ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ

ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ
ΚΑΙ ΦΥΣΙΚΩΝ ΕΠΙΣΤΗΜΩΝ


# ΚΑΤΑΣΚΕΥΗ ΒΑΡΥΤΙΚΩΝ ΜΟΝΤΕΛΩΝ ΩΣ ΘΕΩΡΙΩΝ ΒΑΘΜΙΔΑΣ ΣΕ ΜΗ ΜΕΤΑΘΕΤΙΚΟΥΣ ΧΩΡΟΥΣ

ΔΙΔΑΚΤΟΡΙΚΗ ΔΙΑΤΡΙΒΗ


ΓΕΩΡΓΙΟΥ Ν. ΜΑΝΩΛΑΚΟΥ

Διπλωματούχου Σχολής Εφαρμοσμένων Μαθηματικών
και Φυσικών Επιστημών ΕΜΠ



ΕΠΙΒΛΕΠΩΝ :

ΓΕΩΡΓΙΟΣ ΖΟΥΠΑΝΟΣ

Ομότιμος Καθηγητής Ε.Μ.Π.



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ΤΡΙΜΕΛΗΣ ΣΥΜΒΟΥΛΕΥΤΙΚΗ ΕΠΙΤΡΟΠΗ:

1.............Γεώργιος Ζουπάνος, Ομ. Καθ. ΕΜΠ (Επιβλέπων)

2.............Γεώργιος Κουτσούμπας, Καθ. ΕΜΠ

3.............Νικόλαος Τράκας, Καθ. ΕΜΠ

ΕΠΤΑΜΕΛΗΣ ΣΥΜΒΟΥΛΕΥΤΙΚΗ ΕΠΙΤΡΟΠΗ:

1...........Κων/νος Αναγνωστόπουλος, Αν. Καθ. ΕΜΠ

2...........Νικόλαος Ήργες, Αν. Καθ. ΕΜΠ

3...........Αλέξανδρος Κεχαγιάς, Καθ. ΕΜΠ

4...........Γεώργιος Κουτσούμπας, Καθ. ΕΜΠ

5...........Νικόλαος Τράκας, Καθ. ΕΜΠ

6...........Κωνσταντίνος Φαράκος Καθ. ΕΜΠ

7...........Αθανάσιος Χατζησταυρακίδης, Research associate, Rudjer Bošković

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## Ευχαριστίες

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# Εισαγωγή

Η βαρυτική αλληλεπίδραση στις τρεις και τέσσερις διαστάσεις περιγράφεται επιτυχώς από τη Γενική Θεωρία της Σχετικότητας του Einstein στις οποίες η βαρύτητα θεωρείται ως μία ιδιότητα του χωρόχρονου. Παρόλα αυτά, η περιγραφή της βαρυτικής αλληλεπίδρασης επιδέχεται μία εναλλακτική προσέγγιση, αυτήν της θεωρίας βαθμίδας των ομάδων συμμετριών των θεωρούμενων χωρόχρονων, στις οποίες τα πεδία βαθμίδας ταυτοποιούνται με το vielbein και το spin connection. Η βαρύτητα στις τρεις διαστάσεις ειναι ακριβώς ισοδύναμη με μία θεωρία βαθμίδας τύπου Chern Simons της ομάδας ISO(1,2), ενώ αν περιλαμβάνεται η κοσμολογική σταθερά τότε οι αντίστοιχες ομάδες είναι οι SO(1,3) και SO(2,2), ανάλογα με το πρόσημό της. Η τετραδιάστατη περίπτωση είναι λίγο πιο περίπλοκη, μιας και αν θεωρήσουμε μία θεωρία βαθμίδας ISO(1,3), παρά το γεγονός ότι οι μετασχηματισμοί των πεδίων και οι εκφράσεις των τανυστών καμπυλότητας προκύπτουν ως αναμένεται, υπάρχει ένα κώλυμα στο δυναμικό κομμάτι της θεωρίας, διότι δεν μπορεί να οριστεί με αυτό τον τρόπο κάποια δράση, η μορφή της οποίας να συμπίπτει με την Einstein-Hilbert. Ωστόσο, το παραπάνω πρόβλημα ξεπερνιέται θεωρώντας μία SO(1,4) αναλλοίωτη δράση με την ταυτόχρονη συμπερίληψη ενός βαθμωτού πεδίου στη θεμλιώδη αναπαράσταση. Το πεδίο αυτό επάγει την αυθόρμητη παραβίαση της συμμετρίας και οδηγεί στη ζητούμενη Einstein-Hilbert δράση. Επιπλέον, υπάρχει ένα παρόμοιο πρόγραμμα στο οποίο η βαρύτητα Weyl μεταφράζεται επιτυχώς ως μία θεωρία βαθμίδας της τετραδιάστατης σύμμορφης ομάδας, SO(2,4). Παρομοίως, στην περίπτωση αυτή, κάποιος ξεκινάει με μια δράση τύπου Yang-Mills και με την επιβολή συγκεκριμένων συνδέσμων, σπάει την επιπλέον συμμετρία, καταλήγωντας με μία θεωρία ταυτόσημη με αυτήν της βαρύτητας Weyl.

Οι παραπάνω κατασκευές μπορούν να μεταφερθούν στο πλαίσιο της μη μεταθετικής γεωμετρίας. Πιο συγκεκριμένα, στην περιοχή υψηλών ενεργειών (κλίμακα Planck) η μεταθετικότητα των συντεταγμένων του χώρου μπορεί να θεωρηθεί ότι αίρεται, επομένως οι φυσικές θεωρίες στην περιοχή αυτήν πρέπει να τροποποιηθούν κατάλληλα. Αυτή είναι η ουσία των εργασιών που συνθέτουν την παρούσα διατριβή, δηλαδή η διερεύνηση της βαρυτικής αλληλεπίδρασης στο μη μεταθετικό πλαίσιο εργασίας.

Αυτό επιτυγχάνεται συνδυάζοντας την πετυχημένη περιγραφή της βαρύτητας ως θεωρίας βαθμίδας στις τρεις και τέσσερις διαστάσεις με την καλώς ορισμένη κατασκευή θεωριών βαθμίδας σε μη μεταθετικούς χώρους, με αποτέλεσμα την κατασκευή βαρυτικών μοντέλων ως θεωριών βαθμίδας σε μη μεταθετικούς (ασαφείς) χώρους. Αρχικά, δουλέψαμε στην τρισδιάστατη περίπτωση, τόσο στην Lorentzian, όσο και στην Ευκλείδεια περίπτωση, χρησιμοποιώντας δύο ασαφείς χώρους, οι οποίοι ορίζονται ως φυλλοποιήσεις των τρισδιάστατων Minkowski και Ευκλείδειου χώρων από ασαφή υπερβολοειδή και ασαφείς σφαίρες, αντίστοιχα. Η κατασκευή των θεωριών βαθμίδας οδήγησε στην εξεύρεση των μετασχηματισμών των πεδίων βαθμίδας (vielbein και spin connection) και των εκφράσεων των τανυστών καμπυλότητας καθώς επίσης και στην δράση τύπου Chern-Simons, από την οποία εξάχθηκαν οι εξισώσεις κίνησης. Είναι αξιοσημείωτο ότι όλα τα αποτελέσματα ανάγονται σε αυτά της τρισδιάστατης θεωρίας της βαρύτητας του Einstein κατά την θεώρηση του μεταθετικού ορίου. Έπειτα, επικεντρωθήκαμε στην τετραδιάστατη περίπτωση στην οποία ο μη μεταθετικός χώρος που θεωρήσαμε ήταν η ασαφής εκδοχή του τετραδιάστατου χώρου de Sitter. Παρομοίως με την τρισδιάστατη περίπτωση, ακολουθώντας την καθιερωμένη διαδικασία κατασκευής θεωριών βαθμίδας σε μη μεταθετικούς χώρους, υπολογίζονται οι μετασχηματισμοί των πεδίων βαθμίδας και οι εκφράσεις των τανυστών καμπυλότητας καθώς και ορίζεται αρχικά μία δράση τύπου Yang-Mills, η συμμετρία της οποίας παραβιάζεται από την επιβολή κατάλληλων συνδέσμων. Τα αποτελέσματα και στην περίπτωση αυτή συνάδουν με αυτά της σύμμορφης βαρύτητας στο μεταθετικό όριο.

## Abstract

Gravitational interaction in three and four dimensions is successfully described by Einstein's theory of General Relativity (GR) in which gravity is considered as a geometric property of space and time. However, its description admits an alternative description, that of a gauge theory of the groups of symmetries of the spacetimes considered, in which the gauge fields of the theory are identified as the vielbein and the spin connection. Gravity in three-dimensions is exactly equivalent to a Chern-Simons gauge theory of ISO(1,2), while if cosmological constant is included, the corresponding gauge groups are the SO(1,3) and SO(2,2) ($dS_3$ and $AdS_3$ groups) depending on its sign. In the four-dimensional case, things are more complicated, since considering a gauge theory of ISO(1,3), despite yielding correct expressions for the transformations of the fields and the curvature tensors, there is a drawback in the dynamic part, that is there is no option for an action to recover the Einstein-Hilbert one. Nevertheless, this issue in nicely addressed by considering an SO(1,4) gauge invariant action of Yang-Mills type and include a scalar field in the fundamental representation. Inclusion of the scalar field induces a spontaneous symmetry breaking which leads to the desired Einstein-Hilbert action. Moreover, there is also a similar programme in which Weyl gravity is successfully translated as a gauge theory of the four-dimensional conformal group, SO(2,4). In this case, too, one begins with an action of Yang-Mills type and breaks the redundant symmetry by imposing certain constraints (e.g. the torsionless condition), resulting with a final action which is identical to the one of the Weyl gravity.

The above constructions can be nicely translated in the framework of noncommutative geometry. Specifically, in the large-energy regime (Planck scale), commutativity of the coordinates of the space is naturally assumed to be lifted, therefore, physical theories have to be modified along these lines. This is the essence of the projects that compose this thesis, that is giving insight in the gravitational interaction in this noncommutative regime.

This is achieved by combining the successful description of gravity as gauge theories, in three and four dimensions, with the well-defined construction of gauge theories on noncommutative spaces leading to constructions of gravitational models as gauge theories on noncommutative (fuzzy) spaces. First, we worked in the three-dimensional case, in both Lorentzian and Euclidean signature, employing two fuzzy spaces for each case which are defined as foliations of the three-dimensional Minkowski and Euclidean space by fuzzy hyperboloids and fuzzy spheres, respectively. The construction of the gauge theory led to the transformations of the gauge fields, the curvature tensor expressions and an action of Chern-Simons type, which after variation, produced the equations of motion. It is remarkable that all results reduce to the ones of the Einstein's three-dimensional theory of gravity when the commutative limit is considered. Afterwards, we focused on the four-dimensional case, in which the noncommutative space considered was the four-dimensional fuzzy de Sitter space. Again, following the procedure for constructing the noncommutative gauge theory of gravity, transformations of the fields, curvature tensors and an action of Yang-Mills type were obtained. The results in this case are related to the ones of the gauging of the conformal group, in the commutative limit.

# Introduction

In 1915, Einstein coined the description of gravitational interaction as being mediated by the curvature of the spacetime itself in his theory of General Relativity. General Relativity is an extremely successful theory, passing all tests since then, such us the so-called classical tests of General Relativity (the perihelion precession of Mercury's orbit, the deflection of light by the Sun and the gravitational redshift of light), the recent detection of gravitational waves or the more recent direct observation of a black hole. However, gravity, through the geometric description of General Relativity, is formulated in a completely different way compared to the description of the rest of the interactions (electromagnetic, weak and strong), which is based on the principle of gauge invariance. Therefore, towards the direction of the unification of all interactions, although unification of the three interactions described as gauge theories is, at least, easier to conceive and formulate, gravity is left outside from this picture. Thus, in order to include the gravitational interaction in a unified scheme along with the rest, an alternative approach of gravity, which would also recover the successful results of General Relativity, as a gauge theory seemed like a first step to the direction of unification of all interactions. In middle 1950's, Utiyama pioneered in this field of gauge-theoretic approach of gravity [1], in which he showed that gravity can be regarded as a gauge theory of the Lorentz group, SO(1,3), but the formulation was far from perfect because of the ad hoc introduction of the vierbein. This problem was, at least partly, solved by Kibble [2] taking into consideration the inhomogeneous Lorentz group (Poincaré group ISO(1,3)) as gauge group, identifying both the vierbein and the spin connection as gauge fields of the theory. Still, the whole undertaking of the description of gravity as a gauge theory was no convincing, because there was no action originating from a gauge theory of the Poincaré group that could take the form of Einstein-Hilbert action and therefore dynamics of the General Relativity in four dimensions could not be retrieved. However, in 1980, Stelle and West [3] (see also [4–6]) addressed the above problem, considering an SO(1,4) gauge invariant action of Yang-Mills type along with the introduction of a scalar field in the fundamental representation of the group. By the inclusion of an appropriate potential term, the scalar field induces a spontaneous symmetry breaking leading to an action which is of the desired Einstein-Hilbert form. Therefore, although there is a fundamental difference between gravity as a gauge theory and the rest due to the mixing of the gauge transformations with the spacetime coordinates through the identification of the gauge fields, eventually gravity can fall into a common class with the other interactions.

Also, in the late 1970's, another contribution in the approach of four-dimensional gravity as a gauge theory was made, this time concerning conformal gravity and supergravity [7, 8][1]. Specifically, a gauge theory of the four-dimensional conformal group, SO(2,4) was constructed, with the vierbein and spin connection being identified again as some of the gauge fields. Therefore, it is understood that, in this case, too, the translational part of the internal symmetry has to be be related with the general coordinate transformations. The transformations of all gauge fields and the component curvature tensors were obtained following the standard procedure and the action proposed was of Yang-Mills type making use of the square of the curvature tensor. The SO(2,4) gauge symmetry of this action was broken with the imposition of certain constraints, such as the torsionless condition, leading to the scale invariant Weyl action. Therefore, it was shown that Weyl gravity can be also described as a gauge theory of the conformal group.

[1] For a textbook we propose [9].

Furthermore, another contribution supporting the relation between gravity and gauge theories was the formulation of Einstein's three-dimensional gravity as a gauge theory of the three-dimensional Poincaré group, ISO(1,2) or of the three-dimensional de Sitter and Anti de Sitter groups, SO(1,3) and SO(2,2), respectively, when a cosmological constant is included [10]. The equivalence of the three-dimensional Einstein's gravity to the ISO(1,2) gauge theory was achieved by considering a pure Chern-Simons interaction, instead of the ordinary Yang-Mills interaction. Moreover, in ref. [10] it is claimed that the quantized version of the three-dimensional gravity is a renormalizable theory, a fact that is not evident when gravity is formulated in the standard metric-depended formalism. Also, it is commented that for the four-dimensional gravity that is not the case. Therefore, three-dimensional gravity is equivalent to a Chern-Simons gauge theory.

At this point the discussion about the equivalence between gravity and gauge theories is concluded. Now, since our purpose is to translate the above programme to the noncommutative framework, let us move on with a short introduction on noncommutative geometry [11]. The first implication of spacetime noncommutativity was made in the early days of quantum field theories by the pioneers in the field of quantum mechanics, most notably Heisenberg. The whole idea was based on the postulation that a noncommutative structure for spacetime coordinates at very small scales could lead to the introduction of an effective ultraviolet cutoff [12, 13]. It was claimed that this cutoff, originating from noncommutativity, would regulate the ultraviolet divergences of the quantum field theories such as quantum electrodynamics. However, at the same time, the renormalization programme proved to be successful, therefore, the bad timing made noncommutativity to be set aside for a while. The whole idea of noncommutativity became interesting again in 1980's, when the generalized notion of a differential structure in the noncommutative framework was achieved [14], along with the definition of a generalized integration [15].

In quantum mechanics, a quantum phase space is defined by replacing the variables of the canonical position, $x_i$, and momentum, $p_i$, with Hermitian operators $\hat{x}_i, \hat{p}_i$, respectively, which obey the Heisenberg commutation relations $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$. The notion of a point of a phase space is no longer meaningful, with the notion of the Planck cell replacing it, recovering the ordinary phase space in the $\hbar \to 0$ limit. In analogy to the above quantization of phase space, a noncommutative spacetime is defined by replacing the spacetime coordinates $x^i$ by Hermitian generators $\hat{x}^i$ of a noncommutative $C^*$-algebra of functions [16–32], which obey the commutation relation $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$, in complete analogy to the Heisenberg's commutation relation. Since the coordinates do not commute, they cannot be diagonalized at the same time, therefore the place of the manifold is taken by a Hilbert space of states. In analogy to quantum mechanics, a spacetime uncertainty relation is induced, $\Delta x^i \Delta x^j \geq \frac{1}{2}|\theta^{ij}|$. The noncommutative framework described so far is particularly interesting because it admits the description of both particle physics and gravity models in conditions in which the commutativity of coordinates can be naturally relaxed (e.g. at the Planck scale).

First let us comment on the gravity case. Taking into consideration the relation of gravity and gauge theories as discussed earlier and motivated at the same time by the existence of noncommutative gauge theories [33], it is natural to apply them for the construction of noncommutative gravity models. Such an approach is followed in Refs. [34–38]. Similarly, this has been studied also in three dimensions, making use of the relation to Chern-Simons gauge theory mentioned earlier [39–41]. The above works share a common feature, that is the noncommutative deformation is constant (Moyal-Weyl) and the construction of the theories is made using the corresponding $\star$-product and the Seiberg-Witten map [42]. Alternatively, one can use another type of noncommutative geometries, the matrix geometries, in order to work on quantum gravity [43, 44]. Several approaches have been suggested in recent years, mainly based on Yang-Mills matrix models [45–55], pointing once more at direct relations among noncommutative gauge theories and gravity. For another approach see Refs. [56–58], where a solid indication that the degrees of freedom or basic modes of the resulting theory of gravity can be put in correspondence with those of the noncommutative structure has been presented. In this case, the usual symmetries such as coordinate invariance are built-in, and the commutator of coordinates can have arbitrary dependence on them. In general, attempting to formulate gravity in the noncommutative setup, the price one has to pay is that noncommutative deformations generically break Lorentz invariance. For certain types of noncommutative spaces, it is possible to define deformed symmetries which are preserved, as for example in the case of $\kappa$-Minkowski spacetime [59, 60], which appears as a solution of the Lorentzian IIB matrix model in Ref. [61]. However, there are special types of deformations, in fact some of the very first noncommutative geometries ever considered, that constitute covariant noncommutative spacetimes [12, 13]. This spirit was recently revived in Ref. [62], where the authors discuss a realization of this idea and construct a noncommutative deformation of a general conformal field theory defined on four-dimensional dS or AdS spacetime. Another four-dimensional constructions were pursued in Refs. [63–66][2].

Now, let us say a few words about the particle physics models which can be accommodated in the framework of noncommutative geometry, specifically in the noncommutative gauge-theoretic approach [23] (see also [18, 25, 26]). A very interesting development in the framework of the non-commutative geometry is the programme in which the extra dimensions of higher-dimensional theories are considered to be non-commutative (fuzzy) [67–77]. This programme overcomes the ultraviolet/infrared problematic behaviours of theories defined in noncommutative spaces. A very welcome feature of such theories is that they are renormalizable, versus all known higher-dimensional theories.

The outline of the present thesis is as follows. In chapter one, we write down the necessary preliminaries, that is the non-coordinate basis of the tangent space at a point of a manifold and the vierbein formalism of the theory of general relativity, in which its results are reproduced making use of the vierbein and the spin connection instead of the metric. Then, employing the vierbein formalism, the description of specific gravity theories as gauge theories is reviewed. More specifically, first we recall the three-dimensional Einstein's gravity case, in which the results are exactly reproduced through the construction of a Chern-Simons gauge theory of ISO(1,2) for vanishing cosmological constant and SO(1,3) and SO(2,2) for positive or negative cosmological constant, respectively. The key point in this approach is that the gauge fields of the theory are identified to the dreibein and the spin connection. Then, we recall the corresponding works in the four-dimensional case. Despite the fact that the form of the Einstein-Hilbert action does not admit an ISO(1,3) gauge-theoretic interpretation generalizing the three-dimensional case, alternative ways have been employed. A convincing and straightforward way is to consider the SO(1,4) as the gauge group, in which the gauge fields are again identified as the vierbein and the spin connection. Moreover, a scalar field in the fundamental representation of SO(1,4) has to be introduced in order to induce a spontaneous symmetry breaking, resulting with the Einstein-Hilbert action if the initial action is SO(1,4) gauge invariant of Yang-Mills type. Concluding this

[2] See also [56, 57].

chapter, we review the corresponding case of the Weyl gravity, which is described as a gauge theory of the four-dimensional conformal group, SO(2,4). The transformations of the fields and the curvature tensors are obtained following the standard procedure, but the action is initially defined to be SO(2,4) gauge invariant of Yang-Mills type. In this case too, the symmetry has to be broken and it is induced by the imposition of certain constraints (such as the torsionless condition) rather than spontaneously. The final action is the scale invariant Weyl action.

In chapter two, the framework of noncommutative geometry is studied in the two descriptions, that is the one with functions and the $\star$-product and the other with matrices and the ordinary matrix product, which is the one we use in the construction of our models later. Then, we focus on the very important covariant noncommutative space that is the fuzzy sphere and introduce it in a comparative way to the ordinary sphere. We conclude this chapter with a very important and useful section for our purposes, in which we recall the formulation of gauge theories on noncommutative spaces.

In chapter three, we give some information about the specific noncommutative (fuzzy) spaces we employ in our works, on which we construct our gravitational models. More specifically, we review the definition of the $\mathbb{R}^3_\lambda$ (and its Lorentzian analogue) and also we define a four-dimensional fuzzy de Sitter space [78] and comment on its properties compared to other fuzzy spaces.

In chapter four, we write down the main body of the thesis that is based on our corresponding publications. First, we build a gravitational model in the three-dimensional case [79][3], making use the three-dimensional covariant fuzzy spaces defined in chapter 3. For the construction of the model, we follow the standard procedure of the gauge-theoretic approach, as it is explained in chapter one, obviously translated in the noncommutative framework. The expressions of the transformations of the gauge fields (noncommutative versions of the dreibein and the spin connection) and their corresponding component curvature tensors are obtained and then the action is given along the lines of a Chern-Simons functional. In the four-dimensional case, after the definition of the fuzzy de Sitter space in chapter 3, we went on following the standard procedure generalizing the one of the three-dimensional case [78]. In this case, a symmetry breaking mechanism had to be employed for breaking the symmetry of the Yang-Mills action we considered in the beginning and result with an action with Lorentz symmetry.

The next chapter is devoted to the conclusions of our works. In the appendix, except for appendix A, the rest are calculations and technical (mathematical) details, which were useful for building our models. Appendix A is related to some previous works that we reviewed recently [75–77][4]. More specifically, it consists of works in which the noncommutative geometry framework is employed for the construction of particle physics models. Specifically, a four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is initially considered, consisting of a specific particle spectrum. The particle content is filtered out by an orbifold projection and the resulting theory is a gauge theory with reduced, $\mathcal{N} = 1$ supersymmetry with a particle content consisting only of the fields that survived the projection. From the resulting superpotential, along with the introduction of soft supersymmetry breaking terms, the scalar potential is obtained, which, when minimized, produces a vacuum of the theory that can interpreted as three fuzzy spheres. In other words, after the breaking, the resulting theory mimics the results of a dimensional reduction of a higher-dimensional gauge theory with fuzzy extra dimensions. As for the final gauge group, the most favoured one is the trinification group, that is an $SU(3)^3$

[3]See also [80, 81].

[4]For the original papers see [73, 74].

unified theory that is also chiral.

To sum up, the main part of the thesis is the translation of the programme of describing gravitational theories as gauge theories to the noncommutative framework. Also, for completeness, the reviewed work of a particle physics model using fuzzy spaces as extra dimensions is also included.

# Contents

# Κεφάλαιο 1

# Περίληψη

## 1.1 Εισαγωγή

Το 1915 ο Einstein κατοχύρωσε την περιγραφή της βαρυτικής αλληλεπίδρασης ως να οφείλεται στην καμπυλότητα του χωρόχρονου καθάυτού στην Γενική Θεωρία της Σχετικότητας. Η θεωρία της Σχετικότητας αποτελεί μία από τις πλέον επιτυχημένες θεωρίες, περνώντας όλους τους ελέγχους από την γέννηση της, όπως τα λεγόμενα ¨κλασικά τεστ της Γενικής Θεωρίας της Σχετικότητας¨ (την προπόρευση του περιηλίου του Ερμή, την εκτροπή του φωτός από τον ήλιο και τη βαρυτική μετατόπιση προς στο ερυθρό), η πρόσφατη ανίχνευση των βαρυτικών κυμάτων και η ακόμα πιο πρόσφατη παρατήρηση μελανής οπής. Ωστόσο, η βαρύτητα, μέσω της γεωμετρικής περιγραφής από τη Γενική Θεωρία της Σχετικότητας, διατυπώνεται με έναν εντελώς διαφορετικό τρόπο συγκριτικά με την περιγραφή των υπόλοιπων αλληλεπιδράσεων (ηλεκτρομαγνητική, ασθενής και ισχυρή) η οποία βασίζεται στην αναλλοιώτητα βαθμίδας. Επομένως, στην κατεύθυνση του στόχου πολλών επιστημόνων να ενοποιήσουν όλες τις αλληλεπιδράσεις, παρόλο που η ενοποίηση για τις τρεις αλληλεπιδράσεις οι οποίες περιγράφονται ως θεωρίες βαθμίδας είναι ευκολότερο να κατανοηθεί και να διατυπωθεί, η βαρυτική μένει εκτός αυτής της εικόνας. Επομένως, για να συμπεριληφθεί η βαρυτική αλληλεπίδραση σε ένα πρόγραμμα ενοποίησης μαζί με τις υπόλοιπες αλληλεπιδράσεις, το πρώτο βήμα είναι να αποτυπωθεί μία εναλλακτική προσέγγιση για την περιγραφή της βαρύτητας ως θεωρίας βαθμίδας, η οποία πρέπει να αναπαραγάγει τα αποτελέσματα της Γενικής Θεωρίας της Σχετιότητας. Περί τα μέσα του πρηγούμενου αιώνα, ο Utiyama πρωτοπόρησε στο πεδίο της περιγραφής της βαρύτητας ως θεωρίας βαθμίδας [1] υποστηρίζοντας ότι η βαρύτητα μπορεί να ιδωθεί ως μία θεωρία βαθμίδας της ομάδας Lorentz, SO(1,3). Όμως, η συνολική διατύπωση θεωρήθηκε ως ατελής λόγω της αυθαίρετης εισαγωγής των vierbein. Το πρόβλημα αυτό λύθηκε, τουλάχιστον μερικώς, από τον Kibble, [2] ο οποίος θεώρησε ως ομάδα βαθμίδας την ανομοιογενή ομάδα Lorentz, δηλαδή την ομάδα Poincaré (ISO(1,3)), ταυτοποιώντας τα vierbein και spin connection ως τα πεδία βαθμίδας της θεωρίας. Όμως, ακόμη και μετά από αυτήν την τροποποίηση το συνολικό εγχείρημα δεν κατάφερε να πείσει την κοινότητα ότι η βαρύτητα μπορεί να περιγραφεί ως θεωρία βαθμίδας διότι δεν μπορούσε να παραχθεί η κατάλληλη δράση (Einstein-Hilbert) με έναν συνεπή και μη κατευθυνόμενο τρόπο. Παρόλα αυτά, το 1980 οι Stelle και West αντιμετώπισαν το παραπάνω πρόβλημα θεωρώντας μία SO(1,4) αναλλοίωτη δράση τύπου Yang-Mills με την παράλληλη εισαγωγή ενός βαθμωτού πεδίου στη θεμελιώδη αναπαράσταση της ομάδας [3] (βλέπε επίσης [4–6]). Το βαθμωτό πεδίο επάγει αυθόρμητη παραβίαση της συμμετρίας οδηγώντας σε μία δράση, η μορφή της οποίας είναι η επιθυμητή της Einstein-Hilbert. Επομένως, παρόλο που η περιγραφή της βαρυτικής αλληλεπίδρασης διαφοροποιείται από τις υπόλοιπες υπό την έννοια ότι στην πρώτη αναμιγνύονται ουσιαστικά οι εσωτερικές συμμετρίες με τις χωροχρονικές, εν τέλει η βαρύτητα μπορεί να θεωρηθεί ότι εμπίπτει σε μία κοινή θεώρηση με τις υπόλοιπες αλληλεπιδράσεις.

Επίσης, στα τελη της δεκαετίας του 1970, επετεύχθη ακόμα μία συνεισφορά στην γενικότερη θεώρηση βαρυτικών θεωριών ως θεωριών βαθμίδας, αυτή τη φορά αναφορικά με την σύμμορφη βαρύτητα [7–9]. Πιο συγκεκριμένα, η ομάδα βαθμίδας που χρησιμοποιείται είναι η SO(2,4), με τα vierbein και το spin connection να ταυτοποιούνται, εκτός άλλων, ως διανυσματικά πεδία βαθμίδας. Επομένως, είναι κατανοητό ότι και σε αυτήν την περίπτωση το κομμάτι που σχετίζεται με τους γεννήτορες των μεταθέσεων θα πρέπει να συσχετιστεί με τους γενικούς μετασχηματισούς συντεταγμένων. Οι μετασχηματισμοί των πεδίων και οι εκφράσεις των τανυστών καμπυλότητας αποκτώνται σύμφωνα με την καθιερωμένη διαδικασία ενώ η αρχική δράση που υιοθετείται είναι τύπου Yang-Mills. Η αρχική συμμετρία SO(2,4) παραβιάζεται μέσω της επιβολής συγκεκριμένων συνδέσμων, όπως η συνθήκη μηδενικής στρέψης, με αποτέλεσμα την απόκτηση μίας δράσης η έκφραση της οποίας συμπίπτει με αυτή της δράσης Weyl. Επομένως, αποδείχθηκε ότι η βαρύτητα Weyl μπορεί να περιγραφεί ισοδύναμα ως θεωρία βαθμίδας της τετραδιάστατης σύμμορφης ομάδας.

Επιπλέον, ακόμα μία συνεισφορά προς την κατεύθυνση της συσχέτισης βαρυτικών θεωριών με θεωρίες βαθμίδας αποτελεί η διατύπωση της τρισδιάστατης Einstein βαρύτητας ως θεωρίας βαθμίδας της τρισδιάστατης ομάδας Poincaré, ISO(1,2) ή των τρισδιάστατων ομάδων de Sitter και Anti de Sitter, SO(1,3) και SO(2,2), αντίστοιχα, όταν περιλαμβάνεται κοσμολογική σταθερά [10]. Η αντιστοίχιση ανάμεσα στην τρισδιάστατη βαρύτητα και την ISO(1,2) θεωρία βαθμίδας επετεύχθη με τη θεώρηση μίας Chern-Simons αλληλεπίδρασης, αντί μίας τύπου Yang-Mills. Επομένως, η τρισδιάστατη βαρύτητα μπορεί να περιγραφεί επακριβώς ως μία αμιγής Chern-Simons θεωρία.

Δεδομένου ότι ο σκοπός μας είναι να μεταφράσουμε την παραπάνω συζήτηση για την αντιστοίχιση βαρυτικών θεωριών με θεωρίες βαθμίδας στο μη μεταθετικό πλάισιο εργασίας, ας προχωρήσουμε με την εισαγωγή της έννοιας της μη μεταθετικότητας των συντεταγμένων [11]. Η πρώτη νίξη της μη μεταθετικότητας των συντεταγμένων πραγματοποιήθηκε κατά τη διάρκεια της θεμελίωσης των κβαντικών θεωριών πεδίου από τους πρωτοπόρους της κβαντικής φυσικής με απώτερο σκοπό τη διαχείριση των αναδυόμενων υπεριωδών αποκλίσεων. Υποστηρίχθηκε ότι η εισαγωγή της μη μεταθετικότητας των συντεταγμένων θα μπορούσε να επαγάγει την εισαγωγή ενός ενεργού υπεριώδους ορίου [12, 13]. Ωστόσο, την ίδια περίοδο, το πρόγραμμα της επανακανονικοποίησης έδωσε λύσεις στο παραπάνω πρόβλημα και για το λόγο αυτό η ιδέα της μη μεταθετικότητας πέρασε στην αφάνεια. Η όλη ιδέα επανήλθε στο προσκήνιο κατά τη δεκαετία του 1980 όταν η γενικευμένη έννοια διαφορικού [14] και ολοκληρωτικού λογισμού [15] θεμελιώθηκε για μη μεταθετικούς χώρους.

Ανακαλώντας την περίπτωση της κβαντικής φυσικής, η οποία λειτούργησε σαν πηγή έμπνευσης για τη θεμελίωση του μη μεταθετικού πλαίσιου εργασίας, οι κανονικές συντεταγμένες του χώρου των φάσεων, $x_i$ και $p_j$ αντικαθίστανται από τους ερμιτιανούς πίνακες $\hat{x}_i$ και $\hat{p}_j$, οι οποίοι υπακούουν την περίφημη μεταθετική σχέση του Heisenberg, $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$. Η έννοια του σημείου στο χώρο των φάσεων παύει να έχει σημασία με την έννοια της κυψελίδας Planck να την αντικαθιστά, με την ανάκτηση του συνηθισμένου χώρου να συμβαίνει στο όριο $\hbar \to 0$. Σε αναλογία με την παραπάνω κβάντωση του χώρου των φάσεων, ένας μη μεταθετικός χώρος ορίζεται μέσω της αντικατάστασης των χωροχρονικών συντεταγμένων, $x_i$, από ερμιτιανούς γεννήτορες $\hat{x}^i$ μίας μη μεταθετικής $C^\star$ άλγεβρας συναρτήσεων [16–32], οι οποίοι ικανοποιούν τη μεταθετική σχέση $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$, σε πλήρη αναλογία με τη μεταθετική σχέση του Heisenberg. Εφόσον οι συντεταγμένες δε μετατίθενται, δεν μπορούν να διαγωνοποιηθούν ταυτόχρονα, επομένως η έννοια της πολλαπλότητας αντικαθίσταται από αυτήν ενός χώρου καταστάσεων Hilbert. Σε αναλογία με την κβαντική φυσική, εμφανίζεται μία χωροχρονική σχέση απροσδιοριστίας, $\Delta x^i \Delta x^j \geq \frac{1}{2}|\theta_{ij}|$. Σε ό,τι αφορά τους δικούς μας σκοπούς, οι μη μεταθετικοί χώροι είναι ιδιαίτερα ενδιαφέροντες και χρήσιμοι καθώς μπορούν να φιλοξενήσουν την κατασκευή τόσο σωματιδιακών όσο και βαρυτικών μοντέλων.

Πρώτα, ας ασχοληθούμε με την περίπτωση της βαρύτητας. Λαμβάνοντας υπόψη τη σχέση ανάμεσα στις βαρυτικές θεωρίες και τις θεωρίες βαθμίδας, όπως αυτή συζητήθηκε παραπάνω, με την ταυτόχρονη ύπαρξη καλώς θεμελιωμένων θεωριών βαθμίδας σε μη μεταθετικούς χώρους [33], είναι φυσικό να τις χρησιμοποιήσουμε με σκοπό την κατασευή μη μεταθετικών βαρυτικών μοντέλων. Μία τέτοια προσέγγιση ακολουθήθηκε στις αναφορές [34–38]. Παρομοίως, μελέτες έχουν γίνει και για την τρισδιάστατη περίπτωση συμπεριλαμβάνοντας τη σχέση με τις Chern-Simons θεωρίες βαθμίδας [39–41] στις οποίες αναφερθήκαμε παραπάνω. Όλες οι παραπάνω δουλειές μοιράζονται δύο κοινά χαρακτηριστικά, ότι η μη μεταθετική παραμόρφωση είναι σταθερή (Moyal-Weyl) καθώς και ότι γίνεται χρήση του αντίστοιχου $\star-$γινομένου και η απεικόνιση Seiberg-Witten [42]. Εναλλακτικά, είναι δυνατόν κάποιος να χρησιμοποιήσει έναν διαφορετικό τύπο μη μεταθετικών γεωμετριών, τις γεωμετρίες πινάκων, έτσι ώστε να δουλέψει πάνω στην κβαντική βαρύτητα [43, 44]. Επίσης, αρκετές προσεγγίσεις έχουν προταθεί τα τελευταία χρόνια, κυρίως βασισμένες σε Yang-Mills μοντέλα πινάκων [45–55]. Για μια ακόμα προσέγγιση επί του θέματος βλέπε αναφορές [56–58], στις οποίες παρουσιάζεται μία στέρεη ένδειξη ότι οι βαθμοί ελευθερίας της τελικής θεωρίας βαρύτητας μπορούν να συσχετιστούν με αυτούς μίας μη μεταθετικής κατασκευής. Σε αρκετές περιπτώσεις, οι προσπάθειες να διατυπωθεί η βαρύτητα σε μη μεταθετικούς χώρους εμφανίζουν την παθογένεια ότι οι μη μεταθετικές παραμορφώσεις παραβιάζουν τη συμμετρία Lorentz. Για συγκεκριμένους τύπους μη μεταθετικών χώρων, είναι δυνατόν να οριστούν παραμορφωμένες συμμετρίες οι οποίες διατηρούνται, όπως για παράδειγμα στην περίπτωση του χωρόχρονου $\kappa-$Minkowski [59, 60]. Ωστόσο, υπάρχουν συγκεκριμένα είδη παραμορφώσεων τα οποία αποτελούν συναλλοίωτους μη μεταθετικούς χώρους, μάλιστα τέτοιοι είναι κάποιοι από τους πρώτους που θεμελιώθηκαν [12, 13]. Σε αυτην τη βάση, στην αναφορά [62], οι συγγραφείς συζητούν μία πραγμάτωση της ιδέας αυτής και κατασκευάζουν μία μη μεταθετική παραμόρφωση μίας σύμμορφης θεωρίας πεδίου, ορισμένη στον τετραδιάστατο de Sitter (ή Anti de Sitter) χώρο. Τέλος, για περισσότερες κατασκευές στις τέσσερις διαστάσεις βλέπε αναφορές [63–66][1].

Τώρα, ας περάσουμε στις κατασκευές μοντέλων σωματιδιακής φυσικής οι οποίες φιλοξενούνται στο πλαίσιο εργασίας της μη μεταθετικής γεωμετρίας, συγκεκριμένα ως θεωρίες βαθμίδας [23] (επίσης βλέπε [18, 25, 26]). Μία πολύ ενδιαφέρουσα εξέλιξη στην κατεύθυνση αυτή αποτελεί το πρόγραμμα κατά το οποίο οι έξτρα διαστάσεις μεγαλοδιάστατων θεωριών θεωρούνται μη μεταθετικές (ασαφείς) [67–77] καθώς ξεπερνούν τις προβληματικές συμπεριφορές της ανάμιξης υπεριώδους/υπέρυθρου οι οποίες εμφανίζονται σε θεωρίες σε μη μεταθετικούς χώρους. Επίσης, ένα πολύ ευχάριστο χαρακτηριστικό είναι ότι τέτοιες θεωρίες είναι επανακανονικοποιήσιμες, σε αντίθεση με όλες τις υπόλοιπες μεγαλοδιάστατες θεωρίες.

Ο σκελετός της περίληψης στα ελληνικά έχει ως εξής: Αρχικά αναφέρουμε λίγες βασικές πληροφορίες για την περιγραφή βαρυτικών θεωριών ως θεωριών βαθμίδας. Έπειτα, δίνονται γενικές πληροφορίες για τη μη μεταθετική γεωμετρία και πως αυτή διατυπώνεται και μετά εξειδικεύουμε σε συγκεκριμένους συναλλοίωτους ασαφείς χώρους οι οποίοι είναι απαραίτητοι για την κατασκευή των μοντέλων μας. Αυτά είναι δύο, το πρώτο αφορά μία εκδοχή τρισδιάστατης βαρύτητας ως θεωρία βαθμίδας πάνω τον συναλλοίωτο ασαφή χώρο $\mathbb{R}^3_\lambda$ [79][2] ενώ το δεύτερο αναφέρεται στην τετραδιάστατη βαρυτήτα χτισμένη πάνω σε έναν τετραδιάστατο συναλλοίωτο ασαφή χώρο [78], fuzzy dS,

[1] Επίσης βλέπε [56, 57].

[2] Επίσης βλέπε [80, 81].

τον οποίον κατασκευάσαμε για τον σκοπό αυτό.

Κλείνοντας, η εργασία αφορά στην κατασκευή βαρυτικών μοντέλων σε μη μεταθετικούς χώρους με σκοπό τη διερεύνηση της βαρυτικής αλληλεπίδρασης στις τρεις και τέσσερις διαστάσεις, σε συνθήκες κατά τις οποίες η μη μεταθετικότητα των συντεταγμένων τίθεται σε ισχύ.

## 1.2 Η βαρύτητα ως θεωρία βαθμίδας στις τρεις και τέσσερις διαστάσεις

Ας μελετήσουμε πρώτα την τρισδιάστατη Einstein βαρύτητα και τη σχέση της με θεωρία βαθμίδας [10]. Για να αναδειχθεί η σχέση αυτή, πρέπει να χρησιμοποιηθεί ο (τρισδιάστατος) vielbein φορμαλισμός της Γενικής Θεωρίας της Σχετικότητας, στον οποίο αντί για τη μετρική χρησιμοποιούνται τα vierbein και spin connection ως δυναμικές μετεβλητές. Στις τρεις διαστάσεις, για μία πολλαπλότητα, $M$, η Einstein-Hilbert δράση, απουσία ύλης και κοσμολογικής σταθεράς, σε όρους vielbein και spin connection είναι:

$$S_{\text{EH3}} = \frac{1}{16\pi G}\int_M \epsilon^{\mu\nu\rho} e_\mu{}^a(\partial_\nu\omega_{\rho a} - \partial_\rho\omega_{\nu a} + \epsilon_{abc}\omega_\nu{}^b\omega_\rho{}^c) \ . \tag{1.1}$$

Μεταβολή στην παραπάνω δράση ως προς το πεδίο $e$ οδηγεί στις πεδιακές εξισώσεις του Einstein στο κενό:

$$R_{\mu\nu a} = \partial_\nu\omega_{\rho a} - \partial_\rho\omega_{\nu a} + \epsilon_{abc}\omega_\nu{}^b\omega_\rho{}^c = 0 \ , \tag{1.2}$$

ενώ μεταβολή ως προς το πεδίο $\omega$ οδηγεί στη συνθήκη μηδενικής στρέψης:

$$T_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \epsilon^{abc}\omega_{\mu b}e_{\nu c} - \epsilon^{abc}\omega_{\nu b}e_{\mu c} = D_\mu e_\nu{}^a - D_\nu e_\mu{}^a = 0 \ , \tag{1.3}$$

όπου η $D_\mu e_\nu{}^a$ ορίζεται ως:

$$D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \epsilon^{abc}\omega_{\mu b}e_{\nu c} \ . \tag{1.4}$$

Παραπάνω έχει γίνει χρήση του επαναορισμού $\omega_\mu{}^a = \frac{1}{2}\epsilon^{abc}\omega_{\mu bc}$.

Αν στην παραπάνω δράση, (1.1), θεωρήσουμε ότι τα vielbein και spin connection συμβολίζονται συλλογικά από ένα πεδίο $A$, τότε η δράση γράφεται σαν $AdA+A^3$, μία μορφή η οποία παραπέμπει στη γενική μορφή ενός συναρτησοειδούς Chern-Simons στις τρεις διαστάσεις. Αυτό δείχνει προς την κατεύθυνση συσχετισμού της τρισδιάστατης βαρύτητας με μία Chern-Simons θεωρία βαθμίδας. Απομένει να βρεθεί η κατάλληλη ομάδα βαθμίδας, έτσι ώστε να γραφτεί η αντίστοιχη Chern-Simons δράση και να επιβεβαιωθεί ότι συμπίπτει με την τρισδιάστατη Einstein-Hilbert δράση, (1.1).

Έστω ότι η κατάλληλη ομάδα βαθμίδας είναι η ISO(1,2). Αξίζει να σημειωθεί ότι εν γένει το Chern-Simons συναρτησοειδές ορίζεται για απλές άλγεβρες Lie. Επομένως, δεν είναι προφανές ότι μπορεί να κατασκευαστεί μία Chern-Simons θεωρία βαθμίδας της ISO(1,2), εκτός εάν επιβεβαιωθεί ότι η Chern-Simons αλληλεπίδραση μπορεί να οριστεί για την υποψήφια ομάδα. Με άλλα λόγια, αυτό που επιδιώκεται είναι η εξεύρεση μίας αναλλοίωτης τετραγωνικής μορφής της ISO(1,2) άλγεβρας Lie. Παρόλο που για αυθαίρετη διάσταση, δηλαδή για την ομάδα ISO(1,n-1), κάτι τέτοιο δεν ισχύει, στη συγκεκριμένη περίπτωση όπου $n = 3$ υπάρχει μία αναλλοίωτη και μη εκφυλισμένη μορφή, η οποία είναι η:

$$\text{tr}(J_aP_b) = \delta_{ab} \ , \quad \text{tr}(P_aP_b) = 0 \ , \quad \text{tr}(J_aJ_b) = 0 \ , \tag{1.5}$$

όπου $J_a = \frac{1}{2}\epsilon_{abc}J^{bc}$ είναι οι τρεις γεννήτορες Lorentz και $P_a$ οι τρεις μεταθέσεις, συναποτελώντας τους έξι γεννήτορες της ομάδας ISO(1,2). Οι παραπάνω γεννήτορες ικανοποιούν την παρακάτω άλγεβρα, όπως αυτή δίνεται από τις εξής μεταθετικές σχέσεις:

$$[J_a, J_b] = \epsilon_{abc}J^c \ , \quad [J_a, P_b] = \epsilon_{abc}P^c \ , \quad [P_a, P_b] = 0 \ . \tag{1.6}$$

Το επόμενο βήμα είναι να γραφτεί η συναλλοίωτη παράγωγος:

$$\tilde{D}_\mu = \partial_\mu + [A_\mu, \cdot] \ , \tag{1.7}$$

όπου $A_\mu(x)$ είναι η συνοχή βαθμίδας, η οποία αναπτύσσεται πάνω στους γεννήτορες της άλγεβρας της ISO(1,2), αφού παίρνει τιμές σε αυτή:

$$A_\mu(x) = e_\mu^{\ a}(x)P_a + \omega_\mu^{\ a}(x)J_a \ . \tag{1.8}$$

Στην παραπάνω έκφραση της συνοχής βαθμίδας, $A_\mu$, για κάθε γεννήτορα έχει ανατεθεί ένα διανυσματικό πεδίο. Το vielbein πεδίο έχει επισυναφθεί στις μεταθέσεις, ενώ για το κομμάτι των στροφών το αντίστοιχο πεδίο είναι το spin connection.

Εξ' ορισμού, η παράγωγος $\tilde{D}_\mu$ μετασχηματίζεται συναλλοίωτα, δίνοντας τον κανόνα μετασχηματισμού του $A_\mu$:

$$\delta A_\mu = -\tilde{D}_\mu\epsilon = -\partial_\mu - [A_\mu, \epsilon] \ , \tag{1.9}$$

όπου $\epsilon = \epsilon(x)$ είναι η παράμετρος μετασχηματισμού βαθμίδας, η οποία, αφού είναι στοιχείο της ISO(1,2) άλγεβρας, μπορεί να αναπτυχθεί στους γεννήτορες:

$$\epsilon(x) = \xi^a(x)P_a + \lambda^a(x)J_a \ , \tag{1.10}$$

με $\xi^a(x)$ και $\lambda^a(x)$ να είναι απειροστές παράμετροι. Συνδυάζοντας τις εξισώσεις (1.8), (1.10) με την (1.9) και κάνοντας χρήση της άλγεβρας των γεννητόρων, (1.6), οι κανόνες μετασχηματισμού των διανυσματικών πεδίων $e$ και $\omega$ βρίσκονται:

$$\delta e_\mu^{\ a} = -\partial_\mu\xi^a - \epsilon^{abc}e_{\mu b}\lambda_c - \epsilon^{abc}\omega_{\mu b}\xi_c \ , \tag{1.11}$$

$$\delta\omega_\mu^{\ a} = -\partial_\mu\lambda^a - \epsilon^{abc}\omega_{\mu b}\lambda_c \ . \tag{1.12}$$

Οι παραπάνω μετασχηματισμοί βαθμίδας δεν συμπίπτουν με τους συνήθεις μετασχηματισμούς συντεταγμένων. Παρόλο που οι μεσαίοι όροι των παραπάνω εξισώσεων μπορούν να ταυτοποιηθούν ως τοπικοί μετασχηματισμοί Lorentz, , αφού το $\lambda_c$ έχει αντιστοιχηθεί με τον γεννήτορα Lorentz, $J^c$, στον μετασχηματισμό βαθμίδας, οι υπόλοιποι όροι δεν είναι αναγνωρίσιμοι από την πρώτη ματιά. Αν είναι πιθανό να συσχετιστούν οι παραπάνω εκφράσεις των μετασχηματισμών των πεδίων βαθμίδας με τους διαφορομορφισμούς, τότε θα μπορούν να θεωρηθούν ισοδύναμοι με τους μετασχηματισμούς συντεταγμένων και, την ίδια ώρα, αυτό θα αποτελεί επιβεβαίωση ότι η ISO(1,2) είναι η κατάλληλη ομάδα βαθμίδας για την προσέγγιση της τρισδιάστατης Einstein βαρύτητας ως θεωρίας βαθμίδας. Φυσικά, η δράση που θα προσδιοριστεί θα πρέπει να είναι αναλλοίωτη κατω από τους μετασχηματισμούς βαθμίδας σε αντιστοιχία με την τρισδιάστατη Einstein-Hilbert δράση, η οποία είναι αναλλοίωτη κάτω από τους μετασχηματισμούς συντεταγμένων. Η σχέση ανάμεσα στους διαφορομορφισμούς και τους μετασχηματισμούς βαθμίδας συζητιέται αμέσως μετά τον καθορισμό της δράσης και των αντίστοιχων εξισώσεων κίνησης.

Προχωρώντας στην κατασκευή της θεωρίας βαθμίδας της ISO(1,2), το επόμενο βήμα είναι να υπολογιστούν οι συνιστώντες τανυστές καμπυλότητας των πεδίων βαθμίδας, κάνοντας χρήση της συνηθισμένης φόρμουλας, δηλαδή της χρήσης του μεταθέτη της συναλλοίωτης παραγώγου της θεωρίας βαθμίδας, $\tilde{D}_\mu$:

$$R_{\mu\nu} = [\tilde{D}_\mu, \tilde{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ , \tag{1.13}$$

όπου $A_\mu$ είναι η συνοχή βαθμίδας της εξίσωσης (1.8). Εφόσον ο τανυστής δύναμης πεδίου, $R_{\mu\nu}$, παίρνει τιμές στην άλγεβρα της ISO(1,2), μπορεί να αναπτυχθεί στους γεννήτορές της:

$$R_{\mu\nu} = T_{\mu\nu}{}^a(x) P_a + R_{\mu\nu}{}^a(x) J_a \ , \tag{1.14}$$

όπου οι $T_{\mu\nu}{}^a$ και $R_{\mu\nu}{}^a$ είναι οι συνιστώντες τανυστές καμπυλότητας οι οποίοι σχετίζονται με τα $e$ και $\omega$, αντίστοιχα. Συνδυάζοντας τη φόρμουλα (1.13), με το ανάπτυγμα (1.14) και αντικαθιστώντας το $A_\mu$ με την έκφραση (1.8), αποκτώνται οι εκφράσεις των τανυστών καμπυλότητας:

$$T_{\mu\nu}{}^a = \partial_\mu e_\nu^{\ a} - \partial_\nu e_\mu^{\ a} + \epsilon^{abc}\omega_{\mu b} e_{\nu c} - \epsilon^{abc}\omega_{\nu b} e_{\mu c} \ , \tag{1.15}$$

$$R_{\mu\nu}{}^a = \partial_\mu \omega_{\nu a} - \partial_\nu \omega_{\mu a} + \epsilon_{abc}\omega_\mu^{\ b}\omega_\nu^{\ c} \ , \tag{1.16}$$

οι οποίοι αποτελούν τις τρισδιάστατες εκδοχές των εκφράσεων της στρέψης και καμπυλότητας που δίνονται στις σχέσεις (1.3) και (1.2).

Τέλος, για την ολοκλήρωση της εικόνας, απομένει να καθοριστεί η δράση της θεωρίας. Κατασκευάζοντας μία θεωρία βαθμίδας στις τρεις διαστάσεις, η προφανής επιλογή είναι να θεωρήσουμε το συναρτησοειδές Chern-Simons:

$$S_{\rm CS} = \int_M {\rm tr}(A \wedge dA + A \wedge A \wedge A) = \int_M {\rm tr} A_\mu (\partial_\nu A_\rho - \partial_\rho A_\nu + [A_\nu, A_\rho])\epsilon^{\mu\nu\rho} d^3x \ . \tag{1.17}$$

Αντικαθιστώντας με την έκφραση της συνοχής βαθμίδας, $A_\mu$, όπως δίνεται στη σχέση (1.8), μερικοί όροι της παραπάνω δράσης φιλτράρονται από την εφαρμογή του ίχνους, (1.5), στους γεννήτορες, οδηγώντας στην ακόλουθη έκφραση:

$$\int_M \epsilon^{\mu\nu\rho} e_\mu^{\ a} \Big( (\partial_\nu \omega_{\rho a} - \partial_\rho \omega_{\nu a} + \omega_\nu^{\ b}\omega_\rho^{\ c}\epsilon_{abc}) + (\partial_\nu e_{\rho a} - \partial_\rho e_{\nu a} + (\omega_\nu^{\ b} e_\rho^{\ c} - e_\nu^{\ b}\omega_\rho^{\ c})\epsilon_{abc}) \Big) \ . \tag{1.18}$$

Ο πρώτος όρος είναι αναγνωρίσιμος ως ο τανυστής καμπυλότητας και ο δεύτερος ως ο τανυστής στρέψης, όπως δίνονται στη σχέση (1.16). Λόγω της επιθυμητής SO(1,2) (Lorentz) αναλλοιώτητας της τελικής δράσης, επιβάλλεται η συνθήκη μηδενικής στρέψης, δηλαδή $T_{\nu\rho}{}^a = 0$, κι επομένως η παραπάνω έκφραση της δράσης παίρνει την ακόλουθη μορφή[3]:

$$S_{\rm CS} = \int_M \epsilon^{\mu\nu\rho} e_\mu^{\ a} \Big( \partial_\nu \omega_{\rho a} - \partial_\rho \omega_{\nu a} + \omega_\nu^{\ b}\omega_\rho^{\ c}\epsilon_{abc} \Big) \ , \tag{1.19}$$

η οποία είναι ίδια με τη δράση της τρισδιάστατης βρύτητας της Γενικής Θεωρίας της Σχετικότητας, (1.1). Μεταβολή ως προς το πεδίο $e$ δίνει την αναμενόμενη εξίσωση κίνησης, η οποία είναι ο

[3]Στην αναφορά [10] δίνεται ένας εναλλακτικός και ευρηματικός τρόπος για να αποκτηθεί η δράση Chern-Simons, ξεκινώντας από ένα τοπολογικό αναλλοίωτο της μορφής $\int_Y {\rm tr} R \wedge R$ σε μία τετραδιάστατη πολλαπλότητα, $Y$. Ευθείς υπολογισμοί οδηγούν σε μία ολοκληρώσιμη ποσότητα η οποία γράφεται σαν μία ολική παράγωγος κι επομένως το ολοκληρωμα πάνω στη $Y$ ανάγεται σε ολοκλήρωμα πάνω την $M$, όπου $M$ είναι το τρισδιάστατο όριο της $Y$. Η έκφραση του ολοκληρώματος στην $M$ ταυτοποιείται ως το Chern-Simons συναρτησοειδές το οποίο συμπίπτει με την Einstein-Hilbert δράση. Το πλεονέκτημα της προσέγγισης αυτής είναι ότι η συνθήκη μηδενικής στρέψης δεν χρειάζεται να επιβληθεί.

μηδενισμός του τανυστή καμπυλότητας, $R_{\mu\nu a} = 0$, σε σύμπτωση με την εξίσωση (1.2). Επομένως, μπορεί κάποιος να υποστηρίξει ότι η τρισδιάστατη βαρύτητα μπορεί να περιγραφεί ως μία Chern-Simons θεωρία βαθμίδας της ISO(1,2). Δεν είναι περιττό να αναφερθεί ότι η παραπάνω δράση είναι αναλλοίωτη κάτω από τους μετασχηματισμούς βαθμίδας των διανυσματικών πεδίων $e, \omega$, της εξίσωσης (1.12). Στις παρακάτω γραμμές ακολουθεί η συζήτηση για τη σχέση ανάμεσα στους μετασχηματισμούς βαθμίδας και τους διαφορομορφισμούς η οποία αναβλήθηκε νωρίτερα.

Αρχικά, έστω οι μετασχηματισμοί των vielbein και spin connection κάτω από τους διαφορομορφισμούς που παράγονται από ένα διάνυσμα, $v^\nu$. Η καθιερωμένη παραμετροποίηση των μετασχηματισμών αυτών, $\tilde{\delta} e_\mu{}^a$ και $\tilde{\delta}\omega_\mu{}^a$, γίνεται μέσω των παραγώγων Lie κατά μήκος του διανύσματος $-v^\nu$:

$$\tilde{\delta} e_\mu{}^a = \mathcal{L}_{-v} e_\mu{}^a = -v^\nu \partial_\nu e_\mu{}^a - (\partial_\mu v^\nu) e_\nu{}^a = -v^\nu(\partial_\nu e_\mu{}^a - \partial_\mu e_\nu{}^a) - \partial_\mu (v^\nu e_\nu{}^a) \ , \tag{1.20}$$

$$\tilde{\delta} \omega_\mu{}^a = \mathcal{L}_{-v} \omega_\mu{}^a = -v^\nu \partial_\nu \omega_\mu{}^a - (\partial_\mu v^\nu) \omega_\nu{}^a = -v^\nu(\partial_\nu \omega_\mu{}^a - \partial_\mu \omega_\nu{}^a) - \partial_\mu (v^\nu \omega_\nu{}^a) \ . \tag{1.21}$$

Έστω η διαφορά $\tilde{\delta} e_\mu{}^a - \delta e_\mu{}^a$ και θέτοντας $\xi^a = e_\nu{}^a v^\nu$ και $\lambda^a = \omega_\nu{}^a v^\nu$ υπολογίζεται:

$$\begin{aligned}\tilde{\delta} e_\mu{}^a - \delta e_\mu{}^a &= -v^\nu(\partial_\nu e_\mu{}^a - \partial_\mu e_\nu{}^a) - \partial(v^\nu e_\nu{}^a) + \partial_\mu (e_\nu{}^a v^\nu) + \epsilon^{abc} e_{\mu b}\omega_{\nu c} v^\nu + \epsilon^{abc}\omega_{\mu b} e_{\nu c} v^\nu \\ &= -v^\nu (D_\nu e_\mu{}^a - D_\mu e_\nu{}^a) \ ,\end{aligned} \tag{1.22}$$

όπου χρησιμοποιήθηκε η έκφραση της $D_\mu$, (1.4). Η παραπάνω έκφραση μηδενίζεται από τη συνθήκη της μηδενικής στρέψης η οποία επιβλήθηκε για λόγους αναλλοιώτητας Lorentz της δράσης. Παρομοίως, έστω τώρα η διαφορά $\tilde{\delta}\omega_\mu{}^a - \delta\omega_\mu{}^a$ και θέτοντας $\lambda^a = \omega_\nu{}^a v^\nu$ προκύπτει:

$$\begin{aligned}\tilde{\delta}\omega_\mu{}^a - \delta\omega_\mu{}^a &= -v^\nu(\partial_\nu \omega_\mu{}^a - \partial_\mu \omega_\nu{}^a) - \partial(v^\nu \omega_\nu{}^a) + \partial_\mu (v^\nu \omega_\nu{}^a) + \epsilon^{abc}\omega_{\mu b} v^\nu \omega_{\nu c} \\ &= v^\nu(\partial_\mu \omega_\nu{}^a - \partial_\nu \omega_\mu{}^a + \epsilon^{abc}\omega_{\mu b}\omega_{\nu c}) = v^\nu R_{\mu\nu} \ .\end{aligned} \tag{1.23}$$

Η παραπάνω έκφρασει μηδενίζεται από την εξίσωση κίνησης[4], δηλαδή από τον μηδενισμό του τανυστή καμπυλότητας. Επομένως, συμπεραίνεται ότι οι μετασχηματισμοί βαθμίδας είναι ισοδύναμοι με τους μετασχηματισμούς των διαφορομορφισμών on-shell, που σημαίνει ότι οι μετασχηματισμοί των πεδίων βαθμίδας ισοφαρίζουν για τους γενικούς μετασχηματισμούς συντεταγμένων σε αυτήν την προσέγγιση μέσω θεωριών βαθμίδας. Η αναμενόμενη αναλλοιώτητα της δράσης κάτω από τους μετασχηματισμούς βαθμίδας εξασφαλίζει τη γενική συναλλοιώτητα της θεωρίας με τον ίδιο τρόπο που οι γενικοί μετασχηματισμοί συντεταγμένων αφήνουν αναλλοίωτη την $S_{\text{EH3}}$. Επιπλέον, επιβεβαιώνεται ότι η ISO(1,2) είναι η κατάλληλη ομάδα για την κατασκευή της τρισδιάστατης βαρύτητας ως θεωρίας βαθμίδας[5].

Η παραπάνω ανάλυση πάνω στη σχέση της τρισδιάστατης βαρύτητας και της ISO(1,2) Chern-Simons θεωρίας βαθμίδας μπορεί να γενικευτεί και στην περίπτωση κατά την οποία περιλαμβάνεται κοσμολογική σταθερά. Η παρουσία της καθιστά τον χωρόχρονο ως καμπυλομένο, επομένως υπάρχουν δύο αντίστοιχοι χωρόχρονοι ανάλογα με το πρόσημο της σταθεράς, οι τρισδιάστατοι de Sitter και Anti de Sitter, με ομάδες ισομετριών τις SO(1,3) και SO(2,2), αντίστοιχα. Προκειμένου να συσχετιστεί η τρισδιάστατη βαρύτητα παρουσία κοσμολογικής σταθεράς με θεωρίες βαθμίδας, είναι λογικό να θεωρηθούν οι παραπάνω ομάδες ως ομάδες βαθμίδας, δεδομένου ότι η ISO(1,2) δούλεψε άψογα για την επίπεδη περίπτωση. Η διαδικασία για το χτισίμο των θεωριών αυτών είναι

[4]Στον φορμαλισμό Palatini Στον φορμαλισμό Palatini αυτή είναι η έκφραση της (1.2)

[5]Η ISO(1,2) είναι η ομάδα που περιγράφει τις ισομετρίες του τρισδιάστατου χώρου Minkowski, καθιστώντας την μία όχι και τόσο τυχαία επιλογή.

η ίδια με αυτήν που περιγράφηκε αναλυτικά νωρίτερα για την ISO(1,2). Η κοσμολογική σταθερά υπεισέρχεται στη θεωρία μέσω της μεταθετικής σχέσης των γεννητόρων των μεταθέσεων, η οποία είναι τώρα μη μηδενική. Επίσης, η τροποποίηση αυτή επιφέρει την εισαγωγή ενός παραπάνω όρου στην έκφραση του μετασχηματισμού του πεδίου spin connection της (1.12), δηλαδή:

$$\delta e_\mu{}^a = -\partial_\mu \xi^a - \epsilon^{abc} e_{\mu b} \lambda_c - \epsilon^{abc} \omega_{\mu b} \xi_c \ , \tag{1.24}$$

$$\delta \omega_\mu{}^a = -\partial_\mu \lambda^a - \epsilon^{abc} \omega_{\mu b} \lambda_c - \lambda \epsilon^{abc} e_{\mu b} \xi_c \ , \tag{1.25}$$

καθώς και έναν επιπλέον όρο στην έκφραση του τανυστή καμπυλότητας της (1.16), δηλαδή:

$$T_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \epsilon^{abc} \omega_{\mu b} e_{\nu c} - \epsilon^{abc} \omega_{\nu b} e_{\mu c} \ , \tag{1.26}$$

$$R_{\mu\nu}{}^a = \partial_\mu \omega_{\nu a} - \partial_\nu \omega_{\mu a} + \epsilon_{abc} (\omega_\mu{}^b \omega_\nu{}^c + \lambda e_\mu{}^b e_\nu{}^c) \ . \tag{1.27}$$

Όσον αφορά στη δράση των θεωριών αυτών, αυτή αποκτιέται μέσω του Chern-Simons συναρτησοειδούς, πάλι σε σύμπτωση με την τρισδιάστατη Einstein-Hilbert δράση της Γενικής Θεωρίας της Σχετικότητας στις τρεις διαστάσεις παρουσία κοσμολογικής σταθεράς.

## 1.3 Τετραδιάστατη Einstein βαρύτητα ως θεωρία βαθμίδας

Αν η Γενική Θεωρία της Σχετικότητας στις τέσσερις διαστάσεις μπορεί να περιγραφεί ως θεωρία βαθμίδας αποτελεί ένα αμφιλεγόμενο ζήτημα. Στην αναφορά [10] αναφέρεται ότι η τετραδιάστατη βαρύτητα δεν μπορεί να περιγραφεί ως θεωρία βαθμίδας εξ' αιτίας της μορφής της δράσης Einstein-Hilbert, η οποία είναι της μορφής $\int A \wedge A \wedge (dA + A^2)$ και μία τέτοια δράση δεν μπορεί να εξαχθεί από μία θεωρία βαθμίδας. Παρόλο που μία τέτοια δράση δεν προέρχεται από μία θεωρία βαθμίδας, υπάρχει ένας μη τετριμμένος τρόπος να αποκτηθεί ξεκινώντας από μία δράση τύπου Yang-Mills. Στο κεφάλαιο αυτό, ανακαλούμε την κατασκευή αυτή, δηλαδή την περιγραφή της τετραδιάστατης βαρύτητας ως θεωρίας βαθμίδας.

Όπως στην τρισδιάστατη περίπτωση, πρώτα από όλα πρέπει να επισημανθεί ότι γίνεται χρήση του φορμαλισμού vielbein για την κατασκευή αυτή. Απουσία κοσμολογικής σταθεράς, η ομάδα των ισομετριών του χωρόχρονου Minkowski είναι είναι η ISO(1,3) (η ομάδα Poincaré) και είναι αυτή που θα θεωρηθεί ως η ομάδα βαθμίδας, σε συμφωνία με την τρισδιάστατη περίπτωση. Η άλγεβρα Poincaré αποτελείται από δέκα γεννήτορες, τις τέσσερις μεταθέσεις, $P_a$ και τους έξι μετασχηματισμούς Lorentz, $M_{ab}$, οι οποίοι ικανοποιούν τις παρακάτω μεταθετικές σχέσεις[6]:

$$[M_{ab}, M_{cd}] = 4\eta_{[a[c} M_{d]b]} \ , \quad [P_a, M_{bc}] = 2\eta_{a[b} P_{c]} \ , \quad [P_a, P_b] = 0 \ , \tag{1.28}$$

όπου $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)$ είναι η τετραδιάστατη μετρική του χωρόχρονου Minkowski. Ακολουθώντας την καθιερωμένη διαδικασία, ορίζεται αρχικά η συναλλοίωτη παράγωγος:

$$D_\mu = \partial_\mu + [A_\mu, \cdot] \ , \tag{1.29}$$

όπου $A_\mu(x)$ είναι η συνοχή βαθμίδας. Το ανάπτυγμα της συνοχής βαθμίδας πάνω στους γεννήτορες του ISO(1,3), δίνει την έκφραση:

$$A_\mu(x) = e_\mu{}^a(x) P_a + \omega_\mu{}^{ab}(x) M_{ab} \ , \tag{1.30}$$

[6] Ο συμβολισμός [ ] υπονοεί την αντισυμμετρικότητας των δεικτών που βρίσκονται εντός των αγκυλών, για παράδειγμα $\eta_{a[b} P_{c]} = \frac{1}{2}(\eta_{ab} P_c - \eta_{ac} P_b)$.

όπου τα $e_\mu^{\ a}$ και $\omega_\mu^{\ ab}$ έχουν ταυτοποιηθεί ως τα διανυσματικά πεδία βαθμίδας για τις μεταθέσεις και τους μετασχηματισμούς Lorentz, αντίστοιχα. Εξ' ορισμού, ο μετασχηματισμός της παραγώγου $D_\mu$ είναι συναλλοίωτος, επομένως, ο κανόνας μετασχηματισμού για τη συνοχή βαθμίδας $A_\mu$ δίνεται από τη σχέση:

$$\delta A_\mu = D_\mu \epsilon = \partial_\mu + [A_\mu, \epsilon] \ , \tag{1.31}$$

όπου $\epsilon = \epsilon(x)$ είναι η παράμετρος του μετασχηματισμού βαθμίδας, η οποία ως στοιχείο της άλγεβρας της ISO(1,3), μπορεί να αναπτυχθεί πάνω στους δέκα γεννήτορες:

$$\epsilon(x) = \xi^a(x) P_a + \frac{1}{2}\lambda^{ab}(x) M_{ab} \ , \tag{1.32}$$

όπου $\xi^a(x)$ και $\lambda^{ab}(x)$ απειροστές παράμετροι. Συνδυασμός των (1.30), (1.31) και (1.32) οδηγεί στις εκφράσεις των μετασχηματισμών των πεδίων:

$$\delta e_\mu^{\ a} = \partial_\mu \xi^a + \omega_\mu^{\ ab}\xi_b - \lambda^a_{\ b} e_\mu^{\ b} \ , \tag{1.33}$$

$$\delta \omega_\mu^{\ ab} = \partial_\mu \lambda^{ab} + \lambda^a_{\ c}\omega_\mu^{\ bc} - \lambda^b_{\ c}\omega_\mu^{\ ac} \ . \tag{1.34}$$

Οι τανυστές καμπυλότητας, $T_{\mu\nu}^{\ \ a}$ και $R_{\mu\nu}^{\ \ ab}$, που αντιστοιχούν στα πεδία βαθμίδας, $e$ και $\omega$, αποκτώνται από τον ορισμό του τανυστή δύναμης πεδίου, $R_{\mu\nu}$, του $A_\mu$:

$$R_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ , \tag{1.35}$$

έπειτα από ανάπτυγμά του πάνω στους γεννήτορες:

$$R_{\mu\nu} = T_{\mu\nu}^{\ \ a} P_a + \frac{1}{2} R_{\mu\nu}^{\ \ ab} M_{ab} \ . \tag{1.36}$$

Επομένως, συνδυάζοντας τις σχέσεις (1.30), (1.35) και (1.36), οι εκφράσεις των τανυστών καμπυλότητας είναι οι:

$$\begin{aligned} T_{\mu\nu}^{\ \ a} &= \partial_\mu e_\nu^{\ a} - \partial_\nu e_\mu^{\ a} - \omega_\mu^{\ ab} e_{\nu b} + \omega_\nu^{\ ab} e_{\mu b} \ , \\ R_{\mu\nu}^{\ \ ab} &= \partial_\mu \omega_\nu^{\ ab} - \partial_\nu \omega_\mu^{\ ab} - \omega_\mu^{\ ac}\omega_{\nu c}^{\ \ b} + \omega_\nu^{\ ac}\omega_{\mu c}^{\ \ b} \ , \end{aligned} \tag{1.37}$$

οι οποίες συμπίπτουν με αυτές που αποκτώνται δουλεύοντας με τον vielbein φορμαλισμό της Γενικής Θεωρίας της Σχετικότητας στις τέσσερις διαστάσεις.

Ως εδώ η περιγραφή της τετραδιάστατης βαρύτητας ως θεωρίας βαθμίδας εκτυλίσσεται με έναν ευθύ τρόπο. Προχωρώντας, λοιπόν, στο δυναμικό κομμάτι της θεωρίας, η προφανής επιλογή είναι μία δράση τύπου Yang-Mills της Poincaré ομάδας. Όμως, μία τέτοια επιλογή δεν οδηγεί, έστω έμμεσα, στην απόκτηση της Einstein-Hilbert δράσης, επομένως το εγχείρημα της περιγραφής της τετραδιάστατης Einstein βαρύτητας ως μίας θεωρίας βαθμίδας του ISO(1,3), δείχνει να μην επιτυγχάνεται, όπως υποστηρίζεται στην αναφορά [10]. Από την άλλη μεριά, κάποιος θα μπορούσε να υποστηρίξει ότι η δράση Einstein-Hilbert θα μπορούσε να χτιστεί από αναλλοίωτες ποσότητες, προερχόμενες από τους τανυστές της (1.37). Πιο συγκεκριμένα, το βαθμωτό Ricci θα μπορούσε να προκύψει από τον τανυστή καμπυλότητας, $R_{\mu\nu}^{\ \ ab}$ και κάνοντας χρήση αυτού του αναλλοίωτου κάποιος θα μπορούσε να κατασκευάσει τελικά την Einstein-Hilbert δράση [84]. Παρόλα αυτά, υπάρχει ένας εναλλακτικός και λιγότερο καθοδηγούμενος τρόπος να καταλήξει κάποιος με την Einstein-Hilbert δράση, μεταχειρίζοντας το μεταθετικό και Lorentz κομμάτι με έναν πιο ενιαίο τρόπο, βασισμένος σε πιο διαισθητικά και φυσικά επιχειρήματα.

Προτίστως, πρέπει να σημειωθεί ότι η ζητούμενη δράση οφείλει να είναι αναλλοίωτη κάτω από τους μετασχηματισμούς Lorentz και όχι από την συνολική αρχική συμμετρία η οποία περιέχει και τις μεταθέσεις. Επομένως, προς αυτή την κατεύθυνση, μπορεί να χρησιμοποιηθεί ένας μηχανισμός αυθόρμητης παραβίασης της αρχικής συμμετρίας μέσω της εισαγωγής ενός βαθμωτού πεδίου [3,4]. Με αυτόν τον τρόπο επιτυγχάνεται η ελάττωση των βαθμών ελευθερίας και απομένει να ελεγθεί εάν μπορεί κάποιος να καταλήξει με τη δράση Eistein-Hilbert, επειτα από την παραβίαση της συμμετρίας, ξεκινώντας από μία δράση τύπου Yang-Mills. Για να μπορέσει να συμπεριληφθεί ο μηχανισμός της αυθόρμητης παραβίασης της συμμετρίας, η αρχική συμμετρία της θεωρίας πρέπει να αλλάξει από την Poincaré στην τετραδιάστατη de Sitter. Η επιλογή της ομάδας αυτής είναι στρατηγική διότι περιλαμβάνει τον ίδιο αριθμό γεννητόρων με αυτήν της ομάδας Poincaré, με τη διαφορά ότι όλοι οι γεννήτορες της ομάδας μπορούν να γραφτούν επί ίσοις όροις, με τα αντίστοιχα πεδία βαθμίδας να επιδέχονται έναν ενιαίο συμβολισμό, έστω $\omega^{AB}, A, B = 1 \ldots 5$. Έτσι, το βαθμωτό πεδίο, $\phi^a$, ανατίθεται στη θεμελιώδη αναπαράσταση της SO(1,4) και επάγει την αυθόρμητη παραβίαση της συμμετρίας, από το SO(1,4) στο SO(1,3), δηλαδή η συμμετρία ελαττώνεται στην Lorentz, με τέσσερις από τους δέκα γεννήτορες (τις μεταθέσεις) να σπάνε.

Συγκεκριμένα, κατασκευάζοντας μία αμιγή SO(1,4) θεωρία βαθμίδας, η συνοχή βαθμίδας είναι $A_\mu = \omega_\mu{}^{AB} M_{AB}$, όπου τα $M_{ab}$ είναι οι δέκα SO(1,4) γεννήτορες και ο αντίστοιχος τανυστής δύναμης πεδίου δίνεται από την (1.35) ως:

$$F_{\mu\nu}{}^{AB} = \partial_\mu \omega_\nu{}^{AB} - \partial_\nu \omega_\mu{}^{AB} + \omega_\mu{}^A{}_C \omega_\nu{}^{CB} - \omega_\nu{}^A{}_C \omega_\mu{}^{CB} \ . \tag{1.38}$$

Οι αναλλοίωτες ποσότητες οι οποίες συνιστούν την SO(1,4) αναλλοίωτη δράση πρέπει να κατασκευαστούν σε όρους του παραπάνω τανυστή δύναμης πεδίου. Η μόνη αναλλοίωτη ποσότητα που μπορεί να κατασκευαστεί με αυτόν τον τρόπο και να είναι επίσης πολυωνυμική ως προς το $F_{\mu\nu}$ είναι η παρακάτω τοπολογικά αναλλοίωτη ποσότητα, γνωστή ως δείκτης Pontryagin:

$$S = \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^{AB} F_{\rho\sigma AB} \ . \tag{1.39}$$

Για να φιλοξενήσει τον τομέα της αυθόρμητης παραβίασης της συμμετρίας, η παραπάνω δράση τροποποιείται έτσι ώστε να περιλαμβάνει ένα βαθμωτό πεδίο, $\phi^a$, μαζί με μία παράμετρο $m$ με διαστάσεις αντίστροφου μήκους:

$$S_{\mathrm{SO}(1,4)} = \int d^4x \left( m\phi^A \epsilon_{ABCDE} R_{\mu\nu}{}^{BC} R_{\rho\sigma}{}^{DE} \epsilon^{\mu\nu\rho\sigma} + \lambda(\phi^A \phi_A + m^{-2}) \right) \ , \tag{1.40}$$

όπου η μεταβλητή $\lambda = \lambda(x)$ εξυπηρετεί το ρόλο ενός πολλαπλασιαστή Lagrange, επιβάλλοντας τον σύνδεσμο για το βαθμωτό πεδίο:

$$\phi^A \phi_A = -m^{-2} \ . \tag{1.41}$$

Διαλέγοντας μία συγκεκριμένη βαθμίδα για το βαθμωτό πεδίο:

$$\phi = \phi^0 = (0,0,0,0,m^{-1}) \quad \Leftrightarrow \quad \phi^a(x) = 0 \ \text{ και } \ \phi^5(x) = m^{-1} \ , \tag{1.42}$$

η μη μηδενική τιμή του $\phi^5(x)$, επάγει την παραβίαση της SO(1,4) συμμετρίας, στην SO(1,3). Η δράση, (1.40) ανάγεται στην παρακάτω έκφραση, στην οποία είναι εμφανής η συμμετρία Lorentz:

$$S_{\mathrm{SO}(1,3)} = \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^{ab} F_{\rho\sigma}{}^{cd} \epsilon_{abcd} \ . \tag{1.43}$$

Ορίζοντας τα πεδία βαθμίδας $e_\mu^{\ a} = m^{-1}\omega_\mu^{\ a5}$ και αναπτύσσοντας τον τανυστή δύναμης πεδίου, $F_{\mu\nu} = F_{\mu\nu}^{\ \ AB} M_{AB} = F_{\mu\nu}^{\ \ ab} M_{ab} + F_{\mu\nu}^{\ \ a5} M_{a5}$ της (1.38), αποκτώνται οι παρακάτω εκφράσεις:

$$F_{\mu\nu}^{\ \ a5} = mT_{\mu\nu}^{\ \ a} \ , \tag{1.44}$$

$$F_{\mu\nu}^{\ \ ab} = R_{\mu\nu}^{\ \ ab} - m^2(e_\mu^{\ a} e_\nu^{\ b} - e_\nu^{\ a} e_\mu^{\ b}) \ , \tag{1.45}$$

όπου $T_{\mu\nu}^{\ \ a}$ και $R_{\mu\nu}^{\ \ ab}$ είναι οι ISO(1,3) συνιστώντες τανυστές καμπυλότητας της (1.37). Προφανώς, ο $F_{\mu\nu}^{\ \ a5}$ δεν είναι παρών στη δράση μετά την παραβίαση της συμμετρίας, οπότε τίθεται $T_{\mu\nu}^{\ \ a} = 0$. Ο μηδενισμός του τανυστή της στρέψης οδηγεί σε μία σχέση ανάμεσα στα πεδία βαθμίδας, γράφοντας το spin connection σαν συνάρτηση του vielbein. Για να καταλήξουμε στη σχέση αυτή, προχωρούμε πολλαπλασιάζοντας και τα δύο μέλη της εξίσωσης του μηδενισμού της στρέψης με δύο vielbein:

$$e^\mu_{\ c} e^\nu_{\ d} T_{\mu\nu}^{\ \ a} = e^\mu_{\ c} e^\nu_{\ d} \left( \partial_\mu e_\nu^{\ a} - \partial_\nu e_\mu^{\ a} - \omega_\mu^{\ ab} e_{\nu b} + \omega_\nu^{\ ab} e_{\mu b} \right) = 0 \ , \tag{1.46}$$

έπειτα θεωρώντας κυκλικές μεταθέσεις στους ελεύθερους δείκτες Lorentz, $a, c, d$ και προσθαφαιρώντας τις τρεις σχέσεις που προκύπτουν, η ζητούμενη εκφραση προκύπτει ως:

$$\omega_{\mu ab} = \frac{1}{2} \left( \Omega_{\mu ab} - \Omega_{\mu ba} - \Omega_{ab\mu} \right) \ , \tag{1.47}$$

όπου ο παρακάτω ορισμός έχει χρησιμοποιηθεί:

$$\Omega_{abc} = 2 e^\mu_{\ a} e^\nu_{\ b} \partial_{[\mu} e_{\nu]c} \ . \tag{1.48}$$

Τώρα, αν η παραπάνω έκφραση του $F_{\mu\nu}^{\ \ ab}$, (1.45), εισαχθεί στη δράση $S_{\mathrm{SO}(1,3)}$ της εξίσωσης (1.43), τότε, η έκφραση στην οποία καταλήγει κάποιος μπορεί να γραφτεί στην παρακάτω ομαδοποιημένη μορφή:

$$\begin{aligned} S_{\mathrm{SO}(1,3)} &= \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} \left( R_{\mu\nu}^{\ \ ab} + m^2 (e_\mu^{\ a} e_\nu^{\ b} - e_\mu^{\ b} e_\nu^{\ a}) \right) \left( R_{\rho\sigma}^{\ \ cd} + m^2 (e_\rho^{\ c} e_\sigma^{\ d} - e_\rho^{\ d} e_\sigma^{\ c}) \right) \\ &= \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} \left( R_{\mu\nu}^{\ \ ab} R_{\rho\sigma}^{\ \ cd} + 2m^2 R_{\mu\nu}^{\ \ ab} (e_\rho^{\ c} e_\sigma^{\ d} - e_\rho^{\ d} e_\sigma^{\ c}) + m^4 (e_\mu^{\ a} e_\nu^{\ b} - e_\mu^{\ b} e_\nu^{\ a})(e_\rho^{\ c} e_\sigma^{\ d} - e_\rho^{\ d} e_\sigma^{\ c}) \right) \\ &= \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} \left( \mathcal{L}_{RR} + m^2 \mathcal{L}_{eeR} + m^4 \mathcal{L}_{eeee} \right) \ . \end{aligned} \tag{1.49}$$

Ο πρώτος όρος, $\mathcal{L}_{RR}$, δεν συνεισφέρει στις εξισώσεις κίνησης καθώς αποτελεί έναν Gauss-Bonnet τοπολογικό όρο (βλέπε [85]). Ο δεύτερος όρος, $\mathcal{L}_{eeR}$, σχετίζεται με τη βαθμωτή καμπυλότητα Ricci, ενώ ο τρίτος όρος ταυτοποιείται ως μία κοσμολογική σταθερά τάξης $m^4$. Λόγω της σταθεράς αυτής, η μέγιστα συμμετρική λύση των πεδιακών εξισώσεων είναι ο χώρος de Sitter:

$$F_{\mu\nu}^{\ \ ab} = 0 \ \Rightarrow \ R_{\mu\nu}^{\ \ ab} = m^2 (e_\mu^{\ a} e_\nu^{\ b} - e_\nu^{\ a} e_\mu^{\ b}) \ . \tag{1.50}$$

Σε περίπτωση που η κοσμολογική σταθερά είναι μηδέν, η λύση ανάγεται στον επίπεδο χώρο Minkowski.

Όσον αφορά στη γενική συναλλοιώτητα, αυτή ανακτάται από τους μετασχηματισμούς των πεδίων βαθμίδας (1.31), όπως αυτοί σχετίζονται με τους διαφορομορφισμούς. Ακολουθώντας την ίδια διαδικασία και υπολογισμούς όπως στην τριδιάστατη περίπτωση, καταλήγει κάποιος με την τετραδιάστατη εκδοχή των εξισώεων (1.22) και (1.23). Επομένως, λαμβάνοντας υπόψη τη συνθήκη μηδενικής στρέψης και την εξίσωση κίνησης, η τελική δράση είναι αναλλοίωτη κάτω από τους γενικούς μετασχηματισμούς συντεταγμένων, αφού είναι αναλλοίωτη κάτω από τους μετασχηματισμούς των πεδίων βαθμίδας.

Συνοψίζοντας, είναι πράγματι δυνατό να περιγραφεί η τετραδιάστατη βαρύτητα σαν μία θεωρία βαθμίδας[7]. Οι μετασχηματισμοί των πεδίων $e$ και $\omega$ μπορούν να προκύψουν αν ξεκινήσει κάποιος με μία ISO(1,3) θεωρία βαθμίδας, όμως προκειμένου να καταλήξει με τη ζητούμενη Einstein-Hilbert δράση, είναι απαραίτητο να θεωρήσει ως ομάδα βαθμίδας την de Sitter και να συμπεριλάβει ένα βαθμωτό πεδίο σε μία συγκεκριμένη βαθμίδα για την επαγωγή μίας αυθόρμητης παραβίασης της συμμετρίας, ξεκινώντας από μία πολυωνυμική, ως προς τον τανυστή δύναμης πεδίου δράση, τύπου Yang-Mills. Πράγματι, η τελική δράση είναι αναλλοίωτη κάτω από τους μετασχηματισμούς Lorentz και ταυτοποιείται επιτυχώς ως η δράση Einstein-Hilbert.

## 1.4 Τετραδιάστατη Σύμμορφη βαρύτητα

Στο κεφάλαιο αυτό, ανακαλούμε την προσέγγιση της τετραδιάστατης σύμμορφης βαρύτητας ως θεωρίας βαθμίδας [7,8,86,87]. Συγκεκριμένα, κατασκευάζεται μία θεωρία βαθμίδας της σύμμορφης ομάδας, SO(2,4), και τελικά αποκτιέται η βαρύτητα Weyl. Προκειμένου να καταλήξει κάποιος με την βαρύτητα Weyl, λαμβάνει χώρα μία παραβίαση της αρχικής συμμετρίας βαθμίδας, αυτή τη φορά όχι αυθόρμητα με την εισαγωγή κάποιο βαθμωτού πεδίου, αλλά με την επιβολή συγκεκριμένων συνδέσμων. Όπως στις προηγούμενες περιπτώσεις, έτσι και σε αυτή, η ομάδα που βαθμώνεται είναι η ομάδα των χωροχρονικών συμμετριών, περιλαμβάντας τις μεταθέσεις, οι οποίες σχετίζονται με τους γενικούς μετασχηματισμούς βαθμίδας κάτω από τους οποίους οφείλει να παραμένει αναλλοίωτη η τελική δράση. Η σχέση ανάμεσα στην ¨εσωτερική¨ συμμετρία των μεταθέσεων και τους μετασχηματισμούς συντεταγμένων επιτυγχάνεται με την εισαγωγή του vielbein ως πεδίου βαθμίδας των μεταθέσεων. Η ανάμιξη των εσωτερικών συμμετριών με τις χωροχρονικές είναι ακριβώς αυτό που καθιστά την κατασκευή τέτοιων θεωριών ξεχωριστή σε σχέση με τις γραμμικές θεωρίες βαθμίδας εσωτερικών ομάδων συμμετρίας.

Στην προσέγγιση της τετραδιάστατης σύμμορφης βαρύτητας ως θεωρίας βαθμίδας, ως ομάδα βαθμίδας θεωρείται η SO(2,4), η οποία αποτελείται από δεκαπέντε γεννήτορες, τους έξι μετασχηματισμούς Lorentz, $M_{ab}$, τις τέσσερις μεταθέσεις, $P_a$, τους τέσσερις σύμμορφους μετασχηματισμούς, $K_a$ και τον μετασχηματισμό κλίμακας, $D$. Οι γεννήτορες αυτοί ικανοποιούν τις παρακάτω μεταθετικές σχέσεις οι οποίες καθορίζουν την άλγεβρα:

$$\begin{aligned}
[M_{ab}, M_{cd}] &= \eta_{bc} M_{ad} + \eta_{ad} M_{bc} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} \ , \\
[M_{ab}, P_c] &= \eta_{bc} P_a - \eta_{ac} P_b \ , \\
[M_{ab}, K_c] &= \eta_{bc} K_a - \eta_{ac} K_b \ , \\
[P_a, D] &= P_a \ , \\
[K_a, D] &= -K_a \ , \\
[K_a, P_b] &= -2(\eta_{ab} D + M_{ab}) \ , \qquad (1.51)
\end{aligned}$$

όπου $\eta_{ab}$ είναι η κυρίως θετική τετραδιάστατη μετρική του χωρόχρονου Minkowski. Η κατασκευή της θεωρίας βαθμίδας ξεκινά με τον ορισμό της συναλλοίωτης παραγώγου και με τον προσδιορισμό της συνοχής βαθμίδας, η οποία ως στοιχείο της άλγεβρας SO(2,4), μπορεί να γραφτεί σε όρους των

[7]Ωστόσο, όχι ως μία αμιγής ISO(1,3) θεωρία βαθμίδας.

γεννητόρων:

$$A_\mu = e_\mu{}^a P_a + \frac{1}{2}\omega_\mu{}^{ab} M_{ab} + b_\mu D + f_\mu{}^a K_a \ , \tag{1.52}$$

όπου ένα πεδίο βαθμίδας έχει αντιστοιχηθεί σε κάθε έναν γεννήτορα. Τα πεδία βαθμίδας που σχετίζονται με τις μεταθέσεις έχουν ταυτοποιηθεί με το vielbein, ενώ αυτά που σχετίζονται με τους μετασχηματισμούς Lorentz ταυτοποιούνται με το spin connection, όπως στις προηγούμενες περιπτώσεις. Η συνοχή βαθμίδας, $A_\mu$, υπακούει τον παρακάτω κανόνα μετασχηματισμού:

$$\delta A_\mu = D_\mu \epsilon = \partial_\mu \epsilon + [A_\mu, \epsilon] \ , \tag{1.53}$$

όπου $\epsilon = \epsilon(x)$ είναι μία παράμετρος η οποία ανήκει στην άλγεβρα βαθμίδας και για αυτόν τον λόγο μπορεί να γραφτεί ως:

$$\epsilon = \xi^{\ a} P_a + \frac{1}{2}\lambda^{\ ab} M_{ab} + \kappa D + \rho^{\ a} K_a \ . \tag{1.54}$$

Συνδυάζοντας τις σχέσεις (1.52), (1.53) και (1.54), οδηγείται κάποιος στους κανόνες μετασχηματισμού των διάφορων πεδίων βαθμίδας:

$$\delta e_\mu{}^a = \partial_\mu \xi^a + \omega_\mu{}^a{}_b \xi^b - b_\mu \xi^a - \lambda^a_{\ b} e_\mu{}^b + \kappa e_\mu{}^a \ , \tag{1.55}$$

$$\delta \omega_\mu{}^{ab} = \partial_\mu \lambda^{ab} - 2\omega_\mu{}^{ac}\lambda^b_{\ c} - 4 f_\mu{}^{[a}\xi^{b]} - 4 e_\mu{}^{[a}\rho^{b]} \ , \tag{1.56}$$

$$\delta b_\mu = \partial_\mu \kappa - 2\xi^a f_{\mu a} + 2\rho^a e_{\mu a} \ , \tag{1.57}$$

$$\delta f_\mu{}^a = \partial_\mu \rho^a + \omega_\mu{}^{ab}\rho_b + b_\mu \rho^{\ a} - \lambda^{ab} f_{\mu b} - \kappa f_\mu{}^a \ . \tag{1.58}$$

Ο τανυστής δύναμης πεδίου της θεωρίας δίνεται από την καθιερωμένη φόρμουλα:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ . \tag{1.59}$$

Σε συμφωνία με τη σχέση (1.35), το ανάπτυγμα του τανυστή δύναμης πεδίου πάνω στους γεννήτορες γράφεται ως:

$$F_{\mu\nu} = \tilde{R}_{\mu\nu}{}^a P_a + \frac{1}{2} R_{\mu\nu}{}^{ab} M_{ab} + R_{\mu\nu} D + R_{\mu\nu}{}^a K_a \ . \tag{1.60}$$

Οι συνιστώντες τανυστές καμπυλότητας οι οποίοι συνοδεύουν κάθε γεννήτορα της άλγεβρας υπολογίζονται έπειτα από συνδυασμό των σχέσεων (1.59), (1.52) και (1.60). Οι εκφράσεις τους δίνονται ως εξής:

$$\tilde{R}_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^{ab} e_{\nu b} - \omega_\nu{}^{ab} e_{\mu b} - 2 b_{[\mu} e_{\nu]}^{\ a} \tag{1.61}$$

$$= T^{(0)a}_{\mu\nu} - 2 b_{[\mu} e_{\nu]}^{\ a} \ , \tag{1.62}$$

$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_\nu{}^b{}_c + \omega_\nu{}^{ac}\omega_\mu{}^b{}_c - 8 e_{[\mu}^{\ [a} f_{\nu]}^{\ b]} \tag{1.63}$$

$$= R^{(0)ab}_{\mu\nu} - 8 e_{[\mu}^{\ [a} f_{\nu]}^{\ b]} \ , \tag{1.64}$$

$$R_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu + 4 e_{[\mu}{}^a f_{\nu]a} \ , \tag{1.65}$$

$$R_{\mu\nu}{}^a = \partial_\mu f_\nu{}^a - \partial_\nu f_\mu{}^a + \omega_\mu{}^{ab} f_{\nu b} - \omega_\nu{}^{ab} f_{\mu b} + 2 b_{[\mu} f_{\nu]}^{\ a} \ , \tag{1.66}$$

όπου $T^{(0)a}_{\mu\nu}$ και $R^{(0)ab}_{\mu\nu}$ είνα οι τανυστές στρέψης και καμπυλότητας της τετραδιάστατης Poincaré βαρύτητας, όπως δίνονται στην εξίσωση (1.37).

Όπως αναφέρθηκε νωρίτερα, στην περίπτωση που εξετάζεται, η τελική θεωρία αποκτιέται ύστερα από την επιβολή συγκεκριμένων συνδέσμων, έτσι ώστε να είναι αναλλοίωτη κάτω από τους γενικούς μετασχηματισμούς συντεταγμένων. Για να επιτευχθεί αυτό, πρέπει να συμβεί η ¨ανταλλαγή¨ ανάμεσα στους γενικούς μετασχηματισμούς συντεταγμένων (που συμβολίζονται με $X$) και στις μεταθέσεις, που σημαίνει ότι η τελική ομάδα πρέπει να είναι η αρχική, G=SO(2,4), πλην τις μεταθέσεις, δηλαδή $H = SO(2,4) - \{P\}$ επί $X$. Ακολουθώντας την ίδια διαδικασία όπως στη τριδιάστατη περίπτωση, η διαφορά ανάμεσα στους μετασχηματισμούς των πεδίων και τους διαφορομορφισμούς, $\tilde{\delta} e_\mu{}^a - \delta e_\mu{}^a$, υπολογίζεται κάνοντας χρήση των εξισώσεων (1.22) και (1.55):

$$\tilde{\delta} e_\mu{}^a - \delta e_\mu{}^a = \left(v^\nu \partial_\nu e_\mu{}^a + \partial_\mu (v^\nu e_\nu{}^a) - v^\nu \partial_\mu e_\nu{}^a\right) - \left(\partial_\mu \xi^{\ a} + \omega_\mu{}^a{}_b \xi^b - b_\mu \xi^{\ a} - \lambda^a{}\, b e_\mu^{\ b} + \kappa e_\mu{}^a\right) .$$

Θέτοντας $\xi^a = v^\nu e_\nu{}^a$, $\lambda^a_{\ b} = v^\nu \omega_\nu{}^a{}_b$ και $\kappa = v^\nu b_\nu$, η παραπάνω διαφορά παίρνει την ακόλουθη μορφή:

$$\tilde{\delta} e_\mu{}^a - \delta e_\mu{}^a = v^\nu \left(\partial_\nu e_\mu{}^a - \partial_\mu e_\nu{}^a - \omega_\mu{}^a{}_b e_\nu^{\ b} + \omega_\nu{}^a{}_b e_\mu^{\ b} + b_\mu e_\nu{}^a - b_\nu e_\mu{}^a\right) = -v^\nu \tilde{R}_{\mu\nu}{}^a . \qquad (1.67)$$

Είναι προφανές ότι η συνθήκη που απαιτείται για να απαλλαγεί κάποιος από το μεταθετικό κομμάτι της θεωρίας, με τους μετασχηματισμούς των συντεταγμένων να παίρνουν τη θέση τους, είναι ο μηδενισμός της στρέψης:

$$\tilde{R}_{\mu\nu}{}^a = 0 . \qquad (1.68)$$

Επομένως, θέτοντας τη στρέψη ίση με το μηδέν, οι γεννήτορες της αρχικής ομάδας, SO(2,4), σπάνε και η υποομάδα $H$ έχει ως γεννήτορες τους $M, D$ και $K$. Επιπλέον, προκειμένου να επιτευχθεί η $P \to X$ εναλλαγή για όλα τα πεδία, πρέπει να επιβληθεί και ο παρακάτω σύνδεσμος [7]:

$$R_{\mu\nu}{}^{ab} e_b{}^\nu = 0 . \qquad (1.69)$$

Προχωράμε τώρα με τις λύσεις των παραπάνω συνδέσμων, (1.68) και (1.69), σε όρους των ανεξάρτητων πεδίων, $e_\mu{}^a$ και $b_\mu$.

Για τον πρώτο σύνδεσμο, τον μηδενισμό της στρέψης, (1.68), ακολουθείται η ίδια υπολογιστική πορεία όπως στην τετραδιάστατη Einstein περίπτωση, καταλήγωντας στην παρακάτω σχέση:

$$\omega_\mu{}^{ab} = -\frac{1}{2}\left(\hat{\Omega}_{\mu ab} - \hat{\Omega}_{\mu ba} - \hat{\Omega}_{ab\mu}\right) = -\omega_\mu{}^{ab}(e) + 2b^{[a} e_\mu{}^{b]} , \qquad (1.70)$$

όπου $\hat{\Omega}_{abc} = 2e^\mu{}_a e^\nu{}_b \hat{\partial}_{[\mu} e_{\nu]c}$ είναι το σύμμορφο ανάλογο του $\Omega_{abc}$ της τετραδιάστατης περίπτωσης, (1.48), με τη μερική παράγωγο $\partial_\mu e_\nu{}^a$ να επαναορίζεται ως $\hat{\partial}_\mu e_\nu{}^a = (\partial_\mu + b_\mu) e_\nu{}^a$, δηλαδή την Weyl $(D)$ συναλλοίωτη παράγωγο και $\omega_\mu{}^{ab}(e)$ είναι η έκφραση του spin connection σε όρους του vielbein στην τετραδιάστατη περίπτωση, η οποία δίνεται από τη σχέση (1.47). Επίσης, ορίζεται ο τανυστής $\hat{R}_{\mu\nu}{}^{ab}$, ο οποίος είναι ο τανυστής $R_{\mu\nu}{}^{ab}$ με τη μερική παράγωγο, $\partial_\mu$, να έχει αντικατασταθεί από τη συναλλοίωτη παράγωγο Weyl, $\hat{\partial}_\mu = \partial_\mu + b_\mu$, μιας και θα φανεί χρήσιμος παρακάτω.

Για τον δεύτερο σύνδεσμο, (1.69), λόγω της έκφρασης (1.64), στην οποία ο τανυστής καμπυλότητας $R_{\mu\nu}{}^{ab}$ εκφράζεται σε όρους της καμπυλότητας της βαρύτητας Poincaré συν έναν όρο που περιέχει το πεδίο $f_\mu{}^a$, είναι δυνατό να λυθεί αλγεβρικά ως προς το $f_\mu{}^a$, σε όρους του τανυστή Ricci:

$$R_{\mu\nu}{}^{ab} e_b{}^\nu = 0 \;\Rightarrow\; R_{\mu\nu}^{(0)ab} e^\nu{}_b - 8 e_{[\mu}^{\ [a} f_{\nu]}^{\ b]} = 0 . \qquad (1.71)$$

Λαμβάνοντας υπόψη το ανάπτυγμα Weyl του τανυστή Riemann[8]:

$$R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \frac{2}{n-2}\left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) - \frac{2}{(n-1)(n-2)}Rg_{\mu[\rho}g_{\sigma]\nu} \ , \tag{1.72}$$

όπου $C_{\mu\nu\rho\sigma}$ είναι ο τανυστής Weyl, ο οποίος είναι άιχνος, $R_{\mu\nu}$ είναι ο τανυστής Ricci και $R$ είναι το βαθμωτό Ricci. Αντικαθιστώντας την (1.72) στην (1.71) και πολλαπλασιάζοντας με το $g^{\sigma\nu}$, βρίσκεται ότι:

$$R^{(0)ab}_{\mu\nu}e^{\nu}{}_{b} - 8e_{[\mu}{}^{[a}f_{\nu]}{}^{b]} = 0 \ \Rightarrow \ \frac{1}{(n-1)(n-2)}e_{\mu}{}^{a}R - R_{\mu}{}^{a} = 4f_{\mu}{}^{a} \ , \tag{1.73}$$

όπου $R_{\mu}{}^{a} = R^{(0)ab}_{\mu\nu}e^{\nu}{}_{b}$ και $R = e^{\mu}{}_{a}R_{\mu}{}^{a}$ είναι ίχνη του τανυστή καμπυλότητας της Poincaré περίπτωσης, $R^{(0)ab}_{\mu\nu}$. Για $n = 4$, αποκτιέται η λύση του δεύτερου συνδέσμου:

$$f_{\mu}{}^{a} = -\frac{1}{4}\left(R_{\mu}{}^{a} - \frac{1}{6}e_{\mu}{}^{a}R\right) \ . \tag{1.74}$$

Από τις λύσεις των δύο συνδέσμων, (1.70) και (1.74), προκύπτει ότι $\omega_{\mu}{}^{ab} = \omega_{\mu}{}^{ab}(e,b)$ και $f_{\mu}{}^{a} = f_{\mu}{}^{a}(e,b)$, που σημαίνει ότι τα πεδία βαθμίδας $\omega$ και $f$ έχουν εκφραστεί σε όρους των ανεξάρτητων πεδίων βαθμίδας, $e$ και $b$.

Η αναλλοίωτη δράση κάτω από την υποομάδα $H$ και τους μετασχηματισμούς συντεταγμένων, $X$, είναι:

$$S_W = \frac{1}{8a^2}\int d^4x\epsilon_{abcd}\epsilon^{\mu\nu\rho\sigma}\left(R_{\mu\nu}{}^{ab}R_{\rho\sigma}{}^{ab}\right)^{\omega(e,b)}_{f(e,b)} \ , \tag{1.75}$$

στην οποία οι δύο σύνδεσμοι, (1.70) και (1.74), έχουν συμπεριληφθεί. Αξίζει να σημειωθεί ότι η λύση του δεύτερου συνδέσμου, (1.74), θα μπορούσε να έχει προκύψει από τη μεταβολή της παραπάνω δράσης, (1.75), αν ήταν γραμμένη σε όρους του πεδίου βαθμίδας $f$ (πριν την απαλοιφή του), ως εξίσωση κίνησης. Η έκφραση του τανυστή $R_{\mu\nu}{}^{ab}$, αφού ληφθούν υπόψη οι σχέσεις των $\omega_{\mu}{}^{ab}(e,b)$ και $f_{\mu}{}^{a}(e,b)$, προκύπτει ως εξής:

$$\left[R_{\mu\nu}{}^{ab}\right]^{f(e,b)}_{\omega(e,b)} = -\left(R^{(0)ab}_{\mu\nu} - 2e_{[\mu}{}^{[a}R_{\nu]}{}^{b]} - \frac{1}{3}e_{[\mu}{}^{[a}e_{\nu]}{}^{b]}R\right) = -C^{ab}_{\mu\nu} \ , \tag{1.76}$$

όπου $C^{ab}_{\mu\nu}$ είναι ο τανυστής Weyl και ταυτόχρονα, στην καμπυλότητα Poincaré, $R^{(0)ab}_{\mu\nu}$ της παραπάνω έκφρασης, έχει συμπεριληφθεί ο αντίστοιχος σύνδεσμος, (1.47). Συγκεκριμένα:

$$R^{(0)ab}_{\mu\nu} = R^{(0)ab}_{\mu\nu}\left(\omega(e)\right) = -\left[\hat{R}^{ab}_{\mu\nu}(\omega)\right]^{f_{\mu}{}^{a}=0}_{b_{\mu}=0} \ . \tag{1.77}$$

Στην προτελευταία σχέση, (1.76), είναι προφανές ότι η ο τανυστής καμπυλότητας της θεωρίας, έπειτα από τη συμπερίληψη των συνδέσμων, $\left[R_{\mu\nu}{}^{ab}\right]^{f(e,b)}_{\omega(e,b)}$, είναι ανεξάρτητος από το πεδίο $b_{\mu}$ και, μιας και απαλείφεται, μπορεί να τεθεί $b_{\mu} = 0$. Έπειτα από την επιλογή $b_{\mu} = 0$ (γνωστή και ως K-βαθμίδα), το μόνο ανεξάρτητο πεδίο βαθμίδας στη δράση είναι το vielbein, $e$, επομένως η δράση είναι αναλλοίωτη κάτω από τους μετασχηματισμούς κλίμακας και τους σύμμορφους μετασχηματισμούς, αφού, σε αντίθεση με τους μετασχηματισμούς Lorentz, όλα τα φυσικά πεδία μετασχηματίζονται τετριμμένα κάτω από τους σύμμορφους μετασχηματισμούς των $K_a$ και αυτό συμβαίνει και με το vielbein, $\delta^K e = 0$, όπως είναι εμφανές στη σχέση (1.55). Επίσης, το πεδίο $b_{\mu}$ που σχετίζεται

[8] Στην αναφορά [9], εξίσωση (15.25), μπορεί κάποιος να βρει την απαιτούμενη σχέση, ήδη στην ζητούμενη μορφή.

με τους μετασχηματισμούς κλίμακας είναι το μόνο πεδίο της θεωρίας το οποίο μετασχηματίζεται με μη τετριμμένο τρόπο κάτω από τους μετασχηματισμούς των $K_a$. Επομένως, μιας και η δράση είναι πλέον ανεξάρτητη από το $b_\mu$, είναι και αναλλοίωτη κάτω από τους $K$-μετασχηματισμούς [9]. Λόγω της παρουσίας της $K$-αναλλοιώτητας, κάποιος θα μπορούσε να υποστηρίξει ότι η συμμετρία της τελικής δράσης είναι $X \otimes D \otimes K$. Ωστόσο, έπειτα από την επιλογή $b_\mu = 0$, οι μετασχηματισμοί των σύμμορφων μετασχηματισμών δεν είναι πια ανεξάρτητοι. Πράγματι, αυτό γίνεται κατανοητό από την έκφραση του μετασχηματισμού βαθμίδας του $b_\mu$, (1.57), απουσία του όρου που εμπλέκει τα $\xi^a$, αφού η συμμετρία $X$ των γενικών μετασχηματισμών συντεταγμένων παίρνουν τη θεση τους στην τελική συμμετρία:

$$\delta b_\mu = \partial_\mu \kappa + 2\rho^a e_{\mu a} \;\Rightarrow\; \rho^a = -\frac{1}{2} e^{\mu a} \partial_\mu \kappa \;. \tag{1.78}$$

Η παραπάνω σχέση, η οποία προκύπτει από τη διατήρηση μίας συνθήκης βαθμίδας, ονομάζεται κανόνας αναπτύγματος [9] και εκφράζει μία παράμετρο βαθμίδας που έχει καθοριστεί ($\rho^a$) σε όρους μίας άλλης παραμέτρου βαθμίδας η οποία παραμένει στη θεωρία ($\kappa$). Συνοψίζοντας, η τελική δράση είναι αναλλοίωτη κάτω από τους μετασχηματισμούς $X \otimes D$, οι οποίοι αποτελούν τους γενικούς μετασχηματισμούς συντεταγμένων και τους Weyl μετασχηματισμούς κλίμακας.

Επομένως, χρησιμοποιώντας τη σχέση ανάμεσα στη μετρική και τα vielbein, η έκφραση της δράσης, $S_W$, (1.75) μπορεί να γραφτεί σε όρους του τανυστή Weyl και τελικά προκύπτει η γνωστή δράση Weyl:

$$S_W = \frac{1}{2a^2} \int d^4x \sqrt{g} C_{\lambda\mu\nu\rho} C^{\lambda\mu\nu\rho} = \frac{1}{a^2} \int d^4x \sqrt{g} \left( R^2_{\mu\nu} - \frac{1}{3} R^2 \right) \;. \tag{1.79}$$

Πέρα από την παραπάνω επιλογή της παραβίασης της συμμετρίας, προτείνουμε έναν εναλλακτικό τρόπο παραβίασης της αρχικής συμμετρίας, αυτή τη φορά με κατάληξη τη συμμετρία Lorentz. Αυτό θα μπορούσε να συμβεί με τη συμπερίληψη δύο βαθμωτών πεδίων στη θεμελιώδη αναπαράσταση της SO(2,4) ομάδας βαθμίδας [88]. Αυτός ο τρόπος αποτελεί γενίκευση της μεθοδολογίας που ακολουθήθηκε στην τετραδιάστατη περίπτωση, στην οποία η συμμετρία de Sitter παραβιάζεται με κατάληξη τη Lorentz, από ένα βαθμωτό πεδίο στη θεμελιώδη αναπαράσταση. Επιλέγοντας συγκεκριμένη βαθμίδα για τα δύο αυτά βαθμωτά πεδία θα μπορούσε να επαγάγει την αυθόρμητη παραβίαση της συμμετρίας σε μία πιο πλήρη θεωρία στην οποία περιλαμβάνονται παραπάνω πεδία, πέρα από αυτά της βαθμίδας. Αξίζει να σημειωθεί ότι η τετραδιάστατη σύμμορφη θεωρία βαθμίδας μπορεί να οδηγήσει στη δράση Einstein-Hilbert επιλέγοντας μία άλλη διαδρομή με την επιλογή συνδέσμων, όπως αυτή που περιγράφεται στην αναφορά [89]. Πιο συγκεκριμένα, επιχειρηματολογείται ότι αν οι δύο τανυστές $R(P)$ και $R(K)$ τεθούν ταυτόχρονα ίσοι με το μηδέν, τότε από τους συνδέσμους της θεωρίας προκύπτει ότι τα αντίστοιχα πεδία βαθμίδας, $f_\mu{}^a, e_\mu{}^a$ είναι ανάλογα και $b_\mu = 0$.

Ανακεφαλαιώνοντας, ξεκινώντας με τη σύμμορφη ομάδα, SO(2,4), ως θεωρία βαθμίδας και ακολουθώντας την καθιερωμένη διαδικασία κατασκευής θεωριών βαθμίδας χωροχρονικών συμμετριών, ορίστηκε η συναλλοίωτη παράγωγος μέσω της συνοχής βαθμίδας κι έπειτα σε κάθε γεννήτορα της ομάδας αντιστοιχήθηκε ένα πεδίο βαθμίδας. Στη συνέχεια, υπολογισμοί οδήγησαν στους μετασχηματισμούς των πεδίων βαθμίδας και στις εκφράσεις των αντίστοιχων τανυστών καμπυλότητας.

[9]Στην αναφορά [9], το επιχείρημα για την $K$-αναλλοιώτητα παρουσιάζεται αντίστροφα. Με λίγα λόγια, η $K$-αναλλοιώτητα της $e, b$-εξαρτημένης δράσης επιβάλλεται κατ' αρχήν, επομένως, μιας και το vielbein μετασχηματίζεται με τετριμμένο τρόπο κάτω από τους μετασχηματισμούε των $K$, και το $b_\mu$ όχι, το τελυταίο πρέπει να τεθεί ίσο με το μηδέν ώστε να εξασφαλιστεί η $K$-αναλλοιώτητα της δράσης.

Προκειμένου η τελική θεωρία να είναι αναλλοίωτη κάτω από το μεταθετικό κομμάτι, ο πρώτος σύνδεσμος που επιβλήθηκε ήταν ο μηδενισμός του τενυστή της στρέψης, δίνοντας έτσι τη δυνατότητα να εκφραστούν οι μεταθέσεις μέσω των γενικών μετασχηματισμών των συντεταγμένων. Η λύση του συνδέσμου αυτού κατέστησε το πεδίο spin connection ως εξαρτημένο από τα πεδία $e$ και $b$. Έπειτα, πάλι για λόγους της γενικής συναλλοιώτητας, επιβλήθηκε ένας παραπάνω σύνδεσμος, αυτή τη φορά σχετικός με τον τανυστή καμπυλότητας του $M_{ab}$. Από τον σύνδεσμο αυτόν, το πεδίο βαθμίδας $f$, το οποίο σχετίζεται με τους σύμμορφους μετασχηματισμούς, $K_a$, εκφράστηκε σε όρους του τανυστή και βαθμωτού Ricci. Επομένως, δύο από τις τέσσερις ομάδες των πεδίων βαθμίδας εκφράστηκαν σε όρους των άλλων δύο. Ύστερα από τη συμπερίληψη των αποτελεσμάτων των συνδέσμων, η αρχική δράση τύπου Yang-Mills παίρνει μία μορφή στην οποία δεν συμπεριλαμβάνεται το πεδίο βαθμίδας $b_\mu$ και για τον λόγο αυτό τέθηκε ίσο με το μηδέν. Με αυτόν τον τρόπο, η δράση παύει να περιέχει το μόνο πεδίο που μετασχηματίζεται με μη τετριμμένο τρόπο κάτω από τους σύμμορφους μετασχηματισμούς, που σημαίνει ότι η τελική δράση είναι $K$-αναλλοίωτη. Ωστόσο, η $K$-αναλλοιώτητα δεν αποτελεί ανεξάρτητη συμμετρία έπειτα από την επιλογή βαθμίδας του $b_\mu$ και, για τον λόγο αυτόν, απορροφιέται από την υπόλοιπη τελική συμμετρία και δεν εμφανίζεται ρητά. Επομένως, η συμμετρία της τελικής δράσης (δράση Weyl) αποτελείται από τους γενικούς μετασχηματισμούς συντεταγμένων, $X$, και τους μετασχηματισμούς κλίμακας που παράγονται από τον γεννήτορα $D$. Τελικά, είναι θεμιτό να σημειώσουμε ότι η σύμμορφη βαρύτητα (και με την παραβίαση της συμμετρίας η βαρύτητα Weyl) μπορεί να περιγραφεί επιτυχώς ως θεωρία βαθμίδας της σύμμορφης ομάδας SO(2,4).

## 1.5 Οι μη μεταθετικοί χώροι $S_F^2$, $\mathbb{R}^3_\lambda$ και $dS_F^4$

### 1.5.1 Η ασαφής σφαίρα

Στο κεφάλαιο αυτό συζητάμε την πιο τυπική περίπτωση μη μεταθετικού χώρου Lie-τύπου, δηλαδή της ασαφούς σφάιρας $S_F^2$ [93]. Καταρχάς, ένας ασαφής χώρος ορίζεται ως μία διακριτή προσέγγιση μέσω πινάκων ενός συνεχούς χώρου με την ιδιότητα ότι διατηρούνται οι ισομετρίες. Με άλλα λόγια, είναι ένας μη μεταθετικός χώρος ο οποίος διατηρεί τις ισομετρίες του μεταθετικού ανάλογού του. Ένας αρκετά συνεπής τρόπος να θεμελιωθεί ο χώρος αυτός είναι μέσω μίας συγκριτικής προσέγγισης με τη συνηθισμένη σφαίρα, $S^2$.

Η συνήθης σφαίρα, $S^2$, μπορει να οριστεί σαν μία υποπολλαπλότητα του ευκλείδειου χώρου μίας διάστασης παραπάνω, δηλαδή του $\mathbb{R}^3$, με τις καρτεσιανές συντεταγμένες $x_a, a = 1, 2, 3$, να ικανοποιούν την παρακάτω συνθήκη εμβάπτισης:

$$\sum_{a=1}^{3} x_a^2 = x_1^2 + x_2^2 + x_3^2 = R^2 \ , \tag{1.80}$$

όπου $R$ είναι μία σταθερά η οποία ερμηνεύεται ως η ακτίνα της σφαίρας. Η σφαίρα διέπεται από μία προφανή περιστροφική συμμετρία η οποία παραμετροποιείται από την ομάδα SO(3) (ομάδα ισομετριών). Η ομάδα SO(3) παράγεται από τους τρεις τελεστές στροφορμής οι οποίοι ορίζονται ως $L_a = -i\epsilon_{abc}x_b\partial_c$ και μπορούν επίσης να γραφτούν σε όρους των σφαιρικών συντεταγμένων, $\theta, \phi$, ως $L_a = -\xi_a^i\partial_i$, όπου $i = \theta, \phi$ και $\xi_a^i$ είναι οι συνιστώσες των διανυσμάτων Killing. Ο τελεστής Laplace για τη σφαίρα ορίζεται από τη σχέση:

$$L^2 = -R^2\Delta_{S^2} = -R^2\frac{1}{\sqrt{g}}\partial_i(g^{ij}\sqrt{g}\partial_j) \ , \tag{1.81}$$

όπου $g_{ij}$ είναι ο μετρικός τανυστής της σφαίρας. Τα ιδιοδιανύσματα του τελεστή αυτού είναι οι γνωστές σφαιρικές αρμονικές, $Y_{lm}(\theta, \phi)$, οι οποίες ορίζονται ως:

$$Y_l^m(\theta, \phi) = Ne^{im\phi}P_l^m(\cos\theta) \ , \tag{1.82}$$

όπου $P_l^m$ είναι τα συσχετιζόμενα πολυώνυμα Legendre. Οι σφαιρικές αρμονικές υπακούουν τις παρακάτω συνθήκες ορθοκανονικότητας:

$$\int d\Omega Y_{lm}^\dagger Y_{l'm'} = \delta_{ll'}\delta_{mm'} \ . \tag{1.83}$$

Ας θεωρήσουμε τώρα μία συνάρτηση, $f(\theta, \phi)$, πάνω στη σφαίρα, $S^2$. Δεδομένου ότι οι σφαιρικές αρμονικές αποτελούν ένα πλήρες και ορθογώνιο σύστημα συναρτήσεων, η συνάρτηση $f(\theta, \phi)$ μπορεί αν γραφτεί σαν ανάπτυγμα πάνω στο σύνολο αυτό:

$$f(\theta, \phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm}Y_{lm}(\theta, \phi) \ , \tag{1.84}$$

όπου $c_{lm}$ είναι μιγαδικοί συντελεστές.

Ας προχωρήσουμε τώρα με την ασαφή εκδοχή της σφαίρας, $S_F^2$. Η ασαφής σφαίρα είναι ένας μη μεταθετικός χώρος, που σημαίνει ότι οι συναρτήσεις που εξαρτώνται από τις συντεταγμένες (τελεστές) δεν μετατίθενται κάτω από το σύνηθες γινόμενο. Ας δώσουμε μερικές λεπτομέρειες στο πως κατασκευάζεται ο χώρος αυτός, ξεκινώντας από τις ιδιότητες της κανονικής σφαίρας.

Μία διακριτοποιημένη εκδοχή της σφαίρας μπορεί να προκύψει αντικαθιστώντας την άλγεβρα των συναρτήσεων $Y_{lm}(\theta, \phi)$ πάνω στη σφαίρα, με ένα σύνολο από συντεταγμένες, $\hat{Y}_{lm}(\theta, \phi)$, οι οποίες δεν ξεπερνούν μία συγκεκριμένη τιμή για το $l$, ας τη συμβολίσουμε με $N$. Επομένως, μία συνάρτηση $\hat{f}(\theta, \phi)$, πάνω στη σφαίρα γράφεται ως ανάπτυγμα πάνω στο πεπερασμένο σύνολο των $\hat{Y}_{lm}$:

$$\hat{f} = \sum_{l=0}^{N} \sum_{m=-l}^{l} c_{lm} \hat{Y}_{lm} \ . \tag{1.85}$$

Τώρα, αν θεωρήσουμε ένα γινόμενο δύο τέτοιων συναρτήσεων, τότε αυτό θα συμπεριλαμβάνει όρους με το $l$ να φτάνει μέχρι κάποια συγκεκριμένη τιμή, $j$, η οποία είναι $j = 2N$,[10] η οποία ξεπερνάει το ανώτερο όριο, $N$, πράγμα που συμαίνει ότι η ¨κομμένη¨ άλγεβρα των συναρτήσεων δεν κλείνει κάτω από την πράξη του πολλαπλασιασμού. Ένας αρκετά κομψός και αποτελεσματικός τρόπος να καταλήξει κάποιος με μία άλγεβρα τέτοιων συναρτήσεων είναι να θεωρήσει ένα διαφορετικό τύπο γινομένου, το οποίο είναι μη μεταθετικό, πιο συγκεκριμένα, ένα γινόμενο πινάκων. Επομένως, σε αυτήν τη διακριτοποίηση της σφαίρας, η τοποθέτηση άνω ορίου στην τιμή της στροφορμής και άρα στην πρώην απειροδιάστατη άλγεβρα, αποζημιώνεται από μία πεπερασμένη $(N + 1)$-διάστατη μη μεταθετική άλγεβρα. Αυτός ο διακριτοποιημένος χώρος ορίζεται ως η ασαφής σφαίρα.

Ο πιο ευθύς τρόπος να διατυπωθεί η ασαφής σφαίρα είναι να θεωρηθεί αυτή η ¨κολοβή¨ μη μεταθετική άλγεβρα ως μία άλγεβρα πινάκων σε κάποιον πεπερασμένο διανυσματικό χώρο. Για τον λόγο αυτό, έστω οι τρεις $(N + 1)$-διάστατοι πίνακες $J_a, a = 1, 2, 3$ οι οποίοι σχηματίζουν μία βάση για την $(N+1)$-διάστατη μη αναγωγίσιμη αναπαράσταση της SU(2). Οι γεννήτορες $J_a$ ικανοποιούν την παρακάτω μεταθετική σχεση:

$$[J_a, J_b] = i\epsilon_{abc} J_c \ . \tag{1.86}$$

Επίσης, αφού οι πίνακες που αναπαριστούν τους γεννήτορες $J_a$ θεωρούνται ότι βρίσκονται σε μία μη αναγωγίσιμη αναπαράσταση, η τιμή του τελεστή Casimir σε αυτήν την $(N + 1)$-διάστατη αναπαράσταση είναι:

$$J^2 = J_1^2 + J_2^2 + J_3^2 = \frac{N}{2}\left(\frac{N}{2} + 1\right) \mathbb{1}_{N+1} \ , \tag{1.87}$$

Επομένως, η ασαφής σφάιρα, $S_F^2$, στο επίπεδο ασάφειας $N$, είναι ο μη μεταθετικός χώρος του οποίου οι συντεταγμένες, $\hat{X}_a = \hat{X}^a, a = 1, 2, 3$, ορίζονται ως οι $(N + 1) \times (N + 1)$ ερμιτιανοί πίνακες οι οποίοι είναι ανάλογοι των γεννητόρων, $J_a$, της $(N + 1)$-διάστατης μη αναγωγίσιμης αναπαράστασης της SU(2), δηλαδή:

$$\hat{X}_a = \kappa J_a \ , \tag{1.88}$$

όπου $\kappa$ είναι η σταθερά αναλογίας η οποία προσδιορίζεται από το γεγονός ότι οι $\hat{X}_a$ είναι συντεταγμένες μίας (ασαφούς) σφαίρας και άρα πρέπει να ικανοποιούν τον σύνδεσμο:

$$\sum_{a=1}^{3} \hat{X}_a \hat{X}_a = \hat{X}_1^2 + \hat{X}_2^2 + \hat{X}_3^2 = r^2 \ , \tag{1.89}$$

όπου $r$ είναι η ακτίνα της ασαφούς σφαίρας. Λαμβάνοντας υπόψη την έκφραση του τελεστή Casimir των γεννητόρων $J_a$ της SU(2), όπως αυτή δίνεται στην εξίσωση (1.87) και αντικαθιστώντας

[10] Η τιμή αυτή προκύπτει από τη σύνθεση των μέγιστων τιμών, $N$, των δύο στροφορμών.

τις συντεταγμένες, $\hat{X}_a$, σε όρους των γεννητόρων, όπως δίνονται στην εξίσωση (1.88), αποκτιέται η έκφραση της παραμέτρου αναλογίας, $\kappa$:

$$\kappa = \frac{r}{\sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)}} = \lambda_N r \ , \tag{1.90}$$

όπου $\lambda_N = \frac{1}{\sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)}}$ και επομένως, ξεκινώντας από την εξίσωση (1.88), οι πίνακες-συντεταγμένες, $\hat{X}_a$ γράφονται ως:

$$\hat{X}_a = \kappa J_a = \frac{r}{\sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)}} J_a = \lambda_N r J_a \ . \tag{1.91}$$

Επίσης, η συμπεριφορά του γινομένου δύο τελεστών-συντεταγμένων δίνεται από τη μεταθετική τους σχέση, η οποία προκύπτει από τη μεταθετική σχέση των γεννητόρων $J_a$ της SU(2), (1.86):

$$[\hat{X}_a, \hat{X}_b] = i\kappa\epsilon_{abc}\hat{X}_c = i\lambda_N C_{abc}\hat{X}_c \ , \tag{1.92}$$

όπου έγινε χρήση της σχέσης (1.90) και $C_{abc} = r\epsilon_{abc}$.

Τώρα, είναι δυνατόν οι συντεταγμένες $\hat{X}_a$ να επαναοριστούν ως οι παρακάτω αντιερμιτιανοί πίνακες, $X_a$:

$$X_a = \frac{1}{i\kappa r}\hat{X}_a = \frac{1}{ir}J_a \ . \tag{1.93}$$

Η μεταθετική τους σχέση, (1.92), καθώς και ο σύνδεσμος της ακτίνας, (1.89), επαναορίζονται ως:

$$[X_a, X_b] = C_{abc}X_c \quad \text{and} \quad \sum_{a=1}^{3} X_a X_a = -\frac{\lambda_N^{-2}}{r^2} \ , \tag{1.94}$$

όπου $C_{abc}$ ορίζεται τώρα ως $C_{abc} = \frac{\epsilon_{abc}}{r}$. Η άλγεβρα της ασαφούς σφαίρας περιγράφεται ισοδύναμα και από τις δύο βάσεις.

Επίσης, αξίζει να σημειωθεί ότι οι συναρτήσεις, $\hat{Y}_{lm}$, του πεπερασμένου συνόλου στο οποίο αναπτύσσεται μία συνάρτηση $\hat{f}$, πάνω στην ασαφή σφαίρα, (1.85), είναι γνωστές ως οι ασαφείς σφαιρικές αρμονικές και δίνονται από τις εκφράσεις [94]:

$$\hat{Y}_{lm} = r^{-l}\sum_{\vec{a}} f^{(lm)}_{a_1\ldots a_l}\hat{X}^{a_1}\ldots\hat{X}^{a_l} \ , \tag{1.95}$$

οι οποίες είναι οι ασαφείς ανάλογες εκφράσεις που περιγράφουν τις σφαιρικές αρμονικές σε όρους των καρτεσιανών συντεταγμένων:

$$Y_{lm}(\theta,\phi) = \sum_{\vec{a}} f^{(lm)}_{a_1\ldots a_l}x^{a_1}\ldots x^{a_l} \ . \tag{1.96}$$

Και στις δύο περιπτώσεις, ο $f^{(lm)}_{a_1\ldots a_l}$ είναι ένας άιχνος συμμετρικός τανυστής της SO(3) τάξης $l$. Επίσης, οι ασαφείς σφαιρικές αρμονικές υπακούουν σε μία συνθήκη ορθοκανονικότητας η οποία δίνεται από:

$$\mathrm{Tr}_N\left(\hat{Y}^\dagger_{lm}\hat{Y}_{l'm'}\right) = \delta_{ll'}\delta_{mm'} \ , \tag{1.97}$$

όπου, το $\mathrm{Tr}_N$ συμβολίζει την πράξη της ολοκλήρωσης.

### 1.5.2 Ο ασαφής χώρος $\mathbb{R}^3_\lambda$

Όπως περιγράφηκε στο προηγούμενο κεφάλαιο, η ασαφής σφαίρα είναι ένας μη μεταθετικός χώρος ο οποίος αποτελεί μία προσέγγιση της κανονικής σφαίρας σε όρους πινάκων και οι συντεταγμένες της ικανοποιούν την SU(2) μεταθετική σχέση, (1.92), μαζί με την συνθήκη Casimir, (1.89), η οποία ουσιαστικά αποτελεί τον σύνδεσμο για την ακτίνα. Ο χώρος $\mathbb{R}^3_\lambda$ είναι ένας μη μεταθετικός χώρος η περιγραφή του οποίου βασίζεται στην ασαφή σφαίρα.

Ας θεωρήσουμε την περίπτωση της ασαφούς σφαίρας και ας την τροποποιήσουμε με τον εξής τρόπο: έστω ότι για κάθε συγκεκριμένο $\lambda$ (βλέπε (1.92)), αίρεται η συνθήκη Casimir, (1.89), το οποίο σημαίνει ότι οι πίνακες των συντεταγμένων, $X_a$, επιτρέπεται να φιλοξενούνται από αναγωγίσιμες αναπαραστάσεις της SU(2). Η αναγωγισιμότητα των αναπαραστάσεων επιτρέπει την γραφή των πινάκων, $X_a$, σε μπλοκ διαγώνια μορφή από μη αναγωγίσιμες αναπαραστάσεις, στις οποίες η συνθήκη Casimir ισχύει ακόμα για καθε ξεχωριστό μπλοκ, με άλλα λόγια, κάθε μπλοκ περιγράφει μία ασαφή σφάιρα. Επομένως, ο $\mathbb{R}^3_\lambda$ μπορεί να γραφτεί ως ένα ευθύ άθροισμα από ασαφεις σφαίρες όλων των πιθανών ακτίνων, οι οποίες καθορίζονται από $N \in \mathbb{N}$ ($N$ είναι η τιμή της τροχιακής στροφορμής, $l$): [91, 95–97]:

$$\mathbb{R}^3_\lambda = \sum_{2N\in\mathbb{N}} S^2_N = \bigoplus_{2N\in\mathbb{N}} \mathrm{Mat}(N,\mathbb{C}) \ . \tag{1.98}$$

Επομένως, ο $\mathbb{R}^3_\lambda$ μπορεί να ιδωθεί σαν μία διακριτή φυλλοποίηση του τρισδιάστατου ευκλείδειου χώρου από πολλαπλές ασαφείς σφαίρες, με κάθε ασαφή σφαίρα να είναι ένα φύλλο της φυλλοποίησης. Με ανάλογο τρόπο ορίζεται ο ασαφής χώρος $\mathbb{R}^{1,2}_\lambda$, ο οποίος αποτελεί φυλλοποίηση του τρισδιάστατου χώρου Minkowski από ασαφή υπερβολοειδή [100].

### 1.5.3 Ο ασαφής de Sitter χώρος

Στο κεφάλαιο αυτό κατασκευάζουμε μία ασαφή εκδοχή του χώρου de Sitter, $dS_4$ [78], ο οποίος ορίζεται ως υπόχωρος του πενταδιάστατου χώρου Minkowski με μετρική $\eta_{AB} = \mathrm{diag}(-1,1,1,1,1)$. Συγκεκριμένα, η σχέση εμβάπτισης είναι:

$$\eta^{AB} x_A x_B = R^2 \ , \tag{1.99}$$

όπου $A, B = 0, \ldots, 4$.

Προκειμένου να φτιάξουμε την ασαφή εκδοχή του παραπάνω χώρου, οι συντεταγμένες πρέπει να αντικατασταθούν από τελεστές μίας άλγεβρας $A$, η οποία αναπαρίσταται από πίνακες και επομένως υπακούουν μία μεταθετική σχέση γενικής μορφής:

$$[X_A, X_B] = i\theta_{AB}(X) \ , \tag{1.100}$$

όπου το $\theta_{AB}$ ενσωματώνει τον τύπο της μη μεταθετικότητας. Αν το $\theta_{AB}$ θεωρούνταν ως ένας αντισυμμετρικός, σταθερός τανυστής [12], τότε, η αναλλοιώτητα Lorentz θα παραβιαζόταν, δεδομένου ότι θα υπήρχε προτίμηση στην κατεύθυνση. Ανακαλώντας την περίπτωση της ασαφούς σφαίρας, στην οποία οι συντεταγμένες δεν μετατίθενται σύμφωνα με την (1.92), το $\theta_{AB}$ της παραπάνω γενικής σχέσης θα ήταν $\kappa\epsilon_{AB}^{\ \ C} X_C$. Αυτό σημαίνει ότι ο μεταθέτης των δύο συντεταγμένων, δηλαδή δύο τελεστές ανάλογοι των SU(2) γεννητόρων, παράγει ένα πολλαπλάσιο ενός στοιχείου εντός της SU(2) άλγεβρας, εξασφαλίζοντας την συναλλοιώτητα (οι συντεταγμένες μετασχηματίζονται σαν διανύσματα κάτω από τις περιστροφές, δηλαδή την ομάδα ισομετριών του χώρου). Ακολουθώντας την ίδια μεθοδολογία όπως στην ασαφή σφαίρα, για την κατασκευή του παρόντος ασαφούς χώρου de Sitter, οι μη μετατιθέμενες συντεταγμένες θα έπρεπε να ταυτοποιηθούν με κάποιους γεννήτορες της ομάδας ισομετριών, δηλαδή της SO(1,4). Παρόλα αυτά, μία τέτοια επιλογή δεν είναι επιτυχής επειδή η ταυτοποίηση των συντεταγμένων με γεννήτορες της SO(1,4) παραβιάζει τη συναλλοιώτητα, αφού η άλγεβρα δεν κλείνει, συγκεκριμένα, το $\theta_{AB}$ του δεξιού μέλους της σχέσης (1.100) δεν μπορεί να ανατεθεί σε γεννήτορες της άλγεβρας SO(1,4). Ωστόσο, η διατήρηση της συναλλοιώτητας οφείλει να ληφθεί αξιωματικά, επομένως κάποια ομάδα μεγαλύτερης συμμετρίας πρέπει να χρησιμοποιηθεί, στην οποία θα καταστεί δυνατή η συμπερίληψη όλων των γεννητόρων και της μη μεταθετικότητας εντός αυτής, με την ιδιότητα οι συντεταγμένες να μετασχηματίζονται σαν διανύσματα κάτω από τη δράση της ομάδας Lorentz. Προκειμένου να επιτευχθεί αυτό, η ελάχιστη επέκταση της συμμετρίας οδηγεί στην υιοθέτηση της ομάδας SO(1,5). Χάριν ευκολίας, για τη διατύπωση της παραπάνω μεθοδολογίας, χρησιμοποιούμε την ευκλείδεια περίπτωση, εννοώντας ότι η ομάδα συμμετριών επεκτείνεται από την SO(5) στην SO(6).

Ας θεωρήσουμε τους δεκαπέντε γεννήτορες της SO(6) και ας τους συμβολίσουμε σαν $J_{AB}$, με $A, B = 1, \ldots 6$, ικανοποιώντας την παρακάτω μεταθετική σχέση:

$$[J_{AB}, J_{CD}] = i(\delta_{AC}J_{BD} + \delta_{BD}J_{AC} - \delta_{BC}J_{AD} - \delta_{AD}J_{BC}) \ . \tag{1.101}$$

Αναπτύσσουμε τους παραπάνω γεννήτορες σε έναν SO(4) συμβολισμό και επαναορίζουμε τους γεννήτορες ως:

$$J_{mn} = \frac{1}{\hbar}\Theta_{mn}, \quad J_{m5} = \frac{1}{\lambda}X_m, \quad J_{m6} = \frac{\lambda}{2\hbar}P_m, \quad J_{56} = \frac{1}{2}\mathrm{h} \ , \tag{1.102}$$

όπου $m, n = 1, \ldots, 4$. Η παράμετρος $\lambda$ έχει εισαχθεί για διαστατικούς λόγους και $X_m, P_m, \Theta_{mn}$ ταυτοποιούνται ως οι συντεταγμένες, ορμές και τανυστής μη μεταθετικότητας, αντίστοιχα. Οι συντεταγμένες και οι ορμές ικανοποιούν τις παρακάτω μεταθετικές σχέσεις:

$$[X_m, X_n] = i\frac{\lambda^2}{\hbar}\Theta_{mn}, \quad [P_m, P_n] = 4i\frac{\hbar}{\lambda^2}\Theta_{mn} \tag{1.103}$$

$$[X_m, P_n] = i\hbar\delta_{mn}\mathrm{h}, \quad [X_m, \mathrm{h}] = i\frac{\lambda^2}{\hbar}P_m \tag{1.104}$$

$$[P_m, \mathrm{h}] = 4i\frac{\hbar}{\lambda^2}X_m \ , \tag{1.105}$$

από τις οποίες στην πρώτη, (1.103) γίνεται κατανοητό ότι ο μεταθέτης των συντεταγμένων κλείνει στο SO(4) κομμάτι της SO(6) ομάδας. Η άλγεβρα των μετασχηματισμών των συντεταγμένων είναι η εξής:

$$[X_m, \Theta_{np}] = i\hbar(\delta_{mp}X_n - \delta_{mn}X_p) \tag{1.106}$$

$$[P_m, \Theta_{np}] = i\hbar(\delta_{mp}P_n - \delta_{mn}P_p) \tag{1.107}$$

$$[\Theta_{mn}, \Theta_{pq}] = i\hbar(\delta_{mp}\Theta_{nq} + \delta_{nq}\Theta_{mp} - \delta_{np}\Theta_{mq} - \delta_{mq}\Theta_{np}) \tag{1.108}$$

$$[\mathrm{h}, \Theta_{mn}] = 0 \tag{1.109}$$

Η πρώτη μεταθετική σχέση, (1.103), δείχνει ότι οι συντεταγμένες μετασχηματίζονται σαν διανύσματα κάτω από τη δράση της ομάδας περιστροφών, (1.108), επιβεβαιώνοντας την πλέον σημαντική ιδιότητα της συναλλοιώτητας του χώρου. Η παραπάνω άλγεβρα, σε αντίθεση με την άλγεβρα Heisenberg (βλέπε αναφορά [102]), επιδέχεται αναπαραστάσεις πεπερασμένης διάστασης για τους $X_m$,

$P_m$ και $\Theta_{mn}$, επομένως έχουμε κατασκευάσει έναν χώρο ο οποίος αποτελεί ένα πεπερασμένο κβαντικό σύστημα. Συμπεριλαμβάνοντας την περίπτωση της ασαφούς σφαίρας και του $\mathbb{R}^3_\lambda$ χώροι σαν και αυτόν ονομάζονται ασαφείς συναλλοίωτοι χώροι [62, 63, 103]. Όπως θα περιγράψουμε αργότερα με λεπτομέρεια, κάνουμε χρήση του χώρου αυτού για την κατασκευή του τετραδιάστατου βαρυτικού μοντέλου ως μη μεταθετικής θεωρίας βαθμίδας.

*Συναλλοίωτος τανυστής δύναμης πεδίου του ασαφούς dS$_4$*

Εν γένει, ο τανυστής δύναμης πεδίου γράφεται ως ο μεταθέτης των συναλλοίωτων συντεταγμένων συν έναν έξτρα όρο που συνεισφέρει στην συναλλοιώτητα. Για τις δύο περιπτώσεις της κανονικής και Lie-τύπου μη μεταθετικότητας οι εκφράσεις αυτές είναι οι:

$$T_{ab} = [\hat{X}_a, \hat{X}_b] - i\theta_{ab}\,, \qquad F_{ab} = [\hat{X}_a, \hat{X}_b] - iC_{ab}{}^{c}\hat{X}_c\,. \tag{1.110}$$

Ο παραπάνω όρος σχετίζεται με το δεξί μέλος των μεταθετικών σχέσεων των συντεταγμένων τους:

$$[X_a, X_b] = i\theta_{ab}\,, \qquad [X_a, X_b] = iC_{ab}{}^{c}X_c\,. \tag{1.111}$$

Στην κανονική περίπτωση, στην οποία οι συντεταγμένες μετασχηματίζονται με μη συναλλοίωτο τρόπο, ο έξτρα όρος είναι ένας αντισυμμετρικός σταθερός τανυστής, ενώ στη συναλλοίωτη Lie-τύπου περίπτωση ο έξτρα όρος περιλαμβάνει τις συντεταγμένες με γραμμικό τρόπο. Η παρουσία του έξτρα όρου σε καθεμία από τις περιπτώσεις φαίνεται να χαλάει την αναλογία με τη μεταθετική περίπτωση, ωστόσο είναι απαραίτητος εφόσον σε αυτόν οφείλεται η συναλλοιώτητα του τανυστή.

Εδώ, στην περίπτωση του ασαφούς de Sitter χώρου τον οποίον θεμελιώνουμε [78], ο τανυστής μη μεταθετικότητας είναι ένας σταθερός αντισυμμετρικός τανυστής (γεννήτορας της ομάδας συμμετριών), παραπέμποντας σε μία σχέση με την κανονική περίπτωση, αλλά επίσης αποτελεί έναν συναλλοίωτο μη μεταθετικό χώρο (φτιάχτηκε κατα αυτόν τον τρόπο), παραπέμποντας στην περίπτωση Lie-τύπου. Επομένως, είναι αμφιλεγόμενο σε ποια περίπτωση κατατάσσεται η ο ασαφής χώρος de Sitter. Η απάντηση είναι ότι δεν μπορεί να καταταχθεί σε καμία από τις δύο αυτές περιπτώσεις, επομένως πρέπει να εξεταστεί ξεχωριστά. Όπως φαίνεται στην πρώτη σχέση της (1.103), ο ασαφής χώρος de Sitter ορίζεται ως:

$$[X_a, X_b] = i\frac{\lambda^2}{\hbar}\Theta_{ab} \otimes \mathbf{1}\,, \tag{1.112}$$

όπου $\mathbf{1}$ είναι ένας $p \times p$ μοναδιαίος πίνακας και $p$ είναι η διάσταση της αναπαράστασης της εκάστοτε ομάδας βαθμίδας. Λόγω της ανεξαρτησίας από τις συντεταγμένες του δεξιού μέλους της παραπάνω εξίσωσης, (1.112), των συντεταγμένων, $X_a$, ο προφανής ορισμός του τανυστή δύναμης πεδίου θα ήταν:

$$F_{ab} = [\hat{X}_a, \hat{X}_b] - i\frac{\lambda^2}{\hbar} \otimes \Theta_{ab}\mathbf{1}\,. \tag{1.113}$$

Αν θεωρήσουμε έναν μετασχηματισμό βαθμίδας του τανυστή δύναμης πεδίου, $\delta F_{ab}$, τότε, ευθείς υπολογισμοί οδηγούν στο παρακάτω αποτελέσμα:

$$\delta F_{ab} = [\epsilon, F_{ab}] - i\frac{\lambda^2}{\hbar}[\epsilon, \Theta_{ab} \otimes \mathbf{1}]\,, \tag{1.114}$$

όπου $\epsilon = \epsilon(X)$ είναι μία παράμετρος βαθμίδας. Επίσης, το γεγονός ότι οι συντεταγμένες, $X_a$, και επομένως ο τανυστής μη μεταθετικότητας, $\Theta_{ab}$, είναι αναλλοίωτοι κάτω από τους μετασχηματισμούς βαθμίδας, $\delta X_a = \delta\Theta_{ab} = 0$, έχει ληφθεί υπόψη στον υπολογισμό. Ωστόσο, στην παραπάνω έκφραση είναι εμφανές ότι ο τανυστή δύναμης πεδίου δεν μετασχηματίζεται με συναλλοίωτο τρόπο (εφόσον δεν υπάρχει κανένας λόγος για τον δεύτερο μεταθέτη του δεξιού μέλους να εξαφανίζεται) όπως στις κανονική και Lie-τύπου περιπτώσεις. Για να διορθώσουμε το παραπάνω, μη επιθυμητό αποτέλεσμα, ο ορισμός του τανυστή δύναμης πεδίου πρέπει να τροποποιηθεί κατάλληλα, συγκεκριμένα στην ακόλουθη μορφή:

$$\hat{F}_{ab} = [\hat{X}_a, \hat{X}_b] - \frac{i\lambda^2}{\hbar}\hat{\Theta}_{ab} \ , \tag{1.115}$$

όπου $\hat{\Theta}_{ab}$ είναι ένας τανυστής που ορίζεται ως:

$$\hat{\Theta}_{ab} = \Theta_{ab} \otimes \mathbf{1} + \mathcal{B}_{ab} \ , \tag{1.116}$$

όπου $\mathcal{B}_{ab}$ είναι ένα μη αβελιανό πεδίο 2-μορφής, το οποίο παίρνει επίσης τιμές στην άλγεβρα βαθμίδας της εκάστοτε θεωρίας βαθμίδας. Επομένως, υπολογισμοί οδηγούν στην ακόλουθη έκφραση του (απειροστού) μετασχηματισμού του τανυστή δύναμης πεδίου:

$$\delta\hat{F}_{ab} = i[\epsilon, \hat{F}_{ab}] \ , \tag{1.117}$$

ο οποίος είναι ένας συναλλοίωτος μετασχηματισμός. Εδώ κλείνει το κεφάλαιο στο οποίο γίνεται η περιγραφή των ασαφών χώρων τους οποίους θα χρησιμοποιήσουμε για την κατασκευή θεωριών βαθμίδας.

## 1.6 Τρισδιάστατη βαρύτητα ως θεωρία βαθμίδας σε μη μεταθετικούς χώρους

Στο κεφάλαιο αυτό κατασκευάζουμε ένα τρισδάστατο βαρυτικό μοντέλο ως θεωρία βαθμίδας στο μη μεταθετικό πλαίσιο [79] (βλέπε επίσης [80, 81]) μεταφράζοντας την αντίστοιχη προσέγγιση της τρισδιάστατης Γενικής Θεωρίας της Σχετικότητας όπως αυτή περιγράφεται σαν θεωρία βαθμίδας του ISO(1,2), (κεφάλαιο 1.2), στην οποία οι Poincaré και (A)dS ομάδες θεωρήθηκαν ως ομάδες βαθμίδας. Για το εγχείρημα αυτό, θέτουμε σε εφαρμογή τη γενική μεθοδολογία για την κατασκευή τέτοιων θεωριών βαθμίδας σε μη μεταθετικούς χώρους [33] και εξειδικεύουμε για την περίπτωση του συναλλοίωτου χώρου $\mathbb{R}^3_\lambda$ και του Lorentzian ανάλογού του, $\mathbb{R}^{1,2}_\lambda$.

Όπως συζητήσαμε στο κεφάλαιο 1.2, τα αποτελέσματα της τρισδιάστατης βαρύτητας Einstein ανακτώνται επιτυχώς από μία Chern-Simons θεωρία βαθμίδας της Poincaré ομάδας, ISO(1,2), με τη συνοχή βαθμίδας να κωδικοποιεί την πληροφορία για τα vielbein και spin connection. Εδώ, στοχεύουμε σε ένα τρισδιάστατο (μη μεταθετικό) βαρυτικό μοντέλο με θετική κοσμολογική σταθερά, επομένως η αντίστοιχη ομάδα συμμετρίας η οποία βαθμώνεται είναι η SO(1,3) για τη Lorentzian περίπτωση, ενώ για την ευκλείδεια είναι η SO(4) [104], με αντίστοιχους χώρους που φιλοξενούν τις θεωρίες, τους $\mathbb{R}^3_\lambda$ και $\mathbb{R}^{1,2}_\lambda$. Στις δύο αυτές περιπτώσεις, η πληροφορία για τα vielbein και spin connection εμπεριέχεται στη συναλλοίωτη συντεταγμένη. Έμπνευση για το όλο εγχείρημα αποτέλεσαν οι προηγούμενες δουλειές που περιέχονται στις αναφορές [34–38], στις οποίες οι συγγραφείς θεωρούν ομάδες οι οποίες χρησιμοποιούνται για την κατασκευή τετραδιάστατων μοντέλων χωρίς κοσμολογική σταθερά. Συγκεκριμένα για την τρισδιάστατη περίπτωση, σχετικές δουλειές δίνονται στις παρακάτω αναφορές [39–41, 105]. Συγκριτικά με την προσέγγισή μας, στις παραπάνω δουλειές οι παραμορφώσεις βασίζονται στο $\star$-γινόμενο του κάθε χώρου και χρησιμοποιείται η απεικόνιση Seiberg-Witten [42], ενώ στην δικιά μας περίπτωση χρησιμοποιούνται οι αναπαραστάσεις των τελεστών μέσω πινάκων.

Όπως αναφέραμε παραπάνω, οι ομάδες βαθμίδας που χρησιμοποιούμε είναι οι SO(1,3) και SO(4) ανάλογα με το αν είναι Lorentzian ή ευκλείδειος ο χώρος. Θεωρούμε λοιπόν ισοδύναμα τις σπιν ομάδες Spin(1,3) και Spin(4), οι οποίες είναι με τη σειρά τους ισόμορφες με τις SL(2,$\mathbb{C}$) και SU(2)×SU(2), αντίστοιχα. Θα προχωρήσουμε με την αναλυτική περιγραφή της Lorentzian περίπτωσης. Έπειτα η διαδικασία για την ευκλείδεια είναι η ίδια.

*Η Lorentzian περίπτωση*

Στις μη αβελιανές μη μεταθετικές θεωρίες βαθμίδας, το γεγονός ότι οι αντιμεταθέτες των γεννητόρων της ομαδας δεν κλείνουν, δηλαδή παράγουν τελεστές οι οποίοι δεν αποτελούν στοιχεία της άλγεβρας. Προφανώς, το ίδιο πρόβλημα συναντάται και στην περίπτωη που εξετάζουμε για τους γεννήτορες της SL(2,$\mathbb{C}$) θεωρίας βαθμίδας. Επομένως, για να ξεπεραστεί το πρόβλημα αυτό, το πρώτο βήμα είναι να προσδιορίσουμε την αναπαράσταση στην οποία βρίσκονται οι γεννήτορες, στην περίπτωσή μας είναι η σπινοριακή αναπαράσταση, στην οποία οι έξι γεννήτορες αναπαρίστανται από τους μεταθέτες των Lorentzian $\gamma$-πινάκων, συγκεκριμένα:

$$\Sigma_{AB} = \frac{1}{2}\gamma_{AB} = \frac{1}{4}[\gamma_A, \gamma_B] , \quad A, B = 1, \ldots, 4 . \tag{1.118}$$

Ξεκινώντας από την παρακάτω σχέση [106]:

$$\gamma_{AB}\gamma^{CD} = 2\delta^{[C}_{[B}\delta^{D]}_{A]} + 4\delta^{[C}_{[B}\gamma_{A]}{}^{D]} + i\epsilon_{AB}{}^{CD}\gamma_5 , \tag{1.119}$$

προκύπτουν οι παρακάτω μεταθετικές και αντιμεταθετικές σχέσεις των γεννητόρων:

$$[\gamma_{AB}, \gamma_{CD}] = 8\eta_{A[C}\gamma_{D]B}\ , \tag{1.120}$$

$$\{\gamma_{AB}, \gamma_{CD}\} = 4\eta_{C[B}\eta_{A]D}\mathbf{1} + 2i\epsilon_{ABCD}\gamma_5\ , \tag{1.121}$$

όπου $\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_4$. Στην παραπάνω αντιμεταθετική σχέση, (1.121), τα μόνα στοιχεία που παράγονται από αντιμεταθέτες των γεννητόρων της άλγεβρας σε αυτήν τη συγκεκριμένη αναπαράσταση είνα ο τετραδιάστατος μοναδιαίος πίνακας, $\mathbf{1}$ και ο $\gamma_5$. Επομένως, τα δύο αυτά στοιχεία πρέπει να συμπεριληφθούν στην άλγεβρα, επεκτείνοντάς την σε μία οκταδιάστατη άλγεβρα, η οποία είναι ακριβώς αυτή της GL(2,ℂ) με σύνολο γεννητόρων το $\{\gamma_{AB}, \gamma_5, i\mathbf{1}\}$[11].

Τώρα προχωράμε σε ένα SO(3) ανάπτυγμα των παραπάνω γεννητόρων, για λόγους κατάλληλης ταυτοποίησης των πεδίων βαθμίδας που θα εισαχθούν σε λίγο. Επομένως, ορίζουμε τους γεννήτορες $\tilde{\gamma}^a \equiv \epsilon^{abc}\gamma_{bc}$ και $\bar{\gamma}_a \equiv \gamma_{a4}$, με $a, b, c = 1, 2, 3$. Αυτοί οι επαναορισμοί των γεννητόρων μας επιτρέπουν να ξαναγράψουμε τις μεταθετικές και αντιμεταθετικές σχέσεις των (1.120) και (1.121) σε όρους των SO(3) αναπτυγμάτων των γεννητόρων:

$$[\tilde{\gamma}^a, \tilde{\gamma}^b] = -4\epsilon^{abc}\tilde{\gamma}_c\ , \quad [\bar{\gamma}_a, \tilde{\gamma}_b] = -4\epsilon_{abc}\bar{\gamma}^c\ , \quad [\bar{\gamma}_a, \bar{\gamma}_b] = \epsilon_{abc}\tilde{\gamma}^c \tag{1.122}$$

$$\{\tilde{\gamma}^a, \tilde{\gamma}^b\} = -8\eta_{ab}\mathbf{1}\ , \quad \{\bar{\gamma}_a, \tilde{\gamma}^b\} = 4i\delta_a^b\gamma_5\ , \quad \{\bar{\gamma}_a, \bar{\gamma}_b\} = 2\eta_{ab}\mathbf{1}\ , \tag{1.123}$$

$$[\gamma^5, \tilde{\gamma}^a] = [\gamma^5, \bar{\gamma}^a] = 0\ , \quad \{\gamma^5, \tilde{\gamma}_a\} = 4i\bar{\gamma}_a\ , \quad \{\gamma^5, \bar{\gamma}^a\} = i\tilde{\gamma}^a\ , \tag{1.124}$$

όπου στην τελευταία σειρά χρησιμοποιήθηκαν οι σχέσεις $[\gamma^5, \gamma^{AB}] = 0$ και $\{\gamma^5, \gamma^{AB}\} = i\epsilon^{ABCD}\gamma_{CD}$.

Επομένως, καταλήγουμε σε μία θεωρία με ομάδα βαθμίδας την GL(2,ℂ). Ο τρισδιάστατος χώρος που χρησιμοποιούμε για την κατασκευή της θεωρίας είναι ο $\mathbb{R}^{1,2}_\lambda$, δηλαδή η φυλλοποίηση του τρισδιάστατου χώρου Minkowski από ασαφή υπερβολοειδή, όπου οι τρεις συντεταγμένες, δηλαδή οι τελεστές $X_\mu$, έχουν ταυτοποιηθεί με πολλαπλάσια των γεννητόρων της SU(1,1) σε μία αναγωγίσιμη αναπαράσταση. Προκειμένου να προχωρήσουμε με την κατασκευή της θεωρίας βαθμίδας, εισάγουμε τη συναλλοίωτη συντεταγμένη, η οποία είναι:

$$\hat{X}_\mu = X_\mu + \mathcal{A}_\mu\ , \tag{1.125}$$

όπου $\mu = 0, 1, 2$ είναι οι χωροχρονικοί δείκτες και $\mathcal{A}_\mu$ είναι η συνοχή βαθμίδας. Η τελευταία είναι συνάρτηση των τελεστών-συντεταγμένων, $X_\mu$, η οποία παίρνει τιμές στην άλγεβρα GL(2,ℂ), επομένως, αν για την ώρα συμβολίσουμε τους γεννήτορες συλλογικά ως $T^{\bar{a}}$, όπου $\bar{a} = 1, \ldots, 8$, τότε αναπτύσσεται πάνω σε αυτούς ως:

$$\mathcal{A}_\mu(X) = \mathcal{A}_\mu^{\bar{a}}(X) \otimes T^{\bar{a}}\ . \tag{1.126}$$

Πρέπει να σημειωθεί ότι ανάμεσα στα πεδία βαθμίδας, $\mathcal{A}_\mu^{\bar{a}}(X)$ και τους γεννήτορες $T^{\bar{a}}$, το σύνηθες γινόμενο δεν θα είχε νόημα, για αυτό και αντικαθίσταται από το τανυστικό γινόμενο μιας και τα πεδία βαθμίδας είναι συναρτήσεις των συντεταγμένων, δηλαδή $N \times N$ πίνακες και οι γεννήτορες είναι $4 \times 4$ πίνακες (σπινοριακή αναπαράσταση). Τώρα γράφουμε τη συνοχή βαθμίδας ως ανάπτυγμα πάνω στους γεννήτορες, $T^{\bar{a}} = \{\tilde{\gamma}^a, \bar{\gamma}_a, i\mathbf{1}, \gamma_5\}$:

$$\mathcal{A}_\mu(X) = e_\mu{}^a(X) \otimes \bar{\gamma}_a + \omega_\mu{}^a(X) \otimes \tilde{\gamma}_a + A_\mu(X) \otimes i\mathbf{1} + \tilde{A}_\mu(X) \otimes \gamma_5\ , \tag{1.127}$$

[11]Χρησιμοποιούμε το σύνολο των $\gamma$-πινάκων όπως στην αναφορά [36]. Συμβολίζουμε τον $\gamma_0$ πινακα ως $\gamma_4$ και θεωρούμε το στοιχείο $\eta_{44} = -1$ στην κυρίως θετική τετραδιάστατη μετρική Minkowski που χρησιμοποιούμε. Βλέπε επίσης [107] για περισσότερες λεπτομέρειες.

όπου τα πεδία βαθμίδας που ανατίθενται στους $\bar{\gamma}_a$ και $\tilde{\gamma}_a$ γεννήτορες έχουν ταυτοποιηθεί ως το vielbein, $e_\mu^a(X)$ και το spin connection, ${\omega_\mu}^a(X)$, αντίστοιχα, ακολουθώντας την αντίστοιχη ταυτοποίηση με τη μεταθετική περίπτωση, (1.8). Εδώ, λόγω της μη μεταθετικότητας έχουμε εισαγάγει δύο παραπάνω U(1)-τύπου πεδία βαθμίδας, τα $A_\mu(X)$ και $\tilde{A}_\mu(X)$. Επίσης, θεωρούμε την παράμετρο βαθμίδας, $\epsilon(X)$, η οποία παίρνει τιμές στην ομάδα βαθμίδας και επομένως αναπτύσσεται στους γεννήτορες σαν:

$$\epsilon(X) = \xi^a(X) \otimes \bar{\gamma}_a + \lambda^a(X) \otimes \tilde{\gamma}_a + \epsilon_0(X) \otimes i\mathbf{1} + \tilde{\epsilon}_0(X) \otimes \gamma_5 \ . \tag{1.128}$$

Έχοντας πλέον γράψει τις εκφράσεις της συνοχής βαθμίδας και της παραμέτρου μετασχηματισμού βαθμίδας, προχωράμε με τον προσδιορισμό των μετασχηματισμών των πεδίων βαθμίδας, χρησιμοποιώντας τον κανόνα μετασχηματισμού της συναλλοίωτης συντεταγμένης, $\delta\hat{X} = [\epsilon, \hat{X}]$:

$$\begin{aligned}
\delta {e_\mu}^a &= -i[X_\mu + A_\mu, \xi^a] + 2\{\omega_{\mu b}, \xi_c\}\epsilon^{abc} + 2\{e_{\mu b}, \lambda^c\}\epsilon^{abc} + 2i[\lambda_a, \tilde{A}_\mu] + 2i[\tilde{\epsilon}_0, \omega_{\mu a}] + i[\epsilon_0, e_{\mu a}] \ , \\
\delta {\omega_\mu}^a &= -i[X_\mu + A_\mu, \lambda^a] + 2\{\omega_{\mu b}, \lambda_c\}\epsilon^{abc} - \frac{1}{2}\{e_{\mu b}, \xi_c\}\epsilon^{abc} + \frac{i}{2}[\xi^a, \tilde{A}_\mu] + i[\epsilon_0, {\omega_\mu}^a] + \frac{i}{2}[\tilde{\epsilon}_0, {e_\mu}^a] \ , \\
\delta A_\mu &= -i[X_\mu + A_\mu, \epsilon_0] - i[\xi^a, e_{\mu a}] + 4i[\lambda^a, \omega_{\mu a}] - i[\tilde{\epsilon}_0, \tilde{A}_\mu] \ , \\
\delta \tilde{A}_\mu &= -i[X_\mu + A_\mu, \tilde{\epsilon}_0] + 2i[\xi^a, \omega_{\mu a}] + 2i[\lambda^a, e_{\mu a}] + i[\epsilon_0, \tilde{A}_\mu] \ .
\end{aligned} \tag{1.129}$$

Στο σημείο αυτό, εξετάζουμε δύο σημαντικά όρια που αφορούν στους παραπάνω κανόνες μετασχηματισμού των πεδίων βαθμίδας. Πρώτον, αν είχαμε ξεκινήσει την κατασκευή της θεωρίας βαθμίδας στον χώρο $\mathbb{R}^{1,2}_\lambda$ με μία αβελιανή ομάδα, U(1), τότε η αντίστοιχη συναλλοίωτη συντεταγμένη θα ήταν απλά η $\hat{X}_\mu = X_\mu + A_\mu$ και από τον γνωστό κανόνα μετασχηματισμού, θα αποκτούσαμε τον μετασχηματισμό του πεδίου $\delta A_\mu = -i[X_\mu, \epsilon_0] + i[\epsilon_0, A_\mu]$, όπου $\epsilon_0$ η αντίστοιχη παράμετρος μετασχηματισμού βαθμίδας. Αυτή η αβελιανή θεωρία βαθμίδας υποβόσκει κάτω από την GL(2,$\mathbb{C}$) θεωρία βαθμίδας που χτίζουμε και γίνεται αντιληπτή θέτοντας ${e_\mu}^a, {\omega_\mu}^a, \tilde{A}_\mu = 0$ μαζί με τις αντίστοιχες παραμέτρους ίσες με το μηδέν. Επομένως, ο μόνος μη τετριμμένος μετασχηματισμός της εξίσωσης (1.129) θα ήταν ο $\delta A_\mu = -i[X_\mu, \epsilon_0] + i[\epsilon_0, A_\mu]$, ο οποίος είναι ταυτόσημος με τον κανόνα μετασχηματισμού ενός πεδίου βαθμίδας μίας μη μεταθετικής Maxwell θεωρίας βαθμίδας, όπως αναφέραμε παραπάνω. Επομένως, καταλαβαίνουμε ότι ο τομέας Maxwell είναι πάντα παρών είτε το vielbein είναι τετριμμένο, είτε όχι. Το δεύτερο όριο είναι το μεταθετικό, στο οποίο τα επιπρόσθετα πεδία σε σχέση με αυτά της βαρυτικής θεωρίας αποσυζεύγνυνται και επομένως μπορούμε να θέσουμε $A_\mu = \tilde{A}_\mu = 0$. Επίσης, στο όριο αυτό, η εσωτερική παραγώγιση ανάγεται στη συνηθισμένη σύμφωνα με την απεικόνιση $[X_\mu, f] \ \rightarrow \ -i\partial_\mu f$. Επομένως, οι εκφράσεις των μετασχηματισμών των επιζόντων πεδίων, ${e_\mu}^a, {\omega_\mu}^a$ όπως αποκτήθηκαν στην εξίσωση (1.129), γίνονται:

$$\begin{aligned}
\delta {e_\mu}^a &= -\partial_\mu \xi^a - 4\xi_b \omega_{\mu c}\epsilon^{abc} - 4\lambda_b e_{\mu c}\epsilon^{abc} \ , \\
\delta {\omega_\mu}^{ab} &= -\partial_\mu \lambda^a + \xi_b e_{\mu c}\epsilon^{abc} - 4\lambda_b \omega_{\mu c}\epsilon^{abc} \ .
\end{aligned} \tag{1.130}$$

Οι παραπάνω εκφράσεις θυμίζουν αυτές της προσέγγισης της τρισδιάστατης βαρύτητας ως θεωρίας βαθμίδας με θετική κοσμολογική σταθερά, (1.27) και γίνονται ταυτόσημες με αυτές έπειτα από την θεώρηση των παρακάτω επαναορισμών των γεννητόρων, παραμέτρου μετασχηματισμού βαθμίδας και πεδίων βαθμίδας:

$$\bar{\gamma}_a \ \rightarrow \ \frac{2i}{\sqrt{\lambda}}P_a \ , \ \ \tilde{\gamma}_a \ \rightarrow \ -4J_a \ , \ \ 4\lambda_a \ \rightarrow \ \lambda^a \ , \ \ \xi\frac{2i}{\sqrt{\lambda}} \ \rightarrow \ -\xi^a \ , \ \ {e_\mu}^a \ \rightarrow \ \frac{\sqrt{\lambda}}{2i}{e_\mu}^a \ , \ \ {\omega_\mu}^a \ \rightarrow \ -\frac{1}{4}{\omega_\mu}^a \ . \tag{1.131}$$

Επομένως, στο μεταθετικό όριο, οι μετασχηματισμοί των πεδίων βαθμίδας της τρισδιάσατης θεωρίας βαρύτητας ανακτώνται επιτυχώς.

Προχωρώντας με την κατασκευή της θεωρίας βαθμίδας, το επόμενο βήμα είναι να υπολογιστεί ο τανυστής δύναμης πεδίου ο οποίος δίνει τις εκφράσεις των τανυστών καμπυλότητας. Ο τανυστής δύναμης πεδίου έχει την παρακάτω μορφή:

$$\mathcal{R}_{\mu\nu} = [\hat{X}_\mu, \hat{X}_\nu] - i\lambda C_{\mu\nu}^{\ \ \rho}\hat{X}_\rho \ . \tag{1.132}$$

Ο τανυστής $\mathcal{R}_{\mu\nu}(X)$ παίρνει τιμές στην άλγεβρα, οπότε μπορεί να γραφτεί ως ανάπτυγμα πάνω στους γεννήτορες:

$$\mathcal{R}_{\mu\nu}(X) = T_{\mu\nu}^{\ \ a}(X)\otimes\bar{\gamma}_a + R_{\mu\nu}^{\ \ a}(X)\otimes\tilde{\gamma}_a + F_{\mu\nu}(X)\otimes i\mathbf{1} + \tilde{F}_{\mu\nu}(X)\otimes\gamma_5 \ . \tag{1.133}$$

Συνδυάζοντας τις εξισώσεις (1.125), (1.127), (1.132) και (1.133), αποκτούμε τις παρακάτω εκφράσεις για τους τανυστές καμπυλότητας:

$$\begin{aligned}
T_{\mu\nu}^{\ \ a} &= i[X_\mu + A_\mu, e_\nu^{\ a}] - i[X_\nu + A_\nu, e_\mu^{\ a}] - 2\epsilon^{abc}\left(\{e_{\mu b}, \omega_{\nu c}\} + \{\omega_{\mu b}, e_{\nu c}\}\right) \\
&\quad + 2i\left([\omega_\mu^{\ a}, \tilde{A}_\nu] - [\omega_\nu^{\ a}, \tilde{A}_\mu]\right) - i\lambda C_{\mu\nu}^{\ \ \rho} e_\rho^{\ a} \ , &(1.134)\\
R_{\mu\nu}^{\ \ a} &= i[X_\mu + A_\mu, \omega_\nu^{\ a}] - i[X_\nu + A_\nu, \omega_\mu^{\ a}] + \epsilon^{abc}\left(\tfrac{1}{2}\{e_{\mu b}, e_{\nu c}\} - 2\{\omega_{\mu b}, \omega_{\nu c}\}\right) \\
&\quad + \tfrac{i}{2}\left([e_\mu^{\ a}, \tilde{A}_\nu] - [e_\nu^{\ a}, \tilde{A}_\mu]\right) - i\lambda C_{\mu\nu}^{\ \ \rho}\omega_\rho^{\ a} \ , &(1.135)\\
F_{\mu\nu} &= i[X_\mu + A_\mu, X_\nu + A_\nu] - i[e_\mu^{\ a}, e_{\nu a}] + 4i[\omega_\mu^{\ a}, \omega_{\nu a}] - i[\tilde{A}_\mu, \tilde{A}_\nu] - i\lambda C_{\mu\nu}^{\ \ \rho}(X_\rho + A_\rho) \ , &(1.136)\\
\tilde{F}_{\mu\nu} &= i[X_\mu + A_\mu, \tilde{A}_\nu] - i[X_\nu + A_\nu, \tilde{A}_\mu] + 2i\left([e_\mu^{\ a}, \omega_{\nu a}] + [\omega_\mu^{\ a}, e_{\nu a}]\right) - i\lambda C_{\mu\nu}^{\ \ \rho}\tilde{A}_\rho \ . &(1.137)
\end{aligned}$$

Αξίζει να σημειωθεί ότι αν θεωρήσουμε το μεταθετικό όριο, οι εκφράσεις των δύο πρώτων σχέσεων συμπίπτουν με τις αντίστοιχες της τρισδιάστατης βαρύτητας, (1.27), έπειτα από την υιοθέτηση των επαναορισμών της (1.131).

*Η ευκλείδεια περίπτωση*

Όπως αναφέραμε στην αρχή του κεφαλαίου, η ομάδα βαθμίδας για την περίπτωση αυτή είναι η $SU(2)\times SU(2)$. Παρομοίως, λόγω του ότι οι αντιμεταθέτες των γεννητόρων δεν κλείνουν εντός της άλγεβρας, προσδιορίζουμε την αναπαράσταση και επεκτείνουμε την άλγεβρα με παραπάνω στοιχεία τους τελεστές που προκύπτουν από τους αντιμεταθέτες, καταλήγωντας με τη $U(2)\times U(2)$ ως ομάδα βαθμίδας της θεωρίας. Κάθε $U(2)$ αναπαρίσταται από τους πίνακες Pauli και τον μοναδιαίο πίνακα, επομένως η $U(2)\times U(2)$ ομάδα βαθμίδας θα εμπλέκει τους παρακάτω $4\times 4$ πίνακες:

$$J_a^L = \begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix} \ , \ J_0^L = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \ , \quad \text{και} \quad J_a^R = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_a \end{pmatrix} \ , \ J_0^R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \ . \tag{1.138}$$

Παρόλα αυτά, πρέπει να είμαστε προσεκτικοί κατά την ταυτοποίηση των μη μεταθετικών πεδίων βαθμίδας vielbein και spin connection. Για τη σωστή ταυτοποίηση των πεδίων βαθμίδας, οι γεννήτορες που θεωρούμε είναι οι παρακάτω:

$$P_a = \tfrac{1}{2}(J_a^L - J_a^R) = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix} \ , \quad M_a = \tfrac{1}{2}(J_a^L + J_a^R) = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & \sigma_a \end{pmatrix} \ , \tag{1.139}$$

καθώς επίσης:

$$\mathbf{1} = J_0^L + J_0^R \ , \quad \gamma_5 = J_0^L - J_0^R \ . \tag{1.140}$$

Η παραπάνω μορφή των γεννητόρων ικανοποιούν τις αναμενόμενες μεταθετικές και αντιμεταθετικές σχέσεις, οι οποίες αποκτώνται χρησιμοποιώντας τις αντίστοιχες για τους πίνακες Pauli:

$$\begin{aligned}
&[P_a, P_b] = i\epsilon_{abc}M_c \ , \quad [P_a, M_b] = i\epsilon_{abc}P_c \ , \quad [M_a, M_b] = i\epsilon_{abc}M_c \ , \\
&\{P_a, P_b\} = \tfrac{1}{2}\delta_{ab}\mathbf{1} \ , \quad \{P_a, M_b\} = \tfrac{1}{2}\delta_{ab}\gamma_5 \ , \quad \{M_a, M_b\} = \tfrac{1}{2}\delta_{ab}\mathbf{1} \ , \\
&[\gamma_5, P_a] = [\gamma_5, M_a] = 0 \ , \quad \{\gamma_5, P_a\} = 2M_a \ , \quad \{\gamma_5, M_a\} = 2P_a \ .
\end{aligned} \tag{1.141}$$

Έπειτα, ταυτοποιώντας τον χώρο πάνω στον οποίον χτίζουμε τη θεωρία βαθμίδας ως τον $\mathbb{R}^3_\lambda$, προχωράμε όπως στην Lorentzian περίπτωση ορίζοντας τη συναλλοίωτη συντεταγμένη:

$$\hat{X}_\mu = X_\mu \otimes i\mathbf{1} + e_\mu{}^a \otimes P_a + \omega_\mu{}^a \otimes M_a + A_\mu \otimes i\mathbf{1} + \tilde{A}_\mu \otimes \gamma_5 \ , \tag{1.142}$$

καθώς και την παράμετρο μετασχηματισμού βαθμίδας:

$$\epsilon = \xi^a \otimes P_a + \lambda^a \otimes M_a + \epsilon_0 \otimes i\mathbf{1} + \tilde{\epsilon}_0 \otimes \gamma_5 \ . \tag{1.143}$$

Ο κανόνας μετασχηματισμού της συναλλοίωτης παραγώγου παράγει τους μετασχηματισμούς των πεδίων βαθμίδας, όπως στην εξίσωση (1.129) και κάνοντας χρήση του ορισμού του τανυστή δύναμης πεδίου καταλήγουμε με τους τανυστές καμπυλότητας, παρόμοιους με αυτούς της Lorentzian περίπτωσης, (1.137).

*Η δράση για την τρισδιάστατη μη μεταθετική βαρύτητα*

Για να ολοκληρώσουμε την εικόνα, πρέπει να προσδιορίσουμε τη δράση της θεωρίας. Εφόσον οι χώροι στους οποίους δουλεύουμε είναι τρισδιάστατοι, εμπνεόμενοι από την προσέγγιση της βαρύτητας Einstein ως θεωρίας βαθμίδας, όπως περιγράφηκε στο κεφάλαιο 1.2, η προφανής επιλογή είναι μία δράση τύπου Chern-Simons. Για την Lorentzian περίπτωση, $\mathbb{R}^{1,2}_\lambda$, η δράση [108][12] είναι:

$$S_0 = \frac{1}{g^2}\mathrm{Tr}\ \left(\frac{i}{3}C^{\mu\nu\rho}X_\mu X_\nu X_\rho - m^2 X_\mu X^\mu\right) \ . \tag{1.144}$$

Μεταβολή της παραπάνω δράσης οδηγεί στις εξισώσεις κίνησης:

$$[X_\mu, X_\nu] - 2im^2 C_{\mu\nu}{}^\rho X_\rho = 0 \ , \tag{1.145}$$

η οποία επιδέχεται ως λύση τον χώρο $\mathbb{R}^{1,2}_\lambda$, για $2m^2 = \lambda$. Επίσης, αν είχαμε ξεκινήσει με την ίδια δράση για την ευκλείδεια περίπτωση, $\mathbb{R}^3_\lambda$, η μόνη διαφορά θα ήταν ότι το $C^{\mu\nu\rho}$ θα έπρεπε να αντικατασταθεί από το $\epsilon^{\mu\nu\rho}$ και η παράμετρος θα προέκυπτε $2m^2 = -\lambda$.

Για να εισαγάγουμε τα πεδία βαθμίδας στην παραπάνω δράση, (1.144), είτε πρέπει να θεωρήσουμε διαταραχές στην παραπάνω λύση, (1.145), αντικαθιστώντας τις συντεταγμένες με τις συναλλοίωτες εκδοχές τους, είτε να κάνουμε την αντικατάσταση αυτή στο επίπεδο της δράσης και να καταλήξουμε με τις εξισώσεις κίνησης έπειτα από μεταβολή της δράσης ως προς τα πεδία. Η εμφάνιση των πεδίων συνεπάγεται την εμφάνιση του ίχνους πάνω στους γεννήτορες, $\mathrm{tr}_G$. Τα μη

[12]Παρόμοια δράση είχε προταθεί και στη δουλειά [54] για την περιγραφή βαρύτητας στη fuzzy σφαίρα. Βλέπε επίσης [109].

μηδενιζόμενα ίχνη των γεννητόρων αποκτώνται ξεκινώντας από τις εκφράσεις των αντιμεταθετών στη σχέση (1.124):

$$\mathrm{tr}_G(\bar{\gamma}_a\bar{\gamma}_b) = 4\eta_{ab}\ , \quad \mathrm{tr}_G(\tilde{\gamma}_a\tilde{\gamma}_b) = -16\eta_{ab}\ . \tag{1.146}$$

Επομένως, η δράση μπορεί να γραφτεί σε όρους των πεδίων βαθμίδας ως:

$$S = \frac{1}{g^2}\mathrm{Trtr}_G\left(\tfrac{i}{3}C^{\mu\nu\rho}\hat{X}_\mu\hat{X}_\nu\hat{X}_\rho - \tfrac{\lambda}{2}\hat{X}_\mu\hat{X}^\mu\right)\ , \tag{1.147}$$

όπου το πρώτο ίχνος, $\mathrm{Tr}$ αφορά τους $N \times N$ πίνακες που αναπαριστούν τις συντεταγμένες και το δεύτερο ίχνος $\mathrm{tr}_G$ αφορά τους $4 \times 4$ πίνακες που αναπαριστούν τους γεννήτορες της ομάδας βαθμίδας GL(2,ℂ). Η παραπάνω δράση, (1.147) μπορεί να γραφτεί σε όρους του $\mathcal{R}_{\mu\nu}$ της σχέσης (1.132) σαν:

$$S = \tfrac{1}{6g^2}\mathrm{Trtr}_G\left(iC^{\mu\nu\rho}\hat{X}_\mu\mathcal{R}_{\nu\rho}\right) + S_\lambda\ , \tag{1.148}$$

όπου $S_\lambda = -\frac{\lambda}{6g^2}\mathrm{Trtr}_G(\hat{X}_\mu\hat{X}^\mu)$ και μηδενίζεται στο όριο $\lambda \to 0$. Κάνοντας χρήση των εκφράσεων των ιχνών πάνω στους γεννήτορες, (1.146), στην έκφραση της δράσης (1.148), αποκτούμε την ακόλουθη έκφραση:

$$\begin{aligned} S = &\tfrac{2}{3g^2}\mathrm{Tr}iC^{\mu\nu\rho}\left(e_{\mu a}T_{\nu\rho}{}^a - 4\omega_{\mu a}R_{\nu\rho}{}^a - (X_\mu + A_\mu)F_{\nu\rho} + \tilde{A}_\mu\tilde{F}_{\nu\rho}\right) \\ &- \tfrac{2\lambda}{3g^2}\mathrm{Tr}\left(e_\mu{}^a e^\mu{}_a - 4\omega_\mu{}^a\omega^\mu{}_a - (X_\mu + A_\mu)(X^\mu + A^\mu) + \tilde{A}_\mu\tilde{A}^\mu\right)\ . \end{aligned} \tag{1.149}$$

Αξίζει να σημειωθεί ότι αν θεωρήσουμε το μεταθετικό όριο και εφαρμόσουμε τους επαναορισμούς της (1.131), η παραπάνω έκφραση της δράσης, (1.149), ανάγεται σε αυτήν της τρισδιάστατης Einstein βαρύτητας, η οποία περιγράφεται στο κεφάλαιο 1.2, ειδικά στην (1.19).

Τελειώνοντας την κατασκευή, προχωράμε με τη μεταβολή της παραπάνω δράσης ως προς τα διάφορα πεδία. Οι εξισώσεις κίνησης αποκτώνται και είναι οι παρακάτω:

$$T_{\mu\nu}{}^a = 0\ , \quad R_{\mu\nu}{}^a = 0\ , \quad F_{\mu\nu} = 0 \quad \tilde{F}_{\mu\nu} = 0\ . \tag{1.150}$$

Πάλι, όπως είναι αναμενόμενο, στο μεταθετικό όριο, οι δύο πρώτες ανάγονται σε αυτές της τρισδιάτατης βαρύτητας Einstein.

Συνοψίζοντας, στο κεφάλαιο αυτό κατασκευάσαμε ένα τρισδιάστατο βαρυτικό μοντέλο παρουσία κοσμολογικής σταθεράς ως μία θεωρία βαθμίδας στο μη μεταθετικό πλαίσιο εργασίας. Ακολουθώντας την καθιερωμένη διαδικασία, ορίσαμε τη συναλλοίωτη συντεταγμένη και από τον κανόνα μετασχηματισμού της καταλήξαμε με τους μετασχηματισμούς των πεδίων βαθμίδας της θεωρίας, έπειτα από ένα SO(3) ανάπτυγμα. Έπειτα ορίσαμε τον τανυστή δύναμης πεδίου και αποκτήσαμε τις εκφράσεις των τανυστών καμπυλότητας. Τέλος, κάνοντας χρήση του τανυστή δύναμης πεδίου, προτείναμε μία δράση τύπου Chern-Simons και καταλήξαμε με τις εξισώσεις κίνησης. Αξίζει να τονιστεί ότι τα παραπάνω αποτελέσματα ανάγονται σε αυτά της τρισδιάστατης βαρυτικής θεωρίας Einstein κατά τη θεώρηση του μεταθετικού ορίου.

## 1.7 Τετραδιάστατη βαρύτητα ως θεωρία βαθμίδας σε μη μεταθετικούς χώρους

Στο κεφάλαιο αυτό, επεκτείνουμε το περιεχόμενο του προηγούμενου κεφαλαίου στην τετραδιάστατη περίπτωση [78]. Πιο συγκεκριμένα, ο τετραδιάστατος συναλλοίωτος χώρος που χρησιμοποιούμε είναι ο τετραδιάστατος ασαφής χώρος de Sitter ο οποίος περιγράφτηκε λεπτομερώς στο κεφάλαιο 1.5.3. Όπως αναφέρθηκε στο κεφάλαιο αυτό, στην ευκλείδεια εικόνα, η ομάδα των ισομετριών πρέπει να επεκταθεί στην SO(6) για λόγους συναλλοιώτητας. Οι διάφοροι γεννήτορες της ομάδας αυτής ταυτοποιήθηκαν με τελεστές οι οποίοι αντιστοιχούν σε φυσικές ποσότητες, όπως οι συντεταγμένες, οι ορμές και οι στροφορμές. Προκειμένου να διατυπώσουμε την βαρύτητα ως θεωρία βαθμίδας στον παραπάνω ασαφή χώρο, επιλέγουμε να βαθμώσουμε την SO(5) μέγιστη υποομάδα της SO(6) ομάδας συμμετρίας. Χρησιμοποιώντας τα ενδεδειγμένα εργαλεία και μεθοδολογία, ξεκινάμε να κατασκευάσουμε μία SO(5) θεωρία βαθμίδας όμως λόγω του ότι οι αντιμεταθέτες των γεννητόρων δεν κλείνουν μέσα στην άλγεβρα, η ομάδα βαθμίδας στην οποία καταλήγουμε τελικά είναι η SO(6)×U(1) σε μια συγκεκριμένη αναπαράσταση. Η ομάδα βαθμίδας με την οποία καταλήξαμε για τη διατύπωση της βαρυτικής θεωρίας (κυρίως το κομμάτι SO(6)), φαίνεται να σχετίζεται με τη σύμμορφη ομάδα στην ευκλείδεια εκδοχή της. Επομένως, λόγω της σύμπτωσης αυτής, θα μπορέσουμε να θωρήσουμε ένα μεταθετικό όριο τη μη μεταθετικής θεωρίας βαθμίδας που κατασκευάζουμε και στο όριο αυτό θα συγκρίνουμε τα αποτελέσματά μας με αυτά της σύμμορφης βαρύτητας, όπως περιγράφτηκε στο κεφάλαιο 1.4.

### 1.7.1 Η ομάδα βαθμίδας και η αναπαράστασή της

Σκοπεύουμε να κατασκευάσουμε ένα τετραδιάστατο μη μεταθετικό βαρυτικό μοντέλο ως θεωρία βαθμίδας της ομάδας των συμμετριών ενός τετραδιάστατου συναλλοίωτου ασαφούς χώρου. Ο χώρος που επιλέγουμε είναι ο συναλλοίωτος ασαφής $dS_4$ (όπως περιγράφτηκε στο κεφάλαιο 1.5.3), ο οποίος φέρει τη συμμετρία του μεταθετικού ανάλογού του, δηλαδή την SO(1,4), SO(5) στην ευκλείδεια περίπτωση που χρησιμοποιούμε. Όπως εξηγήσαμε στο 1.5.3, η συμμετρία αυτή πρέπει να επεκταθεί για τη διατήρηση της συναλλοιώτητας. Επομένως, η ομαδα η οποία καταλήγει να φιλοξενεί τους τελεστές των συντεταγμένων είναι η SO(6). Έτσι, κατ' αντιστοιχία με τις περιπτώσεις στις οποίες η βαρύτητα περιγράφεται ως θεωρία βαθμίδας των ομάδων ισομετριών των χώρων στους οποίους κατασκευάζεται, στην περίπτωση αυτή ως ομάδα βαθμίδας επιλέγεται η SO(5) υποομάδα της συνολικής SO(6) στην οποία καταλήξαμε μετά την διεύρυνση της ομάδας ισομετριών.

Παρόλα αυτά, όπως παρατηρήθηκε και στην τρισδιάστατη περίπτωση, η εμπλοκή των αντιμεταθετών στην κατασκευή μη μεταθετικών θεωριών βαθμίδας είναι αναπόφευκτη. Δεδομένου ότι για τυχαία αναπαράσταση των γεννητόρων οι αντιμεταθέτες δεν κλείνουν, επιλέγουμε μια συγκεκριμένη αναπαράσταση και επεκτείνουμε την άλγεβρα συμπεριλαμβάνοντας τους τελεστές που προκύπτουν από τους αντιμεταθέτες ως γεννήτορες. Στην περίπτωσή μας αυτό έχει ως αποτέλεσμα την επέκταση της αρχικής ομάδας βαθμίδας, SO(5) στην SO(6)×U(1), με τους γεννήτορρες να αναπαρίστανται από $4 \times 4$ πίνακες. Συγκεκριμένα, οι πίνακες που αναπαριστούν τους δεκαέξι γεννήτορες κατασκευάζονται σαν συνδυασμοί των τεσσάρων ευκλείδειων $\gamma$-πινάκων οι οποίοι ικανοποιούν την παρακάτω γνωστή αντιμεταθετική σχέση:

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}\mathbf{1} \; , \tag{1.151}$$

όπου $m, n = 1, \ldots 4$. Επίσης, ο πίνακας $\Gamma_5$, ο οποίος ορίζεται ως $\Gamma_5 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4$, πρέπει να συμπεριληφθεί. Επομένως, οι γεννήτορες του SO(6) κομματιού τη ομάδας βαθμίδας είναι:

($i$) Έξι γεννήτορες των στροφών: $\mathrm{M}_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b] = -\frac{i}{2}\Gamma_a\Gamma_b\,, a < b$,

($ii$) Τέσσερις γεννήτορες των σύμμορφων μετασχηματισμών: $\mathrm{K}_a = \frac{1}{2}\Gamma_a$,

($iii$) Τέσσερις γεννήτορες των μεταθέσεων: $\mathrm{P}_a = -\frac{i}{2}\Gamma_a\Gamma_5$,

($iv$) Ένας γεννήτορας των μετασχηματισμών κλίμακας: $\mathrm{D} = -\frac{1}{2}\Gamma_5$

ενώ για το κομμάτι U(1):

($v$) Ο μοναδιαίος πίνακας, $\mathbf{1}$.

Οι $\gamma$-πίνακες χτίζονται από τους πίνακες Pauli:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{1.152}$$

ως τανυστικά γινόμενα των $\sigma_i$:

$$\Gamma_1 = \sigma_1 \otimes \sigma_1, \quad \Gamma_2 = \sigma_1 \otimes \sigma_2, \quad \Gamma_3 = \sigma_1 \otimes \sigma_3 \ , \tag{1.153}$$

$$\Gamma_4 = \sigma_2 \otimes \mathbf{1}, \quad \Gamma_5 = \sigma_3 \otimes \mathbf{1} \ . \tag{1.154}$$

Οι ακριβείς εκφράσεις των γεννητόρων που ορίστηκαν παραπάνω σαν συνδυασμοί των $\Gamma$-πινάκων μπορούν τώρα να γραφτούν σε όρους των πινάκων Pauli. Συγκεκριμένα, οι συνιστώσες του $M_{ab}$ είναι:

$$M_{ij} = -\frac{i}{2}\Gamma_i\Gamma_j = \frac{1}{2}\mathbf{1} \otimes \sigma_k \ , \quad M_{4k} = -\frac{i}{2}\Gamma_4\Gamma_k = -\frac{1}{2}\sigma_3 \otimes \sigma_k \ , \tag{1.155}$$

του $K_a$ είναι:

$$K_i = \frac{1}{2}\Gamma_i \ , \quad K_4 = \frac{1}{2}\Gamma_4 \ , \tag{1.156}$$

του $P_a$ είναι:

$$P_i = -\frac{i}{2}\Gamma_i\Gamma_5 \ , \quad P_4 = -\frac{i}{2}\Gamma_4\Gamma_5 \ , \tag{1.157}$$

όπου $a = i, 4$ και $i, j, k = 1, 2, 3$. Έχοντας εκφράσει τους γεννήτορες σε όρους των πινάκων Pauli, βρίσκουμε τις ακόλουθες μεταθετικές τους σχέσεις:

$$\begin{aligned}
[K_a, K_b] &= iM_{ab} \ , \quad [P_a, P_b] = iM_{ab} \\
[P_a, D] &= iK_a \ , \quad [K_a, P_b] = i\delta_{ab}D \ , \quad [K_a, D] = -iP_a \\
[K_a, M_{bc}] &= i(\delta_{ac}K_b - \delta_{ab}K_c) \\
[P_a, M_{bc}] &= i(\delta_{ac}P_b - \delta_{ab}P_c) \\
[M_{ab}, M_{cd}] &= i(\delta_{ac}M_{bd} + \delta_{bd}M_{ac} - \delta_{bc}M_{ad} - \delta_{ad}M_{bc}) \\
[D, M_{ab}] &= 0 \, .
\end{aligned} \tag{1.158}$$

Επομένως, εφόσον η ομάδα βαθμίδας, οι γεννήτορες και οι μεταθετικές τους σχέσεις έχουν προσδιοριστεί, μπορούμε να συνεχίσουμε με την καθιερωμένη διαδικασία για την κατασκευή της μη μεταθετικής θεωρίας βαθμίδας.

### 1.7.2 Κατασκευή της θεωρίας βαθμίδας

Πρωτίστως, ορίζουμε τη συναλοίωτη συντεταγμένη της θεωρίας, η οποία δίνεται από την παρακάτω έκφραση:

$$\hat{X}_m = X_m \otimes \mathbf{1} + \mathcal{A}_m(X)\,, \tag{1.159}$$

όπου $m = 1, \ldots 4$. Εξ' ορισμού η συντεταγμένη $\hat{X}_m$ μετασχηματίζεται συναλλοίωτα, δηλαδή:

$$\delta\hat{X}_m = i[\epsilon, \hat{X}_m]\,, \tag{1.160}$$

όπου $\epsilon = \epsilon(X)$ είναι η παράμετρος μετασχηματισμού βαθμίδας η οποία είναι συνάρτηση των συντεταγμένων του $dS_4$, οι οποίες είναι $N \times N$ πίνακες, όπου $N$ είναι η διάσταση της αναπαράστασης στην οποία βρίσκονται οι συντεταγμένες. Επίσης, η παράμετρος $\epsilon$ είναι στοιχείο της άλγεβρας SO(6)×U(1), (1.158), η οποία αναπαρίσταται από $4 \times 4$ πίνακες. Για τον λόγο αυτό, μπορούμε να τη γράψουμε σαν ανάπτυγμα πάνω στους δεκαέξι γεννήτορες της άλγεβρας, δηλαδή:

$$\epsilon(X) = \epsilon_0(X) \otimes \mathbf{1} + \xi^a(X) \otimes K_a + \tilde{\epsilon}_0(X) \otimes D + \lambda^{ab}(X) \otimes M_{ab} + \tilde{\xi}^a(X) \otimes P_a\,. \tag{1.161}$$

Κάθε όρος στην παραπάνω έκφραση είναι ένα τανυστικό γινόμενο των $N \times N$ πινάκων (συντεταγμένων) και των $4 \times 4$ πινάκων (γεννητόρων), επομένως, κάθε όρος είναι ένας $4N \times 4N$ πίνακας. Λαμβάνοντας υπόψη ότι οι συντεταγμένες, $X_m$, δεν επηρεάζονται από τον μετασχηματισμό βαθμίδας, δηλαδή $\delta X_m = 0$, βρίσκουμε τον κανόνα μετασχηματισμού της συνοχής βαθμίδας, $\mathcal{A}_m$, η οποία εμφανίστηκε στην εξίσωση (1.160). Η συνοχή βαθμίδας, $\mathcal{A}_m$, είναι μία συνάρτηση των συντεταγμένων-πινάκων, $X_m$, του ασαφούς $dS_4$. Η $\mathcal{A}_m(X)$ παίρνει τιμές στην άλγεβρα SO(6)×U(1) και για τον λόγο αυτόν μπορεί να αναπτυχθεί πάνω στο σύνολο των γεννητόρων, κατά παρόμοιο τρόπο όπως η παράμετρος μετασχηματισμού βαθμίδας, (1.161), δηλαδή:

$$\mathcal{A}_m(X) = e_m{}^a(X) \otimes P_a + \omega_m{}^{ab}(X) \otimes M_{ab}(X) + b_m{}^a(X) \otimes K_a(X) + \tilde{a}_m(X) \otimes D + a_m(X) \otimes \mathbf{1}\,. \tag{1.162}$$

Στην παραπάνω έκφραση, γίνεται κατανοητό ότι έχουμε εισαγάγει ένα πεδίο βαθμίδας για κάθε γεννήτορα. Πλέον, έχοντας προσδιορίσει τη συνοχή βαθμίδας, (1.162), η συναλλοίωτη συντεταγμένη, (1.160) γράφεται ως εξής:

$$\hat{X}_m = X_m \otimes \mathbf{1} + e_m{}^a(X) \otimes P_a + \omega_m{}^{ab}(X) \otimes M_{ab} + b_m{}^a \otimes K_a + \tilde{a}_m \otimes D + a_m \otimes \mathbf{1}\,. \tag{1.163}$$

Επιπλέον, στην SO(6)×U(1) θεωρία βαθμίδας που κατασκευάζουμε, απομένει να καθοριστεί ο τανυστής δύναμης πεδίου. Όπως αναφέραμε στο κεφάλαιο 1.5.3, συγκεκριμένα στην εξίσωση (1.115), ο τανυστής δύναμης πεδίου για τον ασαφή χώρο $dS_4$, δίνεται από την έκφραση:

$$\mathcal{R}_{mn} = [\hat{X}_m, \hat{X}_n] - \frac{i\lambda^2}{\hbar}\hat{\Theta}_{mn}\,. \tag{1.164}$$

Ο παραπάνω τανυστής, $\mathcal{R}_{mn}$, παίρνει τιμές στην άλγεβρα της ομάδας βαθμίδας, επομένως μπορεί να αναπτυχθεί σε όρους των τανυστών καμπυλότητας:

$$\mathcal{R}_{mn}(X) = R_{mn}{}^{ab}(X) \otimes M_{ab} + \tilde{R}_{mn}{}^a(X) \otimes P_a + R_{mn}{}^a(X) \otimes K_a + \tilde{R}_{mn}(X) \otimes D + R_{mn}(X) \otimes \mathbf{1}\,. \tag{1.165}$$

Στην εξίσωση (1.164), υπεισέρχεται το πεδίο 2-μορφή, $\mathcal{B}_{mn}$, για το οποίο συζητήσαμε στην εξίσωση (1.115). Εφόσον το $\mathcal{B}_{mn}$ παίρνει τιμές στην άλγεβα SO(6)×U(1), αυτό σημαίνει ότι μπορεί να αναπτυχθεί στους γεννήτορές της:

$$\mathcal{B}_{mn} = B_{mn} \otimes \mathbf{1} + \tilde{B}_{mn}{}^a \otimes P_a + B_{mn}{}^{ab} \otimes M_{ab} + B_{mn}{}^a \otimes K_a + \tilde{B}_{mn} \otimes D\,, \tag{1.166}$$

το οποίο μετασχηματίζεται συναλλοίωτα:

$$\delta\mathcal{B}_{mn} = i[\epsilon, \hat{\Theta}_{mn}] \tag{1.167}$$

και άρα παίρνουμε τον μετασχηματισμό του $\hat{\Theta}_{mn}$, δηλαδή $\delta\hat{\Theta}_{mn} = i[\epsilon, \hat{\Theta}_{mn}]$. Στο σημείο αυτό έχουμε στην κατοχή μας όλα τα στοιχεία που είναι απαραίτητα για τον καθορισμό των μετασχηματισμών των πεδίων και των εκφράσεων των τανυστών καμπυλότητας. Συνεχίζουμε με την καταγραφή των αποτελεσμάτων μας:

Οι κανόνες των μετασχηματισμών των δεκαέξι πεδίων βαθμίδας είναι:

$$\begin{aligned}
\delta\omega_m^{\ ab} &= -i[X_m, \lambda^{ab}] - i[a_m, \lambda^{ab}] + i[\epsilon_0, \omega_m^{\ ab}] - 2\{\xi^a, b_m^{\ b}\} - \frac{1}{2}\{\lambda^a_{\ c}, \omega_m^{\ bc}\} - \frac{1}{2}\{\tilde{\xi}^a, e_m^{\ b}\} \\
&\quad + i[\xi^c, e_m^{\ d}]\epsilon_{abcd} + \frac{i}{2}[\tilde{\epsilon}_0, \omega_m^{\ cd}]\epsilon_{abcd} + \frac{i}{2}[\lambda^{cd}, \tilde{a}_m]\epsilon_{abcd} - i[\tilde{\xi}^c, b_m^{\ d}]\epsilon_{abcd} \qquad (1.168) \\
\delta e_m^{\ a} &= -i[X_m, \tilde{\xi}^a] - i[a_m, \tilde{\xi}^a] + i[\epsilon_0, e_m^{\ a}] - \{\xi^a, \tilde{a}_m\} + \{\tilde{\epsilon}_0, b_m^{\ a}\} + \frac{1}{4}\{\lambda^a_{\ b}, e_m^{\ b}\} - \frac{1}{4}\{\tilde{\xi}_b, \omega_m^{\ ab}\} \\
&\quad + i[\xi^c, \omega_m^{\ bd}]\epsilon_{abcd} - i[\lambda^{cd}, b_m^{\ b}]\epsilon_{abcd} \qquad (1.169) \\
\delta b_m^{\ a} &= -i[X_m, \xi^a] - i[a_m, \xi^a] + i[\epsilon_0, b_m^{\ a}] - \{\xi_b, \omega_m^{\ ab}\} - 2\{\tilde{\epsilon}_0, e_m^{\ a}\} + \frac{1}{2}\{\lambda^a_{\ b}, b_m^{\ b}\} + \{\tilde{\xi}^a, \tilde{a}_m\} \\
&\quad + i[\lambda^{bc}, e_m^{\ d}]\epsilon_{abcd} + i[\tilde{\xi}^b, \omega_m^{\ cd}]\epsilon_{abcd} \qquad (1.170) \\
\delta a_m &= -i[X_m, \epsilon_0] - i[a_m, \epsilon_0] + i[\xi^a, b_m^{\ a}] + i[\tilde{\epsilon}_0, \tilde{a}_m] + \frac{i}{2}[\lambda_{ab}, \omega_m^{\ ab}] + \frac{i}{2}[\tilde{\xi}_a, e_m^{\ a}] \qquad (1.171) \\
\delta\tilde{a}_m &= -i[X_m, \tilde{\epsilon}_0] - i[a_m, \tilde{\epsilon}_0] + i[\epsilon_0, \tilde{a}_m] + \{\xi_a, e_m^{\ a}\} - \{\tilde{\xi}_a, b_m^{\ a}\} + \frac{i}{2}[\lambda^{ad}, \omega_m^{\ bc}]\epsilon_{abcd}\,. \qquad (1.172)
\end{aligned}$$

Οι κανόνες μετασχημτισμού των πεδίων βαθμίδας 2-μορφή δίνονται ως εξής:

$$\delta B_{mn} = -i[\Theta_{mn}, \epsilon_0] - i[B_{mn}, \epsilon_0] + i[\xi^a, B_{mn}^{\ \ a}] + i[\tilde{\epsilon}_0, \tilde{B}_{mn}] + \frac{i}{2}[\lambda_{ab}, B_{mn}^{\ \ ab}] + \frac{i}{2}[\tilde{\xi}_a, \tilde{B}_{mn}^{\ \ a}] \tag{1.173}$$

$$\delta\tilde{B}_{mn} = -i[\Theta_{mn}, \tilde{\epsilon}_0] - i[B_{mn}, \tilde{\epsilon}_0] + i[\epsilon_0, \tilde{B}_{mn}] + \{\xi_a, \tilde{B}_{mn}^{\ \ a}\} - \{\tilde{\xi}_a, B_{mn}^{\ \ a}\} + \frac{i}{2}[\lambda^{ab}, B_{mn}^{\ \ bc}]\epsilon_{abcd} \tag{1.174}$$

$$\begin{aligned}
\delta\tilde{B}_{mn}^{\ \ a} &= -i[\Theta_{mn}, \tilde{\xi}^a] - i[B_{mn}, \tilde{\xi}^a] + i[\epsilon_0, \tilde{B}_{mn}^{\ \ a}] - \{\xi^a, \tilde{B}_{mn}\} + \{\tilde{\epsilon}_0, B_{mn}^{\ \ a}\} + \frac{1}{4}\{\lambda^a_{\ b}, \tilde{B}_{mn}^{\ \ b}\} \\
&\quad - \frac{1}{4}\{\tilde{\xi}_b, B_{mn}^{\ \ ab}\} + i[\xi^c, B_{mn}^{cd}]\epsilon_{abcd} - i[\lambda^{cd}, B_{mn}^{\ \ b}]\epsilon_{abcd} \qquad (1.175) \\
\delta B_{mn}^{\ \ a} &= -i[\Theta_{mn}, \xi^a] - i[B_{mn}, \xi^a] + i[\epsilon_0, B_{mn}^{\ \ a}] - \{\xi_b, B_{mn}^{\ \ ab}\} - 2\{\tilde{\epsilon}_0, \tilde{B}_{mn}^{\ \ a}\} + \frac{1}{2}\{\lambda^a_{\ b}, B_{mn}^{\ \ b}\} \\
&\quad + \{\tilde{\xi}^a, \tilde{B}_{mn}\} + \frac{i}{2}[\lambda^{bc}, \tilde{B}_{mn}^{\ \ d}]\epsilon_{abcd} + i[\tilde{\xi}^b, B_{mn}^{\ \ cd}]\epsilon_{abcd} \qquad (1.176) \\
\delta B_{mn}^{\ \ ab} &= -i[\Theta_{mn}, \lambda^{ab}] - i[B_{mn}, \lambda^{ab}] + i[\epsilon_0, B_{mn}^{\ \ ab}] - 2\{\xi^a, B_{mn}^{\ \ a}\} - \frac{1}{2}\{\lambda^a_{\ c}, B_{mn}^{\ \ bc}\} - \frac{1}{2}\{\tilde{\xi}^a, \tilde{B}_{mn}^{\ \ b}\} \\
&\quad + i[\xi^c, \tilde{B}_{mn}^{\ \ d}]\epsilon_{abcd} + \frac{i}{2}[\tilde{\epsilon}_0, B_{mn}^{\ \ cd}]\epsilon_{abcd} + \frac{i}{2}[\lambda^{cd}, \tilde{B}_{mn}] - [\tilde{\xi}^c, B_{mn}^{\ \ d}]\epsilon_{abcd}\,. \qquad (1.177)
\end{aligned}$$

Οι εκφράσεις των τανυστών καμπυλότητας είναι:

$$
\begin{aligned}
R_{mn} &= [X_m, a_n] - [X_n, a_m] + [a_m, a_n] + [b_m^a, b_{na}] + [\tilde{a}_m, \tilde{a}_n] \\
&\quad + \frac{1}{2}[\omega_m{}^{ab}, \omega_{nab}] + [e_{ma}, e_n{}^a] - \frac{i\hbar}{\lambda^2} B_{mn} \\
\tilde{R}_{mn} &= [X_m, \tilde{a}_n] + [a_m, \tilde{a}_n] - [X_n, \tilde{a}_m] - [a_n, \tilde{a}_m] - i\{b_{ma}, e_n{}^a\} + i\{b_{na}, e_m{}^a\} \\
&\quad + \frac{1}{2}\epsilon_{abcd}[\omega_m{}^{ab}, \omega_n{}^{cd}] - \frac{i\hbar}{\lambda^2} \tilde{B}_{mn} \\
R_{mn}{}^a &= [X_m, b_n^a] + [a_m, b_n^a] - [X_n, b_m^a] - [a_n, b_m^a] + i\{b_{mb}, \omega_m{}^{ab}\} - i\{b_{nb}, \omega_m{}^{ab}\} \\
&\quad + i\{\tilde{a}_m, e_n{}^a\} - i\{\tilde{a}_n, e_m^a\} + \epsilon_{abcd}([e_m^b, \omega_n{}^{cd}] - [e_n{}^b, \omega_m{}^{cd}]) - \frac{i\hbar}{\lambda^2} B_{mn}{}^a \\
\tilde{R}_{mn}{}^a &= [X_m, e_n{}^a] + [a_m, e_n{}^a] - [X_n, e_m^a] - [a_n, e_m^a] + i\{b_m^a, \tilde{a}_n\} - i\{b_n^a, \tilde{a}_m\} \\
&\quad - ([b_m^b, \omega_n{}^{cd}] - [b_n^b, \omega_m{}^{cd}])\epsilon_{abcd} - i\{\omega_m{}^{ab}, e_{nb}\} + i\{\omega_n{}^{ab}, e_{mb}\} - \frac{i\hbar}{\lambda^2} \tilde{B}_{mn}{}^a \\
R_{mn}{}^{ab} &= [X_m, \omega_n{}^{ab}] + [a_m, \omega_n{}^{ab}] - [X_n, \omega_m{}^{ab}] - [a_n, \omega_m{}^{ab}] + 2i\{b_m^a, b_n^b\} + ([b_m^c, e_n^d] - [b_n^c, e_m^d])\epsilon_{abcd} \\
&\quad + \frac{1}{2}([\tilde{a}_m, \omega_n{}^{cd}] - [\tilde{a}_n, \omega_m{}^{cd}])\epsilon_{abcd} + 2i\{\omega_m{}^{ac}, \omega_n{}^b{}_c\} + 2i\{e_m^a, e_n^b\} - \frac{i\hbar}{\lambda^2} B_{mn}{}^{ab}
\end{aligned}
\tag{1.178}
$$

Οι παραπάνω εκφράσεις των τανυστών καμπυλότητας στρώνουν τον δρόμο για τον ορισμό της δράσης της θεωρίας. Πριν προχωρήσουμε στο κομμάτι της δράσης, αξίζει να σημειώσουμε ότι τα παραπάνω αποτελέσματα (1.172) και (1.178) ανάγονται στα αντίστοιχα αποτελέσματα της σύμμορφης βαρύτητας ως θεωρίας βαθμίδας η οποία περιγράφηκε στο κεφάλαιο 1.4, συγκεκριμένα στις εξισώσεις (1.58) και (1.61)-(1.66), αντίστοιχα, έπειτα από τη θεώρηση του μεταθετικού ορίου. Στο όριο αυτό, το U(1) πεδίο βαθμίδας που συνδέεται με τη μη μεταθετικότητα του χώρου, αποσυζεύγνυται, επομένως η θεωρία βαθμίδας στο μεταθετικό όριο είναι η SO(6), δηλαδή η ομάδα που χρησιμοποιήθηκε για την περιγραφή της σύμμορφης βαρύτητας (στην ευκλείδεια εκδοχή της).

## 1.8 Οι σύνδεσμοι για την παραβίαση της συμμετρίας

Για το δυναμικό κομμάτι της θεωρίας, θα ήταν αναμενόμενο να γραφτεί σε όρους των τανυστών καμπυλότητας, οι οποίοι δίνονται στην εξίσωση (1.178). Απευθείας θεώρηση μίας Yang-Mills δράσης θα περιέγραφε τη θεωρία, αναλλοίωτη κάτω από την SO(6)×U(1) συμμετερία βαθμίδας. Ωστόσο, η συμμετρία βαθμίδας της δράσης με την οποία επιθυμούμε να καταλήξουμε εκφράζεται μέσω της συμμετρίας Lorentz. (στην ευκλείδεια εκδοχή της), επομένως, πρέπει να ελαττώσουμε την πλεονάζουσα συμμετρία. Όπως συζητήθηκε στο κεφάλαιο 1.4 για την περίπτωση της τετραδιάστατης σύμμορφης βαρύτητας, η συμμετρία ελαττώθηκε με την επιβολή συγκεκριμένων συνθηκών, καταλήγωντας σε μία δράση με συμμετρία Weyl. Επίσης, στο ίδιο κεφάλαιο συζητήσαμε ένα εναλλακτικό σενάριο παραβίασης της συμμετρίας με κατάληξη αυτήν τη φορά μία δράση η οποία να σέβεται τη Lorentz συμμετρία, έπειτα από την εισαγωγή δύο βαθμωτών πεδίων στη θεωρία. Δεδομένου ότι εμείς δεν επιθυμούμε να εισαγάγουμε περισσότερα πεδία στη θεωρία, προχωράμε με την επιβολή συγκεκριμένων συνδέσμων, οι οποίοι θέλουμε να μας οδηγήσουν σε μία Lorentz-αναλλοίωτη τελική δράση.

Ο πιο ευθύς τρόπος είναι να πραγματοποιήσουμε την παραπάνω παραβίαση της συμμετρίας είναι να θεωρήσουμε μία περιορισμένη θεωρία, στην οποία οι συνιστώντες τανυστές καμπυλότητας είναι όλοι μηδέν, εκτός από αυτούς που σχετίζονται με την επιθυμητή υπολοιπόμενη συμμετρία.

Οπότε, μιας και θέλουμε να σπάσει η αρχική συμμετρία, SO(6)×U(1) και να καταλήξουμε με μία τελική συμμετρία SO(4)×U(1), οι μόνοι μη μηδενικοί τανυστές θα έπρεπε να είναι οι $R_{mn}^{\ \ ab}(M)$ και $R_{mn}(\mathbf{1})$. Ωστόσο, η προσέγγιση αυτή οδηγεί σε μία υπέρ-περιορισμένη θεωρία, πράγμα το οποίο γίνεται κατανοητό από την καταμέτρηση των βαθμών ελευθερίας που επιζούν από την παραβίαση της συμμετρίας. Επομένως, κρίνεται σωστό να επιβάλουμε μη τετριμμένους συνδέσμους, εξασφαλίζοντας τον σωστό αριθμό των βαθμών ελευθερίας. Η πρώτη συνθήκη είναι:

$$\tilde{R}_{mn}^{\ \ a}(P) = 0 \ , \tag{1.179}$$

δηλαδή η συνθήκη μηδενικής στρέψης, η οποία είναι αναμενόμενη. Επιπλέον, η πιθανή ερμηνεία του $b_m^{\ a}$ ως δεύτερου vielbein της θεωρίας οδηγεί στην πιθανότητα να ερμηνευτεί ως διμετρική, στην οποία όμως δε στοχεύουμε. Επομένως, οδηγούμαστε να λύσουμε τον σύνδεσμο (1.179), θεωρώντας $e_m^{\ a} = b_m^{\ a}$, καταλήγωντας εν τέλει σε μία έκφραση για το spin connection, $\omega_m^{\ ab}$ συναρτήσει των πεδίων $e_m^{\ a}, a_m, \tilde{a}_m$. Για να αποκτήσουμε την ακριβή έκφραση του $\omega_m^{\ ab}$, από την επίλυση του συνδέσμου, (1.179), κάνουμε χρήση των παρακάτω ταυτοτήτων:

$$\delta_{fgh}^{abc} = \epsilon^{abcd}\epsilon_{fghd} \qquad \text{και} \qquad \frac{1}{3!}\delta_{fgh}^{abc}a^{fgh} = a^{[fgh]} \ . \tag{1.180}$$

Επομένως, η συνθήκη (1.179) παίρνει την ακόλουθη μορφή:

$$\epsilon^{abcd}[e_{mb}, \omega_{ncd}] - i\{\omega_m^{\ ab}, e_{nb}\} = -[D_m, e_n^{\ a}] - i\{e_m^{\ a}, \tilde{a}_n\} \ , \tag{1.181}$$

όπου $D_m = X_m + a_m$, δηλαδή, η συναλλοίωτη συντεταγμένη μιας αβελιανής θεωρίας βαθμίδας. Η παραπάνω εξίσωση γράφεται ως:

$$\epsilon^{abcd}[e_m^{\ b}, \omega_n^{\ cd}] = -[D_m, e_n^{\ a}] \qquad \text{και} \qquad \{\omega_m^{\ ab}, e_{nb}\} = \{e_m^{\ a}, \tilde{a}_n\} \ . \tag{1.182}$$

Κάνοντας χρήση των ταυτοτήτων (1.180), οι παραπάνω εξισώσεις οδηγούν στην επιθυμητή σχέση:

$$\omega_n^{\ ac} = -\frac{3}{4}e^m_{\ b}(-\epsilon^{abcd}[D_m, e_{nd}] + \delta^{[bc}\{e_n^{\ a]}, \tilde{a}_m\}) \ . \tag{1.183}$$

Σύμφωνα με την αναφορά [110], ο μηδενισμός του τανυστή δύναμης πεδίου μίας θεωρίας βαθμίδας θα μπορούσε να επαγάγει τον μηδενισμό του αντίστοιχου πεδίου βαθμίδας. Αν το επιχείρημα αυτό ήταν εφαρμόσιμο στην περίπτωσή μας, θα απλοποιούσε τις εκφράσεις των τανυστών καμπυλότητας κι επομένως αυτήν της δράσης. Όμως, αυτό δεν μπορεί να χρησιμοποιηθεί στη δική μας περίπτωση καθώς η ταυτοποίηση του vielbein ως πεδίου βαθμίδας της θεωρίας υπονοεί την ανάμιξη των εσωτερικών συμμετριών με τις χωροχρονικές. Επομένως, δεδομένου ότι το vielbein θεωρείται ότι είναι αντιστρέψιμο σε όλον τον χώρο, η υιοθέτηση του παραπάνω επιχειρήματος (μηδενισμός του vielbein) θα οδηγούσε σε εκφυλισμένο πίνακα vielbein και τελικά σε εκφυλισμένο μετρικό τανυστή [10]. Ωστόσο, θα μπορούσαμε να θέσουμε $\tilde{a}_m = 0$, εφόσον δεν επιδέχεται γεωμετρικής ερμηνείας. Ο προσδιορισμός αυτός του πεδίου βαθμίδας $\tilde{a}_m$ θα τροποποιήσει επίσης την έκφραση του spin connection σε όρους των υπόλοιπων πεδίων, (1.183), καταλήγοντας σε μία ακόμα απλούστερη τελική έκφραση για το spin connection ως προς το U(1) πεδίο βαθμίδας, $a_m$ και το vielbein:

$$\omega_n^{\ ac} = \frac{3}{4}e^m_{\ b}\epsilon^{abcd}[D_m, e_{nd}] \ . \tag{1.184}$$

Ένας εναλλακτικός τρόπος να καταλήξουμε με την επιθυμητή SO(4) συμμετρία μετά την παραβίαση της SO(6), είναι να προβάλουμε το επιχείρημα που αναπτύξαμε στην περίπτωση της σύμμορφης βαρύτητας, στην παρούσα μη μεταθετική περίπτωση, δηλαδή να συμπεριλάβουμε δύο βαθμωτά πεδία στη θεμελιώδη αναπαράσταση της SO(6) επάγοντας μία αυθόρμητη παραβίαση της συμμετρίας. Είμαστε πεπεισμένοι ότι αυτός ο τρόπος παραβίασης της συμμετρίας θα οδηγήσει σε συνδέσμους ισοδύναμους με αυτούς που θεωρήσαμε.

## 1.9 Η δράση

Αφού λοιπόν επιβάλαμε τους συνδέσμους για την παραβίαση της συμμετρίας, είναι σκόπιμο να γράψουμε τις εκφράσεις των τανυστών καμπυλότητας λαμβάνοντάς τους υπόψη, αφού αυτές θα είναι οι εκφράσεις οι οποίες θα χρησιμοποιηθούν στη δράση της θεωρίας. Οι ακριβείς εκφράσεις των επιζώντων τανυστών καμπυλότητας, μετά τις θεωρήσεις $e_\mu{}^a = b_\mu{}^a$ και $\tilde{a}_\mu = 0$, είναι:

$$
\begin{aligned}
R_{mn} &= [X_m, a_n] - [X_n, a_m] + 2[e_\mu{}^a, e_{\nu a}] + \tfrac{1}{2}[\omega_m{}^{ab}, \omega_{\nu ab}] - \tfrac{i\hbar}{\lambda^2} B_{mn}\ , && (1.185)\\
\tilde{R}_{mn} &= \tfrac{1}{2}\epsilon_{abcd}[\omega_m{}^{ab}, \omega_\nu{}^{cd}] - \tfrac{i\hbar}{\lambda^2} B_{mn}\ , && (1.186)\\
R_{mn}{}^a &= [X_m, e_n^a] + [a_m, e_n^a] - [X_n, e_m^a] - [a_n, e_m^a] + i\{e_{mb}, \omega_m{}^{ab}\} - i\{e_{nb}, \omega_m{}^{ab}\}\\
&\quad + \epsilon_{abcd}([e_m{}^b, \omega_n{}^{cd}] - [e_n{}^b, \omega_m{}^{cd}]) - \tfrac{i\hbar}{\lambda^2} B_{mn}{}^a\ , && (1.187)\\
R_{mn}{}^{ab} &= [X_m, \omega_n{}^{ab}] + [a_m, \omega_n{}^{ab}] - [X_n, \omega_m{}^{ab}] - [a_n, \omega_m{}^{ab}] + 4i\{e_m{}^a, e_n{}^b\} + 2i\{\omega_m{}^{ac}, \omega_n{}^b{}_c\} - \tfrac{i\hbar}{\lambda^2} B_{mn}{}^{ab}\ . && (1.188)
\end{aligned}
$$

Οι παραπάνω εκφράσεις στις οποίες το $\omega_m^{ab}$ αντικαθίσταται από την έκφραση (1.184), είναι οι τελικές εκφράσεις των τανυστών έπειτα από την παραβίαση της συμμετρίας. Πριν προχωρήσουμε με τον προσδιορισμό της δράσης της βαρυτικής θεωρίας, ας σχολιάσουμε εν συντομία τη δράση του έξτρα πεδίου 2-μορφής, $\mathcal{B}_{mn}$, η οποία θα συμπεριληφθεί στην τελική δράση.

Έστω ο τανυστής δύναμης πεδίου, $\hat{\mathcal{H}}_{mnp}$, του 2-μορφής πεδίου βαθμίδας:

$$\hat{\mathcal{H}}_{mnp} = \frac{1}{3}\left([\hat{X}_m, \hat{\Theta}_{np}] + [\hat{X}_n, \hat{\Theta}_{pm}] + [\hat{X}_p, \hat{\Theta}_{mn}]\right)\ . \tag{1.189}$$

Ο παραπάνω τανυστής δύναμης πεδίου μεταχηματίζεται συναλλοίωτα κάτω από έναν μετασχηματισμό βαθμίδας. Πράγματι, αυτό μπορεί να αποδειχθεί ξεκινώντας από την έκφραση του μετασχηματισμού του πεδίου:

$$\delta\hat{\mathcal{H}}_{mnp} = \frac{1}{3}\left([\delta\hat{X}_m, \hat{\Theta}_{np}] + [\hat{X}_m, \delta\hat{\Theta}_{np}] + [\delta\hat{X}_n, \hat{\Theta}_{pm}] + [\hat{X}_n, \delta\hat{\Theta}_{pm}] + [\delta\hat{X}_p, \hat{\Theta}_{mn}] + [\hat{X}_p, \delta\hat{\Theta}_{mn}]\right)\ . \tag{1.190}$$

Χρησιμοποιώντας τους κανόνες μετασχηματισμού των $\hat{X}_m$ και $\hat{\Theta}_{mn}$, οι οποίοι δίνονται στις εξισώσεις (1.160) και (1.167), αντίστοιχα, μαζί με την ταυτότητα Jacobi, βρίσκουμε τον παρακάτω κανόνα μετασχηματισμού:

$$\delta\hat{\mathcal{H}}_{mnp} = i[\epsilon, \hat{\mathcal{H}}_m]\ , \tag{1.191}$$

ο οποίος είναι συναλλοίωτος. Κατά τα γνωστά, για να βρεθούν οι ακριβείς εκφράσεις των τανυστών, αναπτύσσουμε τον $\hat{\mathcal{H}}$ στο σύνολο των γεννητόρων της άλγεβρας:

$$\hat{\mathcal{H}}_{mnp} = H_{mnp} \otimes \mathbf{1} + \tilde{H}_{mnp}{}^a \otimes P_a + H_{mnp}{}^{ab} \otimes M_{ab} + H_{mnp}{}^a \otimes K_a + \tilde{H}_{mnp} \otimes D\,, \tag{1.192}$$

και υπολογίζουμε τον κάθε συνιστώντα τανυστή χρησιμοποιώντας τον ορισμό του τανυστή δύναμης πεδίου, $\hat{\mathcal{H}}_{mnp}$. Επομένως, όσον αφορά στη δράση του πεδίου 2-μορφής, θα συμπεριλαμβάνεται μόνο ο κινητικός όρος, ο οποίος είναι:

$$\mathcal{S}_{\mathcal{B}} = \mathrm{Tr}\,\mathrm{tr}\,\hat{\mathcal{H}}_{mnp}\hat{\mathcal{H}}^{mnp}\ . \tag{1.193}$$

Ας επιστρέψουμε τώρα στον προσδιορισμό της συνολικής δράσης της θεωρίας βαθμίδας. Η πιο λογική επιλογή είναι η δράση να είναι τύπου Yang-Mills, επομένως θα γράφεται ως εξής:

$$\mathcal{S} = \mathrm{Tr}\,\mathrm{tr}\,\Gamma_5\left(\mathcal{R}_{mn}\mathcal{R}_{rs}\epsilon^{mnrs} + \hat{\mathcal{H}}_{mnp}\hat{\mathcal{H}}^{mnp}\right)\ , \tag{1.194}$$

όπου $\mathrm{Tr}$ είναι το ίχνος πάνω στους πίνακες που αναπαριστούν τις συντεταγμένες, ενώ $\mathrm{tr}$ είναι το ίχνος πάνω στους γεννήτορες της άλγεβρας. Αξίζει να σημειωθεί ότι η παραπάνω δράση είναι αναλλοίωτη κάτω από τους μετασχηματισμούς βαθμίδας, εφόσον ο τανυστής δύναμης πεδίου μετασχηματίζεται με συναλλοίωτο τρόπο:

$$\delta S = \mathrm{Tr}\Gamma_5(\delta\hat{\mathcal{R}}\hat{\mathcal{R}} + \hat{\mathcal{R}}\delta\hat{\mathcal{R}} + \delta\hat{\mathcal{H}}\mathcal{H} + \mathcal{H}\delta\hat{\mathcal{H}}) = \mathrm{Tr}(i[\epsilon, \mathcal{R}]\mathcal{R} + i\mathcal{R}[\epsilon, \mathcal{R}] + i[\epsilon, \hat{\mathcal{H}}]\hat{\mathcal{H}} + i\hat{\mathcal{H}}[\epsilon, \hat{\mathcal{H}}]) = 0 \,, \tag{1.195}$$

όπου χρησιμοποιήθηκαν οι εξισώσεις (1.117), (1.191) και η ιδιότητα της κυκλικότητας του ίχνους. Επίσης, ο πρώτος όρος της παραπάνω δράσης, (1.194), συμπεριλαμβάνει τον τανυστή δύναμης πεδίου της θεωρίας βαθμίδας, ενώ ο δεύτερος είναι ο (μη τοπολογικός) κινητικός όρος του πεδίου 2-μορφή. Ο τελεστής $\Gamma_5$ έχει συμπεριληφθεί προκειμένου να φιλτραριστούν οι περισσότεροι όροι και, για το SO(4)×U(1) κομμάτι, να διατηρηθεί ο όρος που περιλαμβάνει τον τανυστή καμπυλότητας $R_{mn}^{\ \ ab}$. Επομένως, η δράση (1.194) γίνεται:

$$\mathcal{S} = 2\mathrm{Tr}(R_{mn}^{\ \ ab}R_{rs}^{\ \ cd}\epsilon_{abcd}\epsilon^{mnrs} + 4\tilde{R}_{mn}R_{rs}\epsilon^{mnrs} + \frac{1}{3}H_{mnp}^{\ \ \ ab}H^{mnpcd}\epsilon_{abcd} + \frac{4}{3}\tilde{H}_{mnp}H^{mnp}) \,. \tag{1.196}$$

Αντικαθιστώντας τις εκφράσεις των τανυστών καμπυλότητας των εξισώσεων (1.185)-(1.188) και εκφράζοντας το $\omega$ σε όρους των υπόλοιπων πεδίων, (1.184), τότε μεταβολή της δράσης ως προς τα πεδία βαθμίδας θα οδηγούσε στις εξισώσεις κίνησης. Αξίζει να σημειώσουμε ότι το έξτρα πεδίο 2-μορφής αποσυζεύγνυται κατά τη θεώρηση του μεταθετικού ορίου, συνεπώς δεν αναμένεται να παρατηρηθεί στις χαμηλές ενέργειες.

## 1.10 Συμπεράσματα

Η διατριβή αυτή αποτελείται από τις πιο προσφατες δουλειές μας στις οποίες ασχοληθήκαμε με τον συνδυασμο δύο διαφορετικών πλαισίων εργασίας. Το πρώτο είναι η περιγραφή διάφορων βαρυτικών θεωριών ως θεωριών βαθμίδας και το δεύτερο είναι η μη μεταθετική γεωμετρία. Ο συγκερασμός των δύο παραπάνω πλαισίων επιτυγχάνεται μέσω της ύπαρξης καλά θεμελιωμένων θεωριών βαθμίδας στους μη μεταθετικούς χώρους.

Συγκεκριμένα, κατά τη δική μας άποψη, η δουλειά που αφορά στο τετραδιάστατο βαρυτικό μοντέλο το οποίο κατασκευάσαμε, συνεισφέρει κυρίως σε δύο κατευθύνσεις. Η πρώτη είναι ότι δίνουμε μία επιτυχή κατασκευή ενός τετραδιάστατου συναλλοίωτου ασαφούς χώρου με τρόπο ο οποίος μπορεί να γενικευτεί και σε άλλες περιπτώσεις. Δεύτερον, δώσαμε μία περιγραφή για την βαρυτική αλληλεπίδραση σε καταστάσεις στις οποίες η μη μεταθετικότητα του χώρου μπορεί να δικαιολογηθεί (π.χ. κλίμακα Planck) και καταφέραμε να τη συνδέσουμε με τη σύμμορφη βαρύτητα στο μεταθετικό όριο. Προτεραιότητά μας στο μέλλον είναι να μελετήσουμε την Lorentz αναλλοίωτη δράση στην οποία καταλήξαμε και να προσπαθήσουμε να την συσχετίσουμε με την τετραδιάστατη δράση Einstein-Hilbert.

# Chapter 2

# Gravity as a gauge theory in three and four dimensions

In this chapter we briefly review the correspondence of three-dimensional and four-dimensional gravity theories to gauge theories. We start with recalling the non-coordinate basis and the vielbein formalism, setting up a framework which is independent of the metric tensor. Considering the appropriate symmetry groups for each case taking over the role of the gauge groups, the vielbein and the spin connection are identified as the gauge fields of the theory. Then, the standard procedure for the construction of gauge theories is followed, but also non-trivial techniques are used depending on the particularity of each case.

## 2.1 The non-coordinate basis and vierbein formalism of General Relativity

### 2.1.1 The non-coordinate basis and definition of the vierbein field

The theory of General Relativity (GR) can be reformulated employing the vierbein formalism [82]. In the conventional, coordinate-based, formulation of GR, at any point $p$ of a manifold, there exists a tangent space $T_p$, on which any four-vector $V \in T_p$ can be spanned on a differential basis given by the partial derivatives of the coordinates of $p$, $\hat{e}_\mu = \partial_\mu$. Also, at the point $p$, a cotangent space $T_p^*$ (dual space of $T_p$) is defined and is spanned by the differential forms, $\hat{e}^\mu = dx^\mu$. The above two bases satisfy the relation $\hat{e}^\mu \otimes \hat{e}_\nu = \mathbf{1}^\mu_\nu$.

The freedom of decomposing a vector $V \in T_p$ on any orthonormal basis of $T_p$ leads to the choice of the unit vectors to be independent of the coordinates (tetrad basis), $\hat{e}_a$, with inner product $(\hat{e}_a, \hat{e}_b) = \eta_{ab}$, where $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ is the metric of the Minkowski spacetime.[1] Therefore, an arbitrary set of coordinate-depending vectors, $\hat{e}_\mu(x)$ of the tangent space, $T_p$, can be expressed in terms of the non-coordinate basis, $\hat{e}_a$, as:

$$\hat{e}_\mu(x) = e_\mu^{\ a}(x)\hat{e}_a \ , \tag{2.1}$$

where $e_\mu^{\ a}(x)$ are the components of a 4×4 invertible matrix incorporating all the coordinate information. The whole transformation matrix $e_\mu^{\ a}(x)$ is called *vierbein*. Its components, along

[1] In case the manifold is Riemannian, the inner product of a vector and its dual is $(\hat{e}_a, \hat{e}_b) = \delta_{ab}$. The case studied in the current subsection the manifold is considered to be pseudo-Riemannian.

with the ones of the inverse vierbein field, $e^{\mu}{}_{a}(x)$, satisfy the orthonormality conditions:

$$e^{\mu}{}_{a}(x)e_{\nu}{}^{a}(x) = \delta^{\mu}_{\nu} \ , \qquad e_{\mu}{}^{a}(x)e^{\mu}{}_{b}(x) = \delta^{a}_{b} \ . \tag{2.2}$$

The inner product of the vierbein and the inverse vierbein is given employing the metric tensor, $g_{\mu\nu}$:

$$g_{\mu\nu}(x)e^{\mu}{}_{a}(x)e^{\nu}{}_{b}(x) = \eta_{ab} \ , \quad \text{or} \quad g_{\mu\nu}(x) = e_{\mu}{}^{a}(x)e_{\nu}{}^{b}(x)\eta_{ab} \ , \tag{2.3}$$

where, from the second branch of the above equation, the vierbein field admits the interpretation of the "square root" of the metric. Therefore, in the same sense that a metric tensor and its inverse are used for lowering or raising spacetime tensor indices, the vierbein and its inverse are used for switching manifold (greek) indices to Lorentz (latin) ones and vice versa, respectively. Also, greek and latin indices of the vierbein itself are raised and lowered using the metric tensor, $g_{\mu\nu}$, of the manifold and the Minkowski flat metric, $\eta_{ab}$, respectively.

### 2.1.2 The vierbein formalism and Palatini action

It is known that two vectors defined on a flat space are identical if they have the same magnitude and direction. Therefore, it is understood that if we consider a vector of the space and translate it to a different position, we end up with a vector that shares the same magnitude and direction with the initial one. However, when a vector field is considered on a curved (Riemannian) manifold, the presence of curvature will distort the parallel transport and give a vector of different direction and magnitude. The difference between the initial vector and the one transported is a vector that includes a multiplicative factor, $\Gamma^{\lambda}_{\mu\nu}$, that is called the *affine connection*. It can be easily proven that the above correction to the distortion caused by the curvature of the manifold is included in the definition of the covariant derivative of a tensor, $\nabla_{\mu}T^{\nu\rho}$. For example, the covariant derivative of a vector is:

$$\nabla_{\lambda}V^{\mu} = \partial_{\lambda}V^{\mu} + \Gamma^{\mu}_{\nu\lambda}V^{\nu} \ . \tag{2.4}$$

For a higher rank tensor, there would be terms similar to the last one in the above equation, for every index of the tensor.

The expression of the covariant derivative, involving the affine connection, both defined above, is the one that applies in case the tensors carry exclusively manifold indices. When a tensor bears Lorentz indices (non-coordinate basis), the correction to the partial derivative is not expressed by the affine connection, but from its non-coordinate analogue, that is the *spin connection*. Accordingly, each latin index of the tensor admits a correction term, that is the tensor contracted with the spin connection, as follows:

$$\nabla_{\mu}T^{a}_{\ b} = \partial_{\mu}T^{a}_{\ b} + \omega_{\mu}{}^{a}{}_{c}T^{c}_{\ b} - \omega_{\mu}{}^{c}{}_{b}T^{a}_{\ c} \ . \tag{2.5}$$

Therefore, depending on the basis (coordinate or non-coordinate) on which the tensor is expressed, one has to employ the covariant derivative with the appropriate connection.

Now, if one considers a covariant derivative, $D_{\mu}$, that acts on an object that carries both spacetime and Lorentz indices (e.g. on the vierbein), one would obtain a mixed expression with corrections of both kinds of covariant derivatives, including both the affine and spin connections:

$$D_{\mu}T_{b\nu} = \partial_{\mu}T_{b\nu} - \Gamma^{\lambda}_{\mu\nu}T_{b\lambda} - \omega_{\mu}{}^{c}{}_{b}T_{\nu c} \ . \tag{2.6}$$

The relation between the affine and the spin connection is obtained after considering the covariant derivative of a vector, $V^\nu$, in the coordinate and the non-coordinate bases:

$$\nabla V = \nabla_\mu V^\nu dx^\mu \otimes \partial_\nu = (\partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda) dx^\mu \otimes \partial_\nu \ , \tag{2.7}$$

$$\nabla V = (\nabla_\mu V^a) dx^\mu \otimes \hat{e}_a = (\partial_\mu V^a + \omega_\mu{}^a{}_b V^b) dx^\mu \otimes \hat{e}_a \ . \tag{2.8}$$

Converting the second equation from the above two into the coordinate basis and equating it with the first one, one obtains the relation of the connections:

$$\omega_\mu{}^a{}_b = e_\nu{}^a e^\lambda_b \Gamma^\nu_{\mu\lambda} - e^\lambda_b \partial_\mu e_\lambda{}^a = e^{\nu a} \nabla_\mu e_{b\nu} \ . \tag{2.9}$$

The above expression can be rewritten as:

$$\partial_\mu e_\nu{}^a - e_\sigma{}^a \Gamma^\sigma_{\mu\nu} + \omega_\mu{}^a{}_b e_\nu{}^b = 0 \ \Rightarrow \ D_\mu e_\nu{}^a = 0 \ . \tag{2.10}$$

The last equation obtained above, $D_\mu e_\nu{}^a = 0$, is the well-known *tetrad postulate*, which is often considered axiomatically. Also, if the affine connection is considered as *metric compatible*, i.e. the covariant derivative of the metric is everywhere vanishing, $\nabla_\lambda g_{\mu\nu} = 0$, starting from the relation $\nabla_\mu \eta_{ab} = 0$ and taking into consideration the (2.3) and (2.9), one concludes to $\omega_{\mu ab} = -\omega_{\mu ba}$, that is the antisymmetricity property of the spin connection with respect to the two Lorentz indices.

It can be easily proved that the affine connection, $\Gamma^\lambda_{\mu\nu}$, does not transform as a tensor under the general coordinate transformations, due to an extra term. Therefore, if the connection $\Gamma^\lambda_{\nu\mu}$ is considered, in which the lower indices of $\Gamma^\lambda_{\mu\nu}$ have been interchanged, it is found that the term that renders it as a non-tensor object, coincides with the corresponding one of $\Gamma^\lambda_{\mu\nu}$. Thus, subtracting the above two connections, the part of the transformation that renders each one of them as non-tensor objects is eliminated and therefore their difference produces a tensor that is called the *torsion tensor*. Specifically:

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \ . \tag{2.11}$$

From the above definition, it is obvious that the torsion tensor is antisymmetric in its lower indices. Therefore, if the connection is symmetric, $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, the torsion tensor is identically equal to zero and the connection is called *torsion-free*.

Now, let a manifold endowed with a metric, on which an affine connection is defined. If the properties of the metric compatibility and the symmetricity of the connection (torsion-free connection) are taken into account, then it is proved, in a very straightforward way, both the existence and uniqueness of the affine connection for the specific manifold by obtaining an expression of the connection depending on the metric of the manifold [83]. Specifically:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \ . \tag{2.12}$$

The above expression of the affine connection is the one used in general relativity and it is known as the *Levi-Civita* connection or *Christoffel symbols*.

Returning to the vierbein formalism, in which the notation of differential forms is adopted, the expression of the antisymmetric torsion tensor (torsion two-form) can be easily obtained in terms of the vierbein and spin connection, starting from the tetrad postulate:

$$D_\mu e_\nu{}^b = \partial_\mu e_\nu{}^b - \Gamma^\lambda_{\mu\nu} e_\lambda{}^b + \omega_\mu{}^b{}_c e_\nu{}^c = 0 \ , \tag{2.13}$$

$$D_\nu e_\mu{}^b = \partial_\nu e_\mu{}^b - \Gamma^\lambda_{\nu\mu} e_\lambda{}^b + \omega_\nu{}^b{}_c e_\mu{}^c = 0 \ . \tag{2.14}$$

Subtracting the above two equations and making use of the torsion tensor definition, (2.11), one obtains:

$$T^{\lambda}_{\mu\nu} = e^{\lambda}{}_{a}T_{\mu\nu}{}^{a} = \partial_{\mu}e_{\nu}{}^{a} - \partial_{\nu}e_{\mu}{}^{a} + \omega_{\mu}{}^{a}{}_{b}e_{\nu}{}^{c} - \omega_{\nu}{}^{a}{}_{b}e_{\mu}{}^{c} \ . \tag{2.15}$$

Including explicitly the form notation:

$$e^{a} \equiv e_{\mu}{}^{a}dx^{\mu} \ , \quad \omega^{a}_{\ b} \equiv \omega_{\mu}{}^{a}{}_{b}dx^{\mu} \ , \quad T^{a} \equiv \frac{1}{2}T_{\mu\nu}{}^{a}dx^{\mu} \wedge dx^{\nu} \ , \tag{2.16}$$

one obtains the compact expression of the torsion two-form in terms of the vierbein and spin connection one-forms:

$$T^{a} = de^{a} + \omega^{a}_{\ b} \wedge e^{b} \ . \tag{2.17}$$

The notion of curvature, which, as we mentioned in the beginning of the section, is responsible for the mismatch in the parallel transport and consequently for the introduction of the affine connection, is parametrized by the *Riemann tensor*, $R^{\rho}{}_{\sigma\mu\nu}$. Its expression with respect to the affine connection is obtained after considering the commutator of two covariant derivatives, which measures the difference between parallel transporting a vector (or a tensor in general) first one way and then the other, minus the opposite ordering [83]. Calculations lead to the expression:

$$[\nabla_{\mu}, \nabla_{\nu}]V^{\rho} = (\partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma})V^{\sigma} - (\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu})\nabla_{\lambda}V^{\rho} \ . \tag{2.18}$$

Apparently, the last term is the torsion tensor, (2.11), while the first one is identified as the Riemann tensor:

$$[\nabla_{\mu}, \nabla_{\nu}]V^{\rho} = R^{\rho}{}_{\sigma\mu\nu}V^{\sigma} - T^{\lambda}_{\mu\nu}\nabla_{\lambda}V^{\rho} \ . \tag{2.19}$$

Therefore, the expression of the Riemann tensor is:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \ . \tag{2.20}$$

By definition, the above tensor is antisymmetric in $\mu \leftrightarrow \nu$ and depends exclusively on the affine connection and its derivatives, meaning that it is a general, metric-independent, expression without any extra properties (metric compatibility) considered yet. The Riemann tensor has a number of properties (Bianchi identity, antisymmetricity) that reduces the number of its independent components. Also, consideration of the following trace:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \ , \tag{2.21}$$

that is the contraction of the upper index with the middle lower one, defines the *Ricci tensor*. This tensor is symmetric to the $\mu \leftrightarrow \nu$ interchange, in case the affine connection included is the Christoffel symbol. Now, the trace of the Ricci tensor defines the *Ricci scalar*:

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu}R_{\mu\nu} \ . \tag{2.22}$$

In the vierbein formalism, the analogue of the Riemann tensor is the *curvature* two-form. The starting point is the same with the above analysis for defining the Riemann tensor, that is the commutator of the covariant derivative, acting this time (not on a vector but) on the vierbein, which carries mixed indices:

$$(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu})e_{d\rho} = R_{\rho\sigma\mu\nu}e^{\sigma}{}_{d} + (\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu})\nabla_{\lambda}e_{\rho d} \ . \tag{2.23}$$

Making use of the vierbein for turning two indices of the Riemann tensor from greek to latin:

$$R_{\rho\sigma\mu\nu} = e_\rho^{\ a} e_\sigma^{\ b} R_{ab\mu\nu} \ , \tag{2.24}$$

replacing it to the equation (2.23) and using the relation (2.9), calculations and employment of the second equation of (2.3) lead to the expression of the curvature two-form in terms of the vierbein and the spin connection:

$$R_{ab\mu\nu} = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu ac}\omega_\nu^{\ c}{}_b - \omega_{\nu ac}\omega_\mu^{\ c}{}_b \ . \tag{2.25}$$

Adopting the differential form notation (2.16), along with:

$$R_{ab} \equiv \frac{1}{2} R_{ab\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2} R_{abcd} e^c \wedge e^d \ , \tag{2.26}$$

the expression of the curvature two-form, (2.25), takes the following compact form:

$$R_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega^c_{\ b} \ . \tag{2.27}$$

Obviously, the curvature two-form is antisymmetric under the interchange of its indices.

Now, aiming at the direction of defining the action, the following integral is considered:

$$I = \int \frac{1}{2} \epsilon_{abcd} e^a \wedge e^b \wedge R^{cd} \ . \tag{2.28}$$

The Hodge duality map, that is the "star" operator, in the four-dimensional case, is a map of $n-$forms to $(4-n)-$forms and is defined as:

$$*(e^{a_1} \wedge \ldots \wedge e^{a_n}) \equiv \frac{1}{(4-n)!} \epsilon^{a_1 \ldots a_n}{}_{a_{n+1} \ldots a_4} e^{a_n+1} \wedge \ldots \wedge e^{a_4} \ , \tag{2.29}$$

and, especially for $n = 2$, it can be written as:

$$*(e^{a_1} \wedge e^{a_2}) = \frac{1}{(4-2)!} \epsilon^{a_1 a_2}{}_{a_3 a_4} e^{a_3} \wedge e^{a_4} \ . \tag{2.30}$$

Therefore, applying the above in the $\frac{1}{2}\epsilon_{abcd} e^a \wedge e^b$ part of the integrand of (2.28), the corresponding integral can be written as:

$$I = \int R^{ab} \wedge *(e_a \wedge e_b) \ . \tag{2.31}$$

Using the (2.26), the above expression of the integral becomes:

$$\int \frac{1}{2} R^{ab}{}_{cd} e^c \wedge e^d \wedge *(e_a \wedge e_b) \ , \tag{2.32}$$

where $e^c \wedge e^d \wedge *(e_a \wedge e_b)$ is the inner product of $(e^c \wedge e^d)$ and $(e_a \wedge e_b)$, explicitly:

$$\int \frac{1}{2} R^{ab}{}_{cd} \langle e^c \wedge e^d, e_a \wedge e_b \rangle \epsilon \ , \tag{2.33}$$

where $\epsilon$ is the volume form and it is straightforward to show that it is equal to the Jacobian factor, that is the $\sqrt{-g} d^4 x$. Therefore, the above integral takes the following form:

$$I = \int R^{ab}{}_{cd} (\delta^c_a \delta^d_b - \delta^c_b \delta^d_a) \epsilon = \int d^4 x \sqrt{-g} R \ . \tag{2.34}$$

In the above analysis it is shown that starting from the integral (2.28), one can result with the integral of the Einstein-Hilbert action. Therefore, the complete action could be written as:

$$S = \frac{1}{16\pi G}\int \frac{1}{2}\epsilon_{abcd} e^a \wedge e^b \wedge R^{cd} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \ . \tag{2.35}$$

The initial formulation of general relativity (Einstein gravity), in which the metric tensor is considered as an independent field, is also known as the second order formulation of gravity. The action consisting of the curvature two-form that is, as shown, equivalent to the Einstein-Hilbert action and consists of the frame and spin connection fields, is the tetradic *Palatini* action, also known as a first order formulation of gravity. The latter is a formulation of great importance, especially when fermions are coupled to gravity and have to be added in the Palatini action. Also, for our purpose, the vierbein formalism of general relativity is fundamental because it is the one employed when trying to translate the results of general relativity to a gauge-theoretic language.

## 2.2 Three-dimensional Einstein gravity as a Chern-Simons gauge theory

Let us first study the three-dimensional Einstein gravity and its relation to a gauge theory [10]. In order to designate the above relation, one has to employ the (three-dimensional version of the) vielbein formalism, briefly reviewed in the previous section, in which, instead of the metric tensor, the vielbein and spin connection are the ones considered as the dynamical variables. In the following analysis, the vielbein is considered to be invertible, in relation to the fact that in general relativity the metric tensor (that is related to the vielbein as given in (2.3)) is non-degenerate and therefore the inverse metric tensor exists. The case in which the vielbein is not everywhere invertible is considered to be unphysical (singular) in classical general relativity. Nevertheless, the latter is a key point in the quantization of three-dimensional gravity, making the renormalizability of the theory apparent, as well[2] [10].

In three dimensions, for a manifold $M$, the Einstein-Hilbert action in the vielbein formalism, without the inclusion of cosmological constant and matter, is:

$$S_{\mathrm{EH3}} = \frac{1}{16\pi G}\int_M \epsilon^{\mu\nu\rho} e_\mu{}^a (\partial_\nu \omega_{\rho a} - \partial_\rho \omega_{\nu a} + \epsilon_{abc}\omega_\nu{}^b \omega_\rho{}^c) \ . \tag{2.36}$$

Variation of the above action with respect to the $\omega$ field, yields the torsionless condition, which, using (2.15), can be written as:

$$T_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \epsilon^{abc}\omega_{\mu b} e_{\nu c} - \epsilon^{abc}\omega_{\nu b} e_{\mu c} = D_\mu e_\nu{}^a - D_\nu e_\mu{}^a = 0 \ , \tag{2.37}$$

where $D_\mu e_\nu{}^a$ is defined as:

$$D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \epsilon^{abc}\omega_{\mu b} e_{\nu c} \ . \tag{2.38}$$

Also, variation of the same action with respect to the other fundamental variable, $e$, yields the vacuum Einstein equations of motion:

$$R_{\mu\nu a} = \partial_\nu \omega_{\rho a} - \partial_\rho \omega_{\nu a} + \epsilon_{abc}\omega_\nu{}^b \omega_\rho{}^c = 0 \ . \tag{2.39}$$

[2] Moreover, it is a theory with vanishing beta function (finite).

The expressions of the torsion and curvature 2-forms given in (2.37) and (2.39), are the three-dimensional analogues of the four-dimensional expressions, (2.15) and (2.25), respectively, that are obtained from the respective analysis of three-dimensional general relativity in the dreibein formalism. Also, in this three-dimensional case, the redefinition of the spin connection $\omega_\mu{}^a = \frac{1}{2}\epsilon^{abc}\omega_{\mu bc}$, is admitted and it is also taken into account.

Now, in (2.36), if one denotes collectively the vielbein and spin connection as a gauge field $A$, then the action is written as $AdA + A^3$, which is the general form of a Chern-Simons functional in three dimensions. This is pointing at the direction of the relation of three-dimensional gravity with a Chern-Simons gauge theory. All one has to do, is to find the appropriate gauge group and result with an action that is of Chern-Simons form and coincides with the three-dimensional Einstein-Hilbert action, (2.36).

Let us consider the ISO(1,2) to be the appropriate gauge group. It is crucial to mention that the Chern-Simons functional is defined on simple Lie groups. Therefore, it is not straightforward to develop a Chern-Simons gauge theory of ISO(1,2), unless it is confirmed that the corresponding Chern-Simons interaction exists for the candidate gauge group. In other words, what we seek is an invariant quadratic form on the ISO(1,2) Lie algebra. Although for arbitrary dimensions, i.e. for ISO(1,n-1), this does not hold, for the $n = 3$, ISO(1,2), group there exists an invariant and non-degenerate form, which reads:

$$\mathrm{tr}(J_a P_b) = \delta_{ab} \ , \quad \mathrm{tr}(P_a P_b) = 0 \ , \quad \mathrm{tr}(J_a J_b) = 0 \ , \tag{2.40}$$

where $J_a = \frac{1}{2}\epsilon_{abc}J^{bc}$ are the three Lorentz generators and $P_a$ are the three translations, together comprising the six generators of the ISO(1,2) group. The above generators satisfy the following algebra as given by their commutation relations:

$$[J_a, J_b] = \epsilon_{abc}J^c \ , \quad [J_a, P_b] = \epsilon_{abc}P^c \ , \quad [P_a, P_b] = 0 \ . \tag{2.41}$$

The next step is write down the gauge covariant derivative:

$$\tilde{D}_\mu = \partial_\mu + [A_\mu, \cdot] \ , \tag{2.42}$$

where, $A_\mu(x)$ is the gauge connection, that is expanded on the generators of ISO(1,2), since it is taking values in it:

$$A_\mu(x) = e_\mu{}^a(x)P_a + \omega_\mu{}^a(x)J_a \ . \tag{2.43}$$

In the above expression of the gauge connection, $A_\mu$, for every generator, a component gauge field has been assigned. The vielbein (dreibein) field has been attached to the local translations while for the rotational part (Lorentz transformations), the attached field is the spin connection.

By definition, $\tilde{D}_\mu$ transforms covariantly giving the transformation rule of $A_\mu$:

$$\delta A_\mu = -\tilde{D}_\mu\epsilon = -\partial_\mu - [A_\mu, \epsilon] \ , \tag{2.44}$$

where $\epsilon = \epsilon(x)$ is the gauge transformation parameter, which, being an element of the ISO(1,2) algebra, can be expanded on its generators:

$$\epsilon(x) = \xi^a(x)P_a + \lambda^a(x)J_a \ , \tag{2.45}$$

with $\xi^a(x)$ and $\lambda^a(x)$ being infinitesimal parameters. Combining equations (2.43), (2.45) with (2.44) and making use of the algebra of the generators, (2.41), one obtains the transformation laws of the gauge fields $e$ and $\omega$:

$$\delta e_\mu{}^a = -\partial_\mu \xi^a - \epsilon^{abc} e_{\mu b} \lambda_c - \epsilon^{abc} \omega_{\mu b} \xi_c \ , \tag{2.46}$$

$$\delta \omega_\mu{}^a = -\partial_\mu \lambda^a - \epsilon^{abc} \omega_{\mu b} \lambda_c \ . \tag{2.47}$$

The above gauge transformations do not coincide with the ordinary coordinate transformation law. Although the middle terms of the above equations can be identified as the local Lorentz transformations, since $\lambda_c$ is corresponded to the Lorentz generator $J^c$ in the gauge transformation, the rest of the terms are not recognizable at once. If it is possible to associate the above expressions of the transformations of the gauge fields to diffeomorphisms, then they will be considered equivalent to the coordinate transformations and, at the same time, it will be a confirmation that ISO(1,2) is the appropriate gauge group for the gauge-theoretic approach of three-dimensional Einstein gravity. Of course, after the action of the gauge theory will be given, it will have to be invariant under the above gauge transformations, in the same spirit the three-dimensional Einstein-Hilbert action is invariant under diffeomorphism in the coordinate based formulation of three-dimensional gravity. The relation between diffeomorphisms and gauge transformations is discussed right after the determination of the action and its equations of motion.

Advancing in the construction of the gauge theory of ISO(1,2), the next step is to calculate the component tensors of the gauge fields, using the usual formula, that is the commutator of the covariant derivative of the gauge theory, $\tilde{D}_\mu$:

$$R_{\mu\nu} = [\tilde{D}_\mu, \tilde{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ , \tag{2.48}$$

where $A_\mu$ is the gauge connection given in (2.43). Since $R_{\mu\nu}$ is valued in the algebra of ISO(1,2), it can be expanded on the generators:

$$R_{\mu\nu} = T_{\mu\nu}{}^a(x) P_a + R_{\mu\nu}{}^a(x) J_a \ , \tag{2.49}$$

where the $T_{\mu\nu}{}^a$ and $R_{\mu\nu}{}^a$ are the component curvature tensors, associated to $e$ and $\omega$, respectively. Combining the formula (2.48), with the expansion (2.49) and replacing the $A_\mu$ with its expression, (2.43), one results with the expressions of the component tensors:

$$T_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \epsilon^{abc} \omega_{\mu b} e_{\nu c} - \epsilon^{abc} \omega_{\nu b} e_{\mu c} \ , \tag{2.50}$$

$$R_{\mu\nu}{}^a = \partial_\mu \omega_{\nu a} - \partial_\nu \omega_{\mu a} + \epsilon_{abc} \omega_\mu{}^b \omega_\nu{}^c \ , \tag{2.51}$$

which are the three-dimensional versions of the expressions of the torsion and curvature two-forms given in (2.37) and (2.39).

Finally, to complete the picture, the action of the theory has to be determined. Constructing a gauge theory in three dimensions, the obvious choice is to consider the Chern-Simons action functional:

$$S_{\rm CS} = \int_M \mathrm{tr}(A \wedge dA + A \wedge A \wedge A) = \int_M \mathrm{tr} A_\mu (\partial_\nu A_\rho - \partial_\rho A_\nu + [A_\nu, A_\rho]) \epsilon^{\mu\nu\rho} d^3x \ . \tag{2.52}$$

After replacing with the expression of the gauge connection, $A_\mu$, given in (2.43), some terms of the above action are filtered out by the trace, (2.40), acting on the generators, leading to the following expression:

$$\int_M \epsilon^{\mu\nu\rho} e_\mu{}^a \Big( (\partial_\nu \omega_{\rho a} - \partial_\rho \omega_{\nu a} + \omega_\nu{}^b \omega_\rho{}^c \epsilon_{abc}) + (\partial_\nu e_{\rho a} - \partial_\rho e_{\nu a} + (\omega_\nu{}^b e_\rho{}^c - e_\nu{}^b \omega_\rho{}^c) \epsilon_{abc}) \Big) \ . \tag{2.53}$$

The first term is obviously recognizable as the curvature tensor and the second term of the above relation as the torsion tensor, both given in (2.51). For the sake of the desired SO(1,2) (Lorentz) invariance of the final action, the torsionless condition is imposed, that is $T_{\nu\rho}{}^{a} = 0$, therefore the above expression of the action takes the following form[3]:

$$S_{\text{CS}} = \int_M \epsilon^{\mu\nu\rho} e_\mu{}^a \left( \partial_\nu \omega_{\rho a} - \partial_\rho \omega_{\nu a} + \omega_\nu{}^b \omega_\rho{}^c \epsilon_{abc} \right) , \tag{2.54}$$

which is identical to the action of three-dimensional gravity of general relativity, (2.36). Variation with respect to the $e$ field gives the expected equation of motion, which is the vanishing of the curvature tensor, $R_{\mu\nu a} = 0$, coinciding with the expression of (2.39). Therefore, one can state that three-dimensional gravity can be described as a Chern-Simons gauge theory of ISO(1,2). It is not redundant to mention that the above action is invariant under the gauge transformations of the component fields $e, \omega$, given in (2.47). In the following lines, the discussion about diffeomorphisms and gauge transformations that was postponed earlier is given.

First, let us consider the transformations of the vielbein and the spin connection under a diffeomorphism, which is generated by a vector field, $v^\nu$. The standard parametrization of these transformations, denoted as $\tilde{\delta} e_\mu{}^a$ and $\tilde{\delta}\omega_\mu{}^a$, are given by the Lie derivatives along the vector $-v^\nu$:

$$\tilde{\delta} e_\mu{}^a = \mathcal{L}_{-v} e_\mu{}^a = -v^\nu \partial_\nu e_\mu{}^a - (\partial_\mu v^\nu) e_\nu{}^a = -v^\nu (\partial_\nu e_\mu{}^a - \partial_\mu e_\nu{}^a) - \partial_\mu (v^\nu e_\nu{}^a) , \tag{2.55}$$

$$\tilde{\delta} \omega_\mu{}^a = \mathcal{L}_{-v} \omega_\mu{}^a = -v^\nu \partial_\nu \omega_\mu{}^a - (\partial_\mu v^\nu) \omega_\nu{}^a = -v^\nu (\partial_\nu \omega_\mu{}^a - \partial_\mu \omega_\nu{}^a) - \partial_\mu (v^\nu \omega_\nu{}^a) . \tag{2.56}$$

Next, let us consider the difference $\tilde{\delta} e_\mu{}^a - \delta e_\mu{}^a$ and set $\xi^a = e_\nu{}^a v^\nu$ and $\lambda^a = \omega_\nu{}^a v^\nu$:

$$\begin{aligned} \tilde{\delta} e_\mu{}^a - \delta e_\mu{}^a &= -v^\nu (\partial_\nu e_\mu{}^a - \partial_\mu e_\nu{}^a) - \partial (v^\nu e_\nu{}^a) + \partial_\mu (e_\nu{}^a v^\nu) + \epsilon^{abc} e_{\mu b} \omega_{\nu c} v^\nu + \epsilon^{abc} \omega_{\mu b} e_{\nu c} v^\nu \\ &= -v^\nu (D_\nu e_\mu{}^a - D_\mu e_\nu{}^a) , \end{aligned} \tag{2.57}$$

where the expression, (2.38), of $D_\mu$ has been used. The above expression vanishes by the constraint of vanishing torsion (torsionless condition), which was imposed for reasons of Lorentz invariance of the action[4]. Now, in turn, let us consider the difference $\tilde{\delta}\omega_\mu{}^a - \delta\omega_\mu{}^a$ and set again $\lambda^a = \omega_\nu{}^a v^\nu$:

$$\begin{aligned} \tilde{\delta} \omega_\mu{}^a - \delta \omega_\mu{}^a &= -v^\nu (\partial_\nu \omega_\mu{}^a - \partial_\mu \omega_\nu{}^a) - \partial (v^\nu \omega_\nu{}^a) + \partial_\mu (v^\nu \omega_\nu{}^a) + \epsilon^{abc} \omega_{\mu b} v^\nu \omega_{\nu c} \\ &= v^\nu (\partial_\mu \omega_\nu{}^a - \partial_\nu \omega_\mu{}^a + \epsilon^{abc} \omega_{\mu b} \omega_{\nu c}) = v^\nu R_{\mu\nu} . \end{aligned} \tag{2.58}$$

The above expression vanishes by equation of motion[5], that is the vanishing of the Ricci tensor. Therefore, one concludes that the gauge transformations are equivalent to the diffeomorphism transformations on-shell, which means that the gauge transformations of the fields compensate for the coordinate transformations in this gauge-theoretic approach. The expected invariance of the action under the gauge transformations ensure the general covariance of the theory, in the

[3] In Ref. [10] there is an alternative and heuristic way to obtain the Chern-Simons action beginning with a topological invariant of the form $\int_Y \text{tr} R \wedge R$ on a four-dimensional manifold, $Y$. Straightforward calculations lead to an integrand expressed as a total derivative and therefore the integral on $Y$ reduces to an integral on $M$, where $M$ is a three-dimensional boundary of $Y$. The expression of the integral on $M$ is identified as the Chern-Simons action functional of three-dimensional gravity. The advantage of this approach is that the torsionless condition does not have to be imposed.

[4] In the Palatini formalism, this condition is obtained as an equation of motion, (2.37).

[5] In the Palatini formalism, this is the expression of (2.39)

same spirit the invariance of $S_{\rm EH3}$ does, under the general coordinate transformations. Moreover, it is confirmed that ISO(1,2) is the appropriate group for constructing three-dimensional gravity as a gauge theory[6].

The above analysis of the relation of three-dimensional gravity and ISO(1,2) Chern-Simons gauge theory can be generalized in case a cosmological constant is included. The presence of a cosmological constant renders the spacetime curved, therefore there are two corresponding spacetimes, depending on the sign of the constant, three-dimensional de Sitter and Anti de Sitter, with isometry groups the SO(1,3) and SO(2,2), respectively. In order to relate the three-dimensional gravity with cosmological constant to gauge theories, it is reasonable to consider the above groups as the gauge groups, since ISO(1,2) worked perfectly for the flat case. The procedure of building these gauge theories is the same as the one described before, for ISO(1,2). The cosmological constant is introduced in the gauge theory through the commutation relation of the translations, which is now non-zero. Also, this modification induces the insertion of an extra term in the expression of the transformation of the spin connection, (2.47), namely:

$$\delta e_\mu{}^a = -\partial_\mu \xi^a - \epsilon^{abc} e_{\mu b} \lambda_c - \epsilon^{abc} \omega_{\mu b} \xi_c \ , \tag{2.59}$$

$$\delta \omega_\mu{}^a = -\partial_\mu \lambda^a - \epsilon^{abc} \omega_{\mu b} \lambda_c - \lambda \epsilon^{abc} e_{\mu b} \xi_c \ , \tag{2.60}$$

and another extra term in the expression of the curvature tensor of (2.51), that is:

$$T_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \epsilon^{abc} \omega_{\mu b} e_{\nu c} - \epsilon^{abc} \omega_{\nu b} e_{\mu c} \ , \tag{2.61}$$

$$R_{\mu\nu}{}^a = \partial_\mu \omega_{\nu a} - \partial_\nu \omega_{\mu a} + \epsilon_{abc} (\omega_\mu{}^b \omega_\nu{}^c + \lambda e_\mu{}^b e_\nu{}^c) \ . \tag{2.62}$$

As for the action of the gauge theory, it is obtained after consideration of the Chern-Simons functional, again coinciding with the three-dimensional Einstein-Hilbert action of general relativity in three dimensions with cosmological constant.

## 2.3 Four-dimensional Einstein gravity as a gauge theory

Whether or not general relativity in four dimensions can be described as a gauge theory is a complicated issue. In Ref. [10], it is mentioned that four-dimensional gravity cannot be described as a gauge theory, because of the expression of the Einstein-Hilbert action, which has the general form $\int A \wedge A \wedge (dA + A^2)$ and such an action cannot be retrieved by a gauge theory. Although such an action functional does not originate from a gauge theory, there exists a non-trivial way to obtain the Einstein-Hilbert action in a gauge-theoretic approach, starting from a Yang-Mills-type action functional. In this section, the construction of four-dimensional gravity as a gauge theory is reviewed, so for the kinematics (transformations of the gauge fields), as for the dynamics (action and equations of motion).

Like in the three-dimensional case described in the previous section, first of all, the vierbein formalism has to be employed for the construction of the gauge theory of gravity. In absence of cosmological constant, the isometry group (symmetries of the metric) of the Minkowski spacetime is ISO(1,3) (the Poincaré group) and it is the one that will be considered as the gauge group, in accordance with the three-dimensional case, where isometry groups of the Minkowski, dS

[6]The ISO(1,2) is the group describing the isometries of the three-dimensional Minkowski spacetime, rendering its choice not a random guess.

and AdS were considered as the gauge groups. The Poincaré algebra comprises of ten generators, four local translations, $P_a$ and six Lorentz transformations, $M_{ab}$, satisfying the following commutation relations[7]:

$$[M_{ab}, M_{cd}] = 4\eta_{[a[c}M_{d]b]} \ , \quad [P_a, M_{bc}] = 2\eta_{a[b}P_{c]} \ , \quad [P_a, P_b] = 0 \ , \tag{2.63}$$

where $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)$ is the four-dimensional Minkowski metric. Following the standard procedure, the gauge covariant derivative is defined as:

$$D_\mu = \partial_\mu + [A_\mu, \cdot] \ , \tag{2.64}$$

where $A_\mu(x)$ is the gauge connection. Expansion of the connection on the generators of ISO(1,3) gives the expression:

$$A_\mu(x) = e_\mu{}^a(x)P_a + \omega_\mu{}^{ab}(x)M_{ab} \ , \tag{2.65}$$

where $e_\mu{}^a$ and $\omega_\mu{}^{ab}$ are identified as the component gauge fields for the translations and Lorentz transformations, respectively. By definition, transformation of the $D_\mu$ is covariant, therefore, the transformation law for the gauge connection $A_\mu$ is given by:

$$\delta A_\mu = D_\mu \epsilon = \partial_\mu + [A_\mu, \epsilon] \ , \tag{2.66}$$

where $\epsilon = \epsilon(x)$ is a gauge transformation parameter, which, as an element of ISO(1,3) algebra, it may be written as an expansion on the generators:

$$\epsilon(x) = \xi^a(x)P_a + \frac{1}{2}\lambda^{ab}(x)M_{ab} \ , \tag{2.67}$$

with $\xi^a(x)$ and $\lambda^{ab}(x)$ being infinitesimal parameters. Combination of (2.65), (2.66) and (2.67) leads to the expression of the transformation of the component gauge fields:

$$\delta e_\mu{}^a = \partial_\mu \xi^a + \omega_\mu{}^{ab}\xi_b - \lambda^a{}_b e_\mu{}^b \ , \tag{2.68}$$

$$\delta \omega_\mu{}^{ab} = \partial_\mu \lambda^{ab} + \lambda^a{}_c \omega_\mu{}^{bc} - \lambda^b{}_c \omega_\mu{}^{ac} \ . \tag{2.69}$$

The corresponding field strength tensors, $T_{\mu\nu}{}^a$ and $R_{\mu\nu}{}^{ab}$, of the component fields, $e$ and $\omega$, are obtained by the definition of the field strength tensor, $R_{\mu\nu}$, of $A_\mu$:

$$R_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ , \tag{2.70}$$

after its expansion on the generators:

$$R_{\mu\nu} = T_{\mu\nu}{}^a P_a + \frac{1}{2} R_{\mu\nu}{}^{ab} M_{ab} \ . \tag{2.71}$$

Therefore, combining (2.65), (2.70) and (2.71), the expressions of the component tensors are:

$$\begin{aligned} T_{\mu\nu}{}^a &= \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a - \omega_\mu{}^{ab} e_{\nu b} + \omega_\nu{}^{ab} e_{\mu b} \ , \\ R_{\mu\nu}{}^{ab} &= \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu c}{}^b + \omega_\nu{}^{ac}\omega_{\mu c}{}^b \ , \end{aligned} \tag{2.72}$$

where the above expressions coincide with the ones found for the torsion and curvature two-forms in the vierbein formalism description of general relativity in (2.15) and (2.25).

[7]The [ ] notation implies antisymmetricity of the indices inside the brackets, $\eta_{a[b}P_{c]} = \frac{1}{2}(\eta_{ab}P_c - \eta_{ac}P_b)$.

Until this point, the construction of the gauge-theoretic version of four-dimensional gravity has been unfolding in a straightforward way. Moving on to the dynamical part of the theory, the obvious choice would be an action of Yang-Mills type of the Poincaré group. Nevertheless, in order to claim a successful relation of four-dimensional gravity to a gauge theory, it is necessary to result with the Einstein-Hilbert action, which is, of course, not of Yang-Mills type. In that sense, the answer to the question whether four-dimensional general relativity could be related to a gauge theory would be negative and the corresponding argument in ref. [10] would be confirmed. However, one could claim that the Einstein-Hilbert action could be built out of invariants including the tensors obtained in (2.72). Specifically, the Ricci scalar invariant could be built using the curvature tensor, $R_{\mu\nu}{}^{ab}$ and an action including this invariant could be constructed recovering the Einstein-Hilbert action [84]. However, there is still another -less guided- way to result with Einstein's gravity, treating the Lorentz and translational part in a more unified way, based on more intuitive and physical arguments.

First, it has to be noted that the desired action has to be invariant under the Lorentz transformations and not under the total Poincaré symmetry. Therefore, in order to reduce the symmetry of the action, a spontaneous symmetry breaking mechanism can be employed by the inclusion of a scalar field [3, 4]. The above is the indicated way to reduce the extra degrees of freedom and it is left to be seen if, after the symmetry breaking, it is possible to result with the correct (Einstein-Hilbert) action, from the initial action of Yang-Mills type. For the present purpose, in order to achieve the incorporation of the spontaneous symmetry breaking mechanism, the gauge group, i.e. the gauge symmetry of the action of Yang-Mills type, has to be the de Sitter, SO(1,4), group, instead of the Poincaré[8]. The choice of the de Sitter group is strategic, in the sense that it comprises of the same number of generators as the Poincaré, but carries an extra and useful virtue, that is all generators can be considered on equal footing, denoting them all with a single gauge field, let us say $\omega_\mu{}^{AB}, A, B = 1 \dots 5$, since it is a semisimple group. Therefore, the extra scalar field, $\phi^a$, is assigned to the fundamental representation of SO(1,4) and induces the spontaneous symmetry breaking, from SO(1,4) to SO(1,3), i.e. the symmetry is reduced to the Lorentz with four out of ten generators, the translations, to have been broken.

Specifically, constructing a pure SO(1,4) gauge theory, the gauge connection for the gauge field, $\omega_\mu{}^{AB}$, would be $A_\mu = \omega_\mu{}^{AB} M_{AB}$, where $M_{AB}$ are the ten SO(1,4) generators and the corresponding field strength tensor would be given by (2.70) as:

$$F_{\mu\nu}{}^{AB} = \partial_\mu \omega_\nu{}^{AB} - \partial_\nu \omega_\mu{}^{AB} + \omega_\mu{}^A{}_C \omega_\nu{}^{CB} - \omega_\nu{}^A{}_C \omega_\mu^{CB} \ . \tag{2.73}$$

The invariants which could serve as components for the SO(1,4) action have to be constructed in terms of the above field strength tensor. The only invariant that can be constructed this way and also being polynomial with respect to $F_{\mu\nu}$ is the topological invariant, Pontryagin index:

$$S = \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^{AB} F_{\rho\sigma AB} \ . \tag{2.74}$$

It is worth-noting that along these lines, that is building invariants from the field strength tensor, a non-polynomial action containing square roots of the field strength tensor was proposed in ref. [5], resulting to an expression of an action that was SO(1,3) gauge invariant, after some appropriate gauge fixing. The presence of the square root and the absence of a scaled

[8]Also, the Anti-de Sitter group, SO(2,4), could be employed.

quantity for fixing the vierbein dimensionality [3] was promoted to an alternative and more consistent construction, considering a de Sitter invariant, polynomial action that will eventually lead to an invariant action under the Lorentz group. As we mentioned earlier, the indicated way is to begin with a polynomial action and include a scalar field, $\phi^a$, along with a dimensionful parameter, $m$ (inverse length). The action concentrating the above features comes from a modification of (2.74), specifically:

$$S_{\mathrm{SO}(1,4)} = \int d^4x \left( m\phi^A \epsilon_{ABCDE} R_{\mu\nu}{}^{BC} R_{\rho\sigma}{}^{DE} \epsilon^{\mu\nu\rho\sigma} + \lambda(\phi^A\phi_A + m^{-2}) \right) , \qquad (2.75)$$

where the variable $\lambda = \lambda(x)$ serves as a Lagrange multiplier imposing the constraint for the scalar field[9]:

$$\phi^A\phi_A = -m^{-2} . \qquad (2.76)$$

Picking a specific gauge for the scalar field:

$$\phi = \phi^0 = (0,0,0,0,m^{-1}) \quad \Leftrightarrow \quad \phi^a(x) = 0 \ \text{ and } \ \phi^5(x) = m^{-1} , \qquad (2.77)$$

the non-zero value of $\phi^5(x)$ induces the symmetry breaking of SO(1,4) to the little group, SO(1,3). The action (2.75) reduces to the following expression, in which only the Lorentz symmetry is manifest[10]:

$$S_{\mathrm{SO}(1,3)} = \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^{ab} F_{\rho\sigma}{}^{cd} \epsilon_{abcd} . \qquad (2.78)$$

Defining the scaled gauge fields $e_\mu^{\ a} = m^{-1}\omega_\mu^{\ a5}$ and decomposing the field strength tensor, $F_{\mu\nu} = F_{\mu\nu}{}^{AB} M_{AB} = F_{\mu\nu}{}^{ab} M_{ab} + F_{\mu\nu}^{a5} M_{a5}$ of (2.73), the following expressions are obtained:

$$F_{\mu\nu}^{a5} = m T_{\mu\nu}{}^{a} , \qquad (2.79)$$
$$F_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} - m^2 (e_\mu{}^a e_\nu{}^b - e_\nu{}^a e_\mu{}^b) , \qquad (2.80)$$

where $T_{\mu\nu}{}^a$ and $R_{\mu\nu}{}^{ab}$ are the ISO(1,3) component curvature tensors of (2.72). Of course, the $F_{\mu\nu}^{a5}$ field strength is not present in the action after the symmetry breaking and thus $T_{\mu\nu}{}^a = 0$ [11]. The vanishing of the torsion tensor (torsionless condition) leads to a relation of the gauge fields, writing the spin connection with respect to the vierbein. In order to result with this relation, first one has to contract the spacetime indices:

$$e^\mu{}_c e^\nu{}_d T_{\mu\nu}{}^a = e^\mu{}_c e^\nu{}_d \left( \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a - \omega_\mu{}^{ab} e_{\nu b} + \omega_\nu{}^{ab} e_{\mu b} \right) = 0 , \qquad (2.81)$$

and then do cyclic permutation of the free Lorentz indices $a, c, d$ and add and subtract the three resulting equations. Solving with respect to the spin connection, the result is:

$$\omega_{\mu ab} = \frac{1}{2} \left( \Omega_{\mu ab} - \Omega_{\mu ba} - \Omega_{ab\mu} \right) , \qquad (2.82)$$

where the following definition:

$$\Omega_{abc} = 2 e^\mu{}_a e^\nu{}_b \partial_{[\mu} e_{\nu]c} , \qquad (2.83)$$

[9] It is worth-noting that the above action is even under parity transformations.

[10] This is exactly the SO(1,3) action obtained in ref. [5] and originated from the non-polynomial one. The symmetry breaking version described here [3, 4] gives the same expression but in a more natural and physical way.

[11] It would be present if we were considering a general gauge.

has been employed.

Now, if the above expression of $F_{\mu\nu}{}^{ab}$, (2.80), is inserted into the action $S_{\mathrm{SO}(1,3)}$ of (2.78), then the resulting expression can be grouped into three terms:

$$\begin{aligned}
S_{\mathrm{SO}(1,3)} &= \int d^4x \epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\left(R_{\mu\nu}{}^{ab} + m^2(e_\mu{}^a e_\nu{}^b - e_\mu{}^b e_\nu{}^a)\right)\left(R_{\rho\sigma}{}^{cd} + m^2(e_\rho{}^c e_\sigma{}^d - e_\rho{}^d e_\sigma{}^c)\right) \\
&= \int d^4x \epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\left(R_{\mu\nu}{}^{ab}R_{\rho\sigma}{}^{cd} + 2m^2 R_{\mu\nu}{}^{ab}(e_\rho{}^c e_\sigma{}^d - e_\rho{}^d e_\sigma{}^c) + m^4(e_\mu{}^a e_\nu{}^b - e_\mu{}^b e_\nu{}^a)(e_\rho{}^c e_\sigma{}^d - e_\rho{}^d e_\sigma{}^c)\right) \\
&= \int d^4x \epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\left(\mathcal{L}_{RR} + m^2\mathcal{L}_{eeR} + m^4\mathcal{L}_{eeee}\right) \ . \qquad (2.84)
\end{aligned}$$

The first term, $\mathcal{L}_{RR}$, does not give any contributions to the field equations because it is the integrand of the Gauss-Bonnet topological invariant (see [85]). The second term, $\mathcal{L}_{eeR}$ is the Ricci scalar curvature (Einstein-Hilbert) action, (2.35) and the third is identified as a cosmological constant of order $m^4$. Because of this constant, the maximally symmetric solution of the field equations is a de Sitter space:

$$F_{\mu\nu}{}^{ab} = 0 \ \Rightarrow \ R_{\mu\nu}{}^{ab} = m^2(e_\mu{}^a e_\nu{}^b - e_\nu{}^a e_\mu{}^b) \ . \qquad (2.85)$$

In case the cosmological constant is zero, the solution is flat, or in other words, the trivial maximally symmetric space, that is the Minkowski spacetime.

As for the general covariance, it is recovered by the relation of the gauge transformations, (2.66) and diffeomorphisms. Following the same procedure and calculations as in the three-dimensional case, one ends up with the four-dimensional versions of (2.57) and (2.58). Therefore, taking into consideration the torsionless condition and the equation of motion of vanishing curvature, general covariance is ensured.

Therefore, to conclude, it is possible to describe four-dimensional gravity of general relativity as a gauge theory[12]. The transformations of the gauge fields, $e$ and $\omega$, can be obtained starting with the Poincaré group as the gauge group. However, in order to result with the Einstein-Hilbert action, it is necessary to consider as a gauge group the de Sitter group and include a scalar field in a fixed gauge for the induction of a spontaneous symmetry breaking, starting from a Yang-Mills type, polynomial action with respect to the field strength tensor. Indeed, the resulting action is invariant only under the Lorentz group and is successfully identified as the Einstein-Hilbert action.

## 2.4 Four-dimensional Conformal gravity

In this section, the gauge-theoretic approach of the four-dimensional conformal gravity is briefly reviewed [7, 8, 86, 87]. Specifically, a gauge theory of the conformal group, SO(2,4), is constructed, and eventually, the theory of Weyl gravity is obtained. In order to end up with Weyl gravity, a breaking of the initial gauge symmetry has to take place, this time not spontaneously with the inclusion of some extra scalar field, but with the imposition of certain constraints. Like the two cases of three-dimensional and four-dimensional Einstein gravity, in this case, too, the group to be localized is a spacetime symmetry group including translations, which has to be related to the general coordinate transformations, since the resulting gauge theory

[12] But not as a pure ISO(1,3) gauge theory

must be generally covariant, being a gravitational theory. The relation between the "internal" translational symmetry and coordinate transformations is achieved by introducing the vierbein and identify it as the gauge field associated with the translations. The mixing of internal and spacetime symmetries is precisely what makes the building of such gauge theories special in comparison to the linear gauge theories of internal symmetry groups.

At this point, it is given a rather nice opportunity to discuss a little further the two approaches of the previous cases of three-dimensional and four-dimensional Einstein gravity theories, so far as the methodology followed for resulting with the correct actions and the way the translational part was treated [8]. In the four-dimensional case, for the dynamical part of the theory, the initial gauge group was considered to be the G=SO(1,4). The Lorentz subgroup H=SO(1,3) of $G$ was gauged by the linear gauge fields, while the part of translations was gauged by the non-linear gauge fields, identified as the vierbein. The translations were considered to be a spontaneously broken internal gauge symmetry. As for the invariance under the general coordinate transformations (let us denote them $X$), it was assumed externally. The symmetry of the resulting theory was SO(1,3)$\otimes X$. In the three-dimensional case, another approach was followed. The initial gauge group G=ISO(1,2) was treated as a linear internal symmetry group, without making any distinction between the translation and Lorentz parts. General coordinate transformations, $X$, were externally introduced and the final group was the subgroup H=SO(1,2) times the $X$. Requirement of invariance of the resulting action under the translational part consequently led to imposition of constraints, specifically the torsionless condition. In other words, as it was shown, the constraints gave the possibility to express the translations as a combination of the Lorentz transformations and the diffeomorphisms. Therefore, the translations could be traded with the general coordinate transformations. In the present construction of conformal gravity as a gauge theory, the same strategy is employed as in the three-dimensional case, in which the symmetry breaking takes place by imposing constraints based on physical arguments and not spontaneously with introduction of extra fields.

In the gauge-theoretic approach of the four-dimensional conformal gravity, the gauge group is considered to be the SO(2,4), which comprises of fifteen generators, six Lorentz transformations, $M_{ab}$, four translations, $P_a$, four special conformal transformations (conformal boosts), $K_a$ and the dilatation (scale transformation), $D$. These generators satisfy the following commutation relations which determine the SO(2,4) algebra:

$$\begin{aligned}
[M_{ab}, M_{cd}] &= \eta_{bc}M_{ad} + \eta_{ad}M_{bc} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} \ , \\
[M_{ab}, P_c] &= \eta_{bc}P_a - \eta_{ac}P_b \ , \\
[M_{ab}, K_c] &= \eta_{bc}K_a - \eta_{ac}K_b \ , \\
[P_a, D] &= P_a \ , \\
[K_a, D] &= -K_a \ , \\
[K_a, P_b] &= -2(\eta_{ab}D + M_{ab}) \ ,
\end{aligned} \tag{2.86}$$

where $\eta_{ab}$ is the mostly positive four-dimensional Minkowski metric. The construction of the gauge theory begins with defining the covariant derivative and specifying the gauge connection, which, as an element of the SO(2,4) algebra, can be written in terms of the generators:

$$A_\mu = e_\mu{}^a P_a + \frac{1}{2}\omega_\mu{}^{ab} M_{ab} + b_\mu D + f_\mu{}^a K_a \ , \tag{2.87}$$

where a gauge field has been introduced for each generator. The gauge fields attached to the translations have been identified as the vierbein, while the ones associated to the Lorentz transformations are identified as the spin connection, just like in the previous cases. The gauge connection, $A_\mu$, obeys the following infinitesimal transformation rule:

$$\delta A_\mu = D_\mu \epsilon = \partial_\mu \epsilon + [A_\mu, \epsilon] \ , \tag{2.88}$$

where $\epsilon = \epsilon(x)$ is a parameter that belongs to the gauge algebra and for this reason it can be written as:

$$\epsilon = \xi^{\ a} P_a + \frac{1}{2}\lambda^{\ ab} M_{ab} + \kappa D + \rho^{\ a} K_a \ . \tag{2.89}$$

Combining the relations (2.87), (2.88) and (2.89) leads to the transformation rules of the various component gauge fields:

$$\delta e_\mu^{\ a} = \partial_\mu \xi^a + \omega_\mu^{\ a}{}_b \xi^b - b_\mu \xi^a - \lambda^a_{\ b} e_\mu^{\ b} + \kappa e_\mu^{\ a} \ , \tag{2.90}$$

$$\delta \omega_\mu^{\ ab} = \partial_\mu \lambda^{ab} - 2\omega_\mu^{\ ac}\lambda^b_{\ c} - 4 f_\mu^{\ [a}\xi^{b]} - 4 e_\mu^{\ [a}\rho^{b]} \ , \tag{2.91}$$

$$\delta b_\mu = \partial_\mu \kappa - 2\xi^a f_{\mu a} + 2\rho^a e_{\mu a} \ , \tag{2.92}$$

$$\delta f_\mu^{\ a} = \partial_\mu \rho^a + \omega_\mu^{\ ab}\rho_b + b_\mu \rho^{\ a} - \lambda^{ab} f_{\mu b} - \kappa f_\mu^{\ a} \ . \tag{2.93}$$

The field strength tensor of the theory is given by the standard formula:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ . \tag{2.94}$$

In accordance with equation (2.70), expansion of the field strength tensor on the generators is written down as:

$$F_{\mu\nu} = \tilde{R}_{\mu\nu}^{\ \ a} P_a + \frac{1}{2} R_{\mu\nu}^{\ \ ab} M_{ab} + R_{\mu\nu} D + R_{\mu\nu}^{\ \ a} K_a \ . \tag{2.95}$$

The component curvature tensors that accompany each generator of the algebra can be calculated after combining the relations (2.94), (2.87) and (2.95). Their expressions are given as follows:

$$\tilde{R}_{\mu\nu}^{\ \ a} = \partial_\mu e_\nu^{\ a} - \partial_\nu e_\mu^{\ a} + \omega_\mu^{\ ab} e_{\nu b} - \omega_\nu^{\ ab} e_{\mu b} - 2 b_{[\mu} e_{\nu]}^{\ a} \tag{2.96}$$

$$= T_{\mu\nu}^{(0)a} - 2 b_{[\mu} e_{\nu]}^{\ a} \ , \tag{2.97}$$

$$R_{\mu\nu}^{\ \ ab} = \partial_\mu \omega_\nu^{\ ab} - \partial_\nu \omega_\mu^{\ ab} - \omega_\mu^{\ ac}\omega_\nu^{\ b}{}_c + \omega_\nu^{\ ac}\omega_\mu^{\ b}{}_c - 8 e_{[\mu}^{\ [a} f_{\nu]}^{\ b]} \tag{2.98}$$

$$= R_{\mu\nu}^{(0)ab} - 8 e_{[\mu}^{\ [a} f_{\nu]}^{\ b]} \ , \tag{2.99}$$

$$R_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu + 4 e_{[\mu}^{\ \ a} f_{\nu]a} \ , \tag{2.100}$$

$$R_{\mu\nu}^{\ \ a} = \partial_\mu f_\nu^{\ a} - \partial_\nu f_\mu^{\ a} + \omega_\mu^{\ ab} f_{\nu b} - \omega_\nu^{\ ab} f_{\mu b} + 2 b_{[\mu} f_{\nu]}^{\ a} \ , \tag{2.101}$$

where $T_{\mu\nu}^{(0)a}$ and $R_{\mu\nu}^{(0)ab}$ are the torsion and curvature component tensors of the four-dimensional Poincaré gravity of the previous section, given in (2.72).

As mentioned earlier in this section, in this case the resulting gauge group is found after imposing certain constraints for the sake of invariance of the final theory under the translations, too. In order to achieve this, one has to trade the translations with the general coordinate transformations (denoted by X), which means that the resulting gauge group should be the initial, G=SO(2,4), minus the translations, i.e. $H = SO(2,4) - \{P\}$ times $X$. Following the same procedure as in the three-dimensional case, the difference between a diffeomorphism and the gauge transformation, $\tilde{\delta} e_\mu{}^a - \delta e_\mu{}^a$, is considered using (2.57) and (2.90):

$$\tilde{\delta} e_\mu{}^a - \delta e_\mu{}^a = \left(v^\nu \partial_\nu e_\mu{}^a + \partial_\mu(v^\nu e_\nu{}^a) - v^\nu \partial_\mu e_\nu{}^a\right) - \left(\partial_\mu \xi^{\ a} + \omega_\mu{}^a{}_b \xi^b - b_\mu \xi^{\ a} - \lambda^a{}_b e_\mu{}^b + \kappa e_\mu{}^a\right) .$$

Setting $\xi^a = v^\nu e_\nu{}^a$, $\lambda^a{}_b = v^\nu \omega_\nu{}^a{}_b$ and $\kappa = v^\nu b_\nu$, the above difference takes the following form:

$$\tilde{\delta} e_\mu{}^a - \delta e_\mu{}^a = v^\nu \left(\partial_\nu e_\mu{}^a - \partial_\mu e_\nu{}^a - \omega_\mu{}^a{}_b e_\nu{}^b + \omega_\nu{}^a{}_b e_\mu{}^b + b_\mu e_\nu{}^a - b_\nu e_\mu{}^a\right) = -v^\nu \tilde{R}_{\mu\nu}{}^a . \quad (2.102)$$

It is obvious that the constraint that is needed for getting rid of the translational part of the theory, with a coordinate transformation making up for them, is the vanishing of the torsion:

$$\tilde{R}_{\mu\nu}{}^a = 0 . \quad (2.103)$$

Therefore, setting the torsion to zero, the generators of the initial group, SO(2,4), break and the subgroup $H$ is generated by $M, D$ and $K$. In addition, in order to achieve the $P \to X$ exchange for all fields, the following constraint has to be imposed [7]:

$$R_{\mu\nu}{}^{ab} e_b{}^\nu = 0 . \quad (2.104)$$

Let us now proceed with the solutions of the above constraints, (2.103) and (2.104), in terms of the independent fields, $e_\mu{}^a$ and $b_\mu$.

For the first constraint, the torsionless condition, (2.103), following the same computational procedure as in the four-dimensional case, one ends up with the relation:

$$\omega_\mu{}^{ab} = -\frac{1}{2}\left(\hat{\Omega}_{\mu ab} - \hat{\Omega}_{\mu ba} - \hat{\Omega}_{ab\mu}\right) = -\omega_\mu{}^{ab}(e) + 2b^{[a} e_\mu{}^{b]} , \quad (2.105)$$

where $\hat{\Omega}_{abc} = 2e^\mu{}_a e^\nu{}_b \hat{\partial}_{[\mu} e_{\nu]c}$ is the conformal analogue of $\Omega_{abc}$ of the four-dimensional Einstein case, (2.83), with the partial derivative $\partial_\mu e_\nu^a$ redefined to $\hat{\partial}_\mu e_\nu^a = (\partial_\mu + b_\mu) e_\nu^a$, that is the Weyl $(D)$ covariant derivative, and $\omega_\mu{}^{ab}(e)$ is the expression of the spin connection in terms of the vierbein in the four-dimensional Einstein case, given in (2.82). Also, along the same lines, we can define the tensor $\hat{R}_{\mu\nu}^{ab}$, which is the tensor $R_{\mu\nu}^{ab}$, again with the partial derivative, $\partial_\mu$, to have been replaced with $\hat{\partial}_\mu = \partial_\mu + b_\mu$, that is the Weyl $(D)$ covariant derivative, since it will be useful later.

For the second constraint, (2.104), due to the expression of (2.99), in which the curvature tensor $R_{\mu\nu}^{ab}$ is expressed in terms of the curvature of Poincaré gravity plus an extra term containing $f_\mu{}^a$, it is possible to solve for $f_\mu{}^a$ algebraically, in terms of the Ricci tensor:

$$R_{\mu\nu}{}^{ab} e_b{}^\nu = 0 \;\Rightarrow\; R^{(0)ab}_{\mu\nu} e^\nu{}_b - 8 e_{[\mu}^{[a} f_{\nu]}^{\ b]} = 0 . \quad (2.106)$$

Employing the Weyl decomposition of the Riemann tensor[13]:

$$R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \frac{2}{n-2}\left(g_{\mu[\rho} R_{\sigma]\nu} - g_{\nu[\rho} R_{\sigma]\mu}\right) - \frac{2}{(n-1)(n-2)} R g_{\mu[\rho} g_{\sigma]\nu} , \quad (2.107)$$

[13]In [9] eq. (15.25), one may find the relation already in the desired form.

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, which is traceless, meaning that a contraction on one of its indices is equal to zero, $R_{\mu\nu}$ is the Ricci tensor and $R$ is the Ricci scalar. Replacing the (2.107) in (2.106) and contracting with $g^{\sigma\nu}$, one results with:

$$R^{(0)ab}_{\mu\nu} e^{\nu}{}_{b} - 8 e_{[\mu}{}^{[a} f_{\nu]}{}^{b]} = 0 \;\Rightarrow\; \frac{1}{(n-1)(n-2)} e_{\mu}{}^{a} R - R_{\mu}{}^{a} = 4 f_{\mu}{}^{a} \,, \tag{2.108}$$

where $R_{\mu}{}^{a} = R^{(0)ab}_{\mu\nu} e^{\nu}{}_{b}$ and $R = e^{\mu}{}_{a} R_{\mu}{}^{a}$ are contractions of the Poincaré curvature tensor, $R^{(0)ab}_{\mu\nu}$. For $n = 4$, one results with the solution of the second constraint:

$$f_{\mu}{}^{a} = -\frac{1}{4}\left( R_{\mu}{}^{a} - \frac{1}{6} e_{\mu}{}^{a} R \right) \,. \tag{2.109}$$

The solutions of the two constraints, (2.105) and (2.109), show that $\omega_{\mu}{}^{ab} = \omega_{\mu}{}^{ab}(e,b)$ and $f_{\mu}{}^{a} = f_{\mu}{}^{a}(e,b)$, which means that the gauge fields $\omega$ and $f$ have been expressed in terms of the independent gauge fields, $e$ and $b$. Moreover, the second constraint fixed the gauge field of special conformal transformations, $f$, as understood from the resulting expression, (2.109), rendering it as a non-propagating gauge field.

The invariant action under the subgroup $H$ and the coordinate transformations, $X$, or the action with $H \otimes X$ invariance, is:

$$S_W = \frac{1}{8a^2} \int d^4x \epsilon_{abcd} \epsilon^{\mu\nu\rho\sigma} \left( R_{\mu\nu}{}^{ab} R_{\rho\sigma}{}^{ab} \right)^{\omega(e,b)}_{f(e,b)} \,, \tag{2.110}$$

in which the two constraints, (2.105) and (2.109), have been included. It is worth-noting that the solution of the second constraint, (2.109), could have been derived from variation of the above action, (2.110), if it was written also in terms of the $f$ gauge field (before elimination of $f$), as an equation of motion. The expression of the $R_{\mu\nu}{}^{ab}$ tensor, after taking into consideration the relations for $\omega_{\mu}{}^{ab}(e,b)$ and $f_{\mu}{}^{a}(e,b)$, is found to be:

$$\left[ R_{\mu\nu}{}^{ab} \right]^{f(e,b)}_{\omega(e,b)} = -\left( R^{(0)ab}_{\mu\nu} - 2 e_{[\mu}{}^{[a} R_{\nu]}{}^{b]} - \frac{1}{3} e_{[\mu}{}^{[a} e_{\nu]}{}^{b]} R \right) = -C^{ab}_{\mu\nu} \,, \tag{2.111}$$

where $C^{ab}_{\mu\nu}$ is the Weyl tensor and in the Poincaré curvature $R^{(0)ab}_{\mu\nu}$ in the above equation, the corresponding constraint, (2.82), has been included. Specifically:

$$R^{(0)ab}_{\mu\nu} = R^{(0)ab}_{\mu\nu}\left(\omega(e)\right) = -\left[ \hat{R}^{ab}_{\mu\nu}(\omega) \right]^{f_{\mu}{}^{a}=0}_{b_{\mu}=0} \,. \tag{2.112}$$

In the penultimate relation, (2.111), it is obvious that the expression of the curvature tensor of the theory, after the inclusion of the constraints, $\left[R_{\mu\nu}{}^{ab}\right]^{f(e,b)}_{\omega(e,b)}$, is independent of the field $b_{\mu}$ and, since it drops out, one may set $b_{\mu} = 0$. After the fixing of $b_{\mu} = 0$ (known as the K-gauge), the only independent gauge field of the action is the vierbein, $e$ and, therefore the action is scale and proper conformal invariant, since, in contrast to the Lorentz boosts, all physical fields transform only trivially under the conformal boosts generated by $K_a$ and so does the vierbein, $\delta^{K} e = 0$, as it is evident in (2.90). Also, the dilatation gauge field, $b_{\mu}$ is the only field of the theory which transforms non-trivially under $K_a$. Therefore, since the action is no longer dependent of $b_{\mu}$, the whole action is K-invariant[14]. Due to the presence of K-invariance, one could support that the symmetry of the final theory is $X \otimes D \otimes K$. However, after the gauge fixing of $b_\mu = 0$, the special conformal transformations are no longer independent transformations. In fact, this becomes transparent if the expression of the gauge transformation of $b_\mu$, (2.92), is considered, without the term involving $\xi^a$, since the $X$ symmetry of general covariance makes up for it in the final symmetry:

$$\delta b_\mu = \partial_\mu \kappa + 2\rho^a e_{\mu a} \;\Rightarrow\; \rho^a = -\frac{1}{2} e^{\mu a} \partial_\mu \kappa \; . \tag{2.113}$$

The above relation, originating from the preservation of a gauge condition, is called decomposition law [9] and expresses a gauge-fixed symmetry parameter ($\rho^a$) in terms of gauge parameters of symmetries that remain in the theory ($\kappa$). Concluding, the final action is invariant under the $X \otimes D$ transformations, which consists of the general coordinate transformations and the Weyl transformations.

Therefore, employing the relation between the metric and the vierbein, (2.3), the expression of the action, $S_W$, (2.110), can be written in terms of the Weyl tensor and finally takes the form of the well-known Weyl action:

$$S_W = \frac{1}{2a^2} \int d^4x \sqrt{g} C_{\lambda\mu\nu\rho} C^{\lambda\mu\nu\rho} = \frac{1}{a^2} \int d^4x \sqrt{g} \left( R^2_{\mu\nu} - \frac{1}{3} R^2 \right) \; . \tag{2.114}$$

Besides the above symmetry breaking, we suggest another possible way of breaking the initial symmetry, this time to the Lorentz. This could occur with the inclusion of two scalars in the fundamental representation of the SO(2,4) gauge group [88]. This comes as no surprise, since it is just an extension of the way the de Sitter group (and not the Poincaré for reasons explained in the section 2.3) is broken down to the Lorentz by a scalar in the fundamental of SO(1,4), as described in the previous section for the case of the Einstein gravity. These two scalars could induce a spontaneous symmetry breaking in a complete theory including matter fields, giving rise to the constraints that lead to a resulting four-dimensional action respecting Lorentz symmetry. It is worth-noting that the four-dimensional conformal gauge theory can lead to the Einstein-Hilbert action by choosing a different route of symmetry breaking, that is by imposing constraints, as it is explained in Ref. [89]. More specifically, it is argued that if both tensors $R(P)$ and $R(K)$ are simultaneously set to zero[15], then from the constraints of the theory it is understood that the corresponding gauge fields, $f_\mu{}^a, e_\mu{}^a$ are equal - up to a rescaling factor - and $b_\mu = 0$.

Now, let us recapitulate in order to conclude this section. Starting from the conformal gauge group, SO(2,4), and following the standard procedure of building gauge theories of spacetime symmetry groups, a gauge connection was defined and one gauge field was assigned for each generator. Then, calculations led to the transformation rules of these gauge fields and the expressions of their corresponding component curvature tensors. For the sake of translational invariance of the final theory, the first constraint that was imposed was the vanishing of the torsion tensor, giving the opportunity to express the translations as a combination of the general

[14] In ref. [9], the argument about K-invariance is presented inversely. In other words, K-invariance of the $e, b$-dependent action is imposed in principal, therefore, since the vierbein is trivially transforming under $K$ and the dilatation is not, the latter must be zero in order to result with a K-invariant action.

[15] The consideration of the vanishing of both tensors at the same time is supported by the fact that, since it is desired to result with the Lorentz symmetry out of the initial SO(2,4), the vacuum of the theory can be considered to be directly SO(4) invariant, which means that every other tensor, except for the $R(M)$, has to be vanishing [78].

coordinate transformation and the residual symmetry, $H$, in other words, trading the translations with the general coordinate transformations. The solution of this constraint rendered the spin connection field to depend on $e$ and $b$ fields ($b$ is the dilatation field). Then, again for reasons of general covariance, an additional constraint was imposed, this time related to the curvature tensor of $M_{ab}$. From this constraint the gauge field $f$, which is related to the generators of the conformal boosts, $K_a$, was expressed in terms of the Ricci tensor and scalar. Therefore, two out of the four groups of gauge fields were expressed in terms of the other two. Then, the unique action was written down, in which the dilatation field was cancelled out from the calculations giving the opportunity to set it equal to zero. This way, the action ceases to contain the only field that transforms non-trivially under the conformal boosts, which means that the final action is K-invariant, too. However, K-invariance is not an independent symmetry after the gauge fixing and, for this reason, it was absorbed by the rest of the residual symmetry and did not appear in the final, residual symmetry of the theory. Therefore, the invariance group of the final action (Weyl action) consisted of the general coordinate transformations, $X$, and the Weyl transformations generated by $D$. Eventually, it is legit to remark explicitly that conformal gravity can be successfully described as a gauge theory of the SO(2,4) gauge group.

# Chapter 3

# Noncommutative spaces and gauge theories

In this section we give some information about the framework of noncommutative geometry and then we focus on the description of some specific and important noncommutative spaces, some of which we employ for the building of the models in the next sections. Also, we write down the methodology that has to be followed in order to construct gauge theories on these fuzzy spaces.

## 3.1 The noncommutative framework

Noncommutativity of coordinates is a notion that may be regarded to be contradictive to our perception of space structure. The whole idea that two space coordinates cannot be measured with precision at the same time may seem to defy our intuition. Nevertheless, in general, the notion of noncommutativity is not as radical as may sound, since it has been encountered before in different cases for some other physical quantities. The most common, macroscopic example is the case of the vector of angular momentum, the components of which do not commute with each other. Moreover, in quantum mechanics any two conjugate variables are noncommutative, for example position and momentum, being subjected to an uncertainty relation.

Quantum mechanics is more than just another example in which noncommutativity of variables of physical quantities is encountered; it is the source of inspiration for the foundation of the framework of noncommutative geometry[1] [11]. Specifically, in quantum mechanics, the phase space of canonical position, $x^i$, and momentum, $p_j$, gets quantized, replacing them with their corresponding Hermitian operators $\hat{x}^i, \hat{p}_j$, which apparently do not commute. In contrast, they obey the Heisenberg commutation relation:

$$[\hat{x}_i, \hat{p}^j] = i\hbar\delta_i^{\ j} \ . \tag{3.1}$$

Along these lines of quantizing a classical phase space, the quantization of a space would take place if its coordinates, $x^i$ would be replaced by operators, $\hat{x}^i$, of a $C^*$-algebra[2] of functions

[1]For a more historical and motivation-oriented description of the birth of noncommutativity see the introduction, Section 1.

[2]A $C^*$-algebra is a complex algebra $A$ of continuous linear operators on a complex Hilbert space with two additional properties:

- $A$ is a topologically closed set in the norm topology of operators.

on the space, let us call it $A$, obeying a commutation relation analogous to the Heisenberg commutation relation, that is:

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}(x) \ , \tag{3.2}$$

where $\theta^{ij}(x)$ is a quantity, which in general depends on the coordinates, parametrizing the type of noncommutativity of the space[3]. More details on the nature of $\theta^{ij}$ are given later in the current section.

### 3.1.1 The matrix representation

Let us focus on the $C^*$-algebra $A$ for a while. This algebra is necessarily associative and optionally commutative. So, let us first consider a commutative algebra $A$ and an element of this algebra, which is a configuration of a classical complex scalar field on a space $M$. The most typical example of a commutative and associative algebra is that of functions on a manifold $M$, taking complex values, with addition and multiplication given by:

$$(f+g)(x) = f(x) + g(x) \qquad \text{and} \qquad (f \cdot g)(x) = f(x)g(x) \ . \tag{3.3}$$

Now, the most illustrative example of a noncommutative algebra is that of $N \times N$ matrices with complex entries, that is the algebra $\mathrm{Mat}(N, \mathbb{C})$. Generalizations of the $\mathrm{Mat}(N, \mathbb{C})$ are the matrix algebras of $N \times N$ matrices, with entries elements of the algebra $A$, which are called $\mathrm{Mat}(N, A)$. Addition and multiplication for this algebra are defined in accordance to the matrix addition and multiplication, in terms of those in $A$. This matrix representation of a noncommutative algebra, $A$, with operators as elements, suggests a very smooth way to think of this algebra, since it is quite familiar to the case in quantum mechanic in which the differential operators can be represented by matrices acting on the vectors of the Hilbert space [90].

Now, in order to formulate noncommutative field theories, it is necessary to define the operations of derivation, $e_i$, and integration, $\int \mathrm{Tr}$. These two operations must satisfy the following properties:

- The Leibniz rule: $e_i(AB) = e_i(A)B + Ae_i(B)$. This property combined with linearity means that the derivation of a constant is vanishing.
- The integral of the trace of a total derivative is vanishing: $\int \mathrm{Tr} e_i(A) = 0$.
- The integral of the trace of a commutator is vanishing: $\int \mathrm{Tr}[A, B] = 0$.

Following from the above properties, a nice candidate for the derivation is $e_i(A) = [d_i, A]$, where $d_i$ is an element of the algebra $A$. Since for general nonommutative algebras using matrices separation between the notation of the integral and the trace is not possible, we will denote the integration with the single symbol of the trace.

The above definition of derivations and integrations allows one to study noncommutative field theories, with the operators of the algebra $A$ to be represented by matrices. Indeed, this is the representation we adopt, viewing the operators as matrices, in the construction of such theories in the next sections.

- $A$ is closed under the operation of taking adjoints of operators.

[3]The $i$ in the right-hand side is involved because the commutator of Hermitian operators is anti-Hermitian.

Nevertheless, an alternative way to study noncommutative field theories is to associate the operators of a noncommutative algebra and ordinary multiplication with an algebra of functions of commuting variables equipped with a deformed product. In the next section we review the basics around this subject through the cases of the two very common (and interesting) cases of noncommutative spaces, the canonical and Lie-type.

**The canonical case (Moyal space), the Lie-type case and Weyl quantization**

In the previous section, $\theta^{ij}$ was introduced in equation (3.2) as the noncommutativity parameter, which, regardless its expression, is an antisymmetric tensor, since the left-hand side of the same equation is also antisymmetric. Now, depending on its expression, this parameter defines the kind of noncommutativity of the space.

*The canonical case*

Let us now introduce the canonical case, which is the simplest example of noncommutativity and is defined by $\theta^{ij}(x) = \theta^{ij}$, that is $\theta^{ij}$ is a $N \times N$ constant (complex) antisymmetric tensor, independent of the coordinates. The commutation relation of the coordinates is written down as:

$$[\hat{x}^i, \hat{x}^j] = \theta^{ij} \ , \quad i, j = 1, \ldots, N \ . \tag{3.4}$$

Noncommutative spaces defined by the above relation are denoted as $\mathbb{R}^N_\theta$ and, especially for the $N = 2$ case, $\mathbb{R}^2_\theta$ is called the Moyal space (or Moyal plane).

*The Lie-type case*

Another very interesting case is that of Lie-type noncommutativity, in which the noncommutativity parameter, $\theta_{ij}$, is linearly dependent on the coordinates. The corresponding commutation relation is given by:

$$[\hat{x}^i, \hat{x}^j] = iC^{ij}{}_k \hat{x}^k \ , i, j = 1, \ldots N \ , \tag{3.5}$$

where $C^{ij}{}_k$ are complex numbers. The resemblance to the Lie-algebra structure is striking and that is why the noncommutativity is called Lie-type. A very interesting case of this type is for $N = 3$, where the above relation is actually the definition of the SU(2) algebra. In general, we are going to deal with this $N = 3$ case a lot, since two noncommutative spaces we employ in the building of our models are based on it, namely the fuzzy sphere, $S^2_F$ and $R^3_\lambda$, which we discuss later in detail.

*Weyl quantization*

Previously, we gave some insight on how to work with noncommutative algebras, representing the operators that are elements of $A$ by matrices. There is also an alternative and equivalent way of treating this issue, by associating the operators of the noncommutative algebra with classical (commuting) functions that are subjected to a different kind of multiplication, that is called the Moyal-Weyl $\star$-product. In other words, one can use a mapping from operators that do not commute to functions that commute and upgrade the product from the ordinary for

the operators to the $\star$-product for the functions. This correspondence of operators to functions is one-to-one and is called the Weyl correspondence (or Weyl quantization) [33].
Let $f(x)$ be a function, admitting Fourier expansion, depending on the canonical variables $x^i$. We define an operator $W(f)$ as:

$$W(f) = \frac{1}{(2\pi)^{n/2}} \int d^n k e^{ik_j \hat{x}^j} \tilde{f}(k) \ , \tag{3.6}$$

where $\tilde{f}(k)$ is the Fourier transformation of the function $f(x)$:

$$\tilde{f}(k) = \frac{1}{(2\pi)^{n/2}} \int d^n x e^{-ik_j x^j} f(x) \ , \quad f(x) = \frac{1}{(2\pi)^{n/2}} \int d^n k e^{ik_j x^j} \tilde{f}(k) \tag{3.7}$$

and the operator $\hat{x}$ replaces the variable $x$ in $f$ in the most straightforward way. Multiplication of operators defined in (3.6) yields new operators. The crucial point is to examine whether the product of two such operators, which are associated to classical functions through (3.6), can be also associated to classical functions. Let us assume that such a correspondence exists, i.e. there exists a function which is associated to the product of two operators $W(f)W(g)$ and is denoted as $f \star g$. Then, multiplication of the two operators must give:

$$W(f)W(g) = W(f \star g) = \frac{1}{(2\pi)^n} \int d^n k d^n p e^{ik_i \hat{x}^i} e^{ip_j \hat{x}^j} \tilde{f}(k) \tilde{g}(p) \ . \tag{3.8}$$

If the product of the two exponentials yields a linear combination of $\hat{x}$, then the function $f \star g$ will exist. For the calculation, the Baker-Campbell-Hausdorff formula:

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]+[Y,[Y,X]])-\frac{1}{24}[Y,[X,[X,Y]]]+\dots} \tag{3.9}$$

has to be employed. Therefore, in the canonical case, (3.4), the above formula gives the following result:

$$\begin{aligned} e^{ik_i \hat{x}^i} e^{ip_j \hat{x}^j} &= e^{ik_i \hat{x}^i + ip_j \hat{x}^j + \frac{1}{2}[ik_i \hat{x}^i, ip_j \hat{x}^j] + 0} \\ &= e^{i(k_i+p_i)\hat{x}^i - \frac{1}{2} k_i p_j \theta_{ij}} \ . \end{aligned} \tag{3.10}$$

Therefore, replacing the above result in the expression of the product of the two operators, (3.8), one is led to the expression:

$$W(f \star g) = \frac{1}{(2\pi)^n} \int d^n k d^n p e^{i(k_j+p_j)\hat{x}^j - \frac{i}{2} k_i \theta^{ij} p_j} \tilde{f}(k) \tilde{g}(p) \ . \tag{3.11}$$

The expression of the $f \star g$ is obtained from the above relation, if one replaces the operators $\hat{x}^i$ with the variable $x^i$, that is the inverse substitution of the one employed to define $W(f)$ in (3.6):

$$\begin{aligned} f \star g &= \frac{1}{(2\pi)^n} \int d^n k d^n p e^{i(k_j+p_j)x^j - \frac{i}{2} k_i \theta^{ij} p_j} \tilde{f}(k) \tilde{g}(p) \ \Rightarrow \\ f \star g &= e^{\frac{i}{2} \frac{\partial}{\partial x^i} \theta^{ij} \frac{\partial}{\partial y^i}} f(x) g(x)|_{y \to x} \ . \end{aligned} \tag{3.12}$$

The last equation is the definition of the Moyal-Weyl $\star$-product.

Following the same procedure for the Lie-type case, (3.5), one may result to a similar result. Specifically, the Baker-Campbell-Hausdorff identity, (3.9), in this case leads to the following expression for the product of the two exponentials:

$$e^{ik_i\hat{x}^i}e^{ip_j\hat{x}^j} = e^{P_i(k,p)\hat{x}^i} = e^{k_i+p_i+\frac{1}{2}g_i(k,p)} \; . \tag{3.13}$$

In the above relation, it is implied that every term in the Baker-Campbell-Hausdorff identity gives a result linear with respect to $\hat{x}$. Therefore, $g_i(k,p)$ concentrates all the information about the noncommutativity of the group. Therefore, the $\star$-product for the Lie-type case is found accordingly:

$$\begin{aligned} f \star g &= \frac{1}{(2\pi)^n}\int d^n k d^p e^{iP_i(k,p)x^i}\tilde{f}(k)\tilde{g}(p) \;\Rightarrow \\ f \star g &= e^{\frac{i}{2}x^i g_i(i\frac{\partial}{\partial y}, i\frac{\partial}{\partial z})} f(y)g(z)|_{y\to x}^{z\to x} \; . \end{aligned} \tag{3.14}$$

Starting from (3.12), it is straightforward to find $x^i \star x^j$ for the canonical case[4]:

$$x_i \star x_j = \left(1 + \frac{i}{2}\theta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\right)x_i x_j = x_i x_j + \frac{i}{2}\theta^{ij} \; . \tag{3.15}$$

Also, $x_j \star x_i$ is given by:

$$x_j \star x_i = x_i x_j - \frac{i}{2}\theta^{ij} \; . \tag{3.16}$$

Combining the above equations, (3.15) and (3.16), one obtains the expression of the $\star$-commutator:

$$[x_i \overset{\star}{,} x_j] = i\theta^{ij} \; , \tag{3.17}$$

which is the algebra defined in (3.4), where the operators have been replaced by functions and the ordinary product by the $\star$-product.

It is worth-noting that the choice of the above Weyl correspondence is not unique. Let us focus on the $\star$-product of the canonical case. In equation (3.6), the replacement of canonical variable $x^i$ happened in the most symmetric way, that is with the operator $\hat{x}^i$ (Weyl ordering), or equivalently, $e^{ikx} = (e\star)^{ikx}$. However, one may define infinite number of $\star$-products, all resulting to the same algebra, (3.4). More explicitly, let us add a constant symmetric matrix $A^{ij}$ to the $\theta^{ij}$ in (3.12), or equivalently, take the following modification of the relation (3.15):

$$x^i \star x^j = x^i x^j + \frac{i}{2}\theta^{ij} + A^{ij} \; . \tag{3.18}$$

The above expression of the $\star$-product of two canonical variables is a generalization of (3.15), which, in the operator language, corresponds to a different way of ordering. Indeed, the choice $A^{ij} = \frac{L}{2}\delta^{ij}$ leads to a new $\star$-product, that is $(e\star)^{ikx} = e^{ikx}e^{-Lk^2/4}$, which physically means that instead of using the ordinary waves for the Fourier expansion of the fields, wave packets of width $\sqrt{L/2}$ have been used. The above generalization, (3.18) leads to the same algebra, $[x_i \overset{\star}{,} x_j] = i\theta^{ij}$, therefore different versions of a field theory which correspond to different $\star$-products resulting from different $A_{ij}$, are all related by a field redefinition [91].

Concluding, in the previous section we mentioned that operators belonging to a noncommutative algebra $A$ can be represented by matrices, with multiplication the ordinary matrix

[4] For avoiding confusion, $x_i, x_j$ are elements of the commutative algebra.

product. However, in the current section, we also studied an alternative way of treating the operators of such an algebra, that is replace them by ordinary, commutative functions with multiplication the $\star$-product. Those two different approaches are related in the sense that for an action with fields multiplied with a $\star$-product, it is possible to be viewed as a matrix model. This is true because, the algebras of function equipped with the $\star$-product are actually operators with ordinary multiplication which act on a Hilbert space, therefore they can be represented by matrices [92]. From now on, we employ the matrix representation for the coordinates of the noncommutative spaces on which we work[5].

## 3.2 The fuzzy sphere

*Construction of the Fuzzy sphere*

In this section we discuss the most typical, Lie-type case of noncommutative space, the fuzzy sphere, $S_F^2$ [93], employing the matrix representation of its coordinates. First of all, a fuzzy space is defined as a discrete matrix approximation of a continuous manifold with the additional property of preserving the isometries. In other words, it is a noncommutative space that preserves the isometries of its commutative analogue. A very instructive way to introduce the fuzzy sphere, which is followed here, too, is to formulate it in a comparative way to the ordinary sphere, $S^2$.

The ordinary sphere, $S^2$ can be defined as a submanifold of the Euclidean space of one dimension higher, that is the $\mathbb{R}^3$, with its Cartesian coordinates $x_a, a = 1, 2, 3$, satisfying the constraint:

$$\sum_{a=1}^{3} x_a^2 = x_1^2 + x_2^2 + x_3^2 = R^2 \ , \tag{3.19}$$

where $R$ is a constant identified as the radius of the sphere. The sphere admits an obvious rotational symmetry which is parametrized by the SO(3) group (isometry group). The SO(3) is generated by the three angular momentum operators which are defined as $L_a = -i\epsilon_{abc}x_b\partial_c$ and can be also written in terms of spherical coordinates $\theta, \phi$, as $L_a = -\xi_a^i\partial_i$, where $i = \theta, \phi$ and $\xi_a^i$ are the components of the Killing vectors. The Laplace operator is defined on the sphere by the relation:

$$L^2 = -R^2\Delta_{S^2} = -R^2\frac{1}{\sqrt{g}}\partial_i(g^{ij}\sqrt{g}\partial_j) \ , \tag{3.20}$$

where $g_{ij}$ is the metric tensor of the sphere. The eigenvectors of the above operators are the well-known spherical harmonics, $Y_{lm}(\theta, \phi)$, which are defined as:

$$Y_l^m(\theta, \phi) = Ne^{im\phi}P_l^m(\cos\theta) \ , \tag{3.21}$$

where $P_l^m$ are the associated Legendre polynomials. The spherical harmonics obey the following orthonormality condition:

$$\int d\Omega Y_{lm}^\dagger Y_{l'm'} = \delta_{ll'}\delta_{mm'} \ . \tag{3.22}$$

[5] A methodology to recover the matrix representation starting from the $\star-$ product is given in ref. [92]

Let us now consider a function, $f(\theta, \phi)$, on $S^2$. Since the spherical harmonics form a complete and orthogonal set of functions, the function $f(\theta, \phi)$ can be expanded on this set:

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\theta, \phi) \ , \tag{3.23}$$

where $c_{lm}$ are complex coefficients.

Let us now move on with the fuzzy version of the sphere, $S_F^2$. The fuzzy sphere is a noncommutative space, which means that functions depending on the coordinates (operators) defined on it do not not commute under the ordinary product. Let us give an explanation on how this fuzzy space is derived starting from properties of the ordinary sphere.

A discretized version of the sphere can be achieved by replacing the algebra of functions, $Y_{lm}(\theta, \phi)$ on the sphere with the set of functions, $\hat{Y}_{lm}(\theta, \phi)$, which do not exceed a specific value of $l$, let us denote it $N$. Therefore, a function, $\hat{f}(\theta, \phi)$, on the sphere is written as an expansion on the finite set of $\hat{Y}_{lm}$:

$$\hat{f} = \sum_{l=0}^{N} \sum_{m=-l}^{l} c_{lm} \hat{Y}_{lm} \ . \tag{3.24}$$

Now, if we consider a product of two such functions, then it will involve terms with $l$ up to a specific value, $j$, namely $j = 2N$,[6] exceeding the upper limit, $N$, which means that the truncated algebra of functions does not close under multiplication. A very elegant and efficient way to result with an algebra of those truncated functions is to consider a different type of product which is noncommutative, more specifically, a matrix product[7]. Therefore, in the discretization of the sphere, the truncation replaces an infinite-dimensional commutative algebra of functions with a finite, $(N+1)$-dimensional noncommutative algebra. This discretized space is defined as the fuzzy sphere[8].

The most straightforward way to formulate the fuzzy sphere is to consider this truncated, noncommutative algebra as a matrix algebra on some finite-dimensional vector space. For this reason, let us take the three $(N+1)$-dimensional matrices $J_a, a = 1, 2, 3$ that form a basis for the $(N+1)-$dimensional irreducible representation of SU(2). The generators $J_a$ satisfy the following commutation relation:

$$[J_a, J_b] = i\epsilon_{abc} J_c \ . \tag{3.25}$$

Also, since the matrices representing the generators $J_a$ are considered to be set in an irreducible representation, the value of the Casimir operator in this $(N+1)$-dimensional representation[9] is:

$$J^2 = J_1^2 + J_2^2 + J_3^2 = \frac{N}{2}\left(\frac{N}{2} + 1\right) \mathbf{1}_{N+1} \ , \tag{3.26}$$

Therefore, the fuzzy sphere, $S_F^2$, at fuzziness level $N$ is the noncommutative space whose coordinate functions, $\hat{X}_a = \hat{X}^a, a = 1, 2, 3$ are defined as the $(N+1) \times (N+1)$ Hermitian

[6]This value is obtained from the addition of the maximum values, $N$, of the two angular momenta.

[7]Alternatively, given the equivalence of the matrix realization of the operators of a noncommutative space to the function realization with a modified product, one could define an appropriate $\star$-product in order to introduce noncommutativity.

[8]That is the reason why the fuzzy spaces, in general, are considered as matrix approximations of ordinary spaces.

[9]The standard Casimir operator definition for an $N$-dimensional irreducible representation is $J^2 = \frac{1}{4}(N^2 - 1)$.

matrices which are proportional to the generators, $J_a$, of the $(N+1)$-dimensional irreducible representation of SU(2), that is:

$$\hat{X}_a = \kappa J_a \ , \tag{3.27}$$

where $\kappa$ is the proportionality constant that is determined by the fact that the $\hat{X}_a$ are coordinates of a (fuzzy) sphere, therefore they have to obey the constraint:

$$\sum_{a=1}^{3} \hat{X}_a \hat{X}_a = \hat{X}_1^2 + \hat{X}_2^2 + \hat{X}_3^2 = r^2 \ , \tag{3.28}$$

where $r$ is the radius of the fuzzy sphere. Taking into consideration the expression of the Casimir operator of the generators $J_a$ of the SU(2) given in (3.26) and replacing the expression of the coordinates, $\hat{X}_a$, in terms of the generators given in (3.27), one results with the expression of the proportionality constant, $\kappa$:

$$\kappa = \frac{r}{\sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)}} = \lambda_N r \ , \tag{3.29}$$

where $\lambda_N = \frac{1}{\sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)}}$ and therefore, starting from (3.27), the coordinate matrices, $\hat{X}_a$ are written as:

$$\hat{X}_a = \kappa J_a = \frac{r}{\sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)}} J_a = \lambda_N r J_a \ . \tag{3.30}$$

Also, the behaviour of the product of two coordinate operators is given by their commutation relation, which is calculated using the commutation relation of the $J_a$ generators of SU(2), (3.25):

$$[\hat{X}_a, \hat{X}_b] = i\kappa\epsilon_{abc}\hat{X}_c = i\lambda_N C_{abc}\hat{X}_c \ , \tag{3.31}$$

where the relation (3.29) has been used and $C_{abc} = r\epsilon_{abc}$.

Now, one may redefine the coordinate matrices, $\hat{X}_a$, and work with the following, anti-Hermitian ones, $X_a$:

$$X_a = \frac{1}{i\kappa r}\hat{X}_a = \frac{1}{ir} J_a \ . \tag{3.32}$$

The commutation relation, (3.31), and the radius constraint, (3.28) are redefined as:

$$[X_a, X_b] = C_{abc}X_c \quad \text{and} \quad \sum_{a=1}^{3} X_a X_a = -\frac{\lambda_N^{-2}}{r^2} \ , \tag{3.33}$$

where $C_{abc}$ is now defined as $C_{abc} = \frac{\epsilon_{abc}}{r}$. The algebra of the fuzzy sphere is equivalently described by both bases.

We should also note that the functions, $\hat{Y}_{lm}$, of the finite set which spans a function, $\hat{f}$, on the fuzzy sphere, (3.24), are known as fuzzy spherical harmonics and are given by the expression [94]:

$$\hat{Y}_{lm} = r^{-l}\sum_{\vec{a}} f^{(lm)}_{a_1\ldots a_l}\hat{X}^{a_1}\ldots\hat{X}^{a_l} \ , \tag{3.34}$$

which is the fuzzy sphere analogue of the expression that gives the classical spherical harmonics in terms of the Cartesian coordinates:

$$Y_{lm}(\theta,\phi) = \sum_{\vec{a}} f^{(lm)}_{a_1\ldots a_l} x^{a_1}\ldots x^{a_l} \ . \tag{3.35}$$

In both cases, $f^{(lm)}_{a_1 \dots a_l}$ is a traceless symmetric tensor of SO(3) with rank $l$. Also, the fuzzy spherical harmonics obey an orthonormality condition, which is given by:

$$\mathrm{Tr}_N \left( \hat{Y}^{\dagger}_{lm} \hat{Y}_{l'm'} \right) = \delta_{ll'} \delta_{mm'} \ , \tag{3.36}$$

where, as we explain in the following paragraph, $\mathrm{Tr}_N$ stands for the integration operation.

*Differential calculus and integration on the fuzzy sphere*

Let us now mention some aspects about the differential calculus and the integration on the fuzzy sphere that will be useful in the following sections.

Along the lines of the general discussion in 3.1.1, integration on the fuzzy sphere is defined by the following mapping:

$$\frac{1}{4\pi} \int d\Omega \ \rightarrow \ \frac{1}{N} \mathrm{Tr}_N \ . \tag{3.37}$$

Differential calculus on the fuzzy sphere is three-dimensional and SU(2)-covariant [67]. According to section 3.1.1, the derivation is associated with the commutator of an element of the algebra on the fuzzy sphere. Therefore, the derivations of a function $\hat{f}$, along $X_a$ are given by:

$$e_a(\hat{f}) = [X_a, \hat{f}] \ . \tag{3.38}$$

Consequently, the Lie derivative on the function $\hat{f}$ is given by:

$$\mathcal{L}_a \hat{f} = [X_a, \hat{f}] \ . \tag{3.39}$$

Now, let $\theta^a$ be an 1-form dual to the vector field, $e_a$, that is:

$$\langle e_a, \theta^b \rangle = \delta^b_a \ . \tag{3.40}$$

Therefore, the exterior derivative, $\mathrm{d}$ acting on a function, $\hat{f}$, yields:

$$\mathrm{d}\hat{f} = [X_a, \hat{f}]\theta^a \ . \tag{3.41}$$

Also the action of Lie derivatives on a vector of fuzzy sphere is obtained:

$$\mathcal{L}_a(e_b)\hat{f} = \mathcal{L}_a(e_b\hat{f}) - e_b(\mathcal{L}_a\hat{f}) \overset{3.39}{=} [\mathcal{L}_a, \mathcal{L}_b]\hat{f} = C_{abc}\mathcal{L}_c\hat{f} = C_{abc}e_c\hat{f} \ \Rightarrow \ \mathcal{L}_a e_b = C_{abc}e_c \ , \tag{3.42}$$

since the Lie derivative satisfy the Leibniz rule and the SU(2) Lie algebra commutation relation:

$$[\mathcal{L}_a, \mathcal{L}_b] = C_{abc}\mathcal{L}_c \ . \tag{3.43}$$

Moreover, starting from (3.40), one obtains the action of Lie derivative on 1-forms:

$$\mathcal{L}_a\langle e_b, \theta^c \rangle = 0 \ \Rightarrow \ \langle \mathcal{L}_a e_b, \theta^c \rangle = -\langle e_b, \mathcal{L}_a \theta^c \rangle \ \overset{(3.42)}{\Rightarrow} \ \mathcal{L}_a \theta^b = C_{abc}\theta^c \ . \tag{3.44}$$

The above definitions and calculations of the differential calculus on the fuzzy sphere are very important for the construction of our models.

## 3.3 Gauge theory on fuzzy spaces

In this section the methodology that is followed in order to build gauge theories on fuzzy spaces is described. In the beginning general features applicable on fuzzy spaces in general [33] are presented without specialization, unless necessary.

First, let us consider a scalar field, $\phi(X)$, which is an element of the noncommutative algebra, $A$ and a gauge group, $G$. An infinitesimal gauge transformation is given by the relation:

$$\delta\phi(X) = i\alpha(X)\phi(X) \; , \tag{3.45}$$

which is a covariant transformation rule of the field $\phi(X)$ and $a(X)$ is an infinitesimal gauge parameter depending on the coordinates, being a local gauge transformation. Also, it should be remarked that the transformation $a(X)$ could be Abelian or non-Abelian, depending on where the parameter belongs. Specifically, if $a(X)$ belongs to the algebra $A$, then it is an Abelian transformation, whereas if it belongs in $\mathrm{Mat}(A)$, that is an algebra of matrices with entries elements of $A$, then it is non-Abelian. It is worth-noting that a gauge transformation on the coordinate $X$ does not affect it, meaning that:

$$\delta X_a = 0 \; . \tag{3.46}$$

Recalling the case of the ordinary gauge theories, the transformation of a partial derivative of a field, $(\partial_\mu\phi(x))'$, is not covariant and similarly in the fuzzy case too, the transformation of the product $X_a\phi(X)$:

$$\delta\left(X_a\phi(X)\right) = iX_a a(X)\phi(X) \; , \tag{3.47}$$

is not covariant[10], since $iX_a a(X)\phi(X) \neq ia(X)X_a\phi(X)$, since $X_a$ and $a(X)$ are both elements of the noncommutative algebra, $A$. Therefore, just like in the ordinary gauge theories, in which the next step would be to define a derivative that would transform covariantly, here, too, one has to define the fuzzy analogue of the covariant derivative, which is called *covariant coordinate*, denoted as $\hat{X}_a$[11], which, by definition, fixes the transformation of the product $X_a\phi(X)$ to be covariant:

$$\delta(\hat{X}_a\phi(X)) = ia(X)\hat{X}_a\phi(X) \; . \tag{3.48}$$

From the above equation, one results with the transformation of the covariant coordinate:

$$\delta\hat{X}_a = i[a(x), \hat{X}_a] \; . \tag{3.49}$$

The covariant coordinate, $\hat{X}_a$, is related to the coordinate, $X_a$, as follows:

$$\hat{X}_a = X_a + A_a(X) \; , \tag{3.50}$$

where $A_a(X)$ is an element of the algebra $A$ and is the noncommutative analogue of the gauge potential of the ordinary gauge theories. The transformation of the gauge field, $A_a(X)$, is

[10] It is not a covariant transformation in the sense that it does not transform linearly, as the field $\phi(X)$ does, as obtained in (3.45).

[11] Until now, we had seen that $\hat{X}_a$ was denoting the Hermitian operators of the coordinates of the fuzzy sphere, (3.27). However, throughout the thesis we are using the anti-Hermitian operators, $X_a$, given in (3.32), to denote the coordinates of the fuzzy sphere, therefore, from now on, we commit $\hat{X}_a$ to denote the covariant coordinate of a gauge theory.

obtained in a straightforward way, starting from the transformation property of the covariant coordinate, given in (3.49):

$$\delta A_a = i[a(X), A_a(X)] - i[X_a, a(X)] \ . \tag{3.51}$$

Until this point, the above analysis has been carried out without having specified the fuzzy space and it applies for all cases of noncommutative gauge theories. Moving on, for the construction of gauge-theoretic models, the next object that is necessary to be defined is the field strength tensor. Again, drawing lessons from the ordinary case, the field strength tensor is generally defined as the commutator of the covariant derivatives. However, in the noncommutative framework, this definition is not valid and there is no general formula applicable to all cases. Therefore, one has to define the field strength tensor in different ways, depending on the type of noncommutativity. Here, the field strength tensors for the canonical and Lie-type cases, (3.4) and (3.5), are defined and, later on, the definition of the field strength tensor is provided each time another fuzzy space is introduced.

*Field strength tensor of the canonical case*

The field strength tensor for the canonical case is defined as:

$$T_{ab} = [\hat{X}_a, \hat{X}_b] - i\theta_{ab} \ . \tag{3.52}$$

Replacing in the above expression the equation (3.50), in which the covariant coordinate and the gauge field are related, then one ends up with an alternative expression for $T_{ab}$[12]:

$$T_{ab} = [X_a, A_b] - [X_b, A_a] + [A_a, A_b] \ , \tag{3.53}$$

in which the analogy to the ordinary gauge theories is manifest, judging from (let us say) the (2.70). Moreover, it is important to check the transformation property of the above tensor, $T_{ab}$:

$$\delta T_{ab} = [X_a, \delta A_b] - [X_b, \delta A_a] + [\delta A_a, A_b] + [A_a, \delta A_b] \ . \tag{3.54}$$

Replacing with the expression of the transformation of the gauge field, (3.51) and making use of the Jacobi identity, the final expression for the transformation of $T_{ab}$ is obtained:

$$\delta T_{ab} = i[a, T_{ab}] \ . \tag{3.55}$$

The above result of the transformation of the field strength tensor shows that it transforms covariantly under the gauge transformation.

*Field strength tensor of the Lie-type case*

In the Lie-type case, the corresponding field strength tensor is defined as:

$$F_{ab} = [\hat{X}_a, \hat{X}_b] - iC_{ab}{}^c \hat{X}_c \ , \tag{3.56}$$

[12]From now on we drop the explicit writing of the $X$-dependence for the sake of notational simplicity and will only be recovered in case of possible confusion.

where the $C_{ab}{}^{c}$ is related to the totally antisymmetric tensor. Repeating the procedure followed for the canonical case, first one ends up with the following expression for $F_{ab}$:

$$F_{ab} = [X_a, A_b] - [X_b, A_a] + [A_a, A_b] - iC_{ab}{}^{c}A_c \ , \tag{3.57}$$

and then its gauge transformation rule is obtained:

$$\delta F_{ab} = i[a, F_{ab}] \ , \tag{3.58}$$

which is also covariant.

Instead of working with matrices, one can employ ordinary, commutative functions to construct a gauge theory on a noncommutative space. If so, one has to replace in the above expressions, such as the transformation of the gauge field and the field strength tensor, the operators with functions and the ordinary product with a multiplication given by a $\star$-product defined for each space. The two ways of building gauge theories are equivalent, but we employ, as said before, the matrix realization.

A last and important point we would like to stress concerning the non-Abelian noncommutative gauge theories, is the manipulation of the anticommutators and, consequently, the generic issue regarding the determination of the algebra in which the gauge fields are eventually valued. Let us consider a non-Abelian gauge group, $G$, with generators denoted as $T^a$, the gauge parameter, $\epsilon(X)$ and the gauge fields, $A_m(X)$. The commutator $[\epsilon, A]$ is encountered in the construction of a noncommutative gauge theory, for instance in (3.51) and can be written more explicitly as:

$$[\epsilon, A] = [\epsilon^a T^a, A^b T^b] = \frac{1}{2}\{\epsilon^a, A^b\}[T^a, T^b] + \frac{1}{2}[\epsilon^a, A^b]\{T^a, T^b\} \ , \tag{3.59}$$

where the spacetime indices have been suppressed for simplification of the expression. In the ordinary gauge theories in the commutative regime, the anticommutator of the first term in the right-hand side is reduced to a product of the two elements and the last term is trivially vanishing, since the commutator of two functions, $\epsilon, A$, is zero. Therefore, one does not have to deal with the anticommutator $\{T^a, T^b\}$ at all in the commutative case. However, in the noncommutative case, the commutator of the last term is not vanishing, since functions of $X$ no longer commute and therefore one has to pay attention on the anticommutator $\{T^a, T^b\}$. In general, the anticommutator of the generators of a Lie algebra in an arbitrary representation, does not yield generators of the algebra, but instead, elements outside of it, in other words they do not close. Thus, restriction to the specific initial (matrix) algebra is not achievable and further considerations have to be taken. The first option is to employ the universal enveloping algebra, which means that every element (product of generators) produced by the anticommutator has to be included in the algebra, meaning that one would result with an infinite-dimensional algebra including every possible outcome of the anticommutators. Although this option is valid, the fact that the algebra would be infinite is not desirable for our purposes. The second option to overpass this drawback is to fix the representation of the generators, so that the anticommutators of the generators will produce only a limited number of operators outside the algebra and then include them all in the initial algebra as generators. Therefore, the final gauge group will be enlarged with the outcomes of the anticommutators, all accommodated in a fixed representation[13]. The second option is the one we adopt in the next sections in the construction of our gauge theories on noncommutative spaces.

[13]This way, the enlargement of the algebra is limited leading to a larger but finite set of generators.

# Chapter 4

# The fuzzy spaces $\mathrm{R}^3_\lambda$ and $\mathrm{R}^{1,2}_\lambda$

## 4.1 The fuzzy space $\mathbb{R}^3_\lambda$

As we described previously in section 3.2, the fuzzy sphere is a noncommutative space which is a matrix approximation of the ordinary sphere and its coordinates satisfy the SU(2) commutation relation, (3.31), along with the Casimir relation, (3.28), which is actually the radius constraint. The $\mathbb{R}^3_\lambda$ space is a noncommutative space, the description of which is based on the fuzzy sphere.

Let us consider the fuzzy sphere case and modify it in the following way: for a fixed $\lambda$ (see (3.31)), one relaxes the Casimir condition, (3.28), which means that the matrices of the coordinates, $X_a$, are allowed to live in reducible representations. The reducibility of the representation allows one to write the matrix $X_a$ in a block diagonal form of irreducible representations, in which the Casimir condition still holds for each distinct block, in other words, each block describes a fuzzy sphere. Therefore, $\mathbb{R}^3_\lambda$ can be written as a direct sum of fuzzy spheres with all possible radii determined by $N \in \mathbb{N}$ ($N$ is the value of the angular momentum, $l$) [91, 95–97]:

$$\mathbb{R}^3_\lambda = \sum_{2N\in\mathbb{N}} S^2_N = \bigoplus_{2N\in\mathbb{N}} \mathrm{Mat}(N, \mathbb{C}) \ . \tag{4.1}$$

Therefore, the $\mathbb{R}^3_\lambda$ can be viewed as a discrete foliation of three-dimensional Euclidean space by multiple fuzzy spheres, with each fuzzy sphere being a leaf of the foliation. Although the above matrix description would be enough for our purposes, it is instructive to include the description of $\mathbb{R}^3_\lambda$ through the construction of its $\star$-product.

Let us now pick up the thread from 3.1.1 section, specifically from the Weyl quantization part, in which the Fourier expansion was used for the function $f(x)$ and then the Weyl ordering led to the corresponding $\star$-product of $\mathbb{R}^2_\theta$. Then it was noted (under (3.17)) that the choice of the Weyl correspondence is not unique. Indeed, proving the point, one may define:

$$z = \frac{x^1 + ix^2}{\sqrt{2\theta}} \tag{4.2}$$

and then, using (3.17) and (4.2), it is straightforward to obtain the $\star$-commutator $[z \overset{\star}{,} \bar{z}]$ as:

$$[z \overset{\star}{,} \bar{z}] = 1 \ . \tag{4.3}$$

Then, having the above relation at hand and since any function $f(z, \bar{z})$ can be expanded as:

$$f(z, \bar{z}) = \sum_{n,m} f_{mn} \bar{z}^m z^n \ , \tag{4.4}$$

it is possible to employ, instead of the Fourier expansion of the functions, the above Laurent expansion in order to define an operator, like the one in (3.6). Instead of the Weyl correspondence, this time the harmonic oscillator creation and annihilation operators $a^\dagger, a$ (see Appendix B) are used, obtaining an operator which is "normal ordered". This change in the ordering is the reason why the resulting $\star$-product of the $\mathbb{R}^2_\theta \times \mathbb{R}^2_\theta$ is a different one[1] and, as it will be shown in a while, it is the specific $\star$-product that leads to the one of $\mathbb{R}^3_\lambda$. In order to get the reduced $\star$-product for the $\mathbb{R}^3_\lambda$, a noncommutative version of the Hopf fibration[2] has to be employed, with which it will be shown that the algebra of functions of $\mathbb{R}^3_\lambda$ is equivalent to a sub-algebra of the functions defined on $\mathbb{R}^2_\theta \times \mathbb{R}^2_\theta$, which are also invariant under the transformation:

$$z_1 \to e^{ia} z_1 \ , \quad z_2 \to e^{ia} z_2 \ , \tag{4.5}$$

or equivalently, in the coset space language, $\mathbb{R}^3_\lambda = (\mathbb{R}^2_\theta \times \mathbb{R}^2_\theta)/S^1$. In other words, the desired $\mathbb{R}^3_\lambda$ $\star$-product will be obtained as a reduction from the $\star$-product of the four-dimensional Moyal plane, $\mathbb{R}^2_\theta \times \mathbb{R}^2_\theta$, which is obtained by using the normal ordering. Therefore, summing up the procedure, first the desired $\star$-product on the four-dimensional Moyal plane is obtained, making use of the harmonic oscillator basis and the coherent states and then the $\star$-product of the $\mathbb{R}^3_\lambda$ is obtained as a reduction from the first, employing a noncommutative version of the Hopf fibration.

In order to result with this noncommutative version of the Hopf fibration, the first step is to start with setting the four-dimensional coordinates to be noncommutative, that is to employ the four-dimensional noncommutative space $\mathbb{R}^2_\theta \times \mathbb{R}^2_\theta$ (instead of $\mathbb{R}^4$ or $\mathbb{C}^2$) which is defined by the following commutation relation of its coordinates:

$$[z_a \overset{\star}{,} \bar{z}_b] = \delta_{ab} \ , \tag{4.6}$$

as found in (4.3). As we mentioned before, the $C^\star$-algebra, $A$, is considered to be either a commutative algebra of functions equipped with a $\star$-product, or noncommutative algebra of operators with the ordinary product. Switching to the operator language, the coordinates $z_a, \bar{z}_a$ are replaced by the creation and annihilation operators, $a_a^\dagger, a_b, a, b = 1, 2$, of a system of two uncoupled harmonic oscillators satisfying the commutation relation, $[a_a, a_b^\dagger] = \delta_{ab}$, (B.2), which is actually an operator version of (4.6). The eigenstates of this two-dimensional harmonic oscillator system are denoted as $|n_1 n_2\rangle$ (see equation (B.6)) and are built by acting successively with the creation operator on the vacuum state, defined in (B.8).
Now, as a generalization of the definition of the coherent states[3] for a $z \in \mathbb{C}$, to any vector $\vec{z} \in \mathbb{C}^2$ a coherent state can be assigned:

$$|\vec{z}\rangle \equiv |z_1 z_2\rangle = e^{-\frac{|\vec{z}|^2}{2}} e^{z_a a_a^\dagger} |00\rangle \ , \tag{4.7}$$

[1]Although the $\star$-commutator was given in (4.3), the $z \star \bar{z}$ and $\bar{z} \star z$ are yet to be found.

[2]The Hopf fibration is a predecessor of the fiber bundle and describes a 3-sphere in terms of a 2-sphere and a circle (1-sphere). More specifically, it is a map (projection) from a 3-sphere to a 2-sphere such that every point of the 2-sphere originates from a specific circle of the 3-sphere [98,99]. This means that the 3-sphere is made of fibers (circles) one on every point of the 2-sphere. This scheme is written as: $S^1 \hookrightarrow S^3 \overset{p}{\hookrightarrow} S^2$, meaning that the $S^1$ (fiber space) is embedded in $S^3$ (total space) and $p : S^3 \longrightarrow S^2$ (Hopf's map) projects the $S^3$ on $S^2$ (base space). For further (technical) details see Appendix C.

[3]Necessary information for the present work about the coherent states is given in Appendix D.

where $|\vec{z}|^2 = \bar{z}z = \bar{z}_a z_a$. Also, for this $\mathbb{C}^2$ case, for two coherent states $\vec{z}, \vec{\eta}$, according to (D.8), the non-orthogonality relation is:

$$\langle\vec{\eta}|\vec{z}\rangle = e^{-\frac{|\vec{\eta}|^2}{2} - \frac{|\vec{z}|^2}{2} + \bar{\vec{\eta}}\vec{z}} \, , \tag{4.8}$$

and according to (D.7), the (over-)completeness relation is:

$$\frac{1}{\pi^2} \int d^2\vec{z} |\vec{z}\rangle\langle\vec{z}| = 1 \, , \tag{4.9}$$

where $d^2\vec{z} = d^2 z_1 d^2 z_2$.

Now, having at hand the above machinery, let us proceed with obtaining the desired $\star$-product on $\mathbb{R}^2_\theta \times \mathbb{R}^2_\theta$. As mentioned earlier, for this purpose, it is necessary to associate the operators of the noncommutative algebra to commutative functions with the modified product, in other words, to find a correspondence between them. Let us consider an operator $\hat{f} \in \hat{A}_4$, where $\hat{A}_4$ is the algebra noncommutative algebra generated by the creation and annihilation operators. To this operator a function $f \in A_4$ is associated, where $A_4$ is the algebra of functions on $\mathbb{R}^2_\theta \times \mathbb{R}^2_\theta$, generated by $z_a$ and $\bar{z}_a$:

$$\langle\vec{z}|\hat{f}|\vec{z}\rangle = f(\vec{z}, \bar{\vec{z}}) \, . \tag{4.10}$$

The product of two operators, $\hat{f}, \hat{g}$ will give the $\star$-product $f \star g$:

$$(f \star g)(\vec{z}, \bar{\vec{z}}) = \langle\vec{z}|\hat{f}\hat{g}|z\rangle = \frac{1}{\pi^2} \int d^2\vec{\eta} \langle\vec{z}|\hat{f}|\vec{\eta}\rangle\langle\vec{\eta}\hat{g}|\vec{z}\rangle \, , \tag{4.11}$$

where the completeness relation has been employed. The quantities $\langle\vec{z}|\hat{f}|\vec{\eta}\rangle$ and $\langle\vec{\eta}\hat{g}|\vec{z}\rangle$ are obtained using the translation, acting twice on the functions $f(\vec{z}, \bar{\vec{z}})$ and $g(\vec{z}, \bar{\vec{z}})$, using (4.10) and the fact that $\langle z|z\rangle = 1$:

$$e^{-z_a \frac{\partial}{\partial \eta_a}} e^{\eta_a \frac{\partial}{\partial z_a}} \frac{\langle\vec{z}|\hat{f}|\vec{z}\rangle}{\langle z|z\rangle} = e^{-z_a \frac{\partial}{\partial \eta_a}} \frac{\langle\vec{z}|\hat{f}|\vec{z}+\vec{\eta}\rangle}{\langle\vec{z}|\vec{z}+\vec{\eta}\rangle} = \frac{\langle\vec{z}|\hat{f}|\vec{z}+\vec{\eta}-\vec{z}\rangle}{\langle\vec{z}|\vec{z}+\vec{\eta}-\vec{z}\rangle} = \frac{\langle z|\hat{f}|\vec{\eta}\rangle}{\langle\vec{z}|\vec{\eta}\rangle} \, , \tag{4.12}$$

$$e^{-\bar{z}_a \frac{\partial}{\partial \eta_a}} e^{\bar{\eta}_a \frac{\partial}{\partial z_a}} \frac{\langle\vec{z}|\hat{g}|\vec{z}\rangle}{\langle z|z\rangle} = e^{-\bar{z}_a \frac{\partial}{\partial \eta_a}} \frac{\langle\vec{z}+\vec{\eta}|\hat{g}|\vec{z}\rangle}{\langle\vec{z}+\vec{\eta}|\vec{z}\rangle} = \frac{\langle\vec{z}+\vec{\eta}-\vec{z}|\hat{g}|\vec{z}\rangle}{\langle\vec{z}+\vec{\eta}-\vec{z}|\vec{z}\rangle} = \frac{\langle\vec{\eta}|\hat{g}|\vec{z}\rangle}{\langle\vec{\eta}|\vec{z}\rangle} \, . \tag{4.13}$$

The above double action of the translation operators (on $\eta$-independent functions) can be written in each case as an ordered exponential, as it is shown in the following equations:

$$: e^{(\eta_a - z_a)\frac{\vec{\partial}}{\partial z_a}} : f(\vec{z}, \bar{\vec{z}}) =: e^{(\eta_a - z_a)\frac{\vec{\partial}}{\partial z_a}} : \frac{\langle\vec{z}|\hat{f}|\vec{z}\rangle}{\langle\vec{z}|\vec{z}\rangle} = \frac{\langle\vec{z}|\hat{f}|\vec{\eta}-\vec{z}+\vec{z}\rangle}{\langle\vec{z}|\vec{\eta}-\vec{z}+\vec{z}\rangle} = \frac{\langle\vec{z}|\hat{f}|\vec{\eta}\rangle}{\langle\vec{z}|\vec{\eta}\rangle} \, , \tag{4.14}$$

$$: e^{\frac{\overleftarrow{\partial}}{\partial z_a}(\eta_a - z_a)} : g(\vec{z}, \bar{\vec{z}}) =: e^{\frac{\overleftarrow{\partial}}{\partial z_a}(\eta_a - z_a)} : \frac{\langle\vec{z}|\hat{f}|\vec{z}\rangle}{\langle\vec{z}|\vec{z}\rangle} = \frac{\langle\vec{z}+\vec{\eta}-\vec{z}|\hat{g}|\vec{z}\rangle}{\langle\vec{z}+\vec{\eta}-\vec{z}|\vec{z}\rangle} = \frac{\langle\vec{\eta}|\hat{g}|\vec{z}\rangle}{\langle\vec{\eta}|\vec{z}\rangle} \, . \tag{4.15}$$

In the first relation, the derivatives are ordered to the right in each term in the Taylor expansion and they act also to the right, while in the second relation, the derivatives are ordered to the left in the terms of the expansion and they act also to the left. Replacing the above normal ordered expressions, (4.14) and (4.15), into (4.11), one obtains:

$$(f \star g)(\vec{z}, \bar{\vec{z}}) = \frac{1}{\pi^2} \int d^2\vec{\eta} f(\vec{z}, \bar{\vec{z}}) : e^{\frac{\overleftarrow{\partial}}{\partial z_a}(\eta_a - z_a)} : |\langle\vec{z}|\vec{\eta}\rangle|^2 : e^{(\bar{\eta}_a - \bar{z}_a)\frac{\vec{\partial}}{\partial \bar{z}_a}} : g(z, \bar{\vec{z}}) \, . \tag{4.16}$$

In order to determine the $\star$-product, the measure $d^2\vec{\eta}$ and the scalar product $|\langle\vec{z}|\vec{\eta}\rangle|^2$ are specified. In analogy to the measure in the $z \in \mathbb{C}$ case, (D.6), the $\frac{1}{\pi}d^2\vec{\eta}$ measure is $\frac{1}{\pi^2}dRe(\eta_1)dIm(\eta_1)dRe(\eta_2)$ $dIm(\eta_2)$. The product $|\langle\vec{z}|\vec{\eta}\rangle|^2$ is obtained easily, starting from the non-orthogonality relation, (4.8). Therefore, one obtains:

$$|\langle\vec{z}|\vec{\eta}\rangle|^2 = \left(\overline{e^{-\frac{|\vec{\eta}|^2}{2}-\frac{|\vec{z}|^2}{2}+\bar{\vec{z}}\vec{\eta}}}\right)\left(e^{-\frac{|\vec{\eta}|^2}{2}-\frac{|\vec{z}|^2}{2}+\bar{\vec{z}}\vec{\eta}}\right) = e^{-|\vec{z}-\vec{\eta}|^2} \ . \tag{4.17}$$

Therefore, the integral (4.16) takes the following form:

$$\begin{aligned}(f\star g)(\vec{z},\bar{\vec{z}}) &= \frac{1}{\pi^2}\int d^2\vec{\eta} f(\vec{z},\bar{\vec{z}}) : e^{\frac{\overleftarrow{\partial}}{\partial z_a}(\eta_a - z_a)} : e^{-|\vec{\eta}-\vec{z}|^2} : e^{(\bar{\eta}_a-\bar{z}_a)\frac{\overrightarrow{\partial}}{\partial \bar{z}_a}} : g(z,\bar{\vec{z}}) \\ &= \frac{1}{\pi^2}\int d^2\vec{\eta} f(\vec{z},\bar{\vec{z}}) e^{\frac{\overleftarrow{\partial}}{\partial z_a}\eta_a - |\vec{\eta}|^2 + \bar{\eta}_a\frac{\overrightarrow{\partial}}{\partial \bar{z}_a}} g(z,\bar{\vec{z}}) \ ,\end{aligned} \tag{4.18}$$

where in the last line, a change of variables took place and the ordering of the exponential function was omitted. In the last expression of the above equation, the integrand is Gaussian and the result after the integration is:

$$(f\star g)(\vec{z},\bar{\vec{z}}) = f(\vec{z},\bar{\vec{z}}) e^{\frac{\overleftarrow{\partial}}{\partial z_a}\frac{\overrightarrow{\partial}}{\partial \bar{z}_a}} g(\vec{z},\bar{\vec{z}}) \ . \tag{4.19}$$

This is the resulting expression of the $\star$-product of the four-dimensional Moyal plane, induced by the normal ordering and it is known as Wick-Voros $\star$-product [92]. Using the above relation, the $\star$-product of the coordinates $z_a, \bar{z}_a$ of $\mathbb{R}^2_\theta \times \mathbb{R}^2_\theta$ is found to be:

$$z\star\bar{z} = z_a e^{\frac{\overleftarrow{\partial}}{\partial z_a}\frac{\overrightarrow{\partial}}{\partial \bar{z}_a}} \bar{z}_a = z_a\left(1 + \frac{\overleftarrow{\partial}}{\partial z_a}\frac{\overrightarrow{\partial}}{\partial \bar{z}_a}\right)\bar{z}_a = z_a\bar{z}_a + 1 \ , \tag{4.20}$$

and similarly, $\bar{z}\star z = \bar{z}_a z_a$. The difference of these two results confirms equation (4.3), in which $[z \overset{\star}{,} \bar{z}] = 1$.

Now that the $\star$-product of the four-dimensional Moyal plane is obtained, the calculation of the $\mathbb{R}^3_\lambda$ $\star$-product as a reduction from the Wick-Voros $\star$-product, (4.19), is possible. For this purpose, besides the coherent states, attention is turned in the usual two-dimensional harmonic oscillator basis, $|n_1 n_2\rangle$, defined in (B.9). However, it turns out that the most effective basis for our purpose is the Schwinger basis, in which the general ket, $|jm\rangle$, is given by (B.17). The coherent states can be expanded in the Schwinger basis as:

$$|z\rangle = \sum_{j=0}^{\infty}\frac{e^{-\frac{|z|^2}{2}}}{\sqrt{(2j)!}}\sum_{m=-j}^{m=j} z_1^{j+m} z_2^{j-m}\frac{\sqrt{(2j)!}}{(j+m)!(j-m)!}|jm\rangle \ . \tag{4.21}$$

Next, in order to result with the $\star$-product of the $\mathbb{R}^3_\lambda$, the noncommutative version of the Hopf fibration is employed. For this reason, the subalgebra $\hat{A}_3 \subset \hat{A}_4$ is considered, generated by the $\hat{X}^i = \frac{1}{2}a_a^\dagger\sigma^i_{ab}a_b$ [4] and the corresponding subalgebra of functions of $A_3 \subset A_4$ is given by the relation (C.16), $x^i = \frac{1}{2}\bar{z}_a\sigma^i_{ab}z_b$. Writing $\hat{X}^i$ in the angular momentum basis, given in (B.10),

[4] The relevant discussion is under (4.6).

expressing $\hat{X}^0$ in terms of the number operator[5], (B.12), of Appendix B:

$$\begin{aligned}\hat{X}^0 &= \frac{1}{2}a_a^\dagger \mathbb{1} a_a = \frac{1}{2}N\ ,\\ \hat{X}^1 &= \frac{1}{2}\left(a_a^\dagger a_2 + a_2^\dagger a_1\right) = \frac{1}{2}\left(J_+ + J_-\right)\ ,\\ \hat{X}^2 &= -\frac{i}{2}\left(a_1^\dagger a_2 - a_2^\dagger a_1\right) = \frac{1}{2}\left(J_+ - J_-\right)\ ,\\ \hat{X}^3 &= \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right) = J_z\ ,\end{aligned} \tag{4.22}$$

and calculating the $\sum_{i=1}^3 \hat{X}^i$, one obtains:

$$\begin{aligned}\hat{X}^1 + \hat{X}^2 + \hat{X}^3 &= \frac{1}{4}(J_+ + J_-)^2 - \frac{1}{4}(J_+ - J_-)^2 + J_z^2\\ &= \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2 = \frac{N}{2}\left(\frac{N}{2}+1\right) = \hat{X}^0(\hat{X}^0+1)\ ,\end{aligned} \tag{4.23}$$

where the expression of the Casimir operator, (B.13) has been used. The Casimir operator commutes with every generator of the algebra, therefore, one is led to the crucial conclusion that $\hat{X}^0$ commutes with all $\hat{X}^i$'s and therefore, the subalgebra $\hat{A}_3$ can be defined as the subalgebra of $\hat{A}_4$ of all elements commuting with $\hat{X}^0$:

$$\forall \hat{f}(a^\dagger, a) \in \hat{A}_4 \mid [\hat{X}^0, \hat{f}] = 0 \ \Rightarrow\ \hat{f} \in \hat{A}_3\ . \tag{4.24}$$

Let us translate the above condition for an operator $\hat{f} \in \hat{A}_3$ to the language of functions:

$$\begin{aligned}[\hat{X}^0, \hat{f}] \ \longrightarrow\ [x^0 \overset{\star}{,} f(z,\bar{z})] &= \frac{1}{2}(\bar{z}_a z_a \star f(z,\bar{z}) - f(z,\bar{z}) \star \bar{z}_a z_a)\\ &\overset{(4.19)}{=} \frac{1}{2}\bar{z}_a z_a\left(1 + \frac{\overleftarrow{\partial}}{\partial z_a}\frac{\overrightarrow{\partial}}{\partial \bar{z}_a}\right) f(z,\bar{z}) - \frac{1}{2}f(z,\bar{z})\left(1 + \frac{\overleftarrow{\partial}}{\partial z_a}\frac{\overrightarrow{\partial}}{\partial \bar{z}_a}\right)\bar{z}_a z_a\\ &= \frac{1}{2}\left(\bar{z}_a\bar{\partial}_a - z_a\partial_a\right) f(z,\bar{z}) \equiv \mathcal{L}_0 f(z,\bar{z})\ .\end{aligned} \tag{4.25}$$

Therefore, the elements of the algebra, $A_3$ are functions of $z_a, \bar{z}_a$ which belong to the algebra of functions $A_4$ that are also subjected to the constraint $\mathcal{L}_0 f(z,\bar{z}) = 0$. It is worth-noting that the operator $\mathcal{L}_0$ when acting on a $\star$-product of functions $f(z,\bar{z}), g(z,\bar{z})$ is actually an operator of derivation:

$$\mathcal{L}_0(f \star g) = (\mathcal{L}_0 f) \star g + f \star (\mathcal{L}_0 g)\ , \tag{4.26}$$

which is easily confirmed by straightforward calculations in both sides. Therefore, if the functions $f(z,\bar{z})$, $g(z,\bar{z})$ belong to the subalgebra $A_3$, it is deduced from the above equation, that $\mathcal{L}_0(f \star g) = 0$, since $\mathcal{L}_0 f(z,\bar{z}) = \mathcal{L}_0 g(z,\bar{z}) = 0$. This is an important result because it implies that the subalgebra $A_3$ is closed under the Wick-Voros $\star$-product, i.e. if $f, g \in A_3 \ \Rightarrow\ f \star g \in A_3$. The property of closure along with the fact that all functions of $A_3$ can be expressed in terms of $x^i, x^0$, allow the redefinition of the Wick-Voros $\star$-product in terms of the $x^i$'s and their derivatives. To see this, the derivations on functions of $A_3$ are recalled:

$$\frac{\partial}{\partial z_a} = \frac{1}{2}\bar{z}_b \sigma^i_{ba}\frac{\partial}{\partial x^i}\ ,\quad \frac{\partial}{\partial \bar{z}_a} = \frac{1}{2}\frac{\partial}{\partial x^i}\sigma^i_{ab} z_b\ . \tag{4.27}$$

[5]The function $x^0$ is given as $x^0 = \frac{1}{2}\bar{z}_a z_a$, as given in (C.18)

Also, using the relation for the product of two Pauli matrices, $\sigma^i\sigma^j = \delta^{ij}\,\mathbb{1} + i\epsilon^{ijk}\sigma^k$ and the definitions of $x^0$ and $x^i$, which are $x^0 = \frac{1}{2}\bar{z}z$ and $x^i = \frac{1}{2}\bar{z}_a\sigma^i_{ab}z_b$, respectively, one obtains another useful relation, that is:

$$\bar{z}_b\sigma^i_{ba}\sigma^j_{ac}z_c = 2(\delta^{ij}x^0 + i\epsilon^{ijk}x^k) \ . \tag{4.28}$$

Now, using the above two relations, (4.27) and (4.28), the expression of the Wick-Voros $\star$-product, (4.19), takes the following form in a straightforward way:

$$(f \star g)(\vec{x}) = e^{\frac{1}{2}(\delta^{ij}x^0 + i\epsilon^{ijk}x^k)\frac{\partial}{\partial u^i}\frac{\partial}{\partial v^j}} f(\vec{u})g(\vec{v})|_{u=v=x} \ . \tag{4.29}$$

The above equation gives the $\star$-product between any functions of $A_3$ and, considering its Taylor expansion, produces the following relations:

$$\begin{aligned} x^i \star x^j &= \left[1 + \frac{1}{2}(\delta^{mn}x^0 + i\epsilon^{mnk}x^k)\frac{\partial}{\partial x^m}\frac{\partial}{\partial x^n}\right] x^i x^j = x^i x^j + \frac{1}{2}(\delta^{ij}x^0 + i\epsilon^{ijk}x^k) \ , \\ x^j \star x^i &= x_j x_i + \frac{1}{2}(\delta^{ij}x^0 - i\epsilon^{ijk}x^k) \ . \end{aligned} \tag{4.30}$$

The difference of the above two equations produces the $\star$-commutator of the $A_3$:

$$[x^i \overset{\star}{,} x^j] = i\epsilon^{ijk}x^k \ . \tag{4.31}$$

Also, from the $\star$-product of $A_3$, (4.29), one obtains two more interesting relations, specifically:

$$x^0 \star x^i = x^0 x^i + \frac{1}{2}x^i \ , \quad x^0 \star x^0 = x^0\left(x^0 + \frac{1}{2}\right) \ , \tag{4.32}$$

where the expressions $x^0 = \sqrt{x^i x^i}$ and hence $\frac{\partial x^0}{\partial x^i} = \frac{x^i}{x^0}$. Also, from the first of the above equations it is understood that:

$$[x^0 \overset{\star}{,} x^i] = 0 \ . \tag{4.33}$$

The commutator $[x^i \overset{\star}{,} x^j]$ of (4.31) shows that the subalgebra $A_3$ can be viewed as an algebra of functions on the $\mathbb{R}^3_\lambda$ with the $\star$-product that of (4.29). The commutator of the corresponding coordinates in the operator language, recovering the constant $\theta = \lambda$, would be:

$$[X^i, X^j] = i\lambda\epsilon^{ijk}X^k \ . \tag{4.34}$$

It should be remarked that if one had chosen the Moyal product instead of the Wick-Voros product, then the reduction would be indeed achieved but the corresponding three-dimensional $\star$-product would be expressed in a more complicated way.

## 4.2 The fuzzy space $\mathbb{R}^{1,2}_\lambda$

The above construction of the fuzzy space $\mathbb{R}^3_\lambda$ based on SU(2), that is the foliation of the three-dimensional Euclidean space by fuzzy spheres of different radii, has a direct analogue in the case of the Minkowski signature based on SU(1,1). In this case, the three-dimensional Minkowski space is foliated by a set of fuzzy hyperboloids of different radii.

First, in general, the $n$-dimensional de Sitter space, $dS_n$ is a maximally symmetric Lorentzian non-compact manifold with constant, positive curvature and is the Lorentzian analogue of the $n$-sphere. It is defined as an embedding in the $(n+1)$-dimensional Minkowski space through the relation $\eta_{ab}x^a x^b = r^2$, where $\eta_{ab} = \text{diag}(-1, 1, \ldots, 1)$ is the $(n+1)$-dim Minkowski metric, in the same way as the $n$-sphere is defined as an embedding in the $(n+1)$-dimensional Euclidean space through the equation $\delta_{ab}x^a x^b = r^2$, where $\delta_{ab}$ is the metric of $\mathbb{R}^{n+1}$, $\delta_{ab} = \text{diag}(1, \ldots, 1)$. The $AdS_n$ space is also a maximally symmetric Lorentzian non-compact manifold with constant, negative curvature and is obtained from the same embedding as the $dS_n$ space replacing $r^2 \to -r^2$, and considering the metric $\eta'_{ab} = \text{diag}(-1, 1, \ldots, 1, -1)$. In all above cases, $r$ specifies the curvature radius. Below, there are the three cases of $S_4, dS_4, AdS_4$, specified in the four-dimensional case:

$$\sum_{a,b=1}^{5} = \delta_{ab}x^a x^b = r^2 \;\Rightarrow\; x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = r^2 \qquad (S_4) \tag{4.35}$$

$$\sum_{a,b=1}^{5} = \eta_{ab}x^a x^b = r^2 \;\Rightarrow\; -x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = r^2 \qquad (dS_4) \tag{4.36}$$

$$\sum_{a,b=1}^{5} = \eta'_{ab}x^a x^b = r^2 \;\Rightarrow\; -x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 = -r^2 \qquad (AdS_4) \tag{4.37}$$

Now, let us consider the two-dimensional cases of the non-compact spaces with constant curvature, i.e. the $dS_2$ and $AdS_2$[6]. According to the above, the de Sitter and Anti de Sitter defining relations are:

$$\eta_{ab}x^a x^b = r^2 \;\Rightarrow\; -x_1^2 + x_2^2 + x_3^2 = r^2 \tag{4.38}$$

$$\eta'_{ab}x^a x^b = -r^2 \;\Rightarrow\; -x_1^2 + x_2^2 - x_3^2 = -r^2 \;\Rightarrow\; x_1^2 - x_2^2 + x_3^2 = r^2 \;, \tag{4.39}$$

It is evident that $dS_2$ is obtained from $AdS_2$ by switching $x_1$ and $x_2$ in (4.39) and changing the overall sign of the metric [100] and vice versa. That is the reason why, in the specific two-dimensional case, both cases reduce to one, that is known as the one-sheeted hyperboloid. The isometry group of the hyperboloid is the SO(1,2), or its cover SU(1,1)[7]. The generators of SU(1,1), $J_a$, satisfy the following commutation relations:

$$[J_1, J_2] = -iJ_3 \;, \quad [J_2, J_3] = iJ_1 \;, \quad [J_3, J_1] = iJ_2 \;, \tag{4.40}$$

from which it is possible to extract the structure constants, ${C_{ab}}^c$, of the algebra, namely ${C_{12}}^3 = -{C_{23}}^1 = -{C_{31}}^2 = -1$, with their indices getting lowered or raised by the (mostly positive) three-dimensional Minkowski metric, $\eta_{ab} = \text{diag}(1, 1, -1)$. Therefore, the structure constants with all indices lowered, $C_{abc}$, are determined by the relation:

$$[J_a, J_b] = i{C_{ab}}^c J_c \;\Rightarrow\; [J_a, J_b] = iC_{abc}\eta^{cd}J_d \;, \tag{4.41}$$

and are found to be $C_{123} = C_{231} = C_{312} = 1$. Also, the Casimir operator is given by the relation:

$$J^2 = \sum_{a,b=1}^{3} \eta^{ab}J_a J_b = J_1^2 + J_2^2 - J_3^2 \;. \tag{4.42}$$

[6]There is another one, the Lobachevsky plane, $H^2$, which completes the list with all three such spaces.

[7]The SU(1,1) covers the SO(1,2) group in the same way as the SU(2) double-covers the SO(3).

As in the SU(2) case, in order to study the irreducible representations of the group, it is convenient to define the ladder operators, $J_\pm = J_1 \pm iJ_2$. From this study, it is shown that all unitary irreducible representations are infinite-dimensional, while all finite-dimensional are not unitary [8].

Now, in analogy to the fuzzy sphere, it is possible to define the fuzzy two-dimensional hyperboloid space, $dS_2^F$ in terms of three Hermitian operators (matrices), specifically the rescaled generators of SU(1,1), namely $X_a = \lambda J_a$ (in analogy to (3.27)), and identify them as the coordinate operators (matrices) of the fuzzy space. The commutation relation they satisfy is:

$$[X_a, X_b] = i\lambda C_{ab}{}^c X_c \ , \tag{4.43}$$

where $C_{ab}{}^c$ are the structure constants of the SU(1,1) algebra. However, in order to result with a fuzzy version of the hyperboloid, its coordinates must satisfy an operator-analogue of the radius constraint, given in (4.38). This constraint in the operator language coincides with a modified Casimir relation of the SU(1,1) generators, that is the relation (4.42) in terms of $X_a$:

$$\sum_{a,b=1}^{3} \eta^{ab} X_a X_b = \lambda^2 j(j-1) \ , \tag{4.44}$$

where $j(j-1)$[9] is the eigenvalue of the Casimir operator, $J^2$.

In analogy to the definition of the fuzzy space $\mathbb{R}^3_\lambda$, the fuzzy space $\mathbb{R}^{1,2}_\lambda$ is defined after relaxing the Casimir constraint of the coordinates, (4.44), allowing the coordinates to live in (infinite-dimensional) reducible representations, which means that they can be described in a block diagonal form, with each block being an irreducible representation of SU(1,1), that is a fuzzy hyperboloid. Therefore, representation reducibility induces the definition of $\mathbb{R}^{1,2}_\lambda$ as a foliation of fuzzy hyperboloids of all different radii. This concludes the study of the two three-dimensional fuzzy spaces that will be used for the construction of three-dimensional gravity models as noncommutative gauge theories.

## 4.3 Noncommutative covariant spaces and fuzzy dS$_4$

In this section we construct a fuzzy version of the $dS_4$ space [78], which was defined in (4.36), as a submanifold of the five-dimensional Minkowski spacetime with metric $\eta_{AB} = \mathrm{diag}(-1,1,1,1,1)$. Specifically, the embedding relation reads:

$$\eta^{AB} x_A x_B = R^2 \ , \tag{4.45}$$

where $A, B = 0, \ldots, 4$ and summation of the indices is implied.

Now, in order to obtain the fuzzy version of the above space, its coordinates must be replaced by operators of an algebra $A$, represented by matrices and therefore satisfy a commutation relation of general form of (3.2), that is:

$$[X_A, X_B] = i\theta_{AB}(X) \ , \tag{4.46}$$

[8]For more information about the classification of the unitary irreducible representations of SU(1,1) see [101].

[9]The spin value $j$ is associated with the dimension of the matrix representing the operators of the coordinates, specifically $(2j) = N$.

where $\theta_{AB}$ incorporates the type of noncommutativity. If $\theta_{AB}$ was considered to be a constant antisymmetric tensor (canonical case) [12], then Lorentz invariance would break, since there would be a directional preference. Recalling the fuzzy two-sphere case (Lie-type case), in which the coordinates do not commute according to (3.31), the $\theta_{AB}$ of the above, general relation is $\kappa\epsilon_{AB}^{\ \ C}X_C$. This means that the commutator of two coordinates, that is two rescaled SU(2) generators, produces an (obviously rescaled) element within the SU(2) algebra, ensuring covariance (coordinates transform as vectors under rotations). Following the same methodology as in the fuzzy sphere, for the construction of the present fuzzy de Sitter space, its noncommutative coordinates should be also identified to some generators of its isometry group, that is SO(1,4). Nevertheless, such a choice is not successful, because identification of the coordinates with SO(1,4) generators breaks covariance, because the algebra is not closing, specifically, the $\theta_{AB}$ of the right-hand side of the (4.46) cannot be assigned to generators of the SO(1,4) algebra. However, preservation of covariance has to be taken axiomatically, therefore in order to be preserved in this case, a group of larger symmetry has to be employed, in which we will be able to include all generators and noncommutativity in it, with an appropriate identification, that is the coordinates to transform as vectors under the action of the Lorentz transformations. In order to achieve this, the minimum extension of the symmetry leads to the adoption of the SO(1,5) group. For convenience, to facilitate the formulation of the above scheme, we employ the Euclidean signature from now on, meaning that the symmetry group is extended from SO(5) to SO(6)[10].

Let us consider the fifteen generators of SO(6) by $J_{AB}$, with $A, B = 1, \ldots 6$, satisfying the following commutation relation:

$$[J_{AB}, J_{CD}] = i(\delta_{AC}J_{BD} + \delta_{BD}J_{AC} - \delta_{BC}J_{AD} - \delta_{AD}J_{BC}) \ . \tag{4.47}$$

Now, let us perform a decomposition of the above generators in an SO(4) notation and redefine the generators as:

$$J_{mn} = \frac{1}{\hbar}\Theta_{mn}, \quad J_{m5} = \frac{1}{\lambda}X_m, \quad J_{m6} = \frac{\lambda}{2\hbar}P_m, \quad J_{56} = \frac{1}{2}\mathrm{h} \ , \tag{4.48}$$

where $m, n = 1, \ldots, 4$. The parameter $\lambda$ has been introduced for dimensional reasons and $X_m, P_m, \Theta_{mn}$ are identified as the coordinates, momenta and noncommutativity tensor, respectively. Coordinates and momenta satisfy the following commutation relations:

$$[X_m, X_n] = i\frac{\lambda^2}{\hbar}\Theta_{mn}, \quad [P_m, P_n] = 4i\frac{\hbar}{\lambda^2}\Theta_{mn} \tag{4.49}$$

$$[X_m, P_n] = i\hbar\delta_{mn}\mathrm{h}, \quad [X_m, \mathrm{h}] = i\frac{\lambda^2}{\hbar}P_m \tag{4.50}$$

$$[P_m, \mathrm{h}] = 4i\frac{\hbar}{\lambda^2}X_m \ , \tag{4.51}$$

where the first one, (4.49) is of interest, in which it is manifest that the commutator of coordinates close to the SO(4) part of the total SO(6) symmetry group. The algebra of spacetime transformations is:

$$[X_m, \Theta_{np}] = i\hbar(\delta_{mp}X_n - \delta_{mn}X_p) \tag{4.52}$$

[10]In the Euclidean signature, the study that follows could be viewed as it is about the construction of the fuzzy four-sphere.

$$[P_m, \Theta_{np}] = i\hbar(\delta_{mp}P_n - \delta_{mn}P_p) \tag{4.53}$$

$$[\Theta_{mn}, \Theta_{pq}] = i\hbar(\delta_{mp}\Theta_{nq} + \delta_{nq}\Theta_{mp} - \delta_{np}\Theta_{mq} - \delta_{mq}\Theta_{np}) \tag{4.54}$$

$$[\mathrm{h}, \Theta_{mn}] = 0 \tag{4.55}$$

Again, the first commutation relation, (4.49), shows that coordinates transform as vectors under the action of the Lorentz group (group of rotations, (4.54)), confirming the important virtue of covariance of the space. The above algebra, in contrast to the Heisenberg algebra (see ref. [102]), admits finite dimensional representations for $X_m$, $P_m$ and $\Theta_{mn}$, thus we have obtained a model of spacetime which is a finite quantum system. Including the fuzzy two-sphere, $\mathbb{R}^3_\lambda$ and $\mathbb{R}^{1,2}_\lambda$ spaces like the one above are called fuzzy covariant noncommutative spaces [62, 63, 103]. As we will describe in detail later, this space is employed for the construction of our four-dimensional gravity model as a noncommutative gauge theory.

*Covariant field strength tensor on fuzzy $dS_4$*

As we described in section 3.3, the field strength tensors of the canonical and Lie-type nonommutativity, (3.52) and (3.56) respectively, can be written as the commutator of the covariant coordinates plus an extra term:

$$T_{ab} = [\hat{X}_a, \hat{X}_b] - i\theta_{ab}\,, \qquad F_{ab} = [\hat{X}_a, \hat{X}_b] - iC_{ab}{}^c\hat{X}_c\,. \tag{4.56}$$

This extra term is related to the right-hand side of the commutation relations of their coordinates:

$$[X_a, X_b] = i\theta_{ab}\,, \qquad [X_a, X_b] = iC_{ab}{}^c X_c\,. \tag{4.57}$$

In the canonical case, in which the coordinates commute in a non-covariant way, the extra term is an antisymmetric fixed tensor, while in the covariant Lie-type case the extra term involves the coordinates in a linear way. The presence of this second term in each case, although it seems to spoil the analogy to the commutative case, is a necessary ingredient because it is its contribution that makes the field strength tensor to transform covariantly, as shown in (3.55) and (3.58).

Now, in the present fuzzy de Sitter case we are formulating [78], the noncommutativity tensor is a constant antisymmetric tensor (generator of the symmetry group), pointing at a relation to the canonical case with constant noncommutativity, but also it consists a covariant space (it was built this way), pointing at the Lie-type case. Therefore, it is controversial into which case the fuzzy de Sitter space should fall. The answer is that it cannot be classified into any of these two cases, therefore, it should be examined explicitly. As shown in the first relation of (4.49), the fuzzy de Sitter space is defined as:

$$[X_a, X_b] = i\frac{\lambda^2}{\hbar}\Theta_{ab} \otimes \mathbf{1}\,, \tag{4.58}$$

where $\mathbf{1}$ is a p×p unit matrix, where $p$ is the dimension of the representation of the gauge group[11]. Because of the independence of the right-hand side of the above commutation relation, (4.58), of the coordinates, $X_a$, the obvious definition of the field strength tensor would be:

$$F_{ab} = [\hat{X}_a, \hat{X}_b] - i\frac{\lambda^2}{\hbar} \otimes \Theta_{ab}\mathbf{1}\,. \tag{4.59}$$

[11]This will be clear later when we develop a gauge theory on this space.

If we consider a gauge transformation of the field strength tensor, $\delta F_{ab}$, then straightforward calculations lead to the following result:

$$\delta F_{ab} = [\epsilon, F_{ab}] - i\frac{\lambda^2}{\hbar}[\epsilon, \Theta_{ab} \otimes \mathbb{1}] \ , \tag{4.60}$$

where $\epsilon = \epsilon(X)$ is a gauge parameter. Also, the fact that the coordinates, $X_a$, and consequently the noncommutative tensor, $\Theta_{ab}$, are invariant under the gauge transformation, $\delta X_a = \delta\Theta_{ab} = 0$, has been taken into consideration in the calculation. However, in the above expression it is evident that the field strength tensor does not transform in a covariant way (since there is no reason for the second commutator in the right-hand side to vanish) like in the canonical and Lie-type cases, (3.55) and (3.58). In order to ameliorate the above undesirable result, the definition of the field strength tensor has to be modified appropriately, specifically in the following form:

$$\hat{F}_{ab} = [\hat{X}_a, \hat{X}_b] - \frac{i\lambda^2}{\hbar}\hat{\Theta}_{ab} \ , \tag{4.61}$$

where $\hat{\Theta}_{ab}$ is a tensor defined as:

$$\hat{\Theta}_{ab} = \Theta_{ab} \otimes \mathbb{1} + \mathcal{B}_{ab} \ , \tag{4.62}$$

where $\mathcal{B}_{ab}$ is a non-Abelian 2-form gauge field, which also takes values in the considered gauge group of the theory under construction. Therefore, calculations lead to the following expression of the (infinitesimal) transformation of the field strength tensor:

$$\delta\hat{F}_{ab} = i[\epsilon, \hat{F}_{ab}] \ , \tag{4.63}$$

that is a covariant transformation.

This concludes the section for the description of the fuzzy spaces we are going to employ for constructing gauge theories on them. Also, it includes information about the methodology and steps to follow for their construction and we are going to follow them, too, in building such models (particle and gravity) in the next sections.

# Chapter 5

# Noncommutative three-dimensional gravity as a gauge theory

In this section we construct a three-dimensional gravity model as a gauge theory in the noncommutative framework [79] (see also [80,81]) as a careful translation of the gauge-theoretic approach of the Einstein's three-dimensional general relativity (section 2.2), in which the Poincaré or (A)dS algebras were gauged. For this undertaking, we put into use the general methodology of the construction of gauge theories on noncommutative spaces (section 3.3) and specialize for the two three-dimensional covariant fuzzy spaces, $\mathbb{R}^3_\lambda$ and $\mathbb{R}^{1,2}_\lambda$ (section 4). With this toolkit at hand, we follow the standard procedure and move on with the construction of the model.

As we discussed in section 2.2, the results of the three-dimensional Einstein gravity were recovered by a Chern-Simons gauge-theoretic approach of the ISO(1,2), Poincaré algebra, with the covariant derivative (or more precisely the gauge connection (2.43)) encoding the information of the dreibein and spin connection. Here, we aim at a three-dimensional (noncommutative) gravity model with positive cosmological constant, therefore the symmetry group to be gauged is the SO(1,3) in the Lorentzian case [104] and the SO(4) in the Euclidean one, with the corresponding fuzzy spaces accommodating these gauge theories being the $\mathbb{R}^3_\lambda$ and $\mathbb{R}^{1,2}_\lambda$. In these two cases, the information of the dreibein and spin connection are incorporated into the covariant coordinates. The whole undertaking was initially inspired by the ones considered in refs [34–38] in which the authors consider group structures that refer to four-dimensional cases without cosmological constant. Especially for the three-dimensional case, relevant refs are the [39–41,105]. Compared to our approach, in the above works the deformations are based on the $\star$-product of each space considered and the Seiberg-Witten map is employed [42], while in our approach the matrix representation of the operators of the noncommutative algebra is used.

As we mentioned above, the gauge groups we use in the two signatures are the SO(1,3) and SO(4). We turn to their double groups, which are the spin groups Spin(1,3) and Spin(4), which are, in turn, isomorphic to SL(2,$\mathbb{C}$) and SU(2)$\times$SU(2), respectively. Let us now split the discussion of the two signatures, for better understanding.

*The Lorentzian case*

In the end of section 3.3, we discussed in detail the generic issue encountered in the non-Abelian, noncommutative gauge theories, that is the non-closure of the anticommutators of the generators of the algebra. Obviously, the same feature appears in this case, too, for the generators of the SL(2,ℂ) gauge group. Therefore, according to the general treatment, the first step is to fix the representation, namely the spinor representation, in which the six generators are represented by commutators of the four-dimensional Lorentzian gamma matrices, specifically:

$$\Sigma_{AB} = \frac{1}{2}\gamma_{AB} = \frac{1}{4}[\gamma_A, \gamma_B] \ , \quad A, B = 1, \ldots, 4 \ . \tag{5.1}$$

The commutation and anticommutation relation of the above generators are obtained starting from the following product relation [106]:

$$[\gamma_{AB}, \gamma_{CD}] = 8\eta_{A[C}\gamma_{D]B} \ , \tag{5.2}$$

$$\{\gamma_{AB}, \gamma_{CD}\} = 4\eta_{C[B}\eta_{A]D}\mathbf{1} + 2i\epsilon_{ABCD}\gamma_5 \ , \tag{5.3}$$

where $\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_4$. In the above anticommutation relation, (5.3), the only two elements that are produced by the generators of the algebra in the specific representation are the four-dimensional unit matrix, $\mathbf{1}$ and the $\gamma_5$ matrix. Therefore, according to the indicated treatment, these two elements have to be included in the algebra, extending it to an eight-dimensional one, that is precisely the GL(2,ℂ) generated by the generators $\{\gamma_{AB}, \gamma_5, i\mathbf{1}\}$[1].

Now, we move on with an SO(3) decomposition on the above generators, for reasons of appropriate identification of the gauge fields that will be introduced in a while. Therefore, we define the generators $\tilde{\gamma}^a \equiv \epsilon^{abc}\gamma_{bc}$ and $\bar{\gamma}_a \equiv \gamma_{a4}$, with $a, b, c = 1, 2, 3$. These redefinitions of the generators allow us to rewrite the commutation and anticommutation relations of (5.2) and (5.3) in terms of the SO(3) decomposed generators:

$$[\tilde{\gamma}^a, \tilde{\gamma}^b] = -4\epsilon^{abc}\tilde{\gamma}_c \ , \quad [\bar{\gamma}_a, \tilde{\gamma}_b] = -4\epsilon_{abc}\bar{\gamma}^c \ , \quad [\bar{\gamma}_a, \bar{\gamma}_b] = \epsilon_{abc}\tilde{\gamma}^c \tag{5.4}$$

$$\{\tilde{\gamma}^a, \tilde{\gamma}^b\} = -8\eta_{ab}\mathbf{1} \ , \quad \{\bar{\gamma}_a, \tilde{\gamma}^b\} = 4i\delta_a^b\gamma_5 \ , \quad \{\bar{\gamma}_a, \bar{\gamma}_b\} = 2\eta_{ab}\mathbf{1} \ , \tag{5.5}$$

$$[\gamma^5, \tilde{\gamma}^a] = [\gamma^5, \bar{\gamma}^a] = 0 \ , \quad \{\gamma^5, \tilde{\gamma}_a\} = 4i\bar{\gamma}_a \ , \quad \{\gamma^5, \bar{\gamma}^a\} = i\tilde{\gamma}^a \ , \tag{5.6}$$

where in the last line the relations $[\gamma^5, \gamma^{AB}] = 0$ and $\{\gamma^5, \gamma^{AB}\} = i\epsilon^{ABCD}\gamma_{CD}$ have been used.

Therefore, we have ended up with a theory with gauge group the GL(2,ℂ). The three-dimensional space we employ for the construction of this theory is the $\mathbb{R}^{1,2}_\lambda$, that is the foliation of the three-dimensional Minkowski spacetime by fuzzy hyperboloids, discussed in 4.2, in which the three coordinates, namely the operators $X_\mu$, (4.43), have been identified as the rescaled SU(1,1) generators in a reducible representation. In order to proceed with the construction of the gauge theory, following the steps discussed in 3.3, we introduce the covariant coordinate, (3.50), which is given as:

$$\hat{X}_\mu = X_\mu + \mathcal{A}_\mu \ , \tag{5.7}$$

where $\mu = 0, 1, 2$ is a space index and $\mathcal{A}_\mu$ is the gauge connection. The gauge connection is a function of the operator-coordinates, $X_\mu$, which also takes values in the algebra GL(2,ℂ), therefore, if its generators are denoted collectively as $T^{\bar{a}}$, where $\bar{a} = 1, \ldots, 8$, then it can be expanded on them as:

$$\mathcal{A}_\mu(X) = \mathcal{A}_\mu^{\bar{a}}(X) \otimes T^{\bar{a}} \ . \tag{5.8}$$

It should be noted that between the gauge fields, $\mathcal{A}_\mu^{\bar{a}}(X)$ and the generators $T^{\bar{a}}$, the ordinary product would be meaningless and it is replaced by the tensor product because the gauge fields

[1] We use the set of gamma matrices as in ref. [36]. We denote their $\gamma_0$ matrix as $\gamma_4$ and also consider the $\eta_{44} = -1$ in the mostly positive four-dimensional Minkowski metric we employ. See also [107] for more details.

are now functions of the coordinates which are not mere variables but N×N matrices and the generators are 4×4 matrices (spinor representation). Now, we write down the explicit expansion of the gauge connection onto the generators, $T^{\bar{a}} = \{\tilde{\gamma}^a, \bar{\gamma}_a, i\mathbf{1}, \gamma_5\}$:

$$\mathcal{A}_\mu(X) = e_\mu{}^a(X) \otimes \bar{\gamma}_a + \omega_\mu{}^a(X) \otimes \tilde{\gamma}_a + A_\mu(X) \otimes i\mathbf{1} + \tilde{A}_\mu(X) \otimes \gamma_5 \ , \tag{5.9}$$

where the gauge fields attached to the $\bar{\gamma}_a$ and $\tilde{\gamma}_a$ generators have been identified as the dreibein, $e_\mu^a(X)$ and the spin connection, $\omega_\mu{}^a(X)$, respectively, following the corresponding identification in the commutative case, (2.43). Here, due to noncommutativity, we have introduced two more U(1)-type (Maxwell) gauge fields, the $A_\mu(X)$ and $\tilde{A}_\mu(X)$. We also consider the gauge parameter, $\epsilon(X)$, which is also valued in the algebra of the gauge group and therefore, it also expanded on its generators as:

$$\epsilon(X) = \xi^a(X) \otimes \bar{\gamma}_a + \lambda^a(X) \otimes \tilde{\gamma}_a + \epsilon_0(X) \otimes i\mathbf{1} + \tilde{\epsilon}_0(X) \otimes \gamma_5 \ . \tag{5.10}$$

Having written down explicitly the expressions of the gauge connection and the infinitesimal transformation parameter, we proceed with the determination of the transformations of the component gauge fields, using the standard transformation rule of the covariant coordinate, (3.49)[2]:

$$\begin{aligned}
\delta e_\mu{}^a &= -i[X_\mu + A_\mu, \xi^a] + 2\{\omega_{\mu b}, \xi_c\}\epsilon^{abc} + 2\{e_{\mu b}, \lambda^c\}\epsilon^{abc} + 2i[\lambda_a, \tilde{A}_\mu] + 2i[\tilde{\epsilon}_0, \omega_{\mu a}] + i[\epsilon_0, e_{\mu a}] \ , \\
\delta \omega_\mu{}^a &= -i[X_\mu + A_\mu, \lambda^a] + 2\{\omega_{\mu b}, \lambda_c\}\epsilon^{abc} - \frac{1}{2}\{e_{\mu b}, \xi_c\}\epsilon^{abc} + \frac{i}{2}[\xi^a, \tilde{A}_\mu] + i[\epsilon_0, \omega_\mu{}^a] + \frac{i}{2}[\tilde{\epsilon}_0, e_\mu{}^a] \ , \\
\delta A_\mu &= -i[X_\mu + A_\mu, \epsilon_0] - i[\xi^a, e_{\mu a}] + 4i[\lambda^a, \omega_{\mu a}] - i[\tilde{\epsilon}_0, \tilde{A}_\mu] \ , \\
\delta \tilde{A}_\mu &= -i[X_\mu + A_\mu, \tilde{\epsilon}_0] + 2i[\xi^a, \omega_{\mu a}] + 2i[\lambda^a, e_{\mu a}] + i[\epsilon_0, \tilde{A}_\mu] \ .
\end{aligned} \tag{5.11}$$

At this point we stress two important comments on the above transformation rules of the gauge fields, regarding two limits. First, had we started with the construction of a gauge theory on the $\mathbb{R}^{1,2}_\lambda$ space with an Abelian, U(1), gauge group, its covariant coordinate would be just $\hat{X}_\mu = X_\mu + A_\mu$ and from its standard transformation rule we would obtain the transformation rule $\delta A_\mu = -i[X_\mu, \epsilon_0] + i[\epsilon_0, A_\mu]$ for the gauge field, where $\epsilon_0$ is the corresponding gauge parameter. This Abelian gauge theory lies under the GL(2,$\mathbb{C}$) gauge theory we build and this becomes manifest if we set $e_\mu{}^a, \omega_\mu{}^a, \tilde{A}_\mu = 0$ and their corresponding parameters equal to zero. Therefore, the only non-trivial transformation in (5.11) would be the $\delta A_\mu = -i[X_\mu, \epsilon_0] + i[\epsilon_0, A_\mu]$, which is identical to the transformation rule of a noncommutative Maxwell gauge field, as we mentioned above. Thus, we understand that the Maxwell sector is always present whether the dreibein is trivial or not. The second limit is the commutative one, in which the additional fields to the ones related to the gravity disentangle, so we can set $A_\mu = \tilde{A}_\mu = 0$. Also, in this limit, for the inner derivation becomes the ordinary derivative according to the mapping $[X_\mu, f] \ \rightarrow \ i\partial_\mu f$ (see 3.1.1). Thus, the expressions of the transformations of the surviving fields, $e_\mu{}^a, \omega_\mu{}^a$ obtained in (5.11), become:

$$\begin{aligned}
\delta e_\mu{}^a &= -\partial_\mu \xi^a - 4\xi_b \omega_{\mu c}\epsilon^{abc} - 4\lambda_b e_{\mu c}\epsilon^{abc} \ , \\
\delta \omega_\mu{}^{ab} &= -\partial_\mu \lambda^a + \xi_b e_{\mu c}\epsilon^{abc} - 4\lambda_b \omega_{\mu c}\epsilon^{abc} \ .
\end{aligned} \tag{5.12}$$

[2]Here, due to the definitions of the generators of the gauge group, instead of the (3.49) we use the $\delta\hat{X} = [\epsilon, \hat{X}]$.

The above expressions resemble the ones in the gauge-theoretic approach of the Einstein three-dimensional gravity with positive cosmological constant, (2.62) and become identical to them after the consideration of the following redefinitions of the generators, gauge parameters and gauge fields:

$$\bar{\gamma}_a \;\to\; \frac{2i}{\sqrt{\lambda}}P_a \;,\;\; \tilde{\gamma}_a \;\to\; -4J_a \;,\;\; 4\lambda_a \;\to\; \lambda^a \;,\;\; \xi\frac{2i}{\sqrt{\lambda}} \;\to\; -\xi^a \;,\;\; e_\mu{}^a \;\to\; \frac{\sqrt{\lambda}}{2i}e_\mu{}^a \;,\;\; \omega_\mu{}^a \;\to\; -\frac{1}{4}\omega_\mu{}^a \;. \tag{5.13}$$

Therefore, in the commutative limit, the transformations of the fields of the three-dimensional gravity are successfully recovered.

Moving on with the construction of the gauge theory, the next step is to calculate the field strength tensor which will yield the expressions of the component curvature tensors. As we mentioned in section 3.3, since the fuzzy space we work on is of Lie type, according to equation (3.56), the field strength tensor will have the following form:

$$\mathcal{R}_{\mu\nu} = [\hat{X}_\mu, \hat{X}_\nu] - i\lambda C_{\mu\nu}{}^\rho \hat{X}_\rho \;. \tag{5.14}$$

The curvature tensor $\mathcal{R}_{\mu\nu}(X)$ takes values in the algebra, therefore it may be written as expansion on the generators:

$$\mathcal{R}_{\mu\nu}(X) = T_{\mu\nu}{}^a(X)\otimes\bar{\gamma}_a + R_{\mu\nu}{}^a(X)\otimes\tilde{\gamma}_a + F_{\mu\nu}(X)\otimes i\mathbf{1} + \tilde{F}_{\mu\nu}(X)\otimes\gamma_5 \;. \tag{5.15}$$

Combining the equations (5.7), (5.9), (5.14) and (5.15), we obtain the following expressions of the component curvature tensors:

$$\begin{aligned} T_{\mu\nu}{}^a &= i[X_\mu + A_\mu, e_\nu{}^a] - i[X_\nu + A_\nu, e_\mu{}^a] - 2\epsilon^{abc}\left(\{e_{\mu b}, \omega_{\nu c}\} + \{\omega_{\mu b}, e_{\nu c}\}\right) \\ &\quad + 2i\left([\omega_\mu{}^a, \tilde{A}_\nu] - [\omega_\nu{}^a, \tilde{A}_\mu]\right) - i\lambda C_{\mu\nu}{}^\rho e_\rho{}^a \;, \end{aligned} \tag{5.16}$$

$$\begin{aligned} R_{\mu\nu}{}^a &= i[X_\mu + A_\mu, \omega_\nu{}^a] - i[X_\nu + A_\nu, \omega_\mu{}^a] + \epsilon^{abc}\left(\tfrac{1}{2}\{e_{\mu b}, e_{\nu c}\} - 2\{\omega_{\mu b}, \omega_{\nu c}\}\right) \\ &\quad + \tfrac{i}{2}\left([e_\mu{}^a, \tilde{A}_\nu] - [e_\nu{}^a, \tilde{A}_\mu]\right) - i\lambda C_{\mu\nu}{}^\rho \omega_\rho{}^a \;, \end{aligned} \tag{5.17}$$

$$F_{\mu\nu} = i[X_\mu + A_\mu, X_\nu + A_\nu] - i[e_\mu{}^a, e_{\nu a}] + 4i[\omega_\mu{}^a, \omega_{\nu a}] - i[\tilde{A}_\mu, \tilde{A}_\nu] - i\lambda C_{\mu\nu}{}^\rho(X_\rho + A_\rho) \;, \tag{5.18}$$

$$\tilde{F}_{\mu\nu} = i[X_\mu + A_\mu, \tilde{A}_\nu] - i[X_\nu + A_\nu, \tilde{A}_\mu] + 2i\left([e_\mu{}^a, \omega_{\nu a}] + [\omega_\mu{}^a, e_{\nu a}]\right) - i\lambda C_{\mu\nu}{}^\rho \tilde{A}_\rho \;. \tag{5.19}$$

It is remarkable that, considering the commutative limit, the expressions of the above first two relations coincide with the ones of the three-dimensional Einstein gravity, (2.62), after employing the redefinitions of (5.13).

*The Euclidean case*

As we mentioned in the beginning of the section, the gauge group for this signature is the SU(2)×SU(2). Again, due to the fact that the anticommutators of the group do not close, we have to fix the representation and extend the algebra with the extra elements produced by the anticommutators, resulting with the U(2)×U(2) as the gauge group of the theory. Each U(2) is represented by the Pauli matrices and the unit matrix, therefore, the U(2)×U(2) gauge group will involve the following 4×4 matrices:

$$J_a^L = \begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix} \;,\; J_0^L = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \;, \quad \text{and} \quad J_a^R = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_a \end{pmatrix} \;,\; J_0^R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \;. \tag{5.20}$$

However, one should be careful in identifying correctly the noncommutative dreibein and spin connection gauge fields. For the correct interpretation of the gauge fields, the generators we consider are the following:

$$P_a = \tfrac{1}{2}(J_a^L - J_a^R) = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix} , \quad M_a = \tfrac{1}{2}(J_a^L + J_a^R) = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & \sigma_a \end{pmatrix} , \tag{5.21}$$

and also:

$$\mathbf{1} = J_0^L + J_0^R , \quad \gamma_5 = J_0^L - J_0^R . \tag{5.22}$$

The above form of the generators satisfy the expected commutation and anticommutation relations, which are obtained using the corresponding ones for the Pauli matrices:

$$\begin{aligned} &[P_a, P_b] = i\epsilon_{abc}M_c , \quad [P_a, M_b] = i\epsilon_{abc}P_c , \quad [M_a, M_b] = i\epsilon_{abc}M_c , \\ &\{P_a, P_b\} = \tfrac{1}{2}\delta_{ab}\mathbf{1} , \quad \{P_a, M_b\} = \tfrac{1}{2}\delta_{ab}\gamma_5 , \quad \{M_a, M_b\} = \tfrac{1}{2}\delta_{ab}\mathbf{1} , \\ &[\gamma_5, P_a] = [\gamma_5, M_a] = 0 , \quad \{\gamma_5, P_a\} = 2M_a , \quad \{\gamma_5, M_a\} = 2P_a . \end{aligned} \tag{5.23}$$

Then, identifying the underlying space, on which the above gauge theory is constructed, as the $\mathbb{R}^3_\lambda$ (see section 4.1), one proceeds as in the Lorentzian case with defining the covariant coordinate:

$$\hat{X}_\mu = X_\mu \otimes i\mathbf{1} + e_\mu{}^a \otimes P_a + \omega_\mu{}^a \otimes M_a + A_\mu \otimes i\mathbb{1} + \tilde{A}_\mu \otimes \gamma_5 , \tag{5.24}$$

and then the gauge parameter as an expansion on the generators, too:

$$\epsilon = \xi^a \otimes P_a + \lambda^a \otimes M_a + \epsilon_0 \otimes i\mathbf{1} + \tilde{\epsilon}_0 \otimes \gamma_5 . \tag{5.25}$$

The transformation rule of the covariant coordinate produces the transformations of the component gauge fields, in the same spirit as in the (5.11) and employing the definition of the field strength tensor of Lie-type noncommutativity, one results with the component curvature tensors, similar to the ones in the Lorentzian case, (5.19).

*Action of fuzzy three-dimensional gravity*

In order to complete the picture, we have to determine the action of the theory. Since the fuzzy spaces, on which we construct the gauge theories are three-dimensional, inspired by the gauge-theoretic approach of Einstein's three-dimensional gravity described in section 2.2, the obvious choice is to employ an action of Chern-Simons type. For the Lorentzian case, $\mathbb{R}^{1,2}_\lambda$ the action [108][3] is:

$$S_0 = \frac{1}{g^2}\mathrm{Tr}\left(\frac{i}{3}C^{\mu\nu\rho}X_\mu X_\nu X_\rho - m^2 X_\mu X^\mu\right) . \tag{5.26}$$

Variation of the above action leads to the field equation:

$$[X_\mu, X_\nu] - 2im^2 C_{\mu\nu}{}^\rho X_\rho = 0 , \tag{5.27}$$

which admits as solution the space, $\mathbb{R}^{1,2}_\lambda$, for $2m^2 = \lambda$. Also, had we started with the same action for the Euclidean case, $\mathbb{R}^3_\lambda$, the only difference would be that the $C^{\mu\nu\rho}$ would be replaced by the $\epsilon^{\mu\nu\rho}$ and the parameter would be $2m^2 = -\lambda$ (for details see Appendix E).

[3] A similar action was proposed in Ref. [54] for a gravity theory on the fuzzy sphere. See also [109].

Now, in order to introduce the gauge fields of the theory into the above action, (5.26), one could either consider fluctuations of the above solution, (5.27), replacing the coordinates with the covariant coordinates, or in a less straightforward way, replace the coordinates with the covariant coordinates in the action and then, after variation, obtain the field equations. Eventually, the action would be written in terms of the gauge fields and, therefore, a trace over the gauge indices, $\mathrm{tr}_G$ should be involved. The non-vanishing traces of the generators are obtained starting from the expressions of the anticommutators in (5.6):

$$\mathrm{tr}_G(\bar{\gamma}_a\bar{\gamma}_b) = 4\eta_{ab}\ , \quad \mathrm{tr}_G(\tilde{\gamma}_a\tilde{\gamma}_b) = -16\eta_{ab}\ , \quad . \tag{5.28}$$

Therefore, the action written in terms of the gauge fields is:

$$S = \frac{1}{g^2}\mathrm{Tr}\mathrm{tr}_G\left(\tfrac{i}{3}C^{\mu\nu\rho}\hat{X}_\mu\hat{X}_\nu\hat{X}_\rho - \tfrac{\lambda}{2}\hat{X}_\mu\hat{X}^\mu\right)\ , \tag{5.29}$$

where, the first trace, Tr is over the N×N matrices representing the coordinates and the second trace $\mathrm{tr}_G$ is over the 4×4 matrices representing the generators of the gauge group GL(2,ℂ). The above action, (5.29) can be written in terms of the curvature tensor $\mathcal{R}_{\mu\nu}$ of (5.14) as:

$$S = \tfrac{1}{6g^2}\mathrm{Tr}\mathrm{tr}_G\left(iC^{\mu\nu\rho}\hat{X}_\mu\mathcal{R}_{\nu\rho}\right) + S_\lambda\ , \tag{5.30}$$

where $S_\lambda = -\frac{\lambda}{6g^2}\mathrm{Tr}\mathrm{tr}_G(\hat{X}_\mu\hat{X}^\mu)$ and vanishes in the limit $\lambda \to 0$. Now, using the explicit form of the trace over the generators, (5.28), in the action given in equation (5.30), we obtain the following expression:

$$\begin{aligned} S = {} & \tfrac{2}{3g^2}\mathrm{Tr}iC^{\mu\nu\rho}\left(e_{\mu a}T_{\nu\rho}{}^a - 4\omega_{\mu a}R_{\nu\rho}{}^a - (X_\mu + A_\mu)F_{\nu\rho} + \tilde{A}_\mu\tilde{F}_{\nu\rho}\right) \\ & - \tfrac{2\lambda}{3g^2}\mathrm{Tr}\left(e_\mu{}^a e^\mu{}_a - 4\omega_\mu{}^a\omega^\mu{}_a - (X_\mu + A_\mu)(X^\mu + A^\mu) + \tilde{A}_\mu\tilde{A}^\mu\right)\ . \end{aligned} \tag{5.31}$$

It is worth-noting that if we consider the commutative limit and apply the redefinitions in (5.13), the above action, (5.31), reduces to the one given in the Einstein's three-dimensional gravity, described in section 2.2, specifically in (2.54). However, in the present case, noncommutativity implies the introduction of an additional sector, which cannot be decoupled, unless we consider the commutative limit.

Concluding the construction, we move on with the variation of the above action with respect to the various gauge fields. The equations of motion are obtained to be[4]:

$$T_{\mu\nu}{}^a = 0\ , \quad R_{\mu\nu}{}^a = 0\ , \quad F_{\mu\nu} = 0 \quad \tilde{F}_{\mu\nu} = 0\ . \tag{5.32}$$

Again, as expected, in the commutative limit, the first two reduce to the ones of the three-dimensional Einstein's gravity theory (see section 2.2).

To sum up, in this section we constructed a three-dimensional gravitational theory (with cosmological constant) as a gauge theory in the noncommutative framework. Following the standard procedure, we defined the covariant coordinate and from its transformation rule we obtained the transformations of the component gauge fields of the theory attached to the generators of the (extended due to the non-closure of the anticommutators) algebra, after an SO(3) decomposition. Then, due to the Lie-type classification of the three-dimensional space we used,

[4]For a detailed calculation see Appendix E.

we employed the appropriate expression for the field strength tensor and obtained the expressions of the component curvature tensors of the theory. Then, using the field strength tensor we proposed an action of Chern-Simons type and eventually, from variation with respect to the gauge fields, we obtained the equations of motion. It is worth-noting that the above results reduce to the ones of the three-dimensional Einstein's gravity theory in the gauge-theoretic approach, when the commutative limit is considered.

# Chapter 6

# Noncommutative four-dimensional gravity as a gauge theory

In this section we extend the context of the previous section (the construction of a three-dimensional gravitational model as a noncommutative gauge theory) to the four-dimensional case [78]. More specifically, the four-dimensional covariant space we employ is the fuzzy de Sitter space, fuzzy $dS_4$, which was described in detail in section 4.3. As mentioned there, in the Euclidean signature, the group of isometries must get extended to the SO(6) for reasons of covariance. The various generators of this group were identified with operators which correspond to physical observables, such as the coordinates, momenta and angular momenta. In order to formulate gravity as a gauge theory on the above fuzzy space, we choose to gauge the SO(5) maximal subgroup of the SO(6) symmetry group. Using the standard toolkit and procedure, we begin to construct an SO(5) gauge theory but due to the non-closure property of the anti-commutators of the generators, (3.59), the gauge group we eventually consider, as we explain later, is the SO(6)×U(1) in a fixed representation. The gauge group with which we end up for the formulation of our gravitational theory (specifically the SO(6) part), appears to be related to the conformal group in the Euclidean signature. Therefore, due to this coincidence, we will be able to consider a commutative limit of the noncommutative gauge-theory we build and, at this limit, compare our results with the ones from the gauge-theoretic approach of conformal gravity, described in section 2.4.

## 6.1 The gauge group and its representation

We plan to build a noncommutative four-dimensional gravitational model as a gauge theory of the group of symmetries of a four-dimensional noncommutative covariant space. The space we choose is the covariant fuzzy $dS_4$ space (described in 4.3), which carries the symmetry of its commutative analogue, that is the SO(1,4), SO(5) in the Euclidean signature we employ. As we argued in section 4.3, this symmetry had to be enlarged because the identification of a subset of the generators with the coordinate operators could not provide a commutation relation that would respect covariance. The minimally enlarged symmetry that could fix the problem was the SO(6), which we adopted. Therefore, thinking along the lines of the four-dimensional commutative case of the Poincaré gravity (section 2.3), in which the isometry group was gauged, here, we gauge the isometry group of the fuzzy $dS_4$, i.e. the SO(5), as seen as a subgroup of the

SO(6) group we resulted after the enlargement.

However, as explained in section 3.3, in noncommutative gauge theories, the involvement of the anticommutators of the generators of the algebra is inevitable. Of course, the anticommutators of the generators (in arbitrary representation) of an algebra do not necessarily yield operators belonging to the algebra and this is exactly the case for the generators of the chosen gauge group, SO(5). The indicated prescription for this drawback is to choose a specific representation, in which the generators belong, and include the operators produced by the anticommutators into the algebra, considering them as generators, too. This will result in the extension of the initial gauge group to one with larger symmetry. In our case, application of this recipe leads to the extension of the initial gauge group, SO(5), to the SO(6)×U(1) group[1] with generators being represented by 4x4 matrices, i.e. the representation should be fixed to the 4 of SO(6). Explicitly, the matrices representing the sixteen generators of the SO(6)×U(1) group are constructed, as combinations of the four $\gamma$-matrices[2] (in the Euclidean signature) which satisfy the following well-known anticommutation relation:

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}\mathbf{1} \ , \tag{6.1}$$

where $m, n = 1, \ldots 4$. Also, the $\Gamma_5$ matrix, defined as $\Gamma_5 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4$, has to be included. Therefore, the generators of the SO(6) part of the gauge group are the following:

($i$) Six Lorentz rotation generators: $\mathrm{M}_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b] = -\frac{i}{2}\Gamma_a\Gamma_b \, , a < b$,

($ii$) Four generators for conformal boosts: $\mathrm{K}_a = \frac{1}{2}\Gamma_a$,

($iii$) Four generators for translations: $\mathrm{P}_a = -\frac{i}{2}\Gamma_a\Gamma_5$,

($iv$) One generator for special conformal transformations: $\mathrm{D} = -\frac{1}{2}\Gamma_5$

and for the U(1) part:

($v$) The unit matrix, $\mathbf{1}$, generator.

The $\Gamma$-matrices are built out of the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{6.2}$$

as tensor products of $\sigma_i$ as:

$$\Gamma_1 = \sigma_1 \otimes \sigma_1, \quad \Gamma_2 = \sigma_1 \otimes \sigma_2, \quad \Gamma_3 = \sigma_1 \otimes \sigma_3 \ , \tag{6.3}$$

$$\Gamma_4 = \sigma_2 \otimes \mathbf{1}, \quad \Gamma_5 = \sigma_3 \otimes \mathbf{1} \ . \tag{6.4}$$

The explicit expressions of the generators defined above as combinations of the $\gamma$-matrices can now be written in terms of the Pauli matrices. In particular, the components of $M_{ab}$ are:

$$M_{ij} = -\frac{i}{2}\Gamma_i\Gamma_j = \frac{1}{2}\mathbf{1} \otimes \sigma_k \ , \quad M_{4k} = -\frac{i}{2}\Gamma_4\Gamma_k = -\frac{1}{2}\sigma_3 \otimes \sigma_k \ , \tag{6.5}$$

[1]The extension of the gauge group SO(5) to SO(6)×U(1) is a coincidence with the SO(6) symmetry related to the fuzzy $dS_4$ space and should not be confused.

[2]Notational caption: as space indices we use the latin letters $m, n, \ldots$ while as gauge indices we use the $a, b, \ldots$ latin letters.

of $K_a$ are:

$$K_i = \tfrac{1}{2}\Gamma_i \ , \quad K_4 = \tfrac{1}{2}\Gamma_4 \ , \tag{6.6}$$

of $P_a$ are:

$$P_i = -\tfrac{i}{2}\Gamma_i\Gamma_5 \ , \quad P_4 = -\tfrac{i}{2}\Gamma_4\Gamma_5 \ , \tag{6.7}$$

where $a = i, 4$ and $i, j, k = 1, 2, 3$. Having expressed the explicit expressions of the generators in terms of the Pauli matrices, the following commutation relations are obtained:

$$\begin{aligned}
[K_a, K_b] &= iM_{ab} \ , \quad [P_a, P_b] = iM_{ab} \\
[P_a, D] &= iK_a \ , \quad [K_a, P_b] = i\delta_{ab}D \ , \quad [K_a, D] = -iP_a \\
[K_a, M_{bc}] &= i(\delta_{ac}K_b - \delta_{ab}K_c) \\
[P_a, M_{bc}] &= i(\delta_{ac}P_b - \delta_{ab}P_c) \\
[M_{ab}, M_{cd}] &= i(\delta_{ac}M_{bd} + \delta_{bd}M_{ac} - \delta_{bc}M_{ad} - \delta_{ad}M_{bc}) \\
[D, M_{ab}] &= 0 \, .
\end{aligned} \tag{6.8}$$

Therefore, since the gauge group and the explicit expressions of the generators and their commutation relations are determined, we can now move on with the standard procedure for the building of the noncommutative gauge theory.

## 6.2 Construction of the gauge theory

In order to move on with the construction of the SO(6)×U(1) gauge theory, one has to introduce the covariant coordinate of the theory, which is defined by the following relation:

$$\hat{X}_m = X_m \otimes \mathbf{1} + \mathcal{A}_m(X) \, , \tag{6.9}$$

where $m = 1, \dots 4$. By definition the coordinate $\hat{X}_m$ obeys a covariant gauge transformation rule, that is:

$$\delta\hat{X}_m = i[\epsilon, \hat{X}_m] \, , \tag{6.10}$$

where $\epsilon = \epsilon(X)$ is the infinitesimal gauge parameter which is a function of the coordinates of $dS_4$, which are $N \times N$ matrices, where $N$ is the dimension of the representation in which the coordinates are described. Also, $\epsilon$ takes values in the algebra of SO(6)×U(1), (6.8), which is represented by 4×4 matrices. For this reason, it is possible to express it as an expansion on the sixteen generators of the algebra, that is:

$$\epsilon(X) = \epsilon_0(X) \otimes \mathbf{1} + \xi^a(X) \otimes K_a + \tilde{\epsilon}_0(X) \otimes D + \lambda^{ab}(X) \otimes M_{ab} + \tilde{\xi}^a(X) \otimes P_a \ . \tag{6.11}$$

Each term in the above expression is a tensor product of the $N \times N$ matrices (coordinates) and the $4 \times 4$ matrices (generators), therefore, each term is a $4N \times 4N$ matrix. Now, taking into consideration that the coordinate $X_m$ is not affected by the gauge transformation, i.e. $\delta X_m = 0$, one can find the transformation property of the $\mathcal{A}_m$, introduced in the equation (6.10). In analogy to the commutative case, the transformation property of $\mathcal{A}_m$, (3.51), indicates that it can be interpreted as the potential, that is the gauge connection of the theory. In our case, $\mathcal{A}_m$ is a function of matrices-coordinates, $X_m$, of the fuzzy $dS_4$. The $\mathcal{A}_m(X)$ takes values in the

SO(6)×U(1) algebra, meaning that it can be spanned on its generators, in a similar way to the gauge parameter, (6.11), that is:

$$\mathcal{A}_m(X) = e_m^{\ a}(X)\otimes P_a + \omega_m^{\ ab}(X)\otimes M_{ab}(X) + b_m^{\ a}(X)\otimes K_a(X) + \tilde{a}_m(X)\otimes D + a_m(X)\otimes \mathbf{1}\,. \quad (6.12)$$

In the above expression, it is understood that we have introduced one gauge field for each generator. The component gauge fields depend on the coordinates of the space, $X_m$, meaning that they are $N \times N$ matrices. Again, for the same reason as in the expansion of the gauge parameter, (6.11), each term in the above expansion of the gauge connection, (6.12), is a $4N\times 4N$ matrix. Now, having determined the gauge connection, (6.12), the covariant coordinate, (6.10) is written explicitly as:

$$\hat{X}_m = X_m \otimes \mathbb{1} + e_m^{\ a}(X)\otimes P_a + \omega_m^{\ ab}(X)\otimes M_{ab} + b_m^{\ a}\otimes K_a + \tilde{a}_m \otimes D + a_m \otimes \mathbf{1}\,. \quad (6.13)$$

Furthermore, in the SO(6)×U(1) gauge theory we develop, the field strength tensor remains to be determined. As we argued in section 4.3, equation (4.61), the field strength tensor for the fuzzy $dS_4$ space, is given by the expression:

$$\mathcal{R}_{mn} = [\hat{X}_m, \hat{X}_n] - \frac{i\lambda^2}{\hbar}\hat{\Theta}_{mn}\,. \quad (6.14)$$

The above tensor, $\mathcal{R}_{mn}$, is valued in the algebra of the gauge group, therefore it is expanded in terms of its component curvatures as:

$$\mathcal{R}_{mn}(X) = R_{mn}^{\ \ ab}(X)\otimes M_{ab} + \tilde{R}_{mn}^{\ \ a}(X)\otimes P_a + R_{mn}^{\ \ a}(X)\otimes K_a + \tilde{R}_{mn}(X)\otimes D + R_{mn}(X)\otimes \mathbf{1}\,. \quad (6.15)$$

In (6.14), the introduction of the 2-form field $\mathcal{B}_{mn}$, of which we discussed in (4.61), is taking place. Since it is also valued in the gauge algebra, SO(6)×U(1), it can be expanded on the generators as:

$$\mathcal{B}_{mn} = B_{mn}\otimes \mathbf{1} + \tilde{B}_{mn}^{\ \ a}\otimes P_a + B_{mn}^{\ \ ab}\otimes M_{ab} + B_{mn}^{\ \ a}\otimes K_a + \tilde{B}_{mn}\otimes D\,, \quad (6.16)$$

which transforms covariantly as:

$$\delta\mathcal{B}_{mn} = i[\epsilon, \hat{\Theta}_{mn}] \quad (6.17)$$

and gives the transformation of $\hat{\Theta}_{mn}$, namely $\delta\hat{\Theta}_{mn} = i[\epsilon, \hat{\Theta}_{mn}]$. At this point, all that is necessary for the determination of the transformations of the gauge fields and the expressions of the component curvatures is written down. Here, we write down only their resulting expressions and many intermediate steps and explicit calculations are given in Appendix F.

The gauge transformation rules of the sixteen gauge fields are obtained to be the following:

$$\delta\omega_m^{\ ab} = -i[X_m, \lambda^{ab}] - i[a_m, \lambda^{ab}] + i[\epsilon_0, \omega_m^{\ ab}] - 2\{\xi^a, b_m^{\ b}\} - \frac{1}{2}\{\lambda^a_{\ c}, \omega_m^{\ bc}\} - \frac{1}{2}\{\tilde{\xi}^a, e_m^{\ b}\}$$
$$+ i[\xi^c, e_m^{\ d}]\epsilon_{abcd} + \frac{i}{2}[\tilde{\epsilon}_0, \omega_m^{\ cd}]\epsilon_{abcd} + \frac{i}{2}[\lambda^{cd}, \tilde{a}_m]\epsilon_{abcd} - i[\tilde{\xi}^c, b_m^{\ d}]\epsilon_{abcd} \tag{6.18}$$

$$\delta e_m^{\ a} = -i[X_m, \tilde{\xi}^a] - i[a_m, \tilde{\xi}^a] + i[\epsilon_0, e_m^{\ a}] - \{\xi^a, \tilde{a}_m\} + \{\tilde{\epsilon}_0, b_m^{\ a}\} + \frac{1}{4}\{\lambda^a_{\ b}, e_m^{\ b}\} - \frac{1}{4}\{\tilde{\xi}_b, \omega_m^{\ ab}\}$$
$$+ i[\xi^c, \omega_m^{\ bd}]\epsilon_{abcd} - i[\lambda^{cd}, b_m^{\ b}]\epsilon_{abcd} \tag{6.19}$$

$$\delta b_m^{\ a} = -i[X_m, \xi^a] - i[a_m, \xi^a] + i[\epsilon_0, b_m^{\ a}] - \{\xi_b, \omega_m^{\ ab}\} - 2\{\tilde{\epsilon}_0, e_m^{\ a}\} + \frac{1}{2}\{\lambda^a_{\ b}, b_m^{\ b}\} + \{\tilde{\xi}^a, \tilde{a}_m\}$$
$$+ i[\lambda^{bc}, e_m^{\ d}]\epsilon_{abcd} + i[\tilde{\xi}^b, \omega_m^{\ cd}]\epsilon_{abcd} \tag{6.20}$$

$$\delta a_m = -i[X_m, \epsilon_0] - i[a_m, \epsilon_0] + i[\xi^a, b_m^{\ a}] + i[\tilde{\epsilon}_0, \tilde{a}_m] + \frac{i}{2}[\lambda_{ab}, \omega_m^{\ ab}] + \frac{i}{2}[\tilde{\xi}_a, e_m^{\ a}] \tag{6.21}$$

$$\delta\tilde{a}_m = -i[X_m, \tilde{\epsilon}_0] - i[a_m, \tilde{\epsilon}_0] + i[\epsilon_0, \tilde{a}_m] + \{\xi_a, e_m^{\ a}\} - \{\tilde{\xi}_a, b_m^{\ a}\} + \frac{i}{2}[\lambda^{ad}, \omega_m^{\ bc}]\epsilon_{abcd}\,. \tag{6.22}$$

The gauge transformation rules of the 2-form gauge fields are given as:

$$\delta B_{mn} = -i[\Theta_{mn}, \epsilon_0] - i[B_{mn}, \epsilon_0] + i[\xi^a, B_{mn}^{\ \ a}] + i[\tilde{\epsilon}_0, \tilde{B}_{mn}] + \frac{i}{2}[\lambda_{ab}, B_{mn}^{\ \ ab}] + \frac{i}{2}[\tilde{\xi}_a, \tilde{B}_{mn}^{\ \ a}] \tag{6.23}$$

$$\delta\tilde{B}_{mn} = -i[\Theta_{mn}, \tilde{\epsilon}_0] - i[B_{mn}, \tilde{\epsilon}_0] + i[\epsilon_0, \tilde{B}_{mn}] + \{\xi_a, \tilde{B}_{mn}^{\ \ a}\} - \{\tilde{\xi}_a, B_{mn}^{\ \ a}\} + \frac{i}{2}[\lambda^{ab}, B_{mn}^{\ \ bc}]\epsilon_{abcd} \tag{6.24}$$

$$\delta\tilde{B}_{mn}^{\ \ a} = -i[\Theta_{mn}, \tilde{\xi}^a] - i[B_{mn}, \tilde{\xi}^a] + i[\epsilon_0, \tilde{B}_{mn}^{\ \ a}] - \{\xi^a, \tilde{B}_{mn}\} + \{\tilde{\epsilon}_0, B_{mn}^{\ \ a}\} + \frac{1}{4}\{\lambda^a_{\ b}, \tilde{B}_{mn}^{\ \ b}\}$$
$$- \frac{1}{4}\{\tilde{\xi}_b, B_{mn}^{\ \ ab}\} + i[\xi^c, B_{mn}^{cd}]\epsilon_{abcd} - i[\lambda^{cd}, B_{mn}^{\ \ b}]\epsilon_{abcd} \tag{6.25}$$

$$\delta B_{mn}^{\ \ a} = -i[\Theta_{mn}, \xi^a] - i[B_{mn}, \xi^a] + i[\epsilon_0, B_{mn}^{\ \ a}] - \{\xi_b, B_{mn}^{\ \ ab}\} - 2\{\tilde{\epsilon}_0, \tilde{B}_{mn}^{\ \ a}\} + \frac{1}{2}\{\lambda^a_{\ b}, B_{mn}^{\ \ b}\}$$
$$+ \{\tilde{\xi}^a, \tilde{B}_{mn}\} + \frac{i}{2}[\lambda^{bc}, \tilde{B}_{mn}^{\ \ d}]\epsilon_{abcd} + i[\tilde{\xi}^b, B_{mn}^{\ \ cd}]\epsilon_{abcd} \tag{6.26}$$

$$\delta B_{mn}^{\ \ ab} = -i[\Theta_{mn}, \lambda^{ab}] - i[B_{mn}, \lambda^{ab}] + i[\epsilon_0, B_{mn}^{\ \ ab}] - 2\{\xi^a, B_{mn}^{\ \ a}\} - \frac{1}{2}\{\lambda^a_{\ c}, B_{mn}^{\ \ bc}\} - \frac{1}{2}\{\tilde{\xi}^a, \tilde{B}_{mn}^{\ \ b}\}$$
$$+ i[\xi^c, \tilde{B}_{mn}^{\ \ d}]\epsilon_{abcd} + \frac{i}{2}[\tilde{\epsilon}_0, B_{mn}^{\ \ cd}]\epsilon_{abcd} + \frac{i}{2}[\lambda^{cd}, \tilde{B}_{mn}] - [\tilde{\xi}^c, B_{mn}^{\ \ d}]\epsilon_{abcd}\,. \tag{6.27}$$

The expressions of the component tensors of $\mathcal{R}_{mn}$ of (6.14) are found to be the following:

$$\begin{aligned}
R_{mn} &= [X_m, a_n] - [X_n, a_m] + [a_m, a_n] + [b_m^a, b_{na}] + [\tilde{a}_m, \tilde{a}_n] \\
&\quad + \frac{1}{2}[\omega_m{}^{ab}, \omega_{nab}] + [e_{ma}, e_n{}^a] - \frac{i\hbar}{\lambda^2} B_{mn} \\
\tilde{R}_{mn} &= [X_m, \tilde{a}_n] + [a_m, \tilde{a}_n] - [X_n, \tilde{a}_m] - [a_n, \tilde{a}_m] - i\{b_{ma}, e_n{}^a\} + i\{b_{na}, e_m{}^a\} \\
&\quad + \frac{1}{2}\epsilon_{abcd}[\omega_m{}^{ab}, \omega_n{}^{cd}] - \frac{i\hbar}{\lambda^2}\tilde{B}_{mn} \\
R_{mn}{}^a &= [X_m, b_n^a] + [a_m, b_n^a] - [X_n, b_m^a] - [a_n, b_m^a] + i\{b_{mb}, \omega_m{}^{ab}\} - i\{b_{nb}, \omega_m{}^{ab}\} \\
&\quad + i\{\tilde{a}_m, e_n{}^a\} - i\{\tilde{a}_n, e_m{}^a\} + \epsilon_{abcd}([e_m^b, \omega_n{}^{cd}] - [e_n{}^b, \omega_m{}^{cd}]) - \frac{i\hbar}{\lambda^2} B_{mn}{}^a \\
\tilde{R}_{mn}{}^a &= [X_m, e_n{}^a] + [a_m, e_n{}^a] - [X_n, e_m^a] - [a_n, e_m^a] + i\{b_m^a, \tilde{a}_n\} - i\{b_n{}^a, \tilde{a}_m\} \\
&\quad - ([b_m^b, \omega_n{}^{cd}] - [b_n{}^b, \omega_m{}^{cd}])\epsilon_{abcd} - i\{\omega_m{}^{ab}, e_{nb}\} + i\{\omega_n{}^{ab}, e_{mb}\} - \frac{i\hbar}{\lambda^2}\tilde{B}_{mn}{}^a \\
R_{mn}{}^{ab} &= [X_m, \omega_n{}^{ab}] + [a_m, \omega_n{}^{ab}] - [X_n, \omega_m{}^{ab}] - [a_n, \omega_m{}^{ab}] + 2i\{b_m^a, b_n{}^b\} + ([b_m^c, e_n^d] - [b_n{}^c, e_m^d])\epsilon_{abcd} \\
&\quad + \frac{1}{2}([\tilde{a}_m, \omega_n{}^{cd}] - [\tilde{a}_n, \omega_m{}^{cd}])\epsilon_{abcd} + 2i\{\omega_m{}^{ac}, \omega_n{}^b{}_c\} + 2i\{e_m^a, e_n{}^b\} - \frac{i\hbar}{\lambda^2} B_{mn}{}^{ab}
\end{aligned} \tag{6.28}$$

The above expressions of the component curvatures pave the way for the introduction and explicit expression of the action of the theory. Before we move to the part of the action, we should note that the above results of the transformations of the fields, (6.22), and their corresponding curvatures, (6.28) reduce to the respective results of the gauge-theoretic approach of conformal gravity, described in section 2.4 (modified for the Euclidean curvature), specifically equations (2.93) and (2.96)-(2.101), respectively -up to some tuning of the numerical coefficients- after considering the commutative limit. In this limit, the U(1) gauge field, that was introduced due to noncommutativity, decouples, therefore the gauge theory in the commutative limit is the SO(6), that is the conformal gravity, in euclidean signature.

## 6.3 The constraints for the symmetry breaking

For the dynamic part, the action of the noncommutative theory of gravity should be written down in terms of the curvature tensors, given in (6.28). Straightforward consideration of an action of Yang-Mills type would result to a theory, invariant under the SO(6)×U(1) gauge symmetry. However, the gauge symmetry of the action, with which we would like to end up, is expressed by the Lorentz group (in the Euclidean signature), therefore, we have to reduce the redundant symmetry. As discussed in section 2.4 in the case of the four-dimensional conformal gravity, the symmetry was reduced imposing specific constraints resulting to an action respecting Weyl symmetry. Also, in the same section, we argued that it is possible to result with an action with Lorentz symmetry, after the introduction of two scalars in the fundamental representation of the initial, conformal gauge group, inducing a spontaneous symmetry breaking. In the current noncommutative case, we aim at a Lorentz invariant action, but given that we prefer not to introduce any matter fields, we proceed with adopting the option of the imposition of appropriate constraints that would lead to the desired final symmetry.

The most straightforward way to realize the above breaking is to consider a constrained theory in which the component curvature tensors are all set to zero, except for those related to the remnant symmetry. Since we want to break the initial SO(6)×U(1) symmetry and result with a vacuum with SO(4)×U(1) remnant symmetry, the only non-vanishing tensors would be the $R_{mn}^{\ \ ab}(M)$ and $R_{mn}(\mathbf{1})$. However, this approach leads to an over-constrained theory, that is evident after counting the degrees of freedom that survive the breaking. Therefore, it is rather wise to impose non-trivial constraints, ensuring the correct number of the degrees of freedom. The first constraint we impose is the following:

$$\tilde{R}_{mn}^{\ \ a}(P) = 0 \ , \tag{6.29}$$

that is the torsionless condition, which is rather anticipated, recalling the cases of Einstein and conformal gravity in previous sections. Furthermore, the possible interpretation of $b_m^{\ a}$ as a second vierbein would give a theory with two metrics or two vierbein, which is not desirable in our case. Therefore, we are led to solve the constraint (6.29), considering $e_m^{\ a} = b_m^{\ a}$, leading to an expression of the spin connection $\omega_m^{\ ab}$ as a function of the fields, $e_m^{\ a}, a_m, \tilde{a}_m$. It is worth-mentioning that the imposition of the torsionless condition along with $e_m^{\ a} = b_m^{\ a}$, results in the correct number of degrees of freedom, leading to the noncommutative theory of the four-dimensional gravity with (broken) symmetry SO(4)×U(1).

In order to go on with obtaining the explicit expression resulting from solving the constraint of the torsionless condition, which relates the spin connection field to the rest of the gauge fields, we employ the following two identities:

$$\delta^{abc}_{fgh} = \epsilon^{abcd}\epsilon_{fghd} \qquad \text{and} \qquad \frac{1}{3!}\delta^{abc}_{fgh}a^{fgh} = a^{[fgh]} \ . \tag{6.30}$$

Therefore, the constraint (6.29) takes the following form:

$$\epsilon^{abcd}[e_{mb}, \omega_{ncd}] - i\{\omega_m^{\ ab}, e_{nb}\} = -[D_m, e_n^{\ a}] - i\{e_m^{\ a}, \tilde{a}_n\} \ , \tag{6.31}$$

where $D_m = X_m + a_m$, that is the covariant coordinate of an Abelian gauge theory. The above equation is written as follows:

$$\epsilon^{abcd}[e_m^{\ b}, \omega_n^{\ cd}] = -[D_m, e_n^{\ a}] \qquad \text{and} \qquad \{\omega_m^{\ ab}, e_{nb}\} = \{e_m^{\ a}, \tilde{a}_n\} \ . \tag{6.32}$$

Making use of the identities (6.30), the above two equations lead to the expression of the spin connection with respect to the rest of the fields:

$$\omega_n^{\ ac} = -\frac{3}{4}e^m_{\ b}(-\epsilon^{abcd}[D_m, e_{nd}] + \delta^{[bc}\{e_n^{\ a]}, \tilde{a}_m\}) \, . \tag{6.33}$$

According to ref. [110], the vanishing of a field strength tensor in a gauge theory constructed on a simply connected space means that, locally, its corresponding gauge fields may vanish as well. Should this argument be applicable in our case, it would simplify the expressions of the curvature tensors and, thus, that of the action. Nevertheless, it cannot be applied in our case, because identification of the vierbein as a gauge field of the theory, implies the mixing of gauge theory (internal symmetries) and geometry (spacetime symmetries). Therefore, given that the vierbein is considered to be invertible at every point of the space, adoption of the above argument (setting the vierbein to zero) would lead to degenerate vierbein matrices, inducing a degenerate metric tensor of the space [10]. However, we could set $\tilde{a}_m = 0$, since it does not admit a geometric interpretation. This fixing of the gauge field $\tilde{a}_m$ will also modify the expression of the spin connection with respect to the other fields, (6.33), producing an even simpler and final expression of the spin connection in terms of the vierbein and $U(1)$ gauge field, $a_m$, which reads:

$$\omega_n^{\ ac} = \frac{3}{4} e^m_{\ b} \epsilon^{abcd} [D_m, e_{nd}] \,. \tag{6.34}$$

At this point, it is useful to punctuate that the U(1) field strength tensor, $R_{mn}(\mathbf{1})$ is not chosen to be zero, which means that this U(1) part of the symmetry, which is related to the noncommutativity, remains unbroken in the resulting theory after the breaking, since it is still a theory on a noncommutative space. Its corresponding field, $a_m$, would vanish in the commutative regime in the broken theory, in which noncommutativity is relaxed and $a_m$ decouples. Thus, in this limit, the theory would be just described by the gauge group SO(4).

An alternative way to result with the desired SO(4) as remnant symmetry, after the breaking of SO(6), is to extrapolate the argument we developed in the conformal gravity case to the present noncommutative case, that is to include two scalar fields in the fundamental representation of SO(6) inducing a spontaneous symmetry breaking. We are confident that this kind of symmetry breaking would lead to constraints equivalent to the ones we considered.

## 6.4 The action

Now that we have imposed the constraints for breaking the symmetry, it is helpful to write the expressions of the curvature tensors since it will be them that will be used in the action. The explicit expressions of the non-vanishing tensors, taking into consideration the fixings $e_\mu^{\ a} = b_\mu^{\ a}$ and $\tilde{a}_\mu = 0$ are:

$$R_{mn} = [X_m, a_n] - [X_n, a_m] + 2[e_\mu^{\ a}, e_{\nu a}] + \tfrac{1}{2}[\omega_m^{\ ab}, \omega_{\nu ab}] - \tfrac{i\hbar}{\lambda^2} B_{mn} \ , \tag{6.35}$$

$$\tilde{R}_{mn} = \tfrac{1}{2}\epsilon_{abcd}[\omega_m^{\ ab}, \omega_\nu^{\ cd}] - \tfrac{i\hbar}{\lambda^2} B_{mn} \ , \tag{6.36}$$

$$\begin{aligned} R_{mn}^{\ \ a} &= [X_m, e_n^{\ a}] + [a_m, e_n^{\ a}] - [X_n, e_m^{\ a}] - [a_n, e_m^{\ a}] + i\{e_{mb}, \omega_m^{\ ab}\} - i\{e_{nb}, \omega_m^{\ ab}\} \\ &\quad + \epsilon_{abcd}([e_m^{\ b}, \omega_n^{\ cd}] - [e_n^{\ b}, \omega_m^{\ cd}]) - \tfrac{i\hbar}{\lambda^2} B_{mn}^{\ \ a} \ , \end{aligned} \tag{6.37}$$

$$R_{mn}^{\ \ ab} = [X_m, \omega_n^{\ ab}] + [a_m, \omega_n^{\ ab}] - [X_n, \omega_m^{\ ab}] - [a_n, \omega_m^{\ ab}] + 4i\{e_m^{\ a}, e_n^{\ b}\} + 2i\{\omega_m^{\ ac}, \omega_n^{\ b}{}_c\} - \tfrac{i\hbar}{\lambda^2} B_{mn}^{\ \ ab} \,. \tag{6.38}$$

The above expressions in which the gauge field $\omega_m^{ab}$ is also substituted by the expression (6.34) obtained by the constraint, are the final expressions of the tensors after the symmetry breaking. Before we move on with the determination of the action of the gravitational theory, let us briefly comment on the action of the extra 2-form field, $\mathcal{B}_{mn}$, which will be included in the total action.

Let us define the field strength tensor, $\hat{\mathcal{H}}_{mnp}$, of the 2-form gauge field:

$$\hat{\mathcal{H}}_{mnp} = \frac{1}{3}\left([\hat{X}_m, \hat{\Theta}_{np}] + [\hat{X}_n, \hat{\Theta}_{pm}] + [\hat{X}_p, \hat{\Theta}_{mn}]\right) \,. \tag{6.39}$$

The above field strength tensor transforms covariantly under a gauge transformation. In fact, this can be shown starting from the expression of the transformation of the 2-form field:

$$\delta\hat{\mathcal{H}}_{mnp} = \frac{1}{3}\left([\delta\hat{X}_m, \hat{\Theta}_{np}] + [\hat{X}_m, \delta\hat{\Theta}_{np}] + [\delta\hat{X}_n, \hat{\Theta}_{pm}] + [\hat{X}_n, \delta\hat{\Theta}_{pm}] + [\delta\hat{X}_p, \hat{\Theta}_{mn}] + [\hat{X}_p, \delta\hat{\Theta}_{mn}]\right) \,. \tag{6.40}$$

Making use of the transformation properties of $\hat{X}_m$ and $\hat{\Theta}_{mn}$, given in equations (6.10) and (6.17), respectively, along with the Jacobi identity, we find the following transformation rule:

$$\delta\hat{\mathcal{H}}_{mnp} = i[\epsilon, \hat{\mathcal{H}}_m] \;, \tag{6.41}$$

which is covariant. Next, in order to find the explicit expressions of the tensor, one has to expand the $\hat{\mathcal{H}}$ on the generators of the algebra:

$$\hat{\mathcal{H}}_{mnp} = H_{mnp} \otimes \mathbf{1} + \tilde{H}_{mnp}{}^{a} \otimes P_a + H_{mnp}{}^{ab} \otimes M_{ab} + H_{mnp}{}^{a} \otimes K_a + \tilde{H}_{mnp} \otimes D \,, \tag{6.42}$$

and calculate each component using the definition of the field strength tensor $\hat{\mathcal{H}}_{mnp}$. So, as far as the action of the 2-form is concerned, it will include only the kinetic term of the gauge field, which is:

$$\mathcal{S}_{\mathcal{B}} = \mathrm{Tr}\,\mathrm{tr}\,\hat{\mathcal{H}}_{mnp}\hat{\mathcal{H}}^{mnp} \;. \tag{6.43}$$

Now, we return to the determination of the action of the gauge theory. The most reasonable choice for the action is that of Yang-Mills type, therefore it is given as:

$$\mathcal{S} = \mathrm{Tr}\,\mathrm{tr}\,\Gamma_5 \left( \mathcal{R}_{mn}\mathcal{R}_{rs}\epsilon^{mnrs} + \hat{\mathcal{H}}_{mnp}\hat{\mathcal{H}}^{mnp} \right) \;, \tag{6.44}$$

where the Tr is the trace over the matrices representing the coordinates (takes the role of the integration of the commutative case) whereas the tr is the trace over the generators of the algebra. It should be noted that the above action is gauge invariant, since the field strength tensors transform covariantly:

$$\delta S = \mathrm{Tr}\Gamma_5(\delta\hat{\mathcal{R}}\hat{\mathcal{R}}+\hat{\mathcal{R}}\delta\hat{\mathcal{R}}+\delta\hat{\mathcal{H}}\mathcal{H}+\mathcal{H}\delta\hat{\mathcal{H}}) = \mathrm{Tr}(i[\epsilon,\mathcal{R}]\mathcal{R}+i\mathcal{R}[\epsilon,\mathcal{R}]+i[\epsilon,\hat{\mathcal{H}}]\hat{\mathcal{H}}+i\hat{\mathcal{H}}[\epsilon,\hat{\mathcal{H}}]) = 0 \,, \tag{6.45}$$

where the equations (4.63), (6.41) and the cyclicity property of the trace have been used. Also, the first term of the above action, (6.44), includes the field strength (curvature) tensor of the gauge theory, while the second one is the (non-topological) kinetic term of the 2-form field. The $\Gamma_5$ operator has been included in order to filter out most of the terms and, for the SO(4)×U(1) part, to keep the term including the curvature tensor $R_{mn}{}^{ab}$. The action (6.44) becomes:

$$\mathcal{S} = 2\mathrm{Tr}(R_{mn}{}^{ab}R_{rs}{}^{cd}\epsilon_{abcd}\epsilon^{mnrs} + 4\tilde{R}_{mn}R_{rs}\epsilon^{mnrs} + \frac{1}{3}H_{mnp}{}^{ab}H^{mnpcd}\epsilon_{abcd} + \frac{4}{3}\tilde{H}_{mnp}H^{mnp}) \;, \tag{6.46}$$

where for the above expression of the action written in the desirable SO(4) notation ($a, b, c, d = 1, \ldots 4$), we began from the action expressed in the SO(5) notation ($A, B, C, D = 1, \ldots, 5$) for the gauge indices, calculated the trace of the generators filtering out most of the terms and then decomposed the remaining terms to the SO(4) notation. The commutation and anticommutation relations for the generators of the algebra are given in (F.4).

Replacing with the expressions of the component tensors given in (6.35)-(6.38) and expressing the $\omega$ gauge field in terms of the rest of the surviving gauge fields, as given in (6.34), then variation with respect to the (surviving) gauge fields would lead to the equations of motion. We should remark that the extra 2-form gauge field, that was introduced for the sake of the covariance of the transformation of the field strength tensor of fuzzy $\mathrm{dS}_4$, decouple in the commutative limit and therefore it should not be expected to be observed.

# Chapter 7

# Summary and Conclusions

This thesis consists of our most recent works in which we engaged ourselves with combining two different frameworks. The first is the description of various gravity theories as gauge theories and the second is the framework of noncommutative geometry. The matching of the above two different frameworks is accomplished due to the existence of the well-defined formulation of gauge theories on noncommutative spaces. Also, after our recent reviews on previous works, it was advisable to include (in Appendix A) the construction of a particle physics model involving fuzzy extra dimensions, giving a more complete picture about all interactions of nature and their relation to the framework of noncommutative geometry. In this section, we sum up the whole context of the thesis, we write down our conclusions and we comment on the virtues of our contributions in the subject.

In the beginning we recalled some well-established works in which gravitational theories are interpreted as gauge theories. In general, the procedure of the construction of ordinary non-Abelian gauge theories is followed, but with some modifications due to the peculiarity of the nature of the gravitational interaction. More specifically, the gauge groups that are considered in each case do not parametrize internal symmetries but spacetime ones. In that sense, the vielbein, which is strongly related to the metric of the spacetime, has to be considered as gauge field in all cases that are included in the thesis. Also, diffeomorphism invariance has to be present, therefore it has to be related with the transformations of the fields. The above general coordinate invariance is necessarily respected, therefore the latter functions as a motivation for the imposition of certain constraints in the theories. These constraints break the initial symmetry of the corresponding action in each case, leading to the appropriate ones if one begins with the obvious consideration of action of Yang-Mills type. Also, particularly in the four-dimensional case, the breaking of the symmetry may take place in a spontaneous way, after the inclusion of an extra scalar field in the theory, leading to the Einstein-Hilbert action.

Then, we proceeded with providing the necessary information about the noncommutative framework and its realizations, namely the matrix and $\star-$product ones. Next, we recalled the most typical example of a covariant fuzzy space, as a matrix approximation of the ordinary 2-sphere, which is of major importance in our models. The coordinates on the fuzzy sphere are large N-dimensional matrices and are identified as the rescaled SU(2) generators in the N-dimensional representation while the Casimir operator produces the radius constraint. Then, we recalled the construction of the fuzzy space $\mathbb{R}^3_\lambda$, which is a foliation of the Euclidean space by multiple fuzzy spheres of different radii. The coordinates of this space are parametrized also by the generators of the SU(2), but their representation is considered to be reducible.

This consideration allows the matrices of the coordinates to be written in a block diagonal form of irreducible representations of SU(2) or, in other words, as a "block diagonal form of fuzzy spheres", designating the foliation structure. In this specific space, we describe in detail the methodology for obtaining the corresponding $\star-$product, which can be generalized for the calculation of the $\star-$product of other fuzzy spaces. Also, in analogy to the $\mathbb{R}^3_\lambda$ we included the description of its Lorentzian analogue, namely the $\mathbb{R}^{1,2}_\lambda$, which we also employed in the construction of our models. The latter is a fuzzy space that is a foliation of the three-dimensional Minkowski spacetime by fuzzy hyperboloids. The last fuzzy space which we described in detail is a fuzzy version of the four-dimensional de Sitter space. Along the lines of previous attempts of constructing a four-dimensional covariant spacetime, our construction of fuzzy $dS_4$ was based on considering its isometry group SO(1,4) and try to identify part of its generators as coordinates of the space. However, preservation of covariance was violated, therefore, in order to restore it, we considered a larger group, the SO(1,5) in which such an identification would be possible. In this case, too, the coordinates are represented by N-dimensional matrices, related to the generators of the group. Next, we recalled the methodology of constructing gauge theories on noncommutative spaces, which was crucial for our purposes.

Moving on, we wrote down the construction of our models of noncommutative gravity. First, a three-dimensional gravity model as a noncommutative gauge theory was constructed. The background space that was employed is the $\mathbb{R}^3_\lambda$ and the corresponding gauge group was the U(2)$\times$U(2). Then, after the identification of the gauge fields, the procedure of the construction of gauge theories on noncommutative spaces was followed and the expressions of the transformations of the gauge fields and the component curvature tensors were obtained. Also, an action of Chern-Simons type was proposed and the equations of motion were obtained. It is worth-noting that the above expressions and results reduced to the ones of the corresponding (commutative) three-dimensional Einstein gravity.

Having gained a good amount of experience from the above construction, our next step was to construct a noncommutative gravity model in four dimensions. For this purpose, we considered the four-dimensional fuzzy de Sitter space to be the background space of the theory. The gauge group that was initially considered was the SO(1,4), viewed as a subgroup of the total symmetry, SO(1,5). However, due to noncommutativity, the gauge group that was eventually considered was the SO(1,5)$\times$U(1). Again, following the standard procedure, the expressions of the various gauge fields and the component curvature tensors were obtained. For the dynamic part of the theory, the action that was proposed was of Yang-Mills type, bearing the initial gauge symmetry. However, since we wanted to result with an action with Lorentz symmetry, SO(1,3)$\times$U(1), we imposed specific constraints which broke the gauge symmetry. Again, the commutative limit of the theory produces results that coincide with the ones of conformal gravity.

From our perspective, the above work of the four-dimensional noncommutative gravity contributes a lot in two aspects. The first is that we have accomplished a successful construction of a four-dimensional covariant fuzzy space, in such a way that can be generalized for other spaces, specifically by enlarging the symmetry and introducing a 2-form gauge field. Second, we managed to give a description of the gravitational interaction in a regime in which the coordinates can be considered as noncommutative (e.g. Planck scale), and relate it with the conformal gravity in the commutative, low-energy limit. Our priority in the future is to study the Lorentz invariant action we obtained and attempt to relate it with the four-dimensional Einstein-Hilbert action, with aspirations of connecting the large and low-energy regimes for the gravitational interaction.

# Appendix A

# Fuzzy spaces and particle physics models

In this first appendix, reviewing less recent works [67–74][1], we present the involvement of fuzzy spaces, especially the fuzzy sphere, in particle physics models and stress out some of their important features. First, we describe a dimensional reduction starting from a higher-dimensional theory, in which the extra dimensions are considered to be fuzzy, as an application of the gauge theories on noncommutative spaces. Then, we describe a particle physics model, in which one starts with a four-dimensional gauge theory and the fuzzy extra dimensions are generated dynamically as interpretations of certain vacua of the theory, that is a theory that resembles the latter dimensional reduction of a higher-dimensional theory over fuzzy coset spaces (fuzzy spheres).

## A.1 Dimensional reduction on fuzzy spaces

Let us now start with a noncommutative Yang-Mills gauge theory, with gauge group the U(P) and generators denoted as $\mathcal{T}^I$, on a space that consists of the four-dimensional Minkowski spacetime and a Lie-type fuzzy coset space (like the fuzzy sphere), namely $M^4 \times (S/R)_F$. The action of the theory is given as:

$$\mathcal{S} = \frac{1}{4g^2} \int d^4x k \mathrm{Tr}\, \mathrm{tr}\, F_{MN} F^{MN} \ , \tag{A.1}$$

where $\mathrm{tr}_G$ is the trace over the generators of the gauge group and $k\mathrm{Tr}$ ($k$ is related to the radius of the coset space and the spin value, for details see section 3.1.1) is the integration over the noncommutative coordinates, which are described by N× N matrices. Also, $F_{MN}$ is the field strength tensor which consists of both four-dimensional and extra-dimensional components, written explicitly as $F_{MN} = (F_{\mu\nu}, F_{\mu a}, F_{ab})$. The exclusively extra-dimensional field strength tensor part is given in (3.56) and (3.57), in terms either of the covariant coordinate, we denote it $\phi_a$, or the coordinate, $X_a$ and the gauge connection, $A_a$:

$$F_{ab} = [X_a, A_b] - [X_b, A_a] + [A_a, A_b] - C_{ab}{}^c A_c \ . \tag{A.2}$$

[1] For more recent reviews on these works see [75–77]

The mixed components of the field strength tensor are:

$$F_{\mu a} = \partial_\mu \phi_a + [A_\mu, \phi_a] \equiv D_\mu \phi_a \ , \tag{A.3}$$

where $D_\mu \phi_a$ is the covariant derivative of the ordinary gauge theory, acting on $\phi_a$, which is identified as a scalar field if seen from a four-dimensional point of view. Replacing the above expressions of $F_{\mu a}$ and $F_{ab}$ in the action (A.1), one obtains:

$$\mathcal{S} = \int d^4x \mathrm{Trtr} \left( \tfrac{k}{4g^2} F_{\mu\nu}^2 + \tfrac{k}{2g^2} (D_\mu \phi_a)^2 - V(\phi) \right) \ , \tag{A.4}$$

where $V(\phi)$ is identified as the potential, coming from the $F_{ab}^2$ term, explicitly:

$$\begin{aligned} V(\phi) &= -\tfrac{k}{4g^2} \mathrm{Trtr} F_{ab} F^{ab} \\ &= -\frac{k}{4g^2} \mathrm{Trtr} \left( [\phi_a, \phi_b][\phi^a, \phi^b] - 4C_{abc} \phi^a \phi^b \phi^c + 2R^{-2} \phi^2 \right) \ . \end{aligned} \tag{A.5}$$

The fact that from a four-dimensional point of view the $\phi_a$ can be understood to behave like a scalar field, leads to the observation that the action (A.4) can be naturally interpreted, in total, as an action in the four-dimensional Minkowski spacetime, $M^4$, containing besides the kinetic term of the four-dimensional gauge field, $A_\mu$, a kinetic term of a scalar field $D_\mu \phi_a$ plus mass and interaction terms as understood by the expression of the potential, $V(\phi_a)$. In this four-dimensional interpretation of the above higher-dimensional theory, a gauge transformation of the U(P) gauge theory, $\lambda(x^\mu, X^a)$, on the total space can be also understood as a gauge transformation of a gauge theory in four dimensions, specifically:

$$\lambda(x^\mu, X^a) = \lambda^I(x^\mu, X^a)\mathcal{T}^I = \lambda^{h,I}(x^\mu)\mathcal{T}^h\mathcal{T}^I \ , \tag{A.6}$$

where in the first step the gauge transformation is expanded on the generators $\mathcal{T}^I$, since it takes values in the algebra of U(P). In the second step, the infinitesimal gauge transformation, $\lambda(x^\mu, X^a)$, depends on both $M^4$ coordinates, $x^\mu$ and the extra-dimensional coordinates, $X^a$, or even better $\lambda$ is a function of $X^a$. However, the coordinates $X^a$ are N$\times$N (anti-)Hermitian matrices, therefore they can be expanded on the generators of a U(N) group in the fundamental, N-dimensional representation. Therefore, the $\lambda^{h,I}(x^\mu)$ can be identified with the Kaluza-Klein modes of the $\lambda^I(x^\mu, X^a)$. A very important feature of the above construction becomes manifest after the realization that the $\lambda^{h,I}(x^\mu)$ field can be considered to be taking values in the Lie algebra of Lie(U(N))$\otimes$Lie(U(P)), that is the Lie algebra Lie(U(NP)). The above understanding of the $\lambda^{h,I}(x^\mu)$ as a field in four dimensions taking values in the Lie(U(NP)) can be adopted for the rest of the fields (gauge and scalar fields) of the theory. This means that a dimensional reduction of a gauge theory defined on a $M^4 \times (S/R)_F$ space induces the enhancement of the starting gauge group. Specifically, one may start with a higher-dimensional Abelian gauge theory and achieve symmetry enhancement after a dimensional reduction, resulting with a non-Abelian gauge group in the four-dimensional theory. A rather undesirable feature is that the scalar fields that appear due to the reduction are set in the adjoint representation, rendering the triggering of the electroweak symmetry breaking impossible. In order to overcome this difficulty, one can employ different kinds of dimensional reductions, in which the nice feature of the enlargement of the symmetry still holds (see for example fuzzy CSDR [67] (see also [111])).

## A.2 Dynamical generation of fuzzy spaces as extra dimensions

In this appendix we review a very interesting project, that is the construction of a particle physics model involving fuzzy extra dimensions, which carries the important features of renormalizability, chirality and phenomenological viability. In order to ensure renormalizability, instead of the dimensional reduction of a gauge theory on a space which has fuzzy extra dimensions (see previous subsection A.1) the reverse procedure was considered, that is to start with a renormalizable theory in four dimensions and reproduce the results of a higher-dimensional theory reduced over fuzzy coset spaces [69–72]. More specifically, one starts with an $\mathcal{N}=4$ Supersymmetric Yang-Mills (SYM) theory [112] defined on the four-dimensional Minkowski spacetime, $M^4$, equipped with an appropriate set of scalar fields and a suitable potential, which leads to certain vacua that could be identified as -dynamically generated- fuzzy extra dimensions, including also a finite Kaluza-Klein tower of massive modes. The whole idea of dimensional deconstruction has been introduced earlier, see [113, 114]. Later, an attempt to include fermions took place, but the best one could achieve (for some time) was a model containing mirror fermions in bifundamental representations of the low-energy gauge group [71, 72]. Although mirror fermions do not exclude contact with phenomenology [115], exactly chiral fermions are certainly preferable.

### A.2.1 $\mathcal{N}=4$ SYM field theory and $\mathbb{Z}_3$ orbifolds

Let us consider an $\mathcal{N}=4$ supersymmetric SU(3N) gauge theory defined on Minkowski spacetime with a particle spectrum (in the $\mathcal{N}=1$ language) consisting of an $SU(3N)$ gauge supermultiplet and three adjoint chiral supermultiplets $\Phi^i$, $i=1,2,3$. The component fields of the above supermultiplets are the gauge bosons, $A_\mu$, $\mu=1,\dots,4$, six adjoint real (or three complex) scalars $\phi^a$, $a=1,\dots,6$ and four adjoint Weyl fermions $\psi^p$, $p=1,\dots,4$. The scalars and Weyl fermions belong to the $6$ and $4$ irreducible representations of the SU(4)$_R$ $R$-symmetry of the theory, respectively, while the gauge bosons are singlets. At this point, for reasons of filtering the particle spectrum, the orbifold projection technique is involved (similar to the one developed in [116]), specifically making use of the $\mathbb{Z}_3$ discrete group. For the introduction of the orbifold structure, the $\mathbb{Z}_3$ has to be viewed as a subgroup of SU(4)$_R$ symmetry. The $\mathbb{Z}_3$ can be embedded into SU(4)$_R$ in more than one -not equivalent- ways, with the choice of the embedding affecting the amount of the remnant supersymmetry [116]:

- Maximal embedding of $\mathbb{Z}_3$ into SU(4)$_R$ leads to non-supersymmetric theories, therefore it is excluded.
- Embedding $\mathbb{Z}_3$ into a subgroup of SU(4)$_R$:
  - Embedding into an SU(2) subgroup would lead to $\mathcal{N}=2$ supersymmetric theories with SU(2)$_R$ remnant $R$-symmetry.
  - Embedding into an SU(3) subgroup would lead to $\mathcal{N}=1$ supersymmetric theories with U(1)$_R$ remnant $R$-symmetry.

The desired remnant supersymmetry falls into the last option. Let us consider a generator $g\in\mathbb{Z}_3$, labelled by three integers, $\vec{a}=(a_1,a_2,a_3)$ [117], which satisfy the following relation:

$$a_1+a_2+a_3=0 \bmod 3\ . \tag{A.7}$$

The above relation implies that $\mathbb{Z}_3$ is embedded into the SU(3) subgroup, i.e. the remnant supersymmetry is the desired $\mathcal{N} = 1$ [118]. The various fields of the theory transform differently under SU(4)$_R$, therefore the $\mathbb{Z}_3$ will act on them in a non-trivial way. Gauge and gaugino fields are both singlets under SU(4)$_R$, therefore the geometric action of the $\mathbb{Z}_3$ rotation is trivial. The action of $\mathbb{Z}_3$ on the complex scalar fields is represented by the matrix $\gamma(g)_{ij} = \delta_{ij}\omega^{a_i}$, where $\omega = e^{\frac{2\pi}{3}}$ and the action of $\mathbb{Z}_3$ on the fermions is given by $\gamma(g)_{ij} = \delta_{ij}\omega^{b_i}$, where $b_i = -\frac{1}{2}(a_{i+1} + a_{i+2} - a_i)$, modulo 3. In this case, the three integers describing the generator $g$ are $(1, 1, -2)$, which means that $a_i = b_i$. The matter fields transform non-trivially under a gauge transformation, therefore $\mathbb{Z}_3$ acts on their gauge indices, too. This action is given by the following matrix:

$$\gamma_3 = \begin{pmatrix} \mathbb{1}_N & 0 & 0 \\ 0 & \omega\,\mathbb{1}_N & 0 \\ 0 & 0 & \omega^2\,\mathbb{1} \end{pmatrix} . \tag{A.8}$$

It is not obligatory that these blocks have the same dimensionality (see e.g. [119–121]), however they do, offering to the projected theory the property of anomaly freedom.

After the orbifold projection is performed, the particle spectrum of the theory consists of the fields that remain invariant under the combined action of $\mathbb{Z}_3$ on the "geometric"[2] and gauge indices [117]. Also, concerning the gauge fields, the projection is given by $A_\mu = \gamma_3 A_\mu \gamma_3^{-1}$. Therefore, taking into consideration the expression of the $\gamma_3$ matrix, (A.8), the initial gauge group breaks down to the $H = SU(N) \times SU(N) \times SU(N)$ in the projected theory.

The complex scalar fields transform in a non-trivial way under a gauge symmetry transformation and the $R-$symmetry, therefore the projection is given by $\phi^i_{IJ} = \omega^{I-J+a_i}\phi^i_{IJ}$, where $I, J$ are gauge indices. Therefore, $J = I + a_i$, meaning that the scalar fields that survive the projection have the form $\phi_{I,J+a_i}$ and transform under the gauge group of the projected theory, $H$, as:

$$3 \cdot \left((N, \bar{N}, 1) + (\bar{N}, 1, N) + (1, N, \bar{N})\right) . \tag{A.9}$$

Similarly for the fermions, both gauge group and $R-$symmetry transformations are non-trivial, with the projection being $\psi^i_{IJ} = \omega^{I-J+b_i}\psi^i_{IJ}$. Therefore, the fermions that survive the projection are of the form $\psi^i_{I,I+b_i}$ and are accommodated in the same representation of the scalars, (A.9), manifesting the $\mathcal{N} = 1$ remnant supersymmetry. It is worth-noting that the representation of the scalars or fermions, (A.9), of the projected theory are anomaly free. Therefore, fermions belong to chiral representations of $H$, divided into three generations, since the initial particle spectrum, before the orbifold projection, contains three $\mathcal{N} = 1$ chiral supermultiplets.

Now, let us focus on the part of interactions of the theory after the orbifold projection. The projected theory is an $\mathcal{N} = 1$ supersymmetrc gauge theory, therefore the interactions of the fields that passed through the orbifold filter are all included in the superpotential. However, to specify it explicitly, one has to begin with the interactions of the initial $\mathcal{N} = 4$ SYM theory [112], which are expressed by the following superpotential:

$$W_{\mathcal{N}=4} = \epsilon_{ijk}\mathrm{Tr}(\Phi^i\Phi^j\Phi^k) , \tag{A.10}$$

where $\Phi^i, \Phi^j, \Phi^k$ are the chiral superfields. Therefore, the interactions in the projected theory

[2]In case of ordinary reduction of a 10-dim $\mathcal{N} = 1$ SYM theory, one obtains an $\mathcal{N} = 4$ SYM Yang-Mills theory in four dimensions having a global $SU(4)_R$ symmetry which is identified with the tangent space $SO(6)$ of the extra dimensions [122–125].

$\mathcal{N}=1$ gauge theory are given by the superpotential:

$$W_{\mathcal{N}=1}^{(proj)} = \sum_I \epsilon_{ijk} \Phi^i_{I,I+a_i} \Phi^j_{I+a_i,I+a_i+a_j} \Phi^k_{I+a_i+a_j,I} \ . \tag{A.11}$$

In order to obtain the vacuum of the theory, one has to extract the information for the scalar potential from the above superpotential.

### A.2.2 Dynamical generation of twisted fuzzy spheres

As stated, the superpotential $W_{\mathcal{N}=1}^{proj}$, (A.11), produces the following scalar potential:

$$V_{\mathcal{N}=1}^{proj}(\phi) = \frac{1}{4}\mathrm{Tr}\left([\phi^i,\phi^j]^\dagger[\phi^i,\phi^j]\right) \ , \tag{A.12}$$

where, $\phi^i$ is the scalar part of the superfield, $\Phi^i$. The above scalar potential, $V_{\mathcal{N}=1}^{proj}(\phi)$, is minimized by vanishing vacuum expectation values (vevs) of the fields, therefore, in order to result with non-vanishing solutions admitting interpretation as vacua of a noncommutative geometry, specific modifications have to be made, that is the introduction of soft $\mathcal{N}=1$ supersymmetric terms of the following form[3]:

$$V_{SSB} = \frac{1}{2}\sum_i m_i^2 \phi^{i\dagger}\phi^i + \frac{1}{2}\sum_{i,j,k} h_{ijk}\phi^i\phi^j\phi^k + h.c. \ , \tag{A.13}$$

with $h_{ijk}=0$ unless $i+j+k \equiv 0 \,\mathrm{mod} 3$. The introduction of SSB terms is not disturbing, since an SSB sector is necessary for a supersymmetric model to have phenomenological viability, see e.g. [126]. Also, the $D$-terms of the theory are introduced:

$$V_D = \frac{1}{2}D^2 = \frac{1}{2}D^I D_I \ , \tag{A.14}$$

where $D^I = \phi_i^\dagger T^I \phi^i$ and $T^I$ are the generators of the gauge group, represented by the same representation as the corresponding chiral multiplets. Putting together all terms consisting the potential, the expression of the total potential of the theory is:

$$V = V_{\mathcal{N}=1}^{proj} + V_{SSB} + V_D \ . \tag{A.15}$$

One can choose properly the parameters $m_i^2$ and $h_{ijk}$ of the relation (A.13), specifically $m_i^2 = 1$ and $h_{ijk} = \epsilon_{ijk}$. Thus, the scalar potential, (A.15), takes the following form:

$$V = \frac{1}{4}(F^{ij})^\dagger F^{ij} + V_D \ , \tag{A.16}$$

where the tensor $F^{ij}$ is defined as:

$$F^{ij} = [\phi^i,\phi^j] - i\epsilon^{ijk}(\phi^k)^\dagger \ . \tag{A.17}$$

The first term of the scalar potential, (A.16), is positive, therefore, the minimum of the potential is:

$$[\phi^i,\phi^j] = i\epsilon_{ijk}(\phi^k)^\dagger \ , \quad \phi^i(\phi^i)^\dagger = R^2 \ , \tag{A.18}$$

[3] The SSB terms that will be inserted into $V_{\mathcal{N}=1}^{proj}(\phi)$, are purely scalar. Although this is enough for our purpose, it is obvious that more SSB terms could be involved in order to obtain complete SSB sector [126].

and $[R^2, \phi^i] = 0$. The form of the first relation in (A.18) implies the relation to a fuzzy sphere, (3.31). This gets even more manifest, after the introduction of untwisted fields, $\tilde{\phi}^i$, defined as:

$$\phi^i = \Omega \tilde{\phi}^i \ , \tag{A.19}$$

where $\Omega \neq 1$ satisfying the relations:

$$\Omega^3 = 1 \, , \quad [\Omega, \phi^i] = 0 \, , \quad \Omega^\dagger = \Omega^{-1} \, , \quad (\tilde{\phi}^i)^\dagger = \tilde{\phi}^i \ \Leftrightarrow \ (\phi^i)^\dagger = \Omega \phi^i \, . \tag{A.20}$$

Therefore, from equation (A.18) and making use of the $\tilde{\phi}^i$, the interpretation of a fuzzy sphere becomes evident:

$$[\tilde{\phi}^i, \tilde{\phi}^j] = i \epsilon_{ijk} \tilde{\phi}^k \, , \quad \tilde{\phi}^i \tilde{\phi}^i = R^2 \ , \tag{A.21}$$

confirming the fact that the vacuum of the potential generates a (twisted) fuzzy sphere, $\tilde{S}^2_N$. Next, specific configurations of the twisted fields satisfying the relation (A.18), $\phi^i$, can be obtained. Such a configuration is:

$$\phi^i = \Omega (\mathbf{1}_3 \otimes \lambda^i_{(N)}) \ , \tag{A.22}$$

where $\lambda^i_{(N)}$ are three N×N matrices, representing the three SU(2) generators in the $N$-dimensional irreducible representation and $\Omega$ is the matrix:

$$\Omega = \Omega_3 \otimes \mathbf{1}_N \, , \quad \Omega_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \ , \quad \Omega^3 = \mathbf{1} \, . \tag{A.23}$$

According to the relation (A.19), which expresses the relation between twisted and untwisted fields, the "off-diagonal" orbifold sectors, (A.9), take the following block-diagonal form:

$$\phi^i = \begin{pmatrix} 0 & (\lambda^i_{(N)})_{(N,\bar{N},1)} & 0 \\ 0 & 0 & (\lambda^i_{(N)})_{(1,N,\bar{N})} \\ (\lambda^i_{(N)})_{(\bar{N},1,N)} & 0 & 0 \end{pmatrix} = \Omega \begin{pmatrix} \lambda^i_{(N)} & 0 & 0 \\ 0 & \lambda^i_{(N)} & 0 \\ 0 & 0 & \lambda^i_{(N)} \end{pmatrix} \, . \tag{A.24}$$

The untwisted fields generating the ordinary fuzzy sphere, $\tilde{\phi}^i$, are now in a block-diagonal form. Seperately, each block is identified as a fuzzy sphere, since each one satisfies the corresponding defining commutation relation (A.21). Thus, the vacuum of the theory, (A.24), has taken the form of three fuzzy spheres, with relative angles $2\pi/3$. In other words, in accordance to the orbifold projection, the solution $\phi^i$ is equivalent to the solution of the three fuzzy spheres. It is also worth-noting that the tensor $F^{ij}$ of the equation (A.17), is identified as the field strength tensor of the spontaneously generated fuzzy extra dimensions, that is given in (3.56). The term $V_D$ of the potential causes a change on the radius of the fuzzy sphere (in a similar way to the case of the ordinary fuzzy sphere [69, 72, 127]).

### A.2.3 Chiral models after the orbifold projection - The $SU(3)_c \times SU(3)_L \times SU(3)_R$ model

The gauge group of the initial gauge theory, SU(3N), breaks spontaneously in various ways, therefore, the resulting gauge group after performing the orbifold projection is not unique. The minimal, anomaly free unified models[4] are found to be the $SU(4) \times SU(2) \times SU(2)$, $SU(4)^3$ and

[4]Similar approaches have been studied in the framework of YM matrix models [128]

$SU(3)^3$ of which the case of interest is the last one, that is the trinification group, $SU(3)_c \times SU(3)_L \times SU(3)_R$ [129, 130] (see also [131–135] and for a string theory approach see [136].). First, the integer $N$ is decomposed as $N = n+3$ and then the following embedding is considered:

$$SU(N) \supset SU(n) \times SU(3) \times U(1) \, . \tag{A.25}$$

Thus, the embedding for the gauge group $SU(N)^3$ is the following:

$$SU(N)^3 \supset SU(n) \times SU(3) \times SU(n) \times SU(3) \times SU(n) \times SU(3) \times U(1)^3 \, . \tag{A.26}$$

The three $U(1)$s are not taken into account[5] and as for the representations, they are decomposed according to the embedding of (A.26), as:

$$\begin{aligned} & SU(n) \times SU(n) \times SU(n) \times SU(3) \times SU(3) \times SU(3) \, , \\ & (n, \bar{n}, 1; 1, 1, 1) + (1, n, \bar{n}; 1, 1, 1) + (\bar{n}, 1, n; 1, 1, 1) + (1, 1, 1; 3, \bar{3}, 1) \\ & + (1, 1, 1; 1, 3, \bar{3}) + (1, 1, 1; \bar{3}, 1, 3) + (n, 1, 1; 1, \bar{3}, 1) + (1, n, 1; 1, 1, \bar{3}) \\ & + (1, 1, n; \bar{3}, 1, 1) + (\bar{n}, 1, 1; 1, 1, 3) + (1, \bar{n}, 1; 3, 1, 1) + (1, 1, \bar{n}; 1, 3, 1) \, . \end{aligned} \tag{A.27}$$

Now, taking into consideration the decomposition of (A.25), the gauge group breaks to the $SU(3)^3$, under which the surviving fields from the projection transform as:

$$SU(3) \times SU(3) \times SU(3) \, , \tag{A.28}$$
$$\left( (3, \bar{3}, 1) + (\bar{3}, 1, 3) + (1, 3, \bar{3}) \right) \, , \tag{A.29}$$

which correspond to the desired chiral representations of the trinification gauge group, with the quarks and leptons of the first family transforming as:

$$q = \begin{pmatrix} d & u & h \\ d & u & h \\ d & u & h \end{pmatrix} \sim (3, \bar{3}, 1) \, , q^c = \begin{pmatrix} d^c & d^c & d^c \\ u^c & u^c & u^c \\ h^c & h^c & h^c \end{pmatrix} \sim (\bar{3}, 1, 3) \, , \lambda = \begin{pmatrix} N & E^c & \mathrm{v} \\ E & N^c & e \\ \mathrm{v}^c & e^c & S \end{pmatrix} \sim (1, 3, \bar{3}) \, , \tag{A.30}$$

respectively. It is remarkable that this theory can be upgraded to a two-loop finite theory (for reviews see [137–139]) giving phenomenologically testable predictions [131], too.

To summarize this section, application of an orbifold projection on an $\mathcal{N} = 4$ gauge theory in four dimensions with a specific particle content, leads to another gauge group and different (less) amount of supersymmetry, depending on the way the discrete symmetry of the orbifolding is embedded in the R-symmetry group. Then, the form of the superpotential after the projection leads to a vacuum of the scalar potential which can be interpreted as dynamically generated fuzzy extra dimensions. Eventually, the above scheme leads to a unified theory, accommodating chiral fermions. Concluding, fuzzy extra dimensions can be used for constructing chiral, renormalizable and phenomenologically viable field-theoretical models.

[5] Because of anomalous gaining mass by the Green-Schwarz mechanism and as a result they decouple at the low energy sector of the theory [120].

# Appendix B

# The Schwinger basis

The algebra of angular momentum can be written as an algebra of two independent harmonic oscillators, that is the Schwinger construction [140][1]. Let us consider two simple harmonic oscillators, the $+$-type and the $-$type. For each oscillator, annihilation and creation operators are defined, denoted by $a_+, a_+^\dagger$ and $a_-, a_-^\dagger$, respectively, satisfying the following commutation relations:

$$[a_i, a_j^\dagger] = \delta_{ij} \ , \tag{B.1}$$

where $i, j = +, -$. The above commutation relations are written more explicitly as:

$$[a_+, a_+^\dagger] = 1 \ , \quad [a_-, a_-^\dagger] = 1 \ , \quad [a_+, a_-^\dagger] = [a_-, a_+^\dagger] = 0 \ . \tag{B.2}$$

Also, the number operators are defined as:

$$N_+ = a_+^\dagger a_+ \ , \quad N_- = a_-^\dagger a_- . \tag{B.3}$$

Their commutation relations with the $a_i^\dagger, a_i$ operators are given as:

$$[N_i, a_j] = -a_i \delta_{ij} \ , \quad [N_i, a_j^\dagger] = a_i^\dagger \delta_{ij} \quad (\text{no summation}) \ , \tag{B.4}$$

or, more explicitly as:

$$[N_+, a_+] = -a_+ \ , \quad [N_+, a_+^\dagger] = a_+^\dagger \ , \quad [N_-, a_-] = -a_- \ , \quad [N_-, a_-^\dagger] = a_-^\dagger \ . \tag{B.5}$$

Due to the last relation of (B.2), the number operators, $N_+, N_-$, of the two oscillators commute. Therefore, the two operators share a common set of eigenfunctions, denoted by $|n_+ n_-\rangle$, with eigenvalues $n_+$ and $n_-$, respectively. The eigenvalue equations for $N_\pm$ are:

$$N_+|n_+ n_-\rangle = n_+|n_+ n_-\rangle \ , \quad N_-|n_+ n_-\rangle = n_-|n_+ n_-\rangle \ . \tag{B.6}$$

The creation and annihilation operators, $a_\pm^\dagger, a_\pm$, act on the above eigenstates as:

$$\begin{aligned} a_+^\dagger|n_+ n_-\rangle &= \sqrt{n_+ + 1}|n_+ + 1 n_-\rangle \ , \quad a_-^\dagger|n_+ n_-\rangle = \sqrt{n_- + 1}|n_+ n_- + 1\rangle \ , \\ a_+|n_+ n_-\rangle &= \sqrt{n_+}|n_+ - 1 n_-\rangle \ , \quad a_-|n_+ n_-\rangle = \sqrt{n_-}|n_+ n_- - 1\rangle \ . \end{aligned} \tag{B.7}$$

[1] See also [141, 142]

The most general eigenstates of $N_+, N_-$ are obtained by applying $a^\dagger_\pm$ successively on the vacuum ket, which is defined as:

$$a_+|00\rangle = 0 \ , \quad a_-|00\rangle = 0 \ . \tag{B.8}$$

Therefore, the general eigenstate of $N_+, N_-$ are obtained :

$$|n_+n_-\rangle = \frac{(a^\dagger_+)^{n_+}(a^\dagger_-)^{n_-}}{\sqrt{n_+!}\sqrt{n_-!}}|00\rangle \ . \tag{B.9}$$

Next, in order to make contact with the angular momentum algebra, one defines:

$$J_+ = a^\dagger_+ a_- \ , \quad J_- = a^\dagger_- a_+ \ , \quad J_z = \frac{1}{2}\left(a^\dagger_+ a_+ - a^\dagger_- a_-\right) = \frac{1}{2}\left(N_+ - N_-\right) \ . \tag{B.10}$$

It is straightforward, starting from (B.2), to confirm that the above defined generators satisfy the SU(2) algebra:

$$[J_z, J_\pm] = \pm J_\pm \ , \quad [J_+, J_-] = 2J_z \ . \tag{B.11}$$

Also, for the expression of the Casimir operator, it is convenient to define the total number operator, $N$, to be:

$$N = N_+ + N_- = a^\dagger_+ a_+ + a^\dagger_- a_- \ , \tag{B.12}$$

with eigenvalues. $n_+ + n_-$. Therefore, the Casimir operator is given by the following expression:

$$J^2 = J_z^2 + \frac{1}{2}\left(J_+J_- + J_-J_+\right) = \frac{N}{2}\left(\frac{N}{2} + 1\right) \ . \tag{B.13}$$

The above definitions, besides being motivated by algebraic reasons, admit also a physical interpretation. More specifically, if the quantum unit of the $+$-type oscillator is associated with spin up (particle with $m = 1/2$), while the $-$-type oscillator is associated with spin down (particle with $m = -1/2$) and the eigenvalues $n_+, n_-$ are the corresponding number of spins up and spins down, then the $J_+$ operator, by definition, destroys a spin down unit and creates a spin up unit. This means that the spin angular momentum is increased by 1. Likewise, the $J_-$ operator acts inversely. The $J_z$ operator counts $1/2$ times the difference of $n_+$ and $n_-$, that is the $z-$component of the angular momentum. In general it has been set $\hbar = 1$.

Now, the action of the $J_z, J_\pm$ operators on the eigenstates $|n_+n_-\rangle$, is given after taking into consideration the (B.7):

$$\begin{aligned}
J_+|n_+n_-\rangle &= a^\dagger_+ a_-|n_+n_-\rangle = \sqrt{n_-(n_+ + 1)}|n_+ + 1 n_- - 1\rangle \ , \\
J_-|n_+n_-\rangle &= a^\dagger_- a_+|n_+n_-\rangle = \sqrt{n_+(n_- + 1)}|n_+ - 1 n_- + 1\rangle \ , \\
J_z|n_+n_-\rangle &= \frac{1}{2}(N_+ - N_-)|n_+n_-\rangle = \frac{1}{2}(n_+ - n_-)|n_+n_-\rangle \ .
\end{aligned} \tag{B.14}$$

It is worth-noting that in the above relations of the action of the operators on the eigenstates, the total number of particles, $n_+ + n_-$ is constant. Also, the above relations reduce to the familiar action of $J_z, J_\pm$ operators, after the change:

$$\begin{aligned}
n_+ &\rightarrow j + m \ , \quad n_- \rightarrow j - m \ , \quad \text{or} \\
j &= \frac{n_+ + n_-}{2} \ , \quad m = \frac{n_+ - n_-}{2} \ .
\end{aligned} \tag{B.15}$$

Therefore, taking into consideration the (B.13) and (B.14), one obtains:

$$\begin{aligned}
J_+|n_+n_-\rangle &= \sqrt{(j-m)(j+m+1)}|j+m+1, j-m-1\rangle \ , \\
J_-|n_+n_-\rangle &= \sqrt{(j+m)(j-m+1)}|j+m-1, j-m+1\rangle \ , \\
J_z|n_+n_-\rangle &= J_z|j+m, j-m\rangle = m|j+m\rangle|j-m\rangle \ , \\
J^2|n_+n_-\rangle &= J^2|j+m, j-m\rangle = j(j+1)|j+m, j-m\rangle \ .
\end{aligned} \tag{B.16}$$

The general ket written down in (B.9), can be written now in the following form:

$$|jm\rangle = \frac{(a_+^\dagger)^{j+m}(a_-^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\rangle \ , \tag{B.17}$$

where $|0\rangle$ is the vacuum ket.

# Appendix C

# Quaternions and Hopf fibration

Let us consider first the case of the complex numbers, $\mathbb{C}$. The simplest compact Lie group is the circle, $S^1 \cong$ SO(2), which can be parametrized by $e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's formula)[1] from which it can be understood as the group of elements absolute value equal to one in the complex numbers plane, $\mathbb{C}$, that is the unitary group U(1).

Quartenions is a four-dimensional number system which can be viewed as an extension of the complex numbers and can be understood in analogy to them. Let us consider the compact Lie group of the 3-sphere, $S^3 \cong$SU(2). The picture is completely analogous to the circle case discussed above with the difference that complex numbers have to be upgraded to Hamilton's quartenions, $\mathbb{H}$, meaning that SU(2) can be understood as the group elements of norm equal to one in $\mathbb{H}$, that is the symplectic group Sp(1). A quartenion number is written as:

$$q = a + ib + jc + kd \ , \quad a, b, c, d \in \mathbb{R} \ , \tag{C.1}$$

and multiplication between $i, j, k$ is given by the following matrix:

| | 1 | i | j | k |
|---|---|---|---|---|
| 1 | 1 | i | j | k |
| i | i | -1 | k | -j |
| j | j | -k | -1 | i |
| k | k | j | -i | -1 |

Table C.1: Multiplication table of $i, j, k$.

Extending the notion of associating the ordered pair $(a, b)$ to a complex number or a 2×2 real matrix, giving the opportunity to adopt a sum, product and absolute value for $(a, b)$, to the

[1]An element, $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ of SO(2) behaves the same as the complex number of unit absolute value, $z_\theta = \cos\theta + i\sin\theta$,. This is manifest when $R_\theta$ is written as $R_\theta = \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, where the basis matrices $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ have been used. It is easily checked that $\mathbb{1}^2 = 1$ , $\mathbb{1}\mathbf{i} = \mathbf{i}\mathbb{1} = \mathbf{i}$ , $\mathbf{i}^2 = \mathbb{1}$, which means that the matrices, $\mathbb{1}, \mathbf{i}$ behave the same as the complex numbers $1, i$. In fact, the matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a\mathbb{1} + b\mathbf{i}$ , $a, b \in \mathbb{R}$ behave the same as the complex numbers $a + bi$ under addition and multiplication. Therefore, all complex numbers are represented by 2×2 real matrices, with the determinant giving the absolute value of the complex number [143].

ordered quadrupes $(a, b, c, d)$ of real values, namely the quaternions, it is possible to write them down in a 2×2 complex matrix form, $q = \begin{pmatrix} a+id & -b-ic \\ b-ic & a-id \end{pmatrix} = a\mathbb{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \mathbf{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad \mathbf{j} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} , \quad \mathbf{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} . \tag{C.2}$$

Addition and product of two such quaternion matrices produce matrices of the same form. Also, the squared absolute value of a quaternion, $q$, is retrieved by the determinant of the matrix form of quaternion. From the above matrix representation of $q$, it is understood that, for $q \neq 0$, the explicit formula of the inverse quaternion is $q^{-1} = \frac{1}{a^2+b^2+c^2+d^2}(a - bi - cj - dk)$. For the quaternion $q$, the quaternion conjugate, $\bar{q}$, is defined as $\bar{q} = x_1 - ix_2 - jx_3 - ix_4$ and therefore, it is straightforward to calculate that $q\bar{q}$ to be:

$$q\bar{q} = |q|^2 = (x_1 + ix_2 + jx_3 + kx_4)(x_1 - ix_2 - jx_3 - kx_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 \, . \tag{C.3}$$

The quaternions $q = a+bi+cj+dk$ of unit absolute value (unit quaternions) define the 3-sphere $S^3$ in $\mathbb{R}^4$:

$$S^3 = \{\text{unit quartenions}\} = \left\{q = a + ib + jc + kd \mid a^2 + b^2 + c^2 + d^2 = 1\right\} \, . \tag{C.4}$$

Now, let us define the pure imaginary and pure real quaternions, $p, r$, respectively:

$$p = bi + cj + dk \, , \quad r = a \, , \tag{C.5}$$

which, for the imaginary quaternions, a three-dimensional space, that is denoted as $\mathbb{R}i+\mathbb{R}j+\mathbb{R}k$ ($\mathbb{R}^3$ for short), is formed. It is obvious that the sum of two imaginary quaternions is also imaginary, but this is not true for the product. Let $u = (0, u_1, u_2, u_3)$ and $v = (0, v_1, v_2, v_3)$ be two imaginary quaternions. Their product, using the Table C.1, gives:

$$uv = -(u_1v_1 + u_2v_2 + u_3v_3) + (u_2v_3 - u_3v_2)i - (u_1v_3 - u_3v_1)j + (u_1v_2 - u_2v_1)k \, . \tag{C.6}$$

In the above expression, it is manifest that the product of two imaginary quaternions can be expressed in terms of two other products in $\mathbb{R}^3$, the scalar, $u \cdot v$ and the vector, $u \times v$. Therefore, in terms of these products in $\mathbb{R}^3$, the product of two imaginary quaternions, (C.6), can be written as:

$$uv = -u \cdot v + u \times v \, . \tag{C.7}$$

Since the first term (scalar product) is real, it is understood that the product of two imaginary quaternions is also an imaginary quaternion, if $u \cdot v = 0$, that is in case $u$ and $v$ are orthogonal. Also, since the second term, the cross product, is imaginary, shows that the $uv$ is real only if $u \times v = 0$, that is if $u, v$ are on the same line. In the particular case in which the imaginary $u$ is a unit quaternion, $|u| = 1$, the (C.6) becomes:

$$u^2 = -u \cdot u = -|u|^2 = 1 \, . \tag{C.8}$$

Thus, every unit vector in $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ is a square root of $-1$.

Now, let us consider a quaternion, $s$, with $|s| = 1$. This can be written in a generalized Euler formula, that reads:

$$s = \cos\theta + u \sin\theta \, , \tag{C.9}$$

where $u$ is an imaginary unit vector ($u^2 = -1$). In analogy to the complex number case in which a unit complex number, $z$, corresponds to a rotation in the complex plane, the unit quaternion, $s$, corresponds to a rotation of the $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k$. The difference to the complex numbers case is that this rotation is not induced by mere multiplication, because the multiplication of the quaternion $s$ and an imaginary quaternion, let us call it $w$, is not necessarily an element of $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k$. Instead, the mapping that is employed is the following:

$$w \ \rightarrow \ s^{-1} w s \ , \tag{C.10}$$

which it is shown that it is actually an imaginary quaternion, since $s^{-1} = \cos\theta - u \sin\theta$. The above mapping is called a conjugation by $s$. This is exactly the kind of mapping that is used to project the 3-sphere to a 2-sphere through the Hopf fibration. Let us call this projection $\eta(s)$, that is $S^3 \xrightarrow{\eta} S^2$, which is specifically given by (C.10), for $w = k$[2], that is:

$$\eta(s) = s k s^{-1} \ . \tag{C.11}$$

It is worth-noting at this point that multiplication of quaternions is not commutative. This is the reason why the above function does not reduce to $\eta(s) = k$. Recalling (C.4):

$$S^3 = \left\{ s = a + bi + cj + dk \mid |s|^2 = a^2 + b^2 + c^2 + d^2 = 1 \right\} \ , \tag{C.12}$$

calculations in (C.11), using the Table C.1, lead to:

$$\begin{aligned} \eta(s) = (\eta_0, \eta_1, \eta_2, \eta_3) &= (0, 2ac + 2bd, 2cd - 2ab, a^2 - b^2 - c^2 + d^2) \\ &= 2(ac + bd)i + 2(cd - ab)j + (a^2 - b^2 - c^2 + d^2)k \ . \end{aligned} \tag{C.13}$$

Since the real part of the quaternion is vanished, the $\eta(s)$ is an imaginary quaternion in the subspace $\mathbb{R}^3$. Raising the components of $\eta$ to the square and adding them, the absolute value is obtained:

$$|\eta|^2 = (2ac + 2bd)^2 + (2cd - 2ab)^2 + (a^2 - b^2 - c^2 + d^2)^2 = 1 \ , \tag{C.14}$$

where $|s|^2 = 1$, since $s$ is a unit quaternion, has been used. Therefore, starting from the unit $S^3$ space, that is the points of $\mathbb{R}^4$ that with distance from the origin equal to the unit, described by a unit quaternion, one results with the unit $S^2$, through a specific projection mapping, $\eta(s)$.

The Hopf fibration can be equivalently understood as:

$$S^3 = \{\vec{z} = (z_1, z_2) \in \mathbb{C}^2 \mid \bar{z}_a z_a = 1\} \ \rightarrow \ S^2 = \{\vec{x} = (x^1, x^2, x^3) \in \mathbb{R}^3\} \ , \tag{C.15}$$

with the $x^i$ $(i = 1, 2, 3)$ components being related with the $z_a$ $(a = 1, 2)$ components through the relation:

$$x^i = \frac{1}{2} \bar{z}_a \sigma^i_{ab} z_b \ , \tag{C.16}$$

where $\bar{z}_a$ are the complex conjugates of $z_a$ and $\sigma^i_{ab}$ are the matrix elements of the three Pauli matrices. The above relation leads to the following expressions for the $x^i$s:

$$x^1 = \frac{1}{2} \left( \bar{z}_1 z_2 + \bar{z}_2 z_1 \right) \ , \quad x^2 = -\frac{i}{2} \left( \bar{z}_1 z_2 - \bar{z}_2 z_1 \right) \ , \quad x^3 = \frac{1}{2} \left( \bar{z}_1 z_1 - \bar{z}_2 z_2 \right) \ . \tag{C.17}$$

[2]The $k$ is one of the three "special numbers" used for defining a quaternion.

Using the above expressions for the $x^i$s, one can calculate the following sum:

$$\sum_{i=1}^{3}(x^i)^2 = \frac{1}{4}(\bar{z}_a z_a)^2 = {x^0}^2 \ , \tag{C.18}$$

where $x^0 = \frac{1}{2}\bar{z}_a z_a = 1$ due to the radius constraint in $S^3$. The above description of the Hopf fibration, (C.15), can be derived from the one with the quaternions, (C.12), we described before.

First, the expression of $\eta$, second line of (C.13), has to be rewritten in terms of the Pauli matrices instead of the $i, j, k$, employing their 2×2 matrix representation, (C.2). It is easy to find that:

$$\mathbf{i} = -i\sigma_2 \ , \quad \mathbf{j} = -i\sigma_1 \ , \quad \mathbf{k} = i\sigma_3 \ . \tag{C.19}$$

Therefore, the quaternion $\eta$, (C.13), can be now written as:

$$\eta = 2(ab - cd)i\sigma_1 - 2(ac + bd)i\sigma_2 + (a^2 - b^2 - c^2 + d^2)i\sigma_3 \ , \tag{C.20}$$

where $i\sigma_i$ are the generators of the SU(2) algebra. The components of the imaginary unit quaternion obtained in (C.20) can be redefined in terms of a vector in $\mathbb{C}^2$ with components $z_1, z_2$. Specifically, the definition would be:

$$z_1 = \sqrt{2}(a - id) \ , \quad z_2 = \sqrt{2}(b - ic) \ , \tag{C.21}$$

with $a, b, c, d$ being the real numbers which constitute the quaternion $s$, defined in (C.12). Using the above redefinitions, (C.21), one obtains:

$$\frac{1}{2}\left(\bar{z}_1 z_2 + \bar{z}_2 z_1\right) = 2(ab - cd) \ \Rightarrow \ 2(ab - cd) = \frac{1}{2}(\bar{z}_1 \sigma_1^{12} z_2 + \bar{z}_2 \sigma_1^{21} z_1) = \frac{1}{2}\bar{z}_a \sigma_1^{ab} z_b \ , \tag{C.22}$$

$$\frac{1}{2}\left(-i\bar{z}_1 z_2 + i\bar{z}_2 z_1\right) = -2(ac - bd) \ \Rightarrow \ -2(ac - bd) = \frac{1}{2}(\bar{z}_1 \sigma_2^{12} z_2 + \bar{z}_2 \sigma_2^{21} z_1) = \frac{1}{2}\bar{z}_a \sigma_2^{ab} z_b \ , \tag{C.23}$$

$$\frac{1}{2}\left(\bar{z}_1 z_1 - \bar{z}_2 z_2\right) = a^2 - b^2 - c^2 + d^2 \ \Rightarrow \ a^2 - b^2 - c^2 + d^2 = \frac{1}{2}(\bar{z}_1 \sigma_3^{11} z_1 + \bar{z}_2 \sigma_3^{22} z_2) = \frac{1}{2}\bar{z}\sigma_3^{ab} z_b \ . \tag{C.24}$$

Therefore, the components of the three-dimensional real space coordinates are given by the above relations, in terms of the two vectors, $z_1, z_2$ and the Pauli matrices, (C.16). The constraint to which they are subjected is obtained if we take the sum of the square of the above coordinates, let us call them $x^i, i = 1, 2, 3$:

$$\sum_{i}^{3} {x^i}^2 = \frac{1}{4}\left((\bar{z}_1 z_2 + \bar{z}_2 z_1)^2 + (-i\bar{z}_1 z_2 + i\bar{z}_2 z_1)^2 + (\bar{z}_1 z_1 - \bar{z}_2 z_2)^2\right) = \left(\frac{1}{2}\bar{z}_a z_a\right)^2 \overset{\text{(C.15)}}{=} 1 \tag{C.25}$$

Therefore, using the radius relation of the $S^3$, the radius relation of the $S^2$ space is obtained.

We should also note that the obtained expressions of the $x^i$ coordinates, (C.16) are invariant under the following transformation:

$$z \ \longrightarrow e^{ia} z \ , \quad \bar{z} \longrightarrow e^{-ia} \bar{z} \ , \tag{C.26}$$

specifically shown in the following relation:

$$x^i \ \longrightarrow \ (x^i)' = \frac{1}{2}\bar{z}'_a \sigma^i_{ab} z'_b = \frac{1}{2}e^{-ia} z_a \sigma^i_{ab} e^{ia} z_b = \frac{1}{2}\bar{z}_a \sigma^i_{ab} z_b = x^i \ . \tag{C.27}$$

Therefore, due to this invariance, the above Hopf fibration can be viewed as coordinates on $S^2 = CP^1 = S^3/U(1)$.

# Appendix D

# The Coherent states

The canonical coherent states are defined for each complex number $z \in \mathbb{C}$ by a unitary transformation[1], which, acting on the vacuum state, $|0\rangle$, produces a coherent state, that is an eigenstate of the annihilation operator. Specifically:

$$|z\rangle = e^{za^\dagger - \bar{z}a}|0\rangle \ . \tag{D.1}$$

Using the Baker-Campbell-Hausdorff formula, (3.9), the above formula becomes:

$$|z\rangle = e^{\frac{-|z|^2}{2}} e^{za^\dagger} e^{-\bar{z}a}|0\rangle \ . \tag{D.2}$$

Expanding the exponential, $e^{-\bar{z}a}$ and recalling the definition of the vacuum, (B.8), that is $a|0\rangle = 0$, the above equation takes the following form:

$$|z\rangle = e^{-\frac{|z|^2}{2}} e^{za^\dagger}|0\rangle \ . \tag{D.3}$$

Also, expanding the $e^{za^\dagger}$ in the above equation:

$$|z\rangle = e^{\frac{-|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle \ , \tag{D.4}$$

and introducing the harmonic oscillator eigenstates, $|n\rangle$, of the number operator, (B.6), that is $N|n\rangle = n|n\rangle$ $(N = a^\dagger a)$, which are given by equation (B.9), that is $|n\rangle = \frac{1}{\sqrt{n!}} a^\dagger |0\rangle$, one obtains the following expression of the coherent state:

$$|z\rangle = e^{\frac{-|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle \ . \tag{D.5}$$

Let us now briefly mention some properties of the coherent states, $|z\rangle$. They form an over-complete basis[2] for the Hilbert space, $\mathcal{H}$, meaning that every state can be expressed as a

[1] Specifically, this operator is called displacement operator, $D(z)$, and is defined as $D(z) = e^{za^\dagger - \bar{z}a}$

[2] Instead of the term basis, the tight frame term is technically more accurate [144].

superposition of the coherent states in a non-unique way, that is a single ket can be decomposed in different ways in terms of the same set of vectors[3]. [Martinazzo] In order to obtain the (over-)completeness expression, let us start with considering the following integral:

$$\frac{1}{\pi}\int d^2z|z\rangle\langle z| = \frac{1}{\pi}\sum_{n,m}^{\infty}\frac{|n\rangle\langle m|}{\sqrt{n!m!}}\int e^{-|z|^2}z^n\bar{z}^m d^2z \ , \tag{D.6}$$

where $d^2z \equiv d(Re(z))d(Im(z))$. Introducing the polar coordinates, $z = |z|e^{i\theta}$, then $d^2z = |z|d|z|d\theta$[4] and the above equation becomes:

$$\begin{aligned}\frac{1}{\pi}\int d^2z|z\rangle\langle z| &= \frac{1}{\pi}\sum_{n,m}^{\infty}\frac{|n\rangle\langle m|}{\sqrt{n!m!}}\int_0^{\infty} e^{-|z|^2}|z|^{n+m}|z|d|z|\int_0^{2\pi} e^{i\theta(n-m)}d\theta \\ &= \frac{1}{\pi}\sum_{n,m}^{\infty}\frac{|n\rangle\langle m|}{\sqrt{n!m!}}\int_0^{\infty} e^{-|z|^2}|z|^{n+m}d|z|^2 2\pi\delta_{nm} \\ &= \sum_{n}^{\infty}\frac{|n\rangle\langle n|}{n!}\int_0^{\infty} e^{-|z|^2}|z|^{2n}d|z|^2 \\ &= \sum_{n}^{\infty}\frac{|n\rangle\langle n|}{n!}n! = 1\end{aligned} \tag{D.7}$$

In turn, the property of overcompleteness is responsible for the non-orthogonality of these states[5]. Indeed, definition of the coherent states, (D.5), gives the following non-orthogonality condition for two of them, $z_1, z_2$:

$$\langle z_1|z_2\rangle = \langle n|\left(e^{-\frac{|z_1|^2}{2}}e^{\frac{-|z_2|^2}{2}}\sum_{n=0}^{\infty}\frac{1}{n!}\bar{z}_1^n z_2^n\right)|n\rangle \ \Rightarrow\ \langle z_1|z_2\rangle = e^{-\frac{|z_1|^2}{2}+\bar{z}_1 z_2-\frac{|z_2|^2}{2}} \ . \tag{D.8}$$

The last property of the coherent states being normalized, $\langle z|z\rangle = 1$, concludes the necessary information about the canonical coherent states for our purpose.

[3] This ambiguity, although it seems to be unwieldy, makes the coherent states particularly important and useful in the field of quantum optics.

[4] More explicitly: $d^2z = dRe(|z|e^{i\theta})dIm(|z|e^{i\theta}) = dRe(|z|(1+i\theta))dIm(|z|(1+i\theta)) = d|z|d(|z|\theta) = |z|d|z|d\theta$.

[5] On the contrary, a complete set of states allows to express a function as a superposition in a unique way and thus to be orthogonal.

# Appendix E

# Three-dimensional noncommutative gravity as a gauge theory: Calculations of the field transformations, curvature tensors and equations of motion

Calculations for the expressions of the transformations of the fields, (5.11):

$$
\begin{aligned}
\delta\hat{X}_\mu &= \delta e_\mu{}^a \otimes \bar{\gamma}_a + \delta\omega_\mu{}^a \otimes \tilde{\gamma}_a + \delta A_\mu \otimes i\mathbb{1} + \delta\tilde{A}_\mu \otimes \gamma_5 = [\epsilon, \hat{X}_\mu] \\
&= [\xi^a \otimes \bar{\gamma}_a + \lambda^a \otimes \tilde{\gamma}_a + \epsilon_0 \otimes i\mathbb{1} + \tilde{\epsilon}_0 \otimes \gamma_5, X_\mu \otimes i\mathbb{1} + e_\mu{}^a \otimes \bar{\gamma}_a + \omega_\mu{}^a \otimes \tilde{\gamma}_a + A_\mu \otimes i\mathbb{1} + A_\mu \otimes \gamma_5] \\
&= [\xi^a \otimes \bar{\gamma}_a, X_\mu \otimes i\mathbb{1}] + [\lambda^a \otimes \tilde{\gamma}_a, X_\mu \otimes i\mathbb{1}] + [\epsilon_0 \otimes i\mathbb{1}, X_\mu \otimes i\mathbb{1}] + [\tilde{\epsilon}_0 \otimes \gamma_5, X_\mu \otimes i\mathbb{1}] \\
&\quad + [\xi^a \otimes \bar{\gamma}_a, e_\mu{}^b \otimes \bar{\gamma}_b] + [\xi^a \otimes \bar{\gamma}_a, \omega_\mu{}^b \otimes \tilde{\gamma}_b] + [\xi^a \otimes \bar{\gamma}_a, A_\mu \otimes i\mathbb{1}] + [\xi^a \otimes \bar{\gamma}_a, \tilde{A}_\mu \otimes \gamma_5] \\
&\quad + [\lambda^a \otimes \tilde{\gamma}_a, e_\mu{}^b \otimes \bar{\gamma}_b] + [\lambda^a \otimes \tilde{\gamma}_a, \omega_\mu{}^b \otimes \tilde{\gamma}_b] + [\lambda^a \otimes \tilde{\gamma}_a, A_\mu \otimes i\mathbb{1}] + [\lambda^a \otimes \tilde{\gamma}_a, \tilde{A}_\mu \otimes \gamma_5] \\
&\quad + [\epsilon_0 \otimes i\mathbb{1}, e_\mu{}^a \otimes \bar{\gamma}_a] + [\epsilon_0 \otimes i\mathbb{1}, \omega_\mu{}^a \otimes \tilde{\gamma}_a] + [\epsilon_0 \otimes i\mathbb{1}, A_\mu \otimes i\mathbb{1}] + [\epsilon_0 \otimes i\mathbb{1}, \tilde{A}_\mu \otimes \gamma_5] \\
&\quad + [\tilde{\epsilon}_0 \otimes \gamma_5, e_\mu{}^a \otimes \bar{\gamma}_a] + [\tilde{\epsilon}_0 \otimes \gamma_5, \omega_\mu{}^a \otimes \tilde{\gamma}_a] + [\tilde{\epsilon}_0 \otimes \gamma_5, A_\mu \otimes i\mathbb{1}] + [\tilde{\epsilon}_0 \otimes \gamma_5, \tilde{A}_\mu \otimes \gamma_5] \\
&\overset{(3.59)}{=} -i[X_\mu, \xi^a] \otimes \bar{\gamma}_a - i[X_\mu, \lambda^a] \otimes \tilde{\gamma}_a - i[X_\mu, \epsilon_0] \otimes i\mathbb{1} - i[X_\mu, \tilde{\epsilon}_0] \otimes \gamma_5 \\
&\quad + \tfrac{1}{2}[\xi^a, e_\mu{}^b] \otimes \{\bar{\gamma}_a, \bar{\gamma}_b\} + \tfrac{1}{2}\{\xi^a, e_\mu{}^b\} \otimes [\bar{\gamma}_a, \bar{\gamma}_b] + \tfrac{1}{2}[\xi^a, \omega_\mu{}^b] \otimes \{\bar{\gamma}_a, \tilde{\gamma}_b\} + \tfrac{1}{2}\{\xi^a, \omega_\mu{}^b\} \otimes [\bar{\gamma}_a, \tilde{\gamma}_b] \\
&\quad + i[\xi^a, A_\mu] \otimes \bar{\gamma}_a + \tfrac{1}{2}[\xi^a, \tilde{A}_\mu] \otimes \{\bar{\gamma}_a, \gamma_5\} \\
&\quad + \tfrac{1}{2}[\lambda_a, e_\mu{}^b] \otimes \{\tilde{\gamma}_a, \bar{\gamma}_b\} + \tfrac{1}{2}\{\lambda^a, e_\mu{}^b\} \otimes [\tilde{\gamma}_a, \bar{\gamma}_b] + \frac{1}{2}[\lambda^a, \omega_\mu{}^b] \otimes \{\tilde{\gamma}_a, \tilde{\gamma}_b\} + \tfrac{1}{2}\{\lambda^a, \omega_\mu{}^b\} \otimes [\tilde{\gamma}_a, \tilde{\gamma}_b] \\
&\quad + i[\lambda^a, A_\mu] \otimes \tilde{\gamma}_a + \tfrac{1}{2}[\lambda^a, \tilde{A}_\mu] \otimes \{\tilde{\gamma}_a, \gamma_5\} \\
&\quad + i[\epsilon_0, e_\mu{}^a] \otimes \bar{\gamma}_a + i[\epsilon_0, \omega_\mu{}^a] \otimes \tilde{\gamma}_a + i[\epsilon_0, A_\mu] \otimes i\mathbb{1} + i[\epsilon_0, \tilde{A}_\mu] \otimes \gamma_5 \\
&\quad + \tfrac{1}{2}[\tilde{\epsilon}_0, e_\mu{}^a] \otimes \{\gamma_5, \bar{\gamma}_a\} + \tfrac{1}{2}[\tilde{\epsilon}_0, \omega_\mu{}^a] \otimes \{\gamma_5, \tilde{\gamma}_a\} + i[\tilde{\epsilon}_0, A_\mu] \otimes \gamma_5 - i[\tilde{\epsilon}_0, \tilde{A}_\mu] \otimes i\mathbb{1}
\end{aligned}
$$

$$\overset{(5.4)-(5.6)}{=} -i[X_\mu,\xi^a]\otimes\bar\gamma_a - i[X_\mu,\lambda^a]\otimes\tilde\gamma_a - i[X_\mu,\epsilon_0]\otimes i\mathbb{1} - i[X_\mu,\tilde\epsilon_0]\otimes\gamma_5$$
$$+\tfrac12[\xi^a,e_\mu{}^b]\otimes 2\eta_{ab}\mathbb{1} + \tfrac12\{\xi^a,e_\mu{}^b\}\otimes\epsilon_{abc}\tilde\gamma^c + \tfrac12[\xi^a,\omega_\mu{}^b]\otimes 4i\delta_{ab}\gamma_5 + \tfrac12\{\xi^a,\omega_\mu{}^b\}\otimes(-4\epsilon_{abc}\bar\gamma^c)$$
$$+ i[\xi^a,A_\mu]\otimes\bar\gamma_a + \tfrac12[\xi^a,\tilde A_\mu]\otimes i\tilde\gamma^a$$
$$+\tfrac12[\lambda^a,e_\mu{}^b]\otimes(4i\delta_{ab}\gamma_5) + \tfrac12\{\lambda^a,e_\mu{}^b\}\otimes(-4\epsilon_{abc}\bar\gamma^c) + \tfrac12[\lambda^a,\omega_\mu{}^b]\otimes(-8\eta_{ab}\mathbb{1}) + \tfrac12\{\lambda^a,\omega_\mu{}^b\}\otimes(-4\epsilon^{abc}\tilde\gamma^c)$$
$$+ i[\lambda^a,A_\mu]\otimes\tilde\gamma_a + \tfrac12[\lambda^a,\tilde A_\mu]\otimes 4i\bar\gamma_a$$
$$+ i[\epsilon_0,e_\mu{}^a]\otimes\bar\gamma_a + i[\epsilon_0,\omega_\mu{}^a]\otimes\tilde\gamma_a + i[\epsilon_0,A_\mu]\otimes i\mathbb{1} + i[\epsilon_0,\tilde A_\mu]\otimes\gamma_5$$
$$+\tfrac12[\tilde\epsilon_0,e_\mu{}^a]\otimes i\tilde\gamma_a + \tfrac12[\tilde\epsilon_0,\omega_\mu{}^a]\otimes 4i\bar\gamma_a + i[\tilde\epsilon_0,A_\mu]\otimes\gamma_5 - i[\tilde\epsilon_0,\tilde A_\mu]\otimes i\mathbb{1} \;\Rightarrow$$

$$\begin{aligned}
\delta e_\mu{}^a &= -i[X_\mu + A_\mu,\xi^a] + 2\{\omega_{\mu b},\xi_c\}\epsilon^{abc} + 2\{e_{\mu b},\lambda^c\}\epsilon^{abc} + 2i[\lambda_a,\tilde A_\mu] + 2i[\tilde\epsilon_0,\omega_{\mu a}] + i[\epsilon_0,e_{\mu a}]\ ,\\
\delta\omega_\mu{}^a &= -i[X_\mu + A_\mu,\lambda^a] + 2\{\omega_{\mu b},\lambda_c\}\epsilon^{abc} - \frac12\{e_{\mu b},\xi_c\}\epsilon^{abc} + \frac{i}{2}[\xi^a,\tilde A_\mu] + i[\epsilon_0,\omega_\mu{}^a] + \frac{i}{2}[\tilde\epsilon_0,e_\mu{}^a]\ ,\\
\delta A_\mu &= -i[X_\mu + A_\mu,\epsilon_0] - i[\xi^a,e_{\mu a}] + 4i[\lambda^a,\omega_{\mu a}] - i[\tilde\epsilon_0,\tilde A_\mu]\ ,\\
\delta\tilde A_\mu &= -i[X_\mu + A_\mu,\tilde\epsilon_0] + 2i[\xi^a,\omega_{\mu a}] + 2i[\lambda^a,e_{\mu a}] + i[\epsilon_0,\tilde A_\mu]\ .
\end{aligned}$$

Calculations for the expressions of the component curvature tensors, (5.19):

$$
\begin{aligned}
\mathcal{R}_{\mu\nu} &= T_{\mu\nu}{}^{a}(X)\otimes\bar\gamma_a + R_{\mu\nu}{}^{a}(X)\otimes\tilde\gamma_a + F_{\mu\nu}(X)\otimes i\mathbb{1} + \tilde F_{\mu\nu}(X)\otimes\gamma_5 = [\hat X_\mu,\hat X_\nu] - i\lambda C_{\mu\nu}{}^{\rho}\hat X_\rho \\
&= [X_\mu\otimes i\mathbb{1} + e_\mu{}^a\otimes\bar\gamma_a + \omega_\mu{}^a\otimes\tilde\gamma_a + A_\mu\otimes i\mathbb{1} + \tilde A_\mu\otimes\gamma_5, X_\nu\otimes i\mathbb{1} + e_\nu{}^a\otimes\bar\gamma_a + \omega_\nu{}^a\otimes\tilde\gamma_a \\
&\quad + A_\nu\otimes i\mathbb{1} + \tilde A_\nu\otimes\gamma_5] - i\lambda C_{\mu\nu}{}^{\rho}(X_\rho\otimes i\mathbb{1} + e_\rho{}^a\otimes\bar\gamma_a + \omega_\rho{}^a\otimes\tilde\gamma_a + A_\rho\otimes i\mathbb{1} + \tilde A_\rho\otimes\gamma_5) \\
&= i[X_\mu,X_\nu]\otimes i\mathbb{1} + i[X_\mu,e_\nu{}^a]\otimes\bar\gamma_a + i[X_\mu,\omega_\nu{}^a]\otimes\tilde\gamma_a + i[X_\mu,A_\nu]\otimes i\mathbb{1} + i[X_\mu,\tilde A_\nu]\otimes\gamma_5 \\
&\quad + i[e_\mu{}^a,X_\nu]\otimes\bar\gamma_a + \tfrac12[e_\mu{}^a,e_\nu{}^b]\otimes\{\bar\gamma_a,\bar\gamma_b\} + \tfrac12\{e_\mu{}^a,e_\nu{}^b\}\otimes[\bar\gamma_a,\bar\gamma_b] + \tfrac12[e_\mu{}^a,\omega_\nu{}^b]\otimes\{\bar\gamma_a,\tilde\gamma_b\} \\
&\quad + \tfrac12\{e_\mu{}^a,\omega_\nu{}^b\}\otimes[\bar\gamma_a,\tilde\gamma_b] + i[e_\mu{}^a,A_\nu]\otimes\bar\gamma_a + \tfrac12[e_\mu{}^a,\tilde A_\nu]\otimes\{\bar\gamma_a,\gamma_5\} + \tfrac12\{e_\mu{}^a,\tilde A_\nu\}\otimes[\bar\gamma_a,\gamma_5] \\
&\quad + i[\omega_\mu{}^a,X_\nu]\otimes\tilde\gamma_a + \tfrac12[\omega_\mu{}^a,e_\nu{}^b]\otimes\{\tilde\gamma_a,\bar\gamma_b\} + \tfrac12\{\omega_\mu{}^a,e_\nu{}^b\}\otimes[\tilde\gamma_a,\bar\gamma_b] + \tfrac12[\omega_\mu{}^a,\omega_\nu{}^b]\otimes\{\tilde\gamma_a,\tilde\gamma_b\} \\
&\quad + \tfrac12\{\omega_\mu{}^a,\omega_\nu{}^b\}\otimes[\tilde\gamma_a,\tilde\gamma_b] + i[\omega_\mu{}^a,A_\nu]\otimes\tilde\gamma_a + \tfrac12[\omega_\mu{}^a,\tilde A_\nu]\otimes\{\tilde\gamma_a,\gamma_5\} + \tfrac12\{\omega_\mu{}^a,\tilde A_\nu\}\otimes[\tilde\gamma_a,\gamma_5] \\
&\quad + i[A_\mu,X_\nu]\otimes i\mathbb{1} + [A_\mu,e_\nu{}^a]\otimes\bar\gamma_a + i[A_\mu,\omega_\nu{}^a]\otimes\tilde\gamma_a + i[A_\mu,A_\nu]\otimes i\mathbb{1} + i[A_\mu,\tilde A_\nu]\otimes\gamma_5 \\
&\quad + i[\tilde A_\mu,X_\nu]\otimes\gamma_5 + \tfrac12[\tilde A_\mu,e_\nu{}^a]\otimes\{\gamma_5,\bar\gamma_a\} + \tfrac12\{\tilde A_\mu,e_\nu{}^a\}\otimes[\gamma_5,\bar\gamma_a] + \tfrac12[\tilde A_\mu,\omega_\nu{}^a]\otimes\{\gamma_5,\tilde\gamma_a\} \\
&\quad + \tfrac12\{\tilde A_\mu,\omega_\nu{}^a\}\otimes[\gamma_5,\tilde\gamma_a] + i[\tilde A_\mu,A_\nu]\otimes\gamma_5 - i[\tilde A_\mu,\tilde A_\nu]\otimes i\mathbb{1} - i\lambda C_{\mu\nu}{}^{\rho}\big(X_\rho\otimes i\mathbb{1} + e_\rho{}^a\otimes\bar\gamma_a + \\
&\quad + \omega_\rho{}^a\otimes\tilde\gamma_a + A_\rho\otimes i\mathbb{1} + \tilde A_\rho\otimes\gamma_5\ \big) \\
&= i[X_\mu,X_\nu]\otimes i\mathbb{1} + i[X_\mu,e_\nu{}^a]\otimes\bar\gamma_a + i[X_\mu,\omega_\nu{}^a]\otimes\tilde\gamma_a + i[X_\mu,A_\nu]\otimes i\mathbb{1} + i[X_\mu,\tilde A_\nu]\otimes\gamma_5 \\
&\quad + i[e_\mu{}^a,X_\nu]\otimes\bar\gamma_a + \tfrac12[e_\mu{}^a,e_\nu{}^b]\otimes 2\eta_{ab}\mathbb{1} + \tfrac12\{e_\mu{}^a,e_\nu{}^b\}\otimes\epsilon_{abc}\tilde\gamma^c + \tfrac12[e_\mu{}^a,\omega_\nu{}^b]\otimes 4i\delta_{ab}\gamma_5 \\
&\quad + \tfrac12\{e_\mu{}^a,\omega_\nu{}^b\}\otimes(-4\epsilon_{abc}\bar\gamma^c) + [e_\mu{}^a,A_\nu]\otimes\bar\gamma_a + \tfrac12[e_\mu{}^a,\tilde A_\nu]\otimes i\tilde\gamma^a + i[\omega_\mu{}^a,X_\nu]\otimes\tilde\gamma_a \\
&\quad + \tfrac12[\omega_\mu{}^a,e_\nu{}^b]\otimes 4i\delta_{ab}\gamma_5 + \tfrac12\{\omega_\mu{}^a,e_\nu{}^b\}\otimes(-4\epsilon_{abc}\bar\gamma^c) + \tfrac12[\omega_\mu{}^a,\omega_\nu{}^b]\otimes(-8\eta_{ab}\mathbb{1}) \\
&\quad + \tfrac12\{\omega_\mu{}^a,\omega_\nu{}^b\}\otimes(-4\epsilon_{abc}\tilde\gamma^c) + i[\omega_\mu{}^a,A_\nu]\otimes\tilde\gamma_a + \tfrac12[\omega_\mu{}^a,\tilde A_\nu]\otimes 4i\bar\gamma_a + i[A_\mu,X_\nu]\otimes i\mathbb{1} \\
&\quad + i[A_\mu,e_\nu{}^a]\otimes\bar\gamma_a + i[A_\mu,\omega_\nu{}^a]\otimes\tilde\gamma_a + i[A_\mu,A_\nu]\otimes i\mathbb{1} + i[A_\mu,\tilde A_\nu]\otimes\gamma_5 + i[\tilde A_\mu,X_\nu]\otimes\gamma_5 \\
&\quad + \tfrac12[\tilde A_\mu,e_\nu{}^a]\otimes i\tilde\gamma_a + \tfrac12[\tilde A_\mu,\omega_\nu{}^a]\otimes 4i\bar\gamma_a + i[\tilde A_\mu,A_\nu]\otimes\gamma_5 - i[\tilde A_\mu,\tilde A_\nu]\otimes i\mathbb{1} \\
&\quad - i\lambda C_{\mu\nu}{}^{\rho}\Big(X_\rho\otimes i\mathbb{1} + e_\rho{}^a\otimes\bar\gamma_a + \omega_\rho{}^a\otimes\tilde\gamma_a + A_\rho\otimes i\mathbb{1} + \tilde A_\rho\otimes\gamma_5\Big) \\
&= i[X_\mu,X_\nu]\otimes i\mathbb{1} + i[X_\mu,e_\nu{}^a]\otimes\bar\gamma_a + i[X_\mu,\omega_\nu{}^a]\otimes\tilde\gamma_a + i[X_\mu,A_\nu]\otimes i\mathbb{1} + i[X_\mu,\tilde A_\nu]\otimes\gamma_5 \\
&\quad + i[e_\mu{}^a,X_\nu]\otimes\bar\gamma_a + i[e_\mu{}^a,e_{\nu a}]\otimes i\mathbb{1} + \tfrac12\{e_\mu{}^b,e_\nu{}^c\}\otimes\epsilon_{abc}\tilde\gamma^a + 2i[e_\mu{}^a,\omega_{\nu a}]\otimes\gamma_5 \\
&\quad - 2\{e_\mu{}^b,\omega_\nu{}^c\}\epsilon_{abc}\otimes\bar\gamma^a + i[e_\mu{}^a,A_\nu]\otimes\bar\gamma_a + \tfrac{i}{2}[e_\mu{}^a,\tilde A_\nu]\otimes\tilde\gamma^a + i[\omega_\mu{}^a,X_\nu]\otimes\tilde\gamma_a \\
&\quad + 2i[\omega_\mu{}^a,e_{\nu a}]\otimes\gamma_5 - 2\{\omega_\mu{}^b,e_\nu{}^c\}\epsilon_{abc}\otimes\bar\gamma^a + 4i[\omega_\mu{}^a,\omega_{\nu a}]\otimes i\mathbb{1}) \\
&\quad - 2\{\omega_\mu{}^b,\omega_\nu{}^c\}\epsilon_{abc}\tilde\gamma^a + i[\omega_\mu{}^a,A_\nu]\otimes\tilde\gamma_a + 2i[\omega_\mu{}^a,\tilde A_\nu]\otimes\bar\gamma_a + i[A_\mu,X_\nu]\otimes i\mathbb{1} \\
&\quad + i[A_\mu,e_\nu{}^a]\otimes\bar\gamma_a + i[A_\mu,\omega_\nu{}^a]\otimes\tilde\gamma_a + i[A_\mu,A_\nu]\otimes i\mathbb{1} + i[A_\mu,\tilde A_\nu]\otimes\gamma_5 + i[\tilde A_\mu,X_\nu]\otimes\gamma_5 \\
&\quad + \tfrac{i}{2}[\tilde A_\mu,e_\nu{}^a]\otimes\tilde\gamma_a + 2i[\tilde A_\mu,\omega_\nu{}^a]\otimes\bar\gamma_a + i[\tilde A_\mu,A_\nu]\otimes\gamma_5 - i[\tilde A_\mu,\tilde A_\nu]\otimes i\mathbb{1} \\
&\quad - i\lambda C_{\mu\nu}{}^{\rho}\Big(X_\rho\otimes i\mathbb{1} + e_\rho{}^a\otimes\bar\gamma_a + \omega_\rho{}^a\otimes\tilde\gamma_a + A_\rho\otimes i\mathbb{1} + \tilde A_\rho\otimes\gamma_5\Big)\ \Rightarrow
\end{aligned}
$$

$$
\begin{aligned}
T_{\mu\nu}{}^{a} &= i[X_\mu + A_\mu, e_\nu{}^{a}] - i[X_\nu + A_\nu, e_\mu{}^{a}] - 2\epsilon^{abc}\left(\{e_{\mu b}, \omega_{\nu c}\} + \{\omega_{\mu b}, e_{\nu c}\}\right) \\
&\quad + 2i\left([\omega_\mu{}^{a}, \tilde{A}_\nu] - [\omega_\nu{}^{a}, \tilde{A}_\mu]\right) - i\lambda C_{\mu\nu}{}^{\rho} e_\rho{}^{a} \ , \\
R_{\mu\nu}{}^{a} &= i[X_\mu + A_\mu, \omega_\nu{}^{a}] - i[X_\nu + A_\nu, \omega_\mu{}^{a}] + \epsilon^{abc}\left(\tfrac{1}{2}\{e_{\mu b}, e_{\nu c}\} - 2\{\omega_{\mu b}, \omega_{\nu c}\}\right) \\
&\quad + \tfrac{i}{2}\left([e_\mu{}^{a}, \tilde{A}_\nu] - [e_\nu{}^{a}, \tilde{A}_\mu]\right) - i\lambda C_{\mu\nu}{}^{\rho} \omega_\rho{}^{a} \ , \\
F_{\mu\nu} &= i[X_\mu + A_\mu, X_\nu + A_\nu] - i[e_\mu{}^{a}, e_{\nu a}] + 4i[\omega_\mu{}^{a}, \omega_{\nu a}] - i[\tilde{A}_\mu, \tilde{A}_\nu] - i\lambda C_{\mu\nu}{}^{\rho}(X_\rho + A_\rho) \ , \\
\tilde{F}_{\mu\nu} &= i[X_\mu + A_\mu, \tilde{A}_\nu] - i[X_\nu + A_\nu, \tilde{A}_\mu] + 2i\left([e_\mu{}^{a}, \omega_{\nu a}] + [\omega_\mu{}^{a}, e_{\nu a}]\right) - i\lambda C_{\mu\nu}{}^{\rho}\tilde{A}_\rho \ .
\end{aligned}
$$

Variation of the action (5.27) and determination of the parameter $m^2$ so as the spaces $\mathbb{R}^{1,2}_\lambda$ and $\mathbb{R}^3_\lambda$ are solutions of the field equations derived from the action:

$$
\begin{aligned}
\delta S_0 &= \frac{1}{g^2}\delta \mathrm{Tr}\left(\frac{i}{3}C^{\mu\nu\rho}X_\mu X_\nu X_\rho - m^2 X_\mu X^\mu\right) \\
&= \frac{1}{g^2}\mathrm{Tr}\left[\frac{i}{3}C^{\mu\nu\rho}\left(\delta X_\mu X_\nu X_\rho + X_\mu \delta X_\nu X_\rho + X_\mu X_\nu \delta X_\rho\right) - m^2(\delta X_\mu X^\mu + X_\mu \delta X^\mu)\right] \\
&= \frac{1}{g^2}\mathrm{Tr}\left[\frac{i}{3}C^{\mu\nu\rho}\left(X_\nu X_\rho \delta X_\mu - X_\rho X_\nu \delta X_\mu - X_\rho X_\nu \delta X_\mu\right) - 2m^2 X^\mu \delta X_\mu\right] \\
&= \frac{1}{g^2}\mathrm{Tr}\left[\frac{i}{3}C^{\mu\nu\rho}[X_\nu, X_\rho] - \frac{i}{3}C^{\mu\nu\rho}X_\rho X_\nu - 2m^2 X^\mu\right]\delta X_\mu \\
&= \frac{1}{g^2}\mathrm{Tr}\left[\frac{i}{3}C^{\mu\nu\rho}[X_\nu, X_\rho] + \frac{i}{6}C^{\mu\nu\rho}[X_\nu, X_\rho] - 2m^2 X^\mu\right]\delta X_\mu \\
&= \frac{1}{g^2}\mathrm{Tr}\left[\frac{i}{2}C^{\mu\nu\rho}[X_\nu, X_\rho] - 2m^2 X^\mu\right]\delta X_\mu = 0 \ \Rightarrow
\end{aligned}
$$

$$
\begin{aligned}
&\frac{i}{2}C^{\mu\nu\rho}[X_\nu, X_\rho] - 2m^2 X^\mu = 0 \\
&\frac{i}{2}C_{\kappa\lambda\mu}C^{\mu\nu\rho}[X_\nu, X_\rho] - 2m^2 C_{\kappa\lambda\mu}X^\mu = 0 \\
-&\frac{i}{2}(\delta^\nu_\kappa \delta^\rho_\lambda - \delta^\rho_\kappa \delta^\nu_\lambda)[X_\nu, X_\rho] - 2m^2 C_{\kappa\lambda\mu}X^\mu = 0 \\
&[X_\kappa, X_\lambda] - 2im^2 C_{\kappa\lambda}{}^{\mu}X_\mu = 0
\end{aligned}
\tag{E.1}
$$

Taking into consideration the commutation relation of the coordinates for $\mathbb{R}^{1,2}_\lambda$, $[X_\mu, X_\nu] = i\lambda C_{\mu\nu}{}^{\rho}X_\rho$, one finds $m^2 = \lambda$. Had we started with the $\mathbb{R}^3_\lambda$ action, with the only difference to the above being the structure constants, $\epsilon_{\mu\nu\rho}$, one results with $m^2 = -\lambda$.

Calculations from the action (5.29) to the form of (5.30):

$$\begin{aligned}
S &= \frac{1}{g^2}\mathrm{Trtr}_G\left(\tfrac{i}{3}C^{\mu\nu\rho}\hat{X}_\mu\hat{X}_\nu\hat{X}_\rho - \tfrac{\lambda}{2}\hat{X}_\mu\hat{X}^\mu\right)\\
&= \frac{1}{g^2}\mathrm{Trtr}_G\left(\tfrac{i}{6}C^{\mu\nu\rho}\hat{X}_\mu[\hat{X}_\nu,\hat{X}_\rho] - \tfrac{\lambda}{2}\hat{X}_\mu\hat{X}^\mu\right)\\
&\overset{(5.14)}{=} \frac{1}{g^2}\mathrm{Trtr}_G\left(\tfrac{i}{6}C^{\mu\nu\rho}\hat{X}_\mu(\mathcal{R}_{\nu\rho} + i\lambda C_{\nu\rho\sigma}\hat{X}^\sigma) - \tfrac{\lambda}{2}\hat{X}_\mu\hat{X}^\mu\right)\\
&= \frac{1}{g^2}\mathrm{Trtr}_G\left(\tfrac{i}{6}C^{\mu\nu\rho}\hat{X}_\mu\mathcal{R}_{\nu\rho} - \tfrac{\lambda}{6}C^{\mu\nu\rho}C_{\nu\rho\sigma}\hat{X}_\mu\hat{X}^\sigma - \tfrac{\lambda}{2}\hat{X}_\mu\hat{X}^\mu\right)\\
&= \frac{1}{g^2}\mathrm{Trtr}_G\left(\tfrac{i}{6}C^{\mu\nu\rho}\hat{X}_\mu\mathcal{R}_{\nu\rho} - \tfrac{\lambda}{6}(-2\delta^\mu_\sigma)\hat{X}_\mu\hat{X}^\sigma - \tfrac{\lambda}{2}\hat{X}_\mu\hat{X}^\mu\right)\\
&= \frac{1}{g^2}\mathrm{Trtr}_G\left(\tfrac{i}{6}C^{\mu\nu\rho}\hat{X}_\mu\mathcal{R}_{\nu\rho} - \tfrac{\lambda}{6}\hat{X}_\mu\hat{X}^\mu\right)\\
&= \frac{1}{6g^2}\mathrm{Trtr}_G\left(iC^{\mu\nu\rho}\hat{X}_\mu\mathcal{R}_{\nu\rho}\right) - \tfrac{\lambda}{6g^2}\mathrm{Trtr}_G(\hat{X}_\mu\hat{X}^\mu)\\
&= \tfrac{1}{6g^2}\mathrm{Trtr}_G\left(iC^{\mu\nu\rho}\hat{X}_\mu\mathcal{R}_{\nu\rho}\right) + S_\lambda\ ,
\end{aligned}$$

where we set $S_\lambda = -\frac{\lambda}{6g^2}\mathrm{Trtr}_G(\hat{X}_\mu\hat{X}^\mu)$.

Calculations from (5.30) to (5.31):

$$\begin{aligned}
S &= \frac{1}{6g^2}\mathrm{Trtr}_G\left(iC^{\mu\nu\rho}\hat{X}_\mu\mathcal{R}_{\nu\rho}\right) - \tfrac{\lambda}{6g^2}\mathrm{Trtr}_G(\hat{X}_\mu\hat{X}^\mu)\\
&\overset{(5.9),(5.14)}{=} \tfrac{i}{6g^2}\mathrm{Trtr}_G C^{\mu\nu\rho}\Big((X_\mu\otimes i\mathbb{1} + e_\mu{}^a\otimes\bar\gamma_a + \omega_\mu{}^a\otimes\tilde\gamma_a + A_\mu\otimes i\mathbb{1} + \tilde{A}_\mu\otimes\gamma_5)\cdot\\
&\qquad (T_{\nu\rho}{}^b\otimes\bar\gamma_b + R_{\nu\rho}{}^b\otimes\tilde\gamma_b + F_{\nu\rho}\otimes i\mathbb{1} + \tilde{F}_{\nu\rho}\otimes\gamma_5)\Big)\\
&\quad - \tfrac{\lambda}{6g^2}\mathrm{Trtr}_G\Big((X_\mu\otimes i\mathbb{1} + e_\mu{}^a\otimes\bar\gamma_a + \omega_\mu{}^a\otimes\tilde\gamma_a + A_\mu\otimes i\mathbb{1} + \tilde{A}_\mu\otimes\gamma_5)\cdot\\
&\qquad (X^\mu\otimes i\mathbb{1} + e^{\mu b}\otimes\bar\gamma_a + \omega^{\mu b}\otimes\tilde\gamma_a + A^\mu\otimes i\mathbb{1} + \tilde{A}^\mu\otimes\gamma_5)\Big)\\
&= \frac{i}{6g^2}\mathrm{Tr}C^{\mu\nu\rho}\Big(e_\mu{}^a T_{\nu\rho}{}^b\otimes\mathrm{tr}_G(\bar\gamma_a\bar\gamma_b) + \omega_\mu{}^a R_{\nu\rho}{}^b\otimes\mathrm{tr}_G(\tilde\gamma_a\tilde\gamma_b) - (X_\mu + A_\mu)F_{\nu\rho}\otimes\mathrm{tr}_G\mathbb{1} + \tilde{A}_\mu\tilde{F}_{\nu\rho}\otimes\mathrm{tr}_G\mathbb{1}\Big)\\
&\quad - \tfrac{\lambda}{6g^2}\mathrm{Tr}\Big(e_\mu{}^a e^{\mu b}\otimes\mathrm{tr}_G(\bar\gamma_a\bar\gamma_b) + \omega_\mu{}^a\omega^{\mu b}\otimes\mathrm{tr}_G(\tilde\gamma_a\tilde\gamma_b) - (X_\mu + A_\mu)(X^\mu + A^\mu)\otimes\mathrm{tr}_G\mathbb{1} + \tilde{A}_\mu\tilde{A}^\mu\otimes\mathrm{tr}_G\mathbb{1}\Big)\\
&\overset{(5.28)}{=} \frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}\Big(e_{\mu a}T_{\nu\rho}{}^a - 4\omega_{\mu a}R_{\nu\rho}{}^a - (X_\mu + A_\mu)F_{\nu\rho} + \tilde{A}_\mu\tilde{F}_{\nu\rho}\Big)\\
&\quad - \tfrac{2\lambda}{3g^2}\mathrm{Tr}\Big(e_\mu{}^a e^\mu{}_a - 4\omega_\mu{}^a\omega^\mu{}_a - (X_\mu + A_\mu)(X^\mu + A^\mu) + \tilde{A}_\mu\tilde{A}^\mu\Big)\ .
\end{aligned}$$

Variation of the action (5.31) with respect to the gauge fields. We give the detailed calculations of the variation with respect to the $e$ gauge field and the rest are obtained accordingly.

$$\delta_e S = \frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}(\delta e_{\mu a}T_{\nu\rho}{}^a + e_{\mu a}\delta_e T_{\nu\rho}{}^a - 4\omega_{\mu a}\delta_e R_{\nu\rho}{}^a - (X_\mu + A_\mu)\delta_e F_{\nu\rho} + \tilde{A}_\mu\delta_e\tilde{F}_{\nu\rho}) - \frac{2\lambda}{3g^2}\mathrm{Tr}(2\delta e_\mu{}^a e^\mu{}_a)\ .$$

We break down the above expression and calculate each term separately:

$$
\begin{aligned}
&\frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}e_{\mu a}\delta_e T_{\nu\rho}{}^{a} - \frac{2\lambda}{3g^2}\mathrm{Tr}(2\delta e_{\mu}{}^{a}e^{\mu}{}_{a}) = \\
&= \frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}e_{\mu a}(i[X_\nu + A_\nu],\delta e_{\rho}{}^{a}] - i[X_\rho + A_\rho],\delta e_{\nu}{}^{a}] - 2\epsilon^{abc}(\{\delta e_{\nu b},\omega_{\rho c}\} + \{\omega_{\nu b},\delta e_{\rho c}\} \\
&\quad - i\lambda C_{\nu\rho\sigma}\delta e^{\sigma a}) - \frac{2\lambda}{3g^2}\mathrm{Tr}(2e^{\mu}{}_{a}\delta e_{\mu}{}^{a}) \\
&= \frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}(i[X_\nu + A_\nu, e_{\mu a}\delta e_{\rho}{}^{a}] - i[X_\nu + A_\nu, e_{\mu a}]\delta e_{\rho}{}^{a} - i[X_\rho + A_\rho, e_{\mu a}\delta e_{\nu}{}^{a}] + i[X_\rho + A_\rho, e_{\mu a}]\delta e_{\nu}{}^{a} \\
&\quad - 2\epsilon^{abc}(e_{\mu a}\delta e_{\nu b}\omega_{\rho c} + e_{\mu a}\omega_{\rho c}\delta e_{\nu b} + e_{\mu a}\omega_{\nu b}\delta e_{\rho c} + e_{\mu a}\delta e_{\rho c}\omega_{\nu b})) \\
&\quad - \frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}i\lambda C_{\nu\rho\sigma}e_{\mu a}\delta e^{\sigma a} - \frac{2i}{3g^2}\mathrm{Tr}i\lambda(-2\delta^{\mu}_{\sigma})e^{\sigma}{}_{a}\delta e_{\mu}{}^{a}) \\
&= \frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}(i[X_\nu + A_\nu, e_{\rho a}]\delta e_{\mu}{}^{a} - i[X_\rho + A_\rho, e_{\nu a}]\delta e_{\mu}{}^{a} - 2\epsilon^{abc}(\{e_{\mu a},\omega_{\rho c}\}\delta e_{\nu b} + \{e_{\mu a},\omega_{\nu b}\}\delta e_{\rho c})) \\
&\quad - \frac{2i}{3g^2}\mathrm{Tr}C^{\sigma\nu\rho}i\lambda C_{\nu\rho\mu}e_{\sigma a}\delta e^{\mu a} - \frac{2i}{3g^2}\mathrm{Tr}i\lambda C^{\mu\nu\rho}C_{\nu\rho\sigma}e^{\sigma}{}_{a}\delta e_{\mu}{}^{a} \\
&= \frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}(i[X_\nu + A_\nu, e_{\rho a}]\delta e_{\mu}{}^{a} - i[X_\rho + A_\rho, e_{\nu a}]\delta e_{\mu}{}^{a} - 2\epsilon^{abc}(\{e_{\nu b},\omega_{\rho c}\}\delta e_{\mu a} + \{e_{\rho c},\omega_{\nu b}\}\delta e_{\mu a})) \\
&\quad - \frac{2i}{3g^2}\mathrm{Tr}C_{\mu\nu\rho}C^{\nu\rho\sigma}i\lambda e_{\sigma a}\delta e^{\mu a} - \frac{2i}{3g^2}\mathrm{Tr}i\lambda C^{\mu\nu\rho}C_{\nu\rho\sigma}e^{\sigma}{}_{a}\delta e_{\mu}{}^{a} \\
&= \frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}(i[X_\nu + A_\nu, e_{\rho a}]\delta e_{\mu}{}^{a} - i[X_\rho + A_\rho, e_{\nu a}]\delta e_{\mu}{}^{a} - 2\epsilon_{abc}(\{e_{\nu}{}^{b},\omega_{\rho}{}^{c}\}\delta e_{\mu}{}^{a} + \{e_{\rho}{}^{c},\omega_{\nu}{}^{b}\}\delta e_{\mu}{}^{a})) \\
&\quad - \frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}C_{\nu\rho\sigma}i\lambda e^{\sigma}{}_{a}\delta e_{\mu}{}^{a} - \frac{2i}{3g^2}\mathrm{Tr}i\lambda C^{\mu\nu\rho}C_{\nu\rho\sigma}e^{\sigma}{}_{a}\delta e_{\mu}{}^{a} \\
&= \frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}(i[X_\nu + A_\nu, e_{\rho a}] - i[X_\rho + A_\rho, e_{\nu a}] - 2\epsilon_{abc}(\{e_{\nu}{}^{b},\omega_{\rho}{}^{c}\} + \{e_{\rho}{}^{c},\omega_{\nu}{}^{b}\}) - 2i\lambda C_{\nu\rho\sigma}e^{\sigma}{}_{a})\delta e_{\mu}{}^{a}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}\left(-4\omega_{\mu a}\delta_e R_{\nu\rho}{}^{a}\right) = \\
&\frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}\left(-4\omega_{\mu a}\left(\tfrac{1}{2}\epsilon^{abc}(\{\delta e_{\nu b},e_{\rho c}\} + \{e_{\nu b},\delta e_{\rho c}\}) + \tfrac{i}{2}([\delta e_{\nu}{}^{a},\tilde{A}_\rho] - [\delta e_{\rho}{}^{a},\tilde{A}_\nu])\right)\right) \\
&= \frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}\Big(-2\epsilon^{abc}(\omega_{\mu a}\delta e_{\nu b}e_{\rho c} + \omega_{\mu a}e_{\rho c}\delta e_{\nu b} + \omega_{\mu a}e_{\nu b}\delta e_{\rho c} + \omega_{\mu a}\delta e_{\rho c}e_{\nu b}) \\
&\quad -2i(\omega_{\mu a}[\delta e_{\nu}{}^{a},\tilde{A}_\rho] - \omega_{\mu a}[\delta e_{\rho}{}^{a},\tilde{A}_\nu])\Big) \\
&= \frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}\Big(-2\epsilon^{abc}(\{e_{\rho c},\omega_{\mu a}\}\delta e_{\nu b} + \{e_{\nu b},\omega_{\mu a}\}\delta e_{\rho c}) \\
&\quad -2i([\omega_{\mu a}\delta e_{\nu}{}^{a},\tilde{A}_\rho] - [\omega_{\mu a},\tilde{A}_\rho]\delta e_{\nu}{}^{a} - [\omega_{\mu a}\delta e_{\rho}{}^{a},\tilde{A}_\nu] + [\omega_{\mu a},\tilde{A}_\nu]\delta e_{\rho}{}^{a})\Big) \\
&= \frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}\Big(-2\epsilon_{abc}(\{e_{\rho}{}^{c},\omega_{\nu}{}^{b}\} + \{e_{\nu}{}^{b},\omega_{\rho}{}^{c}\}) - 2i([\omega_{\nu a},\tilde{A}_\rho] - [\omega_{\rho a},\tilde{A}_\nu])\Big)\delta e_{\mu}{}^{a}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}\left(-(X_\mu+A_\mu)\delta_e F_{\nu\rho}\right)= \\
&\quad=\frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}\left(-(X_\mu+A_\mu)(-i[\delta e_\nu{}^a,e_{\rho a}]-i[e_\nu{}^a,\delta e_{\rho a}])\right) \\
&\quad=\frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}\left(-i[e_{\rho a},(X_\mu+A_\mu)\delta e_\nu{}^a]+i[e_{\rho a},X_\mu+A_\mu]\delta e_\nu{}^a+i[e_{\nu a},(X_\mu+A_\mu)\delta e_{\rho a}]-i[e_{\nu a},X_\mu+A_\mu]\delta e_\rho{}^a\right) \\
&\quad=\frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}\left(i[X_\nu+A_\nu,e_{\rho a}]-i[X_\rho+A_\rho,e_{\nu a}]\right)\delta e_\mu{}^a
\end{aligned}
$$

$$
\begin{aligned}
&\frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}\left(\tilde{A}_\mu\delta_e\tilde{F}_{\nu\rho}\right)= \\
&\quad=\frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}\tilde{A}_\mu\left(2i[\delta e_\nu{}^a,\omega_{\rho a}]+[\omega_\nu{}^a,\delta e_{\rho a}]\right) \\
&\quad=\frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}\left(2i[\tilde{A}_\mu\delta e_\nu{}^a,\omega_{\rho a}]-2i[\tilde{A}_\mu,\omega_{\rho a}]\delta e_\nu{}^a+2i[\omega_\nu{}^a,\tilde{A}_\mu\delta e_{\rho a}]-2i[\omega_\nu{}^a,\tilde{A}_\mu]\delta e_\rho{}^a\right) \\
&\quad=\frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}2i\left([\omega_{\nu a},\tilde{A}_\rho]-[\omega_{\rho a},\tilde{A}_\nu]\right)\delta e_\mu{}^a
\end{aligned}
$$

Combining the above calculations, the $\delta_e S$ becomes:

$$
\begin{aligned}
\delta_e S=&\frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}\Big(T_{\nu\rho a}+i[X_\nu+A_\nu,e_{\rho a}]-i[X_\rho+A_\rho,e_{\nu a}]-2\epsilon_{abc}(\{e_\nu{}^b,\omega_\rho{}^c\}+\{e_\rho{}^c,\omega_\nu{}^b\})-2i\lambda C_{\nu\rho\sigma}e^\sigma{}_a \\
&-2\epsilon_{abc}(\{e_\rho{}^c,\omega_\nu{}^b\}+\{e_\nu{}^b,\omega_\rho{}^c\})-2i([\omega_{\nu a},\tilde{A}_\rho]-[\omega_{\rho a},\tilde{A}_\nu]) \\
&+i[X_\nu+A_\nu,e_{\rho a}]-i[X_\rho+A_\rho,e_{\nu a}]+2i[\omega_{\nu a},\tilde{A}_\rho]-2i[\omega_{\rho a},\tilde{A}_\nu]\Big)\delta e_\mu{}^a \\
=&\frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}\Big(T_{\nu\rho a}+2i[X_\nu+A_\nu,e_{\rho a}]-2i[X_\rho+A_\rho,e_{\nu a}]-4\epsilon_{abc}(\{e_\nu{}^b,\omega_\rho{}^c\}+\{e_\rho{}^c,\omega_\nu{}^b\})-2i\lambda C_{\nu\rho\sigma}e^\sigma{}_a \\
&+4i[\omega_{\nu a},\tilde{A}_\rho]-4i[\omega_{\rho a},\tilde{A}_\nu]\Big) \\
=&\frac{2i}{3g^2}\mathrm{Tr}C^{\mu\nu\rho}\left(T_{\nu\rho a}+2T_{\nu\rho a}\right)=0\;\Rightarrow\;T_{\nu\rho a}=0\,.
\end{aligned}
$$

In the above calculations, among others, the antisymmetricity of the structure constants, the trace invariance under cyclic permutations, the vanishing of the trace of a commutator (see section 3.1.1) and the expressions of the component field strength tensors, (5.19), have been used. Variation with respect to the rest of the gauge fields is carried out in a similar way.

# Appendix F

# Four-dimensional noncommutative gravity as a gauge theory: Calculations of the field transformations and curvature tensors

In this appendix, we present steps and calculations for the transformations of the gauge fields and the component curvature tensors for the four-dimensional gravity as a noncommutative gauge theory. In the end, we check whether our results are valid, after the consideration of the commutative limit.

In the main body, in section 6.2, we provided all definitions and necessary information for the set-up of the gauge theory. For a better flow, all calculations that should lie in the text after equation (6.15) are moved in this appendix. So, picking the thread from there, we move on with the calculation of the transformations of the gauge fields and the expressions of the component curvature tensors.

Instead of proceeding with using the transformation rule of the covariant coordinate, (6.10), for a straightforward calculation of the transformation of the sixteen gauge fields of the theory, for calculative reasons, first we employ an SO(5) notation and then, applying a decomposition, we return to the SO(4) notation we have already adopted. As we explained in section 6.1, for reasons of anticommutation closure, the generators of the gauge group SO(6)×U(1) are encountered in a fixed representation given by 4×4 matrices:

$$1, \quad M_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b] = -\frac{i}{2}\Gamma_a\Gamma_b, \quad \frac{1}{2}\Gamma_a, \quad -\frac{1}{2}\Gamma_a\Gamma_5, \quad -\frac{1}{2}\Gamma_5 \ . \tag{F.1}$$

For the upgrade to the SO(5) notation, we introduce the matrices $\Gamma_A$, which satisfy the following anticommutation relation:

$$\{\Gamma_A, \Gamma_B\} = 2\delta_{AB}\mathbb{1} \ , \tag{F.2}$$

with $A, B = 1 \dots 5$. Taking this into consideration, the above generators, (F.1) can be rewritten in the following compact form, in terms of the $\Gamma_A$ matrices of equation (F.2):

$$1, \quad \Gamma_A, \quad M_{AB} = -\frac{i}{4}[\Gamma_A, \Gamma_B] \ . \tag{F.3}$$

In this SO(5) notation, the algebra of the generators and their anticommutation relations are found to be the following [145]:

$$\begin{aligned}
&[M_{AB}, M_{CD}] = i(\delta_{AC}M_{BD} + \delta_{BD}M_{AC} - \delta_{BC}M_{AD} - \delta_{AD}M_{BC})\,, \\
&[\Gamma_M, M_{NP}] = i(\delta_{MP}\Gamma_N - \delta_{MN}\Gamma_P)\,, \\
&\{M_{AB}, \Gamma_C\} = \epsilon_{ABCDE}M_{DE}\,, \\
&\{M_{AB}, M_{CD}\} = \frac{1}{2}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})1 + \epsilon_{ABCDE}\Gamma_E\,.
\end{aligned} \tag{F.4}$$

Next, we turn all components of the gauge theory into the SO(5) notation, specifically, the covariant coordinate is written as:

$$\hat{X}_m = X_m \otimes 1\!\!1 + A_m(X) \otimes 1\!\!1 + A_m{}^B(X) \otimes \Gamma_B + A_m{}^{AB}(X) \otimes M_{AB}\,, \tag{F.5}$$

the gauge parameter as:

$$\epsilon(X) = \epsilon_0(X) \otimes 1\!\!1 + \xi^A(X) \otimes \Gamma_A + \lambda^{AB}(X) \otimes M_{AB} \tag{F.6}$$

and, accordingly, the field strength tensor as:

$$\hat{F}_{mn} = [\hat{X}_m, \hat{X}_n] - \frac{i\hbar}{\lambda^2}\hat{\Theta}_{mn} \otimes 1\!\!1\,, \tag{F.7}$$

which is decomposed on the generators of SO(6)×U(1) in the SO(5) notation:

$$\hat{F}_{mn} = F_{mn}(1\!\!1) \otimes 1\!\!1 + F_{mn}{}^A(\Gamma_A) \otimes \Gamma_A + F_{mn}{}^{AB}(M_{AB}) \otimes M_{AB}\,. \tag{F.8}$$

Now, after rewriting the algebra and the expressions related with the gauge theory, we may proceed with the calculations. The transformation rule of the covariant coordinate is:

$$\delta\hat{X}_m = i[\epsilon, \hat{X}_m]\,. \tag{F.9}$$

Replacing the expressions in (F.5) and (F.6) into the above rule, we obtain the transformations of the component gauge fields in the SO(5) notation:

$$\delta A_m \otimes 1\!\!1 = \left(-i[X_m, \epsilon_0] - i[A_m, \epsilon_0] + i[\xi_A, A_m^A] + \frac{i}{2}[\lambda_{AB}, A_m^{AB}]\right) \otimes 1\!\!1\,, \tag{F.10}$$

$$\begin{aligned}
\delta A_m^A \otimes \Gamma_A = \big(&-i[X_m, \xi^A] - i[A_m, \xi^A] + i[\epsilon_0, A_m^A] - \{\xi_B, A_m^{AB}\} + \{\lambda^A_B, A_m^B\} \\
&+ \frac{i}{2}[\lambda^{BC}, A_m^{DE}]\epsilon_{ABCDE}\Big) \otimes \Gamma_A\,,
\end{aligned} \tag{F.11}$$

$$\begin{aligned}
\delta A_m^{AB} \otimes M_{AB} = \Big(&-i[X_m, \lambda^{AB}] - i[A_m, \lambda^{AB}] + i[\epsilon_0, A_m^{AB}] - 2\{\xi^{\ A}, A_m^B\} + \frac{i}{2}[\xi^{\ C}, A_m^{DE}]\epsilon_{ABCDE} \\
&+ \frac{i}{2}[\lambda^{CD}, A_m^E]\epsilon_{ABCDE} - \frac{1}{2}\{\lambda^A_C, A_m^{BC}\}\Big) \otimes M_{AB}\,.
\end{aligned} \tag{F.12}$$

From the field strength tensor definition, (F.7), its expansion, (F.8) and the definition of the covariant coordinate in the SO(5) notation, (F.5), we obtain the following expressions of the component curvature tensors:

$$F_{mn}\otimes \mathbb{1} = \Big([X_m, A_n]-[X_n,A_m]+[A_m,A_n]+[A_m^{\ A},A_{nA}]+\frac{1}{2}[A_m^{\ AB},A_{nAB}]-\frac{i\hbar}{\lambda^2}B_{mn}\Big)\otimes\mathbb{1} \tag{F.13}$$

$$\begin{aligned}F_{mn}^{\ \ A}(\Gamma_A)\otimes\Gamma_A = \big(&[X_m,A_n^{\ A}]+[A_m,A_n^{\ A}]-[X_n,A_m^{\ A}]-[A_n,A_m^{\ A}]+i\{A_{mB},A_n^{\ AB}\}\\ &-i\{A_m^{\ AB},A_{nB}\}-\frac{1}{2}\epsilon_{ABCDE}[A_m^{\ EB},A_n^{\ CD}]-\frac{i\hbar}{\lambda^2}B_{mn}^{\ \ A}\Big)\otimes\Gamma_A\end{aligned} \tag{F.14}$$

$$\begin{aligned}F_{mn}^{\ \ AB}(M_{AB})\otimes M_{AB} = \big(&[X_m,A_n^{\ AB}]+[A_m,A_n^{\ AB}]-[X_n,A_m^{\ AB}]-[A_n,A_m^{\ AB}]+2i\{A_m^{\ A},A_n^{\ B}\}\\ &+\frac{1}{2}([A_m^{\ C},A_n^{\ DE}]-[A_n^{\ C},A_m^{\ DE}])\epsilon_{ABCDE}+2i\{A_m^{\ AC},A_{n\ C}^{\ B}\}-\frac{i\hbar}{\lambda^2}B_{mn}^{\ \ AB}\Big)\otimes M_{AB}\,.\end{aligned} \tag{F.15}$$

Next, in order to go back to the previous notation and express the above results, (F.12) and (F.15), in the desirable, SO(4) language, we proceed with the following decompositions of the SO(5) generators:

$$\Gamma_A \to (\Gamma_a \equiv 2K_a, \Gamma_5 \equiv -2D)\,, \quad M_{AB}\to \big(M_{ab}=-\tfrac{i}{4}[\Gamma_a,\Gamma_b], M_{a5}=-\tfrac{i}{2}\Gamma_a\Gamma_5\equiv P_a\big)\;. \tag{F.16}$$

Accordingly, we decompose the gauge fields to the SO(4) notation:

$$A_m^{\ AB}\to(A_m^{\ ab}\equiv\omega_m^{\ ab}, A_m^{\ a5}\equiv e_m^{\ a})\,,\quad A_m^{\ A}\to(A_m^{\ a}\equiv b_m^{\ a}, A_m^{\ 5}\equiv\tilde{a}_m)\,,\quad A_m\to a_m\;, \tag{F.17}$$

and also the 2-form gauge field:

$$B_{mn}^{\ \ AB}\to(B_{mn}^{\ \ ab},B_{mn}^{\ \ a5}),\quad B_{mn}^{\ \ A}\to(B_{mn}^{\ \ a},B_{mn}^{\ \ 5})\;, \tag{F.18}$$

as well as the components of the SO(5) gauge parameter:

$$\lambda_{AB}\to(\lambda_{ab},\lambda_{a5}\equiv\tilde{\xi}_a)\,,\quad \xi^A\to(\xi^a,\xi^5\equiv\tilde{\epsilon}_0)\quad \epsilon_0\to\epsilon_0\;. \tag{F.19}$$

Applying all the above decompositions and identifications on the expressions of the transformations of the gauge fields in the SO(5) notation, (F.12), we obtain the corresponding transformations in the desired SO(4) notation:

$$\begin{aligned}\delta\omega_m^{\ ab} &= -i[X_m,\lambda^{ab}]-i[a_m,\lambda^{ab}]+i[\epsilon_0,\omega_m^{\ ab}]-2\{\xi^a,b_m^{\ b}\}-\frac{1}{2}\{\lambda^a_{\ c},\omega_m^{\ bc}\}-\frac{1}{2}\{\tilde{\xi}^a,e_m^{\ b}\}\\ &\quad+i[\xi^c,e_m^{\ d}]\epsilon_{abcd}+\frac{i}{2}[\tilde{\epsilon}_0,\omega_m^{\ cd}]\epsilon_{abcd}+\frac{i}{2}[\lambda^{cd},\tilde{a}_m]\epsilon_{abcd}-i[\tilde{\xi}^c,b_m^{\ d}]\epsilon_{abcd}\end{aligned} \tag{F.20}$$

$$\begin{aligned}\delta e_m^{\ a} &= -i[X_m,\tilde{\xi}^a]-i[a_m,\tilde{\xi}^a]+i[\epsilon_0,e_m^{\ a}]-\{\xi^a,\tilde{a}_m\}+\{\tilde{\epsilon}_0,b_m^{\ a}\}+\frac{1}{4}\{\lambda^a_{\ b},e_m^{\ b}\}-\frac{1}{4}\{\tilde{\xi}_b,\omega_m^{\ ab}\}\\ &\quad+i[\xi^c,\omega_m^{\ bd}]\epsilon_{abcd}-i[\lambda^{cd},b_m^{\ b}]\epsilon_{abcd}\end{aligned} \tag{F.21}$$

$$\begin{aligned}\delta b_m^{\ a} &= -i[X_m,\xi^a]-i[a_m,\xi^a]+i[\epsilon_0,b_m^{\ a}]-\{\xi_b,\omega_m^{\ ab}\}-2\{\tilde{\epsilon}_0,e_m^{\ a}\}+\frac{1}{2}\{\lambda^a_{\ b},b_m^{\ b}\}+\{\tilde{\xi}^a,\tilde{a}_m\}\\ &\quad+i[\lambda^{bc},e_m^{\ d}]\epsilon_{abcd}+i[\tilde{\xi}^b,\omega_m^{\ cd}]\epsilon_{abcd}\end{aligned} \tag{F.22}$$

$$\delta a_m = -i[X_m,\epsilon_0]-i[a_m,\epsilon_0]+i[\xi^a,b_m^{\ a}]+i[\tilde{\epsilon}_0,\tilde{a}_m]+\frac{i}{2}[\lambda_{ab},\omega_m^{\ ab}]+\frac{i}{2}[\tilde{\xi}_a,e_m^{\ a}] \tag{F.23}$$

$$\delta\tilde{a}_m = -i[X_m,\tilde{\epsilon}_0]-i[a_m,\tilde{\epsilon}_0]+i[\epsilon_0,\tilde{a}_m]+\{\xi_a,e_m^{\ a}\}-\{\tilde{\xi}_a,b_m^{\ a}\}+\frac{i}{2}[\lambda^{ad},\omega_m^{\ bc}]\epsilon_{abcd}\,. \tag{F.24}$$

Similar procedure is followed for the calculation of the transformation rules of the 2-form component gauge fields:

$$\delta B_{mn} = -i[\Theta_{mn}, \epsilon_0] - i[B_{mn}, \epsilon_0] + i[\xi^a, B_{mn}{}^a] + i[\tilde{\epsilon}_0, \tilde{B}_{mn}] + \frac{i}{2}[\lambda_{ab}, B_{mn}{}^{ab}] + \frac{i}{2}[\tilde{\xi}_a, \tilde{B}_{mn}{}^a] \tag{F.25}$$

$$\delta \tilde{B}_{mn} = -i[\Theta_{mn}, \tilde{\epsilon}_0] - i[B_{mn}, \tilde{\epsilon}_0] + i[\epsilon_0, \tilde{B}_{mn}] + \{\xi_a, \tilde{B}_{mn}{}^a\} - \{\tilde{\xi}_a, B_{mn}{}^a\} + \frac{i}{2}[\lambda^{ab}, B_{mn}{}^{bc}]\epsilon_{abcd} \tag{F.26}$$

$$\begin{aligned}\delta \tilde{B}_{mn}{}^a &= -i[\Theta_{mn}, \tilde{\xi}^a] - i[B_{mn}, \tilde{\xi}^a] + i[\epsilon_0, \tilde{B}_{mn}{}^a] - \{\xi^a, \tilde{B}_{mn}\} + \{\tilde{\epsilon}_0, B_{mn}{}^a\} + \frac{1}{4}\{\lambda^a{}_b, \tilde{B}_{mn}{}^b\} \\ &\quad - \frac{1}{4}\{\tilde{\xi}_b, B_{mn}{}^{ab}\} + i[\xi^c, B_{mn}^{cd}]\epsilon_{abcd} - i[\lambda^{cd}, B_{mn}{}^b]\epsilon_{abcd}\end{aligned} \tag{F.27}$$

$$\begin{aligned}\delta B_{mn}{}^a &= -i[\Theta_{mn}, \xi^a] - i[B_{mn}, \xi^a] + i[\epsilon_0, B_{mn}{}^a] - \{\xi_b, B_{mn}{}^{ab}\} - 2\{\tilde{\epsilon}_0, \tilde{B}_{mn}{}^a\} + \frac{1}{2}\{\lambda^a{}_b, B_{mn}{}^b\} \\ &\quad + \{\tilde{\xi}^a, \tilde{B}_{mn}\} + \frac{i}{2}[\lambda^{bc}, \tilde{B}_{mn}{}^d]\epsilon_{abcd} + i[\tilde{\xi}^b, B_{mn}{}^{cd}]\epsilon_{abcd}\end{aligned} \tag{F.28}$$

$$\begin{aligned}\delta B_{mn}{}^{ab} &= -i[\Theta_{mn}, \lambda^{ab}] - i[B_{mn}, \lambda^{ab}] + i[\epsilon_0, B_{mn}{}^{ab}] - 2\{\xi^a, B_{mn}{}^a\} - \frac{1}{2}\{\lambda^a{}_c, B_{mn}{}^{bc}\} - \frac{1}{2}\{\tilde{\xi}^a, \tilde{B}_{mn}{}^b\} \\ &\quad + i[\xi^c, \tilde{B}_{mn}{}^d]\epsilon_{abcd} + \frac{i}{2}[\tilde{\epsilon}_0, B_{mn}{}^{cd}]\epsilon_{abcd} + \frac{i}{2}[\lambda^{cd}, \tilde{B}_{mn}] - [\tilde{\xi}^c, B_{mn}{}^d]\epsilon_{abcd}\,.\end{aligned} \tag{F.29}$$

Also, we do the same for the component curvatures. The SO(4)-notation expressions of the component tensors of $\mathcal{R}_{mn}$ of (6.14) are obtained starting from (F.15):

$$\begin{aligned}R_{mn} &= [X_m, a_n] - [X_n, a_m] + [a_m, a_n] + [b_m^a, b_{na}] + [\tilde{a}_m, \tilde{a}_n] \\ &\quad + \frac{1}{2}[\omega_m{}^{ab}, \omega_{nab}] + [e_{ma}, e_n{}^a] - \frac{i\hbar}{\lambda^2}B_{mn}\end{aligned} \tag{F.30}$$

$$\begin{aligned}\tilde{R}_{mn} &= [X_m, \tilde{a}_n] + [a_m, \tilde{a}_n] - [X_n, \tilde{a}_m] - [a_n, \tilde{a}_m] - i\{b_{ma}, e_n{}^a\} + i\{b_{na}, e_m{}^a\} \\ &\quad + \frac{1}{2}\epsilon_{abcd}[\omega_m{}^{ab}, \omega_n{}^{cd}] - \frac{i\hbar}{\lambda^2}\tilde{B}_{mn}\end{aligned} \tag{F.31}$$

$$\begin{aligned}R_{mn}{}^a &= [X_m, b_n^a] + [a_m, b_n^a] - [X_n, b_m^a] - [a_n, b_m^a] + i\{b_{mb}, \omega_m{}^{ab}\} - i\{b_{nb}, \omega_m{}^{ab}\} \\ &\quad + i\{\tilde{a}_m, e_n{}^a\} - i\{\tilde{a}_n, e_m^a\} + \epsilon_{abcd}([e_m^b, \omega_n{}^{cd}] - [e_n{}^b, \omega_m{}^{cd}]) - \frac{i\hbar}{\lambda^2}B_{mn}{}^a\end{aligned} \tag{F.32}$$

$$\begin{aligned}\tilde{R}_{mn}{}^a &= [X_m, e_n^a] + [a_m, e_n^a] - [X_n, e_m^a] - [a_n, e_m^a] + i\{b_m^a, \tilde{a}_n\} - i\{b_n{}^a, \tilde{a}_m\} \\ &\quad - ([b_m^b, \omega_n{}^{cd}] - [b_n{}^b, \omega_m{}^{cd}])\epsilon_{abcd} - i\{\omega_m{}^{ab}, e_{nb}\} + i\{\omega_n{}^{ab}, e_{mb}\} - \frac{i\hbar}{\lambda^2}\tilde{B}_{mn}{}^a\end{aligned} \tag{F.33}$$

$$\begin{aligned}R_{mn}{}^{ab} &= [X_m, \omega_n{}^{ab}] + [a_m, \omega_n{}^{ab}] - [X_n, \omega_m{}^{ab}] - [a_n, \omega_m{}^{ab}] + 2i\{b_m^a, b_n^b\} + ([b_m^c, e_n{}^d] - [b_n{}^c, e_m^d])\epsilon_{abcd} \\ &\quad + \frac{1}{2}([\tilde{a}_m, \omega_n{}^{cd}] - [\tilde{a}_n, \omega_m{}^{cd}])\epsilon_{abcd} + 2i\{\omega_m{}^{ac}, \omega_n{}^b{}_c\} + 2i\{e_m^a, e_n{}^b\} - \frac{i\hbar}{\lambda^2}B_{mn}{}^{ab}\end{aligned} \tag{F.34}$$

# Bibliography


[1] R. Utiyama, “Invariant theoretical interpretation of interaction,” Phys. Rev. **101** (1956) 1597. doi:10.1103/PhysRev.101.1597

[2] T. W. B. Kibble, “Lorentz invariance and the gravitational field,” J. Math. Phys. **2** (1961) 212. doi:10.1063/1.1703702

[3] K. S. Stelle and P. C. West, “Spontaneously Broken De Sitter Symmetry and the Gravitational Holonomy Group,” Phys. Rev. D **21** (1980) 1466. doi:10.1103/PhysRevD.21.1466

[4] T. W. B. Kibble and K. S. Stelle, “Gauge theories of gravity and supergravity,” In Ezawa, H. ( Ed.), Kamefuchi, S. ( Ed.): Progress In Quantum Field Theory, 57-81.

[5] S. W. MacDowell and F. Mansouri, “Unified Geometric Theory of Gravity and Supergravity,” Phys. Rev. Lett. **38** (1977) 739 Erratum: [Phys. Rev. Lett. **38** (1977) 1376]. doi:10.1103/PhysRevLett.38.1376, 10.1103/PhysRevLett.38.739

[6] E. A. Ivanov and J. Niederle, Conference: C80-06-23.3, p.545-551, 1980; “On Gauge Formulations Of Gravitation Theories.,” E. A. Ivanov and J. Niederle, “Gauge Formulation of Gravitation Theories. 1. The Poincare, De Sitter and Conformal Cases,” Phys. Rev. D **25** (1982) 976. doi:10.1103/PhysRevD.25.976; E. A. Ivanov and J. Niederle, “Gauge Formulation of Gravitation Theories. 2. The Special Conformal Case,” Phys. Rev. D **25** (1982) 988. doi:10.1103/PhysRevD.25.988

[7] M. Kaku, P. K. Townsend and P. van Nieuwenhuizen, “Gauge Theory of the Conformal and Superconformal Group,” Phys. Lett. **69B** (1977) 304. doi:10.1016/0370-2693(77)90552-4

[8] E. S. Fradkin and A. A. Tseytlin, “Conformal Supergravity,” Phys. Rept. **119** (1985) 233. doi:10.1016/0370-1573(85)90138-3

[9] D. Z. Freedman and A. Van Proeyen “Supergravity,” Cambridge University Press, 2012

[10] E. Witten, “(2+1)-Dimensional Gravity as an Exactly Soluble System,” Nucl. Phys. B **311** (1988) 46. doi:10.1016/0550-3213(88)90143-5

[11] R. J. Szabo, “Quantum field theory on noncommutative spaces,” Phys. Rept. **378** (2003) 207 doi:10.1016/S0370-1573(03)00059-0 [hep-th/0109162].

[12] H. S. Snyder, Phys. Rev. **71** (1947) 38. doi:10.1103/PhysRev.71.38

[13] C. N. Yang, Phys. Rev. **72** (1947) 874. doi:10.1103/PhysRev.72.874

[14] A. Connes, , Inst. Hautes ´ Etudes Sci. Publ. Math. **62** (1985) 257; S.L. Woronowicz, “Twisted SU(2) Group: An Example of a Noncommutative Differential Calculus” , Publ. Res. Inst. Math. Sci. **23** (1987) 117; , Commun. Math. Phys. **111** (1987) 613.

[15] A. Connes and M. A. Rieffel, “Yang-Mills for noncommutative two-tori,” Contemp. Math. **62** (1987) 237.

[16] G. Landi, “An Introduction to noncommutative spaces and their geometry,” Lect. Notes Phys. Monogr. **51** (1997) 1 doi:10.1007/3-540-14949-X [hep-th/9701078].

[17] J. Madore, “An introduction to noncommutative differential geometry and its physical applications,” Lond. Math. Soc. Lect. Note Ser. **257** (2000) 1.

[18] J. M. Gracia-Bondia, J. C. Varilly and H. Figueroa, “Elements of noncommutative geometry,” Boston, USA: Birkhaeuser (2001) 685 p

[19] T. Filk, “Divergencies in a field theory on quantum space,” Phys. Lett. B **376** (1996) 53. doi:10.1016/0370-2693(96)00024-X

[20] M. Chaichian, A. Demichev and P. Presnajder, “Quantum field theory on noncommutative space-times and the persistence of ultraviolet divergences,” Nucl. Phys. B **567** (2000) 360 doi:10.1016/S0550-3213(99)00664-1 [hep-th/9812180].

[21] S. Minwalla, M. Van Raamsdonk and N. Seiberg, “Noncommutative perturbative dynamics,” JHEP **0002** (2000) 020 doi:10.1088/1126-6708/2000/02/020 [hep-th/9912072].

[22] H. Grosse and R. Wulkenhaar, “Renormalization of phi**4 theory on noncommutative R**4 to all orders,” Lett. Math. Phys. **71** (2005) 13 doi:10.1007/s11005-004-5116-3 [hep-th/0403232].

[23] H. Grosse and H. Steinacker, “Exact renormalization of a noncommutative phi**3 model in 6 dimensions,” Adv. Theor. Math. Phys. **12** (2008) no.3, 605 doi:10.4310/ATMP.2008.v12.n3.a4 [hep-th/0607235].

[24] H. Grosse and H. Steinacker, “Finite gauge theory on fuzzy CP**2,” Nucl. Phys. B **707** (2005) 145 doi:10.1016/j.nuclphysb.2004.11.058 [hep-th/0407089].

[25] A. H. Chamseddine and A. Connes, “Conceptual Explanation for the Algebra in the Noncommutative Approach to the Standard Model,” Phys. Rev. Lett. **99** (2007) 191601 doi:10.1103/PhysRevLett.99.191601 [arXiv:0706.3690 [hep-th]].

[26] M. Dubois-Violette, R. Kerner and J. Madore, “Noncommutative Differential Geometry of Matrix Algebras,” J. Math. Phys. **31** (1990) 316. doi:10.1063/1.528916

[27] A. Connes, M. R. Douglas and A. S. Schwarz, “Noncommutative geometry and matrix theory: Compactification on tori,” JHEP **9802** (1998) 003 doi:10.1088/1126-6708/1998/02/003 [hep-th/9711162].

[28] B. Jurco, S. Schraml, P. Schupp and J. Wess, “Enveloping algebra valued gauge transformations for nonAbelian gauge groups on noncommutative spaces,” Eur. Phys. J. C **17** (2000) 521 doi:10.1007/s100520000487 [hep-th/0006246].

[29] M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari and A. Tureanu, "Noncommutative standard model: Model building," Eur. Phys. J. C **29** (2003) 413 doi:10.1140/epjc/s2003-01204-7 [hep-th/0107055].

[30] X. Calmet, B. Jurco, P. Schupp, J. Wess and M. Wohlgenannt, "The Standard model on noncommutative space-time," Eur. Phys. J. C **23** (2002) 363 doi:10.1007/s100520100873 [hep-ph/0111115].

[31] P. Aschieri, B. Jurco, P. Schupp and J. Wess, "Noncommutative GUTs, standard model and C,P,T," Nucl. Phys. B **651** (2003) 45 doi:10.1016/S0550-3213(02)00937-9 [hep-th/0205214].

[32] W. Behr, N. G. Deshpande, G. Duplancic, P. Schupp, J. Trampetic and J. Wess, "The Z —> gamma gamma, g g decays in the noncommutative standard model," Eur. Phys. J. C **29** (2003) 441 doi:10.1140/epjc/s2003-01207-4 [hep-ph/0202121].

[33] J. Madore, S. Schraml, P. Schupp and J. Wess, "Gauge theory on noncommutative spaces," Eur. Phys. J. C **16** (2000) 161 doi:10.1007/s100520050012 [hep-th/0001203].

[34] A. H. Chamseddine, "Deforming Einstein's gravity," Phys. Lett. B **504** (2001) 33 doi:10.1016/S0370-2693(01)00272-6 [hep-th/0009153].

[35] A. H. Chamseddine, "SL(2,C) gravity with complex vierbein and its noncommutative extension," Phys. Rev. D **69** (2004) 024015 doi:10.1103/PhysRevD.69.024015 [hep-th/0309166].

[36] P. Aschieri and L. Castellani, "Noncommutative D=4 gravity coupled to fermions," JHEP **0906** (2009) 086 doi:10.1088/1126-6708/2009/06/086 [arXiv:0902.3817 [hep-th]].

[37] P. Aschieri and L. Castellani, "Noncommutative supergravity in D=3 and D=4," JHEP **0906** (2009) 087 doi:10.1088/1126-6708/2009/06/087 [arXiv:0902.3823 [hep-th]].

[38] M. Dimitrijević Ćirić, B. Nikolić and V. Radovanović, "Noncommutative $SO(2,3)_\star$ gravity: Noncommutativity as a source of curvature and torsion," Phys. Rev. D **96** (2017) no.6, 064029 doi:10.1103/PhysRevD.96.064029 [arXiv:1612.00768 [hep-th]].

[39] S. Cacciatori, D. Klemm, L. Martucci and D. Zanon, "Noncommutative Einstein-AdS gravity in three-dimensions," Phys. Lett. B **536** (2002) 101 doi:10.1016/S0370-2693(02)01823-3 [hep-th/0201103].

[40] S. Cacciatori, A. H. Chamseddine, D. Klemm, L. Martucci, W. A. Sabra and D. Zanon, "Noncommutative gravity in two dimensions," Class. Quant. Grav. **19** (2002) 4029 doi:10.1088/0264-9381/19/15/310 [hep-th/0203038].

[41] P. Aschieri and L. Castellani, "Noncommutative Chern-Simons gauge and gravity theories and their geometric Seiberg-Witten map," JHEP **1411** (2014) 103 doi:10.1007/JHEP11(2014)103 [arXiv:1406.4896 [hep-th]].

[42] N. Seiberg and E. Witten, "String theory and noncommutative geometry," JHEP **9909** (1999) 032 doi:10.1088/1126-6708/1999/09/032 [hep-th/9908142].

[43] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M theory as a matrix model: A Conjecture,” Phys. Rev. D **55** (1997) 5112 doi:10.1103/PhysRevD.55.5112 [hep-th/9610043].

[44] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, “A Large N reduced model as superstring,” Nucl. Phys. B **498** (1997) 467 doi:10.1016/S0550-3213(97)00290-3 [hep-th/9612115].

[45] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, “Space-time structures from IIB matrix model,” Prog. Theor. Phys. **99** (1998) 713 doi:10.1143/PTP.99.713 [hep-th/9802085].

[46] M. Hanada, H. Kawai and Y. Kimura, “Describing curved spaces by matrices,” Prog. Theor. Phys. **114** (2006) 1295 doi:10.1143/PTP.114.1295 [hep-th/0508211].

[47] K. Furuta, M. Hanada, H. Kawai and Y. Kimura, “Field equations of massless fields in the new interpretation of the matrix model,” Nucl. Phys. B **767** (2007) 82 doi:10.1016/j.nuclphysb.2007.01.003 [hep-th/0611093].

[48] H. S. Yang, “Emergent Gravity from Noncommutative Spacetime,” Int. J. Mod. Phys. A **24** (2009) 4473 doi:10.1142/S0217751X0904587X [hep-th/0611174].

[49] H. Steinacker, “Emergent Geometry and Gravity from Matrix Models: an Introduction,” Class. Quant. Grav. **27** (2010) 133001 doi:10.1088/0264-9381/27/13/133001 [arXiv:1003.4134 [hep-th]].

[50] S. W. Kim, J. Nishimura and A. Tsuchiya, “Expanding (3+1)-dimensional universe from a Lorentzian matrix model for superstring theory in (9+1)-dimensions,” Phys. Rev. Lett. **108** (2012) 011601 doi:10.1103/PhysRevLett.108.011601 [arXiv:1108.1540 [hep-th]].

[51] J. Nishimura, “The Origin of space-time as seen from matrix model simulations,” PTEP **2012** (2012) 01A101 doi:10.1093/ptep/pts004 [arXiv:1205.6870 [hep-lat]].

[52] V. P. Nair, “Gravitational fields on a noncommutative space,” Nucl. Phys. B **651** (2003) 313 doi:10.1016/S0550-3213(02)01061-1 [hep-th/0112114].

[53] Y. Abe and V. P. Nair, “Noncommutative gravity: Fuzzy sphere and others,” Phys. Rev. D **68** (2003) 025002 doi:10.1103/PhysRevD.68.025002 [hep-th/0212270].

[54] P. Valtancoli, “Gravity on a fuzzy sphere,” Int. J. Mod. Phys. A **19** (2004) 361 doi:10.1142/S0217751X04017598 [hep-th/0306065].

[55] V. P. Nair, “The Chern-Simons one-form and gravity on a fuzzy space,” Nucl. Phys. B **750** (2006) 321 doi:10.1016/j.nuclphysb.2006.06.009 [hep-th/0605008].

[56] M. Burić, T. Grammatikopoulos, J. Madore and G. Zoupanos, “Gravity and the structure of noncommutative algebras,” JHEP **0604** (2006) 054 doi:10.1088/1126-6708/2006/04/054 [hep-th/0603044].

[57] M. Burić, J. Madore and G. Zoupanos, “WKB Approximation in Noncommutative Gravity,” SIGMA **3** (2007) 125 doi:10.3842/SIGMA.2007.125 [arXiv:0712.4024 [hep-th]].

[58] M. Burić, J. Madore and G. Zoupanos, "The Energy-momentum of a Poisson structure," Eur. Phys. J. C **55** (2008) 489 doi:10.1140/epjc/s10052-008-0602-x [arXiv:0709.3159 [hep-th]].

[59] J. Lukierski, H. Ruegg, A. Nowicki and V. N. Tolstoi, "Q deformation of Poincare algebra," Phys. Lett. B **264** (1991) 331. doi:10.1016/0370-2693(91)90358-W

[60] J. Lukierski, A. Nowicki and H. Ruegg, "New quantum Poincare algebra and k deformed field theory," Phys. Lett. B **293** (1992) 344. doi:10.1016/0370-2693(92)90894-A

[61] S. W. Kim, J. Nishimura and A. Tsuchiya, "Expanding universe as a classical solution in the Lorentzian matrix model for nonperturbative superstring theory," Phys. Rev. D **86** (2012) 027901 doi:10.1103/PhysRevD.86.027901 [arXiv:1110.4803 [hep-th]].

[62] J. Heckman and H. Verlinde, "Covariant non-commutative space–time," Nucl. Phys. B **894** (2015) 58 doi:10.1016/j.nuclphysb.2015.02.018 [arXiv:1401.1810 [hep-th]].

[63] M. Burić and J. Madore, "Noncommutative de Sitter and FRW spaces," Eur. Phys. J. C **75** (2015) no.10, 502 doi:10.1140/epjc/s10052-015-3729-6 [arXiv:1508.06058 [hep-th]].

[64] M. Sperling and H. C. Steinacker, "Covariant 4-dimensional fuzzy spheres, matrix models and higher spin," J. Phys. A **50** (2017) no.37, 375202 doi:10.1088/1751-8121/aa8295 [arXiv:1704.02863 [hep-th]].

[65] M. Burić, D. Latas and L. Nenadović, "Fuzzy de Sitter Space," arXiv:1709.05158 [hep-th].

[66] H. C. Steinacker, "Emergent gravity on covariant quantum spaces in the IKKT model," JHEP **1612** (2016) 156 doi:10.1007/JHEP12(2016)156 [arXiv:1606.00769 [hep-th]].

[67] P. Aschieri, J. Madore, P. Manousselis and G. Zoupanos, "Dimensional reduction over fuzzy coset spaces," JHEP **0404** (2004) 034 doi:10.1088/1126-6708/2004/04/034 [hep-th/0310072].

[68] P. Aschieri, J. Madore, P. Manousselis and G. Zoupanos, "Unified theories from fuzzy extra dimensions," Fortsch. Phys. **52** (2004) 718 doi:10.1002/prop.200410168 [hep-th/0401200].

[69] P. Aschieri, T. Grammatikopoulos, H. Steinacker and G. Zoupanos, "Dynamical generation of fuzzy extra dimensions, dimensional reduction and symmetry breaking," JHEP **0609** (2006) 026 doi:10.1088/1126-6708/2006/09/026 [hep-th/0606021].

[70] P. Aschieri, H. Steinacker, J. Madore, P. Manousselis and G. Zoupanos, "Fuzzy extra dimensions: Dimensional reduction, dynamical generation and renormalizability," SFIN A **1** (2007) 25 [arXiv:0704.2880 [hep-th]].

[71] H. Steinacker and G. Zoupanos, "Fermions on spontaneously generated spherical extra dimensions," JHEP **0709** (2007) 017 doi:10.1088/1126-6708/2007/09/017 [arXiv:0706.0398 [hep-th]].

[72] A. Chatzistavrakidis, H. Steinacker and G. Zoupanos, "On the fermion spectrum of spontaneously generated fuzzy extra dimensions with fluxes," Fortsch. Phys. **58** (2010) 537 doi:10.1002/prop.201000018 [arXiv:0909.5559 [hep-th]].

[73] A. Chatzistavrakidis, H. Steinacker and G. Zoupanos, “Orbifolds, fuzzy spheres and chiral fermions,” JHEP **1005** (2010) 100 doi:10.1007/JHEP05(2010)100 [arXiv:1002.2606 [hep-th]].

[74] A. Chatzistavrakidis and G. Zoupanos, “Higher-Dimensional Unified Theories with Fuzzy Extra Dimensions,” SIGMA **6** (2010) 063 doi:10.3842/SIGMA.2010.063 [arXiv:1008.2049 [hep-th]].

[75] D. Gavriil, G. Manolakos, G. Orfanidis and G. Zoupanos, “Higher-Dimensional Unification with continuous and fuzzy coset spaces as extra dimensions,” Fortsch. Phys. **63** (2015) 442 doi:10.1002/prop.201500022 [arXiv:1504.07276 [hep-th]].

[76] G. Manolakos and G. Zoupanos, “Higher-Dimensional Unified Theories with continuous and fuzzy coset spaces as extra dimensions,” Springer Proc. Math. Stat. **191** (2016) 203 doi:10.1007/978-981-10-2636-2 13 [arXiv:1602.03673 [hep-th]].

[77] G. Manolakos and G. Zoupanos, “The trinification model $SU(3)^3$ from orbifolds for fuzzy spheres,” Phys. Part. Nucl. Lett. **14** (2017) no.2, 322. doi:10.1134/S1547477117020194

[78] G. Manolakos, P. Manousselis and G. Zoupanos, “Four-dimensional Gravity on a Covariant Noncommutative Space,” arXiv:1902.10922 [hep-th].

[79] A. Chatzistavrakidis, L. Jonke, D. Jurman, G. Manolakos, P. Manousselis and G. Zoupanos, “Noncommutative Gauge Theory and Gravity in Three Dimensions,” Fortsch. Phys. **66** (2018) no.8-9, 1800047 doi:10.1002/prop.201800047 [arXiv:1802.07550 [hep-th]].

[80] D. Jurman, G. Manolakos, P. Manousselis and G. Zoupanos, “Gravity as a Gauge Theory on Three-Dimensional Noncommutative spaces,” PoS CORFU **2017** (2018) 162 doi:10.22323/1.318.0162 [arXiv:1809.03879 [gr-qc]].

[81] G. Manolakos and G. Zoupanos, “Non-commutativity in Unified Theories and Gravity,” Springer Proc. Math. Stat. **263** (2017) 177 doi:10.1007/978-981-13-2715-5-10 [arXiv:1809.02954 [hep-th]].

[82] J. Yepez, “Einstein’s vierbein field theory of curved space,” arXiv:1106.2037 [gr-qc].

[83] S. M. Carroll, “Spacetime and geometry: An introduction to general relativity,” San Francisco, USA: Addison-Wesley (2004) 513 p

[84] P. Ramond, “Field Theory. A Modern Primer,” Front. Phys. **51** (1981) 1 [Front. Phys. **74** (1989) 1].

[85] L. Alvarez-Gaume, A. Kehagias, C. Kounnas, D. Lüst and A. Riotto, “Aspects of Quadratic Gravity,” Fortsch. Phys. **64** (2016) no.2-3, 176 doi:10.1002/prop.201500100 [arXiv:1505.07657 [hep-th]].

[86] A. H. Chamseddine, “Supersymmetry and higher spin fields”, PhD Thesis, (1976)

[87] A. H. Chamseddine and P. C. West, Nucl. Phys. B **129** (1977) 39. doi:10.1016/0550-3213(77)90018-9

[88] L. F. Li, "Group Theory of the Spontaneously Broken Gauge Symmetries," Phys. Rev. D **9** (1974) 1723. doi:10.1103/PhysRevD.9.1723

[89] A. H. Chamseddine, "Invariant actions for noncommutative gravity," J. Math. Phys. **44** (2003) 2534 doi:10.1063/1.1572199 [hep-th/0202137].

[90] M. R. Douglas and N. A. Nekrasov, "Noncommutative field theory," Rev. Mod. Phys. **73** (2001) 977 doi:10.1103/RevModPhys.73.977 [hep-th/0106048].

[91] A. B. Hammou, M. Lagraa and M. M. Sheikh-Jabbari, "Coherent state induced star product on R**3(lambda) and the fuzzy sphere," Phys. Rev. D **66** (2002) 025025 doi:10.1103/PhysRevD.66.025025 [hep-th/0110291].

[92] F. Lizzi and P. Vitale, "Matrix Bases for Star Products: a Review," SIGMA **10** (2014) 086 doi:10.3842/SIGMA.2014.086 [arXiv:1403.0808 [hep-th]].

[93] J. Madore, "The Fuzzy sphere," Class. Quant. Grav. **9** (1992) 69. doi:10.1088/0264-9381/9/1/008

[94] R. P. Andrews and N. Dorey, "Deconstruction of the Maldacena-Nunez compactification," Nucl. Phys. B **751** (2006) 304 doi:10.1016/j.nuclphysb.2006.06.013 [hep-th/0601098].

[95] J. C. Wallet, "Exact partition functions for gauge theories on $\mathbb{R}^3_\lambda$," Nucl. Phys. B **912**, 354 (2016) doi:10.1016/j.nuclphysb.2016.04.001 [arXiv:1603.05045 [math-ph]].

[96] P. Vitale and J. C. Wallet, "Noncommutative field theories on $R^3_\lambda$: Toward UV/IR mixing freedom," JHEP **1304** (2013) 115 Addendum: [JHEP **1503** (2015) 115] doi:10.1007/JHEP04(2013)115, 10.1007/JHEP03(2015)115 [arXiv:1212.5131 [hep-th]].

[97] P. Vitale, "Noncommutative field theory on $\mathbb{R}^3_\lambda$, Fortsch. Phys. **62** (2014) 825 doi:10.1002/prop.201400037 [arXiv:1406.1372 [hep-th]].

[98] H. Hopf, "Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche" H. Math. Ann. (1931) 104: 637. https://doi.org/10.1007/BF01457962

[99] H. Hopf "Über die Abbildungen von Sphären auf Spären Niedrigerer Dimension", Fundamenta Mathematicae 25:427 (1935) DOI: 10.1007/978-3-662-25046-4 7

[100] D. Jurman and H. Steinacker, "2D fuzzy Anti-de Sitter space from matrix models," JHEP **1401** (2014) 100 doi:10.1007/JHEP01(2014)100 [arXiv:1309.1598 [hep-th]].

[101] B. G. Wybourne, "Classical Groups for Physicists," John Wiley & Sons, London (1974).

[102] A. Singh and S. M. Carroll, "Modeling Position and Momentum in Finite-Dimensional Hilbert Spaces via Generalized Clifford Algebra," arXiv:1806.10134 [quant-ph].

[103] A. Barut, " From Heisenberg algebra to Conformal Dynamical Group ' ' in A. Barut, H. D. Doener (Eds) "Conformal Groups and related Symmetries.Physical Results and Mathematical Backgrounf " Lecture Notes in Physics, Springer-Verlag 1985

[104] S. Kováčik and P. Prešnajder, "The velocity operator in quantum mechanics in noncommutative space," J. Math. Phys. **54** (2013) 102103 doi:10.1063/1.4826355 [arXiv:1309.4592 [math-ph]].

[105] M. Banados, O. Chandia, N. E. Grandi, F. A. Schaposnik and G. A. Silva, "Three-dimensional noncommutative gravity," Phys. Rev. D **64** (2001) 084012 doi:10.1103/PhysRevD.64.084012 [hep-th/0104264].

[106] A. Van Proeyen, "Tools for supersymmetry," Ann. U. Craiova Phys. **9** (1999) no.I, 1 [hep-th/9910030].

[107] P. Aschieri and L. Castellani, "Noncommutative gauge fields coupled to noncommutative gravity," Gen. Rel. Grav. **45** (2013) 581 doi:10.1007/s10714-012-1488-3 [arXiv:1205.1911 [hep-th]].

[108] A. Géré, P. Vitale and J. C. Wallet, "Quantum gauge theories on noncommutative three-dimensional space," Phys. Rev. D **90** (2014) no.4, 045019 doi:10.1103/PhysRevD.90.045019 [arXiv:1312.6145 [hep-th]].

[109] A. Y. Alekseev, A. Recknagel and V. Schomerus, "Brane dynamics in background fluxes and noncommutative geometry," JHEP **0005** (2000) 010 doi:10.1088/1126-6708/2000/05/010 [hep-th/0003187].

[110] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology," Cambridge, Uk: Univ. Pr. ( 1987) 596 P. ( Cambridge Monographs On Mathematical Physics)

[111] D. Harland and S. Kurkcuoglu, "Equivariant reduction of Yang-Mills theory over the fuzzy sphere and the emergent vortices," Nucl. Phys. B **821** (2009) 380 doi:10.1016/j.nuclphysb.2009.06.031 [arXiv:0905.2338 [hep-th]].

[112] L. Brink, J. H. Schwarz and J. Scherk, "Supersymmetric Yang-Mills Theories," Nucl. Phys. B **121** (1977) 77. doi:10.1016/0550-3213(77)90328-5

[113] N. Arkani-Hamed, A. G. Cohen and H. Georgi, "Electroweak symmetry breaking from dimensional deconstruction," Phys. Lett. B **513** (2001) 232 doi:10.1016/S0370-2693(01)00741-9 [hep-ph/0105239].

[114] N. Arkani-Hamed, A. G. Cohen and H. Georgi, "(De)constructing dimensions," Phys. Rev. Lett. **86** (2001) 4757 doi:10.1103/PhysRevLett.86.4757 [hep-th/0104005].

[115] J. Maalampi and M. Roos, "Physics of Mirror Fermions," Phys. Rept. **186** (1990) 53. doi:10.1016/0370-1573(90)90095-J.

[116] S. Kachru and E. Silverstein, "4-D conformal theories and strings on orbifolds," Phys. Rev. Lett. **80** (1998) 4855 doi:10.1103/PhysRevLett.80.4855 [hep-th/9802183].

[117] M. R. Douglas, B. R. Greene and D. R. Morrison, "Orbifold resolution by D-branes," Nucl. Phys. B **506** (1997) 84 doi:10.1016/S0550-3213(97)00517-8 [hep-th/9704151].

[118] D. Bailin and A. Love, “Orbifold compactifications of string theory,” Phys. Rept. **315** (1999) 285. doi:10.1016/S0370-1573(98)00126-4

[119] G. Aldazabal, L. E. Ibanez, F. Quevedo and A. M. Uranga, “D-branes at singularities: A Bottom up approach to the string embedding of the standard model,” JHEP **0008** (2000) 002 doi:10.1088/1126-6708/2000/08/002 [hep-th/0005067].

[120] A. E. Lawrence, N. Nekrasov and C. Vafa, “On conformal field theories in four-dimensions,” Nucl. Phys. B **533** (1998) 199 doi:10.1016/S0550-3213(98)00495-7 [hep-th/9803015].

[121] E. Kiritsis, “D-branes in standard model building, gravity and cosmology,” Phys. Rept. **421** (2005) 105 Erratum: [Phys. Rept. **429** (2006) 121] doi:10.1016/j.physrep.2005.09.001 [hep-th/0310001].

[122] P. Manousselis and G. Zoupanos, “Soft supersymmetry breaking due to dimensional reduction over nonsymmetric coset spaces,” Phys. Lett. B **518** (2001) 171 doi:10.1016/S0370-2693(01)01040-1 [hep-ph/0106033];

[123] P. Manousselis and G. Zoupanos, “Supersymmetry breaking by dimensional reduction over coset spaces,” Phys. Lett. B **504** (2001) 122 doi:10.1016/S0370-2693(01)00268-4 [hep-ph/0010141].

[124] P. Manousselis and G. Zoupanos, “Dimensional reduction of ten-dimensional supersymmetric gauge theories in the N=1, D=4 superfield formalism,” JHEP **0411** (2004) 025 doi:10.1088/1126-6708/2004/11/025 [hep-ph/0406207].

[125] M. F. Sohnius, “Introducing Supersymmetry,” Phys. Rept. **128** (1985) 39. doi:10.1016/0370-1573(85)90023-7

[126] A. Djouadi, “The Anatomy of electro-weak symmetry breaking. II. The Higgs bosons in the minimal supersymmetric model,” Phys. Rept. **459** (2008) 1 doi:10.1016/j.physrep.2007.10.005 [hep-ph/0503173].

[127] H. Steinacker, “Gauge theory on the fuzzy sphere and random matrices,” Springer Proc. Phys. **98** (2005) 307 doi:10.1007/3-540-26798-0 31 [hep-th/0409235].

[128] H. Grosse, F. Lizzi and H. Steinacker, “Noncommutative gauge theory and symmetry breaking in matrix models,” Phys. Rev. D **81** (2010) 085034 doi:10.1103/PhysRevD.81.085034 [arXiv:1001.2703 [hep-th]];

[129] S. L. Glashow, “Trinification of All Elementary Particle Forces,” Print-84-0577 (BOSTON).

[130] V. A. Rizov, “A Gauge Model of the Electroweak and Strong Interactions Based on the Group SU(3)$_L$ X SU(3)$_R$ X SU(3)$_c$,” Bulg. J. Phys. **8** (1981) 461.

[131] S. Heinemeyer, E. Ma, M. Mondragon and G. Zoupanos, “Finite SU(3)**3 model,” AIP Conf. Proc. **1200** (2010) no.1, 568 doi:10.1063/1.3327674 [arXiv:0910.0501 [hep-ph]].

[132] E. Ma, M. Mondragon and G. Zoupanos, “Finite SU(N)**k unification,” JHEP **0412** (2004) 026 doi:10.1088/1126-6708/2004/12/026 [hep-ph/0407236].

[133] G. Lazarides and C. Panagiotakopoulos, "MSSM from SUSY trinification," Phys. Lett. B **336** (1994) 190 doi:10.1016/0370-2693(94)00925-2 [hep-ph/9403317].

[134] K. S. Babu, X. G. He and S. Pakvasa, "Neutrino Masses and Proton Decay Modes in SU(3) X SU(3) X SU(3) Trinification," Phys. Rev. D **33** (1986) 763. doi:10.1103/PhysRevD.33.763

[135] G. K. Leontaris and J. Rizos, "A D-brane inspired U(3)(C) x U(3)(L) x U(3)(R) model," Phys. Lett. B **632** (2006) 710 doi:10.1016/j.physletb.2005.11.045 [hep-ph/0510230].

[136] J. E. Kim, "Z(3) orbifold construction of SU(3)**3 GUT with sin**2 (theta0(W)) = 3/8," Phys. Lett. B **564** (2003) 35 doi:10.1016/S0370-2693(03)00567-7 [hep-th/0301177].

[137] S. Heinemeyer, M. Mondragon and G. Zoupanos, "The LHC Higgs boson discovery: Implications for Finite Unified Theories," Int. J. Mod. Phys. A **29** (2014) 1430032 doi:10.1142/S0217751X14300324 [arXiv:1412.5766 [hep-ph]].

[138] S. Heinemeyer, M. Mondragon, N. Tracas and G. Zoupanos, "Reduction of Couplings in Quantum Field Theories with applications in Finite Theories and the MSSM," Springer Proc. Math. Stat. **111** (2014) 177 doi:10.1007/978-4-431-55285-7-11 [arXiv:1403.7384 [hep-ph]].

[139] S. Heinemeyer, M. Mondragon and G. Zoupanos, "Finite Unification: Theory and Predictions," SIGMA **6** (2010) 049 doi:10.3842/SIGMA.2010.049 [arXiv:1001.0428 [hep-ph]].

[140] J. Schwinger, "On angular momentum", in Quantum Theory of Angular momentum, edited by L. C. Biedenharn and H. Van Dam, (Academic Press, New York, 1965).

[141] J. J. Sakurai and J. Napolitano, "Modern quantum physics,"

[142] F. M. Mejía, V. Pleitez, "Schwinger's oscillator method, supersymmetric quantum mechanics and massless particles", arXiv:physics/0207012 [physics.ed-ph].

[143] J. Stillwell, "Geometry of complex numbers and quaternions. In: Naive Lie Theory. Undergraduate Texts in Mathematics". Springer, New York, NY (2008).

[144] F. Parisio, "Coherent-state overcompleteness, path integrals, and weak values," J. Math. Phys. **57** (2016) no.3, 032101 doi:10.1063/1.4943014 [arXiv:1403.3033 [quant-ph]].

[145] L. Smolin and A. Starodubtsev, "General relativity with a topological phase: An Action principle," hep-th/0311163.