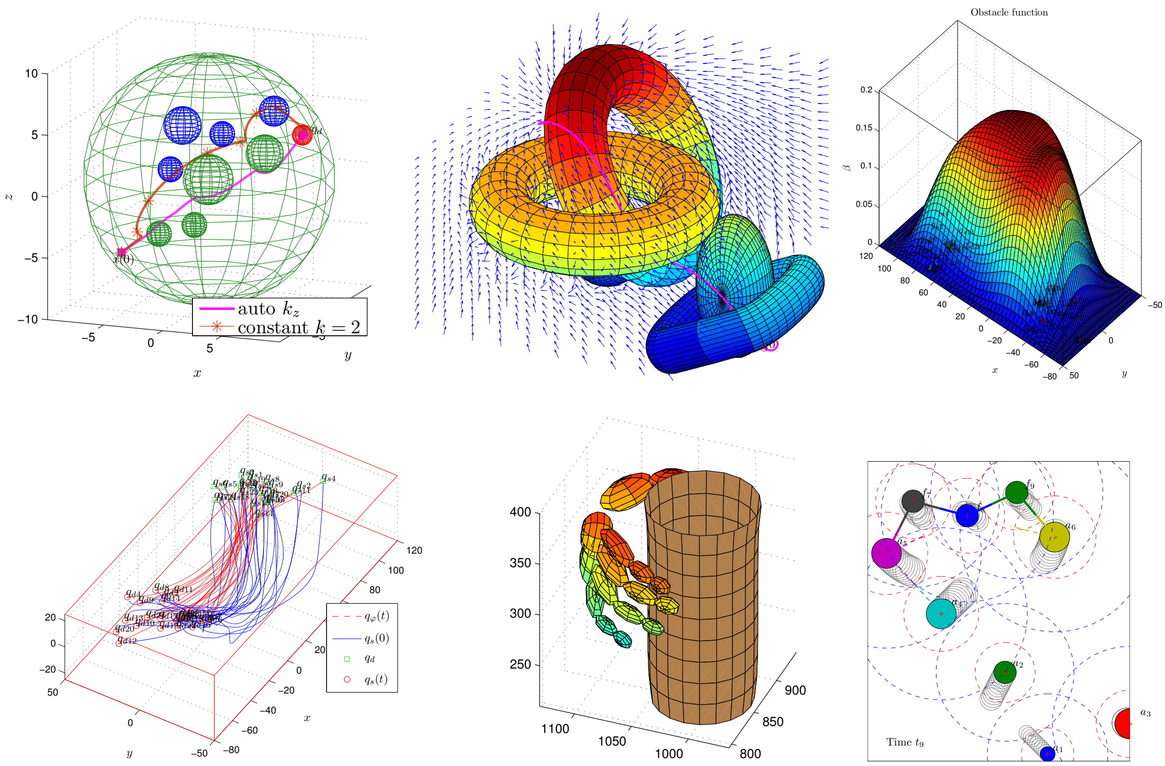
Navigation Functions for Unknown Sphere Worlds, General Geometries, their Inverse Problem and Combination with Formal Methods

by

Ioannis Filippidis



Submitted to the Department of Mechanical Engineering
in partial fulfillment of the requirements for the degree of
Diploma in Engineering
at the

Control Systems Laboratory
Mechanical Engineering Department
National Technical University of Athens
Athens
Greece
jfilippidis@gmail.com

Thesis Supervisor: Konstantinos J. Kyriakopoulos
Professor of Mechanical Engineering
September 2011

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Cover:

- On the upper left navigation of a point agent in an unknown 3-dimensional sphere world using the proposed automatically tuned Navigation Function is shown and compared to a manually tuned Navigation Function trajectory.
- On the upper middle a point agent navigates a known 3-dimensional everywhere partially sufficiently curved world, which includes tori, ellipsoids of one bounded eccentricity and a partially sufficiently curved supertorus.
- On the upper right the obstacle function resulting as the solution of the partial differential equation of the Navigation Function inverse problem is shown. The experimental trajectories used are from grasping experiments.
- On the lower left the trajectories using the previous obstacle function are compared to the experimental ones, within a principal component subspace of the hand configuration space.
- On the lower middle a human hand is driven using the Navigation Function with the B-Spline obstacle function found.
- On the lower right, local Linear Temporal Logic specifications are provided to individual agents of a multi-agent system, each synthesizes a hybrid controller, then decentralized verification occurs and where needed, the connectivity is triggered and maintained utilizing follower agents.

Contents

List of Figures

List of Tables

Chapter 1

Preface

1.1 Abstract

This work has four main contributions:

1. Extending Navigation Functions for exploring unknown sphere worlds;
2. Extending Navigation Functions for everywhere partially sufficiently curved worlds;
3. Formulating and solving the inverse problem of finding an unknown obstacle function corresponding to experimental trajectories to use it in a Navigation Function;
4. Decentralized hybrid control of Multi-Agent systems from local Linear Temporal Logic specifications under limited communication using Navigation Functions.

The first one is provided in Part I and extends Koditschek-Rimon Navigation Functions to unknown sphere worlds, for which automatic tuning of the exponent parameter is developed. This algorithm replaces previous manual tuning with provably correct automatic tuning. The lower bound used is here improved by orders of magnitude compared to the original proof. The computational complexity of updating for discovered obstacles is here examined. An updating algorithm with computational complexity linear in the number of already discovered obstacles is constructed. Moreover, Navigation Functions are extended for unbounded worlds. This work has been published in [125].

The second one, Part II, concerns the extension of Koditschek-Rimon Navigation functions to complicated geometries and topologies without the need for diffeomorphisms. The most general class of worlds to which this type of Navigation Functions is directly applicable is investigated. This leads to a geometric condition characterizing tractable worlds. In particular, those worlds which are everywhere partially sufficiently curved, that is, those worlds for which every obstacle boundary point has at least one sufficient principal curvature. A principal curvature is termed sufficient if its associated tangent sphere with diameter the radius of principal curvature is included within the obstacle. The Navigation Function theory is then reformulated and proved for these worlds. This work has been submitted as [124].

In Part III the Inverse Problem of Navigation Functions is considered. It consists of finding an obstacle function from available feasible trajectories measured in experiments. This obstacle function should be such, that using it in a Koditschek-Rimon Navigation Function will solve the Motion Planning problem similarly to the experimental trajectories. The problem is formulated as the solution of a Partial Differential Equation by gradient minimization of an appropriately selected cost functional. The successful solution depends on the construction of this functional, which requires careful consideration.

The application of this new method is demonstrated using results from human grasping experiments. By training the Navigation Function in the primary principal component subspace of the hand configuration space, grasping hand movements very similar to those produced by humans are achieved. This work has been submitted as [126].

The work in Part IV considers Multi-Agent systems. Local Linear Temporal Logic (LTL) specifications are independently provided to each agent. Then each constructs a hybrid controller comprised of a discrete supervising automaton resulting from the LTL and continuous Navigation Function controllers. Moreover, connectivity maintenance control is implemented between agents requesting it, when triggered by their specifications. Follower agents are utilized to maintain this connectivity. Formal verification of the constructed controllers takes place by Model Checking when agents acquire path-connectedness and can interchange their languages and automata. This work has been submitted as [123].

1.2 Περίληψη

Η εργασία αυτή περιλαμβάνει τέσσερεις κύριες συνεισφορές:

1. Επέκταση της μεθόδου των Συναρτήσεων Πλοήγησης (Navigation Functions) για την εξερεύνηση άγνωστων σφαιρικών κόσμων·
2. Επέκταση της μεθόδου των Συναρτήσεων Πλοήγησης σε κόσμους μερικώς ικανώς καμπύλους σε κάθε συνοριακό σημείο·
3. Διατύπωση και επίλυση του Αντίστροφου Προβλήματος εύρεσης μίας άγνωστης συνάρτησης εμποδίου, αντιστοιχούσας σε πειραματικώς καταγεγραμμένες τροχιές, προς χρήση εντός μίας Συνάρτησης Πλοήγησης·
4. Αποκεντρωμένος υβριδικός έλεγχος πολυ-πρακτορικών συστημάτων από προδιαγραφές διατυπωμένες σε Γραμμική Χρονική Λογική, υπό περιορισμένες δυνατότητες επικοινωνίας, με χρήση Συναρτήσεων Πλοήγησης.

Η πρώτη συνεισφορά αποτελεί το αντικείμενο του Μέρος I και επεκτείνει τη μορφή των Συναρτήσεων Πλοήγησης κατά Koditschek-Rimon σε άγνωστους σφαιρικούς χώρους, για τους οποίους αναπτύσσεται η αυτόματη ρύθμιση της παραμέτρου του εκθέτη. Ο αλγόριθμος αυτός αντικαθιστά την πρώτερη ανθρώπινη ρύθμιση με αποδεδειγμένα ορθή αυτόματη ρύθμιση. Το εδώ χρησιμοποιούμενο ελάχιστο όριο της παραμέτρου είναι βελτιωμένο κατά τάξεις μεγέθους συγκρινόμενο με την αρχική απόδειξη. Εξετάζεται και η υπολογιστική πολυπλοκότητα ανανέωσης για νεο-ευρεθέντα εμπόδια. Κατασκευάζεται ένας αλγόριθμος ανανέωσης με γραμμική υπολογιστική πολυπλοκότητα ως προς το πλήθος των ήδη ανακαλυφθέντων εμποδίων. Επιπροσθέτως, οι Συναρτησεις Πλοήγησης επεκτεινονται και σε μη φραγμένους χώρους. Η εργασία αυτή έχει δημοσιευθεί στο [125].

Η δεύτερη συνεισφορά, Μέρος II, αφορά στην επέκταση των Συναρτήσεων Πλοήγησης τύπου Koditschek-Rimon σε περίπλοκες γεωμετρίες και τοπολογίες δίχως την ανάγκη χρήσης Διαφορίσιμων Απεικονίσεων Χώρου (Diffeomorphisms). Συγκεκριμένα, διερευνάται ποιά είναι η πλέον γενική κατηγορία κόσμων στους οποίους είναι εφαρμόσιμος αυτός ο τύπος Συναρτήσεων Πλοήγησης. Τούτο οδηγεί σε μία γεωμετρική συνθήκη η οποία χαρακτηρίζει τους αποδεκτούς κόσμους. Συγκεκριμένα, πρόκειται για τους χώρους οι οποίοι είναι παντού μερικώς αρκούντως καμπύλοι, δηλαδή εκείνοι στους οποίους κάθε συνοριακό σημείο εμποδίου διαθέτει τουλάχιστον μία ικανή κύρια καμπυλότητα. Μία κύρια καμπυλότητα ονομάζεται ικανή εφόσον η σε αυτή αντιστοιχούσα εφαπτόμενη σφαίρα, με διάμετρο την ακτίνα της κύριας καμπυλότητας, περιλαμβάνεται εντός του εμποδίου. Κατόπιν, η θεωρία των Συναρτήσεων Πλοήγησης επαναδιατυπώνεται και αποδεικνύεται για αυτούς τους κόσμους. Η εργασία αυτή έχει υποβληθεί στο [124].

Στο Μέρος III θεωρείται το Αντίστροφο Πρόβλημα των Συναρτήσεων Πλοήγησης. Συνίσταται στην εύρεση μίας συνάρτησης εμποδίου από διαθέσιμες πραγματοποιήσιμες τροχιές καταγραφείσες σε πειράματα. Η συνάρτηση εμποδίου οφείλει να είναι τέτοια, ώστε η αντικατάστασή εντός μίας Συνάρτησης Πλοήγησης τύπου Koditschek-Rimon να επιλύει το Πρόβλημα Σχεδιασμού Κίνησης παρομοίως με τις πειραματικές τροχιές. Το πρόβλημα διατυπώνεται ως η επίλυση μίας Μερικής Διαφορικής Εξίσωσης με ελαχιστοποίηση διά απότομης καθόδου ενός καταλλήλως επιλεγμένου συναρτησιακού κόστους. Η επιτυχής λύση εξαρτάται από την κατασκευή τούτου του συναρτησιακού, η οποία απαιτεί προσεκτική μελέτη.

Η νέα μέθοδος αναδεικνύεται με εφαρμογή της χρησιμοποιώντας αποτελέσματα από πειράματα αρπαγής αντικειμένων από ανθρώπους. Εκπαιδεύοντας μία Συνάρτηση Πλοήγησης στον πρωτεύοντα υπόχωρο του ιδιοσυστήματος στο χώρο στάσης (Configuration Space) του ανθρωπίνου χεριού, αναπαράγονται αυτομάτως κινήσεις αρπαγής με χέρι πολύ παρόμοιες με τις ανθρώπινες. Η εργασία αυτή έχει υποβληθεί στο [126].

Η εργασία στο Μέρος IV αφορά σε πολυ-πρακτορικά συστήματα. Σε κάθε πράκτορα δίνονται τοπικώς προδιαγραφές διατυπωμένες σε Γραμμική Χρονική Λογική (Linear Temporal Logic - LTL), ανεξαρτήτως μεταξύ τους. Στη συνέχεια, κάθε πράκτορας κατασκευάζει έναν υβριδικό ελεγκτή αποτελούμενο από ένα διακριτό επιβλέπον αυτόματο προκύπτον από τις προδιαγραφές LTL και Συναρτήσεις Πλοήγησης ως συνεχείς ελεγκτές. Επιπροσθέτως, εφαρμόζεται έλεγχος διατήρησης συνδεσιμότητας μεταξύ πρακτόρων που την απαιτούν, όταν αυτό ζητείται από τις προδιαγραφές τους. Ακόλουθοι πράκορες αξιοποιούνται για τη διατήρηση αυτής της συνδεσιμότητας. Τυπική επαλήθευση των κατασκευασμένων ελεγκτών λαμβάνει χώρο με Έλεγχο Μοντέλου (Model Checking) όταν οι πράκτορες αποκτούν συνδεσιμότητα και μπορούν να ανταλλάξουν τις γλώσσες και τα αυτόματά τους. Η εργασία αυτή έχει υποβληθεί στο [123].

1.3 Acknowledgments

I want to thank my advisor Kostas J. Kyriakopoulos (NTUA) for his support and guidance while I have been a member of the Control Systems Lab (CSL). Providing me with the opportunity to join CSL as a 3rd year undergraduate Researcher working on software and hardware for an Autonomous Helicopter started a series of research targets which I have pursued through these three years. They evolved from low-level hardware and software details towards formally specified motion planning performed by autonomous agents. The diversity and coverage of these hierarchically interconnected topics provided me with a very instructive overview of robotics and control. At the same time it gave me the opportunity to work with several collaborators on different subjects.

My sincere thanks go also to Dimos V. Dimarogonas (KTH - Kungliga Tekniska högskolan /Royal Institute of Technology) for inviting me as a Visiting Researcher to the Automatic Control Laboratory of the School of Electrical Engineering at KTH and our collaboration which started there and led to Part IV.

I also want to thank Panagiotis K. Artemiadis (ASU - Arizona State University), Pantelis T. Katsiaris and Minas V. Liarokapis for our collaboration on robotic grasping, to which the method of Part III is applied.

Part I

Adjustable Navigation Functions for Unknown Sphere Worlds

Chapter 2

Navigation Function Tuning

2.1 Introduction

A fundamental problem in robotics is motion planning [2–4, 6]. A great variety of manifestations exists and equally numerous different solution approaches. Among them we may mention sampling-based [5], combinatorial [1] and feedback methods for continuous spaces [14], as well as combinations of methods [66, 67].

The basic motion planning problem over continuous space can be defined as finding a safe path from an initial to a desired configuration [3]. Safety requires avoidance of collisions with obstacles, while a successful plan should also converge to the desired destination. By appropriately constructing a feedback control plan over continuous space, trajectory generation and trajectory tracking can be integrated, leading to closed-loop feedback motion planning.

Artificial Potential Fields are one class of closed-loop feedback motion planning methods to solve the motion planning problem. They were introduced by Khatib [14–17] and utilize a scalar potential field constructed over the workspace, as shown in Fig. 2.1a. The negated gradient of this field repels from obstacles and attracts to the destination. An agent driven by this negated gradient safely reaches the desired configuration. For certain obstacle worlds local minima arise, which can trap the agent and prevent successful attainment of the desired configuration.

Numerous other methods to construct potential fields have followed, as for example harmonic functions constructed through solution of partial differential equations[1] [9–11], harmonic function combined with the panel method [18, 19] and superquadric artificial potential fields [34].

In order to overcome the problem of local minima, Navigation Functions (NF) have been proposed by Rimon and Koditschek [28], Fig. 2.1b. These are also scalar fields over the free space. After showing that complete disappearance of stationary points is unobtainable, they defined an almost globally asymptotically stable scalar potential field. Subject to conditions, only a subset of Lebesgue measure zero traps the agent in the set of remaining saddle points, which are unstable equilibria. But in real applications, finite computation arithmetic renders it practically impossible for an agent to remain in a measure zero set.

The motion planning problem can be abstracted from the geometric to a topological

[1] Note that a *different* harmonic function is constructed using that method, for *each* different destination. This is fundamentally different from the work presented in Part III, where a partial differential equation is solved to find a single obstacle function, which can then be used for *any* destination.

viewpoint. Avoiding obstacles is equivalent to remaining in the same connected component of free space in which the agent started. The path can be first generated in a convenient "model" space which captures the problem's topological structure. As a second step, geometric detail is introduced. Geometrically complicated real obstacles are diffeomorphically mapped to their simpler images in model space. The inverse deiffeomorphism is used to transform the constructed path from the model space to real space. In particular, the NF potential is defined on a sphere world and diffeomorphically mapped to real space.

As discussed in [20], this method can be applied to any spherical agent moving in a workspace with obstacles, whose configuration space connected components are sphere worlds. In the case of a non-point agent, the Minkowski sum of agent with obstacles leads to the configuration space.

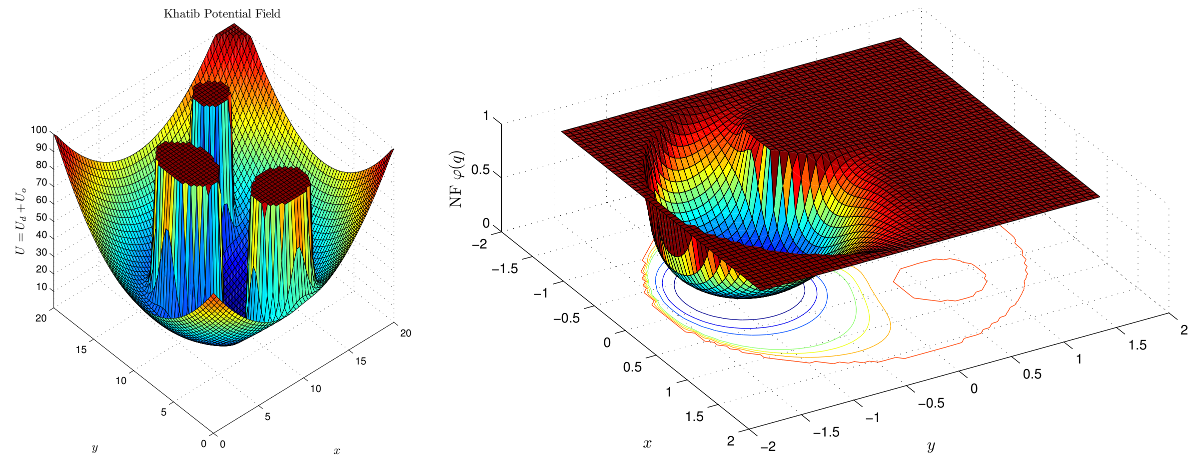
This may lead to loss of configuration space connectivity. Detecting whether initial and desired configurations belong to the same connected component of free space requires running the navigation algorithm and each connected component has been mapped to a sphere world and the algorithm fails, then inital and final configurations belong to different connected components. It may also lead to multiply connected obstacle topologies, which are not diffeomorphic to spheres. Overcoming such a limitation constitutes one of the subjects treated in Part II. Here we are concerned with sphere worlds.

Global knowledge is needed in the original navigation function formulation. This requirement is relaxed in [25, 26] by defining polynomial NFs and in [32] by implementing C^2 switches for multi-agent systems with finite sensing radii.

Tuning hinders implementation. The NF field is shaped by a parameter. As proved in [23] there exists a lower bound on this tuning parameter which clears the field of local minima other than the destination. They become saddles and the potential a NF. In addition to existence, calculation of this lower bound is outlined, but no explicit formula is provided. In consequence, using NFs until now required manual adjustment of the tuning parameter. This is also true for extensions of the NF methodology to multi-agent systems [12, 27, 33].

This work develops an algorithm to calculate the tuning parameter for theoretically guaranteed navigation. The lower bound used is improved compared to the original formulation. The improvement is achieved by cancellation of terms with equivalent effects. Direct substitution of sphere centers and radii suffices to find the desired lower bound.

The above algorithm enables safe tuning globally. The lower bound computation can be rearranged to efficiently update for discovered obstacles. In more detail, initializing constraints for a new obstacle has time computational complexity $\Theta(M_z)$, where M_z the number of the until then known obstacles. Updating those constraints related to already known obstacles upon discovering new obstacles can as well be arranged to require $\Theta(M_z)$. Moreover, there is the option to apply these calculated constraints only when necessary. If this is also implemented, it allows for provably correct locally oriented tuning, for a finite total number of obstacles, in an a priori unknown sphere world.



(a) Khatib potential field $U = U_d + U_o$ (b) Navigation Function

Figure 2.1: An artificial potential field (APF) and a Navigation Function (NF) scalar potential field. The APF is defined over a world with three obstacles: two ellipses and a disk. The NF is over the sphere world of Fig. 2.5 for $k = 2$.

2.2 Definition

2.2.1 Sphere world

A compact connected subset of n-dimensional euclidean space E^n, $n \in \mathbb{N}$, whose boundary is formed by the disjoint union of a finite number of $(n-1)$-dimensional spheres[2] is called a *sphere world*. Let the number of spheres be $M+1$ where $M \in \mathbb{N}$.

Compactness requires a finite sub-cover to exist for every open cover of the sphere world. The sphere world's boundary is formed of spheres. Therefore a finite boundary should be the set covering the sphere world and its internal boundaries. This boundary should be formed of spheres, but since these spheres constitute a *disjoint* set, only a *single* sphere can form the outer boundary. The space bounded by this outer sphere is called the *workspace* and is defined as

$$\mathscr{W} \triangleq \{q \in E^n : \|q\|^2 \leq \rho_0^2\} \tag{2.1}$$

where $0 < \rho_0 \in \mathbb{R}$ is the radius of the bounding sphere, having its center at the origin $0 \in E^n$. The workspace includes both the sphere world and the internal spherical boundaries. There remain M smaller spheres which bound the *obstacles*

$$\mathscr{O}_j \triangleq \{q \in E^n : \|q - q_j\|^2 < \rho_j^2\}, \quad j \in I_1 \triangleq \{1, 2, \ldots, M\} \tag{2.2}$$

where $0 < \rho_j \in \mathbb{R}$ each spherical obstacle's radius, $q_j \in E^n$ its center. Let $I_0 \triangleq \{0, 1, \ldots, M\}$.

The outer spherical boundary $\partial\mathscr{W}$ defines the zero$^{\text{th}}$ obstacle, which is that part of the Euclidean space E^n external to the workspace $\mathscr{W}$

$$\mathscr{O}_0 \triangleq E^n \setminus \mathscr{W} \tag{2.3}$$

The *free space* $\mathscr{F}$ remains after removing all the obstacles $\mathscr{O}_j$ from the workspace $\mathscr{W}$

$$\mathscr{F} \triangleq \mathscr{W} \setminus \bigcup_{j \in I_1} \mathscr{O}_j \tag{2.4}$$

[2]Hereinafter *sphere* will refer to an $(n-1)$-dimensional sphere.

Since the workspace $\mathscr{W}$ is a superset of the sphere world $\mathscr{F}$ the internal spherical boundaries $\partial\mathscr{O}_j$ of $\mathscr{F}$ should be subsets of $\mathscr{W}$. $\mathscr{W}$ is a sphere of radius ρ_0. Each sphere $\mathscr{O}_j \subset \mathscr{W}$ is contained in $\mathscr{W}$. This imposes a constraint on the radii $\rho_j, j \in I_1$.

Suppose $\mathscr{O}_j$ intersects $\mathscr{O}_0$, in which case

$$\begin{aligned} &\mathscr{O}_j \cap \mathscr{O}_0 \neq \varnothing \iff \\ &\qquad \exists q \in E^n : \left\{ \begin{array}{l} q \in \mathscr{O}_j \\ q \in \partial\mathscr{W} \end{array} \right\} \iff \\ &\qquad \exists q \in E^n : \left\{ \begin{array}{l} \|q - q_j\| \leq \rho_j \\ \|q\| = \rho_0 \end{array} \right\} \end{aligned} \tag{2.5}$$

For $\mathscr{O}_j$ and $\mathscr{W}$ to be disjoint $\mathscr{O}_j \cap \mathscr{W} = \varnothing$ the first inequality should *never* be true, this requirement is equivalent to the constraint

$$\|q - q_j\| > \rho_j, \quad \forall j \in I_1 \tag{2.6}$$

It is now shown that $\|q - q_j\| > \rho_j$ is equivalent to $\rho_0 > \|q_j\| + \rho_j$. First let us prove that $\rho_0 > \|q_j\| + \rho_j$ implies $\|q - q_j\| > \rho_j$.

$$\begin{aligned} \rho_0 > \|q_j\| + \rho_j, \quad \forall j \in I_1 &\iff \\ \rho_0 - \|q_j\| > \rho_j, \quad \forall j \in I_1 &\overset{q \in \partial\mathscr{W}}{\implies} \\ \|q\| - \|q_j\| > \rho_j, \quad \forall j \in I_1 &\overset{\|q-q_j\| \geq \|q\| - \|q_j\|}{\implies} \\ \|q - q_j\| \geq \|q\| - \|q_j\| > \rho_j, \quad \forall j \in I_1 &\implies \\ \|q - q_j\| > \rho_j, \quad \forall j \in I_1 & \end{aligned} \tag{2.7}$$

Proof of the opposite, that $\|q - q_j\| > \rho_j$ implies $\rho_0 > \|q_j\| + \rho_j$, requires careful selection of the vector $q \in \partial\mathscr{W}$. Because $\partial\mathscr{W}$ is a spherical boundary, it is always possible to select a q parallel to q_j. Because $q_j \in \mathscr{O}_j \subset \mathscr{W}_j \implies q_j \in \mathscr{W}_j$ and from definition of $\mathscr{W}_j$ it follows that $\|q_j\| \leq \rho_0 = \|q\|$. Let $\lambda \in (0, 1]$. Then

$$q_j = \lambda q \tag{2.8}$$

and it follows that

$$\begin{aligned} \|q\| - \|q_j\| &= \|q\| - \|\lambda q\| = \|q\| - |\lambda|\, \|q\| \overset{\lambda \geq 0}{=} \|q\| - \lambda\, \|q\| = (1 - \lambda)\, \|q\| \overset{1 \geq \lambda}{=} |1 - \lambda|\, \|q\| \\ &= \|(1 - \lambda) q\| = \|q - \lambda q\| = \|q - q_j\| \end{aligned} \tag{2.9}$$

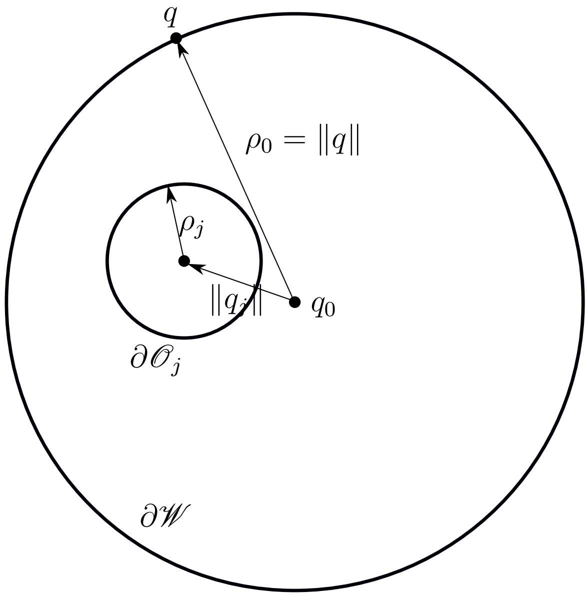
Provided that the center q_j of obstacle $\mathscr{O}_j$ is within the external boundary sphere $\mathscr{O}_0$, as expressed by $q_j = \lambda q$, it is now easy to show that if $\|q - q_j\| > \rho_j$

$$\begin{aligned} \left\{ \begin{array}{l} \|q\| - \|q_j\| = \|q - q_j\| \\ \|q - q_j\| > \rho_j \end{array} \right\} \implies \|q\| - \|q_j\| > \rho_j &\iff \\ \rho_0 - \|q_j\| > \rho_j &\iff \\ \rho_0 > \|q_j\| + \rho_j & \end{aligned} \tag{2.10}$$

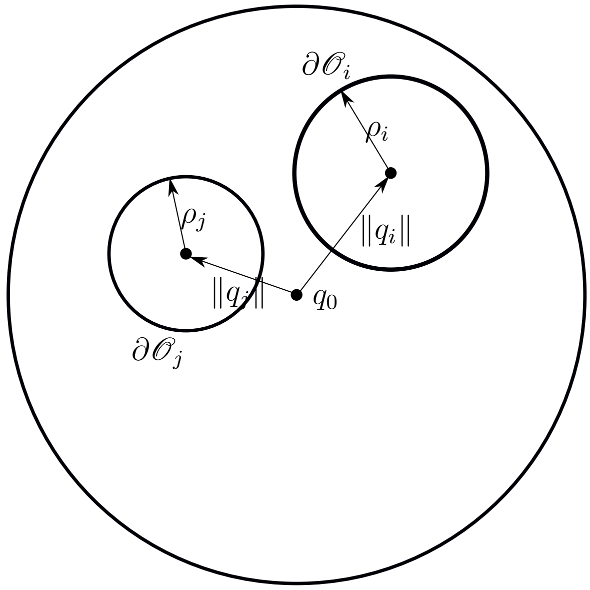
The constraint $\rho_0 > \|q_j\| + \rho_j$ is illustrated graphically in Fig. 2.2a.

Any two C-obstacles $\mathscr{O}_i, \mathscr{O}_j$ are disjoint.

$$\begin{aligned} &\mathscr{O}_i \cap \mathscr{O}_j = \varnothing \iff \\ &\qquad \nexists q \in E^n : \left\{ \begin{array}{l} \|q - q_i\| \leq \rho_i \\ \|q - q_j\| \leq \rho_j \end{array} \right\}, \quad \forall j \in I_1 \implies \\ &\qquad \nexists q \in E^n : \|q - q_i\| + \|q - q_j\| \leq \rho_i + \rho_j, \quad \forall j \in I_1 \iff \\ &\qquad \nexists q \in E^n : \|q_i - q\| + \|q - q_j\| \leq \rho_i + \rho_j, \quad \forall j \in I_1 \end{aligned} \tag{2.11}$$



(a) $\rho_0 > \|q_j\| + \rho_j$



(b) $\rho_0 > \|q_j\| + \rho_j$

Figure 2.2: Graphical illustration of constraints.

Again from the triangular inequality $\|q_i - q\| + \|q - q_j\| \geq \|q_i - q + q - q_j\| = \|q_i - q_j\|$, so

$$\begin{aligned} \nexists q \in E^n : \|q_i - q_j\| \leq \rho_i + \rho_j, \quad \forall j \in I_1 \iff \\ \|q_i - q_j\| > \rho_i + \rho_j, \quad \forall j \in I_1 \end{aligned} \tag{2.12}$$

The opposite can be proved by contradiction. The constraint $\|q_i - q_j\| > \rho_i + \rho_j$ is illustrated graphically in Fig. 2.2b.

2.2.2 Sphere world subsets

Let[3] $\mathscr{B}_i(\varepsilon_i)$ denote the open n-dimensional spherical annulus[4] around a workspace obstacle $\mathscr{O}_j$

$$\mathscr{B}_i(\varepsilon_i) \triangleq \{q \in E^n : 0 < \beta_i(q) < \varepsilon_i\}, \quad i \in I_0 \tag{2.13}$$

where $0 < \varepsilon_i \in \mathbb{R}, \forall i \in I_0$ parameters specifying the annuli widths. Function β_i is defined in subsection 2.3.2 where ε is discussed in more detail. Note that the obstacle's boundary $\partial\mathscr{O}_j = \beta_i^{-1}(0)$ is *not* included, nor the outer boundary of $\beta_i^{-1}(\varepsilon_i)$. Every $\mathscr{B}_i(\varepsilon_i)$ is an open set. The closure of $\mathscr{B}_i(\varepsilon_i)$, that is the union of set $\mathscr{B}_i(\varepsilon_i)$ with its inner boundary $\beta_i^{-1}(0)$ and its outer boundary $\beta_i^{-1}(\varepsilon_i)$ is denoted by $\overline{\mathscr{B}_i(\varepsilon_i)}$, and is defined as

$$\begin{aligned} \overline{\mathscr{B}_i(\varepsilon_i)} &\triangleq \{q \in E^n : 0 \leq \beta_i(q) \leq \varepsilon_i\} \\ &= \mathscr{B}_i(\varepsilon_i) \cup \beta_i^{-1}(0) \cup \beta_i^{-1}(\varepsilon_i) \end{aligned} \tag{2.14}$$

Let us define the outer radius

$$\rho_{\overline{\mathscr{B}}_i} = \left\|\beta^{-1}(\varepsilon_i) - q_i\right\| = \sqrt{\varepsilon_i + \rho_i^2} \tag{2.15}$$

[3]Note that in [23] a global parameter ε is defined. This parameter is obtained as $\min_{i \in \varepsilon_{I_0}} \{\varepsilon_{I_0}\} \leq \varepsilon_i, \forall i \in I_0$. (the subscript i is indicative of the dependence on each obstacle *separately*, the actual subscripts used are slightly different). Therefore they place more severe constraints on the lower bound than required by the problem. To avoid this and obtain a smaller lower bound, separate ε_i are explicitly denoted and used for the spherical annuli $\mathscr{B}_i$.

[4][23], §3.1, p.425: An n-ball "without a core".

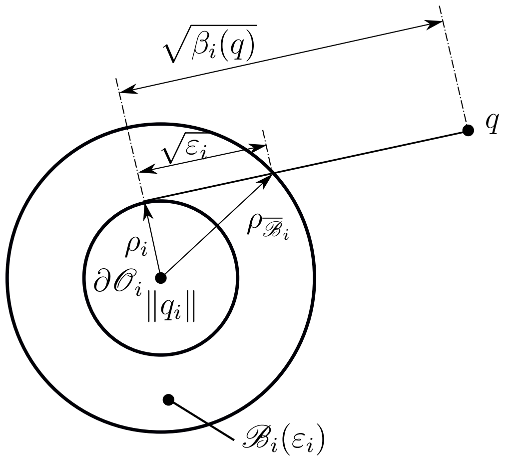


Figure 2.3: The geometric meaning of $\beta_i(\varepsilon_i), \varepsilon_i$ as squares of tangetial linear segments and of the radius $\rho_{\overline{\mathscr{B}}_i}$.

of the closure $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$. This diameter will prove useful for associating geometric meaning to the algebraic expressions. Its geometric meaning is illustrated in Fig. 2.3.

The following sets are defined for convenience and illustrated in Fig. 2.5:

1) Destination point	$\mathscr{F}_d \triangleq \{q_d\}$
2) Free space boundary	$\partial\mathscr{F} \triangleq \beta^{-1}(0) = \bigcup_{i\in I_0} \beta_i^{-1}(0)$
3) Set "near" internal obstacles	$\mathscr{F}_0\left(\varepsilon_{I_1}\right) \triangleq \bigcup_{i\in I_1} \mathscr{B}_i\left(\varepsilon_i\right) \setminus \mathscr{F}_d$
4) Set "near" workspace boundary	$\mathscr{F}_1\left(\varepsilon_{I_0}\right) \triangleq \mathscr{B}_0(\varepsilon_0) \setminus \left(\mathscr{F}_d \cup \mathscr{F}_0\left(\varepsilon_{I_1}\right)\right)$
5) Set "away" from obstacles	$\mathscr{F}_2\left(\varepsilon_{I_0}\right) \triangleq \mathscr{F} \setminus \left(\mathscr{F}_d \cup \partial\mathscr{F} \cup \mathscr{F}_0\left(\varepsilon_{I_1}\right) \cup \mathscr{F}_1\left(\varepsilon_{I_0}\right)\right)$.

where $\varepsilon_{I_0} \triangleq \{\varepsilon_i\}_{i\in I_0}, \varepsilon_{I_1} \triangleq \{\varepsilon_i\}_{i\in I_1}$.

Additionally the assumption is made that each $\varepsilon_i, i \in I_1$ is small enough to guarantee

$$\mathscr{F}_0\left(\varepsilon_{I_1}\right) \subset \mathscr{F} \iff \left\{\mathscr{B}_i\left(\varepsilon_i\right) \cap \mathscr{O}_j = \emptyset, \quad \forall j \in I_0, \quad \forall i \in I_1\right\} \tag{2.16}$$

which is equivalent to the following constraints[5] on each $\varepsilon_i, i \in I_1$

$$\varepsilon_i < \left(\|q_i - q_j\| - \rho_j\right)^2 - \rho_i^2 \triangleq \varepsilon_{i3j}, \quad \forall j \in I_0 \setminus i, \quad \forall i \in I_1 \tag{2.17}$$

This inequality for $j \neq 0$ ensures that internal obstacles $\mathscr{O}_i, i \in I_1$ enlarged by balls $\mathscr{B}_i\left(\varepsilon_i\right)$ do not intersect other internal obstacles $\mathscr{O}_j, j \in I_1 \setminus i$. For $j = 0$ it ensures that internal obstacles $\mathscr{O}_i, i \in I_1$ enlarged by balls $\mathscr{B}_i\left(\varepsilon_i\right)$ do not intersect the 0th obstacle $\mathscr{O}_0$.

The equivalence in now to be proved by application of inequalities (2.10) and (2.12)

[5]In [23] no symbols are assigned to these upper bounds.

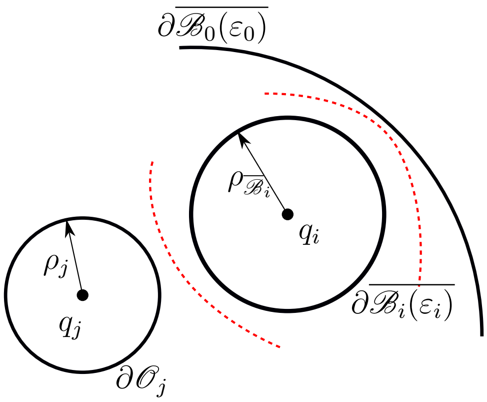


Figure 2.4: The geometric meaning of the constraints ε_{i3j}.

(since $\overline{\mathscr{B}_i(\varepsilon_i)} \cup \mathscr{O}_i$ is a ball as well)

$$
\begin{aligned}
&\mathscr{F}_0(\varepsilon_{I_1}) \subset \mathscr{F} \iff \\
&\left\{ \begin{array}{ll} \mathscr{B}_i(\varepsilon_i) \cap \mathscr{O}_j = \emptyset, & \forall i,j \in I_1 \\ \mathscr{B}_i(\varepsilon_i) \cap \mathscr{O}_0 = \emptyset, & \forall i \in I_1 \end{array} \right\} \iff \\
&\left\{ \begin{array}{ll} \|q_i - q_j\| > \rho_{\overline{\mathscr{B}}_i} + \rho_j, & \forall i,j \in I_1 \\ \|q_i - q_0\| + \rho_{\overline{\mathscr{B}}_i} < \rho_0, & \forall i \in I_1 \end{array} \right\} \iff \\
&\left\{ \begin{array}{ll} \|q_i - q_j\| > \sqrt{\varepsilon_i + \rho_i^2} + \rho_j, & \forall i,j \in I_1 \\ \|q_i\| + \sqrt{\varepsilon_i + \rho_i^2} < \rho_0, & \forall i \in I_1 \end{array} \right\} \overset{q_0 = 0 \in E^n \wedge \varepsilon_i, \rho_i > 0, \forall i \in I_1}{\iff} \\
&\left\{ \begin{array}{ll} \|q_i - q_j\| - \rho_j > \sqrt{\underbrace{\varepsilon_i + \rho_i^2}_{>0}} > 0, & \forall i,j \in I_1 \\ 0 < \sqrt{\underbrace{\varepsilon_i + \rho_i^2}_{>0}} < \rho_0 - \|q_i\|, & \forall i \in I_1 \end{array} \right\} \iff \\
&\left\{ \begin{array}{ll} \left(\|q_i - q_j\| - \rho_j\right)^2 > \varepsilon_i + \rho_i^2, & \forall i,j \in I_1 \\ \varepsilon_i + \rho_i^2 < \left(\rho_0 - \|q_i\|\right)^2, & \forall i \in I_1 \end{array} \right\} \iff \\
&\left\{ \begin{array}{ll} \varepsilon_i < \left(\|q_i - q_j\| - \rho_j\right)^2 - \rho_i^2, & \forall i,j \in I_1 \\ \varepsilon_i < \left(\rho_0 - \|q_i\|\right)^2 - \rho_i^2, & \forall i \in I_1 \end{array} \right\}
\end{aligned}
\tag{2.18}
$$

The geometric equivalent of the above derivation is given in Fig. 2.4.

2.3 Problem Statement

We consider a holonomic agent whose state x is governed by the control law

$$\dot{x}(t) = -\left(\nabla_q \varphi\right)(x(t)) \tag{2.19}$$

where φ is a NF on $\mathscr{F}$ as defined later. As proved in [23] this solves the motion planning problem in $\mathscr{F}$.

We are interested in an algorithm to tune the analytic potential field φ to make it a NF while exploring unknown sphere worlds. It is also desirable to reduce the effect on φ of obstacles distant to the agent, in a provably correct way. This scheme should be also applicable to a priori known worlds diffeomorphic [23, 25, 29] to sphere worlds.

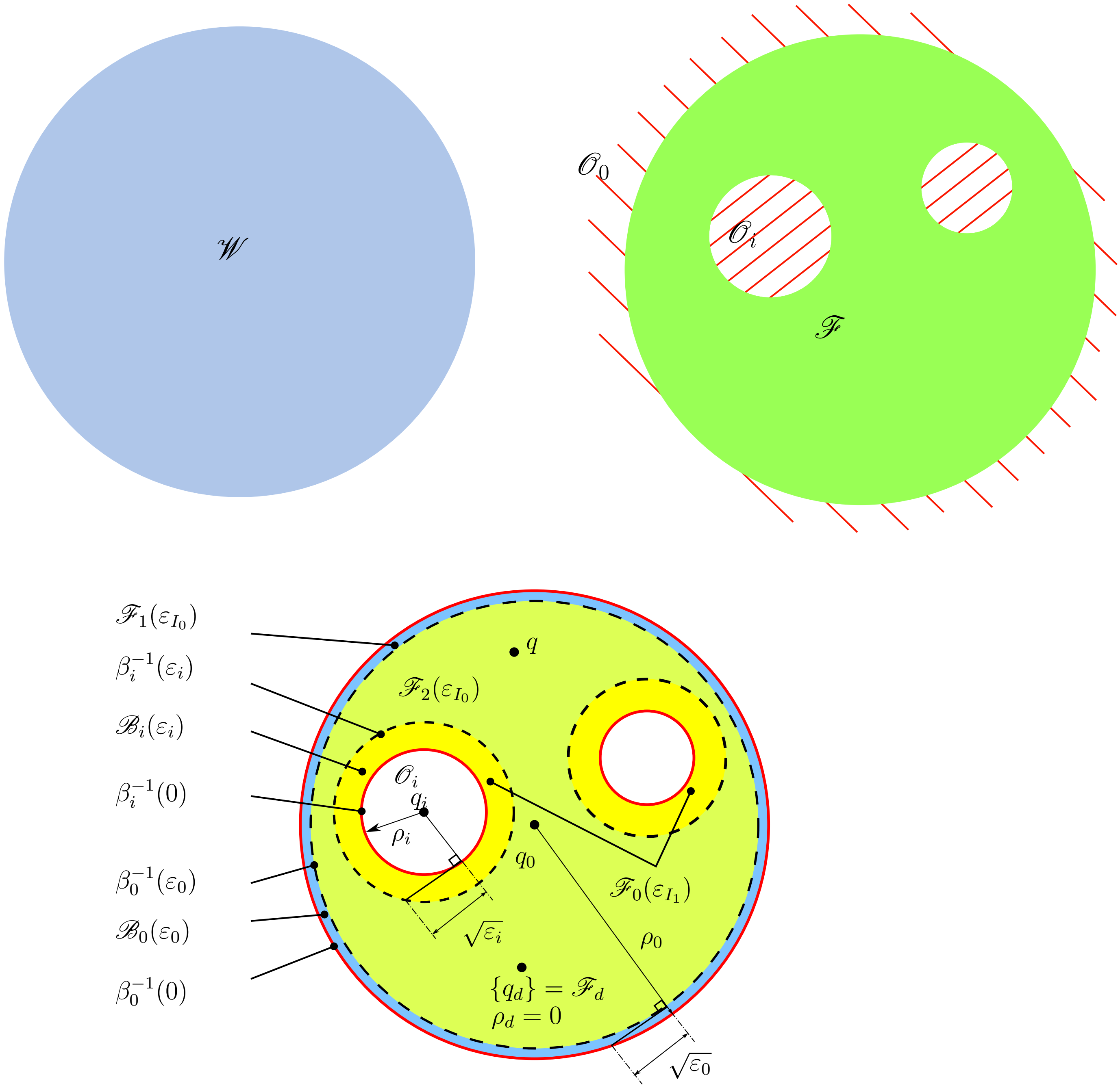


Figure 2.5: Sets defined on a sphere world.

2.3.1 Navigation function definition

A Navigation Function is defined[6] on a compact connected analytic manifold with boundary $\mathscr{M} \subset E^n$ as a map $\varphi : \mathscr{M} \to [0,1]$ which is

1. Analytic[7] on $\mathscr{M}$: locally convergent power series exists (Taylor expansion).
2. Polar on $\mathscr{M}$: unique minimum exists at $q_d \in \mathscr{M}$ (Harold Calvin Marston Morse 1892-1977).
3. Morse on $\mathscr{M}$: all critical points are non-degenerate.
4. Admissible on $\mathscr{M}$: uniformly maximal on $\partial\mathscr{F}$ (Morris W. Hirsch).

2.3.2 The Koditschek-Rimon navigation function

In [23] a navigation function for sphere worlds, $\varphi : \mathscr{F} \to [0,1]$, is proposed which is the composition of three functions

$$\varphi(q) \triangleq \sigma_d \circ \sigma \circ \hat{\varphi}(q) \tag{2.20}$$

The function $\hat{\varphi}$ is polar, almost everywhere Morse, and analytic; it attains a uniform height on $\partial\mathscr{F}$ by blowing up to $+\infty$ there. Its image is "squashed" by the diffeomorphism $\sigma : [0,\infty) \to [0,1]$ defined as

$$\sigma \triangleq \frac{x}{1+x} \tag{2.21}$$

resulting in a polar, admissible, and analytic function which is non-degenerate on $\mathscr{F}$ except at one point - the destination. This last flaw is repaired by σ_d.

They distinguish between "good" and "bad" subsets of $\mathscr{F}$. When a point belongs to the "good" set, we expect the negative gradient lines to lead to it (here it is just the destination $\{q_d\}$). The "bad" subset includes all the boundary points of the free space, and we expect the cost at such a point to be high. Let γ and β denote analytic real valued maps whose zero-levels, i.e. $\gamma^{-1}(0), \beta^{-1}(0)$, are respectively the "good" and "bad" sets.

The function $\hat{\varphi}$ is defined to be

$$\hat{\varphi}(q) \triangleq \frac{\gamma(q)}{\beta(q)} \tag{2.22}$$

where $\gamma : \mathscr{F} \to [0,\infty)$ is[8]

$$\left.\begin{array}{l} \gamma(q) \triangleq \gamma_d^k(q), \quad k \in \mathbb{N} \setminus \{0,1\} \\ \gamma_d(q) \triangleq \|q - q_d\|^2 \end{array}\right\} \implies \gamma(q) = \|q - q_d\|^{2k} \tag{2.23}$$

and $\beta : \mathscr{F} \to [0,\infty)$ is

$$\beta(q) \triangleq \prod_{j \in I_0} \beta_j(q) \geq 0, \quad \forall q \in \mathscr{F} \tag{2.24}$$

where

$$\left.\begin{array}{l} \beta_0(q) \triangleq \rho_0^2 - \|q\|^2 \\ \beta_j(q) \triangleq \|q - q_j\|^2 - \rho_j^2, \quad \forall j \in I_1 \end{array}\right\} \implies \beta_i(q) = \left|\|q - q_i\|^2 - \rho_i^2\right|, \quad \forall i \in I_0 \tag{2.25}$$

[6][23], Definition 1, p.417.
[7]It is important to note that it suffices to require that $\varphi \in C^2[\mathscr{M},[0,1]]$.
[8]The parameter k controls the attractivity of the destination q_d.

The "omitted product" is denoted by

$$\bar{\beta}_i \triangleq \prod_{j \in I_0 \setminus i} \beta_j \geq 0, \quad \forall q \in \mathscr{F}. \tag{2.26}$$

Due to the parameter k in $\hat{\varphi}$, the destination point is a *degenerate* critical point. To counteract this effect, the "distortion" $\sigma_d : [0,1] \to [0,1]$,

$$\sigma_d(x) \triangleq (x)^{\frac{1}{k}} = \sqrt[k]{x}, \quad k \in \mathbb{N} \setminus \{0, 1\} \tag{2.27}$$

is introduced, to change q_d to a non-degenerate critical point.

2.4 Proof of Correctness

2.4.1 Proof overview

Quite informally the whole proof can be summarized as following. Show that k can be linked to obstacle neighbourhood widths ε_{I_0} so that changing ε_{I_0} no critical points escape "away" from obstacles. Any critical points are now trapped near obstacles. Then shrink ε_{I_0} until the obstacle neighbourhoods are so tight around them that no *minima* or *degenerate points* arise[9].

It has been proved that no critical points exist on the free space boundary[10] $\partial\mathscr{F}$ and that the destination q_d is a non-degenerate global minimum[11]. Any other critical points can only exist in $\mathscr{F}_0(\varepsilon_{I_1}) \cup \mathscr{F}_1(\varepsilon_{I_0}) \cup \mathscr{F}_2(\varepsilon_{I_0})$.

Then the set "away" from obstacles $\mathscr{F}_2(\varepsilon_{I_0})$ is cleared of critical points[12]. Specifically for *any* ε_{I_0} the tuning parameter k can *always* be selected such that no critical points remain in $\mathscr{F}_2(\varepsilon_{I_0})$.

So, provided we select $k \geq N(\varepsilon_{I_0})$ any remaining critical points can only arise in $\mathscr{F}_0(\varepsilon_{I_1}) \cup \mathscr{F}_1(\varepsilon_{I_0})$. We can then select $\varepsilon_i, i \in I_0$ to

1. Avoid critical points in $\mathscr{F}_1(\varepsilon_{I_0})$ "near" workspace boundary[13]. This means that any critical points other than q_d can only arise in set $\mathscr{F}_0(\varepsilon_{I_1})$ "near" internal obstacles.
2. Avoid local minima in[14] $\mathscr{F}_0(\varepsilon_{I_1})$. That is, ensure arising critical points are either saddles or local maxima[15].
3. Ensure that all critical points arising in $\mathscr{F}_0(\varepsilon_{I_1})$ are non-degenerate[16]. This guarantees that they can be categorized and they remain disjoint. They are proved to be saddles.

Note that in [23] $\varepsilon'_{0i}, \varepsilon''_{0i}, \varepsilon_1, \varepsilon'_{2i}, \varepsilon''_{2i}, i \in I_1$ are defined. Of these ε_1 applies to $\mathscr{B}_0(\varepsilon_1)$. These indices are changed to better serve the present treatment[17].

The parameters defined here are

$$\varepsilon_i, \varepsilon_{iu}, \quad i \in I_0 \quad \text{and} \quad \varepsilon'_{i0}, \varepsilon''_{i0}, \varepsilon'_{i2}, \varepsilon''_{i2}, \varepsilon_{i3j}, \varepsilon_{i3}, \varepsilon_{i03}, \varepsilon_{i23}, \quad i \in I_1 \tag{2.28}$$

defined as

$$0 < \varepsilon_i < \varepsilon_{iu} \triangleq \begin{cases} \varepsilon_{0u}, & i = 0 \\ \min\{\varepsilon'_{i0}, \varepsilon''_{i0}, \varepsilon'_{i2}, \varepsilon''_{i2}, \varepsilon_{i3}\}, & i \in I_1 \end{cases} \tag{2.29}$$

and

$$\varepsilon_{i3} \triangleq \min_{j \in I_0 \setminus i}\{\varepsilon_{i3j}\}, \quad \varepsilon_{i03} \triangleq \min\{\varepsilon'_{i0}, \varepsilon_{i3}\}, \quad \varepsilon_{i23} \triangleq \min\{\varepsilon'_{i2}, \varepsilon_{i3}\} \tag{2.30}$$

and the definition of $\varepsilon_{0u}, \varepsilon'_{i0}, \varepsilon''_{i0}, \varepsilon'_{i2}, \varepsilon''_{i2}$ will follow in the next sections, while ε_{i3j} have already been defined in (2.17). With this notation ε_i applies to annulus $\mathscr{B}_i$ of obstacle $\mathscr{O}_i, i \in I_0$. The definitions used herein relate to those of [23] as follows

---

[9] In this section firstly $\varepsilon = \min_{i \in I_0}\{\varepsilon_i\}$ is used in a detailed derivation of the Koditschek-Rimon statements. Then the limitations of this derivation are noted and an altered expression is derived.

[10] [23], Proposition 3.3, p.427.

[11] [23], Proposition 3.2, pp.426-427.

[12] [23], Proposition 3.4, p.427.

[13] [23], Proposition 3.7, p.432.

[14] [23], Proposition 3.6, p.429.

[15] Note that the proof leads also to the result that they are never local maxima, so critical points other than q_d are always saddles.

[16] [23], Proposition 3.9, p.433.

[17] The index 1 does not correspond to 0^{th} obstacle index 0.

Table 2.1: Notation used herein compared to [23].

Here	[23]
ε_{0u}	ε_1
ε'_{i0}	ε'_{0i}
ε''_{i0}	ε''_{0i}
ε'_{i2}	ε'_{2i}
ε''_{i2}	ε''_{2i}
$\varepsilon_{iu}, \quad i \in I_1$	none
$\min_{i\in I_1}\{\varepsilon'_{i0}, \varepsilon''_{i0}\}$	ε_0
$\min_{i\in I_1}\{\varepsilon'_{i2}, \varepsilon''_{i2}\}$	ε_2
$\min_{i\in I_0}\{\varepsilon_i\}$	ε
ε_i	none

The upper bounds denoted here by $\varepsilon_{i3}, \varepsilon_{i3j}$ are not assigned any symbols in [23].

In consequence of the above definitions there are two alternatives for definining the sets $\mathscr{B}_i, \mathscr{F}_0, \mathscr{F}_1, \mathscr{F}_2$ as either functions of a *single global*[18] "width" ε, or as functions of the *set* of "widths" ε_i.

In the first case, as developed in [23], the domains are functions of a single parameter ε

$$\mathscr{B}_i\left(\varepsilon_i\right), \mathscr{F}_0(\varepsilon), \mathscr{F}_1(\varepsilon), \mathscr{F}_2(\varepsilon) \tag{2.31}$$

whereas in the second case the domain functions are functions of $M+1$ parameters $\{\varepsilon_i\}_{i\in I_0}$

$$\mathscr{B}_i\left(\varepsilon_i\right), i \in I_0 \quad \mathscr{F}_0\left(\varepsilon_{I_1}\right), \mathscr{F}_1\left(\varepsilon_{I_0}\right), \mathscr{F}_2\left(\varepsilon_{I_0}\right) \tag{2.32}$$

The second formulation appears at first to be computationally more demanding. But since ε results as the minimum of the set ε_{I_0}, this is not true. We need to caclulate all ε_i before determining ε. So there is no additional burden in the second case. Interestingly, as we are to show, the second method leads to better results[19].

2.4.2 Determining a lower bound $N(\varepsilon_{I_0})$ on k

2.4.2.1 Require norm inequality of gradient components

Proposition[20]: For every set ε_{I_0} there exists a positive integer $N\left(\varepsilon_{I_0}\right)$ such that if $k \geq N\left(\varepsilon_{I_0}\right)$ then there are no critical points of $\hat{\varphi}$ in $\mathscr{F}_2\left(\varepsilon_{I_0}\right)$.

[18]Which is selected as $\varepsilon = \min_{i\in I_0}\{\varepsilon_i\}$ to ensure all required constraints are met for the potential to be a navigation function.

[19]Better is to be understood as closer to the real supremum desired.

[20][23], Proposition 3.4, p.428.

At a critical point $q_c \in \mathscr{C}_{\hat{\varphi}} \cap \mathscr{F}_2(\varepsilon_{I_0})$ the gradient is zero

$$
\begin{aligned}
\nabla\hat{\varphi} &= 0 \in E^n \overset{\text{see Appendix for}\nabla\hat{\varphi}}{\iff} \\
\frac{\gamma_d^{k-1}}{\beta^2}\left(k\beta\nabla\gamma_d - \gamma_d\nabla\beta\right) &= 0 \in E^n \overset{\beta,\gamma_d\neq 0,\forall q\in\mathscr{F}_2(\varepsilon_{I_0})}{\iff} \\
k\beta\nabla\gamma_d - \gamma_d\nabla\beta &= 0 \in E^n \iff \\
k\beta\nabla\gamma_d &= \gamma_d\nabla\beta \implies \\
\|k\beta\nabla\gamma_d\| &= \|\gamma_d\nabla\beta\| \overset{\beta,\gamma_d,k>0,\forall q\in\mathscr{F}_2(\varepsilon_{I_0})}{\iff} \\
k\beta\|\nabla\gamma_d\| &= \gamma_d\|\nabla\beta\| \overset{\text{see Appendix for}\|\nabla\gamma_d\|=2\sqrt{\gamma_d}}{\iff} \\
k\beta 2\sqrt{\gamma_d} &= \gamma_d\|\nabla\beta\| \iff \\
2k\beta &= \sqrt{\gamma_d}\|\nabla\beta\| \overset{\beta\neq 0,\forall q\in\mathscr{F}_2(\varepsilon_{I_0})}{\iff} \\
k &= \frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta}
\end{aligned}
\tag{2.33}
$$

A sufficient condition for this equality *not* to hold is

$$\frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta} \neq k, \quad \forall q \in \mathscr{F}_2(\varepsilon_{I_0}) \tag{2.34}$$

There are two alternatives, either

$$k < \frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta}, \quad \forall q \in \mathscr{F}_2(\varepsilon_{I_0}) \tag{2.35}$$

or

$$k > \frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta}, \quad \forall q \in \mathscr{F}_2(\varepsilon_{I_0}) \tag{2.36}$$

and also a mix of the two. Let us examine the first alternative. Since[21] $\inf_{q\in\mathscr{F}_2(\varepsilon_{I_0})}\{\gamma_d\} = 0$ and $\beta, \|\nabla\beta\|$ are both bounded in $\mathscr{F}_2(\varepsilon_{I_0})$ it follows that

$$\left\{\begin{array}{c} k < \frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta}, \quad \forall q \in \mathscr{F}_2(\varepsilon_{I_0}) \\ \inf_{q\in\mathscr{F}_2(\varepsilon_{I_0})}\{\frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta}\} = 0 \end{array}\right\} \implies k < 0 \tag{2.37}$$

which cannot be, since $k \in \mathbb{N} \setminus \{0, 1\}$. As a result, the alternative $k < \frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta}$ is not possible.

The only possible alternative remaining is $k > \frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta}, \quad \forall q \in \mathscr{F}_2(\varepsilon_{I_0})$.

It suffices to find a k always greater than $\frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta}$, without calculating this expression.

Assume that we find an upper bound N on $\frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta}$. Then setting k greater than or equal to this upper bound N, i.e. $N \leq k$, will ensure that k is greater than the expression $\frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta}$.

We seek an upper bound on the left side of the above inequality. Since

$$
\begin{aligned}
\frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta} &= \frac{1}{2}\sqrt{\gamma_d}\frac{\left\|\nabla\prod_{i\in I_0}(\beta_i)\right\|}{\beta} \\
&= \frac{1}{2}\sqrt{\gamma_d}\frac{\left\|\sum_{i\in I_0}\left(\prod_{j\in I_0\setminus i}(\beta_j)\nabla\beta_i\right)\right\|}{\beta} = \frac{1}{2}\sqrt{\gamma_d}\frac{\left\|\sum_{i\in I_0}\left(\bar{\beta}_i\nabla\beta_i\right)\right\|}{\beta}
\end{aligned}
\tag{2.38}
$$

[21]This follows from the fact that *in general* q_d may be a single point excluded from $\mathscr{F}_2(\varepsilon_{I_0})$ and that $\gamma_d(q_d) = 0$.

application of the triangular inequality leads to

$$
\begin{aligned}
\frac{1}{2}\sqrt{\gamma_d}\frac{\left\|\sum_{i\in I_0}\left(\bar{\beta}_i\nabla\beta_i\right)\right\|}{\beta} &\leq \frac{1}{2}\sqrt{\gamma_d}\frac{\sum_{i\in I_0}\left(\left\|\bar{\beta}_i\nabla\beta_i\right\|\right)}{\beta} \overset{\bar{\beta}_i>0,\forall q\in\mathscr{F}_2\left(\varepsilon_{I_0}\right)}{=} \frac{1}{2}\sqrt{\gamma_d}\frac{\sum_{i\in I_0}\left(\bar{\beta}_i\left\|\nabla\beta_i\right\|\right)}{\beta}\\
&= \frac{1}{2}\sqrt{\gamma_d}\sum_{i\in I_0}\left(\frac{\bar{\beta}_i}{\beta}\left\|\nabla\beta_i\right\|\right) = \frac{1}{2}\sqrt{\gamma_d}\sum_{i\in I_0}\left(\frac{\prod_{j\in I_0\setminus i}\beta_j}{\prod_{j\in I_0}\beta_j}\left\|\nabla\beta_i\right\|\right)\\
&= \frac{1}{2}\sqrt{\gamma_d}\sum_{i\in I_0}\left(\frac{1}{\beta_i}\left\|\nabla\beta_i\right\|\right) \implies\\
\frac{1}{2}\sqrt{\gamma_d}\frac{\left\|\sum_{i\in I_0}\left(\bar{\beta}_i\nabla\beta_i\right)\right\|}{\beta} &\leq \frac{1}{2}\sqrt{\gamma_d}\sum_{i\in I_0}\left(\frac{\left\|\nabla\beta_i\right\|}{\beta_i}\right)
\end{aligned}
\tag{2.39}
$$

So *it suffices* to seek an upper bound on $\frac{1}{2}\sqrt{\gamma_d}\sum_{i\in I_0}\left(\frac{\|\nabla\beta_i\|}{\beta_i}\right)$ since this will also be an upper bound on $N(\varepsilon_{I_0})$. This upper bound is $\sup_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)}\left\{\frac{1}{2}\sqrt{\gamma_d}\sum_{i\in I_0}\left(\frac{\|\nabla\beta_i\|}{\beta_i}\right)\right\}$.

Finding the exact supremum is not easily analytically tractable. An alternative would be to select a computational search method, but this would be computationally intensive (requiring time not available in a real-time implementation) and would lack the required guarantees. After all, why search for the global maximum of an auxiliary function, when the original problem was that anyway!

For these reasons we approach to find an *approximation* to the supremum. As expected, there is not a single way for determining such an approximation. In [23] an unfavorable[22] approximation is derived.

This results primarily because the function $\frac{1}{2}\sqrt{\gamma_d}\sum_{i\in I_0}\left(\frac{\|\nabla\beta_i\|}{\beta_i}\right)$ includes $\beta_i(q)$ twice. The one β_i is directly visible in the denominator. The other one is in the norm $\|\nabla\beta_i\|$.

Shortly stated, they bound the maximum of $\max\left\{\frac{a}{b}\right\}$ by $\frac{\max\{a\}}{\min\{b\}}$. But note that β_i arises in both the nominator and denominator. So they end up with (coarsly) $\frac{\max\{\beta_i\}}{\min\{\beta_i\}}$. As expected, this estimate is considerably larger than the alternative.

The alternative would be to *first* cancel the similar terms β_i. Then β_i remains in either the nominator or the denominator (here the denominator). As a result the approximate bound will be only $\max\{\beta_i\}$ or only $\frac{1}{\min\{\beta_i\}}$. Both differ obviously from their product.

In the following two sections both the original and the modified derivations are presented and in the final section of this part they are compared *in the limit*.

2.4.2.2 Koditschek-Rimon formula

Since $\varepsilon = \min_{i\in I_0}\{\varepsilon_i\}$ defines the boundary[23] $\bigcup_{i\in I_0}\beta_i^{-1}(\varepsilon)$ of $\mathscr{F}_2\left(\varepsilon\right)$, for all $\beta_i(q)$

$$
\begin{aligned}
0<\varepsilon\leq\beta_i(q),\quad \forall q\in\mathscr{F}_2\left(\varepsilon\right),\quad \forall i\in I_0 \implies\\
\frac{1}{\beta_i(q)}\leq\frac{1}{\varepsilon},\quad \forall q\in\mathscr{F}_2\left(\varepsilon\right),\quad \forall i\in I_0
\end{aligned}
\tag{2.40}
$$

[22]Their supremum approximation is theoretically perfect, but *computationally* not applicable, because it yields too large k exponent values.

[23]Caution: although the free space $\mathscr{F}$ boundary is $\beta^{-1}(0)$, the boundary of the space "away" from the obstacles $\mathscr{F}_2\left(\varepsilon\right)$ is *not* $\beta^{-1}(\varepsilon)$. The reason for this is that when on the free space boundary, exactly one $\beta_i(q)$ becomes zero, forcing the whole product $\beta(q)$ to become zero (like a veto). On the contrary, when on the boundary of a closed ball $\mathscr{B}_i\left(\varepsilon\right)$ the corresponding $\beta_i(q)=\varepsilon$, but this *cannot* force the product $\beta(q)$ to ε.

Substitution of $\frac{1}{\beta_i(q)} \leq \frac{1}{\varepsilon}$ yields

$$\frac{1}{2}\sqrt{\gamma_d}\sum_{i\in I_0}\left(\frac{\|\nabla\beta_i\|}{\beta_i}\right) \leq \frac{1}{2}\sqrt{\gamma_d}\sum_{i\in I_0}\left(\frac{\|\nabla\beta_i\|}{\varepsilon}\right) = \frac{1}{2}\frac{1}{\varepsilon}\sqrt{\gamma_d}\sum_{i\in I_0}\left(\|\nabla\beta_i\|\right) \tag{2.41}$$

The inequality constraint for k is imposed in the set "away" from the obstacles $q \in \mathscr{F}_2(\varepsilon) \subset \mathscr{F} \subset \mathscr{W}$, so the function values are less than or equal to their maximum values over the workspace $\mathscr{W}$. Note that $\gamma_d < \max_{\mathscr{W}}\{\gamma_d\}, \quad \forall q \in \mathscr{F}_2(\varepsilon)$, because $\max_{\mathscr{W}}\{\gamma_d\}$ is attained on the boundary $\partial\mathscr{B}_0(\varepsilon_0)$ and this boundary is excluded from $\mathscr{F}_2(\varepsilon)$. Therefore instead of writing

$$\frac{1}{2}\frac{1}{\varepsilon}\sqrt{\gamma_d}\sum_{i\in I_0}\left(\|\nabla\beta_i\|\right) \leq \frac{1}{2}\frac{1}{\varepsilon}\max_{\mathscr{W}}\{\sqrt{\gamma_d}\}\sum_{i\in I_0}\max_{\mathscr{W}}\{\|\nabla\beta_i\|\} \tag{2.42}$$

we can replace $\leq$ with $<$ in the previous inequality to obtain

$$\frac{1}{2}\frac{1}{\varepsilon}\sqrt{\gamma_d}\sum_{i\in I_0}\left(\|\nabla\beta_i\|\right) < \frac{1}{2}\frac{1}{\varepsilon}\max_{\mathscr{W}}\{\sqrt{\gamma_d}\}\sum_{i\in I_0}\max_{\mathscr{W}}\{\|\nabla\beta_i\|\} \tag{2.43}$$

Let us now define

$$N_{KR}(\varepsilon) \triangleq \frac{1}{2}\frac{1}{\varepsilon}\max_{\mathscr{W}}\{\sqrt{\gamma_d}\}\sum_{i\in I_0}\max_{\mathscr{W}}\{\|\nabla\beta_i\|\} \tag{2.44}$$

This $N_{KR}(\varepsilon)$ is an upper bound on $\frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta}$, so by setting

$$k \geq N_{KR}(\varepsilon) \tag{2.45}$$

we ensure that all critical points are "pushed" to the set "near" the obstacles[24] $(\{q_d\} \cup \partial\mathscr{F} \cup \mathscr{F}_0(\varepsilon) \cup \mathscr{F}_1(\varepsilon))$.

The expression $\frac{1}{2}\frac{1}{\varepsilon}\max_{\mathscr{W}}\left\{\sqrt{\gamma_d}\right\}\sum_{i\in I_0}\max_{\mathscr{W}}\{\|\nabla\beta_i\|\}$ is not calculable in this form and needs further manipulation. The maxima $\max_{\mathscr{W}}\left\{\sqrt{\gamma_d}\right\}$ and $\max_{\mathscr{W}}\{\|\nabla\beta_i\|\}$ are derived in section A.5. Substituting these in the $N_{KR}(\varepsilon)$ equation results in

$$\begin{aligned}
N_{KR}(\varepsilon) &\triangleq \frac{1}{2}\frac{1}{\varepsilon}\max_{\mathscr{W}}\{\sqrt{\gamma_d}\}\sum_{i\in I_0}\max_{\mathscr{W}}\{\|\nabla\beta_i\|\} \\
&= \frac{1}{2}\frac{1}{\varepsilon}\left(\rho_0 + \|q_d\|\right)\sum_{i\in I_0}\left(2\left(\rho_0 + \|q_i\|\right)\right) \\
&= \frac{1}{\varepsilon}\left(\rho_0 + \|q_d\|\right)\sum_{i\in I_0}\left(\rho_0 + \|q_i\|\right) \\
&= \frac{1}{\varepsilon}\left(\rho_0 + \|q_d\|\right)\left((M+1)\rho_0 + \sum_{i\in I_0}\left(\|q_i\|\right)\right) \overset{\|q_0\|=0}{=} \\
&= \frac{1}{\varepsilon}\left(\rho_0 + \|q_d\|\right)\left((M+1)\rho_0 + \sum_{i\in I_1}\left(\|q_i\|\right)\right) \\
&= \left(\rho_0 + \|q_d\|\right)\sum_{i\in I_0}\frac{\rho_0 + \|q_i\|}{\varepsilon}
\end{aligned} \tag{2.46}$$

Hence the condition to clear $\mathscr{F}_2(\varepsilon)$ of critical points becomes

$$k \geq N_{KR}(\varepsilon) = \left(\rho_0 + \|q_d\|\right)\sum_{i\in I_0}\frac{\rho_0 + \|q_i\|}{\varepsilon} \tag{2.47}$$

[24] Including the workspace boundary, which defines obstacle $\mathscr{O}_0$.

2.4.2.3 Alternative (improved) formula

Let us follow another course and substitute the norms $\|\nabla\beta_i\|, i \in I_0$ as functions of $\beta_i, i \in I_0$

$$\|\nabla\beta_0\| = 2\sqrt{\rho_0^2 - \beta_0}, \quad \|\nabla\beta_i\| = 2\sqrt{\beta_i + \rho_i^2}, \quad \forall i \in I_0 \tag{2.48}$$

in the upper bound of (2.39)

$$\begin{aligned}
\frac{1}{2}\sqrt{\gamma_d}\sum_{i\in I_0}\left(\frac{\|\nabla\beta_i\|}{\beta_i}\right) &= \frac{1}{2}\sqrt{\gamma_d}\left[\frac{2\sqrt{\rho_0^2-\beta_0}}{\beta_0} + \sum_{i\in I_1}\frac{2\sqrt{\beta_i+\rho_i^2}}{\beta_i}\right]\\
&= \sqrt{\gamma_d}\left[\frac{\sqrt{\rho_0^2-\beta_0}}{\beta_0} + \sum_{i\in I_1}\frac{\sqrt{\beta_i+\rho_i^2}}{\beta_i}\right]\\
&= \sqrt{\gamma_d}\left[\sqrt{\frac{\rho_0^2}{\beta_0^2}-\frac{1}{\beta_0}} + \sum_{i\in I_1}\sqrt{\frac{\rho_i^2}{\beta_i^2}+\frac{1}{\beta_i}}\right]\\
&= \sqrt{\gamma_d}\left[\sqrt{\left(\frac{\rho_0}{\beta_0}\right)^2-\frac{1}{\beta_0}} + \sum_{i\in I_1}\sqrt{\left(\frac{\rho_i}{\beta_i}\right)^2+\frac{1}{\beta_i}}\right]\\
&= \sqrt{\gamma_d}\left[\sqrt{\frac{1}{\beta_0}\left(\frac{\rho_0^2}{\beta_0}-1\right)} + \sum_{i\in I_1}\sqrt{\frac{1}{\beta_i}\left(\frac{\rho_i^2}{\beta_i}+1\right)}\right]
\end{aligned} \tag{2.49}$$

where the final three arrangements of the same expression aim to assist further insight.

This bounding function in $\mathscr{F}_2(\varepsilon_{I_0})$ is bounded above by its maximum

$$\begin{aligned}
&\sqrt{\gamma_d}\left[\sqrt{\frac{\rho_0^2}{\beta_0^2}-\frac{1}{\beta_0}} + \sum_{i\in I_1}\sqrt{\frac{\rho_i^2}{\beta_i^2}+\frac{1}{\beta_i}}\right] \le \max_{\mathscr{F}_2(\varepsilon_{I_0})}\left\{\underbrace{\sqrt{\gamma_d}}_{\ge 0, \forall q\in\mathscr{F}_2(\varepsilon_{I_0})}\underbrace{\left[\sqrt{\frac{\rho_0^2}{\beta_0^2}-\frac{1}{\beta_0}} + \sum_{i\in I_1}\sqrt{\frac{\rho_i^2}{\beta_i^2}+\frac{1}{\beta_i}}\right]}_{\ge 0, \forall q\in\mathscr{F}_2(\varepsilon_{I_0})}\right\}\\
&\le \max_{\mathscr{F}_2(\varepsilon_{I_0})}\left\{\sqrt{\gamma_d}\right\}\max_{\mathscr{F}_2(\varepsilon_{I_0})}\left\{\underbrace{\sqrt{\frac{\rho_0^2}{\beta_0^2}-\frac{1}{\beta_0}}}_{\ge 0\forall q\in\mathscr{F}_2(\varepsilon_{I_0})} + \sum_{i\in I_1}\underbrace{\sqrt{\frac{\rho_i^2}{\beta_i^2}+\frac{1}{\beta_i}}}_{\ge 0\forall q\in\mathscr{F}_2(\varepsilon_{I_0})}\right\} \overset{\max\{a+b\}\le\max\{a\}+\max\{b\},\forall a,b\ge 0}{\le}\\
&\le \max_{\mathscr{F}_2(\varepsilon_{I_0})}\left\{\sqrt{\gamma_d}\right\}\left[\max_{\mathscr{F}_2(\varepsilon_{I_0})}\left\{\sqrt{\frac{\rho_0^2}{\beta_0^2}-\frac{1}{\beta_0}}\right\} + \sum_{i\in I_1}\max_{\mathscr{F}_2(\varepsilon_{I_0})}\left\{\sqrt{\frac{\rho_i^2}{\beta_i^2}+\frac{1}{\beta_i}}\right\}\right]\\
&= \max_{\mathscr{F}_2(\varepsilon_{I_0})}\left\{\sqrt{\gamma_d}\right\}\left[\sqrt{\max_{\mathscr{F}_2(\varepsilon_{I_0})}\left\{\frac{\rho_0^2}{\beta_0^2}-\frac{1}{\beta_0}\right\}} + \sum_{i\in I_1}\sqrt{\max_{\mathscr{F}_2(\varepsilon_{I_0})}\left\{\frac{\rho_i^2}{\beta_i^2}+\frac{1}{\beta_i}\right\}}\right]
\end{aligned} \tag{2.50}$$

Now the maxima to be substituted are firstly

$$\begin{aligned}
\max_{\mathscr{F}_2(\varepsilon_{I_0})}\left\{\left(\frac{\rho_0}{\beta_0}\right)^2-\frac{1}{\beta_0}\right\} &\le \left(\max_{\mathscr{F}_2(\varepsilon_{I_0})}\left\{\frac{\rho_0}{\beta_0}\right\}\right)^2 - \min_{\mathscr{F}_2(\varepsilon_{I_0})}\left\{\frac{1}{\beta_0}\right\}\\
&= \left(\frac{\rho_0}{\min_{\mathscr{F}_2(\varepsilon_{I_0})}\{\beta_0\}}\right)^2 - \frac{1}{\max_{\mathscr{F}_2(\varepsilon_{I_0})}\{\beta_0\}} = \left(\frac{\rho_0}{\varepsilon_0}\right)^2 - \frac{1}{\rho_0^2}
\end{aligned} \tag{2.51}$$

because

$$0 < \varepsilon_0 \leq \beta_0, \forall q \in \mathscr{F}_2\left(\varepsilon_{I_0}\right) \implies \min_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)} \{\beta_0\} \geq \varepsilon_0 \tag{2.52}$$

and according to[25] section A.5

$$\mathscr{F}_2\left(\varepsilon_{I_0}\right) \subset \mathscr{W} \implies \max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)} \{\beta_0\} \leq \max_{\mathscr{W}}\{\beta_0\} = \rho_0^2 \tag{2.53}$$

Secondly

$$\begin{aligned}
\max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)} \left\{ \left(\frac{\rho_i}{\beta_i}\right)^2 + \frac{1}{\beta_i} \right\} &= \left(\max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)} \left\{ \frac{\rho_i}{\beta_i} \right\} \right)^2 + \max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)} \left\{ \frac{1}{\beta_i} \right\} \\
&= \left(\frac{\rho_i}{\min_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)} \{\beta_i\}} \right)^2 + \frac{1}{\min_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)} \{\beta_i\}} \\
&= \left(\frac{\rho_i}{\varepsilon_i}\right)^2 + \frac{1}{\varepsilon_i}, \quad \forall i \in I_1
\end{aligned} \tag{2.54}$$

since

$$0 < \varepsilon_i \leq \beta_i, \forall q \in \mathscr{F}_2\left(\varepsilon_{I_0}\right) \implies \min_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)} \{\beta_i\} \geq \varepsilon_i, \quad \forall q \in I_1 \tag{2.55}$$

Note again that

$$\mathscr{F}_2\left(\varepsilon_{I_0}\right) \subseteq \mathscr{W} \setminus \{\partial\mathscr{O}_0 \cup \mathscr{B}_0\left(\varepsilon_0\right)\} \implies \max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)} \{\sqrt{\gamma_d}\} < \max_{\mathscr{W}} \{\sqrt{\gamma_d}\} \tag{2.56}$$

where the maximum $\max_{\mathscr{W}} \left\{\sqrt{\gamma_d}\right\}$ is derived in section A.5. From the previous

$$\begin{aligned}
\frac{1}{2}\sqrt{\gamma_d}\frac{\|\nabla\beta\|}{\beta} &\leq \max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)} \{\sqrt{\gamma_d}\} \left[\sqrt{\max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)} \left\{ \frac{\rho_0^2}{\beta_0^2} - \frac{1}{\beta_0} \right\}} + \sum_{i\in I_1} \sqrt{\max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)} \left\{ \frac{\rho_i^2}{\beta_i^2} + \frac{1}{\beta_i} \right\}} \right] \\
&\leq \left(\rho_0 + \|q_d\|\right) \left[\sqrt{\frac{\rho_0^2}{\varepsilon_0^2} - \frac{1}{\rho_0^2}} + \sum_{i\in I_1} \sqrt{\frac{\rho_i^2}{\varepsilon_i^2} + \frac{1}{\varepsilon_i}} \right]
\end{aligned} \tag{2.57}$$

Let $Q_i : \mathbb{R} \to \mathbb{R}, \forall i \in I_0$

$$\begin{aligned}
Q_0\left(x\right) &\triangleq \sqrt{\frac{\rho_0^2}{x^2} - \frac{1}{x}}, \quad Q_{00} \triangleq \sqrt{\frac{\rho_0^2}{\varepsilon_0^2} - \frac{1}{\rho_0^2}} \\
Q_i\left(x\right) &\triangleq \sqrt{\frac{\rho_i^2}{x^2} + \frac{1}{x}}, \quad Q_{ii} \triangleq Q_i(\varepsilon_i), \quad i \in I_1
\end{aligned} \tag{2.58}$$

Note that

$$Q_0\left(\beta_0\right) = \frac{1}{2}\frac{\|\nabla\beta_0\|}{\beta_0}, \quad Q_i\left(\beta_i\right) = \frac{1}{2}\frac{\|\nabla\beta_i\|}{\beta_i} \tag{2.59}$$

[25]Generally $\max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)} \{\beta_0\}$ and $\max_{\mathscr{W}}\{\beta_0\}$ do not differ substantially and if the wrold center is free space, they are equal.

and while Q_{00} is an upper bound on the maximum of $\max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)}\left\{Q_0\left(\beta_0\right)\right\}$, the $Q_{ii}, \forall i \in I_1$ is equal to the respective maximum $\max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)}\left\{Q_i\left(\beta_i\right)\right\}$

$$
\begin{aligned}
Q_{00} &= \sqrt{\frac{\rho_0^2}{\varepsilon_0^2} - \frac{1}{\rho_0^2}} \geq \max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)}\left\{\frac{1}{2}\frac{2\sqrt{\rho_0^2-\beta_0}}{\beta_0}\right\} = \frac{1}{2}\max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)}\left\{\frac{\|\nabla\beta_0\|}{\beta_0}\right\} = \max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)}\left\{Q_0\left(\beta_0\right)\right\} \\
Q_{ii} &= \sqrt{\frac{\rho_i^2}{\varepsilon_i^2} + \frac{1}{\varepsilon_i^2}} = \max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)}\left\{\frac{1}{2}\frac{2\sqrt{\rho_i^2+\beta_i}}{\beta_i}\right\} = \frac{1}{2}\max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)}\left\{\frac{\|\nabla\beta_i\|}{\beta_i}\right\} = \max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)}\left\{Q_i\left(\beta_i\right)\right\}, \quad \forall i \in I_1
\end{aligned}
\tag{2.60}
$$

therefore

$$
Q_{ii} \geq \frac{1}{2}\max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)}\left\{\frac{\|\nabla\beta_i\|}{\beta_i}\right\} = \max_{\mathscr{F}_2\left(\varepsilon_{I_0}\right)}\left\{Q_i\left(\beta_i\right)\right\}, \quad \forall i \in I_0 \tag{2.61}
$$

Let us define

$$
N(\varepsilon_{I_0}) \triangleq (\rho_0 + \|q_d\|)\left[\sqrt{\frac{\rho_0^2}{\varepsilon_0^2} - \frac{1}{\rho_0^2}} + \sum_{i\in I_1}\sqrt{\frac{\rho_i^2}{\varepsilon_i^2} + \frac{1}{\varepsilon_i}}\right] = (\rho_0 + \|q_d\|)\sum_{i\in I_0} Q_{ii} \tag{2.62}
$$

This $N(\varepsilon_{I_0})$ is an upper bound on $\frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta}$, so by selecting a k satisfying

$$
N(\varepsilon_{I_0}) \leq k \tag{2.63}
$$

we ensure that all critical points are "pushed" to the set "near" the obstacles[26] $(\{q_d\} \cup \partial\mathscr{F} \cup \mathscr{F}_0\left(\varepsilon_{I_1}\right) \cup \mathscr{F}_1\left(\varepsilon_{I_0}\right))$.

2.4.2.4 Comparison of the original and modified formulas

We can compare the two expressions derived as lower bounds for k. This is accomplished by dividing them

$$
\frac{N(\varepsilon_{I_0})}{N_{KR}(\varepsilon)} = \frac{(\rho_0 + \|q_d\|)\left[\sqrt{\frac{\rho_0^2}{\varepsilon_0^2} - \frac{1}{\rho_0^2}} + \sum_{i\in I_1}\sqrt{\frac{\rho_i^2}{\varepsilon_i^2} + \frac{1}{\varepsilon_i}}\right]}{\frac{1}{\varepsilon}(\rho_0 + \|q_d\|)\left((M+1)\rho_0 + \sum_{i\in I_1}(\|q_i\|)\right)} = \frac{\sqrt{\frac{\rho_0^2}{\varepsilon_0^2} - \frac{1}{\rho_0^2}} + \sum_{i\in I_1}\sqrt{\frac{\rho_i^2}{\varepsilon_i^2} + \frac{1}{\varepsilon_i}}}{\sum_{i\in I_0}\frac{\rho_0 + \|q_i\|}{\varepsilon}} \tag{2.64}
$$

Usually the following approximations are valid

$$
\begin{aligned}
\rho_0^2 \gg \varepsilon_0 &\implies \frac{\rho_0^2}{\varepsilon_0} \gg 1 \implies \frac{\rho_0^2}{\varepsilon_0}\frac{1}{\varepsilon_0} \gg \frac{1}{\varepsilon_0} \gg \frac{1}{\rho_0^2} \implies \frac{\rho_0^2}{\varepsilon_0^2} - \frac{1}{\rho_0} \approx \frac{\rho_0^2}{\varepsilon_0^2} \\
\rho_i^2 \gg \varepsilon_i &\implies \frac{\rho_i^2}{\varepsilon_i} \gg 1 \implies \frac{\rho_i^2}{\varepsilon_i}\frac{1}{\varepsilon_i} \gg \frac{1}{\varepsilon_i} \implies \frac{\rho_i^2}{\varepsilon_i^2} + \frac{1}{\varepsilon_i} \approx \frac{\rho_i^2}{\varepsilon_i^2}
\end{aligned}
\tag{2.65}
$$

[26]Including the workspace boundary, which defines obstacle $\mathscr{O}_0$.

Their adoption leads to

$$\begin{aligned}\frac{N(\varepsilon_{I_0})}{N_{KR}(\varepsilon)} &\approx \frac{\sqrt{\frac{\rho_0^2}{\varepsilon_0^2}} + \sum_{i\in I_1}\sqrt{\frac{\rho_i^2}{\varepsilon_i^2}}}{\sum_{i\in I_0}\frac{\rho_0+\|q_i\|}{\varepsilon}} = \frac{\sum_{i\in I_0}\frac{\rho_i}{\varepsilon_i}}{\sum_{i\in I_0}\frac{\rho_0+\|q_i\|}{\varepsilon}} = \sum_{i\in I_0}\left(\frac{\rho_i}{\varepsilon_i}\frac{1}{\sum_{j\in I_0}\frac{\rho_0+\|q_j\|}{\varepsilon}}\right)\\ &= \sum_{i\in I_0}\left(\frac{\varepsilon}{\varepsilon_i}\frac{1}{\sum_{j\in I_0}\frac{\rho_0+\|q_j\|}{\rho_i}}\right) = \sum_{i\in I_0}\left(\frac{\varepsilon}{\varepsilon_i}\frac{1}{\sum_{j\in I_0}\left(\frac{\rho_0}{\rho_i}+\frac{\|q_j\|}{\rho_i}\right)}\right)\\ &= \sum_{i\in I_0}\left(\frac{\min_{i\in I_0}\{\varepsilon_i\}}{\varepsilon_i}\frac{1}{\sum_{j\in I_0}\left(\frac{\rho_0}{\rho_i}+\frac{\|q_j\|}{\rho_i}\right)}\right)\end{aligned} \tag{2.66}$$

Because $\min_{i\in I_0}\{\varepsilon_i\} \leq \varepsilon_i, \forall i \in I_0$ and $\rho_i < \rho_0$ it follows that $N(\{\varepsilon_i\}) < N_{KR}(\varepsilon)$ (provided $\rho_i^2 \gg \varepsilon_i, \forall i \in I_1$). Usually $\min_{i\in I_0}\{\varepsilon_i\} \ll \varepsilon_i, \forall i \in I_0$ and $\rho_i \ll \rho_0$ and as a result $N(\varepsilon_{I_0}) \ll N_{KR}(\varepsilon)$. That the proposed lower limit on k is better has been shown.

2.4.3 ε_{0u} calculation

The squared "width" ε_0 of $\overline{\mathscr{B}_0(\varepsilon_0)}$ will be determined to clear the 0th obstacle neighbourhood $\mathscr{B}_0(\varepsilon_0)$ of critical points[27]. Because we have asserted a changed lower bound $N(\varepsilon_{I_0})$ on k the following Proposition[28] will be shown to still hold.

Proposition 1 (Proposition 3.7 [23])**.** If $k \geq N\left(\varepsilon_{I_0}\right)$, then there exists an ε_{0u} such that $\hat{\varphi}$ has no critical points on $\mathscr{F}_1\left(\varepsilon_{I_0}\right)$, as long as $\varepsilon_0 < \varepsilon_{0u}$.

Proof. It is first convenient to bound $\mathscr{B}_0(\varepsilon_0)$ away from the ball of radius given by the destination point q_d, as follows. If

$$\varepsilon_0 < \rho_0^2 - \|q_d\|^2 \tag{2.67}$$

then because

$$\beta_0 < \varepsilon_0, \quad \forall q \in \mathscr{F}_1\left(\varepsilon_{I_0}\right) \overset{\beta_0 = \rho_0^2 - \|q\|^2}{\Longleftrightarrow} \rho_0^2 - \|q\|^2 < \varepsilon_0, \quad \forall q \in \mathscr{F}_1\left(\varepsilon_{I_0}\right) \tag{2.68}$$

it follows that

$$\|q\| > \|q_d\|, \quad \forall q \in \mathscr{F}_1\left(\varepsilon_{I_0}\right) \tag{2.69}$$

This is a sufficient condition for $\nabla\beta_0$ to point away from the destination, i.e. $\nabla\gamma_d \cdot \nabla\beta_0 < 0$ on $\mathscr{B}_0(\varepsilon_0)$, because

$$\frac{1}{4}\nabla\gamma_d \cdot \nabla\beta_0 = -(q - q_d)\cdot q = q \cdot q_d - \|q\|^2 \leq \|q\|\left(\|q_d\| - \|q\|\right) < 0 \tag{2.70}$$

Now, $\nabla\hat{\varphi}$ is non-vanishing on $\mathscr{F}_1\left(\varepsilon_{I_0}\right)$, since its inner-product with $\nabla\gamma_d$, according to subsection A.3.7 is given by

$$\begin{aligned}\nabla\hat{\varphi} \cdot \nabla\gamma_d &= \frac{\gamma_d^k}{\beta^2}\left(4k\beta - \nabla\beta \cdot \nabla\gamma_d\right) \\ &= \frac{\gamma_d^k}{\beta^2}\left(4k\beta - \left(\beta_0\nabla\bar{\beta}_0 \cdot \nabla\gamma_d + \bar{\beta}_0\nabla\beta_0 \cdot \nabla\gamma_d\right)\right) > \beta_0\frac{\gamma_d^k}{\beta^2}\left(4k\bar{\beta}_0 - \nabla\bar{\beta}_0 \cdot \nabla\gamma_d\right)\end{aligned} \tag{2.71}$$

If k is large enough

$$k > \frac{1}{4}\frac{\nabla\bar{\beta}_0 \cdot \nabla\gamma_d}{\bar{\beta}_0}, \quad \forall q \in \mathscr{F}_1\left(\varepsilon_{I_0}\right) \tag{2.72}$$

the term $\nabla\hat{\varphi} \cdot \nabla\gamma_d$ will be positive. But $k \geq N(\varepsilon_{I_0})$ is sufficient for this to be true, since

$$\begin{aligned}\frac{1}{4}\frac{\nabla\bar{\beta}_0 \cdot \nabla\gamma_d}{\bar{\beta}_0} &\leq \frac{1}{2}\frac{\left\|\nabla\bar{\beta}_0\right\|\sqrt{\gamma_d}}{\bar{\beta}_0} \leq \frac{1}{2}\sqrt{\gamma_d}\sum_{i\in I_1}\frac{\bar{\beta}_i}{\beta}\left\|\nabla\beta_i\right\| \\ &\leq \left(\rho_0 + \|q_d\|\right)\sum_{i\in I_1}\sqrt{\frac{\rho_i^2}{\varepsilon_i^2} + \frac{1}{\varepsilon_i}} = \left(\rho_0 + \|q_d\|\right)\sum_{i\in I_1} Q_{ii} \\ &< \left(\rho_0 + \|q_d\|\right)\sum_{i\in I_0} Q_{ii} = N(\varepsilon_{I_0}) \leq k\end{aligned} \tag{2.73}$$

since by definition of $\mathscr{F}_1\left(\varepsilon_{I_0}\right)$, $\varepsilon_i \leq \beta_i, \forall i \in I_1$. The proof is completed by choosing

$$\varepsilon_{0u} \triangleq \rho_0^2 - \|q_d\|^2 \tag{2.74}$$

□

[27]In [23] ε_1 is used for the variable we have chosen to denote with ε_0 for the sake of clarity, since we are here interested in using all the ε_i in the proposed algorithm.

[28][23], Proposition 3.7, pp.432-433.

2.4.4 ε''_{i2} calculation

In [23] the upper bounds ε''_{i2} on ε_i are derived in the form

$$\frac{1}{4}\frac{\min_{\overline{\mathscr{B}_i(\varepsilon_{i23})}}\left\{\sqrt{\bar{\beta}_i}\left\|\nabla\beta_i\right\|\right\}}{\max_{\overline{\mathscr{B}_i(\varepsilon_{i23})}}\left\{\sqrt{\left|\hat{v}^{\mathrm{T}}D^2\bar{\beta}_i\hat{v}\right|}\right\}} \tag{2.75}$$

which, combined with negative definiteness in the tangent space, ensure non-degeneracy of the critical points in $\mathscr{F}_0(\varepsilon_{I_1})$ (near the internal obstacles), which are the only critical points of $\hat{\varphi}$ remaining.

We can observe that $\bar{\beta}_i$ and $D^2\bar{\beta}_i$ arise in nominator and denominator, respectively. This leads to the same problem as when determining $N_{KR}(\varepsilon)$ in subsection 2.4.2.

From within the terms $\bar{\beta}_i, D^2\bar{\beta}_i$ the various $\beta_j, j\neq i$ come. So we have the *same* β_j in *both* nominator and denominator. After manipulation we end up dividing $\min_{\overline{\mathscr{B}_i(\varepsilon_{i23})}}\{\beta_j\}$ by $\max_{\overline{\mathscr{B}_i(\varepsilon_{i23})}}\{\beta_j\}$, which results in a very ill valued constraint. In the present section an alternative formulation is presented.

What is different here? Observe that if we avoid β_j showing up in both numerator and denominator, the result will not be $\min_{\overline{\mathscr{B}_i(\varepsilon_{i23})}}\{\beta_j\}$ divided by $\max_{\overline{\mathscr{B}_i(\varepsilon_{i23})}}\{\beta_j\}$ any more. To achieve this we can cancel the arising β_j. This can be done by dividing both numerator and benominator by $\bar{\beta}_i$. But this should be done *before* applying $\min\{\}$ and $\max\{\}$.

To do this we return to a previous step in the original proof. There it is required[29] that the following expression[30] be positive

$$\begin{gathered}\left(1-\frac{1}{k}\right)\bar{\beta}_i\left\|\nabla\beta_i\right\|^2-\beta_i^2\left|\hat{r}_i^{\mathrm{T}}D^2\bar{\beta}_i\hat{r}_i\right|-2\beta_i\bar{\beta}_i>0 \iff \\ \underbrace{\left[\frac{1}{2}\left(1-\frac{1}{k}\right)\bar{\beta}_i\left\|\nabla\beta_i\right\|^2-2\beta_i\bar{\beta}_i\right]}_{*}+\underbrace{\left[\frac{1}{2}\left(1-\frac{1}{k}\right)\bar{\beta}_i\left\|\nabla\beta_i\right\|^2-\beta_i^2\left|\hat{r}_i^{\mathrm{T}}D^2\bar{\beta}_i\hat{r}_i\right|\right]}_{**}>0\end{gathered} \tag{2.76}$$

where $\hat{r}_i \triangleq \frac{\nabla\beta_i}{\|\nabla\beta_i\|}$. If we require[31] that $k\geq 2$, then

$$2\leq k \iff 0<\frac{1}{k}\leq\frac{1}{2} \iff -\frac{1}{2}\leq-\frac{1}{k}<0 \iff \frac{1}{2}\leq 1-\frac{1}{k}<1 \tag{2.77}$$

and therefore the term $(*)$ is greater than the expression

$$\frac{1}{2}\left(\frac{1}{2}\right)\bar{\beta}_i\left\|\nabla\beta_i\right\|^2-2\beta_i\bar{\beta}_i\leq\frac{1}{2}\left(1-\frac{1}{k}\right)\bar{\beta}_i\left\|\nabla\beta_i\right\|^2-2\beta_i\bar{\beta}_i \tag{2.78}$$

---

[29][23], p.435.

[30]In the original placing the $||$ and substituting ε for β_i in p.435 are done simultaneously to then seek a positive lower bound for the worst case within $\mathscr{B}_i(\varepsilon_i)$. Here $||$ is placed to enable further manipulation, but β_i is retained and only at the end is bounded by ε''_{i2}.

[31]Note that if we allow $k=1$ then the origin remains degenerate even after diffeomorphism $\sigma_d(x)$. In case there is no boundary $\partial\mathscr{W}$, then for the navigation function to have the radial unboundedness property $q\to\infty \implies \varphi\to\infty$ so that it can serve as a Lyapunov candidate function stricter conditions on k are needed, namely $M<k$ as proved in section 3.1.

Then a sufficient condition we can impose to ensure that term $(*)$ be positive, is

$$
\begin{aligned}
0 < \frac{1}{4}\bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - 2\beta_i\bar{\beta}_i = \bar{\beta}_i \left[\frac{1}{4}\left\|\nabla\beta_i\right\|^2 - 2\beta_i\right] &\overset{\bar{\beta}_i>0, \forall q \in \mathscr{B}_i(\varepsilon_i)}{\iff} \\
0 < \frac{1}{4}\left\|\nabla\beta_i\right\|^2 - 2\beta_i &\iff \\
0 < \frac{1}{8}\left\|\nabla\beta_i\right\|^2 - \beta_i &\iff \\
\beta_i < \frac{1}{8}\left\|\nabla\beta_i\right\|^2 &\iff \\
\beta_i < \frac{1}{8}\left[2\sqrt{\beta_i + \rho_i^2}\right]^2 &\iff \\
\beta_i < \frac{1}{8}4\left(\beta_i + \rho_i^2\right) &\iff \\
\beta_i < \frac{1}{2}\beta_i + \frac{1}{2}\rho_i^2 &\iff \\
\frac{1}{2}\beta_i < \frac{1}{2}\rho_i^2 &\iff \\
\left\|q - q_i\right\|^2 - \rho_i^2 < \rho_i^2 &\iff \\
\left\|q - q_i\right\|^2 < 2\rho_i^2 &\overset{0<\|q-q_i\|, \rho_i, \forall q \in \mathscr{B}_i(\varepsilon_i)}{\iff} \\
\left\|q - q_i\right\| < \rho_i\sqrt{2}
\end{aligned}
\tag{2.79}
$$

In [23] it is required that $\beta_i < \frac{1}{8}\left\|\nabla\beta_i\right\|, \forall q \in \mathscr{B}_i\left(\varepsilon_i\right)$, for which a sufficient condition is imposed

$$
\varepsilon_i < \frac{1}{8}\min_{\mathscr{B}_i(\varepsilon_i)}\left\{\left\|\nabla\beta_i\right\|^2\right\} = \frac{1}{8}(2\rho_i)^2 = \frac{1}{8}4\rho_i^2 = \frac{1}{2}\rho_i^2 \tag{2.80}
$$

and ε'_{i2} is defined as $\frac{1}{2}\rho_i^2$. This leads to

$$
\begin{aligned}
\beta_i < \varepsilon_i < \frac{1}{2}\rho_i^2, \forall q \in \mathscr{B}_i\left(\varepsilon_i\right) &\implies \\
\beta_i < \frac{1}{2}\rho_i^2 \iff \left\|q - q_i\right\|^2 - \rho_i^2 < \frac{1}{2}\rho_i^2 &\iff \\
\left\|q - q_i\right\|^2 < \frac{3}{2}\rho_i^2 \overset{0<\|q-q_i\|, \rho_i, \forall q \in \mathscr{B}_i(\varepsilon_i)}{\iff} \left\|q - q_i\right\| < \rho_i\sqrt{\frac{3}{2}} &= \rho_i\sqrt{2}\frac{1}{\sqrt{2}}\sqrt{\frac{3}{2}} = \left(\rho_i\sqrt{2}\right)\frac{\sqrt{3}}{2}
\end{aligned}
\tag{2.81}
$$

whereas, as already shown, requiring that

$$
\beta_i < \rho_i^2 \iff \left\|q - q_i\right\| < \rho_i\sqrt{2} \tag{2.82}
$$

It is now obvious that

$$
\frac{\rho_i\sqrt{2}}{\left(\rho_i\sqrt{2}\right)\sqrt{\frac{3}{2}}} = \frac{2}{\sqrt{3}} \approx 1.1547 \tag{2.83}
$$

so the selection

$$
\varepsilon'_{i2} \triangleq \rho_i^2, \quad \forall i \in I_1 \tag{2.84}
$$

is slightly better.

Let us now examine the term $(**)$. Since $k \geq 2$

$$
\begin{aligned}
&\frac{1}{2}\left(1-\frac{1}{k}\right)\bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - \beta_i^2 \left|\hat{r}_i^{\mathrm{T}} D^2 \bar{\beta}_i \hat{r}_i\right| \geq \frac{1}{4}\bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - \beta_i^2 \left|\hat{r}_i^{\mathrm{T}} D^2 \bar{\beta}_i \hat{r}_i\right| \\
&\geq \frac{1}{4}\bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - \beta_i^2 2 \sum_{j\in I_0\setminus i}\left(\prod_{l\in I_0\setminus\{i,j\}}\beta_l + \sum_{l\in I_0\setminus\{i,j\}}\left(\prod_{m\in I_0\setminus\{i,j,l\}}\beta_m\right)\left\|\nabla\beta_j\right\|\left\|\nabla\beta_l\right\|\right)
\end{aligned}
\tag{2.85}
$$

The inequality

$$
\left|\hat{r}_i^{\mathrm{T}} D^2 \bar{\beta}_i \hat{r}_i\right| \leq 2 \sum_{j\in I_0\setminus i}\left(\prod_{l\in I_0\setminus\{i,j\}}\beta_l + \sum_{l\in I_0\setminus\{i,j\}}\left(\prod_{m\in I_0\setminus\{i,j,l\}}\beta_m\right)\left\|\nabla\beta_j\right\|\left\|\nabla\beta_l\right\|\right)
\tag{2.86}
$$

is proved in subsection A.6.1. A sufficient condition for the term $(**)$ to be positive is

$$
0 < \frac{1}{4}\bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - \beta_i^2 2 \sum_{j\in I_0\setminus i}\left(\prod_{l\in I_0\setminus\{i,j\}}\beta_l + \sum_{l\in I_0\setminus\{i,j\}}\left(\prod_{m\in I_0\setminus\{i,j,l\}}\beta_m\right)\left\|\nabla\beta_j\right\|\left\|\nabla\beta_l\right\|\right), \forall q \in \mathscr{B}_i\left(\varepsilon_i\right)
\tag{2.87}
$$

This expression can be rearranged as following. It is now that we divide both numerator and denominator by $\bar{\beta}_i$.

$$
\begin{aligned}
&\frac{1}{4}\bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - \beta_i^2 2 \sum_{j\in I_0\setminus i}\left(\prod_{l\in I_0\setminus\{i,j\}}\beta_l + \sum_{l\in I_0\setminus\{i,j\}}\left(\prod_{m\in I_0\{i,j,l\}}\beta_m\right)\left\|\nabla\beta_j\right\|\left\|\nabla\beta_l\right\|\right) \\
&= \frac{1}{4}\bar{\beta}_i 2^2 \sqrt{\beta_i+\rho_i^2}^{\,2} - \beta_i^2 2 \sum_{j\in I_0\setminus i}\left(\prod_{l\in I_0\setminus\{i,j\}}\beta_l + \sum_{l\in I_0\setminus\{i,j\}}\left(\prod_{m\in I_0\setminus\{i,j,l\}}\beta_m\right)\left\|\nabla\beta_j\right\|\left\|\nabla\beta_l\right\|\right) \\
&= \frac{4}{4}\bar{\beta}_i\left(\beta_i+\rho_i^2\right) - 2\beta_i^2 \sum_{j\in I_0\setminus i}\left(\frac{\beta}{\beta_i\beta_j} + \sum_{l\in I_0\setminus\{i,j\}}\left(\frac{\beta}{\beta_i\beta_j\beta_l}\left\|\nabla\beta_j\right\|\left\|\nabla\beta_l\right\|\right)\right) \\
&= \bar{\beta}_i\left(\beta_i+\rho_i^2\right) - 2\beta_i^2 \sum_{j\in I_0\setminus i}\left(\frac{\beta}{\beta_i}\frac{1}{\beta_j} + \frac{\beta}{\beta_i}\frac{\left\|\nabla\beta_j\right\|}{\beta_j}\sum_{l\in I_0\setminus\{i,j\}}\left(\frac{\left\|\nabla\beta_l\right\|}{\beta_l}\right)\right) \\
&= \bar{\beta}_i\left(\beta_i+\rho_i^2\right) - 2\beta_i^2\bar{\beta}_i \sum_{j\in I_0\setminus i}\left(\frac{1}{\beta_j} + \frac{\left\|\nabla\beta_j\right\|}{\beta_j}\sum_{l\in I_0\setminus\{i,j\}}\left(\frac{\left\|\nabla\beta_l\right\|}{\beta_l}\right)\right) \\
&= \bar{\beta}_i\left[\left(\beta_i+\rho_i^2\right) - 2\beta_i^2 \sum_{j\in I_0\setminus i}\left(\frac{1}{\beta_j} + \frac{\left\|\nabla\beta_j\right\|}{\beta_j}\sum_{l\in I_0\setminus\{i,j\}}\left(\frac{\left\|\nabla\beta_l\right\|}{\beta_l}\right)\right)\right] \\
&= \bar{\beta}_i\left[\left(\beta_i+\rho_i^2\right) - 2\beta_i^2\left(\begin{array}{c} \frac{1}{\beta_0} + \frac{2\sqrt{\rho_0^2-\beta_0}}{\beta_0}\sum_{l\in I_1\setminus i}\left(\frac{2\sqrt{\beta_l+\rho_l^2}}{\beta_l}\right) \\ +\sum_{j\in I_1\setminus i}\left(\frac{1}{\beta_j} + \frac{2\sqrt{\beta_j+\rho_j^2}}{\beta_j}\left(\frac{2\sqrt{\rho_0^2-\beta_0}}{\beta_0} + \sum_{l\in I_1\setminus\{i,j\}}\left(\frac{2\sqrt{\beta_l+\rho_l^2}}{\beta_l}\right)\right)\right)\end{array}\right)\right]
\end{aligned}
$$

$$= \bar{\beta}_i \left[\left(\beta_i + \rho_i^2\right) - 2\beta_i^2 \left(\begin{array}{c} \frac{1}{\beta_0} + 4\sqrt{\frac{\rho_0^2}{\beta_0^2} - \frac{1}{\beta_0}} \sum_{l \in I_1 \backslash i} \left(\sqrt{\frac{\rho_l^2}{\beta_l^2} + \frac{1}{\beta_l}} \right) \\ + \sum_{j \in I_1 \backslash i} \left(\frac{1}{\beta_j} + 4\sqrt{\frac{\rho_j^2}{\beta_j^2} + \frac{1}{\beta_j}} \left(\sqrt{\frac{\rho_0^2}{\beta_0^2} - \frac{1}{\beta_0}} + \sum_{l \in I_1 \backslash \{i,j\}} \left(\sqrt{\frac{\rho_l^2}{\beta_l} + \frac{1}{\beta_l}} \right) \right) \right) \end{array} \right) \right]$$

$$= \bar{\beta}_i \left[\beta_i + \rho_i^2 - 2\beta_i^2 \sum_{j \in I_0 \backslash i} \left(\frac{1}{\beta_j} + 4 Q_j\left(\beta_j\right) \sum_{l \in I_0 \backslash \{i,j\}} Q_l\left(\beta_l\right) \right) \right] \tag{2.88}$$

We can now require that this expression be positive

$$\bar{\beta}_i \left[\beta_i + \rho_i^2 - 2\beta_i^2 \sum_{j \in I_0 \backslash i} \left(\frac{1}{\beta_j} + 4 Q_j\left(\beta_j\right) \sum_{l \in I_0 \backslash \{i,j\}} Q_l\left(\beta_l\right) \right) \right] > 0, \forall q \in \mathscr{B}_i\left(\varepsilon_i\right) \overset{\bar{\beta}_i > 0, \forall q \in \mathscr{B}_i(\varepsilon_i)}{\iff}$$
$$\beta_i + \rho_i^2 > 2\beta_i^2 \sum_{j \in I_0 \backslash i} \left(\frac{1}{\beta_j} + 4 Q_j\left(\beta_j\right) \sum_{l \in I_0 \backslash \{i,j\}} Q_l\left(\beta_l\right) \right), \forall q \in \mathscr{B}_i\left(\varepsilon_i\right) \iff$$
$$\frac{\beta_i + \rho_i^2}{2 \sum_{j \in I_0 \backslash i} \left(\frac{1}{\beta_j} + 4 Q_j\left(\beta_j\right) \sum_{l \in I_0 \backslash \{i,j\}} Q_l\left(\beta_l\right) \right)} > \beta_i^2, \forall q \in \mathscr{B}_i\left(\varepsilon_i\right) \iff$$
$$\sqrt{\frac{\beta_i + \rho_i^2}{2 \sum_{j \in I_0 \backslash i} \left(\frac{1}{\beta_j} + 4 Q_j\left(\beta_j\right) \sum_{l \in I_0 \backslash \{i,j\}} Q_l\left(\beta_l\right) \right)}} > \beta_i, \forall q \in \mathscr{B}_i\left(\varepsilon_i\right) \tag{2.89}$$

A sufficient condition for the inequality to hold is

$$\sqrt{\frac{\min_{\overline{\mathscr{B}_i(\varepsilon_{i23})}} \left\{\beta_i + \rho_i^2\right\}}{\max_{\overline{\mathscr{B}_i(\varepsilon_{i23})}} \left\{ 2 \sum_{j \in I_0 \backslash i} \left(\frac{1}{\beta_j} + 4 Q_j\left(\beta_j\right) \sum_{l \in I_0 \backslash \{i,j\}} Q_l\left(\beta_l\right) \right) \right\}}} > \varepsilon_i > \beta_i > 0, \forall q \in \mathscr{B}_i\left(\varepsilon_i\right) \tag{2.90}$$

Let

$$\beta_{ji}^{\min} \triangleq \min_{\overline{\mathscr{B}_i(\varepsilon_{i23})}} \left\{\beta_j\right\}, \quad \beta_{ji}^{\max} \triangleq \max_{\overline{\mathscr{B}_i(\varepsilon_{i23})}} \left\{\beta_j\right\} \tag{2.91}$$

$$Q_{0i} \triangleq \sqrt{\frac{\rho_0^2}{(\beta_{0i}^{\min})^2} - \frac{1}{(\beta_{0i}^{\max})}} \tag{2.92}$$

$$Q_{ji} \triangleq Q_j(\beta_{ji}^{\min}) \tag{2.93}$$

and because

$$\min_{\overline{\mathscr{B}_i(\varepsilon_{i23})}} \left\{\beta_i + \rho_i^2\right\} = \min_{\overline{\mathscr{B}_i(\varepsilon_{i23})}} \left\{\beta_i\right\} + \min_{\overline{\mathscr{B}_i(\varepsilon_{i23})}} \left\{\rho_i^2\right\} \overset{\min_{\overline{\mathscr{B}_i(\varepsilon_{i23})}}\{\beta_i\}=0}{=} \rho_i^2 \tag{2.94}$$

it follows that the above is equivalent to

$$\varepsilon_{i2}'' \triangleq \frac{\rho_i}{\sqrt{2 \sum_{j \in I_0 \backslash i} \left(\frac{1}{\beta_{ji}^{\min}} + 4 Q_{ji} \sum_{l \in I_0 \backslash \{i,j\}} Q_{li} \right)}} > \varepsilon_i > \beta_i > 0, \quad \forall q \in \mathscr{B}_i\left(\varepsilon_i\right), \quad i \in I_1 \tag{2.95}$$

2.4.5 ε''_{i0} calculation

2.4.5.1 Issues with original ε''_{i0}

In [23] the upper bounds $\varepsilon'_{i0}, \varepsilon''_{i0}$ on ε_i are derived as

$$\|q_d - q_i\|^2 - \rho_i^2, \quad \frac{\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\left\{2\,|\nu(q)|\,\bar{\beta}_i^2\right\}}{\max_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\left\{\frac{1}{2}\bar{\beta}_i\nabla\bar{\beta}_i\cdot\nabla\gamma_d + \gamma_d\hat{v}^{\mathrm{T}}\left[\left(1-\frac{1}{k}\right)\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} - \bar{\beta}_i D^2\bar{\beta}_i\right]\hat{v}\right\}} \tag{2.96}$$

respectively. Note that ε'_{i0} in $\overline{\mathscr{B}_i(\cdot)}$ is mandatory to derive the above bound. To proceed further and substitute specific expressions for minima and maxima from the Appendix the additional constraint $\varepsilon_i < \varepsilon_{i3}$ should be placed. This leads to ε_{i03} instead of ε'_{i0} in $\overline{\mathscr{B}_i(\cdot)}$. For ε''_{i2} in subsection 2.4.4 this change has been made from the start.

But here ε'_{i0} is not yet changed because there is an issue associated with the specific selection of ε'_{i0} as $\|q_d - q_i\| - \rho_i^2$ in [23]. Even when ε_{i03} replaces ε'_{i0}, if $\varepsilon'_{i0} \leq \varepsilon_{i3}$ then $\varepsilon_{i03} = \min\{\varepsilon'_{i0}, \varepsilon_{i3}\} = \varepsilon'_{i0}$ so the issue remains. For this reason in what follows firstly ε'_{i0} is redefined to avoid the issue and then ε_{i03} can be used without problems.

There are several issues with (2.96). The first concerns bounding correctly $|\nu(q)|$. This is addressed in subsubsection 2.4.5.2. The second is similar to those treated previously for $N_{KR}(\varepsilon)$ in subsection 2.4.2 and for ε''_{i2} in subsection 2.4.4. Namely appearance of the same terms $\beta_j, j \neq i$ in both nominator and denominator. This again leads to the $\frac{\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\{\beta_j\}}{\max_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\{\beta_j\}}$ problem. If we divide $\bar{\beta}_i^2$ in the denominator *before* taking the fraction $\min\{\}$, we can avoid this problem.

2.4.5.2 Derivation of original ε'_{i0} and ε''_{i0}

The nominator of ε''_{i0} according to [23] is

$$\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\left\{2\,|\nu_i(q)|\,\bar{\beta}_i^2\right\} \geq 2\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\{|\nu_i(q)|\}\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\left\{\bar{\beta}_i^2\right\} \tag{2.97}$$

while for us here it will initially be

$$\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\{2\,|\nu_i(q)|\} = 2\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\{|\nu_i(q)|\} \tag{2.98}$$

and then improved to

$$\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\left\{2\frac{|\nu_i(q)|}{\gamma_d(q)}\right\} = 2\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\left\{\frac{|\nu_i(q)|}{\gamma_d(q)}\right\} \tag{2.99}$$

In cases (2.97) and (2.98) the lower bound $\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\{|\nu_i(q)|\}$ arises. There are two issues in [23] concerning $\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\{|\nu_i(q)|\}$:

1. In p.431, when going from the inequality after (12) to the inequality with $\min\{\cdot\}, \max\{\cdot\}$ the function $\nu_i(q)$ is written $\nu(q)$. Hereinafter it is proved that the missing index i is a typographic error.
2. The selection of $\varepsilon'_{i0} \triangleq \|q_d - q_i\|^2 - \rho_i^2$ leads to $0 \geq \varepsilon_i$ and needs to be altered.

These concerns have been treated in what follows. It is important to note that the correction applies to both the alternative formula developed in this section *and* to that obtained by direct continuation of the Koditschek and Rimon derivation.

It is now necessary to find an expression for the term $\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\{|\nu_i(q)|\}$ appearing in the nominator. To explore the change from $\nu_i(q)$ to $\nu(q)$ and prove that it is a typographic error, the formula for ε''_{i0} is derived in greater detail here.

By taking a $k \geq N(\varepsilon_{I_0})$ we have "pushed" all the critical points[32] q_c out of the set $\mathscr{F}_2\left(\varepsilon_{I_0}\right)$ "away" from the obstacles, to the set $\mathscr{F}_0\left(\varepsilon_{I_1}\right)$ "near" the internal obstacles[33] $\mathscr{O}_j, j \in I_1$.

Now the critical points within $\mathscr{F}_0\left(\varepsilon_{I_1}\right)$ should be further "pushed" towards the obstacles by narrowing $\mathscr{F}_0\left(\varepsilon_{I_1}\right)$. This will place them *so close* to their nearby obstacle, that the steepness of its repulsive effect on the potential will not allow a minimum to form at any of the critical points[34]. Only a maximum or saddle may form[35].

For a non-degenerate critical point q_c *not* to be a local minimum the Hessian matrix $\left(D^2\hat{\varphi}\right)(q_c)$ should posess at least one negative eigenvalue at the critical point q_c.

If the Hessian $(D^2\hat{\varphi})(q)$ (second derivative) is non-degenerate (non-zero determinant), then the function's curvature may be deduced from it. If positive definite (all eigenvalues positive) then the critical point is a local minimum. If negative definite (negative eigenvalues) it is a local maximum and if both positive and negative eigenvalues exist then there are directions with positive curvature and other directions with negative curvature of the function *at the same* critical point, so a saddle forms there.

We will require at least one negative eigenvalue of the Hessian $(D^2\hat{\varphi})(q)$ to arise at the direction defined by the unit vector $\hat{v}$ orthogonal to the repulsive gradient $\nabla\beta_i$ at q_c. The selected test direction is tangential to level sets of β_i. Its unit vector is defined[36] as

$$\hat{t}_i \triangleq \left(\frac{\nabla\beta_i(q)}{\|\nabla\beta_i(q)\|}\right)^{\perp} \tag{2.100}$$

Requiring that at least one negative eigenvalue exists at q_c makes it impossible for q_c to be a local minimum. The only free parameter constrained by this requirement is an ε_i small enough for a negative eigenvalue to exist[37].

We start by writing the requirement of existence of negative eigenvalues at the tangential direction (the direction defined by $\hat{t}_i$)

$$\hat{t}_i^{\mathrm{T}}\left(D^2\hat{\varphi}\right)(q_c)\hat{t}_i < 0 \tag{2.101}$$

[32][23], p.437: $\mathscr{C}_\varphi$ is the set of critical points. Note that in p.430 these points are denoted with q, but when defining $\hat{v}$ a critical point is denoted by q_c (q critical). This is a typographic mistake, occuring also in a previous publication by the same authors, [31]. Here a critical point is denoted only by q_c.

[33]Also remember that ε_{0u} ensures that no critical points exist in $\mathscr{F}_1\left(\varepsilon_{I_0}\right)$ either (the 0^{th} obstacle's zone).

[34]Remember that complete dissapearance of all critical points is impossible, as proved using the Poincare-Hopf theorem, [23], § 2.2, pp.415-417.

[35]Recall that critical point non-degeneracy within $\mathscr{F}_0\left(\varepsilon_{I_1}\right)$ is ensured by ε''_{i2}.

[36]The definition of $\hat{v}$ in [23], p.430 has typographic mistakes. Furthermore note that $\hat{v}$ is used in [23] in three different ways: as any unit tangent vector (normal to β_i gradient $\nabla\beta_i$), as a radial unit vector (parallel to β_i gradient $\nabla\beta_i$) and as any unit vector (to prove a generally used inequality). To avoid ambiguities here we separately define a tangential unit vector wrt β_i as $\hat{t}_i$, a radial unit vector wrt β_i as $\hat{r}_i$ and a unit vector without specified direction as $\hat{v}$. This $\hat{v}$ is used to prove inequalities applying to both $\hat{t}_i$ and $\hat{r}_i$.

[37]It turns out that more than one constraints on ε_i arise from this requirement.

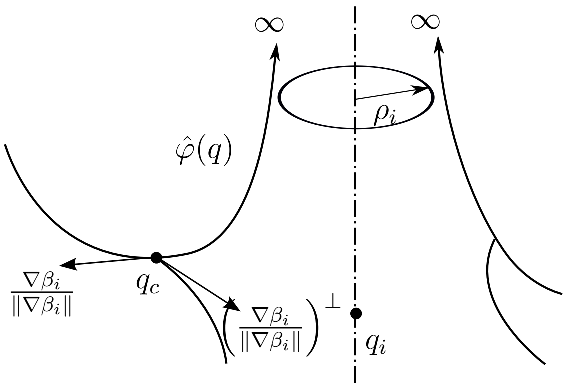


Figure 2.6: Curvatures, saddle and tangential test direction $\hat{t}_i$

Note that

$$\left\{\begin{array}{c}\left\{\begin{array}{c}\gamma_d^{k-1}(q_c) = \left(\|q_c - q_d\|^2\right)^{k-1} = \|q_c - q_d\|^{2k-2} \geq 0 \\ q_c \in \mathscr{F}_0 \implies q_c \neq q_d \iff q_c - q_d \neq 0 \in E^n \iff \|q_c - q_d\| \neq 0 \in \mathbb{R}\end{array}\right\} \\ \implies \gamma_d^{k-1}(q_c) > 0 \\ q_c \in \mathscr{F}_0 \implies \left\{\begin{array}{c} q_c \in \bigcup_{i\in I_1} \mathscr{B}_i(\varepsilon_i) \implies \beta_i(q_c) > 0, \forall i \in I_1 \\ q_c \notin \beta_0^{-1}(0) \implies \beta_0(q_c) > 0 \end{array}\right\} \\ \implies \prod_{i\in I_0} \beta_i(q_c) > 0 \iff \beta(q_c) > 0 \end{array}\right\}$$
$$\implies \frac{\beta^2(q_c)}{\gamma_d^{k-1}(q_c)} > 0 \tag{2.102}$$

By multiplying both sides of inequality (2.101) by $\frac{\beta^2(q_c)}{\gamma_d^{k-1}(q_c)} > 0$, which does not change the inequality direction, we get

$$\frac{\beta^2(q_c)}{\gamma_d^{k-1}(q_c)} \hat{t}_i^{\mathrm{T}} \left(D^2\hat{\varphi}\right)(q_c)\hat{t}_i < 0 \tag{2.103}$$

The above expression is equal to

$$\begin{aligned}\frac{\beta^2}{\gamma_d^{k-1}} \hat{t}_i^{\mathrm{T}} (D^2\hat{\varphi})(q_c)\hat{t}_i =& 2\bar{\beta}_i \underbrace{\left(\frac{1}{4}\nabla\beta_i \cdot \nabla\gamma_d - \gamma_d\right)}_{\nu_i(q_c)} \\ &+ \beta_i \left(\frac{1}{2}\nabla\bar{\beta}_i \cdot \nabla\gamma_d + \gamma_d \hat{t}_i^{\mathrm{T}} \left[\left(1 - \frac{1}{k}\right)\frac{\nabla\bar{\beta}_i \nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - D^2\bar{\beta}_i\right]\hat{t}_i\right)\end{aligned} \tag{2.104}$$

Because $\nu_i(q) = \frac{1}{4}\nabla\beta_i \cdot \nabla\gamma_d - \gamma_d$ substitution in (2.104) yields

$$\begin{aligned}\frac{\beta^2}{\gamma_d^{k-1}} \hat{t}_i^{\mathrm{T}} (D^2\hat{\varphi})(q_c)\hat{t}_i =& 2\bar{\beta}_i \nu_i(q_c) \\ &+ \beta_i \left(\frac{1}{2}\nabla\bar{\beta}_i \cdot \nabla\gamma_d + \gamma_d \hat{t}_i^{\mathrm{T}} \left[\left(1 - \frac{1}{k}\right)\frac{\nabla\bar{\beta}_i \nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - D^2\bar{\beta}_i\right]\hat{t}_i\right)\end{aligned} \tag{2.105}$$

If we now substitute equation (2.105) in inequality (2.103) the constraint takes the form

$$2\bar{\beta}_i \nu_i(q_c) + \beta_i \left(\frac{1}{2}\nabla\bar{\beta}_i \cdot \nabla\gamma_d + \gamma_d \hat{t}_i^{\mathrm{T}} \left[\left(1 - \frac{1}{k}\right)\frac{\nabla\bar{\beta}_i \nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - D^2\bar{\beta}_i\right]\hat{t}_i\right) < 0 \tag{2.106}$$

By definition

$$\begin{aligned} &\beta_j(q) > 0, \quad \forall j \in I_0, \quad \forall q \in \mathscr{F}_0\left(\varepsilon_{I_1}\right) \implies \\ &\bar{\beta}_i(q) = \prod_{j \in I_0 \setminus i} \beta_j(q) > 0, \quad \forall q \in \mathscr{F}_0\left(\varepsilon_{I_1}\right) \end{aligned} \tag{2.107}$$

Multiplying (2.106) by $\bar{\beta}_i(q) > 0$ yields

$$\begin{aligned} \bar{\beta}_i \left(2\bar{\beta}_i \nu_i(q_c) + \beta_i \left(\frac{1}{2} \nabla \bar{\beta}_i \cdot \nabla \gamma_d + \gamma_d \hat{t}_i^{\mathrm{T}} \left[\left(1 - \frac{1}{k} \right) \frac{1}{\bar{\beta}_i} \nabla \bar{\beta}_i \nabla \bar{\beta}_i^{\mathrm{T}} - D^2 \bar{\beta}_i \right] \hat{t}_i \right) \right) < 0 \iff \\ 2\bar{\beta}_i^2 \nu_i(q_c) + \beta_i \left(\frac{1}{2} \bar{\beta}_i \nabla \bar{\beta}_i \cdot \nabla \gamma_d + \gamma_d \hat{t}_i^{\mathrm{T}} \left[\left(1 - \frac{1}{k} \right) \nabla \bar{\beta}_i \nabla \bar{\beta}_i^{\mathrm{T}} - \bar{\beta}_i D^2 \bar{\beta}_i \right] \hat{t}_i \right) < 0 \iff \\ \underbrace{\beta_i}_{>0} \underbrace{\left(\frac{1}{2} \bar{\beta}_i \nabla \bar{\beta}_i \cdot \nabla \gamma_d + \gamma_d \hat{t}_i^{\mathrm{T}} \left[\left(1 - \frac{1}{k} \right) \nabla \bar{\beta}_i \nabla \bar{\beta}_i^{\mathrm{T}} - \bar{\beta}_i D^2 \bar{\beta}_i \right] \hat{t}_i \right)}_{G_i} < -2 \underbrace{\bar{\beta}_i^2}_{>0} \nu_i(q_c) \end{aligned} \tag{2.108}$$

There are four cases of $\nu_i(q_c), G_i$ signs, summarized in Table 2.2, leaving case 0 for later. So if we allow $\nu_i(q_c) > 0$ then G_i can only be negative. On the contrary, if we

Table 2.2: Cases of $\nu_i(q_c)$ and G_i signs and inequality truth value.

Case	$\nu_i(q_c)$	G_i	LHS	RHS	Inequality
1	> 0	> 0	> 0	< 0	FALSE
2	> 0	< 0	< 0	< 0	?
3	< 0	> 0	> 0	> 0	?
4	< 0	< 0	< 0	> 0	TRUE

constrain $\nu_i(q_c) < 0$ then G_i need only be constrained when $G_i > 0$. This is advantageous in that $G_i > 0$ can divide the inequality without changing its sign and there is only one case for which the inequality constraint should be applied. Therefore the expression developed later will ensure the inequality holds in case $G_i > 0$ and it will always be applied as a constraint, without having to determine the sign of G_i.

The constraint $\nu_i(q_c) < 0$ is due to be analyzed both geometrically and analytically. *But* before that, an important note should be made.

To find an upper bound on the denominator G_i of ε''_{i0} it is argued hereinafter that

$$\max_{\mathscr{B}_i(\varepsilon_i)} \{G_i\} \leq \max_{\mathscr{B}_i(\varepsilon_i)} \{|G_i|\} \tag{2.109}$$

therefore a *positive* upper bound is guaranteed to be found. As a result an upper bound constraint will certainly be placed on ε from every obstacle $\mathscr{O}_i$.

But this constraint is not needed if actually $G_i(q_c) < 0$ at the critical point q_c within $\mathscr{B}_i\left(\varepsilon_i\right)$. Because only an upper bound on $|G_i(q_c)|$ is calculated and used and $G_i(q_c)$ remains unknown, that is why the constraint is imposed, while it may not be needed (depending on $G_i(q_c)$). It is very "costly" to determine if it is needed or not.

If $G_i(q_c)$ was computed and resulted negative, then of course it would be impossible to require that

$$\varepsilon_i < 2\frac{-\bar{\beta}_i^2 \nu_i(q_c) > 0}{G_i < 0} < 0 \tag{2.110}$$

because $\varepsilon_i > 0$ by definition. A contradiction?

This apparent contradiction arises due to the fact that in such a case division of the inequality by $G_i < 0$ would change its sign, so that the correct requirement would be

$$\varepsilon_i > \frac{-2\bar{\beta}_i^2 \nu_i(q_c) > 0}{G_i < 0} < 0 \tag{2.111}$$

which is always true[38] since $\varepsilon_i > 0$ by definition.

The conclusion is that unnecessary constraints will most probably be placed on ε by some of the obstacles (those for which $G_i(q_c) < 0$). This is on the safe side.

Note that the case $\nu_i(q_c) = 0$ has been ruled out and that if $G_i(q_c) = 0$ the inequality will be satisfied as in the case that $G_i(q_c) < 0$.

Now the constraint $\nu_i(q) < 0$ is analyzed. The function $\nu_i(q)$ is first defined[39] as

$$\nu_i(q) = \frac{1}{4}\nabla\beta_i \cdot \nabla\gamma_d - \gamma_d = (q_d - q_i) \cdot (q - q_d) \tag{2.112}$$

Its maximum is derived using Lagrange multipliers[40]

$$\max_{\overline{\mathscr{B}_i(\varepsilon_i)}} \{\nu_i(q)\} = \left(\sqrt{\varepsilon + \rho_i^2} - \|q_d - q_i\|\right) \|q_d - q_i\| \tag{2.113}$$

Therefore, if we require

$$\begin{aligned}
&\sqrt{\varepsilon_i + \rho_i^2} < \|q_d - q_i\| \iff \\
&\sqrt{\varepsilon_i + \rho_i^2} - \|q_d - q_i\| < 0 \overset{q_d \neq q_i, \forall i \in I_1 \implies q_d - q_i \neq 0 \implies \|q_d - q_i\| > 0}{\iff} \\
&\left(\sqrt{\varepsilon_i + \rho_i^2} - \|q_d - q_i\|\right) \|q_d - q_i\| < 0 \overset{\max_{\overline{\mathscr{B}_i(\varepsilon_i)}}\{\nu_i(q)\} = \left(\sqrt{\varepsilon_i + \rho_i^2} - \|q_d - q_i\|\right)\|q_d - q_i\|}{\iff} \\
&\max_{\overline{\mathscr{B}_i(\varepsilon_i)}} \{\nu_i(q)\} < 0 \implies \\
&\nu_i(q) < 0, \quad \forall q \in \overline{\mathscr{B}_i(\varepsilon_i)} \overset{\mathscr{B}_i(\varepsilon_i) \subset \overline{\mathscr{B}_i(\varepsilon_i)} \implies \max_{\mathscr{B}_i(\varepsilon_i)}\{\nu_i(q)\} \leq \max_{\overline{\mathscr{B}_i(\varepsilon_i)}}\{\nu_i(q)\}}{\implies} \\
&\nu_i(q) < 0, \quad \forall q \in \mathscr{B}_i(\varepsilon_i) \overset{q_c \in \mathscr{B}_i(\varepsilon_i)}{\implies} \\
&\nu_i(q_c) < 0
\end{aligned} \tag{2.114}$$

the desired constraint is imposed. This is equivalent to requesting for ε_i that, since $\varepsilon_i + \rho_i^2 > 0$ and $\|q_d - q_i\| > 0$

$$\begin{aligned}
\sqrt{\varepsilon_i + \rho_i^2} < \|q_d - q_i\| &\iff \\
\varepsilon_i + \rho_i^2 < \|q_d - q_i\|^2 &\iff \\
\varepsilon_i < \|q_d - q_i\|^2 - \rho_i^2 &
\end{aligned} \tag{2.115}$$

This is essentially the requirement that $q_d \notin \overline{\mathscr{B}_i(\varepsilon_i)}$. It is important to keep this in mind for later.

[38] It is a requirement satisfied without the need to be imposed as a constraint.

[39] [23], p.428.

[40] [23], Lemma 3.5, pp.428-429.

2.4.5.3 Geometry of ε'_{i0}

Here the condition

$$\sqrt{\varepsilon_i + \rho_i^2} < \|q_d - q_i\| \implies \nu_i(q) \quad < 0, \quad \forall q \in \mathscr{B}_i\left(\varepsilon_i\right) \tag{2.116}$$

is illustrated geometrically. First note that $\sqrt{\varepsilon_i + \rho_i^2} = \rho_{\overline{\mathscr{B}}_i}$ as defined in (2.15) so the condition can be written

$$\rho_{\overline{\mathscr{B}}_i} < \|q_d - q_i\| \implies \nu_i(q) \quad < 0, \quad \forall q \in \mathscr{B}_i\left(\varepsilon_i\right) \tag{2.117}$$

Let us start with

$$\nu_i(q) = (q_d - q_i) \cdot (q - q_d) = \|q_d - q_i\| \, \|q - q_d\| \cos(-\theta) \tag{2.118}$$

where $\theta = (\widehat{q - q_d, q_i - q_d})$. By definition

$$\begin{aligned}
\mathscr{B}_i\left(\varepsilon_i\right) &= \left\{q \in E^n | 0 < \beta_i < \varepsilon_i\right\} \\
&= \left\{q \in E^n | 0 < \|q - q_i\|^2 - \rho_i^2 < \varepsilon_i\right\} \implies \\
&\qquad 0 < \|q - q_i\|^2 - \rho_i^2 < \varepsilon_i, \quad \forall q \in \mathscr{B}_i\left(\varepsilon_i\right) \overset{\rho_i > 0, \varepsilon_i + \rho_i^2 > 0, \|q - q_i\| \geq 0}{\Longrightarrow} \\
&\qquad \rho_i < \|q - q_i\| < \sqrt{\varepsilon_i + \rho_i^2}, \quad \forall q \in \mathscr{B}_i\left(\varepsilon_i\right) \iff \\
&\qquad \rho_i < \|q - q_i\| < \rho_{\overline{\mathscr{B}}_i}, \quad \forall q \in \mathscr{B}_i\left(\varepsilon_i\right)
\end{aligned} \tag{2.119}$$

Comparing this result to (2.117) we see that it is equivalent to $q_d \notin \overline{\mathscr{B}_i\left(\varepsilon_i\right)}$. The annulus $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$ has outer radius $\rho_{\overline{\mathscr{B}}_i}$ and center q_i. This outer radius is constrained to be smaller than distance $\|q_d - q_i\|$ of destination q_d from center q_i. As a result $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$ is small enough to not include q_d.

Furthermore the vectors $q_d - q_i$ and $q - q_d$ are shown in Fig. 2.7. Function $\nu_i(q)$ is their inner product. Vector $q_d - q_i$ is constant with respect to q. Since $q \in \mathscr{B}_i\left(\varepsilon_i\right)$ the vector $q - q_d$ remains within the cone $A q_d B$. As long as $\rho_{\overline{\mathscr{B}}_i} < \|q_i - q_d\|$ the cone's aperture remains less than π and $\|q - q_d\| > 0$ so the inner product $\nu_i(q)$ remains negative.

2.4.5.4 Zeroing of original ε''_{i0}

Returning to the required inequality (2.108) when $G_i > 0$, divide both sides by G_i to get

$$\zeta_1 \triangleq \frac{-2\bar{\beta}_i^2 \nu_i(q)}{\frac{1}{2}\bar{\beta}_i \nabla\bar{\beta}_i \cdot \nabla\gamma_d + \gamma_d \hat{t}_i^{\mathrm{T}} \left[\left(1 - \frac{1}{k}\right)\nabla\bar{\beta}_i \nabla\bar{\beta}_i^{\mathrm{T}} - \bar{\beta}_i D^2 \bar{\beta}_i\right] \hat{t}_i} > \beta_i(q_c) \tag{2.120}$$

If we select an $\varepsilon_i > 0$ such that

$$\left\{\begin{array}{c} \zeta_1 \geq \varepsilon_i \\ q_c \in \mathscr{B}_i\left(\varepsilon_i\right) \overset{p.425}{\Longrightarrow} \varepsilon_i > \beta_i(q_c) > 0 \end{array}\right\} \implies \zeta_1 \geq \varepsilon_i > \beta_i(q_c) \implies \zeta_1 > \beta_i(q_c) \tag{2.121}$$

The constraint replacing $\zeta_1 > \beta_i(q_c)$ is

$$\begin{aligned}
&\zeta_1 \geq \varepsilon_i \iff \\
\frac{-2\bar{\beta}_i^2 \nu_i(q)}{\frac{1}{2}\bar{\beta}_i \nabla\bar{\beta}_i \cdot \nabla\gamma_d + \gamma_d \hat{t}_i^{\mathrm{T}} \left[\left(1 - \frac{1}{k}\right)\nabla\bar{\beta}_i \nabla\bar{\beta}_i^{\mathrm{T}} - \bar{\beta}_i D^2 \bar{\beta}_i\right] \hat{t}_i} &\geq \varepsilon_i, \quad q_c \in \mathscr{B}_i\left(\varepsilon_i\right)
\end{aligned} \tag{2.122}$$

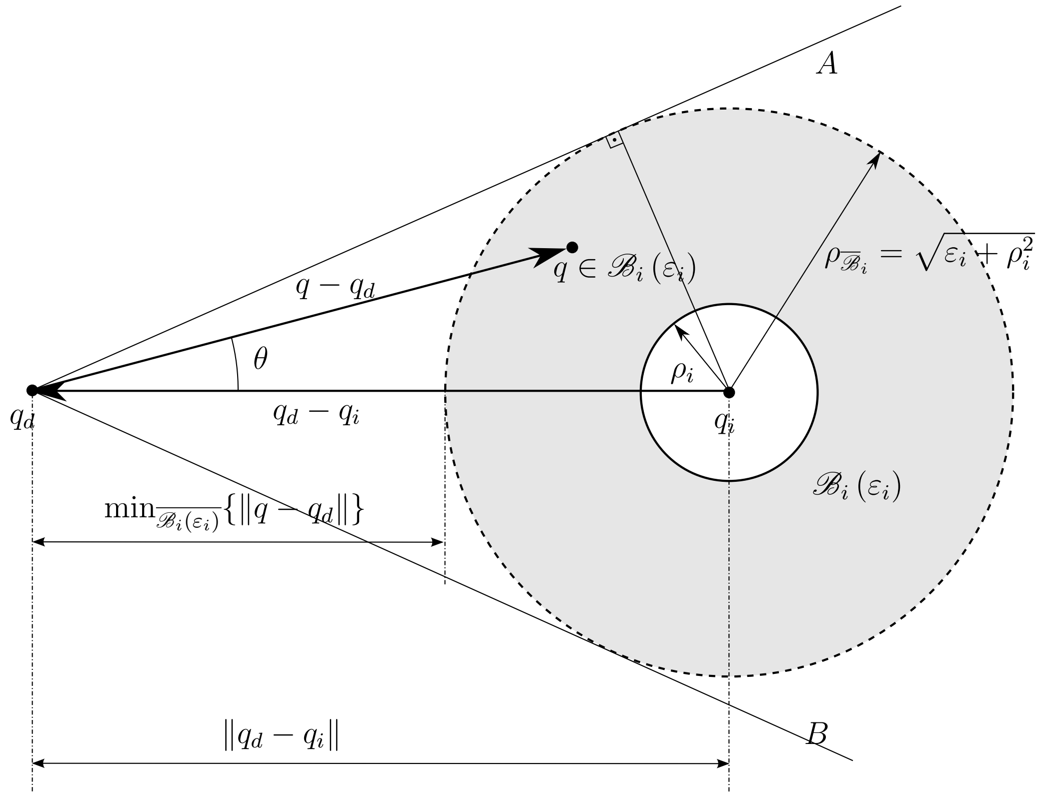


Figure 2.7: Geometry of $\nu_i(q)$.

Note that since $\zeta_1 > 0$ there is no problem of over-constraining ε_i to $0 \geq \varepsilon_i$. This note will prove useful in the sequel.

Because (nom stands for nominator, den for denominator)

$$\zeta_1 = \frac{\mathrm{nom}(q_c) > 0}{\mathrm{den}(q_c) > 0} \geq \frac{|\mathrm{nom}(q_c)| > 0}{\mathrm{den}(q_c) > 0} \geq \min_{\mathscr{B}_i(\varepsilon_i)} \left\{ \frac{|\mathrm{nom}(q_c)| > 0}{\mathrm{den}(q_c) > 0} \right\} \geq \frac{\min_{\mathscr{B}_i(\varepsilon_i)} \{|\mathrm{nom}(q)|\} > 0}{\max_{\mathscr{B}_i(\varepsilon_i)} \{\mathrm{den}(q)\} > 0} \tag{2.123}$$

provided $\varepsilon_i < \|q_d - q_i\|^2 - \rho_i^2$ and

$$\mathscr{B}_i(\varepsilon_i) \subset \overline{\mathscr{B}_i(\varepsilon_i)} \implies \begin{cases} \min_{\mathscr{B}_i(\varepsilon_i)} \{|\mathrm{nom}(q)|\} \geq \min_{\overline{\mathscr{B}_i(\varepsilon_i)}} \{|\mathrm{nom}(q)|\} \geq 0 \\ \max_{\max_{\overline{\mathscr{B}_i(\varepsilon_i)}}\{\mathrm{den}(q)\} \geq \mathscr{B}_i(\varepsilon_i)} \{\mathrm{den}(q)\} > 0 \end{cases} \tag{2.124}$$

it is

$$\zeta_1 \geq \frac{\min_{\mathscr{B}_i(\varepsilon_i)} \{|\mathrm{nom}(q)|\} > 0}{\max_{\mathscr{B}_i(\varepsilon_i)} \{\mathrm{den}(q)\} > 0} \geq \frac{\min_{\overline{\mathscr{B}_i(\varepsilon_i)}} \{|\mathrm{nom}(q)|\} \geq 0}{\max_{\overline{\mathscr{B}_i(\varepsilon_i)}} \{\mathrm{den}(q)\} > 0} \triangleq \zeta_2 \tag{2.125}$$

But *before* replacing the constraint $\zeta_1 \geq \varepsilon_i$ with the constraint $\zeta_2 \geq \varepsilon_i$ we must ensure that

$$\min_{\mathscr{B}_i(\varepsilon_i)} \{|\mathrm{nom}(q)|\} > 0 \tag{2.126}$$

By the requirement applied previously that $q_d \notin \mathscr{B}_i(\varepsilon_i)$ in the form

$$\nu_i(q_c) < 0 \impliedby \varepsilon_i < \|q_d - q_i\|^2 - \rho_i^2 \tag{2.127}$$

$\nu_i(q_c)$ has a negative supremum in $\mathscr{B}_i(\varepsilon_i)$ with an absolute value $\min\{|\nu_i(q_c)|\} < 0$. Notice that had we required just that $\varepsilon_i \leq \|q_d - q_i\|^2 - \rho_i^2$, *even* in the open annulus $\mathscr{B}_i(\varepsilon_i)$ the upper bound would be 0.

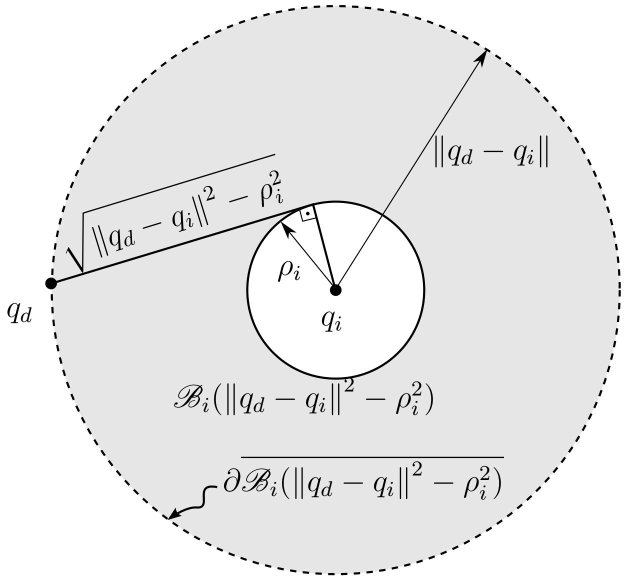


Figure 2.8: Zeroing of ε''_{i0} if ε'_{i0} is set to $\|q_d - q_i\|^2 - \rho_i^2$ and used to find $\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}} \{\nu_i(q)\}$. The minimum is zero because in this case $q_d \in \partial\overline{\mathscr{B}_i(\|q_d - q_i\|^2 - \rho_i^2)}$ and hence $\exists q \in \overline{\mathscr{B}_i(\|q_d - q_i\|^2 - \rho_i^2)}$, namelly $q_d \in \overline{\mathscr{B}_i(\|q_d - q_i\|^2 - \rho_i^2)}$, such that $\|q - q_d\| = \|q_d - q_d\| = 0$, so the inner product $\nu_i(q)$ minimum is $\nu_i(q_d) = 0$.

In [23][41] it is noted that if

$$\varepsilon_{i,KR} < \varepsilon'_{i,KR} \implies \overline{\mathscr{B}_i(\varepsilon_{i,KR})} \subseteq \overline{\mathscr{B}_i(\varepsilon'_{i,KR})} \implies \zeta_2(\varepsilon_{i,KR}) \geq \zeta_2(\varepsilon'_{i,KR}) \tag{2.128}$$

then the constraint $\zeta_2(\varepsilon_{i,KR}) > \varepsilon_i$ can be replaced by the constraint $\zeta_2(\varepsilon'_{i,KR}) > \varepsilon_i$.

But the correct check is whether

$$\zeta_2(\varepsilon_{i,KR}) \geq \zeta_2(\varepsilon'_{i,KR}) > 0 \tag{2.129}$$

Using the argument $\zeta_2(\varepsilon_{i,KR}) \geq \zeta_2(\varepsilon'_{i,KR})$ in [23] the expression

$$\varepsilon'_{i,KR} = \|q_d - q_i\|^2 - \rho_i^2 \tag{2.130}$$

is defined[42] as ε'_{0i}. The concequence is that

$$\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}} \{|\nu_i(q)|\} = \min_{\overline{\mathscr{B}_i(\|q_d - q_i\|^2 - \rho_i^2)}} \{|\nu_i(q)|\} = 0 \tag{2.131}$$

beacuse the closed n-dimensional spherical annulus $\overline{\mathscr{B}_i(\|q_d - q_i\|^2 - \rho_i^2)}$ includes the destination configuration q_d on its *boundary* $\partial\overline{\mathscr{B}_i(\|q_d - q_i\|^2 - \rho_i^2)}$, as can be observed in Fig. 2.8

More rigorously, since $|\nu_i(q)| = |(q_d - q_i) \cdot (q - q_d)|$, if $q \in \overline{\mathscr{B}_i(\|q_d - q_i\|^2 - \rho_i^2)}$, it is possible that $q = q_d$ and then $|\nu_i(q)| = |(q_d - q_i) \cdot (q_d - q_d)| = 0$

If $|\nu_i(q)| = 0$ then $\min_{\overline{\mathscr{B}_i(\|q_d - q_i\|^2 - \rho_i^2)}} \{|\mathrm{nom}(q)|\} = 0 \implies \zeta_2 = 0$. So if the constraint $\zeta_1 \geq \varepsilon_i$ were replaced by $\zeta_2 \geq \varepsilon_i$ then

$$\left\{ \begin{array}{c} \zeta_2 \geq \varepsilon_i \\ \zeta_2 = 0 \end{array} \right\} \implies 0 \geq \varepsilon_i \tag{2.132}$$

[41] The parameter $\varepsilon_{i,KR}$ is an upper bound on ε_i.

[42] [23], p.431.

which cannot be, since by definition $\varepsilon_i > 0$ (If $\varepsilon_i = 0$ the open spherical annuli around the obstacles become of zero width and the sets "near" the obstacles become empty.).

Of course an $\varepsilon_i > \|q_d - q_i\|^2 - \rho_i^2$ would not be allowable even if it did not result in $\varepsilon_i \leq 0$. The reason is that it has been imposed as a constraint to ensure $\nu_i(q_c) < 0$. Therefore we are already confined within $\varepsilon_i < \|q_d - q_i\|^2 - \rho_i^2$.

The conclusion of this analysis is that we cannot replace $\zeta_1(\varepsilon'_{i0})$ by $\zeta_2(\varepsilon'_{i0})$ and set $\varepsilon'_{i0} \triangleq \|q_d - q_i\|^2 - \rho_i^2$. The parameter ε'_{i0} should be smaller than $\|q_d - q_i\|^2 - \rho_i^2$ to avoid q_d from being included in the closure of the open ball $\mathscr{B}_i(\varepsilon'_{i0})$ specified by ε'_{i0}.

2.4.5.5 Correct selection of ε'_{i0}

Unlike Koditschek and Rimon, I select

$$\varepsilon'_{i0} \triangleq \lambda'_{i0}\left(\|q_d - q_i\|^2 - \rho_i^2\right) \tag{2.133}$$

where $\lambda'_{i0} \in (0,1)$ is a scaling factor of our choice. If $\lambda'_{i0} \in (0,1)$ is selected close to 1 then

$$\lambda'_{i0} \to 1^- \implies |\nu_i(q)| \to 0^+ \implies \varepsilon_i \to 0^+ \tag{2.134}$$

an undesired behaviour. If $\lambda'_{i0} \in (0,1)$ is selected close to 0, then

$$\lambda'_{i0} \to 0^+ \implies \varepsilon'_{i0} \to 0^+ \implies \varepsilon_i \to 0^+ \tag{2.135}$$

again the same undesired behaviour. Note also that

$$\varepsilon'_{i0} \to 0^+ \implies |\nu_i(q)| \to \left((\|q_d - q_i\| - \rho_i)\, \|q_d - q_i\|\right)^- \tag{2.136}$$

Tthis particular limit is proved in what follows.

So an intermediate selection is desired. Now the $\nu_i(q_c) < 0$ constrain follows from the constraint (slightly different)

$$\varepsilon_i < \varepsilon'_{i0} = \lambda'_{i0}\left(\|q_d - q_i\|^2 - \rho_i^2\right) < \|q_d - q_i\|^2 - \rho_i^2 \tag{2.137}$$

And it is guaranteed that

$$\nu_i(q) < 0, \quad \forall q \in \overline{\mathscr{B}_i(\varepsilon'_{i0})} \subset \mathscr{B}_i(\|q_d - q_i\|^2 - \rho_i^2) \tag{2.138}$$

so it is also guaranteed that

$$\zeta_1(\varepsilon_{i,KR}) \geq \zeta_1(\varepsilon'_{i0}) \geq \zeta_2(\varepsilon'_{i0}) > 0 \tag{2.139}$$

with[43] $\varepsilon_{i,KR} < \varepsilon'_{0i}$.

So the constraint now imposed on ε_i is

$$\zeta_2(\varepsilon'_{i0}) = \frac{\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\{|\mathrm{nom}(q)|\} > 0}{\max_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\{\mathrm{den}(q)\} > 0} \geq \varepsilon_i \tag{2.140}$$

This means that, provided $q \in \overline{\mathscr{B}_i(\varepsilon_i)}$ and $q_d \notin \overline{\mathscr{B}_i(\varepsilon_i)}$, the desired minimum is

$$\min_{\overline{\mathscr{B}_i(\varepsilon_i)}}\{|\nu_i(q)|\} = -\left(\|q_d - q_i\| - \sqrt{\varepsilon_i + \rho_i^2}\right)\|q_d - q_i\| \tag{2.141}$$

[43] For example $\varepsilon_{i,KR}$ may be an ε_1 or $\varepsilon_2 < \varepsilon'_{0i}$ imposed by other considerations.

Selecting $\varepsilon'_{i0} = \lambda'_{i0}\left(\|q_d - q_i\|^2 - \rho_i^2\right)$ the minimum becomes

$$
\begin{aligned}
\min_{\mathscr{B}_i(\varepsilon'_{i0})}\{|\nu_i(q)|\} &= -\left(\|q_d - q_i\| - \sqrt{\varepsilon'_{i0} + \rho_i^2}\right)\|q_d - q_i\| \\
&= -\left(\|q_d - q_i\| - \sqrt{\lambda'_{i0}\left(\|q_d - q_i\|^2 - \rho_i^2\right) + \rho_i^2}\right)\|q_d - q_i\| \\
&= -\left(\|q_d - q_i\| - \sqrt{\lambda'_{i0}\|q_d - q_i\|^2 + (1 - \lambda'_{i0})\rho_i^2}\right)\|q_d - q_i\|
\end{aligned}
\tag{2.142}
$$

2.4.5.6 Denominator

We have remedied the nominator $\min\{|\nu(q)|\}$ and are now about to divide both nominator and denominator of (2.120) by $\bar{\beta}_i$ to obtain

$$
\begin{aligned}
&\frac{-2\nu_i(q)}{\frac{\frac{1}{2}\bar{\beta}_i\nabla\bar{\beta}_i\cdot\nabla\gamma_d + \gamma_d\hat{t}_i^{\mathrm{T}}\left[\left(1-\frac{1}{k}\right)\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} - \bar{\beta}_i D^2\bar{\beta}_i\right]\hat{t}_i}{\bar{\beta}_i^2}} \\
&= \frac{-2\nu_i(q)}{\frac{1}{2}\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\cdot\nabla\gamma_d + \gamma_d\hat{t}_i^{\mathrm{T}}\left[\left(1-\frac{1}{k}\right)\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\frac{\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - \frac{D^2\bar{\beta}_i}{\bar{\beta}_i}\right]\hat{t}_i}
\end{aligned}
\tag{2.143}
$$

We have justified why we use as a constraint the nominator $\min\{\}$ divided by the denominator $\max\{\}$. The nominator minimum $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{-2\nu_i(q)\} = -2\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\nu_i(q)\} = 2\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{|\nu_i(q)|\}$ has been found in (2.143).

Let us focus on the denominator to cancel similar terms contained in it.

$$
\begin{aligned}
&\frac{1}{2}\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\cdot\nabla\gamma_d + \gamma_d\hat{t}_i^{\mathrm{T}}\left[\left(1-\frac{1}{k}\right)\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\frac{\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - \frac{D^2\bar{\beta}_i}{\bar{\beta}_i}\right]\hat{t}_i \\
&= \frac{1}{2}\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\cdot\nabla\gamma_d + \gamma_d\left(1-\frac{1}{k}\right)\left(\hat{t}_i^{\mathrm{T}}\left[\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\frac{\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i}\right]\hat{t}_i\right) - \gamma_d\hat{t}_i^{\mathrm{T}}\frac{D^2\bar{\beta}_i}{\bar{\beta}_i}\hat{t}_i
\end{aligned}
\tag{2.144}
$$

At this point the term $\hat{t}_i^{\mathrm{T}}D^2\bar{\beta}_i\hat{t}_i$ has appeared. This term had also appeared in subsection 2.4.4. Actually there, its absolute value had appeared. That was because the term had been already substituted by its absolute value to bound the worst case.

Retaining the actual term can prove advantageous. It can be expanded and allow us to split terms

$$
\begin{aligned}
\hat{t}_i^{\mathrm{T}} \frac{D^2\bar{\beta}_i}{\bar{\beta}_i}\hat{t}_i &= \frac{1}{\bar{\beta}_i}\hat{t}_i^{\mathrm{T}} \sum_{j\in I_0\backslash i}\left(2\frac{\bar{\beta}_i}{\beta_j}I + \frac{\nabla\beta_j}{\beta_j}\sum_{l\in I_0\backslash\{i,j\}}\left(\bar{\beta}_i\frac{\nabla\beta_l^{\mathrm{T}}}{\beta_l}\right)\right)\hat{t}_i \\
&= \hat{t}_i^{\mathrm{T}}\left[\sum_{j\in I_0\backslash i}\left(\frac{2I}{\beta_j}+\frac{\nabla\beta_j}{\beta_j}\sum_{l\in I_0\backslash\{i,j\}}\left(\frac{\nabla\beta_l^{\mathrm{T}}}{\beta_l}\right)\right)\right]\hat{t}_i \\
&= \sum_{j\in I_0\backslash i}\left(\frac{2}{\beta_j}\hat{t}_i^{\mathrm{T}}I\hat{t}_i + \hat{t}_i^{\mathrm{T}}\left[\frac{\nabla\beta_j}{\beta_j}\sum_{l\in I_0\backslash\{i,j\}}\left(\frac{\nabla\beta_l^{\mathrm{T}}}{\beta_l}\right)\right]\hat{t}_i\right) \\
&= \sum_{j\in I_0\backslash i}\left(\frac{2}{\beta_j}\left\|\hat{t}_i\right\|^2 + \hat{t}_i^{\mathrm{T}}\left[\frac{\nabla\beta_j}{\beta_j}\sum_{l\in I_0\backslash\{i,j\}}\left(\frac{\nabla\beta_l^{\mathrm{T}}}{\beta_l}\right)\right]\hat{t}_i\right) \\
&= \sum_{j\in I_0\backslash i}\left(\frac{2}{\beta_j} + \hat{t}_i^{\mathrm{T}}\left[\frac{\nabla\beta_j}{\beta_j}\sum_{l\in I_0\backslash\{i,j\}}\left(\frac{\nabla\beta_l^{\mathrm{T}}}{\beta_l}\right)\right]\hat{t}_i\right) \\
&= \sum_{j\in I_0\backslash i}\left(\frac{2}{\beta_j} + \left[\frac{1}{\beta_j}\sum_{l\in I_0\backslash\{i,j\}}\left(\hat{t}_i^{\mathrm{T}}\nabla\beta_j\nabla\beta_l^{\mathrm{T}}\hat{t}_i\frac{1}{\beta_l}\right)\right]\right) \\
&= 2\sum_{j\in I_0\backslash i}\left(\frac{1}{\beta_j}\right) + \sum_{j\in I_0\backslash i}\left(\frac{1}{\beta_j}\sum_{l\in I_0\backslash\{i,j\}}\left(\hat{t}_i^{\mathrm{T}}\nabla\beta_j\nabla\beta_l^{\mathrm{T}}\hat{t}_i\frac{1}{\beta_l}\right)\right)
\end{aligned}
\tag{2.145}
$$

We are now able to return to (2.144) and substitute our result

$$
\begin{aligned}
&\frac{1}{2}\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\cdot\nabla\gamma_d + \gamma_d\left(1-\frac{1}{k}\right)\left(\hat{t}_i^{\mathrm{T}}\left[\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\frac{\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i}\right]\hat{t}_i\right) \\
&\quad - \gamma_d\left[2\sum_{j\in I_0\backslash i}\left(\frac{1}{\beta_j}\right) + \sum_{j\in I_0\backslash i}\left(\frac{1}{\beta_j}\sum_{l\in I_0\backslash\{i,j\}}\left(\hat{t}_i^{\mathrm{T}}\nabla\beta_j\nabla\beta_l^{\mathrm{T}}\hat{t}_i\frac{1}{\beta_l}\right)\right)\right]
\end{aligned}
\tag{2.146}
$$

then group terms (redefining G_i as its previous definition divided by $\bar{\beta}_i^2$)

$$
\underbrace{\left[\begin{array}{c}\frac{1}{2}\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\cdot\nabla\gamma_d \\ +\gamma_d\left(1-\frac{1}{k}\right)\left(\hat{t}_i^{\mathrm{T}}\left[\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\frac{\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i}\right]\hat{t}_i\right) \\ -\gamma_d\sum_{j\in I_0\backslash i}\left(\frac{1}{\beta_j}\sum_{l\in I_0\backslash\{i,j\}}\left(\hat{t}_i^{\mathrm{T}}\nabla\beta_j\nabla\beta_l^{\mathrm{T}}\hat{t}_i\frac{1}{\beta_l}\right)\right)\end{array}\right]}_{A_i} - \underbrace{\left[2\gamma_d\sum_{j\in I_0\backslash i}\left(\frac{1}{\beta_j}\right)\right]}_{B_i} = A_i - B_i
\tag{2.147}
$$

where $B_i = 2\gamma_d\sum_{j\in I_0\backslash i}\frac{1}{\beta_j} > 0, \forall q\in\mathscr{B}_i\left(\varepsilon_i\right)$. We want to find

$$
\max_{\mathscr{B}_i(\varepsilon_{i03})}\{A_i - B_i\} = \max_{\mathscr{B}_i(\varepsilon_{i03})}\{A_i\} + \max_{\mathscr{B}_i(\varepsilon_{i03})}\{-B_i\} \overset{B_i>0}{=} \max_{\mathscr{B}_i(\varepsilon_{i03})}\{A_i\} - \min_{\mathscr{B}_i(\varepsilon_{i03})}\{B_i\}
\tag{2.148}
$$

Recall that $G_i = A_i - B_i$.

We need not impose the constraint under consideration ($\varepsilon''_{i0} > \varepsilon_i$) if $G_i < 0$. In subsubsection 2.4.5.2 our limitation was that we could not check G_i's sign. So we decided to impose the constraint in every case, even if in the cases it is not needed (which we found difficult to examine for the general situation).

But now we are in a position to check *some* cases when $G_i < 0$. If[44] $|A_i| < B_i$ then

$$\left\{ \begin{array}{c} |A_i| < B_i \iff |A_i| - B_i < 0 \\ A_i \leq |A_i| \iff A_i - B_i \leq |A_i| - B_i \end{array} \right\} \implies A_i - B_i < 0 \iff G_i < 0 \tag{2.149}$$

The $\implies$ above stresses the fact that we do not always conclude $G_i < 0$ when it is true, but some times we are able to tell. Anyway, to be able to avoid placing unnecessary constraints in some cases is more useful than never. In such a case it suffices *for that particular obstacle* i, to require just $\varepsilon'_{i0} > \varepsilon_i$ and not also $\varepsilon''_{i0} > \varepsilon_i$.

Continuing with the case $|A_i| > B_i$, for which we still do not know the sign of G_i, an upper bound on $\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \{A_i\}$ would prove useful. Let us first proceed with bounding $|A_i|$ from above. By application of the triangular inequality

$$\begin{aligned} |A_i| &= \frac{1}{2}\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i} \cdot \nabla\gamma_d + \gamma_d \left(1 - \frac{1}{k}\right) \left(\hat{t}_i^{\mathrm{T}} \left[\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i} \frac{\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} \right] \hat{t}_i \right) \\ &\quad - \gamma_d \sum_{j\in I_0\setminus i} \left(\frac{1}{\beta_j} \sum_{l\in I_0\setminus\{i,j\}} \left(\hat{t}_i^{\mathrm{T}} \nabla\beta_j \nabla\beta_l^{\mathrm{T}} \hat{t}_i \frac{1}{\beta_l} \right) \right) \\ &\leq \left| \frac{1}{2}\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i} \cdot \nabla\gamma_d \right| + \left| \gamma_d \left(1 - \frac{1}{k}\right) \left(\hat{t}_i^{\mathrm{T}} \left[\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i} \frac{\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} \right] \hat{t}_i \right) \right| \\ &\quad + \left| \gamma_d \sum_{j\in I_0\setminus i} \left(\frac{1}{\beta_j} \sum_{l\in I_0\setminus\{i,j\}} \left(\hat{t}_i^{\mathrm{T}} \nabla\beta_j \nabla\beta_l^{\mathrm{T}} \hat{t}_i \frac{1}{\beta_l} \right) \right) \right| \end{aligned} \tag{2.150}$$

Each of the three terms comprising this upper bound on $|A_i|$ is bounded individually. By the triangular and Schwarz inequalities

$$\begin{aligned} \left| \frac{1}{2}\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i} \nabla\gamma_d \right| &= \frac{1}{2} \left| \sum_{j\in I_0\setminus i} \left(\frac{\bar{\beta}_i}{\beta_j \bar{\beta}_i} \nabla\beta_j \nabla\gamma_d \right) \right| = \frac{1}{2} \left| \sum_{j\in I_0\setminus i} \left(\frac{\nabla\beta_j}{\beta_j} \nabla\gamma_d \right) \right| \\ &\leq \frac{1}{2} \sum_{j\in I_0\setminus i} \left| \frac{\nabla\beta_j}{\beta_j} \nabla\gamma_d \right| \leq \frac{1}{2} \sum_{j\in I_0\setminus i} \left(\frac{\|\nabla\beta_j\|}{\beta_j} \|\nabla\gamma_d\| \right) \\ &= \frac{1}{2} 2\sqrt{\gamma_d} \left(\frac{2\sqrt{\rho_0^2 - \beta_0}}{\beta_0} + \sum_{j\in I_1\setminus i} \frac{2\sqrt{\beta_j + \rho_j^2}}{\beta_j} \right) = 2\sqrt{\gamma_d} \sum_{j\in I_0\setminus i} Q_j(\beta_j) \end{aligned} \tag{2.151}$$

In subsection A.6.2 the inequality $\left|\hat{v}^{\mathrm{T}} a b^{\mathrm{T}} \hat{v}\right| \leq \|a\| \|b\|$ is proved for any unit vector $\hat{v}$ and vectors $a, b \in E^n$. Since the tangential unit vector $\hat{t}_i$ is a unit vector

$$\left\{ \begin{array}{c} \left|\hat{v}^{\mathrm{T}} a b^{\mathrm{T}} \hat{v}\right| \leq \|a\| \|b\|, \{\forall \hat{v} \in E^n | \, \|\hat{v}\| = 1\} \\ \hat{t}_i \in E^n \wedge \left\|\hat{t}_i\right\| = 1 \end{array} \right\} \implies \left|\hat{t}_i^{\mathrm{T}} a b^{\mathrm{T}} \hat{t}_i\right| \leq \|a\| \|b\|. \tag{2.152}$$

[44] we have shown and know that $B_i > 0$ but the sign of A_i remains undetermined.

Using this inequality we derive

$$
\begin{aligned}
&\left|\gamma_d\left(1-\frac{1}{k}\right)\left(\hat{t}_i^{\mathrm{T}}\left[\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\frac{\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i}\right]\hat{t}_i\right)\right| \le \gamma_d\frac{\left\|\nabla\bar{\beta}_i\right\|^2}{\bar{\beta}_i^2} = \left(\sqrt{\gamma_d}\frac{\left\|\nabla\bar{\beta}_i\right\|}{\bar{\beta}_i}\right)^2 \\
&= \left(\sqrt{\gamma_d}\frac{\left\|\sum_{j\in I_0\backslash i}\frac{\bar{\beta}_i}{\beta_j}\nabla\beta_j\right\|}{\bar{\beta}_i}\right)^2 = \left(\sqrt{\gamma_d}\left\|\sum_{j\in I_0\backslash i}\frac{\nabla\beta_j}{\beta_j}\right\|\right)^2 \le \left(\sqrt{\gamma_d}\sum_{j\in I_0\backslash i}\left\|\frac{\nabla\beta_j}{\beta_j}\right\|\right)^2 \\
&= \left(\sqrt{\gamma_d}\sum_{j\in I_0\backslash i}\frac{\left\|\nabla\beta_j\right\|}{\beta_j}\right)^2 = \left(\sqrt{\gamma_d}\left(\frac{2\sqrt{\rho_0^2-\beta_0}}{\beta_0}+\sum_{j\in I_1\backslash i}\frac{2\sqrt{\beta_j+\rho_j^2}}{\beta_j}\right)\right)^2 \\
&= \left(2\sqrt{\gamma_d}\sum_{j\in I_0\backslash i}Q_j\left(\beta_j\right)\right)^2 = 4\gamma_d\left(\sum_{j\in I_0\backslash i}Q_j\left(\beta_j\right)\right)^2
\end{aligned}
\tag{2.153}
$$

since $\gamma_d > 0, \forall q \in \mathscr{B}_i\left(\varepsilon_i\right) - \{q_d\}$ and $1-\frac{1}{k}\le 1$. Also by $\left|\hat{t}_i^{\mathrm{T}}ab^{\mathrm{T}}\hat{t}_i\right| \le \|a\|\,\|b\|$

$$
\begin{aligned}
&\left|\gamma_d\sum_{j\in I_0\backslash i}\left(\frac{1}{\beta_j}\sum_{l\in I_0\backslash\{i,j\}}\left(\hat{t}_i^{\mathrm{T}}\nabla\beta_j\nabla\beta_l^{\mathrm{T}}\hat{t}_i\frac{1}{\beta_l}\right)\right)\right| \le \gamma_d\sum_{j\in I_0\backslash i}\left(\frac{1}{\beta_j}\sum_{l\in I_0\backslash\{i,j\}}\left(\left|\hat{t}_i^{\mathrm{T}}\nabla\beta_j\nabla\beta_l^{\mathrm{T}}\hat{t}_i\right|\frac{1}{\beta_l}\right)\right) \\
&\le \gamma_d\sum_{j\in I_0\backslash i}\left(\frac{1}{\beta_j}\sum_{l\in I_0\backslash\{i,j\}}\left(\left\|\nabla\beta_j\right\|\left\|\nabla\beta_l\right\|\frac{1}{\beta_l}\right)\right) \\
&\le \gamma_d\sum_{j\in I_0\backslash i}\left(\frac{\left\|\nabla\beta_j\right\|}{\beta_j}\sum_{l\in I_0\backslash\{i,j\}}\frac{\left\|\nabla\beta_l\right\|}{\beta_l}\right) \\
&= \gamma_d\left(\frac{2\sqrt{\rho_0^2-\beta_0}}{\beta_0}\sum_{l\in I_1\backslash i}\left(\frac{2\sqrt{\rho_l^2+\beta_l}}{\beta_l}\right)+\sum_{j\in I_1\backslash i}\left(\frac{2\sqrt{\rho_j^2+\beta_j}}{\beta_j}\left(\frac{2\sqrt{\rho_0^2-\beta_0}}{\beta_0}+\sum_{l\in I_1\backslash\{i,j\}}\frac{2\sqrt{\rho_l^2+\beta_l}}{\beta_l}\right)\right)\right) \\
&= 4\gamma_d\sum_{j\in I_0\backslash i}\left(Q_j\left(\beta_j\right)\sum_{l\in I_0\backslash\{i,j\}}Q_l\left(\beta_l\right)\right)
\end{aligned}
\tag{2.154}
$$

which is the same procedure as followed in subsection A.6.1. Here we wanted to separate $\hat{t}_i^{\mathrm{T}}D^2\bar{\beta}_i\hat{t}_i$, to form A_i and B_i, that is why we did not use directly that result.

Let

$$
\beta_{ji}^{\min} \triangleq \min_{\mathscr{B}_i(\varepsilon_{i03})}\{\beta_j\}, \quad \beta_{ji}^{\max} \triangleq \max_{\mathscr{B}_i(\varepsilon_{i03})}\{\beta_j\} \tag{2.155}
$$

for Q_{0i}, Q_{ji}, and define

$$
\gamma_{di}^{\min} \triangleq \min_{\mathscr{B}_i(\varepsilon_{i03})}\{\gamma_d\}, \quad \gamma_{di}^{\max} \triangleq \max_{\mathscr{B}_i(\varepsilon_{i03})}\{\gamma_d\} \tag{2.156}
$$

The previous results lead to

$$
\begin{aligned}
&\frac{-2\nu_i(q)}{\frac{\frac{1}{2}\bar{\beta}_i\nabla\bar{\beta}_i\cdot\nabla\gamma_d+\gamma_d\hat{t}_i^{\mathrm{T}}\left[\left(1-\frac{1}{k}\right)\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}}-\bar{\beta}_i D^2\bar{\beta}_i\right]\hat{t}_i}{\bar{\beta}_i^2}}\\
&\geq \frac{2\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\nu_i(q)\}}{\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{A_i\}-\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{B_i\}}\\
&\geq \frac{-2\left(\|q_d-q_i\|-\sqrt{\varepsilon_{i0}'+\rho_i^2}\right)\|q_d-q_i\|}{\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\left\{\begin{array}{c}2\sqrt{\gamma_d}\sum\limits_{j\in I_0\backslash i}Q_j\left(\beta_j\right)\\ +\left(2\sqrt{\gamma_d}\sum\limits_{j\in I_0\backslash i}Q_j\left(\beta_j\right)\right)^2\\ +4\gamma_d\sum\limits_{j\in I_0\backslash i}\left(Q_j\left(\beta_j\right)\sum\limits_{l\in I_0\backslash\{i,j\}}Q_l\left(\beta_l\right)\right)\end{array}\right\}-\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\left\{2\gamma_d\sum\limits_{j\in I_0\backslash i}\frac{1}{\beta_j}\right\}}\\
&\geq \frac{-\left(\|q_d-q_i\|-\sqrt{\varepsilon_{i0}'+\rho_i^2}\right)\|q_d-q_i\|}{\sqrt{\gamma_{di}^{\max}}\sum\limits_{j\in I_0\backslash i}Q_{ji}+2\gamma_{di}^{\max}\left(\sum\limits_{j\in I_0\backslash i}Q_{ji}\right)^2+2\gamma_{di}^{\max}\sum\limits_{j\in I_0\backslash i}\left(Q_{ji}\sum\limits_{l\in I_0\backslash\{i,j\}}Q_{li}\right)-\gamma_{di}^{\min}\sum\limits_{j\in I_0\backslash i}\frac{1}{\beta_{ji}^{\max}}}
\end{aligned}
\tag{2.157}
$$

2.4.5.7 Diminishing nominator lower bound

Pairing $\nabla\bar{\beta}_i$ with $\bar{\beta}_i$ by dividing the nominator $\bar{\beta}_i^2$ in the denominator has proved advantageous. It removes $\bar{\beta}_i$ from the nominator and cancels it with $\bar{\beta}_i$ terms in $\left\|\nabla\bar{\beta}_i\right\|$, when bounding the denominator from above.

This relieves the nominator of the unwanted lower bound $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\bar{\beta}_i\}$ and reduces the calculated upper bound on the denominator. As a result the ill-valued fraction $\frac{\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\bar{\beta}_i\}}{\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\bar{\beta}_i\}}$ is avoided. Essentially we avoid ignoring that the same function is embedded in both denominator and nominator.

After advancing with this cancellation we are still left with a nominator $-2\nu_i(q)$. Its lower bound is $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{-2\nu_i(q)\}$. This lower bound has a small value and we would like to replace it. In order to achieve this it is useful to explore this term's behavior.

As already noted $\varepsilon_{i0}' < \|q_d-q_i\|^2-\rho_i^2 \implies \nu_i(q)<0, \forall q\in\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$ so

$$
\begin{aligned}
\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{-\nu_i(q)\} &= -\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\nu_i(q)\}\\
&= \left(\|q_i-q_d\|-\sqrt{\varepsilon_{i0}'+\rho_i^2}\right)\|q_i-q_d\|\\
&= \min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\left\{\sqrt{\gamma_d(q)}\right\}\sqrt{\gamma_d(q_i)}
\end{aligned}
\tag{2.158}
$$

This result should be expected, since

$$\begin{aligned}\nu_i(q) &= \frac{1}{4}\nabla\beta_i\cdot\nabla\gamma_d - \gamma_d \\ &= \frac{1}{4}\left[2(q-q_i)\right]\cdot\left[2(q-q_d)\right] - \|q-q_d\|^2 \\ &= (q-q_i)\cdot(q-q_d) - (q-q_d)\cdot(q-q_d) \\ &= (q_d-q_i)\cdot(q-q_d) \\ &= -\frac{1}{4}\left[2(q_i-q_d)\right]\cdot\left[2(q-q_d)\right] \\ &= -\frac{1}{4}\nabla\gamma_d(q_i)\cdot\nabla\gamma_d(q)\end{aligned} \tag{2.159}$$

Following from this, by application of Schwarz inequality

$$\begin{aligned}|\nu_i(q)| &= \left|-\frac{1}{4}\nabla\gamma_d(q_i)\cdot\nabla\gamma_d(q)\right| \\ &= \frac{1}{4}\left|\nabla\gamma_d(q_i)\cdot\nabla\gamma_d(q)\right| \\ &\leq \frac{1}{4}\left\|\nabla\gamma_d(q_i)\right\|\left\|\nabla\gamma_d(q)\right\| \\ &= \frac{1}{4}2\left\|q_i-q_d\right\|2\left\|q-q_d\right\| \\ &= \sqrt{\gamma_d(q_i)}\sqrt{\gamma_d(q)}\end{aligned} \tag{2.160}$$

The above restatement of $|\nu_i(q)|$ in terms of $\sqrt{\gamma_d(q_i)}$ and $\sqrt{\gamma_d(q)}$ offers valuable insight. It demonstrates that $|\nu_i(q)|$ is bounded from above and below.

For a given obstacle center q_i the euclidean distance between q_i and q_d is fixed and equal to $\sqrt{\gamma_d(q_i)} = \|q_i - q_d\|$. But the second term $\sqrt{\gamma_d(q)} = \|q - q_d\|$, the distance of a point q in $\overline{\mathscr{B}_i(\varepsilon'_{i0})}$ from q_d, is still free to vary between $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\left\{\sqrt{\gamma_d(q)}\right\}$ and $\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\left\{\sqrt{\gamma_d(q)}\right\}$.

The bounds depend on $\overline{\mathscr{B}_i(\varepsilon'_{i0})}$, which is defined by ρ_i and ε'_{i0}, a spherical annulus of inner diameter ρ_i and outer diameter $\rho'_{0i} = \sqrt{\varepsilon'_{i0} + \rho_i^2} < \|q_i - q_d\| = \sqrt{\gamma_d(q_i)}$.

The closer the destination q_d to the obstacle's boundary $\partial\mathscr{O}_i$ the narrower the above bounds. As proved in subsubsection A.5.2.1

$$\begin{aligned}\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\left\{\sqrt{\gamma_d(q)}\right\} &= \|q_i-q_d\| - \sqrt{\varepsilon'_{i0}+\rho_i^2} = \|q_i-q_d\| - \rho'_{0i} \\ \max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\left\{\sqrt{\gamma_d(q)}\right\} &= \|q_i-q_d\| + \sqrt{\varepsilon'_{i0}+\rho_i^2} = \|q_i-q_d\| + \rho'_{0i}\end{aligned} \tag{2.161}$$

These bounds on $\sqrt{\gamma_d(q)}$ within $\overline{\mathscr{B}_i(\varepsilon'_{i0})}$ are illustrated in Fig. 2.9.

The closer we choose destination q_d to obstacle $\mathscr{O}_i$ the smaller the ramaining space available to vary $\overline{\mathscr{B}_i(\varepsilon'_{i0})}$ (reduce it by choosing a smaller ε'_{i0} hence smaller ρ'_{i0}) to increase $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\left\{\sqrt{\gamma_d(q)}\right\}$ provided we have to also satisfy the constraint $q_d \notin \overline{\mathscr{B}_i(\varepsilon'_{i0})}$.

This we would prefer to avoid to prevent ε''_{i0} from becoming impractically small when q_d is close to an $\partial\mathscr{O}_i$.We will accomplish this by recognizing a similar effect in the denominator, as we have already done with $\bar{\beta}_i$, and cancelling these effects.

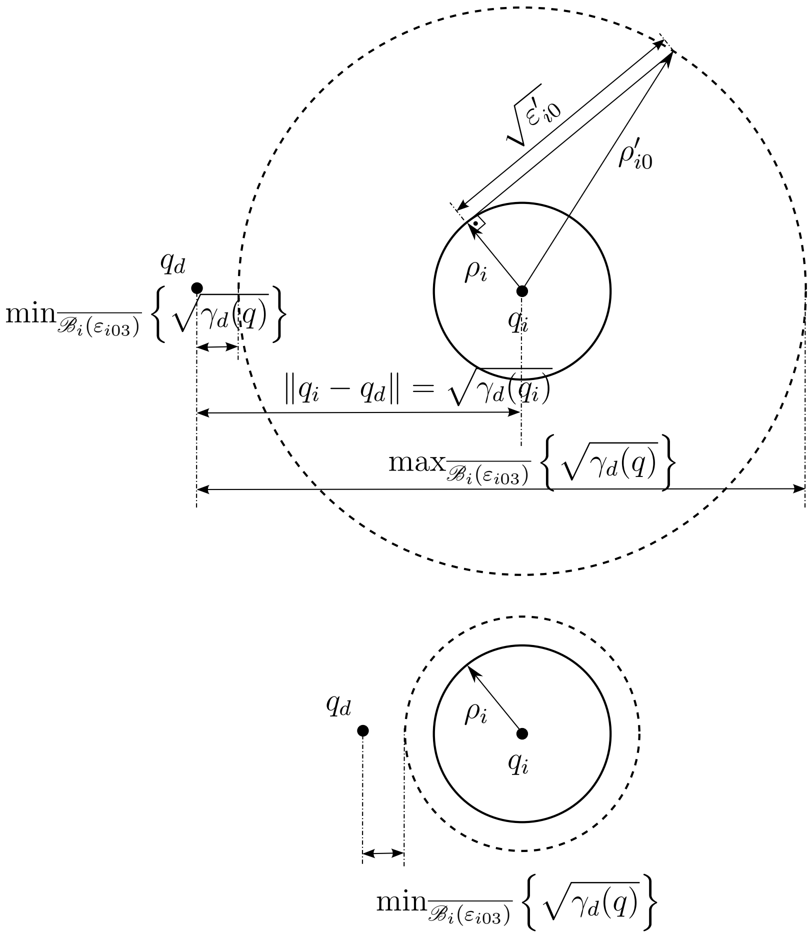


Figure 2.9: Extrema of $\sqrt{\gamma_d(q)}$ within $\overline{\mathscr{B}_i(\varepsilon'_{i0})}$ as q_d is placed closer to an obstacle.

2.4.5.8 Nominator behavior

Let us now look at the denominator to find a $\sqrt{\gamma_d(q)}$

$$\frac{1}{2}\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\cdot\nabla\gamma_d + \gamma_d\hat{t}_i^{\mathrm{T}}\left[\left(1-\frac{1}{k}\right)\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\frac{\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - \frac{D^2\bar{\beta}_i}{\bar{\beta}_i}\right]\hat{t}_i \tag{2.162}$$

We see γ_d and $\nabla\gamma_d$. Obviously $\gamma_d(q)$ incorporates a $\sqrt{\gamma_d(q)}$, but what about $\nabla\gamma_d(q)$?

As already shown when bounding the denominator from above, a term of the denominator upper bound is

$$\left|\frac{1}{2}\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\nabla\gamma_d\right| = 2\sqrt{\gamma_d}\left(\sqrt{\frac{\rho_0^2}{\beta_0^2}-\frac{1}{\beta_0}} + \sum_{j\in I_1\setminus i}\sqrt{\frac{\rho_j^2}{\beta_j^2}+\frac{1}{\beta_j}}\right) \tag{2.163}$$

so if divide the denominator by $\sqrt{\gamma_d(q)}$ it will cancel in the procedure of determining an upper bound.

The nominator lower bound includes $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\left\{\sqrt{\gamma_d(q)}\right\}$. This tends to 0^+ as $q_d \to \partial\mathscr{O}_i$. So we expect that dividing the nominator by $\sqrt{\gamma_d(q)}$ will raise to nonzero the lower bound limit as $q_d \to \partial\mathscr{O}_i$. The new lower bound is

$$\begin{aligned}
\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\left\{\frac{-2\nu_i(q)}{\sqrt{\gamma_d(q)}}\right\} &= \min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\left\{\frac{-2\frac{-1}{4}\nabla\gamma_d(q_i)\cdot\nabla\gamma_d(q)}{\frac{1}{2}2\sqrt{\gamma_d(q)}}\right\} \\
&= \min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\left\{\nabla\gamma_d(q_i)\cdot\left(\frac{\nabla\gamma_d(q)}{\|\nabla\gamma_d(q)\|}\right)\right\} \\
&= \min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\left\{\nabla\gamma_d(q_i)\cdot\hat{v}_d(q)\right\}
\end{aligned} \tag{2.164}$$

were

$$\hat{v}_d(q) \triangleq \frac{\nabla\gamma_d(q)}{\|\nabla\gamma_d(q)\|} \implies \|\hat{v}_d(q)\| = \left\|\frac{\nabla\gamma_d(q)}{\|\nabla\gamma_d(q)\|}\right\| = 1, \forall q \neq q_d \tag{2.165}$$

This expectation is false. Expecting to cancel $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\left\{\sqrt{\gamma_d(q)}\right\}$ in the lower bound by dividing the nominator by $\sqrt{\gamma_d(q)}$ (not its lower bound) does not solve the problem because the nominator has one more effect in it, the *inner product* $\nabla\gamma_d(q_i)\cdot\nabla\gamma_d(q)$.

The inner product prevents us from remedying the lower bound problem by just using a $\sqrt{\gamma_d(q)}$ from the denominator. The angle in the inner product needs more to be annealed.

Having overviewed what is following, we can now examine $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\nabla\gamma_d(q_i)\cdot\hat{v}_d(q)\}$. Our approach will be geometric, Fig. 2.10, saving detailed analytical treatment for the final expression.

It is true that

$$\nabla\gamma_d(q_i)\cdot\hat{v}_d(q) = \|\nabla\gamma_d(q_i)\|\,\|\hat{v}_d(q)\|\cos\theta \tag{2.166}$$

where[45] $\theta = (\widehat{q-q_d, q_i-q_d}) \in [\theta_{\min},\theta_{\max}] \subset (-\frac{\pi}{2},\frac{\pi}{2})$ and in what follows only half of the interval $(-\frac{\pi}{2},\frac{\pi}{2})$, i.e. $[0,\frac{\pi}{2})$, will be considered, due to symmetry.

[45] Note that $\varepsilon_{i0}' < \|q_d - q_i\|^2 - \rho_i^2 \implies q_d \notin \overline{\mathscr{B}_i(\varepsilon_{i0}')} \implies \theta \notin [-\pi,-\frac{\pi}{2}]\cup[\frac{\pi}{2},\pi)$.

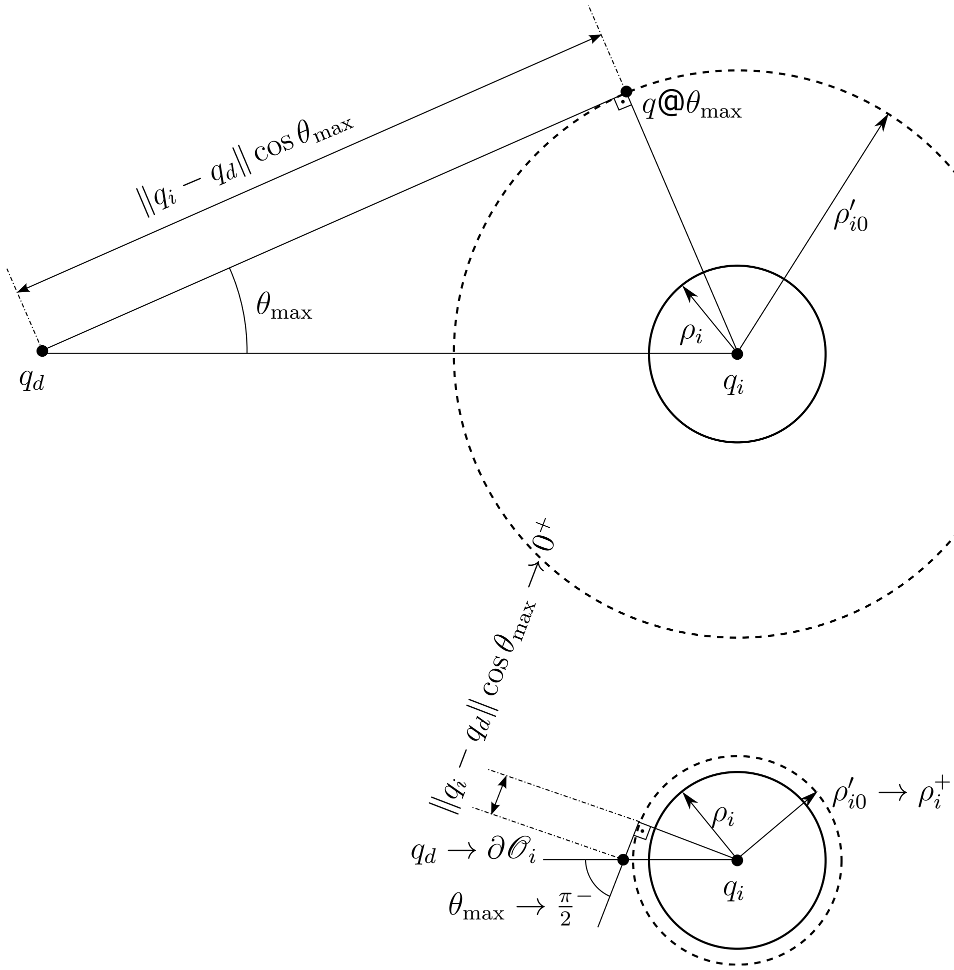


Figure 2.10: Nominator lower bound.

For a given obstacle $\|\nabla\gamma_d(q_i)\| = 2\|q_i - q_d\|$ is fixed, so

$$\begin{aligned}\min_{\mathscr{B}_i(\varepsilon_{i03})} \{\nabla\gamma_d(q_i)\cdot\hat{v}_d(q)\} &= \min_{\mathscr{B}_i(\varepsilon_{i03})} \{2\|q_i - q_d\|\cos\theta\} \\ &= 2\|q_i - q_d\| \min_{\mathscr{B}_i(\varepsilon_{i03})}\{\cos\theta\} \\ &= 2\|q_i - q_d\| \cos \max_{\mathscr{B}_i(\varepsilon_{i03})}\{\theta\}\end{aligned} \tag{2.167}$$

since $\theta \in [0, \theta_{\max}] \subset [0, \frac{\pi}{2}]$.

Clearly $\theta_{\max}$ is the angle between the tangent from q_d to the sphere with center q_i and radius ρ'_{i0} which constitutes the annulus' $\overline{\mathscr{B}_i(\varepsilon'_{i0})}$ outer boundary and the line through q_i and q_d. This implies

$$\theta_{\max} = \arcsin\left(\frac{\rho'_{i0}}{\|q_i - q_d\|}\right) = \arcsin\left(\frac{\rho'_{i0}}{\sqrt{\gamma_d(q_i)}}\right) \tag{2.168}$$

Now that we have expressed the nominator minimum as a function of $\theta_{\max}$

$$\min_{\mathscr{B}_i(\varepsilon_{i03})}\left\{\frac{-2\nu_i(q)}{\sqrt{\gamma_d(q)}}\right\} = 2\|q_i - q_d\|\cos\theta_{\max} \tag{2.169}$$

let us place q_d closer to $\partial\mathscr{O}_i$ and observe what happens tp $\theta_{\max}$, as shown in Fig. 2.10.

We see that

$$\theta_{\max} \to \frac{\pi}{2}^{-} \implies \cos\theta_{\max} \to 0^{+} \tag{2.170}$$

So the nominator lower bound again tends to 0.

2.4.5.9 Nominator lower bound improvement observed

Although the nominator lower bound still tends to zero when q_d goes close to an obstacle, there has been an improvement. This can be shown by considering the lower bound before

$$\begin{aligned}\min_{\mathscr{B}_i(\varepsilon_{i03})}\{-2\nu_i(q)\} &= \left(\|q_i - q_d\| - \sqrt{\varepsilon'_{i0} + \rho_i^2}\right)\|q_i - q_d\| \\ &= \left(\sqrt{\gamma_d(q_i)} - \rho'_{i0}\right)\|q_i - q_d\| \\ &= \left(\sqrt{\gamma_d(q_i)} - \rho'_{i0}\right)\sqrt{\gamma_d(q_i)}\end{aligned} \tag{2.171}$$

and after dividing by $\sqrt{\gamma_d(q)}$

$$\min_{\mathscr{B}_i(\varepsilon_{i03})}\left\{\frac{-2\nu_i(q)}{\sqrt{\gamma_d(q)}}\right\} = \|\nabla\gamma_d(q_i)\|\cos\theta_{\max} = 2\sqrt{\gamma_d(q_i) - (\rho'_{i0})^2} \tag{2.172}$$

where the latter expression can be derived by application of the Pythagorean Theorem for euclidean space, see Fig. 2.11.

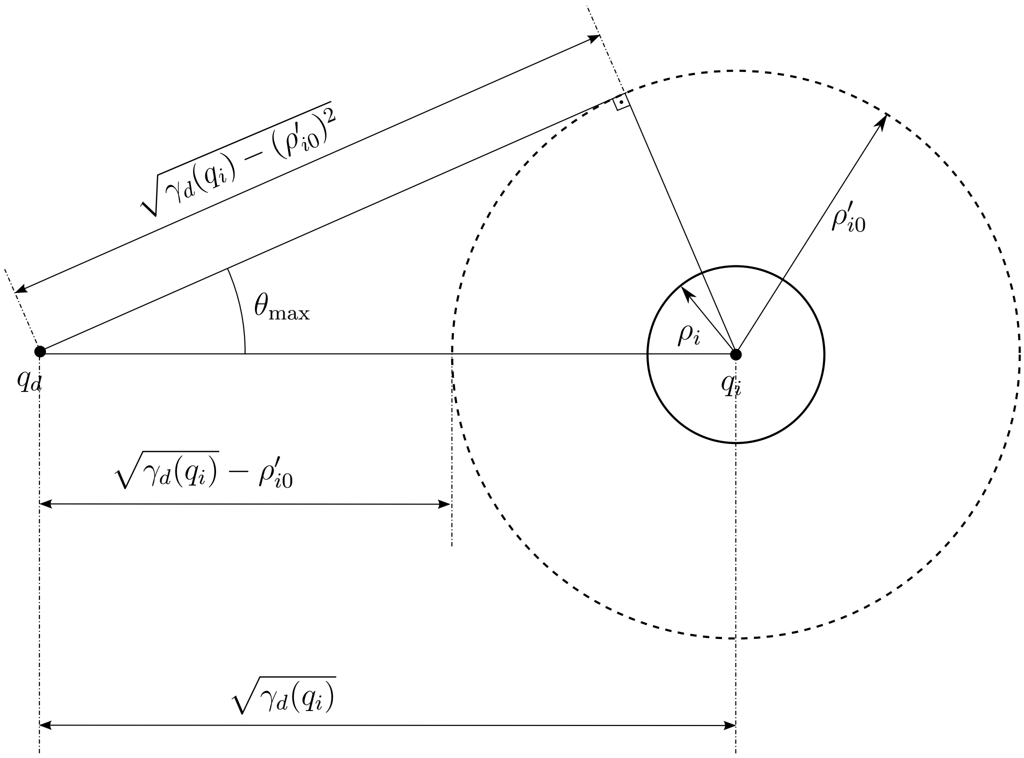


Figure 2.11: Nominator old and new lower bound comparison.

The fraction of lower bounds before and after the change is

$$\begin{aligned}
\frac{\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{-2\nu_i\}}{\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\left\{\frac{-2\nu_i(q)}{\sqrt{\gamma_d(q)}}\right\}} &= \frac{\left(\sqrt{\gamma_d(q_i)} - \rho'_{i0}\right)\sqrt{\gamma_d(q_i)}}{2\sqrt{\gamma_d(q_i) - (\rho'_{i0})^2}} \\
&= \frac{(\sqrt{\gamma_d(q_i)} - \rho'_{i0})\sqrt{\gamma_d(q_i)}}{2\sqrt{\sqrt{\gamma_d(q_i)}^2 - (\rho'_{i0})^2}} \\
&= \frac{\sqrt{\gamma_d(q_i)}}{2}\sqrt{\frac{\sqrt{\gamma_d(q_i)} - \rho'_{i0}}{\sqrt{\gamma_d(q_i)} + \rho'_{i0}}}
\end{aligned} \tag{2.173}$$

so its limit is

$$\lim_{\sqrt{\gamma_d(q_i)}\to(\rho'_{i0})^+}\frac{\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{-2\nu_i\}}{\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\left\{\frac{-2\nu_i(q)}{\sqrt{\gamma_d(q)}}\right\}} = \lim_{\sqrt{\gamma_d(q_i)}\to(\rho'_{i0})^+}\left(\frac{\sqrt{\gamma_d(q_i)}}{2}\sqrt{\frac{\sqrt{\gamma_d(q_i)} - \rho'_{i0}}{\sqrt{\gamma_d(q_i)} + \rho'_{i0}}}\right) = 0^+ \tag{2.174}$$

2.4.5.10 Final nominator improvement

To prevent diminishing of the nominator lower bound we can divide the nominator by $\gamma_d(q)$. This results in an unwanted effect, which will be shown to be less problematic than the initial expression for ε''_{i0}.

The effect is that, unfortunately, the denominator does *not uniformly* incorporate a second $\sqrt{\gamma_d(q)}$. The resulting expression is

$$\frac{-2\frac{\nu_i(q)}{\gamma_d(q)}}{\frac{1}{2}\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\cdot\frac{\nabla\gamma_d}{\gamma_d}+\hat{t}_i^{\mathrm{T}}\left[\left(1-\frac{1}{k}\right)\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\frac{\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i}-\frac{D^2\bar{\beta}_i}{\bar{\beta}_i}\right]\hat{t}_i} \tag{2.175}$$

The $\frac{\nabla\gamma_d(q)}{\gamma_d}$ corresponds to $\frac{1}{\sqrt{\gamma_d(q)}}$ in the denominator upper bound, leading to (because as shown later the nominator lower bound will be 1)

$$\begin{aligned}&\frac{1}{\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{A_i\}\frac{1}{\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\gamma_d(q)\}}+\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{B_i\}}\\ =&\frac{\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\gamma_d(q)\}}{\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{A_i\}+\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{B_i\}\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\gamma_d(q)\}}\end{aligned} \tag{2.176}$$

which is clearly an improvement over the previous expression

$$\frac{\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\gamma_d(q)\}\sqrt{\gamma_d(q_i)}}{\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{A_i\}\sqrt{\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\gamma_d(q)\}}+\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{B_i\}\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\gamma_d(q)\}} \tag{2.177}$$

The nominator improvement is

$$\begin{aligned}-2\frac{\nu_i(q)}{\gamma_d(q)}&=-2\frac{-\frac{1}{4}\nabla\gamma_d(q_i)\cdot\nabla\gamma_d(q)}{\gamma_d(q)}\\ &=2\frac{\nabla\gamma_d(q_i)\cdot\nabla\gamma_d(q)}{2\sqrt{\gamma_d(q)}2\sqrt{\gamma_d(q)}}\\ &=2\frac{\nabla\gamma_d(q_i)}{\|\nabla\gamma_d(q)\|}\underbrace{\frac{\nabla\gamma_d(q)}{\|\nabla\gamma_d(q)\|}}_{\hat{v}_d(q)}\\ &=2\frac{\nabla\gamma_d(q_i)\cdot\hat{v}_d(q)}{\|\nabla\gamma_d(q)\|}\\ &=2\frac{\|\nabla\gamma_d(q_i)\|\,\|\hat{v}_d(q)\|\cos\theta}{\|\nabla\gamma_d(q)\|}\\ &=2\frac{2\,\|q_i-q_d\|\cos\theta}{2\,\|q-q_d\|}=2\frac{r_i}{r}\cos\theta\end{aligned} \tag{2.178}$$

where $r_i \triangleq \|q_i-q_d\|\,, r \triangleq \|q-q_d\|$ and the polar coordinates used are shown in Fig. 2.12. Note that although the problem is defined in the n-dimensional Euclidean space E^n, finding the nominator lower bound reduces to a 2-dimensional subspace problem, because of sphere symmetry.

2.4.5.11 Nominator lower bound: geometric intuition

Before finding the nominator minimum on the semi-annulus D of Fig. 2.12 let us first explore the underlying geometric intuition. Omitting the scaling factor of 2, the function to be minimized over the semi-annulus is

$$f(r,\theta)=\frac{r_i}{r}\cos\theta=\frac{r_i\cos\theta}{r}=\frac{\left\|\mathrm{projection}_{q-q_d}q_i-q_d\right\|}{\|q-q_d\|} \tag{2.179}$$

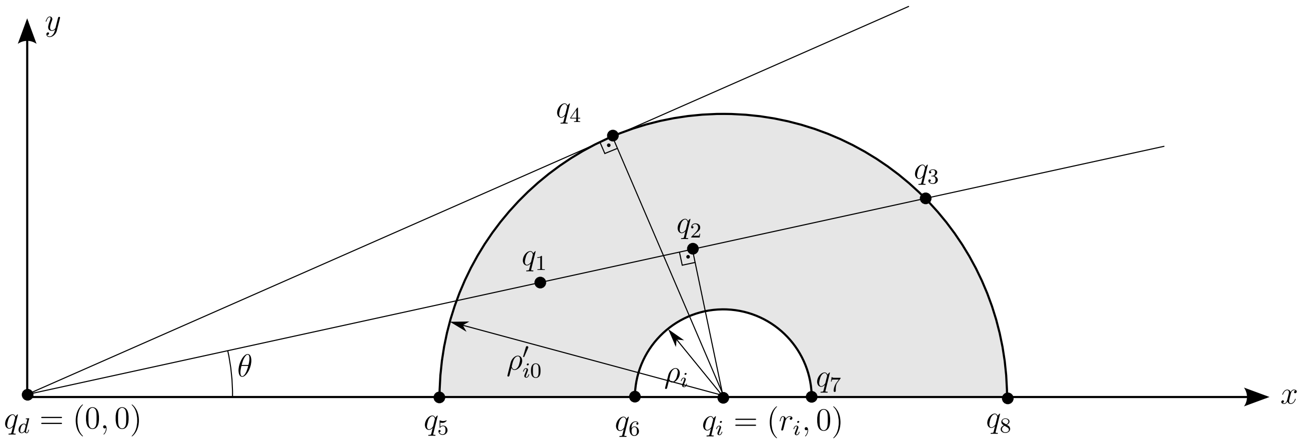


Figure 2.12: Nominator constant lower bound calculation.

Select an angle θ which determines the direction $q - q_d$. Since $q_i - q_d$ is given selecting θ fixes the projection of $q_i - q_d$ on $q - q_d$.

There remains to select $q \in D$ on the semiline whose direction is determined by θ. Observe that selecting

1. $q_1 \implies \|q_1 - q_d\| < \|q_i - q_d\| \cos\theta \implies 1 < f(r,\theta)$
2. $q_2 \implies \|q_2 - q_d\| = \|q_i - q_d\| \cos\theta \implies 1 = f(r,\theta)$
3. $q_3 \implies \|q_3 - q_d\| > \|q_i - q_d\| \cos\theta \implies 1 > f(r,\theta)$

And that q_3 yields the minimum $f(r,\theta)$ on a given direction.

The global minimum over D is attained at $q_8 = (r_i + \rho'_{i0}, 0)$. This will be formally proved in subsubsection 2.4.5.12, where the nomiantor lower bound is found to be

$$2\frac{r_i}{r_i + \rho'_{i0}} \cos 0 = 2\frac{1}{1 + \frac{\rho'_{i0}}{r_i}} \overset{\rho'_{i0} < r_i \implies \frac{\rho'_{i0}}{r_i} < 1}{>} 2\frac{1}{2} = 1 \tag{2.180}$$

2.4.5.12 Nominator lower bound: analytical calculation

We may treat the (symmetric) problem in either polar $(r,\theta) \in \mathbb{R} \times [0,\pi]$ or cartesian coordinates $(x,y) \in \mathbb{R}^2$. Let us choose cartesian coordinates to minimize

$$f(r,\theta) = \frac{r_i}{r}\cos\theta = \frac{r_i}{\sqrt{x^2+y^2}}\frac{x}{\sqrt{x^2+y^2}} = \frac{r_i x}{x^2+y^2} = f(x,y) \tag{2.181}$$

subject to

$$\begin{aligned} &0 \leq y \\ &\rho_i^2 \leq (x - r_i)^2 + y^2 \leq (\rho'_{i0})^2 \end{aligned} \tag{2.182}$$

defining domain D, where $0 < \rho_i < \rho'_{i0} < r_i$.

Domain Interior

$$(0,0) \in D \iff \left\{ \begin{array}{c} y = 0 \leq 0 \\ r_i^2 \leq r_i^2 + 0^2 \leq (\rho'_{i0})^2 \end{array} \right\} \iff \left\{ \begin{array}{c} y = 0 \leq 0 \\ \rho_i \leq r_i \leq \rho'_{i0} \end{array} \right\} \tag{2.183}$$

a contradiction since[46] $\rho_i < \rho'_{i0} < r_i$. Therefore[47] $(0,0) \notin D$ and hence $f(x,y)$ is differentiable everywhere in D with gradient

$$\begin{aligned}\nabla_{x,y} f(x,y) &= \left[\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right] = \left[r_i \frac{y^2 - x^2}{(x^2+y^2)^2}, r_i \frac{2xy}{(x^2+y^2)^2}\right] \\ &= \frac{r_i}{(x^2+y^2)^2}\left[y^2 - x^2, 2xy\right]\end{aligned} \tag{2.184}$$

The gradient is zero at any critical points in the interior of D and since $x > 0 \implies x^2 + y^2 > 0, \forall (x,y) \in D$ and $r_i > 0$, it follows that

$$\nabla_{x,y} f(x,y) = 0 \iff \left\{ \begin{array}{c} y^2 - x^2 = 0 \\ 2xy = 0 \end{array} \right\} \iff x = y = 0 \tag{2.185}$$

which cannot be in D, because $x > 0, \forall (x,y) \in D$.

Line $y = 0$ We are about to examine the boundary. Starting from the x axis, i.e. the line $y = 0$, we have the following constrained minimization problem of $f(x,y)$ in the interior of the linear segments of D on $y = 0$ (and not the corner points).

$$y = 0 \iff g(x,y) = 0 \tag{2.186}$$

The Lagrangian is ($\lambda \in \mathbb{R}$ is a Lagrange multiplier)

$$\Lambda(x,y,\lambda) = f(x,y) + \lambda g(x,y) = r_i \frac{x}{x^2+y^2} + \lambda y \tag{2.187}$$

We require

$$\left\{ \begin{array}{c} \frac{\partial \Lambda}{\partial x}(x,y,\lambda) = 0 \iff r_i \frac{y^2-x^2}{(x^2+y^2)^2} = 0 \overset{r_i>0, x^2+y^2>0, \forall(x,y)\in D}{\iff} \{x = y \vee y = -x\} \\ \frac{\partial \Lambda}{\partial y}(x,y,\lambda) = 0 \iff -r_i \frac{2xy}{(x^2+y^2)^2} + \lambda = 0 \\ \frac{\partial \Lambda}{\partial \lambda}(x,y,\lambda) = 0 \iff g(x,y) = 0 \end{array} \right\} \iff$$
$$x = y = 0 \iff (x,y) \notin D \tag{2.188}$$

So there are no critical points in the interior of the boundary segments of D on $y = 0$.

The corner points are still critical points. Function values at them are

$$\begin{aligned} f(r_i - \rho'_{i0}, 0) &= r_i \frac{r_i - \rho'_{i0}}{(r_i - \rho'_{i0})^2} = \frac{r_i}{r_i - \rho'_{i0}} = \frac{1}{1 - \frac{\rho'_{i0}}{r_i}} \\ f(r_i - \rho_i, 0) &= r_i \frac{r_i - \rho_i}{(r_i - \rho_i)^2} = \frac{r_i}{r_i - \rho_i} = \frac{1}{1 - \frac{\rho_i}{r_i}} \\ f(r_i + \rho_i, 0) &= r_i \frac{r_i + \rho_i}{(r_i - \rho_i)^2} = \frac{r_i}{r_i + \rho_i} = \frac{1}{1 + \frac{\rho_i}{r_i}} \\ f(r_i + \rho'_{i0}, 0) &= r_i \frac{r_i + \rho'_{i0}}{(r_i + \rho'_{i0})^2} = \frac{r_i}{r_i + \rho'_{i0}} = \frac{1}{1 + \frac{\rho'_{i0}}{r_i}} \end{aligned} \tag{2.189}$$

of which $f(r_i + \rho'_{i0}, 0)$ is the minimum value among the four corner points.

[46]Note that $x > 0 \iff (x,y) \notin D$ whereas y can be zero.

[47]This is the constraint $\varepsilon'_{i0} < \|q_d - q_i\|^2 - \rho_i^2$ ensuring $q_d \notin \overline{\mathscr{B}_i(\varepsilon'_{i0})}$.

Semi circles The circles (q_i, ρ_i) and $(q_i, \rho'_{i0}$ are both defined by constraints of the form $(x - r_i)^2 + y^2 - \rho^2 = 0$ so now the Lagrangian is

$$\Lambda(x, y, \lambda) = f(x, y) + \lambda g(x, y) = r_i \frac{x}{x^2 + y^2} + \lambda \left[(x - r_i)^2 + y^2 - \rho^2\right] \tag{2.190}$$

We require

$$\begin{aligned} \frac{\partial \Lambda}{\partial x}(x, y, \lambda) = 0 &\iff r_i \frac{y^2 - x^2}{(x^2 + y^2)^2} + 2\lambda(x - r_i) = 0 \\ \frac{\partial \Lambda}{\partial y}(x, y, \lambda) = 0 &\iff -r_i \frac{2xy}{(x^2 + y^2)^2} + 2\lambda y = 0 \\ \frac{\partial \Lambda}{\partial \lambda}(x, y, \lambda) = 0 &\iff g(x, y) = 0 \end{aligned} \tag{2.191}$$

For $y \neq 0$ (that is, in all points within the semicircles apart from the corner points of D) and $\lambda \neq 0$

$$\begin{aligned} r_i \frac{2xy}{(x^2 + y^2)^2} + 2\lambda y = 0 &\overset{x^2+y^2>0, \forall (x,y) \in D \wedge y \neq 0}{\iff} r_i x + \lambda (x^2 + y^2)^2 = 0 \\ &\overset{\lambda \neq 0}{\iff} (x^2 + y^2)^2 = -\frac{r_i x}{\lambda} \end{aligned} \tag{2.192}$$

Substitution in the first equation yields

$$\begin{aligned} -r_i \frac{y^2 - x^2}{-\frac{r_i x}{\lambda}} + 2\lambda(x - r_i) = 0 &\overset{\lambda \neq 0}{\iff} \frac{y^2 - x^2}{x} + 2(x - r_i) = 0 \iff \\ (x - r_i)^2 + y^2 = r_i^2 &\overset{\rho_i < \rho'_{i0} < r_i}{\iff} (x, y) \notin D \end{aligned} \tag{2.193}$$

so there are no critical points in the interior of the circular boundary segments of D.

The particular cases $y = 0$ and $\lambda = 0$ remain. The case $\lambda = 0 \iff (x, y) = (0, 0) \notin D$ and $y = 0$ on the circles corresponds to the corner points of D. So the only critical points of D are the corner points, which have already been examined previously. The global minimum over D is attained at $(r_i + \rho'_{i0}, 0)$ and is equal to

$$f(r_i + \rho'_{i0}, 0) = \frac{1}{1 + \frac{\rho'_{i0}}{r_i}} \overset{\rho'_{i0} < r_i}{>} \frac{1}{2} \tag{2.194}$$

The plot of f over D is shown in Fig. 2.13. An important note is due here concerning the fact that ε_{i03} is later used in place of ε'_{i0} so that expressions for extrema of involved quantities can be substituted. Since $\varepsilon_{i03} = \min\{\varepsilon'_{i0}, \varepsilon_{i3}\}$ and $\varepsilon'_{i0} < \|q_d - q_i\|^2 - \rho_i^2$ it follows that $\varepsilon_{i03} < \|q_d - q_i\|^2 - \rho_i^2$ as well, hence also $\rho_{i03} < r_i$, so the previous analysis still holds.

2.4.5.13 ε''_{i0} expression

Considering all preceding discussion we are led to define upper bound ε''_{0i} on ε_i as

$$\varepsilon''_{i0} \triangleq \frac{1}{2\frac{1}{\sqrt{\gamma_{di}^{\min}}} \sum_{j \in I_0 \setminus i} Q_{ji} + \left(2 \sum_{j \in I_0 \setminus i} Q_{ji}\right)^2 + 4 \sum_{j \in I_0 \setminus i} \left(Q_{ji} \sum_{l \in I_0 \setminus \{i,j\}} Q_{li}\right) - 2 \sum_{j \in I_0 \setminus i} \frac{1}{\beta_{ji}^{\max}}}, \quad \forall i \in I_1 \tag{2.195}$$

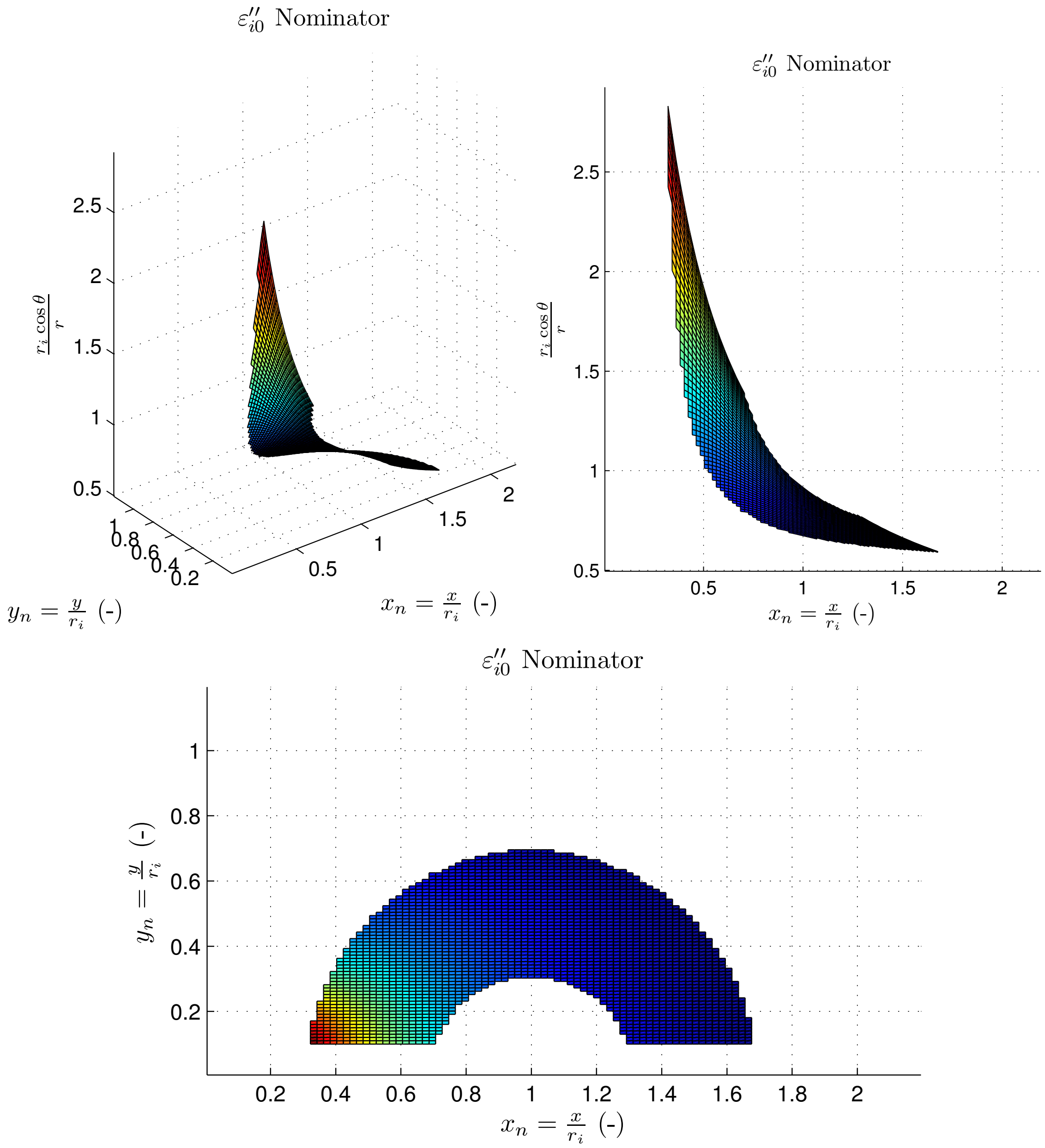


Figure 2.13: The nominator $-2\frac{\nu_i(q)}{\gamma_d(q_c)}$ of ε''_{i0} essentially is a function of two variables $f(r,\theta) = \frac{r_i \cos\theta}{r}$. Here it is illustrated in normalized coordinates $x_n = \frac{x}{r_i}, y_n = \frac{y}{r_i}$.

$$\varepsilon_{i0}'' \leq \frac{\frac{-2\nu_i(q)}{\gamma_d(q)}}{\frac{\frac{1}{2}\bar{\beta}_i\nabla\bar{\beta}_i\cdot\nabla\gamma_d+\gamma_d\hat{t}_i^{\mathrm{T}}\left[\left(1-\frac{1}{k}\right)\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}}-\bar{\beta}_i D^2\bar{\beta}_i\right]\hat{t}_i}{\gamma_d\bar{\beta}_i^2}}, \quad \forall q \in \mathscr{B}_i\left(\varepsilon_i\right), \quad i \in I_1 \tag{2.196}$$

The respective constraint on ε_i is

$$0 < \beta_i < \varepsilon_i < \varepsilon_{i0}'', \quad \forall q \in \mathscr{B}_i\left(\varepsilon_i\right), \quad i \in I_1 \tag{2.197}$$

Chapter 3

Adjustability in Unknown Sphere Worlds

3.1 Upper bound on γ_d when $\mathscr{O}_0$ is unknown

The analysis so far concerned automatically tuning a NF for a sphere world with internal obstacles within $\mathscr{W}$. In section 3.2 we show how to efficiently maintain the NF tuned as new internal obstacles are discovered. But initially no obstacle is known. Any unknown obstacles must be disjoint spheres. So an internal obstacle $\mathscr{O}_i, i \neq 0$ may be discovered before $\mathscr{O}_0$. In this case the workspace is unbounded $\mathscr{O}_0 = \emptyset \implies \mathscr{W} = E^n$. This is not covered by the original NF formulation. We now extend the method of analytic NFs to unbounded worlds with internal spheres.

Propositions 3.2, 3.3 [23] still hold, so no critical points arise on $\partial\mathscr{F}$ and q_d is a local minimum of φ. In the same way as proved in Proposition 2.7 [23] critical points in the interior of $\mathscr{F} \setminus \{q_d\}$ are unaffected by range diffeomorphism $\sigma_d \circ \sigma$ so we can examine critical points of $\hat{\varphi}$.

In case of a single internal obstacle $\mathscr{O}_i, i \neq 0$ Propositions 3.6 and 3.9 [23] hold for

$$\varepsilon_i < \min\{\varepsilon'_{i0}, \varepsilon'_{i2}\} \tag{3.1}$$

since $\varepsilon''_{i0}, \varepsilon''_{i2}, \varepsilon_{i3}$ are not needed whereas ε_{0u} is undefined.

If more internal obstacles are known

$$\varepsilon_i < \min\{\varepsilon''_{i0}, \varepsilon''_{i2}, \varepsilon_{i03}, \varepsilon_{i23}\} \tag{3.2}$$

applies and ε_{0u} is still undefined.

When at least one new obstacle is discovered at $t_m \in [0, +\infty), m \in \mathbb{N} \setminus 0$ the NF is updated. So different NF fields guide the agent before and after t_m. For each discovered obstacle one update is performed, increasing the number $M_z \in \mathbb{N}, z \in \mathbb{N} \setminus 0$ of currently known internal obstacles. Note that $m \leq z$ because several new obstacles may be sensed at t_m. Also note that since the dicovery of a new obstacle triggered the potential update at least one internal obstacle $\mathscr{O}_i, i \in I_{1z} \triangleq \{1, 2, \ldots, M_z\}$ is known, so $M_z \geq 1$. Let $i_{\min} = 1$ if $\mathscr{O}_0$ remains unknown and $i_{\min} = 0$ otherwise. The notation ${}^z\beta$ refers to β when M_z obstacles are known

$${}^z\beta = \prod_{i \in I_{1z}} \beta_i = \prod_{i=1}^{M_z} \beta_i \tag{3.3}$$

Similar notation will follow which has been avoided so far to reduce unnecessary clutter.

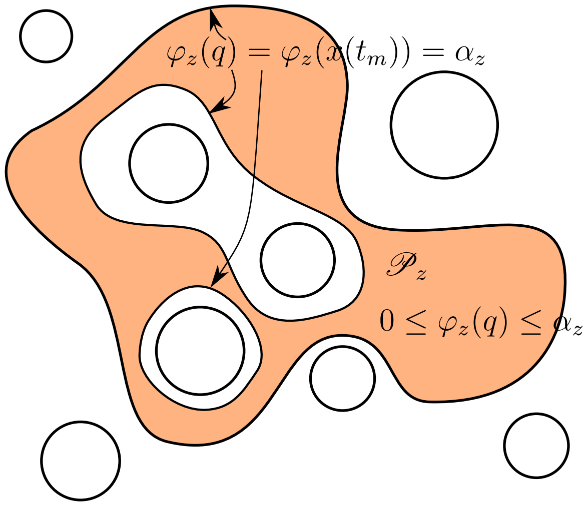


Figure 3.1: Positive invariant set $\mathscr{P}_z$ until next NF update.

The agent at time t_m is positioned at $x(t_m)$. Let

$$\alpha_z \triangleq \varphi_z\left(x(t_m)\right) > 0, \quad z \in \{1, 2, \ldots, N_{\text{transitions}}\} \tag{3.4}$$

be the updated NF potential at the agent's position $x(t_m)$ after the update (but before the agent moves). The number of transitions z is finite $N_{\text{transitions}}$ because each controller is a NF and the number of obstacles is finite. This is equivalent to convergence, which is proved in section 3.6. Also $\alpha_z > 0$ because $\alpha_z = 0 \iff q = q_d$ which is not the present case (the agent here has not converged yet, otherwise we would not bother any more!).

Although φ_z changes due to each added obstacle, it suffices to first add all new obstacles discovered at time t_m, calculating their ε_{iu} and recursively updating ε_{iu} of already known obstacles as detailed later. Following this, a single update of k_z then suffices for each t_m (an update of k_z is redundant for each new obstacle at t_m).

If the agent is in the free space interior when the NF is updated $x(t_m) \in \mathscr{F} \setminus \partial\mathscr{F} = \mathring{\mathscr{F}}$ then $\varphi_z(q) < 1, \forall q \in \mathscr{F} \setminus \partial\mathscr{F} \implies \alpha_z < 1$ ($\partial\mathscr{F}$ has zero measure anyway). Since x is a gradient system $\dot{x} = -\nabla_q \varphi_z$ it cannot overcome α_z until the NF is updated again at t_{m+1}. This is true for the NF after t_m until it changes again (if it first reaches q_d this never happens). So

$$\varphi_z(x(t)) < \alpha_z, \quad \forall t \in [t_m, t_{m+1}] \tag{3.5}$$

Note that if $t_{m+1} = +\infty$ then this interval is $[t_m, +\infty)$.

Let use define the closed set where the potential function φ_z is less than or equal to its value at the agent's initial in $[t_m, t_{m+1}]$ configuration $x(t_m)$

$$\mathscr{P}_z \triangleq \{q \in \mathscr{F} : \varphi_z\left(q\right) \leq \alpha_z\} \tag{3.6}$$

By (3.5) and (3.6) for any initial $x(t_m) \in \mathscr{P}_z$ the agent remains within $\mathscr{P}_z$ for the time interval $[t_m, t_{m+1}]$

$$x(t) \in \mathscr{P}_z, \quad \forall t \in [t_m, t_{m+1}] \tag{3.7}$$

so the set $\mathscr{P}_z$ is positive invariant in the time interval $[t_m, t_{m+1}]$. The agent cannot escape out of it until the next NF update[1]. Set $\mathscr{P}_z$ is schematically shown in Fig. 3.1. By definition

$$\begin{aligned} \varphi_z\left(x(t_m)\right) = \alpha_z &\implies x(t_m) \in \mathscr{P}_z \\ \varphi_z\left(q_d\right) = 0 < \alpha_z &\implies q_d \in \mathscr{P}_z \end{aligned} \tag{3.8}$$

[1] Convergence is guaranteed by the finite total number of unknown obstacles by Theorem 2 so that after a finite number of NF updates $t_{m+1} = +\infty$, that is, no further update occurs

When no obstacle $\mathscr{O}_0$ is known the potential field must ensure the agent remains in a finite region. A potential with $\lim_{\|q\|\to\infty} \varphi_z(q) < \alpha_z$ cannot ensure this. Since $\alpha_z < 1$ if $\lim_{\|q\|\to\infty} \varphi_z(q) = 1$ then $\mathscr{P}_z$ will be always bounded. To ensure $\mathscr{P}_z$ is bounded select

$$k_z > M_z \iff \lim_{\|q\|\to\infty} \varphi_z(q) = 1 \tag{3.9}$$

Proof of this Proposition in subsection A.6.3. Essentially it makes $\hat{\varphi}_z$ radially unbounded.

Hence there exists a sphere

$$\mathscr{Q}_z(\rho_b) \triangleq \{q \in E^n : \|q - q_d\| \leq \rho_b\} \tag{3.10}$$

such that

$$\varphi_z(q) > \alpha_z, \quad \forall q \notin \mathscr{Q}_z \implies \mathscr{P}_z \subseteq \mathscr{Q}_z \tag{3.11}$$

By $\mathscr{P}_z \subseteq \mathscr{Q}_z$ it follows that $\mathscr{P}_z$ is bounded by the sphere $\mathscr{Q}_z(\rho_b)$ of finite radius. Since $\mathscr{P}_z$ is closed by definition and bounded as shown, it is compact.

The limit set for any trajectory of a gradient system on a compact manifold as $\mathscr{P}_z$ is an equilibrium point [23, 49]. As a result the limit set of $\dot{x}(t)$ is the set of equilibrium points in $\mathscr{P}_z$

$$\left\{q \in E^n : \lim_{t\to+\infty} x(t) = q\right\} = \mathscr{C}_\varphi \cap \mathscr{P}_z \tag{3.12}$$

The equilibria may be maxima, minuma, or saddles. Only minima and saddles can constitute the positive limit set of a gradient system. Showing that all saddles are non-degenerate implies that their stable manifold is of measure zero. The remaining equilibria in $\mathscr{P}_z$ which can have an open stable manifold are the minima in $\mathscr{P}_z$. Therefore it suffices that no local minima other than q_d arise within $\mathscr{P}_z$.

$$\nexists q_{\min} \in \mathscr{P}_z \setminus \{q_d\} \tag{3.13}$$

Proposition 3.4 [23] remains to be proved for the case of unknown $\mathscr{O}_0$. But here a serious problem arises, since $\sqrt{\gamma_d}$ is unbounded.

For the lower bound of k_z an upper bound on $\sqrt{\gamma_d}$ is needed within $\mathscr{F}_2(\varepsilon_{I_0})$. But $\mathscr{F}_2(\varepsilon_{I_0})$ is unbounded. Nonetheless the agent can only reach $\mathscr{P}_z \cap \mathscr{F}_2(\varepsilon_{I_0})$ which is bounded (and compact). Since $\mathscr{P}_z \cap \mathscr{F}_2(\varepsilon_{I_0})$ is bounded hence an upper bound on $\sqrt{\gamma_d}$ within it exists.

It suffices to determine an upper bound

$$\rho_a \geq \max_{\mathscr{P}_z}\{\sqrt{\gamma_d}\} \geq \sqrt{\gamma_d}, \quad \forall q \in \mathscr{P}_z \tag{3.14}$$

on $\sqrt{\gamma_d}$ in $\mathscr{P}_z$. This will also be an upper bound on $\sqrt{\gamma_d}$ within the subset $\mathscr{P}_z \cap \mathscr{F}_2(\varepsilon_{I_0})$ of $\mathscr{F}_2(\varepsilon_{I_0})$ which is reachable in time interval $[t_m, t_{m+1}]$.

Proposition 2. If

$$\sqrt{\gamma_d} > \max_i\{\|q_i - q_d\|\} \tag{3.15}$$

and

$$\sqrt{\gamma_d} > a_1^{m_1} a_2^{m_2} \tag{3.16}$$

where

$$a_1 \triangleq \frac{4^{M_z}}{{}^z\beta(x(t_m))}, \quad a_2 \triangleq \gamma_d(x(t_m)) \tag{3.17}$$

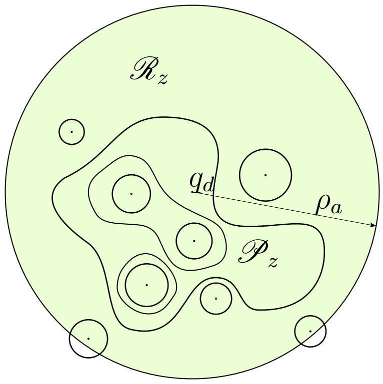


Figure 3.2: Sphere $\mathscr{R}_z$. Note that ρ_a is such that both $\mathscr{R}_z \supseteq \mathscr{P}_z$ and all centers $q_i \in \mathscr{R}_z, i \in I_{1z}$.

and

$$m_1 \triangleq \begin{cases} 0, & a_1 \leq 1 \\ \frac{1}{2}, & a_1 > 1 \end{cases}, \quad m_2 \triangleq \begin{cases} \frac{1}{2}, & a_2 \leq 1 \\ \frac{M_z+1}{2}, & a_2 > 1 \end{cases} \tag{3.18}$$

then $q \notin \mathscr{P}_z$.

Proof. See subsection A.6.4. □

Lemma 3. Let $q \in \mathscr{P}_z$ and suppose

$$\sqrt{\gamma_d(q)} > \rho_a \triangleq \max\{\max_i \{\|q_i - q_d\|\}, \quad a_1^{m_1} a_2^{m_2}\} \tag{3.19}$$

By Proposition 2 it follows that $q \notin \mathscr{P}_z$, a contradiction, hence

$$\sqrt{\gamma_d(q)} \leq \rho_a, \quad \forall q \in \mathscr{P}_z \tag{3.20}$$

Let us define a sphere centered at destination q_d

$$\mathscr{R}_z(\rho_a) \triangleq \{q \in E^n : \|q - q_d\| \leq \rho_a\} \tag{3.21}$$

so[2] by Proposition 1 it includes positive invariant set $\mathscr{P}_z$

$$\left\{q \notin \mathscr{R}_z \implies \|q - q_d\| > \rho_a \iff \sqrt{\gamma_d(q)} > \rho_a \implies q \notin \mathscr{P}_z\right\} \implies \mathscr{P}_z \subseteq \mathscr{R}_z \tag{3.22}$$

hence the agent x does not leave $\mathscr{R}_z$ until the next NF update[3]

$$x(t) \in \mathscr{P}_z \subseteq \mathscr{R}_z, \quad \forall t \in [t_m, t_{m+1}] \tag{3.23}$$

This sphere[4] has $\rho_a \geq \|q_i - q_d\|, \forall i \in I_{1z}$ so it also includes all obstacle centers q_i. Also $\mathscr{Q}_z \subseteq \mathscr{R}_z$.

As noted before, a sufficient inequality for the gradient to be nonzero in the set "away" from obstacles $\mathscr{F}_2(\varepsilon_{I_0})$ is

$$\frac{1}{2}\sqrt{\gamma_d}\frac{\|\nabla\beta\|}{\beta} < k_z, \quad \forall q \in \mathscr{F}_2(\varepsilon_{I_0}) \tag{3.24}$$

[2]If $\mathscr{O}_i \cap \mathscr{P}_z \neq \emptyset$ the it is not $=$.

[3]If no further update occurs $t_{m+1} \to +\infty$.

[4]It is only known that $\varphi_z(q) > \alpha_z, \forall q \notin \mathscr{R}_z$. It is not true that $\varphi_z(q) \leq \alpha_z, \forall q \in \mathscr{R}_z$ because at least one internal obstacle $\mathscr{O}_i, i \in I_{1z}$ is known, whose center belongs to sphere $\mathscr{R}_z$ implying $\mathscr{R}_z \cap \partial\mathscr{F} \neq \emptyset \implies \exists q \in \mathscr{R}_z : \varphi_z(q) = 1 > \alpha_z$.

but when no $\mathscr{O}_0$ is known $\mathscr{F}_2\left(\varepsilon_{I_0}\right)$ is unbounded and $\sqrt{\gamma_d}$ cannot be bounded. What we have proved so far is that for $k_z > M_z$ the positive invariant set $\mathscr{P}_z$ is bounded. As a result we are not concerned with critical points outside $\mathscr{P}_z$ since the agent using this updated NF cannot reach them and be trapped by them.

More directly, if $\mathscr{P}_z$ closed, then if $q_c \notin \mathscr{P}_z \implies \exists \rho_c : \forall q : \|q - q_c\| < \rho_c \implies q \notin \mathscr{P}_z$ so that since $\mathscr{P}_z$ positive invariant, then suppose $\lim_{t\to+\infty} x(t) = q_c \implies \exists t_{\max} : \|x(t) - q_c\| < \rho_c, \forall t > t_{\max} \implies x(t) \notin \mathscr{P}_z, \forall t > t_{\max}$ which contradicts the hypothesis that $\mathscr{P}_z$ is positive invariant. Hence $q_c \notin \mathscr{P}_z$ implies q_c stable manifold $\mathscr{S}_c$ is completely outside $\mathscr{P}_z$, that is $\mathscr{S}_c \cap \mathscr{P}_z = \emptyset$.

We have found an upper bound on $\sqrt{\gamma_d}$ in $\mathscr{R}_z \supseteq \mathscr{P}_z, \forall k_z > M_z$. So we can use it to find a lower bound on k_z within the reachable set $\mathscr{P}_z$.

Substituting the upper bound on $\sqrt{\gamma_d}$ in $\mathscr{R}_z$ in the sufficient inequality

$$\max_{\mathscr{R}_z}\{\sqrt{\gamma_d}\} \sum_{I_{1z}} Q_{ii} < k_z \tag{3.25}$$

to find a lower bound for k_z within the positive invariant set $\mathscr{P}_z$ leads to

$$\rho_a \sum_{I_{1z}} Q_{ii} < k_z \tag{3.26}$$

This prevents critical points from arising "away" from obstacles in $\mathscr{F}_2\left(\varepsilon_{I_0}\right) \cap \mathscr{R}_z$. Since $\mathscr{P}_z \subseteq \mathscr{R}_z$ it also implies that $q_c \notin \left(\mathscr{F}_2\left(\varepsilon_{I_0}\right) \cap \mathscr{P}_z\right)$. Note that the agent is confined in $\mathscr{P}_z$, not in $\mathscr{F}_2\left(\varepsilon_{I_0}\right) \cap \mathscr{P}_z$. But the complement in $\mathscr{P}_z$ of the reachable set "away" from obstacles $\mathscr{P}_z \cap \mathscr{F}_2\left(\varepsilon_{I_0}\right)$ is $\mathscr{P}_z \cap \partial\mathscr{F} \cup \mathscr{F}_0\left(\varepsilon_{I_1}\right)$. These are critical points confined close to internal obstacles. Since $\mathscr{P}_z \subseteq \mathscr{R}_z$ these critical points are within $\mathscr{R}_z$. Any remaining critical points in $\mathscr{R}_z$ are confined near the obstacles[5]. By the ε_i upper bounds these remaining critical points $q_c \neq q_d$ near obstacles are ensured to be non-degenerate saddle points. In $\mathscr{R}_z$ a unique local minimum remains at q_d.

Since $\mathscr{P}_z \subseteq \mathscr{R}_z$, $\mathscr{R}_z$ contains a unique non-degenerate local minimum, all other critical points in $\mathscr{R}_z$ are non-degenerate saddles, and $q_d \in \mathscr{P}_z$, it follows that $\mathscr{P}_z$ contains a unique local minimum, and all other critical points in $\mathscr{P}_z$ are non-degenerate saddles.

So for unknown $\mathscr{O}_0$

$$k_z > \max\left\{\rho_a \sum\nolimits_{I_{1z}} Q_{ii}, \quad M_z\right\} \triangleq N(\varepsilon_{I_{1z}}) \tag{3.27}$$

implies that all equilibria other than destination q_d (local minimum) in $\mathscr{P}_z$ are non-degenerate saddles.

It follows that the only critical point in $\mathscr{P}_z$ with a non-empty stable manifold (dense in $\mathscr{P}_z$) is the local minimum at destination q_d. Almost all (all apart from a set of measure zero) initial in $[t_m, t_{m+1}]$ conditions $x(t_m) \in \mathscr{P}_z$ have q_d as their limit set.

Note that $\mathscr{P}_z$ is connected. We can prove this as follows. Suppose $\mathscr{P}_z$ is not connected. It is closed by definition, so if not connected it will be a union of at least two disjoint closed subsets. The potential φ_z is continuous within each closed subset. As a result φ_z will have a global minimum[6] within each closed subset (global with respect to the

[5]Note that a minimum of saddle outside $\mathscr{R}_z$ may have a stable manifold with common points with $\mathscr{R}_z$, but not with $\mathscr{P}_z$.

[6]Note that since $\mathscr{P}_z$ boundary is a non-zero level set, on the boundary of $\mathscr{P}_z$ the gradient system has non-zero gradient normal to the level set, so no critical points arise on $\partial\mathscr{P}_z$. Any critical points $q_c \in \mathscr{P}_z$ arise in the interior $\mathscr{P}_z \setminus \partial\mathscr{P}_z$ of $\mathscr{P}_z$.

subset). Each minimum in a subset is also a minimum of $\mathscr{P}_z$. Disconnection of $\mathscr{P}_z$ implies at least two disjoint subsets, as already mentioned. Therefore φ_z will have at least two local minima in $\mathscr{P}_z$. This is a contradiction, because we have already shown that, for the selected k_z, function φ_z has in $\mathscr{P}_z$ a unique non-degenerate local minimum at destination q_d.

Local minima or saddles may arise outside $\mathscr{R}_z \supseteq \mathscr{P}_z$ but the agent cannot reach them since their stable manifolds have no common points with $\mathscr{P}_z$. Either it converges to q_d or the NF is again updated at time t_{m+1}.

This replaces Propositions 2.4, 3.4 [23] when no $\mathscr{O}_0$ is known.

3.2 Recursive update of ε''_{i2}

The present analysis applies both when $\mathscr{O}_0$ is known and when it is unknown. For this reason we define the set of indices $I_{1z} \triangleq \{1, 2, \ldots, M_z\}$ of known internal obstacles. The set of indices $I_{i_{\min}z}$ of all obstacles depends on whether $\mathscr{O}_0$ has been discovered. If it remains unknown $I_{i_{\min}z} = I_{1z}$. If $\mathscr{O}_0$ is known then $I_{i_{\min}z} = \{0, 1, \ldots, M_z\} = \{0\} \cup I_{1z}$. For brevity we will denote $I_{i_{\min}z}$ by I_z.

We have defined

$$\varepsilon''_{i2} \triangleq \frac{\rho_i}{\sqrt{2\sum_{j\in I_z\setminus i}\left(\frac{1}{\beta_{ji}^{\min}} + 4Q_{ji}\sum_{l\in I_z\setminus\{i,j\}} Q_{li}\right)}}, \quad i \in I_{1z} \tag{3.28}$$

and our aim is to arrange the denominator calculation in such a way so as to update it with minimal time computational complexity for each new obstacle. An increase in memory requirements is allowed.

A naive first scheme would be to store the obstacle data $\{q_i, \rho_i\}$ and recalculate ε''_{i2} from these each time a new obstacle is discovered. This update is needed for all the until then known obstacles. So the new obstacle causes M_z many ε''_{i2} to be recalculated. Each ε''_{i2} has a time computational complexity of $\Theta(M_z)$. As a result the update complexity becomes $\Theta(M_z^2)$.

Note that the above requires no ε''_{i2} to be stored. No other quantity intermediate in the calculation of ε''_{i2} need to be stored either. But it requires all $\{q_i, \rho_i\}$ to be stored. This has memory complexity $\Theta(M_z)$. Therefore rearranging the calculation to reduce updating time complexity to $\Theta(M_z)$ while keeping the (increased) memory requirements to $\Theta(M_z)$ constitutes an improvement.

Let us examine how to achieve this. The denominator requires computation of

$$\sum_{j\in I_z\setminus i}\left(\frac{1}{\beta_{ji}^{\min}} + 4Q_{ji}\sum_{l\in I_z\setminus\{i,j\}} Q_{li}\right), i \in I_{1z}. \tag{3.29}$$

Assume that additionally to $\{\rho_i, q_i\}$ we also store ε''_{i2} (although it is not needed for the update) and

$$\sum_{j\in I_z\setminus i}\left(\frac{1}{\beta_{ji}^{\min}}\right), \sum_{j\in I_z\setminus i}\left(Q_{ji}\sum_{l\in I_z\setminus\{i,j\}} Q_{li}\right), \sum_{j\in I_z\setminus i} Q_{ji} \tag{3.30}$$

Now assume that a new obstacle $\mathscr{O}_n, n \neq i$ is discovered and I_z becomes I_{z+1}. The new obstacle n is different than the already known i whose ε''_{i2} is updated (hence $n \neq i$). If the new obstacle is $\mathscr{O}_0$ then $n = 0$ and $M_z = M_{z+1}$. If the new obstacle is internal $n \neq 0$ and $M_{z+1} = M_z + 1$. We want to compute the new ε''_{i2} (updated) from the stored quantities from which ε''_{i2} (old) was computed. The update should require minimal time computational complexity.

The new denominator will be

$$\begin{aligned}
&\sum_{j\in I_{z+1}\setminus i}\left(\frac{1}{\beta_{ji}^{\min,\text{new}}} + 4Q_{ji}^{\text{new}}\sum_{l\in I_{z+1}\setminus\{i,j\}} Q_{li}^{\text{new}}\right)\\
&= \sum_{j\in I_{z+1}\setminus i}\left(\frac{1}{\beta_{ji}^{\min,\text{new}}}\right) + 4\sum_{j\in I_{z+1}\setminus i}\left(Q_{ji}^{\text{new}}\sum_{l\in I_{z+1}\setminus\{i,j\}} Q_{li}^{\text{new}}\right)
\end{aligned} \tag{3.31}$$

Before proceeding further let us observe a convenient fact. The terms $\beta_{ji}^{\min,\text{new}}$ and Q_{ji}^{new} are the extrema, minimum and maximum respectively, of β_j and $Q_j\left(\beta_j\right)$ over $\overline{B_i\left(\varepsilon_{i23}^{\text{new}}\right)}$ where $\varepsilon_{i23}^{\text{new}} = \min\{\varepsilon_{i3}^{\text{new}}, \varepsilon_{i2}^{\prime\text{new}}\}$. Since $\varepsilon_{i2}^{\prime\text{new}} = \rho_i^2 = \varepsilon_{i2}^{\prime old}$ a change in $\varepsilon_{i23}^{\text{new}}$ occurs only if $\varepsilon_{i3}^{\text{new}}$ is different from $\varepsilon_{i3}^{\text{old}}$.

But $\varepsilon_{i3}^{\text{new}} = \min_{j\in I_{z+1}\setminus i}\{\varepsilon_{i3j}\}$ and $\{\varepsilon_{i3j}, j \in I_{z+1} \setminus i\} = \{\varepsilon_{i3j}, j \in I_z \setminus i\} \cup \{\varepsilon_{i3n}\}$ from which only ε_{i3n} is new (the rest remain the same). That is

$$\varepsilon_{i3}^{\text{new}} = \min_{j\in I_{z+1}\setminus i}\{\varepsilon_{i3j}\} = \min\{\min_{j\in I_z\setminus i}\{\varepsilon_{i3j}\}, \varepsilon_{i3n}\} = \min\{\varepsilon_{i3}^{\text{old}}, \varepsilon_{i3n}\} \tag{3.32}$$

so only if $\varepsilon_{i3n} < \varepsilon_{i3}^{\text{old}}$ then $\varepsilon_{i3}^{\text{new}}$ is different (and less) than $\varepsilon_{i3}^{\text{old}}$. Summarizing what has been shown so far is that only if $\varepsilon_{i3n} < \varepsilon_{i3}^{\text{old}}$ then $\varepsilon_{i23}^{\text{new}} \neq \varepsilon_{i23}^{\text{new}}$ (ε_{i23} changes). Also

$$\begin{aligned}
\varepsilon_{i3}^{\text{new}} \leq \varepsilon_{i3}^{\text{old}} &\implies \varepsilon_{i23}^{\text{new}} \leq \varepsilon_{i23}^{\text{new}} \implies \overline{B_i\left(\varepsilon_{i23}^{\text{new}}\right)} \subseteq \overline{B_i\left(\varepsilon_{i23}^{\text{old}}\right)} \\
&\implies \begin{cases} Q_{ji}^{\text{new}} & \leq Q_{ji}^{\text{old}} \\ \beta_{ji}^{\min,new} & \geq \beta_{ji}^{\min,\text{old}} \end{cases}, j \in I_z \setminus i
\end{aligned} \tag{3.33}$$

Therefore the old $Q_{ji}^{\text{old}}, j \in I_z$ can serve as upper bounds on $Q_{ji}^{\text{new}}, j \in I_z$ and $\beta_{ji}^{\min,\text{old}}, j \in I_z$ can serve as lower bounds on $\beta_{ji}^{\min,\text{new}}, j \in I_z$.

This allows us to develop the following recursive scheme

$$\sum_{j\in I_{z+1}\setminus i} \frac{1}{\beta_{ji}^{\min,\text{new}}} = \frac{1}{\beta_{ni}^{\min,\text{new}}} + \sum_{j\in I_z\setminus i} \frac{1}{\beta_{ji}^{\min,\text{new}}} \leq \frac{1}{\beta_{ni}^{\min,\text{new}}} + \sum_{j\in I_z\setminus i} \frac{1}{\beta_{ji}^{\min,\text{old}d}} \tag{3.34}$$

and also

$$\begin{aligned}
&\sum_{j\in I_{z+1}\setminus i}\left(Q_{ji}^{\text{new}} \sum_{l\in I_{z+1}\setminus\{i,j\}} Q_{li}^{\text{new}}\right) \\
&= \left(Q_{ni}^{\text{new}} \sum_{l\in I_{z+1}\setminus\{i,n\}} Q_{li}^{\text{new}}\right) + \sum_{j\in I_z\setminus i}\left(Q_{ji}^{\text{new}} \sum_{l\in I_{z+1}\setminus\{i,j\}} Q_{li}^{\text{new}}\right) \\
&= \left(Q_{ni}^{\text{new}} \sum_{l\in I_z\setminus i} Q_{li}^{\text{new}}\right) + \sum_{j\in I_z\setminus i}\left(Q_{ji}^{\text{new}}\left(Q_{ni}^{\text{new}} + \sum_{l\in I_{z+1}\setminus\{i,j,n\}} Q_{li}^{\text{new}}\right)\right) \\
&= \left(Q_{ni}^{\text{new}} \sum_{l\in I_z\setminus i} Q_{li}^{\text{new}}\right) + \sum_{j\in I_z\setminus i}\left(Q_{ji}^{\text{new}} Q_{ni}^{\text{new}}\right) + \sum_{j\in I_z\setminus i}\left(Q_{ji}^{\text{new}} \sum_{l\in I_{z+1}\setminus\{i,j,n\}} Q_{li}^{\text{new}}\right) \\
&= \left(Q_{ni}^{\text{new}} \sum_{l\in I_z\setminus i} Q_{li}^{\text{new}}\right) + \left(Q_{ni}^{\text{new}} \sum_{j\in I_z\setminus i} Q_{ji}^{\text{new}}\right) + \sum_{j\in I_z\setminus i}\left(Q_{ji}^{\text{new}} \sum_{l\in I_z\setminus\{i,j\}} Q_{li}^{\text{new}}\right) \\
&= 2\left(Q_{ni}^{\text{new}} \sum_{j\in I_z\setminus i} Q_{ji}^{\text{new}}\right) + \sum_{j\in I_z\setminus i}\left(Q_{ji}^{\text{new}} \sum_{l\in I_z\setminus\{i,j\}} Q_{li}^{\text{new}}\right)
\end{aligned} \tag{3.35}$$

and since $0 < Q^{\text{new}}_{ji} \leq Q^{\text{old}}_{ji}, \forall j \in I_z \setminus i$ it follows that

$$\begin{aligned} & 2\left(Q^{\text{new}}_{ni} \sum_{j\in I_z\setminus i} Q^{\text{new}}_{ji}\right) + \sum_{j\in I_z\setminus i}\left(Q^{\text{new}}_{ji}\sum_{l\in I_z\setminus\{i,j\}} Q^{\text{new}}_{li}\right) \\ & \leq 2\left(Q^{\text{new}}_{ni} \sum_{j\in I_z\setminus i} Q^{\text{old}}_{ji}\right) + \sum_{j\in I_z\setminus i}\left(Q^{\text{old}}_{ji}\sum_{l\in I_z\setminus\{i,j\}} Q^{\text{old}}_{li}\right) \end{aligned} \tag{3.36}$$

So the update requires computation of new $\frac{1}{\beta^{\min,\text{new}}_{ni}}, Q^{\text{new}}_{ni}$ and storage of old

$$\rho_i, \quad a^{\text{old}}_{i23} \triangleq \sum_{j\in I_z\setminus i} Q^{\text{old}}_{ji}, \qquad a^{\text{old}}_{i22} \triangleq \sum_{j\in I_z\setminus i}\left(Q^{\text{old}}_{ji}\sum_{l\in I_z\setminus\{i,j\}} Q^{\text{old}}_{li}\right), \quad a^{\text{old}}_{i21} \triangleq \sum_{j\in I_z\setminus i}\frac{1}{\beta^{\min,\text{old}}_{ji}}. \tag{3.37}$$

The updating algorithm after computation and loading from memory of these quantities is (a_{i21} is an upper bound on $\sum_{j\in I_z\setminus i}\frac{1}{\beta^{\min}_{ji}}$)

$$\begin{aligned} \sum_{j\in I_{z+1}\setminus i}\frac{1}{\beta^{\min,\text{new}}_{ji}} \geq \frac{1}{\beta^{\min,\text{new}}_{ni}} + \sum_{j\in I_z\setminus i}\frac{1}{\beta^{\min,\text{old}}_{ji}} \implies a^{\text{new}}_{i21} = \frac{1}{\beta^{\min,\text{new}}_{ni}} + a^{\text{old}}_{i21} \\ \sum_{j\in I_{z+1}\setminus i}\left(Q^{\text{new}}_{ji}\sum_{l\in I_{z+1}\setminus\{i,j\}} Q^{\text{old}}_{li}\right) \leq 2\left(Q^{\text{new}}_{ni}\sum_{j\in I_z\setminus i} Q^{\text{old}}_{ji}\right) + \sum_{j\in I_z\setminus i}\left(Q^{\text{old}}_{ji}\sum_{l\in I_z\setminus\{i,j\}} Q^{\text{old}}_{li}\right) \implies \\ a^{\text{new}}_{i22} = 2Q^{\text{new}}_{ni} a^{\text{old}}_{i23} + a^{\text{old}}_{i22} \\ \sum_{j\in I_{z+1}\setminus i} Q^{\text{new}}_{ji} \leq Q^{\text{new}}_{ni} + \sum_{j\in I_z\setminus i} Q^{\text{old}}_{ji} \implies a^{\text{new}}_{i23} = Q^{\text{new}}_{ni} + a^{\text{old}}_{i23} \\ \varepsilon''^{\text{new}}_{i2} = \frac{\rho_i}{\sqrt{2\left(a^{\text{new}}_{i21} + 4a^{\text{new}}_{i22}\right)}} \end{aligned} \tag{3.38}$$

Note that since $a^{\text{new}}_{i21} > a^{\text{old}}_{i21}$ and $a^{\text{new}}_{i22} > a^{\text{old}}_{i22}$ it follows that $\varepsilon''^{\text{new}}_{i2} < \varepsilon^{\text{old}}_{i2}$ so $\varepsilon^{\text{old}}_{i2}$ need not be stored for comparison with the new value. But $\varepsilon^{\text{old}}_{i2}$ is stored because it is needed to calculate

$$\Delta Q_{ii} = Q^{\text{new}}_{ii} - Q^{\text{old}}_{ii} \tag{3.39}$$

when updating k.

3.3 Recursive update of ε''_{i0}

We can work in the same way for

$$\varepsilon''_{i0} \triangleq \frac{1}{2\frac{1}{\sqrt{\gamma^{\min}_{di}}}\sum_{j\in I_z\setminus i} Q_{ji} + 4\left(\sum_{j\in I_z\setminus i} Q_{ji}\right)^2 + 4\sum_{j\in I_z\setminus i}\left(Q_{ji}\sum_{l\in I_z\setminus\{i,j\}} Q_{li}\right) - \sum_{j\in I_z\setminus i}\frac{1}{\beta^{\max}_{ji}}} \tag{3.40}$$

It is important to note that Q_{ji} are calculated over $\overline{\mathscr{B}_i\left(\varepsilon_{i03}\right)}$ where $\varepsilon_{i03} \triangleq \min\{\varepsilon_{i3}, \varepsilon'_{i0}\}$. The quantity ε'_{i0} remains constant. The quantity ε_{i3} can only change due to the new ε_{i3n}, if $\varepsilon_{i3n} < \varepsilon^{\text{old}}_{i03}$.

Let the stored quantities to be used in the recursive update be

$$\frac{1}{\sqrt{\gamma_{di}^{\min}}}, \sum_{j\in I_z\backslash i} Q_{ji}, \sum_{j\in I_z\backslash i}\left(Q_{ji}\sum_{l\in I_z\backslash\{i,j\}} Q_{li}\right), \sum_{j\in I_z\backslash i}\frac{1}{\beta_{ji}^{\max}} \tag{3.41}$$

and also ε''_{i0} to update k when needed.

The update of ε''_{i0} can be arranged as

$$\sum_{j\in I_{z+1}\backslash i} Q_{ji}^{\text{new}} = Q_{ni}^{\text{new}} + \sum_{j\in I_{z+1}\backslash\{i,n\}} Q_{ji}^{\text{new}} = Q_{ni}^{\text{new}} + \sum_{j\in I_z\backslash i} Q_{ji}^{\text{new}} \leq Q_{ni}^{\text{new}} + \sum_{j\in I_z\backslash i} Q_{ji}^{\text{old}} \tag{3.42}$$

and as shown for ε''_{i2} it holds that similarly

$$\sum_{j\in I_{z+1}\backslash i}\frac{1}{\beta_{ji}^{\max,\text{new}}} \geq \frac{1}{\beta_{ni}^{\max,\text{new}}} + \sum_{j\in I_z\backslash i}\frac{1}{\beta_{ji}^{\max,\text{old}}} \tag{3.43}$$

and as already shown for ε''_{i2} it holds that

$$\sum_{j\in I_{z+1}\backslash i}\left(Q_{ji}^{\text{new}}\sum_{l\in I_{z+1}\backslash\{i,j\}} Q_{li}^{\text{new}}\right) \leq 2Q_{ni}^{\text{new}}\sum_{j\in I_z\backslash i} Q_{ji}^{\text{old}} + \sum_{j\in I_z\backslash i}\left(Q_{ji}^{\text{old}}\sum_{l\in I_z\backslash\{i,j\}} Q_{li}^{\text{old}}\right) \tag{3.44}$$

This leads to the following updating scheme. With the computed new $\frac{1}{\beta_{ni}^{\max,\text{new}}}, Q_{ni}^{\text{new}}$ and the stored old

$$\frac{1}{\sqrt{\gamma_{di}^{\min}}}, \quad a_{i03}^{\text{old}} \triangleq \sum_{j\in I_z\backslash i} Q_{ji}^{\text{old}}, \quad a_{i02}^{\text{old}} \triangleq \sum_{j\in I_z\backslash i}\left(Q_{ji}^{\text{old}}\sum_{l\in I_z\backslash\{i,j\}} Q_{li}^{\text{old}}\right), \quad a_{i01}^{\text{old}} \triangleq \sum_{j\in I_z\backslash i}\frac{1}{\beta_{ji}^{\max,\text{old}}} \tag{3.45}$$

the following updating steps constitute the updating algorithm

$$\begin{aligned}
\sum_{j\in I_{z+1}\backslash i}\frac{1}{\beta_{ji}^{\max,\text{new}}} \geq \frac{1}{\beta_{ni}^{\max,\text{new}}} + \sum_{j\in I_z\backslash i}\frac{1}{\beta_{ji}^{\max,\text{old}}} &\implies a_{i01}^{\text{new}} = \frac{1}{\beta_{ni}^{\max,\text{new}}} + a_{i01}^{\text{old}} \\
\sum_{j\in I_{z+1}\backslash i}\left(Q_{ji}^{\text{new}}\sum_{l\in I_{z+1}\backslash\{i,j\}} Q_{li}^{\text{new}}\right) \leq 2Q_{ni}^{\text{new}}\sum_{j\in I_z\backslash i} Q_{ji}^{\text{old}} + \sum_{j\in I_z\backslash i}\left(Q_{ji}^{\text{old}}\sum_{l\in I_z\backslash\{i,j\}} Q_{li}^{\text{old}}\right) &\implies \\
a_{i02}^{\text{new}} = 2Q_{ni}^{\text{new}} a_{i03}^{\text{old}} + a_{i02}^{\text{old}} & \\
\sum_{j\in I_{z+1}\backslash i} Q_{ji}^{\text{new}} \leq Q_{ni}^{\text{new}} + \sum_{j\in I_z\backslash i} Q_{ji}^{\text{old}} &\implies a_{i03}^{\text{new}} = Q_{ni}^{\text{new}} + a_{i03}^{\text{old}} \\
\varepsilon_{i0}^{\prime\prime\text{new}} = \frac{1}{2\frac{1}{\gamma_{di}^{\min}} a_{i03}^{\text{new}} + 4\left(a_{i03}^{\text{new}}\right)^2 + 4a_{i02}^{\text{new}} - a_{i01}^{\text{new}}} &
\end{aligned} \tag{3.46}$$

3.3.1 Tuning updating

3.3.1.1 Tuning calculation summary

By definition $k \in \mathbb{N} \setminus \{0, 1\}$. The condition for selecting k as a function of ε_{I_0} is

$$k \geq N(\varepsilon_{I_0}) \triangleq (\rho_0 + \|q_d\|) \sum_{i \in I_0} Q_{ii} \tag{3.47}$$

List of constraints on ε_i

$$\varepsilon_i < \varepsilon_{i3j} \triangleq (\|q_i - q_j\| - \rho_j)^2 - \rho_i^2, \quad \forall j \in I_0 \setminus i, \quad \forall i \in I_{1z} \tag{3.48}$$

$$\varepsilon_i < \varepsilon_{i3} \triangleq \min_{j \in I_0 \setminus i} \{\varepsilon_{i3j}\} \tag{3.49}$$

$$\varepsilon_i < \varepsilon'_{i2} \triangleq \rho_i^2, \quad \forall i \in I_{1z} \tag{3.50}$$

$$\begin{aligned} \varepsilon_i < \varepsilon''_{i2} &\triangleq \frac{\rho_i}{\sqrt{2 \sum_{j \in I_0 \setminus i} \left(\frac{1}{\beta_{ji}^{\min}} + 4 Q_{ji} \sum_{l \in I_0 \setminus \{i,j\}} Q_{li} \right)}} \\ &\leq \frac{1}{4} \frac{\min_{\overline{\mathscr{B}_i(\varepsilon_{i23})}} \{\|\nabla \beta_i\|\}}{\max_{\overline{\mathscr{B}_i(\varepsilon_{i23})}} \left\{ \sqrt{\frac{\left|\hat{r}_i^{\mathrm{T}} D^2 \bar{\beta}_i \hat{r}_i\right|}{\bar{\beta}_i}} \right\}}, \quad \forall i \in I_{1z} \end{aligned} \tag{3.51}$$

$$\varepsilon_i < \varepsilon'_{i0} \triangleq \lambda'_{i0} \left(\|q_d - q_i\|^2 - \rho_i^2 \right), \quad \lambda'_{i0} \in (0, 1), \quad \forall i \in I_{1z} \tag{3.52}$$

$$\begin{aligned} \varepsilon_i < \varepsilon''_{i0} &\triangleq \frac{1}{2 \frac{1}{\sqrt{\gamma_{di}^{\min}}} \sum_{j \in I_0 \setminus i} Q_{ji} + \left(2 \sum_{j \in I_0 \setminus i} Q_{ji} \right)^2 + 4 \sum_{j \in I_0 \setminus i} \left(Q_{ji} \sum_{l \in I_0 \setminus \{i,j\}} Q_{li} \right) - 2 \sum_{j \in I_0 \setminus i} \frac{1}{\beta_{ji}^{\max}}} \\ &\leq \frac{2 \min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \left\{ \frac{-\nu_i(q)}{\gamma_d} \right\}}{\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \left\{ \frac{1}{2} \frac{\nabla \bar{\beta}_i}{\bar{\beta}_i} \frac{\nabla \gamma_d}{\gamma_d} + \hat{t}_i^{\mathrm{T}} \left[\left(1 - \frac{1}{k}\right) \frac{\nabla \bar{\beta}_i}{\bar{\beta}_i} \frac{\nabla \bar{\beta}_i^T}{\bar{\beta}_i} - \frac{D^2 \bar{\beta}_i}{\bar{\beta}_i} \right] \hat{t}_i \right\}}, \quad \forall i \in I_{1z} \end{aligned} \tag{3.53}$$

$$\varepsilon_0 < \varepsilon_{0u} \triangleq \rho_0^2 - \|q_d\|^2 \tag{3.54}$$

3.3.1.2 Algorithm description

Let $\mathscr{S}(t)$ the agent's open sensing set at time t. Sensing occurs in discrete time $t_{m+1} = t_m + T_s$. Provided $\mathscr{S}(t_m) \cap x(t_{m+1}) \neq \emptyset$ the agent does not venture into unknown territory, ensured by a small enough T_s. To ensure constraints remain valid, k_z is nondecreasing. Initially no obstacle is known, so $I_0 z = 0 = \emptyset, \beta = 1, k_{z=0} = 2$ and $V = \varphi(x(t)) = \sigma_d \circ \sigma \circ \frac{\gamma_d^k}{1}$ does not contain any obstacles.

Next two alternatives exist. Either the system converges to q_d without sensing any obstacles, or an obstacle is discovered, either $\mathscr{O}_0$ or $\mathscr{O}_1$. If only a single internal obstacle is known, $\varepsilon_i < \min\{\varepsilon'_{i0}, \varepsilon'_{i2}\}$ in (3.27). If more internal obstacles are only known $\varepsilon_i < \min\{\varepsilon_{i03}, \varepsilon''_{i0}, \varepsilon_{i23}, \varepsilon''_{i2}\}$ in (3.27).

When $\mathscr{O}_0$ is discovered previous ε_i constraints are updated as described later, and $N(\varepsilon_{I_z}) \leq k_z$ as defined in subsection 2.4.2 instead of (3.27). If only $\mathscr{O}_0$ is known $\varepsilon_0 < \varepsilon_{0u}$

Algorithm 1 Updating the Navigation Function for newly discovered obstacles

procedure New $z+1^{\text{th}}$ discovered $\mathscr{O}_n$
  if $n \neq 0$ **then**
    if $M_z == 0$ and $i_{\min} == 1$ **then**
      $\varepsilon_{1u} \leftarrow \min\{\varepsilon'_{10}, \varepsilon'_{12}\}$
      new ε_1
    else if $M_z == 1$ and $i_{\min} == 1$ **then**
      new $\varepsilon_{iu}, \varepsilon'_{i0}, \varepsilon''_{i0}, \varepsilon'_{i2}, \varepsilon''_{i2}, \varepsilon_{i3}, \varepsilon_i, \forall i \in \{1,2\}$
      update $\varepsilon_{1u}, \varepsilon''_{10}, \varepsilon''_{12}, \varepsilon_{13}$
    else
      new $\varepsilon_{nu}, \varepsilon'_{n0}, \varepsilon''_{n0}, \varepsilon'_{n2}, \varepsilon''_{n2}, \varepsilon_{n3}, \varepsilon_n$
      update $\varepsilon_{iu}, \varepsilon''_{i0}, \varepsilon''_{i2}, \varepsilon_{i3}, i \neq n$
    end if
  else
    $\varepsilon_{0u} \leftarrow \rho_0^2 - \|q_d\|^2$
    new ε_0
    if $M_z > 1$ **then**
      update $\varepsilon_{iu}, \varepsilon''_{i0}, \varepsilon''_{i2}, \varepsilon_{i3}, i \neq 0 = n$
    else if $M_z == 1$ **then**
      new $\varepsilon_{1u}, \varepsilon'_{10}, \varepsilon''_{10}, \varepsilon'_{12}, \varepsilon''_{12}, \varepsilon_{13}, \varepsilon_1$
    end if
  end if
  $k_{z+1} \leftarrow$ update k_z
end procedure

in $N(\varepsilon_{I_z})$. When any new internal obstacle $\mathscr{O}_i$ is discovered calculation of $\varepsilon'_{i0}, \varepsilon''_{i0}, \varepsilon'_{i2}, \varepsilon''_{i2}, \varepsilon_{i3}$ can be performed in time $\Theta(M_z)$, section 3.2. A high level overview of the updating algorithm is provided in Algorithm 1, Algorithm 2, Algorithm 3 and Algorithm 4. For brevity, functions denoted by $f(\cdot)$ are omitted within the algorithm and can be found by the definition of the corresponding variables already provided in the previous sections.

3.3.1.3 Locally oriented tuning of analytic Navigation Functions

Not all constraints need to become effective for provably correct navigation. When an obstacle is discovered, an ε_i can be arbitrarily selected. If used in $N(\varepsilon_{I_z})$, then critical points remain only within $\mathscr{B}_i(\varepsilon_i)$. As long as the agent does not enter $\mathscr{B}_i(\varepsilon_i)$, although updated, $\varepsilon'_{i0}, \varepsilon''_{i0}, \varepsilon'_{i2}, \varepsilon''_{i2}, \varepsilon_{i3}$ need not be applied. This is equivalent to adding "and $\beta_i < \varepsilon_i$" to line 3 of UPDATE k_z.

If for arbitrary ε_i local minima remain within $\mathscr{B}_i(\varepsilon_i)$ and attract the agent, it will eventually enter $\mathscr{B}_i(\varepsilon_i)$. We check this entrance and then apply the calculated constraint $\varepsilon_i < \varepsilon_{ui}$, ensuring those local minima within $\mathscr{B}_i(\varepsilon_i)$ become saddles.

This means a local minimum may still remain close to that obstacle. Its attraction can lead the agent within $\mathscr{B}_i(\varepsilon_i)$. By calculating β_i we can check when it gets within $\mathscr{B}_i(\varepsilon_i)$. Then the maintained constraints become effective, changing ε_i to clear that neighbourhood of local minima.

If we leave the neighbourhood and then discover another obstacle, the previous constraints are updated, as detailed earlier. But the updated values do not become effective unless the agent is lead back within that neighbourhood a second time. This is not prob-

Algorithm 2 Update ε of already known obstacles

procedure update $\varepsilon_{iu}, \varepsilon''_{i0}, \varepsilon''_{i2}, \varepsilon_{i3}$
    for $i \in I_z$ **do**
        $\varepsilon_{i3n} \leftarrow (\|q_i - q_n\| - \rho_n)^2 - \rho_i^2$
        if $\varepsilon_{i3n} < \varepsilon_{i3}$ **then**
            $\varepsilon_{i3} \leftarrow \lambda\varepsilon_{i3n}, \quad \lambda \in (0,1)$
        end if
        if $\varepsilon_{i3} < \varepsilon_{i23}$ **then**
            $\varepsilon_{i23} \leftarrow \varepsilon_{i3}$
        end if
        $\beta_{ni}^{\min} \leftarrow f(\varepsilon_{i23})$
        $Q_{ni} \leftarrow f\left(\beta_{ni}^{\min}\right)$
        $a_{i21} \leftarrow \frac{1}{\beta_{ni}^{\min}} + a_{i21}$
        $a_{i22} \leftarrow 2Q_{ni}a_{i23} + a_{i22}$
        $a_{i23} \leftarrow Q_{ni} + a_{i23}$
        $\varepsilon''_{i2} \leftarrow \frac{\rho_i}{\sqrt{2(a_{i21}+4a_{i22})}}$
        if $\varepsilon_{i3} < \varepsilon_{i03}$ **then**
            $\varepsilon_{i03} \leftarrow \varepsilon_{i3}$
        end if
        $a_{i01} \leftarrow \frac{1}{\beta_{ni}^{\max,\text{new}}} + a_{i01}$
        $a_{i02} \leftarrow 2Q_{ni}^{\text{new}}a_{i03} + a_{i02}$
        $a_{i03} \leftarrow Q_{ni}^{\text{new}} + a_{i03}$
        $\varepsilon_{i0}^{\prime\prime\text{new}} \leftarrow \frac{1}{2\frac{1}{\gamma_{di}^{\min}}a_{i03}+4(a_{i03})^2+4a_{i02}-a_{i01}}$
        $\varepsilon_{iu} \leftarrow \min\left\{\varepsilon_{i23}, \varepsilon_{i03}, \varepsilon''_{i0}, \varepsilon''_{i2}\right\}$
    end for
end procedure

able, since in a sphere world obstacles are convex and for high values of k_z when left behind usually are not encountered further. This scheme reduces the effect of distant obstacles, accounting for the fact that local minima near any obstacles not close to the followed path need never disappear. So smaller k_z values can be achieved.

Algorithm 3 New ε for a newly discovered obstacle

procedure new $\varepsilon_{nu}, \varepsilon'_{n0}, \varepsilon''_{n0}, \varepsilon'_{n2}, \varepsilon''_{n2}, \varepsilon_{n3}, \varepsilon_n$
  $\varepsilon_{n3} \leftarrow \min_{i \in I_z} \left\{ \left(\|q_n - q_i\| - \rho_i \right)^2 - \rho_n^2 \right\}$
  $\varepsilon'_{n2} \leftarrow \frac{1}{2}\rho_n^2$
  $\varepsilon_{n23} \leftarrow \min \left\{ \varepsilon_{n3}, \varepsilon'_{n2} \right\}$
  $\sum Q \leftarrow 0, \quad \sum \frac{1}{\beta} \leftarrow 0$
  for $i \in I_z$ **do**
    $\beta_{in}^{\min} \leftarrow f\left(\varepsilon_{n23}\right)$
    $Q_{in} \leftarrow f\left(\beta_{in}^{\min}\right)$
    $\sum Q \leftarrow \sum Q + Q_{in}$
    $\sum \frac{1}{\beta} \leftarrow \sum \frac{1}{\beta} + \frac{1}{\beta_{in}^{\min}}$
  end for
  $\Sigma_1 \leftarrow 0$
  for $i \in I_z$ **do**
    $\beta_{in}^{\min} \leftarrow f\left(\varepsilon_{n23}\right)$
    $Q_{in} \leftarrow f\left(\beta_{in}^{\min}\right)$
    $\Sigma_1 \leftarrow \Sigma_1 + Q_{in}\left(\sum Q - Q_{in}\right)$
  end for
  $\varepsilon''_{n2} \leftarrow \frac{\rho_n}{\sqrt{2\left(\sum \frac{1}{\beta} + 4\Sigma_1\right)}}$
  $a_{i21} \leftarrow \sum \frac{1}{\beta}, \quad a_{i22} \leftarrow \Sigma_1, \quad a_{i23} \leftarrow \sum Q$
  $\varepsilon'_{n0} \leftarrow \lambda'_{n0}\left(\|q_d - q_n\|^2 - \rho_n^2\right)$
  $\varepsilon_{n03} \leftarrow \min \left\{ \varepsilon_{n3}, \varepsilon'_{n0} \right\}$
  $\gamma_{dn}^{\min} \leftarrow f\left(\varepsilon_{n03}\right)$
  $\sum Q \leftarrow 0, \quad \sum \frac{1}{\beta} \leftarrow 0$
  for $i \in I_z$ **do**
    $\beta_{in}^{\min} \leftarrow f\left(\varepsilon_{n03}\right)$
    $Q_{in} \leftarrow f\left(\beta_{in}^{\min}\right)$
    $\beta_{in}^{\max} \leftarrow f\left(\varepsilon_{n03}\right)$
    $\sum Q \leftarrow \sum Q + Q_{in}$
    $\sum \frac{1}{\beta} \leftarrow \sum \frac{1}{\beta} + \frac{1}{\beta_{in}^{\max}}$
  end for
  $\Sigma_1 \leftarrow 0$
  for $i \in I_z$ **do**
    $\beta_{in}^{\min} \leftarrow f\left(\varepsilon_{i03}\right)$
    $Q_{in} \leftarrow f\left(\beta_{in}^{\min}\right)$
    $\Sigma_1 \leftarrow \Sigma_1 + Q_{in}\left(\sum Q - Q_{in}\right)$
  end for
  $\varepsilon''_{n0} \leftarrow \frac{1}{2 - \frac{1}{\sqrt{\gamma_{dn}^{\min}}} \sum Q + 4\left(\sum Q\right)^2 + 4\Sigma_1 - \sum \frac{1}{\beta}}$
  $a_{i01} \leftarrow \sum \frac{1}{\beta}, \quad a_{i02} \leftarrow \Sigma_1, \quad a_{i03} \leftarrow \sum Q$
  $\varepsilon_{nu} \leftarrow \min \left\{ \varepsilon_{n23}, \varepsilon_{n03}, \varepsilon''_{n2}, \varepsilon''_{n0} \right\}$
  $\varepsilon_n \leftarrow \lambda\left(\|q - q_n\|^2 - \rho_n^2\right), \lambda \in (0, 1)$ ▷ Initialize arbitrarily as half closest distance to that obstacle
end procedure

Algorithm 4 Update k of Navigation Function

procedure update k_z
    for $i = 1 : M_{z+1}$ **do**
        if $\varepsilon_{iu} < \varepsilon_i$ **then** ▷ See subsubsection 3.3.1.3 for "and $\beta_i < \varepsilon_i$"
            $\varepsilon_i^{\text{old}} \leftarrow \varepsilon_i$
            $\varepsilon_i \leftarrow \lambda \varepsilon_{iu}, \quad \lambda \in (0,1)$
            $Q_{ii}^{\text{old}} \leftarrow f(\varepsilon_i^{\text{old}})$
            $Q_{ii}^{\text{new}} \leftarrow f(\varepsilon_i)$
            $\Delta Q_{ii} \leftarrow Q_{ii}^{\text{new}} - Q_{ii}^{\text{old}}$
            $\sum Q_{ii} \leftarrow \sum Q_{ii} + \Delta Q_{ii}$
        end if
    end for
    if $i_{\min} == 0$ **then**
        if $\varepsilon_{0u} < \varepsilon_0$ **then** ▷ See subsubsection 3.3.1.3 for "and $\beta_0 < \varepsilon_0$"
            $\varepsilon_0^{\text{old}} \leftarrow \varepsilon_0$
            $\varepsilon_0 \leftarrow \lambda \varepsilon_{0u}, \quad \lambda \in (0,1)$
            $Q_{00}^{\text{old}} \leftarrow f(\varepsilon_0^{\text{old}})$
            $Q_{00}^{\text{new}} \leftarrow f(\varepsilon_0)$
            $\Delta Q_{00} \leftarrow Q_{00}^{\text{new}} - Q_{00}^{\text{old}}$
            $\sum Q_{ii} \leftarrow \sum Q_{ii} + \Delta Q_{00}$
        end if
        $\sum_{I_0} Q_{ii} \leftarrow \sum Q_{ii}$
        $k_{lb} \leftarrow (\rho_0 + \|q_d\|) \sum_{I_0} Q_{ii}$
    else
        $\sum_{I_1} Q_{ii} \leftarrow \sum Q_{ii}$
        $k_{lb} \leftarrow 1 + \max\left\{\rho_a \sum_{I_1} Q_{ii}, \quad M_z\right\}$
    end if
    $k_{z+1} \leftarrow \max\{2, k_z, k_{lb}\}$
end procedure

3.4 Gradient and normalized gradient descent

3.4.1 Continuous Case

The continuous control law for a holonomic system

$$\dot{q} = -\alpha \frac{\partial V}{\partial q} \tag{3.55}$$

where $\alpha > 0$ and V a Lyapunov function candidate is guaranteed to converge to the goal x_d because

$$\dot{V}(q) = \frac{\partial V}{\partial q}^T \frac{dq}{dt} = \frac{\partial V}{\partial q}^T \left(-\alpha \frac{\partial V}{\partial q}\right) = -\alpha \left\| \frac{\partial V}{\partial q} \right\|^2 < 0 \tag{3.56}$$

because if V is a navigation function, then $\left\| \frac{\partial V}{\partial q} \right\| > 0, \forall q \neq q_d$, so that $\dot{V}(q) < 0, \forall q \neq q_d$ is negatively defined. Aymptotic convergence to the goal q_d is guaranteed by the second Lyapunov method (direct).

The continuous control law for a holonomic system

$$q = -\alpha \frac{\frac{\partial V}{\partial q}}{\left\| \frac{\partial V}{\partial q} \right\|} = -\alpha \frac{\nabla V}{\|\nabla V\|} \tag{3.57}$$

where $\|\nabla V\| \neq 0, \forall q \neq q_d$ if V is a navigation function, $\alpha > 0$ and V a Lyapunov function candidate is also guaranteed to converge to the goal q_d because the integral lines remain the same (no direction change implies no collision with any obstacle) and since

$$\dot{V}(q) = \frac{\partial V}{\partial q}^T \frac{dq}{dt} = \frac{\partial V}{\partial q}^T \dot{q} = \nabla V^T \dot{x} = \nabla V^T \left(-\alpha \frac{\nabla V}{\|\nabla\|}\right) = -\alpha \frac{\|\nabla V\|^2}{\|\nabla V\|} = -\alpha \|\nabla V\| < 0, \forall q \neq q_d \tag{3.58}$$

because $\|\nabla V\| \neq 0 \forall q \neq q_d$, since $V(q)$ is a navigation function. Asymptotic convergence to the goal q_d is guaranteed by the Lyapunov's direct theorem.

From the above we note that using either the gradient field scaled by any positive constant, or the normalized gradient field (unit normal field) scaled by any positive scalar does not affect collision avoidance, nor convergence to the goal. It can be shown that the integral lines remain the same, since the Riemann vector integral $\lim\limits_{n\to+\infty} \sum\limits_{i=1}^{n} \left(\nabla V \frac{\Delta s_i}{\|\nabla V\|}\right) = \lim\limits_{n\to+\infty} \sum\limits_{i=1}^{n} (\nabla V \Delta s'_i)$ because $\exists \lim \|\nabla V\|$.

This is proved in [22], Lemma 7, p.263. and is given here for completeness. Let f_1, f_2 be vector fields on J which differ by a scalar function a, i.e.

$$f_1 = a f_2 \tag{3.59}$$

Then, on the intersection of their respective domains, the flow, F_1^t, generated by f_1 has the relation to the flow, F_2^t, generated by f_2, as follows

$$F_1^t = F_2^{s(t)} \tag{3.60}$$

where $\dot{s} = a$. This fact obtains from simple application of the chain rule

$$\frac{d}{dt} F_2^{s(t)} r_0 = \frac{d}{ds} F_2^{s(t)} r_0 \frac{ds}{dt} = f_2\left(F_2^{s(t)} r_0\right) a = f_1\left(F_2^{s(t)} r_0\right) = \frac{d}{dt} F_1^t r_0 \tag{3.61}$$

3.4.2 Discrete Case (numerical implementation)

The navigation function's numerical implementation aims to find the integral lines of the potential field. The gradient descent is $\dot{q} = -\alpha \nabla V$.

Consider again the previously examined cases of vector velocity fields

$$\dot{q} = -\alpha \nabla V, \quad \dot{q} = -\alpha \frac{\nabla V}{\|\nabla V\|} \tag{3.62}$$

Now let us discuss their (inherently discrete) numerical implementation. The gradient descent evolves in *finite* steps. It has not guaranteed collision avoidance. This is due to the *variable* step size of a gradient descent. If the step in the direction determined by the gradient is taken too large, then a collision can occur.

The position change

$$\Delta q_{i \to i+1} = q_{i+1} - q_i = (q_i + (-\alpha \nabla V)) - q_i = -\alpha \nabla V \implies \|\Delta q_{i \to i+1}\|_2 = \|-\alpha \nabla V\|_2 \overset{\alpha \geq 0}{=} \alpha \|\nabla V\|_2 \tag{3.63}$$

depends on $\|\nabla V\|_2$, which can vary in such a way that $\|\Delta x_{i \to i+1}\|$ becomes too large and leads to a collision[7].

On the contrary the discrete control law

$$\dot{q} = -\alpha \frac{\nabla V}{\|\nabla V\|} \implies \|\Delta q_{i \to i+1}\|_2 = \left\| -\alpha \frac{\nabla V}{\|\nabla V\|} \right\|_2 = \alpha \frac{\|\nabla V\|}{\|\nabla V\|} = \alpha \tag{3.64}$$

which enables control of linear speed. By selecting α we are able to set a *constant* step size

But the above step size is *fixed*. Therefore, although not arbitrarily variable and determined the variation of $\|\nabla V\|$, nonetheless it remains inadequate to ensure collision avoidance[8].

A solution to this is an *adaptive* step size. This is accomplished by the discrete control law

$$\dot{q} = -\alpha(q) \frac{\nabla V}{\|\nabla V\|} \tag{3.65}$$

where $\alpha : E^n \to \mathbb{R}$ is the adaptive step size. Let us select

$$\alpha(q) = \begin{cases} \lambda \min\limits_{i \in I_0} \{|\|q - q_i\| - \rho_i^2|\}, & \min\limits_{i \in I_0} \{|\|q - q_i\| - \rho_i^2|\} < d_{\text{threshold}} \\ d_{\text{threshold}}, & \min\limits_{i \in I_0} \{|\|q - q_i\| - \rho_i^2|\} \geq d_{\text{threshold}} \end{cases} \tag{3.66}$$

with $\lambda \in (0, 1)$. The expressions

$$\rho_0 - \|q\|, \quad \|q - q_i\| - \rho_i, \quad i \in I_1 \tag{3.67}$$

The function

$$\min_{i \in I_0} \left\{ \left| \|q - q_i\| - \rho_i^2 \right| \right\} = \min \left\{ \rho_0^2 - \|q\|, \min_{i \in I_1} \left\{ \|q - q_i\| - \rho_i^2 \right\} \right\} \tag{3.68}$$

[7] Remember: finite step sizes, no continuous update of ∇V here.

[8] The step size in this case can be set small enough to avoid collisions for the *particular* integral path to be found. But for this to be accomplished *a priori*, we need to know the details of the path. Since the path has not been found yet, this is not possible.

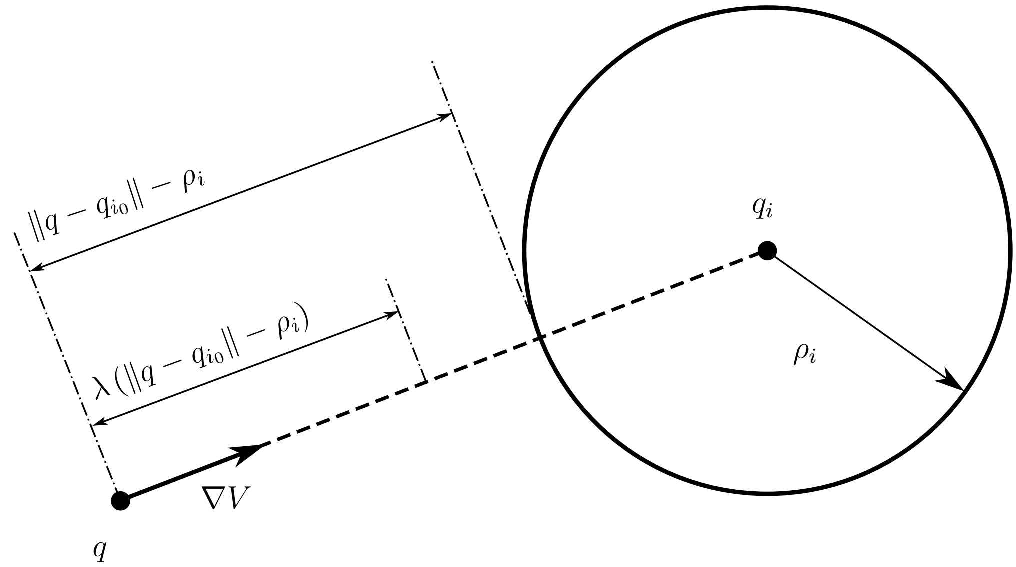


Figure 3.3: Worst case of gradient direction, toward the closest obstacle i_0, shown for the case when the closest obstacle is an internal one, i.e. $i_0 \neq 0$.

is the minimum distance to the closest obstacle. Multiplication by λ yields a step size smaller than the distance to the closest obstacle. Even in the worst case, when the gradient direction $\frac{\nabla V}{\|\nabla V\|}$ points directly to the nearest obstacle i_0 the step size will be

$$\lambda \left| \|q - q_{i_0}\| - \rho_i \right| < \left| \|q - q_{i_0}\| - \rho_i \right| \tag{3.69}$$

the distance to the closest obstacle, Fig. 3.3. This guarantees collision avoidance. It also prevents the step size to increase too much and affect the numerical approximation to the navigation function's potential field integral lines.

But for a discrete implementation convergence cannot be perfect and should be prescribed to a certain error margin. When

It has been shown that the normalized vector field $\frac{\nabla V}{\|\nabla V\|}$ yields the same continuous solutions (paths). By implementing it iwth an appropriate adaptive step it can be numerically calculated for abruptly changing navigation function fields (large k).

This implementation avoids the need to calculate expressions where k arises as in the exponent. Such calculations are not possible when a large lower bound $N(\varepsilon_{I_0})$ on k is caclulated. A further advantage the normalized expression offers is a simple formula where k arises only in a single place as a divisor and its effect on the potential field is clearly deduced.

3.5 Gradient normalization

In this section the gradients of both $\hat{\phi}$ and ϕ are normalized. The second is the one needed. Derivation of the first though is somewhat simpler and guides the second one in concept because the same terms arise. For a detailed derivation of $\nabla\hat{\varphi}, \nabla\varphi$ see subsection A.3.7 and subsection A.3.10.

The normalized gradient of $\hat{\varphi}$ is

$$\begin{aligned}
\frac{\nabla\hat{\varphi}}{\|\nabla\hat{\varphi}\|} &= \frac{\frac{1}{\beta^2}\left[k\beta\gamma_d^{k-1}\nabla\gamma_d - \gamma_d^k\nabla\beta\right]}{\left\|\frac{1}{\beta^2}\left[k\beta\gamma_d^{k-1}\nabla\gamma_d - \gamma_d^k\nabla\beta\right]\right\|} = \frac{\gamma_d^k\left[k\beta\gamma_d^{-1}\nabla\gamma_d - \nabla\beta\right]}{\gamma_d^k\left\|\left[k\beta\gamma_d^{-1}\nabla\gamma_d - \nabla\beta\right]\right\|} \\
&= \frac{\frac{k\beta}{\gamma_d}\nabla\gamma_d - \nabla\beta}{\left\|\frac{k\beta}{\gamma_d}\nabla\gamma_d - \nabla\beta\right\|} = \frac{\frac{\nabla\gamma_d}{\gamma_d} - \frac{1}{k}\frac{\nabla\beta}{\beta}}{\left\|\frac{\nabla\gamma_d}{\gamma_d} - \frac{1}{k}\frac{\nabla\beta}{\beta}\right\|} = \frac{\nabla\gamma_d - \frac{\gamma_d}{k\beta}\nabla\beta}{\nabla\gamma_d - \frac{\gamma_d}{k\beta}\nabla\beta} \\
&= \frac{\nabla\gamma_d - \frac{\gamma_d}{k}\sum\limits_{i=0}^{M}\frac{\nabla\beta_i}{\beta_i}}{\left\|\nabla\gamma_d - \frac{\gamma_d}{k}\sum\limits_{i=0}^{M}\frac{\nabla\beta_i}{\beta_i}\right\|} = \frac{2(q-q_d) - \frac{\|q-q_d\|^2}{k}\sum\limits_{i=0}^{M}\frac{2(q-q_i)}{\left|\|q-q_i\|^2-\rho_i^2\right|}}{\left\|2(q-q_d) - \frac{\|q-q_d\|^2}{k}\sum\limits_{i=0}^{M}\frac{2(q-q_i)}{\left|\|q-q_i\|^2-\rho_i^2\right|}\right\|} \\
&= \frac{(q-q_d) - \frac{\|q-q_d\|^2}{k}\sum\limits_{i=0}^{M}\frac{(q-q_i)}{\left|\|q-q_i\|^2-\rho_i^2\right|}}{\left\|(q-q_d) - \frac{\|q-q_d\|^2}{k}\sum\limits_{i=0}^{M}\frac{(q-q_i)}{\left|\|q-q_i\|^2-\rho_i^2\right|}\right\|}
\end{aligned} \tag{3.70}$$

The normalized gradient of φ is

$$\begin{aligned}
\frac{\nabla\varphi}{\|\nabla\varphi\|} &= \frac{\frac{1}{\left(\sqrt[k]{\gamma_d^k+\beta}\right)^2}\left[\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\nabla\gamma_d - \gamma_d\nabla\left(\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\right)\right]}{\left\|\frac{1}{\left(\sqrt[k]{\gamma_d^k+\beta}\right)^2}\left[\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\nabla\gamma_d - \gamma_d\nabla\left(\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\right)\right]\right\|} \\
&= \frac{\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\nabla\gamma_d - \gamma_d\nabla\left(\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\right)}{\left\|\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\nabla\gamma_d - \gamma_d\nabla\left(\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\right)\right\|}
\end{aligned} \tag{3.71}$$

Note that

$$\begin{aligned}
\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\nabla\gamma_d - \gamma_d\nabla\left(\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\right) &= \left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\nabla\gamma_d - \gamma_d\frac{1}{k}\left(\gamma_d^k+\beta\right)^{\frac{1}{k}-1}\nabla\left(\gamma_d^k+\beta\right) \\
&= \left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\nabla\gamma_d - \gamma_d\frac{1}{k}\left(\gamma_d^k+\beta\right)^{\frac{1}{k}-1}\left[\nabla\left(\gamma_d^k\right)+\nabla\beta\right] \\
&= \left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\nabla\gamma_d - \gamma_d\frac{1}{k}\left(\gamma_d^k+\beta\right)^{\frac{1}{k}-1}\left[k\gamma_d^{k-1}\nabla\gamma_d+\nabla\beta\right] \\
&= \left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\nabla\gamma_d - \gamma_d\frac{1}{k}\left(\gamma_d^k+\beta\right)^{\frac{1}{k}-1}k\gamma_d^{k-1}\nabla\gamma_d - \gamma_d\frac{1}{k}\left(\gamma_d^k+\beta\right)^{\frac{1}{k}-1}\nabla\beta \\
&= \left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\nabla\gamma_d - \left(\gamma_d^k+\beta\right)^{\frac{1}{k}-1}\gamma_d^k\nabla\gamma_d - \frac{1}{k}\gamma_d\left(\gamma_d^k+\beta\right)^{\frac{1}{k}-1}\nabla\beta \\
&= \left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\left[\nabla\gamma_d - \left(\gamma_d^k+\beta\right)^{-1}\gamma_d^k\nabla\gamma_d - \frac{1}{k}\gamma_d\left(\gamma_d^k+\beta\right)^{-1}\nabla\beta\right]
\end{aligned} \tag{3.72}$$

Now note that

$$\begin{aligned}\nabla\gamma_d-\left(\gamma_d^k+\beta\right)^{-1}\gamma_d^k\nabla\gamma_d-\frac{1}{k}\gamma_d\left(\gamma_d^k+\beta\right)^{-1}\nabla\beta&=\left(1-\frac{\gamma_d^k}{\gamma_d^k+\beta}\right)\nabla\gamma_d-\frac{1}{k}\frac{\gamma_d}{\gamma_d^k+\beta}\nabla\beta\\&=\left(\frac{\gamma_d^k+\beta-\gamma_d^k}{\gamma_d^k+\beta}\right)\nabla\gamma_d-\frac{1}{k}\frac{\gamma_d}{\gamma_d^k+\beta}\nabla\beta\\&=\frac{\beta}{\gamma_d^k+\beta}\nabla\gamma_d-\frac{1}{k}\frac{\gamma_d}{\gamma_d^k+\beta}\nabla\beta\\&=\frac{1}{\gamma_d^k+\beta}\left[\beta\nabla\gamma_d-\frac{1}{k}\gamma_d\nabla\beta\right]\end{aligned}\tag{3.73}$$

Combining these expressions

$$\begin{aligned}\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\nabla\gamma_d-\gamma_d\nabla\left(\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\right)&=\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\left[\nabla\gamma_d-\left(\gamma_d^k+\beta\right)^{-1}\gamma_d^k\nabla\gamma_d-\frac{1}{k}\gamma_d\left(\gamma_d^k+\beta\right)^{-1}\nabla\beta\right]\\&=\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\left(\gamma_d^k+\beta\right)^{-1}\left[\beta\nabla\gamma_d-\frac{1}{k}\gamma_d\nabla\beta\right]\\&=\left(\gamma_d^k+\beta\right)^{\frac{1}{k}-1}\left[\beta\nabla\gamma_d-\frac{1}{k}\gamma_d\nabla\beta\right]\end{aligned}\tag{3.74}$$

As a result, the normalization of the gradient now yields

$$\begin{aligned}\frac{\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\nabla\gamma_d-\gamma_d\nabla\left(\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\right)}{\left\|\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\nabla\gamma_d-\gamma_d\nabla\left(\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\right)\right\|}&=\frac{\left(\gamma_d^k+\beta\right)^{\frac{1}{k}-1}\left[\beta\nabla\gamma_d-\frac{1}{k}\gamma_d\nabla\beta\right]}{\left\|\left(\gamma_d^k+\beta\right)^{\frac{1}{k}-1}\left[\beta\nabla\gamma_d-\frac{1}{k}\gamma_d\nabla\beta\right]\right\|}\\&=\frac{\beta\nabla\gamma_d-\frac{1}{k}\gamma_d\nabla\beta}{\left\|\beta\nabla\gamma_d-\frac{1}{k}\gamma_d\nabla\beta\right\|}\\&=\frac{\nabla\gamma_d-\frac{\gamma_d}{k\beta}\nabla\beta}{\left\|\nabla\gamma_d-\frac{\gamma_d}{k\beta}\nabla\beta\right\|}\end{aligned}\tag{3.75}$$

So for $\nabla\beta$ to become effective $k\beta\approx\gamma_d$ and since $\gamma_d\in[0,4\rho_0^2]$ it should be that $\beta\approx\frac{4\rho_0^2}{k}$. For $k\approx 10^n\implies\beta\approx 4\rho_0^2 10^{-n}$.

3.6 Convergence in Unknown Sphere World

Theorem 4. Let $\mathscr{M}$ be a valid sphere world whose sphere obstacles are initially unknown. Let $\mathscr{S}(t)$ the agent's sensing set at time t and assume T_s small enough for the agent to remain in sensed $\bigcup_m \mathscr{S}(t_m)$. If a NF can be found for each intermediate space as obstacles are discovered then the agent converges to the destination q_d.

Proof. At each new sensing time t_m the NF is updated, incorporating newly discovered obstacles. Let $I_a \triangleq \{i_1, i_2, \ldots, i_M\}$ the set of indices of all, known and unknown, obstacles. Let $I_b \subseteq I_a$ the subset of M_z until then discovered obstacles. The NF is defined on a sphere world $\mathscr{M}_z \triangleq E^n \backslash \bigcup_{I_a} \mathscr{O}_i$ comprising of only the until then known obstacles, hence $\mathscr{M}_z \subseteq \mathscr{M}$.

The partially known $\mathscr{M}_z$ is a valid sphere world. Following the adjusted NF on $\mathscr{M}_z$ on it the agent converges to q_d. This is guaranteed by the properties of a NF. Along its trajectory two alternatives exist. Either no new obstacle is discovered and the agent converges, or at least one new obstacle is discovered.

A new obstacle is discovered when $\mathscr{S}(t) \cap \mathscr{O}_i \neq \emptyset$. Because $\mathscr{S}(t)$ is open this is only possible when more than a single point of $\mathscr{O}_i$ can be sensed. Therefore part of the obstacle's spherical boundary is sensed. By hypothesis of an unknown sphere world the radius of curvature ρ_i and center q_i can be found, defining the new sphere obstacle.

Since a NF is updated and followed in the explored sphere world, the only alternative for the agent to not converge is to indefinitely discover new obstacles which change its NF. Each discovered obstacle increases I_b by one, reducing the set of unknown obstacles $I_a \setminus I_b$ by one. By hypothesis a finite number of unknown obstacles exist, so either the agent converges before discovering all of them, or after a finite number of changes, its NF remains constant because all existing obstacles have been sensed and constraints applied. So in all cases the agent converges to q_d. $\square$

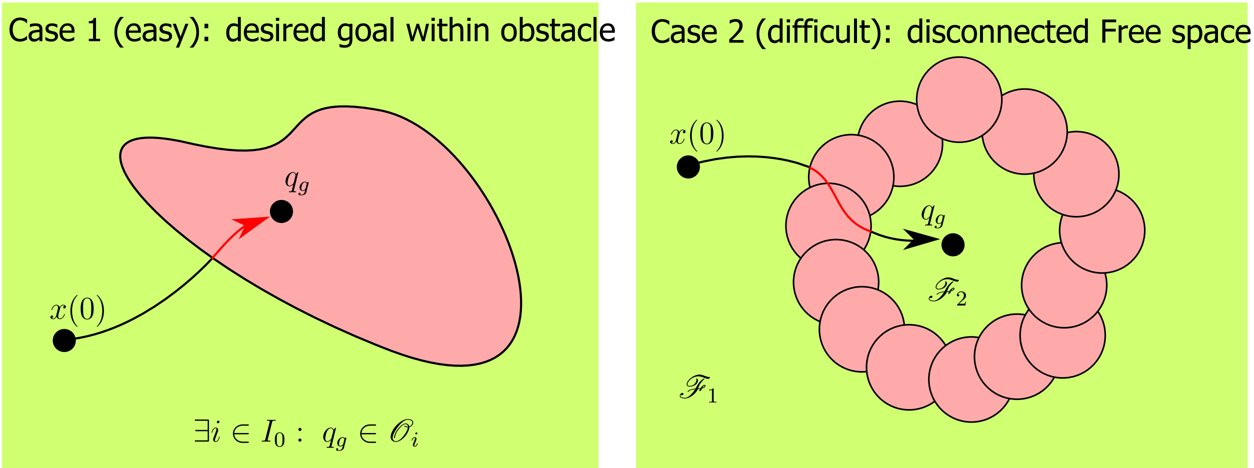


Figure 3.4: Two cases for an unreachable destination q_d.

3.7 Unreachable destination

There are two distinct cases for which the destination q_d is not reachable[9], as shown in Fig. 3.4. The first one is when the destination is within an obstacle. Since the NF methodology is built on the concept of set membership and implicit obstacle functions are used for this purpose, answering this question is quite straightforward. It suffices to check whether the destination belongs to any obstacle set. This is true if and only if $\beta_i(q_d) < 0$ (not necessarily $\beta(q_d) < 0$ is true if intersecting obstacles exist).

On the contrary, it can happen[10] that intersecting obstacles isolate some part of the C-space, by forming a shielding component.

This is not the case for us, because we have assumed that the unknown sphere world is guaranteed to be valid, which requires that the spherical obstacles be disjoint. Giving an answer to the second question constitutes a challenging search problem, because it does not ask for a single feasible answer, but inexistence of any feasible path.

The most direct way of answering this question is actually running a provably correct algorithm and in case it converges to $q' \neq q_d$ then the free space is disconnected[11], Fig. 3.5.

Since this is not our case we are going to analyze the first case and justify the check within the algorithm. Suppose $q_d \in \mathscr{O}_i$, then by definition $\beta_i(q) < 0$.

Proposition 5. In a world in which unions of nonpositive level sets of implicit functions represent obstacle sets

$$\begin{aligned} &\{\exists i \in I_0 : q_g \in \mathscr{O}_i \iff \exists i : \beta_i(q_g) < 0\} \iff \\ &\{q_g \in E^n \setminus \mathscr{F} \iff \exists i : \beta_i(q_g) < 0\} \end{aligned} \tag{3.76}$$

Proof. There are some interesting remarks to be made with respect to these relations. Firstly note that generally

$$\exists i : \beta_i(q_g) < 0 \not\Longleftrightarrow \beta(q_g) < 0 \tag{3.77}$$

(so $q_g \in \mathscr{O}_i \iff q_g \in E^n \setminus \mathscr{F} \not\Longleftrightarrow \beta(q_g) < 0$) because it may be the case that two intersecting obstacles $\mathscr{O}_i, \mathscr{O}_j$ include the desired final point q_g (no more called the "destination",

[9]Equivalently no continuous path between $x(0)$ and q_d exists, or, in other words, the C-space is not path connected.

[10]Especially when exploring an unknown world which is *not* a priori guaranteed to be a valid sphere world.

[11]This requires a critical point searching algorithm in general worlds.

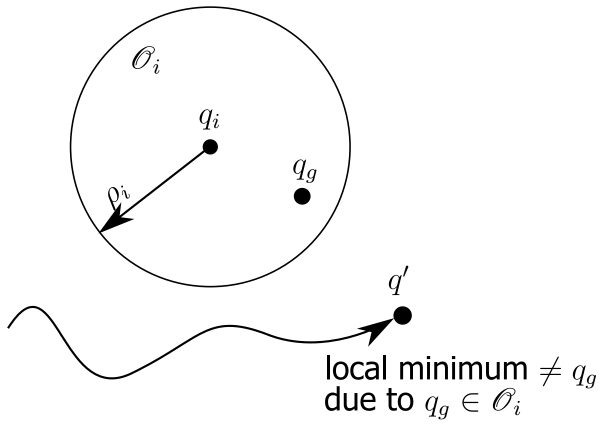


Figure 3.5: Running a provably correct algorithm which converges to $q' \neq q_d$ reveals a disconnected free space.

since it is unattainable), so in such a case

$$\left.\begin{array}{r}\beta_i(q_g) < 0 \wedge \beta_j(q_g) < 0 \\ \beta_k(q_g) > 0, \forall k \in I_o \setminus \{i,j\}\end{array}\right\} \implies \beta(q_g) = \underbrace{\beta_i(q_g)}_{<0}\underbrace{\beta_j(q_g)}_{<0}\underbrace{\prod_{k \in I_o \setminus \{i,j\}} \beta_k(q_g)}_{>0} > 0 \tag{3.78}$$

which continues to hold if q_g belongs to an even number of intersecting obstacles $2r, r \in \mathbb{N}^*$, i.e.

$$\left.\begin{array}{r}\beta_i(q) < 0, \forall i \in I_g, |I_g| = 2r, r \in \mathbb{N}^* \\ \beta_k(q_g) > 0, \forall k \in I_o \setminus I_g\end{array}\right\} \implies \beta(q_g) = \prod_{i \in I_g} \beta_i(q_g) \prod_{k \in I_o \setminus I_g} \beta_k(q_g) > 0 \tag{3.79}$$

But for pairwise disjoint obstacles, as is the case of an (unknown) valid sphere world, no obstacle functions can be simultaneously nonpositive. This is equivalent to

$$\begin{array}{l}\{\nexists q : \beta_i(q) < 0 \wedge \beta_j(q) < 0, i \in I_0, j \in I_0 \setminus \{i\}\} \implies \\ \{\beta_i(q) < 0, i \in I_0 \implies \beta_j(q) \geq 0, \forall j \in I_0 \setminus \{i\}, \forall q \in E^n\}\end{array} \tag{3.80}$$

$$\mathscr{O}_i \cap \mathscr{O}_j = \emptyset, \forall i,j \in I_0, i \neq j \implies \beta_j(q_g) > 0, \forall j \in I_0 \setminus \{i\} \tag{3.81}$$

$$\exists i \in I_0 : q_g \in \mathscr{O}_i \iff \beta_i(q_g) < 0 \tag{3.82}$$

Then

$$\beta(q_g) = \underbrace{\beta_i(q_g)}_{<0}\underbrace{\prod_{j \in I_0 \setminus \{i\}} \beta_j(q_g)}_{>0} < 0 \tag{3.83}$$

□

Also note that in the NF methodology the destination is not allowed to be selected on the free space boundary $\partial\mathscr{F}$. If that was allowed, then

$$\left.\begin{array}{r}\gamma_d(q_g) = 0 \overset{k \in \mathbb{N}\setminus\{0,1\}}{\iff} \gamma_d^k(q_g) = 0 \\ q_g \in \partial\mathscr{F} \iff \exists i : \beta_i(q_g) = 0 \iff \beta(q_g) = 0\end{array}\right\} \implies \gamma_d^k(q_g) + \beta(q_g) = 0$$
$$\overset{k \in \mathbb{N}\setminus\{0,1\}}{\implies} \left(\gamma_d^k(q_g) + \beta(q_g)\right)^{\frac{1}{k}} = 0 \tag{3.84}$$

so the NF denominator will be zero, leading to an undefined φ at the destination q_g (and an undefined $\hat{\varphi}$ as well).

So the intersection of the obstacle closures $\bar{\mathscr{O}}_i \cap \bar{\mathscr{O}}_j$ should be used in condition $\bar{\mathscr{O}}_i \cap \bar{\mathscr{O}}_j = \emptyset$ of non-intersecting obstacle closures. The reason is that obstacles are defined as open sets and their boundary belongs to the free space $\mathscr{F}$. This is for technical reasons, to make $\mathscr{F}$ a manifold with boundary.

For the above reason we check whether in a world with implicit obstacles

$$\exists i \in I_0 : q_g \in \mathscr{O}_i \iff \exists i : \beta_i(q_g) \leq 0 \tag{3.85}$$

where also $\exists i \in I_0 : q_g \in \mathscr{O}_i \iff q_g \in E^n \bar{\setminus} \mathscr{F}$. The above can be stated for worlds with non-intersecting obstacle closures (of which sphere worlds are a special case) as

$$\begin{aligned} q_g \in E^n \bar{\setminus} \mathscr{F} \implies \exists i \in I_0 : q_g \in \bar{\mathscr{O}}_i \iff \\ \exists i : \beta_i(q_g) \leq 0 \overset{\bar{\mathscr{O}}_i \cap \bar{\mathscr{O}}_j = \emptyset, \forall i \in I_0, j \in I_0 \setminus \{i\}}{\iff} \beta(q_g) \leq 0 \end{aligned} \tag{3.86}$$

While exploring a sphere world it suffices to check whether $\beta_n(q_g) > 0, \forall n \in I_0$ where n is the index of a newly discovered obstacle.

Another interesting remark for the cas eof sphere worlds is that (caution, not $q_g \in \bar{\mathscr{O}}_i \implies \gamma_d(q_i) \leq 0$)

$$q_g \in \mathscr{O}_i \iff \|q_g - q_i\| < \rho_i \iff \|q_g - q_i\|^2 < \rho_i^2 \iff \gamma_d(q_i) < \rho_i^2 \tag{3.87}$$

Also $\beta_i(q_i) = \|q_g - q_i\|^2 - \rho_i^2 = -\rho_i^2$ therefore combining these equations we get

$$\begin{aligned} \left.\begin{matrix} \beta_i(q_i) = -\rho_i^2 \\ 0 \leq \gamma_d(q_i) < \rho_i^2 (1) \end{matrix}\right\} \implies -\rho_i^2 \leq \gamma_d(q_i) + \beta_i(q_i) < \rho_i^2 - \rho_i^2 = 0 \iff \\ -\rho_i^2 \leq \gamma_d(q_i) + \beta(q_i) < 0 (2) \end{aligned} \tag{3.88}$$

combining equations (1) and (2) we obtain

$$\frac{\gamma_d(q_i)}{\gamma_d(q_i) + \beta(q_i)} < 0 \tag{3.89}$$

which is interesting, note though that it is *not* φ, *neither* φ without incorporation of other obstacles, because the tuning parameter $k = 1 \notin \mathbb{N} \setminus \{0, 1\}$.

3.8 Simulation Results

The proposed method has been simulated. In Fig. 3.6 navigation in an unknown 2d sphere world with automatically tuned parameter k_z is compared to using manually selected constant $k = 2$ (top) and $k = 10$ (middle). The sensing set is spherical. As $\mathscr{O}_0$ and internal obstacles are gradually discovered, the analytic NF is updated. While a constant k leads to abrupt turns and failure to converge to q_d, use of an updating k_z results in safe and successful navigation, as theoretically guaranteed, with smoother and shorter path. The changing gradient field reveals that the high k_z calculated shapes a NF field which repels only close to obstacles.

A simulation on a 3-dimensional unknown sphere world Fig. 3.7 illustrates applicability to any dimension, a strong advantage of the NF methodology. The adjustive algorithm finds a direct path as guaranteed. Constraints become effective only close to an obstacle, which can be seen during encounter with the first two obstacles. Again for comparison a path with constant $k = 2$ is shown. In this case, for a constant k the agent converges to q_d, although this is not theoretically guaranteed. The reason of convergence here is the smaller ratio of space occupied by obstacles than in Fig. 3.6. For constant k the path followed is changing very abruptly when new obstacles are discovered and is not guaranteed to converge.

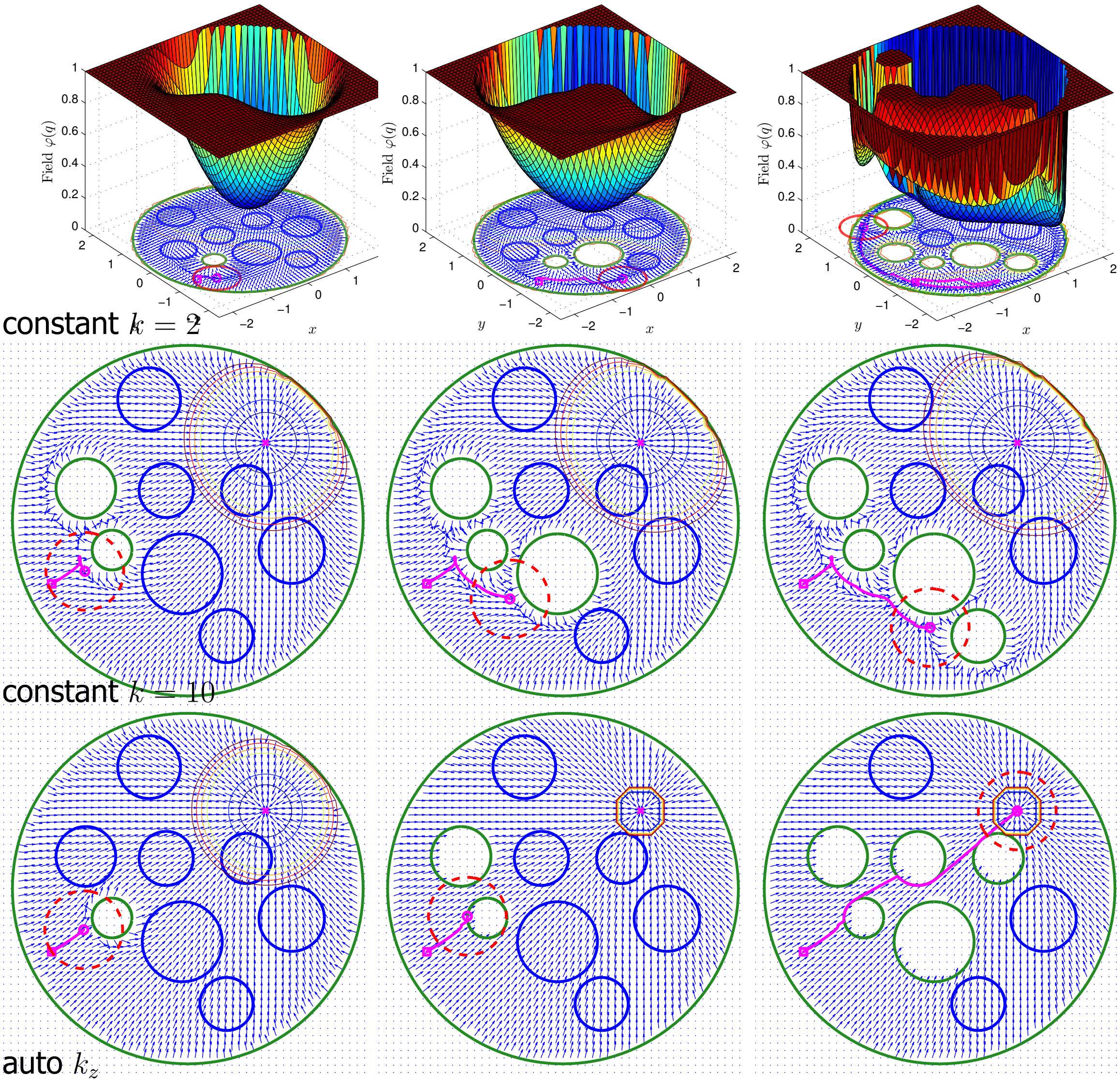


Figure 3.6: Navigation in unknown 2-dimensional sphere world succeeds using automatic k_z (bottom), but fails with constant $k = 2, 10$ (top, middle). Green: sensed, Blue: unsensed obstacles, Red: sensing set.

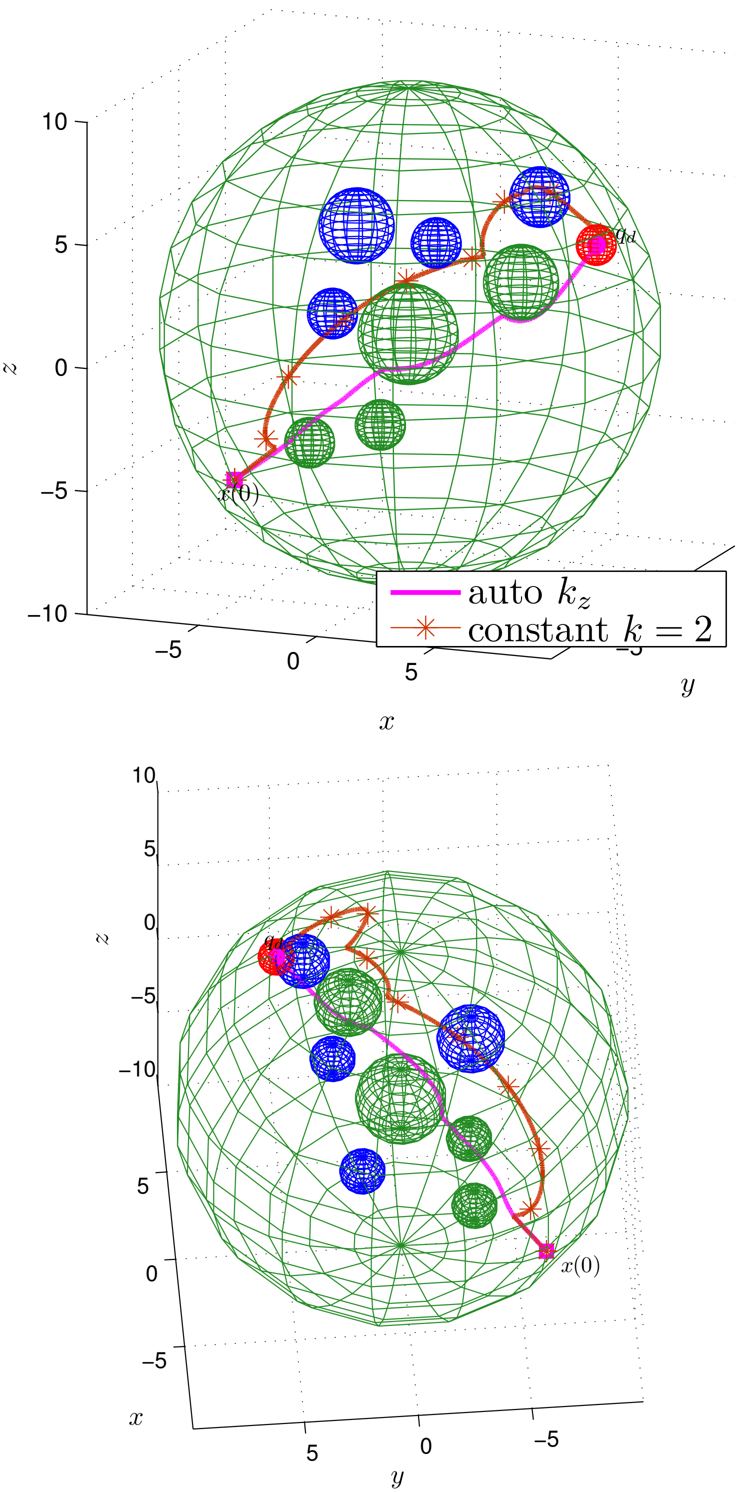


Figure 3.7: Adjustable exploration of unknown 3-dimensional sphere world (2 viewpoints) with constant k and updating k_z. Green: discovered obstacles (auto k_z), Blue: obstacles remaining unsensed (auto k_z), Red: spherical sensing set.

Part II

Navigation Functions for Everywhere Partially Sufficiently Curved Worlds

Chapter 4

Sufficiently Curved Spaces

4.1 Introduction

4.1.1 Necessity of acceptable relative curvature

In this section the proof for sphere worlds [23] is extended to the case of general obstacles β_i. Initially general destination functions γ_d are considered, but due to symmetry considerations, a paraboloid γ_d is selected intermediately.

The sufficient condition associated with $\nu_i(q)$, hence the upper bound ε'_{i0}, is analyzed for general obstacles β_i. This leads to a geometric requirement on the obstacle geometry. It is shown to be a condition on relative level set curvature[1] of β_i and γ_d at a critical point q_c.

For paraboloid γ_d the relative curvature condition obtains a simpler form which suits analysis. It has a particularly interesting and intuitive interpretation. This presentation also relates it to the concepts involved in Meusnier's Theorem [41, 48].

The relative curvature condition depends on the choice of destination $q_d \in \mathscr{F}$. But since destination choice cannot be restricted, it is equivalent to requesting that curvature spheres[2] $\mathscr{S}_{cij}(q)$ be proper subsets of obstacle sets[3] $\mathscr{O}_i$, i.e. $\mathscr{S}_{cij}(q) \subseteq \mathscr{O}_i \cup \{q\}$.

Moreover, the condition is necessary in the following sense. If all principal directions at a point are not sufficiently curved, then two alternatives exist. The first alternative is when all principal curvatures are non-convex. In this case, there exists a $k_{\min}$, such that $\forall k \geq k_{\min}$ if a critical point arises there, it is a local minimum. This precludes use of the same proof procedure. Additionally, it indicates why k tuning alone cannot, in general, make a Kodistchek-Rimon function a Navigation Function in worlds with full non-convexities. For a more detailed discussion, proceed to section 6.3.

On the other hand, the second alternative is when there exists some sufficient principal curvature. Then the NF Hessian has at least one negative eigenvalue. In this case, the critical point is not a local minimum, even when degenerate. It can only be either a saddle, or a local maximum.

Therefore, existence of at least one sufficient principal curvature suffices to ensure

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[1] Relative curvature refers to the relation of level set curvature between the attractive and repulsive fields. If the attractive effect is more curved than the repulsive one, a stable equilibrium (i.e., local minimum) can arise. This minimum can entrap the agent.

[2] A curvature sphere is defined in (4.143) as one tangent to a point q, with center inwardly placed with respect to q and the level set $\beta_i^{-1}(\beta(q))$ through it, and diameter equal to the radius of normal curvature at q.

[3] Note that obstacle sets $\mathscr{O}_i$ are defined as open sets, which do not include their boundary $\partial\mathscr{O}_i$.

that the Navigation Function is Polar (single global minimum at destination). This Polarity is additional to Analyticity and Admissibility (uniformly maximal on free space boundary), both of which are ensured by construction. Note that Propositions 2.7, 3.2 and 3.3 [23] still hold, allowing us to work with the diffeomorphic $\hat{\varphi}$ in $\mathscr{F} \setminus (\partial\mathscr{F} \cup \{q_d\})$.

Furthermore, the existence of at least on sufficiently curved tangent direction suffices to ensure at least one sufficient principal curvature exists. As a result, if the sufficient curvature condition holds for *at least* one tangent direction[4]. Intuitively this corresponds to at least one direction of escape.

But this is not enough to ensure non-degeneracy. Although the result about positive definiteness along $\nabla\beta_i$ of Proposition 3.9 [23] in the case of spheres is extended in section 4.7 to the general case, combining it with negative definiteness along at least one tangential direction is not strong enough. The Hessian may be degenerate.

In the original proof the condition of sufficient curvature is required to hold for all the tangent space. This leads to a direct sum decomposition to two subspaces. In the tangent space negative definiteness is ensured, while in the radial positive definiteness. These suffice by Lemma 3.8 [23] to ensure Hessian non-degeneracy. This is equivalent to local quadratic behavior, so the quadratic form defined by the Hessian can be used to categorize the type of critical point. Considering that the associated quadratic form is continuous in set $\{\hat{v} \in E^n : \|\hat{v}\| = 1\}$ and assumes both negative and positive values, its minima and maxima (which are eigenvalues of the Hessian) are negative and positive respectively, so the critical point is a saddle point.

It is worth noting that existence of at least one direction of negative definiteness and one direction of positive definiteness of the Hessian quadratic form suffice to prove that the critical point is a saddle, even if degenerate [38]. This means that in the general case, sufficient curvature for at least one tangent direction ensures all critical points other than the destination are (possibly degenerate) saddles. Degeneracy is the remaining problem.

Degeneracy means that the function's behavior at a critical point is more complicated than quadratic. Continuity of critical points is possible[5], forming critical sets[6]. Critical sets may be smooth and nondegenerate, in which case Morse-Bott theory applies to them, or non-smooth and possibly degenerate, in which case more general theorems are needed. Another possibility is existence of isolated degenerate critical points, such as a monkey saddle[7], which is illustrated in Fig. 4.1.

Then Morse-Bott theory [37, 40] in combination with Thom's Splitting Lemma [35, 36] can be used to examine the dimensionality of stable sets of degenerate saddle points. In the next chapter it will be proved that if the function has at most one degenerate eigenvalue, then these sets are still of Lebesgue measure zero.

Let us return to the generalization that we make in this chapter. The sufficient curvature condition is less strict than working only with spheres. Spheres satisfy this condition. But other obstacle shapes do so as well.

Requiring that this condition holds along all directions of the tangent space leads to a Navigation Function[8]. This way we can allow obstacle shapes which contain the associated

[4]Tangency is relative to the obstacle level sets implicitly defined by function β_i.

[5]For example due to symmetry, as in the case of a torus. Note that a torus is topologically different from a sphere. This is an important aspect justifying interest in (degenerate) Navigation Functions. Toroidal configuration spaces may arise either due to obstacle topology, or revolute degrees of freedom, as analyzed in chapter 7. More details regarding the thinking behind the original derivation of [23] can be found in [21].

[6]Critical sets are not always submanifolds.

[7][45], pp. 183-204.

[8]Ensuring non-degeneracy, in addition to polarity, analyticity and admissibility.

curvature sphere, at every boundary point.

Examples of such shapes are n-dimensional ellipsoids with an upper bound on eccentricity. The example of ellipses is used here as a demonstration of the theoretical results developed. For eccentricities $e < \sqrt{\frac{1}{2}}$ ellipses satisfy the relative curvature condition. But for greater eccentricities they do not. This also provides an example of shapes that are not acceptable.

4.1.2 World definition

Let $\mathscr{F} \subset E^n$ be a compact connected analytic manifold with boundary, subset of n-dimensional Euclidean space E^n. Each obstacle function β_i is defined on the whole of Euclidean space E^n as the following set membership

$$\beta_i : E^n \to \mathbb{R}, \quad i \in I_0 \triangleq \{0, 1, \ldots, M\}, \quad M \in \mathbb{N} \tag{4.1}$$

It is required to be at least twice continuously differentiable everywhere[9] in free space $\mathscr{F}$

$$\beta_i \in C^{(2)} \left[\mathscr{F},\, [0, +\infty)\right] \tag{4.2}$$

Note that C^2 continuity suffices for the geometric Propositions. Nonetheless, for directly applying Morse-Bott Theory and Thom's Lemma in the next chapter, C^∞ continuity is assumed[10]

The zero level set of β_i defines the obstacle's boundary and its negative coset preimage the obstacle set

$$\begin{aligned} \mathscr{O}_i &\triangleq \{q \in E^n |\, \beta_i(q) < 0\}, \quad \forall i \in I_0 \\ \partial\mathscr{O}_i &\triangleq \{q \in E^n |\, \beta_i(q) = 0\}, \quad \forall i \in I_0 \end{aligned} \tag{4.3}$$

From the range $\mathbb{R}$ of β_i and the above it follows that

$$\beta_i(q) > 0, \quad \forall q \in \mathscr{F} \setminus \partial\mathscr{O}_i \tag{4.4}$$

All obstacle set closures are required to be disjoint[11]

$$\overline{\mathscr{O}_i} \cap \overline{\mathscr{O}_j} = \emptyset, \quad \forall j \in I_0 \setminus \{i\}, \quad \forall i \in I_0 \tag{4.5}$$

and their boundaries $\partial\mathscr{O}_i$ compact[12].

Moreover, we require that no critical points of β_i arise close[13] to obstacle $\mathscr{O}_i$. This is required in a neighborhood $\overline{\mathscr{B}_i(\varepsilon_i)}$ of obstacle $\mathscr{O}_i$

$$\exists \varepsilon_i \in (0, +\infty) : \quad \|\nabla\beta_i(q)\| > 0, \quad \forall q \in \overline{\mathscr{B}_i(\varepsilon_i)}, \quad \forall i \in I_0 \tag{4.6}$$

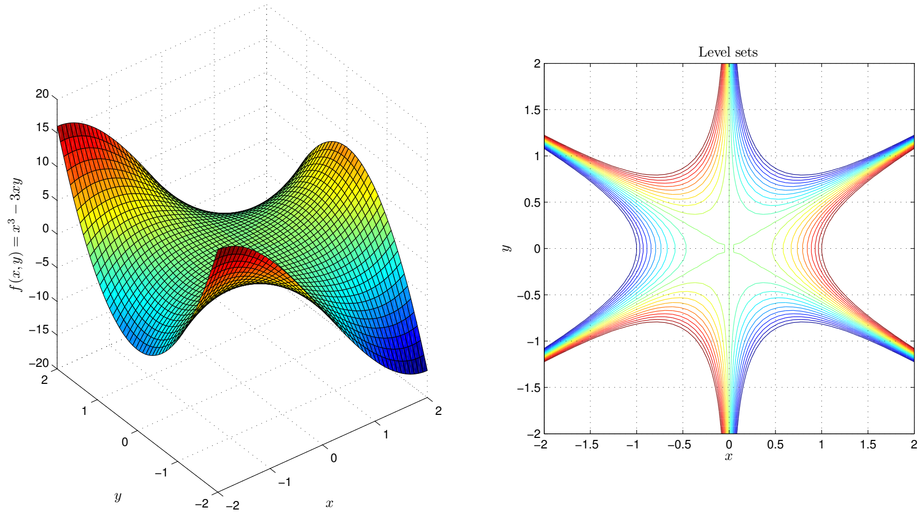
[9]We require C^2 properties everywhere to ensure φ is C^2 everywhere, whereas absence of critical points of its gradient $\nabla\beta_i$ and positive definiteness of its Hessian matrix $D^2\beta_i$ in a neighborhood of $\mathscr{O}_i$ suffices.

[10]Relaxing this is related to the technical details of these Lemmas.

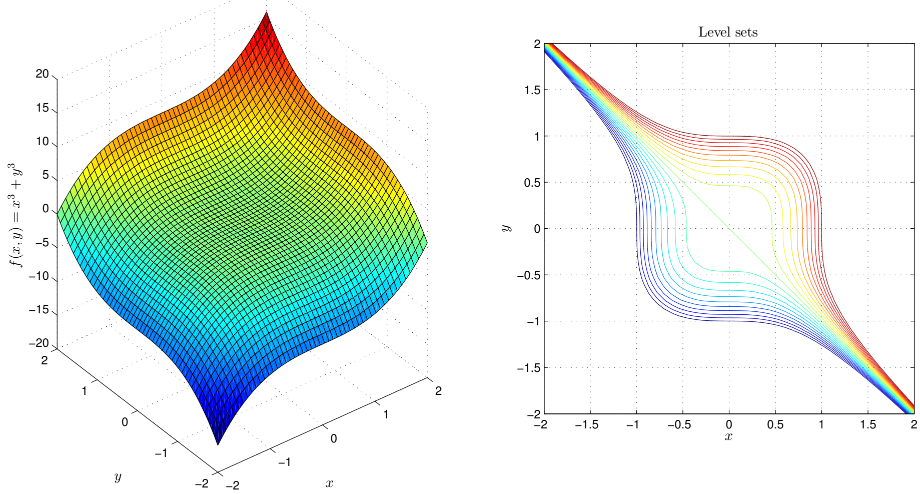
[11]This means that obstacles are not touching. If two or more obstacles $\mathscr{O}_i$ and $\mathscr{O}_j$ touch, then they constitute a single obstacle $\mathscr{O}_m$.

[12]Compact obstacle closure implies that the level sets close to the obstacle are also compact.

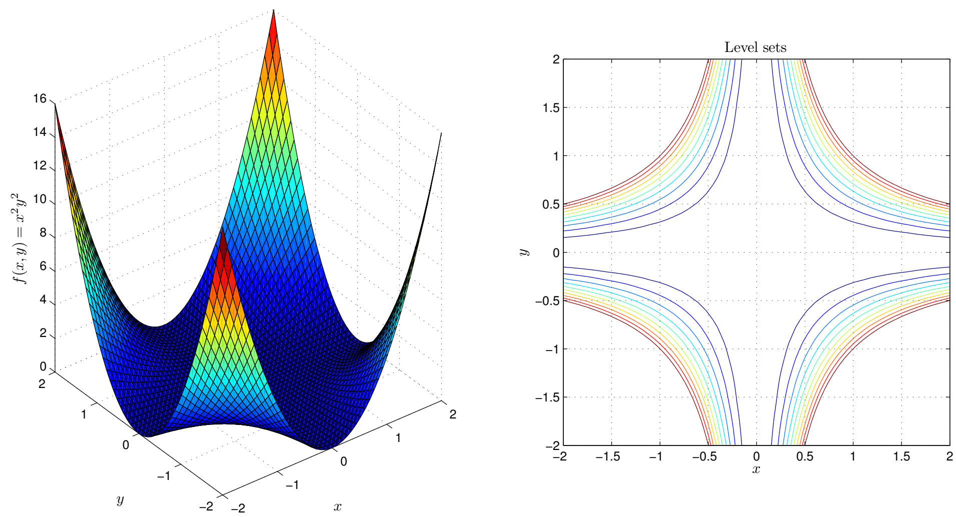
[13]Due to the C^2 property of β_i these requirements "close to obstacle $\mathscr{O}_i$" are equivalent to requiring that they hold on the obstacle's boundary. If they hold on $\partial\mathscr{O}_i$ by C^2 property they extend to an open neighborhood of $\mathscr{O}_i$, so there exists a $\overline{\mathscr{B}_i(\varepsilon_i)}$ in which they hold. However, the converse is also true. According to an extended definition by Rimon and Koditschek [30], obstacles with nonsmooth boundaries are also tractable. In such a case, the requirement applies to the neighborhood and does follow from the boundary properties.



(a) Monkey saddle $f_1(x, y) = x^3 - 3xy^2$



(b) Degenerate with open stable set $f_2(x, y) = x^3 + y^3$



(c) Crossed trough $f_3(x, y) = x^2y^2$

Figure 4.1: All of the above scalar functions f_1, f_2, f_3 have a critical point at the origin and their Hessian matrix is *fully* degenerate there $(D^2z)\left([0\,0]^{\mathrm{T}}\right) = 0_{2\times 2} \in \mathbb{R}^{2\times 2}$. In the first and second cases, the origin is a saddle point, whereas in the third one it is a minimum. But we cannot distinguish between saddle point and minimum based on the Hessian eigenvalues, due to full degeneracy. Also, note that although both f_1 and f_2 are saddle points, f_1 has a stable manifold of Lebesgue measure zero, whereas f_2 has open stable sets.

For the case considered in the present chapter, obstacles should also satisfy the sufficient curvature condition, (4.77), everywhere in a neighborhood of $\mathscr{O}_i$. This condition implies convexity (it is stronger than convexity), i.e., positive definite Hessian matrix

$$\exists \varepsilon_i \in (0, +\infty) : \quad D^2\beta_i(q) > 0, \quad \forall q \in \overline{\mathscr{B}_i(\varepsilon_i)}, \quad \forall i \in I_0 \tag{4.7}$$

This requirement is relaxed in subsequent chapters. Note that initially in this chapter we start without the sufficient curvature requirement and derive it as we proceed. This is the reason for which the above condition is here required from the start.

Obstacle $\mathscr{O}_0$ is called the zero$^{\text{th}}$ obstacle. The whole world, without internal obstacles removed, is a compact connected set

$$\mathscr{W} \triangleq E^n \setminus \mathscr{O}_0 = \{q \in E^n | \, 0 \leq \beta_0(q)\} \tag{4.8}$$

which is bounded by the zero$^{\text{th}}$ obstacle $\mathscr{O}_0$. The $M \in \mathbb{N}$ obstacles

$$\mathscr{O}_i \triangleq \{q \in E^n | \, \beta_i(q) < 0\}, \quad i \in I_1 \triangleq \{1, 2, \ldots, M\} \tag{4.9}$$

are called *internal obstacles*. In the sequel we will refer to both the sets $\mathscr{O}_0$ and their defining functions β_i as "obstacles" interchangeably.

Function γ_d is the destination attractive effect, defined as

$$\begin{gathered} \gamma_d \in C^{(2)}\left[E^n, [0, +\infty)\right] \\ \|\nabla \gamma_d(q)\| > 0, \quad \forall q \in E^n \setminus \{q_d\} \\ D^2\gamma_d(q) > 0, \quad \forall q \in E^n \end{gathered} \tag{4.10}$$

Besides, the specific form of a paraboloid γ_d, which satisfies these conditions, is selected in the course of derivation due to symmetry considerations and in order to enable complete geometric interpretation of the condition.

4.1.3 Navigation Function

The Navigation Function $\varphi : \mathscr{F} \to [0, 1]$ considered here is of the form

$$\varphi \triangleq \frac{\gamma_d}{\left(\gamma_d^k + \beta\right)^{\frac{1}{k}}} \tag{4.11}$$

where $\beta \triangleq \prod_{i \in I_0} \beta_i$ is the aggregate obstacle function and $k \in \mathbb{N} \cap [2, +\infty)$ a tuning parameter. The proof establishes the existence of a sufficient lower bound on k for φ to be a Navigation Function.

Additionally, the following function is defined

$$\hat{\varphi} : \mathscr{F} \setminus \partial\mathscr{F} \to [0, +\infty) \quad \hat{\varphi} \triangleq \frac{\gamma_d^k}{\beta} \tag{4.12}$$

and called the "unsquashed" Navigation Function, defined in the free space interior $\mathscr{F} \setminus \partial\mathscr{F}$.

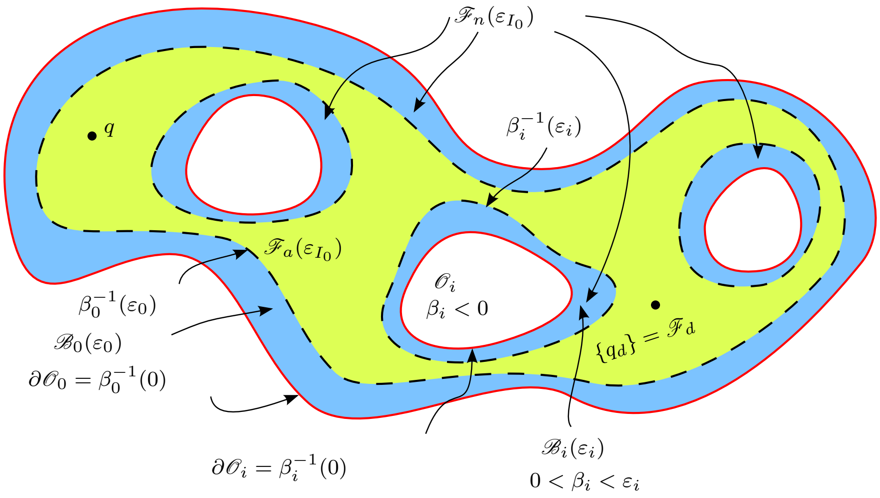


Figure 4.2: Sets defined on a general world.

4.1.4 Definition of world subsets

The following sets are used and illustrated in Fig. 4.2:

1. Destination point

$$\mathscr{F}_d \triangleq \{q_d\}; \tag{4.13}$$

2. Free space boundary

$$\partial\mathscr{F} \triangleq \beta^{-1}(0) = \bigcup_{i\in I_0} \beta_i^{-1}(0); \tag{4.14}$$

3. i^{th} obstacle neighborhood

$$\mathscr{B}_i\left(\varepsilon_i\right) \triangleq \{q \in E^n |\ 0 < \beta_i < \varepsilon_i\}, \quad i \in I_0 \tag{4.15}$$

   and we also require that $\mathscr{B}_i\left(\varepsilon_i\right)$ are pairwise disjoint[14]

$$\begin{aligned} &\mathscr{B}_i\left(\varepsilon_i\right) \cap \mathscr{B}_j\left(\varepsilon_j\right) = \emptyset, \quad \forall j \in I_0 \setminus \{i\}, \quad \forall i \in I_0 \iff \\ &\beta_j(q) \geq \varepsilon_j, \quad \forall q \in \mathscr{B}_i\left(\varepsilon_i\right), \quad \forall j \in I_0 \setminus \{i\}, \quad \forall i \in I_0 \end{aligned} \tag{4.16}$$

   Since obstacle sets $\mathscr{O}_i$ have been defined as pairwise disjoint in (4.5), there always exists a set ε_{I_0} of $0 < \varepsilon_i, i \in I_0$, such that the neighborhoods $\mathscr{B}_i\left(\varepsilon_i\right)$ be pairwise disjoint. In the proof this is addressed by placing the appropriate requirement on the selection of ε_{i3j};
4. "Near" all obstacles (i.e., internal and zero$^{\text{th}}$)

$$\mathscr{F}_n\left(\varepsilon_{I_0}\right) \triangleq \left(\bigcup_{i\in I_0} \mathscr{B}_i\left(\varepsilon_i\right)\right) \setminus \{q_d\}; \tag{4.17}$$

5. Set "away" from all obstacles (i.e., internal and zero$^{\text{th}}$)

$$\mathscr{F}_a\left(\varepsilon_{I_0}\right) \triangleq \mathscr{F} \setminus \left(\mathscr{F}_d\left(\varepsilon_{I_0}\right) \cup \partial\mathscr{F} \cup \mathscr{F}_n\left(\varepsilon_{I_0}\right)\right) \tag{4.18}$$

[14]Note that Koditschek and Rimon enforce this only between their $\mathscr{F}_0$ and the rest $\mathscr{B}_i\left(\varepsilon_i\right), i \in I_1$, by appropriately removing them from $\mathscr{F}_0$ in its definition. This was used in Proposition 3.7, p.432, [23], ensuring that $\beta_i \geq \varepsilon, \forall q \in \mathscr{F}_1(\varepsilon), \forall i \in \{1, \ldots, M\}$. Here this Proposition 3.7 is replaced by a general Proposition which applies to subsets of the neighborhoods of *all* obstacles. This is the reason for which we place this requirement on all obstacles and not only between internal obstacles and the zero$^{\text{th}}$ one.

where $\varepsilon_{I_0} \triangleq \{\varepsilon_i\}_{i \in I_0}$. We define

$$\varepsilon_i, \varepsilon_{iu}, \varepsilon'_{i0}, \varepsilon''_{i0}, \varepsilon'_{i2}, \varepsilon''_{i2}, \varepsilon_{i3j}, \varepsilon_{i3} \triangleq \min_{j \in I_0 \setminus i}\{\varepsilon_{i3j}\}, \varepsilon_{i4}, \varepsilon_{i5}, \quad j \in I_0 \setminus \{i\}, \quad i \in I_0 \tag{4.19}$$

as

$$0 < \varepsilon_i < \varepsilon_{iu} = \frac{1}{2}\min\{\varepsilon'_{i0}, \varepsilon''_{i0}, \varepsilon'_{i2}, \varepsilon''_{i2}, \varepsilon_{i3}, \varepsilon_{i4}, \varepsilon_{i5}\}, \quad i \in I_0. \tag{4.20}$$

With this notation ε_i applies to neighborhood $\mathscr{B}_i$ of obstacle $\mathscr{O}_i$.

For properly defined β_i^{-1} level sets "near" obstacles we require

$$\forall i \in I_0 \quad \exists \varepsilon_{i4} > 0 : \quad \|\nabla \beta_i\| > 0, \quad \forall q \in \overline{\mathscr{B}\left(\varepsilon_{i4}\right)} \tag{4.21}$$

which is needed for radial positive definiteness, in order for $\min\{\|\nabla \beta_i\|\} > 0$ in ε''_{i2}.

In consequence of the above definitions, there are two alternatives for defining sets $\mathscr{B}_i, \mathscr{F}_n, \mathscr{F}_a$ as either functions of a *single global* "width" $\varepsilon \triangleq \min_{i \in I_0}\{\varepsilon_i\}$, or as functions of the *set* ε_{I_0} of "widths" ε_i. Here the sets are functions of $M+1$ parameters ε_{I_0} defined as $\mathscr{B}_i\left(\varepsilon_i\right), i \in I_0, \mathscr{F}_n\left(\varepsilon_{I_0}\right), \mathscr{F}_a\left(\varepsilon_{I_0}\right)$. Note that the above definitions differ from those in [23]. Hereafter sets $\mathscr{F}_i$ are denoted omitting their arguments. Let $\mathscr{C}_f \triangleq \{q_c \in E^n | \nabla f = 0\}$ the critical set of a function f.

4.2 Relative Curvature Function

4.2.1 Overview

General obstacle functions β_i are considered here and a geometric condition they must satisfy is derived.

Propositions 2.7, 3.2 and 3.3 [23] are independent of β_i zero level set shape, i.e., obstacle type. Therefore, they are valid here. Proposition 3.2 ensures that the destination q_d is a nondegenerate local minimum and 3.3 that the free space boundary $\partial\mathscr{F}$ contains no critical points q_c. Then Proposition 2.7 applies range diffeomorphism to $\mathscr{F} \setminus (\partial\mathscr{F} \cup q_d)$. This allows us to work with $\hat{\varphi} \triangleq \frac{\gamma_d^k}{\beta}$ instead of φ in the free space interior $\mathring{\mathscr{F}} \setminus q_d$ for the main part of the proof.

Firstly, Proposition 3.4 [23] continues to hold for general obstacles. It clears the set away from obstacles of critical points. This is achieved by selecting

$$k \geq N(\varepsilon_{I_0}) \triangleq \frac{1}{2} \max_{\mathscr{W}} \{\sqrt{\gamma_d}\} \sum_{i \in I_0} \frac{\max_{\mathscr{W}} \{\|\nabla \beta_i\|\}}{\varepsilon_i} \tag{4.22}$$

so that there are no critical points in $\mathscr{F}_a$. Critical points other than the destination q_d remain only in $\mathscr{F}_n$, i.e., "near" the obstacles.

Next, extending Proposition 3.6 [23] from spheres to general β_i, we are naturally led to the geometric condition of Definition 20. Let $T_q\mathscr{F}$ denote the tangent space of $\mathscr{F}$ at point q. Then, the unit tangent space $UT_q\mathscr{F}$ of $\mathscr{F}$ at point q can be defined as

$$UT_q\mathscr{F} \triangleq \{u \in T_q\mathscr{F} | \ \|u\| = 1\} \tag{4.23}$$

which the set of all unit vectors in the tangent space $T_q\mathscr{F}$ at q.

Let

$$\mathscr{R}_i(q) \triangleq \operatorname{span}\{(\nabla\beta_i)(q)\} \subset T_q\mathscr{F} \tag{4.24}$$

be the "radial" subspace at q spanned by $(\nabla\beta_i)(q)$ at q. Define the orthogonal complement

$$\mathscr{T}_i(q) \triangleq \{u \in T_q\mathscr{F} | \quad u \cdot (\nabla\beta_i)(q) = 0\} \subset T_q\mathscr{F} \tag{4.25}$$

of $\mathscr{R}_i(q)$ in the tangent space $T_q\mathscr{F}$. This is equal to the tangent space of level set $\beta_i^{-1}(c)$, i.e., $\mathscr{T}_i(q) = T_q\beta_i^{-1}(c)$. Also, note that $\mathscr{R}_i(q)$ and $\mathscr{T}_i(q)$ provide a direct sum decomposition

$$T_q\mathscr{F} = \mathscr{R}_i(q) \oplus \mathscr{T}_i(q) \tag{4.26}$$

of tangent space $T_q\mathscr{F}$.

Moreover, let us define the corresponding unit radial space as

$$\begin{aligned} U\mathscr{R}_i(q) &\triangleq \{u \in \mathscr{R}_i(q) | \ \|u\| = 1\} = \{\hat{v} \in UT_q\mathscr{F} | \ \hat{v} \cdot (\nabla\beta_i)(q) = \|\nabla\beta_i\|\} \\ &= \mathscr{R}_i(q) \cap UT_q\mathscr{F} \subset UT_q\mathscr{F} \end{aligned} \tag{4.27}$$

and the unit tangent space of $\beta^{-1}(\beta_i(q))$ as

$$\begin{aligned} U\mathscr{T}_i(q) &\triangleq \{u \in \mathscr{T}_i(q) | \ \|u\| = 1\} = \{\hat{v} \in UT_q\mathscr{F} | \ \hat{v} \cdot (\nabla\beta_i)(q) = 0\} \\ &= \mathscr{T}_i(q) \cap UT_q\mathscr{F} \subset UT_q\mathscr{F} \end{aligned} \tag{4.28}$$

Then, let us define the unit vectors

$$\hat{r}_i \triangleq \frac{(\nabla\beta_i)(q)}{\|(\nabla\beta_i)(q)\|} \in U\mathscr{R}_i(q), \qquad \hat{t}_i \in U\mathscr{T}_i(q). \tag{4.29}$$

These are the "radial" unit vector $\hat{r}_i$ along $\nabla\beta_i$ and the "tangent" unit vector $\hat{t}_i$, which is orthogonal to $\hat{r}_i$. Vector $\hat{t}_i$ is tangent to the i^{th} obstacle function level set $\beta_i^{-1}\left(\beta_i(q)\right)$ which goes through point $q \in E^n$.

Then, starting with the Hessian matrix at a critical point q_c

$$\left(D^2\hat{\varphi}\right)(q_c) = \frac{1}{\beta^2}\left[\beta D^2\left(\gamma_d^k\right) - \gamma_d^k D^2\beta\right] \tag{4.30}$$

and following similar steps with [23], but without the assumption of spherical β_i, we are led to an extended version of equation (11) [23], applying to *any* β_i

$$\begin{aligned}
&\left(\hat{t}_i^{\mathrm{T}}\left(D^2\hat{\varphi}\right)(q_c)\hat{t}_i\right)\frac{\beta^2}{\gamma_d^{k-1}} = \\
&= \gamma_d\bar{\beta}_i\left(\frac{\nabla\beta_i\cdot\nabla\gamma_d}{\left\|\nabla\gamma_d\right\|^2}\left(\hat{t}_i^{\mathrm{T}}D^2\gamma_d\hat{t}_i\right) - \left(\hat{t}_i^{\mathrm{T}}D^2\beta_i\hat{t}_i\right)\right) \\
&\quad + \gamma_d\beta\left(\frac{\nabla\bar{\beta}_i\cdot\nabla\gamma_d}{\left\|\nabla\gamma_d\right\|^2} + \hat{t}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\nabla\bar{\beta}_i\cdot\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - D^2\bar{\beta}_i\right)\hat{t}_i\right) \\
&= \gamma_d\bar{\beta}_i\nu_i(q) \\
&\quad + \gamma_d\beta\left(\frac{\nabla\bar{\beta}_i\cdot\nabla\gamma_d}{\left\|\nabla\gamma_d\right\|^2} + \hat{t}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\nabla\bar{\beta}_i\cdot\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - D^2\bar{\beta}_i\right)\hat{t}_i\right)
\end{aligned} \tag{4.31}$$

where the *relative curvature* function is defined here as

$$\nu_i \triangleq \frac{\nabla\beta_i\cdot\nabla\gamma_d}{\left\|\nabla\gamma_d\right\|^2}\left(\hat{t}_i^{\mathrm{T}}D^2\gamma_d\hat{t}_i\right) - \hat{t}_i^{\mathrm{T}}D^2\beta_i\hat{t}_i \tag{4.32}$$

and for the special case of spheres considered there was $\frac{1}{4}\nabla\beta_i\cdot\nabla\gamma_d - \gamma_d$. The detailed derivation above is now provided.

4.2.2 NF Hessian at critical points

Proposition 6 (NF Hessian matrix at critical points incorporates relative curvature function)**.** At every critical point $q_c \in \mathscr{F}_n \cap \mathscr{C}_{\hat{\varphi}}$, let $\hat{t}_i \in T_{q_c}\beta_i^{-1}(\beta_i(q_c))$ be a vector in the tangent space of level set β_i^{-1} through q_c. Then, the following holds for the Hessian matrix $(D^2\hat{\varphi})(q_c)$ of function $\hat{\varphi}$

$$\begin{aligned}
&\left(\hat{t}_i^{\mathrm{T}}\left(D^2\hat{\varphi}\right)(q_c)\,\hat{t}_i\right)\frac{\beta(q_c)^2}{\gamma_d(q_c)^{k-1}} = \\
&= \gamma_d(q_c)\bar{\beta}_i(q_c)\left(\frac{(\nabla\beta_i)(q_c)\cdot(\nabla\gamma_d)(q_c)}{\|(\nabla\gamma_d)(q_c)\|^2}\left(\hat{t}_i^{\mathrm{T}}\left(D^2\gamma_d\right)(q_c)\hat{t}_i\right)-\left(\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q_c)\hat{t}_i\right)\right)\\
&+\gamma_d(q_c)\beta_i(q_c)\left(\frac{\left(\nabla\bar{\beta}_i\right)(q_c)\cdot(\nabla\gamma_d)(q_c)}{\|(\nabla\gamma_d)(q_c)\|^2}+\hat{t}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\left(\nabla\bar{\beta}_i\right)(q_c)\cdot\left(\nabla\bar{\beta}_i\right)(q_c)^{\mathrm{T}}}{\bar{\beta}_i(q_c)}-\left(D^2\bar{\beta}_i\right)(q_c)\right)\hat{t}_i\right)
\end{aligned} \tag{4.33}$$

where β_i, γ_d are any C^2 and with $\|\nabla\beta_i\| > 0, \forall q \in \mathscr{F}_n$ and $\|\nabla\gamma_d\| > 0, \forall q \neq q_d$.

Proof. For

$$\rho \triangleq \frac{\nu}{\delta}, \quad \nu, \delta \in C^{(2)}\left[E^n, \mathbb{R}\right] \implies D^2\rho|_{\mathscr{C}_\rho} = \frac{1}{\delta^2}\left[\delta D^2\nu - \nu D^2\delta\right]. \tag{4.34}$$

Here we have $\rho = \hat{\varphi}, \nu = \gamma_d^k, \delta = \beta$ so that it follows (derivation of $D^2\left(\gamma_d^k\right)$ in subsection A.3.2)

$$\begin{aligned}
D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} &= \frac{1}{\beta^2}\left[\beta D^2\left(\gamma_d^k\right) - \gamma_d^k D^2\beta\right] \overset{\text{see derivation of } D^2\left(\gamma_d^k\right)}{=} \\
&= \frac{1}{\beta^2}\left[\beta\left(k\gamma_d^{k-1}\left(\frac{k-1}{\gamma_d}\nabla\gamma_d\nabla\gamma_d^{\mathrm{T}} + D^2\gamma_d\right)\right) - \gamma_d^k D^2\beta\right]\\
&= \frac{1}{\beta^2}\left[k\beta\gamma_d^{k-2}\left((k-1)\nabla\gamma_d\nabla\gamma_d^{\mathrm{T}} + \gamma_d D^2\gamma_d\right) - \gamma_d^{k-2}\gamma_d^2 D^2\beta\right]\\
&= \frac{\gamma_d^{k-2}}{\beta^2}\left[k\beta\left(\gamma_d D^2\gamma_d + (k-1)\nabla\gamma_d\nabla\gamma_d^{\mathrm{T}}\right) - \gamma_d^2 D^2\beta\right].
\end{aligned} \tag{4.35}$$

At a critical point

$$\begin{aligned}
&\nabla\hat{\varphi} = 0 \iff \nabla\left(\frac{\gamma_d^k}{\beta}\right) = 0 \iff \frac{\beta\nabla\left(\gamma_d^k\right) - \gamma_d^k\nabla\beta}{\beta^2} = 0 \overset{q\notin\partial\mathscr{F} \implies \beta\neq 0}{\iff}\\
&\beta\nabla\left(\gamma_d^k\right) - \gamma_d^k\nabla\beta = 0 \iff \beta k\gamma_d^{k-1}\nabla\gamma_d - \gamma_d^k\nabla\beta = 0 \overset{q\notin\{q_d\} \implies \gamma_d\neq 0}{\iff} k\beta\nabla\gamma_d = \gamma_d\nabla\beta
\end{aligned} \tag{4.36}$$

Taking the outer product of both sides

$$\begin{aligned}
&(k\beta\nabla\gamma_d)(k\beta\nabla\gamma_d)^{\mathrm{T}} = (\gamma_d\nabla\beta)(\gamma_d\nabla\beta)^{\mathrm{T}} \iff (k\beta)^2\nabla\gamma_d\nabla\gamma_d^{\mathrm{T}} = \gamma_d^2\nabla\beta\nabla\beta^{\mathrm{T}} \overset{q\notin\partial\mathscr{F} \implies \beta\neq 0}{\iff}\\
&k\beta\nabla\gamma_d\nabla\gamma_d^{\mathrm{T}} = \frac{\gamma_d^2}{k\beta}\nabla\beta\nabla\beta^{\mathrm{T}}
\end{aligned} \tag{4.37}$$

and substitution in (4.35) yields

$$\begin{aligned}
D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} &= \frac{\gamma_d^{k-2}}{\beta^2}\left[k\beta\gamma_d D^2\gamma_d + (k-1)\frac{\gamma_d^2}{k\beta}\nabla\beta\nabla\beta^{\mathrm{T}} - \gamma_d^2 D^2\beta\right]\\
&= \frac{\gamma_d^{k-1}}{\beta^2}\left[k\beta D^2\gamma_d + \frac{k-1}{k}\frac{\gamma_d}{\beta}\nabla\beta\nabla\beta^{\mathrm{T}} - \gamma_d D^2\beta\right]\\
&= \frac{\gamma_d^{k-1}}{\beta^2}\left[k\beta D^2\gamma_d + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\nabla\beta\nabla\beta^{\mathrm{T}} - \gamma_d D^2\beta\right]
\end{aligned} \tag{4.38}$$

Let $A \in \mathbb{C}^{n\times n}$ be a matrix. Then its symmetric part is given by $\frac{1}{2}\left(A + A^{\mathrm{T}}\right) = A_{\text{symmetric}}$ abbreviated as A_s. Note that

$$\begin{aligned} \beta = \beta_i\bar{\beta}_i &\implies \nabla\beta \\ &= \beta_i\nabla\bar{\beta}_i + \bar{\beta}_i\nabla\beta_i \implies \\ D^2\beta = D\left[\beta_i\nabla\bar{\beta}_i + \bar{\beta}_i\nabla\beta_i\right] &= \beta_i D^2\bar{\beta}_i + \nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \bar{\beta}_i D^2\beta_i + \nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}} \\ &= \beta_i D^2\bar{\beta}_i + \left[\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}\right] + \bar{\beta}_i D^2\beta_i \end{aligned} \tag{4.39}$$

but since

$$\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}} = \nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)^{\mathrm{T}} = A + A^{\mathrm{T}} = 2A_s \tag{4.40}$$

for $A = \nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}$ so (4.39) can be written as

$$D^2\beta = \beta_i D^2\bar{\beta}_i + 2\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i D^2\beta_i \tag{4.41}$$

Also similarly

$$\begin{aligned} \nabla\beta\nabla\beta^{\mathrm{T}} &= \left(\beta_i\nabla\bar{\beta}_i + \bar{\beta}_i\nabla\beta_i\right)\left(\beta_i\nabla\bar{\beta}_i + \bar{\beta}_i\nabla\beta_i\right)^{\mathrm{T}} \\ &= \left(\beta_i\nabla\bar{\beta}_i + \bar{\beta}_i\nabla\beta_i\right)\left(\beta_i\nabla\bar{\beta}_i^{\mathrm{T}} + \bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right) \\ &= \beta_i\nabla\bar{\beta}_i\beta_i\nabla\bar{\beta}_i^{\mathrm{T}} + \beta_i\nabla\bar{\beta}_i\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \bar{\beta}_i\nabla\beta_i\beta_i\nabla\bar{\beta}_i^{\mathrm{T}} + \bar{\beta}_i\nabla\beta_i\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} \\ &= \beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} + \left(\beta_i\bar{\beta}_i\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \beta_i\bar{\beta}_i\nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}\right) + \bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}} \end{aligned} \tag{4.42}$$

where

$$\beta_i\bar{\beta}_i\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \beta_i\bar{\beta}_i\nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}} = \beta_i\bar{\beta}_i\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}\right) \tag{4.43}$$

and since

$$\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}} = \nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}} + \left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)^{\mathrm{T}} = A + A^{\mathrm{T}} = 2A_s \tag{4.44}$$

again for $A = \nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}$, it follows that

$$\nabla\beta\nabla\beta^{\mathrm{T}} = \beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} + 2\beta_i\bar{\beta}_i\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}} \tag{4.45}$$

Then substitution of $D^2\beta$ from (4.41) and $\nabla\beta\nabla\beta^{\mathrm{T}}$ from (4.45) in (4.38) yields

$$\begin{aligned} D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} =& \frac{\gamma_d^{k-1}}{\beta^2}\Bigg(k\beta D^2\gamma_d \\ &+ \left(1 - \frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} + 2\beta_i\bar{\beta}_i\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}}\right) \\ &- \gamma_d\left(\beta_i D^2\bar{\beta}_i + 2\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i D^2\beta_i\right)\Bigg) \end{aligned} \tag{4.46}$$

Now we are going to evaluate the quadratic form associated with $\left(D^2\hat{\varphi}\right)(q_c)$ in the direction of the unit tangent vector

$$\hat{t}_i \triangleq \left(\frac{\nabla\beta_i\left(q_c\right)}{\|\nabla\beta_i\left(q_c\right)\|}\right)^{\perp} = \left(\nabla\beta_i\left(q_c\right)\right)^{\perp}\frac{1}{\|\nabla\beta_i\left(q_c\right)\|} \tag{4.47}$$

which, treating term by term the expression, yields ($q \neq q_d \implies \gamma_d \neq 0 \wedge q \notin \partial \mathscr{F} \implies \beta \neq 0$)

$$
\begin{aligned}
\hat{t}_i^{\mathrm{T}} D^2 \hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} \hat{t}_i \frac{\beta^2}{\gamma_d^{k-1}} = {} & \hat{t}_i^{\mathrm{T}} \left(k \beta D^2 \gamma_d \right) \hat{t}_i \\
& + \hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k} \right) \frac{\gamma_d}{\beta} \beta_i^2 \nabla \bar{\beta}_i \nabla \bar{\beta}_i^{\mathrm{T}} \right) \hat{t}_i \\
& + \hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k} \right) \frac{\gamma_d}{\beta} 2 \beta_i \bar{\beta}_i \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}} \right)_s \right) \hat{t}_i \\
& + \hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k} \right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \nabla \beta_i \nabla \beta_i^{\mathrm{T}} \right) \hat{t}_i \\
& - \hat{t}_i^{\mathrm{T}} \left(\gamma_d \beta_i D^2 \bar{\beta}_i \right) \hat{t}_i \\
& - \hat{t}_i^{\mathrm{T}} \left(\gamma_d 2 \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}} \right)_s \right) \hat{t}_i \\
& - \hat{t}_i^{\mathrm{T}} \left(\gamma_d \bar{\beta}_i D^2 \beta_i \right) \hat{t}_i
\end{aligned}
\tag{4.48}
$$

and the comprising terms are (term 1)

$$
\hat{t}_i^{\mathrm{T}} \left(k \beta D^2 \gamma_d \right) \hat{t}_i = k \beta \left(\hat{t}_i^{\mathrm{T}} D^2 \gamma_d \hat{t}_i \right) \tag{4.49}
$$

and (term 7)

$$
\hat{t}_i^{\mathrm{T}} \left(\gamma_d \bar{\beta}_i D^2 \beta_i \right) \hat{t}_i = \gamma_d \bar{\beta}_i \left(\hat{t}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i \right) \tag{4.50}
$$

and (term 2)

$$
\hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k} \right) \frac{\gamma_d}{\beta} 2 \beta_i \bar{\beta}_i \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}} \right)_s \right) \hat{t}_i = 2 \left(1 - \frac{1}{k} \right) \gamma_d \hat{t}_i^{\mathrm{T}} \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}} \right)_s \hat{t}_i \tag{4.51}
$$

where

$$
\begin{aligned}
\hat{t}_i^{\mathrm{T}} \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}} \right)_s \hat{t}_i &= \frac{1}{2} \hat{t}_i^{\mathrm{T}} \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}} + \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}} \right)^{\mathrm{T}} \right) \hat{t}_i \\
&= \frac{1}{2} \hat{t}_i^{\mathrm{T}} \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}} + \nabla \beta_i \nabla \bar{\beta}_i^{\mathrm{T}} \right) \hat{t}_i \\
&= \frac{1}{2} \left(\hat{t}_i^{\mathrm{T}} \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}} \right) \hat{t}_i + \hat{t}_i^{\mathrm{T}} \left(\nabla \beta_i \nabla \bar{\beta}_i^{\mathrm{T}} \right) \hat{t}_i \right) \\
&= \frac{1}{2} \left(\left(\hat{t}_i^{\mathrm{T}} \nabla \bar{\beta}_i \right) \cancelto{0}{\left(\nabla \beta_i^{\mathrm{T}} \hat{t}_i \right)} + \cancelto{0}{\left(\hat{t}_i^{\mathrm{T}} \nabla \beta_i \right)} \left(\nabla \bar{\beta}_i \hat{t}_i \right) \right) = 0
\end{aligned}
\tag{4.52}
$$

so that from (4.51)

$$
\hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k} \right) \frac{\gamma_d}{\beta} 2 \beta_i \bar{\beta}_i \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}} \right)_s \right) \hat{t}_i = 0 \tag{4.53}
$$

and (term 4)

$$
\hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k} \right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \nabla \beta_i \nabla \beta_i^{\mathrm{T}} \right) \hat{t}_i = \left(1 - \frac{1}{k} \right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \hat{t}_i^{\mathrm{T}} \left(\nabla \beta_i \nabla \beta_i^{\mathrm{T}} \right) \hat{t}_i \tag{4.54}
$$

where

$$
\hat{t}_i^{\mathrm{T}} \left(\nabla \beta_i \nabla \beta_i^{\mathrm{T}} \right) \hat{t}_i = \cancelto{0}{\left(\hat{t}_i^{\mathrm{T}} \nabla \beta_i \right)} \cancelto{0}{\left(\nabla \beta_i^{\mathrm{T}} \hat{t}_i \right)} = 0 \tag{4.55}
$$

so that from (4.54)

$$
\hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k} \right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \nabla \beta_i \nabla \beta_i^{\mathrm{T}} \right) \hat{t}_i = 0 \tag{4.56}
$$

and (term 6)

$$\hat{t}_i^{\mathrm{T}} \left(\gamma_d 2 \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)_s\right) \hat{t}_i = 2\gamma_d \hat{t}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)_s \hat{t}_i \tag{4.57}$$

where

$$\begin{aligned}
\hat{t}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)_s \hat{t}_i &= \frac{1}{2}\hat{t}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}} + \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)^{\mathrm{T}}\right) \hat{t}_i \\
&= \frac{1}{2}\hat{t}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}} + \nabla\beta_i \nabla\bar{\beta}_i^{\mathrm{T}}\right) \hat{t}_i \\
&= \frac{1}{2}\left(\hat{t}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right) \hat{t}_i + \hat{t}_i^{\mathrm{T}} \left(\nabla\beta_i \nabla\bar{\beta}_i^{\mathrm{T}}\right) \hat{t}_i\right) \\
&= \frac{1}{2}\left(\left(\hat{t}_i^{\mathrm{T}} \nabla\bar{\beta}_i\right) \cancelto{0}{\left(\nabla\beta_i \hat{t}_i\right)} + \cancelto{0}{\left(\hat{t}_i^{\mathrm{T}} \nabla\beta_i\right)} \left(\nabla\bar{\beta}_i^{\mathrm{T}} \hat{t}_i\right)\right) = 0
\end{aligned} \tag{4.58}$$

for the same reason as before. The zero inner products are justified by normality of chosen direction $\hat{t}_i$ to gradient $\nabla\beta_i$ since $\hat{t}_i$ is tangent to level sets

$$\begin{aligned}
\nabla\beta_i^{\mathrm{T}} \hat{t}_i &= \nabla\beta_i \cdot \hat{t}_i = \left(\nabla\beta_i \cdot \nabla\beta_i^{\perp}\right) \frac{1}{\|\nabla\beta_i\|} = 0 \\
\hat{t}_i^{\mathrm{T}} \nabla\beta_i &= \hat{t}_i \cdot \nabla\beta_i = \nabla\beta_i \cdot \hat{t}_i = 0.
\end{aligned} \tag{4.59}$$

So substitution of these terms in (4.48) leads to

$$\begin{aligned}
&\hat{t}_i^{\mathrm{T}} D^2 \hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} \hat{t}_i \frac{\beta^2}{\gamma_d^{k-1}} \\
&= k\beta \left(\hat{t}_i^{\mathrm{T}} D^2 \gamma_d \hat{t}_i\right) - \gamma_d \bar{\beta}_i \left(\hat{t}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i\right) + \hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \beta_i^2 \nabla\bar{\beta}_i \nabla\bar{\beta}_i^{\mathrm{T}} - \gamma_d \beta_i D^2 \bar{\beta}_i\right) \hat{t}_i.
\end{aligned} \tag{4.60}$$

At critical point $q_c \in \mathscr{C}_{\hat{\varphi}}$ the following holds

$$\begin{aligned}
k\beta \nabla\gamma_d = \gamma_d \nabla\beta &\implies k\beta \nabla\gamma_d \cdot \nabla\gamma_d = \gamma_d \nabla\beta \cdot \nabla\gamma_d \iff \\
k\beta \|\nabla\gamma_d\|^2 &= \gamma_d \left(\nabla\left(\bar{\beta}_i \beta_i\right)\right) \cdot \nabla\gamma_d = \gamma_d \left(\bar{\beta}_i \nabla\beta_i + \beta_i \nabla\bar{\beta}_i\right) \cdot \nabla\gamma_d \\
&= \gamma_d \left(\bar{\beta}_i \nabla\beta_i \cdot \nabla\gamma_d + \beta_i \nabla\bar{\beta}_i \cdot \nabla\gamma_d\right) \overset{q\neq q_d \iff \|\nabla\gamma_d\|\neq 0}{\iff} \\
k\beta &= \gamma_d \frac{\bar{\beta}_i \nabla\beta_i \cdot \nabla\gamma_d + \beta_i \nabla\bar{\beta}_i \cdot \nabla\gamma_d}{\|\nabla\gamma_d\|^2}
\end{aligned} \tag{4.61}$$

then the condition for general γ_d, β_i results by substitution in (4.60)

$$\begin{aligned}
\left(\hat{t}_i D^2 \hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} \hat{t}_i\right) \frac{\beta^2}{\gamma_d^{k-1}} =& \gamma_d \frac{\bar{\beta}_i \nabla\beta_i \cdot \nabla\gamma_d + \beta_i \nabla\bar{\beta}_i \cdot \nabla\gamma_d}{\|\nabla\gamma_d\|^2} \left(\hat{t}_i^{\mathrm{T}} D^2 \gamma_d \hat{t}_i\right) - \gamma_d \bar{\beta}_i \left(\hat{t}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i\right) \\
&+ \hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \beta_i^2 \nabla\bar{\beta}_i \nabla\bar{\beta}_i^{\mathrm{T}} - \gamma_d \beta_i D^2 \bar{\beta}_i\right) \hat{t}_i \\
&= \gamma_d \bar{\beta}_i \left(\frac{\nabla\beta_i \cdot \nabla\gamma_d}{\|\nabla\gamma_d\|^2} \left(\hat{t}_i^{\mathrm{T}} D^2 \gamma_d \hat{t}_i\right) - \left(\hat{t}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i\right)\right) \\
&+ \gamma_d \beta_i \left(\frac{\nabla\bar{\beta}_i \cdot \nabla\gamma_d}{\|\nabla\gamma_d\|^2} + \hat{t}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k}\right) \frac{\nabla\bar{\beta}_i \cdot \nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - D^2 \bar{\beta}_i\right) \hat{t}_i\right).
\end{aligned} \tag{4.62}$$

□

4.2.3 Relative curvature function

Let

$$B_i(q) \triangleq \beta_i^{-1}\left(\beta_i(q)\right) \tag{4.63}$$

denote the obstacle β_i implicit level set to which point q belongs. If the level set $\beta_i^{-1}\left(\beta_i(q)\right)$ is disconnected, then $B_i(q)$ is defined as that connected component of this level set, to which point q belongs. Let

$$TB_i = \bigsqcup_{q\in\mathscr{F}} T_qB_i = \bigcup(\{q\}\times T_qB_i) = \bigcup(\{q\}\times\mathscr{T}_i(q)) \tag{4.64}$$

be the tangent bundle of $B_i(q)$. Furthermore, let

$$\begin{aligned} UTB_i &\triangleq \bigsqcup_{q\in\mathscr{F}}\{u\in T_qB_i|\ \|u\|=1\} = \bigsqcup_{q\in\mathscr{F}}\{u\in\mathscr{T}_i(q)|\ \|u\|=1\} \\ &= \bigsqcup_{q\in\mathscr{F}}\{\hat{v}\in U\mathscr{T}_i(q)\} \end{aligned} \tag{4.65}$$

denote the unit tangent bundle of $B_i(q)$.

Definition 7 (Relative Curvature Function $\nu_i(q,\hat{t}_i)$)**.** Let the relative curvature function $\nu_i : UTB_i \to \mathbb{R}$ be defined as

$$\begin{aligned} \nu_i(q,\hat{t}_i) &\triangleq \frac{(\nabla\beta_i)(q)\cdot(\nabla\gamma_d)(q)}{\|(\nabla\gamma_d)(q)\|^2}\left(\hat{t}_i^{\mathrm{T}}\left(D^2\gamma_d\right)(q)\hat{t}_i\right) - \hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i, \\ &\hat{t}_i\in U\mathscr{T}_i(q),\quad q\in\mathscr{F},\quad i\in I_0 \end{aligned} \tag{4.66}$$

which compares the curvature of destination attractive effect level sets γ_d^{-1} to that of the obstacle level sets β_i^{-1}.

Proposition 8 (Relative curvature function ν_i decomposition for paraboloid γ_d)**.** If $\gamma_d(q) = \|q-q_d\|^2$, then at every point $q\in\mathscr{F}$ the relative curvature function ν_i is equal to the sum of two functions $\nu_{i1} : \mathscr{F}\to\mathbb{R}$ and $\nu_{i2} : UTB_i\to\mathbb{R}$, which are defined as

$$\begin{aligned} \nu_{i1}(q) &\triangleq 2\frac{(\nabla\beta_i)(q)\cdot(\nabla\gamma_d)(q)}{\|(\nabla\gamma_d)(q)\|^2}, \\ \nu_{i2}(q,\hat{t}_i) &\triangleq -\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i \end{aligned} \tag{4.67}$$

so that

$$\nu_i(q,\hat{t}_i) = \nu_{i1}(q) + \nu_{i2}(q,\hat{t}_i),\quad \forall\hat{t}_i\in U\mathscr{T}_i(q),\quad \forall q\in\mathscr{F},\quad \forall i\in I_0 \tag{4.68}$$

Proof. If $\gamma_d(q) = \|q-q_d\|^2$ then $(D^2\gamma_d)(q) = 2I, \forall q\in\mathscr{F}$. As a result, if γ_d is paraboloid, then $\hat{t}_i^{\mathrm{T}}(D^2\gamma_d)(q_c)\hat{t}_i = \hat{t}_i^{\mathrm{T}}2I\hat{t}_i = 2, \forall\hat{t}_i\in UT_qB_i$. Substitution in the relative curvature function as defined in (4.66) yields

$$\begin{aligned} \nu_i(q,\hat{t}_i) &\triangleq \frac{(\nabla\beta_i)(q)\cdot(\nabla\gamma_d)(q)}{\|(\nabla\gamma_d)(q)\|^2}2I - \hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i \\ &= 2\frac{(\nabla\beta_i)(q)\cdot(\nabla\gamma_d)(q)}{\|(\nabla\gamma_d)(q)\|^2} - \hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i,\quad \hat{t}_i\in UT_qB_i,\quad q\in\mathscr{F}, i\in I_0 \end{aligned} \tag{4.69}$$

Note that the first term $\nu_{i1}(q)$ on the right hand side is a function *only* of q, whereas the second is the restriction to tangent space $U\mathscr{T}_i(q)$ of the quadratic form $\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i$. Hence, we can define

$$\nu_{i1}(q) \triangleq 2\frac{\left(\nabla\beta_i\right)(q)\cdot\left(\nabla\gamma_d\right)(q)}{\left\|\left(\nabla\gamma_d\right)(q)\right\|^2} \tag{4.70}$$

and

$$\nu_{i2}(q,\hat{t}_i) \triangleq -\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i \tag{4.71}$$

to complete the decomposition and prove the claim. □

Note that for paraboloid γ_d the $\nu_{i1}(q)$ is a function *only* of q (i.e., independent of tangent direction $\hat{t}_i$), therefore common for all tangent directions at q. On the contrary, $\nu_{i2}(q,\hat{t}_i)$ is a function of *both* q and $\hat{t}_i$. But, actually $\nu_{i2}(q,\hat{t}_i)$ is the curvature of level set $\beta_i^{-1}(c)$, scaled by the gradient norm $\|\left(\nabla\beta_i\right)(q)\|$, which is constant for all directions at q.

Proposition 9 (Proportional decomposition of relative curvature function ν_i for paraboloid γ_d)**.** If $\gamma_d(q) = \|q - q_d\|$, then at every point $q \in \mathscr{F}$ the relative curvature function ν_i is equal to the product of the gradient norm $\|\left(\nabla\beta_i\right)(q)\| > 0$ with the sum of two functions $\nu_{i3} : \mathscr{F} \to \mathbb{R}$ and $\nu_{i4} : UTB_i \to \mathbb{R}$, which are defined as

$$\begin{aligned}\nu_{i3}(q) &\triangleq 2\frac{\left(\nabla\beta_i\right)(q)\cdot\left(\nabla\gamma_d\right)(q)}{\left\|\left(\nabla\beta_i\right)(q)\right\|\left\|\left(\nabla\gamma_d\right)(q)\right\|^2}\\ \nu_{i4}(q,\hat{t}_i) &\triangleq -\frac{\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i}{\left\|\left(\nabla\beta_i\right)(q)\right\|}\end{aligned} \tag{4.72}$$

so that

$$\nu_i(q,\hat{t}_i) = \left\|\left(\nabla\beta_i\right)(q)\right\|\left(\nu_{i3}(q) + \nu_{i4}(q,\hat{t}_i)\right), \quad \forall\hat{t}_i \in U\mathscr{T}_i(q), \quad \forall q \in \mathscr{F}, \quad \forall i \in I_0 \tag{4.73}$$

Proof. Using Proposition 8 we have that for paraboloid γ_d it holds that $\nu_i(q,\hat{t}_i) = \nu_{i1}(q) + \nu_{i2}(q,\hat{t}_i)$. Set

$$\begin{aligned}\nu_{i3}(q) &= \frac{\nu_{i1}(q)}{\left\|\left(\nabla\beta_i\right)(q)\right\|}\\ \nu_{i4}(q,\hat{t}_i) &= \frac{\nu_{i2}(q,\hat{t}_i)}{\left\|\left(\nabla\beta_i\right)(q)\right\|}\end{aligned} \tag{4.74}$$

and the claim is proved. □

Proposition 10 (Specific form of relative curvature function ν_i in general)**.** If $\|\left(\nabla\beta_i\right)(q)\| > 0$ then the relative curvature function can be written in the form

$$\nu_i = \|\nabla\beta_i\|\left(\cos(\theta_i)\frac{\hat{t}_i^{\mathrm{T}}D^2\gamma_d\hat{t}_i}{\|\nabla\gamma_d\|} - \frac{\hat{t}_i^{\mathrm{T}}D^2\beta_i\hat{t}_i}{\|\nabla\beta_i\|}\right) \tag{4.75}$$

where

$$\theta_i(q) \triangleq (\left(\nabla\gamma_d\right)\widehat{(q),\left(\nabla\beta_i\right)}(q)) = \frac{\left(\nabla\gamma_d\right)(q)\cdot\left(\nabla\beta_i\right)(q)}{\left\|\left(\nabla\gamma_d\right)(q)\right\|\left\|\left(\nabla\beta_i\right)(q)\right\|} \tag{4.76}$$

is the angle between the two gradients $\left(\nabla\gamma_d\right)(q)$ and $\left(\nabla\beta_i\right)(q)$.

Proof.

$$\nu_i(q, \hat{t}_i) = \frac{\nabla\beta_i \cdot \nabla\gamma_d}{\|\nabla\gamma_d\|^2} \left(\hat{t}_i^{\mathrm{T}} D^2\gamma_d \hat{t}_i\right) - \hat{t}_i^{\mathrm{T}} D^2\beta_i \hat{t}_i \overset{\|\nabla\beta_i\|\neq 0, \forall q \in \mathscr{F}}{=}$$
$$= \|\nabla\beta_i\| \left(\frac{\nabla\beta_i}{\|\nabla\beta_i\|} \cdot \frac{\nabla\gamma_d}{\|\nabla\gamma_d\|} \frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d \hat{t}_i}{\|\nabla\gamma_d\|} - \frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i \hat{t}_i}{\|\nabla\beta_i\|}\right) \tag{4.77}$$
$$= \|\nabla\beta_i\| \left(\cos(\theta_i) \frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d \hat{t}_i}{\|\nabla\gamma_d\|} - \frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i \hat{t}_i}{\|\nabla\beta_i\|}\right)$$

□

As already analyzed, the first term on the right-hand side should be strictly negative, so that an upper bound constraint can be specified, without need to explicitly find actual extremal values of the two terms on the right-side. Therefore the general[15] condition which results in the modified constraint $\varepsilon_i < \varepsilon'_{i0}$ is

$$\nu_i(q) = \frac{\nabla\beta_i \cdot \nabla\gamma_d}{\|\nabla\gamma_d\|^2} \left(\hat{t}_i^{\mathrm{T}} D^2\gamma_d \hat{t}_i\right) - \hat{t}_i^{\mathrm{T}} D^2\beta_i \hat{t}_i < 0 \overset{\|\nabla\beta_i\|\neq 0, \forall q \in \mathscr{F}}{\iff}$$
$$\cos(\theta_i) \frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d \hat{t}_i}{\|\nabla\gamma_d\|} < \frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i \hat{t}_i}{\|\nabla\beta_i\|} \tag{4.78}$$

and when $\left(\hat{t}_i^{\mathrm{T}} D^2\gamma_d \hat{t}_i\right)\left(\hat{t}_i^{\mathrm{T}} D^2\beta_i \hat{t}_i\right)\cos(\theta_i) > 0$

$$\frac{\|\nabla\beta_i\|}{\hat{t}_i^{\mathrm{T}} D^2\beta_i \hat{t}_i} < \frac{\frac{\|\nabla\gamma_d\|}{\hat{t}_i^{\mathrm{T}} D^2\gamma_d \hat{t}_i}}{\cos\theta_i} \tag{4.79}$$

Note that in more detail this is required to hold at a critical point q_c confined within obstacle free space neighborhood $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$

$$\frac{\|\nabla\beta_i(q_c)\|}{\hat{t}_i(q_c)^{\mathrm{T}} D^2\beta_i(q_c) \hat{t}_i(q_c)} < \frac{\frac{\|\nabla\gamma_d(q_c)\|}{\hat{t}_i(q_c)^{\mathrm{T}} D^2\gamma_d(q_c) \hat{t}_i(q_c)}}{\cos\theta_i(q_c)}, \quad q_c \in \overline{\mathscr{B}_i\left(\varepsilon_i\right)} \tag{4.80}$$

4.2.4 Critical point-free neighborhoods

We can "push" the critical points very close to B_i and then $\nabla\beta_i$ dominates $\nabla\bar{\beta}_i$. In this case the existence of critical points is dominated in $\mathscr{B}_i\left(\varepsilon_i\right)$ only by $\nabla\beta_i$ and $-\nabla\gamma_d$. Since $\nabla\gamma_d \cdot \nabla\beta_i \leq 0 \implies 0 \leq (-\nabla\gamma_d) \cdot \nabla\beta_i$, the two vectors $-\nabla\gamma_d, \nabla\beta_i$ have an angle either less or at most equal to $\frac{\pi}{2}$. A direct consequence is that they cannot annihilate each other. Therefore, they cannot cause a critical point q_c. This was an intuitive explanation. Through the formal proof it turns out that, provided $\mathscr{B}_i\left(\varepsilon_i\right)$ are pairwise disjoint, this also holds for the k threshold already imposed in (4.22).

Definition 11 (Good and bad Half-spaces)**.** We need to define two half-spaces[16], separated by the tangent plane $T_q B_i$ of B_i at q. The first one $\mathscr{H}_{i1}(q)$ is the "good" one. When we place the destination q_d in $\mathscr{H}_{i1}(q)$ the inner product $\left(\nabla\beta_i\right)(q) \cdot \left(\nabla\gamma_d\right)(q) \leq 0$. The free subset of the first half-space (i.e., the intersection of this half-space with the free space of allowable destinations)

$$\mathscr{H}_{i1}(q) \triangleq \{q_d \in \mathscr{F} \setminus (\partial\mathscr{F} \cup \{q\}) \mid \left(\nabla\beta_i\right)(q) \cdot \left(\nabla\gamma_d\right)(q) \leq 0\} \tag{4.81}$$

[15]General here refers to any choice of γ_d, β_i.

[16]More exactly: for two half-spaces the intersections of them with the free space.

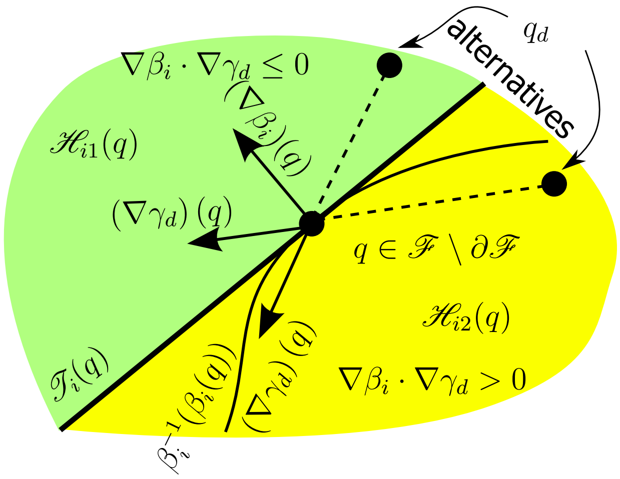


Figure 4.3: Positive/nonpositive inner product half-spaces, depending on q_d.

For a fixed q, it is the half space of possible q_d which render the inner product $(\nabla\beta_i)(q) \cdot (\nabla\gamma_d)(q)$ nonpositive.

The second one is the "bad" half-space

$$\mathscr{H}_{i2}(q) \triangleq \{q_d\mathscr{F} \setminus (\partial\mathscr{F} \cup \{q\}) \mid 0 < (\nabla\beta_i)(q) \cdot (\nabla\gamma_d)(q)\} \tag{4.82}$$

Definition 12 (Subsets of set "near" obstacles)**.** Also, for a given destination q_d, for each q, either

$$(\nabla\beta_i)(q) \cdot (\nabla\gamma_d)(q) \leq 0 \tag{4.83}$$

or

$$(\nabla\beta_i)(q) \cdot (\nabla\gamma_d)(q) > 0 \tag{4.84}$$

This leads us to define two disjoint and complementary subsets of $\mathscr{F}_n$. First let

$$\begin{aligned} \mathscr{A}_{i1}(\varepsilon_i) &\triangleq \{q \in \mathscr{B}_i(\varepsilon_i) \mid \ (\nabla\beta_i)(q) \cdot (\nabla\gamma_d)(q) \leq 0\}, \quad i \in I_0 \\ \mathscr{A}_{i2}(\varepsilon_i) &\triangleq \{q \in \mathscr{B}_i(\varepsilon_i) \mid 0 < (\nabla\beta_i)(q) \cdot (\nabla\gamma_d)(q)\}, \quad i \in I_0 \end{aligned} \tag{4.85}$$

where, it follows by the above definitions that

$$\mathscr{B}_i(\varepsilon_i) = \mathscr{A}_{i1}(\varepsilon_i) \cup \mathscr{A}_{i2}(\varepsilon_i) \tag{4.86}$$

and now we can define their unions

$$\begin{aligned} \mathscr{A}_1(\varepsilon_{I_0}) &\triangleq \bigcup_{i \in I_0} \mathscr{A}_{i1}(\varepsilon_i) = \{q \in \mathscr{F}_n, \, i \in I_0 | \ (\nabla\beta_i)(q) \cdot (\nabla\gamma_d)(q) \leq 0\} \\ \mathscr{A}_2(\varepsilon_{I_0}) &\triangleq \bigcup_{i \in I_0} \mathscr{A}_{i2}(\varepsilon_i) = \{q \in \mathscr{F}_n, \, i \in I_0 | \ 0 < (\nabla\beta_i)(q) \cdot (\nabla\gamma_d)(q)\} \end{aligned} \tag{4.87}$$

where, it follows by these definitions that

$$\mathscr{F}_n = \mathscr{A}_1(\varepsilon_{I_0}) \cup \mathscr{A}_2(\varepsilon_{I_0}) \tag{4.88}$$

Note how these are related to $\mathscr{H}_{i1}(q), \mathscr{H}_{i2}(q)$. Sets $\mathscr{H}_{ij}(q)$ are defined for fixed q and concern all possible q_d selections. On the contrary, sets $\mathscr{A}_1, \mathscr{A}_2$ are defined for a given q_d, as is the case in the whole proof. Therefore, we will use the second pair of sets in our proof, whereas the first pair is useful for one to understand the geometry of the problem.

We will show that the following holds

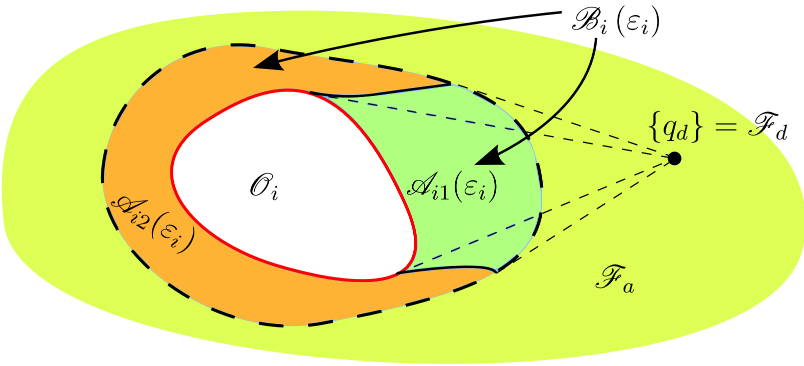


Figure 4.4: Good and bad neighborhoods.

Proposition 13 (No critical points in good subset "near" obstacles)**.** For a given q_d there is a $N(\varepsilon_{I_0})$ (the same as (4.22)), such that if $k > N(\varepsilon_{I_0})$ then $q \in \mathscr{A}_1(\varepsilon_{I_0})$ cannot be a critical point, i.e.

$$\mathscr{C}_{\hat{\varphi}} \cap \mathscr{A}_1(\varepsilon_{I_0}) = \emptyset, \quad \forall k \geq N(\varepsilon_{I_0}) \tag{4.89}$$

This means that by setting $k \geq k_{\min}$ we confine critical points not just in $\bigcup_i \mathscr{B}_i$, but in $\bigcup (\mathscr{B}_i \cap \mathscr{A}_2(\varepsilon_{I_0}))$. The proof is as follows (and is inspired by Proposition 3.7, pp. 432-433, [23], in fact it generalizes that).

Proof. By definition

$$(\nabla \beta_i)(q) \cdot (\nabla \gamma_d)(q) \leq 0, \quad \forall q \in \mathscr{A}_1(\varepsilon_{I_0}) \tag{4.90}$$

The inner product of $\nabla \gamma_d$ with $\nabla \hat{\varphi}$ is (Lemma 3.1 [23])

$$\begin{aligned} \nabla \hat{\varphi} \cdot \nabla \gamma_d &= \frac{\gamma_d^k}{\beta^2} (4k\beta - \nabla \beta \cdot \nabla \gamma_d) \\ &= \frac{\gamma_d^k}{\beta^2} \left(4k\beta - \left(\beta_i \nabla \bar{\beta}_i \cdot \nabla \gamma_d + \bar{\beta}_i \nabla \beta_i \cdot \nabla \gamma_d\right)\right) \\ &= \frac{\gamma_d^k}{\beta^2} \left(4k\beta - \beta_i \nabla \bar{\beta}_i \cdot \nabla \gamma_d - \bar{\beta}_i \nabla \beta_i \cdot \nabla \gamma_d\right) \end{aligned} \tag{4.91}$$

and, since from (4.90)

$$0 \leq -(\nabla \beta_i)(q) \cdot (\nabla \gamma_d)(q), \quad \forall q \in \mathscr{A}_1(\varepsilon_{I_0}) \tag{4.92}$$

it follows that

$$(\nabla \hat{\varphi})(q) \cdot (\nabla \gamma_d)(q) \geq \beta_i \frac{\gamma_d^k}{\beta^2} \left(4k\bar{\beta}_i(q) - \left(\nabla \bar{\beta}_i\right)(q) \cdot (\nabla \gamma_d)(q)\right) \tag{4.93}$$

If k is large enough, i.e.,

$$k > \frac{1}{4} \frac{\left(\nabla \bar{\beta}_i\right)(q) \cdot (\nabla \gamma_d)(q)}{\bar{\beta}_i(q)}, \quad \forall q \in \mathscr{A}_1(\varepsilon_{I_0}) \tag{4.94}$$

then the inner product

$$(\nabla \hat{\varphi})(q) \cdot (\nabla \gamma_d)(q) > 0, \quad \forall q \in \mathscr{A}_1(\varepsilon_{I_0}) \tag{4.95}$$

But this is satisfied by $k > N(\varepsilon_{I_0})$, because

$$\begin{aligned}
\frac{1}{4}\frac{\nabla\bar{\beta}_i \cdot \nabla\gamma_d}{\bar{\beta}_i} &\leq \frac{1}{2}\frac{\left\|\nabla\bar{\beta}_i\right\|\sqrt{\gamma_d}}{\bar{\beta}_i} \\
&\leq \frac{1}{2}\sqrt{\gamma_d}\sum_{j\in I_0\setminus\{i\}}\frac{\|\nabla\beta_j\|}{\beta_j} \\
&= \frac{1}{2}\sqrt{\gamma_d}\left(\sum_{j\in I_0}\left(\frac{\|\nabla\beta_j\|}{\beta_j}\right) - \frac{\|\nabla\beta_i\|}{\beta_i}\right) \\
&< \frac{1}{2}\sqrt{\gamma_d}\sum_{j\in I_0}\frac{\|\nabla\beta_j\|}{\beta_j}
\end{aligned} \tag{4.96}$$

and also

$$\frac{1}{2}\sqrt{\gamma_d(q)}\sum_{j\in I_0}\frac{\|(\nabla\beta_j)(q)\|}{\beta_j(q)} \leq N(\varepsilon_{I_0}) < k, \quad \forall q \in \mathscr{B}_i \tag{4.97}$$

because $\varepsilon_j \leq \beta_j(q), \forall q \in \mathscr{B}_i, \forall j \in I_0 \setminus \{i\}$. As a result, combination of the previous leads to

$$\frac{1}{4}\frac{\left(\nabla\bar{\beta}_i\right)(q)\cdot\left(\nabla\gamma_d\right)(q)}{\bar{\beta}_i(q)} < N(\varepsilon_{I_0}) \leq k, \quad \forall q \in \mathscr{B}_i \tag{4.98}$$

Therefore, since by definition $\forall q \in \mathscr{A}_1(\varepsilon_{I_0}) \implies \exists i \in I_0 :\ q \in \mathscr{B}_i$ it follows that

$$\begin{gathered}
\frac{1}{4}\frac{\left(\nabla\bar{\beta}_i\right)(q)\cdot\left(\nabla\gamma_d\right)(q)}{\bar{\beta}_i(q)} < k, \forall q \in \mathscr{A}_1(\varepsilon_{I_0}) \implies \\
\left(\nabla\hat{\varphi}\right)(q)\cdot\left(\nabla\gamma_d\right)(q) > 0, \quad \forall q \in \mathscr{A}_1(\varepsilon_{I_0})
\end{gathered} \tag{4.99}$$

by the previous equations. □

Lemma 14 (Critical points remain only in $\bigcup_{i\in I_0}\mathscr{A}_2(\varepsilon_{I_0})$ and have $0 < \nu_{i1}(q)$)**.** By Proposition 13 and Propositions 2.7, 3.2, 3.3, 3.4 [23], for every $0 < \varepsilon_i, i \in I_0$ there exists a $N(\varepsilon_{I_0})$, such that for all $k \geq N(\varepsilon_{I_0})$ the only remaining critical points $q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}$, other than the destination, arise in set $\mathscr{A}_2(\varepsilon_{I_0})$, i.e.,

$$\forall \varepsilon_i > 0\ \exists N(\varepsilon_{I_0}) : \quad q_c \in \mathscr{A}_2(\varepsilon_{I_0}), \quad \forall q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}, \quad \forall k \geq N(\varepsilon_{I_0}) \tag{4.100}$$

Moreover, for any point $q \in \mathscr{A}_2(\varepsilon_{I_0})$, therefore also for all remaining critical points, if $\gamma_d(q) = \|q - q_d\|^2$ then

$$0 < \nu_{i1}(q), \quad \forall q \in \mathscr{A}_2(\varepsilon_{I_0}) \tag{4.101}$$

Proof. By Propositions 3.2, 3.3 [23] we know that the only critical points remaining $q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}$ other than the destination q_d cannot arise in $\mathscr{F}_d, \partial\mathscr{F}$. By Proposition 3.4 [23] for any $\varepsilon_i > 0, i \in I_0$ there exists a $N(\varepsilon_{I_0})$ such that for all $k \geq N(\varepsilon_{I_0})$ no critical points $q_c \neq q_d$ exist in $\mathscr{F}_a$.

Then, critical points $q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}$ can arise only in the set "near" obstacles

$$q_c \in \mathscr{F}_n = \mathscr{A}_1(\varepsilon_{I_0}) \cup \mathscr{A}_2(\varepsilon_{I_0}), \quad \forall q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}, \quad \forall k \geq N(\varepsilon_{I_0}) \tag{4.102}$$

By Proposition 13 we have ensured that

$$\left(\nabla\beta_i\right)(q)\cdot\left(\nabla\gamma_d\right)(q) \leq 0 \implies q \notin \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}, \quad \forall k \geq N(\varepsilon_{I_0}) \tag{4.103}$$

By the previous two it follows that

$$\left.\begin{array}{c} q_c \in \mathscr{F}_n = \mathscr{A}_1(\varepsilon_{I_0}) \cup \mathscr{A}_2(\varepsilon_{I_0}), \quad \forall q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}, \quad \forall k \geq N(\varepsilon_{I_0}) \\ q_c \notin \mathscr{A}_1(\varepsilon_{I_0}), \quad \forall q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}, \quad \forall k \geq N(\varepsilon_{I_0}) \end{array}\right\} \implies$$
$$\begin{array}{l} q_c \in \mathscr{A}_2(\varepsilon_{I_0}), \quad \forall q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}, \quad \forall k \geq N(\varepsilon_{I_0}) \implies \\ 0 < (\nabla\beta_i)(q_c) \cdot (\nabla\gamma_d)(q_c), \qquad \forall q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}, \quad \forall k \geq N(\varepsilon_{I_0}) \end{array} \tag{4.104}$$

The above can also be expressed as

$$\mathscr{C}_{\hat{\varphi}} \setminus \{q_d\} \subseteq \mathscr{A}_2(\varepsilon_{I_0}), \quad \forall k \geq N(\varepsilon_{I_0}) \tag{4.105}$$

For a paraboloid attractive effect γ_d by Proposition 8

$$\nu_{i1}(q) = 2\frac{(\nabla\beta_i)(q) \cdot (\nabla\gamma_d)(q)}{\|(\nabla\gamma_d)(q)\|^2} \tag{4.106}$$

which has the same sign as $(\nabla\beta_i)(q) \cdot (\nabla\gamma_d)(q)$, hence

$$\begin{array}{l} 0 < (\nabla\beta_i)(q) \cdot (\nabla\gamma_d)(q), \quad \forall q \in \mathscr{A}_2(\varepsilon_{I_0}) \implies \\ 0 < \nu_{i1}(q), \quad \forall q \in \mathscr{A}_2(\varepsilon_{I_0}) \end{array} \tag{4.107}$$

Therefore, for all $k \geq N(\varepsilon_{I_0})$, at critical points it can only be $0 < \nu_{i1}(q_c)$, i.e.,

$$\left.\begin{array}{c} 0 < \nu_{i1}(q), \forall q \in \mathscr{A}_2(\varepsilon_{I_0}) \\ \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\} \subseteq \mathscr{A}_2(\varepsilon_{I_0}), \forall k \geq N(\varepsilon_{I_0}) \end{array}\right\} \implies$$
$$0 < \nu_{i1}(q), \quad \forall q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}, \quad \forall k \geq N(\varepsilon_{I_0}) \tag{4.108}$$

□

4.3 Sufficient Curvature Condition

4.3.1 Differential Geometry of Implicit Surfaces

We need to interpret term $\nu_{i2}(q, \hat{t}_i)$ in terms of differential geometry. This is provided in the work of Dombrowski [43], who treats the general n-dimensional case. This applies to Navigation Functions, which are defined over n-dimensional space. A simplified derivation for 3-dimensional space is provided by Hughes [46].

Let us denote the normal curvature of a surface along tangent unit vector $\hat{t}_i$ by $\kappa_{i,q}(\hat{t}_i)$. This is given by the second fundamental form II_q at q as

$$\kappa_{i,q}(\hat{t}_i) = \mathrm{II}_q(\hat{t}_i, \hat{t}_i) \tag{4.109}$$

Definition 15 (Weingarten map[47])**.** Let the Weingarten map[17] (or shape operator) at q be

$$\mathrm{L}_q : T_q B_i \to T_q B_i \tag{4.110}$$

Let $n_{B_i}(q) \perp B_i$ be the vector normal to B_i at point q. Suppose $\gamma : [-1, 1] \to B_i$ is a path on (hyper)surface B_i with $\gamma(0) = q$, which has tangent $t_i \in T_q B_i$. The Weingarten map is defined as

$$\mathrm{L}_q(t_i) \triangleq \frac{d\left(n_{B_i}\left(\gamma(t)\right)\right)}{dt}(0) \tag{4.111}$$

so it is the derivative of the surface normal $n_{B_i}(\gamma(t))$ at time $t = 0$, as $\gamma(t)$ passes through q in direction t_i.

Proposition 16 (Weingarten map for Implicit Surfaces [43, 46])**.** For the implicitly defined surface B_i the Weingarten map at q is equal to the linear mapping[18]

$$\mathrm{L}_q(\hat{t}_i) = \frac{1}{\|\nabla\beta_i\|}\left(D^2\beta_i\right)(q)\hat{t}_i, \quad \hat{t}_i \in UT_q B_i \tag{4.112}$$

The Weingarten map is related to the second fundamental form by

$$\mathrm{II}_q(X, Y) = \mathrm{L}_q(X) \cdot Y = X \cdot \mathrm{L}_q(Y), X, Y \in T_q B_i \tag{4.113}$$

This leads to the following expression for the normal curvature of implicit surface B_i at q along $\hat{t}_i$

$$\begin{aligned} \kappa_{i,q}(\hat{t}_i) &= \mathrm{II}_q(\hat{t}_i, \hat{t}_i) = \hat{t}_i \cdot \mathrm{L}_q(\hat{t}_i) \\ &= \hat{t}_i^{\mathrm{T}} \frac{1}{\|\left(\nabla\beta_i\right)(q)\|}\left(D^2\beta_i\right)(q)\hat{t}_i \\ &= \frac{\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i}{\|\left(\nabla\beta_i\right)(q)\|} \in (-\infty, +\infty) = \mathbb{R}, \quad \hat{t}_i \in UT_q B_i \end{aligned} \tag{4.114}$$

This derivation of normal curvature $\kappa_{i,q}$ of an implicitly defined surface connects it to the implicit function β_i defining the surface. This reveals the role of the *restricted* quadratic form $\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i, \hat{t}_i \in UT_q B_i$. Restriction is with respect to the surface's unit tangent space $UT_q B_i$ and is important to avoid misinterpretations. The principal directions are the eigenvectors of the Weingarten map. Hence, they are also the eigenvectors of the restricted quadratic form $\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i|_{\hat{t}_i \in UT_q B_i}$, but they are *not* (necessarily) eigenvectors of the Hessian matrix $\left(D^2\beta_i\right)(q)$.

[17] [47], § 4.7: The Second Fundamental Form and the Weingarten Map, pp.122-127.

[18] [46], § 1.4: The relation between N and ∇G, p.6.

Definition 17 (Radius of Normal Curvature)**.** We can also define the radius of normal curvature $R_{i,q}(\hat{t}_i)$ along tangent direction $\hat{t}_i$, as the inverse of the normal curvature at the same point (allowing $R_{i,q} = \pm\infty$ and understanding that this means flatness of the implicit surface along $\hat{t}_i$ at point q)

$$R_{i,q}(\hat{t}_i) \triangleq \frac{\|(\nabla\beta_i)(q)\|}{\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i} \in [-\infty, 0) \cup (0, +\infty] = \bar{\mathbb{R}} \setminus \{0\}, \quad \hat{t}_i \in UT_qB_i \tag{4.115}$$

Definition 18 (Convex, Nonconvex)**.** It follows that at q, in direction $\hat{t}_i \in UT_qB_i$, the surface B_i can be either

1. Convex if
$$0 < \hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i \implies 0 < \kappa_{i,q}(\hat{t}_i); \tag{4.116}$$
2. Nonconvex if
$$\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{t}_i \leq 0 \implies \kappa_{i,q}(\hat{t}_i) \leq 0. \tag{4.117}$$

Definition 19 (Principal curvatures, principal directions)**.** Let $\kappa_{i,q}(\hat{t}_i)$ be the normal curvature of surface B_i at point q along tangent direction $\hat{t}_i \in UT_qB_i$. The Weingarten map is represented in the tangent space by a linear symmetric operator, which has orthogonal eigenvectors

$$\hat{p}_{ij}(q) \in UT_qB_i, \quad i \in I_0, \quad j \in \{1, 2, \ldots, n\} \tag{4.118}$$

and real eigenvalues

$$\kappa_{ij}(q) \in \mathbb{R}, \quad i \in I_0, \quad j \in \{1, 2, \ldots, n\} \tag{4.119}$$

associated to them. These eigenvectors $\hat{p}_{ij}(q)$ are called principal directions at q and their associated eigenvalues $\kappa_{ij}(q)$ are called principal curvatures at q[19].

From the definition of normal curvature and radius of normal curvature it follows that for an implicitly defined surface β_i, the principal curvatures and principal radii of curvature are related to their associated principal directions as follows

$$\begin{aligned} \kappa_{ij}(q) = \kappa_{i,q}(\hat{p}_{ij}) &= \frac{\hat{p}_{ij}^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{p}_{ij}}{\|\left(\nabla\beta_i\right)(q)\|} \\ R_{ij}(q) = R_{i,q}(\hat{p}_{ij}) &= \frac{\|\left(\nabla\beta_i\right)(q)\|}{\hat{p}_{ij}^{\mathrm{T}}\left(D^2\beta_i\right)(q)\hat{p}_{ij}} \end{aligned} \tag{4.120}$$

[19][47], § 4.8: Principal, Gaussian, Mean, and Normal Curvatures, pp.128-141. In particular Definition: The principal curvatures of a surface M at a point p are the eigenvalues of L_q there. Corresponding unit eigenvectors are called principal directions at p.

4.3.2 Geometric interpretation for any γ_d

Following from the previous definition and the fact that q can be a critical point only for destinations q_d which are in the "bad" set $\mathscr{H}_{i2}(q)$, we provide the following useful definitions.

Definition 20 (Sufficiently curved direction $\hat{t}_i$)**.** A direction $\hat{t}_i \in UT_q\partial\mathscr{O}_i$ is called

1. Sufficiently curved if

$$\nu_i(q, \hat{t}_i) < 0, \quad \forall q_d \in \mathscr{H}_{i2}(q); \tag{4.121}$$

2. Convex but not sufficiently curved if

$$\hat{t}_i^{\mathrm{T}} \left(D^2\beta_i\right)(q)\hat{t}_i > 0 \quad \text{but} \quad \exists q_d \in \mathscr{H}_{i2}(q) : \ \nu_i(q, \hat{t}_i) \geq 0; \tag{4.122}$$

3. Nonconvex if

$$\hat{t}_i^{\mathrm{T}} \left(D^2\beta_i\right)(q)\hat{t}_i \leq 0 \implies 0 < \nu_i(q, \hat{t}_i), \quad \forall q_d \in \mathscr{H}_{i2}(q). \tag{4.123}$$

Note that since we are working with $\mathscr{H}_{i2}(q)$ where $0 < \nu_{i1}(q)$ sufficient curvature implies $0 < \hat{t}_i^{\mathrm{T}} \left(D^2\beta_i\right)(q)\hat{t}_i$ (because if it were $\hat{t}_i^{\mathrm{T}} \left(D^2\beta_i\right)(q) \leq 0$ then $0 < \nu_{i2}(q, \hat{t}_i)$ and hence $\nu_i(q, \hat{t}_i) = \nu_{i1}(q) + \nu_{i2}(q, \hat{t}_i) > 0$, which is contrary to the hypothesis of sufficient curvature).

Also, note that we have covered all possible cases. A direction can be either convex or nonconvex. If nonconvex, then $\nu_i(q, \hat{t}_i)$ can only be nonpositive for $q_d \in \mathscr{H}_{i2}(q)$. If convex, then either $\nu_i(q, \hat{t}_i) < 0, \forall q_d \in \mathscr{H}_{i2}$, or there exists a $q_d \in \mathscr{H}_{i2}$ for which this does not hold. There is no other case left.

The first case is convex sufficiently curved, the second is convex insufficiently curved, the third one is nonconvex hence necessarily insufficiently curved.

Definition 21 (Sufficiently curved point)**.** A point $q \in \partial\mathscr{O}_i$ is called sufficiently curved, with respect to β_i, if every tangent $\hat{t}_i$ at q is sufficiently curved, i.e.,

$$\nu_i(q, \hat{t}_i) < 0, \quad \forall q_d \in \mathscr{H}_{i2}(q), \quad \forall \hat{t}_i \in UT_q\partial\mathscr{O}_i. \tag{4.124}$$

Definition 22 (Everywhere sufficiently curved obstacle)**.** An obstacle β_i is called everywhere sufficiently curved if every boundary point of it is sufficiently curved, i.e.,

$$\nu_i(q, \hat{t}_i) < 0, \quad \forall q_d \in \mathscr{H}_{i2}(q), \quad \forall \hat{t}_i \in UT_q\partial\mathscr{O}_i, \quad \forall q \in \partial\mathscr{O}_i \tag{4.125}$$

Definition 23 (Everywhere sufficiently curved world)**.** A world $\mathscr{F}$ is called everywhere sufficiently curved if all its obstacles $\mathscr{O}_i$ are everywhere sufficiently curved, i.e.,

$$\nu_i(q, \hat{t}_i) < 0, \quad \forall q_d \in \mathscr{H}_{i2}(q), \quad \forall \hat{t}_i \in UT_q\partial\mathscr{O}_i, \quad \forall q \in \partial\mathscr{O}_i, \quad \forall i \in I_0 \tag{4.126}$$

Proposition 24 (Principal curvatures bound curvature)**.** For the restricted quadratic form $\hat{t}_i^{\mathrm{T}} \left(D^2\beta_i\right)(q)$ it holds that

$$\begin{gathered}\hat{p}_{ij_{\min}}(q)^{\mathrm{T}} \left(D^2\beta_i\right)(q)\hat{p}_{ij_{\min}}(q) \leq \hat{t}_i^{\mathrm{T}} \left(D^2\beta_i\right)(q)\hat{t}_i \leq \hat{p}_{ij_{\max}}(q)^{\mathrm{T}} \left(D^2\beta_i\right)(q)\hat{p}_{ij_{\max}}(q), \\ \forall \hat{t}_i \in UT_qB_i, \quad \forall q \in \mathscr{F},\end{gathered} \tag{4.127}$$

where $\hat{p}_{ij_{\min}}(q), \hat{p}_{ij_{\max}}(q)$ are the principal directions at q which correspond to the minimal and maximal principal curvatures $\kappa_{ij_{\min}}(q), \kappa_{ij_{\max}}(q)$, respectively.

Proof. Since $\beta_i \in C^{(2)}[\mathscr{F}, [0, +\infty)]$, consider that principal directions are eigenvectors of the Weingarten map, expressed as

$$\hat{p}_{ij}(q)^{\mathrm{T}} l \hat{p}_{ij}(q) = \frac{1}{\|(\nabla\beta_i)(q)\|} \hat{p}_{ij}(q)^{\mathrm{T}} \left(D^2\beta_i\right)(q) \hat{p}_{ij}(q) \tag{4.128}$$

Then, taking into consideration Proposition 32 about eigenvalues and eigenvectors, provided $\|(\nabla\beta_i)(q)\| > 0$ and because this is constant for all $\hat{t}_i$ at a certain point q, it follows that

$$\begin{aligned} \hat{p}_{ij_{\min}}(q)^{\mathrm{T}} l \hat{p}_{ij_{\min}}(q) &\leq \hat{t}_i^{\mathrm{T}} l \hat{t}_i \\ &\leq \hat{p}_{ij_{\max}}(q)^{\mathrm{T}} l \hat{p}_{ij_{\max}}(q), \quad \forall \hat{t}_i \in UT_q B_i, \quad \forall q \in \mathscr{F} \overset{\|(\nabla\beta_i)(q)\|>0}{\Longrightarrow} \\ \hat{p}_{ij_{\min}}(q)^{\mathrm{T}} \left(D^2\beta_i\right)(q) \hat{p}_{ij_{\min}}(q) &\leq \hat{t}_i^{\mathrm{T}} \left(D^2\beta_i\right)(q) \hat{t}_i \\ &\leq \hat{p}_{ij_{\max}}(q)^{\mathrm{T}} \left(D^2\beta_i\right)(q) \hat{p}_{ij_{\max}}(q), \quad \forall \hat{t}_i \in UT_q B_i, \quad \forall q \in \mathscr{F} \end{aligned} \tag{4.129}$$

where $\hat{p}_{ij_{\min}}(q), \hat{p}_{ij_{\max}}(q)$ are the principal directions which correspond to the minimal and maximal principal curvatures $\kappa_{ij_{\min}}, \kappa_{ij_{\max}}$, respectively. □

Remark 25. Caution is required above, because the inequality holds because $\hat{t}_i$ are eigenvectors of the Weingarten map in the tangent space. This linear operator can be expressed using a matrix l which is an $(n-1) \times (n-1)$ matrix. It has been proved that the two quadratic forms

$$\frac{1}{\|(\nabla\beta_i)(q)\|} \hat{t}_i^{\mathrm{T}} \left(D^2\beta_i\right)(q) \hat{t}_i = \hat{t}_i^{\mathrm{T}} l \hat{t}_i, \quad \hat{t}_i \in UT_q B_i \tag{4.130}$$

are related by the constant at q factor $\frac{1}{\|(\nabla\beta_i)(q)\|}$.

Hence, it seems to appear that $(D^2\beta_i)(q)$ has eigenvectors $\hat{p}_{ij}(q)$. This is not true. The reason is that $(D^2\beta_i)(q)$ is an operator on the whole tangent space $UT_q\mathscr{F}$, not only in UT_qB_i. As a result, it has a different eigensystem. Viewed in another way, the matrix representing $(D^2\beta_i)(q)$ is an $n \times n$ matrix and when acting on $\hat{t}_i$, these are expressed not as $(n-1) \times 1$ vectors in UT_qB_i, but as $n \times 1$ vectors in $UT_q\mathscr{F}$. So $\hat{p}_{ij}(q)$ are eigenvectors of the restricted quadratic form

$$\frac{1}{\|(\nabla\beta_i)(q)\|} \hat{t}_i^{\mathrm{T}} \left(D^2\beta_i\right)(q) \hat{t}_i, \hat{t}_i \in UT_q B_i \tag{4.131}$$

Nonetheless, it is worth noting that if m principal directions $\hat{p}_{ij}(q)$ have e.g. positive principal curvatures, then according to a Proposition proved later, the subspace of T_qB_i spanned by them is positive definite. Hence l is positive definite in it. But also $(D^2\beta_i)(q)$ is positive definite in it. Since it is positive definite in an m dimensional subspace of T_qB_i, it is positive definite in the same m dimensional subspace of $T_q\mathscr{F}$. This implies that the Hessian matrix $(D^2\beta_i)(q)$ also has m positive eigenvalues and m associated eigenvectors in $T_q\mathscr{F}$ at q. But these are not necessarily within T_qB_i.

The above states that principal curvatures are stationary values of curvature in T_qB_i. This allows us to express the inequality in terms of minimal curvature (maximal radius of curvature $R_{i,q}(\hat{p}_{ij_{\min}}(q_c))$) and associated tangential direction $\hat{p}_{ij_{\min}}(q_c)$ of minimal curvature of level set B_i at critical point q_c.

At this point it is important to note that tangential direction $\hat{t}_i$ has been selected as a suitable direction. As already discussed, if the condition holds for *at least one* such direction (not necessarily tangential) then a local minimum cannot arise at q_c.

But this does not guarantee that there exists a direct sum decomposition to two submanifolds with Hessian positive definite in one of them and negative definite in the other one, so that it can be proved to be non-degenerate.

Since the condition is expressed in terms of an arbitrary tangential direction, all radii of curvature between maximal $R_{i,q}\left(\hat{p}_{ij_{\min}}(q_c)\right)$ and minimal $R_{i,q}\left(\hat{p}_{ij_{\max}}(q_c)\right)$ are to be considered.

In particular cases[20] this may be provable. Namely that at least a single direction of negative definiteness suffices as an escape direction because its orthogonal complement[21] is a positive definite submanifold of the Hessian, assuring Hessian $D^2\varphi$ non-degeneracy. In those cases we are interested with what happens for two certain (principal) directions $\hat{t}_i$ and associated curvatures $R_{i,q}(\hat{t}_i)$.

But to obtain a general result that holds for the whole class of implicit functions satisfying the derived condition, from now on we will constrain curvature $\kappa_{i,q}(\hat{t}_i)$ in *all* directions $\hat{t}_i$ at a critical point q_c.

In (4.78) we require

$$\nu_i(q_c) = \frac{\left(\nabla\beta_i\right)(q_c)\cdot\left(\nabla\gamma_d\right)(q_c)}{\left\|\left(\nabla\gamma_d\right)(q_c)\right\|^2}\hat{t}_i^{\mathrm{T}}\left(D^2\gamma_d\right)(q_c)\hat{t}_i - \hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q_c)\hat{t}_i < 0 \tag{4.132}$$

which for q_d in the half-space $\left(\nabla\beta_i\right)(q_c)\cdot\left(\nabla\gamma_d\right)(qc) > 0$ cannot hold if $\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q_c)\hat{t}_i \leq 0$. This is why we initially require that β_i has positive Gaussian curvature close to the obstacle. Then we can select a $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$ small enough for positive definiteness to hold at q_c confined within $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$.

Therefore in order for the relative curvature condition to be proved for all tangential directions, the Hessian $\left(D^2\beta_i\right)(q_c)$ should be positive definite in the tangent space at q_c

$$\hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q_c)\hat{t}_i > 0, \quad \forall\hat{t}_i \in U\mathscr{T}_i(q_c) \tag{4.133}$$

For $\left\|\left(\nabla\beta_i\right)(q_c)\right\| > 0$ this is equivalent to only positive curvature allowable

$$\begin{aligned} &\left\{\begin{array}{c} \hat{t}_i^{\mathrm{T}}\left(D^2\beta_i\right)(q_c)\hat{t}_i > 0, \forall\hat{t}_i \in U\mathscr{T}_i(q_c) \\ \left\|\left(\nabla\beta_i\right)(q_c)\right\| > 0 \end{array}\right\} \iff \\ &\left\{\begin{array}{c} \kappa_{i,q_c}(\hat{t}_i) = \frac{\hat{t}_i(q_c)^{\mathrm{T}}\left(D^2\beta_i\right)(q_c)\hat{t}_i(q_c)}{\left\|\left(\nabla\beta_i\right)(q_c)\right\|} > 0, \forall\hat{t}_i \in U\mathscr{T}_i(q_c) \\ \left\|\left(\nabla\beta_i\right)(q_c)\right\| > 0 \end{array}\right\} \end{aligned} \tag{4.134}$$

Therefore the *minimal* curvature at a critical point should be positive

$$\min_{\hat{t}_i}\left\{\kappa_{i,q_c}(\hat{t}_i)\right\} > 0 \tag{4.135}$$

This is equivalent to all other directional curvatures along $\hat{t}_i$ at q_c being positive, following from Proposition 24.

So all directional curvatures $\kappa_{i,q_c}(\hat{t}_i)$ and as a result radii of curvature $R_{i,q_c}(\hat{t}_i)$ are required to be positive

$$\begin{aligned} \min_{\hat{t}_i}\left\{\kappa_{i,q_c}(\hat{t}_i)\right\} > 0 &\implies \kappa_{i,q_c}(\hat{t}_i) > 0, \quad \forall\hat{t}_i \in U\mathscr{T}_i(q_c) \\ &\implies R_{i,q_c}(\hat{t}_i) = \frac{1}{\kappa_{i,q_c}(\hat{t}_i)} > 0, \quad \forall\hat{t}_i \in U\mathscr{T}_i(q_c) \end{aligned} \tag{4.136}$$

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[20] Such as a hyperboloids: the span of the axis direction and $\nabla\beta_i$ constitutes a positive definite submanifold, whereas the tangential direction of maximal level set curvature defines a negative definite submanifold, their combination is a direct sum decomposition.

[21] Or the negative submanifold may be a superset and its orthogonal complement be a positive definite submanifold.

For the remainder of this section we will work with $R_{i,q_c}(\hat{p}_{ij_{\min}}(q_c))$ which means that a condition will be derived for $D^2\varphi$ to be negative definite in the tangent space at q_c. This can then be combined with positive definiteness in the "radial" submanifold (span of $\nabla\beta_i$) to prove Hessian $D^2\varphi$ non-degeneracy. So (4.79) can be restated in terms of directional radius of curvature as

$$\cos(\theta_i(q_c))R_{i,q_c}(\hat{t}_i) < \frac{\|(\nabla\gamma_d)(q_c)\|}{\hat{t}_i^{\mathrm{T}}(D^2\gamma_d)(q_c)\hat{t}_i} \tag{4.137}$$

A sufficient condition for this to hold in all tangential directions is

$$\cos(\theta_i(q_c))R_{i,q_c}(\hat{p}_{ij_{\min}}(q_c)) < \min_{\hat{t}_i \in U\mathscr{T}_i(q_c)} \left\{ \frac{\|(\nabla\gamma_d)(q_c)\|}{\hat{t}_i^{\mathrm{T}}(D^2\gamma_d)(q_c)\hat{t}_i} \right\} \tag{4.138}$$

or more compactly (remembering evaluation at critical point q_c)

$$\cos(\theta_i)R_{i,q_c} < \min_{\hat{t}_i \in U\mathscr{T}_i(q_c)} \left\{ \frac{\|\nabla\gamma_d\|}{\hat{t}_i^{\mathrm{T}}D^2\gamma_d\hat{t}_i} \right\} \tag{4.139}$$

Condition (4.138) requires that the maximal radius of curvature $R_{i,q_c}(\hat{p}_{ij_{\min}}(q_c))$ of implicit surface $B_i(q_c)$ along tangential direction of minimal curvature $\hat{p}_{ij_{\min}}(q_c) \in UT_qB_i(q_c)$ projected on the normal to implicit surface

$$\Gamma(q) \triangleq \gamma_d^{-1}(\gamma_d(q)) \tag{4.140}$$

should be smaller than $\min_{\hat{t}_i \in U\mathscr{T}_i(q_c)} \left\{ \frac{\|(\nabla\gamma_d)(q_c)\|}{\hat{t}_i^{\mathrm{T}}(D^2\gamma_d)(q_c)\hat{t}_i} \right\}$. Hence

$$\left.\begin{array}{r} R_{i,q_c}(q_c) \in (0, R_{i,q_c}(q_c)], \quad \forall \hat{t}_i \in U\mathscr{T}_i(q_c) \\ R_{i,q_c}(\hat{p}_{ij_{\min}}(q_c)) < \min_{\hat{t}_i \in U\mathscr{T}_i(q_c)} \left\{ \frac{\|(\nabla\gamma_d)(q_c)\|}{\hat{t}_i^{\mathrm{T}}(D^2\gamma_d)(q_c)\hat{t}_i} \right\} \end{array}\right\} \implies R_{i,q_c}(\hat{t}_i) < \min_{\hat{t}_i \in U\mathscr{T}_i(q_c)} \left\{ \frac{\|\nabla\gamma_d(q_c)\|}{\hat{t}_i^{\mathrm{T}}(D^2\gamma_d)(q_c)\hat{t}_i} \right\} \tag{4.141}$$

Note that since $\hat{t}_i \in U\mathscr{T}_i(q_c)$ and not necessarily $\hat{t}_i \in UT_q\Gamma(q_c)$ it follows that quantity

$$\frac{\|(\nabla\gamma_d)(q_c)\|}{\hat{t}_i^{\mathrm{T}}(D^2\gamma_d)(q_c)\hat{t}_i}$$

cannot be readily interpreted as radius of curvature. In subsection 4.4.1 we will see that for a paraboloid selection of attraction γ_d the quantity $\frac{\|\nabla\gamma_d\|}{\hat{t}_i^{\mathrm{T}}D^2\gamma_d\hat{t}_i}$ is the radius of curvature of γ_d level set $\Gamma(q)$ passing through point q_c.

Note that such a sufficient "relative curvature" condition is stronger than mere convexity. Maybe it could be termed a "relative convexity" condition, but "relative curvature" has been selected to better express the role of curvature.

4.4 Sufficient Curvature Condition for paraboloid γ_d

There is the need to replace Lemma 3.5 [23] because it no longer holds. This is treated in subsection 4.4.2, namely in which neighborhood of an obstacle the negativity condition $\nu_i(q, \hat{t}_i) < 0$ holds.

4.4.1 Geometry of Sufficient Curvature for paraboloid γ_d

For $\gamma_d(q) = \|q - q_d\|^2$ sufficient curvature has an interesting interpretation. Let

$$\mathscr{S}(q_a, \rho) \triangleq \{q \in E^n | \ \|q - q_a\| \leq \rho\} \tag{4.142}$$

be a sphere with center q_a, radius ρ. For $\theta_i(q_c) \in [-\pi, +\pi]$ the left hand side

$$\cos(\theta_i(q_c))R_{i,q_c}(\hat{t}_i)$$

in (4.138) defines[22] the boundary of a "curvature sphere"[23].

Definition 26 (Curvature Sphere)**.** We will call curvature sphere along $\hat{t}_i$ the sphere tangent to $B_i(q)$ at q, defined as

$$\mathscr{S}_{ci}(q, \hat{t}_i) \triangleq \mathscr{S}\left(q - \frac{1}{2}R_{i,q}(\hat{t}_i)\hat{r}_i, \quad \frac{1}{2}R_{i,q}(\hat{t}_i)\right). \tag{4.143}$$

of center $q_{ci}(q, \hat{t}_i)$ and radius $\rho_{ci}(q, \hat{t}_i)$ defined as

$$q_{ci}(q, \hat{t}_i) \triangleq q_c - \rho_{ci}(q, \hat{t}_i)\hat{r}_i, \quad \rho_{ci}(q, \hat{t}_i) \triangleq \frac{1}{2}R_{i,q}(\hat{t}_i) \tag{4.144}$$

Proposition 27 (Sufficiently curved $\hat{t}_i \iff \{q_d \notin \mathscr{S}_{ci}(q, \hat{t}_i) \wedge R_{i,q}(\hat{t}_i) > 0\}$)**.** If $\gamma_d(q) = \|q - q_d\|^2$ and $\|\nabla\beta_i\| > 0$, sufficiently curved $\hat{t}_i$ is equivalent to $q_d \notin \mathscr{S}_{ci}(q, \hat{t}_i)$, Fig. 4.5.

Proof. By Proposition 10, Proposition 16 sufficient curvature is written

$$\nu_i = \|\nabla\beta_i\| \left(\cos(\theta_i)\frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d \hat{t}_i}{\|\nabla\gamma_d\|} - \frac{\hat{t}_i^{\mathrm{T}} D^2\beta_i \hat{t}_i}{\|\nabla\beta_i\|}\right) < 0, \quad \forall q_d \in \mathscr{H}_{i2}(q) \tag{4.145}$$

When $\gamma_d = \|q - q_d\|^2$ it is $\frac{\hat{t}_i^{\mathrm{T}} D^2\gamma_d \hat{t}_i}{\|\nabla\gamma_d\|} = \frac{1}{\|q - q_d\|}$. Moreover, $R_{i,q}(\hat{t}_i) = \frac{\nabla\beta_i}{\hat{t}_i^{\mathrm{T}} D^2\beta_i \hat{t}_i}$, hence the previous becomes

$$\nu_i = \|\nabla\beta_i\| \left(\cos(\theta_i)\frac{I}{\|q - q_d\|} - \frac{1}{R_{i,q}(\hat{t}_i)}\right) < 0, \quad \forall q_d \in \mathscr{H}_{i2}(q) \tag{4.146}$$

By definition of ε_{i4} it is $\|\nabla\beta_i\| > 0, \forall q \in \overline{\mathscr{B}_i(\varepsilon_{i4})}$. Also, $\cos(\theta_i)\frac{1}{\|q - q_d\|} > 0$ for $\theta_i \in \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$, therefore $R_{i,q}(\hat{t}_i)$ should be positive. As a result, and since $q \neq q_d \implies \|q - q_d\| > 0$, it follows that

$$\begin{aligned} &\left\{R_{i,q}(\hat{t}_i) > 0 \quad \wedge \quad \cos(\theta_i)R_{i,q}(\hat{t}_i) < \|q - q_d\|, \quad \forall q_d \in \mathscr{H}_{i2}(q)\right\} \iff \\ &\left\{R_{i,q}(\hat{t}_i) > 0 \quad \wedge \quad q_d \notin \mathscr{S}_{ci}\left(q, \hat{t}_i\right), \quad \forall q_d \in \mathscr{H}_{i2}(q)\right\}, \end{aligned} \tag{4.147}$$

which is interestingly related to Meusnier's Theorem [48]. This is schematically illustrated in Fig. 4.5. Level sets involved are shown in Fig. 4.7. □

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[22]This is because two intersecting orthogonal lines through different fixed points -here q and $q - \frac{1}{2}R_{i,q}\hat{r}_i$ at q- define a sphere.

[23]A curvature sphere is usually defined in literature as a sphere with center the center of curvature and radius the radius of curvature. Hence the spheres called "curvature spheres" here are half-radius curvature spheres. Nonetheless here we will call them just curvature spheres.

The above is useful because it specifies a class of acceptable obstacle geometries, since

$$q_d \notin \mathscr{S}_{ci}\left(q,\, \hat{t}_i\right), \quad \forall q_d \in \mathscr{H}_{i2}(q) \iff \mathscr{S}_{ci}\left(q,\, \hat{t}_i\right) \cap \mathscr{H}_{i2}(q) = \emptyset. \tag{4.148}$$

Suppose we want to ensure that for a *given* q_d, there is a neighborhood $\mathscr{B}_i\left(\varepsilon_i\right)$ in which

$$q_d \notin \mathscr{S}_{ci}\left(q,\, \hat{t}_i\right), \quad \forall q \in \mathscr{B}_i\left(\varepsilon_i\right) \tag{4.149}$$

holds for a certain number of principal directions $\hat{p}_{ij}(q)$ at q. By Proposition 31 this is equivalent to the request of sufficient curvature on $\partial\mathscr{O}_i$. We can then ensure the previous if

$$\mathscr{S}_{cij}(q) \triangleq \mathscr{S}_{ci}\left(q,\, \hat{p}_{ij}(q)\right) \subseteq \mathscr{O}_i \cup \{q\}, \quad \forall q \in \partial\mathscr{O}_i \tag{4.150}$$

This requires that at every obstacle boundary point the desired number of *principal* curvature spheres $\mathscr{S}_{cij}(q)$ be subsets of the obstacle.

Note that since by definition $\mathscr{S}_{ci}\left(q,\, \hat{t}_i\right) \cap \overline{\mathscr{O}_i} = \{q\}$ and $q \neq q_d \implies \{q\} \neq \{q_d\}$, as a result

$$\mathscr{S}_{ci}\left(q,\, \hat{t}_i\right) \cap \mathscr{H}_{i2}(q) = \emptyset \iff \left(\mathscr{S}_{ci}\left(q,\, \hat{t}_i\right) \setminus \{q\}\right) \cap \mathscr{H}_{i2}(q) = \emptyset \tag{4.151}$$

For this reason, expressed in terms of the union of curvature spheres, the condition is

$$\begin{aligned} &\mathscr{S}_{ci}\left(q,\, \hat{p}_{ij}(q)\right) \subseteq \mathscr{O}_i \cup \{q\}, \quad \forall q \in \partial\mathscr{O}_i \iff \\ &\qquad \bigcup_{q \in \partial\mathscr{O}_i} \left(\mathscr{S}_{ci}\left(q,\, \hat{p}_{ij}(q)\right) \setminus \{q\}\right) \subseteq \mathscr{O}_i \end{aligned} \tag{4.152}$$

It is interesting to note that condition $\bigcup_{q \in \partial\mathscr{O}_i} \left(\mathscr{S}_{ci}\left(q,\, \hat{p}_{ij}(q)\right)\right) \subseteq \overline{\mathscr{O}_i}$ would have been wrong, because it does not ensure that each sphere has one its q as the unique common point with $\partial\mathscr{O}_i$.

Note that the concept of sufficient curvature can only[24] be applied on $\partial\mathscr{O}_i$, because it refers to all $q_d \in \mathscr{F}$. What *is* meaningful on other level sets is $\nu_i(q, \hat{p}_{ij})$ sign definiteness. Given q_d, this can always be ensured in a neighborhood $\mathscr{B}_i\left(\varepsilon_i\right)$, induced by sufficient curvature on $\partial\mathscr{O}_i$.

[24] For points q on level sets $\beta_i(q) > 0$, any curvature sphere there has non-empty intersection with the free space interior $\mathscr{S}_{ci}\left(q,\, \hat{t}_i\right) \cap \mathscr{F} \neq \emptyset$. As a result, there always exists a $q_d \in \mathscr{F} \cap \mathscr{S}_{ci}\left(q,\, \hat{t}_i\right)$ we can select, leading to $\nu_i(q, \hat{t}_i) \geq 0$.

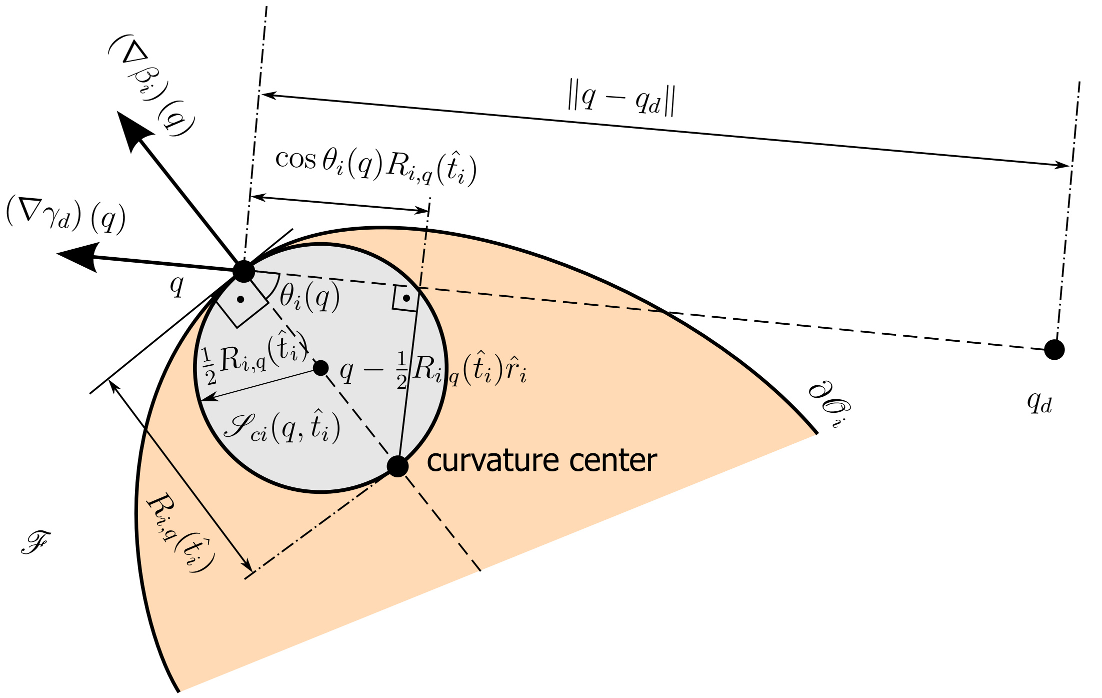


Figure 4.5: Relative curvature constraint at a critical point $q \in \partial\mathscr{O}_i$ for non-spherical obstacle β_i and paraboloid attractive effect γ_d.

4.4.2 Obstacle relative curvature induced to its "bad" neighborhood

By definition every obstacle closure $\overline{\mathscr{O}_i}$ is compact. This implies that the obstacle boundary $\partial\mathscr{O}_i$ is compact. The obstacle boundary is the zero level set $\beta_i^{-1}(0)$ of β_i, so the zero level set is also compact.

As a result, there exists a neighborhood of level sets $\beta_i^{-1}(c_1)$, with $c_1 \in [0, a_1), a_1 > 0$, such that each set $\beta_i^{-1}(c_1)$ is compact.

We set $a_2 = \min\{a_1, \varepsilon_i\}$. Then level set $\beta_i^{-1}(c_2), c_2 \in [0, a_2)$ is in the interior of neighborhood $\mathscr{B}_i(\varepsilon_i)$. For $k \geq N(\varepsilon_{I_0})$ the only critical points other than the destination have been proved to arise in $\mathscr{F}_n \cap \mathscr{A}_2$, the intersection of the neighborhoods with the "bad" set.

If $q_c \neq q_d$ is a critical point, then $q_c \in \mathscr{F}_n \cap \mathscr{A}_2, k \geq N(\varepsilon_{I_0})$. So, there will exist a $i \in I_0$, such that q_c belongs to the intersection of the neighborhood of $\mathscr{B}_i(\varepsilon_i)$ with the "bad" set $\mathscr{A}_2$, i.e., $\forall k \geq N(\varepsilon_{I_0}$ it holds that $\forall q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}\ \exists i \in I_0 :\ q_c \in \mathscr{B}_i(\varepsilon_i) \cap \mathscr{A}_2$.

Then, there will exist a level set $\beta_i^{-1}(\beta_i(q_c))$ through q_c, with $c_2 = \beta_i(q_c) \in (0, a_1)$. Since $q_c \in \mathscr{A}_2$ it follows that $\nu_{i1}(q_c) > 0$. Note that this implies $\nu_{i3}(q_c) > 0$, which important because we will work with $\nu_{i3}(q_c)$ and $\nu_{i4}(q, \hat{t}_i)$.

Proposition 28 (Continuity of ν_i in the tangent bundle)**.** For any point $q \in \beta_i^{-1}(c_2), c_2 \in [0, a_2]$ with $\nu_{i3}(q) > 0$ there exists a continuous function $r(q) > 0$, such that for the closed ball $\overline{B(q, r(q))}$ centered at q with radius r, the following hold

1. $\nu_{i1}(q') > 0, \forall q' \in \overline{B(q, r(q))}$;
2. Points $q, q' \in \overline{B(q, r(q))}$ have the same number of sufficiently curved principal directions $I^-(q) = I^-(q')$ where

$$I^-(q) \triangleq |\{j \in \{1, 2, \ldots, n\} | \nu_i(q, \hat{p}_{ij}(q)) < 0\}| \tag{4.153}$$

and the same number of directions with positive ν_i, $I^+(q) = I^+(q')$ where

$$I^+(q) \triangleq |\{j \in \{1, 2, \ldots, n\} | \nu_i(q, \hat{p}_{ij}(q)) > 0\}| \tag{4.154}$$

Proof. Let $q \in \beta_i^{-1}(c_2), c_2 \in [0, a_2]$ with $\nu_{i3}(q) > 0$. Function $\nu_{i3}(q) = 2\frac{(\nabla\beta_i)(q)\cdot(\nabla\gamma_d)(q)}{\|(\nabla\beta_i)(q)\|\|(\nabla\gamma_d)(q)\|^2}$ is C^2 because both β_i and γ_d are C^2.

Then by continuity for every $\Delta\nu_3 > 0$ there exists an open neighborhood $U_1(\Delta\nu_3) \neq \emptyset$ with $q \in U_1(\Delta\nu_3)$ such that $\nu_{i3}(q')$ is as close to $\nu_{i3}(q)$, as we want, i.e.,

$$\begin{aligned} \forall\Delta\nu_3 > 0\ \exists U_1(\Delta\nu_3) \neq \emptyset :\quad & q \in \mathring{U}_1(\Delta\nu_3) \wedge \\ & |\nu_{i3}(q') - \nu_{i3}(q)| < \Delta\nu_3, \quad \forall q' \in U_1(\Delta\nu_3) \implies \\ & \nu_{i3}(q) - \Delta\nu_3 < \nu_{i3}(q') < \nu_{i3}(q) + \Delta\nu_3, \quad \forall q' \in U_1(\Delta\nu_3) \end{aligned} \tag{4.155}$$

This also implies that there exists a $\Delta\nu_{3\max}$ such that for every $0 < \Delta\nu_3 < \Delta\nu_{3\max}$ functions $\nu_{i3}(q)$ and $\nu_{i1}(q')$ have the same sign $\forall q' \in U_1(\Delta\nu_3)$. Then, $\nu_{i3}(q) > 0 \implies \nu_{i3}(q') > 0$.

The quadratic form associated with the Weingarten map $\hat{t}_i^{\mathrm{T}} l \hat{t}_i$ is equal to $\hat{t}_i^{\mathrm{T}} \left(D^2\beta_i\right)(q)\hat{t}_i$, where $\hat{t}_i \perp \nabla\beta_i$. Since β_i is C^2, also the gradient $\nabla\beta_i$ and the Hessian matrix $\left(D^2\beta_i\right)(q)$ are continuous functions of q. From this it follows that the eigenvalues of the Weingarten map are continuous functions of q.

This essentially states that continuity of the implicit function β_i ensures continuity of the principal curvatures of its level sets (i.e., that level sets which are "close" have principal curvatures which are "close", for pairs of points on each of them). This fact is what we aim to prove so that we can use it.

From the continuity of the Weingarten map principal curvatures, it follows that $\forall\Delta\kappa > 0$ there exists an open neighborhood $U_2(\Delta\kappa) \neq \emptyset$, such that $q \in U_2(\Delta\kappa)$ and there exists a bijective correspondence $m(j)$ of the eigenvalues of the Weingarten map at q, i.e., principal curvatures $\kappa_{ij}(q)$, with the eigenvalues $\kappa_{im(j)}(q')$ of the Weingarten map at $q' \in U_2(\Delta\kappa)$, such that $\kappa_{im(j)}(q')$ is closer to $\kappa_{ij}(q)$ than $\Delta\kappa$, i.e.,

$$\begin{aligned} \forall\Delta\kappa > 0\ \exists U_2(\Delta\kappa) \neq \emptyset :\quad & q \in \mathring{U}_2(\Delta\kappa) \wedge \\ & \left|\kappa_{im(j)}(q') - \kappa_{ij}(q)\right| < \Delta\kappa, \quad \forall j \in \{1, 2, \ldots, n\}, \quad \forall q' \in U_2(\Delta\kappa) \iff \\ & -\Delta\kappa < \kappa_{im(j)}(q') - \kappa_{ij}(q) < \Delta\kappa, \quad \forall j \in \{1, 2, \ldots, n\}, \quad \forall q' \in U_2(\Delta\kappa) \iff \\ & \kappa_{ij}(q) - \Delta\kappa < \kappa_{im(j)}(q') < +\kappa_{ij}(q) + \Delta\kappa, \quad \forall j \in \{1, 2, \ldots, n\}, \quad \forall q' \in U_2(\Delta\kappa) \end{aligned} \tag{4.156}$$

Since $\kappa_{ij}(q) = -\nu_{i4}(q, \hat{p}_{ij}(q))$, it follows that

$$\begin{aligned} & -\nu_{i4}(q, \hat{p}_{ij}(q)) - \Delta\kappa < -\nu_{i4}(q, \hat{p}_{im(j)}(q')) < -\nu_{i4}(q, \hat{p}_{ij}(q)) + \Delta\kappa, \forall j \in \{1, 2, \ldots, n\}, \forall q' \in U_2(\Delta\kappa) \implies \\ & \nu_{i4}(q, \hat{p}_{ij}(q)) - \Delta\kappa < \nu_{i4}(q, \hat{p}_{im(j)}(q')) < \nu_{i4}(q, \hat{p}_{ij}(q)) + \Delta\kappa, \forall j \in \{1, 2, \ldots, n\}, \forall q' \in U_2(\Delta\kappa) \end{aligned} \tag{4.157}$$

Now add (4.155) and (4.157) in the intersection $U_1(\Delta\nu_3) \cap U_2(\Delta\kappa)$

$$\begin{aligned} (\nu_{i3}(q) + \nu_{i4}(q, \hat{p}_{ij}(q))) - (\Delta\nu_3 + \Delta\kappa) &< \nu_{i3}(q') + \nu_{i4}(q, \hat{p}_{im(j)}(q')) \\ &< (\nu_{i3}(q) + \nu_{i4}(q, \hat{p}_{ij}(q))) + (\Delta\nu_3 + \Delta\kappa)\,, \\ & \quad \forall j \in \{1, 2, \ldots, n\}, \quad \forall q' \in U_1(\Delta\nu_3) \cap U_2(\Delta\kappa) \iff \\ \frac{\nu_i(q, \hat{p}_{ij}(q))}{\|(\nabla\beta_i)(q)\|} - (\Delta\nu_3 + \Delta\kappa) &< \frac{\nu_i(q', \hat{p}_{im(j)}(q'))}{\|(\nabla\beta_i)(q')\|} \\ &< \frac{\nu_i(q, \hat{p}_{ij}(q))}{\|(\nabla\beta_i)(q)\|} + (\Delta\nu_3 + \Delta\kappa)\,, \\ & \quad \forall j \in \{1, 2, \ldots, n\}, \quad \forall q' \in U_1(\Delta\nu_3) \cap U_2(\Delta\kappa) \end{aligned} \tag{4.158}$$

Then we can select $0 < \Delta\nu_{3\min} < \Delta\nu_{3\max}$ and $\Delta\kappa_{\min}$ such that for every $\kappa_{ij}(q), j \in$

$\{j_1, j_2, \ldots, j_r\}$ which is sufficiently curved

$$\nu_i(q, \hat{p}_{ij}(q)) < 0 \implies \frac{\nu_i(q, \hat{p}_{ij}(q))}{\|(\nabla\beta_i)(q)\|} < 0, \quad j \in \{j_1, j_2, \ldots, j_r\} \tag{4.159}$$

it holds that

$$\begin{aligned} 0 < \Delta\nu_{3\min} + \Delta\kappa_{\min} < -\frac{\nu_i(q, \hat{p}_{ij}(q))}{\|(\nabla\beta_i)(q)\|}, \quad \forall j \in \{j_1, j_2, \ldots, j_r\} \iff \\ 0 < \Delta\nu_{3\min} + \Delta\kappa_{\min} < \min_{j \in \{j_1, j_2, \ldots, j_r\}} \left\{ \left| \frac{\nu_i(q, \hat{p}_{ij}(q))}{\|(\nabla\beta_i)(q)\|} \right| \right\} \end{aligned} \tag{4.160}$$

Note that we consider only principal directions at q which have negative $\nu_i(q, \hat{p}_{ij}(q))$, hence a positive

$$\min_{j \in \{j_1, j_2, \ldots, j_r\}} \left\{ \left| \frac{\nu_i(q, \hat{p}_{ij}(q))}{\|(\nabla\beta_i)(q)\|} \right| \right\} > 0 \tag{4.161}$$

always exists because $\left| \frac{\nu_i(q, \hat{p}_{ij}(q))}{\|(\nabla\beta_i)(q)\|} \right|, j \in \{j_1, j_2, \ldots, j_r\}$ are always a finite number of positive numbers at q. Then, this ensures that

$$\frac{\nu_i(q', \hat{p}_{im(j)}(q'))}{\|(\nabla\beta_i)(q')\|} < 0 \tag{4.162}$$

and since both $\|(\nabla\beta_i)(q)\| > 0$ and $\|(\nabla\beta_i)(q')\| > 0$ it follows that $\nu_i(q', \hat{p}_{im(j)}(q')) < 0$ is sufficiently curved as well.

For every nonconvex $\hat{t}_i^{\mathrm{T}} l_q \hat{t}_i \geq 0$ principal direction $\hat{p}_{ij}(q), j \in \{j_1, j_2, \ldots, j_w\}$ at q, because[25] $\nu_{i1}(q) > 0$ it also follows that $\nu_i(q, \hat{p}_{ij}(q)) > 0$, so that

$$\nu_i(q, \hat{p}_{ij}(q)) > 0 \implies \frac{\nu_i(q, \hat{p}_{ij}(q))}{\|(\nabla\beta_i)(q)\|} > 0, \quad j \in \{j_1, j_2, \ldots, j_w\} \tag{4.163}$$

We also select $\Delta\nu_{3\min}$ and $\Delta\kappa$ such that for every nonconvex $\kappa_{ij}, j \in \{j_1, j_2, \ldots, j_w\}$ which we have just deduced has $\frac{\nu_i(q, \hat{p}_{ij}(q))}{\|(\nabla\beta_i)(q)\|} > 0$ it holds that

$$\begin{aligned} 0 < \Delta\nu_{3\min} + \Delta\kappa < \frac{\nu_i(q, \hat{p}_{ij}(q))}{\|(\nabla\beta_i)(q)\|}, \quad \forall j \in \{j_1, j_2, \ldots, j_w\} \iff \\ 0 < \Delta\nu_{3\min} + \Delta\kappa < \min_{j \in \{j_1, j_2, \ldots, j_w\}} \left\{ \left| \frac{\nu_i(q, \hat{p}_{ij}(q))}{\|(\nabla\beta_i)(q)\|} \right| \right\} \end{aligned} \tag{4.164}$$

Note that we consider only principal directions at q which have positive $\nu_i(q, \hat{p}_{ij}(q))$, hence a positive

$$\min_{j \in \{j_1, j_2, \ldots, j_w\}} \left\{ \left| \frac{\nu_i(q, \hat{p}_{ij}(q))}{\|(\nabla\beta_i)(q)\|} \right| \right\} > 0 \tag{4.165}$$

[25] This is the tricky part later, because we start from a critical point q_c with $\nu_{i1}(q) > 0$, ensure at least one $q' \in \partial\mathscr{O}_i$ is in a neighborhood of it so that we can induce $\nu_{i1}(q') > 0$ from q to q' and *then* we invert their roles, with $q \in \partial\mathscr{O}_i$ and q' our critical point under consideration q_c. Because we have a priori shrunk the distance between them so that *both* are in a neighborhood of the other in which we can induce properties, we continue by inducing the numbers of sufficiently curved and nonconvex directions from $q \in \partial\mathscr{O}_i$ to q_c. But for ensuring that a nonconvex direction has positive $\nu_{i1}(q)$, we need the initial induction of its positivity from q_c. Because, although we have shown $\nu_{i1}(q_c) > 0$ for the remaining critical points, we cannot deduce this from being or not a critical point for the point $q \in \partial\mathscr{O}_i$, from which we are going to take its curvature properties, since this is on the boundary, so it cannot be a critical point anyway. But we need it to ensure that all nonconvex directions at q will have negative ν_i and not merely nonnegative ν_i.

always exists because $\left|\frac{\nu_i(q,\hat{p}_{ij}(q))}{\|(\nabla\beta_i)(q)\|}\right|, j \in \{j_1, j_2, \ldots, j_w\}$ are always a finite number of positive numbers at q. Then, this ensures the positivity

$$\frac{\nu_i(q', \hat{p}_{im(j)}(q'))}{\|(\nabla\beta_i)(q')\|} > 0 \implies \nu_i(q', \hat{p}_{im(j)}(q')) > 0 \tag{4.166}$$

The intersection of non-empty open neighborhoods $U_3 = U_1(\Delta\nu_{3\min}) \cap U_2(\Delta\kappa_{\min})$ is also a non-empty open neighborhood of q, i.e. $\overset{\circ}{U_3} \neq \emptyset$ and $q \in \overset{\circ}{U_3}$. As a result, for every q there exists an $r(q) > 0$, such that the closed ball $\overline{B(q, r(q))}$ centered at q with radius r is a subset of neighborhood U_3.

This implies that if $\nu_{i1}(q) > 0$, then for all $q' \in \overline{B(q, r(q))}$, the following hold

1. $\nu_{i1}(q') > 0, \forall q' \in \overline{B(q, r(q))}$;
2. Points $q, q' \in \overline{B(q, r(q))}$ have the same number of sufficiently curved principal directions $I^-(q) = I^-(q')$ where

$$I^-(q) \triangleq |\{j \in \{1, 2, \ldots, n\} | \nu_i(q, \hat{p}_{ij}(q)) < 0\}| \tag{4.167}$$

   and the same number of directions with positive ν_i, $I^+(q) = I^+(q')$ where

$$I^+(q) \triangleq |\{j \in \{1, 2, \ldots, n\} | \nu_i(q, \hat{p}_{ij}(q)) > 0\}| \tag{4.168}$$

Since $\beta_i^{-1}([0, a_2])$ is a compact set, the continuous functions $\nu_{i3}(q)$ and $\nu_{i4}(q, \hat{p}_{ij})$ are also uniformly continuous. Uniform continuity implies that for every $\Delta\nu_{3\min}, \Delta\kappa_{\min}$, the neighborhoods $U_2(\Delta\nu_{3\min})$ and $U_3(\Delta\kappa_{\min})$ are bounded from below. Hence, the ball radius $r(q)$ can be selected to be continuous function. □

Proposition 29 (Continuity of ν_i on a level set neighborhood)**.** For the compact set $\beta_i^{-1}([0, a_2])$ there exists an $r_{\min} > 0$, such that every closed ball $\overline{B(q, r(q))}$ has the properties of Proposition 28.

Proof. From Proposition 28, there exists a continuous function $r(q) > 0$, such that the closed ball $\overline{B(q, r(q))}$ around every point q has the desired properties. We can set $r_{\min} = \min_{\beta_i^{-1}([0,a_2])}\{r(q)\}$ because set $\beta_i^{-1}([0, a_2])$ is compact. Because it is compact, by the extreme value theorem it follows that the continuous function $r(q)$ takes on its minimum value at some point in $\beta_i^{-1}([0, a_2])$. Hence $r_{\min} > 0$. □

Lemma 30 (Bidirectional induction between level set points)**.** There exists a level set $0 < \varepsilon'_{i0}$, such that for all level sets $c \in (0, \varepsilon'_{i0})$, induction of properties according to Proposition 29 is valid both

1. from a point $q_1 \in \beta_i^{-1}(c)$ to a point $q_2 \in \beta_i^{-1}(0)$, and
2. from a point $q_2 \in \beta_i^{-1}(0)$ to a point $q_1 \in \beta_i^{-1}(c)$.

Proof. There exists a level set $\beta_i^{-1}(\varepsilon'_{i0}), \varepsilon'_{i0} > 0$, such that $\|q_1 - q_2\| < r_{\min}, \forall q_1 \in \beta_i^{-1}(0), \forall q_2 \in \beta_i^{-1}(\varepsilon'_{i0})$. Then, by Proposition 29 it follows that for any point in the ball $\overline{B(q_1, r_{\min})}$, hence also for q_2, Proposition 28 holds, inducing properties from q_1 to q_2. Also, for any point in ball $\overline{B(q_2, r_{\min})}$, hence also for q_1, properties are induced according to Proposition 28 from q_2 to q_1.

The same holds for level sets $\beta_i^{-1}(z), z \in (0, g)$, since for them $\|q_1 - q_3\| < \|q_1 - q_2\|, q_1 \in \beta_i^{-1}(0), q_2 \in \beta_i^{-1}(\varepsilon'_{i0}), q_3 \in \beta_i^{-1}(z), \forall z$, taking into consideration that function β_i has $\|\nabla\beta_i\| > 0$ outwardly oriented with respect to its level set. □

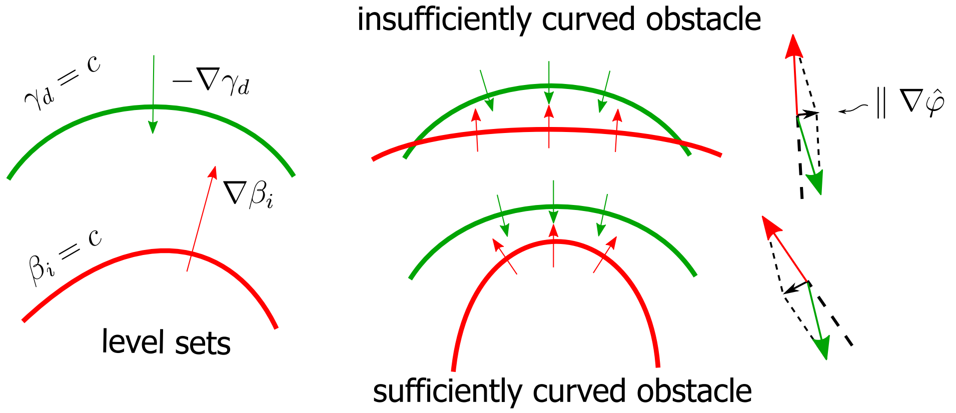


Figure 4.6: KRNF tuning mechanism geometry.

Proposition 31 (Geometry induction from obstacle boundary to any neighborhood critical point)**.** For all $\varepsilon_i < \varepsilon'_{i0}$, every critical point $q_c \in \mathscr{C}_{\hat{\varphi}} \cap \mathscr{B}_i(\varepsilon_i) \cap \mathscr{A}_1$ has at least one corresponding point $q \in \partial\mathscr{O}_i$ such that they have the same number of sufficiently curved principal directions and the same number of principal directions on which $\nu_i(q, \hat{p}_{ij}(q)) > 0$.

Proof. By the previous proposition and the proposition about confinement of critical points, we can induce that for every $q_c \in \mathscr{C}_{\hat{\varphi}} \cap \mathscr{B}_i(\varepsilon_i) \cap \mathscr{A}_1 \implies \nu_{i1}(q_c) > 0$ there exists a boundary point $q \in \partial\mathscr{O}_i$, such that $\nu_{i1}(q) > 0$. Then, again using the previous proposition, q_c has the same number of sufficiently curved principal directions as q and because now we have proved that $\nu_{i1}(q) > 0$, it also has the same number of principal directions on which $\nu_i(q_c, \hat{p}_{ij}(q_c))$, as is the number of nonconvex (including flat) principal directions of q. $\square$

4.4.3 Alternative derivation

For a paraboloid γ_d the same condition can be derived in a shorter way by earlier usage of its specific form. At the critical point $q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}$

$$
k\beta\nabla\gamma_d = \gamma_d\nabla\beta \implies k\beta\nabla\gamma_d\cdot\nabla\gamma_d = \gamma_d\nabla\beta\cdot\nabla\gamma_d \iff k\beta\left\|\nabla\gamma_d\right\|^2 = \gamma_d\nabla\beta\cdot\nabla\gamma_d \overset{\gamma_d=\|q-q_d\|^2}{\implies}
$$

$$
k\beta\left(2\sqrt{\gamma_d}\right)^2 = \gamma_d\nabla\beta\cdot\nabla\gamma_d \iff k\beta 4\gamma_d = \gamma_d\nabla\beta\cdot\nabla\gamma_d \overset{q\neq q_d \iff \gamma_d\neq 0}{\iff}
$$

$$
k\beta = \frac{1}{4}\nabla\beta\cdot\nabla\gamma_d = \frac{1}{4}\left(\bar{\beta}_i\nabla\beta_i\cdot\nabla\gamma_d + \beta_i\nabla\bar{\beta}_i\cdot\nabla\gamma_d\right)
$$

and since also $\hat{t}_i^{\mathrm{T}} D^2\gamma_d\hat{t}_i = 2$ by substitution in (4.60) we get

$$
\begin{aligned}
&\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}}\hat{t}_i\frac{\beta^2}{\gamma_d^{k-1}} \\
=&\frac{1}{4}\nabla\beta\cdot\nabla\gamma_d 2 - \gamma_d\bar{\beta}_i\left(\hat{t}_i^{\mathrm{T}} D^2\beta_i\hat{t}_i\right) + \hat{t}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} - \gamma_d\beta_i D^2\bar{\beta}_i\right)\hat{t}_i \\
=&\frac{1}{2}\left(\bar{\beta}_i\nabla\beta_i\cdot\nabla\gamma_d + \beta_i\nabla\bar{\beta}_i\cdot\nabla\gamma_d\right) - \gamma_d\bar{\beta}_i\left(\hat{t}_i^{\mathrm{T}} D^2\beta_i\hat{t}_i\right) + \hat{t}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} - \gamma_d\beta_i D^2\bar{\beta}_i\right)\hat{t}_i \\
=&\bar{\beta}_i\frac{1}{2}\nabla\beta_i\cdot\nabla\gamma_d - \bar{\beta}_i\gamma_d\left(\hat{t}_i^{\mathrm{T}} D^2\beta_i\hat{t}_i\right) + \beta_i\frac{1}{2}\nabla\bar{\beta}_i\cdot\nabla\gamma_d + \beta_i\hat{t}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\bar{\beta}_i}\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} - \gamma_d D^2\bar{\beta}_i\right)\hat{t}_i \\
=&\bar{\beta}_i\left(\frac{1}{2}\nabla\beta_i\cdot\nabla\gamma_d - \gamma_d\left(\hat{t}_i^{\mathrm{T}} D^2\beta_i\hat{t}_i\right)\right) + \beta_i\left(\frac{1}{2}\nabla\bar{\beta}_i\cdot\nabla\gamma_d + \gamma_d\hat{t}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - D^2\bar{\beta}_i\right)\hat{t}_i\right)
\end{aligned}
$$

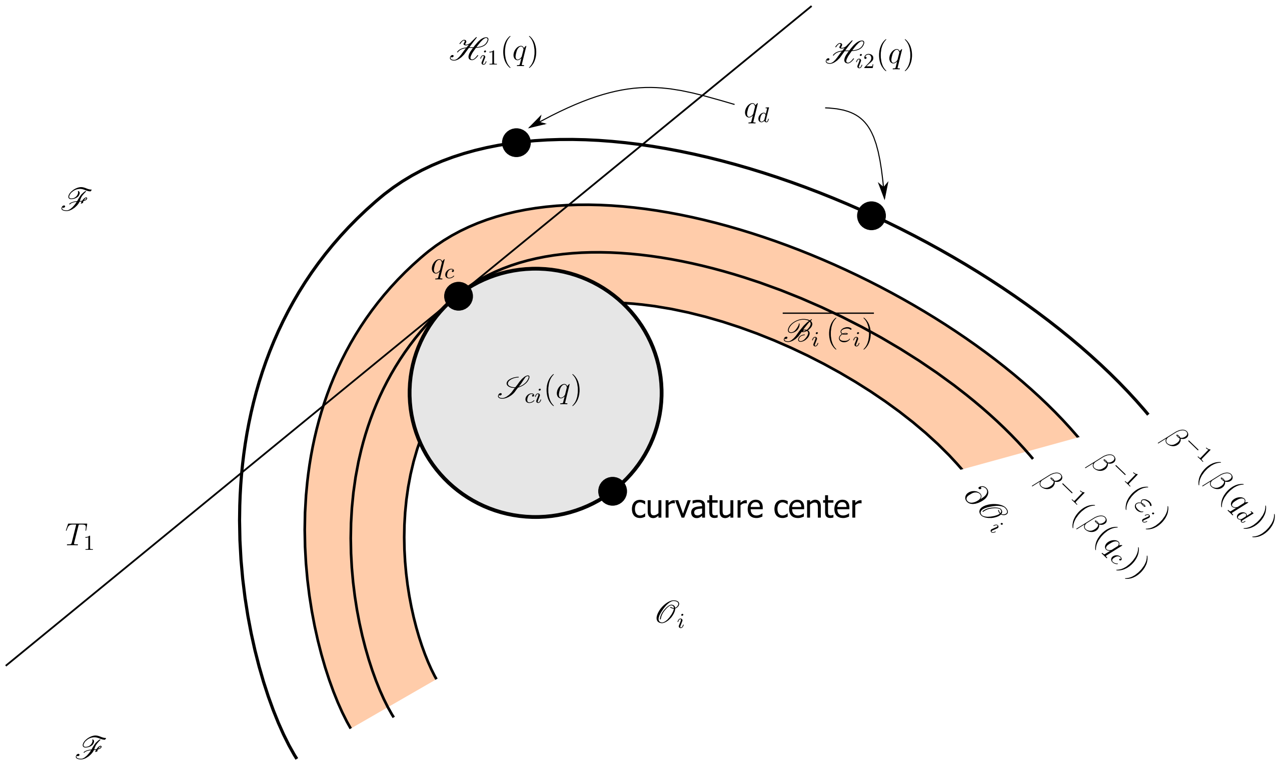


Figure 4.7: Sets involved.

Lemma 3.5 [23] is modified to ensure (for $\gamma_d = \|q - q_d\|^2$) that

$$\nu_i(q) = \frac{1}{2}\nabla\beta_i \cdot \nabla\gamma_d - \gamma_d \left(\hat{t}_i^{\mathrm{T}} D^2\beta_i \hat{t}_i\right) < 0, \forall q \in \overline{\mathscr{B}_i\left(\varepsilon_{i03}\right)}$$

4.5 Curvature of principal direction spans

Proposition 32 (Eigenvalue bounds on Quadratic form in eigen-subspace)**.** Let $H = H^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ be a real symmetric matrix with real eigenvalues $\lambda_i, i \in \{1, 2, \dots, n\}$ and associated eigenvectors $\delta_i, i \in \{1, 2, \dots, n\}$. Consider a subset $\lambda_j, j \in \{1, 2, \dots, l\}, l \in \mathbb{N}$ of its eigenvalues. Then the associated quadratic form $\hat{u}^{\mathrm{T}} H \hat{u}$ is bounded by the minimum and maximum eigenvalues of the selected subset

$$\min_{j\in\{1,2,\dots,l\}} \{\lambda_j\} \leq \hat{u}^{\mathrm{T}} H \hat{u} \leq \max_{j\in\{1,2,\dots,l\}} \{\lambda_j\} \tag{4.169}$$

on the intersection

$$\hat{u} \in S \cap U \tag{4.170}$$

of the unit sphere

$$S \triangleq \{u \in \mathbb{R}^n | \quad \|u\| = 1\} \tag{4.171}$$

with the linear subspace spanned by those eigenvectors

$$U \triangleq \left\{u \in \mathbb{R}^n | \quad u \in \operatorname{span}\left\{\{\delta_j\}_{j\in\{1,2,\dots,l\}}\right\}\right\} \tag{4.172}$$

Proof. Without loss of generality assume the eigenvalues λ_j are numbered in increasing order as

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l \tag{4.173}$$

For each unit vector $\hat{u} \in \mathbb{R}^n, \|\hat{u}\| = 1$ in the linear span

$$\begin{aligned} \hat{u} \in \operatorname{span}\{\delta_1, \delta_2, \dots, \delta_l\} \implies \\ \exists a_j \in \mathbb{R}, j \in \{1, 2, \dots, l\} : \quad \hat{u} = \sum_{j=1}^{l} a_j \delta_j \end{aligned} \tag{4.174}$$

The quadratic form associated with H for $\hat{u}$ is

$$\begin{aligned} \hat{u}^{\mathrm{T}} H \hat{u} &= \left(\sum_{j=1}^{l} a_j \delta_j\right)^{\mathrm{T}} H \left(\sum_{j=1}^{l} a_j \delta_j\right) = \left(\sum_{j=1}^{l} a_j \delta_j^{\mathrm{T}}\right)\left(\sum_{j=1}^{l} a_j H \delta_j\right) \\ &= \sum_{j=1}^{l}\left(a_j \delta_j^{\mathrm{T}} \sum_{p=1}^{l} (a_p \lambda_p \delta_p)\right) = \sum_{j=1}^{l}\sum_{p=1}^{l}\left(a_j a_p \lambda_p \delta_j^{\mathrm{T}} \delta_p\right) \overset{\delta_j^{\mathrm{T}}\delta_p = 0, \forall j\neq p}{=} \\ &= \sum_{j=1}^{l}\left(a_j^2 \lambda_j \delta_j^{\mathrm{T}} \delta_j\right) = \sum_{j=1}^{l}\left(a_j^2 \lambda_j \|\delta_j\|^2\right) \overset{\|\delta_j\|=1}{=} \\ &= \sum_{j=1}^{l} (a_j^2 \lambda_j) \end{aligned} \tag{4.175}$$

since matrix H is symmetric so that its eigensystem is orthogonal, hence the zero inner products $\delta_j^{\mathrm{T}} \delta_p = 0, \forall j \neq p$. Taking into account that

$$\left.\begin{aligned} \|\hat{u}\| = 1 \implies \hat{u}^{\mathrm{T}}\hat{u} = 1 \\ \hat{u} = \textstyle\sum_{j=1}^{l} a_j \delta_j \end{aligned}\right\} \implies \left(\sum_{j=1}^{l} a_j \delta_j\right)^{\mathrm{T}}\left(\sum_{j=1}^{l} a_j \delta_j\right) = 1 \implies \sum_{j=1}^{l} a_j^2 = 1 \tag{4.176}$$

it follows that for all $a_j \neq 0 \implies 0 < a_j^2$ it is

$$\begin{gathered} \lambda_1 \leq \lambda_j \leq \lambda_l, \quad \forall j \in \{1, 2, \ldots, l\} : a_j \neq 0 \overset{0<a_j^2}{\Longrightarrow} \\ a_j^2 \lambda_1 \leq a_j^2 \lambda_j \leq a_j^2 \lambda_l, \quad \forall j \in \{1, 2, \ldots, l\} : a_j \neq 0 \end{gathered} \tag{4.177}$$

and for all $a_j = 0$ it is

$$0 = a_j^2 \lambda_1 = a_j^2 \lambda_2 = \cdots = a_j^2 \lambda_l, \quad \forall j \in \{1, 2, \ldots, l\} : a_j = 0 \tag{4.178}$$

therefore

$$\begin{gathered} \sum_{j=1}^{l} (a_j^2 \lambda_1) \leq \sum_{j=1}^{l} (a_j^2 \lambda_j) \leq \sum_{j=1}^{l} (a_j^2 \lambda_l) \implies \\ \lambda_1 \sum_{j=1}^{l} a_j^2 \leq \sum_{j=1}^{l} (a_j^2 \lambda_j) \leq \lambda_l \sum_{j=1}^{l} a_j^2 \overset{\sum_{j=1}^{l} a_j^2 = 1}{\Longrightarrow} \\ \lambda_1 \leq \sum_{j=1}^{l} (a_j^2 \lambda_j) \leq \lambda_l \end{gathered} \tag{4.179}$$

Substitution of (4.175) in the previous leads to

$$\lambda_1 \leq \hat{u}^{\mathrm{T}} H \hat{u} \leq \lambda_l, \quad \forall \hat{u} \in S \cap U \tag{4.180}$$

which is the desired result, since in (4.173) we have ordered the eigenvalues such that $\lambda_1 = \min_{j \in \{1,2,\ldots,l\}} \{\lambda_j\}$ and $\lambda_l = \lambda_l \max_{j \in \{1,2,\ldots,l\}} \{\lambda_j\}$. □

Let $P_i \triangleq \{\hat{p}_{ij}(q)\}_{j \in I_i}, I_i \triangleq \{j_1, j_2, \ldots, j_r\}, r \in \mathbb{N} \cap [0, n-1]$, to use it as a dummy set and $\mathscr{P}_i \triangleq \operatorname{span}\{P_i\}$.

Proposition 33 (Curvature of subspace spanned by principal directions)**.** Let $\hat{p}_{ij}$ be some principal directions at point q. Then every direction $\hat{t}_i$ in the subspace linearly spanned by these principal directions has normal curvature which is bounded by the minimal and maximal principal curvatures associated with those principal directions.

Proof. The proof follows directly from the previous proposition, taking into account that principal directions are eigenvectors of the matrix form of the Weingarten map and normal curvature is the associated quadratic form of the Wingarten map in the tangent space at q. □

Lemma 34 (Span of convex principal directions is convex)**.** Let $\hat{p}_{ij}$ be some principal directions at point q, which are convex. Then all the directions $\hat{t}_i$ in the subspace spanned by these principal directions are also convex.

Lemma 35 (Span of nonconvex principal directions is nonconvex)**.** Let $\hat{p}_{ij}$ be some principal directions at point q, which are nonconvex. Then all the directions $\hat{t}_i$ in the subspace spanned by these principal directions are also nonconvex.

Proposition 36 (Relative curvature of subspace spanned by principal directions (paraboloid γ_d))**.** Let $\hat{p}_{ij}(q), j \in \{j_1, j_2, \ldots, j_r\}, r \in \mathbb{N} \cap [0, n-1]$ a set of principal directions at point q. If $\gamma_d(q) = \|q - q_d\|^2$, then every direction $\hat{t}_i$ in the subspace linearly spanned by these principal directions has relative curvature $\nu_i(q,\hat{t}_i), \hat{t}_i \in UT_qB_i$ which is bounded by the minimal and maximal relative curvatures of the principal directions considered, i.e.,

$$\begin{aligned}\min_{j\in\{j_1,j_2,\dots,j_r\}} \{\nu_i(q,\hat{p}_{ij}(q))\} \leq \nu_i(q,\hat{t}_i) \leq \max_{j\in\{j_1,j_2,\dots,j_r\}} \{\nu_i(q,\hat{p}_{ij}(q))\}, \\ \forall \hat{t}_i \in \left\{\hat{v} \in \operatorname{span}\left\{\{\hat{p}_{ij}(q)\}_{j\in\{j_1,j_2,\dots,j_r\}}\right\} \mid \|\hat{v}\| = 1\right\}\end{aligned} \tag{4.181}$$

Note that $\operatorname{span}\left\{\{\hat{p}_{ij}(q)\}_{j\in\{j_1,j_2,\dots,j_r\}}\right\} \subset UT_qB_i$.

Proof. Let $\hat{p}_{ij}(q), j \in \{j_1, j_2, \dots, j_r\}, r \in \mathbb{N}\cap[0, n-1]$ be the set of principal directions. Then $\kappa_{ij}(q)$ are the associated principal curvatures which are the eigenvalues of the Weingarten map $\mathrm{L}_q(\hat{t}_i)$ of level set $B_i(q)$ at q. The operator $\mathrm{L}_q(\hat{t}_i)$ is symmetric (self-adjoint), hence its eigenvalues are all real. Therefore, we can always index them in increasing order

$$\kappa_{i1}(q) \leq \kappa_{i2}(q) \leq \cdots \leq \kappa_{ir}(q) \tag{4.182}$$

Let $W = \operatorname{span}\left\{\{\delta_j\}_{j\in\{1,2,\dots,l\}}\right\}$. Since β_i is C^2, ensuring symmetry of the Hessian matrix $D^2\beta_i$, then by Proposition 32 we have that

$$\begin{aligned}\kappa_{i1}(q) \leq \hat{t}_i^{\mathrm{T}} l \hat{t}_i \leq \kappa_{ir}(q), \quad \forall \hat{t}_i \in W \subseteq UT_qB_i \implies \\ \kappa_{i1}(q) \leq \kappa_{n,q}(\hat{t}_i) \leq \kappa_{ir}(q), \quad \forall \hat{t}_i \in W \subseteq UT_qB_i \iff \\ -\nu_{i4}(q,\hat{p}_{i1}(q)) \leq -\nu_{i4}(q,\hat{t}_i) \leq -\nu_{i4}(q,\hat{p}_{ir}(q)), \quad \forall \hat{t}_i \in W \subseteq UT_qB_i \iff \\ \nu_{i4}(q,\hat{p}_{ir}(q)) \leq \nu_{i4}(q,\hat{t}_i) \leq \nu_{i4}(q,\hat{p}_{i1}(q)), \quad \forall \hat{t}_i \in W \subseteq UT_qB_i\end{aligned} \tag{4.183}$$

By Proposition 9 we have that for all directions $\hat{t}_i$ function $\nu_{i3}(q)$ has the same value

$$\nu_{i3}(q) = c \in \mathbb{R}, \quad \forall \hat{t}_i \in UT_qB_i \tag{4.184}$$

Adding this to (4.183), it follows that

$$\begin{aligned}\nu_{i3}(q) + \nu_{i4}(q,\hat{p}_{ir}(q)) \leq \nu_{i3}(q) + \nu_{i4}(q,\hat{t}_i) \leq \nu_{i3}(q) + \nu_{i4}(q,\hat{p}_{i1}(q)), \\ \forall \hat{t}_i \in W \subseteq UT_qB_i \implies \\ \nu_i(q,\hat{p}_{ir}(q)) \leq \nu_i(q,\hat{t}_i) \leq \nu_i(q,\hat{p}_{i1}(q)), \quad \forall \hat{t}_i \in W \subseteq UT_qB_i\end{aligned} \tag{4.185}$$

and the claim has been proved, because by adding $\nu_{i3}(q)$ to (4.183) it follows that

$$\begin{aligned}\nu_i(q,\hat{p}_{ir}(q)) = \min_{j\in\{j_1,j_2,\dots,j_r\}} \{\nu_i(q,\hat{p}_{ij}(q))\} \\ \nu_i(q,\hat{p}_{i1}(q)) = \max_{j\in\{j_1,j_2,\dots,j_r\}} \{\nu_i(q,\hat{p}_{ij}(q))\}\end{aligned} \tag{4.186}$$

□

Lemma 37 (Span of principal directions with negative relative curvature has negative relative curvature)**.** Let β_i be a C^2 obstacle function such that $\|(\nabla\beta_i)(q)\| < 0$. Let $\hat{p}_{ij}(q), j \in \{j_1, j_2, \dots, j_r\}, r \in \mathbb{N}\cap[0, n-1]$ be a set of principal directions at q with $\nu_i(q,\hat{p}_{ij}(q)) < 0$.

If $\gamma_d(q) = \|q - q_d\|^2$, then every direction $\hat{t}_i$ in the subspace linearly spanned by these principal directions has $\nu_i(q,\hat{t}_i) < 0$, i.e.,

$$\nu_i(q,\hat{t}_i) < 0, \quad \forall \hat{t}_i \in \left\{\hat{v} \in \operatorname{span}\left\{\{\hat{p}_{ij}(q)\}_{j\in\{j_1,j_2,\dots,j_r\}}\right\} \mid \|\hat{v}\| = 1\right\} \tag{4.187}$$

Proof. By Proposition 36 it follows that

$$\begin{aligned} \nu_i(q,\hat{t}_i) \leq \max_{j\in\{j_1,j_2,\ldots,j_r\}} \{\nu_i(q,\hat{p}_{ij}(q))\} < 0, \\ \forall \hat{t}_i \in \left\{\hat{v} \in \operatorname{span}\left\{\{\hat{p}_{ij}(q)\}_{j\in\{j_1,j_2,\ldots,j_r\}}\right\} \mid \|\hat{v}\| = 1\right\} \implies \\ \nu_i(q,\hat{t}_i) < 0, \quad \forall \hat{t}_i \in \left\{\hat{v} \in \operatorname{span}\left\{\{\hat{p}_{ij}(q)\}_{j\in\{j_1,j_2,\ldots,j_r\}}\right\} \mid \|\hat{v}\| = 1\right\} \end{aligned} \tag{4.188}$$

□

Lemma 38 (Span of principal directions with positive relative curvature has positive relative curvature)**.** Let β_i be a C^2 obstacle function such that $\|(\nabla\beta_i)(q)\| > 0$. Let $\hat{p}_{ij}(q), j \in \{j_1, j_2, \ldots, j_r\}, r \in \mathbb{N} \cap [0, n-1]$ be a set of principal directions at q with $\nu_i(q, \hat{p}_{ij}(q)) > 0$.

If $\gamma_d(q) = \|q - q_d\|^2$, then every direction $\hat{t}_i$ in the subspace linearly spanned by these principal directions has $\nu_i(q, \hat{t}_i) > 0$, i.e.,

$$\nu_i(q,\hat{t}_i) > 0, \quad \forall \hat{t}_i \in \left\{\hat{v} \in \operatorname{span}\left\{\{\hat{p}_{ij}(q)\}_{j\in\{j_1,j_2,\ldots,j_r\}}\right\} \mid \|\hat{v}\| = 1\right\} \tag{4.189}$$

Proof. The proof is similar with Proposition 37, with reversed signs. □

4.6 Obstacle Geometry Relation to φ Eigenvalues

Let

$$\begin{aligned} I_i^-(q) &\triangleq \{j \in \{1,2,\ldots,n\} |\ \nu_i(q,\hat{p}_{ij}(q)) < 0\}, \\ P_i^-(q) &\triangleq \{\hat{p}_{ij}(q)\}_{j\in I_i^-}, \quad \mathscr{P}_i^-(q) \triangleq \operatorname{span}\{P_i^-(q)\} \end{aligned} \tag{4.190}$$

and define $I_i^+, P_i^+, \mathscr{P}_i^+$ similarly. Also, let

$$I_i^\pm \triangleq I_i^- \cup I_i^+, \quad P_i^\pm \triangleq P_i^- \cup P_i^+, \quad \mathscr{P}_i^\pm \triangleq \mathscr{P}_i^- \cup \mathscr{P}_i^+. \tag{4.191}$$

Hereafter we set $\gamma_d = \|q - q_d\|^2$ and work in $\mathscr{B}_i(\varepsilon_{i4})$ to ensure $\|(\nabla\beta_i)(q)\| > 0$. Using the notions developed so far, it is now possible to generalize Prop. 3.6 [23]. The following connects principal relative curvature $\nu_i(q_c,\hat{p}_{ij}(q_c)), j \in I_i^-$ sign to NF Hessian quadratic form sign on span $\mathscr{P}_i$.

Proposition 39. (At q_c NF Hessian can be made negative (positive) definite on span of negative (positive) principal relative curvatures): There exists an $\varepsilon_{i0}'' > 0$ such that, for all $\varepsilon_i < \varepsilon_{i0}''$ at every critical point $q_c \in \mathscr{C}_{\hat{\varphi}} \cap \mathscr{B}_i(\varepsilon_i)$, if $\nu_i(q_c,\hat{p}_{ij}(q_c)) < 0, \forall j \in I_i^-(q_c) \neq \emptyset$, then the NF Hessian quadratic $\hat{t}_i^{\mathrm{T}}(D^2\hat{\varphi})(q_c)\hat{t}_i < 0, \forall \hat{t}_i \in \mathscr{P}_i^-(q_c)$.

Proof. By hypothesis $I_i^\pm(q_c) \neq \emptyset$, since $\nu_i(q_c,\hat{p}_{ij}(q_c)) < 0, \forall j \in I_i^-(q_c)$. Since $q_c \in \mathscr{B}_i(\varepsilon_i) \subset \overline{\mathscr{B}\left(\frac{1}{2}\varepsilon_{i4}\right)}$ it ensures well definiteness of $\nu_{i\min} \triangleq \min_{j\in I_i^\pm(q), q\in\overline{\mathscr{B}\left(\frac{1}{2}\varepsilon_{i4}\right)}}\{|\nu_i(q,\hat{p}_{ij}(q))|\}$ over compact subset $\bigsqcup_{q\in\overline{\mathscr{B}\left(\frac{1}{2}\varepsilon_{i4}\right)}}(\{q\}\times P_i) \subset \bigsqcup_{q\in\overline{\mathscr{B}\left(\frac{1}{2}\varepsilon_{i4}\right)}} T_qB_i(q)$ on which ν_i is only sign definite, hence also $0 < \nu_{i\min}$. By Proposition 37, Proposition 38 $|\nu_i(q,\hat{p}_{ij}(q))| \leq |\nu_i(q,\hat{t}_i)|, \forall j \in I_i^\pm(q), \forall \hat{t}_i \in \mathscr{P}_i^\pm(q)$, hence $\nu_{i\min} \leq |\nu_i(q,\hat{t}_i)|, \forall j \in I_i^\pm(q), \forall \hat{t}_i \in \mathscr{P}_i^\pm(q)$. Let

$$G_i \triangleq \bar{\beta}_i^{-1}\left(\frac{\nabla\beta_i\cdot\nabla\gamma_d}{\|\nabla\gamma_d\|^2} + \hat{t}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\nabla\bar{\beta}_i\cdot\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - D^2\bar{\beta}_i\right)\hat{t}_i\right) \tag{4.192}$$

and $G_{i\max} \triangleq \max_{\hat{t}_i\in UT_qB_i(q), q\in\overline{\mathscr{B}\left(\frac{1}{2}\varepsilon_{i4}\right)}}\{|G_i(q,\hat{t}_i)|\}$. We can then set the upper bound $\varepsilon_{i0}'' \triangleq \frac{\nu_{i\min}}{G_{i\max}}$. Since, $\beta_i(q) < \varepsilon_i < \varepsilon_{iu} < \varepsilon_{i0}', \varepsilon_{i0}'', \forall q \in \mathscr{B}_i(\varepsilon_i)$, it follows that on $\mathscr{P}_i^-(q), \forall q \in \overline{\mathscr{B}\left(\frac{1}{2}\varepsilon_{i4}\right)}$, in the right hand side of (4.31), its first term $\nu_i(q,\hat{t}_i)$ dominates the second $\beta_i(q)G_i(q,\hat{t}_i)$. As a result, the sum $\nu_i(q,\hat{t}_i)+\beta_i(q)G_i(q,\hat{t}_i)$ has the same sign as $\nu_i(q,\hat{t}_i)$. We have ensured this $\forall \hat{t}_i \in \mathscr{P}_i^-(q), \forall q \in \overline{\mathscr{B}\left(\frac{1}{2}\varepsilon_{i4}\right)}$, so it also holds at q_c, where (4.31) yields $\hat{t}_i^{\mathrm{T}}(D^2\hat{\varphi})(q_c)\hat{t}_i < 0, \forall \hat{t}_i \in \mathscr{P}_i^-(q_c), \forall q_c \in \mathscr{B}_i(\varepsilon_i)$. Since $\nu_{i\min}$ is defined on $I_i^\pm$, the proof applies also to I_i^+. Finally, note that $G_{i\max} = 0$ implies $\varepsilon_{i2}'' = +\infty$, hence no constraint from ε_{i2}'' on k, therefore it is good. □

By Proposition 39 what happens with principal directions $\hat{p}_{ij}(q_c)$ at a critical point carries on to the NF sign definiteness on their spanned subspace. By Proposition 31 we can *control* what happens with $\hat{p}_{ij}(q_c)$, provided we have confined it in $\mathscr{B}_i(\varepsilon_i)$ and set $\varepsilon_i < \varepsilon_{i0}'$. This we do in what follows.

Proposition 40. Every critical point $q_c \in (\mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}) \cap \mathscr{B}_i(\varepsilon_i)$ has at least the number of negative eigenvalues as some boundary point $q \in \partial\mathscr{O}_i$ has sufficiently curved principal directions.

Proof. By Proposition 31, in neighborhood $\mathscr{B}_i(\varepsilon_i) \subset \mathscr{B}(\varepsilon_{i0}')$ it follows that $\forall q \in \mathscr{B}_i(\varepsilon_i)$ there is at least one $q' \in \partial\mathscr{O}_i$ such that $\nu_i(q_c,\hat{p}_{ij}(q_c)) < 0, j \in I_i^-(q_c)$ for as many sufficiently curved principal directions $\hat{p}_{ij}(q')$ as q' has. By Proposition 39, since also $\mathscr{B}_i(\varepsilon_i) \subset \mathscr{B}(\varepsilon_{i0}'')$, the Hessian $(D^2\hat{\varphi})(q_c)$ is negative definite on the subspace spanned by $\hat{p}_{ij}(q_c), j \in I_i^-(q_c)$. As a result, it has at least as many negative eigenvalues. □

Proposition 41. Every critical point has at least the number of positive eigenvalues as an obstacle boundary point has nonconvex directions.

Proof. Same as Proposition 40. □

Lemma 42. If at every boundary point $q \in \partial \mathscr{O}_i$ there exists at least one sufficiently curved principal direction $\hat{p}_{ij}(q_c)$, then for every critical $q_c \in (\mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}) \cap \mathscr{B}_i(\varepsilon_i)$, Hessian matrix $(D^2\hat{\varphi})(q_c)$ has at least one negative eigenvalue.

4.7 Radial positive definiteness

In this section the positive definite submanifold part of Proposition 3.9 [23] is revisited for the case of general β_i, γ_d. For an implicit obstacle function β_i increasing along $\nabla\beta_i$ it can still be proved that the Hessian matrix is positive definite in this direction.

Proposition 43 (Radially positive definite for $\varepsilon_i < \min\{\varepsilon'_{i2}, \varepsilon''_{i2}\}$)**.** If the obstacle function is radially increasing, i.e., $\nabla\beta_i$ is outwardly oriented on β_i^{-1}, at a critical point $q_c \in \mathscr{F}_n \cap \mathscr{C}_{\hat{\varphi}}$, then there exist $0 < \varepsilon'_{i2}, \varepsilon''_{i2}$, such that the Hessian matrix $(D^2\hat{\varphi})(q_c)$ is positive definite in the radial direction $\hat{r}_i$, for all $\varepsilon_i < \min\{\varepsilon'_{i2}, \varepsilon''_{i2}\}$.

Proof. At a critical point of $\hat{\varphi}$ it holds[26] that

$$\begin{aligned}
k\beta\nabla\gamma_d = \gamma_d\nabla\beta \implies (k\beta\nabla\gamma_d)\cdot(k\beta\nabla\gamma_d) &= (\gamma_d\nabla\beta)\cdot(\gamma_d\nabla\beta) \iff \\
(k\beta)^2(\nabla\gamma_d\cdot\nabla\gamma_d) &= \gamma_d^2(\nabla\beta\cdot\nabla\beta) \iff \\
(k\beta)^2\|\nabla\gamma_d\|^2 &= \gamma_d^2\|\nabla\beta\|^2 \overset{\beta\neq 0, \forall q\in\mathscr{B}_i(\varepsilon_i), k\geq 2, q\neq q_d \implies \gamma_d\neq 0, \forall q\in\mathscr{B}_i(\varepsilon_i)}{\implies} \\
k\beta &= \frac{\gamma_d^2}{\left(2\sqrt{\gamma_d}\right)^2}\frac{1}{k\beta}\|\nabla\beta\|^2 \iff \\
k\beta &= \frac{\gamma_d^2}{k\beta}\frac{\|\nabla\beta\|^2}{\|\nabla\gamma_d\|^2}
\end{aligned} \tag{4.193}$$

Taking into consideration that Equation 4.38 holds for any γ_d, β substitution of $k\beta$ from (4.193) in it yields

$$\left.\begin{aligned}
D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} &= \tfrac{\gamma_d^{k-1}}{\beta^2}\left(k\beta D^2\gamma_d + \left(1-\tfrac{1}{k}\right)\tfrac{\gamma_d}{\beta}\nabla\beta\nabla\beta^{\mathrm{T}} - \gamma_d D^2\beta\right) \\
k\beta &= \tfrac{\gamma_d^2}{k\beta}\tfrac{\|\nabla\beta\|^2}{\|\nabla\gamma_d\|^2}
\end{aligned}\right\} \implies$$

$$\begin{aligned}
\frac{\beta^2}{\gamma_d^{k-1}}\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}}\hat{r}_i &= \hat{r}_i^{\mathrm{T}}\left(k\beta D^2\gamma_d\right)\hat{r}_i + \hat{r}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\nabla\beta\nabla\beta^{\mathrm{T}}\right)\hat{r}_i - \hat{r}_i^{\mathrm{T}}\left(\gamma_d D^2\beta\right)\hat{r}_i \\
&= k\beta\left(\hat{r}_i^{\mathrm{T}} D^2\gamma_d\hat{r}_i\right) + \frac{\gamma_d}{\beta}\left(1-\frac{1}{k}\right)\left(\hat{r}_i^{\mathrm{T}}\left(\nabla\beta\nabla\beta^{\mathrm{T}}\right)\hat{r}_i\right) - \gamma_d\left(\hat{r}_i^{\mathrm{T}} D^2\beta\hat{r}_i\right) \\
&= \frac{\gamma_d^2}{\|\nabla\gamma_d\|\, k\beta}\|\nabla\beta\|^2\left(\hat{r}_i^{\mathrm{T}} D^2\gamma_d\hat{r}_i\right) + \frac{\gamma_d}{\beta}\left(1-\frac{1}{k}\right)\left(\hat{r}_i^{\mathrm{T}}\left(\nabla\beta\nabla\beta^{\mathrm{T}}\right)\hat{r}_i\right) - \gamma_d\left(\hat{r}_i^{\mathrm{T}} D^2\beta\hat{r}_i\right)
\end{aligned} \tag{4.194}$$

It is

$$\|\nabla\beta\|^2 = \left\|\nabla\left(\bar{\beta}_i\beta_i\right)\right\|^2 = \left\|\beta_i\nabla\bar{\beta}_i + \bar{\beta}_i\nabla\beta_i\right\|^2 \tag{4.195}$$

and

$$\begin{aligned}
\hat{r}_i^{\mathrm{T}}\left(\nabla\beta\nabla\beta^{\mathrm{T}}\right)\hat{r}_i &= \left(\hat{r}_i^{\mathrm{T}}\right)\left(\nabla\beta^{\mathrm{T}}\hat{r}_i\right) = \left(\hat{r}_i\cdot\nabla\beta\right)\left(\nabla\beta\cdot\hat{r}_i\right) \\
&= \left(\nabla\beta\cdot\hat{r}_i\right)\left(\nabla\beta\cdot\hat{r}_i\right) = \left(\nabla\beta\cdot\hat{r}_i\right)^2 = \left(\nabla\left(\bar{\beta}_i\beta_i\right)\cdot\hat{r}_i\right)^2
\end{aligned} \tag{4.196}$$

Substitution of these in (4.194) yields

$$\begin{aligned}
\frac{\beta^2}{\gamma_d^{k-1}}\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}}\hat{r}_i =& \frac{\gamma_d^2}{k\beta\|\nabla\gamma_d\|^2}\left(\beta_i\nabla\bar{\beta}_i + \bar{\beta}_i\nabla\beta_i\right)\cdot\left(\beta_i\nabla\bar{\beta}_i + \bar{\beta}_i\nabla\beta_i\right)\left(\hat{r}_i^{\mathrm{T}} D^2\gamma_d\hat{r}_i\right) \\
&+ \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\left(\beta_i\nabla\bar{\beta}_i + \bar{\beta}_i\nabla\beta_i\right)\cdot\hat{r}_i\right)^2 \\
&- \gamma_d\left(\hat{r}_i^{\mathrm{T}} D^2\beta\hat{r}_i\right)
\end{aligned} \tag{4.197}$$

[26] [23], Lemma 3.1, p.426.

since

$$\begin{aligned}
&\frac{\gamma_d^2}{\|\nabla\gamma_d\|^2 k\beta}\left(\left(\beta_i\nabla\bar{\beta}_i\right)\cdot\left(\beta_i\nabla\bar{\beta}_i\right)+\left(\beta_i\nabla\bar{\beta}_i\right)\cdot\left(\bar{\beta}_i\nabla\beta_i\right)+\left(\bar{\beta}_i\nabla\beta_i\right)\cdot\left(\beta_i\nabla\bar{\beta}_i\right)+\left(\bar{\beta}_i\nabla\beta_i\right)\cdot\left(\bar{\beta}_i\nabla\beta_i\right)\right)\\
&=\frac{\gamma_d^2}{\|\nabla\gamma_d\|^2 k\beta}\left(\beta_i^2\left\|\nabla\bar{\beta}_i\right\|^2+\beta_i\bar{\beta}_i\left(\nabla\bar{\beta}_i\cdot\nabla\beta_i\right)+\beta_i\bar{\beta}_i\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)+\bar{\beta}_i^2\left\|\nabla\beta_i\right\|^2\right)\\
&=\frac{\gamma_d^2}{\|\nabla\gamma_d\|^2 k\beta}\left(\beta_i^2\left\|\nabla\bar{\beta}_i\right\|^2+2\beta_i\bar{\beta}_i\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)+\bar{\beta}_i^2\left\|\nabla\beta_i\right\|^2\right)\\
&=\frac{\gamma_d^2}{\|\nabla\gamma_d\|^2 k\beta}\left(\beta_i^2\left\|\nabla\bar{\beta}_i\right\|^2+2\beta\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)+\bar{\beta}_i^2\left\|\nabla\beta_i\right\|^2\right)
\end{aligned}\tag{4.198}$$

and

$$\begin{aligned}
\left(\left(\beta_i\nabla\bar{\beta}_i+\bar{\beta}_i\nabla\beta_i\right)\cdot\hat{r}_i\right)^2&=\left(\beta_i\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)+\bar{\beta}_i\left(\nabla\beta_i\cdot\hat{r}_i\right)\right)^2\\
&=\beta_i^2\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)^2+2\beta_i\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)\bar{\beta}_i\left(\nabla\beta_i\cdot\hat{r}_i\right)+\bar{\beta}_i^2\left(\nabla\beta_i\cdot\hat{r}_i\right)^2\\
&=\beta_i^2\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)+2\beta\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)\left(\nabla\beta_i\cdot\hat{r}_i\right)+\bar{\beta}_i^2\left(\nabla\beta_i\cdot\hat{r}_i\right)^2
\end{aligned}\tag{4.199}$$

Note that

$$\nabla\beta_i\cdot\hat{r}_i=\nabla\beta_i\cdot\frac{\nabla\beta_i}{\|\nabla\beta_i\|}=\frac{\nabla\beta_i\cdot\nabla\beta_i}{\|\nabla\beta_i\|}=\frac{\|\nabla\beta_i\|^2}{\|\nabla\beta_i\|}=\|\nabla\beta_i\|\tag{4.200}$$

so substituting in (4.199) yields

$$\beta_i^2\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)^2+2\beta\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)\|\nabla\beta_i\|+\bar{\beta}_i^2\|\nabla\beta_i\|^2\tag{4.201}$$

Substitution of these results in (4.197) leads to

$$\begin{aligned}
&\frac{\beta^2}{\gamma_d^{k-1}}\hat{r}_i^{\mathrm{T}}D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}}\hat{r}_i\\
&\qquad=\frac{\gamma_d^2}{\|\nabla\gamma_d\|^2 k\beta}\left(\beta_i^2\left\|\nabla\bar{\beta}_i\right\|^2+2\beta\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)+\bar{\beta}_i^2\|\nabla\beta_i\|^2\right)\left(\hat{r}_i^{\mathrm{T}}D^2\gamma_d\hat{r}_i\right)\\
&\qquad+\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\beta_i^2\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)^2+2\beta\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)\|\nabla\beta_i\|+\bar{\beta}_i^2\|\nabla\beta_i\|^2\right)\\
&\qquad-\gamma_d\left(\hat{r}_i^{\mathrm{T}}D^2\beta\hat{r}_i\right)
\end{aligned}\tag{4.202}$$

But since

$$\begin{aligned}
\|\nabla\beta_i\|\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)&=\|\nabla\beta_i\|\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)=\left(\|\nabla\beta_i\|\,\hat{r}_i\right)\cdot\nabla\bar{\beta}_i\\
&=\left(\|\nabla\beta_i\|\frac{\nabla\beta_i}{\|\nabla\beta_i\|}\right)\cdot\nabla\bar{\beta}_i=\nabla\beta_i\cdot\nabla\bar{\beta}_i
\end{aligned}\tag{4.203}$$

and also

$$\begin{aligned}
\hat{r}_i^{\mathrm{T}}D^2\beta\hat{r}_i&=\hat{r}_i^{\mathrm{T}}\left(D^2\left(\bar{\beta}_i\beta_i\right)\right)\hat{r}_i=\hat{r}_i^{\mathrm{T}}\left(D\left(\beta_i\nabla\bar{\beta}_i+\bar{\beta}_i\nabla\beta_i\right)\right)\hat{r}_i\\
&=\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}+\beta_iD^2\bar{\beta}_i+\nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}+\bar{\beta}_iD^2\beta_i\right)\hat{r}_i\\
&=\hat{r}_i^{\mathrm{T}}\left(\beta_iD^2\bar{\beta}_i+\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}+\nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}\right)+\bar{\beta}_iD^2\beta_i\right)\hat{r}_i\\
&=\hat{r}_i^{\mathrm{T}}\left(\beta_iD^2\bar{\beta}_i+2\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s+\bar{\beta}_iD^2\beta_i\right)\hat{r}_i
\end{aligned}\tag{4.204}$$

because $\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}+\nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}=\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}+\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)^{\mathrm{T}}=2\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s$. Substitution in

(4.202) yields

$$
\begin{aligned}
&\frac{\beta^2}{\gamma_d^{k-1}}\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}}\hat{r}_i\\
&\quad=\frac{\gamma_d^2}{\|\nabla\gamma_d\|^2 k\beta}\left(\beta_i^2\left\|\nabla\bar{\beta}_i\right\|^2+2\beta\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)+\bar{\beta}_i^2\|\nabla\beta_i\|^2\right)\left(\hat{r}_i^{\mathrm{T}}D^2\gamma_d\hat{r}_i\right)\\
&\qquad+\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\beta_i^2\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)^2+\bar{\beta}_i^2\|\nabla\beta_i\|^2\right)\\
&\qquad+\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}2\beta\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)\\
&\qquad-\gamma_d\hat{r}_i^{\mathrm{T}}\left(\beta_i D^2\bar{\beta}_i+2\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s+\bar{\beta}_i D^2\beta_i\right)\hat{r}_i\\
&\quad=\frac{\gamma_d^2}{\|\nabla\gamma_d\|^2 k\beta}\left(\beta_i^2\left\|\nabla\bar{\beta}_i\right\|^2+2\beta\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)+\bar{\beta}_i^2\|\nabla\beta_i\|^2\right)\left(\hat{r}_i^{\mathrm{T}}D^2\gamma_d\hat{r}_i\right)\\
&\qquad+\gamma_d 2\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)-\frac{1}{k}\frac{\gamma_d}{\beta}2\beta\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)\\
&\qquad+\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\beta_i^2\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)^2+\bar{\beta}_i^2\|\nabla\beta_i\|^2\right)\\
&\qquad-\gamma_d\hat{r}_i^{\mathrm{T}}\left(\beta_i D^2\bar{\beta}_i+2\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s+\bar{\beta}_i D^2\beta\right)\hat{r}_i
\end{aligned}
\tag{4.205}
$$

$$
\begin{aligned}
&=\left(\frac{\gamma_d^2}{\|\nabla\gamma_d\|^2 k\beta}\left(\beta_i^2\left\|\nabla\bar{\beta}_i\right\|^2+\bar{\beta}_i^2\|\nabla\beta_i\|^2\right)+\frac{\gamma_d^2}{\|\nabla\gamma_d\|^2 k\beta}2\beta\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)\right)\left(\hat{r}_i^{\mathrm{T}}D^2\gamma_d\hat{r}_i\right)\\
&\quad+(-1)\frac{\gamma_d^2}{\|\nabla\gamma_d\|^2 k\beta}2\beta\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)\left(\frac{1}{\gamma_d}\|\nabla\gamma_d\|^2\frac{1}{\hat{r}_i^{\mathrm{T}}D^2\gamma_d\hat{r}_i}\right)\left(\hat{r}_i^{\mathrm{T}}D^2\gamma_d\hat{r}_i\right)\\
&\quad+2\gamma_d\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)\\
&\quad+\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\beta_i^2\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)^2+\bar{\beta}_i^2\|\nabla\beta_i\|^2\right)\\
&\quad-\gamma_d\hat{r}_i^{\mathrm{T}}\left(\beta_i D^2\bar{\beta}_i+2\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s+\bar{\beta}_i D^2\beta_i\right)\hat{r}_i\\
&=\frac{\gamma_d^2}{\|\nabla\gamma_d\|^2 k\beta}\left(\beta_i^2\left\|\nabla\bar{\beta}_i\right\|^2+\bar{\beta}_i^2\|\nabla\beta_i\|^2\right)\left(\hat{r}_i^{\mathrm{T}}D^2\gamma_d\hat{r}_i\right)\\
&\quad+\frac{\gamma_d^2}{\|\nabla\gamma_d\|^2 k\beta}2\beta\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)\left(1-\frac{\frac{\|\nabla\gamma_d\|}{\gamma_d}}{\frac{\hat{r}_i^{\mathrm{T}}D^2\gamma_d\hat{r}_i}{\|\nabla\gamma_d\|}}\right)\left(\hat{r}_i^{\mathrm{T}}D^2\gamma_d\hat{r}_i\right)\\
&\quad+2\gamma_d\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)\\
&\quad-\gamma_d\hat{r}_i^{\mathrm{T}}\left(\beta_i D^2\bar{\beta}_i+2\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s+\bar{\beta}_i D^2\beta_i\right)\hat{r}_i\\
&\quad+\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\beta_i^2\left(\hat{r}_i\cdot\nabla\bar{\beta}_i\right)^2+\bar{\beta}_i^2\|\nabla\beta_i\|^2\right)
\end{aligned}
\tag{4.206}
$$

Terms $2\gamma_d\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)$ and $-\gamma_d\hat{r}_i^{\mathrm{T}}2\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s\hat{r}_i$ cancel because

$$2\gamma_d\left(\nabla\beta_i\cdot\nabla\bar{\beta}_i\right)=2\gamma_d\|\nabla\beta_i\|\left(\nabla\bar{\beta}_i\cdot\hat{r}_i\right)\tag{4.207}$$

and

$$
\begin{aligned}
-\gamma_d \hat{r}_i^{\mathrm{T}} 2 \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)_s \hat{r}_i &= -\gamma_d \hat{r}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}} + \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right)^{\mathrm{T}}\right) \hat{r}_i \\
&= -\gamma_d \hat{r}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}} + \nabla\beta_i \nabla\bar{\beta}_i^{\mathrm{T}}\right) \hat{r}_i \\
&= -\gamma_d \left(\hat{r}_i^{\mathrm{T}} \left(\nabla\bar{\beta}_i \nabla\beta_i^{\mathrm{T}}\right) \hat{r}_i + \hat{r}_i^{\mathrm{T}} \left(\nabla\beta_i \nabla\bar{\beta}_i^{\mathrm{T}}\right) \hat{r}_i\right) \\
&= -\gamma_d \left(\left(\hat{r}_i^{\mathrm{T}} \nabla\bar{\beta}_i\right)\left(\nabla\beta_i^{\mathrm{T}} \hat{r}_i\right) + \left(\hat{r}_i^{\mathrm{T}} \nabla\beta_i\right)\left(\nabla\bar{\beta}_i^{\mathrm{T}} \hat{r}_i\right)\right) \\
&= -\gamma_d \left(\left(\hat{r}_i \cdot \nabla\bar{\beta}_i\right)\left(\nabla\beta_i \cdot \hat{r}_i\right) + \left(\hat{r}_i \cdot \nabla\beta_i\right)\left(\nabla\bar{\beta}_i \cdot \hat{r}_i\right)\right) \overset{\nabla\beta_i \cdot \hat{r}_i = \|\nabla\beta_i\|}{=} \\
&= -\gamma_d \left(\left(\hat{r}_i \cdot \nabla\bar{\beta}_i\right)\|\nabla\beta_i\| + \|\nabla\beta_i\|\left(\hat{r}_i \cdot \nabla\bar{\beta}_i\right)\right) \\
&= -\gamma_d 2 \|\nabla\beta_i\| \left(\hat{r}_i \cdot \nabla\bar{\beta}_i\right) \\
&= -2\gamma_d \|\nabla\beta_i\| \left(\hat{r}_i \cdot \nabla\bar{\beta}_i\right)
\end{aligned}
\tag{4.208}
$$

Hence (4.205) becomes

$$
\begin{aligned}
\frac{\beta^2}{\gamma_d^{k-1}} \hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} \hat{r}_i = \frac{\gamma_d^2 \left(\hat{r}_i^{\mathrm{T}} D^2 \gamma_d \hat{r}_i\right)}{k\beta \|\nabla\gamma_d\|^2} &\left(\beta_i^2 \left\|\nabla\bar{\beta}_i\right\|^2 + \bar{\beta}_i^2 \|\nabla\beta_i\|^2 + \left(1 - \frac{\frac{\|\nabla\gamma_d\|}{\gamma_d}}{\frac{\hat{r}_i^{\mathrm{T}} D^2 \gamma_d \hat{r}_i}{\|\nabla\gamma_d\|}}\right)\left(2\beta\left(\nabla\beta_i \cdot \nabla\bar{\beta}_i\right)\right)\right) \\
&+ \left(1 - \frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\beta_i^2 \left(\hat{r}_i \cdot \nabla\bar{\beta}_i\right)^2 + \bar{\beta}_i^2 \|\nabla\beta_i\|^2\right) \\
&- \gamma_d \hat{r}_i^{\mathrm{T}} \left(\beta_i D^2 \bar{\beta}_i + \bar{\beta}_i D^2 \beta_i\right) \hat{r}_i
\end{aligned}
\tag{4.209}
$$

To proceed further we select a symmetric attractive effect $\gamma_d = \|q - q_d\|^2$. It follows that

$$
\frac{\left(\hat{r}_i^{\mathrm{T}} D^2 \gamma_d \hat{r}_i\right)\gamma_d^2}{k\beta \|\nabla\gamma_d\|^2} = \frac{\left(\hat{r}_i^{\mathrm{T}} (2I)\, \hat{r}_i\right)\gamma_d^2}{k\beta \left(2\sqrt{\gamma_d}\right)^2} = \frac{2\gamma_d^2}{k\beta 4\gamma_d} = \frac{\gamma_d}{2k\beta}
\tag{4.210}
$$

and

$$
1 - \frac{\frac{\|\nabla\gamma_d\|}{\gamma_d}}{\frac{\hat{r}_i^{\mathrm{T}} D^2 \gamma_d \hat{r}_i}{\|\nabla\gamma_d\|}} = 1 - \frac{\frac{\left(2\sqrt{\gamma_d}\right)^2}{\gamma_d}}{\hat{r}_i^{\mathrm{T}} (2I)\, \hat{r}_i} = 1 - \frac{4}{2} = 1 - 2 = -1
\tag{4.211}
$$

therefore (4.209) implies

$$
\begin{aligned}
\frac{\beta^2}{\gamma_d^{k-1}} \hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} \hat{r}_i &= \frac{\gamma_d}{2k\beta}\left(\beta_i^2 \left\|\nabla\bar{\beta}_i\right\|^2 + \bar{\beta}_i^2 \|\nabla\beta_i\|^2 - 2\beta\left(\nabla\beta_i \cdot \nabla\bar{\beta}_i\right)\right) \\
&+ \left(1 - \frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\beta_i^2 \left(\hat{r}_i \cdot \nabla\bar{\beta}_i\right)^2 + \bar{\beta}_i^2 \|\nabla\beta_i\|^2\right) \\
&- \gamma_d \hat{r}_i^{\mathrm{T}} \left(\beta_i D^2 \bar{\beta}_i + \bar{\beta}_i D^2 \beta_i\right) \hat{r}_i \\
&= \frac{\gamma_d}{2k\beta}\left(\beta_i^2 \left\|\nabla\bar{\beta}_i\right\|^2 + \bar{\beta}_i^2 \|\nabla\beta_i\|^2 - 2\beta\left(\nabla\beta_i \cdot \nabla\bar{\beta}_i\right)\right) \\
&+ \left(1 - \frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\beta_i^2 \left(\hat{r}_i \cdot \nabla\bar{\beta}_i\right)^2 + \bar{\beta}_i^2 \|\nabla\beta_i\|^2\right) \\
&- \gamma_d \hat{r}_i^{\mathrm{T}} \left(\beta_i D^2 \bar{\beta}_i + \bar{\beta}_i D^2 \beta_i\right) \hat{r}_i
\end{aligned}
\tag{4.212}
$$

Now note that

$$
\begin{aligned}
0 &\leq \left(\beta_i \left\|\nabla\bar{\beta}_i\right\| - \bar{\beta}_i \left\|\nabla\beta_i\right\|\right)^2 = \left(\beta_i \left\|\nabla\bar{\beta}_i\right\| - \bar{\beta}_i \nabla\beta_i\right)\left(\beta_i \left\|\nabla\bar{\beta}_i\right\| - \bar{\beta}_i \nabla\beta_i\right) \\
&= \beta_i^2 \left\|\nabla\bar{\beta}_i\right\|^2 - \bar{\beta}_i \left\|\nabla\beta_i\right\| \beta_i \left\|\nabla\bar{\beta}_i\right\| - \bar{\beta}_i \left\|\nabla\beta_i\right\| \beta_i \left\|\nabla\bar{\beta}_i\right\| + \bar{\beta}_i^2 \left\|\nabla\beta_i\right\|^2 \\
&= \beta_i^2 \left\|\nabla\bar{\beta}_i\right\|^2 - 2\beta \left\|\nabla\beta_i\right\| \left\|\nabla\bar{\beta}_i\right\| + \bar{\beta}_i^2 \left\|\nabla\beta_i\right\|^2 \\
&\leq \beta_i^2 \left\|\nabla\bar{\beta}_i\right\|^2 - 2\beta \left(\nabla\beta_i \cdot \nabla\bar{\beta}_i\right) + \bar{\beta}_i^2 \left\|\nabla\beta_i\right\|^2
\end{aligned}
\tag{4.213}
$$

because

$$
\nabla\beta_i \cdot \nabla\bar{\beta}_i \leq \left\|\nabla\beta_i\right\| \left\|\nabla\bar{\beta}_i\right\| \iff -\left\|\nabla\beta_i\right\| \left\|\nabla\bar{\beta}_i\right\| \leq -\nabla\beta_i \cdot \nabla\bar{\beta}_i
\tag{4.214}
$$

and also note that $\beta_i^2 \left(\hat{r}_i \cdot \nabla\bar{\beta}_i\right)^2 \geq 0$ so that (4.212) implies

$$
\begin{aligned}
\frac{\beta^2}{\gamma_d^{k-1}} \hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}|_{\mathscr{C}_{\hat{\varphi}}} \hat{r}_i =& \Bigg(\frac{\gamma_d}{2k\beta} \left(\beta_i^2 \left\|\nabla\bar{\beta}_i\right\|^2 + \bar{\beta}_i^2 \left\|\nabla\beta_i\right\|^2 - 2\beta\left(\nabla\beta_i \cdot \nabla\bar{\beta}_i\right) + \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \beta_i^2 \left(\hat{r}_i \cdot \nabla\bar{\beta}_i\right)^2 \right) \\
&+ \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \left\|\nabla\beta_i\right\|^2 - \gamma_d \hat{r}_i^{\mathrm{T}} \left(\beta_i D^2 \bar{\beta}_i + \bar{\beta}_i D^2 \beta_i\right) \hat{r}_i \right) \Bigg) \\
\geq& \left(\frac{\gamma_d}{2k\beta} \left(\beta_i \left\|\nabla\bar{\beta}_i\right\| - \bar{\beta}_i \left\|\nabla\beta_i\right\|\right)^2 + \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \beta_i^2 \left(\hat{r}_i \cdot \nabla\bar{\beta}_i\right)^2 \right) \\
&+ \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \left\|\nabla\beta_i\right\|^2 - \gamma_d \hat{r}_i^{\mathrm{T}} \left(\beta_i D^2 \bar{\beta}_i + \bar{\beta}_i D^2 \beta_i\right) \hat{r}_i \right) \\
\geq& \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \left\|\nabla\beta_i\right\|^2 - \gamma_d \hat{r}_i^{\mathrm{T}} \left(\beta_i D^2 \bar{\beta}_i + \bar{\beta}_i D^2 \beta_i\right) \hat{r}_i
\end{aligned}
\tag{4.215}
$$

$$
\begin{aligned}
&= \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta_i} \bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - \frac{\gamma_d}{\beta_i} \hat{r}_i^{\mathrm{T}} \beta_i^2 D^2 \bar{\beta}_i \hat{r}_i - \frac{\gamma_d}{\beta_i} \beta_i \bar{\beta}_i \hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{r}_i \\
&= \frac{\gamma_d}{\beta_i} \left(\left(1 - \frac{1}{k}\right) \bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - \beta_i^2 \hat{r}_i^{\mathrm{T}} D^2 \bar{\beta}_i \hat{r}_i - \beta_i \bar{\beta}_i \left(\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{r}_i\right) \right) \\
&= \frac{\gamma_d}{\beta_i} \Bigg(\left(\frac{1}{2}\left(1 - \frac{1}{k}\right) \bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - \beta_i \bar{\beta}_i \left(\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{r}_i\right) \right) \\
&\quad + \left(\frac{1}{2}\left(1 - \frac{1}{k}\right) \bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - \beta_i^2 \left(\hat{r}_i^{\mathrm{T}} D^2 \bar{\beta}_i \hat{r}_i\right) \right) \Bigg)
\end{aligned}
\tag{4.216}
$$

The term which has changed compared to the sphere world case is $\frac{1}{2}\left(1 - \frac{1}{k}\right) \bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - \beta_i \bar{\beta}_i \left(\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{r}_i\right)$.

Since by definition $2 \leq k \implies \frac{1}{2} \leq 1 - \frac{1}{k}$, and requiring that $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \{\left\|\nabla\beta_i\right\|\} > 0 \implies \left\|\nabla\beta_i\right\| > 0, \forall q \in \mathscr{B}_i\left(\varepsilon_i\right)$ and that $\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{r}_i > 0, \forall q \in \mathscr{B}_i\left(\varepsilon_i\right)$, a sufficient condition for this term to be positive is

$$
\begin{aligned}
0 < \beta_i < \varepsilon_i < \varepsilon_{i2}' < \frac{1}{4} \frac{\left\|\nabla\beta_i\right\|^2}{\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{r}_i}, \forall q \in \mathscr{B}_i\left(\varepsilon_i\right) &\overset{\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{r}_i > 0, \forall q \in \mathscr{B}_i(\varepsilon_i)}{\Longrightarrow} \\
\beta_i \left(\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{r}_i\right) < \frac{1}{4} \left\|\nabla\beta_i\right\|^2, \forall q \in \mathscr{B}_i\left(\varepsilon_i\right) &\overset{\frac{1}{2} \leq 1 - \frac{1}{k}}{\Longrightarrow} \\
\beta_i \left(\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{r}_i\right) < \frac{1}{2}\left(1 - \frac{1}{k}\right) \left\|\nabla\beta_i\right\|^2, \forall q \in \mathscr{B}_i\left(\varepsilon_i\right) &\overset{\bar{\beta}_i > 0, \forall q \in \mathscr{B}_i(\varepsilon_i)}{\Longrightarrow} \\
\beta_i \bar{\beta}_i \left(\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{r}_i\right) < \frac{1}{2}\left(1 - \frac{1}{k}\right) \bar{\beta}_i \left\|\nabla\beta_i\right\|^2, \forall q \in \mathscr{B}_i\left(\varepsilon_i\right) &\iff \\
0 < \frac{1}{2}\left(1 - \frac{1}{k}\right) \bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - \beta_i \bar{\beta}_i \left(\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{r}_i\right), \forall q \in \mathscr{B}_i\left(\varepsilon_i\right)&
\end{aligned}
\tag{4.217}
$$

Provided the implicit obstacle function β_i has no critical points near the obstacle $\|\nabla\beta_i\| \geq \min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\|\nabla\beta_i\|\} > 0$ and the ε'_{i2} can always be selected appropriately greater than 0 in order to satisfy the inequality. $\square$

Lemma 44 (At least one positive eigenvalue for $\varepsilon_i < \varepsilon'_{i2}, \varepsilon''_{i2}$)**.** For $\varepsilon_i < \min\{\varepsilon'_{i2}, \varepsilon''_{i2}\}$ the Hessian matrix $(D^2\hat{\varphi})\,(q_c)$ at a critical point q_c has at least one positive eigenvalue.

4.8 Navigation Functions extended to Everywhere Sufficiently Curved Worlds

Proposition 45 (No local minima other than q_d)**.** In an everywhere sufficiently curved world, if $k \geq N(\varepsilon_{I_0})$, then the NF has no local minima other than q_d.

Proof. By Definition 23 all $\hat{p}_{ij}(q)$ are sufficiently curved $\forall j \in \{1, 2, \ldots, n-1\}, \forall q \in \partial\mathscr{O}_i, \forall i \in I_0$, then by Proposition 40 all remaining q_c have negative definite Hessian on $T_q B_i(q_c)$. $\square$

Proposition 46 (All $q_c \neq q_d$ nondegenerate saddles)**.** In an everywhere sufficiently curved world, if $k \geq N(\varepsilon_{I_0})$, then every critical $q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}$ is a nondegenerate saddle.

Proof. Combining Propositions 40 and 43 at every q_c, $T_q\mathscr{F}_n$ decomposes to positive definite $\mathscr{R}_i(q_c)$ and negative definite $U\mathscr{T}_i(q_c)$, then by Lemma 3.8 [23] the claim is proved. $\square$

Chapter 5

Ellipsoidal obstacles

5.1 General ellipsoid equations

We are going to illustrate the relative curvature condition using ellipsoidal obstacles. The implicit obstacle function for an ellipsoidal obstacle is[1]

$$\beta_i \triangleq (q - q_i)^T A(q - q_i) - 1 \tag{5.1}$$

where $q_i \in E^n$ is the ellipsoid's center, $q \in \mathscr{F}$ and A is a symmetric positive definite matrix $0 < A = A^{\mathrm{T}}$. The obstacle is defined as

$$\mathscr{O}_i = \{q \in E^n : \beta_i(q) < 0\}, \quad \partial\mathscr{O}_i = \{q \in E^n : \beta_i(q) = 0\} \tag{5.2}$$

Note also that

$$\begin{aligned} \nabla\beta_i &= \nabla \left\{ (q - q_i)^{\mathrm{T}} A (q - q_i) - 1 \right\} = 2A(q - q_i) \\ D^2\beta_i &= D\{2A(q - q_i)\} = 2A \end{aligned} \tag{5.3}$$

so that

$$\begin{aligned} \nu_i(q) &= \frac{\nabla\beta_i \cdot \nabla\gamma_d}{\|\nabla\gamma_d\|^2} \left(\hat{t}_i^{\mathrm{T}} D^2\gamma_d \hat{t}_i\right) - \hat{t}_i^{\mathrm{T}} D^2\beta_i \hat{t}_i \\ &= \frac{(2A(q - q_i)) \cdot (2(q - q_d))}{4\|q - q_d\|^2} 2 - \hat{t}_i^{\mathrm{T}} (2A) \hat{t}_i \\ &= 2\left(\frac{(A(q - q_i))^{\mathrm{T}} (q - q_d)}{\|q - q_d\|^2} - \hat{t}_i^{\mathrm{T}} A \hat{t}_i \right) \\ &= 2\left(\frac{(q - q_i)^{\mathrm{T}} A^{\mathrm{T}} (q - q_d)}{\|q - q_d\|^2} - \hat{t}_i^{\mathrm{T}} A \hat{t}_i \right) \overset{A = A^{\mathrm{T}}}{=} \\ &= 2\left(\frac{(q - q_i)^{\mathrm{T}} A (q - q_d)}{(q - q_d)^{\mathrm{T}} (q - q_d)} - \hat{t}_i^{\mathrm{T}} A \hat{t}_i \right) \end{aligned} \tag{5.4}$$

In case $A = \mathrm{diag}\left(\frac{1}{a_{i1}^2}, \frac{1}{a_{i2}^2}, \ldots, \frac{1}{a_{in}^2}\right)$ then the ellipsoid's axes are aligned with the coordinate system and a_{ij} is the j^{th} radius of the i^{th} obstacle.

[1] Its level sets in E^n are ellipsoids and the function is an elliptic paraboloid in $E^n \times [0, +\infty)$.

5.2 Plots and parametric exploration for ellipses

5.2.1 Ellipse minimal curvature

In this subsection we are going to illustrate the theoretical results using ellipses. Ellipses are selected to allow plots of parametric investigations to be created.

The radius of curvature of an ellipse an any point is $R = \frac{(r_1 r_2)^{3/2}}{ab}$ where r_1, r_2 are the distances to the two foci and a, b its radii. The maximum radius of curvature occurs at the end of its minor semi-axis with radius b and is equal to $R = \frac{a^2}{b}$. Requiring that the center of curvature at this point remains within the ellipse is equivalent to the inequality

$$R < 2b \iff \frac{a^2}{b} < 2b \iff a < b\sqrt{2} \iff \frac{a}{b} < \sqrt{2} \tag{5.5}$$

therefore the ellipse should have bounded eccentricity

$$e < e_{\max} = \sqrt{1 - \left(\frac{b}{a}\right)^2} = \sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{1 - \frac{1}{2}} = \sqrt{\frac{1}{2}} \tag{5.6}$$

The curvature spheres of an ellipse for varying eccentricity are shown in Fig. 5.1 were it can be seen that for $\sqrt{\frac{1}{2}} \leq e$ not all curvature spheres are included in $\mathscr{O}_i$

$$\bigcup_{q \in \partial \mathscr{O}_i} \left(\mathscr{S}_{ci}(q, \hat{t}_i) \setminus \{q\}\right) \nsubseteq \mathscr{O}_i \tag{5.7}$$

5.2.2 About necessity or not

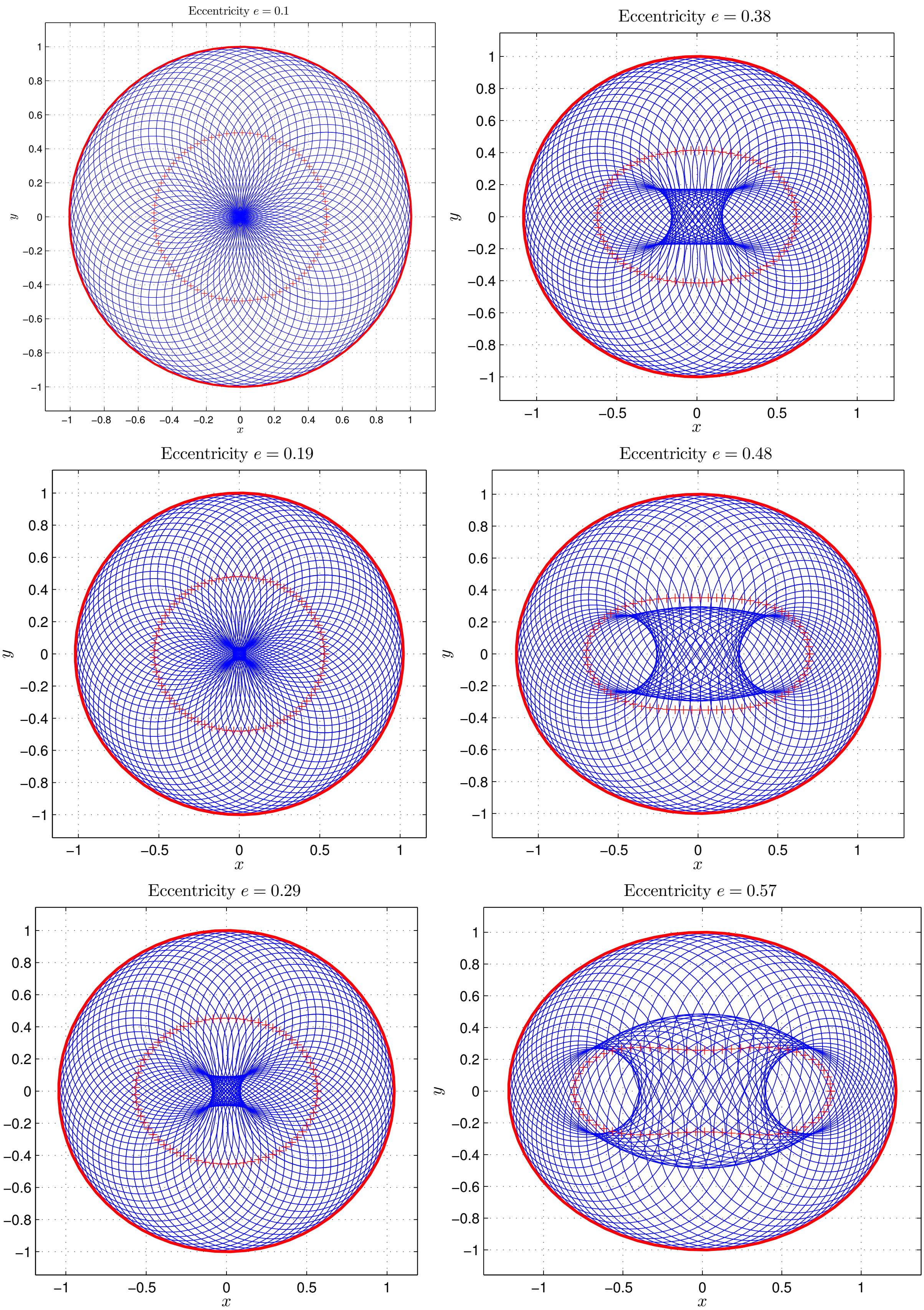
There is only a single internal obstacle. The radial unboundedness condition $1 = M < k = 3$ is satisfied. Therefore the arising local minimum cannot be attributed to this cause.

The relative curvature function $\nu_i(q)$ depends on relative curvature of level sets at a critical point. But existence of critical points has been proved by Koditschek and Rimon. Moreover they can only arise on the obstacle side opposite the destination, because only there do the attractive and repulsive field gradients have negative inner product. Specifically they can only arise only where the gradients have opposite directions. For a single (internal - no world boundary) obstacle this is only possible on the minor axis of the ellipses presented in what follows. So there will be a critical point in the area we discuss, as a result in this simple case the condition proves necessary.

But in general cases with more obstacles this may not be the case. Nonetheless the intuition gained by the examples following helps understand the essence of the relative curvature condition, which, after all, has been formally derived and holds in general settings. Its necessity is not guaranteed in general settings. The related arguments about degeneracy have been discussed at the start of this chapter.

5.2.3 Case studies

Let us now look at the examples. The obstacle's determining characteristic is its minimal curvature. Equivalently center of curvature at the respective boundary point of minimal curvature.



Eccentricity $e = 0.1$
Eccentricity $e = 0.38$
Eccentricity $e = 0.19$
Eccentricity $e = 0.48$
Eccentricity $e = 0.29$
Eccentricity $e = 0.57$
x
y

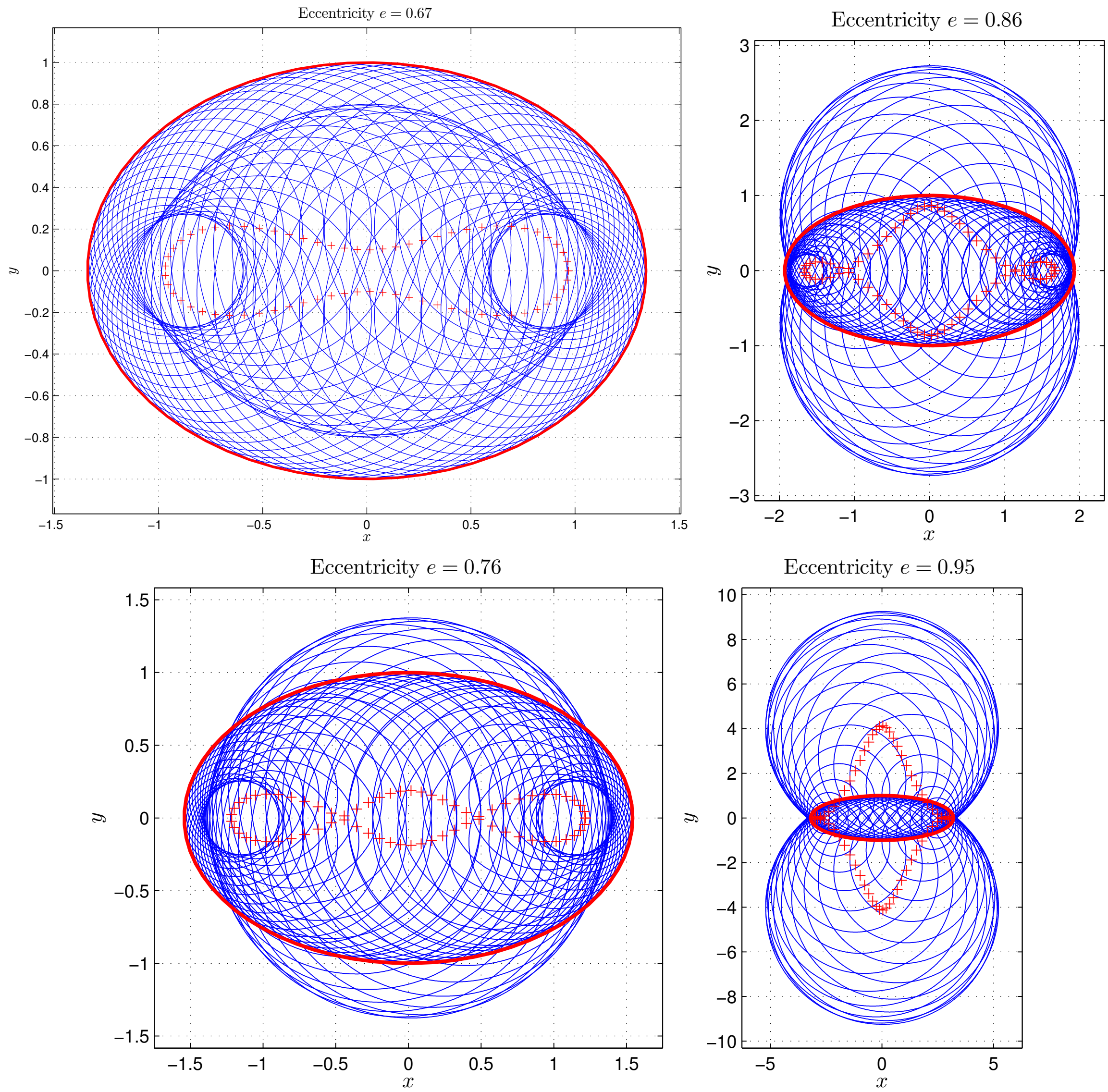


Figure 5.1: Ellipse curvature spheres (disks) for varying eccentricities. The satisfaction of relative curvature condition for $e < \sqrt{\frac{1}{2}} \approx 0.7071$ is visible. So is the existence of q_d that violate the relative curvature condition in the case of insufficiently curved ellipses with $e \geq \sqrt{\frac{1}{2}}$.

In Fig. 5.2a to Fig. 5.3c the destination q_d and $\mathscr{B}_i(\varepsilon_i)$ width ε_i are kept constant and $\nu_i(q), \varphi$ are shown for varying eccentricity e.

As the eccentricity increases, the center of curvature starts from within the ellipse in Fig. 5.2a and Fig. 5.2b, lies on its boundary for critical eccentricity $e = \sqrt{\frac{1}{2}}$ in Fig. 5.2c, exits it for $e > \sqrt{\frac{1}{2}}$ in Fig. 5.3a and goes on past the destination q_d in Fig. 5.3b and Fig. 5.3c. As long as the center of curvature is within the obstacle, we see that $\nu_i(q)$ is positive near the obstacle, so we can select a $\mathscr{B}_i(\varepsilon_i)$ small enough that $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \{\nu_i(q)\} > 0$.

When the curvature center is outside $\mathscr{O}_i$ but closer than destination q_d a ε_i selection still exists to make $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \{\nu_i(q)\} > 0$, Fig. 5.3a. But when the curvature center is farther away than q_d no ε_i exists to make $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \{\nu_i(q)\} > 0$. This is due to $\nu_i(q) < 0$ on the boundary $\partial\mathscr{O}_i$ on the opposite side from q_d, as can be observed in Fig. 5.3b and Fig. 5.3c.

What is important when a curvature center lies outside the ellipse it that a q_d closer to the ellipse than the curvature center can always be selected, making $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \{\nu_i(q)\} < 0$.

This is reflected in the Navigation Function field[2], which is shown for a (relatively small) value of k, for the purpose of emphasizing the difference. A local minimum clearly arises on the ellipse's opposite side from q_d for $\sqrt{\frac{1}{2}} < e$. One may argue that increasing k is the proved way of turning the local minimum to a saddle. But this is not possible here, as has been analytically proved. However large a k we select (equivalently, however small a ε_i) the local minimum remains.

Let us now analyze what happens in more detail. The examples shown do not reveal everything because there are several effects involved. Considering ε_i it does not affect $\nu_i(q)$. What matters if for all valid q_d we can select a ε_i such that $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \{\nu_i(q)\} > 0$.

What primarily matters is the center of curvature q_c. There are three (two really) cases: sufficiently curved, critically curved and insufficiently curved. For the latter there are three relative positions of destination: farther away than q_c, at q_c and closer than q_c (wrt $\mathscr{O}_i$). In all cases when $q_d \in \overline{\mathscr{B}_i(\varepsilon_i)} \implies \min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \{\nu_i(q)\} < 0$, so we should always select $\varepsilon_i < \beta_i(q_d)$.

For a sufficiently curved obstacle in all three cases there exists a $\mathscr{B}_i(\varepsilon_i)$ such that $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \{\nu_i(q)\} > 0$. This is shown for varying q_d in Fig. 5.4a and Fig. 5.4b. In Fig. 5.4a $q_d \in \overline{\mathscr{B}_i(\varepsilon_i)} \implies \min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \{\nu_i(q)\} < 0$ can be observed.

A critically curved obstacle is the limit case of a sufficiently curved one and the same apply, as shown in Fig. 5.5a and Fig. 5.5b.

It is interesting to examine the case of an insufficiently curved obstacle shown in Fig. 5.6a to Fig. 5.7c. In Fig. 5.6a destination $q_d \in \overline{\mathscr{B}_i(\varepsilon_i)} \implies \min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \{\nu_i(q)\} < 0$. Also in Fig. 5.6a to Fig. 5.7a q_d is closer than the curvature center, so that even for $q_d \notin \overline{\mathscr{B}_i(\varepsilon_i)}$ as in Fig. 5.6b to Fig. 5.7a still $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \{\nu_i(q)\} < 0$ for any $\varepsilon_i > 0$. For q_d at the center of curvature $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \{\nu_i(q)\} = 0$ as shown in Fig. 5.7b and for q_d farther away than the curvature center $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \{\nu_i(q)\} > 0$ as shown in Fig. 5.7c.

What is important is that for when the curvature center is outside the obstacle a q_d closer to it can always be selected, so that an analytic Navigation Function of the Koditschek-Rimon form [23] cannot be constructed (of course, as proved in [23], an analytic Navigation Function exists on any analytic manifold with boundary).

In Fig. 5.8a to Fig. 5.9c the minimum of $\nu_i(q)$ in intersection $\overline{\mathscr{B}_i(\varepsilon_i)} \cap [0, +\infty)^2$ of $\overline{\mathscr{B}_i(\varepsilon_i)}$

[2] For these and following figures naming a field as a Navigation Function field does not imply that it does not have local minima, i.e. that it has been (or can be) appropriately tuned.

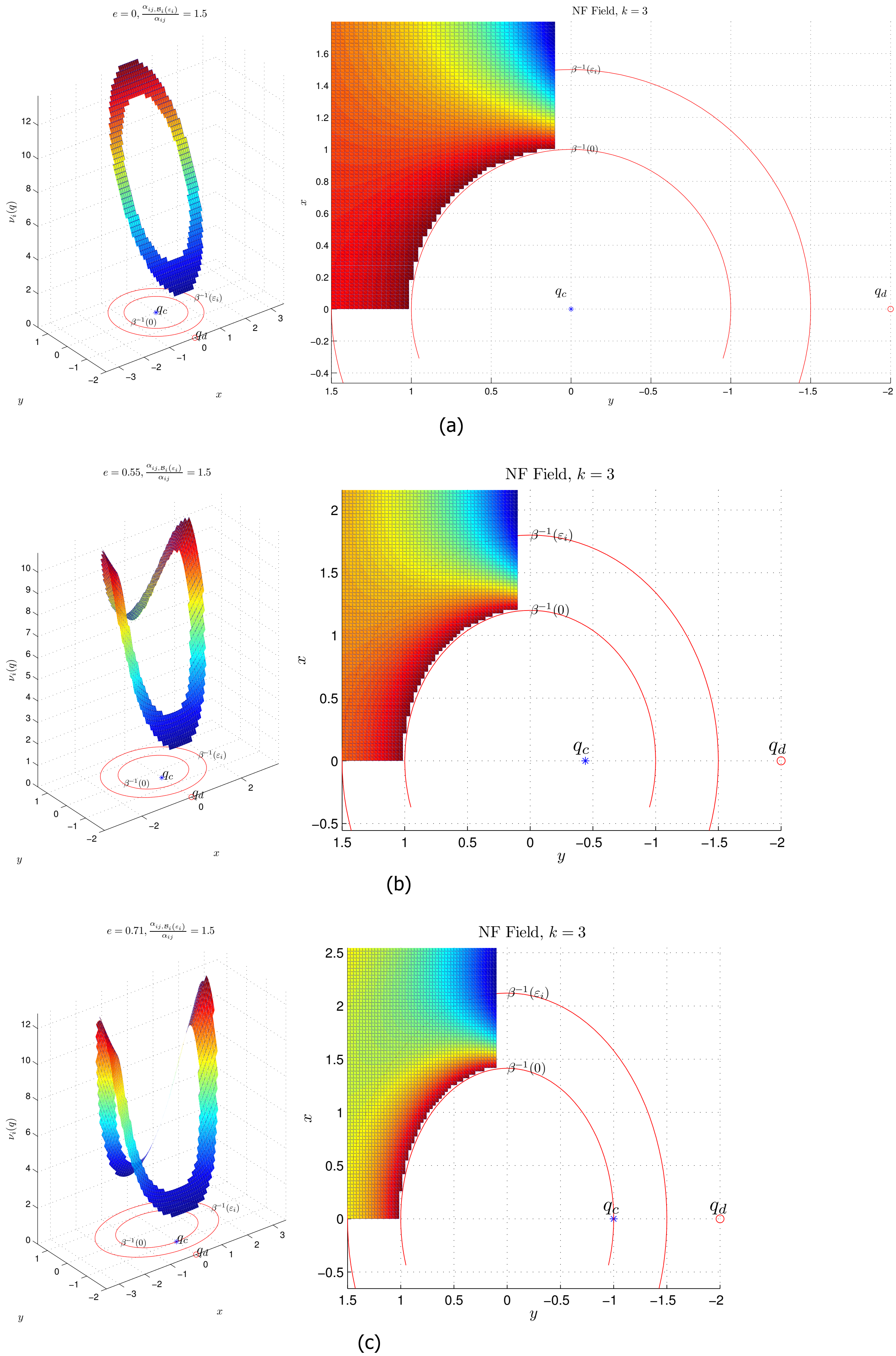


Figure 5.2: Relative curvature function $\nu_i(q)$ within $\overline{\mathscr{B}_i(\varepsilon_i)}$ for varying ellipse eccentricity e and fixed ε_i and destination q_d. The Navigation Function field $\varphi(q)$ is shown as well.

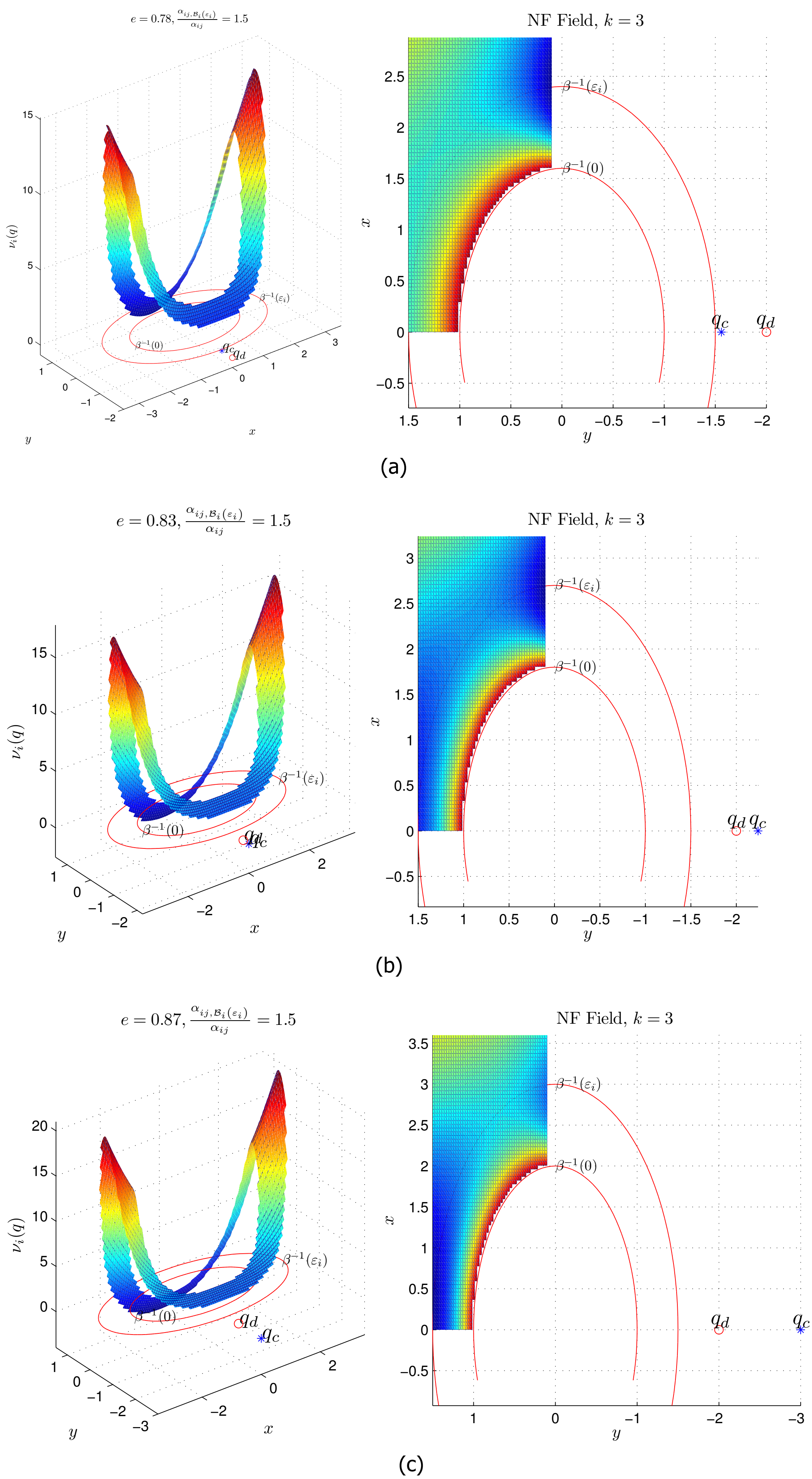


Figure 5.3: Relative curvature function $\nu_i(q)$ within $\overline{\mathscr{B}_i\,(\varepsilon_i)}$ for varying ellipse eccentricity e and fixed ε_i and destination q_d. The Navigation Function field $\varphi(q)$ is shown as well.

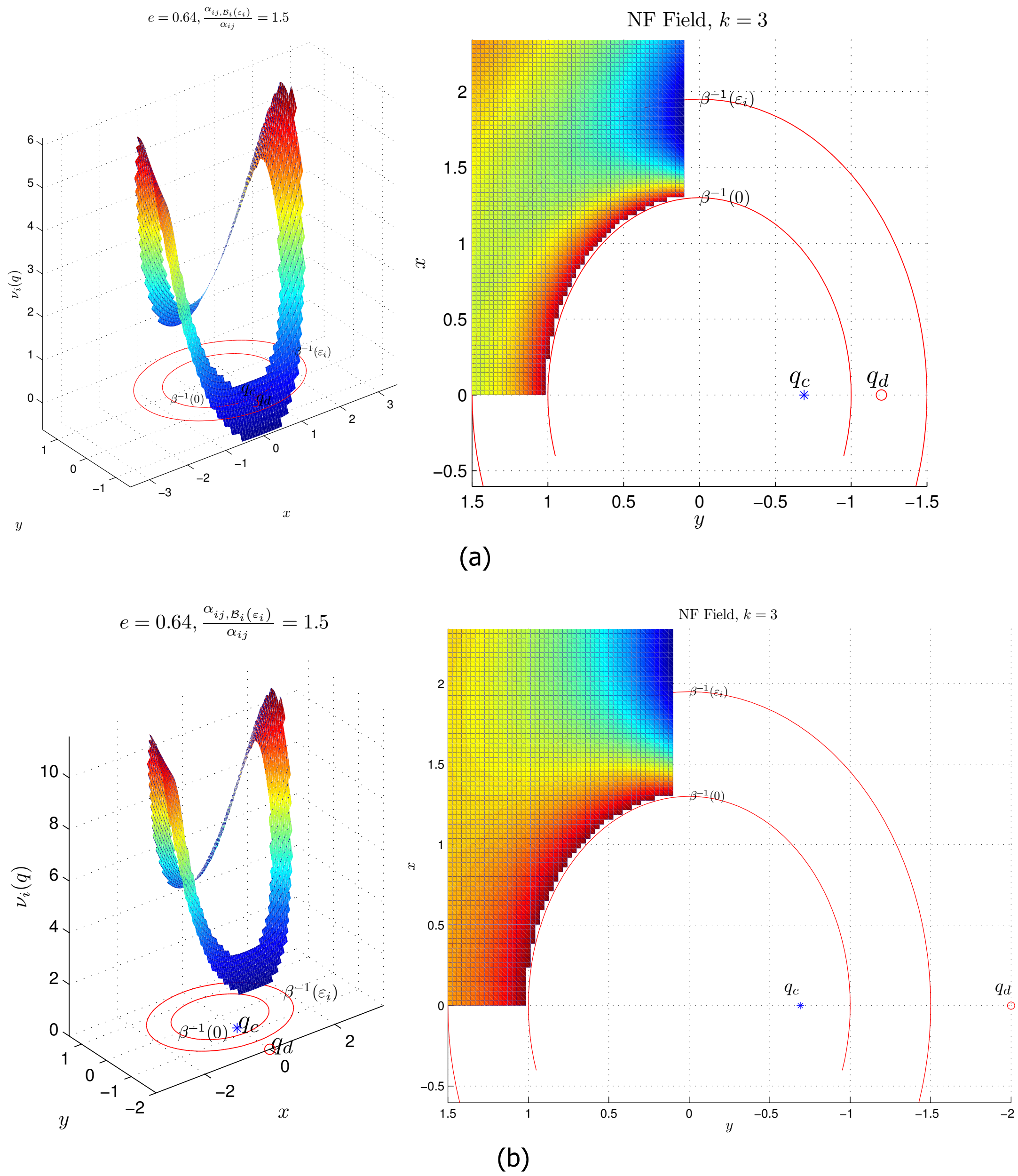
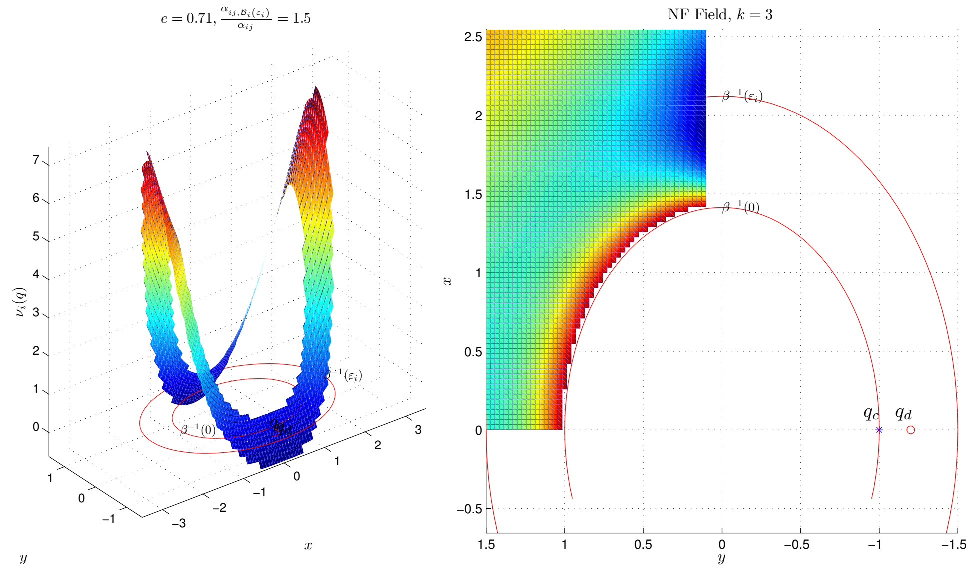
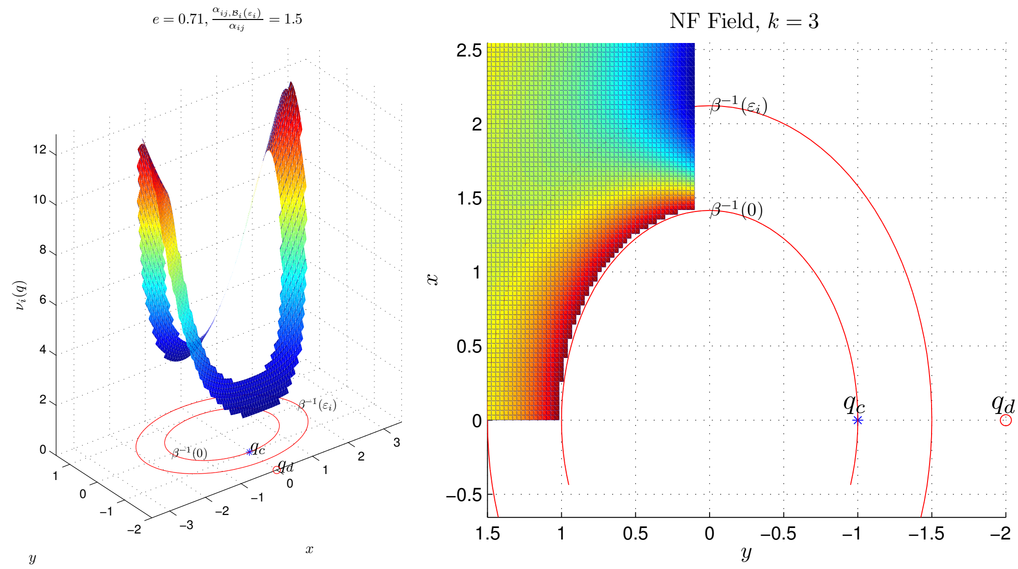


Figure 5.4: Relative curvature function $\nu_i(q)$ within $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$ for varying destination q_d fixed ε_i and sufficient ellipse eccentricity $e = 0.64 < \sqrt{\frac{1}{2}}$. The Navigation Function field $\varphi(q)$ is shown as well.



(a)



(b)

Figure 5.5: Relative curvature function $\nu_i(q)$ within $\overline{\mathscr{B}_i(\varepsilon_i)}$ for varying destination q_d fixed ε_i and critical ellipse eccentricity $e = \sqrt{\frac{1}{2}}$. The Navigation Function field $\varphi(q)$ is shown as well.

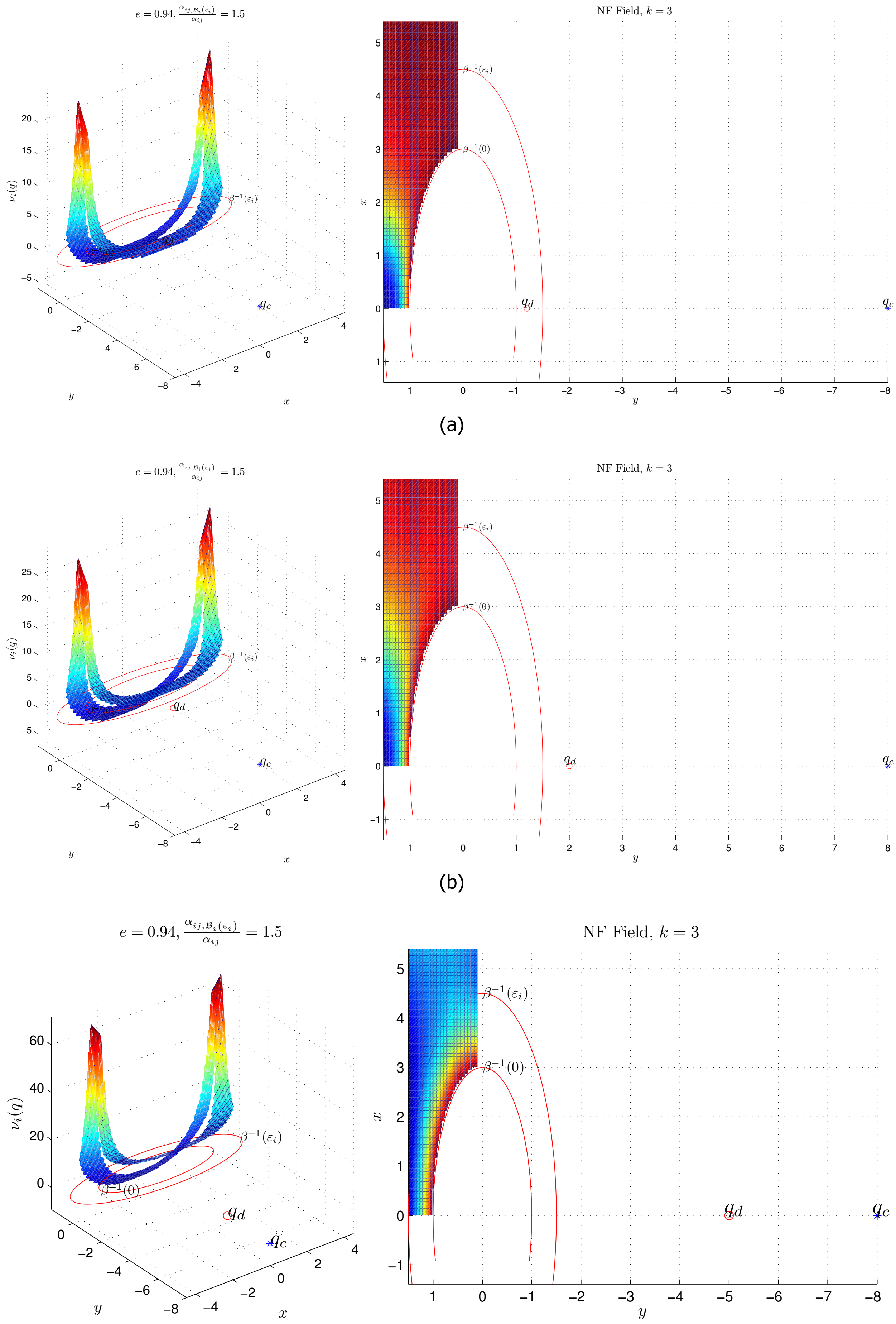


Figure 5.6: Relative curvature function $\nu_i(q)$ within $\overline{\mathscr{B}_i(\varepsilon_i)}$ for varying destination q_d fixed ε_i and insufficient ellipse eccentricity $e = 0.94 > \sqrt{\frac{1}{2}}$. The Navigation Function field $\varphi(q)$ is shown as well.

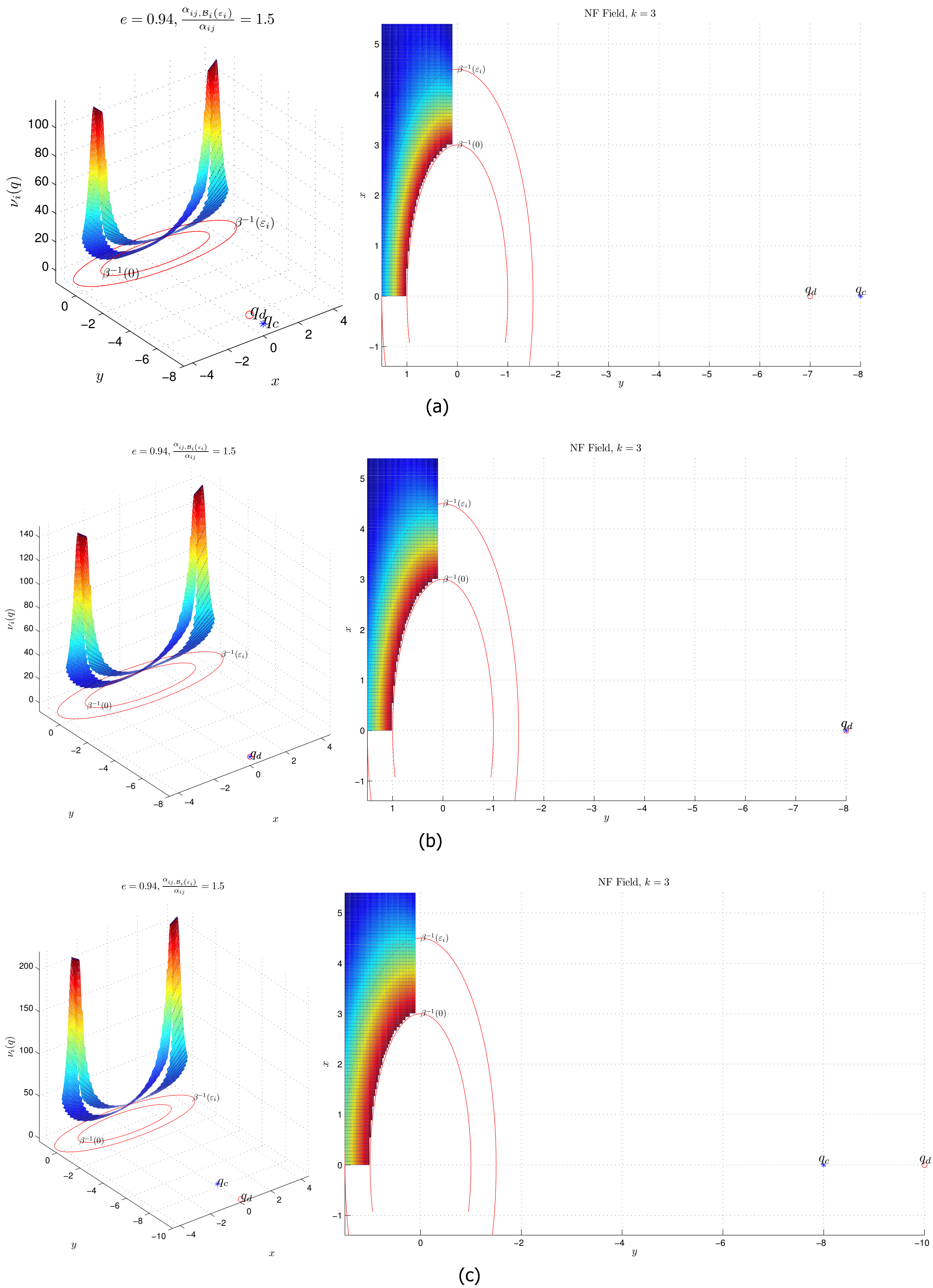


Figure 5.7: Relative curvature function $\nu_i(q)$ within $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$ for varying destination q_d fixed ε_i and insufficient ellipse eccentricity $e = 0.94 > \sqrt{\frac{1}{2}}$. The Navigation Function field $\varphi(q)$ is shown as well.

with first quadrant $[0,+\infty)^2$ is shown for varying destinations q_d over the plane (half-plane shown due to symmetry) and different cases of eccentricity e. The associated level sets are also plotted.

For insufficient eccentricity the zero level set is not confined in $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$ as seen in Fig. 5.9a to Fig. 5.9c. We can also observe that on the same side with q_d (when $q_d \in [0,+\infty)^2$ so $\min_{\overline{\mathscr{B}_i(\varepsilon_i)}\cap[0,+\infty)^2}\{\nu_i(q)\}$ is on the same side) always $\min_{\overline{\mathscr{B}_i(\varepsilon_i)}\cap[0,+\infty)^2}\{\nu_i(q)\} > 0$ for $q_d \notin \overline{\mathscr{B}_i\left(\varepsilon_i\right)}$ because then q_d lies in the negative inner product subspace $\mathscr{X}_1$.

In Fig. 5.10 the results of Fig. 5.8a to Fig. 5.9c are shon as level sets of $\min_{\overline{\mathscr{B}_i(\varepsilon_i)}\cap[0,+\infty)^2}\{\nu_i(q)\}$ in (q_d, e) parameter space.

In Fig. 5.12a to Fig. 5.13c $\min_{\overline{\mathscr{B}_i(\varepsilon_i)}\cap[0,+\infty)^2}\{\nu_i(q)\}$ is plotted for varying destination q_d over the plane, constant eccentricity e and neighborhoods $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$ of various widths ε_i. It becomes clear that the closer q_d is to $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}$ the worst for the relative curvature function minimum. In Fig. 5.11 these results are concatenated in (q_d, e) parameter space.

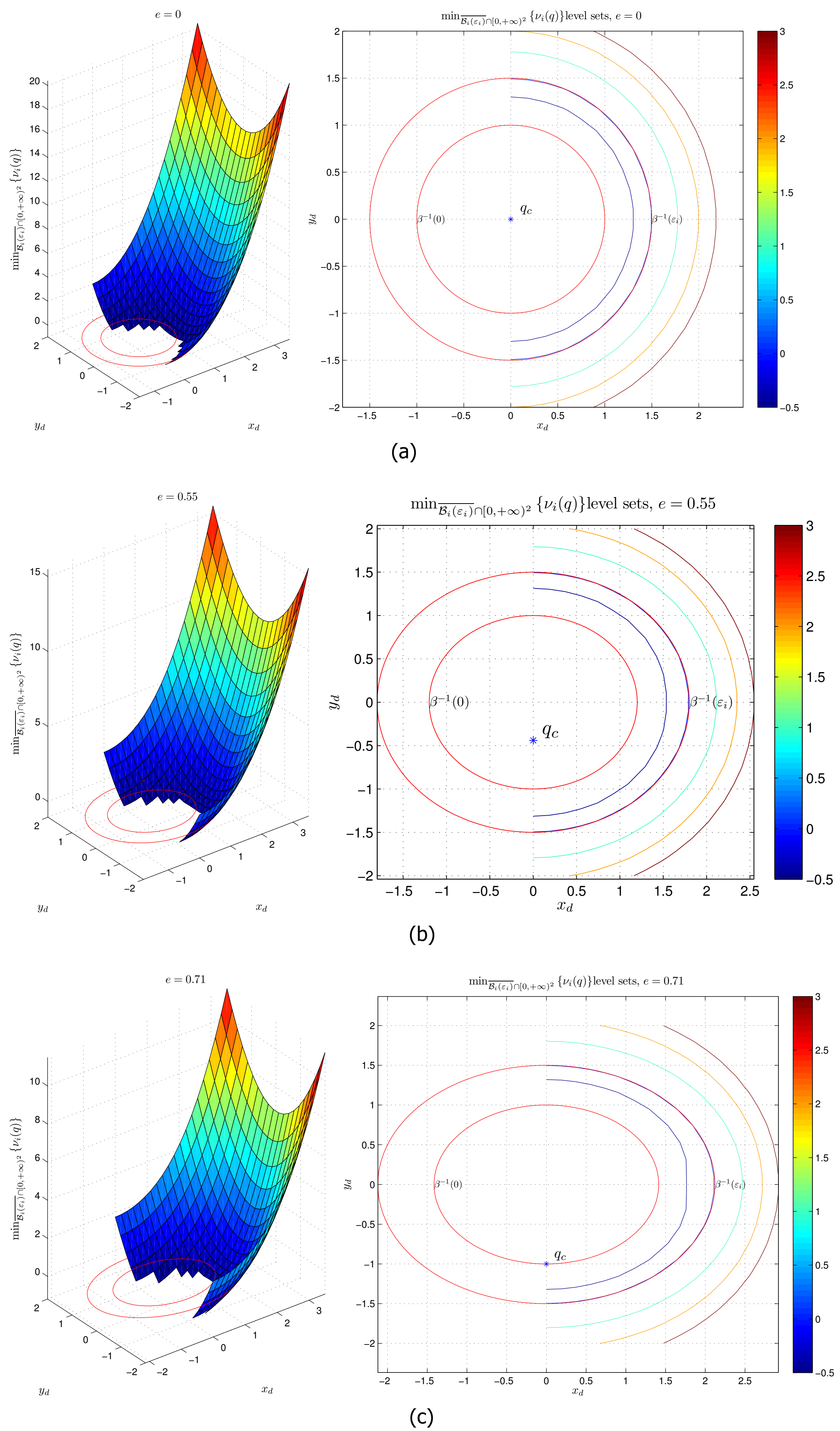
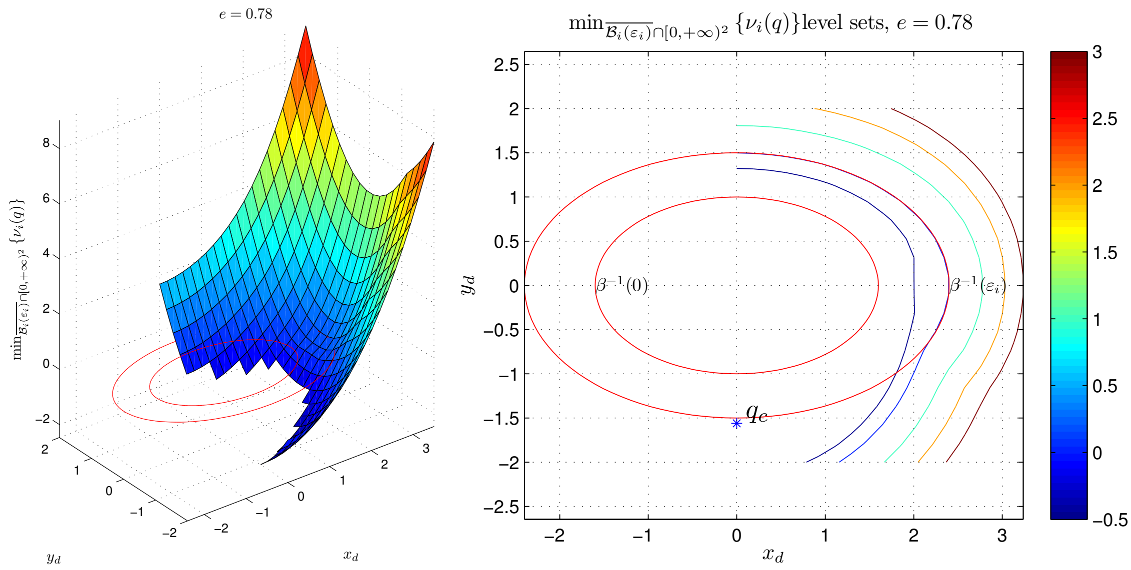
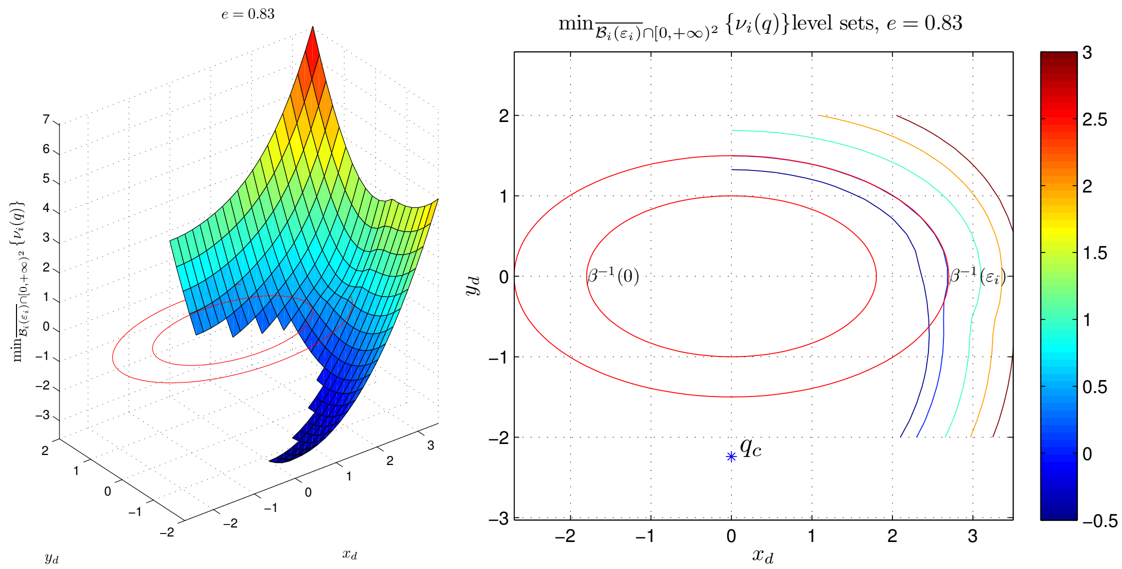


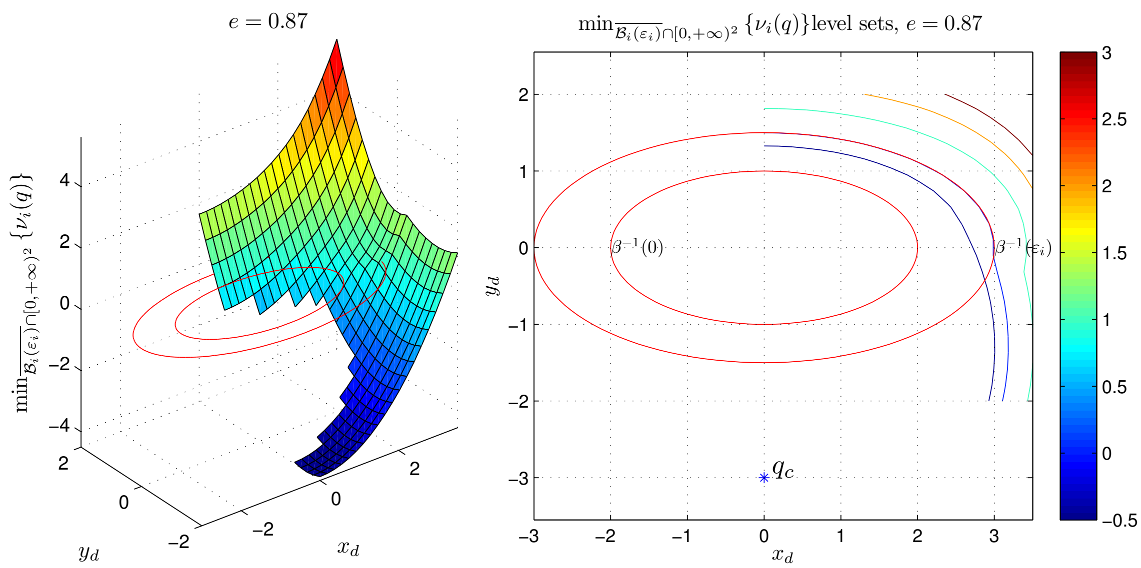
Figure 5.8: Relative curvature function $\nu_i(q)$ minimum in $\overline{\mathscr{B}_i(\varepsilon_i)} \cap [0, +\infty)^2$ for $q_d \in [0, +\infty) \times (-\infty, +\infty)$, constant ε_i and various eccentricities e.



(a)



(b)



(c)

Figure 5.9: Relative curvature function $\nu_i(q)$ minimum in $\overline{\mathscr{B}_i(\varepsilon_i)} \cap [0, +\infty)^2$ for $q_d \in [0, +\infty) \times (-\infty, +\infty)$, constant ε_i and various eccentricities e.

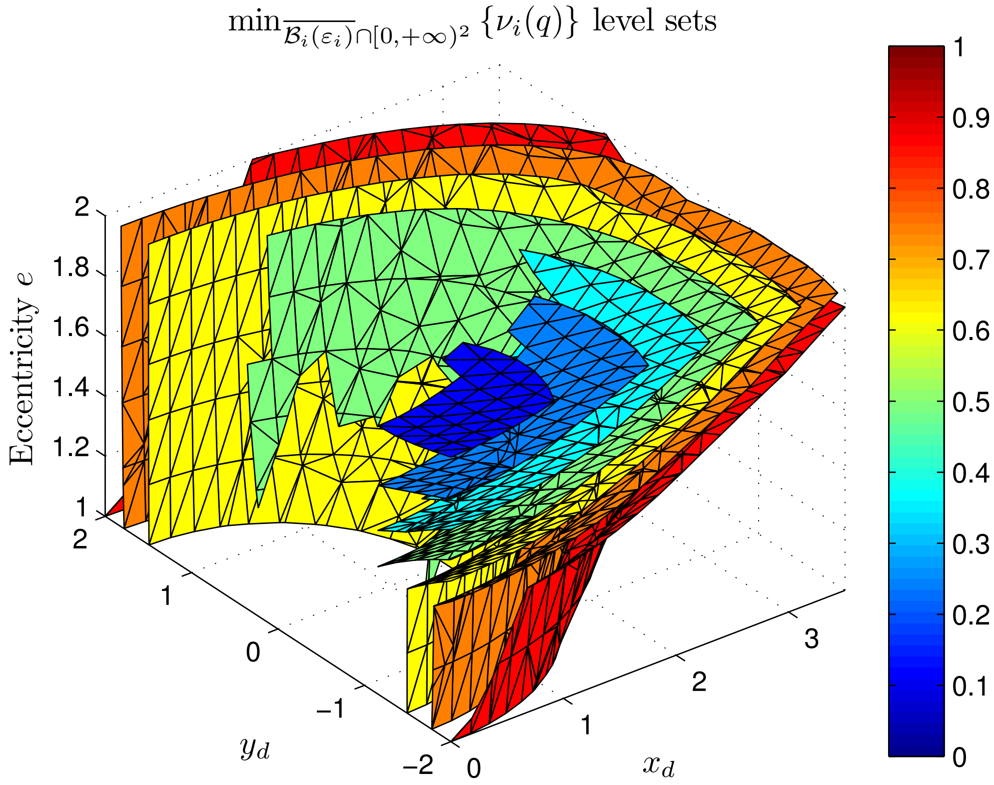


Figure 5.10: Level sets of relative curvature function minimum $\min_{\overline{\mathscr{B}_i(\varepsilon_i)}\cap[0,+\infty)^2} \{\nu_i(q)\}$ in $(q_d, e) \subset \mathbb{R}^3$ space for constant ε_i.

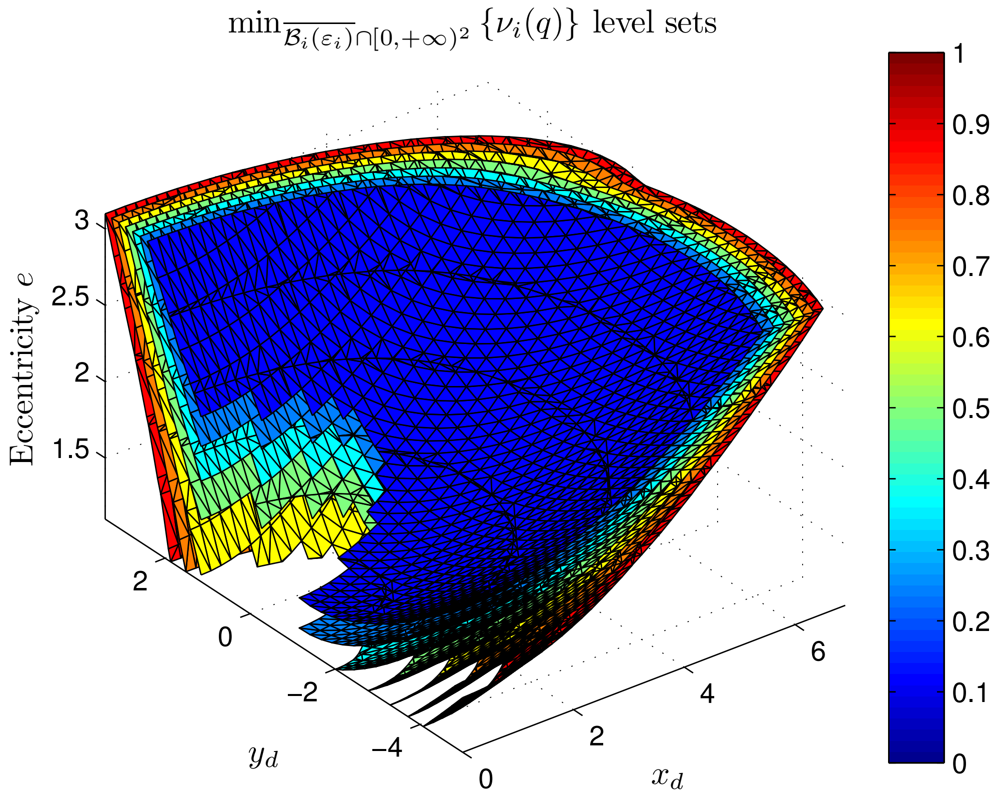


Figure 5.11: Level sets of relative curvature function minimum $\min_{\overline{\mathscr{B}_i(\varepsilon_i)}\cap[0,+\infty)^2} \{\nu_i(q)\}$ in $(q_d, e) \subset \mathbb{R}^3$ space for constant eccentricity $e = 0.87$.

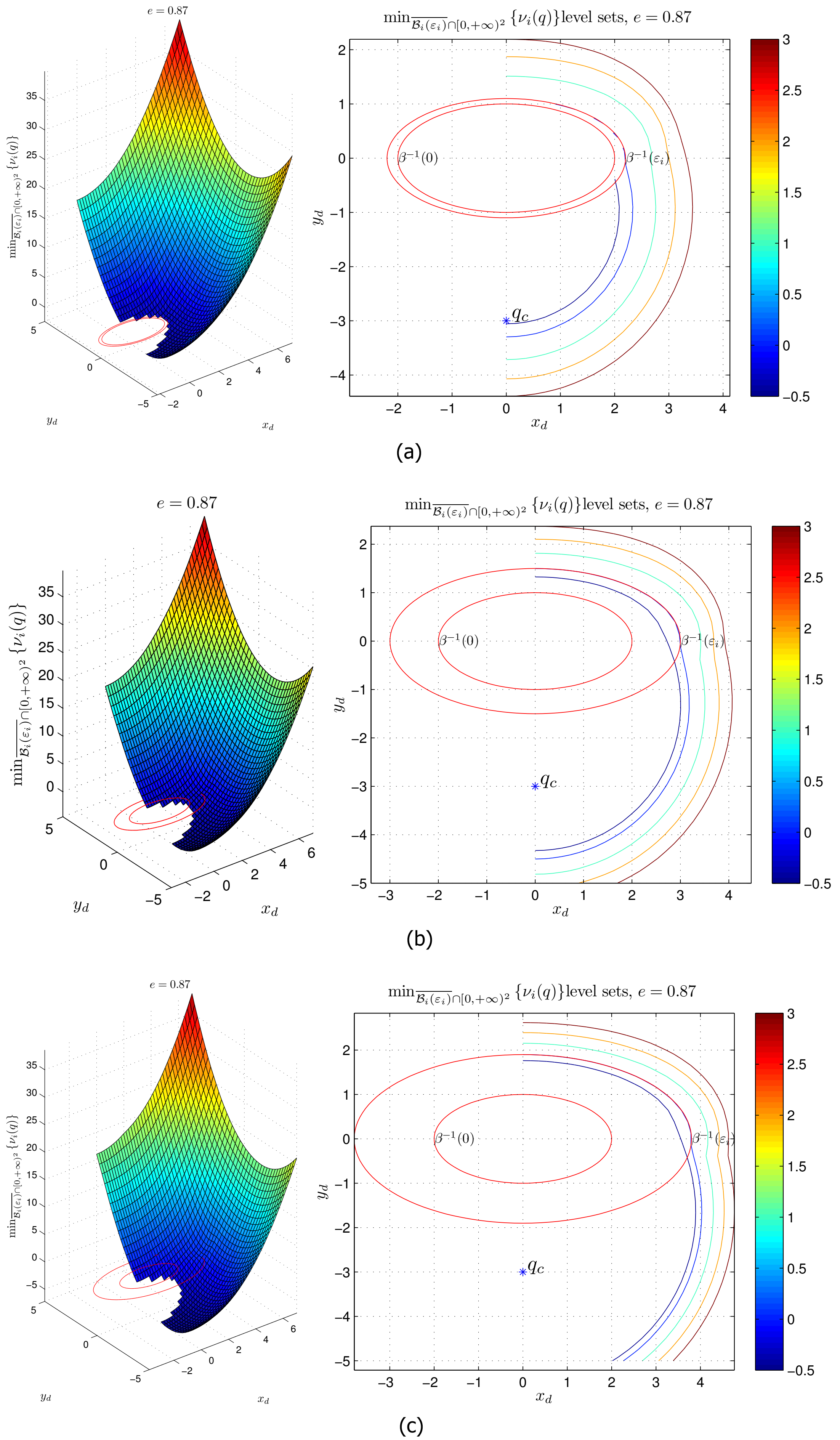


Figure 5.12: Relative curvature function $\nu_i(q)$ minimum in $\overline{\mathscr{B}_i(\varepsilon_i)} \cap [0, +\infty)^2$ for $q_d \in [0, +\infty) \times (-\infty, +\infty)$, eccentricity $e = 0.87$ and various ε_i.

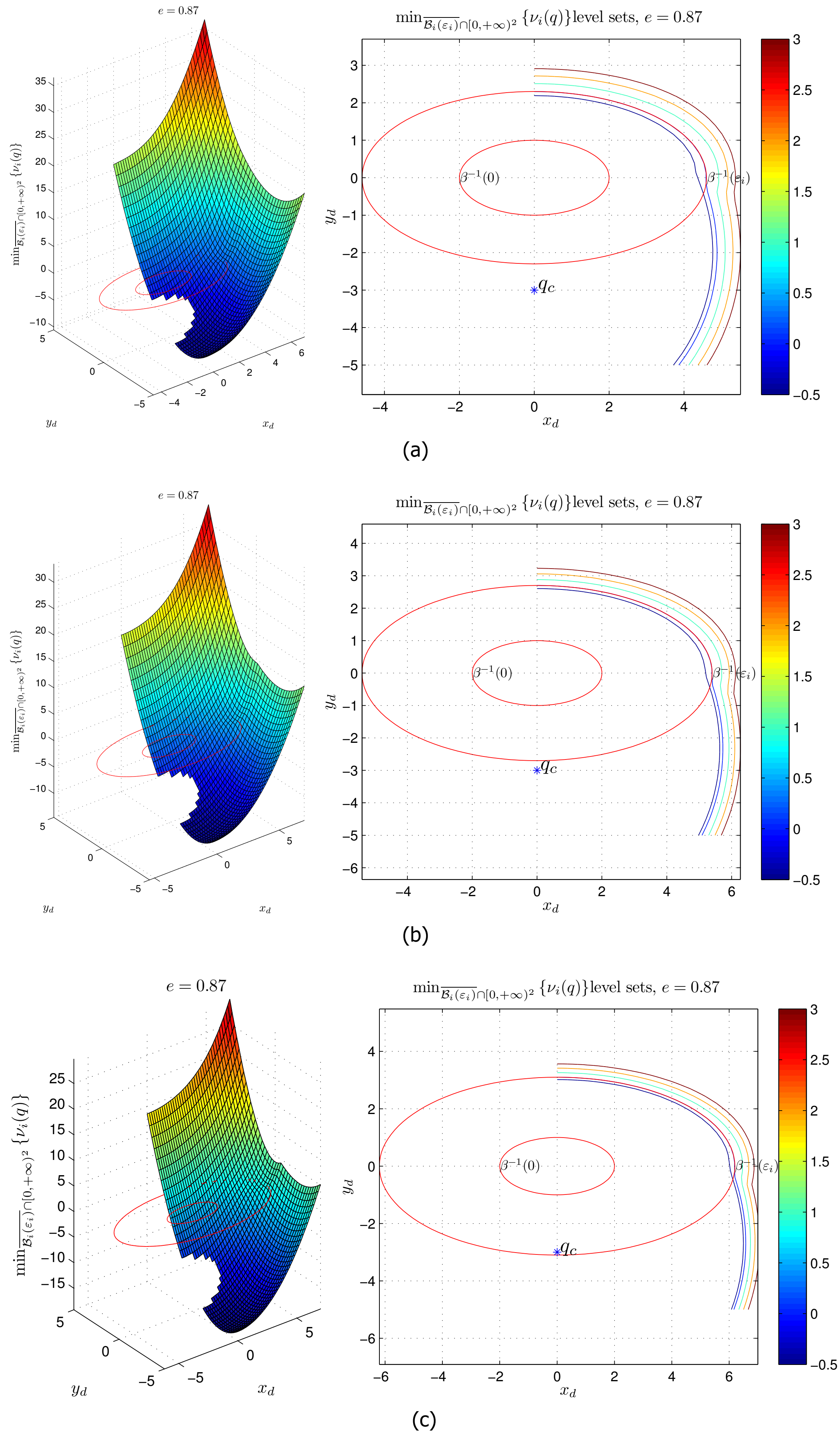


Figure 5.13: Relative curvature function $\nu_i(q)$ minimum in $\overline{\mathscr{B}_i(\varepsilon_i)} \cap [0,+\infty)^2$ for $q_d \in [0,+\infty) \times (-\infty,+\infty)$, eccentricity $e = 0.87$ and various ε_i.

Chapter 6

Partially Sufficiently Curved Spaces

6.1 Partially Nonconvex

The present section concerns spaces which are partially convex and partially sufficiently curved. But no principal curvatures which are convex but not sufficiently curved are treated yet. At least one principal curvature is sufficiently curved (hence also convex) and all principal curvatures are either nonpositive definite, or if positive (i.e., convex) they are sufficiently curved.

Proposition 47. (NF Hessian at $q_c \neq q_d$ can be made positive definite on $\operatorname{span}\left\{P_i^+, \hat{r}_i\right\}$): Let $q_c \in (\mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}) \cap \mathscr{B}_i(\varepsilon_i)$. There exists an $\varepsilon_{i5} > 0$ such that, for all $\varepsilon_i < \varepsilon_{i5}$ at every $q_c \in \mathscr{C}_{\hat{\varphi}} \cap \mathscr{B}_i(\varepsilon_i)$, if $\nu_i(q_c, \hat{p}_{ij}(q_c)) < 0, \forall j \in I_i^+(q_c) \neq \emptyset$, then $\hat{t}_i^{\mathrm{T}} \left(D^2\hat{\varphi}\right)(q_c)\hat{t}_i > 0, \forall \hat{t}_i \in \operatorname{span}\left\{\hat{r}_i(q_c), \hat{p}_{ij_1}(q_c), \ldots, \hat{p}_{ij_r}(q_c)\right\}, j_1, \ldots, j_r \in I_i^+(q_c)$.

Proof. Let the vector spanned by the radial $\hat{r}_i$ and tangential $\hat{t}_i$ vectors be denoted by

$$u_i = \mu \hat{r}_i + \lambda \hat{t}_i \tag{6.1}$$

where[1] $\mu, \lambda \in \mathbb{R} \setminus \{0\}$ are weighting coefficients and the radial and tangential unit vectors are defined with respect to the i^{th} obstacle $\mathscr{O}_i$ implicit function β_i gradient as

$$\hat{r}_i \triangleq \frac{\nabla \beta_i}{\|\nabla \beta_i\|}, \quad \hat{t}_i \triangleq \frac{\nabla \beta_i^{\perp}}{\|\nabla \beta_i\|} \tag{6.2}$$

Note that if $A \in \mathbb{R}^{n \times n}, a \in E^n$ a square real matrix and a euclidean vector respectively, and $b = ca \in E^n, c \in \mathbb{R} \setminus \{0\}$ a vector parallel to a, then for the quadratic form associated to A

$$\begin{gathered} b^{\mathrm{T}} A b = (ca)^{\mathrm{T}} A\,(ca) = ca^{\mathrm{T}} A ca = ca^{\mathrm{T}} c A a = c^2 a^{\mathrm{T}} A a = c^2 \left(a^{\mathrm{T}} A a\right) \overset{c \in \mathbb{R}\setminus\{0\} \Longrightarrow c^2 > 0}{\Longrightarrow} \\ \left\{ \begin{array}{lcl} b^{\mathrm{T}} A b > 0 & \iff & a^{\mathrm{T}} A a > 0 \\ b^{\mathrm{T}} A b = 0 & \iff & a^{\mathrm{T}} A a = 0 \\ b^{\mathrm{T}} A b < 0 & \iff & a^{\mathrm{T}} A a < 0 \end{array} \right\} \end{gathered} \tag{6.3}$$

So it suffices to determine the quadratic form sign on a direction, and it is common for all vectors in that direction.

[1]For our purpose exclusion of 0 from $\mathbb{R}$ is not mandatory.

Let us now at a critical point q_c express the Hessian's associated quadratic form along the direction of u_i. The Hessian matrix at the critical point is

$$\begin{aligned}
D^2\hat{\varphi}\left(q_c\right) = \frac{\gamma_d^{k-1}}{\beta^2}\Bigg(& k\beta D^2\gamma_d + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\left(\beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} + 2\beta_i\bar{\beta}_i\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}}\right) \\
& - \gamma_d\left(\beta_i D^2\bar{\beta}_i + 2\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s + \bar{\beta}_i D^2\beta_i\right)\Bigg) \\
= \frac{\gamma_d^{k-1}}{\beta^2}\Bigg(& k\beta D^2\gamma_d + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}2\beta_i\bar{\beta}_i\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s \\
& + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}} - \gamma_d\beta_i D^2\bar{\beta}_i - 2\gamma_d\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s - \gamma_d\bar{\beta}_i D^2\beta_i\Bigg) \\
= \frac{\gamma_d^{k-1}}{\beta^2}\Bigg(& k\beta D^2\gamma_d + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} \\
& + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}} + 2\gamma_d\left(\nabla\bar{\beta}_i\nabla\bar{\beta}_i\right)_s - 2\frac{1}{k}\gamma_d\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s \\
& - \gamma_d\beta_i D^2\bar{\beta}_i - 2\gamma_d\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s - \gamma_d\bar{\beta}_i D^2\beta_i\Bigg) \\
= \frac{\gamma_d^{k-1}}{\beta^2}\Bigg(& k\beta D^2\gamma_d + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\beta_i^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta}\bar{\beta}_i^2\nabla\beta_i\nabla\beta_i^{\mathrm{T}} \\
& - \frac{2}{k}\gamma_d\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s\gamma_d\beta_i D^2\bar{\beta}_i - \gamma_d\bar{\beta}_i D^2\beta_i\Bigg)
\end{aligned} \tag{6.4}$$

At a critical point the quadratic form along u_i is

$$\begin{aligned}
u_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right) u_i &= \left(\mu\hat{r}_i + \lambda\hat{t}_i\right)^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\left(\mu\hat{r}_i + \lambda\hat{t}_i\right) \\
&= \left(\left(\mu\hat{r}_i\right)^{\mathrm{T}} + \left(\lambda\hat{t}_i\right)^{\mathrm{T}}\right) D^2\hat{\varphi}\left(q_c\right)\left(\mu\hat{r}_i + \lambda\hat{t}_i\right) \\
&= \left(\mu\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right) + \lambda\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\right)\left(\mu\hat{r}_i + \lambda\hat{t}_i\right) \\
&= \left(\mu\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\mu\hat{r}_i\right) + \left(\mu\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\lambda\hat{t}_i\right) + \left(\lambda\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\mu\hat{r}_i\right) + \left(\lambda\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\lambda\hat{t}_i\right) \\
&= \mu^2\left(\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\hat{r}_i\right) + \mu\lambda\left(\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\hat{t}_i\right) + \mu\lambda\left(\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\hat{r}_i\right) + \lambda^2\left(\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\hat{t}_i\right)
\end{aligned} \tag{6.5}$$

Note that by the Clairaut-Schwarz Theorem C^2 continuity of function $\hat{\varphi}$ implies symmetry of its Hessian matrix[2]

$$\hat{\varphi} \in C^2\left(\left[\mathscr{F}\setminus\partial\mathscr{F}, [0,+\infty)\right]\right) \implies D^2\hat{\varphi} = \left(D^2\hat{\varphi}\right)^{\mathrm{T}} \tag{6.6}$$

As a result

$$\begin{aligned}
\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\hat{t}_i &\overset{\left(\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}(q_c)\hat{t}_i\right)\in\mathbb{R}}{=} \left(\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\hat{t}_i\right)^{\mathrm{T}} = \hat{t}_i^{\mathrm{T}}\left(\hat{r}_i D^2\hat{\varphi}\left(q_c\right)\right)^{\mathrm{T}} \\
&= \hat{t}_i^{\mathrm{T}}\left(D^2\hat{\varphi}\left(q_c\right)\right)^{\mathrm{T}}\left(\hat{r}_i^{\mathrm{T}}\right)^{\mathrm{T}} \overset{D^2\hat{\varphi}=\left(D^2\hat{\varphi}\right)^{\mathrm{T}}}{=} \hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\hat{r}_i
\end{aligned} \tag{6.7}$$

[2]In other words the order of partial derivation in mixed derivatives does not matter.

By our previous result

$$u_i^{\mathrm{T}} D^2 \hat{\varphi}\left(q_c\right) u_i = \mu^2 \left(\hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}\left(q_c\right) \hat{r}_i\right) + \lambda^2 \left(\hat{t}_i^{\mathrm{T}} D^2 \hat{\varphi}\left(q_c\right) \hat{t}_i\right) + 2\mu\lambda \left(\hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}\left(q_c\right) \hat{t}_i\right) \tag{6.8}$$

The first two terms $\hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}\left(q_c\right) \hat{r}_i$ and $\hat{t}_i^{\mathrm{T}} D^2 \hat{\varphi}\left(q_c\right) \hat{t}_i$ have been analyzed according to the example set by the original proof. Let us proceed for the third term in the same spirit

$$\begin{aligned}
\hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}\left(q_c\right) \hat{t}_i &= \hat{r}_i^{\mathrm{T}} \left(\frac{\gamma_d^{k-1}}{\beta^2} \left(k\beta D^2 \gamma_d + \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \beta_i^2 \nabla \bar{\beta}_i \nabla \bar{\beta}_i^{\mathrm{T}} \right.\right. \\
&\left.\left. + \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \nabla \beta_i \nabla \beta_i^{\mathrm{T}} - \frac{2}{k} \gamma_d \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}}\right)_s - \gamma_d \beta_i D^2 \bar{\beta}_i - \gamma_d \bar{\beta}_i D^2 \beta_i \right)\right) \hat{t}_i \\
\overset{\frac{\gamma_d^{k-1}}{\beta^2} > 0}{\Longrightarrow} \left(\hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}\left(q_c\right) \hat{t}_i\right) \frac{\beta^2}{\gamma_d^{k-1}} &= \hat{r}_i^{\mathrm{T}} \left(k\beta D^2 \gamma_d + \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \beta_i^2 \nabla \bar{\beta}_i \nabla \bar{\beta}_i^{\mathrm{T}} + \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \nabla \beta_i \nabla \beta_i^{\mathrm{T}} \right. \\
&\left. - \frac{2}{k} \gamma_d \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}}\right)_s - \gamma_d \beta_i D^2 \bar{\beta}_i - \gamma_d \bar{\beta}_i D^2 \beta \right) \hat{t}_i \\
&= \hat{r}_i^{\mathrm{T}} \left(k\beta D^2 \gamma_d\right) \hat{t}_i + \hat{r}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \beta_i^2 \nabla \bar{\beta}_i \nabla \bar{\beta}_i^{\mathrm{T}} \right) \hat{t}_i + \hat{r}_i^{\mathrm{T}} \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \nabla \beta_i \nabla \beta_i^{\mathrm{T}} \right) \hat{t}_i \\
&- \hat{r}_i^{\mathrm{T}} \left(\frac{2}{k} \gamma_d \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}}\right)_s \right) \hat{t}_i - \hat{r}_i^{\mathrm{T}} \left(\gamma_d \beta_i D^2 \bar{\beta}_i\right) \hat{t}_i - \hat{r}_i^{\mathrm{T}} \left(\gamma_d \bar{\beta}_i D^2 \beta_i\right) \hat{t}_i
\end{aligned} \tag{6.9}$$

Now let us find each term separately

$$\begin{aligned}
\hat{r}_i^{\mathrm{T}} \left(k\beta D^2 \gamma_d\right) \hat{t}_i = k\beta \left(\hat{r}_i^{\mathrm{T}} D^2 \gamma_d \hat{t}_i\right) &\overset{\gamma_d = \|q - q_d\|^2 \Longrightarrow D^2 \gamma_d = 2I}{=} k\beta \left(\hat{r}_i^{\mathrm{T}} 2I \hat{t}_i\right) \\
= \left(2k\beta\right) \hat{r}_i^{\mathrm{T}} \left(I \hat{t}_i\right) \overset{I\hat{t}_i = \hat{t}_i}{=} \left(2k\beta\right) \hat{r}_i^{\mathrm{T}} \hat{t}_i = \hat{r}_i \cdot \hat{t}_i &\overset{\hat{r}_i \cdot \hat{t}_i = \frac{\nabla \beta_i \cdot \nabla \beta_i^{\perp}}{\|\nabla \beta_i\|^2}}{=} 0
\end{aligned} \tag{6.10}$$

and

$$\left(\hat{r}_i^{\mathrm{T}} \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \bar{\beta}_i^2 \nabla \beta_i \right) \left(\nabla \beta_i^{\mathrm{T}} \hat{t}_i\right) \overset{\nabla \beta_i^{\mathrm{T}} \hat{t}_i = \nabla \beta_i \cdot \hat{t}_i = \nabla \beta_i \cdot \frac{\nabla \beta_i^{\perp}}{\|\nabla \beta_i\|}}{=} 0 \tag{6.11}$$

As a result

$$\begin{aligned}
\frac{\beta^2}{\gamma_d^{k-1}} \left(\hat{r}_i^{\mathrm{T}} D^2 \hat{\varphi}\left(q_c\right) \hat{t}_i\right) &= \beta_i^2 \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \hat{r}_i^{\mathrm{T}} \left(\nabla \bar{\beta}_i \nabla \bar{\beta}_i^{\mathrm{T}}\right) \hat{t}_i \right) - \beta_i \left(\gamma_d \left(\hat{r}_i^{\mathrm{T}} D^2 \bar{\beta}_i \hat{t}_i\right)\right) \\
&- \frac{2\gamma_d}{k} \hat{r}_i^{\mathrm{T}} \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}}\right)_s \hat{t}_i - \gamma_d \bar{\beta}_i \left(\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i\right) \\
&= \beta_i \left(\left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\bar{\beta}_i} \hat{r}_i^{\mathrm{T}} \left(\nabla \bar{\beta}_i \nabla \bar{\beta}_i^{\mathrm{T}}\right) \hat{t}_i - \gamma_d \left(\hat{r}_i D^2 \bar{\beta}_i \hat{t}_i\right) \right) \\
&- \frac{2\gamma_d}{k} \hat{r}_i^{\mathrm{T}} \left(\nabla \bar{\beta}_i \nabla \beta_i^{\mathrm{T}}\right)_s \hat{t}_i - \gamma_d \bar{\beta}_i \left(\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i\right)
\end{aligned} \tag{6.12}$$

Now observe that

$$
\begin{aligned}
\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)_s\hat{t}_i &= \frac{1}{2}\hat{r}_i\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}+\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\right)^{\mathrm{T}}\right)\hat{t}_i\\
&= \frac{1}{2}\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}+\nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}\right)\hat{t}_i\\
&= \frac{1}{2}\left(\hat{r}_i^{\mathrm{T}}\nabla\bar{\beta}_i\nabla\beta_i^{\mathrm{T}}\hat{t}_i+\hat{r}_i^{\mathrm{T}}\nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}\hat{t}_i\right)\\
&= \frac{1}{2}\hat{r}_i^{\mathrm{T}}\nabla\beta_i\nabla\bar{\beta}_i^{\mathrm{T}}\hat{t}_i\overset{\hat{r}_i=\frac{\nabla\beta_i}{\|\nabla\beta_i\|}}{=}\overset{\|\nabla\beta_i\|\neq 0\,\text{close to}\,\mathscr{O}_i}{=}\frac{1}{2}\frac{\nabla\beta_i^{\mathrm{T}}\nabla\beta_i}{\|\nabla\beta_i\|}\nabla\bar{\beta}_i^{\mathrm{T}}\hat{t}_i\\
&= \frac{1}{2}\frac{\|\nabla\beta_i\|\,\|\nabla\beta_i\|}{\|\nabla\beta_i\|}\nabla\bar{\beta}_i^{\mathrm{T}}\hat{t}_i=\frac{1}{2}\|\nabla\beta_i\|\,\nabla\bar{\beta}_i^{\mathrm{T}}\hat{t}_i\\
&= \frac{1}{2}\|\nabla\beta_i\|\,\nabla\bar{\beta}_i\cdot\hat{t}_i=\frac{1}{2}\|\nabla\beta_i\|\,\hat{t}_i^{\mathrm{T}}\nabla\bar{\beta}_i
\end{aligned}
\tag{6.13}
$$

and therefore

$$
\begin{aligned}
\frac{\beta^2}{\gamma_d^{k-1}}\left(\hat{r}_i^{\mathrm{T}}D^2\hat{\varphi}\left(q_c\right)\hat{t}_i\right) = \beta_i\left(\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\beta_i}\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}}\right)\hat{t}_i-\gamma_d\left(\hat{r}_i^{\mathrm{T}}D^2\bar{\beta}_i\hat{t}_i\right)\right)\\
-\frac{\gamma_d\|\nabla\beta_i\|}{k}\hat{t}_i^{\mathrm{T}}\nabla\bar{\beta}_i-\gamma_d\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}}D^2\beta_i\hat{t}_i\right)
\end{aligned}
\tag{6.14}
$$

From (6.8) we have that the quadratic form associated to the Hessian matrix of function $\hat{\varphi}$ is

$$
u_i^{\mathrm{T}}D^2\hat{\varphi}\left(q_c\right)u_i=\mu^2\left(\hat{r}_i^{\mathrm{T}}D^2\hat{\varphi}\left(q_c\right)\hat{r}_i\right)+\lambda^2\left(\hat{t}_i^{\mathrm{T}}D^2\hat{\varphi}\left(q_c\right)\hat{t}_i\right)+2\mu\lambda\left(\hat{r}_i^{\mathrm{T}}D^2\hat{\varphi}\left(q_c\right)\hat{t}_i\right)
\tag{6.15}
$$

and utilizing the results of previous sections, repeated here

$$
\begin{aligned}
\hat{t}_i^{\mathrm{T}}D^2\hat{\varphi}\hat{t}_i=\frac{\gamma_d^{k-1}}{\beta^2}&\left(\gamma_d\bar{\beta}_i\left(\frac{\nabla\beta_i\cdot\nabla\gamma_d}{\|\nabla\gamma_d\|^2}\left(\hat{t}_i^{\mathrm{T}}D^2\gamma_d\hat{t}_i\right)-\left(\hat{t}_i^{\mathrm{T}}D^2\beta_i\hat{t}_i\right)\right)\right.\\
&\left.+\gamma_d\beta_i\left(\frac{\nabla\bar{\beta}_i\cdot\nabla\gamma_d}{\|\nabla\gamma_d\|^2}+\hat{t}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\nabla\beta_i\cdot\nabla\beta_i^{\mathrm{T}}}{\bar{\beta}_i}-D^2\bar{\beta}_i\right)\hat{t}_i\right)\right)
\end{aligned}
\tag{6.16}
$$

and

$$
\begin{aligned}
\frac{\beta^2}{\gamma_d^{k-1}}\left(\hat{r}_i^{\mathrm{T}}D^2\hat{\varphi}\left(q_c\right)\hat{r}_i\right)&=\frac{\gamma_d}{\beta_i}\left(\left(1-\frac{1}{k}\right)\bar{\beta}_i\|\nabla\beta_i\|^2-\beta_i^2\hat{r}_i^{\mathrm{T}}D^2\bar{\beta}_i\hat{r}_i-\beta_i\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}}D^2\beta_i\hat{r}_i\right)\right)\\
&=\frac{\gamma_d}{\beta_i}\left(\left(1-\frac{1}{k}\right)\bar{\beta}_i\|\nabla\beta_i\|^2-\beta_i^2\left(\hat{r}_i^{\mathrm{T}}D^2\bar{\beta}_i\hat{r}_i\right)-\beta_i\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}}D^2\beta_i\hat{r}_i\right)\right)
\end{aligned}
\tag{6.17}
$$

So substitution of (6.14), (6.16) and (6.17) to (6.8) yields

$$
\begin{aligned}
&u_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right) u_i = \mu^2\left(\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\hat{r}_i\right) + \lambda^2\left(\hat{t}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\hat{t}_i\right) + 2\mu\lambda\left(\hat{r}_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right)\hat{t}_i\right)\\
&= \mu^2\frac{\gamma_d^{k-1}}{\beta^2}\frac{\gamma_d}{\beta_i}\left(\left(1-\frac{1}{k}\right)\bar{\beta}_i\left\|\nabla\beta_i\right\|^2 - \beta_i^2\left(\hat{r}_i^{\mathrm{T}} D^2\bar{\beta}_i\hat{r}_i\right) - \beta_i\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}} D^2\beta_i\hat{r}_i\right)\right)\\
&\quad + \lambda^2\frac{\gamma_d^{k-1}}{\beta^2}\left(\gamma_d\bar{\beta}_i\nu_i(q) + \beta_i\gamma_d\left(\frac{\nabla\bar{\beta}_i\cdot\nabla\gamma_d}{\left\|\nabla\gamma_d\right\|^2} + \hat{t}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - D^2\bar{\beta}_i\right)\hat{t}_i\right)\right)\\
&\quad + 2\mu\lambda\frac{\gamma_d^{k-1}}{\beta^2}\Bigg(\beta_i\left(\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\bar{\beta}_i}\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}}\right)\hat{t}_i - \gamma_d\left(\hat{r}_i^{\mathrm{T}} D^2\bar{\beta}_i\hat{t}_i\right)\right)\\
&\qquad - \frac{\gamma_d\left\|\nabla\beta_i\right\|}{k}\hat{t}_i^{\mathrm{T}}\nabla\bar{\beta}_i - \gamma_d\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}} D^2\beta_i\hat{t}_i\right)\Bigg) \overset{\frac{\gamma_d^{k-1}}{\beta^2}>0}{\Longrightarrow}\\
&\frac{\beta^2}{\gamma_d^{k-1}}\left(u_i^{\mathrm{T}} D^2\hat{\varphi}\left(q_c\right) u_i\right) = \lambda^2\left(\gamma_d\bar{\beta}_i\nu_i(q_c) + \beta_i\gamma_d\left(\frac{\nabla\bar{\beta}_i\cdot\nabla\gamma_d}{\left\|\nabla\gamma_d\right\|^2} + \hat{t}_i^{\mathrm{T}}\left(\left(1-\frac{1}{k}\right)\frac{\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - D^2\bar{\beta}_i\right)\hat{t}_i\right)\right)\\
&+ \lambda\left(2\mu\left(\beta_i\left(\left(1-\frac{1}{k}\right)\frac{\gamma_d}{\bar{\beta}_i}\hat{r}_i^{\mathrm{T}}\left(\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}}\right)\hat{t}_i - \gamma_d\left(\hat{r}_i^{\mathrm{T}} D^2\bar{\beta}_i\hat{t}_i\right)\right) - \frac{\gamma_d\left\|\nabla\beta_i\right\|}{k}\hat{t}_i^{\mathrm{T}}\nabla\bar{\beta}_i - \gamma_d\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}} D^2\beta_i\hat{t}_i\right)\right)\right)\\
&\quad + \left(\mu^2\frac{\gamma_d}{\beta_i}\left(\left(1-\frac{1}{k}\right)\bar{\beta}_i\left\|\nabla\beta_i\right\|^2 - \beta_i^2\left(\hat{r}_i^{\mathrm{T}} D^2\bar{\beta}_i\hat{r}_i\right) - \beta_i\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}} D^2\beta_i\hat{r}_i\right)\right)\right)\\
&= \lambda^2\left(\gamma_d\bar{\beta}_i\nu_i(q_c) + O\left(\beta_i\right)\right)\\
&\quad + \lambda\left(2\mu\left(O\left(\beta_i\right) - \frac{\gamma_d\left\|\nabla\beta_i\right\|}{k}\hat{t}_i^{\mathrm{T}}\nabla\bar{\beta}_i - \gamma_d\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}} D^2\beta_i\hat{t}_i\right)\right)\right)\\
&\quad + \left(\mu^2\frac{\gamma_d}{\beta_i}\left(\left(1-\frac{1}{k}\right)\bar{\beta}_i\left\|\nabla\beta_i\right\|^2 - O\left(\beta_i\right)\right)\right)
\end{aligned}
\tag{6.18}
$$

The above quadratic polynomial has the following discriminant

$$
\begin{aligned}
\Delta &= B^2 - 4AC\\
&= \left(2\mu\left(O\left(\beta_i\right) - \frac{\gamma_d\left\|\nabla\beta_i\right\|}{k}\hat{t}_i^{\mathrm{T}}\nabla\bar{\beta}_i - \gamma_d\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}} D^2\beta_i\hat{t}_i\right)\right)\right)^2\\
&\quad - 4\left(\gamma_d\bar{\beta}_i\nu_i(q_c) + O\left(\beta_i\right)\right)\left(\mu^2\frac{\gamma_d}{\beta_i}\left(\left(1-\frac{1}{k}\right)\bar{\beta}_i\left\|\nabla\beta_i\right\|^2 - O\left(\beta_i\right)\right)\right)\\
&= 4\mu^2\Bigg(\left(O\left(\beta_i\right) - \frac{\gamma_d\left\|\nabla\beta_i\right\|}{k}\hat{t}_i^{\mathrm{T}}\nabla\bar{\beta}_i - \gamma_d\bar{\beta}_i\left(\hat{r}_i^{\mathrm{T}} D^2\beta_i\hat{t}_i\right)\right)^2\\
&\quad - \frac{1}{\beta_i}\left(\gamma_d\bar{\beta}_i\nu_i(q_c) + O\left(\beta_i\right)\right)\left(\gamma_d\left(\left(1-\frac{1}{k}\right)\bar{\beta}_i\left\|\nabla\beta_i\right\|^2 - O\left(\beta_i\right)\right)\right)\Bigg)
\end{aligned}
\tag{6.19}
$$

A positive definite normal subspace is sufficient to ensure that the quadratic form associated with the Hessian matrix is positive on this subspace. Since for $\varepsilon_i < \varepsilon_{i0}''' \implies A > 0$ this is equivalent to ensuring that the quadratic polynomial induced by the quadratic form on this subspace does not have any real roots.

Absence of real roots for $\lambda \in (-\infty, 0) \cup (0, +\infty)$ is equivalent to proving that there is no direction u_i (note that $u_i \neq \hat{t}_i$ and $u_i \neq \hat{r}_i$) in which the quadratic form is zero.

Consequently the quadratic form retains its sign. It can only be positive because the quadratic term coefficient is positive $A > 0$.

A sufficient inequality for absence of real roots is

$$
\begin{gathered}
\Delta < 0 \overset{\mu \in \mathbb{R}\setminus\{0\} \implies \mu^2 \in (0,+\infty)}{\iff} \\
\Bigg[\left(O\left(\beta_i\right) - \frac{\gamma_d \left\|\nabla\beta_i\right\|}{k} \hat{t}_i^{\mathrm{T}} \nabla \bar{\beta}_i - \gamma_d \bar{\beta}_i \left(\hat{r}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i \right) \right)^2 \\
- \frac{1}{\beta_i} \left(\gamma_d \bar{\beta}_i \nu_i(q_c) + O\left(\beta_i\right) \right) \left(\gamma_d \left(\left(1 - \frac{1}{k} \right) \bar{\beta}_i \left\|\nabla\beta_i\right\|^2 - O\left(\beta_i\right) \right) \right) \Bigg] < 0 \iff \\
\left(O\left(\beta_i\right) - A \right)^2 - \frac{1}{\beta_i} \left(B + O\left(\beta_i\right) \right) \left(\Gamma - O\left(\beta_i\right) \right) < 0
\end{gathered}
\tag{6.20}
$$

By selecting a sufficiently small β_i we can make all $O\left(\beta_i\right)$ negligible with respect to their accompanying term's signs A, B, Γ as already proven.

Then the remaining terms determining the discriminant's sign would be

$$
(-A)^2 - \frac{1}{\beta_i} B\Gamma < 0 \iff A^2 - \frac{B\Gamma}{\beta_i} < 0 \tag{6.21}
$$

The above inequality can be satisfied by constraining critical points q_c to a neighborhood $\mathscr{B}_i\left(\varepsilon_i\right)$ by selecting a sufficiently small β_i which is (if $A^2 > 0$, otherwise not needed)

$$
A^2 - \frac{B\Gamma}{\beta_i} < 0 \iff A^2 < \frac{B\Gamma}{\beta_i} \iff \beta_i < \frac{B\Gamma}{A^2} \triangleq \varepsilon_{i4}''' \tag{6.22}
$$

Note also that the half-space $\nabla\beta_i \cdot \nabla\gamma_d > 0$ is of interest, not $\nabla\beta_i \cdot \nabla\gamma_d < 0 \implies \nexists q_c$ there. This yields a further ε_{i4}'. □

Another important note is that the above proof requires that the obstacle has a negative Gaussian curvature (obviously) and that it has negative curvature and positive curvature in two orthogonal complementary subspaces of its tangent space (tangent to level sets in the obstacle's neighborhood). In order to relax the complementary subspaces requirement, an altered proof of the original KR theorem is needed.

A single one-sheet hyperboloid obstacle forms an almost insufficiently curved space for navigation. Therefore a space with a single one-sheet hyperboloid is covered by the proof provided, i.e. that a KRNF exists in it for a high enough k (taking into consideration my proof on γ_d upper bound for unbounded spaces as well). Note that a set of cylindrical pillars as obstacles is also an almost insufficiently curved space, hence navigable with a KRNF as proved.

For two one-sheet hyperboloids which are infinite in size their inevitable intersection renders the space insufficiently curved (and of nonsmooth boundary at the intersection, but that is not out problem). Nevertheless the result of a simulation is successful, Fig. 6.1. Of course had the initial and final configurations be positioned differently with respect to the intersection, then no solution would be possible.

Definition 48. By a partially nonconvex world we refer to every obstacle boundary point having at least one sufficiently curved principal curvature and the rest nonconvex.

Proposition 49. (All $q_c \neq q_d$ nondegenerate saddles) In a partially nonconvex world, there exists a $k \geq N(\varepsilon_{I_0})$, such that every critical $q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}$ is a nondegenerate saddle.

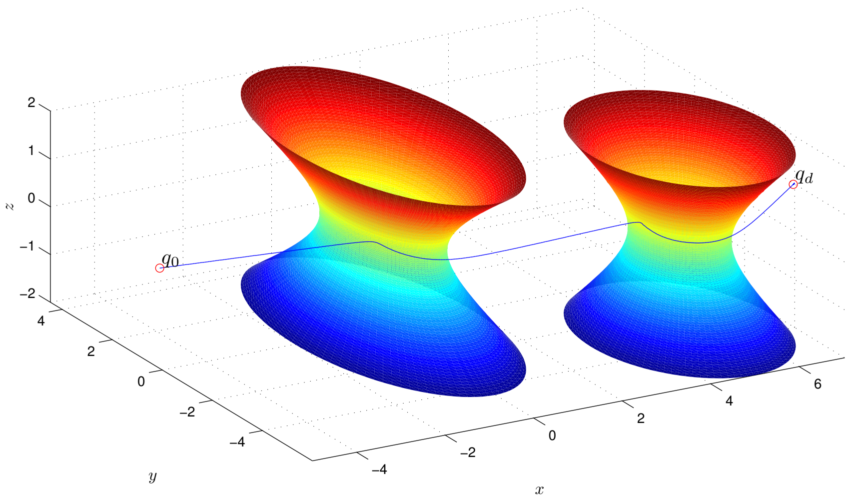


Figure 6.1: A space with a single one-sheet hyperboloid is an almost insufficiently curved space, hence navigable by a KRNF, as proved here. Two one-sheet hyperboloids form an insufficiently curved space due to their intersection.

Proof. By Definition 48 $T_{q_c}\mathscr{F} = \mathscr{P}_i^-(q_c) \oplus \operatorname{span}\left\{P_i^+(q_c), \hat{r}_i\right\}$, by Propositions 40 and 47 Hessian $(D^2\hat{\varphi})(q_c)$ is negative definite on $\mathscr{P}_i^-(q_c)$, positive definite on $\operatorname{span}\left\{P_i^+(q_c), \hat{r}_i\right\}$, so Lemma 3.8 [23] completes proof. $\square$

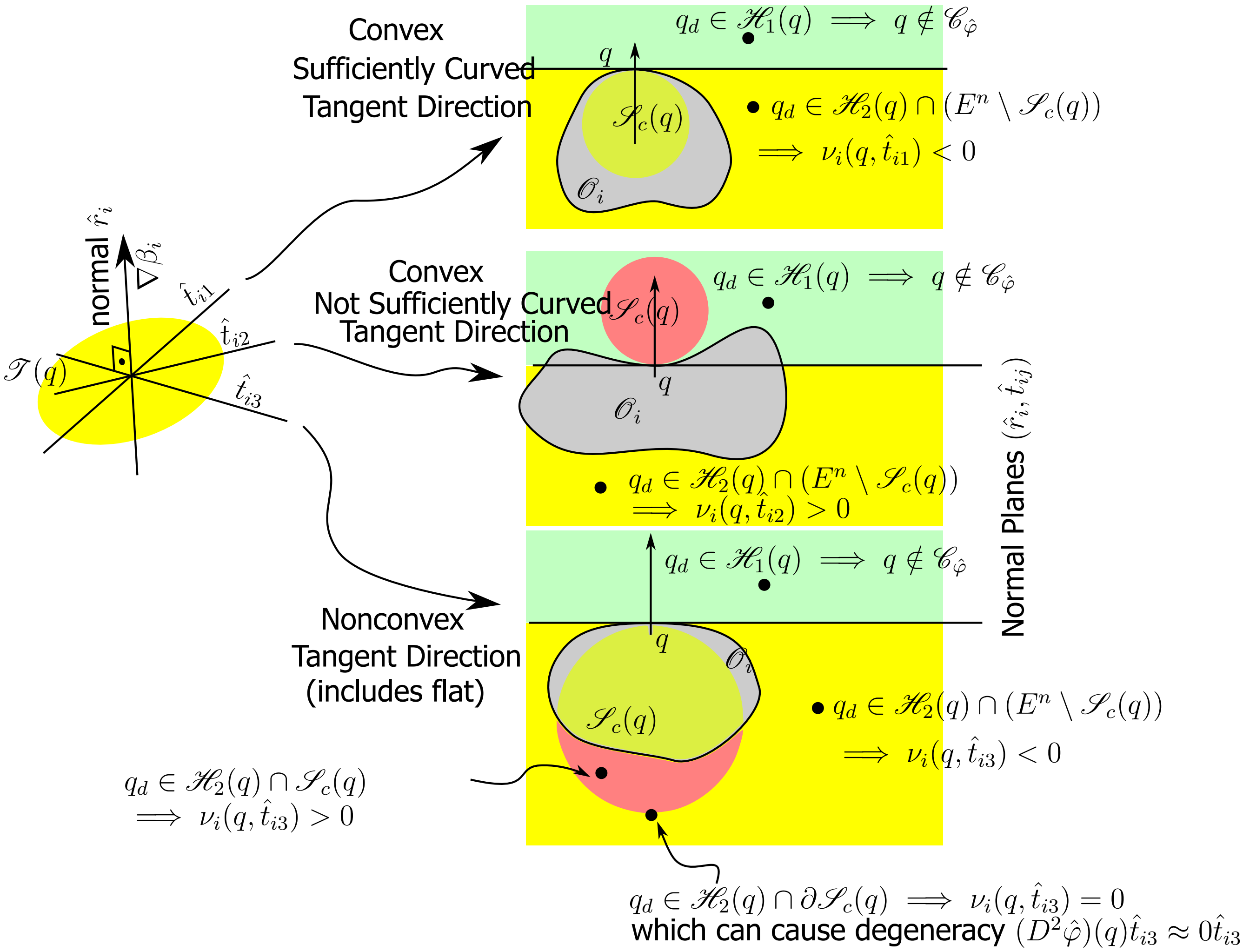


Figure 6.2: Different directions: convex and sufficiently curved, convex but insufficiently curved, and nonconvex.

6.2 Partially Sufficiently Curved Worlds

6.2.1 Intro

Definition 50 (Partially Sufficiently Curved World)**.** We call partially sufficiently curved a world for which $\forall q \in \mathscr{O}_i, \forall i \in I_0$ there is at least one sufficiently curved principal $\kappa_{ij}(q)$, at most one convex but insufficiently curved $\kappa_{ij}(q)$ and the rest $\kappa_{ij}(q)$ are nonconvex.

In chapter 4 and section 6.1 it has been proved that the KR formulation of NFs can be extended to

1. Sufficiently curved spaces, and to
2. Partially Nonconvex Partially Sufficiently Curved Spaces

Both of the above two types of spaces are relatively limited. The first type is limited to obstacles which are in every boundary point curved enough, and, as a result, convex enough. Hence, the first type is a subset of convex obstacles.

The second type of spaces is restricted to partially nonconvex obstacles. These do not include any *closed* surfaces. But we need to generalize to closed surfaces which may not

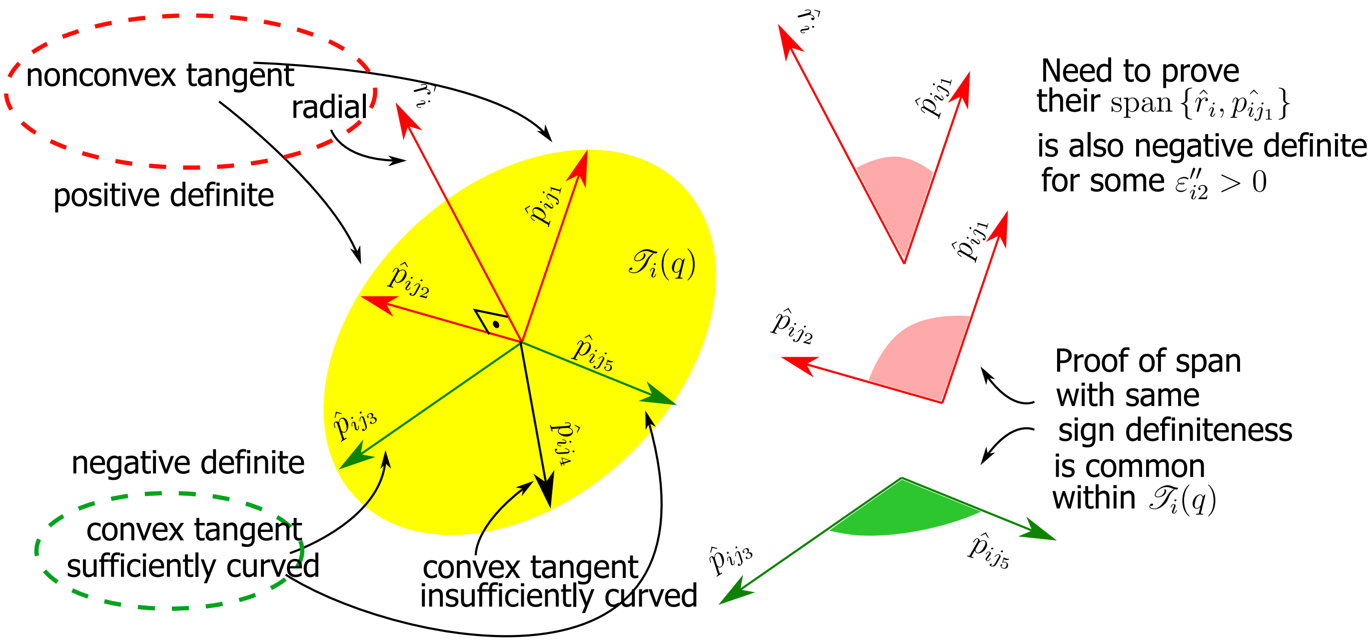


Figure 6.3: Suitable pairs of directions and selection of ε''_{i2} ensures sign definiteness of the corresponding spanned subspaces.

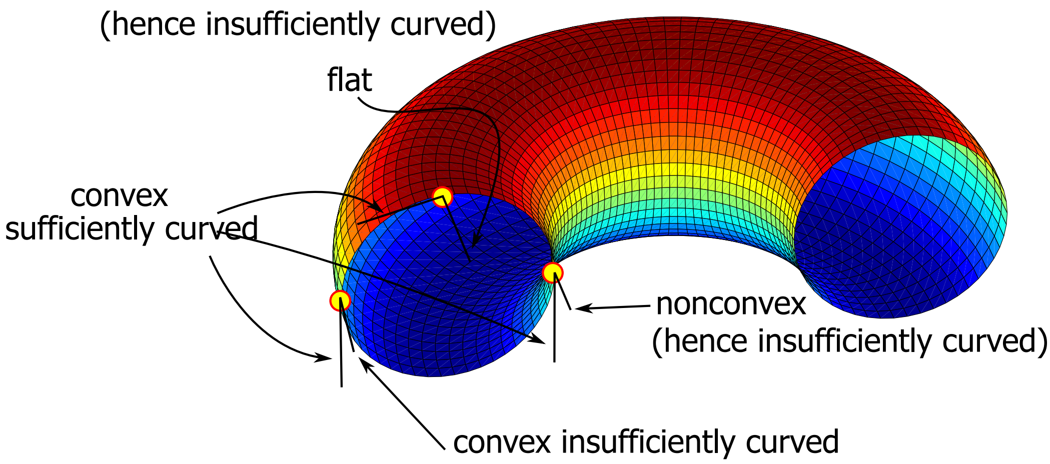


Figure 6.4: Sufficiency of curvatures along principal directions on characteristic points of torus.

be everywhere sufficiently curved, as happens in the first case. We want to treat surfaces which may be partially sufficiently curved and convex.

If a *closed* surface is convex but partially insufficiently curved, or nonconvex and partially sufficiently curved, then curvatures which are neither sufficient nor nonpositive definite[3] arise. This happens by definition in the first case.

For the second case, the following can be shown. No *closed* surface can have both points which are sufficiently curved (convex points) and points which are partially non-convex and partially sufficiently curved in convex directions, without any points which are convex but partially insufficiently curved arising.

This is a consequence Hessian matrix continuity requirements of Hessian matrix continuity requirements (i.e., C^2 continuity of β_i). These can be relaxed in some special cases where obstacle boundary nonsmoothness is allowed, based on an updated definition by Koditschek and Rimon in [28]. But we are interested in general cases, like tori. In these cases C^2-smoothness requires continuous curvature variation over the surface. This leads to convex but partially insufficiently curved points showing up when moving from a sufficiently curved, to a nonconvex partially sufficiently curved point.

Therefore, points of three kinds are differently treated in this study

1. Sufficiently curved points;
2. Partially nonconvex and partially sufficiently curved;
3. Nonconvex partially insufficiently curved and partially sufficiently curved;
4. Partially nonconvex, partially convex but insufficiently curved and partially sufficiently curved.

The last case can be treated similarly to case 3, hence it will not be separately analyzed in what follows (it is covered by the same theorems). Note that it may be better to refer to the convex insufficiently curved tangential directions by the number of associated principal curvatures, since the number of principal curvatures is the one that *does* matter.

The stronger results that are going to be developed in the sequel concern (hyper)surfaces with one convex insufficient principal curvature.

The other results, mentioned in the end, and based on (symmetry breaking) the zero measure of the set of parameter values that lead to degeneracy (nonMorse), allow for any number of convex insufficient principal curvatures, provided there remains at least one sufficient principal curvature.

The reader is reminded that existence of at least one sufficient principal curvature is still required in this case, in order to ensure that all critical points are (possibly degenerate) saddles.

The other principal curvatures relate to degeneracy statements and conclusions.

6.2.2 Insufficiently curved convex directions

The existence of convex but insufficiently curved tangent directions can lead to a degenerate potential field function φ, depending on the choice of destination. Degeneracy of any critical points invalidates the Morse property, which requires that all critical points be nondegenerate.

But the Morse property is a fundamental one in the original NF definition given by Koditschek and Rimon in [23][4]. It is used in the proof of Proposition 2.4, pp. 417-418,

[3] In other words they are both not sufficiently curved and positive definite, that is convex and insufficiently curved.

[4] [23], Definition 1, p.417.

which applies to *any* NF, not only to the KR type. For the KR type the Morse property required in the proof of Proposition 2.4 needs to be proved.

To do this, Lemma 3.8, p.433 is used. This Lemma is then "fed" with both Proposition 3.6 and the proof of Proposition 3.9 related to existence of a positive eigenvalue. In our treatment here, proposition 3.6 is proved as long as there exists at least a single sufficiently curved tangent direction at every obstacle boundary point. It is a condition ensuring a negative (definite) eigenvalue of $D^2\varphi$, the NF Hessian matrix, *does* exist. This is actually the "escape direction".

Hence, that all critical points, other than the destination, are not local minima, but are either saddles or maxima, can be proved solely by requirement of at least one sufficiently curved tangential direction[5].

That all of these are (possibly degenerate) saddles can be proved for radially increasing obstacle functions[6] using the core of Proposition 3.9's proof, of course adapted to be more general (i.e., not only for spheres).

But Proposition 3.9, which uses Lemma 3.8, does not need to hold. Although existence of at least one negative and existence of at least one positive eigenvalue have been proved, the direct sum decomposition required in Proposition 3.9 cannot be used, because there may be other linearly independent subspaces (curvature eigenvectors called principal curvature tangents) in which we have not proved what happens.

In the classic proof by Kodistchek and Rimon, the tangent subspace is shown to be negative definite, while the radial one positive definite.

In the sufficiently curved proof the same is done.

In the partially nonconvex but partially sufficiently curved case, the "pairs" of directions change (subspaces). The tangent directions are separated into those which are sufficiently curved and the rest, which are nonconvex, plus the radial one. These again form a direct sum decomposition, it is just that the sets of durections have changed, and now both eigenvectors from the radial and tangential subspaces participate in the positive definite subspace.

In the convex insufficiently curved case, the method of Lemma 3.8 may be used for a (dense?) set of destination q_d selections, but there will exist another set of destination selections for which the requirements of Lemma 3.8 break down, due to the arisal of $\hat{\varphi}$ Hessian eigenvalues[7]

For those points, the degeneracy proof is not valid any more. For this reason the definition of a NF should be re-examined. Re-consideration of why the Morse property in Definition 1 is needed in the proof of Proposition 2.4 will clarify whether it can be relaxed or not.

The reasons for imposing nondegeneracy requirements are noted within the proof of Proposition 2.4:

``Now suppose that there is some open set of initial conditions in $\mathscr{J}$ whose positive limit set $\omega(\mathscr{J})$ is a saddle point. This would imply that the saddle has a local stable manifold of dimension equal that of $\mathscr{J}$ - a contradiction, since the Hessian is non-degenerate by assumption.'' ([23], p.418)

[5] Which implies at least one sufficiently curved principal curvature, since continuity of the quadratic form associated to the Hessian matrix and constrained to the unit sphere, in combination with the Extreme value Theorem, applied on the sphere, will lead to a minimum eigenvalue which is sufficiently curved. For the related proofs, see the Propositions proved later.

[6] In more detail, those obstacles for which the gradient $\nabla\beta_i$ is outwardly oriented.

[7] We need to prove that a whole subspace of linear combinations is sign definite, because we do not know these are *eigenvalues* of the NF Hessian, we just know their sign.

Also, the reasons are detailed in the comments following the proof of Proposition 2.4, where it is again noted thta with this condition a submanifold of codimension 1 of initial conditions not attracted to q_d can disconnect $\mathscr{F}$ and "block" the flow toward q_d.

Since, the gradient system is a family $\phi_t(x)$ of diffeomorphisms parameterized by time t mapping the initial conditions x_0 to a future point and every x in the compact positively invariant set has an equilibrium as its $\omega(x)$ limit, the above statement is equivalent to the existence of either a codimension 1 manifold of $\omega(x)$ limit points, i.e. equilibria or a set of critical points of other codimension which nonetheless still sttracts a codimension 1 set of initial conditions, hence both blocks the flow and also attracts an open set of initial conditions, of dimension n.

So these are the reasons for imposing nondegeneracy in the first place. We are interested in relaxing this requirement.

Before proceeding further with our case, let us revisit [20] by Koditschek. This work provides a rephrased proof of Proposition 2.4/1990 as Proposition 2.1, p.135. It also provides a more elaborate commentary and, most importantly to our purpose, a comment on possible removal of this requirement[8]. Moreover, note that the proof there is for C^2 functions, as commented in [23].

First of all, the proof uses the argument of the dense character of the complement of a countable union of nowhere dense sets in $\mathscr{J}$. By removing the nondegeneracy requirement, we will be led to an uncountable (continuum) of stable sets, which can result in a union of higher dimension, hence we will be particularly cautious in the dimension of the critical set and of the stable sets of each critical point belonging to the critical manifold.

Secondly, the comments elucidate that what should be prevented is any co-dimension 1 set of *saddles* to disconnect $\mathscr{J}$ (not (initial points) not attracted to q_d, as phrased in [23]).

An important distinction nonetheless is that by allowing degeneracy, we are not allowing *full* degeneracy. In other words, we are going to place some (minimal) restrictions, by allowing only a single eigenvalue to become zero, while requesting that from the rest, at least one is negative and the remaining sign definite (at least one from the remaining must be positive). This will lead us to strong results.

Even weaker *independent* and not subsequent results can be obtained, as noted in the end, which refer to higher order degeneracy and its structural instability. But in the weaker case as well, we still require at least one negative (and nearly optionally, at least one positive) eigenvalues, hence even in the (highly) relaxed case, full degeneracy is avoided.

Taking the previous arguments into consideration, the example provided in [20] which defies the properties we are to prove, does so because it is *fully* degenerate. The same happens with the example referred to therein, i.e., 1.1.3 from Palis-de Melo, Geometric Theory of Dynamical Systems [].

Finally, from this work of Koditschek, in support of our effort it is commented that

``While this condition incurs an undesirable loss of generality the technical problems which result in its relaxation require more attention than worthwhile in this paper.''

This implies that an extension might be possible, although particular technical problems would need to be addressed. Our present work treats this extension.

The original proof by Koditschek and Rimon uses Lemma 3.8 and Proposition 3.9 to advance to nondegenerate saddles at once. This has been done for sufficiently curved

[8] Although from a different viewpoint of why that might be desirable for applications.

spaces in a similar way in chapter 4.

But for our purposes we need the following which is unmentioned in [23] since no degeneracy has been considered there, but could have been used (and is actually as if half of it is implicitly used when deducing that at least one negative eigenvalue implies the point is not a local minimum, i.e., that even in case of degeneracy, this result about the quadratic form implies some things about the actual function behavior).

Definition 51 (Non-semi definite function [38]). Non-semi definite is a homogeneous function $g : K \to \mathbb{R}$ where $K \subseteq \mathbb{R}^n$ is a cone if $x \in K, t \in \mathbb{R} \implies tx \in K$, i.e., a function such that $g(tk) = t^p g(k), \forall t \in \mathbb{R}$ and some fixed p, when there exist $x, y \in K : g(x) > 0 \wedge g(y) < 0$.

Proposition 52 (Nonsemi-definiteness $\implies$ saddle [38]). If F_p is nonsemi-definite, then a is a saddle point of f.

Proof. The proof can be found in [38]. □

where F_p is defined as the first nonzero Taylor form. The kth Taylor form is defined as ($n \in \mathbb{N}$ is the dimension number)

$$\frac{1}{k!} D_x^k f(a) = \frac{1}{k!} \sum_{\substack{i_1, i_2, \ldots, i_n \\ i_j \geq 0, \forall j \in \mathbb{N}^* \cap [1, n] \\ \sum_{j=1}^n i_j = k}} \left(\frac{k!}{i_1! i_2! \cdots i_n!} \left(D_1^{i_1} D_2^{i_2} \cdots D_n^{i_n} f \right)(a) x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \right) \tag{6.23}$$

where $D_j^i \triangleq \frac{\partial^i}{(\partial x_j)^i}$ corresponds to the partial derivative operator. Note that the 2nd Taylor form is obtained for $k = 2$ and includes all 2nd order Terms

$$\frac{1}{2!} D_x^2 f(a) = \frac{1}{2!} \sum_{\substack{i_1, i_2, \ldots, i_n \\ i_j \geq 0, \forall j \in \mathbb{N} \cap [1, n] \\ \sum_{j=1}^n i_j = 2}} \left(\frac{2!}{i_1! i_2! \cdots i_n!} \left(D_1^{i_1} D_2^{i_2} \cdots D_n^{i_n} f \right)(a) x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \right) \tag{6.24}$$

But the exponent constraints

$$\begin{gathered}
\left\{ \begin{array}{c} i_j \geq 0, \forall j \in \mathbb{N} \cap [1, n] \\ \sum_{j=1}^n i_j = 2 \end{array} \right\} \implies \\
\left\{ \begin{array}{c} \{i_{j_1} = 2 \wedge i_{j_m} = 0, \forall j_m \neq j_1\} \vee \\ \{i_{j_1} = 1 \wedge i_{j_2} = 1 \wedge i_{j_m} = 0, \forall j_m \neq j_1, j_2\} \end{array} \right\} \implies \\
\left\{ \begin{array}{c} \left\{ i_1! i_2! \cdots i_n! = \prod_{j=1}^n i_j! = i_{j_1}! \prod_{m=1}^{n-1} i_{j_m}! = 2! \prod_{m=1}^{n-1} 0! = 2 \right\} \vee \\ \left\{ \prod_{j=1}^n i_j! = i_{j_1}! i_{j_2}! \prod_{m=1}^{n-2} i_{j_m}! = 1!1! \prod_{m=1}^{n-2} 0! = 1 \right\} \end{array} \right\}
\end{gathered} \tag{6.25}$$

Therefore, substitution in the 2nd Taylor form yields for the first case

$$\begin{aligned}
&\frac{1}{2!}\left(D_x^2 f\right)(a) \\
&= \sum_{i_{j_1}=2, i_{j_m}=0, \forall j_m \neq j_1} \left(\frac{1}{i_{j_1}! \prod_{m=1}^{n-1}} \left(D_{j_1}^{i_{j_1}} D_{j_{m_1}}^{i_{j_{m_1}}} D_{j_{m_2}}^{i_{j_{m_2}}} \cdots D_{j_{m_{n-1}}}^{i_{j_{m_{n-1}}}} f \right)(a) x_{j_1}^{i_{j_1}} x_{j_{m_1}}^{i_{j_{m_1}}} x_{j_{m_2}}^{i_{j_{m_2}}} \cdots x_{j_{m_{n-1}}}^{i_{j_{m_{n-1}}}} \right) \\
&= \frac{1}{2!0! \cdots 0!} \left(D_{j_1}^2 D_{j_{m_1}}^0 \cdots D_{j_{m_{n-1}}}^0 f \right)(a) x_{j_1}^2 x_{j_{m_1}}^0 x_{j_{m_2}}^0 \cdots x_{j_{m_{n-1}}}^0 \\
&= \frac{1}{2}\left(D_{j_1}^2 f\right)(a) x_{j_1}^2
\end{aligned} \tag{6.26}$$

and for the second case

$$\begin{aligned}
&\frac{1}{2!}\left(D_x^2 f\right)(a) \\
&= \sum_{i_{j_1}=1, i_{j_2}=1, i_{j_m}=0, \forall j_m \neq j_1, j_2} \\
&\left(\frac{1}{i_{j_1}! i_{j_2}! \prod_{m=1}^{n-2} i_{j_m}!} \left(D_{j_1}^{i_{j_1}} D_{j_2}^{i_{j_2}} D_{j_{m_1}}^{i_{j_{m_1}}} D_{j_{m_2}}^{i_{j_{m_2}}} \cdots D_{j_{m_{n-2}}}^{i_{j_{m_{n-2}}}} f \right)(a) x_{j_1}^{i_{j_1}} x_{j_2}^{i_{j_2}} x_{j_{m_1}}^{i_{j_{m_1}}} x_{j_{m_2}}^{i_{j_{m_2}}} \cdots x_{j_{m_{n-2}}}^{i_{j_{m_{n-2}}}} \right) \\
&= \frac{1}{1!1! \prod_{m=1}^{n-2} 0!} \left(D_{j_1}^1 D_{j_2}^1 D_{j_{m_1}}^0 D_{j_{m_2}}^0 \cdots D_{j_{m_{n-2}}}^0 f \right)(a) x_{j_1}^1 x_{j_2}^1 x_{j_{m_1}}^0 x_{j_{m_2}}^0 \cdots x_{j_{m_{n-2}}}^0 \\
&= \left(D_{j_1} D_{j_2} f\right)(a) x_{j1} x_{j2}
\end{aligned} \tag{6.27}$$

Therefore, the kth Taylor form is the polynomial form induced by the kth derivative of f at point a. In other words the terms of the kth Taylor form are those terms of the Taylor series which are of order k.

The first order Taylor form is the linearization, which accurately describes a function in the neighborhood with nonzero first derivative, according to the implicit function theorem.

The second order Taylor form is the quadratic approximation of the function which accurately described it in the neighborhood of a nondegenerate critical point, according to the Morse Lemma [39].

The first nonzero Taylor form refers to the least order of the derivative which is not identically equal to zero. Degeneracy is allowed, as emphasized by the non-semi definiteness condition, but obviously full degeneracy is not allowed, because of the definition of non-semi definiteness.

The following ensures that all these are only (possibly degenerate) saddle points[9].

Proposition 53 (Partial sufficient curvature $\implies$ saddles)**.** For any partially sufficiently curved world, if $k \geq N(\varepsilon_{I_0})$, then any critical $q_c \neq q_d$ is a (possibly degenerate) saddle.

Proof. By Proposition 43, the radial direction is positive definite $\hat{r}_i^{\mathrm{T}} \left(D^2\hat{\varphi}\right)(q_c)\hat{r}_i > 0$, by Definition 50 and Proposition 40, there exists a tangential direction, such that $\hat{t}_i^{\mathrm{T}} \left(D^2\hat{\varphi}\right)(q_c)\hat{t}_i < 0$. Then, set $x = \hat{r}_i, y = \hat{t}_i$ in Definition 51. At q_c the first Taylor form is zero $F_1 \equiv 0$, the second F_2 is the quadratic form associated with the Hessian matrix $\left(D^2\hat{\varphi}\right)(q_c)$. Since $F_2(\hat{r}_i) = \hat{r}_i^{\mathrm{T}} \left(D^2\hat{\varphi}\right)(q_c)\hat{r}_i > 0$, F_2 is not identically equal to zero, so it is the first nonzero

[9]Note that full degeneracy does *not* imply that the Hessian matrix is identically zero, as for example in the case of $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, whereas if the Hessian matrix is identically zero, then obviously it is fully degenerate. Hence, if the Hessian matrix is not fully degenerate, it follows that it cannot be identically zero.

form F_p of $\hat{\varphi}$ at q_c. Since both $F_p(\hat{r}_i) > 0$ and $F_p(\hat{t}_i) < 0$, F_p is nonsemi-definite. By Proposition 52 the result follows. □

The Polar property has been proved. So, by now our function $\hat{\varphi}$ is by construction Analytic or C^2 (depending on our choice, see [23]) and Admissible, and also Polar.

The Morse property remains to be discussed. Note that the diffeomorphism theorem still holds. Hence, we can continue working with $\hat{\varphi}$ instead of φ. Also, our results will be applicable to any diffeomorphic world (so we also extend the NF method to more general topologies as well, namely those which involve multiply connected obstacles).

We haveproved our first results and are now left with saddles which may be degenerate and cause problems. The types of problems we want to avoid have been exposed previously and are open local stable sets of any of these saddles $q_c \in \mathscr{F}_0 \cap \mathscr{C}_{\hat{\varphi}}$.

To advance and continue speaking of NFs, we now need to extend their definition. In the spirit of Appendix I, p.515, [28], we provide the following extended definition. Note that their new definition will allow for $\hat{\varphi}$ to be non-Morse at any critical point other than the destination. This includes the case of sharp corners in the Appendix mentioned.

Definition 54 (Extended Navigation Function)**.** Let $\mathscr{F} \subset E^n$ be a compact connected (analytic or C^2 manifold with boundary. A map $\varphi : \mathscr{F} \to [0, 1]$ is a (possibly degenerate) navigation function if it is

1. C^2 on $\mathscr{F}$ (analytic is stricter but not needed, this ensures uniqueness and existence of closed-loop robot system trajectories);
2. Polar on $\mathscr{F}$, with unique minimum at $q_d \in \mathscr{F} \setminus \partial\mathscr{F}$ (this makes it useful because it ensures convergence);
3. The union of any critical points $q_c \in (\mathscr{F} \cap \mathscr{C}_\varphi) \setminus \{q_d\}$ has a stable set of Lebesgue measure zero (this is the best that can be done with smooth vector fields);
4. Admissible on $\mathscr{F}$ (ensures safety -i.e., collision avoidance- and that transients of the closed loop mechanical system are stable as well, "inheriting" good properties [23]).

Compare this with Definition 1, p.417 [23].

Note, that according to pp. 515-516 [28], the Morse property is used to prove that the resulting feedback control law still guides the physical system correctly. This is noted also in p.418 [23], namely that it permits a straightforward proof that the desirable limiting behaviour of the gradient flow is "inherited" by the ultimate closed loop mechanical system.

Relaxation of the Morse property requires reconsideration of the underlying control theory [28]. Nonetheless, it is also conjectured in p.516 [28] that a simple energy-conservation argument will ensure the physical viability of this extended class.

6.2.3 Exploring degeneracy causes

Before proceeding further with our case, let us explore how degeneracies can arise.

The issue with closed surfaces is that there exist points of insufficiently curved tangential directions, together with sufficiently curved tangential directions, together with sufficiently curved tangential directions, but without any nonconvex tangential directions.

Stated in other words, the tangent space of some points is partially sufficiently curved, partially insufficiently curved, but overall convex.

The reason these points are ineluctable in the case of closed surfaces which are almost insufficiently curved is now going to be elucidated.

First of all, almost insufficiently curved surfaces come in two flavours. Those which are partially nonconvex within their tangent space, and those which are fully convex.

The partially nonconvex surfaces cannot be closed because then points of fully convex tangent space should exists. If we consider surfaces which are both partially nonconvex somewhere and sufficiently curved (implies convex as well) elsewhere, then for these to be C^2, points of convex but almost insufficient curvature will arise intermediately to the two areas. Hence, any closed obstacle which is somewhere partially nonconvex will include points of convex almost insufficient curvature.

Any other closed almost insufficiently curved surface which is fully convex everywhere will include such points by definition, because otherwise it would have been sufficiently curved. It has been explained why convex almost insufficiently curved points always arise on closed surfaces.

At these points there exists a maximum radius of curvature among the tangential direections, such that there are curvature half-spheres with non-empty intersection with the free space $\mathscr{F}$.

As a result, the destination q_d can be placed within the maximal curvature half-sphere at that point, on the maximal curvature half-sphere at that point, or outside the maximal curvature half-sphere. Each case is analysed as follows.

Case 1: If q_d is within the maximal curvature half-sphere, then at that point there exist curvature half-spheres which include q_d, inside them and others which are smaller and do not include it.

In the tangent direction of the first ones the Hessian quadratic form is negative definite $\hat{t}_i^{\mathrm{T}}\left(D^2\varphi\right)(q)\hat{t}_i < 0$ whereas in the tangential direction of the second one the Hessian is positive definite $\hat{t}_i^{\mathrm{T}}\left(D^2\varphi\right)(q)\hat{t}_i > 0$.

As long as these separate the tangent space into a direct sum decomposition, the proof of non-degeneracy is the same as for spaces of negative Gaussian curvature which are almost insufficiently curved[10].

Case 2: If q_d is out of the maximal curvature half-sphere, then it is outside all curvature half-spheres at that point. This implies that $\hat{t}_i^{\mathrm{T}} D^2\varphi\hat{t}_i > 0, \forall\hat{t}_i$ at that point and the classical proof is valid.

Case 3: In this case q_d is on the maximal curvature half-sphere boundary. All other curvature half-spheres at that point are smaller and do not include q_d. Hence, $\hat{t}_i^{\mathrm{T}} D^2\varphi\hat{t}_i > 0$ for these. But $\hat{t}_i^{\mathrm{T}} D^2\varphi\hat{t}_i = 0$ at the tangent direction $\hat{t}_i$ corresponding to the maximal curvature half-sphere.

It is noted again that these conclusions follow from the assumption that the critical point q_c arises in such an area. But since we are not sure where q_c will arise, the existence of such areas is problematic.

The set of such points is open, due to the C^2 property of φ, which implies that curvatures are continuous, therefore there is a neighbourhood of points with maximal curvature half-spheres protruding from the obstacle.

As a result, also the set of destinations q_d which belong to the union of these maximal curvature half-spheres corresponding to these points is an open set.

At those points (case 3) the $\hat{t}_i^{\mathrm{T}} D^2\varphi\hat{t}_i = 0$ can be shown to be the eigenvector of a zero eigenvalue by proving that the quadratic form $\hat{v}^{\mathrm{T}} D^2\varphi\hat{v}$, associated with the Hessian matrix, when restricted to the unit sphere $\|\hat{v}\| = 1$ has a stationary value in this direction.

This can be proved by showing that the discriminant of the restricted quadratic form

---

[10]It interesting to note that radius of curvature can be ordered from $+\infty \to 0^+ \to 0^- \to -\infty$, where the direction is from convex to nonconvex.

in spanning directions of the sphere tangent space is zero.

This can be associated with points of the obstacle where its gradient is an eigenvector of the Hessian matrix (Gradient Extremal Paths), as well as other points. But for a single tangnt direction at them. At all these $\hat{t}_i^{\mathrm{T}} D^2 \beta_i \hat{t}_i = 0$ holds.

In any case, as analysed in a following section, the set of such points is open.

We conclude that the set of points where if a critical point arises q_c it will be on a maximal curvature half-sphere and have zero eigenvalue, is the intersection of two open sets of q_d selections. Therefore it could be an open set.

To treat the case of (possibly arising) degeneracies, as the previous analysis suggests are not always avoidable, we need to consider them by application of more general theorems, namely the Morse-Bott Lemma and Thom Splitting Lemma.

6.2.4 NF General Convergence Proof

Proposition 55 (Extended Proposition 2.4 [23])**.** Let φ be a C^2 function on a compact Riemann manifold $\mathscr{J}$. If the following hold

1. The union of all initial conditions whose positive limit set includes saddle points or maxima is a set of Lebesgue measure zero.
2. The gradient $\nabla\varphi$ is transverse and directed away from the interior $\mathscr{J} \setminus \partial\mathscr{J}$ of set $\mathscr{J}$ on the boundary its boundary $\partial\mathscr{J}$.

Then the negative gradient flow $-\left(\nabla\varphi\right)\left(x(t)\right) = \frac{\partial x}{\partial t}(t)$ has the following properties

1. $\mathscr{J}$ is a positive invariant set;
2. the positive limit set of all initial conditions in $\mathscr{J}$ consists of tje critical points of φ;
3. there is a dense open set $\tilde{\mathscr{J}} \subset J$, whose positive limit set consists of the local minima of φ.

Proof. Claim 1: By hypothesis, the vector field is directed toward the interior of $\mathscr{J}$ on its boundary $\partial\mathscr{J}$. Hence, set $\mathscr{J}$ is positive invariant under the negative gradient flow of function φ.

Note that non-regular points q_c (i.e., those for which $\left(\nabla\varphi\right)(q_c) = 0$ for $q_c \in \partial\mathscr{J}$) on the boundary $\partial\mathscr{J}$ have a gradient which is trivially transverse to the boundary and which does not have a defined direction, since zero.

Claim 2: According to Hirsch and Smale 1974, Theorem 4, p.203, the following holds. Let z be an ω limit point of a trajectory of the negative gradient flow. Then if the trajectory is included in a compact positive invariant set, then z is an equilibrium of the gradient system.

As a result, the positive limit set of $\mathscr{J}$ consists of the critical points of φ. These are either maxima, saddles or minima.

Claim 3: It follows from the hypothesis that no open set of initial conditions is attracted to saddles or maxima. The complement of the set of initial conditions attracted to saddles or maxima is a dense open set. Since all initial conditions have equilibria in their positive limit set, this dense open set has as positive limit set the equilibria which are not saddles, nor maxima, these are the local minima of φ. □

We now need to prove that, under certain assumptions, the KRPF is a NF in the sense of the extended Definition 1 (previously provided). This requires several steps.

Firstly, an understanding of the proofs of Propositions 0,1, and 3, which provide conditions related to sign definiteness of the Hessian matrix $\left(D^2\varphi\right)(q_c)$ at any critical point $q_c \in \mathscr{C}_\varphi \cap \mathscr{F}_0$ in the tangent and radial directions.

Secondly, an understanding of the reason for which degenerate eigenvalues of $(D^2\varphi)(q_c)$ may arise. This depends on the geometry of obstacles and the destination and has already been analysed.

Thirdly, a combination of the previous three, which relates geometric properties of the obstacle's boundary to the NF properties of the Hessian matrix $(D^2\varphi)(q_c)$ eigenvalues which result from them. This is a connection of properties needed to derive our basic result. It is provided by Proposition 6.

Fourthly, the main result shows that there exist a tuning parameter lower bound such, that the KRCNF is a NFaccording to the extended definition, provided obstacles satisfy certain geometrical requirements.

Finally, a comment on the cause of higher order degeneracy is provided.

The following proves that for all critical points in which at most a single Hessian eigenvalue can be degenerate and all others are sign definite with eigenvalues of both signs present, then the set of initial conditions with these critical points in its positive limit set is of measure zero.

Proposition 56. (Single Hessian degeneracy and at least one negative eigenvalue, imply measure zero stable set): If for a subset of critical points $q_c \in \mathscr{C}_{\hat{\varphi}} \setminus \{q_d\}$ all have at least one negative, at least one positive and at most one zero eigenvalues of the Hessian matrix $(D^2\varphi)(q_c)$, then the set of initial conditions of system $\frac{\partial x}{\partial t}(t) = -(\nabla_q\varphi)(x(t))$ which have such a point q_c in their positive limit set is of measure zero.

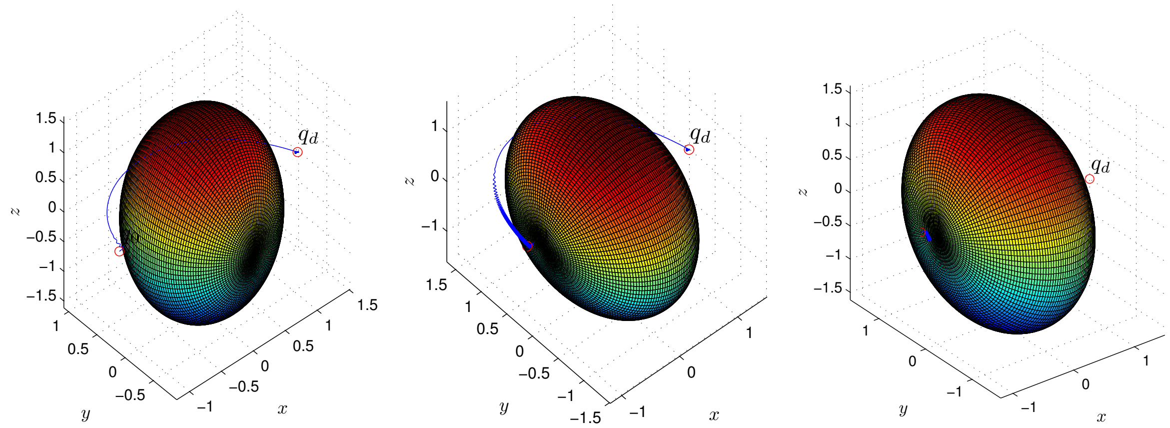
Proof. Since at most a single eigenvalue can be zero, any critical subset is of dimension at most 1. No branching of it can arise. This follows from Thom's Splitting Lemma [36] $\hat{\varphi}(x,y) = \hat{\varphi}_M(x) + \hat{\varphi}_{NM}(y)$, where $\hat{\varphi}_M(x)$ the Morse part on x mapped to $\mathscr{P}^{\pm}(q_c)$ by a smooth change of coordinates and $\hat{\varphi}_{NM}(y)$ the non-Morse part, which is defined on an at most one-dimensional subspace y smoothly mapped to the single degenerate eigenvector $\operatorname{span}\{\hat{p}_{ij_d}(q_c)\}$. To prove it, note that the restriction $(D^2\hat{\varphi}_M)(q_c)|_x$ is nonsingular, hence $\hat{\varphi}$ can remain constant at most along y, limiting the critical set to at most one dimension.

Since 1-dimensional without branching, every critical set is diffeomorphic to either a circle or a line segment. If diffeomorphic to a circle, the critical set is a nondegenerate critical submanifold disjoint from other critical sets, hence the Morse-Bott Lemma to it [40], [37]. If diffeomorphic to a line segment, we break it into its interior and endpoints. To each interior point the Morse-Bott Lemma applies, while to the endpoints Thom's Splitting Lemma. Taking into account that the critical sets are at most of dimension 1, the sign definite subspaces sum to an $(n-1)$-dimensional subspace. On this at least one eigenvalue is negative, hence the stable set is at most $(n-2)$-dimensional at each q_c. The union of stable sets over the critical sets is then at most $(n-2)+1 = (n-1)$-dimensional, hence a Lebesgue measure zero set. □

The following is our main contribution.

Proposition 57. (NF in Partially Sufficiently Curved Worlds) In every partially sufficiently curved world $\mathscr{F}$ there exists a $N(\varepsilon_{I_0})$, such that for all $k \geq N(\varepsilon_{I_0})$ the KRNF φ is a NF Definition 54 on $\mathscr{F}$.

Proof. By Definition 50 and Propositions 40 and 47 the Hessian $(D^2\hat{\varphi})(q_c)$ satisfies the requirements of Proposition 56, hence it is a NF according to the extended Definition 54. □



(a) Partially sufficiently curved. (b) Partially sufficiently curved. (c) Not sufficiently curved.

Figure 6.5: Three different ellipsoids. In the first case the ellipsoid is everywhere partially sufficiently curved. Note that insufficient curvature here arises due to an insufficient convex principal curvature, not a nonconvex one. The agent successfully navigates around the ellipsoid. Increasing the eccentricity along the sufficiently curved tangent direction just below the limit still allows the agent to navigate in the second case. A negligible increase of the smallest principal eccentricity in the third case renders the obstacle insufficiently curved, although convex. The agent clearly cannot navigate it any more.

6.3 Inapplicability to Fully Non-convex Worlds

Insufficiently curved spaces are those which contain at least one obstacle boundary point where the obstacle boundary is not sufficiently curved in any tangential direction $\hat{t}_i$ at that point. The applicability of NFs is depicted in Fig. 6.6.

It is worth emphasizing that insufficiently curved spaces can be both convex and nonconvex. It has been shown that convex worlds which are almost insufficiently curved can be navigated with a KRNF. Partially nonconvex which are almost insufficiently curved (sufficient curvature means at least one direction for which the half-curvature sphere is included in the obstacle, hence this direction is also convex, so any almost insufficiently curved obstacle is also partially convex and cannot be totally nonconvex) have been shown to be navigable as well. But nonconvex are in general not navigable. The reason is to be shown in what follows.

In the following it is shown that fully insufficiently curved spaces do not accept the usual proof. In fact the contrary can be proved for high enough k and certain q_d. Therefore the usual proof is invalid in this case.

This does not formally prove the inexistence of a KRNF in such spaces. But it is inspired by the intuitive reason for which KRNFs work as k increases, and on which the proof is constructed. It is known from experience that for low values of k a KRNF usually does not exist. One would have to show that the upper bound on k for which the KRNF posseses local minima other than q_d is < 2. Then in no case would a KRNF exist in insufficiently curved spaces.

The author's expectation is that such a proof is impossible. The reason is that specially designed insufficiently curved space may be navigable for low k values using a KRNF. But these would be just counterexamples, whereas the general case has been shown to not to be a NF for high k.

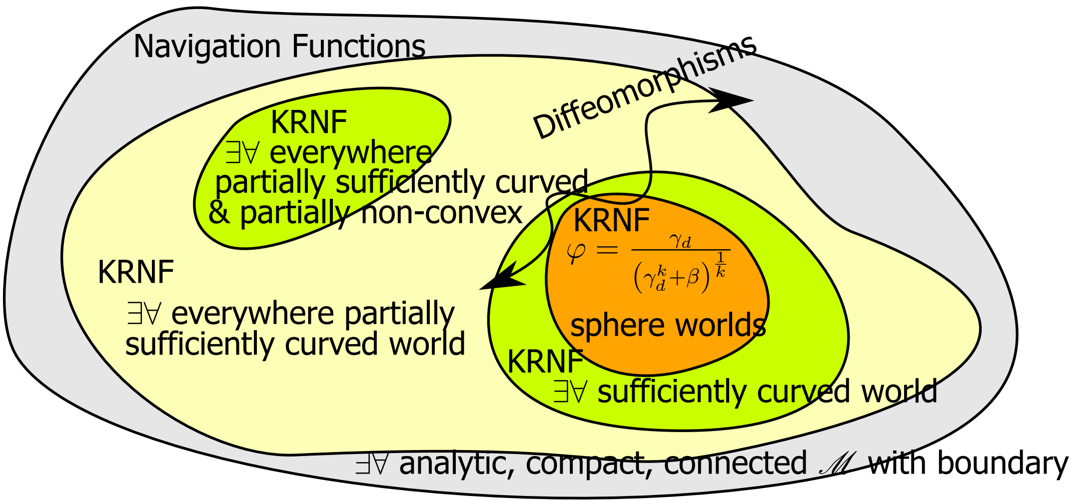


Figure 6.6: Navigation function applicability to different worlds. Note that the $\forall$ everywhere partially sufficiently curved (EPSC) world refers to at most a single convex but not sufficiently curved principal curvature. Nonetheless, extensions of the proof included here towards higher order degeneracy are expected. Moreover, note that in 3-dimensional C-Spaces, every EPSC world can only have one such principal curvature.

There is not much more to be searched in this direction, since it has been shown why further efforts will not be able to proceed as usual, combined with the intuitive impossibility evident.

It is therefore concluded that analytic NFs exist in any manifold with boundary (as proved by KR), but the specific form proposed by KR and herein referred to as KRNF are applicable and tunable to be NFs only in everywhere partially sufficiently curved worlds.

Chapter 7

Application to Superquadric Worlds

7.1 Introduction

Toroidal navigation functions allow us to treat configuration space topologies of any genus. This is due to the fact that obstacles of nonzero genus in such configuration spaces can be diffeomorphically mapped to m-fold tori of the desired genus. Any obstacles of genus 0 can be mapped to spheres, as in the classic NF formulation.

There are various ways in which toroidal obstacles may arise in the model space of a robot. An obvious one is existence of obstacles of genus $g > 0$ in the task space. Their C-space images then may not be of genus 0. If this is the case, the diffeomorphic images in model space of C-obstacles cannot be spheres of genus 0.

A simple example is a point robot in a 3-dimensional task space populated by disjoint 2-tori obstacles. In this case, the C-space is the same as the task space and the obstacles are the same, i.e. 2-tori.

Another case is the possibility of simultaneous collision of a non-point robot with multiple obstacles disjoint in task space. Then their C-space images will be connected. Such connections can lead to multiply connected obstacles, hence genus $g > 0$. Similarly to the previous case, these C-obstacles cannot be diffeomorphically mapped to spheres.

An example is a spherical robot in a 3-dimensional world with disjoint spheres. Suppose that the centers of some of the spheres are located on a circle. It can be the case that the spheres are disjoint in task space, but their C-space images, i.e. their Minkowski sums with the spherical robot, be non-disjoint in C-space. Nevertheless, this can happen so that a genus 1 C-obstacle results.

A further example is the existence of revolute or rotational degrees of freedom in the system. These can produce C-obstacle images of higher genus than the associated task space obstacle. A simple example is an asymmetric oriented holonomic robot amidst spheres on a 2-dimensional Euclidean world. Suppose that the world is such, that simultaneous collision with multiple obstacles is not possible. Still, due to the rotational degree of freedom, the C-obstacles are 2-tori. Each one of them corresponds to a sphere in task space.

In general, if the configuration space is embeddable in a Euclidean space E^n of the same dimension n, then application of KRNFs is possible, although genus 0 obstacles in task space will give rise to higher genus obstacles in configuration space, Fig. 7.1. This increase in genus is caused by the topology of the revolute degrees of freedom.

Nonetheless, note that rotational or revolute degrees of freedom should not necessarily cause such topology changes. A simple solution mentioned in [66] is to parameterize such

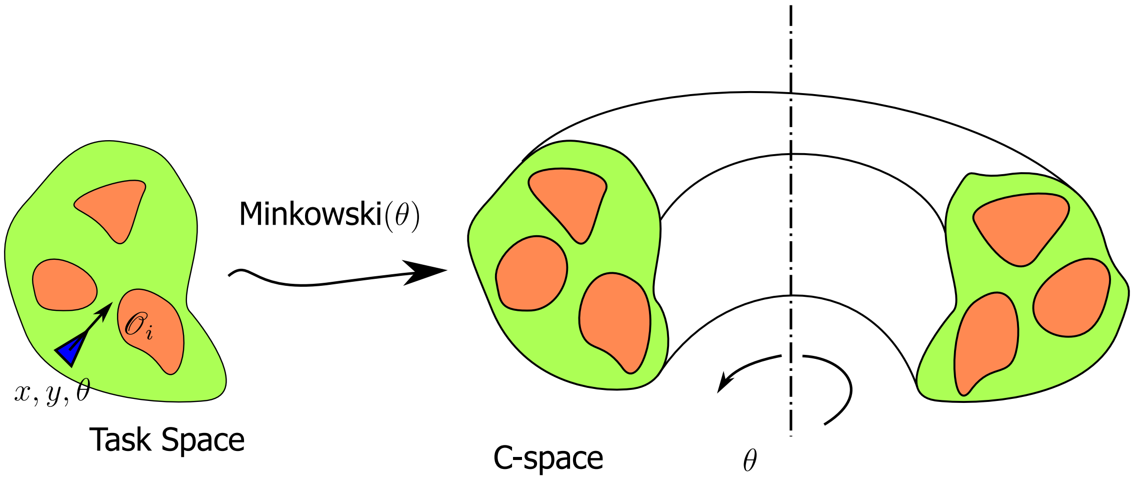


Figure 7.1: Holonomic asymmetric robot in planar world.

degrees of freedom by unbounded real coordinates and use a parameterization periodic in 2π. Even if this is applied, still the previously described cases can still lead to higher genus C-obstacles and associated model space obstacles.

7.2 Tori

7.2.1 Implicit Obstacle Function

A 2-dimensional torus $\Pi^2 = S^1 \times S^1$ centered at the origin with axis z as its rotational axis of symmetry can be defined by the zero level set $\beta^{-1}(0) \triangleq \{q | \beta_i(q) = 0\}$ of the function

$$\beta_i(q) = \left(R - \sqrt{x^2 + y^2}\right)^2 + z^2 - r^2 \tag{7.1}$$

where $q \in \mathbb{R}^3$, $R \in (0, +\infty)$ is the major radius and $r \in (0, R)$ its minor radius[1]. An obstacle $\mathscr{O}_i$ having the 2-torus as its boundary can be represented as

$$\begin{aligned} \partial\mathscr{O}_i &\triangleq \left\{q \in E^3 : \beta_i(q) = 0\right\} \\ \mathscr{O}_i &\triangleq \left\{q \in E^3 : \beta_i(q) < 0\right\} \end{aligned} \tag{7.2}$$

To compensate for differences from the global reference frame, a translation of the origin to the center of the torus, followed by a rotation of its axis suffice. This follows from symmetry considerations.

Let q_i denote the torus center. Firstly, the origin is translated $q' = q - q_i$. Then the rotation is applied to identify the torus axis of symmetry with the z axis. Let $n_i \in \mathbb{R}^3, \|n_i\| = 1$ be the unit vector in the torus axis of symmetry direction, with respect to the global reference frame. Let $n_z \in \mathbb{R}^3, \|n_z\| = 1$ be the unit vector in the z axis direction. If $n_i \neq n_z$ then set $k = n_i \times n_z$ and using the angle of rotation $\theta = \arccos(n_i \cdot n_z)$ around axis k, the rotation matrix to be used with respect to the translated frame of reference is

$${}^0R_1 = R_k(\theta) = I_3 \cos(\theta) + kk^{\mathrm{T}}(1 - \cos\theta) + \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \tag{7.3}$$

[1] The torus aspect ratio $\frac{R}{r} > 1$ here to avoid degeneration a horn or spindle torus, because none of these satisfies the conditions of partially sufficient curvature at every boundary point.

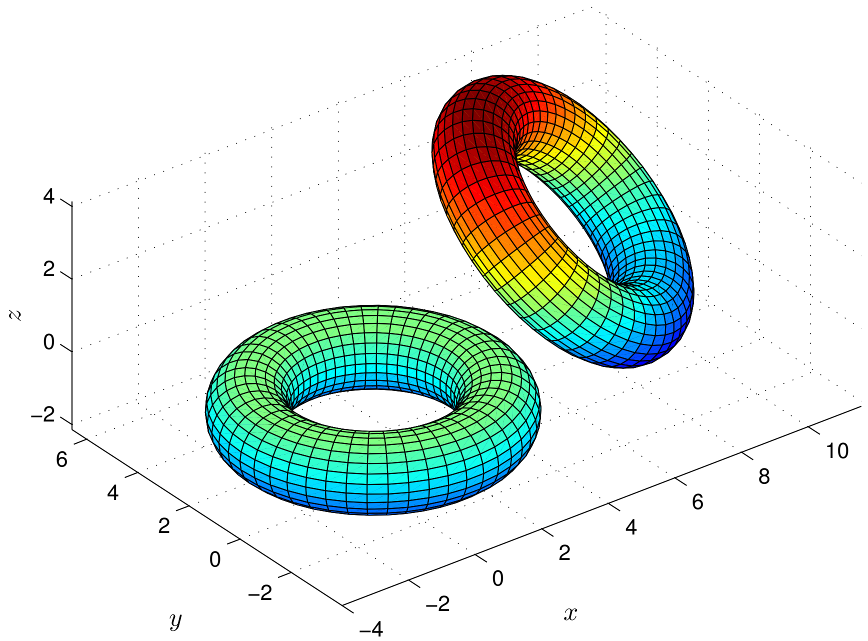


Figure 7.2: Homogenous transform of torus aligned reference frame with respect to global one.

and the transformation is illustrated in Fig. 7.2.

After the appropriate transformation has been applied, the obstacle function partial derivatives in the new (aligned with the torus) coordinate system is

$$
\begin{aligned}
\frac{\partial}{\partial x}\left\{\left(R-\sqrt{x^2+y^2}\right)^2+z^2-r^2\right\} &= \frac{\partial}{\partial x}\left\{\left(R-\sqrt{x^2+y^2}\right)^2\right\}+\frac{\partial}{\partial x}\left\{z^2-r^2\right\} \\
&= 2\left(R-\sqrt{x^2+y^2}\right)\frac{\partial}{\partial x}\left\{R-\sqrt{x^2+y^2}\right\} \\
&= 2\left(R-\sqrt{x^2+y^2}\right)\left(\frac{\partial}{\partial x}\left\{R\right\}-\frac{\partial}{\partial x}\left\{\sqrt{x^2+y^2}\right\}\right) \\
&= -2\left(R-\sqrt{x^2+y^2}\right)\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x}\left\{x^2+y^2\right\} \\
&= -\frac{2x}{\sqrt{x^2+y^2}}\left(R-\sqrt{x^2+y^2}\right) \\
\frac{\partial}{\partial y}\left\{\beta_i(q)\right\} &= -\frac{2y}{\sqrt{x^2+y^2}}\left(R-\sqrt{x^2+y^2}\right) \\
\frac{\partial}{\partial z}\left\{\beta_i(q)\right\} &= \frac{\partial}{\partial z}\left\{\left(R-\sqrt{x^2+y^2}\right)^2+z^2-r^2\right\} = 2z
\end{aligned}
\tag{7.4}
$$

hence obstacle function gradient in the new (aligned with the torus) coordinate system is

$$
\nabla_q \beta_i(q) = \begin{bmatrix} -\frac{2x}{\sqrt{x^2+y^2}}\left(R-\sqrt{x^2+y^2}\right) \\ -\frac{2y}{\sqrt{x^2+y^2}}\left(R-\sqrt{x^2+y^2}\right) \\ 2z \end{bmatrix} \tag{7.5}
$$

We can observe that the gradient $\nabla_q \beta_i(q)$ is not defined at the free space interior point $q = [0,0,0]^{\mathrm{T}}$, the origin, which is the torus center. As a result, this choice of $\beta_i(q)$ to represent a torus is not suitable for building a NF. It is not C^1 at the origin, hence neither C^2 there[2].

Nonetheless, it is interesting to note that

$$\lim_{\|q\|\to 0} \left(\nabla_q \beta_i\right)(q) = \lim_{\|q\|\to 0} \begin{bmatrix} -\frac{2x}{\sqrt{x^2+y^2}}\left(R - \sqrt{x^2+y^2}\right) \\ -\frac{2y}{\sqrt{x^2+y^2}}\left(R - \sqrt{x^2+y^2}\right) \\ 2z \end{bmatrix} = \begin{bmatrix} -2R \\ -2R \\ 0 \end{bmatrix} \tag{7.6}$$

so it is not ill-conditioned in the origin's neighborhood.

Even if we thought about directly defining a NF with such a β_i incorporated in its formula, blowing up of β_i at the torus origin would cause a second global minimum at the torus center q_i (blowing up of φ denominator leads to 0 value, the minimum of its codomain), different than the destination q_d.

For the above reason we are going to use another implicit function, namely the same as above after algebraic elimination of the square root. Let the 2-torus be described by the quartic function

$$\begin{aligned}
\beta_i(q) &= \left(x^2+y^2+z^2+R^2-r^2\right)^2 - 4R^2\left(x^2+y^2\right) \\
&= x^4 + 2x^2\left(y^2+z^2+R^2-r^2\right) + \left(y^2+z^2+R^2-r^2\right) - 4R^2\left(x^2+y^2\right) \\
&= x^4 + 2x^2y^2 + 2x^2z^2 + 2x^2R^2 - 2x^2r^2 + y^4 \\
&\quad + 2y^2\left(z^2+R^2-r^2\right) + \left(z^2+R^2-r^2\right)^2 \\
&\quad - 4R^2x^2 - 4R^2y^2 \\
&= x^4 + 2x^2y^2 + 2x^2z^2 + 2x^2R^2 - 2x^2r^2 + y^4 + 2y^2z^2 \\
&\quad + 2y^2\left(R^2-r^2\right) + z^4 + 2z^2\left(R^2-r^2\right) + \left(R^2-r^2\right)^2 \\
&\quad - 4R^2x^2 - 4R^2y^2
\end{aligned} \tag{7.7}$$

$$\begin{aligned}
&= \underbrace{x^4+y^4+z^4}_{4^{th}\text{ order terms}} \\
&+ \underbrace{-2\left(R^2+r^2\right)x^2 - 2\left(R^2+r^2\right)y^2 + 2\left(R^2-r^2\right)z^2 + 2\left(x^2y^2+y^2z^2+z^2x^2\right)}_{2^{nd}\text{ order terms}} \\
&+ \underbrace{\left(R^2-r^2\right)^2}_{\text{constant term}}
\end{aligned}$$

This is a polynomial in multiple variables, hence a smooth function everywhere $\beta_i \in C^\infty\left([\mathbb{R}^n, \mathbb{R}]\right) \subset C^2\left([\mathbb{R}^n, \mathbb{R}]\right)$ and positive in the free space interior

$$\beta_i \in C^\infty\left([\mathscr{F} \setminus \partial\mathscr{F}, (0,+\infty)]\right) \subset C^2\left([\mathscr{F}, (0,+\infty)]\right)$$

and zero on its boundary on the obstacle $\partial\mathscr{O}_i = \partial\mathscr{F} \cap \mathscr{O}_i$. Hence, it can be used as an obstacle function in a KRNF.

[2]Note that the proof about local minima requires curvature properties in an obstacle's neighborhood. But C^2 properties are required also away from that single obstacle, because such points may belong to the neighborhood used in the proof for local minima near another obstacle.

The partial derivatives of β_i are

$$\begin{aligned}
\frac{\partial}{\partial x}\left\{\beta_i(q)\right\} &= \frac{\partial}{\partial x}\left\{\left(x^2+y^2+z^2+R^2-r^2\right)^2-4R^2\left(x^2+y^2\right)\right\} \\
&= \frac{\partial}{\partial x}\left\{\left(x^2+y^2+z^2+R^2-r^2\right)^2\right\}-4R^2\frac{\partial}{\partial x}\left\{x^2+y^2\right\} \\
&= 2\left(x^2+y^2+z^2+R^2-r^2\right)\frac{\partial}{\partial x}\left\{x^2+y^2+z^2+R^2-r^2\right\}-4R^2 2x \\
&= 4x\left(x^2+y^2+z^2+R^2-r^2\right)-8xR^2 \\
&= 4x\left(x^2+y^2+z^2+R^2-r^2-2R^2\right) \\
&= 4x\left(x^2+y^2+z^2-R^2-r^2\right)
\end{aligned} \tag{7.8}$$

Similarly

$$\frac{\partial}{\partial y}\left\{\beta_i(q)\right\} = 4y\left(x^2+y^2+z^2-R^2-r^2\right) \tag{7.9}$$

and also

$$\begin{aligned}
\frac{\partial}{\partial z}\left\{\beta_i(q)\right\} &= \frac{\partial}{\partial z}\left\{\left(x^2+y^2+z^2+R^2-r^2\right)^2-4R^2\left(x^2+y^2\right)\right\} \\
&= \frac{\partial}{\partial z}\left\{\left(x^2+y^2+z^2+R^2-r^2\right)^2\right\}-\frac{\partial}{\partial z}\left\{4R^2\left(x^2+y^2\right)\right\} \\
&= 2\left(x^2+y^2+z^2+R^2-r^2\right)\frac{\partial}{\partial z}\left\{x^2+y^2+z^2+R^2-r^2\right\} \\
&= 2(2z)\left(x^2+y^2+z^2+R^2-r^2\right) \\
&= 4z\left(x^2+y^2+z^2+R^2-r^2\right)
\end{aligned} \tag{7.10}$$

Therefore the obstacle function gradient with respect to the aligned reference frame is

$$\nabla_q\beta_i(q) = \begin{bmatrix} 4x\left(x^2+y^2+z^2-R^2-r^2\right) \\ 4y\left(x^2+y^2+z^2-R^2-r^2\right) \\ 4z\left(x^2+y^2+z^2+R^2-r^2\right) \end{bmatrix} \tag{7.11}$$

Let us now find the Hessian matrix as well. The second partial derivatives are

$$
\begin{aligned}
\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\left\{\beta_i(q)\right\}\right\} &= \frac{\partial}{\partial x}\left\{4x\left(x^2+y^2+z^2-R^2-r^2\right)\right\} \\
&= \frac{\partial}{\partial x}\left\{4x\right\}\left(x^2+y^2+z^2-R^2-r^2\right)+4x\frac{\partial}{\partial x}\left\{x^2+y^2+z^2-R^2-r^2\right\} \\
&= 4\left(x^2+y^2+z^2-R^2-r^2\right)+4x2x \\
&= 4\left(x^2+y^2+z^2-R^2-r^2\right)+8x^2 \\
&= 4\left(3x^2+y^2+z^2-R^2-r^2\right) \\
\frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}\left\{\beta_i(q)\right\}\right\} &= \frac{\partial}{\partial y}\left\{4x\left(x^2+y^2+z^2-R^2-r^2\right)\right\} \\
&= 4x\frac{\partial}{\partial y}\left\{x^2+y^2+z^2-R^2-r^2\right\} \\
&= 4x2y = 8xy \\
\frac{\partial}{\partial z}\left\{\frac{\partial}{\partial x}\left\{\beta_i(q)\right\}\right\} &= \frac{\partial}{\partial z}\left\{4x\left(x^2+y^2+z^2-R^2-r^2\right)\right\} \\
&= 4x\frac{\partial}{\partial z}\left\{x^2+y^2+z^2-R^2-r^2\right\} \\
&= 4x2z = 8xz \\
\frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y}\left\{\beta_i(q)\right\}\right\} &= 4\left(x^2+3y^2+z^2-R^2-r^2\right) \\
\frac{\partial}{\partial z}\left\{\frac{\partial}{\partial y}\left\{\beta_i(q)\right\}\right\} &= \frac{\partial}{\partial z}\left\{4y\left(x^2+y^2+z^2-R^2-r^2\right)\right\} \\
&= 4y\frac{\partial}{\partial z}\left\{x^2+y^2+z^2-R^2-r^2\right\} \\
&= 4y(2z) = 8yz \\
\frac{\partial}{\partial z}\left\{\frac{\partial}{\partial z}\left\{\beta_i(q)\right\}\right\} &= \frac{\partial}{\partial z}\left\{4z\left(x^2+y^2+z^2+R^2-r^2\right)\right\} \\
&= \frac{\partial}{\partial z}\left\{4z\right\}\left(x^2+y^2+z^2+R^2-r^2\right)+4z\frac{\partial}{\partial z}\left\{x^2+y^2+z^2+R^2-r^2\right\} \\
&= 4\left(x^2+y^2+z^2+R^2-r^2\right)+4z(2z) \\
&= 4\left(x^2+y^2+z^2+R^2-r^2\right)+4(2z^2) \\
&= 4\left(x^2+y^2+3z^2+R^2-r^2\right)
\end{aligned}
\tag{7.12}
$$

Therefore, the Hessiam matrix of this obstacle function is

$$
D^2\beta_i(q) = \begin{bmatrix} 4\left(3x^2+y^2+z^2-R^2-r^2\right) & 8xy & 8xz \\ 8xy & 4\left(x^2+3y^2+z^2-R^2-r^2\right) & 8yz \\ 8xz & 8yz & 4\left(x^2+y^2+3z^2+R^2-r^2\right) \end{bmatrix}
\tag{7.13}
$$

There are essentially two distinct positions for the destination q_d, yielding qualitatively different gradient fields. The first one is when q_d belongs to the $z = 0$ plane. The resulting field is visualized with several trajectories starting from different initial conditions in Fig. 7.3. Codimension-2 saddle critical manifolds form around it and an isolated saddle inside it. These critical manifolds can be ensured to be nondegenerate by a suitably high value of k, as proved in the previous chapters.

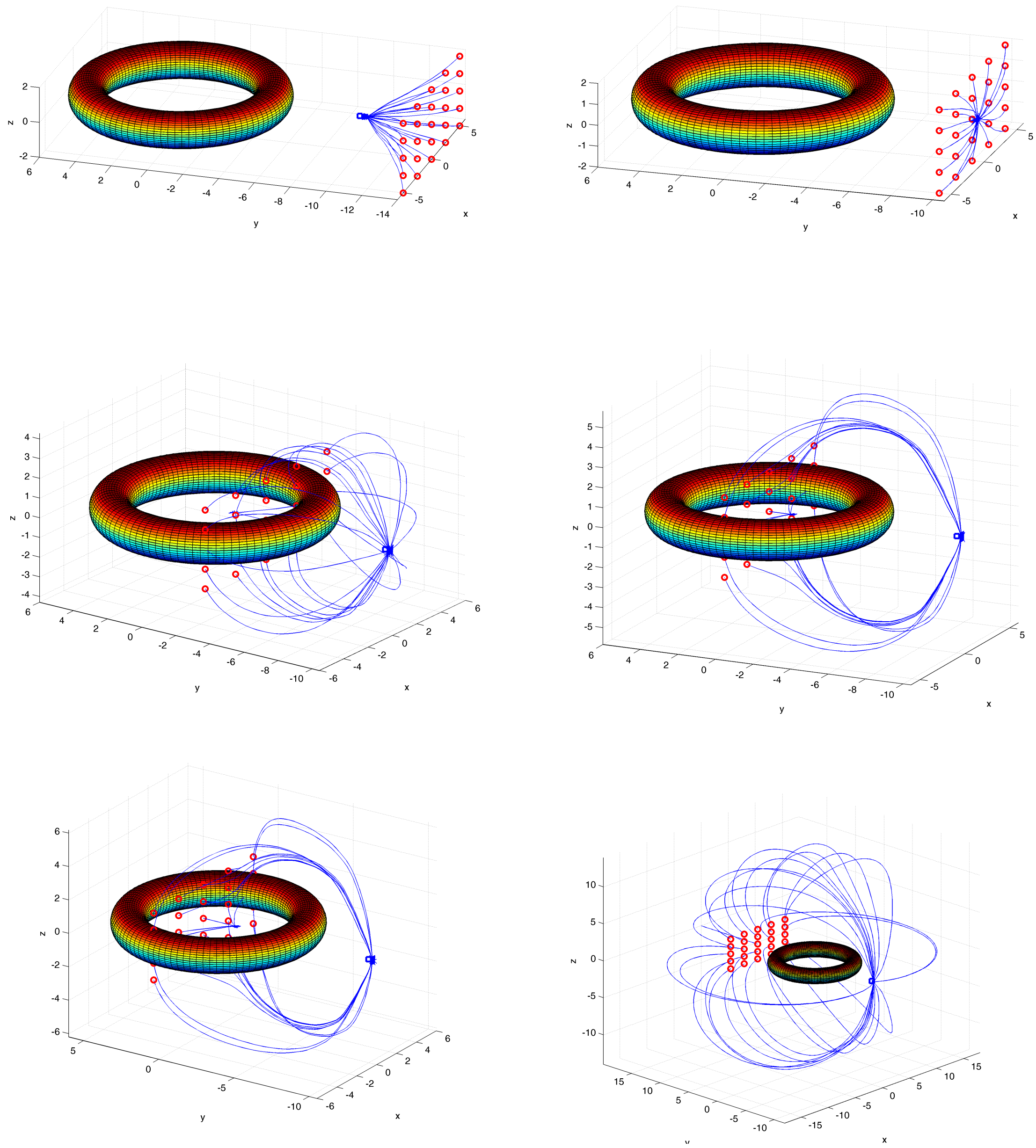


Figure 7.3: Trajectories from different initial conditions, on planes $y = -14, -10, -3, 0, 1, 10$, respectively. In cases $y = -3, 0, 1$, a saddle is visible on the $z = 0$ plane, inside the torus ring, near its center. The destination q_d is on the right.

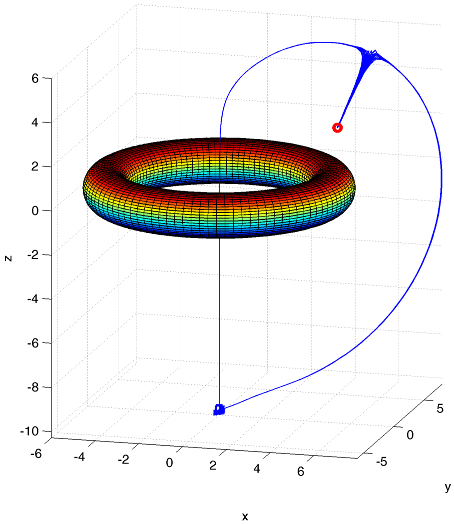


Figure 7.4: Selecting a destination q_d on the torus axis of symmetry leads to a circular nondegenerate critical manifold of saddles (for a single torus on its own of course, when other obstacles are present the symmetry may break, nonetheless for high k close to the torus the situation tends to that when it is on its own, this is how the proof works and hence this is why we consider it alone here).

The other case is when the destination is on the torus axis of symmetry. In this case a circular nondegenerate critical manifold forms, as illustrated in . In the previous chapters it has been proved that such a nondegenerate saddle critical manifold of codimension-1 has a measure zero stable set. In order to further illustrate how symmetry is responsible for this degeneracy, in Fig. 7.4 symmetry breaking is introduced and the corresponding trajectories shown.

7.2.2 Symmetry Breaking

Degeneracies can arise for a torus when q_d is *on* a maximal sphere of curvature (at a maximal point on the torus) not on the symmetry axis, or on the symmetry axis.

The case of degeneracy with q_d not on the symmetry axis leads to isolated degenerate points. It is similar to the case of isolated critical points when q_d is on the symmetry axis of an almost insufficiently curved ellipsoid. Both cases can be analyzed with tools from Catastrophe theory.

Here we are going to work on the degeneracy arising due to symmetry, when the destination q_d belongs to the torus' axis of symmetry (here the z axis of the aligned coordinate system).

It is expected that, although there exists a continuum of critical points forming a critical manifold, rendering such a KRNF degenerate, hence not a NF according to the classical Koditschek-Rimon definition of a general NF, using the Morse-Bott Lemma and the quadratic form expression developed, we can show that the critical set comprises of the union of disjoint smooth connected critical submanifolds and isolated points, hence there can be no open stable manifold.

The stable manifold in this case is of zero measure, which is the reason for requiring non-degeneracy in the first place [21].

A viable alternative to avoid a critical manifold is to break the rotational symmetry of

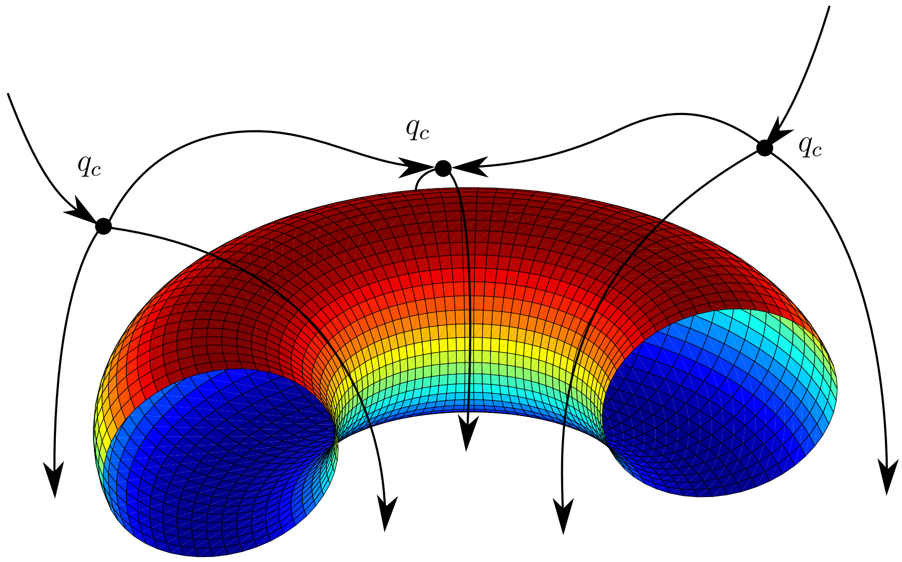


Figure 7.5: Torus symmetry breaking results into four isolated critical points, instead of a critical manifold. Even if the isolated critical points are degenerate, it can be proved that their stable set is of Lebesgue measure zero, suing Thim's Splitting Lemma and the fact that radially the NF is positive definite and in one principal tangent direction the NF is negative definite due to sufficient curvature (for suitably high k).

the positive level sets $\beta_i(q) = C > 0$, but not of the 0 level set, to avoid affecting the obstacle's shape.

This symmetry breaking can be achieved by introducing a rotationally asymmetric term in the implicit function. For level set $\beta_i^{-1}(0)$ to remain unaffected, a multiplicative term is selected, to obtain

$$\beta_i(q) = \underbrace{\cos^2(\theta)}_{\text{symmetry-breaking term}} \left(\left(x^2+y^2+z^2+R^2-r^2\right)^2 - 4R^2\left(x^2+y^2\right)\right) \tag{7.14}$$

where $\theta = \arctan 2(y, x)$. This breaks the degenerate 1-dimensional critical submanifold into four isolated critical points (possibly degenerate, but this again does not matter according to the splitting Lemma applied to the 2-torus), as shown in Fig. 7.5 and the corresponding trajectories in Fig. 7.6.

7.3 Supertoroids

Supertoroids [42] are defined by the implicit function

$$\beta_i(x,y,z) = \left(\left(\left(\frac{x}{a_1}\right)^{\frac{2}{\varepsilon_2}} + \left(\frac{y}{a_2}\right)^{\frac{2}{\varepsilon_2}}\right)^{\frac{\varepsilon_2}{2}} - a_4\right)^{\frac{2}{\varepsilon_1}} + \left(\frac{z}{a_3}\right)^{\frac{2}{\varepsilon_1}} - 1 \tag{7.15}$$

where $\varepsilon_1, \varepsilon_2 \in (0, +\infty)$ are exponent parameters, $a_1, a_2, a_3 > 0$ are the three radii of the supertoroid and $a_4 = \frac{r}{\sqrt{a_1^2+a_2^2}}$, where $r > 0$ is the torus radius. Differentiation yields the

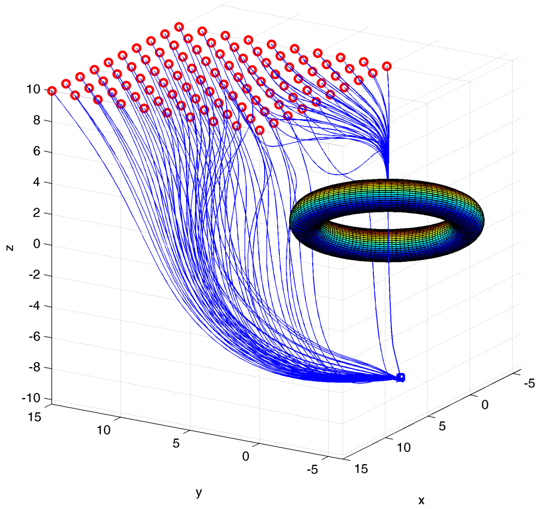


Figure 7.6: Symmetry breaking removes the critical manifold and leads to four isolated critical points.

following gradient

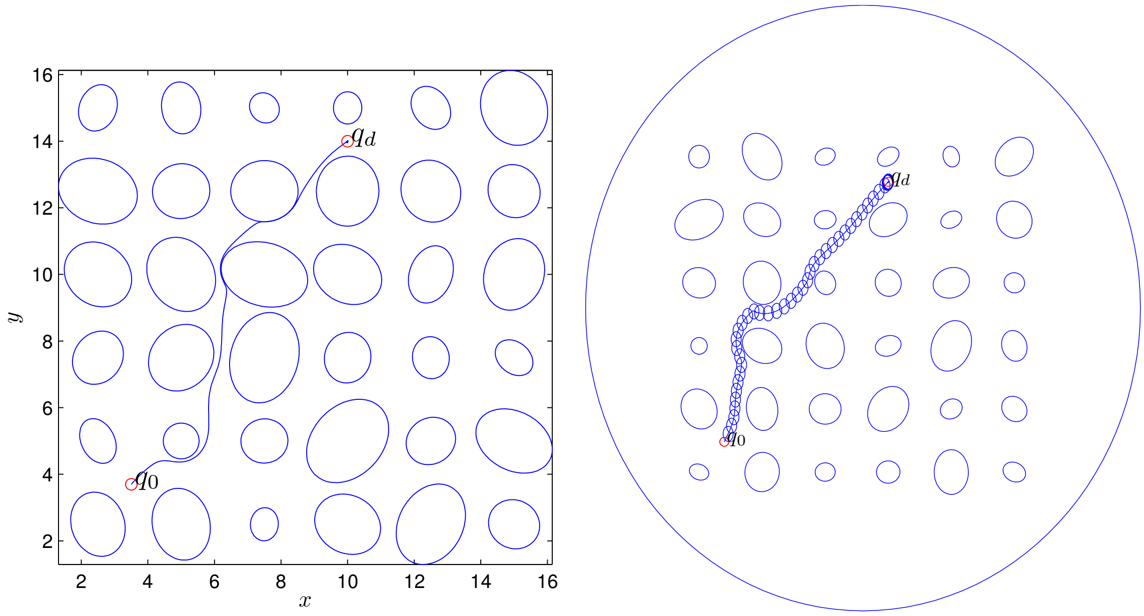
$$(\nabla_q \beta_i)(q) = \begin{bmatrix} \frac{\varepsilon_2}{\varepsilon_1}\left(\left(\left(\frac{x}{a_1}\right)^{\frac{2}{\varepsilon_2}} + \left(\frac{y}{a_2}\right)^{\frac{2}{\varepsilon_2}}\right)^{\frac{\varepsilon_2}{2}} - a_4\right)^{\frac{2}{\varepsilon_1}-1} \left(\left(\frac{x}{a_1}\right)^{\frac{2}{\varepsilon_2}} + \left(\frac{y}{a_2}\right)^{\frac{2}{\varepsilon_2}}\right)^{\frac{\varepsilon_2}{2}-1} \left(\frac{1}{a_1}\right)^{\frac{2}{\varepsilon_2}} x^{\frac{2}{\varepsilon_2}-1} \\ \frac{\varepsilon_2}{\varepsilon_1}\left(\left(\left(\frac{x}{a_1}\right)^{\frac{2}{\varepsilon_2}} + \left(\frac{y}{a_2}\right)^{\frac{2}{\varepsilon_2}}\right)^{\frac{\varepsilon_2}{2}} - a_4\right)^{\frac{2}{\varepsilon_1}-1} \left(\left(\frac{x}{a_1}\right)^{\frac{2}{\varepsilon_2}} + \left(\frac{y}{a_2}\right)^{\frac{2}{\varepsilon_2}}\right)^{\frac{\varepsilon_2}{2}-1} \left(\frac{1}{a_2}\right)^{\frac{2}{\varepsilon_2}} y^{\frac{2}{\varepsilon_2}-1} \\ \left(\frac{1}{a_3}\right)^{\frac{2}{\varepsilon_1}} z^{\frac{2}{\varepsilon_1}-1} \end{bmatrix} \tag{7.16}$$

Note that the order of differentiability (C^2 etc) depends on the values of $\varepsilon_1, \varepsilon_2$. For $\varepsilon_1 \leq 2$ the supertoroid is twice continuously differentiable away from $(x, y) = (0, 0)$, similarly to the torus. Loss of differentiability at the origin can be avoided by changing its definition away from the obstacle (since $(x, y) = (0, 0)$ never belongs to the supertoroid), which is always possible, provided the NF is tuned with a sufficiently high k.

Any torus is everywhere partially sufficiently curved. On the contrary, to obtain everywhere partially sufficiently curved supertoroids, the parameters $\varepsilon_1, \varepsilon_2, a_1, a_2, a_3$ should be appropriately selected. For example, the supertoroid defined by $\varepsilon_1 = 1, \varepsilon_2 = 0.25, a_1 = 0.5, a_2 = 0.5, a_3 = 0.75$ is everywhere partially sufficiently curved. It is illustrated in the example of Fig. 7.9. For these parameter values β_i is not differentiable at the axis $(x, y) = (0, 0)$, but this can be remedied as already commented, because it is away from the obstacle.

7.4 Complicated worlds

In 2d ellipses of limited eccentricity are examples of sufficiently curved obstacles Fig. 5.1. A point agent navigating such an everywhere sufficiently curved world is shown in Fig. 7.7a. No diffeomorphisms are needed, the Koditschek-Rimon Navigation Function is directly defined on the world. Nevertheless, they are still applicable to treat full



(a) Point agent. (b) Elliptic agent.

Figure 7.7: Sufficiently curved worlds. In more detail, a point agent in a sufficiently curved elliptic world and a sufficiently curved elliptical agent in a sufficiently curved elliptic world.

non-convexities. An elliptic agent in the elliptic 2d world of Fig. 7.7b requires an implicit Minkowski sum. We use the derivative of Rvachev conjunction [91, 92] on a set of agent boundary points. This provides $\nabla\beta$, the C-space is sufficiently curved, as the Minkowski sum of sufficiently curved obstacles [44]. In Fig. 7.9 a point agent safely converges to q_d in an everywhere partially sufficiently curved world, illustrating how tori enable treatment of multiply connected obstacles, previously not representable by sphere worlds. The vector field driving it has been visualized in Fig. 7.8. A useful note is that as the C-space dimension increases, the NF method has an advantage, because more directions of "escape" become available and full non-convexity more rare.

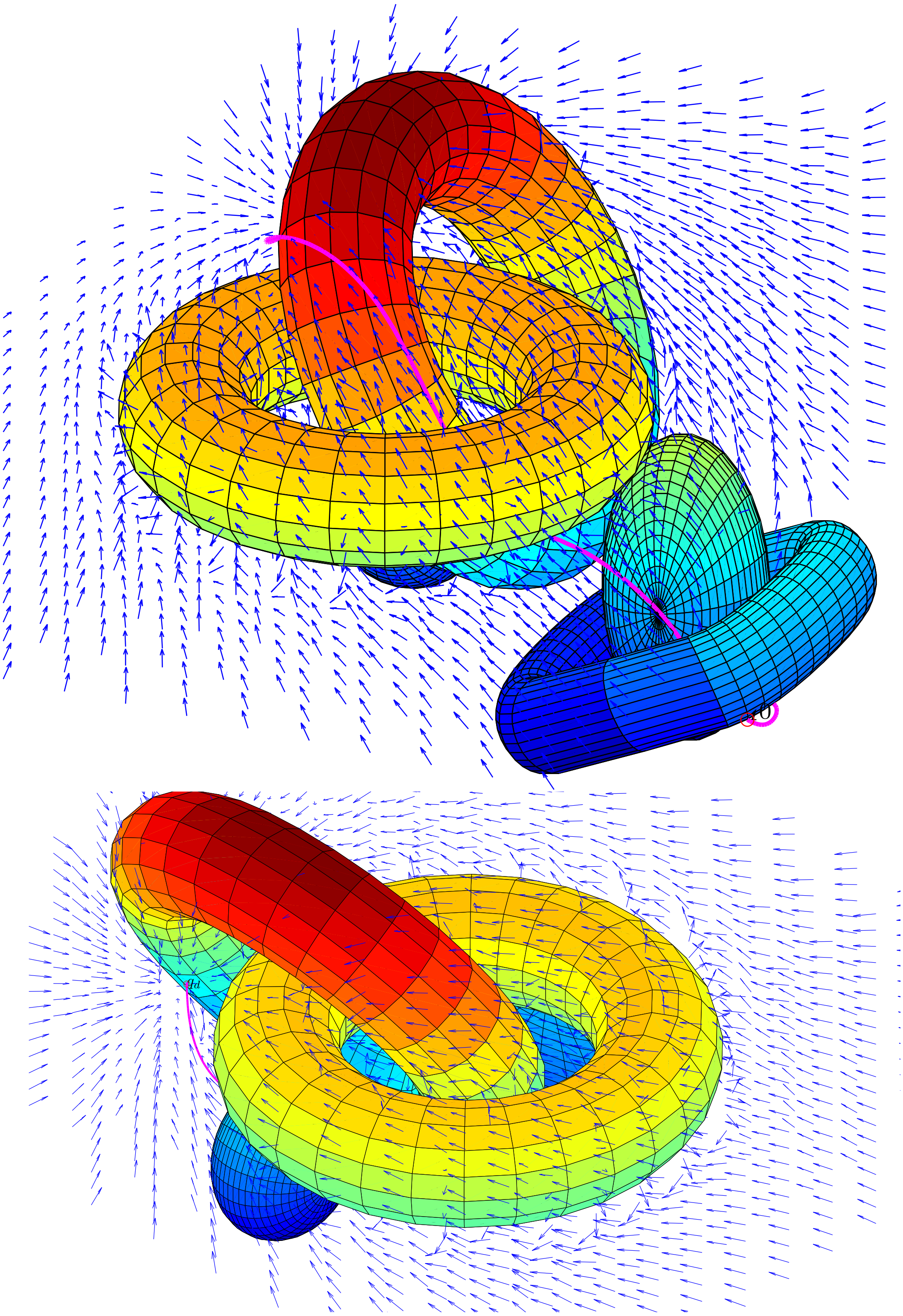


Figure 7.8: Navigation Function gradient field in the world of Fig. 7.9 and the same world without the supertoroid.

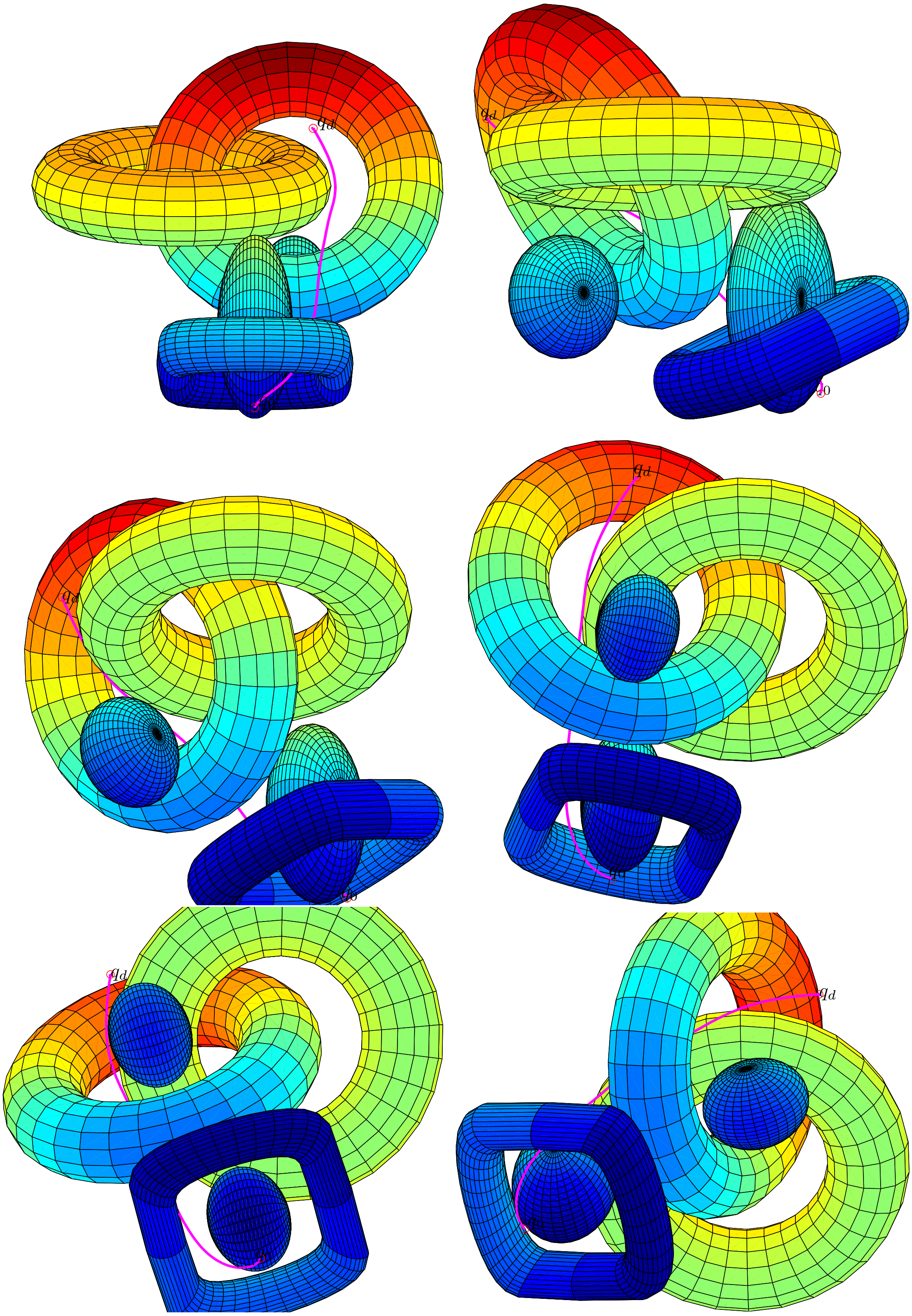


Figure 7.9: Trajectories in complicated everywhere partially sufficiently curved world.

Chapter 8

Navigation Function Simulation Toolbox

The author has developed a Navigation Function Simulation Toolbox for MATLAB. It enables the user to draw circular obstacles, move and resize them, place the agent and its destination as prefered and select a navigation function potential of his choice. Simulation may be run with a user selected parameter k value or with the automatically calculated k, which guarantees obstacle avoidance and convergence to the goal configuration, as analyzed in chapter 2.

A `README` and a `LICENSE` are included in the Toolbox. An `info.xml` and `helptoc.xml`, together with an HTML documentation are provided in the `htmldoc` directory as a reference accessible with the MATLAB Help Browser, or from the MATLAB Start button menu about toolboxes.

To install the toolbox run `installnfsim`, following the instructions contained in the accompanying `README`.

Table 8.1: Developed software metrics[a].

Tool	Files	Code		Comments		Blank		Total
	#	#	%	#	%	#	%	#
nfsim	214	5567	54%	2946	29%	1740	17%	10253
nflearn	41	1478	58%	611	24%	473	18%	2562
ltlmasnf	55	2338	53%	1282	29%	768	18%	4388
Total	310	9383	55%	4839	28%	2981	17%	17203

[a] CLOC has been used to generate these metrics, [119].

8.1 Toolbox structure

8.1.1 Analysis spaces

The usual analysis is performed in C-space. For simple mobile robot systems the task space and C-space are either identical or almost so. In case of spherical agents and obstacles with holonomic constraints, the C-space is just a similar task space where

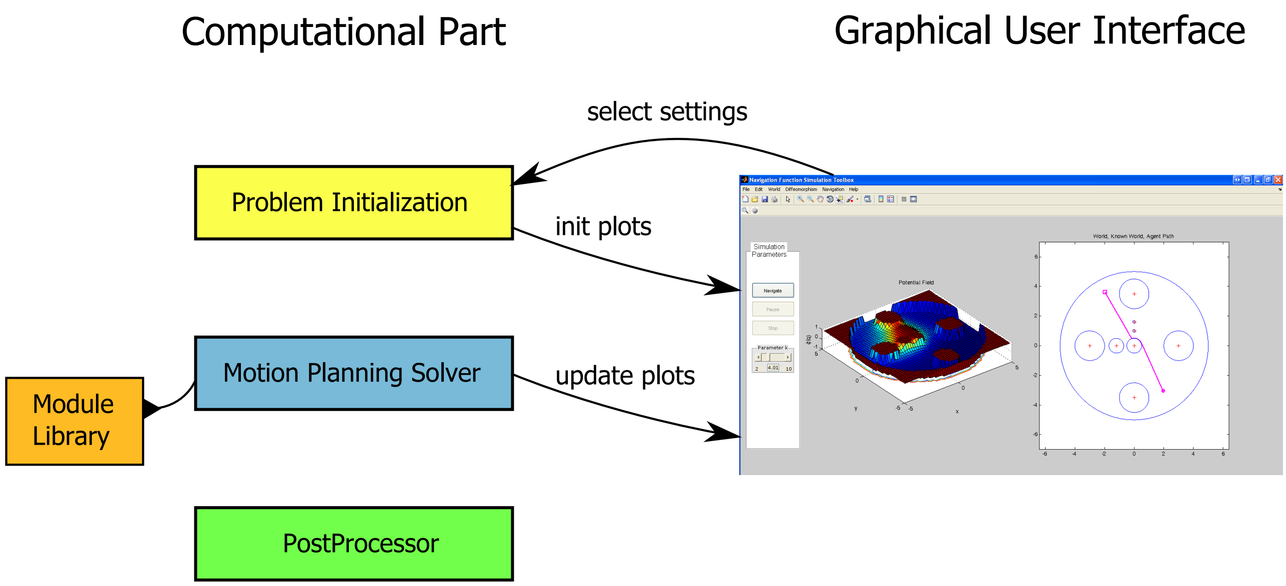


Figure 8.1: Navigation function simulation toolbox architecture.

all bodies other than the agent have originally been spheres, hence their C-images are spheres of increased radii. For KRNF the only constraint is that the augmented spheres do not intersect each other. But for complicated cases, mapping to a Model Space enables use of additional planning methods. Therefore, there are three images of the same problem: Task Space, C-Space, and Model Space, as illustrated in Fig. 8.2.

In case of nonholonomic agents the analysis is performed in C-space and the controller is proved to be a Lyapunov function there, it is not such for purely topological reasons.

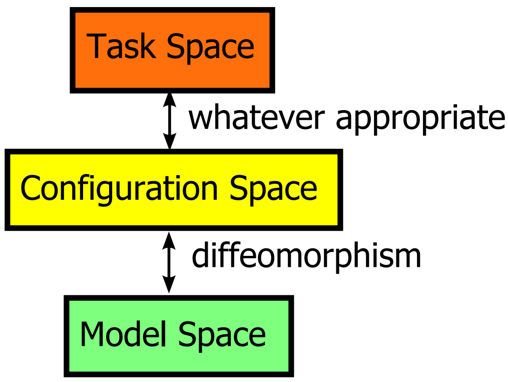


Figure 8.2: Analysis spaces.

In case the Jacobian between C-space (where actuation takes place -?) and Model Space is calculated (either numerically or analytically) then the analytic gradient in Model Space can be used because it will be possible to transform it back to C-space.

The above is also applicable when the C-space is identical with the Model space.

In case the Jacobian is *not* known for the diffeomorphism, then the gradient and Hessian matrix are calculated numerically in the C-space.

There are examples (like dexterous grasping), for which the C-space (i.e., finger configurations) is mapped to Model space (object C-space) through a differentiable mapping whose second derivative requires the Jacobian of kinematics and its derivative and inverse. In such a (rare) case the second derivative of the diffeomorphism is available analytically, e.g. $\ddot{\Theta} = J^{-1}\ddot{x} - J^{-1}\dot{J}\dot{\Theta}$ (Craig p.186, Eq.6.97).

If the obstacle second derivatives are as well twice differentiable, analytically, then the Hessian can be exactly calculated there.

Also, if the obstacles are defined in C-space and the Hessian should be exactly calculated for searching the Model space, then the diffeomorphism's 2nd derivative should be used to calculate the obstacle Hessians in Model space from the obstacle Hessians in C-space.

These can be combined with Hessians of other obstacles which are directly exactly calculated in Model space. Alternatively, they could be calculated in task space and mapped to C-space.

Anyway, the easiest way is to use a numerical Hessian, although not accurate.

Also, the obstacle function β could be numerically differentiated to find $D^2\beta$ and then used in an analytical calculation of the field Hessian.

Note: KRNF require a diffeomorphism to exist because they want to ensure that finite (hence bounded) gradients (=velocities=control inputs) in Model space remain bounded in C-space. If this is independently ensured, one can integrate the trajectory in model space and *then* map the resulting points to C-space (i.e., map the *next* point using the inverse diffeomorphism, hence avoid mapping the gradient using the diffeomorphism's Jacobian (so we do not need to find the Jacobian, nonetheless we need to theoretically prove it is nonsingular everywhere).

After integrating model space the new point can be mapped back with the inverse diffeomorphism (which is certainly easier to calculate than the second derivative of the inverse diffeomorphism).

Then $\frac{new-old}{\Delta t} = u$ is the velocity command in C-space, which due to the theoretical guarantees on the Jacobian is bounded.

The transformation should be invertible in order for the inverse to exist and take us back. This is equivalent to non-singular Jacobian for the forward mapping. Inverse mapping of velocities also requires invertibility of the inverse mapping derivative, which is guaranteed by invertibility of the forward mapping derivative (Jacobian).

Logically enough, the forward 2nd derivative of the mapping should exist for the invariance transform theorem to hold, but no higher derivatives need exist.

8.1.2 Transformations between Analysis Spaces

Depending on the specific problem treated, a custom map is needed to map the task space or configuration space representation to the model world. This step depends on the details of each case. Although it is supported by some functions provided, such as diffeomorphisms, it still relies to a large extent on the problem instant considered.

This is appropriate in order to optimize each implementation appropriately and in fact reduces clutter. The reason is that interfacing to the same model space can vary widely between even similar problems, so no all-over function is provided for this level. By dividing the mapping between C-space and Model space from the planner more transparent data interfaces are achieved.

For data type transformations appropriate converters can be used. Each transformation can act on a world structure and produce another world structure. This can then be parsed to arguments suitable to be provided as input to the chosen planning algorithm. Hence, the need exists for data translator functions.

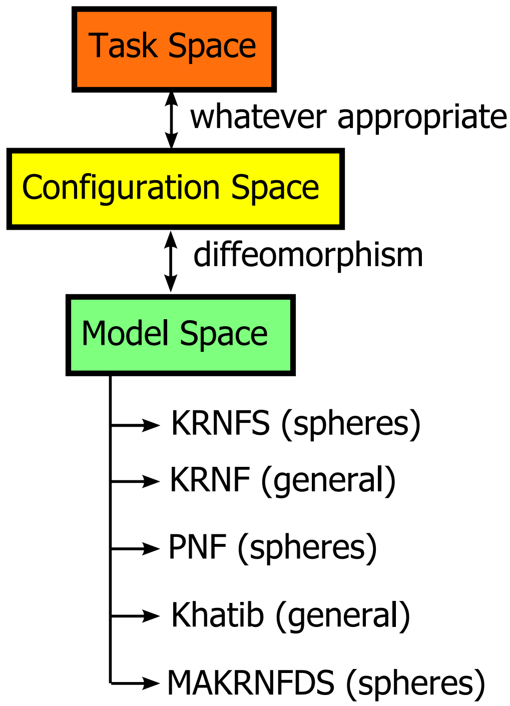


Figure 8.3: Analysis spaces.

After the above procedure, the problem has migrated to the model space. In model space everything is defined using standard primitives, see Fig. 8.3. Therefore, at this level many facilities can be provided. The planning algorithm is called and supplied with geometric arguments in the appropriate format of standardized geometric primitives.

Table 8.2: Function Libraries.

Potential Field Library	Geometry Library
Khatib	Sphere
KRNFS[a]	Ellipse, Ellipsoid
KRNF	One-Sheet Hyperboloid
MAKRNFDS	2-Torus
PNF	Superquadric
	Rvachev operations (CSG)
	General (like splines)

[a] Embedded sphere calculations for efficiency.

8.2 Function Conventions

8.2.1 Potential Field Function

Input $q = [q_1, q_2, \ldots, q_m]$ calculation points.

If the field function only accepts β and does not call any other function to combine β_i of multiple obstacles, then the user should evaluate β from β_i before calling the potential field function. This can be done either with custom code, or by using either `bi2b` or `rvachev`.

Apart from the above exception, the majority of potential field functions require the following standard arguments. In what follows n is the model space dimension, M is the number of all obstacles (including zero$^{\text{th}}$ if such exists) and N is the number of calculation points $q_j \in E^n$ on which the function values are required.

8.2.2 Gradient of Potential Field Function

The user should use the geometry library functions to calculate individual obstacle gradients $\nabla\beta$ and obstacle function values β_i and if `Dbi2Db`, `bi2b` are not incorporated in the field gradient function, the user should also combine them to obtain $\nabla\beta$ in order to pass it as an argument to the selected gradient function.

8.2.3 Hessian matrix of Potential Field Function

The user should independently compute obstacle Hessian $D^2\beta$, gradient $\nabla\beta$ and implicit function β (or for multiple obstacles, if appropriate) and provide them to the Hessian matrix function of the potential field.

Table 8.3: Potential Field Function.

What	Type	Size	Equation
Output	Row array	$1 \times N$	$[\varphi(q_1), \varphi(q_2), \ldots, \varphi(q_N)]$
Calculation points	Matrix	$n \times N$	$[q_1, q_2, \ldots, q_N]$
Destination point	Column vector	$n \times 1$	q_d
Single β at q_j	Row array	$1 \times N$	$[\beta(q_1), \beta(q_2), \ldots, \beta(q_N)]$
Multiple[a] β_i at q_j	Matrix	$M \times N$	$\begin{bmatrix} \beta_1(q_1) & \beta_1(q_2) & \ldots & \beta_1(q_N) \\ \beta_2(q_1) & \beta_2(q_2) & \ldots & \beta_2(q_N) \\ \vdots & \vdots & & \vdots \\ \beta_M(q_1) & \beta_M(q_2) & \ldots & \beta_M(q_N) \end{bmatrix}$

[a] Exceptions are any functions operating in sphere worlds, e.g. `krnfs`, which incorporate sphere obstacle calculations to reduce computational cost. For example, calculation of classic tuning for KRNFS is optimized for spheres.

Table 8.4: Gradient of Potential Field Function.

What	Type	Size	Equation
Output	Matrix	$n \times N$	$[(\nabla\varphi)(q_1), (\nabla\varphi)(q_2), \ldots, (\nabla\varphi)(q_N)]$
Calc. points	Matrix	$n \times N$	$[q_1, q_2, \ldots, q_N]$
Dest. point	Col. vec.	$n \times 1$	q_d
Single β at q_j	Row array	$1 \times N$	$[\beta(q_1), \beta(q_2), \ldots, \beta(q_N)]$
Multiple[a] β_i at q	Matrix	$M \times N$	$\begin{bmatrix} \beta_1(q_1) & \beta_1(q_2) & \ldots & \beta_1(q_N) \\ \beta_2(q_1) & \beta_2(q_2) & \ldots & \beta_2(q_N) \\ \vdots & \vdots & & \vdots \\ \beta_M(q_1) & \beta_M(q_2) & \ldots & \beta_M(q_N) \end{bmatrix}$
Single $\nabla\beta$ at q_j	Matrix	$n \times N$	$[(\nabla\beta_i)(q_1), (\nabla\beta)(q_2), \ldots, (\nabla\beta)(q_N)]$
Multiple $\nabla\beta_i$ at q_j	Cell array	$M \times 1$	$\left\{ \begin{matrix} [(\nabla\beta_1)(q_1), (\nabla\beta_1)(q_2), \ldots, (\nabla\beta_1)(q_N)] \\ [(\nabla\beta_2)(q_1), (\nabla\beta_2)(q_2), \ldots, (\nabla\beta_2)(q_N)] \\ \vdots \\ [(\nabla\beta_M)(q_1), (\nabla\beta_M)(q_2), \ldots, (\nabla\beta_M)(q_N)] \end{matrix} \right\}$

[a] Note that `grad_krnfs, grad_pfs` are exceptions to the above rule, because they are optimized for sphere worlds. The reason is that they are only defined on sphere worlds. In their case, the user provides sphere centers and radii, then the gradient field function takes care of calculating any obstacle data required.

Table 8.5: Hessian matrix of Potential Field Function.

What	Type	Size	Equation
Output	Cell array	$1 \times N$	$\{(D^2\varphi)(q_1), (D^2\varphi)(q_2), \ldots, (D^2\varphi)(q_N)\}$
Calc. points	Matrix	$n \times N$	$[q_1, q_2, \ldots, q_N]$
Dest. point	Column vector	$n \times 1$	q_d
Single β at q_j	Row array	$1 \times N$	$[\beta(q_1), \beta(q_2), \ldots, \beta(q_N)]$
Multiple[a] β_i at q	Matrix	$M \times N$	$\begin{bmatrix} \beta_1(q_1) & \beta_1(q_2) & \ldots & \beta_1(q_N) \\ \beta_2(q_1) & \beta_2(q_2) & \ldots & \beta_2(q_N) \\ \vdots & \vdots & & \vdots \\ \beta_M(q_1) & \beta_M(q_2) & \ldots & \beta_M(q_N) \end{bmatrix}$
Single $\nabla\beta$ at q_j	Matrix	$n \times N$	$[(\nabla\beta_i)(q_1), (\nabla\beta)(q_2), \ldots, (\nabla\beta)(q_N)]$
Multiple $\nabla\beta_i$ at q_j	Cell array	$M \times 1$	$\left\{ \begin{array}{c} [(\nabla\beta_1)(q_1), (\nabla\beta_1)(q_2), \ldots, (\nabla\beta_1)(q_N)] \\ [(\nabla\beta_2)(q_1), (\nabla\beta_2)(q_2), \ldots, (\nabla\beta_2)(q_N)] \\ \vdots \\ [(\nabla\beta_M)(q_1), (\nabla\beta_M)(q_2), \ldots, (\nabla\beta_M)(q_N)] \end{array} \right\}$
Single $D^2\beta$ at $q)j$	Cell array	$1 \times N$	$\{D^2\beta(q_1), D^2\beta(q_2), \ldots, D^2\beta(q_N)\}$
Multiple $D^2\beta_i$ at q_j	Cell Matrix	$M \times N$	$\left\{ \begin{array}{cccc} D^2\beta_1(q_1) & D^2\beta_1(q_2) & \ldots & D^2\beta_1(q_N) \\ D^2\beta_2(q_1) & D^2\beta_2(q_2) & \ldots & D^2\beta_2(q_N) \\ \vdots & & & \\ D^2\beta_M(q_1) & D^2\beta_M(q_2) & \ldots & D^2\beta_M(q_N) \end{array} \right\}$

[a] Note that `hes_krnfs` is an exception as previously described.

Part III

Navigation Functions Learning from Experiments and Application to Anthropomorphic Grasping

Chapter 9

Learning Navigation Functions

9.1 Introduction

We are interested in constructing feedback motion planning controllers in unknown environments. In particular, the selected controllers are of the Navigation Function (NF) type. These controllers are in general functions of the desired destination, the current configuration and the configuration space obstacles. The first two, i.e., destination and current configuration, are always known. What is unknown are the obstacles within the configuration space.

Let us assume that we have a set of experimentally measured feasible trajectories in the configuration space. This offers an indication of which paths to prefer and which to avoid. It incorporates velocity information in the form of both direction and magnitude.

Our aim is to create NF controllers which will navigate from different initial conditions to different desired configurations, while utilizing the information available in the form of the available measured trajectories. This can be achieved by approximating obstacles based on the experimental information.

In more detail, an implicit obstacle function β encodes obstacles in the original NF methodology. Here we formulate the *Inverse Problem of Navigation Functions*. This will lead to a Partial Differential Equation (PDE), which is solved using the experimental trajectories as collocation conditions. By solving this PDE, an approximation of the obstacle function β is obtained, which constitutes an estimate of the unknown obstacles which the measured trajectories tried to avoid.

Then, in chapter 10, the method developed here is applied to anthropomorphic grasping. In this case, the required experimental trajectories come from human hand movements measured during reach-to-grasp movements for a variety of different objects. To reduce the solution space dimension, in this particular application Principal Component Analysis provides a subspace of the hand configuration space, within which to construct the solution. The selected subspace is the one capturing most of the variance observed in the trajectories, in other words the subspace spanned by the principal components corresponding to the n largest eigenvalues of the covariance matrix.

9.2 Problem Definition

9.2.1 Definitions

Assume that a set of $N_e \in \mathbb{N}^* \triangleq \mathbb{N} \setminus \{0\}$ experimentally measured trajectories

$$X_i, \quad i \in I_e \triangleq \{1, 2, \ldots, N_e\} \tag{9.1}$$

is available. Each of them is a set

$$X_i \triangleq \{x_i(t_j)\}_{j \in I_i}, \quad I_i \triangleq \{1, 2, \ldots, N_i\}, \quad i \in I_e \tag{9.2}$$

of $N_i \in \mathbb{N}^*$ configurations $x_i(t_j) \in \mathscr{W} \subset \mathbb{R}^n$ recorded in subsequent time instants $t_j \in [0, +\infty)$, which are indexed in increasing order $t_j < t_{j+1}, \quad \forall j \in I_i \setminus \{N_i\}, \quad \forall i \in I_e$.

Also, assume that the desired destinations $q_{di} \in \mathscr{W}, i \in I_e$ are provided, together with the velocities corresponding to each measured trajectory point

$$u_i(t_j) \triangleq \frac{\partial x_i}{\partial t}(t_j), \quad j \in I_i, \quad i \in I_e. \tag{9.3}$$

Note that both the destinations q_{di} and velocities $u_i(t_j)$ need not be independently provided. If no destinations are provided, then we can set $q_{di} = x_i(t_{N_i})$, provided the trajectories X_i are feasible and had converged successfully. If no velocity sensing is available, the velocities can always be calculated by numerically differentiating the available configuration data

$$u_i(t_j) \triangleq \frac{x_i(t_{j+1}) - x_i(t_j)}{t_{j+1} - t_j}, \quad j \in I_i \setminus \{N_i\}, \quad i \in I_e. \tag{9.4}$$

In such a case, the last configuration lacks a corresponding velocity, so N_i is in this case redefined discarding the last configuration. Let $U_i \triangleq \{u_i(t_j)\}_{j \in I_i}, i \in I_e$ denote the discrete-time samples each velocity function.

9.2.2 Navigation Functions

Let $\mathscr{F} \subset E^n$ be a compact connected C^2 manifold with boundary[1]. For the inverse problem we use the general definition of NFs. The workspace is defined as the compact connected set

$$\mathscr{W} \triangleq \{q \in E^n | \, 0 \leq \beta_0(q)\} \subset E^n \tag{9.5}$$

, which is bounded by the zero$^{\text{th}}$ obstacle defined as

$$\mathscr{O}_0 \triangleq E^n \setminus \mathscr{W} = \{q \in E^n | \, \beta_0(q) < 0\}, \quad \beta_0 \in C^2[E^n, \mathbb{R}] \tag{9.6}$$

Here we are treating the inverse problem, so it suffices to define the aggregate obstacle function $\beta \in C^2[E^n, \mathbb{R}]$ directly, not as the product of individual obstacle functions β_i, each of which corresponds to each connected component (obstacle) $\mathscr{O}_i$ of the free space complement $E^n \setminus \mathscr{F}$. It follows that the negative coset preimage of β is the non-free space, occupied by obstacles

$$E^n \setminus \mathscr{F} = \bigcup_{i \in I_0} \mathscr{O}_i = \beta^{-1}(0) \tag{9.7}$$

[1] Note that boundary non-smoothnesses are tractable, as described in [28].

Individual obstacles can still be defined from β. Each obstacle is a different connected component of the negative coset preimage of β_i. As a result, individual obstacles are by definition disjoint. In any case, we are not going to use individual β_i in the study of the NF inverse problem, only β.

The configuration is denoted by $q \in \mathscr{F} \subseteq E^n$ and the desired destination by $q_d \in \mathscr{F} \setminus \partial\mathscr{F}$. The Koditschek-Rimon NF is a specific form of NF defined as follows

$$\varphi(q, q_d) \triangleq \frac{\gamma_d(q, q_d)}{\left(\gamma_d(q, q_d)^k + \beta(q)\right)^{\frac{1}{k}}} = \varphi\left(\gamma_d(q, q_d), \beta(q)\right) \tag{9.8}$$

where $\gamma_d(q, q_d) = \|q - q_d\|^2 \implies \gamma_d \in C^\infty[\mathscr{W}, [0, +\infty)]$ is the attractive effect of the desired destination q_d and here $\beta \in C^2[\mathscr{W}, \mathbb{R}]$ is the obstacle repulsive effect. Note that β is here only C^2, hence φ is only C^2 and not analytic, but as already noted this suffices [23]. In the NF methodology, the scalar potential field φ is used to control the single integrator holonomic system

$$\frac{\partial x}{\partial t}(t) = u(t) \tag{9.9}$$

with the control law

$$u(t) = -\left(\nabla_q \varphi\right)(x(t),\, q_d) \tag{9.10}$$

yielding the system differential equation

$$\left\{ \begin{array}{c} \frac{\partial x}{\partial t}(t) = u(t) \\ u(t) = -\left(\nabla_q \varphi\right)(x(t),\, q_d) \end{array} \right\} \implies \frac{\partial x}{\partial t}(t) = -\left(\nabla_q \varphi\right)(x(t),\, q_d)\,. \tag{9.11}$$

The construction of φ naturally separates information regarding the known destination in γ_d(liveness/asymptotic stability), from information concerning collision avoidance with (possibly unknown) obstacles in β (safety/stability). This is a key observation leading to the formulation of the inverse problem.

Moreover, the method developed here constructs a β *such that* φ be a NF for the desired trajectories. This is potentially more flexible than only tuning k. The solution obtained later for (9.22) guarantees correct results. This follows from the fact that the solution is *enforced* to reproduce as close as it can the measured speeds over the same paths. As a result, if the experimentally measured speeds do not become zero, this is mathematically guaranteed to yield a NF for the subset of the configuration space which has been experimentally explored.

9.2.3 Working Hypothesis

We make the working hypothesis[2] that a NF function of the form of Equation 9.8 can adequately represent a controller producing the experimental measurements recorded[3]. Taking into account to the controller definition of (9.11) according to the NF methodology, this assumption is equivalent to equating the measured velocity function $u_i(t)$ to the NF gradient

$$\left(\nabla_q \varphi\right)\left(x_i(t),\, q_{di}\right) = \left(\nabla_q \varphi\right)\Big(\gamma_d\big(x_i(t),\, q_{di}\big), \quad \beta\big(x_i(t)\big)\Big) \tag{9.12}$$

[2]The term "working hypothesis" is attributed to Charles S. Peirce, John Dewey.

[3]Equivalently, that there exists a controller of this form, such that it can produce such trajectories.

at the corresponding configuration $x_i(t)$ measured the same time instant t. In continuous time this is expressed as

$$u_i(t) = -\left(\nabla_q \varphi\right)\big(x_i(t), q_{di}\big) = -\left(\nabla_q \varphi\right)\Big(\gamma_d\big(x_i(t),\, q_{di}\big),\quad \beta\big(x(t)\big)\Big) \tag{9.13}$$

In discrete-time, the above becomes

$$u_i(t_j) = -\left(\nabla_q \varphi\right)\big(x_i(t_j),\, q_{di}\big) = -\left(\nabla_q \varphi\right)\Big(\gamma_d\big(x_i(t_j),\, q_{di}\big),\quad \beta\big(x_i(t_j)\big)\Big) \tag{9.14}$$

9.2.4 Problem Statement

The problem can then be stated as follows, Fig. 9.1. Using the above experimental data $E \triangleq \{X_i, U_i, q_{di}\}_{I_e}$ find a function $\beta \in C^2\left([E^n, \mathbb{R}]\right)$ to satisfy equation

$$u(t_j) = -\left(\nabla_q \varphi\right)\big(x_i(t_j),\, q_{di}\big), \quad \forall j \in I_i, \quad \forall i \in I_e \tag{9.15}$$

subject to the positivity constraints on the sampled points

$$\beta\big(x_i(t_j)\big) > 0, \quad \forall j \in I_i, \quad \forall i \in I_e \tag{9.16}$$

and the workspace boundary $\partial\mathscr{W}$ closure requirement

$$\beta(q) \leq 0, \quad \forall q \in \partial\mathscr{W} \tag{9.17}$$

The positivity constraints (9.16) follow from the obstacle function definition

$$\beta(q) > 0, \quad \forall q \in \mathscr{F} \setminus \partial\mathscr{F} \tag{9.18}$$

in the free space interior. The closure at the workspace boundary (9.17) ensures that the trajectories produced by the resulting controller will always remain within $\mathscr{W}$, the domain of our problem.

Note that by now we have departed from our working hypothesis. The problem now is rigorously posed. The experiments specify β on a set of measure zero (union of sampled points). As a result, the solution function should be interpolated in the rest of the domain and a collocation solution method used. Defining the solution over the whole free space is not a priori possible in an unknown world. For this reason, the solution is defined over the whole workspace.

9.3 Inverse Method Formulation

9.3.1 Partial Differential Equation

Note that by (9.14) is the same as (9.15), hence it is equivalent to

$$u_i(t_j) = -\left(\nabla_q \varphi\right)\Big(\gamma_d\big(x_i(t_j),\, q_{di}\big),\quad \beta\big(x_i(t_j)\big)\Big) \tag{9.19}$$

Since

$$\left(\nabla_q \varphi\right)(q, q_d) = \frac{\partial \varphi}{\partial \gamma_d}\big(\gamma_d(q, q_d),\, \beta(q)\big)\left(\nabla_q \gamma_d\right)(q, q_d) + \frac{\partial \varphi}{\partial \beta}\big(\gamma_d(q, q_d),\, \beta(q)\big)\left(\nabla_q \beta\right)(q) \tag{9.20}$$

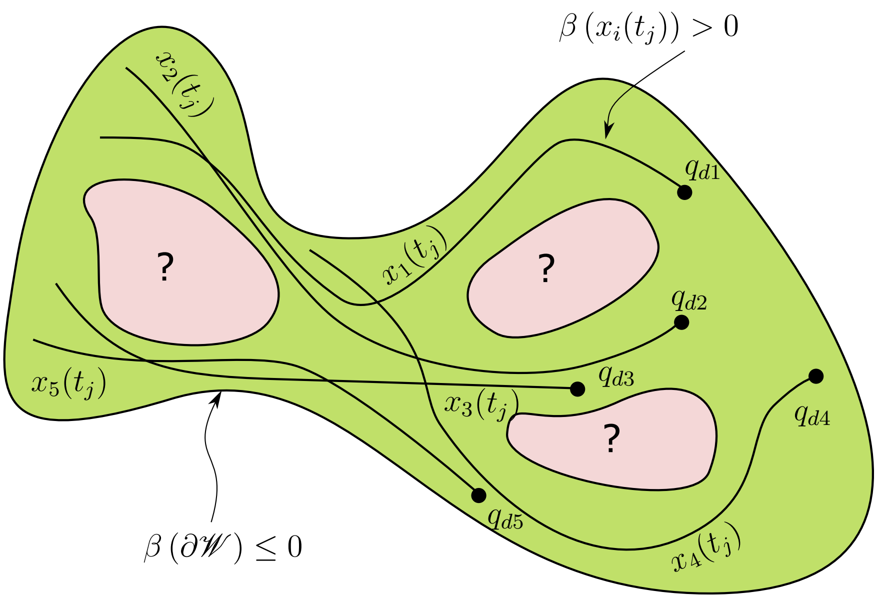


Figure 9.1: Problem definition.

When $\frac{\partial\varphi}{\partial\beta} > 0$, the above equation can be solved with respect to the obstacle function derivative (gradient) $(\nabla_q\beta)(q)$, as follows

$$(\nabla_q\beta)(q) = \frac{(\nabla_q\varphi)(q, q_d) - \frac{\partial\varphi}{\partial\gamma_d}\big(\gamma_d(q, q_d), \beta(q)\big)(\nabla_q\gamma_d)(q, q_d)}{\frac{\partial\varphi}{\partial\beta}\big(\gamma_d(q, q_d), \beta(q)\big)} \tag{9.21}$$

This is a PDE in the configuration q for the unknown obstacle function β. Substituting the experimental results using the problem statement (9.15) in (9.21)

$$(\nabla_q\beta)\big(x_i(t_j)\big) = -\left(\frac{u_i(t_j) + \frac{\partial\varphi}{\partial\gamma_d}\Big(\gamma_d\big(x_i(t_j), q_{di}\big), \beta\big(x_i(t_j)\big)\Big)(\nabla_q\gamma_d)\big(x_i(t_j), q_{di}\big)}{\frac{\partial\varphi}{\partial\beta}\Big(\gamma_d\big(x_i(t_j), q_{di}\big), \beta\big(x_i(t_j)\big)\Big)}\right), \tag{9.22}$$
$$\forall j \in I_i, \quad \forall i \in I_e$$

In this equation $u_i(t_j)$ is known from experimental measurements, $\frac{\partial\varphi}{\partial\gamma_d}, \frac{\partial\varphi}{\partial\beta}$ are also known functions of $\gamma_d(q, q_d)$ and $\beta(q)$, after we have selected a φ, and $\gamma_d, \nabla_q\gamma_d$ are also known functions of the experimental trajectory $x_i(t_j)$ and the known destinations q_{di}. As a result, we can substitute

$$u_i\big(x_i(t_j)\big), \quad \frac{\partial\varphi}{\partial\gamma_d}\Big(\gamma_d\big(x_i(t_j), q_{di}\big), \beta\big(x_i(t_j)\big)\Big), \quad \frac{\partial\varphi}{\partial\beta}\Big(\gamma_d\big(x_i(t_j), q_{di}\big), \beta\big(x_i(t_j)\big)\Big),$$
$$\gamma_d\big(x_i(t_j), q_{di}\big), \quad (\nabla_q\gamma_d)\big(x_i(t_j), q_{di}\big) \tag{9.23}$$

to obtain the PDE coefficients at points $x_i(t_j)$, which contains as unknowns only terms $(\nabla_q\beta)\big(x_i(t_j)\big)$ and $\beta\big(x_i(t_j)\big)$.

This then constitutes a PDE to solve in the unknown obstacle function $\beta : \mathbb{R}^n \to \mathbb{R}$ under the constraints (9.16) and (9.17). When a paraboloid attractive function γ_d is used, then $\nabla\gamma_d(q) = 2(q - q_d)$. We still need to select a NF form, which specifies the form of $\frac{\partial\varphi}{\partial\gamma_d}, \frac{\partial\varphi}{\partial\beta}$.

9.3.2 Selecting a NF form

In what follows two different Navigation Function types are substituted in the PDE, the Koditschek-Rimon NF φ and its non-degenerate unsquashed counterpart $\hat{\varphi}_1$, which are defined as

$$\varphi = \frac{\gamma_d}{\left(\gamma_d^k + \beta\right)^{\frac{1}{k}}}, \quad \hat{\varphi}_1 = \left.\frac{\gamma_d^k}{\beta}\right|_{k=1} = \frac{\gamma_d}{\beta} \tag{9.24}$$

respectively. The results are compared and φ selected for further use. For more details concerning these functions, their derivatives, degeneracies and substitution in the PDE, see Appendix A. Functions $\varphi, \hat{\varphi}$ have parallel gradients, as can be observed in Table A.3. Nonetheless, $\nabla\varphi$ exhibits more nonlinearity than $\nabla\hat{\varphi}$, hence also more nonlinearity than $\nabla\hat{\varphi}_1$, which is evident from

$$\begin{aligned} \nabla\varphi &= \frac{\left(\gamma_d^k + \beta\right)^{\frac{1}{k}} \nabla\gamma_d - \gamma_d \nabla\left(\left(\gamma_d^k + \beta\right)^{\frac{1}{k}}\right)}{\left(\gamma_d + \beta\right)^{\frac{2}{k}}} = \left(\gamma_d^k + \beta\right)^{-\frac{1}{k}-1}\left(\beta\nabla\gamma_d - \frac{1}{k}\gamma_d\nabla\beta\right) \\ \nabla\hat{\varphi} &= \frac{\beta\nabla\left(\gamma_d^k\right) - \gamma_d^k\nabla\beta}{\beta^2} = \frac{\beta k\gamma_d^{k-1}\nabla\gamma_d - \gamma_d^k\nabla\beta}{\beta^2} \end{aligned} \tag{9.25}$$

Substituting the partial derivatives in (9.13) to obtain the specific form of the PDE associated with each of the selected functions, as proved in section A.4, we obtain

$$\begin{aligned} \left(\nabla_q\beta\right)\left(x_i(t_j)\right) = &\left(k\frac{u_i(t_j)}{\gamma_d\left(x_i(t_j), q_{di}\right)}\right)\left(\gamma_d\left(x_i(t_j),\, q_{di}\right)^k + \beta\left(x_i(t_j)\right)\right)^{\frac{1}{k}+1} \\ &+ \left(k\frac{\left(\nabla_q\gamma_d\right)\left(x_i(t_j),\, q_{di}\right)}{\gamma_d\left(x_i(t_j),\, q_{di}\right)}\right)\beta\left(x_i(t_j)\right) \end{aligned} \tag{9.26}$$

and

$$\begin{aligned} \left(\nabla_q\beta\right)\left(x_i(t_j)\right) = &\left(\frac{u_i(t_j)}{\gamma_d\left(x_i(t_j),\, q_{di}\right)^k}\right)\beta\left(x_i(t_j)\right)^2 \\ &+ \left(k\frac{\left(\nabla_q\gamma_d\right)\left(x_i(t_j),\, q_{di}\right)}{\gamma_d\left(x_i(t_j),\, q_{di}\right)}\right)\beta\left(x_i(t_j)\right) \end{aligned} \tag{9.27}$$

for $\varphi, \hat{\varphi}$, respectively, where $q \in \mathscr{F} \setminus \{\partial\mathscr{F} \cup \{q_d\}\} \implies \frac{\gamma_d^k}{\beta} \in (0, +\infty)$.

Equation (9.22) (and its specific forms (9.26), (9.27)) is a first order semi-linear variable-coefficient partial differential equation. The vector coefficients are known at the experimental measurement points. By selecting an approximating candidate function β, the error from expected values can be calculated *at* the sampled points $x_i(t_j), i \in I_e \subset \mathbb{N} \setminus \{0\}$. Therefore, an iterative algorithm can be used for minimization of the PDE satisfaction error (residual) in the parameter space of the β approximation function.

9.3.3 Variable PDE coefficients and k damping

This subsection concerns only the PDE corresponding to function[4] $\hat{\varphi}_1$. Nonetheless, the discussion here offers useful insight for the case of φ as well.

[4]Here, the expressions correspond to $\hat{\varphi}$, but $k = 1$ is used later to avoid degeneracy at the destination.

Consider (9.27). Before substitution of the experimental data, this equation had the form

$$(\nabla_q \beta)(q) = \left(\frac{-(\nabla_q \hat{\varphi}_1)(q, q_d)}{\gamma_d(q, q_d)^k} \right) \beta(q)^2 + \left(k \frac{(\nabla_q \gamma_d)(q, q_d)}{\gamma_d(q, q_d)} \right) \beta(q) \tag{9.28}$$

In this equation it is assumed that the system is controlled by a NF, so that the control action $-(\nabla_q \hat{\varphi}_1)(q, q_d)$ is a function only of q, q_d (where $q = x_i(t_j)$ when considering measurements). Therefore, before substituting experimental measurements, we can define the variable PDE vector-coefficients as functions of only q, q_d, i.e.,

$$A(q, q_d) \triangleq \frac{-(\nabla_q \hat{\varphi}_1)(q, q_d)}{\gamma_d(q, q_d)^k}, \quad B(q, q_d) \triangleq k \frac{(\nabla_q \gamma_d)(q, q_d)}{\gamma_d(q, q_d)} \tag{9.29}$$

But, when experimental measurements are substituted in the PDE, the definition of A should be necessarily changed to

$$A(t_j, q_{di}) \triangleq \frac{u_i(t_j)}{\gamma_d\big(x_i(t_j), q_{di}\big)^k} \tag{9.30}$$

The reason for this is that for experimental data, at the same point q (or its neighborhood in a practical setting), there may be multiple samples at different time samples t_j. Due to the fact that the measured system is not guaranteed to be driven by a NF, but we have assumed that it can be approximated by one, the measured control action $u_i(t_j)$ at different times t_j for which the system passes through the same point $q' = x_i(t_j)$, may be different. In other words, if the control action was really created by a NF, it is a function only of q', but because this is not true for the real system, if it goes through the same point at different times during the experiment, the measured control actions $u_i(t_j)$ at the same q' may be different. For them *not* to be different, it is necessary that

$$w(q) \triangleq \{u_i(t_j) | \, \exists t_j \in [0, +\infty) : x_i(t_j) = q \in E^n\} \tag{9.31}$$

be a function, which is not in general true for the experimental data. This is the reason for which A will not necessarily be a function of q when experimental measurements are substituted. As a result, it should be redefined as a function of t_j, q_{di}.

However, the second coefficient B remains unchanged, i.e., a function only of $q = x_i(t_j), q_{di}$. This is also true in (9.26). It comes from the fact that B does not depend on the measurements, i.e., it is decoupled from the experiment[5]. These comments on the coefficients will be used later.

9.3.3.1 Numerical differentiation

Since only sampled points $x_i(t_j)$ are available in our case the velocity $u_i(t_j)$ is obtained by finite differences, taking into account also the sampling period

$$u_i(t_j) = \frac{x_i(t_{j+1}) - x_i(t_j)}{T} \tag{9.32}$$

where $T = 1\,\mathrm{ms}$ in our case (after re-sampling the experimental measurements) and the units of $x_i(t_j)$ are degrees of angle $[x] = °$.

[5]Coefficient B contains only destination information, whereas coefficient A contains both obstacle *and* destination information.

9.3.3.2 Dimensional considerations

Mentioning units summons the associated issue with the NF formula. Assuming a circular paraboloid γ_d and that its arguments q, q_d are expressed in configuration space ("length") dimensions leads to the following NF gradient units (where $[k] = 1$, i.e. k is assumed unitless)

$$\begin{aligned}[\nabla\hat{\varphi}] &= \left[\frac{\gamma_d^k}{\beta^2}\frac{k\beta\frac{\nabla\gamma_d}{\gamma_d} - \nabla\beta}{\beta^2}\right] = \frac{[\gamma_d]^k}{[\beta]^2}\frac{[k][\beta]\frac{[\nabla\gamma_d]}{[\gamma_d]} - [\nabla\beta]}{[\beta^2]} = \frac{[q]^{2k}}{[\beta]^2}\frac{1[\beta]\frac{[q]}{[q]^2} - \frac{[\beta]}{[q]}}{[q]^4} \\ &= \frac{[q]^{2k}}{[\beta]^2}\frac{\frac{[\beta]}{[q]} - \frac{[\beta]}{[q]}}{[q]^4} = [q]^{2k-4}[\beta]^{-2}\frac{[\beta]}{[q]} = [q]^{2k-4-1}[\beta]^{-2+1} = [q]^{2k-5}[\beta]^{-1}\end{aligned} \tag{9.33}$$

Taking into consideration that in the usual KRNF formulation β has the same "square C-space distance" units as the paraboloid γ_d leads to $[\nabla\hat{\varphi}] = [q]^{2k-5}[q]^{-2} = [q]^{2k-7}$, which does not include any time units.

To achieve unit homogeneity a dimensional constant multiplicative gain K_{NF} with should be used in the controller, even if it possesses unit magnitude

$$u(t) = -K_{NF}\left(\nabla_q\hat{\varphi}\right)(x(t)) \tag{9.34}$$

9.3.3.3 Selection of tuning parameter k

Let us now visualize the vector PDE coefficient values A, B which are calculated from the experimental measurements. These are plotted in Fig. 9.2a and Fig. 9.2b for $k = 1$ and $k = 2$, repsectively. In Fig. 9.3a and Fig. 9.3b the corresponding vector plots are provided.

It can be observed that the last measurements of trajectory near q_d strongly affect the relative order of magnitude of A and B. Furthermore, the inconsistency is even stronger for larger k values.

This happens due to the form of $A = \frac{u}{\gamma_d^k}$ and $B = k\frac{\nabla\gamma_d}{\gamma_d}$ which leads to

$$\frac{\|A\|}{\|B\|} = \frac{\|u\|\,\gamma_d^{-k}}{k\,\|\nabla\gamma_d\|\,\gamma_d^{-1}} = \frac{\|u\|}{\gamma_d^{k-1}\,\|\nabla\gamma_d\|} = \frac{\|u\|}{\gamma_d^{k-1}2\sqrt{\gamma_d}} = \frac{\|u\|}{2\gamma_d^{k-\frac{3}{2}}} \tag{9.35}$$

There are two cases for which different behaviors arise both near q_d and away from it. Before continuing, note that

$$\begin{aligned} q \to q_d &\iff \gamma_d(q) \to 0^+ \\ \|q\| \to +\infty &\iff \gamma_d(q) \to +\infty \end{aligned} \tag{9.36}$$

so the two cases are for $k = 1$ and $k \geq 2$ the following

$$\frac{\|A\|}{\|B\|} = \begin{cases} \frac{1}{2}\|u\|\,\gamma_d^{\frac{1}{2}}, & k = 1 \\ \frac{1}{2}\|u\|\,\gamma_d^{\frac{3}{2}-k}, & k \geq 2 \implies \frac{3}{2} - k \leq \frac{3}{2} - 2 = -\frac{1}{2} < 0 \end{cases} \tag{9.37}$$

If $k = 1$, then away from the destination q_d (start of the experimental trajectory) the ratio $\frac{A}{B}$ is amplified in favor of the experimental measurements, hence strenghtening the information introduced in the PDE by the measurements. This is evident in Fig. 9.2a.

Near the destination q_d (end of the experimental trajectory) the reverse effect results, where $\frac{A}{B}$ is damped. But as observed, A and B are approximately of the same order, so that $k = 1$ is acceptable.

On the contrary, for $k = 2$ away from the destination q_d (start of the experimental trajectory) is seriously damped, as evident in Fig. 9.2b. The damping is heavy for most of the trajectory, essentially "erasing" any experimental information u incorporated only in A and affecting the nonlinear β^2 term of the PDE. Therefore, this results in a PDE practically decoupled from the experiment which assumes the form $\frac{\nabla\beta}{\beta} \approx \frac{\nabla\gamma_d}{\gamma_d}$, therefore we expect that it will yield a solution $\beta \approx \gamma_d$, irrespective of the experiment.

This happens because $u(t)$ is recorded and independent of the distance to the destination. This is divided by γ_d^k, which is larger away from the destination, very small near it, and is tuned by k. On the contrary, B has to similar functions $\sqrt{\gamma_d}$ and γ_d in it nominator and denominator, respectively. As a result, it is not affected by k and remains the same decreasing function of the distance to the destination.

On the contrary, near the destination A is amplified compared to B, as is evident by comparison of Fig. 9.2a and Fig. 9.2b. This is in favor of the experimental measurements near the destination. But the erasing effect of $k = 2$ in most of the trajectory is inadmissible. For the above reasons we select $k = 1$ when using $\hat{\varphi}$.

Also note that the previous analysis indicates that most information about the obstacle function β is provided by the intermediate part of the trajectory, than either by the near or distant field.

9.3.3.4 Experimental trajectory tail rejection

There is a further issue to be addressed, related to the order of magnitude of the PDE vector coefficient norms $\|A\|, \|B\|$ near the destination q_d. It persists even for $k = 1$.

It can be observed in both Fig. 9.2a and Fig. 9.2b, as well as in the vector plots of Fig. 9.3a and Fig. 9.3b, where $\|A\|, \|B\|$ become several orders of magnitude higher near the destination q_d than away from it. This happens because in both coefficient norms

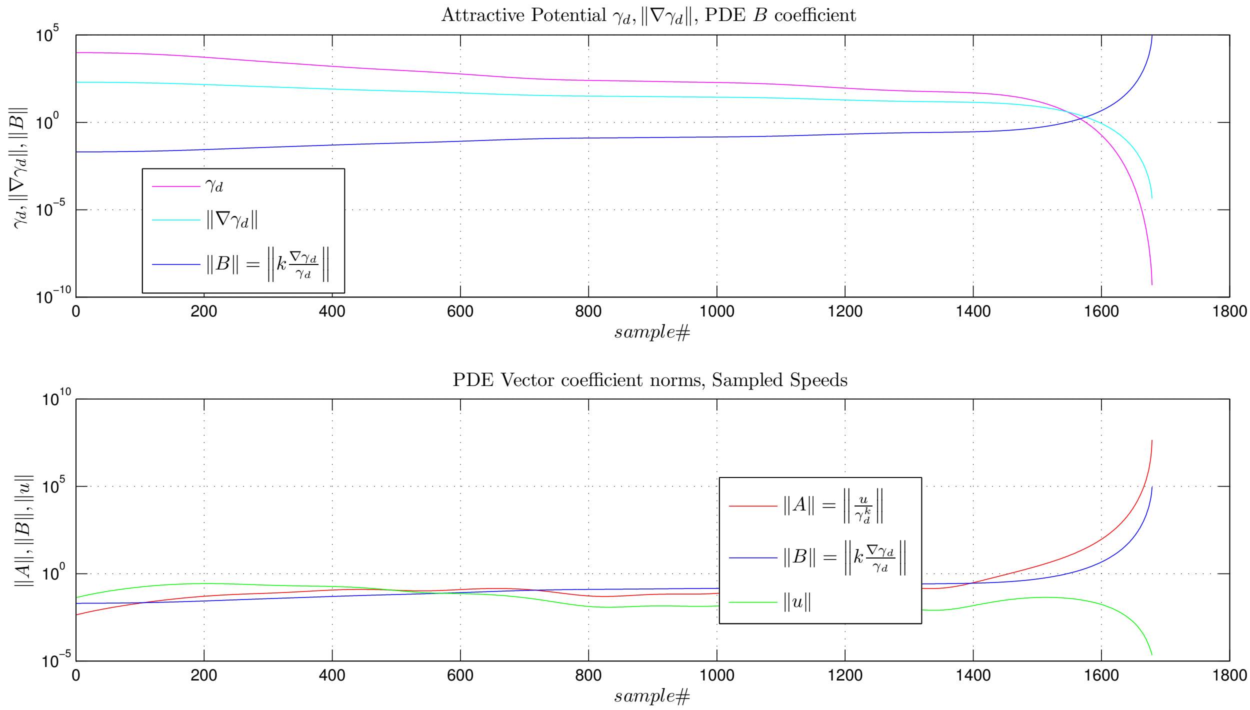
$$\begin{aligned} \|A\| &= \frac{\|u\|}{\gamma_d^k} \\ \|B\| &= k\frac{\|\nabla\gamma_d\|}{\gamma_d} = k\frac{2\sqrt{\gamma_d}}{\gamma_d} = \frac{2k}{\gamma_d^{\frac{1}{2}}} \end{aligned} \tag{9.38}$$

the distance to destination $\gamma_d^{\frac{1}{2}}$ arises in the denominator. Hence, in the goal's neighborhood the denominator vanishes, so that A, B blow up there.

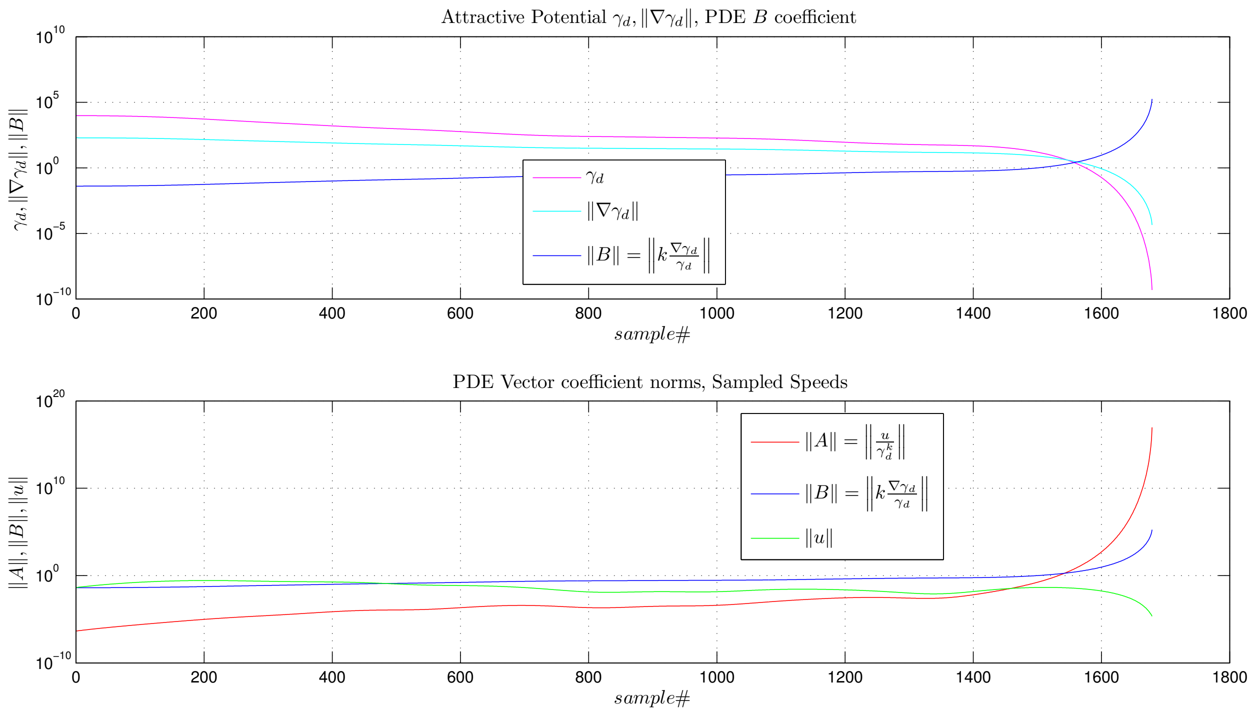
This is unwanted for the numerical solution of the PDE. As explained later, the PDE is solved by iteratively minimizing an error functional J (to be defined). Since the magnitudes of the coefficients differ by several orders of magnitude in the neighborhood of the destination from the major part of the trajectory, the same percentage of error in satisfying the PDE near the goal will result in so large errors, that they will blanket the sum of all the rest of the errors over the whole trajectory.

For these reasons, together with considerations pertaining to numerical stability, the trajectory measurements in the goal's vicinity are not used in the PDE solution. The new norm plots along the trajectory are provided in Fig. 9.4a and Fig. 9.4b, and the corresponding vector plots of the coefficients in Fig. 9.5a and Fig. 9.5b, respectively.

Comparison of Fig. 9.4a to Fig. 9.4b indicates that the issue of diminishing experimental information away from the destination for $k \geq 2$ as compared to $k = 1$, still remains.

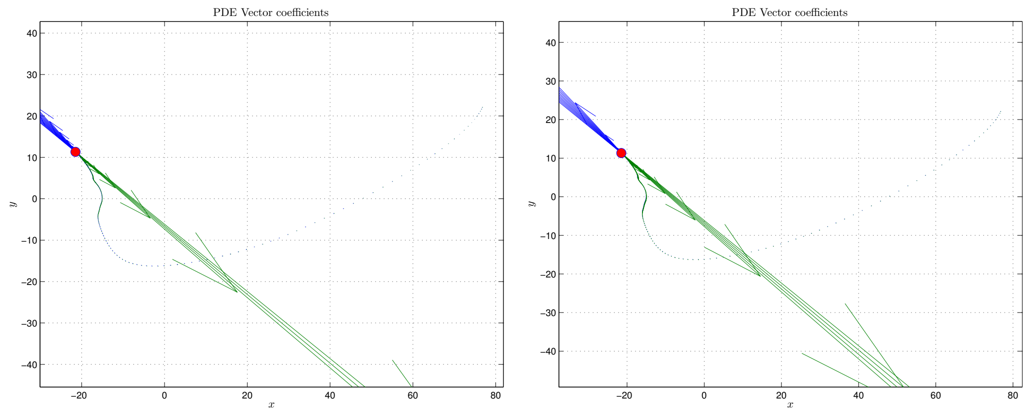


(a) $k = 1$



(b) $k = 2$

Figure 9.2: Using the complete experimental trajectory.



(a) $k = 1$ (b) $k = 2$

Figure 9.3: Using the complete experimental trajectory. Blue vectors denote the first PDE coefficient $A\left(x_1(t_j)\right)$ and green the second $B\left(x_1(t_j)\right)$, on the first experimental trajectory.

Therefore the selection $k = 1$ of subsubsection 9.3.3.3 remains valid for $\hat{\varphi}$.

In sum, we can say that coefficient blowing up *near* the destination led us to reject that part (where for *any* k value the problem does not change), whereas diminishing of experimental measurements from the equation *away* from the destination for $k \geq 2$ led us to select $k = 1$. Since these effects relate to different parts of the trajectory, they are independent, hence both actions are needed.

The comparison away from the destination of the two vector coefficients in the two cases of $k = 1$ and $k = 2$ can be made by reference to the magnified trajectory details illustrated in Fig. 9.5c and Fig. 9.5d. In Fig. 9.5c the sizes of A (blue, contains experimental information) and B (green, does not contain experimental information, only destination information) are comparable, whereas in Fig. 9.5d they are not, with B by far overwhelming A.

9.4 PDE Solution

9.4.1 Basis Selection: Splines

Basis splines (B-Splines) [50] were selected as the solution basis. Therefore, the solution is searched in the finite-dimensional space of B-spline coefficients, where

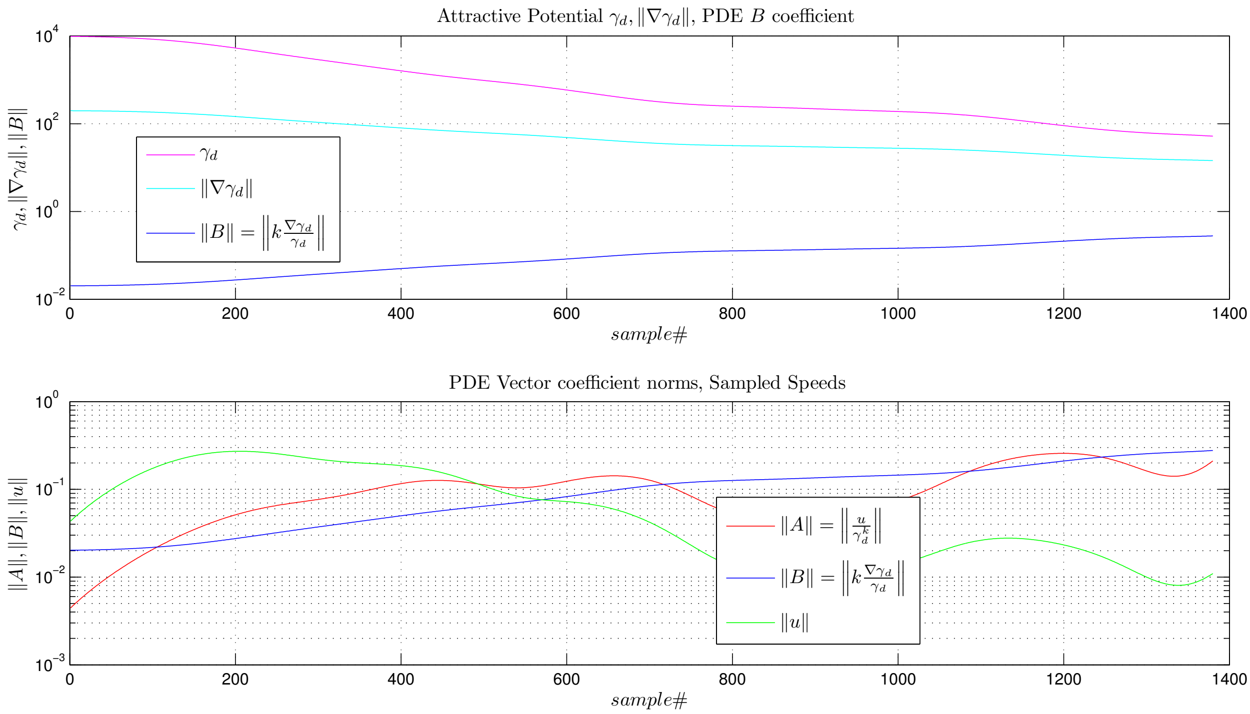
$$\beta(q) = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} \left(c_{i_1 i_2 \dots i_n} \prod_{r=1}^{n} B\left(q_r | t_r\right) \right) \tag{9.39}$$

where $\beta \in C$ is the interpolated obstacle function, $q \in \mathbb{R}^n$ is the system's state, $q_r \in \mathbb{R}$ (only) here denotes the r^{th} component of q,

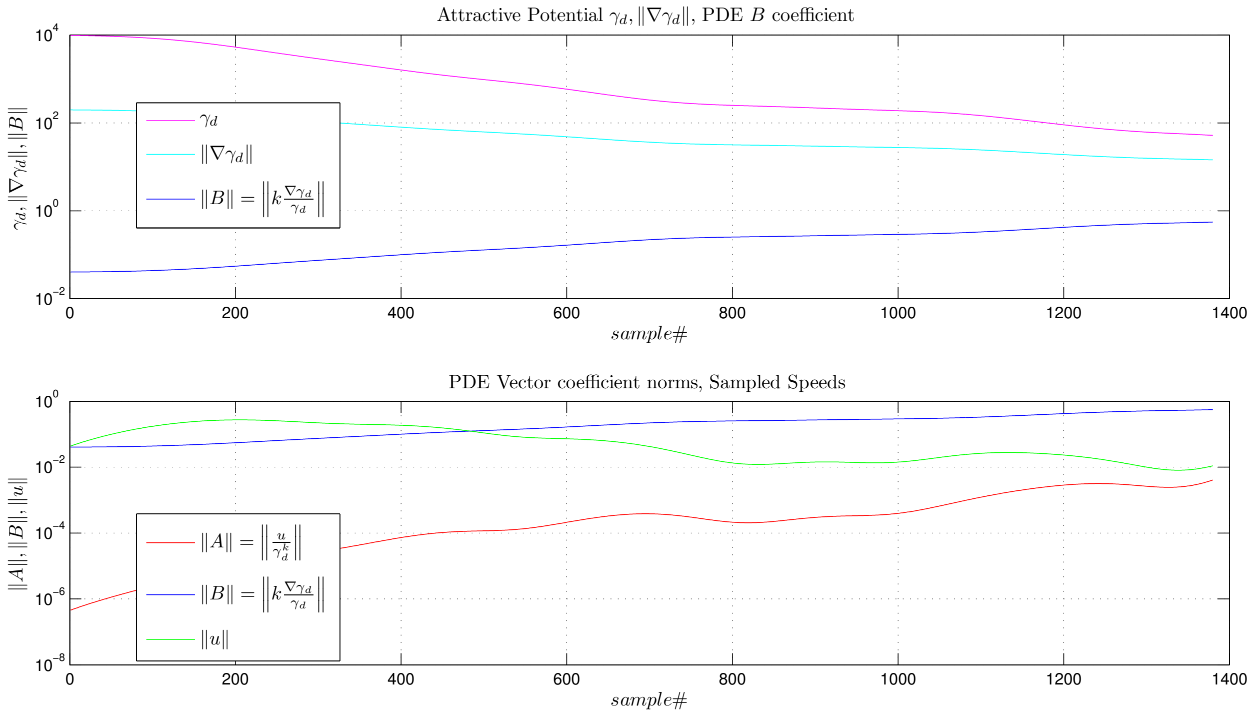
$$C \triangleq \{c_{i_1 i_2 \dots i_n}\}_{i_j \in \{1,2,\dots,m_j\}, j \in \{1,2,\dots,n\}} \in \times_{j \in \{1,2,\dots,n\}} \mathbb{R}^{m_j} \tag{9.40}$$

is the coefficient tensor,

$$t = [t_{r i_r}, t_{r(i_r+1)}, \dots, t_{r(i_r+h_r)}]$$
$$t_{ij} \in \mathbb{R}, \quad i \in \{1, 2, \dots, n\}, \quad j \in \{1, 2, \dots, m_i + h_i\}$$

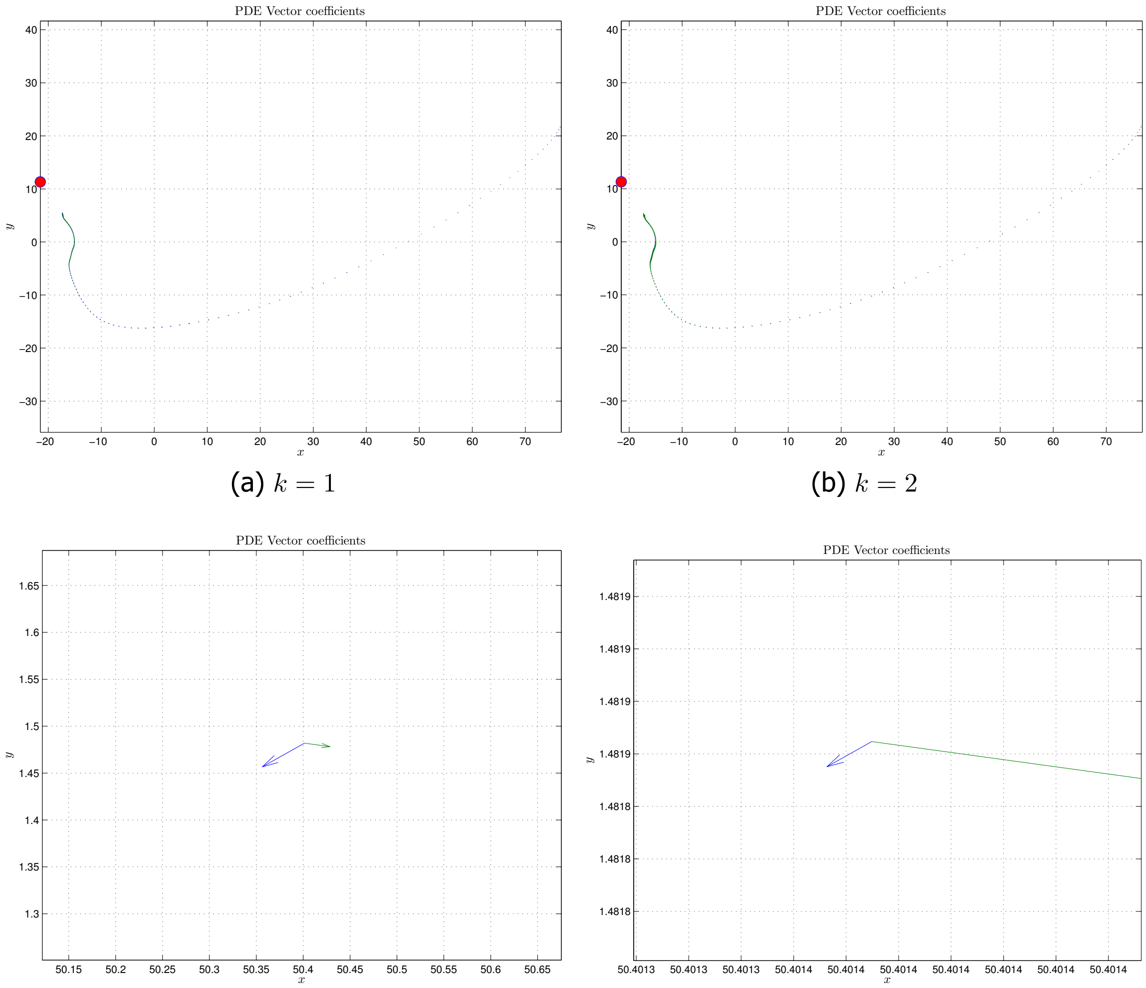


(a) $k = 1$



(b) $k = 2$

Figure 9.4: Using the truncated experimental trajectory.



(a) $k = 1$ (b) $k = 2$

(c) $k = 1$ (d) $k = 2$

Figure 9.5: Using the truncated experimental trajectory.

are the knot sequences of each dimension and $h_i \in \mathbb{R}, i \in \{1, 2, \ldots, n\}$ are the orders of the splines of each dimension and $B(q_r|t_r)$ are the basis functions. Let us also stack the coefficient tensor in a vector of *design variables* for the minimization problem

$$c \triangleq [c_{11\cdots 1}, c_{21\cdots 1}, \ldots, c_{m_1 1\ldots 1}, c_{12\ldots 1}, c_{22\ldots 1}, \ldots, c_{m_1 2\ldots 1}, \ldots, c_{m_1 m_2 \ldots m_n}]^{\mathrm{T}} \in \mathbb{R}^{\sum_{i=1}^{n} m_i}$$

The spline B-form is used instead of the piecewise polynomial representation because it implicitly incorporates smoothness constraints.

The PP-form (Piecewise Polynomial) is utilized for fast calculations, but only for that purpose. We *construct* a spline using the B-form and *use* the constructed spline represented in PP-form (MATLAB Curve Fitting Toolbox).

9.4.1.1 Domain of definition

The domain of definition D is selected based on the variable limits of the problem under consideration. Selection of an appropriate domain is important because if its boundary ∂D is more than a knot away from the closest trajectory point, then the boundary closure (9.17) is implicitly satisfied during solution, provided the initial iteration solution is zero on the boundary.

In this case, perturbations of B-spline coefficients corresponding to boundary knots do not affect the collocation error on the experimental trajectories. As a result, the associated cost functional perturbation ΔJ is zero, hence the respective design variable gradient component remains zero and the coefficient remains constant. Since zero initially, the boundary coefficients remain constantly zero and the boundary closure condition is met.

9.4.1.2 Knot allocation over dimensions and selection

The problem is multidimensional, which requires addressing the issue of allocating the number of knots over dimensions efficiently. Since knots and the associated coefficients are needed mostly where more variance needs to be represented, the dimensions are firstly analysed in terms of experimental trajectory variance. Then the numbers of knots are allocated accordingly. More knots are assigned to the dimensions with maximal variance.

In chapter 10 a principal subspace is selected, before the method is applied. This also serves for the purpose described here. The allocation of knots in the case study is proportional to the associated principal variances. In particular, $10, 6, 3$ knots (excluding boundary knot multiplicity) have been used for each of the primary three Principal Component dimensions, respectively, taking into consideration their principal variances subsection 10.2.2.

9.4.2 Iterative semi-linear PDE system solution

To solve the semi-linear PDE (9.26) (similarly (9.27)) under the positivity constraints (9.16) an iterative gradient descent algorithm [51]

$$c^{N+1} = c^N - \lambda_J \nabla_c J \tag{9.41}$$

has been used, minimizing the error functional J described in subsubsection 9.4.2.2. Here, c^N denotes the B-spline coefficient values at the N^{th} iteration of the minimization algorithm. The positivity constraints are also incorporated in this functional, while the boundary closure constraints (9.17) are implicitly satisfied, as already described.

9.4.2.1 Initial point

A flat obstacle $\beta \equiv 0$ is used as the initial solution. Therefore, during the initial iteration, the main effect is due to the terms J_{sp}, J_{dp} of the cost functional, which are defined in what follows.

9.4.2.2 Optimization Cost Functional J

The appropriate choice of functional $J : C^2([D, \mathbb{R}]) \times D_e \to [0, +\infty)$ is crucial for the successful solution for β. The cost functional used here in the case of discrete samples[6]

$$
\begin{aligned}
J &\triangleq \frac{1}{\sum_{i \in I_e} N_i} \left(w_1 J_{PDE} + w_2 J_{sp} + w_3 J_{dp} + w_4 J_{bn} \right) \\
J_{PDE} &\triangleq \sum_{i \in I_e, j \in I_i} \Delta E_{ij}, \qquad J_{sp} \triangleq \sum_{i \in I_e, j \in I_i} s\left(\beta\left(x_i(t_j)\right) - \beta_t\right) \\
J_{dp} &\triangleq \sum_{q_{di}, i \in I_e} s\left(\beta(q_{di}) - \beta_t\right), \qquad J_{bn} \triangleq \sum_{q_n \in \partial \mathscr{W}} s\left(\beta(q_n)\right)
\end{aligned}
\tag{9.42}
$$

where $w_i \in (0, +\infty), i \in \{1, 2, 3, 4\}$ are weighting factors to select the relative importance of the various terms. The offset β_t serves numerical robustness (practical sign definiteness) by introducing a finite margin, above which β is considered positive. Function $s : \mathbb{R} \to \mathbb{R}$ is a C^2-smooth switch

$$
s(x) \triangleq \begin{cases} x^3, x \leq 0 \\ 0, 0 < x \end{cases}
\tag{9.43}
$$

and the component functionals are described hereafter. The component functionals are now explained.

1. The satisfaction error of PDE system (9.26) is accounted for in J_{PDE} as

$$
\begin{aligned}
\Delta E_{ij} &\triangleq \frac{1}{\gamma_d\big(x_i(t_j), q_{di}\big)^k} \left\| \begin{array}{c} (\nabla_q \beta)\left(x_i(t_j)\right) - \\ - \left(k \frac{u_i(t_j)}{\gamma_d\left(x_i(t_j), q_{di}\right)} \right) \left(\gamma_d\big(x_i(t_j),\, q_{di}\big)^k + \beta\big(x_i(t_j)\big) \right)^{\frac{1}{k}+1} - \\ - \left(k \frac{(\nabla_q \gamma_d)\left(x_i(t_j), q_{di}\right)}{\gamma_d\left(x_i(t_j), q_{di}\right)} \right) \beta\big(x_i(t_j)\big) \end{array} \right\|_2^2 \\
&= \frac{1}{\gamma_d\left(x_{ij}, q_{di}\right)^k} \left\| \begin{array}{c} (\nabla_q \beta)\left(x_{ij}\right) - \left(k \frac{u_{ij}}{\gamma_d(x_{ij}, q_{di})} \right) \left(\gamma_d\left(x_{ij},\, q_{di}\right)^k + \beta\left(x_{ij}\right) \right)^{\frac{1}{k}-1} - \\ - \left(k \frac{(\nabla_q \gamma_d)(x_{ij}, q_{di})}{\gamma_d(x_{ij}, q_{di})} \right) \beta\left(x_{ij}\right) \end{array} \right\|_2^2
\end{aligned}
\tag{9.44}
$$

   where term γ_d^k ensures a fair weighting along the trajectory, for reasons discussed in subsection 9.3.3. In an analogous manner we can define J_{PDE} for the solution of (9.27).
2. Functional J_{sp} enforces positivity $\beta\big(x_i(t_j)\big) > 0$ at the sampled points $x_i(t_j)$, i.e., condition (9.16), because these belong to the free space $\mathscr{F}$. The subscript of J_{sp} is derived from the initials of the words "sampled" and "positivity".
3. Positivity at the destinations is ensured by J_{dp}. This term has been introduced because the trajectory truncation described in subsubsection 9.3.3.4 has removed

[6] Which is always the case for discrete-time measurements.

the destinations q_{di} from the sampled configurations. As a result they are not included in term J_{sp} and hence should be incorporated separately. The subscript of J_{dp} is derived from the initials of the words "destination" and "positivity".

4. Domain closure (9.17) is imposed by the boundary non-positivity functional J_{bn}. Here $w_4 = 0$, because this constraint is implicitly satisfied[7]. The subscript of J_{bn} is derived from the initials of the words "boundary" and "non-positivity".

Discretization of term J_{bn} is essential to render it calculable, but while the trajectory samples are unique and given, the configuration space workspace boundary discretization is subjective. For example the boundary knots and two intermediate points between each pair of such boundary knots could be sampled.

The relative weight of the positivity functional is selected an order of magnitude higher to ensure the search is forced to move in the feasible domain. This is important, because otherwise the search is (possibly) not bounded within the design space. Moreover, it does not correct itself, because negative β leads to reversal of experimental data PDE vector coefficient signs, which in turn leads to negative surface curvature, so non-feasible solutions satisfying the PDE are found ("mirror" β surfaces with respect to $\beta = 0$).

9.4.2.3 PDE Solution Algorithm

The PDE is solved by Algorithm 5. At first all the B-spline coefficients are initialized to be zero. Then in each iteration the following occur.

Firstly, the cost functional J is calculated for the current solution c^N. Then each coefficient $c^N_{i_1 i_2 \dots i_n}$ is perturbed by Δc and the new cost functional value J_p for the corresponding perturbed B-spline is calculated. Then the difference of the two cost functional values $\Delta J = J_p - J$ provides the cost functional perturbation resulting from the single coefficient $c^N_{i_1 i_2 \dots i_n}$ perturbation Δc.

Such a perturbation is performed for each of the design variables $c^N_{i_1 i_2 \dots i_n}$ to calculate the cost functional J gradient $\nabla_c J$ in design space. This gradient is used to perform a gradient descent on J in design space $c^{N+1} = c^N - \lambda_J \nabla_c J$.

The appropriate selection of the spline B-form coefficient perturbation size Δc and the design space cost functional gradient step λ_J (could be adaptive, in general it depends on the optimization scheme implemented) are crucial to obtain correct results.

The optimization for the case study in chapter 10 is shown in Fig. 9.6. It includes the final form of function β, the history of B-spline coefficients c during optimization, the history of the norm of differences between subsequent vectors of coefficients $\|c_{i+1} - c_i\|$, the history of the cost functional design space gradient norm $\|\nabla_c J\|$ and the history of the cost functional J, as they varied in each iteration.

---

[7]This requirement has so far been implicitly satisfied because the zero solution is used as the initial point of optimization in design space. If the spline domain is adequately larger than the experimental trajectories' domain, then the spline B-form coefficients affecting its form near the workspace boundary do not affect it away from this boundary, therefore they do not affect the PDE or positivity constraints on the trajectory, hence their perturbation leads to zero J perturbation, which in turn leads them to remain zero, which actually meets the non-positivity constraint on the configuration workspace boundary. Moreover note that if a C^2 switch s is used in this case, it should not be translated by an offset. This aims to allow zero boundary coefficients to remain zero, because this suffices (remember that in the case of free space sign definiteness is of importance, hence departure from zero is essential, whereas here this is not the case). Not allowing this and forcing them to become negative could also result in unwanted non-zero derivatives at the boundary, without real cause.

Algorithm 5 Inverse Navigation Function Problem PDE Solver

procedure Inverse NF Problem PDE Solver
    Start
    Initialize tensor product solution
    Cost functional J computation
    $f \leftarrow 0, \quad N \leftarrow 1$
    while $f == 0$ **do**
        for $i \in \{1, 2, \ldots, \sum_{j=1}^{n} m_j$ **do**
            Perturb coefficient $c_{i,p}^N \leftarrow c_i^N + \Delta c$
            Compute perturbed cost functional J_p
            Cost functional gradient component $\frac{\partial J}{\partial c_i^N} \leftarrow \frac{J_p - J}{c_{i,p}^N - c_i^N}$
        end for
        Cost functional gradient $g \leftarrow \nabla_c J$
        Design space step $c^{N+1} \leftarrow c^N + \lambda g$
        $f \leftarrow$ Convergence criterion
        $N \leftarrow N + 1$
    end while
end procedure

procedure Cost functional J computation
    Differentiate B-spline
    PDE collocation errors
    Cost functional value
end procedure

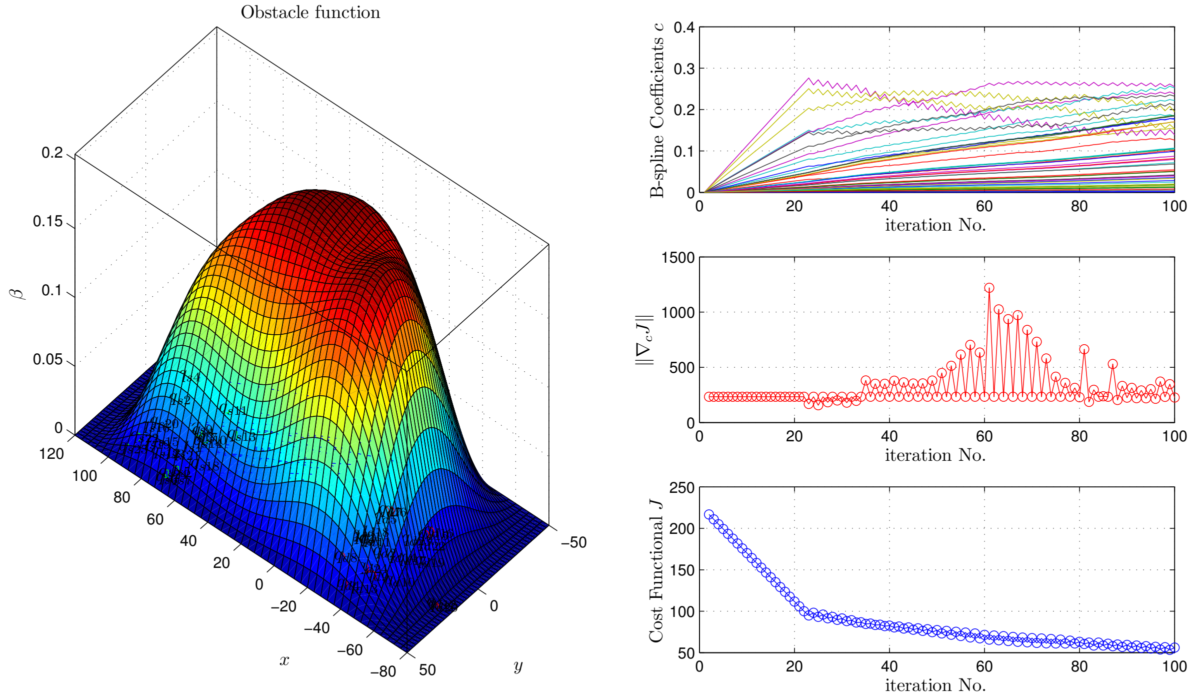


Figure 9.6: Obstacle function β from PDE solution using $n_e = 24$ experiments. This solution has been obtained in the subspace spanned by the two Principal Components with the highest variance. The obstacle function resulting from the solution in the subspace spanned by the three Principal Components with highest variance cannot be plotted in only three dimensions, because its domain is three dimensional. In that case, the induced Navigation Function vector field is shown in Fig. 9.8.

9.4.3 Constructed controller

Select any feasible desired destination $q_d \in \beta^{-1}((0, +\infty))$. Then, the spline obstacle function β resulting from the previously described optimization can be substituted in φ of (9.8) to yield the NF control law (9.10) as

$$\begin{aligned} u_c(t) &= -\left(\nabla_q \varphi\right)(x(t)) \\ &= -\frac{\beta\,(x)\,(\nabla_q \gamma_d)\,(x, q_d) - \frac{\gamma_d(x,q_d)}{k}\,(\nabla_q \beta)\,(x)}{\left(\gamma_d\,(x, q_d)^k + \beta\,(x)\right)^{\frac{1}{k}+1}} \end{aligned} \tag{9.45}$$

where x is the system's state. The potential field and level sets of a 2-dimensional controller for a selected q_d are illustrated in Fig. 9.7. The level set of a 3-dimensional controller are provided in Fig. 9.8.

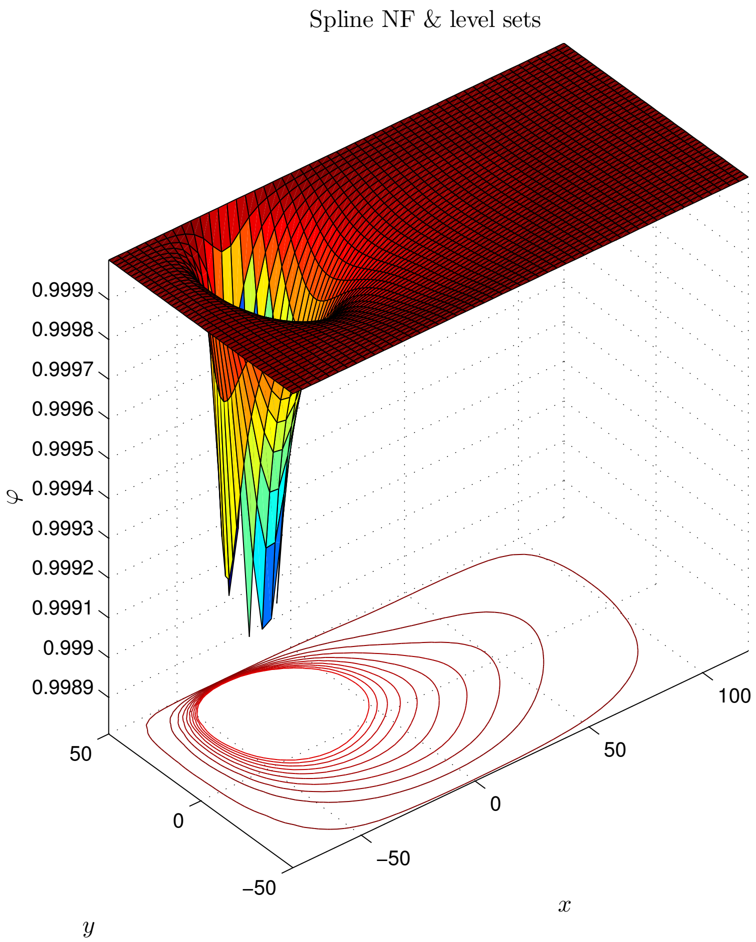


Figure 9.7: Navigation Function and its level sets over 2-d PC space.

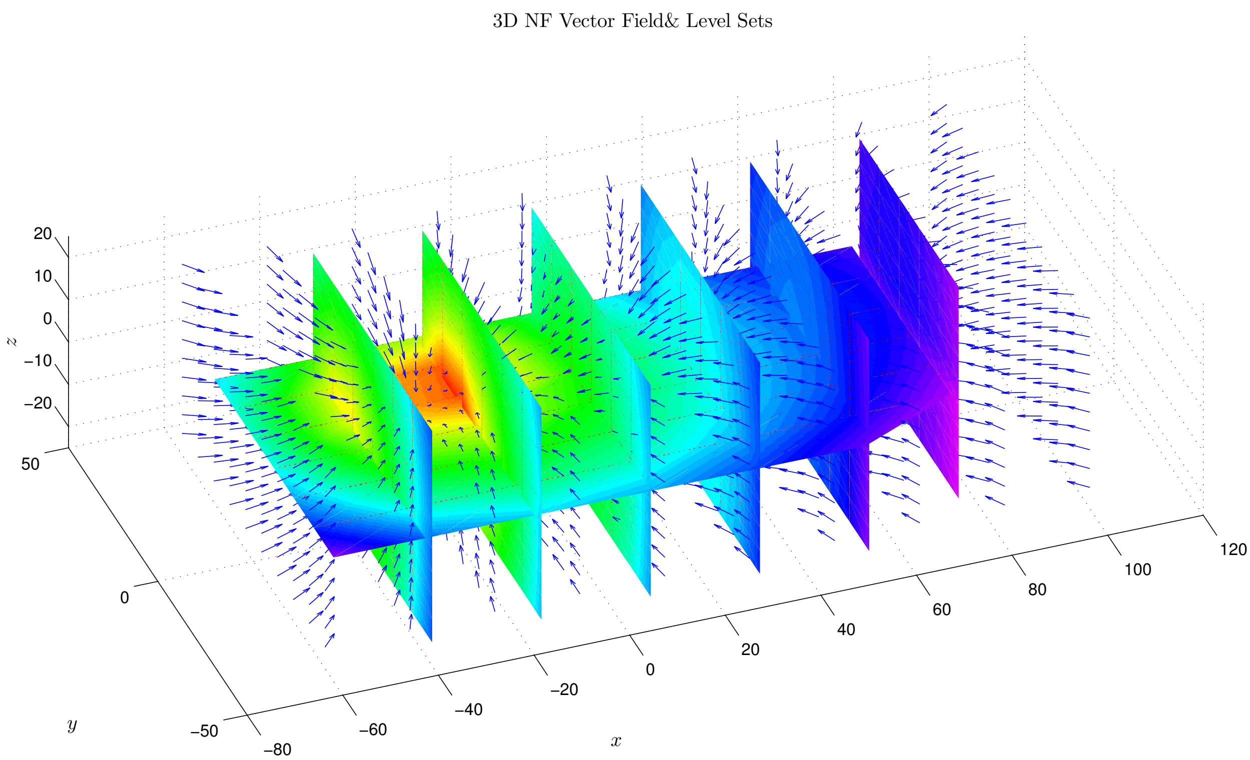


Figure 9.8: Navigation Function vector field and level sets over 3-d PC space.

Chapter 10

Application to Hand Grasping

10.1 Introduction

There has been a sustained and increasing interest in creating autonomous robotic hands similar to the human hand. The motivation behind this is manifold. Contrary to industrial settings, in everyday human environments the majority of tasks involves objects adapted to human manipulation capabilities. Therefore, a robot operating in such settings would need to actuate on them. Since they have been created to suit the human hand, this type of actuator is uniquely suited to handling them.

As a result, developing autonomous, robust and viable robotic hand systems will facilitate robotic applications to human environments. These include prosthetics [79], rehabilitation and teleoperation. Another field of application are dangerous tasks in hazardous or uninhabitable environments, as for example repairs of operating space equipment [74, 78]. There are two main challenges in order to achieve this.

On the one hand, the required hardware needs to be developed. Several efforts witnessed in the past fifteen years started with four fingers, e.g. the Utah/MIT [80], DLR I [71], DIST [72], LMS [76] robotic hands and continued with five fingers, which include the Belgrade/USC [69], Anthrobot [83, 89], Robonaut [86], DLR II [70], Gifu I [81] and II [82, 87], Shadow [88], DLR/HIT I [75, 84] and II [85] hands and the DLR Hand Arm System [77]. Some of the most difficult issues have been the reduction in size, increase of impact strength and elasticity [90], and speed [68]. A comparative overview is provided in [60], a survey in [53].

Operation of these hands requires appropriate controllers. This is a motion planning problem in a configuration space (C-space) of high dimension. Additionally, in many cases, anthropomorphism may be desired for the generated motions. Moreover, studying human motion can provide vital insight, leading to efficient design and control of artificial hands.

There have been several attempts to construct anthropomorphic controllers for robotic hands. The authors in [52, 57] treated a similar problem of anthropomorphic robot arm control by identifying joint dependencies using Dynamic Bayesian Networks. In [64] Bayesian Networks were applied to robot grasping learning. Identifying hand synergies through Principal Component Analysis (PCA) for grasping has been firstly proposed in [63] and *eigengrasps* have been defined in [56].

Controlling a robotic hand in principal subspace has also been considered in [61, 62], where eigengrasps are called Principal Motion Directions. The approach there is different from the one presented here, because free motion of the human hand instead of grasping is recorded, which does not provide information about everyday eigengrasps. Moreover,

Table 10.1: Grasping Experiments

No.	Object	Task
1,2,3,4	Tall glass	Grasp: to drink, from side & move from top & move, from side & rotate
5, 6, 7	Mouse	Grasp to: slide, left click, right click
8,9,10,11	Cup	same as tasks as 1,2,3,4
12	Hammer	Grasp to use
13	Ashtray	Grasp from above to move
14	Cube	Grasp from above to raise
15, 16	Pen	Write, Move
17, 18	Jar	Move, Lid unscrewing
19	Screwdriver	Grasp to operate
20	Book	Grasp from right side to read
21	Mobile phone	Pick up to view
22, 23	Scissor	Grasp to: Move, Use
24	Stapler	Grasp and use

half of the measured configuration dimensions are not used, because PCA is performed after mapping human degrees of freedom (DOF) to robot hand DOF. Here PCA is applied to the full 22 DOF, independently of the robot hand.

For motion planning, Sampling-Based Roadmaps have been used in [61, 62], which provide probabilistic completeness, are computationally intensive and still require the initial and final points to be linked to the roadmap. The NF method is safe by construction, achieves provably correct convergence and offers a closed-loop continuous controller, integrating planning and trajectory tracking. Moreover, Roadmaps cannot capture anthropomorphism *within* the principal subspace. On the contrary, NF can produce similar motions also within this subspace.

10.2 Experiments and Modeling

10.2.1 Experimental procedure

For collecting the trajectories $n_e = 24$ experiments have been conducted with one subject grasping 13 different objects listed in Table 10.1 using its right hand. For 6 of them more than multiple tasks have been performed and for 7 of them one task, as detailed in the table. Snapshots of the experimental setup are provided in Fig. 10.3.

The hand angles have been measured using a CyberGlove data glove [73], which features electric angle sensors with $1°$ resolution and records 22 degrees of freedom at a $100\,\mathrm{Hz}$ sampling rate, 3 flexions/extensions for each finger apart from the thumb, for which they are 2, 1 ab/adduction, palm arch and 2 wrist degrees of freedom. The ElectroMyoGraphic signals of the arm have been measured as well, together with the wrist reference frame and accelerations, although they are not used here.

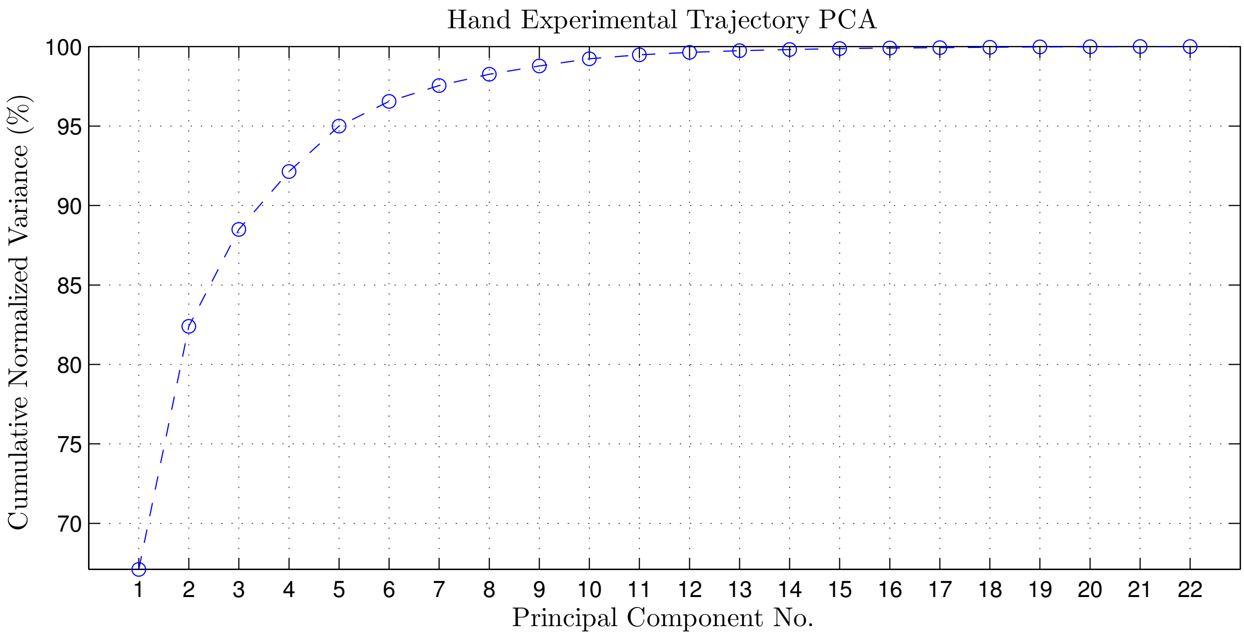


Figure 10.1: Principal component cumulative normalized variances.

10.2.2 Principal Component Analysis

To reduce the high dimension (22 measured angles) Principal Component Analysis [120] has been conducted, as is customary in the analysis of hand and arm systems [52, 56, 63]. This method affinely transforms the coordinate system in the hand C-space (angle space) to one centered at the average of the experiments, and rotated in the eigenvector directions of the covariance matrix.

A subspace of the principal system is used here, comprised of the 3 principal components with the highest variances (covariance matrix eigenvalues). The trajectories for the n_e experiments in this principal subspace are shown in Fig. 10.2.

The selected subspace captures 88.5% of the original movement data variability, Fig. 10.1, hence, the grasping movements of the relatively diverse experiments of Table 10.1 can be reproduced satisfactorily from a C-space of highly reduced dimensionality (3 from the 22).

In particular, as will be shown in what follows, the principal system captures anthropomorphism in a natural way, as has also been observed in [61]. In our case, anthropomorphism is additionally enhanced by the inverse construction of NF we have proposed.

10.2.3 PDE Solution

The B-spline domain used is the (enlarged) (hyper-)parallelepiped

$$\times_{r \in \{1,2,\ldots,n\}} [\lambda_{r,\min} q_{r,\min}, \quad \lambda_{r,\max} q_{r,\max}] \tag{10.1}$$

where

$$q_{r,\min} \triangleq \min \{x_i(t_j)\}, \quad q_{r,\max} \triangleq \max \{x_i(t_j)\} \tag{10.2}$$

are the extremal values per variable over the measured samples of the experimental trajectories. In this case, functions $\min, \max$ are applied element-wise, as they would

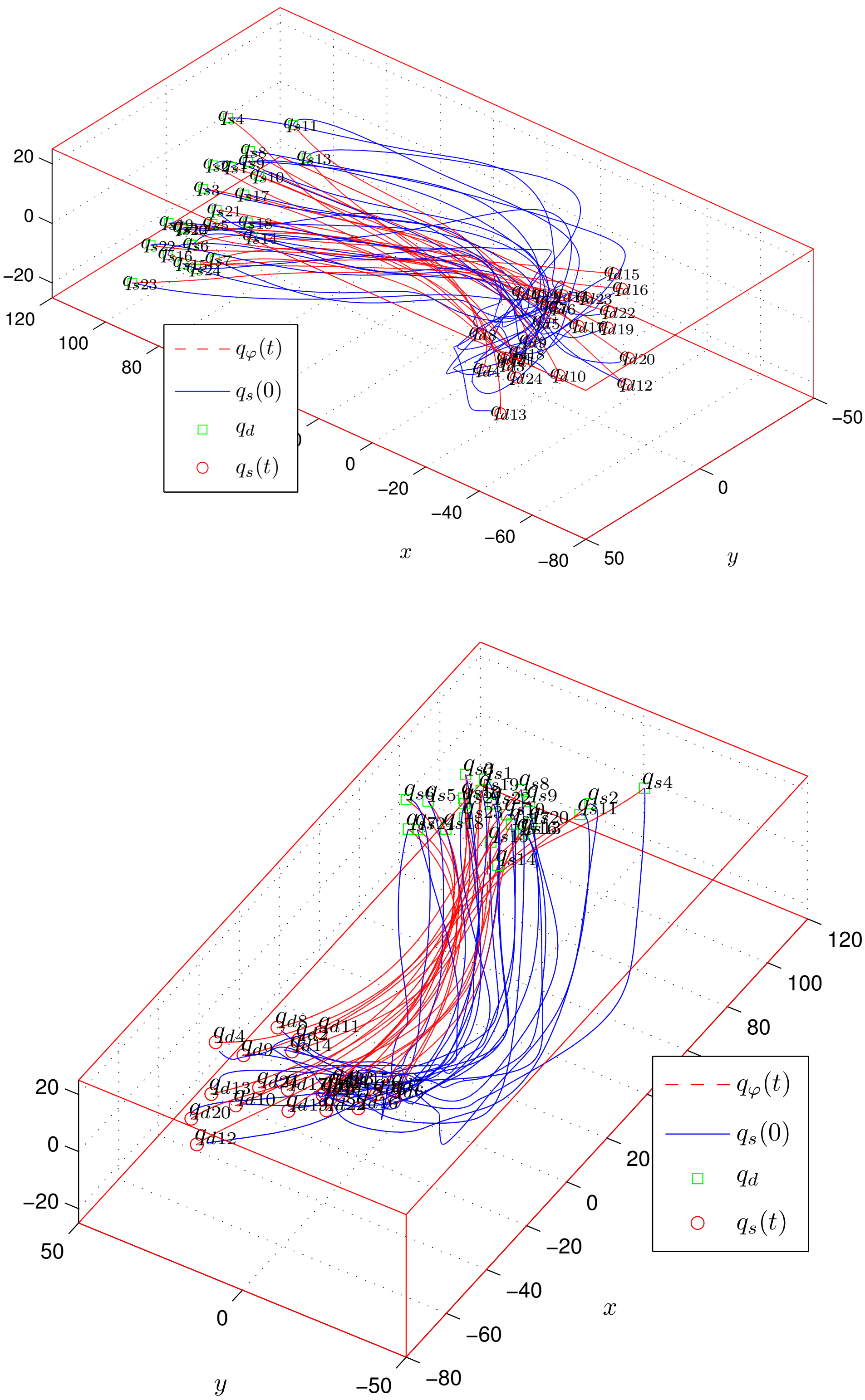


Figure 10.2: Multiple automatically generated trajectories $x_i(t)$ (red dashed) using a Navigation Function φ with $k = 2$ and the *same* obstacle function β constructed over the 3 primary Principal Components. The obstacle function is the solution of the PDE using the experimentally measured trajectories $x_s(t)$ (blue continuous), Table 10.1. Initial conditions are $q_{si}(0)$ (green squares) and the destinations are q_{di} (red circles).

in a vectorized MATLAB implementation issuing the command `min(x, [], 2)`. Also,

$$\lambda_{r,\min} \begin{cases} > 1, & q_{r,\min} < 0 \\ < 1, & q_{r,\min} > 0 \end{cases} \quad , \quad \lambda_{r,\max} \begin{cases} < 1, & q_{r,\max} < 0 \\ > 1, & q_{r,\max} > 0 \end{cases} \tag{10.3}$$

are enlarging multiplicative factors.

Enlargement is important, in order to allow the spline to smoothly change from a positive value at the sampled points (numerically non-negligible, as enforced by offset β_t), to zero value on the configuration workspace boundary $\partial\mathscr{W}$. Note that the domain selection just described is completely automatic, forming part of a seamless algorithm if needed.

10.2.4 Hand Model Definition

The human hand kinematic model described in [59] has been used, with parametrically defined lengths, as functions of the human hand length H_L and hand breadth H_B, in combination with anthropometric data from [55]. The phalanges are modeled as ellipsoids [54]. The human hand kinematic model degrees of freedom and their correspondence to the data glove sensors is provided in Table 10.2. Moreover, in [59] the finger base reference frame distance is provided only for the thumb (I). For this reason these have been calculated here, using data from [55, 59]. The notation is the same as that defined in [55, 59]. In what follows H_L is the hand length and H_B is the hand breadth. For the thumb

$$l_{oo_1} = l_{I-0} - l_{I-1} = 0.118 H_L \tag{10.4}$$

For the index (II) and middle (III) fingers

$$l_{oo_i} = l_{i-0} = (SL)_{i1} + ((BL)_{i1} - (JC)_{i1}) = B_{ij} H_L + l_{i-1}\left(1 - A_{ij}\right), \quad i \in \{II, III\} \tag{10.5}$$

hence

$$\begin{aligned} l_{oo_2} &= 0.463 H_L + \sqrt{(0.0374 H_L)^2 (0.0126 H_B)^2} \\ l_{oo_3} &= 0.4833 H_L \end{aligned} \tag{10.6}$$

For the ring (IV) and little (V) fingers

$$l_{oo_i} = l_{i-0} - l_{i-1} = B_{ij} H_L + l_{i-1}\left(1 - A_{ij}\right) - l_{i1} = B_{ij} H_L - l_{i-1} A_{ij}, \quad i \in \{IV, V\} \tag{10.7}$$

hence

$$\begin{aligned} l_{oo_4} &= 0.421 H_L - \sqrt{(0.3051 H_L)^2 + (0.0693 H_B)^2} \\ l_{oo_5} &= 0.414 H_L - \sqrt{(0.2655 H_L)^2 + (0.1611 H_B)^2} \end{aligned} \tag{10.8}$$

10.3 Comparison of φ to experimental trajectories

A sequence of hand postures automatically generated using the NF on the 3-dimensional Principal subspace of Fig. 10.2 is illustrated in Fig. 10.4. The hand destination configuration has been selected to grasp a tall glass, similarly to the first three experiments. The resultant reach-to-grasp trajectory of the system is smooth and reproduces anthropomorphism in a natural way. As far as arm movement is concerned, it correlates with hand movement [65] and this allows us to combine the methodology proposed here with previous work on

Table 10.2: Degrees of freedom of kinematic hand model from [59] and CyberGlove. The order of DoF possessed by the data glove is the same as the columns within its measurements' log file. Note that the first column in the log file is time.

Part	Degree of Freedom	Name	V[b] [59]	±F/A[c] [59]	±F/A[c] glove
Palm	Palm arch	Palm arch (PA)	N/A	N/A	
Thumb	MCP[a] Joint Flexion/Extension	Flexion 1 (F1)	q_3	-	
	IP[a] Joint Flexion/Extension	Flexion 2 (F2)	q_5	-	
	MCP Joint Ab/Adduction	Abduction (A)	q_4	+	
	CMC[a] Joint Flexion/Extension		q_2	-	N/A
	CMC Joint Ab/Adduction		q_1	-	N/A
Index	MCP Joint Flexion/Extension	Flexion 1 (F1)	q_7	+	-
	PIP Join Flexion/Extension	Flexion 2 (F2)	q_8	+	-
	DIP Joint Flexion/Extension	Flexion 3 (F3)	q_9	+	-
	MCP Joint Ab/Adduction	Abduction (A)	q_6	+	+
Middle	MCP Joint Flexion/Extension	Flexion 1 (F1)	q_{11}	+	-
	PIP Join Flexion/Extension	Flexion 2 (F2)	q_{12}	+	-
	DIP Joint Flexion/Extension	Flexion 3 (F3)	q_{13}	+	-
	MCP Joint Ab/Adduction	Abduction (A)	q_{10}	+	+
Ring	MCP Joint Flexion/Extension	Flexion 1 (F1)	q_{17}	+	-
	PIP Join Flexion/Extension	Flexion 2 (F2)	q_{18}	+	-
	DIP Joint Flexion/Extension	Flexion 3 (F3)	q_{19}	+	-
	MCP Joint Ab/Adduction	Abduction (A)	q_{16}	+	-
	CMC Joint Flexion/Extension		q_{15}	+	N/A
	CMC Joint Ab/Adduction		q_{14}	+	N/A
Little	MCP Joint Flexion/Extension	Flexion 1 (F1)	q_{23}	+	-
	PIP Join Flexion/Extension	Flexion 2 (F2)	q_{24}	+	-
	DIP Joint Flexion/Extension	Flexion 3 (F3)	q_{25}	+	-
	MCP Joint Ab/Adduction	Abduction (A)	q_{22}	+	-
	CMC Joint Flexion/Extension		q_{21}	+	N/A
	CMC Joint Ab/Adduction		q_{20}	+	N/A
Wrist	Flexion/Extension		N/A	N/A	
	Radial/Ulnar		N/A	N/A	

[a] CMC = CarpoMetaCarpal, MCP = MetaCarpoPhalangeal, IP = InterPhalangeal, PIP = ProximalInterPhalangeal, DIP = DistalInterPhalangeal.
[b] Variable name.
[c] Flexion or Adduction sign.

Figure 10.3: Experimental setup during reach to grasp. Hand angles, wrist and object position and orientation in space and EMG signals are recorded.

anthropomorphic arm control [52], for fully automatic control of the complete hand-arm system. Alternative applications include hand prosthesis [56], [58], where the subject provides wrist movement and the controller can select the appropriate configuration on the generated NF trajectory, based on correlations with EMG signals and wrist proximity to object.

10.4 Comparison of $\hat{\varphi}_1$ to experimental trajectories

In this section function $\hat{\varphi}_1$ has been used to find an obstacle function for *individual* experiments. This is contrary to the previous section, where φ has been used for all experiments simultaneously. The present section also aims to illustrate the increased "plasticity" of $\hat{\varphi}_1$, with the associated limitations discussed hereinafter.

In some cases in which the algorithm had converged there followed subtle oscillations, of which abrupt changes of the gradient are characteristic, as visible in Fig. 10.6, Fig. 10.10, Fig. 10.12, Fig. 10.16 and Fig. 10.23. These can be of negligible consequences (local oscillations), but in some cases, as for example Fig. 10.6, if the optimization is allowed to continue after convergence, these subtle oscillations gradually (within the next 100 iterations) alter the obstacle function's shape. They can even cause local minima to arise[1]. This is a general observation, that $\hat{\varphi}_1$ exhibits greater "plasticity" than φ, hence it can take shapes better representing the experimental trajectories. But, this increased "plasticity" is at the same time the disadvantage of $\hat{\varphi}_1$, because local minima can arise, which does *not* happen when using φ.

In Fig. 10.6 to Fig. 10.29 the resulting obstacle functions β for each experiment have been computed *independently*, i.e. with a separate optimization for each experiment. This is contrary to the previous section concerning φ.

In the first couple of figures the domains vary, because the respective trajectories did not require more space. From Fig. 10.14 on the domain is the same for all experiments

[1]This can be attributed to the fact that the trajectory provides only local information to a local interpolant (the spline), hence spline coefficients away from the experimental trajectory can fluctuate without being much constrained by the (single here) trajectory. If this is allowed to continue for a long time (say 65-150 iterations) then the result is gradually deformed. But the cost functional remains practically constant, which is useful to define a convergence check and which also illustrates that there is a quite flat valley of minima in the design space, which extends towards the "insensitive" design variables which are away from the spline.

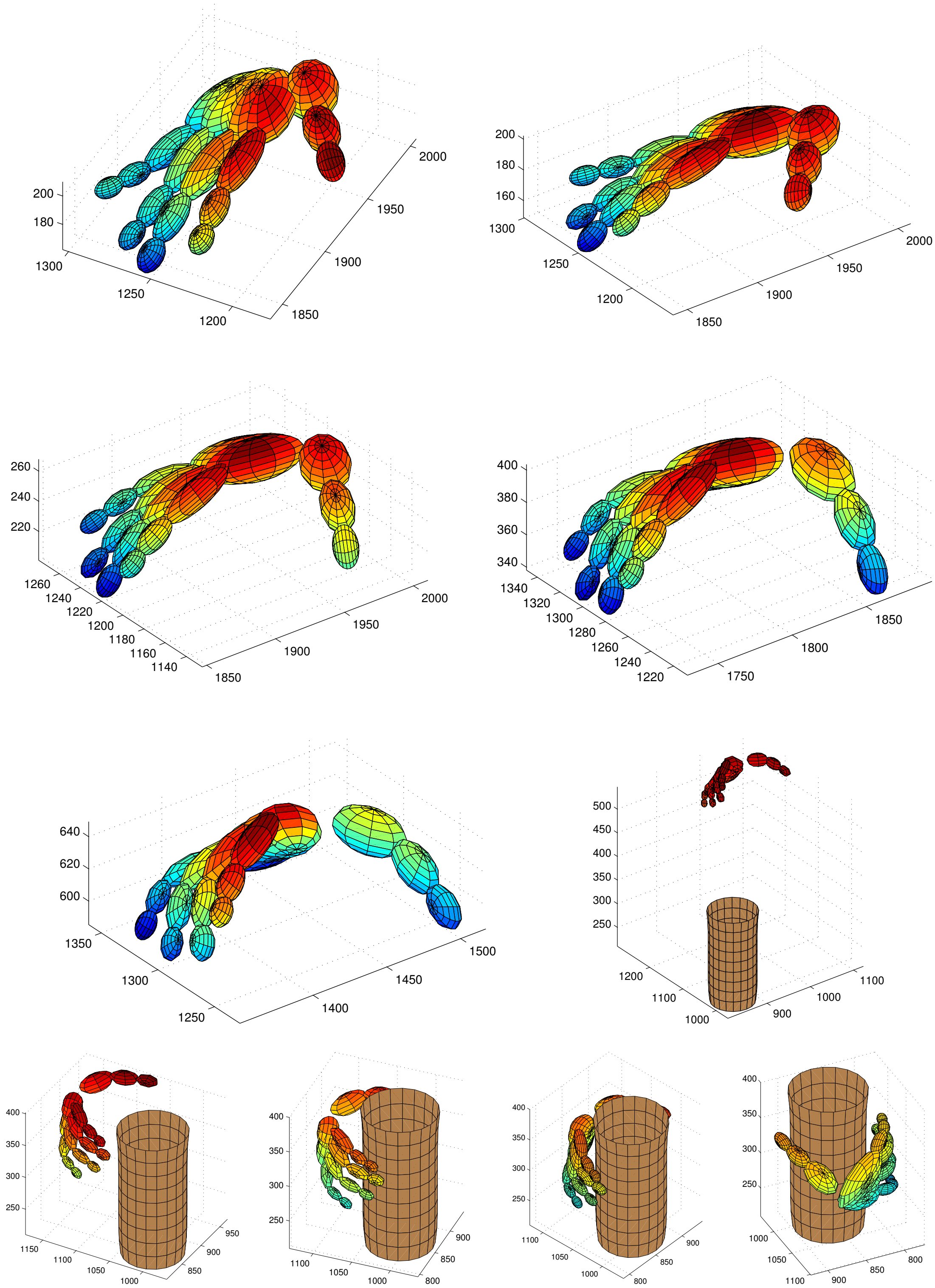


Figure 10.4: Automatically generated grasping movement using Navigation Function in 3-dimensional principal subspace of Fig. 10.2, compare to Fig. 10.3.

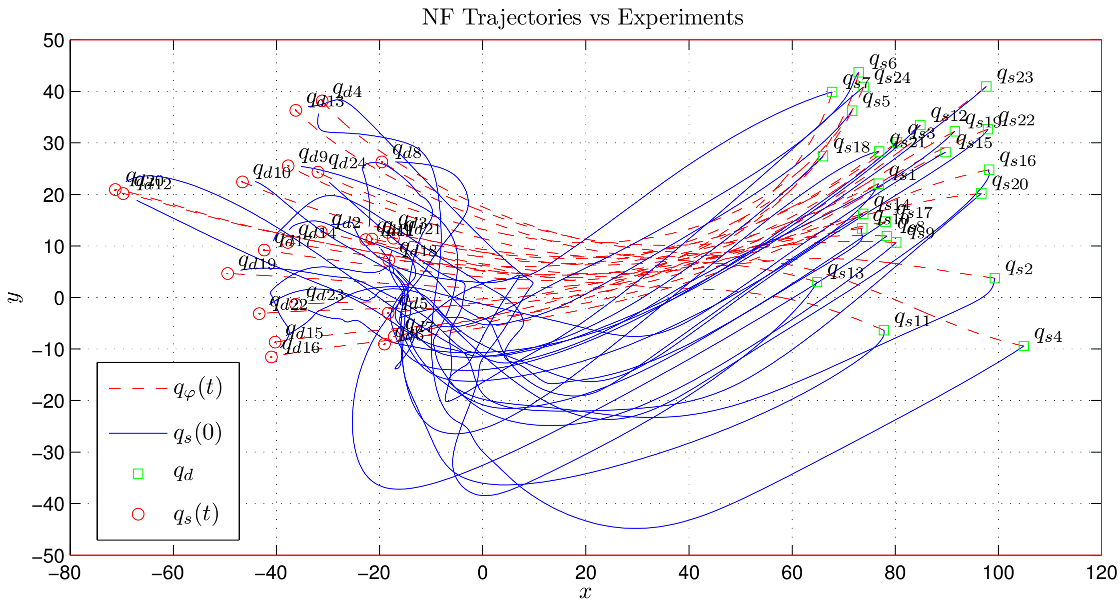


Figure 10.5: Learning NFs with $n = 2$ principal components produce equally good results, compare with Fig. 10.2, with which the legend is the same.

and has been selected by padding the minimum and maximum coordinate dimensions from all experiments.

The β found have been used in a NF $\hat{\varphi}$ and the resulting scalar navigation field is provided in Fig. 10.30 to Fig. 10.32.

Assuming a first order (holonomic) system $u = -\nabla\hat{\varphi}$, the NF of the previous subsection have been used as controllers to guide the system from the same initial states $x_i(0)$ as the corresponding experiments, to the same desired destinations q_{di}. The results are quite encouraging and presented in Fig. 10.33a to Fig. 10.35d.

In the majority of the experiments, individual fitting yields a potential field which successfully navigates from the same initial condition to the same desired destination as the corresponding experiment. Moreover, in many cases the "pattern" is exceptionally close to the experimental one, as can be seen in Fig. 10.33a, Fig. 10.33c, Fig. 10.33d, Fig. 10.33e (the initial linear segments of J are due to the offset β_t), Fig. 10.33f, Fig. 10.33g, Fig. 10.33i, Fig. 10.33j, Fig. 10.34a, Fig. 10.34b, Fig. 10.34d, Fig. 10.34e, Fig. 10.34h, Fig. 10.34i, Fig. 10.34j, Fig. 10.35a (quite good), Fig. 10.35b, Fig. 10.35d (again quite acceptable).

It can be observed that in cases where tight turns arise, the resulting field either follows a shortcut, as in Fig. 10.33h, Fig. 10.34c, Fig. 10.35a, Fig. 10.35d, or, in other cases, local minima arise, as in Fig. 10.33b, Fig. 10.33i, Fig. 10.34f, Fig. 10.34g, Fig. 10.35c. On the contrary, using φ instead of φ, avoids local minima.

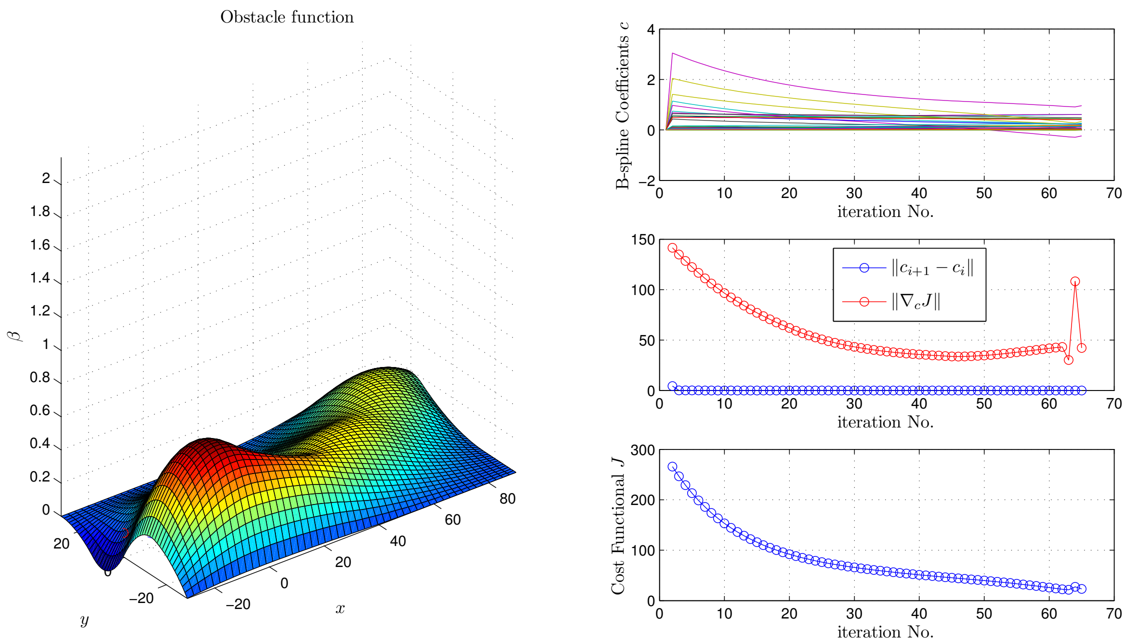


Figure 10.6: Spline fitting optimization: Experiment No.1.

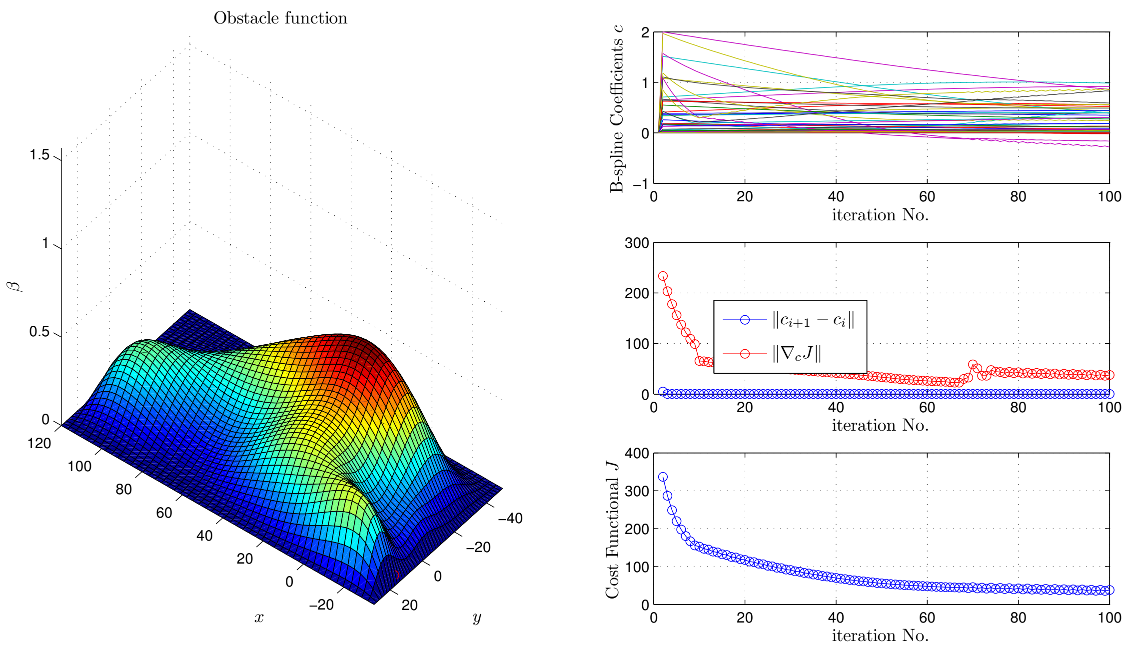


Figure 10.7: Spline fitting optimization: Experiment No.2.

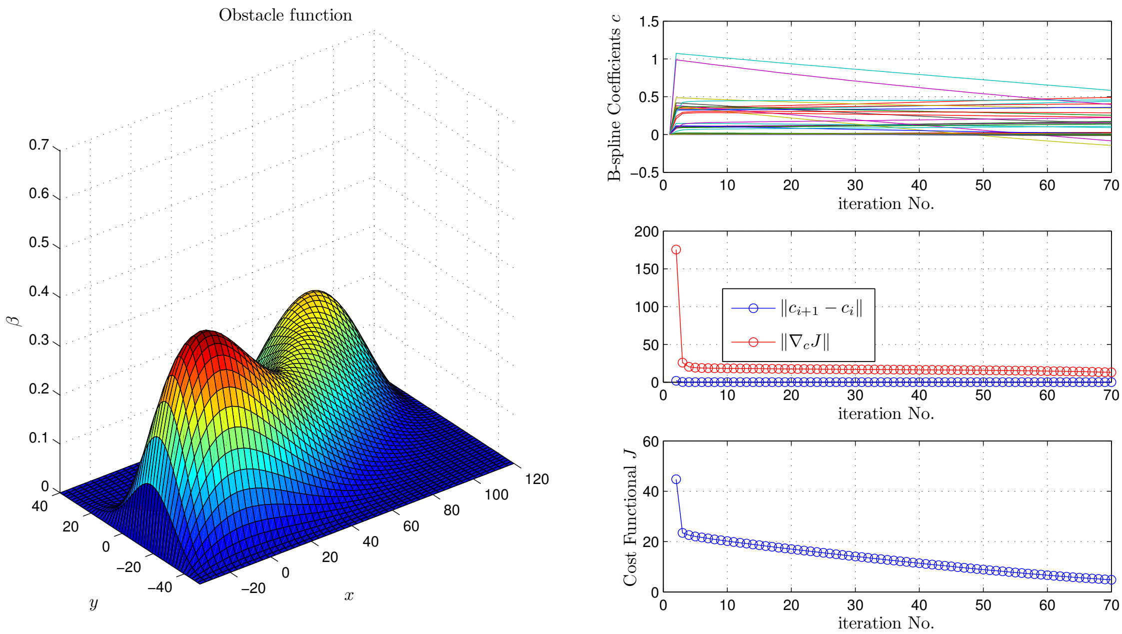


Figure 10.8: Spline fitting optimization: Experiment No.3.

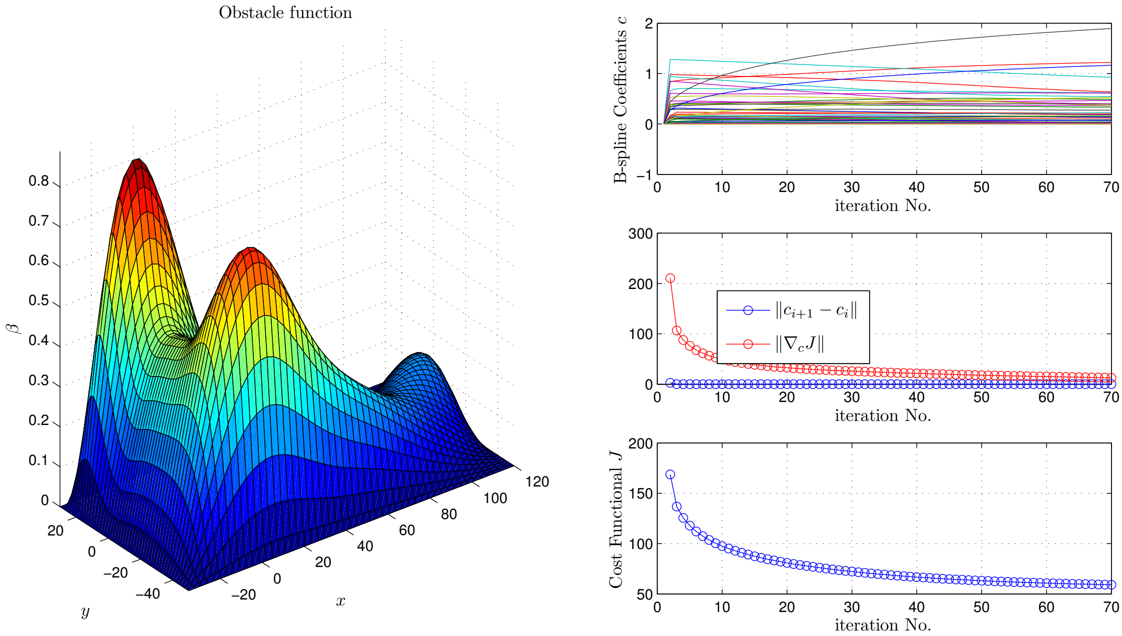


Figure 10.9: Spline fitting optimization: Experiment No.4.

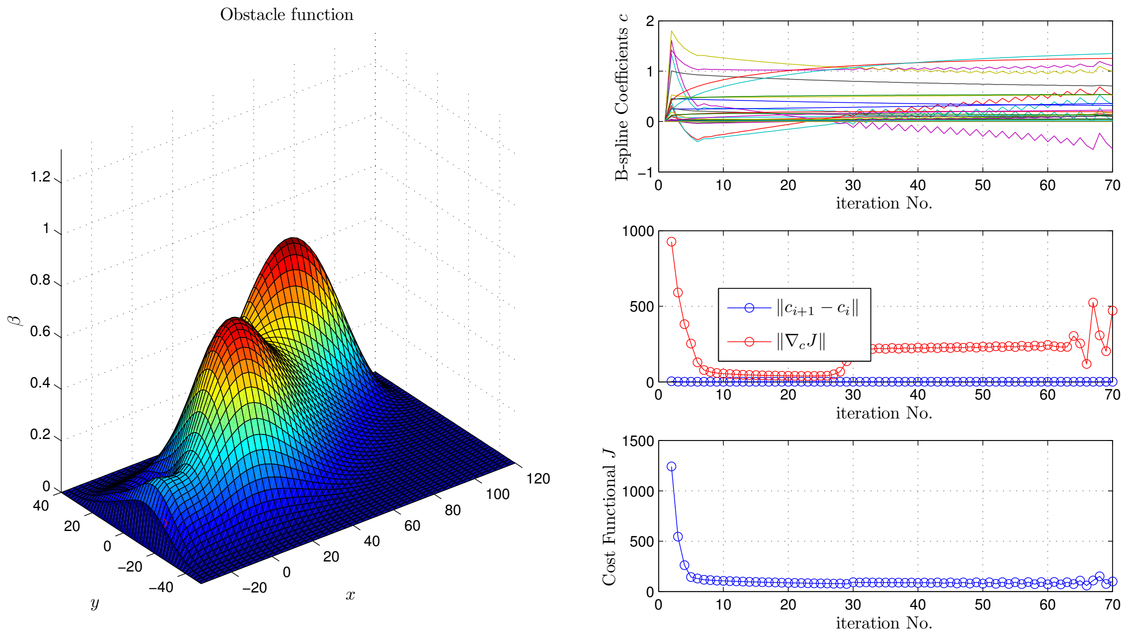


Figure 10.10: Spline fitting optimization: Experiment No.5.

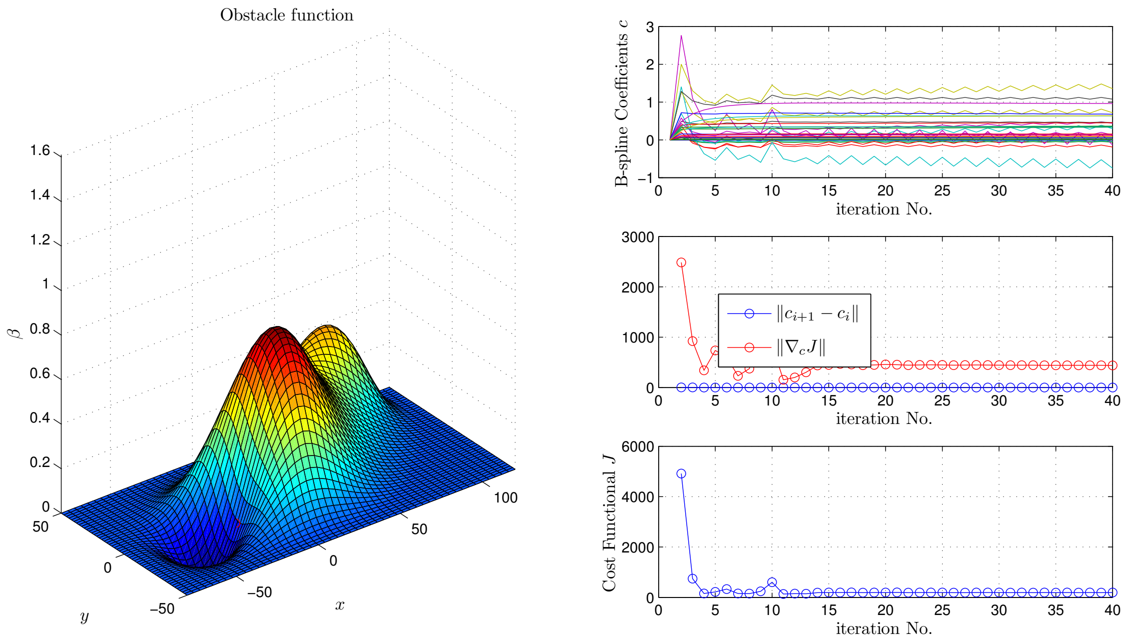


Figure 10.11: Spline fitting optimization: Experiment No.6.

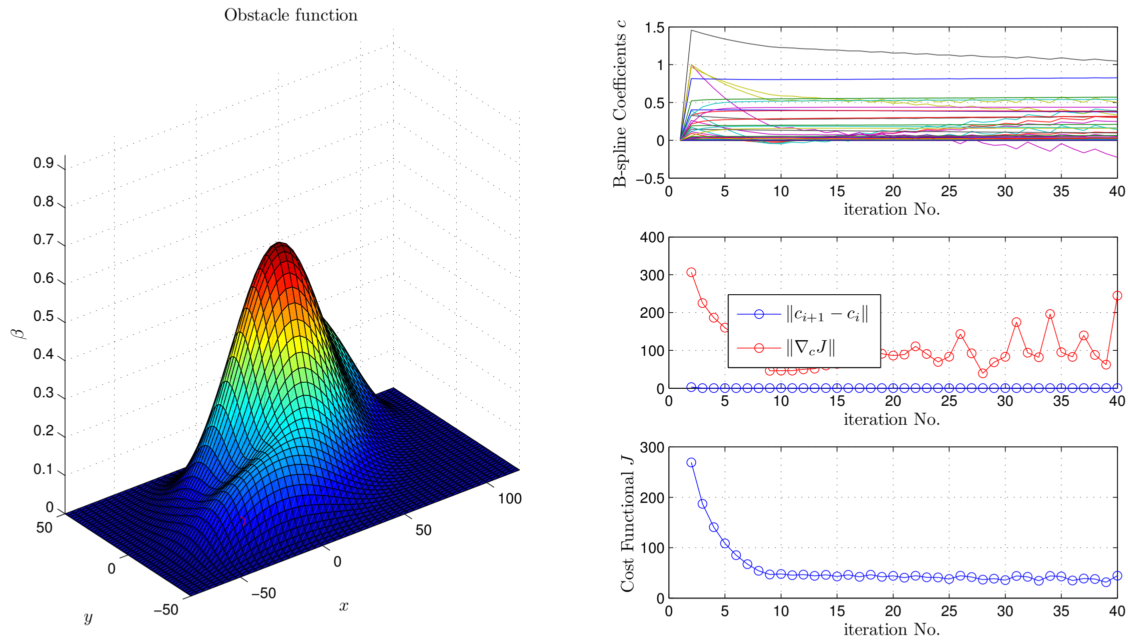


Figure 10.12: Spline fitting optimization: Experiment No.7.

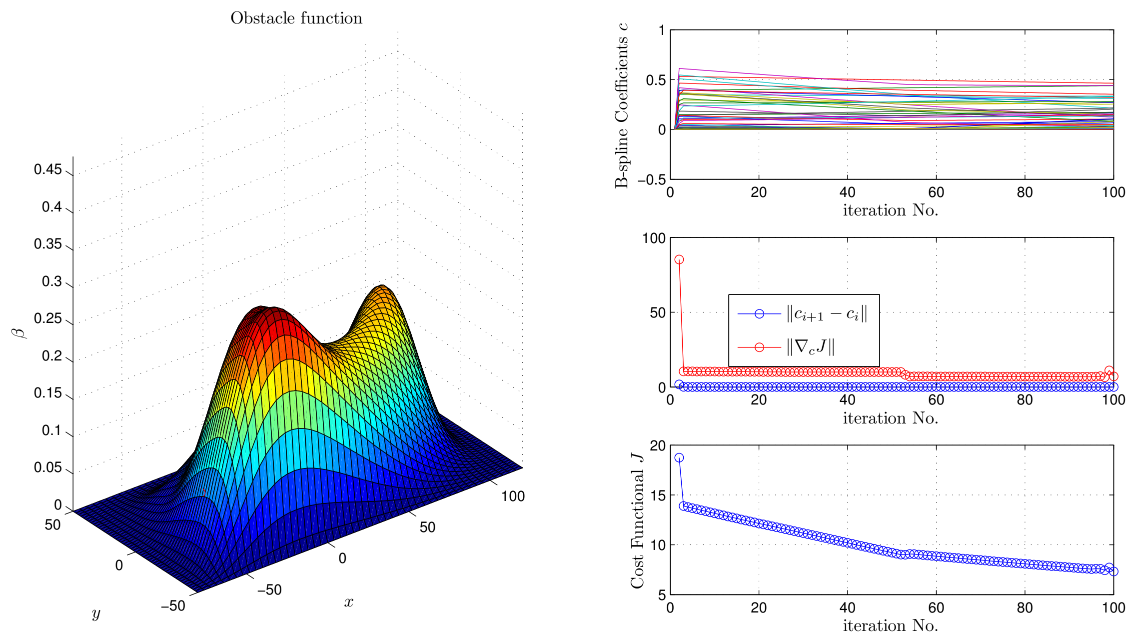


Figure 10.13: Spline fitting optimization: Experiment No.8.

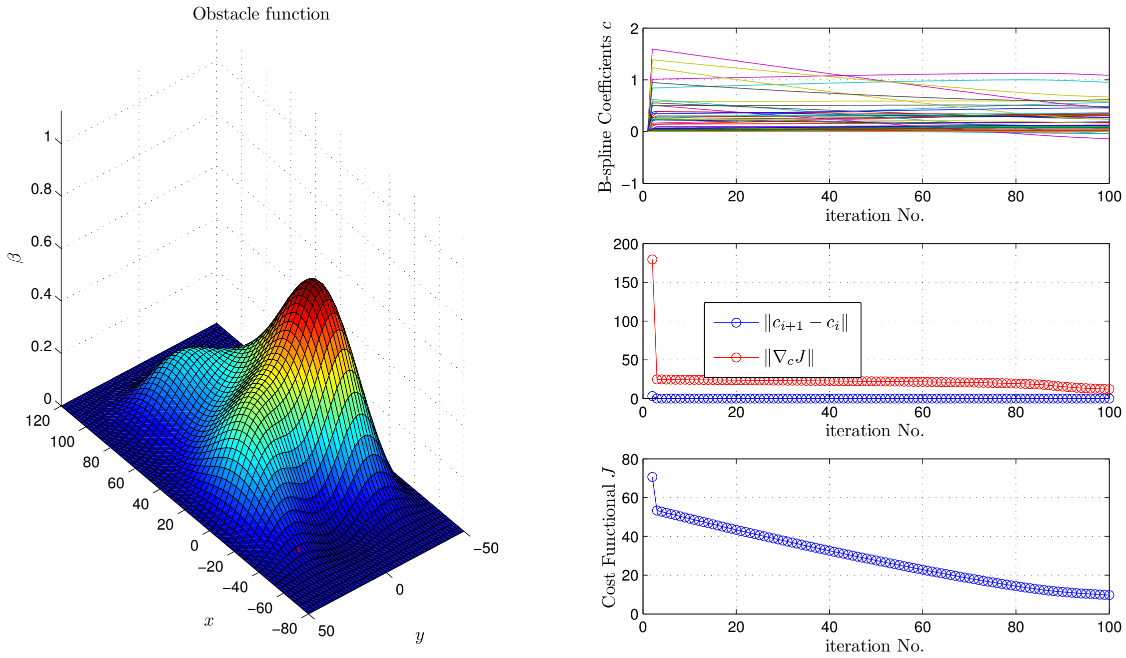


Figure 10.14: Spline fitting optimization: Experiment No.9.

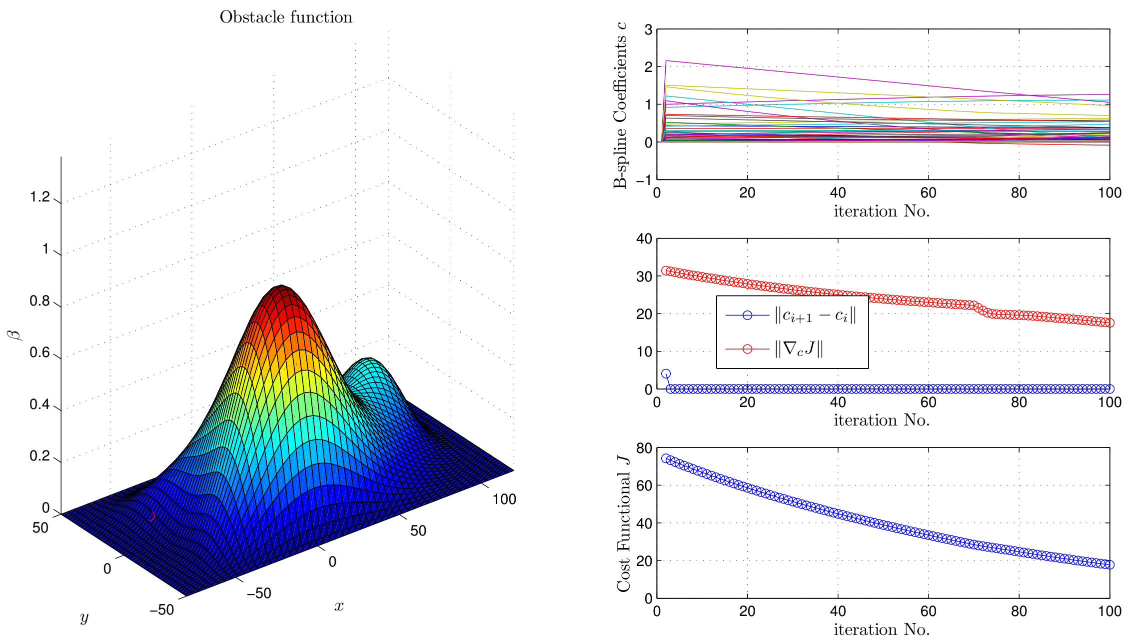


Figure 10.15: Spline fitting optimization: Experiment No.10.

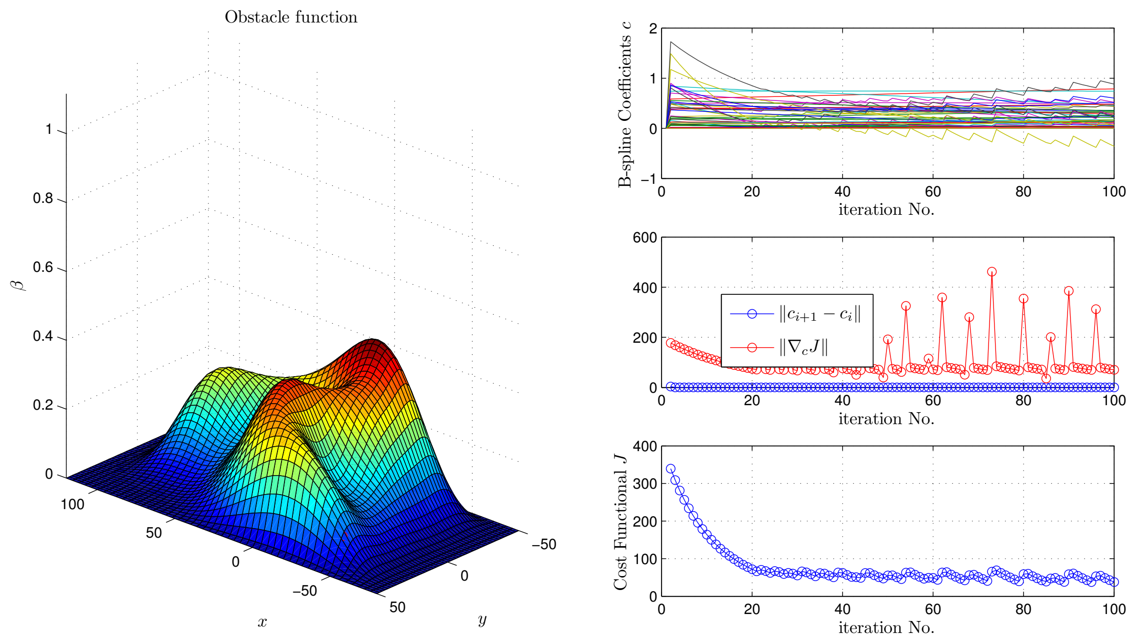


Figure 10.16: Spline fitting optimization: Experiment No.11.

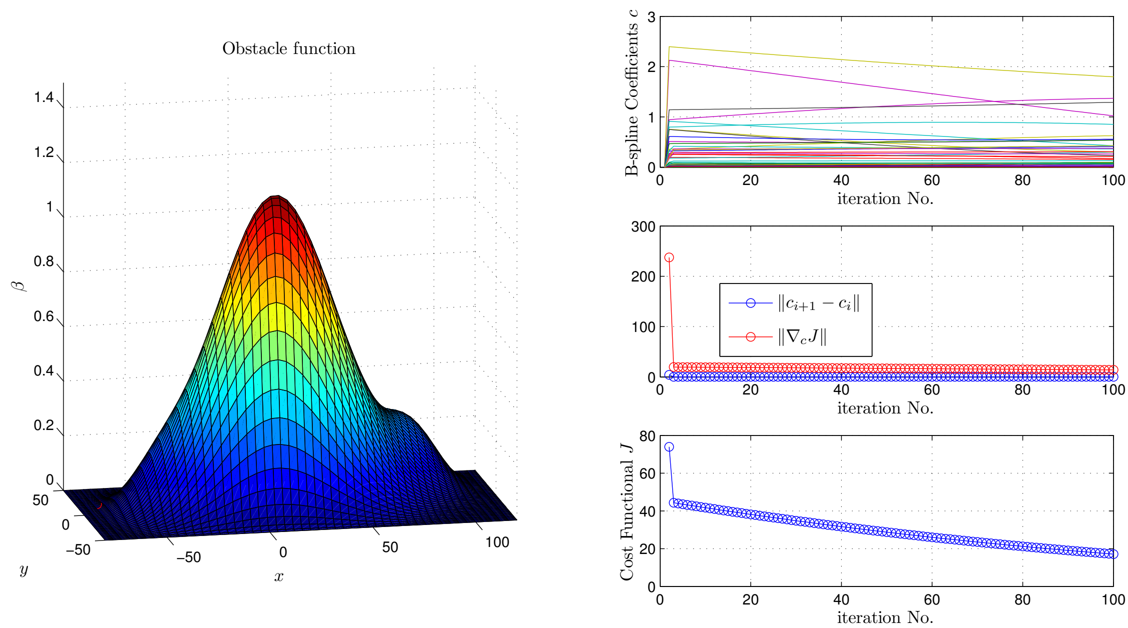


Figure 10.17: Spline fitting optimization: Experiment No.12.

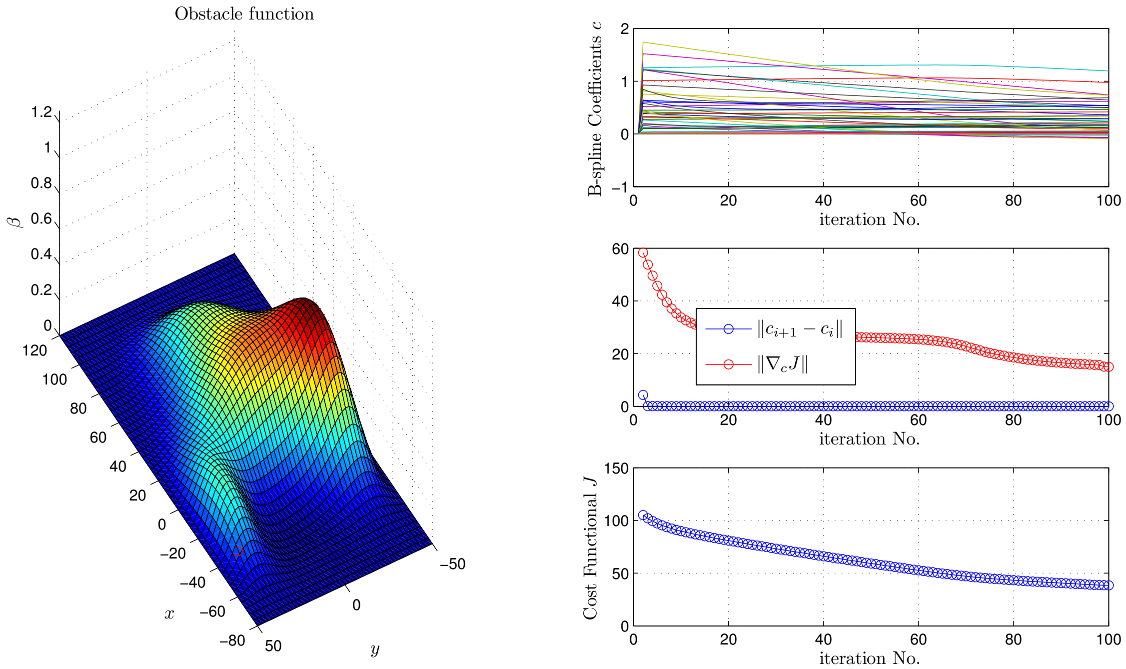


Figure 10.18: Spline fitting optimization: Experiment No.13.

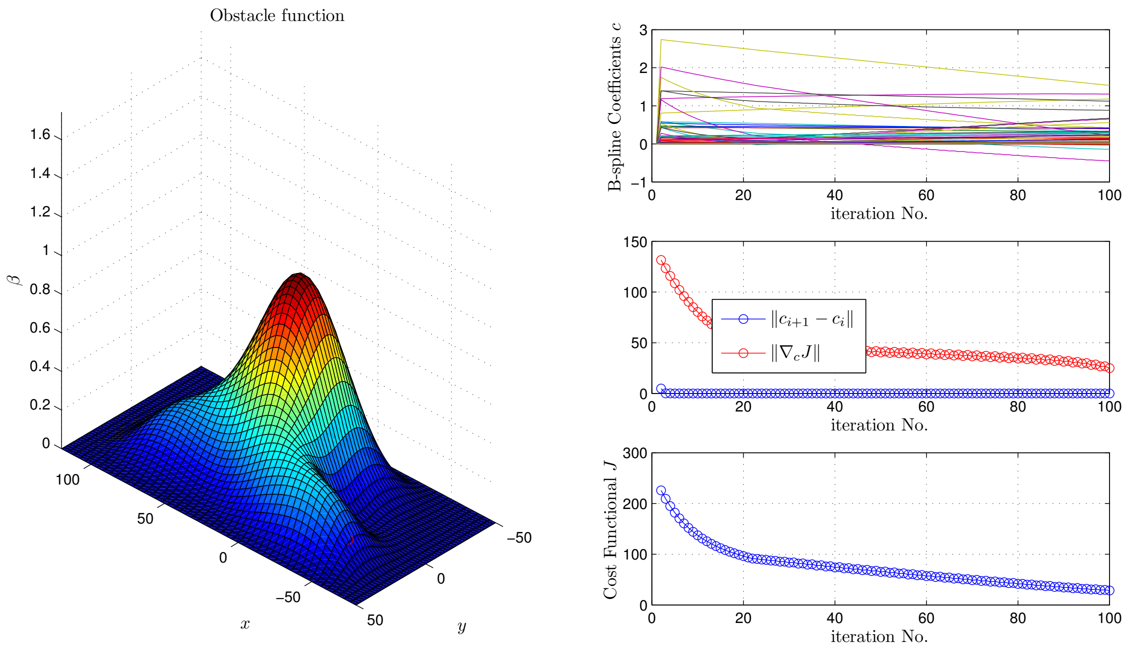


Figure 10.19: Spline fitting optimization: Experiment No.14.

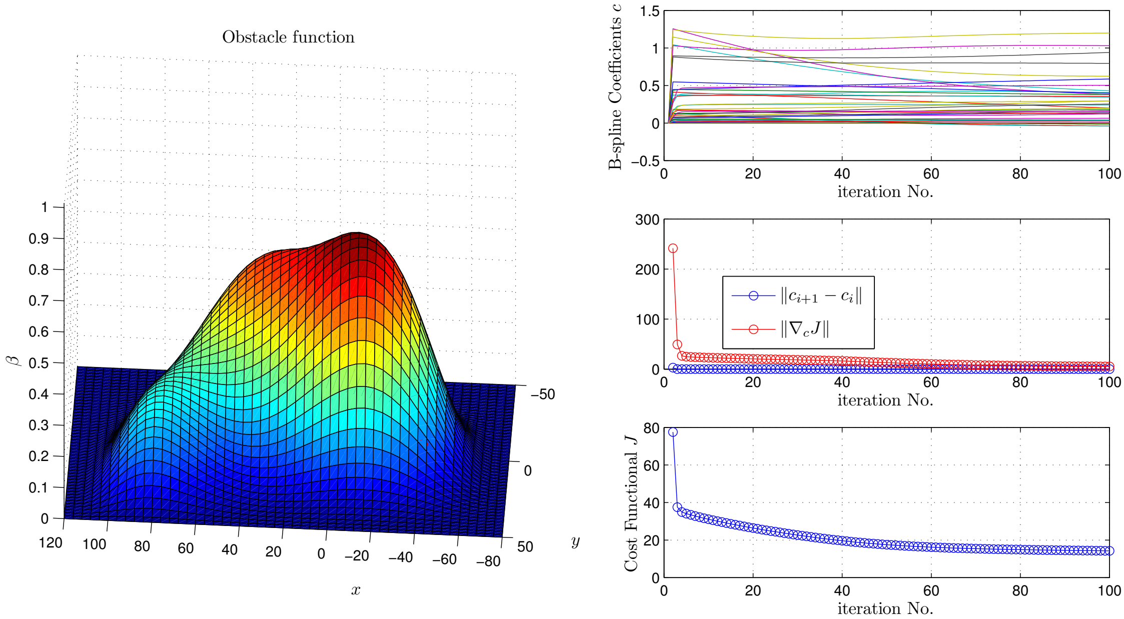


Figure 10.20: Spline fitting optimization: Experiment No.15.

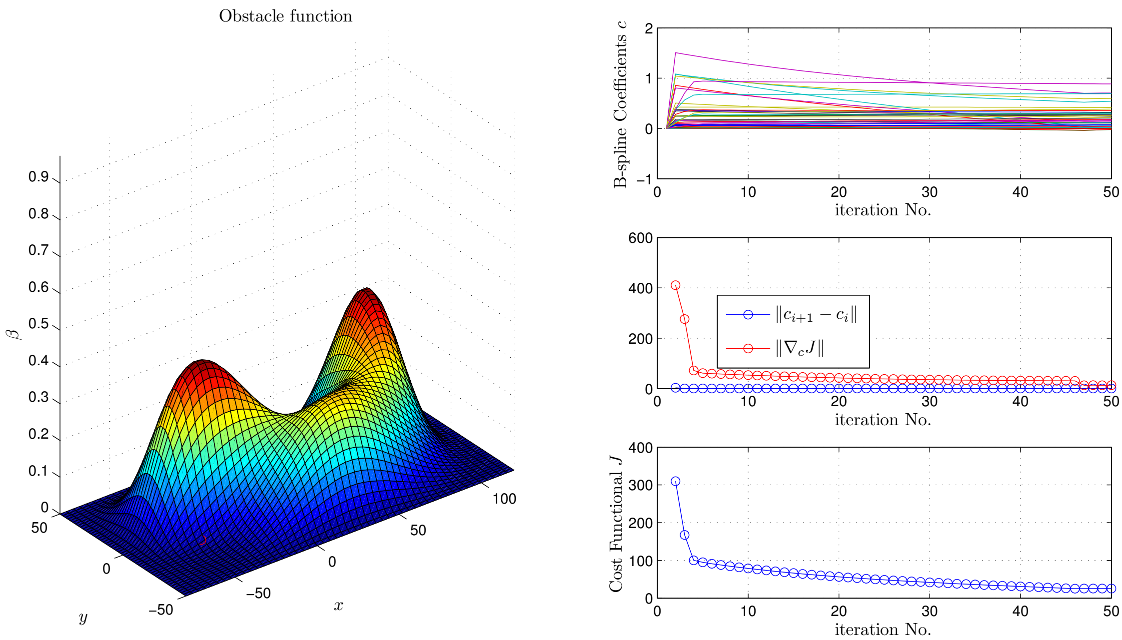


Figure 10.21: Spline fitting optimization: Experiment No.16.

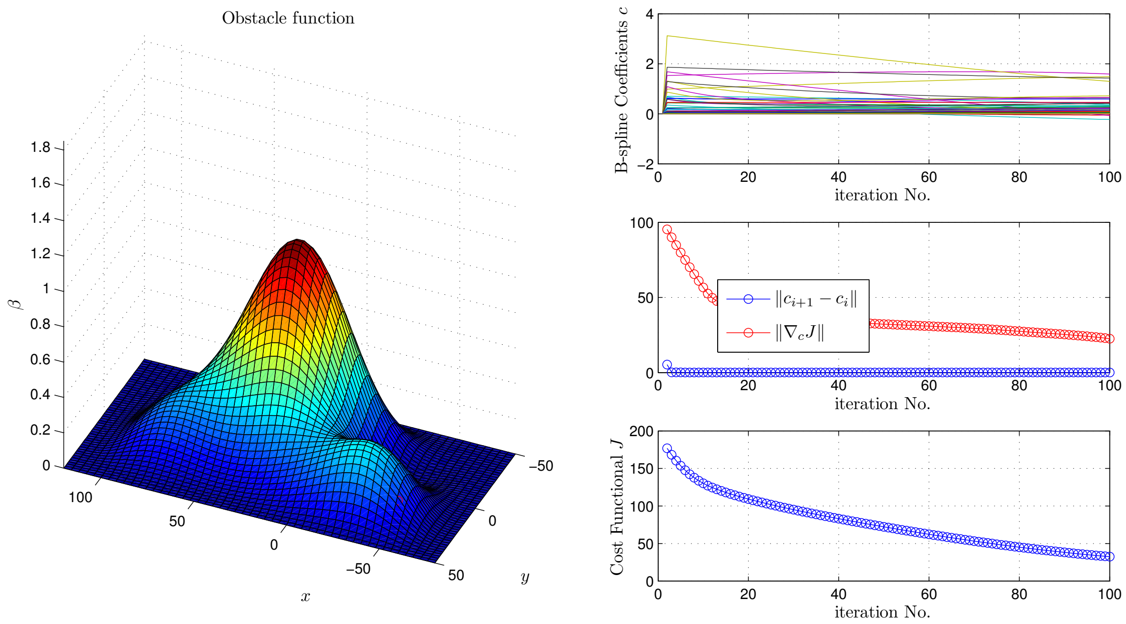


Figure 10.22: Spline fitting optimization: Experiment No.17.

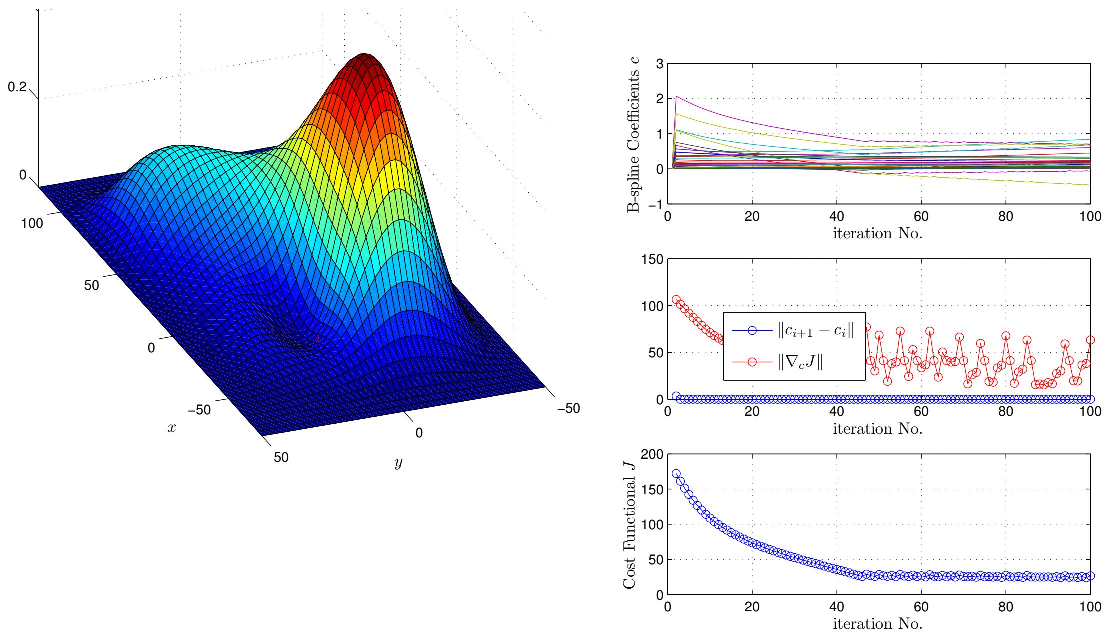


Figure 10.23: Spline fitting optimization: Experiment No.18.

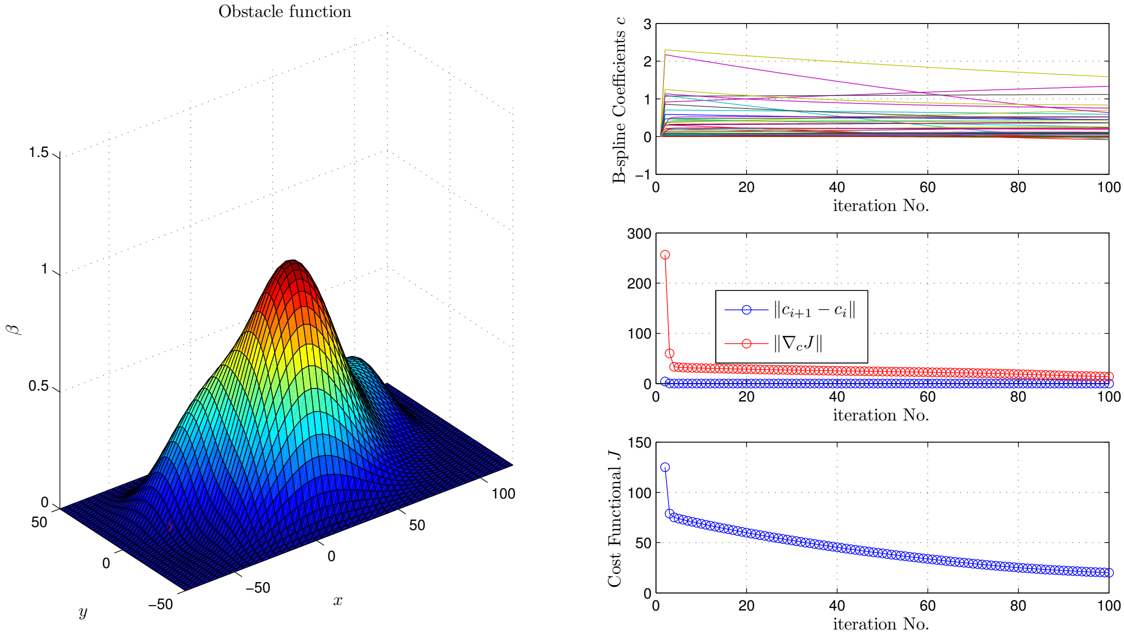


Figure 10.24: Spline fitting optimization: Experiment No.19.

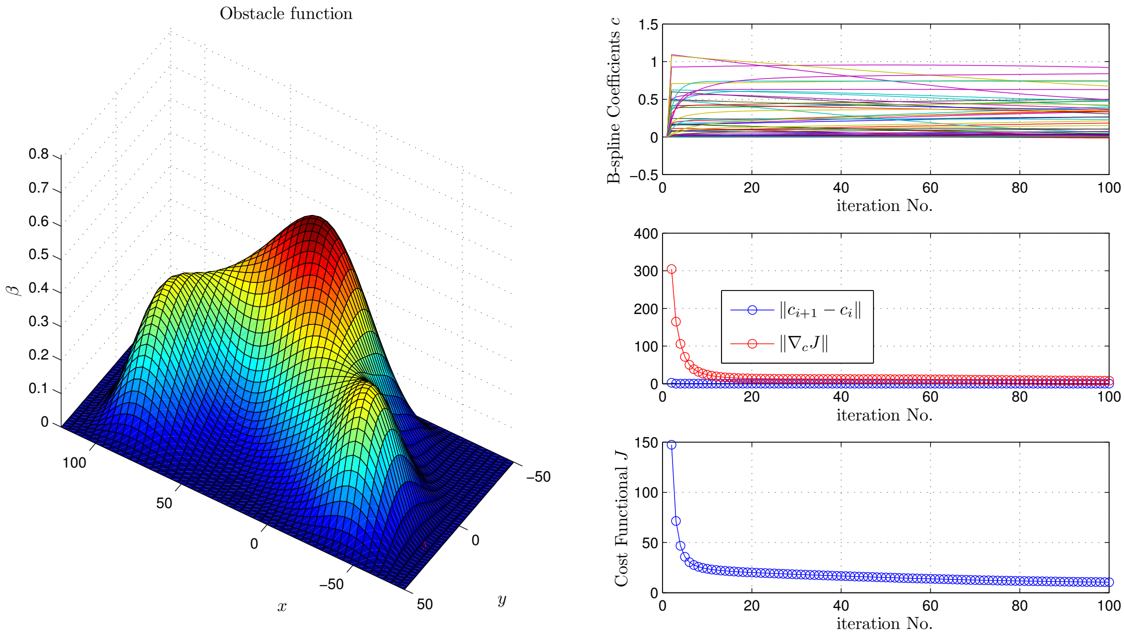


Figure 10.25: Spline fitting optimization: Experiment No.20.

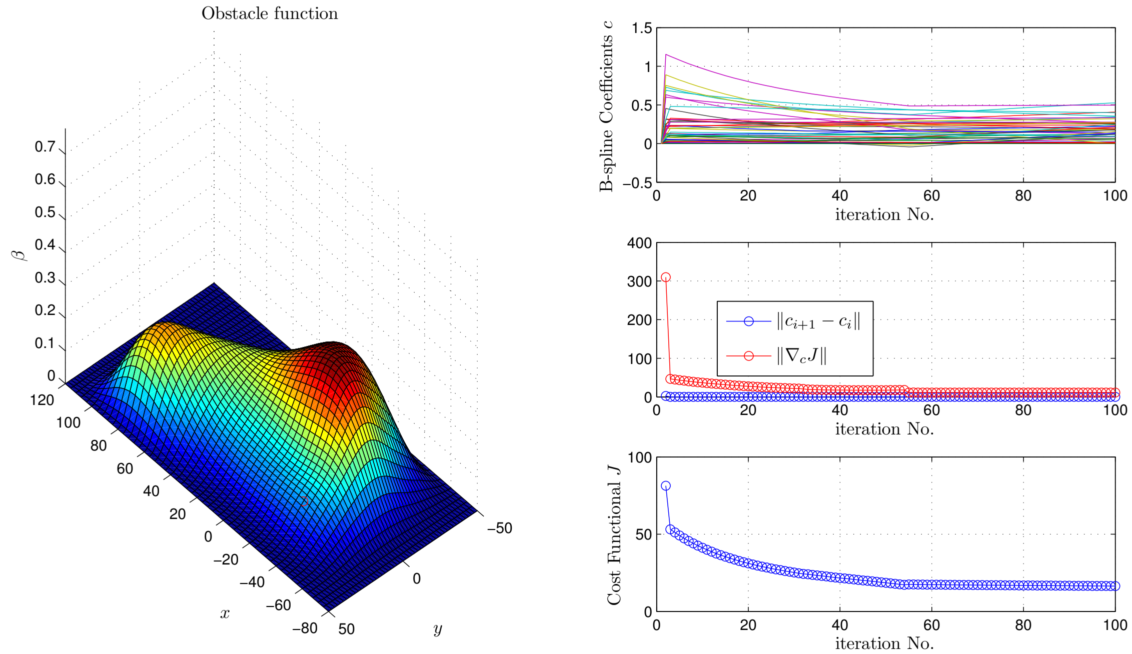


Figure 10.26: Spline fitting optimization: Experiment No.21.

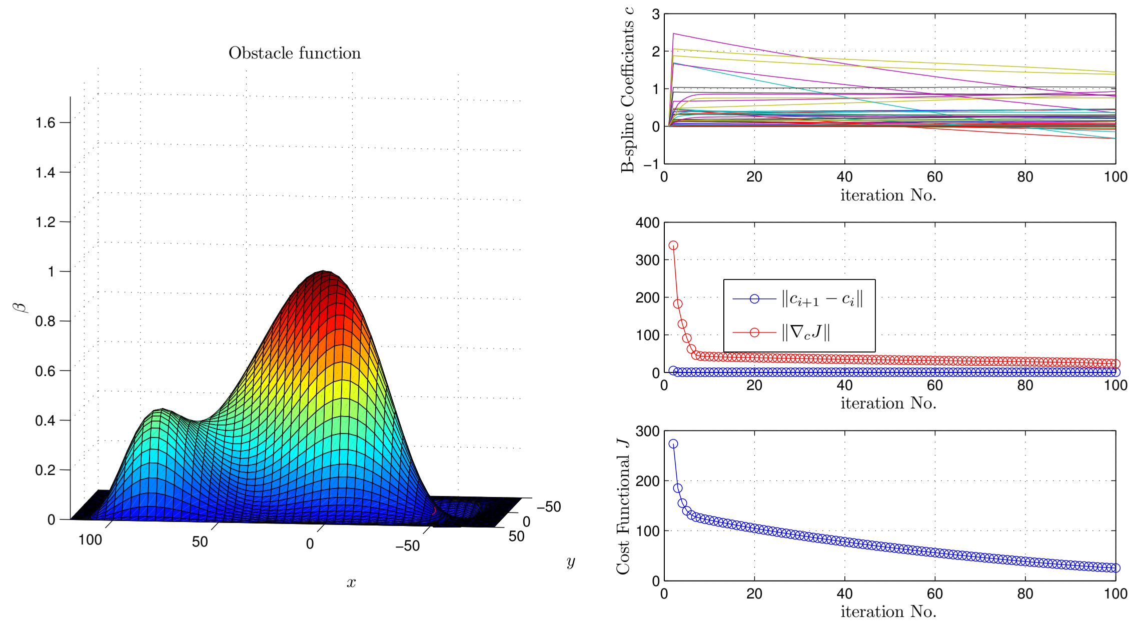


Figure 10.27: Spline fitting optimization: Experiment No.22.

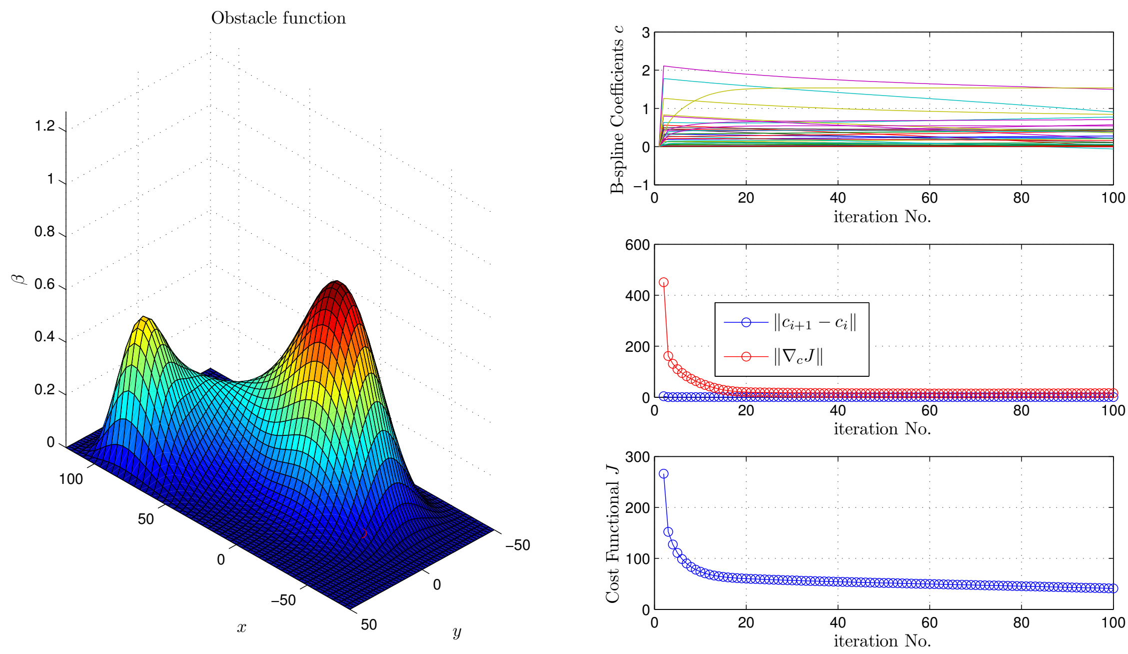


Figure 10.28: Spline fitting optimization: Experiment No.23.

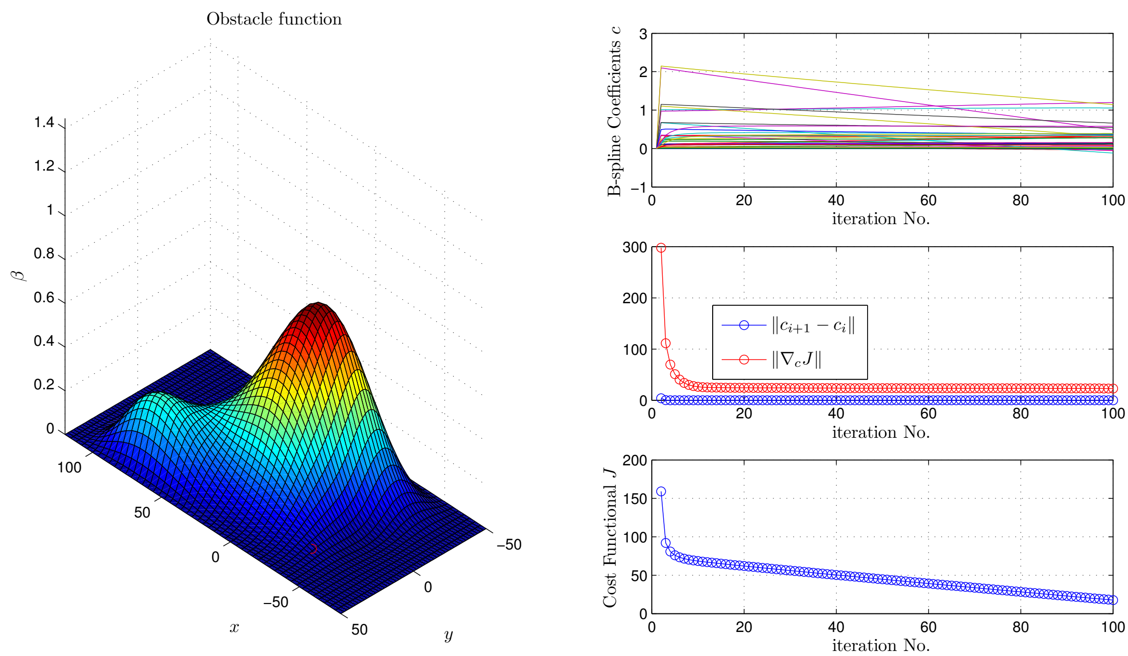
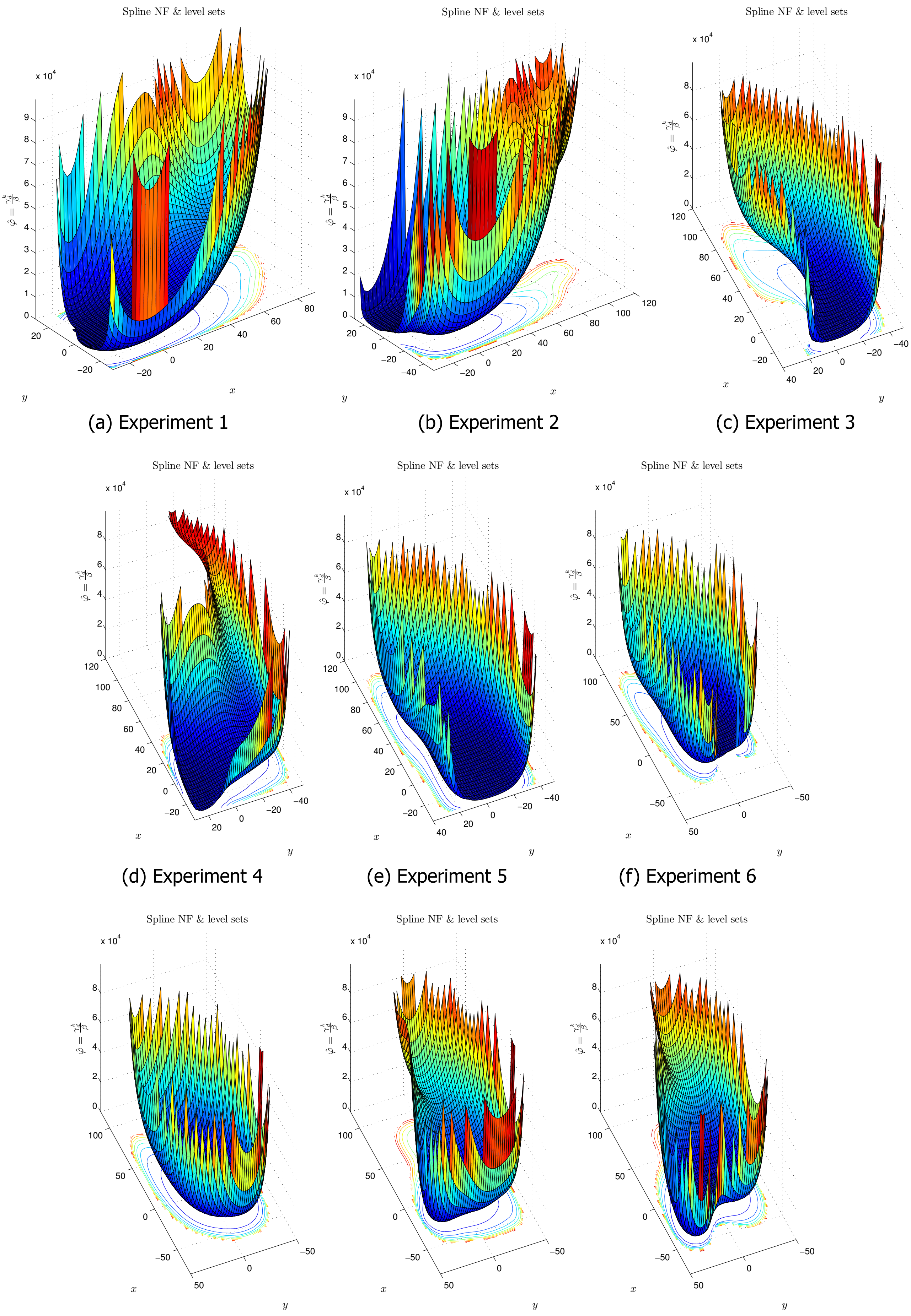


Figure 10.29: Spline fitting optimization: Experiment No.24.

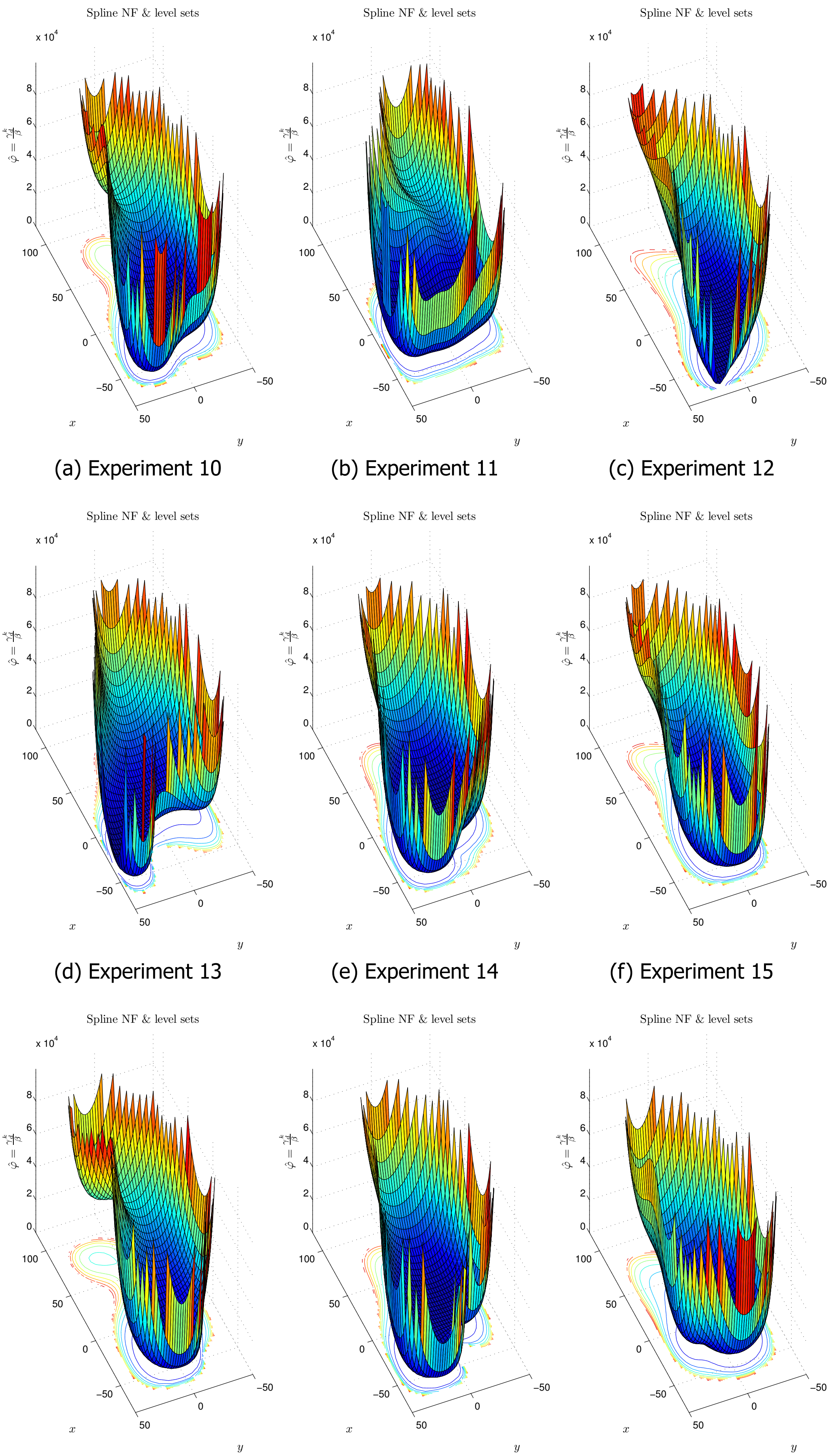


(a) Experiment 1 (b) Experiment 2 (c) Experiment 3

(d) Experiment 4 (e) Experiment 5 (f) Experiment 6

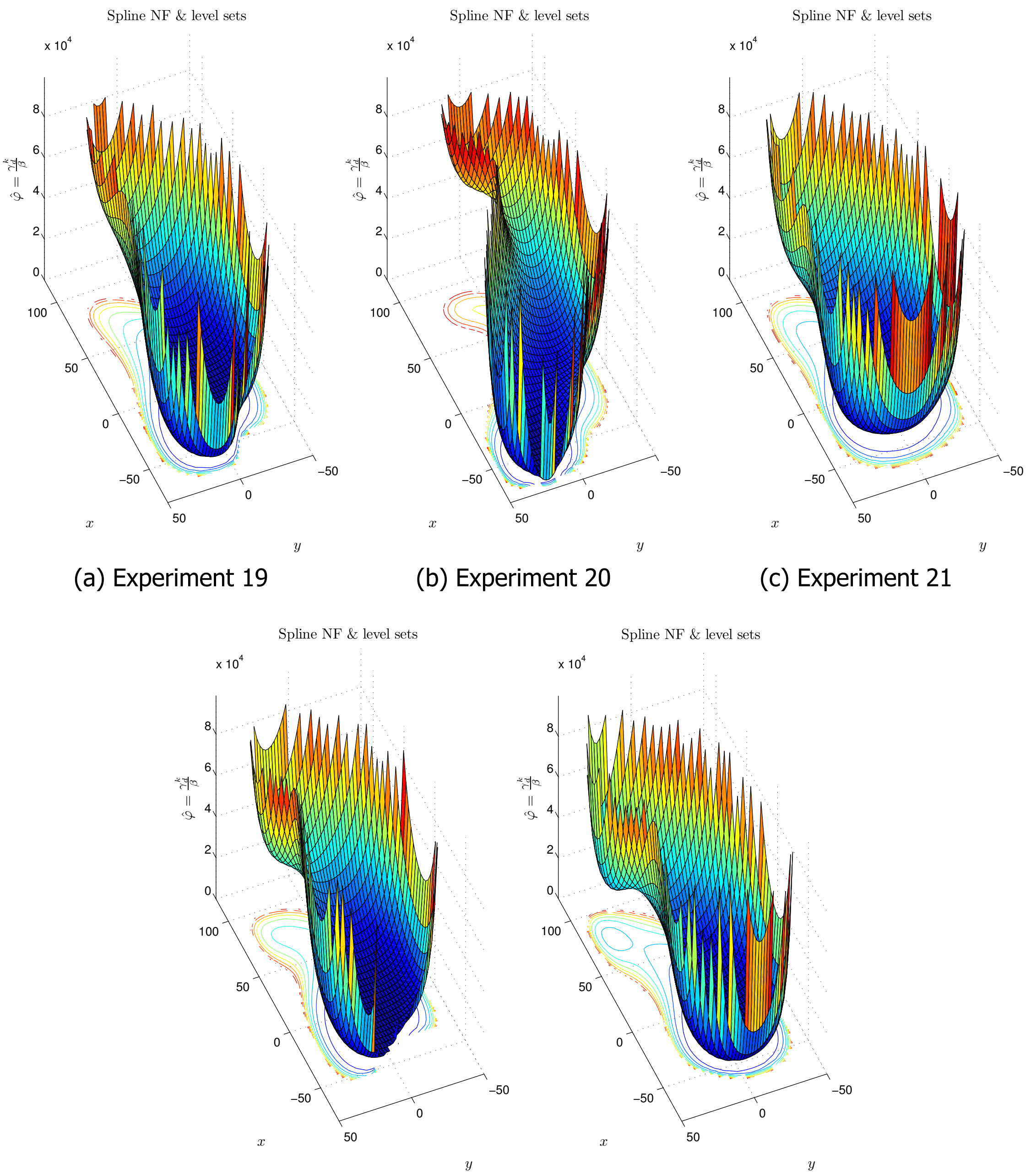
(g) Experiment 7 (h) Experiment 8 (i) Experiment 9

Figure 10.30: Resultant NF in 2-dimensional subspace of first two principal components using $\hat{\varphi}_1$.



(a) Experiment 10
(b) Experiment 11
(c) Experiment 12
(d) Experiment 13
(e) Experiment 14
(f) Experiment 15
(g) Experiment 16
(h) Experiment 17
(i) Experiment 18

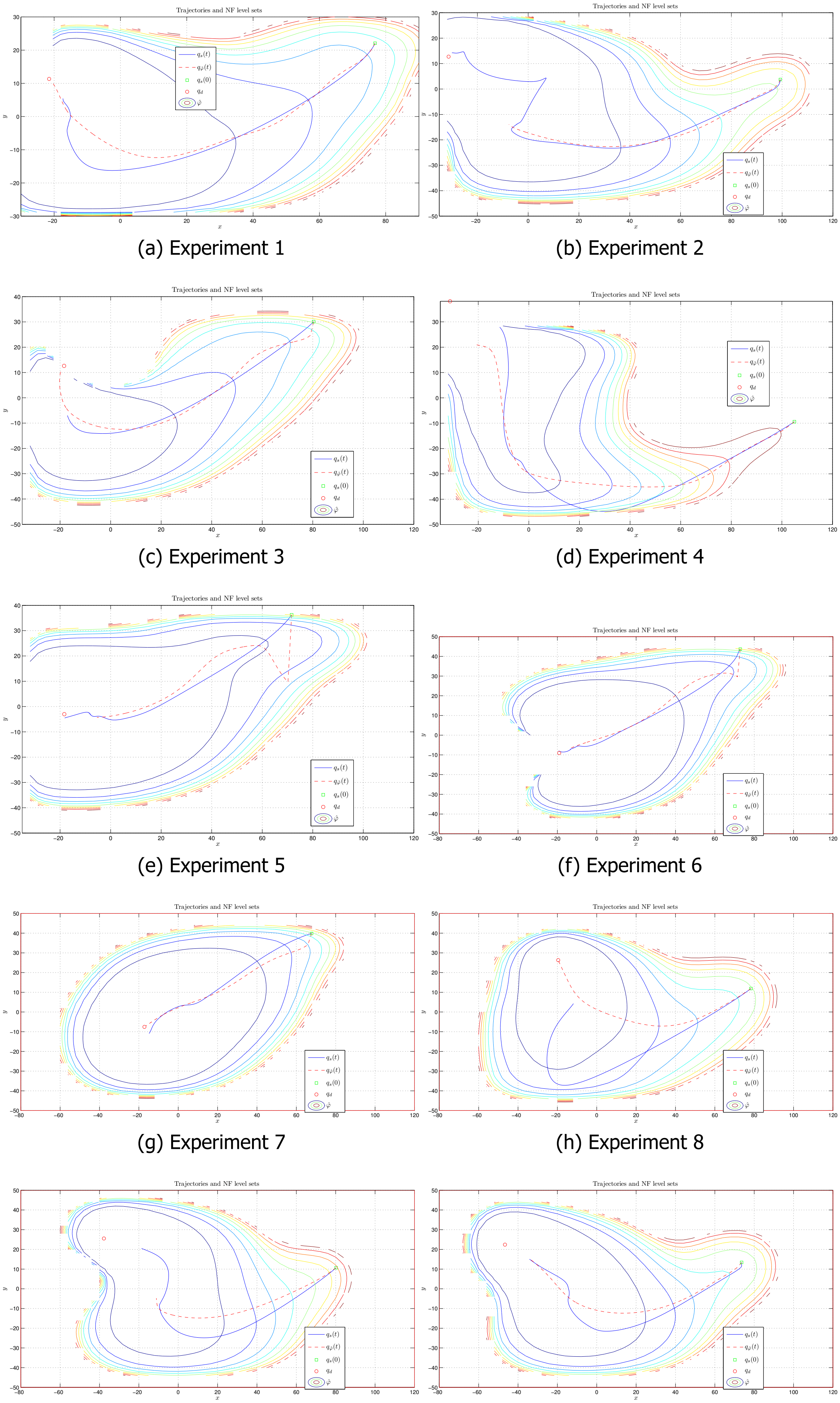
Figure 10.31: Resultant NF in 2-dimensional subspace of first two principal components using $\hat{\varphi}_1$.



(a) Experiment 19 (b) Experiment 20 (c) Experiment 21

(d) Experiment 22 (e) Experiment 23

Figure 10.32: Resultant NF in 2-dimensional subspace of first two principal components using $\hat{\varphi}_1$.



(a) Experiment 1 (b) Experiment 2

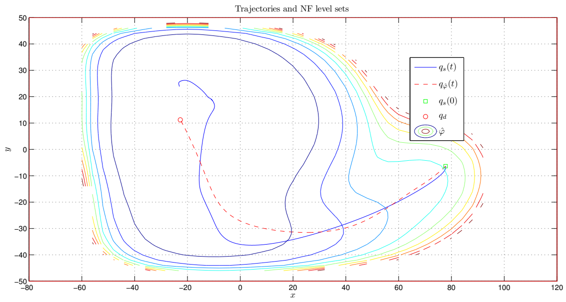
(c) Experiment 3 (d) Experiment 4

(e) Experiment 5 (f) Experiment 6

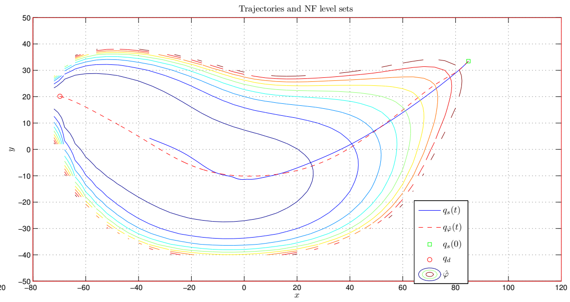
(g) Experiment 7 (h) Experiment 8

(i) Experiment 9 (j) Experiment 10

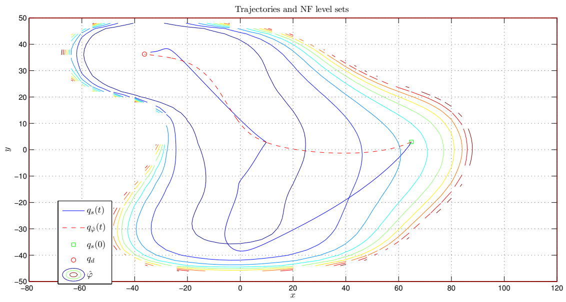
Figure 10.33: Comparison of experimental vs NF trajectories.



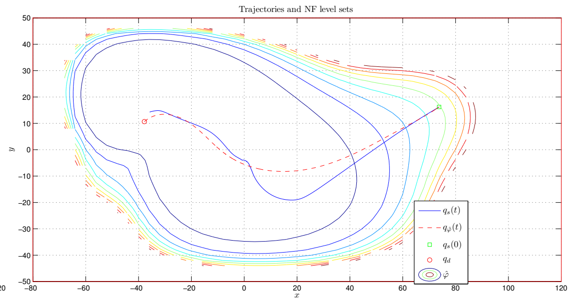
(a) Experiment 11



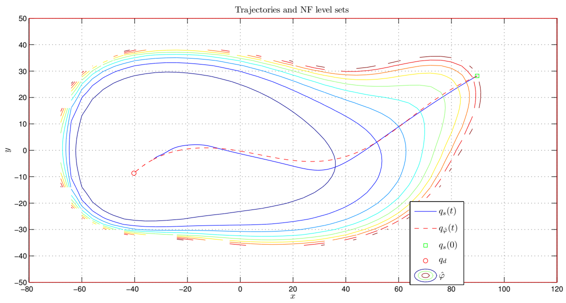
(b) Experiment 12



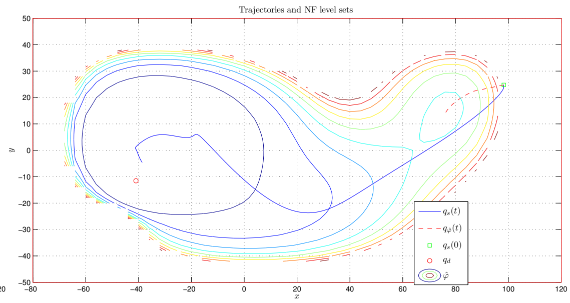
(c) Experiment 13



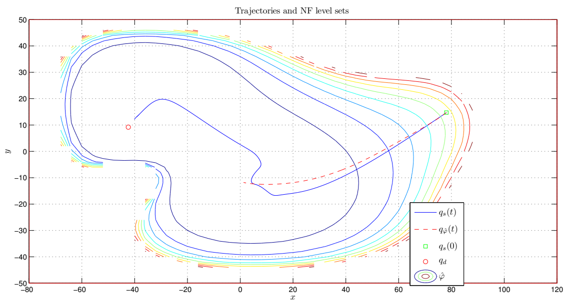
(d) Experiment 14



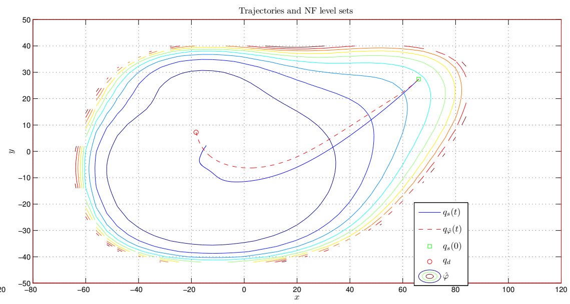
(e) Experiment 15



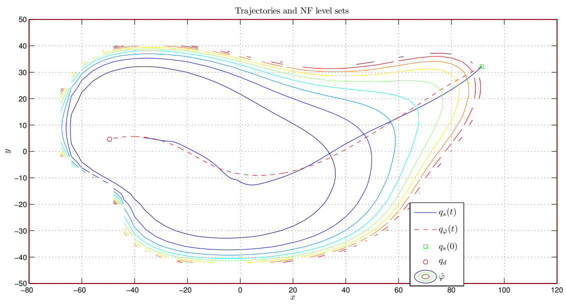
(f) Experiment 16



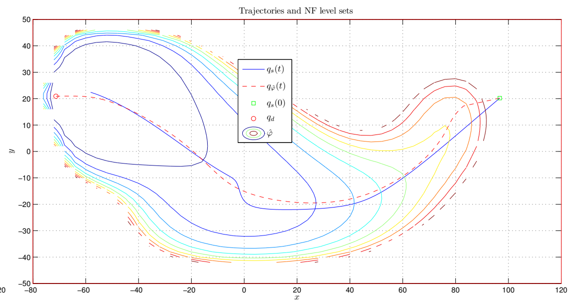
(g) Experiment 17



(h) Experiment 18

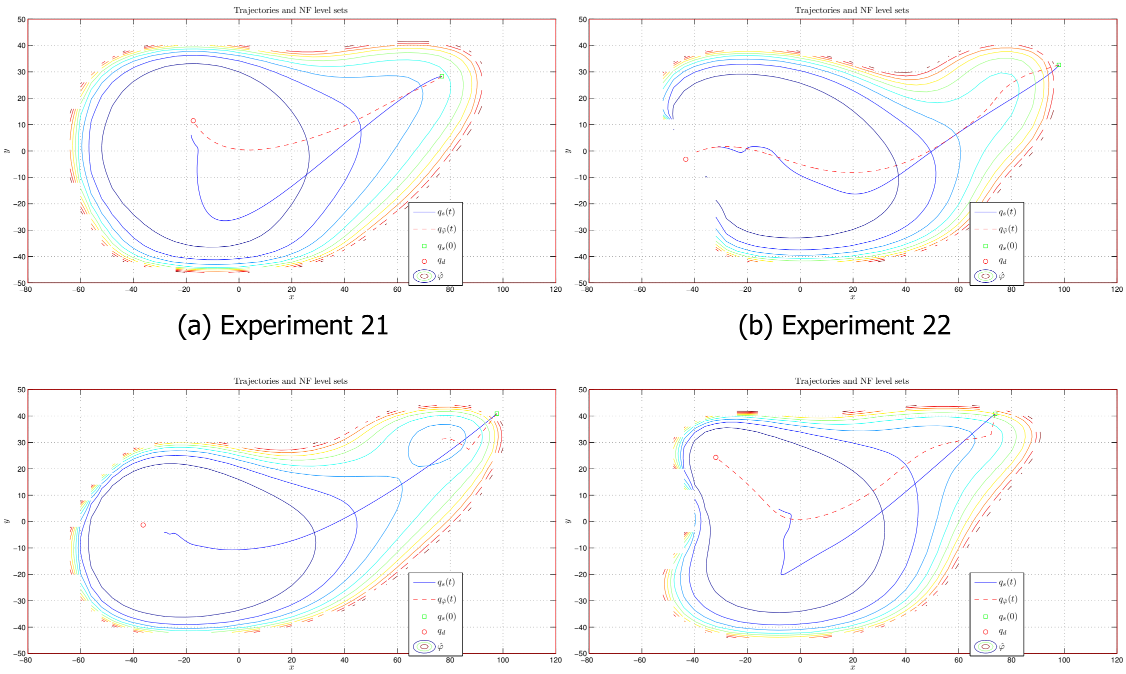


(i) Experiment 19



(j) Experiment 20

Figure 10.34: Comparison of experimental vs NF trajectories.



(a) Experiment 21

(b) Experiment 22

(c) Experiment 23

(d) Experiment 24

Figure 10.35: Comparison of experimental vs NF trajectories.

Chapter 11

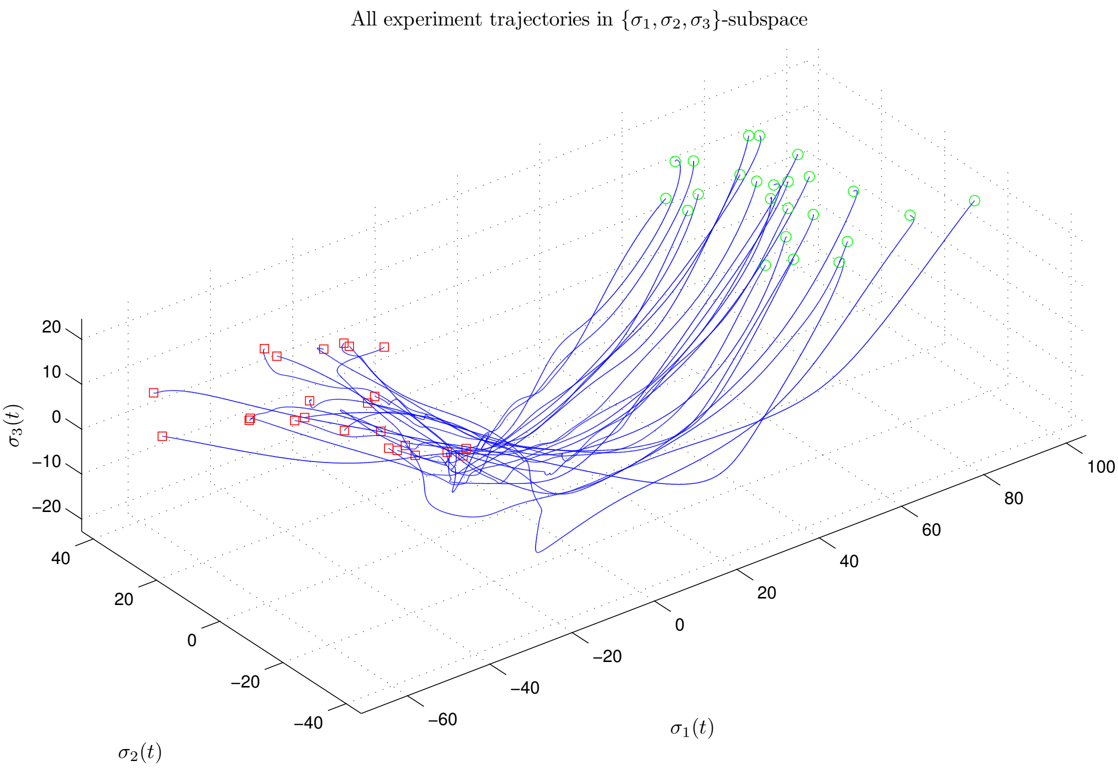
Experiments

11.1 Experimental Data in σ-Space

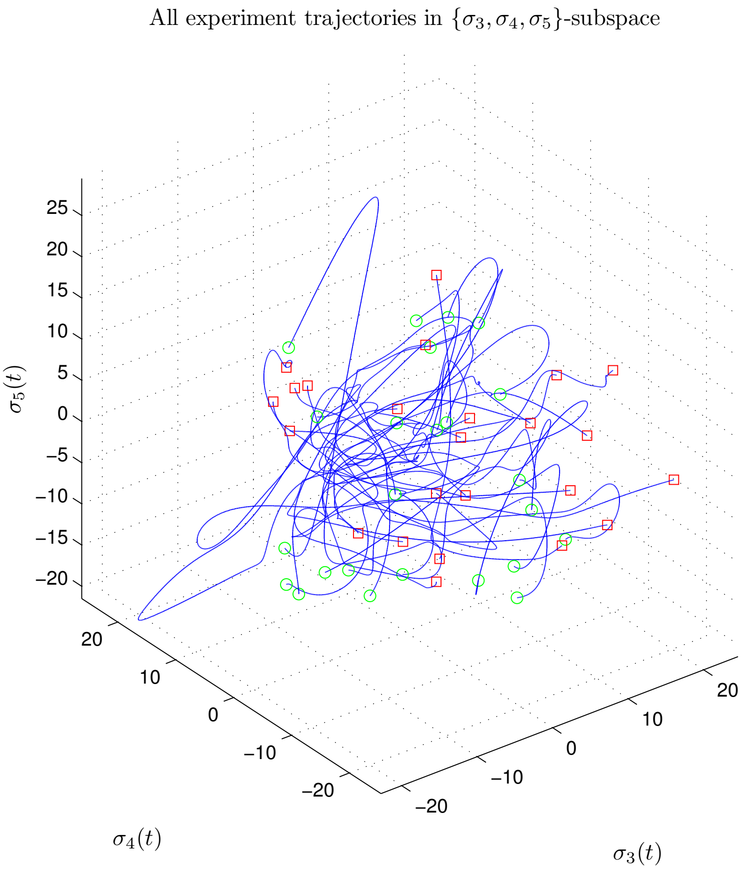
The trajectories in the $\binom{5}{3} = 10$ combinations of 3D subspaces of the 5 principal components have been plotted in Fig. 11.1a to Fig. 11.1b. It is evident that in the first principal components all initial configurations and destinations are in separate neighborhoods and relatively close together. Most travel between the two of them has low path curvature (in the first 3 principal components).

The trajectories in the first 3 principal components are shown in Fig. 11.3a to Fig. 11.5h where the Frenet-Serret frame TNDB has been attached.

The velocity magnitude and path curvature in σ-space are given in Fig. 11.6a to Fig. 11.8h. To put them in the same axes, they have been normalized to their average values (time averages per experiment) and to avoid numerically arising extremities in curvature close to the destination to affect the visible range, the axes are scaled accordingly (so you cannot see curvatures > 4 times the average).

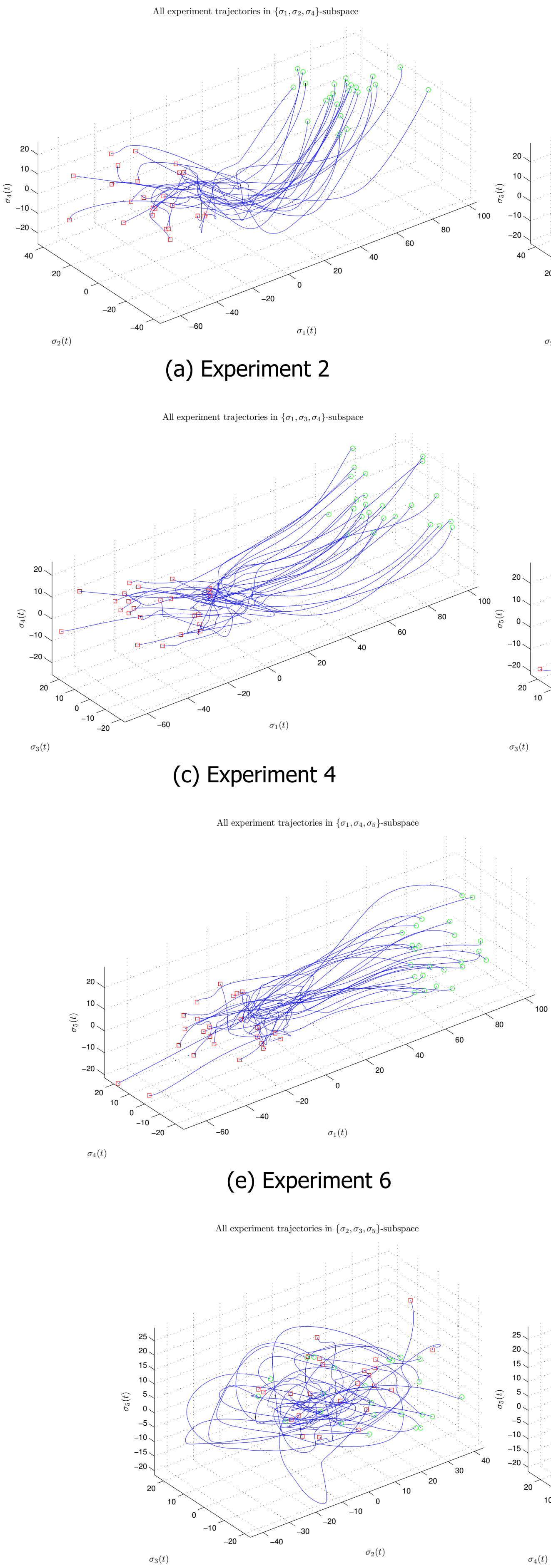


(a) Experiment 1



(b) Experiment 10

Figure 11.1: Experimental trajectories in various different principal subspaces.



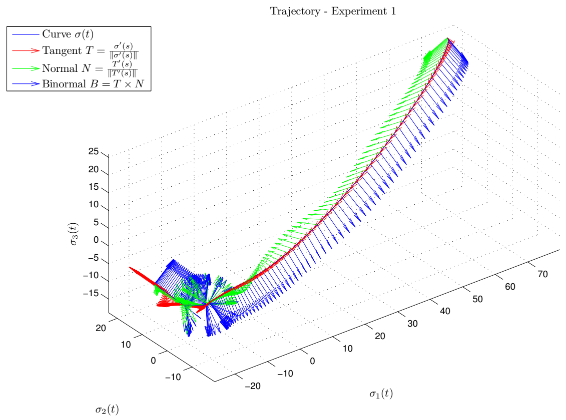
(a) Experiment 2 (b) Experiment 3

(c) Experiment 4 (d) Experiment 5

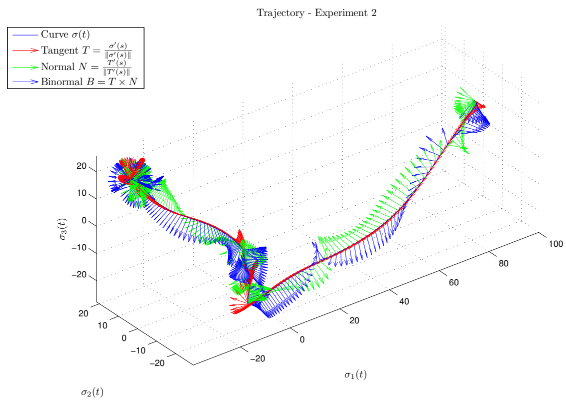
(e) Experiment 6 (f) Experiment 7

(g) Experiment 8 (h) Experiment 9

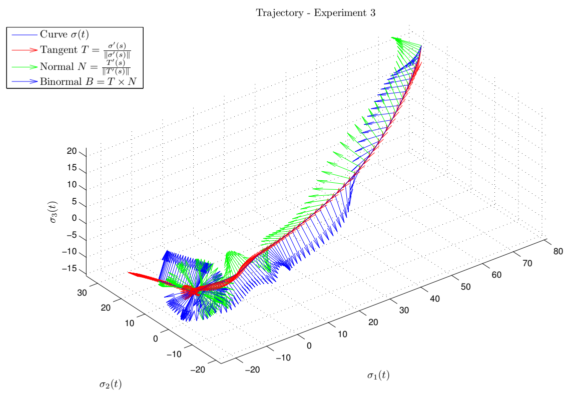
Figure 11.2: Experimental trajectories in various different principal subspaces.



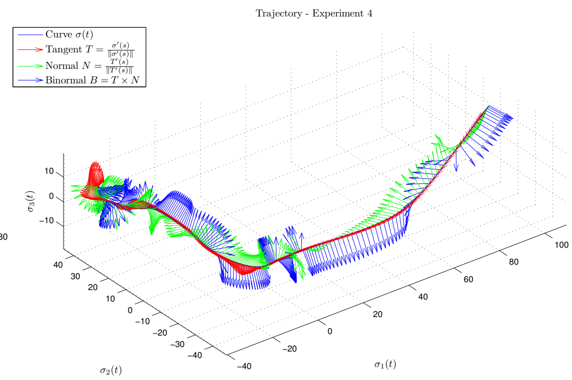
(a) Experiment 1



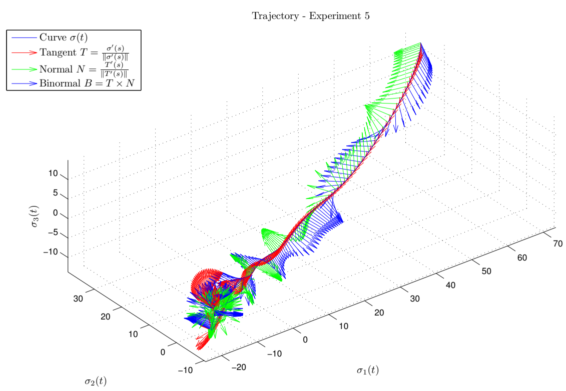
(b) Experiment 2



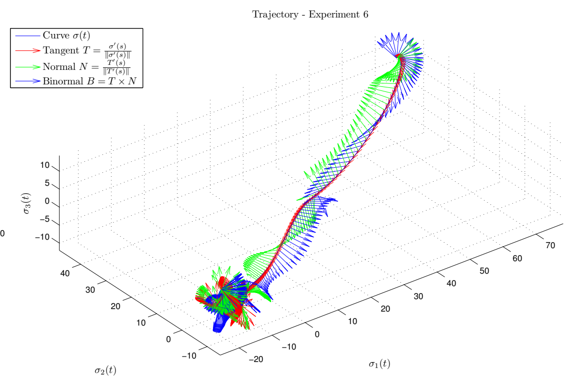
(c) Experiment 3



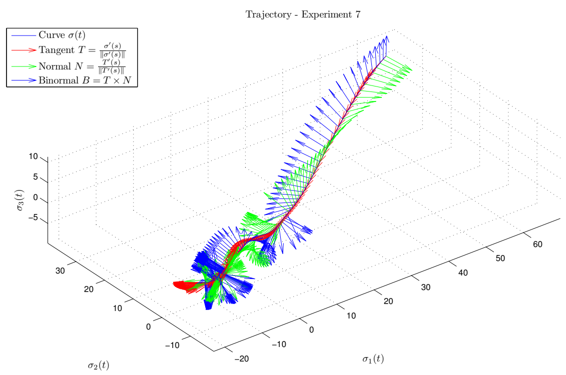
(d) Experiment 4



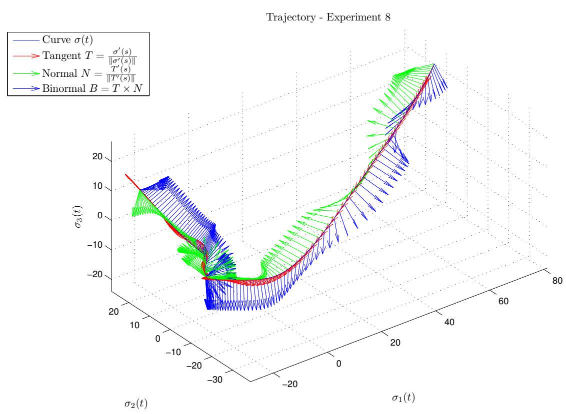
(e) Experiment 5



(f) Experiment 6

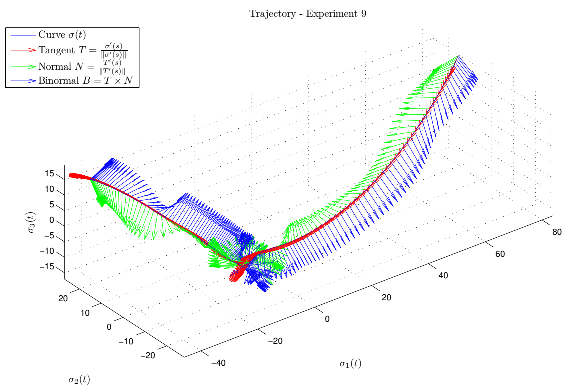


(g) Experiment 7

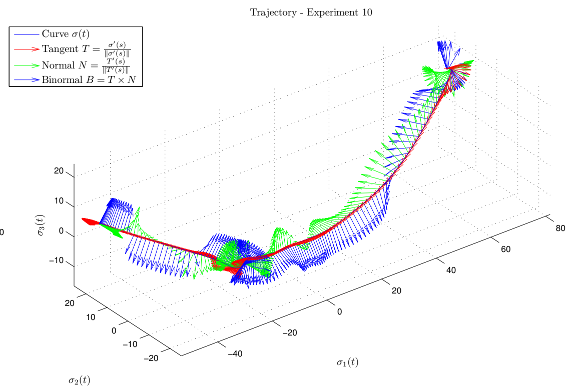


(h) Experiment 8

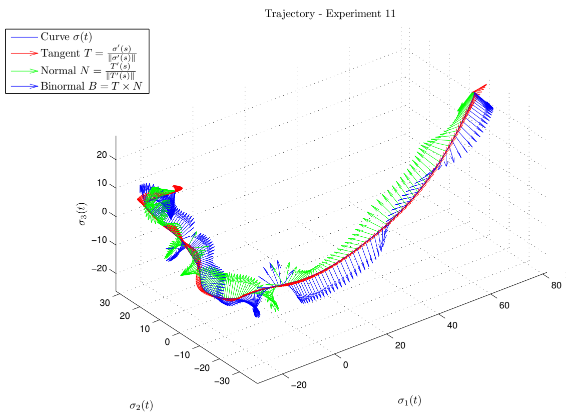
Figure 11.3: Trajectories in principal subspace with Frenet-Serret frame attached.



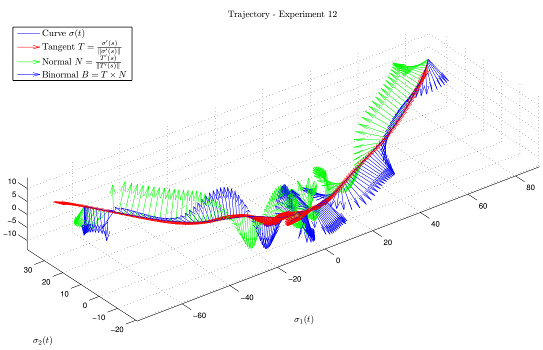
(a) Experiment 9



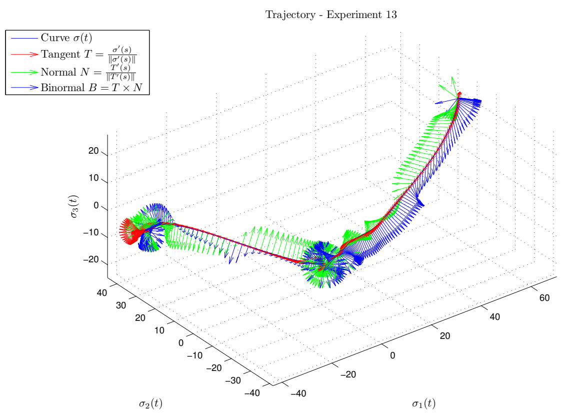
(b) Experiment 10



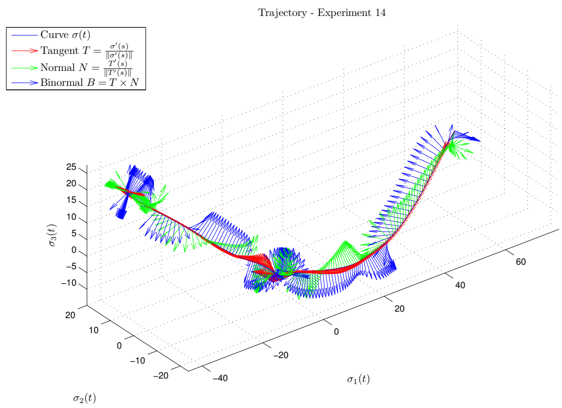
(c) Experiment 11



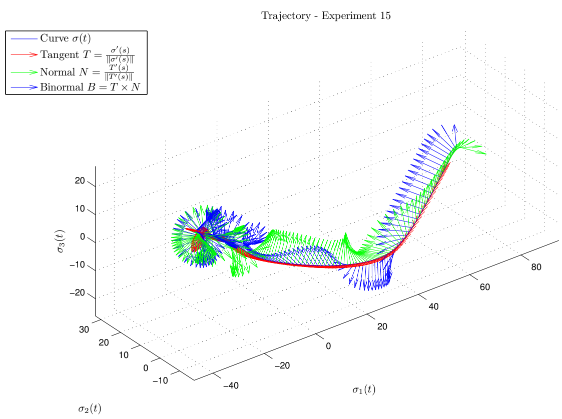
(d) Experiment 12



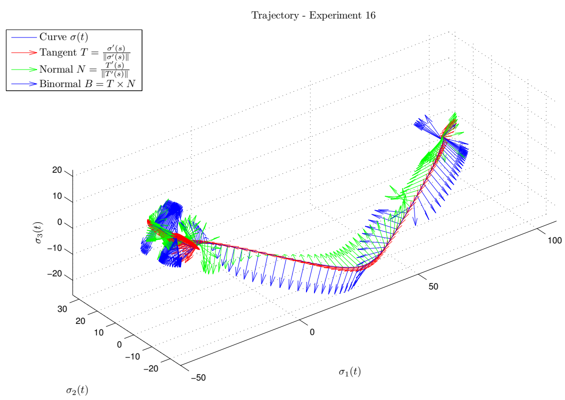
(e) Experiment 13



(f) Experiment 14

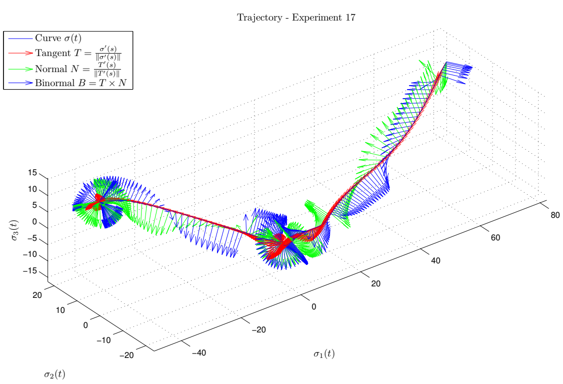


(g) Experiment 15

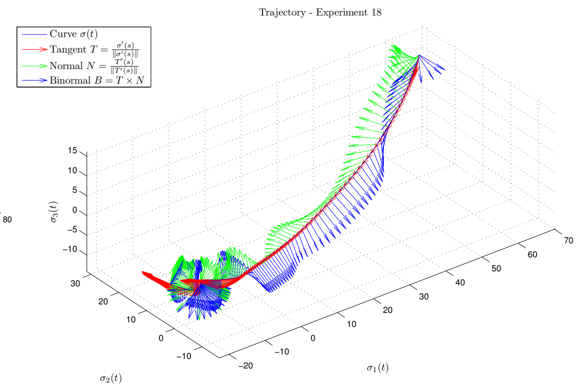


(h) Experiment 16

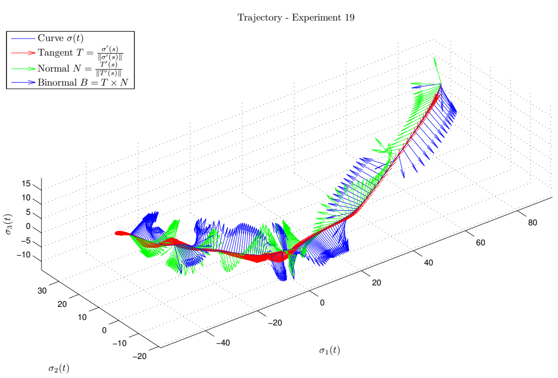
Figure 11.4: Trajectories in principal subspace with Frenet-Serret frame attached.



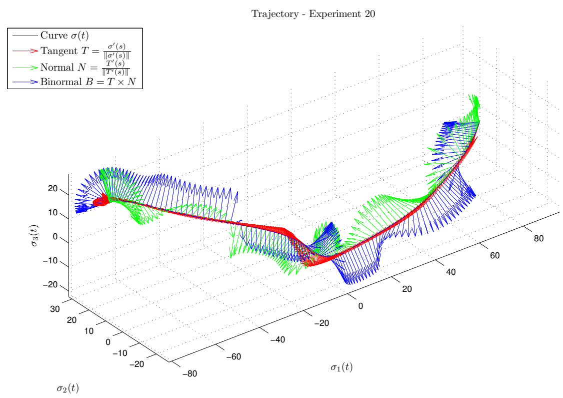
(a) Experiment 17



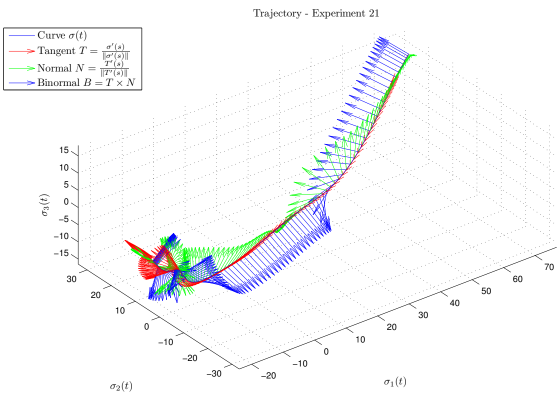
(b) Experiment 18



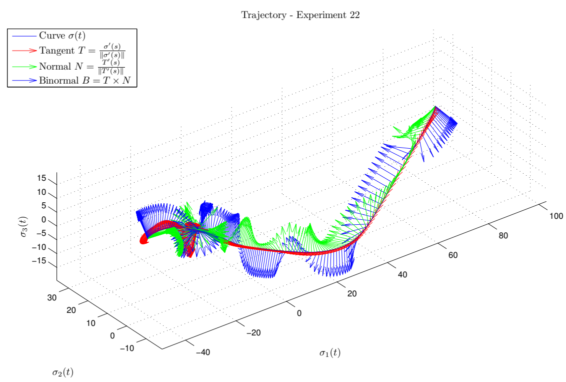
(c) Experiment 19



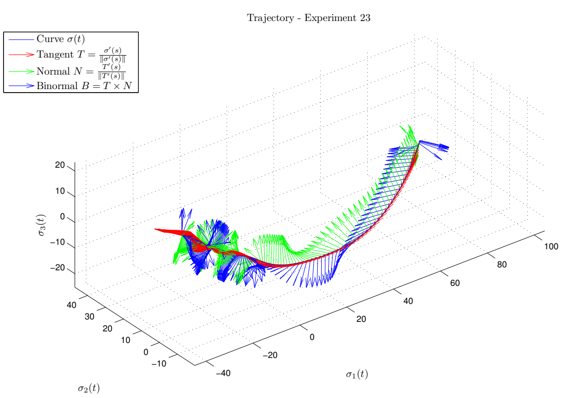
(d) Experiment 20



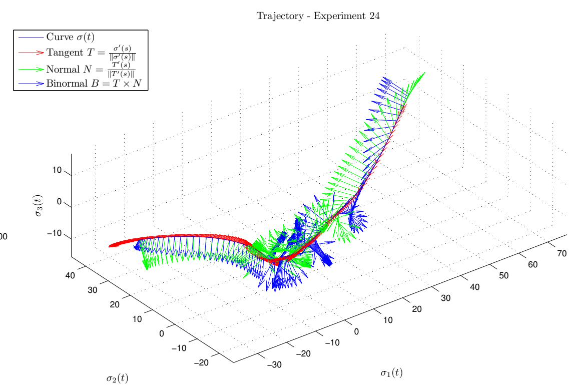
(e) Experiment 21



(f) Experiment 22

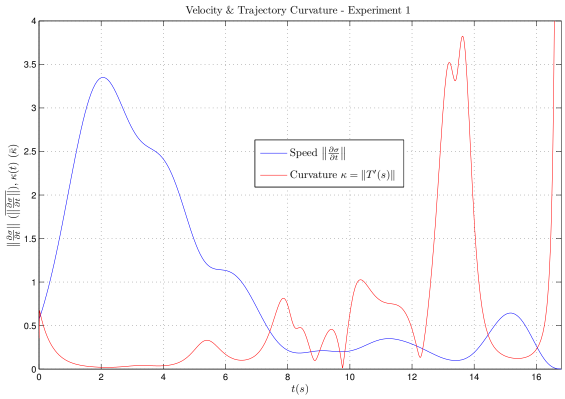


(g) Experiment 23

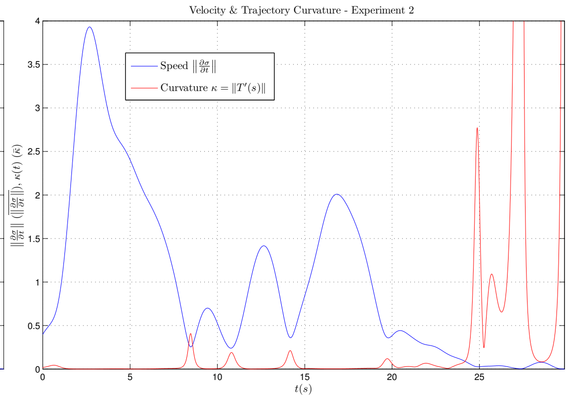


(h) Experiment 24

Figure 11.5: Trajectories in principal subspace with Frenet-Serret frame attached.

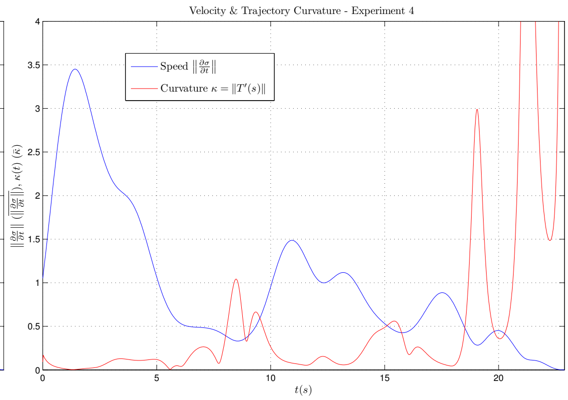


(a) Experiment 1

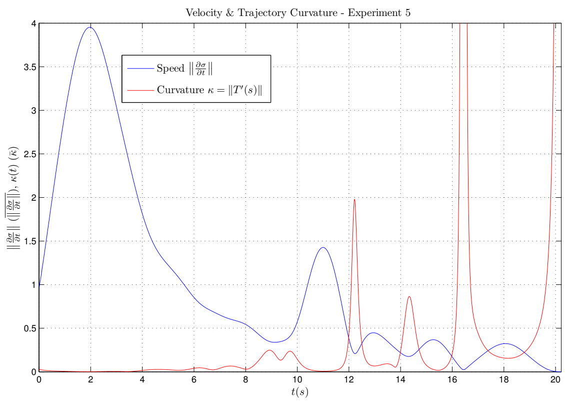


(b) Experiment 2

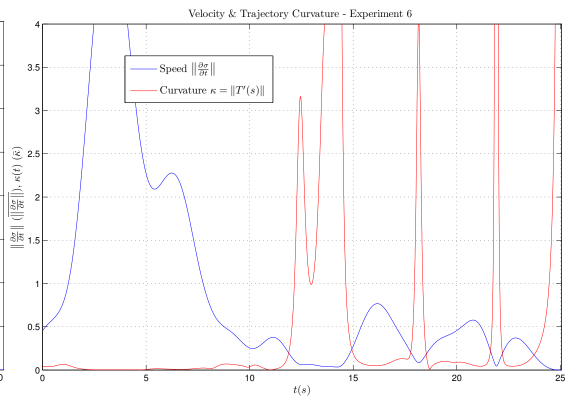
(c) Experiment 3



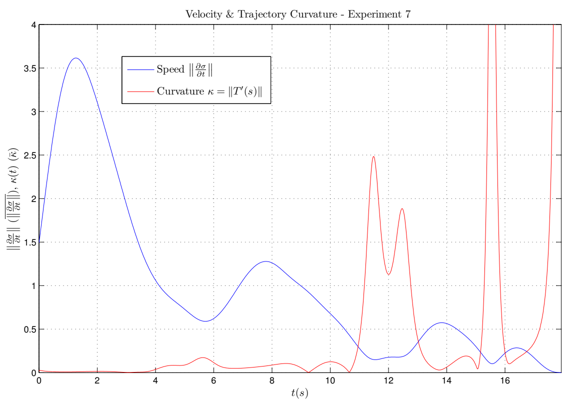
(d) Experiment 4



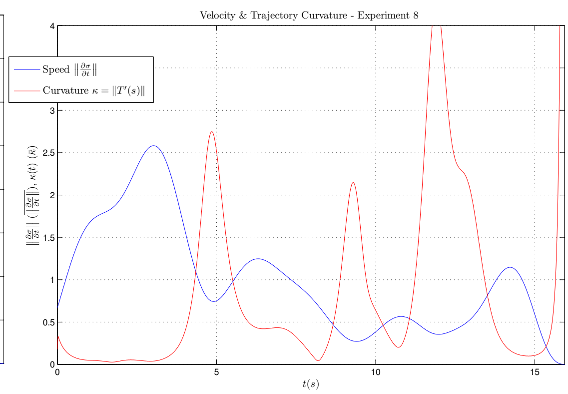
(e) Experiment 5



(f) Experiment 6

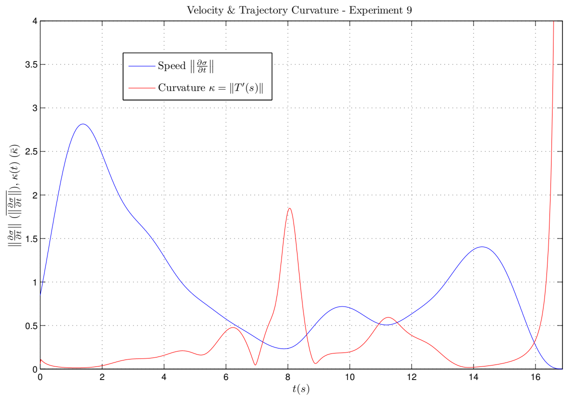


(g) Experiment 7

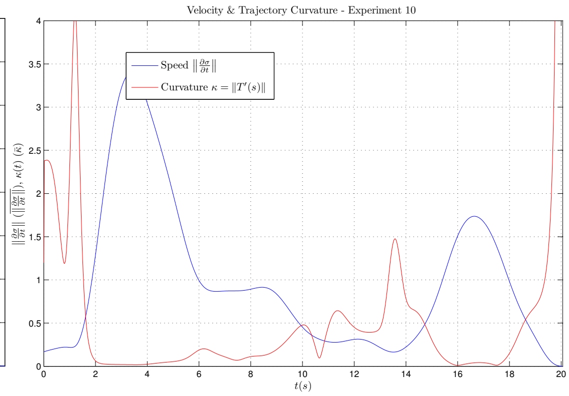


(h) Experiment 8

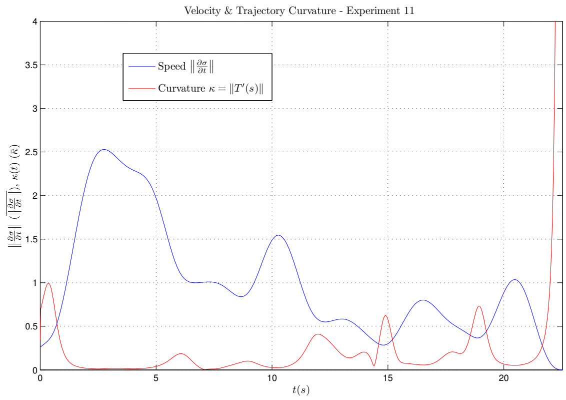
Figure 11.6: Experimental trajectory velocity norms (normalized to average).



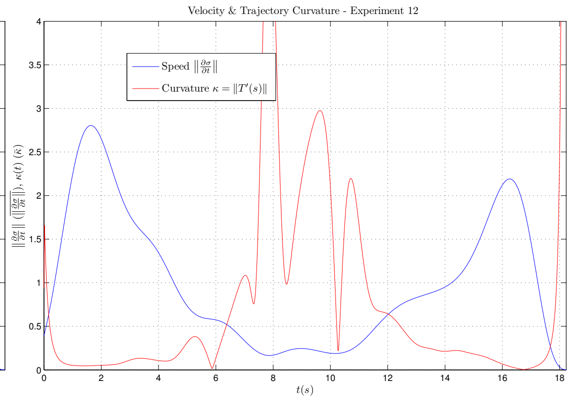
(a) Experiment 9



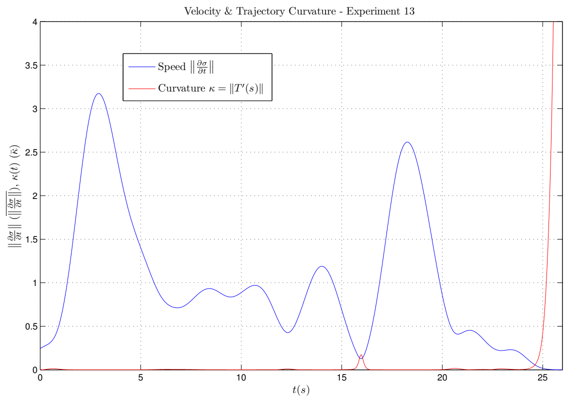
(b) Experiment 10



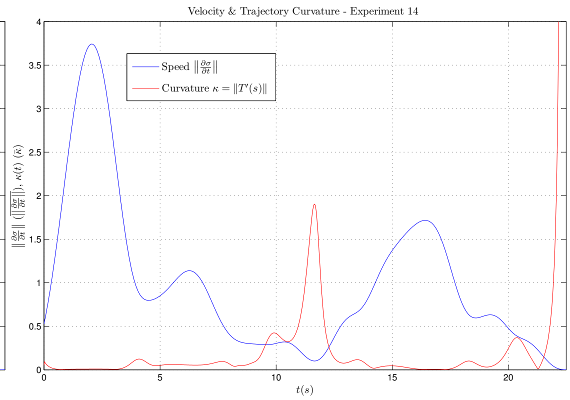
(c) Experiment 11



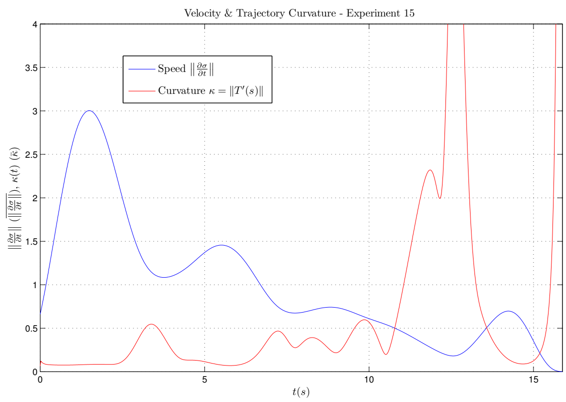
(d) Experiment 12



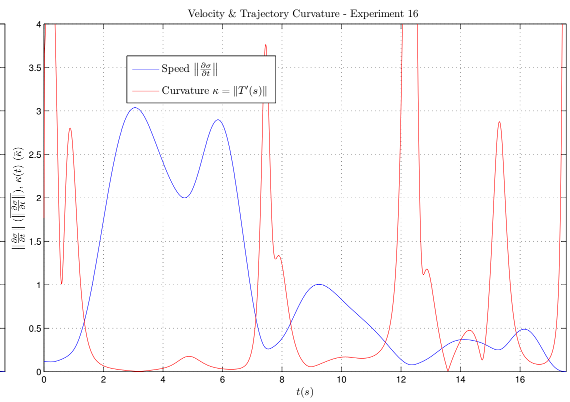
(e) Experiment 13



(f) Experiment 14

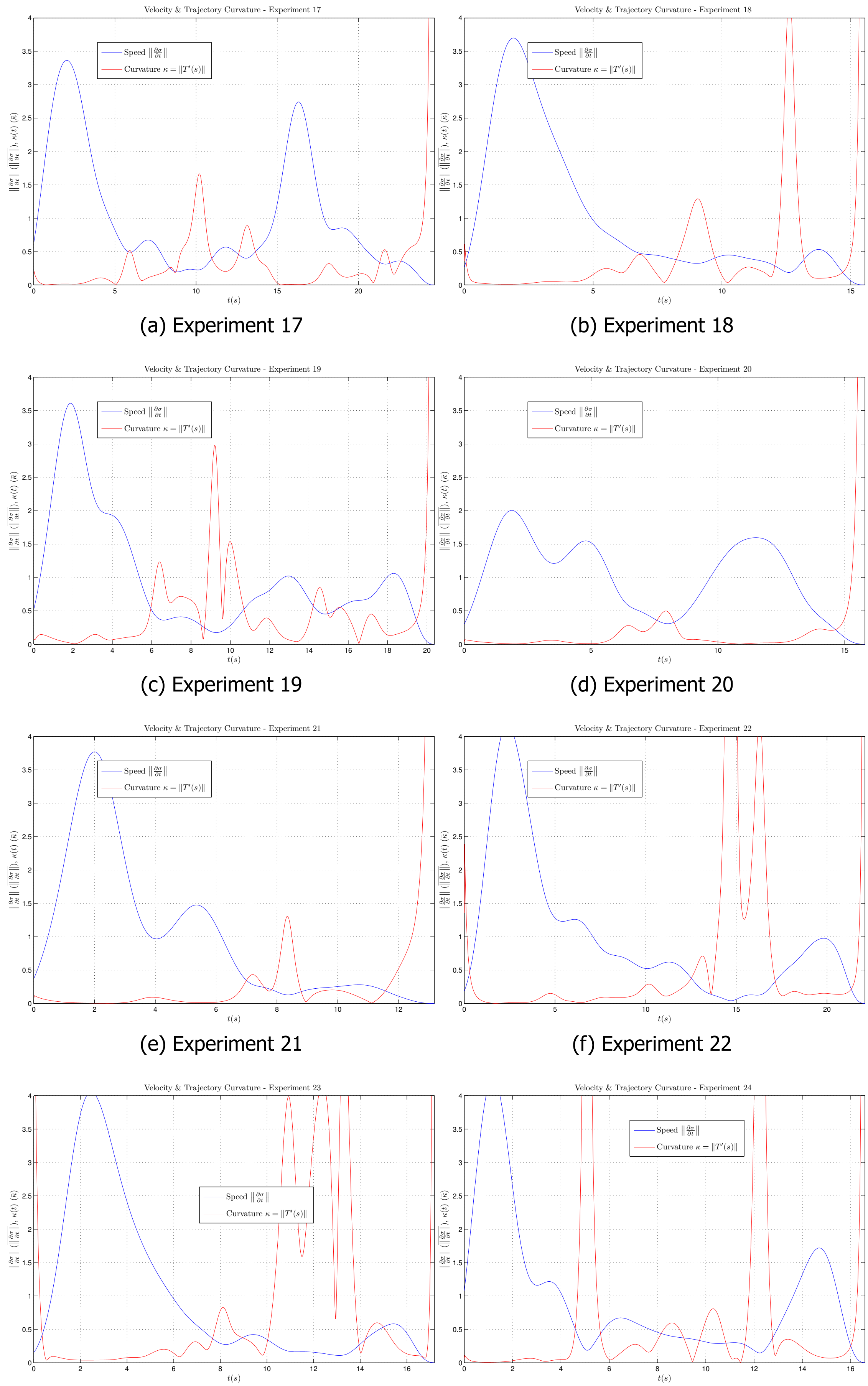


(g) Experiment 15



(h) Experiment 16

Figure 11.7: Experimental trajectory velocity norms (normalized to average).



(a) Experiment 17

(b) Experiment 18

(c) Experiment 19

(d) Experiment 20

(e) Experiment 21

(f) Experiment 22

(g) Experiment 23

(h) Experiment 24

Figure 11.8: Experimental trajectory velocity norms (normalized to average).

Part IV

Decentralized Multi-Agent Control from Local LTL Specifications under Limited Communication

Chapter 12

Formal Methods for Distributed Multi-Agent Systems

12.1 Introduction

There have been multiple approaches to the problem of multi-agent system control. Both classic motion planning [4, 12, 27] and task related methods [95, 96, 109, 118] have been developed. The current effort is oriented towards unification of these two complementary solutions [100–103, 105, 108, 111]. Since the present trend leads to increasingly complex and heterogeneous systems, decentralization is a key ingredient for future scalability.

In addition, safe and guaranteed results are required. Formal methods for specification and automatic synthesis of provably correct controllers can ensure this. The system's specification can be provided in a logic sufficiently expressive for the desired tasks.

In [111] centralized multi-agent systems with perfect information are considered. Synthesis of a single multi-agent motion planning controller is performed from a global LTL specification. This requires a globally connected multi-agent system to ensure information availability.

12.1.1 Decentralization Approaches

12.1.1.1 Computer Science approach

Necessary and sufficient conditions for a global specification to be decomposable to bisimilar local ones are derived in [103]. Decentralization is from top to bottom. A global specification is available and it is then decomposed to local ones. Moreover perfect information availability is assumed for the multiple agents. Therefore no need for addressing communication constraints between them is considered.

In more detail, it is a computer science oriented approach and aims to derive necessary and sufficient conditions for a global specification to be decomposable to bisimilar local specifications. Nonetheless, it concerns specifications in form of deterministic automata, therefore for their approach to be implementable on LTL, it needs to be translated to a deterministic Rabin automaton, instead of the Büchi automaton used in model checking. But acceptance conditions are not treated, as a result it is implied that deterministic accepting traces should be already available for decomposition.

12.1.1.2 Robotics approach

In [107] the issue of communication and synchronization is analyzed a solution diagnozing whether an LTL specification needs communication or not. Similarly to [106], where only communicating agents are allowed to move, this check characterizes the subset of realizable specifications. This does not allow the following type of specifications to be carried out. In particular, those specifications which require that agents be at a distance they can no longer communicate alone, and at the same time need to communicate to decide accordingly in order to meet their specifications. This reduces the realizable scenarios.

Note that the system state space is discrete in [106, 107], whereas continuous in [111]. Discretization (partitioning) of the state space is a difficult task and can considerably increase the state space, leading to stat explosion. On the contrary, utilizing appropriate continuous controllers suited to the problem needs can lead to a reduction in the number of states. This is here pursued by the use of Navigation Function controllers.

12.1.1.3 Issues addressed here

We extend application of formal methods to decentralized multi-agent systems. The method proposed enables each agent to independently synthesize safe controllers, trigger mobile network connectivity when in need of information, verify its plans versus those of others upon meeting them and execute them in a continuous state space using Navigation Functions.

It differs from previous works in decentralization, on-demand mobile network connectivity, decentralized verification and the motion planning controllers. In attempting this, two problems of primary importance need to be solved.

Firstly, LTL specifications provided to the agents are not produced in a centralized way, hence they may be contradicting each other. Secondly, even if mutually satisfiable, we are interested in cases in which long-range communication is not available. If path-connectivity is absent when required by agents, the controllers will fail to act according to their specifications, due to lack of information.

Since we are interested in decentralized systems with limited information, we need to consider the *opposite* approach of [103], that is a bottom-up approach. The solution proposed for the first problem aims to gradually verify that agent specifications are mutually satisfiable. Events of path-connectedness enable exchange of their languages and automata, to allow model checking [94]. Moreover, note that implementing multiple LTL specifications resulting from a top-down decomposition would still require the second aspect described next.

The second problem is critical to the execution of the synthesized controllers. Multi-agent systems in real applications are in many cases scattered over an area. This is in many cases an unavoidable necessity. This leads us to limited communication constraints. Additionally to checking whether local specifications can be carried out without inconsistencies, we need to facilitate their realization. This means that if specific agents have received specifications which require them to be simultaneously out of communication distance *and* decide on LTL which involve atomic propositions (AP) referencing the other agent, then some means of communication between them are needed. These will allow sharing of the needed AP values at the selected times.

We embed in LTL communication requests when information is needed and implement them using additional follower agents under connectivity maintenance control. The followers function as intermediate communication nodes, providing the requested multi-hop path-connectedness between the agents whose LTL specifications require this communication during execution. Just providing communication to everyone would not defer from the assumption of perfect information. What we aim to provide is communication *only* between those agents that need it to carry out their LTL specifications.

For interfacing the discrete controllers to the continuous system state we choose Navigation Functions (NFs) [12, 23]. Navigation Functions are continuous feedback motion planning control laws [4] which ensure collision avoidance by construction and provably correct convergence to the destination. As a result, the specification is formally satisfied in the discrete control level, which in turn is interfaced to the continuous domain via provably correct NF controllers.

12.2 Preliminaries

12.2.1 Linear Temporal Logic

An extension of propositional logic suitable for reasoning about infinite sequences of states is LTL [114]. A set of Atomic Propositions (APs) P is defined [94]. More complex formulae result using propositional and temporal operators.

Here the subset LTL$_{X^-}$is used, omitting operator "next" X. This ensures that all specifications are stutter-invariant by construction, as recommended in [110] for concurrent systems. Any stutter-invariant LTL formula ϕ using X can always be transformed to a LTL$_{X^-}$formula ϕ' [112]. Define the set Φ_P of LTL$_{X^-}$well formed formulas (wff) recursively as

- For all $p \in P$ the expressions $\mathrm{true}, \mathrm{false}, p, \neg p \in \Phi_P$;
- If $\phi_1, \phi_2 \in \Phi_P$ then $\phi_1 \wedge \phi_2 \in \Phi_P$ and $\phi_1 \vee \phi_2 \in \Phi_P$;
- If $\phi_1, \phi_2 \in \Phi_P$ then $\phi_1 U \phi_2 \in \Phi_P$,

where the operator U is read "until" and requires that ϕ_1 be true *until* ϕ_2 becomes true, which is required to happen. Operators $\neg, \wedge, \vee$ are the usual propositional operators for negation, conjunction and disjunction, respectively. Let

- $\Diamond\phi \triangleq \mathrm{true}\, U \phi, \phi \in \Phi_P$, which is read "eventually" and requires that ϕ *eventually* happens at some future point;
- $\Box\phi \triangleq \neg\Diamond\left(\neg\phi\right), \phi \in \Phi_P$, which is read "always" and requires that ϕ be true in all future points;
- $\phi_1 \rightarrow \phi_2 \triangleq (\neg\phi_1) \vee \phi_2, \phi_1, \phi_2 \in \Phi_P$, denoting implication;
- $\phi_1 \leftrightarrow \phi_2 \triangleq (\phi_1 \wedge \phi_2) \vee (\neg\phi_1 \wedge \neg\phi_2), \phi_1, \phi_2 \in \Phi_P$, which denotes equivalence.

The semantics of LTL$_{X^-}$are defined with respect to (wrt) sequences $\sigma : \mathbb{N} \rightarrow 2^P$. Let $\sigma^i(j) \triangleq \sigma(i+j), i, j \in \mathbb{N}$. To obtain the truth *value* of a formula over σ, its interpretation starts from $\sigma(0)$ and is derived according to the following rules, where $p \in P, \phi_1, \phi_2 \in \Phi_P$ and $\sigma \models \phi$ means that sequence σ satisfies wff ϕ

- For all σ we have $\sigma \models \mathrm{true}$ and $\sigma \not\models \mathrm{false}$;
- $\sigma \models p$ if and only if (iff) $p \in \sigma(0)$;
- $\sigma \models \neg p$ if and only if $p \notin \sigma(0)$;
- $\sigma \models \phi_1 \wedge \phi_2$ if and only if $\sigma \models \phi_1$ and $\sigma \models \phi_2$;
- $\sigma \models \phi_1 \vee \phi_2$ if and only if $\sigma \models \phi_1$ or $\sigma \models \phi_2$;
- $\sigma \models \phi_1 U \phi_2$ if and only if $\exists i \in \mathbb{N} : \sigma^i \models \phi_2$ and $\sigma^j \models \phi_1, \forall j \in [0, i) \cap \mathbb{N}$.

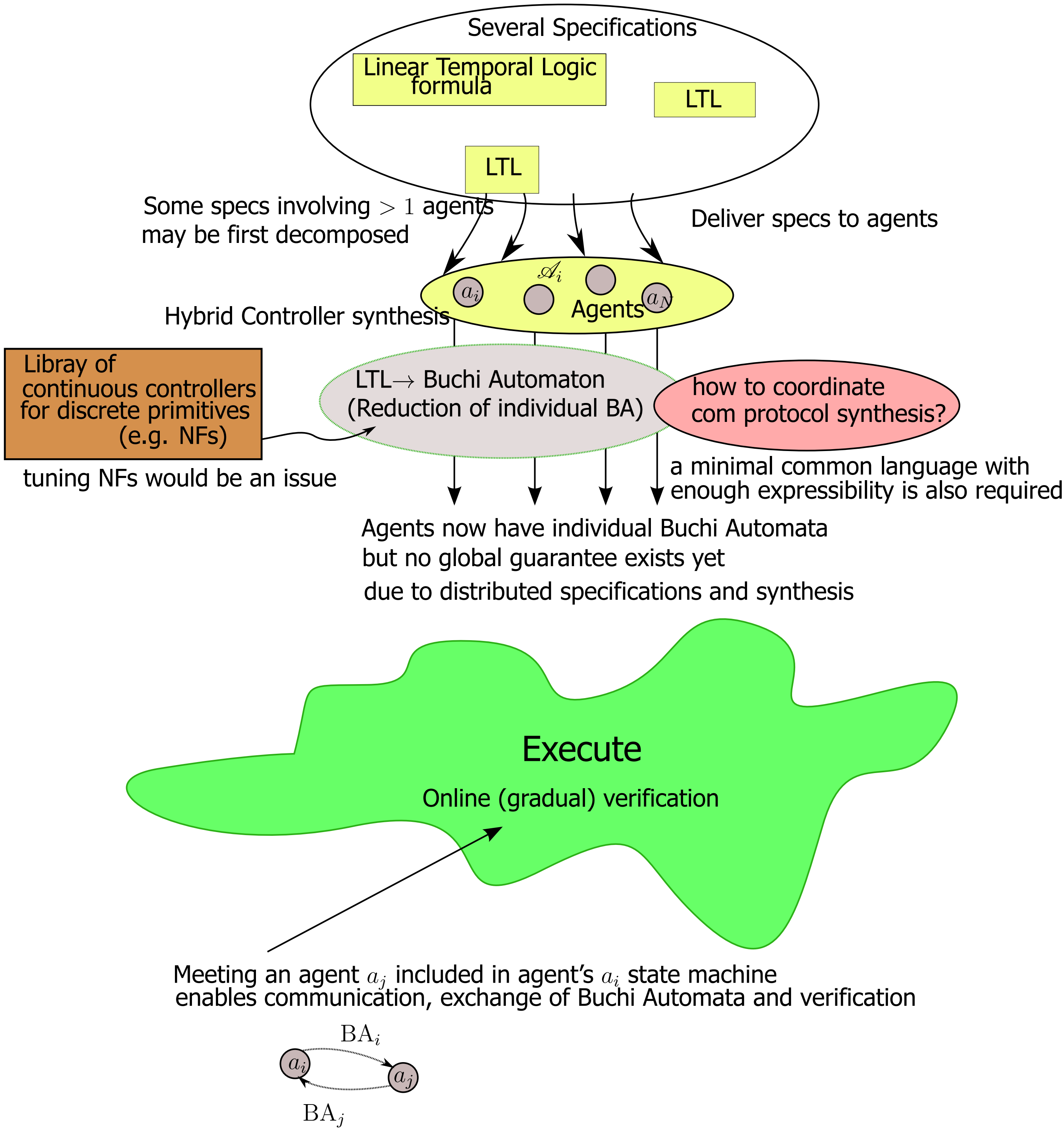


Figure 12.1: General idea.

12.2.2 ω-Automata

Let $\Sigma \triangleq 2^P$ be an alphabet of letters $\sigma(i) \in \Sigma$. An ω-word is an infinite sequence $\sigma \triangleq \sigma(0)\sigma(1)\cdots \in \Sigma^\omega$ over Σ. The set of all possible σ over Σ is denoted by Σ^ω. A Σ^ω subset comprised of those σ satisfying certain rules constitutes a single ω-language $\mathscr{L}_\omega$. Exactly those $\mathscr{L}_\omega$ whose defining rules are expressible in LTL are called ω-regular [94]. Finite transition systems defined on $\mathscr{L}_\omega$ are called ω-automata.

Definition 58 (NBA [94])**.** A Nondeterministic Büchi Automaton (NBA) is a tuple $\mathscr{B} \triangleq \{\Sigma, S, \delta, S_0, F\}$ where

- Σ is a finite alphabet;
- S is a finite set of states;
- $\delta : S \times \Sigma \to 2^S$ is a nondeterministic transition function;
- $S_0 \subseteq S$ is a set of initial states;
- $F \subseteq S$ is a set of accepting states.

Let $\rho : \mathbb{N} \to S$ denote a labeling function of an ω-word by states and $\inf(\rho) \triangleq \{s \in S|\ |\{i : \rho(i) = s\}| = +\infty\}$ the set of states occurring infinitely many times.

Definition 59 (NBA Semantics [94, 117])**.** A $w \in \Sigma^\omega$ is accepted by a NBA $\mathscr{B}$ iff there exists a ρ, such that $\rho(0) \in S_0$, $\rho(i+1) \in \delta(\rho(i), w(i)), \forall i \geq 0$ and $\inf(\rho)$ contains at least one accepting state, i.e., $\inf(\rho) \cap F \neq \emptyset$. Let $\mathscr{L}_\omega(\mathscr{B})$ denote the ω-regular language accepted by $\mathscr{B}$.

According to the complementation result by Büchi [93]:

Theorem 60 (Convert LTL to NBA [97, 117])**.** For every LTL wff $\phi \in \Phi_P$ there exists a NBA $\mathscr{B}$ such that $\mathscr{L}_\omega(\mathscr{B})$ is exactly the same ω-regular language which ϕ defines.

Another type of ω-automaton we will use is

Definition 61 (DRA [115, 116])**.** A Deterministic Rabin Automaton (DRA) is a tuple $\mathscr{R} \triangleq \{\Sigma, S, \gamma, S_0, F\}$ where

- Σ is a finite alphabet;
- S is a finite set of states;
- $\gamma : S \times \Sigma \to S$ is a deterministic transition function;
- $S_0 \triangleq \{s_0\}, s_0 \in S$ is the initial state singleton;
- $F \triangleq \{L_i, U_i\}_{i \in I_{LU}}$ a set of pairs of subsets $L_i, U_i \subseteq S, L_i \cap U_i = \emptyset, \forall i \in I_{LU} \triangleq \{1, 2, \ldots, n_{LU}\}, n_{LU} \in \mathbb{N} \setminus \{0\}$.

Definition 62 (DRA Semantics [115, 116])**.** A $w \in \Sigma^\omega$ is accepted by a DRA $\mathscr{R}$ iff there exists a ρ, such that $\rho(0) = S_0, \rho(i+1) = \gamma(\rho(i), w(i)), \forall i \geq 0$ and for at least one pair i of "good" L_i and "bad" U_i sets, infinitely many from L_i are visited and only finitely many from U_i are visited, i.e., $\exists i \in I_{LU} : \inf(\rho) \cap L_i \neq \emptyset \wedge \inf(\rho) \cap U_i = \emptyset$.

Note that when working with multiple agents later, $L_{ij}, U_{ij}, j \in I_{LU,i}$ will refer to agent a_i. The following holds

Theorem 63 (NBA to DRA [116])**.** For every NBA $\mathscr{B}$ there exists a DRA $\mathscr{R}$ such that they accept exactly the same ω-regular language, i.e., $\mathscr{L}_\omega(\mathscr{B}) = \mathscr{L}_\omega(\mathscr{R})$.

12.3 Problem Definition

12.3.1 Agents, States, State Constraints

Let $\mathscr{A} \triangleq \{a_i\}_{i\in I_a}, I_a \triangleq \{1,2,\ldots,N\}$ be a set of $N \in \mathbb{N}^* \triangleq \mathbb{N} \setminus \{0\}$ leader agents receiving each a (local) specification ϕ_i, defined on Atomic Propositions described in subsection 12.3.2. Each a_i is described by a hybrid state $H_i = x_i \times q_i, x_i \in X_i \subseteq \mathbb{R}^{n_i}, q_i \in Q_i \subseteq \mathbb{N}^{m_i}, i \in I_a, n_i, m_i \in \mathbb{N}$. Here we assume common continuous states $n_i = n \in \mathbb{N}^*, \forall i \in I_a$.

States subject to constraints, as for example continuous dynamics, cannot instantly respond to control actions. As a result, ϕ_i which require immediate changes of observables on constrained states are not in general satisfiable.

Additionally, let $\mathscr{F} \triangleq \{f_i\}_{i\in I_f}, I_f \triangleq \mathbb{N} \cap [N+1, N+n_f], n_f \in \mathbb{N}$ be a set of followers, used to provide on-demand communication as described in section 12.5.

12.3.2 Atomic Propositions

Let $P \triangleq \bigcup_{i\in I_a} (P_{c_i} \cup P_{o_i})$ a set of APs. Each agent a_i can control the values of APs in $P_{c_i} \triangleq \left\{p_{c_{ij}}\right\}_{j\in I_{c_i}}, I_{c_i} \triangleq \{1,2,\ldots,n_{c_i}\}, i \in I_a$. Each $p_{c_{ij}}$ is either true or false when the corresponding continuous or discrete state controller of agent a_i is Active or Not Active, respectively, as described in subsection 12.3.3. Let $f_c : \mathscr{A} \to 2^P$ be a function mapping a_i to its P_{c_i}. Only a_i controls P_{ci}, i.e., $p_{c_{ij}} \in f_c(a_i) \wedge p_{c_{ij}} \notin f_c(a_k), \forall k \in I_a \setminus \{i\}, \forall j \in I_{c_i}, \forall i \in I_a$.

Let $P_{o_i} \triangleq \left\{p_{o_{ij}}\right\}_{j\in I_{o_i}}, I_{o_i} \triangleq \{1,2\ldots,n_{o_i}\}, i \in I_a$ be agent's a_i set of observable APs. Here we use the metric function $\|\cdot\|_2$ to define observations of the form $\|y_1 - y_2\|_2 > | < d_{12}$. Each point y_1, y_2 may be an agent state, e.g., $y_1 = x_3$, or a fixed point wrt a selected reference frame. If $p_{o_{ij}}$ is either defined wrt x_i and x_i is subject to constraints, or wrt $x_j, j \neq i$, then $p_{o_{ij}} \notin P_{c_i}$. If x_i is not subject to constraints, then again $p_{o_{ij}}$ is by definition only observable, but a respective $p_{c_{kr}}$ can indirectly control its value. Let $P_o \triangleq \bigcup_{i\in I_a} P_{o_i}, P_i \triangleq P_{c_i} \cup P_{o_i}, P_c \triangleq \bigcup_{i\in I_a} P_{c_i}$.

The proposed use of metrics facilitate the exchange of languages between meeting agents later to identify common APs and proceed with model checking. The particular choice $\|\cdot\|$ can be readily replaced by more general selections, e.g., set membership functions, depending on the problem treated.

We consider spherical agents of radii $\rho_i, i \in I_a$, with sensing radii $R_{s,i}$. When $\|x_i - x_j\| < R_{s,i}$ then a_i (or f_i) has knowledge of x_j and can receive information from a_j (or f_j). Each agent is assigned a unique $i \in I_a \cup I_f$.

12.3.3 State Controllers

A set of controllers $\mathscr{C}_{ij}$ govern the hybrid state H_i. We select as motion planning controllers $\mathscr{C}_{NF,i}$ decentralized Navigation Functions (NF) [12, 23]. Different $p_{c_{ij}}$ can set a different NF destinations $x_{d_{ij}} \in \mathbb{R}^{n_i}$ in $\mathscr{C}_{NF,i}$. Let $I_{NF,i} \subseteq I_{c_i}$ denote the subset of such $p_{c_{ij}}$. We embed in ϕ_i the requirement $\Box\neg\left(p_{c_{ij}} \wedge p_{c_{ik}}\right), \forall j,k \in I_{NF,i}, \forall t \geq t_0$. Which $p_{c_{ij}}$ becomes true is determined by the discrete controller constructed in section 12.4. This selects the values to assign to $p_{c_{ij}}, j \in I_{c_i}$ in order to enable at least one transition in the automaton $\mathscr{D}_i$, given the current values $p_{o_{ij}}, j \in I_{o_i}$. Note that by defining $x_{d_{ij}} = x_k + c_{d_{ij}}$ for some $j \in I_{c_i}$ wrt another agent, formation control can also be achieved. Other controllers are

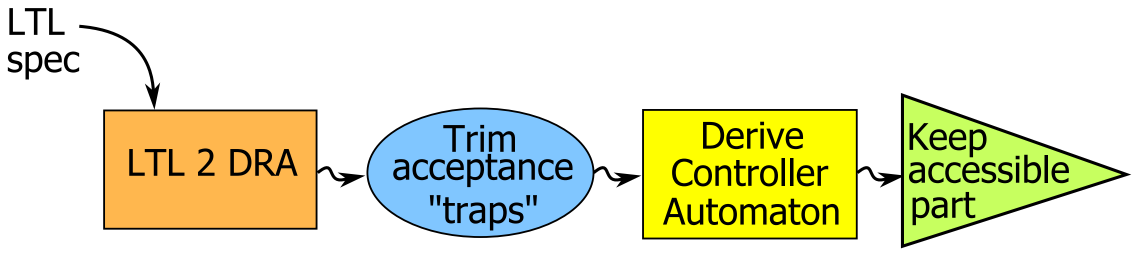


Figure 12.2: From LTL to discrete controller for each agent.

applicable as well, if appropriate[1].

12.3.4 Problem Statement

Each agent a_i receives an LTL$_{X^-}$-specification $\phi_i \in \Phi_{P_i}$, where P_i is defined in subsection 12.3.2. We are interested in an algorithm implemented independently by each agent, to synthesize its hybrid controller $\mathscr{H}_i \triangleq \mathscr{D}_i \times \mathscr{C}_{ij}$ to always satisfy safety specified by ϕ_i and verify liveness triggered by meeting events between agents. It should also provide on-demand long-range path-connectedness as required by ϕ_i, for which it can utilize redundant "follower" agents to maintain connectivity between leaders assigned ϕ_i.

12.4 Discrete Controller Construction

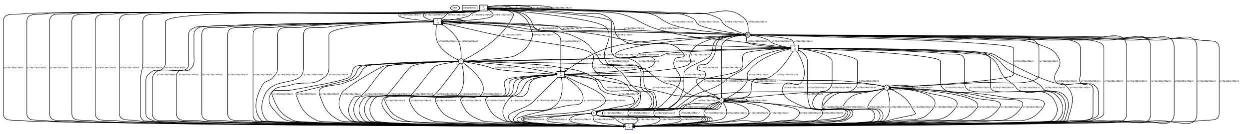
12.4.1 From LTL$_{X^-}$ to Büchi Automata

According to Definition 58, each LTL$_{X^-}$-formula ϕ_i can always be represented by a NBA $\mathscr{B}_i$, which reduces graph searching during model checking. But we want to construct a finite state controller $\mathscr{D}_i$ for each agent which satisfies ϕ_i, so nondeterminism in undesired. A $\mathscr{B}_i$ cannot function as a controller by reacting to $p_{o_{ij}}$ by activating those $p_{c_{ij}}$ which would enable transitions δ_i, for the following reason.

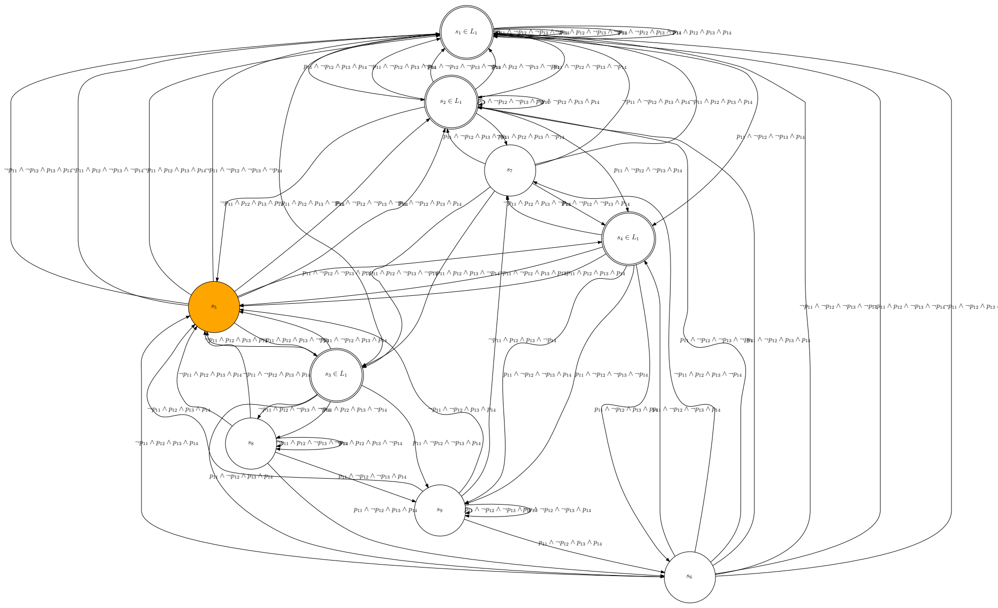
Consider all observable ω-words $w_{o_i} \in \Sigma_{o_i}^{\omega}, \Sigma_{o_i} \triangleq 2^{P_{o_i}}$, such that $\forall w_{o_i}$ there exist corresponding control actions $w_{c_i} \in \Sigma_{c_i}^{\omega}, \Sigma_{c_i} \triangleq 2^{P_{c_i}}$, which, if commanded by the agent, result in an accepted composite $w_i(k) = (w_{c_i}(k) w_{o_i}(k)), k \in \mathbb{N}, w_i \in \mathscr{L}_\omega(\mathscr{B}_i)$. Agent a_i cannot derive its selections of $w_{c_i}(k)$ only from δ_i and $w_{o_i}(k)$. This can always fail to satisfy ϕ_i for *any* given w_{o_i}.

Multiple transitions are possible for the *same* discrete control action $w_{c_i}(k)$. Nevertheless, during execution of a physical system, only a single control action can be selected, not multiple at the same time. On the contrary, a NBA is considered as "copying" itself at such branching points. In other words, it follows *all* possible paths simultaneously. Regardless of whether some lead to deadlocks later on, by Definition 59 it suffices that at least one possible execution exists. In real world executions, following the wrong transition could lead to future violation of ϕ_i, whereas following another one would not. A NBA does not provide a way to select between different transitions, hence a priori knowledge at t_k of future observable suffix $w_{o_i}(k+j), j \in \mathbb{N} \cap [1, +\infty)$ is needed to ensure safety, not available during real world execution.

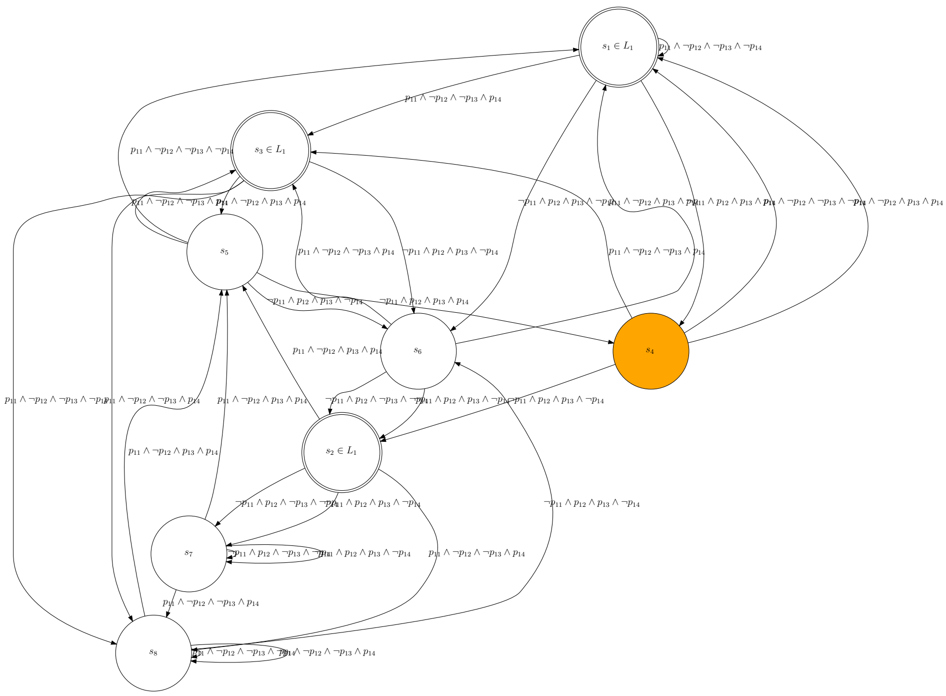
[1]Note that this different from the solution in [111] where control signals from different controllers simultaneously active are mixed. A control signal mixing approach can lead to uncertainty regarding stability, due to relative gains and relative time constants of the continuous controllers.



(a) DRA $\mathscr{R}_i$



(b) Trimmed DRA $\mathscr{T}_i$



(c) Controller automaton $\mathscr{D}_i$

Figure 12.3: Conversion from LTL formula $\square\,(\neg\,(p_{11} \wedge p_{12}) \wedge (p_{13} \rightarrow (p_{12} U p_{14})) \wedge (p_{14} \rightarrow (p_{11} U p_{13})))$ to automata. Orange denotes S_0.

12.4.2 From Büchi Automata to Deterministic Rabin Automata

As described in subsection 12.4.1, a NBA $\mathscr{B}_i$ is not a suitable controller for a physical system. To overcome this limitation, $\mathscr{B}_i$ is determinized into a $\mathscr{R}_i \triangleq \{\Sigma_i, S_i, \gamma_i, S_{0i}, F_i\}, i \in I_a$ by Safra's construction, Theorem 63, using [104]. Although its computational complexity is $2^{n \log n}$, decentralization leads to small size for ϕ_i.

12.4.3 Trimming DRA traps in "bad" states

The $\mathscr{R}_i$ may include entrapping "bad" states $\gamma_i(s_v, l) = s_v \notin L_{ij}, \forall j \in I_{LU,i}, \forall l \in \Sigma_i^\omega$, such that no outgoing transition leading to another state exists, e.g. Fig. 12.3a. In case a_i enters s_v, it remains in s_v infinitely long, violating its ϕ_i.

To prevent a controller based on $\mathscr{R}_i$ from entering such states $s_v \in S_{v,i} \subset S_i$, these are found and removed, yielding a trimmed automaton $\mathscr{T}_i \triangleq \{\Sigma_i, S_i', \gamma_i', S_{0i}', F_i'\}$, where $S_i' \triangleq S_i \setminus S_{v,i}, \gamma_i' \triangleq \gamma_i|_{S_i' \times \Sigma \to S_i'}, S_{0i}' \triangleq S_{0i} \cap S_i', F_i' \triangleq \{L_{ij} \cap S_i', U_{ij} \cap S_i'\}_{j \in I_{UL,i}}$. Accepting runs remain the same, because only w remaining infinitely long in $\bigcup_{j \in I_{LU,i}} U_{ij}$ are removed and these $w \notin \mathscr{L}_\omega(\mathscr{R}_i)$. The trimming algorithm for dead-ends is provided in Algorithm 6. The case of livelock in a closed inescapable cycle through "bad" states is similarly treated. An example is Fig. 12.3b.

Algorithm 6 Trimming entrapping "bad" states

procedure S_i' = Remove Bad TRAPS($\mathscr{R}_i$)
  $Y \leftarrow \{s \in S_i | \nexists j \in I_{LU,i} : s \in L_{ij}\}$
  for $k = 1 : |Y|$ **do**
    $s_k \leftarrow s \in Y$
    if $\nexists l \in \Sigma_i : \gamma_i(s_k, l) \neq s_k$ **then**
      $S_i \leftarrow S_i \setminus \{s_k\}$
    end if
    $Y \leftarrow Y \setminus \{s_k\}$
  end for
end procedure

12.4.4 Discrete Controller from trimmed DRA

Automata have two operating modes. When presented in "reading" mode with a fully specified word w they either accept it if $w \in \mathscr{L}_\omega(\mathscr{R}_i)$, or reject it otherwise. This corresponds to no controllable APs $p_{c_{ij}}$, i.e., $P_{c_i} = \emptyset$. On the other hand, if all APs are controllable, $P_{o_i} = \emptyset$, then the automaton is in "generating" mode. If run according to its rules, it produces exactly $\mathscr{L}_\omega(\mathscr{R}_i)$ [94, 99].

Our case is between accepting and generating modes. It is reacting to observed $p_{o_{ij}}$ values by selecting the controllable values $p_{c_{ij}}$, in such a way so as to avoid deadlock and satisfy Definition 62. Note that $\mathscr{R}_i$ is not deterministic when controllable $p_{c_{ij}}$ exist. We are going to exploit this nondeterminism to design a deterministic controller automaton $\mathscr{D}_i$ from $\mathscr{T}_i$.

The $\mathscr{C}_i$ alone cannot guarantee that an accepting run will be generated if it is used as a controller. For this reason the DRA is now transformed into a discrete event controller

Table 12.1: Next state same as current $s_n = s_c$.

Cost c_{jk}	$s_n \in$					
	$k = j$			$k \neq j$		
$s_c \in$	L_j	U_j	W_j [a]	L_k	U_k	W_k
L_j	+3	-	-	1	-1	0
U_j	-	-3	-	1	-1	0
W_j	-	-	0	1	-1	0

[a] $W_j \triangleq (L_j \cup U_j)^c$.

which "tries its best" to generate an accepting run. The controller selects the best possible transition, judging from the current observable values $p_{o_{ij}}$. To enable the selected transition, it sets the appropriate values of $p_{c_{ij}}$. A criterion for ordering possible transitions and selecting the best is needed. Note that this selection is made only once, when initially designing $\mathscr{C}_i$.

Let us now consider the algorithm applied to a state $s \in S$. Every transition guard contains n_{o_i} observable and n_{c_i} controllable APs. Since $\mathscr{R}_i$ is deterministic in reading mode, so is $\mathscr{T}_i$. So at t_k, the observable vector $\{p_{o_{ij}}\} \in \{0,1\}^{n_{o_i}}$, hence at most $2^{n_{o_i}}$ different observations are possible. Since $\mathscr{T}_i$ is deterministic, for each $\{p_{o_{ij}}\}$, the controller has at most $2^{n_{c_i}}$ choices of transitions and can make only one of them true, by selecting the corresponding control $\{p_{c_{ij}}\}$. A single edge is selected according to the transition ordering later introduced and the remaining are removed. Therefore, if at least one transition was possible for a certain combination of observables, one transition remains possible for that combination, so that no deadlocks are introduced.

Because the $\mathscr{R}_i$ may possess multiple pairs $\{L_{ir}, U_{ir}\}_{r \in I_{LU,i}}$, evaluating each transition is nontrivial. Each transition consists of an ordered pair of states $\{s_c, s_n\}, s_c, s_n \in S_i$, the current s_c and (candidate) next s_n. In turn, s_c, s_n may each belong to both "good" and "bad" sets for different pairs, e.g., $s_c \in L_{ij} \cap U_{ik}, j \in I_{LU,i} \setminus \{k\}, k \in I_{LU,i}$. For this reason tables 12.1 and 12.2 are used to build a matrix $c_{jk} \in \{-3, -2, \ldots, +3\}^{n_{LU,i} \times n_{LU,i}}$ for each possible interpretation $s_c \in L_{ij} \cup U_{ij}, s_n \in L_{ik} \cup U_{ik}$ of the transition $s_c \to s_n$ (in the tables subscript i is omitted to reduce clutter). Then, each $s_c \to s_n$ is assigned a score based on $[c_{jz}]$, according to DRA acceptance of Definition 62.

This can be summarized as $\bigvee_{r \in I_{LU,i}} (L_{r,i} \wedge \neg U_{r,i})$ [116]. If the next state s_n is the same as s_c and "good" in some pair $\exists r \in I_{LU,i} : s_c \in L_{ir}$, then it is obviously not "bad" in that pair, $s_c \in L_{ir} \implies s_c \notin U_{ir}$. This case is assigned $+3$ and dominates all others, because if $s_c \in U_{ij}, j \in I_{LU,i}$ for $j \neq r$, then remaining infinitely long in s_c implies both $\inf(\rho) \cap L_{ir} \neq \emptyset$ and $\inf(\rho) \cap U_{ij} \neq \emptyset$, in which case the first one suffices for LTL ϕ_i satisfaction and thus dominates the second. Similar considerations apply to the other cases as well, leading to $0 < \max\{c_{ij}\} \implies c = \max\{c_{ij}\}$, whereas, if only "bad" and neutral next states s_n are available, the transition is dominated by the worst-case $\max\{c_{ij}\} \leq 0 \implies c = \min\{c_{ij}\}$.

The evaluation resulting from the previous procedure for each transition is used to assign a score according to the following ordering

7. Remains in the same "good" state $s_c \in L_{ir}, r \in I_{LU,i}$;

6. Moves to another "good" state s_n of the same pair $\{L_{ir}, U_{ir}\}, r \in I_{LU,i}$, i.e., $s_n \neq s_c,\ s_n, s_c \in L_{ir}$;

5. Moves to a "good" state s_n of another pair $\{L_{ij}, U_{ij}\}, j \in I_{LU,i}$, i.e., $s_n \neq s_c,\ s_n \in$

Table 12.2: Next state different than current $s_n \neq s_c$.

Cost c_{jk}	$s_n \in$					
	$k = j$			$k \neq j$		
$s_c \in$	L_j	U_j	W_j[a]	L_k	U_k	W_k
L_j	+2	-2	0	1	-1	0
U_j	+2	-2	0	1	-1	0
W_j	+2	-2	0	1	-1	0

[a] $W_j \triangleq (L_j \cup U_j)^c$.

$L_{ir},\ s_c \in L_{ij}, j \neq r$;

4. Moves to a neutral state $s_n \in S_i' : \exists r \in I_{LU,i}\ s_n \notin L_{ir} \cup U_{ir}$ of the same $r = j$ or another $r \neq j$ pair;

3. Moves to a "bad" state s_n of another pair $\{L_{ij}, U_{ij}\}$, i.e., $s_n \neq s_c,\ s_n \in U_{ij},\ s_c \in U_{ij}$;

2. Moves to another "bad" state s_n of the same pair $\{L_{ir}, U_{ir}\}, r \in I_{LU,i}$, i.e., $s_n \neq s_c,\ s_n, s_c \in U_{ir}$;

1. Remains in the same "bad" state $s_c \in U_{ir}, r \in I_{LU,i}$;

During removal of states, it is checked that there remains a path from the initial state to a "good" state and an accepting cycle through it, in order to ensure the controller can still satisfy eventualities in ϕ_i. This leads to $\mathscr{C}_i$ and its sub-automaton accessible from s_0 constitutes $\mathscr{D}_i$, e.g. Fig. 12.3c.

We examine when the above procedure does not remove ω-words from the safe language of observables $\mathscr{L}_\omega^o(\mathscr{R}_i)$.

Proposition 64. If every state $s \in S_{\mathscr{T}_i}$ has outgoing transitions with every combination of observables $p_{o_{ij}}$, then $\mathscr{L}_\omega^o(\mathscr{R}_i) = \mathscr{L}_\omega^o(\mathscr{D}_i)$.

Proof. Since only rejected words $w \notin \mathscr{L}_\omega(\mathscr{R}_i) \implies w_o \notin \mathscr{L}_\omega^o(\mathscr{R}_i)$ are affected by trimming the $\mathscr{R}_i$ to $\mathscr{T}_i$, it follows that $\mathscr{L}_\omega^o(\mathscr{R}_i) = \mathscr{L}_\omega^o(\mathscr{T}_i)$. If every state $s \in S_{\mathscr{T}_i}$ has outgoing transitions for each of the $2^{n_{o_i}}$ combinations of $p_{o_{ij}}$, then the edge removal algorithm maintains exactly one transition per combination. Since every state has transitions for all $P_{o_i} \in \{0,1\}^{n_{o_i}}$, for both $\mathscr{L}_\omega^o(\mathscr{R}_i)$ and $\mathscr{L}_\omega^o(\mathscr{D}_i)$, any observable sequence is safe $\sigma \in \mathscr{L}_\omega^o(\mathscr{R}_i) \wedge \sigma \in \mathscr{L}_\omega^o(\mathscr{D}_i), \forall \sigma \in \Sigma_o^\omega$, hence $\mathscr{L}_\omega^o(\mathscr{R}_i) = \mathscr{L}_\omega^o(\mathscr{D}_i)$. □

The above implies that reactivity in ϕ_i remains unaffected by the proposed algorithm.

An important note is needed at this point. The proposed controller is only locally optimal, with respect to the current state the system is in. It chooses the best move from that state, without considering previous or future moves. This means that in case there exists a strategy to satisfy liveness, which is not locally optimal, the proposed controller may not find it. Therefore, safety properties in ϕ_i are guaranteed to be satisfied, but not necessarily liveness (eventuality) in ϕ_i. Removing this short-coming requires solution of a Rabin game [113], which in our case is both computationally expensive for isolated agents and inappropriate. It is inappropriate because we consider cooperative scenarios.

In a decentralized cooperative setting, agents may not initially know anything about specifications of other agents, which affect their observables. Moreover, there may exist no solution to the adversarial game, in case ϕ_i cannot cope with *any* environment.

Suppose we initially checked eventuality, obtained a negative result, and then prevented the agent from further evolution, declaring a failure. In this case, we disregard both that a decentralized system has initially limited information and that other agents do not constitute *any* environment, but have specific ϕ_j, which a_i can learn and decide about ϕ_i when meeting them during execution.

Thus, verification of the independently executing controllers described in section 12.6 provides an essential check for our system. As is going to be illustrated by the simulation results of section 12.8, the proposed method is successful in practical settings.

We plan to extend the above implementation towards solving the Rabin game when no simpler solution exists. However, we are also interested in determining the class of LTL$_{X-}\phi_i$ which provide a trade-off between multi-agent task expressiveness and Rabin game solution.

12.5 Limited Communication

12.5.1 Specification Structure

It may be the case that an observable subset $\{p_{o_{ij}}\}_{j\in I_k}, I_k \subseteq I_{o_i}$ requires information about another $H_k, k \neq i$. Even if ϕ_i, ϕ_k are mutually satisfiable, in the event that a_i loses connectivity to a_k, it will not have information about H_k. This is an undefined state of information, resulting in ill-defined observables $\{p_{o_{ij}}\}_{j\in I_k}$. To avoid such situations caused by limited communication range, we propose the following scheme.

Let $p_{o_{ij_s}}, j_s \in I_{o_i} \setminus I_k$ be an additional AP in ϕ_i which functions as a switch, being true when a_i, a_k are path-connected in $\{a_i, a_k\} \cup \mathscr{F}'$, and false otherwise. Set $\mathscr{F}' \subseteq \mathscr{F}$ denotes followers not assigned to any leader pair, because only these are free to be immediately committed to $\{a_i, a_k\}$.

If every $\mathscr{D}_i$ transition guard does not depend on the values of $\{p_{o_{ij}}\}_{j\in I_k}$, when $p_{o_{ij_s}} = \mathrm{false}$, then ϕ_i is independent of information regarding agents disconnected from a_i. Such a ϕ_i accounts for limited communication and leaves no possible state of the decentralized multi-agent system undefined.

On the contrary, if ϕ_i does not possess the previous structure, then $\mathscr{D}_i$ is vulnerable to deadlocks caused by lack of information. If all transitions from the current state require knowledge of H_k and a_k is currently disconnected from a_i, then a_i cannot decide what action to apply next.

Nevertheless, certain tasks may need such information in an essential way. In subsection 12.5.2 a method is proposed to trigger connectivity in a formal and controllable way.

12.5.2 On Demand Connectivity Maintenance

Whenever agents a_k and a_z become path-connected, then $p_{o_{kj_s}}$ becomes true. After this event, information about H_z may be required for a certain (finite or infinite) period of time for $p_{o_{kj}}, j \in I_s \subset I_{o_k}$. This is required when, for example, a_k should respond to H_z and they are disconnected.

To maintain this path-connectedness, we introduce mobile network connectivity maintenance controllers [7] within the NF as described in section 12.7, for both leaders $a_k, k \in I_a$ and followers $f_m, m \in I_f$. Each such controller can be supplied a neighbor list $N_k \subseteq$

$I_a \cup I_f$ from the network, as described in what follows. A controllable AP $p_{c_{kj_c}}, j_c \in I_{c_k}$ is also defined, which can be triggered by $p_{o_{kj_s}}, j_s \in I_{o_k}$, according to ϕ_k. The connectivity controller associated to $p_{c_{kj_c}}$ issues periodically a request over the mobile network, as $\{k, z\}, k, z \in I_a$ indexing the agents a_k, a_z to connect.

As defined in subsection 12.3.1, a *leader* is any agent which has received LTL instructions and a *follower* any other agent which has not. Note that this request can only reach the network's connected component to which a_i, a_k both belong, as ensured by $p_{o_{ij_s}}$ which triggered the request.

In each such connected component, all connectedness requests are firstly aggregated. We assume that communication delays are negligible. Then, any available followers in this connected component, which have not yet been assigned to a leader pair, are allocated to the different requests. To resolve which leader pair gets a follower, a utility function is used. Since $p_{o_{ij}}$ defined in subsection 12.3.2 are inequalities using $\|\cdot\|_2$ over X_i, they may reference fixed points $y \in X_i$. The maximal distance $d_{kz\max} \triangleq \max\{\|y_1 - y_2\|\}$ between any two fixed points y_1 referenced by $p_{o_{kj}} \in I_{o_k}$ and y_2 referenced by $p_{o_{zj}}, j \in I_{o_z}$ of the two leaders involved in a pair, constitutes the maximal possible distance which ϕ_k, ϕ_z may require a_k, a_z to reach, while maintaining connectivity. This distance needs to become equal to the sum of communication ranges $\sum_{i \in I_{kz}} R_i$ of the followers $\mathscr{F}_{kz} \triangleq \{f_i\}_{i \in I_{kz}}, I_{kz} \subseteq I_f$ already assigned to the leader pair a_k, a_z. The utility function is defined as $u_{kz} \triangleq 1 - \frac{\sum_{i \in I_{kz}} R_i}{d_{kz\max}}$.

Those pairs $\{a_k, a_z\}$ with higher u_{kz} have relatively fewer followers already assigned. Followers in the connected component are distributed proportionally to u_{kz}. Partitioning according to nearest neighbor distance is used to ensure that each follower subset forms a connected component after assignment. No f_i can belong to more than one $\{a_k, a_z\}$. In this way, when communication between a pair is no more needed, the followers assigned to it can be released again.

After a leader pair is assigned a subset of followers, a chain is formed between a_k and a_z, as follows. An adjacency matrix A with shortest neighbor distances $a_{ij} \triangleq \|x_i - x_j\|$ is formed. The chain between a_k and a_z is initialized as the shortest path in it. Then, recursively each remaining follower $f_i, i \in I_{kz}$ such that $\left\|x_i^f - x_{j_1}\right\| < R_i \wedge \left\|x_i^f - x_{j_2}\right\| < R_i$, is inserted between a_{j_1}, a_{j_2}, replacing their link. If multiple candidate insertions exist, then that with $\max\left\{-\frac{(x_l - x_i^f)\cdot(x_m - x_i^f)}{\left\|x_l - x_i^f\right\|^2 \left\|x_m - x_i^f\right\|^2}\right\}$, which is higher when the two candidate neighbors are closer and opposite positioned to x_i^f. Any agents with only a single neighbor remain connected to it, until they come within range of two neighbors and are then inserted in the chain. Other solutions allowing more flexible manipulation of mobile network links within each subset assigned to each leader pair can be used [8].

12.6 Decentralized Verification

For each agent individual discrete event controllers have been constructed by application of the algorithm described in section 12.4. It is possible that ϕ_i contradicts another ϕ_j.

Therefore, there arises the need to verify that individual controllers constructed can function uncoordinated and still satisfy their respective ϕ_i. For this purpose model checking (MC) is employed. Using the SPIN model checking software [98] and a custom MATLAB interface, each agent is modeled as a separate process. Different processes execute independently. Stutter-invariance is guaranteed because plans are provided in LTL_{X^-}[105, 110, 112], allowing partial-order-reduction methods to be applied [98].

When leader agents become path-connected, they interchange their alphabets (APs). Destinations $x_{d_{ij}}$ in the NF controllers $p_{c_{ij}}$ of one agent a_i are tested as observed states $x_i = x_{d_{ij}}$ in those APs $p_{o_{kj}}$ of the other agent, which depend on x_i. If false, then these observables are initialized for a_j as false and remain unchanged. If true, then a separate environment process is created corresponding to each NF controller $p_{c_{ij}}$, which, when $p_{c_{ij}}$ becomes true, sets to true those observables $p_{o_{kj}}$ which become true for that $x_{d_{ij}}$. By enforcing fairness during MC, this modeling connects controllers of one agent and observables of another with eventual implication. Similar implication is modeled between NF destinations and corresponding observables of a single agent as well. It is provably correct convergence of NF that allows this. Provided that the mechanism of section 12.5 is implemented, x_i is needed by a_K only when path-connected.

After this, each agent performs MC against its own negated specification $\neg\phi_i$ converted to a Büchi automaton expressed as a never claim using [97]. Only liveness is checked, since safety is ensured by construction. If verification succeeds, then the controllers can continue executing independently, whereas if it fails, then refinement is needed. Reconfigurability of the system is in our future goals, as depicted in Fig. 12.4.

12.7 Continuous Controllers

The discrete controllers $\mathscr{D}_i$ are interfaced to the continuous states x_i using Navigation Functions (NF) $\varphi_i(x_i)$ introduced in [23] and extended to decentralized multi-agent systems in [12]. Moreover, due to the need of integrating connectivity maintenance constraints, we also implement [7]. The decentralized NF controllers $\mathscr{C}_{NF,i} \triangleq \{u_i\}$ used by each agent are

$$\dot{x}_i = u_i \tag{12.1}$$

, and

$$u_i \triangleq -\left(\nabla_{x_i}\varphi_i\right)\left(x_i(t)\right) \tag{12.2}$$

, where

$$\varphi_i(x) \triangleq \frac{\gamma_i(x)}{\left(\gamma_i^k(x) + G_i(x)\right)^{\frac{1}{k}}} \tag{12.3}$$

is a potential function of the stack vector x of $x_i, i \in I_a$, which is 1 in collision sets, has a unique minimum $\varphi(\gamma_d^{-1}(0)) = 0$ at the destination q_d and no other local minima, only saddle points. Function

$$\gamma_i(q) \triangleq \frac{1}{2}\sum_{j\in N_i}\left(\|x_i - x_j\|^2\right) \tag{12.4}$$

for a follower to minimize its distance to connected neighbors [7], while

$$\gamma_i(q) \triangleq \frac{1}{2}\left\|x_i - x_{d_{ij}}\right\|^2 \tag{12.5}$$

for a leader to converge to the destination $q_{d_{ij}}$ set by the active controller AP $p_{c_{ij}}, j \in I_{c_i}$. Function

$$G_i(q) \triangleq \prod_{l=1}^{n_L^i}\prod_{j=1}^{n_{R_l}^i} g_{ikl}(\beta_{ij}) \tag{12.6}$$

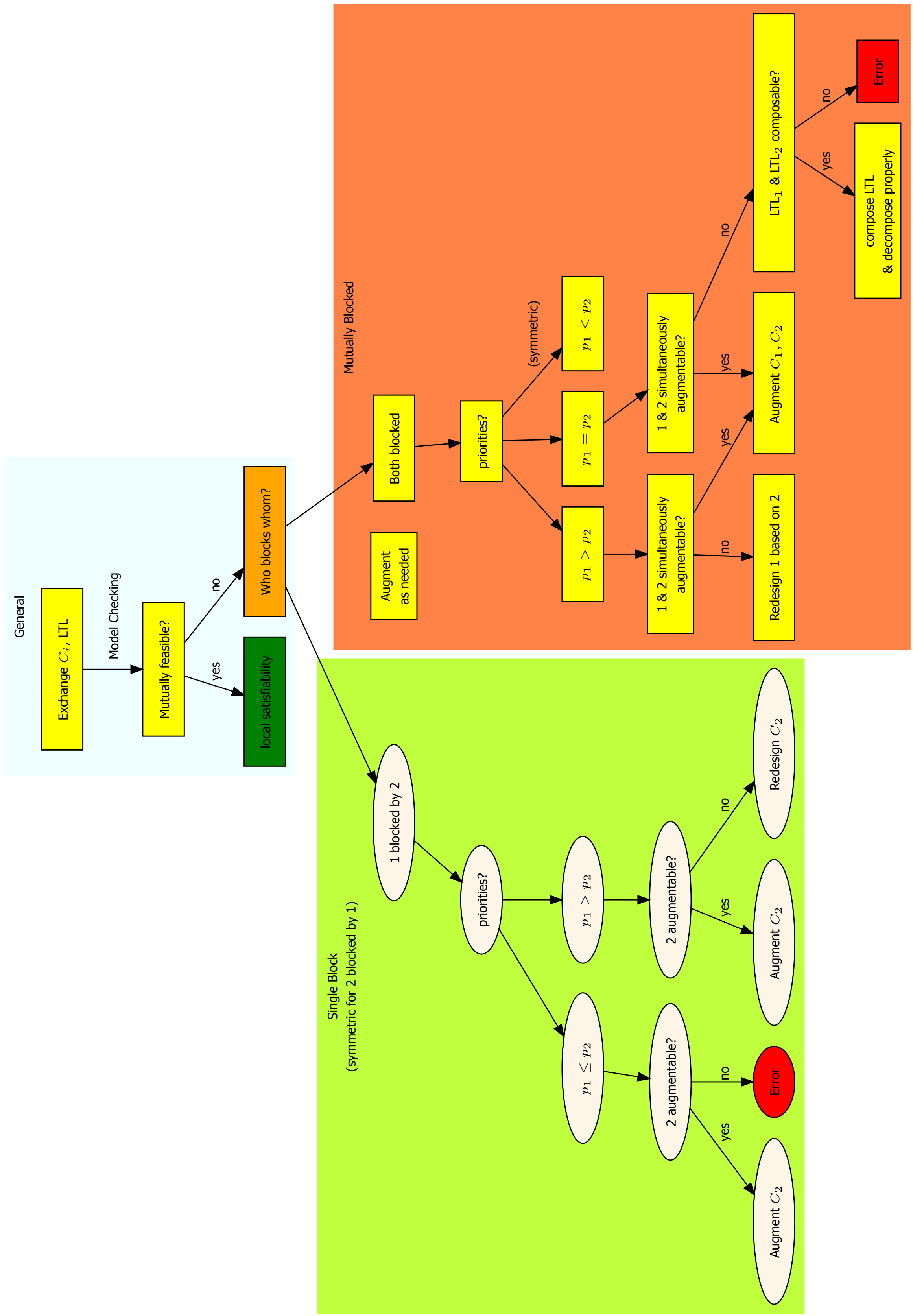


Figure 12.4: Reconfiguration procedure for future work. By p_i priorities of two meeting agents are denoted and by C_i discrete controllers.

encodes collision sets using relation verification functions $g_{ikl}(\beta)$. Functions g_{ikl} determine exactly which combination of $\beta_{ij} \to 0$, i.e. to collision. They are defined as

$$g_{ikl} \triangleq b_{ikl} + \lambda \frac{b_{ikl}}{b_{ikl} + B_{ik^c l}^{\frac{1}{h}}} \tag{12.7}$$

where $b_{ikl} \triangleq \sum_{j \in i_{kl}} \beta_{ij}$ and i_{kl} is the k^{th} set of binary relations $\{a_i, a_j\}$, and $B_{ik^c l} \triangleq \prod_{m \in i_{k^c l}} b_{ml}$, where $i_{k^c l}$ indexes complementary sets of l-level binary relations to set k. In turn, $\beta_{ij}(x_i, x_j)$ is defined differently, depending on whether it relates to another agent with which no connectivity constraints have been imposed, or if it relates to another agent with which connectivity constraints should be maintained. In both cases collision avoidance is incorporated for the spherical agents. Let

$$S = \begin{cases} 1, x \leq 0 \\ -6x^5 + 15x^4 - 10x^3 + 1, 0 < x < 1 \\ 0, 1 \leq x \end{cases} \tag{12.8}$$

be a C^2 switch over $\mathbb{R}$ [25]. Then let

$$\begin{gathered} d_{ij}(x_i, x_j) \triangleq \|x_i - x_j\|_2, \quad A = \frac{d_{ij}^2 - (\rho_i + \rho_j)^2}{d_c^2 - (\rho_i + \rho_j)^2}, \quad B = \frac{d_{ij}^2 - (\rho_i + \rho_j)^2}{d_m^2 - (\rho_i + \rho_j)^2}, \\ \Gamma = \frac{d_c^2 - d_{ij}^2}{d_c^2 - d_m^2}, \quad S_1 = S\left(\frac{d_{ij} - (\rho_i + \rho_j)}{d_c - (\rho_i + \rho_j)}\right), \quad S_2 = S\left(\frac{d_{ij} - (\rho_i + \rho_j)}{d_m - (\rho_i + \rho_j)}\right) \end{gathered} \tag{12.9}$$

so

$$\beta_{ij}(x_i, x_j) \triangleq \begin{cases} S_1 A + 1 - S_1, j \notin N_i \\ S_2 B + (1 - S_2)\Gamma, j \in N_i \end{cases} \tag{12.10}$$

where

$$0 < \rho_i + \rho_j < d_c, \quad d_m \triangleq \sqrt{\frac{d_c^2 + (\rho_i + \rho_j)^2}{2}}. \tag{12.11}$$

Note that limited sensing capabilities are incorporated.

Convergence to the destination can be proved similarly to [7, 13, 24], when tuning parameters $k > 2, \lambda, h > 0$ are selected above a lower bound, provided the agents can reach them without forcing the connected followers to break their connectivity. Therefore, an obvious requirement is that enough followers be available to enable leader convergence while maintaining path-connectedness. Similarly to [8], the leaders try to "do their best" to achieve their objectives.

12.8 Simulation Results

A case study using the proposed algorithm involving $n_a = 6$ leader agents and $n_f = 3$ followers, illustrated in Fig. 12.5, in which agents a_1, a_2 (blue,green) should eventually patrol the lower-right area, visiting infinitely often two points one after the other. Agents a_5, a_6 (magenta,yellow) wait for a_4 (cyan) before requesting connectivity and moving to $x_{d_{51}}, x_{d_{61}}$. Agent a_4 goes first to $x_{d_{41}}$, then to $x_{d_{42}}$. Finally, a_3 (red) goes to $x_{d_{31}}$, waits to see a_2 and then moves to $x_{d_{32}}$. Followers f_7, f_8, f_9 are available to provide path connectivity where needed.

The specifications are defined as

$$\begin{aligned}
&\phi_1 = \Box\big(\neg\left(p_{c_{11}} \wedge p_{c_{12}}\right) \wedge \left(p_{o_{11}} \rightarrow \left(p_{c_{12}} U p_{o_{12}}\right)\right) \wedge \left(p_{o_{12}} \rightarrow \left(p_{c_{11}} U p_{o_{11}}\right)\right)\big)\\
&\phi_2 = \Box\big(\neg\left(p_{c_{21}} \wedge p_{c_{22}}\right) \wedge \left(p_{o_{21}} \rightarrow \left(p_{c_{22}} U p_{o_{22}}\right)\right) \wedge \left(p_{o_{22}} \rightarrow \left(p_{c_{21}} U p_{o_{21}}\right)\right)\big)\\
&\phi_3 = \Box\left(\neg\left(p_{c_{31}} \wedge p_{c_{32}}\right) \wedge \left(p_{c_{31}} U \left(p_{o_{31}} \wedge p_{c_{32}}\right)\right)\right)\\
&\phi_4 = p_{c_{41}} U \left(p_{o_{41}} \wedge p_{c_{42}}\right)\\
&\phi_5 = (((\neg p_{c_{51}}) \wedge (\neg p_{c_{52}}))U(p_{o_{51}} \wedge p_{o_{54}}))\wedge\\
&\qquad \Box((\neg p_{o_{53}}) \rightarrow (\neg p_{c_{52}}))\wedge\\
&\qquad ((\neg(p_{o_{51}} \wedge p_{o_{54}}))U(\Box(p_{o_{53}} \rightarrow (p_{c_{52}} \wedge (p_{c_{51}} U p_{o_{52}})))))\\
&\phi_6 = (((\neg p_{c_{61}}) \wedge (\neg p_{c_{62}}))U(p_{o_{61}} \wedge p_{o_{64}}))\wedge\\
&\qquad \Box((\neg p_{o_{63}}) \rightarrow (\neg p_{c_{62}}))\wedge\\
&\qquad ((\neg(p_{o_{61}} \wedge p_{o_{64}}))U(\Box(p_{o_{63}} \rightarrow (p_{c_{62}} \wedge (p_{c_{61}} U p_{o_{62}})))))
\end{aligned} \tag{12.12}$$

The followers constantly execute a NF with neighbor list as described in subsection 12.5.2. The NF controllers are defined with destinations

$$\begin{aligned}
&x_{d_{11}} = [0,0]^{\mathrm{T}}, x_{d_{12}} = [2,-2]^{\mathrm{T}}, x_{d_{21}} = [0,-1]^{\mathrm{T}}, x_{d_{22}} = [1,1]^{\mathrm{T}}, x_{d_{31}} = [-1,+1]^{\mathrm{T}},\\
&x_{d_{32}} = [4,4]^{\mathrm{T}}, x_{d_{41}} = [0,3]^{\mathrm{T}}, x_{d_{42}} = [-2,1]^{\mathrm{T}}, x_{d_{51}} = [-3,5]^{\mathrm{T}}, x_{d_{61}} = [2,4]^{\mathrm{T}}
\end{aligned} \tag{12.13}$$

and when $p_{c_{52}}, p_{c_{62}}$ are active, they issue connectivity requests to link a_5, a_6.

Let the observable APs be defined as follows

$$\begin{aligned}
&p_{o_{11}} \triangleq \left(\|x_1 - x_{d_{11}}\| < 0.1\right), \quad p_{o_{12}} \triangleq \left(\|x_1 - x_{d_{12}}\| < 0.1\right), \quad p_{o_{21}} \triangleq \left(\|x_2 - x_{d_{21}}\| < 0.1\right),\\
&p_{o_{22}} \triangleq \left(\|x_2 - x_{d_{22}}\| < 0.1\right), \quad p_{o_{31}} \triangleq \left(\|x_2 - x_3\| < 1\right), \quad p_{o_{41}} \triangleq \left(\|x_4 - x_{d_{41}}\| < 0.1\right),\\
&p_{o_{51}} \triangleq \left(\|x_4 - x_{d_{41}}\| < 0.4\right), \quad p_{o_{52}} \triangleq \left(\|x_5 - x_{d_{51}}\| < 0.1\right), \quad p_{o_{61}} \triangleq \left(\|x_4 - x_{d_{41}}\| < 0.4\right),\\
&p_{o_{62}} \triangleq \left(\|x_6 - x_{d_{61}}\| < 0.1\right)
\end{aligned} \tag{12.14}$$

and $p_{o_{53}}, p_{o_{63}}$ detect path-connectivity between a_5, a_6 through followers. APs $p_{o_{54}}, p_{o_{64}}$ detect path-connectivity between a_4 and a_5, a_6, respectively, through any agent and function as information availability switches. Note that $p_{o_{31}}$ requires information about x_2, but it also functions as an information availability switch, because $\|x_2 - x_3\| < 1 < R_{s3}$, so no additional observable is needed in this case.

In the simulation $a_1, \cdots, a_4$ proceed to their objectives avoiding collisions, so that at t_7 agents a_1, a_2 have started patrolling the lower left area and a_3 is heading towards $x_{d_{32}}$. At t_8 agent a_4 comes within distance 0.4 of $x_{d_{41}}$, so that connectivity is triggered (thick continuous lines) and a_5, a_6 begin moving to $x_{d_{51}}, x_{d_{61}}$ respectively, while followers f_7, f_8, f_9 maintain path-connectivity between a_5, a_6. This connectivity is implemented as requested by $p_{c_{52}}, p_{c_{62}}$, which in turn were triggered by $p_{o_{53}}, p_{o_{63}}$ according to ϕ_5, ϕ_6.

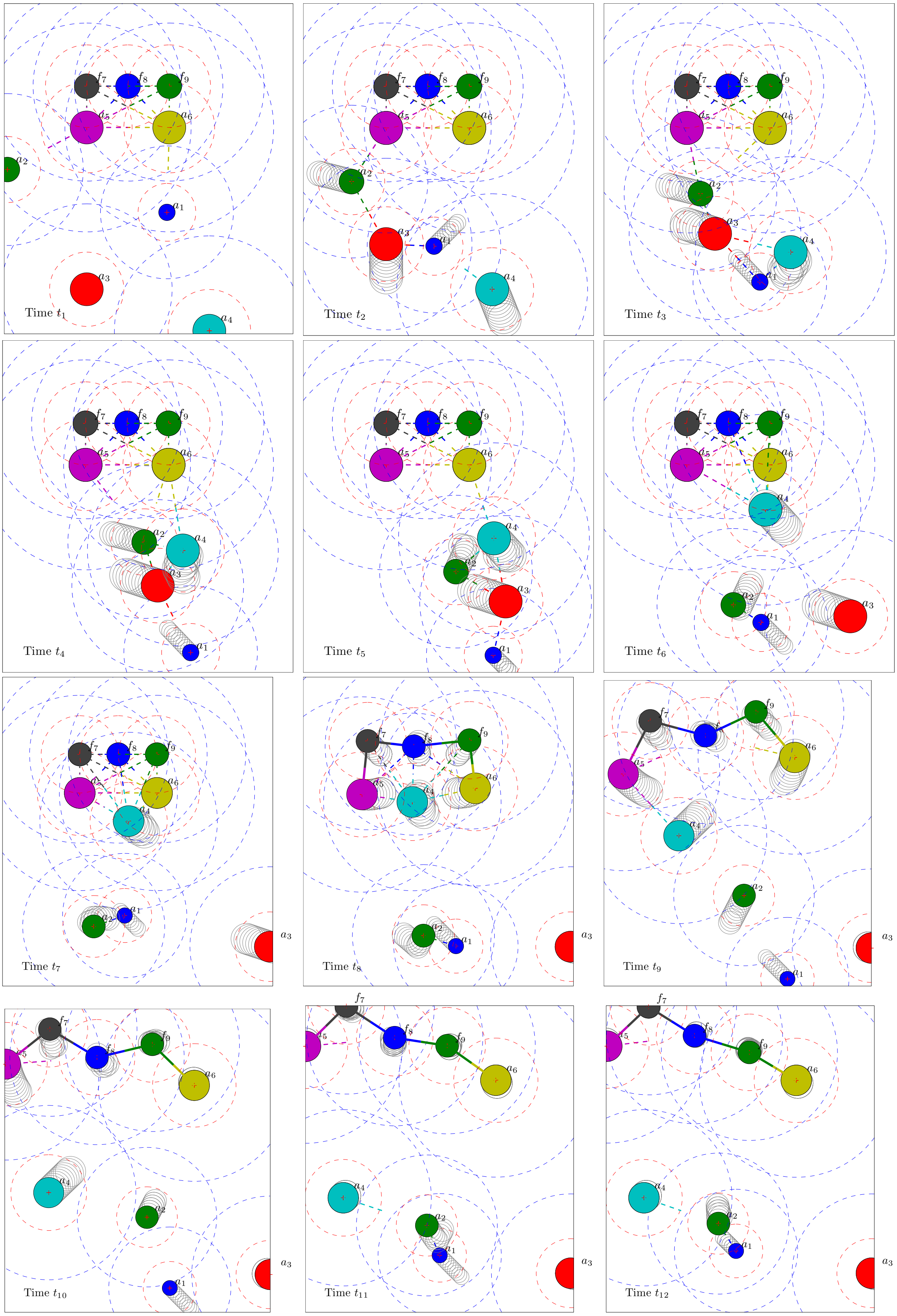


Figure 12.5: Decentralized multi-agent scenario with independent LTL_{X^-}specifications and decentralized Navigation Functions with limited sensing $R_{s,i}$ (blue dashed), collision avoidance distances $d_{c,i}$ (red dashed), thin dashed lines indicate sensing, thick continuous lines denote active connectivity links.

Part V

Appendices

Appendix A

Auxiliary Mathematical Proofs and Derivations

A.1 Notes on Degeneracy

The navigation function tuning parameter k has been set to $k = 1$ for reasons detailed in a previous point. It is worth noting the following about degeneracy at the destination of the various intermediate forms of the Koditschek-Rimon Navigation Function formula, as shown in Table A.2. Therefore there are five candidate non-degenerate at the destination function forms and two degenerate ones.

As shown previously, degeneracy of $\hat{\varphi}$ at the destination q_d causes problems to the PDE coefficient of the inverse obstacle fitting equation. For this reason, initially $\hat{\varphi}_1$ has been used. Unfortunately $\hat{\varphi}_1$ is unsquashed and not tunable. For this reason the alternative forms presented here have been developed.

Any of the nondegenerate forms can be used for inverse obstacle fitting. Tunable forms are preferred and unsquashed as well. The classic φ and the herein proposed φ_β both meet these requirements. Nonetheless, φ_β is not differentiable at the free space boundary. But this is not a problem, because the boundary is by definition uniformly maximal. Since this is a gradient system, it can be shown that for all initial conditions $x(0) \in \mathscr{F} \setminus \partial\mathscr{F}$, there exists a compact positive invariant set with boundary arbitrarily "close" to $\partial\mathscr{F}$.

Hence, working in $\mathscr{F} \setminus \partial\mathscr{F}$, all nondegenerate forms are diffeomorphic to $\hat{\varphi}$, so that the same curvature sufficiency condition applies.

A.2 Derivative Common Structure

Any KRNF has the structure (where V is a wildcard for the selected NF)

$$V(\gamma_d, \beta, k) = (f_2 \circ f_1 \circ \hat{\varphi})(\gamma_d, \beta, k) \tag{A.1}$$

where functions f_2, f_1 may be $\sigma_d(x) = (x)^{\frac{1}{k}}, \sigma(x) = \frac{x}{x+1}$, or the identity function. In the original NF formulation $f_1 = \sigma, f_2 = \sigma_d$, but other alternatives exist and are explored here. In all cases the fundamental building block is $\hat{\varphi} = \frac{\gamma_d^k}{\beta}$, which is a multiplicative form, operating differently than the additive form proposed by Khatib.

In the Khatib potential field formulation addition of repulsive effects yields a composite

Table A.1: Combinations of NF characteristics.

tunable/un	nondegenerate/deg (at q_d)	squashed/un	function
t	n	s	φ, φ_β
t	n	u	$\bar{\varphi}_\beta$
t	d	s	$\bar{\varphi}$
t	d	u	$\hat{\varphi}$
u	n	s	φ_p
u	n	u	$\hat{\varphi}_1$
u	d	s	impossible[a]
u	d	u	impossible[a]

[a] Degeneracy is caused by the exponent k of γ_d, hence simultaneous degeneracy and untunability are not a possible combination.

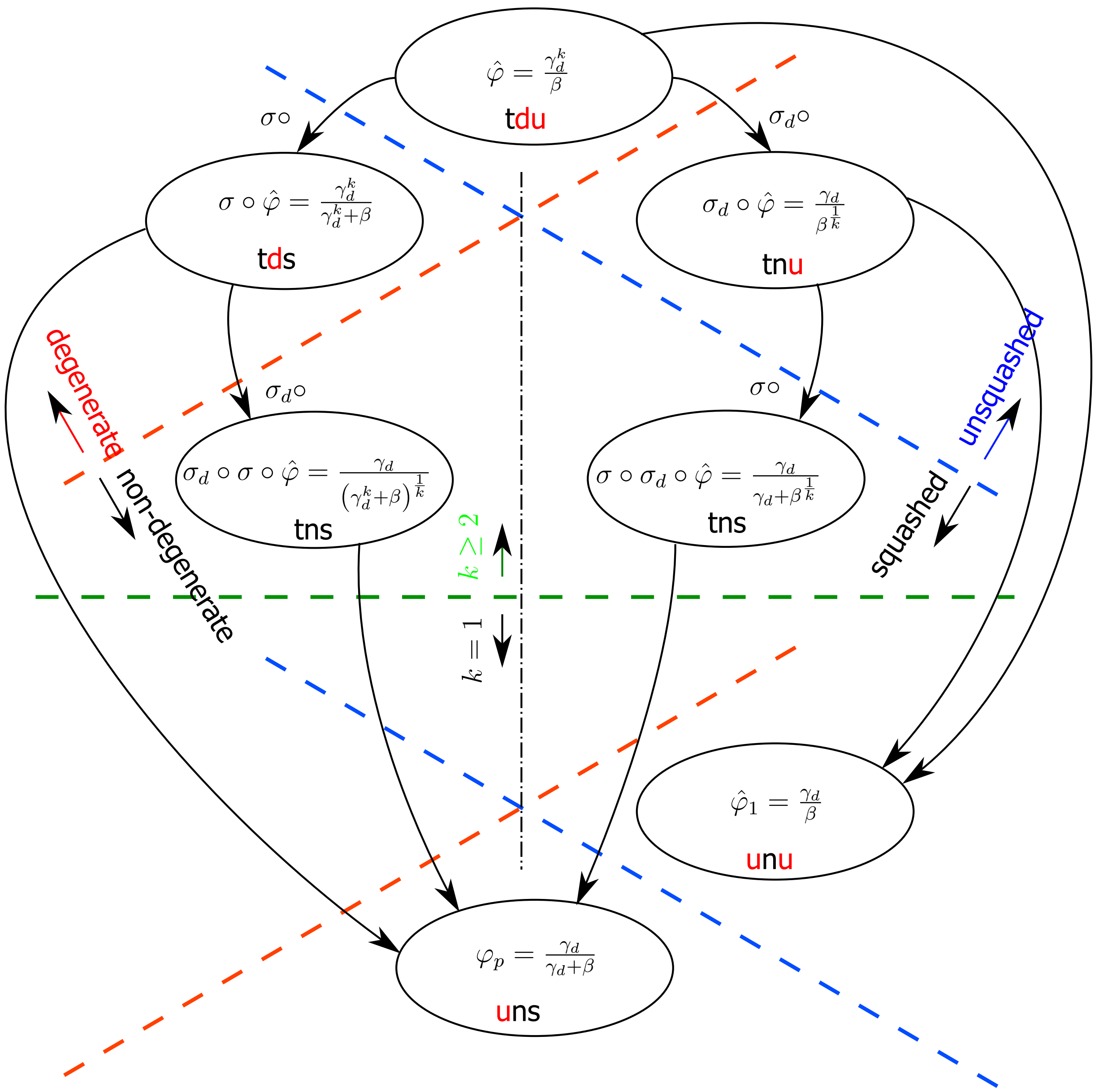


Figure A.1: Navigation Function formulas alternative compositions.

Table A.2: Destination degeneracy for KRNF intermediate formulas.

Composition[c]	Field Function	$\det\left(\left(D^2_{ij}\varphi\right)(q_d)\right)$	Characterization	Comments
$\hat{\varphi}\vert_{k=1}$	$\hat{\varphi}_1 = \frac{\gamma_d}{\beta}$	> 0[a]	untunable, non-degenerate, unsquashed	multiplicative $\neq$ Khatib additive
$\hat{\varphi}$	$\hat{\varphi} = \frac{\gamma_d^k}{\beta}$	0[b]	tunable, degenerate, unsquashed	$k \geq 2$
$\sigma_d \circ \hat{\varphi}$	$\bar{\varphi}_\beta = \frac{\gamma_d}{\beta^{\frac{1}{k}}}$	> 0[a]	tunable, non-degenerate, unsquashed	new
$\sigma \circ \hat{\varphi}\vert_{k=1}$	$\varphi_p = \frac{\gamma_d}{\gamma_d+\beta}$	> 0[a]	untunable, non-degenerate, squashed	Polynomial NF LPK
$\sigma \circ \hat{\varphi}$	$\bar{\varphi} = \frac{\gamma_d^k}{\gamma_d^k+\beta}$	0[b]	tunable, degenerate, squashed	$k \geq 2$
$\sigma_d \circ \sigma \circ \hat{\varphi}$	$\varphi = \frac{\gamma_d}{\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}}$	> 0[a]	tunable, non-degenerate, unsquashed	classic KRNF
$\sigma \circ \sigma_d \circ \hat{\varphi}$	$\varphi_\beta = \frac{\gamma_d}{\gamma_d+\beta^{\frac{1}{k}}}$	> 0[a]	tunable, non-degenerate, squashed	new

[a] $\neq 0$ would denote non-degeneracy. But in our case > 0 is for sure, since q_d is a global minimum by definition, since $\varphi(q) > 0 \iff q \neq q_d$ and $\varphi(q) = 0 \iff q = q_d$. As a result, when non-degenerate all eigenvalues will be > 0, hence their product, i.e. the determinant, will be > 0 as well.

[b] Since even when degenerate, still by definition $\varphi(q) > 0 \iff q \neq q_d$ and $\varphi(q) = 0 \iff q = q_d$ continue to hold, the destination will remain an isolated global minimum. It is degenerate in this case, but this does not affect its stable manifold. Note that Thom's splitting Lemma does not decompose the scalar field to a Morse and a Non-Morse components on complimentary subspaces (up to diffeomorphism) of the manifold's tangent space, because the Hessian matrix tends identically to the zero matrix. As a result in a neighborhood of q_d the function is purely Non-Morse, which essentially means that its Taylor expansion contains only third and higher order terms.

[c] $\sigma_d(x) \triangleq (x)^{\frac{1}{k}}, \sigma(x) \triangleq \frac{x}{x+1}$.

repulsive effect

$$\sum_i \left((\nabla U_i)(q)\right) = \begin{cases} \sum_i \left(\nabla \left(\frac{1}{2}\eta_i \left(\frac{1}{\beta_i(q)} - \frac{1}{\beta_{i0}}\right)^2\right)\right) \\ \quad = -\sum_i \left(\eta_i \left(\frac{1}{\beta_i(q)} - \frac{1}{\beta_{i0}}\right) \frac{1}{\beta_i^2(q)} \nabla \beta_i(q)\right), \quad \beta_i(q) \leq \beta_{i0} \\ 0 \in \mathbb{R}^n, \quad \beta_i(q) > \beta_{i0} \end{cases} \tag{A.2}$$

The effect of $q \to \partial \mathscr{O}_i \implies \beta_i(q) \to 0^+$ is similar to the behavior of a KRNF. But the intermediate field (not too near to a specific obstacle, neither too far outside β_{i0}) is only controllable using individual η_i, β_{i0}. The most straightforward solution is to decouple the obstacles by selecting appropriate β_{i0}, but this needs nontrivial geometric calculations. This is similar to the local diffeomorphisms applied for polynomial NFs.

Nevertheless, in a KRNF a similar procedure is applied, but using the single tuning parameter k, which effectively "decouples" disjoint obstacle effects. The calculations have an analogous flavor of finding disjoint obstacle neighborhoods. The calculations are just easier because we have confined the study to spheres. Of course, there is the benefit of bounded potential and bounded contol input, as well as a single parameter yielding tidier results. Arguably a strong point is that selecting a moderate k for reasonable scenarios uniformly yields satisfactorily results, whereas in the Khatib method different β_{i0} should be selected.

Differentiation of the KRNF form yields

$$\begin{aligned} \frac{\partial V}{\partial \gamma_d}(\gamma_d, \beta, k) &= \frac{\partial f_2}{\partial f_1}(f_1 \circ \hat{\varphi}) \frac{\partial f_1}{\partial \hat{\varphi}}(\hat{\varphi}) \frac{\partial \hat{\varphi}}{\partial \gamma_d}(\gamma_d, \beta, k) \\ \frac{\partial V}{\partial \beta}(\gamma_d, \beta, k) &= \frac{\partial f_2}{\partial f_1}(f_1 \circ \hat{\varphi}) \frac{\partial f_1}{\partial \hat{\varphi}}(\hat{\varphi}) \frac{\partial \hat{\varphi}}{\partial \beta}(\gamma_d, \beta, k) \\ \frac{\partial V}{\partial k}(\gamma_d, \beta, k) &= \frac{\partial f_2}{\partial f_1}(f_1 \circ \hat{\varphi}) \frac{\partial f_1}{\partial \hat{\varphi}}(\hat{\varphi}) \frac{\partial \hat{\varphi}}{\partial k}(\gamma_d, \beta, k) \end{aligned} \tag{A.3}$$

so terms $\frac{\partial f_2}{\partial f_1}, \frac{\partial f_1}{\partial \hat{\varphi}}$ are common and need not be multiply calculated. The alternatives are

$$\begin{aligned} \frac{\partial \sigma_d}{\partial x}(x) &= \frac{\partial}{\partial x}\left\{x^{\frac{1}{k}}\right\} = \frac{1}{k} x^{\frac{1}{k}-1} \\ \frac{\partial \sigma}{\partial x}(x) &= \frac{\partial}{\partial x}\left\{\frac{x}{x+1}\right\} = \frac{\frac{\partial}{\partial x}\{x\}(x+1) - x\frac{\partial}{\partial x}\{x+1\}}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = (x+1)^{-2} \end{aligned} \tag{A.4}$$

For $\hat{\varphi} \triangleq \frac{\gamma_d^k}{\beta}$ the partial derivatives are

$$\begin{aligned} \frac{\partial \hat{\varphi}}{\partial \gamma_d} &= \frac{\partial}{\partial \gamma_d}\left\{\frac{\gamma_d^k}{\beta}\right\} = \frac{1}{\beta}\frac{\partial}{\partial \gamma_d}\left\{\gamma_d^k\right\} = \frac{1}{\beta} k \gamma_d^{k-1} = \frac{k}{\gamma_d}\frac{\gamma_d^k}{\beta} \\ &= \frac{k}{\gamma_d}\hat{\varphi} \\ \frac{\partial \hat{\varphi}}{\partial \beta} &= \frac{\partial}{\partial \beta}\left\{\frac{\gamma_d^k}{\beta}\right\} = \gamma_d^k \frac{\partial}{\partial \beta}\left\{\beta^{-1}\right\} = -\gamma_d^k \beta^{-2} = -\beta^{-1}\frac{\gamma_d^k}{\beta} \\ &= -\frac{1}{\beta}\hat{\varphi} \\ \frac{\partial \hat{\varphi}}{\partial k} &= \frac{\partial}{\partial k}\left\{\frac{\gamma_d^k}{\beta}\right\} = \frac{1}{\beta}\frac{\partial}{\partial k}\left\{\gamma_d^k\right\} = \frac{1}{\beta}\frac{\partial}{\partial k}\left\{e^{k \ln \gamma_d}\right\} = \frac{1}{\beta} e^{k \ln \gamma_d}\frac{\partial}{\partial k}\left\{k \ln \gamma_d\right\} \\ &= \frac{1}{\beta}\gamma_d^k \ln(\gamma_d) = \ln(\gamma_d)\frac{\gamma_d^k}{\beta} \\ &= \ln(\gamma_d)\hat{\varphi} \end{aligned} \tag{A.5}$$

Combining the previous results, the derivatives of the various function forms are

$$
\begin{aligned}
\nabla_q\hat{\varphi} &= \left(\frac{k}{\gamma_d}\hat{\varphi}\right)\nabla_q\gamma_d + \left(-\frac{1}{\beta}\hat{\varphi}\right)\nabla_q\beta + (\ln(\gamma_d)\hat{\varphi})\,\nabla_q k \\
&= \left(\frac{k}{\gamma_d}\nabla\gamma_d - \frac{1}{\beta}\nabla\beta + \ln(\gamma_d)\nabla k\right)\hat{\varphi} \\
&= \left(k\frac{\nabla\gamma_d}{\gamma_d} - \frac{\nabla\beta}{\beta} + \ln(\gamma_d)\nabla k\right)\hat{\varphi} \\
\nabla_q\hat{\varphi}_1 &= \left(1\frac{\nabla_q\gamma_d}{\gamma_d} - \frac{\nabla_q\beta}{\beta} + \ln(\gamma_d)0\right)\hat{\varphi}_1 \\
&= \left(\frac{\nabla\gamma_d}{\gamma_d} - \frac{\nabla\beta}{\beta}\right)\hat{\varphi}_1 \\
\nabla_q\hat{\varphi}_\beta &= \frac{\partial\sigma_d}{\partial\hat{\varphi}}(\hat{\varphi})\nabla_q\hat{\varphi} \\
&= \frac{1}{k}(\hat{\varphi})^{\frac{1}{k}-1}\left(k\frac{\nabla\gamma_d}{\gamma_d} - \frac{\nabla\beta}{\beta} + \ln(\gamma_d)\nabla k\right)\hat{\varphi} \\
&= \frac{1}{k}(\hat{\varphi})^{\frac{1}{k}}\left(k\frac{\nabla\gamma_d}{\gamma_d} - \frac{\nabla\beta}{\beta} + \ln(\gamma_d)\nabla k\right) \\
\nabla_q\bar{\varphi} &= \frac{\partial\sigma}{\partial\hat{\varphi}}(\hat{\varphi})\nabla_q\hat{\varphi} \\
&= (\hat{\varphi}+1)^{-2}\left(k\frac{\nabla\gamma_d}{\gamma_d} - \frac{\nabla\beta}{\beta} + \ln(\gamma_d)\nabla k\right)\hat{\varphi} \\
&= \frac{\hat{\varphi}}{(\hat{\varphi}+1)^2}\left(k\frac{\nabla\gamma_d}{\gamma_d} - \frac{\nabla\beta}{\beta} + \ln(\gamma_d)\nabla k\right) \\
\nabla_q\varphi_p &= \frac{\hat{\varphi}}{(\hat{\varphi}+1)^2}\left(1\frac{\nabla_q\gamma_d}{\gamma_d} - \frac{\nabla_q\beta}{\beta} + \ln(\gamma_d)0\right) \\
&= \frac{\hat{\varphi}}{(\hat{\varphi}+1)^2}\left(\frac{\nabla\gamma_d}{\gamma_d} - \frac{\nabla\beta}{\beta}\right) \\
\nabla_q\varphi &= \frac{\partial\sigma_d}{\partial\sigma}(\sigma\circ\hat{\varphi})\frac{\partial\sigma}{\partial\hat{\varphi}}\nabla_q\hat{\varphi} \\
&= \frac{1}{k}(\sigma\circ\hat{\varphi})^{\frac{1}{k}-1}(\hat{\varphi}+1)^{-2}\nabla_q\hat{\varphi} \\
&= \frac{1}{k}\left(\frac{\hat{\varphi}}{\hat{\varphi}+1}\right)^{\frac{1}{k}-1}(\hat{\varphi}+1)^{-2}\left(k\frac{\nabla\gamma_d}{\gamma_d} - \frac{\nabla\beta}{\beta} + \ln(\gamma_d)\nabla k\right)\hat{\varphi} \\
&= \frac{1}{k}\left(\frac{\hat{\varphi}}{\hat{\varphi}+1}\right)^{\frac{1}{k}-1}\frac{\hat{\varphi}}{\hat{\varphi}+1}\frac{1}{\hat{\varphi}+1}\left(k\frac{\nabla\gamma_d}{\gamma_d} - \frac{\nabla\beta}{\beta} + \ln(\gamma_d)\nabla k\right) \\
&= \frac{1}{k}\left(\frac{\hat{\varphi}}{\hat{\varphi}+1}\right)^{\frac{1}{k}}\frac{1}{\hat{\varphi}+1}\left(k\frac{\nabla\gamma_d}{\gamma_d} - \frac{\nabla\beta}{\beta} + \ln(\gamma_d)\nabla k\right) \\
\nabla_q\varphi_\beta &= \frac{\partial\sigma}{\partial\sigma_d}(\sigma_d\circ\hat{\varphi})\frac{\partial\sigma_d}{\partial\hat{\varphi}}\nabla_q\hat{\varphi} \\
&= (\sigma_d\circ\hat{\varphi}+1)^{-2}\frac{1}{k}\hat{\varphi}^{\frac{1}{k}-1}\nabla\hat{\varphi} \\
&= (\hat{\varphi}^{\frac{1}{k}}+1)^{-2}\frac{1}{k}\hat{\varphi}^{\frac{1}{k}-1}\left(k\frac{\nabla\gamma_d}{\gamma_d} - \frac{\nabla\beta}{\beta} + \ln(\gamma_d)\nabla k\right)\hat{\varphi} \\
&= \frac{1}{k}\frac{\hat{\varphi}^{\frac{1}{k}}}{(\hat{\varphi}^{\frac{1}{k}}+1)^2}\left(k\frac{\nabla\gamma_d}{\gamma_d} - \frac{\nabla\beta}{\beta} + \ln(\gamma_d)\nabla k\right)
\end{aligned}
\tag{A.6}
$$

Table A.3: Summary of derivatives.

Function	Gradient
$\nabla_q\hat{\varphi}$	$\left(k\frac{\nabla\gamma_d}{\gamma_d}-\frac{\nabla\beta}{\beta}+\ln(\gamma_d)\nabla k\right)\hat{\varphi}=\left(k\frac{\nabla\gamma_d}{\gamma_d}-\frac{\nabla\beta}{\beta}+\ln(\gamma_d)\nabla k\right)\frac{\gamma_d^k}{\beta}$
$\nabla_q\hat{\varphi}_1$	$\left(\frac{\nabla\gamma_d}{\gamma_d}-\frac{\nabla\beta}{\beta}\right)\hat{\varphi}_1=\left(\frac{\nabla\gamma_d}{\gamma_d}-\frac{\nabla\beta}{\beta}\right)\frac{\gamma_d^1}{\beta}$
$\nabla_q\hat{\varphi}_\beta$	$\frac{1}{k}(\hat{\varphi})^{\frac{1}{k}}\left(k\frac{\nabla\gamma_d}{\gamma_d}-\frac{\nabla\beta}{\beta}+\ln(\gamma_d)\nabla k\right)=\frac{1}{k}\frac{\gamma_d}{\beta^{\frac{1}{k}}}\left(k\frac{\nabla\gamma_d}{\gamma_d}-\frac{\nabla\beta}{\beta}+\ln(\gamma_d)\nabla k\right)$
$\nabla_q\bar{\varphi}$	$\frac{\hat{\varphi}}{(\hat{\varphi}+1)^2}\left(k\frac{\nabla\gamma_d}{\gamma_d}-\frac{\nabla\beta}{\beta}+\ln(\gamma_d)\nabla k\right)=\frac{\gamma_d^k\beta}{\left(\gamma_d^k+\beta\right)^2}\left(k\frac{\nabla\gamma_d}{\gamma_d}-\frac{\nabla\beta}{\beta}+\ln(\gamma_d)\nabla k\right)$
$\nabla_q\varphi_p$	$\frac{\hat{\varphi}}{(\hat{\varphi}+1)^2}\left(\frac{\nabla\gamma_d}{\gamma_d}-\frac{\nabla\beta}{\beta}\right)=\frac{\gamma_d^k\beta}{\left(\gamma_d^k+\beta\right)^2}\left(\frac{\nabla\gamma_d}{\gamma_d}-\frac{\nabla\beta}{\beta}\right)$
$\nabla_q\varphi$	$\left(\gamma_d^k+\beta\right)^{-\frac{1}{k}-1}\frac{\gamma_d\beta}{k}\left(k\frac{\nabla\gamma_d}{\gamma_d}-\frac{\nabla\beta}{\beta}+\ln\left(\gamma_d\right)\nabla k\right)$ $=\left(\gamma_d^k+\beta\right)^{-\frac{1}{k}-1}\left(\beta\nabla\gamma_d-\frac{\gamma_d}{k}\nabla\beta+\frac{\gamma_d\beta\ln(\gamma_d)}{k}\nabla k\right)$

and since

$$
\begin{aligned}
&\frac{1}{k}\left(\frac{\hat{\varphi}}{\hat{\varphi}+1}\right)^{\frac{1}{k}}\frac{1}{\hat{\varphi}+1}\left(k\frac{\nabla\gamma_d}{\gamma_d}-\frac{\nabla\beta}{\beta}+\ln(\gamma_d)\nabla k\right)=\\
&=\frac{1}{k}\left(\frac{\frac{\gamma_d^k}{\beta}}{\frac{\gamma_d^k}{\beta}+1}\right)^{\frac{1}{k}}\frac{1}{\frac{\gamma_d}{\beta}+1}\left(k\frac{\nabla\gamma_d}{\gamma_d}-\frac{\nabla\beta}{\beta}+\ln(\gamma_d)\nabla k\right)\\
&=\frac{1}{\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}}\frac{1}{\gamma_d^k+\beta}\frac{\gamma_d\beta}{k}\left(k\frac{\nabla\gamma_d}{\gamma_d}-\frac{\nabla\beta}{\beta}+\ln(\gamma_d)\nabla k\right)\\
&=\left(\gamma_d^k+\beta\right)^{-\frac{1}{k}-1}\left(\beta\nabla\gamma_d-\frac{\gamma_d}{k}\nabla\beta+\frac{\gamma_d\beta\ln\left(\gamma_d\right)}{k}\nabla k\right)
\end{aligned}
\tag{A.7}
$$

a summary of the above is provided in the Table A.3.

A.3 Gradients and Hessian matrices

In this section derivations are provided for gradients and Hessian matrices of the functions

$$
\begin{aligned}
&\gamma_d=\|q-q_d\|^2,\quad \gamma_d^k,\quad \beta_i=\|q-q_i\|^2-\rho_i^2, i\in I_1\\
&\beta_0=\rho_0^2-\|q-q_0\|^2=\rho_0^2-\|q\|^2,\quad \beta=\prod_{i=0}^{M}\beta_i,\quad \hat{\varphi}_1=\frac{\gamma_d}{\beta},\quad \hat{\varphi}=\frac{\gamma_d^k}{\beta}
\end{aligned}
\tag{A.8}
$$

and the alternative potential functions (not all navigation functions)

$$
\varphi_p=\frac{\gamma_d}{\gamma_d+\beta},\quad \bar{\varphi}=\frac{\gamma_d^k}{\gamma_d^k+\beta},\quad \varphi=\sqrt[k]{\frac{\gamma_d^k}{\gamma_d^k+\beta}}=\frac{\gamma_d}{\sqrt[k]{\gamma_d^k+\beta}}
\tag{A.9}
$$

at any point q in the free space $\mathscr{F}$ and at any critical point $q_c \in \mathscr{F} \cap \mathscr{C}_f$ (where f is each function considered). Note that Lemma 3.1[1] is applied at critical points.

A.3.1 $\gamma_d = \|q - q_d\|^2$

$$\begin{aligned} \gamma_d(q) &= \|q - q_d\|^2 \implies \\ (\nabla \gamma_d)(q) &= 2(q - q_d) \implies \\ \left(D^2 \gamma_d\right)(q) &= 2I \end{aligned} \tag{A.10}$$

A.3.2 $\gamma_d^k = \|q - q_d\|^{2k}$

$$\begin{aligned} \gamma_d^k(q) &= \|q - q_d\|^{2k} \implies \\ \nabla\left(\gamma_d^k\right) &= k\gamma_d^{k-1}\nabla\gamma_d \implies \\ \left(\nabla\left(\gamma_d^k\right)\right)(q) &= k\gamma_d^{k-1}2(q - q_d) = 2k\|q - q_d\|^{2k-1}\frac{q - q_d}{\|q - q_d\|} \implies \\ D^2\left(\gamma_d^k\right) &= \nabla\gamma_d\nabla\left(k\gamma_d^{k-1}\right)^T + k\gamma_d^{k-1}\nabla^2\gamma_d = k\gamma_d^{k-1}\left(\frac{k-1}{\gamma_d}\nabla\gamma_d\nabla\gamma_d^T + 2I\right) \\ &= k\gamma_d^{k-2}\left((k-1)\nabla\gamma_d\nabla\gamma_d^{\mathrm{T}} + 2I\gamma_d\right) \end{aligned} \tag{A.11}$$

A.3.3 $\beta_i = \|q - q_i\|^2 - \rho_i^2$

$$\begin{aligned} \beta_i(q) &= \|q - q_i\|^2 - \rho_i^2, i \in I_1 \implies \\ (\nabla\beta_i)(q) &= 2(q - q_i) \implies \\ \left(D^2\beta_i\right)(q) &= (D(\nabla\beta_i))(q) = D(2(q - q_i)) = 2D(q - q_i) = 2I \end{aligned} \tag{A.12}$$

Hence

$$\begin{aligned} \|(\nabla\beta_i)(q)\| &= \|2(q - q_i)\| = 2\|q - q_i\| = 2\sqrt{\|q - q_i\|^2} \\ &= 2\sqrt{\underbrace{\left(\|q - q_i\|^2 - \rho_i^2\right)}_{\beta_i} + \rho_i^2} = 2\sqrt{\beta_i + \rho_i^2} \end{aligned} \tag{A.13}$$

A.3.4 $\beta_0 = \rho_0^2 - \|q\|^2$

$$\begin{aligned} \beta_0(q) &= \rho_0^2 - \|q\|^2 \implies \\ (\nabla\beta_0)(q) &= \nabla\left(\rho_0^2 - \|q\|^2\right) = \nabla\left(\rho_0^2\right) - \nabla\left(\|q\|^2\right) = -2q \implies \\ \left(D^2\beta_0\right)(q) &= (D(\nabla\beta_0))(q) = D(-2q) = -2D(q) = -2I \end{aligned} \tag{A.14}$$

[1] [23], Lemma 3.1, p.426.

Hence

$$\begin{aligned}\|(\nabla\beta_0)(q)\| &= \|-2q\| = 2\|q\| = 2\sqrt{\|q\|^2} = 2\sqrt{\rho_0^2-\rho_0^2+\|q\|^2} \\ &= 2\sqrt{\rho^2-\underbrace{\left(\rho^2-\|q\|^2\right)}_{\beta_0}} = 2\sqrt{\rho^2-\beta_0} = 2\sqrt{-\beta_0+\rho_0^2}\end{aligned} \tag{A.15}$$

A.3.5 $\beta = \prod_{i=0}^{M}\beta_i$

$$\begin{aligned}\nabla\beta &= \nabla\left(\prod_{i=0}^{M}\beta_i\right) = \sum_{i=0}^{M}\left(\left(\prod_{j=0,j\neq i}^{M}\beta_i\right)\nabla\beta_i\right) = \sum_{i=0}^{M}\left(\bar{\beta}_i\nabla\beta_i\right) \\ &= \sum_{i=0}^{M}\left(\frac{\beta}{\beta_i}\nabla\beta_i\right) = \beta\sum_{i=0}^{M}\frac{\nabla\beta_i}{\beta_i}\end{aligned} \tag{A.16}$$

and

$$\begin{aligned}D^2(\beta) &= D\left(\beta\sum_{i=0}^{M}\frac{\nabla\beta_i}{\beta_i}\nabla\beta_i\right) = \left(\sum_{i=0}^{M}\frac{\nabla\beta_i}{\beta_i}\nabla\beta_i\right)\nabla\beta^T + \beta\sum_{i=0}^{M}D\left(\frac{\nabla\beta_i}{\beta_i}\nabla\beta_i\right) \\ &= \left(\sum_{i=0}^{M}\frac{\nabla\beta_i}{\beta_i}\nabla\beta_i\right)\beta\sum_{i=0}^{M}\frac{\nabla\beta_i}{\beta_i}\nabla\beta_i + \beta\sum_{i=0}^{M}D\left(\nabla\beta_i\nabla\left(\frac{1}{\beta_i}\right)^T + D^2\beta_i\right) \\ &= \beta\sum_{i=0}^{M}\sum_{j=0}^{M}\left(\frac{\nabla\beta_i\nabla\beta_j^T}{\beta_i\beta_j}\right) + \beta\beta\sum_{i=0}^{M}D\left(\nabla\beta_i\nabla\left(\frac{1}{\beta_i}\right)^T + D^2\beta_i\right) \\ &= \beta\sum_{i=0}^{M}\left(\nabla\beta_i\nabla\left(\frac{1}{\beta_i}\right)^T + D^2\beta_i + \sum_{j=0}^{M}\left(\frac{\nabla\beta_i\nabla\beta_j^T}{\beta_i\beta_j}\right)\right)\end{aligned} \tag{A.17}$$

A.3.6 $\hat{\varphi}_1 = \frac{\gamma_d}{\beta}$

A.3.6.1 Any point

$$\begin{aligned}\nabla\left(\hat{\varphi}_1\right) &= \nabla\left(\frac{\gamma_d}{\beta}\right) = \frac{\beta\nabla\gamma_d - \gamma_d\nabla\beta}{\beta^2} \\ &= \frac{\nabla\gamma_d}{\beta} - \frac{\gamma_d}{\beta}\frac{\nabla\beta}{\beta} = \frac{\nabla\gamma_d}{\beta} - \varphi_1\frac{\nabla\beta}{\beta} = \frac{\nabla\gamma_d}{\beta} - \varphi_1\sum_{i=0}^{M}\frac{\nabla\beta_i}{\beta_i}\end{aligned} \tag{A.18}$$

$$\begin{aligned}D^2\left(\hat{\varphi}_1\right) &= D^2\left(\frac{\gamma_d}{\beta}\right) \\ &= \frac{1}{\beta^2}\left[\beta D^2\gamma_d + \nabla\gamma_d\nabla\beta^T - \nabla\beta\nabla\gamma_d^T - \gamma_d D^2\beta\right] + \beta^2\nabla\varphi_1\left(\nabla\frac{1}{\beta^2}\right)^T\end{aligned} \tag{A.19}$$

A.3.6.2 Critical point

Gradient

$$\begin{aligned}\nabla\left(\frac{\gamma_d}{\beta}\right)(q_c)=0 \iff \beta\nabla\gamma_d=\gamma_d\nabla\beta,\forall q\in\mathscr{F} \implies&\\ \nabla\gamma_d=\frac{\gamma_d}{\beta}\nabla\beta=\varphi_1\nabla\beta,\forall q\in\mathscr{F}-\partial\mathscr{F} \implies&\\ \frac{\nabla\gamma_d}{\gamma_d}=\frac{\nabla\beta}{\beta}=\sum_{i=0}^{M}\frac{\nabla\beta_i}{\beta_i},\forall q\in\mathscr{F}-\partial\mathscr{F}-\{q_d\}&\end{aligned} \tag{A.20}$$

Hessian matrix

$$\begin{aligned}D^2\left(\frac{\gamma_d}{\beta}\right)(q_c)&=\frac{1}{\beta^2}\left[\beta D^2\gamma_d-\gamma_d D^2\beta\right]\\&=\frac{1}{\beta^2}\left[2\beta I-\gamma_d D^2\beta\right]\end{aligned} \tag{A.21}$$

A.3.7 $\hat{\varphi}=\frac{\gamma_d^k}{\beta}$

A.3.7.1 Any point

$$\begin{aligned}\nabla\left(\frac{\gamma_d^k}{\beta}\right)(q)&=\frac{\beta\nabla\left(\gamma_d^k\right)-\gamma_d^k\nabla\beta}{\beta^2}=\frac{\beta k\gamma_d^{k-1}\nabla\gamma_d-\gamma_d^k\nabla\beta}{\beta^2}=k\frac{\gamma_d^k}{\beta}\frac{\nabla\gamma_d}{\gamma_d}-\frac{\gamma_d^k}{\beta}\frac{\nabla\beta}{\beta}\\&=\frac{\gamma_d^k}{\beta}\left(k\frac{\nabla\gamma_d}{\gamma_d}-\frac{\nabla\beta}{\beta}\right)=\hat{\varphi}\left(k\frac{\nabla\gamma_d}{\gamma_d}-\frac{\nabla\beta}{\beta}\right)\\&=\hat{\varphi}\left(k\frac{2(q-q_d)}{\|q-q_d\|^2}-\sum_{i=0}^{M}\frac{\nabla\beta_i}{\beta_i}\right)\end{aligned} \tag{A.22}$$

$$\begin{aligned}D^2\left(\hat{\varphi}\right)&=D^2\left(\frac{\gamma_d^k}{\beta}\right)(q)\\&=\frac{1}{\beta^2}\left[\beta D^2\left(\gamma_d^k\right)+\nabla\left(\gamma_d^k\right)\nabla\beta^T-\nabla\beta\nabla\left(\gamma_d^k\right)^T-\gamma_d^k D^2\beta\right]+\beta^2\nabla\hat{\varphi}\left(\nabla\frac{1}{\beta^2}\right)^T\end{aligned} \tag{A.23}$$

A.3.7.2 Critical point

$$\begin{aligned}D^2\left(\frac{\gamma_d^k}{\beta}\right)(q_c)&=\frac{1}{\beta^2}\left[\beta D^2\left(\gamma_d^k\right)-\gamma_d^k D^2\beta\right]=\frac{1}{\beta}D^2\left(\gamma_d^k\right)-\frac{\gamma_d^k}{\beta}\frac{D^2\beta}{\beta}\\&=\frac{1}{\beta}D^2\left(\gamma_d^k\right)-\hat{\varphi}\frac{D^2\beta}{\beta}\end{aligned} \tag{A.24}$$

At the destination a global minimum exists, where

$$D^2\left(\gamma_d^k\right)(q_d)=k\gamma_d^{k-2}(q_d)\left((k-1)(\nabla\gamma_d)(q_d)(\nabla\gamma_d)(q_d)^{\mathrm{T}}+2I\gamma_d(q_d)\right) \tag{A.25}$$

and since $(\nabla\gamma_d)(q_d)=0\in E^n,\gamma_d(q_d)=0,k\geq 2$ it follows that $D^2\left(\gamma_d^k\right)(q_d)=0\in\mathbb{R}^{n\times n}$. Taking into consideration that also $\hat{\varphi}(q_d)=0$, it follows that $\left(D^2\hat{\varphi}\right)(q_d)=0\in\mathbb{R}^{n\times n}$ and hence $\hat{\varphi}$ is degenerate at the destination q_d.

A.3.8 $\varphi_p = \frac{\gamma_d}{\gamma_d+\beta}$

A.3.8.1 Any point

$$\nabla\varphi(q) = \nabla\left(\frac{\gamma_d}{\gamma_d+\beta}\right) = \frac{(\gamma_d+\beta)\,\nabla\gamma_d - \gamma_d\nabla(\gamma_d+\beta)}{(\gamma_d+\beta)^2} = \frac{\beta\nabla\gamma_d - \gamma_d\nabla\beta}{(\gamma_d+\beta)^2} \tag{A.26}$$

and

$$\begin{aligned} D^2\varphi(q) = {} & \frac{1}{(\gamma_d+\beta)^2}\Big[(\gamma_d+\beta)\,D^2\gamma_d + \nabla\gamma_d\nabla(\gamma_d+\beta)^T - \nabla(\gamma_d+\beta)\,\nabla\gamma_d^T - \gamma_d D^2(\gamma_d+\beta)\Big] + \\ & + (\gamma_d+\beta)^2\,\nabla\varphi\left(\nabla\frac{1}{(\gamma_d+\beta)^2}\right)^T \end{aligned} \tag{A.27}$$

A.3.8.2 Critical point

$$\begin{aligned} D^2\varphi(q_c) &= \frac{1}{(\gamma_d+\beta)^2}\left[(\gamma_d+\beta)\,D^2\gamma_d - \gamma_d D^2(\gamma_d+\beta)\right] \\ &= \frac{1}{(\gamma_d+\beta)^2}\left[(\gamma_d+\beta)\,2I - \gamma_d\left(2I + D^2\beta\right)\right] \\ &= \frac{2\beta I - \gamma_d D^2\beta}{(\gamma_d+\beta)^2} \end{aligned} \tag{A.28}$$

A.3.9 $\bar{\varphi} = \frac{\gamma_d^k}{\gamma_d^k+\beta}$

A.3.9.1 Any point

$$\begin{aligned} \nabla\bar{\varphi} &= \frac{\left(\gamma_d^k+\beta\right)\nabla\left(\gamma_d^k\right) - \gamma_d^k\nabla\left(\gamma_d^k+\beta\right)}{\left(\gamma_d^k+\beta\right)^2} \\ &= \frac{1}{\left(\gamma_d^k+\beta\right)^2}\left[\gamma_d^k\nabla\left(\gamma_d^k\right) + \beta\nabla\left(\gamma_d^k\right) - \gamma_d^k\left(\nabla\left(\gamma_d^k\right) + \nabla\beta\right)\right] \\ &= \frac{1}{\left(\gamma_d^k+\beta\right)^2}\left[\gamma_d^k\nabla\left(\gamma_d^k\right) + \beta\nabla\left(\gamma_d^k\right) - \gamma_d^k\nabla\left(\gamma_d^k\right) - \gamma_d^k\nabla\beta\right] \\ &= \frac{1}{\left(\gamma_d^k+\beta\right)^2}\left[\beta\nabla\left(\gamma_d^k\right) - \gamma_d^k\nabla\beta\right] \end{aligned} \tag{A.29}$$

and

$$\begin{aligned} D^2\bar{\varphi} = {} & \frac{1}{\left(\gamma_d^k+\beta\right)^2}\cdot \\ & \cdot\Big[\left(\gamma_d^k+\beta\right)D^2\left(\gamma_d^k\right) + \nabla\left(\gamma_d^k\right)\nabla\left(\gamma_d^k+\beta\right)^T - \nabla\left(\gamma_d^k+\beta\right)\nabla\left(\gamma_d^k\right)^T - \gamma_d^k D^2\left(\gamma_d^k+\beta\right)\Big] + \\ & + \left(\gamma_d^k+\beta\right)^2\nabla\varphi\left(\nabla\frac{1}{\left(\gamma_d^k+\beta\right)^2}\right)^2 \end{aligned} \tag{A.30}$$

A.3.9.2 Critical point

$$
\begin{aligned}
\left(D^2\bar{\varphi}\right)(q_c) &= \frac{1}{\left(\gamma_d^k+\beta\right)^2}\left[\left(\gamma_d^k+\beta\right)D^2\left(\gamma_d^k\right)-\gamma_d^k D^2\left(\gamma_d^k+\beta\right)\right] \\
&= \frac{1}{\left(\gamma_d^k+\beta\right)^2}\left[\gamma_d^k D^2\left(\gamma_d^k\right)+\beta D^2\left(\gamma_d^k\right)-\gamma_d^k D^2\left(\gamma_d^k\right)+\gamma_d^k D^2(\beta)\right] \\
&= \frac{1}{\left(\gamma_d^k+\beta\right)^2}\left[\beta D^2\left(\gamma_d^k\right)+\gamma_d^k D^2(\beta)\right]
\end{aligned}
\tag{A.31}
$$

In subsection A.3.7 it has been shown that at the destination q_d the Hessian matrix $\left(D^2\left(\gamma_d^k\right)\right)(q_d)=0\in\mathbb{R}^{n\times n}$ is fully degenerate. Also $\gamma_d(q_d)=0, k\geq 2, q_d\notin\partial\mathscr{F} \implies \beta(q_d)$, hence $\left(D^2\bar{\varphi}\right)(q_d)=0\in\mathbb{R}^{n\times n}$, i.e. function $\bar{\varphi}$ is degenerate at the destination q_d.

A.3.10 $\varphi=\frac{\gamma_d}{\sqrt[k]{\gamma_d^k+\beta}}$

A.3.10.1 Any point

$$
\nabla\varphi(q)=\frac{\sqrt[k]{\gamma_d^k+\beta}\,\nabla(\gamma_d)-\gamma_d\nabla\left(\sqrt[k]{\gamma_d^k+\beta}\right)}{\left(\sqrt[k]{\gamma_d^k+\beta}\right)^2}
\tag{A.32}
$$

$$
\begin{aligned}
D^2\varphi(q) = {} & \frac{1}{\left(\sqrt[k]{\gamma_d^k+\beta}\right)^2}\cdot \\
& \cdot\left[\sqrt[k]{\gamma_d^k+\beta}\,D^2\gamma_d+\nabla\gamma_d\nabla\left(\sqrt[k]{\gamma_d^k+\beta}\right)^T-\nabla\left(\sqrt[k]{\gamma_d^k+\beta}\right)\nabla\gamma_d^T-\gamma_d D^2\left(\sqrt[k]{\gamma_d^k+\beta}\right)\right]+ \\
& +\left(\sqrt[k]{\gamma_d^k+\beta}\right)^2\nabla\varphi\left(\nabla\frac{1}{\left(\sqrt[k]{\gamma_d^k+\beta}\right)^2}\right)^T
\end{aligned}
\tag{A.33}
$$

A.3.10.2 Critical point

$$
\begin{aligned}
D^2\varphi(q_c) &= \frac{1}{\left(\gamma_d^k+\beta\right)^{\frac{2}{k}}}\left[\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}D^2\gamma_d-\gamma_d D^2\left(\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\right)\right] \\
&= \frac{1}{\left(\gamma_d^k+\beta\right)^{\frac{2}{k}}}\left[\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}2I-\gamma_d D^2\left(\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}\right)\right]
\end{aligned}
\tag{A.34}
$$

A.4 PDE substitution

For the classic KRNF form $\varphi = \frac{\gamma_d}{\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}}$, the partial derivatives are

$$\begin{aligned}
\frac{\partial\varphi}{\partial\gamma_d} &= \frac{\partial}{\partial\gamma_d}\left\{\frac{\gamma_d}{\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}}\right\} = \left(\gamma_d^k+\beta\right)^{\frac{1}{k}} + \gamma_d\frac{\partial}{\partial\gamma_d}\left\{\left(\gamma_d^k+\beta\right)^{-\frac{1}{k}}\right\} \\
&= \left(\gamma_d^k+\beta\right)^{-\frac{1}{k}} + \gamma_d\left(-\frac{1}{k}\right)\left(\gamma_d^n+\beta\right)^{-\frac{1}{k}-1}\frac{\partial}{\partial\gamma_d}\left\{\gamma_d^k+\beta\right\} \\
&= \left(\gamma_d^k+\beta\right)^{-\frac{1}{k}}\left(1-\frac{\gamma_d^k}{\gamma_d^k+\beta}\right) = \left(\gamma_d^k+\beta\right)^{-\frac{1}{k}}\frac{\gamma_d^k+\beta-\gamma_d^k}{\gamma_d^k+\beta} \\
&= \beta\left(\gamma_d+\beta\right)^{-\frac{1}{k}-1} \\
\frac{\partial\varphi}{\partial\beta} &= \frac{\partial}{\partial\beta}\left\{\frac{\gamma_d}{\left(\gamma_d^k+\beta\right)^{\frac{1}{k}}}\right\} = \gamma_d\frac{\partial}{\partial\beta}\left\{\left(\gamma_d+\beta\right)^{-\frac{1}{k}}\right\} = \gamma_d\left(-\frac{1}{k}\right)\left(\gamma_d^k+\beta\right)^{-\frac{1}{k}-1}\frac{\partial}{\partial\beta}\left\{\gamma_d^k+\beta\right\} \\
&= -\frac{1}{k}\gamma_d\left(\gamma_d^k+\beta\right)^{-\frac{1}{k}-1} = -\frac{1}{k}\frac{\gamma_d}{\left(\gamma_d+\beta\right)^{\frac{1}{k}}}\left(\gamma_d^k+\beta\right)^{-1} = -\frac{1}{k}\left(\gamma_d^k+\beta\right)^{-1}\varphi \\
&= -\frac{1}{k}\gamma_d\left(\gamma_d^k+\beta\right)^{-\frac{1}{k}-1}
\end{aligned} \tag{A.35}$$

Then by substitution in (9.22) we obtain

$$\begin{aligned}
\nabla\beta &= -\left(\frac{u+\frac{\partial\varphi}{\partial\gamma_d}\nabla\gamma_d}{\frac{\partial\varphi}{\partial\beta}}\right) = -\left(\frac{u+\beta\left(\gamma_d^k+\beta\right)^{-\frac{1}{k}-1}\nabla\gamma_d}{-\frac{1}{k}\gamma_d\left(\gamma_d^k+\beta\right)^{-\frac{1}{k}-1}}\right) \\
&= \left(k\frac{u}{\gamma_d}\right)\left(\gamma_d^k+\beta\right)^{\frac{1}{k}+1} + \left(k\frac{\nabla\gamma_d}{\gamma_d}\right)\beta = \frac{k}{\gamma_d}\left(u\left(\gamma_d^k+\beta\right)^{\frac{1}{k}+1}+\nabla\gamma_d\beta\right)
\end{aligned} \tag{A.36}$$

For the form $\hat{\varphi} = \frac{\gamma_d}{\beta}$ the partial derivatives are

$$\begin{aligned}
\frac{\partial\hat{\varphi}}{\partial\gamma_d}\left(\gamma_d,\beta\right) &= \frac{\partial}{\partial\gamma_d}\left\{\frac{\gamma_d^k}{\beta}\right\} = \frac{k\gamma_d^{k-1}}{\beta} \\
\frac{\partial\hat{\varphi}}{\partial\beta}\left(\gamma_d,\beta\right) &= \frac{\partial}{\partial\beta}\left\{\frac{\gamma_d^k}{\beta}\right\} = -\frac{\gamma_d^k}{\beta^2}
\end{aligned} \tag{A.37}$$

so substitution in (9.22) leads to

$$\nabla\beta = -\left(\frac{u+\frac{\partial\hat{\varphi}_\beta}{\partial\gamma_d}\nabla\gamma_d}{\frac{\partial\hat{\varphi}_\beta}{\partial\beta}}\right) = \left(\frac{u}{\gamma_d^k}\right)\beta^2 + \left(k\frac{\nabla\gamma_d}{\gamma_d}\right)\beta = A\beta^2 + B\beta \tag{A.38}$$

For the form $\bar{\varphi}_\beta = \frac{\gamma_d}{\beta^{\frac{1}{k}}}$, the partial derivatives are

$$\begin{aligned}
\frac{\partial\bar{\varphi}_\beta}{\partial\gamma_d} &= \frac{\partial}{\partial\gamma_d}\left\{\frac{\gamma_d}{\beta^{\frac{1}{k}}}\right\} \\
&= \beta^{-\frac{1}{k}} \\
\frac{\partial\bar{\varphi}_\beta}{\partial\beta} &= \frac{\partial}{\partial\beta}\left\{\frac{\gamma_d}{\beta^{\frac{1}{k}}}\right\} = \gamma_d\frac{\partial}{\partial\beta}\left\{\beta^{-\frac{1}{k}}\right\} = \gamma_d\left(-\frac{1}{k}\right)\beta^{-\frac{1}{k}-1} \\
&= -\frac{1}{k}\gamma_d\beta^{-\frac{1}{k}-1}
\end{aligned} \tag{A.39}$$

hence substitution in (9.22) yields

$$\nabla\beta = -\left(\frac{u + \frac{\partial\bar{\varphi}_\beta}{\partial\gamma_d}\nabla\gamma_d}{\frac{\partial\bar{\varphi}_\beta}{\partial\beta}}\right) = -\left(\frac{u + \beta^{-\frac{1}{k}}\nabla\gamma_d}{-\frac{1}{k}\gamma_d\beta^{-\frac{1}{k}-1}}\right) = k\left(\frac{u}{\gamma_d}\beta^{\frac{1}{k}} + \frac{\nabla\gamma_d}{\gamma_d}\beta\right) \tag{A.40}$$

For the form $\varphi_\beta = \frac{\gamma_d}{\gamma_d+\beta^{\frac{1}{k}}}$, the partial derivatives are

$$\begin{aligned}
\frac{\partial\varphi_\beta}{\partial\gamma_d} &= \frac{\partial}{\partial\gamma_d}\left\{\frac{\gamma_d}{\gamma_d+\beta^{\frac{1}{k}}}\right\} = \left(\gamma_d+\beta^{\frac{1}{k}}\right)^{-1} + \gamma_d\frac{\partial}{\partial\gamma_d}\left\{\frac{1}{\gamma_d+\beta^{\frac{1}{k}}}\right\}\\
&= \left(\gamma_d+\beta^{\frac{1}{k}}\right)^{-1}\gamma_d(-1)\left(\gamma_d+\beta^{\frac{1}{k}}\right)^{-2}\frac{\partial}{\partial\gamma_d}\left\{\gamma_d+\beta^{\frac{1}{k}}\right\}\\
&= \left(\gamma_d+\beta^{\frac{1}{k}}\right)^{-1} - \gamma_d\left(\gamma_d+\beta^{\frac{1}{k}}\right)^{-2} = \left(\gamma_d+\beta^{\frac{1}{k}}\right)^{-1}\left(1-\frac{\gamma_d}{\gamma_d+\beta^{\frac{1}{k}}}\right)\\
&= \left(\gamma_d+\beta^{\frac{1}{k}}\right)^{-1}\frac{\gamma_d-\beta^{\frac{1}{k}}-\gamma_d}{\gamma_d+\beta^{\frac{1}{k}}} = \beta^{\frac{1}{k}}\left(\gamma_d+\beta^{\frac{1}{k}}\right)^{-2} &\text{(A.41)}\\
\frac{\partial\varphi_\beta}{\partial\beta} &= \frac{\partial}{\partial\beta}\left\{\frac{\gamma_d}{\gamma_d+\beta^{\frac{1}{k}}}\right\} = \gamma_d\frac{\partial}{\partial\beta}\left\{\left(\gamma_d+\beta^{\frac{1}{k}}\right)^{-1}\right\}\\
&= \gamma_d(-1)\left(\gamma_d+\beta^{\frac{1}{k}}\right)^{-2}\frac{\partial}{\partial\beta}\left\{\gamma_d+\beta^{\frac{1}{k}}\right\}\\
&= -\gamma_d\left(\gamma_d+\beta^{\frac{1}{k}}\right)^{-2}\frac{1}{k}\beta^{\frac{1}{k}-1} = -\frac{1}{k}\gamma_d\beta^{\frac{1}{k}-1}\left(\gamma_d+\beta^{\frac{1}{k}}\right)^{-2}
\end{aligned}$$

Substitution in the PDE (9.22) yields

$$\begin{aligned}
\nabla\beta &= -\left(\frac{u+\beta^{\frac{1}{k}}\left(\gamma_d+\beta^{\frac{1}{k}}\right)^{-2}\nabla\gamma_d}{-\frac{1}{k}\gamma_d\beta^{\frac{1}{k}-1}\left(\gamma_d+\beta^{\frac{1}{k}}\right)^{-2}}\right) \implies\\
\nabla\beta &= \left(k\frac{u}{\gamma_d}\right)\left(\gamma_d+\beta^{\frac{1}{k}}\right)^2\beta^{1-\frac{1}{k}} + \left(k\frac{\nabla\gamma_d}{\gamma_d}\right)\beta
\end{aligned} \tag{A.42}$$

and simplifying this we obtain

$$\begin{aligned}
\nabla\beta &= \left(k\frac{u}{\gamma_d}\right)\left(\gamma_d^2+2\gamma_d\beta^{\frac{1}{k}}+\beta^{\frac{2}{k}}\right)\beta^{1-\frac{1}{k}} + \left(k\frac{\nabla\gamma_d}{\gamma_d}\right)\beta\\
&= \left(k\frac{u}{\gamma_d}\gamma_d^2\right)\beta^{1-\frac{1}{k}} + \left(k\frac{u}{\gamma_d}2\gamma_d\right)\beta^{\frac{1}{k}+1-\frac{1}{k}} + \left(k\frac{u}{\gamma_d}\right)\beta^{\frac{2}{k}+1-\frac{1}{k}} + \left(k\frac{\nabla\gamma_d}{\gamma_d}\right)\beta\\
&= (ku\gamma_d)\,\beta^{1-\frac{1}{k}} + (2ku)\,\beta + \left(k\frac{u}{\gamma_d}\right)\beta^{1+\frac{1}{k}} + \left(k\frac{\nabla\gamma_d}{\gamma_d}\right)\beta &\text{(A.43)}\\
&= \left(k\frac{u}{\gamma_d}\right)\beta^{1+\frac{1}{k}} + \left(k\frac{\nabla\gamma_d}{\gamma_d}+2ku\right)\beta + (ku\gamma_d)\beta^{1-\frac{1}{k}}\\
&= \beta\left(\left(k\frac{u}{\gamma_d}\right)\beta^{\frac{1}{k}} + \left(k\frac{\nabla\gamma_d}{\gamma_d}+2ku\right) + (ku\gamma_d)\,\beta^{-\frac{1}{k}}\right)
\end{aligned}$$

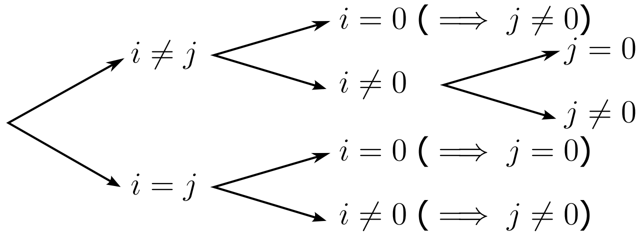


Figure A.2: Combinations of i and j cases treated separately.

A.5 Extrema

The following minima and maxima are needed to calculate an estimate of upper bounds on ε_{I_0}

$$\begin{gathered}\min_{\mathscr{B}_i(\varepsilon_i)}\{\gamma_d\}, \quad \max_{\mathscr{B}_i(\varepsilon_i)}\{\gamma_d\}, \quad \min_{\mathscr{B}_i(\varepsilon_i)}\{\beta_j\}, \quad \max_{\mathscr{B}_i(\varepsilon_i)}\{\beta_j\}, \\ \min_{\mathscr{B}_i(\varepsilon_i)}\{\|\nabla\gamma_d\|\}, \quad \max_{\mathscr{B}_i(\varepsilon_i)}\{\|\nabla\gamma_d\|\}, \quad \min_{\mathscr{B}_i(\varepsilon_i)}\{\|\nabla\beta_j\|\}, \quad \max_{\mathscr{B}_i(\varepsilon_i)}\{\|\nabla\beta_j\|\}, \\ \max_{\mathscr{W}}\{\gamma_d\}, \quad \max_{\mathscr{W}}\{\beta_i\}, \\ \max_{\mathscr{W}}\{\|\nabla\gamma_d\|\}, \quad \max_{\mathscr{W}}\{\|\nabla\beta_i\|\}, i, j \in I_0\end{gathered} \tag{A.44}$$

Note that

$$\min_{\mathscr{W}}\{\gamma_d\} = 0, \quad \min_{\mathscr{W}}\{\beta_i\} = 0, \quad \min_{\mathscr{W}}\{\|\nabla\gamma_d\|\} = 0, \quad \min_{\mathscr{W}}\{\|\nabla\beta_i\|\} = \min_{\mathscr{B}_i(\varepsilon_i)}\{\|\nabla\beta_i\|\} = 2\rho_i \tag{A.45}$$

and need not be considered further. Also[2]

$$\gamma_d = \beta_i|_{q_i=q_d,\rho_i=0} \tag{A.46}$$

so that

$$\begin{aligned}&\min_{\mathscr{B}_i(\varepsilon_i)}\{\gamma_d\}, \quad \max_{\mathscr{B}_i(\varepsilon_i)}\{\gamma_d\}, \quad \min_{\mathscr{B}_i(\varepsilon_i)}\{\|\nabla\gamma_d\|\}, \quad \max_{\mathscr{B}_i(\varepsilon_i)}\{\|\nabla\gamma_d\|\}, i \in I_0 \\ &\max_{\mathscr{W}}\{\gamma_d\}, \quad \max_{\mathscr{W}}\{\|\nabla\gamma_d\|\}\end{aligned} \tag{A.47}$$

are not special cases. The derivations are done for $\beta_j, j \in I_0$ and the results applied to γ_d as well.

A.5.1 $\beta_i, \|\nabla\beta_i\|$ extrema

There are combinations of $\overline{\mathscr{B}_i(\varepsilon_i)}, i \in I_0$ and $\beta_j, j \in I_0$ which need to be considered separately, as shown in Fig. A.2.

A.5.1.1 Case $i \neq j, i = 0$ in $\overline{\mathscr{B}_0(\varepsilon_0)}$

This is case 1 in Fig. A.2.

[2] [23], Appendix B, p.438. This follows from comparing $\gamma_d = \|q - q_d\|^2$ to $\beta_i = \|q - q_i\|^2 - \rho_i^2$.

A.5.1.2 Case $i \neq j, i \neq 0, j = 0$ in $\overline{\mathscr{B}_i(\varepsilon_i)}$

This is case 2 in Fig. A.2. Using Lagrange multipliers it can be proved that the minimum is

$$\min_{\mathscr{B}_i(\varepsilon_i)} \{\beta_0\} = \rho_0^2 - \left(\sqrt{\varepsilon_i + \rho_i^2} + \|q_i\|\right)^2, \quad i \in I_1 \tag{A.48}$$

and the maximum

$$\max_{\mathscr{B}_i(\varepsilon_i)} \{\beta_0\} = \rho_0^2 - \left(\sqrt{\varepsilon_i + \rho_i^2} - \|q_i\|\right)^2, \quad i \in I_1 \tag{A.49}$$

The gradient norm minimum is

$$\begin{aligned}\min_{\mathscr{B}_i(\varepsilon_i)} \{\|\nabla\beta_0\|\} &= \min_{\mathscr{B}_i(\varepsilon_i)} \left\{2\sqrt{-\beta_0 + \rho_0^2}\right\} = 2\sqrt{\rho_0^2 - \max_{\mathscr{B}_i(\varepsilon_i)} \{\beta_0\}} \\ &= 2\sqrt{\rho_0^2 - \rho_0^2 + \left(\sqrt{\varepsilon_i + \rho_i^2} - \|q_i\|\right)^2} \\ &= 2\left|\sqrt{\varepsilon_i + \rho_i^2} - \|q_i\|\right|, \quad i \in I_1\end{aligned} \tag{A.50}$$

and the gradient norm maximum

$$\begin{aligned}\max_{\mathscr{B}_i(\varepsilon_i)} \{\|\nabla\beta_0\|\} &= \max_{\mathscr{B}_i(\varepsilon_i)} \left\{2\sqrt{-\beta_0 + \rho_0^2}\right\} = 2\sqrt{\rho_0^2 - \min_{\mathscr{B}_i(\varepsilon_i)} \{\beta_0\}} \\ &= 2\sqrt{\rho_0^2 - \rho_0^2 + \left(\sqrt{\varepsilon_i + \rho_i^2} + \|q_i\|\right)^2} \\ &= 2\left|\sqrt{\varepsilon_i + \rho_i^2} + \|q_i\|\right|, \quad i \in I_1\end{aligned} \tag{A.51}$$

A.5.1.3 Case $i \neq j, i \neq 0, j \neq 0$ in $\overline{\mathscr{B}_i(\varepsilon_i)}$

This is case 3 in Fig. A.2. Using Lagrange multipliers it can be proved that the minimum is

$$\min_{\mathscr{B}_i(\varepsilon_i)} \{\beta_j\} = \left(\sqrt{\varepsilon_i + \rho_i^2} - \|q_i - q_j\|\right)^2 - \rho_j^2, \quad i, j \in I_1, \quad i \neq j \tag{A.52}$$

and the maximum

$$\max_{\mathscr{B}_i(\varepsilon_i)} \{\beta_j\} = \left(\sqrt{\varepsilon_i + \rho_i^2} + \|q_i - q_j\|\right)^2 - \rho_j^2, \quad i, j \in I_1, \quad i \neq j \tag{A.53}$$

The gradient norm minimum is

$$\begin{aligned}\min_{\mathscr{B}_i(\varepsilon_i)} \{\|\nabla\beta_j\|\} &= \min_{\mathscr{B}_i(\varepsilon_i)} \left\{2\sqrt{\beta_j + \rho_j^2}\right\} = 2\sqrt{\min_{\mathscr{B}_i(\varepsilon_i)} \{\beta_j\} + \rho_j^2} \\ &= 2\sqrt{\left(\sqrt{\varepsilon_i + \rho_i^2} - \|q_i - q_j\|\right)^2 - \rho_j^2 + \rho_j^2} \\ &= 2\left|\sqrt{\varepsilon_i + \rho_i^2} - \|q_i - q_j\|\right|, \quad i, j \in I_1, \quad i \neq j\end{aligned} \tag{A.54}$$

and the gradient norm maximum

$$\begin{aligned}\max_{\mathscr{B}_i(\varepsilon_i)}\{\|\nabla\beta_j\|\} &= \max_{\mathscr{B}_i(\varepsilon_i)}\left\{2\sqrt{\beta_j+\rho_j^2}\right\} = 2\sqrt{\max_{\mathscr{B}_i(\varepsilon_i)}\{\beta_j\}+\rho_j^2} \\ &= 2\sqrt{\left(\sqrt{\varepsilon_i+\rho_i^2}+\|q_i-q_j\|\right)^2-\rho_j^2+\rho_j^2} = 2\left|\sqrt{\varepsilon_i+\rho_i^2}+\|q_i-q_j\|\right| \\ &= 2\left(\sqrt{\varepsilon_i+\rho_i^2}+\|q_i-q_j\|\right), \quad i,j\in I_1, \quad i\neq j\end{aligned} \tag{A.55}$$

A.5.1.4 Case $i = j = 0$ in $\overline{\mathscr{B}_0(\varepsilon_0)}$

This is case 4 in Fig. A.2. It is

$$\min_{\mathscr{B}_0(\varepsilon_0)}\{\beta_0\} = 0, \quad \max_{\mathscr{B}_0(\varepsilon_0)}\{\beta_0\} = \varepsilon_0, \quad i = 0 \tag{A.56}$$

The gradient norm minimum is

$$\min_{\mathscr{B}_0(\varepsilon_0)}\{\|\nabla\beta_0\|\} = \min_{\mathscr{B}_0(\varepsilon_0)}\left\{2\sqrt{\rho_0^2-\beta_0}\right\} = 2\sqrt{\rho_0^2-\max_{\mathscr{B}_0(\varepsilon_0)}\{\beta_0\}} = 2\sqrt{\rho_0^2-\varepsilon_0} \tag{A.57}$$

and the gradient norm maximum

$$\begin{aligned}\max_{\mathscr{B}_0(\varepsilon_0)}\{\|\nabla\beta_0\|\} &= \max_{\mathscr{B}_0(\varepsilon_0)}\left\{2\sqrt{\rho_0^2-\beta_0}\right\} = 2\sqrt{\rho_0^2-\min_{\mathscr{B}_0(\varepsilon_0)}\{\beta_0\}} \\ &= 2\sqrt{\rho_0^2-0} \overset{\rho_i>0,\forall i\in I_1}{=} 2\rho_0\end{aligned} \tag{A.58}$$

A.5.1.5 Case $i = j \neq 0$ in $\overline{\mathscr{B}_i(\varepsilon_i)}$

This is case 5 in Fig. A.2. As in (A.56), also in this case

$$\min_{\mathscr{B}_i(\varepsilon_i)}\{\beta_i\} = 0, \quad \max_{\mathscr{B}_i(\varepsilon_i)}\{\beta_i\} = \varepsilon_i, \quad i\in I_1 \tag{A.59}$$

The gradient norm minimum is

$$\begin{aligned}\min_{\mathscr{B}_i(\varepsilon_i)}\{\|\nabla\beta_i\|\} &= \min_{\mathscr{B}_i(\varepsilon_i)}\left\{2\sqrt{\beta_i+\rho_i^2}\right\} = 2\sqrt{\min_{\mathscr{B}_i(\varepsilon_i)}\{\beta_i\}+\rho_i^2} \\ &= 2\sqrt{0+\rho_i^2} \overset{\rho_i>0,\forall i\in I_1}{=} 2\rho_i, \quad i\in I_1\end{aligned} \tag{A.60}$$

and the gradient norm maximum

$$\max_{\mathscr{B}_i(\varepsilon_i)}\{\|\nabla\beta_i\|\} = \max_{\mathscr{B}_i(\varepsilon_i)}\left\{2\sqrt{\beta_i+\rho_i^2}\right\} = 2\sqrt{\max_{\mathscr{B}_i(\varepsilon_i)}\{\beta_i\}+\rho_i^2} = 2\sqrt{\varepsilon_i+\rho_i^2} \tag{A.61}$$

A.5.1.6 Extrema in $\mathscr{W}$

The upper bounds of β_j in $\mathscr{W}$ are[3]

$$\max_{\mathscr{W}} \{\beta_j(q)\} = (\rho_0 + \|q_j\|)^2 - \rho_j^2, \quad j \in I_1 \tag{A.62}$$

and

$$\max_{\mathscr{W}} \{\beta_0(q)\} = \rho_0^2, \quad j = 0 \tag{A.63}$$

Substitution yields the following maxima of the respective gradient norms

$$\begin{aligned}\max_{\mathscr{W}} \{\|\nabla\beta_j\|\} &= \max_{\mathscr{W}} \left\{2\sqrt{\beta_j + \rho_j^2}\right\} = 2\sqrt{\max_{\mathscr{W}} \{\beta_j\} + \rho_j^2} \\ &= 2\sqrt{(\rho_0 + \|q_j\|)^2 - \rho_j^2 + \rho_j^2} = 2\sqrt{(\rho_0 + \|q_j\|)^2} \overset{\rho_0>0\wedge\|q_j\|>0}{=} \\ &= 2\left(\rho_0 + \|q_j\|\right), \quad j \in I_1\end{aligned} \tag{A.64}$$

and

$$\begin{aligned}\max_{\mathscr{W}} \{\|\nabla\beta_0\|\} &= \max_{\mathscr{W}} \left\{2\sqrt{-\beta_0 + \rho_0^2}\right\} = 2\sqrt{\max_{\mathscr{W}} \{-\beta_0\} + \rho_0^2} \\ &= 2\sqrt{-\min_{\mathscr{W}} \{\beta_0\} + \rho_0^2} \overset{\min_{\mathscr{W}}\{\beta_0\}=0}{=} 2\sqrt{-0 + \rho_0^2} \\ &= 2\sqrt{\rho_0^2} \overset{\rho_0>0}{=} 2\rho_0, \quad j = 0\end{aligned} \tag{A.65}$$

Note that $2\rho_0 = 2\left(\rho_0 + 0\right) \overset{\|q_0\|=0}{=} 2\left(\rho_0 + \|q_0\|\right)$ so that in the general case we can write

$$\max_{\mathscr{W}} \{\|\nabla\beta_i\|\} = 2\left(\rho_0 + \|q_i\|\right), \quad i \in \{0, 1, \ldots, M\} \tag{A.66}$$

A.5.2 γ_d extrema

A.5.2.1 Extrema in $\overline{\mathscr{B}_i\left(\varepsilon_i\right)}, i \neq 0$

The case of γ_d corresponds to $i \neq j$ (since the destination is always in free space) and $j \neq 0$ (since $\gamma_d = \|q - q_d\|^2$ is $\beta_j = \|q - q_j\|^2 - \rho_j^2, j \in I_1$ with $q_j = q_d$ and $\rho_j = 0$, not $\beta_0 = \rho_0^2 - \|q\|^2$).

Also, since $\min_{\overline{\mathscr{B}_0(\varepsilon_0)}} \{\gamma_d\}, \max_{\overline{\mathscr{B}_0(\varepsilon_0)}} \{\gamma_d\}$ are not needed, $i \neq 0$. This is case 3 in Fig. A.2. The results substituting $q_j = q_d$ and $\rho_j = 0$ in the equations of subsubsection A.5.1.3 are

$$\begin{aligned}\min_{\overline{\mathscr{B}_i(\varepsilon_i)}} \{\gamma_d\} &= \left(\sqrt{\varepsilon_i + \rho_i^2} - \|q_i - q_j\|\right)^2 - \rho_j^2 \\ &= \left(\sqrt{\varepsilon_i + \rho_i^2} - \|q_i - q_d\|\right)^2, \quad i \in I_1\end{aligned} \tag{A.67}$$

and the maximum

$$\begin{aligned}\max_{\overline{\mathscr{B}_i(\varepsilon_i)}} \{\gamma_d\} &= \left(\sqrt{\varepsilon_i + \rho_i^2} + \|q_i - q_j\|\right)^2 - \rho_j^2 \\ &= \left(\sqrt{\varepsilon_i + \rho_i^2} + \|q_i - q_d\|\right)^2, \quad i \in I_1\end{aligned} \tag{A.68}$$

[3] [23], Lemma B.1, p.438.

The gradient norm minimum is

$$\begin{aligned}\min_{\mathscr{B}_i(\varepsilon_i)}\{\|\nabla\gamma_d\|\} &= 2\left|\sqrt{\varepsilon_i+\rho_i^2}-\|q_i-q_j\|\right| \\ &= 2\left|\sqrt{\varepsilon_i+\rho_i^2}-\|q_i-q_d\|\right|, \quad i\in I_1\end{aligned} \tag{A.69}$$

and the gradient norm maximum

$$\begin{aligned}\max_{\mathscr{B}_i(\varepsilon_i)}\{\|\nabla\gamma_d\|\} &= 2\left(\sqrt{\varepsilon_i+\rho_i^2}+\|q_i-q_i\|\right) \\ &= 2\left(\sqrt{\varepsilon_i+\rho_i^2}+\|q_i-q_d\|\right), \quad i\in I_1\end{aligned} \tag{A.70}$$

A.5.2.2 Maximum in $\mathscr{W}$

From Lemma B.1 (p.438)

$$\max_{\mathscr{W}}\{\beta_i(q)\} = (\rho_0+\|q_i\|)^2-\rho_i^2, \quad i\in I_1 \tag{A.71}$$

Substituting for γ_d the parameters $q_d, \rho_d = 0$ its maximum over $\mathscr{W}$ is

$$\begin{aligned}\max_{\mathscr{W}}\{\gamma_d\} &= (\rho_0+\|q_d\|)^2-\rho_d^2 = (\rho_0+\|q_d\|)^2-0^2 \\ &= (\rho_0+\|q_d\|)^2\end{aligned} \tag{A.72}$$

Since

$$\gamma_d(q)\le\max_{\mathscr{W}}\{\sqrt{\gamma_d}\}, \forall q\in\mathscr{W} \tag{A.73}$$

and

$$\mathscr{F}_2\subset\mathscr{W} \tag{A.74}$$

it follows that

$$\gamma_d(q)\le\max_{\mathscr{W}}\{\sqrt{\gamma_d}\}, \forall q\in\mathscr{F}_2 \tag{A.75}$$

It is worth noting that for small ε_{I_0} the maxima of γ_d in $\mathscr{W}$ and $\mathscr{F}_2$ do not differ much

$$\lim_{\{\varepsilon_i\to 0\}_{i=0}^M}\left(\max_{\mathscr{W}}\{\gamma_d\}-\max_{\mathscr{F}_2}\{\gamma_d\}\right)=0 \tag{A.76}$$

The gradient norm maximum in $\mathscr{W}$ is

$$\begin{aligned}\max_{\mathscr{W}}\{\|\nabla\gamma_d\|\} &= \max_{\mathscr{W}}\{2\sqrt{\gamma_d}\} = 2\sqrt{\max_{\mathscr{W}}\{\gamma_d\}} = 2\sqrt{(\rho_0+\|q_d\|)^2} \\ &\overset{\rho_0\ge 0\wedge\|q_d\|\ge 0}{=} 2(\rho_0+\|q_d\|)\end{aligned} \tag{A.77}$$

A.6 Other derivations

A.6.1 Bound on $\left|\widehat{\nabla\beta_i}^T \nabla^2 \bar{\beta}_i \widehat{\nabla\beta_i}\right|$

We are now to prove the inequality

$$\left|\hat{\upsilon}^T \nabla^2 \bar{\beta}_i \hat{\upsilon}\right| \leq 2 \sum_{j=0, j\neq i}^{M} \left(\left(\prod_{l=0, l\neq i,j}^{M} \beta_l \right) + \sum_{l=0, l\neq i,j}^{M} \left(\left(\prod_{m=0, m\neq i,j,l}^{M} \beta_m \right) \|\nabla\beta_j\| \, \|\nabla\beta_l\| \right) \right) \quad \text{(A.78)}$$

for a unit vector $\hat{\upsilon}$. In the proof two unit vectors are defined and used, which are *different* from each other. One defines the gradient's $\nabla\beta_i$ direction, while the other the direction tangential to the gradient $\nabla\beta_i^{\perp}$ as

$$\hat{\upsilon} \triangleq \widehat{\nabla\beta_i} \triangleq \frac{\nabla\beta_i}{\|\nabla\beta_i\|} \quad \neq \quad \hat{\upsilon} \triangleq \frac{\nabla\beta_i^{\perp}}{\|\nabla\beta_i\|} \quad \text{(A.79)}$$

Then we have

$$\begin{aligned}
\nabla\bar{\beta}_i &= \nabla\left(\prod_{j=0, j\neq i}^{M} \beta_j\right) = \sum_{j=0, j\neq i}^{M} \left(\left(\prod_{l=0, l\neq i,j}^{M} \beta_l \right) \nabla\beta_j \right) \implies \\
D^2\bar{\beta}_i &= \sum_{j=0, j\neq i}^{M} \left(\left(\prod_{l=0, l\neq i,j}^{M} \beta_l \right) D^2\beta_j + \sum_{l=0, l\neq i,j}^{M} \left(\left(\prod_{m=0, m\neq i,j,l}^{M} \beta_m \right) \nabla\beta_j \nabla\beta_l^T \right) \right) \overset{D^2\beta_j = 2I, \forall j\in I_0}{=} \\
&= \sum_{j=0, j\neq i}^{M} \left(\left(\prod_{l=0, l\neq i,j}^{M} \beta_l \right) 2I + \sum_{l=0, l\neq i,j}^{M} \left(\left(\prod_{m=0, m\neq i,j,l}^{M} \beta_m \right) \nabla\beta_j \nabla\beta_l^T \right) \right)
\end{aligned} \quad \text{(A.80)}$$

multiply this by $\hat{\upsilon}^T$ from left and $\hat{\upsilon}$ from right to get

$$\begin{aligned}
\hat{\upsilon}^T D^2\bar{\beta}_i \hat{\upsilon} &= \hat{\upsilon}^T \sum_{j=0, j\neq i}^{M} \left(\left(\prod_{l=0, l\neq i,j}^{M} \beta_l \right) 2I + \sum_{l=0, l\neq i,j}^{M} \left(\left(\prod_{m=0, m\neq i,j,l}^{M} \beta_m \right) \nabla\beta_j \nabla\beta_l^T \right) \right) \hat{\upsilon} \\
&= \sum_{j=0, j\neq i}^{M} \left(\left(\prod_{l=0, l\neq i,j}^{M} \beta_l \right) \hat{\upsilon}^T 2I \hat{\upsilon} + \sum_{l=0, l\neq i,j}^{M} \left(\left(\prod_{m=0, m\neq i,j,l}^{M} \beta_m \right) \hat{\upsilon}^T \nabla\beta_j \nabla\beta_l^T \hat{\upsilon} \right) \right) \\
&= \sum_{j=0, j\neq i}^{M} \left(\left(\prod_{l=0, l\neq i,j}^{M} \beta_l \right) 2\left\|\hat{\upsilon}\right\|^2 + \sum_{l=0, l\neq i,j}^{M} \left(\left(\prod_{m=0, m\neq i,j,l}^{M} \beta_m \right) \hat{\upsilon}^T \nabla\beta_j \nabla\beta_l^T \hat{\upsilon} \right) \right) \\
&= \sum_{j=0, j\neq i}^{M} \left(2\left(\prod_{l=0, l\neq i,j}^{M} \beta_l \right) + \sum_{l=0, l\neq i,j}^{M} \left(\left(\prod_{m=0, m\neq i,j,l}^{M} \beta_m \right) \hat{\upsilon}^T \nabla\beta_j \nabla\beta_l^T \hat{\upsilon} \right) \right)
\end{aligned} \quad \text{(A.81)}$$

so now

$$\begin{aligned}\left|\hat{v}^T D^2\bar{\beta}_i\hat{v}\right| &\leq \left|\sum_{j=0,j\neq i}^{M}\left(2\left(\prod_{l=0,l\neq i,j}^{M}\beta_l\right)+\sum_{l=0,l\neq i,j}^{M}\left(\left(\prod_{m=0,m\neq i,j,l}^{M}\beta_m\right)\hat{v}^T\nabla\beta_j\nabla\beta_l^T\hat{v}\right)\right)\right|\\
&\leq \sum_{j=0,j\neq i}^{M}\left|2\left(\prod_{l=0,l\neq i,j}^{M}\beta_l\right)+\sum_{l=0,l\neq i,j}^{M}\left(\left(\prod_{m=0,m\neq i,j,l}^{M}\beta_m\right)\hat{v}^T\nabla\beta_j\nabla\beta_l^T\hat{v}\right)\right|\\
&\leq \sum_{j=0,j\neq i}^{M}\left(2\left|\prod_{l=0,l\neq i,j}^{M}\beta_l\right|+\left|\sum_{l=0,l\neq i,j}^{M}\left(\left(\prod_{m=0,m\neq i,j,l}^{M}\beta_m\right)\hat{v}^T\nabla\beta_j\nabla\beta_l^T\hat{v}\right)\right|\right)\\
&\leq \sum_{j=0,j\neq i}^{M}\left(2\left|\prod_{l=0,l\neq i,j}^{M}\beta_l\right|+\sum_{l=0,l\neq i,j}^{M}\left|\left(\prod_{m=0,m\neq i,j,l}^{M}\beta_m\right)\hat{v}^T\nabla\beta_j\nabla\beta_l^T\hat{v}\right|\right)\\
&= \sum_{j=0,j\neq i}^{M}\left(2\left(\prod_{l=0,l\neq i,j}^{M}\beta_l\right)+\sum_{l=0,l\neq i,j}^{M}\left(\left(\prod_{m=0,m\neq i,j,l}^{M}\beta_m\right)\left|\hat{v}^T\nabla\beta_j\nabla\beta_l^T\hat{v}\right|\right)\right)\end{aligned} \tag{A.82}$$

by successive application of the triangular inequality and since $\beta_j(q) \geq 0, \forall q \in \mathring{\mathscr{F}}, \forall j \in I_0$.

It remains to show that $\left|\hat{v}^T\nabla\beta_j\nabla\beta_l^T\hat{v}\right| \leq \|\nabla\beta_j\|\,\|\nabla\beta_l\|$. This is provided in subsection A.6.2, where we set $a = \nabla\beta_j$ and $b = \nabla\beta_l$. As a result

$$\left|\hat{v}^T D^2\bar{\beta}_i\hat{v}\right| \leq \sum_{j=0,j\neq i}^{M}\left(2\left(\prod_{l=0,l\neq i,j}^{M}\beta_l\right)+\sum_{l=0,l\neq i,j}^{M}\left(\left(\prod_{m=0,m\neq i,j,l}^{M}\beta_m\right)\|\nabla\beta_j\|\,\|\nabla\beta_l\|\right)\right) \tag{A.83}$$

It should be noted that in [23], Appendix B.2, pp. 440-441 an overall result is provided, where they have factored 2 out by adding $\sum_{l=0,l\neq i,j}^{M}\left(\prod_{m=0,m\neq i,j,l}^{M}\beta_m\right)\|\nabla\beta_j\|\,\|\nabla\beta_l\|$, which is positive, but this leads to even worse numerical results.

A.6.2 Inequality $\left|\hat{v}^T ab^T\hat{v}\right| \leq \|a\|\,\|b\|$

Matrix multiplication is associative, therefore

$$\begin{aligned}\left(\hat{v}^T\right)\left(ab^T\right)\left(\hat{v}\right) &= \left(\hat{v}^T a\right)\left(b^T\right)\left(\hat{v}\right)\\
&= \left(\hat{v}^T a\right)\left(b^T\hat{v}\right) \implies\\
\left|\left(\hat{v}^T\right)\left(ab^T\right)\left(\hat{v}\right)\right| &= \left|\left(\hat{v}^T a\right)\left(b^T\hat{v}\right)\right|\end{aligned} \tag{A.84}$$

Now note that $\left(\hat{v}^T a\right) \in \mathbb{R}$ and $\left(b^T\hat{v}\right) \in \mathbb{R}$. The absolute value multiplicativeness property allows seperation of the terms

$$\left|\left(\hat{v}^T a\right)\left(b^T\hat{v}\right)\right| = \left|\hat{v}^T a\right|\left|b^T\hat{v}\right| \tag{A.85}$$

Next apply the Cauchy-Bunyakovsky-Schwarz inequality $|\langle x,y\rangle| \leq \|x\|\,\|y\|$ to obtain

$$\begin{aligned}\left|\hat{v}^T a\right| &\leq \left\|\hat{v}^T\right\|\|a\| = 1\cdot\|a\| = \|a\|\\
\left|b^T\hat{v}\right| &\leq \left\|b^T\right\|\|\hat{v}\| = \left\|b^T\right\|\cdot 1 = \|b\|\end{aligned} \tag{A.86}$$

substitution then gives

$$\left|\hat{v}^T ab^T\hat{v}\right| = \left|\left(\hat{v}^T a\right)\left(b^T\hat{v}\right)\right| = \left|\hat{v}^T a\right|\left|b^T\hat{v}\right| \leq \|a\|\,\|b\| \tag{A.87}$$

A.6.3 Bounded $\mathscr{P}_z$ when no $\mathscr{O}_0$ is known

Proposition: If and only if $k_z > M_z$ then $\lim_{\|q\| \to \infty} \varphi_z(q) = 1$.
Proof: We examine the alternative cases

$$\lim_{\|q\|\to\infty} \varphi_z(q) = \lim_{\|q\|\to\infty} \frac{\gamma_d}{\left(\gamma_d^{k_z} + {}^z\beta\right)^{\frac{1}{k_z}}} = \lim_{\|q\|\to\infty} \frac{\|q-q_d\|^2}{\left(\|q-q_d\|^{2k_z} + \prod_{i=1}^{M_z} \|q-q_i\|^2\right)^{\frac{1}{k_z}}}$$
$$= \lim_{\|q\|\to\infty} \frac{\|q\|^2}{\left(\|q\|^{2k_z} + \prod_{i=1}^{M_z} \|q\|^2\right)^{\frac{1}{k_z}}} = \lim_{x\to\infty} \frac{x^2}{\left(x^{2k_z} + \prod_{i=1}^{M_z} x^2\right)^{\frac{1}{k_z}}} = \lim_{x\to\infty} \frac{x^2}{\left(x^{2k_z} + x^{2M_z}\right)^{\frac{1}{k_z}}}$$
$$= \begin{cases} \lim_{x\to\infty} \frac{x^2}{\left(x^{2M_z}\right)^{\frac{1}{k_z}}} = \lim_{x\to\infty} x^{2\left(1-\frac{M_z}{k_z}\right)} = 0^+ < 1, & k_z < M_z \\ \lim_{x\to\infty} \frac{x^2}{\left(x^{2M_z}+x^{2M_z}\right)^{\frac{1}{M_z}}} = \lim_{x\to\infty} \frac{x^2}{\left(2x^{2M_z}\right)^{\frac{1}{M_z}}} = \lim_{x\to\infty} \frac{x^2}{2^{\frac{1}{M_z}} x^2} = \frac{1}{2^{\frac{1}{M_z}}} < 1, & k_z = M_z \\ \lim_{x\to\infty} \frac{x^2}{\left(x^{2k_z}\right)^{\frac{1}{k_z}}} = \lim_{x\to\infty} \frac{x^2}{x^2} = 1, & M_z < k_z \end{cases} \tag{A.88}$$

since

$$2 \leq k_z < M_z \implies 1 - \frac{M_z}{k_z} < 0 \tag{A.89}$$

and

$$M_z \geq 1 \implies \frac{1}{2^{\frac{1}{M_z}}} < 1. \tag{A.90}$$

A.6.4 Bound on γ_d when no $\mathscr{O}_0$ is known

Proposition 1: If $\sqrt{\gamma_d(q)} > \max_i \{\|q_i - q_d\|\}$ and $\sqrt{\gamma_d(q)} > a_1^{m_1} a_2^{m_2}$ where

$$a_1 \triangleq \frac{4^{M_z}}{{}^z\beta(x(t_m))}, \quad a_2 \triangleq \gamma_d(x(t_m)) \tag{A.91}$$

and

$$m_1 \triangleq \begin{cases} 0, & a_1 \leq 1 \\ \frac{1}{2}, & a_1 > 1 \end{cases}, \quad m_2 \triangleq \begin{cases} \frac{1}{2}, & a_2 \leq 1 \\ \frac{M_z+1}{2}, & a_2 > 1 \end{cases} \tag{A.92}$$

then $q \notin \mathscr{P}_z$.
Proof: We have required $k_z > M_z \iff k_z \geq M_z + 1$ to assure $\lim_{\|q\|\to\infty} \varphi_z(q) = 1$. This leads to[4]

$$M_z + 1 \leq k_z \iff 1 \leq k_z - M_z \implies \frac{1}{2(k_z - M_z)} \leq \frac{1}{2} \tag{A.93}$$

and

$$\lim_{k_z\to+\infty} \frac{1}{2(k_z - M_z)} = 0^+ \tag{A.94}$$

so that

$$\frac{1}{2(k_z - M_z)} \in \left(0, \frac{1}{2}\right], \forall k_z \in [M_z + 1, +\infty). \tag{A.95}$$

[4]It is $M_z > 0$ because at least a single internal obstacle has been discovered, leading to a change from the trivial unbounded free space NF.

When $a_1 \leq 1 \implies m_1 = 0$ so $a_1^{m_1} = a_1^0 = 1$ and since $0 < \frac{1}{2(k_z - M_z)}, \forall k_z \in [M_z + 1, +\infty)$ for $a_1 \leq 1 \implies a_1^x \downarrow \forall x \in \mathbb{R}$ it follows that

$$a_1 \leq 1 \implies a_1^{m_1} = a_1^0 = 1 > a_1^{\frac{1}{2(k_z - M_z)}}, \forall k_z \in [M_z + 1, +\infty). \tag{A.96}$$

When $a_1 > 1 \implies m_1 = \frac{1}{2}$ so $a_1^{m_1} = a_1^{\frac{1}{2}}$ and since $\frac{1}{2(k_z - M_z)} \leq \frac{1}{2}, \forall k_z \in [M_z + 1, +\infty)$ for $a_1 > 1 \implies a_1^x \uparrow \forall x \in \mathbb{R}$ it follows that

$$a_1 > 1 \implies a_1^{m_1} = a_1^{\frac{1}{2}} \geq a_1^{\frac{1}{2(k_z - M_z)}}, \forall k_z \in [M_z + 1, +\infty) \tag{A.97}$$

Also the requirement $M_z + 1 \leq k_z$ leads to

$$\begin{aligned} M_z + 1 \leq k_z \iff 0 \leq k_z - 1 - M_z \overset{M_z > 0}{\iff} 0 \leq k_z M_z - M_z - M_z \iff \\ k_z \leq k_z M_z + k_z - M_z^2 - M_z \overset{k_z - M_z \geq 1 > 0}{\iff} \frac{k_z}{2(k_z - M_z)} \leq \frac{M_z + 1}{2} \end{aligned} \tag{A.98}$$

and

$$\lim_{k_z \to +\infty} \frac{k_z}{2(k_z - M_z)} = \frac{1}{2} \tag{A.99}$$

so that[5]

$$\frac{k_z}{2(k_z - M_z)} \in \left(\frac{1}{2}, \frac{1}{2} + \frac{M_z}{2} \right] = \left(\frac{1}{2}, \frac{M_z + 1}{2} \right], \forall k_z \in [M_z + 1, +\infty). \tag{A.100}$$

When $a_2 \leq 1 \implies m_2 = \frac{1}{2}$ so $a_2^{m_2} = a_2^{\frac{1}{2}}$ and since $\frac{1}{2} < \frac{k_z}{2(k_z - M_z)}, \forall k_z \in [M_z + 1, +\infty)$ for $a_2 \leq 1 \implies a_2^x \downarrow \forall x \in \mathbb{R}$ it follows that

$$a_2 \leq 1 \implies a_2^{m_2} = a_2^{\frac{1}{2}} \geq a_2^{\frac{k_z}{2(k_z - M_z)}}, \forall k \in [M_z + 1, +\infty). \tag{A.101}$$

When $1 < a_2 \implies m_2 = \frac{M_z + 1}{2}$ so $a_2^{m_2} = a_2^{\frac{M_z + 1}{2}}$ and since $\frac{k_z}{2(k_z - M_z)} \leq \frac{M_z + 1}{2}, \forall k_z \in [M_z + 1, +\infty)$ for $a_2 > 1 \implies a_2^x \uparrow \forall x \in \mathbb{R}$ it follows that

$$a_2 > 1 \implies a_2^{m_2} = a_2^{\frac{M_z + 1}{2}} > a_2^{\frac{k_z}{2(k_z - M_z)}}, \forall k_z \in [M_z + 1, +\infty). \tag{A.102}$$

Summarizing these inequalities

$$\frac{1}{2(k_z - M_z)} \in \left(0, \frac{1}{2} \right] \implies a_1^{m_1} \geq a_1^{\frac{1}{2(k_z - M_z)}}, \forall k_z > M_z \tag{A.103}$$

and

$$\frac{k_z}{2(k_z - M_z)} \in \left(\frac{1}{2}, \frac{M_z + 1}{2} \right] \implies a_2^{m_2} \geq a_2^{\frac{k_z}{2(k_z - M_z)}}, \forall k_z > M_z. \tag{A.104}$$

[5]Obviously $\frac{1}{2} \leq \frac{M_z + 1}{2}$ even for $0 \leq M_z$ although here it is $1 \leq M_z$.

As a result $a_1^{m_1} a_2^{m_2} \geq a_1^{\frac{1}{2(k_z - M_z)}} a_2^{\frac{k_z}{2(k_z - M_z)}}$ and the condition

$$\begin{aligned}
\sqrt{\gamma_d(q)} > a_1^{m_1} a_2^{m_2} &\implies \sqrt{\gamma_d(q)} > a_1^{\frac{1}{2(k_z - M_z)}} a_2^{\frac{k_z}{2(k_z - M_z)}} \iff \\
\sqrt{\gamma_d(q)} &> \left(\frac{4^{M_z}}{{}^z\beta(x(t_m))} \right)^{\frac{1}{2(k_z - M_z)}} (\gamma_d(x(t_m)))^{\frac{k_z}{2(k_z - M_z)}} \iff \\
\sqrt{\gamma_d(q)} &> \left(\frac{4^{M_z}}{{}^z\beta(x(t_m))} \right)^{\frac{1}{2(k_z - M_z)}} \left(\gamma_d^{k_z}(x(t_m))\right)^{\frac{1}{2(k_z - M_z)}} \iff \\
\sqrt{\gamma_d(q)} &> \left(\frac{4^{M_z}}{{}^z\beta(x(t_m))} \gamma_d^{k_z}(x(t_m)) \right)^{\frac{1}{2(k_z - M_z)}} \iff \\
\left(\sqrt{\gamma_d(q)}\right)^{2(k_z - M_z)} &> \frac{4^{M_z}}{{}^z\beta(x(t_m))} \gamma_d^{k_z}(x(t_m)) \iff \\
\gamma_d^{k_z - M_z}(q) &> \frac{\gamma_d^{k_z}(x(t_m))}{{}^z\beta(x(t_m))} 4^{M_z} \implies \gamma_d^{k_z}(q) > \hat{\varphi}(x(t_m)) \left(4\gamma_d(q)\right)^{M_z}
\end{aligned} \tag{A.105}$$

The triangular inequality yields

$$\begin{aligned}
\sqrt{\beta_i(q)} &= \|q - q_i\| = \|q - q_d + q_d - q_i\| \leq \|q - q_d\| + \|q_d - q_i\| \\
&= \sqrt{\gamma_d(q)} + \|q_i - q_d\|
\end{aligned} \tag{A.106}$$

and for

$$\sqrt{\gamma_d(q)} > \max_i \{\|q_i - q_d\|\} \implies \sqrt{\gamma_d} > \|q_i - q_d\|, \forall i \in I_1, \tag{A.107}$$

as required by hypothesis, it follows that

$$\begin{aligned}
&\sqrt{\beta_i(q)} \leq \sqrt{\gamma_d(q)} + \|q_i - q_d\| < \sqrt{\gamma_d(q)} + \sqrt{\gamma_d(q)} = 2\sqrt{\gamma_d(q)} \implies \\
&\beta_i(q) < 4\gamma_d(q), \forall i \in \{1, 2, \ldots, M_z\} \implies {}^z\beta(q) = \prod_{I_1} \beta_i < (4\gamma_d(q))^{M_z}.
\end{aligned} \tag{A.108}$$

Substitution in (A.105) yields $\gamma_d^{k_z}(q) > \hat{\varphi}(x(t_m))^z\beta(q)$ and since $\partial\mathscr{F} \cap \mathscr{P}_z = \emptyset$ we can examine only the interior $\mathscr{F} \setminus \partial\mathscr{F}$ where ${}^z\beta(q) > 0$ and there the previous is equivalent to

$$\gamma_d^{k_z}(q) > \hat{\varphi}(x(t_m))^z\beta(q) \overset{q \in \mathscr{F} \setminus \partial\mathscr{F} \implies {}^z\beta(q) > 0}{\implies} \frac{\gamma_d^{k_z}(q)}{{}^z\beta(q)} > \hat{\varphi}(x(t_m)) \iff \hat{\varphi}(q) > \hat{\varphi}(x(t_m)) \tag{A.109}$$

and since $\sigma_d \circ \sigma$ is strictly increasing in $[0, +\infty)$ and $q \in \mathscr{F} \setminus \partial\mathscr{F}$

$$\begin{aligned}
\hat{\varphi}(q) > \hat{\varphi}(x(t_m)) \geq 0 \iff \sigma_d \circ \sigma \circ \hat{\varphi}(q) > \sigma_d \circ \sigma \circ \hat{\varphi}(x(t_m)) \implies \\
\varphi_z(q) > \varphi_z(x(t_m)) \implies q \notin \mathscr{P}_z
\end{aligned} \tag{A.110}$$

by definition of $\mathscr{P}_z$.

Appendix B

Note on Polynomial Navigation Functions

A convex obstacle world is a subset of n-dimensional Euclidean space from which M disjoint obstacles have been removed. Each of the obstacles is a simply connected convex subset of E^n with piecewise C^2 boundary. The convex obstacle world is the agent's actual configuration space (C-space).

Note that in case the configuration space is globally non-Euclidean it is not embeddable in a Euclidean space of same dimension as the C-space manifold dimension. It can be embedded only in higher dimensional Euclidean space. This renders ordinary navigation functions inapplicable for such a case.

For example, an ordinary manipulator possesses multiple revolute joints. Its C-space is multidimensional. As long as none of the joints can perform a full turn, the space is diffeomorphic to a subset of a Euclidean space of same dimension. But if any joint performs many revolutions, the space becomes non-Euclidean. It cannot be embedded in a space of same dimension. It needs an ambient space of dimension $n+1$, where n is the number of joints. In that space it is a hypersurface of dimension n, therefore a non-flat subset of Lebesgue measure zero. There is not use in inheritance of the ambient metric for defining a naviagtion function on such a manifold.

Appropriate modifications are made in other chapters of this study to address non-Euclidean spaces and navigation functions for them.

Another matter not addressable by classical KRNFs are obstacles within Euclidean space which are of genus higher than 0. Such worlds are not diffeomorphic to any sphere world (they do not belong to its diffeomorphism class). These obstacles are diffeomorphic to tori. The basic application is a 2-dimensional solid torus. But m-fold tori are also of interest and are treated in another chapter in the 3-dimensional case. This is why Koditschek and Rimon avoid graphs of stars containing loops in [] and request that forest of stars considered form *trees* (acyclic graphs). Taking into consideration the other conditions on disjointness of stars in the forrest reveals that this constarint aims -between other things- to prevent multiply connected obstacles turning up and avoid the need to introduce further constraints/tests to ensure/check this.

B.1 Sphere world

The navigation function on the sphere world is $\varphi_s : E^n \to \mathbb{R}$. The potential's value is $\varphi_s(q)$. The argument $q \in E^n$ is the agent's configuration in the sphere world. The subscript s emphasizes that the navigation function φ_s is defined on the sphere world.

On the contrary, the navigation function $\varphi : \mathscr{C} \to \mathbb{R}$ is defined on the actual configuration space. The function φ is the *image* of φ_s. The image is obtained by applying a mapping. The mapping should be a union of a finite number of diffeomorphisms, which form a homeomorphism. Their union is such, that other conditions are met as well. This results in a continuous, piecewise C^2 navigation function φ.

The sphere world navigation function φ_s is defined by Lionis, Papageorgiou and Kyriakopoulos as

$$\varphi(q) \triangleq \frac{\gamma_d(q)}{\gamma_d(q) + \beta(q)} \tag{B.1}$$

where $\gamma_d : \mathscr{F} \to [0, +\infty)$ is the squared distance to the goal, defined as

$$\gamma_d(q) = \|q - q_d\|^2 \tag{B.2}$$

where the destination configuration is $q_d \in E^n$[1] and $\|\cdot\|$ is the Euclidean norm in E^n.

The function $\beta : \mathscr{F} \to [0, 1]$ becomes zero if and only if the argument q belongs to an obstacle's boundary $\partial\mathscr{O}_i$. It is defined as the product of M obstacle functions β_i

$$\beta(q) = \prod_{i=1}^{M} \beta_i(q) \tag{B.3}$$

where each obstacle function β_i corresponds to one of the M obstacles $\partial\mathscr{O}_i$. Each of these obstacles is a sphere of radius ρ_i. Each one is the image of an obstacle in the actual configuration space $\mathscr{C}$.

To simplify the expressions involved and aid understanding, a different coordinate system is used to define each β_i.

For each obstacle two new coordinate systems are defined.

The first is a cartesian coordinate system. Its origin is the goal configuration q_d. The y_i axis is defined by the unit vector $i_y = q_{c,i} - q_d$ which is the direction from the goal q_d to the ith sphere center $q_{c,i}$.

The second is a polar coordinate system. Its origin is the $i$the sphere center $q_{c,i}$ and the radius $r_{s,i} = \|q - q_{c,i}\|$ measures the distance from that center. The angle $\theta = (q_{c,i} - \widehat{q_d, q} - q_{c,i})$.

The definition of $\beta_i : \mathscr{F} \to [0, 1]$ is

$$\beta_i(q) = \begin{cases} P(z_i) & , z_i \in [0, 1] \\ 1 & , z_i > 1 \end{cases} \tag{B.4}$$

where

$$z_i = \frac{r_{s,i} - \rho_i}{\varepsilon_i} \tag{B.5}$$

is the dimensionless position in the current effect zone[2]. When $\rho_i \leq r_{s,i} \leq \rho_i + \varepsilon_i \iff z_i \in [0, 1]$ the configuration belongs to the annulus. When $\rho_i + \varepsilon_i < r_{s,i} \iff z_i > 1$ the configuration is outside the effect annulus.

Each function β_i is continuous, because $P(1) = 1$ and twice continuously differentiable, because $P'(1) = 0$ and $P''(1) = 0$.

[1]The subscript d denotes *destination*.

[2]The effect zone is always an annulus of width $\varepsilon_i = r_{s,i,\max} - r_{s,i,\min}$ the configuration space.

The gradient of function φ_s is

$$\begin{aligned}\nabla\varphi_s =&\nabla\left(\frac{\gamma_d}{\gamma_d+\beta}\right)\\ &=\frac{(\gamma_d+\beta)\nabla\gamma_d-\gamma_d\nabla(\gamma_d+\beta)}{(\gamma_d+\beta)^2}\\ &=\frac{\gamma_d\nabla\gamma_d+\beta\nabla\gamma_d-\gamma_d(\nabla\gamma_d+\nabla\beta)}{(\gamma_d+\beta)^2}\\ &=\frac{\gamma_d\nabla\gamma_d+\beta\nabla\gamma_d-\gamma_d\nabla\gamma_d-\gamma_d\nabla\beta}{(\gamma_d+\beta)^2}\\ &=\frac{\beta\nabla\gamma_d-\gamma_d\nabla\beta}{(\gamma_d+\beta)^2}\end{aligned} \tag{B.6}$$

and since

$$\nabla\gamma_d=\begin{bmatrix}\frac{\partial}{\partial r}\left(y_{c,i}^2+r_{s,i}^2+2r_{s,i}y_{c,i}\cos\theta_i\right)\\ \frac{\partial}{\partial\theta_i}\left(y_{c,i}^2+r_{s,i}^2+2r_{s,i}y_{c,i}\cos\theta_i\right)\end{bmatrix}=\begin{bmatrix}2r_{s,i}+2y_{c,i}\cos\theta_i\\ -2r_{s,i}y_{c,i}\sin\theta_i\end{bmatrix} \tag{B.7}$$

and

$$\nabla\beta=\nabla\prod_{i=1}^{M}\beta_i=\sum_{i=1}^{M}\left(\nabla\beta_i\prod_{j=1,j\neq i}^{M}\beta_j\right) \tag{B.8}$$

and because only at most one $\nabla\beta_i\neq 0$ (that of the obstacle within whose effect zone the point is) it follows that

$$\nabla\beta=\begin{bmatrix}\frac{\partial}{\partial r_{s,i}}P\left(\frac{r_{s,i}-\rho_i}{\varepsilon_i}\right)\\ \frac{\partial}{\partial\theta_i}P\left(\frac{r_{s,i}-\rho_i}{\varepsilon_i}\right)\end{bmatrix}=\begin{bmatrix}P'\left(\frac{r_{s,i}-\rho_i}{\varepsilon_i}\right)\frac{\partial}{\partial r_{s,i}}\left(\frac{r_{s,i}-\rho_i}{\varepsilon_i}\right)\\ P'\left(\frac{r_{s,i}-\rho_i}{\varepsilon_i}\right)\frac{\partial}{\partial\theta_i}\left(\frac{r_{s,i}-\rho_i}{\varepsilon_i}\right)\end{bmatrix}=\begin{bmatrix}P'(z_i)\frac{1}{\varepsilon_i}\\ P'\left(\frac{r_{s,i}-\rho_i}{\varepsilon_i}\right)\cdot 0\end{bmatrix}=\begin{bmatrix}\frac{1}{\varepsilon_i}P'(z_i)\\ 0\end{bmatrix} \tag{B.9}$$

Substitution of the above yields

$$\nabla\varphi_s=\frac{1}{(\gamma_d+\beta)^2}\left(\beta\begin{bmatrix}2r_{s,i}+2y_{c,i}\cos\theta_i\\ -2r_{s,i}y_{c,i}\sin\theta_i\end{bmatrix}-\gamma_d\begin{bmatrix}\frac{1}{\varepsilon_i}P'(z_i)\\ 0\end{bmatrix}\right) \tag{B.10}$$

and because in the effect annulus of obstacle i only $\beta_i(q)=P(z_i)\neq 1$

$$\begin{aligned}\nabla\varphi_s&=\frac{1}{(\gamma_d+P(z_i))^2}\left(P(z_i)\begin{bmatrix}2r_{s,i}+2y_{c,i}\cos\theta_i\\ -2r_{s,i}y_{c,i}\sin\theta_i\end{bmatrix}-\gamma_d\begin{bmatrix}\frac{1}{\varepsilon_i}P'(z_i)\\ 0\end{bmatrix}\right)\\ &=\frac{1}{(\gamma_d+P(z_i))^2}\left(\begin{bmatrix}P(z_i)\left(2r_{s,i}+2y_{c,i}\cos\theta_i\right)\\ -P(z_i)2r_{s,i}y_{c,i}\sin\theta_i\end{bmatrix}+\begin{bmatrix}-\frac{\gamma_d}{\varepsilon_i}P'(z_i)\\ 0\end{bmatrix}\right)\\ &=\frac{1}{(\gamma_d+P(z_i))^2}\begin{bmatrix}P(z_i)\left(2r_{s,i}+2y_{c,i}\cos\theta_i\right)-\frac{\gamma_d}{\varepsilon_i}P'(z_i)\\ -P(z_i)2r_{s,i}y_{c,i}\sin\theta_i\end{bmatrix}\end{aligned} \tag{B.11}$$

Therefore

$$\frac{\partial\varphi_s}{\partial r_{s,i}}=\frac{P(z_i)\left(2r_{s,i}+2y_{c,i}\cos\theta_i\right)-\frac{\gamma_d}{\varepsilon_i}P'(z_i)}{(\gamma_d+P(z_i))^2},\quad \frac{\partial\varphi_s}{\partial\theta_i}=\frac{-P(z_i)2r_{s,i}y_{c,i}\sin\theta_i}{(\gamma_d+P(z_i))^2} \tag{B.12}$$

Because $P(z_i)>0,\forall z_i>0$ and $r_i>0, y_i>0$ it follows that in the interior of the free space

$$\frac{\partial\varphi_s}{\partial\theta_i}=\frac{-P(z_i)2r_{s,i}y_{c,i}\sin\theta_i}{(\gamma_d+P(z_i))^2}\begin{cases}>0, & \theta_i\in(-\pi,0)\\ =0, & \theta_i\in\{0,\pi\}\\ <0, & \theta_i\in(0,\pi)\end{cases} \tag{B.13}$$

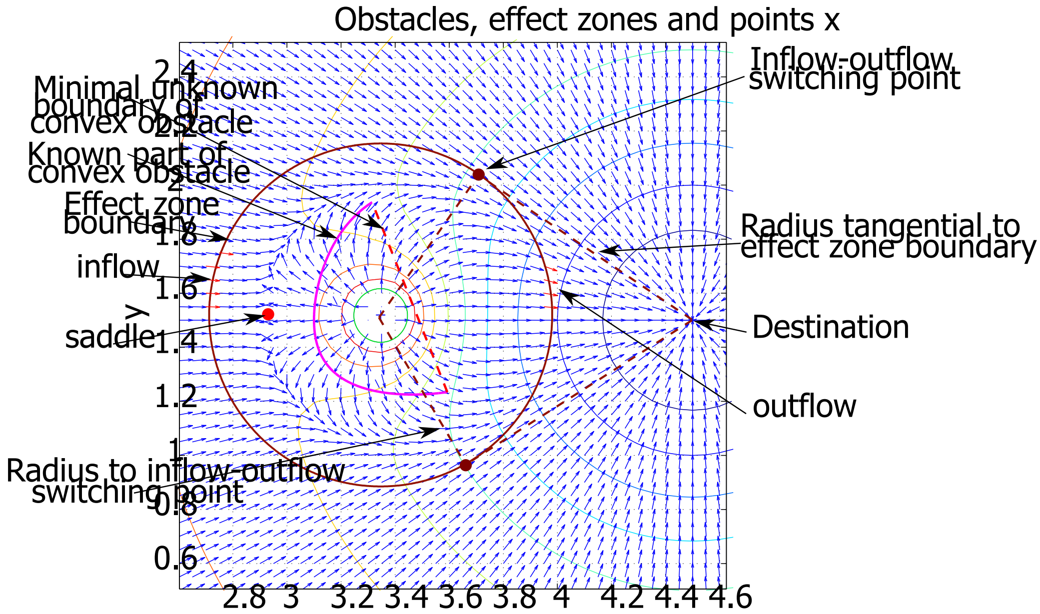


Figure B.1: Polynomial navigation function potential on 2D sphere world, effect zone detail annotated.

and $P(z_i = 0) \implies \frac{\partial \varphi_s}{\partial \theta_i} = 0, \forall z_i = 0$.

This portrait of the tangential partial derivative $\frac{\partial \varphi_s}{\partial \theta_i}$ is symmetric about the y_i axis, outside the effect zones of all other obstacles $j \neq i$. The y_i axis is the line through the goal q_d and the ith obstacle center $q_{c,i}$.

There are two regions, the one in which $\nabla\gamma_d \circ \nabla\beta <$ and the one in which $\nabla\gamma_d \circ \nabla\beta <$. The second one has only radially outward flow. The first is further subdivided into two regions. The inner, where the repulsive effect is stronger than the attractive, resulting in an overall outward flow. The outer, where the repulsive effect is less strong than the attractive effect, resulting in an inward flow. Since in the first region the inner region moves outward and the outer inward, they meet, forming a valley. This valley partially orbits the obstacle, until it leads out of the effect zone.

The navigation function potential is shown in Fig. B.2.

B.2 Diffeomorphism

B.2.1 Conditions to make it a diffeomorphism

The diffeomorphism is defined as

$$T_i(r_i, \theta_i) = \begin{bmatrix} T_i^1 \\ T_i^2 \end{bmatrix} = \begin{bmatrix} S(x_i(q_i))b_i(q_i)\rho_i + (1 - S(x_i(q_i)))r_i \\ \theta_i \end{bmatrix} \tag{B.14}$$

It is desired to select x_i, b_i such that the transformation can be proved to be a diffeomorphism between the closed patch boundaries wchich are C^2 curves.

Note that if an obstacle with covers were to be transformed to a 2-sphere, then $T^1_{r_i}, T^2_{r_i}, T^2_{\theta_i}$ would exist and be the desirable, but $T^1_{\theta_i}$ would *not exist* on the ray through the corner (C^1 discontinuity).

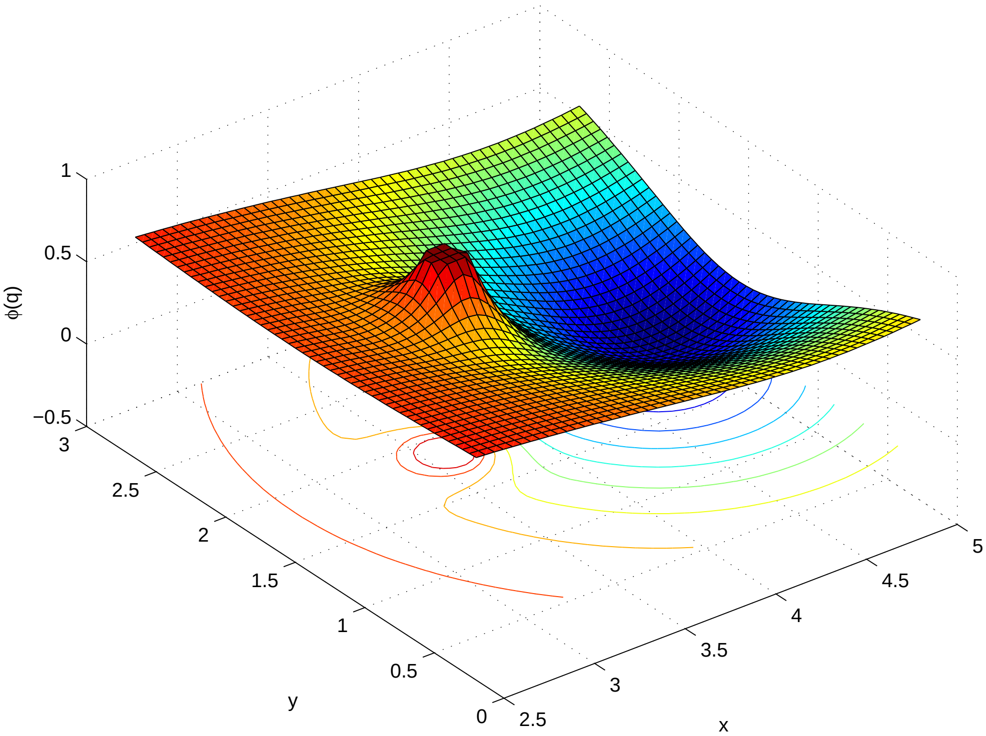


Figure B.2: Polynomial navigation function potential on 2D sphere world.

Therefore it is obvious that just showing some terms have the desired value when the rest are not needed (here the value of $T^1_{\theta_i}$) is not enough. One need to ensure that all the terms exist.

This limitation of a piecewise C^2 obstacle boundary leads us to a different approach. Divide the zone around the obstacle in pieces, of which each one has C^2 boundaries, so that a diffeomorphism can be defined on it to an arc of a circle.

This leads to the requirement of piecewise C^2 curves defining the obstacle boundaries. Then a cover of diffeomorphisms can be constructed.

Let us now show that we can construct such diffeomorphisms as functions of the patch boundaries and the current position.

Define

$$S(x) = \begin{cases} 1, & x \in (-\infty, 0] \\ -6x^5 + 15x^4 - 10x^3 + 1, , & x \in (0,1) \\ 0, & x \in [0, \infty] \end{cases} \tag{B.15}$$

then

$$\frac{\partial}{\partial x} S(x) = -30x^4 + 60x^3 - 30x^2 \tag{B.16}$$

we will write $S'(x) = \frac{\partial}{\partial x} S(x)$ so note that

$$S(0) = 1, S'(0) = 0, S(1) = 0, S'(1) = 0 \tag{B.17}$$

To prove that the transformation is a diffeomorphism it suffices to show that the Jacobian matrix is everywhere on the closed patch differentiable.

The boundaries of interest concerning differentiability are the inner and outer, not the extremal rays corresponding to corners (although if an extension of the obstacle's segment in a C^2 way is constructed, it can be shown that a diffeomorphism can be defined on the open superset).

We will show that differentiability on the inner and outer boundaries of the transformation (and its inverse) holds also on the boundaries, which correspond to the branching points of S).

Let us start by showing existence and invertibility of the Jacobian matrix. Its terms are

$$JT_i(r_i,\theta_i) = \begin{bmatrix} \frac{\partial}{\partial r_i}T_i^1 & \frac{\partial}{\partial \theta_i}T_i^1 \\ \frac{\partial}{\partial r_i}T_i^2 & \frac{\partial}{\partial \theta_i}T_i^2 \end{bmatrix} = \begin{bmatrix} T_{i,r_i}^1 & T_{i,\theta_i}^1 \\ T_{i,r_i}^2 & T_{i,\theta_i}^2 \end{bmatrix} \tag{B.18}$$

where

$$\begin{aligned} \frac{\partial}{\partial r_i}T_i^2(r_i,\theta_i) &= \frac{\partial}{\partial r_i}\theta_i = 0 \\ \frac{\partial}{\partial \theta_i}T_i^2(r_i,\theta_i) &= \frac{\partial}{\partial \theta_i}\theta_i = 1 \end{aligned} \tag{B.19}$$

and

$$\begin{aligned} \frac{\partial}{\partial \theta_i}T_i^1(r_i,\theta_i) &= \frac{\partial}{\partial \theta_i}\left(S(x_i(q_i))b_i(q_i)\rho_i + (1-S(x_i(q_i)))r_i\right) \\ &= S'(x_i(q_i))\frac{\partial}{\partial \theta_i}\{x_i(q_i)\}b_i(q_i)\rho_i - r_iS'(x_i(q_i)))\frac{\partial}{\partial \theta_i}(x_i(q_i)) \\ &= S'(x_i(q_i))\frac{\partial}{\partial \theta_i}x_i(q_i)b_i(q_i)\rho_i + S(x_i(q_i))\frac{\partial}{\partial \theta_i}b_i(q_i)\rho_i - S'(x_i(q_i))\frac{\partial}{\partial \theta_i}\{x_i(q_i)\}r_i \\ &= S'(x_i(q_i))\frac{\partial}{\partial \theta_i}x_i(q_i)\left[b_i(q_i)\rho_i - r_i\right] + S(x_i(q_i))\rho_i\frac{\partial}{\partial \theta_i}b_i(q_i) \end{aligned} \tag{B.20}$$

which exists (and the chain rule can be applied) when $\frac{\partial}{\partial \theta_i}x_i(q_i)$ and $\frac{\partial}{\partial \theta_i}b_i(q_i)$ exist. This depends on the choice of x_i and b_i so it is to be checked when we select them.

The Jacobian matrix is therefore of the form

$$JT_i(r_i,\theta_i) = \begin{bmatrix} T_{i,r_i}^1 & T_{i,\theta_i} \\ 0 & 1 \end{bmatrix} \implies \det JT_i(r_i,\theta_i) = T_{i,r_i}^1 \tag{B.21}$$

As a result, invertibility of $JT_i(r_i,\theta_i)$ is equivalent to $T_{i,r_i}^1 \neq 0$. If we further require that $T_{i,r_i}^1(r_i,\theta_i) > 0$ then it is guaranteed that orientation is preserved and not reverted.

It is (provided that derivatives exist)

$$\begin{aligned} T_{i,r_i}^1(r_i,\theta_i) &= \frac{\partial}{\partial r_i}T_i^1(r_i,\theta_i) \\ &= \frac{\partial}{\partial r_i}\left(S(x_i(q_i)b_i(q_i)\rho_i + (1-S(x_i(q_i)))r_i\right) \\ &= S'(x_i(q_i))\frac{\partial}{\partial r_i}(x_i(q_i))b_i(q_i)\rho_i + S(x_i(q_i))\frac{\partial}{\partial r_i}(b_i(q_i))\rho_i + (1-S(x_i(q_i))) - S'(x_i(q_i))\frac{\partial}{\partial r_i}x_i(q_i)r_i \\ &= S'(x_i(q_i))\frac{\partial}{\partial r_i}(x_i(q_i))\left[b_i(q_i)\rho_i - r_i\right] + S(x_i(q_i))\frac{\partial}{\partial r_i}b_i(q_i)\rho_i + \left[1-S(x_i(q_i))\right] \end{aligned} \tag{B.22}$$

We have to deal with 3 terms

$$\begin{aligned} &S'(x_i(q_i))\frac{\partial}{\partial r_i}(x_i(q_i))\left[b_i(q_i)\rho_i - r_i\right] + \\ &\quad +S(x_i(q_i))\frac{\partial}{\partial r_i}(b_i(q_i))\rho_i + \\ &\quad +1 - S(x_i(q_i)) \end{aligned} \tag{B.23}$$

and all of them are usefull if we are to prove what is desired. This usefullness is to become clear in what follows.

Suppose we select a $x_i(q_i)$ such that

$$\begin{aligned} x_i(q_i) = 0, \quad \forall q_i \in \partial\mathscr{O}_i &\implies \begin{cases} S(x_i(q_i)) = 1, \forall q_i \in \partial\mathscr{O}_i \\ S'(x_i(q_i)) = 0, \forall q_i \in \partial\mathscr{O}_i \end{cases} \\ x_i(q_i) = 1 &\implies \begin{cases} S(x_i(q_i)) = 0, \forall q_i \in \partial P_i - \partial\mathscr{O}_i \\ S'(x_i(q_i)) = 0, \forall q_i \in \partial P_i - \partial\mathscr{O}_i \end{cases} \end{aligned} \tag{B.24}$$

and

$$0 < x_i(q_i) < 1, \quad \forall q_i \in \mathring{P}_i \implies \begin{cases} 0 < S(x_i(q_i)) < 1, \forall q_i \in \mathring{P}_i \\ S'(x_i(q_i)) < 0, \forall q_i \in \mathring{O}_i \end{cases} \tag{B.25}$$

Then $1 - S(x_i(q_i)) > 0, \forall q_i \in \mathring{P}_i$.

$$\exists (JT_i(r_i,\theta_i))^{-1}, \forall q_i \in \mathring{P}_i \iff \det JT_i(r_i,\theta_i) \neq 0, \forall q_i \in \mathring{P}_i \tag{B.26}$$

But because $\det JT_i(r_i,\theta_i) > 0$ for $q_i \in A \subset P_i$ and $\det JT_i(r_i,\theta_i) < 0$ for $q_i \in B \subset P_i$ would mean that the trasnition is discontinuous, hence that $\det JT_i(r_i,\theta_i)$ and so $JT_i(r_i,\theta_i)$ are discontinuous, then this means that $JT_i(r_i,\theta_i)$ would not exist at the discontinuity.

Therefore it must be that $\det JT_i(r_i,\theta_i) > 0$ or $\det JT_i(r_i,\theta_i) < 0$. We choose to require $\det JT_i(r_i,\theta_i) > 0$.

Then at least on eterm should be positive at every point of the closed patch, while the others are allowed to be nonpositive at that same point.

We nontheless require that all terms be positive at every point of the open patch. So this leads us to

$$\underbrace{S(x_i(q_i))}_{>0} \frac{\partial}{\partial r_i} b_i(q_i) \underbrace{\rho_i}_{>0} > 0, \forall q_i \in \mathring{P}_i \iff \frac{\partial}{\partial r_i} b_i(q_i) > 0, \forall q_i \in \mathring{P}_i \tag{B.27}$$

and

$$\underbrace{S'(x_i(q_i))}_{>0} \frac{\partial}{\partial r_i} x_i(q_i) \left[b_i(q_i)\rho_i - r_i \right] > 0, \forall q_i \in \mathring{P}_i \tag{B.28}$$

If we select a x_i such that

$$\frac{\partial}{\partial r_i} x_i(q_i) > 0, \forall q_i \in \mathring{P}_i \tag{B.29}$$

then we should also select a $b_i(q_i)$ such that

$$b_i(q_i)\rho_i - r_i < 0, \forall q_i \in \mathring{P}_i \overset{\rho_i > 0}{\iff} b_i(q_i) < \frac{r_i}{\rho_i}, \forall q_i \in \mathring{P}_i \tag{B.30}$$

the required condition is met[3].

So the conditions required in the interior $\mathring{P}_i$ of P_i have been determined. Let us examine the boundaries.

[3] We could have chosen $\frac{\partial}{\partial r_i} x_i(q_i) < 0, \forall q_i \in \mathring{P}_i, b_i(q_i)\rho_i - r_i > 0, \forall q_i \in \mathring{P}_i$ but this would not allow $x_i(q_i) = 0, \forall q_i \in \partial\mathscr{O}_i \iff x_i(r_{in,i}(\theta_i)) = 0$ and $x_i(q_i) = 1, \forall q_i \in (\partial P_i - \partial\mathscr{O}_i) \iff x_i(r_{out,i}(\theta_i)) = 1$ where $r_{in,i}(\theta_i) < r_{out,i}(\theta_i)$. Therefore this selection is the only possible.

On $r_{in,i}(\theta_i)$ it is $x_i(r_{in,i}(\theta_i)) = 0$ and

$$\begin{aligned} T^1_{i,r_i}(\theta_i) &= S'(x_i(q_i))\frac{\partial}{\partial r_i}x_i(q_i)\left[b_i(q_i)\rho_i - r_i\right] + S(x_i(q_i))\frac{\partial}{\partial r_i}(b_i(q_i))\rho_i + 1 - S(x_i(q_i)) \\ &= \frac{\partial}{\partial r_i}b_i(q_i)\underbrace{\rho_i}_{>0} \end{aligned} \tag{B.31}$$

and if we require $\frac{\partial}{\partial r_i}b_i(q_i) > 0$ on the inner boundary, then $T^1_i(r_{in,i}(\theta_i), \theta_i) > 0$ there.

On $r_{out,i}(\theta_i)$ it is $x_i(r_{out,i}(\theta_i)) = 1$ and

$$\begin{aligned} T^1_{i,r_i}(r_{out,i}(\theta_i), \theta_i) &= S'(x_i(q_i))\frac{\partial}{\partial r_i}(x_i(q_i))\left[b_i(q_i)\rho_i - r_i\right] + S(x_i(q_i))\frac{\partial}{\partial r_i}(b_i(q_i))\rho_i + 1 - S(x_i(q_i)) \\ &= 1 > 0 \end{aligned} \tag{B.32}$$

From the above it becomes clear that not having $b_i(q_i)$ in the product would not allow us to have the term $S(x_i(q_i))\frac{\partial}{\partial r_i}(b_i(q_i))\rho_i$ and if r_i was replaced by $r_{out,i}(\theta_i)$ then neither $q - S(x_i(q_i))$ would arise. Then

$$\begin{aligned} T^1_{i,r_i}(r_{in,i}(\theta_i), \theta_i) &= 0 \\ T^1_{i,r_i}(r_{out,i}(\theta_i), \theta_i) &= 0 \\ T^1_{i,r_i}(r_i, \theta_i) &= \underbrace{S'(x_i(q_i))}_{<0}\underbrace{\frac{\partial}{\partial r_i}(x_i(q_i))}_{>0}\underbrace{[\rho_i - r_i]}_{<0} \end{aligned} \tag{B.33}$$

and the transformation would not be a diffeomorphism on the closed set P_i (or equivalently on the open-closed ∞, inner boundary). Therefore the choice of a linear transformation

$$T^1_i(r_i, \theta_i) = S(x_i(q_i))\rho_i + (1 - S(x_i(q_i)))r_{out,i}(\theta_i) \tag{B.34}$$

would not work.

Note that points on $\partial\mathscr{O}_i$ are transformed to points on the i^{th} circle boundary

$$r_{in,i}(\theta_i) \overset{T_i(r_i,\theta_i)}{\rightarrow} b_i(q_i)\rho_i, q_i \in \mathscr{O}_i \tag{B.35}$$

We want

$$b_i(q_i)\rho_i = \rho_i, \forall q_i \in \mathscr{O}_i \overset{\rho_i > 0}{\iff} b_i(q_i) = 1, \forall q_i \in \mathscr{O}_i \tag{B.36}$$

B.2.1.1 Conditions summary

We want the following to hold for the function $x_i(q_i)$ to be selected

$$\begin{aligned} x_i(q_i) &= 0, \forall q_i \in \partial\mathscr{O}_i \\ x_i(q_i) &= 1, \forall q_i \in (\partial P_i - \partial\mathscr{O}_i) \\ 0 < x_i(q_i) &< 1, \forall q_i \in \mathring{P}_i \\ \frac{\partial x_i}{\partial r_i}(q_i) &> 0, \forall q_i \in \mathring{P}_i \end{aligned} \tag{B.37}$$

and for function $b_i(q_i)$ we require

$$\begin{aligned} b_i(q_i) &= 1, \forall q_i \in \partial\mathscr{O}_i \\ \frac{\partial}{\partial r_i}b_i(q_i) &> 0, \forall q_i \in \mathring{P}_i \\ b_i(q_i) &< \frac{r_i}{\rho_i}, \forall q_i \in \mathring{P}_i \end{aligned} \tag{B.38}$$

Note that the third requirement on $b_i(q_i)$ is on the interior $\mathring{P}_i$ of P_i. If imposed on the closure $\bar{P}_i$ then it is compatible with the first constraint, because

$$b_i q_i = \frac{r_{in,i}(\theta_i)}{\rho_i} > 1, \forall q_i \in \partial\mathscr{O}_i \tag{B.39}$$

B.2.2 Selection 1 for functions $b_i(q), x_i(q)$

Let

$$b_i(q) = \frac{r_{s,i} - r_{s,\text{in},i}(\theta_i)}{\rho_i} + 1 = f(r_i, \theta_i) \tag{B.40}$$

then

$$\begin{aligned} b_i(r_{in,i}(\theta_i)) &= \frac{r_{in,i}(\theta_i) - r_{in,i}(\theta_i)}{\rho_i} + 1 = 1 \\ \frac{\partial}{\partial} b_i(q_i) = \frac{\partial}{\partial}\left(\frac{r_{s,i} - r_{s,\text{in},i}(\theta_i)}{\rho_i} + 1\right) &= \frac{1}{\rho_i} > 0, \forall q_i \in P_i \text{not only} \mathring{P}_i \\ b_i(q_i) = \frac{r_{s,i} - r_{s,\text{in},i}(\theta_i)}{\rho_i} + 1 &= \frac{r_i}{\rho_i} + \underbrace{\left(1 - \frac{r_{in,i}(\theta_i)}{\rho_i}\right)}_{<0} < \frac{r_i}{\rho_i} \end{aligned} \tag{B.41}$$

where it is important to select a circle within the obstacle to ensure

$$r_{in,i}(\theta_i) > \rho_i \underset{\rho_i > 0}{\iff} \frac{r_{in,i}(\theta_i)}{\rho_i} > 1 \iff 1 - \frac{r_{in,i}(\theta_i)}{\rho_i} < 0 \tag{B.42}$$

Then we can select

$$\begin{aligned} x_i(q_i) &= \frac{b_i(q_i) - b_i(\partial\mathscr{O}_i)}{b_i(\partial P - \partial\mathscr{O}_i) - b_i(\partial O_i)} = \frac{b_i(q_i) - 1}{b_i(r_{out,i}(\theta_i)) - 1} \\ &= \frac{\frac{r_i - r_{in,i}(\theta_i)}{\rho_i} + 1 - 1}{\frac{r_{out,i}(\theta_i) - r_{in,i}(\theta_i)}{\rho_i} + 1 - 1} = \frac{r_i - r_{in,i}(\theta_i)}{r_{out,i}(\theta_i) - r_{in,i}(\theta_i)} \end{aligned} \tag{B.43}$$

B.2.3 Selection 2 for functions $b_i(q), x_i(q)$

Select

$$x_i(q_i) = w_i \frac{r_{s,i} - r_{s,\text{in},i}}{r_{s,j} - r_{s,\text{in},j}} \tag{B.44}$$

where $w_i \in (0, 1)$ is the weight of the ith Voronoi cell and j is the second closet obstacle (weighted convex distance Voronoi-wise). The radial partial derivative of x_i is

$$\frac{\partial}{\partial r_{s,i}} x_i(q_i) = \frac{\partial}{\partial r_{s,i}}\left(w_i \frac{r_{s,i} - r_{s,\text{in},i}}{r_{s,j} - r_{s,\text{in},j}}\right) = w_i \frac{1}{r_{s,j} - r_{s,\text{in},j}} \tag{B.45}$$

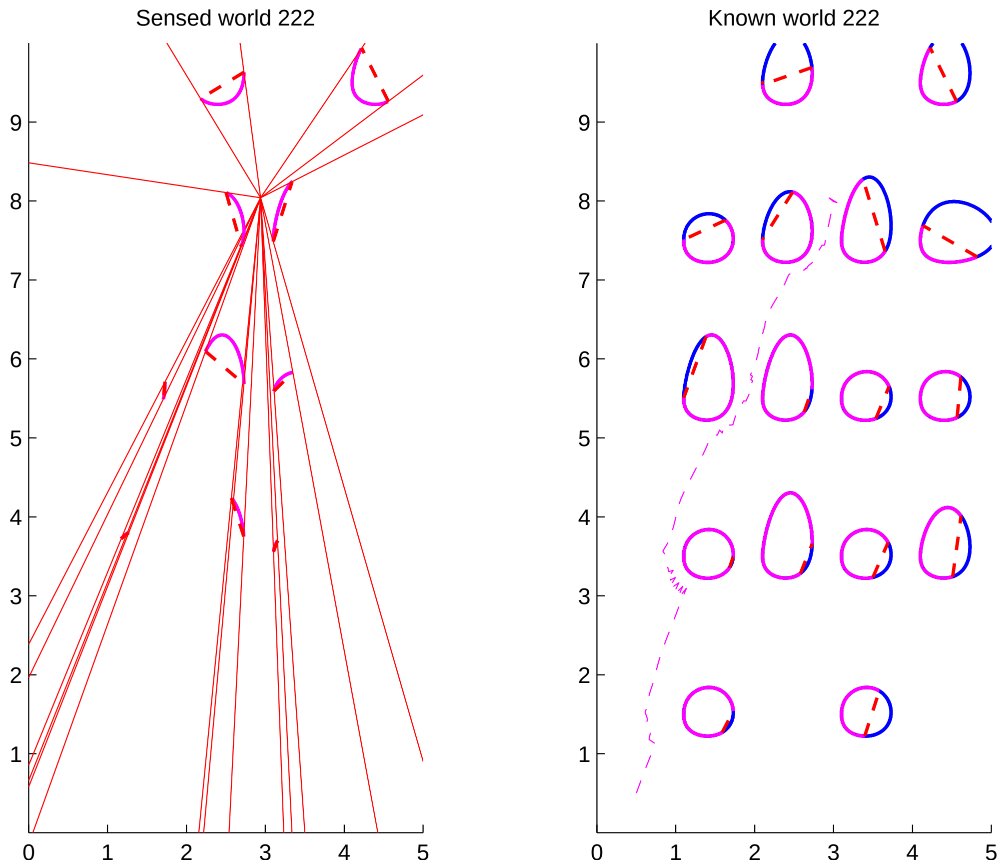


Figure B.3: Navigating a convex obstacle world with Bezier obstacles using an updating polynomial Navigation Function with an updating diffeomorphism.

B.3 Notation and Definitions

Table B.1: Notation and definitions.

Symbol	Meaning	First definition
$\|\cdot\|$	vector norm	
$|\cdot|$	absolute value of real number	
$\mathbb{N}$	set of natural numbers	
$\mathbb{R}$	set of real numbers	
$\mathbb{C}$	set of complex numbers	
∇	gradient of scalar function (1st derivative $D_i\{\cdot\}$)	
D^2	Hessian matrix of scalar function (2nd derivative $D_i D_j\{\cdot\}$)	
n	dimension of Euclidean space	subsection 2.2.1
E^n	n-dimensional Euclidean space	subsection 2.2.1
M	number of internal obstacles (a priori known world)	subsection 2.2.1
$\mathscr{W}$	workspace	(2.1)
q	vector in E^n	(2.1)
ρ_0	radius of obstacle 0	(2.1)
i, j	dummy indices	subsection 2.2.1
$\mathscr{O}_j$	internal obstacle $j \in I_1$	(2.2)
ρ_j	$\mathscr{O}_j$ radius	(2.2)
q_j	$\mathscr{O}_j$ center	(2.2)
I_1	set of internal obstacle indices (a priori known world)	(2.2)
I_0	set of all obstacle indices (a priori known world)	subsection 2.2.1
∂	partial derivative, closed set boundary	
$\partial\mathscr{W}$	workspace $\mathscr{W}$ boundary	subsection 2.2.1
$\mathscr{O}_0$	obstacle 0	(2.3)
$\mathscr{F}$	free space	(2.4)
$\partial\mathscr{O}_j$	$\mathscr{O}_j$ boundary	subsection 2.2.1
λ	auxiliary parameter used in proof	(2.8)
$\mathscr{B}_i(\varepsilon_i)$	open n-dimensional spherical annulus	(2.13)
ε_i	parameters determining the widths of annuli $\mathscr{B}_i(\varepsilon_i)$	(2.13)
$\overline{\mathscr{X}}$	closure of set $\mathscr{X}$ equal to $\mathscr{X} \cup \partial\mathscr{X}$	
$\overline{\mathscr{B}_i(\varepsilon_i)}$	closure of $\mathscr{B}_i(\varepsilon_i)$	(2.14)
$\rho_{\overline{\mathscr{B}}_i}$	annulus $\mathscr{B}_i(\varepsilon_i)$ outer radius	(2.15)
q_d	agent destination in E^n	subsection 2.2.2
$\mathscr{F}_d$	singleton set of destination $\{q_d\}$	subsection 2.2.2
$\partial\mathscr{F}$	free space boundary	subsection 2.2.2
$\mathscr{F}_0$	set "near" internal obstacles	subsection 2.2.2
$\mathscr{F}_1$	set "near" workspace boundary	subsection 2.2.2
$\mathscr{F}_2$	set "away" from obstacles	subsection 2.2.2
ε_{I_0}	set of $\varepsilon_i, i \in I_0$	subsection 2.2.2
ε_{I_1}	set of $\varepsilon_i, i \in I_1$	subsection 2.2.2
ε_{i3j}	constraint on ε_i ensuring $\mathscr{B}_i(\varepsilon_i) \cap \mathscr{O}_j = \emptyset, j \in I_0 \setminus i, i \in I_1$	(2.17)
$\partial\overline{\mathscr{B}_i(\varepsilon_i)}$	boundary of $\overline{\mathscr{B}_i(\varepsilon_i)}, i \in I_0$	Fig. 2.4
$\mathscr{M}$	analytic manifold with boundary	subsection 2.3.1
$\varphi(q)$	general navigation function (a priori known world)	subsection 2.3.1
$\varphi(q)$	Koditschek-Rimon navigation function (a priori known world)	(2.20)

Table B.2: Notation and definitions.

Symbol	Meaning	First definition
$\sigma(x)$	squashing diffeomorphism	(2.21)
x	auxiliary real number	subsection 2.3.2
$\sigma_d(x)$	distortion diffeomorphism	(2.27)
$\hat{\varphi}(q)$	diffeomorphic to navigation function in $\mathscr{F} \setminus \{\partial\mathscr{F} \cup \{q_d\}\}$	(2.22)
$\gamma(q)$	tuned destination attractive effect	(2.23)
$\gamma_d(q)$	destination q_d paraboloid attractive effect	(2.23)
k	navigation function tuning parameter	(2.23)
$\beta(q)$	product of obstacle functions	(2.24)
$\beta_i(q)$	implicit obstacle function	(2.25)
$\bar{\beta}_i(q)$	product of all β_j omitting β_i	(2.26)
$N(\varepsilon_{I_0})$	lower bound on k	(2.62)
ε'_{0i}	defined in [23] as ε'_{i0} here	Table 2.1
ε''_{0i}	defined in [23] as ε''_{i0} here	Table 2.1
ε'_{2i}	defined in [23] as ε'_{i2} here	Table 2.1
ε''_{2i}	defined in [23] as ε''_{i2} here	Table 2.1
ε_1	defined in [23] as ε_{0u} here	Table 2.1
ε_0	defined in [23] as $\min_{i\in I_1}\{\varepsilon'_{i0}, \varepsilon''_{i0}\}$ here	Table 2.1
ε_2	defined in [23] as $\min_{i\in I_1}\{\varepsilon'_{i2}, \varepsilon''_{i2}\}$ here	Table 2.1
ε	defined in [23] as $\min_{i\in I_0}\{\varepsilon_i\}$ here	Table 2.1
ε_{iu}	upper bound on ε_i	(2.29)
ε'_{i0}	constraint on ε_i ensuring $q_d \notin \overline{\mathscr{B}_i(\varepsilon_i)}$ ($q_d \notin \mathscr{B}_i(\varepsilon_i)$ in [23])	(2.29), Table 2.1, (2.133)
ε''_{i0}	constraint on ε_i ensuring $D^2\varphi(q_c) < 0$ in $\nabla\beta_i^{\perp}$	(2.29), Table 2.1, (2.195)
ε'_{i2}	together with ε''_{i2} ensures $D^2\varphi(q_c) > 0$ in $\mathrm{span}\{\nabla\beta_i\}$	(2.29), Table 2.1, (2.84)
ε''_{i2}	together with ε'_{i2} ensures $D^2\varphi(q_c) > 0$ in $\mathrm{span}\{\nabla\beta_i\}$	(2.29), Table 2.1, (2.95)
ε_{i3}	constraint on ε_i ensuring $\mathscr{B}_i(\varepsilon_i) \cap \bigcup_{j\in I_0\setminus i} \mathscr{O}_j = \emptyset, i \in I_1$	(2.30)
ε_{i03}	combined constraints ε'_{i0} and ε_{i3}	(2.30)
ε_{i23}	combined constraints ε'_{i2} and ε_{i3}	(2.30)
q_c	critical point in E^n	subsubsection 2.4.2.1
$\mathscr{C}_{\hat{\varphi}}$	set of critical points of function $\hat{\varphi}$	subsubsection 2.4.2.1
$N_{KR}(\varepsilon)$	lower bound on k as defined in [23]	Equation 2.44
$Q_0(x)$	auxiliary function such that $Q_0(\beta_0) = \frac{1}{2}\frac{\lVert\nabla\beta_0\rVert}{\beta_0}$	(2.58)
Q_{00}	upper bound on $\frac{1}{2}\max_{\mathscr{F}_2}\left\{\frac{\lVert\nabla\beta_0\rVert}{\beta_0}\right\} = \max_{\mathscr{F}_2}\{Q_0(\beta_0)\}$	(2.58)
$Q_i(x)$	auxiliary function such that $Q_i(\beta_i) = \frac{1}{2}\frac{\lVert\nabla\beta_i\rVert}{\beta_i}$	(2.58)
Q_{ii}	$\frac{1}{2}\max_{\mathscr{F}_2}\left\{\frac{\lVert\nabla\beta_i\rVert}{\beta_i}\right\} = \max_{\mathscr{F}_2}\{Q_i(\beta_i)\}, \forall i \in I_1$	(2.58)
ε_{0u}	upper bound on ε_0	(2.74)
$\hat{v}$	unit vector $\{\hat{v} \in E^n \mid \lVert\hat{v}\rVert = 1\}$	subsection 2.4.4
$\hat{r}_i$	unit vector parallel to $\nabla\beta_i$	subsection 2.4.4
$\beta_{ji}^{\min}$	two definitions, as $\min_{\overline{\mathscr{B}_i(\varepsilon_{i23})}}\{\beta_j\}$ and $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\beta_j\}$	(2.91), (2.155)
$\beta_{ji}^{\max}$	two definitions, as $\max_{\overline{\mathscr{B}_i(\varepsilon_{i23})}}\{\beta_j\}$ and $\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\beta_j\}$	(2.91), (2.155)
Q_{0i}	$\sqrt{\frac{\rho_0^2}{(\beta_{0i}^{\min})^2} - \frac{1}{(\beta_{0i}^{\max})}}$ depends on definition of $\beta_{0i}^{\min}$ and $\beta_{0i}^{\max}$	(2.92)
Q_{ji}	$Q_j(\beta_{ji}^{\min})$ depends on definition of $\beta_{0i}^{\min}$	(2.93)
$\nu(q)$	typographic error in [23] intended to be $\nu_i(q)$	(2.96)
$\nu_i(q)$	relative curvature function	(2.104)
$\hat{t}_i$	unit vector normal to $\nabla\beta_i$	(2.100)

Table B.3: Notation and definitions.

Symbol	Meaning	First definition
G_i	multiply defined expression	(2.108)
ζ_1		(2.120)
$\mathrm{nom}(\cdot)$	nominator	(2.123)
$\mathrm{den}(\cdot)$	denominator	(2.123)
ζ_2		(2.125)
$\varepsilon_{i,KR}$		
$\varepsilon'_{i,KR}$		
λ'_{i0}	scaling factor	(2.133)
A_i		
B_i		
a	auxiliary vector	
b	auxiliary vector	
$\gamma_{di}^{\min}$		(2.156)
$\gamma_{di}^{\max}$		(2.156)
ρ'_i		
$\hat{v}_d(q)$		(2.165)
θ	angle of polar coordinate system	(2.166)
$\theta_{\min}$		
$\theta_{\max}$		
r_i		subsubsection 2.4.5.10
r	radius of polar coordinate system	subsubsection 2.4.5.10
D	semi-annulus	subsubsection 2.4.5.11
$f(r,\theta)$	new nominator function in polar coordinates	(2.179)
$q_1 \ldots q_8$	auxiliary points	Fig. 2.12
x	abscissa of cartesian coordinate system	Fig. 2.12
y	ordinate of cartesian coordinate system	Fig. 2.12
$f(x,y)$	function $f(r,\theta)$ as a function of cartesian coordinates	(2.181)
$g(x,y)$	constraint function (multiply defined)	(2.186), (2.190)
$\Lambda(x,y,\lambda)$	Lagrangian for constrained min (multiply defined)	(2.187), (2.190)
λ	Lagrange multiplier	(2.187), (2.190)
x_n	normalized abscissa of cartesian coordinate system $\frac{x}{r_i}$	Fig. 2.13
y_n	normalized ordinate of cartesian coordinate system $\frac{y}{r_i}$	Fig. 2.13
M_z	number of known internal obstacles (exploration)	
I_{1z}	set of internal obstacle indices (exploration)	
I_{0z}	set of all obstacle indices (exploration)	
$\mathring{\mathscr{F}}$	free space interior $\mathscr{F} \setminus \partial\mathscr{F}$	
$\mathscr{P}_z$	positive invariant set for agent controlled by φ_z	
$\varphi_z(q)$	navigation function potential (exploration)	
$\mathscr{F}_n$	set "near" obstacles (both internal and zero$^{\text{th}}$	(4.17)
$\mathscr{F}_a$	set "away" obstacles (both internal and zero$^{\text{th}}$	(4.18)
ε_{i4}	constraint on ε_i ensuring $\|\nabla\beta_i\| > 0$ in $\overline{\mathscr{B}(\varepsilon_{i4})}$	(4.21)
ε_{i5}	constraint on ε_i	
$T_q\mathscr{F}$	Tangent space of $\mathscr{F}$ at q	(4.23)
$UT_q\mathscr{F}$	Unit tangent space of $\mathscr{F}$ at q	(4.23)
$\mathscr{R}_i(q)$	Radial space spanned by $(\nabla\beta_i)(q)$ at q	(4.24)
$U\mathscr{R}_i(q)$	Unit radial singleton $\{\hat{r}_i\}$ at q	(4.27)
$\mathscr{T}_i(q)$	Orthogonal complement of $\mathscr{R}_i(q)$ in $T_q\mathscr{F}$	(4.25)
$U\mathscr{T}_i(q)$	Unit tangent space of $\beta_i^{-1}(\beta_i(q))$	(4.28)

Table B.4: Notation and definitions.

Symbol	Meaning	First definition
$B_i(q)$	connected component of $\beta_i^{-1}(\beta_i(q))$ to which q belongs	(4.63)
TB_i	obstacle level set tangent bundle $\bigsqcup_{q\in\mathscr{F}} T_q B_i$	(4.64)
UTB_i	obstacle level set unit tangent bundle $\bigsqcup_{q\in\mathscr{F}} \{\hat{v} \in U\mathscr{T}_i(q)\}$	(4.65)
$\nu_i(q, \hat{t}_i)$	relative curvature function	(4.66)
$\nu_{i1}(q)$	$\nu_i(q, \hat{t}_i)$ component function for paraboloid γ_d	(4.67)
$\nu_{i2}(q, \hat{t}_i)$	$\nu_i(q, \hat{t}_i)$ component function for paraboloid γ_d	(4.67)
$\nu_{i3}(q)$	$\nu_i(q, \hat{t}_i)$ component function for paraboloid γ_d	(4.72)
$\nu_{i4}(q, \hat{t}_i)$	$\nu_i(q, \hat{t}_i)$ component function for paraboloid γ_d	(4.72)
$\theta_i(q)$	Gradient angle $\left((\nabla\gamma_d)\widehat{(q), (\nabla}\beta_i)(q)\right)$	(4.76)
$\mathscr{H}_{i1}(q)$	"Good" half-space	(4.81)
$\mathscr{H}_{i2}(q)$	"Bad" half-space	(4.82)
$\mathscr{A}_{i1}(\varepsilon_i)$	Single obstacle neighborhood subset	(4.85)
$\mathscr{A}_{i2}(\varepsilon_i)$	Single obstacle neighborhood subset	(4.85)
$\mathscr{A}_1(\varepsilon_{I_0})$	Subset of $\mathscr{F}_n$	(4.87)
$\mathscr{A}_2(\varepsilon_{I_0})$	Subset of $\mathscr{F}_n$	(4.87)
$\kappa_{i,q}(\hat{t}_i)$	Normal curvature at q along $\hat{t}_i$ of (hyper)surface $B_i(q)$	(4.109)
$\mathrm{L}_q(t_i)$	Weingarten map	(4.111)
$n_{B_i}(q)$	vector normal to (hyper)surface $B_i(q)$ at q	(4.111)
$\gamma(t)$	path on (hyper)surface $B_i(q)$	(4.111)
X, Y	tangent vectors in tangent space $T_q B_i$	(4.113)
$R_{i,q}(\hat{t}_i)$	radius of normal curvature at q along $\hat{t}_i$ of (hyper)surface B_i	(4.115)
$\hat{p}_{ij}(q)$	principal direction at q of (hyper)surface $B_i(q)$	(4.118)
$\kappa_{ij}(q)$	principal curvature at q of (hyper)surface $B_i(q)$	(4.119)
$R_{ij}(q)$	radius of principal curvature at q	(4.120)
$\Gamma(q)$	level set of $\gamma_d(q)$	(4.140)
l	Weingarten map matrix representation in tangent space	(4.128)
$\mathscr{S}(q_a, \rho)$	Sphere with center q_a and radius ρ	(4.142)
$\mathscr{S}_{ci}(q, \hat{t}_i)$	Curvature sphere at point q	(4.143)
q_{ci}	Curvature sphere center	(4.144)
ρ_{ci}	Curvature sphere radius	(4.144)
c_1	β_i level set value	subsection 4.4.2
a_1	β_i level set value	subsection 4.4.2
a_2	β_i level set value	subsection 4.4.2
c_2	β_i level set value	subsection 4.4.2
$\overline{B(q, r(q))}$	closed ball around q of radius $r(q)$	Proposition 28
$r(q)$	radius for properties to hold in $\overline{B(q, r(q))}$	Proposition 28
q'	point in ball	subsection 4.4.2
$I^-(q)$	index set of principal curvatures with $\nu_i < 0$ at q	subsection 4.4.2
$I^+(q)$	index set of principal curvatures with $\nu_i > 0$ at q	subsection 4.4.2
$\Delta\nu_3$	Difference of ν_3	subsection 4.4.2
$U_1(\Delta\nu_3)$	open neighborhood function of $\Delta\nu_3$	subsection 4.4.2
$\Delta\kappa$	Difference of κ	subsection 4.4.2
$U_2(\Delta\kappa)$	open neighborhood function of κ	subsection 4.4.2
$r_{\min}$	minimal ball radius in compact neighborhood	Proposition 29
z, g	auxiliary parameters	subsection 4.4.2

Table B.5: Notation and definitions.

Symbol	Meaning	First definition
H	real symmetric matrix	Proposition 32
λ_i	eigenvalues of matrix H	Proposition 32
S	unit sphere in $\mathbb{R}^n$	(4.171)
U	linear span of eigenvectors	(4.172)
δ_j	matrix H eigenvectors	Proposition 32
a_j	matrix H eigenvector weights	Proposition 32
P_i	dummy subset of principal directions	section 4.5
I_i	index set of dummy subset of principal directions	section 4.5
r	index	section 4.5
$\mathscr{P}_i$	span of dummy principal directions' subset P_i	section 4.5
W	linear span of selected eigenvector subset	Proposition 36
$I_i^-(q)$	index set of principal directions with $\nu_i < 0$	section 4.6
$P_i^-(q)$	subset of principal directions with $\nu_i < 0$	section 4.6
$\mathscr{P}_i^-(q)$	span of $P_i^-(q)$	section 4.6
$I_i^+(q)$	index set of principal directions with $\nu_i > 0$	section 4.6
$P_i^+(q)$	subset of principal directions with $\nu_i > 0$	section 4.6
$\mathscr{P}_i^+(q)$	span of $P_i^+(q)$	section 4.6
$I_i^\pm(q)$	index set of principal directions with $\nu_i \neq 0$	section 4.6
$P_i^\pm(q)$	subset of principal directions with $\nu_i \neq 0$	section 4.6
$\mathscr{P}_i^\pm(q)$	span of $P_i^\pm(q)$	section 4.6
G_i	auxiliary function	Proposition 39
A	ellipsoid definition matrix	(5.1)
a_{ij}	ellipsoid radii	(5.1)
a	ellipse major radius	(5.5)
b	ellipse minor radius	(5.5)
e	ellipse eccentricity	(5.6)
u_i	vector in span of a selected radial and tangent unit vector pair	(6.1)
μ	u_i coordinate wrt $\hat{r}_i$	(6.1)
λ	u_i coordinate wrt $\hat{t}_i$	(6.1)
a, b, c	auxiliary variables	section 6.1
g	homogeneous function	Definition 51
K	cone	(51)
x	vector in cone K	(51)
t	scaling factor for x	(51)
p	exponent factor for t	(51)
F_p	First nonzero Taylor form	Proposition 52
$\frac{1}{k!}D_x^k f(a)$	kth Taylor form	(6.23)
D_j^i	Partial derivative operator	(6.23)
$\mathscr{J}$	compact Riemann manifold	Proposition 55
$\hat{\varphi}_M$	Morse part of $\hat{\varphi}$	Proposition 56
$\hat{\varphi}_{NM}$	Non-Morse part of $\hat{\varphi}$	Proposition 56

Table B.6: Notation and definitions.

Symbol	Meaning	First definition
N_e	number of experimental trajectories	(9.1)
I_e	set of indices of experimental trajectories	(9.1)
X_i	single experimental trajectory $\{x_i(t_j)\}_{j\in I_i}$	(9.1)
N_i	number of i^{th} trajectory samples	(9.2)
I_i	set of indices of samples in i^{th} trajectory	(9.2)
t_j	j^{th} time sample	(9.2)
$x_i(t_j)$	configuration of i^{th} trajectory at j^{th} time sample	(9.2)
$u_i(t_j)$	trajectory velocity sample at time t_j	(9.3)
U_i	set of velocity samples for i^{th} trajectory	subsection 9.2.1
q_{di}	destination of i^{th} trajectory	subsection 9.2.1
E	experimental data	subsection 9.2.4
A, B	PDE vector coefficients	subsection 9.3.3
T	sampling period	(9.32)
q_r	r^{th} component of system state q	(9.39)
C	B-spline coefficient tensor	(9.40)
c	vector of B-spline coefficients	(9.4.1)
t, t_{ij}	B-spline knot sequences	(9.4.1)
h_i	order of B-spline in each dimension	(9.4.1)
D	domain of definition	(9.42)
J	optimization cost functional	(9.42)
J_{PDE}	PDE error functional	(9.42)
J_{sp}	sample point obstacle function positivity functional	(9.42)
J_{dp}	destination point obstacle function positivity functional	(9.42)
J_{bn}	domain boundary obstacle function non-positivity functional	(9.42)
$s(x)$	C^2-smooth switch	(9.42)
Δ_{ij}	satisfaction error of PDE system at j^{th} sample of i^{th} trajectory	(9.42)
w_i	relative weight coefficients of component cost functionals	(9.42)

Table B.7: Notation and definitions.

Symbol	Meaning	First definition
ϕ, ϕ'	LTL formulas	subsection 12.2.1
p	atmomic proposition	subsection 12.2.1
P	set of atomic propositions	subsection 12.2.1
Φ_P	set of well formed formulas over set P	subsection 12.2.1
X	LTL operator "next"	subsection 12.2.1
U	LTL operator "until"	subsection 12.2.1
$\Box$	LTL operator "always"	subsection 12.2.1
$\Diamond$	LTL operator "eventually"	subsection 12.2.1
$\neg$	logical negation operator	subsection 12.2.1
$\wedge$	logical conjunction operator	subsection 12.2.1
$\vee$	logical disjunction operator	subsection 12.2.1
σ	sequence of atomic proposition subsets	subsection 12.2.1
$\sigma^i(j)$	sequence suffix	subsection 12.2.1
Σ	alphabet of letters $\sigma(i)$	subsection 12.2.2
Σ^ω	set of all infinite words over Σ	subsection 12.2.2
$\mathscr{L}_\omega$	language of infinite words	subsection 12.2.2
S	finite set of states	subsection 12.2.2
δ	nondeterministic transition function	subsection 12.2.2
S_0	set of initial states	subsection 12.2.2
F	set of accepting states	subsection 12.2.2
ρ	labeling function	subsection 12.2.2
w	infinite word in Σ^ω	subsection 12.2.2
γ	deterministic transition function	subsection 12.2.2
L_i	"good" set of states in Rabin automaton	subsection 12.2.2
U_i	"bad" set of states in Rabin automaton	subsection 12.2.2
I_{LU}	index set of "good"/"bad" pairs in Rabin automaton	subsection 12.2.2
n_{LU}	number of "good"/"bad" pairs in Rabin automaton	subsection 12.2.2
a_i	leader agent i	subsection 12.3.1
$\mathscr{A}_i$	set of leader agents (i.e., with specifications)	subsection 12.3.1
N	number of leader agents	subsection 12.3.1
I_a	index set of leader agents a_i	subsection 12.3.1
ϕ_i	local LTL$_{X^-}$specification given to agent a_i	subsection 12.3.1
H_i	agent a_i hybrid state	subsection 12.3.1
x_i	agent a_i continuous state	subsection 12.3.1
X_i	agent a_i continuous state space	subsection 12.3.1
n_i	dimensionality of continuous state space X_i	subsection 12.3.1
q_i	agent a_i discrete state	subsection 12.3.1
Q_i	agent a_i discrete state space	subsection 12.3.1
m_i	dimensionality of discrete state space Q_i	subsection 12.3.1
f_i	follower agent	subsection 12.3.1
$\mathscr{F}$	set of follower agents f_i (i.e., w/o specs)	subsection 12.3.1
n_f	number of follower agents f_i	subsection 12.3.1
I_f	index set of follower agents f_i	subsection 12.3.1
$p_{c_{ij}}$	j^{th} controllable AP of a_i	subsection 12.3.2
P_{c_i}	set of controllable APs of a_i	subsection 12.3.2
n_{c_i}	number of controllable APs of a_i	subsection 12.3.2
I_{c_i}	index set of APs in P_{c_i}	subsection 12.3.2

Table B.8: Notation and definitions.

Symbol	Meaning	First definition
$p_{o_{ij}}$	j^{th} observable AP of a_i	subsection 12.3.2
$P_{o_{ij}}$	set of observable APs of a_i	subsection 12.3.2
n_{o_i}	number of observable APs of a_i	subsection 12.3.2
I_{o_i}	index set of APs in P_{o_i}	subsection 12.3.2
f_c	function mapping agent a_i to its controllable APs P_{c_i}	subsection 12.3.2
P_c	all agents' controllable APs	subsection 12.3.2
P_o	all agents' observable APs	subsection 12.3.2
P_i	all APs of agent a_i	subsection 12.3.2
ρ_i	radius of spherical agent a_i	subsection 12.3.2
$R_{s,i}$	sensing/communication radius of a_i	subsection 12.3.2
$x_{d_{ij}}$	NF destination corresponding to AP $p_{c_{ij}}$	subsection 12.3.3
$I_{NF,i}$	index set of NF controllable APs, $\subseteq I_{c_i}$	subsection 12.3.3
t_0	initial time of system evolution	subsection 12.3.3
$c_{d_{ij}}$	relative position vector (between agents)	subsection 12.3.3
$\mathscr{H}_i$	agent a_i hybrid controller	subsection 12.3.4
$\mathscr{D}_i$	agent a_i discrete controller	subsection 12.3.4
$\mathscr{C}_{ij}$	agent a_i j^{th} NF controller	subsection 12.3.3
$\mathscr{B}_i$	agent a_i Büchi automaton	section 12.4
$\mathscr{R}_i$	agent a_i Deterministic Rabin automaton	section 12.4
$\mathscr{T}_i$	agent a_i trimmed Rabin automaton	section 12.4
$\mathscr{D}_i$	agent a_i deterministic controller automaton	section 12.4
W_j	complement of union $L_j \cup U_j$	section 12.4
φ_i	NF of an agent	(12.3)
u_i	velocity of an agent	(12.2)
γ_i	agent destination function	(12.4)
G_i, g_{ikl}	functions to build multi-agent proximity relations	(12.6)
b_{ikl}, β_{ij}	agent collision functions	(12.7)
λ, h, k	NF tuning parameters	section 12.7
$S(x)$	C^2-smooth switch	(12.8)
d_{ij}	distance between agents	(12.9)
A, B, Γ	normalized squared distance differences	(12.9)
S_1, S_2	switches on normalized distance differences	(12.9)
d_c	communication distance	(12.9)
d_m	piecewise collision function branching point	(12.9)

Appendix C

References

C.1 Bibliography, Journal and Conference Papers

C.1.1 Motion Planning, Robotics

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