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Spherically Symmetric Solutions in Bi-Metric Gravity

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Contents

Abstract

In this thesis we will examine the different types of Bi-metric theories of gravity. We examine Rosen's Bi-metric theory and the newer type of model, the Scalar-Bi metric gravity. It is in this particular model, studied mainly by M. A. Clayton and J.W. Moffat, that we look for and extract black hole solutions. The model is concerned with two metrics, describing how matter and gravitational fields respectively propagate through space time. In this type of theory, the two metrics in question interact through a scalar field φ. The solutions we are looking for are black hole type metrics, so we impose a certain general, spherically symmetric ansatz on the line element ds^2. Deriving the total action of the system we arrive at a system of second order differential equations that are numerically solved. Due to the complexity of the equations and the freedom of choice arising from the symmetry, we impose some simplifications and restriction on our metric functions and their respective boundary conditions. All our functions and solutions are only r-dependant, since we are looking for static cases.

Chapter 1

Introduction

1.1 General Bi-metric theories and motivation

As the name suggests, a bi-metric theory is a modified version of General relativity that includes not one, but two metric tensors $g_{\mu\nu}$. This addition can be of different effect, as there are many ways to introduce a second metric tensor. The different bi-metric theories through the years have been of various forms, from purely geometrical, to the more modern *massive bi-gravity* theories [14, 13]. The important question one must ask is, why two metrics? The motivation behind the development of bi-metric theories, is to see how two space-times interact with one another. While this is purely geometrical in its formulations, latter bi-metric theories have tackled the problem of how gravitational and matter fields interact, and some have even been used to introduce massive particles (gravitons) in gravity theories. Furthermore, the introduction of the new metric can be used to generate corrections to already calculated properties, such as the speed of gravitational waves and even impose cosmological constrains.

1.2 General Relativity with one metric

We shall begin this thesis by briefly mentioning the standard procedure used in General Relativity in its most basic form, that being with one metric.

In general relativity, the equation -or rather, equations- that describe how space-time and matter interact are the **Einstein's Field Equations** (EFE)

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.1}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor, R is the scalar curvature, Λ is the cosmological constant, G is Newton's gravitational constant, $T_{\mu\nu}$ the stress energy tensor, and $g_{\mu\nu}$ the metric tensor. The metric tensor describes the geometric and causal structure of spacetime, and is used to define things such as distances, time, angles and volumes. In general relativity this tensor describes coordinates in spacetime, generally in (t, x, y, z) notation. However due to the nature of many problems in general relativity such as symmetrical solutions, the metric tensor is usually written in spherical coordinate notation (t, r, θ, ϕ).

The EFE is a tensor equation relating a set of symmetric 4x4 tensors, each with 10 independent components. The Bianchi identities help to reduce the number of equations from 10 to 6, leaving the metric with four degrees of freedom. One can further compact the EFE by defining the *Einstein tensor*

$$\mathcal{E}_{\mu\nu} = R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu} \tag{1.2}$$

which is a function of the metric $g_{\mu\nu}$. The EFE thus become

$$\mathcal{E}_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{1.3}$$

This set of equations along with the *geodesic equation* which describes free-falling matter through space-time, is the core mathematical formulation of general relativity.

To derive these equations, we will start from the **least action principle**. The Einstein-Hilbert action is defined as

$$S = \frac{1}{2\kappa} \int R\sqrt{-g} d^4x \tag{1.4}$$

where $g = det(g_{\mu\nu})$ is the determinant of the metric tensor and R is the Ricci scalar. This however is only the gravitational part of the action. Introducing a term $\mathcal{L}_M$ describing any matter field appearing in the theory we get

$$S = \int \left[\frac{1}{2\kappa} R + \mathcal{L}_M \right] \sqrt{-g} d^4x \tag{1.5}$$

The least action principle tells us that physical laws result from demanding that the variation of this action in respect to the inverse metric is zero. This yields

$$\begin{aligned} dS &= \int \left[\frac{1}{2\kappa} \frac{\delta(\sqrt{-g}R)}{\delta g^{\mu\nu}} + \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} d^4x \\ &= \int \left[\frac{1}{2\kappa} \left(\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \right) + \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} \sqrt{-g} d^4x \\ &= 0 \end{aligned} \tag{1.6}$$

This equation should hold for any variation of the inverse metric, so the expression within the brackets gives

$$\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -2\kappa\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \tag{1.7}$$

This is the equation of the metric field. The right side of the equation is by definition proportional to the stress energy tensor

$$T_{\mu\nu} := \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} = -2\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_M \tag{1.8}$$

To calculate the left side of the equation we need the variations of the Ricci scalar and the metric determinant, which are standard textbook calculations and give:

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu} \tag{1.9}$$

for the Ricci scalar variation and

$$\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu} \tag{1.10}$$

for the metric. Now that we have all the derivations we need, we can plug (1.9) and (1.10) back into equation (1.7) of the metric field and we get

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.11}$$

which is exactly Einstein's equation after we set

$$\kappa = \frac{8\pi G}{c^4} \tag{1.12}$$

If one was to re-introduce the cosmological constant in the action, the process would be exactly similar, and the result would simply be equation (7.1). Without a cosmological constant and in the absence of matter and energy, or in other words when the energy momentum tensor is zero, the set of equations are now the **vacuum field equations**

$$R_{\mu\nu} = 0 \tag{1.13}$$

Solving these equations is the basis on which all the gravitational and cosmological models are generated. In our case however, we want to focus on very specific type of solution, and perhaps the most intriguing and interesting one in all of general relativity, Black holes. In the next section we shall study the normal black hole solution in GR and its properties.

1.3 Black Holes in GR: The Schwarzschild Solution

The most apparent application of a theory of gravity is to calculate a spherically symmetric gravitational field. The obvious motivation behind this is that most relevant object such as the Earth or the Sun, which are also described by a (approximately) spherically symmetrical distribution of mass. We are also mainly concerned with the empty space outside the object and how test particles move through it, as it is more immediately useful. This search leads to the object of interest, black holes.

In classic General Relativity, the physically symmetric solution in vacuum is the famous **Schwarzschild Metric**

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + d\Omega^2 \tag{1.14}$$

where $d\Omega^2$ is the metric on a unit-two sphere,

$$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2 \tag{1.15}$$

M represents the mass of the gravitating object. The original derivation of this metric by Schwarzschild [12] was motivated by searching for solutions *outside* a spherical object, so Einstein's equations take the vacuum form:

$$R_{\mu\nu} = 0 \tag{1.16}$$

The object in question is hypothesized to be static (no evolution with time) and spherically symmetric, so the final solution should also have these properties. The term static in this case simply means that the components of the metric will be time-independent, and there will be no cross product terms such as $dtdx^i + dx^i dt$. This last condition is logical if one imagines a coordinate transformation $t \longrightarrow t'$ should leave the dt^2 unchanged. To impose spherical symmetry, we begin with the Minkowski metric of flat space-time in coordinates $\chi^\mu = (t, r, \theta, \phi)$

$$ds^2_{Minkowski} = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{1.17}$$

The last term, $d\Omega^2$, must maintain the form (1.15) in order for spherical symmetry to be preserved. As for the other metric components, we are free to affix whatever coefficients we want, as long as they remain only r-dependent, such as:

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\gamma(r)}r^2 d\Omega^2. \tag{1.18}$$

The next step is to apply a slight variation to the metric (1.18) before we move on to Einstein's equation itself. This change occurs by defining a new radial coordinate $\bar{r}$ as

$$\bar{r} = e^{\gamma(r)}r \tag{1.19}$$

and consequently

$$d\bar{r} = e^{\gamma}dr + e^{\gamma}r d\gamma = \left(1 + r\frac{d\gamma}{dr}\right)e^{\gamma}dr \tag{1.20}$$

With this new variable, the metric (1.18) becomes

$$ds^2 = -e^{2\alpha(r)}dt^2 + \left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta(r)-2\gamma(r)}d\bar{r}^2 + e^{2\gamma(r)}\bar{r}^2 d\Omega^2, \tag{1.21}$$

Where obviously any function of r is now a function of $\bar{r}$. One can now relabel

$$\bar{r} \longrightarrow r \tag{1.22}$$

$$\left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta(r)-2\gamma(r)} \longrightarrow e^{2\beta} \tag{1.23}$$

without losing information, as none of these two are defined externally and independently of the other components. Now our metric (1.21) becomes

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \tag{1.24}$$

which resembles (1.18) but the $e^2\gamma$ factor has vanished. Note here that this involves no loss of generality, as $e^2\gamma$ has not been set to 1. Setting this to one would imply some information about the geometry, but with the specific coordinate transformation and relabeling, the factor has simply disappeared.

We're now in a position where we can take Einstein's equation and solve for $\alpha(r)$ and $\beta(r)$. As per standard procedure, we calculate the non-vanishing Cristoffel symbols and the components of the Riemann tensor. Contracting the

Riemann tensor as usual, we arrive at the Ricci tensor:

$$
\begin{aligned}
R_{tt} &= e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] \\
R_{rr} &= -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta \\
R_{\theta\theta} &= e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1] + 1 \\
R_{\phi\phi} &= sin^2\theta R_{\theta\theta}
\end{aligned} \tag{1.25}
$$

With the Ricci tensor calculated, we shall set it to zero because of Einstein's equation in vacuum. We can see that R_{tt} and R_{rr} vanish independently, so we write:

$$0 = e^{2(\beta-\alpha)}R_{tt} + R_{rr} = \frac{2}{r}(\partial_r\alpha - \partial_r\beta). \tag{1.26}$$

Taking the last term and equating it to zero we can see that α and β satisfy

$$\alpha = -\beta + c \tag{1.27}$$

where c is a constant. This constant can be set to zero if we rescale with the coordinate transformation $t \to e^{-c}t$. After this we have

$$\alpha = -\beta \tag{1.28}$$

We then calculate the $R_{\theta\theta} = 0$ term of (1.9), which gives

$$e^{2\alpha}(2r\partial_r\alpha + 1) = 1 \tag{1.29}$$

or equivalently

$$\partial_r(re^{2\alpha}) = 1 \tag{1.30}$$

Solving the above we get

$$e^{2\alpha} = 1 - \frac{R_s}{r} \tag{1.31}$$

where R_s is a constant not yet defined. If we now consider this with $\alpha = -\beta$ our metric becomes

$$ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1}dr^2 + r^2d\Omega^2. \tag{1.32}$$

The only freedom we have at this point is to define the constant R_s, called the ***Schwarzschild Radius*** and interpret it in terms of some physical parameter. This is easily done by considering the weak-field limit, in which the tt component of the metric around a point mass object satisfies

$$g_{tt} = -\left(1 - \frac{2GM}{r}\right) \tag{1.33}$$

We can use this because in the formulation of the problem, we stated that far outside the object ($r >> 2GM$) the Schwarzschild metric should reduce to the Minkowski metric, so we approach the weak field limit. So in the case of our g_{tt} we only need to set

$$R_s = 2GM \tag{1.34}$$

The final result is the Schwarzschild metric (7.3). This holds for any spherically symmetric solution to Einstein's vacuum equations. M functions as a parameter, but we know we can interpret it as the conventional Newtonian mass which we would measure by studying orbits far outside the object. We see that as $M \to 0$ we recover the Minkowski space, which is expected because the geometry should become flat in absence of a physical mass. Note that the metric also becomes Minkowski-like as $r \to \infty$. This property is known as asymptotic flatness.

The Schwarzschild metric is not simply a good solution, but also the unique spherically symmetric vacuum solution, due to **Birkhoff's theorem**. The most interesting fact about this metric is obviously the singularities. We can see that the metric coefficients become infinite at $r = 0$ and $r = 2GM$. It is difficult to say however if these represent true physical singularities, or simply a bad coordinate system. We shall not go into great detail as to how this issue is resolved, but of these two, only $r = 0$ represents an actual physical singularity.

The Schwarzschild solution can be thought to be a specific case of a much wider family of black holes. Different solutions arise if we consider, for example, more properties of the black hole (charge, angular momentum) which give Kerr black holes, or different conditions of the whole system (negative Cosmologigal constant) in which case we arrive at solutions such as the de Sitter/ anti de Sitter black holes.

We are now ready to move to the different theories that involve not one, but two metrics, and how they deal with the problem of spherically symmetric solutions.

Chapter 2

Rosen's Bi-Metric theory

2.1 Introduction

Now that we have briefly studied both the normal formulation of GR with one metric and the usual Schwarzschild solution, it is time to move to the main focus of this project, namely Bi-metric theories of gravity. The earliest such theory was created by Nathan Rosen in 1940.

Nathan Rosen proposed his theory of bimetric gravity in his 1940 pair of papers "General Relativity and Flat Space I and II" [10, 11]. The theory proposes that at every point in space-time, a second metric tensor $\gamma_{\mu\nu}$ is introduced along the usual metric $g_{\mu\nu}$. This new tensor corresponds to flat space, i.e for which the Riemann-Christofell tensor vanishes identically everywhere. This can be interpreted in various ways. First, we can suppose that the Riemannian space with the metric $g_{\mu\nu}$ is mapped onto a flat space through the metric $\gamma_{\mu\nu}$. Another way to look at this case is to consider the metrics side by side as a comparison between the normal space with the one that has the gravitational field removed. Note here that the introduction of $\gamma_{\mu\nu}$ does not assume any new properties in space-time.

With this addition, one can define a Euclidean line element in a similar way to general relativity:

$$d\sigma^2 = \gamma_{\mu\nu} d\chi^\mu d\chi^\nu \tag{2.1}$$

One can now define covariant differentiation based on $\gamma_{\mu\nu}$. Since we assumed that the tensor represents flat space with a vanishing Riemann-Christofell tensor, it naturally follows that one can interchange the order of the γ-differentiation so that just the γ-derivatives vanish, while all the ordinary differentiation rules are obeyed. Although it is possible to always choose such a coordinate system due to special relativity, it is not always convenient.

One can then consider the Christofell symbols $\left\{ {\lambda \atop \mu\nu} \right\}$ and $\Gamma^\lambda_{\mu\nu}$ for g and γ differentiation respectively. Then one defines $\Delta^\lambda_{\mu\nu}$ by:

$$\left\{ {\lambda \atop \mu\nu} \right\} = \Gamma^{\lambda}_{\mu\nu} + \Delta^{\lambda}_{\mu\nu} \tag{2.2}$$

Since the difference of two connections is a tensor, $\Delta^{\lambda}_{\mu\nu}$ is a tensor and is found to be given by:

$$\Delta^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}) \tag{2.3}$$

If now, in the usual expression of $R_{\mu\nu}$

$$R_{\mu\nu} = -\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\alpha\mu,\nu} - \Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{\mu\nu} + \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} \tag{2.4}$$

substitute 2.2, it is found that it transforms as:

$$R_{\mu\nu} = -\Delta^{\alpha}_{\mu\nu,\alpha} + \Delta^{\alpha}_{\alpha\mu,\nu} - \Delta^{\alpha}_{\alpha\beta}\Delta^{\beta}_{\mu\nu} + \Delta^{\alpha}_{\beta\mu}\Delta^{\beta}_{\alpha\nu} \tag{2.5}$$

We can see that $R_{\mu\nu}$ is obtainable from the $\Delta^{\lambda}_{\mu\nu}$ tensor through tensor operations. If we compare this form with its usual expression, we can see that $\left\{ {\lambda \atop \mu\nu} \right\}$ has been replaced by $\Delta^{\lambda}_{\mu\nu}$, and ordinary differentiation by γ-differentiation. It follows that one is possible to construct all the quantities in relativity theory so that $\left\{ {\lambda \atop \mu\nu} \right\}$ is replaced by $\Delta^{\lambda}_{\mu\nu}$ and the differentiation replaced in the say way as before. Rosen, through this fact, demonstrates the advantage of his bi-metric theory, since through the Riemann tensor many of the quantities obtain tensor qualities that they previously did not. It is later proved that the quantities that depend on $\gamma_{\mu\nu}$ lose their tensor character. On the other hand, $R_{\mu\nu}$ is independent of $\gamma_{\mu\nu}$ and hence remains a tensor. Obviously, if one wished to keep the tensor character of quantities depending on $\gamma_{\mu\nu}$, one must allow $\gamma_{\mu\nu}$ to transform as a tensor through coordinate transformations.

Although $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ have been considered to exist side by side, so far there has not been any mention of a specific relation between them. Obviously, some relation between them must exist, since it is reasonable to think that if the gravitational field vanished, only the flat space-time $\gamma_{\mu\nu}$ should remain. One such relation can be created by imposing four additional covariant conditions on the field. This is possible because of the set of equations:

$$G_{\mu}{}^{a} : a = 0, \tag{2.6}$$

where (:) denotes differentiation by $g_{\mu\nu}$ (g-differentiation). Einstein too added four such conditions when working with linear approximation of the gravitational equations, in order to eliminate (or at the very least reduce) fields arising from infinitesimal coordinate transformations. These equations serve a similar purpose and they are the ones that provide the relation between the two metrics. They should remove or at least restrict any sort of ambiguity in the form of the solution arising from whatever coordinate transformation is chosen.

2.2 Spherically Symmetric Solutions

Rosen at this point demonstrates the previous statement by showing that a static, spherically symmetric solution arising from the gravitational equations, can be expressed in a number of forms, depending on the choice of the radial variable r.

In detail, one set of reasonable conditions arise from the following considerations: Expression (2.5) can be re-written as:

$$R_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}g_{\mu\nu,\alpha\beta} - \frac{1}{2}S_{\mu;\nu} - \frac{1}{2}S_{\nu;\mu} - \Delta^{\alpha}_{\mu\beta}\Delta^{\alpha^*}_{\nu\beta} - \Delta^{\alpha}_{\mu\beta}\Delta^{\nu^*}_{\alpha\beta^*} - \Delta^{\alpha}_{\nu\beta}\Delta^{\mu^*}_{\alpha\beta*} \quad (2.7)$$

where

$$S_{\mu} \equiv \Delta^{\mu}_{\alpha\alpha*} \equiv g^{\alpha\beta}g_{\alpha\mu,\beta} - \kappa_{\mu}/\kappa \quad (2.8)$$

and asterisks denote raising or lowering indices in respect to $g_{\mu\nu}$. We can further take

$$S_{\mu} = 0 \quad (2.9)$$

as an additional condition as suggested by (2.2). It can be verified these conditions are, to first approximation, the same as Einstein's when considering the linear equations . As a result, the right hand part of (2.7) is transformed, as only the first term now has second order derivatives. To get the spherical solutions, we solve

$$R_{\mu\nu} = 0 \quad (2.10)$$

in free space. Together with (2.2) we finally get

$$ds^2 = \frac{r-m}{r+m}dt^2 - \frac{r+m}{r-m}dr^2 - (r+m)^2(d\theta^2 + sin^2\theta d\phi^2) \quad (2.11)$$

where m is a constant. Another set of possible conditions can be obtained if we set

$$\kappa = 1 \quad (2.12)$$

which simplifies many of the expressions in (2.7) and (2.2). This is also not affected by coordinate transformations, unlike the condition g=-1 that is often used. For the other three necessary conditions, one can take

$$S_{\mu,\nu} - S_{\nu,\mu} = 0 \tag{2.13}$$

as it is by itself equivalent to three expressions, and we already established that S_μ=0 is a good choice. At this point, one can take for $d\sigma^2$ the usual notation

$$d\sigma^2 = dt^2 - dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{2.14}$$

then the final spherically symmetric solution takes the form

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{1}{(1 - \frac{2m}{r})}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{2.15}$$

As we can see, the solution is of Schwarzschild form, so it constitutes a black hole solution.

In conclusion, Rosen's theory of Bi-metric gravity presents us with some advantages, specifically the tensor characteristics that it imposes on many of the classically non-tensor quantities. Further more, a spherically symmetric solution does indeed exist, and it is none other than the Schwarzschild metric. Despite the above mentioned positives, this theory remains purely geometrical - the only thing added is another metric, and no information is given on how matter can interact with the generated space-time. Rosen's bi-metric relativity and general relativity differ when it comes to:

- the propagation of electromagnetic waves
- the external field of a high density star
- the behavior of intense gravitational waves propagating through a strong static gravitational field.

Rosen kept improving on his theory with publications up to 1980, where he tried removing appearing singularities, and in 1989 with two papers regarding how elementary particles behave in bi-metric relativity. Since 1992, however, the predictions of gravitational radiation in Rosen's theory have been shown to be in conflict with observations of the Hulse–Taylor binary pulsar.

Building on Rosen's theory and its limitations, a different type of bi-metric theory emerged since the 1990s. This particular theory, tackles the problem of describing how electromagnetic waves propagate through a bi-metric space-time, and it's the next one that we examine.

Chapter 3

Scalar-Tensor Bi-gravity

3.1 Introduction

The main type of theory that we examine is scalar Bi-metric theory. This theory has been introduced and worked on extensively by M. A. Clayton and J.W. Moffat [5, 4, 3, 2]. This theory builds on the previous models such as Rosen's, as it adds a dynamical nature to the two interacting metrics in the form of a scalar field.

The most general form of this model can be expressed as

$$\hat{g}_{\mu\nu} = A(\varphi)g_{\mu\nu} + B(\varphi)\partial_\mu\varphi\partial_\nu\varphi \tag{3.1}$$

where ϕ is a scalar field, called the "bi-scalar field" and $\partial_\mu\phi = \partial\phi/\partial\chi^\mu$. The inverse metrics $\hat{g}^{\mu\nu}$ and $g^{\mu\nu}$ satisfy the relations

$$\hat{g}^{\mu\alpha}\hat{g}_{\nu\alpha} = \delta^\mu_\nu, \quad g^{\mu\alpha}g_{\nu\alpha} = \delta^\mu_\nu, \tag{3.2}$$

The idea is that the tensor $\hat{g}_{\mu\nu}$ which is referred to as the "matter metric" generates the geometry in which matters fields propagate and interact, and $g_{\mu\nu}$ is the "gravitational" metric and represents the geometry through which gravitational waves propagate. It is immediately obvious that for $A(\phi) = 1$ and $B(\phi) = 1$, the model collapses to the usual GR. An immediate simplification is to set $A(\phi) = 1$ and $B(\phi) = B =$ constant, which eliminates complex contributions to the field equations in the form of reducing derivative dependencies to ϕ. The choice of A=1 is motivated because it has been extensively studied in the Brans-Dicke case (as a conformal factor). It is clear even from this point, the nature of the interacting metrics implies a different propagation speed for matter and gravitational waves, and greatly alters the matter-geometry coupling. Depending on the choice of frame, spacetimes can be viewed as either having a fixed speed of light and a dynamically determined speed of gravitational waves, or a fixed gravitational waves speed with a dynamic speed of light.

Another very interesting aspect of this theory is the choice of the constant B. The choice is a bit particular, as it directly connects the 2 metrics together. The motivation behind its selection on [5, 4] is cosmological in nature, and equal to

$$B = \frac{1}{32} {l_p}^2 \tag{3.3}$$

where l_p is the Planck length and is given by $l_p = \sqrt{Gh/c^3}$. Later on in the papers, B is further constrained as

$$l_p = 10^{-5}\sqrt{12B} \tag{3.4}$$

in order for the Planck scale to affect the CMB spectrum in a way that agree with observations. In a theory as general as this one, however, it is not a good idea to define the form of B just yet. This is because in a general scalar bi-metric theory, electromagnetic and gravitational fields propagate with different speeds, so the Planck length can dynamically vary. This implies that the scale where quantum effects become important will be different for different fields. This variable quantum scale is nonetheless a powerful constraint for scalar bimetric cosmology.

As one can infer, the dynamical nature of the speed of light is very powerful and crucial in formulating cosmology models based on the theory. Such interesting scenarios postulated by Moffat and Clayton include inflation models, CMB spectrum corrections, dimming of stellar events such as supernovae and, solutions to the cosmological problem and many possible results from gravity wave astrophysics. In our case, we will not study the various cosmological models, but instead focus on a topic not yet studied in this kind of scenario, namely the potential black hole solutions of a scalar bi-metric theory.

3.2 The Model

Similar to Moffat and Clayton, we will use the simplified model

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + B\partial_\mu\varphi\partial_\nu\varphi \tag{3.5}$$

We obviously need a matter field that will propagate through the matter metric. Since we are looking for a general static solution, we shall leave the field as $\psi[r]$ and let it be calculated down the line. Just like ψ, all our other functions will only depend on r since we are concerned with static, non evolving solutions.

In our problem, the total action of the system is a sum of the Einstein-Hilbert action $S_{EH}[g_{\mu\nu}]$, the action $S_\varphi[g_{\mu\nu}, \varphi]$ for the scalar field φ and finally the action $S_\psi[g_{\mu\nu}, \psi]$ for the matter field ψ. If we now consider a scalar field which is minimally coupled with the metric $g_{\mu\nu}$ we get the total action

$$S = S_{EH} + S_{\varphi} + S_{\psi} \tag{3.6}$$

Where:

$$S_{EH}[g_{\mu\nu}] = \int d^4x\sqrt{-g}\Big(\frac{R-2\Lambda}{2\kappa}\Big), \tag{3.7}$$

$$S_{\varphi}[g_{\mu\nu},\varphi] = \int d^4x\sqrt{-g}\Bigg[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - V(\varphi)\Bigg], \tag{3.8}$$

$$S_{\psi}[\hat{g_{\mu\nu}},\psi] = \int d^4x\sqrt{-\hat{g}}\Bigg[-\frac{1}{2}g^{\hat{\mu\nu}}\nabla_\mu\psi\nabla_\nu\psi\Bigg]. \tag{3.9}$$

From these 3 components of the total action we can obtain the various tensors of each metric, leading us to the final field equations:

$$\mathcal{E}^{\mu\nu} := G^{\mu\nu} + \Lambda g^{\mu\nu} = \kappa(T^{\mu\nu}_{\varphi} + sT^{\hat{\mu\nu}}) \tag{3.10}$$

$$\Box\varphi - V'(\varphi) = BsT^{\hat{\mu\nu}}\hat{\nabla}_\mu\hat{\nabla}_\nu \tag{3.11}$$

$$\hat{\Box}\psi = 0 \tag{3.12}$$

where $s = \sqrt{-\hat{g}}/\sqrt{-g}$. We will have two energy-momentum tensors $T^{\mu\nu}$ and $T^{\hat{\mu\nu}}$ for $g_{\mu\nu}$ and $\hat{g_{\mu\nu}}$ respectively. These two tensors are:

$$T^{\mu\nu} = \frac{1}{2}g^{\mu\nu}(\nabla\varphi)^2 + \nabla^\mu\varphi\nabla^\nu\varphi - g^{\mu\nu}V(\varphi) \tag{3.13}$$

$$T^{\hat{\mu\nu}} = \frac{1}{2}g^{\hat{\mu\nu}}(\hat{\nabla}\psi)^2 + \hat{\nabla}^\mu\psi\hat{\nabla}^\nu\psi \tag{3.14}$$

Where $(\nabla\varphi)^2 := \nabla^\mu\varphi\nabla_\mu\varphi$, $(\hat{\nabla}\varphi)^2 := \hat{\nabla}^\mu\varphi\hat{\nabla}_\mu\varphi$, $\Box := \nabla^\mu\nabla_\mu$, $\hat{\Box} := \hat{\nabla}^\mu$, $\hat{\nabla}_\mu$. Here is a good point to confirm with the Bianchi identities in order to check whether our field equations are sensible. Considering the general form of $\hat{g}_{\mu\nu}$ we can determine the inverses

$$\hat{g}^{\mu\nu} = g^{\mu\nu} - \frac{B}{I}\nabla^\mu\varphi\nabla^\nu\varphi \tag{3.15}$$

and

$$g^{\mu\nu} = \hat{g}^{\mu\nu} + \frac{B}{K}\nabla^\mu\varphi\nabla^\nu\varphi \tag{3.16}$$

where

$$I = 1 + Bg^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi, \quad K = 1 - B\hat{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \tag{3.17}$$

from which it follows that $IK = 1$. We have already defined $\nabla^\mu\varphi = g^{\mu\nu}\partial_\nu\varphi$ and using the above we get

$$\hat{\nabla}^\mu\varphi = \hat{g}^{\mu\nu}\partial_\nu\varphi = K\nabla^\mu\varphi \tag{3.18}$$

Following these formulations, we are now ready to search for spherically symmetric solutions.

Chapter 4

Spherically Symmetric Solutions

We are interested in spherically symmetric black hole solutions, so we form the most general ansatz that satisfies this symmetry:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2 F dt^2 + \frac{dr^2}{F} + H^2(sin^2\theta d\phi^2 + d\theta^2) \tag{4.1}$$

As stated previously, we are looking for static solutions, so the functions N,F,H,φ and ψ are all only r-dependant. We first calculate the components of the Riemann tensor $R_{\mu\nu}$ [1]

$$\begin{aligned}
R_{tt} &= \frac{2FHN'' + 3HN'F' + 4FN'H' + NHF'' + 2NF'H'}{2N^3FH} \\
R_{rr} &= -\frac{F\left(2FHN'' + 3HN'F' + NHF'' + 2NF'H' + 4NFH''\right)}{2NH} \\
R_{\theta\theta} &= -\frac{FHN'H' + NHF'H' + NFHH'' + NFH'^2 - N}{NH^4} \\
R_{\phi\phi} &= -\frac{\csc^2(\theta)\left(FHN'H' + NHF'H' + NFHH'' + NFH'^2 - N\right)}{NH^4}
\end{aligned} \tag{4.2}$$

Having the Riemann tensor we can now calculate the components of the Einstein tensor:

[1] In both this case and for the Einstein tensor, the metric functions $F(r)$, $N(r)$, and $H(r)$ are displayed as F, N and H. This is mainly to save equation space due to the length of the expressions and because as we mentioned, the functions are only r-dependant

$$
\begin{aligned}
\mathcal{E}_{tt} &= -\frac{HF'H' + 2FHH'' + FH'^2 - 1}{N^2FH^2} \\
\mathcal{E}_{rr} &= \frac{F\left(2FHN'H' + NHF'H' + NFH'^2 - N\right)}{NH^2} \\
\mathcal{E}_{\theta\theta} &= \frac{2\left(F\left(N'H' + NH''\right) + NF'H'\right) + H\left(2FN'' + 3N'F' + NF''\right)}{2NH^3} \\
\mathcal{E}_{\phi\phi} &= \frac{\csc^2(\theta)\left(2FHN'' + 3HN'F' + 2FN'H' + NHF'' + 2NF'H' + 2NFH''\right)}{2NH^3}
\end{aligned}
\tag{4.3}
$$

Solving the Field equations (3.10), we get three non-vanishing terms from $\mathcal{E}^{\mu\nu}$, which are $\mathcal{E}^t_t$, $\mathcal{E}^r_r$ and $\mathcal{E}^\phi_\phi = \mathcal{E}^\theta_\theta$. From these we get:

$$0 = \frac{(F'H' + 2FH'')}{H} + \frac{FH'^2 - 1}{H^2} + \Lambda + \frac{\kappa}{2}F(\varphi'^2 + \sqrt{K}\psi'^2) + \kappa V, \tag{4.4}$$

$$0 = \frac{H'(NF' + 2FN')}{HN} + \frac{FH'^2 - 1}{H^2} + \Lambda - \frac{\kappa}{2}F(\varphi'^2 + K^{3/2}\psi'^2) + \kappa V, \tag{4.5}$$

$$0 = \frac{F''}{2} + \frac{F'H'}{H} + \frac{(3F'N' + 2FN'')}{2N} + \frac{F}{H}(H'' + \frac{H'N'}{N} + \Lambda + \frac{\kappa}{2}F(\varphi'^2 + \sqrt{K}\psi'^2) + \kappa V \tag{4.6}$$

The matter field is described by the Klein-Gordon equation, and considering our kind of ansatz we obtain:

$$\varphi'(r) = \left[\frac{1}{BF}\left(\frac{F^2H^4N^2\psi'^2}{\lambda^2} - 1\right)\right]^{1/2} \tag{4.7}$$

Plugging this equation into (4.4), (4.5), (4.6) and adding 4.4 to 4.5, we can determine the potential V in terms of N, F, H and the field ψ. We can then replace this expression of the potential back into the three equations, which will give us three new equations in which the potential is eliminated from two of them and can be determined explicitly from the third. These new equations are respectively:

$$\frac{F}{HN}(NH'' - H'N') + \frac{\kappa}{4}\left(\frac{2F^2H^4N^2\psi'^2}{B\lambda^2} - \frac{2}{B} + \frac{\lambda\psi'}{H^2N} + \frac{\lambda^3}{F^2H^6N^3\psi'}\right) = 0, \tag{4.8}$$

$$\frac{NF'' + 3F'N' + 2FN''}{2N} + \frac{1 - FH'^2}{H^2} + \frac{\kappa}{4}\left(\frac{2F^2H^4N^2\psi'^2}{B\lambda^2} - \frac{2}{B} + \frac{\lambda\psi'}{H^2N} + \frac{\lambda^3}{F^2H^6N^3\psi'}\right) = 0, \tag{4.9}$$

$$\kappa V + \Lambda - \frac{1 - FH'^2}{H^2} + \frac{H'}{HN}(NF' + 2FN') - \frac{\kappa}{2}\left(\frac{F^2H^4N^2\psi'^2}{B\lambda^2} - \frac{1}{B} + \frac{\lambda\psi'}{H^2N} + \frac{\lambda^3}{F^2H^6N^3\psi'}\right) = 0 \tag{4.10}$$

In order to lower the degree of the third differential equation, we replaced H'' with its form from equation (4.8). Notice we have three equations for the five unknowns H, F, N, V and ψ', since the Klein-Gordon equation of the scalar field

φ becomes redundant due to the Bianchi identities. It is hence clear that the final solution coming from these equations will automatically satisfy the Klein Gordon equation for φ. We must now fix two of the five variables in order to extract a solvable set of equations. The choice of the 2 variables to fix is completely free, since we work with the most general ansatz and we have not yet introduced any constrains of the form of the two metrics. The choices, however, must include at the very least some physical motivation behind them.

4.0.1 Case 1: H(r)=r, N(r)=1

In the first case we consider we fix H and N, so that our , the line element (7.5) takes the form:

$$ds^2 = -Fdt^2 + \frac{dr^2}{F} + r^2(sin^2\theta d\phi^2 + d\theta^2) \tag{4.11}$$

The line element now only depends on F, so it is convenient to find an expression for F from equations (4.8),(4.9) and (4.10). Subtracting (4.8) from (4.9) we get a differential equation we can solve for F. This gives

$$F(r) = \frac{r^2}{l^2} + 1 - \frac{2M}{r} \tag{4.12}$$

We can see this is actually the Black Hole solution of the AdS metric in (3+1) dimensions:

$$ds^2 = -(r^2k^2 + 1 - \frac{C}{r})dt^2 + \frac{dr^2}{(r^2k^2 + 1 - \frac{C}{r})} + r^2 d\Omega^2 \tag{4.13}$$

where C is a constant, and k is the AdS curvature. In our case, M is the respective constant and $l = 1/k$ is the curvature. As for the derivative of the scalar field ψ', it is given by the equation

$$\frac{2r^2\psi'^2(l^2(r-M)+r^3)^2}{B\lambda^2 l^4} - \frac{2}{B} + \frac{\lambda^3 l^4}{r^4\psi'(l^2(r-M)+r^3)^2} + \frac{\lambda\psi'}{r^2} = 0, \tag{4.14}$$

and the potential is given by

$$V(r) = \frac{1}{\kappa}\left(\frac{3}{l^2} - \Lambda\right) + \frac{\lambda^3 l^4}{4r^4\psi'(l^2(r-M)+r^3)^2} - \frac{\lambda\psi'}{4r^2} = 0 \tag{4.15}$$

The solution for ψ' can be analytically obtained from the above, however it is quite complicated. A more promising case with potentially less complicated expressions will be considered next.

4.0.2 Case 2: H(r)=r, V(r)=0

In the second case, we again fix H(r)=r so that the spherical component of the metric will be in its usual form. We also fix the potential V(r) to be zero, so our remaining free variables are F and N. Following a similar process as before, we extract the first of the two needed equations by subtracting (4.8) from (4.9) and we get:

$$\frac{r^2\left(-F''\right)+2F-2}{2r^2}-\frac{3rF'N'+2F\left(rN''+N'\right)}{2rN} \tag{4.16}$$

We now need a second differential equation in order for the system to be solvable for F and N. We can now subtract (4.9) from (4.10)

$$\frac{1}{4}\left(2F''+F'\left(\frac{6N'}{N}+\frac{4}{r}\right)-\frac{k\lambda^3}{r^6F^2N^3\psi'}+\frac{4F\left(rN''+2N'\right)}{rN}+\frac{k\lambda\psi'}{r^2N}+4\Lambda\right) \tag{4.17}$$

At this point we must mention that the length of many of the expressions and the results are simply too big to display effectively on normal equation spaces. Due to this, many of the explicit forms of the equations and the in-between steps will be presented in the Appendix at the end of the thesis. From the above equation we can then solve for the derivative of the scalar field ψ'. The solution is of quadratic form. Plugging one of the two solution (in our first case, the positive one) in equation (4.8), we can extract the second necessary equation for our system. Unfortunately the equation is too big to be displayed effectively, so both itself and the scalar field are posted in the Appendix.

At this point, our system has the form:

$$eq1=f(F,F',F'',N,N',N'',\kappa,\lambda,B,\Lambda) \quad eq2=g(F,F',F'',N,N',N'',\kappa,\lambda,B,\Lambda) \tag{4.18}$$

where κ,λ, B are parameters and Λ is the cosmological constant. Since we will keep the parameters as they are until we plot the resulting functions, we need not worry about them, so the system becomes:

$$eq1=f(F,F',F',N,N',N'') \qquad eq2=g(F,F',F',N,N',N'') \tag{4.19}$$

We see that our two equations now depend only on F, N and their derivatives. We have 6 variables for our two equations, so we must formulate four more. These four, naturally, are going to be the boundary conditions of the differential equations.

Since we have two unknown functions, it is obvious that the boundary conditions will consist of information regarding $F(r_s), F'(r_s), N(r_s)$, and $N'(rs)$ where r_s is the event horizon of our black hole. , all of which will be dynamically connected to one another. After much testing with free choices for all 4 of these conditions, we arrived at extremely inconsistent results that are really hard to work with. For one, the system seems to be unable to be solved at r=1, so our plots give no real information about the behaviour at the event horizon. Furthermore, the system is extremely sensitive to the boundary conditions, and even a slight change of the values gives wildly different results. We hence decided to fix function $N(r)$ in order to simplify the problem. Our choice, $N(r) = 1$, is motivated by the form of the line element (7.5) and its usual description in GR.

4.0.3 Case: H(r)=r, V(r)=0, N(r)=1

From the previous step we have arrived at equation (4.16). Setting N[r]=1 we get:

$$\frac{r^2\left(-F''\right)+2F-2}{2r^2}=0 \tag{4.20}$$

To solve this differential equation we need 2 boundary conditions since it is of second order. Since we want the solution to resemble a black hole, we are motivated to choose that F[1]=0 and F'[1]=1 as a first example. The first derivative allows a wide range of selection, but for this example we chose it to be 1, a general slope. A desired black hole behaviour would be that the function crosses the event horizon (r axis) at some point, and asymptotically approaches negative infinity when r goes to zero. Solving the equation analytically, we get:

$$F[r]=\alpha+\frac{\beta}{r} \tag{4.21}$$

This constitutes the general form of the solution, but it is interesting to see what form it takes for our specific example. The numerical differential solution is the plot:

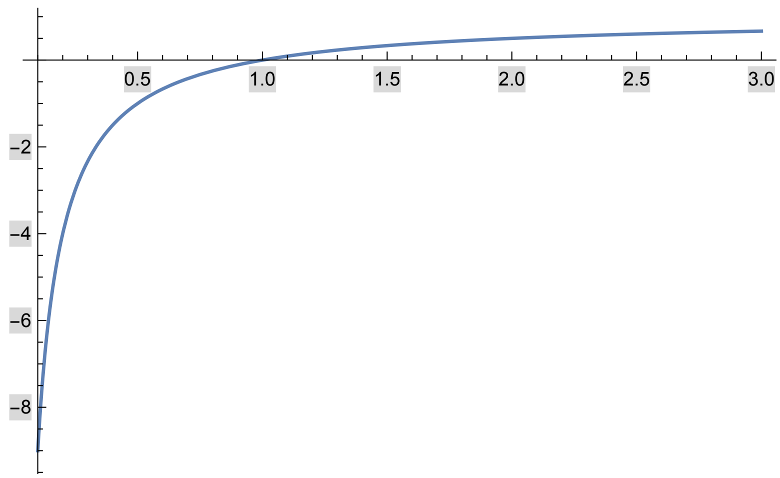

Figure 4.1: F(r) as a result of the numerical solution

Translating the plot into points, we can then approximate the curve by an expression $1 - \frac{1}{r}$ (in other words, a polynomial with powers r^0, r^{-1}) and we finally get:

$$F[r] = 0,9999 - \frac{1,00001}{r} \tag{4.22}$$

Getting more data from the plot helps to increase the accuracy, but it is apparent that we can approximate to

$$F[r] = 1 - \frac{1}{r} \tag{4.23}$$

and thus setting the analytical coefficients as $\alpha = 1$ and $\beta = -1$. Putting any more powers of r into the fitting polynomial reinforces this particular solution, as it sets all their coefficients to basically zero, leaving us only with $1 - \frac{1}{r}$.

Keeping this solution for F'[1]=1, we can then plug this back into the equations (4.8)(4.9)(4.10) and get the derivative of the scalar field as a function of r:

$$\psi'(r) \to -\frac{k\lambda^3}{\sqrt{r^4\left(a+\frac{b}{r}\right)^2\left(r^6\left(a+\frac{b}{r}\right)^2\left(r\left(\frac{2b}{r^3}+2\Lambda\right)-\frac{2b}{r^2}\right)^2+k^2\lambda^4\right)}-r^5\left(a+\frac{b}{r}\right)^2\left(r\left(\frac{2b}{r^3}+2\Lambda\right)-\frac{2b}{r^2}\right)} \tag{4.24}$$

We can now use the expression

$$\varphi'(r) = \left[\frac{1}{BF}\left(\frac{F^2H^4N^2\psi'^2}{\lambda^2} - 1\right)\right]^{1/2} \tag{4.25}$$

to get the derivative of φ from equation (4.24). We finally get

$$\varphi'(r) = 2\sqrt{\frac{\Lambda(r-1)r^5}{B\left(\sqrt{k^2\lambda^4(r-1)^2r^2 + 4\Lambda^2(r-1)^4r^8} - 2\Lambda(r-1)^2r^4\right)}} \quad (4.26)$$

Everything we did up to this point has been mainly trial and error, and putting the boundary conditions in by hand. It would be useful to try and generalize at least the boundary conditions, and even try to reduce the number of them so that we only have one free variable that controls the differential solution.

4.0.4 Expanding and generalizing the boundary conditions

Choosing a different condition for F'[1], we get results that either increase or decrease steadily past the event horizon, such as:

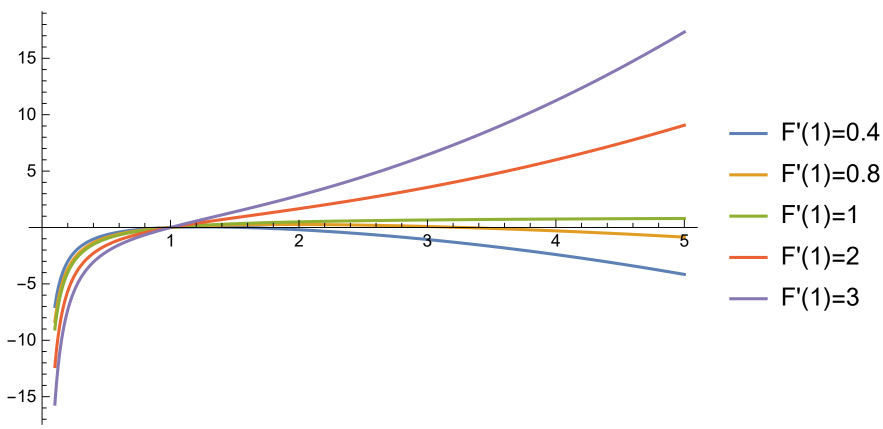

Figure 4.2: Numerical solutions of F with varying F'[1] boundary conditions

For all these cases, the expression of the solutions is of the form

$$F(r) = 1 - \frac{\alpha}{r} + \beta r^2 \quad (4.27)$$

which is the normal Schwartzchild form, with an r^2 term added. Solving for each case, we can see that the constant b is negative for $F'[1] < 1$ and approaches 0 as F'[1] goes to 1. For values $F'[1] > 1$, the constant b is positive. From an initial computing it seems that the two constants satisfy the equations:

$$\beta = \frac{n}{3} \tag{4.28}$$

$$\alpha = (n+3)\frac{1}{3} \tag{4.29}$$

so the function becomes

$$F[r] = 1 - (n+3)\frac{1}{3r} + \frac{nr^2}{3}\ , \ \ n \in \mathcal{R} \tag{4.30}$$

Where n comes from the boundary condition $F'(r_s)$. From the above and trying different conditions, the final form becomes

$$F(r) = 1 - (a+2)\frac{1}{3r} + \frac{(a-1)r^2}{3}\ , \ \ a = F'(r_s) \tag{4.31}$$

The previous solution we calculated, $F(r) = 1 - 1/r$, is now just a specific case for n=0 or $F'(r_s) = 1$, and the various solutions from 7 are results for different values of n.

A good way to check the validity of our boundary conditions while simultaneously generalizing them, is to use (4.16) with a Taylor expansion of F[r] around $r = r_e$. We set

$$F[r] = a_0 + a_1(r - r_e) + a_2(r - re) \tag{4.32}$$

where a_0, a_1 and a_2 are the Taylor expansion constants representing F[r], F'[r] and F"[r] at the point r=1. Plugging this expression into (4.16) and approximating $(r - r_e) \to 0$, we get approximately

$$a_0 = 1.002(0.999 + 0.00099a_1 + 0.4999a_2) \tag{4.33}$$

which for better approximations of $(r - r_e)$, is obviously

$$a_0 = (1 + \frac{a_2}{2}) \tag{4.34}$$

This means that the derivative of F at r=1 is zero, and F[1] can be zero only if $F'(1) = -1/2$. This implies that the function will approach the r axis from minus infinity without crossing it, and then descent back down. Truly, plotting F[r] with this set of conditions yields

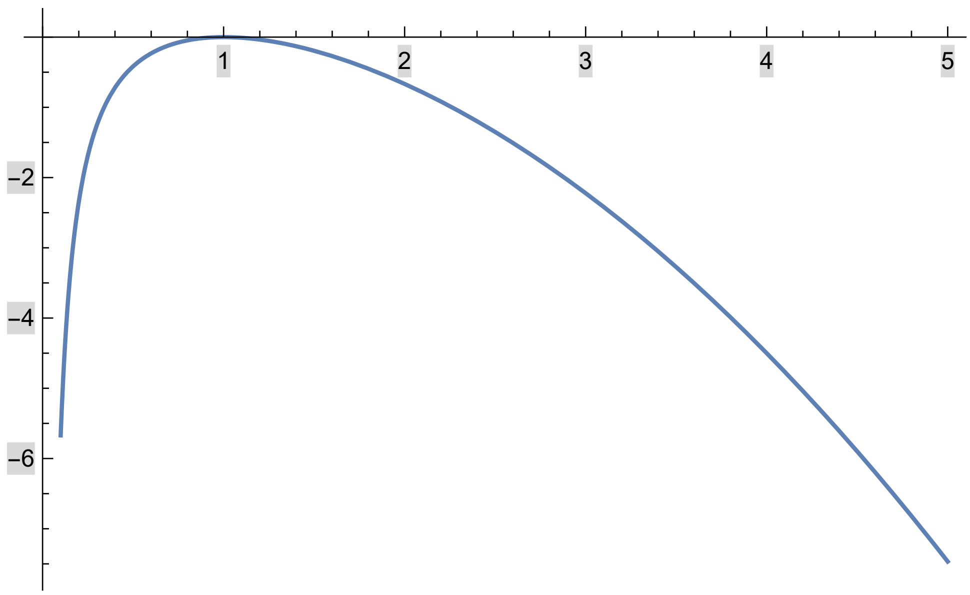

Figure 4.3: Solution to F[r] when F[1]$\neq$ 0

If we keep with our wish to have the event horizon at r=1, we set $a_0 = 0$ and now the condition becomes

$$a_1 = 10^n(1 + \frac{a_2}{2}) \tag{4.35}$$

where n comes from the approximation $(r - r_e) \to 10^{-n}$. As an example, for n=1 and $a_2 = 0$ we have $a_1 = 10$ and equation (4.16) now yields

$$F[r] = 1 - \frac{13}{3r} + \frac{10r^2}{3} \tag{4.36}$$

which is exactly of the form (4.27) we calculated previously.

To calculate our scalar field with our solutions, we use expression (4.27) for F[r] and $\varphi'[r]$ becomes

$$\varphi'[r] = 2\sqrt{\frac{1}{\frac{B\sqrt{r^2(-a+br^3+r)^2\left(4r^6(3b+\Lambda)^2(-a+br^3+r)^2+k^2\lambda^4\right)}}{r^5(3b+\Lambda)(-a+br^3+r)} - \frac{2B(-a+br^3+r)}{r}}} \tag{4.37}$$

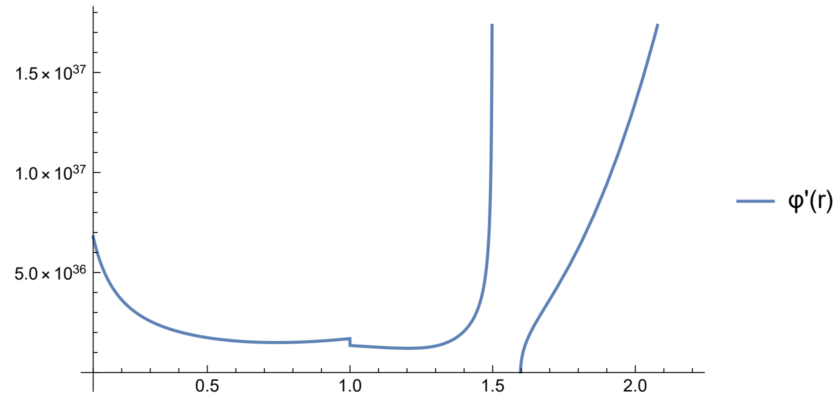

Figure 4.4: φ'(r) for F(r)=1-5$\frac{}{3r+\frac{2r^2}{3},\ B=\frac{l_p^2}{32\pi},\ \ k=\lambda=1}$

One can now plug the analytical constants -which in these cases represent the black hole radius and properties- back into the system and have them relate to B.

We can see how B influences the scalar field by plotting $\varphi'(r)$ for different values of B (while keeping the other constants the same as figure 4.4)

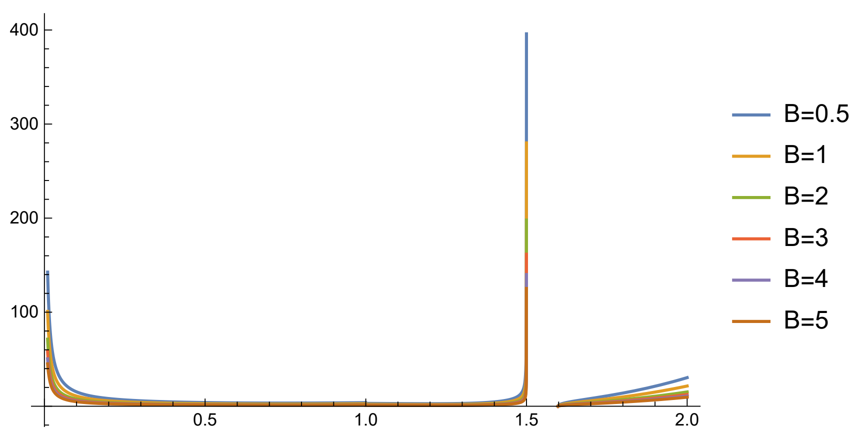

Figure 4.5: $\varphi'(r)$ for different values of B for r $\in$ [0.01,2]

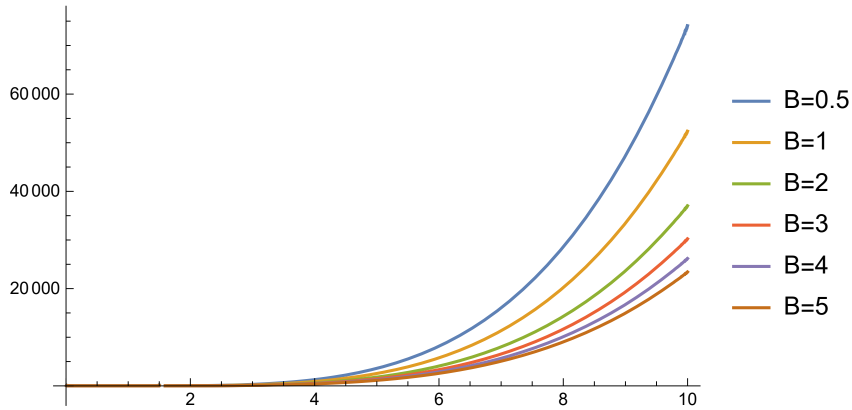

Figure 4.6: $\varphi'(r)$ for different values of B for r $\in$ [0.01,10]

Taking the general solution $F(r) = 1 - \frac{a}{r} + br^2$, the field $\psi'(r)$ becomes:

$$\psi'(r) \to -\frac{k\lambda^3}{\sqrt{r^2\left(-a+br^3+r\right)^2\left(4r^6(3b+\Lambda)^2\left(-a+br^3+r\right)^2+k^2\lambda^4\right)}-2r^4(3b+\Lambda)\left(-a+br^3+r\right)^2} \tag{4.38}$$

We notice that the field does not depend on B, but it does depend on λ, which was introduced through the Klein-Gordon equation of $\varphi(r)$ (**??**). In (**??**) it is introduced as a denominator, so it is safe to plot the field for values other than zero. Plotting for different values of λ [2], we get:

[2]We can see the field also depends on k. This however is the constant introduced through the Einstein equations, so it is again safe to set it something other than zero. Moreover, whether the function is positive or negative is dependant on the product $k\lambda^3$, so the general case is just an extension of our plots.

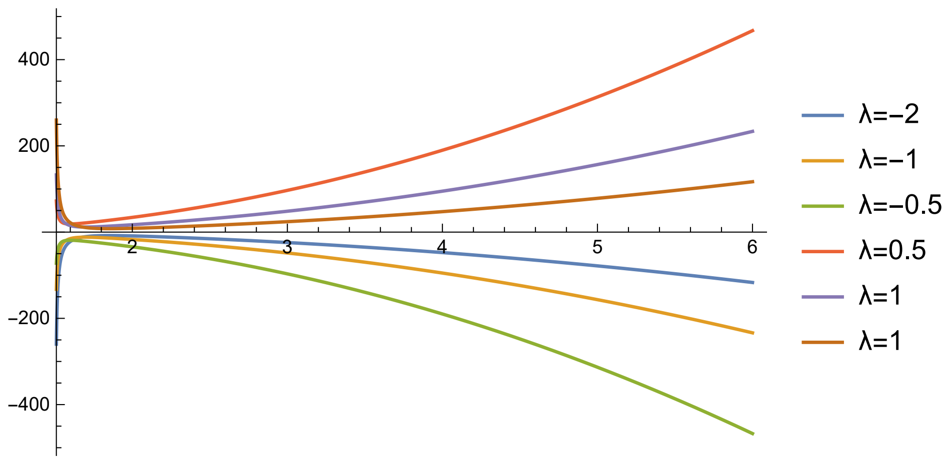

Figure 4.7: $\psi'(r)$ for different values of λ

We see that λ is the main factor that influences the form of the scalar field. Interestingly, the form does not change if we change the cosmological constant. Both positive and negative, as well as zero values produce virtually the same plots, with only the very slightest of changes.

Chapter 5

Conclusions

From the analysis above, we can extract some interesting conclusions about the theory. We have shown that there is indeed a black hole solution to this particular Scalar Bi-metric gravity model, and more specifically of the Anti de-Sitter form:

$$F(r) = \frac{r^2}{l^2} + 1 - \frac{2M}{r} \tag{5.1}$$

. This result was achieved by assuming at least an existing horizon to our function F[r], in other words that F[r]=0 at some point. This was chosen in part to more strongly facilitate our type of desired solution, but further calculations up to (4.35) showed that this condition is compatible with this particular form. In fact, it is shown that this single condition is necessary for a black hole metric to be formed, because as we showed in 4.3, any solution where $F(r_s) \neq 0$ forces the function not to cross the event horizon at any point. In our most general case, we have freedom of choice for one of the boundary conditions, namely $F'(r_s)$, and it's enough due to (4.35).

As for the function itself, the AdS form deviates from the classical GR by an r^2 factor. The main difference with normal black hole solutions in GR, is that the AdS solution does not approach a flat metric past the event horizon (asymptotic flatness). This is caused because in AdS space, the cosmological constant Λ is negative, so even in the absence of matter and energy the space-time has negative curvature. In the case of $F'(r_s) \leq 1$ the function dips back into the r-axis, and for $F'(r_s) \geq 1$ it increases in a parabolic nature.

In regards to B, we have shown that it contributes to the numerical factors of the final solution, although in a quite complicated way. In other words, the addition of B to the second metric changes the constant and the curvature of the AdS black hole. The motivated value $B = \frac{l_p^2}{32\pi}$ is extremely small and thus in the final plots it produces very large values that blow the functions up. As we said, this motivation is purely cosmological, and perhaps more fitting values can be applicable in a project that studies potential black holes. From equation (**??**)

and figures (4.5) and (4.6)we can see that B is acting almost as a scale factor for the steepness of the function, and consequently of the black hole behaviour past the event horizon. Similarly with our solutions for F(r), the scalar field also does not appear to approach a "flat" form as $r \to \infty$.
As for the scalar field $\psi(r)$, it is not directly dependent on B, but changes with regard to λ, which is introduce through the Klein-Gordon equation of $\varphi(r)$. We hence realize that the introduction of the bi-scalar field does change the matter field, or in other words, matter "feels" the field introduced by the second metric in a dynamic way.

Chapter 6

Limitations and extensions of the model

As it has become apparent, our final solution resulted from numerous assumptions and simplifications. This of course is a result of trying to take the most general form of our metric functions and properties, which results in many free choices for the metric functions. The obvious generalization that can me made is our original form of the metric (7.5). For our simplification we set $N(r) = 1$ so that our differential equations will only have one variable function. Obviously for a general case, both F(r) and N(r) would be variables in a system of equations, with specific boundary conditions for both. It is apparent that the boundary conditions will arise from physical restrictions and ansatzes. These could be either properties of the space-time in question, or a different kind of black hole ansatz (e.g. Kerr black holes).

The main obstacle here is that the equations are heavily intertwined, with one's boundary conditions affecting the other's dynamically due to the complexity of the equations. Due to this, the numerical solutions to the metric functions are either lacking vital information we need to make conclusions, or are very inconsistent in regard to their boundary conditions.A way to simplify the system is to use other, more "well-behaved" scalar fields $\psi(r)$ (as for example in [6]) and see if the system becomes more manageable. If simplifying the equations is out of the question, the next way to deal with the problem is to use a better way of solving the system numerically. Higher tolerances, different solving methods and ways to deal with arising infinities and divergences are all ways in which the numerical solving of the system can be optimized. Nevertheless, everything we mentioned above will still require a good physical basis and more information on the behaviour of the function around the important points of the model (event horizon, asymptotically flat region).

Furthermore, the work on this model so far is purely cosmological, and has provided little information on any kind of spherically symmetric solutions. A better formulation of the model with this in mind could provide useful insight on what the set up, methodology or result should look like. Moreover, further cosmological constraints on this model can help the problem of spherically symmetric solutions by imposing limitations on the initial formulation of the model or on constants such as B and the metric functions.

Chapter 7

Summary in Greek

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Spherically Symmetric Solutions in Bi-Metric Gravity

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Ευχαριστίες

Θα ήθελα να ευχαριστήσω των επιβλέπων κύριο Παπανωνόπουλο για την ευκαιρία να εργαστώ σε αυτό το πολύ ενδιαφέρον θέμα, και τους Άκη και Cristian, χωρίς την βοήθεια των οποίων αυτή η εργασία δεν θα ήταν η ίδια.

Abstract

Σε αυτη την εργασία θα αναλύσουμε τις διάφορες μορφές δι-μετρικών θεωριών βαρύτητας. Εξετάζουμε την θεωρία του Rosen και την νεότερη θεωρία, την βαθμωτή δι-μετρική (scalar bi-metric) βαρύτητα. Σε αυτή την θεωρία, ανεπτυγμένη από τους M. A. Clayton και J.W. Moffat αναζητούμε λύσεις μελανών οπών. Το μοντέλο ασχολείται με δύο μετρικές αντί για μία, οι οποίες περιγράφουν το πως βαρυτικά και ηλεκτρομαγνητικά κύματα μεταδίδονται στον χωροχρόνο

Εισαγωγή

Γενικές Δι-μετρικές θεωρίες βαρύτητας κι κίνητρο

Όπως φαίνεται από το όνομα, μία δι-μετρική θεωρία βαρύτητας είναι μία τροποποιημένη μορφή της Γενικής Σχετικότητας η οποία περιέχει όχι μία, αλλά δύο μετρικούς τανυστές $g_{\mu\nu}$. Αυτή η προσθήκη μπορεί να έχει διάφορα αποτελέσματα, καθώς υπάρχουν πολλοί τρόπου να εισάγουμε μία δεύτερη μετρική. Οι ποικίλες δι-μετρικες θεωρίες στην πάροδο των χρόνων ήταν διάφορων μορφών, από πλήρως γεωμετρικές, εώς τις πιο μοντέρνες θεωρίες massive bi-gravity.

Η σημαντικότερη ερώτηση που πρέπει να κανει κανείς είναι, γιατί δύο μετρικές; Το κίνητρο πίσω από την ανάπτυξη τέτοιων θεωριών είναι για να εξετάζουμε πως αλληλεπιδρούν δύο χωροχρώνοι μεταξύ τους. Παρ'ότι αυτό είναι πλήρως γεωμετρικό στην θεμελίωση του, μεταγενέστερες δι-μετρικές θεωρίες έχουν αντιμετωπίσει το πρόβλημα του πώς βαρυτικά και ηλεκτρομαγνητικά πεδία αλληλεπιδρούν, και κάποιες έχουν χρησιμοποιηθεί για την εισαγωγή massive σωματιδίων βαρύτητας (gravitons). Επιπροσθέτως, η εισαγωγή της νέας μετρικής μπορεί να χρησιμοποιηθεί για να δημιουργήσει διορθώσεις σε ήδη υπολογιμένες ποσότητες όπως η ταχύτητα των βαρυτικών κυμάτων και ακόμα και να ορίσει κοσμολογικούς περιορισμούς.

Η Γενική Σχετικότητα με μία μετρική

Στην Γενική Σχετικότητα, η εξίσωση -ή μάλλον, εξισώσεις- που περιγράφουν πως αλληλεπιδρά ο χωροχρόνος και η ύλη είναι οι **Εξισώσεις Πεδίου Einstein**(EFE):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{7.1}$$

όπου $R_{\mu\nu}$ είναι ο τανυστής καμπυλώτητα Ricci, R είναι η βαθμωτή καμπυλώτητα, Λ η κοσμολογική σταθερά, G η βαρυτική σταθερά του Νεύτωνα, $T_{\mu\nu}$ ο τανυστής ενέργειας-ορμής, και $g_{\mu\nu}$ ο μετρικός τανυστής. Ο μετρικός τανυστής περιγράφει την γεωμετρική και αιτιακή δομή του χωροχρόνου, και χρησιμοποιείται για να ορίζει ποσότητες όπως απόσταση, χρόνος, γωνίες και όγκους.Ο τανυστής περιγράφει συντεταγμένες στον χωροχρόνο, συνήθως σε σύμβολισμό (t, x, y, z). Παρ'όλα αυτά, εξ αιτίας τηε φύσης των προβλημάτων στην γενική σχετικότητα όπως η σφαιρικά συμμετρικές λύσεις, ο μετρικός τανυστής γράφεται με συμβολισμό (t, r, θ, ϕ). Οι EFE είναι μία εξίσωση τανυστών η οποία συσχετίζει ένα σύνολο συμμετρικών 4x4 τανυστών, ο καθένας με 10 ανεξλαρτητα στοιχεία. Με τις ταυτότητες Bianchi οι εξισώσεις μειώνονται απο 10 σε 6, αφήνοντας την μετρική με 4 βαθμούς ελευθερίας. Οι εξισώσεις προκύπτουν από την αρχή της ελάχιστης δράσης, μέσω της δράσης Einstein-Hilbert:

$$S = \int \left[\frac{1}{2\kappa} R + \mathcal{L}_M \right] \sqrt{-g} d^4 x \tag{7.2}$$

Μαύρες Τρύπες στην Γενική Σχετικότητα: Η λύση Schwarzschild

Η πιο ξεκάθαρη εφαρμογή μίας θεωρία βαρύτητας είναι ο υπολογισμός ενός σφαιρικά συμμετρικού βαρυτικού πεδίου. Το προφανές κίνητρο πίσω από αυτό είναι πως τα περισσότερα σημαντικά αντικέιμενα όπως η Γη και ο Ήλιος περιγράφονται από μία (σχεδόν) σφαιρικά συμμετρική κατανομή μάζας. Εμείς νοιαζόμαστε για τον κενό χώρο έξω από το αντικέιμενο και πώς δοκιμαστικά σωματίδια κινούνται σε αυτόν. Η αναζήτηση αυτή οδηγεί στα αντικείμενα ενδιαφέροντος, τις μαύρες τρύπες.

Στην Γενική Σχετικότητα, η σφαιρικά συμμετρική λύση στο κενό είναι η διάσημη **Λύση Schwarzschild**

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + d\Omega^2 \tag{7.3}$$

όπου M η νευτώνεια μάζα που μετράμε μελετώντας τροχιές μακριά από το αντικείμενο. Το αντικείμενο σε αυτή την περίπτωση είναι στατικό (καμία χρονική εξέλιξη) και σφαιρικά συμμετρικό. Παρατηρούμε πως για $M \to 0$ λαμβάνουμε τον επίπεδο χώρο Minkowski (ασυμπτωτικό flatness).

Η λύση αυτή δεν είναι απλά μία καλή λύση, αλλά είναι η μοναδική σφαιρικά συμμετρική λύση των εξισώσεων Αινσταιν στο κενό, εξ αιτίας του θεωρήματος Birkhoff. Το πιο ενδιαφέρον κομμάτι αυτής της μετρικής είναι πως εμφανίζονται μοναδικότητες (singularities) για $r = 0$ και $r = 2GM$. Μόνο η $r = 0$ αποτελεί φυσική μοναδικότητα, και είναι αυτό το οποίο δίνει στις μαύρες τρύπες τα ενδιαφέροντα χαρακτηριστικά τους.

Η δι-μετρική θεωρία του Rosen

Ο Nathan Rosen ανέπτυξε την θεωρία του για δι-μετρική βαρύτητα το 1940. Η θεωρία συνιστά πως σε κάθε σημείο του χωροχρόνου εισάγεται μία δεύτερη μετρική $\gamma_{\mu\nu}$. Αυτή η μετρική αντιστοιχεί στον επίπεδο χωροχρόνο.

Το πλεονέκτημα της εισαγωγής αυτής όπως απέδειξε ο Rosen, είναι πως διάφορες ποσότητες πλέον ξαναορίζονται με τανυστικό χαρακτήρα, ενώ πρίν δεν είχαν. Αυτό επιτρέπει μεγαλύτερο έλεγχο πάνω στις ποσότητες, καθώς και συσχετισμό τους με άλλες, όπου προηγουμένος δεν θα γινόταν. Επίσης, η θεωρία του Rosen πραγματικά επιτρέπει λύσεις μαύρων τρυπών, και μάλιστα η μορφή τους δεν είναι άλλη από αυτή του Schwarzschild.

Παρ'ολα τα πλεονεκτήματα της θεωρίας αυτής, παραμένει μία πλήρως γεωμετρική ανάλυση, δηλαδή δεν περιέχει πληροφορίες για πεδία ύλης και πως αλληλεπιδρούν με τον χωροχρόνο. Η θεωρία του Rosen και η γενική σχετικότητα διαφέρουν στα εξής:

- Την διάδοση ηλεκτρομαγνητικών κυμάτων
- το εξωτερικό πεδίο ενός υψηλής πυκνότητας άστρου
- την συμπεριφορά ισχυρών βαρυτικών κυμάτων που διαδίδονται σε ένα ισχυρό στατικό βαρυτικό πεδίο.

Χτίζοντας πάνω στην θεωρία του Rosen, από την δεκαετία του 80-90 κκαι μετά έχει προκύψει ένα νέο είδος θεωρίας δι-βαρύτητας, η βαθμωτή δι-μετρική βαρύτητα.

Scalar δι-μετρική βαρύτητα

Το κύριο είδος θεωρίας που εξετάζουμε είναι η scalar δι-μετρική βαρύτητα. Έχει αναπτυχθεί και μελετάται από τους M. A. Clayton and J.W. Moffat [5, 4, 3, 2], και εισάγει μία δυναμική αλληλεπίδραση μεταξύ των δύο μετρικών με την μορφή ενός βαθμωτού πεδίου.

Η γενική μορφή του μοντέλου είναι:

$$\hat{g}_{\mu\nu} = A(\varphi)g_{\mu\nu} + B(\varphi)\partial_\mu\varphi\partial_\nu\varphi \tag{7.4}$$

όπου ϕ το βαθμωτό πεδίο. Η ιδέα είναι πως η μετρική $\hat{g}_{\mu\nu}$ είναι η "μετρική ύλης" και περιγράφει την διάδοση Η/Μ κυμάτων, ενώ η μετρική $g_{\mu\nu}$ είναι η "βαρυτική μετρική" και περιγράφει την διάδοση των βαρυτικών κυμάτων. Αυτή η συσχέτιση οδηγέι στο συμπέρασμα πως, εν γένει, οι ταχύτητες των βαρυτικών κυμάτων και του φωτός θα είναι διαφορετικές (πράγμα που τα πειράματα ανίχνευσης βαρυτικών κυμάτων δεν υποστιρίζουν). Ένα επίσης ενδιαφέρον κομμάτι της θεωρίας είναι η σταθερά B, η οποία συχετίζει τις δύο μετρικές. Ο περιορισμός της σταθεράς αυτής σε συνδυασμό με τις διαφορετικές ταχύτητες των πεδίων, οδηγούν σε αλλαγή της κβαντικής κλίμακας στην ακτινοβολία υποβάθρου, οπότε έχουμε προβλέψεις για πειραματικές παρατηρήσεις.

Στην δικιά μας περίπτωση, το μοντέλο είναι απλοποιημένο, καθώς επιλέγουμε $B(\varphi) = B =$σταθερά. Επιπλέον, τόσο το πεδίο φ όσο και οι συναρτήσεις που θα ορίσουμε αργότερα δεν έχουν εξέλιξη με τον χρόνο, συνεπώς αναζητούμε στατικές λύσεις.

Σφαιρικά Συμμετρικές λύσεις

Μετά την θεμελίωση του μοντέλου, είμαστε σε θέση να αναζητήσουμε λύσεις μαύρων τρυπών. Παίρνοντας την γενική μετρική με σφαιρική συμμετρία:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2Fdt^2 + \frac{dr^2}{F} + H^2(sin^2\theta d\phi^2 + d\theta^2) \tag{7.5}$$

Μπορούμε να λύσουμε τις εξισώσεις Αινσταιν. Οι τελικές μας εξισώσεις είναι 3, ενώ οι συναρτήσεις μας είναι 5. Για αυτό θα πρέπει να φιξάρουμε δύο συναρτήσεις για να επιλυθεί το σύστημα. Καταλήγουμε λοιπόν στις παρακάτω περιπτώσεις.

Περίπτωση 1: H(r)=r, N(r)=1

Στην πρώτη περίπτωση φιξάρουμε το H(r) και το N(r) έτσι ώστε η μετρική να έχει μία σύνηθης μορφή. Η προσέγγιση αυτή δίνει την λύση:

$$ds^2 = -(r^2k^2 + 1 - \frac{C}{r})dt^2 + \frac{dr^2}{(r^2k^2 + 1 - \frac{C}{r})} + r^2 d\Omega^2 \tag{7.6}$$

Η οποία είναι της μορφής AdS Schwarzschild.

Περίπτωση 2: H(r)=r, V(r)=0

Η δεύτερη περίπτωση απαιτεί από το δυναμικό να είναι μηδέν, και όπως και πριν το ακτινικό κομμάτι παραμένει στην τυπική μορφή του. Παρ'ολα αυτά, το 2x2 σύστημα μας είναι υπερβολικά πολύπλοκο στην λύση, οπότε αναγκαζόμαστε να πάρουμε ακόμα έναν περιορισμό.

Περίπτωση: H(r)=r, V(r)=0, N(r)=1

Σε αυτή την περίπτωση, το σύστημα εξισώσεων δίνει την λύση:

$$F(r) = \frac{r^2}{l^2} + 1 - \frac{2M}{r} \tag{7.7}$$

Η οποία είναι και πάλι της μορφής Η οποία είναι της μορφής AdS Schwarzschild. Η σταθερά M είναι η Νευτώνεια μάζα, και l η καμπυλότητα AdS. Το ενδιαφέρον εδώ είναι αν η κοσμολογική σταθερά εμφανίζεται στην καμπυλότητα, όπως στην θεμελιωμένη θεωρία AdS. Δυστηχώς η πολυπλοκότητα των εξισώσεων δεν επέτρεψαν την εξαγωγή απάντησης στο συγκεκριμένο ερώτημα.

Συμπεράσματα

Από τα παραπάνω προκύπτουν ενδιαφέροντα συμπεράσματα για την θεωρία διμετρικής βαρύτητας. Αρχικά είδαμε πως υπάρχουν λύσεις μελανών οπών στο μοντέλο, και συγκεκριμένα της μορφής AdS Schwarzschild.

Όπως είναι εμφανές, όμως, το μοντέλο αντιμετωπίζει και πολλά προβλήματα. Η δυσκολία των εξισώσεων και η έλλειψη πληροφορίας για τις συναρτήσεις και το σύστημα είναι οι δύο σημαντικότεροι λόγοι που απαγορεύουν την εξαγωγή επιπλέον αποτελεσμάτων.

Appendices

.1 The full system of equations

From equations (4.8,4.9,4.10) we can calculate the derivative of the field $\psi(r)$ by adding (4.8) and (4.9) together, then solving for $\psi'(r)$. The equation is quadratic so it yields

$$\begin{aligned}\psi'(r) = &\mp \frac{k\lambda^3}{\sqrt{r^4F^2N^2\left(r^6F^2N^2\left((3rF'+4F)N'+N\left(r\left(F''+2\Lambda\right)+2F'\right)+2rFN''\right)^2+k^2\lambda^4\right)}} \\ &- r^5F^2N^2\left((3rF'+4F)N'+N\left(r\left(F''+2\Lambda\right)+2F'\right)+2rFN''\right)\end{aligned} \quad (8)$$

We can now plug this expression back into (4.8) and we will have our first final equation:

$$\begin{aligned}0 =&\frac{1}{2}\Bigg(-\frac{k}{B}-\frac{2FN'}{rN} \\ &-\frac{\sqrt{r^4F^2N^2\left(r^6F^2N^2\left((3rF'+4F)N'+N\left(rF''+2F'+2\Lambda r\right)+2rFN''\right)^2+k^2\lambda^4\right)}}{r^6F^2N^3} \\ &+k^3\lambda^4r^4F^2N^2/\Bigg(B\Bigg(-r^5F^2N^2\left((3rF'+4F)N'+N\left(rF''+2F'+2\Lambda r\right)+2rFN''\right) \\ &+\sqrt{r^4F^2N^2\left(r^6F^2N^2\left((3rF'+4F)N'+N\left(rF''+2F'+2\Lambda r\right)+2rFN''\right)^2+k^2\lambda^4\right)}\Bigg)^2\Bigg)\Bigg)\end{aligned} \quad (9)$$

This along with equation (4.16) constitute our system of two equations. We can see that this equation is extremely complex, and solving it in any analytical way is out of the question. We hence only used this expression when solving numerically, with complete freedom of boundary conditions. It was mainly because of this equation that we chose to simplify it by setting N(r)=1

Generalizing the boundary conditions of two functions

If we now consider the general form of the line element, without setting N(r)=1, we can use a similar Taylor expansion as before to extract some information about the boundary condition. Expanding both F and N around a horizon point r_s we get:

$$
\begin{aligned}
F(r) &= f_0 + f_1(r - r_s) + \frac{f_2}{2}(r - r_s)^2 \\
N(r) &= n_0 + n_1(r - r_s) + \frac{n_2}{2}(r - rs)^2
\end{aligned} \tag{10}
$$

where $f_0,\ f_1,\ f_1$ (and similarly for $N(r)$) are the Taylor coefficients that represent $F(r_s)$, $F'(r_s)$ and $F''(r_s)$ respectively. Our demand for a horizon to exist at some point now translates to $f_0 = 0$. We can now re-write equation (4.16) with the expanded functions and we have

$$
\begin{aligned}
0 = &\frac{2\left(\mathrm{f1}(r-\mathrm{rs})+\frac{1}{2}\mathrm{f2}(r-\mathrm{rs})\right)-2}{2r^2} \\
&- \frac{2\left(\mathrm{n1}+\frac{\mathrm{n2}}{2}\right)\left(\mathrm{f1}(r-\mathrm{rs})+\frac{1}{2}\mathrm{f2}(r-\mathrm{rs})\right)+3r\left(\mathrm{f1}+\frac{\mathrm{f2}}{2}\right)\left(\mathrm{n1}+\frac{\mathrm{n2}}{2}\right)}{2r\left(\mathrm{n0}+\mathrm{n1}(r-\mathrm{rs})+\frac{1}{2}\mathrm{n2}(r-\mathrm{rs})\right)}
\end{aligned} \tag{11}
$$

We can demand the function to have a certain behaviour on the horizon, or in mathematical terms when $(r - r_s) \to 0$. We can set both terms to zero and solve for any two of the constants while requiring $(r - rs) \to 0$. The first term yields

$$
f_1 = 1 - \frac{f_2}{2} \tag{12}
$$

Setting the second term to zero now yields

$$
n_1 = -\frac{n_2}{2} \tag{13}
$$

We have hence reduced the number of free variables from five $(f_1,\ f_2,\ n_0,\ n_1,\ n_2)$ to three $(f_1,\ n_0,\ n_1)$. The idea now is to plug these expressions into (9) and derive another equation that relates n_0 to either of the other constants. In that way we will have one boundary condition for each function and we will have much greater control of the numerical solution. This, however, proved extremely difficult due to the complex nature of (9) and its multitude of potential solutions it provided for n_0. It is also unfortunate that we got information only about first and second derivatives, as the numerical system is much better solved with $F(r_s)$ and $N(r_s)$ as the boundary conditions.

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