ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ

ΣΧΟΛΗ ΗΛΕΚΤΡΟΛΟΓΩΝ ΜΗΧΑΝΙΚΩΝ ΚΑΙ ΜΗΧΑΝΙΚΩΝ ΥΠΟΛΟΓΙΣΤΩΝ

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Φιλαλήθεις Μηχανισμοί χωρίς Χρηματικά Ανταλλάγματα

ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

του

ΧΡΗΣΤΟΥ Σ. ΤΖΑΜΟΥ

Επιβλέπων: Δημήτρης Φωτάκης

Λέκτορας Ε.Μ.Π.

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Αθήνα, Ιούνιος 2011

Εθνικό Μετσόβιο Πολυτεχνείο
Σχολή Ηλεκτρολόγων Μηχανικών και Μηχανικών Υπολογιστών
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Αθήνα, Ιούνιος 2011

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Ευχαριστίες

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Περίληψη

Σε αυτήν την διπλωματική εργασία, εξετάζουμε το πρόβλημα του Σχεδιασμού Μηχανισμών στο πλαίσιο της κοινωνικής επιλογής. Λόγω του βασικού θεωρήματος των Gibbard-Satterthwaite, μόνο τετριμμένοι μηχανισμοί είναι φιλαλήθεις στο γενικό μοντέλο, οπότε εξερευνούμε πιο περιοριοριςμένους χώρους όπως οι single-peaked και οι μετρικοί χώροι. Εξετάζουμε το πρόβλημα της τοποθεσίας εγκαταστάσεων (facility location) ώς παιχνίδι, όπου ένα πλήθος από εγκαταστάσεις θα τοποθετηθούν σε ένα μετρικό χώρο με βάση τις τοποθεσίες που ανακοινώθηκαν από στρατηγικούς παίχτες. Ένας μηχανισμός αντιστοιχεί τις θέσεις των παιχτών σε ένα σύνολο από θέσεις για τις εγκαταστάσεις. Κάθε παίχτης στοχεύει να μειώσει το κόστος σύνδεσης του, δηλαδή την απόσταση του από την κοντινότερη εγκατάσταση στην πραγματική του θέση. Ενδιαφερόμαστε για μηχανισμούς που είναι φιλαλήθεις δηλαδή εγγυούνται ότι κανένας παίχτης δεν μπορεί να οφεληθεί δηλώνοντας διαφορετική τοποθεσία από την πραγματική του, δεν χρησιμοποιούν χρήματα και προσσεγγίζουν το βέλτιστο κοινωνικό κόστος. Οι μηχανισμοί μπορούν να είναι είτε ντετερμινιστικοί είτε πιθανοτικοί. Στη διπλωματική αυτή, παρουσιάζουμε διάφορα άνω και κάτω όρια για διάφορες περιπτώσεις: μία εγκατάσταση, σταθερό πλήθος εγκαταστάσεων και μεταβλητό πλήθος εγκαταστάσεων με σταθερό κόστος ανα εγκατάσταση

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Abstract

In this thesis, we consider the problem of Mechanism Design without Money in the Social Choice setting. Due to the Gibbard-Satterthwaite main impossibility result, only trivial mechanisms are strategyproof for the general setting, so we explore more restricted domains like single-peaked preferences and metric spaces. We study the problem as a facility location game, where a number of facilities are to be placed in a metric space based on locations reported by strategic agents. A mechanism maps the agents' locations to a set of facilities. Every agent seeks to minimize her connection cost, namely the distance of her true location to the nearest facility, and may misreport her location. We are interested in mechanisms that are strategyproof, i.e., ensure that no agent can benefit from misreporting her location, do not resort to monetary transfers, and approximate the optimal social cost. The mechanisms can be either deterministic or randomized. We provide upper bounds along with corresponding lower bounds for different cases: single facility, fixed number of facilities and facilities with uniform opening cost.

Keywords

Mechanism design without money, Social choice, Facility Location

Contents

Chapter 1

Introduction

The purpose of this chapter is to make a brief introduction in the concepts of Game Theory and Mechanism Design. We will focus on the case of ”Mechanism Design without money” where there are no monetary transfers and we will present several important results and related work on the field.

1.1 Motivation

Every day, people are called to make decisions in their lives. Those decisions are usually not personal and require two or more people to interact in order to reach an agreement. This is trivial when their interests are similar and a unanimous decision can be reached. But most of the time, their interests are conflicting and a consensus may be hard or impossible to achieve. Such situations are the subject of Game Theory, which provides the mathematical tools to predict the behavior of the participants under the specified conditions. On the other hand, Mechanism Design aims to create procedures through which a decision with certain desirable properties can be achieved.

1.2 Game Theory

Game theory is a branch of mathematics that studies strategic situations where players choose different actions in an attempt to maximize their returns. Due to its very general nature it is used in many different fields in an attempt to model the player behavior such as social sciences (most notably in economics, management, operations research, political science, and social psychology), biology (particularly evolutionary biology and ecology) as well as in formal sciences (logic and computer science).

The strategic situations to be studied are modeled as games where multiple players interact. The games studied in game theory are well-defined mathematical objects. A game consists of a set N of n players (agents), each of whom has a set of possible actions (strategies) S_i available to him. For every possible combination of player strategies a certain payoff per player is specified by a function $h_i : \prod_{i \in N} S_i \rightarrow \mathbb{R}$. The games we will examine are simultaneous, in the sense that every agent chooses a strategy at the same time and a certain outcome depending on all the chosen strategies is achieved.

Game theory tries to predict the outcome of such a game when agents choose their strategies rationally aiming to maximize their own payoffs. These predictions are called "solutions" of the game. Out of the many different solution concepts, the most commonly used are the Nash Equilibrium and the Equilibrium in Dominant Strategies.

1.2.1 Nash Equilibrium

Nash equilibrium is a solution concept of a game involving two or more players which assigns a certain strategy to every player such that every player's best choice is to play his equilibrium strategy assuming that he knows the equilibrium strategies of the others.

We say that a vector of strategies $s^* \in \prod_{i \in N} S_i$ is a Nash equilibrium if no agent can benefit unilaterally changing his or her strategy. Formally, the following must hold:

$$\forall i, s_i \in S_i, s_i \neq s_i^* : h_i(s_i^*, s_{-i}^*) \geq h_i(s_i, s_{-i}^*).$$

A game can have either a pure-strategy or a mixed Nash Equilibrium, (in the latter case each player's strategies are a probability distribution over his pure strategies). Nash proved that if we allow mixed strategies, then every game with a finite number of players in which each player can choose from finitely many pure strategies has at least one Nash equilibrium. This is not the case, however, for pure Nash Equilibrium where it may not exist for several games.

To better understand these concepts consider the game of chicken. The game of chicken models two drivers, both headed for a single lane bridge from opposite directions. The first to swerve away yields the bridge to the other. If neither player swerves, the result is a costly deadlock in the middle of the bridge, or a potentially fatal head-on collision. It is presumed that the best thing for each driver is to stay straight while the other swerves (since the other is the "chicken" while a crash is avoided). Additionally, a crash is presumed to be the worst outcome for both players. This yields a situation where each player, in attempting to secure his best outcome, risks the worst. The following matrix shows the different outcomes depending on the drivers' actions.

	Swerve	Straight
Swerve	(Tie,Tie)	(Lose,Win)
Straight	(Win,Lose)	(Crash,Crash)

Assuming, that Win = 1, Lose = -1, Tie = 0, Crash = -10, we get the following payoff matrix.

	Swerve	Straight
Swerve	(0,0)	(-1,1)
Straight	(1,-1)	(-10,-10)

In this example we can see that the strategy pairs (Swerve, Straight) and (Straight, Swerve) are Nash equilibria since no player can benefit by changing his strategy. The strategy pairs (Swerve, Swerve) or (Straight, Straight) on the other hand are unstable since any player would be better off by changing his strategy.

In the game of Chicken there exist two Pure Nash Equilibria. We now consider a game where no Pure Nash Equilibria Exist. One such game is Rock Paper Scissors. In this game, two players simultaneously choose one gesture out of either Rock, Paper or Scissors. Then the winner is determined under the following rules:

- Rock defeats Scissors.
- Scissors defeats Paper.
- Paper defeats Rock.

If both players choose the same gesture, the game is tied and the players throw again. The corresponding payoff matrix for the game is the following.

	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)

It is obvious that this game has no Pure Nash Equilibrium because if a player knew what the other would play he would want to alter his strategy. However, a mixed Nash Equilibrium exists as always and that is for both players to choose a gesture uniformly at random. Then their expected payoff would be 0 and no player can improve by changing his strategy.

1.2.2 Dominant Strategies Equilibrium

One stronger solution concept than Nash Equilibrium is the Dominant Strategies Equilibrium. An agent's strategy is said to be dominant when that strategy is better than any another strategy for that player, no matter how his opponents may play. Formally, for any player i, a strategy s_i is (weakly) dominant if:

$$\forall s_{-i} \in \prod_{j \in N \setminus \{i\}} S_j, s'_i \in S_i : h_i(s_i, s_{-i}) \geq h_i(s'_i, s_{-i})$$

A vector of strategies $s^* \in \prod_{i \in N} S_i$ is a Dominant Strategies Equilibrium if s^*_i is a dominant strategy for every player i.

As an example of a game where a Dominant Strategy Equilibrium exists consider the situation where an amount of money X is to be shared among some players and each player is asked whether he wants to participate or not. For 2 players and $X = 10$ the payoff matrix becomes:

	Yes	No
Yes	(5,5)	(10,0)
No	(0,10)	(0,0)

In this game, it becomes clear that no matter what the other player(s) would answer, one's best strategy is to say 'Yes'. This notion captures most of the agent's rationality and predicts the outcome since it is clear that all rational players would behave this way.

In the previous example it seemed obvious that saying 'Yes' was the rational thing to do. In several cases however, this is not always so clear, with the most common example being the prisoner's dilemma.

In the prisoner's dilemma, two suspects are arrested by the police. The police doesn't have enough evidence for their conviction, so they begin to interrogate the suspects separately. Each of them is offered a deal. If he confesses and testifies against the other while the other remains silent, he goes free while the other will serve 10 years in prison. If they both confess, they will both serve 9 years while if they both remain silent they will only serve one year in prison. It may seem like the best choice is for them both to stay silent. But this is not what happens if the players are rational. If we look at their payoff matrix, we see that it is a dominant strategy for every player to confess since no matter what the other player would do it is in the player's best interest to confess.

	Stay Silent	Confess
Stay Silent	(-1,-1)	(-10,0)
Confess	(0,-10)	(-9,-9)

From this example it is clear that even when a dominant strategy equilibrium exists it doesn't always guarantee the best possible outcome for everyone. The notion of "the best outcome for everyone" is captured by the concept of Pareto-optimality, that we will define later on.

1.3 Mechanism Design

After establishing several tools to predict the outcome of strategic situations, we will use those to create procedures through which a decision with certain desirable properties can be achieved. This is the area of Mechanism Design.

Mechanism design is the art of designing the rules of a game to achieve a specific outcome. This is done by setting up a structure in which each player has an incentive to behave as the designer intends. The game is then said to implement the desired outcome. The strength of such a result depends on the solution concept used in the game. Throughout this thesis, we will focus on the strongest form of equilibrium in dominated strategies.

In the social choice setting, agents report their preferences in a mechanism and the mechanism outputs a single joint decision. The goal of the mechanism designer is to design a process through which a fair decision can be reached such that the participants are satisfied.

Formally, there is a set A of different alternatives and a set of n voters (the agents) N. Each agent i has a linear order $\succ_i$ over the set A which captures his preferences called his preference profile. The notation $a \succ_i b$ for two alternatives $a, b \in A$ means that voter i prefers alternative a to alternative b. Let us denote by L the set of linear orders on A. Then a function $f : L^n \to A$ that maps the agents' preferences to a single alternative is called social choice function. A mechanism is a social choice function that aggregates the agent preferences into a single joint decision to be implemented.

1.3.1 Desirable Properties and Definitions

In order to be able to reach meaningful decisions, the social choice function must satisfy certain desirable properties.

Definition 1.3.1. *A social choice function f is* onto *if given the appropriate preference profiles any decision out of the set of possible alternatives can be reached. That is, $\forall a \in A, \exists x \in L^n$ such that $f(x) = a$.*

Definition 1.3.2. *A social choice function f is* unanimous *if when all player prefer a certain outcome more than anything else, then that outcome must be the alternative chosen by the mechanism. That is, if $\exists a \in A$ such that $\forall b \in A$ and $i \in N$, $a \succ_i b$ then $f(\succ_1, \ldots, \succ_n) = a$.*

Definition 1.3.3. *A social choice function f is* Pareto optimal *if no other alternative is more preferred by every agent than the alternative chosen by the mechanism. That is, if $f(\succ_1, \ldots, \succ_n) = a$, then $\nexists b \in A$ such that $b \succ_i a$, $\forall i \in N$.*

The onto condition is weaker than the unanimous condition which in turn is weaker than Pareto optimality. Those properties require that the mechanism is efficient in the sense that it computes the most desirable outcome for all the agents. These are desirable properties since we want to ensure that agents will get the best possible outcome and won't have to face situations like the prisoner's dilemma.

However another property is even more important than the properties above. It becomes clear that a social function should be invulnerable to strategic manipulations if our goal is to be fair to the agents. This means that it should be a (possibly weakly) dominant strategy for every agent to report his true preferences and he should never benefit by misreporting them. This notion is captured by the requirement that the mechanism (the social choice function) is *incentive compatible.*

Definition 1.3.4. *A social choice function f can be strategically manipulated by agent i if there exist profiles $\succ_1, \ldots, \succ_n, \succ'_i \in L$ such that $b \succ_i a$ where $a = f(\succ_1$*

$, \dots, \succ_i, \dots, \succ_n)$ *and* $b = f(\succ_1, \dots, \succ_i', \dots, \succ_n)$. *That is agent* i *that prefers* b *to* a *can ensure that* b *gets chosen by strategically misreporting his preferences to be* $\succ_i'$ *rather than* $\succ_i$. *A function* f *that cannot be strategically manipulated by any agent is called* incentive compatible *or* strategy-proof.

If we don't even allow groups of agents to misreport their preferences so that they all achieve a better outcome, we have the stronger requirement of *group strategyproofness*

Definition 1.3.5. *A social choice function* f *can be strategically manipulated by a group of agents* S *if there exist profiles* $\succ_1, \dots, \succ_n \in L$ *and* $\succ_i' \in L$ *for all* $i \in S$ *such that* $b \succ_i a$ *for all* $i \in S$ *where* $a = f(\succ_n)$ *and* $b = f(\succ_S', \succ_{-S})$. *That is the group of agents* S *that prefers* b *to* a *can ensure that* b *gets chosen by strategically misreporting their preferences to be* $\succ_S'$ *rather than* $\succ_S$. *A function* f *that cannot be strategically manipulated by any group of agents is called* group strategy-proof.

We note that an onto, group strategyproof mechanism must always be Pareto optimal otherwise all agents would misreport their preferences such that a more preferable alternative will be chosen. This is not always true however for (not group) strategyproof rules.

If there are only two alternatives ($|A| = 2$), an obvious example of an incentive compatible social choice function is the majority vote. This however is not incentive compatible for more than two alternatives since for example for 3 alternatives an agent that prefers the least preferred alternative among other agents can in some cases achieve a better result by voting for his second most preferred alternative to ensure that his least preferred alternative won't be selected.

Another example of an incentive compatible function is the case of a dictatorship.

Definition 1.3.6. *An agent* i *is a dictator in a social choice function* f *if for all* $\succ_1, \dots, \succ_n \in L$ $f(\succ_1, \dots, \succ_n) = a$ *where* $a \succ_i b$ $\forall b \in L, b \neq a$. f *is called a* dictatorship *if some* i *is a dictator in it.*

Obviously this mechanism despite being incentive compatible and pareto optimal is not preferable since it disregards the preferences of all the agents but the dictator.

Unfortunately, there is an impossibility result that states that when the number of alternatives is larger than 2, only trivial social choice functions are incentive compatible.

1.3.2 Main Impossibility Result - Gibbard-Satterthwaite

We now state the main impossibility result for incentive compatible social choice functions.

Theorem 1.3.1 (Gibbard [1]-Satterthwaite [2])**.** *Let f be an incentive compatible social choice function onto A, where $|A| \geq 3$, then f is a dictatorship.*

The theorem states that every social choice function is either manipulable or a dictatorship. The whole field of Mechanism Design tries to propose alternative routes to escape the negative effects of this theorem. By slightly modifying the model many interesting results can be obtained. There are several solution directions that make it possible to circumvent the previous result. The most typical of them include:

The addition of payments into the model. The main idea behind this concept is that, by including money in the mechanism, agents that are not satisfied with the social decision get somehow compensated and thus they have a limited incentive to lie.

The restriction of the domain of preferences. By limiting the expressiveness of the agents forcing them to choose a preference profile from a structured subset of the set L of linear orders on the different alternatives A, the agents lose some of their strength for manipulating the system. The chosen subset is usually not arbitrary but comes natural based on the limitations of certain problems that don't require the full expressiveness of all different permutations of A.

The addition of randomness into the model. We relax the notion of the mechanism by allowing the output to be a probability distribution over different alternatives rather than a single alternative. We distinguish between randomized and deterministic mechanisms.

In this thesis we will investigate the last two directions that don't require use of money. This is motivated by a number of different important domains, e.g., political elections, organ donations, and school admissions, where monetary transfers are either unethical, strictly prohibited or hard to implement. The mechanism design in these settings requires no payment and therefore is more challenging. In contrast with the mechanisms with money where a socially optimal choice can be enforced as a dominant equilibrium, in our case this is not always possible and a sacrifice in social welfare is a necessity to derive a truthful bidding environment.

Chapter 2

1-facility location

2.1 Single-Peaked preferences

In this section, we examine problems of a simple and restricted domain where there is a natural one-dimensional ordering of the alternatives and the preferences of the agents are single peaked. This domain can be used to model political policies, economic decisions, location problems, or any allocation problem where a single point must be chosen in an interval. The key assumption we make is that agents' preferences are assumed to have a single most preferred point in the interval and that preferences are as one moves away from the peak. As an example consider the following question: "How many assignments should a course have per term? (0-10)". In this setting it is clear that an agent that believes that 7 assignments are most appropriate, he would prefer 8 rather than 10 and 4 rather than 2.

Formally the allocation space (the set of alternatives) is the unit interval $A = [0, 1]$. Each agent i has a preference ordering $\succeq_i$ (weak order) of the alternatives $x \in A$. The preference relation $\succeq_i$ is single-peaked if there exists a single point p_i (the peak) such that for all $x \in A - \{p_i\}$ and all $\lambda \in [0, 1)$, $\lambda x + (1 - \lambda)p_i \succ_i p_i$. Let the class of single peaked preferences be R. We restrict the mechanism in the set R.

Like before we are interested in designing a set of rules for determining the outcome such that no agent has an incentive to lie thus making it weakly dominant strategy to misreport his preference. This is the notion of incentive compatibility we've seen before. From now on we will use the words strategyproof or truthful for the mechanisms that satisfy this property.

One property of truthful mechanisms for single peaked preferences is that the onto requirement is equivalent to the unanimous requirement which is equivalent to the requirement for Pareto optimality. So we have that:

Lemma 2.1.1. *Suppose f is strategyproof. Then f is onto if and only if it is unanimous if and only if it is Pareto-optimal.*

Proof. We assume that f is onto since it is the weaker condition and we will show that is satisfies the other two requirements.

Assuming that f violates the unanimous condition, consider an unanimous profile $\succeq$ such that $p_i = x$ for all the peaks such that $f(\succeq) \neq x$. Since f is onto there must be a profile $\succeq'$ such that $f(\succeq') = x$. By strategyproofness $f(\succeq_1, \succeq'_2, ..., \succeq'_n) = x$ otherwise agent 1 could manipulate f. Repeating the argument we get that $f(\succeq_1, \succeq_2, \succeq'_3, ..., \succeq'_n) = x, ..., f(\succeq) = x$ which is a contradiction.

Now assume that Pareto-optimality is violated. That means that either (i) $f(\succeq) < p_i$ for all $i \in N$ or (ii) $f(\succeq) > p_i$ for all $i \in N$. Without loss of generality, assume that (i) holds and let $m = argminp_i$. For an agent i with $p_i > p_m$ consider the outcome of the mechanism when he switches his preference from $\succeq_i$ to $\succeq'_i$ such that $p'_i = p_m$ and $f(\succeq) \succeq'_i p_i$. By strategyproofness $f(\succeq') \in [\succeq, p_i]$ otherwise agent i with preference $\succeq'_i$ could manipulate the mechanism by reporting $\succeq_i$. Similarly, $f(\succeq') \notin (\succeq, p_i]$, otherwise agent i with preference $\succeq_i$ could manipulate the mechanism by reporting $\succeq'_i$. Therefore, $f(\succeq') = f(\succeq)$. By repeating the argument for every agent i with peak $p_i > p_m$, we get a profile $\succeq'$ such that $f(\succeq') = f(\succeq)$ with all p'_i's equal to p_m. This is a contradiction since f is onto and unanimous as we proved before. □

We now state the characterization of truthful mechanisms for single peaked preferences given by Moulin [3], Barberà and Jackson [4], and Sprumont [5].

Theorem 2.1.2. *A rule f is strategy-proof, onto and anonymous if and only if there exist* $y_1, y_2, \ldots, y_{n-1} \in [0, 1]$ *such that for all* $\succeq \in R^n$,

$$f(\succeq) = median(p_1, p_2, \ldots, p_n, y_1, y_2, \ldots, y_{n-1})$$

To show why these mechanisms are strategyproof, let x be the outcome of the mechanism. Then an agent i with $p_i = x$ wouldn't have an incentive to misreport his preferences. For the case $p_i \neq x$, where without loss of generality we assume $p_i < x$, agent i by changing his preference profile $\succeq_i$ to a profile $\succeq'_i$ with peak at $p'_i \leq x$ cannot alter the outcome of the mechanism. If on the other hand he reports $\succeq'_i$ with peak at $p'_i > x$ as his preference profile, the outcome of the mechanism would be less preferable to him as $f(\succeq_1, ..., \succeq'_i, .., \succeq_n) \geq x$.

In the previous theorem, the points $\vec{y}$ act as phantom agents placed by the mechanism at fixed positions. Choosing the points appropriately (placing $n-k$ points at 0 and $k-1$ points at 1) we can simulate any k-th order statistic of the agent's peaks. However, there are other mechanisms we can create as well such as placing all y_i's at 1/2. Then the mechanism outputs the position of the rightmost peak when all agents have peaks $p_i < 1/2$, the position of the leftmost peak when all agents have peaks $p_i > 1/2$ and 1/2 otherwise.

If we drop the requirement that f is anonymous the complete class of strategyproof mechanisms is called Generalized Median Voter Scheme and is given by the following theorem.

Theorem 2.1.3. *A rule f is strategy-proof and onto* if and only if *there exist points* $a_S \in [0, 1], \forall S \subseteq N$ *such that:*

- $S \subseteq T$ *implies* $a_S \leq a_T$
- $a_\emptyset = 0$ *and* $a_N = 1$
- $\forall \succeq \in R^n,\ f(\succeq) = \max_{S \subseteq N} \min\{a_S, p_i : i \in S\}$

The generalized median is a richer class of mechanisms. The value a_S for a specific set S represents the power of the specific set of agents. For a specific instance each subset is assigned the value $\min\{a_S, p_i : i \in S\}$ and the subset with the highest such value is chosen. By letting $a_S = 0$ if $i \notin S$ and $a_S = 1$ if $i \in S$ for a specific agent i we can simulate the dictatorship. By letting $a_S = 0$ if $|S| < n - k + 1$ and $a_S = 1$ if $|S| \geq n - k + 1$ we can simulate any k-th order statistic since we allow only subsets of specific size to be chosen.

2.2 Facility Location on the line

We note that all strategyproof mechanisms depend only on the peaks of each agent. This allows us to use arbitrary preference profiles. From now on, we simplify the mechanism to depend only on the reported peaks and we assume that agents rank the alternatives according to their distance from the peak. We will use the terms of the facility location problem which is a well known optimization problem.

The problem of Facility Location is somehow classical and has received considerable attention in Operations Research (see e.g. [6]), Algorithms, mostly from the viewpoints of approximation (see e.g. [7, 8, 9]) and online algorithms (see e.g. [10, 11, 12]), Social Choice (see e.g. [3, 4, 5, 13, 14, 15, 16]), and recently, Algorithmic Mechanism Design (see e.g. [17, 18, 19, 20, 21]).

2.2.1 Model

In our basic setting n agents are located on the [0,1] interval and the mechanism must select the location of a public facility; the cost of an agent is its distance to the facility. Each agent i has a location $x_i \in [0, 1]$. We refer to the collection $\vec{x} = \langle x_1, \dots, x_n \rangle$ as the location profile.

A (deterministic) mechanism in this simple setting is a function $f : [0, 1]^n \to [0, 1]$, that is, a function that maps a given location profile to a location of a facility. If the facility is located at y, the cost of agent i is $cost(x_i, y) = |y - x_i|$.

Throughout this thesis we will declare by $lt(\vec{x}), rt(\vec{x}), med(\vec{x})$ for a location profile $\vec{x}$, the leftmost location, the rightmost location and the median location respectively.

Since each agent wants to minimize his facility cost, the condition for strategyproofness means that for all $x \in [0,1]^n$, for each agent i, and for all $x_i' \in [0,1]$, $cost(x_i, f(\vec{x})) \leq cost(x_i, f(x_i', \vec{x}_{-i})$, where $\vec{x}_{-i} = \langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \rangle$ is the vector of the locations of all other agents.

In this setting, it becomes clear that the outcome of the mechanism must be efficient in terms of a certain objective. The most natural and commonly used objectives are to minimize either the sum of the distances (social cost) or the maximum cost. We will measure the efficiency according to the optimal solution for the optimization problem without the condition for strategyproofness. We will try to minimize the approximation ratio to the optimal cost. This idea of approximation fits in the framework of *approximate mechanism design without money*, recently initiated by Procaccia and Tennenholtz [17].

2.2.2 Social cost

Theorem 2.2.1. *The mechanism that selects the median of the reported locations is strategyproof and achieves approximation ratio of 1 for the social cost.*

Proof. From the characterization we gave before for single peaked preferences, we saw that each k-th order statistic is strategyproof, thus the median is strategyproof. We now show that this is also optimal.

Assume that n is odd, $n = 2k + 1$. Any point that is to the left of the median has higher social cost since it is further away from at least $k + 1$ locations and closer to at most k locations, and the same holds for any point to the right of the median. If n is even, $n = 2k$, and without loss of generality $x_1 \leq \cdots \leq x_n$, then any point in the interval $[x_k, x_k + 1]$ is an optimal facility location. In this case the median is considered to be the leftmost point of the optimal interval. □

2.2.3 Maximum cost

When the objective we want to minimize is the maximum cost, the optimal solution is to take the average of the leftmost and rightmost reported location. Unfortunately, this is not strategyproof since the rightmost agent would report a location further to the right and get the facility closer to his real location.

Theorem 2.2.2. *Any k-th order statistic of the reported locations is a strategyproof 2-approximate mechanism for the maximum cost.*

Proof. The strategyproofness follows from the characterization we gave in the previous section. For the approximation we have that:

Let x^* be the optimal facility location that minimizes the maximum cost. We have that $\text{OPT} \geq |x_i - x^*|, \forall i \in N$. Also the cost of selecting the k-th order statistic is $\text{COST} = \max\{|x_i - x_k|\} \leq \max\{|x_i - x^*| + |x^* - x_k|\} \leq 2\text{OPT}$

The approximation ratio is tight since for the instance where 1 agent is at 0 and n-1 agents are at 1 the optimal cost is 0.5 while selecting any k-th order statistic gives a cost of 1. □

On the other hand we have the following matching lower bound.

Theorem 2.2.3. *Any deterministic strategyproof mechanism has an approximation ratio of at least 2 for the maximum cost.*

Proof. Assume for contradiction that $f : R^n \to R$ is a strategyproof mechanism and has an approximation ratio smaller than 2 for the maximum cost. Consider the location profile x where $x_1 = 0$, $x_2 = 1$ and $x_i = 1/2$ for all $i \in N \setminus \{1, 2\}$. Since the mechanism has an approximation ratio smaller than 2 for the maximum cost, then it must locate a facility in $(0, 1)$. Indeed, assume without loss of generality that $f(x) = a, a \geq 1/2$.

Now, consider the profile where $x_1 = 0$, $x_2 = a$ and $x_i = 1/2$ for all $i \in N \setminus \{1, 2\}$. Again, because of approximation the facility must be located somewhere in $(0, a)$. In that case, given the profile x, agent 2 can benefit by reporting $x_2 = 1$, thus moving the facility to a, in contradiction to strategyproofness. □

This is the best we can do in terms of deterministic mechanisms. However if we allow randomization, we get a richer class of strategyproof mechanisms which manages to overcome the lower bound for deterministic mechanisms.

Theorem 2.2.4. *The mechanism that selects the leftmost location $lt(\vec{x})$ with probability 1/4, the rightmost location with probability 1/4 $rt(\vec{x})$, and the average of the leftmost and rightmost location $(lt(\vec{x})+rt(\vec{x}))/2$ with probability 1/2, is strategy-proof and 3/2-approximate for the maximum cost.*

Proof. We first prove the approximation ratio. The cost of selecting any of the two extremes and placing the facility is exactly 2OPT while the cost of selecting the average of the leftmost and rightmost location is exactly OPT. Both cases occur with probability 1/2, so the expected cost is (3/2)OPT.

We now prove the strategyproofness. Without loss of generality consider an agent located at $x \leq (lt(\vec{x}) + rt(\vec{x}))/2$.

- If he reports a location y that is located between $lt(\vec{x})$ and $rt(\vec{x})$ then his expected cost doesn't change.
- If he reports a location y with $y \geq rt(\vec{x})$ his expected cost increases since both the midpoint and the rightmost location are further away.
- If he reports a location $y \leq lt(\vec{x})$ with $d = lt(\vec{x}) - y$ his cost increases by d when the leftmost location is selected and decreases by $d/2$ when the midpoint is selected. So his expected decrease in the cost is $(1/4)(-d) + (1/2)(d/2) = 0$.

□

While the theorem implies that randomization allows us to drop the feasible strategyproof approximation ratio from 2 to 3/2, we can also show that this is as far as randomization can take us.

Theorem 2.2.5. *Any randomized strategyproof mechanism has an approximation ratio of at least 3/2 for the maximum cost.*

2.3 Facility Location on general metric spaces

In this section we investigate the problem of locating a facility in more general metric spaces other than the line/unit interval that we saw previously. The setting is the same as before with the only difference that agents are located in a metric space (M, d) and their cost is the distance of the shortest path to the facility $cost(x_i, y) = d(x_i, y)$. We will focus only on the case of social cost as the case of maximum cost is trivial as picking any arbitrary agent is strategy-proof and gives an approximation ratio of 2.

2.3.1 Tree metrics

A simple extension is the tree metrics. In this class of metric spaces, a similar result to that of the line metric can be obtained by considering the median extended to tree metrics.

Theorem 2.3.1. *The mechanism that selects the median of the reported locations is strategyproof and achieves approximation ratio of 1 for the social cost.*

Consider the following mechanism for finding the median of a tree with respect to the location profile $\vec{x} \in M^n$. We first fix an arbitrary node as the root of the tree. Then, as long as the current location has a subtree that contains more than half of the agents, we smoothly move down this subtree. Finally, when we reach a point where it is not possible to move closer to more than half the agents by continuing downwards, we stop and return the current location.

The fact that the above mechanism is strategyproof is straightforward, and follows from similar arguments as the ones given for a median on a line: an agent can only modify the location of the mechanism's outcome by pushing the returned facility location away from its true location. It can also be verified that the mechanism returns a location that is optimal in terms of the social cost.

2.3.2 Non-tree metrics

Every other metric space other than the tree must contain at least one cycle. Schummer and Vohra [14] prove that in this case any strategy proof mechanism that is onto must is a dictatorship when all agents are located in the cycle.

This result gives us a tight lower bound of $n-1$ on the approximation ratio of any strategyproof mechanism.

Proof. Since any mechanism must have a bounded approximation ratio it must be onto, otherwise if all agents are located in a position that is not in the image set of the mechanism the social cost is greater than zero while the optimal is 0. Since the mechanism is onto the previous result applies and there is a cycle dictator. Place all the agents at a position y on the cycle and the cycle dictator at a position z on the cycle. Then the optimal social cost is $d(y, z)$ while the social cost of any strategyproof mechanism is $(n-1)d(x, y)$. □

The lower bound is tight since any dictatorship is a $(n-1)$-approximate strategyproof mechanism.

By using randomization we manage to break the lower bound by a high margin. In fact the following trivial mechanism allows us to get constant approximation ratio.

Theorem 2.3.2 (Random Dictator)**.** *The mechanism that returns a facility location according to the probability distribution that gives probability 1/n to the location x_i, for all $i \in N$ is strategyproof and $(2-2/n)$-approximate.*

Proof. This mechanism is obviously strategyproof, since by deviating an agent can only lose if its own location is chosen, and does not affect the outcome if another's location is selected. As for the approximation ratio, we have the following: Given the location profile $\vec{x}$ let y be the optimal facility location. We have that the social cost of the mechanism is:

$$
\begin{aligned}
sc(f(\vec{x}), \vec{x}) &= \sum_{i \in N} \frac{1}{n} \sum_{j \in N} d(x_i, x_j) = \frac{1}{n} \sum_{i \in N} \sum_{j \in N - \{i\}} [d(x_i, y) + d(y, x_j)] \\
&= \frac{1}{n} \sum_{i \in N} [(n-1) d(x_i, y) + OPT - d(y, x_i)] = \\
&= \frac{1}{n} \sum_{i \in N} [(n-2) d(x_i, y) + OPT] \\
&= OPT + \frac{n-2}{n} OPT = (2 - \frac{2}{n}) OPT
\end{aligned}
$$

□

The following tables summarize the results for the case of one facility.

For the maximum cost we have:

	Deterministic	Randomized
Line	2	3/2
General	2	2

For the social cost we have:

	Deterministic	Randomized
Line	1	1
General	n-1	2

Chapter 3

Locating more than one facility

3.1 Model and Properties

A natural extension to the setting presented in the previous section is to locate more than one facilities in the metric space. A deterministic mechanism i now a function $f : M^n \to M^k$. Each player is only interested in minimizing the distance to the closest facility so when the k facilities are located at $\vec{y} \in M^k$, his cost becomes $cost(x_i, \vec{y}) = \min_{y_j \in \vec{y}} d(x_i, y_j)$. So the strategyproofness condition is properly adjusted for the new costs.

One interesting property of the facility location games is partial-group strategyproofness.

Definition 3.1.1. *A mechanism is partially group strategyproof if for any group of agents at the same location, each individual cannot benefit if they misreport simultaneously. Formally, given a non-empty set* Sffl $\subset N$*, profile* $\vec{x} = (\vec{x}_S, \vec{x}_{-S}) \in M^n$ *where* $x_S = (x, x, .., x)$ *for some* $x \in M$ *and the misreported locations* $\vec{x}'_S \in M^{|S|}$*, we have:*

$$cost(f(\vec{x}_S, \vec{x}_{-S}), x) \leq cost(f(\vec{x}_S, \vec{x}_{-S}), x)$$

This means that a group of overlapping agents cannot misreport their location at the same time and benefit. It is clear that partial-group strategyproofness is stronger than strategyproofness. However, the other direction holds as well.

Lemma 3.1.1. *In a k-facility game, a strategy-proof mechanism is also partially group strategyproof.*

Proof. To prove the lemma, we iteratively change the locations of every agent in S from x to x_i'. We have that:

$$cost(f(x, ..., x, \vec{x}_{-S}), x) \leq cost(f(x_1', x, ..., x, \vec{x}_{-S}), x)$$

by strategyproofness. Similarly we have that:

$$cost(f(x_1', x, ..., x, \vec{x}_{-S}), x) \leq cost(f(x_1', x_2', x, ..., x, \vec{x}_{-S}), x)$$

iteratively for all $i \in S$. So we get that:

$$cost(f(\vec{x}_S, \vec{x}_{-S}), x) \leq cost(f(\vec{x}_S, \vec{x}_{-S}), x)$$

which completes the proof. □

The previous property will be very useful in the analysis of strategyproof mechanisms.

Another very useful tool, is the image sets. We define the concept of image set. For a given mechanism f, the image set of agent i with respect to a location profile $\vec{x}_{-i}$ is the set of all possible facility locations when agent i varies her reported location:

$$I_i(\vec{x}_{-i}) = \cup_{x_i \in M} f(x_i, \vec{x}_{-i})$$

Any strategy-proof mechanism f must always output some location in $I_i(\vec{x}_{-i})$ that is closest to agent i as shown in the following lemma. Intuitively, the image set represents agent i's power. If f outputs the best solution for agent i within her power, agent i does not have the incentive to lie.

Lemma 3.1.2. *Let f be a strategy-proof mechanism in a k-facility game. We have that for every $\vec{x} \in M^n, i \in N$:*

$$cost(f(\vec{x}), x_i) = inf_{y \in I_i(\vec{x}_{-i})} d(y, x_i)$$

Proof. Assume for contradiction that there exists a $y' \in I_i(\vec{x}_{-i})$ such that $d(y', x_i) < cost(f(\vec{x}), x_i)$. Then, by definition of the imageset there exists a x_i' such that $f(x_i', \vec{x}_{-i}) = y'$. Then, it would be beneficial for i to misreport his location to x_i' which contradicts to strategyproofness. □

As a corollary of the previous lemma we get that the imageset is a closed set of M under the topology induced by the metric $d(.)$.

Due to partial group strategyproofness, we can extend the notion of the imageset to groups of agents and get similar properties.

For a given mechanism f, the image set of agents S with respect to a location profile $\vec{x}_{-S}$ is the set of all possible facility locations when agents S vary their reported locations:

$$I_S(\vec{x}_{-S}) = \cup_{\vec{x}_S \in M^{|S|}} f(\vec{x}_S, \vec{x}_{-i})$$

Using partial group strategyproofness we get the following lemma:

Lemma 3.1.3. *Let f be a strategy-proof mechanism in a k-facility game. We have that for every non empty set $S \subset N$, $\vec{x}_S = (x, ..., x)$ and $\vec{x}_{-S} \in M^{n-|S|}$:*

$$cost(f(\vec{x}_S, \vec{x}_{-S}), x) = inf_{y \in I_S(\vec{x}_{-S})} d(y, x)$$

The proof is the same as before with the difference that now the groups of agents S can misreport their positions and benefit.

Most of the proofs on this section rely on the use of image sets since they essentially capture the agents' power to manipulate the mechanism. We now move on to provide several results for the k-facility game.

3.2 Two facilities

As before we will examine a simple setting first, locating only two facilities on the line metric. As an objective, we will focus primarily on minimizing the social cost in a strategyproof way. In the optimal configuration disregarding strategyproofness, there will be two facility locations y_1, y_2 with $y_1 \leq y_2$. These location separate the locations of the agents into two multisets L, R based on the facility each agent prefers. L stands for the players located on the left of the line while R stands for the players located on the right. As we've seen before in an optimal configuration y_1 must be the median of the points in L and y_2 the median of the points in R respectively. So in order to compute the optimal facility locations, it suffices to optimize over the $n-1$ possible choices of L and R.

This process however of computing the optimal solution is far from being strategyproof since there are cases where an agent would report a different location to alter the multiset he belongs to by including or excluding other agents in order to benefit and thus strategically manipulate the mechanism. This can be made obvious by the following instance: Consider $n-2$ agents located at 0, one agent located at -1 and one agent located at 1. Then the optimal solution places one facility at 0 and the other one at either -1 or 1. Since the setting is symmetric we can assume it is placed at -1. Then the agent located at 1 would report $1+\epsilon$ (for a very small $\epsilon > 0$) as his location forcing the mechanism to place a facility at $1+\epsilon$. This way he manages to alter his multiset $R = \{0, ..., 0, 1\}$ to $R' = \{1+\epsilon\}$ and benefit.

On the other hand there is a very simple mechanism that is strategyproof. Placing the two facilities at the leftmost and rightmost locations is a strategyproof $(n-2)$-approximation mechanism since for any location profile the leftmost location always belongs to the multiset L and the rightmost to multiset R. This mechanism may seem very inefficient but as it turns out this is the only mechanism with bounded approximation ratio as we can see by the following theorem.

Theorem 3.2.1. *Any deterministic strategyproof mechanism for the 2-facility location game on the line that is anonymous, irrelevant to scaling and translation and has a bounded approximation ratio, always places the facilities at the two extremes(left and right).*

Proof. Through the proof we will use instances of the form (n_1, n_2, n_3) where all agents are n_i agents are located in position x_i for all $1 \leq i \leq 3$. We will heavily rely on the use of image sets. An image-set I_{n_1,n_2} is a set that specifies all the points in the line of reals that the mechanism assigns facilities to, when n_1 agents are at position 0, n_2 agents are at position 1 and $n_3 = n - n_1 - n_2$ agents move along the line in the same location.

For the first part of the proof we fix (n_1, n_2, n_3) and only deal with the vector of their respective positions.

Consider an instance where $x_1 < x_2 < x_3$ and a facility is placed in a position $a \in (x_1, x_3)$ with $a \neq x_2$ such that it is closer to x_2 than the other facility. Without loss of generality we assume that a is between x_2 and x_3. Then, the second facility must be placed at a position b with $b < x_2$, otherwise the n_1 agents would move to x_2 and then the two facilities would be placed at positions (x_2, x_3) due to bounded approximation ratio which would be beneficial for them.

Since $b < a < x_3$ the closest facility to the n_3 agents is at a. Assuming $x_1 = 0$ and $x_2 = 1$ then $a \in I_{1,2}$ and $y \notin I_{n_1,n_2} \forall y \in [a, x_3)$. If the n_2 agents move to position $1 - \epsilon$ then due to the scaling of the set I_{n_1,n_2} a facility must be placed at position $a(1 - \epsilon)$ which is beneficial for the n_2 agents, a contradiction. It follows that I_{n_1,n_3} is either full or empty in (0,1).

So for any $a, b \geq 1$ with $a + b < n$, we have that either:

- $(0, 1) \subset I_{a,b}$
- $(0, 1) \cap I_{a,b} = \emptyset$

Now consider the case where for some a, b we have that $(0, 1) \subset I_{a,b}$. Assume that there exists some open interval (x, y) with $x, y \in I_{a,b}$ such that $(x, y) \cap I_{a,b} = \emptyset$. Without loss of generality, $1 \leq x < y$. Then on an instance where a agents are at 0, b agents are at y and the $n - a - b$ agents are at $y - \epsilon$ with $0 < \epsilon < (y - x)/2$ sufficiently small such that the two facilities are assigned one close to 0 and the other at $y - \epsilon$, the b agents would move at a position 1 and the a facility would be assigned at position y due to the imageset $I_{a,b}$. This is a contradiction. We conclude that no such interval exists so $I_{a,b} = \mathbb{R}$ and the $n - a - b$ agents are dictators as a group.

Now consider an instance with three groups of agents a, b, c where none of the groups is a dictator. Then we have that $(0,1) \cap I_{a,b} = \emptyset, (0,1) \cap I_{b,c} = \emptyset, (0,1) \cap I_{a,c} = \emptyset$. Assume that there exists some open interval (x, y) with $x, y \in I_{a,b}$ such that $(x, y) \cap I_{a,b} = \emptyset$. Without loss of generality, $1 \leq x < y$. Then on an instance where a agents are at 0, b agents are at 1 and c agents are at $x + \epsilon$ one facility is located at position x which lies inside the interval (x_a, x_c) which is a contradiction since $(0,1) \cap I_{a,c} = \emptyset$. We conclude that no such interval exists so $I_{a,b} = I_{b,c} = I_{a,c} = \mathbb{R} \setminus (0,1)$. This means that on every instance the facilities are located at the rightmost and leftmost locations.

We conclude that on any instance where agents are located in three distinct locations either there exists a group that is a dictator or the facilities are located at the rightmost and leftmost locations.

Assume that for some instance with three groups of agents a, b, c, the b agents are dictators. We will now show that this must hold for every $a, c > 0$. Assume that this doesn't hold for some a', c'. Then in the instance with groups of agents a', b, c' with $x_{a'} < x_b < x_{c'}$ the facilities are located at $x_{a'}$ and $x_{b'}$. Without loss of generality we assume that $a' < a$. Then on an instance where $x_a = 0, x_b = \epsilon, x_c = 1$, for $\epsilon > 0$ sufficiently small where the facilities are placed one at x_b and one near x_c then $a - a'$ agents located at 0 have an incentive to lie reporting 1 as their location so that a facility is placed at 0. Thus we reach a contradiction.

We conclude that b agents will be dictators no matter how the others are arranged. Obviously more than b agents must be dictators, otherwise some of them would misreport their location so that exactly b of them are in the same place. This indicates that there must be a threshold t such that any instance with three groups of agents where one of them contains t or more agents must always place a facility in their location. Obviously, we must have that $t \geq n/2$ otherwise if we had two groups of dictators and a third group very far away the approximation would be unbounded.

We consider an instance where $a = n - t - 1, b = 2, c = t - 1$ and $x_a < x_b < x_c$. In this instance, every group has less than t agents so there is no dictator. The facilities are placed in x_a and x_c. So no matter how the agents in group b move no facility will ever be place inside x_a and x_c. In any instance $a = n - t - 1, b = 1, c = t - 1, d = 1$ with $x_a < x_b < x_c < x_d$ no facility is allocated between x_a and x_c. However the agent b can force a facility to be allocated either at x_a or at x_c by reporting x_a or x_c respectively as his location. Thus, at an instance where $x_a = 0, x_b = 1, x_c = 3$ and x_d sufficiently large the facilities must be allocated one at 0 and the other near x_d. However, the agents at group c have an incentive to report 1 as their location since they would become dictators and one facility would be placed at 1. This is a contradiction to the fact that there are some instances where a group of agents behave as dictators.

So even in an instance where $a = 1, b = n - 2, c = 1$ where $x_a = 0 < x_b < x_c = 1$ the facilities would be placed at x_a and x_c. This means that no matter what locations the group of $n - 2$ agents report no facility would be placed inside the interval (x_a, x_c). Since the $n - 2$ be misreporting their locations in the interval $[x_a, x_c]$ can generate any possible location profile $\vec{x}$, we conclude that for any possible location profile $\vec{x}$ no facilities are located in the interval $(lt(\vec{x}), rt(\vec{x}))$.

To conclude the proof, we show that for any location profile the facilities are placed in the locations $lt(\vec{x}), rt(\vec{x})$. We show this by induction on the number of distinct locations on the location profile. For the induction basis we have that for three distinct locations all mechanisms place a facility in the leftmost and rightmost location. Assuming it holds for k distinct locations we show that it holds for $k + 1$ distinct locations. Consider an instance where a facility is located at a position greater than $rt(\vec{x})$. Consider the set of agents S located at $rt(\vec{x})$ and the imageset I of $\vec{x}_{-S}$. The image set must contain the location $rt(\vec{x}){-}S$ by the induction step (Otherwise the agents in set S would report $rt(\vec{x}){-}S$ as their location). Assume that there is an interval (a, b) with $a, b \in I$ such that $(a, b) \cap I = \emptyset$. Then if the S agents are located at $a + \epsilon$ for sufficiently small $\epsilon > 0$ then a facility would be placed at a which would be inside the interval $(lt(\vec{x}), rt(\vec{x}))$, a contradiction. $\square$

The conditions for scaling and translation can be removed. A more technical analysis can be found in [22]. The previous theorem indicates that the simple inefficient solution is the best we can achieve in deterministic mechanisms. Fortunately in randomized mechanisms, we can obtain much better results as we will see in the next section.

3.3 Randomized Mechanisms for Two Facilities

In the previous section we saw that the best approximation ratio for the the two facility location game is linear in terms of the number of players. In this section, we deal with randomized mechanisms in order to get better results. Fortunately there exists a mechanism with constant approximation ratio that works for general metric spaces [20].

3.3.1 Proportional Mechanism

Given a location profile $x = (x_1, x_2, ..., x_n)$, the locations of the two facilities are decided by the following random process:

- Round 1: Choose agent i uniformly at random from N. The first facility l_1 is placed at x_i.
- Round 2: Let $d_j = d(l_1, x_j)$ be the distance from agent j to the first facility l_1. Choose agent j with probability $\frac{d_j}{\sum_{k \in N} d_k}$

Theorem 3.3.1. *The Proportional Mechanism for the two-facility game is strategyproof.*

Theorem 3.3.2. *The approximation ratio of the Proportional Mechanism for the two-facility game is at most 4 for any metric space.*

We leave the proofs of the previous two theorems for the next section, where we will revisit the Proportional Mechanism under a slightly modified setting.

We note that the approximation ratio of 4 for the proportional mechanism is tight even on the line. This can be seen if we consider the location profile $\vec{x} = (0, ..., 0, \epsilon, 1)$ for sufficiently large n and $\epsilon \to 0$. The optimal solution has social cost equal to ϵ while the proportional mechanism has social cost equal to:

$$\frac{n-2}{n}\left(\frac{2-\epsilon}{1+\epsilon} + \frac{2(1-\epsilon)}{(n-2)\epsilon + 1 - \epsilon} + \frac{1+\epsilon}{n-2+\epsilon}\right)\epsilon$$

which is equal to $4 - 7/n$ as $\epsilon \to 0$.

The proportional mechanism is the best known randomized mechanism for the two-facility game. It remains an open question whether there exist strategyproof mechanisms with lower approximation ratio. For randomized mechanisms the only known lower bound is the following.

Theorem 3.3.3. *In a two facility game, any randomized strategy-proof mechanism has an approximation ratio of at least* $1 + \frac{\sqrt{2}-1}{12-2\sqrt{2}} - \frac{1}{n-2} \geq 1.045 - \frac{1}{n-2}$ *social cost for any* $n \geq 5$.

The following table summarizes the results for the 2-facility location game in terms of social cost.

	Deterministic	Randomized
Line	n-2	4
General	-	4

3.4 More than two facilities

Since the best strategyproof mechanism possible for the 2-facility game is linear one would not expect to get any better results for more facilities. In fact, the following theorem gives a linear lower bound for the k-facility location game.

Theorem 3.4.1. *Any deterministic strategy-proof mechanism for the k-facility game in the line metric space has an approximation ratio of at least* $\frac{n-1}{2}$ *for* $k \geq 2$.

What's even more interesting however is that no strategyproof mechanism with bounded approximation ratio is known even for the line metric. The following theorem proves that this is not possible for a very large class of almost all natural mechanisms.

Theorem 3.4.2. *Any deterministic anonymous strategy-proof mechanism for the k-facility game on the line that places facilities in the range $[lt(\vec{x}), rt(\vec{x})]$ for every location profile $\vec{x}$ has unbounded approximation ratio.*

Proof. Fix $n-1$ agents at positions $\vec{x}$ and consider the imageset I as an extra agent moves along the line. Since the mechanism must have a bounded approximation ratio the image set is not bounded. Therefore, either (i) $(-\inf, lt(\vec{x})] \subset I$ or (ii) there exists an interval (a, b) such that $a < lt(\vec{x})$ with $(a, b) \cap I = \emptyset$ and $a, b \in I$. Assume case (ii) holds, then on an instance where the agent is located at $a + \epsilon$ for small $\epsilon > 0$ a facility must be placed at a which would be outside of the range of the agents' positions, a contradiction. Thus, for all $\vec{x}$ we have that $(-\inf, lt(\vec{x})] \subset I$ which means that a facility must always be assigned at the leftmost agent.

Now consider an instance where the location profile of the agents is as follows $\vec{x} = (0, 1, 2, 2 + \epsilon)$. Then one facility must be placed at 0 one close to 1 and one close to the last two agents. Consider the imageset $I_{(0,2,2+\epsilon)}$ as agent 2 moves along the line. Let $a = \min\{x | x \in I_{(0,2,2+\epsilon)}, x > 0\}$. This is well defined since at any instance $(0, \epsilon', 2, 2 + \epsilon)$ for small $\epsilon' > 0$ the facilities must be placed one at 0, one near 2 and one near $2 + \epsilon$ in order to have bounded approximation ratio. Now consider an instance $(a - \epsilon', a, 2, 2 + \epsilon)$ for small $\epsilon' > 0$. Here the facilities are placed at $a - \epsilon'$, one near 2 and one near $2 + \epsilon$. In this instance however, the agent 2 at a can manipulate the mechanism by moving to 0 and guarantee that a facility is placed at a. □

The previous theorem leaves almost no space for deterministic strategyproof mechanisms with bounded approximation ratio to exist. So we turn to randomized mechanisms hoping to achieve some positive results. Unfortunately the following natural extension of the proportional mechanism even for 3 facilities is not strategyproof. Allocate the first two facilities the same as in the two facility case, and the third one in some agent with probability proportional to his minimal distance to the first two facilities.

The counter-example to this is as follows: there exist n_0 agents at location 0, n_1 agents at location 1, n_2 agents at location $1 + x$ and 1 agent at location $1 + x + y$. Here n_0 is sufficiently large such that we can assume the first facility l_1 to be always located at 0. In this configuration, let $y = 100$, $x = 10^5$, $n_1 = 50$ and $n_2 = 4$. After a careful calculation one may find out that the agent at location 1 may have the incentive to misreport to location 1 + x.

However for 3 facilities, there exists a strategyproof mechanism with linear approximation ratio that works as follows: The first two facilities are located at the leftmost and the rightmost reported locations. For the third facility, it is randomly chosen among the rest of the agents with probability proportional to their minimal distances to the first two facilities.

With the slight exception of the previous mechanism, no other strategyproof mechanism is known with bounded approximation ratio for the k-facility problem when $k \geq 3$. It still remains an open question to find such mechanisms or to prove that they don't exist.

Chapter 4

Imposing Mechanisms

In the previous section, we saw that finding efficient strategyproof mechanisms when we need to place more than one facility is either impossible or very hard. Due to the lack of positive results, other directions must be followed that allow a more rich set of mechanisms. Such a direction was proposed by Nissim, Smorodinsky, and Tennenholtz [21], who introduced imposing mechanisms.

Imposing mechanisms are a general class of mechanisms that compute a socially efficient outcome but in addition they have the ability restrict how agents exploit this outcome. Restricting the set of allowable post-actions for the agents, the mechanism can penalize liars. For Facility Location games in particular, an imposing mechanism requires that an agent should connect to the facility nearest to her reported location, thus increasing her connection cost if she lies.

Nissim *et al.* using the notion of imposing mechanisms obtained a randomized imposing mechanism for k-Facility Location with a running time exponential in k. The mechanism approximates the optimal average connection cost, namely the optimal connection cost divided n, with an additive term of roughly $n^{2/3}$. However for a large class of instances, the additive approximation guarantee of does not imply any constant approximation ratio for k-Facility Location.

In this section, we will focus on imposing mechanisms and provide several efficient strategyproof mechanisms.

4.1 Model and Definitions

An *imposing mechanism* f maps a location profile $\vec{x}$ to a tuple of non-empty sets $(C, C^1, \ldots C^n)$, where $C \subseteq M$ is the facility set of f and each $C^i \subseteq C$ contains the facilities where agent i should connect. We write $f(\vec{x})$ to denote the facility set of f and $f^i(\vec{x})$ to denote the facility subset of each agent i. For the k-Facility Location game, $|f(\vec{x})| = k$. A *randomized mechanism* is a probability distribution over deterministic mechanisms.

We only consider imposing mechanisms where each agent i should connect to the facility in $f(\vec{x})$ closest to her reported location, namely where $f^i(\vec{x}) = \{z \in f(\vec{x}) : d(z, x_i) = d(x_i, f(\vec{x}))\}$ for each i.

We note that non-imposing mechanisms is a special case where for all location profiles $\vec{x}$ and all agents i, $f^i(\vec{x}) = f(\vec{x})$.

Another, special case of imposing mechanisms is winner imposing mechanisms. A mechanism f is *winner-imposing* if for each agent i, $f^i(\vec{x}) = \{x_i\}$ if $x_i \in f(\vec{x})$, and $f^i(\vec{x}) = f(\vec{x})$ otherwise. For a winner-imposing mechanism f and some location profile $\vec{x}$, we write either that f allocates a facility to agent i or that f places a facility at x_i to denote that f adds x_i in its facility set $f(\vec{x})$. Moreover, we write that f connects agent i to the facility at x_i to denote that $f^i(\vec{x}) = \{x_i\}$, as a result of $x_i \in f(\vec{x})$.

4.2 Deterministic Mechanisms

Imposing mechanisms allow us to design a richer class of strategyproof mechanisms. As we've seen before the only non-imposing anonymous mechanism for the 2-facility location game on the line was to assign facilities at the agents at the two extremes. In the case of imposing mechanisms we are also allowed to do the following:

1st Round: Assign a facility to the agent at the median[$med(\vec{x})$].

2nd Round: If $|lt(\vec{x}) - med(\vec{x})| > |rt(\vec{x}) - med(\vec{x})|$ place a facility at $med(\vec{x}) - \max(|lt(\vec{x}) - med(\vec{x})|, 2|rt(\vec{x}) - med(\vec{x})|)$. Otherwise, place a facility at $med(\vec{x}) + \max(2|lt(\vec{x}) - med(\vec{x})|, |rt(\vec{x}) - med(\vec{x})|)$.

Theorem 4.2.1. *The above mechanism is strategyproof and* $(n-1)$*-approximate.*

We omit the proof since it is similar to the proof for the mechanism of the 3-facility location, we will see next.

The above mechanism fails to make any improvement to the problem but shows instead that a richer class of mechanisms is possible under the imposing setting.

Now, we move on to the 3-facility location game where it was impossible to find any mechanism with bounded approximation before. This is not true however in the case of imposing mechanisms, where we are allowed to do the following:

First place two facilities at the agents at the extremes[$lt(\vec{x}), rt(\vec{x})$]. We denote by x_m the average of the two points ($x_m = (lt(\vec{x}) + rt(\vec{x}))/2$). Let A and B be the set of agents on $[lt(\vec{x}), x_m]$ and $(x_m, rt(\vec{x})]$ respectively. Define $d_A = max_{i \in A}|lt(\vec{x}) - x_i|$ and $d_B = max_{i \in B}|x_i - rt(\vec{x})|$. We allocate the second facility as follows:

If $d_A \geq d_B$, the third facility is placed at $min\{x_m, lt(\vec{x}) + max\{d_A, 2d_B\}\}$.

If $d_A < d_B$, the third facility is placed at $max\{x_m, rt(\vec{x}) - min\{2d_A, d_B\}\}$.

We shall prove that the above mechanism is strategyproof and has a linear approximation ratio.

Theorem 4.2.2. *The imposing mechanism for 3-facility on the line is strategyproof.*

Proof. Assume for contrary that the mechanism is not strategy-proof. Then there must be an agent who benefits by misreporting his location. Since the cost of the agents located at $lt(\vec{x}), rt(\vec{x})$ is 0, they have no incentive to misreport. So we only consider agents located in $(lt(\vec{x}), rt(\vec{x}))$.

If an agent reports a location outside the interval $(lt(\vec{x}), rt(\vec{x}))$ then a facility will be assigned to him and due to the fact that the mechanism is imposing he must connect to that facility. In that case, the cost is higher than the cost of truthfully reporting his location.

Now consider an agent i and assume without loss of generality that he is located at $x_i \in (lt(\vec{x}), x_m]$. We distinguish different cases regarding where the third facility l_3 is placed.

If $l_3 > x_m$ then $d_A < d_B$ and $d_A < (lt(\vec{x}) + x_m)/2$ since otherwise l_3 would be equal to x_m. The closest that x_i can force the facility to approach is $min(2d_B, x_m)$ which is greater than $2d_A$. Since we have that $x_i \leq d_A$, agent i cannot benefit by the third facility since the facility placed at $lt(\vec{x})$ will always be closer.

If $l_3 = x_m$ then $d_A, d_B \geq (lt(\vec{x}) + x_m)/2$ and there is no location that agent i can misreport to so that $l_3 < x_m$.

If $x_i = l_3 < x_m$ then agent i has cost 0 and thus no incentive to misreport.

If $x_i < l_3 < x_m$, then no matter where agent i misreports to, he can only increase the term $min\{x_m, lt(\vec{x}) + max\{d_A, 2d_B\}\}$. □

Theorem 4.2.3. *The imposing mechanism for 3-facility on the line is $(n-2)$-approximate.*

Proof. Assume without loss of generality that $x_A \geq x_B$. Let S_1 be the set of agents covered by the leftmost facility in the optimal solution, S_2 be the set of agents covered by the rightmost facility in the optimal solution and S_3 the set of agents covered by the third facility. Define by I_k the minimum line segment that agents in S_k belong to. We have that $|I_1| + |I_2| + |I_3| \leq OPT$. Also let l_1, l_2, l_3 be the facilities as placed by our mechanism. We define $cost_k = \sum_{i \in S_k} d(f(\vec{x}), x_i)$.

Since $l_1 = lt(\vec{x}) \in I_1$, we have that $cost_1 = \sum_{i \in S_1} d(f(\vec{x}), x_i) \leq \sum_{i \in I_1} d(l_1, x_i) \leq (|S_1| - 1)OPT$. Similarly, $cost_2 \leq (|S_2| - 1)OPT$ since $l_2 = rt(\vec{x}) \in I_2$.

We now prove that $cost_3 \leq |S_3|OPT$.

If $l_3 \in I_3$ then the distance from each agent to his closest facility is at most $|I_3|$, so $cost_3 \leq |S_3|OPT$.

If $l_3 \notin I_3$, let $x_l = lt(\vec{x}) + d_A$ and $x_r = rt(\vec{x}) - d_B$ the locations of the rightmost agent on A and the leftmost agent on B respectively.

If $x_l \in I_1$ or $x_1 \in I_2$ we have that either $|I_1| \geq d_A$ or $|I_2| \geq d_A$ but each agent in S_3 has cost at most d_A. This implies $cost_3 \leq |S_3|d_A \leq |S_3|(|I_1| + |I_2|) \leq |S_3|OPT$.

If $x_l \in I_3$ then $x_r \in I_2$ otherwise if $x_r \in I_3$ then since $l_3 \in [x_l, x_r]$ we get that $l_3 \in I_3$ a contradiction. We deduce that $l_3 > x_l$ so $d_A \leq 2d_B$. For every agent, we have that his distance to the closest facility is less than d_B. So $cost_3 \leq |S_3|d_B \leq |S_3||I_2| \leq |S_3|OPT$.

Summing the costs $cost_1 + cost_2 + cost_3$ together we have, $(|S_1| - 1)OPT + (|S_2| - 1)OPT + |S_3|OPT \leq (n-2)OPT$. $\square$

We see how imposing mechanisms build on and improve the non-imposing ones. Even in this case however no deterministic strategyproof mechanism is known for more than 3-facilities. This is not true for the case of randomized mechanisms where we can obtain very efficient mechanisms for any number of facilities and any metric space.

4.3 Randomized mechanisms

We consider the winner-imposing version of the Proportional Mechanism for the k-Facility Location game. Given a location profile $\vec{x} = (x_i)_{i \in N}$, the Winner-Imposing Proportional Mechanism, or WIProp in short, works in k rounds, fixing the location of one facility in each round. For each $\ell = 1, \ldots, k$, let C_ℓ be the set of the first ℓ facilities of WIProp. Initially, $C_0 = \emptyset$. WIProp proceeds as follows:

1st Round: WIProp selects i_1 uniformly at random from N, places the first facility at x_{i_1}, connects agent i_1 to it, and lets $C_1 = \{x_{i_1}\}$.

ℓ-th Round, $\ell = 2, \ldots, k$: WIProp selects $i_\ell \in N$ with probability $\frac{d(x_{i_\ell}, C_{\ell-1})}{\sum_{i \in N} d(x_i, C_{\ell-1})}$, places the ℓ-th facility at x_{i_ℓ}, connects agent i_ℓ to it, and lets $C_\ell = C_{\ell-1} \cup \{x_{i_\ell}\}$.

The output of the mechanism is C_k, and every agent not allocated a facility is connected to the facility in C_k closest to her true location.

Theorem 4.3.1. WIProp *is a strategyproof* $4k$*-approximation mechanism for the* k*-Facility Location game on any metric space.*

Strategyproofness. Even though the non-imposing version of the Proportional Mechanism is not strategyproof for $k \geq 3$ as we've seen before, WIProp is strategyproof for any k.

Lemma 4.3.2. *For any* $k \geq 1$*,* WIProp *is a strategyproof mechanism for the* k*-Facility Location game.*

Proof. For each $\ell = 0, 1, \ldots, k$, we let $\text{cost}[x_i, f(y, \vec{x}_{-i})|C_\ell]$ be the expected connection cost of an agent i at the end of WIProp, given that i reports location y and that the facility set of WIProp at the end of round ℓ is C_ℓ. For $\ell = k$, $\text{cost}[x_i, f(y, \vec{x}_{-i})|C_k] = d(x_i, C_k)$. For each $\ell = 1, \ldots, k-1$, with probability proportional to $d(y, C_\ell)$ the next facility of WIProp is placed at i's reported location, in which case i is connected to y and incurs a connection cost of $d(x_i, y)$, while for each agent $j \neq i$, with probability proportional to $d(x_j, C_\ell)$ the next facility of WIProp is placed at x_j, in which case the expected connection cost of i is $\text{cost}[x_i, f(y, \vec{x}_{-i})|C_\ell \cup \{x_j\}]$. Therefore:

$$\begin{aligned} \text{cost}[\ x_i, & \ \ f(y, \vec{x}_{-i})|C_\ell] = \\ & = \frac{d(x_i, y)\, d(y, C_\ell) + \sum_{j \neq i} d(x_j, C_\ell)\, \text{cost}[x_i, f(y, \vec{x}_{-i})|C_\ell \cup \{x_j\}]}{d(y, C_\ell) + \sum_{j \neq i} d(x_j, C_\ell)} \end{aligned} \tag{4.1}$$

Similarly, for $\ell = 0$, the expected connection cost of agent i is:

$$\text{cost}[x_i, f(y, \vec{x}_{-i})] = \frac{d(x_i, y) + \sum_{j \neq i} \text{cost}[x_i, f(y, \vec{x}_{-i})|\{x_j\}]}{n} \tag{4.2}$$

By induction on ℓ, we show that for any y, any $\ell = 0, 1, \ldots, k$, and any C_ℓ,

$$\text{cost}[x_i, f(y, \vec{x}_{-i})|C_\ell] \geq \text{cost}[x_i, f(\vec{x})|C_\ell] \tag{4.3}$$

Thus agent i has no incentive to misreport her location, which implies the lemma.

For the basis, we observe that (4.3) holds for $\ell = k$. Indeed, if i's location is not in C_k, her connection cost is $d(x_i, C_k)$ and does not depend on her reported location y, while if i's location is in C_k her connection cost is $d(x_i, y) \geq d(x_i, x_i)$. We inductively assume that (4.3) holds for $\ell+1$ and any facility set $C_{\ell+1}$, and show that (4.3) holds for ℓ and any facility set C_ℓ. If $\ell \geq 1$, we use (4.1) and obtain that:

$$\begin{aligned}\text{cost}[\ x_i,\ \ & f(y, \vec{x}_{-i})|C_\ell] \geq \\ \geq\ \ & \frac{d(x_i, y)\, d(y, C_\ell) + \sum_{j \neq i} d(x_j, C_\ell)\, \text{cost}[x_i, f(\vec{x})|C_\ell \cup \{x_j\}]}{d(y, C_\ell) + \sum_{j \neq i} d(x_j, C_\ell)} \\ =\ \ & \frac{d(x_i, y)\, d(y, C_\ell) + \Big(d(x_i, C_\ell) + \sum_{j \neq i} d(x_j, C_\ell)\Big)\, \text{cost}[x_i, f(\vec{x})|C_\ell]}{d(y, C_\ell) + \sum_{j \neq i} d(x_j, C_\ell)} \end{aligned} \tag{4.4}$$

The inequality follows from (4.1) and the induction hypothesis. For the equality, we apply (4.1) with $y = x_i$. If $d(x_i, C_\ell) \geq d(y, C_\ell)$, (4.4) implies that $\text{cost}[x_i, f(y, \vec{x}_{-i})|C_\ell] \geq \text{cost}[x_i, f(\vec{x})|C_\ell]$. Otherwise, we continue from (4.4) and obtain that:

$$\begin{aligned}\text{cost}[x_i, f(y, \vec{x}_{-i})|C_\ell] &> \frac{d(x_i, y) + d(x_i, C_\ell) + \sum_{j \neq i} d(x_j, C_\ell)}{d(y, C_\ell) + \sum_{j \neq i} d(x_j, C_\ell)}\, \text{cost}[x_i, f(\vec{x})|C_\ell] \\ \text{cost}[x_i, f(\vec{x})|C_\ell] &\geq \text{cost}[x_i, f(\vec{x})|C_\ell]\end{aligned}$$

The first inequality follows from (4.4) using that $d(y, C_\ell) > \text{cost}[x_i, f(\vec{x})|C_\ell]$. For the second inequality, we use that $d(x_i, y) + d(x_i, C_\ell) \geq d(y, C_\ell)$.

If $\ell = 0$, using (4.2) and the induction hypothesis, we obtain that:

$$\text{cost}[x_i, f(y, \vec{x}_{-i})] \geq \frac{1}{n} \sum_{j \neq i} \text{cost}[x_i, f(\vec{x})|\{x_j\}] = \text{cost}[x_i, f(\vec{x})] \tag{4.5}$$

Thus we have established (4.3) for any location y, any $\ell = 0, 1, \ldots, k$, and any C_ℓ.

□

We note that for $k = 2$, the requirement that the mechanism is winner-imposing is not needed. That is why the non-imposing proportional mechanism works for the case of two facilities.

Lemma 4.3.3. *For any $k \geq 1$,* WIProp *achieves an approximation ratio of at most $4k$ for the k-Facility Location game.*

We start by introducing the notation used throughout the proof. We fix a location profile $\vec{x} = (x_i)_{i \in N}$, and compare the cost of WIProp$(\vec{x})$ against the cost of a set $C^* = \{c_1^*, \ldots, c_k^*\}$ of optimal facility locations for $\vec{x}$. C^* partitions N into k clusters, where the p-th optimal cluster, denoted N_p, consists of the agents whose nearest facility in C^* is c_p^*. For each agent i, we let $d_i^* = d(x_i, C^*)$ be i's distance to the nearest facility in C^*. We let $\mathrm{OPT}_p = \sum_{i \in N_p} d_i^*$ denote the optimal cost for agents in N_p, and let $\mathrm{OPT} = \sum_{p \in [k]} \mathrm{OPT}_p$ denote the total cost of the optimal solution.

For a set of agents $N' \subseteq N$ and a non-empty facility set C, we let $D(N', C) = \sum_{i \in N'} d(x_i, C)$ denote the total cost of connecting each agent in N' to the nearest facility in C. For a set of facilities C placed (by WIProp) at the locations of some agents, we let $H(C) = \{p \in [k] : C \cap N_p \neq \emptyset\}$ be the set of indices of the optimal clusters covered by C, and let $U(C) = [k] \setminus H(C)$ be the set of indices of the optimal clusters not covered by C. For a set of indices $I \subseteq [k]$, we let $N(I) = \cup_{p \in I} N_p$ be the set of agents in the optimal clusters indexed by I. For each round ℓ, $1 \leq \ell \leq k$, we let c_ℓ be the facility placed by WIProp at round ℓ, let C_ℓ be the facility set of WIProp at the end of round ℓ, and C_k be the final facility set of WIProp, and let $H_\ell = H(C_\ell)$ and $U_\ell = U(C_\ell)$ be the sets of indices of the optimal clusters covered and not covered, respectively, by WIProp at the end of round ℓ.

To establish the approximation ratio of of WIProp, we observe that

$$\mathbb{E}[D(N, C_k)] = \mathbb{E}[D(N(U_k), C_k)] + \sum_{p \in H_k} \mathbb{E}[D(N_p, C_k) | p \in H_k] \tag{4.6}$$

and analyze the expected cost of covered and uncovered optimal clusters separately. The following lemma, proven in [20, Lemma 4.4], establishes an upper bound on the expected connection cost for the optimal clusters covered by the facilities of WIProp. We include the proof for completeness.

Lemma 4.3.4. *For any optimal cluster N_p, $\mathbb{E}[D(N_p, C_k) | p \in H_k] \leq 4\,\mathrm{OPT}_p$.*

Proof. Let ℓ, $1 \leq \ell \leq k$, be the first round such that $p \in H_\ell$, namely, the round at which the optimal cluster N_p is covered by WIProp for the first time, and let $c_\ell \in N_p$ be the corresponding facility of WIProp. To simplify the notation, for each agent $i \in N_p$, we let $d_i = d(x_i, C_{\ell-1})$ denote the distance of agent i to the nearest facility just before c_ℓ opens. Next, we ignore any subsequent facilities, and upper bound $\mathbb{E}[D(N_p, C_\ell)|c_\ell \in N_p]$. If $\ell = 1$, i.e. N_p is covered at the first round, then:

$$\begin{aligned}
\mathbb{E}[D(N_p, C_1)|c_1 \in N_p] &= \frac{1}{|N_p|}\sum_{i\in N_p}\sum_{j\in N_p} d(x_i, x_j) \\
&\leq \frac{1}{|N_p|}\sum_{i\in N_p}\sum_{j\in N_p} (d_i^* + d_j^*) \\
&\leq 2\,\mathrm{OPT}_p
\end{aligned}$$

If $\ell > 1$, a more careful analysis is required:

$$\begin{aligned}
\mathbb{E}[D(N_p, C_\ell)|c_\ell \in N_p] &= \sum_{i\in N_p} \frac{d_i}{D(N_p, C_{\ell-1})} \sum_{j\in N_p} \min\{d_j, d(x_i, x_j)\} \\
&\leq \sum_{i\in N_p} \frac{d_i}{D(N_p, C_{\ell-1})} \sum_{j\in N_p} \min\{d_j, d_i^* + d_j^*\} \qquad (4.7)
\end{aligned}$$

If $D(N_p, C_{\ell-1}) = \sum_{j\in N_p} d_j \leq \mathrm{OPT}_p$, we use that $\min\{d_j, d_i^* + d_j^*\} \leq d_j$, and obtain that:

$$\mathbb{E}[D(N_p, C_\ell)|c_\ell \in N_p] \leq \sum_{i\in N_p} \frac{d_i}{D(N_p, C_{\ell-1})} \sum_{j\in N_p} d_j = D(N_p, C_{\ell-1}) \leq \mathrm{OPT}_p$$

Otherwise, for each agent $j \in N_p$, we let $s_j = d_j - d_j^*$. Since $D(N_p, C_{\ell-1}) > \mathrm{OPT}_p$, we have that $\sum_{j\in N_p} s_j > 0$. Substituting in (4.7), we obtain that:

$$\begin{aligned}
\mathbb{E}[D(N_p, C_\ell)|c_\ell \in N_p] &\leq \sum_{i\in N_p} \frac{d_i}{D(N_p, C_{\ell-1})} \sum_{j\in N_p} \min\{d_j^* + s_j, d_i^* + d_j^*\} \\
&\leq \sum_{i\in N_p} \frac{d_i}{D(N_p, C_{\ell-1})} \sum_{j\in N_p} d_j^* \\
&\quad + \sum_{i\in N_p} \frac{d_i^*}{D(N_p, C_{\ell-1})} \sum_{j\in N_p} \min\{s_j, d_i^*\} \\
&\quad + \sum_{i\in N_p} \frac{s_i}{D(N_p, C_{\ell-1})} \sum_{j\in N_p} \min\{s_j, d_i^*\}
\end{aligned}$$

We note that the first sum on the rhs of the inequality above is OPT_p. Using that $\min\{s_j, d_i^*\} \leq s_j$ and that $0 < \sum_{j\in N_p} s_j \leq D(N_p, C_{\ell-1})$, we obtain that the second sum is at most OPT_p. For the third sum, we use $\min\{s_j, d_i^*\} \leq d_i^*$, and obtain that:

$$\sum_{i\in N_p} \frac{s_i}{D(N_p, C_{\ell-1})} \sum_{j\in N_p} d_i^* = \sum_{i\in N_p} \frac{s_i\, d_i^*\, |N_p|}{D(N_p, C_{\ell-1})} \leq \sum_{i\in N_p} d_i^* \frac{d(c_p^*, C_{\ell-1})\, |N_p|}{D(N_p, C_{\ell-1})} \leq 2\,\mathrm{OPT}_p\,,$$

where the penultimate inequality follows from $s_i = d(x_i, C_{\ell-1}) - d(x_i, c_p^*) \le d(c_p^*, C_{\ell-1})$, and the ultimate inequality from:

$$d(c_p^*, C_{\ell-1})\,|N_p| \le \sum_{j \in N_p} (d_j^* + d_j) = \mathrm{OPT}_p + D(N_p, C_{\ell-1}) \le 2D(N_p, C_{\ell-1})$$

Thus, in all cases, $\mathbb{E}[D(N_p, C_\ell) | c_\ell \in N_p] \le 4\,\mathrm{OPT}_p$. □

We proceed to bound the connection cost for the optimal clusters not covered by the facilities of WIProp. Given C_ℓ, $\ell \ge 1$, the expected connection cost (at the end of WIProp) for the agents in the optimal clusters not covered by C_ℓ is:

$$\begin{aligned}
\mathbb{E}[D(N(U_\ell), C_k) | C_\ell] \;\le\; & \sum_{p \in H_\ell} \frac{D(N_p, C_\ell)}{D(N, C_\ell)}\, D(N(U_\ell), C_\ell) \\
& + \sum_{p \in U_\ell} \sum_{i \in N_p} \frac{d(x_i, C_\ell)}{D(N, C_\ell)} \sum_{j \in N_p} \min\{d(x_i, x_j), d(x_j, C_\ell)\} \\
& + \sum_{p \in U_\ell} \sum_{i \in N_p} \frac{d(x_i, C_\ell)}{D(N, C_\ell)}\, \mathbb{E}[D(N(U_\ell \setminus \{p\}), C_k) | C_\ell \cup \{x_i\}]
\end{aligned}$$

For the first sum, we use that the connection cost of the agents in $N(U_\ell)$ can only decrease as new facilities open. For each $p \in H_\ell$, the corresponding term in the first sum is at most $D(N_p, C_\ell)$, because $D(N, C_\ell) \ge D(N(U_\ell), C_\ell)$. In the proof of Lemma 4.3.4, we show that for each $p \in U_\ell$, the corresponding sum in the second summation is at most $4\,\mathrm{OPT}_p$. As for the sum in the third summation, we observe that for each $p \in U_\ell$,

$$\sum_{i \in N_p} \frac{d(x_i, C_\ell)}{D(N_p, C_\ell)}\, \mathbb{E}[D(N(U_\ell \setminus \{p\}), C_k) | C_\ell \cup \{x_i\}] = \mathbb{E}[D(N(U_\ell \setminus \{p\}), C_k) | C_\ell \wedge c_{\ell+1} \in N_p]$$

Therefore,

$$\begin{aligned}
\sum_{p \in U_\ell} \sum_{i \in N_p} \frac{d(x_i, C_\ell)}{D(N, C_\ell)}\, \mathbb{E}[D(N(U_\ell \;&\setminus\; \{p\}), C_k) | C_\ell \cup \{x_i\}] \\
&= \sum_{p \in U_\ell} \frac{D(N_p, C_\ell)}{D(N, C_\ell)} \\
&\quad\; \mathbb{E}[D(N(U_\ell \setminus \{p\}), C_k) | C_\ell \wedge c_{\ell+1} \in N_p] \\
&\le \max_{p \in U_\ell} \{\mathbb{E}[D(N(U_\ell \setminus \{p\}), C_k) | C_\ell \wedge c_{\ell+1} \in N_p]\}
\end{aligned}$$

Putting everything together, we have that:

$$\begin{aligned}\mathbb{E}[D(N(U_\ell), C_k)|C_\ell] \;\leq\; & \sum_{p \in H_\ell} D(N_p, C_\ell) + \sum_{p \in U_\ell} 4\,\mathrm{OPT}_p \\ & + \max_{p \in U_\ell}\{\mathbb{E}[D(N(U_\ell \setminus \{p\}), C_k)|C_\ell \wedge c_{\ell+1} \in N_p]\}\end{aligned}$$

We now fix an arbitrary set $U \subseteq [k]$, $|U| \leq k-1$, of the (indices of the) optimal clusters not covered by WIProp, and some round ℓ, $k - |U| \leq \ell \leq k-1$, and condition on all possible facility sets C_ℓ (namely, all possible outcomes of WIProp by round ℓ) with $U_\ell = U$. Applying the inequality above, we get that:

$$\begin{aligned}\mathbb{E}[D(N(U), C_k)|U_\ell = U] \;\leq\; & \sum_{p \notin U} \mathbb{E}[D(N_p, C_\ell)|p \in H_\ell] + \sum_{p \in U} 4\,\mathrm{OPT}_p \\ & + \max_{p \in U}\{\mathbb{E}[D(N(U \setminus \{p\}), C_k)|U_{\ell+1} = U \setminus \{p\}]\}\end{aligned}$$

By Lemma 4.3.4, for each $p \notin U$, $\mathbb{E}[D(N_p, C_\ell)|p \in H_\ell] \leq 4\,\mathrm{OPT}_p$. Therefore, for every set $U \subseteq [k]$, $|U| \leq k-1$, of optimal clusters not covered by WIProp, and every round ℓ, $k - |U| \leq \ell \leq k-1$,

$$\mathbb{E}[D(N(U), C_k)|U_\ell = U] \leq 4\,\mathrm{OPT} + \max_{p \in U}\{\mathbb{E}[D(N(U \setminus \{p\}), C_k)|U_{\ell+1} = U \setminus \{p\}]\} \tag{4.8}$$

To conclude the proof of Lemma 4.3.3, we show that

$$\mathbb{E}[D(N(U_k), C_k)] \leq 4(k-1)\mathrm{OPT} \tag{4.9}$$

Clearly, combining (4.6) with Lemma 4.3.4 and (4.9) implies an approximation ratio of at most $4k$.

To prove (4.9), we show that for every set $U \subseteq [k]$, $|U| \leq k-1$, of optimal clusters not covered by WIProp,

$$\mathbb{E}[D(N(U), C_k)|U_{k-|U|} = U] \leq 4\,|U|\,\mathrm{OPT} \tag{4.10}$$

This implies (4.9), since WIProp covers at least one optimal cluster, and $|U_k| \leq k-1$ for all possible outcomes C_k of WIProp.

The proof of (4.10) is by induction on the cardinality of U. (4.10) holds trivially if $U = \emptyset$. We inductively assume that (4.10) holds for all sets $U' \subseteq [k]$, $|U'| < k-1$, of optimal clusters not covered by WIProp, and show that (4.10) holds for an arbitrary set $U \subseteq [k]$, $|U| = |U'| + 1$, of optimal clusters not covered by WIProp. Indeed, using (4.8) and induction hypothesis, we obtain that:

$$\begin{aligned}
\mathbb{E}[D(N(U), C_k) | U_{k-|U|} = U] &\leq 4\,\mathrm{OPT} + \max_{p \in U}\{\mathbb{E}[D(N(U \setminus \{p\}), C_k)| \\
&\qquad |U_{k-(|U|-1)} = U \setminus \{p\}]\} \\
&\leq 4\,\mathrm{OPT} + \max_{p \in U}\{4(|U| - 1)\mathrm{OPT}\} \\
&= 4|U|\,\mathrm{OPT}
\end{aligned}$$

With a more careful analysis for the case $k = 2$ we can obtain approximation ratio of 4 instead of $4k = 8$.

Chapter 5

Facility Location with Uniform Cost

5.1 Setting

Until now we consider the problem of k-facility location where the number of facilities was fixed. In this section, we consider the case where a variable number of facilities will be placed that depends on the specific instance. We assume that there is a uniform cost for opening a facility. As before, an agent's cost depends only on his distance to the nearest facility.

However, the cost that we need to minimize is the social cost plus the cost to build the facilities which is a constant price multiplied by the number of facilities opened. By appropriately scaling the distances we can assume that the cost of opening a facility is always one. So our objective function becomes:

$$sc(\vec{x}) = \sum_{i \in N} cost(x_i, f^i(\vec{x})) + |f(\vec{x})|$$

As before, a mechanism can be either imposing or non-imposing and we compute the approximation ratio in comparison to the allocation that minimizes the social cost and disregards strategyproofness.

5.2 Deterministic Mechanism for Facility Location on the Line

We begin by presenting a deterministic non-imposing group strategyproof $O(\log n)$-approximate mechanism for the Facility Location game on the real line. As stated earlier, we assume that the cost of opening a new facility is 1. To simplify the presentation, we assume that the agent locations are located in $\mathbb{R}_+ = [0, \infty)$. Our analysis can be easily generalized to the case where the agents are located in $\mathbb{R}$.

The *Line Partitioning* mechanism, or LPart in short, is motivated by the online algorithm for Facility Location on the plane by Anagnostopoulos *et al.* [12]. LPart assumes a hierarchical partitioning of $[0, \infty)$ into intervals with at most $1 + \log_2 n$ levels. The partitioning at level 0 consists of intervals of length 1. Namely, for $p = 0, 1, \ldots$, the p-th level-0 interval is $[p, p+1)$. Each level-ℓ interval $[p\,2^{-\ell}, (p+1)2^{-\ell})$, $\ell = 0, 1, \ldots, \lfloor \log_2 n \rfloor - 1$, is partitioned into two disjoint level-$(\ell+1)$ intervals of length $2^{-(\ell+1)}$, namely $[p\,2^{-\ell}, p\,2^{-\ell} + 2^{-(\ell+1)})$ and $[p\,2^{-\ell} + 2^{-(\ell+1)}, (p+1)\,2^{-\ell})$. A level-0 interval is *active* if it includes the (reported) location of at least one agent, and a level-ℓ interval, $\ell \geq 1$, is active if it includes the (reported) locations of at least $2^{\ell+1}$ agents, and *inactive* otherwise. Intuitively, an interval is active if it includes so many agents that the optimal solution must open a facility nearby.

LPart opens three facilities, two at the endpoints and one at the midpoint, of each level-0 active interval, and one facility at the midpoint of each level-ℓ active interval, for each $\ell \geq 1$. In particular, for each level-0 active interval $[p, p+1)$, LPart opens three facilities at p, at $p + \frac{1}{2}$, and at $p + 1$. For each $\ell \geq 1$ and each level-ℓ active interval $[p\,2^{-\ell}, (p+1)2^{-\ell})$, LPart opens a facility at $p\,2^{-\ell} + 2^{-(\ell+1)}$. LPart is non-imposing, so each agent is connected to the open facility closest to her true location.

In the following, we prove that:

Theorem 5.2.1. LPart *is a group strategyproof $O(\log n)$-approximate mechanism for the Facility Location game on the real line.*

Main Properties. We start the proof of Theorem 5.2.1 with some simple observations regarding the structure of the solution produced by LPart. We observe that if an interval q is active, all the intervals in which q is included are active, and that if an interval q is inactive, all the intervals included in q are inactive. In addition, all level-$\lfloor \log_2 n \rfloor$ intervals are inactive, since each of them contains at most $n < 2^{\lfloor \log_2 n \rfloor + 1}$ agents. So each agent is included in at least one active and at least one inactive interval. In the following, each agent i is associated with the maximal (i.e., that of the smallest level) inactive interval, denoted q_i, that includes her true location. The maximal inactive intervals q_i, q_j of two agents i, j either coincide with each other or are disjoint.

A simple induction shows that each active interval q has three open facilities, two at its endpoints and one at its midpoint. Moreover, if an active level-ℓ interval contains an inactive level-$(\ell + 1)$ subinterval q', q' has two open facilities at its endpoints. Therefore, the connection cost of each agent i is equal to the distance of her true location to the nearest endpoint of her maximal inactive interval q_i. Furthermore, i's connection cost is at least as large as the distance of her true location to the nearest endpoint of any inactive interval including her true location.

Group Strategyproofness. The above properties of LPart immediately imply the following:

Lemma 5.2.2. LPart *is group strategyproof.*

Proof. Let $S \subseteq N$, $S \neq \emptyset$, be any coalition of agents who misreport their locations so as to improve their connection cost, and let $\vec{x}_S = (x_i)_{i \in S}$ and $\vec{y}_S = (y_i)_{i \in S}$ be the profiles with their true and their misreported locations respectively. If for some agent i, i's maximal inactive interval q_i contains the same number of agents in LPart$(\vec{x}_S, \vec{x}_{-S})$ and in LPart$(\vec{y}_S, \vec{x}_{-S})$, q_i is inactive in LPart$(\vec{y}_S, \vec{x}_{-S})$ as well, and i's connection cost does not improve. On the other hand, if q_i contains more agents in LPart$(\vec{y}_S, \vec{x}_{-S})$ than in LPart$(\vec{x}_S, \vec{x}_{-S})$, there are some agents in S whose maximal inactive interval is disjoint to q_i in LPart$(\vec{x}_S, \vec{x}_{-S})$, and is included in q_i in LPart$(\vec{y}_S, \vec{x}_{-S})$. Therefore, there is some agent $j \in S$ whose maximal inactive interval q_j contains less agents in LPart$(\vec{y}_S, \vec{x}_{-S})$ than in LPart$(\vec{x}_S, \vec{x}_{-S})$. Thus q_j is inactive in LPart$(\vec{y}_S, \vec{x}_{-S})$ as well, and j's connection cost does not improve due to the agents in S deviating from $\vec{x}_S$ to $\vec{y}_S$. □

Approximation Ratio. The analysis of the approximation ratio proceeds along the lines of [12, Theorem 1]. We first prove that the optimal solution must have a facility close to each active interval.

Proposition 5.2.3. *Let $q = [p\,2^{-\ell}, (p+1)2^{-\ell})$ be an active level-ℓ interval, for some $\ell \geq 0$. Then, the optimal solution has a facility in $[(p-1)2^{-\ell}, (p+2)2^{-\ell})$. i.e. either in q, or in the next level-ℓ interval on the left, or in the next level-ℓ interval on the right.*

Proof. Let $q_l = [(p-1)2^{-\ell}, p\,2^{-\ell})$ be the interval next to q on the left, let $q_r = [(p+1)2^{-\ell}, (p+2)2^{-\ell})$ be the interval next to q on the right, and let n_q be the number of agents in q. For sake of contradiction, we assume that the optimal solution does not have a facility in $q_l \cup q \cup q_r$. Then the connection cost of the agents in q is greater than $n_q 2^{-\ell}$. If $\ell = 0$, placing an optimal facility at the location of some agent in q costs 1 and decreases the connection cost of the agents in q to at most $n_q - 1$. If $\ell \geq 1$, placing an optimal facility at the midpoint of q decreases the connection cost of the agents in q to at most $n_q 2^{-(\ell+1)}$. Since q is active and $n_q \geq 2^{\ell+1}$ ($n_q \geq 1$ for $\ell = 0$), the total cost in the later case is less than the connection cost of the agents in q to a facility outside $q_l \cup q \cup q_r$, which contradicts the hypothesis that the optimal solution does not have a facility in $q_l \cup q \cup q_r$. □

Lemma 5.2.4. LPart *has an approximation ratio of $O(\log n)$.*

Proof. Let k be the number of facilities in the optimal solution. By Proposition 5.2.3, there are at most 3 active intervals per optimal facility at each level $\ell = 0, 1, \ldots, \lfloor \log n \rfloor - 1$. The total facility cost for the three (neighboring) active level-0 intervals is 7, and the facility cost for each active level-ℓ interval, $\ell \geq 1$, is 1. Therefore, the number of active intervals is at most $3k \log_2 n$, and the total facility cost of LPart is at most $4k + 3k \log_2 n$.

To bound the connection cost of LPart, we consider the set of maximal inactive intervals that include the location at least one agent (i.e., they are non-empty). This accounts for the connection cost of all agents, since each agent i is associated with her maximal inactive interval q_i. Each maximal inactive interval q at level ℓ, $\ell \geq 1$, contains less than $2^{\ell+1}$ agents and has two facilities at its endpoints. Thus the total connection cost for the agents in q is at most $2^{\ell+1}2^{-\ell}/2 = 1$. Furthermore, q is included in some active level-$(\ell-1)$ interval. Thus, the total number of non-empty maximal inactive intervals, and thus the total connection cost of LPart, is at most $6k \log_2 n$. Overall, the total cost of LPart is at most $4k + 9k \log_2 n$, i.e. $O(\log_2 n)$ times the optimal cost. □

Theorem 5.2.1 follows immediately from Lemma 5.2.2 and Lemma 5.2.4.

5.3 Randomized Winner-Imposing Mechanism for Facility Location

Next we consider the winner-imposing version of Meyerson's randomized online algorithm for Facility Location [10], and show that it is strategyproof. Combining this with [10, Theorem 2.1], we obtain a randomized winner-imposing strategyproof 8-approximate mechanism for the Facility Location game.

Meyerson's algorithm, or OFL in short, process the agents one-by-one in a random order, and places a facility at the location of the each agent with probability equal to her distance to the nearest facility available divided by the facility opening cost (which we assume to be 1). In the following, we assume for simplicity that the agents are indexed according to the random permutation chosen by OFL. Also we let C_i denote the facility set of OFL just after agent i is processed.

Formally, given the locations $\vec{x} = (x_i)_{i \in N}$ of a randomly permuted set of agents, the (winner-imposing) OFL mechanism first places a facility at x_1, connects agent 1 to it, and lets $C_1 = \{x_1\}$. Then, for each $i = 2, \ldots, n$, with probability $d(x_i, C_{i-1})$, OFL opens a facility at x_i, connects agent i to it, and lets $C_i = C_{i-1} \cup \{x_i\}$. Otherwise, OFL lets $C_i = C_{i-1}$. The output of the mechanism is C_n, and every agent not allocated a facility is connected to the facility in C_n closest to her true location.

Lemma 5.3.1. *The winner-imposing version of* OFL *is strategyproof for the Facility Location game.*

Proof. We fix an arbitrary permutation of N, and assume that the agents are indexed according to it. Let i be any agent, let x_i be i's true location, and let C_{i-1} be the set of facilities just before agent i is processed by OFL. If $i = 1$ or $d(x_i, C_{i-1}) \geq 1$, OFL places a facility at x_i with certainty, so i has no incentive to lie about her location. So we can restrict our attention to the case where $d(x_i, C_{i-1}) < 1$.

Let $\mathrm{cost}[x_i, f(y, x_{i+1}, \ldots, x_n)|C]$ be the expected connection cost of agent i at the end of OFL, given that i reports location y, and that just before i's location is processed, the set of facilities is C. Similarly, let $\mathrm{cost}[x_i, f(x_{i+1}, \ldots, x_n)|C]$ be the expected connection cost of agent i at the end of OFL, given that just after i's location is processed, the set of facilities is C. Next we show that for any agent i located at x_i, for any location y, and for any C_{i-1},

$$\mathrm{cost}[x_i, f(x_i, x_{i+1}, \ldots, x_n)|C_{i-1}] \leq \mathrm{cost}[x_i, f(y, x_{i+1}, \ldots, x_n)|C_{i-1}]\,, \tag{5.1}$$

which implies the lemma. For the lhs and the rhs of (5.1), we observe that:

$$\begin{aligned} \mathrm{cost}[x_i, f(x_i, x_{i+1}, \ldots, x_n)|C_{i-1}] &= (1 - d(x_i, C_{i-1}))\,\mathrm{cost}[x_i, f(x_{i+1}, \ldots, x_n)|C_{i-1}] \\ \mathrm{cost}[x_i, f(y, x_{i+1}, \ldots, x_n)|C_{i-1}] &= (1 - d(y, C_{i-1}))\,\mathrm{cost}[x_i, f(x_{i+1}, \ldots, x_n)|C_{i-1}] \\ &\quad + d(y, C_{i-1})\, d(x_i, y) \end{aligned}$$

Therefore, (5.1) holds iff

$$(d(y, C_{i-1}) - d(x_i, C_{i-1}))\,\mathrm{cost}[x_i, f(x_{i+1}, \ldots, x_n)|C_{i-1}] \leq d(x_i, y)\, d(y, C_{i-1}) \tag{5.2}$$

Clearly (5.1) holds if $d(y, C_{i-1}) \leq d(x_i, C_{i-1})$, since then the lhs of (5.2) becomes non-positive. On the other hand, if $d(y, C_{i-1}) > d(x_i, C_{i-1})$, (5.1) holds because $d(y, C_{i-1}) - d(x_i, C_{i-1}) \leq d(x_i, y)$ and $\mathrm{cost}[x_i, f(x_{i+1}, \ldots, x_n)|C_{i-1}] \leq d(x_i, C_{i-1}) < d(y, C_{i-1})$. $\square$

Combining Lemma 5.3.1 with [10, Theorem 2.1], we obtain that:

Theorem 5.3.2. OFL *is a strategyproof* 8*-approximation mechanism for the Facility Location game on any metric space.*

Remark. We should highlight that the proof of Lemma 5.3.1 does not apply to the non-imposing version of OFL. In particular, if when OFL opens a facility at the location of an agent i, i is not forced to be connected to it, but can connect to the facility in C_n closest to her true location, there are instances where for a particular order of agents, some of them can improve her expected connection cost by lying about her location. For example, consider a simple instance with n agents on the real line. The first agent is located at $-1/2$, the second agent is located at 0, the third agent is located at $1/2 - \varepsilon$, for some small $\varepsilon > 0$, and the remaining $n - 3$ agents are located at 0. For appropriately chosen n and ε, the third agent can improve her expected connection cost in the non-imposing version of OFL by reporting $1/2$ instead of 0. However, this can only happen for that particular order. Hence our example demonstrates that the argument in the proof of Lemma 5.3.1 fails to establish that the non-imposing version of OFL is strategyproof. On the other hand, no agent has an incentive to lie if the expectation of her connection cost is also taken over all random permutations of agents. Thus, our example does not exclude the possibility that the non-imposing version of OFL is strategyproof for the Facility Location game. ◻

Chapter 6

Conclusion

6.1 Remarks

Most of the results on Imposing Mechanisms and Facility Location with Uniform facility opening cost that are found in chapters 5 and 6 were published in the Proceedings of the 6th Workshop on Internet and Network Economics [23].

For the non-imposing case our results on the characterization of strategyproof efficient mechanisms for the case where 2 facilities are placed on the line along with the result on the unbounded approximation ratio for 3 or more facilities even on the line can be found at [22].

6.2 Future Directions

The results for deterministic mechanisms in the case where multiple facilities must be located are very restrictive and show that no efficient mechanisms are strategyproof even on simple cases. We expect that similar results hold for the case of deterministic imposing mechanisms, although no results available yet quantifying their inefficiency.

One possible direction is to investigate randomized mechanisms. The winner-imposing proportional mechanism that works for any value of k shows that there is some space for improvement in that direction. Also, it would be interesting to have several results on the non-imposing case where no mechanism with bounded approximation ratio is known yet for more than three facilities.

Another direction to consider is to alter the requirement of the mechanisms to be efficient in the sense of multiplicative approximation and investigate efficient mechanisms with additive approximation ratio. This essentially removes the requirement that the mechanism is unanimous and allows some space for improvement since the negative results found in chapter 3 don't apply. The imposing mechanisms in [21] use this notion of approximation to obtain several positive results.

Finally, this thesis only covers a certain part of mechanism design without money, the case of facility location games. Since the field of mechanism design without money is becoming larger and more problems are explored, it would be interesting to apply several of the results and ideas presented here to those problems.

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