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Variational & Asymptotic Methods in the Study of Nonlinear, Free-Surface Waves

(Diploma Thesis)

by

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"*Who in the world am I ?*" wonders Lewis Carroll. My belief is that, for each of us, that question can truly be answered after the end of our journey. But what I already know is that, whoever I am or whoever I will ever be, whatever I have accomplished or whatever I will ever achieve, it is not merely my doing; it is the efforts of all the wonderful people that make this life worth living. This diploma thesis is, of course, no exception. The least I can do, therefore, is express my gratitude to all the people responsible for its completion, as nothing would have ever been possible without them.

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Constantinos P. Mavroeidis

Abstract

In the present thesis, we investigate the application of certain asymptotic and variational methods to the classical water-wave problem, when the assumptions of weak nonlinearity and periodicity or narrowbandedness are made. Particularly, the methods of interest are the Multiple-Scales Method (MSM) [(Nayfeh 2008; Holmes 2012)] and Whitham's Averaged Variational Principle (AVP) [(Whitham 1965a, 1965b, 1974; Jeffrey and Kawahara 1982)].

Our focus lies primarily in the derivation, via those methods, of simpler, but nonlinear nonetheless, model equations that govern the propagation of weakly nonlinear, narrow-banded (i.e. slowly modulated) wavetrains, such as the nonlinear Schrödinger (NLS) equation. Although there is a standard, and well understood, procedure to achieve that by implementing the MSM [(Mei, Stiassnie, and Yue 2005)], in the case of the AVP there occur some issues that render its applicability and its connection with other, established methods, such as the MSM, unclear. The matter of that connection has been answered to a great degree by (Yuen and Lake 1975) and (Sedletsky 2012, 2013, 2015, 2016), who showed that, given suitable ansatzes, the AVP leads to the NLS and other evolutionary equations, for waves in water of either infinite or arbitrary depth. However, in both cases, the vertical dependence of the velocity potential is a priori incorporated into the respective ansatz, and is inspired by the results of other formal perturbation methods. Namely, up to now, it seems that the AVP is not self-contained and, in order to ensure its consistency with other acknowledged results, a significant part of the solution (i.e. the vertical structure of the potential) has to be supplemented by "external" means.

The main result of our work is that, in fact, the AVP is self-contained, as it can yield the appropriate vertical dependence by considering the admissible variations of arbitrary vertical functions. First, we apply our modification of the method to the case of weakly nonlinear, uniform wavetrains (Stokes waves), where we rederive the results of (Fenton 1985) variationally. Interestingly, in the context of the AVP, the two definitions of (Stokes 1847), regarding the wave celerity, arise naturally. Next, we do the same for slowly modulated wavetrains, where, relying again solely on the AVP, we conclude to results that are in complete agreement with those of (Sedletsky 2012, 2013). Therefore, our approach may be considered as a generalization of the works of Yuen, Lake and Sedletsky that renders Whitham's AVP an autonomous and consistent method for the study of periodic or nearly periodic waves.

Contents

Acknowledgements .... **i**
Abstract .... **iii**
Contents .... **v**
Abbreviations .... **vii**

Part I: Weakly nonlinear waves in water of intermediate depth **1**

Chapter 1: The Water-Wave Problem 3
- 1.1. Introduction 3
- 1.2. The classical problem of surface gravity waves 5
- 1.3. The problem of steady periodic waves 9
- 1.4. Some methods for the analysis of nonlinear water waves 16

Chapter 2: A consistent, autonomous approach to Stokes waves, using a variational method .. 18
- 2.1. Introduction 18
- 2.2. Derivation of Stokes waves by means of the AVP 22

Chapter 3: The Multiple-Scales Method in weakly nonlinear, narrow-banded wavetrains 39
- 3.1. Introduction 39
- 3.2. The effects of weak nonlinearity and narrowbandedness in the classical WWP ... 41
- 3.3. The MSM in the governing equations of the problem 44
- 3.4. Derivation of the NLS equation 57

Chapter 4: The Averaged Variational Principle in weakly nonlinear, narrow-banded wavetrains 61
- 4.1. Introduction 61
- 4.2. Derivation of the NLS equation 63

Part II: Appendices **77**

Appendix A: Implementation of the AVP for Stokes waves using Wolfram® Mathematica (Ch. 2) 79

Appendix B: Implementation of the AVP for slowly modulated wavetrains using Wolfram® Mathematica (Ch. 4) 103

Appendix C : Supplementary equations of Chapter 3 121

Appendix D: Proof that the second derivative of the linear dispersion relation is always negative.. 129

Conclusions and future directions **133**

Bibliography **135**

Abbreviations

AVP	:	Averaged Variational Principle
BFI	:	Benjamin-Feir Instability
BR	:	Benney-Roskes
DS	:	Davey-Stewartson
EL	:	Euler-Lagrange
MSM	:	Multiple-Scales Method
NLS	:	Nonlinear Schrödinger
ode	:	ordinary differential equation
pde	:	partial differential equation
rhs	:	right-hand side
WWP	:	Water-Wave Problem

Part I:

Weakly nonlinear waves in water of intermediate depth

Chapter 1

The Water-Wave Problem

“…the next waves of interest, that are easily seen by everyone and which are usually used as an example of waves in elementary courses, are water waves. As we shall soon see, they are the worst possible example, because they are in no respects like sound and light; they have all the complications that waves can have.”

FEYNMAN et al., The Feynman Lectures on Physics, Vol. I, Ch. 51, p. 7

1.1. Introduction

The study of gravity water waves (free-surface waves) constitutes, for many reasons, a scientific subject of fundamental theoretical, physical and practical importance. The efforts to understand their motion, under different circumstances and assumptions, have led, throughout the years, to numerous interesting and demanding mathematical problems [(Johnson 1997; Lannes 2013)], whose treatment has been requiring not only the use of the existing advanced mathematical results and techniques, but the development of new ones in various branches of mathematics. The propagation of water waves, further, defined by the interactions of the water with various factors, such as the seabed and the atmosphere, is associated with a vast amount of striking phenomena that render the *Water-Wave Problem* (WWP) one of the main topics of fluid dynamics and the theory of nonlinear dispersive waves [(Debnath 1994)]. Apart from those aspects, though, understanding the motion of water waves is also very important to applications of naval, marine, coastal and civil engineering or oceanography, among other fields, where the presence of such waves induces significant ramifications that need to be taken into account [(Stoker 1957; Newman 1977; Massel 1989; Mei, Stiassnie, and Yue 2005)]. Indicative examples are the design and optimization of fixed or floating offshore structures, vessels and wave-energy converters, the safety of life at sea, the protection of the marine environment and the study of the effects of wave motion on coastal zones or on the physical properties of the ocean. As a consequence of their complex nature and their impact on such a broad range of scientific disciplines, water waves remain a very active field of research, for which new contributions appear constantly, including involved mathematical analysis and proofs, the derivation of new or enhanced model equations, numerical computations and experiments.

The WWP, even when the simplifying assumptions of an incompressible, inviscid and irrotational fluid are adopted, which allow for the description of the fluid motion via a scalar field (i.e. the velocity potential), is mathematically very difficult to handle. The reason for this is its inherent nonlinearity, owing to the nonlinear boundary conditions on the upper boundary of the fluid domain, i.e. the free surface, and to the fact that that boundary is unknown as well [(Stoker 1957; Debnath 1994)]. Thus, although the WWP is an old problem, tracing back to 1687, when

Newton published his monumental work “Philosophiæ Naturalis Principia Mathematica”, and despite the great minds that dealt with it since then, up to the first half of the 19th century almost all the relevant contributions concerned the theory of the linearized problem. Authors of those contributions were, among others, Cauchy, Euler, Lagrange, Laplace and Poisson, and, further, Russel with his famous observation regarding the solitary wave, and Airy. Extensive information as to that era of the WWP can be found in the historical survey papers of (Darrigol 2003) and (Craik 2004). A milestone, for the study of the nonlinear WWP, was the work of (Stokes 1847), who, via a perturbative approach, presented solutions for the steady periodic problem, in the case of small, but finite nonetheless, waves (see (Craik 2005), for an overview of the life and work of Stokes, and Sec. 2.1, for a very brief outline of his approach and references to the consequent literature). Thereafter, many important works followed in the same direction, where the main goal was the derivation, under certain assumptions, of simplified models, which could be handled, up to a point, by the mathematical means available at the time. Such examples are the works of (Saint-Venant 1871), (Boussinesq 1872) and (Korteweg and De Vries 1895). A comprehensive account on the contributions of that period can be found in (Stoker 1957) and (Wehausen and Laitone 1960).

The scientific production, in regard to the WWP, reached a peak in the 20th century, resulting in a countless number of contributions, and continues to this day with unabated intensity. During that period, rigorous mathematical results and new techniques of exact or asymptotic nature have been provided, striking features of water waves have been discovered and new simplified models, or improved versions of the existing ones, have been derived. A complete review of all those works, if possible, is out of the scope of this section. An extensive briefing on the subject, though, can be found in the books mentioned above and, also, in (Whitham 1974; Massel 1996; Dingemans 1997; Hunt 1997; Kuznetsov et al. 2002; Kharif, Pelinovsky, and Slunyaev 2008; Holthuijsen 2010; Osborne 2010; Ablowitz 2011; Constantin 2011; Bridges, Groves, and Nicholls 2016; Chalikov 2016). However, due to what follows later, we particularly refer to key contributions regarding the variational formulation of the classical WWP. The first relevant effort is that of (Petrov 1964), followed by (Luke 1967), who, in contrast to Petrov, introduced a kinematically unconstrained variational principle. Subsequently, (Zakharov 1968) presented the Hamiltonian formulation of the problem, which was later refined by (Craig and Sulem 1993) via the introduction of a proper Dirichlet to Neumann (DtN) operator. Closing that brief review, we should highlight the most distinct feature of that latter period; the rapid evolution of computers that has allowed for the development of numerical schemes, capable of treating even the fully nonlinear WWP. State-of-the-art schemes, suitable for the fully nonlinear 3D problem are, e.g., the Zakharov/Craig-Sulem method [(Craig and Sulem 1993; Guyenne and Nicholls 2005)], the Higher Order Spectral method [(Dommermuth and Yue 1987; Liu and Yue 1998)], the Boundary Element Method [(Grilli et al. 1994; Grilli, Guyenne, and Dias 2001)], the method of Finite Differences [(Bingham and Zhang 2007; Engsig-Karup, Bingham, and Lindberg 2009)] and the Hamiltonian Coupled Mode Theory [(Athanassoulis and Papoutsellis 2015; Papoutsellis, Charalampopoulos, and Athanassoulis 2018)], which has evolved from the Consistent Coupled-Mode System [(Athanassoulis and Belibassakis 1999, 2007)].

In this era of high-performance computing, where the numerical treatment of the fully nonlinear WWP is feasible, the utilization of asymptotic methods can still be very useful. Specifically, the model equations that result from such approaches can provide proper initializations for the various advanced numerical schemes and, additionally, being computationally “inexpensive”, can be exploited for their enhancement. Motivated by that, in the present thesis we are concerned with the application of asymptotic methods to the WWP, under the assumption of waves of weak nonlinearity and narrow spectrum, towards the derivation of simpler model equations.

1.2. The classical problem of surface gravity waves

In this work, we are mainly interested in the study of periodic or almost periodic progressive waves. Those waves, in the context of a homogeneous, incompressible and inviscid fluid, and in absence of surface tension, constitute special cases of the *classical gravity WWP*. Thus, to lay the foundations of what follows, we begin with the formulation of the latter.

(a) Differential formulation of the problem

Let Oxz be a Cartesian coordinate system, where x denotes the horizontal variable and z the vertical one, pointing upwards. We assume a uniform seabed of depth h, placed at $z=-h$, so that $z=0$ expresses the still surface of the fluid, when it is at rest. Further, $\eta=\eta(x,t)$ denotes the free-surface elevation of the waves, around $z=0$. As a consequence, in the presence of waves, the fluid domain is defined by

$$D = D_h^{\eta}(t) = \{(x,z)\in\mathbb{R}^2: \quad x\in X, \quad -h<z<\eta(x,t), \quad t\geq t_0\}, \tag{1}$$

where $X\subseteq\mathbb{R}$ is the horizontal region that the fluid occupies and t_0 the starting time of the wave motion. Subsequently, the seabed and the free-surface boundaries are represented by

$$\Gamma_h = \{(x,z)\in\mathbb{R}^2: \quad x\in X, \quad z=-h\} \tag{2}$$

and

$$\Gamma_\eta = \Gamma_\eta(t) = \{(x,z)\in\mathbb{R}^2: \quad x\in X, \quad z=\eta(x,t), \quad t\geq t_0\}, \tag{3}$$

respectively. Then, the 2D classical gravity WWP, for water of constant density ρ, over uniform bathymetry h, and under external pressure p_0, can be stated as follows [(Stoker 1957; Whitham 1974; Johnson 1997; Mei, Stiassnie, and Yue 2005)]:

Given the water depth h, the water density ρ, the external pressure p_0 and the functions $\eta_0(x)$ and $\Phi_0(x)$,

find the free-surface elevation field $\eta=\eta(x;t)$ and the velocity potential field $\Phi=\Phi(x,z;t)$ that satisfy the equations

$$\Delta\Phi = \frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial z^2} = 0, \qquad \text{in } D, \tag{4}$$

$$\frac{\partial\Phi}{\partial z} = 0, \qquad \text{on } \Gamma_h, \tag{5}$$

$$\frac{\partial\eta}{\partial t}+\frac{\partial\Phi}{\partial x}\frac{\partial\eta}{\partial x}-\frac{\partial\Phi}{\partial z} = 0, \qquad \text{on } \Gamma_\eta \tag{6}$$

and

$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right\}+g\eta = \frac{p_0}{\rho}, \qquad \text{on } \Gamma_\eta, \tag{7}$$

supplemented by appropriate lateral boundary conditions and initial conditions

$$\eta(x;t_0) = \eta_0(x), \qquad \Phi(x,z=\eta_0(x);t_0) = \Phi_0(x). \tag{8}$$

Each of the above Eqs. (4)-(7) reflects an assumed physical feature of the fluid motion. In particular, Eq. (4) expresses the incompressibility of the flow, given its irrotationality, Eq. (5) the impermeability of the seabed and Eqs. (6) & (7) constitute the kinematic and dynamic boundary conditions of the free surface, respectively. The kinematic condition states that the free surface consists of the same fluid particles, at all times, and, therefore, its motion is determined by the motion of those particles. The dynamic condition, on the other hand, being Bernoulli's law on the free surface, ensures the continuity of the pressure field.

The introduction of appropriate lateral boundary conditions and initial conditions of the form of Eq. (8) is necessary, as it renders the problem's system of equations closed. Without those conditions, the fields η and Φ cannot be uniquely defined, as no information is contained in Eqs. (4)-(7), neither for their behavior on the lateral boundaries, nor their initial state, which is required by the t-differentiations of Eqs. (6) & (7).

In absence of specific distribution of applied pressure along the free surface, and because the fluid is considered incompressible, p_0 may be a function of the temporal variable t only, without affecting the fluid flow. Concurrently, the physical meaning of the velocity potential lies in its gradient. Thus, with a suitable transformation, it can "absorb" any function of t alone, preserving its physical essence [(Stoker 1957)]. We deduce, therefore, that Φ can be redefined, so that $p_0(t)/\rho$ is incorporated into it. Indeed, introducing

$$\hat{\Phi}(x,z;t) = \Phi(x,z;t) - \frac{1}{\rho}\int_{t_0}^{t} p_0(s)\,ds, \tag{9}$$

Eqs. (4)-(7) are rewritten as

$$\Delta\hat{\Phi} = \frac{\partial^2\hat{\Phi}}{\partial x^2} + \frac{\partial^2\hat{\Phi}}{\partial z^2} = 0, \qquad \text{in } D, \tag{4'}$$

$$\frac{\partial\hat{\Phi}}{\partial z} = 0, \qquad \text{on } \Gamma_h, \tag{5'}$$

$$\frac{\partial\eta}{\partial t} + \frac{\partial\hat{\Phi}}{\partial x}\frac{\partial\eta}{\partial x} - \frac{\partial\hat{\Phi}}{\partial z} = 0, \qquad \text{on } \Gamma_\eta \tag{6'}$$

and

$$\frac{\partial\hat{\Phi}}{\partial t} + \frac{1}{2}\left\{\left(\frac{\partial\hat{\Phi}}{\partial x}\right)^2 + \left(\frac{\partial\hat{\Phi}}{\partial z}\right)^2\right\} + g\eta = 0, \qquad \text{on } \Gamma_\eta, \tag{7'}$$

while the velocity field remains unchanged ($\nabla\hat{\Phi} = \nabla\Phi$) [(Stoker 1957; Clamond 2017)]. Henceforth, the uniformity of p_0 is assumed, namely $p_0 = p_0(t)$. Thus, both Φ and $\hat{\Phi}$ are used, as they are equivalent from a physical viewpoint, with the understanding that the latter contains an additional dependence on t, as shown in Eq. (9).

(b) Variational formulation of the problem

The classical gravity WWP admits an equivalent variational formulation, in terms of Luke's unconstrained variational principle, due to (Luke 1967). Specifically, defining the Lagrangian density

$$\mathcal{L}[\eta(x;t),\widehat{\Phi}(x,\bullet;t)] = \int_{-h}^{\eta}\left\{\frac{\partial\widehat{\Phi}}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\widehat{\Phi}}{\partial x}\right)^2+\left(\frac{\partial\widehat{\Phi}}{\partial z}\right)^2\right]+gz\right\}dz, \quad (10)$$

and integrating it over the time interval $T=[t_0,t_1]$ and the unbounded horizontal domain X, results in the action functional ([1])

$$\mathscr{S}[\eta,\widehat{\Phi}] = \int_T\int_X \mathcal{L}[\eta(x;t),\widehat{\Phi}(x,\bullet;t)]\,dx\,dt. \quad (11)$$

Then, Eqs. (4')-(7') arise as the Euler-Lagrange (EL) equations of the variational equation

$$\delta\mathscr{S}[\eta,\widehat{\Phi};\delta\eta,\delta\widehat{\Phi}] = \delta_\eta\mathscr{S}[\eta,\widehat{\Phi};\delta\eta]+\delta_{\widehat{\Phi}}\mathscr{S}[\eta,\widehat{\Phi};\delta\widehat{\Phi}] = 0, \quad (12)$$

$\forall\,(\delta\eta,\delta\widehat{\Phi})\in\mathcal{S}_{adm}$, where $\mathcal{S}_{adm}$ is the space of admissible variations and $\delta_\eta\mathscr{S}$ and $\delta_{\widehat{\Phi}}\mathscr{S}$ are the partial Gâteaux derivatives of $\mathscr{S}$, with respect to the fields η and $\widehat{\Phi}$, and in the directions $\delta\eta$ and $\delta\widehat{\Phi}$, respectively. In particular, calculating those derivatives [(Luke 1967; Whitham 1974)], it is true that

$$\delta_\eta\mathscr{S} = \int_T\int_X\left[\frac{\partial\widehat{\Phi}}{\partial t}+\frac{1}{2}\left\{\left(\frac{\partial\widehat{\Phi}}{\partial x}\right)^2+\left(\frac{\partial\widehat{\Phi}}{\partial z}\right)^2\right\}+gz\right]_{z=\eta}\delta\eta\,dx\,dt \quad (13)$$

and

$$\begin{aligned}\delta_{\widehat{\Phi}}\mathscr{S} &= \int_T\int_X\int_{-h}^{\eta}\{\delta\widehat{\Phi}_t+\widehat{\Phi}_x\delta\widehat{\Phi}_x+\widehat{\Phi}_z\delta\widehat{\Phi}_z\}\,dz\,dx\,dt = \\ &= \int_T\int_X\left\{\frac{\partial}{\partial t}\int_{-h}^{\eta}\delta\widehat{\Phi}\,dz+\frac{\partial}{\partial x}\int_{-h}^{\eta}\frac{\partial\widehat{\Phi}}{\partial x}\delta\widehat{\Phi}\,dz\right\}dx\,dt \\ &\quad -\int_T\int_X\int_{-h}^{\eta}\left\{\frac{\partial^2\widehat{\Phi}}{\partial x^2}+\frac{\partial^2\widehat{\Phi}}{\partial z^2}\right\}\delta\widehat{\Phi}\,dz\,dx\,dt \\ &\quad -\int_T\int_X\left[\left(\frac{\partial\eta}{\partial t}+\frac{\partial\widehat{\Phi}}{\partial x}\frac{\partial\eta}{\partial x}-\frac{\partial\widehat{\Phi}}{\partial z}\right)\delta\widehat{\Phi}\right]_{z=\eta}dx\,dt \\ &\quad -\int_T\int_X\left[\frac{\partial\widehat{\Phi}}{\partial z}\delta\widehat{\Phi}\right]_{z=-h}dx\,dt.\end{aligned} \quad (14)$$

Hence, considering, in Eq. (12), arbitrary variations $\delta\eta$ and $\delta\widehat{\Phi}$ that vanish on the boundaries of T (i.e. $t=t_0$, $t=t_1$) and X (i.e. $|x|\to\infty$) ([2]), and implementing the standard procedure of the calculus of variations [(Gelfand and Fomin 1963; Sagan 1969)], Eqs. (4')-(7') are recovered as follows. Since Eq. (12) holds $\forall\,(\delta\eta,\delta\widehat{\Phi})\in\mathcal{S}_{adm}$, it must also hold for varia-

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([1]) Assuming that the functional is well-defined.
([2]) In that case, the first integral of the rhs of the second equality of Eq. (14) vanishes, as it integrates out to the boundaries of T and X.

tions $\delta\eta$ or $\delta\widehat{\Phi}$ alone. Starting with the admissible variations $\delta\widehat{\Phi}$ in Eq. (14), we initially take those arbitrary variations that vanish on the boundaries Γ_h and Γ_η (i.e. on the seabed and on the free surface). As a consequence, by the standard argument of the variational calculus,

$$\delta\widehat{\Phi}\Big|_D : \qquad \Delta\widehat{\Phi} = \frac{\partial^2\widehat{\Phi}}{\partial x^2} + \frac{\partial^2\widehat{\Phi}}{\partial z^2} = 0, \qquad \text{in } D.$$

is derived. Given that, we next take into account the arbitrary variations that vanish on the free surface, which lead to

$$\delta\widehat{\Phi}\Big|_{\Gamma_h} : \qquad \frac{\partial\widehat{\Phi}}{\partial z} = 0, \qquad \text{on } \Gamma_h,$$

whereas, in a similar manner, variations that vanish on the seabed yield

$$\delta\widehat{\Phi}\Big|_{\Gamma_\eta} : \qquad \frac{\partial\eta}{\partial t} + \frac{\partial\widehat{\Phi}}{\partial x}\frac{\partial\eta}{\partial x} - \frac{\partial\widehat{\Phi}}{\partial z} = 0, \qquad \text{on } \Gamma_\eta.$$

Finally, considering the admissible variations $\delta\eta$, from Eq. (13) we obtain

$$\delta\eta \quad : \qquad \frac{\partial\widehat{\Phi}}{\partial t} + \frac{1}{2}\left\{\left(\frac{\partial\widehat{\Phi}}{\partial x}\right)^2 + \left(\frac{\partial\widehat{\Phi}}{\partial z}\right)^2\right\} + g\eta = 0, \qquad \text{on } \Gamma_\eta,$$

which completes the recovery of the problem's differential formulation from its variational one. More details on the subject can be found in (Luke 1967), (Whitham 1974) and (Debnath 1994).

1.3. The problem of steady periodic waves

Based on the previous section, a special case of the classical gravity WWP is that of steady progressive waves. That is, travelling waves that propagate uniformly in the horizontal direction, with (phase) speed c, so as the wave motion appears to be independent of time (steady) in a reference frame moving with c along the x axis.

For our purposes, steady and, in addition, periodic waves are particularly important. Hence, in this section, with the general problem of Sec. 1.2 as a starting point, we formulate the *steady periodic WWP*.

(a) Differential formulation of the problem

Towards the pursuit of steady progressive waves, we assume that the respective fields η and Φ, which, in general, have to satisfy Eqs. (4)-(7) of Sec.1.2, are of the form [(Stoker 1957)]

$$\Phi = \Phi(\xi, z) \qquad \text{and} \qquad \eta = \eta(\xi), \tag{1}$$

where

$$\xi = x - ct. \tag{2}$$

In that instance, deploying the chain rule, it is easily verified that

$$\frac{\partial}{\partial t}\{\eta,\Phi\} = \frac{\partial}{\partial \xi}\{\eta,\Phi\}\frac{\partial \xi}{\partial t} = -c\frac{\partial}{\partial \xi}\{\eta,\Phi\}, \tag{3a}$$

and

$$\frac{\partial}{\partial x}\{\eta,\Phi\} = \frac{\partial}{\partial \xi}\{\eta,\Phi\}\frac{\partial \xi}{\partial x} = \frac{\partial}{\partial \xi}\{\eta,\Phi\}, \tag{3b}$$

while the z-differentiation remains unaffected. Moreover, since the flow is supposed to be steady with respect to the moving frame ([1]), the external pressure p_0, in Eq. (7, Sec. 1.2) has to be independent of time. Namely,

$$p_0 / \rho = R, \tag{4}$$

R being a Bernoulli constant ([2]) [(Landau and Lifshitz 1987; Vasan and Deconinck 2013; Clamond 2017)].

As said before, we are interested in waves that are not only steady, but also periodic. Consequently, introducing a characteristic length λ (wave length), we assume that the fields η and Φ are such that

$$\eta(\xi) = \eta(\xi + \lambda) \qquad \text{and} \qquad \Phi(\xi, z) = \Phi(\xi + \lambda, z). \tag{5}$$

In other words, we consider the wave fields to be ξ-periodic, with period λ.

Naturally, searching for solutions of steady periodic waves affects the fluid domain D. To begin with, because of the form of the fields η and Φ in Eq. (1), it becomes

$$D_{\Xi} = \{(\xi, z) \in \mathbb{R}^2 : \quad \xi \in \Xi, \quad -h < z < \eta(\xi)\}. \tag{6}$$

([1]) For the flow to be steady, its physical quantities (e.g. velocity, pressure) must be independent of time.

([2]) As a result, via Eq. (9, Sec. 1.2), we deduce that, up to an arbitrary constant, $\hat{\Phi} = \hat{\Phi}(\xi, z; t) = \Phi(\xi, z) - Rt$, which is the general form of the redefined potential, in the case of steady waves.

Further, due to the ξ-periodicity of the waves, the domain Ξ is confined within their periodic cell. Namely, for some $\xi_0 \in \mathbb{R}$, $\Xi = [\xi_0, \xi_0 + \lambda]$. Without loss of generality, therefore, we set

$$\Xi = [0, \lambda]. \tag{7}$$

As for the vertical boundaries of D_Ξ, we keep the notations Γ_h and Γ_η, for the seabed and the free surface, respectively, with the understanding that they refer to the periodic cell of the waves.

Taking into account the above, and substituting Eqs. (1)-(4) into Eqs. (4)-(7) of Sec. 1.2, we formulate the steady periodic WWP as follows:

<u>Given</u> the three lengths h (water depth), λ (wave length) and H (wave height),

<u>find</u>

- *the constants c and R,*
- *the ξ-periodic free-surface elevation field $\eta = \eta(\xi)$,*

  $\eta(\xi) = \eta(\xi + \lambda)$,
- *and the ξ-periodic wave potential field $\Phi = \Phi(\xi, z)$,*

  $\Phi(\xi, z) = \Phi(\xi + \lambda, z)$,

that satisfy the equations

$$\Delta_\xi \Phi = \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0, \qquad \text{in } D_\Xi, \tag{8}$$

$$\frac{\partial \Phi}{\partial z} = 0, \qquad \text{on } \Gamma_h, \tag{9}$$

$$-c\,\frac{\partial \eta}{\partial \xi} + \frac{\partial \Phi}{\partial \xi}\frac{\partial \eta}{\partial \xi} - \frac{\partial \Phi}{\partial z} = 0, \qquad \text{on } \Gamma_\eta, \tag{10}$$

$$-c\,\frac{\partial \Phi}{\partial \xi} + \frac{1}{2}\left\{\left(\frac{\partial \Phi}{\partial \xi}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right\} + g\eta = R, \qquad \text{on } \Gamma_\eta, \tag{11}$$

and that are subject to the constraints

$$\int_\Xi \eta(\xi)\, d\xi = 0, \qquad \eta_{\max} - \eta_{\min} = \eta(0) - \eta(\lambda/2) = H. \tag{12a,b}$$

Each constraint has a certain meaning. Eq. (12a) is the conservation of mass (for uniform ρ) within the periodic cell, whereas Eq. (12b) essentially constitutes the definition of H, constraining η. The second equality of the latter equation is a choice that places the wave crest at the origin of the ξ-axis (i.e. $\xi = 0$) of the periodic cell's coordinate system.

(b) <u>Variational formulation of the problem</u>

As with the classical gravity WWP, the steady periodic one may be variationally reformulated, too. Given the Lagrangian density of Eq. (10, Sec. 1.2) and the subsequent action functional of Eq. (11, Sec. 1.2), we search for steady periodic solutions. We assume, therefore, the validity of Eqs. (1)-(7) & (12). As a result, the action functional $\mathscr{S}$ is confined within the periodic cell of the problem and the change of variables

$$(x,t) \to (\xi,\tau), \qquad X \to \Xi, \qquad T \to \mathrm{T} = [\tau_0, \tau_1]$$

occurs in it, with $\tau = t$ and ξ given by Eq. (2). Thus, $\mathscr{S}$ is transformed as

$$\int_T \int_X \mathcal{L}(x,t)\,dx\,dt = \int_{\mathrm{T}} \int_{\Xi} \mathcal{L}(x(\xi,\tau), t(\xi,\tau)) \left|\frac{\partial(x,t)}{\partial(\xi,\tau)}\right| d\xi\,d\tau\,,$$

the Jacobian being

$$\frac{\partial(x,t)}{\partial(\xi,\tau)} = \begin{vmatrix} x_\xi & x_\tau \\ t_\xi & t_\tau \end{vmatrix} = \begin{vmatrix} 1 & c \\ 0 & 1 \end{vmatrix} = 1.$$

Further, in that case, the redefined potential $\widehat{\Phi}$ is shaped as (see Footnote 2)

$$\widehat{\Phi} = \widehat{\Phi}(\xi,z;\tau) = \Phi(\xi,z) - R\tau\,. \tag{13}$$

Hence, for the action functional, we have that

$$\begin{aligned} \mathscr{S} &= \int_{\mathrm{T}} \int_{\Xi} \mathcal{L}[\eta(\xi), \Phi(\xi,\bullet) - R\tau]\,d\xi\,d\tau = \qquad (14) \\ &= \int_{\mathrm{T}} \int_{\Xi} \int_{-h}^{\eta} \left\{ -c\,\frac{\partial\Phi}{\partial\xi} + \frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial\xi}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2 \right] + gz - R \right\} dz\,d\xi\,d\tau = \\ &= \int_{\mathrm{T}} d\tau \int_{\Xi} \int_{-h}^{\eta} \left\{ -c\,\frac{\partial\Phi}{\partial\xi} + \frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial\xi}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2 \right] + gz - R \right\} dz\,d\xi = \\ &= (\tau_1 - \tau_0) \int_{\Xi} \int_{-h}^{\eta} \left\{ -c\,\frac{\partial\Phi}{\partial\xi} + \frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial\xi}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2 \right] + gz - R \right\} dz\,d\xi. \end{aligned}$$

The constant $\tau_1 - \tau_0 \neq 0$, though, is indifferent regarding the variational equation of $\mathscr{S}$ and, thus, can be omitted. Consequently, without any damage, we set

$$\mathscr{S} = \int_{\Xi} \int_{-h}^{\eta} \left\{ -c\,\frac{\partial\Phi}{\partial\xi} + \frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial\xi}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2 \right] + gz - R \right\} dz\,d\xi\,. \tag{15}$$

Rendering $\mathscr{S}$ in the above form stationary, demanding, that is, the satisfaction of the variational equation

$$\delta\mathscr{S}[\eta,\Phi;\delta\eta,\delta\Phi] = \delta_\eta \mathscr{S}[\eta,\Phi;\delta\eta] + \delta_\Phi \mathscr{S}[\eta,\Phi;\delta\Phi] = 0\,, \tag{16}$$

$\forall\ (\delta\eta,\delta\Phi) \in \mathcal{S}_{adm}$, leads, as shown in the proof below, to

$$\delta\Phi\big|_{D_\Xi}: \qquad \Delta_\xi \Phi = \frac{\partial^2\Phi}{\partial\xi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0\,, \qquad \text{in } D_\Xi\,,$$

$$\delta\Phi\big|_{\Gamma_h}: \qquad \frac{\partial\Phi}{\partial z} = 0\,, \qquad \text{on } \Gamma_h\,,$$

$$\delta\Phi\big|_{\Gamma_\eta} \quad : \qquad -c\frac{\partial\eta}{\partial\xi}+\frac{\partial\Phi}{\partial\xi}\frac{\partial\eta}{\partial\xi}-\frac{\partial\Phi}{\partial z}=0, \qquad \text{on } \Gamma_\eta,$$

and

$$\delta\eta \quad : \qquad -c\frac{\partial\Phi}{\partial\xi}+\frac{1}{2}\left\{\left(\frac{\partial\Phi}{\partial\xi}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right\}+g\eta=R, \quad \text{on } \Gamma_\eta,$$

which are no other than Eqs. (8)-(11). Therefore, the combination of Eqs. (15) & (16) with the constraints of Eq. (12), as essential conditions, reproduces the steady periodic WWP.

Proof: Since the functional of Eq. (15) is constructed from the general functional of the classical gravity WWP, assuming steady periodic waves, it is already anticipated that it can provide us with their governing equations. Nevertheless, we prove that this is the case indeed.

We consider the variational Eq. (16), with the functional $\mathscr{S}$ given by Eq. (15). Calculating the functional derivative $\delta_\Phi\mathscr{S}$, we obtain

$$\begin{aligned}\delta_\Phi\mathscr{S}[\eta,\Phi;\delta\Phi] &= \int_\Xi\int_{-h}^{\eta}\{-c\,\delta\Phi_\xi+\Phi_\xi\,\delta\Phi_\xi+\Phi_z\,\delta\Phi_z\}dz\,d\xi = \\ &= \int_\Xi\left\{\int_{-h}^{\eta}(\Phi_\xi-c)\,\delta\Phi_\xi\,dz+\int_{-h}^{\eta}\Phi_z\,\delta\Phi_z\,dz\right\}d\xi.\end{aligned} \tag{a}$$

Utilizing, further, Leibnitz's integral rule, we find that

$$\begin{aligned}\frac{d}{d\xi}\int_{-h}^{\eta}(\Phi_\xi-c)\,\delta\Phi\,dz &= \int_{-h}^{\eta}\frac{\partial}{\partial\xi}\{(\Phi_\xi-c)\,\delta\Phi\}dz+\eta_\xi\big[(\Phi_\xi-c)\,\delta\Phi\big]_{z=\eta} = \\ &= \int_{-h}^{\eta}\{(\Phi_\xi-c)\,\delta\Phi_\xi+\Phi_{\xi\xi}\,\delta\Phi\}dz+\eta_\xi\big[(\Phi_\xi-c)\,\delta\Phi\big]_{z=\eta}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{-h}^{\eta}(\Phi_\xi-c)\,\delta\Phi_\xi\,dz &= \frac{d}{d\xi}\int_{-h}^{\eta}(\Phi_\xi-c)\,\delta\Phi\,dz \\ &\quad -\int_{-h}^{\eta}\Phi_{\xi\xi}\,\delta\Phi\,dz-\eta_\xi\big[(\Phi_\xi-c)\,\delta\Phi\big]_{z=\eta}.\end{aligned} \tag{b}$$

Moreover, since

$$\Phi_z\,\delta\Phi_z=\frac{\partial}{\partial z}(\Phi_z\,\delta\Phi)-\Phi_{zz}\,\delta\Phi,$$

it is true that

$$\int_{-h}^{\eta}\Phi_z\,\delta\Phi_z\,dz=\int_{-h}^{\eta}\left\{\frac{\partial}{\partial z}(\Phi_z\,\delta\Phi)-\Phi_{zz}\,\delta\Phi\right\}dz=$$

$$= \left[\Phi_z \,\delta\Phi\right]_{z=\eta} - \left[\Phi_z \,\delta\Phi\right]_{z=-h} - \int_{-h}^{\eta} \Phi_{zz}\,\delta\Phi\, dz. \qquad \text{(c)}$$

Therefore, substituting Eqs. (b) & (c) into Eq. (a), the latter is shaped as

$$\delta_\Phi \mathscr{S} = \int_\Xi \left\{ \begin{aligned} &\frac{d}{d\xi}\int_{-h}^{\eta} (\Phi_\xi - c)\,\delta\Phi\, dz - \int_{-h}^{\eta} (\Phi_{\xi\xi} + \Phi_{zz})\,\delta\Phi\, dz \\ &+ \left[\Phi_z\,\delta\Phi - \eta_\xi(\Phi_\xi - c)\,\delta\Phi\right]_{z=\eta} - \left[\Phi_z\,\delta\Phi\right]_{z=-h} \end{aligned} \right\} d\xi\,.$$

Though, the term

$$\int_\Xi \frac{d}{d\xi}\int_{-h}^{\eta} (\Phi_\xi - c)\,\delta\Phi\, dz\, d\xi$$

integrates out to the boundaries of Ξ . Consequently, choosing variations $\delta\Phi$ such that

$$\delta\Phi(\xi = 0, z) = \delta\Phi(\xi = \lambda, z) = 0\,,$$

that term vanishes. Hence, after all,

$$\delta_\Phi \mathscr{S} = -\int_\Xi \left\{ \begin{aligned} &\int_{-h}^{\eta} (\Phi_{\xi\xi} + \Phi_{zz})\,\delta\Phi\, dz + \left[\Phi_z\,\delta\Phi\right]_{z=-h} \\ &\qquad - \left[\Phi_z\,\delta\Phi - \eta_\xi(\Phi_\xi - c)\,\delta\Phi\right]_{z=\eta} \end{aligned} \right\} d\xi\,. \qquad \text{(d)}$$

With $\delta_\Phi \mathscr{S}$ dictated by Eq. (d), the variational equation of the action functional becomes

$$\int_\Xi \left\{ \begin{aligned} &\int_{-h}^{\eta} (\Phi_{\xi\xi} + \Phi_{zz})\,\delta\Phi\, dz + \left[\Phi_z\,\delta\Phi\right]_{z=-h} \\ &\qquad - \left[\Phi_z\,\delta\Phi - \eta_\xi(\Phi_\xi - c)\,\delta\Phi\right]_{z=\eta} \end{aligned} \right\} d\xi = 0\,, \qquad \text{(e)}$$

$\forall\, \delta\Phi \in \mathcal{S}_{adm}$, when admissible variations $\delta\Phi$ are considered. So, following the standard procedure of the calculus of variations, we initially take those variations that vanish on the boundaries Γ_h and Γ_η . As a result,

$$\int_\Xi \int_{-h}^{\eta} (\Phi_{\xi\xi} + \Phi_{zz})\,\delta\Phi\, dz\, d\xi = 0$$

must be true for arbitrary variations $\delta\Phi$, within the fluid domain D_Ξ . Thus, using the fundamental lemma of the variational calculus, we conclude that

$$\Phi_{\xi\xi} + \Phi_{zz} = 0 \qquad \text{in } D_\Xi\,. \qquad \text{(f)}$$

Given the above equation, Eq. (e) turns into

$$\int_{\Xi} \left[\Phi_z\,\delta\Phi\right]_{z=-h} - \left[\Phi_z\,\delta\Phi - \eta_\xi(\Phi_\xi - c)\,\delta\Phi\right]_{z=\eta}\; d\xi \;=\; 0\,. \tag{e1}$$

Taking, therefore, into account those arbitrary variations that vanish on the free surface yields

$$\int_{\Xi} \left[\Phi_z\right]_{z=-h}\;\delta\Phi_{\;z=-h}\, d\xi \;=\; 0\,.$$

Hence, exploiting, once more, the arbitrariness of the admissible variations and the fundamental lemma results in

$$\left[\Phi_z\right]_{z=-h} \;=\; 0\,. \tag{g}$$

Then, inserting Eq. (g) into Eq. (e1), there remains

$$\int_{\Xi} \left[\Phi_z - \eta_\xi(\Phi_\xi - c)\right]_{z=\eta}\;\delta\Phi_{\;z=\eta}\, d\xi \;=\; 0\,. \tag{e2}$$

Thus, in a similar manner, we are led to

$$\left[\Phi_z - \eta_\xi(\Phi_\xi - c)\right]_{z=\eta} \;=\; 0\,. \tag{h}$$

Moving on to the functional derivative $\delta_\eta \mathscr{S}$, we instantly find that

$$\delta_\eta \mathscr{S}[\eta,\Phi;\delta\eta] = \int_{\Xi} \left[-c\Phi_\xi + \frac{1}{2}(\Phi_\xi{}^2 + \Phi_z{}^2) + g\,z - R\right]_{z=\eta} \delta\eta\, d\xi\,.$$

As a result, working as before, we deduce that, for admissible variations $\delta\eta$, the variational equation of $\mathscr{S}$ gives

$$\left[-c\Phi_\xi + \frac{1}{2}(\Phi_\xi{}^2 + \Phi_z{}^2) + g\,z - R\right]_{z=\eta} \;=\; 0\,. \tag{i}$$

Eqs. (f), (g), (h) & (i), which are derived as the EL equations of the variational principle of Eqs. (15) & (16), coincide with Eqs. (8)-(11). Thus, the proof is complete. ■

A slightly different, but equivalent, variational reformulation, for the same problem, is obtained as follows. Instead of using Eq. (13) for $\widehat{\Phi}$, we adopt, for it, the form

$$\widehat{\Phi} = \widehat{\Phi}(\xi,z) = \Phi(\xi,z)\,, \tag{17}$$

leaving aside the additional t-dependence that arises from Eq. (9, Sec. 1.2). In other words, given Eqs. (10, Sec. 1.2) & (11, Sec. 1.2), we assume nothing more than fields η and $\widehat{\Phi}$ that correspond to steady periodic waves. Furthermore, introducing the Lagrange multiplier Q, we incorporate Eq. (12a) (mass conservation), divided by λ ([3]), into the Lagrangian density $\mathcal{L}$. Thus, in the place of Eq. (14), we now have ([4])

$$\mathscr{S} \;=\; \int_{\mathrm{T}}\int_{\Xi} \left\{\mathcal{L}[\eta(\xi),\Phi(\xi,\bullet)] - \frac{Q}{\lambda}\int_{\Xi}\eta(\xi)\,d\xi\right\} d\xi\, d\tau \;=$$

([3]) As will be seen below, there is a strong connection between the Lagrange multiplier Q and the Bernoulli constant R. Thus, dividing (both sides of) Eq. (12a) by λ aims at matching the dimensions of Q with those of R.
([4]) Omitting again the indifferent constant $\tau_1 - \tau_0$.

$$= \int_{\mathrm{T}} d\tau \int_{\Xi} \left\{ \mathcal{L}[\eta(\xi), \Phi(\xi,\bullet)] - \frac{Q}{\lambda} \int_{\Xi} \eta(\xi)\, d\xi \right\} d\xi \ =$$

$$= \int_{\Xi} \{ \mathcal{L}[\eta(\xi), \Phi(\xi,\bullet)] - Q\,\eta(\xi) \} d\xi \ =$$

$$= \int_{\Xi} \left\{ \int_{-h}^{\eta} \left[-c\, \frac{\partial \Phi}{\partial \xi} + \frac{1}{2} \left(\left(\frac{\partial \Phi}{\partial \xi} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right) + g\, z \right] dz - Q\,\eta \right\} d\xi \,. \qquad (18)$$

With $\mathscr{S}$ given as in Eq. (18), and following once more the usual arguments of calculus of variations, its $\delta\Phi$-variation recovers Eqs. (8)-(10). The $\delta\eta$-variation, on the other hand, leads to

$$-c\, \frac{\partial \Phi}{\partial \xi} + \frac{1}{2} \left\{ \left(\frac{\partial \Phi}{\partial \xi} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right\} + g\eta = Q\,, \qquad \text{on } \Gamma_{\eta}\,.$$

Comparing the above equation with Eq. (11), we see that the Lagrange multiplier of the conservation-of-mass constraint coincides with the Bernoulli constant R (i.e. we can set $Q = R$). Furthermore, viewing Q as a function that is constant throughout Ξ, the functional $\mathscr{S}$ can be regarded as

$$\mathscr{S} = \mathscr{S}[\eta, \Phi, Q]\,.$$

Then, Eq. (12a) emerges from the variation δQ of $\mathscr{S}$. As a result, the stationarity of the functional of Eq. (18), together with Eq. (12b) as an essential condition (since Eq. (12a) is embedded into $\mathscr{S}$), constitutes an alternative version of the variational formulation of the steady periodic problem.

1.4. Some methods for the analysis of nonlinear water waves

The mathematical treatment of the classical gravity WWP constitutes a very difficult task, due to the fact that, not only the upper boundary of the fluid domain - i.e. the free-surface - is unknown, but also the (boundary) conditions on it are nonlinear (see Eqs. (6) & (7) of Sec. 1.2). Thus, in general, the involvement of an appropriate numerical scheme is required, to obtain solutions for the problem.

Attacking the full WWP is not the only option, though. In fact, a very popular approach is the derivation of simpler model equations, out of the complete formulation of the WWP, by using asymptotic methods, under certain assumptions of smallness. Such assumptions may refer to the water depth, the seabed's variation, the nonlinearity of the waves or the rate of their modulation [(Papoutsellis 2016)]. The range of validity of the resulting simplified models is, of course, limited, but they are easier to solve and less demanding computationally.

A full account on the various simplified models for the WWP or the asymptotic methods used for their derivation, is out of the scope of the present thesis. Our main interest, after all, as mentioned earlier, lies in the implementation of such methods under the assumptions of weakly nonlinear and narrow-banded wavetrains, aiming at their exploitation in the enhancement of numerical schemes and the determination of appropriate initializations for them. Nevertheless, some notable references on those subjects are the books of (Whitham 1974), (Jeffrey and Kawahara 1982), (Debnath 1994), (Dingemans 1997), (Johnson 1997), (Mei, Stiassnie, and Yue 2005), (Ablowitz 2011) and (Lannes 2013). Here, the methods we focus on are the Multiple-Scales Method (MSM) and the Averaged Variational Principle (AVP), which are suitable for the study of weakly nonlinear, narrow-banded (i.e. slowly modulated) wavetrains.

The MSM is a singular perturbation method [(Nayfeh 2008; Holmes 2012)], which can be used for the asymptotic treatment of weakly nonlinear differential equations, ordinary or partial, of oscillatory type [(Murdock 1999)], in which the weak nonlinearity has a non-negligible, cumulative effect, inducing phenomena that evolve in different temporal and/or spatial scales [(Kevorkian and Cole 2012)]. Therefore, the main idea of the method is the introduction of appropriate scales, which are viewed as independent variables, and the assumption that the solution of the problem may be expressed as a function of those scales [(Sanchez 1996; Holmes 2012)]. That function is then expressed as a perturbation expansion, with respect to a perturbation parameter that represents the weak nonlinearity. As a result, substituted into the initial differential equation, it leads to a perturbation hierarchy of differential equations, which can be solved sequentially. Usually, nonuniformities (secular terms) arise during the latter procedure, which, unresolved, lead to the breakdown of the orders of the asymptotic solution, rendering it nonuniform. The added independent variables (i.e. the introduced scales), though, increase the degrees of freedom, providing us with the capability of imposing proper solvability conditions that eliminate the aforementioned nonuniformities [(Jeffrey and Kawahara 1982; Holmes 2012)]. Those conditions usually impose restrictions to a -yet- arbitrary, fundamental amplitude, which occurs from the recursive solution of the perturbation hierarchy and governs the solution's oscillatory part. Their combination leads to a, so-called, amplitude equation [(Bender and Orszag 1984; Moloney and Newell 2004)], which determines that amplitude. The nature of the solvability conditions, the arguments that are used to derive them and the way they affect the asymptotic solution are not unique. On the contrary, they heavily depend on the problem and its possible nonuniformities [(Nayfeh 2011)]. There are several variants of the MSM. Here, we adopt the derivative-expansion method, which is thoroughly analyzed in the book of (Nayfeh 2008), alongside an extensive literature review on the MSM and its applications. Other general books about perturbation methods, that feature the derivative-expansion method, are those of (Bender and Orszag 1984), (Hinch 1991), (Murdock 1999) and (Holmes 2012), among many others. One more valuable reference is the work of (Jeffrey and Kawahara 1982), which deals with the asymptotic methods that are useful in the study of nonlinear waves, including the MSM as well.

The other method of interest, the AVP, was devised by (Whitham 1965a, 1965b), aiming at the study of slowly varying dispersive wavetrains, and is generally applicable to periodic or nearly periodic waves. For its implementation, a variational formulation of the desired problem is required and suitable ansatzes, referring to its unknown governing fields [(Whitham 1974; Jeffrey and Kawahara 1982)]. Those ansatzes are functions of a variable that represents the fast, oscillatory part of the wave motion, i.e. the phase function [(Mei, Stiassnie, and Yue 2005)], and other quantities, which may be constant or slowly varying functions in space and time, depending on whether the waves are exactly or nearly periodic. Given those, the essence of the method is the following [(Whitham 1967, 1974; Karpman 1975; Jeffrey and Kawahara 1982)]. First, the ansatzes are substituted into the Lagrangian density. Then, since the latter becomes a function of the (fast) phase and other, slowly varying functions, it is averaged over that phase, under the assumption that, during that integration, the other quantities remain approximately constant [(Whitham 1974; Jeffrey and Kawahara 1982)]. In that way the, so called, averaged Lagrangian of the problem is obtained, which is a function of the aforementioned slowly varying functions alone. Replacing the Lagrangian with the averaged Lagrangian, the initial variational principle of the problem turns into Whitham's AVP, whose action functional is now a functional in terms of the unknown, slowly varying functions contained in the ansatzes [(Debnath 1994)]. According to Whitham, taking the variations of that functional with respect to those functions, results in relations that determine the latter. Following the work of Whitham, (Lighthill 1965) examined the range of applicability of the former's theory, whereas (Luke 1966), extending the method of (Kuzmak 1959) to pdes, partially justified Whitham's approach by relating it to his systematic perturbation method. Not much later, (Bretherton 1968) examined the validity of the method in a wide class of linear systems, while (Hoogstraten 1968, 1969) also contributed to the justification of the AVP for the cases of waves in shallow and deep water. (Whitham 1970), moreover, given that Luke worked with the Euler equations of the problem, applied the same ideas, for the justification of his method, directly on its variational principle. Subsequently, (Kurylev 1981), using the Klein-Gordon equation, highlighted the connection between the AVP and the WKB method, providing some results that further add to the validity of the former. A full justification, though, is still an open matter. Of course, numerous contributions have been made in regard to applications and extensions of Whitham's method. Several of them are mentioned in Chapter 4. Here we confine ourselves with the work of (Kawahara 1977), who introduced a systematized version of the AVP, by essentially introducing the MSM into the variational formalism. Some valuable references, for a complete exposition to the method and a wider bibliographical guidance, are the books of (Whitham 1974), (Karpman 1975), (Jeffrey and Kawahara 1982), (Debnath 1994), (Dingemans 1997) and (Berdichevsky 2009).

Henceforth, the MSM and the AVP are considered known to some extent, and the same applies to the usual asymptotic arguments and definitions that are required (see the general books about perturbation methods, above). Nevertheless, attention has been paid to the sufficiently detailed explanation of the various steps, during the implementation of the methods, so that even a reader with little familiarity with them may be able to follow the text.

Chapter 2

A consistent, autonomous approach to Stokes waves, using a variational method

"Distress not yourself if you cannot at first understand the deeper mysteries of Spaceland. By degrees they will dawn upon you."

E. A. ABBOTT - Flatland: A Romance of Many Dimensions

2.1. Introduction

In his groundbreaking work, (Stokes, 1847) was the first to consider nonlinear solutions to the steady periodic WWP. He did so by implementing a systematic perturbation scheme, under the assumption that the steepness of the waves in search is small (but not infinitesimal). The essential steps of his method are the following. Given the equations of the steady periodic WWP, and exploiting the aforementioned assumption of small wave motion, the boundary conditions on the free surface are expanded into Taylor series around the known still-water level. In this way, the nonlinearity that occurs from the normally unknown upper boundary of the fluid domain ceases to exist. Moreover, for the wave fields (i.e. the free-surface elevation and the velocity potential), given their periodic nature, Fourier representations are adopted, with the particularity that their unknown coefficients constitute perturbation expansions in terms of the supposed small steepness. Inserting those representations into the equations of the wave motion, with the free-surface boundary conditions being in their Taylor-expanded forms, a perturbation hierarchy is produced, which can be solved successively for the perturbed Fourier coefficients. Doing that, approximate solutions of various orders can be obtained for the steady periodic WWP, provided that the included nonlinearity is weak. Such solutions are known as (perturbative) *Stokes waves* [(Whitham, 1974; Massel, 1989; Debnath, 1994; Dingemans, 1997; Johnson, 1997)].

Stokes' theory alone does not contain any information about the observational frame of reference of the waves under examination. Thus, the celerity c, which generally appears in the modelling equations as an unknown quantity, is rendered indeterminate, due to the fact that there exists an infinite number of possible reference frames [(Clamond, 2017)]. One option, of course, is to study the waves with respect to a frame moving with c, where, as said in Sec. 1.3, the motion is steady and, consequently, the celerity is no longer explicitly involved in the problem's equations. However, other, 'fixed' frames are of theoretical and practical importance, too, and their use is necessary. Then, c reappears and needs to be determined [(Fenton, 1985, 1990)]. For that to be accomplished, though, an additional condition that specifies the appropriate 'fixed' frame is required. Stokes, to remedy that issue, introduced two definitions for the celerity, each of which is suitable for different applications [(Stokes, 1847; Dingemans, 1997; Clamond, 2017)]. His *first definition* requires the vanishing of the mean horizontal fluid velocity at a point below the trough level ([1]). Hence, the observational reference frame is either stationery, in absence of any uniform current, or moving with that current's speed in case it exists. His *second definition* requires the vanishing of the mean horizontal mass transport, and is the proper one in laboratory conditions, where the waves are generated in wave tanks. There, the usually existent mass transport, in the direction of the wave propagation, is counterbalanced by a flow in the opposite direction. So, via the second definition, the laboratory becomes the observational reference frame. The use of either definition, therefore, specifies the frame in regards to which the waves are studied, allowing for the determination of their celerity.

Although their nonlinearity is weak, Stokes waves reveal many remarkable features of the nonlinear wave motion, which the linear theory neglects. For example, their profile does not coincide with the sinusoidal shape encountered in linear waves, but is characterized by sharper crests and flatter troughs. Namely, it resembles a *trochoid*. Another very important feature is that the nonlinear dispersion relation is a function, not only of the wavenumber, but of the wave amplitude, too. Specifically, the dispersion relation of Stokes waves consists of the linear dispersion relation modified by higher order corrections that contain their amplitude. As a result, waves of the same wavenumber propagate with different phase speed for different values of their steepness. Those corrections occur due to the rise of secular terms, during the implementation of Stokes' procedure, and their subsequent elimination, which is achieved by expressing the wave frequency as a perturbation expansion. That treatment of the secular terms, introduced by Stokes, is now known as the Lindstedt-Poincaré method [(Nayfeh, 2008)]. Stokes was also the first to examine another nonlinear effect known as *Stokes drift*. In essence, based on the results of his work, he found out that the paths of the fluid particles are open curves. That is, the start and end points of the position of a particle, after a period, do not coincide, as is the case in the linear theory. The reason for this is the mass-transport velocity, which is a second-order mean velocity in the direction of propagation.

The method of Stokes rests heavily on the assumed perturbation parameter, i.e. the slope of the waves, and that affects its range of validity as to the depth of the seabed. Generally, it is accurate for waves in water of arbitrary or infinite depth. In shallow water, on the contrary, it fails to yield correct results, due to the fact that the effective perturbation parameter is different [(Sobey *et al.*, 1987)]. In that case, cnoidal theory is the appropriate one to employ [(Fenton, 1990)]. A more precise criterion, for the decision between the two theories, has been given by (Hedges, 1995) in terms of the Ursell parameter.

The originality and innovation of Stokes' initial contribution have led, since then, to a great amount of efforts towards this direction. Except for that first work, in which he presented a third-order solution for deep-water waves and a second-order one for waves in water of arbitrary depth, (Stokes, 1880) also devised an alternative, inverse method, where the velocity potential and the stream function are viewed as the independent variables of the problem and the spatial coordinates as functions of them. In both of his works, Stokes used the term ka for the wave steepness, where k is the wavenumber and a the lowest-order amplitude of the free-surface elevation. His latter approach has the advantage of requiring greatly fewer calculations, and that allowed him to extend his previous results by an order [(Stokes, 1880; Massel, 1989; Craik, 2005)]. It suffers, however, from a smaller radius of convergence [(Drennan, Hui and Tenti,

([1]) Which point is not important, as long as it is always immersed in the water, as the assumption of potential flow and the horizontal seabed result in a mean horizontal flow that is independent of the vertical position of the point [(Constantin, 2006)].

1992)] and is further limited in the sense that its generalization in three-dimensional fluid domains or the case of uneven seabeds is not possible [(Dingemans, 1997)]. Following Stokes, (Wilton, 1914) extended his deep-water results to higher orders, without avoiding some errors, while (De, 1955) derived a fifth-order solution in the case of arbitrary depth and so did (Chappelear, 1961), pointing also out a mistake in De's work. A fifth-order solution for water of arbitrary depth was obtained by (Skjelbreia and Hendrickson, 1960), too, via the physical plane this time. In all those approaches, Stokes' first definition for the celerity was used. The second definition was utilized by (Tsuchiya and Yamaguchi, 1972), who gave a fourth-order solution based on it. Deviating from the established workflow, (Schwartz, 1974) treated the coefficients of the Fourier representations numerically. Thus, he was able to reach very high orders, and, in the process, deduced that the convergence of Stokes' expansions is limited when ka plays the role of the perturbation parameter, rendering the theory incapable of yielding the so-called highest wave. More importantly, he showed that expressing, instead, the steepness via the wave height, as $kH/2$, restores that issue, with the aid, though, of Padè approximants. The approach of Schwartz was adopted, for deep water, by (Longuet-Higgins, 1975), who dealt with the calculation of the wave integral properties, for waves up to the highest. Further, (Cokelet, 1977), working similarly, addressed the same problem, but in the case of water of arbitrary depth. Adding to the already existent Stokes solutions, (Tsuchiya and Yasuda, 1981) concluded to a solution by imposing conditions other than Stokes' definitions for the wave speed. Later, (Fenton, 1985), being aware of the various disadvantages and, sometimes, mistakes of the previous attempts, developed a fifth-order theory free of them. In his version of Stokes waves, the perturbation parameter (i.e. the wave slope) is expressed as $kH/2$, while the solution is solely presented in terms of the perturbation parameter and kh, h being the (constant) depth of the seabed. Moreover, both of Stokes' celerity definitions may be used, as well as uniform current may be incorporated, if existent. Fenton also examined the application of his theory to various practical problems, and gave guidelines as to its correct implementation in each case. Ever since its publication, his work is widely acknowledged and very popular in engineering applications.

The perturbative Fourier series that Stokes introduced gave, as is natural, rise to the matter of their convergence, as an aspect of the more general one regarding the existence of such waves (periodic, of permanent form). While (Nekrasov, 1921) showed that a solution to the steady WWP exists, provided a sufficiently small steepness, (Levi-Civita, 1925) proved, under the same assumption and in the deep-water case, that the aforementioned series converge. The latter result was extended to water of arbitrary depth in the work of (Struik, 1926), for which several corrections were made by (Hunt, 1953). Subsequently, (Krasovskii, 1961) proved, independently of the depth of the seabed, the existence of steady waves for all the steepness values for which the inclination of the free surface does not exceed $\pi/6$. (Keady and Norbury, 1978), afterwards, obtained for the same problem a broader set of solutions that contains Krasovskii's, whereas (Toland, 1978) showed that a solution to the steady WWP exists even for waves of the greatest height. An account on the theoretical aspects of Stokes waves can be found in (Toland, 1996).

Provided the existence of Stokes waves, another very important and independent matter is their stability. Postponed until the 1960s, that question was mainly answered by the work of (Benjamin and Feir, 1967), via a perturbation procedure, and those of (Lighthill, 1965, 1967) and (Whitham, 1967), via the AVP. From them, it turned out that Stokes waves are in fact unstable in deep water. Benjamin and Feir, in particular, discovered that they are characterized by a sideband instability, now known as the Benjamin-Feir Instability (BFI), which occurs when $kh > 1.363$. That finding led them to conjecture that, under the influence of sideband perturbations, Stokes waves disintegrate. A decade later, though, (Lake *et al.*, 1977), studying the long-term evolution of uniform wavetrains affected by side-band perturbations, both experimentally and theoretically, concluded that, except for the BFI those wavetrains initially undergo, they are generally governed by the Fermi-Pasta-Ulam recurrence. More on the instability of Stokes waves can be found in (Yuen and Lake, 1982), (Craik, 1988), (Debnath, 1994), (Mei, Stiassnie and Yue, 2005) and (Zakharov and Ostrovsky, 2009), among other sources ([2]).

Despite the extensive literature on Stokes waves, especially in regard to the various methodological approaches used throughout the years, to our knowledge there does not exist any systematic procedure for their variational rederivation. A promising step towards the accomplishment of such an endeavor, however, seems to be the utilization of Whitham's AVP. This is exactly our aim in what follows: to examine how the AVP handles the weakly nonlinear steady periodic WWP and to compare its outcome with previous, accepted results, like Fenton's.

([2]) Here, we avoid any references to the nonlinear Schrödinger equation and its connection to the instability of uniform wavetrains, due to the fact that it constitutes the central element of the following chapters.

2.2. Derivation of Stokes waves by means of the AVP

As mentioned before, we are interested in the study of the steady periodic WWP, via the AVP, when waves of small slope over uniform bathymetry h are considered (i.e. perturbative Stokes waves).

2.2.1. Prerequisites and assumptions for the unknowns of the problem

A requirement, for the implementation of the AVP, is the existence of a variational formulation of the problem at hand [(Whitham 1974; Karpman 1975)]. As stated in Section 1.3, the steady periodic WWP admits such a formulation,

$$\begin{aligned} &\delta \mathscr{S}[\eta,\Phi,R;\delta\eta,\delta\Phi,\delta R] = \\ &\quad = \delta_{\eta}\mathscr{S}[\eta,\Phi,R;\delta\eta] + \delta_{\Phi}\mathscr{S}[\eta,\Phi,R;\delta\Phi] + \delta_{R}\mathscr{S}[\eta,\Phi,R;\delta R] = 0, \end{aligned} \tag{1}$$

$\forall\, (\delta\eta,\delta\Phi,\delta R) \in \mathcal{S}_{adm}$, in which the respective action functional reads

$$\mathscr{S}[\eta,\Phi,R] = \int_{\Xi} \mathcal{L}[\eta(\xi),\Phi(\xi,\bullet),R]\, d\xi\,, \tag{2}$$

with

$$\mathcal{L} = \int_{-h}^{\eta} \left\{ -c\,\frac{\partial\Phi}{\partial\xi} + \frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial\xi}\right)^{2} + \left(\frac{\partial\Phi}{\partial z}\right)^{2} \right] + g\,z \right\} dz - R\,\eta \tag{3}$$

being its constrained, by the mass conservation within the periodic cell, Lagrangian. In those equations, $\eta = \eta(\xi)$ is the free-surface elevation, $\Phi = \Phi(\xi,z)$ the velocity potential and R the Lagrange multiplier of the mass constraint, which plays the role of a Bernoulli constant. Further, $\xi = x - ct$, so that the fields η and Φ express steady progressive waves with phase speed (celerity) c and, also, $\Xi = [0,\lambda]$ is confined within the assumed periodic cell.

Another prerequisite is the introduction of proper trial functions for the problem's unknown fields η and Φ. Thus, in accordance with the waves under examination, i.e. small-slope, periodic waves (see, also, Section 2.1), we introduce the Fourier perturbation expansions [(Whitham 1967, 1974)]

$$\tilde{\eta}(\theta) = \frac{1}{k}\sum_{i=1}^{N} \varepsilon^{i} \left\{ \zeta_{i} + \sum_{j=1}^{i} a_{ij}\cos j\theta \right\} + O(\varepsilon^{N+1}) \tag{4}$$

and

$$\tilde{\Phi}(\theta,z) = \sqrt{\frac{g}{k^{3}}}\sum_{i=1}^{N}\varepsilon^{i}\sum_{j=1}^{i} b_{ij}K_{ij}(z)\sin j\theta + O(\varepsilon^{N+1})\,, \tag{5}$$

respectively. As usual, $\varepsilon \ll 1$ constitutes a perturbation parameter,

$$k = 2\pi/\lambda$$

is the wave number, g the acceleration of gravity and

$$\theta = k(x-ct) = k\xi \qquad \text{(phase)} \tag{6}$$

the phase of the waves [(Mei, Stiassnie, and Yue 2005)], which, given the definition of k, brings the ξ-period (equal to λ) of the actual fields η, Φ and the θ-period (equal to 2π) of the trigonometric representations $\tilde{\eta}$ and $\tilde{\Phi}$ into agreement [(French 1971)]. Further, we postulate that the celerity is also expressed in terms of a perturbation expansion, i.e.

$$c = \sqrt{\frac{g}{k}} \sum_{i=0}^{N} \varepsilon^i c_i, \qquad \text{(celerity)} \tag{7}$$

and so does the Bernoulli constant R. That is,

$$R = \frac{g}{k} \sum_{i=1}^{N} \varepsilon^i R_i \qquad \text{(Bernoulli constant)} \tag{8}$$

[(Fenton 1985)]. As for the rest of the dimensionless coefficients of Eqs. (4) & (5):

$\zeta_{\{i\}}$ allow for wave-induced mean elevation,

$a_{\{ij\}}$ and $b_{\{ij\}}$ are the coefficients of the harmonics, and

$K_{\{ij\}}(z)$ are unknown vertical functions.

Note that, the assumption is made that the vertical dependence of Φ (at each harmonic and order) may be separated from the horizontal one (see Eq. (5)). Which is the exact form of that vertical dependence, though, is a question left to be answered by the AVP.

2.2.2. Remarks on the perturbation parameter in use

The perturbation parameter ε expresses the smallness of the slope, and for its physical implementation there exist various possibilities [(Schwartz 1974; Fenton 1985; Sobey et al. 1987)]. In the present work,

$$\varepsilon = \frac{kH}{2}, \tag{9}$$

where H is the wave height. That choice, of course, affects the ansatz of $\tilde{\eta}$, Eq. (4). To demonstrate that, we assume, as an example, a perturbation expansion of $\tilde{\eta}$ up to $O(\varepsilon^3)$. In that case, from Eq. (4),

$$\tilde{\eta} = \frac{1}{k}\Big\{\varepsilon(\zeta_1 + a_{11}\cos\theta) + \varepsilon^2(\zeta_2 + a_{21}\cos\theta + a_{22}\cos 2\theta) \\ + \varepsilon^3(\zeta_3 + a_{31}\cos\theta + a_{32}\cos 2\theta + a_{33}\cos 3\theta)\Big\} + O(\varepsilon^4).$$

But, via the definition of the wave height,

$$H = \tilde{\eta}_{\max} - \tilde{\eta}_{\min} = \tilde{\eta}(0) - \tilde{\eta}(\pi) = \frac{2}{k}\{\varepsilon a_{11} + \varepsilon^2 a_{21} + \varepsilon^3(a_{31} + a_{33})\} \iff \\ \iff \frac{kH}{2} = \varepsilon a_{11} + \varepsilon^2 a_{21} + \varepsilon^3(a_{31} + a_{33}).$$

Thus, to satisfy Eq. (9), we have to set

$$a_{21} = 0, \qquad a_{31} = -a_{33} \tag{10}$$

and, also, $a_{11} = 1$. In what follows, though, the latter substitution is postponed until after the completion of the procedure dictated by the AVP. The reason for this is that the (nonvanishing) coefficient a_{11} indicates the effect of the fundamental component of $\tilde{\eta}$ in the so called averaged Lagrangian of the AVP. As will be clarified later, that effect is very important, as it is related with the determination of the celerity and, hence, the dispersion relation of the waves [(Whitham 1967, 1974; Bhakta 1988)].

2.2.3. Relation between the AVP and the variational principle of exactly periodic waves

Generally, the AVP method, for the periodic or nearly periodic WWP, consists of three main steps, given the problem's variational formulation and appropriate ansatzes for its unknown fields. Those steps are [(Whitham 1974; Jeffrey and Kawahara 1982)]:

a) the insertion of the ansatzes into the Lagrangian,
b) the averaging of the Lagrangian, over the variable in which the periodicity occurs, and
c) the substitution of the result of the averaging, namely the averaged Lagrangian, into the initial action functional of the problem, which is then viewed as a functional defined on the unknown parameters/coefficients of the ansatzes.

However, in the case of exactly periodic waves, as those considered in this section, the two last steps become redundant.

To justify the above statement, we first assume that suitable trial functions $\tilde{\eta} = \tilde{\eta}(\xi)$ and $\tilde{\Phi} = \tilde{\Phi}(\xi, z)$ are available, for the fields η and Φ, and, then, we use those functions in both the initial variational principle of the problem (see Eq. (1)) and its averaged one. In that spirit, let the vector $\boldsymbol{P} = (p_i)_{i=1}^{I}$, $I \in \mathbb{N}$, denote the unknown constant coefficients, contained in the ansatzes $\tilde{\eta}$ and $\tilde{\Phi}$, and the Bernoulli constant R. Starting with the AVP, we insert $\tilde{\eta}$ and $\tilde{\Phi}$ into $\mathcal{L}$. Thus,

$$\mathcal{L} = \mathcal{L}[\tilde{\eta}(\xi), \tilde{\Phi}(\xi, \bullet); \boldsymbol{P}]. \tag{11}$$

Then, the averaged Lagrangian, defined as

$$\bar{\mathcal{L}}[\boldsymbol{P}] = \frac{1}{\lambda} \int_{\Xi} \mathcal{L}[\tilde{\eta}(\xi), \tilde{\Phi}(\xi, \bullet); \boldsymbol{P}] \, d\xi,$$

is substituted into the initial action functional (see Eq. (2)), so that

$$\mathscr{S} = \mathscr{S}[\boldsymbol{P}] = \int_{\Xi} \bar{\mathcal{L}}[\boldsymbol{P}] \, d\xi = \lambda \bar{\mathcal{L}}[\boldsymbol{P}] = \int_{\Xi} \mathcal{L}[\tilde{\eta}(\xi), \tilde{\Phi}(\xi, \bullet); \boldsymbol{P}] \, d\xi.$$

But, skipping the steps (b) and (c), using, that is, Eq. (11) in Eq. (2) immediately, we arrive at the exact same result, that

$$\mathscr{S} = \mathscr{S}[\boldsymbol{P}] = \int_{\Xi} \mathcal{L}[\tilde{\eta}(\xi), \tilde{\Phi}(\xi, \bullet); \boldsymbol{P}] \, d\xi. \tag{12}$$

As a consequence, when periodic waves are considered, whose ansatzes are such that their Lagrangian is of the form of Eq. (11), the, reshaped by the ansatzes, initial action functional coincides with that of the AVP. Hence, in that case, the introduction of proper trial functions into the variational principle of the problem is equivalent to the implementation of the AVP.

Remark: In Eq. (12), $\mathscr{S}$ is a functional with respect to the scalar elements of $\boldsymbol{P}$. Its stationarity, therefore, is achieved via the satisfaction of the variational equation

$$\delta\mathscr{S}[p_{\{i\}};\delta p_{\{i\}}] = \sum_{i=1}^{I}\delta_{p_i}\mathscr{S}[p_{\{i\}};\delta p_i] = 0\,, \tag{13}$$

$\forall\,(\delta p_{\{i\}}) \in \mathcal{S}_{adm}$. In view of Eq. (13), the elements of $\boldsymbol{P}$, i.e. the constant coefficients of the ansatzes and the Bernoulli constant, are regarded, throughout Ξ, as constant functions. Thus, in that instance, $\mathcal{S}_{adm}$ consists of variations constant in Ξ.

2.2.4. Implementation of the AVP

After all the above considerations, we are ready to proceed with our main task. Namely, the study of the aforementioned small-slope waves via the AVP. In order to facilitate the demonstration of the procedure, we confine ourselves with waves up to $O(\varepsilon^3)$. However, higher order waves may be considered, without any methodological complicacies, but with the price of extensive algebraic calculations ([1,2]).

As stated in the previous subsection, to implement the AVP in periodic waves, it suffices to introduce the trial functions of their fields into the variational principle that governs them. For the periodic waves of small slope we are interested in, those functions are provided by Eqs. (4) & (5). Hence, they are of the form $\tilde{\eta} = \tilde{\eta}(\theta)$ and $\tilde{\Phi} = \tilde{\Phi}(\theta, z)$, with $\theta = k\,\xi$. Inserting them into the Lagrangian of the problem, Eq. (3), and noting that

$$\frac{\partial\tilde{\Phi}}{\partial\xi} = \frac{\partial\tilde{\Phi}}{\partial\theta}\frac{\partial\theta}{\partial\xi} = k\,\frac{\partial\tilde{\Phi}}{\partial\theta}$$

the latter becomes

$$\mathcal{L} = \int_{-h}^{\tilde{\eta}}\left\{-ck\frac{\partial\tilde{\Phi}}{\partial\theta} + \frac{1}{2}\left[k^2\left(\frac{\partial\tilde{\Phi}}{\partial\theta}\right)^2 + \left(\frac{\partial\tilde{\Phi}}{\partial z}\right)^2\right] + g\,z\right\}dz - R\,\tilde{\eta}\,. \tag{14}$$

Consequently, the respective action functional, Eq. (2), is shaped as

$$\mathscr{S} = \int_0^{\lambda}\mathcal{L}[\tilde{\eta}(k\,\xi), \tilde{\Phi}(k\,\xi, \bullet), R]\,d\xi = \frac{\lambda}{2\pi}\int_0^{2\pi}\mathcal{L}[\tilde{\eta}(\theta), \tilde{\Phi}(\theta, \bullet), R]\,d\theta\,, \tag{15}$$

where its second form ([3]) occurs from the change of variables $\theta = k\,\xi$. Eqs. (14) & (15) are the general expressions of $\mathcal{L}$ and $\mathscr{S}$, when $\tilde{\eta}$ and $\tilde{\Phi}$ are functions of θ, instead of ξ.

The specific forms of the trial functions, searching for solutions up to $O(\varepsilon^3)$ and taking into account Eqs. (9) & (10), are

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([1]) As is commonly known, perturbation methods are generally accompanied by fearsome algebra. Although the AVP reduces, up to a point, the required calculations [(Whitham 1967, 1974)], still, without the aid of software of symbolic mathematics, higher-order perturbation expansions constitute a painful and tedious task.
([2]) For the algebra of the present section, Wolfram® Mathematica has been utilized. The required computational steps and the respective code can be found in Appendix A.
([3]) The term $\lambda/2\pi$ is just a (known) constant and, thus, indifferent in the context of rendering $\mathscr{S}$ stationary. Due to that, we can omit it. However, the presence of $1/2\pi$ is desired, as it simplifies the form of the integrand, in the case where trigonometric representations are used for the unknown fields. So, hereafter, λ is neglected.

$$\tilde{\eta}(\theta) = \frac{1}{k}\Big\{\varepsilon(\zeta_1 + a_{11}\cos\theta) + \varepsilon^2(\zeta_2 + a_{22}\cos 2\theta) + \varepsilon^3[\zeta_3 + a_{32}\cos 2\theta - a_{33}(\cos\theta - \cos 3\theta)]\Big\} + O(\varepsilon^4), \tag{16}$$

for the free-surface elevation, and

$$\tilde{\Phi}(\theta,z) = \sqrt{\frac{g}{k^3}}\,\Big\{\varepsilon b_{11} K_{11}(z)\sin\theta + \varepsilon^2[b_{21} K_{21}(z)\sin\theta + b_{22} K_{22}(z)\sin 2\theta] + \varepsilon^3[b_{31} K_{31}(z)\sin\theta + b_{32} K_{32}(z)\sin 2\theta + b_{33} K_{33}(z)\sin 3\theta]\Big\} + O(\varepsilon^4), \tag{17}$$

for the velocity potential, and are obtained by truncating the terms of $O(\varepsilon^4)$ or higher in Eqs. (4) & (5). Normally, our next step would be to substitute Eqs. (16) & (17) into Eqs. (14) & (15). Though, in our case, observing Eq. (17), the additional difficulty of the unknown vertical dependence of $\tilde{\Phi}$ occurs, via the unknown functions $K_{\{ij\}}(z)$. As stated earlier, our intention is to let the AVP determine that dependence. In that direction, we work as follows.

The vertical problem

We view the problem of the determination of the functions $K_{\{ij\}}(z)$ as independent from the total one. In that context, the free-surface elevation $\tilde{\eta}$, in Eq. (14), can be considered as a parameter of the integration domain, rather than an unknown. Thus, for that vertical problem, we assume the Lagrangian density

$$\mathcal{L}_{vert} = -ck\frac{\partial\tilde{\Phi}}{\partial\theta} + \frac{1}{2}\left\{k^2\left(\frac{\partial\tilde{\Phi}}{\partial\theta}\right)^2 + \left(\frac{\partial\tilde{\Phi}}{\partial z}\right)^2\right\} + gz. \tag{18}$$

Given Eq. (18), we put into effect the AVP. First, we insert Eq. (17) in Eq. (18), to rewrite $\mathcal{L}_{vert}$ in terms of the functions $K_{\{ij\}}(z)$, and then define the averaged vertical Lagrangian

$$\bar{\mathcal{L}}_{vert}(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{L}_{vert}(\theta,z)\,d\theta\,. \tag{19}$$

Doing the necessary algebra, the latter takes the shape

$$\bar{\mathcal{L}}_{vert} = gz + \varepsilon^2\bar{\mathcal{L}}_{vert,2} + \varepsilon^3\bar{\mathcal{L}}_{vert,3} + \varepsilon^4\bar{\mathcal{L}}_{vert,4} + \varepsilon^5\bar{\mathcal{L}}_{vert,5} + \varepsilon^6\bar{\mathcal{L}}_{vert,6}, \tag{20}$$

where

$$\bar{\mathcal{L}}_{vert,2} = \frac{g}{4k^3} b_{11}{}^2\{k^2 K_{11}{}^2(z) + [K_{11}{}'(z)]^2\}, \tag{20a}$$

$$\bar{\mathcal{L}}_{vert,3} = \frac{g}{2k^3} b_{11} b_{21}\{k^2 K_{11}(z) K_{21}(z) + K_{11}{}'(z) K_{21}{}'(z)\}, \tag{20b}$$

$$\bar{\mathcal{L}}_{vert,4} = \frac{g}{4k^3}\Big\{b_{21}{}^2\{k^2 K_{21}{}^2(z) + [K_{21}{}'(z)]^2\} + b_{22}{}^2\{4k^2 K_{22}{}^2(z) + [K_{22}{}'(z)]^2\} \tag{20c}$$

$$+ 2b_{11} b_{31} [k^2 K_{11}(z) K_{31}(z) + K_{11}{}'(z) K_{31}{}'(z)]\Big\},$$

$$\begin{aligned} \bar{\mathcal{L}}_{vert,5} &= \frac{g}{2k^3} \Big\{ b_{21} b_{31} [k^2 K_{21}(z) K_{31}(z) + K_{21}{}'(z) K_{31}{}'(z)] \\ &\quad + b_{22} b_{32} [4k^2 K_{22}(z) K_{32}(z) + K_{22}{}'(z) K_{32}{}'(z)] \Big\} \end{aligned} \tag{20d}$$

and

$$\begin{aligned} \bar{\mathcal{L}}_{vert,6} &= \frac{g}{4k^3} \Big\{ b_{31}{}^2 \{k^2 K_{31}{}^2(z) + [K_{31}{}'(z)]^2\} \\ &\quad + b_{32}{}^2 \{4k^2 K_{32}{}^2(z) + [K_{32}{}'(z)]^2\} \\ &\quad + b_{33}{}^2 \{9k^2 K_{33}{}^2(z) + [K_{33}{}'(z)]^2\} \Big\}. \end{aligned} \tag{20e}$$

Afterwards, with $\bar{\mathcal{L}}_{vert}$ known, we examine the subsequent vertical action functional

$$\mathscr{S}_{vert}[K_{\{ij\}}] = \int_{-h}^{\eta} \bar{\mathcal{L}}_{vert}(K_{\{ij\}}(z), K_{\{ij\}}{}'(z))\, dz \tag{21}$$

and its variational equation

$$\delta \mathscr{S}_{vert} = \delta \int_{-h}^{\eta} \bar{\mathcal{L}}_{vert}\, dz = 0 \iff \sum_{i=2}^{6} \varepsilon^i \left(\delta \int_{-h}^{\eta} \bar{\mathcal{L}}_{vert,i}\, dz \right) = 0, \tag{22}$$

$\forall\, (\delta K_{\{ij\}}) \in \mathcal{S}_{adm}$. From that equation, we deduce that, essentially, the use of the perturbation expansion of Eq. (17) leads, via its insertion in the Lagrangian of the problem, to a perturbation hierarchy of variational equations. Accordingly, starting with the lowest-order variational equation of that hierarchy, we solve the latter recursively. Therefore, we initially consider the variational equation

$$\delta \int_{-h}^{\eta} \bar{\mathcal{L}}_{vert,2}\, dz = 0. \tag{22,i}$$

Taking its δK_{11} variation gives

$$\delta K_{11}: \quad \frac{d}{dz}\left(\frac{\partial \bar{\mathcal{L}}_{vert,2}}{\partial K_{11}{}'} \right) - \frac{\partial \bar{\mathcal{L}}_{vert,2}}{\partial K_{11}} = 0 \iff K_{11}{}''(z) - k^2 K_{11}(z) = 0. \tag{23}$$

Thus, solving the above linear ordinary differential equation, we find that

$$K_{11}(z) = A_1 e^{kz} + A_2 e^{-kz}, \tag{24}$$

where A_1 and A_2 are arbitrary constants. Carrying that result to the higher orders of Eq. (22), it is easy to verify that the EL equation that arises from the variational principle

$$\delta \int_{-h}^{\eta} \bar{\mathcal{L}}_{vert,3}\, dz = 0 \tag{22,ii}$$

is satisfied identically, yielding nothing new. In particular, with $K_{11}(z)$ given by Eq. (24), the above principle admits only variations δK_{21}. But those simply lead to

$$\delta K_{21}: \quad \frac{d}{dz}\left(\frac{\partial \bar{\mathcal{L}}_{vert,3}}{\partial K_{21}{}'}\right) - \frac{\partial \bar{\mathcal{L}}_{vert,3}}{\partial K_{21}} = 0 \iff K_{11}{}''(z) - k^2 K_{11}(z) = 0 \iff$$
$$\iff (C_1 e^{kz} + C_2 e^{-kz})'' - k^2 (C_1 e^{kz} + C_2 e^{-kz}) = 0 \iff$$
$$\iff \quad 0 = 0,$$

which confirms the above statement. Hence, we proceed with the fourth-order variational equation of the perturbation hierarchy,

$$\delta \int_{-h}^{\eta} \bar{\mathcal{L}}_{vert,4}\, dz = 0. \tag{22,iii}$$

As is evident from Eq. (20c), that equation admits variations δK_{21}, δK_{22} and δK_{31}. From the first two of those variations, there occur, respectively, the EL equations

$$\delta K_{21}: \quad \frac{d}{dz}\left(\frac{\partial \bar{\mathcal{L}}_{vert,4}}{\partial K_{21}{}'}\right) - \frac{\partial \bar{\mathcal{L}}_{vert,4}}{\partial K_{21}} = 0 \iff K_{21}{}''(z) - k^2 K_{21}(z) = 0 \tag{25}$$

and

$$\delta K_{22}: \quad \frac{d}{dz}\left(\frac{\partial \bar{\mathcal{L}}_{vert,4}}{\partial K_{22}{}'}\right) - \frac{\partial \bar{\mathcal{L}}_{vert,4}}{\partial K_{22}} = 0 \iff K_{22}{}''(z) - 4k^2 K_{22}(z) = 0. \tag{26}$$

Regarding the latter variation, δK_{31}, as in the case of the δK_{21}-variation of Eq. (22,ii), it results in an indifferent, identically satisfied, EL equation. Solving Eqs. (25) & (26), we conclude that

$$K_{21}(z) = B_1 e^{kz} + B_2 e^{-kz} \tag{27}$$

and

$$K_{22}(z) = C_1 e^{2kz} + C_2 e^{-2kz}, \tag{28}$$

with B_1, B_2, C_1 and C_2 being arbitrary constants. Inserting, next, the above findings into Eq. (22) renders the variational equation

$$\delta \int_{-h}^{\eta} \bar{\mathcal{L}}_{vert,5}\, dz = 0 \tag{22,iv}$$

unimportant, as its variations, δK_{31} and δK_{32}, merely produce identities. As a consequence, there remains the last order of Eq. (22),

$$\delta \int_{-h}^{\eta} \bar{\mathcal{L}}_{vert,6}\, dz = 0, \tag{22,v}$$

whose variations δK_{31}, δK_{32} and δK_{33} yield, respectively,

$$\delta K_{31}: \quad \frac{d}{dz}\left(\frac{\partial \bar{\mathcal{L}}_{vert,6}}{\partial K_{31}{}'}\right) - \frac{\partial \bar{\mathcal{L}}_{vert,6}}{\partial K_{31}} = 0 \iff K_{31}{}''(z) - k^2 K_{31}(z) = 0, \tag{29}$$

$$\delta K_{32}: \quad \frac{d}{dz}\left(\frac{\partial \bar{\mathcal{L}}_{vert,6}}{\partial K_{32}{}'}\right) - \frac{\partial \bar{\mathcal{L}}_{vert,6}}{\partial K_{32}} = 0 \iff K_{32}{}''(z) - 4k^2 K_{32}(z) = 0 \tag{30}$$

and

$$\delta K_{33}: \quad \frac{d}{dz}\left(\frac{\partial \bar{\mathcal{L}}_{vert,6}}{\partial K_{33}{}'}\right) - \frac{\partial \bar{\mathcal{L}}_{vert,6}}{\partial K_{33}} = 0 \iff K_{33}{}''(z) - 9k^2 K_{33}(z) = 0. \tag{31}$$

The general solutions of Eqs. (29)-(31) are, in turn,

$$K_{31}(z) = D_1 e^{kz} + D_2 e^{-kz}, \tag{32}$$

$$K_{32}(z) = E_1 e^{2kz} + E_2 e^{-2kz} \tag{33}$$

and

$$K_{33}(z) = F_1 e^{3kz} + F_2 e^{-3kz}, \tag{34}$$

where, once more, $D_{\{1,2\}}$, $E_{\{1,2\}}$ and $F_{\{1,2\}}$ are arbitrary integration constants. In this way, the z-dependence of the functions $K_{\{i\,j\}}$, which are contained in $\tilde{\Phi}$, becomes fully known.

Summing up, the isolation of the vertical problem and its treatment with the AVP method is a procedure capable of providing us with the form of the vertical functions $K_{\{i\,j\}}$, up to some arbitrary constants. It turns out that

$$K_{11}(z) = A_1 e^{kz} + A_2 e^{-kz}, \tag{24}$$

$$K_{21}(z) = B_1 e^{kz} + B_2 e^{-kz} \tag{27}$$

$$K_{22}(z) = C_1 e^{2kz} + C_2 e^{-2kz}, \tag{28}$$

$$K_{31}(z) = D_1 e^{kz} + D_2 e^{-kz}, \tag{32}$$

$$K_{32}(z) = E_1 e^{2kz} + E_2 e^{-2kz} \tag{33}$$

and

$$K_{33}(z) = F_1 e^{3kz} + F_2 e^{-3kz}. \tag{34}$$

The exact knowledge of the z-dependence of $K_{\{i\,j\}}$ allows us, as will be clearly seen below, to implement the AVP in the total problem considered and, hence, determine the unknown parameters of the fields $\tilde{\eta}$ and $\tilde{\Phi}$. In regard to the aforementioned arbitrary constants, they are viewed as parameters of $\tilde{\Phi}$, just like $b_{\{i\,j\}}$. Thus, finding them is also a matter for the AVP.

<u>The total problem</u>

With the vertical structure of $\tilde{\Phi}$ known, we return to the whole steady periodic WWP for waves of small slope. As a first step, given Eqs. (7) & (8), for the celerity c and the Bernoulli constant R, and Eqs. (24), (27), (28) & (32)-(34), for the functions $K_{\{ij\}}$, we insert the trial functions of Eqs. (16) & (17) into Eq. (14). Consequently, defining the vector

$$\boldsymbol{P} = (R_{\{i\}}, \zeta_{\{i\}}, a_{\{ij\}}, b_{\{ij\}}, A_{\{1,2\}}, B_{\{1,2\}}, C_{\{1,2\}}, D_{\{1,2\}}, E_{\{1,2\}}, F_{\{1,2\}}) \tag{35}$$

of the parameters of the wave-field ansatzes and R, the Lagrangian is shaped as

$$\mathcal{L} = \mathcal{L}[\tilde{\eta}(\theta), \tilde{\Phi}(\theta,\bullet); \boldsymbol{P}].$$

Afterwards, we substitute $\mathcal{L}$ into Eq. (15). As a result, the action functional of the problem becomes

$$\mathscr{S} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{L}[\tilde{\eta}(\theta), \tilde{\Phi}(\theta,\bullet); \boldsymbol{P}] \, d\theta . \tag{36}$$

In it, because of the z-integration of Eq. (14), terms of the form

$$S_n \equiv e^{nk(h+\tilde{\eta}(\theta))} = e^{nkh} e^{nk\tilde{\eta}(\theta)} \tag{37}$$

are present. Thus, initially, it is not possible to express the outcome of the θ-integration of Eq. (36) in terms of elementary functions. However, the distinction between orders, via the perturbation parameter ε, and the fact that $\tilde{\eta} = O(\varepsilon)$ allow us to approximate the quantities $e^{nk\tilde{\eta}(\theta)}$ with their Maclaurin series. Accordingly, we set

$$e^{nk\tilde{\eta}} = \sum_{m=0}^{M} \frac{(nk)^m}{m!} \tilde{\eta}^m + O(\varepsilon^{M+1}) . \tag{38}$$

Using Eq. (38) into Eq. (36), the previous difficulty of the θ-integration is overcome. So, after the necessary algebraic calculations and keeping terms up to $O(\varepsilon^6)$, we conclude to

$$\mathscr{S} = \mathscr{S}[\boldsymbol{P}] = \sum_{i=0}^{6} \varepsilon^i \mathscr{S}_i[\boldsymbol{P}] + O(\varepsilon^7) , \tag{39}$$

where

$$\mathscr{S}_0 = -\frac{gh^2}{2} , \tag{39a}$$

$$\mathscr{S}_1 = 0 , \tag{39b}$$

$$\begin{aligned} \mathscr{S}_2 = \frac{g}{4k^2} \big\{ & a_{11}{}^2 - 2c_0 a_{11} b_{11} (A_1 + A_2) + 2\zeta_1{}^2 \\ & - 4\zeta_1 R_1 + b_{11}{}^2 (1 - e^{-2kh})(A_1{}^2 + A_2{}^2 e^{2kh}) \big\} , \end{aligned} \tag{39c}$$

$$\begin{aligned} \mathscr{S}_3 = \frac{g}{2k^2} \big\{ & a_{11} b_{11} [c_0 \zeta_1 (A_2 - A_1) - c_1 (A_1 + A_2)] - 2\zeta_1 R_2 \\ & - 2\zeta_2 R_1 + b_{11} b_{21} [A_1 B_1 (1 - e^{-2kh}) + A_2 B_2 (e^{2kh} - 1)] \\ & + 2\zeta_1 \zeta_2 + \zeta_1 b_{11}{}^2 (A_1{}^2 + A_2{}^2) - c_0 a_{11} b_{21} (B_1 + B_2) \big\} , \end{aligned} \tag{39d}$$

and

$$
\begin{aligned}
\mathscr{S}_4 = \frac{g}{16k^2}\Big\{ & 8a_{33}(c_0 b_{11} A_2 - a_{11}) + 16a_{11} b_{11} b_{22}(A_1 C_1 + A_2 C_2) \\
& + 8c_0 b_{22}[a_{11}{}^2(C_2 - C_1) - 2a_{22}(C_1 + C_2)] + 4a_{22}{}^2 \\
& + 8b_{22}{}^2(1 - e^{-4kh})(C_1{}^2 + e^{4kh} C_2{}^2) + b_{11} A_1[8a_{22} b_{11} A_2 \\
& + 4b_{11} A_1(a_{11}{}^2 + 2\zeta_1{}^2 + 2\zeta_2) + 8b_{31} D_1(1 - e^{-2kh}) \\
& + c_0(8a_{33} - a_{11}{}^3)] - 4a_{11} b_{11}\{A_1[c_0(a_{22} + \zeta_1{}^2 + 2\zeta_2) \\
& + 2c_1\zeta_1 + 2c_2] + A_2[c_0(\zeta_1{}^2 - 2\zeta_2 - a_{22}) - 2c_1\zeta_1 \\
& + 2c_2] + a_{11} b_{11} A_2{}^2\} + b_{11} A_2[8b_{11} A_2(\zeta_2 - \zeta_1{}^2) \\
& - c_0 a_{11}{}^3] + 8b_{31}[b_{11} A_2 D_2(e^{2kh} - 1) - c_0 a_{11}(D_1 + D_2)] \\
& + 4b_{21} B_1[b_{21} B_1(1 - e^{-2kh}) - 2a_{11}(c_1 + c_0\zeta_1) + 4\zeta_1 b_{11} A_1] \\
& + 4b_{21} B_2[b_{21} B_2(e^{2kh} - 1) - 2a_{11}(c_1 - c_0\zeta_1) + 4\zeta_1 b_{11} A_2] \\
& + 8(2\zeta_1\zeta_3 + \zeta_2{}^2 - 2\zeta_1 R_3 - 2\zeta_2 R_2 - 2\zeta_3 R_1)\Big\},
\end{aligned}
\tag{39e}
$$

while, avoiding their full expressions, due to their large size,

$$
\mathscr{S}_5 = \mathscr{S}_5[\boldsymbol{P} \text{ excluding } \{R_1, b_{33}, F_1, F_2\}] \tag{39f}
$$

and

$$
\mathscr{S}_6 = \mathscr{S}_6[\boldsymbol{P} \text{ excluding } \{R_1, R_2\}]. \tag{39g}
$$

Next, given the above, we demand the vanishing of the first variation of $\mathscr{S}$, i.e.

$$
\delta\mathscr{S} = \delta\sum_{i=0}^{6}\varepsilon^i \mathscr{S}_i = 0 \iff \sum_{i=0}^{6}\varepsilon^i \delta\mathscr{S}_i = 0, \tag{40}
$$

for each admissible variation, in the sense of Eq. (13). That is, as with the vertical problem, owing to the appearance of orders within $\mathscr{S}$, its stationarity occurs as a perturbation hierarchy of variational equations. Observing, therefore, that $\mathscr{S}_0$ and $\mathscr{S}_1$ have zero contributions, we begin with the variational equation

$$
\delta\mathscr{S}_2 = 0. \tag{40i}
$$

Taking its admissible variations, we obtain

$$
\delta\zeta_1: \quad \frac{\partial\mathscr{S}_2}{\partial\zeta_1} = 0 \iff R_1 = \zeta_1, \tag{41'}
$$

$$
\delta R_1: \quad \frac{\partial\mathscr{S}_2}{\partial R_1} = 0 \iff \zeta_1 = 0, \tag{42}
$$

$$
\delta b_{11}: \quad \frac{\partial\mathscr{S}_2}{\partial b_{11}} = 0 \iff b_{11} = \frac{e^{2kh}[\coth(kh) - 1](A_1 + A_2)}{2(A_1{}^2 + e^{2kh} A_2{}^2)} c_0 a_{11}, \tag{43'}
$$

$$
\delta A_1: \quad \frac{\partial\mathscr{S}_2}{\partial A_1} = 0 \iff A_1 = \frac{e^{2kh}[\coth(kh) - 1]}{2} c_0 \frac{a_{11}}{b_{11}} \tag{44'}
$$

and ([4])

$$\delta a_{11}: \quad \frac{\partial \mathscr{S}_2}{\partial a_{11}} = 0 \iff c_0 = \frac{a_{11}}{(A_1 + A_2) b_{11}} . \tag{45'}$$

Then, combining Eqs. (41) & (42), we deduce that

$$R_1 = 0 , \tag{41}$$

while, at the same time, the system of Eqs. (43)-(45) yields

$$b_{11} = \frac{\coth(kh) - 1}{2\sqrt{\coth(kh)}} \frac{a_{11}}{A_2} , \tag{43}$$

$$A_1 = e^{2kh} A_2 \tag{44}$$

and

$$c_0 = \sqrt{\tanh(kh)} . \tag{45}$$

Substituting the above results into the higher orders of Eq. (40), $\mathscr{S}_3$ becomes

$$\mathscr{S}_3 = -\frac{g\sqrt{\coth(kh)}}{2k^2} c_1 a_{11}{}^2 .$$

Thus, from the variational equation

$$\delta \mathscr{S}_3 = 0 , \tag{40ii}$$

we get

$$\delta a_{11}: \quad \frac{\partial \mathscr{S}_3}{\partial a_{11}} = 0 \iff c_1 = 0 . \tag{46}$$

Continuing in the same manner, namely, updating the subsequent orders of Eq. (40) with the findings of the previous ones and deriving their EL equations, the variational equation

$$\delta \mathscr{S}_4 = 0 , \tag{40iii}$$

gives

$$\delta \zeta_2: \quad \frac{\partial \mathscr{S}_4}{\partial \zeta_2} = 0 \iff R_2 = \frac{\operatorname{csch}(2kh)}{2} a_{11}{}^2 , \tag{47}$$

$$\delta R_2: \quad \frac{\partial \mathscr{S}_4}{\partial R_2} = 0 \iff \zeta_2 = 0 , \tag{48}$$

$$\delta b_{21}: \quad \frac{\partial \mathscr{S}_4}{\partial b_{21}} = 0 \iff b_{21} = 0 , \tag{49}$$

([4]) The averaged action functional is defined on the parameters that are introduced by the ansatzes of the wave fields. Thus, no variations with respect to the celerity can be taken, as it preexists from the formulation of the steady periodic WWP. However, it remains unknown. Its determination, and the closure of the system of equations for the problem's unknowns, comes from the δa_{11}-variations at each order [(Whitham 1967, 1974; Jeffrey and Kawahara 1982; Bhakta 1988; Dingemans 1997; Y. V. Sedletsky 2012, 2013)]. That is here the role of a_{11}, which is otherwise unnecessary, as the amplitude of the elevation's first harmonic is truly defined via ε. Namely, it just constitutes a convenient indicator as to the effect of that amplitude in the averaged action functional.

$$\delta a_{22}: \quad \frac{\partial \mathscr{S}_4}{\partial a_{22}} = 0 \iff a_{22} = \frac{5\cosh(kh) + \cosh(3kh)}{8\sinh^3(kh)} a_{11}{}^2, \tag{50}$$

$$\delta b_{22}: \quad \frac{\partial \mathscr{S}_4}{\partial b_{22}} = 0 \iff b_{22} = \frac{3e^{2kh}\sqrt{\coth(kh)}}{(e^{2kh}-1)^3(e^{2kh}+1)} \frac{a_{11}{}^2}{C_2}, \tag{51}$$

$$\delta C_1: \quad \frac{\partial \mathscr{S}_4}{\partial C_1} = 0 \iff C_1 = e^{4kh} C_2 \tag{52}$$

and

$$\delta a_{11}: \quad \frac{\partial \mathscr{S}_4}{\partial a_{11}} = 0 \iff c_2 = \frac{[8 + \cosh(4kh)]\sqrt{\coth(kh)}}{8\sinh(2kh)\sinh^2(kh)} a_{11}{}^2. \tag{53}$$

Further, from

$$\delta \mathscr{S}_5 = 0, \tag{40iv}$$

we infer that

$$\delta a_{11}: \quad \frac{\partial \mathscr{S}_5}{\partial a_{11}} = 0 \iff c_3 = 0, \tag{54}$$

whereas the remaining variational equation

$$\delta \mathscr{S}_6 = 0 \tag{40v}$$

leads to

$$\delta \zeta_3: \quad \frac{\partial \mathscr{S}_6}{\partial \zeta_3} = 0 \iff R_3 = 0, \tag{55}$$

$$\delta R_3: \quad \frac{\partial \mathscr{S}_6}{\partial R_3} = 0 \iff \zeta_3 = 0, \tag{56}$$

$$\delta b_{31}: \quad \frac{\partial \mathscr{S}_6}{\partial b_{31}} = 0 \iff$$

$$\iff b_{31} = \frac{7\cosh(2kh) - 10\cosh(4kh) - \cosh(6kh) - 23}{e^{-6kh}(e^{2kh}-1)^7\sqrt{\coth(kh)}} \frac{a_{11}{}^3}{D_2}, \tag{57}$$

$$\delta D_1: \quad \frac{\partial \mathscr{S}_6}{\partial D_1} = 0 \iff D_1 = e^{2kh} D_2, \tag{58}$$

$$\delta a_{32}: \quad \frac{\partial \mathscr{S}_6}{\partial a_{32}} = 0 \iff a_{32} = 0, \tag{59}$$

$$\delta b_{32}: \quad \frac{\partial \mathscr{S}_6}{\partial b_{32}} = 0 \iff b_{32} = 0, \tag{60}$$

$$\delta a_{33}: \quad \frac{\partial \mathscr{S}_6}{\partial a_{33}} = 0 \iff$$

$$\Longleftrightarrow \quad a_{33} = \frac{14+15\cosh(2kh)+6\cosh(4kh)+\cosh(6kh)}{256\sinh^6(kh)}\,3{a_{11}}^3, \tag{61}$$

$$\delta b_{33}: \quad \frac{\partial \mathscr{S}_6}{\partial b_{33}} = 0 \iff b_{33} = \frac{e^{4kh}[11-2\cosh(2kh)]}{(e^{2kh}-1)^7\sqrt{\coth(kh)}}\frac{{a_{11}}^3}{F_2} \tag{62}$$

and

$$\delta F_1: \quad \frac{\partial \mathscr{S}_6}{\partial F_1} = 0 \iff F_1 = e^{6kh}F_2. \tag{63}$$

In this way, the determination of the wave parameters is completed. Hence, the need for the presence of a_{11} ceases to exist (see Subsection 2.2.2) and, as a consequence, we may now set

$$a_{11} = 1, \tag{64}$$

in order to finally satisfy the desired definition of ε (see Eq. (9)).

Gathering all the above results, namely Eqs.(41)-(64), it is apparent that each one of the previously unknown wave parameters is now given by an expression provided by the AVP. Inserting those findings into Eqs. (16) & (17), without forgetting Eqs. (24), (27), (28) & (32)-(34) regarding the vertical dependence, we have, up to $O(\varepsilon^3)$, that

$$\tilde{\eta}(\theta) = \frac{1}{k}\Big\{[\varepsilon+\varepsilon^3 Q_3(kh)]\cos\theta + \varepsilon^2 Q_2(kh)\cos 2\theta - \varepsilon^3 Q_3(kh)\cos 3\theta\Big\} \tag{65}$$

and

$$\begin{aligned}\tilde{\Phi}(\theta,z) = \sqrt{\frac{g}{k^3}}\,\Big\{&[\varepsilon S_1(kh)+\varepsilon^3 S_{31}(kh)]\cosh[k(z+h)]\sin\theta \\ &+ \varepsilon^2 S_2(kh)\cosh[2k(z+h)]\sin 2\theta \\ &+ \varepsilon^3 S_{33}(kh)\cosh[3k(z+h)]\sin 3\theta\Big\},\end{aligned} \tag{66}$$

where

$$Q_2(kh) = \frac{5\cosh(kh)+\cosh(3kh)}{8\sinh^3(kh)}, \tag{67a}$$

$$Q_3(kh) = -\frac{3}{256}\,\frac{14+15\cosh(2kh)+6\cosh(4kh)+\cosh(6kh)}{\sinh^6(kh)}, \tag{67b}$$

$$S_1(kh) = \sqrt{2\operatorname{csch}(2kh)}, \tag{67c}$$

$$S_2(kh) = \frac{3}{8}\frac{\sqrt{\tanh(kh)}}{\sinh^4(kh)}, \tag{67d}$$

$$S_{31}(kh) = \frac{7\cosh(2kh)-10\cosh(4kh)-\cosh(6kh)-23}{64\sinh^7(kh)\sqrt{\coth(kh)}} \tag{67e}$$

and

$$S_{33}(kh) = \frac{11-2\cosh(2kh)}{64\sinh^7(kh)\sqrt{\coth(kh)}}. \tag{67f}$$

Moreover, the celerity, Eq. (7), is shaped as

$$c = \sqrt{\frac{g}{k}}\left(\sqrt{\tanh(kh)} + \varepsilon^2 \frac{[8+\cosh(4kh)]\sqrt{\coth(kh)}}{8\sinh(2kh)\sinh^2(kh)}\right), \tag{68}$$

while the Bernoulli constant, Eq. (8), becomes

$$R = \varepsilon^2 \frac{g\,\mathrm{csch}(2kh)}{2k}. \tag{69}$$

Those solutions are the same as the ones obtained by (Fenton 1985), in the case where Stokes' first approximation for the wave celerity is used (see Sec. 2.1).

Remark: Generally, the solutions of Fenton are expressed with respect to a reference frame moving with the wave celerity. In that frame, the undisturbed state of the fluid is a uniform flow in the negative direction and with magnitude the celerity. Thus, except for c, the only other quantities that can be directly compared with Fenton's respective results are $\tilde{\eta}$ and $\tilde{\Phi}$, which he presents in terms of a stationary frame, too.

That, however, is not the case for the Bernoulli constant. To compare its value with that of Fenton, we have to express it in the moving frame.

To do so, we work as follows. The existence of R implies that, essentially, the velocity potential contains an additional, linear t-dependence (see, also, Sec. 1.3). Thus, we introduce the redefined potential

$$\hat{\Phi}(\theta,z;t) = \tilde{\Phi}(\theta,z) - Rt$$

and, afterwards, via a suitable Galilean transformation [(Clamond 2017)], we express it in terms of the moving frame as

$$\hat{\Phi}_m = \hat{\Phi}(\theta,z;t) - cx + \frac{c^2}{2}t\Big|_{x=\breve{x}+ct} = \tilde{\Phi}(k\breve{x},z) - c\breve{x} - \left(R + \frac{c^2}{2}\right)t,$$

where $\breve{x}$ denotes the spatial variable in that frame. Therefore, in that instance, the Bernoulli constant becomes

$$R_m = R + c^2/2.$$

Inserting Eqs. (68)-(69) into the above, and keeping terms up to $O(\varepsilon^3)$, we conclude that

$$R_m = \frac{g}{2k}\tanh(kh) + \varepsilon^2 \frac{g}{8k}\frac{6+2\cosh(2kh)+\cosh(4kh)}{\sinh^2(kh)\sinh(2kh)},$$

which coincides with the result given by Fenton.

2.2.5. The case of wave-induced mean flow

Up to now, we had assumed waves in absence of any current. For most situations, nonetheless, that is unrealistic, as some sort of current is almost always present.

A special case of particular interest is the coexistence of waves with a mean flow induced by the waves themselves. Instances like that occur in closed wave tanks, among others, and, in the context of Stokes' theory, they are treated by using Stokes' second approximation for the celerity (see Sec. 2.1). Hence, it is important to examine how the AVP handles such problems.

In contrast with Eq. (5), we now allow for a wave-induced, and, thus, small, mean flow in the horizontal direction. Consequently, the velocity potential takes the form

$$\begin{aligned}\tilde{\Phi}_{mf}(\theta,z;x) &= \sqrt{\frac{g}{k^3}} \sum_{i=1}^{N} \varepsilon^i \left(k\,\psi_i\, x + \sum_{j=1}^{i} b_{ij} K_{ij}(z) \sin j\theta \right) + O(\varepsilon^{N+1}) = \\ &= \sqrt{\frac{g}{k^3}} \sum_{i=1}^{N} \varepsilon^i k\,\psi_i\, x + \tilde{\Phi}(\theta,z) + O(\varepsilon^{N+1}),\end{aligned} \tag{70}$$

where the dimensionless coefficients $\psi_{\{i\}}$ denote the wave-induced current. Although $\tilde{\Phi}_{mf}$ contains an explicit dependence on x, it does not cause any complicacies in the implementation of the AVP, as has been described so far.

To show that, we note that, given the form of $\tilde{\Phi}_{mf}$, during the passing from the classical gravity WWP to the steady periodic one (see Sec. 1.3), its x-differentiation gives

$$\frac{\partial \tilde{\Phi}_{mf}}{\partial x} = \sqrt{\frac{g}{k}} \sum_{i=1}^{N} \varepsilon^i \psi_i + k \frac{\partial \tilde{\Phi}}{\partial \theta}.$$

So, instead of Eq. (14), the Lagrangian density of the problem is shaped as

$$\mathcal{L} = \int_{-h}^{\tilde{\eta}} \left\{ -ck \frac{\partial \tilde{\Phi}}{\partial \theta} + \frac{1}{2} \left[\left(\sqrt{\frac{g}{k}} \sum_{i=1}^{N} \varepsilon^i \psi_i + k \frac{\partial \tilde{\Phi}}{\partial \theta} \right)^2 + \left(\frac{\partial \tilde{\Phi}}{\partial z} \right)^2 \right] + gz \right\} dz - R\tilde{\eta}.$$

As a result, $\boldsymbol{P}$ (see Eq. (35)) is enriched with the additional elements $\psi_{\{i\}}$, while the form $\mathcal{L} = \mathcal{L}[\tilde{\eta}(\theta), \tilde{\Phi}(\theta,\bullet); \boldsymbol{P}]$ and, hence, the form $\mathscr{S} = \mathscr{S}[\boldsymbol{P}]$ (see Eq. (36)) are preserved. That preservation enables us to apply the AVP in the same way, as before.

Repeating the procedure of Subsection 2.2.4, and considering the additional variations $\delta\psi_{\{i\}}$, we conclude to the exact same results, excluding, of course, the previously absent coefficients $\psi_{\{i\}}$, and, also, the celerity's coefficient c_2. Specifically, the coefficients of the wave-induced current turn out to be

$$\delta\psi_1: \quad \frac{\partial \mathscr{S}_2}{\partial \psi_1} = 0 \iff \psi_1 = 0, \tag{71}$$

$$\delta\psi_2: \quad \frac{\partial \mathscr{S}_4}{\partial \psi_2} = 0 \iff \psi_2 = -\frac{\sqrt{\coth(kh)}}{2kh} \tag{72}$$

and

$$\delta\psi_3: \quad \frac{\partial \mathscr{S}_6}{\partial \psi_3} = 0 \iff \psi_3 = 0. \tag{73}$$

Namely, only ψ_2 is different from zero, expressing a second-order mean flow in the negative direction (i.e. opposite to the wave propagation). Moreover, for the coefficient c_2 we now have that

$$\delta a_{11}: \quad \frac{\partial \mathscr{S}_4}{\partial a_{11}} = 0 \iff$$
$$\iff c_2 = \frac{[8+\cosh(4kh)]kh + 2\sinh(2kh) - \sinh(4kh)}{8kh\sinh(2kh)\sinh^2(kh)\sqrt{\tanh(kh)}}. \tag{74}$$

Thus, the celerity is altered, and the reason for that is the presence of the wave-induced current. To verify that statement, it suffices to observe Eqs. (53), (72) & (74) (remembering Eq. (64)). Then, it can be easily deduced that

$$c_2\Big|_{\text{with wave-induced flow}} = c_2\Big|_{\text{without wave-induced flow}} + \psi_2 . \tag{75}$$

This time, the results of the AVP correspond to those of (Fenton 1985) when Stokes' second approximation for the celerity is used.

2.2.6. The case of (known) uniform current

A different, but similar, and frequently encountered, situation is the presence of a known uniform current $U = O(1)$ within the fluid domain [(Fenton 1990)]. If so, since we suppose that the value of that current is given, the consideration of separate coefficients $\psi_{\{i\}}$ is meaningless, as every horizontal flow, no matter its origin, has to be included in (the measured current) U ([5]). Accordingly, in that case, the potential is expressed as

$$\tilde{\Phi}_U(\theta,z;x) = Ux + \tilde{\Phi}(\theta,z) + O(\varepsilon^{N+1}). \tag{76}$$

Then, the Lagrangian of the problem becomes

$$\mathcal{L} = \int_{-h}^{\tilde{\eta}} \left\{ -ck\frac{\partial \tilde{\Phi}}{\partial \theta} + \frac{1}{2}\left[\left(U + k\frac{\partial \tilde{\Phi}}{\partial \theta} \right)^2 + \left(\frac{\partial \tilde{\Phi}}{\partial z} \right)^2 \right] + gz \right\} dz - R\tilde{\eta}, \tag{77}$$

and, as a consequence, the AVP can once more be deployed without any complicacies. The only disparity, in comparison to the case of wave-induced flow, is that, because U is of $O(1)$, a component $R_0 = O(1)$ is also included in the Bernoulli constant, Eq. (8), so as to take into account the dynamic pressure that the current induces.

Implementing the AVP, we rederive, for each wave parameter, the findings of Subsection 2.2.4, with the exception of R_0 and c_0. Particularly, for the latter, we get

$$\delta\zeta_1: \quad \frac{\partial \mathscr{S}_1}{\partial \zeta_1} = 0 \iff R_0 = \frac{kU^2}{2g} \tag{78}$$

and

$$\delta a_{11}: \quad \frac{\partial \mathscr{S}_2}{\partial a_{11}} = 0 \iff c_0 = \frac{kU}{\sqrt{gk}} + \sqrt{\tanh(kh)} . \tag{79}$$

([5]) If we do include the coefficients $\psi_{\{i\}}$ in $\tilde{\Phi}$, following the AVP, we conclude that either $\psi_{\{i\}} = 0$ or $U = 0$.

Thus, in this instance, remembering Eqs. (7) & (8), the celerity is increased by the current's speed U, while the quantity $U^2 / 2$ is added to the Bernoulli constant. Obviously, that is a generalization of the pure wave motion of Subsection 2.2.4, which can be recovered by setting $U = 0$.

2.2.7. Concluding remarks

The AVP provides us with a systematic approach for the study of steady periodic waves of small steepness, leading to results that are in complete agreement with those obtained conventionally. Furthermore, it does so by requiring the inclusion of much less information within the ansatzes of the wave fields.

For example, the a priori introduction of an appropriate vertical dependence is not needed. Instead, its determination can efficiently, and almost effortlessly, be achieved by isolating the respective vertical problem and treating it by means of that method.

The same also applies to the wave celerity. No matter the specific case under examination (e.g. no current, unknown wave-induced current, known current), it is not necessary to assume, for c, any of Stokes' approximations. It rather suffices to configure the corresponding trial functions ($\tilde{\Phi}$, in particular) accordingly, so as to reflect the various situations, and the rest is left to the AVP.

Of course, neither the choice of Eq. (9) for the perturbation parameter is mandatory, nor the pursuit of solutions up to $O(\varepsilon^3)$. Alternative expressions for ε (e.g. $\varepsilon = k a$) can be used in the same way and, also, higher-order waves can be considered, without any complications as to the consistency of the results of the AVP with the respective ones from other methods.

As a final note, we mention that, in the context of the AVP, the consideration of small-slope waves, which allows for order separation, perturbation hierarchies and asymptotic approximations, should not be indispensable. When (exactly) periodic waves are examined, the AVP is equivalent to the introduction of suitable ansatzes (e.g. Fourier expansions) into the problem's Lagrangian. Thus, in principle, the method could yield a closed system of equations, for the wave parameters, without the adoption of orders. Of course, it would be complicated enough to demand the use of numerical methods, for the final evaluation of the ansatzes' coefficients, under certain given data (e.g. k, h, etc.).

Chapter 3

The Multiple-Scales Method in weakly nonlinear, narrow-banded wavetrains

"Not ignorance, but ignorance of ignorance is the death of knowledge."

A. N. WHITEHEAD

3.1. Introduction

The striking findings of (Lighthill, 1965, 1967), (Benjamin and Feir, 1967) and (Whitham, 1967), regarding the instability of uniform wavetrains (see, also, Sec. 2.1), led, as is natural, in the subsequent years, to an increased scientific interest for the evolution of weakly nonlinear, narrow-banded (i.e. slowly modulated) wave packets [(Mei, Stiassnie and Yue, 2005)]. A vast amount of efforts has been put into that direction and different approaches have been deployed, the most popular being the MSM, the AVP of Whitham and Zakharov's Hamiltonian formalism [(Sedletsky, 2012)]. Here, we mostly focus on contributions whose main task is the derivation of the model equations of such wavetrains, deploying either the MSM or the AVP. The emphasis of this section, in particular, is on results obtained via the former method. Relevant works that use the approach of Whitham are the main subject of Section 4.1. A comprehensive overview of the developments and the literature on this fascinating topic in general can be found in the books of (Dingemans, 1997), (Johnson, 1997), (Sulem and Sulem, 1999), (Mei, Stiassnie and Yue, 2005), (Ablowitz, 2011), (Lannes, 2013) and (Bridges, Groves and Nicholls, 2016). Valuable other contributions, for the same purpose, are, among others, the works of (Hammack and Henderson, 1993) and (Dias and Bridges, 2005), and the review paper of (Zakharov and Ostrovsky, 2009).

The endeavor, for the derivation of satisfying model equations, via systematic perturbation methods, that represent the evolution of slowly modulated wavetrains, lasts for decades. (Benney and Newell, 1967) were the ones to pioneer the nonlinear Schrödinger (NLS) equation for weakly nonlinear dispersive waves in general. A year later, (Zakharov, 1968) derived the NLS, too, but via an alternative route. Specifically, he did so by using canonical variables in the context of his Hamiltonian formalism. Afterwards, (Benney and Roskes, 1969) obtained the, so called, Benney-Roskes (BR) system of equations, which refers to the slow modulation of 3-dimensional wave packets, whereas (Davey, 1972) studied the propagation of weakly nonlinear, narrow-banded wavetrains in the case of a medium that is, in addition, dissipative. (Hasimoto and Ono, 1972), subsequently, derived the NLS equation for slowly varying waves in water of arbitrary depth. In a similar line of work, (Kawahara, 1973) implemented the derivative-expansion version of the MSM (see Sec. 1.4 and (Nayfeh, 2008), for example) systematically in cases that lead to the NLS equation. (Davey and Stewartson, 1974), on the other hand, derived a system of evolutionary equations for 3-dimensional wavetrains, which is now known as the Davey-Stewartson (DS) system of equations. That system essentially constitutes an extension of the NLS equation to wavetrains that propagate in the horizontal plane of 3-dimentional fluid domains. Later, (Dysthe, 1979) improved the ordinary NLS equation and its range of validity, in the case of deep water, by taking the perturbation expansions of the wave fields to the next order. His version of the NLS equation usually goes by the name "modified NLS equation". (Peregrine, 1983), further, dealt with the evolutionary equations that describe slowly varying wavetrains and highlighted the cases in which their model equations can be reduced to the NLS. Moreover, he presented several analytical solutions of the NLS equation and investigated them. It should be mentioned, at this point, that the NLS equation is very important, not only because of its relative simplicity and the phenomena it brings out, but also due to the fact that it can be analytically solved via the inverse scattering transform [(Shabat and Zakharov, 1972; Zakharov and Shabat, 1973)]. Returning to the derivation of model equations, (Lo and Mei, 1985) introduced minor modifications to Dysthe's equation, while (Brinch-Nielsen and Jonsson, 1986) extended his results to 3-dimensional wavetrains in water of arbitrary depth. A decade later, (Trulsen and Dysthe, 1996) extended the modified NLS equation by relaxing the narrow-bandwidth constraint, while (Trulsen *et al.*, 2000) further improved their work via the use of pseudodifferential operators that capture the full linear dispersive behavior. In a similar attempt, to enhance the modified NLS equation, (Sedletsky, 2003) extended the original work of Dysthe to waves in water of arbitrary depth and, in his turn, (Slunyaev, 2005) generalized the work of Sedletsky even more. It should be noted that, except for the work of Zakharov, all the above contributions, in regard to the derivation of the aforementioned model equations, involve some version of the MSM.

Of course, a very important matter is whether or not all those results, obtained heuristically, are or can be mathematically justified. Although that topic diverges from our main goal, we should mention the works of (Craig, Sulem and Sulem, 1992), (Craig, Schanz and Sulem, 1997) and (Schneider, 1998). The first two rigorously rederived the model equations of 2-dimensional and 3-dimensional slowly varying wave packets, while the third provided a rigorous derivation of the NLS equation from the Korteweg-de-Vries equation. For more information on this subject, the interested reader is referred to (Dias and Bridges, 2005) and the books of (Sulem and Sulem, 1999) and (Lannes, 2013).

In the remainder of this chapter, our interest lies in the derivation of model equations for weakly nonlinear, narrow-banded wavetrains in terms of the MSM. Our attention is especially focused on the intuition behind the implementation of the method to the WWP and to the assumptions that lead to the emergence of the NLS equation.

3.2. The effects of weak nonlinearity and narrowbandedness in the classical WWP

As stated earlier, our intention is to use the MSM to study the evolution of weakly nonlinear, narrow-banded wavetrains. We do so for waves in a 3D fluid domain. Thus, first, we extend the 2D differential formulation of the classical gravity WWP, given in Sec. 1.2, to the 3D case [(Stoker 1957; Johnson 1997; Mei, Stiassnie, and Yue 2005)]. Towards that, let $O\boldsymbol{x}z$ be a Cartesian coordinate system, where $\boldsymbol{x}=(x,y)$ expresses the horizontal variables and z the vertical one, pointing upwards. Once more, we assume a seabed of constant depth h, placed at $z=-h$, and we denote the free-surface elevation, measured from the still-water level $z=0$, as $\eta=\eta(\boldsymbol{x};t)$. Then, the 3D fluid domain is expressed as

$$D = D_h^{\eta}(t) = \{(\boldsymbol{x},z)\in\mathbb{R}^3: \quad \boldsymbol{x}\in X, \quad -h<z<\eta(\boldsymbol{x},t), \quad t\geq t_0\}, \tag{1}$$

where $X\subseteq\mathbb{R}^2$ is the horizontal region of the fluid and t_0 the wave motion's initialization time. Moreover, the boundary of the seabed is now represented by

$$\Gamma_h = \{(\boldsymbol{x},z)\in\mathbb{R}^3: \quad \boldsymbol{x}\in X, \quad z=-h\}, \tag{2}$$

whereas the free-surface boundary is denoted with

$$\Gamma_\eta = \Gamma_\eta(t) = \{(\boldsymbol{x},z)\in\mathbb{R}^3: \quad \boldsymbol{x}\in X, \quad z=\eta(\boldsymbol{x},t), \quad t\geq t_0\}. \tag{3}$$

Consequently, instead of Eqs. (4')-(7') of Sec. 1.2, the differential formulation of the 3D classical gravity WWP consists of the following equations

$$\Delta\Phi = \frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2} = 0, \qquad \text{in } D, \tag{4}$$

$$\frac{\partial\Phi}{\partial z} = 0, \qquad \text{on } \Gamma_h, \tag{5}$$

$$\frac{\partial\eta}{\partial t}+\frac{\partial\Phi}{\partial x}\frac{\partial\eta}{\partial x}+\frac{\partial\Phi}{\partial y}\frac{\partial\eta}{\partial y}-\frac{\partial\Phi}{\partial z} = 0, \qquad \text{on } \Gamma_\eta \tag{6}$$

and

$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right\}+g\eta = 0, \qquad \text{on } \Gamma_\eta, \tag{7}$$

supplemented, again, by appropriate initial and lateral boundary conditions. As is usual the case, $\Phi=\Phi(\boldsymbol{x},z;t)$ refers to the velocity potential. Alternatively, the kinematic and dynamic boundary conditions of the free surface can be combined into [(Mei, Stiassnie, and Yue 2005)]

$$\frac{\partial^2\Phi}{\partial t^2}+g\frac{\partial\Phi}{\partial z}+\frac{\partial\boldsymbol{u}^2}{\partial t}+\frac{1}{2}\boldsymbol{u}\nabla\boldsymbol{u}^2 = 0, \qquad \text{on } \Gamma_\eta, \tag{8}$$

where $\boldsymbol{u}=\nabla\Phi$ is the velocity field of the fluid domain. In that instance, of course, the need for a second condition on the free surface doesn't cease to exist. What is gained, though, is that, supposing that the unknown boundary can be somehow treated, one can first solve a boundary value problem that involves only $\Phi(\boldsymbol{x},z;t)$ and then, with $\Phi(\boldsymbol{x},z;t)$ known, find the free-surface elevation via the free surface's dynamic condition, for example. That set-up is very useful for our purpose, as will be seen clearly in what follows.

The main features of the wavetrains we want to study are their weak nonlinearity and the narrowness of their bandwidth. As mentioned before, the nonlinearity of the WWP arises from the unknown upper boundary of the fluid domain, namely the free surface, and the (nonlinear) boundary conditions on it. Therefore, for a wave motion to be weakly nonlinear, the fields that determine it have to be small [(Debnath 1994)]. From Ch. 2, where Stokes waves are the main subject, it is known that weakly nonlinear periodic waves can be expressed in terms of their (small) steepness. Assuming, therefore, a characteristic wavenumber k, since the waves under examination are also narrow-banded, and a characteristic amplitude A [(Bretherton and Garrett 1968)], their weak nonlinearity implies that $\boldsymbol{u}$, Φ and η are of $O(kA)$, where the steepness kA is small. The smallness of those fields allows for the Taylor expansion of the velocity potential and the free-surface boundary conditions around $z=0$ [(Mei, Stiassnie, and Yue 2005)]. Since powers of Φ, η and the components of $\boldsymbol{u}$ occur from that process, powers of the small steepness appear into those expansions. Up to $O(kA)^3$, the expansion of Φ, Eq. (7) and Eq. (8) yields, respectively,

$$\Phi\Big|_{z=\eta} = \Phi\Big|_{z=0} + \eta\left[\frac{\partial\Phi}{\partial z}\right]_{z=0} + \frac{\eta^2}{2}\left[\frac{\partial^2\Phi}{\partial z^2}\right]_{z=0} + O(kA)^4, \tag{9}$$

$$\begin{aligned}
&\left[\frac{\partial\Phi}{\partial t} + \frac{1}{2}\boldsymbol{u}^2\right]_{z=0} + \eta\left[\frac{\partial^2}{\partial z\partial t}\Phi + \frac{1}{2}\frac{\partial}{\partial z}\boldsymbol{u}^2\right]_{z=0} + \\
&\qquad + \frac{\eta^2}{2}\left[\frac{\partial^3}{\partial z^2\partial t}\Phi\right]_{z=0} + O(kA)^4 = -g\eta \iff \\
&\iff \left(\left[\frac{\partial\Phi}{\partial t}\right]_{z=0} + g\eta\right) + \left(\eta\left[\frac{\partial^2}{\partial z\partial t}\Phi\right]_{z=0} + \frac{1}{2}\left[\boldsymbol{u}^2\right]_{z=0}\right) + \\
&\qquad + \left(\frac{\eta}{2}\left[\frac{\partial}{\partial z}\boldsymbol{u}^2\right]_{z=0} + \frac{\eta^2}{2}\left[\frac{\partial^3}{\partial z^2\partial t}\Phi\right]_{z=0}\right) + O(kA)^4 = 0
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&\left[\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} + \frac{\partial\boldsymbol{u}^2}{\partial t} + \frac{1}{2}\boldsymbol{u}\nabla\boldsymbol{u}^2\right]_{z=0} + \\
&\qquad + \eta\left[\frac{\partial^3}{\partial z\partial t^2}\Phi + g\frac{\partial^2\Phi}{\partial z^2} + \frac{\partial^2}{\partial z\partial t}\boldsymbol{u}^2\right]_{z=0} + \\
&\qquad + \frac{\eta^2}{2}\left[\frac{\partial^4}{\partial z^2\partial t^2}\Phi + g\frac{\partial^3\Phi}{\partial z^3}\right]_{z=0} + O(kA)^4 = 0 \iff \\
&\iff \left[\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z}\right]_{z=0} + \left[\eta\left(\frac{\partial^3}{\partial z\partial t^2}\Phi + g\frac{\partial^2\Phi}{\partial z^2}\right) + \frac{\partial\boldsymbol{u}^2}{\partial t}\right]_{z=0} + \\
&\qquad + \left[\frac{1}{2}\boldsymbol{u}\nabla\boldsymbol{u}^2 + \eta\frac{\partial^2}{\partial z\partial t}\boldsymbol{u}^2 + \frac{\eta^2}{2}\left(\frac{\partial^4}{\partial z^2\partial t^2}\Phi + g\frac{\partial^3\Phi}{\partial z^3}\right)\right]_{z=0} + \\
&\qquad + O(kA)^4 = 0.
\end{aligned} \tag{11}$$

Thus, the assumed weak nonlinearity results in the appearance of scales in the nonlinear equations of the problem, via the appearance of powers of the steepness. Furthermore, it is known that in linear problems, where the principle of superposition stands, the superposition of harmonics with similar frequencies leads to the formation of an envelope [(Stoker 1957; Debnath 1994; Mei, Stiassnie, and Yue 2005)]. Inspired by that, given the narrowbandedness of the wavetrains and that the weak nonlinearity should lead to slow energy exchange between the modes during their interaction, such wavetrains can be viewed as slowly modulated ones, around a carrier wave described by k and A [(Whitham 1974; Debnath 1994; Johnson 1997)]. Those observations constitute adequate indications for the use of the MSM.

3.3. The MSM in the governing equations of the problem

This section is dedicated to the implementation of the MSM in the problem of weakly nonlinear, narrow-banded wavetrains. During that procedure, we follow closely (Mei, Stiassnie, and Yue 2005).

3.3.1. General set–up and derivation of the corresponding perturbation hierarchy

As explained previously, we consider the wavetrains we want to study as wavetrains around a carrier wave of wavenumber k and amplitude A that experience slow spatiotemporal modulations. Hence, considering the steepness kA as the problem's perturbation parameter ε ([1]) (i.e. $\varepsilon = kA$), we put into effect the MSM [(Nayfeh 2008)]. First, therefore, we introduce, as independent variables, the temporal scales

$$T_i = \varepsilon^i t, \qquad i \in \mathbb{N}_0, \tag{1a}$$

and, assuming that the wavetrains propagate in the x direction, the spatial ones

$$X_j = \varepsilon^j x, \qquad j \in \mathbb{N}_0. \tag{1b}$$

Allowing, also, for slow modulation alongside the y axis, we additionally define the scales

$$Y_k = \varepsilon^k y, \qquad k \in \mathbb{N}. \tag{1c}$$

Thus, hypothesizing that the unknown fields Φ and η depend on the above variables, we postulate, for them, the existence of asymptotic solutions of the form

$$\Phi(x,y,z;t;\varepsilon) = \tilde{\Phi}(X_0,X_1,X_2,...,Y_1,Y_2,...,z;T_0,T_1,T_2,...;\varepsilon) \tag{2}$$

and

$$\eta(x,y;t;\varepsilon) = \tilde{\eta}(X_0,X_1,X_2,...,Y_1,Y_2,...;T_0,T_1,T_2,...;\varepsilon). \tag{3}$$

For those solutions, we further assume that they can be expressed by means of the asymptotic expansions

$$\tilde{\Phi} = \sum_{n=1} \varepsilon^n \varphi_n \tag{4}$$

and

$$\tilde{\eta} = \sum_{n=1} \varepsilon^n \eta_n, \tag{5}$$

where, $\forall\, n = 1,2,...,$ $\varphi_n = \varphi_n(X_{\{j\}},Y_{\{k\}},z;T_{\{i\}})$ and $\eta_n = \eta_n(X_{\{j\}},Y_{\{k\}};T_{\{i\}})$.

The above Eqs. (2)-(5) are substituted into Eqs. (4), (5), (10) & (11) of the previous section and, also, the various derivatives contained in the latter acquire new forms, according to the chain rule. Specifically, using Eqs. (1)-(3), and with $(Q,\tilde{Q})$ denoting either $(\Phi,\tilde{\Phi})$ or $(\eta,\tilde{\eta})$, it applies that

---

([1]) The problem of weakly nonlinear, narrow-banded wavetrains is governed by three different kinds of scales, owed to the (small) steepness, the narrowness of the bandwidth and the assumed slow modulation. Generally, the determination of the relative importance of the various scales is not a trivial matter [(Dingemans 1997; Johnson 1997)]. In the present context, nevertheless, the balancing of the effects of nonlinearity and dispersion is achieved by expressing them all via the same perturbation parameter ε [(Benney and Roskes 1969; Slunyaev 2005)].

$$\begin{pmatrix}\frac{\partial Q}{\partial x}\\ \frac{\partial Q}{\partial y}\\ \frac{\partial Q}{\partial t}\end{pmatrix}=\begin{pmatrix}\frac{\partial \tilde{Q}}{\partial x}\\ \frac{\partial \tilde{Q}}{\partial y}\\ \frac{\partial \tilde{Q}}{\partial t}\end{pmatrix}\iff\begin{pmatrix}\frac{\partial Q}{\partial x}\\ \frac{\partial Q}{\partial y}\\ \frac{\partial Q}{\partial t}\end{pmatrix}=\begin{pmatrix}\frac{\partial \tilde{Q}}{\partial X_0}\frac{\partial X_0}{\partial x}+\frac{\partial \tilde{Q}}{\partial X_1}\frac{\partial X_1}{\partial x}+...\\ \frac{\partial \tilde{Q}}{\partial Y_1}\frac{\partial Y_1}{\partial y}+\frac{\partial \tilde{Q}}{\partial Y_2}\frac{\partial Y_2}{\partial y}+...\\ \frac{\partial \tilde{Q}}{\partial T_0}\frac{\partial T_0}{\partial t}+\frac{\partial \tilde{Q}}{\partial T_1}\frac{\partial T_1}{\partial t}+...\end{pmatrix}\iff$$

$$\iff\begin{pmatrix}\partial_x Q\\ \partial_y Q\\ \partial_t Q\end{pmatrix}=\begin{pmatrix}\partial_{X_0}\tilde{Q}+\varepsilon\partial_{X_1}\tilde{Q}+\varepsilon^2\partial_{X_2}\tilde{Q}+...\\ \varepsilon\partial_{Y_1}\tilde{Q}+\varepsilon^2\partial_{Y_2}\tilde{Q}+...\\ \partial_{T_0}\tilde{Q}+\varepsilon\partial_{T_1}\tilde{Q}+\varepsilon^2\partial_{T_2}\tilde{Q}+...\end{pmatrix}. \tag{6a}$$

In the same fashion,

$$\begin{pmatrix}\partial_{xx} Q\\ \partial_{yy} Q\\ \partial_{tt} Q\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x}(\partial_{X_0}\tilde{Q}+\varepsilon\partial_{X_1}\tilde{Q}+\varepsilon^2\partial_{X_2}\tilde{Q}+...)\\ \frac{\partial}{\partial y}(\varepsilon\partial_{Y_1}\tilde{Q}+\varepsilon^2\partial_{Y_2}\tilde{Q}+...)\\ \frac{\partial}{\partial t}(\partial_{T_0}\tilde{Q}+\varepsilon\partial_{T_1}\tilde{Q}+\varepsilon^2\partial_{T_2}\tilde{Q}+...)\end{pmatrix}\iff$$

$$\iff\begin{pmatrix}\partial_{xx} Q\\ \partial_{yy} Q\\ \partial_{tt} Q\end{pmatrix}=\begin{pmatrix}\partial_{X_0X_0}\tilde{Q}+\varepsilon 2\partial_{X_0X_1}\tilde{Q}+\varepsilon^2(\partial_{X_1X_1}\tilde{Q}+2\partial_{X_0X_2}\tilde{Q})+...\\ \varepsilon^2\partial_{Y_1Y_1}\tilde{Q}+...\\ \partial_{T_0T_0}\tilde{Q}+\varepsilon 2\partial_{T_0T_1}\tilde{Q}+\varepsilon^2(\partial_{T_1T_1}\tilde{Q}+2\partial_{T_0T_2}\tilde{Q})+...\end{pmatrix}. \tag{6b}$$

Next, the aforementioned substitution is followed by some algebra and the grouping of the terms of like powers of ε, up to $O(\varepsilon^3)$. As a result, the Laplace equation becomes

$$\begin{aligned}&\varepsilon(\partial_{X_0X_0}\varphi_1+\partial_{zz}\varphi_1)+\varepsilon^2(\partial_{X_0X_0}\varphi_2+\partial_{zz}\varphi_2+2\partial_{X_0X_1}\varphi_1)+\\ &\quad+\varepsilon^3(\partial_{X_0X_0}\varphi_3+\partial_{zz}\varphi_3+2\partial_{X_0X_1}\varphi_2+\\ &\qquad+\partial_{X_1X_1}\varphi_1+\partial_{Y_1Y_1}\varphi_1+2\partial_{X_0X_2}\varphi_1)+O(\varepsilon^4)=0,\end{aligned} \tag{7}$$

while the bottom's impermeability condition turns into

$$\varepsilon\partial_z\varphi_1+\varepsilon^2\partial_z\varphi_2+\varepsilon^3\partial_z\varphi_3+O(\varepsilon^4)=0. \tag{8}$$

As for the dynamic and combined boundary conditions of the free surface, they are respectively shaped, on $z=0$, as

$$\begin{aligned}&\varepsilon(\partial_{T_0}\varphi_1+g\eta_1)+\varepsilon^2\Big(\partial_{T_0}\varphi_2+g\eta_2+\partial_{T_1}\varphi_1+\\ &\quad+\frac{1}{2}\{(\partial_{X_0}\varphi_1)^2+(\partial_z\varphi_1)^2\}+\eta_1\partial_{T_0,z}\varphi_1\Big)+\varepsilon^3\Big(\partial_{T_0}\varphi_3+g\eta_3+\\ &\quad+\partial_{T_2}\varphi_1+\partial_{T_1}\varphi_2+\partial_{X_0}\varphi_1\partial_{X_1}\varphi_1+\partial_{X_0}\varphi_1\partial_{X_0}\varphi_2+\end{aligned}$$

$$+\partial_z\varphi_1\partial_z\varphi_2+\eta_1\partial_{T_1,z}\varphi_1+\eta_2\partial_{T_0,z}\varphi_1+\eta_1\partial_{T_0,z}\varphi_2+$$
$$\left.+\eta_1\partial_{X_0}\varphi_1\partial_{X_0,z}\varphi_1+\eta_1\partial_z\varphi_1\partial_{zz}\varphi_1+\frac{1}{2}\eta_1^{\,2}\partial_{T_0,zz}\varphi_1\right)+O(\varepsilon^4)=0 \tag{9}$$

and

$$\varepsilon(\partial_{T_0T_0}\varphi_1+g\partial_z\varphi_1)+\varepsilon^2(\partial_{T_0T_0}\varphi_2+g\partial_z\varphi_2+2\partial_{T_0T_1}\varphi_1+$$
$$+2\partial_{X_0}\varphi_1\partial_{X_0T_0}\varphi_1+2\partial_z\varphi_1\partial_{T_0,z}\varphi_1+\eta_1\partial_{T_0T_0,z}\varphi_1+g\eta_1\partial_{zz}\varphi_1)+$$
$$+\varepsilon^3\left(\partial_{T_0T_0}\varphi_3+g\partial_z\varphi_3+2\partial_{T_0T_1}\varphi_2+\partial_{T_1T_1}\varphi_1+2\partial_{T_0T_2}\varphi_1+\right.$$
$$+\eta_1\partial_{T_0T_0,z}\varphi_2+2\eta_1\partial_{T_0T_1,z}\varphi_1+g\eta_1\partial_{zz}\varphi_2+\eta_2\partial_{T_0T_0,z}\varphi_1+$$
$$+g\eta_2\partial_{zz}\varphi_1+2\partial_{X_0}\varphi_1\partial_{X_0T_1}\varphi_1+2\partial_z\varphi_1\partial_{T_1,z}\varphi_1+2\partial_{X_0T_0}\varphi_1\partial_{X_0}\varphi_2+$$
$$+2\partial_{X_0T_0}\varphi_2\partial_{X_0}\varphi_1+2\partial_{X_0T_0}\varphi_1\partial_{X_1}\varphi_1+2\partial_{X_1T_0}\varphi_1\partial_{X_0}\varphi_1+$$
$$+2\partial_{T_0,z}\varphi_1\partial_z\varphi_2+2\partial_{T_0,z}\varphi_2\partial_z\varphi_1+(\partial_{X_0}\varphi_1)^2\partial_{X_0X_0}\varphi_1+$$
$$+(\partial_z\varphi_1)^2\partial_{zz}\varphi_1+2\partial_{X_0}\varphi_1\partial_z\varphi_1\partial_{X_0,z}\varphi_1+2\eta_1\partial_{X_0T_0}\varphi_1\partial_{X_0,z}\varphi_1+ \tag{10}$$
$$+2\eta_1\partial_{X_0T_0,z}\varphi_1\partial_{X_0}\varphi_1+2\eta_1\partial_{T_0,z}\varphi_1\partial_{zz}\varphi_1+2\eta_1\partial_{T_0,zz}\varphi_1\partial_z\varphi_1+$$
$$\left.+\frac{1}{2}\eta_1^{\,2}\partial_{T_0T_0,zz}\varphi_1+\frac{1}{2}\eta_1^{\,2}g\partial_{zzz}\varphi_1\right)+O(\varepsilon^4)=0.$$

Consequently, we conclude to the perturbation hierarchy

$$O(\varepsilon^1):\begin{cases}\partial_{X_0X_0}\varphi_1+\partial_{zz}\varphi_1=0, & -h<z<0\\ \partial_z\varphi_1=0, & z=-h\\ \partial_{T_0T_0}\varphi_1+g\partial_z\varphi_1=0, & z=0\\ \partial_{T_0}\varphi_1+g\eta_1=0, & z=0\end{cases} \tag{11a}$$

$$O(\varepsilon^2):\begin{cases}\partial_{X_0X_0}\varphi_2+\partial_{zz}\varphi_2=-2\partial_{X_0X_1}\varphi_1, & -h<z<0\\ \partial_z\varphi_2=0, & z=-h\\ \partial_{T_0T_0}\varphi_2+g\partial_z\varphi_2=-2\partial_{T_0T_1}\varphi_1-2\partial_{X_0}\varphi_1\partial_{X_0T_0}\varphi_1 & \\ \qquad -2\partial_z\varphi_1\partial_{T_0,z}\varphi_1-\eta_1\partial_{T_0T_0,z}\varphi_1-g\eta_1\partial_{zz}\varphi_1, & z=0\\ \partial_{T_0}\varphi_2+g\eta_2=-\partial_{T_1}\varphi_1-\eta_1\partial_{T_0,z}\varphi_1-\frac{1}{2}\{(\partial_{X_0}\varphi_1)^2+(\partial_z\varphi_1)^2\}, & z=0\end{cases} \tag{11b}$$

$$
O(\varepsilon^3):\begin{cases}
\partial_{X_0X_0}\varphi_3+\partial_{zz}\varphi_3 = -2\partial_{X_0X_1}\varphi_2-\partial_{X_1X_1}\varphi_1 \\
\qquad -\partial_{Y_1Y_1}\varphi_1-2\partial_{X_0X_2}\varphi_1, & -h<z<0 \\
\partial_z\varphi_3 = 0, & z=-h \\
\partial_{T_0T_0}\varphi_3+g\partial_z\varphi_3 = -2\partial_{T_0T_1}\varphi_2-\partial_{T_1T_1}\varphi_1-2\partial_{T_0T_2}\varphi_1 \\
\quad -\eta_1\partial_{T_0T_0,z}\varphi_2-2\eta_1\partial_{T_0T_1,z}\varphi_1-g\eta_1\partial_{zz}\varphi_2-\eta_2\partial_{T_0T_0,z}\varphi_1 \\
\quad -g\eta_2\partial_{zz}\varphi_1-2\partial_{X_0}\varphi_1\partial_{X_0T_1}\varphi_1-2\partial_z\varphi_1\partial_{T_1,z}\varphi_1 \\
\quad -2\partial_{X_0T_0}\varphi_1\partial_{X_0}\varphi_2-2\partial_{X_0T_0}\varphi_2\partial_{X_0}\varphi_1-2\partial_{X_0T_0}\varphi_1\partial_{X_1}\varphi_1 \\
\quad -2\partial_{X_1T_0}\varphi_1\partial_{X_0}\varphi_1-2\partial_{T_0,z}\varphi_1\partial_z\varphi_2-2\partial_{T_0,z}\varphi_2\partial_z\varphi_1 \\
\quad -(\partial_{X_0}\varphi_1)^2\partial_{X_0X_0}\varphi_1-2\partial_{X_0}\varphi_1\partial_z\varphi_1\partial_{X_0,z}\varphi_1 \\
\quad -(\partial_z\varphi_1)^2\partial_{zz}\varphi_1-2\eta_1\partial_{X_0T_0}\varphi_1\partial_{X_0,z}\varphi_1-2\eta_1\partial_{T_0,z}\varphi_1\partial_{zz}\varphi_1 \\
\quad -2\eta_1\partial_{T_0,zz}\varphi_1\partial_z\varphi_1-2\eta_1\partial_{X_0T_0,z}\varphi_1\partial_{X_0}\varphi_1 \\
\quad -\frac{1}{2}\eta_1{}^2(\partial_{T_0T_0,zz}\varphi_1+g\partial_{zzz}\varphi_1), & z=0 \\
\partial_{T_0}\varphi_3+g\eta_3 = -\partial_{T_2}\varphi_1-\partial_{T_1}\varphi_2-\partial_{X_0}\varphi_1\partial_{X_1}\varphi_1-\partial_{X_0}\varphi_1\partial_{X_0}\varphi_2 \\
\quad -\partial_z\varphi_1\partial_z\varphi_2-\eta_2\partial_{T_0,z}\varphi_1-\eta_1\partial_{T_1,z}\varphi_1-\eta_1\partial_{T_0,z}\varphi_2 \\
\quad -\eta_1\partial_{X_0}\varphi_1\partial_{X_0,z}\varphi_1-\eta_1\partial_z\varphi_1\partial_{zz}\varphi_1-\frac{1}{2}\eta_1{}^2\partial_{T_0,zz}\varphi_1, & z=0.
\end{cases} \tag{11c}
$$

Hence, at each order of the perturbation parameter ε, and given that we search for periodic solutions with respect to T_0, a boundary value problem for the respective potential component arises, composed of that order's first three equations. Those boundary value problems, independently of the order of ε, are governed by the linear operators

$$\Delta_{X_0,z}(\bullet)\equiv\partial_{X_0X_0}(\bullet)+\partial_{zz}(\bullet), \qquad -h<z<0, \tag{12a}$$

$$\partial_z(\bullet), \qquad z=-h, \tag{12b}$$

and

$$\Gamma(\bullet)\equiv\partial_{T_0T_0}(\bullet)+g\partial_z(\bullet), \qquad z=0. \tag{12c}$$

Provided the velocity potential of an order, the corresponding elevation can be determined via the respective free surface's dynamic boundary condition (last equation of each order).

The first problem of the hierarchy is actually the linearized WWP and has, therefore, the known solution [(Stoker 1957; Debnath 1994; Johnson 1997)]

$$\varphi_1 = C_1 - j\frac{g}{\omega}A_1\frac{\cosh[k(z+h)]}{\cosh(kh)}e^{j(kX_0-\omega T_0)}+(*),$$

where C_1 and A_1 are functions of the slow scales $(X_1,X_2,...,Y_1,Y_2,...,T_1,T_2,...)$. That, and the form of the nonlinear forcing terms on the free surface, leads to higher order solutions that contain not only the fundamental harmonic $\exp[j(kX_0-\omega T_0)]$, but also higher ones, based on it. Thus, we represent each φ_n as ([2]) [(Davey and Stewartson 1974; Mei, Stiassnie, and Yue 2005)]

$$\varphi_n = \sum_{m=-n}^{n} \varphi_{nm} e^{jm(kX_0-\omega T_0)}, \tag{13}$$

where $\{\varphi_{nm}\}_{m=-n}^{n}$ is a set of appropriate complex amplitudes. For those, it applies that $\varphi_{nm}=\varphi_{nm}(X_1,X_2,...,Y_1,Y_2,...,z;T_1,T_2,...)$ and, additionally, $\varphi_{n,-m}=\overline{\varphi_{nm}}$, so that φ_n is real.

The new assumed expressions for the potentials $\varphi_{\{n\}}$, namely Eqs. (13), are inserted into the perturbation hierarchy and, hence, reshape it. In particular, for the first order of the latter we conclude to

$$O(\varepsilon^1):\begin{cases}\partial_{zz}\varphi_{10}+\\ \quad+(\partial_{zz}\varphi_{11}-k^2\varphi_{11})e^{j(kX_0-\omega T_0)}+\\ \quad+(\partial_{zz}\varphi_{1,-1}-k^2\varphi_{1,-1})e^{-j(kX_0-\omega T_0)}=0, & -h<z<0\\ \\ \partial_z\varphi_{10}+\\ \quad+\partial_z\varphi_{11}e^{j(kX_0-\omega T_0)}+\\ \quad+\partial_z\varphi_{1,-1}e^{-j(kX_0-\omega T_0)}=0, & z=-h\\ \\ g\partial_z\varphi_{10}+\\ \quad+(g\partial_z\varphi_{11}-\omega^2\varphi_{11})e^{j(kX_0-\omega T_0)}+\\ \quad+(g\partial_z\varphi_{1,-1}-\omega^2\varphi_{1,-1})e^{-j(kX_0-\omega T_0)}=0, & z=0.\end{cases} \tag{14a}$$

Regarding the other orders of the hierarchy, due to the size of the equations they contain, they are included in Eqs. (14b) & (14c) of the Appendix C. As mentioned before, furthermore, the spatiotemporal scales, on which the unknown fields depend, are considered as independent variables [(Mei, Stiassnie, and Yue 2005; Nayfeh 2008; Holmes 2012)]. Thus, the φ_{nm}-expressions, which are functions of the slow scales only and accompany the harmonics (functions of the fast scales X_0,T_0) can be viewed as coefficients of the latter. As a consequence, the various equations, contained in Eqs. (14), are valid only when those coefficients are equal to zero, since each equation consists of a zero rhs (linear independence of the harmonics). Accordingly, there arises a boundary value problem for each φ_{nm}. In that spirit, from the first order of the perturbation hierarchy we obtain

([2]) Another approach is to solve for φ_n, at each order of the perturbation hierarchy, and, then, eliminate all the secular (that is, non–periodic) terms, since the solutions in search are supposed to be periodic. Ultimately, the outcome is the same, but this approach is obviously more involved in terms of the required calculations [(Johnson 1997)].

$$O(\varepsilon^1):\ \begin{cases} \partial_{zz}\varphi_{10} = 0, & -h<z<0 \\ \partial_z\varphi_{10} = 0, & z=-h \\ g\,\partial_z\varphi_{10} = 0, & z=0 \\ & \\ \partial_{zz}\varphi_{11} - k^2\varphi_{11} = 0, & -h<z<0 \\ \partial_z\varphi_{11} = 0, & z=-h \\ g\,\partial_z\varphi_{11} - \omega^2\varphi_{11} = 0, & z=0 \\ & \\ \partial_{zz}\varphi_{1,-1} - k^2\varphi_{1,-1} = 0, & -h<z<0 \\ \partial_z\varphi_{1,-1} = 0, & z=-h \\ g\,\partial_z\varphi_{1,-1} - \omega^2\varphi_{1,-1} = 0, & z=0. \end{cases} \tag{15a}$$

Similar results are derived for the other orders of the hierarchy, which are expressed via Eqs. (15b) & (15c) of the Appendix C.

3.3.2. Determination of the complex amplitudes $\{\varphi_{nm}\}$ via the successive solution of the perturbation hierarchy

We move forward with the solution of the boundary value problems of Eq. (15a), which happen to be homogenous. As for φ_{10}, it is easily seen that

$$\varphi_{10}(X_{\{\geq 1\}},Y_{\{\geq 1\}},z;T_{\{\geq 1\}}) = C_{10}(X_{\{\geq 1\}},Y_{\{\geq 1\}};T_{\{\geq 1\}})\,. \tag{16}$$

That is, it is an arbitrary function of the slow scales. Next, there remain φ_{11} and $\varphi_{1,-1}$, where it suffices to find one of them, since the other is its complex conjugate. Working with φ_{11}, and given that in our case k is real, from

$$\partial_{zz}\varphi_{11} - k^2\varphi_{11} = 0,$$

it occurs that its general solution is

$$\varphi_{11} = C_1 e^{kz} + C_2 e^{-kz},$$

where $C_1 = C_1(X_{\{\geq 1\}},Y_{\{\geq 1\}};T_{\{\geq 1\}})$ and $C_2 = C_2(X_{\{\geq 1\}},Y_{\{\geq 1\}};T_{\{\geq 1\}})$. Fitting, afterwards, the seabed's boundary condition gives

$$\left[\partial_z\varphi_{11}\right]_{z=-h} = 0 \iff \left[kC_1 e^{kz} - kC_2 e^{-kz}\right]_{z=-h} = 0 \iff$$
$$\iff kC_1 e^{-kh} - kC_2 e^{kh} = 0 \iff C_2 = C_1 e^{-2kh}.$$

Thus, after some algebraic manipulations,

$$\varphi_{11}(X_{\{\geq 1\}},Y_{\{\geq 1\}},z;T_{\{\geq 1\}}) = C_{11}(X_{\{\geq 1\}},Y_{\{\geq 1\}};T_{\{\geq 1\}})\cosh[k(z+h)]. \tag{17}$$

Using, then, the above expression into the boundary condition of the free surface, we obtain

$$\left[g\,\partial_z\varphi_{11} - \omega^2\varphi_{11}\right]_{z=0} = 0 \iff C_{11}[gk\sinh(kh) - \omega^2\cosh(kh)] = 0\,.$$

Accordingly, for nontrivial solutions, we require that

$$\omega^2 = g\,k\tanh(k\,h), \tag{18}$$

which, of course, is the (known) dispersion relation that arises in the context of the linearized WWP ([3]) [(Stoker 1957; Whitham 1974; Debnath 1994)].

Before proceeding with the solution of the boundary value problems of $O(\varepsilon^2)$, it is possible to further shape φ_{11}, by taking advantage of the dynamic condition of Eq. (11a). In particular, via Eqs. (13), (16) & (17),

$$\varphi_1 = C_{10} + \{C_{11}\cosh[k(z+h)]e^{j(kX_0-\omega T_0)} + (*)\}.$$

As a consequence, inserting φ_1 into the aforementioned condition yields

$$\eta_1 = -\frac{1}{g}\left[\partial_{T_0}\varphi_1\right]_{z=0} \iff \eta_1 = j\frac{\omega}{g}C_{11}\cosh(k\,h)e^{j(kX_0-\omega T_0)} + (*).$$

Though, in general, η_1 can be also written in the form

$$\eta_1(X_{\{\geq 0\}},Y_{\{\geq 1\}};T_{\{\geq 0\}}) = A(X_{\{\geq 1\}},Y_{\{\geq 1\}};T_{\{\geq 1\}})e^{j(kX_0-\omega T_0)} + (*), \tag{19}$$

where A is an arbitrary, slowly modulated amplitude. With that being the desired expression for the elevation η_1, we obtain

$$j\frac{\omega}{g}C_{11}\cosh(k\,h) = A \iff C_{11} = -j\frac{g}{\omega}\frac{1}{\cosh(k\,h)}A,$$

so that, after all,

$$\varphi_{11}(X_{\{\geq 1\}},Y_{\{\geq 1\}},z;T_{\{\geq 1\}}) = -j\frac{g}{\omega}A(X_{\{\geq 1\}},Y_{\{\geq 1\}};T_{\{\geq 1\}})\frac{\cosh[k(z+h)]}{\cosh(k\,h)}. \tag{17'}$$

That, of course, leads to the final form

$$\varphi_1 = C_{10} - \left\{j\frac{g}{\omega}A\frac{\cosh[k(z+h)]}{\cosh(k\,h)}e^{j(kX_0-\omega T_0)} + (*)\right\}. \tag{20}$$

for the first-order component of the velocity potential.

Next in line is the determination of the complex amplitudes of $O(\varepsilon^2)$. The first we encounter is φ_{20}, which is governed by the boundary value problem (see Appendix C)

$$\begin{cases} \partial_{zz}\varphi_{20} = 0, & -h < z < 0 \\ \partial_z\varphi_{20} = 0, & z = -h \\ g\,\partial_z\varphi_{20} = 0, & z = 0. \end{cases} \tag{15b,i}$$

Thus, as in the case of φ_{10} (see Eq. (16)), φ_{20} constitutes an arbitrary function of the slow scales, too. In other words,

$$\varphi_{20}(X_{\{\geq 1\}},Y_{\{\geq 1\}},z;T_{\{\geq 1\}}) = C_{20}(X_{\{\geq 1\}},Y_{\{\geq 1\}};T_{\{\geq 1\}}). \tag{21}$$

([3]) As mentioned earlier, the problem of Eq. (11a) or, equivalently, the problems of Eq. (15a), coincides with the linear WWP. As a consequence, the result of Eq. (18) is, essentially, a priori expected.

Moving on, the amplitude φ_{21} results from the solution of

$$\begin{cases} \partial_{zz}\varphi_{21} - k^2\varphi_{21} = -2\dfrac{gk}{\omega}\partial_{X_1}A\dfrac{\cosh[k(z+h)]}{\cosh(kh)}, & -h<z<0 \\ \partial_z\varphi_{21} = 0, & z=-h \\ g\partial_z\varphi_{21} - \omega^2\varphi_{21} = 2g\partial_{T_1}A, & z=0. \end{cases} \qquad \text{(15b,ii)}$$

The above is clearly an inhomogeneous boundary value problem. Due to that, whether it has a solution or not is under question. In particular, invoking Fredholm's alternative theorem [(Evans 1998)], since the respective homogeneous problem has the nontrivial solution φ_{11} (see Eqs. (15a) & (17')), the problem of Eq. (15b,ii) is solvable if and only if the *solvability condition* [(Mei, Stiassnie, and Yue 2005; Ablowitz 2011; Nayfeh 2011; Holmes 2012)]

$$\int_{-h}^{0}\left[(\partial_{zz}\varphi_{21} - k^2\varphi_{21})\varphi_{11} - \varphi_{21}(\partial_{zz}\varphi_{11} - k^2\varphi_{11})\right]dz = \left[\partial_z\varphi_{21}\varphi_{11} - \varphi_{21}\partial_z\varphi_{11}\right]_{-h}^{0}$$

is satisfied. The above is, of course, the application of Green's theorem to φ_{11} and φ_{21}. But it further applies that

$$\partial_{zz}\varphi_{11} - k^2\varphi_{11} = 0, \qquad -h<z<0$$

and

$$\partial_{zz}\varphi_{21} - k^2\varphi_{21} = f_{21}(z), \qquad -h<z<0,$$

where $f_{21}(z)$ is the forcing term of the problem (15b,ii), i.e. the rhs of the differential equation towards φ_{21}. Thus, the solvability condition becomes

$$\int_{-h}^{0} f_{21}\varphi_{11}\,dz = \left[\partial_z\varphi_{21}\varphi_{11} - \varphi_{21}\partial_z\varphi_{11}\right]_{-h}^{0} \iff$$

$$\iff \int_{-h}^{0} f_{21}\varphi_{11}\,dz = \left[\partial_z\varphi_{21}\right]_0\left[\varphi_{11}\right]_0 - \left[\varphi_{21}\right]_0\left[\partial_z\varphi_{11}\right]_0 \iff$$

$$\iff \int_{-h}^{0} f_{21}\left(\frac{\cosh[k(z+h)]}{\cosh(kh)}\right)dz = \left[\partial_z\varphi_{21}\right]_0 - \left[\varphi_{21}\right]_0 k\tanh(kh) \iff$$

$$\iff \int_{-h}^{0} f_{21}\left(\frac{\cosh[k(z+h)]}{\cosh(kh)}\right)dz = \left[\partial_z\varphi_{21} - \varphi_{21}k\tanh(kh)\right]_0.$$

Moreover, exploiting the dispersion relation of Eq. (18), we finally get

$$\int_{-h}^{0} f_{21}\left(\frac{\cosh[k(z+h)]}{\cosh(kh)}\right)dz = \frac{1}{g}\left[g\partial_z\varphi_{21} - \omega^2\varphi_{21}\right]_0, \qquad (22)$$

where the bracketed expression is the free surface's boundary condition. Given Eq. (22), after making the necessary substitutions for f_{21} and the aforementioned condition, we are led to

$$\partial_{T_1} A + \frac{\omega}{2k}\left(1 + \frac{2kh}{\sinh(2kh)}\right)\partial_{X_1} A = 0, \tag{23}$$

which is a necessary condition regarding the amplitude A, so that the final perturbation expansion of the potential may be uniformly valid. However, from the theory of the linearized WWP, and using Eq. (18) again, it is known that the group velocity C_g is [(Stoker 1957; Whitham 1974; Debnath 1994)]

$$C_g \equiv \frac{d\omega}{dk} = \frac{\omega}{2k}\left(1 + \frac{2kh}{\sinh(2kh)}\right).$$

Thereby, Eq. (23) can alternatively be written as

$$\partial_{T_1} A + C_g \partial_{X_1} A = 0. \tag{23'}$$

Requiring the satisfaction of the above equation, the boundary value problem of Eq. (15b,ii) has a solution, which is the sum of φ_{11} and a particular solution. To find one, using the method of undetermined coefficients [(Boyce and DiPrima 2012)], we postulate that it is of the form

$$\varphi_{21,p} = M\{k(z+h)\}\cosh[k(z+h)] + N\{k(z+h)\}\sinh[k(z+h)],$$

where M and N are the "coefficients" (functions of the slow scales, but constant towards the variable z of the differential equation) that need to be determined. Inserting the above equation into Eq. (15b,ii) yields

$$\begin{aligned}
&\partial_{zz}\varphi_{21,p} - k^2\varphi_{21,p} = -2\frac{gk}{\omega}\partial_{X_1} A \frac{\cosh[k(z+h)]}{\cosh(kh)} \iff \\
&\iff \partial_{zz}\ M\{k(z+h)\}\cosh[k(z+h)] + N\{k(z+h)\}\sinh[k(z+h)] \\
&\qquad -k^2\ M\{k(z+h)\}\cosh[k(z+h)] + N\{k(z+h)\}\sinh[k(z+h)] = \\
&\qquad = -2\frac{gk}{\omega}\partial_{X_1} A \frac{\cosh[k(z+h)]}{\cosh(kh)} \iff \\
&\iff M\sinh[k(z+h)] + N\cosh[k(z+h)] = -\frac{g}{\omega k}\frac{\partial_{X_1} A}{\cosh(kh)}\cosh[k(z+h)].
\end{aligned}$$

So, equating the coefficients of the like functions of z (hyperbolic sines and cosines), we get

$$M = 0 \qquad \text{and} \qquad N = -\frac{g}{\omega k}\frac{\partial_{X_1} A}{\cosh(kh)} = -\frac{\omega}{k^2}\frac{\partial_{X_1} A}{\sinh(kh)}.$$

As a result,

$$\varphi_{21,p} = -\frac{\omega}{k^2}\frac{\partial_{X_1} A}{\sinh(kh)}\{k(z+h)\}\sinh[k(z+h)]. \tag{24}$$

The complex amplitude φ_{22}, onwards, is determined via the solution of

$$\begin{cases}
\partial_{zz}\varphi_{22} - 4k^2\varphi_{22} = 0, & -h < z < 0 \\
\partial_z \varphi_{22} = 0, & z = -h \\
g\partial_z\varphi_{22} - 4\omega^2\varphi_{22} = j\dfrac{g^2k^2}{\omega}\dfrac{3A^2}{\cosh^2(kh)}, & z = 0.
\end{cases} \tag{15b,iv}$$

As before, we once more have to deal with an inhomogeneous boundary value problem. This time, though, the differential operator is not the same as in the other instances. Thus, we have to examine whether or not the respective homogeneous problem admits nontrivial solutions and proceed accordingly. In that direction, the latter's general solution is

$$\varphi_{22,h} = C_1 e^{2kz} + C_2 e^{-2kz},$$

where $C_{1,2} = C_{1,2}(X_{\{\geq 1\}}, Y_{\{\geq 1\}}; T_{\{\geq 1\}})$. Fitting the condition at the bottom results in

$$\left[\partial_z \varphi_{22,h}\right]_{z=-h} = 0 \iff 2kC_1 e^{-2kh} - 2kC_2 e^{2kh} = 0 \iff C_2 = C_1 e^{-4kh}.$$

So,

$$\varphi_{22,h} = C_1 e^{-2kh}\left(e^{2k(z+h)} + e^{-2k(z+h)}\right) \iff \varphi_{22,h} = C_{22}\cosh[2k(z+h)],$$

where, again, $C_{22} = C_{22}(X_{\{\geq 1\}}, Y_{\{\geq 1\}}; T_{\{\geq 1\}})$. Of course, $\varphi_{22,h}$ must also satisfy the homogeneous boundary condition of the free surface. Namely, for a nontrivial solution it is necessary that

$$\left[g\partial_z \varphi_{22,h} - 4\omega^2 \varphi_{22,h}\right]_{z=0} = 0 \iff \omega^2 = gk\frac{\tanh(2kh)}{2}.$$

But that is opposed to Eq. (18), which must remain valid [(Mei, Stiassnie, and Yue 2005)]. As a consequence, it is not possible for the homogeneous problem to admit nontrivial solutions. Due to that, Fredholm's alternative theorem states that the problem of Eq. (15b,iv) has a unique solution for each choice of data. Here the forcing is zero, which means that

$$\varphi_{22} = C_{22}\cosh[2k(z+h)].$$

To satisfy, subsequently, the free surface's boundary condition, it must be true that

$$\left[g\partial_z \varphi_{22} - 4\omega^2 \varphi_{22}\right]_{z=0} = j\frac{g^2k^2}{\omega}\frac{3A^2}{\cosh^2(kh)} \iff$$

$$\iff C_{22}\{2gk\sinh(2kh) - 4\omega^2\cosh(2kh)\} = j\frac{g^2k^2}{\omega}\frac{3A^2}{\cosh^2(kh)}.$$

Exploiting, in addition, Eq. (18), and the identities

$$\sinh(2kh) = 2\sinh(kh)\cosh(kh)$$

and

$$\cosh(2kh) = \sinh^2(kh) + \cosh^2(kh),$$

the above expression is shaped as

$$-4\omega^2\sinh^2(kh)C_{22} = j\frac{g^2k^2}{\omega}\frac{3A^2}{\cosh^2(kh)} \iff C_{22} = -j\frac{3}{4}\omega\frac{A^2}{\sinh^4(kh)}.$$

Consequently,

$$\varphi_{22} = -j\frac{3}{4}\omega\frac{A^2}{\sinh^4(kh)}\cosh[2k(z+h)]. \tag{25}$$

Given the above complex amplitudes, the combination of Eqs. (21), (24) & (25), while remembering Eq. (13), yields

$$\varphi_2 = C_{20} - \left\{\left(j\frac{g}{\omega}A_2\frac{\cosh[k(z+h)]}{\cosh(kh)}\right.\right.$$

$$
\begin{aligned}
&+\frac{\omega}{k}(z+h)\partial_{X_1}A\frac{\sinh[k(z+h)]}{\sinh(kh)}\Bigg)e^{j(kX_0-\omega T_0)}+(*)\Bigg\} \\
&-\left\{j\frac{3}{4}\omega\frac{A^2}{\sinh^4(kh)}\cosh[2k(z+h)]e^{j2(kX_0-\omega T_0)}+(*)\right\}.
\end{aligned} \tag{26}
$$

As for the elevation η_2, inserting Eqs. (20) & (26) into the free–surface condition of Eq. (11b), we obtain

$$
\begin{aligned}
\eta_2 = &-\frac{1}{2g}\partial_{T_1}C_{10}-\frac{k}{\sinh(2kh)}A\bar{A} \\
&+\left(A_2-j\tanh(kh)h\partial_{X_1}A+j\frac{1}{\omega}\partial_{T_1}A\right)e^{j(kX_0-\omega T_0)} \\
&+\frac{k\cosh(kh)\{2\cosh^2(kh)+1\}}{2\sinh^3(kh)}A^2e^{j2(kX_0-\omega T_0)}+(*).
\end{aligned} \tag{27}
$$

At this point, there remains the treatment of the boundary value problems of the third order of the perturbation hierarchy. It should be noted that, now, the primary goal is not to actually find the amplitudes $\{\varphi_{3m}\}$. Our main interest, instead, lies in deriving the appropriate solvability conditions that will allow as to fully determine the asymptotic expansions $\tilde{\Phi}$ and $\tilde{\eta}$ up to $O(\varepsilon^2)$ and render them uniformly valid.

Before proceeding with the derivation of those conditions, it is very important to highlight some laborsaving facts. In particular, as seen in Eq. (26), there exists an arbitrary amplitude A_2 as part of the general solution of the problem (15b,ii). The same equation contains, additionally, the arbitrary function C_{20}, which constitutes the solution of the problem (15b,i). However, using Eqs. (20) & (26) into Eq. (4), it can be easily seen that $\tilde{\Phi}$ contains a total mean-flow component

$$\tilde{C} = C_{10}+\varepsilon C_{20}$$

and a total amplitude of the fundamental harmonic

$$\tilde{A} = A+\varepsilon A_2 .$$

One can write the fields $\tilde{\Phi}$ and $\tilde{\eta}$, and their solvability conditions (Eq. (23) and the ones to be determined from $O(\varepsilon^3)$), in terms of $\tilde{C}$ and $\tilde{A}$. Then, using asymptotic arguments, it can be revealed that, after all, the influence of A_2 and C_{20}, up to the desired $O(\varepsilon^2)$, is included in A and C_{10}, respectively ([4]) [(Nayfeh 2008; Holmes 2012)]. Consequently, from now on we will be taking

$$A_2 = C_{20} = 0 . \tag{28}$$

([4]) Specifically, using $\tilde{A}$ and $\tilde{C}$, $\tilde{\Phi}$, $\tilde{\eta}$ and their solvability conditions, up to $O(\varepsilon^2)$, acquire the same forms as before, with the difference that A_2 and C_{20} vanish and, additionally, $\tilde{A}$ takes the place of A. Equivalently, in the existing equations, one can set A_2 and C_{20} equal to zero and keep A as it is (with the understanding that, in essence, it expresses $\tilde{A}$).

Moving forward, with the complex amplitudes $\{\varphi_{1m}\}$ and $\{\varphi_{2m}\}$ known, the boundary value problem that governs φ_{30} is shaped as

$$\begin{cases} \partial_{zz}\varphi_{30} = -\partial_{X_1X_1}C_{10} - \partial_{Y_1Y_1}C_{10}, & -h < z < 0 \\ \partial_z\varphi_{30} = 0, & z = -h \\ g\partial_z\varphi_{30} = -\partial_{T_1T_1}C_{10} + S_1\partial_{X_1}(A\bar{A}) - S_2\partial_{T_1}(A\bar{A}), & z = 0, \end{cases} \tag{15c,i}$$

where, using partially Eq. (23), as in (Mei, Stiassnie, and Yue 2005), to reduce the length of the subsequent expressions,

$$S_1 = S_1(k,h) \equiv \frac{2\omega^3}{k\tanh^2(kh)} \tag{s1}$$

and

$$S_2 = S_2(k,h) \equiv \frac{\omega^2}{\sinh^2(kh)}. \tag{s2}$$

Obviously, the above problem is inhomogeneous and, additionally, the respective homogeneous problem has the nontrivial solution

$$\varphi_{30,h}(X_{\{\geq 1\}},Y_{\{\geq 1\}},z;T_{\{\geq 1\}}) = C_{30}(X_{\{\geq 1\}},Y_{\{\geq 1\}};T_{\{\geq 1\}}).$$

As a consequence, again via Fredholm's alternative theorem, for the problem of Eq. (15c,i) to be solvable, a suitable condition must be imposed. That arises, as before, from the use of Green's theorem, to C_{30} and φ_{30} this time. Therefore, it must be satisfied that

$$\int_{-h}^{0}\left\{\partial_{zz}\varphi_{30}C_{30} - \varphi_{30}\partial_{zz}C_{30}\right\}dz = \left[\partial_z\varphi_{30}C_{30} - \varphi_{30}\partial_z C_{30}\right]_{-h}^{0} \iff$$

$$\iff \int_{-h}^{0}(-\partial_{X_1X_1}C_{10} - \partial_{Y_1Y_1}C_{10})C_{30}\,dz = \left[\partial_z\varphi_{30}C_{30}\right]_0 \iff$$

$$\iff (-\partial_{X_1X_1}C_{10} - \partial_{Y_1Y_1}C_{10})C_{30}\,h = \left[\partial_z\varphi_{30}\right]_0 C_{30},$$

which, after some trivial calculations, becomes

$$\partial_{T_1T_1}C_{10} - gh(\partial_{X_1X_1}C_{10} + \partial_{Y_1Y_1}C_{10}) = S_1\partial_{X_1}(A\bar{A}) - S_2\partial_{T_1}(A\bar{A}). \tag{29}$$

The above equation constitutes a solvability condition that affects the derivatives of C_{10} in relation to those of A.

Keeping in mind that our objective, regarding the amplitudes of $O(\varepsilon^3)$, is only the establishment of solvability conditions, the last boundary value problem that we need to deal with is that of φ_{31}. That is because, as with the case of φ_{22}, the homogeneous versions of the boundary value problems that correspond to φ_{32} and φ_{33} do not have any nontrivial solutions, due to the dispersion relation of Eq. (18), whose validity has to be preserved. Thus, no solvability conditions are derived in the process of solving for those amplitudes.

The complex amplitude φ_{31} is defined by the problem

$$\begin{cases} \partial_{zz}\varphi_{31} - k^2\varphi_{31} = f_{31}, & -h < z < 0, \\ \partial_z\varphi_{31} = 0, & z = -h, \\ g\partial_z\varphi_{31} - \omega^2\varphi_{31} = Q_1, & z = 0, \end{cases} \tag{15c,ii}$$

where

$$f_{31} = \left(j\frac{g}{\omega}(\partial_{X_1X_1}A + \partial_{Y_1Y_1}A) - \frac{2gk}{\omega}\partial_{X_2}A \right)\frac{\cosh[k(z+h)]}{\cosh(kh)} + j2\omega\partial_{X_1X_1}A(z+h)\frac{\sinh[k(z+h)]}{\sinh(kh)}$$

and

$$Q_1 = 2g\partial_{T_2}A + j\frac{g}{\omega}\partial_{T_1T_1}A - j2\frac{\omega^2 h}{k}\partial_{T_1X_1}A + j2gkA\partial_{X_1}C_{10} - j\frac{2\omega k}{\sinh(2kh)}A\partial_{T_1}C_{10} + jS_3A^2\bar{A},$$

with

$$S_3 = S_3(k,h) \equiv \frac{\omega^3 k\cosh(kh)}{16\sinh^5(kh)}\{\cosh(4kh) + 8 - 2\tanh^2(kh)\}. \tag{s3}$$

Because of the same structure of the problems (15b,ii) and (15c,ii), the solvability condition of the latter arises from adjusting Eq. (22) as

$$\int_{-h}^{0} f_{31}\left(\frac{\cosh[k(z+h)]}{\cosh(kh)}\right)dz = \frac{1}{g}\left[g\partial_z\varphi_{31} - \omega^2\varphi_{31}\right]_0.$$

After some calculations, the above equation yields the solvability condition

$$2\partial_{T_2}A + 2C_g\partial_{X_2}A - j\frac{C_g}{k}\partial_{Y_1Y_1}A - j\frac{gh}{\omega}\partial_{X_1X_1}A + j\frac{1}{\omega}\partial_{T_1T_1}A - j2h\tanh(kh)\partial_{T_1X_1}A + j2kA\partial_{X_1}C_{10} - j\frac{k^2}{\omega\cosh^2(kh)}A\partial_{T_1}C_{10} + j\frac{1}{g}S_3A^2\bar{A} = 0. \tag{30}$$

3.4. Derivation of the NLS equation

At this point, a summary of the results obtained in the previous section should be made, for easier reference. In that spirit, the asymptotic representation of the velocity potential of slowly modulated wavetrains is

$$\tilde{\Phi} = \varepsilon \frac{C_{10}}{2} - \left(\varepsilon j \frac{g}{\omega} A \frac{\cosh[k(z+h)]}{\cosh(kh)} + \varepsilon^2 \frac{\omega}{k}(z+h)\partial_{X_1} A \frac{\sinh[k(z+h)]}{\sinh(kh)} \right) e^{j(kX_0 - \omega T_0)} - \varepsilon^2 j \frac{3}{4}\omega \frac{A^2}{\sinh^4(kh)} \cosh[2k(z+h)] e^{j2(kX_0 - \omega T_0)} + (*) + O(\varepsilon^3), \tag{4'}$$

whereas the respective representation for the free-surface elevation has the form

$$\tilde{\eta} = -\varepsilon^2 \left(\frac{1}{2g}\partial_{T_1} C_{10} + \frac{k}{\sinh(2kh)} A\bar{A} \right) + \left(\varepsilon A - \varepsilon^2 j \tanh(kh) h \partial_{X_1} A + \varepsilon^2 j \frac{1}{\omega}\partial_{T_1} A \right) e^{j(kX_0 - \omega T_0)} + \varepsilon^2 \frac{k\cosh(kh)\{2\cosh^2(kh)+1\}}{2\sinh^3(kh)} A^2 e^{j2(kX_0 - \omega T_0)} + (*) + O(\varepsilon^3). \tag{5'}$$

As for the unknown functions C_{10} and A, moreover, they are subject to the solvability conditions

$$\partial_{T_1} A + C_g \partial_{X_1} A = 0, \tag{23'}$$

$$\partial_{T_1 T_1} C_{10} - gh(\partial_{X_1 X_1} C_{10} + \partial_{Y_1 Y_1} C_{10}) = S_1 \partial_{X_1}(A\bar{A}) - S_2 \partial_{T_1}(A\bar{A}) \tag{29}$$

and

$$2\partial_{T_2} A + 2C_g \partial_{X_2} A - j\frac{C_g}{k}\partial_{Y_1 Y_1} A - j\frac{gh}{\omega}\partial_{X_1 X_1} A + j\frac{1}{\omega}\partial_{T_1 T_1} A - j2h\tanh(kh)\partial_{T_1 X_1} A + j2kA\partial_{X_1} C_{10} - j\frac{k^2}{\omega\cosh^2(kh)} A\partial_{T_1} C_{10} + j\frac{1}{g} S_3 A^2 \bar{A} = 0, \tag{30}$$

where S_1, S_2 and S_3 are known expressions of the wavenumber k and the uniform depth h (see Eqs. (s1)-(s3) of Sec. 3.3). Those conditions are exactly the means with which we can derive the evolutionary equations that govern the slowly varying functions C_{10} and A.

Several additional assumptions can be made for the wavetrains under examination, which lead to different model equations for them. One direction is to ignore the dependence of C_{10} and A on the very slow spatiotemporal scales (T_2, X_2 and Y_2) and, then, solve Eq. (23') for $\partial_{T_1} A$, substituting the subsequent result into the higher-order Eqs. (29) & (30). Afterwards, adding Eq. (23') to the latter as a final step, a new form of Eqs. (29) & (30) is obtained that corresponds to the BR system of evolutionary equations [(Benney and Roskes 1969; Mei, Stiassnie, and Yue

2005)]. Another approach is to preserve the dependence of C_{10} and A on T_2, excluding only their dependence on X_2 and Y_2. Then, setting

$$\xi = X_1 - C_g T_1 \tag{31}$$

and assuming that $A = A(\xi, Y_1; \tau)$ and $C_{10} = C_{10}(\xi, Y_1; \tau)$, where $\tau = T_2$, results in a modification of Eqs. (29) & (30), which, this time, is equivalent to the DS system of equations [(Davey and Stewartson 1974)].

In addition to the above assumptions that lead to the DS equations, let us now ignore the dependence of C_{10} and A on Y_1 as well, so that no modulation occurs alongside the y axis. As a consequence, the aforementioned slowly varying functions acquire the forms

$$A = A(\xi;\tau) \qquad \text{and} \qquad C_{10} = C_{10}(\xi;\tau). \tag{32}$$

It should be noted that the motivation for the introduction of the variable ξ emerges from Eq. (23'), which states that the modulational disturbance of the weakly nonlinear, narrow-banded wavetrains propagates with nearly ([1]) C_g [(Hasimoto and Ono 1972)]. Establishing, besides, Eqs. (32) leads to the immediate satisfaction of Eq. (23'). Taking into account those considerations, and using the chain rule, it follows that ([2])

$$\partial_{X_1} Q = \frac{\partial Q}{\partial \xi}\frac{\partial \xi}{\partial X_1} = \partial_\xi Q, \tag{33a}$$

and

$$\partial_{T_1} Q = \frac{\partial Q}{\partial \xi}\frac{\partial \xi}{\partial T_1} = -C_g \partial_\xi Q, \tag{33b}$$

where Q denotes either C_{10} or A. Thus, using Eqs. (33) into Eqs. (29) & (30), the latter acquire the forms

$$\begin{aligned} &{C_g}^2 \partial_{\xi\xi} C_{10} - gh\partial_{\xi\xi} C_{10} = S_1 \partial_\xi (A\bar{A}) + C_g S_2 \partial_\xi (A\bar{A}) \\ &\iff ({C_g}^2 - gh)\partial_{\xi\xi} C_{10} = (S_1 + C_g S_2)\partial_\xi (A\bar{A}) \end{aligned} \tag{29'}$$

and

$$\begin{aligned} &2\partial_\tau A - j\frac{gh}{\omega}\partial_{\xi\xi} A + j\frac{1}{\omega}{C_g}^2 \partial_{\xi\xi} A + j2h\tanh(kh) C_g \partial_{\xi\xi} A \\ &+ j2kA\partial_\xi C_{10} + j\frac{k^2}{\omega\cosh^2(kh)} C_g A \partial_\xi C_{10} + j\frac{1}{g} S_3 A^2 \bar{A} = 0, \end{aligned} \tag{30'}$$

respectively. Integrating Eq. (29') once with respect to ξ, yields

$$\partial_\xi C_{10} = \frac{S_1 + C_g S_2}{{C_g}^2 - gh} |A|^2 + K(\tau),$$

$K(\tau)$ being an arbitrary function of τ, due to the ξ-integration. As a result, inserting the above into Eq. (30'), we obtain

---

([1]) To the $O(\varepsilon^2)$ approximation, within which the solvability condition of Eq. (23') occurs.

([2]) The variables (scales) are still considered independent with each other (initial assumption of the MSM). Namely, τ is independent of T_1.

$$
\begin{aligned}
&2\partial_\tau A - j\frac{gh}{\omega}\partial_{\xi\xi}A + j\frac{1}{\omega}{C_g}^2\partial_{\xi\xi}A \\
&\quad + j2h\tanh(kh)C_g\partial_{\xi\xi}A + j2kA\left(\frac{S_1 + C_g S_2}{{C_g}^2 - gh}|A|^2 + K(\tau)\right) \\
&\quad + j\frac{k^2}{\omega\cosh^2(kh)}C_g A\left(\frac{S_1 + C_g S_2}{{C_g}^2 - gh}|A|^2 + K(\tau)\right) + j\frac{1}{g}S_3 A^2\bar{A} = 0 \iff \\
\iff\ &2\partial_\tau A + j\left(\frac{1}{\omega}{C_g}^2 - \frac{gh}{\omega} + 2h\tanh(kh)C_g\right)\partial_{\xi\xi}A \\
&\quad + j\left(2k\frac{S_1 + C_g S_2}{{C_g}^2 - gh} + \frac{k^2 C_g}{\omega\cosh^2(kh)}\frac{S_1 + C_g S_2}{{C_g}^2 - gh} + \frac{1}{g}S_3\right)|A|^2 A \\
&\quad + j\left(2k + \frac{k^2 C_g}{\omega\cosh^2(kh)}\right)K(\tau)A = 0 \iff \\
\iff\ &-j\partial_\tau A + \left(\frac{1}{2\omega}{C_g}^2 - \frac{gh}{2\omega} + h\tanh(kh)C_g\right)\partial_{\xi\xi}A \\
&\quad + \left(k\frac{S_1 + C_g S_2}{{C_g}^2 - gh} + \frac{k^2 C_g}{2\omega\cosh^2(kh)}\frac{S_1 + C_g S_2}{{C_g}^2 - gh} + \frac{1}{2g}S_3\right)|A|^2 A \\
&\quad + \left(k + \frac{k^2 C_g}{2\omega\cosh^2(kh)}\right)K(\tau)A = 0.
\end{aligned}
\tag{34}
$$

The above Eq. (34) constitutes the cubic NLS equation, which can further be written in the more compact form [(Mei, Stiassnie, and Yue 2005)]

$$
-j\partial_\tau A + \alpha\partial_{\xi\xi}A + \beta|A|^2 A + \gamma A = 0, \tag{35}
$$

provided that

$$
\alpha = \frac{{C_g}^2}{2\omega} + h\tanh(kh)C_g - \frac{gh}{2\omega} = -\frac{1}{2}\frac{d^2\omega}{dk^2} = -\frac{1}{2}\frac{dC_g}{dk}, \tag{36a}
$$

$$
\beta = k\left(1 + \frac{kC_g}{2\omega\cosh^2(kh)}\right)\frac{S_1 + C_g S_2}{{C_g}^2 - gh} + \frac{1}{2g}S_3, \tag{36b}
$$

and

$$
\gamma = k\left(1 + \frac{kC_g}{2\omega\cosh^2(kh)}\right)K(\tau). \tag{36c}
$$

The parameter α is always positive, since ω'' is always negative (see (Mei, Stiassnie, and Yue 2005) and Appendix D). The sign of β, on the other hand, depends on the value of kh, affecting critically the NLS equation's predictions [(Sulem and Sulem 1999)], and its change occurs for $kh = 1.363$, which corresponds to the BFI [(Benjamin and Feir 1967)]. Moreover, introducing the transformation [(Hasimoto and Ono 1972; Mei, Stiassnie, and Yue 2005)]

$$
A(\xi;\tau) = B(\xi;\tau)\exp\left(-j\int\gamma\, d\tau\right), \tag{37}
$$

and using it into Eq. (35), the same equation can also be expressed as

$$-j\partial_{\tau}B+\alpha\partial_{\xi\xi}B+\beta\left|B\right|^{2}B=0, \tag{38}$$

in terms of the redefined amplitude $B(\xi;\tau)$.

Introducing an appropriate transformation of the complex amplitude, which essentially means to write $A(\xi;\tau)$ or $B(\xi;\tau)$ in polar form, the NLS equation can also be formulated in terms of real functions, as a coupled system involving the magnitude and phase of that amplitude. For more on that, the interested reader is referred to (Mei, Stiassnie, and Yue 2005).

The use of the NLS equation in the study of wavetrains, quantitatively or qualitatively, is out of the scope of the present thesis. Though, extensive relevant information is provided in the given literature of Sec. 3.1.

Chapter 4

The Averaged Variational Principle in weakly nonlinear, narrow-banded wavetrains

"I shall be telling this with a sigh
Somewhere ages and ages hence:
Two roads diverged in a wood, and I-
I took the one less traveled by,
And that has made all the difference."

ROBERT FROST - The Road Not Taken

4.1. Introduction

In Ch. 3 we derived, under various assumptions, the evolutionary equations that govern the slow modulation of weakly nonlinear, narrow-banded wavetrains, by using the MSM. That method, however, is not the only one that can be implemented for the study of such wavetrains. Another, as also mentioned in Sec. 1.4, is Whitham's AVP.

The question arises, therefore, as to how the problem is viewed in terms of the latter method and how its results compare to those obtained otherwise. In other words, the matter in regard to the connection between the two methods is set.

Except for the relevant literature of Sec. 1.4, many other contributions have been made throughout the years on the application of the AVP to the WWP ([1]). (Lighthill, 1967) used the AVP to study various special cases. (Whitham, 1967), concurrently, applied his method to linear and slowly modulated wavetrains, examined in more detail the derived evolutionary equations and, further, used the AVP to investigate the stability of periodic waves, relating his findings to those of (Benjamin and Feir, 1967). Subsequently, (Bretherton, 1968) studied various cases of wavetrains propagating in inhomogeneous moving media, whereas (Simmons, 1969) dealt with weak, resonant wave interactions, paying special attention to the interaction of capillary-gravity waves. (Chu and Mei, 1970), on the other hand, were the first to introduce the dispersive term now known as "Chu-Mei quotient" and noted the need for its inclusion to the theory of Whitham. Not much later, (Hayes, 1973) extended the results of Whitham to the case of waves that propagate in two spatial dimensions, while (Dysthe, 1974) slightly modified the AVP, as it had already been indicated by Whitham, to remedy the inadequacies of the latter's approach in regard to the BFI. Until then, however, the precise connection between the AVP and other formal perturbation methods, in the context of the derivation of evolutionary equations such as the NLS, had not been clarified completely. Significant progress in that direction was made by

([1]) The AVP has also been applied to problems of other scientific areas, but that is out of our scope.

(Yuen and Lake, 1975, 1982), who managed to derive the NLS equation for waves in deep water, via the AVP, by using an appropriate ansatz, for the velocity potential, that contained derivatives of the amplitude. Since then, the AVP has been applied to various other cases. For example, (Jimenez and Whitham, 1976) modified it to study problems that include small dissipation, (Peregrine and Thomas, 1979) investigated the evolution of wavetrains on large-scale currents, (Easwaran, 1986) dealt with capillary-gravity Stokes waves in water of arbitrary depth and (Kirby, 1986), considering Stokes waves as well, examined their gradual reflection in varying seabed topography. In addition, (Bhakta, 1988) used the AVP to derive a nonlinear dispersion relation for the NLS. Years later, (Sedletsky, 2012), introducing some improvements to the approach of Whitham, concluded to the use of appropriate ansatzes that allowed him to derive the NLS equation for water in arbitrary depth, generalizing essentially the work of Yuen and Lake. In particular, he enriched Whitham's trial functions by considering corrections that allow for phase shifting and, also, by taking into account the slow spatiotemporal variation of all the amplitudes that are included in the ansatzes, and not only the variation of the free-surface elevation's fundamental amplitude. Subsequently, (Sedletsky, 2013) refined his earlier work, while (Sedletsky, 2015) reimplemented it using complex amplitudes. That allowed (Sedletsky, 2016) to derive the DS system of equations in terms of the AVP.

Despite the notable amount of contributions, regarding the study of the WWP via the AVP, up to now, and to our knowledge, there seems to exist a yet unresolved issue, which, in a sense, renders the method insufficient. As is known (see Sec. 1.4 and Ch. 2), the implementation of the AVP requires the use of appropriate ansatzes for the fields that govern the wavetrains under examination. But, to obtain the averaged Lagrangian for waves in water of either infinite or arbitrary depth, an integration with respect to the vertical coordinate is involved. Thus, the apparent need for the a priori knowledge of the velocity potential's vertical dependence arises. To handle that matter, (Whitham, 1967, 1974), among others, utilized Stokes' expansions, while (Yuen and Lake, 1975, 1982) used an ansatz, in which there was already embedded a suitable vertical structure. Similarly, (Sedletsky, 2012, 2013, 2015, 2016) introduced a trial function for the potential, where the dependence on the vertical coordinate was inspired by the results of (Slunyaev, 2005), which were obtained by using the MSM. In other words, before proceeding with the actual implementation of the AVP, it seems that the vertical dependence has to be already known by means of an "external" source.

The above matter has, obviously, a negative impact on the AVP, as it renders it nonautonomous and restricts its applicability. Dealing, though, with the problem of uniform wavetrains (see Ch. 2), we found out that, in that case at least, the AVP is in truth capable of yielding the vertical dependence by considering, first, the respective "vertical problem". It is our intention, therefore, to examine whether or not that is true for slowly modulated wavetrains as well and, further, to evaluate the outcome of that approach, in comparison to the findings of Sedletsky, which coincide with the established results obtained via perturbation methods such as the MSM (see, also, Ch. 3).

4.2. Derivation of the NLS equation

In what follows, we implement the AVP in the case of weakly nonlinear and narrow-banded (i.e. slowly modulated) wavetrains. We do so for waves centered around a carrier wave of wave-number k and frequency ω, over a seabed of uniform depth h.

4.2.1. Prerequisites and assumptions for the unknowns of the problem

To utilize the AVP, we need a variational formulation of the problem at hand, which, in contrast with that of Ch. 2, is not steady. This time, therefore, we consider the general variational formulation of Sec. 1.2, for the WWP, which reads

$$\delta\mathscr{S}[\eta,\Phi;\delta\eta,\delta\Phi] = \delta_\eta \mathscr{S}[\eta,\Phi;\delta\eta] + \delta_\Phi \mathscr{S}[\eta,\Phi;\delta\Phi] = 0, \quad (1)$$

$\forall\, (\delta\eta,\delta\Phi) \in \mathcal{S}_{adm}$, where the action functional is

$$\mathscr{S}[\eta,\Phi] = \int_T \int_X \mathcal{L}[\eta(x;t),\Phi(x,\bullet;t)]\,dx\,dt \quad (2)$$

and, regarding the Lagrangian density,

$$\mathcal{L} = \int_{-h}^{\eta} \left\{ \frac{\partial\Phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right] + g z \right\} dz. \quad (3)$$

As usual, η denotes the free-surface elevation and Φ the velocity potential. Moreover, $X \subseteq \mathbb{R}$ is the horizontal region of interest of the fluid domain and, respectively, $T = [t_0, t_1]$ the time interval within which the problem is studied.

Before proceeding with the determination of appropriate ansatzes for the fields η and Φ, which is also required by the method, a remark should be noted first. As is evident, three *smallness* parameters (parameters of $o(1)$,that is) occur in the problem of narrow-banded wavetrains of weak nonlinearity, due to (see (Mei, Stiassnie, and Yue 2005), (Sedletsky 2012, 2013) and, also, Ch. 3):

- the assumption of small wave motion (small steepness),
- the narrowness of the spectrum, around the supposed carrier wave, and
- the introduction of slow spatiotemporal scales, in which the waves are modulated.

Generally, the relative values of those parameters depend on the proper balancing between the nonlinearity, the dispersion and the spectrum narrowness [(Debnath 1994; Ablowitz 2011)]. Referring, however, to Ch. 3, and assuming consistency between the MSM and the AVP, we express them all via the same perturbation parameter ε ([1]) [(Sedletsky 2012, 2013)].

Returning to the matter of the wave fields' trial functions, we adopt as a basis the general form of the Fourier perturbation expansions of Sec. 2.2 (see Eqs. (4) & (5)). Yet, to comply with the nature of the problem, we generalize them allowing for the slow spatiotemporal modulation of the wave parameters. Namely, the unknown Fourier "coefficients" are now slowly varying function in space and time. That is not the only generalization we introduce, though. We additionally

([1]) In deriving the NLS equation via the MSM, the balancing of the various effects is achieved by using a single perturbation parameter. For more details, see also Footnote 1 of Sec. 3.3.

allow for possible phase shifts at higher orders ([2]). Thus, dropping the use of dimensionless unknowns and keeping terms up to $O(\varepsilon^2)$, we conclude to the trial functions

$$\tilde{\eta}(\theta;\varepsilon x,\varepsilon t) = \varepsilon^2\zeta(\varepsilon x,\varepsilon t) + \varepsilon a_{11}(\varepsilon x,\varepsilon t)\cos\theta + \varepsilon^2\sum_{j=1}^{2}\{a_{2j}(\varepsilon x,\varepsilon t)\cos j\theta + \hat{a}_{2j}(\varepsilon x,\varepsilon t)\sin j\theta\} + O(\varepsilon^3) \tag{4}$$

and

$$\tilde{\Phi}(\theta,z;\varepsilon x,\varepsilon t) = \varepsilon\psi(\varepsilon x,\varepsilon t) + \varepsilon K_{11}(z,\varepsilon x,\varepsilon t)b_{11}(\varepsilon x,\varepsilon t)\sin\theta + \varepsilon^2\sum_{j=1}^{2}K_{2j}(z,\varepsilon x,\varepsilon t)\{b_{2j}(\varepsilon x,\varepsilon t)\sin j\theta + \hat{b}_{2j}(\varepsilon x,\varepsilon t)\cos j\theta\} + O(\varepsilon^3). \tag{5}$$

Those ansatzes generalize, in essence, the improved approach of (Sedletsky 2012, 2013). Evidently, $\{a,\hat{a}\}_{\{ij\}}$ and $\{b,\hat{b}\}_{\{ij\}}$ constitute the amplitudes of the harmonics, some of which may be phase-shifted, while the unknown vertical functions $K_{\{ij\}}$ may generally depend on the slow spatiotemporal scales as well. The reason for this is that, since the other unknowns are slowly varying functions of x and t, the possible presence of such terms in the EL equations of the vertical functions cannot be excluded. As for the phase θ, given the narrowness of the spectrum around the values (k,ω), it reads [(Yuen and Lake 1975; Sedletsky 2012, 2013)]

$$\theta(x,t) = kx - \omega t + \varepsilon\tilde{\theta}(x,t), \tag{6}$$

where for the function $\tilde{\theta}$ we impose the constraint that its derivatives experience slow spatiotemporal variations. Namely,

$$\tilde{\theta}_x = \tilde{\theta}_x(\varepsilon x,\varepsilon t) \quad \text{and} \quad \tilde{\theta}_t = \tilde{\theta}_t(\varepsilon x,\varepsilon t). \tag{7}$$

As a consequence, we end up with the generalized wavenumber

$$\theta_x = \theta_x(\varepsilon x,\varepsilon t) = k + \varepsilon\tilde{\theta}_x(\varepsilon x,\varepsilon t) \tag{8a}$$

and the generalized wave frequency

$$\theta_t = \theta_t(\varepsilon x,\varepsilon t) = -\omega + \varepsilon\tilde{\theta}_t(\varepsilon x,\varepsilon t). \tag{8b}$$

Thus, the x,t-derivatives of $\tilde{\theta}$ act as slow modulations of the basic values k and ω, respectively ([3]). Regarding ζ, it constitutes a second-order, wave-induced mean elevation, whereas

([2]) The same freedom, with regards to the effect of phase shifting, can be considered in the case of the uniform wavetrains of Ch. 2, too. Though, it turns out that there do not occur any phase shifts in that instance and, as a result, the additional coefficients just vanish. First-order phase shifts can further be introduced as well. The phase difference between the respective terms of η and Φ, however, is always that of the linear case ($\pi/2$). So, the use of $\cos(\bullet)$ in the first-order approximation of the one field induces the presence of $\sin(\bullet)$ in that of the other.

([3]) Alternatively, and in a more straightforward manner, introducing the functions $\tilde{k}(\varepsilon x,\varepsilon t)$ and $\tilde{\omega}(\varepsilon x,\varepsilon t)$, one can express θ as

$$\theta(x,t) = [k+\varepsilon\tilde{k}(\varepsilon x,\varepsilon t)]x - [\omega+\varepsilon\tilde{\omega}(\varepsilon x,\varepsilon t)]t = kx - \omega t + \varepsilon[\tilde{k}(\varepsilon x,\varepsilon t)x - \tilde{\omega}(\varepsilon x,\varepsilon t)t].$$

But, then,

$$\theta_x = \theta_x(\varepsilon x,\varepsilon t) = k + \varepsilon\tilde{k}(\varepsilon x,\varepsilon t) + \varepsilon[\varepsilon x\tilde{k}_{\varepsilon x}(\varepsilon x,\varepsilon t) - \varepsilon t\tilde{\omega}_{\varepsilon x}(\varepsilon x,\varepsilon t)] = k + \varepsilon f_1(\varepsilon x,\varepsilon t)$$

ψ is a function that incorporates, in this case of nonuniform wavetrains, the modulated extensions of both the wave-induced mean flow and the Bernoulli constant ([4]) [(Whitham 1967, 1974; Dingemans 1997; Sedletsky 2012, 2013)]. The absence of other like terms is owed to the exploitation of the results of Ch. 2, where the first-order elements of the mean elevation, the Bernoulli constant and the wave-induced mean flow vanish trivially ([5,6]).

4.2.2. Implementation of the AVP

At this point, given the problem's variational formulation and the above ansatzes for its unknown fields, we are able to follow the procedure dictated by the AVP. As in the case of Stokes waves (Ch. 2) though, before addressing the total problem, first we have to deal with the complicacy of the unknown vertical dependence of $\tilde{\Phi}$, via the functions $K_{\{ij\}}$. That necessity arises, as seen before, from the fact that, without that knowledge, the z-integration contained in the Lagrangian $\mathcal{L}$, Eq. (3), can't be carried out. Accordingly, unless the presence of z in $K_{\{ij\}}$ is known explicitly, it is not possible to move on with the study of the desired wavetrains.

The vertical problem

To overcome that difficulty, we initially consider the vertical problem, i.e. the problem of finding the functions $K_{\{ij\}}$, independently (see, also, Sec. 2.2). Therefore, viewing η merely as a parameter of the integration domain, we introduce the vertical Lagrangian

$$\mathcal{L}_{vert} = \frac{\partial \Phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right] + g\,z\,. \tag{9}$$

Inserting Eq. (5) into Eq. (9), and averaging the latter with respect to the phase function, we obtain the averaged vertical Lagrangian

$$\bar{\mathcal{L}}_{vert}(z;\varepsilon x,\varepsilon t) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{L}_{vert}(\theta,z;\varepsilon x,\varepsilon t)\,d\theta\,. \tag{10}$$

During the integration over θ, the implicit assumption is made that the functions that vary slowly in x and t experience very little change within a period of the fast oscillations of the

---

and a similar result applies for θ_t. That is, $\theta_t = \theta_t(\varepsilon x,\varepsilon t) = -\omega + \varepsilon f_2(\varepsilon x,\varepsilon t)$. Setting, therefore,

$$\tilde{\theta}(x,t) = \tilde{k}(\varepsilon x,\varepsilon t)x - \tilde{\omega}(\varepsilon x,\varepsilon t)t\,,$$

those two ways of defining the phase are equivalent. However, leaving $\tilde{\theta}$ completely arbitrary results in an unknown parameter less and, also, in more compact relations [(Sedletsky 2012, 2013)].

([4]) As is known, the velocity potential of periodic waves can be written in an alternative form that contains the Bernoulli constant (see, also, Eq. (13, Sec. 1.3)). Then, in the case of slowly modulated (nearly periodic) wavetrains, allowing additionally for wave-induced mean flow, instead of considering terms like

$$\tilde{\psi}(\varepsilon x,\varepsilon t)x - \tilde{r}(\varepsilon x,\varepsilon t)t\,,$$

at each order of the potential's ansatz, we equivalently use a single function $\psi(\varepsilon x,\varepsilon t)$, as with $\tilde{\theta}$.

([5]) At first order, the linear theory is recovered, which does not include those (higher-order) effects [(Whitham 1967, 1974; Debnath 1994)].

([6]) The Lagrangian contains only the derivatives of the potential, and not the potential itself, as only them have physical meaning. Upon differentiation of the potential with respect to x and t, using the chain rule, the derivatives ψ_x and ψ_t appear as second-order components (and not as first-order ones, like ψ).

wave motion. Accordingly, while the integration is carried out, those slowly varying functions are viewed approximatively as constants [(Whitham 1974; Jeffrey and Kawahara 1982)]. That is an element of asymptotic nature, which characterizes the AVP when nearly periodic (instead of exactly periodic) waves are studied. After some calculations, and omitting the arguments of the various functions in an attempt to enhance the readability of the subsequent equations, Eq. (10) can be written as

$$\bar{\mathcal{L}}_{vert} = g z + \varepsilon^2 \bar{\mathcal{L}}_{vert,2} + \varepsilon^3 \bar{\mathcal{L}}_{vert,3} + \varepsilon^4 \bar{\mathcal{L}}_{vert,4} + O(\varepsilon^5), \tag{11}$$

where

$$\bar{\mathcal{L}}_{vert,2} = \psi_{\hat{t}} + \frac{1}{4} b_{11}{}^2 (k^2 K_{11}{}^2 + K_{11,z}{}^2), \tag{11a}$$

$$\bar{\mathcal{L}}_{vert,3} = \frac{1}{2} b_{11} \{ k b_{11} K_{11}{}^2 \tilde{\theta}_{\hat{x}} + b_{21} (k^2 K_{11} K_{21} + K_{11,z} K_{21,z}) \} \tag{11b}$$

and

$$\begin{aligned} \bar{\mathcal{L}}_{vert,4} = \frac{1}{4} \{ & 4 k b_{11} b_{21} K_{11} K_{21} \tilde{\theta}_{\hat{x}} + b_{11}{}^2 K_{11}{}^2 \tilde{\theta}_{\hat{x}}{}^2 - 2 k \hat{b}_{21} K_{11} K_{21} b_{11,\hat{x}} \\ & + K_{11}{}^2 b_{11,\hat{x}}{}^2 + 2 \psi_{\hat{x}}{}^2 + b_{11}{}^2 K_{11,\hat{x}}{}^2 + 2 b_{11} [- k \hat{b}_{21} K_{21} K_{11,\hat{x}} \\ & + K_{11} (k K_{21} \hat{b}_{21,\hat{x}} + b_{11,\hat{x}} K_{11,\hat{x}} + k \hat{b}_{21} K_{21,\hat{x}})] \\ & + b_{21}{}^2 (k^2 K_{21}{}^2 + K_{21,z}{}^2) + \hat{b}_{21}{}^2 (k^2 K_{21}{}^2 + K_{21,z}{}^2) \\ & + (b_{22}{}^2 + \hat{b}_{22}{}^2)(4 k^2 K_{22}{}^2 + K_{22,z}{}^2) \}. \end{aligned} \tag{11c}$$

Generally, after the averaging of the Lagrangian density, only slowly varying functions remain, regarding the variables x and t. Thus, in the above equations, and throughout the remainder of this chapter, the notation $\hat{x}$, $\hat{t}$ is used for the slow scales εx and εt. Namely, the x,t-differentiations contained in Eqs. (11) are differentiations in terms of the respective slow variables. With $\bar{\mathcal{L}}_{vert}$ known, the vertical action functional is shaped as

$$\mathscr{S}_{vert}[K_{\{ij\}}] = \int_T \int_X \int_{-h}^{\eta} \bar{\mathcal{L}}_{vert} (K_{\{ij\}}(z,\hat{x},\hat{t}), K_{\{ij\},z}(z,\hat{x},\hat{t}), \ldots) \, dz \, d\hat{x} \, d\hat{t} \,, \tag{12}$$

where it should be reminded that, in the context of the assumed independent vertical problem, the free-surface elevation η is not treated as an unknown field. In that way, we are led to the variational equation

$$\begin{aligned} \delta \mathscr{S}_{vert} &= \delta \int_T \int_X \int_{-h}^{\eta} \bar{\mathcal{L}}_{vert} \, dz \, d\hat{x} \, d\hat{t} = 0 \iff \\ & \iff \sum_{i=2}^{4} \varepsilon^i \left(\delta \int_T \int_X \int_{-h}^{\eta} \bar{\mathcal{L}}_{vert,i} \, dz \, d\hat{x} \, d\hat{t} \right) = 0, \end{aligned} \tag{13}$$

$\forall \, (\delta K_{\{ij\}}) \in \mathcal{S}_{adm}$. Hence, as in Sec. 2.2, the asymptotic character of the assumed trial function for the velocity potential results, in the context of the AVP, in a variational formulation that, in fact, consists of a perturbation hierarchy of variational equations. Naturally, we have to deal with that hierarchy sequentially. Starting with the lowest-order variational equation

$$\delta \int_T \int_X \int_{-h}^{\eta} \bar{\mathcal{L}}_{vert,2}\, dz\, d\hat{x}\, d\hat{t} = 0, \tag{13,i}$$

and taking its δK_{11} variation, we obtain

$$\delta K_{11}: \qquad \frac{\partial}{\partial z}\left(\frac{\partial \bar{\mathcal{L}}_{vert,2}}{\partial K_{11,z}}\right) - \frac{\partial \bar{\mathcal{L}}_{vert,2}}{\partial K_{11}} = 0 \iff K_{11,zz} - k^2 K_{11} = 0. \tag{14}$$

The solution of the above differential equation leads to

$$K_{11}(z,\hat{x},\hat{t}) = A_1(\hat{x},\hat{t})e^{kz} + A_2(\hat{x},\hat{t})e^{-kz}, \tag{15}$$

where A_1 and A_2 are arbitrary functions that vary slowly in x and t. Substituting Eq. (15) into the higher orders of Eq. (13), it is easy to verify that the δK_{21} variation of the next in line variational equation

$$\delta \int_T \int_X \int_{-h}^{\eta} \bar{\mathcal{L}}_{vert,3}\, dz\, d\hat{x}\, d\hat{t} = 0 \tag{13,ii}$$

produces an EL equation that is satisfied identically. Thus, there remains the last equation of the variational perturbation hierarchy

$$\delta \int_T \int_X \int_{-h}^{\eta} \bar{\mathcal{L}}_{vert,4}\, dz\, d\hat{x}\, d\hat{t} = 0. \tag{13,iii}$$

Considering, first, its variation with respect to K_{21}, the EL equation

$$\begin{aligned}\delta K_{21}: \qquad &\frac{\partial}{\partial z}\left(\frac{\partial \bar{\mathcal{L}}_{vert,4}}{\partial K_{21,z}}\right) + \frac{\partial}{\partial \hat{x}}\left(\frac{\partial \bar{\mathcal{L}}_{vert,4}}{\partial K_{21,\hat{x}}}\right) - \frac{\partial \bar{\mathcal{L}}_{vert,4}}{\partial K_{21}} = 0 \iff \\ &\iff K_{21,zz} - k^2 K_{21} = -f_1 e^{kz} - f_2 e^{-kz}\end{aligned} \tag{16}$$

is yielded, where

$$f_1 = 2k \frac{b_{11}\hat{b}_{21} A_{1,\hat{x}} + A_1(-b_{11} b_{21} \tilde{\theta}_{\hat{x}} + \hat{b}_{21} b_{11,\hat{x}})}{b_{21}{}^2 + \hat{b}_{21}{}^2} = f_1(\hat{x},\hat{t}) \tag{17a}$$

and

$$f_2 = 2k \frac{b_{11}\hat{b}_{21} A_{2,\hat{x}} + A_2(-b_{11} b_{21} \tilde{\theta}_{\hat{x}} + \hat{b}_{21} b_{11,\hat{x}})}{b_{21}{}^2 + \hat{b}_{21}{}^2} = f_2(\hat{x},\hat{t}), \tag{17b}$$

under, of course, the assumption that the amplitudes b_{21} and $\hat{b}_{21}$ do not vanish simultaneously. With the help of the method of undetermined coefficients [(Boyce and DiPrima 2012)], the solution of Eq. (16) turns out to be

$$K_{21}(z,\hat{x},\hat{t}) = e^{kz}\{B_1(\hat{x},\hat{t}) + zB_2(\hat{x},\hat{t})\} + e^{-kz}\{B_3(\hat{x},\hat{t}) + zB_4(\hat{x},\hat{t})\}, \tag{18}$$

in which $B_{\{i\}}$ are arbitrary, slowly varying functions. To be precise, in the actual solution via the method of undetermined coefficients, in the place of each $B_{\{i\}}$ a combination of the forcing terms of Eqs. (17) and the arbitrary functions of the respective homogeneous solution occur. Eq. (18) is the result of the redefinition of those arbitrary functions, given that the absorption of the various specific terms of Eqs. (17) into them has no effect on the solution of the total problem ([7]). The reason we proceed in that way is to simplify the form of the equations that follow in the next phase of the procedure, where, having determined the z dependence explicitly, the AVP is applied to the total problem of slowly modulated wavetrains. Moving on with the δK_{22} variation of Eq. (13,iii), we get

$$\delta K_{22}: \quad \frac{\partial}{\partial z}\left(\frac{\partial \bar{\mathcal{L}}_{vert,4}}{\partial K_{22,z}}\right) - \frac{\partial \bar{\mathcal{L}}_{vert,4}}{\partial K_{22}} = 0 \iff K_{22,zz} - 4k^2 K_{22} = 0\,, \tag{19}$$

whose solution is

$$K_{22}(z,\hat{x},\hat{t}) = C_1(\hat{x},\hat{t})\,e^{2kz} + C_2(\hat{x},\hat{t})\,e^{-2kz}\,, \tag{20}$$

where, again, C_1 and C_2 are arbitrary functions of the slow spatiotemporal variables. Consequently, the vertical dependence of the potential's trial function is given by Eqs. (15), (18) & (20), which are substituted into Eq. (5).

<u>The total problem</u>

At this point, we are able to return to the total problem and study it via the AVP. Thus, we first insert Eqs. (4) & (5) into Eq. (3), remembering that $K_{\{ij\}}$ are expressed as above, so that the Lagrangian density becomes

$$\mathcal{L} = \mathcal{L}[\tilde{\eta}(\theta;\hat{x},\hat{t}),\tilde{\Phi}(\theta,\bullet;\hat{x},\hat{t});\boldsymbol{P}]\,, \tag{21}$$

$\boldsymbol{P}$ being a vector of all the unknown wave parameters, in the ansatzes, that have to be found by the AVP. Namely,

$$\boldsymbol{P} = (\tilde{\theta},\zeta,\psi,a_{\{ij\}},\hat{a}_{\{ij\}},b_{\{ij\}},\hat{b}_{\{ij\}},A_{\{1,2\}},B_{\{1,2,3,4\}},C_{\{1,2\}})\,. \tag{22}$$

Then, carrying out the integration with respect to z, we subsequently introduce the averaged Lagrangian

$$\bar{\mathcal{L}}\,(\boldsymbol{P}(\hat{x},\hat{t})) = \frac{1}{2\pi}\int_0^{2\pi}\mathcal{L}(\theta,\boldsymbol{P}(\hat{x},\hat{t}))\,d\theta\,. \tag{23}$$

As in Sec. 2.2, quantities of the form

$$S_n \equiv e^{nk(h+\tilde{\eta})} = e^{nkh}\,e^{nk\tilde{\eta}} \tag{24}$$

appear within the θ-integral. So, to find the latter in terms of elementary functions, we approximate the quantities $e^{nk\tilde{\eta}}$ with their Maclaurin series, i.e.

$$e^{nk\tilde{\eta}} = \sum_{m=0}^{M}\frac{(nk)^m}{m!}\tilde{\eta}^m + O(\varepsilon^{M+1})\,, \tag{25}$$

([7]) As seen earlier (Sec. 2.2), the vertical problem aims only at finding the z-structure of the solution, whereas, knowing that structure, the determination of the unknown (and slowly varying, in this instance) terms of the ansatzes is achieved by reapplying the AVP to the total problem. Hence, since upon the reimplementation of the AVP the dependence on z is known explicitly, the form of the unknown arbitrary functions is unimportant, as, either way, the variational principle, containing all the information that defines the problem, will shape them accordingly.

exploiting the fact that $\tilde{\eta} = O(\varepsilon)$ and, also, the order separation that the perturbative nature of the problem induces. After the use of Eq. (25) into Eq. (23), the θ-integration is easily carried out, resulting in

$$\bar{\mathcal{L}} = -\frac{gh^2}{2} + \varepsilon^2 \bar{\mathcal{L}}_2 + \varepsilon^3 \bar{\mathcal{L}}_3 + \varepsilon^4 \bar{\mathcal{L}}_4 + O(\varepsilon^5), \tag{26}$$

where

$$\begin{aligned} \bar{\mathcal{L}}_2 = \frac{1}{4}\{g a_{11}{}^2 - 2\omega(A_1 + A_2) a_{11} b_{11} \\ + k(1 - e^{-2kh})(A_1{}^2 + e^{2kh} A_2{}^2) b_{11}{}^2 + 4h\psi_{\hat{t}}\}, \end{aligned} \tag{26a}$$

$$\begin{aligned} \bar{\mathcal{L}}_3 = \frac{1}{4}\Big[b_{11} \; - 2\{A_2[\omega a_{21} + k(hB_2 + B_3) b_{21}] \\ + A_1[\omega a_{21} - k(B_1 + hB_4) b_{21}]\} \\ + (A_1{}^2 + 4khA_1A_2 - A_2{}^2) b_{11} \tilde{\theta}_{\hat{x}} \\ + e^{2kh} A_2\{2k(B_3 - hB_4) b_{21} + A_2 b_{11} \tilde{\theta}_{\hat{x}}\} \\ - e^{-2kh} A_1\{2k(B_1 - hB_2) b_{21} + A_1 b_{11} \tilde{\theta}_{\hat{x}}\} \\ + 2a_{11}\{g a_{21} - \omega(B_1 + B_3) b_{21} + (A_1 + A_2) b_{11} \tilde{\theta}_{\hat{t}}\}\Big] \end{aligned} \tag{26b}$$

and

$$\begin{aligned} \bar{\mathcal{L}}_4 = \bar{\mathcal{L}}_4(\tilde{\omega}, \tilde{k}, \zeta, \psi_x, \psi_t, a_{\{ij\}}, \hat{a}_{\{ij\}}, b_{\{ij\}}, b_{11,\{x,t\}}, \hat{b}_{\{ij\}}, \hat{b}_{21,\{x,t\}}, \\ A_{\{1,2\}}, A_{\{1,2\},\{x,t\}}, B_{\{1,3\}}, B_{\{1,3\},\{x,t\}}, B_{\{2,4\}}, B_{\{2,4\},x}, C_{\{1,2\}}), \end{aligned} \tag{26c}$$

omitting its full form due to its size. Having found $\bar{\mathcal{L}}$, as a next step we introduce the corresponding action functional

$$\mathscr{S}[\boldsymbol{P}] = \int_T \int_X \bar{\mathcal{L}}(\boldsymbol{P}(\hat{x}, \hat{t})) d\hat{x} d\hat{t} \tag{27}$$

of the averaged problem, whose stationarity is supposed to yield the evolutionary equations of the unknown, slowly modulated wave parameters [(Whitham 1974; Karpman 1975; Jeffrey and Kawahara 1982; Debnath 1994)]. Accordingly, we obtain the variational equation

$$\delta\mathscr{S} = \delta \int_T \int_X \bar{\mathcal{L}} \, d\hat{x} d\hat{t} = 0 \iff \sum_{i=2}^{4} \varepsilon^i \left(\delta \int_T \int_X \bar{\mathcal{L}}_i \, d\hat{x} d\hat{t} \right) = 0, \tag{28}$$

for each admissible variation of the elements of $\boldsymbol{P}$. Once more, therefore, because of the orders that appear in $\bar{\mathcal{L}}$, we essentially end up with a perturbation hierarchy of variational equations.

The orders of the perturbation hierarchy are treated successively.

Beginning with the lowest-order equation

$$\delta \int_T \int_X \bar{\mathcal{L}}_2 \, d\hat{x} d\hat{t} = 0, \tag{28,i}$$

we derive, considering its admissible variations, the EL equations

$$\delta b_{11}: \quad \frac{\partial}{\partial \hat{x}}\left(\frac{\partial \bar{\mathcal{L}}_2}{\partial b_{11,\hat{x}}}\right)+\frac{\partial}{\partial \hat{t}}\left(\frac{\partial \bar{\mathcal{L}}_2}{\partial b_{11,\hat{t}}}\right)-\frac{\partial \bar{\mathcal{L}}_2}{\partial b_{11}} = 0 \iff$$
$$\iff b_{11} = \frac{\omega}{2k} e^{2kh}[\coth(kh)-1]\frac{A_1+A_2}{A_1{}^2+e^{2kh}A_2{}^2}a_{11}, \tag{29'}$$

$$\delta A_1: \quad \frac{\partial}{\partial \hat{x}}\left(\frac{\partial \bar{\mathcal{L}}_2}{\partial A_{1,\hat{x}}}\right)+\frac{\partial}{\partial \hat{t}}\left(\frac{\partial \bar{\mathcal{L}}_2}{\partial A_{1,\hat{t}}}\right)-\frac{\partial \bar{\mathcal{L}}_2}{\partial A_1} = 0 \iff$$
$$\iff A_1 = \frac{\omega}{2k} e^{2kh}[\coth(kh)-1]\frac{a_{11}}{b_{11}}, \tag{30',i}$$

$$\delta A_2: \quad \frac{\partial}{\partial \hat{x}}\left(\frac{\partial \bar{\mathcal{L}}_2}{\partial A_{2,\hat{x}}}\right)+\frac{\partial}{\partial \hat{t}}\left(\frac{\partial \bar{\mathcal{L}}_2}{\partial A_{2,\hat{t}}}\right)-\frac{\partial \bar{\mathcal{L}}_2}{\partial A_2} = 0 \iff$$
$$\iff A_2 = \frac{\omega}{2k}[\coth(kh)-1]\frac{a_{11}}{b_{11}} \tag{30',ii}$$

and

$$\delta a_{11}: \quad \frac{\partial}{\partial \hat{x}}\left(\frac{\partial \bar{\mathcal{L}}_2}{\partial a_{11,\hat{x}}}\right)+\frac{\partial}{\partial \hat{t}}\left(\frac{\partial \bar{\mathcal{L}}_2}{\partial a_{11,\hat{t}}}\right)-\frac{\partial \bar{\mathcal{L}}_2}{\partial a_{11}} = 0 \iff$$
$$\iff \omega = g\frac{a_{11}}{(A_1+A_2)b_{11}}, \tag{31'}$$

given that the EL equation that corresponds to the $\delta\psi$ variation is satisfied identically. From Eqs. (30'), we deduce that

$$A_1 = e^{2kh}A_2. \tag{30}$$

Using the above into Eq. (29'), we get

$$b_{11} = \frac{\omega}{2k}[\coth(kh)-1]\frac{a_{11}}{A_2}. \tag{29}$$

Hence, via Eqs. (29) & (30), Eq. (31') becomes

$$\omega^2 = gk\tanh(kh), \tag{31}$$

stating that the wavenumber and the frequency of the assumed carrier wave are connected with the linear dispersion relation. Those $O(\varepsilon^2)$ results, i.e. Eqs. (29)-(31), are substituted into the higher orders of Eq. (28) [(Sedletsky 2012, 2013)].

The next-order variational equation is

$$\delta\int_T\int_X \bar{\mathcal{L}}_3 \, d\hat{x}\, d\hat{t} = 0, \tag{28,ii}$$

in which the Lagrangian density $\bar{\mathcal{L}}_3$ is reshaped, owing to the use of Eqs. (29)-(31), as

$$\bar{\mathcal{L}}_3 = \frac{\omega}{4k^2}\{\omega[\coth(kh)+kh\,\mathrm{csch}^2(kh)]\tilde{\theta}_{\hat{x}}+2k\coth(kh)\tilde{\theta}_{\hat{t}}\}a_{11}{}^2. \tag{26b'}$$

Consequently, from the admissible variations δa_{11} and $\delta\tilde{\theta}$ of Eq. (28,ii), we are led to the EL equations

$$\delta a_{11}: \quad \frac{\partial}{\partial \hat{x}}\left(\frac{\partial \bar{\mathcal{L}}_3}{\partial a_{11,\hat{x}}}\right)+\frac{\partial}{\partial \hat{t}}\left(\frac{\partial \bar{\mathcal{L}}_3}{\partial a_{11,\hat{t}}}\right)-\frac{\partial \bar{\mathcal{L}}_3}{\partial a_{11}} = 0 \iff$$
$$\iff \tilde{\theta}_{\hat{t}} + \frac{\omega}{2k}\left(1+\frac{2kh}{\sinh(2kh)}\right)\tilde{\theta}_{\hat{x}} = 0 \tag{32'}$$

and

$$\delta\tilde{\theta}: \quad \frac{\partial}{\partial \hat{x}}\left(\frac{\partial \bar{\mathcal{L}}_3}{\partial \tilde{\theta}_{\hat{x}}}\right)+\frac{\partial}{\partial \hat{t}}\left(\frac{\partial \bar{\mathcal{L}}_3}{\partial \tilde{\theta}_{\hat{t}}}\right) = 0 \iff$$
$$\iff a_{11,\hat{t}} + \frac{\omega}{2k}\left(1+\frac{2kh}{\sinh(2kh)}\right)a_{11,\hat{x}} = 0. \tag{33'}$$

But, as is known, the quantity

$$C_g = \frac{\omega}{2k}\left(1+\frac{2kh}{\sinh(2kh)}\right) \tag{34}$$

is the group velocity of the linearized WWP [(Stoker 1957; Whitham 1974; Debnath 1994)]. Namely, after all,

$$\tilde{\theta}_{\hat{t}} + C_g\,\tilde{\theta}_{\hat{x}} = 0 \tag{32}$$

and

$$a_{11,\hat{t}} + C_g\, a_{11,\hat{x}} = 0, \tag{33}$$

meaning that the modulational disturbance of the slowly varying wavetrains under examination propagates with C_g, up to the $O(\varepsilon^3)$ approximation ([8,9]).

Moving on, there remains the last variational equation of the hierarchy

$$\delta\int_T\int_X \bar{\mathcal{L}}_4\, d\hat{x}\, d\hat{t} = 0, \tag{28,iii}$$

in which the form of $\bar{\mathcal{L}}_4$ occurs from the use of the lower-order results for the wave parameters, i.e. Eqs. (29)-(33), into Eq. (26c). In other words, the followed procedure is that, each time a variational equation of a certain order is considered, the respective Lagrangian density is up-dated with the results of the previous orders. Taking, first, into account the variations $\delta\zeta$ and $\delta\psi$ of Eq. (28,iii), we obtain

$$\delta\zeta: \quad \frac{\partial}{\partial \hat{x}}\left(\frac{\partial \bar{\mathcal{L}}_4}{\partial \zeta_{\hat{x}}}\right)+\frac{\partial}{\partial \hat{t}}\left(\frac{\partial \bar{\mathcal{L}}_4}{\partial \zeta_{\hat{t}}}\right)-\frac{\partial \bar{\mathcal{L}}_4}{\partial \zeta} = 0 \iff$$
$$\iff \zeta = -\frac{k\tanh(kh)}{4\omega^2}\left[\omega^2 \operatorname{csch}^2(kh)a_{11}{}^2 + 4\psi_{\hat{t}}\right] \tag{35}$$

and

([8]) See (Hasimoto and Ono 1972), (Sedletsky 2012, 2013) and Sec. 3.4.
([9]) Of interest is the connection with the conservation equations of Whitham [(Whitham 1974; Debnath 1994)].

$$\delta\psi: \quad \frac{\partial}{\partial\hat{x}}\left(\frac{\partial\bar{\mathcal{L}}_4}{\partial\psi_{\hat{x}}}\right)+\frac{\partial}{\partial\hat{t}}\left(\frac{\partial\bar{\mathcal{L}}_4}{\partial\psi_{\hat{t}}}\right)-\frac{\partial\bar{\mathcal{L}}_4}{\partial\psi}=0 \iff \tag{36}$$

$$\iff \zeta_{\hat{t}}+\omega\coth(kh)a_{11}a_{11,\hat{x}}+h\psi_{\hat{x}\hat{x}}=0,$$

which constitute a system of equations that governs the coupled evolution of ζ and ψ ([10]). The rest of the admissible variations, except for the variations with respect to a_{11} and $\tilde{\theta}$, lead to two systems of equations, through which we are able to determine the respective slowly varying functions. Specifically, the one system consists of the EL equations that correspond to the variations

$$\{\delta a_{21},\delta\hat{a}_{21},\delta b_{21},\delta\hat{b}_{21},\delta B_1,\delta B_2,\delta B_3,\delta B_4\}, \tag{s1}$$

while the other involves the EL equations of the variations

$$\{\delta a_{22},\delta\hat{a}_{22},\delta b_{22},\delta\hat{b}_{22},\delta C_1,\delta C_2\}. \tag{s2}$$

From the system of Eq. (s1), we get ([11])

$$\delta a_{21}: \quad \frac{\partial}{\partial\hat{x}}\left(\frac{\partial\bar{\mathcal{L}}_4}{\partial a_{21,\hat{x}}}\right)+\frac{\partial}{\partial\hat{t}}\left(\frac{\partial\bar{\mathcal{L}}_4}{\partial a_{21,\hat{t}}}\right)-\frac{\partial\bar{\mathcal{L}}_4}{\partial a_{21}}=0 \iff \tag{37'}$$

$$\iff a_{21}=\frac{k}{\omega}(1-e^{-2kh})B_1 b_{21}+\frac{1+2kh[\coth(2kh)-1]}{2k}\tilde{\theta}_{\hat{x}}a_{11},$$

$$\delta\hat{a}_{21}: \quad \frac{\partial}{\partial\hat{x}}\left(\frac{\partial\bar{\mathcal{L}}_4}{\partial\hat{a}_{21,\hat{x}}}\right)+\frac{\partial}{\partial\hat{t}}\left(\frac{\partial\bar{\mathcal{L}}_4}{\partial\hat{a}_{21,\hat{t}}}\right)-\frac{\partial\bar{\mathcal{L}}_4}{\partial\hat{a}_{21}}=0 \iff \tag{38'}$$

$$\iff \hat{a}_{21}=\left(\frac{k}{\omega}(1-e^{-2kh})\frac{B_1 b_{21}}{\tilde{\theta}_{\hat{x}}a_{11}}+\frac{1+2kh[\coth(2kh)-1]}{2k}\right)a_{11,\hat{x}},$$

$$\delta\hat{b}_{21}: \quad \frac{\partial}{\partial\hat{x}}\left(\frac{\partial\bar{\mathcal{L}}_4}{\partial\hat{b}_{21,\hat{x}}}\right)+\frac{\partial}{\partial\hat{t}}\left(\frac{\partial\bar{\mathcal{L}}_4}{\partial\hat{b}_{21,\hat{t}}}\right)-\frac{\partial\bar{\mathcal{L}}_4}{\partial\hat{b}_{21}}=0 \iff \hat{b}_{21}=-b_{21}\frac{a_{11,\hat{x}}}{\tilde{\theta}_{\hat{x}}a_{11}}, \tag{39}$$

$$\delta B_2: \quad \frac{\partial}{\partial\hat{x}}\left(\frac{\partial\bar{\mathcal{L}}_4}{\partial B_{2,\hat{x}}}\right)+\frac{\partial}{\partial\hat{t}}\left(\frac{\partial\bar{\mathcal{L}}_4}{\partial B_{2,\hat{t}}}\right)-\frac{\partial\bar{\mathcal{L}}_4}{\partial B_2}=0 \iff \tag{40}$$

$$\iff B_2=\frac{\omega e^{2kh}(\coth(kh)-1)}{2kb_{21}}\tilde{\theta}_{\hat{x}}a_{11},$$

$$\delta B_3: \quad \frac{\partial}{\partial\hat{x}}\left(\frac{\partial\bar{\mathcal{L}}_4}{\partial B_{3,\hat{x}}}\right)+\frac{\partial}{\partial\hat{t}}\left(\frac{\partial\bar{\mathcal{L}}_4}{\partial B_{3,\hat{t}}}\right)-\frac{\partial\bar{\mathcal{L}}_4}{\partial B_3}=0 \iff \tag{41'}$$

$$\iff B_3=e^{-2kh}B_1-\frac{\omega h[\coth(kh)-1]}{kb_{21}}\tilde{\theta}_{\hat{x}}a_{11}$$

([10]) Obviously, Eqs. (35) & (36) can be further combined into a single equation that involves only ψ and its derivatives. Solving that equation for ψ, ζ can be readily determined via Eq. (35).

([11]) For details, see the symbolic code of Appendix B.

and

$$\delta B_4: \quad \frac{\partial}{\partial \hat{x}}\left(\frac{\partial \bar{\mathcal{L}}_4}{\partial B_{4,\hat{x}}}\right)+\frac{\partial}{\partial \hat{t}}\left(\frac{\partial \bar{\mathcal{L}}_4}{\partial B_{4,\hat{t}}}\right)-\frac{\partial \bar{\mathcal{L}}_4}{\partial B_4}=0 \iff \iff B_4=-\frac{\omega[\coth(kh)-1]}{2kb_{21}}\tilde{\theta}_{\hat{x}}\,a_{11}. \tag{42}$$

Similarly, the solution of the system of Eq. (s2) yields

$$\delta a_{22}: \quad \frac{\partial}{\partial \hat{x}}\left(\frac{\partial \bar{\mathcal{L}}_4}{\partial a_{22,\hat{x}}}\right)+\frac{\partial}{\partial \hat{t}}\left(\frac{\partial \bar{\mathcal{L}}_4}{\partial a_{22,\hat{t}}}\right)-\frac{\partial \bar{\mathcal{L}}_4}{\partial a_{22}}=0 \iff \iff a_{22}=\frac{5\cosh(kh)+\cosh(3kh)}{\sinh^3(kh)}\frac{k}{8}a_{11}{}^2, \tag{43}$$

$$\delta \hat{a}_{22}: \quad \frac{\partial}{\partial \hat{x}}\left(\frac{\partial \bar{\mathcal{L}}_4}{\partial \hat{a}_{22,\hat{x}}}\right)+\frac{\partial}{\partial \hat{t}}\left(\frac{\partial \bar{\mathcal{L}}_4}{\partial \hat{a}_{22,\hat{t}}}\right)-\frac{\partial \bar{\mathcal{L}}_4}{\partial \hat{a}_{22}}=0 \iff \hat{a}_{22}=0, \tag{44}$$

$$\delta b_{22}: \quad \frac{\partial}{\partial \hat{x}}\left(\frac{\partial \bar{\mathcal{L}}_4}{\partial b_{22,\hat{x}}}\right)+\frac{\partial}{\partial \hat{t}}\left(\frac{\partial \bar{\mathcal{L}}_4}{\partial b_{22,\hat{t}}}\right)-\frac{\partial \bar{\mathcal{L}}_4}{\partial b_{22}}=0 \iff \iff b_{22}=\frac{3\omega e^{2kh}}{(e^{2kh}-1)^4 C_2}a_{11}{}^2, \tag{45}$$

$$\delta \hat{b}_{22}: \quad \frac{\partial}{\partial \hat{x}}\left(\frac{\partial \bar{\mathcal{L}}_4}{\partial \hat{b}_{22,\hat{x}}}\right)+\frac{\partial}{\partial \hat{t}}\left(\frac{\partial \bar{\mathcal{L}}_4}{\partial \hat{b}_{22,\hat{t}}}\right)-\frac{\partial \bar{\mathcal{L}}_4}{\partial \hat{b}_{22}}=0 \iff \hat{b}_{22}=0 \tag{46}$$

and

$$\delta C_1: \quad \frac{\partial}{\partial \hat{x}}\left(\frac{\partial \bar{\mathcal{L}}_4}{\partial C_{1,\hat{x}}}\right)+\frac{\partial}{\partial \hat{t}}\left(\frac{\partial \bar{\mathcal{L}}_4}{\partial C_{1,\hat{t}}}\right)-\frac{\partial \bar{\mathcal{L}}_4}{\partial C_1}=0 \iff C_1=e^{4kh}C_2. \tag{47}$$

The reason that no equations are obtained for A_2, C_2 and b_{21} is that their role in the ansatz of the potential (Eq. (5)) is essentially redundant. In particular, the expressions of the vertical functions $K_{\{ij\}}$, as derived by considering the vertical problem (Eqs. (15), (18) & (20)), introduce additional unknown slowly varying functions in Eq. (5) that, in fact, can be absorbed into the existing ones, without any impact on the freedom of that ansatz. Our choice to keep all those functions reflects the aforementioned absence of equations for the unnecessary ones. That is not a problem, though, at all. Upon substitution of the wave parameters into the ansatzes, the functions A_2, C_2 and b_{21} just vanish (see below), having no effect on the solution ([12,13]). However, that is not the case with B_1. Not only it is preserved in the final forms of $\tilde{\eta}$ and $\tilde{\Phi}$, but it also affects the evolutionary equations of a_{11} and $\tilde{\theta}$, which occur by considering the respective admissible variations of the latter. The exact role of B_1, and the physical meaning of the

([12]) The same happens in the study of Stokes waves via the AVP (Sec. 2.2).
([13]) Evidently, which unknown function becomes redundant is a matter of choice. For example, if Eq. (30) is solved for A_2, then A_1 will play that role and vanish, but that is not of any importance, as nothing will change with respect to the final solution. The AVP will just shape the rest of the unknowns accordingly.

additional freedom it induces, is unclear and its investigation should be a subject of future work. When that term vanishes, though, all the final results obtained via our approach coincide with those of (Sedletsky 2012, 2013). Thus, in what follows, we take

$$B_1 = 0. \tag{48}$$

Consequently, Eqs. (37'), (38') & (41') reduce to

$$a_{21} = \frac{1+2kh[\coth(2kh)-1]}{2k}\tilde{\theta}_{\hat{x}}\, a_{11}, \tag{37}$$

$$\hat{a}_{21} = \frac{1+2kh[\coth(2kh)-1]}{2k} a_{11,\hat{x}} \tag{38}$$

and

$$B_3 = -\frac{\omega h[\coth(kh)-1]}{k b_{21}}\tilde{\theta}_{\hat{x}}\, a_{11}. \tag{41}$$

As for the remaining variations δa_{11} and $\delta\tilde{\theta}$, inserting first Eqs. (37)-(48) into $\bar{\mathcal{L}}_4$, terms of the form

$$Q(\omega,k,h)\, a_{11}\, a_{11,\hat{x}\hat{x}} = Q(\omega,k,h)\{(a_{11}\, a_{11,\hat{x}})_{\hat{x}} - a_{11,\hat{x}}{}^2\}$$

appear in it. Yet, since we consider admissible variations that vanish on the boundaries, we can make the substitution

$$a_{11}\, a_{11,\hat{x}\hat{x}} \;\rightarrow\; -a_{11,\hat{x}}{}^2$$

inside the Lagrangian, without any effect on the subsequent EL equation that corresponds to the variation δa_{11} ([14]) [(Gelfand and Fomin 1963; Sedletsky 2012, 2013)]. Our gain from that action is a simplified form for $\bar{\mathcal{L}}_4$, but is nonetheless not necessary. Finally, we conclude to the EL equations

$$\delta a_{11}: \quad \frac{\partial}{\partial\hat{x}}\left(\frac{\partial\bar{\mathcal{L}}}{\partial a_{11,\hat{x}}}\right) + \frac{\partial}{\partial\hat{t}}\left(\frac{\partial\bar{\mathcal{L}}}{\partial a_{11,\hat{t}}}\right) - \frac{\partial\bar{\mathcal{L}}}{\partial a_{11}} = 0 \iff$$

$$\iff \tilde{\theta}_{\hat{t}} + C_g\tilde{\theta}_{\hat{x}} + \varepsilon\left\{\frac{1}{16}\omega k^2[8+\cosh(4kh)]\operatorname{csch}^4(kh)\, a_{11}{}^2 + k[\omega\operatorname{csch}(2kh)\zeta + \psi_{\hat{x}}] + \frac{1}{2}C_g{}'\left(\tilde{\theta}_{\hat{x}}{}^2 - \frac{a_{11,\hat{x}\hat{x}}}{a_{11}}\right)\right\} \tag{49}$$

and

$$\delta\tilde{\theta}: \quad \frac{\partial}{\partial\hat{x}}\left(\frac{\partial\bar{\mathcal{L}}}{\partial\tilde{\theta}_{\hat{x}}}\right) + \frac{\partial}{\partial\hat{t}}\left(\frac{\partial\bar{\mathcal{L}}}{\partial\tilde{\theta}_{\hat{t}}}\right) = 0 \iff$$

$$\iff a_{11,\hat{t}} + C_g\, a_{11,\hat{x}} + \varepsilon\frac{1}{2}C_g{}'(2\tilde{\theta}_{\hat{x}}\, a_{11,\hat{x}} + \tilde{\theta}_{\hat{x}\hat{x}}\, a_{11}), \tag{50}$$

([14]) Note that, if $S_1 = -Q(\omega,k,h)\, a_{11,\hat{x}}{}^2$ and $S_2 = Q(\omega,k,h)\, a_{11}\, a_{11,\hat{x}\hat{x}}$, then the respective EL equations, towards the variation δa_{11}, coincide. Namely,

$$\frac{\partial}{\partial\hat{x}}\left(\frac{\partial S_1}{\partial a_{11,\hat{x}}}\right) = -\frac{\partial^2}{\partial\hat{x}^2}\left(\frac{\partial S_2}{\partial a_{11,\hat{x}\hat{x}}}\right) - \frac{\partial S_2}{\partial a_{11}} = -2Q(\omega,k,h)\, a_{11,\hat{x}\hat{x}}.$$

where, as mentioned before, C_g is the linear group velocity and $C_g{}'$ its derivative with respect to ω. Eqs. (49) & (50) constitute the coupled evolutionary equations of a_{11} and $\tilde{\theta}$, and are the same with those obtained by (Sedletsky 2012, 2013). It should be noted that, for those two equations, the total averaged Lagrangian $\bar{L}$ is considered, in order to combine the $O(\varepsilon^3)$ and $O(\varepsilon^4)$ contributions, regarding the evolution of a_{11} and $\tilde{\theta}$, into a single equation, for each variation.

At this point, after all the above steps, the unknown functions of the wave-fields' ansatzes are expressed in terms of the fundamental amplitude a_{11} and the phase modulation $\tilde{\theta}$. Using Eqs. (15), (18), (20), (29), (30) & (37)-(48) into Eqs. (4) & (5), the ansatzes of the free-surface elevation and the velocity potential are shaped, respectively, as

$$\begin{aligned}\tilde{\eta} = {} & \varepsilon^2\zeta + \varepsilon a_{11}\cos\theta \\ & + \varepsilon^2\left\{\frac{1}{2k}\left(1 - 2kh + \frac{2kh}{\tanh(2kh)}\right)(\tilde{\theta}_{\hat{x}}\, a_{11}\cos\theta + a_{11,\hat{x}}\sin\theta)\right. \\ & \left. + \frac{5\cosh(kh) + \cosh(3kh)}{8\sinh^3(kh)}\, k\, a_{11}{}^2\cos 2\theta\right\} + O(\varepsilon^3)\end{aligned}$$

and

$$\begin{aligned}\tilde{\Phi} = {} & \varepsilon\psi + \varepsilon\frac{\omega}{k}\frac{\cosh[k(h+z)]}{\sinh(kh)} a_{11}\sin\theta \\ & + \varepsilon^2\left\{\frac{\omega}{k}\frac{(z+h)\sinh[k(h+z)] - h\cosh[k(h+z)]}{\sinh(kh)}(\tilde{\theta}_{\hat{x}}\, a_{11}\sin\theta - a_{11,\hat{x}}\cos\theta)\right. \\ & \left. + \frac{3\cosh[2k(h+z)]}{8\sinh^4(kh)}\,\omega\, a_{11}{}^2\sin 2\theta\right\} + O(\varepsilon^3).\end{aligned}$$

As already stated earlier, the evolution of the fields ζ and ψ, which are included in them, is governed by Eqs. (35) & (36), whereas the evolution of a_{11} and $\tilde{\theta}$ is determined by Eqs. (49) & (50).

4.2.3. Final remarks

In conclusion, our approach, in which we use much more generic trial functions, leaving the vertical dependence arbitrary, leads to the exact same results as those of (Sedletsky 2012, 2013), who incorporates in his ansatzes an a priori known vertical structure that is derived by means of the MSM (see, also, Sec. 4.1). That consistency refers not only to the obtained evolutionary equations, but also to the vertical dependence of the potential and the form of the ansatzes in general. Besides, the findings of this chapter are in agreement with those of Ch. 3, where, again, we study the problem of weakly nonlinear, narrow-banded wavetrains, but via the MSM.

The above evolutionary equations, for the slowly modulated functions a_{11}, $\tilde{\theta}$, ζ and ψ of the wave-field ansatzes, produce, under certain assumptions and combinations [(Yuen and Lake 1975; Sedletsky 2012, 2013, 2016)] well known model equations for the propagation of slowly modulated wavetrains, such as the NLS or the BR and DS systems of equations. In contrast to previous efforts, here we derived those equations based solely on the AVP, driven by the belief that variational principles, when correctly formulated, contain all the necessary information for the problem, so that no “external” help should be needed. Hence, our proposed implementation of the method could be regarded as a self-contained and consistent way to derive the NLS equation variationally.

Part II:
Appendices

Appendix A

Implementation of the AVP for Stokes waves using Wolfram® Mathematica (Ch.2)

The AVP in Stokes Waves

Setup

Ansatzes

```
$Assumptions = {g > 0, k > 0, h > 0};

c = Sqrt[g / k] * (c0 + eps * c1 + eps^2 * c2 + eps^3 * c3);
R = (g / k) * (R0 + eps * R1 + eps^2 * R2 + eps^3 * R3);

phi1 = k * psiX1 * x + K11[z] * ( b11 * Sin[k * (x - c * t)] );
phi2 = k * psiX2 * x + K21[z] * ( b21 * Sin[k * (x - c * t)] ) + K22[z] * ( b22 * Sin[2 * k * (x - c * t)] );
phi3 = k * psiX3 * x + K31[z] * ( b31 * Sin[k * (x - c * t)] ) +
   K32[z] * ( b32 * Sin[2 * k * (x - c * t)] ) + K33[z] * ( b33 * Sin[3 * k * (x - c * t)] );

eta1 = a11 * Cos[k * (x - c * t)];
eta2 = a21 * Cos[k * (x - c * t)] + a22 * Cos[2 * k * (x - c * t)];
eta3 = a31 * Cos[k * (x - c * t)] + a32 * Cos[2 * k * (x - c * t)] + a33 * Cos[3 * k * (x - c * t)];

phi = U * x + Sqrt[g / k^3] * (eps * phi1 + eps^2 * phi2 + eps^3 * phi3);

eta = ( eps * (zeta1 + eta1) + eps^2 * (zeta2 + eta2) + eps^3 * (zeta3 + eta3) ) / k;
```

Perturbation parameter

If we want eps = k*H / 2, then

k*H / 2 = k*(etaMax - etaMin) / 2 = eps

must be true. The above results in relations for a21, a31 & a33.

```
s = Collect[k * ( (eta /. k * (x - c * t) → 0) - (eta /. k * (x - c * t) → Pi) ) / 2, eps, Simplify];

FullSimplify[ Coefficient[s, eps, 1] == a11 ]
True

a21 = Solve[ Coefficient[s, eps, 2] == 0, a21 ] [[1, 1, 2]]
0

a31 = Solve[ Coefficient[s, eps, 3] == 0, a31 ] [[1, 1, 2]]
-a33
```

Vertical problem

Procedure

Lagrangian Lz of the vertical problem

```
Lz = D[phi, t] + (D[phi, x]^2 + D[phi, z]^2) / 2 + g * z;
```

Rewrite Lz in terms of theta & omit terms of O(eps^7) or higher

```
Lz1 = Lz /. Sin[q_] :> Sin[ FullSimplify[q / (k * (x - c * t))] * theta ];
```

```
Lz2 = Lz1 /. Cos[q_] :> Cos[ FullSimplify[q / (k * (x - c * t))] * theta ];

Lz3 = Collect[Lz2, eps] /. eps^x_ /; x ≥ 7 → 0;
```

Averaged Lagrangian AvLz of the vertical problem

```
AvLz = Collect[Integrate[Lz3, {theta, 0, 2 * Pi}] / 2 / Pi, eps, Simplify]
```

$$\text{eps}\sqrt{\frac{g}{k}}\ \text{psiX1 U} + \frac{U^2}{2} + g\,z + \frac{1}{4\,k^3}\text{eps}^2\left(2\,g\,k^2\,\text{psiX1}^2 + 4\sqrt{g\,k^5}\ \text{psiX2 U} + \text{b11}^2\,g\,\left(k^2\,\text{K11}[z]^2 + \text{K11}'[z]^2\right)\right) +$$
$$\frac{1}{2\,k^3}\text{eps}^3\left(2\,g\,k^2\,\text{psiX1 psiX2} + 2\sqrt{g\,k^5}\ \text{psiX3 U} + \text{b11 b21}\,g\,k^2\,\text{K11}[z]\,\text{K21}[z] + \text{b11 b21}\,g\,\text{K11}'[z]\,\text{K21}'[z]\right) +$$
$$\frac{1}{4\,k^3}\text{eps}^4\,g\,\left(2\,k^2\,\left(\text{psiX2}^2 + 2\,\text{psiX1 psiX3}\right) + \text{b21}^2\,k^2\,\text{K21}[z]^2 + 4\,\text{b22}^2\,k^2\,\text{K22}[z]^2 + \text{b21}^2\,\text{K21}'[z]^2 + \text{b22}^2\,\text{K22}'[z]^2 + 2\,\text{b11 b31}\,\left(k^2\,\text{K11}[z]\,\text{K31}[z] + \text{K11}'[z]\,\text{K31}'[z]\right)\right) +$$
$$\frac{1}{2\,k^3}\text{eps}^5\,g\,\left(2\,k^2\,\text{psiX2 psiX3} + \text{b21 b31}\,k^2\,\text{K21}[z]\,\text{K31}[z] + 4\,\text{b22 b32}\,k^2\,\text{K22}[z]\,\text{K32}[z] + \text{b21 b31}\,\text{K21}'[z]\,\text{K31}'[z] + \text{b22 b32}\,\text{K22}'[z]\,\text{K32}'[z]\right) + \frac{1}{4\,k^3}\text{eps}^6\,g$$
$$\left(2\,k^2\,\text{psiX3}^2 + \text{b31}^2\,k^2\,\text{K31}[z]^2 + 4\,\text{b32}^2\,k^2\,\text{K32}[z]^2 + 9\,\text{b33}^2\,k^2\,\text{K33}[z]^2 + \text{b31}^2\,\text{K31}'[z]^2 + \text{b32}^2\,\text{K32}'[z]^2 + \text{b33}^2\,\text{K33}'[z]^2\right)$$

EL equations of Lav up to O (eps^2)

```
eq1 = FullSimplify[
   D[ D[ Coefficient[AvLz, eps, 2], Derivative[1][K11][z] ], z ] -
     D[ Coefficient[AvLz, eps, 2], K11[z] ] == 0
                 ];

DSolve[eq1, K11[z], z]
```

$\{\{\text{K11}[z] \to e^{k\,z}\,C[1] + e^{-k\,z}\,C[2]\}\}$

Update AvLz

```
AvLz1 = Block[{K11}, K11[z_] := e^(k z) A1 + e^(-k z) A2 ; AvLz];
```

EL equations of Lav up to O (eps^4)

```
eq2 = FullSimplify[
   D[ D[ Coefficient[AvLz1, eps, 4], Derivative[1][K21][z] ], z ] -
     D[ Coefficient[AvLz1, eps, 4], K21[z] ] == 0
                 ];

DSolve[eq2, K21[z], z]
```

$\{\{\text{K21}[z] \to e^{k\,z}\,C[1] + e^{-k\,z}\,C[2]\}\}$

Update AvLz

```
AvLz2 = Block[{K21}, K21[z_] := e^(k z) B1 + e^(-k z) B2 ; AvLz1];

eq3 = FullSimplify[
   D[ D[ Coefficient[AvLz2, eps, 4], Derivative[1][K22][z] ], z ] -
     D[ Coefficient[AvLz2, eps, 4], K22[z] ] == 0
                 ];

DSolve[eq3, K22[z], z]
```

$\{\{\text{K22}[z] \to e^{2\,k\,z}\,C[1] + e^{-2\,k\,z}\,C[2]\}\}$

Update AvLz

```
AvLz3 = Block[{K22}, K22[z_] := e^(2 k z) C1 + e^(-2 k z) C2 ; AvLz2];
```

EL equations of Lav up to O (eps^6)

```
eq4 = FullSimplify[
   D[ D[ Coefficient[AvLz3, eps, 6], Derivative[1][K31][z] ], z ] -
     D[ Coefficient[AvLz3, eps, 6], K31[z] ] == 0
                  ];
```

```
DSolve[eq4, K31[z], z]
```

$\{\{K31[z] \to e^{k z} C[1] + e^{-k z} C[2]\}\}$

Update AvLz

```
AvLz4 = Block[{K31}, K31[z_] := e^(k z) D1 + e^(-k z) D2 ; AvLz3];
```

```
eq5 = FullSimplify[
   D[ D[ Coefficient[AvLz4, eps, 6], Derivative[1][K32][z] ], z ] -
     D[ Coefficient[AvLz4, eps, 6], K32[z] ] == 0
                  ];
```

```
DSolve[eq5, K32[z], z]
```

$\{\{K32[z] \to e^{2 k z} C[1] + e^{-2 k z} C[2]\}\}$

Update AvLz

```
AvLz5 = Block[{K32}, K32[z_] := e^(2 k z) E1 + e^(-2 k z) E2 ; AvLz4];
```

```
eq6 = FullSimplify[
   D[ D[ Coefficient[AvLz5, eps, 6], Derivative[1][K33][z] ], z ] -
     D[ Coefficient[AvLz5, eps, 6], K33[z] ] == 0
                  ];
```

```
DSolve[eq6, K33[z], z]
```

$\{\{K33[z] \to e^{3 k z} C[1] + e^{-3 k z} C[2]\}\}$

Update AvLz

```
AvLz6 = Block[{K33}, K33[z_] := e^(3 k z) F1 + e^(-3 k z) F2 ; AvLz5];
```

Final results

```
K11[z_] := e^(k z) A1 + e^(-k z) A2 ;
K21[z_] := e^(k z) B1 + e^(-k z) B2 ;
K22[z_] := e^(2 k z) C1 + e^(-2 k z) C2 ;
K31[z_] := e^(k z) D1 + e^(-k z) D2 ;
K32[z_] := e^(2 k z) E1 + e^(-2 k z) E2 ;
K33[z_] := e^(3 k z) F1 + e^(-3 k z) F2 ;
```

Uniform & periodic WWP

Procedure

Lagrangian L

```
t0Indef = Integrate[Coefficient[Lz, eps, 0], z];
t0i = Limit[t0Indef, z → -h];
t0f = Limit[t0Indef, z → eta];
t0 = t0f - t0i;
```

```
t1Indef = Integrate[Coefficient[Lz, eps, 1], z];
t1i = Limit[t1Indef, z → -h];
t1f = Limit[t1Indef, z → eta];
t1 = t1f - t1i;

t2Indef = Integrate[Coefficient[Lz, eps, 2], z];
t2i = Limit[t2Indef, z → -h];
t2f = Limit[t2Indef, z → eta];
t2 = t2f - t2i;

t3Indef = Integrate[Coefficient[Lz, eps, 3], z];
t3i = Limit[t3Indef, z → -h];
t3f = Limit[t3Indef, z → eta];
t3 = t3f - t3i;

t4Indef = Integrate[Coefficient[Lz, eps, 4], z];
t4i = Limit[t4Indef, z → -h];
t4f = Limit[t4Indef, z → eta];
t4 = t4f - t4i;

t5Indef = Integrate[Coefficient[Lz, eps, 5], z];
t5i = Limit[t5Indef, z → -h];
t5f = Limit[t5Indef, z → eta];
t5 = t5f - t5i;

t6Indef = Integrate[Coefficient[Lz, eps, 6], z];
t6i = Limit[t6Indef, z → -h];
t6f = Limit[t6Indef, z → eta];
t6 = t6f - t6i;

L = t0 + eps * t1 + eps^2 * t2 + eps^3 * t3 + eps^4 * t4 +
   eps^5 * t5 + eps^6 * t6 - R * ( eps * zeta1 + eps^2 * zeta2 + eps^3 * zeta3 ) / k;
```

Rewrite L in terms of theta & omit terms of O(eps^7) or higher

```
Lr1 = L /. Sin[q_] :> Sin[ FullSimplify[q / (k * (x - c * t))] * theta ];

Lr2 = Lr1 /. Cos[q_] :> Cos[ FullSimplify[q / (k * (x - c * t))] * theta ];

(*Lr3 = Lr2 /. Csc[q_] :> Csc[   FullSimplify[q/(k*(x - c*t))]*theta   ];*)

Lr4 = Collect[Lr2, eps] /. eps^x_ /; x ≥ 7 → 0;
```

Substitute the small-argument exponential functions with their MacLaurin expansions

```
Lr5 = Lr4 /. Exp[x_] /; FreeQ[x, eps] == False → 1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120 + x^6/720;

Lr6 = Collect[Lr5, eps] /. eps^x_ /; x ≥ 7 → 0;
```

Averaged Lagrangian AvL

```
q0 = Integrate[Coefficient[Lr6, eps, 0], {theta, 0, 2 * Pi}] / 2 / Pi;

q1 = Integrate[Coefficient[Lr6, eps, 1], {theta, 0, 2 * Pi}] / 2 / Pi;

q2 = Integrate[Coefficient[Lr6, eps, 2], {theta, 0, 2 * Pi}] / 2 / Pi;

q3 = Integrate[Coefficient[Lr6, eps, 3], {theta, 0, 2 * Pi}] / 2 / Pi;

q4 = Integrate[Coefficient[Lr6, eps, 4], {theta, 0, 2 * Pi}] / 2 / Pi;

q5 = Integrate[Coefficient[Lr6, eps, 5], {theta, 0, 2 * Pi}] / 2 / Pi;

q6 = Integrate[Coefficient[Lr6, eps, 6], {theta, 0, 2 * Pi}] / 2 / Pi;

AvL = q0 + eps * q1 + eps^2 * q2 + eps^3 * q3 + eps^4 * q4 + eps^5 * q5 + eps^6 * q6;
```

Final result

$$
\mathrm{AvL} = -\frac{g\,h^2}{2} + \frac{h\,U^2}{2} + \frac{1}{2\,k^2}\,\mathrm{eps}\left(2\,h\,k\,\sqrt{g\,k}\;\mathrm{psiX1}\,U - 2\,g\,R0\,\mathrm{zeta1} + k\,U^2\,\mathrm{zeta1}\right) +
$$

$$
\frac{1}{4\,k^2}\,e^{-2\,h\,k}\,\mathrm{eps}^2\left(A1^2\,b11^2\left(-1+e^{2\,h\,k}\right)g - 2\,A1\,a11\,b11\,e^{2\,h\,k}\left(c0\,g - \sqrt{g\,k}\;U\right) +\right.
$$

$$
e^{2\,h\,k}\left(a11^2\,g + A2^2\,b11^2\left(-1+e^{2\,h\,k}\right)g + 2\,a11\,A2\,b11\left(-c0\,g + \sqrt{g\,k}\;U\right) + 2\left(g\left(h\,k\,\mathrm{psiX1}^2 - 2\,R1\right.\right.\right.
$$

$$
\left.\left.\left.\left.\mathrm{zeta1} + \mathrm{zeta1}^2 - 2\,R0\,\mathrm{zeta2}\right) + U\left(2\,h\,k\,\sqrt{g\,k}\;\mathrm{psiX2} + 2\,\sqrt{g\,k}\;\mathrm{psiX1}\,\mathrm{zeta1} + k\,U\,\mathrm{zeta2}\right)\right)\right)\right) +
$$

$$
\frac{1}{2\,\pi}\,\mathrm{eps}^4\left(\frac{a22^2\,g\,\pi}{2\,k^2} - \frac{a11\,a33\,g\,\pi}{k^2} + \frac{A1^2\,a11^2\,b11^2\,g\,\pi}{2\,k^2} - \frac{a11^2\,A2^2\,b11^2\,g\,\pi}{2\,k^2} + \frac{A1\,A2\,a22\,b11^2\,g\,\pi}{k^2} + \frac{B1^2\,b21^2\,g\,\pi}{2\,k^2} -\right.
$$

$$
\frac{B2^2\,b21^2\,g\,\pi}{2\,k^2} - \frac{A1\,a11^3\,b11\,c0\,g\,\pi}{8\,k^2} - \frac{a11^3\,A2\,b11\,c0\,g\,\pi}{8\,k^2} - \frac{A1\,a11\,a22\,b11\,c0\,g\,\pi}{2\,k^2} + \frac{a11\,A2\,a22\,b11\,c0\,g\,\pi}{2\,k^2} +
$$

$$
\frac{A1\,a33\,b11\,c0\,g\,\pi}{k^2} + \frac{A2\,a33\,b11\,c0\,g\,\pi}{k^2} - \frac{a11\,B1\,b21\,c1\,g\,\pi}{k^2} - \frac{a11\,B2\,b21\,c1\,g\,\pi}{k^2} + \frac{2\,A1\,a11\,b11\,b22\,C1\,g\,\pi}{k^2} -
$$

$$
\frac{a11^2\,b22\,c0\,C1\,g\,\pi}{k^2} - \frac{2\,a22\,b22\,c0\,C1\,g\,\pi}{k^2} + \frac{b22^2\,C1^2\,g\,\pi}{k^2} - \frac{A1\,a11\,b11\,c2\,g\,\pi}{k^2} - \frac{a11\,A2\,b11\,c2\,g\,\pi}{k^2} +
$$

$$
\frac{2\,a11\,A2\,b11\,b22\,C2\,g\,\pi}{k^2} + \frac{a11^2\,b22\,c0\,C2\,g\,\pi}{k^2} - \frac{2\,a22\,b22\,c0\,C2\,g\,\pi}{k^2} - \frac{b22^2\,C2^2\,g\,\pi}{k^2} + \frac{A1\,b11\,b31\,D1\,g\,\pi}{k^2} -
$$

$$
\frac{a11\,b31\,c0\,D1\,g\,\pi}{k^2} - \frac{A2\,b11\,b31\,D2\,g\,\pi}{k^2} - \frac{a11\,b31\,c0\,D2\,g\,\pi}{k^2} - \frac{b22^2\,C1^2\,e^{-4\,h\,k}\,g\,\pi}{k^2} - \frac{B1^2\,b21^2\,e^{-2\,h\,k}\,g\,\pi}{2\,k^2} -
$$

$$
\frac{A1\,b11\,b31\,D1\,e^{-2\,h\,k}\,g\,\pi}{k^2} + \frac{B2^2\,b21^2\,e^{2\,h\,k}\,g\,\pi}{2\,k^2} + \frac{A2\,b11\,b31\,D2\,e^{2\,h\,k}\,g\,\pi}{k^2} + \frac{b22^2\,C2^2\,e^{4\,h\,k}\,g\,\pi}{k^2} + \frac{a11\,B1\,b21\,g\,\pi\,\mathrm{psiX1}}{k^2} +
$$

$$
\frac{a11\,B2\,b21\,g\,\pi\,\mathrm{psiX1}}{k^2} + \frac{A1\,a11\,b11\,g\,\pi\,\mathrm{psiX2}}{k^2} + \frac{a11\,A2\,b11\,g\,\pi\,\mathrm{psiX2}}{k^2} + \frac{g\,h\,\pi\,\mathrm{psiX2}^2}{k} + \frac{2\,g\,h\,\pi\,\mathrm{psiX1}\,\mathrm{psiX3}}{k} +
$$

$$
\frac{1}{8}\,A1\,a11^3\,b11\,\sqrt{\frac{g}{k^3}}\;\pi\,U + \frac{1}{8}\,a11^3\,A2\,b11\,\sqrt{\frac{g}{k^3}}\;\pi\,U + \frac{1}{2}\,A1\,a11\,a22\,b11\,\sqrt{\frac{g}{k^3}}\;\pi\,U - \frac{1}{2}\,a11\,A2\,a22\,b11\,\sqrt{\frac{g}{k^3}}\;\pi\,U -
$$

$$
A1\,a33\,b11\,\sqrt{\frac{g}{k^3}}\;\pi\,U - A2\,a33\,b11\,\sqrt{\frac{g}{k^3}}\;\pi\,U + a11^2\,b22\,C1\,\sqrt{\frac{g}{k^3}}\;\pi\,U + 2\,a22\,b22\,C1\,\sqrt{\frac{g}{k^3}}\;\pi\,U -
$$

$$
a11^2\,b22\,C2\,\sqrt{\frac{g}{k^3}}\;\pi\,U + 2\,a22\,b22\,C2\,\sqrt{\frac{g}{k^3}}\;\pi\,U + a11\,b31\,D1\,\sqrt{\frac{g}{k^3}}\;\pi\,U + a11\,b31\,D2\,\sqrt{\frac{g}{k^3}}\;\pi\,U +
$$

$$
\frac{2\,A1\,B1\,b11\,b21\,g\,\pi\,\mathrm{zeta1}}{k^2} + \frac{2\,A2\,b11\,B2\,b21\,g\,\pi\,\mathrm{zeta1}}{k^2} - \frac{a11\,B1\,b21\,c0\,g\,\pi\,\mathrm{zeta1}}{k^2} + \frac{a11\,B2\,b21\,c0\,g\,\pi\,\mathrm{zeta1}}{k^2} -
$$

$$
\frac{A1\,a11\,b11\,c1\,g\,\pi\,\mathrm{zeta1}}{k^2} + \frac{a11\,A2\,b11\,c1\,g\,\pi\,\mathrm{zeta1}}{k^2} + \frac{A1\,a11\,b11\,g\,\pi\,\mathrm{psiX1}\,\mathrm{zeta1}}{k^2} - \frac{a11\,A2\,b11\,g\,\pi\,\mathrm{psiX1}\,\mathrm{zeta1}}{k^2} +
$$

$$
\frac{2\,g\,\pi\,\mathrm{psiX1}\,\mathrm{psiX2}\,\mathrm{zeta1}}{k^2} - \frac{2\,g\,\pi\,R3\,\mathrm{zeta1}}{k^2} + a11\,B1\,b21\,\sqrt{\frac{g}{k^3}}\;\pi\,U\,\mathrm{zeta1} - a11\,B2\,b21\,\sqrt{\frac{g}{k^3}}\;\pi\,U\,\mathrm{zeta1} +
$$

$$
\frac{2\,\sqrt{\frac{g}{k}}\;\pi\,\mathrm{psiX3}\,U\,\mathrm{zeta1}}{k} + \frac{A1^2\,b11^2\,g\,\pi\,\mathrm{zeta1}^2}{k^2} - \frac{A2^2\,b11^2\,g\,\pi\,\mathrm{zeta1}^2}{k^2} - \frac{A1\,a11\,b11\,c0\,g\,\pi\,\mathrm{zeta1}^2}{2\,k^2} -
$$

$$
\frac{a11\,A2\,b11\,c0\,g\,\pi\,\mathrm{zeta1}^2}{2\,k^2} + \frac{1}{2}\,A1\,a11\,b11\,\sqrt{\frac{g}{k^3}}\;\pi\,U\,\mathrm{zeta1}^2 + \frac{1}{2}\,a11\,A2\,b11\,\sqrt{\frac{g}{k^3}}\;\pi\,U\,\mathrm{zeta1}^2 +
$$

$$\frac{A1^2\ b11^2\ g\ \pi\ zeta2}{k^2}+\frac{A2^2\ b11^2\ g\ \pi\ zeta2}{k^2}-\frac{A1\ a11\ b11\ c0\ g\ \pi\ zeta2}{k^2}+\frac{a11\ A2\ b11\ c0\ g\ \pi\ zeta2}{k^2}+$$

$$\frac{g\ \pi\ psiX1^2\ zeta2}{k^2}-\frac{2\ g\ \pi\ R2\ zeta2}{k^2}+A1\ a11\ b11\sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta2-a11\ A2\ b11\sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta2+$$

$$\left.\frac{2\sqrt{g\ k}\ \pi\ psiX2\ U\ zeta2}{k^2}+\frac{g\ \pi\ zeta2^2}{k^2}-\frac{2\ g\ \pi\ R1\ zeta3}{k^2}+\frac{2\sqrt{g\ k}\ \pi\ psiX1\ U\ zeta3}{k^2}+\frac{2\ g\ \pi\ zeta1\ zeta3}{k^2}\right)+$$

$$\frac{1}{2\ \pi}\ eps^5\left(\frac{a22\ a32\ g\ \pi}{k^2}+\frac{A1\ A2\ a32\ b11^2\ g\ \pi}{k^2}+\frac{A1\ a11^2\ B1\ b11\ b21\ g\ \pi}{k^2}+\frac{A2\ a22\ B1\ b11\ b21\ g\ \pi}{k^2}-\frac{a11^2\ A2\ b11\ B2\ b21\ g\ \pi}{k^2}+\right.$$

$$\frac{A1\ a22\ b11\ B2\ b21\ g\ \pi}{k^2}-\frac{A1\ a11\ a32\ b11\ c0\ g\ \pi}{2\ k^2}+\frac{a11\ A2\ a32\ b11\ c0\ g\ \pi}{2\ k^2}-\frac{a11^3\ B1\ b21\ c0\ g\ \pi}{8\ k^2}-\frac{a11\ a22\ B1\ b21\ c0\ g\ \pi}{2\ k^2}+$$

$$\frac{a33\ B1\ b21\ c0\ g\ \pi}{k^2}-\frac{a11^3\ B2\ b21\ c0\ g\ \pi}{8\ k^2}+\frac{a11\ a22\ B2\ b21\ c0\ g\ \pi}{2\ k^2}+\frac{a33\ B2\ b21\ c0\ g\ \pi}{k^2}-\frac{A1\ a11^3\ b11\ c1\ g\ \pi}{8\ k^2}-$$

$$\frac{a11^3\ A2\ b11\ c1\ g\ \pi}{8\ k^2}-\frac{A1\ a11\ a22\ b11\ c1\ g\ \pi}{2\ k^2}+\frac{a11\ A2\ a22\ b11\ c1\ g\ \pi}{2\ k^2}+\frac{A1\ a33\ b11\ c1\ g\ \pi}{k^2}+\frac{A2\ a33\ b11\ c1\ g\ \pi}{k^2}+$$

$$\frac{2\ a11\ B1\ b21\ b22\ C1\ g\ \pi}{k^2}-\frac{2\ a32\ b22\ c0\ C1\ g\ \pi}{k^2}-\frac{a11^2\ b22\ c1\ C1\ g\ \pi}{k^2}-\frac{2\ a22\ b22\ c1\ C1\ g\ \pi}{k^2}-\frac{a11\ B1\ b21\ c2\ g\ \pi}{k^2}-$$

$$\frac{a11\ B2\ b21\ c2\ g\ \pi}{k^2}+\frac{2\ a11\ B2\ b21\ b22\ C2\ g\ \pi}{k^2}-\frac{2\ a32\ b22\ c0\ C2\ g\ \pi}{k^2}+\frac{a11^2\ b22\ c1\ C2\ g\ \pi}{k^2}-\frac{2\ a22\ b22\ c1\ C2\ g\ \pi}{k^2}-$$

$$\frac{A1\ a11\ b11\ c3\ g\ \pi}{k^2}-\frac{a11\ A2\ b11\ c3\ g\ \pi}{k^2}+\frac{B1\ b21\ b31\ D1\ g\ \pi}{k^2}-\frac{a11\ b31\ c1\ D1\ g\ \pi}{k^2}-\frac{B2\ b21\ b31\ D2\ g\ \pi}{k^2}-$$

$$\frac{a11\ b31\ c1\ D2\ g\ \pi}{k^2}-\frac{B1\ b21\ b31\ D1\ e^{-2\ h\ k}\ g\ \pi}{k^2}+\frac{B2\ b21\ b31\ D2\ e^{2\ h\ k}\ g\ \pi}{k^2}+\frac{2\ A1\ a11\ b11\ b32\ E1\ g\ \pi}{k^2}-\frac{a11^2\ b32\ c0\ E1\ g\ \pi}{k^2}-$$

$$\frac{2\ a22\ b32\ c0\ E1\ g\ \pi}{k^2}+\frac{2\ b22\ b32\ C1\ E1\ g\ \pi}{k^2}-\frac{2\ b22\ b32\ C1\ e^{-4\ h\ k}\ E1\ g\ \pi}{k^2}+\frac{2\ a11\ A2\ b11\ b32\ E2\ g\ \pi}{k^2}+\frac{a11^2\ b32\ c0\ E2\ g\ \pi}{k^2}-$$

$$\frac{2\ a22\ b32\ c0\ E2\ g\ \pi}{k^2}-\frac{2\ b22\ b32\ C2\ E2\ g\ \pi}{k^2}+\frac{2\ b22\ b32\ C2\ e^{4\ h\ k}\ E2\ g\ \pi}{k^2}+\frac{A1\ a11^3\ b11\ g\ \pi\ psiX1}{8\ k^2}+\frac{a11^3\ A2\ b11\ g\ \pi\ psiX1}{8\ k^2}+$$

$$\frac{A1\ a11\ a22\ b11\ g\ \pi\ psiX1}{2\ k^2}-\frac{a11\ A2\ a22\ b11\ g\ \pi\ psiX1}{2\ k^2}-\frac{A1\ a33\ b11\ g\ \pi\ psiX1}{k^2}-\frac{A2\ a33\ b11\ g\ \pi\ psiX1}{k^2}+$$

$$\frac{a11^2\ b22\ C1\ g\ \pi\ psiX1}{k^2}+\frac{2\ a22\ b22\ C1\ g\ \pi\ psiX1}{k^2}-\frac{a11^2\ b22\ C2\ g\ \pi\ psiX1}{k^2}+\frac{2\ a22\ b22\ C2\ g\ \pi\ psiX1}{k^2}+$$

$$\frac{a11\ b31\ D1\ g\ \pi\ psiX1}{k^2}+\frac{a11\ b31\ D2\ g\ \pi\ psiX1}{k^2}+\frac{a11\ B1\ b21\ g\ \pi\ psiX2}{k^2}+\frac{a11\ B2\ b21\ g\ \pi\ psiX2}{k^2}+\frac{A1\ a11\ b11\ g\ \pi\ psiX3}{k^2}+$$

$$\frac{a11\ A2\ b11\ g\ \pi\ psiX3}{k^2}+\frac{2\ g\ h\ \pi\ psiX2\ psiX3}{k}+\frac{1}{2}\ A1\ a11\ a32\ b11\sqrt{\frac{g}{k^3}}\ \pi\ U-\frac{1}{2}\ a11\ A2\ a32\ b11\sqrt{\frac{g}{k^3}}\ \pi\ U+$$

$$\frac{1}{8}\ a11^3\ B1\ b21\sqrt{\frac{g}{k^3}}\ \pi\ U+\frac{1}{2}\ a11\ a22\ B1\ b21\sqrt{\frac{g}{k^3}}\ \pi\ U-a33\ B1\ b21\sqrt{\frac{g}{k^3}}\ \pi\ U+\frac{1}{8}\ a11^3\ B2\ b21\sqrt{\frac{g}{k^3}}\ \pi\ U-$$

$$\frac{1}{2}\ a11\ a22\ B2\ b21\sqrt{\frac{g}{k^3}}\ \pi\ U-a33\ B2\ b21\sqrt{\frac{g}{k^3}}\ \pi\ U+2\ a32\ b22\ C1\sqrt{\frac{g}{k^3}}\ \pi\ U+2\ a32\ b22\ C2\sqrt{\frac{g}{k^3}}\ \pi\ U+$$

$$a11^2\ b32\ E1\sqrt{\frac{g}{k^3}}\ \pi\ U+2\ a22\ b32\ E1\sqrt{\frac{g}{k^3}}\ \pi\ U-a11^2\ b32\ E2\sqrt{\frac{g}{k^3}}\ \pi\ U+2\ a22\ b32\ E2\sqrt{\frac{g}{k^3}}\ \pi\ U+$$

$$\frac{A1^2\ a11^2\ b11^2\ g\ \pi\ zeta1}{k^2}+\frac{a11^2\ A2^2\ b11^2\ g\ \pi\ zeta1}{k^2}+\frac{B1^2\ b21^2\ g\ \pi\ zeta1}{k^2}+\frac{B2^2\ b21^2\ g\ \pi\ zeta1}{k^2}-$$

$$\frac{A1\ a11^3\ b11\ c0\ g\ \pi\ zeta1}{8\ k^2}+\frac{a11^3\ A2\ b11\ c0\ g\ \pi\ zeta1}{8\ k^2}-\frac{A1\ a11\ a22\ b11\ c0\ g\ \pi\ zeta1}{2\ k^2}-\frac{a11\ A2\ a22\ b11\ c0\ g\ \pi\ zeta1}{2\ k^2}+$$

$$\frac{A1\ a33\ b11\ c0\ g\ \pi\ zeta1}{k^2}-\frac{A2\ a33\ b11\ c0\ g\ \pi\ zeta1}{k^2}-\frac{a11\ B1\ b21\ c1\ g\ \pi\ zeta1}{k^2}+\frac{a11\ B2\ b21\ c1\ g\ \pi\ zeta1}{k^2}+$$

$$\frac{6\ A1\ a11\ b11\ b22\ C1\ g\ \pi\ zeta1}{k^2}-\frac{2\ a11^2\ b22\ c0\ C1\ g\ \pi\ zeta1}{k^2}-\frac{4\ a22\ b22\ c0\ C1\ g\ \pi\ zeta1}{k^2}+\frac{4\ b22^2\ C1^2\ g\ \pi\ zeta1}{k^2}-$$

$$\frac{A1\ a11\ b11\ c2\ g\ \pi\ zeta1}{k^2}+\frac{a11\ A2\ b11\ c2\ g\ \pi\ zeta1}{k^2}-\frac{6\ a11\ A2\ b11\ b22\ C2\ g\ \pi\ zeta1}{k^2}-\frac{2\ a11^2\ b22\ c0\ C2\ g\ \pi\ zeta1}{k^2}+$$

$$\frac{4\ a22\ b22\ c0\ C2\ g\ \pi\ zeta1}{k^2}+\frac{4\ b22^2\ C2^2\ g\ \pi\ zeta1}{k^2}+\frac{2\ A1\ b11\ b31\ D1\ g\ \pi\ zeta1}{k^2}-\frac{a11\ b31\ c0\ D1\ g\ \pi\ zeta1}{k^2}+$$

$$\frac{2\ A2\ b11\ b31\ D2\ g\ \pi\ zeta1}{k^2}+\frac{a11\ b31\ c0\ D2\ g\ \pi\ zeta1}{k^2}+\frac{a11\ B1\ b21\ g\ \pi\ psiX1\ zeta1}{k^2}-\frac{a11\ B2\ b21\ g\ \pi\ psiX1\ zeta1}{k^2}+$$

$$\frac{A1\ a11\ b11\ g\ \pi\ psiX2\ zeta1}{k^2}-\frac{a11\ A2\ b11\ g\ \pi\ psiX2\ zeta1}{k^2}+\frac{g\ \pi\ psiX2^2\ zeta1}{k^2}+\frac{2\ g\ \pi\ psiX1\ psiX3\ zeta1}{k^2}+$$

$$\frac{1}{8}\ A1\ a11^3\ b11\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta1-\frac{1}{8}\ a11^3\ A2\ b11\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta1+\frac{1}{2}\ A1\ a11\ a22\ b11\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta1+$$

$$\frac{1}{2}\ a11\ A2\ a22\ b11\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta1-A1\ a33\ b11\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta1+A2\ a33\ b11\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta1+$$

$$2\ a11^2\ b22\ C1\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta1+4\ a22\ b22\ C1\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta1+2\ a11^2\ b22\ C2\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta1-$$

$$4\ a22\ b22\ C2\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta1+a11\ b31\ D1\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta1-a11\ b31\ D2\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta1+$$

$$\frac{2\ A1\ B1\ b11\ b21\ g\ \pi\ zeta1^2}{k^2}-\frac{2\ A2\ b11\ B2\ b21\ g\ \pi\ zeta1^2}{k^2}-\frac{a11\ B1\ b21\ c0\ g\ \pi\ zeta1^2}{2\ k^2}-\frac{a11\ B2\ b21\ c0\ g\ \pi\ zeta1^2}{2\ k^2}-$$

$$\frac{A1\ a11\ b11\ c1\ g\ \pi\ zeta1^2}{2\ k^2}-\frac{a11\ A2\ b11\ c1\ g\ \pi\ zeta1^2}{2\ k^2}+\frac{A1\ a11\ b11\ g\ \pi\ psiX1\ zeta1^2}{2\ k^2}+\frac{a11\ A2\ b11\ g\ \pi\ psiX1\ zeta1^2}{2\ k^2}+$$

$$\frac{1}{2}\ a11\ B1\ b21\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta1^2+\frac{1}{2}\ a11\ B2\ b21\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta1^2+\frac{2\ A1^2\ b11^2\ g\ \pi\ zeta1^3}{3\ k^2}+$$

$$\frac{2\ A2^2\ b11^2\ g\ \pi\ zeta1^3}{3\ k^2}-\frac{A1\ a11\ b11\ c0\ g\ \pi\ zeta1^3}{6\ k^2}+\frac{a11\ A2\ b11\ c0\ g\ \pi\ zeta1^3}{6\ k^2}+\frac{1}{6}\ A1\ a11\ b11\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta1^3-$$

$$\frac{1}{6}\ a11\ A2\ b11\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta1^3+\frac{2\ A1\ B1\ b11\ b21\ g\ \pi\ zeta2}{k^2}+\frac{2\ A2\ b11\ B2\ b21\ g\ \pi\ zeta2}{k^2}-$$

$$\frac{a11\ B1\ b21\ c0\ g\ \pi\ zeta2}{k^2}+\frac{a11\ B2\ b21\ c0\ g\ \pi\ zeta2}{k^2}-\frac{A1\ a11\ b11\ c1\ g\ \pi\ zeta2}{k^2}+\frac{a11\ A2\ b11\ c1\ g\ \pi\ zeta2}{k^2}+$$

$$\frac{A1\ a11\ b11\ g\ \pi\ psiX1\ zeta2}{k^2}-\frac{a11\ A2\ b11\ g\ \pi\ psiX1\ zeta2}{k^2}+\frac{2\ g\ \pi\ psiX1\ psiX2\ zeta2}{k^2}-\frac{2\ g\ \pi\ R3\ zeta2}{k^2}+$$

$$a11\ B1\ b21\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta2-a11\ B2\ b21\ \sqrt{\frac{g}{k^3}}\ \pi\ U\ zeta2+\frac{2\ \sqrt{\frac{g}{k}}\ \pi\ psiX3\ U\ zeta2}{k}+\frac{2\ A1^2\ b11^2\ g\ \pi\ zeta1\ zeta2}{k^2}-$$

$$\frac{2\ A2^2\ b11^2\ g\ \pi\ zeta1\ zeta2}{k^2}-\frac{A1\ a11\ b11\ c0\ g\ \pi\ zeta1\ zeta2}{k^2}-\frac{a11\ A2\ b11\ c0\ g\ \pi\ zeta1\ zeta2}{k^2}+$$

$$\text{A1 a11 b11} \sqrt{\frac{g}{k^3}}\, \pi\, \text{U zeta1 zeta2} + \text{a11 A2 b11} \sqrt{\frac{g}{k^3}}\, \pi\, \text{U zeta1 zeta2} + \frac{\text{A1}^2\, \text{b11}^2\, g\, \pi\, \text{zeta3}}{k^2} +$$

$$\frac{\text{A2}^2\, \text{b11}^2\, g\, \pi\, \text{zeta3}}{k^2} - \frac{\text{A1 a11 b11 c0}\, g\, \pi\, \text{zeta3}}{k^2} + \frac{\text{a11 A2 b11 c0}\, g\, \pi\, \text{zeta3}}{k^2} + \frac{g\, \pi\, \text{psiX1}^2\, \text{zeta3}}{k^2} - \frac{2\, g\, \pi\, \text{R2 zeta3}}{k^2} +$$

$$\left. \text{A1 a11 b11} \sqrt{\frac{g}{k^3}}\, \pi\, \text{U zeta3} - \text{a11 A2 b11} \sqrt{\frac{g}{k^3}}\, \pi\, \text{U zeta3} + \frac{2 \sqrt{g\, k}\, \pi\, \text{psiX2 U zeta3}}{k^2} + \frac{2\, g\, \pi\, \text{zeta2 zeta3}}{k^2} \right) +$$

$$\frac{1}{2\, \pi} \text{eps}^6 \left(\frac{\text{a32}^2\, g\, \pi}{2\, k^2} + \frac{\text{a33}^2\, g\, \pi}{k^2} + \frac{\text{A1}^2\, \text{a11}^4\, \text{b11}^2\, g\, \pi}{8\, k^2} - \frac{\text{a11}^4\, \text{A2}^2\, \text{b11}^2\, g\, \pi}{8\, k^2} + \frac{\text{A1}^2\, \text{a11}^2\, \text{a22 b11}^2\, g\, \pi}{2\, k^2} + \right.$$

$$\frac{\text{a11}^2\, \text{A2}^2\, \text{a22 b11}^2\, g\, \pi}{2\, k^2} + \frac{\text{A1}^2\, \text{a22}^2\, \text{b11}^2\, g\, \pi}{2\, k^2} - \frac{\text{A2}^2\, \text{a22}^2\, \text{b11}^2\, g\, \pi}{2\, k^2} - \frac{\text{A1}^2\, \text{a11 a33 b11}^2\, g\, \pi}{k^2} + \frac{\text{a11 A2}^2\, \text{a33 b11}^2\, g\, \pi}{k^2} +$$

$$\frac{\text{A2 a32 B1 b11 b21}\, g\, \pi}{k^2} + \frac{\text{A1 a32 b11 B2 b21}\, g\, \pi}{k^2} + \frac{\text{a11}^2\, \text{B1}^2\, \text{b21}^2\, g\, \pi}{2\, k^2} + \frac{\text{a22 B1 B2 b21}^2\, g\, \pi}{k^2} - \frac{\text{a11}^2\, \text{B2}^2\, \text{b21}^2\, g\, \pi}{2\, k^2} -$$

$$\frac{\text{A1 a11}^5\, \text{b11 c0}\, g\, \pi}{192\, k^2} - \frac{\text{a11}^5\, \text{A2 b11 c0}\, g\, \pi}{192\, k^2} - \frac{\text{A1 a11}^3\, \text{a22 b11 c0}\, g\, \pi}{12\, k^2} + \frac{\text{a11}^3\, \text{A2 a22 b11 c0}\, g\, \pi}{12\, k^2} - \frac{\text{A1 a11 a22}^2\, \text{b11 c0}\, g\, \pi}{4\, k^2} -$$

$$\frac{\text{a11 A2 a22}^2\, \text{b11 c0}\, g\, \pi}{4\, k^2} + \frac{\text{A1 a11}^2\, \text{a33 b11 c0}\, g\, \pi}{4\, k^2} + \frac{\text{a11}^2\, \text{A2 a33 b11 c0}\, g\, \pi}{4\, k^2} - \frac{\text{a11 a32 B1 b21 c0}\, g\, \pi}{2\, k^2} +$$

$$\frac{\text{a11 a32 B2 b21 c0}\, g\, \pi}{2\, k^2} - \frac{\text{A1 a11 a32 b11 c1}\, g\, \pi}{2\, k^2} + \frac{\text{a11 A2 a32 b11 c1}\, g\, \pi}{2\, k^2} - \frac{\text{a11}^3\, \text{B1 b21 c1}\, g\, \pi}{8\, k^2} - \frac{\text{a11 a22 B1 b21 c1}\, g\, \pi}{2\, k^2} +$$

$$\frac{\text{a33 B1 b21 c1}\, g\, \pi}{k^2} - \frac{\text{a11}^3\, \text{B2 b21 c1}\, g\, \pi}{8\, k^2} + \frac{\text{a11 a22 B2 b21 c1}\, g\, \pi}{2\, k^2} + \frac{\text{a33 B2 b21 c1}\, g\, \pi}{k^2} + \frac{9\, \text{A1 a11}^3\, \text{b11 b22 C1}\, g\, \pi}{4\, k^2} +$$

$$\frac{\text{a11}^3\, \text{A2 b11 b22 C1}\, g\, \pi}{12\, k^2} + \frac{3\, \text{A1 a11 a22 b11 b22 C1}\, g\, \pi}{k^2} + \frac{\text{a11 A2 a22 b11 b22 C1}\, g\, \pi}{k^2} - \frac{2\, \text{A1 a33 b11 b22 C1}\, g\, \pi}{k^2} +$$

$$\frac{2\, \text{A2 a33 b11 b22 C1}\, g\, \pi}{k^2} - \frac{\text{a11}^4\, \text{b22 c0 C1}\, g\, \pi}{3\, k^2} - \frac{2\, \text{a11}^2\, \text{a22 b22 c0 C1}\, g\, \pi}{k^2} - \frac{2\, \text{a32 b22 c1 C1}\, g\, \pi}{k^2} + \frac{4\, \text{a11}^2\, \text{b22}^2\, \text{C1}^2\, g\, \pi}{k^2} -$$

$$\frac{\text{A1 a11}^3\, \text{b11 c2}\, g\, \pi}{8\, k^2} - \frac{\text{a11}^3\, \text{A2 b11 c2}\, g\, \pi}{8\, k^2} - \frac{\text{A1 a11 a22 b11 c2}\, g\, \pi}{2\, k^2} + \frac{\text{a11 A2 a22 b11 c2}\, g\, \pi}{2\, k^2} + \frac{\text{A1 a33 b11 c2}\, g\, \pi}{k^2} +$$

$$\frac{\text{A2 a33 b11 c2}\, g\, \pi}{k^2} - \frac{\text{a11}^2\, \text{b22 C1 c2}\, g\, \pi}{k^2} - \frac{2\, \text{a22 b22 C1 c2}\, g\, \pi}{k^2} + \frac{\text{A1 a11}^3\, \text{b11 b22 C2}\, g\, \pi}{12\, k^2} + \frac{9\, \text{a11}^3\, \text{A2 b11 b22 C2}\, g\, \pi}{4\, k^2} -$$

$$\frac{\text{A1 a11 a22 b11 b22 C2}\, g\, \pi}{k^2} - \frac{3\, \text{a11 A2 a22 b11 b22 C2}\, g\, \pi}{k^2} + \frac{2\, \text{A1 a33 b11 b22 C2}\, g\, \pi}{k^2} - \frac{2\, \text{A2 a33 b11 b22 C2}\, g\, \pi}{k^2} +$$

$$\frac{\text{a11}^4\, \text{b22 c0 C2}\, g\, \pi}{3\, k^2} - \frac{2\, \text{a11}^2\, \text{a22 b22 c0 C2}\, g\, \pi}{k^2} - \frac{2\, \text{a32 b22 c1 C2}\, g\, \pi}{k^2} + \frac{\text{a11}^2\, \text{b22 c2 C2}\, g\, \pi}{k^2} - \frac{2\, \text{a22 b22 c2 C2}\, g\, \pi}{k^2} -$$

$$\frac{4\, \text{a11}^2\, \text{b22}^2\, \text{C2}^2\, g\, \pi}{k^2} - \frac{\text{a11 B1 b21 c3}\, g\, \pi}{k^2} - \frac{\text{a11 B2 b21 c3}\, g\, \pi}{k^2} + \frac{\text{A1 a11}^2\, \text{b11 b31 D1}\, g\, \pi}{k^2} + \frac{\text{A2 a22 b11 b31 D1}\, g\, \pi}{k^2} -$$

$$\frac{\text{a11}^3\, \text{b31 c0 D1}\, g\, \pi}{8\, k^2} - \frac{\text{a11 a22 b31 c0 D1}\, g\, \pi}{2\, k^2} + \frac{\text{a33 b31 c0 D1}\, g\, \pi}{k^2} + \frac{2\, \text{a11 b22 b31 C1 D1}\, g\, \pi}{k^2} - \frac{\text{a11 b31 c2 D1}\, g\, \pi}{k^2} +$$

$$\frac{\text{b31}^2\, \text{D1}^2\, g\, \pi}{2\, k^2} - \frac{\text{a11}^2\, \text{A2 b11 b31 D2}\, g\, \pi}{k^2} + \frac{\text{A1 a22 b11 b31 D2}\, g\, \pi}{k^2} - \frac{\text{a11}^3\, \text{b31 c0 D2}\, g\, \pi}{8\, k^2} + \frac{\text{a11 a22 b31 c0 D2}\, g\, \pi}{2\, k^2} +$$

$$\frac{\text{a33 b31 c0 D2}\, g\, \pi}{k^2} - \frac{\text{a11 b31 c2 D2}\, g\, \pi}{k^2} + \frac{2\, \text{a11 b22 b31 C2 D2}\, g\, \pi}{k^2} - \frac{\text{b31}^2\, \text{D2}^2\, g\, \pi}{2\, k^2} - \frac{\text{b31}^2\, \text{D1}^2\, e^{-2\, h\, k}\, g\, \pi}{2\, k^2} +$$

$$\frac{\text{b31}^2\, \text{D2}^2\, e^{2\, h\, k}\, g\, \pi}{2\, k^2} + \frac{2\, \text{a11 B1 b21 b32 E1}\, g\, \pi}{k^2} - \frac{2\, \text{a32 b32 c0 E1}\, g\, \pi}{k^2} - \frac{\text{a11}^2\, \text{b32 c1 E1}\, g\, \pi}{k^2} - \frac{2\, \text{a22 b32 c1 E1}\, g\, \pi}{k^2} +$$

$$\frac{b32^2\,E1^2\,g\,\pi}{k^2}-\frac{b32^2\,e^{-4hk}\,E1^2\,g\,\pi}{k^2}+\frac{2\,a11\,B2\,b21\,b32\,E2\,g\,\pi}{k^2}-\frac{2\,a32\,b32\,c0\,E2\,g\,\pi}{k^2}+\frac{a11^2\,b32\,c1\,E2\,g\,\pi}{k^2}-$$

$$\frac{2\,a22\,b32\,c1\,E2\,g\,\pi}{k^2}-\frac{b32^2\,E2^2\,g\,\pi}{k^2}+\frac{b32^2\,e^{4hk}\,E2^2\,g\,\pi}{k^2}+\frac{3\,A1\,a11^2\,b11\,b33\,F1\,g\,\pi}{k^2}+\frac{3\,A1\,a22\,b11\,b33\,F1\,g\,\pi}{k^2}-$$

$$\frac{9\,a11^3\,b33\,c0\,F1\,g\,\pi}{8\,k^2}-\frac{9\,a11\,a22\,b33\,c0\,F1\,g\,\pi}{2\,k^2}-\frac{3\,a33\,b33\,c0\,F1\,g\,\pi}{k^2}+\frac{6\,a11\,b22\,b33\,C1\,F1\,g\,\pi}{k^2}+\frac{3\,b33^2\,F1^2\,g\,\pi}{2\,k^2}-$$

$$\frac{3\,b33^2\,e^{-6hk}\,F1^2\,g\,\pi}{2\,k^2}-\frac{3\,a11^2\,A2\,b11\,b33\,F2\,g\,\pi}{k^2}+\frac{3\,A2\,a22\,b11\,b33\,F2\,g\,\pi}{k^2}-\frac{9\,a11^3\,b33\,c0\,F2\,g\,\pi}{8\,k^2}+$$

$$\frac{9\,a11\,a22\,b33\,c0\,F2\,g\,\pi}{2\,k^2}-\frac{3\,a33\,b33\,c0\,F2\,g\,\pi}{k^2}+\frac{6\,a11\,b22\,b33\,C2\,F2\,g\,\pi}{k^2}-\frac{3\,b33^2\,F2^2\,g\,\pi}{2\,k^2}+\frac{3\,b33^2\,e^{6hk}\,F2^2\,g\,\pi}{2\,k^2}+$$

$$\frac{A1\,a11\,a32\,b11\,g\,\pi\,psiX1}{2\,k^2}-\frac{a11\,A2\,a32\,b11\,g\,\pi\,psiX1}{2\,k^2}+\frac{a11^3\,B1\,b21\,g\,\pi\,psiX1}{8\,k^2}+\frac{a11\,a22\,B1\,b21\,g\,\pi\,psiX1}{2\,k^2}-$$

$$\frac{a33\,B1\,b21\,g\,\pi\,psiX1}{k^2}+\frac{a11^3\,B2\,b21\,g\,\pi\,psiX1}{8\,k^2}-\frac{a11\,a22\,B2\,b21\,g\,\pi\,psiX1}{2\,k^2}-\frac{a33\,B2\,b21\,g\,\pi\,psiX1}{k^2}+$$

$$\frac{2\,a32\,b22\,C1\,g\,\pi\,psiX1}{k^2}+\frac{2\,a32\,b22\,C2\,g\,\pi\,psiX1}{k^2}+\frac{a11^2\,b32\,E1\,g\,\pi\,psiX1}{k^2}+\frac{2\,a22\,b32\,E1\,g\,\pi\,psiX1}{k^2}-$$

$$\frac{a11^2\,b32\,E2\,g\,\pi\,psiX1}{k^2}+\frac{2\,a22\,b32\,E2\,g\,\pi\,psiX1}{k^2}+\frac{A1\,a11^3\,b11\,g\,\pi\,psiX2}{8\,k^2}+\frac{a11^3\,A2\,b11\,g\,\pi\,psiX2}{8\,k^2}+$$

$$\frac{A1\,a11\,a22\,b11\,g\,\pi\,psiX2}{2\,k^2}-\frac{a11\,A2\,a22\,b11\,g\,\pi\,psiX2}{2\,k^2}-\frac{A1\,a33\,b11\,g\,\pi\,psiX2}{k^2}-\frac{A2\,a33\,b11\,g\,\pi\,psiX2}{k^2}+$$

$$\frac{a11^2\,b22\,C1\,g\,\pi\,psiX2}{k^2}+\frac{2\,a22\,b22\,C1\,g\,\pi\,psiX2}{k^2}-\frac{a11^2\,b22\,C2\,g\,\pi\,psiX2}{k^2}+\frac{2\,a22\,b22\,C2\,g\,\pi\,psiX2}{k^2}+$$

$$\frac{a11\,b31\,D1\,g\,\pi\,psiX2}{k^2}+\frac{a11\,b31\,D2\,g\,\pi\,psiX2}{k^2}+\frac{a11\,B1\,b21\,g\,\pi\,psiX3}{k^2}+\frac{a11\,B2\,b21\,g\,\pi\,psiX3}{k^2}+$$

$$\frac{g\,h\,\pi\,psiX3^2}{k}+\frac{1}{192}\,A1\,a11^5\,b11\,\sqrt{\frac{g}{k^3}}\,\pi\,U+\frac{1}{192}\,a11^5\,A2\,b11\,\sqrt{\frac{g}{k^3}}\,\pi\,U+\frac{1}{12}\,A1\,a11^3\,a22\,b11\,\sqrt{\frac{g}{k^3}}\,\pi\,U-$$

$$\frac{1}{12}\,a11^3\,A2\,a22\,b11\,\sqrt{\frac{g}{k^3}}\,\pi\,U+\frac{1}{4}\,A1\,a11\,a22^2\,b11\,\sqrt{\frac{g}{k^3}}\,\pi\,U+\frac{1}{4}\,a11\,A2\,a22^2\,b11\,\sqrt{\frac{g}{k^3}}\,\pi\,U-$$

$$\frac{1}{4}\,A1\,a11^2\,a33\,b11\,\sqrt{\frac{g}{k^3}}\,\pi\,U-\frac{1}{4}\,a11^2\,A2\,a33\,b11\,\sqrt{\frac{g}{k^3}}\,\pi\,U+\frac{1}{2}\,a11\,a32\,B1\,b21\,\sqrt{\frac{g}{k^3}}\,\pi\,U-$$

$$\frac{1}{2}\,a11\,a32\,B2\,b21\,\sqrt{\frac{g}{k^3}}\,\pi\,U+\frac{1}{3}\,a11^4\,b22\,C1\,\sqrt{\frac{g}{k^3}}\,\pi\,U+2\,a11^2\,a22\,b22\,C1\,\sqrt{\frac{g}{k^3}}\,\pi\,U-\frac{1}{3}\,a11^4\,b22\,C2\,\sqrt{\frac{g}{k^3}}\,\pi\,U+$$

$$2\,a11^2\,a22\,b22\,C2\,\sqrt{\frac{g}{k^3}}\,\pi\,U+\frac{1}{8}\,a11^3\,b31\,D1\,\sqrt{\frac{g}{k^3}}\,\pi\,U+\frac{1}{2}\,a11\,a22\,b31\,D1\,\sqrt{\frac{g}{k^3}}\,\pi\,U-a33\,b31\,D1\,\sqrt{\frac{g}{k^3}}\,\pi\,U+$$

$$\frac{1}{8}\,a11^3\,b31\,D2\,\sqrt{\frac{g}{k^3}}\,\pi\,U-\frac{1}{2}\,a11\,a22\,b31\,D2\,\sqrt{\frac{g}{k^3}}\,\pi\,U-a33\,b31\,D2\,\sqrt{\frac{g}{k^3}}\,\pi\,U+2\,a32\,b32\,E1\,\sqrt{\frac{g}{k^3}}\,\pi\,U+$$

$$2\,a32\,b32\,E2\,\sqrt{\frac{g}{k^3}}\,\pi\,U+\frac{9}{8}\,a11^3\,b33\,F1\,\sqrt{\frac{g}{k^3}}\,\pi\,U+\frac{9}{2}\,a11\,a22\,b33\,F1\,\sqrt{\frac{g}{k^3}}\,\pi\,U+3\,a33\,b33\,F1\,\sqrt{\frac{g}{k^3}}\,\pi\,U+$$

$$\frac{9}{8}\,a11^3\,b33\,F2\,\sqrt{\frac{g}{k^3}}\,\pi\,U-\frac{9}{2}\,a11\,a22\,b33\,F2\,\sqrt{\frac{g}{k^3}}\,\pi\,U+3\,a33\,b33\,F2\,\sqrt{\frac{g}{k^3}}\,\pi\,U+\frac{2\,A1\,a11^2\,B1\,b11\,b21\,g\,\pi\,zeta1}{k^2}+$$

$$\frac{2\,a11^2\,A2\,b11\,B2\,b21\,g\,\pi\,zeta1}{k^2}-\frac{A1\,a11\,a32\,b11\,c0\,g\,\pi\,zeta1}{2\,k^2}-\frac{a11\,A2\,a32\,b11\,c0\,g\,\pi\,zeta1}{2\,k^2}-$$

$$\frac{a11^3\,B1\,b21\,c0\,g\,\pi\,\mathrm{zeta1}}{8\,k^2}-\frac{a11\,a22\,B1\,b21\,c0\,g\,\pi\,\mathrm{zeta1}}{2\,k^2}+\frac{a33\,B1\,b21\,c0\,g\,\pi\,\mathrm{zeta1}}{k^2}+\frac{a11^3\,B2\,b21\,c0\,g\,\pi\,\mathrm{zeta1}}{8\,k^2}-$$

$$\frac{a11\,a22\,B2\,b21\,c0\,g\,\pi\,\mathrm{zeta1}}{2\,k^2}-\frac{a33\,B2\,b21\,c0\,g\,\pi\,\mathrm{zeta1}}{k^2}-\frac{A1\,a11^3\,b11\,c1\,g\,\pi\,\mathrm{zeta1}}{8\,k^2}+\frac{a11^3\,A2\,b11\,c1\,g\,\pi\,\mathrm{zeta1}}{8\,k^2}-$$

$$\frac{A1\,a11\,a22\,b11\,c1\,g\,\pi\,\mathrm{zeta1}}{2\,k^2}-\frac{a11\,A2\,a22\,b11\,c1\,g\,\pi\,\mathrm{zeta1}}{2\,k^2}+\frac{A1\,a33\,b11\,c1\,g\,\pi\,\mathrm{zeta1}}{k^2}-\frac{A2\,a33\,b11\,c1\,g\,\pi\,\mathrm{zeta1}}{k^2}+$$

$$\frac{6\,a11\,B1\,b21\,b22\,C1\,g\,\pi\,\mathrm{zeta1}}{k^2}-\frac{4\,a32\,b22\,c0\,C1\,g\,\pi\,\mathrm{zeta1}}{k^2}-\frac{2\,a11^2\,b22\,c1\,C1\,g\,\pi\,\mathrm{zeta1}}{k^2}-\frac{4\,a22\,b22\,c1\,C1\,g\,\pi\,\mathrm{zeta1}}{k^2}-$$

$$\frac{a11\,B1\,b21\,c2\,g\,\pi\,\mathrm{zeta1}}{k^2}+\frac{a11\,B2\,b21\,c2\,g\,\pi\,\mathrm{zeta1}}{k^2}-\frac{6\,a11\,B2\,b21\,b22\,C2\,g\,\pi\,\mathrm{zeta1}}{k^2}+\frac{4\,a32\,b22\,c0\,C2\,g\,\pi\,\mathrm{zeta1}}{k^2}-$$

$$\frac{2\,a11^2\,b22\,c1\,C2\,g\,\pi\,\mathrm{zeta1}}{k^2}+\frac{4\,a22\,b22\,c1\,C2\,g\,\pi\,\mathrm{zeta1}}{k^2}-\frac{A1\,a11\,b11\,c3\,g\,\pi\,\mathrm{zeta1}}{k^2}+\frac{a11\,A2\,b11\,c3\,g\,\pi\,\mathrm{zeta1}}{k^2}+$$

$$\frac{2\,B1\,b21\,b31\,D1\,g\,\pi\,\mathrm{zeta1}}{k^2}-\frac{a11\,b31\,c1\,D1\,g\,\pi\,\mathrm{zeta1}}{k^2}+\frac{2\,B2\,b21\,b31\,D2\,g\,\pi\,\mathrm{zeta1}}{k^2}+\frac{a11\,b31\,c1\,D2\,g\,\pi\,\mathrm{zeta1}}{k^2}+$$

$$\frac{6\,A1\,a11\,b11\,b32\,E1\,g\,\pi\,\mathrm{zeta1}}{k^2}-\frac{2\,a11^2\,b32\,c0\,E1\,g\,\pi\,\mathrm{zeta1}}{k^2}-\frac{4\,a22\,b32\,c0\,E1\,g\,\pi\,\mathrm{zeta1}}{k^2}+\frac{8\,b22\,b32\,C1\,E1\,g\,\pi\,\mathrm{zeta1}}{k^2}-$$

$$\frac{6\,a11\,A2\,b11\,b32\,E2\,g\,\pi\,\mathrm{zeta1}}{k^2}-\frac{2\,a11^2\,b32\,c0\,E2\,g\,\pi\,\mathrm{zeta1}}{k^2}+\frac{4\,a22\,b32\,c0\,E2\,g\,\pi\,\mathrm{zeta1}}{k^2}+\frac{8\,b22\,b32\,C2\,E2\,g\,\pi\,\mathrm{zeta1}}{k^2}+$$

$$\frac{A1\,a11^3\,b11\,g\,\pi\,\mathrm{psiX1}\,\mathrm{zeta1}}{8\,k^2}-\frac{a11^3\,A2\,b11\,g\,\pi\,\mathrm{psiX1}\,\mathrm{zeta1}}{8\,k^2}+\frac{A1\,a11\,a22\,b11\,g\,\pi\,\mathrm{psiX1}\,\mathrm{zeta1}}{2\,k^2}+$$

$$\frac{a11\,A2\,a22\,b11\,g\,\pi\,\mathrm{psiX1}\,\mathrm{zeta1}}{2\,k^2}-\frac{A1\,a33\,b11\,g\,\pi\,\mathrm{psiX1}\,\mathrm{zeta1}}{k^2}+\frac{A2\,a33\,b11\,g\,\pi\,\mathrm{psiX1}\,\mathrm{zeta1}}{k^2}+$$

$$\frac{2\,a11^2\,b22\,C1\,g\,\pi\,\mathrm{psiX1}\,\mathrm{zeta1}}{k^2}+\frac{4\,a22\,b22\,C1\,g\,\pi\,\mathrm{psiX1}\,\mathrm{zeta1}}{k^2}+\frac{2\,a11^2\,b22\,C2\,g\,\pi\,\mathrm{psiX1}\,\mathrm{zeta1}}{k^2}-$$

$$\frac{4\,a22\,b22\,C2\,g\,\pi\,\mathrm{psiX1}\,\mathrm{zeta1}}{k^2}+\frac{a11\,b31\,D1\,g\,\pi\,\mathrm{psiX1}\,\mathrm{zeta1}}{k^2}-\frac{a11\,b31\,D2\,g\,\pi\,\mathrm{psiX1}\,\mathrm{zeta1}}{k^2}+$$

$$\frac{a11\,B1\,b21\,g\,\pi\,\mathrm{psiX2}\,\mathrm{zeta1}}{k^2}-\frac{a11\,B2\,b21\,g\,\pi\,\mathrm{psiX2}\,\mathrm{zeta1}}{k^2}+\frac{A1\,a11\,b11\,g\,\pi\,\mathrm{psiX3}\,\mathrm{zeta1}}{k^2}-\frac{a11\,A2\,b11\,g\,\pi\,\mathrm{psiX3}\,\mathrm{zeta1}}{k^2}+$$

$$\frac{2\,g\,\pi\,\mathrm{psiX2}\,\mathrm{psiX3}\,\mathrm{zeta1}}{k^2}+\frac{1}{2}\,A1\,a11\,a32\,b11\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\mathrm{zeta1}+\frac{1}{2}\,a11\,A2\,a32\,b11\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\mathrm{zeta1}+$$

$$\frac{1}{8}\,a11^3\,B1\,b21\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\mathrm{zeta1}+\frac{1}{2}\,a11\,a22\,B1\,b21\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\mathrm{zeta1}-a33\,B1\,b21\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\mathrm{zeta1}-$$

$$\frac{1}{8}\,a11^3\,B2\,b21\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\mathrm{zeta1}+\frac{1}{2}\,a11\,a22\,B2\,b21\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\mathrm{zeta1}+a33\,B2\,b21\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\mathrm{zeta1}+$$

$$4\,a32\,b22\,C1\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\mathrm{zeta1}-4\,a32\,b22\,C2\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\mathrm{zeta1}+2\,a11^2\,b32\,E1\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\mathrm{zeta1}+$$

$$4\,a22\,b32\,E1\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\mathrm{zeta1}+2\,a11^2\,b32\,E2\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\mathrm{zeta1}-4\,a22\,b32\,E2\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\mathrm{zeta1}+$$

$$\frac{A1^2\,a11^2\,b11^2\,g\,\pi\,\mathrm{zeta1}^2}{k^2}-\frac{a11^2\,A2^2\,b11^2\,g\,\pi\,\mathrm{zeta1}^2}{k^2}+\frac{B1^2\,b21^2\,g\,\pi\,\mathrm{zeta1}^2}{k^2}-\frac{B2^2\,b21^2\,g\,\pi\,\mathrm{zeta1}^2}{k^2}-$$

$$\frac{A1\,a11^3\,b11\,c0\,g\,\pi\,\mathrm{zeta1}^2}{16\,k^2}-\frac{a11^3\,A2\,b11\,c0\,g\,\pi\,\mathrm{zeta1}^2}{16\,k^2}-\frac{A1\,a11\,a22\,b11\,c0\,g\,\pi\,\mathrm{zeta1}^2}{4\,k^2}+\frac{a11\,A2\,a22\,b11\,c0\,g\,\pi\,\mathrm{zeta1}^2}{4\,k^2}+$$

$$\frac{A1\,a33\,b11\,c0\,g\,\pi\,\mathrm{zeta1}^2}{2\,k^2}+\frac{A2\,a33\,b11\,c0\,g\,\pi\,\mathrm{zeta1}^2}{2\,k^2}-\frac{a11\,B1\,b21\,c1\,g\,\pi\,\mathrm{zeta1}^2}{2\,k^2}-\frac{a11\,B2\,b21\,c1\,g\,\pi\,\mathrm{zeta1}^2}{2\,k^2}+$$

$$\frac{9\,A1\,a11\,b11\,b22\,C1\,g\,\pi\,\mathrm{zeta1}^2}{k^2}-\frac{2\,a11^2\,b22\,c0\,C1\,g\,\pi\,\mathrm{zeta1}^2}{k^2}-\frac{4\,a22\,b22\,c0\,C1\,g\,\pi\,\mathrm{zeta1}^2}{k^2}+$$

$$\frac{8\,\text{b22}^2\,\text{C1}^2\,g\,\pi\,\text{zeta1}^2}{k^2}-\frac{\text{A1}\,\text{a11}\,\text{b11}\,\text{c2}\,g\,\pi\,\text{zeta1}^2}{2\,k^2}-\frac{\text{a11}\,\text{A2}\,\text{b11}\,\text{c2}\,g\,\pi\,\text{zeta1}^2}{2\,k^2}+\frac{9\,\text{a11}\,\text{A2}\,\text{b11}\,\text{b22}\,\text{C2}\,g\,\pi\,\text{zeta1}^2}{k^2}+$$

$$\frac{2\,\text{a11}^2\,\text{b22}\,\text{c0}\,\text{C2}\,g\,\pi\,\text{zeta1}^2}{k^2}-\frac{4\,\text{a22}\,\text{b22}\,\text{c0}\,\text{C2}\,g\,\pi\,\text{zeta1}^2}{k^2}-\frac{8\,\text{b22}^2\,\text{C2}^2\,g\,\pi\,\text{zeta1}^2}{k^2}+\frac{2\,\text{A1}\,\text{b11}\,\text{b31}\,\text{D1}\,g\,\pi\,\text{zeta1}^2}{k^2}-$$

$$\frac{\text{a11}\,\text{b31}\,\text{c0}\,\text{D1}\,g\,\pi\,\text{zeta1}^2}{2\,k^2}-\frac{2\,\text{A2}\,\text{b11}\,\text{b31}\,\text{D2}\,g\,\pi\,\text{zeta1}^2}{k^2}-\frac{\text{a11}\,\text{b31}\,\text{c0}\,\text{D2}\,g\,\pi\,\text{zeta1}^2}{2\,k^2}+\frac{\text{a11}\,\text{B1}\,\text{b21}\,g\,\pi\,\text{psiX1}\,\text{zeta1}^2}{2\,k^2}+$$

$$\frac{\text{a11}\,\text{B2}\,\text{b21}\,g\,\pi\,\text{psiX1}\,\text{zeta1}^2}{2\,k^2}+\frac{\text{A1}\,\text{a11}\,\text{b11}\,g\,\pi\,\text{psiX2}\,\text{zeta1}^2}{2\,k^2}+\frac{\text{a11}\,\text{A2}\,\text{b11}\,g\,\pi\,\text{psiX2}\,\text{zeta1}^2}{2\,k^2}+$$

$$\frac{1}{16}\,\text{A1}\,\text{a11}^3\,\text{b11}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta1}^2+\frac{1}{16}\,\text{a11}^3\,\text{A2}\,\text{b11}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta1}^2+\frac{1}{4}\,\text{A1}\,\text{a11}\,\text{a22}\,\text{b11}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta1}^2-$$

$$\frac{1}{4}\,\text{a11}\,\text{A2}\,\text{a22}\,\text{b11}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta1}^2-\frac{1}{2}\,\text{A1}\,\text{a33}\,\text{b11}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta1}^2-\frac{1}{2}\,\text{A2}\,\text{a33}\,\text{b11}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta1}^2+$$

$$2\,\text{a11}^2\,\text{b22}\,\text{C1}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta1}^2+4\,\text{a22}\,\text{b22}\,\text{C1}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta1}^2-2\,\text{a11}^2\,\text{b22}\,\text{C2}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta1}^2+$$

$$4\,\text{a22}\,\text{b22}\,\text{C2}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta1}^2+\frac{1}{2}\,\text{a11}\,\text{b31}\,\text{D1}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta1}^2+\frac{1}{2}\,\text{a11}\,\text{b31}\,\text{D2}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta1}^2+$$

$$\frac{4\,\text{A1}\,\text{B1}\,\text{b11}\,\text{b21}\,g\,\pi\,\text{zeta1}^3}{3\,k^2}+\frac{4\,\text{A2}\,\text{b11}\,\text{B2}\,\text{b21}\,g\,\pi\,\text{zeta1}^3}{3\,k^2}-\frac{\text{a11}\,\text{B1}\,\text{b21}\,\text{c0}\,g\,\pi\,\text{zeta1}^3}{6\,k^2}+\frac{\text{a11}\,\text{B2}\,\text{b21}\,\text{c0}\,g\,\pi\,\text{zeta1}^3}{6\,k^2}-$$

$$\frac{\text{A1}\,\text{a11}\,\text{b11}\,\text{c1}\,g\,\pi\,\text{zeta1}^3}{6\,k^2}+\frac{\text{a11}\,\text{A2}\,\text{b11}\,\text{c1}\,g\,\pi\,\text{zeta1}^3}{6\,k^2}+\frac{\text{A1}\,\text{a11}\,\text{b11}\,g\,\pi\,\text{psiX1}\,\text{zeta1}^3}{6\,k^2}-\frac{\text{a11}\,\text{A2}\,\text{b11}\,g\,\pi\,\text{psiX1}\,\text{zeta1}^3}{6\,k^2}+$$

$$\frac{1}{6}\,\text{a11}\,\text{B1}\,\text{b21}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta1}^3-\frac{1}{6}\,\text{a11}\,\text{B2}\,\text{b21}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta1}^3+\frac{\text{A1}^2\,\text{b11}^2\,g\,\pi\,\text{zeta1}^4}{3\,k^2}-\frac{\text{A2}^2\,\text{b11}^2\,g\,\pi\,\text{zeta1}^4}{3\,k^2}-$$

$$\frac{\text{A1}\,\text{a11}\,\text{b11}\,\text{c0}\,g\,\pi\,\text{zeta1}^4}{24\,k^2}-\frac{\text{a11}\,\text{A2}\,\text{b11}\,\text{c0}\,g\,\pi\,\text{zeta1}^4}{24\,k^2}+\frac{1}{24}\,\text{A1}\,\text{a11}\,\text{b11}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta1}^4+$$

$$\frac{1}{24}\,\text{a11}\,\text{A2}\,\text{b11}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta1}^4+\frac{\text{A1}^2\,\text{a11}^2\,\text{b11}^2\,g\,\pi\,\text{zeta2}}{k^2}+\frac{\text{a11}^2\,\text{A2}^2\,\text{b11}^2\,g\,\pi\,\text{zeta2}}{k^2}+\frac{\text{B1}^2\,\text{b21}^2\,g\,\pi\,\text{zeta2}}{k^2}+$$

$$\frac{\text{B2}^2\,\text{b21}^2\,g\,\pi\,\text{zeta2}}{k^2}-\frac{\text{A1}\,\text{a11}^3\,\text{b11}\,\text{c0}\,g\,\pi\,\text{zeta2}}{8\,k^2}+\frac{\text{a11}^3\,\text{A2}\,\text{b11}\,\text{c0}\,g\,\pi\,\text{zeta2}}{8\,k^2}-\frac{\text{A1}\,\text{a11}\,\text{a22}\,\text{b11}\,\text{c0}\,g\,\pi\,\text{zeta2}}{2\,k^2}-$$

$$\frac{\text{a11}\,\text{A2}\,\text{a22}\,\text{b11}\,\text{c0}\,g\,\pi\,\text{zeta2}}{2\,k^2}+\frac{\text{A1}\,\text{a33}\,\text{b11}\,\text{c0}\,g\,\pi\,\text{zeta2}}{k^2}-\frac{\text{A2}\,\text{a33}\,\text{b11}\,\text{c0}\,g\,\pi\,\text{zeta2}}{k^2}-\frac{\text{a11}\,\text{B1}\,\text{b21}\,\text{c1}\,g\,\pi\,\text{zeta2}}{k^2}+$$

$$\frac{\text{a11}\,\text{B2}\,\text{b21}\,\text{c1}\,g\,\pi\,\text{zeta2}}{k^2}+\frac{6\,\text{A1}\,\text{a11}\,\text{b11}\,\text{b22}\,\text{C1}\,g\,\pi\,\text{zeta2}}{k^2}-\frac{2\,\text{a11}^2\,\text{b22}\,\text{c0}\,\text{C1}\,g\,\pi\,\text{zeta2}}{k^2}-\frac{4\,\text{a22}\,\text{b22}\,\text{c0}\,\text{C1}\,g\,\pi\,\text{zeta2}}{k^2}+$$

$$\frac{4\,\text{b22}^2\,\text{C1}^2\,g\,\pi\,\text{zeta2}}{k^2}-\frac{\text{A1}\,\text{a11}\,\text{b11}\,\text{c2}\,g\,\pi\,\text{zeta2}}{k^2}+\frac{\text{a11}\,\text{A2}\,\text{b11}\,\text{c2}\,g\,\pi\,\text{zeta2}}{k^2}-\frac{6\,\text{a11}\,\text{A2}\,\text{b11}\,\text{b22}\,\text{C2}\,g\,\pi\,\text{zeta2}}{k^2}-$$

$$\frac{2\,\text{a11}^2\,\text{b22}\,\text{c0}\,\text{C2}\,g\,\pi\,\text{zeta2}}{k^2}+\frac{4\,\text{a22}\,\text{b22}\,\text{c0}\,\text{C2}\,g\,\pi\,\text{zeta2}}{k^2}+\frac{4\,\text{b22}^2\,\text{C2}^2\,g\,\pi\,\text{zeta2}}{k^2}+\frac{2\,\text{A1}\,\text{b11}\,\text{b31}\,\text{D1}\,g\,\pi\,\text{zeta2}}{k^2}-$$

$$\frac{\text{a11}\,\text{b31}\,\text{c0}\,\text{D1}\,g\,\pi\,\text{zeta2}}{k^2}+\frac{2\,\text{A2}\,\text{b11}\,\text{b31}\,\text{D2}\,g\,\pi\,\text{zeta2}}{k^2}+\frac{\text{a11}\,\text{b31}\,\text{c0}\,\text{D2}\,g\,\pi\,\text{zeta2}}{k^2}+\frac{\text{a11}\,\text{B1}\,\text{b21}\,g\,\pi\,\text{psiX1}\,\text{zeta2}}{k^2}-$$

$$\frac{\text{a11}\,\text{B2}\,\text{b21}\,g\,\pi\,\text{psiX1}\,\text{zeta2}}{k^2}+\frac{\text{A1}\,\text{a11}\,\text{b11}\,g\,\pi\,\text{psiX2}\,\text{zeta2}}{k^2}-\frac{\text{a11}\,\text{A2}\,\text{b11}\,g\,\pi\,\text{psiX2}\,\text{zeta2}}{k^2}+\frac{g\,\pi\,\text{psiX2}^2\,\text{zeta2}}{k^2}+$$

$$\frac{2\,g\,\pi\,\text{psiX1}\,\text{psiX3}\,\text{zeta2}}{k^2}+\frac{1}{8}\,\text{A1}\,\text{a11}^3\,\text{b11}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta2}-\frac{1}{8}\,\text{a11}^3\,\text{A2}\,\text{b11}\,\sqrt{\frac{g}{k^3}}\,\pi\,U\,\text{zeta2}+$$

$$\frac{1}{2} A1\, a11\, a22\, b11 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta2 + \frac{1}{2} a11\, A2\, a22\, b11 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta2 - A1\, a33\, b11 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta2 +$$

$$A2\, a33\, b11 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta2 + 2\, a11^2\, b22\, C1 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta2 + 4\, a22\, b22\, C1 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta2 +$$

$$2\, a11^2\, b22\, C2 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta2 - 4\, a22\, b22\, C2 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta2 + a11\, b31\, D1 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta2 -$$

$$a11\, b31\, D2 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta2 + \frac{4\, A1\, B1\, b11\, b21\, g\, \pi\, zeta1\, zeta2}{k^2} - \frac{4\, A2\, b11\, B2\, b21\, g\, \pi\, zeta1\, zeta2}{k^2} -$$

$$\frac{a11\, B1\, b21\, c0\, g\, \pi\, zeta1\, zeta2}{k^2} - \frac{a11\, B2\, b21\, c0\, g\, \pi\, zeta1\, zeta2}{k^2} - \frac{A1\, a11\, b11\, c1\, g\, \pi\, zeta1\, zeta2}{k^2} -$$

$$\frac{a11\, A2\, b11\, c1\, g\, \pi\, zeta1\, zeta2}{k^2} + \frac{A1\, a11\, b11\, g\, \pi\, psiX1\, zeta1\, zeta2}{k^2} + \frac{a11\, A2\, b11\, g\, \pi\, psiX1\, zeta1\, zeta2}{k^2} +$$

$$a11\, B1\, b21 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta1\, zeta2 + a11\, B2\, b21 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta1\, zeta2 + \frac{2\, A1^2\, b11^2\, g\, \pi\, zeta1^2\, zeta2}{k^2} +$$

$$\frac{2\, A2^2\, b11^2\, g\, \pi\, zeta1^2\, zeta2}{k^2} - \frac{A1\, a11\, b11\, c0\, g\, \pi\, zeta1^2\, zeta2}{2\, k^2} + \frac{a11\, A2\, b11\, c0\, g\, \pi\, zeta1^2\, zeta2}{2\, k^2} +$$

$$\frac{1}{2} A1\, a11\, b11 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta1^2\, zeta2 - \frac{1}{2} a11\, A2\, b11 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta1^2\, zeta2 + \frac{A1^2\, b11^2\, g\, \pi\, zeta2^2}{k^2} -$$

$$\frac{A2^2\, b11^2\, g\, \pi\, zeta2^2}{k^2} - \frac{A1\, a11\, b11\, c0\, g\, \pi\, zeta2^2}{2\, k^2} - \frac{a11\, A2\, b11\, c0\, g\, \pi\, zeta2^2}{2\, k^2} + \frac{1}{2} A1\, a11\, b11 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta2^2 +$$

$$\frac{1}{2} a11\, A2\, b11 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta2^2 + \frac{2\, A1\, B1\, b11\, b21\, g\, \pi\, zeta3}{k^2} + \frac{2\, A2\, b11\, B2\, b21\, g\, \pi\, zeta3}{k^2} -$$

$$\frac{a11\, B1\, b21\, c0\, g\, \pi\, zeta3}{k^2} + \frac{a11\, B2\, b21\, c0\, g\, \pi\, zeta3}{k^2} - \frac{A1\, a11\, b11\, c1\, g\, \pi\, zeta3}{k^2} + \frac{a11\, A2\, b11\, c1\, g\, \pi\, zeta3}{k^2} +$$

$$\frac{A1\, a11\, b11\, g\, \pi\, psiX1\, zeta3}{k^2} - \frac{a11\, A2\, b11\, g\, \pi\, psiX1\, zeta3}{k^2} + \frac{2\, g\, \pi\, psiX1\, psiX2\, zeta3}{k^2} - \frac{2\, g\, \pi\, R3\, zeta3}{k^2} +$$

$$a11\, B1\, b21 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta3 - a11\, B2\, b21 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta3 + \frac{2 \sqrt{\frac{g}{k}}\, \pi\, psiX3\, U\, zeta3}{k} + \frac{2\, A1^2\, b11^2\, g\, \pi\, zeta1\, zeta3}{k^2} -$$

$$\frac{2\, A2^2\, b11^2\, g\, \pi\, zeta1\, zeta3}{k^2} - \frac{A1\, a11\, b11\, c0\, g\, \pi\, zeta1\, zeta3}{k^2} - \frac{a11\, A2\, b11\, c0\, g\, \pi\, zeta1\, zeta3}{k^2} +$$

$$\left. A1\, a11\, b11 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta1\, zeta3 + a11\, A2\, b11 \sqrt{\frac{g}{k^3}}\, \pi\, U\, zeta1\, zeta3 + \frac{g\, \pi\, zeta3^2}{k^2} \right) + \frac{1}{2\, k^2}\, e^{-2\, h\, k}\, eps^3$$

$$\Big(A1^2\, b11^2\, e^{2\, h\, k}\, g\, zeta1 + A1\, b11 \left(B1\, b21 \left(-1 + e^{2\, h\, k}\right) g - a11\, e^{2\, h\, k} \left(c1\, g - g\, psiX1 + c0\, g\, zeta1 - \sqrt{g\, k}\, U\, zeta1\right)\right) +$$

$$e^{2\, h\, k} \Big(2\, g\, h\, k\, psiX1\, psiX2 + 2\, h \sqrt{g\, k^3}\, psiX3\, U - a11\, (B1 + B2)\, b21 \left(c0\, g - \sqrt{g\, k}\, U\right) +$$

$$A2^2\, b11^2\, g\, zeta1 + g\, psiX1^2\, zeta1 - 2\, g\, R2\, zeta1 + 2 \sqrt{g\, k}\, psiX2\, U\, zeta1 +$$

$$A2\, b11 \left(B2\, b21 \left(-1 + e^{2\, h\, k}\right) g + a11 \left(-c1\, g + g\, psiX1 + c0\, g\, zeta1 - \sqrt{g\, k}\, U\, zeta1\right)\right) -$$

$$2\, g\, R1\, zeta2 + 2 \sqrt{g\, k}\, psiX1\, U\, zeta2 + 2\, g\, zeta1\, zeta2 - 2\, g\, R0\, zeta3 + k\, U^2\, zeta3\Big)\Big);$$

Cases

Case 1: no wave-induced flow, no current

```
psiX1 = 0; psiX2 = 0; psiX3 = 0;  U = 0; R0 = 0;
```

EL-equations up to O(eps^2)

Variation zeta1

```
FullSimplify[
 Solve[
  D[Coefficient[AvL, eps, 2], zeta1] == 0, R1
      ]
            ]
```

{{R1 → zeta1}}

Update AvL

```
AvL1 = AvL /. R1 → zeta1;
```

Variation R1

```
FullSimplify[
 Solve[
  D[Coefficient[AvL, eps, 2], R1] == 0, zeta1
      ]
            ]
```

{{zeta1 → 0}}

Update AvL

```
AvL2 = AvL1 /. zeta1 → 0;
```

Variation b11

```
FullSimplify[
 Solve[
  D[Coefficient[AvL2, eps, 2], b11] == 0, b11
      ]
            ]
```

$$\left\{\left\{b11 \to \frac{a11\ (A1 + A2)\ c0\ (1 + \mathrm{Coth}[h\,k])}{2\ \left(A1^2 + A2^2\ e^{2hk}\right)}\right\}\right\}$$

Update AvL

$$AvL3 = AvL2\ /.\ b11 \to \frac{a11\ (A1 + A2)\ c0\ (1 + \mathrm{Coth}[h\,k])}{2\ \left(A1^2 + A2^2\ e^{2hk}\right)};$$

Variation A1

```
FullSimplify[
 Solve[
  D[Coefficient[AvL3, eps, 2], A1] == 0, A1
      ]
            ]
```

$$\{\{A1 \to -A2\},\ \{A1 \to A2\ e^{2hk}\}\}$$

Update AvL

```
AvL4 = AvL3 /. A1 → A2 ⅇ^(2 h k);
```

Variation a11

```
FullSimplify[
 Solve[
  D[ Coefficient[AvL4, eps, 2], a11 ] == 0, c0
    ]
         ]
```

$\{\{c0 \to -\sqrt{\mathrm{Tanh}[h\,k]}\}, \{c0 \to \sqrt{\mathrm{Tanh}[h\,k]}\}\}$

Update AvL

```
AvL5 = AvL4 /. c0 → Sqrt[Tanh[h k]] ;
```

EL-equations up to O(eps^3)

Variation a11

```
FullSimplify[
 Solve[
  D[ Coefficient[AvL5, eps, 3], a11 ] == 0, c1
    ]
         ]
```

{{c1 → 0}}

Update AvL

```
AvL6 = AvL5 /. c1 → 0;
```

EL-equations up to O(eps^4)

Variation b21

```
FullSimplify[
 Solve[
  D[Coefficient[AvL6, eps, 4], b21] == 0,  b21
    ]
         ]
```

{{b21 → 0}}

Update AvL

```
AvL7 = AvL6 /. b21 → 0;
```

Variation b22

```
FullSimplify[
 Solve[
  D[Coefficient[AvL7, eps, 4], b22] == 0,  b22
    ]
         ]
```

$\{\{b22 \to -\left(\left((C1 + C2)\, e^{5hk} \left(a11^2 \mathrm{Cosh}[h\,k] - 2\, a22\, \mathrm{Sinh}[h\,k]\right)\right) \Big/ \left(\left(-1 + e^{4hk}\right)^{3/2} \left(C1^2 + C2^2 e^{4hk}\right)\right)\right)\}\}$

Update AvL

$AvL8 = AvL7\ /.\ b22 \to -\left(\left((C1 + C2)\, e^{5hk} \left(a11^2 \mathrm{Cosh}[h\,k] - 2\, a22\, \mathrm{Sinh}[h\,k]\right)\right) \Big/ \left(\left(-1 + e^{4hk}\right)^{3/2} \left(C1^2 + C2^2 e^{4hk}\right)\right)\right);$

Variation C1

```
FullSimplify[
 Solve[
  D[Coefficient[AvL8, eps, 4], C1] == 0, C1
      ]
           ]
```

$\{\{C1 \to -C2\}, \{C1 \to C2\, e^{4 h k}\}\}$

Update AvL

```
AvL9 = AvL8 /. C1 → C2 ⅇ^(4 h k);
```

Variation a22

```
FullSimplify[
 Solve[
  Simplify[ D[Coefficient[AvL9, eps, 4], a22] ] == 0, a22
      ]
           ]
```

$\left\{\left\{a22 \to \frac{1}{8}\, a11^2\, (5\, \mathrm{Cosh}[h\, k] + \mathrm{Cosh}[3\, h\, k])\, \mathrm{Csch}[h\, k]^3\right\}\right\}$

Update AvL

```
AvL10 = AvL9 /. a22 → 1/8 a11^2 (5 Cosh[h k] + Cosh[3 h k]) Csch[h k]^3;
```

Variation zeta2

```
FullSimplify[
 Solve[
  D[Coefficient[AvL10, eps, 4], zeta2] == 0, R2
      ]
           ]
```

$\left\{\left\{R2 \to zeta2 + \frac{1}{2}\, a11^2\, \mathrm{Csch}[2\, h\, k]\right\}\right\}$

Update AvL

```
AvL11 = AvL10 /. R2 → zeta2 + 1/2 a11^2 Csch[2 h k];
```

Variation R2

```
FullSimplify[
 Solve[
  D[Coefficient[AvL10, eps, 4], R2] == 0, zeta2
      ]
           ]
```

$\{\{zeta2 \to 0\}\}$

Update AvL

```
AvL12 = AvL11 /. zeta2 → 0;
```

Variation a11

```
FullSimplify[
 Solve[
  D[ Coefficient[AvL12, eps, 4], a11 ] == 0, c2
      ]
           ]
```

$$\left\{\left\{c2 \to \frac{1}{16}\, a11^2\, (8 + \mathrm{Cosh}[4\, h\, k])\, \sqrt{\mathrm{Coth}[h\, k]}\, \mathrm{Csch}[h\, k]^3\, \mathrm{Sech}[h\, k]\right\}\right\}$$

Update AvL

```
AvL13 = AvL12 /. c2 → 1/16 a11^2 (8 + Cosh[4 h k]) Sqrt[Coth[h k]] Csch[h k]^3 Sech[h k];
```

EL-equations up to O(eps^5)

Variation a11

```
FullSimplify[
 Solve[
  D[ Coefficient[AvL13, eps, 5], a11] == 0, c3
      ]
           ]
```

{{c3 → 0}}

Update AvL

```
AvL14 = AvL13 /. c3 → 0;
```

EL-equations up to O(eps^6)

Variation zeta3

```
FullSimplify[
 Solve[
  D[Coefficient[AvL14, eps, 6], zeta3] == 0, R3
      ]
           ]
```

{{R3 → zeta3}}

Update AvL

```
AvL15 = AvL14 /. R3 → zeta3;
```

Variation R3

```
FullSimplify[
 Solve[
  D[Coefficient[AvL14, eps, 6], R3] == 0, zeta3
      ]
           ]
```

{{zeta3 → 0}}

Update AvL

```
AvL16 = AvL15 /. zeta3 → 0;
```

Variation a32

```
FullSimplify[
 Solve[
  D[Coefficient[AvL16, eps, 6], a32] == 0, a32
      ]
           ]
```

$$\left\{\left\{a32 \to 2\, b32\, (E1 + E2)\, \sqrt{\mathrm{Tanh}[h\, k]}\right\}\right\}$$

Update AvL

```
AvL17 = AvL16 /. a32 → 2 b32 (E1 + E2) √Tanh[h k] ;
```

Variation b32

```
FullSimplify[
 Solve[
  D[Coefficient[AvL17, eps, 6], b32] == 0, b32
      ]
           ]
```

$$\{\{b32 \to 0\}\}$$

Update AvL

```
AvL18 = AvL17 /. b32 → 0;
```

Variation b31

```
FullSimplify[
 Solve[
  Simplify[ D[Coefficient[AvL18, eps, 6], b31] == 0 ], b31
      ]
           ]
```

$$\Big\{\Big\{b31 \to \Big((D1 + D2)\, e^{6 h k} \Big(-2\, e^{h k} \sqrt{-1 + e^{4 h k}}\, \big(-13\, a11^3 + 24\, a33 + 8\, (3\, a11^3 - 4\, a33)\, \mathrm{Cosh}[2 h k]\big)\, \mathrm{Sinh}[h k] + \mathrm{Cosh}[4 h k]\, \Big(-(5\, a11^3 + 8\, a33)\, (-1 + e^{2 h k}) \sqrt{-1 + e^{4 h k}} + 4\, a11^3\, (1 + e^{2 h k})^2\, \mathrm{Tanh}[h k]^{3/2}\Big)\Big)\Big) \Big/ \Big(4\, (-1 + e^{2 h k})^4\, (1 + e^{2 h k})^2\, (D1^2 + D2^2\, e^{2 h k}) \sqrt{-1 + e^{4 h k}}\, \mathrm{Tanh}[h k]^{3/2}\Big)\Big\}\Big\}$$

Update AvL

```
AvL19 =
  AvL18 /. b31 → ((D1 + D2) e^(6 h k) (-2 e^(h k) √(-1 + e^(4 h k)) (-13 a11^3 + 24 a33 + 8 (3 a11^3 - 4 a33) Cosh[2 h k]) Sinh[h k] +
        Cosh[4 h k] (-(5 a11^3 + 8 a33) (-1 + e^(2 h k)) √(-1 + e^(4 h k)) + 4 a11^3 (1 + e^(2 h k))^2 Tanh[h k]^(3/2)))) /
     (4 (-1 + e^(2 h k))^4 (1 + e^(2 h k))^2 (D1^2 + D2^2 e^(2 h k)) √(-1 + e^(4 h k)) Tanh[h k]^(3/2));
```

Variation D1

```
FullSimplify[
 Solve[
  Simplify[ D[Coefficient[AvL19, eps, 6], D1] == 0 ], D1
      ]
           ]
```

$$\left\{\{D1 \to -D2\},\ \left\{D1 \to D2\, e^{2 h k}\right\}\right\}$$

Update AvL

```
AvL20 = AvL19 /. D1 → D2 e^(2 h k);
```

Variation b33

```
FullSimplify[
 Solve[
  D[Coefficient[AvL20, eps, 6], b33] == 0,  b33
     ]
          ]
```

$$\left\{\left\{b33 \to -\left(\left(e^{10hk}(F1+F2)\left(13\,a11^3-24\,a33+32\left(a11^3+a33\right)\mathrm{Cosh}[2hk]+\left(3\,a11^3-8\,a33\right)\mathrm{Cosh}[4hk]\right)\right)\Big/\left(4\left(-1+e^{2hk}\right)^{9/2}\sqrt{1+e^{2hk}}\left(1+e^{2hk}+e^{4hk}\right)\left(F1^2+e^{6hk}F2^2\right)\right)\right)\right\}\right\}$$

Update AvL

```
AvL21 =
  AvL20 /. b33 → -((e^(10 h k) (F1 + F2) (13 a11^3 - 24 a33 + 32 (a11^3 + a33) Cosh[2 h k] + (3 a11^3 - 8 a33) Cosh[4 h k])) /
      (4 (-1 + e^(2 h k))^(9/2) Sqrt[1 + e^(2 h k)] (1 + e^(2 h k) + e^(4 h k)) (F1^2 + e^(6 h k) F2^2)));
```

Variation F1

```
FullSimplify[
 Solve[
  Simplify[ D[Coefficient[AvL21, eps, 6], F1] == 0 ],  F1
     ]
          ]
```

$$\{\{F1 \to -F2\},\ \{F1 \to e^{6hk}\,F2\}\}$$

Update AvL

```
AvL22 = AvL21 /. F1 → e^(6 h k) F2;
```

Variation a33

```
FullSimplify[
 Solve[
  Simplify[ D[Coefficient[AvL22, eps, 6], a33] == 0 ],  a33
     ]
          ]
```

$$\left\{\left\{a33 \to \frac{3}{256}\,a11^3\,(14+15\,\mathrm{Cosh}[2hk]+6\,\mathrm{Cosh}[4hk]+\mathrm{Cosh}[6hk])\,\mathrm{Csch}[hk]^6\right\}\right\}$$

Case 2: allowing for wave-induced flow

Case 3: uniform current

Check of solutions (Fenton 1985)

Check of Case 1

Final results

$$\text{psiX1} = 0;\ \text{psiX2} = 0;\ \text{psiX3} = 0;\ \text{U} = 0;\ \text{R0} = 0;$$

$$\text{a11} = 1;$$

$$\text{zeta1} = 0;\ \ \text{zeta2} = 0;\ \ \text{zeta3} = 0;$$

$$\text{R1} = \text{zeta1};\ \text{R2} = \text{zeta2} + \frac{1}{2}\,\text{a11}^2\,\text{Csch}[2\,h\,k];\ \text{R3} = \text{zeta3};$$

$$\text{A1} = \text{A2}\,e^{2hk};\ \text{C1} = \text{C2}\,e^{4hk};\ \text{D1} = \text{D2}\,e^{2hk};\ \text{F1} = e^{6hk}\,\text{F2};$$

$$\text{c0} = \sqrt{\text{Tanh}[h\,k]}\ ;\ \text{c1} = 0;\ \text{c2} = \frac{1}{16}\,\text{a11}^2\,(8 + \text{Cosh}[4\,h\,k])\,\sqrt{\text{Coth}[h\,k]}\ \text{Csch}[h\,k]^3\,\text{Sech}[h\,k];\ \text{c3} = 0;$$

$$\text{b11} = \frac{\text{a11}\,(\text{A1} + \text{A2})\,\text{c0}\,(1 + \text{Coth}[h\,k])}{2\left(\text{A1}^2 + \text{A2}^2\,e^{2hk}\right)};$$

$$\text{b21} = 0;$$

$$\text{a22} = \frac{1}{8}\,\text{a11}^2\,(5\,\text{Cosh}[h\,k] + \text{Cosh}[3\,h\,k])\,\text{Csch}[h\,k]^3;$$

$$\text{b22} = -\left(\left((\text{C1} + \text{C2})\,e^{5hk}\left(\text{a11}^2\,\text{Cosh}[h\,k] - 2\,\text{a22}\,\text{Sinh}[h\,k]\right)\right)\Big/\left(\left(-1 + e^{4hk}\right)^{3/2}\left(\text{C1}^2 + \text{C2}^2\,e^{4hk}\right)\right)\right);$$

$$\text{b32} = 0;$$

$$\text{a32} = 2\,\text{b32}\,(\text{E1} + \text{E2})\,\sqrt{\text{Tanh}[h\,k]}\ ;$$

$$\text{a33} = \frac{3}{256}\,\text{a11}^3\,(14 + 15\,\text{Cosh}[2\,h\,k] + 6\,\text{Cosh}[4\,h\,k] + \text{Cosh}[6\,h\,k])\,\text{Csch}[h\,k]^6;$$

$$\begin{aligned}\text{b31} = \Big(&(\text{D1} + \text{D2})\,e^{6hk}\Big(-2\,e^{hk}\sqrt{-1 + e^{4hk}}\ \left(-13\,\text{a11}^3 + 24\,\text{a33} + 8\left(3\,\text{a11}^3 - 4\,\text{a33}\right)\text{Cosh}[2\,h\,k]\right)\text{Sinh}[h\,k] + \\ &\text{Cosh}[4\,h\,k]\left(-\left(5\,\text{a11}^3 + 8\,\text{a33}\right)\left(-1 + e^{2hk}\right)\sqrt{-1 + e^{4hk}} + 4\,\text{a11}^3\left(1 + e^{2hk}\right)^2\text{Tanh}[h\,k]^{3/2}\right)\Big)\Big)\Big/ \\ &\left(4\left(-1 + e^{2hk}\right)^4\left(1 + e^{2hk}\right)^2\left(\text{D1}^2 + \text{D2}^2\,e^{2hk}\right)\sqrt{-1 + e^{4hk}}\ \text{Tanh}[h\,k]^{3/2}\right);\end{aligned}$$

$$\begin{aligned}\text{b33} = -\Big(&\left(e^{10hk}\,(\text{F1} + \text{F2})\left(13\,\text{a11}^3 - 24\,\text{a33} + 32\left(\text{a11}^3 + \text{a33}\right)\text{Cosh}[2\,h\,k] + \left(3\,\text{a11}^3 - 8\,\text{a33}\right)\text{Cosh}[4\,h\,k]\right)\right)\Big/ \\ &\left(4\left(-1 + e^{2hk}\right)^{9/2}\sqrt{1 + e^{2hk}}\ \left(1 + e^{2hk} + e^{4hk}\right)\left(\text{F1}^2 + e^{6hk}\,\text{F2}^2\right)\right)\Big);\end{aligned}$$

Wave fields, celerity, Bernoulli constant

Celerity

```
Collect[c, eps, FullSimplify]
```

$$\frac{1}{16}\,\text{eps}^2\,(8 + \text{Cosh}[4\,h\,k])\,\sqrt{\frac{g\,\text{Coth}[h\,k]}{k}}\ \text{Csch}[h\,k]^3\,\text{Sech}[h\,k] + \sqrt{\frac{g\,\text{Tanh}[h\,k]}{k}}$$

Bernoulli constant

```
Collect[R, eps, FullSimplify]
```

$$\frac{\text{eps}^2\, g\, \text{Csch}[2 h k]}{2 k}$$

Velocity potential

```
phiN = Collect[phi /. Sin[q_] :> Sin[ FullSimplify[q / (k * (x - c * t))] * theta ], eps, FullSimplify]
```

$$\left(e^{-k z}\left(1+e^{2k(h+z)}\right)\text{eps}\sqrt{\frac{g}{k}}\,(-1+\text{Coth}[h k])\,\text{Sin}[\text{theta}]\right)\Big/\left(2 k\sqrt{\text{Coth}[h k]}\right)+$$
$$\left(3 e^{2k(h-z)}\left(1+e^{4k(h+z)}\right)\text{eps}^2\sqrt{\frac{g}{k}}\,\text{Sin}[2\,\text{theta}]\right)\Big/\left(\left(-1+e^{2hk}\right)^4 k\sqrt{\text{Coth}[h k]}\right)+$$
$$\Big(e^{6hk-kz}\left(1+e^{2k(h+z)}\right)\text{eps}^3\sqrt{\frac{g}{k}}\,(-(23-7\,\text{Cosh}[2 h k]+10\,\text{Cosh}[4 h k]+\text{Cosh}[6 h k])\,\text{Sin}[\text{theta}]-$$
$$(-11+2\,\text{Cosh}[2 h k])\,(-1+2\,\text{Cosh}[2 k(h+z)])\,\text{Sin}[3\,\text{theta}])\Big)\Big/\left(\left(-1+e^{2hk}\right)^7 k\sqrt{\text{Coth}[h k]}\right)$$

Free-surface elevation

```
etaN = Collect[eta /. Cos[q_] :> Cos[ FullSimplify[q / (k * (x - c * t))] * theta ], eps, FullSimplify]
```

$$\frac{\text{eps}\,\text{Cos}[\text{theta}]}{k}+\frac{1}{8k}\text{eps}^2\,\text{Cos}[2\,\text{theta}]\,(5\,\text{Cosh}[h k]+\text{Cosh}[3 h k])\,\text{Csch}[h k]^3-$$
$$\frac{1}{64 k}3\,\text{eps}^3\,\text{Cos}[\text{theta}]\,(14+15\,\text{Cosh}[2 h k]+6\,\text{Cosh}[4 h k]+\text{Cosh}[6 h k])\,\text{Csch}[h k]^6\,\text{Sin}[\text{theta}]^2$$

Comparison with Fenton's results

Comparison of celerity, potential, free-surface elevation

```
{
 FullSimplify[ c == cF ],

 FullSimplify[etaN == (etaF /. Cos[q_] :> Cos[ FullSimplify[q / (k * (x - cF * t))] * theta])
              ],

 FullSimplify[phiN == (phiF /. Sin[q_] :> Sin[ FullSimplify[q / (k * (x - cF * t))] * theta])
              ]
}
{True, True, True}
```

Comparison of Bernoulli constant

1. Check of the validity of the dynamic b.c:

Before comparing with Fenton, we verify that the dynamic b.c. on the free surface is, indeed, verified. That is, we check the validity of the equality :

(D[phiN, t] + (D[phiN, x]^2 + D[phiN, z]^2)/2 + g*z) /. z -> etaN == R

```
B1 = D[phiN, theta] * D[k * (x - c * t), t] +
    ( (D[phiN, theta] * D[k * (x - c * t), x])^2 + D[phiN, z]^2) / 2 + g * z /. z → etaN;

B2 = B1 /. Exp[x_] :> ExpT[Coefficient[x, eps, 0]] * Exp[Simplify[x - Coefficient[x, eps, 0]]] ;
```

```
B3 = B2 /. Exp[x_] /; FreeQ[x, eps] == False → 1 + x + x^2/2 + x^3/6;

B4 = B3 /. ExpT[x_] :→ Exp[x];

B5 = B4 /. eps^x_ /; x ≥ 4 → 0;

B6 = B5 /. Cosh[x_] :→ Sinh[Coefficient[x, eps, 0]] * Sinh[FullSimplify[x - Coefficient[x, eps, 0]]] +
     Cosh[Coefficient[x, eps, 0]] * Cosh[FullSimplify[x - Coefficient[x, eps, 0]]];

B7 = B6 /. Sinh[x_] /; FreeQ[x, eps] == False → x + x^3 / 6;

B8 = B7 /. Cosh[x_] /; FreeQ[x, eps] == False → 1 + x^2 / 2;
```

The equality has to be valid for each theta. Since the rhs is a constant, we integrate both sides wrt to theta, from 0 to 2*pi. That is equivalent to:

```
B = FullSimplify[

  Integrate[ Coefficient[B8, eps, 0] + eps * Coefficient[B8, eps, 1] + eps^2 * Coefficient[B8, eps, 2] +
     eps^3 * Coefficient[B8, eps, 3],    {theta, 0, 2 * Pi}   ] / 2 / Pi

                ]
```

$$\frac{\text{eps}^2\, g\, \text{Csch}[2\, h\, k]}{2\, k}$$

Comparing the above result with the previously obtained R:

```
FullSimplify[ R == B ]
```

True

2. Comparison with Fenton:

The results of Fenton, regarding the Bernoulli constant, are given wrt the moving frame, where the supposed undisturbed state is a uniform flow of speed cF (celerity) in the negative direction. Also, the existence of R implies (essentially) an additional time dependence of phi (redefined potential). Thus, first we calculate our redefined potential and, next, we express it in terms of the moving frame (Galilean transformation).

```
phiRedef = phi - R * t;

phiRedefMovingFr = phiRedef  - c * x + c^2 * t / 2  /.  x → (x + c * t);
```

Having obtained the redefined potential in the moving frame, the Bernoulli constant, in terms of that frame, is equal to

- D[phiRedefMovingFr, t] :

```
BmovingFr = - Collect[ D[phiRedefMovingFr, t], eps, FullSimplify ] /. eps^x_ /; x ≥ 4 → 0
```

$$\frac{1}{16\, k}\, \text{eps}^2\, g\, (6 + 2\, \text{Cosh}[2\, h\, k] + \text{Cosh}[4\, h\, k])\, \text{Csch}[h\, k]^3\, \text{Sech}[h\, k] + \frac{g\, \text{Tanh}[h\, k]}{2\, k}$$

Checking our Bernoulli constant with that of Fenton:

```
FullSimplify[ BmovingFr == RF ]
```

True

Check of Case 2

Fenton' s results for Stokes waves (up to eps^3)

Dimensionless coefficients

```
S = Sech[2*k*h];
SH = Sinh[k*h];
CTH = Coth[k*h];
TH = Tanh[k*h];
C0 = Sqrt[TH];
C2 = Sqrt[TH] * (2 + 7*S^2) / 4 / (1 - S)^2;

A11 = SH^(-1);
A22 = 3*S^2/2/(1 - S)^2;
A31 = (- 4 - 20*S + 10*S^2 - 13*S^3) / 8 / SH / (1 - S)^3;
A33 = (- 2*S^2 + 11*S^3) / 8 / SH / (1 - S)^3;

B11 = a11;
B22 = FullSimplify[CTH * (1 + 2*S) / 2 / (1 - S)];
B31 = FullSimplify[(-3) * (1 + 3*S + 3*S^2 + 2*S^3) / 8 / (1 - S)^3];
B33 = -B31;

D2 = - Sqrt[Coth[k*h]] / 2;

E2 = Tanh[k*h] * (2 + 2*S + 5*S^2) / 4 / (1 - S)^2;
```

Wave fields, celerity, Bernoulli constant

Celerity

```
(* 1st definition *)   c1F = Sqrt[g/k] * ( C0 + eps^2*C2 );
(* 2nd definition *)   c2F = Sqrt[g/k^3] * ( C0*k + eps^2*(C2*k + D2/h) );

(* choice of definition*) cF = c1F;
```

Bernoulli constant

```
RF = (g/k) * (C0^2/2 + eps^2*E2);
```

Free-surface elevation

```
etaF = (  eps*B11*Cos[k*(x - cF*t)] + eps^2*B22*Cos[2*k*(x - cF*t)] +
      eps^3*(  B31*Cos[k*(x - cF*t)] + B33*Cos[3*k*(x - cF*t)]  )   ) / k;
```

Velocity potential

```
phiF = C0*Sqrt[g/k^3] *
    (  eps*A11*Cosh[k*(z + h)]*Sin[k*(x - cF*t)] + eps^2*A22*Cosh[2*k*(z + h)]*
       Sin[2*k*(x - cF*t)] + eps^3*( A31*Cosh[k*(z + h)]*Sin[k*(x - cF*t)] +
         A33*Cosh[3*k*(z + h)]*Sin[3*k*(x - cF*t)] )   );
```

Wave-induced mean flow

```
UmwF = FullSimplify[ cF - c1F ];
```

Appendix B

Implementation of the AVP for slowly modulated wavetrains using Wolfram® Mathematica (Ch.4)

The AVP in slowly modulated wavetrains

Setup

Ansatzes

```
$Assumptions = {g > 0, k0 > 0, h > 0, w0 > 0};

phi1 = psi[eps * x, eps * t] +
   b11[eps * x, eps * t] * K11[z, eps * x, eps * t] * Sin[ (k0 * x - w0 * t) + eps * thetaTilda[x, t] ];
phi2 = K21[z, eps * x, eps * t] * (  b21[eps * x, eps * t] * Sin[ (k0 * x - w0 * t) + eps * thetaTilda[x, t] ] +
      b21s[eps * x, eps * t] * Cos[ (k0 * x - w0 * t) + eps * thetaTilda[x, t] ]   ) +
   K22[z, eps * x, eps * t] * (  b22[eps * x, eps * t] * Sin[ 2 * ((k0 * x - w0 * t) + eps * thetaTilda[x, t]) ] +
      b22s[eps * x, eps * t] * Cos[ 2 * ((k0 * x - w0 * t) + eps * thetaTilda[x, t]) ]   );
eta1 = a11[eps * x, eps * t] * Cos[ (k0 * x - w0 * t) + eps * thetaTilda[x, t] ];
eta2 = zeta[eps * x, eps * t]  + a21[eps * x, eps * t] * Cos[ (k0 * x - w0 * t) + eps * thetaTilda[x, t] ] +
   a21s[eps * x, eps * t] * Sin[ (k0 * x - w0 * t) + eps * thetaTilda[x, t] ] +
   a22[eps * x, eps * t] * Cos[ 2 * ((k0 * x - w0 * t) + eps * thetaTilda[x, t]) ] +
   a22s[eps * x, eps * t] * Sin[ 2 * ((k0 * x - w0 * t) + eps * thetaTilda[x, t]) ];

phi = eps * phi1 + eps^2 * phi2 ;
eta = eps * eta1 + eps^2 * eta2;
```

Vertical problem

Procedure

Lagrangian Lz of the vertical problem

```
Lz = D[phi, t] + (D[phi, x]^2 + D[phi, z]^2) / 2 + g * z;
```

Rewrite Lz in terms of theta & omit terms of O(eps^5) or higher

```
Lz1 = Lz /. Sin[q_] :> Sin[ FullSimplify[q / ( (k0 * x - w0 * t) + eps * thetaTilda[x, t] )] * theta ];

Lz2 = Lz1 /. Cos[q_] :> Cos[ FullSimplify[q / ((k0 * x - w0 * t) + eps * thetaTilda[x, t] )] * theta ];

Lz3 = Collect[Lz2, eps] /. eps^x_ /; x ≥ 5 → 0;
```

The derivatives of thetaTilda are slowly varying functions

```
Lz4 = Lz3 /. { Derivative[1, 0][thetaTilda][x, t] → kTilda[eps * x, eps * t],
    Derivative[0, 1][thetaTilda][x, t] → - wTilda[eps * x, eps * t] };
```

Averaged Lagrangian AvLz of the vertical problem

```
AvLz = Collect[Integrate[Lz4, {theta, 0, 2*Pi}] / 2 / Pi, eps, Simplify]
```

```
g z + eps^2 (psi^(0,1)[eps x, eps t] + 1/4 b11[eps x, eps t]^2 (k0^2 K11[z, eps x, eps t]^2 + K11^(1,0,0)[z, eps x, eps t]^2)) +
 1/2 eps^3 b11[eps x, eps t] (k0 b11[eps x, eps t] K11[z, eps x, eps t]^2 kTilda[eps x, eps t] + b21[eps x, eps t]
    (k0^2 K11[z, eps x, eps t] K21[z, eps x, eps t] + K11^(1,0,0)[z, eps x, eps t] K21^(1,0,0)[z, eps x, eps t])) +
 1/4 eps^4 (4 k0 b11[eps x, eps t] b21[eps x, eps t] K11[z, eps x, eps t] K21[z, eps x, eps t] kTilda[eps x, eps t] +
    b11[eps x, eps t]^2 K11[z, eps x, eps t]^2 kTilda[eps x, eps t]^2 - 2 k0 b21s[eps x, eps t] K11[z, eps x, eps t]
     K21[z, eps x, eps t] b11^(1,0)[eps x, eps t] + K11[z, eps x, eps t]^2 b11^(1,0)[eps x, eps t]^2 +
    2 psi^(1,0)[eps x, eps t]^2 + b11[eps x, eps t]^2 K11^(0,1,0)[z, eps x, eps t]^2 +
    2 b11[eps x, eps t] (-k0 b21s[eps x, eps t] K21[z, eps x, eps t] K11^(0,1,0)[z, eps x, eps t] +
       K11[z, eps x, eps t] (k0 K21[z, eps x, eps t] b21s^(1,0)[eps x, eps t] +
          b11^(1,0)[eps x, eps t] K11^(0,1,0)[z, eps x, eps t] + k0 b21s[eps x, eps t] K21^(0,1,0)[z, eps x, eps t])) +
    b21[eps x, eps t]^2 (k0^2 K21[z, eps x, eps t]^2 + K21^(1,0,0)[z, eps x, eps t]^2) +
    b21s[eps x, eps t]^2 (k0^2 K21[z, eps x, eps t]^2 + K21^(1,0,0)[z, eps x, eps t]^2) +
    (b22[eps x, eps t]^2 + b22s[eps x, eps t]^2) (4 k0^2 K22[z, eps x, eps t]^2 + K22^(1,0,0)[z, eps x, eps t]^2))
```

```
AvLz1 = AvLz /. {eps*x → x, eps*t → t};
(*During the derivation of the EL equations, x & t are considered as slow variables*)
```

EL equations of Lav up to O (eps^2)

```
eq1 = FullSimplify[
   D[ D[ Coefficient[AvLz1, eps, 2], Derivative[1, 0, 0][K11][z, x, t] ], z ] +
     D[ D[ Coefficient[AvLz1, eps, 2], Derivative[0, 1, 0][K11][z, x, t] ], x ] +
     D[ D[ Coefficient[AvLz1, eps, 2], Derivative[0, 0, 1][K11][z, x, t] ], t ] -
     D[ Coefficient[AvLz1, eps, 2], K11[z, x, t] ] == 0
               ];
```

```
DSolve[eq1, K11[z, x, t], {z, x, t}]
```

```
{{K11[z, x, t] → e^(k0 z) C[1][x, t] + e^(-k0 z) C[2][x, t]}}
```

Update AvLz

```
AvLz2 = Block[{K11}, K11[z_, x_, t_] := e^(k0 z) A1[x, t] + e^(-k0 z) A2[x, t] ; AvLz1];
```

EL equations of Lav up to O (eps^4)

```
eq2 = FullSimplify[
   D[ D[ Coefficient[AvLz2, eps, 4], Derivative[1, 0, 0][K21][z, x, t] ], z ] +
     D[ D[ Coefficient[AvLz2, eps, 4], Derivative[0, 1, 0][K21][z, x, t] ], x ] +
     D[ D[ Coefficient[AvLz2, eps, 4], Derivative[0, 0, 1][K21][z, x, t] ], t ] -
     D[ Coefficient[AvLz2, eps, 4], K21[z, x, t] ] == 0
               ];
```

```
FullSimplify[ DSolve[eq2, K21[z, x, t], {z, x, t}] ]
```

```
{{K21[z, x, t] → 1/(2 k0 (b21[x, t]^2 + b21s[x, t]^2)) e^(-k0 z) (2 k0 b21[x, t]^2 C[2][x, t] + 2 k0 b21s[x, t]^2 C[2][x, t] +
      e^(2 k0 z) (2 k0 (b21[x, t]^2 + b21s[x, t]^2) C[1][x, t] + (1 - 2 k0 z) b11[x, t] b21s[x, t] A1^(1,0)[x, t]) +
      b11[x, t] b21s[x, t] A2^(1,0)[x, t] + 2 k0 z b11[x, t] b21s[x, t] A2^(1,0)[x, t] +
      e^(2 k0 z) (-1 + 2 k0 z) A1[x, t] (b11[x, t] b21[x, t] kTilda[x, t] - b21s[x, t] b11^(1,0)[x, t]) -
      (1 + 2 k0 z) A2[x, t] (b11[x, t] b21[x, t] kTilda[x, t] - b21s[x, t] b11^(1,0)[x, t]))}}
```

```
eq4 = FullSimplify[
   D[ D[ Coefficient[AvLz2, eps, 4], Derivative[1, 0, 0][K22][z, x, t] ], z ] +
     D[ D[ Coefficient[AvLz2, eps, 4], Derivative[0, 1, 0][K22][z, x, t] ], x ] +
     D[ D[ Coefficient[AvLz2, eps, 4], Derivative[0, 0, 1][K22][z, x, t] ], t ] -
     D[ Coefficient[AvLz2, eps, 4], K22[z, x, t] ] == 0
                  ];

FullSimplify[ DSolve[eq4, K22[z, x, t], {z, x, t}] ]
```

$\{\{K22[z, x, t] \to e^{-2 k0 z} (e^{4 k0 z} C[1][x, t] + C[2][x, t])\}\}$

Final results

```
(* Reintroduncton of unknown slowly varying functions,
to simplify the subsequent expressions, but preserve the z-structure. *)
```

$K11[z\_, x\_, t\_] := A1[x, t] e^{k0 z} + e^{-k0 z} A2[x, t];$

$K21[z\_, x\_, t\_] := e^{k0 z} (B1[x, t] + z B2[x, t]) + e^{-k0 z} (B3[x, t] + z B4[x, t]);$

$K22[z\_, x\_, t\_] := e^{-2 k0 z} (e^{4 k0 z} C1[x, t] + C2[x, t]);$

Total problem of slowly modulated wavetrains

Procedure

```
etaTh = eta /. {  Cos[q_] :> Cos[ FullSimplify[q / ((k0 * x - w0 * t) + eps * thetaTilda[x, t])] * theta ],
    Sin[q_] :> Sin[ FullSimplify[q / ( (k0 * x - w0 * t) + eps * thetaTilda[x, t] )] * theta ]  };
```

Lagrangian L

```
t0Indef = Integrate[Coefficient[Lz4, eps, 0], z];
t0i = Limit[t0Indef, z → -h];
t0f = Limit[t0Indef, z → etaTh ];
t0 = t0f - t0i;

t1Indef = Integrate[Coefficient[Lz4, eps, 1], z];
t1i = Limit[t1Indef, z → -h];
t1f = Limit[t1Indef, z → etaTh];
t1 = t1f - t1i;

t2Indef = Integrate[Simplify[ Coefficient[Lz4, eps, 2] ], z];
t2i = Limit[t2Indef, z → -h];
t2f = Limit[t2Indef, z → etaTh];
t2 = t2f - t2i;

t3Indef = Simplify[ Integrate[Simplify[ Coefficient[Lz4, eps, 3] ], z] ];
t3i = Limit[t3Indef, z → -h];
t3f = Limit[t3Indef, z → etaTh];
t3 = t3f - t3i;

t4Indef = Simplify[ Integrate[Simplify[ Coefficient[Lz4, eps, 4] ], z] ];
t4i = Limit[t4Indef, z → -h];
t4f = Limit[t4Indef, z → etaTh];
t4 = t4f - t4i;

L = t0 + eps * t1 + eps^2 * t2 + eps^3 * t3 + eps^4 * t4;
```

Substitute the small-argument exponential functions with their MacLaurin expansions

```
Lr1 = L /. Exp[x_] /; FreeQ[x, eps] == False → 1 + x + x^2/2 + x^3/6 + x^4/24;

Lr2 = Collect[Lr1, eps] /. eps^x_ /; x ≥ 5 → 0;
```

Averaged Lagrangian AvL

```
q0 = Integrate[Coefficient[Lr2, eps, 0], {theta, 0, 2 * Pi}] / 2 / Pi;

q1 = Integrate[Coefficient[Lr2, eps, 1], {theta, 0, 2 * Pi}] / 2 / Pi;

q2 = Integrate[Coefficient[Lr2, eps, 2], {theta, 0, 2 * Pi}] / 2 / Pi;

q3 = Integrate[Coefficient[Lr2, eps, 3], {theta, 0, 2 * Pi}] / 2 / Pi;

q4 = Integrate[Coefficient[Lr2, eps, 4], {theta, 0, 2 * Pi}] / 2 / Pi;

AvL = q0 + eps * q1 + eps^2 * q2 + eps^3 * q3 + eps^4 * q4 ;
```

Result

(*) After having obtained the Averaged Lagrangian, all the functions (wave parameters & AvL) are functions of the slow variables. So, remembering that, epsilon inside the arguments can be omitted. Further, and more crucially, rewritting AvL in that way facilitates the correct derivation of the EL equations (which obtain x,t-differentiations) by means of symbolic software.

```
AvL1 = Simplify[ AvL /. {eps * x → x, eps * t → t} ]
```

```
1/4 (-2 g h^2 +
   eps^3 (e^(-2 h k0) b11[x, t] ((-1 + e^(2 h k0)) A1[x, t]^2 b11[x, t] kTilda[x, t] + e^(2 h k0) A2[x, t] (-2 w0 a21[x, t] - 2 h
            k0 B2[x, t] b21[x, t] - 2 k0 b21[x, t] B3[x, t] + 2 e^(2 h k0) k0 b21[x, t] B3[x, t] - 2 e^(2 h k0) h k0
            b21[x, t] B4[x, t] - A2[x, t] b11[x, t] kTilda[x, t] + e^(2 h k0) A2[x, t] b11[x, t] kTilda[x, t]) +
         2 A1[x, t] (-e^(2 h k0) w0 a21[x, t] + k0 ((-1 + e^(2 h k0)) B1[x, t] b21[x, t] + h B2[x, t]
               b21[x, t] + e^(2 h k0) h (b21[x, t] B4[x, t] + 2 A2[x, t] b11[x, t] kTilda[x, t])))) +
      2 a11[x, t] (g a21[x, t] - w0 B1[x, t] b21[x, t] - w0 b21[x, t] B3[x, t] - A1[x, t] b11[x, t]
         wTilda[x, t] - A2[x, t] b11[x, t] wTilda[x, t])) +
   eps^2 (g a11[x, t]^2 - 2 w0 a11[x, t] (A1[x, t] + A2[x, t]) b11[x, t] + e^(-2 h k0) (-1 + e^(2 h k0)) k0
      (A1[x, t]^2 + e^(2 h k0) A2[x, t]^2) b11[x, t]^2 + 4 h psi^(0,1)[x, t]) +
   1/(4 k0) eps^4 (4 g k0 a21[x, t]^2 + 4 g k0 a21s[x, t]^2 + 4 g k0 a22[x, t]^2 + 4 g k0 a22s[x, t]^2 -
      k0^3 w0 A1[x, t] a11[x, t]^3 b11[x, t] - k0^3 w0 a11[x, t]^3 A2[x, t] b11[x, t] -
      4 k0^2 w0 A1[x, t] a11[x, t] a22[x, t] b11[x, t] + 4 k0^2 w0 a11[x, t] A2[x, t] a22[x, t] b11[x, t] +
      4 k0^4 A1[x, t]^2 a11[x, t]^2 b11[x, t]^2 - 4 k0^4 a11[x, t]^2 A2[x, t]^2 b11[x, t]^2 +
      8 k0^3 A1[x, t] A2[x, t] a22[x, t] b11[x, t]^2 + 4 k0^2 B1[x, t]^2 b21[x, t]^2 - 4 e^(-2 h k0) k0^2 B1[x, t]^2 b21[x, t]^2 +
      8 e^(-2 h k0) h k0^2 B1[x, t] B2[x, t] b21[x, t]^2 + 2 B2[x, t]^2 b21[x, t]^2 - 2 e^(-2 h k0) B2[x, t]^2 b21[x, t]^2 -
      4 e^(-2 h k0) h^2 k0^2 B2[x, t]^2 b21[x, t]^2 + 4 k0^2 B1[x, t]^2 b21s[x, t]^2 - 4 e^(-2 h k0) k0^2 B1[x, t]^2 b21s[x, t]^2 +
      8 e^(-2 h k0) h k0^2 B1[x, t] B2[x, t] b21s[x, t]^2 + 2 B2[x, t]^2 b21s[x, t]^2 - 2 e^(-2 h k0) B2[x, t]^2 b21s[x, t]^2 -
      4 e^(-2 h k0) h^2 k0^2 B2[x, t]^2 b21s[x, t]^2 - 8 h k0^2 B2[x, t] b21[x, t]^2 B3[x, t] -
      8 h k0^2 B2[x, t] b21s[x, t]^2 B3[x, t] - 4 k0^2 b21[x, t]^2 B3[x, t]^2 + 4 e^(2 h k0) k0^2 b21[x, t]^2 B3[x, t]^2 -
      4 k0^2 b21s[x, t]^2 B3[x, t]^2 + 4 e^(2 h k0) k0^2 b21s[x, t]^2 B3[x, t]^2 + 8 h k0^2 B1[x, t] b21[x, t]^2 B4[x, t] +
      8 h k0 B2[x, t] b21[x, t]^2 B4[x, t] + 8 h k0^2 B1[x, t] b21s[x, t]^2 B4[x, t] +
      8 h k0 B2[x, t] b21s[x, t]^2 B4[x, t] - 8 e^(2 h k0) h k0^2 b21[x, t]^2 B3[x, t] B4[x, t] -
      8 e^(2 h k0) h k0^2 b21s[x, t]^2 B3[x, t] B4[x, t] - 2 b21[x, t]^2 B4[x, t]^2 + 2 e^(2 h k0) b21[x, t]^2 B4[x, t]^2 +
      4 e^(2 h k0) h^2 k0^2 b21[x, t]^2 B4[x, t]^2 - 2 b21s[x, t]^2 B4[x, t]^2 + 2 e^(2 h k0) b21s[x, t]^2 B4[x, t]^2 +
      4 e^(2 h k0) h^2 k0^2 b21s[x, t]^2 B4[x, t]^2 - 8 k0^2 w0 a11[x, t]^2 b22[x, t] C1[x, t] -
      16 k0 w0 a22[x, t] b22[x, t] C1[x, t] + 16 k0^3 A1[x, t] a11[x, t] b11[x, t] b22[x, t] C1[x, t] +
      16 k0 w0 a22s[x, t] b22s[x, t] C1[x, t] + 8 k0^2 b22[x, t]^2 C1[x, t]^2 -
```

$8\, e^{-4 h k0}\, k0^2\, b22[x,t]^2\, C1[x,t]^2 + 8\, k0^2\, b22s[x,t]^2\, C1[x,t]^2 - 8\, e^{-4 h k0}\, k0^2\, b22s[x,t]^2\, C1[x,t]^2 +$

$8\, k0^2\, w0\, a11[x,t]^2\, b22[x,t]\, C2[x,t] - 16\, k0\, w0\, a22[x,t]\, b22[x,t]\, C2[x,t] +$

$16\, k0^3\, a11[x,t]\, A2[x,t]\, b11[x,t]\, b22[x,t]\, C2[x,t] + 16\, k0\, w0\, a22s[x,t]\, b22s[x,t]\, C2[x,t] -$

$8\, k0^2\, b22[x,t]^2\, C2[x,t]^2 + 8\, e^{4 h k0}\, k0^2\, b22[x,t]^2\, C2[x,t]^2 - 8\, k0^2\, b22s[x,t]^2\, C2[x,t]^2 +$

$8\, e^{4 h k0}\, k0^2\, b22s[x,t]^2\, C2[x,t]^2 + 8\, k0\, A1[x,t]\, B1[x,t]\, b11[x,t]\, b21[x,t]\, kTilda[x,t] -$

$8\, e^{-2 h k0}\, k0\, A1[x,t]\, B1[x,t]\, b11[x,t]\, b21[x,t]\, kTilda[x,t] + 16\, h\, k0^2\, A2[x,t]\, B1[x,t]$

$b11[x,t]\, b21[x,t]\, kTilda[x,t] - 4\, A1[x,t]\, b11[x,t]\, B2[x,t]\, b21[x,t]\, kTilda[x,t] +$

$4\, e^{-2 h k0}\, A1[x,t]\, b11[x,t]\, B2[x,t]\, b21[x,t]\, kTilda[x,t] + 8\, e^{-2 h k0}\, h\, k0\, A1[x,t]\, b11[x,t]$

$B2[x,t]\, b21[x,t]\, kTilda[x,t] - 8\, h^2\, k0^2\, A2[x,t]\, b11[x,t]\, B2[x,t]\, b21[x,t]\, kTilda[x,t] +$

$16\, h\, k0^2\, A1[x,t]\, b11[x,t]\, b21[x,t]\, B3[x,t]\, kTilda[x,t] - 8\, k0\, A2[x,t]\, b11[x,t]\, b21[x,t]$

$B3[x,t]\, kTilda[x,t] + 8\, e^{2 h k0}\, k0\, A2[x,t]\, b11[x,t]\, b21[x,t]\, B3[x,t]\, kTilda[x,t] -$

$8\, h^2\, k0^2\, A1[x,t]\, b11[x,t]\, b21[x,t]\, B4[x,t]\, kTilda[x,t] - 4\, A2[x,t]\, b11[x,t]\, b21[x,t]$

$B4[x,t]\, kTilda[x,t] + 4\, e^{2 h k0}\, A2[x,t]\, b11[x,t]\, b21[x,t]\, B4[x,t]\, kTilda[x,t] -$

$8\, e^{2 h k0}\, h\, k0\, A2[x,t]\, b11[x,t]\, b21[x,t]\, B4[x,t]\, kTilda[x,t] + 2\, A1[x,t]^2\, b11[x,t]^2\, kTilda[x,t]^2 -$

$2\, e^{-2 h k0}\, A1[x,t]^2\, b11[x,t]^2\, kTilda[x,t]^2 + 8\, h\, k0\, A1[x,t]\, A2[x,t]\, b11[x,t]^2\, kTilda[x,t]^2 -$

$2\, A2[x,t]^2\, b11[x,t]^2\, kTilda[x,t]^2 + 2\, e^{2 h k0}\, A2[x,t]^2\, b11[x,t]^2\, kTilda[x,t]^2 -$

$8\, k0\, a11[x,t]\, B1[x,t]\, b21[x,t]\, wTilda[x,t] - 8\, k0\, a11[x,t]\, b21[x,t]\, B3[x,t]\, wTilda[x,t] - 8\, k0$

$a21[x,t]\, (w0\, B1[x,t]\, b21[x,t] + w0\, b21[x,t]\, B3[x,t] + (A1[x,t] + A2[x,t])\, b11[x,t]\, wTilda[x,t]) -$

$8\, k0^2\, w0\, A1[x,t]\, a11[x,t]\, b11[x,t]\, zeta[x,t] + 8\, k0^2\, w0\, a11[x,t]\, A2[x,t]\, b11[x,t]\, zeta[x,t] +$

$8\, k0^3\, A1[x,t]^2\, b11[x,t]^2\, zeta[x,t] + 8\, k0^3\, A2[x,t]^2\, b11[x,t]^2\, zeta[x,t] + 8\, g\, k0\, zeta[x,t]^2 +$

$8\, k0\, a11[x,t]\, b21s[x,t]\, B1^{(0,1)}[x,t] + 8\, k0\, a21s[x,t]\, \big(w0\, B1[x,t]\, b21s[x,t] + w0\, b21s[x,t]\, B3[x,t] +$

$b11[x,t]\, A1^{(0,1)}[x,t] + b11[x,t]\, A2^{(0,1)}[x,t] + A1[x,t]\, b11^{(0,1)}[x,t] + A2[x,t]\, b11^{(0,1)}[x,t]\big) +$

$8\, k0\, a11[x,t]\, B1[x,t]\, b21s^{(0,1)}[x,t] + 8\, k0\, a11[x,t]\, B3[x,t]\, b21s^{(0,1)}[x,t] +$

$8\, k0\, a11[x,t]\, b21s[x,t]\, B3^{(0,1)}[x,t] + 16\, k0\, zeta[x,t]\, psi^{(0,1)}[x,t] -$

$4\, k0\, B1[x,t]\, b11[x,t]\, b21s[x,t]\, A1^{(1,0)}[x,t] + 4\, e^{-2 h k0}\, k0\, B1[x,t]\, b11[x,t]\, b21s[x,t]\, A1^{(1,0)}[x,t] +$

$2\, b11[x,t]\, B2[x,t]\, b21s[x,t]\, A1^{(1,0)}[x,t] - 2\, e^{-2 h k0}\, b11[x,t]\, B2[x,t]\, b21s[x,t]\, A1^{(1,0)}[x,t] -$

$4\, e^{-2 h k0}\, h\, k0\, b11[x,t]\, B2[x,t]\, b21s[x,t]\, A1^{(1,0)}[x,t] - 8\, h\, k0^2\, b11[x,t]\, b21s[x,t]\, B3[x,t]\, A1^{(1,0)}[x,t] +$

$4\, h^2\, k0^2\, b11[x,t]\, b21s[x,t]\, B4[x,t]\, A1^{(1,0)}[x,t] + 2\, b11[x,t]^2\, A1^{(1,0)}[x,t]^2 -$

$2\, e^{-2 h k0}\, b11[x,t]^2\, A1^{(1,0)}[x,t]^2 - 8\, h\, k0^2\, B1[x,t]\, b11[x,t]\, b21s[x,t]\, A2^{(1,0)}[x,t] +$

$4\, h^2\, k0^2\, b11[x,t]\, B2[x,t]\, b21s[x,t]\, A2^{(1,0)}[x,t] + 4\, k0\, b11[x,t]\, b21s[x,t]\, B3[x,t]\, A2^{(1,0)}[x,t] -$

$4\, e^{2 h k0}\, k0\, b11[x,t]\, b21s[x,t]\, B3[x,t]\, A2^{(1,0)}[x,t] + 2\, b11[x,t]\, b21s[x,t]\, B4[x,t]\, A2^{(1,0)}[x,t] -$

$2\, e^{2 h k0}\, b11[x,t]\, b21s[x,t]\, B4[x,t]\, A2^{(1,0)}[x,t] + 4\, e^{2 h k0}\, h\, k0\, b11[x,t]\, b21s[x,t]\, B4[x,t]\, A2^{(1,0)}[x,t] +$

$8\, h\, k0\, b11[x,t]^2\, A1^{(1,0)}[x,t]\, A2^{(1,0)}[x,t] - 2\, b11[x,t]^2\, A2^{(1,0)}[x,t]^2 + 2\, e^{2 h k0}\, b11[x,t]^2\, A2^{(1,0)}[x,t]^2 +$

$4\, k0\, A1[x,t]\, b11[x,t]\, b21s[x,t]\, B1^{(1,0)}[x,t] - 4\, e^{-2 h k0}\, k0\, A1[x,t]\, b11[x,t]\, b21s[x,t]\, B1^{(1,0)}[x,t] +$

$8\, h\, k0^2\, A2[x,t]\, b11[x,t]\, b21s[x,t]\, B1^{(1,0)}[x,t] - 4\, k0\, A1[x,t]\, B1[x,t]\, b21s[x,t]\, b11^{(1,0)}[x,t] +$

$4\, e^{-2 h k0}\, k0\, A1[x,t]\, B1[x,t]\, b21s[x,t]\, b11^{(1,0)}[x,t] - 8\, h\, k0^2\, A2[x,t]\, B1[x,t]\, b21s[x,t]\, b11^{(1,0)}[x,t] +$

$2\, A1[x,t]\, B2[x,t]\, b21s[x,t]\, b11^{(1,0)}[x,t] - 2\, e^{-2 h k0}\, A1[x,t]\, B2[x,t]\, b21s[x,t]\, b11^{(1,0)}[x,t] -$

$4\, e^{-2 h k0}\, h\, k0\, A1[x,t]\, B2[x,t]\, b21s[x,t]\, b11^{(1,0)}[x,t] + 4\, h^2\, k0^2\, A2[x,t]\, B2[x,t]\, b21s[x,t]\, b11^{(1,0)}[x,t] -$

$8\, h\, k0^2\, A1[x,t]\, b21s[x,t]\, B3[x,t]\, b11^{(1,0)}[x,t] + 4\, k0\, A2[x,t]\, b21s[x,t]\, B3[x,t]\, b11^{(1,0)}[x,t] -$

$4\, e^{2 h k0}\, k0\, A2[x,t]\, b21s[x,t]\, B3[x,t]\, b11^{(1,0)}[x,t] + 4\, h^2\, k0^2\, A1[x,t]\, b21s[x,t]\, B4[x,t]\, b11^{(1,0)}[x,t] +$

$2\, A2[x,t]\, b21s[x,t]\, B4[x,t]\, b11^{(1,0)}[x,t] - 2\, e^{2 h k0}\, A2[x,t]\, b21s[x,t]\, B4[x,t]\, b11^{(1,0)}[x,t] +$

$4\, e^{2 h k0}\, h\, k0\, A2[x,t]\, b21s[x,t]\, B4[x,t]\, b11^{(1,0)}[x,t] + 4\, A1[x,t]\, b11[x,t]\, A1^{(1,0)}[x,t]\, b11^{(1,0)}[x,t] -$

$4\, e^{-2 h k0}\, A1[x,t]\, b11[x,t]\, A1^{(1,0)}[x,t]\, b11^{(1,0)}[x,t] + 8\, h\, k0\, A2[x,t]\, b11[x,t]\, A1^{(1,0)}[x,t]\, b11^{(1,0)}[x,t] +$

$8\, h\, k0\, A1[x,t]\, b11[x,t]\, A2^{(1,0)}[x,t]\, b11^{(1,0)}[x,t] - 4\, A2[x,t]\, b11[x,t]\, A2^{(1,0)}[x,t]\, b11^{(1,0)}[x,t] +$

$4\, e^{2 h k0}\, A2[x,t]\, b11[x,t]\, A2^{(1,0)}[x,t]\, b11^{(1,0)}[x,t] + 2\, A1[x,t]^2\, b11^{(1,0)}[x,t]^2 -$

$2\, e^{-2 h k0}\, A1[x,t]^2\, b11^{(1,0)}[x,t]^2 + 8\, h\, k0\, A1[x,t]\, A2[x,t]\, b11^{(1,0)}[x,t]^2 - 2\, A2[x,t]^2\, b11^{(1,0)}[x,t]^2 +$

$2\, e^{2 h k0}\, A2[x,t]^2\, b11^{(1,0)}[x,t]^2 - 2\, A1[x,t]\, b11[x,t]\, b21s[x,t]\, B2^{(1,0)}[x,t] + 2\, e^{-2 h k0}\, A1[x,t]$

$b11[x,t]\, b21s[x,t]\, B2^{(1,0)}[x,t] + 4\, e^{-2 h k0}\, h\, k0\, A1[x,t]\, b11[x,t]\, b21s[x,t]\, B2^{(1,0)}[x,t] -$

$4\, h^2\, k0^2\, A2[x,t]\, b11[x,t]\, b21s[x,t]\, B2^{(1,0)}[x,t] + 4\, k0\, A1[x,t]\, B1[x,t]\, b11[x,t]\, b21s^{(1,0)}[x,t] -$

$4\, e^{-2 h k0}\, k0\, A1[x,t]\, B1[x,t]\, b11[x,t]\, b21s^{(1,0)}[x,t] + 8\, h\, k0^2\, A2[x,t]\, B1[x,t]\, b11[x,t]\, b21s^{(1,0)}[x,t] -$

$2\, A1[x,t]\, b11[x,t]\, B2[x,t]\, b21s^{(1,0)}[x,t] + 2\, e^{-2 h k0}\, A1[x,t]\, b11[x,t]\, B2[x,t]\, b21s^{(1,0)}[x,t] +$

$4\, e^{-2 h k0}\, h\, k0\, A1[x,t]\, b11[x,t]\, B2[x,t]\, b21s^{(1,0)}[x,t] -$

$4\, h^2\, k0^2\, A2[x,t]\, b11[x,t]\, B2[x,t]\, b21s^{(1,0)}[x,t] + 8\, h\, k0^2\, A1[x,t]\, b11[x,t]\, B3[x,t]\, b21s^{(1,0)}[x,t] -$

$4\, k0\, A2[x,t]\, b11[x,t]\, B3[x,t]\, b21s^{(1,0)}[x,t] + 4\, e^{2 h k0}\, k0\, A2[x,t]\, b11[x,t]\, B3[x,t]\, b21s^{(1,0)}[x,t] -$

$4\, h^2\, k0^2\, A1[x,t]\, b11[x,t]\, B4[x,t]\, b21s^{(1,0)}[x,t] - 2\, A2[x,t]\, b11[x,t]\, B4[x,t]\, b21s^{(1,0)}[x,t] +$

$2\, e^{2 h k0}\, A2[x,t]\, b11[x,t]\, B4[x,t]\, b21s^{(1,0)}[x,t] - 4\, e^{2 h k0}\, h\, k0\, A2[x,t]\, b11[x,t]\, B4[x,t]\, b21s^{(1,0)}[x,t] +$

$8\, h\, k0^2\, A1[x,t]\, b11[x,t]\, b21s[x,t]\, B3^{(1,0)}[x,t] - 4\, k0\, A2[x,t]\, b11[x,t]\, b21s[x,t]\, B3^{(1,0)}[x,t] +$

$4\, e^{2 h k0}\, k0\, A2[x,t]\, b11[x,t]\, b21s[x,t]\, B3^{(1,0)}[x,t] - 4\, h^2\, k0^2\, A1[x,t]\, b11[x,t]\, b21s[x,t]\, B4^{(1,0)}[x,t] -$

2 A2[x, t] b11[x, t] b21s[x, t] $B4^{(1,0)}$[x, t] + 2 $e^{2 h k0}$ A2[x, t] b11[x, t] b21s[x, t] $B4^{(1,0)}$[x, t] -
4 $e^{2 h k0}$ h k0 A2[x, t] b11[x, t] b21s[x, t] $B4^{(1,0)}$[x, t] + 8 $k0^2$ A1[x, t] a11[x, t] b11[x, t] $psi^{(1,0)}$[x, t] +
8 $k0^2$ a11[x, t] A2[x, t] b11[x, t] $psi^{(1,0)}$[x, t] + 8 h k0 $psi^{(1,0)}$[x, t]^2))

EL equations

EL-equations up to O(eps^2)

Variation b11

```
FullSimplify[
 Solve[
  D[ D[Coefficient[AvL1, eps, 2], Derivative[1, 0][b11][x, t]], x ] +
    D[ D[Coefficient[AvL1, eps, 2], Derivative[0, 1][b11][x, t]], t ] -
    D[ Coefficient[AvL1, eps, 2], b11[x, t] ]  == 0 ,  b11[x, t]
      ]
           ]
```

$$\left\{\left\{b11[x, t] \to \frac{e^{2 h k0}\, w0\, a11[x, t]\, (A1[x, t] + A2[x, t])\, (-1 + Coth[h k0])}{2\, k0\, \left(A1[x, t]^2 + e^{2 h k0} A2[x, t]^2\right)}\right\}\right\}$$

Update AvL

$$AvL2 = Block\left[\{b11\}, b11[x\_, t\_] := \frac{e^{2 h k0}\, w0\, a11[x, t]\, (A1[x, t] + A2[x, t])\, (-1 + Coth[h k0])}{2\, k0\, \left(A1[x, t]^2 + e^{2 h k0} A2[x, t]^2\right)};\ AvL1\right];$$

Variation A1

```
FullSimplify[
 Solve[
  D[ D[Coefficient[AvL2, eps, 2], Derivative[1, 0][A1][x, t]], x ] +
    D[ D[Coefficient[AvL2, eps, 2], Derivative[0, 1][A1][x, t]], t ] -
    D[ Coefficient[AvL2, eps, 2], A1[x, t] ]  == 0 ,  A1[x, t]
      ]
           ]
```

$\{\{A1[x, t] \to -A2[x, t]\}, \{A1[x, t] \to e^{2 h k0} A2[x, t]\}\}$

Update AvL

$AvL3 = Block[\{A1\}, A1[x\_, t\_] := e^{2 h k0} A2[x, t];\ AvL2];$

Variation psi

```
FullSimplify[

 D[ D[Coefficient[AvL3, eps, 2], Derivative[1, 0][psi][x, t]], x ] +
   D[ D[Coefficient[AvL3, eps, 2], Derivative[0, 1][psi][x, t]], t ] -
   D[ Coefficient[AvL3, eps, 2], psi[x, t] ]  == 0

           ]
```

True

Variation a11

```
FullSimplify[
 Solve[
  D[ D[Coefficient[AvL3, eps, 2], Derivative[1, 0][a11][x, t]], x ] +
    D[ D[Coefficient[AvL3, eps, 2], Derivative[0, 1][a11][x, t]], t ] -
    D[ Coefficient[AvL3, eps, 2], a11[x, t] ]  == 0 ,  g
      ]
           ]
```

$$\left\{\left\{g \to \frac{w0^2\,\text{Coth}[h\,k0]}{k0}\right\}\right\}$$

Update AvL

```
AvL4 = AvL3 /. g → w0^2 Coth[h k0] / k0;
```

EL-equations up to O(eps^3)

Variation a11

```
FullSimplify[
 Solve[
  D[ D[Coefficient[AvL4, eps, 3], Derivative[1, 0][a11][x, t]], x ] +
    D[ D[Coefficient[AvL4, eps, 3], Derivative[0, 1][a11][x, t]], t ] -
    D[ Coefficient[AvL4, eps, 3], a11[x, t] ]  == 0,  wTilda[x, t]
      ]
           ]
```

$$\left\{\left\{\text{wTilda}[x, t] \to \frac{w0\,(1 + 2\,h\,k0\,\text{Csch}[2\,h\,k0])\,\text{kTilda}[x, t]}{2\,k0}\right\}\right\}$$

Variation thetaTilda

```
FullSimplify[
 Solve[
  D[ D[Coefficient[AvL4, eps, 3], kTilda[x, t]], x ] -
    D[ D[Coefficient[AvL4, eps, 3], wTilda[x, t]], t ]  == 0 , a11^(0,1)[x, t]
      ]
           ]
```

$$\left\{\left\{a11^{(0,1)}[x, t] \to -\frac{w0\,(1 + 2\,h\,k0\,\text{Csch}[2\,h\,k0])\,a11^{(1,0)}[x, t]}{2\,k0}\right\}\right\}$$

List of the O3 - results (that concern derivatives)

$$\text{lsO3} = \left\{\text{wTilda}[x, t] \to \frac{w0\,(1 + 2\,h\,k0\,\text{Csch}[2\,h\,k0])\,\text{kTilda}[x, t]}{2\,k0},\; a11^{(0,1)}[x, t] \to -\frac{w0\,(1 + 2\,h\,k0\,\text{Csch}[2\,h\,k0])\,a11^{(1,0)}[x, t]}{2\,k0}\right\};$$

EL-equations up to O(eps^4)

Variation zeta

```
evEq1 = FullSimplify[
  Solve[
   D[ D[Coefficient[AvL4, eps, 4], Derivative[1, 0][zeta][x, t]], x ] +
     D[ D[Coefficient[AvL4, eps, 4], Derivative[0, 1][zeta][x, t]], t ] -
     D[ Coefficient[AvL4, eps, 4], zeta[x, t] ] == 0, zeta[x, t]
    ]
          ]
```

$$\left\{\left\{\text{zeta}[x,t]\to -\frac{\text{k0}\,\text{Tanh}[h\,\text{k0}]\left(\text{w0}^2\,\text{a11}[x,t]^2\,\text{Csch}[h\,\text{k0}]^2+4\,\text{psi}^{(0,1)}[x,t]\right)}{4\,\text{w0}^2}\right\}\right\}$$

Variation a21

```
FullSimplify[
 Solve[
  Simplify[  D[ D[Coefficient[AvL4, eps, 4], Derivative[1, 0][a21][x, t]], x ] +
      D[ D[Coefficient[AvL4, eps, 4], Derivative[0, 1][a21][x, t]], t ] -
      D[ Coefficient[AvL4, eps, 4], a21[x, t] ]  /. ls03   ] == 0 ,  a21[x, t]
      ]
           ]
```

$$\left\{\left\{\text{a21}[x,t]\to \frac{\text{a11}[x,t]\,(1+2\,h\,\text{k0}\,\text{Csch}[2\,h\,\text{k0}])\,\text{kTilda}[x,t]}{2\,\text{k0}}+\frac{\text{k0}\,\text{b21}[x,t]\,(\text{B1}[x,t]+\text{B3}[x,t])\,\text{Tanh}[h\,\text{k0}]}{\text{w0}}\right\}\right\}$$

Update AvL

```
AvL5 = Block[{a21},
   a21[x_, t_] := a11[x, t] (1 + 2 h k0 Csch[2 h k0]) kTilda[x, t] / (2 k0) + k0 b21[x, t] (B1[x, t] + B3[x, t]) Tanh[h k0] / w0;
   AvL4];
```

Variation a21s

```
FullSimplify[
 Solve[
  Simplify[  D[ D[Coefficient[AvL5, eps, 4], Derivative[1, 0][a21s][x, t]], x ] +
      D[ D[Coefficient[AvL5, eps, 4], Derivative[0, 1][a21s][x, t]], t ] -
      D[ Coefficient[AvL5, eps, 4], a21s[x, t] ]  /. ls03   ] == 0 ,  a21s[x, t]
      ]
           ]
```

$$\left\{\left\{\text{a21s}[x,t]\to -\frac{\text{k0}\,\text{b21s}[x,t]\,(\text{B1}[x,t]+\text{B3}[x,t])\,\text{Tanh}[h\,\text{k0}]}{\text{w0}}+\frac{(1+2\,h\,\text{k0}\,\text{Csch}[2\,h\,\text{k0}])\,\text{a11}^{(1,0)}[x,t]}{2\,\text{k0}}\right\}\right\}$$

Update AvL

```
AvL6 = Block[{a21s},
   a21s[x_, t_] := - k0 b21s[x, t] (B1[x, t] + B3[x, t]) Tanh[h k0] / w0 + (1 + 2 h k0 Csch[2 h k0]) a11^(1,0)[x, t] / (2 k0);
   AvL5];
```

Variation b21

```
h1 = FullSimplify[
    (D[ D[Coefficient[AvL6, eps, 4], Derivative[1, 0][b21][x, t]], x ] + D[ D[Coefficient[AvL6, eps, 4],
        Derivative[0, 1][b21][x, t]], t ] - D[ Coefficient[AvL6, eps, 4], b21[x, t] ])  /. lsO3
            ] ==
  0;
```

Variation b21s

```
h2 = FullSimplify[
    (D[ D[Coefficient[AvL6, eps, 4], Derivative[1, 0][b21s][x, t]], x ] + D[ D[Coefficient[AvL6, eps, 4],
        Derivative[0, 1][b21s][x, t]], t ] - D[ Coefficient[AvL6, eps, 4], b21s[x, t] ])  /. lsO3
            ] ==
  0;
```

Variation B1

```
h3 = FullSimplify[
    (D[ D[Coefficient[AvL6, eps, 4], Derivative[1, 0][B1][x, t]], x ] + D[ D[Coefficient[AvL6, eps, 4],
        Derivative[0, 1][B1][x, t]], t ] - D[ Coefficient[AvL6, eps, 4], B1[x, t] ]) /. lsO3
            ] ==
  0;
```

Variation B2

```
h4 = FullSimplify[
    (D[ D[Coefficient[AvL6, eps, 4], Derivative[1, 0][B2][x, t]], x ] + D[ D[Coefficient[AvL6, eps, 4],
        Derivative[0, 1][B2][x, t]], t ] - D[ Coefficient[AvL6, eps, 4], B2[x, t] ]) /. lsO3
            ] ==
  0 ;
```

Variation B3

```
h5 = FullSimplify[
    (D[ D[Coefficient[AvL6, eps, 4], Derivative[1, 0][B3][x, t]], x ] + D[ D[Coefficient[AvL6, eps, 4],
        Derivative[0, 1][B3][x, t]], t ] - D[ Coefficient[AvL6, eps, 4], B3[x, t] ]) /. lsO3
            ] ==
  0 ;
```

Variation B4

```
h6 = FullSimplify[
    (D[ D[Coefficient[AvL6, eps, 4], Derivative[1, 0][B4][x, t]], x ] + D[ D[Coefficient[AvL6, eps, 4],
        Derivative[0, 1][B4][x, t]], t ] - D[ Coefficient[AvL6, eps, 4], B4[x, t] ]) /. lsO3
            ] ==
  0 ;
```

Solution to the system of equations for B1[x, t], B2[x, t], B3[x, t] & B4[x, t]

```
Simplify[
 Solve[{h3, h4, h5, h6}, { B1[x, t], B2[x, t], B3[x, t], B4[x, t]}]
      ]
```

$$\{\{B2[x,t]\to \left(e^{2hk0}\,w0\left(a11[x,t]\,b21[x,t]\,kTilda[x,t]-b21s[x,t]\,a11^{(1,0)}[x,t]\right)\right)/\left(\left(-1+e^{2hk0}\right)k0\left(b21[x,t]^2+b21s[x,t]^2\right)\right),$$

$$B3[x,t]\to e^{-2hk0}B1[x,t]-\left(2\,h\,w0\left(a11[x,t]\,b21[x,t]\,kTilda[x,t]-b21s[x,t]\,a11^{(1,0)}[x,t]\right)\right)/\left(\left(-1+e^{2hk0}\right)k0\left(b21[x,t]^2+b21s[x,t]^2\right)\right),$$

$$B4[x,t]\to \left(-w0\,a11[x,t]\,b21[x,t]\,kTilda[x,t]+w0\,b21s[x,t]\,a11^{(1,0)}[x,t]\right)/\left(\left(-1+e^{2hk0}\right)k0\left(b21[x,t]^2+b21s[x,t]^2\right)\right)\}\}$$

```
lsB = {B2[x, t] → (ⅇ^(2 h k0) w0 (a11[x, t] b21[x, t] kTilda[x, t] - b21s[x, t] a11^(1,0)[x, t])) /
      ((-1 + ⅇ^(2 h k0)) k0 (b21[x, t]^2 + b21s[x, t]^2)),
   B3[x, t] → ⅇ^(-2 h k0) B1[x, t] - (2 h w0 (a11[x, t] b21[x, t] kTilda[x, t] - b21s[x, t] a11^(1,0)[x, t])) /
       ((-1 + ⅇ^(2 h k0)) k0 (b21[x, t]^2 + b21s[x, t]^2)),
   B4[x, t] → (-w0 a11[x, t] b21[x, t] kTilda[x, t] + w0 b21s[x, t] a11^(1,0)[x, t]) /
      ((-1 + ⅇ^(2 h k0)) k0 (b21[x, t]^2 + b21s[x, t]^2))};
```

Solution to the system of equations for b21[x,t] & b21s[x,t]

```
h1n = Simplify[h1 /. lsB];
h2n = Simplify[h2 /. lsB];

Simplify[
 Solve[{h1n, h2n}, {b21[x, t], b21s[x, t]}]
       ]
```

$$\left\{\left\{b21s[x,t] \to \frac{a11[x,t]\, b21[x,t]\, kTilda[x,t]}{a11^{(1,0)}[x,t]}\right\}, \left\{b21s[x,t] \to -\frac{b21[x,t]\, a11^{(1,0)}[x,t]}{a11[x,t]\, kTilda[x,t]}\right\}\right\}$$

```
AvL7 = Block[{B2}, B2[x_, t_] := (ⅇ^(2 h k0) w0 (a11[x, t] b21[x, t] kTilda[x, t] - b21s[x, t] a11^(1,0)[x, t])) /
      ((-1 + ⅇ^(2 h k0)) k0 (b21[x, t]^2 + b21s[x, t]^2));
   AvL6];

AvL8 = Block[{B3},
   B3[x_, t_] := ⅇ^(-2 h k0) B1[x, t] - (2 h w0 (a11[x, t] b21[x, t] kTilda[x, t] - b21s[x, t] a11^(1,0)[x, t])) /
       ((-1 + ⅇ^(2 h k0)) k0 (b21[x, t]^2 + b21s[x, t]^2));
   AvL7];

AvL9 = Block[{B4}, B4[x_, t_] := (-w0 a11[x, t] b21[x, t] kTilda[x, t] + w0 b21s[x, t] a11^(1,0)[x, t]) /
      ((-1 + ⅇ^(2 h k0)) k0 (b21[x, t]^2 + b21s[x, t]^2));
   AvL8];

AvL10 = Block[{b21s}, b21s[x_, t_] := - (b21[x, t] a11^(1,0)[x, t]) / (a11[x, t] kTilda[x, t]); AvL9];
```

Variation a22s

```
FullSimplify[
 Solve[
  D[ D[Coefficient[AvL10, eps, 4], Derivative[1, 0][a22s][x, t]], x ]  +
    D[ D[Coefficient[AvL10, eps, 4], Derivative[0, 1][a22s][x, t]], t ]  -
    D[ Coefficient[AvL10, eps, 4], a22s[x, t] ]  == 0 ,  a22s[x, t]
      ]
           ]
```

$$\left\{\left\{a22s[x,t] \to -\frac{2\, k0\, b22s[x,t]\, (C1[x,t] + C2[x,t])\, Tanh[h\, k0]}{w0}\right\}\right\}$$

Update AvL

```
AvL11 = Block[{a22s}, a22s[x_, t_] := - (2 k0 b22s[x, t] (C1[x, t] + C2[x, t]) Tanh[h k0]) / w0;
   AvL10];
```

Variation b22s

```
FullSimplify[
 Solve[
  Simplify[  D[ D[Coefficient[AvL11, eps, 4], Derivative[1, 0][b22s][x, t]], x ] +
      D[ D[Coefficient[AvL11, eps, 4], Derivative[0, 1][b22s][x, t]], t ] -
      D[ Coefficient[AvL11, eps, 4], b22s[x, t] ]  /. ls03    ] == 0 ,  b22s[x, t]
      ]
            ]
```

```
{{b22s[x, t] → 0}}
```

Update AvL

```
AvL12 = Block[{b22s}, b22s[x_, t_] := 0; AvL11];
```

Variation b22

```
FullSimplify[
 Solve[
  Simplify[  D[ D[Coefficient[AvL12, eps, 4], Derivative[1, 0][b22][x, t]], x ] +
     D[ D[Coefficient[AvL12, eps, 4], Derivative[0, 1][b22][x, t]], t ] -
     D[ Coefficient[AvL12, eps, 4], b22[x, t] ] ] == 0 ,  b22[x, t]
      ]
            ]
```

$$\left\{\left\{b22[x,t]\to-\left(\left(e^{5 h k0}\, w0\, (C1[x,t]+C2[x,t])\, \left(k0\, a11[x,t]^2 \operatorname{Cosh}[h\, k0]-2\, a22[x,t] \operatorname{Sinh}[h\, k0]\right)\right)\Big/\left(\left(-1+e^{2 h k0}\right)^2 \left(1+e^{2 h k0}\right) k0 \left(C1[x,t]^2+e^{4 h k0}\, C2[x,t]^2\right)\right)\right)\right\}\right\}$$

Update AvL

```
AvL13 = Block[{b22},
   b22[x_, t_] := -((e^(5 h k0) w0 (C1[x, t] + C2[x, t]) (k0 a11[x, t]^2 Cosh[h k0] - 2 a22[x, t] Sinh[h k0])) /
       ((-1 + e^(2 h k0))^2 (1 + e^(2 h k0)) k0 (C1[x, t]^2 + e^(4 h k0) C2[x, t]^2)));
   AvL12];
```

Variation C1

```
FullSimplify[
 Solve[
  D[ D[Coefficient[AvL13, eps, 4], Derivative[1, 0][C1][x, t]], x ] +
    D[ D[Coefficient[AvL13, eps, 4], Derivative[0, 1][C1][x, t]], t ] -
    Simplify[D[ Coefficient[AvL13, eps, 4], C1[x, t] ]] == 0 ,  C1[x, t]
      ]
            ]
```

$$\left\{\{C1[x,t]\to-C2[x,t]\},\left\{C1[x,t]\to e^{4 h k0}\, C2[x,t]\right\}\right\}$$

Update AvL

```
AvL14 = Block[{C1}, C1[x_, t_] := e^(4 h k0) C2[x, t]; AvL13];
```

Variation a22

```
FullSimplify[
 Solve[
  D[ D[Coefficient[AvL14, eps, 4], Derivative[1, 0][a22][x, t]], x ] +
    D[ D[Coefficient[AvL14, eps, 4], Derivative[0, 1][a22][x, t]], t ] -
    D[ Coefficient[AvL14, eps, 4], a22[x, t] ] == 0 ,  a22[x, t]
      ]
           ]
```

$$\left\{\left\{a22[x,t] \to \frac{1}{8} k0\, a11[x,t]^2 (5 \operatorname{Cosh}[h\, k0] + \operatorname{Cosh}[3 h\, k0]) \operatorname{Csch}[h\, k0]^3\right\}\right\}$$

Update AvL

```
AvL15 = Block[{a22}, a22[x_, t_] := 1/8 k0 a11[x, t]^2 (5 Cosh[h k0] + Cosh[3 h k0]) Csch[h k0]^3;
   AvL14];
```

Variation psi

```
evEq2 = FullSimplify[

  D[ D[Coefficient[AvL15, eps, 4], Derivative[1, 0][psi][x, t]], x ] +
    D[ D[Coefficient[AvL15, eps, 4], Derivative[0, 1][psi][x, t]], t ] -
    D[ Coefficient[AvL15, eps, 4], psi[x, t] ]  == 0


            ]
```

$$zeta^{(0,1)}[x,t] + w0\, a11[x,t] \operatorname{Coth}[h\, k0]\, a11^{(1,0)}[x,t] + h\, psi^{(2,0)}[x,t] == 0$$

Update of AvL so that it is determined within the accuracy of the complete derivative of an arbitrary function & use of the O(eps^3) results

```
AvL16t = Coefficient[AvL15, eps, 0] + eps * Coefficient[AvL15, eps, 1] +
   eps^2 * Coefficient[AvL15, eps, 2] + eps^3 * Coefficient[AvL15, eps, 3] +
   eps^4 * (Coefficient[AvL15, eps, 4] /. {a11^(0,1)[x, t] -> -(w0 (1 + 2 h k0 Csch[2 h k0]) a11^(1,0)[x, t])/(2 k0),
       a11^(0,2)[x, t] -> -1/(2 k0) w0 (1 + 2 h k0 Csch[2 h k0]) (-(w0 (1 + 2 h k0 Csch[2 h k0]) a11^(2,0)[x, t])/(2 k0)),
       a11^(1,1)[x, t] -> -(w0 (1 + 2 h k0 Csch[2 h k0]) a11^(2,0)[x, t])/(2 k0),
       wTilda[x, t] -> (w0 (1 + 2 h k0 Csch[2 h k0]) kTilda[x, t])/(2 k0)});

AvL16 = Expand[AvL16t] /. a11^(2,0)[x, t] -> - a11^(1,0)[x, t]^2 / a11[x, t];

AvL17 = AvL16 /. D[ kTilda[x, t], t ] -> - D[(w0 (1 + 2 h k0 Csch[2 h k0]) kTilda[x, t])/(2 k0), x];
```

Vanishing of the arbitrary function B1[x,t]

```
AvL18 = Collect[  Block[{B1}, B1[x_, t_] := 0; AvL17],  eps,  Simplify  ];
```

Variation a11

```
evEq3 = FullSimplify[

  D[ D[AvL18, Derivative[1, 0][a11][x, t]], x ] +
   D[ D[AvL18, Derivative[0, 1][a11][x, t]], t ] - D[ AvL18, a11[x, t] ]

         ]
```

$$
\begin{aligned}
&-\frac{1}{8\left(-1+e^{2\,h\,k0}\right)^3 k0^3} e^{3\,h\,k0}\,eps^3\,w0\,\text{Sech}[h\,k0]\,\Big(4\,eps\,k0^4\,w0\,\text{a11}[x,t]^3\,(8+\text{Cosh}[4\,h\,k0])\,\text{Coth}[h\,k0]^2+\\
&\quad \text{a11}[x,t]\,\Big(8\,k0\,w0\,\text{kTilda}[x,t]\,\text{Sinh}[2\,h\,k0]\,(2\,h\,k0+\text{Sinh}[2\,h\,k0])+\\
&\qquad eps\,w0\,\text{kTilda}[x,t]^2\,\big(1+8\,h^2\,k0^2-\text{Cosh}[4\,h\,k0]+8\,h\,k0\,(-2\,h\,k0\,\text{Cosh}[2\,h\,k0]+\text{Sinh}[2\,h\,k0])\big)+\\
&\qquad 16\,k0^2\,\text{Sinh}[2\,h\,k0]\,\big(eps\,k0\,w0\,\text{zeta}[x,t]+\text{Sinh}[2\,h\,k0]\,\big(-\text{wTilda}[x,t]+eps\,k0\,\text{psi}^{(1,0)}[x,t]\big)\big)\Big)+\\
&\quad eps\,w0\,\big(-1-8\,h^2\,k0^2+\text{Cosh}[4\,h\,k0]+8\,h\,k0\,(2\,h\,k0\,\text{Cosh}[2\,h\,k0]-\text{Sinh}[2\,h\,k0])\big)\,\text{a11}^{(2,0)}[x,t]\Big)
\end{aligned}
$$

Variation thetaTilda

```
evEq4 = FullSimplify[

  D[ D[AvL18, kTilda[x, t]], x ] - D[ D[AvL18, wTilda[x, t]], t ]

         ]
```

$$
\begin{aligned}
&\frac{1}{4\left(-1+e^{2\,h\,k0}\right)^3\left(1+e^{2\,h\,k0}\right) k0^3}\\
&e^{4\,h\,k0}\,eps^3\,w0\,\text{a11}[x,t]\,\Big(16\,k0^2\,\text{Sinh}[2\,h\,k0]^2\,\text{a11}^{(0,1)}[x,t]+2\,w0\,\Big(4\,k0\,\text{Sinh}[2\,h\,k0]\,(2\,h\,k0+\text{Sinh}[2\,h\,k0])+\\
&\qquad eps\,\text{kTilda}[x,t]\,\big(1+8\,h^2\,k0^2-\text{Cosh}[4\,h\,k0]+8\,h\,k0\,(-2\,h\,k0\,\text{Cosh}[2\,h\,k0]+\text{Sinh}[2\,h\,k0])\big)\Big)\,\text{a11}^{(1,0)}[x,t]+\\
&\quad eps\,w0\,\text{a11}[x,t]\,\big(1+8\,h^2\,k0^2-\text{Cosh}[4\,h\,k0]+8\,h\,k0\,(-2\,h\,k0\,\text{Cosh}[2\,h\,k0]+\text{Sinh}[2\,h\,k0])\big)\,\text{kTilda}^{(1,0)}[x,t]\Big)
\end{aligned}
$$

Final solution

Rearranging of the evolutionary equations

Group velocity & its derivative

```
FullSimplify[  D[  Sqrt[g * k0 * Tanh[k0 * h]],  k0  ] /. g → w0^2 Coth[h k0]/k0  ]
```

$$\frac{w0}{2\,k0}+h\,w0\,\text{Csch}[2\,h\,k0]$$

```
FullSimplify[  D[  Sqrt[g * k0 * Tanh[k0 * h]],  k0 ,  k0 ]  /. g → w0^2 Coth[h k0]/k0  ]
```

$$-\frac{w0\,(1-2\,h\,k0\,\text{Csch}[2\,h\,k0])^2}{4\,k0^2}-h^2\,w0\,\text{Sech}[h\,k0]^2$$

Evolution of psi & zeta

```
{evEq1,  evEq2}
```

$$
\begin{aligned}
&\Big\{\Big\{\Big\{\text{zeta}[x,t]\to-\frac{k0\,\text{Tanh}[h\,k0]\,\big(w0^2\,\text{a11}[x,t]^2\,\text{Csch}[h\,k0]^2+4\,\text{psi}^{(0,1)}[x,t]\big)}{4\,w0^2}\Big\}\Big\},\\
&\ \text{zeta}^{(0,1)}[x,t]+w0\,\text{a11}[x,t]\,\text{Coth}[h\,k0]\,\text{a11}^{(1,0)}[x,t]+h\,\text{psi}^{(2,0)}[x,t]=0\Big\}
\end{aligned}
$$

Evolution of thetaTilda

```
evEq3a = Collect[ - evEq3 / Coefficient[ Coefficient[evEq3, eps, 3], wTilda[x, t] ] / eps^3,
    eps, FullSimplify ] /. w0/(2 k0) + h w0 Csch[2 h k0] → Cg
```

$$Cg\,kTilda[x,t] - wTilda[x,t] + \frac{1}{16\,k0^2\,a11[x,t]}\,eps\,\Big(k0^4\,w0\,a11[x,t]^3\,(8+Cosh[4\,h\,k0])\,Csch[h\,k0]^4 + a11[x,t]\Big(-w0\,Csch[2\,h\,k0]^2\,kTilda[x,t]^2\,\big(-1-8\,h^2\,k0^2+Cosh[4\,h\,k0]+8\,h\,k0\,(2\,h\,k0\,Cosh[2\,h\,k0]-Sinh[2\,h\,k0])\big) + 16\,k0^3\,\big(w0\,Csch[2\,h\,k0]\,zeta[x,t]+psi^{(1,0)}[x,t]\big)\Big) + w0\,Csch[2\,h\,k0]^2\,\big(-1-8\,h^2\,k0^2+Cosh[4\,h\,k0]+8\,h\,k0\,(2\,h\,k0\,Cosh[2\,h\,k0]-Sinh[2\,h\,k0])\big)\,a11^{(2,0)}[x,t]\Big)$$

```
evEq3b = Coefficient[evEq3a, eps, 0] +
  eps * Collect[ Coefficient[evEq3a, eps, 1], {a11[x, t], kTilda[x, t], a11^(2,0)[x, t]}, FullSimplify ]
```

$$Cg\,kTilda[x,t] - wTilda[x,t] + eps\,\Big(\frac{1}{16}\,k0^2\,w0\,a11[x,t]^2\,(8+Cosh[4\,h\,k0])\,Csch[h\,k0]^4 + \frac{1}{8\,k0^2}\,w0\,kTilda[x,t]^2\,\big(-1+2\,h\,k0\,Csch[2\,h\,k0]\,(2-h\,k0\,(3+Coth[h\,k0]^2)\,Tanh[h\,k0])\big) + k0\,\big(w0\,Csch[2\,h\,k0]\,zeta[x,t]+psi^{(1,0)}[x,t]\big) + \frac{1}{16\,k0^2\,a11[x,t]}\,w0\,Csch[2\,h\,k0]^2\,\big(-1-8\,h^2\,k0^2+Cosh[4\,h\,k0]+8\,h\,k0\,(2\,h\,k0\,Cosh[2\,h\,k0]-Sinh[2\,h\,k0])\big)\,a11^{(2,0)}[x,t]\Big)$$

```
FullSimplify[ Coefficient[ Coefficient[evEq3b, eps, 1], kTilda[x, t]^2 ] ==
  ( - (w0 (1 - 2 h k0 Csch[2 h k0])^2)/(4 k0^2) - h^2 w0 Sech[h k0]^2 ) / 2 ]
```

True

```
FullSimplify[ Coefficient[ Coefficient[evEq3b, eps, 1], a11^(2,0)[x, t] / a11[x, t] ] ==
  - ( - (w0 (1 - 2 h k0 Csch[2 h k0])^2)/(4 k0^2) - h^2 w0 Sech[h k0]^2 ) / 2 ]
```

True

```
evEq3c = Collect[ evEq3b /. {Coefficient[ Coefficient[evEq3b, eps, 1], kTilda[x, t]^2 ] → dCg / 2,
    Coefficient[ Coefficient[evEq3b, eps, 1], a11^(2,0)[x, t] / a11[x, t] ] → - dCg / 2}, eps ]
```

$$Cg\,kTilda[x,t] - wTilda[x,t] + eps\,\Big(\frac{1}{16}\,k0^2\,w0\,a11[x,t]^2\,(8+Cosh[4\,h\,k0])\,Csch[h\,k0]^4 + \frac{1}{2}\,dCg\,kTilda[x,t]^2 + k0\,\big(w0\,Csch[2\,h\,k0]\,zeta[x,t]+psi^{(1,0)}[x,t]\big) - \frac{dCg\,a11^{(2,0)}[x,t]}{2\,a11[x,t]}\Big)$$

Evolution of a11

```
evEq4a = Collect[ evEq4 / Coefficient[ Coefficient[evEq4, eps, 3], a11^(0,1)[x, t] ] / eps^3,
    eps, FullSimplify ] /. w0/(2 k0) (1 + 2 k0 h Csch[2 h k0]) → Cg
```

$$a11^{(0,1)}[x,t] + Cg\,a11^{(1,0)}[x,t] - \frac{1}{16\,k0^2}\,eps\,w0\,Csch[2\,h\,k0]^2\,\big(-1-8\,h^2\,k0^2+Cosh[4\,h\,k0]+8\,h\,k0\,(2\,h\,k0\,Cosh[2\,h\,k0]-Sinh[2\,h\,k0])\big)\,\big(2\,kTilda[x,t]\,a11^{(1,0)}[x,t]+a11[x,t]\,kTilda^{(1,0)}[x,t]\big)$$

$$\text{FullSimplify}\left[-\frac{1}{16\,\text{k0}^2}\,\text{w0}\,\text{Csch}[2\,\text{h}\,\text{k0}]^2\left(-1-8\,\text{h}^2\,\text{k0}^2+\text{Cosh}[4\,\text{h}\,\text{k0}]+8\,\text{h}\,\text{k0}\,(2\,\text{h}\,\text{k0}\,\text{Cosh}[2\,\text{h}\,\text{k0}]-\text{Sinh}[2\,\text{h}\,\text{k0}])\right) == \left(-\frac{\text{w0}\,(1-2\,\text{h}\,\text{k0}\,\text{Csch}[2\,\text{h}\,\text{k0}])^2}{4\,\text{k0}^2}-\text{h}^2\,\text{w0}\,\text{Sech}[\text{h}\,\text{k0}]^2\right)\Big/\,2\right]$$

True

$$\text{evEq4b} = \text{evEq4a} \;/.\; -\frac{\text{w0}\,\text{Csch}[2\,\text{h}\,\text{k0}]^2\left(-1-8\,\text{h}^2\,\text{k0}^2+\text{Cosh}[4\,\text{h}\,\text{k0}]+8\,\text{h}\,\text{k0}\,(2\,\text{h}\,\text{k0}\,\text{Cosh}[2\,\text{h}\,\text{k0}]-\text{Sinh}[2\,\text{h}\,\text{k0}])\right)}{16\,\text{k0}^2} \to \text{dCg}/2$$

$$\text{a11}^{(0,1)}[\text{x},\text{t}]+\text{Cg}\,\text{a11}^{(1,0)}[\text{x},\text{t}]+\frac{1}{2}\,\text{dCg}\,\text{eps}\left(2\,\text{kTilda}[\text{x},\text{t}]\,\text{a11}^{(1,0)}[\text{x},\text{t}]+\text{a11}[\text{x},\text{t}]\,\text{kTilda}^{(1,0)}[\text{x},\text{t}]\right)$$

Final form of the wave fields

Resulting expressions for the unknown slowly varying functions

$$\text{b11}[\text{x\_},\text{t\_}] := \frac{e^{2\,\text{h}\,\text{k0}}\,\text{w0}\,\text{a11}[\text{x},\text{t}]\,(\text{A1}[\text{x},\text{t}]+\text{A2}[\text{x},\text{t}])\,(-1+\text{Coth}[\text{h}\,\text{k0}])}{2\,\text{k0}\left(\text{A1}[\text{x},\text{t}]^2+e^{2\,\text{h}\,\text{k0}}\,\text{A2}[\text{x},\text{t}]^2\right)}$$

$$\text{A1}[\text{x\_},\text{t\_}] := e^{2\,\text{h}\,\text{k0}}\,\text{A2}[\text{x},\text{t}]$$

$$\text{g} = \frac{\text{w0}^2\,\text{Coth}[\text{h}\,\text{k0}]}{\text{k0}};$$

$$\text{a21}[\text{x\_},\text{t\_}] := \frac{\text{a11}[\text{x},\text{t}]\,(1+2\,\text{h}\,\text{k0}\,\text{Csch}[2\,\text{h}\,\text{k0}])\,\text{kTilda}[\text{x},\text{t}]}{2\,\text{k0}}+\frac{\text{k0}\,\text{b21}[\text{x},\text{t}]\,(\text{B1}[\text{x},\text{t}]+\text{B3}[\text{x},\text{t}])\,\text{Tanh}[\text{h}\,\text{k0}]}{\text{w0}}$$

$$\text{a21s}[\text{x\_},\text{t\_}] := \frac{1}{\left(-1+e^{4\,\text{h}\,\text{k0}}\right)\text{k0}\,\text{w0}}$$
$$e^{2\,\text{h}\,\text{k0}}\left(-4\,\text{k0}^2\,\text{b21s}[\text{x},\text{t}]\,(\text{B1}[\text{x},\text{t}]+\text{B3}[\text{x},\text{t}])\,\text{Sinh}[\text{h}\,\text{k0}]^2+\text{w0}\,(2\,\text{h}\,\text{k0}+\text{Sinh}[2\,\text{h}\,\text{k0}])\,\text{a11}^{(1,0)}[\text{x},\text{t}]\right)$$

$$\text{B1}[\text{x\_},\text{t\_}] := 0$$

$$\text{B2}[\text{x\_},\text{t\_}] := \frac{e^{2\,\text{h}\,\text{k0}}\,\text{w0}\left(\text{a11}[\text{x},\text{t}]\,\text{b21}[\text{x},\text{t}]\,\text{kTilda}[\text{x},\text{t}]-\text{b21s}[\text{x},\text{t}]\,\text{a11}^{(1,0)}[\text{x},\text{t}]\right)}{\left(-1+e^{2\,\text{h}\,\text{k0}}\right)\text{k0}\left(\text{b21}[\text{x},\text{t}]^2+\text{b21s}[\text{x},\text{t}]^2\right)}$$

$$\text{B3}[\text{x\_},\text{t\_}] := e^{-2\,\text{h}\,\text{k0}}\,\text{B1}[\text{x},\text{t}]-\frac{2\,\text{h}\,\text{w0}\left(\text{a11}[\text{x},\text{t}]\,\text{b21}[\text{x},\text{t}]\,\text{kTilda}[\text{x},\text{t}]-\text{b21s}[\text{x},\text{t}]\,\text{a11}^{(1,0)}[\text{x},\text{t}]\right)}{\left(-1+e^{2\,\text{h}\,\text{k0}}\right)\text{k0}\left(\text{b21}[\text{x},\text{t}]^2+\text{b21s}[\text{x},\text{t}]^2\right)}$$

$$\text{B4}[\text{x\_},\text{t\_}] := \frac{-\text{w0}\,\text{a11}[\text{x},\text{t}]\,\text{b21}[\text{x},\text{t}]\,\text{kTilda}[\text{x},\text{t}]+\text{w0}\,\text{b21s}[\text{x},\text{t}]\,\text{a11}^{(1,0)}[\text{x},\text{t}]}{\left(-1+e^{2\,\text{h}\,\text{k0}}\right)\text{k0}\left(\text{b21}[\text{x},\text{t}]^2+\text{b21s}[\text{x},\text{t}]^2\right)}$$

$$\text{b21s}[\text{x\_},\text{t\_}] := -\frac{\text{b21}[\text{x},\text{t}]\,\text{a11}^{(1,0)}[\text{x},\text{t}]}{\text{a11}[\text{x},\text{t}]\,\text{kTilda}[\text{x},\text{t}]}$$

$$\text{a22s}[\text{x\_},\text{t\_}] := -\frac{2\,\text{k0}\,\text{b22s}[\text{x},\text{t}]\,(\text{C1}[\text{x},\text{t}]+\text{C2}[\text{x},\text{t}])\,\text{Tanh}[\text{h}\,\text{k0}]}{\text{w0}}$$

$$\text{b22s}[\text{x\_},\text{t\_}] := 0$$

$$\text{b22}[\text{x\_},\text{t\_}] := -\frac{e^{5\,\text{h}\,\text{k0}}\,\text{w0}\,(\text{C1}[\text{x},\text{t}]+\text{C2}[\text{x},\text{t}])\left(\text{k0}\,\text{a11}[\text{x},\text{t}]^2\,\text{Cosh}[\text{h}\,\text{k0}]-2\,\text{a22}[\text{x},\text{t}]\,\text{Sinh}[\text{h}\,\text{k0}]\right)}{\left(-1+e^{2\,\text{h}\,\text{k0}}\right)^2\left(1+e^{2\,\text{h}\,\text{k0}}\right)\text{k0}\left(\text{C1}[\text{x},\text{t}]^2+e^{4\,\text{h}\,\text{k0}}\,\text{C2}[\text{x},\text{t}]^2\right)}$$

$$\text{C1}[\text{x\_},\text{t\_}] := e^{4\,\text{h}\,\text{k0}}\,\text{C2}[\text{x},\text{t}]$$

$$\text{a22}[\text{x\_},\text{t\_}] := \frac{1}{8}\,\text{k0}\,\text{a11}[\text{x},\text{t}]^2\,(5\,\text{Cosh}[\text{h}\,\text{k0}]+\text{Cosh}[3\,\text{h}\,\text{k0}])\,\text{Csch}[\text{h}\,\text{k0}]^3$$

Field eta

```
Collect[
 eta /. {  Cos[q_] :> Cos[ FullSimplify[q / ((k0 * x - w0 * t) + eps * thetaTilda[x, t])] * theta ],
   Sin[q_] :> Sin[ FullSimplify[q / ( (k0 * x - w0 * t) + eps * thetaTilda[x, t] )] * theta ]  },
 {eps, theta},  FullSimplify
      ]
```

$$\text{eps a11[eps x, eps t] Cos[theta]} + \frac{1}{8\,\text{k0}}\,\text{eps}^2 \big(\text{k0}^2\,\text{a11[eps x, eps t]}^2\,\text{Cos[2 theta] (5 Cosh[h k0] + Cosh[3 h k0]) Csch[h k0]}^3 + 4\,\text{a11[eps x, eps t] Cos[theta] (1 - 2 h k0 + 2 h k0 Coth[2 h k0]) kTilda[eps x, eps t]} + 8\,\text{k0 zeta[eps x, eps t]} + 4\,\text{(1 - 2 h k0 + 2 h k0 Coth[2 h k0]) Sin[theta] a11}^{(1,0)}\text{[eps x, eps t]}\big)$$

Field phi

```
lsTheta = {  Cos[q_] :> Cos[ FullSimplify[q / ((k0 * x - w0 * t) + eps * thetaTilda[x, t])] * theta ],
   Sin[q_] :> Sin[ FullSimplify[q / ( (k0 * x - w0 * t) + eps * thetaTilda[x, t] )] * theta ]  };

eps * FullSimplify[  ExpToTrig[  Coefficient[phi /. lsTheta, eps, 1] ]  ] + eps^2 * Collect[
   ExpToTrig[ Coefficient[phi /. lsTheta, eps, 2] ], {Sin[theta], Sin[2 * theta]}, FullSimplify  ]
```

$$\text{eps}\left(\text{psi[eps x, eps t]} + \frac{\text{w0 a11[eps x, eps t] Cosh[k0 (h + z)] Csch[h k0] Sin[theta]}}{\text{k0}}\right) + \text{eps}^2\left(\frac{3}{8}\,\text{w0 a11[eps x, eps t]}^2\,\text{Cosh[2 k0 (h + z)] Csch[h k0]}^4\,\text{Sin[2 theta]} + \frac{1}{\text{k0}}\,\text{w0 a11[eps x, eps t] Csch[h k0] kTilda[eps x, eps t] Sin[theta] (-h Cosh[k0 (h + z)] + (h + z) Sinh[k0 (h + z)])} + \frac{1}{\text{k0}}\,\text{w0 Cos[theta] Csch[h k0] (h Cosh[k0 (h + z)] - (h + z) Sinh[k0 (h + z)]) a11}^{(1,0)}\text{[eps x, eps t]}\right)$$

Appendix C

Supplementary equations of Chapter 3

Notation : $E(n) \equiv e^{jn(kX_0 - \omega T_0)}$

Equation (14b):

$$O(\varepsilon^2): \begin{cases} \frac{1}{2}\partial_{zz}\varphi_{20} + (\partial_{zz}\varphi_{21} - k^2\varphi_{21} + j2k\partial_{X_1}\varphi_{11})E(1) + (\partial_{zz}\varphi_{22} - 4k^2\varphi_{22})E(2) + (*) = 0, & -h < z < 0 \\ \frac{1}{2}\partial_z\varphi_{20} + \partial_z\varphi_{21}E(1) + \partial_z\varphi_{22}E(2) + (*) = 0, & z = -h \\ \frac{1}{2}\left(g\partial_z\varphi_{20} - j\frac{\omega^3}{g}\varphi_{11}\partial_z\varphi_{1,-1} + j\frac{\omega^3}{g}\partial_z\varphi_{11}\varphi_{1,-1} - j\omega\partial_{zz}\varphi_{11}\varphi_{1,-1} + j\omega\varphi_{11}\partial_{zz}\varphi_{1,-1}\right) & \\ \quad + (g\partial_z\varphi_{21} - \omega^2\varphi_{21} - j2\omega\partial_{T_1}\varphi_{11} - j2\omega\partial_z\varphi_{10}\partial_z\varphi_{11} + j\omega\varphi_{11}\partial_{zz}\varphi_{10})E(1) & \\ \quad + \left(g\partial_z\varphi_{22} - 4\omega^2\varphi_{22} - j2\omega(\partial_z\varphi_{11})^2 + j2k^2\omega\varphi_{11}{}^2 + j\omega\varphi_{11}\partial_{zz}\varphi_{11} - j\frac{\omega^3}{g}\varphi_{11}\partial_z\varphi_{11}\right)E(2) + (*) = 0, & z = 0 \end{cases}$$

Equation (14c):

$$O(\varepsilon^3): \begin{cases} \frac{1}{2}(\partial_{zz}\varphi_{30} + \partial_{X_1X_1}\varphi_{10} + \partial_{Y_1Y_1}\varphi_{10}) & \\ \quad + (\partial_{zz}\varphi_{31} - k^2\varphi_{31} + \partial_{X_1X_1}\varphi_{11} + \partial_{Y_1Y_1}\varphi_{11} + j2k\partial_{X_1}\varphi_{21} + j2k\partial_{X_2}\varphi_{11})E(1) & \\ \quad + (\partial_{zz}\varphi_{32} - 4k^2\varphi_{32} + j4k\partial_{X_1}\varphi_{22})E(2) + (\partial_{zz}\varphi_{33} - 9k^2\varphi_{33})E(3) + (*) = 0, & -h < z < 0 \\ \frac{1}{2}\partial_z\varphi_{30} + \partial_z\varphi_{31}E(1) + \partial_z\varphi_{32}E(2) + \partial_z\varphi_{33}E(3) + (*) = 0, & z = -h \\ \frac{1}{2}(g\partial_z\varphi_{30} - Q_0) + (g\partial_z\varphi_{31} - \omega^2\varphi_{31} - Q_1)E(1) & \\ \quad + (g\partial_z\varphi_{32} - 4\omega^2\varphi_{32} - Q_2)E(2) + (g\partial_z\varphi_{33} - 9\omega^2\varphi_{33} - Q_3)E(3) + (*) = 0, & z = 0 \end{cases}$$

where

$$Q_0 = \frac{1}{2}\Big(\partial_{T_1}\varphi_{10}\,\partial_{zz}\varphi_{10} - 2\partial_z\varphi_{10}\,\partial_{T_1,z}\varphi_{10} - \partial_{T_1T_1}\varphi_{10} - \frac{1}{2}(\partial_z\varphi_{10})^2\,\partial_{zz}\varphi_{10}$$

$$+ k^2\,\partial_{zz}\varphi_{10}\,\varphi_{11}\,\varphi_{1,-1} - \frac{1}{g}\omega^2\,\partial_{zzz}\varphi_{10}\,\varphi_{1,-1}\,\varphi_{11} - \partial_{zz}\varphi_{10}\,\partial_z\varphi_{11}\,\partial_z\varphi_{1,-1} - \frac{2}{g}\omega^2\,\partial_z\varphi_{10}\,\partial_z\varphi_{11}\,\partial_z\varphi_{1,-1}\Big)$$

$$+ j\frac{1}{g}\omega^3\varphi_{11}\,\partial_z\varphi_{2,-1} + j\omega\varphi_{2,-1}\,\partial_{zz}\varphi_{11} - j\omega\varphi_{11}\,\partial_{zz}\varphi_{2,-1} + j\frac{1}{g}\omega^3\varphi_{21}\,\partial_z\varphi_{1,-1} - \partial_z\varphi_{10}\,\partial_z\varphi_{1,-1}\,\partial_{zz}\varphi_{11}$$

$$- \frac{1}{g}\omega^2\,\partial_z\varphi_{11}\,\partial_{T_1}\varphi_{1,-1} + \frac{1}{g}\omega^2\varphi_{11}\,\partial_z\varphi_{1,-1}\,\partial_{zz}\varphi_{10} + \partial_{T_1}\varphi_{1,-1}\,\partial_{zz}\varphi_{11} - 2k^2\varphi_{11}\,\partial_{T_1}\varphi_{1,-1} + \frac{2}{g}\omega^2\varphi_{11}\,\partial_{T_1,z}\varphi_{1,-1}$$

$$- 2\partial_z\varphi_{11}\,\partial_{T_1,z}\varphi_{1,-1} - 2k\omega\varphi_{11}\,\partial_{X_1}\varphi_{1,-1} + 2k\omega\varphi_{1,-1}\,\partial_{X_1}\varphi_{11} - 2k^2\varphi_{1,-1}\,\partial_z\varphi_{10}\,\partial_z\varphi_{11} + \frac{2}{g}\omega^2\varphi_{1,-1}\,\partial_{zz}\varphi_{11}\,\partial_z\varphi_{10} + (*)$$

$$Q_1 = -j\frac{4}{g}\omega^3\varphi_{1,-1}\,\partial_z\varphi_{22} + j2\omega\partial_z\varphi_{1,-1}\,\partial_z\varphi_{22} + j\omega\varphi_{1,-1}\,\partial_{zz}\varphi_{22} + j2\omega\partial_{T_1}\varphi_{21} + j2\omega\partial_{T_2}\varphi_{11} - j\omega\varphi_{11}\,\partial_{zz}\varphi_{20} - j\omega\varphi_{21}\,\partial_{zz}\varphi_{10}$$

$$+ j\frac{2}{g}\omega^3\varphi_{22}\,\partial_z\varphi_{1,-1} - j2\omega\varphi_{22}\,\partial_{zz}\varphi_{1,-1} + j4k^2\omega\varphi_{1,-1}\varphi_{22} + j2\omega\partial_z\varphi_{11}\,\partial_z\varphi_{20} + j2\omega\partial_z\varphi_{10}\,\partial_z\varphi_{21} + \partial_{T_1}\varphi_{11}\,\partial_{zz}\varphi_{10}$$

$$- \partial_z\varphi_{10}\,\partial_z\varphi_{11}\,\partial_{zz}\varphi_{10} - \partial_{T_1T_1}\varphi_{11} + \left(\frac{1}{g^2}\omega^4 - 2k^2\right)(\partial_z\varphi_{11})^2\varphi_{1,-1} - \frac{1}{g}\omega^2\,\partial_{T_1}\varphi_{10}\,\partial_z\varphi_{11} + \partial_{T_1}\varphi_{10}\,\partial_{zz}\varphi_{11} - 2\partial_{T_1,z}\varphi_{10}\,\partial_z\varphi_{11}$$

$$- 2k\omega\varphi_{11}\,\partial_{X_1}\varphi_{10} - \frac{3}{2g}\omega^2(\partial_z\varphi_{11})^2\,\partial_z\varphi_{1,-1} - \frac{1}{2}(\partial_z\varphi_{11})^2\,\partial_{zz}\varphi_{1,-1} - \frac{5}{g}k^2\omega^2\varphi_{11}\varphi_{1,-1}\,\partial_z\varphi_{11} + \left(k^2 + \frac{\omega^4}{g^2}\right)\varphi_{11}\varphi_{1,-1}\,\partial_{zz}\varphi_{11}$$

$$- \frac{1}{g}\omega^2\varphi_{11}\varphi_{1,-1}\,\partial_{zzz}\varphi_{11} + \frac{1}{2g}k^2\omega^2\varphi_{11}{}^2\,\partial_z\varphi_{1,-1} + k^4\varphi_{11}{}^2\varphi_{1,-1} - \frac{1}{2}\left(k^2 + \frac{\omega^4}{g^2}\right)\varphi_{11}{}^2\,\partial_{zz}\varphi_{1,-1} + \frac{1}{2g}\omega^2\varphi_{11}{}^2\,\partial_{zzz}\varphi_{1,-1}$$

$$- \frac{1}{2g}\omega^2(\partial_z\varphi_{10})^2\,\partial_z\varphi_{11} - \frac{1}{2}(\partial_z\varphi_{10})^2\,\partial_{zz}\varphi_{11} - \partial_z\varphi_{11}\,\partial_z\varphi_{1,-1}\,\partial_{zz}\varphi_{11} + \frac{3}{g}\omega^2\varphi_{1,-1}\,\partial_z\varphi_{11}\,\partial_{zz}\varphi_{11} - 2\partial_z\varphi_{10}\,\partial_{T_1,z}\varphi_{11}$$

$$- \frac{1}{g}\omega^2\varphi_{11}\,\partial_z\varphi_{1,-1}\,\partial_{zz}\varphi_{11} + \frac{1}{g}\omega^2\varphi_{11}\,\partial_z\varphi_{11}\,\partial_{zz}\varphi_{1,-1}$$

$$
\begin{aligned}
Q_2 = {} & j4\omega\partial_{T_1}\varphi_{22} + j4\omega\partial_z\varphi_{22}\partial_z\varphi_{10} - j2\omega\varphi_{22}\partial_{zz}\varphi_{10} + j\frac{1}{g}\omega^3\varphi_{11}\partial_z\varphi_{21} + j\frac{1}{g}\omega^3\varphi_{21}\partial_z\varphi_{11} - j\omega\varphi_{11}\partial_{zz}\varphi_{21} \\
& - j\omega\varphi_{21}\partial_{zz}\varphi_{11} - j4k^2\omega\varphi_{11}\varphi_{21} - \frac{1}{g}\omega^2(\partial_z\varphi_{11})^2\partial_z\varphi_{10} - \frac{1}{2}(\partial_z\varphi_{11})^2\partial_{zz}\varphi_{10} - \frac{2}{g}\omega^2\varphi_{11}\partial_{T_1,z}\varphi_{11} \\
& - \frac{1}{g}\omega^2\partial_z\varphi_{11}\partial_{T_1}\varphi_{11} + \partial_{T_1}\varphi_{11}\partial_{zz}\varphi_{11} - 2\partial_z\varphi_{11}\partial_{T_1,z}\varphi_{11} + 2k^2\varphi_{11}\partial_{T_1}\varphi_{11} - \partial_z\varphi_{10}\partial_z\varphi_{11}\partial_{zz}\varphi_{11} + 2k^2\varphi_{11}\partial_z\varphi_{10}\partial_z\varphi_{11} \\
& - \frac{2}{g}\omega^2\varphi_{11}\partial_z\varphi_{10}\partial_{zz}\varphi_{11} - \frac{1}{g}\omega^2\varphi_{11}\partial_z\varphi_{11}\partial_{zz}\varphi_{10} - \frac{1}{2}k^2\varphi_{11}{}^2\partial_{zz}\varphi_{10} + \frac{1}{2g}\omega^2\varphi_{11}{}^2\partial_{zzz}\varphi_{10} - 4k\omega\varphi_{11}\partial_{X_1}\varphi_{11}
\end{aligned}
$$

$$
\begin{aligned}
Q_3 = {} & - j12k^2\omega\varphi_{11}\varphi_{22} + j\frac{2}{g}\omega^3\varphi_{22}\partial_z\varphi_{11} - j2\omega\varphi_{22}\partial_{zz}\varphi_{11} + j\frac{4}{g}\omega^3\varphi_{11}\partial_z\varphi_{22} + j6\omega\partial_z\varphi_{11}\partial_z\varphi_{22} - j\omega\varphi_{11}\partial_{zz}\varphi_{22} \\
& - \frac{1}{2g}\omega^2(\partial_z\varphi_{11})^3 + \left(2k^2 - \frac{\omega^4}{g^2}\right)\varphi_{11}(\partial_z\varphi_{11})^2 - \frac{1}{2}(\partial_z\varphi_{11})^2\partial_{zz}\varphi_{11} + \frac{9}{2g}k^2\omega^2\varphi_{11}{}^2\partial_z\varphi_{11} - \frac{3}{g}\omega^2\varphi_{11}\partial_z\varphi_{11}\partial_{zz}\varphi_{11} \\
& - \frac{1}{2}\left(k^2 + \frac{\omega^4}{g^2}\right)\varphi_{11}{}^2\partial_{zz}\varphi_{11} + \frac{1}{2g}\omega^2\varphi_{11}{}^2\partial_{zzz}\varphi_{11} - k^4\varphi_{11}{}^3
\end{aligned}
$$

Equation (15b):

$$O(\varepsilon^2): \begin{cases}
\partial_{zz}\varphi_{20} = 0, & -h < z < 0 \\
\partial_z \varphi_{20} = 0, & z = -h \\
g\partial_z \varphi_{20} = j\dfrac{\omega^3}{g}\varphi_{11}\partial_z\varphi_{1,-1} - j\dfrac{\omega^3}{g}\partial_z\varphi_{11}\varphi_{1,-1} + j\omega\partial_{zz}\varphi_{11}\varphi_{1,-1} - j\omega\varphi_{11}\partial_{zz}\varphi_{1,-1}, & z = 0 \\
\\
\partial_{zz}\varphi_{21} - k^2\varphi_{21} = -j2k\partial_{X_1}\varphi_{11}, & -h < z < 0 \\
\partial_z \varphi_{21} = 0, & z = -h \\
g\partial_z\varphi_{21} - \omega^2\varphi_{21} = j2\omega\partial_{T_1}\varphi_{11} + j2\omega\partial_z\varphi_{10}\partial_z\varphi_{11} - j\omega\varphi_{11}\partial_{zz}\varphi_{10}, & z = 0 \\
\\
\partial_{zz}\varphi_{2,-1} - k^2\varphi_{2,-1} = j2k\partial_{X_1}\varphi_{1,-1}, & -h < z < 0 \\
\partial_z\varphi_{2,-1} = 0, & z = -h \\
g\partial_z\varphi_{2,-1} - \omega^2\varphi_{2,-1} = j\omega\varphi_{1,-1}\partial_{zz}\varphi_{10} - j2\omega\partial_z\varphi_{10}\partial_z\varphi_{1,-1} - j2\omega\partial_{T_1}\varphi_{1,-1}, & z = 0 \\
\\
\partial_{zz}\varphi_{22} - 4k^2\varphi_{22} = 0, & -h < z < 0 \\
\partial_z\varphi_{22} = 0, & z = -h \\
g\partial_z\varphi_{22} - 4\omega^2\varphi_{22} = j2\omega(\partial_z\varphi_{11})^2 - j2k^2\omega\varphi_{11}{}^2 - j\omega\varphi_{11}\partial_{zz}\varphi_{11} + j\dfrac{\omega^3}{g}\varphi_{11}\partial_z\varphi_{11}, & z = 0 \\
\\
\partial_{zz}\varphi_{2,-2} - 4k^2\varphi_{2,-2} = 0, & -h < z < 0 \\
\partial_z\varphi_{2,-2} = 0, & z = -h \\
g\partial_z\varphi_{2,-2} - 4\omega^2\varphi_{2,-2} = j2k^2\omega\varphi_{1,-1}{}^2 - j2\omega(\partial_z\varphi_{1,-1})^2 - j\dfrac{\omega^3}{g}\varphi_{1,-1}\partial_z\varphi_{1,-1} + j\omega\varphi_{1,-1}\partial_{zz}\varphi_{1,-1}, & z = 0
\end{cases}$$

Equation (15c):

$$O(\varepsilon^3):\begin{cases}
\partial_{zz}\varphi_{30} = -\partial_{X_1X_1}\varphi_{10} - \partial_{Y_1Y_1}\varphi_{10}, & -h < z < 0\\
\partial_z\varphi_{30} = 0, & z = -h\\
g\partial_z\varphi_{30} = Q_0, & z = 0\\
\\
\partial_{zz}\varphi_{31} - k^2\varphi_{31} = -\partial_{X_1X_1}\varphi_{11} - \partial_{Y_1Y_1}\varphi_{11} - j2k\partial_{X_1}\varphi_{21} - j2k\partial_{X_2}\varphi_{11}, & -h < z < 0\\
\partial_z\varphi_{31} = 0, & z = -h\\
g\partial_z\varphi_{31} - \omega^2\varphi_{31} = Q_1, & z = 0\\
\\
\partial_{zz}\varphi_{3,-1} - k^2\varphi_{3,-1} = -\partial_{X_1X_1}\varphi_{1,-1} - \partial_{Y_1Y_1}\varphi_{1,-1} + j2k\partial_{X_1}\varphi_{2,-1} + j2k\partial_{X_2}\varphi_{1,-1}, & -h < z < 0\\
\partial_z\varphi_{3,-1} = 0, & z = -h\\
g\partial_z\varphi_{3,-1} - \omega^2\varphi_{3,-1} = \bar{Q}_1, & z = 0\\
\\
\partial_{zz}\varphi_{32} - 4k^2\varphi_{32} = -j4k\partial_{X_1}\varphi_{22}, & -h < z < 0\\
\partial_z\varphi_{32} = 0, & z = -h\\
g\partial_z\varphi_{32} - 4\omega^2\varphi_{32} = Q_2, & z = 0\\
\\
\partial_{zz}\varphi_{3,-2} - 4k^2\varphi_{3,-2} = j4k\partial_{X_1}\varphi_{2,-2}, & -h < z < 0\\
\partial_z\varphi_{3,-2} = 0, & z = -h\\
g\partial_z\varphi_{3,-2} - 4\omega^2\varphi_{3,-2} = \bar{Q}_2, & z = 0
\end{cases}$$

$$
O(\varepsilon^3): \begin{cases}
\partial_{zz}\varphi_{33} - 9k^2\varphi_{33} = 0, & -h < z < 0 \\
\partial_z \varphi_{33} = 0, & z = -h \\
g\partial_z \varphi_{33} - 9\omega^2\varphi_{33} = Q_3, & z = 0 \\
 & \\
\partial_{zz}\varphi_{3,-3} - 9k^2\varphi_{3,-3} = 0, & -h < z < 0 \\
\partial_z \varphi_{3,-3} = 0, & z = -h \\
g\partial_z \varphi_{3,-3} - 9\omega^2\varphi_{3,-3} = \bar{Q}_3, & z = 0
\end{cases}
$$

Appendix D

Proof that the second derivative of the linear dispersion relation is always negative

The linear dispersion relation reads

$$\omega = \omega(k) = \sqrt{g k \tanh(kh)}\,. \tag{1}$$

Consequently, the group velocity $C_g(k) \equiv \omega'(k)$ is equal to

$$\begin{aligned} C_g &= \frac{g}{2\omega}\left(\tanh(kh) + \frac{kh}{\cosh^2(kh)}\right) = \frac{\omega}{2k}\left(1 + \frac{kh}{\tanh(kh)\cosh^2(kh)}\right) = \\ &= \frac{\omega}{2k}\left(1 + \frac{2kh}{\sinh(2kh)}\right). \end{aligned} \tag{2}$$

Given the above result, for the second derivative of ω we have that

$$\begin{aligned} \omega'' &= C_g{}' = \frac{d}{dk}\left(\frac{\omega}{2k} + \frac{\omega h}{\sinh(2kh)}\right) = \\ &= \frac{2kC_g - 2\omega}{4k^2} + \frac{C_g h \sinh(2kh) - 2\omega h^2 \cosh(2kh)}{\sinh^2(2kh)} = \\ &= C_g\left(\frac{1}{2k} + \frac{h}{\sinh(2kh)}\right) - \frac{\omega}{2k^2} - \frac{2\omega h^2 \cosh(2kh)}{\sinh^2(2kh)} = \\ &= \omega\left(\frac{1}{2k} + \frac{h}{\sinh(2kh)}\right)^2 - \frac{\omega}{2k^2} - \frac{2\omega h^2 \cosh(2kh)}{\sinh^2(2kh)} = \\ &= -\frac{\omega}{4k^2}\left\{2 + \frac{8k^2h^2\cosh(2kh)}{\sinh^2(2kh)} - 4k^2\left(\frac{1}{2k} + \frac{h}{\sinh(2kh)}\right)^2\right\}. \end{aligned} \tag{3}$$

Using, though, the identities

$$\sinh(2x) = 2\sinh x \cosh x \tag{4a}$$

and

$$\cosh(2x) = \sinh^2 x + \cosh^2 x\,, \tag{4b}$$

Eq. (3) can further be shaped as

$$\begin{aligned} \omega'' &= -\frac{\omega}{4k^2}\left\{\begin{matrix} 2 + 2k^2h^2\dfrac{\sinh^2(kh) + \cosh^2(kh)}{\sinh^2(kh)\cosh^2(kh)} \\ -4k^2\left(\dfrac{1}{4k^2} + \dfrac{h}{k\sinh(2kh)} + \dfrac{h^2}{\sinh^2(2kh)}\right) \end{matrix}\right\} = \\ &= -\frac{\omega}{4k^2}\left\{\begin{matrix} 2 + \dfrac{2k^2h^2}{\cosh^2(kh)} + \dfrac{2k^2h^2}{\sinh^2(kh)} \\ -1 - \dfrac{4kh}{\sinh(2kh)} - \dfrac{4k^2h^2}{\sinh^2(2kh)} \end{matrix}\right\} = \\ &= -\frac{\omega}{4k^2}\left\{1 + \frac{2k^2h^2}{\cosh^2(kh)} + \frac{2k^2h^2}{\sinh^2(kh)} - \frac{4kh}{\sinh(2kh)} - \frac{4k^2h^2}{\sinh^2(2kh)}\right\}. \end{aligned} \tag{5}$$

Exploiting, now, the fact that

$$-\frac{4k^2h^2}{\sinh^2(2kh)} = \frac{4k^2h^2}{\sinh^2(2kh)} - \frac{8k^2h^2}{\sinh^2(2kh)} \tag{6a}$$

and

$$\frac{2k^2h^2}{\cosh^2(kh)} = \frac{4k^2h^2}{\cosh^2(kh)} - \frac{2k^2h^2}{\cosh^2(kh)}, \tag{6b}$$

Eq. (5) becomes

$$\begin{aligned}\omega'' &= -\frac{\omega}{4k^2}\left\{\begin{aligned}&\left(1-\frac{4kh}{\sinh(2kh)}+\frac{4k^2h^2}{\sinh^2(2kh)}\right)+\frac{4k^2h^2}{\cosh^2(kh)}\\ &+2k^2h^2\left(\frac{1}{\sinh^2(kh)}-\frac{1}{\cosh^2(kh)}\right)-\frac{8k^2h^2}{\sinh^2(2kh)}\end{aligned}\right\} = \\ &= -\frac{\omega}{4k^2}\left\{\begin{aligned}&\left(1-\frac{2kh}{\sinh(2kh)}\right)^2+\left(\frac{2kh}{\cosh(kh)}\right)^2\\ &+8k^2h^2\frac{\cosh^2(kh)-\sinh^2(kh)}{\sinh^2(2kh)}-\frac{8k^2h^2}{\sinh^2(2kh)}\end{aligned}\right\}.\end{aligned} \tag{7}$$

But,

$$\cosh^2 x - \sinh^2 x = 1. \tag{8}$$

Thus, finally,

$$\omega'' = -\frac{\omega}{4k^2}\left\{\left(1-\frac{2kh}{\sinh(2kh)}\right)^2+\left(\frac{2kh}{\cosh(kh)}\right)^2\right\}. \tag{9}$$

Since the nature of the WWP demands ω to take positive values, it is obvious from Eq. (9) that ω'' is always negative.

Conclusions and future directions

In the present thesis, we mainly examined the AVP, implementing it in a manner that, as we saw, renders it self-contained. Particularly, motivated by the fact that variational principles should contain all the necessary information regarding the problem they correspond to, instead of using an explicit, "externally" obtained, vertical dependence for the velocity potential, we introduced arbitrary vertical functions, whose determination we left to the AVP. Doing so, we deduced that, both in Stokes waves and weakly nonlinear, narrow-banded wavetrains, the results of the AVP are in agreement with the respective findings in the context of systematic perturbation methods. Hence, following that approach, the AVP seems to become an autonomous and consistent method for the study of periodic or nearly periodic wavetrains of weak nonlinearity.

We believe that the full potential of Whitham's method is not yet completely clear, especially after the new insights in regard to its capability in determining, each time, the appropriate vertical structure of the problem. Nevertheless, some advantages are readily noticeable. First, the required calculations, in the context of the AVP, seem to be fewer. A truly important merit of the method, though, is that the variations with respect to the components that determine the first-order free-surface elevation (fundamental amplitude and phase modulation) correspond to the solvability conditions we impose within the MSM [(Whitham 1970; Kurylev 1981)]. That is, the otherwise necessary invocation of Fredholm's alternative theorem is replaced by certain admissible variations. An immediate consequence is that the symbolic implementation of the AVP constitutes a simpler and much more straightforward task, allowing for the relatively unhampered extension of the method to higher orders.

Given our findings, the question arises as to future directions towards the exploitation of the AVP. Some of the ideas under our consideration are as follows. The AVP arises naturally when periodic waves are considered (see Ch. 2); it suffices to assume field representations (for the free-surface elevation and the velocity potential) in terms of the wave phase (variable in which the periodicity occurs). Thus, it would be interesting to use ansatzes without orders and investigate how the resulting (nonlinear systems of) EL equations for the unknowns compare with the (fully nonlinear) Fourier approximation method of (Rienecker and Fenton 1981). Another application refers to the study of the side-band instability of steady progressive waves via the AVP, and the relation of the subsequent results with those of (Benjamin and Feir 1967). (Whitham 1967) dealt with that matter, but not without inconsistencies. It is noteworthy, therefore, to determine whether or not the improvements introduced by (Sedletsky 2012, 2013) and us restore that issue. One more direction of work, of great scientific and practical importance, is to apply the AVP in the case of weakly nonlinear and narrow-banded wavetrains over a seabed that exhibits slow spatial variation (see, for example, the relevant works of (Berkhoff 1973), (Kirby 1986), (Massel 1993), (Porter and Staziker 1995) and (Dingemans 1997)).

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