Εθνικό Μετσόβιο Πολυτεχνείο
Σχολή Εφαρμοσμένων Μαθηματικών και Φυσικών Επιστημών

ΕΚΕΦΕ "Δημόκριτος"
Ινστιτούτο Πυρηνικής και Σωματιδιακής Φυσικής

Διατμηματικό Πρόγραμμα Μεταπτυχιακών Σπουδών
"Φυσική και Τεχνολογικές Εφαρμογές"

Τοπικές λύσεις στην Complete Brans-Dicke θεωρία

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ
ΤΟΥ
ΧΡΙΣΤΌΦΟΡΟΥ ΒΛΆΧΟΥ

Επιβλέπων: Ελευθέριος Παπαντωνόπουλος

ΑΘΗΝΑ, ΟΚΤΏΒΡΙΟΣ 2018

Declaration of Authorship

I, Christoforos Vlachos, declare that this thesis titled, "Local Solutions in Complete Brans-Dicke Theory" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: ____

Date: ____

"*Failures are the pillars of success* "

Dave Barry

Ευχαριστίες

Σε αυτό το σημείο θα ήθελα να ευχαριστήσω θερμά τον επιβλέποντα της παρούσας εργασίας κ. *Ε.Παπαντωνόπουλο* για της καλές συμβουλές, τις ευκαιρίες και την στήριξή του, τόσο σε επιστημονικό/πρακτικό όσο και σε ψυχολογικό επίπεδο όταν τα πράγματα έδειχναν να φτάνουν σε ένα τέλμα. Επίσης, ευχαριστώ τον καθηγητή *Γ.Κοφινά* για τις χρήσιμες συμβουλές του καθόλη τη διάρκεια εκπόνησης αυτής της εργασίας.

Επιπλέον, θα ήθελα να ευχαριστήσω θερμά όλους τους κοντινούς μου ανθρώπους, τους φίλους μου και την κοπέλα μου, χωρίς των οποίων τη στηρίξη, αγάπη και κατανόηση δε θα είχα φτάσει ποτέ σε αυτό το σημείο. Και εννοείται φυσικά ότι δεν θα βρισκόμουν εδώ που βρίσκομαι, εάν δεν είχα την υποστήριξη και την αμέριστη αγάπη του αδερφού μου και των γονιών μου. Τους οφείλω τον άνθρωπο στον οποίο έχω εξελιχθεί μέχρι σήμερα, και σε τελική ανάλυση, τα λόγια δεν είναι αρκετά για να τους εκφράσω την ευγνωμοσύνη μου.

Abstract

The present thesis mainly consists of two separate parts. The first part introduces and elaborates on the mathematical apparatus needed in order for one to begin studying the General Theory of Relativity(GR). We try to cover a decent part of old and modern differential geometry (at least those subjects that are crucial to the understanding of the last chapters of this thesis), in order to give a consistent bottom to top (as from structurally poorest to richest notions) image of the geometry of, what we call, spacetime. That is why we begin by rigorously defining topological spaces in the first chapter. In the next few chapters, we equip these topological spaces with extra structure and properties until they are upgraded into smooth manifolds which is the basic mathematical ingredient one needs so as to built GR upon it. Along the way, various pictures and diagrams are being provided in order to help the reader in an intuitive manner. We also try to emphasize in the relation between purely mathematical definitions/properties and their physical consequences wherever possible. In the last few chapters of the first part, physics comes into play in an increasing way since the majority of the necessary mathematical tools has already been presented. The part closes with the introduction of General Relativity and the derivation of Einstein's equations using the variational principle i.e. the "Hilbert" way.

The second part of this thesis is mostly based on a research project that was carried out by me and the professors *E.Papantonopoulos*, *G.Kofinas*. The main object of this part is the careful examination and analysis of a local solution based on a modified gravity model *The Complete Brans-Dicke Theory*. The task was to find whether the solution can describe a new class of black hole solutions or if it just reproduces the already known solutions of scalar-tensor gravity. The analysis of metric components, areal radius and scalar curvature is been carried out mostly in the isotropic chart. Furthermore, due to the fact that the transformation to Schwarzschild-like coordinates does not exist, we express the metric functions in terms of the conformal factor in an attempt to undrestand whether the solution is horizonless or not. In any case, the new parameter ν of the model does not play a crucial role in the behaviour of the solution as it just defining the scale.

List of Acronyms

ctm chart transition map

mfd manifold

wrt with respect to

st such that

i.e. id est (Latin for “that is”)

rhs right hand side

lhs left hand side

Contents

I Mathematical Structure of Spacetime V

1 Topological Spaces 2

1.1 Continuous Maps 3

1.2 Inheriting a Topology 4

2 Manifolds 5

2.1 Chart Transition Maps 6

3 Multilinear Algebra 9

3.1 Vector Spaces 9

3.2 Dual Vector Spaces 10

3.3 Vectors and Covectors as Tensors 11

4 Differentiable Manifolds 13

4.1 Compatible charts 13

4.2 Diffeomorphisms 15

5 Tangent Spaces 17

5.1 Tangent Vector Space 18

5.2 Components of a vector w.r.t. a chart 20

5.3 Change of vector components under a change of chart 22

5.4 Cotangent Spaces 23

5.5 Change of components of a covector under a change of chart 23

6 Fields **25**

6.1 Tangent Bundle of a smooth manifold 27

6.2 Tensor Fields and the $C^\infty(M)$-module $\Gamma(TM)$ 29

7 Connections **32**

7.1 Directional Derivatives of Tensor Fields 33

7.2 Changing of $\Gamma's$ under change of chart 36

8 Parallel Transport and Curvature **38**

8.1 Parallelity of Vector Fields and Curves 39

8.2 Autoparallel Equation 40

8.3 Torsion and Curvature 42

9 Newtonian Spacetime is Curved **46**

9.1 The Full Wisdom of Newton I 47

9.2 The foundations of the geometric formulation of Newton's axiom 49

10 Metric Manifolds **53**

10.1 Metrics 54

10.2 Length of a curve 56

10.3 Geodesics 57

11 Symmetry **60**

11.1 Push-Forward and Pull-Back Map 61

11.2 Flow of a complete vector field 63

11.3 Lie subalgebras of the Lie algebra $(\Gamma(TM), [\cdot, \cdot])$ of vector fields 64

11.4 Symmetry 65

11.5 Lie derivative 66

12 Integration on Manifolds **68**

12.1 Review of integration on $\mathbb{R}^d$ 68

12.2 Integration on one chart . . . 70

12.3 Volume forms . . . 71

12.4 Integration on M . . . 72

13 Relativistic Spacetime **75**

13.1 Time Orientation . . . 75

13.2 Observers . . . 77

13.3 Role of the Lorentz Transformations . . . 80

14 Matter and Einstein Gravity **82**

14.1 Point Matter . . . 82

14.2 Field Matter . . . 84

14.3 Energy-Momentum Tensor of Matter Fields . . . 84

14.4 Variation of Hilbert Action . . . 86

II Modified Gravity **90**

15 Einstein Gravity with Minimally Coupled Scalar Field **91**

15.0.1 Investigation of the Second Solution . . . 94

16 Complete Brans-Dicke Theory **96**

16.1 Branch $\epsilon < 0$ Solution . . . 99

16.1.1 Scalar field . . . 99

16.1.2 Metric Components . . . 99

16.1.3 Areal Radius & Ricci Scalar . . . 102

16.1.4 Metric in Terms of the Conformal Factor . . . 105

16.2 Branch $\epsilon > 0$ Solution . . . 108

16.2.1 Scalar field . . . 108

16.2.2 Metric Components . . . 109

16.2.3 Areal Radius & Ricci Scalar . . . 111

16.2.4 Metric in Terms of the Conformal Factor 114

Part I

Mathematical Structure of Spacetime

Definition of Spacetime

It is well known by now, that the Einstein's equations connect the matter contents in the universe with gravity in the universe. Of course matter gravitates but, what is new in General Relativity(GR) is that the gravitational effect of matter is encoded in a change of the structure of spacetime, namely in the curvature. One side of the Einstein's equations is referring to matter and the other side talks about gravity. But before one starts talking about relativistic gravity and relativistic matter, there is an underlying notion that needs to be understood, and that is the notion of *spacetime*.

Definition 0.0.1. ***Spacetime*** *is a 4*-dim *topological manifold with a smooth atlas, carrying a torsion-free connection, compatible with a Lorentzian metric and a time orientation, satisfying the Einstein equations*

We will begin our journey by trying to put together all the necessary mathematical pieces in order to understand how we came to define spacetime in such a way. That is why we are going to start from a purely mathematical point of view and, as the journey continues, physics will come into play in an increasingly fashion. Let us begin. . .

Chapter 1

Topological Spaces

At the coarsest level, spacetime is a set it consists of points . But this structure is not enough to talk about even the simplest notions we would like to talk about in classical physics, namely the notion of continuity (of maps). Roughly speaking, we would like continuity of maps because in classical physics we have the idea that curves do not jump. For example, we would not like to have discontinuous trajectories of particles.

The weakest structure that can be established on a set which allows a good definition of continuity of maps is called a *topology*

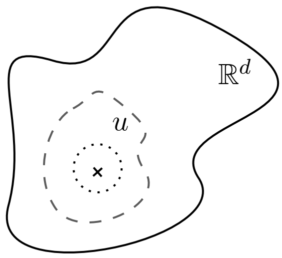
Mathematicians in general investigate the structure of various kinds of spaces, by studying structure-respecting maps between them. The kind of map that respects the structure of a topological space is a continuous map. With that in mind, we begin by providing the rigorous mathematical definition of topology, which is going to allow us to further define continuous maps.

Definition 1.0.1. *Let M be a set. A* ***Topology*** *is a subset $\mathcal{O} \subseteq \mathcal{P}(M) \equiv \{set\ of\ all\ subsets\ of\ M\}$ satisfying:*

- $\emptyset \in \mathcal{O}$ & $M \in \mathcal{O}$.
- $\forall\, u \in \mathcal{O},\ v \in \mathcal{O} \Rightarrow u \bigcap v \in \mathcal{O}$.
- $\forall\, a \in A\,,\ u_a \in \mathcal{O} \Rightarrow \left(\bigcup_{a \in A} u_a\right)\ \in\ \mathcal{O}$.

Example 1.0.1. $M = \mathbb{R}^d = \mathbb{R} \times \mathbb{R} \times \ldots \mathbb{R} = \{(p_1, p_2, \ldots, p_d)\,|p_i\ \in\ \mathbb{R}\}$
$\mathcal{O}_{standard} \subseteq \mathcal{P}(\mathbb{R}^d)$*:*

- ***Soft-ball set****:*
  $\forall\, r\ \in\ \mathbb{R}^+,\ p\ \in\ \mathbb{R}^d : B_r(p) := \{(q_1, q_2, \ldots, q_d)\,|\,\sum_{i=1}^{d} (q_i - p_i)^2 < r^2\}$.
- $u\ \in\ \mathcal{O}_{standard} \Longleftrightarrow \forall\, p\ \in\ u : \exists\, r\ \in\ \mathbb{R}^+ : B_r(p) \subseteq u$.



Usually $\mathbb{R}$ is equipped with $\mathcal{O}_{standard}$ without being said.

Terminology:

- M is called a set
- $\mathcal{O}$ is called a topology $:=$ set of open sets
- $(M, \mathcal{O})$ is called a **topological space**.
- $u \in \mathcal{O} \Leftrightarrow u \subseteq M$ is an **open set**.
- $M \; A \in \mathcal{O} \Leftrightarrow A \subseteq M$is a **closed set**.

We want to promote our set M into a topological space $(M, \mathcal{O}_M)$ because now we can talk about continuous maps. *Topology yields a notion of continuity.*

1.1 Continuous Maps

In order to characterize whether a map $f : M \to N$ is surjective(onto) or injective("1-1") we do not need to provide the sets M, N with extra structure. However, if we would like to talk about the continuity or not, of f then we must provide each of the sets with a topology.

Definition 1.1.1. *Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ topological spaces. Then a map $f : M \to N$ is called **continuous**(with respect to the topologies) if $\forall\ v \in \mathcal{O}_N : preim_f(v) \in \mathcal{O}_M$ where $preim_f(v) := \{m \in M \mid f(m) \in v\}$*

Whether a map is continuous depends on the chosen topologies on M and N. Basically, the above definition states that "*a map is continuous iff the preimages of (all) open sets (in the target N) are open sets (in the domain M)*".

Theorem 1.1.1. *Let the maps f-continuous and g-continuous .Then $g \circ f$- continuous.*

1.2 Inheriting a Topology

In general there are many ways of inheriting a topology from a given topological space. However, given a topological space $(M, \mathcal{O}_M)$, one way of inheriting a topology from it, that is important to spacetime physics, is the subspace topology.

Theorem 1.2.1. *If $(M, \mathcal{O}_M)$ is a topological space and $S \subseteq M$, then the set $\mathcal{O}|_S \subseteq \mathcal{P}(S)$ such that $\mathcal{O}|_S := \{S \cap U | U \in \mathcal{O}_M\}$ is a topology. $\mathcal{O}|_S$ is called the* ***subspace topology*** *inherited from $\mathcal{O}_M$.*

Proof.

1. $\emptyset, S \in \mathcal{O}|_S \because \emptyset = S \cap \emptyset, S = S \cap M$.
2. $S_1, S_2 \in \mathcal{O}|_S \implies \exists U_1, U_2 \in \mathcal{O}_M : S_1 = S \cap U_1, S_2 = S \cap U_2 \implies U_1 \cap U_2 \in \mathcal{O}_M$ $\implies S \cap (U_1 \cap U_2) \in \mathcal{O}|_S \implies (S \cap U_1) \cap (S \cap U_2) \in \mathcal{O}|_S \implies S_1 \cap S_2 \in \mathcal{O}|_S$.
3. Let $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is an index set. Then $S_\alpha \in \mathcal{O}|_S \implies \exists U_\alpha \in \mathcal{O}_M : S_\alpha = S \cap U_\alpha$.
   Further, let $\mathcal{U} = \left(\bigcup_{\alpha \in \mathcal{A}} U_\alpha\right)$. Therefore, $\mathcal{U} \in \mathcal{O}_M$.
   Now, $\left(\bigcup_{\alpha \in \mathcal{A}} S_\alpha\right) = \left(\bigcup_{\alpha \in \mathcal{A}} (S \cap U_\alpha)\right) = S \cap \left(\bigcup_{\alpha \in \mathcal{A}} U_\alpha\right) = S \cap \mathcal{U} \implies \left(\bigcup_{\alpha \in \mathcal{A}} S_\alpha\right) \in \mathcal{O}|_S$.

□

Theorem 1.2.2. *If $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ are topological spaces, and $f : M \longrightarrow N$ is continuous wrt $\mathcal{O}_M$ and $\mathcal{O}_N$, then the restriction of f to $S \subseteq M, f|_S : S \longrightarrow N$ s.t. $f|_S(s \in S) = f(s)$, is continuous wrt $\mathcal{O}|_S$ and $\mathcal{O}_N$.*

Proof. Let $V \in \mathcal{O}_N$. Then, $\mathrm{preim}_f(V) \in \mathcal{O}_M$.
Now $\mathrm{preim}_{f|_S}(V) = S \cap \mathrm{preim}_f(V) \implies \mathrm{preim}_{f|_S}(V) \in \mathcal{O}|_S \implies f|_S$ is continuous. □

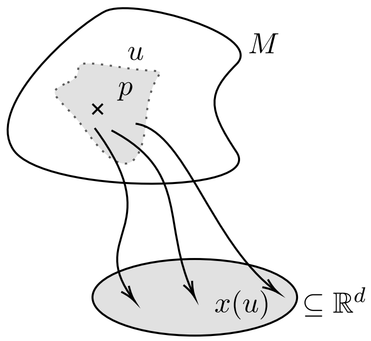
Chapter 2

Manifolds

There are so many topological spaces tha mathematicians cannot even classify them. For spacetime physics we may focus on topological spaces that can be *charted*, analogously to how the surface of the earth can be charted in an atlas.

Definition 2.0.1. *A topological space* $(M, \mathcal{O})$ *is called a* ***d****-dim* ***topological manifold*** *if* $\forall\, p \in M : \exists\, u \in \mathcal{O} : \exists$ *map* $x : u \to x(u) \subseteq \mathbb{R}^d$ *such that :*

- x *is invertible,* $x^{-1} : x(u) \to u$
- x *is continuous*
- x^{-1} *is continuous*



Remark 1. *Two spaces* M, N *are called* ***homeomorphic*** *if there is a bijection* f *(a map or a function that is "1-1" and onto) where* f *and* f^{-1} *are continuous. The corresponding bijection is called* ***homeomorphism****. Hence, the map* x *in the above definition is a homeomorphism.*

The pair (u, x) is called a **chart**. We require that for every point on the manifold, $p \in M$, there exists a chart (u, x) that contains this point. The whole manifold is covered with charts. Moreover, the homeomorphism x maps the entire (open) set u into a subset of $\mathbb{R}^d$.

Terminology:

- The pair (u, x) is called a **chart** of $(M, \mathcal{O})$.
- $\mathcal{A} := \left\{ \left(u_{(a)}, x_{(a)}\right) \mid a \in A \right\}$ is an **Atlas** of $(M, \mathcal{O})$ if $\bigcup_{a \in A} u_a = M$.
- $x : u \to x(u) \subseteq \mathbb{R}^d$ is called a **chart map** .
- The map $x : u \to \mathbb{R}^d$ is the equivalent of d-maps $x^i : u \to \mathbb{R}$, namely given a chart (u, x):

$$x : u \to \mathbb{R}^d \Longleftrightarrow \begin{cases} x^1 : u \to \mathbb{R} \\ x^2 : u \to \mathbb{R} \\ \quad\vdots \\ x^d : u \to \mathbb{R} \end{cases}$$

  The maps x^i are called **coordinate maps** or **component maps**.

- Let $p \in u$, then

  $x^1(p)$ is the first coordinate of the point p with respect to the chosen chart (u, x).
  $x^2(p)$ is the second coordinate of the point p with respect to the chosen chart (u, x).

Remark 2. $\{1, 2\} = \{2, 1\}$. *In sets the is no order. A set is just a collection of it's elements. The set $\mathbb{R}^2$ by itself is "like a powder". We could change the relative positions of it's elements, rearrange them, and the set would still be the same. It has not the structure of a plane as one may imagine.*

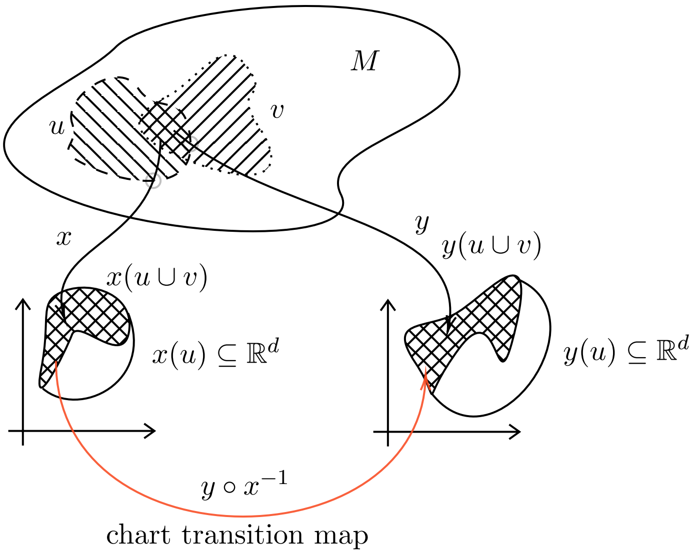
By equipping a set with a topology we are giving it extra structure. Like a "rubber", or a "piece of paper". We could compress it , stretch it but, not rip it apart because this would violate the continuity of the whole structure.

2.1 Chart Transition Maps

Definition 2.1.1. *Let $(u, x), (v, y)$ two charts of $(M, \mathcal{O})$ with overlapping regions, thus $u \cap v \neq 0$. The map*

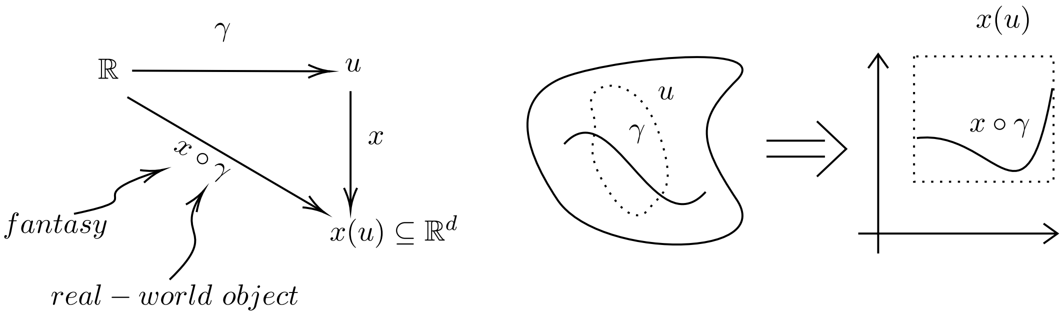
$$\mathbb{R}^d \supseteq x\,(u \cap v) \xrightarrow{y \circ x^{-1}} y\,(u \cap v) \subseteq \mathbb{R}^d \tag{2.1}$$

*is called **chart transition map (ctm)**.*



Manifold philosophy:

Often it is desirable (or indeed the only way) to define properties (such as continuity) of real-world objects (e.g. curve $\gamma : \mathbb{R} \to M$) by judging suitable conditions, not on the real-world object itself but on a **chart-representative(fantasy)** of that real-world object($x \circ \gamma$). One **disadvantage** of this philosophy is that, a property may be ill-defined (because an arbitrary choice of chart is employed). We need to make sure that the defined property does not change if we afford another chart.



In order to explain the form of the **ctm** in grater detail, let $\gamma : \mathbb{R} \to M$ be a curve on the manifold and $(u, x), (v, y)$ two charts with $u \cap v \neq 0$. Suppose that the region $u \cap v \neq 0$ contains a part of the curve γ. Then, we can choose to represent that part of γ in either the two charts by using the definition (2.0.1). Thus we can choose either $(x \circ \gamma)$ or $(y \circ \gamma)$. By trying to relate the two chart-representatives, we find the form of **ctm**.

$$y \circ \gamma = y \circ \left(x^{-1} \circ x\right) \circ \gamma = \overbrace{\left(y \circ x^{-1}\right)}^{chart\,transition\,map} \circ x \circ \gamma \tag{2.2}$$

$$
\begin{tikzcd}
 & y(u) \subseteq \mathbb{R}^d \\
\mathbb{R} \arrow[ur, "y \circ \gamma"] \arrow[r, "\gamma"] \arrow[dr, "x \circ \gamma"'] & u \arrow[u, "y"'] \arrow[d, "x"] \\
 & x(u) \subseteq \mathbb{R}^d
\end{tikzcd}
$$

The **ctm** is continuous as a composition of continuous maps. Informally, it contains the information in how to glue together the charts of an Atlas.

The continuity of the ***ctm*** *makes sure that the defined property is chart-independent which is what we wanted in the first place.*

This is because the underlying laws of physics do not change, if we change coordinates(charts). Physics is chart independent.

Chapter 3

Multilinear Algebra

The object of study of multilinear algebra is vector spaces. However, we will not equip space(-time) with a vector space structure. If physical spacetime carried a vector space structure then one would be able to add positions, or multiply them by numbers. That is like calculating things like "$5 \cdot$ (position of Paris)" or "(position of Vienna)+(position of Paris)". There is just no such notion on physical position space. It's the tangent spaces T_pM to smooth manifolds that carry a vector space structure, and once one has a vector space structure then one has a derived notion of a tensor. Tensors are mathematical objects that can be defined in any vector space.

3.1 Vector Spaces

Definition 3.1.1. *An $\mathbb{R}$-vector space $(V, +, \cdot)$ is a triplet consisting of a set V and two operations "$+, \cdot$" where :*

- $+ : V \times V \to V$ *"addition"*
- $\cdot : \mathbb{R} \times V \to V$ *"S-multiplication"*

Satisfying:

- ***C**ommutative:* $w + v = v + w$
- ***A**ssociative:* $(u + v) + w = u + (v + w)$
- ***N**eutral:* $\exists\, 0 \in V : \forall\, v \in V :\ v + 0 = v$
- ***I**nverse:* $\forall\, v \in V : \exists\, (-v) \in V :\ v + (-v) = 0$

- ***A**ssosiative:* $\lambda \cdot (\mu \cdot \nu) = (\lambda \cdot \mu) \cdot \nu\,,\ \forall \lambda, \mu \in \mathbb{R}$
- ***D**istributive:* $(\lambda + \mu) \cdot \nu = \lambda \cdot \nu + \mu \cdot \nu$

- ***Distributive:*** $\lambda \cdot \nu + \lambda \cdot \mu = \lambda \cdot (\nu + \mu)$
- ***Unitary:*** $1 \cdot \nu = \nu$

An element of a vector space is often referred to, informally, as a *vector*.

Linear Maps

In topology we studied continuous maps as they respect the structure of topological spaces. By following the same general philosophy we would like to study linear maps because they are the structure-respecting maps between vector spaces.

Definition 3.1.2. *Let $(V, +_V, \cdot_W)$ and $(W, +_W, \cdot_W)$ vector spaces. Then a map $\phi : V \to W$ is called* ***linear*** *if*

- $\phi(v + \tilde{v}) = \phi(v) + \phi(\tilde{v})$
- $\phi(\lambda \cdot v) = \lambda \cdot \phi(v)$

Notation: *ϕ-linear* $\Rightarrow \phi : V \xrightarrow{\sim} W$

Example 3.1.1. *The differentiation operator is a linear map*

$$\begin{aligned} \delta :& P \longrightarrow P \\ & p \mapsto \delta(p) := p^{'} \end{aligned}$$

$$\begin{aligned} i)\ \delta(p+q) &= (p+q)^{'} = p^{'} + q^{'} = \delta(p) + \delta(q) \\ ii)\ \delta(\lambda \cdot p) &= (\lambda \cdot p)^{'} \quad = \lambda \cdot p^{'} \quad = \lambda \cdot \delta(p) \end{aligned}$$

therefore $\delta : P \xrightarrow{\sim} P$

Theorem 3.1.1. *If ϕ and ψ linear maps then their composition $(\phi \circ \psi)$ is a linear map.*

Remark 3. *If we want to define* **"continuity"** *of a map then we need to equip it's domain and target with a* **topology** *while, If we want the notion of* **"linearity"** *we need to equip them with the operations of* **addition** *and* **S-multiplication**.

3.2 Dual Vector Spaces

Definition 3.2.1. *Let $(V, +, \cdot)$ and $(W, +, \cdot)$ be vector spaces. We define the set of homomorphisms from V to W as*

$$\hom(V, W) := \{\phi : V \xrightarrow{\sim} W\}. \tag{3.1}$$

We can promote this set into a vector space by equipping it with the operations

$$\bigoplus : \hom(V, W) \times \hom(V, W) \longrightarrow \hom(V, W)$$

$$\bigodot : \mathbb{R} \times \hom(V, W) \longrightarrow \hom(V, W)$$

Hence $\big(\hom(V, W), \bigoplus, \bigodot\big)$ *is a* ***vector space***.

Definition 3.2.2. *Consider the set* $V^* := \{\phi : V \xrightarrow{\sim} \mathbb{R}\} = \hom(V, \mathbb{R})$. *Then the triplet* $(V^*, +, \cdot)$ *is called the* ***dual*** *vector space to* V.

An object $\phi \in V^*$ is informally called a *covector.*

Definition 3.2.3. *Let* $(V, +, \cdot)$ *be a vector space. An (r,s)-tensor over* V *is a multilinear map*

$$T : V^* \times V^* \times \ldots \times V^* \times V \times V \ldots \times V \xrightarrow{\sim} \mathbb{R} \tag{3.2}$$

Remark 4. *A map from* $V \xrightarrow{\sim} V$ *contains the same data as a map from* $v \times V^* \xrightarrow{\sim} \mathbb{R}$

3.3 Vectors and Covectors as Tensors

Theorem 3.3.1. $\phi \in V^* \iff \phi : V \xrightarrow{\sim} \mathbb{R} \iff \phi$ *is a* $(0,1)$*-tensor informally called a* ***covector****.*

Theorem 3.3.2. $v \in V = (V^*)^* \iff v : V^* \xrightarrow{\sim} \mathbb{R} \iff v$ *is a* $(1,0)$*-tensor informally called a* ***vector****.*

Definition 3.3.1. *Let* $(V, +, \cdot)$ *be a vector space. A subset* $B \subset V$ *is called* ***basis*** *if*
$\forall\, v \in V\, \exists!\, finite\ F \subset B : \exists!\, v^1, v^2, \ldots, v^n \in \mathbb{R} :\ v = v^1 f_1 + v^2 f_2 + \ldots + v^n f_n$

Definition 3.3.2. *if* $\exists$ *basis* B *with finitely many elements, say d-many, then we call* $d := \dim V$

Remark 5. *Let* $(V, +, \cdot)$ *be a finite dimensional vector space. Having chosen a basis* $e_1, \ldots, e_n$ *of* $(V, +, \cdot)$ *we may uniquely associate* $v \xrightarrow{with} (v^1, v^2, \ldots, v^n)$ *called the* ***components*** *of* v *with respect to the chosen basis, where* $v = v^1 e_1 + v^2 e_2 + \ldots + v^n e_n$.

Once we choose a basis $e_1, e_2, \ldots, e_n$ for V then we can arbitrarily choose a basis $\epsilon^1, \epsilon^2, \ldots, \epsilon^n$ for V^*. However, it is more economical to require that, once a basis $e_1, e_2, \ldots, e_n$ for V has been chosen, the basis $\epsilon^1, \epsilon^2, \ldots, \epsilon^n$ for V^* satisfies $\epsilon^a(e_b) = \delta^a_b$. This uniquely determines the choice of the basis of the dual space from the choice of basis in V.

Definition 3.3.3. *If a basis* $\epsilon^1, \epsilon^2, \ldots, \epsilon^n$ *for* V^* *satisfies* $\epsilon^a(e_b) = \delta^a_b$*, then it is called* ***dual basis*** *(of the dual space).*

Example 3.3.1. *consider a vector space* $\mathcal{P} := \{p : (-1,1) \longrightarrow \mathbb{R} | p(x) = \sum\limits_{n=0}^{N} p_n x^n\}$, $N = 3$. *The set* $e_a := x^a$, $a = 0, 1, 2, 3$ *constitutes a basis of* $\mathcal{P}$ *while* $\epsilon^a := \frac{1}{a!}\,\partial^a|_{x=0}$ *is a dual basis of the dual space* $\mathcal{P}^*$.

Proof: $\epsilon^a\,(e_b) = \frac{1}{a!}\partial^a x^b = \delta^a_b$

Components of a Tensor

Definition 3.3.4. *Let* T *be a an* $(r,s)-$*tensor over a finite-dim vector space* V *and* $e_1, \ldots, e_n$ *a basis. Then we define the* $(r+s)^{\dim(V)}$ *many real numbers:*

$$T^{i_1, i_2, \ldots, i_r}{}_{j_1, j_2, \ldots, j_s} \in \mathbb{R}\,,\ where\ i_1, \ldots, i_r, j_1, \ldots, j_s \in \{1, \ldots \dim V\}$$

$$T^{i_1, i_2, \ldots, i_r}{}_{j_1, j_2, \ldots, j_s} := T\left(\epsilon^{i_1}, \ldots, \epsilon^{i_r}, e_{j_1}, \ldots, e_{j_s}\right)$$

Useful definition because knowing the components (w.r.t. a basis) and the basis, one can reconstruct the entire tensor.

Example 3.3.2. *Say T is a $(1,1)-$tensor. Then it's components are $T^i_{\ j} = T(\epsilon^i, e_j)$. By reconstruction we mean that one is able to calculate any image*

$$T(\phi, v) = T\left(\sum_{i=0}^{\dim V} \phi_i\, \epsilon^i\ ,\ \sum_{j=0}^{\dim V} v^j\, e_j\right)$$

$$= \sum_{i=0}^{\dim V}\sum_{j=0}^{\dim V} \phi_i\, v^j\, T\,(\epsilon_i, e_j) = \sum_{i=0}^{\dim V}\sum_{j=0}^{\dim V} \phi_i\, v^j\, T^i_{\ j} =: \sum_{i=0}^{\dim V}\sum_{j=0}^{\dim V} \phi_i\, v^j\, T^i_{\ j}$$

Chapter 4

Differentiable Manifolds

So far we have introduced the notion of topological manifolds $(M, \mathcal{O})$ which is a special kind of topological spaces, that can be covered with *charts*. In order to define certain properties of real world objects (such as continuity of curves/ trajectrories), we required suitable conditions on the chart-representatives of those real-world objects

Now we wish to establish the notion of differentiability of curves $(\mathbb{R} \to M)$, functions $(M \to \mathbb{R})$ or even maps $(M \to N)$ on a manifold. But the underlying mathematical structure is not enough. We need to add further structure to our topological manifold so as to start talking about differentiability, just as we added extra structure on our set M(the topology $\mathcal{O}_M$) in order to introduce the notion of continuity .

Consider two charts (u, x) and (v, y) with overlapping domains $u \bigcup v \neq 0$, and a curve $\gamma : \mathbb{R} \to M$. The chart-representatives of this curve will be $(x \circ \gamma)$ and $(y \circ \gamma)$, respectively. The question we wish to answer is " if the map $(x \circ \gamma)$ is differentiable , will $(y \circ \gamma)$ be necessarily differentiable ? If the answer is yes then we know that the properties defined in a chart (let's say (u, x)) will also hold in any other chart (v, y). The map

$$y \circ \gamma = (y \circ x^{-1}) \circ (x \circ \gamma)$$

is the composition of a continuous map $(y \circ x^{-1})$ and a differentiable one $(x \circ \gamma)$, which is not necessarily differentiable. Hence, at first glance our strategy does not work. But there is a remedy (compatible charts).

4.1 Compatible charts

In the first section we used any imaginable charts of the topological manifold $(M, \mathcal{O}_M)$. In order to emphasize that we considered any possible charts, we say that we took u and v from the **maximal atlas** of $(M, \mathcal{O}_M)$. We remind that an atlas is the collection of enough chart such that, the pre-chart regions overlap the entire manifold. The maximal atlas on the other hand, has many more redundant charts. Any possible chart belongs to the maximal atlas. In order to define the differentiability of a curve we may reduce the maximal atlas such that all those charts which have between them a non-differentiable ctm's are thrown away. Thus, *we keep a subset of charts that still covers the entire manifold but all ctm are differentiable.*

Definition 4.1.1. *Two charts* (u,x) *and* (v,y) *of a topological manifold are called* ♠*-compatible if either:*

$$\begin{aligned} &i)\ u \cap v = \emptyset \ \ or \\ &ii)\ u \cup v \neq \emptyset \ \text{such that the ctm's satisfy} \\ &\quad y \circ x^{-1} : x(u \cap v) \longrightarrow y(u \cap v) \\ &\quad x \circ y^{-1} : y(u \cap v) \longrightarrow x(u \cap v) \end{aligned}$$

The point is that because the maps $y \circ x^{-1}$ and $x \circ y^{-1}$ are maps from $\mathbb{R}^d$ to $\mathbb{R}^d$, we can use any notion of ♠−property on $\mathbb{R}^d$ in order to define ♠−compatibility on the manifold. For instance, if we have a notion of $\mathbb{R}^d$−differentiability we call two charts differentialy-compatible if the ctm's between them are differetialy-compatible

Definition 4.1.2. *An atlas* $\mathcal{A}_{♠}$ *is called a* ♠−*compatible atlas if any two charts in* $\mathcal{A}_{♠}$ *are* ♠−*compatible.*

Definition 4.1.3. *A* ♠−*manifold is a triple* $(M, \mathcal{O}, \mathcal{A}_{♠})$ *where* $\mathcal{A}_{♠} \subseteq \mathcal{A}_{maximal}$

Various types of compatibility

- C^o : continuous ctm's w.r.t. the topology.
- C^1 : ctm's are differentiable (once) and the result is continuous.

$$\vdots$$

- C^k : k-times continuously differentiable ctm's.
- D^k : k-times differentiable ctm's, but the result is not continuous.
- C^∞ : continuously differentiable arbitrarily many times. A manifold with such ctm's is called **"smooth"**.
- C^ω : $\exists$ a multi-dimensional Taylor expansion. This is a much stronger restriction than the case C^∞ because not everything in physics can be Taylor expanded.
- $\mathbb{C}^\infty$: The ctm's pairwise satisfy the Cauchy-Riemann equations(only for even-dim manifolds). The corresponding manifold is called **complex manifold**.

Therefore a curve on our manifold can be called k-times continuously differentiable when we manage to find an atlas that is a C^k−atlas. In general, the more fancy stuff we want of our objects on the manifold, the more restrictive we have to be on our choice of atlas.

Theorem 4.1.1. *Any* $C^{k\geq 1}$−*atlas* $\mathcal{A}_{C^{k\geq 1}}$ *of a topological manifold contains a* C^∞−*atlas.*

For the manifold that we can achieve that all ctm's are at least once continuously differentiable , and we have an atlas, we can remove more and more charts until we have a $C^{\infty}-$atlas. So the difficult step is from C^o to C^1. With that in mind we may always consider, without loss of generality, C^{∞}(smooth)-manifolds, unless we wish to define Taylor expandability/complex differentiability.

Definition 4.1.4. *A* **smooth manifold** *is a triple* $(M, \mathcal{O}_M, \mathcal{A})$ *where* $\mathcal{A} = c^{\infty}$.

4.2 Diffeomorphisms

Definition 4.2.1. *Two sets M,N(without further structure) are called* **isomorphic** *if* $\exists$ *bijection("1-1" and onto)* $\phi : M \to N$*. Then we write* $M \cong N$*. The map* ϕ *is called isomorphism.*

Example 4.2.1. $\mathbb{N} \cong \mathbb{Z}$, $\mathbb{N} \cong \mathbb{Q}$, $\mathbb{N} \ncong \mathbb{R}$

Now by adding further structure we can say that two topological spaces $(M, \mathcal{O}_M) \cong (N, \mathcal{O}_N)$ are **topologically isomorphic** or **Homeomorphic**. At the set level they are isomorphic but additionally $\exists$ bijection $\phi : M \to N$ such that ϕ, ϕ^{-1} are continuous maps(because it is a structure preserving property).

Along similar lines we can write $(V, +_N, \cdot_V) \cong (W, +_W, \cdot_W)$, **homeomorphic vector spaces** which implies that $\exists$ bijection $\phi : M \xrightarrow{\sim} N$ with ϕ, ϕ^{-1} linear maps.

Definition 4.2.2. *Two* C^{∞} *manifolds* $(M, \mathcal{O}_M, \mathcal{A}_M)$ *and* $(N, \mathcal{O}_N, \mathcal{A}_N)$ *are* **diffeomorphic** *if* $\exists$ *bijection* $\phi : M \to N$ *such that the maps* ϕ, ϕ^{-1} *are* $C^{\infty}-$*maps* .

$$\begin{tikzcd}
\mathbb{R}^d \arrow[rr, "\tilde{y} \circ \phi \circ \tilde{x}^{-1}"'] & & \mathbb{R}^e \\
M \supset u \arrow[u, "\tilde{x}"'] \arrow[d, "x"] \arrow[rr, "\phi"] & & v \supset N \arrow[u, "\tilde{y}"] \arrow[d, "y"'] \\
\mathbb{R}^d \arrow[rr, "y \circ \phi \circ \tilde{x}^{-1}"'] \arrow[uu, bend left=60, "\text{chart transition map}\quad C^{\infty}"] & & \mathbb{R}^e \arrow[uu, bend right=60, "C^{\infty}\quad \text{chart transition map}"']
\end{tikzcd}$$

It is evident that whether the two manifolds are diffeomorphic or not, should not depend on our choice of chart. If we can prove that the manifolds are diffeomorphic using the charts x, y then the same must hold for any other combination of charts $\tilde{x}, \tilde{y}$.

If ϕ is a C^{∞} map and M, N are smooth then the map $y \circ \phi \circ x^{-1}$ is also C^{∞}. Moreover the ctm , $(x^{-1} \circ \tilde{x})$ and $(y^{-1} \circ \tilde{y})$, are smooth as a composition of smooth maps hence, the map $\tilde{y} \circ \phi \circ \tilde{x}^{-1}$ is also smooth. This means that the property of diffeomorphism is preserved under a change of charts.

A careful reader might ask why are we interested in diffeomorphisms. The answer is that, in general, if two topological manifolds differ by a diffeomorphism we consider them the same.

At this level of structure,$(M, \mathcal{O}, \mathcal{A})$, things do not have a shape yet unless there are "edges". What we mean by that is that for a topologist, a ball and an ellipsoid are practically the same smooth manifold. They differ by a diffeomorphism. On the other hand, a sphere and a second sphere which has been folded at some point creating an "edge", are not the same smooth manifolds[1] $(M_{sphere}, \mathcal{O}_{sphere}, \mathcal{A}_{sphere}) \neq (M_{edge}, \mathcal{O}_{edge}, \mathcal{A}_{edge})$. However, they are the same topological manifolds $(M_{sphere}, \mathcal{O}_{edge}) = (M_{edge}, \mathcal{O}_{edge})$. They differ by a homeomorphism but not by a diffeomorphism.

Theorem 4.2.1. *$\#=$ number of C^∞-manifolds one can make of given C^o-manifold(if any) up to diffeomorphism. Thus it always depends on the level of structure that we look at.*

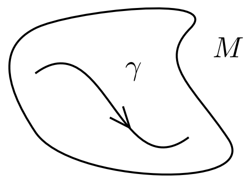
$\dim M$	$\#$
1	1
2	1
3	1
4	$infinitely$ $uncountably$ $many$
5	$finite$
6	$finite$
$\vdots$	$finite$

[1] the differentiability of the ctm of $\mathcal{A}_{edge}$ is destroyed at the points where the "edge" is located. On the contrary, the continuity w.r.t. the $\mathcal{O}_{edge}$ is still preserved. This is the reason that two mathematical structures can be different at level of smooth manifolds but entirely the same at the level of topological manifolds.

Chapter 5

Tangent Spaces

Lead question: What is the velocity of a curve γ at a point p?



We should emphasize that "velocity" and "speed" and two different things in this framework. Velocity can be directly defined on a curve γ of a differentiable manifold whereas speed cannot be defined before a so called *metric* is defined on a manifold.

It is a tradition to consider the velocity a vector and in fact it is going to be a vector in our formalism. However it is important , at this point, to completely erase anything we have learned about "velocities" because the goal of this chapter is to rediscover them from scratch. We this in mind we start by providing it's mathematical definition

Velocities

Definition 5.0.1. *Let $(M, \mathcal{O}, \mathcal{A})$ be a smooth manifold and a curve $\gamma : \mathbb{R} \to M$ which is at least C^1(once continuously differentiable w.r.t. the charts of $\mathcal{A}$). Moreover, suppose that $\gamma(\lambda_o) = p$. The* **velocity** *of γ at p is the* **linear map***:*

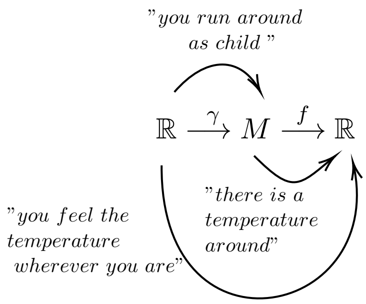
$$u_{\gamma,p}(f) : C^\infty(M) \xrightarrow{\sim} \mathbb{R} \tag{5.1}$$

$$f \mapsto u_{\gamma,p}(f) := (f \circ \gamma)'(\lambda_o) \tag{5.2}$$

where C^∞ and $\mathbb{R}$ are both vector spaces. By $\mathbb{R}$ we do not mean the set of real numbers but the vector space $(\mathbb{R}, +_\mathbb{R}, \cdot_\mathbb{R})$ whose elements are of course the real numbers. $C^\infty(M)$ is the vector space of smooth functions defined as

$$\begin{aligned} C^\infty :=& \{\, f : M \to \mathbb{R} \,|\, f - smooth function\} \text{ equipped with} \\ & (f \oplus g)(p) = f(p) + g(p) \\ & (\lambda \odot g) = \lambda \cdot g(p) \end{aligned}$$

Intuition:



therefore $(f \circ \gamma)'$ = *rate of change of temperature as you run along*

Directional derivative of f ,
as you run along the curve γ*,*
if you are at the point p

With all of the above in mind one could say that *in differential geometry, vectors survive as the directional derivatives they induce.*

5.1 Tangent Vector Space

Definition 5.1.1. *For each point* $p \in M$ *we define the* **set**

$$T_pM := \{u_{\gamma,p} \,|\, \gamma\text{-smooth curve}\} \tag{5.3}$$

called the ***tangent space to* M *at the point* p***.*

This is simply the collection of all possible tangent vectors to all possible smooth curves through the point p. We should point out that we defined the tangent spaces as the set of all possible velocities. But none of these constructions made any reference to an ambient space around our manifold. We may imagine that a tangent space T_pM is a plane that lies outside of M but this is not entirely true. It has also been proven whether we define objects intrinsically or extrinsically(using embedding theorems) does not play a significant role in drawing conclusions. Nevertheless, we are going to use the intrinsic perspective because if M is going to be the universe(spacetime) it would be uneconomical to refer to something around the universe.

Observation: T_pM can be promoted into o vector space. We begin by defining

$$\begin{aligned} \oplus :& T_pM \times T_pM \longrightarrow \hom(C^\infty(M), \mathbb{R}) \\ & (u_{\gamma,p} \oplus u_{\delta,p})(f) := u_{\gamma,p}(f) +_{\mathbb{R}} u_{\delta,p}(f) \end{aligned}$$

$$\begin{aligned} \odot :& \mathbb{R} \times T_pM \longrightarrow \hom(C^\infty(M), \mathbb{R}) \\ & (\alpha \odot u_{\gamma,p})(f) := \alpha \cdot_{\mathbb{R}} U_{\gamma,p} \,,\, \forall \alpha \in \mathbb{R} \end{aligned}$$

And now for the completion of the promotion, it remains to be shown that:

1. $\exists\, \tau$ curve : $\alpha \odot U_{\gamma,p} = u_{\tau,p}$
2. $\exists\, \sigma$ curve : $(u_{\gamma,p} \oplus u_{\delta,p})(f) = u_{\sigma,p}$

Proof. (1.) let curve

$$\begin{aligned} \tau :& \mathbb{R} \to M \\ & \lambda \to \tau(\lambda) := \gamma(\alpha \cdot \lambda + \lambda_o) = (\gamma \circ \mu_\alpha)(\lambda) \end{aligned}$$

where

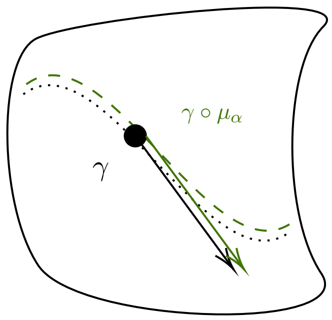
$$\begin{aligned}\mu_\alpha :\, & \mathbb{R} \to \mathbb{R}\\ & r \mapsto \alpha \cdot r + \lambda_o\end{aligned}$$

we claim that the above curve does the trick and we check.

$$\tau(0) = \gamma(\lambda_o) = p - point$$

$$\begin{aligned} u_{\tau,p} &= (f\circ\tau)'(0) = (f\circ\gamma\circ\mu_\alpha)'(0) = \frac{\partial f}{\partial\gamma}\frac{\partial\gamma}{\partial\mu_\alpha}\frac{\partial\mu_\alpha}{\partial\lambda}\bigg|_{\lambda_o=0}\\ &= (f\circ\gamma)'(\mu_\alpha(0))\cdot\alpha = (f\circ\gamma)'(\lambda_o)\cdot\alpha = \alpha\cdot u_{\gamma,p}\end{aligned}$$

□



The function μ_α produced a "double" velocity $(\alpha \cdot u_{\gamma,p})$ by running through the curve at "double" parameter speed. which means that the velocity of the curve depends on it's parametrization.

Proof. (2.) In order to construct the curve σ we need to do something very ugly... We have to make a choice of chart. Nevertheless, our final conclusion has to be chart independent.
Let chart (u, x) and point $p \in u$. We construct a candidate curve $\sigma : \mathbb{R} \to M$, by using this chart and we claim that σ does the trick.

$$\sigma_x(\lambda) = x^{-1}\left(\overbrace{(x\circ\gamma)(\lambda_o+\lambda)}^{\mathbb{R}\to\mathbb{R}^d} + \overbrace{(x\circ\delta)(\lambda_1+\lambda)}^{\mathbb{R}\to\mathbb{R}^d} - \overbrace{(x\circ\gamma)(\lambda_o)}^{constant}\right)$$

where $\gamma(\lambda_o) = p$, $\delta(\lambda_1) = p$. **since:**

$$\sigma_x(0) = x^{-1}\left((x\circ\gamma)(\lambda_o) + (x\circ\delta)(\lambda_1) - (x\circ\gamma)(\lambda_o)\right) = x^{-1}(x\circ\delta)(\lambda_1) = \delta(\lambda_1) = p$$

now:

$$u_{\sigma_x,p}(f) := (f\circ\sigma_x)'(0) = (f\circ x^{-1}\circ x\circ\sigma_x)'(0) = \left(\overbrace{(f\circ x^{-1})}^{\mathbb{R}^d\to\mathbb{R}} \circ \overbrace{(x\circ\sigma_x)}^{\mathbb{R}\to\mathbb{R}^d}\right)'(0) =$$

$$
\begin{tikzcd}
\mathbb{R} \arrow[r, "\sigma_x", "curve"'] \arrow[dr, "x\circ\sigma_x"'] & u(\subseteq M\ ) \arrow[r, "f", "function"'] \arrow[d, "chart"', "x"] & \mathbb{R} \\
& x(u)\subseteq\mathbb{R}^d \arrow[ur, "f\circ x^{-1}"'] &
\end{tikzcd}
$$

$$
\begin{aligned}
&\overset{i-component}{=} \big(\partial_i(f\circ x^{-1})\big) \cdot_{\mathbb{R}} (x\circ\sigma_x)^{i\prime}(0) \left(x(\overbrace{\sigma_x(0)}^{p}) \right) = \\
&\big(\partial_i(f\circ x^{-1})\big) \cdot_{\mathbb{R}} \Big[(x\circ\gamma)^{i\prime}(\lambda_o) + (x\circ\delta)^{i\prime}(\lambda_1) - 0\Big]\,(x(\sigma_x(0))) = \\
&(x\circ\gamma)^{i\prime}\,\big(\partial_i(f\circ x^{-1})\big)\,(x(p)) + (x\circ\delta)^{i\prime}(\lambda_1)\,\big(\partial_i(f\circ x^{-1})\big)\,(x(p)) = \\
&(f\circ x^{-1}\circ x\circ\gamma)'(\lambda_o) + (f\circ x^{-1}\circ x\circ\delta)'(\lambda_1) = (f\circ\gamma)'(\lambda_o) + (f\circ\delta)'(\lambda_1) = \\
&(u_{\gamma,p} \oplus u_{\delta,p})(f)\,, \quad \forall\, f \in C^\infty(M)
\end{aligned}
$$

$\square$

which is what we wanted in the first place. Although we intermediately made a choice of chart, neither the left nor the right hand side depends on it.

Remark 6. *A "plus" does not exists on a manifold. We cannot add curves... that would mean adding the trajectories of particles and would amount to adding position vectors which does not exist. What we can add is the velocities since we just promoted T_pM into a vector space.*

5.2 Components of a vector w.r.t. a chart

Definition 5.2.1. *Let a chart $(u,x) \in \mathcal{A}_{smooth}$ and a curve $\gamma : \mathbb{R} \to u \subseteq M$, where $\gamma(0) = p$. Then we calculate*

$$
\begin{aligned}
u_{\gamma,p}(f) &\coloneqq (f\circ\gamma)'(0) = \big((f\circ x^{-1})\circ(x\circ\gamma)\big)'(0) = \underbrace{(x\circ\gamma)^{i\prime}(0)}_{\dot{\gamma_x}^i(0)} \cdot \underbrace{\partial_i(f\circ x^{-1})(x(p))}_{\left(\frac{\partial f}{\partial x^i}\right)_p} \\
&= \dot{\gamma_x}^i(0)\cdot\left(\frac{\partial}{\partial x^i}\right)_p f\,, \quad \forall\, f \in C^\infty(M)
\end{aligned}
$$

therefore

$$
u_{\gamma,p}(f) \overset{chart}{=} \dot{\gamma_x}^i(0)\cdot\left(\frac{\partial}{\partial x^i}\right)_p \tag{5.4}
$$

where the last part of the calculation is just *notation*.

Remark 7.

- ${\gamma_x}^i$ *: position of a curve w.r.t a chart*
- $\dot{\gamma_x}^i(0)$ *: chart-representative of the curve γ w.r.t. to x(the i-th component of it), which after we took the i-th component we derive and we evaluate at the point zero (0). Hence, it is the i-th component of $u_{\gamma,p}$ w.r.t. the chart x*
- $\left(\frac{\partial}{\partial x^i}\right)_p$ *: basis elements of T_pM w.r.t. which the components need to be understood.*
- *The "i" has nothing to do with the chart x. It is just bad notation.*
- *When we see $\left(\frac{\partial f}{\partial x^i}\right)_p$ we need to immediately translate it into $\partial_i(f \circ x^{-1})(x(p))$, because it not truly a partial derivative(although it can be proven later on).*

Chart Induced Basis

Theorem 5.2.1. *If $(u, x) \in \mathcal{A}_{smooth}$ then $\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^d}\right)_p \in T_pu \subseteq T_pM$ constitute a* **basis** *of T_pu*

Proof. It remains to be shown that they are linearly independent. Taking the definition of linear independence we require $\lambda_i \left(\frac{\partial}{\partial x^i}\right)_p \overset{!}{=} 0$ iff $\forall\, i = 1, \dots, d\ \lambda_i = 0$.

$\lambda_i \left(\frac{\partial}{\partial x^i}\right)_p \overbrace{(x^j)}^{x^j : u \to \mathbb{R}} \overset{notation}{=} \lambda_i \cdot \partial_i (x^j \circ x^{-1})(x(p)) = \lambda^i {\delta^i}_j = \lambda^j$, $j = 1, \dots, d$.
Hence, iff $\lambda_i = 0\ \forall\, i's$ then $\lambda_i \left(\frac{\partial}{\partial x^i}\right)_p = 0$ which means that the basis elements $\left(\frac{\partial}{\partial x^i}\right)_p$ are indeed linearly independent. □

The basis vectors$\left(\frac{\partial}{\partial x^i}\right)_p$ act on functions $f : M \to \mathbb{R}$ from the definition of velocity (5.0.1). That is why we plugged in the component maps $x^j : u \to \mathbb{R}$.

Corollary 1.

- $\dim T_pM = d$ *because we have $\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^d}\right)_p$ d basis vectors.*
- $\dim M = d$ *because we have a collection*

$$\left\{ \begin{array}{l} x^1 : u \to \mathbb{R} \\ \vdots \\ x^1 : u \to \mathbb{R} \end{array} \right\} \Longleftrightarrow x : u \to \mathbb{R}^d \tag{5.5}$$

of d coordinate maps

Therefore $\underset{top.mfd}{\dim M} = d = \underset{vector\ space}{\dim T_pM}$

5.3 Change of vector components under a change of chart

A vector does not change under a change of chart. It is an **abstract object**. The velocity of a bird does not change just because we our thinking about the coordinate system.

Terminology:

$$\begin{tikzcd}
 & \exists\, \gamma : \mathbb{R} \to M : \quad X = u_{\gamma,p} \\
X \in T_pM \arrow[ur] \arrow[dr] & \\
 & \begin{array}{l} \exists\, x^1, \dots, x^d : \; X = X^i \; \left(\frac{\partial}{\partial\, x^i}\right)_p \\ where\ X^i = \; \dot{\gamma}^i_x(\lambda_o) \; , \; \gamma(\lambda_o) = p \end{array}
\end{tikzcd}$$

Let (u, x) and (v, y) be overlapping charts and $p \in u \cap v$. Let also $X \in T_pM$. Then we know that we can write

$$X^j_{(y)} \left(\frac{\partial}{\partial y^j}\right)_p = X = X^i_{(x)} \left(\frac{\partial}{\partial x^i}\right)_p \tag{5.6}$$

In order to derive the formula for the change of components we have to relate the two charts

$$\left(\frac{\partial}{\partial x^i}\right)_p f = \partial_i (f \circ x^{-1})(x(p)) = \partial_i (f \circ y^{-1} \circ y \circ x^{-1})(x(p)) = \partial_i \left((f \circ y^{-1}) \circ (y \circ x^{-1})\right)(x(p))$$

$$\overset{j-comp}{=} \left(\partial_i (y \circ x^{-1})^j\right)(x(p)) \cdot \partial_j (f \circ y^{-1})(y(p)) = \left(\partial_i \underbrace{(y^j \circ x^{-1})}_{ctm}\right)(x(p)) \cdot \partial_j (f \circ y^{-1})(y(p))$$

$$= \left(\frac{\partial y^j}{\partial x^i}\right)_p \cdot_{\mathbb{R}} \left(\frac{\partial f}{\partial y^j}\right)_p$$

therefore

$$\left(\frac{\partial}{\partial x^i}\right)_p = \left(\frac{\partial y^j}{\partial x^i}\right)_p \cdot \left(\frac{\partial}{\partial y^j}\right)_p \tag{5.7}$$

and the equation (5.6) becomes

$$X^j_{(y)} = \left(\frac{\partial y^j}{\partial x^i}\right)_p X^i_{(x)} \tag{5.8}$$

The last equation shows the correlation of components of the same physical vector in two different charts. The transformation from one chart to another can be wildly non-linear . *We can have a non-linear dependence between x and y* . however, in the transformation of the components we evaluate the term $\left(\frac{\partial y^j}{\partial x^i}\right)$ at a specific point p, therefore it takes a particular value and hence becomes a constant. That means the components transform in a linear fashion even if the "global" transformation is non linear.

Remark 8. *A lot of times in Special Relativity we are using the term "Minkowski vector space". If the Minkowski space(position space) had a vector space structure, it would mean that we can add positions and trajectories which cleary is not the case in General Relativity. The truth is*

Minkowski space *is* ***not a vector space*** *either. The formulas only work because we restrict ourselves to changing from one coordinate system to another by linear transformations. Which we further restrict to be Lorentz transformations. The fact is* ***Velocities transform under Lorentz transformations****(not positions) in each point of the tangent space by a linear map* $\left(\frac{\partial y^j}{\partial x^i}\right)_p$ *. That* does not *mean that we mix the points of our space only by linear maps. We want to be able to do physics in ,let's say polar coordinates, if we wish so. Indeed we can do it. We can change from Cartesian coordinates to polar coordinates however, in the velocity spaces (T_pM) it will induce a linear map*

It is an over-structuralization to equip the position space of special relativity with a vector space structure because then we restrict ourselves to transform position vectors only by linear maps (structure preserving maps of vector spaces).

5.4 Cotangent Spaces

Definition 5.4.1. *For each point $p \in M$ we define the dual of T_pM:*

$$(T_pM)^* \coloneqq \{\phi : T_pM \xrightarrow{\sim} \mathbb{R}\} \tag{5.9}$$

called the ***cotangent space to M at the point*** *p.The elements of the cotangent space are called* ***covectors****.*

We would to investigate whether there are any objects we are interested in that lie in it.

Example 5.4.1. *Let a smooth function $f \in C^\infty$. We define the gradient of f as*

$$\begin{aligned}(df)_p \,:\, T_pM &\xrightarrow{\sim} \mathbb{R}\\ X &\mapsto (df)_p(X) \coloneqq Xf\end{aligned}$$

i.e. the gradient of a function $f \in C^\infty$ at a point p is an element of the cotangent space.

Corollary 2. *The gradient is defined independently of the choice of chart*

The components of gradient w.r.t. the chart induced basis (u, x) can be calculated as

$$((df)_p)_j \coloneqq (df)_p\left(\left(\frac{\partial}{\partial x^j}\right)_p\right) = \left(\frac{\partial f}{\partial x^j}\right)_p = \partial_j(f \circ x^{-1})(x(p))$$

5.5 Change of components of a covector under a change of chart

Theorem 5.5.1. *Consider chart $(u, x) \Rightarrow x^i : u \to \mathbb{R}$ coordinate maps. Then the collection $(dx^1)_p, \ldots, (dx^d)_p$ constitutes a* ***basis*** *of T_p^*M.*

In fact, it can be easily proven that the above collection is

$$(dx^\alpha)\left(\left(\frac{\partial}{\partial x^b}\right)_p\right) = \left(\frac{\partial x^\alpha}{\partial x^b}\right)_p = \delta^\alpha_b$$

the *dual basis of the dual space.* If we wanted we could have constructed a basis in the cotangent space that is totally independent from the basis of the tangent space. In conclusion, $(dx^i)_p$ is the dual basis (on T_p^*M) of the chart induced basis $\left(\frac{\partial}{\partial x^j}\right)_p$ (on T_pM).

Now, let $\omega \in T_p^*M$ be a covector. Then we can write

$$\omega_{(y)j}(dy^j)_p = \omega = \omega_{(x)i}(dx^i)_p \tag{5.10}$$

where the change of components is given by(same proof as for the vectors)

$$\omega_{(y)i} = \left(\frac{\partial x^j}{\partial y^i}\right)_p \omega_{(x)j} \tag{5.11}$$

Chapter 6

Fields

Up until now, we have technically focused on a single tangent space. We managed to equip it with a vector space structure and found all the necessary tools to define vectors in that particular space, but nevertheless we were always at *one point*. As physicists we are also interested in vector fields. That is, we would like to think that at any point there is a given vector. The notion of *field* was introduced into physics by Michael Faraday. He was the first one to speak of such an entity. In fact, Faraday did not know any mathematics by modern standards and so he invented the fields to try and deal with that fact.

Roughly speaking, the following analysis can be understood by just *drawing pictures* just like Faraday did however, in order to really work with this new notion we will need some mathematics that underly the pictures in a "$1-1$" manner. This area of mathematics is called *Bundle theory.*

Definition 6.0.1. *A **bundle** is an entity consisting of three pieces of data E, π and M such that*

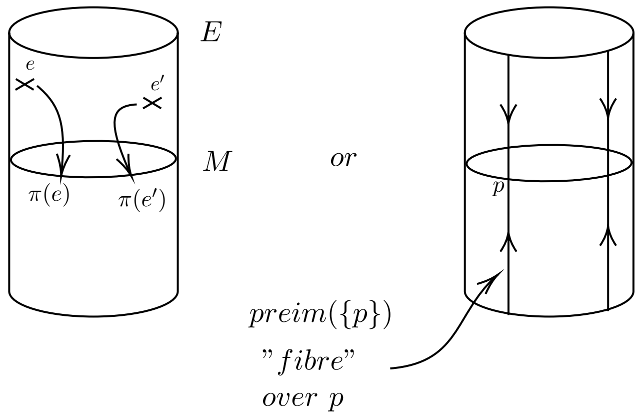
$$E \xrightarrow{\pi} M \tag{6.1}$$

where E is a smooth mfd[1] *called "total space" (of the bundle), M is also a smooth mfd called "base space" (of the bundle) and, π is a smooth and surjective map between them that goes by the name "projection map".*

Any such three pieces of data, relating in this way, are called a bundle.

Example 6.0.1. *Consider E to be a two dimensional cylinder, M to be a one dimensional circle, both equipped with a tolopogy, a smooth atlas etc. and, π a smooth surjective map $\pi : E \to M$. Let us now provide a more intuitive approach by drawing two possible pictures of that bundle.*

[1] a mathematician would say that this is called a "smooth bundle" but we will only consider such.



One can invent any map he wishes as long as the map is smooth and surjective i.e. it maps all the point of E to all the points of M in an infinitely continuously differentiable manner. The two pictures differ only by the choice of map π. As we can observe, spaces E and M remain as they were, but the mapping of the points from E to M has changed.

The afforementioned example, naturally leads us to a new definition.

Definition 6.0.2. *If $E \xrightarrow{\pi} M$ consists a bundle and $p \in M$ a point on the base space, we define a* ***fiber over p*** *as the preimage of the set with the element "p" wrt the projection map that is,*

$$preim_{\pi}(\{p\}) \tag{6.2}$$

Now comes the basic idea that is going to allow us, after some more work, to establish the notion of a field.

Definition 6.0.3. *A* ***section σ*** *of a bundle $E \xrightarrow{\pi} M$, is a map $M \xrightarrow{\sigma} E$ with the extra condition that*

$$\pi \circ \sigma = id_M \tag{6.3}$$

Notice that a section of a bundle requires all the structure a bundle has to give. It requires the pair E,M as the target and the domain, respectively but, also the map π by the afforementioned compatibility condition. The condition is chosen such that **projecting down**(to the base space) using the map π, **after having gone up**(to the total space) using the map σ, **is the identity** on the base space. This prevents the point $\sigma(p) \in$ *total space*, where $p \in$ *base space*, to lie in a different fibre than the base point ρ. This extra condition is what separates a simple map from M to E from a *section*.

Now as physicists, imagine that we were able to construct the ***tangent space T_pM as the fibre over p*** (i.e. the fibre would consist of all the tangent vectors at the point p). Then what would a ***section*** be? What would it do? ...It would go to the point $p \in M$ and ***pick an element $\sigma(p)$ of that particular fibre over p*** (otherwise it would not be a section but just a map (6.3)). ***That means*** it would pick ***a tangent vector*** $X \in T_pM$ iff the fibre is indeed the tangent space.

This is the reason we need (6.3) to be true because, if σ mapped the point p in a different fibre (lets say a fibre over q) it would be like stating that a tangent vector on p (i.e. $\sigma(p)$) belongs in a different tangent space T_qM.

6.1 Tangent Bundle of a smooth manifold

Let $(M, \mathcal{O}_M, \mathcal{A})$ be a smooth mfd, which is going to be the base space of the bundle we are going to construct.

1. Firstly as a set, the *tangent bundle* is defined as

$$TM \coloneqq \dot{\bigcup_{p \in M}} T_pM \tag{6.4}$$

where by $\dot{\bigcup}$ we denote the *disjoint union* which is there so as to emphasize that elements of T_pM are different from elements of T_qM. It is just a reminder that there are no elements to be identified from the different sets. The TM is supposed to be the *total space* of the bundle we are constructing.

2. The second piece of data we need is the *projection map*

$$\pi : TM \longrightarrow M \tag{6.5}$$
$$X \mapsto p \tag{6.6}$$

where p is the unique point $p \in M : X \in T_pM$. Furthermore, since TM includes every possible tangent space, the map π hits every point in M and so it is a *surjective* map.

To recapitulate, so far we have constructed the set TM along with a surjective map $\pi : TM \longrightarrow M$ where M is a smooth mfd by assumption. We cannot judge wether the map π is smooth because we need to promote the set TM into a smooth mfd first. If we do not do it we will not have a *bundle structure*.

---

3. In order to make TM into a smooth mfd we firstly need to promote it into a *topological mfd* and then discard pages from the atlas until we have smooth mfd. Therefore

   - firstly, we equip the set TM with a topology. We can construct the coarsest topology st the map π is just continuous (sometimes called "initial" topology wrt π), that is

   $$\mathcal{O}_{TM} \coloneqq \{preim_\pi(u) \mid u \in \mathcal{O}\} \tag{6.7}$$

   so now we have the topological space $(TM, \mathcal{O}_{TM})$.

   - Secondly, we want to construct an atlas $\mathcal{A}_{TM}$ st we will always going to be able to choose charts with smooth transition maps between them. Thus, we construct a C^∞-atlas on TM from the C^∞-atlas $\mathcal{A}$ on M.

   $$\mathcal{A}_{TM} \coloneqq \{(Tu, \xi_x)\} \tag{6.8}$$

   where

   $$\xi_x : Tu \longrightarrow \mathbb{R}^{2\dim M} \tag{6.9}$$

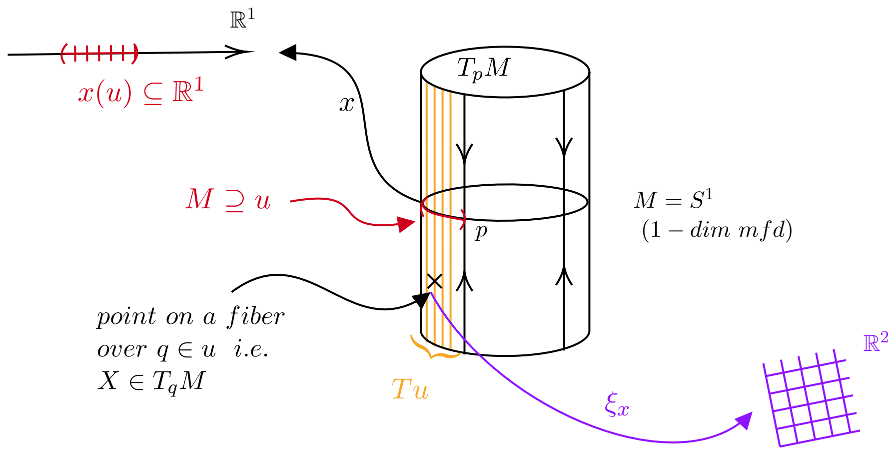
   $$X \longmapsto \left(\underbrace{(x^1 \circ \pi)(X), \ldots, (x^d \circ \pi)(X)}_{(u,x)-coords\, of\, the\, "base"\, point\, \pi(X)}, \underbrace{(dx^1)_{\pi(X)}(X), \ldots, (dx^d)_{\pi(X)}(X)}_{components\, X^i\, of\, vector\, wrt\, chosen\, chart} \right). \tag{6.10}$$

The first d-components stored inside ξ_x are just the coordinates of our base point $\pi(X)$ wrt the chart (u, x). The next d-components are the components of the tangent vector X also wrt the chosen chart. This can be seen if we expand the abstract vector X wrt to the chart induced basis $\left(\frac{\partial}{\partial x^i}\right)$ and then apply the elements of the dual basis of the dual space $\left(dx^i\right)$. Hence, we expand

$$X := X^i_{(x)} \left(\frac{\partial}{\partial x^i}\right)_{\pi(X)} \tag{6.11}$$

and apply

$$\left(dx^j\right)_{\pi(X)}(X) = \left(dx^j\right)_{\pi(X)} \left(X^i_{(x)} \left(\frac{\partial}{\partial x^i}\right)_{\pi(X)}\right) = X^i_{(x)} \delta^j_{\ i} = X^j_{(x)} \tag{6.12}$$



$$\begin{tikzcd}
T_{\pi(X)}M \ni X \arrow[r, "\pi"] \arrow[dr, "\pi \circ x^i"'] & u(\subseteq M) \arrow[d, "chart\ x^i"] \\
& x^{\ i}(u) \subseteq \mathbb{R}^1
\end{tikzcd}$$

$i - th\ component\ of\ \pi(X)$

Now the question is "is this construction of ξ_x a chart map?...does it do the job?" If we think of chart maps and changes of charts then we need to think the inverse of a chart map.

Note: The inverse of ξ_x is defined by

$$\xi_x^{-1} : \xi_x(Tu) \longrightarrow Tu \tag{6.13}$$

$$(a^1, \ldots, a^d, \beta^1, \ldots, \beta^d) := \beta^i \left(\frac{\partial}{\partial x^i}\right)_{\underbrace{x^{-1}(a^1, \ldots, a^d)}_{\pi(X)}} \tag{6.14}$$

which means that the inverse of ξ_x reconstructs the tangent vector. Therefore now all that is left is to check whether the chart transition maps of $\mathcal{A}_{TM}$ are smooth. We calculate

$$(\xi_y \circ \xi_x^{-1})(a^1, \ldots, a^d, \beta^1, \ldots, \beta^d) = \xi_y \left(\beta^i \left(\frac{\partial}{\partial x^i} \right)_{x^{-1}(a^1,\ldots,a^d)} \right) =$$

$$\left(\ldots, (y^i \circ \pi) \left(\beta^j \left(\frac{\partial}{\partial x^j} \right)_{x^{-1}(a^1,\ldots,a^d)} \right), \ldots, \right.$$
$$\left. \ldots, (dy^i)_{x^{-1}(a^1,\ldots,a^d)} \left(\beta^j \left(\frac{\partial}{\partial x^j} \right)_{x^{-1}(a^1,\ldots,a^d)} \right), \ldots \right) =$$

where the map π just selects the base point $\pi(X) \equiv x^{-1}(a^1, \ldots, a^d)$ and afterwards y^i acts on that point, thus

$$\left(\ldots, (y^i \circ x^{-1})(a^1, \ldots, a^d), \ldots, \ldots, \beta^j (dy^i)_{x^{-1}(a^1,\ldots,a^d)} \left(\left(\frac{\partial}{\partial x^j} \right)_{x^{-1}(a^1,\ldots,a^d)} \right), \ldots \right) =$$

$$\left(\ldots, (y^i \circ x^{-1})(a^1, \ldots, a^d), \ldots, \beta^j \left(\frac{\partial y^i}{\partial x^j} \right)_{x^{-1}(a^1,\ldots,a^d)}, \ldots \right) =$$

$$\left(\ldots, (y^i \circ x^{-1})(a^1, \ldots, a^d), \ldots, \beta^j \partial_j (y^i \circ x^{-1})(a^1, \ldots, a^d), \ldots \right)$$

where the first d-components are just the ctms on the base space hence these components are indeed smooth. The last d-components are also smooth as they consist of real numbers β^j times the derivative of ctms which are still smooth.

Hence, we have constructed a smooth atlas $\mathcal{A}_{TM}$.

After this long procedure, we have constructed a triple $TM \xrightarrow{\pi} M$ where M is smooth by assumption, TM is also smooth as we just proved and the map π is also smooth as it is a projection on the base space of the first d-components. That means we have finally constructed the so called ***Tangent Bundle***.

6.2 Tensor Fields and the $C^\infty(M)$-module $\Gamma(TM)$

Definition 6.2.1. *A* ***smooth vector field*** *X is a smooth map on the bundle $TM \xrightarrow{\pi} M$ that is also a section. That is $TM \xleftarrow{X} M$, with the extra condition $\pi \circ X = id_M$*

Vector fields are just sections of a bundle. To study them we do not need the total space TM to be smooth at all. However in order to derive the notion of a smooth vector field we need smoothness on the spaces that the map(section) is acting. That is why we put so much effort into deriving that the total space is a smooth mfd.

Now recall that $C^\infty(M)$ is the collection of all smooth functions promoted into a vector space as we defined in chapter (5). That is written as $(C^\infty(M), +, \bullet)$ where the "fat" multiplication sign denotes that , besides the S-multiplication with $\mathbb{R}$, we can multiply two smooth functions i.e. two elements of the vector space, and get a smooth function. The addition operator is commutative, associative and there is a neutral and an inverse element wrt it. On the other hand, the multiplication operator is commutative,associative, there is a neutral element wrt it however, *there is not an inverse element.* We cannot multiply two smooth, non-zero, functions and get a zero result. This small detail is what separates a *"vector space"* from a *"ring"* as a mathematician would say. *Although it is not evident at this point, the fact that the inverse element wrt the multiplication does not exist, is very important.*

Let us first define what $\Gamma(TM)$ is and then we are going to elaborate on why $(C^\infty(M), +, \bullet)$ being a ring is important.

Definition 6.2.2. $\Gamma(TM)$ *is defined as the collection of the sections over the total space* TM *(tangent bundle) that is, the collection of all vector fields.*

$$X \, : \Gamma(TM) \coloneqq \{M \longrightarrow TM \mid \textit{smooth section}\} \tag{6.15}$$

By equipping this set with an addition $\oplus$ *and a multiplication* $\odot$ *s.t.*

$$1. \, (X \oplus \bar{X})(f) :- (Xf) +_{C^\infty(M)} (\bar{X}f) \tag{6.16}$$

$$2. \; (\alpha \odot X)(f) :- \alpha \cdot X(f) \, , \, \alpha \in \mathbb{R} \tag{6.17}$$

$$3. \; (g \odot X)(f) :- g \bullet_{C^\infty(M)} X(f) \, , \, g \in C^\infty(M) \tag{6.18}$$

we define a new structure $(\Gamma(TM), \oplus, \odot)$ *which is called a* $C^\infty(M)$***module****.*

If we kept only 1. and 2. from the above properties then we would have an authentic vector space. The whole difference lies in the third property. It allows us to multiply a vector field by a smooth function, which means we can *rescale our vector field in an arbitrary manner i.e. scale it differently at every point.* Basically, we have just defined a ***vector space over a ring*** and not just over $\mathbb{R}$, and that is the definition of a *module.* A module satisfies almost all the vector space axioms but the underline scalars $g \in C^\infty(M)$ for the "S-multiplication" are just a ring. Thus, the upshot is that ***the set of all smooth vector fields can be made into a*** $\boldsymbol{C^\infty(M) - }$***module i.e. a vector space over the ring*** $\boldsymbol{C^\infty(M)}$.

This small mathematical modification in the structure is going to have major consequences. The fact that every vector space has a basis depends on the choice of axioms (Z.F.C.) of our set theory. Basically, it is the axiom of Choice (C. in Z.F.C.) that permits us to define a basis. It is literally a shame that no such result exists for modules. $\Gamma(TM)$ does have a global basis. It is a sure thing that if we define a global basis on a module, it will vanish at some point hence it will lose its power. When it comes to modules, we can define bases only in a local manner. Therefore the general strategy, from the point of view of a physicist, is that we will deal with modules as if they are vector spaces but we will not define bases globally.

Therefore, so far we have constructed the set of all smooth sections over the tangent bundle $\Gamma(TM)$ which constitutes a $C^\infty(M)$-module. Similarly, we can construct the set of all smooth sections over the cotangent bundle i.e. $\Gamma(T^*M)$ which will also constitute a $C^\infty(M)$-module. Now, take your attention away from the Γ's and recall that T_pM ***and*** T_p^*M ***were the basic building blocks from which we understood every tensor. Analogously,*** TM ***and*** T^*M ***are the basic building blocks from which we are going to understand every vector field.***

Definition 6.2.3. *An (r,s) tensor field T is a $C^{\infty}(M)$ multilinear map*

$$T\,:\,\underbrace{\Gamma(T^{*}M)\times\ldots\times\Gamma(T^{*}M)}_{r}\times\underbrace{\Gamma(TM)\times\ldots\times\Gamma(TM)}_{s}\xrightarrow{\sim}C^{\infty}(M) \tag{6.19}$$

thus it is a multilinear map between modules.

Example 6.2.1. *Let $f\in C^{\infty}(M)$ be a smooth function on a mfd M i.e. $f:M\longrightarrow\mathbb{R}$. Then we define the gradient of f as*

$$df\,:\,\Gamma(TM)\xrightarrow{\sim}C^{\infty}(M) \tag{6.20}$$

$$X\longmapsto df(X):-X\,f\equiv(Xf)(p)\equiv X(p)f \tag{6.21}$$

where $p\in M$ and recall that $X:M\longrightarrow TM$. Writing Xf is just weird notation to denote that X acts on the function f. What we actually mean is $X(p)f$ where, $X(p)\in T_pM$ and recall from chapter (5) that every element $u_{\gamma,p}$ of T_pM is a map $u_{\gamma,p}\,:\,C^{\infty}(M)\xrightarrow{\sim}\mathbb{R}$.

It is easy to check that the gradient map is $C^{\infty}(M)$-linear. We just rescale the vector field X by an arbitrary function g and feed it inside df

$$gX\longmapsto df(gX):-gXf\equiv(gXf)(p)\equiv\underbrace{g}_{\in C^{\infty}(M)}\cdot\left(\underbrace{X(p)f}_{\in\mathbb{R}}\right) \tag{6.22}$$

therefore the gradient is indeed linear which means that it qualifies as a (0,1)-tensor.

Chapter 7

Connections

In this chapter we are going to provide further structure on our smooth mfd, which comes by the name "connection". This new structure is different in a way, because it will eventually be determined by the Einstein equations as opposed to whatever structure we established so far (i.e. the notions of *set, topology, topological mfd, smooth mfd, bundle* etc.) which was *introduced by hand*. Following this course is inevitable since there are no equations that determine what happened so far therefore, one has to make all of these aforementioned mathematical assumptions in order to start doing physics as we know it.

At this point we are to introduce yet another stucture, the connection, which also goes by the name "covariant derivative" however, there is a slight difference between the two but, for our purposes we will not elaborate in this difference. Roughly speaking a connection is the slightly more general notion.

As being said, the connection in spacetime will be determined by the Einstein equations (through the metric) and the reader should keep in mind that *everything from now on will be objects that are the subject of Einstein equations.* They will be determined by the equations of GR, depending on the matter content in the universe. Having said that, we will continue in the same mathematical fashion.

So far, we saw that a vector field X can be used to provide a directional derivative of a function $f \in C^{\infty}(M)$:

$$\nabla_X f :- Xf$$

where we just introduced a new way of denoting it. $\nabla_X f$ basically denotes "the derivative of f in the direction of the vector field X" and we should underline the fact that *so far, the directional derivative acts only in functions*, see for example (6.21). The addition of this new symbolism seems like an absolute ***notational overkill*** since, we have introduced three different notations ($\nabla_X f \equiv Xf \equiv (df)(X)$) to refer to the same thing precisely. The reason for this is that they are not quite the same thing

$$\nabla_X f \overset{?}{\equiv} Xf \overset{?}{\equiv} (df)(X)$$

because the objects that appear are not exactly the same, namely

$$X : C^{\infty}(M) \longrightarrow C^{\infty}(M). \tag{7.1}$$

That is what happens when we write Xf. On the other hand df takes a slightly different perspective on what is going on since

$$df : \Gamma(TM) \longrightarrow C^\infty(M) \tag{7.2}$$

however, $\nabla_X f$ is precisely the same as Xf and is there just to emphasize that this whole thing is a derivative. Thus the use of ∇_x is indeed a notational overkill but it is introduced because

$$\nabla_X f : C^\infty(M) \longrightarrow C^\infty(M)$$

can be generalised to "eating" an arbitrary (p,q) tensor field and yield an arbitrary (p,q) tensor field whereas X can only act on functions. We want to extend the action of a directional derivative to acting on any (p,q) tensor field. Moreover note that, it would not be necessary to invent a new symbol if this extension came for free i.e. it was automatically defined on any mfd. However, this is not the case. *One needs to provide extra structure on a mfd in order to define a directional derivative that acts on objects other than functions.* Thus, this the reason we invented $\boldsymbol{\nabla_x}$ because it ***will stand for this additional structure***. That means that ∇_X includes also the case Xf but it can also act on more general tensor fields.

7.1 Directional Derivatives of Tensor Fields

Now we wish to find what kind of structure we should provide to our mfd so as to define ∇_x. First, we formulate a wish-list of properties which the directional derivative acting on a tensor field should have. We are going to put the list into a form of a definition however, after we have completed this list, there may be many structures that satisfy it. It will be important for us to find out how many or how to "parametrise" these different structures because, depending on the situation, we will have to pick only one. Hence, we will have to identify the type and amount of information we will need to provide as extra structure (beyond $(M, \mathcal{O}, \mathcal{A})$) in order to fix a particular covariant derivative.

Definition 7.1.1. *A **connection**[1] ∇ on a smooth mfd $(M, \mathcal{O}, \mathcal{A})$ is a map that takes a pair consisting of a vector (field) X and a (p,q) tensor field T and sends them to a (p,q) tensor (field) $\nabla_X T$, satisfying:*

1. $\nabla_X f = Xf \; , \; \forall \, f \in C^\infty(M)$ *i.e.* $\forall \, (0,0)$*-tensor fields*
2. $\nabla_X(T+S) = \nabla_X T + \nabla_X S$ *, additive in the higher entry.*
3. $\nabla_X T(\omega, Y) = (\nabla_X T)(\omega, Y) + T(\nabla_X \omega, Y) + T(\omega, \nabla_X Y)$ *"Leibniz Rule"*

   Where $T : \Gamma(T^*M) \times \Gamma(TM) \longrightarrow C^\infty(M)$ *, therefore* $T(\omega, Y) \in C^\infty(M)$.

   The Leibniz rule applies analogously for any (p,q) tensor field.
4. $\nabla_{fX+Z} T = \nabla_{fX} T + \nabla_Z T = f \nabla_X T + \nabla_Z T$ *,* $C^\infty(M)$*-linear in the lower entry.*

Remark 9. ∇_X *is the extension of X. ∇ is the extension of d (gradient operator).*

A ***manifold with connection*** is a quadruple of structures $(M, \mathcal{O}, \mathcal{A}, \nabla)$. At this point we would like to identify the remaining freedom we have in choosing a particular ∇.

[1] In the literature there is also the name "linear connection" which is more precise but we will not look at non-linear ones for our purposes. It mostly goes by the name "covariant derivative" which definitely emphasizes the way in which we use it... Some people also use the term "affine connection".

New structure on $(M, \mathcal{O}, \mathcal{A})$ required to fix ∇

The questions we want to answer basically are "how many such structures are there?" and "what is the new structure required to fix the covariant derivative?". It turns out that it is useful to first consider the simplest case i.e. what do we need in order to be able to make ∇ to act on a *vector field*, beyond acting on a function ? We will consider this case and then move on to more general ones thus. . . let X, Y be vector fields. We calculate

$$\begin{aligned}
\nabla_X Y \overset{(u,x)}{=} \nabla_{X^i \frac{\partial}{\partial x^i}} \left(Y^m \frac{\partial}{\partial x^m} \right) &= X^i \nabla_{\frac{\partial}{\partial x^i}} \left(Y^m \frac{\partial}{\partial x^m} \right) \\
=& X^i \left(\nabla_{\frac{\partial}{\partial x^i}} Y^m \right) \frac{\partial}{\partial x^m} + X^i Y^m \underbrace{\nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^m} \right)}_{vector\, field} \\
=& X^i \left(\frac{\partial}{\partial x^m} Y^m \right) \frac{\partial}{\partial x^m} + X^i \, Y^m \, \Gamma^q_{(x)\, mi} \frac{\partial}{\partial x^q}
\end{aligned} \tag{7.3}$$

where Γ^q_{mi} are the connection coefficient functions (on M) of ∇ wrt the chart (u, x).

Definition 7.1.2. *Let $(M, \mathcal{O}, \mathcal{A}, \nabla)$ and $(u, x) \in \mathcal{A}$. Then the **connection coefficient functions** wrt (u, x) are the $(\dim M)^3$-many, chart dependent functions*

$$\begin{aligned}
\Gamma^i_{(x)\, jk} \, : \, u &\longrightarrow \mathbb{R} \\
p &\longmapsto dx^i \left(\nabla_{\frac{\partial}{\partial x^k}} \left(\frac{\partial}{\partial x^j} \right) \right) (p)
\end{aligned} \tag{7.4}$$

where recall that $\nabla_{\frac{\partial}{\partial x^k}} \left(\frac{\partial}{\partial x^j} \right)$ is a vector (tensor) field thus, we extract its i-th component by acting with the covector field dx^i.

Therefore, $\nabla_X Y$ is a vector field with components

$$(\nabla_X Y)^i = X^m \left(\frac{\partial}{\partial x^m} Y^i \right) + \Gamma^i_{(x)\, nm} Y^n X^m \tag{7.5}$$

Hence if we are provided with all the $(\dim M)^3$ many functions $\Gamma^i_{(x)\, jk}$ then we are able to calculate the directional derivative of Y wrt X, in the chart (u, x). This is the freedom that is left. That is all the information we need in order to calculate covariant derivatives of vector fields. In the next step we will see that the use of the same $(\dim M)^3$ functions enables us to calculate covariant derivatives of any tensor field , thus. . .

Remark 10. *On a chart domain u the choice of the $(\dim M)^3$ functions $\Gamma^i_{(x)\, jk}$ suffices to fix the action of ∇ on a vector field. Fortunately, the same $(\dim M)^3$ fix the action of ∇ on any tensor field.*

The *key observation* to see that we don not need any more information, is to check whether we have to require new $\Gamma's$ (let's call them $\Sigma's$) when we calculate the action of ∇ on a covector field. Obviously the calculation will be the same as before up to the point where we arrive at the elementary question

$$\nabla_{\frac{\partial}{\partial x^m}} dx^i \overset{?}{=} \Sigma^i_{(x)\, qm} dx^q \tag{7.6}$$

but now consider that

$$\nabla_{\frac{\partial}{\partial x^m}}\left(dx^i\left(\frac{\partial}{\partial x^j}\right)\right) = \frac{\partial}{\partial x^m}\left(\delta^i_{\ j}\right) = 0\,. \tag{7.7}$$

However, if we apply the Leibniz rule, the same object is written as

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x^m}}\left(dx^i\left(\frac{\partial}{\partial x^j}\right)\right) &= \left(\nabla_{\frac{\partial}{\partial x^m}} dx^i\right)\left(\frac{\partial}{\partial x^j}\right) + dx^i\left(\nabla_{\frac{\partial}{\partial x^m}}\frac{\partial}{\partial x^j}\right)\\
&= \left(\nabla_{\frac{\partial}{\partial x^m}} dx^i\right)\left(\frac{\partial}{\partial x^j}\right) + dx^i\left(\Gamma^q_{(x)\,jm}\frac{\partial}{\partial x^q}\right)\\
&= \left(\nabla_{\frac{\partial}{\partial x^m}} dx^i\right)\left(\frac{\partial}{\partial x^j}\right) + \Gamma^q_{(x)\,jm} dx^i\left(\frac{\partial}{\partial x^q}\right)\\
&= \left(\nabla_{\frac{\partial}{\partial x^m}} dx^i\right)\left(\frac{\partial}{\partial x^j}\right) + \Gamma^q_{(x)\,jm}\delta^i_{\ q}\\
&= \left(\nabla_{\frac{\partial}{\partial x^m}} dx^i\right)_j + \Gamma^i_{(x)\,jm}
\end{aligned} \tag{7.8}$$

thus, combining the last two relations we arrive at

$$\left(\nabla_{\frac{\partial}{\partial x^m}} dx^i\right)_j = -\Gamma^i_{(x)\,jm}\,. \tag{7.9}$$

Now recall that we assumed

$$\nabla_{\frac{\partial}{\partial x^m}} dx^i = \Sigma^i_{(x)\,qm} dx^q\,. \tag{7.10}$$

From the last two equations we see that

$$\Sigma^i_{\ jm} = -\Gamma^i_{\ jm} \tag{7.11}$$

which shows that indeed we do not need any more functions than the already established $\Gamma' s$. Summarizing, so far we saw that

$$\begin{aligned}
(\nabla_X Y)^i &= X(Y^i) + \Gamma^i_{\ jm} Y^j X^m \quad \text{:(0,1)-tensor} &(7.12)\\
(\nabla_X \omega)_i &= X(\omega_i) + \Gamma^j_{\ im}\,\omega_j\, X^m \quad \text{:(1,0)-tensor}\,, &(7.13)
\end{aligned}$$

similarly by further application of the Leibniz rule one is able to calculate the covariant derivative of any tensor field irrespectively of its rank. For example in the case of a (1,2) tensor field we get

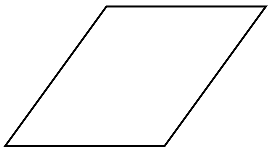
$$(\nabla_X T)^i_{\ jk} = X(T^i_{\ jk}) + \Gamma^i_{\ sm} T^s_{\ jk} X^m - \Gamma^s_{\ jm} T^i_{\ sk} X^m - \Gamma^s_{\ km} T^i_{\ js} X^m\,. \tag{7.14}$$

Question: If we consider a flat/Euclidean space, the $\Gamma' s$ all vanish, in a (then existing) global chart?

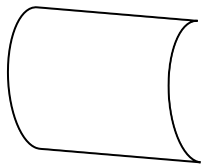
Answer: Yes! But what is flat/Euclidean space? ... At the level $(M = \mathbb{R}^3, \mathcal{O}_{st.}, \mathcal{A})$ it is just a smooth manifold. $\Gamma' s$ are not even defined at this level. However, if we assume a (global) chart $(u, x) = (\mathbb{R}^3, id_{\mathbb{R}^3}) \in \mathcal{A}$ and also that in this particular chart $\Gamma^i_{(x)\,jk} = dx^i\left(\nabla_{E\,\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^j}\right) \stackrel{!}{=} 0$ then yes, this is a flat/Euclidean space. All this happens at the level $(M = \mathbb{R}^3, \mathcal{O}_{st.}, \mathcal{A}, \nabla_E)$. Which means that *flat/Euclidean space is one in which we are* *able to find a (global) coordinate system where the* $\Gamma' s$ *all vanish (globally).* If choose another chart (one in which the mapping is non-linear e.g. a polar coordinate chart) then we will find non-vanishing $\Gamma' s$, however the underlying structure of the that space remains the same .

Besides the above, if we also choose a different connection on the same smooth mfd i.e. choose $(M = \mathbb{R}^3, \mathcal{O}_{st.}, \mathcal{A}, \nabla_{Hyperbolic})$, then we may find that there is no global chart in which the $\Gamma' s$ vanish, globally. There may be a local chart but definitely not a global. Intuitively, we provide the following picture

$\left(\mathbb{R}^2, \mathcal{O}_{st.}, \mathcal{A}, \nabla_{Euclidean}\right)$ $\qquad$ $\left(\mathbb{R}^2, \mathcal{O}_{st.}, \mathcal{A}, \nabla_{Hyperbolic}\right)$



,



which is like stating that the choice of the connection will have consequences on the curvature of our mfd, as we will see in the next chapters.

Definition 7.1.3. *Let X be a vector field on $(M, \mathcal{O}, \mathcal{A}\nabla)$. Then the divergence* ***divergence of the vector field X*** *is the chart dependent function:*

$$div(X) :- \left(\nabla_{\frac{\partial}{\partial x^i}} X\right)^i \tag{7.15}$$

7.2 Changing of $\Gamma' s$ under change of chart

Let overlapping charts (u, x) and (v, y) $\in \mathcal{A}$, $u \bigcap v \neq 0$. The connection coefficient functions in these charts are related by

$$\begin{aligned}
\Gamma^i_{(y)\,jk} :- dy^i\left(\nabla_{\frac{\partial}{\partial x^k}}\left(\frac{\partial}{\partial x^j}\right)\right) &= \frac{\partial y^i}{\partial x^q} dx^q \left(\nabla_{\frac{\partial x^p}{\partial y^k}\frac{\partial}{\partial x^p}}\left(\frac{\partial x^s}{\partial y^j}\frac{\partial}{\partial x^s}\right)\right) \\
&= \frac{\partial y^i}{\partial x^q} dx^q \frac{\partial x^p}{\partial y^k}\left[\left(\nabla_{\frac{\partial}{\partial x^p}}\frac{\partial x^s}{\partial y^j}\right)\frac{\partial}{\partial x^s} + \frac{\partial x^s}{\partial y^j}\left(\nabla_{\frac{\partial}{\partial x^p}}\frac{\partial}{\partial x^s}\right)\right] \\
&= \frac{\partial y^i}{\partial x^q}\frac{\partial x^p}{\partial y^k}\frac{\partial}{\partial x^p}\left(\frac{\partial x^s}{\partial y^j}\right)\delta^q_s + \frac{\partial y^i}{\partial x^q}\frac{\partial x^s}{\partial y^j}\frac{\partial x^p}{\partial y^k}\underbrace{dx^q\left(\nabla_{\frac{\partial}{\partial x^p}}\frac{\partial}{\partial x^s}\right)}_{\Gamma^q_{(x)\,sp}}
\end{aligned} \tag{7.16}$$

hence,

$$\Gamma^i_{(y)\,jk} = \frac{\partial y^i}{\partial x^q}\frac{\partial x^s}{\partial y^j}\frac{\partial x^p}{\partial y^k}\Gamma^q_{(x)\,sp} + \frac{\partial y^i}{\partial x^q}\frac{\partial^2 x^q}{\partial y^k \partial y^j} \tag{7.17}$$

which destroys the tensor component transformation law due to the appearance of the last term. That is, $\Gamma' s$ are not tensors. Recall that we defined $\Gamma' s$ as ***chart dependent functions*** (7.4). This becomes evident from the form of (7.17) as, even if in one chart x, $\Gamma^q_{(x)\,sp} = 0$, we can always find another chart y so that $\Gamma^i_{(y)\,jk}(\neq 0) = \frac{\partial^2 x^q}{\partial y^k \partial y^j}$ because of the second term in the transformation law.

Normal Coordinates

Theorem 7.2.1. *Let $p \in M$ of $(M, \mathcal{O}, \mathcal{A}, \nabla)$. Then one can construct a chart (u, x) with $p \in u$ s.t. the symmetric part of the connection coefficient functions vanishes at p i.e.*

$$\Gamma^i_{(x)(jk)} = 0 \tag{7.18}$$

at the point p but, ***not*** *in any neighbourhood.*

Proof. Let chart (v, y) and point $p \in v$. Thus, in general $\Gamma^i_{(y)jk} \neq 0$. Then consider a new chart (u, x) to which we transit by virtue of

$$(x \circ y^{-1})^i(a^1, \ldots, a^d) :- a^i + \frac{1}{2}\Gamma^i_{(y)(jk)} a^j a^k \tag{7.19}$$

then

$$\left(\frac{\partial x^i}{\partial y^j}\right)_p :- \partial_j(x^i \circ y^{-1}) = \delta^i_j + \Gamma^i_{(y)(mj)}(p) a^m \,, \tag{7.20}$$

$$\left(\frac{\partial^2 x^i}{\partial y^k \partial y^j}\right)_p = \Gamma^i_{(y)(kj)}(p) \tag{7.21}$$

but if , without loss of generality, we choose the point p to have coordinates wrt the chart (v, y) as :

$$y(p) = (y^1(p), \ldots, y^d(p)) \equiv (a^1, \ldots, a^d) = (0, \ldots, 0)$$

then

$$\left(\frac{\partial x^i}{\partial y^j}\right)_p = \delta^i_j \,, \tag{7.22}$$

$$\left(\frac{\partial^2 x^i}{\partial y^k \partial y^j}\right)_p = \Gamma^i_{(y)(jk)}(p) \tag{7.23}$$

and the transformation law (7.17) gives

$$\Gamma^i_{(y)\,jk}(p) = \frac{\partial y^i}{\partial x^q}\,\frac{\partial x^s}{\partial y^j}\,\frac{\partial x^p}{\partial y^k}\,\Gamma^q_{(x)\,sp}(p) + \frac{\partial y^i}{\partial x^q}\,\frac{\partial^2 x^q}{\partial y^k \partial y^j} \tag{7.24}$$

$$= \delta^i_q\,\delta^s_j\,\delta^p_k\,\Gamma^q_{(x)sp}(p) + \delta^i_q\,\Gamma^q_{(y)(kj)}(p) \Longrightarrow \tag{7.25}$$

$$\Gamma^i_{(x)jk}(p) = \Gamma^i_{(y)jk}(p) - \Gamma^i_{(y)(jk)}(p) \tag{7.26}$$

$$\equiv \Gamma^i_{(y)[kj]} \tag{7.27}$$

which means that in the new chart (u, x) we have *removed the symmetric part of the connection coefficient functions.* Thus indeed

$$\Gamma^i_{(x)(jk)} = 0 \,. \tag{7.28}$$

□

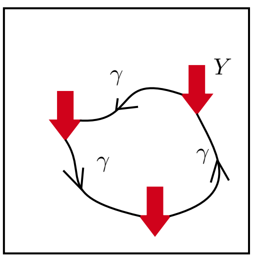
Terminology: The chart (u, x) is called a ***normal coordinate chart*** of the connection ∇ at the point $p \in M$.

Chapter 8

Parallel Transport and Curvature

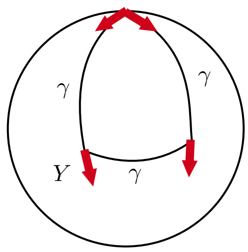
Thought Experiment: Consider that we have an arrow Y, representing a vector, and that we move it along a closed curve γ keeping its direction unchanged. Mathematically this requirement is written $\nabla_{u_\gamma} Y \overset{!}{=} 0$ i.e. the directional derivative of the arrow[1] wrt to the curve γ is zero. Intuitively, as can be seen in the following picture, the arrow is still in the direction it was when it "started to move", when the space is flat. If we now consider the same situation but in a "curved" space like the surface of a round sphere then we will realize that the direction of the arrow will not remain the same as it was in the beginning of its "movement". This means that $\nabla_{u_\gamma} Y \neq 0$ when we are in a curved space. Therefore by "giving the arrow" the same "instructions" we obtain a different result. This shows that *the covariant derivative* which gives a directional derivative on tensors, in particular on vectors, *has something to do with curvature. Inside the covariant derivative is encoded somehow, the information about the curvature of space.*

flat space



$\nabla_{u_\gamma} Y \overset{!}{=} 0$

curved space



$(S^2, \mathcal{O}, \mathcal{A}, \nabla)$
with a very specific connection :
"round sphere"

$\nabla_{u_\gamma} Y \neq 0$

All of the above is just our motivation that in the covariant derivative there must be information on what we intuitively call curvature. The purpose of this chapter is to make these statements precise.

[1]Note that, if we move the "stable arrow" so as to cover every point of the given space then, it defines a vector field.

8.1 Parallelity of Vector Fields and Curves

Definition 8.1.1. *Let $(M, \mathcal{O}, \mathcal{A}, \nabla)$ be a smooth manifold with connection ∇.*

(1) A vector field X on M is said to be ***parallely transported*** *along a smooth curve $\gamma : \mathbb{R} \longrightarrow M$ if*

$$\nabla_{u_\gamma} X = 0 \tag{8.1}$$

i.e. if we evaluate at each point of the curve $\gamma(\lambda)$, we may write

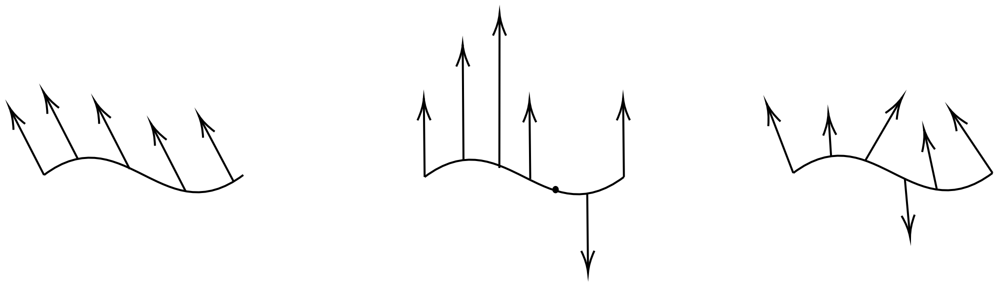
$$\left(\nabla_{u_{\gamma,\gamma(\lambda)}} X\right)_{\gamma(\lambda)} = 0$$

*(2) A slightly weaker condition (it allows more) is that a vector field being "**parallel**" along a curve γ if, for $\mu : \mathbb{R} \longrightarrow \mathbb{R}$,*

$$\left(\nabla_{u_{\gamma,\gamma(\lambda)}} X\right)_{\gamma(\lambda)} = \mu(\lambda) X_{\gamma(\lambda)} \tag{8.2}$$

Note that although "parallely transported" sounds like an action, it is actually a property. This property is satisfied if, the directional derivative of the vector field in the direction of the tangent of the curve, always equals to zero.

Example 8.1.1. *Consider the Euclidean plane in the sense we defined it earlier, $(\mathbb{R}^2, \mathcal{O}, \mathcal{A}, \nabla_E)$. The connection ∇_E enables us to define a specific (global) chart in which all the Γ's vanish, like before. But lets not go into any chart. If we "draw pictures in the real world" then*



X parallely transported along γ

X parallel along γ

X neither parallel nor parallely transported along γ

One should be very careful as one may be tempted to talk about the "lenght" of vectors, but there is no notion of length yet, only "affine length"(lenght wrt the parameter of the curve). All this is made just by our connection ∇_E. Therefore keep in mind that the pictures despite the fact that they can provide some intuition, they can also be misleading when one studies more general spaces other than the Euclidean plane.

Definition 8.1.2. *(1) A curve $\gamma : \mathbb{R} \longrightarrow M$ is called* ***autoparallely transported*** *if*

$$\nabla_{u_\gamma} u_\gamma = 0 \Longleftrightarrow \left(\nabla_{u_{\gamma,\gamma(\lambda)}} u_\gamma\right)_{\gamma(\lambda)} = 0 \tag{8.3}$$

*(2) The weaker condition of a curve being **autoparallel**[2] is defined by*

$$\nabla_{u_\gamma} u_\gamma = \mu u_\gamma \tag{8.4}$$

Example 8.1.2. *Again consider the Euclidean plane $(\mathbb{R}^2, \mathcal{O}, \mathcal{A}, \nabla_E)$, where we have good intuition...*

autoparallely transported | *autoparallel*

In physics, the curve on the left would describe a uniform, straight motion whereas the curve on the right would describe just straight motion. Recall that Newton's first law talks about uniform, straight motion. In our context, Newton's first law is a measurement prescription. If you take something and throw it (and there is no additional force) then, whatever you see as the path it takes, is an autoparallel(-ly transported) curve. Therefore, the first axiom of Newton's tells us to do an experiment and conclude what the connection is. It is a measurement prescription for our geometry.

8.2 Autoparallel Equation

Consider the portion of an autoparallely transported curve γ which lies in the chart $(u, x) \in \mathcal{A}$. We would like to find the chart representative of the condition (8.3) i.e. express it terms of objects of the chart.

$$\begin{aligned}
0 &= \left(\nabla_{u_\gamma} u_\gamma\right) \\
&= \left(\nabla_{\left(\dot\gamma^m_{(x)} \frac{\partial}{\partial x^m}\right)} \dot\gamma^n_{(x)} \frac{\partial}{\partial x^n}\right) && \text{recall that } \gamma^m_{(x)} :- x^m \circ \gamma \\
&= \dot\gamma^m \left(\nabla_{\left(\frac{\partial}{\partial x^m}\right)} \dot\gamma^n\right) \frac{\partial}{\partial x^n} + \dot\gamma^m \dot\gamma^n \left(\nabla_{\left(\frac{\partial}{\partial x^m}\right)} \frac{\partial}{\partial x^n}\right) \\
&= \dot\gamma^m \left(\frac{\partial}{\partial x^m} \dot\gamma^n\right) \frac{\partial}{\partial x^n} + \dot\gamma^m \dot\gamma^n \left(\nabla_{\left(\frac{\partial}{\partial x^m}\right)} \frac{\partial}{\partial x^n}\right) \\
&= \dot\gamma^m \left(\frac{\partial}{\partial x^m} \dot\gamma^q\right) \frac{\partial}{\partial x^q} + \dot\gamma^m \dot\gamma^n \left(\Gamma^q{}_{nm} \frac{\partial}{\partial x^q}\right) && \text{change of index in 1st term} \\
&= \left(\dot\gamma^m \frac{\partial}{\partial x^m} \dot\gamma^q + \dot\gamma^m \dot\gamma^n \Gamma^q{}_{nm}\right) \frac{\partial}{\partial x^q} \\
&= \left(\ddot\gamma^q + \dot\gamma^m \dot\gamma^n \Gamma^q{}_{nm}\right) \frac{\partial}{\partial x^q}
\end{aligned}$$

[2]Usually in the literature they use the this term but they actually refering to the autoparallely-transported curves. From now on we will make the same abuse of terminology thus when we write "autoparallel" we mean "autoparallely transported".

One can perform the last part of the calculation in a cleaner way however, for our purposes we will just provide an intuitive explanation. The term $\frac{\partial \dot{\gamma}^q}{\partial x^m}$ is the tangent vector components to the curve, derived in all possible directions. But this vector field is not "everywhere", it is just along the curve γ. Hence, strictly speaking that term alone is not well defined. However, the whole term is $\dot{\gamma}^m \frac{\partial \dot{\gamma}^q}{\partial x^m}$ that is, there is immediately the projection of $\frac{\partial \dot{\gamma}^q}{\partial x^m}$ to the direction of the curve i.e. a projection of a derivative in the tangent space (space of directional derivatives) thus the whole term gives a second derivative. The first derivative is $\dot{\gamma}^q$ and the projection in the direction of the curve is $\dot{\gamma}^m \frac{\partial}{\partial x^m}$.

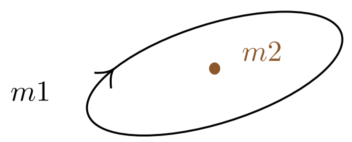
From the last line of the calculation, we find that the ***chart representative of an autoparallely transported curve γ*** satisfies [3]

$$\ddot{\gamma}^m_{(x)} + \Gamma^m_{(x)ab}\left(\gamma(\lambda)\right)\dot{\gamma}^a_{(x)}(\lambda)\dot{\gamma}^b_{(x)}(\lambda) = 0 \tag{8.5}$$

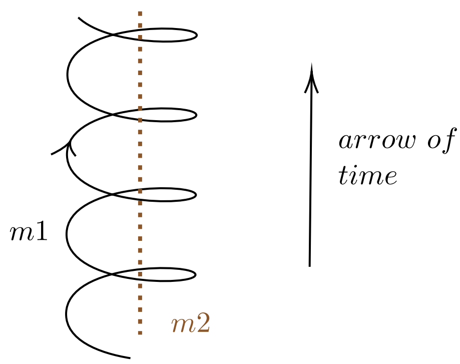
This is one of most important equations to understanding general relativity. As we will see in the next chapter, Newtonian spacetime carries a connection, in such a way that we can trasform gravity from a force into a curvature of that Newtonian spacetime. In fact one must do that conceptual transition because of Newton's first axiom.

The axiom states *that under the influence of* ***no force****, a particle will move in a uniform and straight fashion i.e. on an autoparallely transported curve.* Now consider that there is **at least one other particle** in our universe (also recall that in Newtonian mechanics all particles have positive mass, there are no massless particles etc.). The question is ***"how are we suppose to use the first axiom if we know that all particles interact gravitationaly up to infinite distance?"*** In such a universe, the first axiom would be totally out of work *if gravity was a force* since the condition is "look at a particle on which no force acts. . . ".

However, *if we interpret gravity as curvature and not a force then, in a universe with at least two particles, Newton's axiom will maintain its credibility.* Let $m1, m2$ be the masses of the particles, with $m1 \ll m2$. We know that due to gravity, we are going to observe an elliptical motion of $m1$ around $m2$. In this case *we have got to declare that this path is consists an autoparallely transported curve.* Not solely in space as we shall see tomorrow but in space-time this is possible.



elliptical motion in space



autoparallely transported curve in spacetime.

[3] Note that the only acceptable reparametrization of λ that still satisfies the autoparallel equation is $\tilde{\lambda} = \alpha\lambda + b$. Parameters relating this way are called **"affine"**

Therefore, in order not to make the first axiom useless we must not describe gravity as a force and, this is the way out. Then the axiom will tell us what the autoparallels are, by experiment and, if a path of particle deviates from the autoparallels then we will know that this due to an external force.

Example 8.2.1. *(1) We already know that in a Euclidean plane having a chart $(U = \mathbb{R}^2, x = id_{\mathbb{R}^2})$, then $\Gamma^i_{jk}(x) = 0 \implies \ddot{\gamma}^m_{(x)} = 0$ which reminds us the equation of a straight line. This is the autoparallel equation in this particular chart of the Euclidean plane. The solution reads $\gamma^m_{(x)}(\lambda) = a^m\lambda + b^m$, where $a, b \in \mathbb{R}^d$.*

(2) Consider the round sphere $(S^2, \mathcal{O}, \mathcal{A}, \nabla_{round})$, i.e., the sphere $(S^2, \mathcal{O}, \mathcal{A})$ with the connection ∇_{round}. Consider the chart $x(p) = (\theta, \phi)$ where $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$. In this chart ∇_{round} is given by

$$\Gamma^1_{22}(x)\left(x^{-1}(\theta,\phi)\right) := -\sin\theta\cos\theta$$
$$\Gamma^2_{(x)12}\left(x^{-1}(\theta,\phi)\right) = \Gamma^2_{(x)21}\left(x^{-1}(\theta,\phi)\right) := \cot\theta$$

All other Γs vanish. Then, using the sloppy notation (familiar to us from classical mechanics) i.e., $x^1(p) = \theta(p)$ and $x^2(p) = \phi(p)$, the autoparallel equation is

$$\left.\begin{aligned}\ddot{\theta} + {\Gamma^1}_{(x)22}\dot{\phi}\dot{\phi} = 0\\ \ddot{\phi} + 2{\Gamma^2}_{(x)12}\dot{\theta}\dot{\phi} = 0\end{aligned}\right\} \implies \qquad \begin{aligned}\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}\dot{\phi} = 0\\ \ddot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0\end{aligned}$$

It can be seen that the above equations are satisfied at the equator where $\theta(\lambda) = \pi/2$, and $\phi(\lambda) = \omega\lambda + \phi_0$. This means that running around the equator at constant angular speed ω is an autoparallel curve. The autoparallel curves are the straightest curves wrt ∇_{round}. However, $\phi(\lambda) = \omega\lambda^2 + \phi_0$ would not be an autoparallel.

8.3 Torsion and Curvature

Torsion

Now, having understood that a connection on a mfd gives us a notion of autoparallels, we are ready to proceed to another question that at first sound abstract but in fact is very concrete. The question is *"can one use ∇ to define tensors on $(M, \mathcal{O}, \mathcal{A}, \nabla)$?*

We know that ∇ is given by non tensorial objects ($\Gamma's$ do not transform as tensors) but nevertheless there is some tensorial information inside a connection, as we shall see.

Definition 8.3.1. *The **torsion** of a connection ∇ is the $(1,2)$-tensor field*

$$T(\omega, X, Y) :- \omega(\nabla_X Y - \nabla_Y X - [X,Y]) \tag{8.6}$$

where $[X,Y]$, called the commutator of X and Y is a vector field defined by $[X,Y]f := X(Yf) - Y(Xf)$.

Proof. We shall check that T is a tensor field i.e. T is C^∞-linear in each entry.

$$\begin{aligned}
T(f\omega, X, Y) &= f\omega(\nabla_X Y - \nabla_Y(X) - [X,Y]) \\
&= fT(\omega, X, Y) \\
T(\omega+\psi, X, Y) &= (\omega+\psi)(\nabla_X Y - \nabla_Y(X) - [X,Y]) \\
&= T(\omega, X, Y) + T(\psi, X, Y) \\
T(\omega, fX, Y) &= \omega(\nabla_{fX} Y - \nabla_Y(fX) - [fX,Y]) \\
&= \omega(f\nabla_X Y - (\nabla_Y(f))X - f(\nabla_Y X) - [fX,Y]) \\
&= \omega(f\nabla_X Y - (Yf)X - f(\nabla_Y X) - [fX,Y])
\end{aligned}$$

But

$$\begin{aligned}
&[fX,Y]g = fX(Yg) - Y(fX)g = fX(Yg) - (Yf)(Xg) - fY(Xg) \\
\implies &[fX,Y] = f[X,Y] - (Yf)X
\end{aligned}$$

Therefore

$$\begin{aligned}
T(\omega, fX, Y) &= \omega(f\nabla_X Y - (Yf)X - f(\nabla_Y X) - f[X,Y] + (Yf)X) \\
&= \omega(f\nabla_X Y - f(\nabla_Y X) - f[X,Y]) \\
&= f\omega(\nabla_X Y - (\nabla_Y X) - [X,Y]) = fT(\omega, X, Y)
\end{aligned}$$

Further, $T(\omega, X, Y) = -T(\omega, Y, X)$, which means scaling in the last factor need not to be checked separately. Additivity in the last two factors can also be checked. $\square$

Definition 8.3.2. *A mfd $(M, \mathcal{O}, \mathcal{A}, \nabla)$ is called* ***torsion-free*** *if the torsion of its connection is zero. That is, $T = 0$.*

The torsion components wrt a chart are given by

$$T^i_{ab} :- T\left(dx^i, \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) = \left(\nabla_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^b} - \nabla_{\frac{\partial}{\partial x^b}} \frac{\partial}{\partial x^a} - \left[\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right]\right) dx^i \tag{8.7}$$

$$= \Gamma^q{}_{ab} \frac{\partial}{\partial x^q} dx^i - \Gamma^q{}_{ba} \frac{\partial}{\partial x^q} dx^i - \frac{\partial}{\partial x^a}\left(\frac{\partial}{\partial x^b} dx^i\right) + \frac{\partial}{\partial x^b}\left(\frac{\partial}{\partial x^a} dx^i\right) \tag{8.8}$$

$$= 2\Gamma^i{}_{[ab]} - \frac{\partial}{\partial x^a}(\delta^i_b) - \frac{\partial}{\partial x^b}(\delta^i_a) \implies \tag{8.9}$$

$$T^i_{ab} = 2\Gamma^i{}_{[ab]} \tag{8.10}$$

From now on, we only use torsion-free connections.

Curvature

Definition 8.3.3. *The* ***Riemann curvature*** *of a connection ∇ is the $(1,3)$-tensor field*

$$Riem(\omega, Z, X, Y) :- \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \tag{8.11}$$

It can be shown that C^∞-linear in each slot i.e. a tensor field.

The components wrt to a chart (u,x) can be calculated.

$$\begin{aligned}\nabla_X\nabla_Y Z &= \nabla_X((Y^\mu\frac{\partial}{\partial x^\mu}Z^\rho + \Gamma^\rho_{\mu\nu}Z^\mu Y^\nu)\frac{\partial}{\partial x^\rho})\\ &= (X^\alpha\frac{\partial}{\partial x^\alpha}(Y^\mu\frac{\partial}{\partial x^\mu}Z^\rho + \Gamma^\rho_{\mu\nu}Z^\mu Y^\nu) + \Gamma^\rho_{\alpha\beta}(Y^\mu\frac{\partial}{\partial x^\mu}Z^\alpha + \Gamma^\alpha_{\mu\nu}Z^\mu Y^\nu)X^\beta)\frac{\partial}{\partial x^\rho}\end{aligned}$$

For $X = \partial_a$, $Y = \partial_b$, $Z = \partial_j$, then the partial derivatives of the coefficients of the input vectors become zero.

$$\implies \nabla_{\partial_a}\nabla_{\partial_b}\partial_j = \frac{\partial}{\partial x^a}(\Gamma^i_{jb}) + \Gamma^i_{\kappa a}\Gamma^\kappa_{jb}$$

Now

$$[X,Y]^i = X^j\frac{\partial}{\partial x^j}Y^i - Y^j\frac{\partial X^i}{\partial x^j}$$

For coordinate vectors, $[\partial_i,\partial_j] = 0\ \forall\, i,j = 0,1\dots d$.

Thus,

$$R^i{}_{jab} = \frac{\partial}{\partial x^a}\Gamma^i_{jb} - \frac{\partial}{\partial x^b}\Gamma^i_{ja} + \Gamma^i_{\kappa a}\Gamma^\kappa_{jb} - \Gamma^i_{\kappa b}\Gamma^\kappa_{ja}$$

Algebraic relevance of *Riem*:
We ask whether there is difference in applying the two directional derivatives in different order. From the definition of curvature we get

$$\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z = Riem(\cdot, Z, X, Y) + \nabla_{[X,Y]}Z$$

In a chart (U,x) and by using the chart induced basis (denoting $\nabla_{\frac{\partial}{\partial x^a}}$ by ∇_a[4])

$$(\nabla_a\nabla_b Z)^m - (\nabla_b\nabla_a Z)^m = Riem^m{}_{nab}Z^n + \nabla_{\underbrace{\left[\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^b}\right]}_{=0}}Z$$

where the last term vanishes[5]. Therefore in a *coordinate induced chart*, the Riemann tensor components

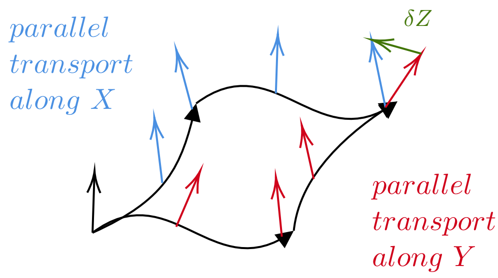
$$Riem^m{}_{nab}Z^n = (\nabla_a\nabla_b Z)^m - (\nabla_b\nabla_a Z)^m \tag{8.12}$$

contain all the information of how the covariant derivatives fail to commute when they act on a vector field. If they act on a tensor field, there are several terms on the rhs like the one term above; if they act on a function, of course they commute. Being a tensor, *Riem* vanishes in all coordinate systems if it vanishes in one coordinate system, as it does in flat spaces.

Geometric significance of *Riem*: If we parallel transport a vector X along two different paths, the resulting vectors at the final point are different in general.

[4]**Dangerous notation** if one changes charts.

[5]The commutator vanishes because the basis is "holonomic". If one chose a more general basis, for example a tetrad basis, then in general the commutator would not vanish.



The difference δZ for very small parameter distances $(\delta\lambda, \delta\sigma)$, if $T = 0$, is given by

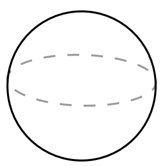
$$(\delta Z)^m = Riem^m{}_{nab} X^a Y^b Z^n + O(\delta\lambda^2\delta\sigma, \delta\lambda\delta\sigma^2)$$

If, however, we parallel transport a vector in a Euclidean space, where the parallel transport is defined in our usual sense, the resulting vector does not depend on the path along which it has been parallel transported. We expect that this non-integrability of parallel transport characterizes the intrinsic notion of curvature, which does not depend on the special coordinates chosen. Hence, the *Riemann tensor control of how much, parallel transport along one path fails to match parallel transport along another path. Whenever we observe this effect, we have detected curvature.* Curvature is just tensorial information contained in a connection. The basis notion is still the connection.

To recapitulate, we first introduced a **covariant derivative**, which induces a notion of ***parallel transport*** (not a tensorial object). However, there is ***still some tensorial information*** in there. . . the *torsion* which we set to zero (because of the later use we are going to make of the whole structure) but, the ***Riemann tensor*** will not be set to zero.

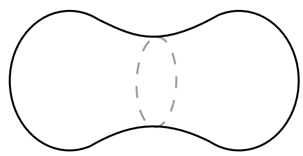
We shall see in the next chapters that,a certain combination of the Riemann tensor including some metric, will be equivalent to the *Stress-Energy tensor* of matter (Einstein's equations) therefore, the *matter* tells us something about the *curvature* of spacetime. Moreover, the curvature tells us something about the underlying *geometric structure*, which is not a connection but a special kind of it. . . a *connection that comes from a metric.* Hence, matter will determine in an indirect manner the underlying connection.

"round sphere"



$(\underbrace{M, \mathcal{O}, \mathcal{A}}_{\text{prescribed by hand}}, \nabla_{round})$

"peanut"



$(M, \mathcal{O}, \mathcal{A}, \nabla_{peanut})$

Which is right ? ∇_{round} or ∇_{peanut} ?
The matter will determine
(via the Einstein equations)

Chapter 9

Newtonian Spacetime is Curved

In the standard formulation newtonian spacetime is not curved of course; and this formulation still holds up. But if we inspect the axioms properly, newtonian spacetime must be considered curved, and in fact this curvature absorbs the effect of a gravitational force, so that no longer gravity can be considered a force. With that in mind we proceed by writing down Newton's axioms of classical mechanics.

Newton I: A body on which *no force* acts moves *uniformly* along a *straight* line.

Newton II: *Deviation* of a body's motion from such uniform straight motion is effected by a *force*, reduced by a factor of the body's reciprocal mass.

We can read both axioms as statements about what a particle does under various circumstances. However, if we read the first axiom as a postulate about what a particle does, it is merely a special case of the second axiom. But it is evident that Newton was not naive. Why would he need to emphasize this special case into an extra axiom. Therefore the nature of the first axiom must be something else. The *idea* is that, *one assumes that no force acts on the particle and then one checks what a straight line is i.e. one is testing the geometry of space(time).* Therefore we conclude that:

(1) In order for the first axiom to be relevant, it must read as a measurement prescription for the geometry of space.

*(2) Since gravity, universally acts on every particle, in a universe with at least 2 particles, gratity must not be considered a force if **Newton I** is supposed to remain applicable.*

The first to think about this was Laplace...

Laplace's Question

Question: Can gravity be encoded in a curvature of space, such that its effects show if particles under the influence of (no other) force we postulated to more along straight lines in this curved space?

The answer is no. We are going to see why because it is instructive to see what does not work. That way can repair it.

Proof. By describing gravity as a force one has to consider that the force in Newton's second law , $F^\alpha = m\ddot{x^\alpha}(t)$ is equal with the force in his law of gravitation $F^\alpha = -\left(\frac{Gm_i}{r^2}\right)^\alpha m \equiv f^\alpha m$

$$m\ddot{x}^\alpha(t) = \underbrace{mf^\alpha}_{force:F^\alpha}(x(t))$$

where $-\partial_\alpha f^\alpha = 4\pi G\rho$ (Poisson); $\rho =$ mass density of matter.
The same m appearing on both sides of the equation is an *experimental fact*, also known as the ***weak equivalence principle***. Therefore,

$$\ddot{x}^\alpha(t) - f^\alpha(x(t)) = 0\,. \tag{9.1}$$

Laplace asks: Is this last equation of the form $\ddot{x}^\alpha(t) + \Gamma^\alpha{}_{\beta\gamma}(x(t))\dot{x}^\beta(t)\dot{x}^\gamma(t) = 0$? That is, is it possible to take the form of autoparallel equation?

The answer is no. The Γ can only depend on the point x and if we assume that

$$f^\alpha = \Gamma^\alpha{}_{\beta\gamma}(x(t))\dot{x}^\beta(t)\dot{x}^\gamma(t)$$

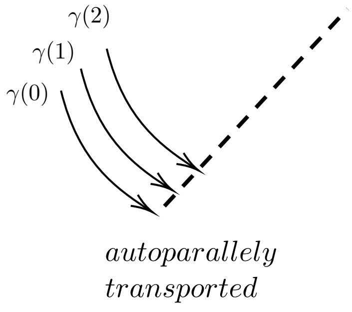
then we will run into a problem because the last equation implies

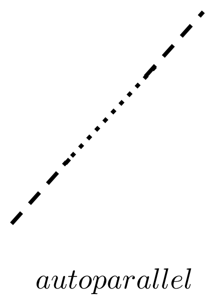
$$\Gamma^\alpha{}_{\beta\gamma}(x(t)) = \frac{f^\alpha}{\dot{x}^\beta(t)\dot{x}^\gamma(t)} \Longrightarrow \Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma}(x(t), \dot{x}(t))$$

which against the definition of Γs. Therefore the Γs cannot take care of the f^α in the preceding equation. Had there been such Γs, we would be able to find the notion of straight line that could have absorbed the effect we usually attribute to a force. Conclusively, one cannot find Γs such that Newton's equation takes the form of an autoparallel equation. □

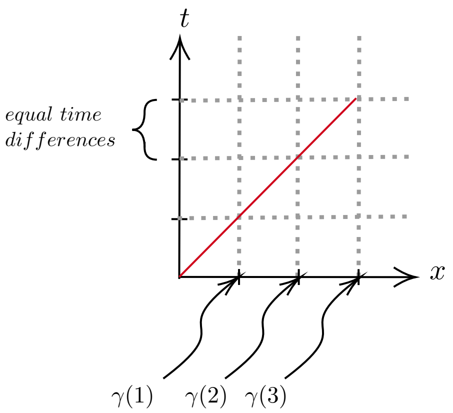
9.1 The Full Wisdom of Newton I

What Laplace basically asked is: Can we find a curvature of space st particles move along straight lines. He did not read *Newton I* properly. *Newton I* talks about straight lines; but he also talks about *uniform* motion and Laplace did not take this into account. Therefore, one must use this extra information too. Recall that

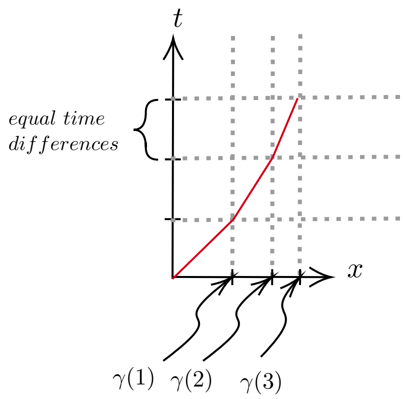




the curve on the left represents uniform and straight motion while the curve on the right depicts just straight motion, not uniform. This is an example which shows that *a curve is more that the set of its points*; ***it is the set of its points and its parametrization***, which makes the whole difference. In order to talk about this difference easily we have to introduce an appropriate setting. The appropriate setting would be one in which we do not have to remember the parametrization of the curves, unlike the prevously displayed images. To achieve that, we attach a *time* axis and we store the data concerning the parametrization there. Consider a curve γ in a 1-dimensional world, whose dimension we are going to denote by x. The curve will of course lie on this x-axis. A graphical illustration of uniform and non-uniform motion is given by the following picture.



$uniform\ motion$



$non - uniform\ motion$
$(getting\ slower)$

Now all the information about the parametrization of the curve is stored in the t-axis. We observe that, ***uniform and straight motion in space is simply straight motion in spacetime***. This gives us a *hint*. *Laplace's idea might work*, if you only want to implement *straight lines, but* it will work in *space-time*. In Newtonian spacetime particularly. This has **nothing to do with *relativity*** it is just representing the data differently.

So lets try the same idea in spacetime.

$Let\ x : \mathbb{R} \to \mathbb{R}^3\ be\ a\ particle's\ trajectory\ in\ space$ $\longleftrightarrow$ $The\ worldline\ (history)\ of\ the\ particle\ is\ defined\ as$

$$\begin{aligned} X : \mathbb{R} &\to \mathbb{R}^4 \\ t &\mapsto \big(t, x^1(t), x^2(t), x^3(t)\big) \\ &\qquad =: \big(X^0, X^1(t), X^2(t), X^3(t)\big) \end{aligned}$$

where the Xs are called *worldline coordinates*. Now we have stored the information regarding *uniformity*, namely the parametrization of the curve, into the X^0 coordinate. Let's see why this works.

In section (**??**) we saw that in space (e.g. little xs) one cannot write the equation $\ddot{x}^\alpha(t) -$

$f^\alpha(x(t)) = 0$ into the form of an autoparallel. We simply rewrite, using the new notation

$$\begin{aligned} X^0(t) = t \Rightarrow \dot{X}^0(t) = 1 \\ \Rightarrow \ddot{X}^0(t) = 0 \end{aligned} \tag{9.2}$$

and now we assume that the equivalence principle is still true i.e.

$$\ddot{x}^\alpha(t) - f^\alpha(x(t)) = 0$$

only for $\alpha = 1, 2, 3$. By multiplying with $\dot{X}^0(t) = 1$ we get

$$\ddot{x}^\alpha(t) - f^\alpha(x(t))\dot{X}^0(t)\dot{X}^0(t) = 0 \Longleftrightarrow \ddot{X}^\alpha(t) - f^\alpha(X(t))\dot{X}^0(t)\dot{X}^0(t) = 0\,. \tag{9.3}$$

Note that $f^\alpha(X(t))$ is just bad notation since it is a different function than the previously defined f^α in the sence that it ignores its first entry, namely the X^0 coordinate. We take (9.2) and (9.3)

$$\begin{aligned} \ddot{X}^0(t) = 0 \\ \ddot{X}^\alpha(t) - f^\alpha(X(t))\dot{X}^0(t)\dot{X}^0(t) = 0 \end{aligned}$$

and we observe that together, they are of the form of an autoparallel

$$\ddot{X}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{X}^\beta\dot{X}^\gamma = 0.$$

We simply choose $\Gamma^\alpha{}_{00} = -f^\alpha$ and all others to vanish. But now one may wonder if this is a coordinate artifact i.e. if we can make go to a chart where the connection coefficient functions vanish altogether, using (7.17). The answer is no since we can calculate a non-zero Riemann curvature $R^\alpha{}_{0\beta0} = -\frac{\partial f^\alpha}{\partial x^\beta}$[1] (only non-vanishing components) but also

$$R_{00} = R^m{}_{0m0} = -\partial_\alpha f^\alpha = 4\pi G\rho \tag{9.4}$$

where we used the Poisson's law. Since the *curvature being non-zero is independent of the coordinates*, the Γs cannot be removed by a coordinate transformation. Hence, we have managed to brought (9.1) into the form of an autoparallel. Conclusively, *Laplace's idea works in spacetime despite the fact it does not in space.* This becomes obvious when one checks that $R^\alpha_{\beta\gamma\delta} = 0$ for $\alpha, \beta, \gamma, \delta = 1, 2, 3$ meaning that the curvature components regarding only spatial dimensions vanish. That is roughly speaking, the "space curvature" is zero. On the other hand equation (9.4) shows that "spacetime curvature" is not.

Moreover, note that neither Γ nor $\ddot{X}$ transform as tensors in such a way that the autoparallel equation as a whole does. It is the ***acceleration vector components***

$$\underbrace{\ddot{X}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{X}^\beta\dot{X}^\gamma = 0}_{\underbrace{(\nabla_{u_x}u_x)^\alpha}_{a^\alpha:-\text{acceleration}}}$$

9.2 The foundations of the geometric formulation of Newton's axiom

Definition 9.2.1. *A **Newtonian spacetime** is a quintuple $(M, \mathcal{O}, \mathcal{A}, \nabla, t)$ where $(M, \mathcal{O}, \mathcal{A})$ is a 4-dimensional smooth manifold, and $t : M \longrightarrow \mathbb{R}$ a smooth function on the mfd satisfying:*

[1] This is known as the *tidal force tensor*, it's minus the Hessian of the gravitational potential

(1) *"There is an absolute space"* $\quad (dt)_p \neq 0 \qquad \forall\, p \in M$

(2) *"Absolute time flows uniformly" i.e.*

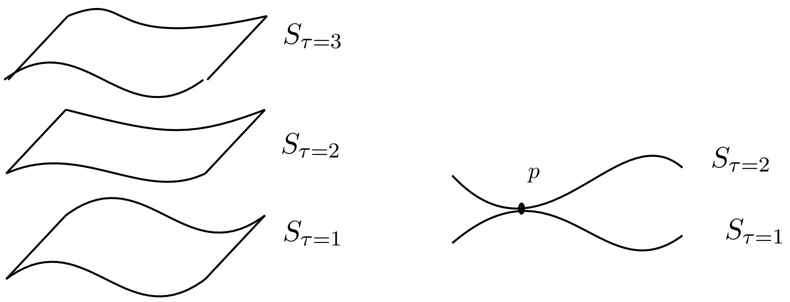
$$\underbrace{\nabla dt}_{\text{(0,2)-tensor field}} = 0 \qquad \textit{everywhere}$$

(3) *The connection is* $\nabla = 0$ *torsion free*

Definition 9.2.2. *The* ***absolute space*** *at a time* τ *is defined as*

$$S_\tau :- \{p \in M | t(p) = \tau\} \ .$$

Remark 11. *From the condition* $dt \neq 0$ *follows that* $M = \dot{\bigcup} S_\tau$ *which means that spacetime can be decomposed into purely spatial slices* S_{τ_i}*.*



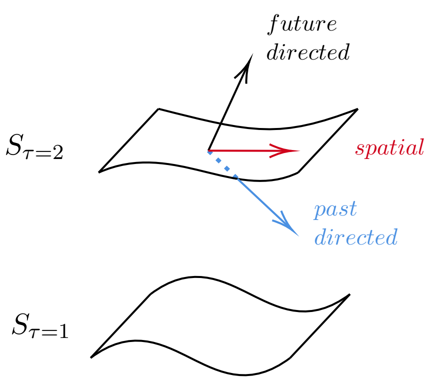
$dt \neq 0 \qquad\qquad (dt)_p = 0$

Definition 9.2.3. *A vector* $X \in T_pM$ *is called*

(a) ***future-directed****, if* $dt(X) > 0$

(b) ***spatial****, if* $dt(X) = 0$

(c) ***past-directed****, if* $dt(X) < 0$



Now we can rewrite **Newton I** using what we learnt.

Newton I: The ***worldline*** of a particle under the influence of no force (gravity is not) is a ***future directed autoparallel*** (wrt to the connection we have). That is it satisfies

$$\nabla_{u_x} u_x = 0 \quad , \quad dt(u_x) > 0$$

Newton II: The *deviation* from such motion is effected by a *force* i.e.

$$\nabla_{u_x} u_x = \frac{F}{m}$$

where F is a spatial vector field $dt(F) = 0$[2]. The last, in term of components, is rewriten as

$$m a^\alpha = F^\alpha \qquad\qquad , \text{where} \quad a^\alpha = \ddot{X}^\alpha + \Gamma^\alpha{}_{\beta\gamma} \dot{X}^\beta \dot{X}^\gamma$$

Hence we did not change Newton's law, we just generalized the notion acceleration to incorporate what we call "gravity".

At this point, let us introduce a very practical **convention**. We restrict attention to atlases $\mathcal{A}_{stratified}$ whose charts (u, x) have the property

$$\left.\begin{array}{c} x^0 : u \to \mathbb{R} \\ x^1 : u \to \mathbb{R} \\ \vdots \\ x^i : u \to \mathbb{R} \end{array}\right\} \longrightarrow x^0 = t|_u \text{ therefore } \nabla t = 0 \to \left(\nabla_{\frac{\partial}{\partial x^\alpha}} dx^0\right)_\beta = -\Gamma^0{}_{\beta\alpha} \text{ in this stratified atlas.}$$

Let's evaluate **Newton II** in a chart (U, x) of a stratified atlas $\mathcal{A}_{sheet}$.

$$\nabla_{v_X} v_X = \frac{F}{m} \quad \text{gives the equations}$$

$$\Rightarrow (X^0)'' + \Gamma^0{}_{\cancel{\alpha\beta}}\cancel{(X^\alpha)'(X^\beta)'}^{\text{stratified atlas}} = 0 \ \text{ and}$$

$$\Rightarrow (X^\alpha)'' + \Gamma^\alpha{}_{\gamma\delta} X^{\gamma'} X^{\delta'} + \Gamma^\alpha{}_{00} X^{0'} X^{0'} + 2\Gamma^\alpha{}_{\gamma 0} X^{\gamma'} X^{0'} = \frac{F^\alpha}{m} \qquad \alpha, \gamma, \delta = 1, 2, 3$$

The solution of the first equation gives

$$(X^0)''(\lambda) = 0 \Longrightarrow X^0(\lambda) = a\lambda + b$$

where a, b constants. But in a stratified atlas

$$X^0(\lambda) :- (x^0 \circ X)(\lambda) \overset{\text{stratified}}{=} (t \circ X)(\lambda)$$

meaning the time of a point along a curve runs proportionaly with the λ parameter of the worldline curve X. This enables us to parametrize the worldline by absolute time, if we wish so since, $\frac{d}{d\lambda} = a \frac{d}{dt}$. Then the second equation from **Newton II** takes the form

$$\underbrace{\ddot{X}^\alpha + \Gamma^\alpha{}_{\gamma\delta} \dot{X}^\gamma \dot{X}^\delta + \Gamma^\alpha{}_{00} \dot{X}^0 \dot{X}^0 + 2\Gamma^\alpha{}_{\gamma 0} \dot{X}^\gamma \dot{X}^0}_{a^\alpha - acceleration} = \frac{1}{a^2} \frac{F^\alpha}{m} \, .$$

[2]Since a force is suppose to accelerate a body only in spatial dimensions.

We saw that in the presence of gravity ${\Gamma^\alpha}_{00} = -f^\alpha$ =*gravitational force* thus they correspond to the acceleration due to gravity and as we also saw, they cannot be transformed away. If there is no gravity then we can find a suitable chart (section (**??**)) where all Γs vanish. The components ${\Gamma^\alpha}_{\beta\gamma}$, containing only spatial indices, are non-zero when we choose a curvilinear coordinate system (e.g. polar coordinates). On the other hand, in a rotating system ${\Gamma^\alpha}_{00}$ will be the *centrifugal pseudo-acceleration* and ${\Gamma^\alpha}_{\gamma 0}$ will be the *coriolis pseudo-accelaration.* Why do we call them *pseudo-accelerations*?…Because they are coordinate artifacts, not real accelerations. Only their sum is.

Chapter 10

Metric Manifolds

We would like to establish a structure on a smooth manifold that allows one to assign vectors in each tangent space a length (and an angle between vectors in the same tangent space). A geometric structure on a vector space that induces a *notion of both* ***angle and length*** is an ***inner product*** (if we only wanted length we would establish a *norm*). However, we do not want a inner product on just one tangent space, we want it on *each tangent space*. That's an *inner product field* if you like. An inner product is basically a $(0,2)-$tensor with certain properties. That's the so called ***metric***

From this structure, one can then define a ***notion of length of a curve***. Because if we know how to measure the velocity to a curve, that is to make the velocity into a number (section (5)), we can call that number "speed". By integrating speed over the parameter distance we will get the "distance". Then we can look at ***shortest curves*** (which will be called **geodesics**).

In flat space it is evident that the shortest curves are always the straight curves. We know how to talk about straight curves if we have a covariant derivative. But if we now introduce such a structure (which will be called a metric), we will get from it the notion of shortest curves.

If we establish fully independently the two structures (connection and metric) then the shortest curves won't be necessarily the straight curves. However, requiring such condition i.e. the shortest curves (wrt metric) coincide with the straight curves (wrt ∇), will result in ∇ being determined by the metric structure[1]. Hence the metric will further determine the curvature.

$$metric \xrightarrow[T=0]{\substack{shortest \\ = \\ straightest}} \nabla \longrightarrow Riem$$

[1]That is true only in the presence of zero torsion and is called "metric compatibolity" condition

10.1 Metrics

Definition 10.1.1. *A* ***metric*** *g on a smooth manifold $(M, \mathcal{O}, \mathcal{A})$ is a $(0,2)$-tensor field satisfying*

(i) ***symmetry:*** *$g(X,Y) = g(Y,X) \quad \forall X, Y$ vector fields*

(ii) ***non-degeneracy:*** *that is, the* musical map *(also called "flat")*

$$\begin{aligned} \flat : \Gamma(TM) &\longrightarrow \Gamma(T^*M) \\ X &\mapsto \flat(X)\,, \text{ where } \underbrace{\flat(X)}_{\in \Gamma(T^*M)}(Y) := g(X,Y) \end{aligned}$$

must be a $C^\infty-$isomorphism, in other words it is invertible.

One can think the musical map as $\flat(X) :- g(X, \cdot)$. Note that by virtue of the metric, now we have a way of converting a vector field into a covector field.

Definition 10.1.2. *The $(2,0)$-tensor field $g^{\text{“}-1\text{”}}$ with respect to a metric g is the symmetric*

$$\begin{aligned} g^{\text{“}-1\text{”}} : \Gamma(T^*M) \times \Gamma(T^*M) &\xrightarrow{\sim} C^\infty(M) \\ (\omega, \sigma) &\longmapsto \omega(\flat^{-1}(\sigma)) \quad , \quad \flat^{-1}(\sigma) \in \Gamma(TM)) \end{aligned}$$

In a chart the components of the, so called, inverse metric satisfy $(g^{-1})^{am} g_{mb} = \delta^a_b$.

We call g^{-1} "inverse metric" due to the condition its components satisfy in a chart. However, stricktly speaking a rigorous definition of an inverse metric should include a mapping like $C^\infty(M) \xrightarrow{\sim} \Gamma(TM) \times \Gamma(TM)$ which not the case.

Remark 12. *As previously stated, due the the metric we now have a means of converting vectors field into covector field and vice versa. In a chart*

$$(\flat(X))_\alpha :- g_{\alpha\mu} X^\mu \quad , \quad \left(\flat^{-1}(\omega)\right)^\alpha :- (g^{-1})^{\alpha\mu} \omega_\mu \,.$$

These operations are known as "lowering/raising the index" by applying the metric. In the literature one is probably going to encounter the above relations in the form

$$X_\alpha = g_{\alpha\mu} X^\mu \quad , \quad \omega^\alpha = g^{\alpha\mu} \omega_\mu$$

which is a dangerous notation because it does not show if, for example, the object X^α is an original covector or a covector constructed from a vector by a metric.

Example 10.1.1. *Consider $(S^2, \mathcal{O}, \mathcal{A})$ and the chart (u, x)*

$$\varphi \in (0, 2\pi), \qquad \theta \in (0, \pi)$$

Then

$$g_{ij}(x^{-1}(\theta, \varphi)) = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{bmatrix}_{ij}$$

where $R \in \mathbb{R}^+$ is a constant, defines the metric of the round sphere of radius R. Recall that at the level $(S^2, \mathcal{O}, \mathcal{A}, \nabla)$ the round spheres (as mfds with connection) had a fixed shape , depending on the connection, but not a fixed size. As mfds with a metric the round spheres have eveything fixed, intuitively speaking. Moreover note that one could establish a different metric on the same smooth space $(S^2, \mathcal{O}, \mathcal{A})$ and provide it with a completely different shape.

Signature

Recall from linear algebra the eigenvalue problem

$$Av = \lambda v \longrightarrow A^{\alpha}_{\ \mu} v^{\mu} = \lambda v^{\alpha}$$

where A is a (1,1) tensor. The eigenvalue form of A is

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

The same problem is not well defined for $(0,2)$ tensor .

$$g_{\alpha\mu} v^{\mu} \stackrel{?}{=} \lambda v^{\alpha}$$

Notice that on the lhs we have a covector whereas on the rhs just a vector. An eigevalue is just not a well defined notion on a $(0,2)$ tensor. Eigenvalues and eigenvectors are meaningful for mixed tensors, but the metric is a symmetric bilinear form that must be expressed by a purely co- or contra-variant tensor. In matrix notation, mixed tensors, under a change of basis, would transform as $\tilde{M} = PMP^{-1}$, but symmetric bilinear forms as $\tilde{S} = PSP^{T}$. Under the former, eigenvalues are invariant; but under the latter, only the numbers of positive, negative, and null eigenvalues are invariant. The closest to an eigenvalue form one can bring the metric to is

$$\begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & -1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & -1 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix}.$$

However, there is still something that is well defined and invariant under a change of basis, on such a tensor and it is called a *signature*. Thus,

- A (1,1) tensor has eigenvalues.
- A (0,2) tensor has signature (p,q) where $p :- \# of\, 1$ and $q :- \# of\, -1$. The number of the remaining zeros is of course $\dim V - p - q$.
- The condition that $\flat$ is an isomorphism means that there will be no zeros, hence $\dim V = p + q$

Example 10.1.2. *In three dimensions there are* $\left.\begin{matrix} (+++) \\ (++-) \\ (+--) \\ (---) \end{matrix}\right\}$ $d+1$ *possible signatures.*

Definition 10.1.3. *A metric is called* ***Riemannian*** *if its signature is* $(++\cdots+)$ *or* $(--\cdots-)$*, and* ***Lorentzian*** *if it is* $(+-\cdots-)$ *or* $(-+\cdots-)$ *etc.. Every signature other than Riemannian is called* ***Pseudo-Riemannian****.*

Due to their signature, only the Riemannian metrics define in each tangent space an inner product of which one property is that $g(X,X)=0$ iff $X=0$ i.e. *every non zero vector has a non zero length.* This is not the case for Pseudo-Riemannian metrics.

Now that we have defined our metric we finally can talk about *length* of curves.

10.2 Length of a curve

Let γ be a smooth curve. Then we know its veloctiy $v_{\gamma,\gamma(\lambda)}$ at each point $\gamma(\lambda)\in M$.

Definition 10.2.1. *On a Riemannian metric manifold* $(M,\mathcal{O},\mathcal{A},g)$*, the* ***speed*** *of a curve at* $\gamma(\lambda)$ *is the number*

$$s(\lambda)=\left(\sqrt{g(v_\gamma,v_\gamma)}\right)_{\gamma(\lambda)} \tag{10.1}$$

The speed is defined only in a metric mfd and not just in a smooth mfd like velocity. Thus, velocity is a more fundamental object that speed. Furthermore, notice that you do not see a connection ∇ anywhere. That is because we do not have to imply any relation between shortest and straight curves; that is an additional assumption.

Remark 13. *The physical dimensions of the velocity vector components are* $[v^a]=\frac{1}{T}$ *i.e. they are not "speeds" in different directions. They are defined in an arbitrary chart in which the notion of distance is not established if a metric is not introduced. Therefore, in a sense the physical dimensions of the metric components are* $[g_{\alpha\beta}]=L^2$*. Only then we observe that* $[\sqrt{g_{\alpha\beta}v^\alpha v^\beta}]=\sqrt{\frac{L^2}{T^2}}=\frac{L}{T}$ *which reconciles us with the classical notion of speed. The trick is that* v^i*s transform as vector components,* $g_{\alpha\beta}$ *as (0,2) tensor components but they cancel out and the final result is invariant. Furthermore* $[g_{\alpha\beta}]=L^2$ *shows that the* ***metric carries the information on how to translate coordinate distances into real length****. Hence, without a metric, coordinate distance has nothing to do with real distance.*

Definition 10.2.2. *Let* $\gamma:(0,1)\longrightarrow M$ *a smooth curve. Then the* ***length of*** γ*,* $L[\gamma]\in\mathbb{R}$ *is the number*

$$L[\gamma]:=\int_0^1 d\lambda\, s(\lambda)=\int_0^1 d\lambda\sqrt{(g(u_\gamma,u_\gamma))_{\gamma(\lambda)}}\,. \tag{10.2}$$

Due to its definition Λ *is called a functional. It maps a function to a number.*

Given the above definition we realise that **velocity is more fundamental than speed and speed is more fundamental that length**.

Example 10.2.1. *Reconsider the round sphere of radius R. Consider its equator. In a chart* (u,x) *the chart representatives (the "first" and the "second" coordinate) of* γ *are*

$$\theta(\lambda):=(x^1\circ\gamma)(\lambda)=\frac{\pi}{2}\,,\quad \varphi(\lambda):=(x^2\circ\gamma)(\lambda)=2\pi\lambda^3$$
$$\textit{hence,}\quad \theta'(\lambda)=0\,,\quad \varphi'(\lambda)=6\pi\lambda^2$$

where we have parametrize in a non trivial way on purpose. After all we can parametrize however we want. On the same chart $g_{ij} = \begin{bmatrix} R^2 & \\ & R^2\sin^2\theta \end{bmatrix}$

The length functional is given by

$$
\begin{aligned}
L[\gamma] &= \int_0^1 d\lambda \sqrt{g_{ij}(x^{-1}(\theta(\lambda), \varphi(\lambda)))(x^i \circ \gamma)'(\lambda)(x^j \circ \gamma)'(\lambda)} \\
&= \int_0^1 d\lambda \sqrt{R^2 \cdot 0 + R^2 \sin^2(\theta(\lambda)) 36\pi^2\lambda^4} \\
&= 6\pi R \int_0^1 d\lambda\lambda^2 = 6\pi R[\frac{1}{3}\lambda^3]_0^1 = 2\pi R
\end{aligned}
$$

where along the way of the calculation, some "funny" factors appeared due to the weird parametrization but in the end the correct result was produced. In this example it becomes clear that the parametrization does not matter.

Theorem 10.2.1. *Let* $\gamma : (0,1) \longrightarrow M$ *be a smooth curve and* $\sigma : (0,1) \longrightarrow (0,1)$ *be a smooth bijective and* increasing *map.*[2] *The length functional is independent of the parametrization i.e.*

$$L[\gamma] = L[\gamma \circ \sigma]$$

Now that we have defined the length of any smooth curve on a mfd, it would be interesting to look at *extremal curves.*

10.3 Geodesics

Definition 10.3.1. *A curve* $\gamma : (0,1) \longrightarrow M$ *is called a* ***geodesic*** *on a Riemannian manifold* $(M, \mathcal{O}, \mathcal{A}, g)$ *if it is a* ***stationary*** *curve with respect to a length functional* L*.*

Theorem 10.3.1. *A curve* γ *is geodesic iff it satisfies the Euler-Lagrange equations for the Lagrangian*

The Lagrangian is just a function on the tangent bundle i.e.

$$
\begin{aligned}
\mathcal{L} : TM &\longrightarrow \mathbb{R} \\
X &\mapsto \sqrt{g(X,X)}
\end{aligned}
$$

In a chart, the Euler Lagrange equations[3] take the form

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right)^{\cdot} - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0\,.$$

Here the chart representative of $\mathcal{L}$ is

$$\mathcal{L}(\gamma^\alpha, \dot{\gamma}^\alpha) = \sqrt{g_{\alpha\xi}(\gamma(\lambda))\dot{\gamma}^\alpha(\lambda)\dot{\gamma}^\xi(\lambda)}\,.$$

[2] Basically it is a "reparametrization" map. That why it has to be increasing. Otherwise the new parameter would run backwards and the value of length wrt it would change.

[3] They are just a reformulation of the stationarity condition.

Apart from the square root, this is what we would write in classical mechanics. Here, the square root is important so as to make the Lagrangian parametrization invariant. In classical mechanics we do not want this invariance because we want uniform parameter. The difference is that we are doing classical mechanics, we are doing geometry. Having said that, now it is important to extract the Euler-Lagrange equations. Therefore we calculate

$$\frac{\partial \mathcal{L}}{\partial \dot{\gamma}^\mu} = \frac{1}{\sqrt{\cdots}} g_{\mu\xi}(\gamma(\lambda))\dot{\gamma}^\xi(\lambda)$$

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{\gamma}^\mu}\right)^{\cdot} = \left(\frac{1}{\sqrt{\cdots}}\right)^{\cdot} g_{\mu\xi}(\gamma(\lambda)) \cdot \dot{\gamma}^\xi(\lambda) + \frac{1}{\sqrt{\cdots}}\Big(g_{\mu\xi}(\gamma(\lambda))\ddot{\gamma}^\xi(\lambda) + \dot{\gamma}^\sigma(\partial_\sigma g_{\mu\xi})\dot{\gamma}^\xi(\lambda)\Big)$$

where we have used the chain rule in the last term[4]. Now what remains is to calculate the derivative $\left(\frac{1}{\sqrt{\cdots}}\right)^{\cdot}$. However, we saw that the length functional does not depend on the parametrization. Thus we can impose a suitable condition on our parameter to make the calculations simpler. The condition we require is

$$g(\dot{\gamma}, \dot{\gamma}) = 1$$

i.e we choose a parameter such that at every point the velocity is equal to unity. Hence $\sqrt{g(\dot{\gamma}, \dot{\gamma})} = 1$ and $\left(\sqrt{g(\dot{\gamma}, \dot{\gamma})}\right)^{\cdot} = 0$ for any choice of our λ parameter. Moreover we calculate

$$\frac{\partial \mathcal{L}}{\partial \gamma^\mu} = \frac{1}{2\sqrt{\cdots}} \partial_\mu g_{\alpha\xi}(\gamma(\lambda))\dot{\gamma}^\alpha(\lambda)\dot{\gamma}^\xi(\lambda)$$

and by putting of these together, the E-L equations will give

$$\begin{aligned}
g_{\mu\xi}\ddot{\gamma}^\xi + \partial_\sigma g_{\mu\xi}\dot{\gamma}^\sigma\dot{\gamma}^\xi - \frac{1}{2}\partial_\mu g_{\alpha\xi}\dot{\gamma}^\alpha\dot{\gamma}^\xi &= 0 \;\; || \cdot (g^{-1})^{\rho\mu} \\
\ddot{\gamma}^\rho + (g^{-1})^{\rho\mu}(\partial_\alpha g_{\mu\xi} - \frac{1}{2}\partial_\mu g_{\alpha\xi})\dot{\gamma}^\alpha\dot{\gamma}^\xi &= 0 \\
\ddot{\gamma}^\rho + (g^{-1})^{\rho\mu}(\partial_\alpha g_{\mu\xi} - \frac{1}{2}\partial_\mu g_{\alpha\xi})\dot{\gamma}^{(\alpha}\dot{\gamma}^{\xi)} &= 0 \\
\ddot{\gamma}^\rho + (g^{-1})^{\rho\mu}\frac{1}{2}(\partial_\alpha g_{\mu\xi} + \partial_\xi g_{\mu\alpha} - \partial_\mu g_{\alpha\xi})\dot{\gamma}^\alpha\dot{\gamma}^\xi &= 0
\end{aligned}$$

which is the ***geodesic equation for a curve*** γ in a chart. We observe that it is equivalent with equation (8.5) if we choose

$$(g^{-1})^{\rho\mu}\frac{1}{2}(\partial_\alpha g_{\mu\xi} + \partial_\xi g_{\mu\alpha} - \partial_\mu g_{\alpha\xi}) =: {}^{\text{L.C.}}\Gamma^\rho{}_{\alpha\xi}(\gamma(\lambda))\,.$$

The choice of such connection coefficient functions implies that shortest curves wrt the metric, coincide with straight curves wrt the connection.

Note: We wrote the E-L equations,in a chart, in the standard notation as we do in classical mechanics. However, because the Lagrangian is a function on the tangent bundle , a more mathematically rigorous way to write down the E-L equations would be

$$\left(\frac{\partial \mathcal{L}}{\partial \xi_x^{a+\dim M}}\right)^{\cdot}_{\sigma(x)} - \left(\frac{\partial \mathcal{L}}{\partial \xi_x^a}\right)_{\sigma(x)} = 0$$

where ξ_x is a chart of the atlas on the tangent bundle TM which was constructed from the atlas $\mathcal{A}$ on the underlying mfd M (see §6.1). $\sigma(x)$ is the curve $\gamma(\lambda)$ also lifted on the tangent

[4] $\dot{g_{\mu\lambda}}(\gamma(\lambda)) = \dot{\gamma}^\sigma(\partial_\sigma g_{\mu\lambda})$

bundle TM. That would be the precise formulation where the $\mathcal{L}$ is a function on TM. This also gives a hint on why, in the standard Lagrangian formulation, we treat $\dot{\gamma}$ as being independent of γ. They are not but we treat them this way because in the preceding relation we plug in the curves σ only after the derivation has been done. The final result would of course be the same.

Definition 10.3.2. ${}^{L.C.}\Gamma$ *are the connection coefficient functions of the so-called Levi-Civita connection* ${}^{L.C.}\nabla$. *They are also called* ***Christoffel symbols****.*

We usually make this choice of ∇ if g is given.

$$(M, \mathcal{O}, \mathcal{A}, g) \longrightarrow (M, \mathcal{O}, \mathcal{A}, g, {}^{\text{L.C.}}\nabla)$$

It is just a particular choice of a connection. Namely a connection which ***identifies the goedesics with the autoparallels***.

Definition 10.3.3. *(a) The* ***Riemann-Christoffel curvature*** *is defined by*

$$R_{\alpha\beta\gamma\delta} := g_{\alpha\mu} R^{\mu}{}_{\beta\gamma\delta} \tag{10.3}$$

while the

(b) ***Ricci*** *curvature is given by*

$$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta} \tag{10.4}$$

and last but not least the

(c) ***(Ricci) scalar*** *curvature is*

$$R = g^{\alpha\beta} R_{\alpha\beta} \tag{10.5}$$

where we have adopted the heavily used ***notation***, $\boldsymbol{g^{\alpha\beta} :- (g^{-1})^{\alpha\beta}}$.

Definition 10.3.4. *The* ***Einstein curvature*** *of* $(M, \mathcal{O}, \mathcal{A}, g)$ *is defined as*

$$G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R\,. \tag{10.6}$$

Chapter 11

Symmetry

In this chapter we are going to talk about symmetry but we will pick a number of elementary techniques in differential geometry that we will need in Einstein's theory. We shall motivate these techniques by appealing to the feeling that the round sphere $(S^2, \mathcal{O}, \mathcal{A}, g^{\text{round}})$ has rotational symmetry, while the potato $(S^2, \mathcal{O}, \mathcal{A}, g^{\text{potato}})$ does not.

All this seems simple enough however, it is quite different from the ideas of symmetry we typically consider. So far you have probably considered rotational symmetry as an $SO(3)$ group. Meaning that, so far we have considered symmetry by having established an inner product first, and then demanding that wrt that inner product we can classify linear maps A acting on vectors X and Y such that inner product of AX and AY results in inner product XY. Therefore these As are linear transformations in a vector space that respect the inner product.

Here we talk about something different altogether nevertheless, it is the same idea. First of all, we must realize that the distinction between $(S^2, \mathcal{O}, \mathcal{A}, g^{\text{round}})$ and $(S^2, \mathcal{O}, \mathcal{A}, g^{\text{potato}})$ is entirely contained in g. Up to the level $(S^2, \mathcal{O}, \mathcal{A})$ there is no metric thus there is no inner product. Secondly, once we have a metric, since it is the only thing that distinguishes these two objects, we talk about *a symmetry of the metric itself* and not, given a metric we have additional transformations that respect something. Hence *symmetry is a property of the metric.* One more difference is that a metric provides an inner product on each tangent space and; since there are many different tangent spaces with many different inner products, one could use them to redefine for example $SO(3)$ (or any other group) with respect to some of those inner products.

Recall that g talks about the *distribution of these inner products over the mfd* (sphere in this case). That distribution is, in some sense, rotationally invariant or not.

Therefore, the question is: *How to describe the symmetries of a metric?* This is important because nobody has solved Einstein's Equations without assuming some sort of additional assumptions such as symmetry of the solution. Hence this is not an "academic" question but a very important one in a technical sence.

In order to begin talking about these symmetries there are a number of technical concepts that have to be introduced.

11.1 Push-Forward and Pull-Back Map

Push-Forward

Definition 11.1.1. *Let M and N be smooth manifolds with tangent bundles TM and TN respectively. Let $\phi : M \longrightarrow N$ be a smooth map. Then, the* ***push-forward map*** *ϕ_* (induced from ϕ) is the map*

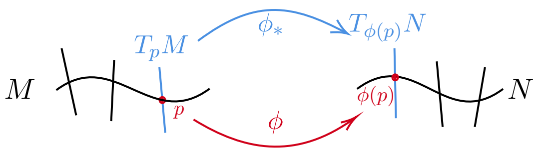
$$\begin{tikzcd} TM \arrow[r, "\phi_*"] \arrow[d, "\pi_{TM}"'] & TN \arrow[d, "\pi_{TN}"] & \\ M \arrow[r, "\phi"] & N \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

defined by $X \mapsto \phi_*(X) \in TN$ where for any function $f \in C^\infty(N)$: $\phi_*(X)f :- X(f \circ \phi)$

Notice that the push forward map ϕ_* takes a vector $X \in T_pM$ in the tangent space at the point $p \in M$ to the vector $\phi_*(X) \in T_qN$ in the tangent space at the point $\phi(p) = q \in N$, such that the action of $\phi_*(X)$ on any smooth function $f \in C^\infty(N)$ gives the same result as the action of X on the function $(f \circ \phi)$.

Note: From this construction we can see that if we apply the push forward to an entire fiber T_pM, over the point p then, the result lies inside the fiber $T_{\phi(p)}N$. That is

$$\phi_*(T_pM) \subseteq T_{\phi(p)} \,.$$



In order to remember what the push forward does there is a little mnemonic that says

"vectors are pushed forward".

Having defined the map abstractly we can now find its **components wrt *two* charts** (since the whole construction involves two separate mfds). Consider $(u, x) \in \mathcal{A}_M$ and $(v, y) \in \mathcal{A}_N$. We know that $\frac{\partial}{\partial x^i}{}_p$ is a vector i.e. an element of the tangent bundle TM. Then $\phi_*(\frac{\partial}{\partial x^i})_p$ is a vector in N i.e. an element of TN. We can extract the ath-component of this vector by applying the dual base vector dy^a as follows:

$$\underbrace{dy^a\left(\phi_*\left(\left(\frac{\partial}{\partial x^i}\right)_p\right)\right)}_{:-\phi^a_{*i}} \overset{grad-def}{:-} \phi_*\left(\left(\frac{\partial}{\partial x^i}\right)_p\right) y^a \overset{\phi_*-def}{=} \left(\frac{\partial}{\partial x^i}\right)_p (y^a \circ \phi)$$

$$= \left(\frac{\partial}{\partial x^i}\right)_p (y \circ \phi)^a :- \left(\frac{\partial \hat{\phi}^a}{\partial x^i}\right)_p \tag{11.1}$$

One must be very careful of the notation because ϕ^a_{*i} are not tensor components. $a : 1, 2, \ldots, \dim N$ whereas $i : 1, 2, \ldots, \dim M$. The two indices refer to different mfds. Moreover, the map $\hat{\phi}$ is defined as

$$\begin{tikzcd} M \supseteq u \arrow[r, "\phi"] \arrow[d, "x"'] \arrow[dr, "\hat{\phi}"] & v \subseteq N \arrow[d, "y"] \\ \mathbb{R}^d \supseteq x(u) \arrow[r, "x^{-1} \circ \phi \circ y"'] & y(v) \subseteq \mathbb{R}^c \end{tikzcd}$$

chart representative of ϕ

thus now we have completely defined the push forward map. In order to understand it in greater depth it is a clever idea to consider that the point p lies on a curve $\gamma : \mathbb{R} \to M$. We know that the tangent vector of γ at p is a map $u_{\gamma,p} : C^\infty(M) \to \mathbb{R}$. By using ϕ we can map all the points of γ into the mfd N. Hence the map $\phi \circ \gamma$ is the image of γ under the map ϕ in N. The claim is that while ϕ "pushes" the curves; the push forward map ϕ_* "pushes" the tangent vectors of curves γ to tangent vectors of the pushed-forward curves. Therefore we have the following theorem…

Theorem 11.1.1. *If $\gamma : \mathbb{R} \longrightarrow M$ is a curve in M and $\phi \circ \gamma : \mathbb{R} \longrightarrow N$ is a curve in N then, ϕ_* pushes the tangent to a curve γ to the tangent to the curve $(\phi \circ \gamma)$ i.e.,*

$$\phi_*\left(v_{\gamma,p}\right) = v_{(\phi\circ\gamma),\phi(p)} \,. \tag{11.2}$$

Proof. Let $p = \gamma(\lambda_0)$. Then $\forall\, f \in C^\infty(N)$,

$$\begin{aligned}\phi_*\left(v_{\gamma,p}\right) f = v_{\gamma,p}(f\circ\phi) = ((f\circ\phi)\circ\gamma)'(\lambda_0) = (f\circ(\phi\circ\gamma))'(\lambda_0) = v_{(\phi\circ\gamma),\phi(\gamma(\lambda_0))} f \\ = v_{(\phi\circ\gamma),\phi(p)} f\end{aligned}$$

□

The importance of the aforementioned theorem lies in the fact that it provides a simple and geometric picture of what the push forward does.

Pull-Back Map

Definition 11.1.2. *Let M and N be smooth manifolds with cotangent bundles T^*M and T^*N respectively. Let $\phi : M \longrightarrow N$ be a smooth map. Then, the* ***pull-back map*** *ϕ^* (induced from ϕ) is the map*

$$\begin{aligned} \phi^* : T^*N &\longrightarrow T * M \\ \omega &\mapsto \phi^*(\omega) \end{aligned}$$

where $\phi^*(\omega)(X) := \omega(\phi_*(X))$ for any $X \in T_pM$.

Its **components wrt two charts** $(u, x) \in \mathcal{A}_M$ and $(v, y) \in \mathcal{A}_N$ are given by

$$\begin{aligned} \phi^{*a}{}_i &:= \phi^* \left((dy^a)_{\phi(p)} \right) \left(\left(\frac{\partial}{\partial x^i} \right)_p \right) \\ &= (dy^a)_{\phi(p)}\, \phi_* \left(\left(\frac{\partial}{\partial x^i} \right)_p \right) = \left(\frac{\partial \hat{\phi}^a}{\partial x^i} \right)_p = \phi^a_{*i} \end{aligned}$$

Thus, the components of the push-forward and pull-back maps are exactly the same

$$\begin{aligned} (\phi_* (X))^a &= \phi^a_{*i}\, X^i \\ (\phi^* (\omega))_i &= \phi^{*a}{}_i\, \omega_a = \phi^a_{*i}\, \omega_a\, . \end{aligned}$$

The mnemonic phrase regarding map ϕ^* goes

"covectors are pulled back".

Remark 14. *This remark is about an* ***important application*** *regarding these two maps. Let M and N be smooth mfds and $\phi : M \to N$ be an injective map. Suppose that $\dim M < \dim N$, which corresponds to the embedding of M in N, and that we have metric g in the "bigger" mfd N . Then, usign g, we can induce a metric on the "smaller" mfd M which is given by*

$$g_M(X, Y) :- g\left(\phi_*(X), \phi_*(Y)\right) \tag{11.3}$$

for any $X, Y \in T_pM$. In terms of component we get

$$\left((g_M)_{ij} \right)_p = (g_{ab})_{\phi(p)} \left(\frac{\partial \hat{\phi}^a}{\partial x^i} \right)_{\phi(p)} \left(\frac{\partial \hat{\phi}^b}{\partial x^j} \right)_{\phi(p)}\, . \tag{11.4}$$

Notice that the induced metric depends on the embedding map ϕ.

11.2 Flow of a complete vector field

Definition 11.2.1. *Let X be a vector field on a smooth manifold $(M, \mathcal{O}, \mathcal{A})$. A curve $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ is called an* ***integral curve of*** *X if*

$$v_{\gamma,\gamma(\lambda)} = X_{\gamma(\lambda)}$$

Thus, for a curve γ to be called *integral*, its tangent vector u_γ at each point $\gamma(\lambda)$ must reproduce the given vector X at the point $\gamma(\lambda)$. Consequently a curve γ can be an integral curve wrt a vector field X but not wrt another vector field Y.

As an analogy, one may think that the vector field X is the velocity of water molecules in a river. Then an integral curve γ would the the trajectory of a ship that has no propulsion and just flows with the river.

Definition 11.2.2. *A vector field X is **complete** if all integral curves have $I = \mathbb{R}$.*

Definition 11.2.3. *The **flow of a complete vector field** X on a manifold M is a 1-parameter family*

$$\begin{aligned} h^X : &\mathbb{R} \times M \longrightarrow M \\ &(\lambda, p) \mapsto \gamma_p(\lambda) \end{aligned}$$

where $\gamma_p : \mathbb{R} \longrightarrow M$ is the integral curve of X with $\gamma(0) = p$.

Then for fixed $\lambda \in \mathbb{R}$ we get a map $h^X_\lambda : M \longrightarrow M$ that is smooth. This map takes every point on the mfd and pushes it, for a parameter distance λ, according to the integral curves of the field X. Thus it can be considered as a special case of the map ϕ from which we induced the push-forward and the pull-back maps.

Notice that we have not used a metric in this "flow business". Somebody somehow gave us a vector field and now we can flow the points of our mfd along that vector field. One can give many different vector fields on a smooth mfd. If we push the points using h^X i.e. let them flow along X, then we can look at the pushed-forward vectors or the pulled-back covectors. Even better we can check the induced metric that we have, let it flow, and then check what the result is. If the induced metric hasn't change then it is *symmetric.*

The concept of flows is the key to understanding what symmetry is. The next step is to understand how to distinguish the different types of symmetries.

11.3 Lie subalgebras of the Lie algebra $(\Gamma(TM), [\cdot, \cdot])$ of vector fields

Recall that $\Gamma(TM) = \{$ set of all vector fields $\}$, which can be seen as a $C^\infty(M)$-module (since we can multiply field with $C^\infty(M)$ functions). However we can restrict ourselves such that $\Gamma(TM)$ is an $\mathbb{R}-$vector space. Then

Definition 11.3.1. *If we have two vector fields X, Y then the **commutator** $[X, Y] \in \Gamma(TM)$ is defined by*

$$[X, Y]f :- X(Yf) - Y(Xf)$$

satisfying

(i) ***Anticommutativity:*** $[X, Y] = -[Y, X]$

(ii) $\mathbb{R}$***-Linearity:*** $[\lambda X + Z, Y] = \lambda[X, Y] + [Z, Y]$ *where* $\lambda \in \mathbb{R}$

(iii) ***Jacobi identity****:* $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$

Any vector space equipped with a map $[X, Y]$ *is called a* ***Lie algebra***, therefore

Definition 11.3.2. *The vector space* $(\,\Gamma(TM), [\cdot, \cdot]\,)$ *constitutes the Lie algebra of all vector fields on* M*.*

Now as a next step lets consider the following:

Let $X_1, \ldots, X_s$ be s (many) vector fields on M, such that

$$\forall i, j \in \{1, \ldots, s\} \quad [X_i, X_j] = \underbrace{C^k_{ij} X_k}_{\text{linear combination of } X_k s}$$

where $C^k_{ij} \in \mathbb{R}$ are called **structure constants**.

Moreover let $\mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_s\}$ be the set of all linear combinations of X_k (thus they can define a sub-vector space). Then

Definition 11.3.3. *The vector space* $L :- (\mathit{span}_{\mathbb{R}}\{X_1, \ldots, X_s\}, [\cdot, \cdot])$ *is called a* ***Lie subalgebra*** *of* $(\Gamma(TM), [\cdot, \cdot])$*.*

Example 11.3.1. *In* S^2*, assume that the vector fields* X_1, X_2, X_3 *satisfy*

$$\begin{aligned}[X_1, X_2] &= X_3, \\ [X_2, X_3] &= X_1 \\ [X_3, X_1] &= X_2\,.\end{aligned}$$

Then $(\mathit{span}_{\mathbb{R}}\{X_1, X_2, X_3\}, [\cdot, \cdot])$ *(=* $SO(3)$*) is a Lie subalgebra. An instance of vector fields satisfying these conditions (with* X_i, θ, ϕ *all taken at a point* p*, and* $x^1 = \theta, x^2 = \phi$*) is*

$$\begin{aligned}X_1 &= -\sin\phi \frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi} \\ X_2 &= \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi} \\ X_3 &= \frac{\partial}{\partial\phi}\end{aligned}$$

Note that the above is defined on a merely smooth manifold without any additional structure like metric. These are just vector fields given in a particular chart on S^2*.*

Now we can use this to define what we mean by symmetry of the metric.

11.4 Symmetry

Definition 11.4.1. *A finite-dimensional Lie subalgebra* $(L, [\cdot, \cdot])$ *is said to be a* ***symmetry*** *of a metric tensor field* g *if* $\forall\, X$ *(complete vector field)* $\in L$*,* $\lambda \in \mathbb{R}$*,* $A, B \in T_pM$

$$g\left(\left(h^X_\lambda\right)_* (A), \left(h^X_\lambda\right)_* (B)\right) = g(A, B)$$

In another formulation (using pullback), one would write $\left(h_\lambda^X\right)^* g = g$. Recalling the definition of pull-back of $\phi : M \longrightarrow M$ we see that for a $(0,2)$-tensor field g the pull-back is defined by

$$(\phi^* g)(A, B) := g(\phi_*(A), \phi_*(B)) \, .$$

The map h_λ^X is called an **isometry**. Here we have a 1-parameter family of isometries generated from each vector field $X \in L$.

11.5 Lie derivative

The Lie derivative must be introduced at this point since it makes the job of checking for symmetries easy.

It is obvious that $\forall X \in L$ (symmetry subalgebra) then $\left(h_\lambda^X\right)^* g - g = 0$. Therefore we also have that iff

$$\mathcal{L}_X g := \lim_{\lambda \longrightarrow 0} \frac{\left(h_\lambda^X\right)^* g - g}{\lambda} = 0$$

then L is a symmetry of the metric g.

Definition 11.5.1. *The **Lie derivative** $\mathcal{L}$ on a smooth manifold $(M, \mathcal{O}, \mathcal{A})$ defined by*

$$\mathcal{L}_X g := \lim_{\lambda \longrightarrow 0} \frac{\left(h_\lambda^X\right)^* g - g}{\lambda}$$

takes a pair of a vector field X and a (p,q)-tensor field T to a (p,q)-tensor field such that

(i) $\mathcal{L}_X f = Xf \quad \forall f \in C^\infty M$

(ii) $\mathcal{L}_X Y = [X, Y]$ *where X, Y are vector fields*

(iii) $\mathcal{L}_X (T + S) = \mathcal{L}_X T + \mathcal{L}_X S$ *where T, S are (p,q)-tensor fields of the same valence*

(iv) ***Leibnitz rule:*** $\mathcal{L}_X T(\omega_1, \dots, \omega_p, Y_1, \dots, Y_q) = (\mathcal{L}_X T)(\omega_1, \dots, \omega_p, Y_1, \dots, Y_q)$
$+ T(\mathcal{L}_X \omega_1, \dots, \omega_p, Y_1, \dots, Y_q) + \cdots + T(\omega_1, \dots, \mathcal{L}_X \omega_p, Y_1, \dots, Y_q)$
$+ T(\omega_1, \dots, \omega_p, \mathcal{L}_X Y_1, \dots, Y_q) + \cdots + T(\omega_1, \dots, \omega_p, Y_1, \dots, \mathcal{L}_X Y_q)$ *where T is a (p,q)-tensor*

(v) $\mathcal{L}_{X+Y} T = \mathcal{L}_X T + \mathcal{L}_Y T$

Note that conditions $(i), (iii), (iv), (v)$ are less restrictive than the conditions on the covariant derivative. This means more freedom. Also recall that for the covariant derivative we had to put extra structure to fix that freedom. Therefore it must be condition (ii) that lets us do whole thing without introducing extra structure.

The careful reader might wonder why we didn't define it before since it doesn't need additional structure (whereas ∇ needed). Roughly speaking there is a kind of cheating because $\mathcal{L}_X$ sucks in the information, of how the field X behaves away from the point where we do the derivative. Indeed we want to calculate the derivative at a particular point $(\mathcal{L}_X)_p = [X, Y]_p$ but this does not suffice to prescribe the value of X at that point, in order to calculate the whole thing $(\mathcal{L}_X Y)_p$. We need the whole information of how X behaves in the neighbourhood, in order to make the flow, in order to calculate the Lie derivative. This is reflected on $\mathcal{L}_X$ not being C^∞-linear.

The proof of finding the components of $\mathcal{L}_X Y$ is very similar with that of the covariant derivative. We proceed by just writting down that in a chart (u, x)

$$(\mathcal{L}_X Y)^i = X^m \frac{\partial}{\partial x^m}(Y^i) - \frac{\partial}{\partial x^s}(X^i) Y^s \tag{11.5}$$

whereas for the covariant derivative we had

$$(\nabla_X Y)^i = X^m \frac{\partial}{\partial x^m}(Y^i) + \Gamma^i{}_{sm} X^m Y^s$$

and now the point we emphasized earlier becomes evident. Because we added extra structure for the covariant derivative, we do not need a vector field just a vector at a point. That's why in the second term we have the Γs and not a derivative. On the other hand, notice the term $\frac{\partial}{\partial x^s}(X^i)$ in the Lie derivative. This is a derivative of the components and just like every derivative it requires knowledge of the behaviour of X in the neighbouhood of a point and not just at the point. This is the price that we paid for not imposing extra structure.

In general

$$(\mathcal{L}_\mathcal{X} Y)^i{}_j = X^m \frac{\partial}{\partial x^m}(T^i{}_j) - \frac{\partial x^i}{\partial x^s} T^s{}_j + \frac{\partial x^s}{\partial x^j} T^i{}_s \tag{11.6}$$

As above, it is easy to calculate components of Lie derivative of metric g, $L_X g$. Thus, by checking if the derivative equals 0 or not, it can be determined whether a metric features a symmetry.

Chapter 12

Integration on Manifolds

In this chapter we are going to complete the mathematical apparatus that we need in order to proceed to physics. So far we have managed to lift differentiation, parallelism etc. to the manifold level sometimes, with the cost of adding extra structure. This chapter will be the completion of our "lift" of analysis on charts to the manifold level.

We want to be able to integrate a function f over a manifold M. This $\int_M f$ will be an important tool for writing down the action which produces the Einstein Equations. However, to define such an integral we need a *mild new structure* on our smooth manifold $(M, \mathcal{O}, \mathcal{A})$. It requires

(i) a choice of a certain tensor field, the so-called **volume form** and

(ii) a restriction on the atlas $\mathcal{A}$, which is called '**orientation**'.

12.1 Review of integration on $\mathbb{R}^d$

We review this because, after all, this is what happens in a chart; and we want to use this knowledge to have a well-defined integration on manifolds.

a) If $F : \mathbb{R} \longrightarrow \mathbb{R}$, we assume a notion of integration is known. We define an integral over an interval (a, b) by using the Riemann integral as follows:

$$\int_{(a,b)} F := \int_a^b dx\, F(x)\,.$$

b) If we have slightly more interesting function $F : \mathbb{R}^d \longrightarrow \mathbb{R}$, then we define its integral in two steps

(1) On a box-shaped domain, $Box = (a, b) \times (c, d) \times \cdots \times (u, v) \subseteq \mathbb{R}^d$, the integral is defined by

$$\int_{Box} F := \int_a^b dx^1 \int_c^d dx^2 \cdots \int_u^v dx^d\, F(x^1, x^2, \ldots, x^d)$$

therefore it reduces to single integrals over the variables and one has to execute one after another.

(2) for other domains, $G \subseteq \mathbb{R}^d$, we first must introduce an **indicator function** $\mu_G : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that

$$\mu_G(x) = \begin{cases} 1, & x \in G \\ 0, & x \notin G \end{cases}$$

and then define

$$\int_G F := \int_{-\infty}^{+\infty} dx^1 \int_{-\infty}^{+\infty} dx^2 \cdots \int_{-\infty}^{+\infty} dx^d \, \mu_G(x) \cdot F(x^1, x^2, \ldots, x^d)$$

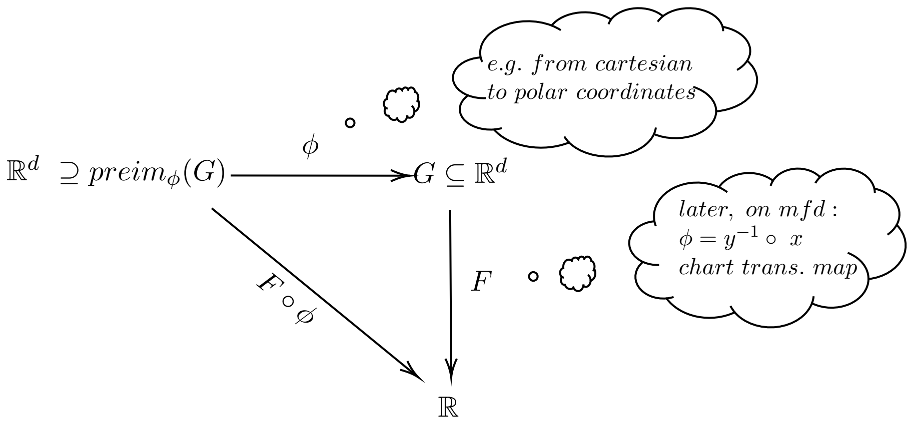
While this may not be a practical definition, it tells us what we mean by an integral over a function from $\mathbb{R}^k$ to $\mathbb{R}$ over an arbitrary

***Note**: All of the above comes with the disclaimer '**if the integral exists**' since there could be many issues that do not allow the existence of the integral as defined above.*

Change of Variables

Theorem 12.1.1. *If $F : G \longrightarrow \mathbb{R}$ and $\phi : preim_\phi(G) \longrightarrow G$, then*

$$\int_G F(x) = \int_{preim_\phi(G)} \underbrace{|det(\partial_\cdot \phi^\cdot)(y)|}_{Jacobian\ of\ \phi} \cdot (F \circ \phi)(y)$$



Example 12.1.1. *Consider the domain $G \subset \mathbb{R}^2$, which includes the entire R^2 except the x-axis. Let*

$$\phi : \mathbb{R}^+ \times \{(0, \pi) \cup (\pi, 2\pi)\} \longrightarrow G$$
$$(r, \varphi) \mapsto (r \cos\varphi, r \sin\varphi)$$

Thus, G is in Cartesian coordinates and the $preim_\phi(G)$ is in polar coordinates. Let us calculate *the Jacobian.*

$$\left(\partial_a x^b\right)(r,\varphi) = \begin{vmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{vmatrix}$$

$$\det\left(\partial_a x^b\right)(r,\varphi) = r$$

$$\Longrightarrow \int_G \underbrace{dx^1\, dx^2}_{\textit{volume element}} \; F(x^1,x^2) = \int_0^\infty \int_0^{2\pi} \underbrace{dr\, d\varphi\, r}_{\textit{volume element}} \; F(r\cos\varphi, r\sin\varphi)$$

In our context $\mathbb{R}^d$ will be the image of some chart. Our goal is take that theorem and apply it in differential geometry .

12.2 Integration on one chart

Let $(M,\mathcal{O},\mathcal{A})$ be a smooth manifold, $f : M \longrightarrow \mathbb{R}$ be an integrable function and choose charts $(u,x),(u,y) \in \mathcal{A}$.

$$\begin{tikzcd}
\mathbb{R}^d \supseteq y(u) \arrow[dr, "f_{(y)} := f \circ y^{-1}"] & \\
M \supseteq u \arrow[u] \arrow[d] \arrow[r, "f"] & \mathbb{R} \\
\mathbb{R}^d \supseteq x(u) \arrow[ur, "f_{(x)} := f \circ x^{-1}"'] \arrow[uu, bend left, "\phi = y \circ x^{-1}"] &
\end{tikzcd}$$

Now lets consider integrating over a chart domain and by using the preceding theorem we will try to make the transition to the other chart. We have to do that check since we need an integration method that is independent of our choice of chart. We begin by considering an integration using the chart (u,y). Thus,

$$\begin{aligned}
\int_{y(u)} d^d\beta\, f_{(y)}(\beta) &= \int_{x(u)} d^d\alpha\, \left|\det\left(\partial_i(y^j \circ x^{-1})(\alpha)\right)\right| \left(f_{(y)} \circ (y \circ x^{-1})\right)(\alpha) \\
&= \int_{x(u)} d^d\alpha \left|\det\left(\frac{\partial y^j}{\partial x^i}\right)_{x^{-1}(\alpha)}\right| \left(f \circ y^{-1} \circ y \circ x^{-1}\right)(\alpha) \neq \int_{x(u)} d^d\alpha\, f_{(x)}(\alpha)
\end{aligned}$$

Hence using an integral over a chart in this way, is ill-defined. This means our attempt to define $\int_u f$ as $\int_{x(u)} d^d\alpha\, f_{(x)}(\alpha)$ is wrong. Then why did we do it one might wonder. . . We showed this explicitly because inside the failure there is an insight. The term $\left|\det\left(\frac{\partial y^j}{\partial x^i}\right)_{x^{-1}(\alpha)}\right|$ is what causes the problem. Thus, if manage to find an object which transforms in an opposite manner, i.e. creates the term $\frac{1}{|\det(\ldots)|}$, our problem is solved. We will then put this object additionally inside the definition of integration and all our integrals will result in something that is chart independent.

Unfortunately there is no such object coming from the smooth structure $(M, \mathcal{O}, \mathcal{A})$ alone. Therefore we are going to have to introduce new structure.

12.3 Volume forms

Definition 12.3.1. *On a smooth manifold $(M, \mathcal{O}, \mathcal{A})$, a $(0, dimM)$-tensor field Ω is called a* ***volume form*** *if*

(a) Ω vanishes nowhere (i.e. $\Omega \neq 0 \; \forall p \in M$)

(b) is totally antisymmetric

$$\Omega(\ldots, \underbrace{X}_{ith}, \ldots, \underbrace{Y}_{jth} \ldots) = -\Omega(\ldots, \underbrace{Y}_{ith}, \ldots, \underbrace{X}_{jth} \ldots)$$

Due to antisymmetry its components in a chart have the property:

$$\Omega_{i_1 \ldots i_d} = \Omega_{[i_1 \ldots i_d]}$$

The volume form contains all the extra structure that is needed . As we will see in the example, the "volume form structure" may not be included into smooth mfd however it is included into a metric mfd structure.

Example 12.3.1. *Consider $(M, \mathcal{O}, \mathcal{A}, g)$ metric manifold. Then one can construct a volume form Ω from the metric g. In <u>any</u> chart: (u, x)*

$$\Omega_{(x)i_1 \ldots i_d} := \sqrt{det(g_{(x)ij})} \epsilon_{i_1 \ldots i_d}$$

where $\epsilon_{i_1 \ldots i_d}$ is called ***Levi-Civita symbol*** *(independent of charts) and is defined as*

$$\epsilon_{123 \ldots d} = +1$$
$$\epsilon_{1 \ldots d} = \epsilon_{[i_1 \ldots i_d]} \, .$$

We make the claim that the rhs transforms as a $(0, d)-$tensor. We have to check it by going into another chart and observe how it transforms.

Proof. In a different chart (u, y) we write

$$\begin{aligned}
\Omega(y)_{i_1 \ldots i_d} &= \sqrt{\det(g_{(y)ij})} \epsilon_{i_1 \ldots i_d} = \\
&= \sqrt{\det\left(g_{(x)mn} \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j}\right)} \frac{\partial y^{m_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{m_d}}{\partial x^{i_d}} \epsilon_{[m_1 \ldots m_d]} = \\
&= \sqrt{|\det g_{(x)ij}|} \left| \det\left(\frac{\partial x}{\partial y}\right) \right| \det\left(\frac{\partial y}{\partial x}\right) \epsilon_{i_1 \ldots i_d} = \sqrt{\det g_{(x)ij}} \, \epsilon_{i_1 \ldots i_d} \operatorname{sgn}\left(\det\left(\frac{\partial x}{\partial y}\right)\right)
\end{aligned}$$

where in the last step we used $\det\left(\frac{\partial y}{\partial x}\right) = \frac{1}{\det\left(\frac{\partial x}{\partial y}\right)}$ [1] and $|x| = \mathrm{sgn}(x)\, x$. The result shows that Ω is well defined iff $\det\left(\frac{\partial x}{\partial y}\right) > 0$ for every pair of charts (u, x) and (v, y). $\square$

We can require such a property by further restricting our choice of $\mathcal{A}$. Therefore we make the following ***requirement***. Lets restrict our smooth atlas $\mathcal{A}$ to a ***subatlas***

$$\mathcal{A}^{\uparrow} \subseteq \mathcal{A}$$

such that any two charts $(u, x), (v, y)$ have chart transition maps $y \circ x^{-1}$, $x \circ y^{-1}$ such that

$$\det\left(\frac{\partial y}{\partial x}\right) > 0\,.$$

Such an atlas is called an ***oriented atlas***.

Hence we cannot define a volume form Ω from g using the structure $(M, \mathcal{O}, \mathcal{A}, g)$. We need to further restrict our atlas to $\mathcal{A}^{\uparrow}$ and use the structure $(M, \mathcal{O}, \mathcal{A}^{\uparrow}, g)$.

If we just have a smooth mfd it may not be orientable; in those manifolds one cannot integrate.

Definition 12.3.2. *Let Ω be a volume form on $(M, \mathcal{O}, \mathcal{A}^{\uparrow})$ and consider a chart (u, x). We then define*

$$\omega_{(x)} :- \Omega_{i_1 \dots i_d}\, \epsilon^{i_1 \dots i_d}$$

where $\epsilon^{i_1 \dots i_d}$ is defined exactly as $\epsilon_{i_1 \dots i_d}$. One can show that

$$\omega_{(x)} = \det\left(\frac{\partial x}{\partial y}\right) \omega_{(y)}\,.$$

Such an object is called ***scalar density***.

12.4 Integration on M

Integration on one chart domain u

Definition 12.4.1. *The integral over a chart domain u of a function $f : M \longrightarrow \mathbb{R}$ on a manifold is defined as*

$$\int_u f \overset{(u,x)}{:-} \int_{x(u)} d^d\alpha\ \omega_{(x)}(x^{-1}(\alpha))\, f_{(x)}(\alpha)$$

Now we need to check whether it is well defined, by going into another chart.

[1] Recall that $\frac{\partial x}{\partial y} :- \partial(x \circ y^{-1})$ is map from $\mathbb{R}^d$ to $\mathbb{R}^d$ i.e. an Endomorphism. The determinant is well defined abstractly for Endomorphisms and; the inverse map would of course be $y \circ x^{-1} = \frac{\partial y}{\partial x}$. This is why we can write $\det\left(\frac{\partial y}{\partial x}\right) = \frac{1}{\det\left(\frac{\partial x}{\partial y}\right)}$

Proof.

$$\int_U f \overset{(u,y)}{:=} \int_{y(u)} d^d\beta \; \omega_{(y)}(y^{-1}(\beta))\, f_{(y)}(\beta)$$

$$= \int_{x(u)} \int d^d\alpha \left| \det\left(\frac{\partial y}{\partial x}\right) \right| f_{(x)}(\alpha)\, \omega_{(x)}(x^{-1}(\alpha)) \det\left(\frac{\partial x}{\partial y}\right)$$

$$= \int_{x(u)} d^d\alpha \; \omega_{(x)}(x^{-1}(x))\, f_{(x)}(\alpha)$$

□

Therefore, by substituting ω and then $\Omega_{i_1 \dots i_d}$, ***on an oriented metric manifold*** $(M, \mathcal{O}, \mathcal{A}^{\uparrow}, g)$ ***the integral is defined by***

$$\int_u f := \int_{x(u)} d^d\alpha \underbrace{\sqrt{\det(g_{ij}(x))(x^{-1}(\alpha))}}_{\sqrt{g}} f_{(x)}(\alpha)$$

Integration on the entire manifold

Lets consider an oriented metric mfd. If we choose a chart (u, x) we know we can integrate a function over the chart region u. After that, we can use the chart transition maps so as to cover and integrate over the entire mfd. However, there is a problem with that strategy. The problem is that we are going to double-count the overlapping regions between the various charts. Moreover, we cannot cut out the intersections between the charts, so as not to double-count, because the intersections are not open sets. We need a different idea.

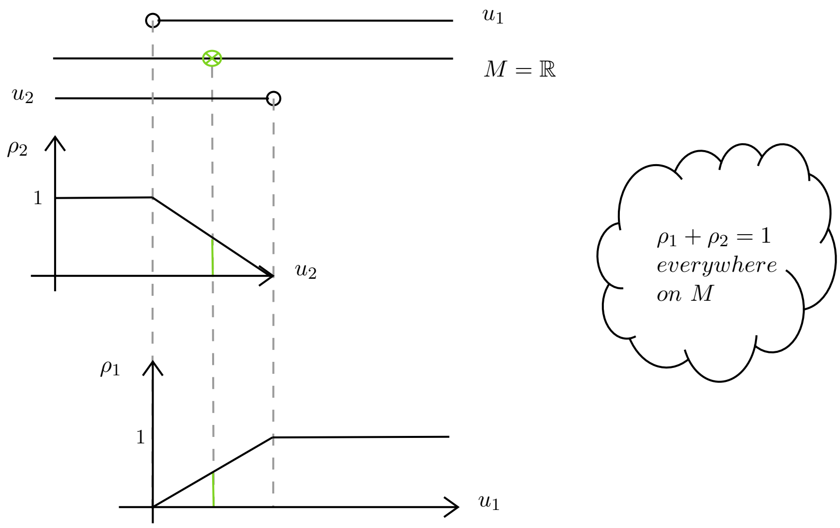
The idea is that we require that the mfd admits a so-called **partition of unity**. Roughly speaking, by partition of unity we mean that :

for any finite subatlas $\mathcal{A}' \subseteq \mathcal{A}^{\uparrow}$ there exist continuous functions $\rho_i : u_i \longrightarrow \mathbb{R}$ such that:

$$\forall\, p \in M : \sum_{p \in u_i} \rho_i(p) = 1$$

This will act as weight distribution function that equally distributes a weight when the integration takes place in an overlapping region. Thus in the overlapping region we will take a smaller contribution (from both charts) in such a way that if we weight the function with the various $\rho_i(p)$s, we will not get an overcounting.

Example 12.4.1. *Let us consider $M = \mathbb{R}$ and only two charts with overlapping regions u_1, u_2.*



The integral over the entire mfd is then written

Definition 12.4.2. $\int_M f :- \sum_{i=1}^{finite} \int_{u_i} (\rho_i \cdot f)$.

The atlas $\mathcal{A}'$ must be finite in order for the sum to be finite otherwise we will run into convergence issues.

Chapter 13

Relativistic Spacetime

All the previous chapters dealt with the mathematical foundations of general relativity. From this chapter and on the character of our course will change significantly i.e the physics begin. We are going to employ all the mathematical tools we introduced in order to describe relativistic spacetime. We should emphasize that from now on some of our arguments will be more vague, because we will have to deal with physics and physics is more complicated than mathematics. In mathematics everything is more clear due to the fact that mathematics are self-referencial; and that their power. On the other hand, in physics we need to relate our notions to the world and in the end we need make predictions. Therefore a lot more handwaving and arguing is required.

Having said that we would like from the reader to recall the *definition of Newtonian spacetime*:

$$(M, \mathcal{O}, \mathcal{A}, \nabla, t)$$

where ∇ a torsion free connection, $t \in C^\infty(M)$, $dt \neq 0$ (purely time slices exist) and $\nabla dt = 0$ (time flows uniformly). Also recall the definition of relativistic spacetime which in the language of mathematics is given by

$$(M, \mathcal{O}, \mathcal{A}, \nabla, g, T)$$

where ∇ torsion free, g Lorentzian metric and is the so-called *time-orientation*. We will define exactly what we mean by time-orientation but for the moment we would like to stress that the role played in Newtonian spacetime by the absolute time function t is now being played by the interplay of two additional structures; the Lorentzian metric and the time-orientation.

13.1 Time Orientation

Definition 13.1.1. *Let $(M, \mathcal{O}, \mathcal{A}^\uparrow, g)$ be a Loretzian mfd. Then a **time orientation** is given by a smooth vector field T that:*

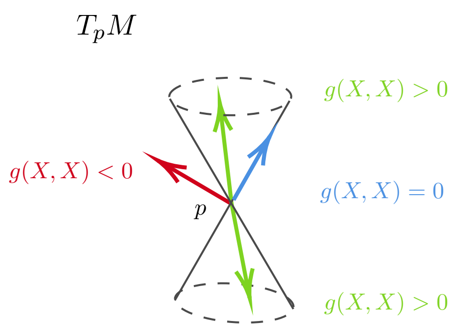
(i) Does not vanish anywhere

(ii) $g(T, T) > 0$

In Newtonian spacetime we had the absolute t which satisfied $dt \neq 0$. That enabled us to "draw" purely spatial slices of equal time. Then we defined the future directed vectors as those satisfying the condition $dt(X) > 0$. All vectors live in a tangent space and; by selecting only those that satisfy $dt(X) > 0$, we are basically selecting only "half" of that tangent space. The space of all future directed vectors which is of course a subset of a T_pM.

In relativistic spacetime we do not have an absolute time but instead we have a Lorentzian metric, which produces a "double-cone" structure (in a tangent space) similarly like $dt \neq 0$ produced equal time slices. Combining such a metric with a time orientation produces a distribution of "single-cones" of the tangent space, similarly like $dt(X) > 0$ produced a distribution of "half-spaces" of the tangent space, everywhere across the manifold.

Besides the Lorentzian metric, a time orientation is also critical if we want to make a *particle definition*. Because in order to define what a particle is we have to talk about future and past. We want particles to run forward in time rather than backward. Hence, we need to eliminate one of the cones. However, the metric does not distinguish between the cones therefore, we need an extra choice of vector field (T) that lies in one of the cones. That is why we can select only the future directed cones.

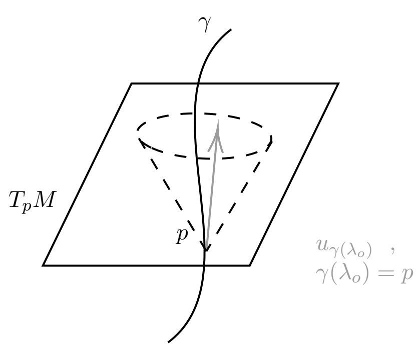


This definition of relativistic spacetime has been made to enable the following physical postulates:

(P1) The worldline γ of a ***massive*** particle satisfies:

(i) $g_{\gamma(\lambda)}\left(u_{\gamma,\gamma(\lambda)}\,,\,u_{\gamma,\gamma(\lambda)}\right) > 0$ i.e. for every point the tangent to the worldline lies inside the cone.

(ii) $g_{\gamma(\lambda)}\left(T_{\gamma(\lambda)}\,,\,u_{\gamma,\gamma(\lambda)}\right) > 0$ i.e. we choose only the future directed cones.



(P2) The worldline of a ***massless particle*** satisfies:

   (i) $g_{\gamma(\lambda)}\left(u_{\gamma,\gamma(\lambda)}\,,\,u_{\gamma,\gamma(\lambda)}\right) = 0$ i.e. for every point the tangent vector lies on the boundary of the cone.

   (ii) $g_{\gamma(\lambda)}\left(T_{\gamma(\lambda)}\,,\,u_{\gamma,\gamma(\lambda)}\right) > 0$

Having established these two postulates, we can now distinguish between these different kinds of particles. Usually one says that *"nothing runs faster than the speed of light"*. But "faster" refers to speed and we haven't talk about that yet. It is very important to distinguish velocity and speed. They are not the same thing. *Speed is what an observer sees in his laboratory, velocity is not.* For instance, we could measure the length of the velocity vector at some point i.e. $g_{\gamma(\lambda)}\left(u_{\gamma,\gamma(\lambda)}\,,\,u_{\gamma,\gamma(\lambda)}\right)$, and we would get a positive number if the particle is massive. But that is not its speed one would measure. The only restriction on the velocity vector is that it must lie inside or on the boundary of the cone for a massive or a massless particle, respectively.

13.2 Observers

From now on we will always assume that the underlying structure is the *relativistic spacetime* $(M, \mathcal{O}, \mathcal{A}^{\uparrow}, \nabla, g, T)$ unless we state otherwise.

Definition 13.2.1. *An **observer**[1] is a worldline γ with*

(i) $g(u_\gamma, u_\gamma) > 0$

(ii) $g(T, u_\gamma) > 0$

*together with a choice of **basis***

$$e_0(\lambda), e_1(\lambda), e_2(\lambda), e_3(\lambda)$$

of each $T_{\gamma(\lambda)}M$ where the observer worldline passes, if $u_{\gamma,\gamma(\lambda)} = e_0(\lambda)$ and $g(e_\alpha(\lambda), e_\beta(\lambda)) = \eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$.

(P3) A ***clock*** carried by an observer (γ, e) will measure a ***time***

$$\tau := \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{g_{\gamma(\lambda)}\left(u_{\gamma,\gamma(\lambda)}, u_{\gamma,\gamma(\lambda)}\right)}$$

   between the two "events"
   $\gamma(\lambda_0)$ = "start the clock" and $\gamma(\lambda_1)$ = "stop the clock".

[1] A more precise formulation states: An observer is a smooth curve in the **frame bundle** LM over M. The frame bundle is a bundle whose fibers are no longer the various tangent spaces. That is, in the frame bundle, a fiber at p no longer contains all tangent vectors at p; that is the tangent bundle. In the frame bundle, a point of a fiber at p is specific choice of a basis at p. This means a fiber at p of the frame bundle, contains all the possible quadruples $e_0(\lambda), e_1(\lambda), e_2(\lambda), e_3(\lambda)$ of tangent vectors that constitute a basis.

This is the first instance the word time enters the discussion. In this context time is derived notion. We do not just talk about time, we talk about time as the time that a clock measures. Moreover, notice that according to (P3) there is no time associated to an individual event as there is also no time difference associated to two events.

There is a time difference associated to a certain path, that a clock takes, between two events.

One can see that τ is indeed in the real world and not a coordinate artifact since it is defined in any chart...we did not use any components. In the literature τ is mostly reffered to as *propertime* and, it is nothing more than the length functional we introduced in §10.2.

Example 13.2.1. *(Application):*
Consider $M = \mathbb{R}^4$, $\mathcal{O}_=\mathcal{O}_{st.}$, $\mathcal{A}^\uparrow \ni (\mathbb{R}^4, id_{\mathbb{R}^4})$ *and also take that in chart, the metric is given by* $g : g_{(x)ij} = \eta_{ij}$ *and the time orientation by* $T^i_{(x)} = (1,0,0,0)^i$.

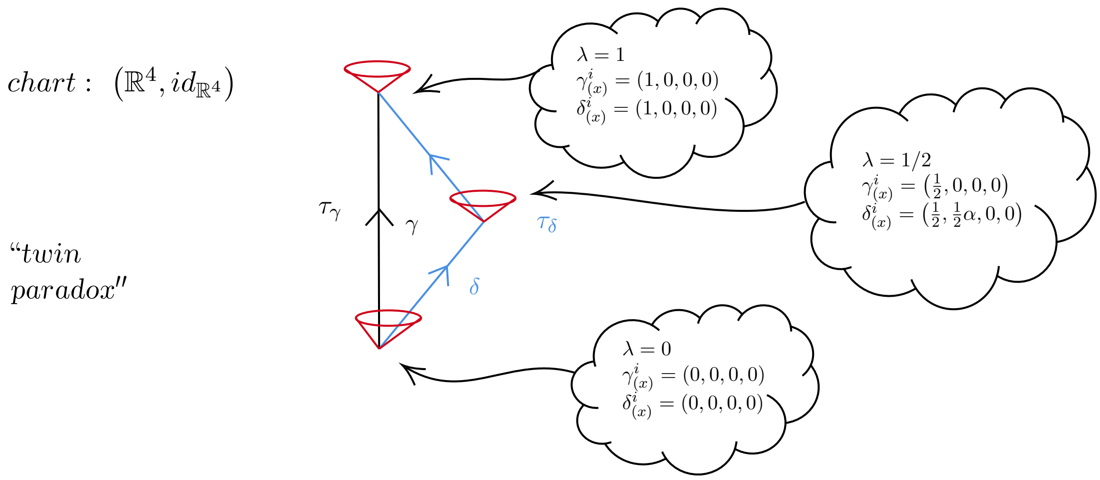
If the chart $(\mathbb{R}^4, id_{\mathbb{R}^4})$ *exists and* $g_{(x)ij} = \eta_{ij}$ *then we know from a previous example that* ${\Gamma^i}_{(x)jk} = 0$ *everywhere. This means in our spacetime* $(M, \mathcal{O}, \mathcal{A}^\uparrow, g, \nabla, T) \Longrightarrow Riem = 0$. *That's because the metric induced connection* ∇ *has vanishing connection coefficient functions, in a chart that covers the whole mfd; and not just a portion of it.* $Riem = 0$ *means the spacetime is flat. This situation is called Special Relativity.*

Now, in that chart, consider two observers:

$$\gamma : (0,1) \longrightarrow M \ , \ \gamma^i_{(x)} = (\lambda, 0,0,0)$$

and

$$\delta : (0,1) \longrightarrow M \ , \ \ \alpha \in (0,1) \ : \ \delta^i_{(x)} = \begin{cases} (\lambda, \alpha\lambda, 0, 0)^i & \lambda \leq \frac{1}{2} \\ (\lambda, (1-\lambda)\alpha, 0, 0)^i & \lambda > \frac{1}{2} \end{cases}$$



We would like to find the time elapsing of the clock of γ *and* δ *and compare them. We calculate*

$$\tau_\gamma :- \int_0^1 d\lambda \sqrt{g_{(x)ij}\dot{\gamma}^i_{(x)}\dot{\gamma}^j_{(x)}} = \int_0^1 d\lambda \, 1 = 1$$

$$\tau_\delta :- \int_0^{1/2} d\lambda \sqrt{1-\alpha^2} + \int_{1/2}^1 d\lambda \sqrt{1^2 - (-\alpha)^2} = \int_0^1 \sqrt{1-\alpha^2} d\lambda = \sqrt{1-\alpha^2} < \tau_\gamma$$

which shows that the more α goes to 1, the more the δ observer goes to the boundary of the cone and his propertime goes to zero i.e. he doesn't age that much. Notice that we didn't talk about relativity. This is just the definition of time shown by a clock.

(P4) Let (γ, e) be an ***observer*** and δ be a ***massive particle*** worldline that is parametrized such that $g(u_\delta, u_\delta) = 1$ [2]. Suppose the observer and the particle meet at a point in spacetime

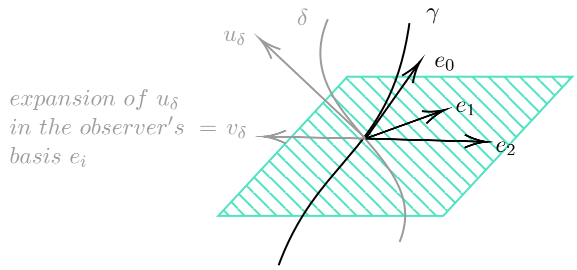
$$\delta(\tau_2) = p = \gamma(\tau_1)$$

This observer measures the 3-velocity (spatial velocity) vector of *this* particle as

$$v_{\delta,\delta(\tau_2)} := \epsilon^a \left(u_{\delta,\delta(\tau_2)}\right) e_a \quad , \quad a = 1, 2, 3$$

where $\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3$ is the unique basis, dual of the basis e^0, e^1, e^2, e^3.

Basically v_δ is a expansion in the spatial basis vectors e^1, e^2, e^3 of our observer. The covector ϵ^a acts on the 4-vector $u_{\delta,\delta(\tau_2)}$ and produces the component wrt the basis vector e_a.



The concept of 3-velocity is derived in our context. The 4-velocity u_δ is objective, it is in the real world because the worldline itself is objective and u_δ is just a tangent vector to it. However, the 3-velocity v_δ we had to construct using an observer who sees this 3-velocity. *Another observer with another frame will extract from the same objective 4-velocity, a different 3-velocity.* That is the whole secret explaining why different observers see different things. On the other hand it is a universal truth that nothing massive exceeds the speed of light. This upper limit constraints the spatial 3-velocities an observer can see. However, in the spacetime picture, it has to do with the 4-velocity not lying outside the cone.

Consequence: An observer(γ, e) will extract quantities measurable in his laboratory from objective spacetime quantities always in that fashion.

Example 13.2.2. *Consider the* $(0, 2)$ *Faraday tensor of electromagnetism*

$$F_{\alpha,\beta} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & -B_2 & -B_1 & 0 \end{bmatrix} = F(e_\alpha, e_\beta)$$

[2] I parametrize the worldlines according to the time a clock would display that travels with the particle. We do this to avoid unnecessary normalizing factors in the following definitions. Recall that worldlines are curves in spacetime and we know we can parametrize a curve however we want.

where e_α, e_β are the basis vectors of a particular observer. This also explains why different observers see different E and B fields. According to F those fields given by

$$E_\alpha :- F(e_0, e_\alpha) \quad , \quad B^i :- F(e_j, e_k)\epsilon^{ijk}$$

where ϵ^{ijk} is the Levi-Civita symbol and $i, j, k = 1, 2, 3$. It is evident that the fields E_α, B^i are highly observer dependent. Therefore, the objective spacetime quantity is the Faraday tensor F which is unimpressed by charts and observers. It is just a consequence of the Maxwell's equations. However, if we want to go the lab and measure the Faraday tensor, then we measure E_α, B^i because our lab is basically an observer.

13.3 Role of the Lorentz Transformations

Lorentz transformations emerge as follows:

Let (γ, e) and $(\tilde{\gamma}, \tilde{e})$ be observers with $\gamma(\tau_1) = \tilde{\gamma}(\tau_2)$ i.e. at some point in spacetime they meet. For simplicity let us reparametrise the two worldline st $\gamma(0) = \tilde{\gamma}(0)$. We know that

$$\begin{aligned} &e_o, e_1, e_2, e_3 \quad , \quad \text{at } \tau = 0 \quad \text{and,} \\ &\tilde{e_o}, \tilde{e_1}, \tilde{e_2}, \tilde{e_3} \quad , \quad \text{at } \tau = 0 \end{aligned}$$

are both bases for the same $T_{\gamma(0)}M$. Thus if they are both bases, we can express one in terms of the other by using a certain transformation. That is

$$\tilde{e_\alpha} = \Lambda^\beta{}_\alpha \, e_\beta$$

where $\Lambda \in GL(4)$ a general linear map (thus invertible) otherwise we cannot send a basis to a basis. Moreover, due to the observer's definition we know that

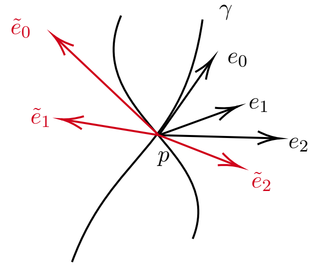
$$\eta_{\alpha\beta} = g\left(\tilde{e}_\alpha, \tilde{e}_\beta\right) = g\left(\Lambda^\kappa{}_\alpha \, e_\kappa \, , \, \Lambda^\lambda{}_\beta \, e_\lambda\right) = \Lambda^\kappa{}_\alpha \, \Lambda^\lambda{}_\beta \, g(e_\kappa, e_\lambda) = \Lambda^\kappa{}_\alpha \, \Lambda^\lambda{}_\beta \, \eta_{\kappa\lambda} \Rightarrow$$

$$\eta_{\alpha\beta} = \Lambda^\kappa{}_\alpha \, \Lambda^\lambda{}_\beta \, \eta_{\kappa\lambda}$$

i.e. Λ is an element of the Lorentz transformation group, $\Lambda \in O(1, 3)$. The result is that

Lorentz transformations relate the frames of any two observers, at the same point.

They do *not* act on spacetime. They act on *one* tangent space of spacetime, and relate *observers who meet there.*



Sometimes in the literature one might encounter the statement: *" $\tilde{x}^\mu = \Lambda^\mu_\nu \, x^\nu$ where $\tilde{x}^\mu$ is the new spacetime point and x^ν is the old spacetime point "*. Such a statement is simply *not* *true* since it implies that Lorentz transformations act on spacetime points. It is the General Diffeomorphisms that can be chosen in order to act on spacetime points, hence go to different charts etc. Not Lorentz transformations. No matter what curvature spacetime has. Thus,

both in special and on general relativity, Lorentz transformations relate observer frames at the same point and; General Diffeomorphisms can be chose so as to go to different charts.

An example of a general diffeomorphism but not a Lorentz transformation is the transformation from Cartesian to polar coordinates.

Chapter 14

Matter and Einstein Gravity

Matter in general relativity is a very big topic for which one chapter is definitely not enough to be covered. We will just cover what we will need in order to continue with what we are interested in thus, this chapter will be a simple introduction to the subject.

Essentially there are two theoretical models of matter. Point matter, like a particle of mass m and, field matter like the electromagnetic field. Basically, in the real world none of the two exists since they are both classical types of matter. However, it is well known that GR works very well even without introducing a quantum mechanical type of matter. From the point of view of GR, field matter is the fundamental type whereas point matter plays a big role in the phenomenology of the theory. For instance, it will be field matter that generates the curvature of spacetime.

On the other hand, one can always take point matter and transform it in a field type of matter by considering its continuous limit. All this, simply shows that this is more of an academic distinction. Sometimes it easier to express physical ideas using point matter and and other times it comes in handy to use field matter to write down equations. Physicists are not claiming there are actually two different types of matter.

14.1 Point Matter

Our postulates (P1) and (P2) already constrain the possible particle worldlines. The question is how does a particular particle really moves? What is the precise law of motion in presence of "forces" ? To find an answer let us first consider a more basic question. What the precise law of motion without the forces?

Without External Forces

The action of a massive particle worldline is given by

$$S_{massive}[\gamma] :- m \cdot \int d\lambda \sqrt{g_{\gamma(\lambda)} \left(u_{\gamma,\gamma(\lambda)}, u_{\gamma,\gamma(\lambda)}\right)}$$

where $g_{\gamma(\lambda)}\left(T_{\gamma(\lambda)}\,,\,u_{\gamma,\gamma(\lambda)}\right) > 0$. One can discard the square root by reparametrizing the curve γ such that $g_{\gamma(\lambda)}\left(u_{\gamma,\gamma(\lambda)}, u_{\gamma,\gamma(\lambda)}\right) = 1$ all the time . The dynamical law will be given by the Euler-Lagrange equations of the preceding action.

Similarly the action for the massless particle is

$$S_{massive}[\gamma,\mu] :- \int d\lambda\, \mu\, g_{\gamma(\lambda)}\left(u_{\gamma,\gamma(\lambda)}, u_{\gamma,\gamma(\lambda)}\right)$$

where μ is a Lagrange multiplier. Thus it will produce a constrain on the equations of motion. Varying wrt μ gives

$$\delta\mu\ :\ g\left(u_{\gamma,\gamma(\lambda)}, u_{\gamma,\gamma(\lambda)}\right) = 0 \qquad\qquad \text{null curve}$$

and wrt γ will give the corresponding eoms.

The reason we describe eoms by actions, is that composite systems have an action that is the sum of actions of the parts of that system, possibly including interaction terms. For example the action for two particles that interact with each other would be of the form

$$S[\gamma] + S[\delta] + S_{int}[\gamma,\delta]\,.$$

Presence of External Forces

As we mentioned earlier, fields are more fundamental than particles in GR. In the language of fields the presence of an external force translates into the *presence of a field to which a particle "couples"*. That simply means there is an interaction term with the external field that represents the force.

As an example consider a *massive point particle* that interacts with an external field. The total action will be given by

$$S[\gamma\,;A] :- m\cdot \int d\lambda\, \sqrt{g_{\gamma(\lambda)}\left(u_{\gamma,\gamma(\lambda)}, u_{\gamma,\gamma(\lambda)}\right)} + q\cdot A(u_{\gamma,\gamma(\lambda)})$$

where A an, assumed to be given, covector field on the mfd (e.g. the electromagnetic potential). It is assumed to be given because it is external. Note that if the charge q is zero, the particle does not feel the field even if it is there. The Euler-Lagrange equations for this action yield

$$m\,\left(\nabla_{u_\gamma} u_\gamma\right)_\alpha + \left(\frac{\partial L_{int}}{\partial \dot\gamma^\alpha_{(x)}}\right)^{\bullet} - \frac{\partial L_{int}}{\partial \gamma^\alpha_{(x)}} = 0$$

where in a chart $L_{int} = q\,A_{(x)\mu}\,\dot\gamma^\mu_{(x)}$. We calculate

$$\frac{\partial L_{int}}{\partial \dot\gamma^\alpha_{(x)}} = q\,A_{(x)\mu} \quad , \quad \left(\frac{\partial L_{int}}{\partial \dot\gamma^\alpha_{(x)}}\right)^{\bullet} = q\,\dot A_{(x)\alpha} = q\cdot \frac{\partial}{\partial x^\mu}\left(A_{(x)\alpha}\right)\cdot \dot\gamma^\mu_{(x)}$$

where the last result is due to the fact that $A_{(x)\alpha}$ depends on the position along the worldline. We further calculate

$$\frac{\partial L_{int}}{\partial \gamma^\alpha_{(x)}} = q\cdot \frac{\partial}{\partial x^\alpha}\left(A_{(x)\mu}\right)\cdot \dot\gamma^\mu_{(x)}$$

and substituting in E-L equations gives

$$m\left(\nabla_{u_\gamma}u_\gamma\right)_\alpha + q\left(\frac{\partial A_{(x)\alpha}}{\partial x^\mu} - \frac{\partial A_{(x)\mu}}{\partial x^\alpha}\right)\dot\gamma^\mu_{(x)} \Rightarrow$$

$$m\left(\nabla_{u_\gamma}u_\gamma\right)^\alpha = -q\,F^\alpha_{\ \mu}\,\dot\gamma^\mu$$

where $F_{\alpha\mu}$ is the Faraday tensor. The whole rhs can be recognized as the *Lorentz force.*

14.2 Field Matter

For our purposes we consider that **classical field matter** is any tensor field on spacetime whose eom derive from an action. The only classical field theory we have is Maxwell's theory. It is given by the action

$$S_{Maxwell}[A;g] = \frac{1}{4}\int_M d^4x\sqrt{-g}\left(F_{\alpha\beta}F^{\gamma\delta}g^{\alpha\gamma}g^{\beta\delta} + A(j)\right).$$

For simplicity we assume that one chart covers the whole mfd. Otherwise we will have to do a partition of unity etc. Moreover note that we assume a fixed curved background g, that what the semicolon denotes. The Faraday tensor equals to $F_{\alpha\beta} :- 2\partial_{[\alpha}A_{\beta]} = 2\nabla_{[\alpha}A_{\beta]}$ The equations of motion arising from this action, in a chart, are given by

$$0 = \frac{\partial\mathcal{L}}{\partial A_\mu} - \frac{\partial}{\partial x^\sigma}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\sigma A_\mu)}\right) - \frac{\partial}{\partial x^\sigma}\frac{\partial}{\partial x^\tau}\left(\frac{\partial^2\mathcal{L}}{\partial(\partial_\tau\partial_\sigma A_\mu)}\right)\dots$$

We vary wrt A because the real degree of freedom of electrodynamics is not the E and B fields but, the underlying potential. By $\mathcal{L}$ we denote the *Lagrangian density* $\sqrt{-g}F_{\alpha\beta}F^{\gamma\delta}g^{\alpha\gamma}g^{\beta\delta}$. It is a density due to the square root of the determinant of the metric. After some calculations Maxwell's action yields the inhomogeneous Maxwell equations

$$\left(\nabla_{\frac{\partial}{\partial x^\mu}}F\right)^{\mu\alpha} = j^\alpha.$$

If the current comes from point charged particle then $j = q\,u_\gamma$. The homogeneous Maxwell equations are

$$\left(\nabla_{[\alpha}F\right)_{\beta\gamma]} = 0$$

and they arise directly from $F_{\alpha\beta} :- 2\partial_{[\alpha}A_{\beta]} = 2\nabla_{[\alpha}A_{\beta]}$.

14.3 Energy-Momentum Tensor of Matter Fields

At some point we want to write down an action for the metric tensor itself so as to find the equations of motion of spacetime curvature i.e. Einstein equations. However, this action $S_{grav}[g]$ will added any $S_{matter}[A,\phi,\dots]$ in order to describe the total system. Therefore we will have an action of the form

$$S_{total}[g,A] = S_{grav}[g] + S_{Maxwell}[A,g]$$

Varying wrt A will yield the Maxwell equations while varying wrt g will give eoms of the form

$$\{contribution\,from\,S_{grav}\} + \{contribution\,from\,S_{Maxwell}\} = 0$$

where the second term will be the energy momentum tensor of a Maxwell field.

Definition 14.3.1. *If $S_{matter}[\Phi, g]$ is the matter action then the so-called* ***energy momentum tensor*** *is given by*

$$T^{\alpha\beta} :- \frac{\pm 2}{\sqrt{-g}} \left(\frac{\partial L_{matter}}{\partial g_{\alpha\beta}} - \partial_\sigma \frac{\partial L_{matter}}{\partial(\partial_\sigma g_{\alpha\beta})} + \ldots \right)$$

The sign isn't fixed because it depends on the convention one will consider. However, it wise to choose all sign convention such that $T(\epsilon^0, \epsilon^0) > 0$ where ϵ^0 is an element of the unique dual basis of an observer frame basis. Therefore $T(\epsilon^0, \epsilon^0)$ with what an observer sees. Basically this is a positive energy requirement.

Example 14.3.1. *For $S_{Maxwell}$ we get*

$$T^{Maxwell}_{\alpha\beta} = F_{\alpha\mu} F_{\beta\nu} g^{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g_{\alpha\beta}$$

which of course changes from point to point depending on the form of the electromagnetic fields. Some observer dependent quantities are

$$T(e_0, e_0) = E^2 + B^2$$

which is the energy density and,

$$T(e_0, e_\alpha) = (E \times B)_\alpha \,.$$

An important fact is that we usually do not specify the fundamental action for matter but we are rather satisfied to assume certain properties of general forms of $T_{\alpha\beta}$. For example, cosmology , in a homogeneous and isotropic universe it is usually considered that in the large scales the galaxies and galaxy clusters form a *perfect fluid* of pressure p and density ϱ. This is modeled by an energy-momentum tensor of the form[1]

$$T^{\alpha\beta} = (\rho + p) u^\alpha u^\beta - p\, g^{\alpha\beta}$$

and one does not care about saying from what action does it come. This is phenomenology, it is our idea of how the contents of the universe look like at very large scales.

[1] This is in component notation; in terms of tensor operations the same equation is writen

$$\mathbf{T} = (\rho + p)\mathbf{u} \otimes \mathbf{u} - p\mathbf{g}$$

Einstein Gravity

Recall that in Newtonian spacetime we were able to re-formulate Poisson's Law $\Delta\phi = 4\pi G_N \rho$ in terms of the Newtonian spacetime curvature as

$$R_{00} = 4\pi G_N \rho \,.$$

Roughly speaking, this prompted Einstein to postulate that the relativistic field equations for the Lorentzian metric g of (relativistic) spacetime ought to be

$$R_{\alpha\beta} = 8\pi G_N T^{\alpha\beta}$$

However, this equation suffers from a problem. If one formulates the matter action in a chart independent way, it can be shown that there is a conserved Noether current that gives $(\nabla_\alpha T)^{\alpha\beta} = 0$; which is a conservation law of energy-matter. However, the situation is different on the lhs because in general $(\nabla_\alpha R)^{\alpha\beta} \neq 0$. Thus if we take the divergence on both sides of the equation it would lead to a contradiction. Einstein tried to argue this problem away ... nevertheless it is wrong. The solution for this came of course by Einstein but at the same time by Hilbert, a famous mathematician who was a specialist on variational principles.

Hilbert

As we saw in the previous chapter, the energy-momentum tensor generally arises from the variation of $\mathcal{L}_{matter}$ of an action. Hilbert thought since $T^{\alpha\beta}$ comes from an action, why not to derive also the lhs from an action.

$$S_{Hilbert}[g] = \int_M \sqrt{-g}\, R_{\alpha\beta}\, g^{\alpha\beta}, .$$

From Riemannian geometry it can be shown that $, R_{\alpha\beta}\, g^{\alpha\beta}$ is the simplest function one can built from a metric , its 1^{st} and 2^{nd} derivatives. The aim is to vary this action wrt metric $g_{\alpha\beta}$ and result in some tensor $G_{\alpha\beta} = 0$. We would like to find the specific form of $G_{\alpha\beta}$.

14.4 Variation of Hilbert Action

We require $\delta S_{Hilbert}[g] \stackrel{!}{=} 0$ and we calculate

$$\delta S_{Hilbert}[g] = \int_M \left[\delta\sqrt{-g} g^{\alpha\beta} R_{\alpha\beta} + \sqrt{-g} g^{\alpha\beta}\, R_{\alpha\beta} + \sqrt{-g} g^{\alpha\beta} R_{\alpha\beta}\right]$$

First Term:

$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}\delta(-g) = -\frac{1}{2}\frac{\sqrt{-g}}{-g}\delta g = \frac{1}{2}\sqrt{-g}\,\delta(\ln g) :- \frac{1}{2}\sqrt{-g}\,\delta\left(\ln(\det g)\right)$$

and by choosing a basis in which g is diagonal (it can also be proven using an arbitrary basis) we get

$$\begin{aligned}\delta\sqrt{-g} &= \frac{1}{2}\sqrt{-g}\delta\left(\ln\left(\prod_{i=0}^{3} g_{ii}\right)\right) = \frac{1}{2}\sqrt{-g}\delta\left(\sum_{i=0}^{3}\ln g_{ii}\right) = \frac{1}{2}\sqrt{-g}\sum_{i=0}^{3}\delta(\ln g_{ii}) = \frac{1}{2}\sqrt{-g}\sum_{i=0}^{3}\frac{1}{g_{ii}}\delta g_{ii}\\ &= \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta}\end{aligned}$$

Second Term:

$$\begin{aligned}&g^{\alpha\beta}g_{\alpha\gamma} = \delta^{\alpha}_{\gamma} \to \delta g^{\alpha\beta}\,g_{\alpha\gamma} + g^{\alpha\beta}\,\delta g_{\alpha\gamma} = 0 \to \delta g^{\alpha\beta}\,g_{\alpha\gamma} = -g^{\alpha\beta}\,\delta g_{\alpha\gamma} \to\\ &\delta g^{\alpha\beta} = -g^{\alpha\mu}\,g^{\beta\nu}\delta g_{\mu\nu}\end{aligned}$$

where we multiplied both sides by $g^{\alpha\mu}$ and remaned the indices.

Third Term:

If use normal coordinates we know that the connection coefficient function will vanish. Then the variation of the Ricci tensor will give

$$\begin{aligned}\delta R_{\alpha\beta} &= \delta\partial_{\beta}\Gamma^{\mu}{}_{\alpha\mu} - \delta\partial_{\mu}\Gamma^{\mu}{}_{\alpha\beta} = \partial_{\beta}\delta\Gamma^{\mu}{}_{\alpha\mu} - \partial_{\mu}\delta\Gamma^{\mu}{}_{\alpha\beta}\\ &= \nabla_{\beta}\delta\Gamma^{\mu}{}_{\alpha\mu} - \nabla_{\mu}\delta\Gamma^{\mu}{}_{\alpha\beta}\end{aligned}$$

Let us explain what happened here. We know that $\delta\Gamma$ is basically a term of the form

$$\Gamma^{i}{}_{(x)jk} - \tilde{\Gamma}^{i}_{(x)jk}$$

which is a tensor because the non-tensorial terms that arise when we change charts, cancel out due to the subtraction. Hence $\delta\Gamma$ is a tensor. In the next step we were able to lift the partial derivatives into covariant derivatives because their only difference is some extra terms that contain the Γs. But since we are using normal coordinates all the Γs vanish.[2] Therefore

$$\delta R_{\alpha\beta} = \nabla_{\beta}(\delta\Gamma)^{\mu}_{\alpha\mu} - \nabla_{\mu}(\delta\Gamma)^{\mu}_{\alpha\beta} :- \delta\Gamma^{\mu}{}_{\alpha\mu;\beta} - \delta\Gamma^{\mu}{}_{\alpha\beta;\mu}$$

where we used the *notation* $(\nabla_{\beta}A)^{i}_{j} :- A^{i}_{j;\beta}$. Now the whole third term takes the form

$$\begin{aligned}\sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta} &\overset{\nabla g=0}{=} \sqrt{-g}\left(\underbrace{g^{\alpha\beta}\delta\Gamma^{\mu}{}_{\alpha\mu}}_{A^{\beta}}\right)_{;\beta} - \sqrt{-g}\left(\underbrace{g^{\alpha\beta}\delta\Gamma^{\mu}{}_{\alpha\beta}}_{B^{\mu}}\right)_{;\mu}\\ &= \sqrt{-g}A^{\beta}_{;\beta} - \sqrt{-g}B^{\mu}_{;\mu} = \left(\sqrt{-g}A^{\beta}\right)_{,\beta} - \left(\sqrt{-g}B^{\mu}\right)_{,\mu}\end{aligned}$$

[2] The fact we are using these particular coordinates, greatly simplifies the calculations. One can follow the same procedure in an arbitrary chart - so as to be sure the result is chart independent - and will arrive in the same results, in the end.

Collecting the terms one obtains

$$\delta S_{Hilbert} = \int_M \left[\frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}R - \sqrt{-g}g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu}R_{\alpha\beta} + \left(\sqrt{-g}A^{\beta}\right)_{,\beta} - \left(\sqrt{-g}B^{\mu}\right)_{,\mu}\right].$$

The last two terms are basically surface terms due to Gauss theorem. Recall from the variations in classical mechanics that the initial and final points were fixed, in order for the variational principle to hold. Similarly, the proper way to do this field theory calculation is to fix an initial an a final surface and let the fields vary in between. Therefore in the action there may be surface terms that survive. However, the surface term do not make a difference to the equations of motion[3]. Hence we get

$$\delta S_{Hilbert} = \int_M \delta g_{\mu\nu}\left[\frac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right]$$

which must be zero for an arbitrary variation $\delta g_{\mu\nu}$. Therefore the terms inside the bracket must vanish. Thus,

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\,.$$

From this mathematical argument Hilbert concluded that one may take

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu}$$

and in fact Einstein by physical arguments, arrived at the same result. The last equation is the famous ***Einstein equation*** while the action from which we derived it is mostly known in the literature as $S_{Einstein-Hilbert}$.

As far as the solution of the $(\nabla_\alpha T)^{\alpha\beta} = 0$ issue is concerned, one can show that the Einstein curvature $G_{\alpha\beta}$ satisfies the so-called *contracted Bianchi identities*

$$(\nabla_\alpha G)^{\alpha\beta} = 0\,.$$

Note that the Einstein field equations can be written down also in slightly different way. If we multiply by $g^{\mu\nu}$ we get

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu} \xrightarrow{\cdot g^{\mu\nu}} R - 2R = T :- T_{\mu\nu}g^{\mu\nu} \longrightarrow R = -T$$

and now we substitute this result back and we get

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T$$

so Einstein was not that wrong after all; he just took the wrong version of the energy-momentum tensor.

Now just for completeness we mention that if one wants to introduce the so-called *cosmological constant* into the game, the Einstein-Hilbert action becomes

$$S_{E-H} = \int_M \sqrt{-g}\left(R + 2\Lambda\right)\,.$$

[3]for a careful treatment of the surface terms see [6].

Mathematically Λ is nothing mysterious, it is simply a constant over the mfd. Around 1915 Einstein stated that the cosmological constant must exist and also that $\Lambda < 0$ it is negative. He made that claim because that is the only way to produce cosmological solutions that describe a static universe. After some years Hubble observed that the universe is indeed expanding which is perfectly compatible with $\Lambda = 0$. That mistake on the behaviour of Λ was what Einstein called "his biggest blunder". Today the cosmological constant is supposed to satisfy $\Lambda > 0$ to account for the accelerated expansion of the universe. The problem is that according to our measurements it must be positive but at the same time , very small compared to any effect we could imagine that produces a non-vanishing Λ.

Taking a closer look at S_{E-H} we see that $\Lambda \neq 0$ can be interpreted as a contribution to the energy momentum tensor of matter in spacetime. Specifically a $-\frac{1}{2}\Lambda g_{\mu\nu}$ contribution of one does the calculations. That means constantly over all the universe there energy provided by Λ. This energy is recently known as *dark energy.* It does not interact with anything but it contribute to the curvature of spacetime. That why we call it *dark*.

Part II

Modified Gravity

Chapter 15

Einstein Gravity with Minimally Coupled Scalar Field

In this section we are going to derive the local solution for the Einstein gravity with a minimally coupled scalar field based on the work of [12].

The line element in isotropic coordinates is given by

$$ds^2 = -e^{f(\rho)}dt^2 + e^{-h(\rho)}\left[d\rho^2 + r\rho^2 d\Omega_2^2\right] \tag{15.1}$$

while the Klein-Gordon equation for the scalar field gives

$$\nabla_\mu \nabla^\mu \phi = 0 \longrightarrow \ddot{\phi} + \dot{\phi}\left(\frac{\dot{f}}{2} - \frac{\dot{h}}{2} + \frac{2}{\rho}\right) = 0$$

$$\left(e^{f/2-h/2}\rho^2 \phi_{,\rho}\right)_{,\rho} = 0 \tag{15.2}$$

which in D-dimensions is written $\left(e^{f/2+(3-D)h/2}\,\rho^{D-2}\phi_{,\rho}\right)_{,\rho} = 0$. After performing an integration, equation (15.2) gives

$$\dot{\phi} = \frac{C}{\rho^{D-2}} \exp\left[\frac{(D-3)h - f}{2}\right]. \tag{15.3}$$

From the Einstein equations $R_{\mu\nu} = k\nabla_\mu\phi\nabla_\nu\phi$ only the $(\rho,\rho)-$component will survive, since we have assumed a *static and spherically symmetric solution.* Hence, $R_{\rho\rho} = k\dot{\phi}^2$ and $R_{tt} = R_{ii} = 0$. Let us now consider the relation

$$R^t_{\ t} + (D-3)R^i_{\ i} = 0$$

$$\Rightarrow \left\{\ddot{f} - (D-3)\ddot{h}\right\} + \frac{1}{2}\left\{\dot{f}^2 - 2\dot{h}\dot{f}(D-3) + (D-3)^2\dot{h}^2\right\} + \frac{(2D-5)}{\rho}\left\{\dot{f} - (D-3)\dot{h}\right\} = 0 \tag{15.4}$$

which can be written as a first order differential equation with respect to y, where $y \equiv \dot{f} - (D-3)\dot{h}$.

$$\dot{y} + \frac{1}{2}y^2 + \frac{(2D-5)}{\rho}y = 0 \tag{15.5}$$

The solution of the above equation is

$$y = \frac{4(D-3)A}{\rho\left(\rho^{2D-6}+A\right)}, \tag{15.6}$$

where $A = -\frac{1}{4C_1(D-3)}$ an integration constant. After substituting y back and performing an integration, one finds a relation between f and h.

$$\dot{f} - (D-3)\dot{h} = \frac{4(D-3)A}{\rho\left(\rho^{2D-6}+A\right)} \to f - (D-3)h = \ln\left(\frac{A+\rho^{2D-6}}{\hat{B}\rho^{2D-6}}\right)^2, \tag{15.7}$$

where $C_2 = -\ln\left(\hat{B}\right)^2$ an integration constant. By introducing the relation (15.7) into the scalar field equation (15.3), the last is written

$$\dot{\phi} = \frac{C}{\rho^{D-2}}\exp\left[\frac{(D-3)h-f}{2}\right] \xrightarrow{(15.7)} \dot{\phi} = C\hat{B}\frac{\rho^{D-4}}{\rho^{2D-6}+A}. \tag{15.8}$$

Now we wish to write the $(t,t)-$component of the Einstein equation, in order to produce a second relation between f and h.

$$R_{tt} = 0 \xrightarrow{\dot{f}\neq 0} \dot{f}\left[\frac{\ddot{f}}{\dot{f}} + \frac{\dot{f}}{2} - \frac{\dot{h}}{2} + \frac{2}{\rho}\right] = 0 \tag{15.9}$$

whose generalization in $D-$dimensions is $\dot{f}\left[\frac{\ddot{f}}{\dot{f}} + \frac{\dot{f}}{2} + \frac{\dot{h}}{2} + \frac{D-2}{r}\left(1-\frac{r\dot{h}}{2}\right)\right] = 0$. In order to find a differential equation purely with respect to f, one has to solve (15.6) with respect to $\dot{h}$ and substitute in (15.9). Equation (15.6) gives

$$\dot{h} = \frac{4A}{\rho\left(\rho^{2D-6}+A\right)} + \frac{\dot{f}}{D-3} \tag{15.10}$$

therefore for arbitrary number of dimensions $D \geq 4$ equation (15.9) is written as

$$\dot{f}\left[\frac{\ddot{f}}{\dot{f}} + \frac{\dot{f}}{2} + \frac{\dot{h}}{2} + \frac{D-2}{\rho}\left(1-\frac{\rho\dot{h}}{2}\right)\right] = 0 \xrightarrow{(15.10)} \ddot{f} - \frac{2(D-3)A-(D-2)(\rho^{2D-6}+A)}{\rho(\rho^{2D-6}+A)}\dot{f} = 0 \tag{15.11}$$

which, after an integration gives

$$\dot{f} = C_3\frac{\rho^{D+2}}{A\rho^6+\rho^{2D}} \longrightarrow \dot{f} = \hat{B}G\frac{\rho^{D-4}}{\rho^{2D-6}+A} \tag{15.12}$$

$$\tag{15.13}$$

where $C_3 = \hat{B}G$ another integration constant. Thus, now we have produced three decoupled differential equations with respect to ϕ, f and h, respectively. By plugging the relations (15.6),(15.8) and (15.12) in the $(\rho,\rho)-$component of the Einstein equations, we obtain an identity about the various integration constants encountered so far.

$$R_{\rho\rho} = k\dot{\phi}^2 \longrightarrow 4A(D-3)(D-2) = -\left[\hat{B}^2C^2k + \frac{1}{4}\frac{D-2}{D-3}\hat{B}^2G^2\right] \tag{15.14}$$

This relation indicates that A must be negative and that is why we shall denote it by $A = -\rho_o^{2D-6}$. This new information about the constant A puts us in a position where we are able to solve (15.8),(15.10) and (15.12) explicitly. Note that (15.8),(15.12) are practically the same equation. Let us now consider (15.8)

$$\dot{\phi} = C\hat{B}\frac{\rho^{D-4}}{\rho^{2D-6}+A} \,. \tag{15.15}$$

It can be verified that

$$\frac{d}{d\rho}\left\{-\frac{\hat{B}C}{\rho_o^{D-3}(D-3)}\tanh^{-1}\left(\frac{\rho_o^{D-3}}{\rho^{D-3}}\right)\right\} = \frac{d}{d\rho}\left\{-\frac{\hat{B}C}{\rho_o^{D-3}(D-3)}\tanh^{-1}\left(\frac{\rho^{D-3}}{\rho_o^{D-3}}\right)\right\} = C\hat{B}\frac{\rho^{D-4}}{\rho^{2D-6}-\rho_o^{2D-6}} \tag{15.16}$$

hence, (15.8) admits two solutions in the intervals $\left|\frac{\rho_o^{D-3}}{\rho^{D-3}}\right| < 1$ and $\left|\frac{\rho^{D-3}}{\rho_o^{D-3}}\right| < 1$, respectively. For now we will be concerned only with the first solution, for reasons that will be explained later on. Thus the solution reads

$$\phi = -\frac{\hat{B}C}{\rho_o^{D-3}(D-3)}\tanh^{-1}\left(\frac{\rho_o^{D-3}}{\rho^{D-3}}\right)$$

and by taking advantage of

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \ , \ |x| < 1 \tag{15.17}$$

we arrive at

$$\phi = \frac{\hat{B}C}{2\rho_o^{D-3}(D-3)}\ln\left(\frac{\rho^{D-3}-\rho_o^{D-3}}{\rho^{D-3}+\rho_o^{D-3}}\right) . \tag{15.18}$$

Following the same procedure, the solution of (15.12) will be given by

$$f = \frac{\hat{B}G}{2\rho_o^{D-3}(D-3)}\ln\left(\frac{\rho^{D-3}-\rho_o^{D-3}}{\rho^{D-3}+\rho_o^{D-3}}\right) \tag{15.19}$$

and we have completely determined the form of the function f, we can proceed by solving (15.10)

$$\dot{h} = \frac{-4\rho_o^{2D-6} + \frac{\hat{B}G}{(D-3)}\rho^{D-3}}{\rho\left(\rho^{2D-6}-\rho_o^{2D-6}\right)} \longrightarrow h = -\frac{\hat{B}G}{2\rho_o^{D-3}(D-3)^2}\ln\left(\frac{\rho^{D-3}+\rho_o^{D-3}}{\rho^{D-3}-\rho_o^{D-3}}\right) + \frac{2}{D-3}\ln\left(\frac{\rho^{2D-6}}{\rho^{2D-6}-\rho_o^{2D-6}}\right) . \tag{15.20}$$

These last three relations constitute the solutions of the scalar field and Einstein equations, respectively. One can bring these solution on a more elegant form by replacing constants $\hat{B}$ and G with a new parameter, lets say γ given by

$$4(D-3)\rho_o^{D-3}\gamma = \hat{B}G \,. \tag{15.21}$$

Now (15.19) and (15.20) take the form

$$f = \ln\left(\frac{\rho^{D-3}-\rho_o^{D-3}}{\rho^{D-3}+\rho_o^{D-3}}\right)^{2\gamma} , \tag{15.22}$$

$$h = \ln\left[\left(\frac{\rho^{D-3}-\rho_o^{D-3}}{\rho^{D-3}+\rho_o^{D-3}}\right)^{2\gamma/(D-3)} \cdot \left(\frac{\rho^{2D-6}-\rho_o^{2D-6}}{\rho^{2D-6}}\right)^{2/(D-3)}\right] \tag{15.23}$$

By replacing (15.21) inside (15.14) and making the substitution $A = \rho_o^{2D-6}$,the coefficient of the logarithm in (15.18) gives

$$4A(D-3)(D-2) = -\left[\hat{B}^2C^2k + \frac{1}{4}\frac{D-2}{D-3}\hat{B}^2G^2\right] \xrightarrow{(15.21)}$$

$$\hat{B}^2C^2 = \frac{4\rho_o^{2D-6}}{k}(D-2)(D-3)(1-\gamma^2)\,. \tag{15.24}$$

Consequently the relation (15.18) takes the form

$$\phi = \left[\frac{1}{k}\frac{D-2}{D-3}(1-\gamma^2)\right]^{1/2}\ln\left(\frac{\rho^{D-3}-\rho_o^{D-3}}{\rho^{D-3}+\rho_o^{D-3}}\right) \equiv \tilde{\gamma}\ln\left(\frac{\rho^{D-3}-\rho_o^{D-3}}{\rho^{D-3}+\rho_o^{D-3}}\right). \tag{15.25}$$

Thus the final form of the solution for the metric and scalar field is

$$e^f = \left(\frac{\rho^{D-3}-\rho_o^{D-3}}{\rho^{D-3}+\rho_o^{D-3}}\right)^{2\gamma} \tag{15.26}$$

$$e^{-h} = \left(\frac{\rho^{D-3}+\rho_o^{D-3}}{\rho^{D-3}-\rho_o^{D-3}}\right)^{2\gamma/(D-3)} \cdot \left(1-\frac{\rho_o^{2D-6}}{\rho^{2D-6}}\right)^{2/(D-3)} \tag{15.27}$$

$$\phi = \tilde{\gamma}\ln\left(\frac{\rho^{D-3}-\rho_o^{D-3}}{\rho^{D-3}+\rho_o^{D-3}}\right)\ ,\ \tilde{\gamma} = \left[\frac{1}{k}\frac{D-2}{D-3}(1-\gamma^2)\right]^{1/2} \tag{15.28}$$

$$\tag{15.29}$$

15.0.1 Investigation of the Second Solution

Now lets examine the solution of (15.8), defined in the interval $\left|\frac{\rho^{D-3}}{\rho_o^{D-3}}\right| < 1$. We write

$$\dot{f} = G\hat{B}\frac{\rho^{D-4}}{\rho^{2D-6}+A} \xrightarrow{0<\rho<\rho_o} f = \frac{G\hat{B}}{2\rho_o^{D-3}(D-3)}\ln\left(\frac{\rho_o^{D-3}-\rho^{D-3}}{\rho_o^{D-3}+\rho^{D-3}}\right) = 2\gamma\ln\left(\frac{\rho_o^{D-3}-\rho^{D-3}}{\rho_o^{D-3}+\rho^{D-3}}\right)$$

hence the corresponding metric component takes the form

$$e^f = \left(\frac{\rho_o^{D-3}-\rho^{D-3}}{\rho_o^{D-3}+\rho^{D-3}}\right)^{2\gamma}\,. \tag{15.30}$$

Equation (15.10) now yields

$$\dot{h} = \frac{4A}{\rho(\rho^{2D-6}+A)} + \frac{\dot{f}}{D-3} \Longrightarrow \dot{h} = \frac{4A + \frac{G\hat{B}}{(D-3)}\rho^{D-3}}{\rho(\rho^{2D-6}+A)} \xrightarrow{0<\rho<\rho_o}$$

$$h = 2\gamma\ln\left(\frac{\rho_o^{D-3}-\rho^{D-3}}{\rho_o^{D-3}+\rho^{D-3}}\right) + \frac{2}{D-3}\ln\left(\frac{\rho^{2D-6}}{\rho_o^{2D-6}-\rho^{2D-6}}\right) \tag{15.31}$$

thus,

$$e^{-h} = \left(\frac{\rho_o^{D-3}+\rho^{D-3}}{\rho_o^{D-3}-\rho^{D-3}}\right)^{2\gamma}\left(\left(\frac{\rho_o}{\rho}\right)^{2D-6}-1\right)^{2/(D-3)} \tag{15.32}$$

and the whole solution takes the form

$$e^f = \left(\frac{\rho_o^{D-3} - \rho^{D-3}}{\rho_o^{D-3} + \rho^{D-3}} \right)^{2\gamma} \tag{15.33}$$

$$e^{-h} = \left(\frac{\rho_o^{D-3} + \rho^{D-3}}{\rho_o^{D-3} - \rho^{D-3}} \right)^{2\gamma/(D-3)} \cdot \left(\frac{\rho_o^{2D-6}}{\rho^{2D-6}} - 1 \right)^{2/(D-3)} \tag{15.34}$$

$$\phi = \tilde{\gamma} \ln \left(\frac{\rho_o^{D-3} - \rho^{D-3}}{\rho_o^{D-3} + \rho^{D-3}} \right) \ , \ \tilde{\gamma} = \left[\frac{1}{k} \frac{D-2}{D-3} (1 - \gamma^2) \right]^{1/2} . \tag{15.35}$$

The transition to Schwarzschild coordinates is given by

$$r = \rho e^{-h/2} \tag{15.36}$$

which in $D = 4$ dimensions yields

$$r = \rho \left(\frac{\rho_o + \rho}{\rho_o - \rho} \right)^{\gamma} \left(\left(\frac{\rho_o}{\rho} \right)^2 - 1 \right) \ , \ 0 < \rho < \rho_o \, . \tag{15.37}$$

The areal radius r is found to be a decreasing function of ρ in $0 < \rho < \rho_o$. Moreover, $r \xrightarrow{\rho \to \rho_o} 0$ while $r \xrightarrow{\rho \to 0} \infty$.

Along similar lines, the radial coordinate is well defined in both coordinate systems (isotropic and Schwarzschild) for the solution (15.26)-(15.28). It is given by

$$r = \rho e^{-h/2} = \rho \left(\frac{\rho + \rho_o}{\rho - \rho_o} \right)^{\gamma} \left(1 - \left(\frac{\rho_o}{\rho} \right)^2 \right) \ , \ \rho > \rho_o > 0 \tag{15.38}$$

and is increasing with ρ. In this case, $r \xrightarrow{\rho \to \rho_o} 0$ while $r \xrightarrow{\rho \to \infty} \infty$. Hence, both solutions have an asymptotically flat limit in the isotropic chart, together with the fact that they correspond to $r = 0$ when $\rho \to \rho_o$.

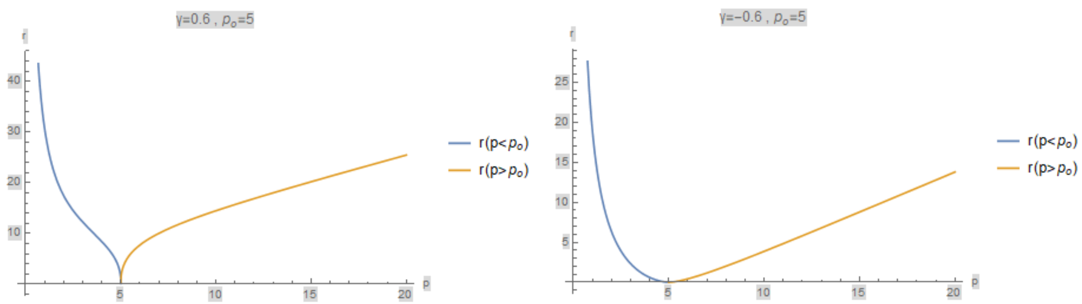


Figure 15.1: Plot of Areal radius r vs the isotropic radius for both solutions (15.26) -(15.28) and (15.33)-(15.35). The point $\rho = \rho_o$ always corresponds to vanishing r.

Chapter 16

Complete Brans-Dicke Theory

We will consider a modified Brans-Dicke theory presented in [13] and described by the following equations

$$G^{\mu}{}_{\nu} = \frac{8\pi}{\phi}(T^{\mu}{}_{\nu} + \mathcal{T}^{\mu}{}_{\nu}) \tag{16.1}$$

$$\begin{aligned} T^{\mu}{}_{\nu} =& \frac{\phi}{2\lambda(\nu+8\pi\phi^2)^2}\Big\{2\big[(1+\lambda)\nu+4\pi(2-3\lambda)\phi^2\big]\phi^{;\mu}\phi_{;\nu} - \big[(1+2\lambda)\nu+4\pi(2-3\lambda)\phi^2\big]\delta^{\mu}{}_{\nu}\phi^{;\rho}\phi_{;\rho}\Big\} \\ &+\frac{\phi^2}{\nu+8\pi\phi^2}\big(\phi^{;\mu}{}_{;\nu} - \delta^{\mu}{}_{\nu}\Box\phi\big) \end{aligned} \tag{16.2}$$

$$\Box\phi = 4\pi\lambda\mathcal{T} \tag{16.3}$$

$$\mathcal{T}^{\mu}{}_{\nu;\mu} = \frac{\nu}{\phi(\nu+8\pi\phi^2)}\mathcal{T}^{\mu}{}_{\nu}\phi_{;\mu}\,. \tag{16.4}$$

The new characteristic of these equations compared to the standard Brans-Dicke theory is the appearance of the dimensionfull parameter (with dimensions mass to the fourth) ν which is encountered in the gravitational field equation (16.1) and at the same time it violates the exact conservation of the matter energy-momentum tensor $\mathcal{T}^{\mu}_{\nu}$ in (16.4). For $\nu = 0$ the system (16.1)-(16.4) reduces to the Brans-Dicke equations of motion (in units with unit velocity of light)

$$G^{\mu}{}_{\nu} = \frac{8\pi}{\phi}(T^{\mu}{}_{\nu} + \mathcal{T}^{\mu}{}_{\nu}) \tag{16.5}$$

$$T^{\mu}{}_{\nu} = \frac{2-3\lambda}{16\pi\lambda\phi}\Big(\phi^{;\mu}\phi_{;\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}\phi^{;\rho}\phi_{;\rho}\Big) + \frac{1}{8\pi}\big(\phi^{;\mu}{}_{;\nu} - \delta^{\mu}{}_{\nu}\Box\phi\big) \tag{16.6}$$

$$\Box\phi = 4\pi\lambda\mathcal{T} \tag{16.7}$$

$$\mathcal{T}^{\mu}{}_{\nu;\mu} = 0\,, \tag{16.8}$$

which is described by the action

$$S_{BD} = \frac{1}{16\pi}\int d^4x\,\sqrt{-g}\,\Big(\phi R - \frac{\omega_{BD}}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\Big) + \int d^4x\,\sqrt{-g}\,L_m\,, \tag{16.9}$$

where $L_m(g_{\kappa\lambda}, \Psi)$ is the matter Lagrangian depending on some extra fields Ψ. Equations (16.1)-(16.4) are the unique construction under the assumption of the simple form of the scalar field equation (16.3) and given that the energy-momentum tensor $T^{\mu}{}_{\nu}$ of ϕ is made from terms each of which involves two derivatives of one or two ϕ fields, and ϕ itself. The parameter ν appears as an integration constant in this construction. The parameter $\lambda \neq 0$ is related to the standard Brans-Dicke parameter $\omega_{BD} = \frac{2-3\lambda}{2\lambda}$.

Here, we will be interested in the vacuum theory with $\mathcal{T}^{\mu}{}_{\nu} = 0$. Although the extra matter vanishes, it leaves an impact on the vacuum equation (16.1) through the parameter ν, and this is the novel difference compared to the vacuum Brans-Dicke equation (16.5). This vacuum theory arises from the action [14]

$$S = \frac{\eta}{2(8\pi)^{3/2}} \int d^4x \sqrt{-g} \left[\sqrt{|\nu+8\pi\phi^2|}\, R - \frac{8\pi}{\lambda} \frac{\nu+4\pi(2-3\lambda)\phi^2}{|\nu+8\pi\phi^2|^{3/2}} g^{\mu\nu} \phi_{,\mu}\phi_{,\nu} \right], \tag{16.10}$$

where $\eta = \mathrm{sgn}(\phi)$. It is useful for the following analysis of spherically symmetric solutions to transform the action (16.10) to its canonical form. Let the conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^2(\phi) g_{\mu\nu}\,, \tag{16.11}$$

where

$$\Omega = \left(\frac{|\nu+8\pi\phi^2|}{8\pi} \right)^{\frac{1}{4}}, \tag{16.12}$$

together with a field redefinition from the field $\phi(x)$ to the new field $\sigma(x)$ defined by

$$\frac{d\phi}{d\sigma} = \sqrt{\frac{|\lambda|}{16\pi}} \sqrt{|\nu+8\pi\phi^2|}\,. \tag{16.13}$$

The action (16.10) takes the form

$$S = \frac{\eta}{16\pi} \int d^4x \sqrt{-\tilde{g}} \left(\tilde{R} - \frac{1}{2}\epsilon\epsilon_\lambda \tilde{g}^{\mu\nu} \sigma_{,\mu}\sigma_{,\nu} \right), \tag{16.14}$$

where $\epsilon = \mathrm{sgn}(\nu+8\pi\phi^2)$, $\epsilon_\lambda = \mathrm{sgn}(\lambda)$. The Lagrangian (16.14) refers to the Einstein frame where the gravitational coupling is a true constant and the field σ behaves as a usual scalar field. In order for σ not to be a ghost, and so to behave as a normal field with positive energy, it should be $\epsilon\epsilon_\lambda > 0$. This is achieved even if the kinetic term in (16.10) is positive. Therefore, we assume throughout that $\epsilon\epsilon_\lambda = 1$. For $\epsilon > 0$, the integration of equation (16.13) gives

$$\sigma = \sqrt{\frac{2}{|\lambda|}} \ln \left| 4\pi\phi + \sqrt{2\pi}\sqrt{\nu+8\pi\phi^2} \right|, \tag{16.15}$$

where an additive integration constant σ_0 has been absorbed into σ. Inversely,

$$\phi = \frac{s}{8\pi} \left(e^{\sqrt{\frac{|\lambda|}{2}}\sigma} - 2\pi\nu e^{-\sqrt{\frac{|\lambda|}{2}}\sigma} \right), \tag{16.16}$$

where $s = \mathrm{sgn}(4\pi\phi+\sqrt{2\pi}\sqrt{\nu+8\pi\phi^2}) = \mathrm{sgn}\left(e^{\sqrt{\frac{|\lambda|}{2}}\sigma} + 2\pi\nu e^{-\sqrt{\frac{|\lambda|}{2}}\sigma}\right)$. The conformal factor Ω in terms of the new field σ takes the form

$$\Omega = \frac{1}{\sqrt{8\pi}} \left| e^{\sqrt{\frac{|\lambda|}{2}}\sigma} + 2\pi\nu e^{-\sqrt{\frac{|\lambda|}{2}}\sigma} \right|^{\frac{1}{2}}. \tag{16.17}$$

For the physically more interesting case with $\phi > 0$, the absolute value in (16.17) disappears. For $\epsilon < 0$, the integration of equation (16.13) gives

$$\sigma = \sqrt{\frac{2}{|\lambda|}} \arcsin \left(\sqrt{\frac{8\pi}{|\nu|}}\,\phi \right), \tag{16.18}$$

where again an additive integration constant σ_0 has been absorbed into σ and it is $-\frac{\pi}{2} < \sqrt{\frac{|\lambda|}{2}}\,\sigma < \frac{\pi}{2}$. Inversely,

$$\phi = \sqrt{\frac{|\nu|}{8\pi}}\,\sin\left(\sqrt{\frac{|\lambda|}{2}}\,\sigma\right). \tag{16.19}$$

The conformal factor Ω in terms of the new field σ takes the form

$$\Omega = \left(\frac{|\nu|}{8\pi}\right)^{\frac{1}{4}}\left[\cos\left(\sqrt{\frac{|\lambda|}{2}}\,\sigma\right)\right]^{\frac{1}{2}}. \tag{16.20}$$

After the solution of the fields $\tilde{g}_{\mu\nu}, \sigma$ governed by the action (16.14) has been derived, the solution for the initial fields $g_{\mu\nu}, \phi$ is found through the equations (16.11), (16.16), (16.19) as functions of ρ. The action (16.14) defines Einstein gravity minimally coupled to a scalar field whose equations of motion are

$$\tilde{G}_{\mu\nu} = \frac{1}{2}\sigma_{,\mu}\sigma_{,\nu} - \frac{1}{4}\tilde{g}_{\mu\nu}\tilde{g}^{\kappa\lambda}\sigma_{,\kappa}\sigma_{,\lambda} \tag{16.21}$$

$$\tilde{\Box}\sigma = 0\,. \tag{16.22}$$

The solution of this system, assuming spherical symmetry, has been found in [12]. In the Einstein frame we consider a static spherically symmetric line element in isotropic coordinates

$$d\tilde{s}^2 = -e^f dt^2 + e^{-h}\left[d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right], \tag{16.23}$$

where f, h are functions of the radial coordinate ρ (we keep the symbol r for the radius in the standard coordinates). Due to the symmetry it is also $\sigma(\rho)$. The solution of the system (16.21), (16.22) is the following [12]

$$\begin{aligned}
\sigma &= 2\sqrt{1-\gamma^2}\ \ln\frac{\rho-\rho_o}{\rho+\rho_o}\\
e^f &= \left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{2\gamma}\\
e^{-h} &= \left(1-\frac{\rho_o^2}{\rho^2}\right)^2\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{-2\gamma},
\end{aligned}$$

where $\rho_o > 0, \gamma$ are integration constants. In the Jordan frame the line element is given

$$ds^2 = -\Omega^{-2}e^f dt^2 + \Omega^{-2}e^{-h}(d\rho^2 + \rho^2 d\Omega^2). \tag{16.24}$$

in isotropic coordinates (t, ρ, θ, ϕ), where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element of the unit 2-sphere. Again the exponentials are given by

$$\begin{aligned}
e^f &= \left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{2\gamma},\\
e^{-h} &= \left(1-\frac{\rho_o^2}{\rho^2}\right)^2\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{-2\gamma} = \frac{1}{\rho^4}\frac{(\rho+\rho_o)^{2(\gamma+1)}}{(\rho-\rho_o)^{2(\gamma-1)}} = \frac{(1+\rho_o/\rho)^{2(\gamma+1)}}{(1-\rho_o/\rho)^{2(\gamma-1)}},
\end{aligned} \tag{16.25}$$

where it must be $\rho > \rho_o > 0$ and $0 \leq \gamma^2 \leq 1$[12]. The conformal factors Ω as functions of ρ are given by the relations

$$\Omega = \left(\frac{|\nu|}{8\pi}\right)^{\frac{1}{4}}\left[\cos\left(\sqrt{2|\lambda|(1-\gamma^2)}\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right)\right]^{\frac{1}{2}} \qquad , \text{for } \epsilon < 0 \tag{16.26}$$

$$\Omega = \frac{1}{\sqrt{8\pi}}\left|\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{\sqrt{2|\lambda|(1-\gamma^2)}} + 2\pi\nu\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{\sqrt{2|\lambda|(1-\gamma^2)}}\right|^{\frac{1}{2}} \qquad , \text{for } \epsilon > 0. \tag{16.27}$$

The $\lambda \neq 0$ parameter is related to the standard Brans-Dicke $\omega = \frac{1}{\lambda} - \frac{3}{2}$. Hence in our model, λ and ν are parameters of the theory and ρ_o and γ are the parameters of this specific family of solution. Moreover, note that (16.24) is the solution of Einstein gravity with a minimally coupled scalar field pulled in the Jordan frame by applying an inverse conformal transformation.

16.1 Branch $\epsilon < 0$ Solution

16.1.1 Scalar field

The Brans-Dicke scalar is given by the relation

$$\phi = \sqrt{\frac{|\nu|}{8\pi}} \sin\left(\sqrt{2|\lambda|(1-\gamma^2)} \ln \frac{\rho - \rho_o}{\rho + \rho_o}\right). \tag{16.28}$$

Note that ϕ becomes constant when $\gamma = \pm 1$, and the theory reduces to GR. Moreover, the scalar field vanishes in the limits $\rho \to \infty$ or $\lambda \to 0 (i.e.\ \omega \to \infty)$, which means the effective gravitational constant diverges.

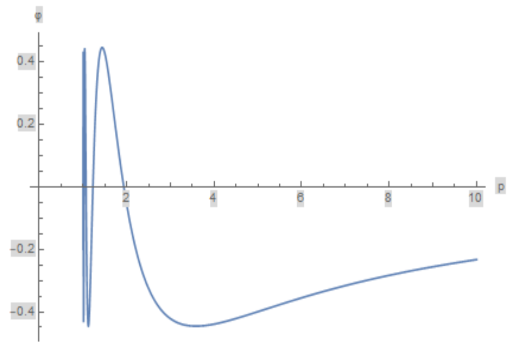


Figure 16.1: Behaviour of scalar field for $\gamma \neq \pm 1,\ \rho_o = 1,\ \neq 0\,,\ \lambda \neq 0$. ϕ vanishes at infinity as well as for $\lambda = 0$ or $\gamma = \pm 1$.

16.1.2 Metric Components

The metric components in isotropic coordinates are given by

$$\begin{aligned} g_{tt} =& -\left(\frac{|\nu|}{8\pi}\right)^{-\frac{1}{2}} \left[\cos\left(\sqrt{2|\lambda|(1-\gamma^2)} \ln \frac{\rho - \rho_o}{\rho + \rho_o}\right)\right]^{-1} \left(\frac{\rho - \rho_o}{\rho + \rho_o}\right)^{2\gamma}, \\ =& -\left(\frac{|\nu|}{8\pi}\right)^{-\frac{1}{2}} \sec\left(\sqrt{2|\lambda|(1-\gamma^2)} \ln \frac{\rho - \rho_o}{\rho + \rho_o}\right) \left(\frac{\rho - \rho_o}{\rho + \rho_o}\right)^{2\gamma} \\ g_{\rho\rho} =& \left(\frac{|\nu|}{8\pi}\right)^{-\frac{1}{2}} \left[\cos\left(\sqrt{2|\lambda|(1-\gamma^2)} \ln \frac{\rho - \rho_o}{\rho + \rho_o}\right)\right]^{-1} \frac{1}{\rho^4} \frac{(\rho + \rho_o)^{2(\gamma+1)}}{(\rho - \rho_o)^{2(\gamma-1)}} \\ =& \left(\frac{|\nu|}{8\pi}\right)^{-\frac{1}{2}} \sec\left(\sqrt{2|\lambda|(1-\gamma^2)} \ln \frac{\rho - \rho_o}{\rho + \rho_o}\right) \frac{1}{\rho^4} \frac{(\rho + \rho_o)^{2(\gamma+1)}}{(\rho - \rho_o)^{2(\gamma-1)}}. \end{aligned}$$

The appearance of the secant function prevents g_{tt} and $g_{\rho\rho}$ from vanishing, however it causes divergences at the points that satisfy the equation $\sqrt{2|\lambda|(1-\gamma^2)}\ln\frac{\rho-\rho_o}{\rho+\rho_o} = \frac{\pi}{2}+n\pi$ where $n \in Z$, if $\gamma \neq \pm 1$ [1]. The local minimums and local maximums of the secant function can be found by solving the equations $\sec\left(\sqrt{2|\lambda|(1-\gamma^2)}\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right) = 1$ and $\sec\left(\sqrt{2|\lambda|(1-\gamma^2)}\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right) = -1$, respectively. Note that due to the minus sign in g_{tt}, the minimum of the secant function will correspond to a local maximum of g_{tt} and vice versa.

It is rather easy to check the asymptotic behavior of our metric. By expanding the metric functions around $\frac{\rho_o}{\rho} \approx 0$ we get

$$g_{tt} = \left(\frac{8\pi}{|\nu|}\right)^{1/2}\left[\frac{1-\gamma\frac{\rho_o}{\rho}+\frac{1}{2}(\gamma-1)\gamma\left(\frac{\rho_o}{\rho}\right)^2+\mathcal{O}\left(\left(\frac{\rho_o}{\rho}\right)^3\right)}{1+\gamma\frac{\rho_o}{\rho}+\frac{1}{2}(\gamma-1)\left(4(\gamma+1)|\lambda|+\gamma\right)\left(\frac{\rho_o}{\rho}\right)^2+\mathcal{O}\left(\left(\frac{\rho_o}{\rho}\right)^3\right)}\right]^2$$

and

$$g_{\rho\rho} = \left(\frac{8\pi}{|\nu|}\right)^{1/2}\left[\frac{1+\frac{\gamma+1}{2}\frac{\rho_o}{\rho}+\frac{1}{8}(\gamma^2-1)\left(\frac{\rho_o}{\rho}\right)^2+\mathcal{O}\left(\left(\frac{\rho_o}{\rho}\right)^3\right)}{1-\frac{1-\gamma}{2}\frac{\rho_o}{\rho}+\frac{1}{8}(\gamma-1)\left(8(\gamma+1)|\lambda|+\gamma-3\right)\left(\frac{\rho_o}{\rho}\right)^2+\mathcal{O}\left(\left(\frac{\rho_o}{\rho}\right)^3\right)}\right]^4 .$$

Thus, a straightforward calculation gives

$$\lim_{\rho\to\infty} g_{tt} = -\sqrt{\frac{8\pi}{|\nu|}} \quad , \quad \lim_{p\to\infty} g_{\rho\rho} = \sqrt{\frac{8\pi}{|\nu|}} .$$

The fact that ν is a parameter of a theory and not a dynamic variable , means that we can absorb the above factors in the line element by redefining dt and $d\rho$. Thus, in this case the line element (16.24) with Ω given by (16.26), describes an ***asymptotically flat spacetime***.

Let us now consider three particular cases.

- If $\gamma = 1$ then

$$g_{tt} = -\left(\frac{|\nu|}{8\pi}\right)^{-\frac{1}{2}}\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^2 \tag{16.29}$$

$$g_{\rho\rho} = \left(\frac{|\nu|}{8\pi}\right)^{-\frac{1}{2}}\left(1+\frac{\rho_o}{\rho}\right)^4 . \tag{16.30}$$

  Note that $g_{tt} \xrightarrow{\rho\to\rho_o} 0$ while it remains negative for $\rho \neq \rho_o$. Furthermore, the radial component $g_{\rho\rho}$ diverges as $\rho \to 0$ and, $g_{tt} \xrightarrow{\rho\to\rho_o} 2^4\left(\frac{8\pi}{|\nu|}\right)^{1/2} \in \mathbb{R}$. The Brans-Dicke scalar becomes constant for $\gamma = 1$ and as can be seen from the forms of g_{tt} and $g_{\rho\rho}$, the solution reduces to the standard ***Schwarzschild metric in isotropic coordinates, with mass*** $M = 2\rho_o$.

- In the case $\gamma = -1$ one can find that

$$g_{tt} = -\left(\frac{|\nu|}{8\pi}\right)^{-\frac{1}{2}}\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^2 \xrightarrow{\rho\to\rho_o} -\infty \tag{16.31}$$

[1] a graphical depiction of this situation can be seen in the plot for $\gamma = 0$.

and

$$g_{\rho\rho} = \left(\frac{|\nu|}{8\pi}\right)^{-\frac{1}{2}} \left(1 - \frac{\rho_o}{\rho}\right)^4 \xrightarrow{\rho \to \rho_o} 0 \tag{16.32}$$

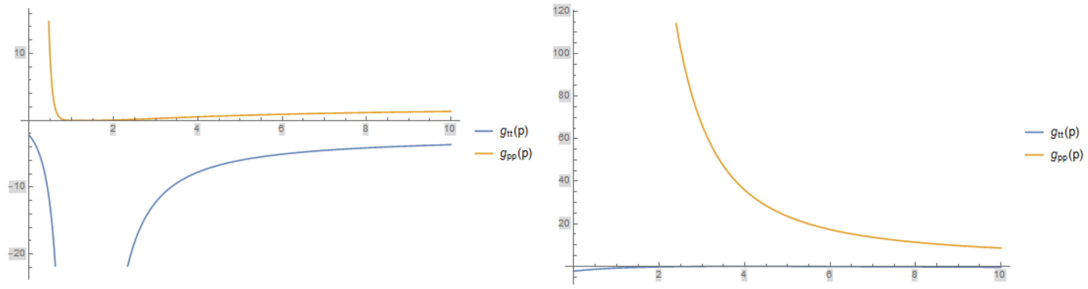
which remains positive for every $\rho > 0$. On the other hand it diverges as $\rho \to 0$ however, as we shall see in the next section, the range $0 < \rho < \rho_o$ is unphysical when $\gamma = -1$. The scalar field is again constant thus the solution corresponds to ***Schwarzschild spacetime with a negative mass*** $M = -2\rho_o$.

- When $\gamma = 0$ the metric components take the form

$$\begin{aligned} g_{tt} &= -\left(\frac{|\nu|}{8\pi}\right)^{-\frac{1}{2}} \sec\left(\sqrt{2|\lambda|}\ln\frac{\rho - \rho_o}{\rho + \rho_o}\right) \\ g_{\rho\rho} &= \left(\frac{|\nu|}{8\pi}\right)^{-\frac{1}{2}} \sec\left(\sqrt{2|\lambda|}\ln\frac{\rho - \rho_o}{\rho + \rho_o}\right)\left(1 - \frac{\rho_o^2}{\rho^2}\right)^2 . \end{aligned}$$

  Due to the secant function both components exhibit divergences as discussed earlier, however none of them vanishes at any particular point. Hence, in this case the solution is ***horizonless i.e. a naked singularity***. Although $g_{\rho\rho}$ seems to vanish as $\rho \to \rho_o$, its limit is actually undetermined.

The features discussed above can be seen in the diagrams below.



$\gamma = -1$. $\gamma = 1$.

Figure 16.2: $\rho_o = 2,\ \nu \neq 0,\ \lambda \neq 0$

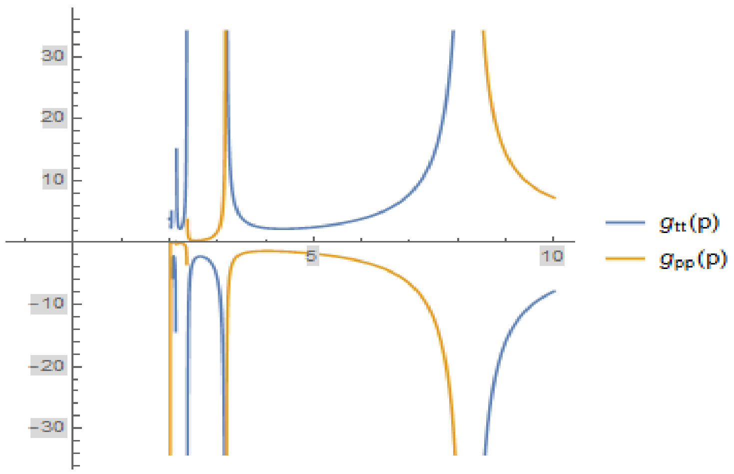


Figure 16.3: $\gamma = 0,\ \rho_o = 2,\ \nu = 5\,,\ \lambda = 2$

16.1.3 Areal Radius & Ricci Scalar

Now we wish to analyze the behavior of the Areal Radius since it can give us extra information about the geometry but also, help us deduce about which ranges of the spatial coordinate ρ are physically meaningful. Basically, what we call Areal Radius is just the radial coordinate of the spherical(Schwarschild) coordinates and that is why we are going to denote it by r. On the other hand, the study of scalar quantities helps us detect real spacetime singularities since they do not depend on our choice of coordinates. They are invariants and thus, if we manage to find a point where a scalar curvature diverges we know that it will correspond to a true spacetime singularity.

The areal radius is read off the line element(16.24)

$$r = \rho\Omega^{-1}e^{-\frac{h}{2}}$$

and in this case is

$$\begin{aligned} r =&\rho\Omega^{-1}e^{\frac{h}{2}} = \Omega^{-1}\frac{1}{\rho}\frac{(\rho+\rho_o)^{\gamma+1}}{(\rho-\rho_o)^{\gamma-1}} \\ =&\Big(\frac{|\nu|}{8\pi}\Big)^{-1/4}\left[\cos\left(\sqrt{2|\lambda|(1-\gamma^2)}\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right)\right]^{-\frac{1}{2}}\frac{1}{\rho}\frac{(\rho+\rho_o)^{\gamma+1}}{(\rho-\rho_o)^{\gamma-1}} \end{aligned} \tag{16.33}$$

while its derivative is given by

$$\begin{aligned} \frac{dr}{d\rho} =& \sqrt[4]{\frac{2\pi}{|\nu|}}\frac{1}{p^2}\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{\gamma}\left[\sqrt{2}\left(\rho^2+\rho_o^2-2\gamma\rho\rho_o\right)\cos\left(\alpha(\lambda,\gamma)\ln\left(\frac{p-\rho_o}{\rho+\rho_o}\right)\right)+\right. \\ &\left.\sqrt{2}\alpha(\lambda,\gamma)\rho\rho_o\sin\left(\alpha(\lambda,\gamma)\ln\left(\frac{p-\rho_o}{\rho+\rho_o}\right)\right)\right]\cos^{-\frac{3}{2}}\left(\alpha(\lambda,\gamma)\ln\left(\frac{p-\rho_o}{\rho+\rho_o}\right)\right) \end{aligned} \tag{16.34}$$

where we have denoted $\alpha(\lambda,\gamma) \equiv \sqrt{2\,|\lambda|\,(1-\gamma^2)}$ for brevity. Here a few points should be stressed.

- As $\rho \to \rho_o^+$ the areal radius approaches $r \to 0$(unless $\gamma = 1$) since if we express r in terms of $\frac{\rho_o}{\rho}$ and expand around one we get

$$\begin{aligned} r =&\rho_o\Big(\frac{|\nu|}{8\pi}\Big)^{-1/4}\left[\cos\left(\alpha(\lambda,\gamma)\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right)\right]^{-\frac{1}{2}}\cdot \\ &\left[2^{\gamma+1}\left(1-\frac{\rho_o}{\rho}\right)^{1-\gamma}-2^{\gamma}(\gamma+1)\left(1-\frac{\rho_o}{\rho}\right)^{2-\gamma}+2^{\gamma-2}(\gamma+1)\gamma\left(1-\frac{\rho_o}{\rho}\right)^{3-\gamma}+\mathcal{O}\left(\left(\frac{\rho_o}{\rho}\right)^3\right)\right] \end{aligned} \tag{16.35}$$

  which goes to zero if $\rho = \rho_o$.

- Areal radius approaches infinity if $\rho \longrightarrow \infty$ or $\rho = \rho_o\frac{e^K+1}{e^K-1}$ where $K = \frac{\pi\alpha(\lambda,\gamma)}{2}$ and has a point of minimun value which satisfies the equation

$$\frac{dr}{d\rho} = 0 \longrightarrow -\frac{1}{2^{1/2}\alpha(\lambda,\gamma)}\left(\frac{\rho_o}{\rho}+\frac{\rho}{\rho_o}-2\gamma\right) = \tan\left(\alpha(\lambda,\gamma)\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right)$$

- If $\gamma = 1$ then

$$r = \left(\frac{8\pi}{|\nu|}\right)^{1/4} \frac{(\rho + \rho_o)^2}{\rho} \, , \tag{16.36}$$

$$\frac{dr}{d\rho} = \left(\frac{8\pi}{|\nu|}\right)^{1/4} \left(1 - \frac{\rho_o^2}{\rho^2}\right) \tag{16.37}$$

whereas if $\gamma = -1$

$$r = \left(\frac{8\pi}{|\nu|}\right)^{1/4} \frac{(\rho - \rho_o)^2}{\rho} \, , \tag{16.38}$$

$$\frac{dr}{d\rho} = \left(\frac{8\pi}{|\nu|}\right)^{1/4} \left(1 - \frac{\rho_o^2}{\rho^2}\right) \tag{16.39}$$

which means that for these two particular cases, the areal radius decreases for $0 < \rho < \rho_o$, has an absolute minimum at $\rho = \rho_o$(whose value is $r = \left(\frac{8\pi}{|\nu|}\right)^{1/4} 4\rho_o > 0$ if $\gamma = 1$, and $r = 0$ if $\gamma = -1$), and increases for $\rho > \rho_o$. Thus, for $\gamma = -1$ the range $0 < \rho < \rho_o$ is unphysical.

Additionally, note that $r \to +\infty$ in the limits $\rho \to 0$ and $\rho \to \infty$, for both cases $\gamma = \pm 1$. Therefore the region near $\rho \to 0$ corresponds to a second asymptotically flat region of spacetime.

- If $\gamma = 0$ then

$$r = \left(\frac{|\nu|}{8\pi}\right)^{-1/4} \left[\cos\left(\sqrt{2|\lambda|} \ln \frac{\rho - \rho_o}{\rho + \rho_o}\right)\right]^{-1/2} \frac{1}{\rho} \left(\rho^2 - \rho_o^2\right) \tag{16.40}$$

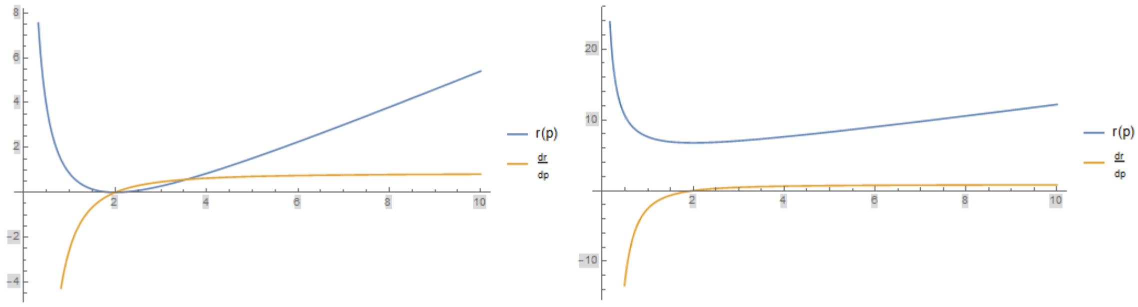
$$\frac{dr}{d\rho} = \left(\frac{2\pi}{|\nu|}\right)^{1/4} \frac{1}{p^2} \Bigg[\sqrt{2}\left(\rho^2 + \rho_o^2\right) \cos\left(\sqrt{2|\lambda|} \ln\left(\frac{p - \rho_o}{\rho + \rho_o}\right)\right) + $$

$$2\sqrt{|\lambda|}\rho\rho_o \sin\left(\sqrt{2|\lambda|} \ln\left(\frac{p - \rho_o}{\rho + \rho_o}\right)\right)\Bigg] \cos^{-\frac{3}{2}}\left(\sqrt{2|\lambda|} \ln\left(\frac{\rho - \rho_o}{\rho + \rho_o}\right)\right) \tag{16.41}$$

whereas their limits for $\rho \to \rho_o$ are undefined due to the arguments of the trigonometric functions.

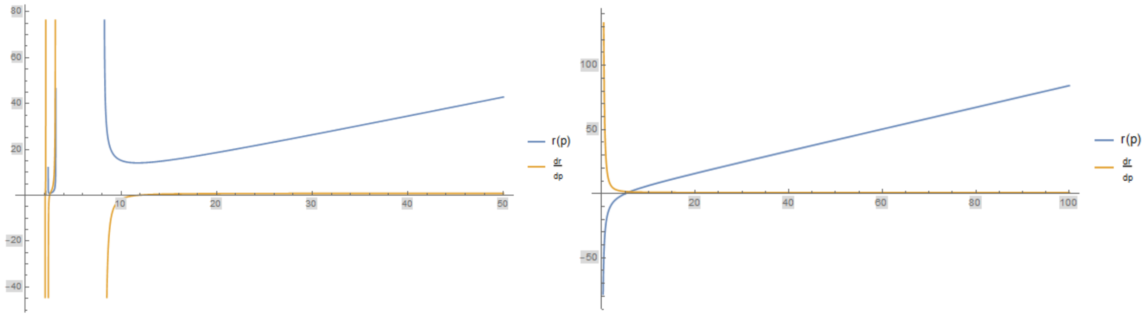
- The new parameter ν does not play a significant role in the behavior of the areal radius as it is just a multiplicative factor but, on the other hand it defines the scale.

A few diagrams are given below, in order to depict this behavior.



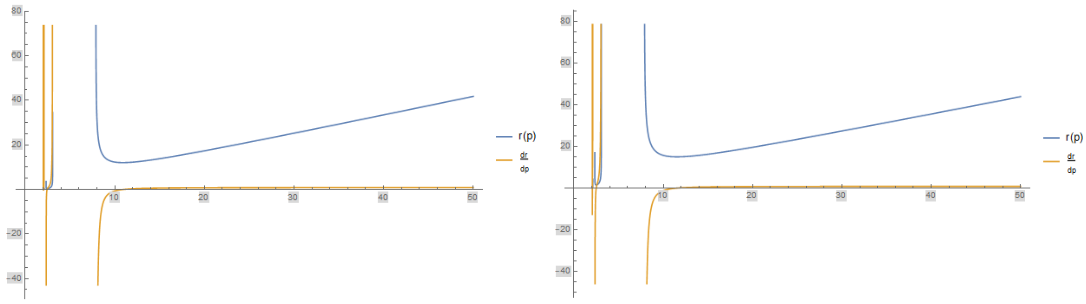
$\gamma = -1$ $\gamma = 1$

Figure 16.4: $\rho_o = 2,\ \nu \neq 0\,,\ \lambda \neq 0$



$\lambda \neq 0$ $\lambda = 0$

Figure 16.5: $\rho_o = 2,\ \nu \neq 0\,,\ \gamma = 0$



$\gamma = -0.3$ $\gamma = 0.3$

Figure 16.6: $\rho_o = 2,\ \nu \neq 0\,,\ \lambda \neq 0$

Now we proceed in the calculation of the Ricci scalar. In terms of the functions (16.25) and the conformal factor, the Ricci scalar is written

$$\mathcal{R} = -\frac{e^h}{2\rho}\Bigg\{ -6\Omega\Big[\Big(\rho\dot{f} - \rho\dot{h} + 4\Big)\,\dot{\Omega} + 2\rho\ddot{\Omega}\Big] + \\ \Omega^2\Big[2\rho\ddot{f} + \dot{f}\,\Big(4 - \rho\dot{h}\Big) + \rho\dot{f}^{\,2} - 4\rho\ddot{h} + \rho\dot{h}^2 - 8\dot{h}\Big] + 24\rho\dot{\Omega}^{\,2}\Bigg\} \tag{16.42}$$

and by substituting the relation (16.25) we arrive at

$$\mathcal{R} = \frac{2}{\rho^4}\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{2\gamma}\left\{\Omega\Big[3\ddot{\Omega}\left(\rho^2-\rho_o^2\right)^2 - 4\rho_o^2(\gamma^2-1)\Omega\Big] - 6\left(\rho^2-\rho_o^2\right)^2(\dot{\Omega})^2 + 6\rho\,\Omega\,\dot{\Omega}(\rho^2-\rho_o^2)\right\}. \tag{16.43}$$

Note that in the limit $\rho \to \rho_o$ it is $\Omega \to \infty$ in the case $\epsilon > 0$, while Ω remains finite for $\epsilon < 0$.

Now in order to obtain the Ricci scalar in terms of $\rho, \rho_o, \gamma, \nu, \lambda$, we just substitute the corresponding Ω i.e., relation (16.26).

$$\begin{aligned}\mathcal{R} =& \sqrt{\frac{|\nu|}{2\pi}}\frac{(\rho-\rho_o)^{2(\gamma-2)}}{(\rho+\rho_o)^{2(\gamma+2)}}(\gamma^2-1)\sec\left(\alpha(\lambda,\gamma)\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right)\cdot \\ &\cdot\left\{4\cos^4\left(\alpha(\lambda,\gamma)\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right) + 3|\lambda|\Big[\cos\left(2\alpha(\lambda,\gamma)\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right) - 5\Big]\right\}\end{aligned} \tag{16.44}$$

Thus unless $\gamma = \pm 1$ we get a naked singularity at $\rho = \rho_o$.

16.1.4 Metric in Terms of the Conformal Factor

Recall that the metric in isotropic coordinates is given by

$$ds^2 = -\Omega^{-2}e^f dt^2 + \Omega^{-2}e^{-h}(d\rho^2 + \rho^2 d\Omega^2)\,. \tag{16.45}$$

The exponential are given by the relations (16.25). The metric in standard coordinates is obtained by performing the transformation

$$r = \rho\Omega^{-1}e^{-h/2} = r(\rho) \tag{16.46}$$

and then

$$ds^2 = -\Omega^{-2}e^f dt^2 + \frac{dr^2}{\left(1 - \frac{\rho}{\Omega}\frac{d\Omega}{d\rho} - \frac{\rho}{2}\frac{dh}{d\rho}\right)^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{16.47}$$

or

$$ds^2 = -\Omega^{-2}e^f dt^2 + \frac{r^2}{\rho^2}\frac{dr^2}{\left(\frac{dr}{d\rho}\right)^2} + r^2(d\theta^2 + \sin\theta^2 d\varphi^2)\,. \tag{16.48}$$

The inverse of the transformation (16.46) does not exist and thus, we have no way of writing the metric purely with respect to r. What can be done is to find the inverse of (16.26) so as to construct a relation of the form $\rho(\Omega)$. That way we can write the whole metric with respect to the conformal factor. Equation (16.26) yields

$$\Omega = \left(\frac{|\nu|}{8\pi}\right)^{\frac{1}{4}}\left[\cos\left(\sqrt{2|\lambda|(1-\gamma^2)}\ln\frac{\rho-\rho_o}{\rho+\rho_o}\right)\right]^{\frac{1}{2}} \Rightarrow \left(\frac{\rho-\rho_o}{\rho+\rho_o}\right) = \exp\left\{\pm\frac{\arccos\left[\left(\frac{8\pi}{|\nu|}\right)^{1/2}\Omega^2\right]}{\sqrt{2\,|\lambda|\,(1-\gamma^2)}}\right\} \equiv e^{K(\Omega)} \tag{16.49}$$

where we denote

$$\frac{\arccos\left[\left(\frac{8\pi}{|\nu|}\right)^{1/2}\Omega^2\right]}{\sqrt{2\,|\lambda|\,(1-\gamma^2)}} \equiv K(\Omega)\,. \tag{16.50}$$

The function $\arccos(x)\,:[-1,1]\to[0,\pi]$ imposes certain bounds on the conformal factor

$$-\left(\frac{|\nu|}{8\pi}\right)^{1/4} \leq \Omega \leq \left(\frac{|\nu|}{8\pi}\right)^{1/4} \tag{16.51}$$

but the argument of arccos is always positive and hence we can either ignore the positive or negative Ω's as they will produce the same values. We shall see that only the positive range is needed because only this will produce positive values of the areal radius. Therefore

$$0 \leq \Omega \leq \left(\frac{|\nu|}{8\pi}\right)^{1/4} \tag{16.52}$$

and consequently

$$0 \leq K(\Omega) \leq \frac{\pi/2}{\sqrt{2\,|\lambda|\,(1-\gamma^2)}} \longrightarrow\ 1 \leq e^{K(\Omega)} \leq e^{\frac{\pi/2}{\sqrt{2|\lambda|(1-\gamma^2)}}}\,. \tag{16.53}$$

$$\longrightarrow e^{-\frac{\pi/2}{\sqrt{2|\lambda|(1-\gamma^2)}}} \leq e^{-K(\Omega)} \leq 1 \tag{16.54}$$

Solving with respect to ρ, equation (16.49) gives the solutions

$$\rho = \rho_o \frac{1+e^{K(\Omega)}}{1-e^{K(\Omega)}} \tag{16.55}$$

and

$$\rho = \rho_o \frac{1+e^{-K(\Omega)}}{1-e^{-K(\Omega)}}\,. \tag{16.56}$$

The solution (16.55) is not well defined because, as can be seen from (16.53) it implies that $\rho \leq \rho_o$. Therefore we only accept (16.56) as the solution of the system. From (16.25) and (16.56) it is found that

$$e^f = e^{-2\gamma K(\Omega)}\,, \tag{16.57}$$

$$e^{-h} = \frac{16}{\left(1+e^{-K(\Omega)}\right)^4}\, e^{2(1-\gamma)K(\Omega)} \tag{16.58}$$

therefore the metric functions and the areal radius, respectively take the following form

$$g_{tt} = -\Omega^{-2}\cdot e^{-2\gamma K(\Omega)} \tag{16.59}$$

$$g_{\rho\rho} = \frac{16}{\Omega^2}\cdot\frac{e^{2(1-\gamma)K(\Omega)}}{\left(1+e^{-K(\Omega)}\right)^4} \tag{16.60}$$

$$r(\Omega) = \frac{4\rho_o}{\Omega}\cdot\frac{e^{(\gamma-1)K(\Omega)}}{1-e^{-2K(\Omega)}}\,. \tag{16.61}$$

At this point it is evident that only when Ω is positive and within the bounds (16.52) the areal radius is also positive . The derivatives of (16.59),(16.60) and (16.61) with respect to Ω respectively, are given by

$$\frac{dg_{tt}}{d\Omega} = \frac{2\left(\sqrt{|\lambda|(1-\gamma^2)(|\nu|-8\pi\Omega^4)} - 4\gamma\sqrt{\pi}\Omega^2\right)}{\Omega^3\sqrt{|\lambda|(1-\gamma^2)(|\nu|-8\pi\Omega^4)}} \cdot e^{-2\gamma K(\Omega)} \tag{16.62}$$

$$\frac{dg_{\rho\rho}}{d\Omega} = -32e^{2(3-\gamma)K(\Omega)} \cdot \left[\frac{\sqrt{|\lambda|(1-\gamma^2)(|\nu|-8\pi\Omega^4)}\left(e^{K(\Omega)}+1\right) + 4\sqrt{\pi}\Omega^2\left((1-\gamma)e^{K(\Omega)}+(3-\gamma)\right)}{\Omega^3\sqrt{|\lambda|(1-\gamma^2)(|\nu|-8\pi\Omega^4)}\left(e^{K(\Omega)}+1\right)^5}\right] \tag{16.63}$$

$$\frac{dr}{d\Omega} = -4\rho_o e^{(\gamma-3)K(\Omega)} \cdot \left[\frac{\sqrt{|\lambda|(1-\gamma^2)(|\nu|-8\pi\Omega^4)}\left(e^{2K(\Omega)}-1\right) + 4\sqrt{\pi}\Omega^2\left((1-\gamma)e^{2K(\Omega)}+(3-\gamma)\right)}{\Omega^2\sqrt{|\lambda|(1-\gamma^2)(|\nu|-8\pi\Omega^4)}\left(e^{2K(\Omega)}-1\right)^2}\right]. \tag{16.64}$$

Within the range $0 < \Omega < \left(\frac{|\nu|}{8\pi}\right)^{1/4}$, the derivative of g_{tt} is always positive and those of $g_{\rho\rho}$ and $r(\Omega)$ are always negative. Lets again consider three special cases

- If $\gamma = 0$ then

$$g_{tt} = -\frac{1}{\Omega^2},$$

$$g_{\rho\rho} = \frac{16}{\Omega^2}\left(1+\exp\left(-\frac{\arccos\left[\left(\frac{8\pi}{|\nu|}\right)^{1/2}\Omega^2\right]}{\sqrt{2\,|\lambda|}}\right)\right)^{-4} \cdot \exp\left(\sqrt{\frac{2}{|\lambda|}}\arccos\left[\left(\frac{8\pi}{|\nu|}\right)^{1/2}\Omega^2\right]\right),$$

$$r(\Omega) = \left(\frac{4\rho_o}{\Omega}\right) \cdot \frac{\exp\left(-\sqrt{\frac{1}{2|\lambda|}}\arccos\left[\left(\frac{8\pi}{|\nu|}\right)^{1/2}\Omega^2\right]\right)}{1-\exp\left(\sqrt{\frac{2}{|\lambda|}}\arccos\left[\left(\frac{8\pi}{|\nu|}\right)^{1/2}\Omega^2\right]\right)}.$$

- For $\gamma = -1$ we cannot perform explicit calculations however, we can calculate the limits of the metric functions and the areal radius as $\gamma \to -1^+$. Thus,

$$\lim_{\gamma\to-1^+} g_{tt} = \lim_{\gamma\to-1^+} -\left(\frac{1}{\Omega^2}\right)\exp\left(-\frac{2\gamma}{\sqrt{2|\lambda|(1-\gamma^2)}}\arccos\left[\left(\frac{8\pi}{|\nu|}\right)^{1/2}\Omega^2\right]\right) = -\infty,$$

$$\lim_{\gamma\to-1^+} g_{\rho\rho} = \lim_{\gamma\to-1^+}\left(\frac{16}{\Omega^2}\right) \cdot \frac{\exp\left(\sqrt{\frac{2(1-\gamma)}{|\lambda|(1+\gamma)}}\arccos\left[\left(\frac{8\pi}{|\nu|}\right)^{1/2}\Omega^2\right]\right)}{\left(1+\exp\left(-\frac{\arccos\left[\left(\frac{8\pi}{|\nu|}\right)^{1/2}\Omega^2\right]}{\sqrt{2|\lambda|(1-\gamma^2)}}\right)\right)^4} = \infty,$$

$$\lim_{\gamma\to-1^+} r(\Omega) = \lim_{\gamma\to-1^+}\left(\frac{4\rho_o}{\Omega}\right) \cdot \frac{\exp\left(-\sqrt{\frac{(1-\gamma)}{2|\lambda|(1+\gamma)}}\arccos\left[\left(\frac{8\pi}{|\nu|}\right)^{1/2}\Omega^2\right]\right)}{1-\exp\left(-\sqrt{\frac{2}{|\lambda|(1-\gamma^2)}}\arccos\left[\left(\frac{8\pi}{|\nu|}\right)^{1/2}\Omega^2\right]\right)} = 0.$$

- Again for $\gamma = 1$ we calculate the limits as

$$\lim_{\gamma\to 1^-} g_{tt} = \lim_{\gamma\to 1^-} = -\left(\frac{1}{\Omega^2}\right)\exp\left(-\frac{2\gamma}{\sqrt{2|\lambda|(1-\gamma^2)}}\arccos\left[\left(\frac{8\pi}{|\nu|}\right)^{1/2}\Omega^2\right]\right) = 0\,,$$

$$\lim_{\gamma\to 1^-} g_{\rho\rho} = \lim_{\gamma\to 1^-}\left(\frac{16}{\Omega^2}\right)\cdot\frac{\exp\left(\sqrt{\frac{2(1-\gamma)}{|\lambda|(1+\gamma)}}\arccos\left[\left(\frac{8\pi}{|\nu|}\right)^{1/2}\Omega^2\right]\right)}{\left(1+\exp\left(-\frac{\arccos\left[\left(\frac{8\pi}{|\nu|}\right)^{1/2}\Omega^2\right]}{\sqrt{2|\lambda|(1-\gamma^2)}}\right)\right)^4} = \frac{16}{\Omega^2}$$

$$\lim_{\gamma\to 1^-} r(\Omega) = \lim_{\gamma\to 1^-}\left(\frac{4\rho_o}{\Omega}\right)\cdot\frac{\exp\left(-\sqrt{\frac{(1-\gamma)}{2|\lambda|(1+\gamma)}}\arccos\left[\left(\frac{8\pi}{|\nu|}\right)^{1/2}\Omega^2\right]\right)}{1-\exp\left(-\sqrt{\frac{2}{|\lambda|(1-\gamma^2)}}\arccos\left[\left(\frac{8\pi}{|\nu|}\right)^{1/2}\Omega^2\right]\right)} = \frac{4\rho_o}{\Omega}\,.$$

The diagrams of the metric and the areal radius with respect to the conformal factor, for all possible values of the parameter space are of the form.

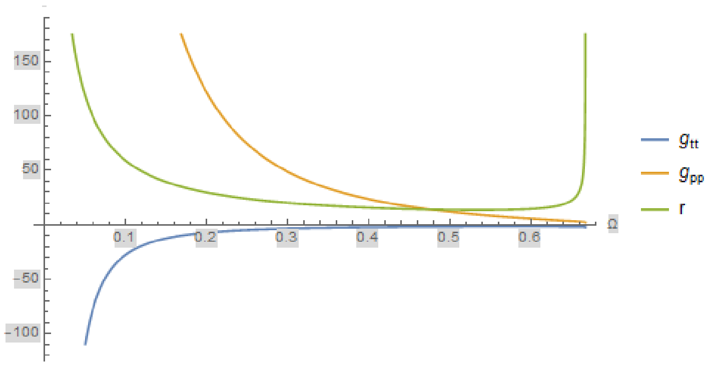


Figure 16.7: Plot of the metric functions and the areal radius with respect to the conformal factor.

16.2 Branch $\epsilon > 0$ Solution

16.2.1 Scalar field

In this case the Brans-Dicke scalar takes the form

$$\phi = \frac{s}{8\pi}\left|\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{\sqrt{2|\lambda|(1-\gamma^2)}} - 2\pi\nu\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{\sqrt{2|\lambda|(1-\gamma^2)}}\right|, \tag{16.65}$$

$$s = sgn(4\pi\phi + \sqrt{2\pi(\nu + 8\pi\phi^2)})\,. \tag{16.66}$$

Just as in the case $\epsilon < 0$, ϕ becomes constant for $\gamma = \pm 1$ or $\lambda \to 0$ (i.e. $\omega \to \infty$), therefore the theory reduces to GR. The scalar field becomes also constant as $\rho \to \infty$ and diverges in the limit $\rho \to \rho_o$ hence,the effective gravitational constant vanishes.

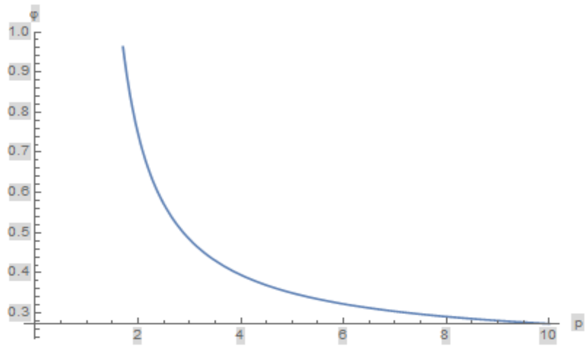


Figure 16.8: Behaviour of scalar field for $\gamma \neq \pm 1,\ \rho_o = 1,\ \neq 0\,,\ \lambda \neq 0$. ϕ becomes constant at infinity as well as for $\lambda = 0$ or $\gamma = \pm 1$.

16.2.2 Metric Components

The metric components are given by

$$\begin{aligned} g_{tt} =& -8\pi\left|\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{\sqrt{2|\lambda|(1-\gamma^2)}} + 2\pi\nu\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{\sqrt{2|\lambda|(1-\gamma^2)}}\right|^{-1}\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{2\gamma} \\ =& -8\pi\left|\frac{\left(\rho^2-\rho_o^2\right)^{\alpha(\lambda,\gamma)}}{(\rho-\rho_o)^{2\alpha(\lambda,\gamma)}+2\pi\nu(\rho+\rho_o)^{2\alpha(\lambda,\gamma)}}\right|\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{2\gamma} \end{aligned} \tag{16.67}$$

$$\begin{aligned} g_{\rho\rho} =& 8\pi\left|\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{\sqrt{2|\lambda|(1-\gamma^2)}} + 2\pi\nu\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{\sqrt{2|\lambda|(1-\gamma^2)}}\right|^{-1}\frac{1}{\rho^4}\frac{(\rho+\rho_o)^{2(\gamma+1)}}{(\rho-\rho_o)^{2(\gamma-1)}} \\ =& 8\pi\left|\frac{\left(\rho^2-\rho_o^2\right)^{\alpha(\lambda,\gamma)}}{(\rho-\rho_o)^{2\alpha(\lambda,\gamma)}+2\pi\nu(\rho+\rho_o)^{2\alpha(\lambda,\gamma)}}\right|\frac{1}{\rho^4}\frac{(\rho+\rho_o)^{2(\gamma+1)}}{(\rho-\rho_o)^{2(\gamma-1)}}\,. \end{aligned} \tag{16.68}$$

Similar to the case $\epsilon < 0$, expansion of the metric functions around $\frac{\rho_o}{\rho} \approx 0$ yields

$$g_{tt} = -8\pi\left[\frac{1-\gamma\,\frac{\rho_o}{\rho}+\frac{1}{2}(\gamma-1)\gamma\left(\frac{\rho_o}{\rho}\right)^2+\mathcal{O}\left(\left(\frac{\rho_o}{\rho}\right)^3\right)}{|1+2\pi\nu|^{1/2}+\frac{|1+2\pi\nu|^{1/2}}{(1+2\pi\nu)}\left(\alpha(\lambda,\gamma)(2\pi\nu-1)+\gamma(1+2\pi\nu)\right)\left(\frac{\rho_o}{\rho}\right)+\dots}\right]^2$$

and

$$g_{\rho\rho} = 8\pi\left[\frac{1+\frac{\gamma+1}{2}\,\frac{\rho_o}{\rho}+\frac{1}{8}(\gamma^2-1)\left(\frac{\rho_o}{\rho}\right)^2+\mathcal{O}\left(\left(\frac{\rho_o}{\rho}\right)^3\right)}{|1+2\pi\nu|^{1/4}+\frac{1}{2}\frac{|1+2\pi\nu|^{1/4}}{(1+2\pi\nu)}\left(\alpha(\lambda,\gamma)(2\pi\nu-1)-(\gamma-1)(1+2\pi\nu)\right)\frac{\rho_o}{\rho}+\dots}\right]^4\,.$$

Hence, as $\rho \longrightarrow \infty$ we calculate

$$\lim_{\rho\to\infty} g_{tt} = -\frac{8\pi}{|2\pi\nu+1|} \quad , \quad \lim_{\rho\to\infty} g_{\rho\rho} = \frac{8\pi}{|2\pi\nu+1|}\,.$$

Again, the asymptotic behavior of $g_{tt}, g_{\rho\rho}$ depends only on the parameter ν which means the ***spacetime becomes Minkowskian in the large distance limit***. Moreover we can observe that

- If $\gamma = 1$ then

$$g_{tt} = -\frac{8\pi}{|1+2\pi\nu|}\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^2 \xrightarrow{\rho\to\rho_o} 0\,, \tag{16.69}$$

$$g_{\rho\rho} = \frac{8\pi}{|1+2\pi\nu|}\left(1+\frac{\rho_o}{\rho}\right)^4 \xrightarrow{\rho\to\rho_o} \frac{8\pi}{|1+2\pi\nu|}2^4 \tag{16.70}$$

Note that $g_{\rho\rho}$ diverges as $\rho \to 0$. The Brans-Dicke scalar becomes constant for $\gamma = 1$ and the solution reduces to the ***Schwarzschild solution with*** $M = 2\rho_o$.

- In the case $\gamma = -1$ one finds

$$g_{tt} = -\frac{8\pi}{|1+2\pi\nu|}\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^2 \xrightarrow{\rho\to\rho_o} -\infty\,, \tag{16.71}$$

$$g_{\rho\rho} = \frac{8\pi}{|1+2\pi\nu|}\left(1-\frac{\rho_o}{\rho}\right)^4 \xrightarrow{\rho\to\rho_o} 0 \tag{16.72}$$

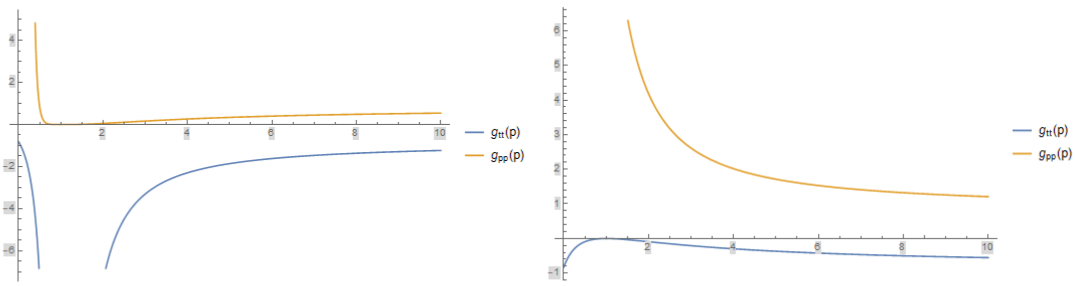
where again, the radial component "blows-up" as $\rho \to 0$. It is evident from the form of the metric functions that this is again the ***Schwarzschild solution with negative mass*** $M = -\rho_o$.

- If $\gamma = 0$ then

$$g_{tt} = -8\pi\left|\frac{\left(\rho^2-\rho_o^2\right)^{\sqrt{2|\lambda|}}}{(\rho-\rho_o)^{2\sqrt{2|\lambda|}}+2\pi\nu(\rho+\rho_o)^{2\sqrt{2|\lambda|}}}\right| \xrightarrow{\rho\to\rho_o} 0\,, \tag{16.73}$$

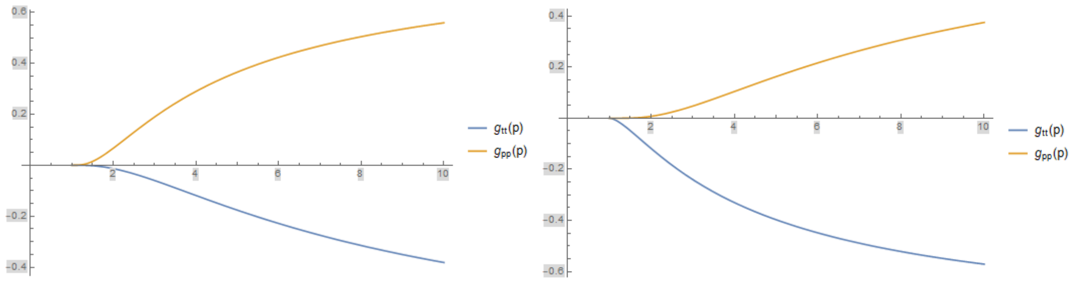
$$g_{\rho\rho} = 8\pi\left|\frac{\left(\rho^2-\rho_o^2\right)^{\sqrt{2|\lambda|}}}{(\rho-\rho_o)^{2\sqrt{2|\lambda|}}+2\pi\nu(\rho+\rho_o)^{2\sqrt{2|\lambda|}}}\right|\left(1-\frac{\rho_o^2}{\rho^2}\right)^2 \xrightarrow{\rho\to\rho_o} 0 \tag{16.74}$$

Therefore, the coefficient g_{tt} exhibits the standard behavior of a black hole horizon, as $\rho \to \rho_o$, in the cases $\gamma = 0, 1$ and as we shall see the point $\rho = \rho_o$ corresponds to finite areal radius for $\gamma = 1$, however when $\gamma = 0$ then $r \xrightarrow{\rho\to\rho_o} 0$. Thus, the case $\gamma = 0$ produces a ***naked singularity***.



$\gamma = -1$. $\gamma = 1$.

Figure 16.9: $\rho_o = 1,\ \nu \neq 0\,,\ \lambda \neq 0$



$\gamma = 0.5.$ $\qquad\qquad$ $\gamma = -0.5$

Figure 16.10: $\rho_o = 1,\ \nu \neq 0\,,\ \lambda \neq 0$

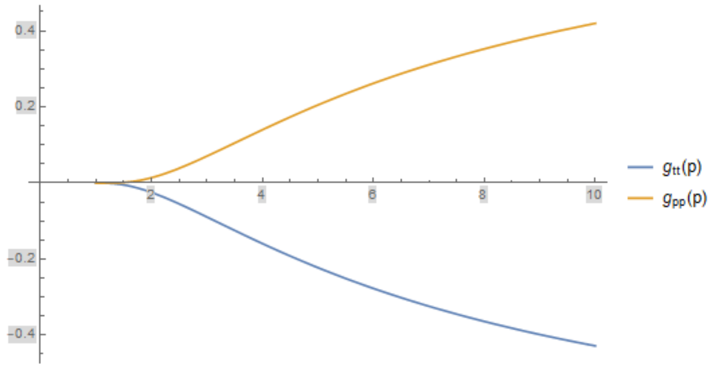


Figure 16.11: $\gamma = 0,\ \rho_o = 1,\ \nu \neq 0\,,\ \lambda \neq 0$

16.2.3 Areal Radius & Ricci Scalar

By substituting the corresponding Ω, the areal radius and its derivative now take the form

$$
\begin{aligned}
r =& \rho\Omega^{-1} e^{\frac{h}{2}} = \Omega^{-1} \frac{1}{\rho} \frac{(\rho+\rho_o)^{\gamma+1}}{(\rho-\rho_o)^{\gamma-1}} \\
=& \sqrt{8\pi} \left| \left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{\sqrt{2|\lambda|(1-\gamma^2)}} + 2\pi\nu \left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{\sqrt{2|\lambda|(1-\gamma^2)}} \right|^{-\frac{1}{2}} \frac{1}{\rho} \frac{(\rho+\rho_o)^{\gamma+1}}{(\rho-\rho_o)^{\gamma-1}} \\
=& \frac{\sqrt{8\pi}}{\rho} \frac{(\rho+\rho_o)^{\gamma+1}}{(\rho-\rho_o)^{\gamma-1}} \left\{ \left[\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{\alpha(\lambda,\gamma)} + 2\pi\nu \left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{\alpha(\lambda,\gamma)} \right]^2 \right\}^{-1/4}
\end{aligned}
\tag{16.75}
$$

$$
\begin{aligned}
\frac{dr}{d\rho} =& \left(\frac{1}{8\pi}\right) \left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{\gamma} \frac{1}{\rho^2\Omega^3} \cdot \\
& \cdot \left\{ (\rho^2 + \rho_o^2 - 2\rho\rho_o\gamma) 8\pi\Omega^2 - \rho\rho_o\alpha(\lambda,\gamma) \left[\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{\alpha(\lambda,\gamma)} - 2\pi\nu \left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{\alpha(\lambda,\gamma)} \right] \right\}
\end{aligned}
\tag{16.76}
$$

One can observe the following:

- If $\gamma = 1$ then

$$r = \left(\frac{8\pi}{1+2\pi\nu}\right)^{1/2} \frac{(\rho+\rho_o)^2}{\rho}\,, \tag{16.77}$$

$$\frac{dr}{d\rho} = \left(\frac{8\pi}{1+2\pi\nu}\right)^{1/2} \left(1-\frac{\rho_o^2}{\rho^2}\right)\,. \tag{16.78}$$

If $\gamma = -1$ we get

$$r = \left(\frac{8\pi}{1+2\pi\nu}\right)^{1/2} \frac{(\rho-\rho_o)^2}{\rho}\,, \tag{16.79}$$

$$\frac{dr}{d\rho} = \left(\frac{8\pi}{1+2\pi\nu}\right)^{1/2} \left(1-\frac{\rho_o^2}{\rho^2}\right)\,. \tag{16.80}$$

Hence, just like in case $\epsilon < 0$, if $\gamma = \pm 1$ then the areal radius is an decreasing function for $0 < \rho < \rho_o$, has an absolute minimum at $\rho = \rho_o$ and increases for $\rho > \rho_o$. If $\gamma = -1$ its minimum value is $r(\rho = \rho_o) = 0$ whereas, if $\gamma = 1$ then $r(\rho = \rho_o) = \left(\frac{8\pi}{1+2\pi\nu}\right)^{1/2} 4\rho_o > 0$. Therefore, for $\gamma = -1$ the range $0 < \rho < \rho_o$ of the isotropic radius is unphysical. Moreover, $r \to \infty$ in both limits $\rho \to 0^+$ and $\rho \to \infty$.

Again,due to the fact that ϕ vanishes, the solution turns out to be Schwarschild with mass $M = \pm 2\rho_o$ for $\gamma = \pm 1$, respectively.

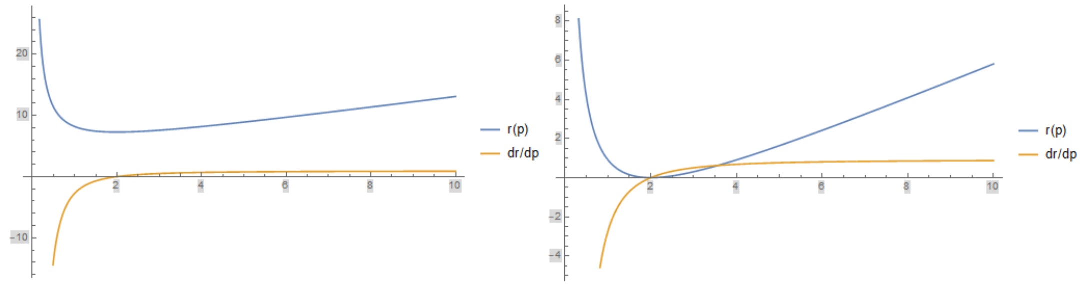
- In the case $\gamma = 0$ one gets the, not so elegant, expressions

$$r = (8\pi)^{1/2} \left\{ \frac{(\rho^2-\rho_o^2)^{\sqrt{2|\lambda|}+2}}{(\rho-\rho_o)^{\sqrt{2|\lambda|}} + 2\pi\nu(\rho+\rho_o)^{\sqrt{2|\lambda|}}} \right\}^{1/2} \xrightarrow{\rho\to\rho_o} 0 \tag{16.81}$$

$$\begin{aligned} \frac{dr}{d\rho} =& \frac{8\pi}{\rho^2}\left[(\rho-\rho_o)^{\sqrt{2|\lambda|}} + 2\pi\nu(\rho+\rho_o)^{\sqrt{2|\lambda|}}\right]^{-3/2} \cdot \\ & \cdot \left\{ \left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{\sqrt{2|\lambda|}} (\rho^2+\rho_o^2-\rho\rho_o\sqrt{2|\lambda|}) + 2\pi\nu\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{\sqrt{2|\lambda|}} (\rho^2+\rho_o^2+\rho\rho_o\sqrt{2|\lambda|}) \right\} > 0 \end{aligned} \tag{16.82}$$

for $\rho > \rho_o$. Again, the range $0 < \rho < \rho_o$ is unphysical and, on top of that, the above relations are not even well defined in this particular range, when $\gamma \neq \pm 1$.

To illustrate the situation, a few diagrams are been given.



$\gamma = -1$ $\gamma = -1$

Figure 16.12: $\rho_o = 2,\ \nu \neq 0,\ \lambda \neq 0$

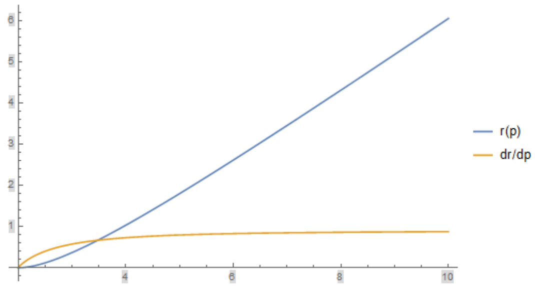


Figure 16.13: $\gamma = 0.5$

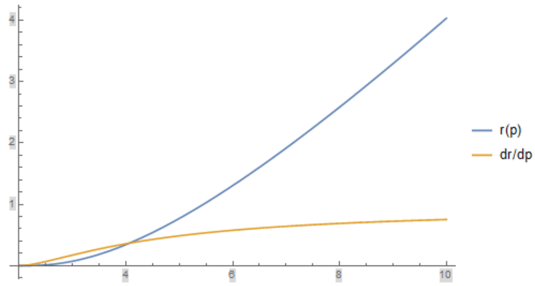


Figure 16.14: $\gamma = -0.5$

Figure 16.15: $\rho_o = 2,\ \nu \neq 0,\ \lambda \neq 0$

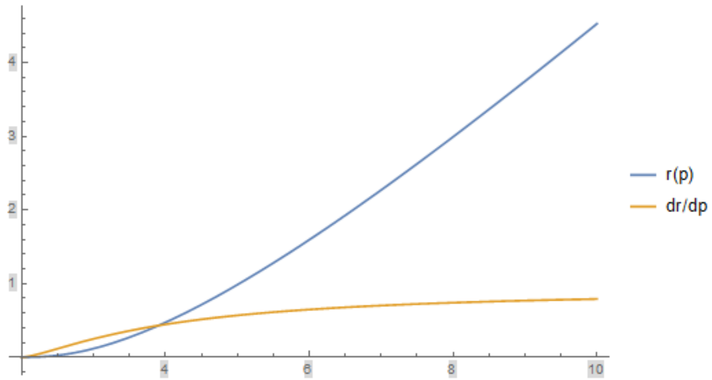


Figure 16.16: $\gamma = 0,\ \rho_o = 2,\ \nu \neq 0\,,\ \lambda \neq 0$

Now lets a closer look at the scalar curvature. Recall that the Ricci scalar is given by (16.43). Therefore by substituting the corresponding Ω we get

$$\begin{aligned}\mathcal{R} =& (\gamma^2 - 1)\rho^4 \rho_o^2 \frac{(\rho - \rho_o)^{2(\gamma-2)}}{(\rho + \rho_o)^{2(\gamma+2)}} \cdot \\ & \frac{(3|\lambda| - 2)\left[\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{2\alpha(\lambda,\gamma)} + 4\pi^2\nu^2\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{2\alpha(\lambda,\gamma)}\right] - 4\pi\nu(15|\lambda| + 2)}{2\pi\left|\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{\alpha(\lambda,\gamma)} + 2\pi\nu\left(\frac{\rho+\rho_o}{\rho-\rho_o}\right)^{\alpha(\lambda,\gamma)}\right|}\end{aligned} \tag{16.83}$$

Again if $\gamma = \pm 1$ the solution again, corresponds to Schwarzschild spacetime. In all other cases we observe that there is a ***naked singularity*** at $\rho = \rho_o$.

16.2.4 Metric in Terms of the Conformal Factor

Just like in the previous section we would like to construct a relation of the form $\rho(\Omega)$ and then express the whole metric with respect to the conformal factor. We begin by manipulating the the expression (16.27)

$$\Omega = \frac{1}{\sqrt{8\pi}} \left| \left(\frac{\rho - \rho_o}{\rho + \rho_o}\right)^{\alpha(\lambda,\gamma)} + 2\pi\nu \left(\frac{\rho + \rho_o}{\rho - \rho_o}\right)^{\alpha(\lambda,\gamma)} \right|^{1/2} .$$

By denoting $\left(\frac{\rho-\rho_o}{\rho+\rho_o}\right)^{\alpha(\lambda,\gamma)} \equiv \kappa > 0$ the relation takes the form

$$\begin{aligned} 8\pi\Omega^2 = \left|\kappa + 2\pi\nu\kappa^{-1}\right| \Rightarrow 8\pi\Omega^2 = \left(\kappa + 2\pi\nu\kappa^{-1}\right) \cdot sgn(\kappa + 2\pi\nu\kappa^{-1}) \Rightarrow \\ 8\pi\Omega^2 = \pm\left(\kappa + 2\pi\nu\kappa^{-1}\right) \Rightarrow \\ \kappa^2 \mp 8\pi\Omega\kappa + 2\pi\nu = 0 \end{aligned} \tag{16.84}$$

and thus we have to solve to two separate equations

Case $\kappa + 2\pi\nu\kappa^{-1} > 0$

The above inequality is true for

$$\nu \leq 0 \ \ \& \ \ \kappa > \sqrt{2\pi|\nu|} \text{ or } \nu > 0 \ \ \& \ \ \kappa > 0\,. \tag{16.85}$$

and equation (16.84) is solved by

$$\kappa^2 - 8\pi\Omega\kappa + 2\pi\nu = 0 \Longrightarrow \kappa_{1,2} = 4\pi\Omega^2 \pm \sqrt{(4\pi\Omega^2)^2 - 2\pi\nu}$$

$$\kappa_1 = 4\pi\Omega^2 + \sqrt{(4\pi\Omega^2)^2 - 2\pi\nu} \tag{16.86}$$

$$\kappa_2 = 4\pi\Omega^2 - \sqrt{(4\pi\Omega^2)^2 - 2\pi\nu} \tag{16.87}$$

where if $\nu > 0$ then, κ_1 and κ_2 are positive for $\Omega \geq \left(\frac{\nu}{8\pi}\right)^{1/4}$ while if $\nu \leq 0$, κ_1 satisfies (16.85) for every $\Omega > 0$ and κ_2 does not satisfy them at all.

Case $\kappa + 2\pi\nu\kappa^{-1} < 0$

Now the last inequality holds if

$$\nu \leq 0 \ \ \& \ \ 0 < \kappa < \sqrt{2\pi|\nu|} \tag{16.88}$$

where recall that the definition of κ is valid only for positive values.

In this case (16.84) is solved by

$$\kappa^2 + 8\pi\Omega\kappa + 2\pi\nu = 0 \Longrightarrow \kappa_{3,4} = -4\pi\Omega^2 \pm \sqrt{(4\pi\Omega^2)^2 - 2\pi\nu}$$

$$\kappa_3 = -4\pi\Omega^2 + \sqrt{(4\pi\Omega^2)^2 - 2\pi\nu} \tag{16.89}$$

$$\kappa_4 = -4\pi\Omega^2 - \sqrt{(4\pi\Omega^2)^2 - 2\pi\nu}\,. \tag{16.90}$$

The solution κ_3 satisfies the bounds (16.88) for every $\Omega > 0$ whereas, κ_4 cannot satisfy them unless Ω is imaginary, therefore this solution is disregarded.

By now, we have found three expressions $\kappa_i, i = 1, 2, 3$ in terms of the conformal factor and the new parameter ν. We are going to use them to construct the inverse of (16.27) i.e. a relation $\rho(\Omega)$. Using the definition of κ one gets

$$\left(\frac{\rho - \rho_o}{\rho + \rho_o}\right)^{\alpha(\lambda,\gamma)} = \kappa_i \Rightarrow \ \rho_i(\Omega) = \rho_o \left(\frac{1 + \kappa_i^{1/\alpha(\lambda,\gamma)}}{1 - \kappa_i^{1/\alpha(\lambda,\gamma)}}\right) \tag{16.91}$$

where the condition $\rho > \rho_o$ implies

$$\left(\frac{1 + \kappa_i^{1/\alpha(\lambda,\gamma)}}{1 - \kappa_i^{1/\alpha(\lambda,\gamma)}}\right) > 1 \Rightarrow \ \alpha(\lambda, \gamma) > 0 \ \& \ 0 < \kappa_i < 1 \tag{16.92}$$

which further restricts the range of κ_i. Note that $\kappa_i \to 1$ corresponds to $\rho_i(\Omega) \to \infty$ while, $\kappa_i \to 0$ corresponds to $\rho_i(\Omega) \to \rho_o$.

If we now combine the results from (16.85) and (16.88), with the extra condition (16.92), we will get the ranges of Ω and ν on which the equation (16.91) is well defined, for every κ_i. The combination of the bounds for κ_1 gives

$$\nu > 0 \ \ \& \ \ 0 < \kappa_1 < 1 \longrightarrow 0 < \nu < \frac{1}{2\pi} \ \ \& \ \ \left(\frac{\nu}{8\pi}\right)^{1/4} \leq \Omega < \left(\frac{2\pi\nu + 1}{8\pi}\right)^{1/2}, \tag{16.93}$$

$$\nu \leq 0 \ \ \& \ \ \sqrt{2\pi|\nu|} < \kappa_1 < 1 \longrightarrow -\frac{1}{2\pi} < \nu < 0 \ \ \& \ \ 0 \leq \Omega < \left(\frac{2\pi\nu + 1}{8\pi}\right)^{1/2}, \tag{16.94}$$

where the restriction on ν in (16.94) is there to ensure that $\sqrt{2\pi|\nu|} < 1$. For κ_2 one obtains

$$\nu > 0 \ \ \& \ \ 0 < \kappa_2 < 1 \longrightarrow 0 < \nu \leq \frac{1}{2\pi} \ \ \& \ \ \Omega \geq \left(\frac{\nu}{8\pi}\right)^{1/4}, \tag{16.95}$$

$$\nu > \frac{1}{2\pi} \ \ \& \ \ \Omega > \left(\frac{2\pi\nu + 1}{8\pi}\right)^{1/2} \tag{16.96}$$

and for κ_3 we get

$$\nu \leq 0 \ \ \& \ \ \left\{0 < \kappa_3 < \sqrt{2\pi|\nu|} \text{ or } 0 < \kappa_3 < 1\right\} \longrightarrow \nu \leq -\frac{1}{2\pi} \ \ \& \ \ \Omega > \left(\frac{2\pi|\nu| - 1}{8\pi}\right)^{1/2}, \tag{16.97}$$

$$-\frac{1}{2\pi} < \nu < 0 \ \ \& \ \ \Omega > 0\,. \tag{16.98}$$

Note that in the bounds of κ_3, $\nu \leq -\frac{1}{2\pi}$ means that $\sqrt{2\pi|\nu|} > 1$ and the true restriction is $0 < \kappa_3 < 1$ whereas, the condition $-\frac{1}{2\pi} < \nu < 0$ leads to $\sqrt{2\pi|\nu|} < 1$ thus the relation that must be satisfied is $0 < \kappa_3 < \sqrt{2\pi|\nu|}$. At this point we know all the acceptable ranges for the conformal factor and thus, we are in position to calculate the metric functions and the areal radius with respect to Ω.

$$g_{tt} = -\Omega^{-2} e^{f_i} = -\Omega^{-2} \left(\frac{\rho_i - \rho_o}{\rho_i + \rho_o}\right)^{2\gamma} = -\Omega^{-2} \kappa_i^{2\gamma/\alpha(\lambda,\gamma)} , \tag{16.99}$$

$$g_{\rho\rho} = \Omega^{-2} e^{-h_i} = \Omega^{-2} \left(1 - \frac{\rho_o^2}{\rho_i^2}\right)^2 \left(\frac{\rho_i + \rho_o}{\rho_i - \rho_o}\right)^{2\gamma} = \Omega^{-2} \left(1 - \frac{1 - \kappa_i^{1/\alpha(\lambda,\gamma)}}{1 + \kappa_i^{1/\alpha(\lambda,\gamma)}}\right)^2 \kappa_i^{-2\gamma/\alpha(\lambda,\gamma)}$$

$$= \Omega^{-2} \left(\frac{2\kappa_i^{\frac{1-\gamma}{\alpha(\lambda,\gamma)}}}{1 + \kappa_i^{\frac{1}{\alpha(\lambda,\gamma)}}}\right)^2 , \tag{16.100}$$

$$r(\Omega) = \rho_i \Omega^{-1} e^{-h_i/2} = \frac{2\rho_o}{\Omega} \left(\frac{\kappa_i^{\frac{1-\gamma}{\alpha(\lambda,\gamma)}}}{1 - \kappa_i^{\frac{1}{\alpha(\lambda,\gamma)}}}\right) , \quad i = 1, 2, 3 \tag{16.101}$$

From the bounds for the various κ_i's i.e. the inequalities (16.93),(16.94),(16.95) and (16.97), is evident that g_{tt} does not vanish hence, there is no sign of a horizon. Only if $\kappa_i = 0$ there would be a horizon but, expect the fact that this is a forbidden value, it also corresponds to $r(\Omega) = 0$, which means that the singularity would not be covered by it. Moreover, $g_{\rho\rho}$ does not diverge unless $\Omega \to 0$ so there is no sign of the singularity either.

When $\gamma \to 1$ we find that

$$g_{tt} \xrightarrow{\gamma\to 1} 0 , \tag{16.102}$$

$$g_{\rho\rho} \xrightarrow{\gamma\to 1} 4\Omega^{-2} , \tag{16.103}$$

$$r(\Omega) \xrightarrow{\gamma\to 1} \frac{2\rho_o}{\Omega} . \tag{16.104}$$

In this case $g_{\rho\rho}$ diverges s $\Omega \to 0$ but this singularity is actually pushed to infinity due to $r(\Omega)$.

On the other hand if $\gamma \to -1$ we get

$$g_{tt} \xrightarrow{\gamma\to -1} -\infty , \tag{16.105}$$

$$g_{\rho\rho} \xrightarrow{\gamma\to -1} 0 , \tag{16.106}$$

$$r(\Omega) \xrightarrow{\gamma\to -1} 0 . \tag{16.107}$$

The transformation from the isotropic to the areal radius becomes degenerate, along with the fact that neither g_{tt} vanishes nor $g_{\rho\rho}$ diverges at any particular Ω.

Bibliography

[1] Albert Einstein. *Zur Elektrodynamik bewegter Körper.* (German) [*On the electrodynamics of moving bodies*]. Annalen der Physik, 322(10):891–921, 1905.

[2] J. M. Lee. *Introduction to Smooth Manifolds.* Springer-Verlag, New York, 2 edition, 2012.

[3] J. M. Lee. *Riemannian Manifolds: An Introduction to Curvature.* Springer-Verlag, New York, 1 edition, 1997.

[4] M. Nakahara. *Geometry, Topology and Physics.* IOP Publishing, Bristol-Philadelphia, 2 edition, 2003.

[5] R. W. Sharpe. *Differential Geometry.* Springer-Verlag, New York, 1 edition, 1997.

[6] E. Poisson. *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics.* Cambridge University Press, 1 edition, 2004.

[7] S. M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity.* San Francisco, USA: Addison-Wesley, 1 edition, 2004

[8] C. Brans and R. H. Dicke. *Mach's principle and a Relativistic Theory of Gravitation.* Phys. Rev. 124 (1961) 925. doi:10.1103/PhysRev.124.925

[9] T.Kolyvaris. *Μελέτη μελανών οπών συζευγμένων με βαθμωτά πεδία* (Greek) [*Study of Black Hole Solutions coupled with Scalar fields*]. Ph.D Thesis, NTU Athens, September 2013.

[10] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis *Modifed Gravity and Cosmology.* Phys. Rept. 513 (2012) 1 doi:10.1016/j.physrep.2012.01.001 [arXiv:1106.2476 [astro-ph]].

[11] R. G. Cai and Y. S. Myung. *Black holes in the Brans-Dicke-Maxwell Theory.* Phys. Rev. D 56 (1997) 3466 doi:10.1103/PhysRevD.56.3466 [gr-qc/9702037].

[12] B. C. Xanthopoulos and T. Zannias. *Einstein Gravity Coupled to a Massless Scalar Field in Arbitrary Space-time Dimensions.* Phys. Rev. D 40 (1989) 2564. doi:10.1103/PhysRevD.40.2564

[13] G. Kofinas.*The Complete Brans-Dicke Theory.* [arXiv:1510.06845 [gr-qc]]

[14] G. Kofinas and M. Tsoukalas.*The action Complete Brans-Dicke Theory.* [arXiv:1512.04786 [gr-qc]]

[15] C. H. Brans. *Mach's Principle and a Relativistic Theory of Gravitation. II.* Phys. Rev. 125, 2194 (1962)

[16] V. Faraoni, F. Hammad and S. D. Belknap-Keet. *Revisiting the Brans solutions of scalar-tensor gravity*. Phys. Rev. D 94 (2016) 104019 doi:10.1103/PhysRevD.94.104019 [arXiv:1609.02783 [gr-qc]]