The philosophical aspect of quantum mechanics: a critical evaluation on various interpretations and problems

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2019

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1 Introduction

Quantum mechanics is a physical theory developed in the 1920s in order to describe the phenomena of matter at the atomic scale. It was an effort by several different people that were searching for a theory which could explain unsolved problems such as the spectrum of electromagnetic radiation of a hot object, the photoelectric effect, as well as the issue of why the hydrogen radiates photons of certain wavelength. From that time onwards it has been the most empirically successful theory in the history of physics ([54] p.198). However, attempts to understand such a theory are difficult due to lack of a sufficient way of interpreting the physical meaning of the mathematical symbols.

The wish and the curse of quantum mechanics comes from the fact that the state of the system is a sum of several distinct physical states that interact with each other. All of these terms contain simultaneously existing versions of reality and we grasp only one of those possible versions. Still, quantum theory treats all of them equally without telling us which of those we are going to observe. It is as if the system exists in several states; yet when someone makes a measurement the result is only one of these. We are thus left unable to understand why we perceive a particular version of all the possible ones.

The fact that measurement cannot privilege one of the terms over others, makes the problem even more challenging, because we have to find a rule that helps us determine which of those terms is more likely to exist. At this point, we need to integrate in the theory the concept of probability so as to foresee the possible outcomes of the measurement. Until now, we had been unable to understand this "rolling-the-dice" feature of the theory.

It follows from the above that is a clear need to interpret quantum mechanics. In this dissertation, I consider a number of interpretations. The first one is the *Copenhagen interpretation* , which was first proposed by Niels Bohr and Werner Heisenberg. The Copenhagen interpretation claims that the quantum state is not to be taken as a description of the physical system, but rather as a summary of what we can expect if we take measurements of the system. A way out of the multiple version of reality problem is to think that the wavefunction is not real *per se*, in a sense that because we do not actually observe a wave. The wavefunction could instead be seen as a recipe of how different realities correspond to their possibilities, with only one ever existing, the one we end up seeing.

According to another interpretation, called *hidden variable theories*, the quantum state is a partial description of the system,with the rest of the description ling in the values of the hidden variables. The hidden variables choose one from the possible physical states as the actual one. It could be said that this situation bear some similarities with statistical thermodynamics, in where large systems constituted by many particles are described in a probabilistic way yet each one of these particles behaves deterministically and according to the classical equations of motion. In other words, quantum mechanics[1] is not the

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[1]At this point the word quantum mechanics is used as the description of the micro-level

ultimate description of nature. There is a hidden reality beneath quantum mechanics, that is described by the variables of a more fundamental theory, which can always predict the outcome of a measurement with certainty.

Each interpretation portrays a different physical reality, and the question of the optimum one is still open. These realities differ from one another depending on whether quantum mechanics is deterministic or not. Two of these interpretations are described below.

2 The Copenhagen Interpretation

2.1 Introduction - The Origins

In order to understand how a theory could put forth an interpretation, we have to briefly review the history of classical mechanics. At around the end of the 17th and the beginning of the 18th century, Newton developed an empirically-sound mathematical scheme according to which atoms move through three-dimensional space. As we know well by know, the picture painted by Newton prove to be highly compatible with our everyday view of the world. Following this important theoretical breakthrough, few phenomena were left unexplained, which is why many physicists were quick to think that this was the end of the story.

While successful at describing the majority of phenomena, classical electro-dynamics yields a problematic set of predictions about the following phenomena: the photoelectric effect, the electromagnetic energy emitted in a hallow cavity, and the mechanics of the alpha particles fired at a thin gold foil. Starting with the first of these phenomena, the photoelectric effect is the emission of electrons when a beam of light shines on metal. According to the classical electromagnetic theory, the photoelectric effect can be attributed to the transfer of energy from the light to the electron. The classical theory also makes a couple of problematic, as we said above, predictions: i) that the intensity of light will be correlated with the kinetic energy of electrons; and ii) that a low intensity light will cause a time delay in the emission of electrons. As it turns out, both of these predictions are disproven by experimental data, which show the energy of the electrons depends on the frequency of the incident light alone.

Let us now consider the second of the aforementioned phenomena, black-body radiation, an important phenomenon that challenged the theoretical apparatus available to physicists at the beginning of the 20th century. A black-body is a surface that is able to absorb all the incident electromagnetic radiation, which is why the surface of this body appears black. To make such a peculiar object, we have to take any large cavity and heat it at some temperature. Then, when a thermal equilibrium is established, we can let radiation escape from a small hole on the side of the cavity. In principle, our theories should have been able to predict the spectrum of that radiation with great accuracy. Yet, classical physics fails to make such a prediction.

In order to address this theoretical shortcoming, Max Planck conceived a system as a collection of radiating harmonic oscillators in thermal equilibrium. His first try with classical electromagnetic theory failed. He then turned to thermodynamics, seeking to identify a formula that could describe the distribution of thermal energy at equilibrium. After some unsuccessful efforts, Planck decided to employ statistical-mechanical methods, according to which the total energy of the system should be seen existing in discrete amounts. What he found was that the distribution of radiation of the emitted light can be reproduced if, and only if, the oscillators absorb and emit energy in discrete packets.

In a talk given to the German Physical Society on 14 December 1900, Max

Planck proposed a law [56] describing the energy distribution in black-body radiation. According to Planck, light is electromagnetic radiation that can only be emitted or absorbed in fixed units of hf, where $h = 6.62 \times 10^{-36} \frac{m^2 kg}{sec}$ is a constant and f the frequency of light. This hypothesis suggests that light is a discrete rather than continuous quantity. In 1905, Albert Einstein [29] used the same constant in the hypothesis that light is carried by discrete quantized packets, which he advanced in order to explain the experimental data of photoelectric effect. Electrons are emitted if the incident light surpasses a threshold frequency, whereas no electrons dislodge below that frequency regardless of the intensity of the light or the time of exposure.

Planck found that the emission and absorption of energy has to be quantized, but that the method he had developed could not be applied to the energy of electromagnetic radiation itself. The reason behind this difficulty was the ability of the electromagnetic field to carry the energy of a continuously varying wave. Today, we do quantize the electromagnetic field through a method that is called *quantization*. According to this method, which is compatible with special relativity, the electromagnetic field's quanta are called *photons*. Interestingly, when radiation consists of a large amount of quanta, it behaves like a wave.

According to the idea of the quantization of energy, photons consist of a unique amount of energy which correlates with different colours in the spectrum – or otherwise seen, with different types of electromagnetic radiation. The relation of energy and frequency is given by the formula:

$$E = hf \qquad\qquad \text{quantization of energy}$$

Having said this, the time is now due to move to the third of the aforementioned phenomena – that is, the mechanics of the alpha particles fired at a thin gold foil. In 1911, Ernest Rutherford [57] suggested that there is a positive charge in the atom, which is concentrated in a nucleus large enough to reflect an alpha particle. He also proposed that the electrons orbit the nucleus at certain permissible orbits or energy levels. Given the fact that energy can only take particular values, the bouncing of electrons between these orbits can only emit certain wavelengths of light – a fact that explains why the light spectrum consists of discrete wavelengths. It should thus follows from this that, much like the phenomena previously considered, this third phenomenon can also be explained by the hypothesis about the quantization of energy.

In 1913, Niels Bohr [13] developed a theory which could reproduce Rutherford's results as well as the spectrum of the hydrogen atom. The theory was based on two strange assumptions:

1. Every atomic system has a set of states, called *stationary states*, with discrete energies.

2. The possibility for an atom to absorb or emit radiation is a function of the energy difference $h\nu$ between two states. As an electron passes from one state to another, it either emits or absorbs radiation.

Bohr's model introduced discontinuity between the electron's orbits around the nucleus. He specifically proposed that electrons cannot exist between states. An electron can be transmitted from one orbit to another, but there is no place for it between states. Electrons therefore populate the ground state that is closest to the nucleus, or an excited state if an impact forced them to leave their ground state. As Rutherford pointed out, the electron appears to "know" the final energy it is moving towards, and thus emits a photon with the appropriate frequency.

The time in which and the way through which the transition takes place is a matter of probability. There are no external or internal causes that interrupt the electron's state. Any excited electron can spontaneously move to a lower state or to the ground state. From 1913 until 1925, Bohr, Arnold Sommerfeld [59] [60] [61] [62] and others improved Bohr's model with the introduction of extra postulates, such as the exclusion principle of Wolfgang Pauli. The revised model was able to describe the basic chemical elements and shed new light on the way through which they are created.

It is important, however, to acknowledge here that the acceptance of the quantization of energy introduces as many problems as it solves. Firstly, why matter is so stable, and secondly what stops electrons from spiralling down towards the nucleus of the atom as they continuously lose energy from emitting photons? To address these issues, Louis de Broglie [24] developed a theory, in 1923, according to which particles are wave-like entities – that is, every particle follows the trajectory of the associated wave. He was committed to a continuous wave-like description of the quantum phenomena, treating the wave as a fundamental physical entity. In 1924, [25] he explained the stability problem of the atom with the same conjecture – i.e., by stating that electrons are wave-like entities whose energy is a function of their wavelength, so only certain wavelengths can fit in orbit around the nucleus without leaving a remainder .

Yet the most important idea/implication of de Broglie's theory was that the atom cannot exist in all possible states, only in a series of discrete stationary states. The energy that is emitted when an electron passes from one energy level to another is proportional to the energy difference between the two states. This difference $E_n - E_m = h\nu_{nm}$ is the light quantum. It follows from this that an electron cannot exist in every point in space, only in those permitted by the electron's frequency.

At the Fifth Solvay Congress, De Broglie presented the pilot wave theory and gave a particularly good explanation of how a pilot wave actually guides a particle. This theory stipulates that the particle is guided by its pilot wave, and that the wavefunction can only evolve deterministically; there is no collapse. In order for the theory to hold, de Broglie had to introduce *non local interactions* – that is, instantaneous, long range influences between the subsystems of a system. He also proposed that there is a clock-like periodic process inside the particle. In the *rest frame* of the particle, the clock would have a frequency $\omega_0 = mc^2/\hbar$. As special relativity defines it, the rest frame of an object is the coordinate system in which the object is at rest. It is through this very assumption that

he derived what is known as the *Bohr-Sommerfeld relationship*:

$$\oint p \cdot dx = nh \tag{1}$$

which is a condition in order for the clock to remain in phase with the pilot wave.

In 1926, Erwin Schrödinger [58] came up with an equation that governs these material waves. These breakthroughs ultimately provided the foundations for what we call today quantum mechanics.

$$i\hbar \frac{\partial}{\partial t} |\Psi> = H|\Psi> \tag{2}$$

This is the Schrödinger's time-dependent equation in its most general form. The constant $\hbar$ is the Planck constant $\hbar = \frac{h}{2\pi}$. To employ Schrödinger's equation, we have to set up a Hamiltonian operation H consisting of the kinetic and potential energy of a system evolving in time. The solution of the resulting differential equation is the wavefunction $|\Psi>$ which yields information about the system. This equation plays a fundamental role in quantum theory. An important property of the wave equation is that there are states that are the sum of states. This property is called *superposition*.

The same year, Max Born[17] proposed a rule that can be used to predict the possibility of an experiment's outcome. What this rule helped do is close the gap between formalism and experiment.

The Born rule states that if a system is in a state $|\Psi> = \sum_i c_i |e_i>$, then the probability $P(A|\Psi) = |<\Psi|\Psi>|^2 = |c_i^* c_i|^2$

Within a year of Schrödinger's equation formulation, Clinton Davisson and Lester Germer [23] had discovered that electrons exhibit interference patterns, much like light waves. They were able to demonstrate that in a nickel crystal within which electrons are knocked around by arranged atoms, their waves build up in some directions while being annihilated in others. This is why more electrons are detected in some places, while fewer are found in others. Starting in 1927, the theory of quantum mechanics began gaining more and more traction. In the subsequent years, experimental data came to verify the postulates of quantum mechanics, the most notable example being Paul Dirac's [26] prediction, in 1931, of the electron's antiparticle. Following the postulates of Quantum mechanics, Quantum Electrodynamics continued the tradition of accurate predictions when it described the magnetic moment of the electron with an accuracy of about one part in a trillion [53]. Quantum mechanics has proven to be an especially successful theory in all of the experiments that have taken place since its beginning, [54] p.198.

Yet on an interpretive level, the theory was found to be less than adequate. Usually, when a physical theory is ready to reach a broader audience, the disseminating scientists have a clear understanding of its ontological position. In other words, they have reached a consensus regarding the nature of the objects introduced by the theory and the laws guiding their relations. For example, what is an electron, and what is the relationship with its wave? Given that

quantum mechanics is a theory about the propagation of waves through space, there should be no difference from other scientific schemes. Nevertheless, there are various reasons why some interpretations insist that these waves are not physical entities.

One of the reasons is the fact that the electrons found in a particular region of space can be explained through the interference pattern between the different parts of the wavefunction. However, when we take a measurement, electrons are observed not as waves, but as particles. As Max Born [16] explained in 1926, the intensity of the wavefunction gives the probability of a particle to be found somewhere. While it may be true that the objects described by quantum mechanics can be represented by waves, in reality we do not really encounter them as waves but as instantiations of particles. According to Born's rule, wavefunctions assign probabilities to the possible outcomes of measurements as follows: If an observable with operator A is measured in a system with wavefunction ψ, then

- the result will be the eigenvalue λ_i of A.

- the probability of measuring the eigenvalue of A will be $< \psi | P_i | \psi >$ where $P_i = |\lambda_i > < \lambda_i|$ and if the eigenspace is one-dimensional then the probability $< \psi | P_i | \psi > = < \psi | \lambda i > < \lambda_i | \psi > = |< \lambda_i | \psi >|^2$ and the number $< \lambda_i | \psi >$ is called the probability amplitude.

Another reason why the reality of the wavefunction can be challenged is because quantum waves are not transmitted through three dimensional space but through $3n$ dimensions, where n is the number of particles of the system under consideration. Therefore, it is not clear if the ontology propagating through space is of a wave form.

Since wavefunction gives us the probability of a particle to occupy a location, it is easy to think of the quantum state as potential knowledge for the system and adopt Copenhagen interpretation without to worry about some fancy philosophical position about the cosmos. However, this position comes at a price, as we then have to accept that quantum mechanics is incomplete as a description about the world because we handle it only as calculating device. As Einstein suggested, if we are not going to worry about the ontology of the symbols we use, then we must get used to the fact that these tools are not the ultimate description of nature.

The main idea upon which the original version of quantum mechanics was founded was the assumption that energy comes in discrete packets. The old quantum theory has its roots in work by Max Planck on black-body radiation, and was further developed through Albert Einstein's work on the photoelectric effect (i.e., the absorption of photons from electrons). In 1926, Erwin Schrödinger found a differential wave equation which could describe all the quantum phenomena with great empirical success. In fact, while Schrödinger's wave mechanics and Heisenberg's matrix mechanics had been separately developed, these methods were later proven to be equivalent. As we can see then, quantum mechanics was not developed in a linear progressive manner;some roads/aspects

matured earlier than others, many introduced with different ways and there were many setbacks. It should thus be easy to understand why, among the physical theories, quantum theory is in dire need of an interpretation that converges with what mathematics has already revealed.

2.2 Welcome to Copenhagen

The term *Copenhagen interpretation* was first advanced as a singular position of ideas by Heisenberg in 1955 [44]. In this section, we plan on considering some important concepts associated with the Copenhagen school of quantum mechanics, starting of course with a little background. The Copenhagen interpretation, developed through the work of Niels Bohr and Werner Heisenberg during 1920s, is generally considered the first serious attempt to understand the meaning of quantum theory. What provided fertile ground for the Copenhagen interpretation to blossom was the conflict between matrix mechanics and wave mechanics. On the one hand, Born, Heisenberg and Jordan [16] developed a theory based on matrix mechanics in 1925. The central equation of matrix mechanics was the commutation relation, also known as the uncertainty principle [39]:

$$\sum_k (X_{nk}P_{km} - P_{nk}X_{km}) = \frac{ih}{2\pi}\delta_{nm} \tag{3}$$

where X is the matrix of the position and P is the matrix of the momentum of the quantum particle. The results of experiments are the eigenvalues of these matrices. At the time of the measurement, the state of the system is described by the eigenvector that corresponds to the measured eigenvalue. The equation states that we cannot determine simultaneously the position and momentum of a given particle because there are no common eigenvectors between the two matrices X,P. Through the help of Kramers [49], Heisenberg was able to develop a formalism according to which the transition probabilities in quantum phenomena are not like classical probabilities. He also replaced the coefficients in the Fourier series of the classical probabilities with a matrix of coefficients. In classical mechanics, the Fourier coefficients yield the intensity of the emitted radiation, whereas in Heisenberg's formalism the magnitude of the matrix elements of a particular observable gives the intensity of radiation. In addition, the algebra of these matrices differs from its classical counterpart because they satisfy certain non-commutative laws, much like the ones seen in 3.

It is interesting to note, at this point, that the Copenhagen interpretation had been influenced by logical positivism, a philosophical view that was flourishing at the time [6]. According to the positivist school of thought, a physical theory can only make empirical predictions about the results of experiments; its place is not to speculate about the ontological status of the world. Influenced by the positivists, Heisenberg introduced the uncertainty condition. To illustrate what this condition entails, he considered the way in which a microscope measures the position of an electron. As he argued, one would have to use shorter wavelengths (e.g., gamma rays) if they wanted to make their measurements more accurate. But for such wavelengths, the collision between the photon and the electron would disturb electron's path, and thus affect its position. According to the uncertainty condition, the more accurately is the position of the particle known, the more indeterminate its momentum is. The particle is more likely to be found where the fluctuations of the wave are more intense; yet the more intense the waves' fluctuations are, the more unexpected the measurement of

the momentum-defining frequency is. The uncertainty principle expresses our inability to accurately measure certain incompatible quantities. It might then be fair to say that the uncertainty principle declares in a way the limits of our knowledge about reality.

On the other hand, Bohr's view was for the most part in line with Kant's [8], [35], [43], [51], [33]. Kant believed that we conceive reality in a certain way through our senses, and that the universe is not biddable to such conceptualization. In fact, we conceptualise the world as objects moving in three-dimensional space. Consequently, concepts as space and time, energy and position, are not real but a priori structures projected by the mind onto the physical world. This line of thinking appears to be much in common with Bohr's idea of a world in which the quantum realm exists independently from us, and where we are unable to fully grasp it because we try to make sense of it through classical instruments and theories. It follows from this that we should not try to interpret quantum mechanics through our conceptions of everyday life, because doing so will prove fruitless.

Despite their similarities, Faye [33] tells us that Bohr's ideas differed from Kant's, in that he believed objective knowledge can only be acquired by separating the experienced object from the observing subject. This is a move that would preclude the experiencing subject from imposing a priori structures upon the object. In order for an observer to distance himself/herself from the object, he/she must be able to recognise the significance of his/her experiences. This distancing can be, specifically, achieved if the subject employs causal and spatial-temporal concepts to classify the data gathered by the measurement apparatus. As real-world events take place in space-time, humans as isolated entities observe them, and compose a causal space-time description which is eventually what makes them real. Bohr called the constituents of this description, such as position, time, momentum, and energy, *classical concepts*. It is worth recognising of course that these concepts are also part of our language, and as such preconditions of our collective knowledge. It should thus not be surprising that any effort to understand nature should made through recourse to these concepts. It is important to note here that Bohr, as had Heisenberg, identified a principle that plays major role in the development of a consistent theory of atoms. According to the *correspondence* principle, a transition between atomic states corresponds to a harmonic movement of the macroscopic object. Following this principle, Bohr proposed that, in the case of hydrogen atom, the frequencies of radiation from states of high quantum numbers (i.e., those far from the nucleus) exhibit the same numerical values as values predicted by classical electrodynamics. For as he thought, the experimental results are expected to coincide with those of classical physics in areas where Planck's constant h could be neglected (i.e. in the limit of large quantum numbers).

The correspondence principle stems from the idea that classical concepts are inseparable from our understanding of the world. Given that position, momentum, space-time, and other such concepts are invaluable means for grasping reality, the only way we can compare different experiences is by describing classical or quantum phenomena in these terms. This relation between classical

concepts and the correspondence principle was highlighted by Bohr in his 1934 monograph *Atomic Theory and the Description of Nature* [14]. A strong advocate for the principle of correspondence, Bohr believed that quantum mechanics is a generalisation of classical physics. It could be said that the classical concepts are so integrated in our thinking that newer theories are expected to reproduce older ones to some extent. This is not of course to imply that classical concepts are not in need of refining when it comes to their application in newer theories such as quantum mechanics. Having made clear why classical concepts need refinement, let us now turn to another idea which played a significant role in the development of Copenhagen interpretation.

2.2.1 Complementarity

As the literature suggests (e.g., [40], [35], [21]), Bohr[12] adhered to the principle of *complementarity*, an assumption according to which an atomic object has kinematic and dynamic properties that are complementary. It should be clarified here that what is denoted by the term 'dynamic properties' are so-called 'claims of causality', such as the conservation of energy and momentum, whereas the term 'kinematic properties' refers to 'space-time descriptions'. According to the complementarity principle, quantum objects have pairs of complementary properties which cannot be measured simultaneously. Otherwise put, the experimental arrangements are mutually exclusive. Case in point, consider how microscopic phenomena have both wave and particle aspects, yet these aspects are never realised together and the realisation of each aspect depends on the experimental context. Interestingly, Bohr figured out that complementarity can be quantified by way of Heisenberg's uncertainty principle. He was also able to account for the quantization of energy states in the hydrogen atom by emphasising the importance of discontinuity – that is, discontinuity in the stationary states that electrons rotating around the nucleus and discontinuity in the transitions of electrons between the orbits. De Broglie[24], on the other hand, thought of the electron not as a particle but as a wave bundle around the nucleus. While these takes on the quantization of energy in hydrogen states may initially appear incongruous with one another, the complementarity principle tells us that they are in fact deeply entwined aspects of the same reality. See for example in figure 1

Based on this line of thinking, Bohr was able to claim that objects do not have intrinsic properties – that is, properties independent of determination from a measuring device. All properties of a given entity are context-dependent. As he saw it, the principle of complementarity is as fundamental for quantum mechanics as the requirement of relativity for the general theory of relativity. He believed that both concepts came from studying the measurement problem in different contexts. The relativity principle is the result of the limit of light's maximum velocity, while the principle of complementarity is the eventual outcome of the minimum of action in the domain of quantum mechanics. It is due to this universal lower limit in the theory of relativity, that we cannot separate between space and time without referring to an observer. It is likewise

The De Broglie Wavelength

$$\lambda \; = \; \frac{h}{p} \; = \; \frac{h}{mv}$$

λ = wavelength
h = Planck's constant (6.63×10^{-34} J · s)
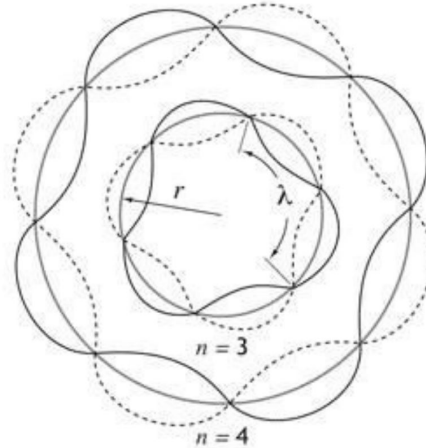p = momentum
m = mass
v = speed

Figure 1: De Broglie's extension of the concept of particle-wave duality from photons to include all forms of matter allowed the interpretation of electrons in Bohr's model as standing electron waves. De Broglie's work marked the start of development of wave mechanics.

impossible, in quantum mechanics, to separate the quantum object from the means of observation. As we can see, the observed object and the observer are distinguished in both of these cases [12] p.105. Perhaps then, the principle of relativity may be somehow linked to Heisenberg's uncertainty principle. Having identified the measurement problem as the connection between quantum mechanics and the theory of relativity, Bohr proceeded to criticize them both for their apparent lack of pictoriality [33]. Despite gaining widespread approval by the scientific community, these theories were not capable enough – in his view – of painting an accurate image of the world. While they are more than capable of accurately predicting the results of well-defined experiments, they could provide no insight into the world beyond our senses (or measuring apparatuses, for that matter). With that said, Bohr's views on complementarity and the interpretation of quantum mechanics can be summarised in a series of points:

- The interpretation of physical theories must be in accordance with the results yielded by experiments.

- Experiments require integrating previously developed methods and practices into a measurement apparatus.

- Scientific methods and practices entail notions (i.e., separation, orientation, and identification) which depend on common spatial and temporal relations (e.g., position and the change of position, duration and change

of duration, and cause and effect).

- These relations are the initial conditions for objective knowledge and concepts.

- Classical concepts are an invaluable tool not only for describing our experience, but communicating it to others as well. This does not mean however that they are not in need of refinement, especially in light of Planck's empirical discovery of the quantization of action.

- In cases where the quantization of action does play an important role, the application of classical concepts does not mean that the properties of the objects participating in the measurement are independent from it. In these cases, a non-contextual description of independent (from a specific experimental set up), absolute kinematic, and dynamic properties of the object is problematic.

- The same quantum object has different experiment-dependent way of being manifestations. It should thus follow that mutually exclusive properties cannot be measured simultaneously because they require different experimental set ups. At the same time though, the knowledge gained from various experiments conducted at different times starts exhausting the information available for any given object.

- In order to be interpret the phenomenon of an object's measurement, we must resort to classical concepts, because we are essentially trying to describe an uncontrollable interaction between an object and a classical machine.

- The use of classical concepts in the process of measurement must be in accordance the experimental setting. Thus, the setting furnishes the conditions in order to apply the kinematic and dynamic concepts to the quantum system under consideration.

- The kinematic and dynamic properties are mutually exclusive, because each one of them needs a special experimental arrangement in order to be measured. They are therefore called *epistemically incompatible*.

- The description of the object through quantum mechanics is dictated by laws that are different from those guiding the measuring apparatus. This is a difference that has to be reflected in the description of the measurement process by separating the object from the instrument of measurement. That said, it should be kept in mind that the line of separation is not straight, but fuzzy, which is why some parts of the machine may be considered as parts of the quantum object.

- Born believed that quantum mechanics does not supply any kind of a descriptive representation of the wave function. It is only the square of the absolute value of the wave function that has a counterpart in reality

as a wave of probability of existence in space and time, called probability amplitude. As he saw it, the fact that the time-dependent wave equation of Schrödinger contains an imaginary number means that the equation can only have symbolic character. This is a law that can be used to make predictions about the outcomes of measurement, provided of course that there is a set of initial conditions for concepts like position, momentum, time, and energy.

In closing our overview of Bohr's work, we would like to note how he was influenced by the thought-experiment of Einstein, Podolsky and Rosen (EPR). Before 1935, he would talk about the physical disturbance of the atomic object by the measuring instrument, arguing that it would be wrong to attribute intrinsic and measurement-independent properties to quantum objects which are inaccessible to us, on the basis of the fact that they are disturbed by our instruments. In light of insights gained by the EPR though, he gave up this picture as misleading. As he started to believe, it is not only the instrument that can affect an object – it is also the object that can affect the apparatus, even when the two are light years away from one another. Having recognised the bidirectionality in the relation between objects and apparatuses, he came to the conclusion that the process of measurement is context-dependent in both in quantum mechanics and theory of relativity. On this note, let us now turn to the EPR, an experiment with serious implications for quantum mechanics.

2.2.2   EPR's Argument

Driven by his disappointment for quantum mechanics' failure to provide a definite description of reality during the moment of observation, Albert Einstein co-wrote, in 1935, a paper with Boris Podolsky, and Nathan Rosen [30]. Known as the "EPR" (from the initials of its authors' surnames), their paper on the completeness of quantum mechanics drew the attention of the physics community, and remained in the spotlight till the present day. In this paper, the authors identified indeterminacy as one of the main reasons behind the incompleteness of quantum mechanics. As we saw above, probabilities in the Copenhagen interpretation of quantum mechanics are fundamental. The fact that probabilities play such important role in a physical theory, is called indeterminism. Indeterminism can be quantified through the probability distribution on the set of a measurement's possible outcomes. Probabilities are therefore inextricably linked with our knowledge of the quantum object. In this light, Einstein started to doubt the completeness of quantum theory. He was not fond of the fact that quantum theory was, at the time, considered a final theory (i.e., a theory of everything). Yet this overgeneralization seemed dangerous in his eyes since the theory entailed the features of irrealism, indeterminism, and complementarity. As he saw it, determining whether quantum mechanics was indeed a final theory should depend on the matter of completeness alone. EPR subsequently proceed to discuss what kind of theory they count as complete. From their point of view, the necessary condition for the theory to be

complete is the following: "every element of the physical reality must have a counterpart in the physical theory". In other words, the physical theory must include a quantity for each and every element comprising physical reality. The term "element" is not defined explicitly in EPR's text. Yet one can infer that the term "element" corresponds to values of physical quantities like position, momenta etc. What this means is that for every "element of physical reality" there exists a "real physical state" that assigns it with "definite values". This relation between states and values of quantities is functional, in the mathematical sense. It thus follows that, if no change occurs in the state of the system then quantities do not change in any way. We are thus naturally faced with the issue of establishing when is it that a quantity has a definite value. To address this issue, EPR presented an adequate condition:

Criterion of Reality: If we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity without in any way disturbing a system, then there exists an element of reality corresponding to that quantity.

EPR mainly based their argument on the logical consequences of two premises. The first one concerned the completeness of quantum mechanics – that is, the fact that quantum mechanics is incomplete. The other one related to the incompatible variables, such as the position x and the momentum p of an electron. As we have already noted, these variables cannot have simultaneous realities or, to be more precise, definite values at the same time. According to the authors, there is a disjunction between these two premises; they cannot both hold true at the same time. It follows from this that, either quantum mechanics is incomplete and incompatible variables cannot have definite values at the same time, or that quantum mechanics is complete and incompatible variables can have simultaneous values.

In evaluating these possibilities, EPR start by taking the first premise to be true and the second one to be false. They do this by considering a quantum state $|\psi>$, which is an *eigenstate* of a quantity. When this is true, the probability of observing that state is equal to one. Then, from the criterion of reality we can deduce that those quantities have a definite value so the first premise must be true. However, it is easy to show that the wave function cannot be an eigenstate of both position and momentum of a system. Therefore, EPR's conclusion is that since wave function is unable to provide definite values to the incompatible variables then the description of quantum mechanics is incomplete. Afterwards, EPR attempt to show that if quantum mechanics is complete, then incompatible quantities can have simultaneous definite values. However, they do this not by assuming completeness and on that basis to show that incompatible variables can have simultaneously definite values, but by providing a case (i.e., thought experiment) where incompatible variables do have simultaneously definite values and in this way leaving the question of completeness open, implying that quantum mechanics need not to be complete.

To make that happen, EPR continued by presenting a thought experiment which became "iconic" and is widely discussed till today. In this experiment,

there are two quantum subsystems (i.e., particles) which become spatially isolated from one another after a short time interacting. As they discovered, the positions and the momenta of the two subsystems are related through the total wave function of the system. If we apply the law of momentum conservation, then the total momentum of the system along the x-axis is zero. For instance, if the momentum of the subsystem called Molly's was to be found p along the x-axis, then the momentum of the subsystem called Sally's would be $-p$, so that the total momentum equals zero. Similarly, the positions of the subsystems along the x-axis are also determined at the same time. As long as the subsystems do not interact with the rest of the universe, their elements of reality remain connected with each other.

With that in mind, the authors proceeded to outline a pair of naturally occurring assumptions. The first one is called *separability* and hypothesizes that, when two subsystems are far from each other, each subsystem must have its own physical reality separate from the other. They essentially assumed that each subsystem is surrounded by its own special environment and its state defines its physical quantities – that is, the elements of its reality.

The second assumption they made is known as the notion of *locality*. This assumption postulates that, when the subsystems are far apart, "no real change can take place" in the second subsystem, as a consequence of anything that may have happened in the first subsystem. According to the principle of locality, these do not interact in any way when either of the subsystems is measured. This assumption sought to guarantee that the measurement of one subsystem cannot interrupt what is real to the second subsystem, a subsystem whose independent reality is guaranteed by the assumption of separability. Yet, despite these expectations, it turned out that the elements of the two subsystems' realities stay correlated in such a way that the measurement of one defines the value of the other. Clearly then, the assumptions of separability and locality proved to be violated. Given that the Copenhagen interpretation was unable to explain this behaviour, EPR were led to conclude that the quantum mechanical description of the system in terms of its wave function is incomplete.

After the publication of EPR's paper, Bohr's position in [15] was that the state of the object, composed from the two subsystems, and the state of the measuring device are inseparable during the measurement. To put it in another way, there is no inherent, Cartesian subject-object distinction. For this reason, when we measure a subsystem, the assignment of certain values to variables reflects the relationship between the experimental apparatus and the composite system. In other words, there are "influences" that relate the particles' variables to the experimental context that we dictate. These "influences" are not some "mechanical disturbance", but a new non local interaction, which is transmitted instantaneously at any distance. Case in point, by taking a measurement of Sally's particle, we can predict the value of a particular quantity in Molly's particle on the basis of the "influences" existing between Sally's experimental device and Molly's system. In this light, Bohr maintained that the post-measurement joint state of the object and the measuring device are entangled in the same way as EPR's subsystems before they are measured. Having said that, the main fea-

tures of EPR's argument about the completeness of quantum mechanics can be summarized in the following few points: The main features of EPR:

- The authors try to determine whether the wave function is able to describe all the phenomena of nature. As they see it, the description of quantum mechanics is not complete.

- In order for a physical theory to be considered complete, it must have a variable corresponding to each and every element of reality.

- The criterion of reality, in turn, establishes that these variables have definite values.

- The authors subsequently posit that one of the following premises must hold: either quantum mechanical description is incomplete and there are no definite values for incompatible variables, or quantum mechanics is complete and there are values for incompatible variables.

- After giving it much thought, EPR arrive to the understanding that the wave function is unable to assign definite values to incompatible variables. It then follows from this that the quantum mechanical description is incomplete.

- To this extent, they put forward a thought experiment about a pair of subsystems departing from one another. According to the principle of separability, spatially separated subsystems have independent physical realities.

- They also delineate a notion of locality, according to which the measurement of the one subsystem does not affect the reality of the other.

- It then follows from the criterion of reality and the notions of separability and locality, that if quantities in separated subsystems have correlated values, then those quantities have definite values.

- The separated subsystems of the experiment have definite position and momentum values at the same time. EPR infer from this fact that the notions of separability and locality are violated.

2.2.3   Summary of the main points

Despite being met with approval by most theoretical physicists, the Copenhagen interpretation's issue of whether the formalism corresponds to something real or whether it is just a mathematical invention used to calculate probabilities of outcomes remained a matter of contention. As we saw above, the Copenhagen interpretation is made up from different and usually opposing points of view which flourished during the second quarter of 20th century. That being said, let us summarize its main features and most important drawbacks:

Main Aspects:

  I Quantum objects exhibit a duality of existence which is either wave-like or particle-like. It is the experimental disposition that affects which kind of existence will be eventually observed.

 II Quantum objects are not characterised by strictly defined properties prior to the moment of the measurement. These are instead determined upon the completion of measurement.

III The wave functions evolve in two ways. The first one occurs when no measurement takes place and is characterized by a continuous, deterministic change of the state of the system according to Schrödinger equation $\frac{\partial \psi}{\partial t} = U\psi$, where U is a linear operator. The second way takes place during the process of measurement, which is distinguished by a indeterministic, discontinuous transition (collapse of the wave function) between the initial state ψ and the final state ϕ_j. According to Born's rule, the probability of this transition to occur is $|(\psi, \phi_j)|^2$.

IV The Hilbert space becomes the canvas upon which the whole theory of quantum mechanics is developed. It is a vector space with an inner product, which allows the estimate of angle and length.

 V Quantum probabilities are ontic, which implies that randomness is an inherent property of our reality. There is no deeper, more fundamental theory than quantum mechanics, which could explain randomness as a deterministic process of cause and effect.

Drawbacks

  I Although Bohr's view was that the quantum object and the experimental apparatus are inseparable, quantum mechanics could not describe the process of measurement. For this reason, John von Neumann [66] introduced a "cut" between quantum object and measurement apparatus.

 II Quantum mechanics loses its descriptive power as a result of the reduction of general predictive laws to a series of numerical probabilities concerning individual outcomes.

III The way in which potentialities (infinite possible states) become a unique actuality (a measured state) – that is, the *objectification problem* – still remains an open question.

2.3 The Formalism

In this section, we are going to present the formalism of quantum mechanics. Familiarity with the formalism requires an introduction of the dynamical principles, the central mathematical tools that are used in quantum mechanics, as well as the axioms of the theory. Let us start with the quantization of energy states, which forces us to represent the quantum states as vectors in Hilbert space. Hilbert spaces are kinds of a vector space where mathematical entities are collected. In our case, vector spaces are made up of vectors. A vector has two properties: length and direction. We follow Dirac's notation in symbolising vectors as $|\psi>$. These vectors live in the Hilbert space, which is closed under vector additions and scalar multiplication. In other words, if $|\psi>$ and $|\phi>$ are elements in a Hilbert space, then so is their sum $|\psi>+|\phi>$ and the multiplication of $|\psi>$ by a number (i.e., scalar).

The result of the multiplication of a vector by a scalar λ is a new vector pointing in the same direction, but with a length λ times as long. An important feature of a Hilbert space is that a dot product can be defined. The dot product is an operation that multiplies two vectors and yields a number. In Dirac's notation, the dot product can be written as $<\psi|\phi>$. The introduction of Hilbert space and vectors enables us to represent the states of a quantum system by normalized vectors in a Hilbert space. Normalised vectors have length 1, and are thus unit vectors. Yet, vectors are also useful because the observables of a system are represented by operators that act on vectors, with the purpose of obtaining definite values.

Another important observable besides those already mentioned is the property of *spin*, which was introduced in 1925 by Samuel Goudsmit and George Uhlenbeck [64]. The reason it was introduced was to make Bohr's model more accurate regarding the atomic spectra. We can think of spin as the intrinsic angular momentum of a particle – albeit, this account is not exactly accurate. In order to accurately talk about spin, we have to focus our attention towards a particular direction, let us say the z-direction. An electron in this direction has only two possible spin states, the z-spin up and the z-spin down, although other particles might have more possibilities. Another example of a spin state could be a state vector that does not represent a spin up or down, but a vector that points halfway between the previous two vectors. This state is called a singlet.

The variety of Hilbert space we are going to utilize; depends on the system we study. For this reason, the dimensions of a Hilbert space are determined by the allowable values of the state of the system. For example, if we are interested in position (or momentum, energy, etc.,) then the number of possible values of position specify the dimensions of the space. By the law of addition of states, we can extract the result:

$$|\psi>=\frac{1}{\sqrt{2}}|z-spin \quad up>+\frac{1}{\sqrt{2}}|z-spin \quad down> \tag{4}$$

As we see, this vector has half of its magnitude in the z-spin up direction and half in the z-spin down direction. This state (i.e. vector) is called *superposition*
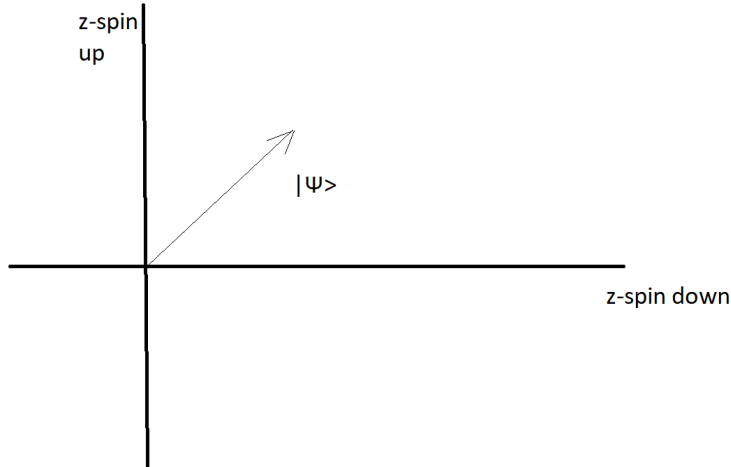
Figure 2: Singlet state

of z-spin states. Furthermore, it lives in a two-dimensional Hilbert space due to the two possible states of spin (up or down). If we had a system with infinite directions, each contributing a small magnitude, the state would be:

$$|\Psi >= a_1|\psi_1 > +a_2|\psi_2 > +... + a_n|\psi_n >= \sum_{i=1}^{n} a_i|\psi_i > \qquad (5)$$

For instance, this state would represent a particle with infinite number of possible positions to be. When states are used to define the dimensions of a Hilbert space, then one determines a *basis* of that space. In particular, the unit vectors $|\psi_i >$ constitute a basis. In this way, every state can be described by this basis. Furthermore, the utilization of a basis makes possible the expression of states which are not eigenstates of a particular observable. At the same time, there are infinite number of possible bases to choose from. Let us consider as an example the position state of a particle confined to a one-dimensional space with infinite length. The position state is a function that takes as an input a point in space, and gives as an output a number. Given the uncountable nature of the set of possible positions and their superposition between these, the wave-function can be a curve spread out over the space of possible positions like for an example a Gaussian function. Let us examine another example in order to understand how the formalism of quantum mechanics and experiment are connected.

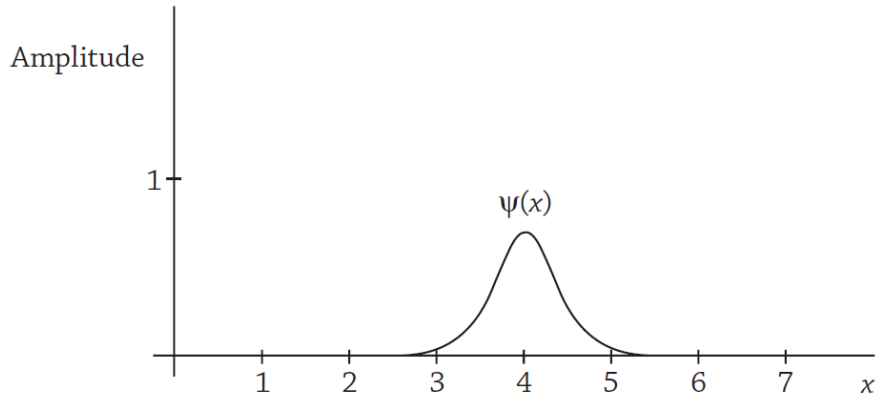$$\Psi = a|A > +b|B > \qquad (6)$$

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Figure 3: Gaussian wave-function

This wave-function is normalized so that $a^2 + b^2 = 1$. If we would like to make a measurement to the system in order to see in which state is in, we would found that the probability of finding the system in state $|A>$ will be a^2, and the probability of finding the system in state $|B>$ will be b^2. For instance, if our system constitutes from an electron in this state, then there is a^2 chance of being in the state $|A>$ and b^2 chance of being in the state $|B>$. In addition, for a system that is represented by the wave-function Ψ, the absolute value of that function squared is $|\Psi|^2$. With that in mind, the integration of the absolute value over a region of physical space, enable us to calculate the probability of finding the system in that particular region of space. This result is called *Born's rule.*

According to Born, the wave function is the probability density for particles to have a property. For example, if one makes a measurement of position of a particle, then the wave function bears the information about where it is most probable to be. The standard formalism accommodates empirical predictions and experimental facts that have been developed in the period 1900-1935, ranging from Schrödinger [58], Dirac [27] to von Neumann [65]. Also, the formalism most of the time is given in an axiomatic form.

2.3.1 Axiom 1

With every quantum system there is an associated complex Hilbert space $(\mathcal{H}, +, \cdot, < \cdot, \cdot >)$. The states of the system are all positive trace-class linear maps $\rho : \mathcal{H} \to \mathcal{H}$ for which $Tr\rho = 1$.

Remark: The normalized (i.e., $< \psi | \psi >= 1$) elements $\psi \in \mathcal{H}$ are the "states" of the quantum system.

Terminology: A state is called *pure* if there is $\psi \in \mathcal{H}$ such that $\rho : \mathcal{H} \to \mathcal{H}$.

Definitions: Complex Hilbert space $(\mathcal{H}, +, \cdot, < \cdot, \cdot >)$

1. $\mathcal{H}$ is a set satisfying the following axioms:
   $+ : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$
   $* : \mathbb{C} \times \mathcal{H} \to \mathcal{H}$
   $C : x + y = y + x, x * y = y * x$
   $A : (x + y) + z = x + (y + z), x * (y * z) = (x * y) * z$
   $N : 0 + x = x + 0 = x, x * 1 = x$
   $I : (-x) + x = x + (-x) = 0, x * x^{-1} = 1$
   A norm on a vector space V over K is a function from V to $\mathbb{R}$ denoted $|| \cdot ||$, satisfying for all $v, w \in V$ and $k \in K$:

   (a) $||v|| = 0$ iff $v = 0$

   (b) $||kv|| = |k| ||v||$

   (c) $||v + w|| \leq ||v|| + ||w||$

   It follows that $||v|| \geq 0$ for all $v \in V$. The norm defines a topology on V: the open balls are sets $\{x | \quad ||x - v|| < r\}$ for some $v \in V$ and $r \in \mathbb{R}$. A Hilbert space is an inner product space that is complete with respect to the norm topology meaning that the limit of any sequence of vectors is itself contained in the space.

2. Sesquilinear form $< \cdot, \cdot >: \mathcal{H} \times \mathcal{H} \to \mathbb{C}$

   i) $< \phi, \psi > = < \bar{\psi, \phi} >$

   ii) $< \phi, \psi_1 + a\psi_2 > = < \phi, \psi_1 > + a < \phi, \psi_2 >$

   iii) $< \psi, \psi > \geq 0, \forall \psi \in \mathcal{H}$, and $= 0$ iff $\psi = 0 \in \mathcal{H}$

3. Completeness
   If one has a sequence in $\mathcal{H}$, which satisfies the Cauchy property namely:
   $\forall \epsilon \geq 0, \exists N \in \mathbb{N}, \forall n, m \geq N : ||\phi_n - \phi_m|| \leq \epsilon$
   $||\psi|| = \sqrt{< \psi, \psi >}$
   $|| \cdot || : \mathcal{H} \to \mathbb{R}$
   One may already conclude that the sequence converges in $\mathcal{H}$, in other words
   $\exists \phi \in \mathcal{H} : \forall \epsilon \geq 0, \exists N \in \mathbb{N}, \forall n \geq N : ||\phi - \phi_n|| \leq \epsilon$

4. Linear map A: $\mathcal{D}_A \subseteq \mathcal{H} \to \mathcal{H}$ $A(\phi + a\psi) = A(\phi) + aA(\psi) \forall a \in \mathbb{C}$

   i) Notation $A\psi := A(\psi)$

   ii) positive linear map: $\forall \psi \in \mathcal{D}_A :< \psi, A\psi \geq 0$

   iii) A is trace class: if for any orthonormal basis e_n of the $\mathcal{H}$ the sum/series
   $\sum_n < e_n, Ae_n > < \infty$ then TrA:=$\sum_n < e_n, Ae_n >$

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2.3.2 Axiom 2

The observables of a system are the self-adjoint linear maps $A : \mathcal{D}_A \to \mathcal{H}$.

Definitions:

1. $A : \mathcal{D}_A \to \mathcal{H}$ is called self-adjoint if it coincide with its adjoint map $A^* : \mathcal{D}_{A^*} \to \mathcal{H}$. Coincide means that $\mathcal{D}_{A^*} = \mathcal{D}_A$ and $A^*\psi = A\psi \quad \forall \psi \in \mathcal{D}_A$

2. The adjoint $A^* : \mathcal{D}_{A^*} \to \mathcal{H}$ of a linear map $A : \mathcal{D}_A \to \mathcal{H}$ is defined by

   i) $D_A := \{\psi \in \mathcal{H} | \forall a \in D_A, \exists \eta \in \mathcal{H} :< \psi, Aa >=< \eta, a >\}$

   ii) $A^*(\psi) := \eta$

2.3.3 Axiom 3

The probability that a measurement of an observable A on a system that is in a the state ρ yields a result in the Borel set $E \subseteq \mathbb{R}$ is given by:

$$\mu_\rho^A(E) = Tr(P_A(E) \cdot \rho) \tag{7}$$

where $P_A : Borel(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ bounded operator. This unique projection-value measure is associated with a self-adjoint map A according to the spectral theorem:

$$A = \int_\mathbb{R} id_\mathbb{R} dP_A \equiv \int \lambda P_A(d\lambda) \tag{8}$$

Definitions:

1. If a vector space V has a norm $||\cdot||$, then F is _bounded_ if and only if there is some $r \in \mathbb{R}$ such that $||Fv|| \le r||v||$ for all $v \in V$.

2. Given some set S, a $\sigma - algebra$ over S is a family of subsets of S closed under complement, countable union and countable intersection. The _Borel algebra_ over $\mathbb{R}$ is the smallest $\sigma - algebra$ containing the open sets of $\mathbb{R}$. A _Borel set_ of real numbers is an element of the Borel algebra over $\mathbb{R}$.

3. $\mathcal{L}(\mathcal{H})$ is a Banach space of bounded linear maps on $\mathcal{H}$

   (a) A Banach space is a $(V, +, \cdot, ||\cdot||)$ vector space with a norm $||\cdot|| : V \to \mathbb{R}$

      i. $||\lambda f|| = |\lambda| \cdot ||f||, \forall \lambda \in \mathbb{C}$

      ii. $||f + g|| \le ||f|| + ||g||$

      iii. $||f|| \ge 0, = 0$ iff $f = 0 \in V$

      which is complete norm.

   (b) A map A is said bounded if $\sup_{f \in V} \frac{||Af||_W}{||f||_V} < \infty$, usually $\sup ||f|| = 1$

4. A Banach space is a normed vector space that is complete with respect to the norm topology meaning that the limit of any sequence of vectors is itself contained in the space.

5. Projections-valued measures (PVMs): A map $P : \sigma(\mathcal{O}_{\mathbb{R}}) \to \mathcal{L}(\mathcal{H})$ is called a PVM if

projection value $\begin{cases} \forall \Omega \in \sigma(\mathcal{O}_{\mathbb{R}}) : P(\Omega)^* = P(\Omega) \\ \forall \Omega \in \sigma(\mathcal{O}_{\mathbb{R}}) : P(\Omega) \circ P(\Omega) = P(\Omega) \end{cases}$

measure part $\begin{cases} P(\mathbb{R}) = id_{\mathcal{H}} \\ \Omega = \cup_{n \geq 1} \Omega_n \Rightarrow \forall \ \psi \in \mathcal{H} : \quad P(\Omega)\psi = \sum_{n \geq 1} P(\Omega_n)\psi \end{cases}$

Properties of a PVM:

(a) $P(\varnothing) = \varnothing_{\mathcal{H}}$ where $\varnothing_{\mathcal{H}}; \mathcal{H} \to \mathcal{H}$

(b) $P(\mathbb{R} \setminus \Omega) = id_{\mathcal{H}} - P(\Omega)$ for any $\Omega \in \sigma(\mathcal{O}_{\mathbb{R}})$

(c) $P(\Omega_1 \cup \Omega_2) + P(\Omega_1 \cap \Omega_2) = P(\Omega_1) + P(\Omega_2)$

(d) $P(\Omega_1 \cap \Omega_2) = P(\Omega_1) \circ P(\Omega_2)$

(e) $\Omega_1 \subseteq \Omega_2 \Rightarrow ran P(\Omega_1) \subseteq ran P(\Omega_2)$

<u>Axiom 4</u>: Unitary dynamics: during time interval (t_1, t_2) which no measurement occurs.

States $\begin{cases} \rho(t_2) \text{ at time } t_2 \\ \rho(t_2) \text{ at time } t_1 \end{cases}$

are related through:

$$\rho(t_2) = U(t_2 - t_1)\rho(t_1)U^{-1}(t_2 - t_1) \tag{9}$$

$$U(t) := exp(-\frac{i}{\hbar}Ht) \tag{10}$$

Where $f : \mathbb{R} \to \mathbb{C}$

$$f(A) := \int_{-\infty}^{+\infty} f(\lambda)dP_A(\lambda) \tag{11}$$

<u>Axiom 5</u>: Projective dynamics: when a measurement occurs at time t_m, the state ρ_{after} is

$$\rho_{after} = \frac{P_A(E)\rho_{before}P_A(E)}{Tr(P_A(E)\rho_{before}P_A(E))} \tag{12}$$

Where E is the smallest Borel set in which the outcome of the measurement happened to lie.

2.4 The aftermath of EPR paper

Let us now continue with a very interesting property of multiparticle systems, *entanglement*. The term was coined by Schrödinger [52], who conceived entanglement as an interaction between systems that were previously part of the same system. Let us consider, for the purposes of illustration, a simple case of entanglement in a two-particle system. After interacting once, the wave function of the two particles is:

$$\Psi = \frac{1}{\sqrt{2}}|x_1 >_1 |x_2 >_2 + \frac{1}{\sqrt{2}}|x_2 >_1 |x_1 >_2 \qquad (13)$$

The subscripts in brackets indicate which particle is being described. This state describes a 50% chance of particle 1 to be in position x_1 and a 50% chance to be in position x_2. The same, of course, holds for the second particle. A peculiar property of such a system is that the wave function of the whole system contains more information about the system than the sum of the two parts' states. This is due to the fact that the entangled state cannot be written as a product of the individual systems' states. If we make a measurement in order to find out the locations of the particles, we are going to discover that, if the first particle is at position x_1, then the second particle will certainly be at x_2, and vice versa. If we do not measure the system, however, we still know that the particles of the whole system are separated by a distance of $|x_1 - x_2$, though we do not know where exactly the individual parts are.

The implications of EPR experiments, such as the above, are of prime significance. Should we opt for an interpretation according to which the wave function is a state of microscopic ordinary three dimensional objects which occupy a certain position in space, we will soon be faced with a paradox regarding entangled states, since we cannot appoint a position to each particle prior to measurement. To understand further the result of entanglement we are going to examine Bohm's version of EPR's experiment. Consider two electrons being in the entangled state of x-spin known as singlet state:

$$\Psi_{singlet} = \frac{1}{\sqrt{2}}|x-spinup >_1 |x-spindown >_2 - \frac{1}{\sqrt{2}}|x-spindown >_1 |x-spinup >_2$$
$$(14)$$

This state demonstrates our ignorance about the spin direction of the individual electrons, in a similar way to the previous case, in which we could not know the exact positions of particles. In the same vein, what can be confidently said without measuring is that the electrons spins are opposite to one another. For instance, if we measure electron 1 and find it to have x-spin up, then the second electron can be expected to have x-spin down, and vice versa. What we can also tell based on Bohm's theoretical experiment about a pair of separated electrons is that it would take light an hour to travel from one electron to the other. This experiment requires making two measurements: the first one involves measuring electron 1 in order to determine its spin state, whereas the second one takes place five minutes later at the second electron, for the purpose

of establishing whether its state satisfies the entanglement property and, by extension, the spin conversation law according to which the total spin equals zero. The results of various experiments [**?**], [**?**] confirmed entanglement, and more precisely, the fact that the spin orientation is always antithetic in a pair of entangled particles. A 2015 experiment [41] actually showed that entangled particles are found 143 kilometres apart from each other. Experiments such as these show that the measurement of one side could not have possibly affected the result of the second particle in any way, seeing how not even light could have travelled fast enough to notify the second electron about the spin orientation it should choose. Furthermore, if measurements took place around the same time, a moving observer would not have been able to tell which took place first, as a consequence of the relativity principle. This is the reason why the entanglement between the two particles is not considered causal. In other words, the result of the first particle's measurement is not the cause of the result of the second particle's measurement. Clearly then, these experiments help demonstrate the non-local nature of quantum phenomena.

EPR could not accept that non-locality governs entangled particles. They thus opted for the view that there are facts about the states of the electrons other than those incorporated in the wave function. One could, for instance, say that there is a reality about the spins of the electrons that we ignore. As they claimed in their famous paper [30], there are additional details waiting to be described by a more fundamental approach that will be able to account for the instantaneous reaction of the two particles. These yet undiscovered facts are called by EPR *hidden variables*. In 1964, John Bell [3] constructed an experiment that could determine whether the quantum mechanical description of the EPR experiment is incomplete (i.e., whether there are hidden variables explaining the phenomenon) and/or nonlocal. It became clear through his experiments that nature is inherently non-local. Thus, even if hidden variables do exist, they are not sufficiently capable of dismissing a non-local explanation of the phenomenon. In other words, hidden variables are not able to restore locality. To prove this fact, Bell [3] sketched a theory that Einstein would approve – that is, one that is local and uses hidden variables to explain entanglement. In what follows, Bell proves that a local hidden variable theory cannot predict the results of the EPR experiment as well as quantum mechanics. This phenomenological failure of local hidden variable theory is described by inequality relations, which Bell managed to figure out. All these results are summarised in Bell's theorem, which states:

Theorem 1 *No physical theory of local hidden variables can ever agree with all of the predictions of quantum mechanics.*

It follows from this that a local, hidden variables theory is not in line with experimental results, and hence not a good candidate for a physical theory of nature. Bell's work provided an answer on whether quantum mechanics can be reinterpreted as a local theory with the addition of hidden variables. As it demonstrated, any future theory that will seek to incorporate all experimental

verifications of quantum theory will have to be non-local in character. Having said that, let us turn our attention to the source of all quantum mechanical problems, the *measurement problem*.

The next issue that we must consider is the measurement process itself. Let us take, for instance, the measurement of an electron's spin in the x-direction. This can be accomplished by a measuring device called an "x-spin detector", which has a knob that can point towards three distinct positions: i) "ready", which means that the device is ready to make a new measurement, ii) "x-spin up", which signifies that the detector has measured a particle with a spin up in the x-direction and iii) "x-spin down" which respectively shows that the detector has measured a particle with a spin down in the x-direction. When the device measures a particle's spin, it will indicate "x-spin up" or "x-spin down" depending on the state of the particle. After the measurement, the machine-particle system will evolve as follows:

$$|ready>_m |x-spin \quad up>_e \rightarrow |"x-spin \quad up">_m |x-spin \quad up>_e \quad (15)$$

Similarly,

$$|ready>_m |x-spin \quad down>_e \rightarrow |"x-spin \quad down">_m |x-spin \quad down>_e \quad (16)$$

The mathematics symbolise that the device (m) is initially in the ready state and that the particle's state, which is indicated by subscript (e), is x-spin up in the x-direction. The particle subsequently enters the device and the measuring apparatus points towards the "x-spin down/up" position. This process helps establish that the device is actually working as intended. For the purposes of the actual experiment, the detector is then fed with an electron that is known to be in a state of spin up in the z-direction. As is verified by quantum mechanics, it is indeed possible to describe a z-spin up state as a superposition of x-spin down and x-spin up states:

$$|z-spin \quad up>_e = \frac{1}{\sqrt{2}}|x-spin \quad up>_e + \frac{1}{\sqrt{2}}|x-spin \quad down>_e \quad (17)$$

If then we supply that electron in the machine, we will get:

$$|ready>_m |z-spin \quad up>_e = |ready>_m \{\frac{1}{\sqrt{2}}|x-spin \quad up>_e +$$

$$\frac{1}{\sqrt{2}}|x-spin \quad down>_e\} = \frac{1}{\sqrt{2}}|ready>_m |x-spin \quad up>_e +$$

$$\frac{1}{\sqrt{2}}|ready>_m|x-spin \quad down>_e$$

Given what was said about a properly working machine, the state has to evolve

into:

$$\frac{1}{\sqrt{2}}|ready>_m \, |x-spin \quad up>_e + \frac{1}{\sqrt{2}}|ready>_m \, |x-spindown>_e \rightarrow$$

$$\frac{1}{\sqrt{2}}|"x-spin \quad up">_m \, |x-spin \quad up>_e +$$

$$\frac{1}{\sqrt{2}}|"x-spin \quad down">_m \, |x-spin \quad down>_e$$

Hence, the final state of the electron combined with the measuring device must be:

$$\Psi_{m,e} = \frac{1}{\sqrt{2}}|"x-spin \quad up">_m \, |x-spin \quad up>_e +$$

$$\frac{1}{\sqrt{2}}|"x-spin \quad down">_m \, |x-spin \quad down>_e$$

This state describes the experimental device being in superposition due to pointing in two positions simultaneously. Yet does not seem to be in accordance with what we would expect. In reality, what are actually observed are states that are either $|"x-spinup">_m \, |x-spinup>_e$ with probability 50% or $|"x-spin \quad down">_m \, |x-spin \quad down>_e$ with the same probability. This contradiction between what is predicted and what is actually observed is often called the *measurement problem* [**?**]. The pointer is incapable of being in superposition between two positions, unless quantum mechanics predicts the unreasonable. Hence, the problem is the following: why are macroscopic objects like the measuring apparatus not in superposition of states as quantum mechanics would have predicted. If we want to make sense of the theory (i.e., to be realists), we have to address this problem. In light of formalism, this issue can be described as follows:

$$\Psi_{m,e} = \frac{1}{\sqrt{2}}|"x-spin \quad up">_m \, |x-spinup>_e + \frac{1}{\sqrt{2}}|"x-spin \quad down">_m \, |x-spin \quad down>_e \tag{18}$$

However, in reality we observe either:

$$|"x-spin \quad up>_m \, |x-spin \quad up>_e \tag{19}$$

with probability 50%, or:

$$|"x-spin \quad down">_m \, |x-spin \quad down>_e \tag{20}$$

with the same probability.

The first attempt to resolve this issue was made by John von Neumann [**?**], who added an additional postulate which introduces an *ad hoc* collapse of the wave function in one of the two choices of the superposition. This attempt points towards the fact that the usual unitary evolution of quantum mechanics is incapable of describing the process of measurement where a more reductive

mechanism is taking place. The interaction between a quantum system and an external entity, like a measuring device, is nonlinear and stochastic, and therefore different from the deterministic and linear evolution given by Schrödinger's equation. This new interaction is often called *collapse* of the wave function or reduction of the wave packet. According to von Neumann, there are two processes in nature: one that is deterministic, unitary, and evolves according to the Schrödinger equation, and one that is instantaneously, non-linear, non-deterministic, and unfolds via the measurement process. Besides this categorization of processes, we must provide a criterion that will help us to identify which processes can be taken as measurements and which cannot. We also have to account for the point in which the wave function of a particle collapses. Is it when the electron enters the device, or is it when the experimenter observes the pointer? This is usually mentioned as the quantum *time of arrival problem* [31].

In response to the above questions, we should note that, following the publication in 1932 of von Neumann's work, the scientific community was divided in two opposing sides: there were those who believed that the collapse of the wave function is a result of conscious observation, and those who believed that the collapse is the result of physical, material interactions. The Copenhagen interpretation is often linked with John von Neumann's solution of the measurement problem by way of the collapse postulate. Von Neumann [?] made a distinction between the nonlinear collapse of the wave function as type 1 processes, and the deterministic processes as type 2 processes. He also noted that the transition between type 1 and type 2 processes happens only in the presence of the observer's consciousness. A non-physical consciousness is necessary for the wave function to collapse, since it is the only entity which cannot be described by quantum mechanics. Von Neumann was thus led to treat the mind of the observer as the only true measuring apparatus, an apparatus that causes the collapse of the wave function. That said, Von Neumann did not really describe how the material domain is affected by the mental at the moment of the wave function's collapse. To put it slightly differently, we do not know how a system that consists of the quantum object (e), the measuring apparatus (m), and the observer (o) should evolve. We certainly do know that the interactions involving the consciousness of observer, should not be governed by Schrödinger's equation. Granted this constraint, the postulate of "collapse" fails to give an analytic process of measurement. It thus becomes apparent that the *ad hoc* introduction of a postulate of "collapse" conceals our ignorance about the moment of measurement. As it turns out, we do not know much about what physics can do in order to describe the state of awareness of conscious human beings when they are, for instance, perceiving a pointer on the board of an experimental apparatus. Bell [5], on the other hand, expresses the view that the "collapse" of the wave function is the result of material interactions alone. He additionally claims that we cannot determine exactly when or where the measurement process occurs, due to the non-separability of the quantum object and the measuring apparatus at the time of measurement. To make things worse, the meaning of the word measurement is vague. When we measure something in reference to some

pre-existing property of an object, we are effectively forcing the object to take determinate values, even though quantum mechanics tell us the opposite. Bell also believes that quantum mechanics should not be about organisms as complex as human observers. He therefore concludes that the formalism describing the measurement process should be independent of whether the scientist is actually observing the pointer. Like von Neumann, Eugene Wigner [32] defines measurement as the interaction between a physical system and an irreducible consciousness. He proposes that consciousness has a direct effect on physical phenomena, and that one is destined to fail if they try to isolate consciousness for the purpose of studying it. This is why knowledge about the consciousness of another being can only be acquired indirectly, through its impact on the physical world. Wigner's distinction between systems involving a conscious observer and systems that do not, helps us decide whether to follow the nonlinear collapse of wave function or adhere to a deterministic take on process.

Having said that, the measurement problem can be briefly put as follows: According to the basic laws of quantum mechanics, the interaction between two quantum systems will be a superposition of each system's states, with the compound system evolving in time through a deterministic differential equation. If a measuring apparatus, constituted by many quantum particles, interacts with the quantum object, then the joint state (including the apparatus) should be in another superposition. Yet, we never observe superpositions but definite, absolute results. This contradiction between formalism and reality has been called the measurement problem. Apart from this issue, the Copenhagen interpretation is also tied to the idea that consciousness causes the collapse of the wave function. This view became subject of numerous criticisms, which in turn led to alternative research roads and interpretations.

3   Hidden Variable Theories

In a lecture that took place in Oxford in 1926, Born [16] pointed out that if someone is not satisfied with the probabilistic approach of a given physical theory, one may assume additional parameters which describe an individual event in a deterministic way. As an example, Born considered a gas as a collection of molecules, for which we cannot know the exact position and momentum of each molecule. In this situation, we are obliged to use a probabilistic method in order to predict statistical values of physical quantities, which characterise the gas. The lesson, from the use of statistical methods, is that one can convey the information as parameters that express the trend of the ensemble. We can therefore quantify our ignorance for a particular fact by introducing specific variables that are called *hidden variables*. These variables allow us to describe phenomena with extreme accuracy and without the use of probabilities, a move that constitutes the physical theory non-deterministic. Being aware that such variables exist does not, however, mean that our computing systems are in a position to calculate the equations of 10^{23} variables, which would respectively account for the number of gas molecules. This is why we are not always able to use theories based on hidden variables in order to make predictions.

In his 1932 book, Von Neumann [65] articulates a proof of an argument for the nonexistence of theories with hidden variables. According to this statement, hidden variables cannot possibly exist in a physical theory that is formulated, like quantum mechanics, entirely by probabilities. Most physicists and philosophers, at the time, accepted von Neumann's argument. John Bell [4], however, criticised von Neumann's assumptions as unreasonable, later stating that "the proof of von Neumann is not merely false but foolish" (Interview in Omni, May, 1988, p. 88.). Even though, the scientific community now considers von Neumann's argument wrong, there are other mathematical theorems that impose severe conditions in the kind of hidden variables a theory can have, such as Gleason's [36], Bell's [3], and Kochen and Specker's [47]. These theorems require that a hidden variable theory must be "nonlocal" and "contextual". A model that not only meets all requirements imposed by these theorems, but also agrees with the predictions made by quantum mechanics, was actually put forth by David Bohm [10] in 1952.

Like Born, Bohm [11] believed that randomness is caused by our ignorance of each and every interaction taking place in large congregates of particles. That is why, as mentioned earlier, randomness can be studied through the use of hidden variables. Additionally, the measurement cannot be, as Bohr supported, unpredictable, uncontrollable and not subject to a rational analysis. As he saw it, a physical theory that describes quantum phenomena should be expected to account for the behaviour of matter at the quantum level, while also being able to provide a detailed description of all processes. A theory that can accommodate this kind of knowledge should take into account additional parameters as well as the equations that bring them together.

In order to illustrate this, take on hidden variables, Bohm [9] asked us to consider the example of an automobile accident. It is generally accepted that,

in order for an accident to occur, several factors must combine in a very precise way. It should be fair to say that a small perturbation affecting any of the initial conditions, might well lead to the accident being avoided. Naturally then, we can say that accidental events depend on chance. That said, we should also note that a rising number of accidents starts to reveal certain patterns, such as the effects of bad weather or alcohol consumption. These patterns are called *statistical regularities*. It is interesting to note that these regularities are not fixed; they can instead be changed by affecting factors that lead to accidents, such as the kind of punishment that careless driving incurs, thus effecting a reduction in the number of accidents taking place in a certain area. Yet, in the case of the individual motorist, we can never confidently predict whether an accident is impending or not. Given our inability to say with certainty whether an actual accident is bound to happen, the factors determining the possibility that two vehicles collide are this scenario's hidden variables.

When a large number of seemingly random phenomena are observed, we start seeing statistical laws that can be, in turn, used to predict their behaviour. Having recognised this, Bohm implemented the hidden variable theory in order to unveil the reality of the quantum phenomena, while doing away with the wave function's collapse. Bohmian mechanics is thus an example of a theory depending on hidden variables. Seeing how wave function is only a partial description of a system, Bohmian mechanics tell us that the description is only ever complete if we take into account the positions of the particles, which are in this case the theory's hidden variables.

One of the main equations of Bohmian mechanics is the *guiding equation*. This equation governs the evolution of particles' positions of a system under consideration, as well as the positions of every apparatus or any other object that takes place in the process of measurement. This equation links the velocities (i.e., the first derivative of the particle's position) and the wave function. In this interpretation, the positions of the particles come first, while the wave function is second [37]. The guiding equation suggests that the evolution of a particle's position, which is its velocity, is affected by the wave function. This affection is non-local in character. To see how guiding equation is applied, we consider the example of a single particle system of mass m:

$$\frac{dQ}{dt} = \frac{\hbar}{m} Im \frac{\psi^* \nabla \psi}{\psi^* \psi} \tag{21}$$

Where Q are the particles' positions and ψ is the wave function. Another central equation of Bohm's theory is the usual Schrödinger's equation which for the single particle system takes the following form:

$$i\hbar \frac{d\psi}{dt} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(x)\psi \tag{22}$$

ψ is a complex-valued wave function on a 3-dimensional Euclidean space, which can be written as $\psi(x,t) = \psi_r(x,t) + i\psi_i(x,t)$. We can express the wave function

as two real-valued functions R and S, in the following way:

$$\psi = Re^{iS/\hbar} \tag{23}$$

$$R^2(x,t) = \psi_r^2(x,t) + \psi_i^2(x,t) \tag{24}$$

$$S(x,t) = \hbar \arctan \frac{\psi_i(x,t)}{\psi_r(x,t)} \tag{25}$$

The quantum action S is not well defined in the regions where $\psi_r(x,t) = \psi_i(x,t) = 0$, but due to the fact that $R(x,t) = 0$, there are no particles to reach those regions. By inserting this form of the wave function into the Schrödinger's equation and separating the real from the imaginary part, we obtain two equations:

$$\frac{dR}{dt} = -\frac{1}{2m}(R\nabla^2 S + 2\nabla R \cdot \nabla S) \tag{26}$$

$$\frac{dS}{dt} = -(\frac{(\nabla S)^2}{2m} + V(x) - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}) \tag{27}$$

By setting P for R^2 we take:

$$\frac{dP}{dt} + \nabla(P\frac{\nabla S}{m}) = 0 \tag{28}$$

$$\frac{dS}{dt} + \frac{(\nabla S)^2}{2m} + V(x) - \frac{\hbar^2}{4m}\{\frac{\nabla^2 P}{P} - \frac{1}{2}\frac{(\nabla P)^2}{P^2}\} = 0 \tag{29}$$

For $\hbar = 0$, equation 29 becomes the classical Hamilton-Jacobi equation. We can identify the velocity of a particle passing through a given surface $S = const.$ with $\nabla S(x)/m$, so equation 28 becomes:

$$v(t) = \frac{\nabla S(x,t)}{m} \tag{30}$$

$$\frac{dP}{dt} + \nabla(P\mathbf{v}) = 0 \tag{31}$$

The interpretations of symbols are the following: Pv symbolize the current of the particle, and $P(x)$ the density. The equation 31 is the continuity equation.

For $\hbar > 0$, Bohm extended his interpretation to include, apart from the classical potential, the quantum potential U which also acts on the particle and is given by equation 29:

$$U(x) = -\frac{\hbar^2}{4m}(\frac{\nabla^2 P}{P} - \frac{1}{2}\frac{(\nabla P)^2}{P^2}) = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} \tag{32}$$

In this interpretation, the position and momentum of the particle have always definite values, but at the same time these values fluctuate and change under the action of the quantum potential $U(x)$. It is generally accepted that for a given potential there is always a force that is acting on the particle. This force is obtained not only from the classical potential but also from the quantum potential $U(x)$.

From the above formulation of Bohm's theory, becomes evident the fact that the particle is always accompanied by a quantum field ψ. The wave function ψ is a real field that guides the particle in a similar way the electromagnetic field acts on an electric charge. In view of this fact, we may say that the particle and the field are causally connected. It should be pointed out that unlike any other material field, the quantum field does not have sources, nor does it have any other way to be directly affected by the positions of the particles. Given that, the wave function causes changes in particle's positions and no the other way around. This kind of unilateral causation is an important difference between the quantum field and the fields that have been used in physical theories.

In order to better understand the mathematical symbols' interpretation, let us see another example of Berndl, Daumer, Dürr, Goldstein and Zanghí [7]. They investigate the wave function of N particles. In Bohmian mechanics, the state of a system consisted of N particles, can be defined by the wave function $\psi = \psi(q_1, ..., q_N) = \psi(q)$. This wave function, as we can easily infer, is a function of all possible configurations of the system. Besides the wave function, we have to specify the theory's hidden variables, namely an actual configuration Q, which is the particles' actual positions till the present moment. Thus, a system can be described by the pair:

$$(Q, \psi) \tag{33}$$

Where $Q = (\mathbf{Q}_1, \mathbf{Q}_2, ..., \mathbf{Q}_N) \in \mathbf{R}^{3N}$, and $\psi = \psi(q) = \psi(q_1, q_2, ..., q_N)$ is the wave function. There are two evolution equations in the theory of Bohmian mechanics, which govern the dynamics of the system. The first one is the equation of motion for the wave function:

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \tag{34}$$

$$i\hbar \frac{\partial \psi}{\partial t} = -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k} \Delta_k \psi + V\psi \tag{35}$$

The second one is the evolution equation for the positions of particles, which is defined through the velocities of particles:

$$\frac{dQ}{dt} = v^{\psi}(Q) \tag{36}$$

where $v^k = (v_1 ... v_N)$ are the velocities of the particles. The role of the wave function is to guide the particles through space-time. For this reason, we define velocity to depend on the wave function ψ. Additionally, we must require Galilean and time-reversal invariance. The simplest choice for the definition of velocity that satisfies the aforementioned requirements is:

$$v^{\psi} = \frac{\hbar}{m} Im \frac{\psi^* \nabla \psi}{\psi^* \psi} \tag{37}$$

and for a N-particle system:

$$v_k^{\psi} = \frac{\hbar}{m_k} Im \frac{\psi^* \nabla_k \psi}{\psi^* \psi} \tag{38}$$

$$\frac{dQ_k}{dt} = \frac{\hbar}{m_k} Im \frac{\psi^* \partial_k \psi}{\psi^* \psi}(Q_1, ..., Q_N) \qquad (39)$$

Here, m_k is the mass of the k-th particle. The $\partial_k = (\partial/\partial x_k, \partial/\partial y_k, \partial/\partial z_k)$ is the gradient with respect to the coordinates $q_k = (x_k, y_k, z_k)$ of the k-th particle. In the case of ψ being a *spinor* [2], the products involving ψ are scalar products. Moreover, in the case where external fields act upon the particles, the gradient should be a covariant derivative involving the vector potential of the external field. The nodes of ψ might cause problems regarding the global existence and uniqueness of dynamics. These issues are discussed in [7], [63].

It is important to mention at this point that Bohmian mechanics is capable of having the same predictions as Copenhagen's interpretation of quantum mechanics. To mention a few of them, spectral lines and quantum interference experiments to scattering theory and superconductivity. Copenhagen's experimental machinery of probabilities and amplitudes which is given by the square of the absolute value of the wave function, in Bohmian mechanics emerge as a consequence of the two equations of motion, namely the Schrödinger equation and the guiding equation. The main difference is that the measurement process is not upgraded to some kind of special interaction needing a non-material entity observing the measuring apparatus.

In order to obtain the probabilistic results of the Copenhagen interpretation, it is enough to suppose that the density[3] ρ of our system at some initial time is given by the square of the absolute value of the wave function $|\psi|^2 = \psi * \psi$. In this supposition, the configuration Q becomes random, meaning that the initial positions of particles are randomly distributed in space. This fact makes Bohmian mechanics reproduce every probabilistic result of the Copenhagen interpretation without presupposing any inherent randomness in nature. If the system does not interact with its environment, then the configuration Q will remain random at a later time. It is at the heart of Bohmian mechanics the idea that the quantum randomness arises from our ignorance about the configuration Q, and not from inherent randomness of the configuration.

To talk about an experiment in Bohmian mechanics, we have to take into consideration the combined system, which consists of the system under observation as well as the instruments and other devices that have significant role in the process. By including the measuring apparatus in our analysis, along with the initial random configuration of the composite system, we obtain the hidden variable model of an experiment. By defining the guiding equation of the composite system, we obtain the final configuration of the system including the final orientation of the instruments. Thus, the deterministic hidden variable model exhibits the same predictions as a Copenhagen version of the experiment.

In 2009, Dürr, Goldstein, Zanghì [28] showed that, the velocity in Bohm's theory can be measured through a method called *weak measurement*. In such

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[2] In physics a spinor is defined as a $\frac{1}{2}$-spin particle

[3] The probability density is defined by the continuity equation that is associated with the Schrödinger equation

a measurement, the wave function is not affected significantly. This method has been developed by Aharonov, Albert, and Vaidman [1]. To experimentally test Bohmian mechanics, Kocsis et al. [48] used weak measurements to measure the trajectories of the particles in the double slit experiment (two-slit interferometer). This experiment confirmed the theoretically calculated trajectories of the De Broglie-Bohm model, as shown in the figure below. In figure 4, we see
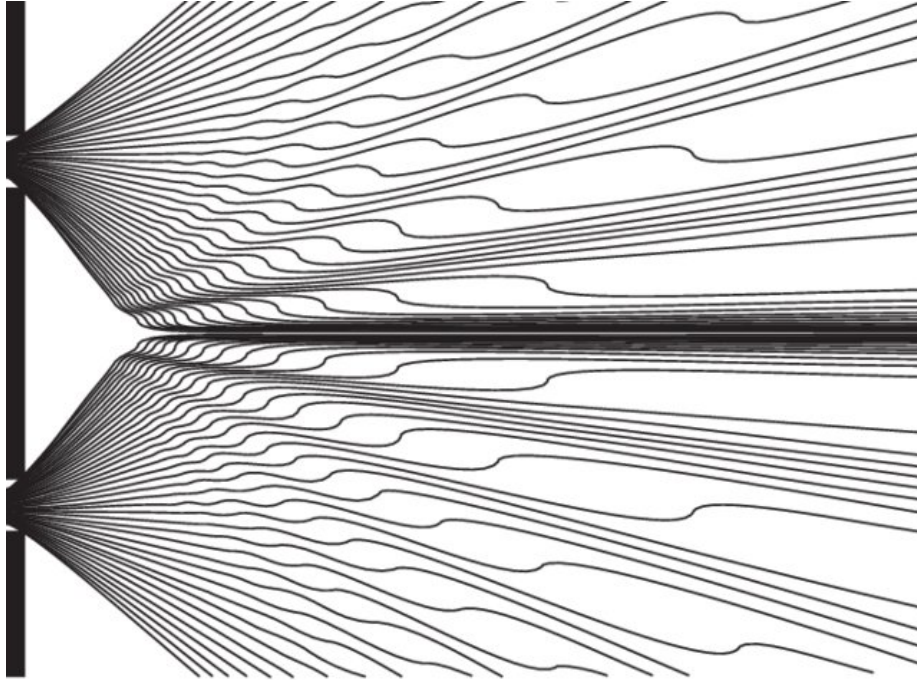

Figure 4: Possible Bohmian trajectories in the double-slit experiment (from C. Philippidis, C. Dewdney and B.J. Hiley, Il Nuovo Cimento 52, 15 (1979)) [55]

a family of Bohmian trajectories for the two-slit experiment. Each trajectory passes through only one slit, but the wave function passes through both slits. In this case, we try to determine the slit from which the particle passes, then according to Bohm, we must include the interaction with the system which makes the determination of the slit (i.e., the measuring apparatus). In 2016, Mahler et al. [50], by the means of weak measurement, tested and confirmed the Bohmian trajectories of entangled photons. Having said that, let us turn into a different formulation of Bohm's theory which would shed light on quantum phenomena from a different perspective.

3.1 Reformulation of de Broglie-Bohm theory

Till now, we presented Bohm theory as a first-order theory, meaning that the fundamental observables like velocity are first order derivatives of position,
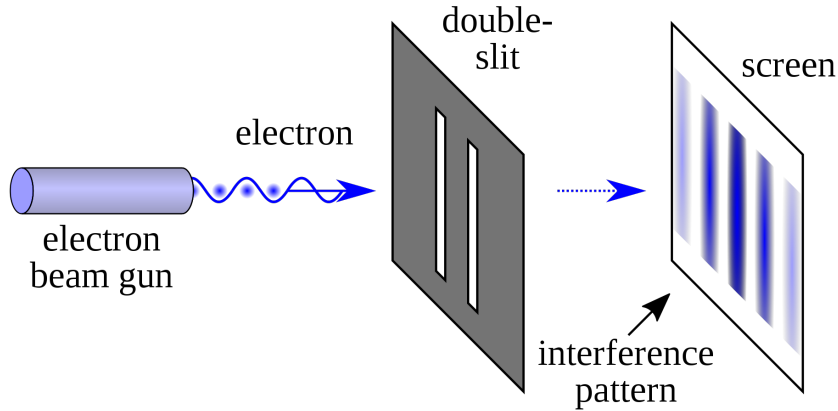
Figure 5: Interference pattern when we do not know from which slit the electron passes through (from Wikipedia)

unlike acceleration, force, work and energy which are second order Newtonian concepts. Bohm [10], however, in his first paper on hidden variables considered the second order version of his theory. He wrote the wave function in polar form, namely $\psi = Re^{iS/\hbar}$, where R, S are real and R non-negative. Through this assumption, he arrived on a pair of coupled evolution equations: the continuity equation for $\rho = R^2$, and an altered Hamilton-Jacobi equation for S which differs from the usual classical Hamilton-Jacobi equation by the appearance of an extra term that is called *quantum potential*.

$$U = -\sum_k \left(\frac{\hbar^2}{2m_k}\right)\left(\frac{\partial_k^2 R}{R}\right) \tag{40}$$

Particle trajectories can be defined through the modified Hamilton-Jacobi equation, in the same way as in classical version of Hamilton-Jacobi equation. We identify $\partial_k S$ with $m_k v_k$:

$$\frac{dQ_k}{dt} = \frac{\partial_k S}{m_k} \tag{41}$$

The motion obtained from these equations is a result of the quantum potential and the usual classical forces. This formulation of the de Broglie-Bohm theory is convenient in order to see in which way Newtonian mechanics emerge from Bohmian mechanics in the classical limit (i.e., $\hbar = 0$). Writing Schrödinger's equation in terms of R, S, however, can increase complexity. The Schrödinger's equation is much more simpler than the modified Hamilton-Jacobi equation, which is nonlinear. Additionally, the modified Hamilton-Jacobi equation requires the continuity equation for R in order for the problem to be well defined. Furthermore, the *ad hoc* introduction of the quantum potential and its form is

neither simple nor natural, meaning that there is a price when one interprets a non-classical phenomenology with classical concepts.

The reformulation of Bohmian mechanics via the quantum potential, induces the theory to account "realistically" for the quantum phenomena. The reason for the introduction of the quantum potential was the non-local character of the quantum phenomena. The first impression from reformulating the theory through quantum potential is that we encounter quantum phenomena through methods of classical mechanics. However, Bohmian mechanics are not merely classical mechanics with an additional quantum potential. In Bohm's theory, in contrast to classical mechanics, the velocities of the particles are not separate from their positions and together they are constrained by the guiding equation. In classical Hamilton-Jacobi theory the term that includes velocity can be eliminated, and thus we take an equation between the positions and the momenta of the particles, like Newton's or Hamilton's equations.

3.2   Active Information

The fact that the quantum potential depends on the form and not on the amplitude of the quantum field is of crucial importance. According to Bohm and Hiley *bohm2006undivided*, effects of this kind are encountered in ordinary experience whenever we are dealing with *information*. For instance, we consider a ship guided by radio waves. One may say that these waves inform the captain about the ship's environment. Similarly, Bohm's theory explains the interference properties in double split experiment by supposing that the quantum field ψ provides information about where the slits are. Thus, the information of particle's environment is included in the movements of the particle. For this reason, they introduce a new concept called *active information*. This concept describes the phenomenon of having very little energy (e.g., radio waves, quantum field) which directs a much greater energy (e.g., the ship, the particle).

Bohm and Hiley define a different kind of information, that is not related to our own knowledge or lack of it. Rather, they suppose information that is capable of determining the movement of the electron. They take the literal meaning of the word, that is to in-form, which is to actively put form into something or to infuse something with form. For example, in the case of the ship and radio waves where the information that transmits the radio wave is the form of the wave. The batteries of the radio provide essentially unformed energy which, subsequently, is given a form by the pattern of the circuit of the radio. This process is entirely objective and has nothing to do with our knowledge of the details of the process. However, we would like to understand active information independently of structures that are designed by human beings. For this reason, let us consider the example of the DNA molecule. The DNA is constituted by a code, while the meaning of the code is expressed by the various processes, such as the protein making activities that are implied by particular sections of the DNA molecule. The notion of active information is also applied here. In the process of cell growth, it is only the form of the DNA molecule that counts while the energy is supplied by the rest of the cell and the environment as

a whole. Thus, unformed energy is becoming in-formed by the DNA molecule. At any moment, only a part of the DNA is being 'read' and giving rise to activity while the rest is potentially active.

To see how the concept of active information is applied in quantum mechanics, we have to examine once again the double slit experiment. In this experiment, Bohm and Hiley *bohm2006undivided* observed that the electron is capable of doing work due to the active information provided by the quantum field ψ. As the electron reaches certain points in front of the slit, it is informed to change its movement according to the obstacles. The electron or any other elementary particle is guided by the information that is contained in the quantum field. This fact suggests that elementary particles may have a complex inner structure. This idea of inner structure goes against the tradition of modern physics which assumes that quantum matter is not analysable into smaller parts and their structure is elementary. However, this inner structure is not always penetrable by our physical theories, for example a large crowd can be treated by simple statistical laws, whereas individually their behaviour is immensely more complex. Similarly, large aggregations of matter can be described by Newtonian laws, whereas molecules and atoms may have a more complex inner structure. As we can see, Bohmian mechanics indicates a different quantum world than previously thought.

A very important implication of active information is that in order to examine the measurement process of a quantum object, we have to take into consideration the entire experimental set up as an undivided whole. This is due to the fact that the motion of the particles can be strongly affected by distant features of the environment, such as the slits or a measuring apparatus. The motion, therefore, of the particles cannot be discussed in abstraction from the total experimental arrangement. This fact makes Bohmian theory "contextual", as the Kochen and Specker's theorem [47] requires. Let us concentrate the key points:

1. Particles have a well-defined position $x(t)$, which varies continuously and is causally determined by the wave function.

2. Particles are inextricably connected with a new type of field that is fundamental and is given by $\psi = R\,exp(iS/\hbar)$. This field (i.e., the wave function) satisfies Schrödinger's equation.

3. Particles satisfy an equation of motion:

$$m\frac{dv}{dt} = -\nabla(V) - \nabla(Q) \tag{42}$$

This equation states that the forces acting on a particle are not only the classical force $-\nabla V$, but also the quantum force $-\nabla Q$

4. According to Bohm's theory, randomness is not inherent property of nature. Rather it is caused by our ignorance of the initial conditions of the ensemble.

5. The wave-like and the particle-like aspects of matter are given special position in Bohmian mechanics. This is done by the introduction of the guiding equation, which combines the positions and the wave function in a single equation.

6. The notion of active information makes Bohmian mechanics contextual. By that we mean that the theory assigns definite values only to certain variables according to the experimental arrangement. Consequent of that fact is that each context (i.e., experimental arrangement) is capable of providing knowledge about reality from a certain point of view.

That being said, let us turn our attention on how Bohm's theory provides solutions to Copenhagen's most difficult conceptual problems, starting with the measurement problem.

3.3   The Measurement Problem

In Bohmian mechanics, as previously discussed, measurement is treated as a reciprocal and irreducible relationship between the measuring instrument and the observed object. In further detail, such process can be divided into two stages, the first stage is the interaction between the measuring apparatus and the observed system where the wave function of the combined system breaks into (i.e., discretized) a sum of non-overlapping packets. These packets correspond to the possible results of the measurement. In the second stage, a significant overlap and interfere is created between the packets of the after-measurement wave function of the combined system. Meanwhile, the apparatus magnifies one of the packets, and therefore the others become physically insignificant. The result of our observation is this particular packet (i.e., component of the wave function). Regarding the first stage of the measurement process, we can treat the wave function in the same way as von Neumann did in his approach. The initial wave function is defined by:

$$\psi_i^{'}(x) = \sum_n C_n \psi_n(x) \tag{43}$$

where $\psi_n(x)$ are the eigenfunctions of the operator O which is measured. We also define the wave function of the measuring apparatus as $\phi_0(y)$, which symbolize the suitable wave packet that describes the position of the pointer on the measuring apparatus. The initial wave function of the combined system is:

$$\Psi_i(x, y) = \phi_0(y) \sum_n C_n \psi_n(x) \tag{44}$$

The quantum object is represented by the wave function with the x subscript, and its complement or the environment of the object by y subscript. During the measurement, the interactions between the system and the environment induce the total wave function to analysed into a sum of the possible results:

$$\Psi(x, y, t) = \sum_n C_n \psi_n(x) \phi_0(y - \lambda O_n t) \tag{45}$$

where O_n are the eigenvalues of the operator O. After the interaction is over, which lasted Δt, the wave function is:

$$\Psi_f = \sum_n C_n \psi_n(x) \phi_0(y - \lambda O_n \Delta t) \qquad (46)$$

The $\phi_n = \phi_0(y - \lambda O_n \Delta t)$ are distinct and non-overlapping packets that yield the various possible results of the measurement. During the measurement, however, the previously distinct packets, as 45 suggests, overlap and interfere chaotically. As a result, the motion of the particle, that is governed by the guiding equation which contains a very complex and rapidly fluctuating wave function, becomes irregular and uncontrolled. The apparatus y magnifies one of the possible wave packets, say m. For this reason, the guidance equation of the positions of the particles will be determined only by the packet $\psi_m(x)\phi_m(y)$, and all the other packets will have zero contribution.

So far, we have not given sufficient arguments for the explanation of the irreversibility of the measurement process. It is Bohm's theory position ([11] chapter 5.2) that in the interaction of a system with a background, which contains a large number of particles like a measuring apparatus, the transition of the wave function becomes irrevocable. This is due to the fact that the information in an unoccupied channel will lose its potential for becoming active due to restrictions of the environment, so that it will remain permanently inactive after interaction with the measuring apparatus or with any complex background. We came to understand that the channels corresponding to actual or potentially active information are constantly narrowing down as the system interacts with its surroundings. However, at the same time the Schrödinger's equation states that the wave function of a system is constantly spreading out, and thus the channels are becoming ever more separated. These two processes are antithetical and the average effect is the irreversible result we observe in nature.

3.4 Schrödinger's cat paradox

As we saw in a previous section, most of the paradoxes of the Copenhagen interpretation arise from the assumption that the wave function provides a complete description of reality. In Bohm's theory, however, that is not true. The mathematical description of a system is completed with the definition of the particles' positions. In this way, we can avoid many of the problems of the Copenhagen interpretation. In order to make things clearer, in this section we examine the Schrödinger's cat experiment. In this example a cat is isolated in a box with a gun pointing at it. The gun fires when an individual electron strikes it. As the electron is headed to strike the detonator will pass through a beam splitter. The purpose of that splitter is to divide the wave function of the electron into two coherent states $\psi_1(x)$ and $\psi_2(x)$. The first state does not provoke the gun as it is absorbed, but the second state goes straight to the detonator and fires the gun. Overall, we expect that there is 50

Let $z_1...z_N$ be the positions of particles constituting the cat and let $\psi_L(z_1...z_N)$ represent a living cat while $\psi_D(z_1...z_N)$ represents a dead cat. The positions

of particles constituting the gun, the bullet and the firing device are represented by $y_1...y_N$. Moreover, we can symbolize the unfired state of the system by $\psi_U(y_1...y_N)$, while $\psi_F(y_1...y_N)$ represent the fired state. When the electron passes the beam splitter the wave function of the total system would be:

$$\Psi(x,y,z) = \frac{1}{\sqrt{2}}(\psi_1(x) + \psi_2(x))\psi_U(y_1...y_N)\psi_L(z_1...z_N) \tag{47}$$

After the process has come to an end, the wave function is:

$$\Psi = \frac{1}{\sqrt{2}}\psi_1(x)\psi_U(y_1...y_M)\psi_L(z_1...z_N) + \frac{1}{\sqrt{2}}\psi_2(x)\psi_F(y_1...y_M)\psi_D(z_1...z_N) \tag{48}$$

Schrödinger could not accept the fact that, according to Copenhagen interpretation, an observation must take place in order to determine if the cat is alive or dead. However, in Bohmian mechanics this issue is resolved because we no longer assume that the wave function is the complete description of reality. A complete description would contain the positions of all the entities participating in the experiment. This fact provides us with the information that, when the cat is alive, many of its particles would be in different positions than they were if the cat were dead. Thus, we can define the actual state, that is the state of the cat (i.e., $\psi_L or \psi_D$), without the need of an observation. All in all, the state depends on the positions of the particles that compose the system and this fact enables us to have mutually exclusive states without the need of measurement.

3.5 Nonlocality in Bohmian Mechanics

Very often non-locality is discussed through the EPR experiment. To see how Bohmian mechanics incorporate non-locality, let us consider an equivalent experiment of the EPR experiment in which a molecule is disintegrated in atoms that are separated by a large distance. Suppose that we measure the spin of the atom A in the z-direction and thus predict the spin of the atom B is opposite. From the moment that we measure spin and disturb atom A, we cannot suppose that the spin of atom A is an element of reality. However, due to the fact that our predictions of spin of atom B can be made with probability equal to unity and without disturbance of the system in any way, then we can conclude that the spin of atom B is and always was an element of reality. Subsequently, we can measure every component of the spin of atom A and in this way to predict every component of the spin of atom B without disturbing the system at all. It follows that every component of the spin of atom B is an element of reality. Furthermore, through the application of Heisenberg uncertainty between the components of spin of the atom B, it follows that they cannot be defined at the same time. Thus, the components of the spin of the atom B cannot be all elements of reality at the same time. Copenhagen's interpretation, therefore, is incomplete because it cannot reflect this fact in any way.

In this equivalent experiment, suppose that there is an atom A with centre of mass R_A and an internal electron coordinate that is r_A. Similarly, there is

an atom B whose variables are R_B, r_B and the whole system of atoms compose a single molecule with total angular momentum zero. The atoms have wave functions given by $\phi_A(R_A), \phi_B(R_B)$. Where $g_A(r_A)$ is the wave function of the particle A and $g(r_B)$ is the wave function of particle B, $g_A(r_A)$ is a state which for example describes the particle A being far away to the left of the original centre of mass of the molecule, while $g(r_B)$ being a state where the particle B is far away to the right. The wave functions of the atoms are $g_A(r_A)\psi_i(r_A)$ and $g_B(r_B)\psi_i(r_B)$.

$$\psi_1(r_A) = -\frac{x_A + iy_A}{\sqrt{2}r_A} \tag{49}$$

$$\psi_0(r_A) = \frac{z_A}{r_A} \tag{50}$$

$$\psi_{-1}(r_A) = \frac{x_A - iy_A}{\sqrt{2}r_A} \tag{51}$$

These represent the three possible eigenvalues (i.e., 1, 0, -1) of angular momentum operator along the z-direction. The combined wave function is:

$$\Psi = N\phi_A(R_A)\phi_B(R_b)g_A(r_A)g_B(r_B)\times \tag{52}$$

$$\times \frac{1}{\sqrt{3}}(\psi_1(r_A)\psi_{-1}(r_B) - \psi_0(r_A)\psi_0(r_B) + \psi_{-1}(r_A)\psi_1(r_B)) \tag{53}$$

$$= -N\phi_A(R_A)\phi_B(R_b)g_A(r_A)g_B(r_B)\frac{1}{\sqrt{3}}\frac{r_A \cdot r_B}{r_A r_B} \tag{54}$$

The above wave function corresponds to total angular momentum zero and N is a normalisation constant. Let us make a measurement of the angular momentum of particle A in any direction. Due to isotropy of the wave function, we can choose the z-direction. We assume that the interaction of measurement in described by the Hamiltonian:

$$H_I = \lambda L_{z_A}\frac{\partial}{\partial y} \tag{55}$$

Where y is the coordinate of the measuring apparatus and λ is the intensity of the measurement. The z-component of the angular momentum is given by:

$$L_{z_A} = -i(x_A\frac{\partial}{\partial y_A} - y_A\frac{\partial}{\partial x_A}) = -i\frac{\partial}{\partial \phi_A} \tag{56}$$

Where ϕ_A is the azimuthal angle of particle A. If $\Phi_A(y)$ is the initial wave packet of the apparatus, then the total initial wave function is:

$$\mathcal{X}_0 = \Phi_A\Psi \tag{57}$$

During the interaction the wave function would be:

$$\xi = \sum_j \psi_j(r_A)\psi_{-j}(r_B)\Phi_A(y - ja\Delta t) \tag{58}$$

where $a \sim \lambda$. For a measurement to occur, $a\Delta t$ must be large enough allowing the $\Phi_A(y - ja\Delta t)$ to not overlap for different j. This fact ensures that there will be no interference between terms of different j and the apparatus particles will enter one of the packets corresponding to a certain value of the angular momentum $j'\hbar$. From then on, it will remain in that packet and the process measurement will be as if it collapsed to the wave function of the system $\psi_{j'}(r_A)\psi_{-j'}(r_B)$. The particle A will have angular momentum $j'\hbar$ and the particle B will have the opposite angular momentum $-j'\hbar$. As Bohmian mechanics shows in [11] the correlation of the two particles is both context dependent and non-local. Particularly, particle B is dependent on the apparatus that measures particle A so the context of B is provided by the apparatus on A and thus the context of B is non-local.

We begin by expanding the wave function ξ:

$$\xi = \sin\theta_A \sin\theta_B (\Phi_A(y - a\Delta t)e^{i(\phi_A - \phi_B)} + \tag{59}$$

$$+ \Phi_A(y + a\Delta t)e^{-i(\phi_A - \phi_B)}) + 2\cos\theta_A \cos\theta_B \Phi_A(y) \tag{60}$$

According to angular momentum operator the z-component of particle A would be $p_{\phi_A} = \partial S/\partial\phi_A$ and for particle B would be $p_{\phi_B} = \partial S/\partial\phi_B$ where S is the phase of the wave function. The general form of the z-component angular momentum of a particle will be given:

$$p_{\phi_A} = \frac{1}{2mi|\xi|^2}(\xi^* \frac{\partial\xi}{\partial\phi_A} - \xi\frac{\partial\xi^*}{\partial\phi_A}) \tag{61}$$

a similar equation holds for particle B. Assuming that $\Phi_A(y)$ is real we can calculate the following expression:

$$|\xi|^2 p_{\phi_A} = Re\{\frac{\sin\theta_A \sin\theta_B}{m}(\xi^*(\Phi_A(y - a\Delta t)e^{i(\phi_A - \phi_B)} - \tag{62}$$

$$- \Phi_A(y + a\Delta t)e^{-i(\phi_A - \phi_B)})\} \tag{63}$$

Due to the fact that the wave function ξ depends on the difference $\phi_A - \phi_B$ we can interchange the indexes so that:

$$p_{\phi_A} = \frac{\partial S}{\partial\phi_A} = -\frac{\partial S}{\partial\phi_B} = -p_{\phi_B} \tag{64}$$

At the beginning, that is when $\Delta t = 0$ the $p_{\phi_A} = p_{\phi_B} = 0$. Subsequently, after a short time the contribution of $\Phi_A(y - a\Delta t)$ and $\Phi_A(y + a\Delta t)$ no longer is zero so the values of p_{ϕ_A}, p_{ϕ_B} will depend on y, which has random distribution with probability $|\Phi_A(y)|^2$. The particle B will depend on the value of y but also depends on θ_A, θ_B and on $\phi_A - \phi_B$. It becomes clear that the behaviour of particle B is rest on the initial values of θ_A and ϕ_A as well as those of the apparatus. Thus, we have a nonlocal interaction and we should also note that the particle A will depend on the initial properties of particle B. Namely, the result of an individual measurement of A is determined by the way particle B starts out. For this reason, nonlocality is reciprocal.

As a result of $p_{\phi_A} = -p_{\phi_B}$ and the separation of the wave packets, one of the particles will end up with a unit of angular momentum and the other with the opposite unit. A possible measurement of B will depend not only on the variables of particle A but also on those of the apparatus that measures A. So it becomes clear that the experiment not only involves nonlocality but a dependence on a non-local context.

3.5.1 Bell's inequalities

After the formulation of Bohmian mechanics, Bell [5] was led to ask whether nonlocality is a necessary feature for all interpretations that reproduce the predictions of quantum mechanics through hidden variables. So, he considered the EPR experiment representing the elements of reality by a set of hidden variables λ. These variables along with the wave function of the system and the measuring apparatus will determine the results of each measurement process. In the case of the EPR experiment, we define orientations parameters a,b which characterise the orientation of the measuring apparatuses that measure the spin of the particle A, B. Additionally, we define hidden variables $\mu_{a,b}$ associated with the piece of measuring apparatus as well as variables $\lambda_{a,b}$ that belong respectively to particles A, B and a further set λ which is associated with the observed system as a whole.

We also introduce a symbol to represent the result of measurement of the spin of the particle A, which is $A = +1$ for a positive spin and $A = -1$ for negative spin, while B is then corresponding symbol for the result of measurement on particle B. It is clear that the result of a measurement will depend on the set of variables that we defined above:

$$A = A(a, \mu_a, b, \mu_b, \lambda_A, \lambda_B, \lambda) \tag{65}$$

$$B = B(a, \mu_a, b, \mu_b, \lambda_A, \lambda_B, \lambda) \tag{66}$$

A and B will also depend on the wave function of the composite system nevertheless we suppress that dependence for the sake of clarity. At this point we have to take into consideration the context dependence of each measurement performed in each section of the experiment. Initially, we are going to assume that only local context is allowed, that is, the result of measuring particle A will depend only on the context of particle A i.e. the measuring device that measures the spin of particle A and not on the context of particle B. Thus, the interaction between A and B will be local if the result A depends only on μ_a, λ_A and not on μ_b, λ_B. We are going to show that a local hidden variables theory cannot reproduce the predictions of quantum mechanics. Therefore,

$$A = A(a, \mu_a, \lambda_A, \lambda) \tag{67}$$

$$B = B(b, \mu_b, \lambda_B, \lambda) \tag{68}$$

It is quite possible to conceive the EPR experiment by means of local explanations of forces that are transmitted at the speed of light or less provided that the measurements were made with space-like separation and held only when there

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was sufficient time for signals to be transmitted back and forth between A and B. However, Bell's version of the EPR experiment is based on the assumption that there is on time for a signal to allow the results of a measurement of one of the particles to depend on the context of the other measuring apparatus.

In order to derive Bell's inequality we have to consider the distribution of hidden variables. Locality dictates that the distributions of μ_a and μ_b are independent as well as when we change the orientation of the apparatus will greatly alter the hidden variables $\lambda_A, \lambda_B, \lambda$ of the two particle system. Due to the fact that the hidden variables μ_a are independent of the orientation of the apparatus, we can replace the symbol μ_a by μ_A which express that the same set of parameters can be used regardless of the orientation of the measuring device and μ_b by μ_B. Consequently, we define three distributions $P_A(\mu_A), P_B(\mu_B)$.

The averages of A and B over the hidden variables μ_A and μ_B respectively are defined by:

$$\bar{A}(a, \lambda_A, \lambda) = \int P_A(\mu_A) A(a, \mu_A, \lambda_A, \lambda) d\mu_A \qquad (69)$$

$$\bar{B}(b, \lambda_A, \lambda) = \int P_B(\mu_B) B(b, \mu_B, \lambda_B, \lambda) d\mu_B \qquad (70)$$

Since

$$|A(a, \mu_A, \lambda_A, \lambda)| = |B(b, \mu_B, \lambda_B, \lambda)| = 1 \qquad (71)$$

Then it follows that

$$|\bar{A}(a, \lambda_A, \lambda)| \leq 1 \qquad |\bar{B}(b, \lambda_B, \lambda)| \leq 1 \qquad (72)$$

We are going to consider the dependence of the correlations of the experimental results of A and B from the various orientations a and b. These correlations are measured in a real experiment and they will provide a test for locality. A typical correlation is defined as:

$$P(a, b) = \int \rho(\lambda_A, \lambda_B, \lambda) \bar{A}(a, \lambda_A, \lambda) \bar{B}(b, \lambda_B, \lambda) d\lambda_A d\lambda_B d\lambda \qquad (73)$$

To derive Bell inequality we consider the difference between two correlations:

$$P(a, b) - P(a, c) = \int \rho(\lambda_A, \lambda_B, \lambda) \qquad (74)$$

$$\times [\bar{A}(a, \lambda_A, \lambda) \bar{B}(b, \lambda_B, \lambda) - \bar{A}(a, \lambda_A, \lambda) \bar{B}(c, \lambda_B, \lambda)] d\lambda_A d\lambda_B d\lambda \qquad (75)$$

$$= \int \rho(\lambda_A, \lambda_B, \lambda) [\bar{A}(a, \lambda_A, \lambda) \bar{B}(b, \lambda_B, \lambda) \{1 \pm \bar{A}(d, \lambda_A, \lambda) \bar{B}(c, \lambda_B, \lambda)\} \qquad (76)$$

$$- \bar{A}(a, \lambda_A, \lambda) \bar{B}(c, \lambda_B, \lambda) \{1 \pm \bar{A}(d, \lambda_A, \lambda) \bar{B}(b, \lambda_B, \lambda)\}] \qquad (77)$$

Taking into consideration that:

$$|\bar{A}(a, \lambda_A, \lambda) \bar{B}(b, \lambda_B, \lambda)| \leq 1 \qquad (78)$$

$$|\bar{A}(a, \lambda_A, \lambda) \bar{B}(c, \lambda_B, \lambda)| \leq 1 \qquad (79)$$

By taking the absolute value of both sides of 74 we obtain:

$$|P(ab) - P(ac)| \leq 2 \pm (P(dc) + P(db)) \qquad (80)$$

Then it is straightforward that

$$|P(ab) - P(ac)| + |P(dc) + P(db)| \leq 2 \qquad (81)$$

This is Bell's inequality which must be satisfied by a local hidden variable theory for a system of two particles with spin [22]. Through these inequalities is possible to test locality in experiments of four sets of correlations. Quantum theory predicts that

$$P(ab) = -a \cdot b \qquad (82)$$

Bell's inequality has been tested in a large number of experiments and has been found to be violated. There is a set of angles for which 82 has been found to does not satisfy the inequality 81. So if we define the angles $\alpha = b - a, \beta = c - d$ and $\gamma = d - c$ and insert them in 81 we get

$$|P(\alpha) - P(\alpha + \beta)| + |P(\gamma) + P(\beta + \gamma)| \leq 2 \qquad (83)$$

If $\alpha = 60, \beta = 60$ and $\gamma = 0$ then the inequality on the left hand side is $5/2$ which is not less than 2 so the inequality is violated.

The most thorough set of experiments has been performed by Aspect *et al.* [2]. In particular these experiments were able to test whether the correlations were maintained even when the events of detection of the two photons were outside each other's light cones. Aspect *et al.* found that the inequality was violated and this implies that we have experimental proof that if hidden variables exist they must be nonlocal.

3.6 Implicate order

Till now we have been concerned mainly with showing a consistent ontological interpretation of quantum mechanics. Bohmian mechanics raise the question of going beyond the current quantum theory and sketches different proposals for doing this. Bohm and Hiley [11] introduce a new concept of order, which they call the *implicate order* or the *enfolded order*. This is to be contrasted with our current concepts of order which are based on the ideas of Descartes who introduced coordinate systems precisely for the purpose of describing and representing order in physical processes with the aid of Cartesian grid and the extended version the curvilinear coordinates which describe what is essentially a local order. However, in quantum domain this order shows its incapacity of representing physical properties which are attributed to well-defined structures and processes in space-time while remaining within the Hilbert space. For example, the uncertainty principle implies that it is not possible to give a definite space-time order to the motion of a particle along its trajectory. The fact that we still use Cartesian coordinates leads to a certain difficulty in talking about the subject of order in quantum mechanics.

The Cartesian grid and the curvilinear coordinates are constant features of physics over the past few centuries. In quantum physics though, physical properties cannot be defined independently from objects and processes so for example the uncertainty principle expresses the inadequacy of defining both the position and the momentum of the particle at the same time. This disparity between the physical concepts (e.g. particle, wave, position, momentum) and the implications of the mathematical equations arises because the physical concepts are involved with the Cartesian notion of order and this violates the essential content of quantum mechanics. What is needed is a notion of order that incorporates all of our concepts, and such a coherent way will be realised with the aid of the notion of implicate order and shall be developed here.

It is generally accepted that general relativity and quantum theory have not been unified in a consistent way. This seems logical by noting that the basic orders implied by relativity theory and quantum mechanics are in complete contradiction. Thus, relativity requires continuity, causality and locality in the order of the movement of particles and fields, and quantum mechanics imply exactly the opposite. Could it be possible that this contradiction of basic concepts of relativity and quantum theory lead to a qualitative new idea that would bridge the contrasts and resolve all the difficulties? Maybe the clue to what this new idea might be, would be the common elements and not the differences between these concepts. The element that they have in common is the quality of *unbroken wholeness*.

To better explain what this means, let us first consider relativity. In relativity all structures have to be understood as forms in a generalised field which is a function of all space-time points. In this context, a particle has to be treated as a stable pulse of finite extent. The pulse from its centre decreases with the distance, but it never goes to zero. Therefore, the fields of all the particles will merge to form a single structure that is an unbroken whole. Similarly, in quantum mechanics even the conventional interpretations speak of quantum processes that link different systems in an unanalysable way. These links can in principle be extended to the whole universe but for practical reasons their effects can be neglected on a large scale, so in some classical approximation we could use a simplified world as made up from separate parts in interaction. For this reason, the definition of the fields is through the concept of space-time points in a Cartesian grid, so it is this order that make it possible to define concepts such as local interaction between fields and general operators, which play a fundamental role in expressing the movements of these fields. To sum up, relativity and quantum theory share a new notion of order that will encompass different kinds of wholeness, which could lead to a physical content that includes relativity and quantum theory as limits.

The next step is to find an appropriate example in our experience that can express the order that gives rise to wholeness that we observe in nature. The ordinary Cartesian order that applies to separate points can be found in the function of the lens. What lens does is to create a correspondence between the points of the object and the points on its image. On the other side the hologram makes possible an image of an object but in a totally different way. In
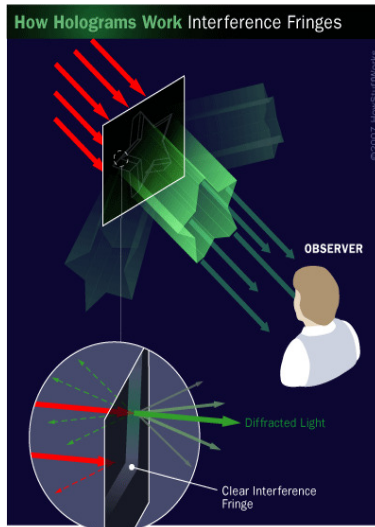
Figure 6: A portion of the incident light scattered in all directions, creates an image in the process and another portion of the light is reflected, from https://science.howstuffworks.com

a hologram we do not have correspondence between points but a combination of remote regions. The regions of the hologram do not look like the object at all but give rise to an image only when they are suitably illuminated. The hologram seems to have no significant order in it but a more careful inspection shows that the whole object is enfolded in each region of the hologram rather than being a point-to-point correspondence.

We shall therefore say that the order of the hologram is *implicate* that it means something that is "enfolded". The order of the object as well as the image, will then be unfolded and we shall call it *explicate*. The process of which the one order is conveyed from the object to the hologram will be called *enfoldment* or *implication*. However, the process in which the order in the hologram becomes manifest to the viewer in an image will be called *unfoldment* or *explication*. These processes are encountered quite often, for example whenever we are in a room the order of the whole room is enfolded in a very small region of space which is inside of our eye, this information is processed by our brain and nervous system to give rise to a conscious awareness of the order of the room. Correspondingly, the order of the whole universe is enfolded into small regions of telescopes and measuring devices which is then unfolded to a conscious awareness like us. The fact that light and other substances, which are the means of observing the whole universe, constantly enfold and unfold the whole is crucial for our ability to learn about the universe. According to [11], almost all laws of movement in quantum mechanics do correspond to enfoldment and unfoldment. Particularly, the evolution of the wave function at one time $\psi(x,t)$ is determined

be the propagator or the Green's function $K(x-x',t-t')$ through the equation:

$$\psi(x,t) = \int K(x-x',t-t')\psi(x',t')dx' \qquad (84)$$

The wave function at x,t is evaluated as the sum of contributions from the whole of x' at an earlier time t' weighted with the factor $K(x-x',t-t')$. We may, therefore, deduct that the region near x enfolds contributions from all over space at other times. The inverse is also true, each region near x' will unfold into the whole space x with the weighting factor $K(x-x',t-t')$. The picture of the movement of a particle is that waves from the whole space enfold and interfere in order to guide the particle into each region and waves from each region or particle radiate or unfold back into the whole space (figure **??** Huygen's lthough Green's function has been derived from the Cartesian order by solving differential equations, we can adopt a point of view that Green's functions are more basic than differential equations and so the order of enfoldment and unfoldment will be fundamental while the Cartesian order will have limited significance. The process of enfoldment and unfoldment is very close with an already well known principle of Huygens where waves from each point unfold and at the same time waves from many points are enfolding to give rise to a new wave front. Thus, the one process includes both enfoldment and unfoldment, only when we focus on one part we are led to consider them as distinct. It is the Huygens' construction that is the basis of the Feynman graphs which are widely used.

To explain the connection between the Huygens' principle and Feynman's graphs let us consider the following example of propagating waves from point P to point Q. As we see in the figure, in an interval of time Δt there is a possible path from P to P'' and in a second interval of time from P' to P'' and so on. Through this construction, the path eventually arrives at Q. The Huygens' principle suggests that the waves that arrive at Q from P are developed from the contributions from every possible path. This is the starting point of Feynman's graphs. We see that Feynman in [34] wanted to regard these paths as actual trajectories of the particles but this fact would not be consistent as waves from various contributions can interfere destructively as well as constructively. Rather, we can regard each path that represents a contribution to the field amplitude as Huygens' principal suggests. And thus, all paths enfold towards Q or alternatively all paths unfold from P and some interfere constructively to lead to P' and so on.
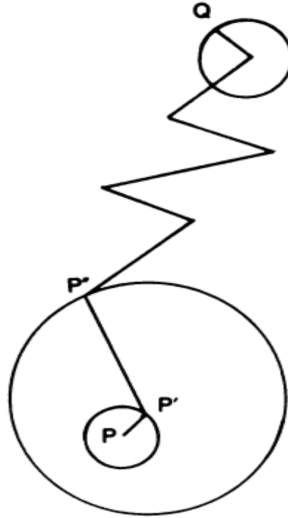
Figure 7: Feynman's graphs and Huygens' principle from [11] p. 356

4 Epilogue

In this work we intended to show the most important issues that trouble the scientists and philosophers of quantum mechanics. It has become evident that these issues are far from being solved. However, a plethora of solutions have been proposed and some of them have been presented in this text, such as Bohm's theory. In this section we are going to make a summary of these issues and draw some conclusions. Starting with Bohr, he was an advocate of two views. The first one was that an atom is stable only in certain energy levels. The second view was that the transition from one level to another is responsible for the emission or the absorption of a photon with energy $h\nu$. He also spoke about the notion of complementarity which claims that there are phenomena that are complementary meaning that they cannot occur at the same time. By phenomena he meant the measurement of the value of a particular physical quantity. As an example the complementarity states that the measurement of position and momentum cannot be actualized at the same time and this fact is expressed by Heisenberg's uncertainty condition. Bohr believed that complementarity is of crucial importance in understanding quantum phenomena.

The next issue that puzzled scientists in the 1920's was the measurement problem. They could not explain the fact that the theory of quantum mechanics predicts that the state of the measuring device should be in superposition of different results. To tackle this problem, John von Neumann proposed a solution in which the wave function "collapses" instantaneously in one of the possible results. According to von Neumann this kind of process happens whenever the observer' consciousness measures a quantum object. This is due to the fact that

mental interactions are the only kind of interactions that cannot be described by quantum mechanics and thus only those are able to induce the collapse of the wave function. Copenhagen interpretation is often connected with von Neumann's solution to the measurement problem.

However, the Copenhagen interpretation does not explain with accuracy which parts form the measuring apparatus and which the quantum system itself. Bell argued that the separation between the quantum system and the measuring device is arbitrary. Thus, what we consider as a system can always change so we can enlarge it to include what was previously defined as an apparatus, but then we need something external to the enlarged system to provide the collapse of the wave function of the enlarged system. The next topic we dived into was the EPR argument. Based partly on incompatible variables (i.e., variables satisfying the uncertainty relation) and on arguments about the locality and separability of the theory, EPR attempted to show that quantum mechanics is incomplete. They arrive to the understanding that the wave function is unable to assign definite values to incompatible variables. For this reason, they were led to conclude that the quantum mechanical description of the system in terms of its wave function is incomplete. They thus supported the view that there are facts about the states of the particles that we do not know and are not incorporated in the wave function. These facts called *hidden variables*. It follows that Copenhagen's interpretation of quantum mechanics can say little about reality itself. Philosophically speaking, it does not give what can be called an ontology for the quantum system. On the other side, the ontology of Bohmian mechanics is very simple: everything is made up from particles with a well-defined position, guided by waves. For Bohm position plays a fundamental role in his theory which is why he proved that the measurements of variables are essentially position measurements. Additionally, electrons are particles whose trajectories are guided by a pilot field which is the wave function solution of the Schödinger equation. The uncertainties in experiments are coming from the uncertainties in the initial conditions of the trajectories of the particles. Also, the non-local character of Bohmian mechanics is due to the quantum potential of the system of N particles. The quantum potential is responsible for the instantaneous and non-local changes in the trajectories of the particles. To make things clearer, let us see some differences and similarities between Bohmian mechanics and classical physics:

1. Differences

    (a) **Quantum wholeness:** Due to the presence of the term $Q(x,t)$ in the quantum Hamilton-Jacobi equation, we can imply that Bohmian trajectories depend not only on the classical potential $V(x,t)$ but also on the quantum potential $Q(x,t)$, which is a function of $R(x,t)$. It is, thus, the shape and not the absolute value of $R(x,t)$ that acts on each individual quantum trajectory. Classical trajectories, on the contrary, cannot be computed from the shape of an ensemble.

2. Similarities

(a) In the classical version as well as the quantum Hamilton-Jacobi equation only the initial position has to be fixed because the initial velocity is determined by the spatial derivative of the action function S. However, one can argue that we also need to fix the initial wave function that fixes the initial velocity. Moreover, for a particular Hamiltonian even if one fixes a particular position then it is plausible to obtain different initial velocities.

In Bohmian mechanics measurement process is treated like any other quantum process of interacting particles that occupy certain positions. Thus, the previous measurement difficulties of Copenhagen interpretation dissolved. The quantum system is a collection of a many-particle trajectory as well as a many-particle wave function. The wave function and the trajectory are associated with the entire system, that is, the quantum system plus the measuring apparatus. So, there is a dynamical law for the evolution of the wave function and another for the evolution of the trajectory:

1. The dynamical law of Schödinger equation involves the Hamiltonian of the quantum system plus the measuring apparatus and determines the time evolution of the wave function independently of whether a measurement process takes place or not.

2. The movement of the particle, which is its trajectory, is determined by the time integration of the Bohmian velocity. It is independent of whether a measurement process takes place or not.

For example, let us consider that some kind of pointer specify the measured quantity. The Hamiltonian of the system will also include the degrees of freedom of the particles that make up the pointer. When the trajectories which are associated with the positions of the pointer are known, then the value of the measurement is predicted. We just need knowledge of the position of the pointer particles. Thus, we have to introduce into the Hamiltonian the interaction of the particles of the pointer with the rest of the particles of the system. However, a pointer in a measuring device is a macroscopic object with a number of particles on the order of Avogadro's number ($\approx 10^{23}$ particles) so it would be impossible to simulate such a large number of particles and know exactly all the "positions of the instrument pointers".

Let us turn our attention to the Cartesian order which has proven its limits due to its inability to give a concrete description of quantum reality without leading in paradoxes. Cartesianism has shown its inadequacy to keep up with the evolution of modern theories like quantum mechanics and theory of relativity due to their non-mechanistic point of view. Considering the historical evolution of scientific theories of the 20th century we can find out that the changes that took place in those theories can be closely correlated. The correlations are not excused just from coincidence but they have arisen from tumultuous processes of scientific research and paradoxes in mathematics and physics. The most characteristic correlation has to do with the role of observer in quantum

mechanics as well as in intuitionistic mathematics of Brouwer. In both areas the observer becomes an inseparable element and an important factor of the scientific activity through the act of observation in physics or as a part in the creation of mathematical ideas. For Brouwer [18] mathematics is considered a mental activity of humans and thus the truth of a mathematical statement is a subjective claim. This view about mathematics was in contrast with the dominant view in the early years of the 20th century. For an intuitionist, the discovery of a mathematical object is independent of whether that object exists in reality, mathematics does not necessarily reveal some deep reality of the world. On top of that, the intuitionist is forced to reject some of the assumptions of the classical logic. This is due to the fact that the existence of an object cannot be proven by refuting its non-existence. This fact led Arend Heyting [42] on the discovery of intuitionistic logic. The intuitionistic logic is a system of symbolic logic which rejects the law of the excluded middle and the double negation elimination, and thus is a more general system of abductive reasoning.

The same narrative can be applied in the case of physical theories. Physical theories are discovery of the human cognition. Thus, they are an understanding of the world from a specific perspective. This simple statement is able to reformulate physical theories on a more realistic base in which contextuality plays a significant role. A reformulation of physical theories on the basis of intuitionistic logic provides us with a more general and contextual background on which we can do physics. This reformulation was initiated by C.J. Isham, J. Butterfield and J. Hamilton [46], [45],[19], [38], [20] and they used the mathematical framework of topos theory. This theory is a branch of mathematics that is called category theory. Topos theory and by extension category theory is a general mathematical theory of structures which is composed of objects and arrows between them. These theories provide us with categories or universes of discourse and methods for passing from one to the other.

In this work we presented the historical evolution of quantum mechanics in the first half of 20th century. Through this endeavour we came to understand that the truthfulness of an argument about a quantum object depends on how it is measured. This fact about the quantum phenomena tell us that there are no prior to measurement properties but the quantum object together with the measuring apparatus create a physical reality which is unpredictable. Therefore, every description that does not take into account that the quantum reality is in constant formation and development by the rest of the universe, cannot be complete. For this reason, we believe that a study of a categorical approach to physics could enable us to understand physical theories in different experimental contexts.

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