 c_4^2 \frac{I_6 G}{\rho} \frac{1}{1-v} k^4 = 0 \Rightarrow$$

$$B_1\, 2c_4^2 \frac{I_6 G}{\rho}\frac{1}{1-v}k^4 - B_1 I_0\,\omega^2 - B_1 c_4^2 I_6\,\omega^2 k^2 + B_1 \frac{G}{\rho}K_0\,k^2 -$$
$$- i\,B_2 \frac{G}{\rho}K_0\,k - i\,B_2\, c_4 J_4\,\omega^2 k + i\,B_2\, c_4 J_4 \frac{2G}{\rho}\frac{1}{1-v}k^3 = 0 \quad \Rightarrow$$

$$B_1\left(c_4^2 \frac{I_6}{\rho}\frac{2G}{1-v}k^4 - I_0\,\omega^2 - c_4^2 I_6\,\omega^2 k^2 + \frac{G}{\rho}K_0\,k^2\right) +$$
$$+ i\,B_2\left(c_4 J_4 \frac{2G}{\rho}\frac{1}{1-v}k^3 - \frac{G}{\rho}K_0\,k - c_4 J_4\,\omega^2\,k\right) = 0 \qquad \textbf{(7a)}$$

From the Eq. **(6b)**,

$$-K_2\,\omega^2 B_2 + c_4 J_4\, i\,\omega^2 B_1 k + \frac{G}{\rho}K_0 \left(B_2 + i\,B_1 k\right) -$$
$$- \frac{2G}{\rho(1-v)}c_4 J_4\, i\,B_1 k^3 + K_2 \frac{2G}{\rho(1-v)}B_2 k^2 = 0 \Rightarrow$$

$$c_4 J_4\, i\,\omega^2 B_1 k + i\,B_1 \frac{G}{\rho}K_0\, k - \frac{2G}{\rho(1-v)}c_4 J_4\, i\,B_1 k^3 +$$
$$+ K_2 \frac{2G}{\rho(1-v)}B_2 k^2 - K_2\,\omega^2 B_2 + \frac{G}{\rho}K_0\, B_2 = 0 \Rightarrow$$

$$i\,B_1\left(c_4 J_4\,\omega^2\,k + \frac{G}{\rho}K_0\,k - \frac{2G}{\rho(1-v)}c_4 J_4\,k^3\right) +$$
$$+ B_2\left(K_2 \frac{2G}{\rho(1-v)}k^2 - K_2\,\omega^2 + \frac{G}{\rho}K_0\right) = 0 \qquad \textbf{(7b)}$$

Subsequently, we note that the above three equations consist a 2x2 system with respect to three unknowns B_1, B_2, namely the amplitudes of the displacements w_0, ϕ_x respectively.

Equating now the determinant of the coefficients B_1, B_2 to zero in the above system yields the dispersion relation, as will be shown on end of this section. This is rational because the sufficient and necessary condition for the existence of non-trivial solution of the aforementioned system, is its zero determinant.

However, to simplify the form of the above equations, we set specific symbols for the quantities multiplied with the coefficients B_1, B_2,

$$\Psi_{11} = c_4^2 \frac{I_6}{\rho}\frac{2G}{1-v}k^4 - I_0\,\omega^2 - c_4^2 I_6\,\omega^2 k^2 + \frac{G}{\rho}K_0\,k^2$$
$$\Psi_{12} = i\left(c_4 J_4 \frac{2G}{\rho}\frac{1}{1-v}k^3 - \frac{G}{\rho}K_0\,k - c_4 J_4\,\omega^2\,k\right)$$

$$\Psi_{21} = i\left(c_4 J_4 \omega^2 k + \frac{G}{\rho} K_0 k - \frac{2G}{\rho(1-v)} c_4 J_4 k^3\right)$$

$$\Psi_{22} = K_2 \frac{2G}{\rho(1-v)} k^2 - K_2 \omega^2 + \frac{G}{\rho} K_0$$

By this way the 2x2 system is simplified to the following form,

$$\begin{array}{l} \Psi_{11} B_1 + \Psi_{12} B_2 = 0 \\ \Psi_{21} B_1 + \Psi_{22} B_2 = 0 \end{array} \quad \text{or} \quad \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0 \Leftrightarrow \boldsymbol{\Psi} \boldsymbol{B} = 0$$

Now the determinant is clear to be written,

$$det(\boldsymbol{\Psi}) = \underbrace{\Psi_{11}\Psi_{22}}_{\Psi_A} - \underbrace{\Psi_{12}\Psi_{21}}_{\Psi_B} = 0$$

First, we calculate the quantities inside the brackets of each part Ψ_A, Ψ_B of the determinant. Second, we try to isolate some specific expressions (which do not contain the wave number k or the angular frequency ω) inside the aforementioned parts in order to make it easier on the total relation of the zero determinant to express explicitly the angular frequency through the wavenumber.

As for the first part (Ψ_A),

$$\Psi_A = \Psi_{11}\Psi_{22} =$$

$$= c_4^2 \frac{I_6}{\rho} \frac{2G}{1-v} K_2 \frac{2G}{\rho(1-v)} k^6 - c_4^2 \frac{I_6}{\rho} \frac{2G}{1-v} K_2 \omega^2 k^4 + c_4^2 \frac{I_6}{\rho} \frac{2G}{1-v} \frac{G}{\rho} K_0 k^4 -$$

$$- I_0 K_2 \frac{2G}{\rho(1-v)} \omega^2 k^2 + I_0 K_2 \omega^4 - I_0 \frac{G}{\rho} K_0 \omega^2 -$$

$$- c_4^2 I_6 K_2 \frac{2G}{\rho(1-v)} \omega^2 k^4 + c_4^2 I_6 K_2 \omega^4 k^2 - c_4^2 I_6 \frac{G}{\rho} K_0 \omega^2 k^2 +$$

$$+ \frac{G}{\rho} K_0 K_2 \frac{2G}{\rho(1-v)} k^4 - \frac{G}{\rho} K_0 K_2 \omega^2 k^2 + \frac{G}{\rho} K_0 \frac{G}{\rho} K_0 k^2 =$$

(and grouping together the terms with the same size of exponent on the k and ω)

$$= c_4^2 I_6 K_2 \left(\frac{2G}{\rho(1-v)}\right)^2 k^6 - c_4^2 \frac{I_6}{\rho} \frac{4G}{1-v} K_2 \omega^2 k^4 -$$

$$- \left(I_0 K_2 \frac{2G}{\rho(1-v)} + c_4^2 I_6 \frac{G}{\rho} K_0 + \frac{G}{\rho} K_0 K_2\right) \omega^2 k^2 +$$

$$+ c_4^2 I_6 K_2 \omega^4 k^2 + I_0 K_2 \omega^4 - I_0 \frac{G}{\rho} K_0 \omega^2 +$$

$$+ \left(c_4^2 I_6 \frac{2G^2}{\rho^2(1-v)} K_0 + K_0 K_2 \frac{2G^2}{\rho^2(1-v)}\right) k^4 + \frac{G}{\rho} K_0 \frac{G}{\rho} K_0 k^2$$

As for the second part (Ψ_B),

$$\Psi_B = \Psi_{12}\Psi_{21} =$$

$$= - {c_4 J_4}^2 \frac{2G}{\rho(1-v)} \omega^2 k^4 - c_4 J_4 \frac{2G^2}{\rho^2(1-v)} K_0 k^4 + \left(c_4 J_4 \frac{2G}{\rho(1-v)}\right)^2 k^6 +$$

$$+ \frac{G}{\rho} K_0 c_4 J_4 \omega^2 k^2 + \frac{G^2}{\rho^2} K_0^2 k^2 - K_0 \frac{2G^2}{\rho^2(1-v)} c_4 J_4 k^4 +$$

$$+ {c_4 J_4}^2 \omega^4 k^2 + c_4 J_4 \frac{G}{\rho} K_0 \omega^2 k^2 - {c_4 J_4}^2 \frac{2G}{\rho(1-v)} \omega^2 k^4 =$$

(and grouping together the terms with the same size of exponent on the k and ω)

$$= - {c_4 J_4}^2 \frac{4G}{\rho(1-v)} \omega^2 k^4 - c_4 J_4 \frac{4G^2}{\rho^2(1-v)} K_0 k^4 + \left(c_4 J_4 \frac{2G}{\rho(1-v)}\right)^2 k^6 +$$

$$+ \frac{2G}{\rho} K_0 c_4 J_4 \omega^2 k^2 + \frac{G^2}{\rho^2} K_0^2 k^2 + {c_4 J_4}^2 \omega^4 k^2$$

Now combining the last two expressions, we get

$$det(\boldsymbol{\Psi}) = \Psi_A - \Psi_B = 0 \Rightarrow$$

$$c_4^2 I_6 K_2 \left(\frac{2G}{\rho(1-v)}\right)^2 k^6 - \left(c_4 J_4 \frac{2G}{\rho(1-v)}\right)^2 k^6 - c_4^2 \frac{I_6}{\rho}\frac{4G}{1-v} K_2 \omega^2 k^4 + c_4^2 J_4^2 \frac{4G}{\rho(1-v)} \omega^2 k^4 -$$

$$- \left(I_0 K_2 \frac{2G}{\rho(1-v)} + c_4^2 I_6 \frac{G}{\rho} K_0 + \frac{G}{\rho} K_0 K_2\right) \omega^2 k^2 - \frac{2G}{\rho} K_0 c_4 J_4 \omega^2 k^2 +$$

$$+ c_4^2 I_6 K_2 \omega^4 k^2 - {c_4 J_4}^2 \omega^4 k^2 + I_0 K_2 \omega^4 - I_0 \frac{G}{\rho} K_0 \omega^2 +$$

$$+ \left(c_4^2 I_6 \frac{2G^2}{\rho^2(1-v)} K_0 + K_0 K_2 \frac{2G^2}{\rho^2(1-v)}\right) k^4 + c_4 J_4 \frac{4G^2}{\rho^2(1-v)} K_0 k^4 = 0 \Rightarrow$$

$$\begin{aligned}
& c_4^2 \left(\frac{2G}{\rho(1-v)}\right)^2 I_6 K_2 - J_4^2 \; k^6 - c_4^2 \; I_6 K_2 - J_4^2 \; \frac{4G}{\rho(1-v)} \omega^2 k^4 - \\
& - \frac{G}{\rho}\left(I_0 K_2 \frac{2}{1-v} + K_0 \; c_4^2 I_6 + K_2 + 2 c_4 J_4\right) \omega^2 k^2 + \\
& + c_4^2 \; I_6 K_2 - J_4^2 \; \omega^4 k^2 + I_0 K_2 \omega^4 - I_0 \frac{G}{\rho} K_0 \omega^2 + \\
& + K_0 \frac{2G^2}{\rho^2(1-v)} \; c_4^2 I_6 + K_2 + 2 c_4 J_4 \; k^4 = 0
\end{aligned} \qquad (8)$$

Now in order to proceed to the non-dimensionality of the above dispersion relation, we calculate separately some coefficients which are repeated constantly and also include terms used previously for the abbreviation of the governing equations of the model of TSDT. By this way these terms can be written extensively again and after that can be simplified in order to result to new coefficients that appear explicitly the fundamental units. Thus,

$$c_4^2 I_6 + K_2 + 2c_4 J_4 = c_4^2 I_6 + I_2 - 2c_4 I_4 + c_4^2 I_6 + 2c_4 I_4 - 2c_4^2 I_6 =$$

$$= I_2 = \frac{\rho h^3}{12}$$

$$K_2 I_6 - J_4^2 = \; I_2 - 2c_4 I_4 + c_4^2 I_6 \;\; I_6 - \;\; I_4 - c_4 I_6 \;\; {}^2 =$$

$$= I_2 I_6 - 2c_4 I_4 I_6 + c_4^2 I_6^2 - \; I_4^2 + 2c_4 I_4 I_6 - c_4^2 I_6^2 =$$

$$= I_2 I_6 - I_4^2 = \frac{\rho h^3}{12}\frac{\rho h^7}{448} - \frac{\rho^2 h^{10}}{6400} \simeq 2.98\ 10^{-5} \rho^2 h^{10}$$

Note that the second calculated term form the above two, is of much smaller order since due to the initial assumption of the moderately thick plate, $h^{10} \ll h^3$. However we proceed to the total calculation of the coefficients of k^6, $\omega^2 k^4$, $\omega^2 k^2$, $\omega^4 k^2$, ω^4, ω^2, k^4 in order to decide about the size of the contribution of each term to dispersion relation, Eq. (8).
Consequently, we proceed to a specific notation of the aforementioned coefficients through which we separate the net numbers (dimensionless quantities) from the dimensional parameters of the material or the geometry of the plate, such as G, ρ, h. Thus, we insert the coefficients a_{60}, a_{42}, a_{22}^1, a_{22}^2, a_{24}, a_{04}, a_{02}, a_{40}. The rationality of the sub indexes of these coefficients a_{ij} is that the first index i declares the exponent of the wavenumber (k^i, $i = 2, 4, 6$) and the second shows the exponent of the angular frequency (ω^j, $j = 2, 4$). By this way it becomes easier to distinguish the quantities that contribute to the elimination of the "dimension" from the Eq. (8).

The coefficient of the k^6,

$$c_4^2 \left(\frac{2G}{\rho(1-v)}\right)^2 \; I_6 K_2 - J_4^2 \; = \frac{16}{9\cancel{h^4}} \frac{4G^2}{\cancel{\rho^2}(1-v)^2} 2.98\ 10^{-5} \cancel{\rho^2} h^{\cancel{10}6} =$$

$$= \frac{64G^2}{9(1-v)^2} 2.98\ 10^{-5} h^6 =$$

$$= \underbrace{\frac{64}{9} 2.98\ 10^{-5}}_{a_{60}} \frac{h^6 G^2}{(1-v)^2} = a_{60} \frac{h^6 G^2}{(1-v)^2}$$

The coefficient of $\omega^2 k^4$,

$$c_4^2 \; I_6 K_2 - J_4^2 \; \frac{4G}{\rho(1-v)} = \frac{16}{9\cancel{h^4}} 2.98\ 10^{-5} \; \rho^{\cancel{2}} h^{\cancel{10}6} \frac{4G}{\cancel{\rho}(1-v)} =$$

$$= \frac{4G}{1-v} \frac{16}{9} 2.98\ 10^{-5} \rho h^6 =$$

$$= \underbrace{21.19\ 10^{-5}}_{a_{42}} \frac{G\rho h^6}{1-v} = a_{42} \frac{G\rho h^6}{1-v}$$

The third term $\omega^2 k^2$,

$$\frac{G}{\rho}\left(I_0 K_2 \frac{2}{1-v} + K_0 \;\; c_4^2 I_6 + K_2 + 2c_4 J_4 \right) =$$

$$= \frac{G}{\rho}\left(I_0 \;\; I_2 - 2c_4 I_4 + c_4^2 I_6 \;\; \frac{2}{1-v} + \;\; I_0 - 2c_2 I_2 + c_2^2 I_4 \;\; I_2 \right) =$$

$$= \frac{G}{\rho}\left(\;\; I_0 I_2 - 2c_4 I_0 I_4 + c_4^2 I_0 I_6 \;\; \frac{2}{1-v} + \;\; I_0 I_2 - 2c_2 I_2^2 + c_2^2 I_4 I_2 \;\; \right) =$$

$$= \frac{G}{\rho}\left(\left(\frac{\rho^2 h^4}{12} - \frac{\rho^2 h^4}{30} + \frac{\rho^2 h^4}{252} \right) \frac{2}{1-v} + \left(\frac{\rho^2 h^4}{12} - \frac{\rho^2 h^4}{18} + \frac{\rho^2 h^4}{60} \right) \right) \simeq$$

$$\simeq \frac{G}{\rho}\left(0.054\, \rho^2 h^4 \frac{2}{1-v} + 0.044\, \rho^2 h^4 \right) =$$

$$= G\left(0.054 \frac{2}{1-v} + 0.044 \right) \rho h^4 = 0.054 \frac{2\rho h^4 G}{1-v} + 0.044\, G \rho h^4 =$$

$$= \underbrace{10.8\; 10^{-2}}_{a_{22}^{(1)}} \frac{\rho h^4 G}{1-v} + \underbrace{4.4\; 10^{-2}}_{a_{22}^{(2)}} G \rho h^4 =$$

$$= a_{22}^{(1)} \frac{G \rho h^4}{1-v} + a_{22}^{(2)} G \rho h^4$$

The coefficient of the $\omega^4 k^2$,

$$c_4^2 \;\; I_6 K_2 - J_4^2 \;\; = \frac{16}{9 \cancel{h^4}} 2.98\; 10^{-5} \rho^2 h^{\cancel{10}\,6} = \underbrace{5.298\; 10^{-5}}_{a_{24}} \rho^2 h^6 = a_{24} \;\; \rho^2 h^6$$

As for the coefficient of the ω^4,

$$I_0 K_2 = I_0 \;\; I_2 - 2c_4 I_4 + c_4^2 I_6 \;\; = \;\; I_0 I_2 - 2c_4 I_0 I_4 + c_4^2 I_0 I_6 \;\; =$$

$$= \rho h \frac{\rho h^3}{12} - 2\frac{4}{3h^2} \rho h \frac{\rho h^5}{80} + \frac{16}{9h^4} \rho h \frac{\rho h^7}{448} = \left(\frac{1}{12} - \frac{1}{30} + \frac{1}{252} \right) \rho^2 h^4 = \underbrace{5.4\; 10^{-2}}_{a_{04}} \rho^2 h^4$$

As for the coefficient of the ω^2,

$$I_0 \frac{G}{\rho} K_0 = I_0 \frac{G}{\rho} \;\; I_0 - 2c_2 I_2 + c_2^2 I_4 \;\; = G\left(\rho h^2 - \frac{8}{h^2} \frac{\rho h^4}{12} + \frac{16}{h^4} \frac{\rho h^6}{80} \right) =$$

$$= G\left(\rho h^2 - \frac{2}{3} \rho h^2 + \frac{\rho h^2}{5} \right) = \frac{8}{15} G \rho h^2 = a_{02}\, G \rho h^2$$

The coefficient of the k^4,

$$K_0 \frac{2G^2}{\rho^2(1-\nu)} \quad c_4^2 I_6 + K_2 + 2c_4 J_4 \quad =$$

$$= \quad I_0 - 2c_2 I_2 + c_2^2 I_4 \; \frac{2G^2}{\rho^2(1-\nu)} I_2 \; =$$

$$= \frac{2G^2}{\rho^2(1-\nu)} \quad I_0 I_2 - 2c_2 I_2^2 + c_2^2 I_4 I_2 \quad =$$

$$= \frac{2G^2}{\cancel{\rho^2}(1-\nu)}\left(\cancel{\rho} h \frac{\cancel{\rho} h^3}{12} - \frac{8}{h^2}\frac{\cancel{\rho^2} h^6}{144} + \frac{16}{h^4}\frac{\cancel{\rho} h^5}{80}\frac{\cancel{\rho} h^3}{12}\right) = \frac{4}{45}\frac{G^2 h^4}{1-\nu} = a_{40}\frac{G^2 h^4}{1-\nu}$$

Thus, substituting the final and smarter form of the above coefficients into the Eq. (8),

$$a_{60}\frac{h^6 G^2}{(1-\nu)^2}k^6 + a_{40}\frac{G^2 h^4}{1-\nu}k^4 - a_{42}\frac{G\rho h^6}{1-\nu}k^4\omega^2 + a_{24}\,\rho^2 h^6 k^2 \omega^4 -$$
$$-\left(a_{22}^{(1)}\frac{G\rho h^4}{1-\nu} + a_{22}^{(2)}\,G\rho h^4\right)k^2\omega^2 + a_{04}\,\rho^2 h^4\omega^4 - a_{02}\,G\rho h^2\omega^2 = 0 \qquad (8`)$$

As a next step we consider the non-dimensional wavenumber $\tilde{k}$ and the non-dimensional angular frequency $\tilde{\omega}$, as exactly shown for the Kirchhoff's Plate Theory. Thus,

$$\tilde{\omega} = \frac{\omega}{\Omega} \qquad \text{and} \qquad \tilde{k} = \frac{k}{K}$$

and by substituting the above relations to the Eq. (8`), we get

$$a_{60}\frac{h^6 G^2 K^6}{(1-\nu)^2}\tilde{k}^6 + a_{40}\frac{G^2 h^4 K^4}{1-\nu}\tilde{k}^4 - a_{42}\frac{G\rho h^6 K^4 \Omega^2}{1-\nu}\tilde{k}^4\tilde{\omega}^2 +$$
$$+ a_{24}\,\rho^2 h^6 K^2 \Omega^4\,\tilde{k}^2\tilde{\omega}^4 - \left(a_{22}^{(1)}\frac{\rho h^4 G K^2 \Omega^2}{1-\nu} + a_{22}^{(2)}\,G\rho h^4 K^2 \Omega^2\right)\tilde{k}^2\tilde{\omega}^2 +$$
$$+ a_{04}\,\rho^2 h^4 \Omega^4\,\tilde{\omega}^4 - a_{02}\,G\rho h^2 \Omega^2\,\tilde{\omega}^2 = 0 \quad \xrightarrow{\div\, a_{02}\,G\rho h^2\Omega^2}$$

$$\frac{a_{60}\,h^4 G K^6}{a_{02}\,\rho\,\Omega^2(1-\nu)^2}\tilde{k}^6 + \frac{a_{40}\,G h^2 K^4}{a_{02}\,\rho\,\Omega^2(1-\nu)}\tilde{k}^4 - \frac{a_{42}\,h^4 K^4}{a_{02}\,(1-\nu)}\tilde{k}^4\tilde{\omega}^2 +$$
$$+ \frac{a_{24}\,\rho h^4 K^2 \Omega^2}{a_{02}\,G}\tilde{k}^2\tilde{\omega}^4 - \left(\frac{a_{22}^{(1)}\,h^2 K^2}{a_{02}\,(1-\nu)} + \frac{a_{22}^{(2)}\,h^2 K^2}{a_{02}}\right)\tilde{k}^2\tilde{\omega}^2 + \qquad (9)$$
$$+ \frac{a_{04}\,\rho h^2 \Omega^2}{a_{02}\,G}\tilde{\omega}^4 - \tilde{\omega}^2 = 0$$

For the sake of convenience, we repeat the values of the constants a_{60}, a_{42}, $a_{22}^{(1)}$, $a_{22}^{(2)}$, a_{24}, a_{04}, a_{02} and a_{40} below.

a_{60}	$21.19\ 10^{-5}$
a_{42}	$21.19\ 10^{-5}$
$a_{22}^{(1)}$	$10.80\ 10^{-2}$
$a_{22}^{(2)}$	$4.40\ 10^{-2}$
a_{24}	$5.298\ 10^{-5}$
a_{04}	$5.40\ 10^{-2}$
a_{02}	$5.33\ 10^{-1}$
a_{40}	$8.88\ 10^{-2}$

Subsequently, from the Eqs. (5a), (5b) of the section 1 of the present part (Part C), we have

$$K^2 = \frac{12}{h^2}\ , \qquad , \qquad K^4 = \frac{12^2}{h^4}\ , \qquad K^6 = \frac{12^3}{h^6}\ ,$$

$$\Omega^2 = \frac{12\,E}{(1-v^2)\rho\, h^2}\ , \qquad \Omega^4 = \frac{12^2\,E^2}{(1-v^2)^2\rho^2\, h^4}\,.$$

Thus, substituting the above dimensional quantities into the coefficients of each terms of the Eq. (9), namely $\tilde{k}^6$, $\tilde{\omega}^2\tilde{k}^4$, $\tilde{\omega}^2\tilde{k}^2$, $\tilde{\omega}^4\tilde{k}^2$, $\tilde{\omega}^4$, $\tilde{\omega}^2$, $\tilde{k}^4$, we get the following separate results, before we derive the final non-dimensional form of the Eq. (9).

For the coefficient of the term involving $\tilde{k}^6$,

$$\frac{a_{60}\,h^4\,G\,K^6}{a_{02}\,\rho\,\Omega^2(1-v)^2} = \frac{a_{60}\,h^4\,G}{a_{02}\,\rho\,(1-v)^2}\,\frac{12^3}{h^6}\,\frac{(1-v^2)\rho\,h^2}{12\,E} =$$

$$= \frac{a_{60}\,h^4\,E}{a_{02}\,\rho\;2\,(1+v)(1-v)^2}\,\frac{12^3}{h^6}\,\frac{(1-v^2)\rho\,h^2}{12\,E} = \frac{a_{60}}{a_{02}}\,\frac{12^2}{2\,(1-v)}$$

For the coefficient of the term including $\tilde{k}^4$,

$$\frac{a_{40}\,G\,h^2\,K^4}{a_{02}\,\rho\,\Omega^2(1-v)} = \frac{a_{40}\,E\,h^2}{a_{02}\,\rho\;2\,(1+v)(1-v)}\,\frac{12^2}{h^4}\,\frac{(1-v^2)\rho\,h^2}{12\,E} = \frac{6\,a_{40}}{a_{02}}$$

The coefficient of $\tilde{k}^4\,\tilde{\omega}^2$ becomes,

$$\frac{a_{42}\,h^4\,K^4}{a_{02}\,(1-v)} = \frac{a_{42}\,h^4}{a_{02}\,(1-v)}\,\frac{12^2}{h^4} = \frac{12^2 a_{42}}{a_{02}\,(1-v)}$$

And the coefficient of the term including $\tilde{k}^2\,\tilde{\omega}^4$ is converted to,

$$\frac{a_{24}\,\rho\,h^4 K^2\,\Omega^2}{a_{02}\,G} = \frac{a_{24}\,\rho\,h^4 2\,(1+v)}{a_{02}\,E}\,\frac{12}{h^2}\,\frac{12\,E}{(1-v^2)\rho\,h^2} = \frac{a_{24}}{a_{02}}\,\frac{2\cdot 12^2}{1-v}$$

As for the coefficient of the term including $\tilde{k}^2\,\tilde{\omega}^2$,

$$\left(\frac{a_{22}^{(1)}\,h^2}{a_{02}\,(1-v)}+\frac{a_{22}^{(2)}\,h^2}{a_{02}}\right)K^2=\left(\frac{a_{22}^{(1)}\,h^2}{a_{02}\,(1-v)}+\frac{a_{22}^{(2)}\,h^2}{a_{02}}\right)\frac{12}{h^2}=\left(\frac{a_{22}^{(1)}}{a_{02}\,(1-v)}+\frac{a_{22}^{(2)}}{a_{02}}\right)12$$

And finally for the coefficient multiplied with $\tilde{\omega}^4$,

$$\frac{a_{04}\,\rho\,h^2\,\Omega^2}{a_{02}\,G}=\frac{a_{04}\,\rho\,h^2}{a_{02}\,G}\,\frac{12\,E}{(1-v^2)\rho\,h^2}=\frac{a_{04}}{a_{02}}\,\frac{2\cdot 12}{1-v}$$

Note that all the above results of the coefficients are non-dimensional and the only parameter which remains is the Poisson's ratio v. Now, it is time to substitute the above results to the Eq. (9).

$$\frac{a_{60}}{a_{02}}\,\frac{12^2}{2\,(1-v)}\,\tilde{k}^6+\frac{6\,a_{40}}{a_{02}}\,\tilde{k}^4-\frac{a_{42}}{a_{02}}\,\frac{12^2}{(1-v)}\,\tilde{k}^4\,\tilde{\omega}^2+\frac{a_{24}}{a_{02}}\,\frac{2\cdot 12^2}{1-v}\,\tilde{k}^2\,\tilde{\omega}^4-$$
$$-\left(\frac{a_{22}^{(1)}}{a_{02}\,(1-v)}+\frac{a_{22}^{(2)}}{a_{02}}\right)12\,\tilde{k}^2\,\tilde{\omega}^2+\frac{a_{04}}{a_{02}}\,\frac{2\cdot 12}{1-v}\,\tilde{\omega}^4-\tilde{\omega}^2=0$$

Subsequently, we use another notation for the coefficients of $\tilde{k}^6$, $\tilde{\omega}^2\,\tilde{k}^4$, $\tilde{\omega}^2\,\tilde{k}^2$, $\tilde{\omega}^4\,\tilde{k}^2$, $\tilde{\omega}^4$, $\tilde{\omega}^2$, $\tilde{k}^4$, in order to simplify the form of the last dispersion relation. Thus,

$$C_{\tilde{k}^6}=\frac{a_{60}}{a_{02}}\,\frac{12^2}{2\,(1-v)}=\frac{28.62\;10^{-5}}{1-v}\;,\qquad C_{\tilde{k}^4}=\frac{6\,a_{40}}{a_{02}}=0.999\simeq 1\;,$$

$$C_{\tilde{k}^4\,\tilde{\omega}^2}=\frac{12^2 a_{42}}{a_{02}\,(1-v)}=\frac{57.25\;10^{-5}}{1-v}\;,\qquad C_{\tilde{k}^2\,\tilde{\omega}^4}=\frac{a_{24}}{a_{02}}\,\frac{2\cdot 12^2}{1-v}=\frac{28.63\;10^{-5}}{1-v}\;,$$

$$C_{\tilde{k}^2\,\tilde{\omega}^2}=\left(\frac{a_{22}^{(1)}}{a_{02}\,(1-v)}+\frac{a_{22}^{(2)}}{a_{02}}\right)12=\frac{2.436}{1-v}+0.996\simeq\frac{2.436}{1-v}+1\;,$$

$$C_{\tilde{\omega}^4}=\frac{a_{04}}{a_{02}}\,\frac{2\cdot 12}{1-v}=\frac{2.432}{1-v}$$

And after this step, we get the following form of the non-dimensional dispersion relation for the TSDT,

$$\boxed{C_{\tilde{k}^6}\,\tilde{k}^6+C_{\tilde{k}^4}\,\tilde{k}^4-C_{\tilde{k}^4\tilde{\omega}^2}\,\tilde{k}^4\,\tilde{\omega}^2+C_{\tilde{k}^2\tilde{\omega}^4}\,\tilde{k}^2\,\tilde{\omega}^4-C_{\tilde{k}^2\tilde{\omega}^2}\,\tilde{k}^2\,\tilde{\omega}^2+C_{\tilde{\omega}^4}\,\tilde{\omega}^4-\tilde{\omega}^2=0}\qquad\textbf{(10)}$$

The next important stage, is to express the non-dimensional angular frequency $\tilde{\omega}$ explicitly as a function of the non-dimensional wavenumber $\tilde{k}$, in order to illustrate their relation on specific plots. To accomplish the previous, we choose to diminish the grade of the polynomial equation (10) as for the $\tilde{\omega}$. Thus, we set $\tilde{\omega}^2=y$ and substitute into the Eq. (10).

$$C_{\tilde{k}^6}\,\tilde{k}^6 + C_{\tilde{k}^4}\,\tilde{k}^4 - C_{\tilde{k}^4\tilde{\omega}^2}\,\tilde{k}^4\,y + C_{\tilde{k}^2\tilde{\omega}^4}\,\tilde{k}^2\,y^2 - C_{\tilde{k}^2\tilde{\omega}^2}\,\tilde{k}^2\,y + C_{\tilde{\omega}^4}\,y^2 - y = 0 \;\Rightarrow$$

And grouping appropriately the terms in order to solve the binomial equations as to $y = \tilde{\omega}^2$,

$$\underbrace{C_{\tilde{k}^2\tilde{\omega}^4}\,\tilde{k}^2 + C_{\tilde{\omega}^4}}_{A}\;y^2 \;-\; \underbrace{C_{\tilde{k}^4\tilde{\omega}^2}\,\tilde{k}^4 + C_{\tilde{k}^2\tilde{\omega}^2}\,\tilde{k}^2 + 1}_{B}\;y \;+\; \underbrace{C_{\tilde{k}^6}\,\tilde{k}^6 + C_{\tilde{k}^4}\,\tilde{k}^4}_{C} = 0$$

The discriminant of the above binomial equation is, $\Delta = B^2 - 4AC$ and the solutions of the above equation are

$$y = \frac{-B \pm \sqrt{\Delta}}{2A} \;\Leftrightarrow\; y = \frac{-B + \sqrt{\Delta}}{2A} \;\text{ or }\; y = \frac{-B - \sqrt{\Delta}}{2A} \;\Leftrightarrow$$

$$\Leftrightarrow\; \tilde{\omega}^2 = \frac{-B + \sqrt{\Delta}}{2A} \;\text{ or }\; \tilde{\omega}^2 = \frac{-B - \sqrt{\Delta}}{2A} \;\Leftrightarrow$$

$$\Leftrightarrow \left(\tilde{\omega} = \sqrt{\frac{-B + \sqrt{\Delta}}{2A}} \;\text{ or }\; -\sqrt{\frac{-B + \sqrt{\Delta}}{2A}}\right) \;\text{ or }\; \left(\tilde{\omega} = \sqrt{\frac{-B - \sqrt{\Delta}}{2A}} \;\text{ or }\; -\sqrt{\frac{-B - \sqrt{\Delta}}{2A}}\right)$$

However now we have to examine which of the four relations between the $\tilde{\omega}$ and $\tilde{k}$ give positive values for the angular frequency $\tilde{\omega}$, because the negative values of the $\tilde{\omega}$ have no physical interpretation. Consequently, we mean to keep only the branches with positive values for the $\tilde{\omega}$ and we are going to reject those which give negative values for the aforementioned quantity. Further, we have to investigate the sign of the quantities under the square roots and the sign of the discriminant in order to find if we have real or/and imaginary angular frequencies.

As for the coefficients of the binomial equation A, B, C, it is obvious that $A > 0$, $B < 0$ and $C > 0$. Subsequently, we investigate the sign of the discriminant $\Delta = B^2 - 4AC$, which is an eighth-order polynomial as shown below.

$$\Delta = \; C^2_{\tilde{k}^4\tilde{\omega}^2} - 4C_{\tilde{k}^2\tilde{\omega}^4}\,C_{\tilde{k}^6}\;\tilde{k}^8 + \;2\;C_{\tilde{k}^4\tilde{\omega}^2} - 2C_{\tilde{k}^2\tilde{\omega}^4}\,C_{\tilde{k}^4} - 2C_{\tilde{\omega}^4}\,C_{\tilde{k}^6}\;\tilde{k}^6 +$$

$$+\; C^2_{\tilde{k}^2\tilde{\omega}^2} + 2C_{\tilde{k}^4\tilde{\omega}^2} - 4C_{\tilde{\omega}^4}\,C_{\tilde{k}^4}\;\tilde{k}^4 + \;2C_{\tilde{k}^2\tilde{\omega}^2}\tilde{k}^2 + 1$$

By the program Matlab R2013a, choosing the value of Poisson's ratio $v = 0.3$ (usual value for a wide range of materials) and the range of non-dimensional wavenumber $\tilde{k} \in [0,1]$, we find that the discriminant Δ is positive for all values of $\tilde{k}$ inside the interval $[0,1]$. Thus, $\Delta > 0$. Note also that if we substitute the values of the coefficients inside the brackets of the above discriminant, we find that

$$\Delta = -2.3367\,\tilde{k}^8 + 6.1729\,\tilde{k}^4 + 8.96\,\tilde{k}^2 + 1$$

which shows that the discriminant could take negative values for extremely large wavenumbers, since the only negative coefficient is this of $\tilde{k}^8$. However wavenumbers of extremely large size will not occupy us on the wave propagation through plate, since they have not so clear physical impact on these applications.

Further, we investigate the sign of the under square root quantities, $-B+\sqrt{\Delta}$ and $-B-\sqrt{\Delta}$, in order to conclude if we have real or imaginary angular frequencies. The first one relation,

$$y_1 = -B+\sqrt{\Delta} = C_{\tilde{k}^4\tilde{\omega}^2}\tilde{k}^4 + C_{\tilde{k}^2\tilde{\omega}^2}\tilde{k}^2 + 1 + \\ + \sqrt{\begin{array}{c} \left(C_{\tilde{k}^4\tilde{\omega}^2}^2 - 4C_{\tilde{k}^2\tilde{\omega}^4}C_{\tilde{k}^6}\right)\tilde{k}^8 + 2\left(C_{\tilde{k}^4\tilde{\omega}^2} - 2C_{\tilde{k}^2\tilde{\omega}^4}C_{\tilde{k}^4} - 2C_{\tilde{\omega}^4}C_{\tilde{k}^6}\right)\tilde{k}^6 + \\ + \left(C_{\tilde{k}^2\tilde{\omega}^2}^2 + 2C_{\tilde{k}^4\tilde{\omega}^2} - 4C_{\tilde{\omega}^4}C_{\tilde{k}^4}\right)\tilde{k}^4 + 2C_{\tilde{k}^2\tilde{\omega}^2}\tilde{k}^2 + 1 \end{array}}$$

is obviously positive ($y_1 > 0$) since we have $\Delta > 0$ and $B < 0 \Rightarrow -B > 0$.

As for the second relation,

$$y_2 = -B-\sqrt{\Delta} = C_{\tilde{k}^4\tilde{\omega}^2}\tilde{k}^4 + C_{\tilde{k}^2\tilde{\omega}^2}\tilde{k}^2 + 1 - \\ - \sqrt{\begin{array}{c} \left(C_{\tilde{k}^4\tilde{\omega}^2}^2 - 4C_{\tilde{k}^2\tilde{\omega}^4}C_{\tilde{k}^6}\right)\tilde{k}^8 + 2\left(C_{\tilde{k}^4\tilde{\omega}^2} - 2C_{\tilde{k}^2\tilde{\omega}^4}C_{\tilde{k}^4} - 2C_{\tilde{\omega}^4}C_{\tilde{k}^6}\right)\tilde{k}^6 + \\ + \left(C_{\tilde{k}^2\tilde{\omega}^2}^2 + 2C_{\tilde{k}^4\tilde{\omega}^2} - 4C_{\tilde{\omega}^4}C_{\tilde{k}^4}\right)\tilde{k}^4 + 2C_{\tilde{k}^2\tilde{\omega}^2}\tilde{k}^2 + 1 \end{array}}$$

it is not clear if y_2 takes positive or negative values inside the interval $\tilde{k} \in [0,1]$. Consequently, by the aid of the mathematical package Matlab R2013a once again, we find that $y_2 > 0$ for all values of $\tilde{k}$ inside the interval $[0,1]$.

Finally, we conclude that we have only real values of the non-dimensional angular frequencies and further we choose to illustrate only the two positive of the totally four, namely the

$$\tilde{\omega}_{Shear} = \sqrt{\frac{-B+\sqrt{\Delta}}{2A}} \quad (11)$$

and

$$\tilde{\omega}_{flex} = \sqrt{\frac{-B-\sqrt{\Delta}}{2A}} \quad (12)$$

The relation (11) is illustrated by the red curve, which is the shear branch of the TSDT and the relation (12) is illustrated by the green curve, which is the flexural branch of the TSDT, shown on the following figure (Figure 6).

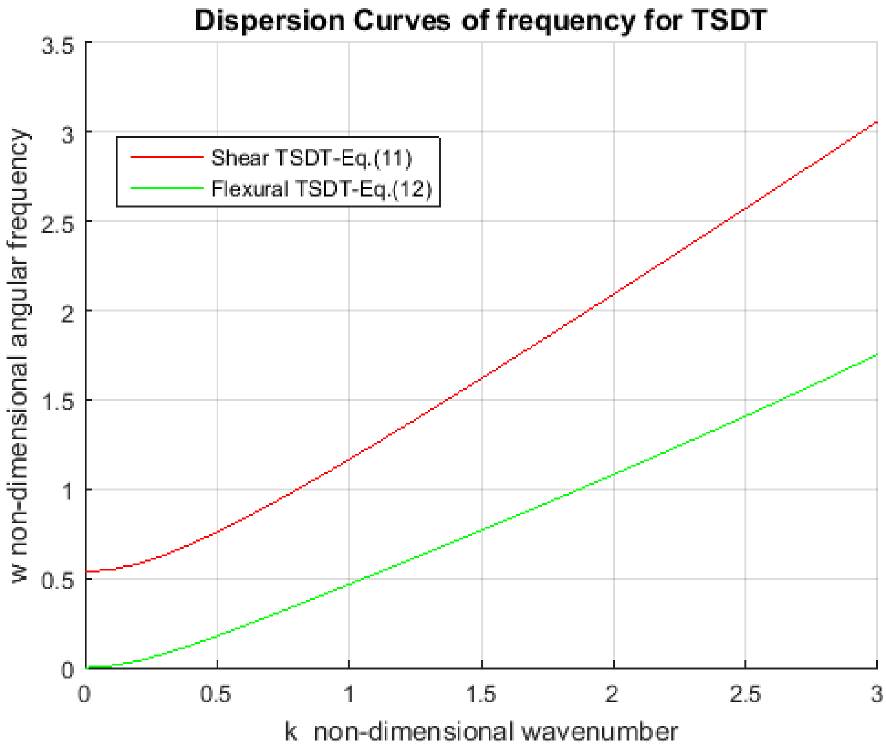

Figure 6: Dispersion Curves of frequency-wavenumber as for the Third-Order Shear Deformation Plate Theory.

The next step is to present the form of phase and group velocity of the wave propagation. Thus, it is shown the relation between the non-dimensional phase velocity of the wave propagation and the non-dimensional wavenumber and subsequently the relation between the non-dimensional group velocity of the wave propagation and the non-dimensional wavenumber. However, here we have two branches as for the phase velocity, namely the shear and the flexural due to the existence of two branches for the angular frequency. For the same reason, we have two branches for the group velocity, one shear and one flexural. Thus, as for the phase velocities we get

$$\tilde{c}_{p\ Shear} = \frac{\tilde{\omega}_{Shear}}{\tilde{k}} = \frac{1}{\tilde{k}}\sqrt{\frac{-B+\sqrt{\Delta}}{2\,A}} \tag{13}$$

and
$$\tilde{c}_{p\ flex} = \frac{\tilde{\omega}_{flex}}{\tilde{k}} = \frac{1}{\tilde{k}}\sqrt{\frac{-B-\sqrt{\Delta}}{2\,A}} \tag{14}$$

As for the group velocities, we derive

$$\tilde{c}_{g\ Shear} = \frac{d\tilde{\omega}_{Shear}}{d\tilde{k}} = \frac{1}{\sqrt{2A}}\,\frac{1}{\sqrt{-B+\sqrt{\Delta}}}\left(\frac{\partial B}{\partial \tilde{k}} + \frac{1}{2\sqrt{\Delta}}\,\frac{\partial \Delta}{\partial \tilde{k}}\right) \tag{15}$$

$$\tilde{c}_{g\ flex} = \frac{d\tilde{\omega}_{flex}}{d\tilde{k}} = \frac{1}{\sqrt{2A}}\,\frac{1}{\sqrt{-B-\sqrt{\Delta}}}\left(-\frac{\partial B}{\partial \tilde{k}} - \frac{1}{2\sqrt{\Delta}}\,\frac{\partial \Delta}{\partial \tilde{k}}\right) \tag{16}$$

where, $\dfrac{\partial \Delta}{\partial \tilde{k}} = 2\,B\dfrac{\partial B}{\partial \tilde{k}} - 4\left(\dfrac{\partial A}{\partial \tilde{k}}C + A\dfrac{\partial C}{\partial \tilde{k}}\right)$ and $\dfrac{\partial A}{\partial \tilde{k}} = 2\,C_{\tilde{k}^2\tilde{\omega}^4}\,\tilde{k}$,

$$\frac{\partial B}{\partial \tilde{k}} = -\ 4\,C_{\tilde{k}^4\,\tilde{\omega}^2}\ \tilde{k}^{\,3} + 2\,C_{\tilde{k}^2\,\tilde{\omega}^2}\ \tilde{k} \qquad \text{and} \qquad \frac{\partial C}{\partial \tilde{k}} = 6\,C_{\tilde{k}^6}\ \tilde{k}^{\,5} + 4\,C_{\tilde{k}^4}\ \tilde{k}^{\,3}.$$

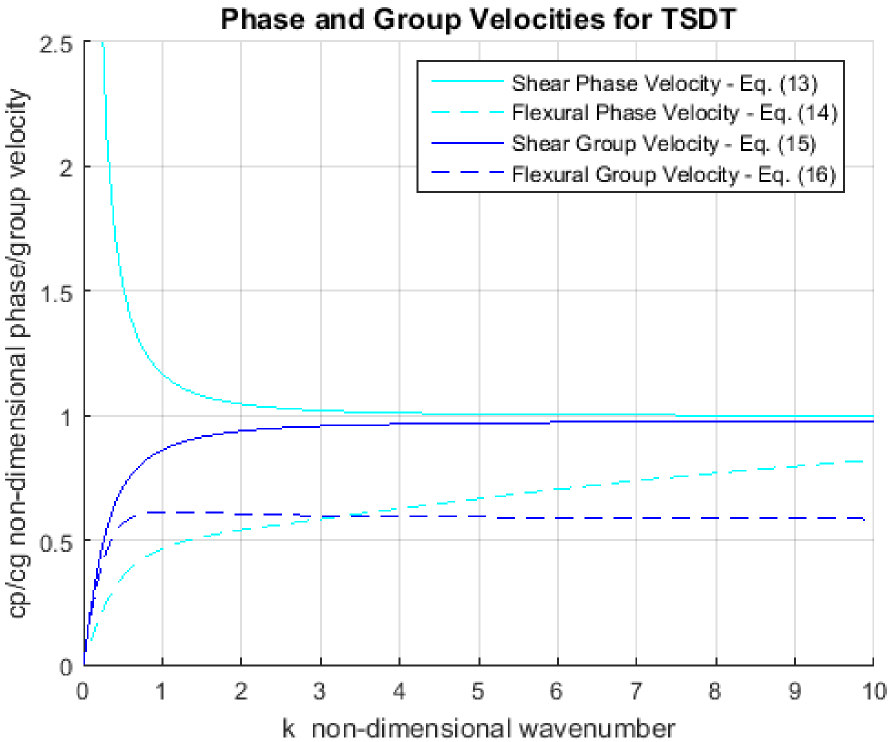

Figure 7: Dispersion Curves of phase/group velocity-wavenumber as for the TSDT.

In conclusion, we derive dimensionless dispersion relations of the model of Third-Order Shear Deformable Plate, which essentially coincide with the model of Bickford-Reddy Beam, due to the assumption of one-dimensional wave propagation along the infinite dimension of the plate.

3. Comparison of the dispersion curves of the models of CPT, FSDT and TSDT

Finally, on the present section taking into account the plots of the previous two sections (section 1 and 2 of the Part C) and the corresponding plots of the APPENDIX B (for the Mindlin's Plate), we set them into the same figures as for the kind of dispersion relation (frequency, phase or group velocity) in order to compare the results of the three plate models (CPT, FSDT and TSDT). To compare with more accuracy the models and to acquire a better sight as for their asymptotic behavior for larger wavenumbers, we are going to present the aforementioned plots for (an appropriate per case) different range of the wavenumbers. The last change will not influence the sign of the discriminant and the quantities existing under the square roots of the frequencies (section 2 of the Part C and APPENDIX B).

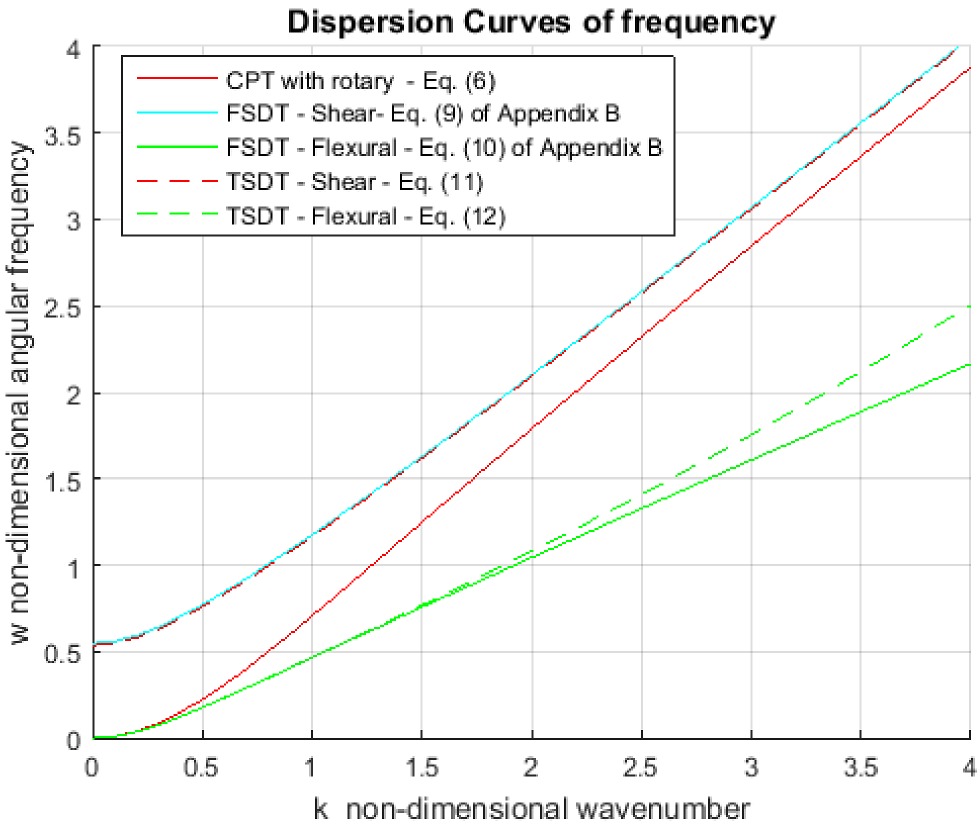

Figure 8: Comparison of the frequency-wavenumber dispersion curves.

In order to compare the frequencies of the different "Plate Theories", we choose the range of the wavenumber $\tilde{k} \in [0,4]$. This choice is on purpose because the flexural branch of the TSDT gives good results for small values of $\tilde{k}$. Especially for $\tilde{k} < 3$ the results of the flexural TSDT coincide with that of the flexural part of FSDT. Thus, the model of FSDT gives better results in comparison with the model of TSDT for larger wavenumbers.

Remark also that the two higher-order plate theories examined in the context of this dissertation have identically similar behavior as for the shear branches for a wide range of wavenumbers.

Further note that the CPT overpredicts the values of frequencies but coincidence with those of the higher-order plate theories for very small wavenumber near the zero. This fact is obvious from the above figure (Figure 8), since the unique branch of the CPT is compared with the flexural branches of the higher-order plate theories. The last is justified and rational because the dispersion relation of the CPT contains only the variation w_0, which affects the vertical motion (flexural response) of the plate, whereas the dispersion curves of the FSDT and TSDT are divided to two branches, one related to the vertical displacement δw_0 (flexural branch) and the other associated to the in-plane motion (shear branch) through the displacement $\delta\phi_x$.

In the sequel, is given the Figure 9 in which are illustrated the phase velocities of the three plate models.

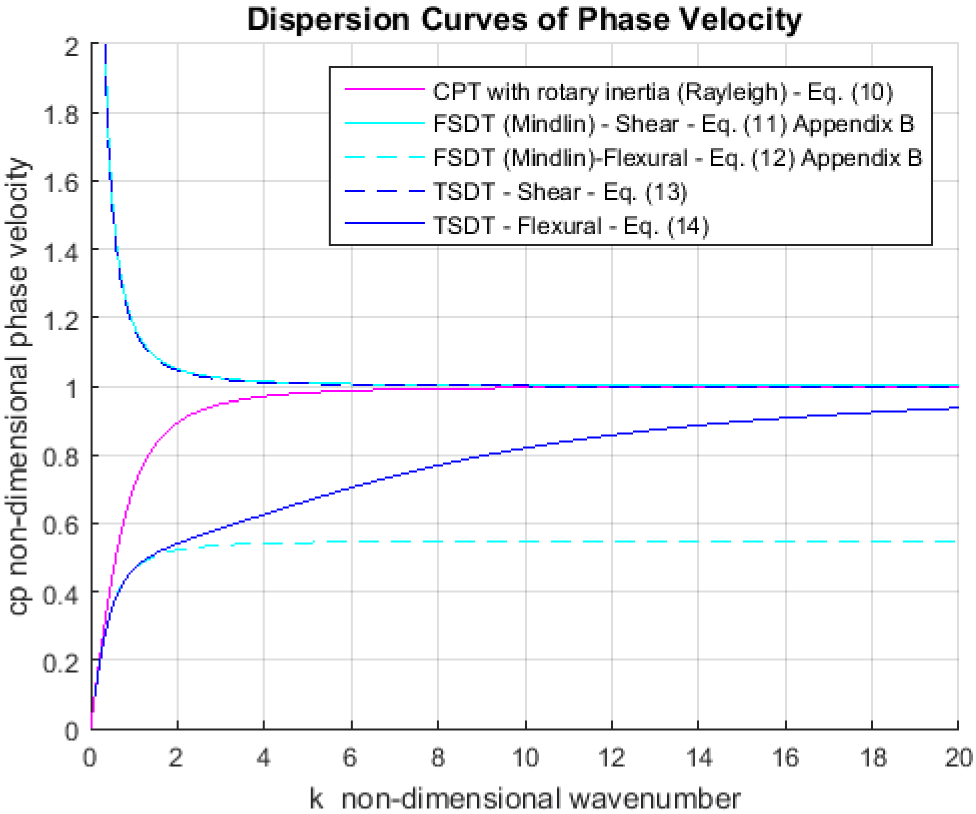

Figure 9: Comparison of the phase velocity-wavenumber dispersion curves.

As for the comparison of the phase velocity, the shear as well as the flexural branches of the FSDT, TSDT follow the behavior of the dispersion curve of CPT as $\tilde{k} \to \infty$. This behavior is regarded well because for large values of wavenumbers the curves are bounded, fact that assures the rightness of the non-dimensionalization used previously on this part (Part C) and also the successful choice of the shear correction factor of the model of FSDT. Thus, remark that the curves of CPT, the shear branch of the FSDT and the TSDT and the flexural branch of the TSDT converge to the unit as $\tilde{k} \to \infty$, whereas the flexural branch of the FSDT converges to the 0.5 as $\tilde{k} \to \infty$.

However, the shear and flexural branches have remarkably different behavior around the region of zero wavenumber. This limit process as $\tilde{k} \to 0$ is shown explicitly for the phase velocities of the FSDT on the Appendix B. In the sequel, it will be presented for the phase velocities of the TSDT.

Let $\tilde{k} \to 0$ first for the Eq. (13),

$$\lim_{\tilde{k}\to 0} \tilde{c}_{p\,Shear} = \lim_{\tilde{k}\to 0} \frac{1}{\tilde{k}} \sqrt{\frac{-B+\sqrt{\Delta}}{2\,A}} = \lim_{\tilde{k}\to 0} \frac{1}{\tilde{k}} \lim_{\tilde{k}\to 0} \sqrt{\frac{-B+\sqrt{\Delta}}{2\,A}} = \lim_{\tilde{k}\to 0} \frac{1}{\tilde{k}} \sqrt{\frac{1-v}{2.432}} = \infty$$

Let now $\tilde{k} \to 0$ for the Eq. (14),

$$\lim_{\tilde{k}\to 0} \tilde{c}_{p\ flex} = \lim_{\tilde{k}\to 0} \frac{1}{\tilde{k}} \sqrt{\frac{-B-\sqrt{\Delta}}{2\,A}} = \frac{\lim_{\tilde{k}\to 0} \sqrt{\frac{-B-\sqrt{\Delta}}{2\,A}}}{\lim_{\tilde{k}\to 0} \tilde{k}} \xrightarrow{\text{L}' \text{ Hospital's Rule}}$$

[due to the indeterminate forms of the numerator and denominator $\frac{0}{0}$ as $\tilde{k} \to 0$, we perform the so called L' Hospital's Rule]

$$\lim_{\tilde{k}\to 0} \tilde{c}_{p\ flex} = \frac{\lim_{\tilde{k}\to 0} \frac{\partial}{\partial \tilde{k}} \left(\sqrt{\frac{-B-\sqrt{\Delta}}{2\,A}} \right)}{1} = \lim_{\tilde{k}\to 0} \frac{\partial}{\partial \tilde{k}} \left(\sqrt{\frac{-B-\sqrt{\Delta}}{2\,A}} \right) =$$

$$= \lim_{\tilde{k}\to 0} \frac{\frac{\partial}{\partial \tilde{k}} \sqrt{-B-\sqrt{\Delta}}\ \sqrt{2\,A} - \sqrt{-B-\sqrt{\Delta}}\ \frac{\partial}{\partial \tilde{k}} \sqrt{2\,A}}{2\,A}$$

where,

$$\frac{\partial}{\partial \tilde{k}} \sqrt{-B-\sqrt{\Delta}} = \frac{1}{2\sqrt{-B-\sqrt{\Delta}}} \left(-\frac{\partial B}{\partial \tilde{k}} - \frac{1}{2\sqrt{\Delta}} \left\{ 2\,B \frac{\partial B}{\partial \tilde{k}} - 4 \frac{\partial A}{\partial \tilde{k}} C - 4 \frac{\partial C}{\partial \tilde{k}} A \right\} \right)$$

Thus,

$$\lim_{\tilde{k}\to 0} \tilde{c}_{p\ flex} = \frac{\sqrt{2\,A} \left[\left(-\frac{\partial B}{\partial \tilde{k}} - \frac{1}{2\sqrt{\Delta}} \left\{ 2\,B \frac{\partial B}{\partial \tilde{k}} - 4 \frac{\partial A}{\partial \tilde{k}} C - 4 \frac{\partial C}{\partial \tilde{k}} A \right\} \right) \right] - 2\ \ -B-\sqrt{\Delta}\ \ \frac{1}{\sqrt{2\,A}} \frac{\partial A}{\partial \tilde{k}}}{4\,A\sqrt{-B-\sqrt{\Delta}}}$$

[due to the indeterminate forms of the numerator and denominator $\frac{0}{0}$ as $\tilde{k} \to 0$, we perform once again the so called L' Hospital's Rule]

Taking apart the derivatives of the numerator and denominator respectively, we have

$$Num = \sqrt{2\,A} \left[\left(-\frac{\partial B}{\partial \tilde{k}} - \frac{1}{2\sqrt{\Delta}} \left\{ 2\,B \frac{\partial B}{\partial \tilde{k}} - 4 \frac{\partial A}{\partial \tilde{k}} C - 4 \frac{\partial C}{\partial \tilde{k}} A \right\} \right) \right] - 2\ \ -B-\sqrt{\Delta}\ \ \frac{1}{\sqrt{2\,A}} \frac{\partial A}{\partial \tilde{k}} =$$

$$= \frac{1}{\sqrt{2\,A}} \left(-\frac{\partial B}{\partial \tilde{k}} - \frac{1}{2\sqrt{\Delta}} \left\{ 2\,B \frac{\partial B}{\partial \tilde{k}} - 4 \frac{\partial A}{\partial \tilde{k}} C - 4 \frac{\partial C}{\partial \tilde{k}} A \right\} \right) +$$

$$+ \sqrt{2\,A} \left(\begin{array}{l} -\frac{\partial^2 B}{\partial^2 \tilde{k}} + \frac{1}{4\Delta^{2/3}} \left\{ 2\,B \frac{\partial B}{\partial \tilde{k}} - 4 \frac{\partial A}{\partial \tilde{k}} C - 4 \frac{\partial C}{\partial \tilde{k}} A \right\} - \\ - \frac{1}{2\sqrt{\Delta}} \left\{ 2 \left(\frac{\partial B}{\partial \tilde{k}} \right)^2 + 2\,B \frac{\partial^2 B}{\partial^2 \tilde{k}} - 4 \left(\frac{\partial^2 A}{\partial^2 \tilde{k}} C + 2 \frac{\partial A}{\partial \tilde{k}} \frac{\partial C}{\partial \tilde{k}} + A \frac{\partial^2 C}{\partial^2 \tilde{k}} \right) \right\} \end{array} \right) +$$

$$+ 2 \left(\frac{\partial B}{\partial \tilde{k}} + \frac{1}{2\sqrt{\Delta}} \frac{\partial \Delta}{\partial \tilde{k}} \right) \frac{1}{\sqrt{2\,A}} \frac{\partial A}{\partial \tilde{k}} + (B+\sqrt{\Delta}) \frac{1}{\sqrt{2}A^{2/3}} \frac{\partial A}{\partial \tilde{k}} + 2(B+\sqrt{\Delta}) \frac{1}{\sqrt{2\,A}} \frac{\partial^2 A}{\partial^2 \tilde{k}}$$

And taking the limit of the numerator as $\tilde{k} \to 0$, we have $\lim_{\tilde{k} \to 0} Num = 0$.

The derivative of the denominator is,

$$Denom = 4\frac{\partial A}{\partial \tilde{k}}\sqrt{-B-\sqrt{\Delta}} + 4A\frac{1}{2\sqrt{-B-\sqrt{\Delta}}}\left(\frac{\partial B}{\partial \tilde{k}} - \frac{1}{2\sqrt{\Delta}}\frac{\partial \Delta}{\partial \tilde{k}}\right)$$

And similarly taking the limit of the above, we have $\lim_{\tilde{k} \to 0} Denom = \infty$.

Finally, for the phase velocity of the flexural branch we get $\lim_{\tilde{k} \to 0} \tilde{c}_{p\ flex} = \left[\frac{0}{\infty}\right] = 0$.

Subsequently, are presented the group velocities of the three plate models on the following figure (Figure 10).

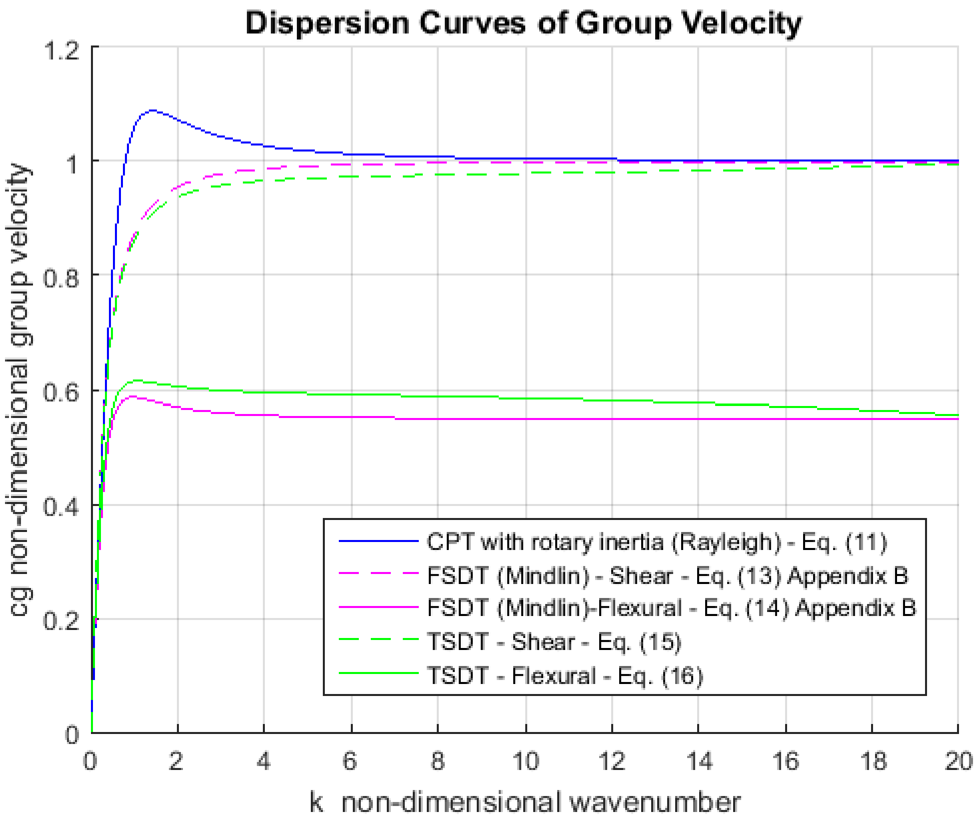

Figure 10: Comparison of the group velocity-wavenumber dispersion curves for $\tilde{k} \in [0,20]$.

Comparing the shear branches of the FSDT, TSDT, we note that they follow the behavior of the dispersion curve of CPT as $\tilde{k} \to \infty$. This behavior is regarded well because for large values of wavenumbers the curves are bounded, fact that similarly to the Figure 9 assures the rightness of the non-dimensionalization used previously on this part (Part C) and also the successful choice of the shear correction factor of the model of FSDT. Thus, remark that the curves of CPT, the shear branch of the FSDT and the TSDT converge to the unit as $\tilde{k} \to \infty$, whereas the flexural branches of the FSDT and TSDT converges to the 0.5 as $\tilde{k} \to \infty$. The range of wavenumber $\tilde{k} \in [0,20]$ is chosen purposely in order to illustrate better the behavior of the group velocities for infinite wavenumbers, as $\tilde{k} \to \infty$.

However, the shear and flexural branches have remarkably different behavior around the region of zero wavenumber. This limit process as $\tilde{k} \to 0$ is shown explicitly for the group velocities of the FSDT on the Appendix B. In the sequel, it will be presented for the group velocities of the TSDT. To acquire a better insight of the limit process near the zero wavenumbers, we represent the same curves again but with a shorter range of the wavenumber on the horizontal axis, namely $\tilde{k} \in [0,10]$ (Figure 11).

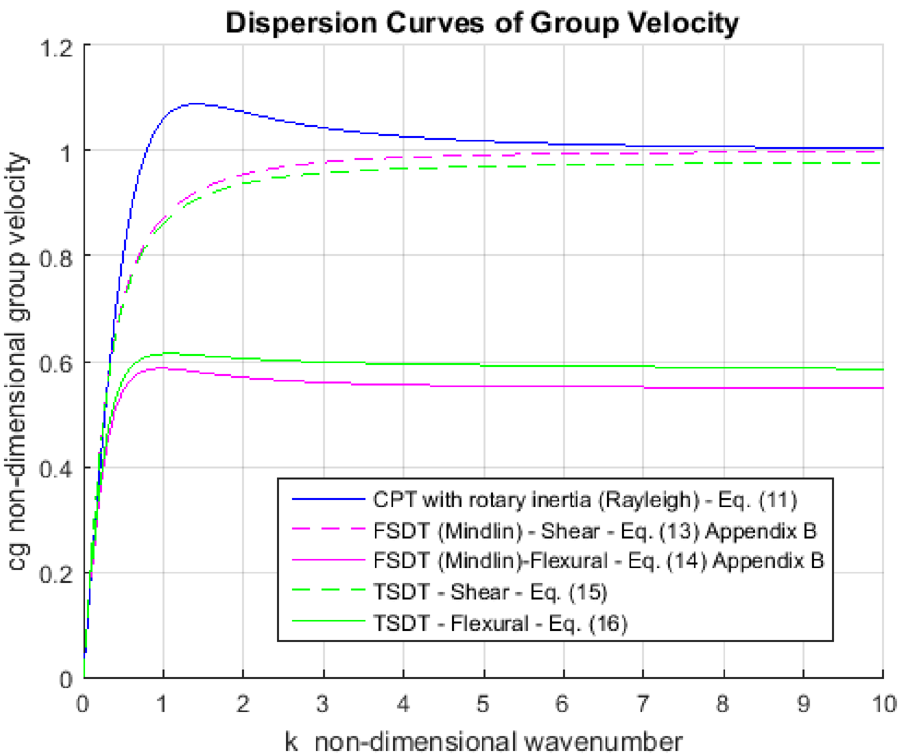

Figure 11: Comparison of the group velocity-wavenumber dispersion curves for $\tilde{k} \in [0,10]$.

Let $\tilde{k} \to 0$ on the Eq. (15), then

$$\lim_{\tilde{k}\to 0} \tilde{c}_{g\,Shear} = \lim_{\tilde{k}\to 0} \frac{1}{\sqrt{2A}} \frac{1}{\sqrt{-B+\sqrt{\Delta}}} \left(\frac{\partial B}{\partial \tilde{k}} + \frac{1}{2\sqrt{\Delta}} \frac{\partial \Delta}{\partial \tilde{k}} \right) = 0 \,,$$

which is shown easily, since $\lim_{\tilde{k}\to 0} \frac{1}{\sqrt{2A}} = \sqrt{\frac{1-v}{4.864}}$, $\lim_{\tilde{k}\to 0} \frac{1}{\sqrt{-B+\sqrt{\Delta}}} = \frac{1}{\sqrt{2}}$, and

$$\lim_{\tilde{k}\to 0} \left(\frac{\partial B}{\partial \tilde{k}} + \frac{1}{2\sqrt{\Delta}} \frac{\partial \Delta}{\partial \tilde{k}} \right) = 0 \,.$$

Let $\tilde{k} \to 0$ on the Eq. (16), then

$$\lim_{\tilde{k}\to 0} \tilde{c}_{g\;flex} = \lim_{\tilde{k}\to 0} \frac{1}{\sqrt{2A}} \frac{1}{\sqrt{-B-\sqrt{\Delta}}} \left(-\frac{\partial B}{\partial \tilde{k}} - \frac{1}{2\sqrt{\Delta}} \frac{\partial \Delta}{\partial \tilde{k}} \right) \xrightarrow[\text{L' Hospital}]{\frac{0}{0}}$$

$$\lim_{\tilde{k}\to 0} \tilde{c}_{g\ flex} = \lim_{\tilde{k}\to 0} \frac{1}{\frac{\partial}{\partial \tilde{k}}\sqrt{-2A(B+\sqrt{\Delta})}}\left(-\frac{\partial^2 B}{\partial^2 \tilde{k}} + \frac{1}{2\,\Delta^{3/2}}\frac{\partial \Delta}{\partial \tilde{k}} - \frac{1}{2\sqrt{\Delta}}\frac{\partial^2 \Delta}{\partial^2 \tilde{k}}\right) =$$

$$= \lim_{\tilde{k}\to 0} \frac{\sqrt{-2A(B+\sqrt{\Delta})}\left(-\frac{\partial^2 B}{\partial^2 \tilde{k}} + \frac{1}{2\,\Delta^{3/2}}\frac{\partial \Delta}{\partial \tilde{k}} - \frac{1}{2\sqrt{\Delta}}\frac{\partial^2 \Delta}{\partial^2 \tilde{k}}\right)}{\frac{\partial A}{\partial \tilde{k}}(B+\sqrt{\Delta}) + A\left(\frac{\partial B}{\partial \tilde{k}} + \frac{1}{2\sqrt{\Delta}}\frac{\partial \Delta}{\partial \tilde{k}}\right)}$$

We perform once again L' Hospital's Rule, since the limit continues to be of indeterminate form $\left[\frac{0}{0}\right]$. Taking apart the derivatives of the numerator and denominator of the previous limit, we find that the indeterminate form of the limit has not been eliminated yet. Thus, we conclude that after a few more iterations of the L' Hospital's Rule we reach the conclusion that

$$\lim_{\tilde{k}\to 0} \tilde{c}_{g\ flex} = 0 .$$

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Appendix A. Transformation from the Cartesian to the local boundary normal-tangent co-ordinate system

[References: *J.N. Reddy "Theory and Analysis of Elastic Plates and Shells"*, Chapter 1.4/ 3.5 and the *Lecture Notes, G.A. Athanassoulis* (*2016*), *"Invariances and transformation of physical quantities under rotations of the reference system"*].

First, we transform the appropriate boundary expressions in terms of the displacements, forces and moments over the edge of the plate (and specifically the arbitrary curve Γ surrounding the mid-surface of the plate). For this purpose the Cartesian orthogonal coordinate system (x_1, x_2, x_3) is transformed to a local coordinate system (n, s, z), which "follows" the shape of the arbitrary curve Γ on the lateral surface of the plate. The expression "follows", denotes that the coordinate system (n, s, z) moves upon the curve Γ, so that the n-axis be normal to the lateral boundary (with a unit normal $\hat{\boldsymbol{n}}$) and s-axis be tangential to the same curve (with a unit tangential vector $\hat{\boldsymbol{s}}$). These vectors projected on the Cartesian coordinate system (x_1, x_2, x_3), are expressed as

$$\hat{\boldsymbol{n}} = n_{x_1} \boldsymbol{e}_{x_1} + n_{x_2} \boldsymbol{e}_{x_2} \tag{1}$$

$$\hat{\boldsymbol{s}} = s_{x_1} \boldsymbol{e}_{x_1} + s_{x_2} \boldsymbol{e}_{x_2} \tag{2}$$

Further, we suppose that the unit normal $\hat{\boldsymbol{n}}$ is oriented at an angle ϑ clockwise from the positive x_1-axis, then its direction cosines are $n_{x_1} = \cos\vartheta$ and $n_{x_2} = \sin\vartheta$. Similarly, the direction cosines of the vector $\hat{\boldsymbol{s}}$ are $s_{x_1} = -n_{x_2} = -\sin\vartheta$ and $s_{x_2} = n_{x_1} = \cos\vartheta$.
The Eqs. (1) and (2) can be expressed in matrix form as seem below,

$$\begin{Bmatrix} \hat{\boldsymbol{n}} \\ \hat{\boldsymbol{s}} \end{Bmatrix} = \begin{bmatrix} n_{x_1} & n_{x_2} \\ s_{x_1} & s_{x_2} \end{bmatrix} \begin{Bmatrix} \boldsymbol{e}_{x_1} \\ \boldsymbol{e}_{x_2} \end{Bmatrix} = \begin{bmatrix} n_{x_1} & n_{x_2} \\ -n_{x_2} & n_{x_1} \end{bmatrix} \begin{Bmatrix} \boldsymbol{e}_{x_1} \\ \boldsymbol{e}_{x_2} \end{Bmatrix} \tag{A}$$

and

$$\begin{Bmatrix} \boldsymbol{e}_{x_1} \\ \boldsymbol{e}_{x_2} \end{Bmatrix} = \begin{bmatrix} n_{x_1} & -n_{x_2} \\ n_{x_2} & n_{x_1} \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{n}} \\ \hat{\boldsymbol{s}} \end{Bmatrix} \tag{A`}$$

Also the transverse normal coordinate x_3 is parallel to the z-axis and the both used coordinate systems $[(x_1, x_2, x_3)$ and $(n, s, z)]$ are right-hand side.
The entire above are illustrated clear on the following figure (**Figure 6**),

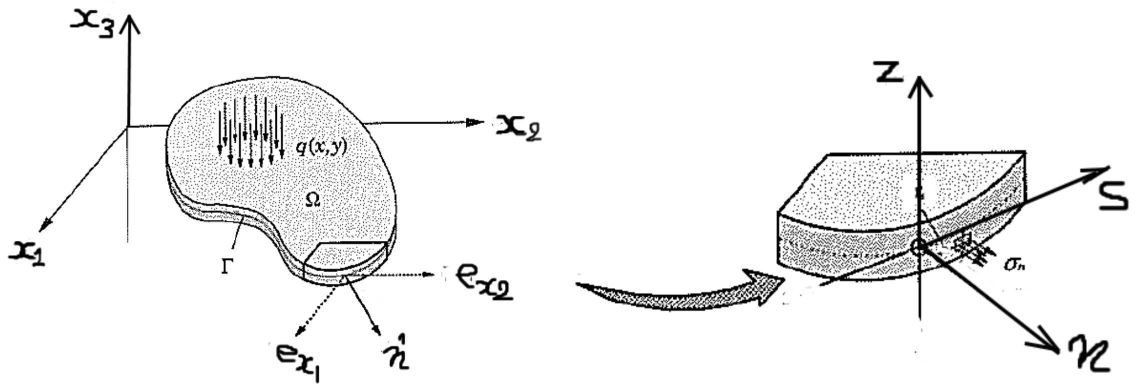

Figure 6: Transformation from the global coordinate system $(x_1\, x_2\, x_3)$ to the local coordinate system $(n s z)$.

Subsequently, we set the Figure 7, where is presented the top view of the plate in order to show the transformation of the components of the displacement field $\boldsymbol{u}$.
Note that the vector (first order tensor) of the displacement field $\boldsymbol{u}$, is invariant and independent from the coordinate system on which is expressed. However, its components are frame-dependent, which means that their values with respect to a reference frame differ from the others with respect to a different frame.

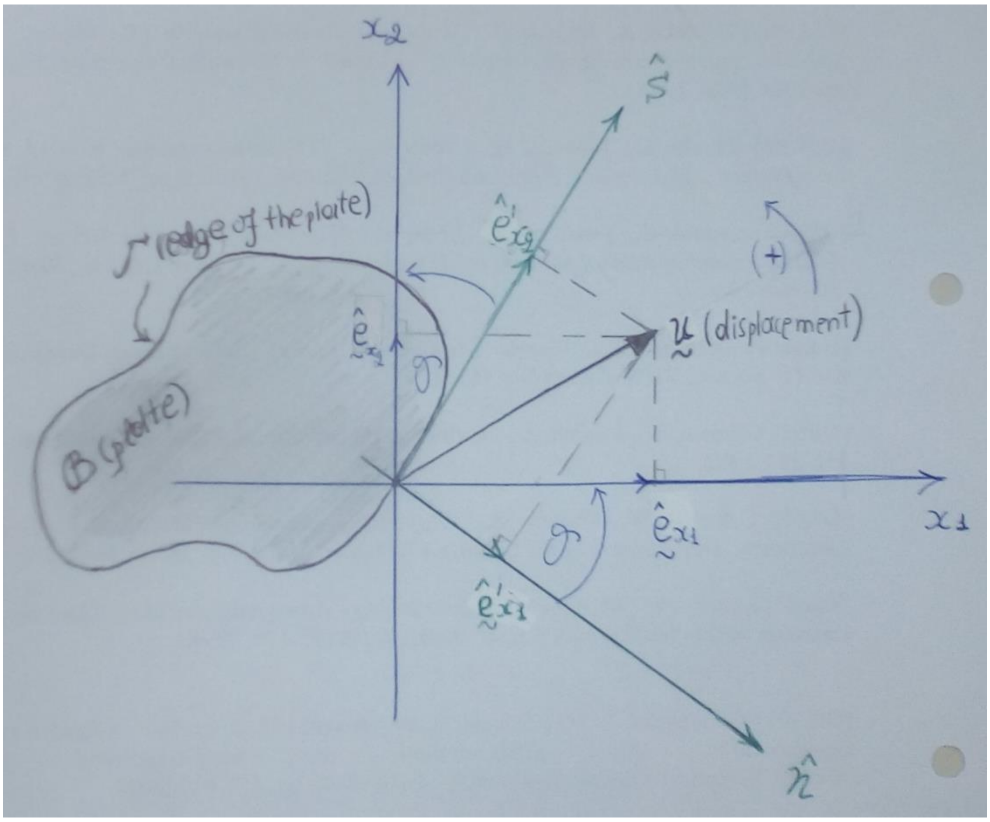

Figure 7: Transformation from the global coordinate system $(x_1\, x_2\, x_3)$ to the local coordinate system shown on the mid- plane $(x_1, x_2, 0)$.

On the context of our problem of CPT, we assume that the transformation from $(x_1\, x_2\, x_3)$ to $(n\, s\, z)$ is clockwise, whereas the transformation from $(n\, s\, z)$ to $(x_1\, x_2\, x_3)$ is counterclockwise. This fact is going to be verified from the sign of the determinant of the rotation matrix.

Thus, the displacement field is expressed as

$$\boldsymbol{u} = u_0\, \boldsymbol{e}_{x_1} + v_0\, \boldsymbol{e}_{x_2} + w_0\, \boldsymbol{e}_{x_3} \tag{3}$$

$$\boldsymbol{u} = u_{0n}\, \boldsymbol{e}'_{x_1} + u_{0s}\, \boldsymbol{e}'_{x_2} + w_0\, \boldsymbol{e}'_{x_3} \tag{4}$$

Multiplying both sides of Eq. (3) by $\boldsymbol{e}'_{x_1}, \boldsymbol{e}'_{x_2}, \boldsymbol{e}'_{x_3}$, respectively, we obtain

$$\begin{aligned} \boldsymbol{e}'_{x_1}\cdot\boldsymbol{u} &= u_0(\boldsymbol{e}'_{x_1}\cdot\boldsymbol{e}_{x_1}) + v_0(\boldsymbol{e}'_{x_1}\cdot\boldsymbol{e}_{x_2}) + w_0(\boldsymbol{e}'_{x_1}\cdot\boldsymbol{e}_{x_3}) = \\ &= u_0 \cos\vartheta + v_0 \cos(90^o + \vartheta) + w_0 \cancel{\cos(90^o)} = \\ &= u_0 \cos\vartheta - v_0 \sin\vartheta \end{aligned} \tag{5}$$

$$\begin{aligned} \boldsymbol{e}'_{x_2}\cdot\boldsymbol{u} &= u_0(\boldsymbol{e}'_{x_2}\cdot\boldsymbol{e}_{x_1}) + v_0(\boldsymbol{e}'_{x_2}\cdot\boldsymbol{e}_{x_2}) + w_0(\boldsymbol{e}'_{x_2}\cdot\boldsymbol{e}_{x_3}) = \\ &= u_0 \cos[-(90^o - \vartheta)] + v_0 \cos\vartheta + \cancel{w_0 \cos(90^o)} = \\ &= u_0 \sin\vartheta + v_0 \cos\vartheta \end{aligned} \tag{6}$$

$$\begin{aligned} \boldsymbol{e}'_{x_3}\cdot\boldsymbol{u} &= u_0(\boldsymbol{e}'_{x_3}\cdot\boldsymbol{e}_{x_1}) + v_0(\boldsymbol{e}'_{x_3}\cdot\boldsymbol{e}_{x_2}) + w_0(\boldsymbol{e}'_{x_3}\cdot\boldsymbol{e}_{x_3}) = \\ &= u_0 \cancel{\cos(90^o)} + v_0 \cancel{\cos(90^o)} + w_0 \cos(0^o) = w_0 \end{aligned} \tag{7}$$

Further the left-hand side of the Eq. (5), (6) and (7) due to (4),

$$\boldsymbol{e}'_{x_1}\cdot\boldsymbol{u} = u_{0n} \tag{8}$$

$$\boldsymbol{e}'_{x_2}\cdot\boldsymbol{u} = u_{0s} \tag{9}$$

$$\boldsymbol{e}'_{x_3}\cdot\boldsymbol{u} = w_0 \tag{10}$$

Finally substituting the Eqs. (8), (9), (10) to the Eqs. (5), (6), (7) respectively,

$$(5) \xrightarrow{(8)} u_{0n} = u_0 \cos\vartheta - v_0 \sin\vartheta$$

$$(6) \xrightarrow{(9)} u_{0s} = u_0 \sin\vartheta + v_0 \cos\vartheta$$

$$(7) \xrightarrow{(10)} w_0 \equiv 1\cdot w_0$$

Consequently, the rotation matrix of the components of the displacement field $\boldsymbol{u} = (u_0, v_0, w_0)$ from the Cartesian coordinate system $(x_1\, x_2\, x_3)$ to the local coordinate system $(n\, s\, z)$, is

$$\boldsymbol{a} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And as an additional expression in matrix form,

$$\begin{Bmatrix} u_{0n} \\ u_{0s} \\ w_0 \end{Bmatrix} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \end{Bmatrix} = \begin{bmatrix} n_{x_1} & -n_{x_2} & 0 \\ n_{x_2} & n_{x_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \end{Bmatrix} \qquad \textbf{(B)}$$

To verify that the above matrix $\boldsymbol{a}$ is a rotational orthogonal matrix, examine its determinant,

$\det(\boldsymbol{a}) = \cos^2\vartheta + \sin^2\vartheta = 1.$

The sign of the determinant defines the orientation of the orthogonal reference system, which in our case is positive, so the (n, s, z) system is right-which coincidence with our initial assumption.

Proceeding to the inverse transformation, we multiply both sides of Eq. (4) by $\boldsymbol{e}_{x_1}, \boldsymbol{e}_{x_2}, \boldsymbol{e}_{x_3}$, respectively,

$$\begin{aligned} \boldsymbol{e}_{x_1}\cdot\boldsymbol{u} &= u_{0n}(\boldsymbol{e}_{x_1}\cdot\boldsymbol{e}'_{x_1}) + u_{0s}(\boldsymbol{e}_{x_1}\cdot\boldsymbol{e}'_{x_2}) + w_0(\boldsymbol{e}_{x_1}\cdot\boldsymbol{e}'_{x_3}) = \\ &= u_{0n}\cos(-\vartheta) + u_{0s}\cos(90^o-\vartheta) + w_0\,\cancel{\cos(90^o)} = \\ &= u_{0n}\cos\vartheta + u_{0s}\sin\vartheta \end{aligned} \qquad (11)$$

$$\begin{aligned} \boldsymbol{e}_{x_2}\cdot\boldsymbol{u} &= u_{0n}(\boldsymbol{e}_{x_2}\cdot\boldsymbol{e}'_{x_1}) + u_{0s}(\boldsymbol{e}_{x_2}\cdot\boldsymbol{e}'_{x_2}) + w_0(\boldsymbol{e}_{x_2}\cdot\boldsymbol{e}'_{x_3}) = \\ &= u_{0n}\cos\left[-(90^o+\vartheta)\right] + u_{0s}\cos(-\vartheta) + w_0\,\cancel{\cos(90^o)} = \\ &= -u_{0n}\sin\vartheta + u_{0s}\cos\vartheta \end{aligned} \qquad (12)$$

$$\begin{aligned} \boldsymbol{e}_{x_3}\cdot\boldsymbol{u} &= u_{0n}(\boldsymbol{e}_{x_3}\cdot\boldsymbol{e}'_{x_1}) + u_{0s}(\boldsymbol{e}_{x_3}\cdot\boldsymbol{e}'_{x_2}) + w_0(\boldsymbol{e}_{x_3}\cdot\boldsymbol{e}'_{x_3}) = \\ &= u_{0n}\,\cancel{\cos(90^o)} + u_{0s}\,\cancel{\cos(90^o)} + w_0\cos(0^o) = w_0 \end{aligned} \qquad (13)$$

The left-hand side of the Eqs. (11), (12) and (13) due to the Eq. (3),

$$\boldsymbol{e}_{x_1}\cdot\boldsymbol{u} = u_0 \qquad (14)$$

$$\boldsymbol{e}_{x_2}\cdot\boldsymbol{u} = v_0 \qquad (15)$$

$$\boldsymbol{e}_{x_3}\cdot\boldsymbol{u} = w_0 \qquad (16)$$

Finally, substituting the Eqs. (14), (15), (16) to (11), (12) and (13), we get

$$(11) \xrightarrow{(14)} u_0 = u_{0n}\cos\vartheta + u_{0s}\sin\vartheta$$

$$(12) \xrightarrow{(15)} u_0 = -u_{0n}\sin\vartheta + u_{0s}\cos\vartheta$$

$$(13) \xrightarrow{(16)} w_0 \equiv 1\cdot w_0$$

Consequently, the inverse rotation matrix from $(x_1\, x_2\, x_3)$ coordinate system to $(n\, s\, z)$, is

$$\boldsymbol{a}^T = \begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And in matrix form we have,

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \end{Bmatrix} = \begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{0n} \\ u_{0s} \\ w_0 \end{Bmatrix} = \begin{bmatrix} n_{x_1} & n_{x_2} & 0 \\ -n_{x_2} & n_{x_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{0n} \\ u_{0s} \\ w_0 \end{Bmatrix} \qquad \textbf{(C)}$$

Also, as before we have the determinant $\det(\boldsymbol{a}^T) = \cos^2\vartheta + \sin^2\vartheta = 1$.

In addition, the same rotation law is valid for the variations of the displacements. Thus,

$$\begin{Bmatrix} \delta u_0 \\ \delta v_0 \\ \delta w_0 \end{Bmatrix} = \begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \delta u_{0n} \\ \delta u_{0s} \\ \delta w_0 \end{Bmatrix} = \begin{bmatrix} n_{x_1} & n_{x_2} & 0 \\ -n_{x_2} & n_{x_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \delta u_{0n} \\ \delta u_{0s} \\ \delta w_0 \end{Bmatrix} \qquad \textbf{(C`)}$$

Further recalling the boundary terms of the variational equation (3) of the section 4.5, we notice that there also the spatial derivatives of the variation δw_0 on the lateral boundary of the plate, which need to be transformed to the curvilinear coordinate system $(n\, s\, z)$. Subsequently using the transformation law **(C`)**, we get

$$\begin{Bmatrix} \partial\delta w_0 / \partial x_1 \\ \partial\delta w_0 / \partial x_2 \end{Bmatrix} = \begin{bmatrix} n_{x_1} & n_{x_2} \\ -n_{x_2} & n_{x_1} \end{bmatrix} \begin{Bmatrix} \partial\delta w_0 / \partial n \\ \partial\delta w_0 / \partial s \end{Bmatrix} \qquad \textbf{(D)}$$

In addition, remark that the given surface tractions $\hat{T}_1$, $\hat{T}_2$, $\hat{T}_3$ defined on the section 4.4 of this dissertation, are first-order tensors or vectors, which are dependent from the functions $a_{T0}(x_1, x_2)$, $b_{T0}(x_1, x_2)$, $a_{T1}(x_1, x_2)$, $b_{T1}(x_1, x_2)$, $c_{T0}(x_1, x_2)$. Consequently these functions follow the transformation law of the vector and are transformed to $a_{T0n}(s)$, $a_{T0s}(s)$, $a_{T1n}(s)$, $a_{T1s}(s)$, $c_{T0}(s)$, as shown thoroughly above. Thus,

$$\begin{Bmatrix} a_{T0} \\ b_{T0} \\ c_{T0} \end{Bmatrix} = \begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} a_{T0n} \\ a_{T0s} \\ c_{T0} \end{Bmatrix} = \begin{bmatrix} n_{x_1} & n_{x_2} & 0 \\ -n_{x_2} & n_{x_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} a_{T0n} \\ a_{T0s} \\ c_{T0} \end{Bmatrix} \qquad \textbf{(T0)}$$

and

$$\begin{Bmatrix} a_{T1} \\ b_{T1} \end{Bmatrix} = \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix} \begin{Bmatrix} a_{T1n} \\ a_{T1s} \end{Bmatrix} = \begin{bmatrix} n_{x_1} & n_{x_2} \\ -n_{x_2} & n_{x_1} \end{bmatrix} \begin{Bmatrix} a_{T1n} \\ a_{T1s} \end{Bmatrix} \quad \textbf{(T1)}$$

At this point, it is essential to remark the curvilinear dependence of the functions $a_{T0n}(s)$, $a_{T0s}(s)$, $a_{T1n}(s)$, $a_{T1s}(s)$, $c_{T0}(s)$. In contrast to the respective functions on the Cartesian coordinate system, the functions a_{T0n}, a_{T0s}, a_{T1n}, a_{T1s} and c_{T0} are dependent from the variable s. This variable counts the length of the curve Γ, which declares the position of a point around the edge of the plate as to a specific principal point.

In everyday language, a curve is a subset of $\mathbb{R}^2$ (plane) or $\mathbb{R}^3$ (geometric space) equipped with a specific structure. The most of the curves of the $\mathbb{R}^2$, which concern now our problem, can be expressed as graphs of functions, namely

$$s, f(s): s \in I = [0, l] \quad ,$$

where $f(s)$ is a well-posed function, inside the field $[0, l]$ and l is the length of the curve Γ.

However, the analytic description of the curve, which is valid for all the curve which is valid for all the curves inside the spaces $\mathbb{R}^2$, $\mathbb{R}^3$ and in general inside the space $\mathbb{R}^N$, is

$$\boldsymbol{r} = \boldsymbol{r}(s), \quad s \in I$$

where $\boldsymbol{r}(s) = (x_1(s), x_2(s), ..., x_N(s))$ and especially for a two dimensional-curve we get,

$$\begin{cases} x_1 = x_1(s) \\ x_2 = x_2(s) \\ \quad s \in I \end{cases}$$

The above description is usually called in the literature, parametric representation of the curve. Consequently, the same rationality follows the notation of the functions a_{T0n}, a_{T0s}, a_{T1n}, a_{T1s} and c_{T0}.

Now we examine the transformation law of stresses, which is identical to the transformation law of the thickness-integrated forces and moments. This is justified due to the fact that these forces and moments are integrated along the thickness of a thin plate in the context of the problem of CPT and are explicitly dependent from the vertical (x_3) spatial variable.

It is meaningful to note that the stress matrix (σ_{ij}) is a second-order tensor, which leads to the need of the rotation matrix of two vectors in order to define its rotation.

This transformation has to do with the rotation of the system about the vertical axis $x_3 = z$ at an angle ϑ, which is the simplest case of transformation (planar rotation). Consequently, the transformation law of the stresses, is given by the following relation

$$\boldsymbol{\sigma} = \boldsymbol{a}\boldsymbol{\sigma}'\boldsymbol{a}^T, \tag{17}$$

where $\boldsymbol{\sigma}$: are the components of stresses on the Cartesian coordinate system (x_1, x_2, x_3) and $\boldsymbol{\sigma}'$: are the components of stresses on the local coordinate system $(n\,s\,z)$.

Further, due to the proof of the Lecture Notes, G.A. Athanassoulis (2016), *"Invariances and transformation of physical quantities under rotations of the reference system"*, the rotation matrix $\boldsymbol{a}$ and is transpose $\boldsymbol{a}^T$ are these of the above relations **(B)** and **(C)**.

The notation of the components of the two stress matrices is,

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \quad \textbf{(E)} \quad \text{and} \quad \boldsymbol{\sigma}' = \begin{pmatrix} \sigma_{nn} & \sigma_{ns} & \sigma_{nz} \\ \sigma_{sn} & \sigma_{ss} & \sigma_{sz} \\ \sigma_{zn} & \sigma_{zs} & \sigma_{zz} \end{pmatrix} \quad \textbf{(E`)}$$

We calculate the right-hand side of the Eq. (17), substituting the Eqs. **(B)**, **(C)** and **(E`)**,

$$\boldsymbol{\sigma} = \begin{bmatrix} n_{x_1} & -n_{x_2} & 0 \\ n_{x_2} & n_{x_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \sigma_{nn} & \sigma_{ns} & \sigma_{nz} \\ \sigma_{sn} & \sigma_{ss} & \sigma_{sz} \\ \sigma_{zn} & \sigma_{zs} & \sigma_{zz} \end{pmatrix} \begin{bmatrix} n_{x_1} & n_{x_2} & 0 \\ -n_{x_2} & n_{x_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{pmatrix} n_{x_1}\sigma_{nn} + n_{x_2}\sigma_{sn} & n_{x_1}\sigma_{ns} + n_{x_2}\sigma_{ss} & n_{x_1}\sigma_{nz} + n_{x_2}\sigma_{sz} \\ -n_{x_2}\sigma_{nn} + n_{x_1}\sigma_{sn} & -n_{x_2}\sigma_{ns} + n_{x_1}\sigma_{ss} & -n_{x_2}\sigma_{nz} + n_{x_1}\sigma_{sz} \\ \sigma_{zn} & \sigma_{zs} & \sigma_{zz} \end{pmatrix} \begin{bmatrix} n_{x_1} & n_{x_2} & 0 \\ -n_{x_2} & n_{x_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{pmatrix} n_{x_1}^2\sigma_{nn} + n_{x_1}n_{x_2}\sigma_{sn} + n_{x_1}n_{x_2}\sigma_{ns} + n_{x_2}^2\sigma_{ss} & -n_{x_1}n_{x_2}\sigma_{nn} - n_{x_2}^2\sigma_{sn} + n_{x_1}^2\sigma_{ns} + n_{x_1}n_{x_2}\sigma_{ss} & n_{x_1}\sigma_{nz} + n_{x_2}\sigma_{sz} \\ -n_{x_1}n_{x_2}\sigma_{nn} + n_{x_1}^2\sigma_{sn} - n_{x_2}^2\sigma_{ns} + n_{x_1}n_{x_2}\sigma_{ss} & n_{x_2}^2\sigma_{nn} - n_{x_1}n_{x_2}\sigma_{sn} - n_{x_1}n_{x_2}\sigma_{ns} + n_{x_1}^2\sigma_{ss} & -n_{x_2}\sigma_{nz} + n_{x_1}\sigma_{sz} \\ n_{x_1}\sigma_{zn} + n_{x_2}\sigma_{zs} & -n_{x_2}\sigma_{zn} + n_{x_1}\sigma_{zs} & \sigma_{zz} \end{pmatrix}$$

Due to the symmetry of the stress matrix in the context of our problem, we get

$$\boldsymbol{\sigma} = \begin{pmatrix} n_{x_1}^2\sigma_{nn} + 2n_{x_1}n_{x_2}\sigma_{ns} + n_{x_2}^2\sigma_{ss} & \sigma_{ss} - \sigma_{nn}\; n_{x_1}n_{x_2} + \; n_{x_1}^2 - n_{x_2}^2\; \sigma_{ns} & n_{x_1}\sigma_{nz} + n_{x_2}\sigma_{sz} \\ \sigma_{ss} - \sigma_{nn}\; n_{x_1}n_{x_2} + \; n_{x_1}^2 - n_{x_2}^2\; \sigma_{ns} & n_{x_2}^2\sigma_{nn} - 2n_{x_1}n_{x_2}\sigma_{ns} + n_{x_1}^2\sigma_{ss} & -n_{x_2}\sigma_{nz} + n_{x_1}\sigma_{sz} \\ n_{x_1}\sigma_{zn} + n_{x_2}\sigma_{zs} & -n_{x_2}\sigma_{zn} + n_{x_1}\sigma_{zs} & \sigma_{zz} \end{pmatrix}$$

Also due to the initial assumptions of the problem of CPT as for the displacement, strain and stress field of the plate in conjunction with the aforementioned stress matrix $\boldsymbol{\sigma}$,

$$\boldsymbol{\sigma} = \begin{pmatrix} n_{x_1}^2\sigma_{nn} + 2n_{x_1}n_{x_2}\sigma_{ns} + n_{x_2}^2\sigma_{ss} & (\sigma_{ss}-\sigma_{nn})n_{x_1}n_{x_2} + (n_{x_1}^2 - n_{x_2}^2)\sigma_{ns} & n_{x_1}\cancel{\sigma_{nz}} + n_{x_2}\cancel{\sigma_{sz}} \\ (\sigma_{ss}-\sigma_{nn})n_{x_1}n_{x_2} + (n_{x_1}^2 - n_{x_2}^2)\sigma_{ns} & n_{x_2}^2\sigma_{nn} - 2n_{x_1}n_{x_2}\sigma_{ns} + n_{x_1}^2\sigma_{ss} & -n_{x_2}\cancel{\sigma_{nz}} + n_{x_1}\cancel{\sigma_{sz}} \\ n_{x_1}\cancel{\sigma_{zn}} + n_{x_2}\cancel{\sigma_{zs}} & -n_{x_2}\cancel{\sigma_{zn}} + n_{x_1}\cancel{\sigma_{zs}} & \cancel{\sigma_{zz}} \end{pmatrix} \Rightarrow$$

$$\boldsymbol{\sigma} = \begin{pmatrix} n_{x_1}^2\sigma_{nn} + 2n_{x_1}n_{x_2}\sigma_{ns} + n_{x_2}^2\sigma_{ss} & (\sigma_{ss}-\sigma_{nn})n_{x_1}n_{x_2} + (n_{x_1}^2 - n_{x_2}^2)\sigma_{ns} & 0 \\ (\sigma_{ss}-\sigma_{nn})n_{x_1}n_{x_2} + (n_{x_1}^2 - n_{x_2}^2)\sigma_{ns} & n_{x_2}^2\sigma_{nn} - 2n_{x_1}n_{x_2}\sigma_{ns} + n_{x_1}^2\sigma_{ss} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

And finally we derive the below equations,

$$\sigma_{11} = n_{x_1}^2\,\sigma_{nn} + 2n_{x_1}n_{x_2}\sigma_{ns} + n_{x_2}^2\,\sigma_{ss} \tag{18a}$$

$$\sigma_{22} = n_{x_2}^2\,\sigma_{nn} - 2n_{x_1}n_{x_2}\sigma_{ns} + n_{x_1}^2\,\sigma_{ss} \tag{18b}$$

$$\sigma_{12} = \sigma_{21} = n_{x_1}n_{x_2}\;\; \sigma_{ss} - \sigma_{nn} \;+\; n_{x_1}^2 - n_{x_2}^2 \;\; \sigma_{ns} \tag{18c}$$

Expressing the Eqs. (18a), (18b) and (18c) in matrix form,

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} n_{x_1}^2 & n_{x_2}^2 & 2n_{x_1}n_{x_2} \\ n_{x_2}^2 & n_{x_1}^2 & -2n_{x_1}n_{x_2} \\ -n_{x_1}n_{x_2} & n_{x_1}n_{x_2} & n_{x_1}^2 - n_{x_2}^2 \end{bmatrix} \begin{Bmatrix} \sigma_{nn} \\ \sigma_{ss} \\ \sigma_{ns} \end{Bmatrix} \tag{F}$$

Further, we are going to use the inverse transformation of the stress matrix, in order to express the components of the stress of the curvilinear coordinate system in terms of the components of the Cartesian coordinate system. Thus, from rotation law (17) we get the following.

$$\boldsymbol{\sigma} = \boldsymbol{a}\boldsymbol{\sigma}'\boldsymbol{a}^T \xrightarrow[\text{from the left}]{\times \boldsymbol{a}^T} \boldsymbol{a}^T\boldsymbol{\sigma} = (\boldsymbol{a}^T\boldsymbol{a})\boldsymbol{\sigma}'\boldsymbol{a}^T \Rightarrow \boldsymbol{a}^T\boldsymbol{\sigma} = \boldsymbol{I}\boldsymbol{\sigma}'\boldsymbol{a}^T \Rightarrow$$

$$\Rightarrow \boldsymbol{a}^T\boldsymbol{\sigma}\boldsymbol{a} = \boldsymbol{\sigma}'(\boldsymbol{a}^T\boldsymbol{a}) \xrightarrow[\text{from the right}]{\times \boldsymbol{a}} \boldsymbol{a}^T\boldsymbol{\sigma}\boldsymbol{a} = \boldsymbol{\sigma}'\boldsymbol{I} \Rightarrow \boldsymbol{\sigma}' = \boldsymbol{a}^T\boldsymbol{\sigma}\boldsymbol{a}$$

Consequently, after a few calculations and the use unit matrix $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we have the inverse transformation law of the stresses,

$$\boldsymbol{\sigma}' = \boldsymbol{a}^T\boldsymbol{\sigma}\boldsymbol{a} \tag{19}$$

We calculate now the right-hand side of the Eq. (19), substituting the Eqs. **(B)**, **(C)** and **(E)**,

$$\boldsymbol{\sigma}' = \begin{bmatrix} n_{x_1} & n_{x_2} & 0 \\ -n_{x_2} & n_{x_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{bmatrix} n_{x_1} & -n_{x_2} & 0 \\ n_{x_2} & n_{x_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{pmatrix} n_{x_1}^2\sigma_{11} + 2n_{x_1}n_{x_2}\sigma_{12} + n_{x_2}^2\sigma_{22} & (\sigma_{22}-\sigma_{11})n_{x_1}n_{x_2} + (n_{x_1}^2 - n_{x_2}^2)\sigma_{12} & n_{x_1}\sigma_{13} + n_{x_2}\sigma_{23} \\ (\sigma_{22}-\sigma_{11})n_{x_1}n_{x_2} + (n_{x_1}^2 - n_{x_2}^2)\sigma_{12} & n_{x_2}^2\sigma_{11} - 2n_{x_1}n_{x_2}\sigma_{12} + n_{x_1}^2\sigma_{22} & -n_{x_2}\sigma_{13} + n_{x_1}\sigma_{23} \\ n_{x_1}\sigma_{31} + n_{x_2}\sigma_{32} & -n_{x_2}\sigma_{31} + n_{x_1}\sigma_{32} & \sigma_{33} \end{pmatrix}$$

Also due to the initial assumptions of the problem of CPT as for the displacement, strain and stress field of the plate in conjunction with the aforementioned stress matrix $\boldsymbol{\sigma}'$,

$$\boldsymbol{\sigma}' = \begin{pmatrix} n_{x_1}^2\sigma_{11} + 2n_{x_1}n_{x_2}\sigma_{12} + n_{x_2}^2\sigma_{22} & (\sigma_{22}-\sigma_{11})n_{x_1}n_{x_2} + (n_{x_1}^2 - n_{x_2}^2)\sigma_{12} & n_{x_1}\cancel{\sigma_{13}} + n_{x_2}\cancel{\sigma_{23}} \\ (\sigma_{22}-\sigma_{11})n_{x_1}n_{x_2} + (n_{x_1}^2 - n_{x_2}^2)\sigma_{12} & n_{x_2}^2\sigma_{11} - 2n_{x_1}n_{x_2}\sigma_{12} + n_{x_1}^2\sigma_{22} & -n_{x_2}\cancel{\sigma_{13}} + n_{x_1}\cancel{\sigma_{23}} \\ n_{x_1}\cancel{\sigma_{31}} + n_{x_2}\cancel{\sigma_{32}} & -n_{x_2}\cancel{\sigma_{31}} + n_{x_1}\cancel{\sigma_{32}} & \cancel{\sigma_{33}} \end{pmatrix} \Rightarrow$$

$$\boldsymbol{\sigma}' = \begin{pmatrix} n_{x_1}^2\sigma_{11} + 2n_{x_1}n_{x_2}\sigma_{12} + n_{x_2}^2\sigma_{22} & (\sigma_{22}-\sigma_{11})n_{x_1}n_{x_2} + (n_{x_1}^2 - n_{x_2}^2)\sigma_{12} & 0 \\ (\sigma_{22}-\sigma_{11})n_{x_1}n_{x_2} + (n_{x_1}^2 - n_{x_2}^2)\sigma_{12} & n_{x_2}^2\sigma_{11} - 2n_{x_1}n_{x_2}\sigma_{12} + n_{x_1}^2\sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{Bmatrix} \sigma_{nn} \\ \sigma_{ss} \\ \sigma_{ns} \end{Bmatrix} = \begin{pmatrix} n_{x_1}^2 & n_{x_2}^2 & 2n_{x_1}n_{x_2} \\ n_{x_2}^2 & n_{x_1}^2 & -2n_{x_1}n_{x_2} \\ -n_{x_1}n_{x_2} & n_{x_1}n_{x_2} & n_{x_1}^2 - n_{x_2}^2 \end{pmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \qquad \textbf{(G)}$$

Similarly, are expressed the thickness-integrated forces and moments in matrix form,

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \begin{bmatrix} n_{x_1}^2 & n_{x_2}^2 & 2n_{x_1}n_{x_2} \\ n_{x_2}^2 & n_{x_1}^2 & -2n_{x_1}n_{x_2} \\ -n_{x_1}n_{x_2} & n_{x_1}n_{x_2} & n_{x_1}^2 - n_{x_2}^2 \end{bmatrix} \begin{Bmatrix} N_{nn} \\ N_{ss} \\ N_{ns} \end{Bmatrix} \qquad \textbf{(H)}$$

$$\begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \begin{bmatrix} n_{x_1}^2 & n_{x_2}^2 & 2n_{x_1}n_{x_2} \\ n_{x_2}^2 & n_{x_1}^2 & -2n_{x_1}n_{x_2} \\ -n_{x_1}n_{x_2} & n_{x_1}n_{x_2} & n_{x_1}^2 - n_{x_2}^2 \end{bmatrix} \begin{Bmatrix} M_{nn} \\ M_{ss} \\ M_{ns} \end{Bmatrix} \qquad \textbf{(I)}$$

and the inverse transformation law of the **(H)** and **(I)**, is exactly the same with the inverse transformation law of the stress field, namely the **(G)**.

APPENDIX B: First-Order Shear Deformable Plate Theory (Mindlin's Plate Theory) – Governing Equations and Dispersion Curves

[References: *Liew K.M., Wang C.M., Xiang Y., Kitipornchai S. (1998), "Vibration of Mindlin Plates - Programming the p-Version Ritz Method", Reddy J.N. (2007), "Theory of Elastic Plates and Shells", Reddy J.N. (2004), "Mechanics of Laminated Composite Plates and Shells- Theory and Analysis"*].

The purpose of the present appendix is to extract the non-dimensional form of the dispersion relation of a one-dimensional wave propagation through an infinite medium, but at this moment the medium is an elastic plate subjected to the assumptions of the so called Mindlin's Plate Theory or First-Order Shear Deformable Plate Theory (FSDT) without externally applied loads (free surface).

The reason of the above effort is the comparison of the dispersion curves jointly of the three plate theories (CPT, FSDT, TSDT), which the most commonly used on the analysis of the motion of the vibrating plates. Except from the above reason, the main parts of this diploma thesis are dedicated the CPT and TSDT and there is no facts about the FSDT, which the intermediate (to the two previous) plate theory as for number geometric constraints during the deformation of the plate. Thus, the dispersion curve of the FSDT is a way to establish better our results for the CPT and the TSDT and further to give comments about the advantages and the disadvantages of each one theory.

According to the literature, and especially taking into account the J.N. Reddy's results for the governing equations of the plate in the context of FSDT, we have five equations of motion. However, as exactly on the TSDT the three of these equations which include the displacements (w_0, ϕ_x, ϕ_y) are going to occupy us on the problem of wave propagation through infinite medium. Thus, regarding the Eqs. (10.1.33), (10.1.34), (10.1.35) of the book of *J.N. Reddy (2007), "Theory and Analysis of Elastic Plates and Shells"* (Chap. 10.1, pp. 366) and neglecting terms which insert elastic foundation and thermal effects (since there is no such assumption in the context of our problem) and assuming isotropic material, we get the following governing equations of motion,

$$\kappa^2 G h \left(\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{\partial \phi_x}{\partial x_1} \right) + \kappa^2 G h \left(\frac{\partial^2 w_0}{\partial^2 x_2} + \frac{\partial \phi_y}{\partial x_2} \right) + q = I_0 \ddot{w}_0 \tag{1}$$

$$D \frac{\partial^2 \phi_x}{\partial^2 x_1} + v D \frac{\partial^2 \phi_y}{\partial x_1 \partial x_2} + \frac{G h^3}{12} \left(\frac{\partial^2 \phi_x}{\partial^2 x_2} + \frac{\partial^2 \phi_y}{\partial x_1 \partial x_2} \right) - \kappa^2 G h \left(\frac{\partial w_0}{\partial x_1} + \phi_x \right) = I_2 \ddot{\phi}_x \tag{2}$$

$$D \frac{\partial^2 \phi_y}{\partial^2 x_2} + v D \frac{\partial^2 \phi_x}{\partial x_1 \partial x_2} + \frac{G h^3}{12} \left(\frac{\partial^2 \phi_x}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi_y}{\partial^2 x_1} \right) - \kappa^2 G h \left(\frac{\partial w_0}{\partial x_2} + \phi_y \right) = I_2 \ddot{\phi}_y \tag{3}$$

where κ^2 is the shear correction factor which is included in the Mindlin's Plate Theory in order to correct the distribution of the shear stresses along the thickness of the elastic plate.

According to the chapter 2.3 of the reference *Liew K.M., Wang C.M., Xiang Y., Kitipornchai S. (1998), "Vibration of Mindlin Plates - Programming the p-Version Ritz Method"*, the values of the shear correction factor are dependent on those of the Poisson's ratio, namely the shear correction factor is relates to the kind of material. For instance, regarding an isotropic plate with Poisson's ration $\nu = 0.3$ (which is a usual value for a wide range of materials), the shear correction factor is $\kappa^2 = 0.86$.

Note also that the above equations of motion (1), (2) and (3) are found on the book of *Graff Karl F. (1975) "Wave Motion in Elastic Solids"* on the chapter 8.3 *"Approximate theories for waves in plates, rods and shells"* [Eqs (8.3.30), (8.331) pp. 488], which after expansion of the Laplace operators and substitution of Marcus moment $\Phi = \partial\phi_x / \partial x_1 + \partial\phi_y / \partial x_2$ conclude exactly to the same relations. However, in order to establish better our results we extract the single equation of $\ddot{w}_0$, which includes the previous three [Eqs (1), (2) and (3)] and represent the flexural response of the Mindlin's plate and after that we are going to investigate the values of shear correction factor, which give the Kirchhoff's model. Thus, by differentiating the Eq. (2) as for x_1 and the Eq. (3) as for x_2 and adding the results, we eliminate the Marcus moment $\Phi = \partial\phi_x / \partial x_1 + \partial\phi_y / \partial x_2$ and after that we conclude to the single equation of motion, given below. This process is also prescribed on the aforementioned reference, *Graff Karl F. (1975) "Wave Motion in Elastic Solids"* on the chapter 8.3 *"Approximate theories for waves in plates, rods and shells"* pp.488-492 and the Appendix of the reference, *"Ship Dynamics" (2012), G.A. Athanassoulis, K.A. Belibassakis.*

$$D\Delta^2 w_0 - \frac{\rho h^3}{12}\Delta\ddot{w}_0 - \frac{\rho D}{\kappa^2 G}\Delta\ddot{w}_0 + \frac{\rho}{\kappa^2 G}\frac{\rho h^3}{12}\ddddot{w}_0 + \rho h\ddot{w}_0 = q - \frac{D}{\kappa^2 G h}\Delta q + \frac{\rho h^2}{12\kappa^2 G}\ddot{q} \tag{4}$$

Let the shear corrector factor be inclined to infinity $\kappa^2 \to 0$. Then it is obvious that we conclude to the Kirchhoff's third governing equation of motion related to the vertical vibration of the plate.

$$D\Delta^2 w_0 - \frac{\rho h^3}{12}\Delta\ddot{w}_0 + \rho h\ddot{w}_0 = q \Rightarrow I_0\ddot{w}_0 - I_2\Delta\ddot{w}_0 + D\Delta^2 w_0 = q$$

At this point, we can get the dispersion curve of the wave propagation through the Mindlin's plate by substituting the harmonic function of the vertical displacement into the Eq. (4).
However for the sake of consistency with the corresponding section 2 of the Part C, we use directly the three equations of motion, Eqs (1), (2), (3) and substitute into them the harmonic functions of the displacements (w_0, ϕ_x, ϕ_y), which due to the previous assumption of the one-dimensional wave propagation take the following form

$$w = B_1 e^{i(kx_1 - \omega t)}, \qquad \phi_x = B_2 e^{i(kx_1 - \omega t)}, \qquad \phi_y = B_3 e^{i(kx_1 - \omega t)}$$

Further, the boundary of the one-directional infinite plate is free. Thus, $q = 0$.
Taking into account the above, the Eqs (1), (2) and (3) are converted to

$$\kappa^2 G h\left(\frac{\partial^2 w_0}{\partial^2 x_1} + \frac{\partial\phi_x}{\partial x_1}\right) = I_0\ddot{w}_0 \tag{4a}$$

$$D\frac{\partial^2\phi_x}{\partial^2 x_1} - \kappa^2 G h\left(\frac{\partial w_0}{\partial x_1} + \phi_x\right) = I_2\ddot{\phi}_x \tag{4b}$$

$$\frac{G h^3}{12}\frac{\partial^2\phi_y}{\partial^2 x_1} - \kappa^2 G h\phi_y = I_2\ddot{\phi}_y \tag{4c}$$

Remark that the above Eqs. (4a), (4b) and (4c) are decoupled. Namely the first two Eqs are coupled and both of them includes two degrees of freedom w_0 and ϕ_x, whereas the third one is decoupled from the aforementioned equations because it includes only the ϕ_y variable. This fact was expected due to our initial assumption of the infinite plate along one direction, here the x_1-direction. This consideration leads to the elimination of the lateral (to the direction of the wave propagation) distortion of the plate, namely $\phi_y = 0$ and finally we treat it as a beam or so called plate strip (since the length of the plate is regarded here very large in comparison with its breadth). Thus, we are going to occupy with a 2x2 system and through the two degrees of freedom w_0 and ϕ_x and the Eqs. (4a) and (4b). After substituting the harmonic functions of the w_0 and ϕ_x into the Eqs. (4a) and (4b),

$$B_1 \; I_0\,\omega^2 - \kappa^2 G h k^2 \; + B_2\, i\,\kappa^2 G h k = 0 \qquad \textbf{(5a)}$$

$$B_1\, i\,\kappa^2 G h k + B_2 \; D k^2 - \kappa^2 G h - I_2\,\omega^2 \; = 0 \qquad \textbf{(5b)}$$

Subsequently, we note that the above three equations consist a 2x2 system with respect to three unknowns B_1, B_2, namely the amplitudes of the displacements w_0, ϕ_x respectively.

Equating now the determinant of the coefficients B_1, B_2 to zero in the above system yields the dispersion relation. This is rational because the sufficient and necessary condition for the existence of non-trivial solution of the aforementioned system, is its zero determinant.

At this point, to simplify the form of the above equations, we set specific symbols for the quantities multiplied with the coefficients B_1, B_2,

$$\Psi_{11} = I_0\,\omega^2 - \kappa^2 G h k^2 \,, \qquad \Psi_{12} = i\,\kappa^2 G h k$$

$$\Psi_{21} = i\,\kappa^2 G h k \,, \qquad \Psi_{22} = D k^2 - \kappa^2 G h - I_2\,\omega^2$$

By this way the 2x2 system is simplified to the following form,

$$\begin{array}{l} \Psi_{11} B_1 + \Psi_{12} B_2 = 0 \\ \Psi_{21} B_1 + \Psi_{22} B_2 = 0 \end{array} \quad \text{or} \quad \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0 \;\Leftrightarrow\; \boldsymbol{\Psi}\,\boldsymbol{B} = 0$$

Now the determinant is clear to be written,

$$det(\boldsymbol{\Psi}) = \Psi_{11}\Psi_{22} - \Psi_{12}\Psi_{21} = 0$$

And after performing a few calculations, the dispersion relation which derives from the governing equations relates to the vertical motion (vibration) of a plate is the following

$$\boxed{\frac{D}{\rho h} k^4 - \left(\frac{h^2}{12} + \frac{D}{\kappa^2 G h}\right)\omega^2 k^2 - \omega^2 + \frac{h^2}{12}\frac{\rho}{\kappa^2 G}\omega^4 = 0} \qquad \textbf{(6)}$$

As for the residual quantities involved in the Eq. (6), they are explicitly explained on the main part of this dissertation.

Note also that the Eq. (6) exist on the reference: *Graff F. Karl, (1975) "Wave Motion in Elastic Solids"*, on the Chapter 8.3 pp. 492 and can be shown exactly if we expand the Laplacian and Biharmonic Operators.

At this moment, we have to extract the non-dimensional dispersion curve of the FSDT. Because of the fact that we expect to compare the results of this appendix to those gained by the previous work on the Part C (section 1 and 2), it is essential to choose the same non-dimensional quantities. Consequently, we have the non-dimensional angular frequency and the non-dimensional wavenumber given as

$$\tilde{\omega} = \frac{\omega}{\Omega} \qquad \text{and} \qquad \tilde{k} = \frac{k}{K}$$

and substituting the above into the Eq. (6), we get

$$\frac{D}{\rho h}\tilde{k}^4 K^4 - \left(\frac{h^2}{12} + \frac{D}{\kappa^2 G h}\right)\Omega^2 K^2 \tilde{\omega}^2 \tilde{k}^2 - \Omega^2 \tilde{\omega}^2 + \frac{h^2}{12}\frac{\rho}{\kappa^2 G}\Omega^4 \tilde{\omega}^4 = 0 \Rightarrow$$

[and using the expressions of K^2, K^4 Ω^2 and Ω^4 of the section 2- Part C]

$$\frac{D}{\rho h}\tilde{k}^4 \frac{12^2}{h^4} - \left(\frac{h^2}{12} + \frac{D}{\kappa^2 G h}\right)\frac{12\,E}{(1-\nu^2)\rho h^2}\frac{12}{h^2}\tilde{\omega}^2 \tilde{k}^2 -$$

$$- \frac{12\,E}{(1-\nu^2)\rho h^2}\tilde{\omega}^2 + \frac{h^2}{12}\frac{\rho}{\kappa^2 G}\frac{12^2 E^2}{(1-\nu^2)^2 \rho^2 h^4}\tilde{\omega}^4 = 0 \Rightarrow$$

[we examine isotropic material, $G = E/2(1+\nu)$ and $D = E h^3/12(1-\nu^2)$]

$$\frac{E h^3}{12(1-\nu^2)\rho h}\tilde{k}^4 \frac{12^2}{h^4} - \left(\frac{h^2}{12} + \frac{E h^3 2(1+\nu)}{12(1-\nu^2)\kappa^2 E h}\right)\frac{12\,E}{(1-\nu^2)\rho h^2}\frac{12}{h^2}\tilde{\omega}^2 \tilde{k}^2 -$$

$$- \frac{12\,E}{(1-\nu^2)\rho h^2}\tilde{\omega}^2 + \frac{h^2}{12}\frac{\rho\, 2(1+\nu)}{\kappa^2 E}\frac{12^2 E^2}{(1-\nu^2)^2 \rho^2 h^4}\tilde{\omega}^4 = 0 \Rightarrow$$

$$\tilde{k}^4 - \left(1 + \frac{2}{(1-\nu)\kappa^2}\right)\tilde{\omega}^2 \tilde{k}^2 - \tilde{\omega}^2 + \frac{2}{\kappa^2(1-\nu)}\tilde{\omega}^4 = 0 \qquad \textbf{(7)}$$

The last expression [Eq. (7)] is the non-dimensional form of the dispersion relation of the Mindlin's Plate Theory.

Subsequently, we have to express the non-dimensional angular frequency $\tilde{\omega}$ explicitly as a function of the non-dimensional wavenumber $\tilde{k}$, in order to illustrate their graph. The last takes place by the reduction of the grade of the grade of the polynomial equation (7) as for the $\tilde{\omega}$. Thus, we set $\tilde{\omega}^2 = y$ and substitute into the Eq. (7).

$$\tilde{k}^4 - \left(1 + \frac{2}{(1-\nu)\kappa^2}\right) y \tilde{k}^2 - y + \frac{2}{\kappa^2(1-\nu)} y^2 = 0 \quad \Rightarrow$$

And grouping appropriately the terms in order to solve the binomial equations as to $y = \tilde{\omega}^2$,

$$\underbrace{\frac{2}{\kappa^2(1-\nu)}}_{A} y^2 - \underbrace{\left(\tilde{k}^2 + \frac{2\tilde{k}^2}{(1-\nu)\kappa^2} + 1\right)}_{B} y + \underbrace{\tilde{k}^4}_{C} = 0 \qquad (8)$$

The discriminant of the above binomial equation is, $\Delta = B^2 - 4AC$ and the solutions of the above equation are

$$y = \frac{-B \pm \sqrt{\Delta}}{2A} \Leftrightarrow y = \frac{-B + \sqrt{\Delta}}{2A} \text{ or } y = \frac{-B - \sqrt{\Delta}}{2A} \Leftrightarrow$$

$$\Leftrightarrow \tilde{\omega}^2 = \frac{-B + \sqrt{\Delta}}{2A} \text{ or } \tilde{\omega}^2 = \frac{-B - \sqrt{\Delta}}{2A} \Leftrightarrow$$

$$\Leftrightarrow \left(\tilde{\omega} = \sqrt{\frac{-B + \sqrt{\Delta}}{2A}} \text{ or } -\sqrt{\frac{-B + \sqrt{\Delta}}{2A}}\right) \text{ or } \left(\tilde{\omega} = \sqrt{\frac{-B - \sqrt{\Delta}}{2A}} \text{ or } -\sqrt{\frac{-B - \sqrt{\Delta}}{2A}}\right)$$

However now we have to examine which of the four relations between the $\tilde{\omega}$ and $\tilde{k}$ give positive values for the angular frequency $\tilde{\omega}$, because the negative values of the $\tilde{\omega}$ have no physical interpretation. Consequently, we mean to keep only the branches with positive values for the $\tilde{\omega}$ and we are going to rejected those which give negative values for the aforementioned quantity. By the aid of the mathematical package Matlab R2013 a, we find two acceptable branches. This fact was expected because the one branch describes the shear waves and the other the flexural waves. Further, we have to investigate the sign of the quantities under the square roots and the sign of the discriminant in order to find if we are have real or/imaginary angular frequencies.

As for the coefficients of the binomial equation A, B, C, it is apparent that $A > 0$, $B < 0$ and $C > 0$. In the sequel, we investigate the sign of the discriminant $\Delta = B^2 - 4AC$, which is a fourth-order polynomial as shown below.

$$\Delta = \left[\left(1 + \frac{2}{(1-\nu)\kappa^2}\right)^2 - \frac{8}{(1-\nu)\kappa^2}\right]\tilde{k}^4 + 2\left(1 + \frac{2}{(1-\nu)\kappa^2}\right)\tilde{k}^2 + 1$$

For usual material in engineering applications, we take the value of Poisson's ratio and shear correction factor, $\nu = 0.3$ and $\kappa^2 = 0.86$ respectively and after that the discriminant is converted to,

$$\Delta = 5.3929\,\tilde{k}^4 + 8.6445\,\tilde{k}^2 + 1 > 0, \qquad \forall\ \tilde{k} \in [0,1]$$

Further, we investigate the sign of the under square root quantities, $-B + \sqrt{\Delta}$ and $-B - \sqrt{\Delta}$, in order to conclude if we have real or imaginary frequencies. As for the first one,

$$y_1 = -B + \sqrt{\Delta} = \tilde{k}^2 + \frac{2\tilde{k}^2}{(1-\nu)\kappa^2} + 1 +$$

$$+\sqrt{\left[\left(1 + \frac{2}{(1-\nu)\kappa^2}\right)^2 - \frac{8}{(1-\nu)\kappa^2}\right]\tilde{k}^4 + 2\left(1 + \frac{2}{(1-\nu)\kappa^2}\right)\tilde{k}^2 + 1}$$

which is obviously positive ($y_1 > 0$) since we have $B < 0 \Rightarrow -B > 0$ and $\Delta > 0$.

As for the second relation,

$$y_2 = -B - \sqrt{\Delta} = \tilde{k}^2 + \frac{2\tilde{k}^2}{(1-\nu)\,\kappa^2} + 1 -$$

$$-\sqrt{\left[\left(1+\frac{2}{(1-\nu)\,\kappa^2}\right)^2 - \frac{8}{(1-\nu)\,\kappa^2}\right]\tilde{k}^4 + 2\left(1+\frac{2}{(1-\nu)\,\kappa^2}\right)\tilde{k}^2 + 1}$$

it is not clear if y_2 takes positive or negative values inside the interval $\tilde{k} \in [0,1]$. Consequently, by the aid of the mathematical package Matlab R2013a once again, we find that $y_2 > 0$ for all values of $\tilde{k}$ inside the interval $[0,1]$.

Finally, we conclude that we have only real values of the non-dimensional angular frequencies and further we choose to illustrate only the two positive of the totally four, namely the

$$\tilde{\omega}_{Shear} = \sqrt{\frac{-B+\sqrt{\Delta}}{2\,A}} \tag{9}$$

and

$$\tilde{\omega}_{flex} = \sqrt{\frac{-B-\sqrt{\Delta}}{2\,A}} \tag{10}$$

The relation (9) is illustrated by the cyan curve, which is the shear branch of the FSDT and the relation (10) is illustrated by the green curve, which is the flexural branch of the FSDT, shown on the following figure (Figure 1).

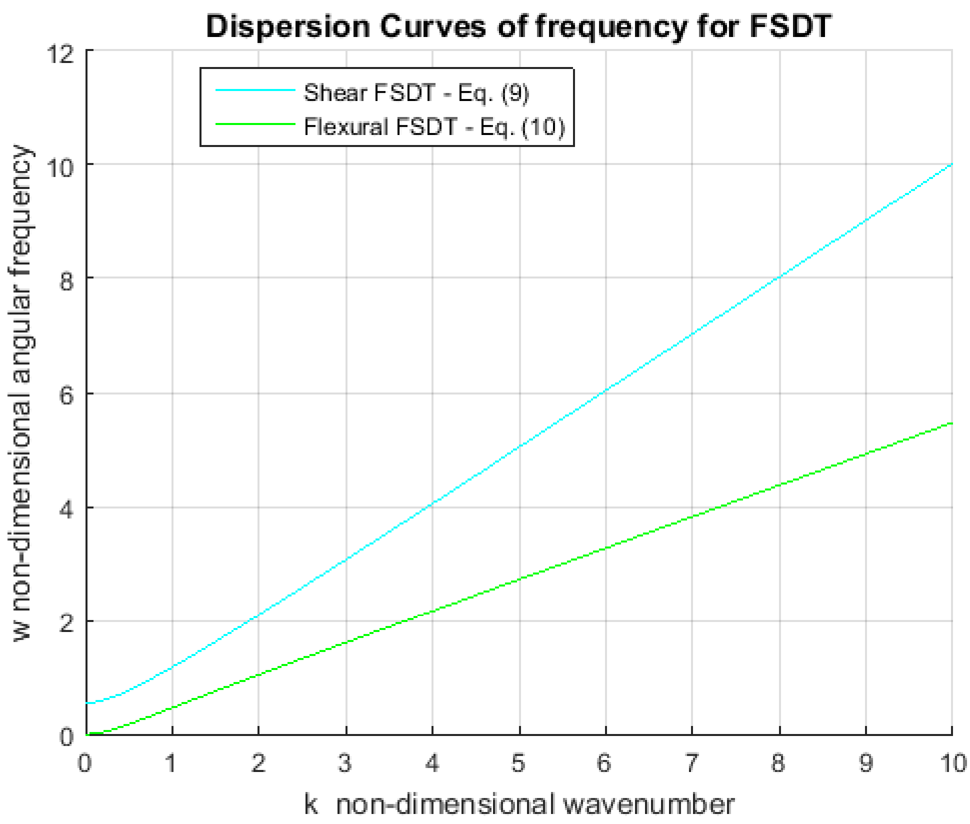

Figure 1: Dispersion Curves of frequency-wavenumber as for the TSDT (Mindlin's Plate Theory).

Further, we give the dispersion relation between the non-dimensional face velocity and the non-dimensional wavenumber. Also we extract the respective dispersion relation between the dimensionless group velocity of the wave propagation and the dimensionless wavenumber. The definition of these velocities is given below and is explained extensively on the Lecture Notes of *Triantafillou G., Belibassakis K.A. (2015), "Basic Principles of Naval and Marine Hydrodynamics", NTUA*.

However, here we have two branches as for the phase velocity, namely the shear and the flexural due to the existence of two branches for the angular frequency. For the same reason, we have two branches for the group velocity, one shear and one flexural. Thus, as for the phase velocities we get

$$\tilde{c}_{p\ Shear} = \frac{\tilde{\omega}_{Shear}}{\tilde{k}} = \frac{1}{\tilde{k}}\sqrt{\frac{-B+\sqrt{\Delta}}{2A}} \tag{11}$$

and
$$\tilde{c}_{p\ flex} = \frac{\tilde{\omega}_{flex}}{\tilde{k}} = \frac{1}{\tilde{k}}\sqrt{\frac{-B-\sqrt{\Delta}}{2A}} \tag{12}$$

As for the group velocities, we derive

$$\tilde{c}_{g\ Shear} = \frac{d\tilde{\omega}_{Shear}}{d\tilde{k}} = \frac{1}{\sqrt{2A}}\frac{1}{\sqrt{-B+\sqrt{\Delta}}}\left(\frac{\partial B}{\partial \tilde{k}} + \frac{1}{2\sqrt{\Delta}}\frac{\partial \Delta}{\partial \tilde{k}}\right) \tag{13}$$

$$\tilde{c}_{g\ flex} = \frac{d\tilde{\omega}_{flex}}{d\tilde{k}} = \frac{1}{\sqrt{2A}}\frac{1}{\sqrt{-B-\sqrt{\Delta}}}\left(-\frac{\partial B}{\partial \tilde{k}} - \frac{1}{2\sqrt{\Delta}}\frac{\partial \Delta}{\partial \tilde{k}}\right) \tag{14}$$

where, $\frac{\partial \Delta}{\partial \tilde{k}} = 2B\frac{\partial B}{\partial \tilde{k}} - 4\left(\frac{\partial A}{\partial \tilde{k}}C + A\frac{\partial C}{\partial \tilde{k}}\right)$ and $\frac{\partial A}{\partial \tilde{k}} = 0$,

$\frac{\partial B}{\partial \tilde{k}} = 2\tilde{k} + \frac{4\tilde{k}}{(1-\nu)\kappa^2}$ and $\frac{\partial C}{\partial \tilde{k}} = 4\tilde{k}^3$.

Finally, we have to study the behavior of the curves in region near the zero wavenumber, $\tilde{k} \to 0$.

Let $\tilde{k} \to 0$ first on the Eq. (11), then

$$\lim_{\tilde{k}\to 0} \tilde{c}_{p\ Shear} = \lim_{\tilde{k}\to 0} \frac{1}{\tilde{k}}\sqrt{\frac{-B+\sqrt{\Delta}}{2A}} = \frac{\lim_{\tilde{k}\to 0}\sqrt{\frac{-B+\sqrt{\Delta}}{2A}}}{\lim_{\tilde{k}\to 0}\tilde{k}}$$

We study the limits of the numerator and denominator of the above fraction separately. Thus, as for the limit $\lim_{\tilde{k}\to 0}\sqrt{\frac{-B+\sqrt{\Delta}}{2A}}$, we examine its terms isolate.

$$\lim_{\tilde{k}\to 0}(-B) = \lim_{\tilde{k}\to 0}\left(\tilde{k}^2 + \frac{2\tilde{k}^2}{(1-\nu)\kappa^2} + 1\right) = 1,$$

$$\lim_{\tilde{k}\to 0}\sqrt{\Delta} = \lim_{\tilde{k}\to 0}\sqrt{5.3929\,\tilde{k}^4 + 8.6445\,\tilde{k}^2 + 1} = 1,$$

$$\lim_{\tilde{k}\to 0} 2A = \lim_{\tilde{k}\to 0}\frac{4}{\kappa^2(1-\nu)} = \frac{4}{\kappa^2(1-\nu)}.$$

Consequently, we get $\lim_{\tilde{k}\to 0}\sqrt{\frac{-B+\sqrt{\Delta}}{2A}} = \sqrt{\frac{\kappa^2(1-\nu)}{2}}$ [constant]

and finally $\lim_{\tilde{k}\to 0}\tilde{c}_{p\ Shear} = \lim_{\tilde{k}\to 0}\frac{1}{\tilde{k}}\sqrt{\frac{\kappa^2(1-\nu)}{2}} = \infty$.

Second, let $\tilde{k}\to 0$ on the Eq. (12), then

$$\lim_{\tilde{k}\to 0}\tilde{c}_{p\ flex} = \lim_{\tilde{k}\to 0}\frac{1}{\tilde{k}}\sqrt{\frac{-B-\sqrt{\Delta}}{2A}} = \frac{\lim_{\tilde{k}\to 0}\sqrt{\frac{-B-\sqrt{\Delta}}{2A}}}{\lim_{\tilde{k}\to 0}\tilde{k}} \xrightarrow[\text{L' Hospital}]{\frac{0}{0}}$$

$$\lim_{\tilde{k}\to 0}\tilde{c}_{p\ flex} = \frac{\lim_{\tilde{k}\to 0}\frac{\partial}{\partial\tilde{k}}\sqrt{\frac{-B-\sqrt{\Delta}}{2A}}}{1} = \lim_{\tilde{k}\to 0}\frac{\partial}{\partial\tilde{k}}\frac{\sqrt{-B-\sqrt{\Delta}}}{\sqrt{2A}} =$$

$$= \lim_{\tilde{k}\to 0}\frac{\frac{\partial}{\partial\tilde{k}}\sqrt{-B-\sqrt{\Delta}}\ \sqrt{2A} - \frac{\partial}{\partial\tilde{k}}\sqrt{2A}\ \sqrt{-B-\sqrt{\Delta}}}{2A} = 0$$

Subsequently, let $\tilde{k}\to 0$ on the Eq. (13), then

$\lim_{\tilde{k}\to 0}\tilde{c}_{g\ Shear} = \lim_{\tilde{k}\to 0}\frac{1}{\sqrt{\frac{-B+\sqrt{\Delta}}{2A}}}\left(\frac{\partial B}{\partial\tilde{k}} + \frac{1}{2\sqrt{\Delta}}\frac{\partial\Delta}{\partial\tilde{k}}\right)$ and examine isolate the denominator and numerator, we get $\lim_{\tilde{k}\to 0}\sqrt{\frac{-B+\sqrt{\Delta}}{2A}} = \sqrt{\frac{\kappa^2(1-\nu)}{2}}$ [constant] and

$$\lim_{\tilde{k}\to 0}\left(\frac{\partial B}{\partial\tilde{k}} + \frac{1}{2\sqrt{\Delta}}\frac{\partial\Delta}{\partial\tilde{k}}\right) = \lim_{\tilde{k}\to 0}\frac{\partial B}{\partial\tilde{k}} + \lim_{\tilde{k}\to 0}\frac{1}{2\sqrt{\Delta}}\frac{\partial\Delta}{\partial\tilde{k}}.$$

Subsequently, taking apart the terms of the above limit, we derive the following

$$\lim_{\tilde{k}\to 0}\frac{\partial B}{\partial\tilde{k}} = \lim_{\tilde{k}\to 0}\left(2\tilde{k} + \frac{4\tilde{k}}{(1-\nu)\kappa^2}\right) = 0 \quad \text{and}$$

$$\lim_{\tilde{k}\to 0}\frac{1}{2\sqrt{\Delta}}\frac{\partial\Delta}{\partial\tilde{k}} = \lim_{\tilde{k}\to 0}\frac{1}{2\sqrt{\Delta}}\left[2B\frac{\partial B}{\partial\tilde{k}} - 4\left(\frac{\partial A}{\partial\tilde{k}}C + A\frac{\partial C}{\partial\tilde{k}}\right)\right] = \lim_{\tilde{k}\to 0}\frac{1}{\sqrt{\Delta}}\left[B\frac{\partial B}{\partial\tilde{k}} - 2\left(\frac{\partial A}{\partial\tilde{k}}C + A\frac{\partial C}{\partial\tilde{k}}\right)\right] =$$

$$= \lim_{\tilde{k}\to 0}\frac{1}{\sqrt{\Delta}}\left[-\left(\tilde{k}^2 + \frac{2\tilde{k}^2}{(1-\nu)\kappa^2} + 1\right)\left(2\tilde{k} + \frac{4\tilde{k}}{(1-\nu)\kappa^2}\right) - 2\left(0\cancel{\tilde{k}^4} + \frac{2}{\kappa^2(1-\nu)}4\tilde{k}^3\right)\right] =$$

$$= \lim_{\tilde{k}\to 0} \frac{1}{\sqrt{\Delta}} \left[-\left(\tilde{k}^2 + \frac{2\tilde{k}^2}{(1-\nu)\kappa^2} + 1 \right) \left(2\tilde{k} + \frac{4\tilde{k}}{(1-\nu)\kappa^2} \right) - 2\left(0\cdot\tilde{k}^4 + \frac{2}{\kappa^2(1-\nu)} 4\tilde{k}^3 \right) \right] =$$

$$= \frac{-1\cdot 0 - 2\cdot 0}{1} = 0$$

And taking into account the previous limits, the limit value of the shear group velocity near the zero is,

$$\lim_{\tilde{k}\to 0} \tilde{c}_{g\,Shear} = \lim_{\tilde{k}\to 0} \frac{1}{\sqrt{\dfrac{-B+\sqrt{\Delta}}{2A}}} \left(\frac{\partial B}{\partial \tilde{k}} + \frac{1}{2\sqrt{\Delta}} \frac{\partial \Delta}{\partial \tilde{k}} \right) = (0+0)\sqrt{\frac{2}{\kappa^2(1-\nu)}} = 0.$$

Last but not least, we examine the limit value on the same region (near the zero wavenumber) for the flexural group velocity.

$$\lim_{\tilde{k}\to 0} \tilde{c}_{g\ flex} = \lim_{\tilde{k}\to 0} \frac{1}{\sqrt{2A}} \frac{1}{\sqrt{-B-\sqrt{\Delta}}} \left(-\frac{\partial B}{\partial \tilde{k}} - \frac{1}{2\sqrt{\Delta}} \frac{\partial \Delta}{\partial \tilde{k}} \right) = \lim_{\tilde{k}\to 0} \frac{1}{\sqrt{\dfrac{-B-\sqrt{\Delta}}{2A}}} \left(-\frac{\partial B}{\partial \tilde{k}} - \frac{1}{2\sqrt{\Delta}} \frac{\partial \Delta}{\partial \tilde{k}} \right)$$

As for the numerator of the limit $\lim_{\tilde{k}\to 0} \tilde{c}_{g\ flex}$, we have $\lim_{\tilde{k}\to 0} \left(-\frac{\partial B}{\partial \tilde{k}} - \frac{1}{2\sqrt{\Delta}} \frac{\partial \Delta}{\partial \tilde{k}} \right) = 0$ and as for the denominator we derive $\lim_{\tilde{k}\to 0} \sqrt{\frac{-B-\sqrt{\Delta}}{2A}} = 0$. Now, we are going to perform the L' Hospital's rule which uses the derivatives of the numerator and denominator in order to evaluate limits involving indeterminate forms. Thus, taking apart the derivatives of the numerator and denominator to make the calculations easier,

$$\lim_{\tilde{k}\to 0} \left(-\frac{\partial^2 B}{\partial^2 \tilde{k}} - \frac{1}{2\sqrt{\Delta}} \frac{\partial^2 \Delta}{\partial^2 \tilde{k}} + \frac{1}{4\sqrt{\Delta}^3} \frac{\partial \Delta}{\partial \tilde{k}} \right) = -\lim_{\tilde{k}\to 0} \left(2 + \frac{4}{(1-\nu)\kappa^2} \right) - \lim_{\tilde{k}\to 0} \frac{1}{2\sqrt{\Delta}} \frac{\partial^2 \Delta}{\partial^2 \tilde{k}} + \lim_{\tilde{k}\to 0} \frac{1}{4\sqrt{\Delta}^3} \frac{\partial \Delta}{\partial \tilde{k}}$$

where,

$$\lim_{\tilde{k}\to 0} \frac{1}{2\sqrt{\Delta}} \frac{\partial^2 \Delta}{\partial^2 \tilde{k}} = \frac{\lim\limits_{\tilde{k}\to 0} \dfrac{\partial^2 \Delta}{\partial^2 \tilde{k}}}{\lim\limits_{\tilde{k}\to 0} 2\sqrt{\Delta}} = \frac{\lim\limits_{\tilde{k}\to 0} \left(2\dfrac{\partial B}{\partial \tilde{k}} \dfrac{\partial B}{\partial \tilde{k}} + 2B\dfrac{\partial^2 B}{\partial^2 \tilde{k}} - 4\left(\dfrac{\partial^2 A}{\partial^2 \tilde{k}} C + 2\dfrac{\partial A}{\partial \tilde{k}} \dfrac{\partial C}{\partial \tilde{k}} + A\dfrac{\partial C}{\partial \tilde{k}} \right) \right)}{\lim\limits_{\tilde{k}\to 0} 2\sqrt{\Delta}} =$$

$$= \frac{2\left(2\cdot 0 + \dfrac{4\cdot 0}{(1-\nu)\kappa^2} \right)^2 + 2\cdot 1\left(2 + \dfrac{4}{(1-\nu)\kappa^2} \right) - 4\left(0\cdot 0 + 2\,0\cdot 0 + \dfrac{2}{(1-\nu)\kappa^2}\cdot 0 \right)}{2\cdot 1} = 2 + \frac{4}{(1-\nu)\kappa^2}$$

and

$$\lim_{\tilde{k}\to 0} \frac{1}{4\sqrt{\Delta}^3} \frac{\partial \Delta}{\partial \tilde{k}} = \frac{\lim\limits_{\tilde{k}\to 0} \dfrac{\partial \Delta}{\partial \tilde{k}}}{\lim\limits_{\tilde{k}\to 0} 4\sqrt{\Delta}^3} = \frac{0}{4} = 0.$$

Consequently, we get for the limit of the derivative of the numerator,

$$\lim_{\tilde{k}\to 0}\left(-\frac{\partial^2 B}{\partial^2\tilde{k}}-\frac{1}{2\sqrt{\Delta}}\frac{\partial^2\Delta}{\partial^2\tilde{k}}+\frac{1}{4\sqrt{\Delta}^3}\frac{\partial\Delta}{\partial\tilde{k}}\right)=-2-\frac{4}{(1-\nu)\,\kappa^2}-\left(2+\frac{4}{(1-\nu)\,\kappa^2}\right)=-4-\frac{8}{(1-\nu)\,\kappa^2}$$

As for the derivative of the denominator's limit $\lim_{\tilde{k}\to 0}\tilde{c}_{g\ flex}$, we get

$$\lim_{\tilde{k}\to 0}\frac{\partial}{\partial\tilde{k}}\left(\sqrt{\frac{-B-\sqrt{\Delta}}{2A}}\right)=\frac{1}{2\sqrt{-B-\sqrt{\Delta}}}\left(-\frac{\partial B}{\partial\tilde{k}}-\frac{1}{2\sqrt{\Delta}}\frac{\partial\Delta}{\partial\tilde{k}}\right)$$

Note again that the previous limit is of indeterminate form, since $\lim_{\tilde{k}\to 0}2\sqrt{-B-\sqrt{\Delta}}=0$ and $\lim_{\tilde{k}\to 0}\left(-\frac{\partial B}{\partial\tilde{k}}-\frac{1}{2\sqrt{\Delta}}\frac{\partial\Delta}{\partial\tilde{k}}\right)=0$. Consequently, we proceed again to the application of the L' Hospital's rule and we get

$$\lim_{\tilde{k}\to 0}\frac{\partial}{\partial\tilde{k}}\left(\sqrt{\frac{-B-\sqrt{\Delta}}{2A}}\right)=\frac{1}{2\sqrt{-B-\sqrt{\Delta}}}\left(-\frac{\partial B}{\partial\tilde{k}}-\frac{1}{2\sqrt{\Delta}}\frac{\partial\Delta}{\partial\tilde{k}}\right)=\infty.$$ Finally taking all the previous results into account, the limit of flexural group velocity near the zero is

$$\lim_{\tilde{k}\to 0}\tilde{c}_{g\ flex}=\lim_{\tilde{k}\to 0}\frac{1}{\sqrt{\frac{-B-\sqrt{\Delta}}{2A}}}\left(-\frac{\partial B}{\partial\tilde{k}}-\frac{1}{2\sqrt{\Delta}}\frac{\partial\Delta}{\partial\tilde{k}}\right)=0.$$

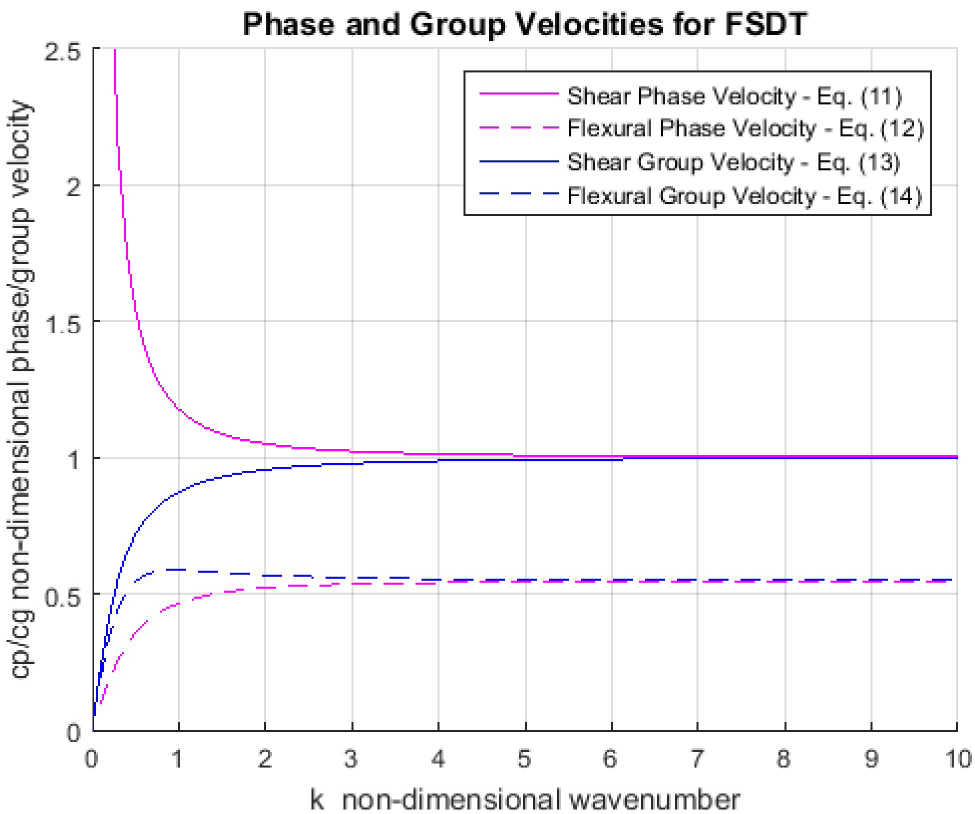

Figure 2: Dispersion Curves of velocity-wavenumber as for the FSDT (Mindlin's Plate Theory).