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Fréchet derivatives of the eigenelements of regular Sturm-Liouville problems

Master Thesis

by

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Abstract

The present master thesis is about the variation of the eigenvalues and the eigenfunctions of regular Sturm-Liouville (S-L) boundary value problems (BVPs). In the first Chapter we present some basic results about S-L BVPs which will be needed in the sequel (see Zettl 2005, Haberman 2004, Ch.5, Walter, 1998, pp.268-281 and Coodington& Carlson, 1997, pp.255-262). In the Chapter 2, firstly we study the dependence of the eigenvalues of a regular S-L problem on the boundary points and we obtain the formulas of the derivatives of the eigenvalues with respect to the boundary. The arguments of the proof are based on those of Kong Q. & Zettl A. (1999) in Theorems 3.2,3.3,3.4. Subsequently, we assume that the boundary points remain unchanged and we prove the formulas of the variation of the eigenvalues with respect to the coefficient-functions (see also Kong Q., Zettl A., 1996).Then, we solve a constant-coefficient S-L BVP and we calculate the variations of its eigenvalues with respect to the boundary points by using Implicit Function's Theorem. We also apply the formulas proved in Sec.2.1 to the constant-coefficient problem and as we expected they are identical to the variations we found by straightforward differentiation. The last content of the 2ndChapter, is an introduction to the problem of the variation of the eigenfunctions with respect to the coefficient-functions. The formulation of this problem shows the emergence of using the theory of the non-homogeneous BVPs (NHBVPs).

In Chapter 3, Green's Function is exploited in order to express the solution of the non-homogeneous BVP. At first, we construct Green's Function when the parameter λ is not an eigenvalue, $\lambda \neq \lambda_n$ and we prove the expression and the uniqueness of the solution (see Stone &Goldbart 2009, Sec.5.2, Butkov, 1973, pp.508-514). When $\lambda = \lambda_n$, Green's Function which is referred as Generalized Green's Function (GGF) or Modified Green's Function satisfies a certain non-homogeneous problem, which is solved by the method of variation of the parameters(see Stakgolt, 2011, Sec.3.5, Haberman,2004, Sec.9.4.3, Stone &Goldbart,2009, Sec. 5.2.3).The expression of Green's Function in this case, is not symmetric but it is possible to be, according to the instructions of Haberman, in Haberman R., Green's Function, Lecture 7, about the construction of GGF in a general case and the way to make it symmetric. In Sec.3.4, we prove the expression and the existence of the solution of the NHBVP and we conclude that the solution is not unique (see also Haberman, 2004, pp.409-411). In the last Section of Chapter 3 we present the basic theory which is needed to solve a fully non-homogeneous BVP since the variation of the eigenfunctions satisfies a BVP of this type. The technique we follow is to reduce the latter problem to an equivalent non-homogeneous BVP.

In Section 4.1, since we presented the appropriate theory in Chapter 3, we are ready to solve the fully non-homogeneous BVP which is satisfied by the variation of the eigenfunctions. As we mentioned in Chapter 3, the solution of the problem is not unique, which is expected since the eigenfunctions are not unique. Hence, we assume that the eigenfunctions of the initial and the perturbed BVP are normalized and this fact renders the variation of the eigenfunctions uniquely defined as well. As far as the last Section of the present thesis is concerned, which is about the variation of the eigenfunctions with respect to the endpoints, there are no related sources. In order to solve this problem we applied a transformation to the perturbed problem

in order to reduce the problem of the variation of the domain to the problem of the variation of the coefficients which is already solved in the previous section.

Περίληψη

Η παρούσα διπλωματική εργασία ασχολείται με τις συναρτησιακές παραγώγους ιδιοτιμών και ιδιοσυναρτήσεων ομαλών προβλημάτων συνοριακών τιμών, τύπου Sturm-Liouville (S-L). Στο πρώτο κεφάλαιο, παρουσιάζουμε κάποια βασικά αποτελέσματα για τα προβλήματα S-Lτα οποία θα χρειαστούν στην συνέχεια (βλ. Zettl 2005, Haberman 2004, Ch.5, Walter, 1998, pp.268-281 και Coodington & Carlson, 1997, pp. 255-262). Στο δεύτερο κεφάλαιο, αρχικά εξετάζουμε την εξάρτηση των ιδιοτιμών του προβλήματος από τα άκρα του διαστήματος στο οποίο ορίζεται το πρόβλημα και εξάγουμε τις σχέσεις για τις παραγώγους των ιδιοτιμών ως προς αυτά. Τα επιχειρήματα της απόδειξης αυτής, στηρίζονται σε αυτά των Kong Q. & Zettl A. (1999) στα Θεωρήματα 3.2,3.3,3.4. Στη συνέχεια, θεωρούμε ότι τα άκρα του διαστήματος παραμένουν σταθερά και αποδεικνύουμε τις σχέσεις των μεταβολών των ιδιοτιμών ως προς τις συναρτήσεις-δεδομένα (βλ. επίσης Kong Q., Zettl A., 1996).

Έπειτα, λύνουμε ένα πρόβλημα S-L με σταθερούς συντελεστές και υπολογίζουμε τις παραγώγους των ιδιοτιμών ως προς τα άκρα του συνόρου χρησιμοποιώντας το Θεώρημα Πεπλεγμένης Συνάρτησης. Εφαρμόζουμε επίσης τους τύπους των μεταβολών των ιδιοτιμών στο πρόβλημα σταθερών συντελεστών και όπως ήταν αναμενόμενο ταυτίζονται με τις μεταβολές που βρήκαμε με απευθείας παραγώγηση. Το τελευταίο περιεχόμενο του 2ου κεφαλαίου είναι μία εισαγωγή στο πρόβλημα της μεταβολής των ιδιοσυναρτήσεων ως προς τις συναρτήσεις-δεδομένα. Η διατύπωση αυτού του προβλήματος αναδεικνύει την χρησιμότητα της θεωρίας σχετικά με τα μη-ομογενή προβλήματα.

Στο Κεφάλαιο 3, χρησιμοποιείται η συνάρτηση Green ώστε να εκφράσουμε την λύση του μη ομογενούς προβλήματος. Αρχικά, κατασκευάζουμε την συνάρτηση Green όταν η παράμετρος λ δεν είναι ιδιοτιμή, $\lambda \neq \lambda_n$ και αποδεικνύουμε την έκφραση της λύσης και την μοναδικότητα της (βλ. Stone & Goldbart 2009, Sec.5.2, Butkov, 1973, pp.508-514).Όταν $\lambda = \lambda_n$, η συνάρτηση Green όπου αναφέρεται ως Generalized Green's Function (GGF) ή Modified Green's Function ικανοποιεί ένα συγκεκριμένο μη ομογενές πρόβλημα το οποίο λύνουμε με την μέθοδο μεταβολής των παραμέτρων (βλ. Stakgolt, 2011, Sec.3.5, Haberman, 2004, Sec.9.4.3, Stone & Goldbart, 2009, Sec. 5.2.3). Η έκφραση της συνάρτησης Green σε αυτήν την περίπτωση δεν είναι συμμετρική αλλά είναι δυνατόν να γίνει σύμφωνα με τις υποδείξεις του Haberman στο Haberman R.,Green's Function, Lecture 7, σχετικά με την κατασκευή της GGF και τον τρόπο να γίνει συμμετρική. Στην Ενότητα 3.4 αποδεικνύουμε την ύπαρξη και την έκφραση της λύσης του μη-ομογενούς προβλήματος και συμπεραίνουμε ότι η λύση δεν είναι μοναδική (βλ. Haberman, 2004, pp. 409-411). Στην τελευταία ενότητα του κεφαλαίου 3, παρουσιάζουμε την βασική θεωρία που χρειάζεται για να λυθεί ένα πλήρως μη-ομογενές πρόβλημα, αφού η μεταβολή των ιδιοσυναρτήσεων ικανοποιεί ένα πρόβλημα τέτοιου τύπου. Η μέθοδος που ακολουθούμε είναι να ανάγουμε το προαναφερθέν πρόβλημα σε ένα ισοδύναμο μη-ομογενές.

Στην Ενότητα 4.1, αφού παρουσιάσαμε την κατάλληλη θεωρία στο Κεφάλαιο 3, μπορούμε να λύσουμε το πλήρως μη-ομογενές πρόβλημα που ικανοποιείται από την μεταβολή των ιδιοσυναρτήσεων. Όπως αναφέραμε στο Κεφάλαιο 3, η λύση του προβλήματος δεν είναι μοναδική, κάτι που είναι αναμενόμενο αφού οι ιδιοσυναρτήσεις δεν είναι μοναδικές. Έτσι, υποθέτουμε ότι οι ιδιοσυναρτήσεις του αρχικού και του διαταραγμένου προβλήματος συνοριακών τιμών είναι κανονικοποιημένες στην μονάδα και αυτή η υπόθεση καθιστά την

μεταβολή των ιδιοσυναρτήσεων μονοσήμαντα ορισμένη. Όσον αφορά την τελευταία ενότητα της παρούσας εργασίας, που ασχολείται με την μεταβολή των ιδιοσυναρτήσεων ως προς τα άκρα του διαστήματος, δεν υπάρχουν βιβλιογραφικά στοιχεία. Για να λύσουμε αυτό το πρόβλημα εφαρμόζουμε έναν μετασχηματισμό στο διαταραγμένο πρόβλημα και με αυτόν τον τρόπο ανάγουμε το πρόβλημα μεταβολής των ιδιοσυναρτήσεων ως προς το σύνορο στο πρόβλημα της μεταβολής των ιδιοσυναρτήσεων ως προς τις συναρτήσεις-δεδομένα το οποίο έχει ήδη λυθεί στην προηγούμενη ενότητα.

Table of Contents

Introduction

Jacques Charles Francois Sturm (1803-1855) and Joseph Liouville (1809-1882), in 1836-1837 published a series of articles and created a subject in Mathematical Analysis. The theory which is known as Sturm-Liouville (S-L) Theory deals with the second order differential equation

$$-\left(p(z)u'(z)\right)' + q(z)u(z) = \lambda w(z)u(z), \quad \text{in } (a,b), \qquad -\infty < a < b < \infty, \tag{1}$$

with the imposed boundary conditions (BCs)

$$(pu')(a) - \nu_0 u(a) = 0, \tag{2a}$$

$$(pu')(b) - \mu_0 u(b) = 0 \tag{2b}$$

The differential equation (1) accompanied by the boundary conditions (2), consist a S-L boundary value problem (BVP).Many historical notes about the development of S-L Theory can be found in Lutzen's book (Lutzen, 1982).The selection of the function space which each function-data belongs, it depends on the purpose we have in each case. An obvious choice is the following

$p(z) \in C^1[a,b]$, $q(z), w(z) \in C[a,b]$ and $\lambda \in \mathbb{C}$ (in the particular work, we will be restricted to self adjoint problems, thus, $\lambda \in \mathbb{R}$ in our case). An important fact, which makes S-L Theory widely applied is that every second order differential equation can be reduced to a S-L equation (see Walter, 1998, pp.245-246).The BVP (1)-(2) only allows non trivial solutions, called eigenfunctions for certain values of λ, the eigenvalues.

In the Chapter 1 of this Thesis, we state and prove some basic theorems which will be utilized in the next Chapters. In the following Chapter, our purpose is to find the functional derivatives of the eigenvalues of the BVP(1). This work is already done by Kong & Zettl (1996, 1999). The case of Neumann BVP is studied by Dauge&Helffer(1993). As we can notice, an eigenvalue λ of the above BVP depends on the data-functions $p(x), q(x), w(x)$, the constants ν_0, μ_0 and the boundary points a, b. Hence, it makes sense to use the notation

$$\lambda = \lambda\left(p(\cdot), q(\cdot), w(\cdot), \nu_0, \mu_0, a, b\right). \tag{3}$$

As we can see from Eq.(3), λ is a function of ν_0, μ_0, a, b and a functional on $p(z), q(z), w(z)$.Firstly, we assume that all the arguments of λ remain unchanged except from the endpoints a and b, and we prove the formulas of the variation of the eigenvalues with respect to a and b respectively. We also state some remarks which include special cases of the BCs. The arguments we used for the proof are based on those of Kong and Zettl in Kong & A. Zettl , (1999) in Theorems 3.2, 3.3, 3.4 (see also Dauge&Helffer, 1993, for the Neumann case). Next, we assume that the boundary points are not perturbed and we prove the formulas of the variation of the eigenvalues with respect to $p(\cdot), q(\cdot), w(\cdot), \nu_0, \mu_0$. (See also KongQ., Zettl A., 1996) A different approach which leads to the same variation formulas is given by Farshad M. (1973).We also solve a constant-coefficient S-L BVP and we calculate the variations of its eigenvalues with respect to the boundary points by using the Implicit Function's Theorem. Then, we apply the formulas proved in Sec.2.1 for the constant-coefficient problem and as we expected they are identical to the variations we found by straightforward differentiation. The last content of the 2nd Chapter, is a preamble about the problem of the variation of the eigenfunctions with respect to the coefficient-functions. In particular, consider the following S-L problem

$$\partial_z^2 Z_n(z) + \left(\lambda_n - Q(z)\right) Z_n(z) = 0, \qquad a < z < b, \tag{4}$$

$$B^b Z_n \equiv \partial_z Z_n(b) - \mu_0 Z_n(b) = 0, \tag{5a}$$

$$B_a Z_n \equiv \partial_z Z_n(a) - \nu_0 Z_n(a) = 0. \tag{5b}$$

By assuming that $Q(z), \mu_0, \nu_0$ are perturbed, the eigenvalues and the eigenfunctions are also changed. In order to determine the variation of the eigenfunctions with respect to the coefficients it is necessary to solve a fully non homogeneous BVP which in the non-homogeneous part of the differential equation contains the variation of the eigenvalues with respect to the coefficients. Hence, the variation of eigenvalues is necessary to precede the solution of the variational problem of the eigenfunctions. The non-homogeneous problem, which is satisfied by the variation of the eigenvalues, is the motivation for recalling the basic theory for the solution of non-homogeneous BVPs. Green's Function is exploited in order to express the solution of the non-homogeneous BVP. At first, we construct Green's Function when the parameter λ is not an eigenvalue, $\lambda \neq \lambda_n$. (See Stone &Goldbart, 2009 Sec.5.2, Butkov, 1973pp.508-514, I. Stackgolt, M. Holst 2011 pp.193-196 and E.L. Ince, 1956,pp.255-256). Then, we prove the expression and the existence of the solution, in the same case, and at last, we prove that the solution is unique. When $\lambda = \lambda_n$, at first we present two negative results about the construction of Green's Function, which in this case is called Generalized Green's Function (GGF) or Modified Green's Function. (see Stackgolt ,2011, Sec. 3.5, Haberman ,2004, Sec.9.4.3, Stone & Goldbart,2009, Sec. 5.2.3). However, Green's Function satisfies a certain non-homogeneous problem, which is solved by the method of variation of the parameters. The expression of Green's Function, is not symmetric but it is possible to be, according to the instructions ofHaberman R., Green's Function, Lecture 7, about the construction of GGF in a general case and the way to make it symmetric. In this Section, we prove the expression and the existence of the solution of the NHBVP but the solution is not unique (see also Stakgolt,2011, pp.193-196, Haberman, 2004, pp. 409-411). In the last Section of Chapter 4 we present the basic theory needed to solve a fully non-homogeneous BVP. The technique we follow is to reduce the latter problem to an equivalent non-homogeneous BVP and when λ is an eigenvalue, $(\lambda = \lambda_n)$ we prove that the compatibility condition of the initial and of the reduced problem is the same, as we expected, since the problems are equivalent. In Section 4.1, we apply the appropriate theory from Chapter 3 and we solve the fully non-homogeneous BVP which is satisfied by the variation of the eigenfunctions. As we mentioned in the previous Chapter, the solution of the problem is not unique, which is expected since the eigenfunctions are not unique. Hence, by assuming that the eigenfunctions of the initial and the perturbed BVP are normalized, the constant which the variation of the eigenfunction contains can be uniquely determined. The variation of the eigenfunctions with respect to the coefficients can also be found in Farshad M., (1973). As far as the last Section of the present thesis is concerned, it is about the variation of the eigenfunctions with respect to the endpoints, there are no related sources. In order to solve this problem we applied a transformation to the perturbed problem and we managed to be defined in the same interval as the unperturbed problem. Thus, we reduced the problem of the variation of the domain to the already solved problem of the variation of the coefficients.

1. Sturm-Liouville problems

This Chapter contains some basic results about the specific category of the problems, called Sturm-Liouville problems, which will be utilized next.

Definition 1
A **Sturm-Liouville** (S-L) **equation** is a second order homogeneous linear differential equation of the form

$$-(p(x)u'(x))' + q(x)u(x) = \lambda w(x)u(x)\,, \qquad (1)$$

where $u'(x) = \dfrac{d\,u(x)}{dx}$, λ is a parameter , p, q, w are real-valued functions of x.
The functions q and w are assumed to be continuous and p continuously differentiable.

A Sturm-Liouville equation can be categorized according to the properties of the data-functions $p(x), q(x), w(x)$. Thus, we present the definition of a regular S-L equation.

Definition 2
The S-L equation is called **regular** in a closed interval $[a,b]$ when the functions $p(x)$, and $w(x)$ are strictly positive for $x \in [a,b]$.

Since the functions $p(x), q(x), w(x)$ are continuous, they are bounded in the interval $[a,b]$.

At the majority of the physical problems, the differential equation is accompanied by a number of boundary conditions. Hence, we give the following definition

Definition 3
A regular **Sturm-Liouville system/problem** is a S-L equation in a finite closed interval $[a,b]$ together with two separated boundary conditions (BCs) of the form

$$(pu')(a) - \nu_0 u(a) = 0\,, \qquad (2a)$$

$$(pu')(b) - \mu_0 u(b) = 0\,. \qquad (2b)$$

Where ν_0, μ_0 are given real numbers.

An important question is when a second order differential equation can be reduced to a S-L equation. The answer is that every second order differential equation, and this fact points out the importance of S-L problems (see also W. Walter, 1998, pp.245-246, G. Birkhoff, G.-C. Rota, 1989, pp. 320-322).
The general form of a second order differential equation is the following

$$u''(x) + a_1(x)u'(x) + a_0(x)u(x) = 0\,. \qquad (3)$$

Equation (3) can always transformed into the S-L form, Eq.(1), by multiplying by the factor
$p(x) = e^{\int a_1(x)\,dx}$.

Indeed, Eq.(3) takes the form

$$\left(p(x)u'(x)\right)' + p(x)a_0(x)u(x) = 0\,. \tag{4}$$

By choosing $p(x)a_0(x) = -q(x)$, Eq.(4) becomes identical to Eq.(1).

The most usual problem is to search for non-trivial solutions of a S-L BVP. Especially, we search for a function $u_n(x)$ which satisfies Eq.(1) and the BCs (2) and also for a corresponding value of the parameter $\lambda = \lambda_n$ which also satisfies Eq.(1).

Definition 3

A non-trivial solution of a S-L system is called an **eigenfunction** and the corresponding λ is called its **eigenvalue**. The set of all eigenvalues of a regular S-L system is called the **spectrum** of the system.

1.1 Important Identities and implications

Let u and υ be both twice differentiable functions, satisfying the following equations

$$Lu = -\left(p(x)u'(x)\right)' + q(x) \tag{1}$$

$$L\upsilon = -\left(p(x)\upsilon'(x)\right)' + q(x)\,. \tag{2}$$

By multiplying Eqs.(1),(2) by υ and u respectively and subtracting Eq.(2) from (1), we obtain

$$\upsilon Lu - uL\upsilon = -\upsilon(x)\left(p(x)u'(x)\right)' + u(x)\left(p(x)\upsilon'(x)\right)' \tag{3}$$

Where, the last is known as Lagrange's Identity.

Now, by integrating Eq.(3) over the interval $[a,b]$ we obtain

$$\int_a^b \upsilon(x)Lu(x) - u(x)L\upsilon(x)\; dx = \left[p(x)\upsilon(x)u'(x) - p(x)u(x)\upsilon'(x)\right]_a^b, \tag{4}$$

which is Green's Identity.

If the functions u,υ satisfy also the BCs (2a,b) of the previous section, then

$\left[p(x)\upsilon(x)u'(x) - p(x)u(x)\upsilon'(x)\right]_a^b = 0\,.$

Proof.

Firstly, we write the bracket in the following form, in order to utilize the BCs.

$$\left[p(x)\upsilon(x)u'(x) - p(x)u(x)\upsilon'(x)\right]_a^b = p(b)\upsilon(b)u'(b) - p(b)u(b)\upsilon'(b) - p(a)\upsilon(a)u'(a) + p(a)u(a)\upsilon'(a).$$

Assume that the BCs at the endpoint b are Neumann, ($\mu_0 = 0$).This, yields to

$$p(b)\upsilon(b)u'(b) - p(b)u(b)\upsilon'(b) = 0\,. \tag{5}$$

Note that in the case of Dirichlet BCs, ($u(b)=0, \upsilon(b)=0$), Eq.(5) is valid.
In case of general BCs at the endpoint b we have

$$p(b)\upsilon(b)u'(b) - p(b)u(b)\upsilon'(b) = \mu_0 u(b)\upsilon(b) - p(b)u(b)\upsilon'(b), \tag{6}$$

since u satisfies the BC (2b) of the previous section.
By factorizing the right hand side of Eq.(6), he derive

$$p(b)\upsilon(b)u'(b) - p(b)u(b)\upsilon'(b) = u(b)\left(\mu_0\upsilon(b) - p(b)\upsilon'(b)\right) = 0. \tag{7}$$

Where, for the last equation we utilized the fact that $\upsilon(x)$ also satisfies the BCs.
Hence, in every case $p(x)\upsilon(x)u'(x) - p(x)u(x)\upsilon'(x)$ is zero at $x=b$.
By using the same arguments, $p(x)\upsilon(x)u'(x) - p(x)u(x)\upsilon'(x)$ is also zero at $x=a$.
Therefore, we conclude that $\left[p(x)\upsilon(x)u'(x) - p(x)u(x)\upsilon'(x)\right]_a^b = 0$.

Using the result of the previous proof, Green's Identity for u,υ satisfying the S-L problem (1)-(2) has the form

$$\int_a^b \upsilon(x)Lu(x) - u(x)L\upsilon(x)\, dx = 0. \tag{8}$$

The above equation, allow us to conclude that the operator L is self-adjoint, which implies the characterization of the BVP as a S-L regular, self-adjoint problem (see Ince, 1956 Chapters 9.31, 9.4)

1.2 Properties of eigenvalues and eigenfunctions.

Theorem 1(E. Coddington, R. Carlson, 1977, Theorem 8.3 p.257)
If L is self-adjoint, then

(i) each eigenvalue of L, λ is real,
(ii) eigenfunctions corresponding to distinct eigenvalues are orthogonal. ■

Proof.

(i) Let λ be a non-zero eigenvalue of the operator L.Then,

$$Lx = \lambda x,\ \|x\| > 0. \tag{1}$$

Because of the self-adjointness of the operator L we have

$$0 = \left(Lx, x\right) - \left(x, Lx\right) = \left(\lambda x, x\right) - \left(x, \lambda x\right) = \tag{2}$$
$$= \lambda\left(x,x\right) - \bar{\lambda}\left(x,x\right) = \left(\lambda - \bar{\lambda}\right)\|x\|^2.$$

Thus, $\lambda = \bar{\lambda}$, since $\|x\| > 0$.

(ii) If x, y are eigenfunctions of L corresponding to the eigenvalues $\lambda, \nu \in \mathbb{R}, \lambda \neq \nu$ respectively, then Eq.(2) gives us

$$0 = \left(y, Lx\right) - \left(x, Ly\right) = \left(y, \lambda x\right) - \left(x, \nu y\right) =$$
$$= \lambda\left(y,x\right) - \nu\left(x,y\right) = \left(\lambda - \nu\right)\left(x,y\right).$$

$$\Rightarrow (x,y)=0,$$

which implies the orthogonality of x, y.

Theorem 2 (E.Coddington, R.Carlson 1977)
Each eigenvalue, λ, of the operator L is simple. ■

Proof:
Assume that u,υ are eigenfunctions corresponding to λ. Since u,υ satisfying the BC (2a), the vectors $(u(a),u'(a))$, $(\upsilon(a),\upsilon'(a)) \in \mathbb{R}^2$ are both orthogonal to the non-zero vector $(-\nu_0,1)$. Hence, they lie in a one-dimensional subspace of $\mathbb{R}^2$
If $\mathbf{X}=(u,\upsilon)$, then the Wronskian W_{X} at a is

$$W_{\mathrm{X}}(a)=\det\begin{pmatrix} u(a) & \upsilon(a) \\ u'(a) & \upsilon'(a)\end{pmatrix} \tag{3}$$

Since the columns are linearly dependent, $W_{\mathrm{X}}(a)=0$.
Consequently,(see Coodington Carlson 1997, Th.2.10,p.37)

$$W_{\mathrm{X}}(t)=0, \forall t\in[a,b]. \tag{4}$$

so u,υ are linearly dependent.
This implies that $\dim Ker(L-\lambda \mathrm{I})=1$, which is equivalent to the fact that the eigenvalue λ is simple.

Let P and $\tilde{P}$ be the following BVPs

$$P \qquad \begin{aligned} &-(p(x)u'(x))'+q(x)u(x)=\lambda w(x)u(x), \quad a<x<b \\ &(pu')(a)-\nu_0 u(a) = 0 \\ &(pu')(b)-\mu_0 u(b) = 0 \end{aligned}$$

$$\tilde{P} \qquad \begin{aligned} &-(\tilde{p}(x)\tilde{u}'(x))'+\tilde{q}(x)\tilde{u}(x)=\tilde{\lambda}\tilde{w}(x)\tilde{u}(x), \quad \tilde{a}<x<\tilde{b} \\ &(\tilde{p}\tilde{u}')(\tilde{a})-\tilde{\nu}_0 \tilde{u}(\tilde{a}) = 0 \\ &(\tilde{p}\tilde{u}')(\tilde{b})-\tilde{\mu}_0 \tilde{u}(\tilde{b}) = 0 \end{aligned}$$

Theorem 3 (Continuity of the eigenvalues)
For the above BVPs P and $\tilde{P}$ we have
$\forall \varepsilon>0$, $\exists \delta>0$ such that, if

$$|a-\tilde{a}|+|b-\tilde{b}|+|\nu_0-\tilde{\nu}_0|+|\mu_0-\tilde{\mu}_0|+\int_a^b |p(x)-\tilde{p}(x)|+|q(x)-\tilde{q}(x)|+|w(x)-\tilde{w}(x)|dx<\delta$$

then,

$$|\lambda-\tilde{\lambda}|<\varepsilon. \qquad ■$$

Proof:
See Theorem 3.1 of Q.Kong, A. Zettl , (1996).

By a normalized eigenfunction u of a S-L problem we mean an eigenfunction u that satisfies

$$\int_a^b u^2(x)w(x) = 1. \tag{5}$$

Theorem 4

Assume that the eigenvalue $\tilde{\lambda}$ of the BVP $\tilde{P}$ is simple and let $\tilde{u}$ denote a normalized eigenfunction of $\tilde{\lambda}$. Then there exist normalized eigenfunction u corresponding to the eigenvalue λ of the BVP P such that

$u \to \tilde{u}$ and $p\tilde{u}' \to pu'$, as $\tilde{P} \to P$. ■

Proof:
See Theorem 3.2 of Q. Kong, A. Zettl , (1996).

2. Variation of eigenvalues of regular Sturm-Liouville problems

Consider the regular, self-adjoint Sturm-Liouville (S-L) problem

$$-\left(p(z)u'(z)\right)' + q(z)u(z) = \lambda\, w(z)u(z), \quad \text{in } (a,b), \qquad -\infty < a < b < \infty, \tag{1}$$

$$(pu')(a) - \nu_0\, u(a) = 0, \tag{2a}$$

$$(pu')(b) - \mu_0\, u(b) = 0 \tag{2b}$$

where $p(z)\in C^1[a,b], p(z) > 0$, $q(z), w(z)\in C[a,b], w(z) > 0$ and ν_0, μ_0 are given real numbers.

Assume that the parameter λ is an eigenvalue of the problem. As we have noticed, each eigenvalue λ depends on the data of the problem, that is, the functions $p(z), q(z), w(z)$, the constants ν_0, μ_0 and on the domain $[a,b]$. Since, in this work, we are interested in studying this dependence, it is expedient to use the following notation for the eigenvalue

$$\lambda = \lambda\left(a,b,\nu_0,\mu_0, p(\cdot), q(\cdot), w(\cdot)\right). \tag{3}$$

That means that λ is a usual function of a, b, μ_0, ν_0, and a functional on the functions $p(z)$, $q(z), w(z)$.

2.1 Variation of eigenvalues with respect to the boundary points

At first, we assume that all coefficients of the BVP (1)-(2) of Section 2, remain unchanged, except of the endpoints a and b. Thus, we state and prove the following theorem

Theorem 1 [Q. Kong, A. Zettl , (1999)]**.**The variation (shape derivative) of the eigenvalue λ, of the BVP (1), (2) of Sec. 2, considered as a function of the endpoints a and b, respectively, is given by

(i) $$\frac{d\lambda(a)}{da} = \frac{1}{p(a)}\,(pu')^2(a) - u^2(a)\left[\, q(a) - \lambda(a)\, w(a)\,\right], \tag{1a}$$

(ii) $$\frac{d\lambda(b)}{db} = -\frac{1}{p(b)}(pu')^2(b) + u^2(b)\left[\, q(b) - \lambda(b)\, w(b)\,\right], \tag{1b}$$

where u is the corresponding normalized ($\| u \|_{L^2(w\,dx)} = 1$) eigenfunction of λ. ■

Proof:

The arguments of the proof are based on those of Kong's and Zettl's(Theorems 3.2 and 3.3) in Q. Kong, A. Zettl , (1999)

(i) Denoting by $u(\cdot\,, a)$ the normalized eigenfunction belonging to the eigenvalue $\lambda(a)$, we have

$$-\left(p(z)\, u'(z,a)\right)' + q(z)\, u(z,a) = \lambda(a)\, w(z)\, u(z,a)\,. \tag{2}$$

Multiplying Eq.(2) by $u(\cdot\,, a+h)$, the eigenfunction corresponding to $\lambda(a+h)$, where $h \neq 0$ and significantly small, we obtain

$$-\left(p(z)u'(z,a)\right)' u(z,a+h) + q(z)\,u(z,a)\,u(z,a+h) \;=\; \lambda(a)\,w(z)\,u(z,a)\,u(z,a+h)\,. \tag{3}$$

In terms of the notation $u(\cdot\,,a)$ for the eigenfunction, the BCs (2a,b) take the form

$$(pu')(a,a) - \nu_0\,u(a,a) \;=\; 0\,, \tag{4a}$$

$$(pu')(b,a) - \mu_0\,u(b,a) \;=\; 0 \tag{4b}$$

Similarly, for $\lambda(a+h)$ and $u(\cdot\,,a+h)$ we obtain

$$-\left(p(z)\,u'(z,a+h)\right)' u(z,a) + q(z)\,u(z,a+h)\,u(z,a) = \lambda(a+h)\,w(z)\,u(z,a+h)\,u(z,a)\,, \tag{5}$$

$$(pu')(a+h,a+h) - \nu_0\,u(a+h,a+h) \;=\; 0\,, \tag{6a}$$

$$(pu')(b,a+h) - \mu_0\,u(b,a+h) \;=\; 0 \tag{6b}$$

By subtracting Eq. (3) from Eq. (5), and integrating over the interval $[a,b]$, we find

$$\begin{aligned}\left(\lambda(a+h) - \lambda(a)\right)\int_a^b u(s,a)u(s,a+h)\,w(s)\,ds \;=\;& \\ =\int_a^b -\left(pu'(s,a+h)\right)' u(s,a)\,ds + \int_a^b \left(pu'(s,a)\right)' u(s,a+h)\,ds.&\end{aligned} \tag{7}$$

Performing an integration by parts in each integral of the right-hand side of Eq.(7), we obtain

$$\begin{aligned}\left(\lambda(a+h) - \lambda(a)\right)\int_a^b u(s,a)\,u(s,a+h)\,w(s)\,ds \;=\;& \\ =\;-\left[(pu')(s,a+h)\,u(s,a) - (pu')(s,a)\,u(s,a+h)\right]_a^b&\end{aligned} \tag{8}$$

At the unperturbed endpoint $z=b$, we have that the difference $(pu')(b,a+h)u(b,a) - (pu')(b,a)u(b,a+h)$ vanishes. Indeed, if $\mu_0 = 0$(Neumann boundary conditions), BCs (4b) and (6b) become $u'(b,a) = 0$ and $u'(b,a+h) = 0$, thus, $(pu')(b,a+h)u(b,a) - (pu')(b,a)u(b,a+h) = 0$ trivially. Also, if the BC at the endpoint b is Dirichlet, means that $u(b,a) = 0$ and $u(b,a+h) = 0$. This results again in $(pu')(b,a+h)u(b,a) - (pu')(b,a)u(b,a+h) = 0$.On the other hand, if $\mu_0 \neq 0$, by using BC (4b), we find

$$\begin{aligned}-(pu')(b,a+h)\,u(b,a) + (pu')(b,a)\,u(b,a+h) &\overset{(4b)}{=} \\ =-(pu')(b,a+h)\,u(b,a) \;+\; \mu_0\,u(b,a)\,u(b,a+h) &= \\ =\; u(b,a)\left[-(pu')(b,a+h) + \mu_0\,u(b,a+h)\right] &\overset{(6b)}{=}\; 0\,.\end{aligned} \tag{9}$$

Thus, Eq.(8) becomes

$$\big(\lambda(a+h)-\lambda(a)\big)\int_a^b u(s,a)\,u(s,a+h)\,w(s)\,ds = \\ = u(a,a)\,(pu')(a,a+h)-u(a,a+h)\,(pu')(a,a). \tag{10}$$

In this step we will work separately for each term of the right hand side of Eq.(10).We start from $(pu')(a,a+h)$ of the first term.

If the BC at the endpoint a is Dirichlet, $u(a,a)=0$, thus the term $u(a,a)\big((pu')(a,a+h)\big)$ vanishes.

In any other case, we subtract BC (6a) from $(pu')(a,a+h)$ in order to replace the derivative of the perturbed eigenfunction $(pu')(a,a+h)$ by proper integrals and we have

$$(pu')(a,a+h)=(pu')(a,a+h)-(pu')(a+h,a+h)+\nu_0 u(a+h,a+h)\,. \tag{11}$$

By applying the fundamental theorem of calculus to the first two terms of the right hand side of the above equation and using Eq.(1) of Section 3 we derive

$$(pu')(a,a+h) = -\int_a^{a+h}(pu')'(s,a+h)\,ds+\nu_0 u(a+h,a+h)$$

$$(pu')(a,a+h) = -\int_a^{a+h}\big[q(s)u(s,a+h)-\lambda(a+h)u(s,a+h)\,w(s)\big]ds+\nu_0 u(a+h,a+h)$$

By adding and subtracting the terms $q(s)u(s,a)$ and $\lambda(a+h)u(s,a)w(s)$ into the integrant we have

$$\begin{aligned}(pu')(a,a+h) &= -\int_a^{a+h} q(s)u(s,a)\;ds+\int_a^{a+h} q(s)\big[u(s,a)-u(s,a+h)\big]\;ds+ \\ &+\lambda(a+h)\int_a^{a+h} u(s,a)w(s)\;ds-\lambda(a+h)\int_a^{a+h}\big[u(s,a)-u(s,a+h)\big]w(s)\;ds+ \\ &+\nu_0 u(a+h,a+h)\end{aligned} \tag{12}$$

Now we study $u(a,a+h)$ from the second term of the right hand side of Eq.(10).Notice that if $\nu_0=0$ the term $u(a,a+h)\big(pu'(a,a)\big)$ vanishes.

For $\nu_0\neq 0$, we subtract from $u(a,a+h)$ the B.C (6a), in order to create the difference $u(a,a+h)-u(a+h,a+h)$ and

$$u(a,a+h)=u(a,a+h)-u(a+h,a+h)+\frac{1}{\nu_0}(pu')(a+h,a+h).$$

In this case we cannot create the second derivative in order to utilize Eq.(1), so we have to work differently. By applying the fundamental theorem of calculus to the first two terms of the above equation and also multiplying and dividing $u'(s,a+h)$ by $p(s)$ we obtain

$$u(a,a+h)=-\int_a^{a+h}\frac{1}{p(s)}(pu')(s,a+h)\,ds+\frac{1}{\nu_0}(pu')(a+h,a+h).$$

By adding and subtracting the term $\dfrac{1}{p(s)}(pu')(s,a)$ into the integrant, in order to eliminate the term $(pu')(s,a+h)$ we have

$$u(a,a+h) = -\int_a^{a+h} \frac{1}{p(s)}(pu')(s,a)\,ds + \int_a^{a+h} \frac{1}{p(s)}\big[(pu')(s,a) - (pu')(s,a+h)\big]ds + \\ + \frac{1}{\nu_0}(pu')(a+h,a+h). \tag{13}$$

By substituting Eqs. (12) and (13) into the right hand side of Eq. (10) and dividing both sides of the resulting equation by h we obtain

$$\frac{\lambda(a+h)-\lambda(a)}{h}\int_a^b u(s,a)u(s,a+h)w(s)\,ds = \underbrace{-\,u(a,a)\,\frac{1}{h}\int_a^{a+h} q(s)\,u(s,a)\,ds}_{A} + \\ + \underbrace{u(a,a)\,\frac{1}{h}\int_a^{a+h} q(s)\big[u(s,a)-u(s,a+h)\big]ds}_{B} + \underbrace{\lambda(a+h)\,u(a,a)\,\frac{1}{h}\int_a^{a+h} u(s,a)\,w(s)\,ds}_{C} - \\ - \underbrace{\lambda(a+h)\,u(a,a)\,\frac{1}{h}\int_a^{a+h}\big[u(s,a)-u(s,a+h)\big]\,w(s)\,ds}_{D} + \frac{1}{h}\,\nu_0\,u(a+h,a+h)\,u(a,a) + \\ + \underbrace{(pu')(a,a)\,\frac{1}{h}\int_a^{a+h}\frac{1}{p(s)}\,(pu')(s,a)\,ds}_{E} - \underbrace{(pu')(a,a)\,\frac{1}{h}\int_a^{a+h}\frac{1}{p(s)}\,\big[(pu')(s,a)-(pu')(s,a+h)\big]ds}_{F} + \\ - \frac{1}{h}\frac{1}{\nu_0}(pu')(a+h,a+h)\,(pu')(a,a) \tag{14}$$

The sum of the boundary terms of the above equation is equal to zero, as we confirm below, by using the BCs (4a) and (6a).

$$\nu_0\,u(a+h,a+h)\,u(a,a) - \frac{1}{\nu_0}(pu')(a+h,a+h)\,(pu')(a,a) = \\ \underset{(6a)}{=} \nu_0\,u(a+h,a+h)\,u(a,a) - u(a+h,a+h)(pu')(a,a) = \\ = u(a+h,a+h)\Big(\nu_0\,u(a,a) - (pu')(a,a)\Big) \underset{(4a)}{=} 0. \tag{15}$$

By using Lebesque's Differentiation Theorem to calculate the terms A,B,C,D,E,F we obtain

$$A = -\,u(a,a)\lim_{h\to 0}\left[\frac{1}{h}\int_a^{a+h} q(s)u(s,a)\,ds\right] = \\ = -\,u(a,a)\big(q(a)u(a,a)\big) = -\,u^2(a,a)\,q(a). \tag{16}$$

$$B = u(a,a)\lim_{h\to 0}\left[\frac{1}{h}\int_a^{a+h} q(s)\big[u(s,a) - u(s,a+h)\big]\, ds\right] =$$

$$= u(a,a)\Big(q(a)[u(a,a) - \lim_{h\to 0} u(a,a+h)]\Big) =$$

$$= u(a,a)\big(q(a)[u(a,a) - u(a,a)]\big) = 0, \tag{17}$$

since $u(s,a+h) \to u(s,a)$, as h tends to zero.

$$C = \lim_{h\to 0}\left[\lambda(a+h)u(a,a)\frac{1}{h}\int_a^{a+h} u(s,a)\, w(s)\, ds\right] =$$

$$= \lambda(a)u(a,a)\big[u(a,a)\, w(a)\big] = \lambda(a)u^2(a,a)\, w(a). \tag{18}$$

$$D = \lim_{h\to 0}\left[\lambda(a+h)u(a,a)\frac{1}{h}\int_a^{a+h} \big[u(s,a) - u(s,a+h)\big]\, ds\right] =$$

$$= \lim_{h\to 0}\left[\lambda(a+h)u(a,a)\frac{1}{h}\int_a^{a+h} \big[u(s,a) - u(s,a+h)\big]\, ds\right] =$$

$$= \lambda(a)u(a,a)\Big(u(a,a) - \lim_{h\to 0} u(a,a+h)\Big) = \lambda(a)u(a,a)\big(u(a,a) - u(a,a)\big) = 0\,, \tag{19}$$

since λ is continuous at a.

$$E = \lim_{h\to 0}\left[(pu')(a,a)\ \frac{1}{h}\int_a^{a+h} \frac{1}{p(s)}(pu')(s,a)\, ds\right] =$$

$$= (pu')(a,a)\frac{1}{p(a)}(pu')(a,a) = \frac{1}{p(a)}(pu')^2(a,a). \tag{20}$$

$$F = (pu')(a,a)\lim_{h\to 0}\left[\frac{1}{h}\int_a^{a+h} \frac{1}{p(s)}\big[(pu')(s,a) - (pu')(s,a+h)\big]\ ds\right] =$$

$$= (pu')(a,a)\left[\frac{1}{p(a)}(pu')(a,a) - \frac{1}{p(a)}\lim_{h\to 0}(pu')(a,a+h)\right] =$$

$$= (pu')(a,a)\left[\frac{1}{p(a)}(pu')(a,a) - \frac{1}{p(a)}(pu')(a,a)\right] = 0. \tag{21}$$

By taking the limit as h tends to zero, the left-hand side of Eq.(14) becomes

$$\lim_{h\to 0}\left[\frac{\lambda(a+h) - \lambda(a)}{h}\int_a^b u(s,a)u(s,a+h)w(s)\ ds\right] = \frac{d\lambda(a)}{da}. \tag{22}$$

Because $u(\cdot,a+h) \to u(\cdot,a)$, as h tends to zero and because of the normalization condition

$$\int_a^b u^2(s,a)\, w(s)\ ds = 1.$$

Taking into account Eqs. (15)-(22), we obtain the wanted formula

(ii) The proof is similar to (i).

Remark 1 Since we used Lebesque's Differentiation Theorem a choice of function spaces which is demanding by the theorem is the following
$w(z), q(z) \in L^1[a,b]$, $u'(z) \in L^1[a,b]$

Remark 2 Notice that in case of Dirichlet BCs, the proof is the same until Eq.(9). In the right-hand side of Eq.(10) there is only one term since the other satisfies the BC $\left(u(a,a)=0\right)$ .Hence, are implied the following corollaries

Corollary 1. If the boundary conditions of the BVP (1)-(2) are Dirichlet at the endpoint a and either Dirichet or Neumann at the endpoint b, the variation of the eigenvalue λ with respect to a takes the form

$$\frac{d\lambda(a)}{da} = p(a)\left(u'(a)\right)^2. \tag{23}$$

If the boundary conditions are Neumann at the endpoint a and either Dirichlet or Neumann at the endpoint b, the variation of the eigenvalue λ as a function of the endpoint a is given by

$$\frac{d\lambda(a)}{da} = -\, u^2(a)[q(a) - \lambda(a)w(a)]. \tag{24}$$

Corollary 2. If the boundary conditions of the BVP (1)-(2) are either Dirichlet or Neumann at the endpoint a and Dirichletat b, the variation of the eigenvalue λ with respect to b takes the form

$$\frac{d\lambda(b)}{db} = -\frac{1}{p(b)}(pu')^2(b). \tag{25}$$

If the boundary conditions are either Neumann or Dirichlet at the endpoint a and Neumann at the endpoint b, the variation of the eigenvalue λ as a function of the endpoint b is given by

$$\frac{d\lambda(b)}{db} = u^2(b)\left[\, q(b) - \lambda(b)\, w(b)\,\right]. \tag{26}$$

Application of the variation formulas to the problem with the differential equation in the standard Liouville normal form

Our purpose is to apply the formulas which proved above, to the following problem

$$\partial_z^2 Z_n(z) + \left(\lambda_n - Q(z)\right) Z_n(z) = 0, \qquad a < z < b, \tag{27}$$

$$B^b Z_n \equiv \partial_z Z_n(b) - \mu_0 Z_n(b) = 0, \tag{28a}$$

$$B_a Z_n \equiv \partial_z Z_n(a) - \nu_0 Z_n(a) = 0. \tag{28b}$$

We observe that Eq.(1), becomes identical to Eq.(27) under the choice of $p(z)=1, q(z)=Q(z)$ and $w(z)=1$. Moreover, the BCs (2a,b) are identified with BCs (28a,b) under the choice of $p(z)=1$.

Recall also that the formulas are proved for normalized eigenfunctions, hence $u(z)$ will be replaced by $\dfrac{Z_n(z)}{\|Z_n\|}$.

Thus, the variation of the eigenvalues of the problem with respect to the endpoint a is given by

$$\frac{d\lambda_n(a)}{da} = \frac{\left(Z_n'(a)\right)^2 - Z_n^2(a)\left(Q(a)-\lambda_n(a)\right)}{\|Z_n\|^2}. \tag{29}$$

Similarly, the variation of the eigenvalues λ_n, with respect to the endpoint b has the form

$$\frac{d\lambda_n(b)}{db} = \frac{-\left(Z_n'(b)\right)^2 + Z_n^2(b)\left(Q(b)-\lambda_n(b)\right)}{\|Z_n\|^2}. \tag{30}$$

2.2.Variation of eigenvalues with respect to the coefficients.

Consider the regular, self-adjoint Sturm-Liouville problem

$$-\left(pu'(z)\right)' + q(z)u(z) = \lambda w(z)u(z), \qquad \text{on } (a,b), \qquad -\infty < a < b < \infty, \tag{1}$$

$$(pu')(a) - \nu_0 u(a) = 0, \tag{2a}$$

$$(pu')(b) - \mu_0 u(b) = 0, \tag{2b}$$

Assume now that the domain remains unchanged and the data $p(z), q(z), w(z)\ \nu_0, \mu_0$ are perturbed. To start with, we consider the following BVP, which is a perturbed version of BVP (1)-(2)

$$-\left(\tilde{p}\tilde{u}'(z)\right)' + \tilde{q}(z)\tilde{u}(z) = \tilde{\lambda}\tilde{w}(z)\tilde{u}(z), \qquad \text{on } (a,b), \qquad -\infty < a < b < \infty, \tag{3}$$

$$(\tilde{p}\tilde{u}')(a) - \tilde{\nu}_0 \tilde{u}(a) = 0, \tag{4a}$$

$$(\tilde{p}\tilde{u}')(b) - \tilde{\mu}_0 \tilde{u}(b) = 0. \tag{4b}$$

We shall now derive some identities connecting the eigenvalues and the eigenfunctions of the two problems (1)-(2) and (3)-(4), which will be repeatedly used in the sequel.

By multiplying Eq.(1) by $\tilde{u}(z)$ and Eq. (2) by $u(z)$ we obtain, respectively,

$$-\left(pu'(z)\right)'\tilde{u}(z) + q(z)u(z)\tilde{u}(z) = \lambda w(z)u(z)\tilde{u}(z), \tag{5}$$

$$-\left(\tilde{p}\tilde{u}'(z)\right)' u(z) + \tilde{q}(z)\tilde{u}(z)u(z) = \tilde{\lambda}\tilde{w}(z)\tilde{u}(z)u(z). \tag{6}$$

Subtracting Eq.(5) from (6), results in

$$-\left(\tilde{p}\,\tilde{u}'(z)\right)'u(z)+\left(pu'(z)\right)'\tilde{u}(z)+\left(\tilde{q}(z)-q(z)\right)\tilde{u}(z)u(z)=\tilde{\lambda}\,\tilde{w}(z)\tilde{u}(z)u(z)-\lambda\,w(z)\tilde{u}(z)u(z)$$

By integrating the above equation over the interval $[a,b]$,we take the following identity:

$$\int_a^b\left(\tilde{\lambda}\,\tilde{w}(z)\tilde{u}(z)u(z)-\lambda\,w(z)\tilde{u}(z)u(z)\right)dz=\int_a^b-\left(\tilde{p}\,\tilde{u}'(z)\right)'u(z)\,dz+\int_a^b\left(pu'(z)\right)'\tilde{u}(z)\,dz$$
$$+\int_a^b\left(\tilde{q}(z)-q(z)\right)\tilde{u}(z)u(z)\,dz$$

Performing an integration by parts in the first two integrals of the right-hand side of the above equation, we obtain

$$\begin{aligned}\int_a^b\left(\tilde{\lambda}\,\tilde{w}(z)\tilde{u}(z)u(z)-\lambda\,w(z)\tilde{u}(z)u(z)\right)dz&=\left[-(\tilde{p}\tilde{u}')(z)u(z)+(pu')(z)\tilde{u}(z)\right]_a^b+\\&+\int_a^b\left(\tilde{p}(z)-p(z)\right)\tilde{u}'(z)u'(z)\,dz+\int_a^b\left(\tilde{q}(z)-q(z)\right)\tilde{u}(z)u(z)\,dz.\end{aligned}\tag{7}$$

If the eigenfunctions $u(z)$ and $\tilde{u}(z)$ satisfy the same boundary conditions (2a,b),which means that the constants ν_0,μ_0 remain unchanged, the bracket$\left[-(\tilde{p}\tilde{u}')(z)u(z)+(pu')(z)\tilde{u}(z)\right]_a^b$, vanishes and Eq.(7) takes the form

$$\boxed{\int_a^b\left(\tilde{\lambda}\,\tilde{w}(z)\tilde{u}(z)u(z)-\lambda\,w(z)\tilde{u}(z)u(z)\right)dz=\int_a^b\left(\tilde{p}(z)-p(z)\right)\tilde{u}'(z)u'(z)\,dz+\int_a^b\left(\tilde{q}(z)-q(z)\right)\tilde{u}(z)u(z)\,dz.}\tag{8}$$

If the constants ν_0,μ_0 are also perturbed, then, taking into account that $u(z)$ satisfies the BCs (2a,b) and $\tilde{u}(z)$ satisfies the BCs (4a,b), the bracket expands as follows

$$\begin{aligned}\left[-(\tilde{p}\tilde{u}')(z)u(z)+(pu')(z)\tilde{u}(z)\right]_a^b&=-(\tilde{p}\tilde{u}')(b)u(b)+(pu')(b)\tilde{u}(b)+(\tilde{p}\tilde{u}')(a)u(a)-(pu')(a)\tilde{u}(a)=\\&=-\tilde{\mu}_0\,\tilde{u}(b)u(b)+\mu_0\,u(b)\tilde{u}(b)+\tilde{\nu}_0\,\tilde{u}(a)u(a)-\nu_0\,u(a)\tilde{u}(a)=\\&=\tilde{u}(b)u(b)\left(\mu_0-\tilde{\mu}_0\right)+\tilde{u}(a)u(a)\left(\tilde{\nu}_0-\nu_0\right).\end{aligned}$$

By utilizing the above equation, Eq.(7) is written as

$$\boxed{\begin{aligned}\int_a^b\left(\tilde{\lambda}\,\tilde{w}(z)\tilde{u}(z)u(z)-\lambda\,w(z)\tilde{u}(z)u(z)\right)dz&=\tilde{u}(b)u(b)\left(\mu_0-\tilde{\mu}_0\right)+\tilde{u}(a)u(a)\left(\tilde{\nu}_0-\nu_0\right)+\\&+\int_a^b\left(\tilde{p}(z)-p(z)\right)\tilde{u}'(z)u'(z)\,dz+\int_a^b\left(\tilde{q}(z)-q(z)\right)\tilde{u}(z)u(z)\,dz.\end{aligned}}\tag{9}$$

Theorem 1 [Q.Kong, A. Zettl , (1996)**].** The Fréchet derivative of the eigenvalue λ of the BVP (1)-(2), considered as a functional acting on $\frac{1}{p(z)}$, $p(z)$, $q(z)$, $w(z)$, and as a function of ν_0 and μ_0, respectively, is given by

(i) $$d\lambda\left(\frac{1}{p}\right)h = -\int_a^b \left|(pu')(s)\right|^2 h(s)\,ds, \tag{10}$$

(ii) $$d\lambda(p)h = \int_a^b \left|u'(s)\right|^2 h(s)\,ds, \tag{11}$$

(iii) $$d\lambda(q)h = \int_a^b \left|u(s)\right|^2 h(s)\,ds, \tag{12}$$

(iv) $$d\lambda(w)h = -\lambda\int_a^b \left|u(s)\right|^2 h(s)\,ds, \tag{13}$$

(v) $$\frac{d\lambda(\nu_0)}{d\nu_0} = u^2(a), \tag{14}$$

(vi) $$\frac{d\lambda(\mu_0)}{d\mu_0} = -u^2(b), \tag{15}$$

where u is the corresponding normalized eigenfunction belonging to λ. ■

Proof: The arguments of the proof presented below are based on the proof of Theorem 4.2 of Q. Kong, A. Zettl, (1996)

(i) Applying Eq.(8) for the eigenvalues,
$\lambda = \lambda\left(1/p(\cdot)\right)$ and $\tilde{\lambda} = \lambda\left(1/p(\cdot) + h(\cdot)\right)$, with $\|h\| \neq 0$ but sufficiently small, we have

$$\left[\lambda\left(\frac{1}{p}+h\right) - \lambda\left(\frac{1}{p}\right)\right]\int_a^b u(s)\tilde{u}(z)w(s)\,ds = \int_a^b (p_h\tilde{u})(s)u'(s) - (pu')(s)\tilde{u}(s)\,ds, \tag{16}$$

where $$\frac{1}{p_h} = \frac{1}{p} + h. \tag{17}$$

Taking into account that $p - p_h = p\,p_h\,h$, we obtain

$$\left[\lambda\left(\frac{1}{p}+h\right) - \lambda\left(\frac{1}{p}\right)\right]\int_a^b u(s)\tilde{u}(s)w(s)\,ds = -\int_a^b (pu')(s)(p_h\tilde{u})(s)h(s)\,ds, \tag{18}$$

Recalling that $\tilde{u} \to u$ and $\frac{1}{p_h} \to \frac{1}{p}$ as $\|h\| \to 0$, and because of the normalization of the eigenfunctions, we finally obtain

$$d\lambda\left(\frac{1}{p}\right)h = -\int_a^b \left|(pu')(s)\right|^2 h(s)\,ds.$$

(ii) Applying Eq.(8) for the eigenvalues $\lambda = \lambda\left(p(\cdot)\right)$ and $\tilde{\lambda} = \lambda\left(p(\cdot) + h(\cdot)\right)$, with $\|h\| \neq 0$, we obtain

$$\left(\lambda\left(p + h\right) - \lambda\left(p\right)\right) \int_a^b u(s)\tilde{u}(s)\,w(s)\; ds \;=\; \int_a^b h(s)\tilde{u}'(s)u'(s)\; ds\,. \tag{19}$$

Taking into account that $\tilde{u}' \to u'$ as $\|h\|$ tends to zero and the normalization condition, the Fréchet derivative of λ with respect to the coefficient $p(z)$ takes the form

$$d\lambda(p)\,h \;=\; \int_a^b \left|u'(s)\right|^2 h(s)\,ds.$$

(iii) The proof is similar to (ii)

(iv) By applying Eq.(8) for the eigenvalues $\lambda = \lambda\left(w(\cdot)\right)$ and $\tilde{\lambda} = \lambda\left(w(\cdot) + h(\cdot)\right)$,we take

$$\int_a^b \left[\lambda(w+h)\left(w(s)+h(s)\right)u(s)\tilde{u}(s) - \lambda(w)w(s)u(s)\tilde{u}(s)\right] ds = 0\,. \tag{20}$$

In order to create the difference $\lambda(w+h) - \lambda(w)$ we make the following calculations

$$\lambda(w+h)\int_a^b \left(w(s) + h(s)\right)u(s)\tilde{u}(s)\; ds \;-\lambda(w)\int_a^b w(s)u(s)\tilde{u}(s)\; ds = 0 \tag{21}$$

$$\left(\lambda(w+h)-\lambda(w)\right)\int_a^b w(s)u(s)\tilde{u}(s)\; ds \;=\; -\,\lambda(w+h)\int_a^b h(s)u(s)\tilde{u}(s)\; ds\,. \tag{22}$$

By using the continuity of eigenvalues and eigenfunctions and the normalization condition, we obtain

$$d\lambda(w)\,h \;=\; -\lambda\int_a^b \left|u(s)\right|^2 h(s)\,ds\,.$$

(v) Since the constant ν_0 is perturbed, use is made of Eq.(9). By applying it to the eigenvalues $\lambda = \lambda(\nu_0)$ and $\tilde{\lambda} = \lambda(\nu_0 + h)$, with $h \neq 0$,we derive

$$\left[\lambda\left(\nu_0 + h\right) - \lambda\left(\nu_0\right)\right]\int_a^b u(s)\tilde{u}(s)\,w(s)\,ds = h(s)\,u(a)\,\tilde{u}(a). \tag{23}$$

Dividing both sides of Eq.(23) by h, taking the limit as h tends to zero, and because of the continuity and the normalization of the eigenfunctions, Eq.(23) leads to the desired formula

$$\frac{d\lambda\left(\nu_0\right)}{d\left(\nu_0\right)} = u^2(a)\,.$$

(vi) The proof is similar to (v).

Application of the variation formulas to the problem with the differential equation in the standard Liouville normal form

Our purpose is to apply the formulas which proved above, to the following problem

$$\partial_z^2 Z_n(z) + \big(\lambda_n - Q(z)\big) Z_n(z) = 0, \qquad a < z < b, \tag{24}$$

$$B^b Z_n \equiv \partial_z Z_n(b) - \mu_0 Z_n(b) = 0, \tag{25a}$$

$$B_a Z_n \equiv \partial_z Z_n(a) - \nu_0 Z_n(a) = 0. \tag{25b}$$

We notice that Eq.(1) can be identified with Eq.(24) under the choice of $p(z) = 1$, $q(z) = Q(z)$, and $w(z) = 1$. Moreover, the BCs (2a,b) are identified with BCs (25a,b) if $p(z) = 1$.
According to the Theorem 1 of Section 2.2, the eigenfunctions are assumed normalized, hence we replace $u(z)$ by $\dfrac{Z_n(z)}{\|Z_n\|}$.

The above identification, implies that the variation of the eigenvalues of the problem with respect to the function $Q(z)$ is given by

$$d\lambda(Q)h = \int_a^b \frac{\left|Z_n(s)\right|^2}{\left\|Z_n\right\|^2} h(s)\, ds, \tag{26}$$

Moreover, the variation of the eigenvalues λ_n, with respect to μ_0 is the following

$$\frac{d\lambda(\mu_0)}{d\mu_0} = -\frac{Z_n^2(b)}{\left\|Z_n\right\|^2}. \tag{27}$$

Last, the variation of the eigenvalues with respect to ν_0 has the form

$$\frac{d\lambda(\nu_0)}{d\nu_0} = \frac{Z_n^2(a)}{\left\|Z_n\right\|^2}. \tag{28}$$

2.3 Solution of the constant-coefficient, Sturm-Liouville problems

Our goal herein is to solve the constant-coefficient S-L problem and find the variation of the eigenvalues with respect to the boundary points, by straightforward differentiation in order to compare them to the formulas proved in Section 2.1.We proceed now to derive the analytical solution to the constant-coefficient, SLP:

Find the (complex) constants k_n for which the following homogeneous boundary value problem admits of nontrivial solutions $Z_n = Z_n(z) \neq 0$:

$$\partial_z^2 Z_n(z) + \left(k_n^2 + Q\right) Z_n(z) = 0, \quad -h < z < \eta, \tag{1}$$

$$B^{\eta} Z_n \equiv [(\partial_z - \mu_0) Z_n]_{z=\eta} = 0, \qquad (2a)$$

$$B_h Z_n \equiv [(\partial_z - \nu_0) Z_n]_{z=-h} = 0. \qquad (2b)$$

To simplify the algebraic manipulations, we set

$$\rho_n^2 \stackrel{\text{def}}{=} k_n^2 + Q \quad \Leftrightarrow \quad \rho_n = \sqrt{k_n^2 + Q}\,. \qquad (3)$$

There are two possible cases $\rho_n \in \mathbb{R}$ and $\rho_n \in i\mathbb{R}$
The general solution of Eq. (1) is given by

$$Z_n(z) = A\cos\big(\rho_n (z+h)\big) + B\sin\big(\rho_n (z+h)\big), \qquad (4a)$$

from which we obtain the following expressions for the boundary values at $z = \eta$ and $z = -h$:

$$Z_n(z=\eta) = A\cos\big(\rho_n H\big) + B\sin\big(\rho_n H\big), \qquad (4b)$$

$$Z_n(z=-h) = A, \qquad (4c)$$

where $H = \eta + h$. The derivative $\partial_z Z_n$ of Z_n, and its boundary values, are easily calculated, as follows:

$$\partial_z Z_n(z) = -A\,\rho_n \sin\big(\rho_n (z+h)\big) + B\,\rho_n \cos\big(\rho_n (z+h)\big), \qquad (5a)$$

$$\partial_z Z_n(z=\eta) = -A\,\rho_n \sin\big(\rho_n H\big) + B\,\rho_n \cos\big(\rho_n H\big), \qquad (5b)$$

$$\partial_z Z_n(z=-h) = B\,\rho_n. \qquad (5c)$$

Using Eqs. (4c) and (5c), the boundary condition (2b) takes the form

$$B_h Z_n \equiv [(\partial_z - \nu_0) Z_n]_{z=-h} \equiv B\,\rho_n - A\,\nu_0 = 0. \qquad (6)$$

Assuming (tentatively) that $A = 0$, we conclude that $B\,\rho_n = 0$ and, thus, using also Eq. (5a), we find $\partial_z Z_n(z) = 0 \Rightarrow Z_n(z) = const$. Then, invoking Eq. (4c), we see that $Z_n(z) = 0$. This conclusion is unacceptable, since we are looking for nontrivial solutions. Thus, the assumption $A = 0$ should be rejected, obtaining $A \neq 0$. Accordingly, Eq. (6) can be written as

$$\frac{B}{A} = \frac{\nu_0}{\rho_n}. \qquad (7)$$

Similarly, using Eqs. (4b) and (5b), the boundary condition (2a) takes the form

$$\begin{aligned} B^{\eta} Z_n &\equiv [(\partial_z - \mu_0) Z_n]_{z=\eta} = \\ &= -A\,\rho_n \sin\big(\rho_n H\big) + B\,\rho_n \cos\big(\rho_n H\big) - A\,\mu_0 \cos\big(\rho_n H\big) - B\,\mu_0 \sin\big(\rho_n H\big) \\ &= -\big(A\,\rho_n + B\,\mu_0\big)\sin\big(\rho_n H\big) - \big(A\,\mu_0 - B\,\rho_n\big)\cos\big(\rho_n H\big) = 0 \quad \Rightarrow \\ &-\left(\rho_n + \frac{B}{A}\mu_0\right)\sin\big(\rho_n H\big) - \left(\mu_0 - \frac{B}{A}\rho_n\right)\cos\big(\rho_n H\big) = 0. \end{aligned}$$

Substituting B/A, from Eq. (7), into the above equation, we obtain the following general condition (dispersion relation) for the determination of ρ_n's:

$$-\left(\rho_n + \frac{\nu_0}{\rho_n}\,\mu_0\right)\sin\left(\rho_n H\right) - \left(\mu_0 - \frac{\nu_0}{\rho_n}\,\rho_n\right)\cos\left(\rho_n H\right) = 0\,,$$

or

$$\left(\rho_n + \frac{\mu_0\,\nu_0}{\rho_n}\right)\tan\left(\rho_n H\right) + \left(\mu_0 - \nu_0\right) = 0\,. \tag{7a}$$

Alternative equivalent forms of the dispersion relation (7) are also given below:

$$\tan\left(\rho_n H\right) = -\frac{\mu_0 - \nu_0}{\rho_n + \dfrac{\mu_0\,\nu_0}{\rho_n}} = -\frac{\left(\mu_0 - \nu_0\right)\rho_n}{\mu_0\,\nu_0 + \rho_n^2}\,, \tag{7b}$$

$$\frac{\nu_0}{\rho_n}\tan\left(\rho_n H\right) = \frac{\nu_0^2 - \mu_0\,\nu_0}{\rho_n^2 + \mu_0\,\nu_0}\,. \tag{7c}$$

At this point, we have to distinguish two cases

(i) If $\rho_n \in \mathbb{R}$,

The nontrivial solutions of the SLP (1), (2) are the eigenfunctions

$$Z_n(z) = M\left(\cos\left(\rho_n(z+h)\right) + \frac{\nu_0}{\rho_n}\sin\left(\rho_n(z+h)\right)\right) \tag{8}$$

where, $M \neq 0$ is an arbitrary constant (to be specified by appropriate normalization) and ρ_n are the roots of the (local) dispersion relation (7a)

(ii) If $\rho_n^2 = -\sigma_n^2 < 0, \Longleftrightarrow \rho_n = i\sigma_n,\ \sigma_n > 0$,

The dispersion relation (7b) takes the form

$$\tanh\left(\sigma_n H\right) = \frac{\sigma_n\left(\nu_0 - \mu_0\right)}{-\sigma_n^2 + \nu_0\mu_0}\,. \tag{9}$$

Or, in terms of ρ_n

$$\tan\left(\rho_n H\right) = \frac{\rho_n\left(\nu_0 - \mu_0\right)}{\rho_n^2 + \nu_0\mu_0}\,. \tag{10}$$

The non-trivial solutions in this case, are the following eigenfunctions

$$Z_n(z) = C\left(\cosh\left(\sigma_n(z+h)\right) + \frac{\nu_0}{\sigma_n}\sinh\left(\sigma_n(z+h)\right)\right) \tag{11}$$

where $C \neq 0$ is an arbitrary constant and σ_n are the roots of the (local) dispersion relation (9).

2.3a The L^2 – norm of the eigenfunctions.

In this subsection, since our aim is to compare the formulas of the Section 2.1 with those which will be derived by straightforward differentiation for the constant coefficient problem, we have to compute the L^2 – norm of the eigenfunctions of the problem.

We will work separately with the two different cases of ρ_n.

a). Setting tentatively $a = \nu_0 / \rho_n$, and assuming that a and ρ_n are real quantities, we obtain

$$\left\|\hat{Z}_n(z)\right\|_{L^2}^2 = \int_{-h(x)}^{\eta(x)} \left[\cos\left(\rho_n(z+h)\right) + a\sin\left(\rho_n(z+h)\right)\right]^2 \, dz \tag{1}$$

The right hand side of Eq.(1) can be written as the sum of the following integrals:

(i) $$\int_{-h(x)}^{\eta(x)} \cos^2\left(\rho_n(z+h)\right) dz = \int_0^H \frac{1+\cos\left(2\rho_n(z+h)\right)}{2} \, dz = \int_0^H \frac{1+\cos\left(2\rho_n u\right)}{2} \, du$$

$$= \left[\frac{1}{2}u\right]_{u=0}^{u=H} + \frac{1}{2}\left[\frac{\sin\left(2\rho_n u\right)}{2\rho_n}\right]_{u=0}^{u=H} =$$

$$= \frac{1}{2}H + \frac{1}{4\rho_n}\sin(2\rho_n H). \tag{2}$$

(ii) $$\int_{-h(x)}^{\eta(x)} 2a\cos\left(\rho_n(z+h)\right)\sin\left(\rho_n(z+h)\right) dz = \int_0^H 2a\cos\left(\rho_n u\right)\sin\left(\rho_n u\right) du =$$

$$= a\int_0^H \sin\left(2\rho_n u\right) du = a\left[\frac{-\cos\left(2\rho_n u\right)}{2\rho_n}\right]_{u=0}^{u=H} = \frac{a}{2\rho_n}\left(1-\cos(2\rho_n H)\right). \tag{3}$$

(iii) $$\int_{-h(x)}^{\eta(x)} a^2\sin^2\left(2\rho_n(z+h)\right) dz = \int_0^H a^2\sin^2\left(2\rho_n u\right) du =$$

$$= a^2\int_0^H \frac{1-\cos\left(2\rho_n u\right)}{2} \, du = \frac{a^2}{2}\int_0^H 1-\cos\left(2\rho_n u\right) du =$$

$$= \frac{a^2}{2}H - \frac{a^2}{2}\left[\frac{\sin\left(2\rho_n u\right)}{2\rho_n}\right]_{u=0}^{u=H} = \frac{a^2}{2}H - \frac{a^2}{4\rho_n}\sin(2\rho_n H). \tag{4}$$

Substituting Eqs.(2),(3),(4) into (1) we obtain

$$\left\|\hat{Z}_n(z)\right\|_{L^2}^2 = \left(\frac{1+a^2}{2}\right)H + \frac{1-a^2}{4\rho_n}\sin(2\rho_n H) + \frac{a}{2\rho_n}\left(1-\cos(2\rho_n H)\right). \tag{5}$$

The above equation can be expressed in a form free of transcendental functions, by exploiting the dispersion relation and the following trigonometric identities

$$\sin\left(2\rho_n H\right) = \frac{2\tan\left(\rho_n H\right)}{1+\tan^2\left(\rho_n H\right)}, \tag{6}$$

$$\cos\left(2\rho_n H\right) = \frac{1-\tan^2\left(\rho_n H\right)}{1+\tan^2\left(\rho_n H\right)}, \tag{7}$$

$$1-\cos\left(2\rho_n H\right) = 1-\frac{1-\tan^2\left(\rho_n H\right)}{1+\tan^2\left(\rho_n H\right)} = \frac{2\tan^2\left(\rho_n H\right)}{1+\tan^2\left(\rho_n H\right)}. \tag{8}$$

Thus, by substituting the dispersion relation into Eqs.(6) and (8), we obtain, respectively,

$$\sin\left(2\rho_n H\right) = \frac{2\dfrac{(\nu_0-\mu_0)\rho_n}{\mu_0\nu_0+\rho_n^2}}{1+\dfrac{(\nu_0-\mu_0)^2\rho_n^2}{\left(\mu_0\nu_0+\rho_n^2\right)^2}} = \frac{2\dfrac{(\nu_0-\mu_0)\rho_n}{\cancel{\mu_0\nu_0+\rho_n^2}}}{\dfrac{\left(\mu_0\nu_0+\rho_n^2\right)^2+(\nu_0-\mu_0)^2\rho_n^2}{\left(\mu_0\nu_0+\rho_n^2\right)^{\cancel{2}}}} =$$

$$= \frac{2\left(\nu_0-\mu_0\right)\left(\mu_0\nu_0+\rho_n^2\right)\rho_n}{\left(\mu_0\nu_0+\rho_n^2\right)^2+\left(\nu_0-\mu_0\right)^2\rho_n^2} = \frac{2\left(\nu_0-\mu_0\right)\left(\mu_0\nu_0+\rho_n^2\right)\rho_n}{\rho_n^4+\left(\mu_0^2+\nu_0^2\right)\rho_n^2+\mu_0^2\nu_0^2}$$

$$1-\cos\left(2\rho_n H\right) = \frac{2\dfrac{\left(\nu_0-\mu_0\right)^2\rho_n^2}{\left(\mu_0\nu_0+\rho_n^2\right)^2}}{1+\dfrac{\left(\nu_0-\mu_0\right)^2\rho_n^2}{\left(\mu_0\nu_0+\rho_n^2\right)^2}} = \frac{2\dfrac{\left(\nu_0-\mu_0\right)^2\rho_n^2}{\left(\mu_0\nu_0+\rho_n^2\right)^2}}{\dfrac{\left(\mu_0\nu_0+\rho_n^2\right)^2+\left(\nu_0-\mu_0\right)^2\rho_n^2}{\left(\mu_0\nu_0+\rho_n^2\right)^2}} =$$

$$= \frac{2\left(\nu_0-\mu_0\right)^2\rho_n^2}{\left(\mu_0\nu_0+\rho_n^2\right)^2+\left(\nu_0-\mu_0\right)^2\rho_n^2}$$

Taking also into account that $a=\dfrac{\nu_0}{\rho_n}$ we find the following formula

$$\left\|\hat{Z}_n(z)\right\|_{L^2}^2 = \frac{\left(\rho_n^2+\nu_0^2\right)H}{2\rho_n^2} + \frac{\left(\rho_n^2-\nu_0^2\right)\left(\nu_0-\mu_0\right)\left(\mu_0\nu_0+\rho_n^2\right)}{2\rho_n^2\left(\rho_n^4+\left(\mu_0^2+\nu_0^2\right)\rho_n^2+\mu_0^2\nu_0^2\right)} +$$

$$+\frac{\nu_0\left(\nu_0-\mu_0\right)^2}{\left(\rho_n^4+\left(\mu_0^2+\nu_0^2\right)\rho_n^2+\mu_0^2\nu_0^2\right)} \tag{9}$$

Notice that the factorization of the following expression is

$$\rho_n^4+\left(\nu_0^2+\mu_0^2\right)\rho_n^2+\mu_0\nu_0 = \rho_n^4+\nu_0^2\rho_n^2+\mu_0^2\rho_n^2+\mu_0^2\nu_0^2 =$$
$$= \rho_n^2\left(\rho_n^2+\nu_0^2\right)+\mu_0^2\left(\rho_n^2+\nu_0^2\right) = \left(\rho_n^2+\nu_0^2\right)\left(\rho_n^2+\mu_0^2\right)$$

so the L^2-norm of the eigenfunctions can be simplified as below

$$\left\|\hat{Z}_n(z)\right\|_{L^2}^2 = \frac{H\left(\rho_n^2+\nu_0^2\right)^2\left(\rho_n^2+\mu_0^2\right)+\left(\rho_n^2-\nu_0^2\right)\left(\nu_0-\mu_0\right)\left(\mu_0\nu_0+\rho_n^2\right)+2\nu_0\left(\nu_0-\mu_0\right)^2\rho_n^2}{2\rho_n^2\left(\rho_n^2+\nu_0^2\right)\left(\rho_n^2+\mu_0^2\right)}. \tag{10}$$

(b) Now, in case of $\rho_n = i\sigma_n$ we will construct again the L^2-norm of the eigenfunctions

$$\hat{Z}_n(z) = \left(\cosh\left(\sigma_n(z+h)\right) + \frac{\nu_0}{\sigma_n}\sinh\left(\sigma_n(z+h)\right)\right).$$

By setting $a=\nu_0/\sigma_n$, we obtain

$$\left\|\hat{Z}_n(z)\right\|_{L^2}^2 = \int_{-h(x)}^{\eta(x)}\left[\cosh\left(\sigma_n(z+h)\right) + a\sinh\left(\sigma_n(z+h)\right)\right]^2 dz \tag{11}$$

The right hand side of Eq.(11) can be written as the sum of the following integrals:

(i) $$\int_{-h(x)}^{\eta(x)}\cosh^2\left(\sigma_n(z+h)\right)dz = \int_0^H\frac{1+\cosh\left(2\sigma_n(z+h)\right)}{2}dz = \int_0^H\frac{1+\cosh\left(2\sigma_n u\right)}{2}du$$

$$=\left[\frac{1}{2}u\right]_{u=0}^{u=H}+\frac{1}{2}\left[\frac{\sinh\left(2\sigma_n u\right)}{2\sigma_n}\right]_{u=0}^{u=H} =$$

$$=\frac{1}{2}H+\frac{1}{4\sigma_n}\sinh(2\sigma_n H). \tag{12}$$

(ii) $$\int_{-h(x)}^{\eta(x)} 2a\cosh\left(\sigma_n(z+h)\right)\sinh\left(\sigma_n(z+h)\right)\,dz = \int_0^H 2a\cosh\left(\sigma_n u\right)\sinh\left(\sigma_n u\right)\,du =$$

$$= a\int_0^H \sinh\left(2\sigma_n u\right)\,du = a\left[\frac{\cosh\left(2\sigma_n u\right)}{2\sigma_n}\right]_{u=0}^{u=H} = \frac{a}{2\sigma_n}\left(\cosh(2\sigma_n H) - 1\right). \qquad (13)$$

(iii) $$\int_{-h(x)}^{\eta(x)} a^2\sinh^2\left(2\sigma_n(z+h)\right)\,dz = \int_0^H a^2\sinh^2\left(2\sigma_n u\right)\,du =$$

$$= a^2\int_0^H \frac{\cosh\left(2\sigma_n u\right) - 1}{2}\,du = \frac{a^2}{2}\int_0^H \cos\left(2\sigma_n u\right) - 1\,du =$$

$$= \frac{a^2}{2}\left[\frac{\sinh\left(2\sigma_n u\right)}{2\sigma_n}\right]_{u=0}^{u=H} - \frac{a^2}{2}H = \frac{a^2}{4\sigma_n}\sinh(2\sigma_n H) - \frac{a^2}{2}H. \qquad (14)$$

Substituting Eqs.(12),(13),(14) into (11) we obtain

$$\left\|\hat{Z}_n(z)\right\|_{L^2}^2 = \left(\frac{1-a^2}{2}\right)H + \left(\frac{1+a^2}{4\sigma_n}\right)\sinh(2\sigma_n H) + \frac{a}{2\sigma_n}\left(\cosh(2\sigma_n H) - 1\right). \qquad (15)$$

The above equation can be expressed in a form free of transcendental functions, by exploiting the dispersion relation and the trigonometric identities

$$\sinh\left(2\sigma_n H\right) = \frac{2\tanh\left(\sigma_n H\right)}{1-\tanh^2\left(\sigma_n H\right)}, \qquad (16)$$

$$\cosh\left(2\sigma_n H\right) = \frac{1+\tanh^2\left(\sigma_n H\right)}{1-\tanh^2\left(\sigma_n H\right)}, \qquad (17)$$

$$\cosh\left(2\sigma_n H\right) - 1 = \frac{1+\tanh^2\left(\sigma_n H\right)}{1-\tanh^2\left(\sigma_n H\right)} - 1 = \frac{2\tanh^2\left(\sigma_n H\right)}{1-\tanh^2\left(\sigma_n H\right)} \qquad (18)$$

By substituting the dispersion relation, which in this case is Eq.(9), into Eqs.(16) and (18), respectively, we obtain

$$\sinh\left(2\sigma_n H\right) = \frac{2\,\dfrac{\sigma_n\left(\nu_0 - \mu_0\right)}{-\sigma_n^2 + \nu_0\mu_0}}{1 - \dfrac{\sigma_n^2\left(\nu_0 - \mu_0\right)^2}{\left(-\sigma_n^2 + \nu_0\mu_0\right)^2}} = \frac{2\,\dfrac{\sigma_n\left(\nu_0 - \mu_0\right)}{\cancel{-\sigma_n^2 + \nu_0\mu_0}}}{\dfrac{\left(-\sigma_n^2 + \nu_0\mu_0\right)^2 - \sigma_n^2\left(\nu_0 - \mu_0\right)^2}{\left(-\sigma_n^2 + \nu_0\mu_0\right)^{\cancel{2}}}}$$

$$\sinh\left(2\sigma_n H\right) = \frac{2\,\sigma_n\left(\nu_0 - \mu_0\right)\left(-\sigma_n^2 + \nu_0\mu_0\right)}{\left(-\sigma_n^2 + \nu_0\mu_0\right)^2 - \sigma_n^2\left(\nu_0 - \mu_0\right)^2}$$

$$\cosh\left(2\sigma_n H\right) - 1 = \frac{2\dfrac{\sigma_n^2\left(\nu_0 - \mu_0\right)^2}{\cancel{\left(-\sigma_n^2 + \nu_0\mu_0\right)^2}}}{\dfrac{\left(-\sigma_n^2 + \nu_0\mu_0\right)^2 - \sigma_n^2\left(\nu_0 - \mu_0\right)^2}{\cancel{\left(-\sigma_n^2 + \nu_0\mu_0\right)^2}}} = \frac{2\sigma_n^2\left(\nu_0 - \mu_0\right)^2}{\left(-\sigma_n^2 + \nu_0\mu_0\right)^2 - \sigma_n^2\left(\nu_0 - \mu_0\right)^2}.$$

Taking also into account that $a = \dfrac{\nu_0}{\sigma_n}$ we find the following formula

$$\left\|\hat{Z}_n(z)\right\|_{L^2}^2 = \left(\frac{\sigma_n^2 - \nu_0^2}{2\sigma_n^2}\right)H + \frac{\left(\sigma_n^2 + \nu_0^2\right)\left(\nu_0 - \mu_0\right)\left(-\sigma_n^2 + \nu_0\mu_0\right)}{2\sigma_n^2\left[\left(-\sigma_n^2 + \nu_0\mu_0\right)^2 - \sigma_n^2\left(\nu_0 - \mu_0\right)^2\right]} + \\ + \frac{2\sigma_n^2\nu_0\left(\nu_0 - \mu_0\right)^2}{2\sigma_n^2\left[\left(-\sigma_n^2 + \nu_0\mu_0\right)^2 - \sigma_n^2\left(\nu_0 - \mu_0\right)^2\right]}.$$

The following factorization

$$2\sigma_n^2\left[\left(-\sigma_n^2 + \nu_0\mu_0\right)^2 - \sigma_n^2\left(\nu_0 - \mu_0\right)^2\right] = 2\sigma_n^2\left(\mu_0^2 - \sigma_n^2\right)\left(\nu_0^2 - \sigma_n^2\right),$$

leads to

$$\left\|\hat{Z}_n(z)\right\|_{L^2}^2 = \frac{-H\left(\mu_0^2 - \sigma_n^2\right)\left(\nu_0^2 - \sigma_n^2\right)^2 + \left(\sigma_n^2 + \nu_0^2\right)\left(\nu_0 - \mu_0\right)\left(-\sigma_n^2 + \nu_0\mu_0\right) + 2\nu_0\left(\nu_0 - \mu_0\right)^2\sigma_n^2}{2\sigma_n^2\left(\mu_0^2 - \sigma_n^2\right)\left(\nu_0^2 - \sigma_n^2\right)} \tag{19}$$

2.3b. Derivatives of the eigenvalues with respect to the endpoints

Setting $G_n(\rho_n, H) = \tan(\rho_n H) + \dfrac{\left(\mu_0 - \nu_0\right)\rho_n}{\mu_0\,\nu_0 + \rho_n^2}$, the dispersion relation (7b), implicitly defining the function $\rho_n = \rho_n(H)$ is written in the form $G_n(\rho_n, H) = 0$. The implicit function theorem implies that the derivative of ρ_n with respect to H has the following form

$$\partial_H \rho_n = -\frac{\partial_H G_n}{\partial_{\rho_n} G_n}, \tag{1}$$

where, $\partial_H G_n = \rho_n \sec^2(\rho_n H)$ (2)

$$\partial_{\rho_n} G_n = H\sec^2(\rho_n H) + \frac{\mu_0 - \nu_0}{\mu_0 \nu_0 + \rho_n^2} - \frac{2\rho_n^2\left(\mu_0 - \nu_0\right)}{\left(\mu_0\nu_0 + \rho_n^2\right)^2}. \tag{3}$$

By substituting (10) into (9) we have

$$\partial_H \rho_n = -\frac{\partial_H G_n}{\partial_{\rho_n} G_n} = -\frac{\rho_n \sec^2(\rho_n H)}{H\sec^2(\rho_n H) + \frac{\mu_0 - \nu_0}{\mu_0 \nu_0 + \rho_n^2} - \frac{2\rho_n^2\left(\mu_0 - \nu_0\right)}{\left(\mu_0\nu_0 + \rho_n^2\right)^2}}. \tag{4}$$

Recalling that $\sec^2(\rho_n H) = 1 + \tan^2\left(\rho_n H\right)$ and using the dispersion relation (7b), we obtain

$$\partial_H \rho_n = -\frac{\rho_n\left(1+\tan^2(\rho_n H)\right)}{H\left(1 + \tan^2(\rho_n H)\right) + \frac{\mu_0 - \nu_0}{\mu_0 \nu_0 + \rho_n^2} - \frac{2\rho_n^2\left(\mu_0 - \nu_0\right)}{\left(\mu_0\nu_0 + \rho_n^2\right)^2}}$$

$$\partial_H \rho_n = -\frac{\rho_n\left(1+\frac{\left(\mu_0 - \nu_0\right)^2 \rho_n^2}{\left(\mu_0\nu_0 + \rho_n^2\right)^2}\right)}{H\left(1+\frac{\left(\mu_0 - \nu_0\right)^2 \rho_n^2}{\left(\mu_0\nu_0 + \rho_n^2\right)^2}\right) + \frac{\mu_0 - \nu_0}{\mu_0 \nu_0 + \rho_n^2} - \frac{2\rho_n^2\left(\mu_0 - \nu_0\right)}{\left(\mu_0\nu_0 + \rho_n^2\right)^2}}$$

In order to simplify the expression, we make the following algebraic manipulations

$$\partial_H \rho_n = -\frac{\rho_n\left(\frac{\left(\mu_0\nu_0 + \rho_n^2\right)^2}{\left(\mu_0\nu_0 + \rho_n^2\right)^2}+\frac{\left(\mu_0 - \nu_0\right)^2 \rho_n^2}{\left(\mu_0\nu_0 + \rho_n^2\right)^2}\right)}{H\left(\frac{\left(\mu_0\nu_0 + \rho_n^2\right)^2}{\left(\mu_0\nu_0 + \rho_n^2\right)^2}+\frac{\left(\mu_0 - \nu_0\right)^2 \rho_n^2}{\left(\mu_0\nu_0 + \rho_n^2\right)^2}\right) + \frac{\left(\mu_0 - \nu_0\right)\left(\mu_0 \nu_0 + \rho_n^2\right)}{\left(\mu_0 \nu_0 + \rho_n^2\right)^2} - \frac{2\rho_n^2\left(\mu_0 - \nu_0\right)}{\left(\mu_0\nu_0 + \rho_n^2\right)^2}}$$

$$\partial_H \rho_n = -\frac{\frac{\rho_n\left(\mu_0\nu_0 + \rho_n^2\right)^2 + \left(\mu_0 - \nu_0\right)^2 \rho_n^3}{\cancel{\left(\mu_0\nu_0 + \rho_n^2\right)^2}}}{\frac{H\left(\mu_0\nu_0 + \rho_n^2\right)^2 + H\left(\mu_0 - \nu_0\right)^2 \rho_n^2 + \left(\mu_0 - \nu_0\right)\left(\mu_0 \nu_0 + \rho_n^2\right) - 2\rho_n^2\left(\mu_0 - \nu_0\right)}{\cancel{\left(\mu_0\nu_0 + \rho_n^2\right)^2}}}$$

Thus, we result in

$$\partial_H \rho_n = -\frac{\rho_n\left(\mu_0 \nu_0 + \rho_n^2\right)^2 + \left(\mu_0 - \nu_0\right)^2 \rho_n^3}{H\left(\mu_0 \nu_0 + \rho_n^2\right)^2 + H\left(\mu_0 - \nu_0\right)^2 \rho_n^2 + \left(\mu_0 - \nu_0\right)\left(\mu_0 \nu_0 + \rho_n^2\right) - 2\rho_n^2\left(\mu_0 - \nu_0\right)}. \quad (5)$$

The derivatives of ρ_n with respect to η and $-h$ are trivially obtained from $\partial_H \rho_n$, since $\partial_\eta \rho_n = \partial_H \rho_n \, \partial_\eta H = \partial_H \rho_n$ and similarly $\partial_{-h} \rho_n = \partial_H \rho_n \, \partial_{-h} H = -\partial_H \rho_n$.

We present the derivative of ρ_n with respect to $-h$, since the derivative of ρ_n with respect to η is also given by Eq.(5)

$$\partial_{-h} \rho_n = \frac{\rho_n\left(\mu_0 \nu_0 + \rho_n^2\right)^2 + \left(\mu_0 - \nu_0\right)^2 \rho_n^3}{H\left(\mu_0 \nu_0 + \rho_n^2\right)^2 + H\left(\mu_0 - \nu_0\right)^2 \rho_n^2 + \left(\mu_0 - \nu_0\right)\left(\mu_0 \nu_0 + \rho_n^2\right) - 2\rho_n^2\left(\mu_0 - \nu_0\right)}. \quad (6)$$

2.4 Comparison of the general variation formula for the eigenvalues with respect to the boundary points with the corresponding analytical solution in the case of constant-coefficient problem

Now we are going to apply the formulas of the variation of the eigenfunctions with respect to the endpoints, which is proved in Section 2.1 to the special constant-coefficient problem. Next, we will compare them to the formulas of Section 2.3b which were derived by straightforward differentiation.

2.4a The derivative of the eigenvalues with respect to the endpoint *-h*

In order to compare the formulas we are going to distinguish two different cases

1) Case $\rho_n > 0$.

Recall that the Sturm-Liouville problem is the following

$$\partial_z^2 Z_n(z) + \rho_n^2 Z_n(z) = 0, \quad -h < z < \eta \quad (1)$$

$$B^\eta Z_n \equiv [(\partial_z - \mu_0) Z_n]_{z=\eta} = 0 \quad (2a)$$

$$B_h Z_n \equiv [(\partial_z - \nu_0) Z_n]_{z=-h} = 0 \quad (2b)$$

Recall also that the eigenfunctions Z_n take the form

$$Z_n(z) = M\left(\cos\left(\rho_n(z+h)\right) + \frac{\nu_0}{\rho_n}\sin\left(\rho_n(z+h)\right)\right) \quad (3a)$$

or

$$\hat{Z}_n(z) = \cos\left(\rho_n(z+h)\right) + \frac{\nu_0}{\rho_n}\sin\left(\rho_n(z+h)\right) \quad (3b)$$

As we proved in Section 2.1, when we applied the variation formulas to the problem with the differential equation in the standard Liouville normal form, the variation of the eigenvalues λ_n, with respect to the endpoint a has the form

$$\frac{d\lambda_n(a)}{da} = \frac{\left(Z_n'(a)\right)^2 - Z_n^2(a)\left(Q(a) - \lambda_n(a)\right)}{\left\|Z_n\right\|^2}. \tag{4}$$

Therefore, in terms of the BVP (1)-(2), Eq.(4) becomes

$$\frac{d\rho_n^2(-h)}{d(-h)} = (\tilde{Z}_n')^2(-h) + \tilde{Z}_n{}^2(-h)\,\rho_n^2, \tag{5}$$

where $\tilde{Z}_n(z) = \dfrac{\hat{Z}_n(z)}{\left\|\hat{Z}_n(z)\right\|_{L^2}}$

Taking into account the above expression for $\tilde{Z}_n$, Eq.(5) takes the form

$$\frac{d\rho_n^2(-h)}{d(-h)} = \frac{(\hat{Z}_n')^2(-h)}{\left\|\hat{Z}_n(z)\right\|_{L^2}^2} + \frac{\hat{Z}_n{}^2(-h)}{\left\|\hat{Z}_n(z)\right\|_{L^2}^2}\rho_n^2 = \frac{\nu_0^2}{\left\|\hat{Z}_n(z)\right\|_{L^2}^2} + \frac{\rho_n^2}{\left\|\hat{Z}_n(z)\right\|_{L^2}^2}. \tag{6}$$

Substituting the L^2 – norm of the eigenfunctions, which is given in Section 2.3a, into Eq.(6) we have

$$\frac{d\rho_n^2(-h)}{d(-h)} = -\frac{2\rho_n^2\left(\rho_n^2+\nu_0^2\right)^2\left(\rho_n^2+\mu_0^2\right)}{H\left(\rho_n^2+\nu_0^2\right)^2\left(\rho_n^2+\mu_0^2\right)+\left(\rho_n^2-\nu_0^2\right)\left(\nu_0-\mu_0\right)\left(\mu_0\nu_0+\rho_n^2\right)+2\nu_0\left(\nu_0-\mu_0\right)^2\rho_n^2}$$

Our next step is to simplify the above complicated fraction. For this purpose we work separately with the denominator, denoted by D, of the above fraction in order to simplify the expression for the derivative. Consider first the following part of the denominator:

$$D_2 = \left(\rho_n^2-\nu_0^2\right)\left(\nu_0-\mu_0\right)\left(\mu_0\nu_0+\rho_n^2\right)+2\nu_0\left(\nu_0-\mu_0\right)^2\rho_n^2.$$

Extracting $\left(\nu_0-\mu_0\right)$ as common factor, performing simple algebraic manipulations, and factorizing once again, we find

$$\begin{aligned}
D_2 &= \left(\nu_0-\mu_0\right)\left[\left(\rho_n^2-\nu_0^2\right)\left(\mu_0\nu_0+\rho_n^2\right)+2\nu_0\left(\nu_0-\mu_0\right)\rho_n^2\right] = \\
&= \left(\nu_0-\mu_0\right)\left[\rho_n^2\mu_0\nu_0+\rho_n^4-\mu_0\nu_0^3-\nu_0^2\rho_n^2+2\nu_0^2\rho_n^2-2\rho_n^2\mu_0\nu_0\right] = \\
&= \left(\nu_0-\mu_0\right)\left[-\rho_n^2\mu_0\nu_0+\rho_n^4-\mu_0\nu_0^3+\nu_0^2\rho_n^2\right] = \\
&= \left(\nu_0-\mu_0\right)\left[-\mu_0\nu_0\left(\rho_n^2+\nu_0^2\right)+\rho_n^2\left(\rho_n^2+\nu_0^2\right)\right] = \\
&= \left(\nu_0-\mu_0\right)\left(\rho_n^2-\mu_0\nu_0\right)\left(\rho_n^2+\nu_0^2\right).
\end{aligned}$$

Thus, the denominator takes the form

$$D = D_1 + D_2 = H\left(\rho_n^2 + \nu_0^2\right)^2\left(\rho_n^2 + \mu_0^2\right) + \left(\nu_0 - \mu_0\right)\left(\rho_n^2 - \mu_0\nu_0\right)\left(\rho_n^2 + \nu_0^2\right) =$$

$$= \left(\rho_n^2 + \nu_0^2\right)\left[H\left(\rho_n^2 + \nu_0^2\right)\left(\rho_n^2 + \mu_0^2\right) + \left(\nu_0 - \mu_0\right)\left(\rho_n^2 - \mu_0\nu_0\right)\right]. \tag{7}$$

and the expression of the derivative takes the form

$$\frac{d\rho_n^2(-h)}{d(-h)} = \frac{2\rho_n^2\left(\rho_n^2+\nu_0^2\right)^2\left(\rho_n^2+\mu_0^2\right)}{\left(\rho_n^2+\nu_0^2\right)\left[H\left(\rho_n^2+\nu_0^2\right)\left(\rho_n^2+\mu_0^2\right)+\left(\nu_0-\mu_0\right)\left(\rho_n^2-\mu_0\nu_0\right)\right]} \tag{8}$$

We need the first derivative of ρ_n in order to compare the formula to this we calculated in the previous Section. Hence, we use the following derivation rule

$$\frac{d\rho_n^2(-h)}{d(-h)} = 2\rho_n\frac{d\rho_n(-h)}{d(-h)}$$

and we have

$$\frac{d\rho_n(-h)}{d(-h)} = \frac{\rho_n\left(\rho_n^2+\nu_0^2\right)\left(\rho_n^2+\mu_0^2\right)}{H\left(\rho_n^2+\nu_0^2\right)\left(\rho_n^2+\mu_0^2\right)+\left(\nu_0-\mu_0\right)\left(\rho_n^2-\mu_0\nu_0\right)} \tag{9}$$

By using the equation

$\left(\mu_0\nu_0 + \rho_n^2\right)^2 + \left(\mu_0 - \nu_0\right)^2\rho_n^2 = \left(\rho_n^2 + \nu_0^2\right)\left(\rho_n^2 + \mu_0^2\right)$, Eq.(9) yields

$$\frac{d\rho_n(-h)}{d(-h)} = \frac{\rho_n\left(\mu_0\nu_0+\rho_n^2\right)^2+\left(\mu_0-\nu_0\right)^2\rho_n^3}{H\left(\mu_0\nu_0+\rho_n^2\right)^2+H\left(\mu_0-\nu_0\right)^2\rho_n^2+\left(\nu_0-\mu_0\right)\left(\rho_n^2-\mu_0\nu_0\right)} \tag{10}$$

Eq.(10) should be equal to the following formula which was calculated in Section 2.3b

$$\partial_{-h}\rho_n = \frac{\rho_n\left(\mu_0\nu_0+\rho_n^2\right)^2+\left(\mu_0-\nu_0\right)^2\rho_n^3}{H\left(\mu_0\nu_0+\rho_n^2\right)^2+H\left(\mu_0-\nu_0\right)^2\rho_n^2+\left(\mu_0-\nu_0\right)\left(\mu_0\nu_0+\rho_n^2\right)-2\rho_n^2\left(\mu_0-\nu_0\right)}. \tag{11}$$

In order to prove that the two derivatives are equal we work separately with the term

$\left(\mu_0-\nu_0\right)\left(\mu_0\nu_0+\rho_n^2\right)-2\rho_n^2\left(\mu_0-\nu_0\right)$ of the denominator of the right hand side of Eq.(11)

$$\left(\mu_0-\nu_0\right)\left(\mu_0\nu_0+\rho_n^2\right)-2\rho_n^2\left(\mu_0-\nu_0\right) = \mu_0^2\nu_0 + \mu_0\rho_n^2 - \mu_0\nu_0^2 - \nu_0\rho_n^2 - 2\rho_n^2\mu_0 + 2\rho_n^2\nu_0 =$$

$$= \underline{\underline{\mu_0^2\nu_0}}\,\underline{-\mu_0\rho_n^2}\,\underline{\underline{-\mu_0\nu_0^2}}\,\underline{+\rho_n^2\nu_0} = \rho_n^2\left(\nu_0-\mu_0\right) - \mu_0\nu_0\left(\nu_0-\mu_0\right) =$$

$$= \left(\nu_0 - \mu_0\right)\left(\rho_n^2 - \mu_0 \nu_0\right).$$

Thus, we conclude that expressions (10) and (11) are identical, which was the desired result.

2) Case $\rho_n^2 = -\sigma_n^2 < 0 \Rightarrow$

$\rho_n = i\,\sigma_n\,,\ \sigma_n > 0.$

In this case, the BVP takes the form

$$\partial_z^2 Z_n(z) - \sigma_n^2 Z_n(z) = 0, \quad -h < z < \eta \tag{1}$$

$$B^\eta Z_n \equiv [(\partial_z - \mu_0) Z_n]_{z=\eta} = 0 \tag{2a}$$

$$B_h Z_n \equiv [(\partial_z - \nu_0) Z_n]_{z=-h} = 0 \tag{2b}$$

The general solution of Eq. (1) is given by

$$\hat{Z}_n(z) = \cosh\left(\sigma_n(z+h)\right) + \frac{\nu_0}{\sigma_n}\sinh\left(\sigma_n(z+h)\right) \tag{3}$$

By utilizing Eq.(4) from the previous case, the formula for the derivative, in terms of the certain BVP takes the form

$$-\frac{d\sigma_n^2(-h)}{d(-h)} = (\tilde{Z}'_n)^2(-h) - \tilde{Z}_n{}^2(-h)\sigma_n^2 \tag{4}$$

where $\tilde{Z}_n(z) = \dfrac{\hat{Z}_n(z)}{\left\|\hat{Z}_n(z)\right\|_{L^2}}$

By substituting $\tilde{Z}_n$, Eq.(7) takes the form

$$\frac{d\sigma_n^2(-h)}{d(-h)} = -\frac{(\hat{Z}'_n)^2(-h)}{\left\|\hat{Z}_n(z)\right\|_{L^2}^2} + \frac{\hat{Z}_n{}^2(-h)}{\left\|\hat{Z}_n(z)\right\|_{L^2}^2}\sigma_n^2 =$$

$$= -\frac{\left(\sigma_n \sinh\left(\sigma_n H\right) + \nu_0 \cosh\left(\sigma_n H\right)\right)^2}{\left\|\hat{Z}_n(z)\right\|_{L^2}^2} + \frac{\left(\cosh\left(\sigma_n H\right) + \dfrac{\nu_0}{\sigma_n}\sinh\left(\sigma_n H\right)\right)^2}{\left\|\hat{Z}_n(z)\right\|_{L^2}^2}\sigma_n^2 =$$

$$= \frac{-\sigma_n^2 \sinh^2\left(\sigma_n H\right) - \nu_0^2 \cosh^2\left(\sigma_n H\right) + \sigma_n^2 \cosh^2\left(\sigma_n H\right) + \nu_0^2 \sinh^2\left(\sigma_n H\right)}{\left\|\hat{Z}_n(z)\right\|_{L^2}^2}$$

$$= \frac{-\nu_0^2 \left(\cosh^2\left(\sigma_n H\right) - \sinh^2\left(\sigma_n H\right)\right) + \sigma_n^2 \left(\cosh^2\left(\sigma_n H\right) - \sinh^2\left(\sigma_n H\right)\right)}{\left\| \hat{Z}_n(z) \right\|_{L^2}^2}. \tag{5}$$

The trigonometric identity $\cosh^2\left(\rho_n H\right) - \sinh^2\left(\rho_n H\right) = 1$ implies

$$\frac{d\sigma_n^2(-h)}{d(-h)} = \frac{-\left(\nu_0^2 - \sigma_n^2\right)}{\left\| \hat{Z}_n(z) \right\|_{L^2}^2}. \tag{6}$$

The L^2- norm of the eigenfunctions is given by

$$\left\| \hat{Z}_n(z) \right\|_{L^2}^2 = \frac{-H\left(\nu_0^2 - \sigma_n^2\right)^2\left(\mu_0^2 - \sigma_n^2\right) + \left(\sigma_n^2 + \nu_0^2\right)\left(\nu_0 - \mu_0\right)\left(\mu_0 \nu_0 - \sigma_n^2\right) + 2\nu_0\left(\nu_0 - \mu_0\right)^2 \sigma_n^2}{2\sigma_n^2\left(\nu_0^2 - \sigma_n^2\right)\left(\mu_0^2 - \sigma_n^2\right)}.$$

Hence, Eq.(6) becomes

$$\frac{d\sigma_n^2(-h)}{d(-h)} = -\frac{2\sigma_n^2\left(\nu_0^2 - \sigma_n^2\right)^2\left(\mu_0^2 - \sigma_n^2\right)}{-H\left(\nu_0^2 - \sigma_n^2\right)^2\left(\mu_0^2 - \sigma_n^2\right) + \left(\sigma_n^2 + \nu_0^2\right)\left(\nu_0 - \mu_0\right)\left(\mu_0 \nu_0 - \sigma_n^2\right) + 2\nu_0\left(\nu_0 - \mu_0\right)^2 \sigma_n^2}. \tag{7}$$

The factorized expression of the denominator of the above fraction, denoted by D, is the following which is attempted in the same way as in the first case

$$D = -\left(\nu_0^2 - \sigma_n^2\right)\left[H\left(\nu_0^2 - \sigma_n^2\right)\left(\mu_0^2 - \sigma_n^2\right) - \left(\nu_0 - \mu_0\right)\left(\mu_0 \nu_0 + \sigma_n^2\right)\right]. \tag{8}$$

By substituting Eq.(8), into Eq.(7), we have

$$\frac{d\sigma_n^2(-h)}{d(-h)} = \frac{2\sigma_n^2\left(\nu_0^2 - \sigma_n^2\right)^2\left(\mu_0^2 - \sigma_n^2\right)}{\left(\nu_0^2 - \sigma_n^2\right)\left[H\left(\nu_0^2 - \sigma_n^2\right)\left(\mu_0^2 - \sigma_n^2\right) - \left(\nu_0 - \mu_0\right)\left(\mu_0 \nu_0 + \sigma_n^2\right)\right]}. \tag{9}$$

As in the previous case, we use the following derivation rule, in order to find the formula for the first derivative of σ_n with respect to $-h$

$$\frac{d\sigma_n^2(-h)}{d(-h)} = 2\sigma_n \frac{d\sigma_n(-h)}{d(-h)}$$

and we obtain

$$\frac{d\sigma_n(-h)}{d(-h)} = \frac{\sigma_n\left(\nu_0^2 - \sigma_n^2\right)^2\left(\mu_0^2 - \sigma_n^2\right)}{\left(\nu_0^2 - \sigma_n^2\right)\left[H\left(\nu_0^2 - \sigma_n^2\right)\left(\mu_0^2 - \sigma_n^2\right) - \left(\nu_0 - \mu_0\right)\left(\mu_0\nu_0 + \sigma_n^2\right)\right]}. \tag{10}$$

Notice that the Eq. (10) can be reduced to

$$\frac{d\sigma_n(-h)}{d(-h)} = \frac{\sigma_n\left(\nu_0^2 - \sigma_n^2\right)\left(\mu_0^2 - \sigma_n^2\right)}{H\left(\nu_0^2 - \sigma_n^2\right)\left(\mu_0^2 - \sigma_n^2\right) - \left(\nu_0 - \mu_0\right)\left(\mu_0\nu_0 + \sigma_n^2\right)}. \tag{11}$$

Recall that $\sigma_n = -i\rho_n$

$$\frac{d\rho_n(-h)}{d(-h)} = -\frac{\rho_n\left(\rho_n^2 + \nu_0^2\right)\left(\rho_n^2 + \mu_0^2\right)}{H\left(\nu_0^2 + \rho_n^2\right)\left(\mu_0^2 + \rho_n^2\right) - \left(\nu_0 - \mu_0\right)\left(\mu_0\nu_0 - \rho_n^2\right)}. \tag{12}$$

The above derivative should be the same with the following formula which was derived in Section 2.3b

$$\frac{d\rho_n(-h)}{d(-h)} = -\frac{\rho_n\left(\mu_0\nu_0 + \rho_n^2\right)^2 + \left(\mu_0 - \nu_0\right)^2\rho_n^3}{H\left(\mu_0\nu_0 + \rho_n^2\right)^2 + H\left(\mu_0 - \nu_0\right)^2\rho_n^2 + \left(\mu_0 - \nu_0\right)\left(\mu_0\nu_0 + \rho_n^2\right) - 2\rho_n^2\left(\mu_0 - \nu_0\right)}. \tag{13}$$

Notice that the factorization of the two following expressions is

$$\left(\mu_0\nu_0 + \rho_n^2\right)^2 + \left(\mu_0 - \nu_0\right)^2\rho_n^2 = \left(\rho_n^2 + \nu_0^2\right)\left(\rho_n^2 + \mu_0^2\right)$$

$$\left(\mu_0 - \nu_0\right)\left(\mu_0\nu_0 + \rho_n^2\right) - 2\rho_n^2\left(\mu_0 - \nu_0\right) = \left(\nu_0 - \mu_0\right)\left(\rho_n^2 - \mu_0\nu_0\right) = -\left(\nu_0 - \mu_0\right)\left(\mu_0\nu_0 - \rho_n^2\right)$$

which allow as to conclude that the Eqs. (12) and (13) are equal.

2.4b The derivative of the eigenvalues with respect to the endpoint η

As in the previous Section we are going to distinguish two different cases

(1) Case $\rho_n^2 > 0$

Recall that the eigenfunctions Z_n of the BVP (1)-(2) of the previous section take the form

$$Z_n(z) = M\left(\cos\left(\rho_n(z+h)\right) + \frac{\nu_0}{\rho_n}\sin\left(\rho_n(z+h)\right)\right) \tag{1a}$$

or

$$\hat{Z}_n(z) = \cos\left(\rho_n(z+h)\right) + \frac{\nu_0}{\rho_n}\sin\left(\rho_n(z+h)\right) \tag{1b}$$

As we proved in Sec.2.1, in Application of the variation formulas to the problem with the differential equation in the standard Liouville normal form, the variation of the eigenvalue λ_n, with respect to the endpoint b has the form

$$\frac{d\lambda_n(b)}{db} = \frac{-\left(Z_n'(b)\right)^2 + Z_n^2(b)\left(Q(b) - \lambda_n(b)\right)}{\left\|Z_n\right\|^2}. \tag{2}$$

Which in terms of the BVP (1)-(2) of Sec. 2.4a is given by

$$\frac{d\rho_n^2(\eta)}{d\eta} = -\left((\tilde{Z}_n')^2(\eta) + \tilde{Z}_n{}^2(\eta)\,\rho_n^2\right), \tag{3}$$

where $\tilde{Z}_n(z) = \dfrac{\hat{Z}_n(z)}{\left\|\hat{Z}_n(z)\right\|_{L^2}}$.

By substituting $\tilde{Z}_n$, Eq.(3) takes the form

$$\frac{d\rho_n^2(\eta)}{d\eta} = -\left(\frac{(\hat{Z}_n')^2(\eta)}{\left\|\hat{Z}_n(z)\right\|_{L^2}^2} + \frac{\hat{Z}_n{}^2(\eta)}{\left\|\hat{Z}_n(z)\right\|_{L^2}^2}\rho_n^2\right) =$$

$$= -\left(\frac{\left(-\rho_n\sin\left(\rho_n H\right) + \nu_0\cos\left(\rho_n H\right)\right)^2}{\left\|\hat{Z}_n(z)\right\|_{L^2}^2} + \frac{\left(\cos\left(\rho_n \mathrm{H}\right) + \dfrac{\nu_0}{\rho_n}\sin\left(\rho_n \mathrm{H}\right)\right)^2\rho_n^2}{\left\|\hat{Z}_n(z)\right\|_{L^2}^2}\right) =$$

$$= -\frac{\left(-\rho_n\sin\left(\rho_n H\right) + \nu_0\cos\left(\rho_n H\right)\right)^2 + \left(\cos\left(\rho_n \mathrm{H}\right) + \dfrac{\nu_0}{\rho_n}\sin\left(\rho_n \mathrm{H}\right)\right)^2\rho_n^2}{\left\|\hat{Z}_n(z)\right\|_{L^2}^2}.$$

The first and the second terms of the numerator are expanded as

$$\left(\cdots\right)^2 = \rho_n^2\sin^2\left(\rho_n H\right) + \nu_0^2\cos^2\left(\rho_n H\right) - 2\rho_n\nu_0\sin\left(\rho_n H\right)\cos\left(\rho_n H\right),$$
$$\left(\cdots\right)^2\rho_n^2 = \rho_n^2\cos^2\left(\rho_n H\right) + \nu_0^2\sin^2\left(\rho_n H\right) + 2\rho_n\nu_0\sin\left(\rho_n H\right)\cos\left(\rho_n H\right),$$

leading to

$$\frac{d\rho_n^2(\eta)}{d\eta} = -\frac{\rho_n^2 + \nu_0^2}{\left\|Z_n(z)\right\|_{L^2}^2}. \tag{4}$$

Recall that, the L^2- norm of the eigenfunctions is given by

$$\left\|\hat{Z}_n(z)\right\|_{L^2}^2 =$$

$$= \frac{H\left(\rho_n^2+\nu_0^2\right)^2\left(\rho_n^2+\mu_0^2\right)+\left(\rho_n^2-\nu_0^2\right)\left(\nu_0-\mu_0\right)\left(\mu_0\nu_0+\rho_n^2\right)+2\nu_0\left(\nu_0-\mu_0\right)^2\rho_n^2}{2\rho_n^2\left(\rho_n^2+\nu_0^2\right)\left(\rho_n^2+\mu_0^2\right)}.$$

Using the above equation, the derivative of ρ_n^2 with respect to the endpoint η is expressed by the following complicated fraction:

$$\frac{d\rho_n^2(\eta)}{d\eta} = -\frac{2\rho_n^2\left(\rho_n^2+\nu_0^2\right)^2\left(\rho_n^2+\mu_0^2\right)}{H\left(\rho_n^2+\nu_0^2\right)^2\left(\rho_n^2+\mu_0^2\right)+\left(\rho_n^2-\nu_0^2\right)\left(\nu_0-\mu_0\right)\left(\mu_0\nu_0+\rho_n^2\right)+2\nu_0\left(\nu_0-\mu_0\right)^2\rho_n^2}. \quad (5)$$

The denominator of the above fraction, denoted by D, can be equivalently written as

$$D=\left(\rho_n^2+\nu_0^2\right)\left[H\left(\rho_n^2+\nu_0^2\right)\left(\rho_n^2+\mu_0^2\right)+\left(\nu_0-\mu_0\right)\left(\rho_n^2-\mu_0\nu_0\right)\right]. \quad (6)$$

Combining now Eqs. (5) and (6), we obtain

$$-\frac{d\rho_n^2(\eta)}{d\eta} = \frac{2\rho_n^2\left(\rho_n^2+\nu_0^2\right)\left(\rho_n^2+\mu_0^2\right)}{H\left(\rho_n^2+\nu_0^2\right)\left(\rho_n^2+\mu_0^2\right)+\left(\nu_0-\mu_0\right)\left(\rho_n^2-\mu_0\nu_0\right)}. \quad (7)$$

To compare this formula of the derivative to this was calculated in previous Sec.2.3b, we need to find the first derivative of ρ_n with respect to $\eta, d\rho_n(\eta)/d\eta$. For this purpose, we use the formula

$$\frac{d\rho_n^2(\eta)}{d\eta} = 2\rho_n\frac{d\rho_n(\eta)}{d\eta},$$

from which we obtain

$$\frac{d\rho_n(\eta)}{d\eta} = -\frac{\rho_n\left(\rho_n^2+\nu_0^2\right)\left(\rho_n^2+\mu_0^2\right)}{H\left(\rho_n^2+\nu_0^2\right)\left(\rho_n^2+\mu_0^2\right)+\left(\nu_0-\mu_0\right)\left(\rho_n^2-\mu_0\nu_0\right)}. \quad (8)$$

Using the following equation

$\left(\mu_0\nu_0+\rho_n^2\right)^2+\left(\mu_0-\nu_0\right)^2\rho_n^2=\left(\rho_n^2+\nu_0^2\right)\left(\rho_n^2+\mu_0^2\right)$, Eq. (8) becomes

$$\frac{d\rho_n(\eta)}{d\eta} = -\frac{\rho_n\left(\mu_0\nu_0+\rho_n^2\right)^2+\left(\mu_0-\nu_0\right)^2\rho_n^3}{H\left(\mu_0\nu_0+\rho_n^2\right)^2+H\left(\mu_0-\nu_0\right)^2\rho_n^2+\left(\nu_0-\mu_0\right)\left(\rho_n^2-\mu_0\nu_0\right)} \quad (9)$$

The above derivative should be the same with the following formula which was calculated in Sec.2.3b

$$\frac{d\rho_n(\eta)}{d\eta} = -\frac{\rho_n\left(\mu_0 \nu_0 + \rho_n^2\right)^2 + \left(\mu_0 - \nu_0\right)^2 \rho_n^3}{H\left(\mu_0 \nu_0 + \rho_n^2\right)^2 + H\left(\mu_0 - \nu_0\right)^2 \rho_n^2 + \left(\mu_0 - \nu_0\right)\left(\mu_0 \nu_0 + \rho_n^2\right) - 2\rho_n^2\left(\mu_0 - \nu_0\right)}. \quad (10)$$

By using the equation

$$\left(\mu_0 - \nu_0\right)\left(\mu_0 \nu_0 + \rho_n^2\right) - 2\rho_n^2\left(\mu_0 - \nu_0\right) = \left(\nu_0 - \mu_0\right)\left(\rho_n^2 - \mu_0 \nu_0\right),$$

We conclude that Eqs.(9) and (10) are the same, which was the desired result.

(2) Case $\rho_n^2 = -\sigma_n^2 < 0 \iff \rho_n = i\,\sigma_n,\ \sigma_n > 0$

Recall that in this case, the BVP takes the form

$$\partial_z^2 Z_n(z) - \sigma_n^2 Z_n(z) = 0, \quad -h < z < \eta \quad (1)$$

$$B^\eta Z_n \equiv [(\partial_z - \mu_0) Z_n]_{z=\eta} = 0 \quad (2a)$$

$$B_h Z_n \equiv [(\partial_z - \nu_0) Z_n]_{z=-h} = 0 \quad (2b)$$

The eigenfunctions are given by

$$\hat{Z}_n(z) = \cosh\left(\sigma_n (z+h)\right) + \frac{\nu_0}{\sigma_n} \sinh\left(\sigma_n (z+h)\right) \quad (3)$$

By recalling Eq.(3) of the first case, the expression of the derivative of σ_n^2 has the form

$$\frac{d\sigma_n^2(\eta)}{d\eta} = (\tilde{Z}'_n)^2(\eta) - \tilde{Z}_n{}^2(\eta)\sigma_n^2 \quad (4)$$

where $\tilde{Z}_n(z) = \dfrac{\hat{Z}_n(z)}{\left\|\hat{Z}_n(z)\right\|_{L^2}}$

By substituting $\tilde{Z}_n$, Eq.(4) takes the form

$$\frac{d\sigma_n^2(\eta)}{d\eta} = \frac{(\hat{Z}'_n)^2(\eta)}{\left\|\hat{Z}_n(z)\right\|_{L^2}^2} - \frac{\hat{Z}_n{}^2(\eta)}{\left\|\hat{Z}_n(z)\right\|_{L^2}^2}\sigma_n^2 =$$

$$= \frac{\left(\sigma_n \sinh\left(\sigma_n H\right) + \nu_0 \cosh\left(\sigma_n H\right)\right)^2}{\left\| \hat{Z}_n(z) \right\|_{L^2}^2} - \frac{\left(\cosh\left(\sigma_n H\right) + \dfrac{\nu_0}{\sigma_n} \sinh\left(\sigma_n H\right)\right)^2}{\left\| \hat{Z}_n(z) \right\|_{L^2}^2} \sigma_n^2 =$$

$$= \frac{\sigma_n^2 \sinh^2\left(\sigma_n H\right) + \nu_0^2 \cosh^2\left(\sigma_n H\right) - \sigma_n^2 \cosh^2\left(\sigma_n H\right) - \nu_0^2 \sinh^2\left(\sigma_n H\right)}{\left\| \hat{Z}_n(z) \right\|_{L^2}^2}$$

$$= \frac{\nu_0^2 \left(\cosh^2\left(\sigma_n H\right) - \sinh^2\left(\sigma_n H\right)\right) - \sigma_n^2 \left(\cosh^2\left(\sigma_n H\right) - \sinh^2\left(\sigma_n H\right)\right)}{\left\| \hat{Z}_n(z) \right\|_{L^2}^2}.$$

From the trigonometric identity $\cosh^2\left(\rho_n H\right) - \sinh^2\left(\rho_n H\right) = 1$ we derive

$$\frac{d\sigma_n^2(\eta)}{d\eta} = \frac{\nu_0^2 - \sigma_n^2}{\left\| \hat{Z}_n(z) \right\|_{L^2}^2}. \tag{5}$$

The L^2- norm of the eigenfunctions is given by

$$\left\| \hat{Z}_n(z) \right\|_{L^2}^2 = \frac{-H\left(\nu_0^2 - \sigma_n^2\right)^2 \left(\mu_0^2 - \sigma_n^2\right) + \left(\sigma_n^2 + \nu_0^2\right)\left(\nu_0 - \mu_0\right)\left(\mu_0 \nu_0 - \sigma_n^2\right) + 2\nu_0 \left(\nu_0 - \mu_0\right)^2 \sigma_n^2}{2\sigma_n^2 \left(\nu_0^2 - \sigma_n^2\right)\left(\mu_0^2 - \sigma_n^2\right)}.$$

Hence, Eq.(5) becomes

$$\frac{d\sigma_n^2(\eta)}{d\eta} = \frac{2\sigma_n^2 \left(\nu_0^2 - \sigma_n^2\right)^2 \left(\mu_0^2 - \sigma_n^2\right)}{-H\left(\nu_0^2 - \sigma_n^2\right)^2 \left(\mu_0^2 - \sigma_n^2\right) + \left(\sigma_n^2 + \nu_0^2\right)\left(\nu_0 - \mu_0\right)\left(\mu_0 \nu_0 - \sigma_n^2\right) + 2\nu_0 \left(\nu_0 - \mu_0\right)^2 \sigma_n^2}. \tag{6}$$

The factorized form of the denominator of the above fraction, denoted by D, is the following,

$$D = -\left(\nu_0^2 - \sigma_n^2\right)\left[H\left(\nu_0^2 - \sigma_n^2\right)\left(\mu_0^2 - \sigma_n^2\right) - \left(\nu_0 - \mu_0\right)\left(\mu_0 \nu_0 + \sigma_n^2\right)\right]. \tag{7}$$

Combining Eqs. (6) and (7) we obtain

$$\frac{d\sigma_n^2(\eta)}{d\eta} = \frac{2\sigma_n^2 \left(\nu_0^2 - \sigma_n^2\right)^2 \left(\mu_0^2 - \sigma_n^2\right)}{-\left(\nu_0^2 - \sigma_n^2\right)\left[H\left(\nu_0^2 - \sigma_n^2\right)\left(\mu_0^2 - \sigma_n^2\right) - \left(\nu_0 - \mu_0\right)\left(\mu_0 \nu_0 + \sigma_n^2\right)\right]}. \tag{8}$$

We use the following derivation rule, in order to find the formula for the first derivative of σ_n with respect to η

$$\frac{d\sigma_n^2(\eta)}{d\eta} = 2\sigma_n \frac{d\sigma_n(\eta)}{d\eta}.$$

Hence, Eq.(8) becomes

$$\frac{d\sigma_n(\eta)}{d\eta} = \frac{\sigma_n\left(\nu_0^2 - \sigma_n^2\right)^2\left(\mu_0^2 - \sigma_n^2\right)}{-\left(\nu_0^2 - \sigma_n^2\right)\left[H\left(\nu_0^2 - \sigma_n^2\right)\left(\mu_0^2 - \sigma_n^2\right) - \left(\nu_0 - \mu_0\right)\left(\mu_0\nu_0 + \sigma_n^2\right)\right]}. \tag{9}$$

Notice that $\left(\nu_0^2 - \sigma_n^2\right)$ is common factor of the numerator and the denominator of the above fraction. Therefore, Eq.(9) can be reduced to

$$\frac{d\sigma_n(\eta)}{d\eta} = -\frac{\sigma_n\left(\nu_0^2 - \sigma_n^2\right)\left(\mu_0^2 - \sigma_n^2\right)}{H\left(\nu_0^2 - \sigma_n^2\right)\left(\mu_0^2 - \sigma_n^2\right) - \left(\nu_0 - \mu_0\right)\left(\mu_0\nu_0 + \sigma_n^2\right)}. \tag{10}$$

Recall that $\sigma_n = -i\rho_n$ and Eq.(10) takes the form

$$\frac{d\rho_n(\eta)}{d\eta} = -\frac{\rho_n\left(\rho_n^2 + \nu_0^2\right)\left(\rho_n^2 + \mu_0^2\right)}{H\left(\nu_0^2 + \rho_n^2\right)\left(\mu_0^2 + \rho_n^2\right) - \left(\nu_0 - \mu_0\right)\left(\mu_0\nu_0 - \rho_n^2\right)}. \tag{11}$$

The above derivative should be the same with the following formula of Sec.2.3b

$$\frac{d\rho_n(\eta)}{d\eta} = -\frac{\rho_n\left(\mu_0\nu_0 + \rho_n^2\right)^2 + \left(\mu_0 - \nu_0\right)^2\rho_n^3}{H\left(\mu_0\nu_0 + \rho_n^2\right)^2 + H\left(\mu_0 - \nu_0\right)^2\rho_n^2 + \left(\mu_0 - \nu_0\right)\left(\mu_0\nu_0 + \rho_n^2\right) - 2\rho_n^2\left(\mu_0 - \nu_0\right)}. \tag{12}$$

Notice the factorization of the two following expressions

$$\left(\mu_0\nu_0 + \rho_n^2\right)^2 + \left(\mu_0 - \nu_0\right)^2\rho_n^2 = \left(\rho_n^2 + \nu_0^2\right)\left(\rho_n^2 + \mu_0^2\right)$$

$$\left(\mu_0 - \nu_0\right)\left(\mu_0\nu_0 + \rho_n^2\right) - 2\rho_n^2\left(\mu_0 - \nu_0\right) = \left(\nu_0 - \mu_0\right)\left(\rho_n^2 - \mu_0\nu_0\right) = -\left(\nu_0 - \mu_0\right)\left(\mu_0\nu_0 - \rho_n^2\right)$$

which yields to the fact that Eqs. (11) and (12) are equivalent.

2.5 Preamble about the problem of variation of eigenfunctions

Consider the regular Sturm-Liouville BVP in the standard Liouville normal form

$$\partial_z^2 Z_n(z) + \left(\lambda_n - Q(z)\right)Z_n(z) = 0, \qquad a < z < b, \tag{1}$$

$$B^b Z_n \equiv \partial_z Z_n(b) - \mu_0 Z_n(b) = 0, \tag{2a}$$

$$B_a Z_n \equiv \partial_z Z_n(a) - \nu_0 Z_n(a) = 0. \tag{2b}$$

Assume that the eigenvalues and the eigenfunctions,

$$\lambda_n = \lambda_n[Q(\bullet), \mu_0, \nu_0] \tag{3a}$$

$$Z_n(z) = Z_n[Q(\bullet), \mu_0, \nu_0](z), \tag{3b}$$

of the problem are known. Assume further that each datum of the problem is perturbed as follows

$$Q(z) \to \tilde{Q}(z) = Q(z) + \delta Q(z),$$
$$\mu_0 \to \tilde{\mu}_0 = \mu_0 + \delta\mu_0,$$
$$\nu_0 \to \tilde{\nu}_0 = \nu_0 + \delta\nu_0,$$

where $\delta Q(z), \delta\mu_0, \delta\nu_0$ are small quantities, formally considered as first-order infinitesimals. Note that the smallness of the function $\delta Q(z)$ should be eventually measured by an appropriate norm. Under the above perturbation of the data of the problem, the eigenvalues and the eigenfunctions are also changed,

$$\lambda_n \to \tilde{\lambda}_n = \lambda_n + \delta\lambda_n, \tag{4a}$$

$$Z_n(z) \to \tilde{Z}_n(z) = Z_n(z) + \delta Z_n(z). \tag{4b}$$

According to the Continuity Theorem of the eigensystem $\left(\lambda_n, Z_n(\bullet)\right)$ with respect to the data $\left(Q(\bullet), \mu_0, \nu_0\right)$ (see Q. Kong & A. Zettl , 1996, Theorem 3.1, Lemma 3.1), we can consider that the variations $\delta\lambda_n$, $\delta Z_n(z)$ are small, since the variations of the data $\delta Q(z), \delta\mu_0, \delta\nu_0$ were assumed small. Recall further, that the variation of each eigenvalue, $\delta\lambda_n$, has been explicitly expressed in terms of the variation of the data; see Section 2.2 (see also Q.Kong& A. Zettl , 1996, Theorem 4.2.).

The variation of eigenvalues $\delta\lambda_n$, has the following form:

$$\delta\lambda_n = \int_a^b \left|\hat{Z}_n(z)\right|^2 \delta Q(z)\, dz + \hat{Z}_n^2(a)\delta\nu_0 - \hat{Z}_n^2(b)\delta\mu_0 \tag{5}$$

where $\hat{Z}_n(z) = \dfrac{Z_n(z)}{\left\|Z_n(z)\right\|}$.

Our goal herein (in Chapter 4) is to quantify the variation of each eigenfunction, $\delta Z_n(z)$, by relating it explicitly with the variations of the data. That is, to calculate the functional derivative of $Z_n[Q(\bullet), \mu_0, \nu_0](z)$ with respect to $Q(\bullet), \mu_0, \nu_0$. For this purpose, we formulate the perturbed problem,

$$\partial_z^2 \tilde{Z}_n(z) + \left(\tilde{\lambda}_n - \tilde{Q}(z)\right)\tilde{Z}_n(z) = 0, \quad a < z < b, \tag{5}$$

$$B^b \tilde{Z}_n \equiv \partial_z \tilde{Z}_n(b) - \tilde{\mu}_0 \tilde{Z}_n(b) = 0, \tag{6a}$$

$$B_a \tilde{Z}_n \equiv \partial_z \tilde{Z}_n(a) - \tilde{\nu}_0 \tilde{Z}_n(a) = 0, \tag{6b}$$

from which we obtain a non-homogeneous BVP for the variation $\delta Z_n(z)$.Using Eqs. (4a,b), Eq. (5) becomes

$$\partial_z^2 \big(Z_n(z) + \delta Z_n(z)\big) + \big(\lambda_n + \delta\lambda_n - Q(z) - \delta Q(z)\big)\big(Z_n(z) + \delta Z_n(z)\big) = 0 \;\Rightarrow$$

$$\partial_z^2 Z_n(z) + \partial_z^2 \delta Z_n(z) + \big(\lambda_n - Q(z)\big) Z_n(z) + \big(\lambda_n - Q(z)\big)\delta Z_n(z) +$$
$$+ \big(\delta\lambda_n - \delta Q(z)\big) Z_n(z) + \big(\delta\lambda_n - \delta Q(z)\big)\delta Z_n(z) = 0.$$

The sum of the red terms is zero, because they satisfy Eq.(1); the blue term can be neglected, since it is of second order. Thus, we obtain the following differential equation for $\delta Z_n(z)$:

$$\partial_z^2 \delta Z_n(z) + \big(\lambda_n - Q(z)\big)\delta Z_n(z) = -\big(\delta\lambda_n - \delta Q(z)\big) Z_n(z). \tag{7}$$

By substituting $\tilde{Z}_n = Z_n(z) + \delta Z_n(z)$ and $\tilde{\mu}_0 = \mu_0 + \delta\mu_0$ into Eq.(6a), we obtain

$$\partial_z \big(Z_n(b) + \delta Z_n(b)\big) - \big(\mu_0 + \delta\mu_0\big)\big(Z_n(b) + \delta Z_n(b)\big) = 0 \quad \Rightarrow$$

$$\partial_z Z_n(b) + \partial_z \delta Z_n(b) - \mu_0 Z_n(b) - \delta\mu_0 Z_n(b) - \mu_0 \delta Z_n(b) - \delta\mu_0 \delta Z_n(b) = 0.$$

Again, the sum of the red terms is zero because of the boundary condition (2a), and the blue term is omitted as a second order term. Hence, we obtain

$$B^b(\delta Z_n) \equiv \partial_z(\delta Z_n(b)) - \mu_0 (\delta Z_n(b)) = \delta\mu_0\, Z_n(b). \tag{8a}$$

Working similarly with the boundary condition (6b), we obtain

$$B_a(\delta Z_n) \equiv \partial_z(\delta Z_n(a)) - \nu_0 (\delta Z_n(a)) = \delta\nu_0\, Z_n(a). \tag{8b}$$

Therefore, $\delta Z(z)$ satisfies the inhomogeneous problem (7)-(8).
In order to solve the BVP (7)-(8) we need to apply the theory for non-homogeneous BVPs. Hence, the following chapter presents the theory we need to find the variation of the eigenfunctions with respect to the coefficients.

3. Green's Function and the non-homogeneous boundary-value problem ([1])

This Chapter includes the presentation of the developed theory which is widely used to solve non homogeneous BVPs. At first we are going to construct Green's Function and then to express the solution of the NHBVP. The later theory will be applied in Chapter 4 in order to solve the variational problem of the eigenfunctions.

Consider the boundary value problem (BVP)

([1]) Main references used in writing this chapter include Stakgolt I., M. Holst 2011 pp.193-196 ,. Ince E.L, 1956, pp.255-256 and Stone &Goldbart, 2009, Sec.5.2 .

$$\mathsf{L}_{\lambda}[u](x) \equiv \left(p(x)u'(x)\right)' - q(x)u(x) + \lambda u(x) = \delta(x-\xi), \qquad a < x < b, \tag{1}$$

$$\mathsf{B}_a[u] \equiv (pu')(a) - \nu_0 u(a) = 0, \tag{2a}$$

$$\mathsf{B}_b[u] \equiv (pu')(b) - \mu_0 u(b) = 0, \tag{2b}$$

under the standard assumptions on the coefficient functions $p(\,\bullet\,), q(\,\bullet\,)$ and the constants ν_0, μ_0. In Eq. (1), $\delta(x-\xi)$ is the Dirac (generalized) function, which requires the differential equation (1) to be considered in the sense of generalized functions. Nevertheless, a classical treatment, based only on the following two formal properties of the Dirac function,

$$\delta(x-\xi) = 0, \quad \text{for } x \neq \xi \text{ and } \int_{x=\xi-d_-}^{x=\xi+d_+} \delta(x-\xi)\,dx = 1 \tag{3a,b}$$

is possible, as we shall see in the sequel. In Eq. (3b), d_- and d_+ are positive numbers.

The solution of the BVP (1), (2), if it exists, is denoted by $G(x,\xi;\lambda)$ and is called the **Green Function of the problem** (1), (2).

According to Eq. (3a), the differential equation (1) implies

$$\left(p(x)\,G'(x,\xi;\lambda)\right)' - q(x)\,G(x,\xi;\lambda) + \lambda G(x,\xi;\lambda) = 0, \quad \text{for } \begin{cases} a < x < \xi, \\ \quad \text{and} \\ \xi < x < b. \end{cases} \tag{4}$$

Thus, if $u_1(x) = u_1(x;\lambda), u_2(x) = u_2(x;\lambda)$ are two linearly independent solutions of the homogeneous version of Eq. (1), Green's function $G(x,\xi;\lambda)$, should it exists, will be given by the expression

$$G(x,\xi;\lambda) = \begin{cases} G_<(x,\xi;\lambda) = a_1(\xi)\,u_1(x) + a_2(\xi)\,u_2(x), & a \le x < \xi, \\ G_>(x,\xi;\lambda) = b_1(\xi)\,u_1(x) + b_2(\xi)\,u_2(x), & \xi < x \le b. \end{cases} \tag{5}$$

Note that, all functions a_i, b_i, u_i are also dependent on λ. The construction of the Green function will be accomplished if we are able to calculate the four quantities $a_i(\xi), b_i(\xi)$, $i = 1, 2$, in a way ensuring that Eqs. (1) – (2) are satisfied.

Equation (4) is identical to Eq. (1) for $x \neq \xi$. To complete the treatment of Eq. (1), we have to capture the effect of $\delta(x-\xi)$ when x lies in the vicinity of ξ.In order to avoid considerations of generalized functions theory, we integrate Eq. (1) from $x = \xi - \varepsilon$ to $x = \xi + \varepsilon$, where $\varepsilon > 0$ (and sufficiently small), taking into account the property (3b). This yields

$$\left[p(x) \, \frac{dG(x,\xi;\lambda)}{dx} \right]_{\xi-\varepsilon}^{\xi+\varepsilon} + \int\limits_{\xi-\varepsilon}^{\xi+\varepsilon} (\lambda - q(x)) \, G(x,\xi;\lambda) \, dx = 1 . \qquad (6)$$

Recalling that $q(x)$ is continuous in $[a,b]$ (and thus at $x=\xi$), and assuming that $G(x,\xi)$ is continuous,

$$G(\xi+0,\xi;\lambda) = G(\xi-0,\xi;\lambda) , \qquad (7)$$

we conclude that

$$\lim_{\varepsilon \to 0} \int\limits_{\xi-\varepsilon}^{\xi+\varepsilon} (\lambda - q(x)) \, G(x,\xi;\lambda) \, dx = 0 .$$

Then, Eq. (6) implies

$$\lim_{\varepsilon \to 0} \left[p(x) \, \frac{dG(x,\xi;\lambda)}{dx} \right]_{\xi-\varepsilon}^{\xi+\varepsilon} = 1 ,$$

which, on the basis of the continuity of $p(x)$, is equivalent to

$$\frac{dG(\xi+0,\xi;\lambda)}{dx} - \frac{dG(\xi-0,\xi;\lambda)}{dx} = \frac{1}{p(\xi)} . \qquad (8)$$

Eq. (8) shows that, if $G(x,\xi;\lambda)$ exists, its first derivative at $x=\xi$ must have a jump discontinuity of magnitude $1/p(\xi)$. Of course, if the Green function exists, it has to satisfy the boundary conditions (2), which, according to Eq. (5), implies

$$p(a) \, G'_{<}(a,\xi;\lambda) - \nu_0 G_{<}(a,\xi;\lambda) = 0 , \qquad (9)$$

$$p(b) \, G'_{>}(b,\xi;\lambda) - \mu_0 \, G_{>}(b,\xi;\lambda) = 0 . \qquad (10)$$

Now, we have at our disposal four equations, namely, Eqs. (7) - (10), for the determination of the four quantities $a_i(\xi), b_i(\xi)$, $i=1,2$, which, hopefully, can provide the values of the latter. Let us write these equations in explicit form. Eqs. (7) and (8), in conjunction with Eq. (5), lead to

$$a_1 u_1(\xi) + a_2 u_2(\xi) - b_1 u_1(\xi) - b_2 u_2(\xi) = 0 ,$$

$$a_1 u'_1(\xi) + a_2 u'_2(\xi) - b_1 u'_1(\xi) - b_2 u'_2(\xi) = -\frac{1}{p(\xi)} ,$$

or

$$\left(a_1 - b_1\right) u_1(\xi) + \left(a_2 - b_2\right) u_2(\xi) = 0 , \qquad (11a)$$

$$\left(a_1 - b_1\right) u'_1(\xi) + \left(a_2 - b_2\right) u'_2(\xi) = -\frac{1}{p(\xi)} . \qquad (11b)$$

The determinant $\begin{vmatrix} u_1(\xi) & u_2(\xi) \\ u_1'(\xi) & u_2'(\xi) \end{vmatrix}$ is the Wronskian $W\left(u_1, u_2\right)$. Since u_1, u_2 are two linearly independent solutions of Eq. (1), $W\left(u_1, u_2\right) \neq 0$ for any $\xi \in [a, b]$ and, thus, the system (11) is uniquely solvable with respect to $a_1 - b_1$ and $a_2 - b_2$:

$$a_1 - b_1 = \frac{u_2(\xi) / p(\xi)}{W\left(u_1, u_2\right)}, \quad a_2 - b_2 = -\frac{u_1(\xi) / p(\xi)}{W\left(u_1, u_2\right)}. \tag{12a,b}$$

Consider now the couple of Eqs. (9) and (10). Using Eq. (5), these two equations can be written in the form

$$a_1\left(p(a)\, u_1'(a) - \nu_0 u_1(a)\right) + a_2\left(p(a)\, u_2'(a) - \nu_0 u_2(a)\right) = 0,$$
$$b_1\left(p(b)\, u_1'(b) - \mu_0 u_1(b)\right) + b_2\left(p(b)\, u_2'(b) - \mu_0 u_2(b)\right) = 0,$$

or

$$a_1\, \mathsf{B}_a[u_1] + a_2\, \mathsf{B}_a[u_2] = 0, \tag{13a}$$
$$b_1\, \mathsf{B}_b[u_1] + b_2\, \mathsf{B}_b[u_2] = 0. \tag{13b}$$

Note that these two equations do not form a system, since each one of them contains two different unknowns. However, combining Eqs. (13) withEqs. (12), we are able to find all coefficients and, thus, the Green function. To proceed with we have to consider two distinct cases.

3.1. Construction of Green's Function when λ is not an eigenvalue $(\lambda \neq \lambda_n)$

Assume first that λ is not an eigenvalue ($\lambda \neq \lambda_n$). Then, neither of solutions $u_1 = u_1(x;\lambda), u_2 = u_2(x;\lambda)$ can satisfy both BCs (2a,b). On the other hand, we always consider that a solution (either u_1 or u_2) satisfies one of the BCs (2a,b) ([2]). Accordingly, without loss of generality, we can assume

$$\begin{aligned} &\mathsf{B}_a[u_1] = 0, \qquad \mathsf{B}_b[u_1] \neq 0, \\ &\mathsf{B}_a[u_2] \neq 0, \qquad \mathsf{B}_b[u_2] = 0. \end{aligned} \tag{1}$$

In this case, Eq. (13a) of Section 3implies $a_2 = 0$ and Eq. (13b) implies $b_1 = 0$. Then, Eqs. (12) of Section 3 result in

$$a_1 = \frac{u_2(\xi) / p(\xi)}{W\left(u_1, u_2\right)}, \qquad b_2 = \frac{u_1(\xi) / p(\xi)}{W\left(u_1, u_2\right)}. \tag{2}$$

Substituting the values of the coefficients a_1, a_2, b_1, b_2, found above to Eq. (5) of the previous Section, we obtain

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([2]) Since each solution of Eq. (1) is identified by specifying initial values for the function its derivative, we can always chose a solution, say $u_1(x)$, such that $u_1(a)$ and $u_1'(a)$ satisfy $p(a)\, u_1'(a) - \nu_0\, u_1(a) = 0$.

$$G(x,\xi;\lambda) = \begin{cases} G_<(x,\xi;\lambda) = \dfrac{u_1(x;\lambda)\,u_2(\xi;\lambda)}{p(\xi)\,W\big(u_1,u_2;\lambda\big)}, & a \le x < \xi, \\ G_>(x,\xi;\lambda) = \dfrac{u_1(\xi;\lambda)\,u_2(x;\lambda)}{p(\xi)\,W\big(u_1,u_2;\lambda\big)}, & \xi < x \le b. \end{cases} \tag{3}$$

Notice that Green's function in this case, is symmetric with respect to its arguments. Thus,

$$G(x,\xi;\lambda) = G(\xi,x;\lambda). \tag{4}$$

3.2 Solution of non-homogeneous problem when λ is not an eigenvalue ($\lambda \neq \lambda_n$)

Consider now the non-homogeneous BVP with general excitation $g \in C[a,b]$ in the differential equation, under the assumption that λ is not an eigenvalue:

$$\mathsf{L}_\lambda[u](x) \equiv Lu(x) + \lambda u(x) = \big(p(x)\,u'(x)\big)' - q(x)\,u(x) + \lambda\,u(x) = g(x), \quad a < x < b, \tag{1}$$

$$\mathsf{B}_a[u] \equiv p(a)u'(a) - \nu_0\,u(a) = 0, \tag{2a}$$

$$\mathsf{B}_b[u] \equiv p(b)u'(b) - \mu_0\,u(b) = 0. \tag{2b}$$

In this case, Green's function is well and uniquely defined, see Eq. (3) of Section 3.1, and it will be exploited to construct the solution of the above problem. For this purpose, we first recall Green's Identity for the operator $\mathsf{L}_\lambda[u]$:

$$\begin{aligned} &\int_a^b \Big(\upsilon(x)\,\mathsf{L}_\lambda[u](x) - u(x)\,\mathsf{L}_\lambda[\upsilon](x)\Big)\,dx = \\ &= \int_a^b \big(\upsilon(x)\,Lu(x) - u(x)\,L\upsilon(x)\big)\,dx = \big[\,\upsilon(x)\,pu'(x) - u(x)\,p\upsilon'(x)\,\big]_a^b \end{aligned}. \tag{3}$$

Assume now that there exists a C^2-function $u(x)$ satisfyingEqs. (1) and (2), that is a classical solution of our BVP, and apply Green's identity (3) to the functions $u(x) = u(x;\lambda)$ and $G(x,\xi;\lambda)$. Since both functions satisfy the boundary conditions, the right-hand side of Eq. (3) becomes zero, and thus we have

$$\int_a^b \Big(G(x,\xi;\lambda)\,\mathsf{L}_\lambda[u](x) - u(x)\,\mathsf{L}_\lambda[G(\cdot,\xi;\lambda)](x)\Big)\,dx = 0. \tag{4}$$

Taking into account that $u(x)$ satisfies the equation $\mathsf{L}_\lambda[u](x) = g(x)$, and $G(x,\xi;\lambda)$ satisfies the equation $\mathsf{L}_\lambda[G(\cdot,\xi;\lambda)](x) = \delta(x-\xi)$, Eq.(4) results in

$$\int_a^b \left(G(x,\xi;\lambda)\, g(x) - u(x;\lambda)\, \delta(x-\xi) \right) dx \;=\; 0\,. \tag{5}$$

The linearity of the definite integral and the reproducing property of the δ function,

$$\int_a^b f(x)\, \delta(x-\xi)\, dx = f(\xi)\,,$$

imply $\int_a^b G(x,\xi;\lambda)\, g(x)\, dx - u(\xi;\lambda) \;=\; 0$, from which we obtain

$$u(x;\lambda) \;=\; \int_a^b G(\xi,x;\lambda)\, g(\xi)\, d\xi \;=\; \int_a^b G(x,\xi;\lambda)\, g(\xi)\, d\xi\,. \tag{6}$$

The substitution of $G(\xi,x;\lambda)$ by $G(x,\xi;\lambda)$, in the rightmost equation above is justified by the symmetry of Green function with respect to its arguments; see Eq. (4) of Section 3.1.

The solution given by Eq. (6) has been based on the assumption that such a solution exists (the analysis part of the proof of existence). Now we proceed to the justification of the assumption of the existence, by proving that the function $u(x;\lambda)$, as defined by Eq. (6), is indeed a solution.

Theorem 1. When λ is not an eigenvalue, the function $u(x;\lambda)$, as defined by Eq. (6), is a solution to the nonhomogeneous BVP (1), (2), for any given excitation $g \in C[a,b]$. ■

Remark. Before proving the above Theorem 1, we note that the whole procedure (including the derivation of Eq. (6)) can be considered as an existence theorem, constructed by the scheme [analysis $\to$ construction $\to$ proof/justification]. Theorem 1 refers to the last step.

***Proof*:** Observe first that the Green function $(x,\xi) \to G(x,\xi;\lambda)$, as defined by Eq. (3) of Section 3.1, is continuous on $[a,b]\times[a,b]$, and it has continuous first and second derivatives in the domain $[a,b]\times[a,b] - \left\{ (\zeta,\zeta),\, \zeta \in [a,b] \right\}$. At $x = \xi$, there exist the unilateral limits

$$\lim_{x\to\xi-} \frac{\partial G(x,\xi;\lambda)}{\partial x}, \quad \lim_{x\to\xi+} \frac{\partial G(x,\xi;\lambda)}{\partial x}, \quad \lim_{x\to\xi-} \frac{\partial^2 G(x,\xi;\lambda)}{\partial x^2}, \quad \lim_{x\to\xi+} \frac{\partial^2 G(x,\xi;\lambda)}{\partial x^2}\,.$$

Thus, for any $x \in [a,b]$, the integrals

$$\int_{\xi=a}^{\xi=x} G(x,\xi;\lambda)\, g(\xi)\, d\xi\,, \qquad \int_{\xi=x}^{\xi=b} G(x,\xi;\lambda)\, g(\xi)\, d\xi$$

can be differentiated with respect to x, at least two times. Taking these facts into account, we proceed with the calculation of the first and second derivatives of the function $u(x;\lambda)$, as defined by Eq. (6).

$$\begin{aligned}\frac{du(x;\lambda)}{dx} &= \frac{\partial}{\partial x}\int_{\xi=a}^{\xi=b} G(x,\xi;\lambda)\,g(\xi)\,d\xi = \\ &= \frac{\partial}{\partial x}\left\{\int_{\xi=a}^{\xi=x} G(x,\xi;\lambda)\,g(\xi)\,d\xi + \int_{\xi=x}^{\xi=b} G(x,\xi;\lambda)\,g(\xi)\,d\xi\right\} = \\ &= \frac{\partial}{\partial x}\int_{\xi=a}^{\xi=x} G(x,\xi;\lambda)\,g(\xi)\,d\xi + \frac{\partial}{\partial x}\int_{\xi=x}^{\xi=b} G(x,\xi;\lambda)\,g(\xi)\,d\xi\,.\end{aligned}$$

Since $u(x;\lambda)$ has two branches, see Eq. (3), Section 3.1, we substitute the appropriate branch into each integral, and then we apply Leibnitz's rule of differentiation of integrals:

$$\frac{du(x;\lambda)}{dx} = \frac{\partial}{\partial x}\int_{\xi=a}^{\xi=x} G_{>}(x,\xi;\lambda)\,g(\xi)\,d\xi + \frac{\partial}{\partial x}\int_{\xi=x}^{\xi=b} G_{<}(x,\xi;\lambda)\,g(\xi)\,d\xi =$$

$$= G_{>}(x,x-0;\lambda)\,g(x) + \int_{\xi=a}^{\xi=x-} \frac{\partial G_{>}(x,\xi;\lambda)}{\partial x}\,g(\xi)\,d\xi$$

$$- G_{<}(x,x+0;\lambda)\,g(x) + \int_{\xi=x+}^{\xi=b} \frac{\partial G_{<}(x,\xi;\lambda)}{\partial x}\,g(\xi)\,d\xi\,.$$

The continuity of $G(x,\xi;\lambda)$ at $x=\xi$ implies that $G_{>}(x,x-0;\lambda) - G_{<}(x,x+0;\lambda) = 0$,and thus, the first derivative of $u(x;\lambda)$ is given by

$$\frac{du(x;\lambda)}{dx} = \int_{\xi=a}^{\xi=x} \frac{\partial G_{>}(x,\xi;\lambda)}{\partial x}\,g(\xi)\,d\xi + \int_{\xi=x}^{\xi=b} \frac{\partial G_{<}(x,\xi;\lambda)}{\partial x}\,g(\xi)\,d\xi. \tag{7}$$

We proceed now to the calculation of the second derivative of $u(x;\lambda)$, by differentiating Eq.(7):

$$\frac{d^2u(x;\lambda)}{dx^2} = \frac{\partial}{\partial x}\int_{\xi=a}^{\xi=x} \frac{\partial G_{>}(x,\xi;\lambda)}{\partial x}\,g(\xi)\,d\xi + \frac{\partial}{\partial x}\int_{\xi=x}^{\xi=b} \frac{\partial G_{<}(x,\xi;\lambda)}{\partial x}\,g(\xi)\,d\xi.$$

Note that, even though $\partial G(x,\xi;\lambda)/\partial x$ has a jump discontinuity at $x=\xi$, it is continuous and has a continues derivative in the intervals $[a,\xi)$ and $(\xi,b]$.This fact is confirmed by the expression of Green's function, Eq. (3) of Section 3, since the $x-$dependence of $G(x,\xi;\lambda)$ comes only from the $x-$dependence of solutions $u_1(x;\lambda), u_2(x;\lambda)$, which are C^2 functions. By using again Leibnitz's rule, we obtain

$$\frac{d^2u(x;\lambda)}{dx^2} = \frac{\partial G_{>}(x,x-0;\lambda)}{\partial x}\,g(x) - \frac{\partial G_{<}(x,x+0;\lambda)}{\partial x}\,g(x) +$$

$$+ \int_{\xi=a}^{\xi=x} \frac{\partial^2 G_{>}(x,\xi;\lambda)}{\partial x^2}\,g(\xi)\,d\xi + \int_{\xi=x}^{\xi=b} \frac{\partial^2 G_{<}(x,\xi;\lambda)}{\partial x^2}\,g(\xi)\,d\xi\,.$$

Taking into account the jump discontinuity of the first derivative of $G(x,\xi;\lambda)$, Eq. (8) of Section 3, and its symmetry with respect to the arguments x,ξ, Eq. (4)of Section 3.1, we conclude

$$\frac{\partial G_>(x,x-0;\lambda)}{\partial x} - \frac{\partial G_<(x,x+0;\lambda)}{\partial x} = \frac{1}{p(x)}.$$

Thus, we obtain the following expression for the second derivative of $u(x;\lambda)$

$$\frac{d^2u(x;\lambda)}{dx^2} = \int_{\xi=a}^{\xi=x} \frac{\partial^2 G_>(x,\xi;\lambda)}{\partial x^2} g(\xi)\, d\xi + \int_{\xi=x}^{\xi=b} \frac{\partial^2 G_<(x,\xi;\lambda)}{\partial x^2} g(\xi)\, d\xi + \frac{g(x)}{p(x)}. \qquad (8)$$

Now we are ready to prove that the function $u(x;\lambda)$, as defined by Eq. (6), solves the nonhomogeneous BVP (1), (2). Substituting the expressions for the function $u(x;\lambda)=u(x)$, Eq. (6), and its derivatives, as calculated by Eqs. (7) and (8) above, into Eq.(1), we get

$$\begin{aligned}
&\big(p(x)\,u'(x)\big)' - q(x)\,u(x) + \lambda\, u(x) = p(x)\,u''(x) + p'(x)\,u'(x) + \big(\lambda - q(x)\big)u(x) = \\
&\quad = p(x)\left[\int_{\xi=a}^{\xi=x} \frac{\partial^2 G_>(x,\xi;\lambda)}{\partial x^2} g(\xi)\, d\xi + \int_{\xi=x}^{\xi=b} \frac{\partial^2 G_<(x,\xi;\lambda)}{\partial x^2} g(\xi)\, d\xi + \frac{g(x)}{p(x)}\right] + \\
&\qquad + p'(x)\left[\int_{\xi=a}^{\xi=x} \frac{\partial G_>(x,\xi;\lambda)}{\partial x} g(\xi)\, d\xi + \int_{\xi=x}^{\xi=b} \frac{\partial G_<(x,\xi;\lambda)}{\partial x} g(\xi)\, d\xi\right] - \\
&\qquad + \big(\lambda - q(x)\big)\left[\int_{\xi=a}^{\xi=x} G_>(x,\xi;\lambda)\, g(\xi)\, d\xi + \int_{\xi=x}^{\xi=b} G_<(x,\xi;\lambda)\, g(\xi)\, d\xi\right] = \\
&= \int_{\xi=a}^{\xi=x}\left(p(x)\frac{\partial^2 G_>(x,\xi;\lambda)}{\partial x^2} + p'(x)\frac{\partial G_>(x,\xi;\lambda)}{\partial x} + \big(\lambda - q(x)\big) G_>(x,\xi;\lambda)\right) g(\xi)\, d\xi + \\
&\quad + \int_{\xi=x}^{\xi=b}\left(p(x)\frac{\partial^2 G_<(x,\xi;\lambda)}{\partial x^2} + p'(x)\frac{\partial G_<(x,\xi;\lambda)}{\partial x} + \big(\lambda - q(x)\big) G_<(x,\xi;\lambda)\right) g(\xi)\, d\xi + \\
&\qquad\qquad + p(x)\frac{g(x)}{p(x)} = \\
&\qquad\qquad = g(x),
\end{aligned}$$

Since each branch of Green's function satisfy the homogeneous differential equation. This completes the proof of the claim that $u(x;\lambda)$, as defined by Eq. (6), satisfies Eq. (1).

Besides, we have to show that this function $u(x;\lambda)$ satisfies the boundary conditions as well. At the endpoint $x=a$, we have

$$\begin{aligned}
\mathsf{B}_a[u] &\equiv p(a)u'(a) - \nu_0\, u(a) = \\
&= \int_{\xi=a}^{\xi=b} p(a)\frac{\partial G_<(a,\xi;\lambda)}{\partial x} g(\xi)\, d\xi - \nu_0 \int_{\xi=a}^{\xi=b} G_<(a,\xi;\lambda)\, g(\xi)\, d\xi = \\
&= \int_{\xi=a}^{\xi=b}\left(p(a)\frac{\partial G_<(a,\xi;\lambda)}{\partial x} - \nu_0\, G_<(a,\xi;\lambda)\right) g(\xi)\, d\xi = 0,
\end{aligned}$$

since $G_<(x,\xi;\lambda)$ satisfies the BC at $x=a$. The satisfaction of the BC at $x=b$ is proved similarly. The proof of Theorem 1 is now completed.

In this case, the solution is unique. An easy proof is provided by the following

Theorem 2. The solution to the non-homogeneous BVP, Eqs, (1), (2), provided by Eq. (6), is unique. ■

***Proof*:**Indeed, let $u\,,\upsilon$ two different classical solutions of this problem. Applying Green's identity for these two functions, we obtain

$$\int_a^b \Big(\upsilon(x)\,\mathcal{L}_{\lambda}[u](x) - u(x)\,\mathcal{L}_{\lambda}[\upsilon](x)\Big)\,dx = \big[\,\upsilon(x)\,pu'(x) - u(x)\,p\upsilon'(x)\,\big]_a^b.$$

The right hand side is zero, since both functions satisfy the BCs (2).Using also the fact that both $u\,,\upsilon$ satisfy Eq.(1), Green's identity becomes

$$\int_a^b \big(\upsilon(x)\,g(x) - u(x)\,g(x)\big)\,dx \equiv \int_a^b \big(\upsilon(x) - u(x)\big)\,g(x)\,dx = 0. \tag{9}$$

Applying Eq. (9) to the case $g = u - \upsilon \in C[a\,,b]$ we have

$$\int_a^b \big(u(x) - \upsilon(x)\big)^2\,dx = 0. \tag{10}$$

This implies that $u(x) = \upsilon(x),\quad \forall\, x \in [a,b]$. The proof of uniqueness is completed.

3.3 Construction of Green's Function when λ is an eigenvalue $(\lambda = \lambda_n)$

Green's functions for compatible differential systems consisting of a single nth-order linear differential equation with continuous coefficients together with boundary conditions at two points may be traced back to Hilbert. In the bibliography, Green's function in this case is referred as Generalized Green's function (GGF) or Modified Green's function. In particular, in Stakgolt& Holst (2011) in Sec.3.5, is referred as Modified Green's function and there is presented the basic obstacle which is a solvability issue. The expression of Green's function in this case is not given for the general case, but there is a specific example. There is also a discussion about symmetry, which in this case does not always satisfied, but even though a Modified Green's function may be not symmetric, it is possible to become.In Stone &Goldbart (2009), where is also referred as Modified Green's function is given an example where someone can understand conceptual details. Moreover, in Haberman (2004), in Sec.9.4.3 there is a more detailed a general presentation of Generalized Green's function, which describes and proves its properties such as symmetry and there is also an example. Moreover, in Haberman R., Green's Function, Lecture 7, there are detailed steps about the construction of GGF, in the general case. There is also a general way of making symmetric a GGF.

The construction of Green's function in Section 3.1 (for the case $\lambda \neq \lambda_n$) was based on the assumption that each of two linearly independent solutions, $u_1(x;\lambda)$, $u_2(x;\lambda)$, of the homogeneous DE satisfies one of the two BCs and does not satisfy the other BC. This assumption is not valid in the present case, where λ is an eigenvalue. Also, the non-homogeneous BVP may have not a solution in this case. These two negative results make questionable the construction of Green's Function, for the case $\lambda = \lambda_n$. Before proceeding to examine this question, we shall deal with the two afore-mentioned negative results in detail, in order to clarify the constraints we have to face in this case.

3.3a Two negative results

Let us consider, once again, the BVP

$$\big((pu')(x)\big)' + \big(\lambda - q(x)\big)u(x) = 0, \qquad a < x < b \tag{1}$$

$$\mathsf{B}_a[u] \equiv p(a)u'(a) - \nu_0 u(a) = 0, \tag{2a}$$

$$\mathsf{B}_b[u] \equiv p(b)u'(b) - \mu_0 u(b) = 0. \tag{2b}$$

Lemma 1: Let $u_1(x;\lambda)$, $u_2(x;\lambda)$ be two linearly independent solutions of the homogeneous DE (1). Then, the following identity holds true:

$$\begin{aligned} p(a)u_1(a;\lambda)\,u_2'(a;\lambda) \;-\; p(a)u_1'(a;\lambda)\,u_2(a;\lambda) &= \\ = p(b)u_1(b;\lambda)\,u_2'(b;\lambda) \;-\; p(b)u_1'(b;\lambda)\,u_2(b;\lambda). \end{aligned} \tag{3}$$

Proof: We start by examining some properties of the Wronskian of the two solutions $u_1(x;\lambda)$ and $u_2(x;\lambda)$, which is defined by

$$\begin{aligned} W\big(u_1, u_2;\lambda\big)(x) = W\big(u_1, u_2;\lambda\big) &= \begin{vmatrix} u_1(x;\lambda) & u_2(x;\lambda) \\ u_1'(x;\lambda) & u_2'(x;\lambda) \end{vmatrix} \\ &= u_1(x;\lambda)\,u_2'(x;\lambda) \;-\; u_1'(x;\lambda)\,u_2(x;\lambda). \end{aligned} \tag{a}$$

The derivative of the Wronskian with respect to x, denoted by $W'\big(u_1, u_2;\lambda\big)(x)$, can be expressed as follows:

$$\begin{aligned} W'\big(u_1, u_2;\lambda\big)(x) &= \big(u_1(x;\lambda)\,u_2'(x;\lambda) \;-\; u_1'(x;\lambda)\,u_2(x;\lambda)\big)' = \\ &= u_1(x;\lambda)\,u_2''(x;\lambda) \;-\; u_1''(x;\lambda)\,u_2(x;\lambda) \end{aligned} \tag{b}$$

From the DE we obtain

$$u''(x) = -\frac{p'(x)}{p(x)}u'(x) + \frac{q(x)-\lambda}{p(x)}u(x) \tag{c}$$

By substituting Eq.(c) into Eq.(b), we find

$$W'\left(u_1, u_2; \lambda\right) = u_1(x;\lambda)\, u_2''(x;\lambda) - u_1''(x;\lambda)\, u_2(x;\lambda) =$$
$$= u_1(x;\lambda)\left(-\frac{p'(x)}{p(x)} u_2'(x;\lambda) + \frac{q(x)-\lambda}{p(x)} u_2(x;\lambda)\right) -$$
$$-\left(-\frac{p'(x)}{p(x)} u_1'(x;\lambda) + \frac{q(x)-\lambda}{p(x)} u_1(x;\lambda)\right) u_2(x;\lambda) =$$
$$= -\frac{p'(x)}{p(x)}\left(u_1(x;\lambda)\, u_2'(x;\lambda) - u_1'(x;\lambda)\, u_2(x;\lambda)\right) = -\frac{p'(x)}{p(x)} W\left(u_1, u_2; \lambda\right).$$

From the above expression we obtain

$$W\left(u_1, u_2; \lambda\right)(x) - W\left(u_1, u_2; \lambda\right)(s) = \exp\left(-\int_s^x \frac{p'(s)}{p(s)}\, ds\right).$$

But, since

$$\exp\left(-\int_s^x \frac{p'(s)}{p(s)}\, ds\right) = \exp\left(-\int_s^x \frac{dp}{p}\right) = \exp\left(\log\left(\frac{p(s)}{p(x)}\right)\right) = \frac{p(s)}{p(x)},$$

the following equation is implied

$$p(x)\, W\left(u_1, u_2; \lambda\right)(x) = p(s)\, W\left(u_1, u_2; \lambda\right)(s). \tag{4}$$

Applying the above equation to the case $x = a$, $s = b$, we find

$$p(a) W\left(u_1, u_2; \lambda\right)(a) = p(b) W\left(u_1, u_2; \lambda\right)(b). \tag{5}$$

Expanding the Wronskians in Eq. (5), we obtain Eq. (3). This completes the proof of the lemma.

Assume now that $u_1(x;\lambda)$ is identified by the choice

$$u_1(a;\lambda) = \text{given}, \quad \text{and}\; u_1'(a;\lambda) = \frac{\nu_0\, u_1(a;\lambda)}{p(a)} \Rightarrow \mathsf{B}_a[u_1] = 0, \tag{6a}$$

while $u_2(x;\lambda)$ is identified by the choice

$$u_2(b;\lambda) = \text{given}, \quad \text{and}\; u_2'(b;\lambda) = \frac{\mu_0\, u_2(b;\lambda)}{p(b)} \Rightarrow \mathsf{B}_b[u_2] = 0. \tag{6b}$$

Then,

$$p(a) u_2'(a;\lambda) - \nu_0\, u_2(a;\lambda) = \frac{u_2(b;\lambda)}{u_1(a;\lambda)}\left(u_1(b;\lambda)\, \mu_0 - p(b) u_1'(b;\lambda)\right)$$

which is equivalently written as

$$\mathrm{B}_a[u_2] = \frac{u_2(b;\lambda)}{u_1(a;\lambda)} \mathrm{B}_b[u_1]. \tag{7}$$

Now, we distinguish the two cases $\lambda \neq \lambda_n$ and $\lambda = \lambda_n$.

Assume first that $\lambda \neq \lambda_n$. Then, the additional assumption that $\mathrm{B}_a[u_2] = 0$ would mean that $u_2(x;\lambda)$ is an eigenfunction, since it satisfies the differential equation and the BCs. However, this is impossible, since, in the present case, the homogeneous problem has only the trivial solution. That is, in this case, we have $\mathrm{B}_a[u_2] \neq 0$ and, according to Eq.(7), $\mathrm{B}_b[u_1] \neq 0$, as well.

Corollary 1: When $\lambda \neq \lambda_n$, it is always possible to choose the two linearly independent solutions of the homogeneous DE, $u_1(x;\lambda)$, $u_2(x;\lambda)$, so that

$$\mathrm{B}_a[u_1] = 0 \quad \wedge \quad \mathrm{B}_b[u_1] \neq 0, \quad \text{and} \tag{8a}$$

$$\mathrm{B}_a[u_2] \neq 0 \quad \wedge \quad \mathrm{B}_b[u_2] = 0. \tag{8b}$$

Assume now that $\lambda = \lambda_n$. In this case there is an eigenfunction, say $u_n(x)$. This can be expressed as a linear combination of the two linearly independent solutions $u_1(x;\lambda)$ and $u_2(x;\lambda)$, that is $u_n(x) = C_1\, u_1(x;\lambda) + C_2\, u_2(x;\lambda)$. Then, we have

$$\mathrm{B}_a[u_n] = C_1\, \mathrm{B}_a[u_1] + C_2\, \mathrm{B}_a[u_2] = 0, \tag{9a}$$

$$\mathrm{B}_b[u_n] = C_1\, \mathrm{B}_b[u_1] + C_2\, \mathrm{B}_b[u_2] = 0. \tag{9b}$$

Assume now that it is possible to choose $u_1(x;\lambda)$, $u_2(x;\lambda)$ so that to satisfy Eqs. (8a,b). This leads to a contradiction, since $\mathrm{B}_a[u_1] = 0$ and Eq. (9a) imply $\mathrm{B}_a[u_2] = 0$ and, similarly, we also find $\mathrm{B}_b[u_1] = 0$.

Corollary 2: When $\lambda = \lambda_n$, it is not possible to choose the two linearly independent solutions of the homogeneous DE, $u_1(x;\lambda)$, $u_2(x;\lambda)$, so that to satisfy Eqs. (8a,b).

Remark. In the case $\lambda = \lambda_n$, it is possible to choose the two linearly independent solutions of DE (1), $u_1(x;\lambda)$, $u_2(x;\lambda)$, as follows:

Either $\mathrm{B}_a[u_1] = \mathrm{B}_b[u_1] = 0$ and $\mathrm{B}_a[u_2] \neq 0 \;\wedge\; \mathrm{B}_b[u_2] \neq 0$,

Or $\mathrm{B}_a[u_1] = \mathrm{B}_a[u_2] = 0$ and $\mathrm{B}_b[u_1] \neq 0 \;\wedge\; \mathrm{B}_b[u_2] \neq 0$,

Or $\mathrm{B}_a[u_1] \neq 0 \;\wedge\; \mathrm{B}_a[u_2] \neq 0$ and $\mathrm{B}_b[u_1] \neq 0 \;\wedge\; \mathrm{B}_b[u_2] \neq 0$.

Another negative result is that, in the case of $\lambda = \lambda_n$, the non-homogeneous BVP is not always solvable. Indeed, let the following NHBVP

$$\mathsf{L}_{\lambda_n}[u](x) \equiv Lu(x) + \lambda_n u(x) \equiv \left(p(x)\, u'(x)\right)' - q(x)\, u(x) + \lambda_n\, u(x) = g(x), \quad a < x < b, \tag{10}$$

$$\mathsf{B}_a[u] \equiv p(a)\, u'(a) - \nu_0\, u(a) = 0, \tag{11a}$$

$$\mathsf{B}_b[u] \equiv p(b)\, u'(b) - \mu_0\, u(b) = 0. \tag{11b}$$

In this case, the eigenfunction $u_n(x)$, belonging to the eigenvalue λ_n, satisfies the homogeneous problem, that is the problem (10)-(11), with $g(x) = 0$. In the context of the analysis part of our investigation, <u>we start by assuming</u> that <u>there exists a solution</u> to this problem, denoted by $u(x) = u(x;\lambda_n)$. Then, both functions $u(x;\lambda_n)$ and $u_n(x)$ satisfy the BCs (11a,b) and thus Green's identity applied to $u(x;\lambda_n)$ and $u_n(x)$, yields

$$\int_a^b \Big(u_n(x)\, \mathsf{L}_{\lambda_n}[u(\cdot\,;\lambda_n)](x) - u(x;\lambda_n)\, \mathsf{L}_{\lambda_n}[u_n](x) \Big)\, dx = 0. \tag{12}$$

Since $\mathsf{L}_{\lambda_n}[u_n](x) = 0$ and $\mathsf{L}_{\lambda_n}[u(\cdot\,;\lambda_n)](x) = g(x)$, the above equation leads to the necessary condition

$$\int_a^b u_n(x)\, g(x)\, dx = 0. \tag{13}$$

This necessary condition will (sometimes) be referred to as **compatibility condition** (for the BVP (10)-(11)). We can summarize the above finding in the form of the following

Theorem 1. In order that the non-homogeneous BVP (10), (11) has a solution, when $\lambda = \lambda_n$ is an eigenvalue, it is necessary the excitation term $g(x)$ to be orthogonal to the corresponding eigenfunction $u_n(x)$. ■

3.3b Construction of a generalized Green's Function in the case $\lambda = \lambda_n$

However tempting may be to use the Green's function method (in this case as well), in order to construct the solution to the non-homogeneous BVP (10)-(11) of the previous section, Theorem 1 poses a serious obstacle. The difficulty arises from the fact that Green's function is a solution of a specific non-homogeneous problem, with excitation $\delta(x-\xi)$. In the present case, we have the necessary condition (13) of Sec. 3.3a, which, when applied to the excitation $g(x) = \delta(x-\xi)$, leads to the requirement

$$\int_a^b u_n(x)\,\delta(x-\xi)\,dx = u_n(\xi) = 0\,, \tag{1}$$

which is not valid. A way out of this controversy is to try to define the Green's function in a different way, so that the compatibility condition to be valid. To motivate such a new definition of Green's function, we shall come back to the general non-homogeneous BVP (10)-(11) of Sec. 3.3a, and modify the excitation so that to satisfy the compatibility condition.

From Eq. (13) of the previous section, we understand that the source of the problem is the non-vanishing of the projection of the excitation function $g(x)$ on the subspace generated by the eigenfunction $u_n(x)$. Let us subtract this projection from the excitation, and replace the latter by

$$\begin{aligned} \tilde{g}(x) &= g(x) - u_n(x)\,\frac{\int_a^b u_n(\xi)\,g(\xi)\,d\xi}{\int_a^b u_n^2(\xi)\,d\xi} = \\ &= g(x) - u_n(x)\,\frac{<u_n\,,g>}{\left\|u_n\right\|^2}\,. \end{aligned} \tag{2}$$

Then, the right hand side of Eq.(13) of Sec. 3.3a, becomes

$$\begin{aligned} &\int_a^b u_n(x)\left(g(x) - u_n(x)\,\frac{\int_a^b u_n(\xi)\,g(\xi)\,d\xi}{\int_a^b u_n^2(\xi)\,d\xi}\right)dx = \\ &= \int_a^b u_n(x)\,g(x)\,dx - \int_a^b u_n^2(x)\,dx\,\frac{\int_a^b u_n(\xi)\,g(\xi)\,d\xi}{\int_a^b u_n^2(\xi)\,d\xi} = 0\,, \end{aligned}$$

that is, the compatibility condition is now satisfied! If there is a specific solution $\tilde{u}(x;\lambda_n)$ to the new problem, having excitation $\tilde{g}(x)$ (this remains to be proved), then, any function of the form $\tilde{u}(x;\lambda_n) + C\,u_n(x)$ will also satisfy the same problem, for any value of $C \in \mathbb{R}$ or $C \in \mathbb{C}$. Thus, in this case the solution is not unique.

The same ideas will be now applied to formulate the problem, Eqs. (3) – (4) below, defining the sought-for (new) Green's function, which will be called the *generalized Green's function* (GGF) of the non-homogeneous BVP in the case of $\lambda = \lambda_n$:

$$\mathsf{L}_{\lambda_n}[\tilde{G}_{(\lambda_n)}(\cdot,\xi)](x) \equiv L\tilde{G}_{(\lambda_n)}(x,\xi) + \lambda_n \tilde{G}_{(\lambda_n)}(x,\xi) \equiv$$

$$\equiv \left(p(x)\tilde{G}'_{(\lambda_n)}(x,\xi)\right)' - q(x)\tilde{G}_{(\lambda_n)}(x,\xi) + \lambda_n \tilde{G}_{(\lambda_n)}(x,\xi) = \delta(x-\xi) - \frac{u_n(x)\,u_n(\xi)}{\|u_n\|^2}, \tag{3}$$

$$\mathsf{B}_a[\tilde{G}_{(\lambda_n)}] \equiv p(a)\tilde{G}'_{(\lambda_n)}(a,\xi) - \nu_0 \tilde{G}_{(\lambda_n)}(a,\xi) = 0, \tag{4a}$$

$$\mathsf{B}_b[\tilde{G}_{(\lambda_n)}] \equiv p(b)\tilde{G}'_{(\lambda_n)}(b,\xi) - \mu_0 \tilde{G}_{(\lambda_n)}(b,\xi) = 0. \tag{4b}$$

As in Sec.3 we are going to derive the basic properties of GGF's although we have not its analytic expression. Recall the following two properties of delta function

$$\delta(x-\xi) = 0, \quad \text{for } x \neq \xi \text{ and } \int_{x=\xi-d_-}^{x=\xi+d_+} \delta(x-\xi)\,dx = 1, \tag{5a,b}$$

where d_- and d_+ are positive numbers. According to the property (5a), the differential equation (3) can be written, separately in each subinterval (a,ξ) and (ξ,b), as follows:

$$\left(p(x)\tilde{G}'_{(\lambda_n)}(x,\xi)\right)' - q(x)\tilde{G}_{(\lambda_n)}(x,\xi) - \lambda_n \tilde{G}_{(\lambda_n)}(x,\xi) = -\frac{u_n(x)\,u_n(\xi)}{\|u_n\|^2}, \tag{6}$$

$$\text{for} \begin{cases} a < x < \xi, \\ \quad \text{and} \\ \xi < x < b. \end{cases}$$

Since Eq. (6) does not apply to the point $x = \xi$, we need interface (matching) conditions for the sought-for function $\tilde{G}_{(\lambda_n)}(x,\xi)$ at $x = \xi$. First, we impose continuity of $\tilde{G}_{(\lambda_n)}(x,\xi)$, by assuming that

$$\tilde{G}_{(\lambda_n)}(\xi+0,\xi) = \tilde{G}_{(\lambda_n)}(\xi-0,\xi). \tag{7}$$

In order to find a matching condition for the first derivative of the GGF, use will be made of the property (5b).Integrating both members of Eq. (3) from $x = \xi-\varepsilon$ to $x = \xi+\varepsilon$, where $\varepsilon > 0$ (and sufficiently small), we obtain

$$\left[p(x)\frac{d\tilde{G}_{(\lambda_n)}(x,\xi)}{dx}\right]_{\xi-\varepsilon}^{\xi+\varepsilon} + \int_{\xi-\varepsilon}^{\xi+\varepsilon} \left(\lambda_n - q(x)\right)\tilde{G}_{(\lambda_n)}(x,\xi)\,dx = 1 - \frac{u_n(\xi)}{\|u_n\|^2}\int_{\xi-\varepsilon}^{\xi+\varepsilon} u_n(x)\,dx$$

Recalling that $q(x)$ and $u_n(x)$ are continuous in $[a,b]$ (and thus at $x = \xi$), and having assumed that $\tilde{G}_{(\lambda_n)}(x,\xi)$ is also continuous in $[a,b]$, we conclude that

$$\lim_{\varepsilon\to 0}\int_{\xi-\varepsilon}^{\xi+\varepsilon}\left(\lambda_n - q(x)\right)\tilde{G}_{(\lambda_n)}(x,\xi)\,dx = 0 \text{ and } \lim_{\varepsilon\to 0}\int_{\xi-\varepsilon}^{\xi+\varepsilon} u_n(x)\,dx = 0 .$$

Hence, we obtain

$$\lim_{\varepsilon\to 0}\left[p(x)\frac{d\tilde{G}_{(\lambda_n)}(x,\xi)}{dx}\right]_{\xi-\varepsilon}^{\xi+\varepsilon} = 1, \tag{8}$$

as in the case $\lambda \neq \lambda_n$. Taking into account the continuity of $p(x)$, we finally derive the condition

$$\frac{d\tilde{G}_{(\lambda_n)}(\xi+0,\xi)}{dx} - \frac{d\tilde{G}_{(\lambda_n)}(\xi-0,\xi)}{dx} = \frac{1}{p(\xi)} . \tag{9}$$

Eq. (9) means that the first derivative of GGF must have a jump discontinuity of magnitude $1/p(\xi)$ at the point $x=\xi$.

To find the GGF we have to find solution(s) of the following two problems

$$\mathsf{L}_{\lambda_n}[\tilde{G}^{<}_{(\lambda_n)}(\cdot,\xi)](x) = -\frac{u_n(x)\,u_n(\xi)}{\left\|u_n\right\|^2}, \qquad a<x<\xi, \tag{10}$$

$$\mathsf{B}_a[\tilde{G}^{<}_{(\lambda_n)}] \equiv p(a)\tilde{G}'^{<}_{(\lambda_n)}(a,\xi) - \nu_0\tilde{G}^{<}_{(\lambda_n)}(a,\xi) = 0 \tag{11}$$

and

$$\mathsf{L}_{\lambda_n}[\tilde{G}^{>}_{(\lambda_n)}(\cdot,\xi)](x) = -\frac{u_n(x)\,u_n(\xi)}{\left\|u_n\right\|^2}, \qquad \xi<x<b, \tag{12}$$

$$\mathsf{B}_b[\tilde{G}^{>}_{(\lambda_n)}] \equiv p(b)\tilde{G}'^{>}_{(\lambda_n)}(b,\xi) - \mu_0\tilde{G}^{>}_{(\lambda_n)}(b,\xi) = 0 \tag{13}$$

and then to construct a function $\tilde{G}_{(\lambda_n)}(x,\xi)$.

Each one of Eqs (10) and (12) can be solved as follows.

The general solution of a second order non-homogeneous DE is given by

$$u(x) = C_1 u_1(x) + C_2 u_2(x) + u_p , \tag{14}$$

Where $C_1u_1 + C_2u_2(x)$ is the general solution of the corresponding homogeneous DE, and u_p is a particular solution of a non-homogeneous DE, defined on the interval (a,b).The particular solution is given in the form (see Appendix A for a detailed derivation)

$$u_p(x) = \frac{u_1(x)}{p(x)W(u_1,u_2)(x)}\int_x^{\xi} u_2(s)\,g(s)\,ds - \frac{u_2(x)}{p(x)W(u_1,u_2)(x)}\int_x^{\xi} u_1(s)\,g(s)\,ds , \tag{15}$$

where $g(x) = -\,u_n(x)\,u_n(\xi)/\left\|u_n\right\|^2$ is the non-homogeneous part of the DE. Note that, as proved in Lemma 1, $p(x)\,W(u_1,u_2)(x) = \text{constant}$; thus, in the sequel we shall simplify the notation writing $p\,W$ instead of $p(x)W(u_1,u_2)(x)$.

Applying Eqs. (14) and (15) to DEs (10) and (12) (which are defined in different domains), we obtain

$$\tilde{G}^{<}_{(\lambda_n)}(x,\xi) = C_1 u_1(x) + C_2 u_2(x) + \frac{u_1(x)}{p\,W}\int_x^{\xi} u_2(s)g(s)ds - \frac{u_2(x)}{p\,W}\int_x^{\xi} u_1(s)g(s)ds \tag{16}$$

and

$$\tilde{G}^{>}_{(\lambda_n)}(x,\xi) = C_3 u_1(x) + C_4 u_2(x) \; + \frac{u_1(x)}{p\,W}\int_x^{\xi} u_2(s)g(s)ds - \frac{u_2(x)}{p\,W}\int_x^{\xi} u_1(s)g(s)ds. \tag{17}$$

In order to determine the constants C_1, C_2, C_3, C_4 use is made of Eqs. (7),(9),(11),(13).

At first, the first derivative of the function $\tilde{G}^{<}_{(\lambda_n)}(x,\xi)$ with respect to x is

$$\frac{d\tilde{G}^{<}_{(\lambda_n)}(x,\xi)}{dx} = C_1 u_1'(x) + C_2 u_2'(x) + \frac{u_1'(x)}{p\,W}\int_x^{\xi} u_2(x)g(x)dx \; - \cancel{\frac{u_2(x)u_1(x)g(x)}{p\,W}} \; - $$
$$- \frac{u_2'(x)}{p\,W}\int_x^{\xi} u_1(x)g(x)dx + \cancel{\frac{u_2(x)u_1(x)g(x)}{p\,W}}\,.$$

Hence, the above expression is reduced to

$$\frac{d\tilde{G}^{<}_{(\lambda_n)}(x,\xi)}{dx} = C_1 u_1'(x) + C_2 u_2'(x) + \frac{u_1'(x)}{p\,W}\int_x^{\xi} u_2(x)g(x)dx - \frac{u_2'(x)}{p\,W}\int_x^{\xi} u_1(x)g(x)dx. \tag{18}$$

Since $\tilde{G}^{<}_{(\lambda_n)}(x,\xi)$ satisfies the BC at the endpoint $x = a$, Eq.(11), we use Eqs.(16) and (18) and we have

$$p(a)\tilde{G}'^{<}_{(\lambda_n)}(a,\xi) - \nu_0\,\tilde{G}^{<}_{(\lambda_n)}(a,\xi) \;=\; 0\,.$$

Using the fact that the one term of the right hand side of Eq.(18) vanishes when we set $x = a$, we have

$$\Rightarrow p(a)\left(C_1 u_1'(a) + C_2 u_2'(a) + \frac{u_1'(a)}{pW}\int_a^{\xi} u_2(x)\,g(x)\,dx - \frac{u_2'(a)}{pW}\int_a^{\xi} u_1(x)\,g(x)\,dx\right) -$$

$$-\nu_0\left(C_1 u_1(a) + C_2 u_2(a) + \frac{u_1(a)}{pW}\int_a^{\xi} u_2(x)\,g(x)\,dx - \frac{u_2(a)}{pW}\int_a^{\xi} u_1(x)\,g(x)\,dx\right) = 0$$

The above equation is reduced to

$$C_1\Big(p(a)u_1'(a) - \nu_0 u_1(a)\Big) + C_2\Big(p(a)u_2'(a) - \nu_0 u_2(a)\Big) = -\frac{p(a)u_1'(a) - \nu_0 u_1(a)}{pW}\int_a^{\xi} u_2(x)\,g(x)\,dx +$$

$$+\frac{p(a)u_2'(a) - \nu_0 u_2(a)}{pW}\int_a^{\xi} u_1(x)\,g(x)\,dx\,. \tag{19}$$

Working in the same way for the function $\tilde{G}^{>}_{(\lambda_n)}(x,\xi)$ we result in

$$C_3\Big(p(b)u_1'(b) - \mu_0 u_1(b)\Big) + C_4\Big(p(b)u_2'(b) - \mu_0 u_2(b)\Big) = -\frac{p(b)u_1'(b) - \mu_0 u_1(b)}{pW}\int_b^{\xi} u_2(x)\,g(x)\,dx +$$

$$+\frac{p(b)u_2'(b) - \mu_0 u_2(b)}{pW}\int_b^{\xi} u_1(x)\,g(x)\,dx\,. \tag{20}$$

At this point, we are going to use Eqs.(7) and (9).

The continuity of Green's Function at the point $x = \xi$, which is Eq.(7), implies

$$\Big(C_3 - C_1\Big)u_1(\xi) + \Big(C_4 - C_2\Big)u_2(\xi) = 0 \tag{21}$$

The jump discontinuity of the first derivative with respect to x of Green's Function, at the point $x = \xi$ which is particularly Eq.(9) gives us

$$\Big(C_3 - C_1\Big)u_1'(\xi) + \Big(C_4 - C_2\Big)u_2'(\xi) = \frac{1}{p(\xi)} \tag{22}$$

From now on, we shall omit the arguments in the integrals in order to simplify the expressions; for example,

$$\int_a^{\xi} u_1(x)g(x)\,dx = \int_a^{\xi} u_1 g\,.$$

By solving the system of Eqs.(21)-(22), we obtain

$$C_3 - C_1 = -\frac{u_2(\xi)}{pW} \tag{23}$$

and

$$C_4 - C_2 = \frac{u_1(\xi)}{pW}. \tag{24}$$

Assume now that $u_1(x) = u_n(x)$ is the eigenfunction belonging to the eigenvalue λ_n and $u_2(x) = \upsilon_n(x)$ is the second (linearly independent from $u_n(x)$) solution of the homogeneous problem. Recall that $\upsilon_n(x)$ can be expressed in terms of u_n (see Appendix B). Taking into account the above clarifications, Eq.(19) takes the form

$$C_2 = \frac{\int_a^\xi u_n g}{pW}, \tag{25}$$

since $u_n(x)$,as an eigenfunction, satisfies both BCs and $\upsilon_n(x)$ cannot satisfy neither of the BCs, according to Corollary 2, of the previous section, 3.3a.

Using the same arguments as above, Eq.(20) becomes

$$C_4 = \frac{\int_b^\xi u_n g}{pW}. \tag{26}$$

We also write Eqs.(23) and (24) taking into account the choice $u_1(x) = u_n(x), u_2(x) = \upsilon_n(x)$ and we have

$$C_3 - C_1 = -\frac{\upsilon_n(\xi)}{pW}, \tag{27}$$

$$C_4 - C_2 = \frac{u_n(\xi)}{pW}, \tag{28}$$

Notice that Eqs.(25).(26) and (28) give us two expressions for the constants C_2 and C_4. Although these two expressions cannot be identical for arbitrary $g(x)$, in the present case, where $g(x) = -u_n(x)\, u_n(\xi)/\|u_n\|^2$ they are equivalent. This is proved as follows. Substituting $g(x)$ by its specific form in the right-hand side of Eq. (25), we obtain

$$C_2 = -\frac{u_n(\xi)\int\limits_a^\xi u_n^2}{pW\|u_n\|^2} \tag{29a}$$

Doing the same with Eq. (26), we find

$$C_4 = -\frac{u_n(\xi)\int\limits_b^\xi u_n^2}{pW\|u_n\|^2}. \tag{29b}$$

Substituting the above equations into the left hand-side of Eq.(28), we take

$$C_4 - C_2 = -\frac{u_n(\xi)\int\limits_b^\xi u_n^2}{pW\|u_n\|^2} + \frac{u_n(\xi)\int\limits_a^\xi u_n^2}{pW\|u_n\|^2} = \frac{u_n(\xi)\int\limits_\xi^b u_n^2}{pW\|u_n\|^2} + \frac{u_n(\xi)\int\limits_a^\xi u_n^2}{pW\|u_n\|^2}.$$

Taking into account the equation

$$\|u_n\|^2 = \int\limits_a^b u_n^2 = \int\limits_a^\xi u_n^2 + \int\limits_\xi^b u_n^2,$$

we conclude that Eq.(28) is satisfied for the specific values of the constants C_2 and C_4.

Therefore, the constants are defined as below

$$C_1 = \frac{\upsilon_n(\xi)}{\|u_n\|^2 pW} + C, \tag{30a}$$

$$C_2 = -\frac{u_n(\xi)\int\limits_a^\xi u_n^2}{pW\|u_n\|^2}, \tag{30b}$$

$C_3 = C$,arbitrary constant (30c)

$$C_4 = \frac{u_n(\xi)}{pW} - \frac{u_n(\xi)}{pW\|u_n\|^2}\int\limits_a^\xi u_n^2 \tag{30d}$$

Hence, Eqs.(16) and (17) for $u_1(x) = u_n(x), u_2(x) = \upsilon_n(x)$ become

$$\tilde{G}^{<}_{(\lambda_n)}(x,\xi) = C u_n(x) + \frac{\upsilon_n(\xi)u_n(x)}{pW} - \frac{\upsilon_n(x)u_n(\xi)}{pW\|u_n\|^2}\int_a^{\xi} u_n^2 - \frac{u_n(x)u_n(\xi)}{pW\|u_n\|^2}\int_x^{\xi}\upsilon_n u_n + \frac{\upsilon_n(x)u_n(\xi)}{pW\|u_n\|^2}\int_x^{\xi} u_n^2 , \tag{31}$$

$$\tilde{G}^{>}_{(\lambda_n)}(x,\xi) = C u_n(x) + \frac{u_n(\xi)\upsilon_n(x)}{pW} - \frac{u_n(\xi)\upsilon_n(x)}{pW\|u_n\|^2}\int_a^{\xi} u_n^2 - \frac{u_n(x)u_n(\xi)}{pW\|u_n\|^2}\int_x^{\xi}\upsilon_n u_n + \frac{\upsilon_n(x)u_n(\xi)}{pW\|u_n\|^2}\int_x^{\xi} u_n^2 . \tag{32}$$

At this point we are going to work separately with $-\frac{\upsilon_n(x)u_n(\xi)}{pW\|u_n\|^2}\int_a^{\xi} u_n^2 + \frac{\upsilon_n(x)u_n(\xi)}{pW\|u_n\|^2}\int_x^{\xi} u_n^2$ which is included in both Eqs.

$$-\frac{\upsilon_n(x)u_n(\xi)}{pW\|u_n\|^2}\int_a^{\xi} u_n^2 + \frac{\upsilon_n(x)u_n(\xi)}{pW\|u_n\|^2}\int_x^{\xi} u_n^2 = \frac{\upsilon_n(x)u_n(\xi)}{pW\|u_n\|^2}\int_{\xi}^{a} u_n^2 + \frac{\upsilon_n(x)u_n(\xi)}{pW\|u_n\|^2}\int_x^{\xi} u_n^2 = \frac{\upsilon_n(x)u_n(\xi)}{pW\|u_n\|^2}\int_x^{a} u_n^2 \tag{33}$$

Hence, taking into account Eq.(33), Eqs.(31) and (32) which are the branches of GGF, become

$$\tilde{G}^{<}_{(\lambda_n)}(x,\xi) = = C u_n(x) + \frac{\upsilon_n(\xi)u_n(x)}{pW} + \frac{\upsilon_n(x)u_n(\xi)}{pW\|u_n\|^2}\int_x^{a} u_n^2 - \frac{u_n(x)u_n(\xi)}{pW\|u_n\|^2}\int_x^{\xi}\upsilon_n u_n , \tag{34a}$$

$$\tilde{G}^{>}_{(\lambda_n)}(x,\xi) = C u_n(x) + \frac{u_n(\xi)\upsilon_n(x)}{pW} + \frac{u_n(\xi)\upsilon_n(x)}{pW\|u_n\|^2}\int_x^{a} u_n^2 - \frac{u_n(x)u_n(\xi)}{pW\|u_n\|^2}\int_x^{\xi}\upsilon_n u_n . \tag{34b}$$

Thus, the Generalized Green's Function (GGF), is given by

$$\tilde{G}_{(\lambda_n)}(x,\xi) = C u_n(x) + \frac{u_n(\xi)\upsilon_n(x)}{pW\|u_n\|^2}\int_x^{a} u_n^2 - \frac{u_n(x)\,u_n(\xi)}{pW\,\|u_n\|^2}\int_x^{\xi}\upsilon_n u_n + \begin{cases} \dfrac{\upsilon_n(\xi)\,u_n(x)}{pW}, & a < x < \xi \\ \dfrac{\upsilon_n(x)\,u_n(\xi)}{pW}, & \xi < x < b \end{cases} \tag{35}$$

Note that, the GGF, as given by Eq. (35), is not symmetric. Nevertheless, it is possible to obtain a symmetric form of GGF, by adding to the right-hand side of Eq. (35) an appropriate term of the form $c(\xi)\,u_n(x)$, which is always permissible. See also Haberman R., Green's Function, Lecture 7. One convenient choice of the extra term $c(\xi)\,u_n(x)$ is the following

$$c(\xi)\, u_n(x) \;=\; -C\, u_n(x) + \frac{u_n(x)\, \upsilon_n(\xi)}{pW \left\| u_n \right\|^2} \int_{\xi}^{a} u_n^2 \;-\; 2\frac{u_n(x)\, u_n(\xi)}{pW \left\| u_n \right\|^2} \int_{\xi}^{a} u_n\, \upsilon_n \;. \tag{36}$$

Adding the right-hand side of Eq. (36) to the right-hand side of Eq. (35) has the following effects: First, the term $C u_n(x)$ cancels out. Second, the term $\frac{u_n(x)\, \upsilon_n(\xi)}{pW \left\| u_n \right\|^2} \int_{\xi}^{a} u_n^2$ provides the symmetric of the similar term in Eq. (38). Third, one of the two $\frac{u_n(x)\, u_n(\xi)}{pW \left\| u_n \right\|^2} \int_{\xi}^{a} u_n\, \upsilon_n$ combines with $\frac{u_n(x)\, u_n(\xi)}{pW \left\| u_n \right\|^2} \int_{x}^{\xi} \upsilon_n\, u_n$ providing the new term $\frac{u_n(x)\, u_n(\xi)}{pW \left\| u_n \right\|^2} \int_{x}^{a} \upsilon_n\, u_n$, and the other one remains as it is, providing the symmetric term to the previous one. The resulting symmetric form of GGF, is as follows:

$$\begin{aligned}\tilde{G}_{(\lambda_n)}(x,\xi) \;=\; & \frac{\upsilon_n(x)\, u_n(\xi)}{pW \left\| u_n \right\|^2} \int_{x}^{a} u_n^2 + \frac{u_n(x)\, \upsilon_n(\xi)}{pW \left\| u_n \right\|^2} \int_{\xi}^{a} u_n^2 - \frac{u_n(x)\, u_n(\xi)}{pW \left\| u_n \right\|^2} \int_{x}^{a} \upsilon_n\, u_n \,- \\ & - \frac{u_n(x)\, u_n(\xi)}{pW \left\| u_n \right\|^2} \int_{\xi}^{a} u_n\, \upsilon_n \;+\; \begin{cases} \dfrac{\upsilon_n(\xi)\, u_n(x)}{pW}, & a < x < \xi, \\ \dfrac{\upsilon_n(x)\, u_n(\xi)}{pW}, & \xi < x < b. \end{cases}\end{aligned} \tag{37}$$

3.4 Solution of non-homogeneous problem when λ is an eigenvalue $(\lambda = \lambda_n)$

By utilizing the above expression of GGF we are going to find the solution of the following NHBVP, when λ is an eigenvalue $\left(\lambda = \lambda_n\right)$,

$$\mathsf{L}_{\lambda_n}[u](x) \;\equiv\; L u(x) + \lambda_n u(x) \equiv \left(p(x)\, u'(x)\right)' - q(x)\, u(x) + \lambda_n\, u(x) \;=\; g(x), \quad a < x < b, \tag{1}$$

$$\mathsf{B}_a[u] \;\equiv\; p(a)\, u'(a) - \nu_0\, u(a) \;=\; 0, \tag{2a}$$

$$\mathsf{B}_b[u] \;\equiv\; p(b)\, u'(b) - \mu_0\, u(b) \;=\; 0, \tag{2b}$$

assuming that the compatibility condition

$$\int_{a}^{b} u_n(x)\, g(x)\, dx \;=\; 0\,, \tag{3}$$

is satisfied.

In conformity with Eq.(3),the necessary condition for the NHBVP (1)-(2) to be compatible (that is, to have a solution) holds true. We shall show subsequently that this condition is also sufficient.

Let us denote, for simplicity, the solution $u(x,\lambda_n)$ of the BVP (1)-(2)by $u(x)$. By using Green's Identity for $u(x)$ and $\tilde{G}_{(\lambda_n)}(x,\xi)$ we have

$$\int_a^b \left(\tilde{G}_{(\lambda_n)}(x,\xi)\, \mathsf{L}_{\lambda_n}[u](x) - u(x)\, \mathsf{L}_{\lambda_n} \tilde{G}_{(\lambda_n)}(x,\xi) \right) dx = 0 ,$$

Since both Green's Function and the solution $u(x)$ satisfy the BCs. Using also the fact that $\mathsf{L}_{\lambda_n}[u](x) = g(x)$ and $\mathsf{L}_{\lambda_n}\left[\tilde{G}_{(\lambda_n)}\right](x,\xi) = \delta(x-\xi) - u_n(x)\, u_n(\xi) / \|u_n\|^2$, we obtain (see also Haberman, 2004, pp.409-410)

$$\int_a^b \left[\tilde{G}_{(\lambda_n)}(x,\xi)\, g(x) - u(x) \left(\delta(x-\xi) - \frac{u_n(x)\, u_n(\xi)}{\|u_n\|^2} \right) dx \right] = 0 . \tag{4}$$

$$\int_a^b \tilde{G}_{(\lambda_n)}(x,\xi)\, g(x)\, dx = \int_a^b u(x) \delta(x-\xi)\, dx - \int_a^b \frac{u(x) u_n(x)\, u_n(\xi)}{\|u_n\|^2} dx \tag{5}$$

Recalling the property $\int_a^b u(x)\delta(x-\xi)\, dx = u(\xi)$, Eq.(5) becomes

$$\int_a^b \tilde{G}_{(\lambda_n)}(x,\xi)\, g(x)\, dx = u(\xi) - \int_a^b \frac{u(x) u_n(x)\, u_n(\xi)}{\left\|u_n\right\|^2} dx . \tag{6}$$

solving Eq.(6) with respect to $u(\xi)$ and replacing argument ξ by x we obtain

$$u(x) = \int_a^b \frac{u(\xi) u_n(\xi)\, u_n(x)}{\left\|u_n\right\|^2} d\xi + \int_a^b \tilde{G}_{(\lambda_n)}(\xi,x)\, g(\xi)\, d\xi \tag{7}$$

$$u(x) = u_n(x) \int_a^b \frac{u(\xi) u_n(\xi)}{\left\|u_n\right\|^2}\, d\xi + \int_a^b \tilde{G}_{(\lambda_n)}(\xi,x)\, g(\xi)\, d\xi \tag{8}$$

$$u(x) = u_n(x) \int_a^b \frac{u(\xi) u_n(\xi)}{\left\|u_n\right\|^2}\, d\xi + \int_a^b \tilde{G}_{(\lambda_n)}(x,\xi)\, g(\xi)\, d\xi , \tag{9}$$

where in the last step use is made of the symmetry of GGF. Since the first term of the right hand-side of the above equation is a multiple of the corresponding homogeneous BVP (1)-(2), a particular solution of it has the following form

$$u(x) = \int_a^b \tilde{G}_{(\lambda_n)}(x,\xi)\, g(\xi)\, d\xi \tag{10}$$

By substituting the expression of GGF (see Sec.3.3, Eq.(37)) into Eq. (10) we obtain

$$u(x) = u_n(x) \int_a^b \frac{\upsilon_n(\xi) g(\xi)}{pW \|u_n\|^2} \left(\int_\xi^a u_n^2(s)\, ds \right) d\xi + \frac{\upsilon_n(x)}{pW} \int_a^x u_n(\xi) g(\xi) d\xi + \frac{u_n(x)}{pW} \int_x^b \upsilon_n(\xi) g(\xi)\, d\xi$$

where we have utilized the compatibility condition. Therefore, the final expression of the solution is

$$u(x) = C u_n(x) + \frac{\upsilon_n(x)}{pW} \int_a^x u_n(\xi) g(\xi) d\xi + \frac{u_n(x)}{pW} \int_x^b \upsilon_n(\xi) g(\xi)\, d\xi \,. \tag{11}$$

Theorem 1 When λ is an eigenvalue the function $u(x) = u(x, \lambda_n)$ as defined by Eq.(11) is a solution of the NHBVP (1)-(2) when the compatibility condition, Eq.(3), is satisfied. ■

Proof:
By differentiating Eq.(11) twice, and substituting the first and the second derivative of $u(x)$ into the non-homogeneous DE (1) we conclude that $u(x)$ satisfies it. The function $u(x)$ also satisfies the BCs (2). Therefore, $u(x)$ is a solution of the NHBVP (1)-(2).

Remark 1: The solution in this case is not unique, which is obvious from the C-dependence of it, as we see in Eq.(11).

3.5 The Solution of the fully non-homogeneous BVP

Since the variation of the eigenfunctions with respect to the coefficients satisfies a fully-non-homogeneous BVP we are interested to express the solution of a BVP of this type. The fully non-homogeneous BVP contains a non-homogeneous differential equation which is accompanied by two non-homogeneous BCs. As in the previous sections we distinguish two different cases. At the first case we present the method which is used to solve this type of BVPs which is the same for both cases but when the parameter λ is an eigenvalue, also a compatibility condition has to be satisfied.

3.5a The case $\lambda \neq \lambda_n$

The most general non-homogeneous BVP is

$$\mathsf{L}_\lambda[u](x) = u''(x) - q(x)u(x) + \lambda u(x) = g(x), \quad a < x < b \tag{1}$$

$$\mathsf{B}_a[u] = u'(a) - \nu_0 u(a) = r_1, \tag{2a}$$

$$\mathsf{B}_b[u] \equiv u'(b) - \mu_0 u(b) = r_2 \,. \tag{2b}$$

In order to solve the BVP (1)-(2) we follow the steps below
We are looking for a solution $u(x)$ of the following form

$$u(x) = \upsilon(x) + u_1(x) + u_2(x) \tag{3}$$

Where $u_1(x), u_2(x)$ are specific, as simple as possible and they satisfy

$$\mathrm{B}_a[u_1] = r_1 \ , \tag{4a}$$

$$\mathrm{B}_b[u_1] = 0, \tag{4b}$$

$$\mathrm{B}_a[u_2] = 0, \tag{4c}$$

$$\mathrm{B}_b[u_2] = r_2. \tag{4d}$$

u_1, u_2 could be linear functions of x of the form

$$u_1(x) = Ax + B, \tag{5}$$

$$u_2(x) = Cx + D. \tag{6}$$

Since $u_1(x)$ satisfies the BC (4a) we have

$$u_1'(a) - \nu_0 u_1(a) = r_1$$

$$A - \nu_0 (Aa + B) = r_1$$

$$A(1 - \nu_0 a) - B\nu_0 = r_1. \tag{7a}$$

Similarly, substituting the expression of $u_1(x)$ into Eq.(4b) we obtain

$$A(1 - \mu_0 b) - \mu_0 B = 0. \tag{7b}$$

The solutions of the system (7a)-(7b) are

$$A = \frac{\mu_0 r_1}{\mu_0 (1 - \nu_0 a) - \nu_0 (1 - \mu_0 b)}, \tag{8a}$$

$$B = \frac{(1 - \mu_0 b) r_1}{\mu_0 (1 - \nu_0 a) - \nu_0 (1 - \mu_0 b)}. \tag{8b}$$

Thus, taking into account Eqs.(8), Eq.(5) takes the form

$$u_1(x) = \frac{r_1 (\mu_0 x + 1 - \mu_0 b)}{\mu_0 (1 - \nu_0 a) - \nu_0 (1 - \mu_0 b)}. \tag{9a}$$

Working in the same way for the function $u_2(x)$ we obtain

$$u_2(x) = \frac{- r_2 (\nu_0 x + 1 - \nu_0 a)}{\mu_0 (1 - \nu_0 a) - \nu_0 (1 - \mu_0 b)}. \tag{9b}$$

Hence, the sum of the above functions is

$$u_1(x) + u_2(x) = \frac{r_1 (\mu_0 x + 1 - \mu_0 b) - r_2 (\nu_0 x + 1 - \nu_0 a)}{\mu_0 (1 - \nu_0 a) - \nu_0 (1 - \mu_0 b)}. \tag{10}$$

It remains to determine, which satisfies the following problem of the form

$$\mathsf{L}_{\lambda}[\upsilon](x) = \mathsf{L}_{\lambda}[u](x) - \mathsf{L}_{\lambda}[u_1 + u_2](x) \tag{11}$$

Using the fact that $\mathsf{L}_{\lambda}[u](x) = g(x)$, Eq.(11) becomes

$$\mathsf{L}_{\lambda}[\upsilon](x) = g(x) - \mathsf{L}_{\lambda}[u_1 + u_2](x). \tag{12}$$

The function $\upsilon(x)$ also satisfies the BCs

$$\mathsf{B}_a[\upsilon] = 0 \tag{13a}$$

$$\mathsf{B}_b[\upsilon] = 0 \tag{13b}$$

The solution of the BVP (12)-(13), according to Section 3.2 is given by

$$\upsilon(x) = \int_a^b G(x,\xi,\lambda)\Big(g(\xi) - \mathsf{L}_{\lambda}[u_1 + u_2](\xi)\Big)\, d\xi \tag{14}$$

where the expression of Green's Function, in the case of $\lambda \neq \lambda_n$ is given by Eq.(3) of Sec.3.1

By setting

$$k(\xi) = \mathsf{L}_{\lambda}[u_1 + u_2](\xi) = (\lambda - q(\xi))\frac{r_1\big(\mu_0\,\xi + 1 - \mu_0\, b\big) - r_2\big(\nu_0\,\xi + 1 - \nu_0\, a\big)}{\mu_0\big(1 - \nu_0\, a\big) - \nu_0\big(1 - \mu_0\, b\big)}. \tag{15}$$

Substituting Eqs.(10) (14) and (15) into Eq.(3) we obtain the expression of the solution of the fully non-homogeneous BVP (1)-(2)

$$u(x) = \int_a^b G(x,\xi,\lambda)\big(g(\xi) - k(\xi)\big)\, d\xi + \frac{r_1\big(\mu_0\, x + 1 - \mu_0\, b\big) - r_2\big(\nu_0\, x + 1 - \nu_0\, a\big)}{\mu_0\big(1 - \nu_0\, a\big) - \nu_0\big(1 - \mu_0\, b\big)}. \tag{16}$$

Hence, we managed to reduce the fully non-homogeneous problem to a non-homogeneous problem which has a non-homogeneous term only in the DE.

3.5b The case $\lambda = \lambda_n$

In order to examine in the case of $\lambda = \lambda_n$ the solvability of the fully non-homogeneous BVP(1)-(2) of the previous section we will use Green's Formula. Assume that $u(x;\lambda_n)$ is the solution of the latter BVP. Assume further that $u_n(x)$ is an eigenfunction belonging to the eigenvalue λ_n. As in the previous cases, we will use Green's identity for $u(x;\lambda)$ and $u_n(x)$ as follows

$$\int_a^b \Big(u_n(x)\mathsf{L}_{\lambda_n}[u](x) - u(x)\mathsf{L}_{\lambda_n}[u](x)\Big)\, dx = \Big[u_n(x)u'(x) - u(x)u_n'(x)\Big]_a^b. \tag{1}$$

By working separately with the right hand side of Eq.(1) we have

$$\Big[u_n(x)u'(x) - u(x)u_n'(x)\Big]_a^b = u_n(b)u'(b) - u(b)u_n'(b) - u_n(a)u'(a) + u(a)u_n'(a).$$

The fact that $u(x;\lambda)$ satisfies the BCs (2) of Section 3.5a and $u_n(x)$ the corresponding homogeneous implies

$$\left[u_n(x)u'(x) - u(x)u_n'(x)\right]_a^b = u_n(b)\left(r_2 + \mu_0 u(b)\right) - u(b)\mu_0 u_n(b) -$$

$$- u_n(a)\left(r_1 + \nu_0 u(a)\right) + u(a)\nu_0 u_n(a).$$

Combining the like terms of the right hand side of the above equation, we have

$$\left[u_n(x)u'(x) - u(x)u_n'(x)\right]_a^b = r_2\, u_n(b) - r_1 u_n(a)\,. \tag{2}$$

Taking into account Eq.(2), we return in Eq.(1), and we have

$$\int_a^b \left(u_n(x)\mathrm{L}_{\lambda_n}[u](x) - u(x)\mathrm{L}_{\lambda_n}[u](x)\right) dx = r_2 u_n(b) - r_1 u_n(a)\,.$$

Recall that $\mathrm{L}_{\lambda_n}[u](x) = g(x)$ and $\mathrm{L}_{\lambda_n}[u_n](x) = 0$. Thus, we derive

$$\int_a^b u_n(x)g(x)\,dx = r_2 u_n(b) - r_1 u_n(a)\,. \tag{3}$$

The above is the compatibility condition for the fully non-homogeneous BVP (1)-(2), of Section 3.5a, when λ is an eigenvalue.

At this point we are going to examine when the reduced BVP (12)-(13) of Section 3.5a in the case of $\lambda = \lambda_n$ is compatible.

The solvability condition, will once again arise by using Green's identity and assuming that $\upsilon(x;\lambda_n)$ is a solution of the reduced BVP and $u_n(x)$ is an eigenfunction.

Hence, Green's Identity implies

$$\int_a^b \left(u_n(x)\,\mathrm{L}_{\lambda_n}[\upsilon](x) - \upsilon(x)\mathrm{L}_{\lambda_n}[u_n](x)\right) dx = \left[u_n(x)(\upsilon')(x) - \upsilon(x)(u_n')(x)\right]_a^b.$$

$$\int_a^b u_n(x)\left(g(x) - \mathrm{L}_{\lambda_n}[u_1](x) - \mathrm{L}_{\lambda_n}[u_2](x)\right) dx = 0.$$

The right hand side of the above equation is zero, since $u_n(x)$ and $\upsilon(x)$ satisfying the BCs (2) of Section 3.5a. As far as the left hand side is concerned, we used the fact that $\mathrm{L}_{\lambda_n}\left[u_n\right](x) = 0$ and also that $\upsilon(x)$ satisfies Eq.(12) of the previous section.

$$\int_a^b u_n(x)\,g(x)\,dx = \int_a^b \mathrm{L}_{\lambda_n} u_1(x)u_n(x) + \mathrm{L}_{\lambda_n} u_2(x)u_n(x)dx\,. \tag{4}$$

In order to use Green's Identity we write the right hand side of Eq.(4) in the form

$$\int_a^b u_n(x)\,g(x)\,dx = \int_a^b \mathcal{L}_{\lambda_n}\big(u_1(x)+u_2(x)\big)u_n(x) - \cancel{\big(u_1(x)+u_2(x)\big)\mathcal{L}_{\lambda_n}[u_n](x)}\,dx \tag{5}$$

$$\int_a^b u_n(x)\,g(x)\,dx = \Big[u_n(x)\big(u_1(x)+u_2(x)\big)' - \big(u_1(x)+u_2(x)\big)u_n'(x)\Big]_a^b \tag{6}$$

Working only with the right hand side of Eq.(6), gives us

$$\Big[u_n(x)\big(u_1(x)+u_2(x)\big)' - \big(u_1(x)+u_2(x)\big)u_n'(x)\Big]_a^b =$$

$$= u_n(b)\big(u_1(b)+u_2(b)\big)' - \big(u_1(b)+u_2(b)\big)u_n'(b) -$$
$$- u_n(a)\big(u_1(a)+u_2(a)\big)' + \big(u_1(a)+u_2(a)\big)u_n'(a) =$$

$$= u_n(b)\big(\mu_0 u_1(b) + r_2 + \mu_0 u_2(b)\big) - \big(u_1(b)+u_2(b)\big)\mu_0 u_n(b) -$$
$$- u_n(a)\big(r_1 + \nu_0 u_1(a) + \nu_0 u_2(a)\big) + \big(u_1(a)+u_2(a)\big)\nu_0 u_n(a),$$

where in the last step we used the fact that $u_1(x), u_2(x)$ satisfy the BCs (4) of the Section 4.6a and $u_n(x)$ the homogeneous BCs.

Combining the like terms, we result in

$$\Big[u_n(x)\big(u_1(x)-u_2(x)\big)' - \big(u_1(x)-u_2(x)\big)u_n'(x)\Big]_a^b = r_2\,u_n(b) - r_1\,u_n(a).$$

Therefore, Eq.(6) becomes

$$\int_a^b u_n(x)\,g(x)\,dx = r_2 u_n(b) - r_1 u_n(a)\,. \tag{7}$$

As we expected, the above condition is equivalent to

$$\int_a^b u_n(x)\,h(x)\,dx = 0\,. \tag{8}$$

Thus, Eq.(7) is the compatibility condition for the reduced problem which is the same with Eq.(3).The latter is the expected result, because the fully non homogeneous BVP and the reduced BVP are equivalent.

The method we follow to solve this problem is the same as in the previous case, but the solution of the reduced BVP (12)-(13) of the previous section when λ is an eigenvalue is different. As we proved in Sec. 3.4 the solution $\upsilon(x)$, for $p(x)=1$ is given by

$$\upsilon(x) = C u_n(x) + \frac{\upsilon_n(x)}{W}\int_a^x u_n(s)h(s)ds + \frac{u_n(x)}{W}\int_x^b \upsilon_n(s)h(s)\,ds\,, \tag{9}$$

where, $u_n(x)$ is the eigenfunction corresponding to the eigenvalue λ_n and $\upsilon_n(x)$ is another solution of the corresponding homogeneous DE, linearly independent from $u_n(x)$ which can be expressed in terms of $u_n(x)$ (see also Appendix B)

and

$$h(s) = g(s) - \mathrm{L}_{\lambda_n}[u_1 + u_2](s) = g(s) - \left(\lambda_n - q(s)\right)\frac{r_1\left(\mu_0 s + 1 - \mu_0 b\right) - r_2\left(\nu_0 s + 1 - \nu_0 a\right)}{\mu_0\left(1 - \nu_0 a\right) - \nu_0\left(1 - \mu_0 b\right)}.$$

Thus, taking into account Eq.(8), the solution of the fully non-homogeneous BVP, in the case of $\lambda = \lambda_n$ has the form

$$\begin{aligned} u(x) = C u_n(x) + \frac{\upsilon_n(x)}{W}\int_a^x u_n(s)h(s)ds + \frac{u_n(x)}{W}\int_x^b \upsilon_n(s)h(s)\,ds\, + \\ + \frac{r_1\left(\mu_0 x + 1 - \mu_0 b\right) - r_2\left(\nu_0 x + 1 - \nu_0 a\right)}{\mu_0\left(1 - \nu_0 a\right) - \nu_0\left(1 - \mu_0 b\right)}. \end{aligned} \tag{10}$$

4. Variation of the Eigenfunctions with respect to the coefficients

As we saw in the Section 2.5, the variation of the eigenfunctions, δZ_n satisfies the following non-homogeneous BVP

$$\mathrm{L}_{\lambda_n}[\delta Z_n](z) = \partial_z^2 \delta Z_n(z) + \left(\lambda_n - Q(z)\right)\delta Z_n(z) = -\left(\delta\lambda_n - \delta Q(z)\right)Z_n(z). \tag{1}$$

$$\mathrm{B}_b(\delta Z_n) \equiv \partial_z(\delta Z_n(b)) - \mu_0\,(\delta Z_n(b)) = \delta\mu_0\, Z_n(b), \tag{2a}$$

$$\mathrm{B}_a(\delta Z_n) \equiv \partial_z(\delta Z_n(a)) - \nu_0\,(\delta Z_n(a)) = \delta\nu_0\, Z_n(a). \tag{2b}$$

As we notice from the non-homogeneous part of Eq.(1), it is necessary to know the variation of the eigenvalues in order to solve the above BVP. We present again the formula of $\delta\lambda_n$ (see also Section 2.2)

$$\delta\lambda_n = \frac{1}{\|Z_n\|^2}\left(\int_a^b |Z_n(z)|^2\,\delta Q(z)\,dz + Z_n^2(a)\,\delta\nu_0 - Z_n^2(b)\,\delta\mu_0\right). \tag{3}$$

We have to check the solvability of the fully non-homogeneous BVP (1)-(2). As we saw in Eq.(3) of the Section 3.5b,the following equation should be satisfied

$$\int_a^b u_n(x)g(x)\,dx = r_2\,p(b)u_n(b) - r_1\,p(a)u_n(a).$$

This equation, in terms of our problem, becomes

$$\int_a^b Z_n(z)\left(\delta Q(z) - \delta\lambda_n\right)Z_n(z)\,dz = \delta\mu_0\,Z_n^2(b) - \delta\nu_0\,Z_n^2(a). \tag{4}$$

We start from the left-hand side of Eq.(4) in order to prove its validity

$$\int_a^b Z_n(z)\left(\delta Q(z) - \delta\lambda_n\right)Z_n(z)\,dz = \int_a^b \delta Q(z)Z_n^2(z)\,dz - \int_a^b \delta\lambda_n Z_n^2(z)\,dz. \tag{5}$$

By substituting the variation of the eigenvalues, Eq.(3), into Eq.(5), we have

$$\int_a^b Z_n(z)\left(\delta Q(z) - \delta\lambda_n\right)Z_n(z)\,dz = \int_a^b \delta Q(z)Z_n^2(z)\,dz -$$

$$-\int_a^b \left[\frac{1}{\|Z_n\|^2}\left(\int_a^b |Z_n(z)|^2\,\delta Q\,dz + Z^2{}_n(a)\,\delta\nu_0 - Z^2{}_n(b)\,\delta\mu_0\right)\right]Z_n^2(z)\,dz =$$

$$= \int_a^b \delta Q(z)Z_n^2(z)\,dz - \frac{1}{\|Z_n\|^2}\int_a^b |Z_n(z)|^2\,\delta Q(z)\,dz \int_a^b Z_n^2(z)\,dz -$$

$$- \frac{1}{\left\|Z_n\right\|^2}\int_a^b Z_n^2(a)\,\delta\nu_0 Z_n^2(z)\,dz \;+\; \frac{1}{\left\|Z_n\right\|^2}\int_a^b Z_n^2(b)\,\delta\mu_0\, Z_n^2(z)\,dz \;=$$

$$= \int_a^b \delta Q(z) Z_n^2(z)\,dz \;-\; \cancel{\frac{1}{\left\|Z_n\right\|^2}}\int_a^b \left|Z_n(z)\right|^2 \delta Q(z)\,dz\;\; \cancel{\int_a^b Z_n^2(z)\,dz} \;-$$

$$- \;Z_n^2(a)\,\delta\nu_0\, \xcancel{\frac{1}{\left\|Z_n\right\|^2}\int_a^b Z_n^2(z)\,dz} \;+\; Z_n^2(b)\,\delta\mu_0\, \xcancel{\frac{1}{\left\|Z_n\right\|^2}\int_a^b Z_n^2(z)\,dz} \;=$$

$$= \int_a^b \delta Q(z) Z_n^2(z)\,dz \;-\; \int_a^b \left|Z_n(z)\right|^2 \delta Q(z)\,dz \;-\; Z_n^2(a)\,\delta\nu_0 \;+\; Z_n^2(b)\,\delta\mu_0 .$$

Hence, we finally obtain

$$\int_a^b Z_n(z)\left(\delta Q(z) - \delta\lambda_n\right) Z_n(z)\,dx = \delta\mu_0\, Z_n^2(b) - \delta\nu_0\, Z_n^2(a),$$

which is the desired result. Thus, we conclude that it makes sense to search for a solution of the BVP (1)-(2).

4.1 Solution of the Variational Problem of the Eigenfunctions

As we have already proved in the Sec.3.5b, when the solvability condition is satisfied, the solution of the fully non homogeneous BVP in the case of $\lambda = \lambda_n$ has the form

$$\begin{aligned} u(x) = C\,u_n(x) + \frac{\upsilon_n(x)}{W}\int_a^x u_n(s)h(s)ds + \frac{u_n(x)}{W}\int_x^b \upsilon_n(s)h(s)\,ds \;+ \\ + \frac{r_1\left(\mu_0\, x + 1 - \mu_0\, b\right) - r_2\left(\nu_0\, x + 1 - \nu_0\, a\right)}{\mu_0\left(1 - \nu_0\, a\right) - \nu_0\left(1 - \mu_0\, b\right)}. \end{aligned} \qquad (1)$$

where

$$h(s) = g(s) - \left(\lambda_n - q(s)\right)\frac{r_1\left(\mu_0\, s + 1 - \mu_0\, b\right) - r_2\left(\nu_0\, s + 1 - \nu_0\, a\right)}{\mu_0\left(1 - \nu_0\, a\right) - \nu_0\left(1 - \mu_0\, b\right)}.$$

The terms of Eq.(1) are identified with the terms of our problem by the following choices

$$u_n(z) = Z_n(z)$$

$$\upsilon_n(z) = U_n(z),$$

which is the second linearly independent solution of the corresponding homogeneous DE, expressed in terms of $Z_n(z)$ by using an order-reduction and solving the obtained first-order equation; see Appendix B,

$C = \delta C$,since it is expected to be a sufficiently small term

$g(z) = -\left(\delta\lambda_n - \delta Q(z)\right)Z_n(z)$,

$r_1 = \delta\nu_0 Z_n(a)$, $r_2 = \delta\mu_0 Z_n(b)$,

and

$$h(z) = -\delta\lambda_n Z_n(z) + \delta Q(z) Z_n(z) - \\ -\left(\lambda_n - Q(z)\right)\frac{\delta\nu_0 Z_n(a)\left(\mu_0 z + 1 - \mu_0 b\right) - \delta\mu_0 Z_n(b)\left(\nu_0 z + 1 - \nu_0 a\right)}{\mu_0\left(1 - \nu_0 a\right) - \nu_0\left(1 - \mu_0 b\right)}. \tag{2}$$

Since $h(z)$ is proportional to the variation of each data, the most appropriate notation is

$$\delta h(z) = -Z_n(z)\delta\lambda_n + \delta Q(z) Z_n(z) - \\ -\left(\lambda_n - Q(z)\right)\frac{\delta\nu_0 Z_n(a)\left(\mu_0 z + 1 - \mu_0 b\right) - \delta\mu_0 Z_n(b)\left(\nu_0 z + 1 - \nu_0 a\right)}{\mu_0\left(1 - \nu_0 a\right) - \nu_0\left(1 - \mu_0 b\right)}. \tag{3}$$

Taking into account the above equations,the solution of the fully non-homogeneous BVP(1)-(2) of the Sec.4 is given by

$$\delta Z_n(z) = \delta C\, Z_n(z) + \frac{U_n(z)}{W}\int_a^z Z_n(s)\delta h(s)ds + \frac{Z_n(z)}{W}\int_z^b U_n(s)\delta h(s)\, ds + \\ + \frac{\delta\nu_0 Z_n(a)\left(\mu_0 z + 1 - \mu_0 b\right) - \delta\mu_0 Z_n(b)\left(\nu_0 z + 1 - \nu_0 a\right)}{\mu_0\left(1 - \nu_0 a\right) - \nu_0\left(1 - \mu_0 b\right)} \tag{4}$$

As we notice from Eq.(4) the solution of the problem is not unique, which is an expected result since the eigenfunction $Z_n(z)$ is not unique.

Assume now that the eigenfunctions are both normalized, that is

$$\left\| Z_n(z) \right\| = 1, \qquad \left\| \tilde{Z}_n(z) \right\| = \left\| Z_n(z) + \delta Z_n(z) \right\| = 1. \tag{5a,b}$$

Then, both $Z_n(z)$ and the perturbed eigenfunction $\tilde{Z}_n(z) = Z_n(z) + \delta Z_n(z)$ are uniquely defined. This fact renders the variation $\delta Z_n(z)$ uniquely defined as well. The condition ensuring the uniqueness of $\delta Z_n(z)$ (that is, the determination of the value of the constant C in Eq. (4)), is easily obtained as follows:

$$
\begin{aligned}
\tilde{Z}_n(z) &= Z_n(z) + \delta Z_n(z) \quad \Rightarrow \\
\Rightarrow \quad \left\| \tilde{Z}_n(z) \right\|^2 &= \left\| Z_n(z) + \delta Z_n(z) \right\|^2 \quad \Rightarrow \\
\Rightarrow \quad 1 &= \left\| Z_n(z) + \delta Z_n(z) \right\|^2 \quad \Rightarrow \\
\Rightarrow \quad & \left\langle Z_n(z) + \delta Z_n(z) \, , \, Z_n(z) + \delta Z_n(z) \right\rangle = 1 \quad \Rightarrow \\
\Rightarrow \quad & \left\langle Z_n(z) \, , \, Z_n(z) \right\rangle + \left\langle \delta Z_n(z) \, , \, Z_n(z) \right\rangle + \left\langle Z_n(z) \, , \, \delta Z_n(z) \right\rangle + \\
& \qquad + \left\langle \delta Z_n(z) \, , \, \delta Z_n(z) \right\rangle = 1 \quad \Rightarrow \\
\Rightarrow \quad & \left\langle \delta Z_n(z) \, , \, Z_n(z) \right\rangle + \left\langle Z_n(z) \, , \, \delta Z_n(z) \right\rangle + \left\| \delta Z_n(z) \right\|^2 = 0 \, .
\end{aligned}
$$

Omitting the 2nd-order term $\left\| \delta Z_n(z) \right\|^2$, we obtain

$$
\left\langle \delta Z_n(z) \, , \, Z_n(z) \right\rangle + \left\langle Z_n(z) \, , \, \delta Z_n(z) \right\rangle \equiv 2 \, \mathrm{Re} \left\{ \left\langle Z_n(z) \, , \, \delta Z_n(z) \right\rangle \right\} = 0 \, ,
$$

that is,

$$
\boxed{ \mathrm{Re} \left\{ \left\langle Z_n(z) \, , \, \delta Z_n(z) \right\rangle \right\} = 0 } \, . \tag{6}
$$

The introduction of the expression of $\delta Z_n(z)$, where also $\delta h(z)$ and the normalization condition (5a) have been considered, will determine the constant δC .

4.2. Variation of the eigenfunctions with respect to the boundary points

Now are going to find the variation of the eigenfunctions when the domain changes and the coefficients remain unchanged.
Recall that the BVP is

$$
\partial_z^2 Z_n(z) + \left(\lambda_n - Q(z) \right) Z_n(z) = 0, \qquad a < z < b \, , \tag{1}
$$

$$
B^b Z_n \equiv \partial_z Z_n(b) - \mu_0 Z_n(b) = 0 \, , \tag{2a}
$$

$$
B_a Z_n \equiv \partial_z Z_n(a) - \nu_0 Z_n(a) = 0 \, . \tag{2b}
$$

Assume that the domain $\left(a,b \right)$ is extended to $\left(a,b + \varepsilon \right)$. Assume further that the function $Q(z)$ can be extended up to $z = b + \varepsilon$ and the new BVP is given by

$$
\partial_z^2 Z_n^\varepsilon(z) + \left(\lambda_n^\varepsilon - Q(z) \right) Z_n^\varepsilon(z) = 0, \qquad a < z < b + \varepsilon \, , \tag{3}
$$

$$
B^{b+\varepsilon} Z_n^\varepsilon \equiv \partial_z Z_n^\varepsilon(b + \varepsilon) - \mu_0 Z_n^\varepsilon(b + \varepsilon) = 0 \, , \tag{4a}
$$

$$B_a Z_n^\varepsilon \equiv \partial_z Z_n^\varepsilon(a) - \nu_0 Z_n^\varepsilon(a) = 0. \tag{4b}$$

The question is to estimate/calculate the first-order variation $\delta Z_n(z) = Z_n^\varepsilon(z) - Z_n(z)$ of the eigenfunctions $Z_n(z)$, due to an $\varepsilon-$variation of the end point b, in terms of the eigensystem $\{\lambda_n, Z_n\}$.

Our purpose is to reduce the problem of the variation of the domain to the already solved problem of the variation of the coefficients.

1st step: Consider the BVP (3)-(4), defined on $a < z < b+\varepsilon$, and introduce the following transformation (of independent and dependent variables)

$$\tilde{z} - a \underset{\text{of transf.}}{\overset{\text{def}}{=}} \frac{b-a}{b-a+\varepsilon}(z-a), \qquad [\text{that is } a \le \tilde{z} \le b, \text{ when } a \le z \le b+\varepsilon], \tag{5a}$$

$$\tilde{Z}_n^\varepsilon \equiv \tilde{Z}_n^\varepsilon(\tilde{z}) \underset{\text{of transf.}}{\overset{\text{def}}{=}} Z_n^\varepsilon(z(\tilde{z})). \tag{5b}$$

Setting $H = b - a$, we have

$$z - a = \frac{b-a+\varepsilon}{b-a}(\tilde{z}-a) = (1+\varepsilon/H)(\tilde{z}-a) \text{ and } \frac{\partial z(\tilde{z})}{\partial \tilde{z}} = 1 + \varepsilon/H. \tag{6a,b}$$

2nd step: Applying this transformation, we obtain the transformed BVP problem, which is defined in the interval $a < \tilde{z} < b$. We have to find out the exact form of the transformed problem. We start by calculating the derivatives $\partial_{\tilde{z}} \tilde{Z}_n^\varepsilon, \partial_{\tilde{z}}^2 \tilde{Z}_n^\varepsilon$ in terms of derivatives $\partial_z Z_n^\varepsilon, \partial_z^2 Z_n^\varepsilon$:

$$\partial_{\tilde{z}} \tilde{Z}_n^\varepsilon \equiv \frac{\partial \tilde{Z}_n^\varepsilon(\tilde{z})}{\partial \tilde{z}} = \frac{\partial Z_n^\varepsilon(z(\tilde{z}))}{\partial \tilde{z}} = \frac{\partial Z_n^\varepsilon(z(\tilde{z}))}{\partial z}\frac{\partial z(\tilde{z})}{\partial \tilde{z}} = \partial_z Z_n^\varepsilon (1+\varepsilon/H), \tag{7a}$$

$$\partial_{\tilde{z}}^2 \tilde{Z}_n^\varepsilon = \partial_z^2 Z_n^\varepsilon (1+\varepsilon/H)^2 \tag{7b}$$

Defining, further the function $\tilde{Q}(\tilde{z}) = Q(z(\tilde{z}))$, we obtain the following formulation for the problem, by introducing the transformed quantities in the BVP(3)-(4)

$$\partial_{\tilde{z}}^2 \tilde{Z}_n^\varepsilon (1+\varepsilon/H)^{-2} + \left(\lambda_n^\varepsilon - \tilde{Q}(\tilde{z})\right)\tilde{Z}_n^\varepsilon = 0, \quad a < \tilde{z} < b, \tag{8}$$

$$\tilde{B}^b \tilde{Z}_n^\varepsilon \equiv \partial_{\tilde{z}} \tilde{Z}_n^\varepsilon(b)(1+\varepsilon/H)^{-1} - \mu_0 \tilde{Z}_n^\varepsilon(b) = 0, \tag{9a}$$

$$\tilde{B}_a \tilde{Z}_n^\varepsilon \equiv \partial_{\tilde{z}} \tilde{Z}_n^\varepsilon(a)(1+\varepsilon/H)^{-1} - \nu_0 \tilde{Z}_n^\varepsilon(a) = 0, \tag{9b}$$

or

$$\partial_{\tilde{z}}^2 \tilde{Z}_n^\varepsilon + \left(\lambda_n^\varepsilon (1+\varepsilon/H)^2 - \tilde{Q}(\tilde{z})(1+\varepsilon/H)^2\right)\tilde{Z}_n^\varepsilon = 0, \quad a < \tilde{z} < b, \tag{10}$$

$$\tilde{B}^b \tilde{Z}_n^\varepsilon \equiv \partial_{\tilde{z}} \tilde{Z}_n^\varepsilon(b) - \mu_0\left(1+\varepsilon/H\right)\tilde{Z}_n^\varepsilon(b) = 0, \tag{11a}$$

$$\tilde{B}_a \tilde{Z}_n^\varepsilon \equiv \partial_{\tilde{z}} \tilde{Z}_n^\varepsilon(a) - \nu_0\left(1+\varepsilon/H\right)\tilde{Z}_n^\varepsilon(a) = 0. \tag{11b}$$

Comparing the BVP (10)-(11) to the initial BVP (1)-(2),

$$\partial_z^2 Z_n(z) + \left(\lambda_n - Q(z)\right) Z_n(z) = 0, \qquad a < z < b,$$

$$B^b Z_n \equiv \partial_z Z_n(b) - \mu_0 Z_n(b) = 0,$$

$$B_a Z_n \equiv \partial_z Z_n(a) - \nu_0 Z_n(a) = 0,$$

we observe that the problems are now defined in the same interval and they differ in the function $Q(z)$ and the constants ν_0, μ_0. Hence, we managed to reduce the problem of the variation of the eigenfunctions with respect to the domain to the already solved problem of the variation of the eigenvalues with respect to the coefficients. Since the two problems are defined in the same interval, we can use the notation z for the independent variable $\tilde{z}$, but we have to change it in the end. At this point, we assume that the eigenfunctions of the initial and the perturbed problem are normalized. Let us expand the perturbed quantities as follows

$$\tilde{Z}_n^\varepsilon(z) = Z_n(z) + \delta\tilde{Z}_n^\varepsilon(z), \tag{12a}$$

$$\tilde{Q}(z)\left(1+\varepsilon/H\right)^2 = Q(z) + \delta Q(z) + 2\frac{\varepsilon}{H}Q(z) = Q(z) + \delta\tilde{Q}(z), \tag{12b}$$

$$\lambda_n + \delta\lambda_n^\varepsilon + 2\frac{\varepsilon}{H} = \lambda_n + \delta\tilde{\lambda}_n^\varepsilon, \tag{12c}$$

where the variation of the eigenvalues is given by

$$\delta\tilde{\lambda}_n^\varepsilon = \int_a^b \left|Z_n(z)\right|^2 \delta\tilde{Q}(z)\,dz + Z_n^2(a)\delta\nu_0 - Z_n^2(b)\delta\mu_0, \tag{12d}$$

$$\mu_0\left(1+\varepsilon/H\right) = \mu_0 + \mu_0\frac{\varepsilon}{H} = \mu_0 + \delta\mu_0, \tag{12e}$$

$$\nu_0\left(1+\varepsilon/H\right) = \nu_0 + \delta\nu_0, \tag{12f}$$

By substituting Eqs. (12a,b,c,d) into the DE (10) we have

$$\partial_z^2\left(Z_n(z) + \delta\tilde{Z}_n^\varepsilon(z)\right) + \left(\left(\lambda_n + \delta\tilde{\lambda}_n^\varepsilon\right)\right)\left(Z_n(z) + \delta\tilde{Z}_n^\varepsilon(z)\right) - \left(Q(z) + \delta\tilde{Q}(z)\right)\left(Z_n(z) + \delta\tilde{Z}_n^\varepsilon(z)\right) = 0$$

Expanding further the above equation and neglecting second order terms $\left(\delta\lambda_n^\varepsilon\,\delta\tilde{Z}_n^\varepsilon(z), \delta\tilde{Q}(z)\delta\tilde{Z}_n^\varepsilon(z)\right)$ we result in

$$\partial_z^2 Z_n(z) + \partial_z^2 \delta\tilde{Z}_n^\varepsilon(z) + \lambda_n Z_n(z) + \delta\lambda_n^\varepsilon Z_n(z) + \lambda_n \delta\tilde{Z}_n^\varepsilon(z) - $$
$$- Q(z)Z_n(z) - \delta\tilde{Q}(z)Z_n(z) - Q(z)\delta\tilde{Z}_n^\varepsilon(z) = 0,$$

where, since $Z_n(z)$ satisfies the DE (1) the sum of the red terms is equal to zero.

Therefore, the variation $\delta\tilde{Z}_n^\varepsilon(\tilde{z})$ satisfies the following DE

$$\partial_z^2 \delta\tilde{Z}_n^\varepsilon(z) + \lambda_n \delta\tilde{Z}_n^\varepsilon(z) - Q(z)\delta\tilde{Z}_n^\varepsilon(z) = -\delta\lambda_n^\varepsilon Z_n(z) + \delta\tilde{Q}(z)Z_n(z). \tag{13}$$

By substituting also $\tilde{Z}_n^\varepsilon(z)$ and Eq.(12e) into the BCs (11), we obtain

$$\partial_z Z_n(b) + \partial_z \delta\tilde{Z}_n^\varepsilon(b) - \left(\mu_0 + \delta\mu_0\right)\left(Z_n(b) + \delta\tilde{Z}_n^\varepsilon(b)\right) = 0,$$

$$\partial_z Z_n(b) + \partial_z \delta\tilde{Z}_n^\varepsilon(b) - \mu_0 Z_n(b) - \delta\mu_0 Z_n(b) - \mu_0 \delta\tilde{Z}_n^\varepsilon(b) - \cancel{\delta\mu_0\delta\tilde{Z}_n^\varepsilon(b)} = 0$$

where the sum of the red terms is zero since $Z_n(z)$ satisfies the BC (2a) and the other is neglected as a second order term.

Hence, the variation $\delta\tilde{Z}_n^\varepsilon(\tilde{z})$ satisfies the following BC at the endpoint b

$$\tilde{B}^b \delta\tilde{Z}_n^\varepsilon = \partial_{\tilde{z}} \delta\tilde{Z}_n^\varepsilon(b) - \mu_0 \delta\tilde{Z}_n^\varepsilon(b) = \delta\mu_0 Z_n(b). \tag{14a}$$

Using the same arguments, the BC which satisfies $\delta\tilde{Z}_n^\varepsilon(\tilde{z})$ at the endpoint a is

$$\tilde{B}_a \delta\tilde{Z}_n^\varepsilon \equiv \partial_{\tilde{z}} \delta\tilde{Z}_n^\varepsilon(a) - \nu_0\, \delta\tilde{Z}_n^\varepsilon(a) = \delta\nu_0 Z_n(a). \tag{14b}$$

The BVP (13)-(14) has a solution since the compatibility condition is satisfied, as we can confirm below

$$\int_a^b Z_n(z)g(z)\,dz = \int_a^b -\delta\lambda_n^\varepsilon \left|Z_n(z)\right|^2 + \delta\tilde{Q}(z)\left|Z_n(z)\right|^2\,dz = -\delta\lambda_n^\varepsilon \cancel{\int_a^b \left|Z_n(z)\right|^2 dz} + \int_a^b \delta\tilde{Q}(z)\left|Z_n(z)\right|^2\,dz$$

$$\int_a^b Z_n(z)g(z)\,dz = -\int_a^b \delta\tilde{Q}(z)\left|Z_n(z)\right|^2\,dz + \int_a^b \delta\tilde{Q}(z)\left|Z_n(z)\right|^2\,dz + Z_n^2(a)\delta\nu_0 - Z_n^2(b)\delta\mu_0$$

$$\Rightarrow \int_a^b Z_n(z)g(z)\,dz = Z_n^2(a)\delta\nu_0 - Z_n^2(b)\delta\mu_0$$

where, use is made of the variation formula of the eigenvalues, Eq.(12d).

In the previous Section, we derived the solution of the general problem of the variation of the eigenfunctions with respect to the coefficients. Hence, Eq.(4) of Sec.4.1, which is the solution of the latter problem, in terms of our problem has the form

$$\delta\tilde{Z}_n^\varepsilon(\tilde{z}) = \delta C\, Z_n(\tilde{z}) + \frac{U_n(\tilde{z})}{W}\int_a^{\tilde{z}} Z_n(\tilde{s})\delta h(\tilde{s})d\tilde{s} + \frac{Z_n(\tilde{z})}{W}\int_{\tilde{z}}^b U_n(\tilde{s})\delta h(\tilde{s})\,d\tilde{s} +$$
$$+ \frac{\delta\nu_0 Z_n(a)\left(\mu_0\tilde{z} + 1 - \mu_0 b\right) - \delta\mu_0 Z_n(b)\left(\nu_0\tilde{z} + 1 - \nu_0 a\right)}{\mu_0\left(1 - \nu_0 a\right) - \nu_0\left(1 - \mu_0 b\right)}. \tag{16}$$

Substituting the variations $\delta\tilde{\lambda}_n^{\varepsilon}, \delta\tilde{Q}, \delta\nu_0, \delta\mu_0$ into Eq.(16) and also replacing the transformed quantities by the initial, we obtain the desired variation of the eigenfunctions with respect to the endpoint b, $\delta Z_n^{\varepsilon}(z)$.

Appendix A. Particular solution of a non-homogeneous DE using the method of the variation of parameters

Consider the following second order non homogeneous differential equation with non-constant coefficients

$$\big(p(x)u'(x)\big)' + \big(\lambda_n - q(x)\big)u(x) = g(x), \qquad a < x < b \tag{1a}$$

or equivalently

$$p'(x)u'(x) + p(x)u''(x) + \big(\lambda_n - q(x)\big)u(x) = g(x). \tag{1b}$$

Our purpose is to find a particular solution of the above equation. According to the method of the variation of the parameters, we are going to search this solution in the following form (see also Haberman, 2004, Sec.9.3.2)

$$u_p(x) = v_1(x)u_1(x) + v_2(x)u_2(x), \tag{2}$$

where $u_1(x)$, $u_2(x)$ are two linearly independent solutions of the corresponding homogeneous DE.

In order to plague our purposed solution into DE (1b) we need its derivatives. Thus, the first derivative is

$$u_p'(x) = v_1'(x)u_1(x) + v_1(x)u_1'(x) + v_2'(x)u_2(x) + v_2(x)u_2'(x). \tag{3a}$$

In order to simplify the form of the above derivative we make the following assumption

$$v_1'(x)u_1(x) + v_2'(x)u_2(x) = 0. \tag{3b}$$

Hence, the simplified version of Eq.(3a) is

$$u_p'(x) = v_1(x)u_1'(x) + v_2(x)u_2'(x). \tag{4}$$

By differentiating both members of Eq.(4) we derive the second derivative which is

$$u_p''(x) = v_1'(x)u_1'(x) + v_1(x)u_1''(x) + v_2'(x)u_2'(x) + v_2(x)u_2''(x). \tag{5}$$

Substituting Eqs.(2),(4),(5) into (1) we have

$$p'(x)\big(v_1(x)u_1'(x) + v_2(x)u_2'(x)\big) + p(x)\big(v_1'(x)u_1'(x) + v_1(x)u_1''(x) + v_2'(x)u_2'(x) + v_2(x)u_2''(x)\big) -$$

$$+ \big(\lambda_n - q(x)\big)\big(v_1(x)u_1(x) + v_2(x)u_2(x)\big) = g(x)$$

Taking into account that $u_1(x)$ and $u_2(x)$ satisfy the corresponding homogeneous DE, we obtain

$$p(x)\big(v_1'(x)u_1'(x) + v_2'(x)u_2'(x)\big) = g(x), \tag{6}$$

or

$$v_1'(x)u_1'(x) + v_2'(x)u_2'(x) = \frac{g(x)}{p(x)}. \tag{7}$$

Equations (3b) and (7) constitute a system and its solutions are

$$v_1'(x) = -\frac{u_2(x)\,g(x)}{p(x)\big(u_1(x)u_2'(x) - u_1'(x)u_2(x)\big)} = -\frac{u_2(x)\,g(x)}{p(x)W(u_1,u_2)(x)}, \tag{9a}$$

$$v_2'(x) = \frac{u_1(x)\,g(x)}{p(x)\big(u_1(x)u_2'(x) - u_1'(x)u_2(x)\big)} = \frac{u_1(x)\,g(x)}{p(x)W(u_1,u_2)(x)}. \tag{9b}$$

By integrating both parts of Eq.(9a) in the interval $\left(x,b\right)$ we obtain

$$\int_x^b v_1'(s)\,ds = -\int_x^b \frac{u_2(s)\,g(s)}{p(s)W(u_1,u_2)(s)}\,ds\,,$$

$$v_1(b) - v_1(x) = -\int_x^b \frac{u_2(s)\,g(s)}{p(s)W(u_1,u_2)(s)}\,ds$$

$$v_1(x) = \int_x^b \frac{u_2(s)\,g(s)}{p(s)W(u_1,u_2)(s)}\,ds - v_1(b)\,. \tag{10a}$$

Similarly, by integrating both parts of Eq.(9b) in the interval (a,x) we have

$$v_2(x) = \int_a^x \frac{u_1(s)\,g(s)}{p(s)W(u_1,u_2)(s)}\,ds + v_2(a)\,. \tag{10b}$$

Notice that

$$\begin{aligned}\frac{d}{dx}\Big(p(x)W(u_1,u_2)(x)\Big) &= p'(x)\Big(u_1(x)u_2'(x) - u_2(x)u_1'(x)\Big) + \\ &+ p(x)\Big(u_1'(x)u_2'(x) + u_1(x)u_2''(x) - u_2'(x)u_1'(x) - u_2(x)u_1''(x)\Big)\end{aligned}\,. \tag{11}$$

Recall that $u_1(x)$, $u_2(x)$ satisfy the corresponding homogeneous differential equation and particularly the following equations hold

$$p(x)u_1''(x) = -p'(x)u_1'(x) - \Big(\lambda_n - q(x)\Big)u_1(x)\,, \tag{12a}$$

$$p(x)u_2''(x) = -p'(x)u_2'(x) - \Big(\lambda_n - q(x)\Big)u_2(x)\,. \tag{12b}$$

Substituting Eqs.(12) into (11) we obtain

$$\frac{d}{dx}\Big(p(x)W(u_1,u_2)(x)\Big) = 0\,. \tag{13}$$

which means that the term $p(x)W(u_1,u_2)$ is independent of x, hence it does not participate in the integration with respect to x. Therefore, taking into account Eq.(13) and substituting Eqs.(10) into Eq.(2) we have

$$u_p(x) = \left(\int_x^b \frac{u_2(s)\,g(s)}{p(s)W(u_1,u_2)(s)}\,ds - v_1(b)\right)u_1(x) + \left(\int_a^x \frac{u_1(s)\,g(s)}{p(s)W(u_1,u_2)(s)}\,ds + v_2(a)\right)u_2(x) \tag{14}$$

By rearranging the terms of the right hand-side of Eq.(14) we take

$$u_p(x) = \frac{u_2(x)}{p(x)W(u_1,u_2)(x)}\int_a^x u_1(s)\,g(s)\,ds + \frac{u_1(x)}{p(x)W(u_1,u_2)(x)}\int_x^b u_2(s)\,g(s)\,ds + v_2(a)u_2(x) - v_1(b)u_1(x)$$

Since the two final terms are solutions of the homogeneous DE, they are already contained in the expression of the solution of the non-homogeneous DE. Thus, the final form of the particular solution of the non-homogeneous DE (1) is given by

$$u_p(x) = \frac{u_2(x)}{p(x)W(u_1,u_2)(x)} \int_a^x u_1(s)\, g(s)\, ds + \frac{u_1(x)}{p(x)W(u_1,u_2)(x)} \int_x^b u_2(s)\, g(s)\, ds\,. \tag{15}$$

Appendix B. Construction of the second solution of a homogeneous 2nd-order DE by means of reduction of the order

Consider the following DE

$$\partial_z^2 Z_n(z) + \big(\lambda_n - Q(z)\big) Z_n = 0\,. \tag{1}$$

Assume that $Z_n(z)$ is a known solution. Our purpose is to find another solution of Eq. (1), linearly independent from $Z_n(z)$. To start with, we assume that the second solution, say $W_n(z)$, can be written in the form

$$W_n(z) = v(z) Z_n(z)\,. \tag{2}$$

At this point, we calculate W_n's first and second derivatives

$$W_n'(z) = v'(z) Z_n(z) + v(z) Z_n'(z)\,,$$

$$W_n''(z) = v''(z) Z_n(z) + 2v'(z) Z_n'(z) + v(z) Z_n''(z)\,,$$

we plug them into Eq.(1), and obtain

$$v''(z) Z_n(z) + 2v'(z) Z_n'(z) + v(z) Z_n''(z) + \big(\lambda_n - Q(z)\big) v(z) Z_n(z) = 0\,.$$

Since $Z_n(z)$ satisfies Eq.(1) the above equation is reduced to

$$v''(z) Z_n(z) + 2v'(z) Z_n'(z) = 0\,. \tag{3}$$

In order to solve the above DE, we make the following change of variable

$$u(z) = v'(z)\,, \tag{4}$$

and we have

$$u'(z) Z_n(z) + 2u(z) Z_n'(z) = 0\,. \tag{5}$$

The solution of the above DE is

$$u(z) = \frac{C_0}{Z_n^2(z)} \overset{(4)}{\Rightarrow} v'(z) = \frac{C_0}{Z_n^2(z)} \;\Rightarrow\; v(z) = C_0 \int \frac{dz}{Z_n^2(z)} \;+\; C_1\,. \tag{6}$$

Taking into account the above expression, Eq.(2) becomes

$$W_n(z) = C_0\, Z_n(z) \int \frac{dz}{Z_n^2(z)} \;+\; C_1 Z_n(z)\,. \tag{7}$$

Without loss of generality, we can assume that $C_0 = 1$ and $C_1 = 0$, taking as second solution the function

$$W_n(z) = Z_n(z) \int \frac{dz}{Z_n^2(z)}\,. \tag{8}$$

Example. Let us consider the case $Q(z) = Q$.

In this case, the solution has the following form.

$$Y_n(z) = A\cos\left(\sqrt{Q-\lambda_n}\, z\right) + B\sin\left(\sqrt{Q-\lambda_n}\, z\right).$$

Hence, the one solution of the DE in the case of $Q(z) = Q$ is $Z_n(z) = \cos\left(\sqrt{Q-\lambda_n}\, z\right)$.

Let us use Eq.(8) to determine the other solution

$$W_n(z) = Z_n(z)\int^z \frac{ds}{Z_n^2(s)} = \cos\left(\sqrt{Q-\lambda_n}\, z\right)\int^z \frac{ds}{\cos^2\left(\sqrt{Q-\lambda_n}\, s\right)}$$

$$W_n(z) = \cos\left(\sqrt{Q-\lambda_n}\, z\right)\frac{\tan\left(\sqrt{Q-\lambda_n}\, z\right)}{\sqrt{Q-\lambda_n}} = \frac{\sin\left(\sqrt{Q-\lambda_n}\, z\right)}{\sqrt{Q-\lambda_n}}$$

The denominator can be neglected as a constant, hence the solution $W_n(z)$ is

$W_n(z) = \sin\left(\sqrt{Q-\lambda_n}\, z\right)$,as we expected.

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