Εθνικο Μετσοβιο Πολυτεχνειο
Σχολη Ηλεκτρολογων Μηχανικων Και Μηχανικων Υπολογιστων
Τομεας Τεχνολογιας Πληροφορικης και Υπολογιστων

Αναζητώντας παρεμποδίσεις k-απόγειων γραφημάτων για κλάσεις με φραγμένο βαθμό

ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

του

ΚΟΣΜΑ Ζ. ΠΑΛΗΟΥ

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Καθηγητής Ε.Κ.Π.Α.

Αθήνα, Ιούλιος 2018

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(Υπογραφή)	*(Υπογραφή)*	*(Υπογραφή)*
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Νικόλαος Παπασπύρου	Δημήτριος Θηλυκός	Αριστείδης Παγουρτζής
Αναπλ. Καθηγητής Ε.Μ.Π.	Καθηγητής Ε.Κ.Π.Α.	Αναπλ. Καθηγητής Ε.Μ.Π.

Αθήνα, Ιούλιος 2018

(Υπογραφή)

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ΚΟΣΜΑΣ Ζ. ΠΑΛΗΟΣ

Διπλωματούχος Ηλεκτρολόγος Μηχανικός και Μηχανικός Υπολογιστών Ε.Μ.Π.

Ευχαριστίες

Θα ήθελα να ευχαριστήσω όσους συνέβαλαν στην πραγματοποίηση αυτής της εργασίας. Τους καθηγητές μου, ιδιαίτερα τον κ. Δημήτριο Θηλυκό και τον κ. Νικόλα Παπασπύρου, για την βοήθεια και την στήριξή τους. Τους γονείς μου και τους φίλους μου για την αμέριστη συμπαράστασή τους. Και φυσικά, την Ιωάννα, χωρίς την οποία η εργασία αυτή δεν θα είχε πραγματοποιηθεί.

Εκτεταμένη ελληνική περίληψη

Θα αντιμετωπίσουμε προβλήματα που αφορούν την αναζήτηση παρεμποδίσεων κλάσεων γραφημάτων. Οι παρεμποδίσεις μιας κλάσης ορίζονται ως τα ελαχιστοτικά, ως προς τη σχέση του ελάσσονος, γραφήματα που δεν ανήκουν σε μια κλάση γραφημάτων κλειστή ως προς ελάσσονα. Η συνεισφορά της εργασίας μας μπορεί να χωριστεί σε 2 μέρη. Ένα περισσότερο θεωρητικό, που περιλαμβάνει την απόδειξη ενός τετραγωνικού (ως προς k και d) φράγματος για το μέγεθος των παρεμποδίσεων των k-απόγειων της κλάσης των γραφημάτων με μέγιστο βαθμό μικρότερο από d, αλλά και ενός φράγματος για την ειδική περίπτωση $d = 2$. Ακόμα, υπάρχει ένα μέρος περισσότερο πρακτικό, που συνίσταται στην εύρεση και παρουσίαση των παρεμποδίσεων 4 κλάσεων, εκ των οποίων οι 3 είναι k-απόγεια των γραφημάτων με μέγιστο βαθμό μικρότερο από d, για τα 3 ζεύγη τιμών των (k, d), (1,2),(2,2) και (1,3). Η τέταρτη κλάση είναι μια υποκλάση της τρίτης, που εισάγει τον περιορισμό της ακυκλικότητας.

Θεωρητικό Υπόβαθρο

Αρχικά, ορίζουμε την πράξη της σύνθλιψης ακμής ενός γραφήματος.

Ορισμός 1 (Σύνθλιψης ακμής)**.** *Έστω γράφημα G. Ορίζουμε τη σύνθλιψη μιας ακμής e του G, ως την πράξη που μας δίνει ένα καινούριο γράφημα G', στον οποίο οι κορυφές x, y (μαζί με τις ακμές που συνδέονται σε αυτές) αντικαθίστανται με μια νέα κορυφή z, η οποία ενώνεται με κάθε κορυφή στην γειτονιά των x και y.*

Έτσι μπορεί να οριστεί η σχέση του ελάσσονος στα γραφήματα:

Ορισμός 2 (Έλασσον γράφημα). *Ένα γράφημα H λέγεται ότι είναι έλασσον ενός γραφήματος G όταν και μόνο όταν το H μπορεί να παραχθεί από το G, μέσα από μηδέν ή περισσότερες από τις ακόλουθες πράξεις: διαγραφή ακμής, διαγραφή κορυφής και σύνθλιψη ακμής. Συμβολίζουμε $H \leq_m G$.*

Μπορούμε τώρα να παρουσιάσουμε το πρώτο από τα δύο θεμελιώδη θεωρήματα αυτής της ενότητας.

Θεώρημα 1 (Θεώρημα του Wagner). *Ένα γράφημα δεν είναι επίπεδο όταν, και μόνο όταν, περιέχει ως ελάσσονα ένα K_5 ή ένα $K_{3,3}$.*

Το θεώρημα αυτό διατυπώθηκε από τον Wagner, αλλά είχε διατυπωθεί και νωρίτερα από τους Kuratowski και Pontryagin (ανεξάρτητα μεταξύ τους), χωρίς την έννοια του ελάσσονος. Ήταν η πρώτη φορά που χαρακτηρίστηκε μια κλάση γραφημάτων με βάση τα «απαγορευμένα» της γραφήματα. Σήμερα, χρησιμοποιείται συχνότερα ο όρος «παρεμπόδιση», και «σύνολο παρεμπόδισης»:

Ορισμός 3 (Σύνολο παρεμπόδισης). *Έστω $\mathcal{C}$ μια κλάση γραφημάτων κλειστή ως προς ελάσσονα. Τότε, ορίζουμε το* σύνολο παρεμπόδισης *της $\mathcal{C}$, $\mathbf{obs}(\mathcal{C})$, ως το σύνολο των ελαχιστοτικών (ως προς τη σχέση του ελάσσονος) γραφημάτων που δεν ανήκουν στη $\mathcal{C}$. Τα μέλη του συνόλου παρεμπόδισης λέγονται* παρεμποδίσεις *της $\mathcal{C}$.*

Δηλαδή, το προηγούμενο θεώρημα μπορεί ισοδύναμα να διατυπωθεί και ως εξής: «Το σύνολο παρεμπόδισης της κλάσης των επίπεδων γραφημάτων είναι το $\{K_5, K_{3,3}\}$». Ωστόσο, η έννοια της παρεμπόδισης μπορεί να γενικευτεί και ως προς άλλες σχέσεις - ή καλύτερα, άλλες μερικές διατάξεις στα γραφήματα. Μια συνηθισμένη παραλλαγή της παρεμπόδισης είναι η παρεμπόδιση με την έννοια του υπογραφήματος:

Ορισμός 4 (Σύνολο παρεμπόδισης με την έννοια του υπογραφήματος). *Έστω $\mathcal{C}$ μια κλάση γραφημάτων κλειστή ως προς την σχέση του υπογραφήματος. Τότε, ορίζουμε το* σύνολο παρεμπόδισης με την έννοια του υπογραφήματος *της $\mathcal{C}$, $\mathbf{obs}_{\subseteq}(\mathcal{C})$, ως το σύνολο των ελαχιστοτικών (ως προς τη σχέση του υπογραφήματος) γραφημάτων που δεν ανήκουν στη $\mathcal{C}$. Τα μέλη του συνόλου παρεμπόδισης λέγονται* παρεμποδίσεις με την έννοια του υπογραφήματος *ή απλά* παρεμποδίσεις-υπογραφήματα *της $\mathcal{C}$.*

Στο υπόλοιπο κείμενο, όταν αναφερόμαστε σε παρεμποδίσεις χωρίς να διευκρινίσουμε για ποιο είδος μιλάμε, εννοείται ότι αναφερόμαστε σε παρεμποδίσεις με την έννοια του ελάσσονος.

Έχουν όλες οι κλάσεις σύνολα παρεμπόδισης; Κατ' αρχήν, για να ορίζεται ένα σύνολο παρεμπόδισης, πρέπει η κλάση υπό συζήτηση να είναι κλειστή ως προς τη σχέση του ελάσσονος (ή προς όποια άλλη σχέση χρησιμοποιούμε για να ορίσουμε την παρεμπόδιση). Ένα άλλο ερώτημα που προκύπτει είναι εάν είναι πάντα πεπερασμένα τα σύνολα παρεμπόδισης. Η απάντηση σε αυτό το ερώτημα ήρθε το 2004, από τους Robertson και Seymour, οι οποίοι απέδειξαν το ακόλουθο θεώρημα.

Θεώρημα 2 (Robertson & Seymour, 2004 [28])**.** *Κάθε κλάση γραφημάτων κλειστή ως προς τη σχέση του ελάσσονος έχει πεπερασμένο σύνολο παρεμπόδισης.*

Αυτή η ανακάλυψη θεωρήθηκε σταθμός στην ιστορία του επιστημονικού πεδίου και παρότρυνε πολλούς ερευνητές να μελετήσουν τα σύνολα παρεμπόδισης διαφόρων κλάσεων. Μια ακόμη θεωρητική συνεισφορά των Robertson και Seymour ήταν η εύρεση ενός αλγορίθμου, χρονικής πολυπλοκότητας $O(n^3)$, ο οποίος μπορεί να αποφανθεί εάν, δοθέντων 2 γραφημάτων, το ένα είναι έλασσον του άλλου. Συνδυάζοντας τον αλγόριθμο αυτόν με το προηγούμενο θεώρημα, προκύπτει ότι κάθε πρόβλημα που μπορεί να κωδικοποιηθεί σαν πρόβλημα συμμετοχής ενός γραφήματος σε μια συγκεκριμένη κλάση, κλειστή ως προς ελάσσονα, έχει πολυωνυμική πολυπλοκότητα!

Ακόμα μια έννοια-κλειδί στην παρούσα εργασία είναι οι κλάσεις-απόγεια.

Ορισμός 5 (k-απόγειο κλάσης)**.** *Έστω $\mathcal{C}$ μια κλάση γραφημάτων. Ορίζουμε το k-απόγειο της κλάσης $\mathcal{C}$,το οποίο συμβολίζουμε $\mathcal{A}_k(\mathcal{C})$, ως*

$$\{G \in \mathcal{G} \mid \exists S \subseteq V(G) : |S| \leq k \land G \setminus S \in \mathcal{C}\}.$$

Προτού συνεχίσουμε, πρέπει να ορίσουμε ακόμα δύο έννοιες.

Ορισμός 6 (Κλειστότητα ως προς διακεκριμένη ένωση)**.** *Έστω κλάση γραφημάτων $\mathcal{C}$. Λέμε ότι η $\mathcal{C}$ είναι* κλειστή ως προς διακεκριμένη ένωση*, όταν για κάθε 2 γραφήματα G_1,G_2 που ανήκουν στη $\mathcal{C}$, η διακεκριμένη ένωση τους ανήκει και αυτή στη $\mathcal{C}$.*

Ορισμός 7 (**exc**). *Έστω σύνολο γραφημάτων S. Ορίζουμε την κλάση γραφημάτων $\mathbf{exc}_{\leq}(S)$ (αντίστοιχα $\mathbf{exc}_{\subseteq}(S)$) ως την κλάση όλων των γραφημάτων που δεν έχουν έλασσον (αντίστοιχα υπογράφημα) κάποιο από τα γραφήματα στο S.*

Το ακόλουθο πόρισμα του Dinneen [8] είναι σημαντικό για τη διαδικασία αναζήτησης παρεμποδίσεων κλάσεων-απόγειων.

Θεώρημα 3 «Θεώρημα μη συνεκτικών παρεμποδίσεων κλάσεων-απόγειων». *Έστω η κλάση γραφημάτων $\mathcal{A}_k(\mathcal{C})$, όπου $\mathcal{C}$ κλάση γραφημάτων κλειστή ως προς ελάσσονα και ως προς την πράξη της διακεκριμένης ένωσης. Τότε, το σύνολο των μη συνεκτικών παρεμποδίσεων της $\mathcal{A}_k(\mathcal{C})$ ταυτίζεται με το σύνολο των γραφημάτων $G = C_1 \cup C_2 \cup \ldots C_m$, όπου C_i είναι συνεκτική παρεμπόδιση της κλάσης $\mathcal{A}_{k_i}(\mathcal{C})$, έτσι ώστε $k_i < k$ και $\sum_{1 \leq i \leq m} (k_i + 1) = k + 1$.*

Με βάση αυτό το θεώρημα, μπορούμε να «ξεχάσουμε» τις μη συνεκτικές παρεμποδίσεις των κλάσεων-απόγειων που μελετάμε, καθώς προκύπτουν από τις συνεκτικές των κλάσεων με χαμηλότερη παράμετρο k.

Έχοντας ορίσει τις θεμελιώδεις έννοιες αυτής της διπλωματικής εργασίας, μπορούμε να προχωρήσουμε στην παρουσίαση των αποτελεσμάτων της.

Θεωρήματα για το μέγεθος και το μέγιστο βαθμό των παρεμποδίσεων

Στην παρούσα ενότητα θα αποδείξουμε ορισμένα θεωρητικά αποτελέσματα σχετικά με τα απόγεια μιας παραμετρικής κλάσης. Συγκεκριμένα, για δεδομένα $k, d \in \mathbb{N}$, θα ασχοληθούμε με τα k-απόγεια της κλάσης $\mathcal{B}_d$, που ορίζεται ως η κλάση των γραφημάτων με μέγιστο βαθμό το πολύ $d - 1$. Συμβολίζουμε $\mathcal{A}_k^{(d)} = \mathcal{A}_k(\mathcal{B}_d)$.

Η κλάση $\mathcal{A}_k^{(d)}$ είναι μια παραμετρική κλάση, που αποτελεί γενίκευση της κλάσης των γραφημάτων που επιδέχονται ένα k-κάλυμμα κορυφών (περίπτωση $d = 1$). Ακόμα, για $d \in \{1, 2, 3\}$, έχουμε $\mathcal{A}_k(\mathbf{exc}(K_{1,d})) = \mathcal{A}_k^{(d)}$. Στα λήμματα και θεωρήματα που ακολουθούν χρησιμοποιούμε τον όρο «παρεμπόδιση» χωρίς κάποια περαιτέρω διευκρίνιση. Τα αποτελέσματα αυτά δεν ισχύουν μόνο για τις παρεμποδίσεις με την έννοια του ελάσσονος, αλλά και για τις παρεμποδίσεις-υπογραφήματα, αφού η μόνη πράξη στην οποία βασίζονται είναι η διαγραφή ακμής.

Αρχικά, για το θεώρημα που ακολουθεί, είναι απαραίτητο να γίνουν ορισμένα ενδιάμεσα βήματα. Θα αποδείξουμε τα εξής λήμματα:

Λήμμα 1. *Έστω G παρεμπόδιση της $\mathcal{A}_k^{(d)}$. Τότε, δεν υπάρχει ακμή $e = \{u, v\} \in E(G)$ τέτοια ώστε $\mathbf{deg}(u) < d$ και $\mathbf{deg}(v) < d$.*

Απόδειξη. Θα αποδείξουμε το ζητούμενο δια της ατόπου απαγωγής. Έστω ότι υπάρχουν 2 γειτονικές κορυφές $u, v \in V(G)$ ώστε $\mathbf{deg}(u) < d$ και $\mathbf{deg}(v) < d$. Καθώς το G είναι παρεμπόδιση, υπάρχει ένα υποσύνολο A_e του $V(G)$ ώστε το γράφημα $G' = \{G \backslash e\} \backslash A_e$ να έχει μέγιστο βαθμό έως και $d-1$. Παρατηρούμε ότι στο G', $\mathbf{deg}'_G(u) \leq d-2$ και $\mathbf{deg}'_G(v) \leq d-2$. Επίσης, έστω $G'' = G' \setminus A_e$, το γράφημα που προκύπτει αφού επαναφέρουμε την ακμή e στο G'. Η επαναφορά αυτή επηρεάζει το βαθμό μονάχα των u, v. Όμως, βλέπουμε ότι το γράφημα εξακολουθεί να έχει μέγιστο βαθμό έως και $d-1$, αφού οι νέοι βαθμοί των u και v είναι αυξημένοι κατά 1 σε σχέση με πριν, δηλαδή είναι το πολύ $d-1$. Αυτό είναι άτοπο, γιατί έρχεται σε αντίφαση με το ότι το G δεν ανήκει στην κλάση $\mathcal{A}_k^{(d)}$. □

Λήμμα 2. *Ο αριθμός ανεξαρτησίας ενός γραφήματος G είναι το πολύ ίσος με $n(G) - \frac{m(G)}{\Delta(G)}$.*

Απόδειξη. Έστω ότι $I \subseteq V(G)$ είναι ένα ανεξάρτητο σύνολο του G. Τότε το σύνολο $S = V(G) \backslash I$ είναι σύνολο κάλυψης στο G, αφού κάθε ακμή προσκείμενη σε κορυφή στο

I έχει το άλλο της άκρο στο S, αλλιώς δεν θα ήταν το I πραγματικό ανεξάρτητο σύνολο. Αυτό σημαίνει ότι

$$m \leq |S| \cdot \Delta(G) \Leftrightarrow$$

$$|S| \geq \frac{m}{\Delta(G)} \Leftrightarrow^{(S=V(G)\backslash I)}$$

$$|I| \leq n - \frac{m}{\Delta(G)}$$

□

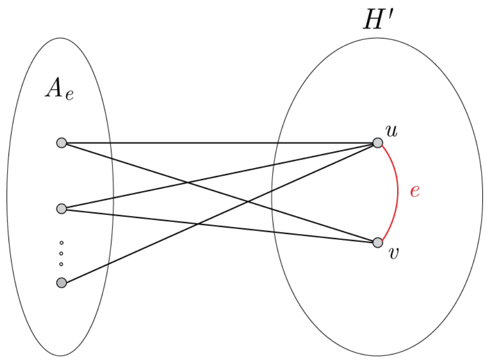
Μπορούμε πλέον να προχωρήσουμε στην απόδειξη του γενικού θεωρήματός μας:

Θεώρημα 4. *Έστω G παρεμπόδιση της $\mathcal{A}_k^{(d)}$. Τότε:*

1. *Ο μέγιστος βαθμός της είναι $\Delta(G) \leq k + d$.*

2. *Για το πλήθος των κορυφών της G έχουμε $V(G) \leq k(d+1)(d+k)+2d+k$*

Απόδειξη. Πρώτα θα δείξουμε την πρόταση (1), δηλαδή ότι ο μέγιστος βαθμός είναι $\Delta(G) = k + d$.

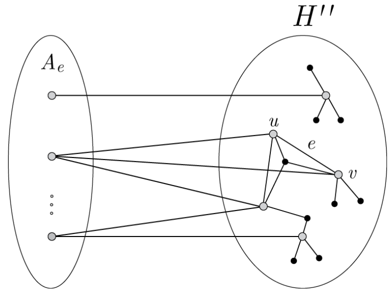


Σχήμα 1: Η αφαίρεση της ακμής e μας αφήνει με ένα γράφημα που διαμερίζουμε στο A_e (το σύνολο των απόγειων κορυφών), στα αριστερά, και στις υπόλοιπες κορυφές, στα δεξιά.

Αρκεί να δείξουμε ότι για κάθε $u \in V(G)$, $\mathbf{deg}_G(u) \leq d + k$.
Έστω $u \in V(G)$. Διαλέγουμε στην τύχη κάποιο γείτονα v της u (σε περίπτωση που δεν έχει γείτονες η u, έχουμε τελειώσει, αφού $\mathbf{deg}_G(u) = 0 \leq d + k$).

Αφαιρούμε την ακμή $e = \{u, v\}$ από το G. Συμβολίζουμε $H = G \setminus e$. Υπάρχει ένα σύνολο A_e το πολύ k κορυφών απόγειων των οποίων η διαγραφή από το H μας δίνει το H', επαγόμενο υπογράφημα του H, όπου $\Delta(H') \leq d-1$. Ας σημειωθεί ότι καμιά από τις u, v δεν μπορεί να ανήκει στις k απόγειες κορυφές, αφού εάν ανήκαν, τότε θα είχαμε ότι $G \setminus A_e = H \setminus A_e$, δηλαδή η G θα ανήκε στην κλάση της οποίας είναι παρεμπόδιση.

Εφόσον η επαναφορά των κορυφών του A_e στο γράφημα αυξάνει το βαθμό του u το πολύ κατά $|A_e|$, έχουμε $\mathbf{deg}_H(u) \leq \mathbf{deg}_{H'}(u) + k$. Αρά, $\mathbf{deg}_H(u) \leq d - 1 + k$. Τέλος, καθώς $\mathbf{deg}_G(u) = \mathbf{deg}_H(u) + 1$, ισχύει ότι $\mathbf{deg}_G(u) \leq d - 1 + k + 1 = d + k$. Έτσι αποδεικνύεται το ζητούμενο.



Σχήμα 2: Οι κορυφές στο γράφημα $\{G \setminus e\} \setminus A_e$ διαμερίζονται στα σύνολα S_1 (γκρι κορυφές) και S_2 (μαύρες κορυφές)

Προχωράμε τώρα στην απόδειξη της πρότασης (2). Αρχικά θα θεωρήσουμε ότι το G είναι συνεκτικό. Για άλλη μια φορά, ορίζουμε $H = G \setminus e$. Ως γνωστόν, $H \in \mathcal{A}_k^{(d)}$, δηλαδή υπάρχει σύνολο $A(e)$ μεγέθους k, ώστε το $H \setminus A_e$ να έχει μέγιστο βαθμό έως και $d-1$. Χωρίζουμε το γράφημα G σε 2 κομμάτια, $G_1 = G[A_e]$ και $G_2[V(G) \setminus A_e]$, όπως φαίνεται στο σχήμα. Χωρίζουμε τις κορυφές του $V(G) \setminus A_e$ σε δύο σύνολα: στο σύνολο S_1, που περιέχει τις κορυφές u, v και τις κορυφές που γειτνιάζουν με το $A(e)$ στο G, και στο σύνολο S_2, με τις υπόλοιπες. Θα επιχειρήσουμε μια εκτίμηση του μεγέθους κάθε συνόλου.

1. Το μέγεθος του S_1 είναι το πολύ $2 + k\Delta(G)$. Ο πρώτος όρος εκπροσωπεί τις κορυφές u και v, για τις οποίες δεν γνωρίζουμε εάν είναι γειτονικές με το A_e. Ο δεύτερος όρος είναι το μέγιστο μέγεθος της γειτονιάς του A_e, και μπορεί να γραφτεί και ως $2 + k(k + d)$, σύμφωνα με το πρώτο σκέλος του θεωρήματος(1).

2. Αυτό που γνωρίζουμε για το S_2 είναι ότι περιέχει κορυφές των οποίων ο βαθμός δεν αλλάζει μετά τη διαγραφή των e και A_e από το γράφημα. Επομένως, $\forall u \in S_2$, $\mathbf{deg}(u) \leq d - 1$. Επιπροσθέτως, όπως λέει και το Λήμμα 1, το S_2 είναι ανεξάρτητο σύνολο στο G, και άρα στο G_2. Ακόμα, χάρη στο Λήμμα 2, γνωρίζουμε ότι κάθε ανεξάρτητο σύνολο στο G_2 έχει το πολύ $n(G_2) - \frac{m(G_2)}{\Delta(G_2)}$ κορυφές. Για να εκμεταλλευτούμε αυτό το φράγμα, θα φράξουμε το πλήθος των ακμών $m(G_2)$ από κάτω, και θα φράξουμε άνω το μέγιστο βαθμό του G_2. Γνωρίζουμε ότι το $m(G_2)$, σε γράφημα με $n(G_2)$ κορυφές, είναι τουλάχιστον $n(G) - l$, οπού l το πλήθος των μεγιστοτικών συνεκτικών συνιστωσών του G_2. Η αφαίρεση k κορυφών από ένα συνεκτικό γράφημα G μπορεί να το κατακερματίσει σε

$k\Delta(G)$ συνεκτικές συνιστώσες το πολύ, άρα το G_2 έχει το πολύ $k(k+d)$ συνιστώσες. Ακόμα, γνωρίζουμε ότι $\Delta(G_2) = d$. Εν τέλει, καταλήγουμε στην ανισότητα $|S_2| \leq n(G_2) - \frac{n(G_2)-k(k+d)}{d}$ (2).

Πλέον, μπορούμε να ενώσουμε τα κομμάτια και να εξάγουμε το συμπέρασμα μας. Αρχικά ισχύει ότι $|V(G)| = |V(G_1)| + |V(G_2)| = |A_e| + |S_1| + |S_2|$. Έστω $n = V(G)$. Συνυπολογίζοντας ότι $n(G_2) = n - k$, καθώς και τις εξισώσεις (1) και (2) καταλήγουμε στο

$$n \leq k + 2 + k(k+d) + (n-k) - \frac{n-k-k(k+d)}{d} \Leftrightarrow$$

$$\frac{n-k-k(k+d)}{d} \leq 2 + k(k+d)] \Leftrightarrow$$

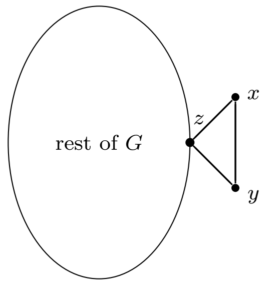
$$n - k - k(k+d) \leq 2d + kd(k+d). \Leftrightarrow$$

$$n \leq k(d+1)(d+k) + 2d + k$$

Αποδείξαμε την πρόταση για τις συνεκτικές παρεμποδίσεις, για τις οποίες βρήκαμε το φράγμα $b(k,d) = k(d+1)(d+k) + 2d + k$. Για να επεκτείνουμε την απόδειξη και στις μη συνεκτικές, θα καταφύγουμε στο θεώρημα του Dinneen (Θεώρημα 3): αν G μια μη συνεκτική παρεμπόδιση της κλάσης $\mathcal{A}_k^{(d)}$ με συνεκτικές συνιστώσες $C_1, C_2, \ldots, C_r$, κάθε συνεκτική συνιστώσα C_i του G είναι παρεμπόδιση μιας κλάσης $\mathcal{A}_{k_i}^{(d)}$, έτσι ώστε $\sum\limits_{i\in\{1,\ldots,r\}} (k_i + 1) = k + 1$. Άρα, $\sum\limits_{i\in\{1,\ldots,r\}} k_i < k$. Έτσι, αθροίζοντας τα φράγματα για κάθε συνεκτική συνιστώσα του G έχουμε $n(G) \leq \sum\limits_{i\in\{1,\ldots,r\}} (b(k_i,d)) \leq b\Big(\sum\limits_{i\in\{1,\ldots,r\}} k_i, d\Big) < b(k,d)$. Αποδεικνύεται έτσι το φράγμα και για τις μη συνεκτικές παρεμποδίσεις. $\square$

Στη συνέχεια θα ασχοληθούμε με την ειδική περίπτωση $d = 2$. Θα ξεκινήσουμε με μερικά χρήσιμα λήμματα.

Σχήμα 3: Το τρίγωνο της απόδειξης του Λήμματος 3.



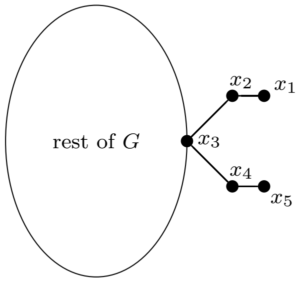
Λήμμα 3. *Έστω $G \in \mathbf{obs}(\mathcal{A}_k^{(2)})$. Τότε το G δεν μπορεί να έχει τρίγωνο x, y, z όπου 2 κορυφές να έχουν βαθμό ακριβώς 2.*

Απόδειξη. Έστω x, y οι κορυφές του τριγώνου με βαθμό 2. Η αφαίρεση της ακμής $e = \{x, y\}$ από το γράφημα μας δίνει ένα σύνολο απόγειων κορυφών A_e, των οποίων η αφαίρεση κάνει το μέγιστο βαθμό του εναπομένοντος γραφήματος ίσο με 1. Αυτό σημαίνει ότι μια από τις x, y, και z ανήκει στο $A(e)$, αλλιώς η z θα εξακολουθεί να διατηρεί βαθμό μεγαλύτερο του 1. Έτσι όμως, καλύπτεται και η ακμή e, δηλαδή βλέπουμε ότι ακόμα και χωρίς τη διαγραφή της, το ίδιο σύνολο A_e μας δίνει ένα γράφημα μέγιστου βαθμού 1. Οδηγούμαστε σε άτοπο, γιατί το G είναι παρεμπόδιση και όχι μέλος της $\mathcal{A}_k^{(2)}$. □

Λήμμα 4. *Έστω $G \in \mathbf{obs}(\mathcal{A}_k^{(2)})$. Τότε το G δεν μπορεί να έχει ένα υπογράφημα P_5 όπου μόνο η κεντρική κορυφή να έχει βαθμό μεγαλύτερο του 2.*

Απόδειξη. Η αφαίρεση της ακμής $e = \{x_1, x_2\}$ από το γράφημα μας δίνει ένα σύνολο απόγειων κορυφών A_e, των οποίων η αφαίρεση κάνει το μέγιστο βαθμό του εναπομένοντος γραφήματος ίσο με 1. Αυτό σημαίνει ότι μια από τις x_3, x_4, x_5 ανήκει στο $A(e)$, αλλιώς η x_4 θα εξακολουθεί να διατηρεί βαθμό μεγαλύτερο του 1. Χωρίς βλάβη της γενικότητας, θεωρούμε ότι η x_3 ανήκει στο

Σχήμα 4: Η μορφή γραφήματος του Λήμματος 4



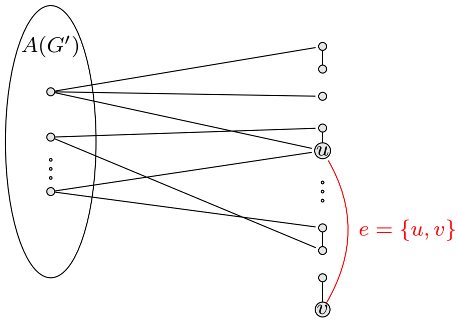
A_e (οποιαδήποτε από τις x_4, x_5 θα μπορούσε να αντικατασταθεί από τη x_3 διατηρώντας το μέγιστο βαθμό). Για άλλη μια φορά όμως, βλέπουμε ότι ακόμα και χωρίς τη διαγραφή της e, το ίδιο σύνολο A_e μας δίνει ένα γράφημα μέγιστου βαθμού 1. Οδηγούμαστε σε άτοπο, γιατί το G είναι παρεμπόδιση και όχι μέλος της $\mathcal{A}_k^{(2)}$. □

Θεώρημα 5. *Έστω $G \in \mathbf{obs}(\mathcal{A}_k^{(2)})$. Τότε το G έχει το πολύ $k^2 + 4k + 3$ κορυφές.*

Απόδειξη. Όπως και στην προηγούμενη απόδειξη, θα αποδείξουμε το φράγμα πρώτα για τα συνεκτικά γραφήματα, και στη συνέχεια θα το επεκτείνουμε και στα μη συνεκτικά. Θα προσπαθήσουμε, μετά την αφαίρεση μιας οποιασδήποτε ακμής e, να μετρήσουμε το πλήθος των κορυφών στο γράφημα $H'' = G \setminus A_e$, όπου A_e είναι ένα σύνολο κορυφών-απόγειων στο $G \backslash e$. Πώς μοιάζει το γράφημα αυτό; Μια απεικόνιση του φαίνεται στο Σχήμα 5.

Το H'', το οποίο βλέπουμε στα δεξιά, αν εξαιρέσουμε τη συνεκτική συνιστώσα στην οποία ανήκει η e, αποτελείται είτε από απομονωμένες κορυφές είτε από απομονωμένες ακμές. Αυτό ισχύει γιατί κατά την αφαίρεση της e και του A_e από το G, έχουμε μέγιστο βαθμό 1. Επειδή το G είναι συνεκτικό, πρέπει κάθε συνεκτική συνιστώσα στο H'' να έχει τουλάχιστον ένα γείτονα στο A_e. Η A_e από την άλλη, έχει το πολύ $k(k+2)$ γείτονες, αφού $\Delta(G) = k+d$. Αυτό σημαίνει ότι δεν μπορούμε να έχουμε πάνω από $k(k+2)$ συνεκτικές συνιστώσες στην H'', και άρα το πολύ $2k(k+2)+2 = 2k^2+4k+2$ κορυφές. Είναι όμως αυτό το φράγμα «σφιχτό»;

Σχήμα 5: Με τη διαγραφή της e έχουμε ένα γράφημα που χωρίζεται στο $G[A_e]$, και ένα γράφημα με συνιστώσες P_1 και P_2.



Μιλώντας πάντα για το H'', ας υποθέσουμε ότι το πλήθος των απομονωμένων ακμών (αντίστοιχα κορυφών) με έναν μόνο γείτονα στο A_e είναι p_1 (αντίστοιχα l_1), ότι το πλήθος των απομονωμένων ακμών (αντίστοιχα κορυφών) με παραπάνω από έναν γείτονες στο A_e είναι p_2 (αντίστοιχα l_2).

Παρατηρούμε ότι μια απομονωμένη ακμή που έχει μόνο έναν γείτονα στο A_e συνδέεται μαζί του μόνο διαμέσου μιας εκ των δύο κορυφών της. Η αντίθετη περίπτωση είναι αδύνατη λόγω του Λήμματος 3. Έστω τώρα ότι 2 απομονωμένες ακμές έχουν έναν κοινό γείτονα στο A_e και κανέναν άλλον. Έχουμε και σε αυτή την περίπτωση άτοπο, διότι οι 2 αυτές συνιστώσες μαζί με τον κοινό τους γείτονα δημιουργούν σχηματισμό απαγορευμένο από το Λήμμα 4. Επομένως, κάθε κορυφή στο A_e έχει το πολύ μια απομονωμένη ακμή να συνδέεται αποκλειστικά σε αυτήν, και άρα η μέγιστη τιμή του p_1 είναι k.

Όσον αφορά τα l_1,p_2,l_2, παρατηρούμε ότι αφορούν συνιστώσες οι οποίες έχουν γείτονες στο A_e που στο πλήθος είναι περισσότεροι ή ίσοι με το μέγεθος τους (1 ή 2 κορυφές). Δηλαδή, οι συνιστώσες που έχουν μέγεθος 1 (l_1, l_2) έχουν τουλάχιστον 1 γείτονα. Εξ ορισμού, οι συνιστώσες που αντιστοιχούν στο p_2 έχουν 2 ή παραπάνω γείτονες στο A_e.

Τέλος, ξεκινάμε την άθροιση. Το σύνολο A_e γειτνιάζει με έναν αριθμό από συνιστώσες του H''. Ορισμένες από αυτές έχουν περισσότερες από έναν γείτονα στο A_e. Θεωρούμε ότι κάθε απόγεια κορυφή έχει βαθμό $k + d$, δηλαδή το μέγιστο δυνατό. Δηλαδή από το επαγόμενο υπογράφημα $G[A_e]$, έχουμε $k(k + 2)$ εξερχόμενες ακμές. Μια από αυτές φτάνει στη συνιστώσα της e, που έχει έως και 4 κορυφές. Το πολύ k από αυτές φτάνουν σε απομονωμένες ακμές που έχουν ακριβώς ένα γείτονα στο A_e. Οι υπόλοιπες $k(k+2)-1-k$ φτάνουν σε συνιστώσες που αθροιστικά, έχουν το πολύ ισάριθμες κορυφές. Άρα αθροιστικά, οι κορυφές προκύπτουν: $|V(H'')| = 4+2k+k(k+1)-1 = k^2+3k+3$, και τελικά $|V(G)| \leq k^2+4k+3$. □

Προετοιμασία αναζήτησης: αποδεικνύοντας τα φράγματα

Σε αυτήν την εργασία παρουσιάζονται ορισμένα σύνολα παρεμπόδισης, για συγκεκριμένες κλάσεις από την οικογένεια που μελετάμε. Ασχολούμαστε με τις κλάσεις:

	$d=2$	$d=3$
$k=1$	$\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$	$\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}\}))$, $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}, K_3\}))$
$k=2$	$\mathcal{A}_2(\mathbf{exc}(\{K_{1,2}\}))$	-

Συγκεκριμένα, για κάθε μία από τις $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$, $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}\}))$ και $\mathcal{A}_2(\mathbf{exc}(\{K_{1,2}\}))$ βρήκαμε 2 σύνολα παρεμπόδισης, ένα για τον ορισμό της παρεμπόδισης ως προς τη σχέση του ελάσσονος και ένα για τον ορισμό της ως προς την σχέση του υπογραφήματος. Ακόμα, για την κλάση $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}, K_3\}))$ βρήκαμε μόνο τις παρεμποδίσεις με την έννοια του ελάσσονος.

Η αναζήτηση των παρεμποδίσεων, για να είναι εφικτή από άποψη χρόνου, πρέπει να γίνει σε σύνολο γραφημάτων αρκετά μικρό. Τα γενικά θεωρήματα που αποδείξαμε πιο πάνω δεν δίνουν αρκετά καλά αποτελέσματα όσον αφορά το πλήθος των κορυφών της κλάσης $\mathcal{A}_k^{(d)}$. Συγκεκριμένα, για $d=2, k=2$ έχουμε μέχρι 15 κορυφές, και για $d=3, k=1$ έχουμε μέχρι και 27 κορυφές. Για να γίνουν αισθητά τα μεγέθη αυτά, μπορεί κανείς να σκεφτεί ότι το σύνολο όλων των γραφημάτων μεγέθους 15 και με μέγιστο βαθμό το μέχρι και 4, είναι στο 8.788.983.173 στο πλήθος!

Για να μικρύνουμε τον χώρο αναζήτησης, μελετήσαμε τις ιδιαιτερότητες συγκεκριμένων κλάσεων. Καταλήξαμε στα ακόλουθα θεωρήματα:

Θεώρημα 6. *Έστω $G \in \mathbf{obs}_{\subseteq}(\mathcal{A}_2^{(2)})$. Τότε $n(G) \leq 10$.*

Θεώρημα 7. *Έστω* $G \in \mathbf{obs}_{\subseteq}(\mathcal{A}_1^{(3)})$ *ή* $G \in \mathbf{obs}_{\leq}(\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}, K_3\})))$ *. Τότε:*

- *Εάν* $\Delta(G) = 4$, $n \leq 7$,
- *Διαφορετικά,* $n(G) \leq 15$.

Εύρεση παρεμποδίσεων

Χάρη στο Θεώρημα 3, μπορούμε να παράγουμε με τυποποιημένο τρόπο τις μη συνεκτικές παρεμποδίσεις των κλάσεων που μας ενδιαφέρουν, μόνο από τις συνεκτικές χαμηλότερων τάξεων. Επομένως, για το υπόλοιπο του κεφαλαίου, ασχολούμαστε μόνο με τις συνεκτικές παρεμποδίσεις. Η αναζήτηση τους έγινε σε ένα πεπερασμένο σύνολο γραφημάτων, χάρη στα άνω φράγματα του μεγέθους και του βαθμού που βρήκαμε στην προηγούμενη ενότητα. Ένα μεγάλο μέρος της προσπάθειας μας αφορούσε τη βελτιστοποίηση των φραγμάτων αυτών, ώστε η αναζήτηση να είναι πρακτικά εφικτή.

Οι Dinneen και Xiong [9] ακολουθούν μια μέθοδο παρόμοια με τη δική μας.

Κατ' αρχήν, θέλουμε έναν αλγόριθμο που αποφαίνεται εάν ένα γράφημα G είναι παρεμπόδιση της $\mathcal{F}$. Σύμφωνα με τον ορισμό της παρεμπόδισης μιας κλάσης γραφημάτων $\mathcal{F}$, υπάρχει ένας απλός αλγόριθμος για αυτό. Αρκείται απλά στο να ελέγχει αφ' ενός, εάν το γράφημα ανήκει ή όχι στην κλάση υπό συζήτηση (εάν ανήκει τότε δεν είναι παρεμπόδισή της) και αφ' ετέρου, εάν όλα τα ελάσσονα του είναι μέλη της κλάσης (εάν κάποιο δεν είναι τότε πάλι το γράφημα εισόδου δεν είναι παρεμπόδιση). Ο αλγόριθμος δίνεται σε ψευδοκώδικα παρακάτω. Δέχεται ως είσοδο μια συνάρτηση συμμετοχής (membership algorithm) και το γράφημα υπό εξέταση.

```
1: function isObstruction(MembershipAlgorithm isMember, Graph G)
2:     if isMember(G) then return false
3:     else
4:         for all e ∈ E(G) do
5:             G' ← remove_edge(e, G)
6:             if not(isMember(G')) then return false
7:             end if
8:             G'' ← contract_edge(e, G)
9:             if not(isMember(G")) then return false
10:            end if
11:        end for
12:    end if
13:    return true
14: end function
```

Στη δική μας περίπτωση, αναζητούμε και τις παρεμποδίσεις με την έννοια του απλού υπογραφήματος. Για τον έλεγχο αυτών των παρεμποδίσεων, απλώς παραλείπουμε την πράξη σύμπτυξης ακμής, δηλαδή τις γραμμές 8, 9, και 10 του ψευδοκώδικα.

Όσον αφορά τη συνάρτηση συμμετοχής, υλοποιήσαμε μια εξαντλητική αναζήτηση στα k-υποσύνολα του συνόλου κορυφών, με σκοπό να βρούμε εάν υπάρχει απόγειο σύνολο η αφαίρεση του οποίου μας αφήνει με ένα γράφημα που να ανήκει στην κλάση-όρισμα του τελεστή $\mathcal{A}_k()$. Η τελευταία, είναι είτε τα γραφήματα βαθμού μικρότερου από d, είτε τα άκυκλα γραφήματα βαθμού μικρότερου από d. Και για τις δύο κλάσεις, ο έλεγχος είναι γραμμικός.

```
function isMember(Graph G,int k,int d)
    for all S| S is a k-subset of V(G) do
        G' ← remove_vertices(S, G)
        if max_degree(G)< d then
            if is_acyclic(G) then
                return true
            end if
        end if
    end for
    return false
end function
```

Ο παραπάνω αλγόριθμος είναι η συνάρτηση συμμετοχής στην κλάση $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}, K_3\}))$, για $d \in \{1, 2, 3\}$. Φυσικά, με την αφαίρεση του ελέγχου για την ύπαρξη κύκλων, έχουμε τον αλγόριθμο συμμετοχής στην κλάση $\mathcal{A}_k(\mathbf{exc}(\{K_{1,d}\}))$, για $d \in \{1, 2, 3\}$.

Ας κοιτάξουμε λίγο πιο προσεκτικά τους αλγορίθμους. Έστω n και m το πλήθος των κορυφών και των ακμών του γραφήματος αντίστοιχα. Η χρονική πολυπλοκότητα του καθορίζεται πρωτίστως από το πλήθος των επαναλήψεων της δομής for: $\binom{n}{k}$ επαναλήψεις. Το σώμα του βρόχου από μόνο του εκτελείται σε γραμμικό (ως προς το μέγεθος του γραφήματος) χρόνο: ο υπολογισμός του μέγιστου βαθμού απαιτεί ακριβώς n συγκρίσεις, και η συνάρτηση που εξετάζει την ύπαρξη κύκλου γίνεται σε χρόνο $O(n+m)$, αφού βασίζεται στον αναζήτηση κατά βάθος. Έτσι, τελικά έχουμε χρονική πολυπλοκότητα $O(n^{k+1}m)$. Θεωρώντας το k σταθερό, έχουμε πολυωνυμικό χρόνο. Η χωρική πολυπλοκότητα είναι και αυτή γραμμική: στην ουσία, αποθηκεύουμε απλά τα γραφήματα G και G', ένα υποσύνολο των κορυφών S, καθώς και λίγες βοηθητικές μεταβλητές.

Προκύπτει λοιπόν η χρονική πολυπλοκότητα του πρώτου αλγορίθμου: καλείται $m + 1$ φορές η συνάρτηση $isMember()$, άρα το χρονικό κόστος είναι $O(m^2 n^{k+1})$. Η χωρική πολυπλοκότητα είναι πάλι γραμμική.

Δεν ασχοληθήκαμε ιδιαίτερα με τη βελτίωση της χρονικής πολυπλοκότητας κάθε αλγορίθμου μας. Η επιλογή μας αυτή δικαιολογείται από τα μεγέθη των εισόδων στο πρόβλημά μας: Το k θεωρήθηκε μικρό, τουλάχιστον για τα πρώτα βήματα αυτής της έρευνας (κομμάτι των οποίων αποτελεί και η παρούσα εργασία). Τα γραφήματα είναι όλα πολύ μικρά, με $n < 15$, με μέγιστο βαθμό

$\Delta(G) \leq k + d$ και άρα $m \leq \frac{(k+d)n}{2}$. Ο περιορισμός του μεγέθους του n φυσικά είναι περιορισμός στο χρόνο και στο χώρο που απαιτεί μια εξαντλητική αναζήτηση στο χώρο των γραφημάτων, ακόμα και αν περιορίσουμε το μέγιστο βαθμό του γραφήματος σε 3 ή 4. Φυσικά, ένα ακόμα κίνητρο για τη χρήση αυτών των αλγορίθμων ήταν η απλότητα. Ένας πολύ πιο περίπλοκος αλγόριθμος θα μας προκαλούσε δυσκολίες, τόσο θεωρητικές (θα έπρεπε να αποδείξουμε την ορθότητα του), όσο και πρακτικές (δύσκολη υλοποίηση, σφάλματα).

Η υλοποίηση των αλγορίθμων αυτών ήταν το πιο κρίσιμο κομμάτι της συγγραφής των προγραμμάτων μας, στη γλώσσα C++ . Χρησιμοποιήσαμε όσο το δυνατόν απλούστερες και αφαιρετικότερες δομές δεδομένων για λόγους βελτιστοποίησης. Ένα γράφημα αναπαρίσταται ως δομή (struct), που περιέχει το πλήθος των κορυφών, το πλήθος των ακμών, έναν πίνακα γειτνίασης (adjacency matrix) και ένα πίνακα με τους βαθμούς των κορυφών, λόγω της συχνής πρόσβασης σε αυτούς. Στο πνεύμα της βελτιστοποίησης από άποψη χρόνου, οι πίνακες αυτοί είναι στατικοί, πράγμα που επιτρέπει το μικρό μέγεθος των γραφημάτων εισόδου.

Η αναζήτηση των παρεμποδίσεων έγινε με τη βοήθεια του ταχύτατου εργαλείου *geng* της σουίτας *gtools* [23]. Πρόκειται για ένα πρόγραμμα γραμμής εντολών για Linux, το οποίο, με είσοδο n, απαριθμεί όλα τα μη ισομορφικά γραφήματα με n κορυφές. Σημαντικές παράμετροι που δέχεται είναι το μέγιστο και ελάχιστο πλήθος ακμών, ο μέγιστος βαθμός, ο ελάχιστος βαθμός και η συνεκτικότητα. Έτσι, με πολύ απλό και γρήγορο τρόπο, γίνεται εφικτή η σύνθεση αρχείων με όλα τα συνεκτικά γραφήματα μεγέθους έως και n με μέγιστο βαθμό έως και Δ, για n, Δ αρκούντως μικρά. Παραδείγματος χάρη, όλα τα συνεκτικά γραφήματα μέγιστου βαθμού 4 με 11 κορυφές υπολογίστηκαν από το *geng* σε λιγότερο από 2 δευτερόλεπτα. Παρεμπιπτόντως, με αυτούς τους περιορισμούς έχουμε μόνο 739.335 γραφήματα.

```
$geng -cd1D4 n=11 e=10-22
>739335 graphs generated in 1.22 sec
```

Συνοψίζοντας, για την αναζήτηση παρεμποδίσεων ακολουθήσαμε τα εξής βήματα.

1. Υπολογίσαμε άνω φράγματα για το πλήθος των κορυφών και το μέγιστο

βαθμό των συνεκτικών παρεμποδίσεων της κλάσης.

2. Χρησιμοποιήσαμε το *geng* για τη δημιουργία αρχείου εισόδου με όλα τα συνεκτικά γραφήματα που είναι "κάτω από τα φράγματα" αυτά.

3. Εκτελέσαμε ένα απλό πρόγραμμα *C++* που καλεί τη συνάρτηση $isObstruction(G)$ της εκάστοτε κλάσης και του εκάστοτε τύπου παρεμποδίσεων (υπογραφήματος ή ελάσσονος), σειριακά για όλα τα γραφήματα στο αρχείο, εμφανίζοντας στην έξοδο όσα είναι παρεμποδίσεις.

Τα όρια του υπολογισμού

Η μέθοδος μας, αυτή της εξαντλητικής αναζήτησης, δεν μπορεί να εφαρμοστεί σε κάθε πρόβλημα. Για παράδειγμα, για κλάσεις με παρεμποδίσεις μεγάλου μεγέθους είμαστε αναγκασμένοι να χρησιμοποιήσουμε (και) άλλα μέσα. Αυτό οφείλεται κυρίως στο γεγονός ότι το πλήθος των γραφημάτων με n κορυφές αυξάνει πάρα πολύ γρήγορα καθώς αυξάνεται το n. Ενδεικτικά, αναφέρουμε ότι το πλήθος των γραφημάτων (χωρίς ετικέτες) μεγέθους 15 είναι περισσότερα από 3×10^{19} [29], ξεπερνώντας κατά πολύ τα όρια του πρακτικά υπολογίσιμου.

Όμως, πέρα από την αξία που έχει η ανακάλυψη παρεμποδίσεων σε κόσμους όπου το n είναι ακόμα χαμηλό, η μέθοδος της απαρίθμησης με υπολογιστή μπορεί να προεκτείνει τις δυνατότητες παρατήρησης μας πάνω σε οποιαδήποτε κλάση, βοηθώντας τους ερευνητές να δουν τα γραφήματα από ένα σημείο με καλύτερη «θέα». Για παράδειγμα, η εύρεση όλων των παρεμποδίσεων με τάξη έως και 10, μπορεί να «προδώσει» ορισμένες μέχρι πρότινος άγνωστες ιδιότητες της κλάσης που μελετάμε. Η αναζήτηση με τη βοήθεια υπολογιστή, παρόλο που δεν είναι ικανή να παράγει αυτούσια «σκληρά» μαθηματικά αποτελέσματα, έχει φανεί χρήσιμο εργαλείο για την εκτέλεση πειραμάτων, και τον έλεγχο υποθέσεων. Μάλιστα, έχει καταστήσει δυνατή τη διατύπωση γραφοθεωρητικών εικασιών από προγράμματα υπολογιστή [12].

Χαρακτηριστικά, ο Paul Halmos αναφέρει [34] :

> «*Τα μαθηματικά δεν είναι μία αφαιρετική επιστήμη — αυτό είναι το κλισέ. Όταν προσπαθείς να αποδείξεις ένα θεώρημα, δεν απαριθμείς τις υποθέσεις και μετά ξεκινάς τους συλλογισμούς. Αυτό που κάνεις*

είναι δοκιμές και σφάλματα, πειραματισμούς, εικασίες. Θέλεις να ανακαλύψεις την πραγματικότητα, και υπό αυτή την έννοια, η δουλειά σου είναι παρόμοια με αυτή ενός τεχνικού σε εργαστήριο.»

Abstract

An *apex set* of a graph G, with respect to a graph class $\mathcal{C}$, is a set of vertices $A \subseteq V(G)$, such that $G \setminus A \in \mathcal{C}$. Similarly, a k-apex class of a class $\mathcal{C}$, denoted $\mathcal{A}_k(\mathcal{C})$, is defined as the class of all graphs that have an apex set w.r.t. $\mathcal{C}$, of size at most k. Let $\mathcal{B}_d$ denote the class of graphs of maximum degree at most $d-1$. We establish a $O(k^2)$ and $O(d^2)$ bound on the size of the obstructions of $\mathcal{A}_k(\mathcal{B}_d)$. This result is, to our knowledge, new. Moreover, we study some special cases of apices of d-bounded classes, where, by "d-bounded", we refer to their property of being disjoint-union-closed and of maximum degree up to $d-1$. Such classes are studied in [25]. We present 7 obstruction sets of 4 classes belonging to this class family. We obtain these obstruction sets through exhaustive enumeration by computer. Finally, we make some philosophical observations on the increasingly important role of the computer in mathematical research.

Keywords

graph theory, graph minors, forbidden graph characterization, obstruction set, obstructions, apex graphs, kuratowski characterization, kuratowski graphs, exhaustive search, computer-aided proof, experimental mathematics,k-vertex cover,subgraph obstructions, forbidden subgraphs

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Chapter 1

Introduction

We could say that this M.Sc. thesis is a work that a certain amount of energy has been spent on. Hopefully, these joules did not go to waste. There are 3 intended outcomes of this thesis:

- First, several concrete obstruction sets, i.e. sets of graphs that are minimal non-members of specific graph classes.
- Second, some theorems on the size and the qualities of obstructions of a certain family of graph classes.
- Third, the experience of a collaboration between a mathematician and a candidate computer engineer.

We will now address every objective in the reverse order.

The third one refers to a results on a more personal level. As this thesis was born from ideas worked in collaboration with professor Dimitrios M. Thilikos, it has produced, along with its expected output, something perhaps just as important: the experience and lessons gained through hours of brainstorming, thought-provoking discussion and fruitful exchange of ideas from two different schools of thought.

The second one, is the establishment of a number of bounds on parameters of the obstructions of several parametrized classes. These bounds serve to delimit the set of possible graphs that one must search through, in order to find those with the rare property of being an obstruction of one of the previously mentioned classes. Additionally, these bounds have some value in and of themselves and they leave room for improvement.

The first one, is depicted in the Appendix, where we give the complete characterization of graph classes $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$, $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}\}))$, $\mathcal{A}_2(\mathbf{exc}(\{K_{1,2}\}))$, and $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}, K_3\}))$ through their forbidden minors and for the first 3 classes, through their forbidden subgraphs too. These forbidden graphs were found by a computer program which examined every graph from a large list (our *search space*) and returned those that satisfied a set of restrictions. This was made possible by bounds like the ones mentioned above, that limited the search space satisfactorily.

This technique could not have been applicable without the existence of a computer, as the number of graphs examined is too large (roughly 10^5 graphs), but also because it is not possible to enumerate all possible graphs up to a certain size and degree by hand.

Computer-aided searching techniques have been used in various cases during the last years. They are part of the recent tradition of computer-assisted proofs, which first received attention with the proof of the Four-Colour Theorem [2], then followed by a number of other results: the computer solution to the 17-point Happy Ending Problem [31], the proof that at least 17 clues are necessary in a sudoku puzzle [22] and finally, the troubled proving of Kepler's conjecture [16]

1.1 The role of the computer in mathematical research

Mechanization tends to emphasize practice rather than theory, deeds rather than words, explicit answers rather than existence statements, definitions that are formalized rather than behaviouristic, local rather than global phenomena, the limited rather than the infinite, the concrete rather than the abstract, and one could almost say, the scientific rather than the artistic.
D.H. Lehmer [19]

There exist a number of dangers in relying on the machine to prove mathematical statements. One of the first critiques of computer-aided proving was made by Thomas Tymoczko. He stated that proofs with exhaustive enumeration by complex computer programs are non-surveyable, i.e. they hide part of the reasoning in computer generated arguments that cannot be verified by hand. What's more, in such proofs, experimentation is substituted for deduction, which goes against the tradition of mathematics being a purely analytical discipline.

> *[the] use of computers in mathematics, as in the [Four-Color Theorem], introduces empirical experiments into mathematics. Whether or not we choose to regard the [Four-Color Theorem] as proved, we must admit that the current proof is no traditional proof, no a priori deduction of a statement from premises. It is a traditional proof with a . . . gap, which is filled by the results of a well-thought-out experiment.*
>
> Thomas Tymoczko - The Four-Color Problem and its Mathematical Significance ([32])

Indeed, the proof of the four-colour theorem was not well-received by the contemporary mathematical community. Apart from the philosophical objections, mathematicians expressed concerns on the correctness of the programs employed, as programs have bugs, occasionally very costly ones. This issue is of course present in all computer-assisted proofs. How can one be absolutely certain that the implementation of a complex algorithm has no faults? Apart from conventional software testing, an answer to this question could be the solutions offered by the field of formal verification, where programs are checked for their correctness rigorously through a combination of purely mathematical methods and automated proof verifiers. Of course, the latter are also computer programs! Which raises the question: "Who will guard the guards themselves?" (translation of "*Quis custodiet ipsos custodes?*")

As for the the critique regarding the undermining of the analytical character of the mathematical proofs: John Von Neumann argues in his article ([33]) that mathematics contain a good deal of empiricism. He narrates a sequence of events, as described in the following paragraphs.

In the last decade of the 19th century, the discovery of contradictions in Cantor's naive set theory made it clear that mathematical results were not as indisputable as they were thought to be. This was one of the main causes that lead to Hilbert's program, an attempt to formalize mathematics, in order to save them from inconsistencies. This was a hefty task, considering the varying degrees of "robustness" in proofs of classical mathematics, which, in some cases, were very close to their roots on experience and experimentation. For example, von Neumann tells us that "Kepler's first attempts at integration were formulated as 'dolichometry' - measurement of kegs - that is, volumetry for bodies with curved surfaces". At the same time, a school of mathematics competitive to Hilbert's formalism, would attempt to fit mathematics into an intuitionistic system, to validate, in its own way, the past results.

However, both attempts did not succeed. At best, they had limited results. This failure was, to some extent, a result of Gödel's incompleteness theorems, by which he proved the following: if a system of mathematics does not lead into contradiction, then this fact cannot be demonstrated with the procedures of that system. As a result, the projects of justifying classical mathematics lost their main hope. However that did not stop mathematicians from using that system anyway. After all, it still worked in producing results that were both elegant and useful.

This story might serve as a reminder that it is hardly possible to believe in the existence of an absolute, immutable concept of mathematical rigour, dissociated from human experience. On the other hand, not all mathematical ideas originate in empirical subjects. In fact, a number of fields have been completely cut off from their empirical roots. Von Neumann's view on the matter is interesting:

> As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from "reality" it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely I'art pour I'art. This need not be bad, if the field is surrounded by correlated subjects, which still have closer empirical connections, or if the discipline is under the influence of men with an exceptionally well-developed taste. But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant branches, and that the discipline will become a disorganized mass of details and complexities. In other words, at a great distance from its empirical source, or after much "abstract" inbreeding, a mathematical subject is in danger of degeneration. [...] [W]henever this stage is reached, the only remedy seems to me to be the rejuvenating return to the source: the re-injection of more or less directly empirical ideas. I am convinced that this was a necessary condition to conserve the freshness and the vitality of the subject and that this will remain equally true in the future.

Perhaps a computer used as a tool for mathematical experimentation, but also for producing partial results, could provide even the most abstract mathematical field with a means of connecting with the world of experience.

1.2 Outline of this thesis

This thesis is organized as follows: in Chapter 2 we introduce basic elements of graph theory and graph minor theory. In Chapter 3 we give our definition of a *k-apex class* (or simply *k-apex*), explain a theorem that is crucial for our later analysis, and mention several important results on apex classes. Chapter 4 and Chapter 5 describe our contribution. More specifically, Chapter 4 includes 2 theorems on the degree and size of the obstructions of specific cases of k-apices of d-bounded classes. Chapter 5 is about our method in hunting the obstructions of $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$, $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}\}))$, $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}, K_3\}))$ and $\mathcal{A}_2(\mathbf{exc}(\{K_{1,2}\})$. Finally, we conclude in Chapter 6, with possible improvements of our work and a section about the emerging field of experimental mathematics.

Chapter 2

Graph Minors

2.1 Basic definitions and notation

We define a simple undirected graph G as a pair (V, E), where E consists of two-subsets of V. We call the members of V vertices, and the members of E edges. Also, the sets themselves are referred to as vertex set and edge set respectively. Throughout this thesis we will depict a graph as dots or circles (the vertices) connected by curves, mostly line segments (the edges). The vertex set (resp. edge set) of a graph G is referred to as $V(G)$ (resp. $E(G)$). Two important characteristics of a graph G are $n(G)$ and $m(G)$: we are referring to the sizes of $V(G)$ and $E(G)$ respectively. When the context can provide for unambiguity, we refer to these characteristics by simply the letters n and m.

Given $S \subseteq V(G)$, we define $G[S]$, the subgraph of G induced by S, as follows: $G[S] = (S, \{\{u, v\} \mid \{u, v\} \in \binom{S}{2} \cap E(G)\}$. We denote by $G \setminus S$ the graph $G[V(G) \setminus S]$. Given a graph G, we say that a vertex $u \in V(G)$ is adjacent to another vertex $v \in V(G)$ if and only if $\{u, v\} \in E(G)$. Throughout this paper the relation "adjacent" is over-loaded: we also say that a vertex u is adjacent to an edge e if $e = \{u, v\}$, for some $v \in V(G)$.

For this paragraph, we consider a graph G. Every edge or vertex mentioned belongs to G. We define the neighbourhood of vertex u, $N_G(u)$, as the set of all vertices in G that are adjacent to u. More generally, we define the neighbourhood of a set of vertices S in G, as $N_G(S) = \bigcup_{u \in S} N_G(u)$.

We also define the degree of a vertex u: $\mathbf{deg}_G(u) = |N_G(u)|$. A vertex is called isolated when it has degree zero. An edge is called isolated when its two adjacent vertices have degree 1. We define a (simple) path in G as a sequence of vertices v_i, $i \in \{1, 2, \ldots, k\}$ and edges e_i, $i \in \{1, 2, \ldots, k-1\}$ of the form $v_1, e_1, v_2, e_2, v_3, e_3, \ldots, e_{k-1}, v_k$, where $\forall\ i$, $e_i = \{v_i, v_{i-1}\}$ and $\forall\ i, j$, v_i and v_j are pairwise distinct. Vertices v_1 and v_k are called the endpoints of the path. Intuitively, it is indeed a path between two endpoints. The length of a path is the number of edges it includes. A cycle is defined as a path, but with the restriction of the first and last vertex being the same. A graph without cycles is called acyclic. The distance between two vertices u and v, is denoted $d\,(u, v)$ and is the length of the shortest path connecting u and v.

A graph is called *connected* if there exists a path in G between any 2 vertices in $V(G)$. If a graph is not connected, it has two or more connected components, which are defined as connected induced subgraphs of G. A connected acyclic graph is called a tree. A graph that whose connected components are all trees is called a forest.

We define the path graph P_n as the graph that consists of a simple path of length $n-1$. Similarly, we define the cycle graph C_n, as a graph that consists of only one cycle with n vertices. In addition, we define the clique of n vertices as K_n, that is, the graph of n vertices where every vertex is connected to every other.

Finally, given two graphs G_1 and G_2, we define $G_1 \cup G_2$, their disjoint union (sometimes called "union" in this thesis) as $G = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$.

2.2 Graph Minors

In this section we will provide the definitions required to understand the specific graph-theoretical problem we tackle in this thesis. First of all, we define some basic operations on graphs

1. The *deletion* of edge e in graph G, which results in $G \setminus e$, a graph differing from G only by edge e.

2. The *deletion* of vertex v in graph G, which results in $G \setminus v = G[V(G) \setminus \{v\}]$.

3. The *contraction* of edge $e = \{x, y\}$ in G, which results in a new graph G', where we replace x, y with a new vertex z, which is adjacent to every vertex in the $N_G(\{x, y\})$, the neighbourhood of x and y.

4. The *dissolution* of a degree-2 vertex v in G, where v is deleted from the graph, and it is substituted with an edge connecting its two neighbours. It is equivalent to the contraction of any of v's two edges.

The first two operations can produce every subgraph of a graph.

Consider the following fundamental theorem:

Theorem 1 (Kuratowksi's Theorem)**.** *A graph G is non-planar if and only if it contains a subgraph that can be reduced to either K_5 or $K_{3,3}$ through a number of vertex dissolutions.*

The above theorem is about the problem of planarity of a graph: a graph is planar when it can be drawn on a plane surface without any two edges crossing with each other. It is a theorem of great significance because it is presumably the first that presents us with a complete description of a class, namely the class of planar graphs, using a number of "forbidden" substructures. It was proven by Lev Pontryagin and – independently – by Kazimierz Kuratowski in 1930. A few years later Wagner proved a similar theorem, but this time using the notion of graph minors. What is a minor of a graph? In [24], an informal definition is given: "A minor of a graph G describes a substructure of G that is more general than a subgraph. If we take a subgraph of G and then contract some connected pieces in this subgraph to single points, the resulting graph is called a minor of G." Or to put it more formally

Definition 1. *A graph H is called a* minor *of another graph G if and only if H can be obtained from G through zero or more vertex deletions, edge deletions or edge contractions. We denote $H \leq_m G$.*

Finally, the result of Wagner, which uses the above term is

Theorem 2. *A graph G is planar if and only if it does not contain K_5 and $K_{3,3}$ as minors.*

The difference between this theorem and Kuratowski's one can be seen as a difference on the definition of edge contractions. Kuratowski uses *vertex dissolutions*, which is equivalent to contractions of edges adjacent to some vertex of degree 2. From this restricted version of a contraction, emerges the notion of the *topological minor*, which is a special type of minor.

These two theorems have been a breakthrough in the field of Graph Theory, as they have shown that it is possible to fully characterize empirically defined properties (such as planarity) by excluding a number of, possibly very simple, specific graph structures. The definition of a class of graphs through a number of "forbidden" substructures is called *forbidden graph characterization.* A lot of steps have been taken towards the discovery of forbidden graph characterizations (or Kuratowski characterizations, as they are sometimes called) of many fundamental graph classes. Our work in this thesis is essentially part of this endeavour.

2.3 More about Graph Minors

We will now examine more closely the minor relation. First of all the minor relation on graphs is reflexive (every graph G is a (zero-step) minor of itself) and transitive ($G_1 \leq_m G_2$ and $G_2 \leq_m G_3 \Rightarrow G_1 \leq_m G_3$). This makes it a quasi-order.

Definition 2. *An ordering is called a* quasi-ordering *if and only if it is reflexive and transitive.*

Definition 3. *A quasi-ordering $\preceq$ is called a* well-quasi-ordering *on a set X if and only if, for every infinite sequence $x_0, x_1, x_2 \ldots$ of elements of X, there exist two elements x_i and x_j such that $x_i \preceq x_j$.*

Wagner, in 1970, conjectured that the minor relation is a well-quasi-ordering on $\mathcal{G}$, the class of all graphs. Before we proceed to show how the proving of this conjecture played a major role in the research on graph minors and Kuratowski characterizations, we need to introduce the notion of the obstruction set.

Definition 4. *Given a graph class $\mathcal{C}$ that is closed under minors, its* minor-obstruction set $\mathbf{obs}_{\leq}(\mathcal{C})$ *is defined as the set of all minor-minimal graphs not in $\mathcal{C}$. The members of the minor-obstruction set are called* minor obstructions*.*

For the rest of this paper, when we refer to the notion of the obstruction or the obstruction set, without specifying further, we are always referring to minor-obstructions. Similarly, when we omit the index of **obs**, we are always referring to minor-obstructions.

Compare the definition of the minor-obstruction set with that of the Kuratowski characterization. It can be proved easily that the minor-obstruction set of a minor-closed class $\mathcal{C}$ coincides with its Kuratowski characterization.

Statement 1. *A graph G is a member of $\mathcal{C}$ if and only if it has an minor-obstruction of $\mathcal{C}$ as a minor.*

Proof. Equivalently, we prove that every graph in $\mathcal{C}$ has no obstruction of $\mathcal{C}$ as its minor and every graph not in $\mathcal{C}$ has an obstruction of $\mathcal{C}$ as its minor. The first statement is easy to prove, because of the property of $\mathcal{C}$ of being closed under minors. Now, we proceed to the second statement. Every graph G not in $\mathcal{C}$ is either an obstruction of $\mathcal{C}$, or it is not. We will prove that, in both cases, G has an obstruction as a minor. If it is an obstruction, then it is a minor of itself, and we are done. If not, consider the following recursive process: 1. G is not minor-minimal, therefore it must have a (strict) minor H that is not in $\mathcal{C}$. 2. If H is minor-minimal then it must be an obstruction. Which means that we have found a minor that is an obstruction, and we stop here. Otherwise, substitute G with H and re-do step 1. The above process will always terminate, since graph G is finite. □

We also need to define the inverse of the **obs**() function.

Definition 5. *Let $\mathcal{S}$ be a set of graphs. We define* $\mathbf{exc}(\mathcal{S})$ *as the class of graphs who do not have any graph in $\mathcal{S}$ as a minor.*

In 2004, Wagner's conjecture was proved to be true by Robertson and Seymour, after twenty long papers, starting in 1983. This was an extremely important breakthrough in the field of structural graph theory.

Theorem 3 (Robertson-Seymour Theorem, 2004 [28])**.** *Graphs are well-quasi-ordered by the graph minor relation.*

In other words, the Robertson-Seymour theorem (abbreviated RST), states that there is no infinite antichain for the minor relation, by definition of the well-quasi-orderings. The obstruction sets, defined above, are antichains in the minor relation. Therefore, all obstruction sets of minor-closed classes are finite. The theorem that follows is another powerful result of Robertson and Seymour.

Theorem 4 (Robertson and Seymour, 1995 [27])**.** *For any two graphs G, H there exists an $O(n^3)$ algorithm that decides if $H \leq_m G$, where $n = n(G)$.*

Combining this theorem with the RST, we get a $O(n^3)$ membership algorithm for every minor-closed graph class $\mathcal{C}$ - simply by checking if any of the $|\mathbf{obs}(\mathcal{C})|$ forbidden minors of $\mathcal{C}$ is a minor of the candidate graph. This result has a significant impact on complexity theory too, as it implies a priori knowledge that every decision problem that can be formulated as a membership problem of a graph to some minor-closed class is in P.

Unfortunately, things are not so bright regarding the task of finding the obstructions of minor ideals (another term for minor-closed classes). Theorem 3 is not constructive and in fact, it cannot be ([13])! In addition, the obstruction sets may have tremendously large sizes. Already, a number of results indicate this, as shown in the following chapter. A large amount of work is done only to partially characterize some obstruction sets, while other essays focus on bounding the size of obstructions of certain minor ideals. Of course, the complete obstruction sets of several graph classes have been found, and for others, the obstruction hunting continues. Not surprisingly, the interest in forbidden minors has greatly increased after the advent of the RST.

In the previous paragraph, we introduced the term "minor ideal". A more general term, "lower ideal", is used to describe graph families that are closed relative to a partial order.

2.4 Obstructions: beyond minors

The notion of the obstruction set has been generalized to include more partial orders. For example, we have seen, in a number of papers, a rising interest on obstruction sets with respect to immersion order (e.g. [14]), the topological order, as well as the subgraph order (e.g. [10]). We denote $\mathbf{obs}_{\leq}(\mathcal{C})$, $\mathbf{obs}_{\leq_{im}}(\mathcal{C})$, $\mathbf{obs}_{\leq_{tp}}(\mathcal{C})$, $\mathbf{obs}_{\subseteq}(\mathcal{C})$ the obstruction set of a class $\mathcal{C}$ with respect to the minor order, the immersion order, the topological order, and the subgraph order respectively.

Apart from minor obstructions, in this essay we present new results on subgraph obstructions, i.e. the obstructions defined with respect to the subgraph order. For a given class $\mathcal{C}$, $\mathbf{obs}_{\subseteq}(\mathcal{C})$ is not always finite, in contrast with $\mathbf{obs}_{\leq}(\mathcal{C})$. For example, the set of acyclic graphs is not well-quasi-ordered by the subgraph relation, as its obstruction set must contain the infinite antichain of C_n graphs, for all positive n. Fortunately, there exists a theorem by Ding [15] that characterizes classes well-quasi-ordered by the subgraph relation:

Theorem 5 (Theorem (Ding 1992))**.** *A graph class $\mathcal{C}$ is a subgraph ideal if and only if $\mathcal{C}$ contains finitely many C_n and F_n graphs (paths of n vertices with two pendant edges attached to each of their endpoints).*

Chapter 3

Hunting obstructions of Apex classes

In this chapter we describe the notion of an apex class and we provide the background required to understand the results of this essay. Also, we present some important results regarding the obstruction sets of apex classes.

Definition 6. *Let $\mathcal{F}$ be a class of graphs. Given an integer $k \geq 0$, we define the k-*apex class *of $\mathcal{F}$, or simply k-*apex *of $\mathcal{F}$, denoted $\mathcal{A}_k(\mathcal{F})$, as*

$$\{G \in \mathcal{G} \,|\, \exists S \subseteq V(G) : |S| \leq k \wedge G \setminus S \in \mathcal{F}\}$$

In addition:

1. *$\mathcal{A}_0(\mathcal{F}) = \mathcal{F}$.*
2. *$\mathcal{F}$ is called the* base class *of $\mathcal{A}_k(\mathcal{F})$.*
3. *In a graph $G \in \mathcal{A}_k(\mathcal{F})$, any set of at most k vertices S such that $G \setminus S \in \mathcal{F}$ is called an* apex set *with respect to $\mathcal{F}$ and the vertices in S are called* apex vertices*. When there is no possibility of ambiguity, we simply use the terms apex set/vertex.*

Note that this is a generalized version of older definitions. In the Wikipedia article titled "apex graph", the term is used to describe $\mathcal{A}_1\{\textit{planar graphs}\}$. Then, there is a more general definition, which is not limited to planar graphs,

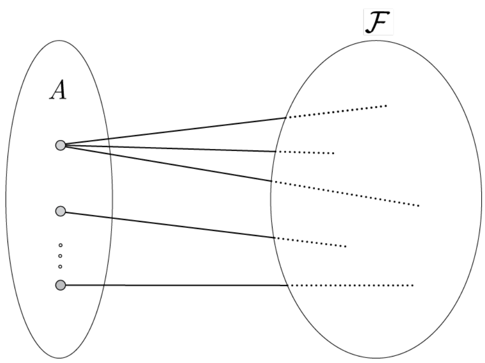


Figure 3.1: The deletion of the apex vertices (left) leaves us with a graph in $\mathcal{F}$

but to any graph class C, with the notation *apex*-$\mathcal{C}$, which in our language, means $\mathcal{A}_1(\mathcal{C})$. Our general definition for apices has also been found in other papers, but with different names: Graphs within k vertices of $\mathcal{C}$, $\mathcal{W}_k(\mathcal{C})$ as well as k-$\mathcal{C}$-deletion graphs are what we call $\mathcal{A}_k(\mathcal{C})$.

Apex graphs themselves belong to the family of parametrized graph classes, which are discussed by Dinneen in [8]. His results about disconnected obstructions are crucial for the obstruction hunt of this essay.

3.1 Disconnected Obstructions

3.1.1 Obstructions as forbidden minors

In this section we will explain the results of Dinneen M.J. [8] about disconnected obstructions. He studied classes defined by a parameter k and a function on graphs, $\lambda() : \mathcal{G} \Rightarrow \mathbb{N}$, as follows:

$$\mathcal{F}[k] = \{G \mid \lambda(G) \leq k\}$$

Additionally, he posed the following restrictions on $\lambda()$:

1. For graphs G_1, G_2, $\lambda(G_1 \cup G_2) = \lambda(G_1) + \lambda(G_2)$
2. For any minor H of G, $\lambda(H) \leq G$
3. Every graph G has a minor H such that $\lambda(H) \geq \lambda(G) - 1$

In this setting, he proved the following theorem about $\mathcal{F}[k]$.

Theorem 6. *(Dinneen 1997) If $G = C_0 \cup C_1$ is an obstruction for $\mathcal{F}[k]$, $k \geq 0$, then each C_i is an obstruction for $F[\mathbf{ad}_{\mathcal{F}}(C_i) - 1]$.*

Furthermore, it is easy to prove that, for graphs C_0,C_1, obstructions for $\mathcal{F}[k_0], \mathcal{F}[k_1]$ respectively, graph $G = C_0 \cup C_1$ is an obstruction of $\mathcal{F}[k_0+k_1-1]$. As a consequence, we have the following powerful corollary of Theorem 6.

Corollary 6.1. *The set of disconnected obstructions of $\mathcal{F}[k]$, is exactly all graphs of the form $G = C_1 \cup C_2 \cup \ldots C_m$, where C_i is a connected obstruction of $\mathcal{F}[k_i]$, such that $k_i < k$ and $\sum_{1 \leq i \leq m} (k_i + 1) = k + 1$.*

Thanks to these results, we are able to focus on the search of connected obstructions for the apex classes under examination. The disconnected ones will simply emerge from the obstructions of lower order classes.

3.1.2 Obstructions as forbidden subgraphs

Here we will modify the results of Dinneen so as to describe disconnected subgraph obstructions. For the purposes of this section, the term "obstruction" will always refer to a subgraph obstruction.

Suppose we have $\mathcal{A}_k(\mathcal{F})$, the class of graphs within k vertices of some class $\mathcal{F}$, where $\mathbf{obs}_{\subseteq}(\mathcal{F})$ is finite and $\mathcal{F}$ closed under the disjoint union of graphs. From this point onwards we will use $\mathcal{F}$ to denote a graph obeying these two restrictions.

The question here is: under what circumstances is $\mathbf{obs}_{\subseteq}(\mathcal{A}_k(\mathcal{F}))$ finite? We conjecture that it is always finite. However, we cannot prove it, so instead we take a different path. We will proceed to prove that if we have finitely many connected obstructions of $\mathcal{A}_k(\mathcal{F})$, then, the cardinality of $\mathbf{obs}_{\subseteq}(\mathcal{F})$ is finite too.

To do this, we define

$$\mathbf{ad}_{\mathcal{F}}(G) = \min\{|V'| : G \setminus V' \in \mathcal{F}, \text{ where } V' \subseteq V\}.$$

We will work with this function exactly as Dinneen has worked with $\lambda()$ (see previous section). With this function we express the *apex distance* from $\mathcal{F}$

of G. Using this function, we can give an equivalent definition of the apex graph operator:

$$\mathcal{A}_k(\mathcal{F}) = (G \in G \mid \mathbf{ad}_{\mathcal{F}}(G) \le k)$$

Following the flow of [8], we prove the following lemma.

Lemma 1. *Let $\mathcal{F}$ be a class of which is closed under the disjoint union of graphs. For any graph $G \in \mathcal{F}$ we have the following properties*

1. $\mathbf{ad}_{\mathcal{F}}(G_1 \cup G_2) = \mathbf{ad}_{\mathcal{F}}(G_1) + \mathbf{ad}_{\mathcal{F}}(G_2)$

2. *For $H \subseteq G$, $\mathbf{ad}_{\mathcal{F}}(G) \le \mathbf{ad}_{\mathcal{F}}(G)$*

3. *There exists a subgraph $H \subseteq G$, $\mathbf{ad}_{\mathcal{F}}(H) \ge \mathbf{ad}_{\mathcal{F}}(G) - 1$.*

Proof. For proposition (1): Clearly, $\mathbf{ad}_{\mathcal{F}}(G_1 \cup G_2) \le \mathbf{ad}_{\mathcal{F}}(G_1) + \mathbf{ad}_{\mathcal{F}}(G_2)$ since if $G_1 \setminus V_1 \in \mathcal{F}$ and $G_2 \setminus V_2 \in \mathcal{F}$ then $(G_1 \cup G_2) \setminus V_{1,2} \in \mathcal{F}$. Likewise, $\mathbf{ad}_{\mathcal{F}}(G_1) + \mathbf{ad}_{\mathcal{F}}(G_2) \le \mathbf{ad}_{\mathcal{F}}(G_1 \cup G_2)$ as, if $(G_1 \cup G_2) \setminus V_{1,2} \in \mathcal{F}$, then $G_1 \setminus V(G_1) \cap V_{1,2} \in \mathcal{F}$ and $G_2 \setminus V(G_2) \cap V_2 \in \mathcal{F}$.

For proposition (2): If $G \setminus V_{\text{apex}} \in \mathcal{F}$, and $G' \subseteq G$, then $G' \setminus (V(G') \cap V_{\text{apex}}) \subseteq G \setminus V_{\text{apex}}$, and since $\mathcal{F}$ is closed under the subgraph relation, $G' \setminus (V(G') \cap V_{\text{apex}}) \in \mathcal{F}$. Therefore $\mathbf{ad}_{\mathcal{F}}(G') \le |V(G') \cap V_{\text{apex}}| \le |V_{\text{apex}}| \le \mathbf{ad}_{\mathcal{F}}(G)$.

For proposition (3): It follows from the fact that, for all subgraphs H of G that emerge from the deletion of one vertex, $\mathbf{ad}_{\mathcal{F}}(H)$ can be at most one less than $\mathbf{ad}_{\mathcal{F}}(G)$. □

Now, we can proceed to prove that:

Theorem 7 (Theorem). *If $G = C_0 \cup C_1$ is an obstruction for $\mathcal{A}_k(\mathcal{F})$, then each C_i is an obstruction for $\mathcal{A}_{\mathbf{ad}_{\mathcal{F}}(C_i)-1}$. Conversely, every disjoint union of obstructions of lower order apex classes, i.e. every graph $G = C_0 \cup C_1$, where every C_i is an obstruction of $\mathcal{A}_{\mathbf{ad}_{\mathcal{F}}(C_i)-1}(\mathcal{F})$, is an obstruction for $\mathcal{A}_{\mathbf{ad}_{\mathcal{F}}(C_1)+\mathbf{ad}_{\mathcal{F}}(C_2)-1}(\mathcal{F})$.*

Proof. We will prove the first part by contradiction. Suppose one C_i is not an obstruction for $\mathcal{A}_{\mathbf{ad}_{\mathcal{F}}(C_i)-1}$. Then, C_i has a subgraph C_i' for which $\mathbf{ad}_{\mathcal{F}}(C_i') =$

$\mathbf{ad}_{\mathcal{F}}(C_i)$. Furthermore, $C_i' \cup C_{1-i}$ is a subgraph of G. Finally, we can deduce that

$$\mathbf{ad}_{\mathcal{F}}(C_i' \cup C_{1-i}) = \mathbf{ad}_{\mathcal{F}}(C_i') + \mathbf{ad}_{\mathcal{F}}(C_{1-i}) \Rightarrow$$

$$\mathbf{ad}_{\mathcal{F}}(C_i' \cup C_{1-i}) = \mathbf{ad}_{\mathcal{F}}(C_i) + \mathbf{ad}_{\mathcal{F}}(C_{1-i}) \Rightarrow$$

$$\mathbf{ad}_{\mathcal{F}}(C_i' \cup C_{1-i}) = \mathbf{ad}_{\mathcal{F}}(C_0 \cup C_1) \Rightarrow$$

$$\mathbf{ad}_{\mathcal{F}}(C_i' \cup C_{1-i}) = \mathbf{ad}_{\mathcal{F}}(G)$$

which is impossible, as G is an obstruction.

Now, we will prove the second statement. Let $G = C_0 \cup C_1$. We will prove that, firstly, G is outside of $\mathcal{A}_{\mathbf{ad}_{\mathcal{F}}(C_1)+\mathbf{ad}_{\mathcal{F}}(C_2)-1}(\mathcal{F})$, and secondly, that whatever edge or vertex we delete from G, we get a graph of lower distance from $\mathcal{F}$. Obviously, since $\mathbf{ad}_{\mathcal{F}}(G) = \mathbf{ad}_{\mathcal{F}}(C_0) + \mathbf{ad}_{\mathcal{F}}(C_1)$, G is outside the aforementioned class. Now, consider the removal of an edge e from G. Without loss of generality, suppose $e \in E(C_0)$. Hence, $G \backslash e$ is isomorphic to $G' = \{C_0 \backslash e\} \cup C_1$. But $\mathbf{ad}_{\mathcal{F}}(G') = \mathbf{ad}_{\mathcal{F}}(C_0 \backslash e) + \mathbf{ad}_{\mathcal{F}}(C_1) \leq \mathbf{ad}_{\mathcal{F}}(C_0) - 1 + \mathbf{ad}_{\mathcal{F}}(C_1)$. In other words, $\mathbf{ad}_{\mathcal{F}}(G') \leq \mathbf{ad}_{\mathcal{F}}(G) - 1$, so G' is inside the aforementioned class. □

The following is corollary of Theorem 7.

Corollary 7.1. *Let* $G = \bigcup_{i \in \{1,\ldots,r\}} C_i$ *be a graph which consists of* r *maximal connected components, which we call* C_i*.* G *is an obstruction for* $\mathcal{A}_k(\mathcal{F})$*, if and only if every* C_i *is a obstruction of* $\mathcal{A}_{\mathbf{ad}_{\mathcal{F}}(C_i)-1}(\mathcal{F})$*.*

Proof. Apply Theorem 7 $r - 1$ times. □

Corollary 7.2. *For every* k*,* $\mathcal{A}_k(\mathcal{F})$ *has finitely many disconnected obstructions, if* $\mathcal{A}_{k'}(\mathcal{F})$ *has finitely many connected obstructions, for all* $k' < k$*.*

Proof. Using Corollary 7.1 above, we see that every disconnected obstruction of $\mathcal{A}_k(\mathcal{F})$ is a disjoint union of connected obstructions of $\mathcal{A}_{k'}(\mathcal{F})$ where $k' < k$. Due to this, we would have infinite distinct graphs that are combinations of connected obstructions of previous classes. This is impossible, as the connected obstructions of previous classes are finitely many, so their powerset is finite too. □

Corollary 7.3. *The disconnected subgraph obstructions for* $\mathcal{A}_k(\mathcal{F})$ *are exactly the set of all graphs* $G = \bigcup_{i \in \{1,\ldots,r\}} C_i$*, each* C_i *being an obstruction of* $A_{k_i}(\mathcal{F})$*, such that* $k_i < k$ *and* $\sum_{1 \le i \le r}(k_i + 1) = k + 1$*.*

Observe that we have come to the exact same conclusion for subgraph obstructions, as the corollary of Dinneen for minor obstructions. This permits us to use the same method to find both subgraph and minor obstructions for class $\mathcal{A}_k(\mathcal{F})$.

3.2 Well-known results on the obstructions of apex classes

Although the Graph Minor Theorem (Theorem 3) promises that the obstruction sets of minor ideals are finite, there is no other guarantee about their size. Examples show that, for many k-parametrized parametrized families, the number of obstructions grows very fast. For example, graphs with pathwidth up to k, we have 2 obstructions for $k = 1$, 110 for $k = 2$ and more than 60 million for $k = 3$ [8].

In the case of k-apices of lower ideals closed under disjoint union, such as the classes treated in this work, the obstructions grow superpolynomially as k increases, else the polynomial time hierarchy collapses to $\sum_3^P$ [8]. This result should not surprise us because, as shown in the previous section, only the disconnected obstructions of $F[k]$ are at least as many as the partitions of $k + 1$.

A number of apex minor ideals have been studied deeply throughout the last 20 years. One is the class of graphs admitting a k-Vertex Cover (see the next chapter for more details), equivalently defined as $\mathcal{A}_k(\mathbf{exc}(\{K_{1,1}\}))$. Then, there is the class of graphs that have a k-Feedback Vertex Set. A feedback vertex set of a graph G is a subset of $V(G)$ whose deletion will leave the graph without cycles. Equivalently, it is defined as $\mathcal{A}_k(\mathbf{exc}(\{C_3\})$. Here, for $k = 1, 2$ the forbidden minors are known. Also, all outerplanar obstructions of this class can be effectively enumerated, for all $k \geq 1$ ([30]).

More results include the obstruction set of $\mathcal{A}_1(\{$*outerplanar graphs*$\})$ [6] as well as of $\mathcal{A}_1(\{$*cactus graphs*$\})$ [11]. In [11], we are informed that the hunt for the obstructions of $\mathcal{A}_1((\{\textit{series-parallel graphs}\})$ is underway. Also, a group of students from NKUA, Athens, Greece, have found the obstructions for $\mathcal{A}_1(\{\textit{monocyclic}\})$ (personally communicated). However, despite all this progress, we have yet to see the obstruction set of $\mathcal{A}_1(\{\textit{planar graphs}\})$ in its entirety. We only have partial results about it ([3], [21],[26]). In [26], it is explained how various techniques were used to find 109 obstructions of this class.

Finally, apart from these class-specific results, we have 2 more general ones. One is that of Adler, Grohe and Kreutzer ([1]) which states that there is an algorithm that given a finite set of graphs $\mathcal{F}$ and a non-negative integer k, outputs $\mathbf{obs}(\mathcal{A}_k(\mathbf{exc}(\mathcal{F})))$. The second is from [25], where it is proven that for any bounded-degree, minor-closed, disjoint-union-closed class C, the size of $\mathcal{A}_k(C)$ is $O(tk^7 + t^7k^2)$, where t is a bound on the size of obstructions for C.

Chapter 4

Bounding the size of obstructions for $\mathcal{A}_k^{(d)}$

4.1 Class $\mathcal{A}_k^{(d)}$

Definition 7 ($\mathcal{A}_k^{(d)}$)**.** *Let $\mathcal{B}_d$ be the class of* all *graphs with maximum degree up to $d-1$. We define the parametrized class $\mathcal{A}_k^{(d)} = \mathcal{A}_k(\mathcal{B}_d)$, with parameters k and d.*

In this chapter we establish some bounds on the obstruction size of $\mathcal{A}_k^{(d)}$. Some special cases of this parametrized class have already been extensively studied. In fact, if we set parameter d to 1, we get a special case of the well-known Vertex Cover decision problem:

Vertex Cover
Instance: Graph G and integer k
Question: Does G have a vertex cover of size at most k?

To remind the reader, a vertex cover is defined as a set of vertices $S \subseteq V(G)$ such that all edges in G are adjacent to at least one of the vertices in S. It is easy to see that this problem is equivalent to the following:

Vertex Cover (equivalent definition)
Instance: Graph G and integer k

Question: Can G be reduced to a graph without edges, after the deletion of at most k vertices?

This problem is NP-Complete [18]. On the contrary, if we fix k, then the k-Vertex Cover decision problem admits a polynomial solution, as it is Fixed Parameter Tractable with k as the parameter.

k-Vertex Cover Problem

Instance: Graph G

Question: Can G be transformed to a graph without edges, after the deletion of at most k vertices?

The fact that this problem is in P can be independently proved by using the fact that the graphs that have a vertex cover of size k are closed under taking minors and therefore, have a finite obstruction set. Indeed, one way to solve the problem would be to test if any obstructions for the class of graphs with a vertex cover of size at most k is a minor of the input graph G. This solution has polynomial time complexity, as testing if a graph is a minor of another can be done in polynomial time, and the number of obstructions is a constant. Nevertheless, this is not the way to go, if one wants to solve the problem efficiently. For large values of k, the number of obstructions may be too many (e.g. for $k > 100$, we have more than 190 million non-connected graphs [8]) and the known polynomial algorithm that checks the minor relation between two graphs has astronomically large hidden constants. The obstruction sets of this parametrized class have been found for all $1 \leq k \leq 7$, in a series of papers [4] [9] [7]. The techniques employed are all based on the same method: first bounding certain parameters of the obstructions (e.g order and maximum degree) and then performing an exhaustive search on the finite search space defined by these bounds. Up to date, the obstruction set for $k = 8$ remains unknown.

In Chapter 5, we find the obstruction sets of 3 instances of $\mathcal{A}_k^{(d)}$. These are illustrated in Table 4.1. Furthermore, note that, for $d \in \{1, 2, 3\}$, $\mathcal{A}_k^{(d)} = \mathcal{A}(K_{1,d})$. Therefore, using similar arguments, we can also study graph class

	$d=1$	$d=2$	$d=3$	$d=4$	...
$k=1$	2 obstructions, found in [5]	2 connected and 1 disc. obstruction	12 connected + 1 disc. obstruction		
$k=2$	4 obstructions, found in [5]	13 connected + 3 disconnected			
$k=3$	8 obstructions, found in [5]				
$k=4$	18 obstructions, found in [5]				
$k=5$	56 obstructions, found in [5]				
$k=6$	260 obstructions, found in [9]				
$k=7$	2250 obstructions, found in [7]				
$k=8$					
...					

Table 4.1: The number of obstructions $\mathcal{A}_k^{(d)}$, where found. Cells of the green column have obstruction size at most (k^2+4k+3),the rest of the cells are have a higher bound of $k(d+1)(d+k)+2d+k$.

$\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}, C_3\})$, which is a subclass of $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}\})) = A_1^{(3)}$ (observe that we only add acyclicity as desired property).

4.2 A general bound on obstruction size

In this section, we prove some upper bounds on the size and degree of obstructions for the parametrized class we are discussing. Note that, the theorems proved in this section, as well as in the following, refer to *both subgraph obstructions as well as minor obstructions*, as they include only edge deletions in their arguments.

For our first theorem, it is necessary to prove some useful lemmas.

Lemma 2. *Let G be a subgraph or minor obstruction of $\mathcal{A}_k^{(d)}$. Then, there does not exist a pair of adjacent vertices $u, v \in V(G)$ such that $\mathbf{deg}_G(u) < d$ and $\mathbf{deg}(v) < d$.*

Proof. We will prove the statement by contradiction. Suppose there exist two adjacent vertices $u, v \in V(G)$ with $\mathbf{deg}(u) < d$ and $\mathbf{deg}(v) < d$. We denote $e = \{u, v\}$. As G is an obstruction, there exists a set A_e of k vertices such that $G' = \{G \setminus e\} \setminus A_e$ has maximum degree at most d-1. Note that in G', $\mathbf{deg}(u) \leq d-2$ and $\mathbf{deg}(v) \leq d-2$ (due to the deletion of e). Now consider $G'' = G' + e$, the graph that emerges after re-inserting e in G'. The re-insertion of e does not affect any other vertex other than u and v, which now have their degree incremented by one (which is still small enough). This graph has also maximum degree at most d-1. Observe that this graph is $G \setminus A_e$, so G is in $\mathcal{A}_k^{(d)}$), which contradicts our hypothesis that G is an obstruction. □

Lemma 3. *The independence number of a graph G is not greater than $n(G) - \frac{m(G)}{\Delta(G)}$.*

Proof. Suppose $I \subseteq V(G)$ is an independent set in G. Then, $S = V(G) \setminus I$ is a vertex cover of G, as every edge of I is adjacent to some vertex in S (otherwise it would not be an independent set). And since S is a vertex cover, we get

$$m \leq |S| \cdot \Delta(G) \Leftrightarrow$$

$$|S| \geq \frac{m}{\Delta(G)} \Leftrightarrow^{(S=V(G)\setminus I)}$$

$$|I| \leq n - \frac{m}{\Delta(G)}$$

∎

We now have all the pieces required to prove our result:

Theorem 8 (Theorem on the order and degree of obstructions of $\mathcal{A}_k^{(d)}$). *Let G be a (subgraph or minor) obstruction of $\mathcal{A}_k^{(d)}$. Then:*

1. $\Delta(G) \leq k + d$
2. $V(G) \leq k(d+1)(d+k) + 2d + k$

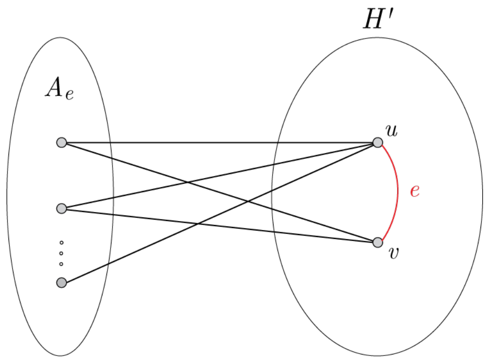


Figure 4.1: The deletion of e and a set of apex vertices with respect to e (A_e, left) leaves us with a graph of bounded degree.

Proof. For the first statement: we will prove that for every $u \in V(G)$, $\mathbf{deg}_G(u) \leq d + k$. Take $u \in V(G)$. Randomly select a neighbour v of u. If it has no neighbours we are done, as $\mathbf{deg}_G(u) = 0 \leq d + k$.

Otherwise, remove edge $e = \{u, v\}$ from G. We denote $H = G \setminus e$. Then, there exists a set of at most k apex vertices whose removal results into H', an induced subgraph of H, where $\Delta(H') \leq d - 1$ (see Figure 4.1). Note that, u or v cannot be among the k apex vertices of H, because if they were, then the same vertices would be apex vertices for G, and G would not be an obstruction.

Since the re-addition of these vertices to the graph increases u's degree by the number of edges coming from apex vertices to u, we get that $\mathbf{deg}_H(u) \leq \mathbf{deg}_{H'}(u) + k$. Therefore, $\mathbf{deg}_H(u) \leq d - 1 + k$. Finally, since $\mathbf{deg}_G(u) = \mathbf{deg}_H(u) + 1$, $\mathbf{deg}_G(u) \leq d - 1 + k + 1 = d + k$. This concludes our proof that every vertex u has degree $\mathbf{deg}_G(u) \leq d + k$, and therefore $\Delta(G) \leq d + k$.

For the second statement: We will first prove the statement for connected obstructions.

Consider a slight variant of H', $H'' = H' \cup e$. Let us take a closer look at H''. We will split the vertices in H'', (equivalently, the vertices in $V(G) \setminus A_e$) into two sets (see Figure 4.2): S_1, the vertices adjacent to A_e, plus the vertices of e, u and v. And S_2, the rest of the vertices, which are all non-adjacent to A_e. We will try to estimate the size of every set.

1. The size of S_1 is at most $2 + k \cdot \Delta(G)$. The first summand represents u and v, which might or might not be adjacent to A_e. As for the rest of the vertices in S_1, we get $k \cdot \Delta(G)$ vertices at most. This holds because we have k vertices in A_e, each of which is adjacent to at most $\Delta(G)$ vertices. Taking into consideration the first part of the theorem, we get $|S_1| \leq k(k + d)$ (1).

2. Notice that S_2 contains vertices whose degree does not change after the deletion of the vertices in A_e. Also, neither u nor v is here. Thus, $\forall\, u \in S_2$, $\mathbf{deg}(u) \leq d - 1$. In addition, due to Lemma 2, S_2 is an independent set in G, therefore in H'' too. Finally, using lemma 3, we know that for every independent set I in H'', $|I| \leq n(H'') - \frac{m(H'')}{\Delta(H'')}$. To use this inequality, we will bound $m(H'')$ from below and $\Delta(H'')$ from above. We have to consider that $m(H'')$, in a graph of fixed order, is equal to at least $n(G) - l$, where l is the number of maximal connected components. The removal of k vertices from a graph can break the it into at most $k \cdot \Delta(G)$ connected components, so H'' has at most $k(k+d)$ maximal connected components. Also, $\Delta(H'') = d$. Eventually, we get that $|S_2| \leq n(H'') - \frac{n(H'')-k(k+d)}{d}$ (2).

Now we can join all the parts together: $|V(G)| = |A_e| + |V(H'')| = |A_e| + |S_1| + |S_2|$. We define $n = V(G)$. Considering that $n(H'') = n - k$, as well as equations (1) and (2) we get

$$n \leq k + 2 + k(k + d) + (n - k) - \frac{n - k - k(k + d)}{d} \Leftrightarrow$$

$$\frac{n - k - k(k + d)}{d} \leq 2 + k(k + d) \Leftrightarrow$$

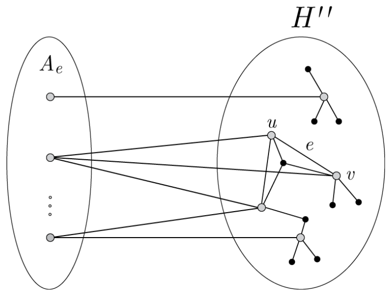


Figure 4.2: In H'' we split the vertices into sets S_1 (gray vertices) and S_2 (black vertices)

$$n - k - k(k+d) \leq 2d + kd(k+d) \Leftrightarrow$$
$$n \leq k(d+1)(d+k) + 2d + k$$

For the case of disconnected obstructions, it suffices to recall the results by Dinneen in the previous chapter: every disconnected obstruction can be written as $G = \bigcup_{i=1}^{r} C_i$, where every C_i is an obstruction of $\mathcal{A}_{k_i}^{(d)}$, such that $0 \leq k_i < k$ and $\sum_{i \in \{1,\ldots,r\}} (k_i + 1) = k + 1$. As the bound for the connected case is a superadditive function of k, we can assert that the disconnected obstructions are bounded from above by same function. □

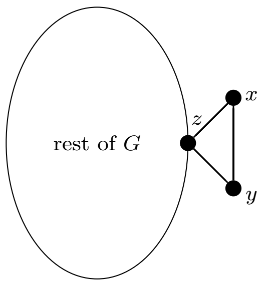
This bound is, to the best of our knowledge, new. It a better, but less general version of the bound in [25], which is $O(tk^7 + t^7k^2)$, where t is a bound on the size of obstructions for C.

4.3 An improvement of the bound, for $d = 2$

In this section we will work on the special case $d = 2$. Once more, we begin with some useful lemmas.

Lemma 4. *Let G be a subgraph obstruction or minor obstruction of $\mathcal{A}_k^{(2)}$. Then G cannot have a cycle of three vertices x, y, z, out of which 2 have degree equal to exactly 2.*

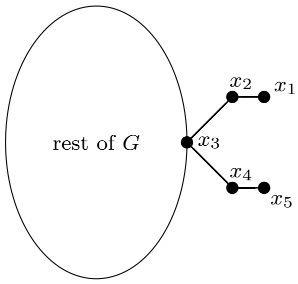
Figure 4.3: The "triangular tail" is forbidden in the obstructions of $\mathcal{A}_k^{(2)}$.



Proof. Let x, y be the vertices of the triangle with degree $\mathbf{deg}_G(x) = \mathbf{deg}_G(y) = 2$. The removal of $e = \{x, y\}$ from the graph makes it possible to reduce the maximum degree to 1, after the removal of the set of apex vertices, A_e, This means that one of x, y or z belongs to A_e, otherwise z would still have two neighbours, namely x and y, hence a degree of at least 2. However, the deletion of any of these vertices from G would leave x and y with a degree of 1, or it would completely remove the edge. This means that, since $\{G \setminus e\} \setminus A_e$ has maximum degree at most 1, so has $G \setminus A_e$. This is of course a contradiction, as G is an obstruction of $\mathcal{A}_k^{(2)}$. □

Lemma 5. *Let G be a (subgraph or minor) obstruction of $\mathcal{A}_k^{(2)}$. Then G cannot have a P_5 subgraph where only the central vertex has degree in G greater than 2 and whose endpoints have both degree 1 in G. ("double tail")*

Figure 4.4: The "double tail" is forbidden in obstructions of $\mathcal{A}_k^{(2)}$.



Proof. As in the previous proof, we choose an edge e and prove that the deletion of A_e reduces the graph's degree sufficiently, even without the deletion of e. Let e be $\{x_1, x_2\}$. One of x_3, x_4, x_5 must be a member of A_e, otherwise x_4 will not have a degree of 1 or 0 after the deletion of e and A_e. Without loss of generality, we suppose x_3 is a member of A_e (if any of x_4, x_5 was a member of A_e instead of x_3, we could replace it with x_3 in the apex set, maintaining the property of being an apex set). However, once more, we see that there is no need to delete e; simply by deleting the vertices in A_e we reduce the maximum degree of G to 1, which contradicts G being an obstruction. □

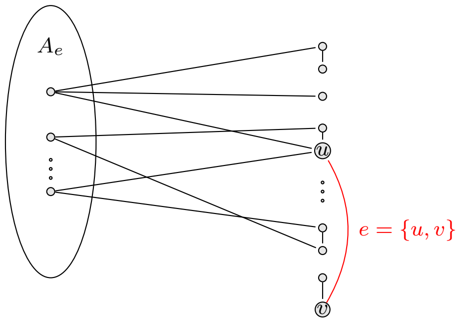
Theorem 9. *Let G be a (subgraph or minor) obstruction of $\mathcal{A}_k^{(2)}$. Then G has at most $k^2 + 4k + 3$ vertices.*

Proof. Similarly to the previous proof, we only need to argue for the connected obstruction size.

Here we will attempt, after the selection of an edge e, to count the number of vertices in graph $H = G \setminus A_e$. How does this graph look like? We can see it roughly depicted in Figure 4.5.

Observe that H, on the right side of the figure, obviously consists of the connected component of e, plus a number of isolated edges or vertices.

Figure 4.5: The removal of e results in a graph which can be partitioned in A_e and a disjoint union of P_2 and P_1 graphs



Equivalently, H is a graph where all vertices apart from u, or v or both have degree zero or one. As G is connected, every connected component in H must have at least one neighbour in A_e. On the other hand, A_e has at most $k(k+2)$ components, since $\Delta(G) = k + d$, as shown in Theorem 8. This means that we cannot have more than $k(k+2)$ connected components in H, so, we cannot have more than $2k(k+2)+2 = 2k^2+4k+2$ vertices in H (the last summand is for the extra vertices in the connected component of e). Is this bound tight? Can we improve it?

We will examine H a bit more closely. Suppose that the number of isolated edges (resp. vertices) with only one neighbour in A_e is p_1 (resp. l_1), and that the number of isolated edges (resp.vertices) with more than one neighbours in A_e is p_2 (resp. l_2).

We can show that an isolated edge having only one neighbour in A_e shares only one edge with this neighbour (i.e not both vertices of the isolated edge are connected with the neighbour). The contrary is impossible because of Lemma 4. Also, consider that no two isolated edges with only one neighbour can share this one neighbour in A_e, as this would form a subgraph forbidden by Lemma 5. As a result, every apex vertex can be connected to at most one isolated edge with only one neighbour in A_e. Consequently, $p_1 \leq k$.

As regards p_2, l_1 and l_2, we observe that they correspond to components that have at least as many neighbours in A_e as the number of vertices they consist of. Therefore, in total, the number of edges going to these components are equal or more than the number of vertices in these components.

Finally, we will count how many vertices there can be in H, in the worst case. As said before, the entire set of apex vertices has, in total, $k(k+d)$ neighbours outside of A_e. To better illustrate our argument, we will think in terms of edges leaving $G[A_e]$ and entering H. We will see how many vertices can exist in H by calculating how many edges of $G[A_e]$ reach every one of the types of connected components of H. One edge (or more) leaves $G[A_e]$ and enters the component of e. We have at most k edges going to isolated edges. And the rest of the $k(k+2)-1-k$ edges go to the $p_2 \cdot 2+l_1+l_2$ vertices that, as we explained in the previous paragraph, are no more than the edges they receive. In total we have $V(G) = |A_e| + |V(H)| = k+4+2p_1+2p_2+l_1+l_2$. Therefore

$$V(G) \leq k+4+2k+k(k+2)-1-k \Leftrightarrow$$

$$V(G) \leq k^2+4k+3$$

□

Chapter 5

Specific obstruction Sets

5.1 Concrete classes

It is now time to look at some fixed classes. We have stated some theorems trying to attack the problem globally, i.e. we managed to limit the search space where an obstruction hunter must search to characterize class $\mathcal{A}_k^{(d)}$, for *any* two values of k and d. However, apart from their independent theoretical interest, these results alone do not suffice for the purposes of an exhaustive search for the obstructions of a concrete class, as the search space is still too big. For example, we see that, for values as humble as 2 for both k and d, we may have to test through a multitude of graphs of 15 vertices. For 15 vertices, even if we restrict ourselves to graphs of maximum degree 4, the number of graphs we have to look at is no less than 8.788.983.173 graphs (found using the graph generator by [23]).

In this chapter, we focus on specific instances of $\mathcal{A}_k^{(d)}$ (and also one acyclic variant). Firstly, we find the minor obstructions (as opposed to subgraph obstructions) of class $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$, through a purely analytical method (Section 5.2). For the rest of the obstruction sets, we use exhaustive search programs. The exact process is described in Section 5.5. To make the search possible, we reduce the search space significantly through improving the bounds on the obstruction size for specific pairs of k and d (Section 5.3). We limit our search space to connected graphs only. To find the disconnected obstructions, we combine connected obstructions of classes of lower distance k from the base class (Section 5.4). Finally, we describe the simple algorithms and tools employed in our hunting of obstructions (Section 5.5).

In this essay, we find the obstruction sets of the classes displayed in the table below:

	$d=2$	$d=3$
$k=1$	$\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$	$\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}\}))$, $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3},K_3\}))$
$k=2$	$\mathcal{A}_2(\mathbf{exc}(\{K_{1,2}\}))$	-

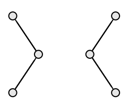
Where possible, we also calculate the corresponding subgraph obstruction sets.

5.2 Finding the obstructions of $\mathcal{A}_1^{(2)}$

In this section we find the obstructions of $A_1^{(2)} = \mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$ without the use of a computer.

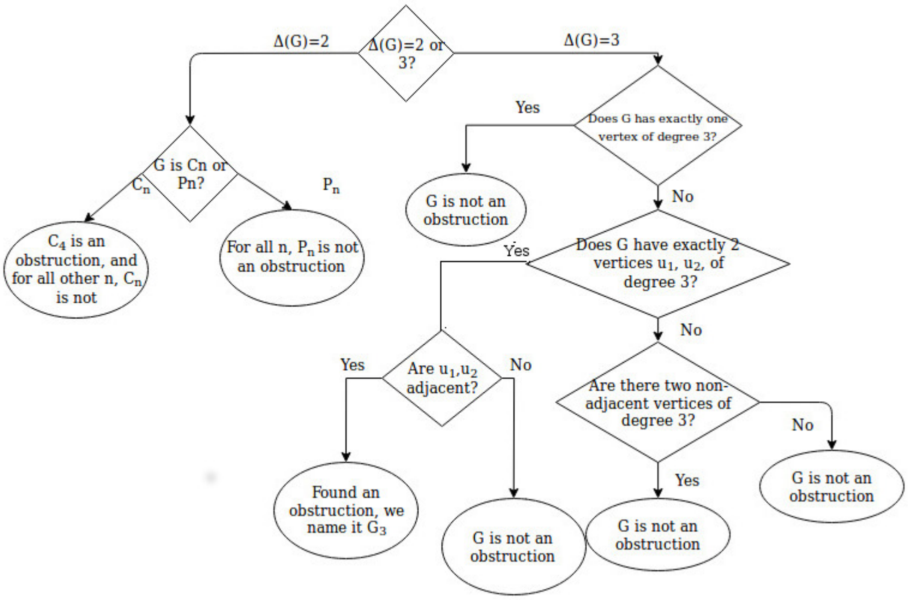
First we will make use of a theorem from [8], according to which every disconnected obstruction of $\mathcal{A}_k(\mathcal{F})$ is the disjoint union of 2 graphs, which are obstructions of $\mathcal{A}_{k'}(\mathcal{F})$ and $\mathcal{A}_{k''}(\mathcal{F})$ resp., for some positive integers $k' < k$, $k'' < k$, such that $k' + k'' = k - 1$ So according to this result, we get that we have only one disconnected obstruction of $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$: the disjoint union of two $K_{1,2}$ graphs. We call this G_1.

Figure 5.1: G_1, the only disconnected obstruction of $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$



Applying Theorem 8 to our case, we get that $\Delta(G) \leq 3$. Therefore, $\Delta(G)$ can be either equal to 0,1,2 or 3. But since G is an obstruction, its maximum degree should be greater than 1, otherwise it would belong to class $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$. So obstructions can either have maximum degree equal to i) 2 or ii) 3.

Perhaps it is better if we now provide a schema of our obstruction hunt here, in case the reader prefers to get a glimpse of the bigger picture before



following the proof's train of thought. We start by trying to imagine a connected obstruction G, and continue as follows:

Now, on to the obstruction hunt for class $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$. Let us consider a connected obstruction G where $\Delta(G) = 2$. G is either a cycle or a path. First we suppose it is a cycle, C_n. Is C_n an obstruction for some $n = n_0$? If yes, then all C_n graphs with $n > n_0$ are not obstructions, because they contain C_{n_0} as a minor, and all C_n graphs with $n < n_0$ belong to the class, because they can be generated by a series of edge contractions on C_{n_0}.

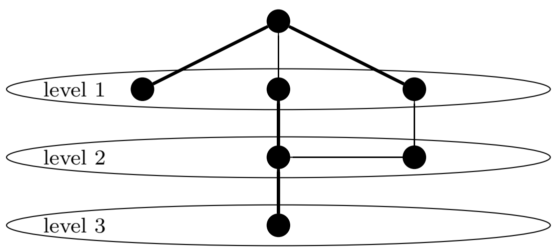
After a bit of trial and error, we discover that C_4 is an obstruction! That is, the removal of one vertex cannot give us a graph of disjoint P_2 components, or, in other words, C_4 does not belong to $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$. However, after removing an edge or contracting any edge e (which one does not matter, because of the symmetry of the graph) leaves us with a graph in $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$. The reader can easily confirm this.

As for the path graph, we can see that for $n < 6$ we have P_n belonging to the class (just remove the middle vertex). However P_6 does not belong to the class. Is P_6 an obstruction? No, because the removal of the "middle edge" leaves us with a graph still outside of class. In fact, it leaves us with the disconnected obstruction found previously! Since a minor of P_6 is an obstruction, and for $n > 6$, P_6 is a minor of P_n, we know that P_n, $n > 5$, is not an obstruction, as it would not be minor-minimal.

Now, we have found the only obstruction G with maximum degree 2. What about the obstructions with max. degree 3?

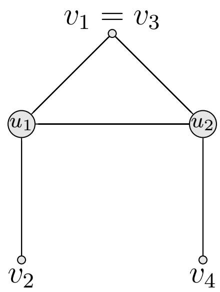
Suppose we have only one vertex u of degree 3, the rest of the vertices having degree 2 or 1. What could this graph be like? We will imagine the BFS traversal of this graph, starting from u. When does the traversal stop? It cannot stop after 1 or 2 levels, otherwise the deletion of u would render our graph $K_{1,2}$-free (and G would belong to $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$). So it must have at least 3 levels. However, in that case, as shown in Figure 5.2, we will always have 2 disjoint $K_{1,2}$ in G: one will range across three levels of the BFS, starting from level 1, and the other will include u and its neighbours which are not included in the first $K_{1,2}$ structure. Hence, once again, G cannot be an obstruction. So there must be at least two vertices of degree 3 in G, say u_1, u_2.

Figure 5.2: The presence of two disjoint $K_{1,2}$ subgraphs (in thick lines) contradict the hypothesis of G being an obstruction, as it contains G_1.



In the case where we have exactly two vertices of degree 3, we have to consider how they are related. First of all, we must not have two disjoint $K_{1,2}$ subgraphs. Therefore u_1 and u_2 should have distance at most 2. If their distance is 1, i.e. they are connected, how could the obstruction look like? Denote u_1's non-u_2 neighbours v_1, v_2. Also, denote the two neighbours of u_2 - apart from u_1 - v_3,v_4. If v_1, v_2, v_3, v_4 are all distinct vertices, then, once again we have a G_1 minor. Thus, u_1 and u_2 must have a neighbour in common. Hence, the below graph is formed:

Figure 5.3: G_3, another obstruction.



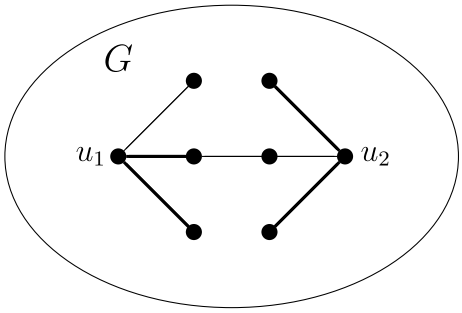
After examination, we observe that the graph in Figure 5.3 is an obstruction! Can we find another? Resuming our exploration journey, we observe that in case u_1 and u_2 have two common neighbours a C_4 is formed.

We have exhausted every possible configuration of two connected vertices of degree 3 each: they either have zero, one, or two common neighbours.

Now suppose they have distance 2. Again, we have to see how many common neighbours they have. Since they have distance 2, they must have at least one common neighbour. However they cannot have 2 or 3 common neighbours, as a C_4 is formed. Therefore they must have one common neighbour only. However, in that case, there exist two $K_{1,2}$ disjoint subgraphs in G, as shown in Figure 5.4 (see thick lines). Therefore, no matter what, we cannot have two vertices of degree 3 that are not adjacent in an obstruction.

To finish, we now prove that a graph with 3 vertices of degree 3, say u_1, u_2, u_3 cannot be an obstruction. To prove this, we take two cases: if two of these vertices are not adjacent, then as shown previously, they cannot be part of an obstruction. If they are all pairwise connected, they cause another problem. Imagine that every one of them has three neighbours, two which belong to $\{u_1, u_2, u_3\}$ and another one, say u'_i for each u_i. If $u'_1 = u'_2 = u'_3 = u_c$, then we have a K_4 which contains a C_4 subgraph. Otherwise, if two of the three vertices in $\{u'_1, u'_2, u'_3\}$ are distinct then we can

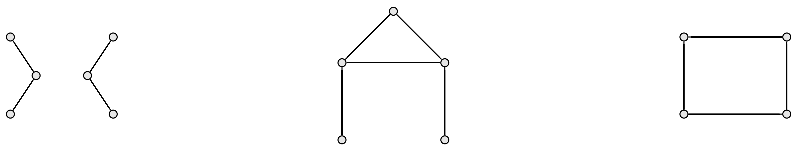
Figure 5.4: Two disjoint $K_{1,2}$ subgraphs can be seen if the 2 vertices of degree 3 that have one common neighbour.



see G_3 being formed. In all cases, a graph with 3 vertices of degree 3 contains an obstruction already discovered as a minor.

We have now exhausted all cases, and can present the three obstructions of $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$.

Figure 5.5: We have found all three obstructions of $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$.



5.3 Bounding the search space

In this section we will prove some bounds that will help us perform the exhaustive search for obstructions of the classes under consideration. Our goal is to improve the bounds already found, using a more fine-grained analysis. We prove these bounds for minor obstructions as well as for subgraph obstructions (where defined).

5.3.1 Obstructions of $A_2^{(2)}$

In this section, our analysis will be the same in both the case of minor obstructions, and that of subgraph obstructions. Note that, thanks to Corollary 7.2 and Theorem 8, we know that $A_k^{(d)}$ has always a finite subgraph obstruction set, for any k and d. Also, consider the following: since, by definition, every minor obstruction is a subgraph obstruction, it would be enough to only establish bounds for the case of subgraph obstructions.

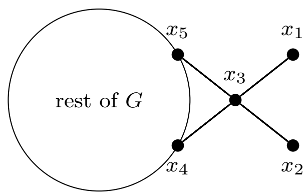
Before proving our theorem about the bound on the size of obstructions, we need the following useful lemma:

Lemma 6. *Let G be an obstruction (minor or subgraph) of $A_k^{(2)}$. Then G cannot contain:*

1. *$k+1$ distinct $K_{1,2}$ subgraphs (they must not have any common vertices).*
2. *a cycle of three vertices, suppose x, y, z, with $\mathbf{deg}_G(x) = \mathbf{deg}_G(y) = 2$ ("triangular tail", see Figure 4.3).*
3. *a P_5 subgraph where only the central vertex has degree in G greater than 2 and whose endpoints have both degree 1 in G ("double tail", see Figure 4.4)*
4. *2 P_3 subgraphs sharing their central vertex, each P_3 having a vertex of degree 1 in G ("x-shaped tail").*

The last forbidden substructure can be seen in Figure 5.6

Figure 5.6: The "x-shaped tail" is forbidden in obstructions of $\mathcal{A}_k^{(2)}$.

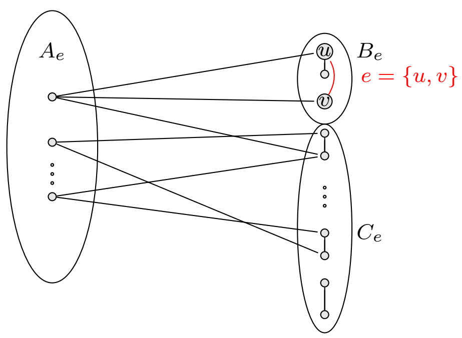


Proof. Statement 1 holds because $K_{1,2}$ is a connected obstruction of $\mathcal{A}_0^{(2)}$ (see Section 3.1 about disconnected obstructions). Statements 2 and 3 have been previously proved, in Lemmas 4 and 5.

For Statement 4, the proof is similar to that of the previous ones. Suppose we have an obstruction G with an x-shaped tail, composed by x_1, x_2, x_3, x_4, and x_5. x_1 and x_2 have degree 1 in G. Also x_3 is connected with the rest of x_i, and $\mathbf{deg}_G(x_3) = 4$. Consider the removal of $e = \{x_4, x_3\}$. Since G is an obstruction, there exists an apex set A_e, whose deletion leaves G without vertices of degree greater than 1. As x_3 is adjacent to x_1 and x_2, one of x_1,x_2, and x_3 must be an apex vertex. Without loss of generality, consider that x_3 is a member of A_e (if any of x_1 and x_2 was a member of A_e instead of x_3, we could replace it with x_3 in the apex set, while maintaining its property of being an apex set). However, the existence of x_3 in the apex set nullifies the effect of deleting e: even without deleting e, $\Delta(G \setminus A_e) < d$. But this contradicts our hypothesis that G is an obstruction. □

Theorem 10. *Let G be an obstruction (minor or subgraph) of* $A_2^{(2)}$. *G cannot have more than* 10 *vertices.*

Figure 5.7: The selection of e yields a partition of V(G) in 3 parts



Proof. Like in the proof of Theorem 8, suppose we have an obstruction G. The arguments that follow prove the theorem for the case of subgraph obstructions, but, as all minor obstructions are also subgraph obstructions, this proof covers both definitions of the obstruction.

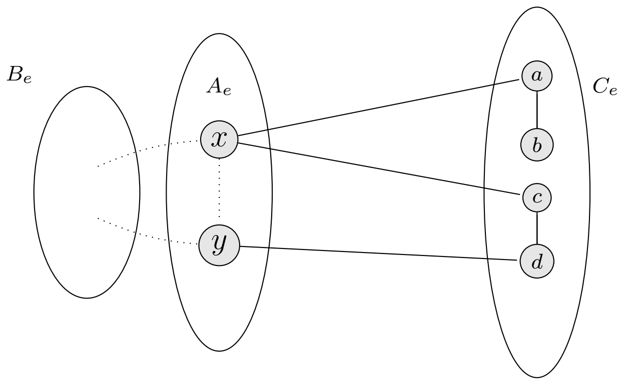
The removal of any edge $e = (u, v)$ gives us a graph within class $\mathcal{A}_2^{(2)}$. The selection of e yields a partition of $V(G)$ in 3 parts: A_e, a set of apex vertices if we remove e, B_e, the vertices of the connected component of e, after the removal of the apex vertices, and C_e, the vertices that remain, which form either P_2 or P_1 subgraphs after the removal of the apex vertices.

We know that $|A_e| = 2$, and $|B_e| \leq 4$. Hence, it suffices to prove that C_e has at most 4 vertices.

We will prove this statement by contradiction. We suppose that C_e has 5 vertices. Among these vertices one can either find: 2 P_2 components, 1 P_2 component and 3 disconnected vertices, or 5 disconnected vertices. We tackle each case seperately. Also, it should be considered that, as proven in the previous section, the maximum degree of every obstruction is less or equal to 4.

Case 1: C_e has two P_2's

Figure 5.8: C_e has two P_2's



Suppose $G[C_e]$ has two distinct P_2 subgraphs. We prove that this is impossible.

(1) Observe that in $G[B_e]$ there is one P_3 component. In case two P_2 components of C_e are connected to distinct vertices in A_e, we can find 2 more P_3 disconnected subgraphs in G, which is impossible (Lemma 6.(1)). Thus, all P_2 components are connected to the same vertex in A_e.

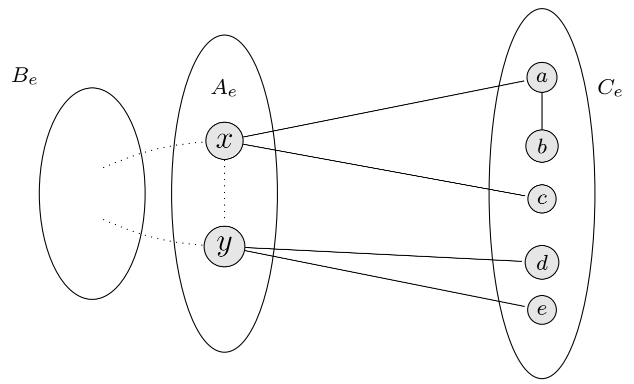
(2) Remember that both P_2 components are connected to A_e, otherwise the graph would be disconnected. It follows from 1. that both P_2 components are connected to A_e through only one vertex, named x. We

cannot have x connected to both ends of some P_2 component, as this would violate Lemma 6 ("triangular tail"). Therefore, x is connected to only one of the ends of every P_2.

(3) However, if x is connected to only one of the ends of every P_2, and every P_2 is not otherwise connected to A_e (as shown in 1. above), then there is again an objection: x connected to these two P_2's form a P_5, of the form forbidden Lemma 6 ("double tail"). This means that G is not an obstruction, which contradicts our hypothesis.

Case 2: C_e has one P_2 and 3 disconnected vertices

Figure 5.9: The last of the possible forms of this graph is once more a contradiction, as there are three $K_{1,2}$ graphs



Suppose $G[C_e]$ has vertices a, b, c, d, e, and the edge a, b. We prove that this is impossible.

(1) As before, observe that in $G[B_e]$ there is one P_3 induced subgraph. Also, remember that every one of of the four connected components in $G[C_e]$ must be connected to either x or y, or both. Below, we break down the case of the figure above in sub-cases, all of which are proven to be impossible.

(2) If x is connected to three of the the four connected components of $G[C_e]$ and y is not connected to any of these three components, then we have the impossible formation 4 ("x-shaped tail").

(3) If x is connected to three of the four components of $G[C_e]$ and y is connected to any of these three components, we can find two P_3 in $G[A \cup C_e]$, one with x in the middle and one with y.

(4) Because of (2) and (3), we have that x is not connected to 3 components. If x is connected to one component of $G[C_e]$, we have that y is connected to the other three, which case is exactly symmetrical to the union of the two previous cases.

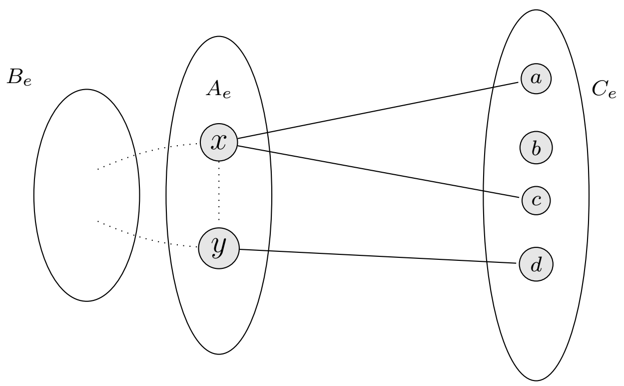
(5) If x is connected to zero components of $G[C_e]$, we have that y is connected to four, therefore also to three, which is impossible from (2) and (3).

(6) That leaves us with one final case: If two components are connected to x and the remaining two connected to y, then we have two independent P_3's, which along with the one in $G[B_e]$, gives us a contradiction.

The above set of cases cover all the possibilities. Therefore, we have proven that the formation depicted in figure 5.9 is impossible.

Case 3: C_e has four disconnected vertices

Figure 5.10: Case 3 of Theorem



Suppose $G[C_e]$ has four disconnected vertices, namely a, b, c, and d. Again we have $G[C_e]$ forming four components. We can argue exaclty as in the previous case to prove the contradiction. □

5.3.2 Obstructions of $\mathcal{A}_1^{(3)}$ and $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}, K_3\}))$

Theorem 11. *Let G be a minor or subgraph obstruction of $\mathcal{A}_1^{(3)}$ or a minor obstruction of $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}, K_3\}))$. Then*

- *If $\Delta(G) = 4$ then $n(G) \leq 7$*
- *Otherwise, $n(G) < 16$*

Proof. We will first prove the theorem for the obstructions of $\mathcal{A}_1^{(3)}$. Similarly to the proof of Theorem 10, we only bother with proving for connected obstructions.

Case 1: There exists a vertex u of degree 4 in G.

(1) Select any edge e at random, so long as it is not adjacent to u.

(2) Let $H = G/u$. Since G is not into the class, the removal of any vertex should result in a graph where at least one vertex has degree 3, because otherwise G would belong to $\mathcal{A}_1^{(3)}$.

(3) From (1) above, we gather that the removal of any edge in H will leave with a graph of maximum degree 2. Due to this, H has no vertex of degree 4, because the deletion of an edge cannot reduce a degree of a vertex by 2. Furthermore, the deletion of any edge results to the reduction of the degree of all vertices with degree 3.

(4) Therefore, H can only be one of $K_{1,3}$, $K_{1,2}$, and $K_{1,1}$. Thus G can have at most 7 vertices.

Case 2: All vertices have degree 3 or lower.

(1) Now, select any edge e from G. We will call the apex vertex for e u_a. Let $H = G \setminus u_a$. G can be partitioned into the apex vertex u_a, the connected component of e in H, denoted B_e, and C_e, the rest of connected components of H.

(2) Firstly, we know that H has at most 3 connected components (because u_a has degree up to 3). With the deletion of e, we possibly create one more. So, in total, there could be as many as 4 connected components in $H \setminus e$.

(3) We know that in these 4 components, the maximum degree is 2. So, the possible formations there can be are cycles or paths.

(4) We distinguish between vertices with degree at most 2 in G, and vertices with greater degree. We call the first type *light vertices* and the second type *heavy vertices*.

(5) A heavy vertex is adjacent to either u_a or e. As a consequence, we may have at most 5 heavy vertices.

(6) On the other hand, light vertices cannot neighbour one another because of Lemma 2, introduced in a previous section.

(7) Based on the last two observations we can have at most $2 \cdot 5 + 4 + 1 = 15$ vertices: 5 pairs of light and heavy vertices, plus 1 extra for every connected component (in case it is a path of odd size), plus u_a.

Finally, as regards the proof for the obstructions of $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}, K_3\}))$, the proof is identical to the proof for the obstructions of $\mathcal{A}_1^{(3)}$, except for one difference: step (6) uses Lemma 2, which does not cover the acyclic version. However, it is easy to adapt Lemma 2 just for class $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}, K_3\}))$. One must only contract edge e instead of deleting it, and notice that the contraction of any edge e between 2 vertices of degree 2 does not reduce the number of cycles in a graph, so the acyclicity property depends only on the deletion of A_e.

□

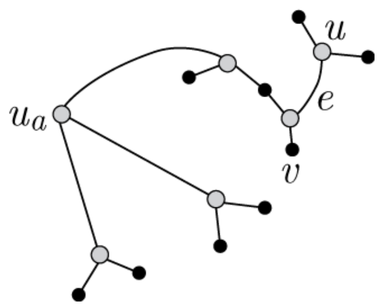


Figure 5.11: Worst case scenario (15 vertices) in proof of Theorem 11

5.4 Disconnected obstructions

As promised, we now discuss the way of determining the disconnected obstructions of the classes we are occupied with. We will use the notation $\mathbf{obs}_c(\mathcal{C})$ to refer to the connected obstructions of a class $\mathcal{C}$. We call $\mathbf{obs}_c(\mathcal{C})$ the *connected obstruction set* of $\mathcal{C}$.

Using the result of Dinneen as presented in 3.1, we gather that, fixing on a disjoint-union-closed base graph class $\mathcal{C}$ of bounded degree, we can generate the disconnected obstructions of $\mathcal{A}_k(\mathcal{C})$ as follows: each disconnected obstruction of $\mathcal{A}_k(\mathcal{C})$ is equal to $G = \bigcup_{i \in \{1,\ldots,r\}} C_i$, for $r > 1$, where every C_i is a member of $\mathbf{obs}_c(A_{k_i}(\mathcal{C}))$, with $k_i \geq 0$ and $\sum_{i \in \{1,\ldots,r\}} (k_i + 1) = k + 1$.

We can easily generate all disconnected obstructions of $\mathcal{A}_k(\mathcal{C})$, using the following algorithm.

function DISC(*Array*<*ObstructionSet*> lowerOrderConnObs, *int* k)
  IntegerPartitionSet ps ← GENERATEALLPARTITIONS(k+1)
  for all (*IntegerPartition* p in ps) **do**
    PRINTALLELEMENTSOF(lowerOrderConnObs,currentPart)
  end for
end function

Some notes on the algorithm:

1. We define a function, DISC(). It receives an integer k and a list of the connected obstruction sets of $\mathcal{A}_{k'}(\mathcal{C})$, for all k' in $\{0, 1, \ldots, k-1\}$. It prints out every disconnected obstruction of $\mathcal{A}_k(\mathcal{C})$.

2. Function GENERATEALLPARTITIONS(a) generates all partitions of the integer a, apart from the trivial partition $\{a\}$.

3. For every partition, we use function PRINTALLELEMENTSOF, which receives a list of obstruction sets denoted $O[\ ]$ (using C-style array syntax) and a multiset of integers $\{k_1, k_2, \ldots, k_r\}$, where $k_i < k$, for $i \in \{1, \ldots, r\}$ and returns all elements of the set $O[k_1] \times O[k_2] \times \ldots O[k_r]$.

4. The number of disconnected obstructions grow very quickly with k: one only has to take into account how the number of partitions increases.

To make things worse, this number is probably much lower that the number of disconnected obstructions, considering that, for every k, we normally have more than one connected obstructions in $\mathcal{A}_k(\mathcal{C})$.

5.5 Searching for connected obstructions: algorithms and implementation

The obstruction searching was carried out in a finite set of graphs, thanks to the upper bounds for the order and degree we found in the previous sections. Our method resembles that of Dinneen and Xiong in [9]. Before we start, we would like to present the graph classes for which we found the obstructions.

	$d = 2$	$d = 3$
$k = 1$	$\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$	$\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}\}))$, $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}, K_3\}))$
$k = 2$	$\mathcal{A}_2(\mathbf{exc}(\{K_{1,2}\}))$	-

Table 5.1: Concrete classes

For the 4 classes in Table 5.1, we found 7 obstruction sets: we found the minor obstructions for all four, but we found the subgraph obstructions only for three of the classes. See Table 5.2.

	minor obstructions	subgraph obstructions
$\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\}))$	3 obstructions	4 obstructions
$\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}\}))$	13 obstructions	17 obstructions
$\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}, K_3\}))$	8 obstructions	(not available)
$\mathcal{A}_2(\mathbf{exc}(\{K_{1,2}\}))$	16 obstructions	18 obstructions

Table 5.2: The obstruction sets found

Let $\mathcal{F}$ be a class of graphs. First of all, if we want to track down the (minor) obstructions of $\mathcal{F}$, we need an algorithm that decides whether a graph

G is an obstruction of $\mathcal{F}$. There exists a simple algorithm that performs this task, which follows the logic of the definition of an obstruction. It simply checks two conditions: first, if G is a member of $\mathcal{F}$ (if it is, G is not an obstruction), and secondly, if every one of its one-step minors is a member of $\mathcal{F}$ (if any one-step minor is not, then G is not an obstruction). This algorithm is given in pseudocode below. It receives as input a membership algorithm and the input graph.

```
1: function isObstruction(MembershipAlgorithm isMember, Graph G)
2:     if isMember(G) then return false
3:     else
4:         for all e in E(G) do
5:             H ← remove_edge(e,G)
6:             if not(isMember(H)) then return false
7:             end if
8:             H ← contract_edge(e,G)
9:             if not(isMember(H)) then return false
10:            end if
11:        end for
12:    end if
13:    return true
14: end function
```

Of course, we are also interested in subgraph obstructions of $\mathcal{F}$. To check for this kind of obstructions, we simply remove line 8,9 and 10, that perform the edge contractions.

As for the membership algorithm, in our case, we implemented an exhaustive search through all k-subsets of $V(G)$, in order to find out if there exists an apex set whose deletion would leave a graph which belongs to the base class. The base class here is either the (maximal) set of graphs of maximum degree at most $d-1$, or its acyclic variant. The test for membership in the base class is, in both cases, linear.

```
function isMember(Graph G, int k, int d)
    for all S | S is a k-subset of V(G) do
        G′ ← remove_vertices(S,G)
        if max_degree(G′)<d then
            if is_acyclic(G′) then
                return true
            end if
        end if
    end for
    return false
end function
```

The above algorithm works for the case where the base class is acyclic, i.e. for $\mathcal{A}_k(\{G \mid \Delta(G) < d \ \wedge \ G \textit{ is acyclic}\})$. By removing the is_acyclic(G) function, we get the membership algorithm for class $\mathcal{A}_k^{(d)}$.

Let us study the algorithms presented up to this point in detail. Let us denote $n = |V(G)|$ and $m = |E(G)|$. As regards our membership algorithm, time complexity is primarily affected by the number of times the for-loop is executed, which is $\binom{n}{k}$ times. The body of the loop itself is executed in linear time, with respect to the size of the input graph: the computation of $\Delta(G)$ requires only n comparisons, if the input graph is accompanied by a degree table. The function that checks for acyclicity, terminates after $O(n{+}m)$ steps, since it is a variation of the linear Breadth-First Search algorithm. Hence, the time complexity of our membership function is $O(n^{k+1}m)$. Considering k a constant, this is polynomial time. Space complexity is linear: all we store is graphs G and G', a subset of vertices S, as well some auxiliary variables.

The time complexity of function isObstruction() can now be computed. It is roughly equal to the time spent to execute isMember() $m+1$ times. All other operations take constant time. So we have $O(m^2 n^{k+1})$ time. Again here, space complexity is linear.

We did not invest a lot in improving these algorithms. This is justified by the small magnitude of the input values (this was more or less known beforehand). The value of k would be at most 4. All graphs were small, with order well under 20, and maximum degree $\Delta(G) < k + d$, and therefore number of edges $m \leq \frac{(k+d)n}{2}$. After all, any improvement of these complexities would be overshadowed by the main restriction we faced, which was the size of the search space. Another motive to stay with these algorithms was their simplicty. Possibly, it would much more difficult to prove the correctness of a more complex algorithm. Furthermore, it would probably have a longer development cycle (considering the debugging and testing required). We decided that a slower algorithm is a good trade-off for a faster one that would require much more theoretical, as well as technical work.

The implementation is in the C++ programming language, mainly for reasons of speed. To preserve resources, we used data structures that were as simple as possible. A graph is represented as a simple C-struct. It only includes the number of vertices of the graph, the number of edges, its adjacency matrix, and an array with the degree of every vertex, to save time from repeated calculation. In the spirit of optimizing the execution time, every array is static, which is possible thanks to the limited graph size.

The generation of our search space was made with the help of `geng`, a tool from the `gtools` suite [23]. It is a very fast Linux command line program, which can, given an input n, output all non-isomorphic graphs with n vertices. It allows restrictions on the maximum and minimum number of edges, the maximum and minimum degree, as well as connectivity of the graphs enumerated. This made it very easy for us to generate the search space needed, which was a collection of graphs of bounded order and maximum degree. It is also surprisingly fast. To give an example, it can generate all 739.335 connected graphs of order 11 and maximum degree 4 in less than 2 seconds. Also, after several hours, it was able to compute the number of connected graphs of order 15 and maximum degree up to 4.

```
$ geng 11 -c -D4
>A geng -cd1D4 n=11 e=10-22
>739335 graphs generated in 1.22 sec
$geng 15 -c D4
>A geng -cd1D4 n=15 e=14-30
>8788983173 graphs generated in 18903.26 sec
```

To summarize, our method to find the obstructions of a class was the following.

1. Calculate upper bounds for the order and maximum degree of the obstructions.

2. Generate all graphs satisfying these restrictions with *geng*.

3. Execute a simple C++ program that calls the variant of function *isObstruction* corresponding to the chosen class and obstruction type (subgraph or minor) for all graphs in the input file in a serial fashion, printing on the spot any obstructions detected.

Chapter 6

Conclusion and possible next steps

6.1 Future work

As it is common in scientific research, the completion of this thesis does not close the subject it is about. On the contrary, it is more open than it was before. For every question answered, we have many new raised.

The bound of Theorem 8 is a square bound, with respect to both k and d. Can we improve it? This is hard to answer. Considering the analysis in our proof, one reason the bound is square is the number of connected components of H'', the graph that emerges after the removal of the apex vertices. The case where we have $k(k+d)$ connected components is a far-fetched situation which, in reality, cannot exist, because it violates the condition of connectivity of the graph. But what else can we say about the connectivity of the graph? If the graph we are examining is k-connected, then we only have 2 connected components instead of $k(k+d)$. We believe that the bound can be improved. We conjecture that it can be linear with respect to d.

What about the obstructions of $\mathcal{A}_k^{(d)}$, for specific (k,d) pairs? Can we improve the bound for $(k,d) = (2,3)$ or $(k,d) = (3,2)$, which are exactly outside the boundaries of the classes we have already characterized? Perhaps, with a case analysis such as the one carried out for Theorem 10, one could get a bound that is good enough and go on and find the obstructions of several “neighbouring” classes, such as the ones with the added restriction of acyclicity of the base class, like $\mathcal{A}_2(\mathbf{exc}(\{K_{1,3}, K_3\}))$.

Finally, the software used to perform the search can be generalized to receive any pair of k, d as parameters, as well as the available bounds for the obstructions and if possible, return all obstructions of $\mathcal{A}_k^{(d)}$, or else return all obstructions found after a certain time period. Furthermore, we could draw some ideas from the more advanced t-parses method used by Dinneen and Versteegen in [7].

6.2 Experimental Mathematics

The method of exhaustive enumeration that we use cannot be generalized to apply to every obstruction set. For classes with obstructions with many vertices we are obliged to resort to more advanced mathematics. This is mainly due to the rapid growth of the number of existing graphs, as the number of vertices increase. Already at 15 vertices, we have $3 \cdot 10^{19}$ ([29]) unlabeled graphs, exceeding by far the number of graphs any computer can check in a lifetime.

However, searching obstructions with a computer can be very useful, even outside of worlds where we have few vertices. For example, having a list of all obstructions with size up to a certain integer, can help uncover previously unknown properties of the class under examination. The computer can serve as powerful tool for experimentation and exploration. It can extend the researcher’s vision, providing, in a sense, “a seat with a better view“ of the mathematical objects. For graph theory, the most basic example would be the visualization of graphs, represented in various settings, surfaces, or embeddings. On the other hand, the most extreme example would be having the computer validating a proof’s correctness or even making conjectures, as in the case of Graffiti ([12]). Such works are said to pertain to the emerging field of experimental mathematics. Nevertheless, experimentation has always been part of mathematical research:

> “Mathematics is not a deductive science—that’s a cliché. When you try to prove a theorem, you don’t just list the hypotheses,

> and then start to reason. What you do is trial and error, experimentation, guesswork. You want to find out what the facts are, and what you do is in that respect similar to what a laboratory technician does."
>
> (Paul Halmos - I Want to be a Mathematician: An Automathography [17])

We see an increase in the popularity of computer-assisted mathematics. However, this in turn increases the need for attention to the computer engineering part of the research process. The route from the abstract idea of an algorithm to a computer program is long. It starts with the careful design of the algorithm; it includes analysis and improvements of running time and space requirements; selection of (sometimes mathematically very involved) data structures; and programming. It takes optimizing resource management, specific to the execution unit, the inputs, the compiler. This is where the computer engineering part might change the outcome of the experiment. Consequently, as the interplay between mathematicians and machines is steadily increasing, there is plenty of room for mathematically educated engineers and tech-savvy mathematicians.

Many division lines exist among mathematicians alone [20]; if we consider the difference between the world of a computer engineer and that of a mathematician, it should take considerable effort for these two worlds to meet. But from this meeting, new discoveries and knowledge can emerge; with luck, the kind of knowledge that can extend human capabilities and aid the building of the necessary conditions for a free human society.

Appendix: Obstructions sets found by exhaustive search

Minor Obstructions

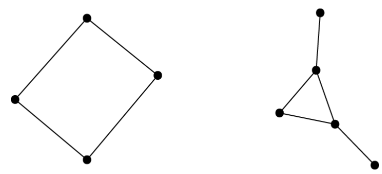


Figure 6.1: Connected obstructions of $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\})$

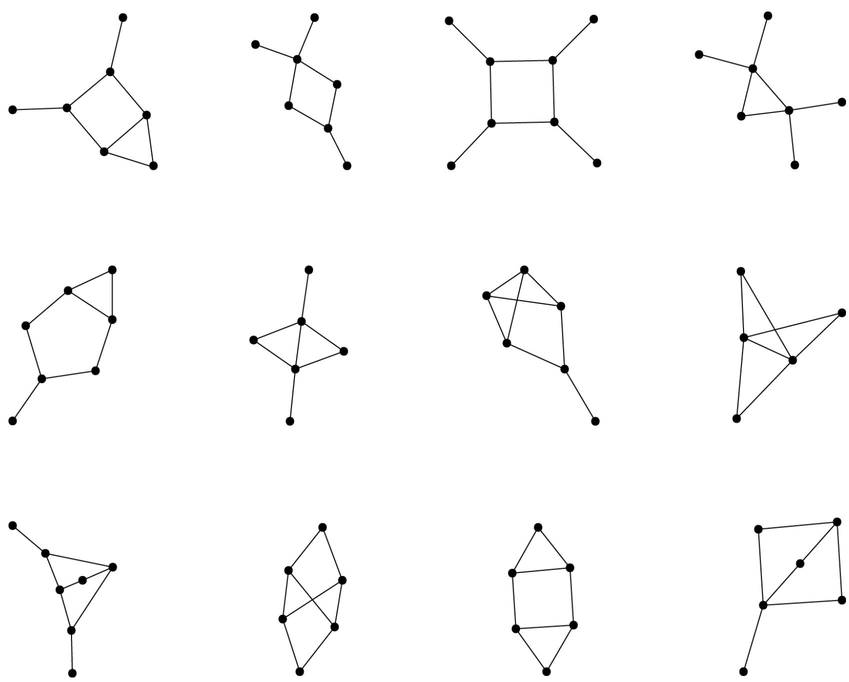


Figure 6.2: Connected obstructions of $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}\})$

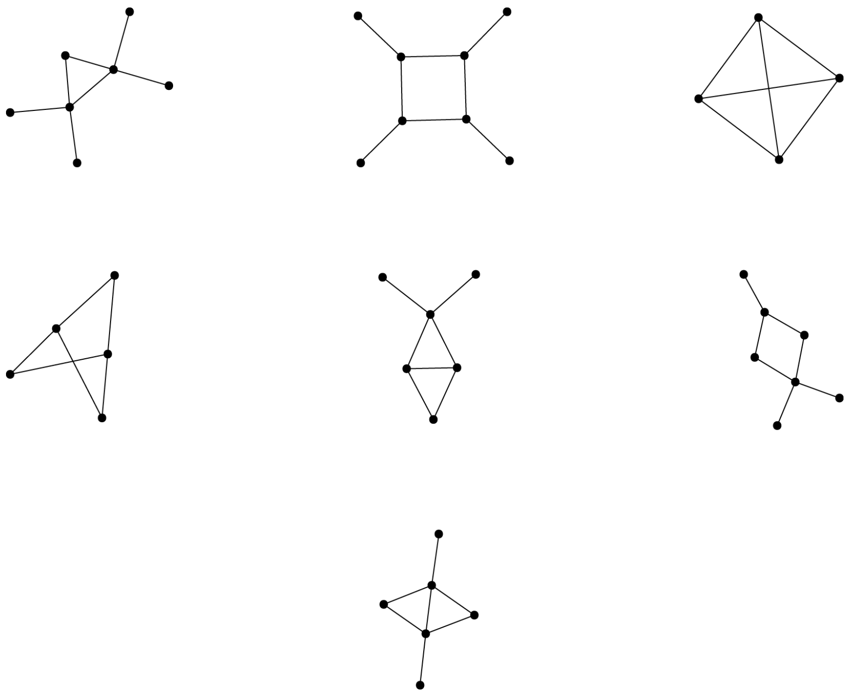


Figure 6.3: Connected obstructions of $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}, C_3\})$

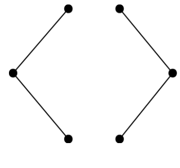


Figure 6.4: The only disconnected obstruction of $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\})$, $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}\})$ and $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}, C_3\})$

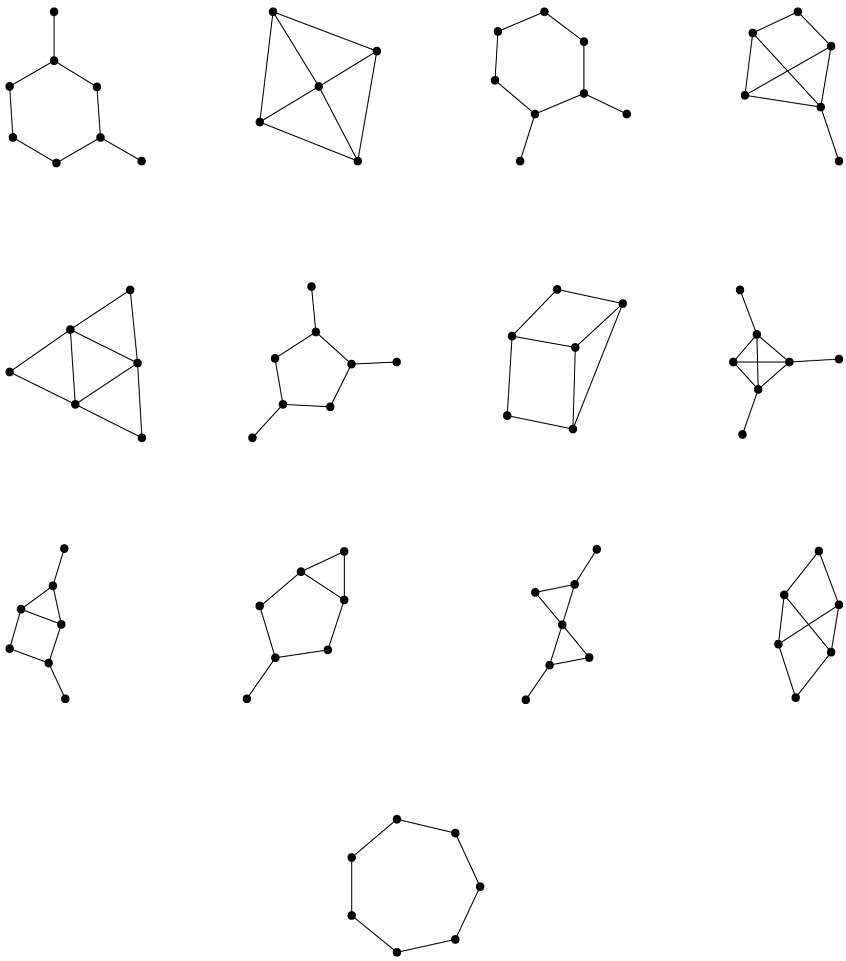


Figure 6.5: Connected obstructions of $\mathcal{A}_2(\mathbf{exc}(\{K_{1,2}\})$

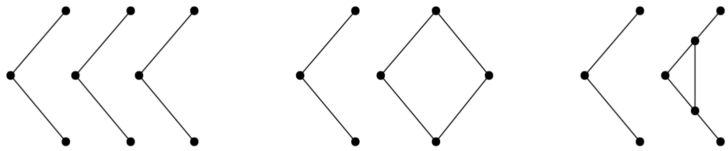


Figure 6.6: The disconnected obstructions of $\mathcal{A}_2(\mathbf{exc}(\{K_{1,2}\})$

Subgraph Obstructions

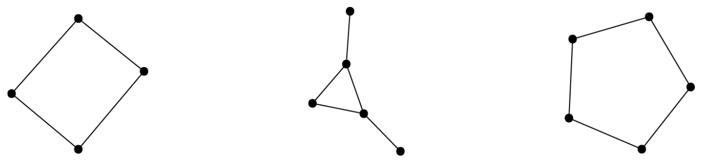


Figure 6.7: Connected subgraph obstructions of $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\})$

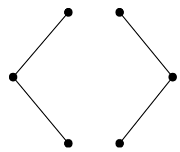


Figure 6.8: The only disconnected subgraph obstruction of $\mathcal{A}_1(\mathbf{exc}(\{K_{1,2}\})$ and $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}\})$

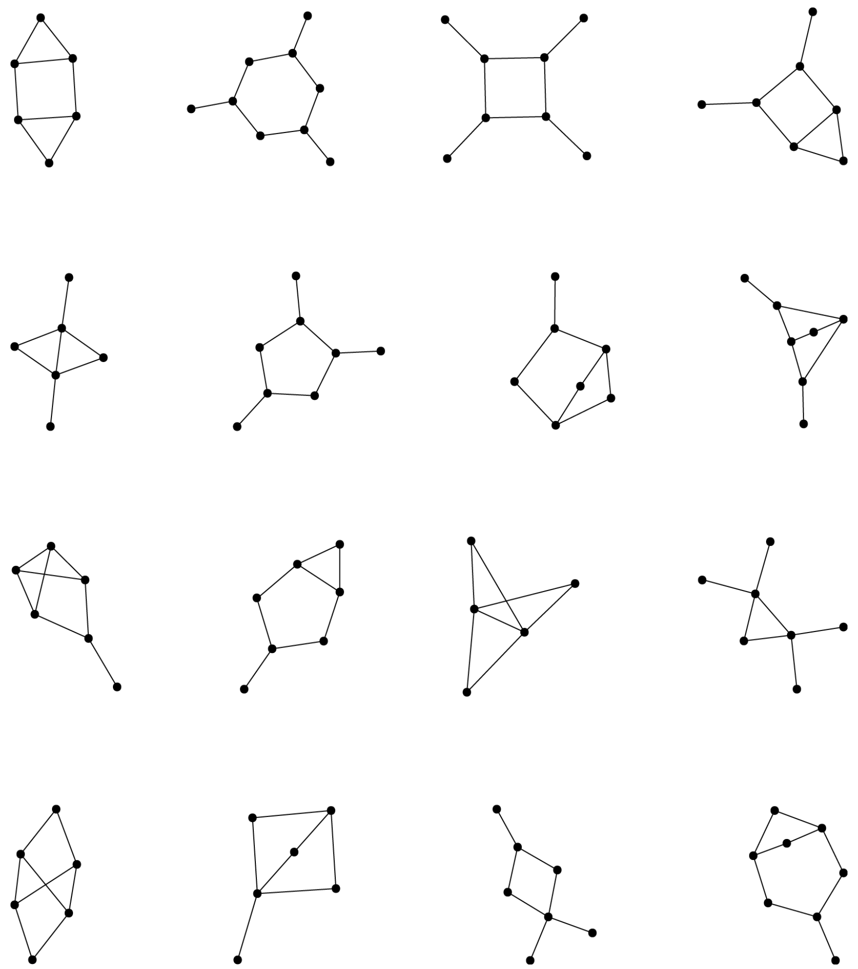


Figure 6.9: Connected subgraph obstructions of $\mathcal{A}_1(\mathbf{exc}(\{K_{1,3}\})$

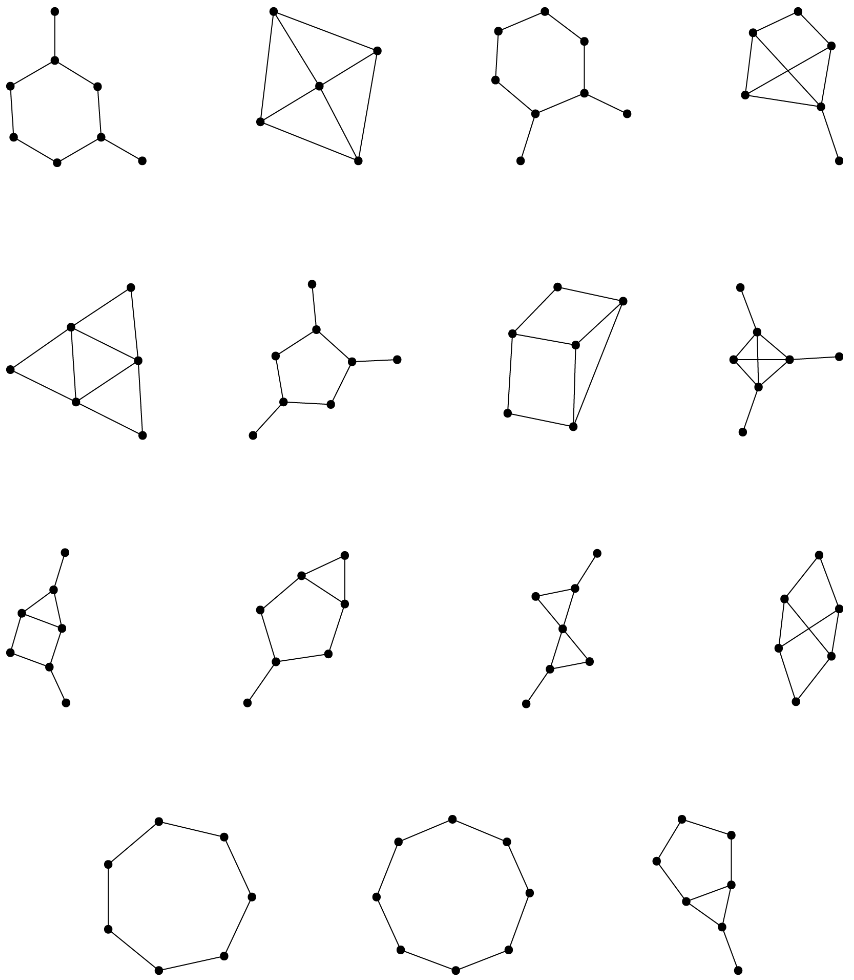


Figure 6.10: Connected subgraph obstructions of $\mathcal{A}_2(\mathbf{exc}(\{K_{1,2}\})$

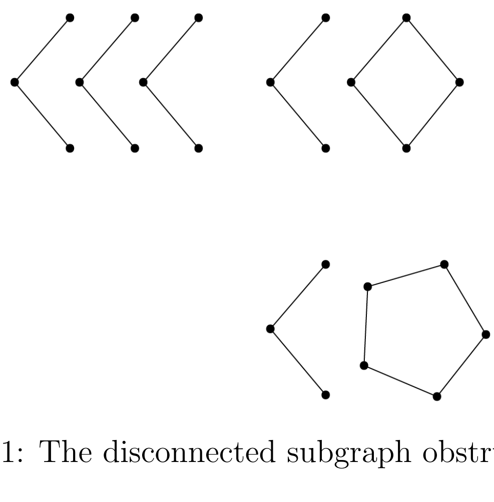


Figure 6.11: The disconnected subgraph obstructions of $\mathcal{A}_2(\mathbf{exc}(\{K_{1,2}\})$

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