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Probabilistic description of responses of nonlinear dynamical systems under colored Gaussian excitation

by

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Synopsis

The determination of the probabilistic structure of the response of nonlinear dynamical systems excited by general stochastic noise, is a question of the utmost importance for numerous applications in structural dynamics, statistical physics, material sciences, environmental systems and elsewhere, thereby constituting the basis of uncertainty quantification. In many cases, the random excitations have to be considered smoothly-correlated (colored) noises. Hence, the complete probabilistic structure of the responses is defined by an infinite hierarchy of probability density functions (pdfs). Abandoning the assumption of white-noise excitation has profound effects on the needed theoretical background, since the Markovian character of responses is lost, and thus, all standard tools (e.g. Itô Calculus and Fokker-Planck-Kolmogorov equation) are not applicable. An alternative, efficient approach is to derive pdf evolution equations corresponding to the system.

This thesis aims at extending a methodology used to develop evolution equations for the response pdf of nonlinear dynamical systems subjected to colored Gaussian excitation in order to account for second-order pdfs corresponding to such systems. Following this approach, we commence with representing the sought-for pdf as the average of a random delta function, i.e. the *delta projection method*. Then, by carrying out simple algebraic manipulations a stochastic alternative of the Liouville equation is obtained. This equation, called Stochastic Liouville Equation (SLE), is non-closed due to terms depending on both the response and the excitation of the examined system, and is further evaluated by employing an appropriate correlation splitting. The said correlation splitting is performed via the appropriate Novikov-Furutsu theorem for which a collection of novel extensions as well as the manner in which they can be formulated and proven is presented in Chapter 2.

In Chapter 3 and in particular section 3.1, the main steps of these methodology are outlined for the response pdf of a nonlinear random differential equation (RDE) under additive, colored Gaussian excitation in order to present in a comprehensive manner the foundation upon which the derivation of second-order pdf evolution equations is established. Moreover, a more intricate case for the response pdf of a RDE subject to both additive and multiplicative excitation is presented in section 3.2.

The first major result of this thesis is produced in Chapter 4, in which evolution equations for second-order pdfs are derived, namely for the one-time, joint response-excitation(s) pdfs. More specifically, in section 4.1, the case of a nonlinear, additively excited RDE is considered while in section 4.2, we examine the case of a both additively and multiplicatively excited RDE. Subsequently, in Chapter 5, for the former case, two-time response pdf evolution equations are formulated. Last, in Chapter 6, once for the case of a nonlinear, additively excited RDE, evolution equations for the two-time pdf of the response and its derivative are derived. In all the aforementioned chapters, we examine the potency of this methodology for a linear, additively excited RDE and see if the derived results correspond to the correct ones, obtained using other approaches.

Keywords: stochastic dynamics, nonlinear random differential equations, colored noise excitation, second-order pdfs, uncertainty quantification, generalized Fokker-Planck-Kolmogorov equations

Σύνοψη (Synopsis in Greek)

Ο προσδιορισμός της πιθανοθεωρητικής δομής της απόκρισης μη γραμμικών δυναμικών συστημάτων που διεγείρονται από γενικό στοχαστικό θόρυβο είναι ένα ερώτημα ύψιστης σημασίας για πληθώρα εφαρμογών σε προβλήματα δομικής μηχανικής, στατιστικής φυσικής, επιστήμης υλικών, περιβαλλοντικά συστήματα και αλλού, καθιστώντας το, έτσι τη βάση της ποσοτικοποίησης της αβεβαιότητας. Σε πολλές περιπτώσεις, οι τυχαίες διεγέρσεις πρέπει να θεωρούνται ομαλώς συσχετισμένοι (χρωματισμένοι) θόρυβοι. Συνεπώς, η πλήρης πιθανοθεωρητική δομή των αποκρίσεων καθορίζεται από μια άπειρη ιεραρχία συναρτήσεων πυκνότητας πιθανότητας (σππ). Η εγκατάλειψη της υπόθεσης για διέγερση λευκού θορύβου έχει σημαντικές επιπτώσεις στο απαιτούμενο θεωρητικό υπόβαθρο, καθώς ο Μαρκοβιανός χαρακτήρας των αποκρίσεων χάνεται και έτσι όλα τα βασικά εργαλεία (π.χ. Itô Άλγεβρα and Fokker-Planck-Kolmogorov εξίσωση) δεν είναι εφαρμόσιμα. Μια εναλλακτική, αποτελεσματική πρακτική είναι να εξαχθούν εξελικτικές εξισώσεις σππ που αντιστοιχούν στο σύστημα.

Η παρούσα διπλωματική εργασία στοχεύει στη διεύρυνση μια μεθοδολογίας που χρησιμοποιείται για την παραγωγή εξελικτικών εξισώσεων για τη σππ της απόκρισης μη γραμμικών δυναμικών συστημάτων που υπόκεινται σε χρωματισμένη Γκαουσιανή διέγερση, ούτως ώστε να συμπεριλαμβάνει δευτέρας τάξης σππ που αντιστοιχούν σε τέτοια συστήματα. Ακολουθώντας αυτή τη προσέγγιση, αρχίζουμε με την αναπαράσταση της αναζητούμενης σππ ως τη μέση τιμή μια τυχαίας συνάρτησης δέλτα, δηλαδή τη μέθοδο δέλτα-προβολών. Έπειτα, εκτελώντας απλούς αλγεβρικούς χειρισμούς ένα στοχαστικό ανάλογο της εξίσωσης Liouville εξάγεται. Αυτή η εξίσωση, που ονομάζεται στοχαστική εξίσωση Liouville (ΣEL) είναι μη-κλειστή λόγω όρων που εξαρτώνται τόσο από την απόκριση όσο και από τη διέγερση του εξεταζόμενου συστήματος, και αναλύονται εφαρμόζοντας το κατάλληλο θεώρημα Novikov Furutsu για το οποίο μια συλλογή από νέες επεκτάσεις, καθώς και ο τρόπος με τον οποίο αυτές σχηματίζονται και αποδεικνύονται, παρουσιάζεται στο Κεφάλαιο 2.

Στο Κεφάλαιο 3, και πιο συγκεκριμένα στην Ενότητα 3.1, τα βασικά βήματα αυτής της μεθοδολογίας περιγράφονται για τη σππ της απόκρισης μιας μη γραμμικής, τυχαίας διαφορικής εξίσωσης (ΤΔΕ) υπό αθροιστική, χρωματισμένη Γκαουσιανή διέγερση για να παρουσιάσουμε με ένα σαφή τρόπο τη βάση πάνω στην οποία η εξαγωγή δευτέρας τάξεως εξελικτικών εξισώσεων σππ θεμελιώνεται. Επιπλέον, μια πιο περίπλοκη περίπτωση για τη σππ της απόκρισης μιας ΤΔΕ που υπόκειται και σε αθροιστική και σε πολλαπλασιαστική διέγερση παρουσιάζεται στην Ενότητα 3.2.

Το πρώτο σημαντικό αποτέλεσμα αυτής της διπλωματικής εργασίας παράγεται στο Κεφάλαιο 4, στο οποίο εξάγονται εξελικτικές εξισώσεις για σππ δευτέρας τάξης, συγκεκριμένα για τις ενός χρόνου, από κοινού σππ απόκρισης-διέγερσης (διεγέρσεων). Ειδικότερα, στην Ενότητα 4.1, εξετάζεται η περίπτωση μιας μη-γραμμικής, προσθετικά διεγερμένης ΤΔΕ, ενώ στην Ενότητα 4.2 εξετάζουμε την περίπτωση μιας μη-γραμμικής, προσθετικά και πολλαπλασιαστικά διεγερμένης ΤΔΕ. Στη συνέχεια, στο Κεφάλαιο 5, για την πρώτη περίπτωση, διατυπώνονται εξελικτικές εξισώσεις για τη σππ της απόκρισης σε δύο χρόνους. Τέλος, στο Κεφάλαιο 6, εκ νέου για τη περίπτωση της μη-γραμμικής, προσθετικά διεγερμένης ΤΔΕ, εξάγονται εξελικτικές εξισώσεις για τη δύο χρόνων σππ της απόκρισης και της παραγώγου της. Σε όλα τα προαναφερθέντα κεφάλαια, εξετάζουμε την αποτελεσματικότητα αυτής της μεθοδολογίας για μια γραμμική, προσθετικά διεγερμένη ΤΔΕ και βλέπουμε εάν τα προκύπτοντα αποτελέσματα αντιστοιχούν στα σωστά, όπως αυτά λαμβάνονται χρησιμοποιώντας άλλες προσεγγίσεις.

Λέξεις-Κλειδιά: στοχαστική δυναμική, μη γραμμικές διαφορικές εξισώσεις, χρωματισμένος θόρυβος, δευτέρας τάξης σπδ, ποσοτικοποίηση της αβεβαιότητας, γενικευμένες εξισώσεις Fokker-Planck-Kolmogorov

Contents

Chapter 1

Introduction

The determination of the probabilistic structure of the response of nonlinear dynamical systems excited by general stochastic noise, is a question of the utmost importance for numerous applications in structural dynamics, statistical physics, material sciences, environmental systems and elsewhere. In macroscopic stochastic dynamics it constitutes the basis of uncertainty quantification. In many cases, e.g. in economy, finance, biology, signal processing etc., where the systems are considered subject to white noise excitation, the problem is very well examined. More specifically, in these cases, the response is Markovian and thus, its probabilistic structure is majorly encapsulated in the transition pdf corresponding to the system; the evolution of the said pdf is governed by the Fokker-Planck-Kolmogorov (FPK) equation (Cáceres, 2017; Risken, 1996; Stratonovich, 1989; Sun, 2006, Chapter 6). However, in problems in which the random excitations are smoothly correlated (colored) noises and thus, the response is inherently non-Markovian, this convenient description, via a partial differential equation, is not pertinent (van Kampen, 1998). The relevance of colored noise excitation in numerous applications as well as the theoretical complicacies it engenders, are thoroughly discussed in a considerable number of works, including (P. Hänggi & Jung, 1995; Francesc Sagués, Sancho, & García-Ojalvo, 2007; Sapsis & Athanassoulis, 2008; van Kampen, 2007; Venturi, Sapsis, Cho, & Karniadakis, 2012a). Regardless of the intrinsic intricacies that are associated with the study of systems under colored noise excitation, numerous methods have been developed and proposed as an efficient means to address such systems.

A brute force approach that can be applied to these problems is the Monte Carlo simulation (MCS). Although, the MSC approach is very versatile, it is a method of high computational cost, especially when high-dimensional systems are considered. Another useful methodology, which does not present high computational cost, is the stochastic linearization technique (Crandall, 2006; Roberts & Spanos, 2003). Despite some particular benefits of this technique – e.g. that is does not require precise knowledge of the excitation – it does not result in very accurate findings. One more notable approach, which also emphasizes some of the complicacies associated with these systems, is the formulation and solution of moment equations corresponding to the systems (Athanassoulis, Tsantili, & Kapelonis, 2015).

The most straightforward approach, especially prominent in engineering applications, is the filtering approach. In this approach, the colored noise is approximated by being determined as an output of a "filter" equation which, in turn, is excited by white noise. This filter results in an augmentation of the original system of RDEs but, also, admits an exact FPK description. This approach – also referred to as Markovianization by extension (Krée, 1985) or embedding in a Markovian process of higher dimensions (P. Hänggi & Jung, 1995, sec. V. C.) – is the starting point of the unified colored noise approximation introduced in (P. Hänggi & Jung, 1987). Moreover, this approach has been used to further enhance and extend a Wiener path integral (WPI) technique, first developed by Kougioumtzoglou & Spanos (I. A. Kougioumtzoglou &

Spanos, 2012; Ioannis A. Kougioumtzoglou & Spanos, 2014), to account for non-white, i.e. colored, excitations (Psaros, Brudastova, Malara, & Kougioumtzoglou, 2018; Psaros, Kougioumtzoglou, & Petromichelakis, 2018). Although, this approach has led to notable and useful techniques that produce great results, its inherent drawback is that it leads to an inflation of the degrees of freedom in the FPK equation and as such, an analogous increase in the computational cost.

Last, an alternative approach is formulating pdf evolution equations analogous to FPK equations while both taking into account the given colored excitation as well as maintaining the natural degrees of freedom of the examined system. A major complicacy that resides within this approach refers to the emergence of terms dependent on the whole time-history of response even in the simplest of cases, e.g. for one-time response pdf evolution equations; this complicacy will be discussed in detail subsequently. The derivation of these equations, also referred to as generalized FPK equations (Cetto, de la Peña, & Velasco, 1984) since their counterpart for white noise excitation is the classical FPK equation, is not a recent venture. Already from the 70's by the works of (Fox, 1977; van Kampen, 1975) and later on by (Fox, 1986; P Hänggi, 1978), this approach has been implemented in numerous cases ranging from energy harvesting (Harne & Wang, 2014) to medical applications (Zeng & Wang, 2010) and more.

More recently, this methodology has been revisited, generalized and presented in a more systematic and comprehensive manner by Mamis, Athanassoulis *et al.* in (Athanassoulis & Mamis, 2019; Mamis & Athanassoulis, 2016; Mamis, Athanassoulis, & Kapelonis, 2019; Mamis, Athanassoulis, & Papadopoulos, 2018). Since this thesis aims to further extend these works and showcase their versatility, it is, now useful to concisely describe some of its fundamental parts.

Recapitulation of Mamis et al., 2019

Following (Mamis et al., 2019), we consider the prototype case of a scalar, nonlinear additively excited RDE:

$$\dot{X}(t;\theta) = h(X(t;\theta)) + \kappa\Xi(t;\theta), \qquad X(t_0;\theta) = X_0(\theta), \qquad (1.1a,b)$$

where θ is the stochastic argument, the overdot denotes differentiation with respect to time, $h(x)$ is a deterministic continuous function modelling the nonlinearities (restoring term), and κ is a constant. Initial value $X_0(\theta)$ and excitation $\Xi(t;\theta)$ are considered correlated and jointly Gaussian with non-zero mean values m_{X_0}, $m_{\Xi}(t)$, autocovariances $C_{X_0X_0}$, $C_{\Xi\Xi}(t,s)$ and cross-covariance $C_{X_0\Xi}(t)$. Then, we represent the response pdf of the random initial value problem (RIVP) Eqs. (1.1a,b) as the average of a random delta function:

$$f_{X(t)}(x) = \mathbb{E}^{\theta}\left[\delta(x - X(t;\theta))\right], \qquad (1.2)$$

where $\mathbb{E}^{\theta}[\bullet]$ is the ensemble average operator. This representation of the pdf is prevalent in statistical mechanics (van Kampen, 2007, Chapter XVI sec. 5), where it is called Van Kampen's lemma, and the theory of turbulence (Lundgren, 1967) where it is known as the pdf method. Herein, the term delta projection method is used reminiscent of the manner that it is derived (see e.g. sec. 3.1). Then by differentiating Eq. (1.2) and performing some simple manipulations, we find

$$\frac{\partial f_{X(t)}(x)}{\partial t} + \frac{\partial}{\partial x}\left(h(x)\, f_{X(t)}(x)\right) = -\kappa\, \frac{\partial}{\partial x}\left(\mathbb{E}^{\theta}\left[\Xi(t;\theta)\,\delta(x - X(t;\theta))\right]\right). \quad (1.3)$$

Eq. (1.3) is called the stochastic Liouville equation and its derivation is shown in detail in sec. 3.1. For reasons that will become clear in the said section, Eq. (1.3) is non-closed and nonlocal due to the averaged term appearing on its right-hand side. This term depends on the whole time-history of the response and the excitation and thus, an appropriate correlation splitting technique must be employed, namely an appropriate extension of the Novikov-Furutsu (NF) theorem. After the application of the NF theorem, a novel approximation scheme, similar to the one presented in (Mamis et al., 2019), is utilized for the nonlocal terms. Thus, we obtain a closed, approximate pdf evolution equation of the form

$$\begin{aligned} &\frac{\partial f_{X(t)}(x)}{\partial t} + \frac{\partial}{\partial x}\left[\left(h(x) + \kappa m_{\Xi}(t)\right) f_{X(t)}(x)\right] = \\ &= \frac{\partial^2}{\partial x^2}\left\{\left[\sum_{m=0}^{M}\frac{1}{m!} D_m^{\text{eff}}\left[R_{h'}(\bullet), t\right] \varphi_{h'}^{m}\left(x\,;R_{h'}(t)\right)\right] f_{X(t)}(x)\right\}. \end{aligned} \quad (1.4)$$

Through the coefficients D_m^{eff}, called the ***generalized effective noise intensities***, as well as the terms $\varphi_{h'}^{m}$ defined in paragraph 3.1.3, the pdf equation retains a trackable amount of nonlocality (in time) and nonlinearity, reflecting the non-Markovian character of the response. In the present case, this approximation has been shown to provide good results even for large correlation times and noise intensities.

Main contributions of this thesis

Based on the approach outlined above, in this thesis, we first showcase its versatility by implementing it in the case of the following RDE which is excited by both additive and multiplicative Gaussian excitations.

$$\dot{X}(t;\theta) = h(X(t;\theta)) + q(X(t;\theta))\Xi_1(t;\theta) + \kappa\,\Xi_0(t;\theta), \quad (1.5)$$

The consideration of an RDE subject to multiplicative colored excitation has been considered before by many authors, see e.g. (Cetto et al., 1984; Fox, 1986; San Miguel & Sancho, 1980b; Sancho, San Miguel, Katz, & Gunton, 1982). Nevertheless, herein we also consider an additional additive excitation and investigate the nuances that arise. To that end, we formulate second-order evolution equations not only for the one-time response pdf corresponding to RDE (1.5) but, also, the one-time joint response-excitations. The derivation of one-time joint response-excitation pdf evolution equation is also carried out for the RIVP (1.1a,b) in chapter 4. The consideration of such joint pdfs does not only possesses practical value (Venturi et al., 2012a) but it can also serve as a better approximation for response pdfs, since, as it is shown in this work, some terms which introduce complicacies in the response pdf evolution equations can easily be addressed when we take into account joint response-excitation(s) pdf evolution equations. Of course, in order to obtain the said equations, additional extensions to the NF theorem as well as suitable approximation schemes are, for the first time, introduced in the subsequent chapters.

Furthermore, we formulate two-time pdf evolution equations corresponding to RIVP (1.1a,b). In particular, we focus on the derivation of an evolution equation for the two-time response pdf. This problem has been considered before in (Hernandez-Machado, Sancho, San Miguel, & Pesquera, 1983; F. Sagués, San Miguel, & Sancho, 1984; Sancho & San Miguel, 1989), albeit for a multiplicatively excited RDE. In these works, the authors follow a similar approach but focus on the stationary properties of the evolution equation that they derive as well as the stationary covariance of the response obtained via the said equation. Herein, we focus more on the derivation of a computable, yet approximate, two-time response pdf evolution equation. Finally, we explore the efficiency of this methodology for the case of the two-time joint pdf of the response and its derivative, a problem with particular significance in first-passage problems (Verechtchaguina, Sokolov, & Schimansky-Geier, 2006) and outline some of the intricacies that arise.

Chapter 2

The Novikov-Furutsu theorem

In the present chapter, we are going to discuss one of the most fundamental tools towards the derivation of the pdf evolution equations, i.e. the Novikov-Furutsu (NF) theorem. The NF theorem is a well-known mathematical tool used for correlation splitting, that is, for evaluating the mean value of the product of a random functional with a Gaussian argument multiplied by the argument itself. Its classical form, was independently proven by (Furutsu, 1963; Novikov, 1965) in their works on electromagnetic waves and turbulence, respectively. Recently, in (Athanassoulis & Mamis, 2019), the authors extended this theorem to account for mappings (function-functionals) of two Gaussian arguments having non-zero mean value and being correlated; a result of particular significance for the study of random differential equations of the form presented throughout this work. Therefore, in this chapter, we are going to concisely describe the extended NF theorem as established by the aforementioned authors as well as formulate and prove some other extensions needed in the subsequent chapters 4-7.

2.1 The mean value of random, nonlinear (function-) functionals

Following (Athanassoulis & Mamis, 2019), we shall first discuss the deterministic counterparts of the random functionals and FFℓs that are considered subsequently as well as their analytical properties. A more thorough discussion regarding averages of random FFℓs can also be found in (Mamis, 2020).

Consider a real-valued function-functional $\mathcal{G}[\upsilon\,;u(\bullet|_{t_0}^{t})]:\ \mathbb{R}\times\mathscr{Z}\to\mathbb{R}$, where $\mathscr{Z}=C\big([t_0,t]\to\mathbb{R}\big)$ is the space of continuous functions. The said FFℓ is assumed to have derivatives of any order, with respect to both the scalar argument υ and the function argument $u(\bullet)$, and it is expandable in Volterra-Taylor series, jointly with respect to υ and $u(\bullet)$, around a fixed pair $\big(\upsilon_0\,;u_0(\bullet)\big)$. Henceforth, the aforementioned smoothness properties of $\mathcal{G}[\upsilon\,;u(\bullet|_{t_0}^{t})]$ will be concisely denoted as C^∞. Accordingly, consider a real-valued functional of two arguments $\mathcal{J}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]:\mathscr{Z}_0\times\mathscr{Z}_1\to\mathbb{R}$ possessing analogous properties to the previous one, i.e. $\mathcal{J}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]:\in C^\infty\big(\mathscr{Z}_0\times\mathscr{Z}_1\to\mathbb{R}\big)$.

The appropriate random FFℓ, $\mathcal{G}$, is obtained by replacing the argument $\big(\upsilon\,;u(\bullet)\big)$ by the random element $\big(X_0(\theta);\Xi(\bullet;\theta)\big)$ which, as discussed in section 1.1, is fully described by the infinite-dimensional joint probability measure $\mathbf{P}_{X_0\Xi}$. Accordingly, the functional of two random arguments, $\mathcal{J}$, is acquired by substituting into $\left(\upsilon(\bullet);u(\bullet)\right)$ the random element $\left(\Xi_0(\bullet;\theta);\Xi_1(\bullet;\theta)\right)$, which, in turn, is fully described by joint probability measure $\mathbf{P}_{\Xi_0\Xi_1}$. Since, $\left(X_0(\theta);\Xi(\bullet;\theta)\right):\theta\to\mathbb{R}\times\mathscr{Z}$ and $\left(\Xi_0(\bullet;\theta);\Xi_1(\bullet;\theta)\right):\theta\to\mathscr{Z}_0\times\mathscr{Z}_1$ are Borel measurable and the deterministic quantities $\mathcal{G}$, $\mathcal{J}$ are C^∞, then their stochastic counterparts $\mathcal{G}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]$ and $\mathcal{J}[\Xi_0(\bullet|_{t_0}^{t});\Xi_1(\bullet|_{t_0}^{t})]$ will also be Borel measurable.

As such, by definition, the mean value of $\mathcal{G}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]$ is expressed as

$$\mathbb{E}^{\theta}_{\mathbf{P}_{X_0\Xi}}\left[\mathcal{G}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\right]=\int\limits_{\mathbb{R}\times\mathscr{Z}}\mathcal{G}[\chi;\xi(\bullet)]\,\mathbf{P}_{X_0\Xi}\left(d\chi\times d\xi(\bullet)\right),\tag{2.1}$$

while, the mean value of $\mathcal{J}[\Xi_0(\bullet|_{t_0}^{t});\Xi_1(\bullet|_{t_0}^{t})]$ is expressed as

$$\mathbb{E}^{\theta}_{\mathbf{P}_{\Xi_0\Xi_1}}\left[\mathcal{J}[\Xi_0(\bullet|_{t_0}^{t});\Xi_1(\bullet|_{t_0}^{t})]\right]=\int\limits_{\mathscr{Z}_0\times\mathscr{Z}_1}\mathcal{J}[\xi_0(\bullet);\xi_1(\bullet)]\mathbf{P}_{\Xi_0\Xi_1}\left(d\,\xi_0(\bullet)\times d\,\xi_1(\bullet)\right).\tag{2.2}$$

The above definitions involve a path integral over an infinite-dimensional space; a fact which constitutes their calculations inherently difficult in the majority of cases. Thus, as a more workable alternative, we shall express, in the following Theorems 1 and 2, the mean value of a random FFℓ and a random functional of two arguments via the probabilistic structure of their arguments. This will be accomplished by making use of the following expressions:

$$\varphi_{X_0\Xi}[\upsilon;u(\bullet|_{t_0}^{t})]=\mathbb{E}^{\theta}_{\mathbf{P}_{X_0\Xi}}\left[\exp\left(i\,X_0(\theta)\,\upsilon+i\int_{t_0}^{t}\Xi(s;\theta)\,u(s)\,ds\right)\right],\tag{2.3}$$

$$\varphi_{\Xi_0\Xi_1}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]=\mathbb{E}^{\theta}_{\mathbf{P}_{\Xi_0\Xi_1}}\left[\exp\left(i\int_{t_0}^{t}\Xi_0(s;\theta)\upsilon(s)\,ds+i\int_{t_0}^{t}\Xi_1(s;\theta)u(s)\,ds\right)\right].\tag{2.4}$$

Eq. (2.3) is the joint characteristic function-functional of $X_0(\theta)$, $\Xi(\bullet;\theta)$ while Eq. (2.4) is the joint characteristic functional of $\Xi_0(\bullet;\theta)$, $\Xi_1(\bullet;\theta)$. Note that in both the above expressions, the random elements may be dependent, having ***any prescribed probability distribution***.

Theorem 1 [Mean value of a random FFℓ]: Let $\mathcal{G}[\upsilon;u(\bullet|_{t_0}^{t})]$ be a sufficiently smooth FFℓ, and consider the random FFℓ, generated from $\mathcal{G}[\upsilon;u(\bullet|_{t_0}^{t})]$ by replacing υ by a scalar random variable $X_0(\theta)$ and $u(\bullet)$ by a scalar random function $\Xi(\bullet;\theta)$. The mean value of the random function-functional $\mathcal{G}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]$ is expressed by the formula

$$\mathbb{E}^{\theta}\Big[\mathcal{G}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\Big] = \varphi_{\hat{X}_0\hat{\Xi}}\left(\frac{\partial}{i\,\partial\upsilon};\frac{\delta}{i\,\delta u(\bullet)}\right)\mathcal{G}[\upsilon;u(\bullet|_{t_0}^{t})]\Big|_{\substack{\upsilon=m_{X_0}\\ u(\bullet)=m_{\Xi}(\bullet)}} =$$
$$= \mathbb{E}^{\theta}\left[\exp\left(\hat{X}_0(\theta)\frac{\partial}{\partial\upsilon}+\int_{t_0}^{t}ds\,\hat{\Xi}(s;\theta)\frac{\delta}{\delta u(s)}\right)\right]\mathcal{G}[\upsilon;u(\bullet|_{t_0}^{t})]\Big|_{\substack{\upsilon=m_{X_0}\\ u(\bullet)=m_{\Xi}(\bullet)}} \tag{2.5}$$

where $\mathbb{E}^{\theta}[\,\bullet\,]\equiv\mathbb{E}^{\theta}_{\mathbf{P}_{X_0\Xi}}[\,\bullet\,]$, m_{X_0} and $m_{\Xi}(s)$ are the mean values of $X_0(\theta)$ and $\Xi(\bullet;\theta)$, respectively; $\partial/\partial\upsilon$ denotes partial differentiation with respect to υ, and $\delta/\delta u(s)$ denotes Volterra functional differentiation with respect to the function $u(\bullet)$ at s. Further, the quantities

$$\hat{X}_0(\theta) = X_0(\theta)-m_{X_0}, \qquad \hat{\Xi}(s;\theta) = \Xi(s;\theta)-m_{\Xi(\bullet)}(s) \tag{2.6a,b}$$

are the fluctuations of the random elements $X_0(\theta)$ and $\Xi(s;\theta)$ around their mean values, and $\varphi_{\hat{X}_0\hat{\Xi}(\bullet)}[\upsilon;u(\bullet)]$ is the joint characteristic FFℓ of the said fluctuations. The operator appearing in the right-hand side of Eq. (2.5) is called the function-functional shift operator and it is established in (Athanassoulis & Mamis, 2019, sec. 3). ■

Theorem 2 [Mean value of a random functional of two arguments]: Let $\mathcal{J}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ be a sufficiently smooth functional, and consider the random functional, generated from $\mathcal{J}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ by replacing $\upsilon(\bullet)$ by a scalar random function $\Xi_0(\bullet;\theta)$ and $u(\bullet)$ by a scalar random function $\Xi_1(\bullet;\theta)$. The mean value of the random functional $\mathcal{J}[\Xi_0(\bullet|_{t_0}^{t});\Xi_1(\bullet|_{t_0}^{t})]$ is expressed in the form

$$\mathbb{E}^{\theta}\Big[\mathcal{J}[\Xi_0(\bullet|_{t_0}^{t});\Xi_1(\bullet|_{t_0}^{t})]\Big] =$$
$$= \varphi_{\hat{\Xi}_0(\bullet)\hat{\Xi}_1(\bullet)}\left[\frac{\delta}{i\delta\upsilon(s)};\frac{\delta}{i\delta u(s)}\right]\mathcal{J}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\Big|_{\substack{\upsilon(\bullet)=m_{\Xi_0}(\bullet)\\ u(\bullet)=m_{\Xi_1}(\bullet)}} = \tag{2.7}$$
$$= \mathbb{E}^{\theta}\left[\exp\left(\int_{t_0}^{t}ds\,\hat{\Xi}_0(s;\theta)\frac{\delta}{\delta\upsilon(s)}+\int_{t_0}^{t}ds\,\hat{\Xi}_1(s;\theta)\frac{\delta}{\delta u(s)}\right)\right]\mathcal{J}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\Big|_{\substack{\upsilon(\bullet)=m_{\Xi_0}(\bullet)\\ u(\bullet)=m_{\Xi_1}(\bullet)}}$$

where $\mathbb{E}^{\theta}[\,\bullet\,]\equiv\mathbb{E}^{\theta}_{\mathbf{P}_{\Xi_0\Xi_1}}[\,\bullet\,]$, $m_{\Xi_0}(\bullet)$ and $m_{\Xi_1}(\bullet)$ are the mean values and $\hat{\Xi}_0(\bullet;\theta)$, $\hat{\Xi}_1(\bullet;\theta)$ accordingly; $\delta/\delta\upsilon(s_0)$ and $\delta/\delta u(s_1)$ denotes Volterra functional differentiation with respect to the functions $\upsilon(\bullet)$ at s_0 and $u(\bullet)$ at s_1, respectively. Further, the quantities

$$\hat{\Xi}_i(s_i;\theta) = \Xi_i(s_i;\theta)-m_{\Xi_i}(\bullet), \qquad \text{for} \qquad i=0,1 \tag{2.8}$$

are the fluctuations of the random elements $\Xi_0(s_0;\theta)$ and $\Xi_1(s_1;\theta)$ around their mean values and $\varphi_{\hat{\Xi}_0\hat{\Xi}_1}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ is the joint characteristic functional of said fluctuations. The exact meaning of the operator appearing in the right-hand side of Eq. (2.7) will be defined below. ■

Theorem 1 has first been formulated and proven by (Athanassoulis & Mamis, 2019) in a comprehensive manner using the Volterra technique of passing form the discrete to continuous and as such, its proof is omitted. However, in the ensuing paragraph 2.1.1, we are going to apply this approach in order to prove Theorem 2. Before we are able to proceed with the said proof, we must introduce the following, alternative shift operator presented in the aforementioned paper, namely the shift operator of a functional of two random arguments and its exponential form:

$$J[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})] = J[\upsilon_0(\bullet|_{t_0}^{t}) + \hat{\upsilon}(\bullet|_{t_0}^{t});u_0(\bullet|_{t_0}^{t}) + \hat{u}(\bullet|_{t_0}^{t})] =$$
$$= T_{\hat{\upsilon}_0(\bullet);\hat{u}_0(\bullet)}\,\mathcal{G}[\upsilon_0(\bullet|_{t_0}^{t});u_0(\bullet|_{t_0}^{t})] =$$
$$= \exp\left(\int_{t_0}^{t}\hat{\upsilon}(s)\frac{\delta}{\delta\upsilon(s)}ds + \int_{t_0}^{t}\hat{u}(s)\frac{\delta}{\delta u(s)}ds\right)J[\upsilon_0(\bullet|_{t_0}^{t});u_0(\bullet|_{t_0}^{t})]. \tag{2.9}$$

In the above expression, $\left(\upsilon_0(\bullet|_{t_0}^{t});u_0(\bullet|_{t_0}^{t})\right)$ is the pair around which the Volterra-Taylor expansion of J is employed. The proof of Eq. (2.9) is omitted herein since it is almost identical to the one presented in detail in the aforementioned paper.

2.1.1 Proof of Theorem 2

Equation (2.9) is the essential deterministic prerequisite for the proof of Theorem 1. Substituting, in Eq. (2.9), the arguments $\upsilon(\bullet)$, $u(\bullet)$ by the random arguments $\Xi_0(\bullet;\theta)$, $\Xi_1(\bullet;\theta)$, we obtain the following representation of $\mathcal{G}[\Xi_0(\bullet|_{t_0}^{t});\Xi_1(\bullet|_{t_0}^{t})]$:

$$\mathcal{G}[\Xi_0(\bullet|_{t_0}^{t});\Xi_1(\bullet|_{t_0}^{t})] \equiv G[m_{\Xi_0(\bullet)}(\bullet|_{t_0}^{t}) + \hat{\Xi}_0(\bullet|_{t_0}^{t});m_{\Xi_1(\bullet)}(\bullet|_{t_0}^{t}) + \hat{\Xi}_1(\bullet|_{t_0}^{t})]$$
$$\equiv T_{\hat{\Xi}_0(\bullet)\hat{\Xi}_1(\bullet)}\,\mathcal{G}[m_{\Xi_0(\bullet)}(\bullet|_{t_0}^{t});m_{\Xi_1(\bullet)}(\bullet|_{t_0}^{t})] = \tag{2.10}$$
$$= \exp\left(\int_{t_0}^{t}ds\,\hat{\Xi}_0(s;\theta)\frac{\delta}{\delta\upsilon(s)} + \int_{t_0}^{t}ds\,\hat{\Xi}_1(s;\theta)\frac{\delta}{\delta u(s)}\right)\mathcal{G}[m_{\Xi_0(\bullet)}(\bullet|_{t_0}^{t});m_{\Xi_1(\bullet)}(\bullet|_{t_0}^{t})].$$

Recall that $m_{\Xi_0}(\bullet)$, $m_{\Xi_1}(\bullet)$ are the mean values and $\hat{\Xi}_0(\bullet;\theta)$, $\hat{\Xi}_1(\bullet;\theta)$ are the fluctuations of the random elements $\Xi_0(\bullet;\theta)$, $\Xi_1(\bullet;\theta)$ around their mean values, see Eq. (2.8).

By averaging, now, both sides of Eq. (2.10), we obtain

$$\mathbb{E}^{\theta}\left[\mathcal{G}[\Xi_0(\bullet|_{t_0}^{t});\Xi_1(\bullet|_{t_0}^{t})]\right] =$$
$$= \mathbb{E}^{\theta}\left[\exp\left(\int_{t_0}^{t}ds\,\hat{\Xi}_0(s;\theta)\frac{\delta}{\delta\upsilon(s)} + \int_{t_0}^{t}ds\,\hat{\Xi}_1(s;\theta)\frac{\delta}{\delta u(s)}\right)\right]\mathcal{G}[m_{\Xi_0(\bullet)}(\bullet|_{t_0}^{t});m_{\Xi_1(\bullet)}(\bullet|_{t_0}^{t})]. \tag{2.11}$$

In Eq. (2.11) the averaged term is called the averaged shift operator. Recalling the form of the joint characteristic functional, Eq. (2.4), we see that Eq. (2.11) can also be written as

$$\mathbb{E}^{\theta}\left[\mathcal{G}[\Xi_0(\bullet|_{t_0}^{t});\Xi_1(\bullet|_{t_0}^{t})]\right] = \\ = \varphi_{\hat{\Xi}_0(\bullet)\hat{\Xi}_1(\bullet)}\left[\frac{\delta}{i\delta\upsilon(s)};\frac{\delta}{i\delta u(s)}\right]\mathcal{G}[m_{\Xi_0(\bullet)}(\bullet|_{t_0}^{t});m_{\Xi_1(\bullet)}(\bullet|_{t_0}^{t})], \tag{2.12}$$

which, when combined with Eq. (2.9), is the exact form in which Theorem 2 is written. This concludes the proof of Theorem 2. ■

2.2 Extensions of the Novikov-Furutsu theorem

At this point, it is useful to reiterate that all results presented thus far hold true regardless of the distribution of the random arguments of the FFℓ or the functional of two arguments. Nevertheless, by considering different forms of the aforementioned $\mathcal{G}$, J (function-) functionals and specifying their corresponding joint characteristic FFℓ and functional as Gaussian ones, we will be able to derive various formulas that extend the classical Novikov-Furutsu theorem and are instrumental in the derivation of the pdf evolution equations presented in this thesis.

More specifically, the Gaussian form of the joint characteristic FFℓ $\varphi_{X_0\Xi}[\upsilon;u(\bullet|_{t_0}^{t})]$ reads as follows:

$$\varphi^{\text{Gauss}}_{X_0\Xi}[\upsilon;u(\bullet|_{t_0}^{t})] = \\ = \exp\left(i\int_{t_0}^{t} m_{\Xi}(s)u(s)\,ds - \frac{1}{2}\int_{t_0}^{t}\int_{t_0}^{t} C_{\Xi\Xi}(s_1,s_2)\,u(s_1)\,u(s_2)\,ds_1 ds_2\right)\times \\ \times\exp\left(i\,m_{X_0}\,\upsilon - \frac{1}{2}\,C_{X_0X_0}\,\upsilon^2\right)\cdot\exp\left(-\,\upsilon\int_{t_0}^{t} C_{X_0\Xi}(s)\,u(s)\,ds\right). \tag{2.13}$$

Accordingly, the Gaussian joint characteristic functional $\varphi_{\Xi_0\Xi_1}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ reads

$$\varphi^{Gauss}_{\Xi_0(\bullet)\Xi_1(\bullet)}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})] = \\ = \exp\left(i\int_{t_0}^{t} m_{\Xi_0}(s)\upsilon(s)\,ds - \frac{1}{2}\int_{t_0}^{t}\int_{t_0}^{t} C_{\Xi_0\Xi_0}(s_1,s_2)\upsilon(s_1)\upsilon(s_2)\,ds_1\,ds_2\right)\times \\ \times\exp\left(i\int_{t_0}^{t} m_{\Xi_1}(s)u(s)\,ds - \frac{1}{2}\int_{t_0}^{t}\int_{t_0}^{t} C_{\Xi_1\Xi_1}(s_1,s_2)u(s_1)u(s_2)\,ds_1\,ds_2\right)\times \\ \times\exp\left(-\int_{t_0}^{t}\int_{t_0}^{t} C_{\Xi_0\Xi_1}(s_0,s_1)\upsilon(s_0)u(s_1)\,ds_0\,ds_1\right). \tag{2.14}$$

Eqs. (2.13), (2.14) are essential for the proof of the sought-for extensions. The derivation of these expressions is detailed in the ensuing paragraph.

2.2.1 Derivation of the Gaussian joint characteristic FFℓ and functional

In this paragraph, we are going to focus on the derivation of Eq. (2.14) for the joint characteristic functional of two Gaussian argument; the procurement of Eq. (2.13) is similar and slightly simpler. For this, we must begin from the discrete analogue. We first consider the joint characteristic function of a random vector $\boldsymbol{\Xi}(\theta)$. This is straightforwardly obtained by simple manipulations of a $(2\mathrm{N})-$ dimensional characteristic function (see e.g. (Lukacs & Laha, 1964)) resulting into

$$
\begin{aligned}
\varphi_{\Xi}^{Gauss}(\mathbf{u}) &= \exp\left(i\sum_{n=1}^{2N}\left(\mathbf{m}_{\Xi}\right)_n u_n - \frac{1}{2}\sum_{n=1}^{2N}\sum_{m=1}^{2N}\left(\mathbf{C}_{\Xi\Xi}\right)_{nm} u_n u_m \right) = \\
&= \exp\left(i\sum_{n=1}^{N}\left(\mathbf{m}_{\Xi}\right)_n u_n + i\sum_{n=N+1}^{2N}\left(\mathbf{m}_{\Xi}\right)_n u_n \right)\times \\
&\quad\times \exp\left(-\frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N}\left(\mathbf{C}_{\Xi\Xi}\right)_{nm} u_n u_m - \frac{1}{2}\sum_{n=1}^{N}\sum_{m=N+1}^{2N}\left(\mathbf{C}_{\Xi\Xi}\right)_{nm} u_n u_m \right)\times \\
&\quad\times \exp\left(-\frac{1}{2}\sum_{n=N+1}^{2N}\sum_{m=1}^{N}\left(\mathbf{C}_{\Xi\Xi}\right)_{nm} u_n u_m - \frac{1}{2}\sum_{n=N+1}^{2N}\sum_{m=N+1}^{2N}\left(\mathbf{C}_{\Xi\Xi}\right)_{nm} u_n u_m \right),
\end{aligned}
\tag{2.15}
$$

where $\mathbf{m}_{\Xi}$, $\mathbf{C}_{\Xi\Xi}$ are the mean value and autocovariance of the random vector $\boldsymbol{\Xi}(\theta)$, respectively. In order to transform the above equation into a more suitable form for our case we a) denote the first N elements of the vector by $\boldsymbol{\Xi}_0$ and the respective arguments of the characteristic function by υ and, in similar fashion, the remaining terms by $\boldsymbol{\Xi}_1$ and u, accordingly; b) take advantage of the commutation of the sums. Thus, we find

$$
\begin{aligned}
\varphi_{\Xi_0\Xi_1}^{Gauss}(\upsilon;u) &= \exp\left(i\sum_{n=1}^{N}\left(\mathbf{m}_{\Xi_0}\right)_n \upsilon_n - \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N}\left(\mathbf{C}_{\Xi_0\Xi_0}\right)_{nm} \upsilon_n \upsilon_m \right)\times \\
&\quad\times \exp\left(i\sum_{n=N+1}^{2N}\left(\mathbf{m}_{\Xi_1}\right)_n u_n - \frac{1}{2}\sum_{n=N+1}^{2N}\sum_{m=N+1}^{2N}\left(\mathbf{C}_{\Xi_1\Xi_1}\right)_{nm} u_n u_m \right)\times \\
&\quad\times \exp\left(-\sum_{n=1}^{N}\sum_{m=N+1}^{2N}\left(\mathbf{C}_{\Xi_0\Xi_1}\right)_{nm} \upsilon_n u_m \right),
\end{aligned}
\tag{2.16}
$$

where $\mathbf{m}_{\Xi_0}$, $\mathbf{C}_{\Xi_0\Xi_0}$ denote the mean value and the autocovariance of the random vector $\boldsymbol{\Xi}_0$; $\mathbf{m}_{\Xi_1}$, $\mathbf{C}_{\Xi_1\Xi_1}$ are the mean value and the autocovariance of the random vector $\boldsymbol{\Xi}_1$; and $\mathbf{C}_{\Xi_0\Xi_1}$ denotes the cross-covariance of the two random vectors. By setting, now, $\Xi_n(\theta)=\Xi_n(t_n;\theta)$, $\upsilon_n=\upsilon_n(t_n)$, $u_n=u_n(t_n)$ Volterra's principle of passing from the discrete to the continuous (see e.g. (Mamis, 2020, Appendix A)), we obtain the required Eq. (2.14). Note that the derivation of Eq. (2.13) is similarly accomplished by considering its discrete analogue, i.e. the $(1+N)-$ dimensional joint Gaussian characteristic function of a random variable $X_0(\theta)$ and a random vector $\boldsymbol{\Xi}(\theta)$, and repeating the above process.

2.2.2 Extensions for a random function-functional

Now, by setting in Eq. (2.5)

$$\begin{aligned} \mathcal{G}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)] &\equiv \mathcal{G}[\cdots] = \\ &= \Xi(t;\theta)\,\mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)] \equiv \Xi(t;\theta)\,\mathcal{F}[\cdots], \end{aligned} \tag{2.17}$$

taking into account the Gaussian joint characteristic FFℓ Eq. (2.13) and calculating the action of the operator $\varphi_{\hat{X}_0\hat{\Xi}}\left(\dfrac{\partial}{i\,\partial\upsilon};\dfrac{\delta}{i\,\delta u(\bullet)}\right)$ on the FFℓ $u(t)\,\mathcal{F}[\upsilon;u(\bullet|_{t_0}^{t})]$, the following extension of the NF theorem for FFℓs is obtained.

Theorem 3 [Extension I of the Novikov-Furutsu theorem]: For a sufficiently smooth FFℓ of the form $\mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)] \equiv \mathcal{F}[\cdots]$, whose arguments $X_0(\theta)$, $\Xi(\bullet;\theta)$ are jointly Gaussian, the following formula holds true:

$$\begin{aligned} \mathbb{E}^{\theta}\big[\Xi(t;\theta)\,\mathcal{F}[\cdots]\big] &= \\ &= m_{\Xi}(t)\,\mathbb{E}^{\theta}\big[\mathcal{F}[\cdots]\big] + C_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\left[\frac{\partial\mathcal{F}[\cdots]}{\partial X_0(\theta)}\right] + \\ &\quad + \int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\frac{\delta\mathcal{F}[\cdots]}{\delta\Xi(s;\theta)}\right]d\tau\,. \end{aligned} \tag{2.18}$$ ■

Eq. (2.18) was first derived in (Athanassoulis & Mamis, 2019) and can be promptly seen as a generalization to the classical form. More specifically, by setting $m_{\Xi}(t)=0$ and assuming that $X_0(\theta)$ and $\Xi(t;\theta)$ are uncorrelated the classical form is retrieved. Moreover, following the aforementioned paper's approach, the following simple, yet very useful, generalization of the above theorem can be obtained

Theorem 4 [Extension II of the Novikov-Furutsu theorem]: For a sufficiently smooth FFℓ of the form $\mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)] \equiv \mathcal{F}[\cdots]$, whose arguments $X_0(\theta)$, $\Xi(\bullet;\theta)$ are jointly Gaussian, the following formula holds true:

$$\begin{aligned} \mathbb{E}^{\theta}\big[\Xi(s;\theta)\,\mathcal{F}[\cdots]\big] &= \\ &= m_{\Xi}(s)\,\mathbb{E}^{\theta}\big[\mathcal{F}[\cdots]\big] + C_{X_0\Xi}(s)\,\mathbb{E}^{\theta}\left[\frac{\partial\mathcal{F}[\cdots]}{\partial X_0(\theta)}\right] + \\ &\quad + \int_{t_0}^{s} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\frac{\delta\mathcal{F}[\cdots]}{\delta\Xi(\tau;\theta)}\right]d\tau\,, \end{aligned} \tag{2.19}$$ ■

where $t_0 \le s \le t$. The derivation of Eq. (2.19) will be thoroughly discussed in the ensuing section 2.2.

Further, by setting in Eq. (2.5)

$$\mathcal{G}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)] \equiv \mathcal{G}[\cdots] = \\ = \dot{\Xi}(s;\theta)\,\mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)] \equiv \dot{\Xi}(s;\theta)\,\mathcal{F}[\cdots], \tag{2.20}$$

and calculating the action of the operator $\varphi_{\hat{X}_0\hat{\Xi}}\left(\dfrac{\partial}{i\,\partial\upsilon};\dfrac{\delta}{i\,\delta u(\bullet)}\right)$, this time, on the FFℓ $\dot{u}(s)\,\mathcal{F}[\upsilon;u(\bullet|_{t_0}^{t})]$, the following extension of the NF theorem for FFℓs is obtained. Note that the overdot denotes the first temporal derivative.

Theorem 5 [Extension III of the Novikov-Furutsu theorem]: For a sufficiently smooth FFℓ of the form $\mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)] \equiv \mathcal{F}[\cdots]$, whose arguments $X_0(\theta)$, $\Xi(\bullet;\theta)$ are jointly Gaussian, the following formula holds true:

$$\mathbb{E}^{\theta}\left[\dot{\Xi}(s;\theta)\,\mathcal{F}[\cdots]\right] = \\ = \dot{m}_{\Xi}(s)\,\mathbb{E}^{\theta}\left[\mathcal{F}[\cdots]\right] + \dot{C}_{X_0\Xi}(s)\,\mathbb{E}^{\theta}\left[\frac{\partial\mathcal{F}[\cdots]}{\partial X_0(\theta)}\right] + \\ + \int_{t_0}^{t}\partial_s C_{\Xi\Xi}(s,\tau)\,\mathbb{E}^{\theta}\left[\frac{\delta\mathcal{F}[\cdots]}{\delta\Xi(\tau;\theta)}\right]d\tau, \tag{2.21}$$ ■

where $t_0 \le s \le t$ and $\partial_s C_{\Xi\Xi}(s,\tau) = \partial C_{\Xi\Xi}(s,\tau)/\partial s$. The proof of theorems 4 and 5 will be outlined in the next section 2.3

2.2.3 Extensions for a functional of two random arguments

Proceeding in the same fashion as before, by setting in Eq. (2.6)

$$\mathcal{J}[\Xi_0(\bullet|_{t_0}^{t});\Xi_1(\bullet|_{t_0}^{t})] = \mathcal{J}[\cdots] = \\ = \Xi_1(s;\theta)\mathcal{F}[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)] \equiv \Xi_1(s;\theta)\mathcal{F}[\cdots], \tag{2.22}$$

taking into account the joint Gaussian characteristic functional Eq. (2.14) and calculating the action of operator $\varphi_{\hat{\Xi}_0\hat{\Xi}_1}\left[\dfrac{\delta}{i\delta\upsilon(\tau_0)};\dfrac{\delta}{i\delta u(\tau_1)}\right]$ on the functional $u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$, the following extension of the NF theorem is obtained.

Theorem 6 [Extension IV of the Novikov–Furutsu theorem]: For a sufficiently smooth functional of the form $\mathcal{F}[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)] \equiv \mathcal{F}[\cdots]$, whose arguments $\Xi_0(\bullet|_{t_0}^{t};\theta)$, $\Xi_1(\bullet|_{t_0}^{t};\theta)$ are jointly Gaussian, the following formula holds true:

$$\mathbb{E}^{\theta}\left[\Xi_1(s;\theta)]\mathcal{F}[\cdots]\right] =$$
$$= m_{\Xi_1}(s)\mathbb{E}^{\theta}\left[\mathcal{F}[\cdots]\right] + \int_{t_0}^{t} C_{\Xi_0\Xi_1}(\tau_0, s)\mathbb{E}^{\theta}\left[\frac{\delta\mathcal{F}[\cdots]}{\delta\Xi_0(\tau_0;\theta)}\right]d\tau_0 + \tag{2.23}$$
$$+ \int_{t_0}^{t} C_{\Xi_1\Xi_1}(s,\tau_1)\mathbb{E}^{\theta}\left[\frac{\delta\mathcal{F}[\cdots]}{\delta\Xi_1(\tau_1;\theta)}\right]d\tau_1. \quad \blacksquare$$

Accordingly, by setting in Eq. (2.6)

$$J[\Xi_0(\bullet|_{t_0}^{t});\Xi_1(\bullet|_{t_0}^{t})] = J[\cdots] =$$
$$= \dot{\Xi}_1(s;\theta)\mathcal{F}[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)] \equiv \dot{\Xi}_1(s;\theta)\mathcal{F}[\cdots], \tag{2.24}$$

and calculating the action of operator $\varphi_{\hat{\Xi}_0\hat{\Xi}_1}\left[\frac{\delta}{i\delta\upsilon(\tau_0)};\frac{\delta}{i\delta u(\tau_1)}\right]$ on the functional $\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$, a different extension of the NF theorem is derived.

Theorem 7 [Extension V of the Novikov–Furutsu theorem]: For a sufficiently smooth functional of the form $\mathcal{F}[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)] \equiv \mathcal{F}[\cdots]$, whose arguments $\Xi_0(\bullet|_{t_0}^{t};\theta)$, $\Xi_1(\bullet|_{t_0}^{t};\theta)$ are jointly Gaussian, the following formula holds true:

$$\mathbb{E}^{\theta}\left[\dot{\Xi}_1(s;\theta)]\mathcal{F}[\cdots]\right] =$$
$$= \dot{m}_{\Xi_1}(s)\mathbb{E}^{\theta}\left[\mathcal{F}[\cdots]\right] + \int_{t_0}^{t} \partial_s C_{\Xi_0\Xi_1}(\tau_0, s)\mathbb{E}^{\theta}\left[\frac{\delta\mathcal{F}[\cdots]}{\delta\Xi_0(\tau_0;\theta)}\right]d\tau_0 + \tag{2.25}$$
$$+ \int_{t_0}^{t} \partial_s C_{\Xi_1\Xi_1}(s,\tau_1)\mathbb{E}^{\theta}\left[\frac{\delta\mathcal{F}[\cdots]}{\delta\Xi_1(\tau_1;\theta)}\right]d\tau_1. \quad \blacksquare$$

2.3 Proof of the extensions of the Novikov-Furutsu theorem

In this section, we are going to follow the approach showcased in (Athanassoulis & Mamis, 2019) in order to prove the aforementioned extensions of the Novikov-Furutsu theorem.

2.2.1 Proof of the extensions for a random function-functional

Since in the presented extensions, Theorems 3-5, the arguments of the FFℓ are assumed jointly Gaussian, then their fluctuations defined by Eqs. (2.6a,b) will also be jointly Gaussian with zero mean values and the same central moments, i.e. $C_{\hat{X}_0\hat{X}_0} = C_{X_0X_0}$, $C_{\hat{X}_0\hat{\Xi}}(\bullet) = C_{X_0\Xi}(\bullet)$, $C_{\hat{\Xi}\hat{\Xi}}(\bullet,\bullet) = C_{\Xi\Xi}(\bullet,\bullet)$. As such, through Eq. (2.13), we find

$$\varphi^{\text{Gauss}}_{\hat{X}_0\hat{\Xi}}[\upsilon\,;u(\bullet|^{t}_{t_0})] = \exp\left(-\frac{1}{2}\int_{t_0}^{t}\int_{t_0}^{t} C_{\Xi\Xi}(\tau_1,\tau_2)\,u(\tau_1)\,u(\tau_2)\,d\tau_1\,d\tau_2\right)\times$$
$$\times\exp\left(i\,m_{X_0}\,\upsilon-\frac{1}{2}\,C_{X_0X_0}\,\upsilon^2\right)\cdot\exp\left(-\upsilon\int_{t_0}^{t}C_{X_0\Xi}(\tau)\,u(\tau)\,d\tau\right). \tag{2.26}$$

Having obtained the above expression, it is easy to observe that the averaged shift operator $\varphi_{\hat{X}_0\hat{\Xi}}\left[\partial/i\,\partial\upsilon\,;\delta/i\,\delta u(\tau)\right]$ can be expressed as the product of three operators

$$\varphi_{\hat{X}_0\hat{\Xi}}\left[\frac{\partial\bullet}{i\,\partial\upsilon}\,;\frac{\delta\bullet}{i\,\delta u(\tau)}\right] = \left(\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\bullet\right)\left(\bar{\mathcal{T}}_{\hat{X}_0\hat{\Xi}}\bullet\right)\left(\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\bullet\right), \tag{2.27}$$

defined by:

$$\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\bullet = \exp\left(\frac{1}{2}\,C_{X_0X_0}\,\frac{\partial^2\bullet}{\partial\upsilon^2}\right), \tag{2.28a}$$

$$\bar{\mathcal{T}}_{\hat{X}_0\hat{\Xi}}\bullet = \exp\left(\int_{t_0}^{t}C_{X_0\Xi}(\tau)\,\frac{\partial\bullet}{\partial\upsilon}\,\frac{\delta\bullet}{\delta u(\tau)}\,d\tau\right), \tag{2.28b}$$

$$\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\bullet = \exp\left(\frac{1}{2}\int_{t_0}^{t}\int_{t_0}^{t}C_{\Xi\Xi}(\tau_1,\tau_2)\,\frac{\delta^2\bullet}{\delta u(\tau_1)\,\delta u(\tau_2)}\,d\tau_1\,d\tau_2\right). \tag{2.28c}$$

These $\bar{\mathcal{T}}-$operators can be considered as second-order versions of the shift operators and, thus, termed as ***quadratic averaged shift operators***. Using Eqs. (2.27) and (2.28), Theorem 1 for Gaussian arguments takes the form

$$\mathbb{E}^{\theta}\left[\mathcal{G}[X_0(\theta)\,;\Xi(\bullet|^{t}_{t_0};\theta)]\right] = \left[\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\,\mathcal{G}[\upsilon\,;u(\bullet|^{t}_{t_0})]\right]_{\substack{\upsilon=m_{X_0}\\ u(\bullet)=m_{\Xi}(\bullet)}}. \tag{2.29}$$

Therefore, it is readily understood that the proof of the presented extensions of the NF theorem for a random FFℓ $\mathcal{G}[X_0(\theta)\,;\Xi(\bullet|^{t}_{t_0};\theta)]$ is largely encapsulated on the determination of the action of these operators on the appropriate forms of the deterministic counterparts of the said FFℓ, that is $\mathcal{G}[\upsilon\,;u(\bullet|^{t}_{t_0})]$. For this, it is necessary to first examine the properties of these operators.

Properties of the $\bar{\mathcal{T}}-$operators. On C^{∞} function-functionals, $\bar{\mathcal{T}}-$operators are well-defined and they have the following properties, which are needed subsequently for the proof of the extended NF theorem. Note that the proof of the lemmata presented in this paragraph can be found in (Mamis, 2020). Herein, we will only provide the proof of the properties of the analogous lemmata corresponding to the functional of two random arguments

Lemma 2.1: $\bar{\mathcal{T}}-$operators are linear. That is, for any two C^{∞} functionals $\mathcal{F}[\upsilon;u(\bullet|^{t}_{t_0})]$, $\mathcal{G}[\upsilon;u(\bullet|^{t}_{t_0})]$ it holds true that

$$\bar{\mathcal{T}}\left[\alpha\mathcal{G}[\upsilon;u(\bullet|_{t_0}^{t})]+\beta\mathcal{F}[\upsilon;u(\bullet|_{t_0}^{t})]\right]= \\ =\alpha\bar{\mathcal{T}}\left[\mathcal{G}[\upsilon;u(\bullet|_{t_0}^{t})]\right]+\beta\bar{\mathcal{T}}\left[\mathcal{F}[\upsilon;u(\bullet|_{t_0}^{t})]\right], \tag{2.30}$$

where $\bar{\mathcal{T}}\bullet$ stands for any of the operators $\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\bullet$, $\bar{\mathcal{T}}_{\hat{X}_0\hat{\Xi}}\bullet$, $\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\bullet$, and a, β are scalars or scalar functions having argument(s) different than the differentiation argument(s) appearing in the corresponding $\bar{\mathcal{T}}\bullet$ operator.

Lemma 2.2: $\bar{\mathcal{T}}$ – operators commute with υ– and $u(\tau)$– differentiation. That is, for a C^∞ FFℓ $\mathcal{G}[\upsilon;u(\bullet|_{t_0}^{t})]$, and for $\bar{\mathcal{T}}\in\left\{\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\bullet,\bar{\mathcal{T}}_{\hat{X}_0\hat{\Xi}}\bullet,\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\bullet\right\}$,

$$\frac{\partial}{\partial\upsilon}\left[\bar{\mathcal{T}}\mathcal{G}[\upsilon;u(\bullet|_{t_0}^{t})]\right]=\bar{\mathcal{T}}\left[\frac{\partial\mathcal{G}[\upsilon;u(\bullet|_{t_0}^{t})]}{\partial\upsilon}\right], \tag{2.31a}$$

and

$$\frac{\delta}{\delta u(\tau)}\left[\bar{\mathcal{T}}\mathcal{G}[\upsilon;u(\bullet|_{t_0}^{t})]\right]=\bar{\mathcal{T}}\left[\frac{\delta\mathcal{G}[\upsilon;u(\bullet|_{t_0}^{t})]}{\delta u(\tau)}\right]. \tag{2.31b}$$

Lemma 2.3: $\bar{\mathcal{T}}$ – operators commute with each other. That is, for any C^∞ FFℓ $\mathcal{G}[\upsilon;u(\bullet|_{t_0}^{t})]$, it holds true that

$$\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\bar{\mathcal{T}}_{\hat{X}_0\hat{\Xi}}\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\,\mathcal{G}[\upsilon;u(\bullet|_{t_0}^{t})]=\bar{\mathcal{T}}_{\hat{X}_0\hat{\Xi}}\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\,\mathcal{G}[\upsilon;u(\bullet|_{t_0}^{t})]=\cdots \\ \cdots=\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\bar{\mathcal{T}}_{\hat{X}_0\hat{\Xi}}\,\mathcal{G}[\upsilon;u(\bullet|_{t_0}^{t})]. \tag{2.32}$$

In other words, the product of the three $\bar{\mathcal{T}}$ – operators under any permutation of their order, has the same action on the FFℓ $\mathcal{G}[\upsilon;u(\bullet|_{t_0}^{t})]$.

Proof of extensions II, III of the NF theorem. For this, we specify the FFℓ $\mathcal{G}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]$ as $\Xi(s;\theta)\,\mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]$. Then, using Lemmata 2.1-2.3 and Eq. (2.29) we find:

$$\mathbb{E}^{\theta}\left[\Xi(s;\theta)\mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\right]= \\ =\left\{\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\left[u(s)\mathcal{F}[\upsilon;u(\bullet|_{t_0}^{t})]\right]\right\}_{\substack{\upsilon=m_{X_0}\\ u(\bullet)=m_{\Xi}(\bullet)}}. \tag{2.33}$$

Eq. (2.33) is the most convenient form to calculate the result of the successive application of the three $\bar{\mathcal{T}}$ – operators on $u(s)\mathcal{F}[\upsilon;u(\bullet|_{t_0}^{t})]$ and thus, prove the required extensions of the NF theorem. This is performed using the following lemmata.

Lemma 2.4. The action of operator $\bar{\mathcal{T}}_{\hat{X}_0 \hat{X}_0}$ on $u(s)\,\mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]$ is given by

$$\bar{\mathcal{T}}_{\hat{X}_0 \hat{X}_0}\left[u(s)\,\mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]\right] = u(s)\,\bar{\mathcal{T}}_{\hat{X}_0 \hat{X}_0}\left[\mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]\right]. \tag{2.34}$$

At this point, by setting the term $\bar{\mathcal{T}}_{\hat{X}_0 \hat{X}_0}\left[\mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]\right]$ as a new functional $\mathcal{F}_1[\upsilon\,;u(\bullet|_{t_0}^{t})]$ and applying the operator $\bar{\mathcal{T}}_{\hat{X}_0 \hat{\Xi}}$ on both sides of the above equation, yields:

$$\bar{\mathcal{T}}_{\hat{X}_0 \hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0 \hat{X}_0}\left[u(s)\,\mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]\right] = u(s)\,\bar{\mathcal{T}}_{\hat{X}_0 \hat{\Xi}}\left[\mathcal{F}_1[\upsilon\,;u(\bullet|_{t_0}^{t})]\right]. \tag{2.35}$$

The right-hand side of Eq. (2.36) can be calculated using the following result:

Lemma 2.5. The action of operator $\bar{\mathcal{T}}_{\hat{X}_0 \hat{\Xi}}$ on $u(s)\,\mathcal{F}_1[\upsilon\,;u(\bullet|_{t_0}^{t})]$ is given by

$$\begin{aligned}\bar{\mathcal{T}}_{\hat{X}_0 \hat{\Xi}}\left[u(s)\,\mathcal{F}_1[\upsilon\,;u(\bullet|_{t_0}^{t})]\right] &= u(s)\,\bar{\mathcal{T}}_{\hat{X}_0 \hat{\Xi}}\left[\mathcal{F}_1[\upsilon\,;u(\bullet|_{t_0}^{t})]\right] + \\ &+ C_{X_0\Xi}(s)\,\bar{\mathcal{T}}_{\hat{X}_0 \hat{\Xi}}\left[\frac{\partial \mathcal{F}_1[\upsilon\,;u(\bullet|_{t_0}^{t})]}{\partial \upsilon}\right].\end{aligned} \tag{2.36}$$

Then, by applying the $\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}$ operator on both sides of the above expression, employing Eq. (2.35) for the left-hand side of Eq. (2.36) and designating the term $\bar{\mathcal{T}}_{\hat{X}_0 \hat{\Xi}}\left[\mathcal{F}_1[\upsilon\,;u(\bullet|_{t_0}^{t})]\right]$ as $\mathcal{F}_2[\upsilon\,;u(\bullet|_{t_0}^{t})]$, we find

$$\begin{aligned}\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0 \hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0 \hat{X}_0}\left[u(s)\,\mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]\right] &= \bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\left[u(s)\,\mathcal{F}_2[\upsilon\,;u(\bullet|_{t_0}^{t})]\right] + \\ &+ \bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\left\{C_{X_0\Xi}(s)\,\bar{\mathcal{T}}_{\hat{X}_0 \hat{\Xi}}\left[\frac{\partial \mathcal{F}_1[\upsilon\,;u(\bullet|_{t_0}^{t})]}{\partial \upsilon}\right]\right\}.\end{aligned} \tag{2.37}$$

We shall now elaborate on the two terms appearing in the right-hand side of Eq. (27), separately. First, the second term in the right-hand side of Eq. (2.37) can equivalently be expressed in terms of the FFℓ $\mathcal{F}[\upsilon\,;u(\bullet)]$ by taking advantage of: a) the linearity of $\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}$ (Lemma 2.1); b) the commutation of $\bar{\mathcal{T}}_{\hat{X}_0 \hat{X}_0}$ with the $\upsilon-$ derivative (Lemma 2.2); c) the definition of $\mathcal{F}_1[\upsilon\,;u(\bullet)]$, as

$$\begin{aligned}\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\left\{C_{X_0\Xi}(s)\,\bar{\mathcal{T}}_{\hat{X}_0 \hat{\Xi}}\left[\frac{\partial \mathcal{F}_1[\upsilon\,;u(\bullet|_{t_0}^{t})]}{\partial \upsilon}\right]\right\} &= \\ = C_{X_0\Xi}(s)\,\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0 \hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0 \hat{X}_0}&\left[\frac{\partial \mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]}{\partial \upsilon}\right].\end{aligned} \tag{2.38}$$

Concerning the first term in the right-hand side of Eq. (2.37), we need the following Lemma:

Lemma 2.6. The action of operator $\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}$ on $u(s)\,\mathcal{F}_2[\upsilon\,;u(\bullet|_{t_0}^{t})]$ is given by

$$\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\left[u(s)\,\mathcal{F}_2[\upsilon\,;u(\bullet|_{t_0}^{t})]\right] = u(s)\,\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\left[\mathcal{F}_2[\upsilon\,;u(\bullet|_{t_0}^{t})]\right] + \\ + \int_{t_0}^{t} C_{\Xi\Xi}(s,\tau)\,\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\left[\frac{\delta\mathcal{F}_2[\upsilon\,;u(\bullet|_{t_0}^{t})]}{\delta u(s)}\right]d\tau\,. \tag{2.39}$$

Combining, now, Eqs. (2.37)-(2.39) and the definition of $\mathcal{F}_2[\upsilon\,;u(\bullet)]$, and employing the commutation of $\bar{\mathcal{T}}$ – operators with the $u(\tau)$ – derivative (Lemma 2), we obtain

$$\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\left[u(s)\,\mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]\right] = \\ = u(s)\,\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\left[\mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]\right] + \\ + C_{X_0\Xi}(s)\,\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\left[\frac{\partial\mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]}{\partial\upsilon}\right] + \\ + \int_{t_0}^{t} C_{\Xi\Xi}(s,\tau)\,\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\left[\frac{\delta\mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]}{\delta u(\tau)}\right]d\tau\,. \tag{2.40}$$

Finally, setting $\upsilon = m_{X_0}$, and $u(\bullet) = m_{\Xi}(\bullet)$ in Eq. (2.40), and applying Eq. (2.29) to each term of the form $\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\,[\cdots]$ in both sides of Eq. (2.70), we obtain the extended NF theorem, Eq. (2.19). The proof is now completed. Moreover, Theorem 3, Eq. (2.18), can be seen as a special case of Eq. (2.19) and thus, its proof is also concluded.

Proof of extension IV of the NF theorem. For this, we specify in Eq. (2.29) the FFℓ $\mathcal{G}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]$ as $\dot{\Xi}(s;\theta)\,\mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]$, where the overdot denotes the first temporal derivative. Then, using Lemmata 2.1-2.3 and Eq. (2.29) we find

$$\mathbb{E}^{\theta}\left[\dot{\Xi}(s;\theta)\mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\right] = \\ = \left\{\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0\hat{\Xi}}\,\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\left[\dot{u}(s)\mathcal{F}[\upsilon;u(\bullet|_{t_0}^{t})]\right]\right\}_{\substack{\upsilon=m_{X_0}\\ u(\bullet)=m_{\Xi}(\bullet)}}. \tag{2.41}$$

Therefore, it is seen that, this time, we must evaluate the action of the $\bar{\mathcal{T}}$ – operators on the deterministic counterpart of $\dot{\Xi}(s;\theta)\,\mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]$, i.e. $\dot{u}(s)\mathcal{F}[\upsilon;u(\bullet|_{t_0}^{t})]$. This is easily obtained by considering the following alternatives of Lemmata 4-6.

$$\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\left[\dot{u}(s)\mathcal{F}[\upsilon;u(\bullet|_{t_0}^{t})]\right] = \dot{u}(s)\,\bar{\mathcal{T}}_{\hat{X}_0\hat{X}_0}\left[\mathcal{F}[\upsilon;u(\bullet|_{t_0}^{t})]\right], \tag{2.42}$$

$$\bar{\mathcal{T}}_{\hat{X}_0 \hat{\Xi}}\left[\dot{u}(s)\,\mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]\right] = \dot{u}(s)\,\bar{\mathcal{T}}_{\hat{X}_0 \hat{\Xi}}\left[\mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]\right] + \\ + \dot{C}_{X_0 \Xi}(s)\,\bar{\mathcal{T}}_{\hat{X}_0 \hat{\Xi}}\left[\frac{\partial \mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]}{\partial \upsilon}\right], \tag{2.43}$$

$$\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\left[\dot{u}(s)\,\mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]\right] = \dot{u}(s)\,\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\left[\mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]\right] + \\ + \int_{t_0}^{t} \partial_s C_{\Xi\Xi}(s,\tau)\,\bar{\mathcal{T}}_{\hat{\Xi}\hat{\Xi}}\left[\frac{\delta \mathcal{F}[\upsilon\,;u(\bullet|_{t_0}^{t})]}{\delta u(\tau)}\right] d\tau\,, \tag{2.44}$$

where $\partial_s C_{\Xi\Xi}(s,\tau) = \partial C_{\Xi\Xi}(s,\tau)/\partial s$. Using Eqs. (2.42), (2.43), (2.44) and following the process described above for the proof of extensions I and II of the NF theorem, the proof Eq. (2.21) is also completed.

2.2.2 Proof of the extensions for a functional of two random arguments

As already discussed in paragraph 2.2.1, since in Theorems 4-6 the arguments of the functional of two random arguments are then their fluctuations defined by Eq. (2.8) will also be jointly Gaussian with zero mean values and the same central moments, i.e. $C_{\hat{\Xi}_i \hat{\Xi}_j}(\bullet,\bullet) = C_{\Xi_i \Xi_j}(\bullet,\bullet)$ with $i,\ j = 0, 1$. As such, through Eq. (2.14) we find

$$\varphi^{Gauss}_{\hat{\Xi}_0 \hat{\Xi}_1}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})] = \exp\left(-\frac{1}{2}\int_{t_0}^{t}\int_{t_0}^{t} C_{\Xi_0\Xi_0}(\tau_1,\tau_2)\,\upsilon(\tau_1)\,\upsilon(\tau_2)\,d\tau_1\,d\tau_2\right)\times \\ \times \exp\left(-\frac{1}{2}\int_{t_0}^{t}\int_{t_0}^{t} C_{\Xi_1\Xi_1}(\tau_1,\tau_2)\,u(\tau_1)\,u(\tau_2)\,ds_1\,ds_2\right)\times \\ \times \exp\left(-\int_{t_0}^{t}\int_{t_0}^{t} C_{\Xi_0\Xi_1}(\tau_0,s_1)\,\upsilon(\tau_0)\,u(\tau_1)\,d\tau_0\,d\tau_1\right). \tag{2.45}$$

Thus, in this case, the averaged shift operator $\varphi_{\hat{\Xi}_0 \hat{\Xi}_1}\left[\delta/i\,\delta\upsilon(\tau_0)\,;\delta/i\,\delta u(\tau_1)\right]$ takes the form

$$\varphi_{\hat{\Xi}_0 \hat{\Xi}_1}\left[\frac{\delta\,\bullet}{i\,\delta\upsilon(\tau_0)};\frac{\delta\,\bullet}{i\,\delta u(\tau_1)}\right] = \left(\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_0}\bullet\right)\left(\bar{\mathcal{T}}_{\hat{\Xi}_1\hat{\Xi}_1}\bullet\right)\left(\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\bullet\right), \tag{2.46}$$

with the three ***quadratic averaged shift operators*** defined by

$$\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_0}\bullet = \exp\left(\frac{1}{2}\int_{t_0}^{t}\int_{t_0}^{t} C_{\Xi_0\Xi_0}(\tau_0,\tau_1)\,\frac{\delta^2\bullet}{\delta\upsilon(\tau_0)\delta\upsilon(\tau_1)}\,d\tau_0\,d\tau_1\right), \tag{2.47a}$$

$$\bar{\mathcal{T}}_{\hat{\Xi}_1 \hat{\Xi}_1} \bullet = \exp\left(\frac{1}{2} \int_{t_0}^{t} \int_{t_0}^{t} C_{\Xi_1 \Xi_1}(\tau_0, \tau_1) \frac{\delta^2 \bullet}{\delta u(\tau_0) \delta u(\tau_1)} d\tau_0 \, d\tau_1 \right), \tag{2.47b}$$

$$\bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_1} \bullet = \exp\left(\int_{t_0}^{t} \int_{t_0}^{t} C_{\Xi_0 \Xi_1}(\tau_0, \tau_1) \frac{\delta^2 \bullet}{\delta \upsilon(\tau_0) \delta u(\tau_1)} d\tau_0 \, d\tau_1 \right). \tag{2.47c}$$

Using Eqs. (2.46), (2.47) Theorem 2 for Gaussian arguments takes the form

$$\mathbb{E}^{\theta}\Big[J[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)] \Big] = \Big[\bar{\mathcal{T}}_{\hat{\Xi}_1 \hat{\Xi}_1} \bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_1} \bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_0} J[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})] \Big]_{\substack{\upsilon = m_{X_0} \\ u(\bullet) = m_{\Xi}(\bullet)}} . \tag{2.48}$$

Therefore, by considering the appropriate forms of the functional $J[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)]$ and calculating the action of the $\bar{\mathcal{T}}-$ operators on its corresponding deterministic counterparts $J[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ the required generalizations of the NF theorem are obtained. Proceeding in the same fashion as in paragraph 2.2.1, let us examined the properties of these operators

Properties of the $\bar{\mathcal{T}}-$operators. On C^{∞} functionals, $\bar{\mathcal{T}}-$ operators are well-defined and they have the following properties; the proof of the following lemmata is provided in Appendix A.

Lemma 2.7. $\bar{\mathcal{T}}$ – operators are linear. That is, for any two C^{∞} functionals $\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$, $\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ it holds true that

$$\begin{aligned} &\bar{\mathcal{T}}\Big[\alpha \mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})] + \beta \mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})] \Big] = \\ &\qquad = \alpha \bar{\mathcal{T}}\Big[\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})] \Big] + \beta \bar{\mathcal{T}}\Big[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})] \Big], \end{aligned} \tag{2.49}$$

where $\bar{\mathcal{T}}\bullet$ stands for any of the tree operators $\bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_0}\bullet$, $\bar{\mathcal{T}}_{\hat{\Xi}_1 \hat{\Xi}_1}\bullet$, $\bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_1}\bullet$, and a , β are scalars or scalar functions having argument(s) different than the differentiation argument(s) appearing in the corresponding $\bar{\mathcal{T}}\bullet$ operator.

Lemma 2.8. $\bar{\mathcal{T}}$ – operators commute with $\upsilon(\tau_0)-$ and $u(\tau_1)-$ differentiation. That is, for any C^{∞} functional $\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$, it holds true that

$$\frac{\delta}{\delta \upsilon(\tau_0)} \Big[\bar{\mathcal{T}} \mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})] \Big] = \bar{\mathcal{T}}\left[\frac{\delta \mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]}{\delta \upsilon(\tau_1)} \right], \tag{2.50a}$$

and

$$\frac{\delta}{\delta u(\tau_1)} \Big[\bar{\mathcal{T}} \mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})] \Big] = \bar{\mathcal{T}}\left[\frac{\delta \mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]}{\delta u(\tau_1)} \right], \tag{2.50b}$$

where $\bar{\mathcal{T}} \in \left\{ \bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_0} \bullet, \bar{\mathcal{T}}_{\hat{\Xi}_1 \hat{\Xi}_1} \bullet, \bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_1} \bullet \right\}$.

Lemma 2.9: $\bar{\mathcal{T}}$ – operators commute with each other. That is, for any C^∞ functional $\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$, it holds true that

$$\begin{aligned} &\bar{\mathcal{T}}_{\hat{\Xi}_1 \hat{\Xi}_1} \bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_1} \bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_0} \mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})] = \\ &\qquad = \bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_1} \bar{\mathcal{T}}_{\hat{\Xi}_1 \hat{\Xi}_1} \bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_0} \mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})] = \cdots \\ &\qquad \cdots = \bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_0} \bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_1} \bar{\mathcal{T}}_{\hat{\Xi}_1 \hat{\Xi}_1} \mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]. \end{aligned} \tag{2.51}$$

In other words, the product of the three $\bar{\mathcal{T}}$ – operators under any permutation of their order, has the same action on the functional $\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$.

Proof of extensions IV, V of the NF theorem. At this point, by specifying the functional $\mathcal{J}[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)]$ as $\Xi_1(s;\theta)\,\mathcal{F}[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)]$, using Lemmata 2.7-2.9 and Eq. (2.48) we find the appropriate expression for the proof of Eq. (2.23),

$$\begin{aligned} &\mathbb{E}^\theta\left[\Xi_1(s;\theta)\mathcal{F}[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)]\right] = \\ &\qquad = \left\{\bar{\mathcal{T}}_{\hat{\Xi}_1 \hat{\Xi}_1} \bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_1} \bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_0} \left[u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]\right\}_{\substack{\upsilon(\bullet)=m_{\Xi_0}(\bullet)\\ u(\bullet)=m_{\Xi_1}(\bullet)}}. \end{aligned} \tag{2.52}$$

Accordingly, setting $\mathcal{J}[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)] = \dot{\Xi}_1(s;\theta)\,\mathcal{F}[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)]$, we obtain the following, appropriate form for the proof of Eq. (2.24):

$$\begin{aligned} &\mathbb{E}^\theta\left[\dot{\Xi}_1(s;\theta)\mathcal{F}[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)]\right] = \\ &\qquad = \left\{\bar{\mathcal{T}}_{\hat{\Xi}_1 \hat{\Xi}_1} \bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_1} \bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_0} \left[\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]\right\}_{\substack{\upsilon(\bullet)=m_{\Xi_0}(\bullet)\\ u(\bullet)=m_{\Xi_1}(\bullet)}}. \end{aligned} \tag{2.53}$$

Similar to the profs presented in the previous paragraph, we must again calculate the action of these $\bar{\mathcal{T}}$ – operators on $u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ and $\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$. This is accomplished be making use of the following lemmata.

Lemma 2.10. The action of operator $\bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_0}$ on $u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ is given by

$$\bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_0}\left[u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right] = u(s)\bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_0}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]. \tag{2.54}$$

Accordingly, the action of operator $\bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_0}$ on $\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ is given by

$$\bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_0}\left[\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right] = \dot{u}(s)\bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_0}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]. \tag{2.55}$$

Lemma 2.11. The action of operator $\bar{\mathcal{T}}_{\hat{\Xi}_0 \hat{\Xi}_1}$ on $u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ is given by

$$\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\left[u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right] = u(s)\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right] + \\ + \int_{t_0}^{t} C_{\hat{\Xi}_0\hat{\Xi}_1}(\tau_0, s)\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\left[\frac{\delta\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]}{\delta\upsilon(\tau_0)}\right]d\tau_0 . \tag{2.56}$$

Accordingly, the action of operator $\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}$ on $\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ is given by

$$\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\left[\dot{u}(t)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right] = \dot{u}(s)\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right] + \\ + \int_{t_0}^{t} \partial_s C_{\Xi_0\Xi_1}(\tau_0, s)\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\left[\frac{\delta\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]}{\delta\upsilon(\tau_0)}\right]d\tau_0 . \tag{2.57}$$

Lemma 2.12. The action of operator $\bar{\mathcal{T}}_{\hat{\Xi}_1\hat{\Xi}_1}$ on $u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ is given by

$$\bar{\mathcal{T}}_{\hat{\Xi}_1\hat{\Xi}_1}\left[u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right] = u(s)\bar{\mathcal{T}}_{\hat{\Xi}_1\hat{\Xi}_1}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right] + \\ + \int_{t_0}^{t} C_{\Xi_1\Xi_1}(s,\tau_1)\bar{\mathcal{T}}_{\hat{\Xi}_1\hat{\Xi}_1}\left[\frac{\delta\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]}{\delta u(\tau_1)}\right]d\tau_1 . \tag{2.58}$$

Accordingly, the action of operator $\bar{\mathcal{T}}_{\hat{\Xi}_1\hat{\Xi}_1}$ on $\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ is given by

$$\bar{\mathcal{T}}_{\hat{\Xi}_1\hat{\Xi}_1}\left[\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right] = \dot{u}(s)\bar{\mathcal{T}}_{\hat{\Xi}_1\hat{\Xi}_1}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right] + \\ + \int_{t_0}^{t} \partial_s C_{\Xi_1\Xi_1}(s,\tau_1)\bar{\mathcal{T}}_{\hat{\Xi}_1\hat{\Xi}_1}\left[\frac{\delta\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]}{\delta u(\tau_1)}\right]d\tau_1 . \tag{2.59}$$

Finally, repeating the process presented in paragraph 2.2.1 for $u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ as well as $\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ yields the sought-for extensions Theorem 4 and 5, respectively. This concludes the proofs of the said theorems.

2.4 Generalizations of Theorems 6 and 7

In this section, we are going to present the generalizations of Theorems 6, 7. More specifically, we shall consider the cases in which $\mathcal{J}$ is both a function of $X_0(\theta)$ and a functional of excitations $\Xi_0(\bullet|_{t_0}^{t})$ and $\Xi_1(\bullet|_{t_0}^{t})$, i.e. $\mathcal{J}[\cdots] = \mathcal{J}[X_0(\theta);\Xi_0(\bullet|_{t_0}^{t},\theta);\Xi_1(\bullet|_{t_0}^{t},\theta)]$.

Specifying, thus, $\mathcal{J}$ as

$$\mathcal{J}[\cdots] = \mathcal{J}[X_0(\theta);\Xi_0(\bullet|_{t_0}^{t},\theta);\Xi_1(\bullet|_{t_0}^{t},\theta)] = \\ = \Xi_1(s;\theta)\mathcal{F}[X_0(\theta);\Xi_0(\bullet|_{t_0}^{t},\theta);\Xi_1(\bullet|_{t_0}^{t},\theta)] = \Xi_1(s;\theta)\mathcal{F}[\cdots], \tag{2.60}$$

where $t_0 \le s \le t$. Then, taking into account its corresponding joint characteristic FFℓ and the appropriate operator, the following theorem can be proven:

Theorem 8 [Extension VI of the Novikov-Furutsu theorem]: For a sufficiently smooth functional of the form $\mathcal{F}[X_0(\theta);\Xi_0(\bullet|_{t_0}^{t},\theta);\Xi_1(\bullet|_{t_0}^{t},\theta)] \equiv \mathcal{F}[\cdots]$, whose arguments $X_0(\theta)$, $\Xi_0(\bullet|_{t_0}^{t};\theta)$, $\Xi_1(\bullet|_{t_0}^{t};\theta)$ are jointly Gaussian, the following formula holds true:

$$\begin{aligned}\mathbb{E}^{\theta}\left[\Xi_1(s;\theta)]\mathcal{F}[\cdots]\right] &= \\ &= m_{\Xi_1}(s)\mathbb{E}^{\theta}\left[\mathcal{F}[\cdots]\right] + C_{X_0\Xi_0}(s)\mathbb{E}^{\theta}\left[\frac{\partial\mathcal{F}[\cdots]}{\partial X_0(\theta)}\right] + \\ &\quad + C_{X_0\Xi_1}(s)\mathbb{E}^{\theta}\left[\frac{\partial\mathcal{F}[\cdots]}{\partial X_0(\theta)}\right] + \int_{t_0}^{t} C_{\Xi_0\Xi_1}(\tau_0,s)\mathbb{E}^{\theta}\left[\frac{\delta\mathcal{F}[\cdots]}{\delta\Xi_0(\tau_0;\theta)}\right]d\tau_0 + \\ &\quad + \int_{t_0}^{t} C_{\Xi_1\Xi_1}(s,\tau_1)\mathbb{E}^{\theta}\left[\frac{\delta\mathcal{F}[\cdots]}{\delta\Xi_1(\tau_1;\theta)}\right]d\tau_1. \quad \blacksquare\end{aligned} \tag{2.61}$$

Accordingly, by setting:

$$\begin{aligned}\mathcal{J}[\cdots] &= \mathcal{J}[X_0(\theta);\Xi_0(\bullet|_{t_0}^{t},\theta);\Xi_1(\bullet|_{t_0}^{t},\theta)] = \\ &= \dot{\Xi}_1(s;\theta)\mathcal{F}[X_0(\theta);\Xi_0(\bullet|_{t_0}^{t},\theta);\Xi_1(\bullet|_{t_0}^{t},\theta)] = \dot{\Xi}_1(s;\theta)\mathcal{F}[\cdots],\end{aligned} \tag{2.62}$$

the following theorem can be proven.

Theorem 9 [Extension VII of the Novikov-Furutsu theorem]: For a sufficiently smooth functional of the form $\mathcal{F}[X_0(\theta);\Xi_0(\bullet|_{t_0}^{t},\theta);\Xi_1(\bullet|_{t_0}^{t},\theta)] \equiv \mathcal{F}[\cdots]$, whose arguments $X_0(\theta)$, $\Xi_0(\bullet|_{t_0}^{t};\theta)$, $\Xi_1(\bullet|_{t_0}^{t};\theta)$ are jointly Gaussian, the following formula holds true:

$$\begin{aligned}\mathbb{E}^{\theta}\left[\dot{\Xi}_1(s;\theta)]\mathcal{F}[\cdots]\right] &= \\ &= \dot{m}_{\Xi_1}(s)\mathbb{E}^{\theta}\left[\mathcal{F}[\cdots]\right] + \dot{C}_{X_0\Xi_0}(s)\mathbb{E}^{\theta}\left[\frac{\partial\mathcal{F}[\cdots]}{\partial X_0(\theta)}\right] + \\ &\quad + \dot{C}_{X_0\Xi_1}(s)\mathbb{E}^{\theta}\left[\frac{\partial\mathcal{F}[\cdots]}{\partial X_0(\theta)}\right] + \int_{t_0}^{t} \partial_s C_{\Xi_0\Xi_1}(\tau_0,s)\mathbb{E}^{\theta}\left[\frac{\delta\mathcal{F}[\cdots]}{\delta\Xi_0(\tau_0;\theta)}\right]d\tau_0 + \\ &\quad + \int_{t_0}^{t} \partial_s C_{\Xi_1\Xi_1}(s,\tau_1)\mathbb{E}^{\theta}\left[\frac{\delta\mathcal{F}[\cdots]}{\delta\Xi_1(\tau_1;\theta)}\right]d\tau_1. \quad \blacksquare\end{aligned} \tag{2.63}$$

The proofs of theorems 8, 9 are not presented herein but can easily be proven by following the approach presented in section 2.2.

Chapter 3

One-time response pdf evolution equations

The present chapter will serve as an outset for demonstrating in a comprehensive manner the methodology upon which the extensions presented in this thesis are founded. More specifically, we showcase the fundamental steps towards apprehending one-time pdf evolution equations for the response of nonlinear systems under colored Gaussian excitation. It must be noted that the results presented in paragraph 3.1 have been first presented in (Mamis et al., 2019) while the ones presented in paragraph 3.2 are derived by employing the same methodology in a different case, namely that in which the RDE is excited by both additive and multiplicative colored Gaussian noise.

3.1 The case of a scalar, nonlinear, additively excited RDE

We commence from the study of the following scalar, nonlinear, additively excited RDE presented in Chapter 1:

$$\dot{X}(t;\theta) = h(X(t;\theta)) + \kappa\Xi(t;\theta), \qquad X(t_0;\theta) = X_0(\theta). \tag{3.1a,b}$$

As was also presented in the said chapter, in the above random initial value problem (RIVP) θ is the stochastic argument, the overdot denotes differentiation with respect to time, $h(x)$ is a deterministic continuous function modelling the nonlinearities (restoring term), and κ is a constant. Initial value $X_0(\theta)$ and excitation $\Xi(t;\theta)$ are considered correlated and jointly Gaussian with non-zero mean values m_{X_0}, $m_\Xi(t)$, autocovariances $C_{X_0X_0}$, $C_{\Xi\Xi}(t,s)$ and cross-covariance $C_{X_0\Xi}(t)$.

3.1.1 The corresponding stochastic Liouville equation

The starting point of our analysis in all cases is the ***delta projection method***. As presented in section 2.1, by virtue of probability measure $\mathbf{P}_{X(\cdot)\Xi(\cdot)}$, the mean value of a $\mathscr{B}(\mathscr{X}\times\mathscr{Z})$– measurable functional of the response and excitation, $\mathcal{G}[X(\bullet|_{t_0}^{t};\theta);\Xi(\bullet|_{t_0}^{t};\theta)]$, is defined as

$$\mathbb{E}^{\theta}_{\mathbf{P}_{X(\cdot)\Xi(\cdot)}}\left[\mathcal{G}[X(\bullet|_{t_0}^{t};\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\right] = \\ = \int_{\mathscr{X}\times\mathscr{Z}} \mathcal{G}[\chi(\bullet);\xi(\bullet)]\,\mathbf{P}_{X(\cdot)\Xi(\cdot)}\big(d\chi(\bullet)\times d\xi(\bullet)\big). \tag{3.2}$$

Let us, now, consider its discrete analogue

$$\begin{aligned}&\mathbb{E}^{\theta}\left[G(\boldsymbol{X}(\theta);\boldsymbol{\Xi}(\theta))\right] = \\ &\quad = \mathbb{E}^{\theta}\left[G\big(X(\tau_1;\theta),\dots,X(\tau_m;\theta);\Xi(s_1;\theta),\dots,\Xi(s_n;\theta)\big)\right] = \\ &\quad\quad = \int_{\mathscr{X}\times\mathscr{Z}} G\big(\chi(\tau_1),\dots,\chi(\tau_m;\theta);\xi(s_1),\dots,\xi(s_n)\big)\,\mathbf{P}_{X(\bullet)\Xi(\bullet)}\big(d\chi(\bullet)\times d\xi(\bullet)\big),\end{aligned} \tag{3.3}$$

where $G(\boldsymbol{X}(\theta);\boldsymbol{\Xi}(\theta))$ is a $\mathscr{B}(\mathscr{X}\times\mathscr{Z})-$measurable function, and $\boldsymbol{X}(\theta)$, $\boldsymbol{\Xi}(\theta)$ are the $m-$ and $n-$dimensional random vectors defined as the response and excitation in multiple (fixed) time instances $s_1,\dots,s_n$, $\tau_1,\dots,\tau_m\in[t_0,t]$, respectively. Note that Eq. (3.2) can be obtained via Eq. (3.3) by applying Volterra's passing from the discrete to the continuous (Athanassoulis & Mamis, 2019; Mamis, 2020; Venturi et al., 2012a). Since the integrand on the right-hand side of Eq. (3.3) depends only on the specific values of the path functions $\chi(\bullet)$ and $\xi(\bullet)$, the infinite-dimensional integral in Eq. (3.2) is reduced to a $(n+m)-$dimensional one, with respect to marginal, $(n+m)-$point measure $\mathbf{P}_{X(\tau_1)\cdots X(\tau_m)\Xi(s_1)\cdots\Xi(s_n)}$

$$\mathbb{E}^{\theta}\left[G(\boldsymbol{X}(\theta);\boldsymbol{\Xi}(\theta))\right] = \int_{\mathbb{R}^{n+m}} G(\boldsymbol{w};\boldsymbol{z})\,\mathbf{P}_{X(\tau_1)\cdots X(\tau_m)\Xi(s_1)\cdots\Xi(s_n)}\big(d\boldsymbol{w}\times d\boldsymbol{z}\big). \tag{3.4}$$

Under the assumption that the point measure $\mathbf{P}_{X(\tau_1)\cdots X(\tau_m)\Xi(s_1)\cdots\Xi(s_n)}$ is smoothly distributed, i.e. the joint pdf $f_{X(\tau_1)\cdots X(\tau_m)\Xi(s_1)\cdots\Xi(s_n)}(\boldsymbol{w},\boldsymbol{z})$ exists, Eq. (3.4) can be written as

$$\mathbb{E}^{\theta}\left[G(\boldsymbol{X}(\theta);\boldsymbol{\Xi}(\theta))\right] = \int_{\mathbb{R}^{n+m}} G(\boldsymbol{w};\boldsymbol{z})\,f_{X(\tau_1)\cdots X(\tau_m)\Xi(s_1)\cdots\Xi(s_n)}(\boldsymbol{w},\boldsymbol{z})\,d\boldsymbol{w}\,d\boldsymbol{z}\,. \tag{3.5}$$

By considering Volterra's passing, in the opposite direction, Eq. (3.5) for $n, m\to\infty$ gives rise to

$$\begin{aligned}&\mathbb{E}^{\theta}_{\mathbf{P}_{X(\bullet)\Xi(\bullet)}}\left[\mathcal{G}[X(\bullet|_{t_0}^{t};\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\right] = \\ &\quad\quad = \int_{\mathscr{X}\times\mathscr{Z}} \mathcal{G}[\chi(\bullet);\xi(\bullet)]\,f_{X(\bullet)\Xi(\bullet)}[\chi(\bullet);\xi(\bullet)]\,d\chi(\bullet)\,d\xi(\bullet),\end{aligned} \tag{3.6}$$

which is an equivalent expression of Eq. (3.2), under the assumption that the infinite-dimensional joint response-excitation probability density functional $f_{X(\bullet)\Xi(\bullet)}[\chi(\bullet);\xi(\bullet)]$ exists, see e.g. (Fox, 1986).

Using, now, Eq (3.6) to express the average of the random delta function $\delta(x-X(t;\theta))$, we find

$$\mathbb{E}^{\theta}\left[\delta(x-X(t;\theta))\right] = \int_{\mathbb{R}}\delta(x-w)\,f_{X(t)}(w)\,dw, \tag{3.7}$$

which by employing the identity for the delta function is transformed into

$$f_{X(t)}(x) = \mathbb{E}^{\theta}\left[\delta(x-X(t;\theta))\right]. \tag{3.8}$$

Then, by differentiation both sides of the above equation we obtain

$$\frac{\partial f_{X(t)}(x)}{\partial t} = \frac{\partial}{\partial t}\mathbb{E}^{\theta}\left[\delta(x-X(t;\theta))\right] = \mathbb{E}^{\theta}\left[\frac{\partial\delta(x-X(t;\theta))}{\partial X(t;\theta)}\,\dot{X}(t;\theta)\right]. \tag{3.9}$$

The rightmost side of Eq. (3.9) is derived by interchanging differentiation and expectation operators and using chain rule in differentiation. Now, substituting Eq. (3.1a) into Eq. (3.3) results in

$$\frac{\partial f_{X(t)}(x)}{\partial t} = \mathbb{E}^{\theta}\left[\frac{\partial \delta(x - X(t;\theta))}{\partial X(t;\theta)} h(X(t;\theta))\right] + \kappa \mathbb{E}^{\theta}\left[\frac{\partial \delta(x - X(t;\theta))}{\partial X(t;\theta)} \Xi(t;\theta)\right]. \tag{3.10}$$

Each term on the right-hand side of the above expression is subsequently evaluated using ***the delta projection method's formalism***, as follows:

$$\mathbb{E}^{\theta}\left[\frac{\partial \delta(x - X(t;\theta))}{\partial X(t;\theta)} h(X(t;\theta))\right] = \int_{\mathbb{R}} \frac{\partial \delta(x-w)}{\partial w} h(w)\, f_{X(t)}(w)\, dw = \\ = -\frac{\partial}{\partial x}\left(h(x)\, f_{X(t)}(x)\right). \tag{3.11}$$

$$\mathbb{E}^{\theta}\left[\frac{\partial \delta(x - X(t;\theta))}{\partial X(t;\theta)} \Xi(t;\theta)\right] = \int_{\mathbb{R}^2} \frac{\partial \delta(x-w)}{\partial w} z\, f_{X(t)\Xi(t)}(w, z)\, dw dz = \\ = -\frac{\partial}{\partial x}\int_{\mathbb{R}} z\, f_{X(t)\Xi(t)}(x, z)\, dz = \\ = -\frac{\partial}{\partial x}\int_{\mathbb{R}^2} \delta(x-w)\, z\, f_{X(t)\Xi(t)}(w, z)\, dw dz = \\ = -\frac{\partial}{\partial x}\mathbb{E}^{\theta}\left[\delta(x - X(t;\theta))\, \Xi(t;\theta)\right]. \tag{3.12}$$

For Eqs. (3.11), (3.12) to be valid, the function $h(x)$ of RDE (3.1a) as well as pdfs $f_{X(t)}(x)$, $f_{X(t)\Xi(t)}(x, y)$ should possess continuous first derivatives. At this point, substituting Eqs. (3.11) and (3.12) into Eq. (3.10) provides us with

$$\frac{\partial f_{X(t)}(x)}{\partial t} + \frac{\partial}{\partial x}\left(h(x)\, f_{X(t)}(x)\right) = -\kappa \frac{\partial}{\partial x}\left(\mathbb{E}^{\theta}\left[\Xi(t;\theta)\, \delta(x - X(t;\theta))\right]\right). \tag{3.13}$$

Eq. (3.13) is called the ***stochastic Liouville equation*** (SLE) pertaining to RIVP (3.1a,b), a term introduced by Kubo in (Kubo, 1963). This equation has been derived by many authors in the past, using various approaches, e.g.(Cetto et al., 1984; Fox, 1986; P Hänggi, 1978; San Miguel & Sancho, 1980a). Moreover, the initial condition of SLE (3.5) is easily determined through the data of RIVP (3.1a,b) to

$$f_{X(t_0)}(x) = f_{X_0}(x). \tag{3.14}$$

At this point, it is readily seen that the SLE (3.13) is exact, yet non-closed due to the term $\mathcal{N}_{\Xi X} = \mathbb{E}^{\theta}\left[\Xi(t;\theta)\, \delta(x - X(t;\theta))\right]$, appearing on its right-hand side. Thus, in order to proceed and obtain a more workable alternative to SLE (3.13), the explicit dependence of the $\mathcal{N}_{\Xi X}$ over the excitation $\Xi(t;\theta)$ must be eliminated.

Transformed SLE. Before we continue with the treatment of SLE (3.13), it must be noted that the response $X(t;\theta)$ is regarded, through the solution of RIVP (3.1), as a function-functional

(FFℓ) on the initial value $X_0(\theta)$ and the time history of the excitation $\Xi(\bullet;\theta)$, from the initial time t_0 to the current time t; a perception which also emphasizes the non-Markovian character of the response. The notation $X(t;\theta) = X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]$ is used subsequently, whenever it is needed to remember the dependence of the response on $X_0(\theta)$ and $\Xi(\bullet;\theta)$. The above discussion makes clear that SLE (3.13) is not only non-closed but also nonlocal by virtue of the dependence of $\mathcal{N}_{\Xi X}$ on the whole history of the excitation.

Returning to the treatment of SLE (3.13), the averaged term, $\mathcal{N}_{\Xi X}$, can equivalently be written as

$$\begin{aligned}\mathcal{N}_{\Xi X} &= \mathbb{E}^{\theta}\left[\Xi(t;\theta)\,\delta(x - X(t;\theta))\right] = \\ &= \mathbb{E}^{\theta}\left[\Xi(t;\theta)\,\delta\Big(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\Big)\right],\end{aligned} \tag{3.15}$$

which is also the appropriate form for the application of ***Extension I of the NF theorem***, Eq. (2.18) as presented in sec. 2.2 under the comprehension that the random delta function $\delta(x - X(t;\theta)) = \delta\Big(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\Big)$ is considered a FFℓ like $\mathcal{F} = \mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]$. Thus, applying the NF theorem to the non-local term $\mathcal{N}_{\Xi X}$ yields

$$\begin{aligned}&\mathbb{E}^{\theta}\left[\Xi(t;\theta)\,\delta\Big(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\Big)\right] = \\ &= m_{\Xi}(t)\,\mathbb{E}^{\theta}\left[\delta\Big(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\Big)\right] + \\ &\quad + C_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\left[\frac{\partial\delta\Big(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\Big)}{\partial X_0(\theta)}\right] + \\ &\quad + \int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\frac{\delta\delta\Big(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\Big)}{\delta\Xi(\tau;\theta)}\right]d\tau\,.\end{aligned} \tag{3.16}$$

The averages in the last two terms of Eq. (3.16) can be further evaluated by making use of the ***chain rule*** for the derivatives of the random delta function as

$$\begin{aligned}&\mathbb{E}^{\theta}\left[\Xi(t;\theta)\,\delta\big(x - X(t;\theta)\big)\right] = \\ &= m_{\Xi}(t)\,\mathbb{E}^{\theta}\left[\delta\Big(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\Big)\right] + \\ &\quad + C_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\left[\frac{\partial\delta\Big(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\Big)}{\partial X(t;\theta)}\,V_{X_0}(t;\theta)\right] + \\ &\quad + \int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\frac{\delta\delta\Big(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\Big)}{\delta X(t;\theta)}\,V_{\Xi(\tau)}(t;\theta)\right]d\tau\,,\end{aligned} \tag{3.17}$$

where $V_{X_0}(t;\theta)$, $V_{\Xi(\tau)}(t;\theta)$ are defined as the derivatives of the response with respect to initial value and excitation, respectively:

$$V_{X_0}(t;\theta) = \frac{\partial X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]}{\partial X_0(\theta)}, \tag{3.18a}$$

$$V_{\Xi(\tau)}(t;\theta) = \frac{\delta X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]}{\delta \Xi(\tau;\theta)}. \tag{3.18b}$$

and are collectively called the ***variational derivatives of the response***. Despite the response being the solution to a nonlinear RDE, its variational derivatives are easily calculated by formulating and solving the corresponding variational equations as it is performed in the subsequent paragraph 3.1.2.

Heretofore, we have considered the response as an FFℓ with Gaussian arguments, $X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]$, in order to be able to employ the NF theorem and the chain rule. We shall now revert to considering the response as a random function per se, $X(t;\theta)$; doing so, simplifies the notation and allows us to carry out some simple manipulations of the delta projection method for the averaged terms of Eq. (3.17). Thus, we find

$$\begin{aligned} \mathbb{E}^{\theta}\left[\Xi(t;\theta)\,\delta\big(x - X(t;\theta)\big)\right] &= m_{\Xi}(t)\; f_{X(t)}(x) - \\ &- C_{X_0\Xi}(t)\,\frac{\partial}{\partial x}\,\mathbb{E}^{\theta}\left[\delta\big(x - X(t;\theta)\big)\,V_{X_0}(t;\theta)\right] - \\ &- \frac{\partial}{\partial x}\int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\delta\big(x - X(t;\theta)\big)\,V_{\Xi(\tau)}(t;\theta)\right]d\tau. \end{aligned} \tag{3.19}$$

Last, combining Eqs. (3.13) and (3.19) results in the following ***transformed SLE for the one-time response pdf*** $f_{X(t)}(x)$:

$$\begin{aligned} \frac{\partial f_{X(t)}(x)}{\partial t} + \frac{\partial}{\partial x}\left[\big(h(x) + \kappa m_{\Xi}(t)\big)\; f_{X(t)}(x)\right] &= \\ = \kappa\, C_{X_0\Xi}(t)\,\frac{\partial^2}{\partial x^2}\,\mathbb{E}^{\theta}\left[\delta\big(x - X(t;\theta)\big)\,V_{X_0}(t;\theta)\right] &+ \\ + \kappa\frac{\partial^2}{\partial x^2}\int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\delta\big(x - X(t;\theta)\big)\,V_{\Xi(\tau)}(t;\theta)\right]d\tau&. \end{aligned} \tag{3.20}$$

By comparing the transformed SLE (3.20) to its previous form, Eq. (3.13), we observe that the use of the NF theorem results in: **i)** an augmented drift term, which can be identified as the right-hand side of RDE (3.1) with excitation replaced by its mean value, **ii)** the appearance of second order $x-$derivatives in the right-hand side of the equation, and **iii)** the appearance of the averages of the random delta function multiplied by the variational derivatives of the response with respect to initial value and excitation.

3.1.2 Formulation and solution of the variational equations

In the present paragraph, we formulate and solve the initial value problems governing the variational derivates $V_{X_0}(t;\theta)$ and $V_{\Xi(\tau)}(t;\theta)$. Before we begin, it must be noted that all manipulations performed here regarding the solution $X(t;\theta)$ of RIVP (3.1) are of purely analytic character. Hence, they can be executed path-wise, i.e. for every value of the stochastic argument θ separately. On a practical level, this equals to discarding the stochastic argument θ, and working with the deterministic initial value problem

$$\dot{X}(t) = h(X(t)) + \kappa\Xi(t), \qquad X(t_0) = X_0. \tag{3.21a,b}$$

Assuming that all appropriate conditions ensuring the existence and the uniqueness of solution of IVP (3.21a,b) hold true, we are interested in the dependence of the solution on the initial value X_0 and the excitation function $\Xi(\bullet|_{t_0}^{t})$. Thus, the solution is considered as a function-functional on initial value X_0 and excitation $\Xi(\bullet|_{t_0}^{t})$, denoted by $X[X_0;\Xi(\bullet|_{t_0}^{t})]$.

Being parameters of the solution, we may calculate the derivatives of the solution with respect to X_0 and $\Xi(\bullet|_{t_0}^{t};\theta)$, by formulating and solving the corresponding variational equations along the solution, see e.g. (Amann, 1990; Anosov & Arnold, 1987; Grigorian, 2008).

(a) *Variational IVP with respect to initial value and its solution*

By applying the differential operator $\partial\bullet/\partial X_0$ on both sides of Eqs. (3.21a,b) and under the assumption that, for a given θ, excitation $\Xi(t)$ is not functionally dependent on X_0, we obtain

$$\dot{V}_{X_0}(t) = h'(X(t;\theta))\,V_{X_0}(t), \qquad V_{X_0}(t_0) = 1, \tag{3.22a,b}$$

where the prime denotes the first derivative of $h(\bullet)$ with respect to its argument. In turn, IVP (3.22a,b) is recognized as a linear ordinary differential equation for the sought-for variational derivative which can easily be solved as

$$V_{X_0}(t) = \exp\left(\int_{t_0}^{t} h'\big(X(u)\big)\,du\right). \tag{3.23}$$

Returning to the notation of RIVP (3.1a,b), Eq. (3.23) is written as

$$V_{X_0}(t;\theta) = \exp\left(\int_{t_0}^{t} h'\big(X(u;\theta)\big)\,du\right). \tag{3.24}$$

(b) *Variational IVP with respect to excitation and its solution*

Working in similar fashion as above, we apply the differential operator $\delta\bullet/\delta\Xi(\tau)$, $\tau\in[t_0,t]$ on both sides of Eqs. (3.21a,b). For this, we assume that Volterra derivative has analogous properties to the usual partial derivative, i.e. is linear, can be interchanged with the temporal derivative and obeys the usual chain rule of differentiation. Taking account of the aforementioned assumptions, application of the Volterra differential operator on Eqs. (3.21a,b) yields

$$\dot{V}_{\Xi(\tau)}(t) = h'(X(t))\, V_{\Xi(\tau)}(t) + \kappa \frac{\delta \Xi(t)}{\delta \Xi(\tau)}, \tag{3.25}$$

along with the initial condition

$$V_{\Xi(\tau)}(t_0) = 0. \tag{3.26a}$$

Moreover, since for a given path function $\Xi(\bullet)$, the value of $\Xi(t)$ does not functionally depend on the value $\Xi(\tau)$, for $\tau \neq t$, Eq. (3.25) can be written as

$$\dot{V}_{\Xi(\tau)}(t) = h'(X(t))\, V_{\Xi(\tau)}(t) + \kappa\, \delta(t-\tau), \tag{3.26b}$$

where $\delta(t-\tau)$ denotes the Dirac delta function. Since, by causality, any perturbation $\delta\Xi(\tau)$, in excitation at time τ, cannot result in a perturbation $\delta X(t)$ for $t < \tau$, we have $V_{\Xi(\tau)}(t) = 0$ for $t < \tau$. By integrating, now, Eq. (3.26b) over $[\tau - \varepsilon, t]$, for small $\varepsilon > 0$, and taking the limit $\varepsilon \to 0$, we obtain

$$V_{\Xi(\tau)}(t) = \int_{\tau}^{t} h'(X(u))\, V_{\Xi(\tau)}(u)\, \mathrm{du} + \kappa. \tag{3.27}$$

Eq. (3.27) is a Volterra integral equation of the second kind, equivalent to the linear IVP (Polyanin & Manzhirov, 2008)

$$\dot{V}_{\Xi(\tau)}(t) = h'(X(t))\, V_{\Xi(\tau)}(t), \qquad V_{\Xi(\tau)}(\tau) = \kappa. \tag{3.28a,b}$$

Thus, IVP (3.28a,b) is linear ODE which can easily be solved as

$$V_{\Xi(\tau)}(t) = \kappa \exp\left(\int_{\tau}^{t} h'\big(X(u)\big)\, du \right), \tag{3.29}$$

which is the required variational derivative with respect to excitation.

Last, returning to the notation of the RIVP gives rise to

$$V_{\Xi(\tau)}(t;\theta) = \kappa \exp\left(\int_{\tau}^{t} h'\big(X(u;\theta)\big)\, du \right). \tag{3.30}$$

3.1.3 One-time response pdf evolution equations

Having obtained expressions (3.24) and (3.30) for the variational derivatives $V_{X_0}(t;\theta)$ and $V_{\Xi(\tau)}(t;\theta)$ respectively, we can readily identify them as functionals of the response. In addition, in order to simplify the notation, we set

$$\mathcal{I}_{h'}[X(\bullet|_{\tau}^{t};\theta)] = \int_{\tau}^{t} h'\big(X(u;\theta)\big)\, du. \tag{3.31}$$

Subsequently, substituting Eqs. (3.24) and (3.30) into Eq. (3.20) and using the above notation, we obtain the following ***exact, non-closed one-time response pdf evolution equation***:

$$\frac{\partial f_{X(t)}(x)}{\partial t} + \frac{\partial}{\partial x}\Big[\big(h(x)+\kappa m_{\Xi}(t)\big)\, f_{X(t)}(x)\Big] =$$
$$= \kappa\, C_{X_0 \Xi}(t)\, \frac{\partial^2}{\partial x^2}\, \mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big)\exp\Big(\mathcal{I}_{h'}[X(\bullet|_{t_0}^{t};\theta)]\Big)\Big] + \qquad (3.32)$$
$$+ \kappa^2 \frac{\partial^2}{\partial x^2}\int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\, \mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big)\exp\Big(\mathcal{I}_{h'}[X(\bullet|_{\tau}^{t};\theta)]\Big)\Big] d\tau .$$

Under this notation, the fact that the derived SLE is non-closed and nonlocal becomes apparent by the presence of the two nonlocal terms inside the averages, carrying the time history of the response $X(t;\theta)$, that multiply the random delta function. This form, despite being exact, is of little practical use since its analytical, numerical solution is virtually impossible, from a computational cost point of view. As such, in this paragraph, we are going to employ and concisely describe an approximation scheme, similar to the one introduced in (Mamis et al., 2019), in order to obtain a closed, computable, albeit approximate alternative to SLE (3.32).

First, a decomposition of the effect of nonlinearity, $h'\big(X(u;\theta)\big)$, is performed as follows:

$$\exp\left(\int_{\tau}^{t} h'\big(X(u;\theta)\big)\, du\right) = \exp\left(\int_{\tau}^{t} R_{h'}(u)\, du\right)\cdot \exp\left(\int_{\tau}^{t} \varphi_{h'}\big(X(u;\theta)\big)\, du\right) \qquad (3.33)$$

in which $R_{h'}(u) = \mathbb{E}^{\theta}\big[h'\big(X(u;\theta)\big)\big]$ is the mean effect and $\varphi_{h'} \equiv \varphi_{h'}\big(X(u;\theta)\,;\,R_{h'}(u)\big) = h'\big(X(u;\theta)\big) - R_{h'}(u)$ is the fluctuation. Afterwards, a current-time approximation for the fluctuation integral is utilized, which is efficient under the assumption that the fluctuation is small:

$$\exp\left(\int_{\tau}^{t} h'\big(X(u;\theta)\big)\, du\right) \cong \exp\left(\int_{\tau}^{t} R_{h'}(u)\, du\right)\cdot \exp\Big(\varphi_{h'}\big(X(t;\theta)\big)(t-\tau)\Big). \qquad (3.34)$$

Last, we take the Taylor expansion of the fluctuation exponential, truncated at M-th term:

$$\exp\left(\int_{\tau}^{t} h'\big(X(u;\theta)\big)\, du\right) \cong \exp\left(\int_{\tau}^{t} R_{h'}(u)\, du\right)\cdot \sum_{m=0}^{M} \frac{\varphi_{h'}^{m}\big(X(t;\theta)\big)}{m!}(t-\tau)^m . \qquad (3.35)$$

Substituting, the above approximation scheme into SLE (3.32) results in the following ***closed, approximate, one-time pdf evolution equation*** corresponding to RIVP (3.1a,b):

$$\frac{\partial f_{X(t)}(x)}{\partial t} + \frac{\partial}{\partial x}\Big[\big(h(x)+\kappa m_{\Xi}(t)\big)\, f_{X(t)}(x)\Big] =$$
$$= \frac{\partial^2}{\partial x^2}\left\{\left[\sum_{m=0}^{M}\frac{1}{m!} D_m^{\text{eff}}\big[R_{h'}(\bullet), t\big]\, \varphi_{h'}^{m}\big(x\,;\,R_{h'}(t)\big)\right] f_{X(t)}(x)\right\}, \qquad (3.36)$$

where

$$D_m^{\text{eff}}\left[R_{h'}(\bullet),t\right]=\kappa\exp\left(\int_{t_0}^{t}R_{h'}(u)\,du\right)C_{X_0\Xi}(t)\,(t-t_0)^m+\\+\kappa^2\int_{t_0}^{t}\exp\left(\int_{\tau}^{t}R_{h'}(u)\,du\right)C_{\Xi\Xi}(t,\tau)\,(t-\tau)^m\,d\tau\,. \tag{3.37}$$

Through the coefficients D_m^{eff}, called the ***generalized effective noise intensities***, and the terms $\varphi_{h'}^m$, the pdf equation retains a trackable amount of nonlocality (in time) and nonlinearity, reflecting the non-Markovian character of the response.

Remark 3.1. Although the assumption for the fluctuation being small seems somewhat restrictive, it has been shown (Mamis et al., 2019), that in present case, this approximation scheme is more effective compared to other methods (e.g. Fox' s approximation (Fox, 1986), small correlation time approach (Sancho et al., 1982), Hänggi's ansatz (P. Hänggi & Jung, 1995)) even for large correlation times and noise intensities. A more thorough description of the approximation schemes mentioned as well as the potency of the one presented herein is given in (Mamis, 2020; Mamis et al., 2019)

3.1.4 Exact response pdf for a linear, additively excited RDE

By considering $h(x)=\eta x$, with $\eta<0$ for stability purposes, and $q(x)=\kappa=$ constant, RIVP (3.1a,b) becomes linear and additively excited

$$\dot{X}(t;\theta)=\eta X(t;\theta)+\kappa\Xi(t;\theta),\qquad X(t_0;\theta)=X_0(\theta). \tag{3.38a,b}$$

In this case, the variational derivatives of RIVP (3.38) are independent from the response and the excitation and as such, formulae (3.24), (3.30) can be explicitly calculated as

$$V_{X_0}(t;\theta)=e^{\eta(t-t_0)},\qquad V_{\Xi(s)}(t;\theta)=\kappa e^{\eta(t-s)} \tag{3.39a,b}$$

Substituting the above expressions into SLE (3.20) results in the following ***pdf evolution equation***:

$$\frac{\partial f_{X(t)}(x)}{\partial t}+\frac{\partial}{\partial x}\left[\left(\eta x+\kappa m_{\Xi}(t)\right)f_{X(t)}(x)\right]=D^{\text{eff}}(t)\,\frac{\partial^2 f_{X(t)}(x)}{\partial x^2}\,, \tag{3.40}$$

where the effective noise intensity, $D^{\text{eff}}(t)$, is given by

$$D^{\text{eff}}(t)=\kappa\,e^{\eta(t-t_0)}\,C_{X_0\Xi}(t)+\kappa^2\int_{t_0}^{t}e^{\eta(t-\tau)}\,C_{\Xi\Xi}(t,\tau)\,d\tau\,. \tag{3.41}$$

Comparing pdf evolution equation Eq. (3.40) to SLE (3.20), it is easy to observe that the former is not only ***exact*** but also ***closed*** and thus, does not require the implementation of any approximation scheme for its solution. In fact, in the present paragraph, this is achieved by making use of the ***Fourier transform*** for Eq. (3.40) as well as the supplementary Gaussian initial condition

$$f_{X(t_0)}(x) = f_{X_0}(x) = \frac{1}{\sqrt{2\pi\sigma_{X_0}^2}} \exp\left[-\frac{1}{2} \frac{(x - m_{X_0})^2}{\sigma_{X_0}^2} \right], \tag{3.42}$$

in which m_{X_0}, $\sigma_{X_0}^2$ are the initial mean value and initial variance, respectively.

Before we proceed with the solution of RIVP (3.40) and (3.42), a well-known result must be designated; the response process of any linear system, with Gaussian initial distribution, to an additive Gaussian excitation (either colored or white) is also a Gaussian process. Furthermore, the mean value $m_X(t)$ and variance $\sigma_X^2(t)$ of the response $X(t;\theta)$ can be determined as the solutions to the respective moment equations, derived directly from RIVP (3.38a,b) (see e.g. (Sun, 2006) or (Athanassoulis et al., 2015)). This task is performed in Appendix B, in which Eqs. (B.3) and (B.25) read

$$m_X(t) = m_{X_0} e^{\eta(t-t_0)} + \kappa \int_{t_0}^{t} m_\Xi(\tau)\, e^{\eta(t-\tau)}\, d\tau, \tag{3.43}$$

and

$$\sigma_X^2(t) = \sigma_{X_0}^2\, e^{2\eta(t-t_0)} + 2\kappa \int_{t_0}^{t} C_{X\Xi}(\tau,\tau)\, e^{2\eta(t-\tau)}\, d\tau. \tag{3.44}$$

Remark 3.2: Connection between effective noise intensity and cross-correlation. Eq. (B.22) for response-excitation cross-covariance reads

$$C_{X\Xi}(t,s) = C_{X_0\Xi}(s)\, e^{\eta(t-t_0)} + \kappa \int_{t_0}^{t} C_{\Xi\Xi}(\tau,s)\, e^{\eta(t-\tau)}\, d\tau. \tag{3.45}$$

Comparing Eq. (3.45) to Eq. (3.41), effective noise intensity can be expressed in terms of the one-time response-excitation cross-covariance as

$$D^{\text{eff}}(t) = \kappa C_{X\Xi}(t,t). \tag{3.46}$$

Under Eq. (3.46), Eq. (3.44) for the variance of the response is expressed equivalently as

$$\sigma_X^2(t) = \sigma_{X_0}^2\, e^{2\eta(t-t_0)} + 2 \int_{t_0}^{t} D^{\text{eff}}(\tau)\, e^{2\eta(t-\tau)}\, d\tau. \tag{3.47}$$

Solution of Eq. (3.45) using Fourier transform. Employing the Fourier transform, $\varphi_{X(t)}(y) = \int_{\mathbb{R}} e^{iyx} f_{X(t)}(x)\, dx$ for Eq. (3.45) leads to the following equation of first partial derivatives for the characteristic function $\varphi_{X(t)}(y)$

$$\frac{\partial \varphi_{X(t)}(y)}{\partial t} = \eta y \frac{\partial \varphi_{X(t)}(y)}{\partial y} + \left(i\kappa m_\Xi(t)\, y - D^{\text{eff}}(t)\, y^2 \right) \varphi_{X(t)}(y), \tag{3.48a}$$

supplemented with the transformed initial condition (3.42)

$$\varphi_{X(t_0)}(y) = \exp\left(i m_{X_0}\, y - \frac{1}{2} \sigma_{X_0}^2\, y^2 \right). \tag{3.48b}$$

Initial value problem (3.49a,b) is solved (Polyanin, Zaitsev, & Moussiaux, 2001) sec. 3.1, by first determining the characteristic curve $w(y,t) = ye^{\eta t}$ as the solution of the characteristic equation $dt = -dy/(\eta y)$. Then, we seek a solution of the form $g(w)\exp\left(\int_{t_0}^{t} h(w,t)\,dt\right)$, where $g(w)$ is a function of the characteristic curve, to be defined by the initial condition (3.53b), and $h(w,t) = i\kappa m_{\Xi}(t)\, we^{-\eta t} - D^{\text{eff}}(t)\, w^2 e^{-2\eta t}$, that is the coefficient multiplying $\varphi_{X(t)}(y)$ in Eq. (3.48a), rewritten in terms of w, t. Finally, by returning to the original variables y, t, we obtain the solution

$$\begin{aligned}\varphi_{X(t)}(y) = \exp&\left[i \left(m_{X_0} e^{\eta(t-t_0)} + \kappa \int_{t_0}^{t} m_{\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau \right) y \right] \times \\ &\times \exp\left[-\frac{1}{2} \left(\sigma_{X_0}^2 e^{2\eta(t-t_0)} + 2\int_{t_0}^{t} D^{\text{eff}}(\tau)\, e^{2\eta(t-\tau)}\, d\tau \right) y^2 \right].\end{aligned} \tag{3.49}$$

Note that this methodology for the solution of Eq. (3.49) can also been found in paragraph 4.4.4, albeit for a more convoluted case. Employing the inverse Fourier transform for Eq. (3.49), and utilizing Eqs. (3.43), (3.44) for $m_X(t)$ and $\sigma_X^2(t)$, results in

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi\sigma_X^2(t)}} \exp\left[-\frac{1}{2} \frac{\left(x - m_X(t)\right)^2}{\sigma_X^2(t)} \right], \tag{3.50}$$

which is the expected Gaussian distribution. This result constitutes the verification of the response pdf evolution Eq. (3.36).

Remark 3.3. The ***uniqueness of Gaussian solution*** (3.49) is ensured by the injectivity of Fourier transform for absolutely integrable functions, and the uniqueness of solution for transformed problem (3.48a,b), see (Polyanin et al., 2001), sec. 10.1.2. What is more, the uniqueness of solution for Eq. (3.50) is also proven directly, without resorting to Fourier transform, in (Mamis, 2020, Appendix E).

3.2 The case of an RDE subject to both additive and multiplicative excitation

In this section, we are going to examine the applicability and versatility of the developed methodology by employing it in a different case. More specifically, we consider the following additively and multiplicatively excited RDE:

$$\dot{X}(t;\theta) = h(X(t;\theta)) + q(X(t;\theta))\Xi_1(t;\theta) + \kappa\, \Xi_0(t;\theta), \tag{3.51a}$$

along with the initial condition

$$X(t_0;\theta) = a. \tag{3.51b}$$

Note that in the present case, initial condition a is considered a ***scalar*** and not a ***random variable***. This is done for reasons of brevity since the ensuing, apprehended expressions are quite lengthy. Regardless, all of the presented results can be comprehensively derived even in the case of a random initial condition. Let us once more state that the overdot denotes differentiation with respect to time, $h(x)$ and $q(x)$ are deterministic continuous functions modelling the nonlinearities, and κ is a constant. Excitations $\Xi_0(t;\theta)$ and $\Xi_1(t;\theta)$ are considered correlated and jointly Gaussian with non-zero mean values $m_{\Xi_0}(t)$, $m_{\Xi_1}(t)$, autocovariances $C_{\Xi_0\Xi_0}(t,s)$, $C_{\Xi_1\Xi_1}(t,s)$ and cross-covariance $C_{\Xi_0\Xi_1}(t,s)$.

3.2.1 The corresponding stochastic Liouville equation

Commencing, as in section 3.1, from the delta projection method, we write

$$f_{X(t)}(x) = \mathbb{E}^{\theta}\left[\delta\left(x - X(t;\theta)\right)\right]. \tag{3.52}$$

Then, by differentiating both sides of Eq. (3.52) with respect to time t and using Eq. (3.51a), we obtain

$$\begin{aligned} \frac{\partial f_{X(t)}(x)}{\partial t} = \mathbb{E}^{\theta}\left[\frac{\partial\delta\left(x - X(t;\theta)\right)}{\partial X(t;\theta)}\, h(X(t;\theta))\right] + \\ + \mathbb{E}^{\theta}\left[\frac{\partial\delta\left(x - X(t;\theta)\right)}{\partial X(t;\theta)}\, q(X(t;\theta))\, \Xi_1(t;\theta)\right] + \\ + \kappa\, \mathbb{E}^{\theta}\left[\frac{\partial\delta\left(x - X(t;\theta)\right)}{\partial X(t;\theta)}\, \Xi_0(t;\theta)\right]. \end{aligned} \tag{3.53}$$

Each averaged term on the right-hand side of Eq. (3.53) can be further evaluated by making use of the ***delta projection method's formalism***. The first averaged term appearing has already been evaluated by Eq. (3.11). For the other two terms, we work accordingly, as follows:

$$\begin{aligned} &\mathbb{E}^{\theta}\left[\frac{\partial\delta\left(x - X(t;\theta)\right)}{\partial X(t;\theta)}\, q(X(t;\theta))\, \Xi_1(t;\theta)\right] = \\ &= \int_{\mathbb{R}^2} \frac{\partial\delta(x-w)}{\partial w}\, q(w)\, z_1\, f_{X(t)\Xi_1(t)}(w, z_1)\, dw dz_1 = \\ &= -\frac{\partial}{\partial x}\, q(x) \int_{\mathbb{R}} z_1\, f_{X(t)\Xi_1(t)}(x, z_1)\, dx dz_1 = \\ &= -\frac{\partial}{\partial x}\left(q(x) \int_{\mathbb{R}^2} \delta(x-w) z_1\, f_{X(t)\Xi_1(t)}(w, z_1)\, dx dz_1\right) = \\ &= -\frac{\partial}{\partial x}\left(q(x)\, \mathbb{E}^{\theta}\left[\delta\left(x - X(t;\theta)\right)\Xi_1(t;\theta)\right]\right), \end{aligned} \tag{3.54}$$

$$\begin{aligned} &\mathbb{E}^{\theta}\left[\frac{\partial\delta(x-X(t;\theta))}{\partial X(t;\theta)}\, \Xi_0(t;\theta)\right] = \int_{\mathbb{R}^2} \frac{\partial\delta(x-w)}{\partial w}\, z_0\, f_{X(t)\Xi_0(t)}(w, z_0)\, dw dz_0 = \\ &= -\frac{\partial}{\partial x}\int_{\mathbb{R}} z_0\, f_{X(t)\Xi_0(t)}(x, z_0)\, dz_0 = \\ &= -\frac{\partial}{\partial x}\int_{\mathbb{R}^2} \delta(x-w)\, z_0\, f_{X(t)\Xi_0(t)}(w, z_0)\, dw dz_0 = \end{aligned}$$

$$= -\frac{\partial}{\partial x}\left(\mathbb{E}^{\theta}\left[\delta\big(x - X(t;\theta)\big)\,\Xi_0(t;\theta)\right]\right). \tag{3.55}$$

Substituting, now, the above formulae into Eq. (3.53), transforms the latter into

$$\frac{\partial f_{X(t)}(x)}{\partial t} + \frac{\partial}{\partial x}\Big(h(x)\,f_{X(t)}(x)\Big) =$$
$$= -\frac{\partial}{\partial x}\left(q(x)\mathbb{E}^{\theta}\left[\delta(x-X(t;\theta))\,\Xi_1(t;\theta)\right]\right) - \tag{3.56}$$
$$-\kappa\frac{\partial}{\partial x}\left(\mathbb{E}^{\theta}\left[\delta(x-X(t;\theta))\,\Xi_0(t;\theta)\right]\right).$$

Eq. (3.56) is the ***one-time response SLE*** corresponding to RDE (3.41a). As was the case for SLE (3.13), an additional assumption regarding the smoothness of functions $h(x)$ and $q(x)$ of RIVP (3.51a,b) must be introduced; namely that these two functions have at least continuous first derivatives. In addition, the initial condition of the response (3.51b) specifies the initial condition of SLE (3.56) into

$$f_{X(t_0)}(x) = \delta(x-\alpha). \tag{3.57}$$

Note that in the special case in which $\kappa = 0$, the adjusted SLE (3.57) coincides with the one derived by many authors, using various approaches; see e.g. (Cetto et al., 1984; Fox, 1986; P Hänggi, 1978; Sancho & San Miguel, 1980; Sancho et al., 1982). Moreover, SLE (3.56) is non-closed due to the terms $\mathcal{N}_{\Xi_0 X} = \mathbb{E}^{\theta}\left[\Xi_0(t;\theta)\,\delta(x-X(t;\theta))\right]$, $\mathcal{N}_{\Xi_1 X} = \mathbb{E}^{\theta}\left[\Xi_1(t;\theta)\,\delta(x-X(t;\theta))\right]$ and thus, a correlation splitting, similar to the one carried out for SLE (3.13), must be performed.

Transformed SLE. At this point, a similar discussion to the one carried out in 3.1.1, regarding the "nature" of response $X(t;\theta)$ must be conducted. More specifically, the response, $X(t;\theta)$, is regarded through RIVP (3.51a,b) as a functional over the time-history of both excitations $\Xi_0(\bullet;\theta)$, $\Xi_1(\bullet;\theta)$, from the initial time t_0 to the current time t; written according to the familiar notation as $X(t;\theta) = X[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)]$. Note that this time, there is no dependence, as a function, on initial value since the latter is a scalar. Using the familiar notation, the averaged terms, $\mathcal{N}_{\Xi_i X}$, with $i = 0, 1$, can equivalently be written as

$$\begin{aligned}\mathcal{N}_{\Xi_i X} &= \mathbb{E}^{\theta}\left[\Xi_i(t;\theta)\,\delta(x-X(t;\theta))\right] = \\ &= \mathbb{E}^{\theta}\left[\Xi_i(t;\theta)\,\delta\Big(x - X[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)]\Big)\right],\end{aligned} \tag{3.58}$$

which is the appropriate form for the implementation of **Extension IV of the NF theorem**, Eq. (2.23) under the understanding that the random delta function $\delta(x-X(t;\theta)) = \delta(x-X[\cdots]) = \delta\Big(x - X[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)]\Big)$ is considered as a random functional $\mathcal{F}[\cdots] = \mathcal{F}[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)]$.

Thus, by applying the said extended NF theorem to the nonlocal term $\mathcal{N}_{\Xi_1 X}$, we obtain

$$\mathbb{E}^{\theta}\Big[\Xi_1(t;\theta)]\mathcal{F}[\cdots]\Big] = m_{\Xi_1}(t)\mathbb{E}^{\theta}\big[X[\cdots]\big] + \\ + \int_{t_0}^{t} C_{\Xi_0\Xi_1}(\tau_0, t)\, \mathbb{E}^{\theta}\left[\frac{\delta X[\cdots]}{\delta \Xi_0(\tau_0;\theta)}\right] d\tau_0 + \\ + \int_{t_0}^{t} C_{\Xi_1\Xi_1}(t, \tau_1)\, \mathbb{E}^{\theta}\left[\frac{\delta X[\cdots]}{\delta \Xi_1(\tau_1;\theta)}\right] d\tau_1, \tag{3.59}$$

which can be further evaluated by employing the chain rule for the Volterra and the properties of the delta projection method as

$$\mathbb{E}^{\theta}\Big[\Xi_1(t;\theta)]\mathcal{F}[\cdots]\Big] = m_{\Xi_1}(t)\, f_{X(t)}(x) - \\ - \frac{\partial}{\partial x}\int_{t_0}^{t} C_{\Xi_0\Xi_1}(\tau_0, t)\, \mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big) V_{\Xi_0(\tau_0)}(t;\theta)\Big] d\tau_0 - \\ - \frac{\partial}{\partial x}\int_{t_0}^{t} C_{\Xi_1\Xi_1}(\tau_1, t)\, \mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big) V_{\Xi_1(\tau_1)}(t;\theta)\Big] d\tau_1, \tag{3.60}$$

where $V_{\Xi_i(\tau_i)}(t;\theta)$ are the variational derivatives of the response with respect to excitation at time instance τ_i, defined by

$$V_{\Xi_i(\tau_i)}(t;\theta) = \frac{\delta X[\Xi_0(\bullet|_{t_0}^{t};\theta);\Xi_1(\bullet|_{t_0}^{t};\theta)]}{\delta \Xi_i(\tau_i;\theta)}, \quad i = 0, 1. \tag{3.61}$$

These variational derivatives can be explicitly calculated by formulating and solving their corresponding initial value problems, as presented in paragraph 3.1.2. Thus, solving the said IVPs results in

$$V_{\Xi_0(\tau_0)}(t;\theta) = \kappa \exp\left(\int_{\tau_0}^{t}\Big[h'\big(\mathrm{X}(u;\theta)\big) + q'\big(\mathrm{X}(u;\theta)\big)\Xi_1(u;\theta)\Big] du\right), \tag{3.62}$$

$$V_{\Xi_1(\tau_1)}(t;\theta) = q\big(\mathrm{X}(\tau_1;\theta)\big)\exp\left(\int_{\tau_1}^{t}\Big[h'\big(\mathrm{X}(u;\theta)\big) + q'\big(\mathrm{X}(u;\theta)\big)\Xi_1(u;\theta)\Big] du\right). \tag{3.63}$$

Finally, implementing the extended NF theorem for the nonlocal term $\mathcal{N}_{\Xi_0 X}$ and substituting the evaluated non-local terms in Eq. (3.56) results in the following ***transformed SLE for the one-time response pdf***:

$$\frac{\partial f_{X(t)}(x)}{\partial t} + \frac{\partial}{\partial x}\Big[\Big(h(x) + \kappa m_{\Xi_0}(t) + q(x) m_{\Xi_1}(t)\Big) f_{X(t)}(x)\Big] =$$
$$= \kappa \frac{\partial^2}{\partial x^2}\int_{t_0}^{t} C_{\Xi_0\Xi_0}(\tau_0, t)\mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big) V_{\Xi_0(\tau_0)}(t;\theta)\Big] d\tau_0 +$$
$$+ \kappa \frac{\partial^2}{\partial x^2}\int_{t_0}^{t} C_{\Xi_0\Xi_1}(t, \tau_1)\mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big) V_{\Xi_1(\tau_1)}(t;\theta)\Big] d\tau_1 + \tag{3.64}$$
$$+ \frac{\partial}{\partial x}\left(q(x)\frac{\partial}{\partial x}\int_{t_0}^{t} C_{\Xi_0\Xi_1}(\tau_0, t)\mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big) V_{\Xi_0(\tau_0)}(t;\theta)\Big] d\tau_0 \right) +$$
$$+ \frac{\partial}{\partial x}\left(q(x)\frac{\partial}{\partial x}\int_{t_0}^{t} C_{\Xi_1\Xi_1}(t, \tau_1)\mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big) V_{\Xi_1(\tau_1)}(t;\theta)\Big] d\tau_1 \right).$$

The variational derivatives inside the averaged terms of the above expression constitute SLE (3.64) non-closed and nonlocal and as such, in this form, its solution seems as a rather arduous task. Thus, a closure of SLE (3.64) is introduced in the subsequent paragraph 3.2.2 in order to obtain an approximate, yet computable alternative of the above expression.

3.2.2 One-time response pdf evolution equations

Introducing, now, the following notation for the variational derivatives:

$$J_{h,q}^{\tau_i}[X(\bullet|_{\tau_i}^{t};\theta)] = \int_{\tau_i}^{t}\Big[h'\big(\mathrm{X}(u;\theta)\big) + q'\big(\mathrm{X}(u;\theta)\big)\Xi_1(u;\theta)\Big] du, \quad i = 0, 1, \tag{3.65}$$

and using it to rewrite Eq. (3.64), we obtain the following ***pdf evolution equation for the one-time response pdf*** $f_{X(t)}(x)$:

$$\frac{\partial f_{X(t)}(x)}{\partial t} + \frac{\partial}{\partial x}\Big[\Big(h(x) + \kappa m_{\Xi_0}(t) + q(x) m_{\Xi_1}(t)\Big) f_{X(t)}(x)\Big] =$$
$$= \kappa^2 \frac{\partial^2}{\partial x^2}\int_{t_0}^{t} C_{\Xi_0\Xi_0}(\tau_0, t)\mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big) \exp\Big(J_{h,q}^{\tau_0}[X(\bullet|_{\tau_0}^{t};\theta]\Big)\Big] d\tau_0 +$$
$$+ \kappa \frac{\partial^2}{\partial x^2}\int_{t_0}^{t} C_{\Xi_0\Xi_1}(t, \tau_1)\mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big)\, q\big(\mathrm{X}(\tau_1;\theta)\big) \exp\Big(J_{h,q}^{\tau_1}[X(\bullet|_{\tau_1}^{t};\theta]\Big)\Big] d\tau_1 +$$
$$+ \kappa \frac{\partial}{\partial x}\left(q(x)\frac{\partial}{\partial x}\int_{t_0}^{t} C_{\Xi_0\Xi_1}(\tau_0, t)\mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big) \exp\Big(J_{h,q}^{\tau_0}[X(\bullet|_{\tau_0}^{t};\theta]\Big)\Big] d\tau_0 \right) +$$
$$+ \frac{\partial}{\partial x}\left(q(x)\frac{\partial}{\partial x}\int_{t_0}^{t} C_{\Xi_1\Xi_1}(t,\tau_1)\mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big)\, q\big(\mathrm{X}(\tau_1;\theta)\big) \exp\Big(J_{h,q}^{\tau_1}[X(\bullet|_{\tau_1}^{t};\theta]\Big)\Big] d\tau_1 \right). \tag{3.66}$$

Under this notation, it becomes clear that the terms multiplying the random delta function $\delta\big(x - X(t;\theta)\big)$ inside the averaged terms of the above expression are not of the same form as the ones defined by Eq. (3.31). First, the term $q\big(\mathrm{X}(\tau_1;\theta)\big)$ appears, whose time-argument τ_1 does not match with the current time t and thus, it is not manageable by the delta projection method; for this term, a current-time approximation must be implemented. Second, the most notable complicacy stems from the J terms inside the averaged on the right-hand side of Eq. (3.66). More specifically, the integrand of the said term depends not only on the response, as was the case with I defined by Eq. (3.66), but also the multiplicative excitation $\Xi_1(u;\theta)$. As such, it becomes clear that in order to procure a closed, computable alternative of pdf evolution Eq. (3.66), some concessions must be made. For this, we shall write J in terms of I, as follows:

$$\begin{aligned} J_{h,q}^{\tau_i}[X(\bullet|_{\tau_i}^{t};\theta)] &= \int_{\tau_i}^{t} h'\big(\mathrm{X}(u;\theta)\big)\,du + \int_{\tau_i}^{t} q'\big(\mathrm{X}(u;\theta)\big)\,\Xi_1(u;\theta)\,du = \\ &= I_{h'}[X(\bullet|_{\tau_i}^{t};\theta)] + \int_{\tau_i}^{t} q'\big(\mathrm{X}(u;\theta)\big)\,\Xi_1(u;\theta)\,du\,, \end{aligned} \tag{3.67}$$

as well as the following, current time approximation for $q\big(\mathrm{X}(\tau_1;\theta)\big)$:

$$\begin{aligned} q\big(X(\tau_1;\theta)\big) &\cong q(X(t;\theta)) - q'(X(t;\theta))\,\dot{X}(t;\theta)\,(t-\tau_1) = q(X(t;\theta)) - \\ &- q'(X(t;\theta))\big(h(X(t;\theta)) + q(X(t;\theta))\,\Xi_0(t;\theta) + \kappa\,\Xi_1(t;\theta)\big)(t-\tau_1)\,. \end{aligned} \tag{3.68}$$

Then, by introducing the assumptions that excitation $\Xi_1(u;\theta)$ is of small intensity and that we are working for small correlation times, we shall consider only the terms on the rightmost sides of Eqs. (3.67) and (3.68), respectively, which do not entail the excitations. This assumption is not novel in this thesis; a similar one, for the case of an RDE excited by solely multiplicative noise, has been introduced in (Fox, 1986). In this regard, following the approximation scheme presented in paragraph 3.1.3 for Eq. (3.66), results in

$$\begin{aligned} &\frac{\partial f_{X(t)}(x)}{\partial t} + \frac{\partial}{\partial x}\Big[\Big(h(x) + \kappa m_{\Xi_0}(t) + q(x) m_{\Xi_1}(t)\Big) f_{X(t)}(x)\Big] = \\ &= \frac{\partial^2}{\partial x^2}\left\{\left[\sum_{m=0}^{M}\frac{1}{m!}\Big(\kappa^2\, D_m^{(1)}[R_{h'}(\bullet|_{\tau_0}^{t});t] + \kappa\, D_m^{(2)}[R_{h'}(\bullet|_{\tau_1}^{t});t]\Big)\right] f_{X(t)}(x)\right\} + \\ &+ \frac{\partial}{\partial x}\left\{q(x)\,\frac{\partial}{\partial x}\left[\sum_{m=0}^{M}\frac{1}{m!}\Big(D_m^{(3)}[R_{h'}(\bullet|_{\tau_1}^{t});t] + \kappa\, D_m^{(4)}[R_{h'}(\bullet|_{\tau_0}^{t});t]\Big)\right] f_{X(t)}(x)\right\}. \end{aligned} \tag{3.69}$$

where

$$D_m^{(1)}[R_{h'}(\bullet|_{\tau_0}^{t});t] = \int_{t_0}^{t} C_{\Xi_0\Xi_0}(\tau_0, t)\,\exp\left(\int_{\tau_0}^{t} R_{h'}(u)\,du\right)(t-\tau_0)^m\,d\tau_0\,, \tag{3.70a}$$

$$D_m^{(2)}[R_{h'}(\bullet|_{\tau_1}^{t});t] = q(x)\int_{t_0}^{t} C_{\Xi_0\Xi_1}(t,\tau_1)\exp\left(\int_{\tau_1}^{t} R_{h'}(u)\,du\right)(t-\tau_1)^m\,d\tau_1 - $$
$$- q'(x)h(x)\int_{t_0}^{t} C_{\Xi_0\Xi_1}(t,\tau_1)\exp\left(\int_{\tau_1}^{t} R_{h'}(u)\,du\right)(t-\tau_1)^{m+1}\,d\tau_1\,, \tag{3.70b}$$

$$D_m^{(3)}[R_{h'}(\bullet|_{\tau_1}^{t});t] = q(x)\int_{t_0}^{t} C_{\Xi_1\Xi_1}(t,\tau_1)\exp\left(\int_{\tau_1}^{t} R_{h'}(u)\,du\right)(t-\tau_1)^m\,d\tau_1 - $$
$$- q'(x)h(x)\int_{t_0}^{t} C_{\Xi_1\Xi_1}(t,\tau_1)\exp\left(\int_{\tau_1}^{t} R_{h'}(u)\,du\right)(t-\tau_1)^{m}\,d\tau_1\,, \tag{3.70c}$$

$$D_m^{(4)}[R_{h'}(\bullet|_{\tau_0}^{t});t] = \int_{t_0}^{t} C_{\Xi_0\Xi_1}(\tau_0,t)\exp\left(\int_{\tau_0}^{t} R_{h'}(u)\,du\right)(t-\tau_0)^m\,d\tau_0\,. \tag{3.70d}$$

Eq. (3.69) is closed and computable and retains, through the D_m terms, a substantial amount of the original nonlocality and nonlinearity of the exact Eq. (3.66), despite the concessions made.

In order to obtain a seemingly more accurate, approximate pdf evolution equation corresponding to a RIVP subject to both additive and multiplicative excitation, it is necessary to consider the joint response-excitation pdf evolution equations, as it is shown in Chapter 4. More specifically, regarding the examined RIVP (3.51a,b), we must formulate the evolution equation for the joint pdf of the response $X(t;\theta)$ and both excitations $\Xi_0(t;\theta)$, $\Xi_1(t;\theta)$ in order to be able to employ both terms of the current-time approximation introduced in Eq. (3.68).

Chapter 4

One-time response-excitation pdf evolution equations

In the present chapter, the first major extensions of the already presented methodology is performed. More specifically, by keeping in mind the steps taken towards formulating one-time response pdf evolution equations for systems under colored Gaussian excitation, we showcase a straightforward generalization for higher order pdfs, namely one-time joint response excitation pdfs. As already discussed in the Chapter 1, the formulation and solution of these pdfs is of the utmost importance and practicality since they can be directly applied in some cases or constitute even better approximations (through their marginalization) than their counterparts, as derived in Chapter 3.

4.1 The case of a scalar, nonlinear, additively excited RDE

Let us consider once more the scalar, nonlinear, additively RIVP, presented in section 3.1

$$\dot{X}(t;\theta) = h(X(t;\theta)) + \kappa\Xi(t;\theta), \qquad X(t_0;\theta) = X_0(\theta). \tag{4.1a,b}$$

4.1.1 The corresponding stochastic Liouville equation

Our starting point is representing the one-time, joint response-excitation pdf as the average of the product of two random delta functions, i.e. the ***delta projection method***. This is readily achieved by employing Eq. (3.6) for the product of random delta functions $\delta(x - X(t;\theta))\,\delta(u - \Xi(t;\theta))$, as follows:

$$\mathbb{E}^{\theta}\left[\delta(x - X(t;\theta))\,\delta(u - \Xi(t;\theta))\right] = \int_{\mathbb{R}^2} \delta(x-w)\,\delta(u-z)\, f_{X(t)\Xi(t)}(w,z)\, dw dz,$$

which results in the following representation for the joint response-excitation pdf:

$$f_{X(t)\Xi(t)}(x,u) = \mathbb{E}^{\theta}\left[\delta(x - X(t;\theta))\,\delta(u - \Xi(t;\theta))\right]. \tag{4.2}$$

Proceeding in the fashion showcased in sections 3.1 and 3.2, we differentiate both sides of Eq. (4.2) and use the chain rule

$$\frac{\partial f_{X(t)\Xi(t)}(x,u)}{\partial t} = \mathbb{E}^{\theta}\left[\frac{\partial \delta(x - X(t;\theta))}{\partial X(t;\theta)}\,\delta(u - \Xi(t;\theta))\,\dot{X}(t;\theta)\right] + \\ + \mathbb{E}^{\theta}\left[\delta(x - X(t;\theta))\,\frac{\partial \delta(u - \Xi(t;\theta))}{\partial \Xi(t;\theta)}\,\dot{\Xi}(t;\theta)\right]. \tag{4.3}$$

At this point, we are going to elaborate separately on the two averaged terms on the rightmost side of Eq. (4.3) Thus, making use of Eq. (4.1a) the first averaged term can be rewritten as

$$\begin{aligned}\mathbb{E}^{\theta}\left[\frac{\partial\delta(x-X(t;\theta))}{\partial X(t;\theta)}\,\delta(u-\Xi(t;\theta))\,\dot{X}(t;\theta)\right] &= \\ = \mathbb{E}^{\theta}\left[\frac{\partial\delta(x-X(t;\theta))}{\partial X(t;\theta)}\,h(X(t;\theta))\,\delta(u-\Xi(t;\theta))\right] &+ \\ +\kappa\,\mathbb{E}^{\theta}\left[\frac{\partial\delta(x-X(t;\theta))}{\partial X(t;\theta)}\,\delta(u-\Xi(t;\theta))\,\Xi(t;\theta)\right]&,\end{aligned} \tag{4.4}$$

which can be further evaluated by making use of the delta projection formalism for the two averaged terms appearing on its right-hand side. Under this formalism, the first averaged term can be written as

$$\begin{aligned}\mathbb{E}^{\theta}\left[\frac{\partial\delta(x-X(t;\theta))}{\partial X(t;\theta)}\,h(X(t;\theta))\,\delta(u-\Xi(t;\theta))\right] &= \\ = \int_{\mathbb{R}^2}\frac{\partial\delta(x-w)}{\partial w}\,h(w)\,\delta(u-z)\,f_{X(t)\Xi(t)}(w,z)\,dwdz &= \\ = -\frac{\partial}{\partial x}\left(h(x)\int_{\mathbb{R}}\delta(u-z)\,f_{X(t)\Xi(t)}(x,z)\,dz\right) &= \\ = -\frac{\partial}{\partial x}\Big(h(x)\;f_{X(t)\Xi(t)}(x,u)\Big)&.\end{aligned} \tag{4.5}$$

Accordingly, for the second averaged term on the right-hand side of Eq. (4.4), we work in the following manner:

$$\begin{aligned}\mathbb{E}^{\theta}\left[\frac{\partial\delta(x-X(t;\theta))}{\partial X(t;\theta)}\,\delta(u-\Xi(t;\theta))\,\Xi(t;\theta)\right] &= \\ = \int_{\mathbb{R}^2}\frac{\partial\delta(x-w)}{\partial w}\,\delta(u-z)\,z\,f_{X(t)\Xi(t)}(w,z)\,dwdz &= \\ = -\frac{\partial}{\partial x}\left(\int_{\mathbb{R}}\delta(u-z)\,z\;f_{X(t)\Xi(t)}(x,z)\,dz\right) &= \\ = -\frac{\partial}{\partial x}\Big(u\;f_{X(t)\Xi(t)}(x,u)\Big)&.\end{aligned} \tag{4.6}$$

Note that for the above expressions to hold true, the additional assumptions that $h(x)$ and $f_{X(t)\Xi(t)}(x,u)$ are $C^1\big([t_0,t]\to\mathbb{R}\big)$ must also be introduced. Combining, thus, Eqs. (4.5) and (4.6), we find that

$$\begin{aligned}\mathbb{E}^{\theta}\left[\frac{\partial\delta(x-X(t;\theta))}{\partial X(t;\theta)}\,\delta(u-\Xi(t;\theta))\,\dot{X}(t;\theta)\right] &= \\ = -\frac{\partial}{\partial x}\Big[\big(h(x)+\kappa\,u\big)\;f_{X(t)\Xi(t)}(x,u)\Big]&.\end{aligned} \tag{4.7}$$

Let us, now, continue with the examination of the second averaged term of Eq. (4.3). This can be calculated by again employing the delta projection formalism as well as using the following convenient, yet formal expression for $\dot{\Xi}(t;\theta)$:

$$\dot{\Xi}(t;\theta) = \int_{t_0}^{t} \delta'(t-s)\,\Xi(s;\theta)\,ds, \tag{4.8}$$

where $\delta'(t-s) \equiv -\partial\delta(t-s)/\partial s$. In Eq. (4.8), $\dot{\Xi}(t;\theta)$ is formally treated as a functional of integral type with singular kernel. Hence, by utilizing the usual delta projection formalism we find

$$\begin{aligned}\mathbb{E}^{\theta}\left[\delta(x - X(t;\theta))\,\frac{\partial\delta(u-\Xi(t;\theta))}{\partial\Xi(t;\theta)}\,\dot{\Xi}(t;\theta)\right] &= \\ = -\frac{\partial}{\partial u}\,\mathbb{E}^{\theta}\Big[\delta(x - X(t;\theta))\,\delta(u - \Xi(t;\theta))\,\dot{\Xi}(t;\theta)\Big]&.\end{aligned} \tag{4.9}$$

Last, substituting Eqs. (4.7) and (4.9) into Eq. (4.3) results in the following, ***one-time response excitation stochastic Liouville equation***:

$$\begin{aligned}\frac{\partial f_{X(t)\Xi(t)}(x,u)}{\partial t} + \frac{\partial}{\partial x}\Big[\big(h(x) + \kappa\,u\big)\,f_{X(t)\Xi(t)}(x,u)\Big] &= \\ = -\frac{\partial}{\partial u}\,\mathbb{E}^{\theta}\Big[\delta(x - X(t;\theta))\,\delta(u - \Xi(t;\theta))\,\dot{\Xi}(t;\theta)\Big]&.\end{aligned} \tag{4.10}$$

SLE (4.10) has also been derived in (Venturi, Sapsis, Cho, & Karniadakis, 2012b) for a multiplicatively excited RDE using a different, more convoluted approach. As in the case of SLE (3.13), the initial condition for the one-time response excitation SLE can easily be derived by the data of the initial problem

$$f_{X(t_0)\Xi(t_0)}(x,u) = f_{X_0\,\Xi(t_0)}(x,u). \tag{4.11}$$

Transformed SLE. Similar to the previous examined cases, SLE (4.10) in non-closed due to the averaged term in its right-hand side. By recalling the dependence of the response $X(t;\theta)$ on initial value $X_0(\theta)$ and excitation $\Xi(t;\theta)$, and using the familiar notation, the averaged term can be expressed as

$$\begin{aligned}\mathbb{E}^{\theta}\big[\delta(x - X(t;\theta))\,\delta(u - \Xi(t;\theta))\big] &= \\ = \mathbb{E}^{\theta}\Big[\delta(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)])\,\delta(u - \Xi(t;\theta))\Big]&.\end{aligned} \tag{4.12}$$

Under this notation, it is readily understood that Eq. (4.11) is in the appropriate form for the application of the extended NF theorem, Eq. (2.21), in which the product of random delta functions $\delta(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)])\,\delta(u - \Xi(t;\theta))$ is regarded as a FFℓ like $\mathcal{F} = \mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]$. Thus, the averaged term can be evaluated to

$$
\begin{aligned}
&\mathbb{E}^{\theta}\left[\dot{\Xi}(t;\theta)\ \delta(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)])\ \delta(u - \Xi(t;\theta))\right] = \\
&\quad = \dot{m}_{\Xi}(t)\,\mathbb{E}^{\theta}\left[\delta(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)])\ \delta(u - \Xi(t;\theta))\right] + \\
&\qquad + \dot{C}_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\left[\frac{\partial\left\{\delta(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)])\ \delta(u - \Xi(t;\theta))\right\}}{\partial X_0(\theta)}\right] + \\
&\qquad + \int_{t_0}^{t}\partial_t C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\frac{\delta\left\{\delta(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)])\ \delta(u - \Xi(t;\theta))\right\}}{\delta\Xi(\tau;\theta)}\right] d\tau .
\end{aligned}
\tag{4.13}
$$

Using the product rule for the derivatives, while taking into account that the paths of excitation are functionally independent from the initial value, we find

$$
\begin{aligned}
&\mathbb{E}^{\theta}\left[\dot{\Xi}(t;\theta)\ \delta(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)])\ \delta(u - \Xi(t;\theta))\right] = \\
&\quad = \dot{m}_{\Xi}(t)\,\mathbb{E}^{\theta}\left[\delta(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)])\ \delta(u - \Xi(t;\theta))\right] + \\
&\qquad + \dot{C}_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\left[\frac{\partial\,\delta(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)])}{\partial X_0(\theta)}\ \delta(u - \Xi(t;\theta))\right] + \\
&\qquad + \int_{t_0}^{t}\partial_t C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\frac{\delta\,\delta(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)])}{\delta\Xi(\tau;\theta)}\ \delta(u - \Xi(t;\theta))\right] d\tau + \\
&\qquad + \int_{t_0}^{t}\partial_t C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\delta\Big(x - \delta(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)])\Big)\ \frac{\delta\,\delta(u - \Xi(t;\theta))}{\delta\Xi(\tau;\theta)}\right] d\tau .
\end{aligned}
\tag{4.14}
$$

Proceeding according to the previous examined cases, we apply the chain rule to the above expression and simplify the notation by setting $X[\cdots] = X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]$, resulting in

$$
\begin{aligned}
&\mathbb{E}^{\theta}\left[\dot{\Xi}(t;\theta)\ \delta(x - X[\cdots])\ \delta(u - \Xi(t;\theta))\right] = \\
&\quad = \dot{m}_{\Xi}(t)\,\mathbb{E}^{\theta}\left[\delta(x - X[\cdots])\ \delta(u - \Xi(t;\theta))\right] - \\
&\qquad + \dot{C}_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\left[\frac{\partial\,\delta(x - X[\cdots])}{\partial X(t;\theta)}\ \delta(u - \Xi(t;\theta))\,V_{X_0}(t;\theta)\right] + \\
&\qquad + \int_{t_0}^{t}\partial_t C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\frac{\delta\,\delta(x - X[\cdots])}{\delta\Xi(t;\theta)}\ \delta(u - \Xi(t;\theta))\,V_{\Xi(\tau)}(t;\theta)\right] d\tau + \\
&\qquad + \int_{t_0}^{t}\partial_t C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\delta\big(x - \delta(x - X[\cdots])\big)\ \frac{\delta\,\delta(u - \Xi(t;\theta))}{\delta\Xi(t;\theta)}\ \frac{\delta\Xi(t;\theta)}{\delta\Xi(\tau;\theta)}\right] d\tau ,
\end{aligned}
\tag{4.15}
$$

where $V_{X_0}(t;\theta)$ and $V_{\Xi(\tau)}(t;\theta)$ are the variational derivatives with respect to initial value and excitation, defined by Eqs. (3.18a) and (3.18b), respectively. In addition, since the value of

$\Xi(t;\theta)$ does not functionally depend on the value $\Xi(\tau;\theta)$, for $\tau \neq t$, we can write $\delta\Xi(t;\theta)/\delta\Xi(\tau;\theta) = \delta(t-\tau)$. Then, returning to the treatment of the response $X(t;\theta)$ as a function and performing the familiar manipulations of the delta projection method, Eq. (4.15) becomes

$$\begin{aligned}
&\mathbb{E}^{\theta}\Big[\dot{\Xi}(t;\theta)\ \delta\big(x - X[\cdots]\big)\,\delta(u-\Xi(t;\theta))\Big] = \\
&\quad = \dot{m}_{\Xi}(t)\,\mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big)\,\delta(u-\Xi(t;\theta))\Big] - \\
&\quad - \dot{C}_{X_0\Xi}(t)\,\frac{\partial}{\partial x}\,\mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big)\,\delta(u-\Xi(t;\theta))\,V_{X_0}(t;\theta)\Big] - \\
&\quad - \frac{\partial}{\partial x}\int_{t_0}^{t}\partial_t C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big)\,\delta(u-\Xi(t;\theta))\,V_{\Xi(\tau)}(t;\theta)\Big]d\tau - \\
&\quad - \frac{\partial}{\partial u}\int_{t_0}^{t}\partial_t C_{\Xi\Xi}(t,\tau)\,\delta(t-\tau)\mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big)\,\delta(u-\Xi(t;\theta))\Big]d\tau .
\end{aligned} \tag{4.16}$$

The following, final form of the NF theorem for the nonlocal averaged term is derived by employing in Eq. (4.16) the identity of the Dirac delta function.

$$\begin{aligned}
&\mathbb{E}^{\theta}\Big[\dot{\Xi}(t;\theta)\,\delta(x-X(t;\theta))\,\delta(u-\Xi(t;\theta))\Big] = \dot{m}_{\Xi}(t)\ f_{X(t)\Xi(t)}(x,u) - \\
&\quad - \dot{C}_{X_0\Xi}(t)\,\frac{\partial}{\partial x}\,\mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta))\,\delta(u-\Xi(t;\theta))\,V_{X_0}(t;\theta)\Big] - \\
&\quad - \frac{\partial}{\partial x}\int_{t_0}^{t}\partial_t C_{\Xi\Xi}(t,s)\,\mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta)\,\delta(u-\Xi(t;\theta))\,V_{\Xi(s)}(t;\theta)\Big]ds - \\
&\quad - \partial_t C_{\Xi\Xi}(t,s)\Big|_{s=t}\,\frac{\partial f_{X(t)\Xi(t)}(x,u)}{\partial u},
\end{aligned} \tag{4.17}$$

in which the term $\partial_t C_{\Xi\Xi}(t,s)\Big|_{s=t}$ can be further calculated as

$$\partial_t C_{\Xi\Xi}(t,s)\Big|_{s=t} = C_{\dot{\Xi}\Xi}(t,t) = \mathbb{E}^{\theta}[\dot{\Xi}(t;\theta)\,\Xi(t;\theta)] = \frac{1}{2}\frac{\partial}{\partial t}\mathbb{E}^{\theta}[\Xi^2(t;\theta)] = \frac{1}{2}\,\dot{\sigma}_{\Xi}^2(t). \tag{4.18}$$

Last, combining Eqs. (4.17), (4.18) and substituting them in Eq. (4.10) results in the following ***transformed SLE of the one-time response-excitation pdf***:

$$\begin{aligned}
&\frac{\partial f_{X(t)\Xi(t)}(x,u)}{\partial t} + \frac{\partial}{\partial x}\Big[\big(h(x)+q(x)u\big)\ f_{X(t)\Xi(t)}(x,u)\Big] + \dot{m}_{\Xi}(t)\,\frac{\partial f_{X(t)\Xi(t)}(x,u)}{\partial u} = \\
&\quad = \dot{C}_{X_0\Xi}(t)\,\frac{\partial^2}{\partial x\partial u}\,\mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta))\,\delta(u-\Xi(t;\theta))\,V_{X_0}(t;\theta)\Big] + \\
&\quad + \frac{\partial^2}{\partial x\partial u}\int_{t_0}^{t}\partial_t C_{\Xi\Xi}(t,s)\,\mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta)\,\delta(u-\Xi(t;\theta))\,V_{\Xi(s)}(t;\theta)\Big]ds + \\
&\quad + \frac{1}{2}\,\dot{\sigma}_{\Xi}^2(t)\,\frac{\partial^2 f_{X(t)\Xi(t)}(x,u)}{\partial u^2}.
\end{aligned} \tag{4.19}$$

As in all of our previous cases, SLE (4.19) is non-closed due to nonlocal terms, depending on the history of the response and excitation, which are identified as the variational derivatives. As such, in order to procure a closed and thus, computable equation an approximation scheme must be implemented, as it is shown in the ensuing paragraph 4.1.2.

4.1.2 Novel, one-time evolution equations for the joint response-excitation pdf

Substituting Eqs. (3.24) and (3.30) for the variational derivatives appearing inside the averaged term of SLE (4.17) and using the notation of Eq. (3.31), SLE (4.19) can be rewritten as

$$\frac{\partial f_{X(t)\Xi(t)}(x,u)}{\partial t} + \frac{\partial}{\partial x}\Big[\big(h(x)+\kappa\, u\big)\, f_{X(t)\Xi(t)}(x,u)\Big] + \dot{m}_{\Xi}(t)\,\frac{\partial f_{X(t)\Xi(t)}(x,u)}{\partial u} =$$

$$= \dot{C}_{X_0\Xi}(t)\,\frac{\partial^2}{\partial x \partial u}\,\mathbb{E}^{\theta}\Big[\delta(x - X(t;\theta))\,\delta(u-\Xi(t;\theta))\,\exp\Big(\mathcal{I}_{h'}[X(\bullet|_{t_0}^{t};\theta)]\Big)\Big] +$$

$$+ \kappa\,\frac{\partial^2}{\partial x \partial u}\int_{t_0}^{t}\partial_t C_{\Xi\Xi}(t,s)\,\mathbb{E}^{\theta}\Big[\delta(x - X(t;\theta)\,\delta(u-\Xi(t;\theta))\,\exp\Big(\mathcal{I}_{h'}[X(\bullet|_{s}^{t};\theta)\Big)\Big]\,ds +$$

$$+ \frac{1}{2}\,\dot{\sigma}^2_{\Xi}(t)\,\frac{\partial^2 f_{X(t)\Xi(t)}(x,u)}{\partial u^2}. \tag{4.20}$$

Further, employing the approximation scheme already presented in paragraph 3.1.3 for the above expression, we obtain the following, ***closed, approximate, one-time, joint response-excitation pdf evolution equation***:

$$\frac{\partial f_{X(t)\Xi(t)}(x,u)}{\partial t} + \frac{\partial}{\partial x}\Big[\big(h(x)+\kappa\, u\big)\, f_{X(t)\Xi(t)}(x,u)\Big] + \dot{m}_{\Xi}(t)\,\frac{\partial f_{X(t)\Xi(t)}(x,u)}{\partial u} =$$

$$= \frac{\partial^2}{\partial x \partial u}\left\{\left[\sum_{m=0}^{M}\frac{1}{m!}G_m\big[R_{h'}(\bullet),t\big]\,\varphi_{h'}^{m}\big(x\,;R_{h'}(t)\big)\right] f_{X(t)\Xi(t)}(x,u)\right\} + \tag{4.21}$$

$$+ \frac{1}{2}\,\dot{\sigma}^2_{\Xi}(t)\,\frac{\partial^2 f_{X(t)\Xi(t)}(x,u)}{\partial u^2},$$

where

$$G_m\big[R_{h'}(\bullet),t\big] = \exp\left(\int_{t_0}^{t}R_{h'}(u)\,du\right)\dot{C}_{X_0\Xi}(t)\,(t-t_0)^m + $$
$$+ \kappa^2\int_{t_0}^{t}\exp\left(\int_{s}^{t}R_{h'}(u)\,du\right)\partial_t C_{\Xi\Xi}(t,s)\,(t-s)^m\,ds. \tag{4.22}$$

The G_m terms of the above equation are a simple generalization of the generalized effective noise intensities, D_m^{eff}, for the case of the joint response-excitation pdf evolution equation. Through these terms, the approximate Eq. (4.21) retains a tangible amount of nonlinearity and nonlocality in time, thus preserving the non-Markovian character of the initial problem.

4.1.3 Check of compatibility for the closed, one-time, joint response-excitation pdf evolution equation

In this paragraph, we are going to examine the compatibility of the approximate Eq. (4.22) in terms of the marginal pdf evolution equations that can be derived from it. More specifically, we are going to compare these one-time marginal pdfs with the ones obtained in Chapter 3 in order to see if the extended methodology established throughout this chapter is consistent with the one it is founded upon.

(a) Marginal response pdf evolution equation $f_{X(t)}(x)$ coincides with the SLE

By integrating both sides of Eq. (4.21) with respect to u, and under the plausible assumptions

$$f_{X(t)\Xi(t)}(x, \pm\infty) = 0, \qquad \frac{\partial f_{X(t)\Xi(t)}(x, \pm\infty)}{\partial u} = 0, \tag{4.23a,b}$$

we obtain

$$\frac{\partial f_{X(t)}(x)}{\partial t} + \frac{\partial}{\partial x}\Big(h(x)\ f_{X(t)}(x)\Big) = -\kappa \frac{\partial}{\partial x}\int_{\mathbb{R}} u\ f_{X(t)\Xi(t)}(x,u)\,du\,. \tag{4.24}$$

Eq. (4.24) is a form of the exact stochastic Liouville equation for the one-time response pdf. Under the manipulation

$$\begin{aligned}\int_{\mathbb{R}} u\ f_{X(t)\Xi(t)}(x,u)\,du &= \int_{\mathbb{R}} u\ \mathbb{E}^{\theta}\big[\delta(x-X(t;\theta))\,\delta(u-\Xi(t;\theta))\big]\,du = \\ &= \mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta))\int_{\mathbb{R}} u\ \delta(u-\Xi(t;\theta))\,du\Big] = \mathbb{E}^{\theta}\big[\delta(x-X(t;\theta))\,\Xi(t;\theta)\big],\end{aligned} \tag{4.25}$$

stochastic Liouville Eq. (4.23) is written in the more familiar form

$$\frac{\partial f_{X(t)}(x)}{\partial t} + \frac{\partial}{\partial x}\Big(h(x)\ f_{X(t)}(x)\Big) = -\kappa \frac{\partial}{\partial x}\ \mathbb{E}^{\theta}\big[\delta(x-X(t;\theta))\,\Xi(t;\theta)\big]. \tag{4.26}$$

Having obtain Eq. (4.26), it is easy to see that it is identical to Eq. (3.13) as derived in paragraph 3.1.1. This occurrence reaffirms the potency of both the methodology as well as the approximation scheme used to derive the approximate Eq. (4.21).

(b) Marginal excitation pdf evolution equation $f_{\Xi(t)}(u)$

Before proceeding with the evolution equation for the marginal excitation pdf $f_{\Xi(t)}(u)$, we have to prove the following lemma.

Lemma 1: Evolution equation for a Gaussian pdf. The one-time pdf of the Gaussian random function $\Xi(t;\theta)$

$$f_{\Xi(t)}(u) = \frac{1}{\sqrt{2\pi\sigma_{\Xi}^{2}(t)}}\exp\left[-\frac{1}{2}\frac{\big(u-m_{\Xi}(t)\big)^{2}}{\sigma_{\Xi}^{2}(t)}\right], \tag{4.27}$$

with differentiable, with respect to t, mean $m_{\Xi}(t)$ and variance $\sigma_{\Xi}^{2}(t)$, satisfies the equation

$$\frac{\partial f_{\Xi(t)}(u)}{\partial t} + \dot{m}_{\Xi}(t)\frac{\partial f_{\Xi(t)}(u)}{\partial u} = \frac{1}{2}\dot{\sigma}^2_{\Xi}(t)\frac{\partial^2 f_{\Xi(t)}(u)}{\partial u^2}. \tag{4.28}$$

Proof. By differentiations of Gaussian pdf (4.27) we obtain:

$$\frac{\partial f_{\Xi(t)}(u)}{\partial t} = \frac{f_{\Xi(t)}(u)}{\left(\sigma^2_{\Xi}(t)\right)^2}\,\mathrm{A}\left(u\,;\,m_{\Xi}(t),\,\sigma^2_{\Xi}(t)\right), \tag{4.29a}$$

$$\frac{\partial f_{\Xi(t)}(u)}{\partial u} = \frac{f_{\Xi(t)}(u)}{\left(\sigma^2_{\Xi}(t)\right)^2}\,\mathrm{B}\left(u\,;\,m_{\Xi}(t),\,\sigma^2_{\Xi}(t)\right), \tag{4.29b}$$

$$\frac{\partial^2 f_{\Xi(t)}(u)}{\partial u^2} = \frac{f_{\Xi(t)}(u)}{\left(\sigma^2_{\Xi}(t)\right)^2}\,\Gamma\left(u\,;\,m_{\Xi}(t),\,\sigma^2_{\Xi}(t)\right) \tag{4.29c}$$

where

$$\begin{aligned} \mathrm{A} = \mathrm{A}\left(u\,;\,m_{\Xi}(t),\,\sigma^2_{\Xi}(t)\right) &= \frac{1}{2}\dot{\sigma}^2_{\Xi}(t)\,u^2 + \\ &+ \left(\sigma^2_{\Xi}(t)\,\dot{m}_{\Xi}(t) - m_{\Xi}(t)\,\dot{\sigma}^2_{\Xi}(t)\right)u + \\ &+ \frac{1}{2}m_{\Xi}(t)\,\dot{\sigma}^2_{\Xi}(t) - m_{\Xi}(t)\,\sigma^2_{\Xi}(t)\,\dot{m}_{\Xi}(t) - \frac{1}{2}\sigma^2_{\Xi}(t)\,\dot{\sigma}^2_{\Xi}(t), \end{aligned} \tag{4.30a}$$

$$\mathrm{B} = \mathrm{B}\left(u\,;\,m_{\Xi}(t),\,\sigma^2_{\Xi}(t)\right) = -\sigma^2_{\Xi}(t)\,u + m_{\Xi}(t)\,\sigma^2_{\Xi}(t), \tag{4.30b}$$

$$\Gamma = \Gamma\left(u\,;\,m_{\Xi}(t),\,\sigma^2_{\Xi}(t)\right) = u^2 - 2\,m_{\Xi}(t)\,u + m^2_{\Xi}(t) - \sigma^2_{\Xi}(t). \tag{4.30c}$$

By substituting Eqs. (4.29) into Eq. (4.28) we obtain the algebraic relation

$$\mathrm{A} + \dot{m}_{\Xi}(t)\,\mathrm{B} = \frac{1}{2}\dot{\sigma}^2_{\Xi}(t)\,\Gamma. \tag{4.31}$$

Via Eqs. (4.30), it is easy to see that Eq. (4.31) is always satisfied:

$$\begin{aligned} &\frac{1}{2}\dot{\sigma}^2_{\Xi}(t)\,u^2 + \sigma^2_{\Xi}(t)\,\dot{m}_{\Xi}(t)\,u - m_{\Xi}(t)\,\dot{\sigma}^2_{\Xi}(t)\,u + \\ &\quad + \frac{1}{2}m^2_{\Xi}(t)\,\dot{\sigma}^2_{\Xi}(t) - m_{\Xi}(t)\,\sigma^2_{\Xi}(t)\,\dot{m}_{\Xi}(t) - \frac{1}{2}\sigma^2_{\Xi}(t)\,\dot{\sigma}^2_{\Xi}(t) - \\ &\quad - \dot{m}_{\Xi}(t)\sigma^2_{\Xi}(t)\,u + m_{\Xi}(t)\,\dot{m}_{\Xi}(t)\,\sigma^2_{\Xi}(t) - \frac{1}{2}\dot{\sigma}^2_{\Xi}(t)\,u^2 + \\ &\quad + \dot{\sigma}^2_{\Xi}(t)\,m_{\Xi}(t)\,u - \frac{1}{2}\dot{\sigma}^2_{\Xi}(t)\,m^2_{\Xi}(t) + \frac{1}{2}\dot{\sigma}^2_{\Xi}(t)\,\sigma^2_{\Xi}(t) = 0. \end{aligned} \tag{4.32}$$

This concludes the lemma's proof. ■

Returning, now, to the derivation of the required marginal, we integrate both sides of Eq. (4.21) with respect to x resulting in

$$\frac{\partial f_{\Xi(t)}(u)}{\partial t} + \dot{m}_{\Xi}(t)\frac{\partial f_{\Xi(t)}(u)}{\partial u} = C_{\Xi\Xi}(t,t)\frac{\partial^2 f_{\Xi(t)}(u)}{\partial u^2}. \tag{4.33}$$

Eq. (4.33) is retrieved under the following assumptions regarding the behaviour at the boundaries:

$$h(\pm\infty)\; f_{X(t)\Xi(t)}(\pm\infty, u) = 0\,, \tag{4.34a}$$

$$f_{X(t)\Xi(t)}(\pm\infty, u) = 0\,, \tag{4.34b}$$

$$\varphi_{h'}^{m}\left(\pm\,\infty\,; R_{h'}(t)\right)\; f_{X(t)\Xi(t)}(\pm\infty, u) = 0\,. \tag{4.34c}$$

Thus, it is promptly seen that Eq. (4.33) is identical to Eq. (4.28) a fact that further supports the legitimacy of our approach.

4.1.4 Exact response-excitation pdf for a linear, additively excited RDE

In this paragraph, we are going to examine the validity of our methodology in the case of the linear, additively excited RDE, i.e. $h(x) = \eta\, x$, with $\eta < 0$ and see if the correct Gaussian form for $f_{X(t)\,\Xi(t)}(x, u)$ is retrieved.

As previously discussed, in the linear case the variational derivatives are independent from the time history of the response and thus, can be specified, by Eqs. (3.44a,b), into $V_{X_0}(t\,;\theta) = e^{\eta(t-t_0)}$, $V_{\Xi(s)}(t\,;\theta) = \kappa\, e^{\eta(t-s)}$. By substituting these expressions into SLE (4.19), the following ***exact response-excitation pdf evolution equation*** is obtained, in ***closed form***:

$$\begin{aligned}\frac{\partial f_{X(t)\Xi(t)}(x, u)}{\partial t} + \frac{\partial}{\partial x}\left[\left(\eta\, x + \kappa u\right)\, f_{X(t)\Xi(t)}(x, u)\right] + \dot{m}_{\Xi}(t)\,\frac{\partial f_{X(t)\Xi(t)}(x, u)}{\partial u} &= \\ = G(t)\,\frac{\partial^2 f_{X(t)\Xi(t)}(x, u)}{\partial x \partial u} + \frac{1}{2}\,\dot{\sigma}_{\Xi}^{2}(t)\,\frac{\partial^2 f_{X(t)\Xi(t)}(x, u)}{\partial u^2}&\,,\end{aligned} \tag{4.35}$$

where

$$G(t) \;=\; \dot{C}_{X_0\Xi}(t)\, e^{\eta\,(t\,-\,t_0)} \;+\, \kappa \int_{t_0}^{t} \partial_t\, C_{\Xi\,\Xi}(t, s)\, e^{\eta\,(t\,-\,s)}\; ds\,. \tag{4.36}$$

Eq. (4.35) is a first-order, linear partial differential equation, which can be readily solved by making use of its Fourier transform as it is subsequently shown. However, before we commence with its solution, we are going to provide the following useful result.

Connection between $G(t)$ and $D^{\text{eff}}(t)$. By using Eq. (4.34), as well as the definition relation (4.35) for the effective noise intensity $D^{\text{eff}}(t)$ that appears in the exact response pdf evolution Eq. (4.34), it is easily derived that

$$\dot{D}^{\text{eff}}(t) \;=\; \eta\, D^{\text{eff}}(t) + \kappa^2\, \sigma_{\Xi}^{2}(t) + \kappa G(t)\,, \quad D^{\text{eff}}(t_0) \;=\; \kappa C_{X_0\Xi}(t_0)\,, \tag{4.37a,b}$$

and by solving the above IVP

$$D^{\text{eff}}(t) \;=\; \kappa\, C_{X_0\Xi}(t_0)\, e^{\eta\,(t\,-\,t_0)} + \int_{t_0}^{t}\left(\kappa^2\, \sigma_{\Xi}^{2}(\tau) + \kappa\, G(\tau)\right) e^{\eta\,(t\,-\,\tau)}\; d\tau\,. \tag{4.38}$$

Relation (4.38) will be proven quite useful in validating the moments obtained from the solution of Eq. (4.35).

Solution of Eq. (4.35) using Fourier transform. As performed, in paragraph 4.1.1, for Eq. (3.45), response-excitation evolution Eq. (4.35) is solved by utilizing the two-dimensional Fourier transform; $\varphi_{X(t)\Xi(t)}(y_1, y_2) \equiv \varphi_{X(t)\Xi(t)}(\boldsymbol{y}) = \int_{\mathbb{R}^2} e^{i(y_1 x + y_2 u)} \; f_{X(t)\Xi(t)}(x, u)\, dx\, du$, resulting in the equation

$$\frac{\partial \varphi_{X(t)\Xi(t)}(\boldsymbol{y})}{\partial t} = \eta\, y_1 \frac{\partial \varphi_{X(t)\Xi(t)}(\boldsymbol{y})}{\partial y_1} + \kappa\, y_1 \frac{\partial \varphi_{X(t)\Xi(t)}(\boldsymbol{y})}{\partial y_2} + + \left(i\, \dot{m}_\Xi(t)\, y_2 - G(t)\, y_1 y_2 - \frac{1}{2}\dot{\sigma}^2_\Xi(t)\, y_2^2 \right) \varphi_{X(t)\Xi(t)}(\boldsymbol{y}), \tag{4.39a}$$

supplemented by the transformed Gaussian initial condition

$$\varphi_{X(t_0)\Xi(t_0)}(\boldsymbol{y}) = \exp\left(i\, m_{X_0}\, y_1 + i\, m_\Xi(t_0)\, y_2 - \frac{1}{2}\sigma^2_{X_0}\, y_1^2 - \frac{1}{2}\sigma^2_\Xi(t_0)^2\, y_2^2 - C_{X_0\Xi}(t_0)\, y_1 y_2 \right). \tag{4.39b}$$

IVP (4.39a,b) is a first-order PDE problem which can be readily solved using the method of characteristics. Following (Polyanin et al., 2001, sec. 4.1), let us first consider the homogeneous variant of Eq. (4.39a)

$$\frac{\partial \varphi_{X(t)\Xi(t)}(\boldsymbol{y})}{\partial t} - \eta\, y_1 \frac{\partial \varphi_{X(t)\Xi(t)}(\boldsymbol{y})}{\partial y_1} - \kappa\, y_1 \frac{\partial \varphi_{X(t)\Xi(t)}(\boldsymbol{y})}{\partial y_2} = 0, \tag{4.40}$$

and its corresponding characteristic system

$$\frac{dt}{1} = -\frac{dy_1}{\eta\, y_1} = -\frac{dy_2}{\kappa\, y_1}. \tag{4.41}$$

The solution of the characteristic system (4.41) determines the characteristic curves $\upsilon_1(t, y_1, y_2) = \upsilon_1 = y_1 e^{\eta t}$, $\upsilon_2(t, y_1, y_2) = \upsilon_2 = \kappa\, y_1 - \eta\, y_2$. In turn, these curves dictate the change of variables $y_1 = \upsilon_1 e^{-\eta t}$, $y_2 = \frac{\kappa}{\eta} y_1 - \frac{1}{\eta}\upsilon_2 = \frac{\kappa}{\eta}\upsilon_1 e^{-\eta t} - \frac{1}{\eta}\upsilon_2$, under which Eq. (4.39a) is transformed into

$$\frac{\partial \varphi_{X(t)\Xi(t)}(\boldsymbol{\upsilon})}{\partial t} = \varphi_{X(t)\Xi(t)}(\boldsymbol{\upsilon}) \times \times \left[\frac{1}{\eta} \left(i\, \dot{m}_\Xi(t) - G(t)\, \upsilon_1 e^{-\eta t} \right) \left(\kappa \upsilon_1 e^{-\eta t} - \upsilon_2 \right) - \frac{\dot{\sigma}^2_\Xi(t)}{2\eta^2} \left(\kappa \upsilon_1 e^{-\eta t} - \upsilon_2 \right)^2 \right]. \tag{4.42}$$

Accordingly, initial condition, Eq. (4.39b), is expressed under this change of variables as

$$\begin{aligned} \varphi_{X(t_0)\Xi(t_0)}(\boldsymbol{\upsilon}) = {} & \exp\left[i \left(m_{X_0} + \frac{\kappa}{\eta} m_\Xi(t_0) \right) \upsilon_1 e^{-\eta t_0} - i \frac{1}{\eta} m_\Xi(t_0)\, \upsilon_2 \right] \times \\ & \times \exp\left[-\frac{1}{2} \left(\sigma^2_{X_0} + \frac{\kappa^2}{\eta^2}\sigma^2_\Xi(t_0) + 2\frac{\kappa}{\eta} C_{X_0\Xi}(t_0) \right) \upsilon_1^2 e^{-2\eta t_0} \right] \times \\ & \times \exp\left[-\frac{1}{2\eta^2}\sigma^2_\Xi(t_0)\, \upsilon_2^2 + \left(\frac{\kappa}{\eta^2}\sigma^2_\Xi(t_0) + \frac{1}{\eta} C_{X_0\Xi}(t_0) \right) \upsilon_1 \upsilon_2 e^{-\eta t_0} \right]. \end{aligned} \tag{4.43}$$

Solution to IVP (4.42), (4.43) is, thus, easily determined to

$$\varphi_{X(t)\Xi(t)}(\boldsymbol{\upsilon}) = \varphi_{X(t_0)\Xi(t_0)}(\boldsymbol{\upsilon}) \exp\left[\frac{1}{\eta}\int_{t_0}^{t}\left(i\dot{m}_{\Xi}(\tau) - G(\tau)\,\upsilon_1 e^{-\eta\tau}\right)\left(\kappa\upsilon_1 e^{-\eta\tau} - \upsilon_2\right)d\tau\right]\times$$
$$\times \exp\left[-\frac{1}{2\eta^2}\int_{t_0}^{t}\dot{\sigma}_{\Xi}^2(\tau)\left(\kappa\upsilon_1 e^{-\eta\tau} - \upsilon_2\right)^2 d\tau\right], \tag{4.44}$$

Performing, now, some simple manipulations for the integrals inside the averages, substituting initial value $\varphi_{X(t_0)\Xi(t_0)}(\boldsymbol{\upsilon})$ by Eq. (4.43) and then, returning to the initial variables $\upsilon_1 = y_1 e^{\eta t}$, $\upsilon_2 = \kappa y_1 - \eta y_2$, solution (4.44) is written as

$$\varphi_{X(t)\Xi(t)}(\boldsymbol{y}) = \exp\left[i y_1\left(m_{X_0} e^{\eta(t-t_0)} + \kappa\int_{t_0}^{t} m_{\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau\right) + i y_2\, m_{\Xi}(t) - \frac{1}{2} y_2^2\, \sigma_{\Xi}^2(t)\right]\times$$
$$\times \exp\left\{-\frac{1}{2} y_1^2\left[\sigma_{X_0}^2 e^{2\eta(t-t_0)} + \frac{2\kappa}{\eta}\left(C_{X_0\Xi}(t_0)(e^{2\eta(t-t_0)} - e^{\eta(t-t_0)}) + \right.\right.\right.$$
$$\left.\left.\left. + \int_{t_0}^{t}\left(\kappa\sigma_{\Xi}^2(\tau) + G(\tau)\right)(e^{2\eta(t-\tau)} - e^{\eta(t-\tau)})\, d\tau\right)\right]\right\}\times \tag{4.45}$$
$$\times \exp\left[-y_1 y_2\left(C_{X_0\Xi}(t_0) e^{\eta(t-t_0)} + \int_{t_0}^{t}\left(\kappa\sigma_{\Xi}^2(\tau) + G(\tau)\right) e^{\eta(t-\tau)}\, d\tau\right)\right].$$

From Eq. (4.45), we identify the first and second moments of $X(t;\theta)$, $\Xi(t;\theta)$. First, since the moments of the excitation are data of the problem, Eq. (4.45) returns the trivial relations $m_{\Xi}(t) = m_{\Xi}(t)$, $\sigma_{\Xi}^2(t) = \sigma_{\Xi}^2(t)$. Moving now to the moments of response, as well as the response-excitation covariance, we retrieve, from Eq. (4.45), the relations

$$m_X(t) = m_{X_0} e^{\eta(t-t_0)} + \kappa\int_{t_0}^{t} m_{\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau, \tag{4.46a}$$

$$\sigma_X^2(t) = \sigma_{X_0}^2 e^{2\eta(t-t_0)} +$$
$$+ \frac{2\kappa}{\eta}\left(C_{X_0\Xi}(t_0)(e^{2\eta(t-t_0)} - e^{\eta(t-t_0)}) + \int_{t_0}^{t}\left(\kappa\sigma_{\Xi}^2(\tau) + G(\tau)\right)(e^{2\eta(t-\tau)} - e^{\eta(t-\tau)})\, d\tau\right), \tag{4.46b}$$

$$C_{X\Xi}(t,t) = C_{X_0\Xi}(t_0) e^{\eta(t-t_0)} + \int_{t_0}^{t}\left(\kappa\sigma_{\Xi}^2(\tau) + G(\tau)\right) e^{\eta(t-\tau)}\, d\tau. \tag{4.46c}$$

Eq. (4.46a) is validated as the solution for the mean value of the response, see Eq. (B.3) of Appendix B, while, by employing Eq. (4.38) and after some algebraic manipulations, Eqs. (4.46b,c) are expressed equivalently as

$$\sigma_X^2(t) = \sigma_{X_0}^2 e^{2\eta(t-t_0)} + 2\int_{t_0}^{t} D^{\text{eff}}(\tau)\, e^{2\eta(t-\tau)}\, d\tau , \tag{4.46b′}$$

$$C_{X\Xi}(t,t) = \frac{1}{\kappa} D^{\text{eff}}(t) . \tag{4.46c′}$$

Eqs. (4.46b′,c′) are correct, since they coincide with the validated relations (3.51), (3.52) for $C_{X\Xi}(t,t)$, $\sigma_X^2(t)$. Thus, by solving Eq. (4.39a), the expected Gaussian solution was obtained.

4.2 The case of an RDE subject to both additive and multiplicative excitation

Consider, once more, the case of an RDE both additively and multiplicatively excited

$$\dot{\mathrm{X}}(t;\theta) = h(\mathrm{X}(t;\theta)) + q(\mathrm{X}(t;\theta))\Xi_1(t;\theta) + \kappa\,\Xi_0(t;\theta), \tag{4.47a}$$

along with the initial condition

$$X(t_0;\theta) = a . \tag{4.47b}$$

As already discussed in section 3.2, in the present section we are going to formulate pdf evolution equation for the joint, one-time response excitations pdf evolution equation $f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)$ since, through marginalization, it can provide a more accurate approximation for the one time pdf $f_{X(t)}(x)$.

4.2.1 The corresponding stochastic Liouville equation

As always, we begin with delta representation method which, in this case, gives rise to the following representation for the one-time response excitation pdf:

$$f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1) = \mathbb{E}^{\theta}[\delta(x - X(t;\theta))\delta(u_0 - \Xi_0(t;\theta))\delta(u_1 - \Xi_1(t;\theta))], \tag{4.48}$$

and will be more concisely denoted as

$$f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1) = \mathbb{E}^{\theta}\left[\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))\right]. \tag{4.49}$$

Eqs. (4.48), (4.49) are similar to the one used for the nonlinear, additively excited RIVP (4.1a,b). Thus, by differentiating both sides of the above expression with respect to time t and employing the product and chain rules for derivatives, we obtain

$$\frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)}{\partial t} = E_1 + E_2 + E_3 , \tag{4.50}$$

where

$$E_1 = \mathbb{E}^{\theta}\left[\frac{\partial\delta(x - X(t;\theta))}{\partial X(t;\theta)}\,\delta(u_0 - \Xi_0(t;\theta))\,\delta(u_1 - \Xi_1(t;\theta))\,\dot{X}(t;\theta)\right], \tag{4.51a}$$

$$E_2 = \mathbb{E}^{\theta}\left[\delta(x - X(t;\theta))\,\frac{\partial\delta(u_0 - \Xi_0(t;\theta))}{\partial\Xi_0(t;\theta)}\,\delta(u_1 - \Xi_1(t;\theta))\,\dot{\Xi}_0(t;\theta)\right], \tag{4.51b}$$

$$E_3 = \mathbb{E}^{\theta}\left[\delta(x - X(t;\theta))\,\delta(u_0 - \Xi_0(t;\theta))\,\frac{\partial\delta(u_1 - \Xi_1(t;\theta))}{\partial\Xi_1(t;\theta)}\,\dot{\Xi}_1(t;\theta)\right]. \tag{4.51c}$$

Each term on the right-hand side of Eq. (4.50) is subsequently elaborated separately. For the first averaged term, we use RDE (4.47a) resulting in

$$\begin{aligned}
E_1 &= \mathbb{E}^{\theta}\left[\frac{\partial\delta(x - X(t;\theta))}{\partial X(t;\theta)}\,\delta(u_0 - \Xi_0(t;\theta))\,\delta(u_1 - \Xi_1(t;\theta))\,\dot{X}(t;\theta)\right] = \\
&= \mathbb{E}^{\theta}\left[\frac{\partial\delta(x - X(t;\theta))}{\partial X(t;\theta)}\,\delta(u_0 - \Xi_0(t;\theta))\,\delta(u_1 - \Xi_1(t;\theta))\,h(\mathrm{X}(t;\theta))\right] + \\
&+ \mathbb{E}^{\theta}\left[\frac{\partial\delta(x - X(t;\theta))}{\partial X(t;\theta)}\,\delta(u_0 - \Xi_0(t;\theta))\,\delta(u_1 - \Xi_1(t;\theta))\,q(\mathrm{X}(t;\theta))\Xi_1(t;\theta)\right] + \\
&+ \kappa\,\mathbb{E}^{\theta}\left[\frac{\partial\delta(x - X(t;\theta))}{\partial X(t;\theta)}\,\delta(u_0 - \Xi_0(t;\theta))\,\delta(u_1 - \Xi_1(t;\theta))\,\Xi_0(t;\theta)\right].
\end{aligned} \tag{4.52}$$

Eq. (4.52) can be further evaluated using the familiar delta projection formalism. For reasons of brevity and clarity, as an example, we are going to present only the manipulation of the second averaged term on the right-hand side of Eq. (4.52).

$$\begin{aligned}
&\mathbb{E}^{\theta}\left[\frac{\partial\delta(x - X(t;\theta))}{\partial X(t;\theta)}\,\delta(u_0 - \Xi_0(t;\theta))\,\delta(u_1 - \Xi_1(t;\theta))\,q(\mathrm{X}(t;\theta))\Xi_1(t;\theta)\right] = \\
&= \int_{\mathbb{R}^3}\frac{\partial\delta(x - w)}{\partial w}\,\delta(u_0 - z_0)\,\delta(u_1 - z_1)\,q(w) z_1\, f_{X(t)\Xi_0(t)\Xi_1(t)}(w, z_0, z_1)\,dw dz_0\,dz_1 = \\
&= -\frac{\partial}{\partial x}\left(q(x)\int_{\mathbb{R}^2}\delta(u_0 - z_0)\,\delta(u_1 - z_1) z_1\, f_{X(t)\Xi_0(t)\Xi_1(t)}(x, z_0, z_1)\,dz_0\,dz_1\right) = \\
&= -\frac{\partial}{\partial x}\left(q(x) u_1\, f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1)\right).
\end{aligned} \tag{4.53}$$

Working accordingly for the other two terms, E_1 is transformed into

$$\begin{aligned}
E_1 = &-\frac{\partial}{\partial x}\left(h(x)\, f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1)\right) - \\
&-\frac{\partial}{\partial x}\left(q(x) u_1\, f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1)\right) = \\
&-\kappa\,\frac{\partial}{\partial x}\left(u_0\, f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1)\right).
\end{aligned} \tag{4.54}$$

Under similar treatment, the other two averaged terms on the right-hand side of Eq. (4.50) can be equivalently written as

$$E_2 = -\frac{\partial}{\partial u_0}\,\mathbb{E}^{\theta}\left[\delta(x, u_0, u_1; X(t), \Xi_0(t), \Xi_1(t))\,\dot{\Xi}_0(t;\theta)\right], \tag{4.55}$$

$$E_3 = -\frac{\partial}{\partial u_1}\mathbb{E}^{\theta}\left[\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))\,\dot{\Xi}_1(t;\theta)\right]. \tag{4.56}$$

where $\dot{\Xi}_0(t;\theta)$, $\dot{\Xi}_1(t;\theta)$ are treated as functionals of integral type with a singular kernel, as shown in Eq. (4.8). Last, substituting Eqs. (4.54)-(4.56) into Eq. (4.50) results in the ***SLE for the one-time, joint response excitations pdf*** $f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)$:

$$\begin{aligned}\frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)}{\partial t} &+ \frac{\partial}{\partial x}\left[\left(h(x)+q(x)u_1+\kappa u_0\right)f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)\right] = \\ &= -\frac{\partial}{\partial u_0}\mathbb{E}^{\theta}\left[\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))\dot{\Xi}_0(t;\theta)\right] - \\ &\quad -\frac{\partial}{\partial u_1}\mathbb{E}^{\theta}\left[\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))\dot{\Xi}_1(t;\theta)\right].\end{aligned} \tag{4.57}$$

In consistence with the previous case, SLE (4.57) is non-closed due to the averaged terms on its right-hand side and thus, the appropriate correlation splitting must be conducted by employing the appropriate extensions of the Novikov-Furutsu theorem. Further, SLE (4.57) is supplemented by the initial condition obtained by the data of RIVP (4.47a,b)

$$f_{X(t_0)\Xi_0(t_0)\Xi_1(t_0)}(x,u_0,u_1) = \delta(x-a)\,f_{\Xi_0(t_0)\Xi_1(t_0)}(u_0,u_1). \tag{4.58}$$

Transformed SLE. Recalling, at this point, the discussion regarding the dependence of the response $X(t;\theta)$ of RDE (4.47a) on the history of both excitations $\Xi_0(\bullet;\theta)$, $\Xi_1(\bullet;\theta)$ over the time interval $[t_0,t]$ and using the familiar notation, the product of random delta functions inside the averaged terms of Eq. (4.57) can be written as

$$\begin{aligned}&\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t)) \equiv \\ &\quad\equiv \delta(x-X(t;\theta))\,\delta(u_0-\Xi_0(t;\theta))\,\delta(u_1-\Xi_1(t;\theta)) = \\ &\quad= \delta(x-X[\cdots])\,\delta(u-\Xi_1(t;\theta))\,\delta(u_1-\Xi_1(t;\theta)) = \\ &\quad= \delta(x-X[\Xi_0(\bullet\big|_{t_0}^{t};\theta);\Xi_1(\bullet\big|_{t_0}^{t};\theta)])\,\delta(u_0-\Xi_0(t;\theta)\,\delta(u_1-\Xi_1(t;\theta)).\end{aligned} \tag{4.59}$$

Under this notation, the above product can be regarded as functional like $\mathcal{F}[\cdots]$ and thus, Eq. (2.25) of the NF theorem can be implemented to the two nonlocal, averaged terms. Since both terms are similar, we are going to present in detail only the implementation of Eq. (2.25) for the second averaged term on the right-hand side of Eq. (4.57):

$$\begin{aligned}&\mathbb{E}^{\theta}\left[\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))\,\dot{\Xi}_1(t;\theta)\right] = \\ &\quad= \dot{m}_{\Xi_1}(t)\mathbb{E}^{\theta}\left[\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))\right] + \\ &\quad+ \int_{t_0}^{t}\partial_t C_{\Xi_0\Xi_1}(\tau_0,t)\mathbb{E}^{\theta}\left[\frac{\delta\left\{\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))\right\}}{\delta\Xi_0(\tau_0;\theta)}\right]d\tau_0 + \\ &\quad+ \int_{t_0}^{t}\partial_t C_{\Xi_1\Xi_1}(t,\tau_1)\mathbb{E}^{\theta}\left[\frac{\delta\left\{\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))\right\}}{\delta\Xi_1(\tau_1;\theta)}\right]d\tau_1.\end{aligned} \tag{4.60}$$

By using the product and chain rules for the derivatives while, also, taking into account that there is not any functional dependence between the two excitations, the above expression takes the form

$$\begin{aligned}
&\mathbb{E}^{\theta}\left[\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))\,\dot{\Xi}_1(t;\theta)\right] = \\
&= \dot{m}_{\Xi_1}(t)\mathbb{E}^{\theta}\left[\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))\right] + \\
&+ \int_{t_0}^{t}\partial_t C_{\Xi_0\Xi_1}(\tau_0,t)\mathbb{E}^{\theta}\left[\frac{\delta\delta(x-X(t;\theta))}{\delta X(t;\theta)}\,\delta(u_0-\Xi_0(t;\theta))\,\delta(u_1-\Xi_1(t;\theta))\,V_{\Xi_0(\tau_0)}(t;\theta)\right]d\tau_0 + \\
&+ \int_{t_0}^{t}\partial_t C_{\Xi_0\Xi_1}(\tau_0,t)\mathbb{E}^{\theta}\left[\delta(x-X(t;\theta))\,\frac{\delta\delta(u_0-\Xi_0(t;\theta))}{\delta\Xi_0(t;\theta)}\,\delta(u_1-\Xi_1(t;\theta))\,\frac{\delta\Xi_0(t;\theta)}{\delta\Xi_0(\tau_0;\theta)}\right]d\tau_0 + \\
&+ \int_{t_0}^{t}\partial_t C_{\Xi_1\Xi_1}(t,\tau_1)\mathbb{E}^{\theta}\left[\frac{\delta\delta(x-X(t;\theta))}{\delta X(t;\theta)}\,\delta(u_0-\Xi_0(t;\theta))\,\delta(u_1-\Xi_1(t;\theta))\,V_{\Xi_1(\tau_1)}(t;\theta)\right]d\tau_1 + \\
&+ \int_{t_0}^{t}\partial_t C_{\Xi_1\Xi_1}(t,\tau_1)\mathbb{E}^{\theta}\left[\delta(x-X(t;\theta))\,\delta(u_0-\Xi_0(t;\theta))\,\frac{\delta\delta(u_1-\Xi_1(t;\theta))}{\delta\Xi_1(t;\theta)}\,\frac{\delta\Xi_1(t;\theta)}{\delta\Xi_1(\tau_1;\theta)}\right]d\tau_1 .
\end{aligned} \tag{4.61}$$

where the variational derivatives $V_{\Xi_0(\tau_0)}(t;\theta)$ and $V_{\Xi_1(\tau_1)}(t;\theta)$ are given by Eqs. (3.67) and (3.68), respectively. Finally, by recognizing that $\delta\Xi_i(t;\theta)/\delta\Xi_i(\tau_i;\theta)=\delta(t-\tau_i)$, with $\mathrm{i}=0,1$, utilizing the identity for the delta function and performing the usual manipulations of the delta projection method, Eq. (4.61) is transformed into

$$\begin{aligned}
&\mathbb{E}^{\theta}\left[\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))\,\dot{\Xi}_1(t;\theta)\right] = \\
&= \dot{m}_{\Xi_1}(t)\mathbb{E}^{\theta}\left[\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))\right] + \\
&- \frac{\partial}{\partial x}\int_{t_0}^{t}\partial_t C_{\Xi_0\Xi_1}(\tau_0,t)\mathbb{E}^{\theta}\left[\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))\,V_{\Xi_0(\tau_0)}(t;\theta)\right]d\tau_0 - \\
&\qquad - \partial_t C_{\Xi_0\Xi_1}(\tau_0,t)\Big|_{\tau_0=t}\,\frac{\partial}{\partial u_0}\,\mathbb{E}^{\theta}\left[\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))\right] - \\
&- \frac{\partial}{\partial x}\int_{t_0}^{t}\partial_t C_{\Xi_1\Xi_1}(t,\tau_1)\mathbb{E}^{\theta}\left[\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))\,V_{\Xi_1(\tau_1)}(t;\theta)\right]d\tau_1 - \\
&\qquad - \partial_t C_{\Xi_1\Xi_1}(t,\tau_1)\Big|_{\tau_1=t}\,\frac{\partial}{\partial u_1}\,\mathbb{E}^{\theta}\left[\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))\right].
\end{aligned} \tag{4.62}$$

In Eq. (4.62), the terms $\partial_t C_{\Xi_1\Xi_1}(t,\tau_1)\Big|_{\tau_1=t}$ and $\partial_t C_{\Xi_0\Xi_1}(\tau_0,t)\Big|_{\tau_0=t}$ can equivalently be written as follows:

$$\partial_t C_{\Xi_1\Xi_1}(t,\tau_1)\Big|_{\tau_1=t} = C_{\dot{\Xi}_1\Xi_1}(t,t) = \mathbb{E}^{\theta}[\dot{\Xi}_1(t;\theta)\,\Xi_1(t;\theta)] = \frac{1}{2}\frac{\partial}{\partial t}\mathbb{E}^{\theta}[\Xi_1^2(t;\theta)] = \frac{1}{2}\,\dot{\sigma}^2_{\Xi_1}(t). \tag{4.63a}$$

$$\partial_t C_{\Xi_0\Xi_1}(\tau_0, t)\Big|_{\tau_0 = t} = C_{\dot{\Xi}_0\Xi_1}(t, t). \tag{4.63b}$$

Using, the above two expressions Eq. (4.62) takes the following, final form:

$$\begin{aligned}
&\mathbb{E}^{\theta}\Big[\delta(x, u_0, u_1; X(t), \Xi_0(t), \Xi_1(t))\,\dot{\Xi}_1(t;\theta)\Big] = \\
&= \dot{m}_{\Xi_1}(t)\, f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1) + \\
&- \frac{\partial}{\partial x}\int_{t_0}^{t} \partial_t C_{\Xi_0\Xi_1}(\tau_0, t)\mathbb{E}^{\theta}\Big[\delta(x, u_0, u_1; X(t), \Xi_0(t), \Xi_1(t))\, V_{\Xi_0(\tau_0)}(t;\theta)\Big] d\tau_0 - \\
&- C_{\Xi_0\dot{\Xi}_1}(t, t)\, \frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1)}{\partial u_0} - \\
&- \frac{\partial}{\partial x}\int_{t_0}^{t} \partial_t C_{\Xi_1\Xi_1}(t, \tau_1)\mathbb{E}^{\theta}\Big[\delta(x, u_0, u_1; X(t), \Xi_0(t), \Xi_1(t))\, V_{\Xi_1(\tau_1)}(t;\theta)\Big] d\tau_1 - \\
&- \frac{1}{2}\,\dot{\sigma}^2_{\Xi_1}(t)\, \frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1)}{\partial u_1}.
\end{aligned} \tag{4.64}$$

Accordingly, the other averaged term on the right-hand side of Eq. (4.57) is evaluated into

$$\begin{aligned}
&\mathbb{E}^{\theta}\Big[\delta(x, u_0, u_1; X(t), \Xi_0(t), \Xi_1(t))\,\dot{\Xi}_0(t;\theta)\Big] = \\
&= \dot{m}_{\Xi_0}(t)\, f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1) + \\
&- \frac{\partial}{\partial x}\int_{t_0}^{t} \partial_t C_{\Xi_0\Xi_1}(t, \tau_1)\mathbb{E}^{\theta}\Big[\delta(x, u_0, u_1; X(t), \Xi_0(t), \Xi_1(t))\, V_{\Xi_1(\tau_1)}(t;\theta)\Big] d\tau_0 - \\
&- C_{\dot{\Xi}_0\Xi_1}(t, t)\, \frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1)}{\partial u_1} - \\
&- \frac{\partial}{\partial x}\int_{t_0}^{t} \partial_t C_{\Xi_0\Xi_0}(\tau_0, t)\mathbb{E}^{\theta}\Big[\delta(x, u_0, u_1; X(t), \Xi_0(t), \Xi_1(t))\, V_{\Xi_0(\tau_0)}(t;\theta)\Big] d\tau_1 - \\
&- \frac{1}{2}\,\dot{\sigma}^2_{\Xi_0}(t)\, \frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1)}{\partial u_0}.
\end{aligned} \tag{4.65}$$

Finally, by substituting Eqs. (4.64) and (4.65) into SLE (4.57), we obtain

$$\begin{aligned}
&\frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1)}{\partial t} + \frac{\partial}{\partial x}\Big[\Big(h(x) + q(x)u_1 + \kappa u_0\Big) f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1)\Big] + \\
&\qquad + \dot{m}_{\Xi_0}(t)\frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1)}{\partial u_0} + \dot{m}_{\Xi_1}(t)\frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1)}{\partial u_1} =
\end{aligned}$$

$$= \frac{\partial^2}{\partial x \partial u_0} \int_{t_0}^{t} \partial_t C_{\Xi_0 \Xi_1}(t, \tau_1) \mathbb{E}^{\theta}\Big[\delta(x, u_0, u_1; X(t), \Xi_0(t), \Xi_1(t))\, V_{\Xi_1(\tau_1)}(t;\theta) \Big] d\tau_1 +$$

$$+ \frac{\partial^2}{\partial x \partial u_0} \int_{t_0}^{t} \partial_t C_{\Xi_0 \Xi_0}(\tau_0, t) \mathbb{E}^{\theta}\Big[\delta(x, u_0, u_1; X(t), \Xi_0(t), \Xi_1(t))\, V_{\Xi_0(\tau_0)}(t;\theta) \Big] d\tau_0 +$$

$$+ \frac{1}{2} \dot{\sigma}^2_{\Xi_1}(t) \frac{\partial^2 f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1)}{\partial u_1^2} + \frac{1}{2} \dot{\sigma}^2_{\Xi_0}(t) \frac{\partial^2 f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1)}{\partial u_0^2} +$$

$$+ \frac{\partial^2}{\partial x \partial u_1} \int_{t_0}^{t} \partial_t C_{\Xi_0 \Xi_1}(\tau_0, t) \mathbb{E}^{\theta}\Big[\delta(x, u_0, u_1; X(t), \Xi_0(t), \Xi_1(t))\, V_{\Xi_0(\tau_0)}(t;\theta) \Big] d\tau_0 +$$

$$+ \frac{\partial^2}{\partial x \partial u_1} \int_{t_0}^{t} \partial_t C_{\Xi_1 \Xi_1}(t, \tau_1) \mathbb{E}^{\theta}\Big[\delta(x, u_0, u_1; X(t), \Xi_0(t), \Xi_1(t))\, V_{\Xi_1(\tau_1)}(t;\theta) \Big] d\tau_1 +$$

$$+ C_{\dot{\Xi}_0 \Xi_1}(t, t) \frac{\partial^2 f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1)}{\partial u_0 \partial u_1} + C_{\Xi_0 \dot{\Xi}_1}(t, t) \frac{\partial^2 f_{X(t)\Xi_0(t)\Xi_1(t)}(x, u_0, u_1)}{\partial u_0 \partial u_1} . \tag{4.66}$$

Eq. (4.66) is the ***transformed SLE for the one-time, joint response excitation pdf*** $f_{X(t)\Xi_1(t)}(x, u)$. At this point, is becomes clear that even though the derivation of Eq. (4.66) is not difficult to follow, the apprehended equation is exact and still not closed. Thus, an appropriate approximation scheme must be employed so to obtain an approximate, yet computable alternative of Eq. (4.66).

4.2.2 Novel, one-time evolution equations for the joint response-excitations pdf

As was presented in paragraph 3.2.1, the variational derivatives appearing inside the averaged terms of Eq. (4.66) have been specified to

$$V_{\Xi_0(\tau_0)}(t;\theta) = \kappa \exp\Big(J^{\tau_0}_{h,q}[X(\bullet|^{t}_{\tau_0};\theta)] \Big), \tag{4.67}$$

$$V_{\Xi_1(\tau_1)}(t;\theta) = q\big(\mathrm{X}(\tau_1;\theta)\big) \exp\Big(J^{\tau_1}_{h,q}[X(\bullet|^{t}_{\tau_1};\theta)] \Big), \tag{4.68}$$

in which the following, convenient notation has been utilized:

$$J^{\tau_i}_{h,q}[X(\bullet|^{t}_{\tau_i};\theta)] = \int_{\tau_i}^{t} \Big[h'\big(\mathrm{X}(u;\theta)\big) + q'\big(\mathrm{X}(u;\theta)\big) \Xi_1(u;\theta) \Big] du , \quad i = 0, 1. \tag{4.69}$$

At this point, substituting Eqs. (4.67), (4.68) into Eq. (4.66) results in the following, ***exact, non-closed one-time evolution equation for the joint response-excitations pdf***:

$$\frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)}{\partial t} + \frac{\partial}{\partial x}\Big[\big(h(x)+q(x)u_1+\kappa u_0\big) f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)\Big] +$$
$$+\dot{m}_{\Xi_0}(t)\frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)}{\partial u_0} + \dot{m}_{\Xi_1}(t)\frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)}{\partial u_1} =$$
$$= \frac{\partial^2}{\partial x \partial u_0}\int_{t_0}^{t}\partial_t C_{\Xi_0\Xi_1}(t,\tau_1)\mathbb{E}^\theta\Big[\delta(\cdots)\, q\big(X(\tau_1;\theta)\big)\exp\Big(J_{h,q}^{\tau_1}[X(\bullet|_{\tau_1}^{t};\theta)]\Big)\Big]d\tau_1 +$$
$$+\kappa\frac{\partial^2}{\partial x \partial u_0}\int_{t_0}^{t}\partial_t C_{\Xi_0\Xi_0}(\tau_0,t)\mathbb{E}^\theta\Big[\delta(\cdots)\exp\Big(J_{h,q}^{\tau_0}[X(\bullet|_{\tau_0}^{t};\theta)]\Big)\Big]d\tau_0 + \tag{4.70}$$
$$+\frac{1}{2}\dot{\sigma}^2_{\Xi_1}(t)\frac{\partial^2 f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)}{\partial u_1^2} + \frac{1}{2}\dot{\sigma}^2_{\Xi_0}(t)\frac{\partial^2 f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)}{\partial u_0^2} +$$
$$+\kappa\frac{\partial^2}{\partial x \partial u_1}\int_{t_0}^{t}\partial_t C_{\Xi_0\Xi_1}(\tau_0,t)\mathbb{E}^\theta\Big[\delta(\cdots)\exp\Big(J_{h,q}^{\tau_0}[X(\bullet|_{\tau_0}^{t};\theta)]\Big)\Big]d\tau_0 +$$
$$+\frac{\partial^2}{\partial x \partial u_1}\int_{t_0}^{t}\partial_t C_{\Xi_1\Xi_1}(t,\tau_1)\mathbb{E}^\theta\Big[\delta(\cdots)\, q\big(X(\tau_1;\theta)\big)\exp\Big(J_{h,q}^{\tau_1}[X(\bullet|_{\tau_1}^{t};\theta)]\Big)\Big]d\tau_1 +$$
$$+C_{\dot{\Xi}_0\Xi_1}(t,t)\frac{\partial^2 f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)}{\partial u_0\partial u_1} + C_{\Xi_0\dot{\Xi}_1}(t,t)\frac{\partial^2 f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)}{\partial u_0\partial u_1},$$

where the abbreviation $\delta(x,u_0,u_1;X(t),\Xi_0(t),\Xi_1(t))=\delta(\cdots)$ has been used. Under this notation, it also becomes clear that Eq. (4.70) is non-closed due to the nonlocal terms that have been introduced by the variational derivatives and depend on the whole time history of the response $X(\bullet;\theta)$ and the excitation $\Xi_1(\bullet;\theta)$ .Since inside the product of random delta function the current time of the response and the excitations appears explicitly, we can apply an analogue of the novel, approximation scheme introduced in paragraph 3.1.3 for the exponential terms. First, the integrand of J is decomposed into its mean value

$$R(u) = R_{h'}(u) + R_{q'\Xi}(u) = \mathbb{E}^\theta\big[h'(X(u;\theta))\big] + \mathbb{E}^\theta\big[q'(X(u;\theta))\,\Xi_1(u;\theta)\big], \tag{4.71}$$

and the fluctuations

$$\varphi\big(X(u;\theta),\Xi(u;\theta);R(u)\big) = h'(X(u;\theta)) + q'(X(u;\theta))\,\Xi_1(u;\theta) - R(u). \tag{4.72}$$

Under this decomposition, the nonlocal, exponential terms are equivalently expressed as

$$\exp\Big(J_{h,q}^{\tau_i}[X(\bullet|_{\tau_i}^{t};\theta)]\Big) = \exp\left(\int_{\tau_i}^{t}R(u)\,du\right)\exp\left(\int_{\tau_i}^{t}\varphi\big(X(u;\theta),\Xi_1(u;\theta);R(u)\big)\,du\right). \tag{4.73}$$

Then, a current-time approximation for the fluctuation's integral is utilized,

$$\exp\Big(J_{h,q}^{\tau_i}[X(\bullet|_{\tau_i}^{t};\theta)]\Big) \cong \exp\left(\int_{\tau_i}^{t}R(u)\,du\right)\exp\Big(\varphi\big(X(t;\theta),\Xi_1(t;\theta);R(t)\big)(t-\tau_i)\Big),$$

which is valid under the assumption that the fluctuations are small. Last, a Taylor expansion of the exponential containing the fluctuations is employed

$$\exp\left(J_{h,q}^{\tau_i}[X(\bullet|_{\tau_i}^{t};\theta)]\right) \cong \exp\left(\int_{\tau_i}^{t} R(u)\,du\right)\sum_{m=0}^{M}\frac{(t-\tau_i)^m}{m!}\,\varphi^m\left(X(t;\theta),\,\Xi_1(t;\theta);R(t)\right). \tag{4.74}$$

Eq. (4.74) constitutes the analogue of the novel approximation for the case of an RDE excited by both additive and multiplicative noise. Further, as we have already discussed in paragraph 3.2.2, a current-time approximation should be also performed on the $q\left(X(\tau_1;\theta)\right)$. This is performed via a Taylor expansion around current time t

$$\begin{aligned} q\left(X(\tau_1;\theta)\right) &\cong q(X(t;\theta)) - q'(X(t;\theta))\dot{X}(t;\theta)(t-\tau_1) = q(X(t;\theta)) - \\ &- q'(X(t;\theta))\left(h(X(t;\theta)) + q(X(t;\theta))\,\Xi_0(t;\theta) + \kappa\,\Xi_1(t;\theta)\right)(t-\tau_1). \end{aligned} \tag{4.75}$$

Finally, substituting approximations Eq. (4.74) and (4.75) into Eq. (4.70) result in the following, ***closed, approximate evolution equation for the one-time joint response excitations pdf***:

$$\begin{aligned} &\frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)}{\partial t} + \frac{\partial}{\partial x}\Big[\left(h(x) + q(x)u_1 + \kappa u_0\right) f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)\Big] + \\ &\quad + \dot{m}_{\Xi_0}(t)\frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)}{\partial u_0} + \dot{m}_{\Xi_1}(t)\frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)}{\partial u_1} = \\ &= \frac{\partial^2}{\partial x\,\partial u_0}\left\{ f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)\left[\sum_{m=0}^{M}\frac{1}{m!}\left(\kappa\,D_m^{(1)}[R(\bullet|_{\tau_0}^{t});t] + D_m^{(2)}[R(\bullet|_{\tau_1}^{t});t]q(x) - \right.\right.\right. \\ &\quad \left.\left.\left. - D_m^{(3)}[R(\bullet|_{\tau_1}^{t});t]\,q'(x)\left(h(x)+q(x)u_0+\kappa\,u_1\right)\right)\varphi^m\left(x,u_1;R(t)\right)\right]\right\} + \\ &+ \frac{1}{2}\dot{\sigma}_{\Xi_1}^2(t)\frac{\partial^2 f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)}{\partial u_1^2} + \frac{1}{2}\dot{\sigma}_{\Xi_0}^2(t)\frac{\partial^2 f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)}{\partial u_0^2} + \\ &+ \frac{\partial^2}{\partial x\,\partial u_1}\left\{ f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)\left[\sum_{m=0}^{M}\frac{1}{m!}\left(\kappa\,D_m^{(4)}[R(\bullet|_{\tau_0}^{t});t] + D_m^{(5)}[R(\bullet|_{\tau_1}^{t});t]q(x) - \right.\right.\right. \\ &\quad \left.\left.\left. - D_m^{(6)}[R(\bullet|_{\tau_1}^{t});t]\,q'(x)\left(h(x)+q(x)u_0+\kappa\,u_1\right)\right)\varphi^m\left(x,u_1;R(t)\right)\right]\right\} + \\ &+ C_{\dot{\Xi}_0\Xi_1}(t,t)\frac{\partial^2 f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)}{\partial u_0\,\partial u_1} + C_{\Xi_0\dot{\Xi}_1}(t,t)\frac{\partial^2 f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)}{\partial u_0\,\partial u_1}, \end{aligned} \tag{4.76}$$

where

$$D_m^{(1)}[R(\bullet|_{\tau_0}^{t});t] = \int_{t_0}^{t}\partial_t C_{\Xi_0\Xi_0}(\tau_0,t)\exp\left(\int_{\tau_0}^{t}R(u)\,du\right)(t-\tau_0)^m\,d\tau_0, \tag{4.77a}$$

$$D_m^{(2)}[R(\bullet|_{\tau_1}^t);t] = \int_{t_0}^{t} \partial_t C_{\Xi_0\Xi_1}(t,\tau_1) \exp\left(\int_{\tau_1}^{t} R(u)\,du\right)(t-\tau_1)^m\,d\tau_1\,, \tag{4.77b}$$

$$D_m^{(3)}[R(\bullet|_{\tau_1}^t);t] = \int_{t_0}^{t} \partial_t C_{\Xi_0\Xi_1}(t,\tau_1) \exp\left(\int_{\tau_1}^{t} R(u)\,du\right)(t-\tau_1)^{m+1}\,d\tau_1\,, \tag{4.77c}$$

$$D_m^{(4)}[R(\bullet|_{\tau_0}^t);t] = \int_{t_0}^{t} \partial_t C_{\Xi_0\Xi_1}(\tau_0,t) \exp\left(\int_{\tau_0}^{t} R(u)\,du\right)(t-\tau_0)^m\,d\tau_0\,, \tag{4.77d}$$

$$D_m^{(5)}[R(\bullet|_{\tau_1}^t);t] = \int_{t_0}^{t} \partial_t C_{\Xi_1\Xi_1}(t,\tau_1) \exp\left(\int_{\tau_1}^{t} R(u)\,du\right)(t-\tau_1)^m\,d\tau_1\,, \tag{4.77e}$$

$$D_m^{(6)}[R(\bullet|_{\tau_1}^t);t] = \int_{t_0}^{t} \partial_t C_{\Xi_1\Xi_1}(t,\tau_1) \exp\left(\int_{\tau_1}^{t} R(u)\,du\right)(t-\tau_1)^{m+1}\,d\tau_1\,. \tag{4.77f}$$

At this point, it is useful to point out that even though Eq. (4.76) is lengthy and thereby, difficult to follow, its terms are very similar to the one-time response pdf evolution equation (3.74). Thus, it is readily understood that through Eqs. (4.77), pdf evolution equation (4.76) maintains a tractable amount of probabilistic nonlocality due to the terms $\int_{\tau_i}^{t} R(u)\,du$ which depend on the time-history of the unknown response pdf $f_{X(t)}(x)$. Further, the φ^m terms through $R(t)$ introduce a kind of probabilistic nonlinearity since $R(t)$ depends on the unknown response pdf at the current time t.

In order to present a succinct and thereby, more comprehensive form of pdf evolution equation (4.76), let us set $\mathscr{A}(x,u_0,u_1) = h(x) + q(x)u_1 + \kappa u_0$, $f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1) = f(\cdots)$ and rewrite the said equation as follows:

$$\frac{\partial f(\cdots)}{\partial t} + \frac{\partial}{\partial x}\Big[\mathscr{A}(x,u_0,u_1)\,f(\cdots)\Big] + \dot{m}_{\Xi_0}(t)\frac{\partial f(\cdots)}{\partial u_0} + \dot{m}_{\Xi_1}(t)\frac{\partial f(\cdots)}{\partial u_1} =$$
$$= \frac{\partial^2}{\partial x \partial u_0} f(\cdots)\,\mathscr{B}_0(x,u_0,u_1,t,R[\bullet]) + \frac{\partial^2}{\partial x \partial u_1} f(\cdots)\,\mathscr{B}_1(x,u_0,u_1,t,R[\bullet]) +$$
$$+ \frac{1}{2}\dot{\sigma}^2_{\Xi_1}(t)\frac{\partial^2 f(\cdots)}{\partial u_1^2} + \frac{1}{2}\dot{\sigma}^2_{\Xi_0}(t)\frac{\partial^2 f(\cdots)}{\partial u_0^2} + \Big(C_{\dot{\Xi}_0\Xi_1}(t,t) + C_{\Xi_0\dot{\Xi}_1}(t,t)\Big)\frac{\partial^2 f(\cdots)}{\partial u_0 \partial u_1}, \tag{4.78}$$

where

$$\mathscr{B}_0(x,u_0,u_1,t,R[\bullet]) = \sum_{m=0}^{M}\frac{1}{m!}\Big(\kappa\, D_m^{(1)}[R(\bullet|_{\tau_0}^t);t] + D_m^{(2)}[R(\bullet|_{\tau_1}^t);t]\,q(x) -$$
$$- D_m^{(3)}[R(\bullet|_{\tau_1}^t);t]\,q'(x)\,\mathscr{A}(x,u_0,u_1)\Big)\,\varphi^m\big(x,u_1;R(t)\big), \tag{4.79a}$$

$$\mathscr{B}_1(x,u_0,u_1,t,R[\bullet]) = \sum_{m=0}^{M} \frac{1}{m!}\Big(\kappa\, D_m^{(4)}[R(\bullet|_{\tau_0}^{t});t] + D_m^{(5)}[R(\bullet|_{\tau_1}^{t});t]\,q(x) - \\ - D_m^{(6)}[R(\bullet|_{\tau_1}^{t});t]\,q'(x)\mathscr{A}(x,u_0,u_1)\Big)\varphi^m\big(x,u_1;R(t)\big). \tag{4.79b}$$

Remark 4.1: The marginal response pdf evolution equation coincides with SLE (3.56). Working accordingly to paragraph 4.1.3, we are going to investigate the compatibility of Eq. (4.76) in terms of the marginal response pdf evolution equations that can be derive from it. Integrating both sides of Eq. (4.76) with respect to u_0 and u_1 simultaneously, we obtain

$$\frac{\partial f_{X(t)}(x)}{\partial t} + \frac{\partial}{\partial x}\Big[h(x)\,f_{X(t)}(x)\Big] = \\ = -q(x)\int_{\mathbb{R}^2} u_1\ f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)\,du_0\,du_1 - \\ - \kappa\int_{\mathbb{R}^2} u_0\ f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)\,du_0\,du_1\,. \tag{4.80}$$

Eq. (4.80) is derived under the plausible assumptions

$$f_{X(t)\Xi_0(t)\Xi_1(t)}(x,\pm\infty,\pm\infty) = 0, \tag{4.81a}$$

$$\frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x,\pm\infty,u_1)}{\partial u_0} = 0, \qquad \frac{\partial f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,\pm\infty)}{\partial u_1} = 0. \tag{4.81b,c}$$

Let us, now, evaluate the right-most integral of Eq. (4.80) by using the delta projection formalism

$$\int_{\mathbb{R}^2} u_0\ f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0,u_1)\,du_0\,du_1 = \int_{\mathbb{R}} u_0\ f_{X(t)\Xi_0(t)\Xi_1(t)}(x,u_0)\,du_0 = \\ = \int_{\mathbb{R}} u_0\ \mathbb{E}^{\theta}[\delta(x-X(t;\theta))\,\delta(u_0-\Xi_0(t;\theta))]\,du_0 = \\ = \mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta))\int_{\mathbb{R}} u_0\ \delta(u_0-\Xi_0(t;\theta))\,du_0\Big] = \\ = \mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta))\,\Xi_0(t;\theta)\Big]. \tag{4.82}$$

Operating accordingly for the other integral, Eq. (4.80) is equivalently written as

$$\frac{\partial f_{X(t)}(x)}{\partial t} + \frac{\partial}{\partial x}\Big(h(x)\,f_{X(t)}(x)\Big) = \\ = -\frac{\partial}{\partial x}\Big(q(x)\mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta))\,\Xi_1(t;\theta)\Big]\Big) - \\ - \kappa\,\frac{\partial}{\partial x}\Big(\mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta))\,\Xi_0(t;\theta)\Big]\Big). \tag{4.83}$$

Thus, it is readily seen that Eq. (4.83) is the same as SLE (3.56). This correspondence serves as a preliminary validation for the efficiency of pdf evolution equation (4.76).

Chapter 5

Two-time response pdf evolution equations

In the present chapter we are going to focus on the examination of the scalar, nonlinear additively excited RDE and present another significant extension of this thesis. In particular, we formulate evolution equations governing the joint, two-time response pdf of the system. As it was also explained in the introduction, the consideration of such a problem is not novel to this thesis. Nevertheless, most results were concerned with the stationary properties of the said equations or the two-time correlation of the response e.g. (Hernandez-Machado et al., 1983). Herein, a more holistic approach to the problem is presented that aims to provide computable equations for the joint two-time response pdf.

5.1 The corresponding stochastic Liouville equation

Let us consider once more the scalar, nonlinear, additively RIVP

$$\dot{X}(t;\theta) = h(X(t;\theta)) + \kappa\Xi(t;\theta), \qquad X(t_0;\theta) = X_0(\theta), \qquad (5.1a,b)$$

Commencing in similar fashion, as in sec. 4.1, we represent the sought-for two-time response pdf as the average of the product of two random delta functions. However, in this case, both the random delta functions have as their random argument the response of RIVP (5.1a,b). As such, the two-time response pdf can be expressed as

$$\mathbb{E}^{\theta}\Big[\delta(x_1 - X(t;\theta))\,\delta(x_2 - X(s;\theta))\Big] = \\ = \int_{\mathbb{R}^2} \delta(x_1 - w_1)\,\delta(x_2 - w_2)\, f_{X(t)X(s)}(w_1, w_2)\, dw_1 dw_2 = f_{X(t)X(s)}(x_1, x_2). \qquad (5.2)$$

where the two time instances t, s are considered different; $s \neq t$. Note that Eq. (5.2) is not valid for $s = t$, since, in this case, the delta representation reads

$$\mathbb{E}^{\theta}\Big[\delta(x_1 - X(t;\theta))\,\delta(x_2 - X(t;\theta))\Big] = \int_{\mathbb{R}} \delta(x_1 - w_1)\,\delta(x_2 - w_1)\, f_{X(t)}(w_1)\, dw_1, \qquad (5.3)$$

and, in the right-hand side of Eq. (5.3), the single integral containing the two delta functions is not defined. On the other hand, the fact that delta projection (5.3) fails for $t = s$ does not diminish the importance of formulating evolution equations for $f_{X(t)X(s)}(x_1, x_2)$, since it is the $t \neq s$ case that interests us. For $t = s$, pdf $f_{X(t)X(t)}(x_1, x_2)$ is just a duplication of the one-time response pdf $f_{X(t)}(x)$, whose evolution equation has already been specified into Eq. (3.36).

Let us, now, differentiate both sides of Eq. (5.2) with respect to time t, while time s is treated as parameter

$$\frac{\partial f_{X(t)X(s)}(x_1, x_2)}{\partial t} = \mathbb{E}^{\theta}\left[\frac{\partial \delta(x_1 - X(t;\theta))}{\partial t}\, \delta(x_2 - X(s;\theta))\right]. \tag{5.4}$$

Then, by using the chain rule and substituting $\dot{X}(t;\theta)$ from RDE (5.1a), we obtain

$$\begin{aligned}\frac{\partial f_{X(t)X(s)}(x_1, x_2)}{\partial t} &= \mathbb{E}^{\theta}\left[\frac{\partial \delta(x_1 - X(t;\theta))}{\partial X(t;\theta)}\, h(X(t;\theta))\delta(x_2 - X(s;\theta))\right] + \\ &+ \mathbb{E}^{\theta}\left[\frac{\partial \delta(x_1 - X(t;\theta))}{\partial X(t;\theta)}\, q(X(t;\theta))\, \delta(x_2 - X(s;\theta))\, \Xi(t;\theta)\right].\end{aligned} \tag{5.5}$$

In the right-hand side of Eq. (5.5), both averaged terms can be expressed and further evaluated by making use of the delta projection formalism as follows:

$$\begin{aligned}&\mathbb{E}^{\theta}\left[\frac{\partial \delta(x_1 - X(t;\theta))}{\partial X(t;\theta)}\, h(X(t;\theta))\, \delta(x_2 - X(s;\theta))\right] = \\ &\quad = \int_{\mathbb{R}^2} \frac{\partial \delta(x_1 - w_1)}{\partial w_1}\, h(w_1)\, \delta(x_2 - w_2)\, f_{X(t)X(s)}(w_1, w_2)\, dw_1 dw_2 = \\ &\quad = -\frac{\partial}{\partial x_1}\left(h(x_1)\, f_{X(t)X(s)}(x_1, x_2)\right)\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}&\mathbb{E}^{\theta}\left[\frac{\partial \delta(x_1 - X(t;\theta))}{\partial X(t;\theta)}\, q(X(t;\theta))\, \delta(x_2 - X(s;\theta))\, \Xi(t;\theta)\right] = \\ &\quad = \int_{\mathbb{R}^3} \frac{\partial \delta(x_1 - w_1)}{\partial w_1}\, q(w_1)\, \delta(x_2 - w_2)\, z\, f_{X(t)X(s)\Xi(t)}(w_1, w_2, z)\, dw_1 dw_2 dz = \\ &\quad = -\frac{\partial}{\partial x_1}\left(q(x_1)\int_{\mathbb{R}^2} \delta(x_2 - w_2)\, z\, f_{X(t)X(s)\Xi(t)}(x_1, w_2, z)\, dw_2 dz\right) = \\ &\quad = -\frac{\partial}{\partial x_1}\left(q(x_1)\int_{\mathbb{R}^3} \delta(x_1 - w_1)\, \delta(x_2 - w_2)\, z\, f_{X(t)X(s)\Xi(t)}(w_1, w_2, z)\, dw_1 dw_2 dz\right) = \\ &\quad = -\frac{\partial}{\partial x_1}\left(q(x_1)\, \mathbb{E}^{\theta}\left[\delta(x_1 - X(t;\theta))\, \delta(x_2 - X(s;\theta))\, \Xi(t;\theta)\right]\right).\end{aligned} \tag{5.7}$$

Substitution of Eqs. (5.6), (5.7) into Eq. (5.5) results into the following ***stochastic Liouville equation for the two-time response pdf*** $f_{X(t)X(s)}(x_1, x_2)$ pertaining to RIVP (5.1a,b):

$$\begin{aligned}&\frac{\partial f_{X(t)X(s)}(x_1, x_2)}{\partial t} + \frac{\partial}{\partial x_1}\left(h(x_1)\, f_{X(t)X(s)}(x_1, x_2)\right) = \\ &\quad = -\kappa\, \frac{\partial}{\partial x_1}\, \mathbb{E}^{\theta}\left[\delta\big(x_1 - X(t;\theta)\big)\, \delta\big(x_2 - X(s;\theta)\big)\, \Xi(t;\theta)\right], \qquad t \neq s.\end{aligned} \tag{5.8}$$

Consistent with the delta representation (5.2), SLE (5.8) is a differential equation with respect to time t, while time s enters a parameter. As a consequence, the initial condition needed to solve SLE (5.8)[1] should also be parametric with respect to s

$$f_{X(t_0)\,X(s)}(x_1, x_2) = f_{X_0\,X(s)}(x_1, x_2). \tag{5.9}$$

In Eq. (5.9), and contrary to the initial conditions (3.14) and (4.11) of the SLEs for one-time response and one-time response-excitation pdfs, respectively, we observe that the joint response-initial value pdf $f_{X_0\,X(s)}(x_1, x_2)$, is not part of the data of RIVP (5.1a,b), since it models the statistical dependence of the response (at time t) and its initial value. Thus, for determining pdf $f_{X_0\,X(s)}(x_1, x_2)$, we have to solve another pdf evolution equation, starting by formulating the stochastic Liouville equation for the response-initial value pdf in the following section 5.2.

Eq. (5.8) is the same SLE for two-times also derived in (Hernandez-Machado et al., 1983). However, in the aforementioned work, as well as in others of the same research team (F. Sagués et al., 1984; Sancho & San Miguel, 1989), parameter time s is always considered before evolution time t, $s < t$. While such an assumption may be conceptually more convenient, the delta projection method also works for $s > t$. Note also that, under the assumption $s < t$, the aforementioned works consider as initial condition not Eq. (5.9), but $f_{X(s)\,X(s)}(x_1, x_2) = f_{X(s)}(x_1)\,\delta(x_1 - x_2)$, i.e. for $t = s$, under the understanding that $f_{X(s)}(x_1)$ can be calculated by solving the appropriate one-time evolution equation. Thus, for two-time response pdfs, the scheme of calculating the initial condition by solving another evolution equation is present both in the existing literature and in our approach.

Transformed SLE. Consistent with all the prior examined cases, SLE (5.8) in non-closed due to the averaged term on its rightmost side. Thus, anew revoking the dependence of the response $X(t;\theta)$ on initial value and excitation, the averaged term can be equivalently expressed using the familiar notation as

$$\begin{aligned} &\mathbb{E}^{\theta}\Big[\delta\big(x_1 - X(t;\theta)\big)\,\delta\big(x_2 - X(s;\theta)\big)\,\Xi(t;\theta)\Big] = \\ &= \mathbb{E}^{\theta}\Big[\delta\Big(x_1 - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\Big)\,\delta\Big(x_2 - X[X_0(\theta);\Xi(\bullet|_{t_0}^{s};\theta)]\Big)\,\Xi(t;\theta)\Big]. \end{aligned} \tag{5.10}$$

Expressing the averaged term as in Eq. (5.10) is of pivotal importance for the application of the required, extended NF theorem. Nevertheless, before we are able to proceed with the implementation of the theorem, the product of the two random delta functions must be expressed as an appropriate FFℓ. This is easily achieved by considering the product of random delta functions as a FFℓ of initial value $X_0(\theta)$ and excitation $\Xi(\bullet;\theta)$ over the time interval $[t_0, t_1]$, with $t_1 = \max(t, s)$. Under this convenient notation, and regardless of the time ordering of t, s, the product of random delta functions can always be regarded as a FFℓ like $\mathcal{F} = \mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t_1};\theta)]$. Thus, by employing Eq. (2.19) the averaged term can be calculated as follows:

[1] More specifically, its closed solvable approximation which will be derived in Section 5.3.

$$
\begin{aligned}
&\mathbb{E}^{\theta}\Big[\Xi(t;\theta)\,\delta\big(x_1 - X(t;\theta)\big)\delta\big(x_2 - X(s;\theta)\big)\Big] = \\
&\qquad = m_{\Xi}(t)\,\mathbb{E}^{\theta}\Big[\delta\big(x_1 - X(t;\theta)\big)\delta\big(x_2 - X(s;\theta)\big)\Big] + \\
&\qquad\qquad + C_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\left[\frac{\partial\big\{\delta\big(x_1 - X(t;\theta)\big)\delta\big(x_2 - X(s;\theta)\big)\big\}}{\partial X_0(\theta)}\right] + \\
&\qquad\qquad + \int_{t_0}^{t_1} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\frac{\delta\big\{\delta\big(x_1 - X(t;\theta)\big)\delta\big(x_2 - X(s;\theta)\big)\big\}}{\delta\Xi(\tau;\theta)}\right] d\tau .
\end{aligned}
\tag{5.11}
$$

Using, now, the product and chain rules for the random delta functions, the above expression can be further evaluated into

$$
\begin{aligned}
&\mathbb{E}^{\theta}\Big[\delta(x_1 - X(t;\theta))\,\delta(x_2 - X(s;\theta))\,\Xi(t;\theta)\Big] = m_{\Xi}(t)\; f_{X(t)X(s)}(x_1, x_2) + \\
&\qquad + C_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\left[\frac{\partial\delta(x_1 - X(t;\theta))}{\partial X(t;\theta)}\,\delta(x_2 - X(s;\theta))\,V_{X_0}(t;\theta)\right] + \\
&\qquad + C_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\left[\delta(x_1 - X(t;\theta))\,\frac{\partial\delta(x_2 - X(s;\theta))}{\partial X(s;\theta)}\,V_{X_0}(s;\theta)\right] + \\
&\quad + \int_{t_0}^{t_1} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\frac{\partial\delta(x_1 - X(t;\theta))}{\partial X(t;\theta)}\,\delta(x_2 - X(s;\theta))\,V_{\Xi(\tau)}(t;\theta)\right] d\tau + \\
&\quad + \int_{t_0}^{t_1} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\delta(x_1 - X(t;\theta))\,\frac{\partial\delta(x_2 - X(s;\theta))}{\partial X(s;\theta)}\,V_{\Xi(\tau)}(s;\theta)\right] d\tau ,
\end{aligned}
\tag{5.12}
$$

where the usual variational derivatives $V_{X_0}(t;\theta)$ and $V_{\Xi(\tau)}(t;\theta)$, also, appear. Furthermore, due to causality, variational derivatives $V_{\Xi(\tau)}(t;\theta) = \delta X(t;\theta)/\delta\Xi(\tau;\theta)$ and $V_{\Xi(\tau)}(s;\theta) = \delta X(s;\theta)/\delta\Xi(\tau;\theta)$, are zero for $t < \tau$ and $s < \tau$, respectively since a variation of excitation, $\delta\Xi(\bullet;\theta)$, at a certain time instance τ cannot result in variation of the response in previous time instances. Thus, the upper limits of the integrals in Eq. (5.12) are adjusted accordingly to

$$
\begin{aligned}
&\mathbb{E}^{\theta}\Big[\delta(x_1 - X(t;\theta))\,\delta(x_2 - X(s;\theta))\,\Xi(t;\theta)\Big] = m_{\Xi}(t)\; f_{X(t)X(s)}(x_1, x_2) + \\
&\qquad + C_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\left[\frac{\partial\delta(x_1 - X(t;\theta))}{\partial X(t;\theta)}\,\delta(x_2 - X(s;\theta))\,V_{X_0}(t;\theta)\right] + \\
&\qquad + C_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\left[\delta(x_1 - X(t;\theta))\,\frac{\partial\delta(x_2 - X(s;\theta))}{\partial X(s;\theta)}\,V_{X_0}(s;\theta)\right] + \\
&\quad + \int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\frac{\partial\delta(x_1 - X(t;\theta))}{\partial X(t;\theta)}\,\delta(x_2 - X(s;\theta))\,V_{\Xi(\tau)}(t;\theta)\right] d\tau + \\
&\quad + \int_{t_0}^{s} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\delta(x_1 - X(t;\theta))\,\frac{\partial\delta(x_2 - X(s;\theta))}{\partial X(s;\theta)}\,V_{\Xi(\tau)}(s;\theta)\right] d\tau .
\end{aligned}
\tag{5.13}
$$

Finally, substituting Eq. (5.13) into SLE (5.8) and using the delta projection method's formalism, we obtain the following, ***transformed SLE for the two-time response pdf*** $f_{X(t)X(s)}(x_1, x_2)$:

$$\frac{\partial f_{X(t)X(s)}(x_1, x_2)}{\partial t} + \frac{\partial}{\partial x_1}\left[\left(h(x_1) + \kappa m_{\Xi}(t)\right) f_{X(t)X(s)}(x_1, x_2)\right] =$$
$$= \kappa C_{X_0\Xi}(t) \frac{\partial^2}{\partial x_1^2} \mathbb{E}^{\theta}\left[\delta\left(x_1 - X(t;\theta)\right)\delta\left(x_2 - X(s;\theta)\right) V_{X_0}(t;\theta)\right] +$$
$$+ \kappa C_{X_0\Xi}(t) \frac{\partial^2}{\partial x_1 \partial x_2} \mathbb{E}^{\theta}\left[\delta\left(x_1 - X(t;\theta)\right)\delta\left(x_2 - X(s;\theta)\right) V_{X_0}(s;\theta)\right] + \tag{5.14}$$
$$+ \kappa \frac{\partial^2}{\partial x_1^2} \int_{t_0}^{t} C_{\Xi\Xi}(t, \tau)\, \mathbb{E}^{\theta}\left[\delta\left(x_1 - X(t;\theta)\right)\delta\left(x_2 - X(s;\theta)\right) V_{\Xi(\tau)}(t;\theta)\right] d\tau +$$
$$+ \kappa \frac{\partial^2}{\partial x_1 \partial x_2} \int_{t_0}^{s} C_{\Xi\Xi}(t, \tau)\, \mathbb{E}^{\theta}\left[\delta\left(x_1 - X(t;\theta)\right)\delta\left(x_2 - X(s;\theta)\right) V_{\Xi(\tau)}(s;\theta)\right] d\tau .$$

As expected, the nonlocal variational derivatives appearing inside the averages terms of the above expression constitute SLE (5.14) non-closed. Thus, an approximation that results in a computable alternative must be implemented; this is performed in section 5.3.

5.2 The auxiliary stochastic Liouville equation for the initial value-response pdf

In this section, we are going to formulate the stochastic Liouville equation for the joint response-initial value pdf that supplements SLE (5.14). Our starting point, is once more the delta representation which in this case reads

$$\mathbb{E}^{\theta}\left[\delta(x_0 - X_0(\theta))\,\delta(x_1 - X(t;\theta))\right] =$$
$$= \int_{\mathbb{R}^2} \delta(x_0 - w_0)\,\delta(x_1 - w_1)\, f_{X_0 X(t)}(w_0, w_1)\, dw_0\, dw_1 = f_{X_0 X(t)}(x_0, x_1). \tag{5.15}$$

Then, differentiation of Eq. (5.15) with respect to t, yields

$$\frac{\partial f_{X_0 X(t)}(x_0, x_1)}{\partial t} = \mathbb{E}^{\theta}\left[\delta(x_0 - X_0(\theta)) \frac{\partial \delta(x_1 - X(t;\theta))}{\partial t}\right], \tag{5.16}$$

which can be further evaluated using the chain rule for the derivative of the delta function as well as employing RDE (5.1a) into

$$\frac{\partial f_{X_0 X(t)}(x_0, x_1)}{\partial t} = \mathbb{E}^{\theta}\left[\delta(x_0 - X_0(\theta)) \frac{\partial \delta(x_1 - X(t;\theta))}{\partial X(t;\theta)} h(X(t;\theta))\right] +$$
$$+ \kappa\, \mathbb{E}^{\theta}\left[\delta(x_0 - X_0(\theta)) \frac{\partial \delta(x_1 - X(t;\theta))}{\partial X(t;\theta)} \Xi(t;\theta)\right]. \tag{5.17}$$

Each of the averaged terms on the right-hand side of Eq. (5.17) are subsequently calculated by making use of the usual delta projection method manipulations:

$$\mathbb{E}^{\theta}\left[\delta(x_0 - X_0(\theta))\,\frac{\partial\delta(x_1 - X(t;\theta))}{\partial X(t;\theta)}\,h(X(t;\theta))\right] =$$

$$= \int_{\mathbb{R}^2}\delta(x_0 - w_0)\,\frac{\partial\delta(x_1 - w_1)}{\partial w_1}\,h(w_1)\,f_{X_0\,X(t)}(w_0,\,w_1)\,dw_0\,dw_1 =$$

$$= -\frac{\partial}{\partial x_1}\Big(h(x_1)\,f_{X_0\,X(t)}(x_0,\,x_1)\Big). \tag{5.18}$$

and

$$\mathbb{E}^{\theta}\left[\delta(x_0 - X_0(\theta))\,\frac{\partial\delta(x_1 - X(t;\theta))}{\partial X(t;\theta)}\,\Xi(t;\theta)\right] =$$

$$= \int_{\mathbb{R}^3}\delta(x_0 - w_0)\,\frac{\partial\delta(x_1 - w_1)}{\partial w_1}\,z\,f_{X_0\,X(t)\,\Xi(t)}(w_0,\,w_1,\,z)\,dw_0\,dw_1\,dz =$$

$$= -\frac{\partial}{\partial x_1}\int_{\mathbb{R}^2}\delta(x_0 - w_0)\,z\,f_{X_0\,X(t)\,\Xi(t)}(w_0,\,x,\,z)\,dw_0\,dz =$$

$$= -\frac{\partial}{\partial x_1}\int_{\mathbb{R}^3}\delta(x_0 - w_0)\,\delta(x_1 - w_1)\,z\,f_{X_0\,X(t)\,\Xi(t)}(w_0,\,w_1,\,z)\,dw_0\,dw_1\,dz =$$

$$= -\frac{\partial}{\partial x_1}\,\mathbb{E}^{\theta}\Big[\delta(x_0 - X_0(\theta))\,\delta(x_1 - X(t;\theta))\,\Xi(t;\theta)\Big]. \tag{5.19}$$

Combining now Eqs. (5.17) – (5.20) results in

$$\frac{\partial f_{X_0 X(t)}(x_0,\,x_1)}{\partial t} + \frac{\partial}{\partial x_1}\Big[h(x_1)\;f_{X_0 X(t)}(x_0,\,x_1)\Big] =$$
$$= -\,\kappa\,\frac{\partial}{\partial x_1}\,\mathbb{E}^{\theta}\Big[\delta\Big(x_0 - X_0(\theta)\Big)\,\delta\Big(x_1 - X(t;\theta)\Big)\,\Xi(t;\theta)\Big]. \tag{5.20}$$

Eq. (5.20) is the required ***initial value-response stochastic Liouville equation***, which is also supplemented with the following initial condition:

$$f_{X_0 X(t_0)}(x_0,\,x_1) = f_{X_0 X_0}(x_0,\,x_1). \tag{5.21}$$

From Eq. (5.21), it becomes apparent that the initial condition, pdf $f_{X_0\,X_0}(x_0,\,x_1)$, is just the duplication of initial value pdf $f_{X_0}(x_0)$. By identifying the conditional probability distribution as $f_{X_0|X_0}(x_1|\,x_0) = \delta(x_0 - x_1)$, Eq. (5.21) is elaborated as

$$f_{X_0\,X(t_0)}(x_0,\,x_1) = f_{X_0\,X_0}(x_0,\,x_1) = f_{X_0|X_0}(x_1|\,x_0)\;f_{X_0}(x_0) =$$
$$= f_{X_0}(x_0)\,\delta(x_0 - x_1). \tag{5.22}$$

Thus, it is readily seen that the initial distribution for SLE (5.20) is the one-dimensional $f_{X_0}(x_0)$, placed on the diagonal $x_0 = x_1$ of the two-dimensional plane $(x_0,\,x_1)$.

Moreover, for SLE (5.20), the long-time behavior of the joint initial value-response pdf can also be recognized as $f_{X_0 X(t)}(x_0, x_1) = f_{X_0}(x_0)\, f_{X(t)}(x_1)$. This "final" condition states the fact that after an adequately long time-interval the effects of the initial value $X_0(\theta)$ will have no impact to the response $X(t;\theta)$, thereby constituting their pdfs $f_{X_0}(x_0)$, $f_{X(t)}(x_1)$ uncorrelated.

Transformed auxiliary SLE. At this point, we are going to proceed, as per usual, with the evaluation of the non-closed averaging on the right-hand side of Eq. (5.20). This is easily performed by reverting to the familiar notation for representing the response as a random FFℓ. Hence, the product of the two random delta functions can be rewritten as $\delta\big(x_0 - X_0(\theta)\big)\, \delta\Big(x_1 - X[X_0(\theta); \Xi(\bullet|_{t_0}^{t}; \theta)]\Big)$ and thus, collectively regarded as a functional like $\mathcal{F} = \mathcal{F}[X_0(\theta); \Xi(\bullet|_{t_0}^{t_1}; \theta)]$. This allows us to implement Eq. (2.19) of the extended NF theorem which yields

$$\begin{aligned}
&\mathbb{E}^{\theta}\Big[\delta(x_0 - X_0(\theta))\, \delta(x_1 - X(t;\theta))\, \Xi(t;\theta)\Big] = \\
&= m_{\Xi}(t)\, f_{X_0 X(t)}(x_0, x_1) + C_{X_0 \Xi}(t)\, \mathbb{E}^{\theta}\left[\frac{\partial \delta(x_0 - X_0(\theta))}{\partial X_0(\theta)}\, \delta(x_1 - X(t;\theta))\right] + \\
&+ C_{X_0 \Xi}(t)\, \mathbb{E}^{\theta}\left[\delta(x_0 - X_0(\theta))\, \frac{\partial \delta(x_1 - X(t;\theta))}{\partial X(t;\theta)}\, V_{X_0}(t;\theta)\right] + \\
&+ \int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\, \mathbb{E}^{\theta}\left[\delta(x_0 - X_0(\theta))\, \frac{\partial \delta(x_1 - X(t;\theta))}{\partial X(t;\theta)}\, V_{\Xi(\tau)}(t;\theta)\right] d\tau.
\end{aligned} \tag{5.23}$$

Finally, by carrying out the regular manipulations of the delta projection and substituting the apprehended into SLE (5.20), we obtain

$$\begin{aligned}
&\frac{\partial f_{X_0 X(t)}(x_0, x_1)}{\partial t} + \frac{\partial}{\partial x_1}\Big[\big(h(x_1) + \kappa\, m_{\Xi}(t)\big)\, f_{X_0 X(t)}(x_0, x_1)\Big] = \\
&= \kappa\, C_{X_0 \Xi}(t)\, \frac{\partial^2 f_{X_0 X(t)}(x_0, x_1)}{\partial x_0\, \partial x_1} + \\
&+ \kappa\, C_{X_0 \Xi}(t)\, \frac{\partial^2}{\partial x_1^2}\, \mathbb{E}^{\theta}\Big[\delta\big(x_0 - X_0(\theta)\big)\, \delta\big(x_1 - X(t;\theta)\big)\, V_{X_0}(t;\theta)\Big] + \\
&+ \kappa\, \frac{\partial^2}{\partial x_1^2} \int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\, \mathbb{E}^{\theta}\Big[\delta\big(x_0 - X_0(\theta)\big)\, \delta\big(x_1 - X(t;\theta)\big)\, V_{\Xi(\tau)}(t;\theta)\Big] d\tau.
\end{aligned} \tag{5.24}$$

Eq. (5.24) is the ***transformed SLE for the initial value-response pdf*** $f_{X_0 X(t)}(x_0, x_1)$. As in all of our previous cases, the appearance of the variational derivatives inside the two averaged terms on the rightmost side of SLE (5.24) constitute the latter nonlocal and non-closed. Therefore, in order to obtain a computable alternative to both SLEs (5.14), (5.24) an effective approximation scheme must be utilized, as is performed in the following section.

5.3 Novel, two-time response pdf evolution equations

Observing SLEs (5.14) and (5.24), it is easy to see that the variational derivatives are the same with the ones calculated in paragraph 4.1.2. Thus, using Eqs. (3.24) and (3.30) as well as the notation

$$I_{h'}[X(\bullet|_{\tau}^{t};\theta)] = \int_{\tau}^{t} h'\big(X(u;\theta)\big)\,du\,, \tag{5.25}$$

the aforementioned SLEs can be further transformed into

$$\begin{aligned}
&\frac{\partial f_{X(t)X(s)}(x_1,x_2)}{\partial t} + \frac{\partial}{\partial x_1}\Big[\big(h(x_1)+\kappa m_{\Xi}(t)\big)\, f_{X(t)X(s)}(x_1,x_2)\Big] = \\
&= \kappa C_{X_0\Xi}(t)\,\frac{\partial^2}{\partial x_1^2}\,\mathbb{E}^{\theta}\Big[\delta\big(x_1-X(t;\theta)\big)\,\delta\big(x_2-X(s;\theta)\big)\exp\Big(I_{h'}[X(\bullet|_{t_0}^{t};\theta)]\Big)\Big] + \\
&+ \kappa C_{X_0\Xi}(t)\,\frac{\partial^2}{\partial x_1\partial x_2}\,\mathbb{E}^{\theta}\Big[\delta\big(x_1-X(t;\theta)\big)\,\delta\big(x_2-X(s;\theta)\big)\exp\Big(I_{h'}[X(\bullet|_{t_0}^{s};\theta)]\Big)\Big] + \\
&+ \kappa^2\,\frac{\partial^2}{\partial x_1^2}\int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\Big[\delta\big(x_1-X(t;\theta)\big)\,\delta\big(x_2-X(s;\theta)\big)\exp\Big(I_{h'}[X(\bullet|_{\tau}^{t};\theta)]\Big)\Big]\,d\tau + \\
&+ \kappa^2\,\frac{\partial^2}{\partial x_1\partial x_2}\int_{t_0}^{s} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\Big[\delta\big(x_1-X(t;\theta)\big)\,\delta\big(x_2-X(s;\theta)\big)\exp\Big(I_{h'}[X(\bullet|_{\tau}^{s};\theta)]\Big)\Big]\,d\tau
\end{aligned} \tag{5.26}$$

and

$$\begin{aligned}
&\frac{\partial f_{X_0X(t)}(x_0,x_1)}{\partial t} + \frac{\partial}{\partial x_1}\Big[\big(h(x_1)+\kappa m_{\Xi}(t)\big)\, f_{X_0X(t)}(x_0,x_1)\Big] = \\
&= \kappa\, C_{X_0\Xi}(t)\,\frac{\partial^2 f_{X_0X(t)}(x_0,x_1)}{\partial x_0\,\partial x_1} + \\
&+ \kappa\, C_{X_0\Xi}(t)\,\frac{\partial^2}{\partial x_1^2}\,\mathbb{E}^{\theta}\Big[\delta\big(x_0-X_0(\theta)\big)\,\delta\big(x_1-X(t;\theta)\big)\exp\Big(I_{h'}[X(\bullet|_{t_0}^{t};\theta)]\Big)\Big] + \\
&+ \kappa^2\,\frac{\partial^2}{\partial x_1^2}\int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\Big[\delta\big(x_0-X_0(\theta)\big)\,\delta\big(x_1-X(t;\theta)\big)\exp\Big(I_{h'}[X(\bullet|_{\tau}^{t};\theta)]\Big)\Big]\,d\tau.
\end{aligned} \tag{5.27}$$

In this notation, it becomes obvious that the nonlocal terms inside the averages of the above expressions are of the same form to the ones appeared in the one-time response pdf evolution equation $f_{X(t)}(x)$, Eq. (3.32). Thus, the identical approximation scheme presented in paragraph 3.1.3 can be employed. For the transformed SLE (5.26), this results in the following ***closed, approximate two-time pdf evolution equation***, which is valid for $t \neq s$:

$$\frac{\partial f_{X(t)X(s)}(x_1,x_2)}{\partial t} + \frac{\partial}{\partial x_1}\Big[\big(h(x_1)+\kappa m_{\Xi}(t)\big)\, f_{X(t)X(s)}(x_1,x_2)\Big] =$$

$$= \frac{\partial^2}{\partial x_1^2} \left\{ \left[\sum_{m=0}^{M} \frac{1}{m!} D_m^{\text{eff}} \left[R_{h'}(\bullet), t, t \right] \varphi_{h'}^m \left(x_1 ; R_{h'}(t) \right) \right] f_{X(t)X(s)}(x_1, x_2) \right\} + \tag{5.28}$$

$$+ \frac{\partial^2}{\partial x_1 \partial x_2} \left\{ \left[\sum_{m=0}^{M} \frac{1}{m!} D_m^{\text{eff}} \left[R_{h'}(\bullet), t, s \right] \varphi_{h'}^m \left(x_2 ; R_{h'}(s) \right) \right] f_{X(t)X(s)}(x_1, x_2) \right\},$$

where

$$D_m^{\text{eff}} \left[R_{h'}(\bullet), t, s \right] = \kappa \exp\left(\int_{t_0}^{s} R_{h'}(u)\, du \right) C_{X_0 \Xi}(t)\, (s - t_0)^m + \\ + \kappa^2 \int_{t_0}^{s} \exp\left(\int_{\tau}^{s} R_{h'}(u)\, du \right) C_{\Xi\Xi}(t, \tau)\, (s - \tau)^m \, d\tau . \tag{5.29}$$

Note that in Eq. (5.28) the term $D_m^{\text{eff}} \left[R_{h'}(\bullet), t, t \right]$ coincides with the generalized effective noise intensity Eq. (3.37). Accordingly, application of the approximation scheme for the transformed SLE (5.27) yields the following ***closed, approximate initial value-response pdf evolution equation***:

$$\frac{\partial f_{X_0 X(t)}(x_0, x_1)}{\partial t} + \frac{\partial}{\partial x_1} \left[\left(h(x_1) + \kappa m_{\Xi}(t) \right) f_{X_0 X(t)}(x_0, x_1) \right] = \\ = \kappa\, C_{X_0 \Xi}(t) \frac{\partial^2 f_{X_0 X(t)}(x_0, x_1)}{\partial x_0 \partial x_1} + \\ + \frac{\partial^2}{\partial x_1^2} \left\{ \left[\sum_{m=0}^{M} \frac{1}{m!} D_m^{\text{eff}} \left[R_{h'}(\bullet), t, t \right] \varphi_{h'}^m \left(x_1 ; R_{h'}(t) \right) \right] f_{X_0 X(t)}(x_0, x_1) \right\}. \tag{5.30}$$

As was explained in many previous cases, through the coefficients D_m and the terms $\varphi_{h'}^m$, the novel pdf equations (5.28), (5.30) retain a trackable amount of nonlocality (in time) and nonlinearity, reflecting the non-Markovian character of the response.

Example of compatibility. Let us now scrutinize the compatibility of the methodology used to derive two-time pdf evolution equations with its counterpart for one-time pdf evolution equations. This is performed by integrating both sides of Eq. (5.28) with respect to x_2 and under the plausible assumption $f_{X(t)X(s)}(x_1, \pm\infty) = 0$. Thus, we find

$$\frac{\partial f_{X(t)}(x_1)}{\partial t} + \frac{\partial}{\partial x_1} \left[\left(h(x_1) + \kappa m_{\Xi}(t) \right) f_{X(t)}(x_1) \right] = \\ = \frac{\partial^2}{\partial x_1^2} \left\{ \left[\sum_{m=0}^{M} \frac{1}{m!} D_m \left[R_{h'}(\bullet), t, t \right] \varphi_{h'}^m \left(x_1 ; R_{h'}(t) \right) \right] f_{X(t)}(x_1) \right\}, \tag{5.31}$$

which is identical to pdf evolution equation (3.36). Furthermore, Eq. (5.31) is also retrieved by integrating both sides of Eq. (5.30) with respect to x_0. The correspondence of the pdf evolution equation, obtained as a marginal of the two-time response pdf evolution Eq. (5.28) as well as of the initial value-response pdf evolution equation (5.30) with Eq. (3.36) derived in paragraph

3.1.3 demonstrates the compatibility of the novel, extended methodology presented in this chapter with the one it is founded upon.

5.4 Exact pdfs for a linear, additively excited RDE

In this section, we are going to examine the results of this methodology in the linear case. By setting $h(x) = \eta x$, $\eta < 0$, in RDE (5.1a), we obtain the RIVP

$$\dot{X}(t;\theta) = \eta X(t;\theta) + \kappa\, \Xi(t;\theta), \qquad X(t_0;\theta) = X_0(\theta). \tag{5.32a,b}$$

As was already discussed, in this case the variational derivatives are explicitly calculated as $V_{X_0}(t;\theta) = e^{\eta(t-t_0)}$ and $V_{\Xi(\tau)}(t;\theta) = \kappa\, e^{\eta(t-\tau)}$.

It is a well-known result that the response process of any linear system, with Gaussian initial distribution, to an additive Gaussian excitation (either colored or white) is also a Gaussian process. Hence, the joint pdf of two Gaussian processes can likewise be identified as a Gaussian process whose moments can be specified by formulating and solving their corresponding moment problems, as it is performed in sec. B.2.

5.4.1 Exact, auxiliary initial value-response pdf

Using the aforementioned calculated variational derivatives, the joint initial value-response SLE (5.24) is specified into

$$\begin{aligned}\frac{\partial f_{X_0 X(t)}(x_0, x_1)}{\partial t} &+ \frac{\partial}{\partial x_1}\left[\left(h(x_1) + \kappa\, m_\Xi(t)\right) f_{X_0 X(t)}(x_0, x_1)\right] = \\ &= \kappa\, C_{X_0\Xi}(t)\, \frac{\partial^2 f_{X_0 X(t)}(x_0, x_1)}{\partial x_0 \partial x_1} + D^{\text{eff}}(t)\, \frac{\partial^2 f_{X_0 X(t)}(x_0, x_1)}{\partial x_1^2},\end{aligned} \tag{5.33}$$

where $D^{\text{eff}}(t)$ is the effective noise intensity given by Eq. (3.46). In contrast with SLE (5.24) and in accordance with all the linear cases examined so far, Eq. (5.33) is closed and exact. Further, it is supplemented with the initial condition Eq. (5.22) in which the initial value pdf $f_{X_0}(x_0)$ is the following Gaussian distribution:

$$f_{X_0}(x_0) = \frac{1}{\sqrt{2\pi\sigma_{X_0}^2}} \exp\left[-\frac{1}{2}\frac{(x_0 - m_{X_0})^2}{\sigma_{X_0}^2}\right]. \tag{5.34}$$

Proceeding in the usual manner, we employ the two-dimensional Fourier transform $\varphi_{X_0 X(t)}(y_0, y_1) = \int_{\mathbb{R}^2} e^{i(y_0 x_0 + y_1 x_1)}\, f_{X_0 X(t)}(x_0, x_1)\, dx_1 dx_0$, which results in the transformed Eq. (5.33)

$$\begin{aligned}\frac{\partial \varphi_{X_0 X(t)}(y_0, y_1)}{\partial t} &= \eta\, y_1 \frac{\partial \varphi_{X_0 X(t)}(y_0, y_1)}{\partial y_1} + \\ &+ \left(i\kappa\, m_\Xi(t)\, y_1 - \kappa\, C_{X_0\Xi}(t)\, y_0 y_1 - D^{\text{eff}}(t)\, y_1^2\right) \varphi_{X_0 X(t)}(y_0, y_1).\end{aligned} \tag{5.35}$$

Under the above Fourier transform, initial condition (5.22) is respectively expressed as

$$\varphi_{X_0 X(t_0)}(y_0, y_1) = \iint_{\mathbb{R}^2} e^{i(y_0 x_0 + y_1 x_1)} \delta(x_0 - x_1) f_{X_0}(x_0)\, dx_0\, dx_1 = \\ = \int_{\mathbb{R}} e^{i x_0 y_0} \left(\int_{\mathbb{R}} e^{i x_1 y_1} \delta(x_0 - x_1)\, dx_1 \right) f_{X_0}(x_0)\, dx_0 . \tag{5.36}$$

By employing, now, the identity of delta function and at the same time, keeping in mind that the characteristic function of initial value is defined as $\varphi_{X_0}(y) = \int_{\mathbb{R}} e^{iyx} f_{X_0}(x)\, dx$, we obtain

$$\varphi_{X_0 X(t_0)}(y_0, y_1) = \int_{\mathbb{R}} e^{i(y_0+y_1)x_0} f_{X_0}(x_0)\, dx_0 = \varphi_{X_0}(y_0 + y_1) . \tag{5.37}$$

Moreover, since the Fourier transform of Gaussian pdf (5.34) has already been calculated into Eq. (3.53b) as

$$\varphi_{X_0}(y) = \exp\left(i m_{X_0} y - \frac{1}{2} \sigma^2_{X_0} y^2 \right),$$

transformed initial condition (5.37) is expressed in its final form:

$$\varphi_{X_0 X(t_0)}(y_0, y_1) = \exp\left(i m_{X_0} (y_0 + y_1) - \frac{1}{2} \sigma^2_{X_0} (y_0 + y_1)^2 \right) = \\ = \exp\left(i m_{X_0} y_0 + i m_{X_0} y_1 - \frac{1}{2} \sigma^2_{X_0} y_0^2 - \frac{1}{2} \sigma^2_{X_0} y_1^2 - \sigma^2_{X_0} y_0 y_1 \right). \tag{5.38}$$

As usual, IVP (5.35), (5.36) is solved using the method of characteristics. First, considering the homogenous variant of Eq. (5.35), the characteristic curve $\upsilon(y_1, t) = y_1 e^{\eta t}$ is obtained as the solution of the characteristic equation $dt = -dy/(\eta y_1)$. Then, the acquired characteristic curve dictates the change of variable from (y_0, y_1) to (y_0, υ) under which the transformed Eq. (5.35) becomes a linear ODE with respect to time t. Last, by solving the ODE and returning to the initial variables, we obtain the unique solution

$$\varphi_{X_0 X(t)}(y_0, y_1) = \exp\left[i m_{X_0} y_0 + i y_1 \left(m_{X_0} e^{\eta(t-t_0)} + \kappa \int_{t_0}^{t} m_{\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau \right) \right] \times \\ \times \exp\left[-\frac{1}{2} \sigma^2_{X_0} y_0^2 - \frac{1}{2} y_1^2 \left(\sigma^2_{X_0} e^{2\eta(t-t_0)} + 2 \int_{t_0}^{t} D^{\text{eff}}(\tau)\, e^{2\eta(t-\tau)}\, d\tau \right) \right] \times \\ \times \exp\left[-y_0 y_1 \left(\sigma^2_{X_0} e^{\eta(t-t_0)} + \kappa \int_{t_0}^{t} C_{X_0 \Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau \right) \right]. \tag{5.39}$$

At this point, as performed for the derivation of the one-time response excitation pdf, from Eq. (5.39) the first and second moments of initial value $X_0(\theta)$ and response $X(t;\theta)$ as well as their cross-covariance are readily identified into

$$m_X(t) = m_{X_0} e^{\eta(t-t_0)} + \kappa \int_{t_0}^{t} m_{\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau , \tag{5.40a}$$

$$\sigma_X^2(t) = \sigma_{X_0}^2\, e^{2\eta(t-t_0)} + 2\int_{t_0}^{t} D^{\text{eff}}(\tau)\, e^{2\eta(t-\tau)}\, d\tau, \tag{5.40b}$$

$$C_{X_0 X(t)}(t) = \sigma_{X_0}^2\, e^{\eta(t-t_0)} + \kappa\int_{t_0}^{t} C_{X_0\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau. \tag{5.40c}$$

Furthermore, by recalling the connection between effective noise intensity and cross-correlation Eq. (5.40b) is equivalently written as

$$\sigma_X^2(t) = \sigma_{X_0}^2\, e^{2\eta(t-t_0)} + 2\kappa\int_{t_0}^{t} C_{X\Xi}(\tau,\tau)\, e^{2\eta(t-\tau)}\, d\tau. \tag{5.40b'}$$

Having obtained Eqs. (5.40), it becomes obvious that they are identical to formulae (B.3), (B.23) and (B.27) as derived from their corresponding moment problems. Thus, we straightforwardly conclude that Eq. (5.40) gives the expected Gaussian form for $\varphi_{X_0 X(t)}(y_0, y_1)$, thereby verifying the validity of response-initial value pdf evolution Eq. (5.33). Finally, using the above expression Eq. (5.39) can equivalently be written in the more concise form

$$\varphi_{X_0 X(t)}(y_0, y_1) = \exp\left(i m_{X_0} y_0 + i m_X(t) y_1 - \frac{1}{2}\sigma_{X_0}^2 y_0^2 - \frac{1}{2}\sigma_X^2(t) y_1^2 - C_{X_0 X}(t) y_0 y_1 \right). \tag{5.41}$$

5.4.2 Exact, two-time response pdf

Let us now attend to the two-time response SLE in the linear case. Thus, substituting in Eq. (5.14) $h(x) = \eta x$, and calculating the variational derivatives to $V_{X_0}(t;\theta) = e^{\eta(t-t_0)}$, $V_{\Xi(s)}(t;\theta) = \kappa\, e^{\eta(t-s)}$, we obtain the following exact, two-time response pdf evolution equation:

$$\frac{\partial f_{X(t)X(s)}(x_1, x_2)}{\partial t} + \frac{\partial}{\partial x_1}\Big[\big(h(x_1) + \kappa m_\Xi(t)\big)\, f_{X(t)X(s)}(x_1, x_2)\Big] = \\ = D(t,t)\,\frac{\partial^2 f_{X(t)X(s)}(x_1, x_2)}{\partial x_1^2} + D(t,s)\,\frac{\partial^2 f_{X(t)X(s)}(x_1, x_2)}{\partial x_1 \partial x_2}, \tag{5.42}$$

where

$$D(t,s) = \kappa\, C_{X_0\Xi}(t)\, e^{\eta(s-t_0)} + \kappa^2\int_{t_0}^{s} C_{\Xi\Xi}(t,\tau)\, e^{\eta(s-\tau)}\, d\tau. \tag{5.43}$$

As discussed in section 5.1, the initial condition that supplements Eq. (5.42) is $f_{X(t_0)X(s)}(x_1, x_2) = f_{X_0 X(s)}(x_1, x_2)$. In the present case, this pdf has already been determined as the solution of the response-initial value pdf evolution Eq. (5.33). Before we proceed with the solution of Eq. (5.42), it is easily recognized through Eq. (5.43) that the term $D(t,t)$ is identical to $D^{\text{eff}}(t)$ defined by Eq. (3.46); thus, the latter, more familiar writing is used subsequently in our calculations.

Solution of Eq. (5.42) using Fourier transform. As for the previous exact pdf equations, we employ the Fourier transform $\varphi_{X(t)X(s)}(\boldsymbol{y}) = \int_{\mathbb{R}^2} e^{i(y_1x_1+y_2x_2)} \, f_{X(t)X(s)}(x_1, x_2)\, dx_1dx_2$, which results in the transformed equation

$$\frac{\partial\varphi_{X(t)X(s)}(\boldsymbol{y})}{\partial t} = \eta\, y_1 \frac{\partial\varphi_{X(t)X(s)}(\boldsymbol{y})}{\partial y_1} + \\ + \left(i\kappa m_{\Xi}(t)y_1 - D(t,s)y_1y_2 - \kappa D^{\mathrm{eff}}(t)y_1^2\right)\varphi_{X(t)X(s)}(\boldsymbol{y}). \tag{5.44}$$

Initial condition for Eq. (5.44) has already been specified in the previous paragraph 5.4.1, as the solution of initial value-response pdf, Eq. (5.41), written for $t = s$:

$$\varphi_{X_0X(s)}(\boldsymbol{y}) = \exp\left(i m_{X_0}\, y_1 + i m_X(s)y_2 - \frac{1}{2}\sigma^2_{X_0}\, y_1^2 - \frac{1}{2}\sigma^2_X(s)y_2^2 - C_{X_0X}(s)y_1y_2\right). \tag{5.45}$$

Following the same procedure to the one presented in paragraph 5.1.4, we begin be considering the homogenous variant of Eq. (5.44), which in turn prescribes the characteristic system

$$\frac{dt}{1} = -\frac{dy_1}{\eta\, y_1}. \tag{5.46}$$

The solution of this system determines the characteristic curve $\upsilon(t, y_1, y_2) = \upsilon = y_1 e^{\eta t}$. Thus, by applying the change of variable $y = \upsilon e^{-\eta t}$ to Eqs. (5.44) and (5.45), we obtain

$$\frac{\partial\varphi_{X(t)X(s)}(\upsilon, y_2)}{\partial t} = \varphi_{X(t)X(s)}(\upsilon, y_2)\times \\ \times\left(i\kappa m_{\Xi}(t)\upsilon\, e^{-\eta t} - D(t,s)\upsilon y_2\, e^{-\eta t} - \kappa D^{\mathrm{eff}}(t)\upsilon^2 e^{-2\eta t}\right). \tag{5.47}$$

$$\varphi_{X_0X(s)}(\upsilon, y_2) = \exp\left(i m_{X_0}\upsilon e^{-\eta t_0} + i m_X(s)y_2 - \frac{1}{2}\sigma^2_{X_0}\upsilon^2 e^{-2\eta t_0}\right)\times \\ \times\exp\left(-\frac{1}{2}\sigma^2_X(s)y_2^2 - C_{X_0X}(s)\upsilon y_2 e^{-\eta t_0}\right). \tag{5.48}$$

Finally, by solving the IVP (5.47)-(5.48) and returning to the initial variables $\upsilon = y_1 e^{\eta t}$, the solution of the two-time pdf evolution equation (5.44) is procured:

$$\varphi_{X(t)X(s)}(\boldsymbol{y}) = \exp\left[i y_1\left(m_{X_0}\, e^{\eta(t-t_0)} + \kappa\int_{t_0}^{t} m_{\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau\right) + i y_2 m_X(s)\right]\times \\ \times\exp\left[-\frac{1}{2}y_1^2\left(\sigma^2_{X_0}\, e^{2\eta(t-t_0)} + 2\int_{t_0}^{t} D^{\mathrm{eff}}(t)\, e^{2\eta(t-\tau)}\, d\tau\right) - \frac{1}{2}y_2^2\sigma^2_X(s)\right]\times \\ \times\exp\left[-y_1y_2\left(C_{X_0X}(s)\, e^{\eta(t-t_0)} + \int_{t_0}^{t} D(\tau, s)\, e^{\eta(t-\tau)}\, d\tau\right)\right]. \tag{5.49}$$

By utilizing Eq. (B.3) for $m_X(t)$, the verified Eq. (3.52) for $\sigma_X^2(t)$, as well as Eq. (B.24) for $C_{XX}(t,s)$ in conjunction with the definition relation (5.43) for $D(t,s)$, solution (5.49) is written equivalently as

$$\varphi_{X(t)X(s)}(\boldsymbol{y}) = \exp\left(i m_X(t) y_1 + i m_X(s) y_2 - \frac{1}{2}\sigma_X^2(t) y_1^2 - \frac{1}{2}\sigma_X^2(s) y_2^2 - C_{XX}(t,s) y_1 y_2 \right). \tag{5.50}$$

Eq. (5.50) is the expected Gaussian form of the two-time response pdf, yielding both the one-time; $m_X(t)$, $m_X(s)$, $\sigma_X^2(t)$, $\sigma_X^2(s)$, and the two-time, $C_{XX}(t,s)$, moments correctly. This positive result for the linear case, as well as the other similar results obtained in previous sections, constitute an indication in support of the conjecture that the SLEs, under an appropriate closure scheme, could also be well-posed and yield satisfactory results for the case of non-linear random dynamical systems.

Chapter 6

Two-time pdf evolution equations for the response and its derivative

In the present chapter, we are going to formulate pdf evolution equation s that entail both the response and its first derivative; a pursuit of special interest in first passage problems (Verechtchaguina et al., 2006). For this, we consider the familiar scalar, nonlinear, additively excited RIVP

$$\dot{X}(t;\theta) = h(X(t;\theta)) + \kappa\,\Xi(t;\theta), \qquad X(t_0;\theta) = X_0(\theta). \tag{6.1a,b}$$

In order to formulate our sought-for pdf evolution equation, it is necessary to also take into account the auxiliary RDE,

$$\ddot{X}(t;\theta) = h'(X(t;\theta))\,\dot{X}(t;\theta) + \kappa\,\dot{\Xi}(t;\theta), \tag{6.2a}$$

whose initial value is determined by setting in Eq. (6.1a) $t = t_0$, thus resulting in:

$$\dot{X}(t_0;\theta) = h(X_0(\theta)) + \kappa\,\Xi(t_0;\theta). \tag{6.2b}$$

All the results presented in this chapter will be derived by making use of our usual approach for both the RIVP (6.1a,b) as well as the auxiliary Eqs. (6.2a,b).

6.1 First variant of the evolution equations for the two-time pdf of the response and its derivative

Commencing in the same fashion presented in all the previous cases, we represent the sought-for two-time pdf as the average of the product of two random delta functions, i.e. the delta projection method:

$$f_{X(s)\dot{X}(t)}(x, y) = \mathbb{E}^{\theta}\Big[\delta(x - X(s;\theta))\,\delta(y - \dot{X}(t;\theta))\Big]. \tag{6.3}$$

In Eq. (6.3), the two time instances t, s are considered different; $s \neq t$, while $\dot{X}(t;\theta)$ is regarded as an abbreviation of the right-hand side of RDE (6.1a). In this section, we are going to examine the problem in which the time argument of the derivative of the response, t, is treated as the evolution time and the time argument of the response, s, enters as a parameter. The problem in which the time argument of the response is regarded as the evolution time is presented in the ensuing section 6.2.

6.1.1 The corresponding stochastic Liouville equation

Differentiating both sides of Eq. (6.3) with respect to time t and substituting Eq. (6.2a) into the apprehended expression results in

$$\frac{\partial f_{X(s)\dot{X}(t)}(x,\, y)}{\partial t} =$$
$$= \mathbb{E}^{\theta}\left[\delta(x - X(s;\theta))\, \frac{\partial \delta(y - \dot{X}(t;\theta))}{\partial \dot{X}(t;\theta)}\, h'(X(t;\theta))\, \dot{X}(t;\theta)\right] + \tag{6.4}$$
$$+ \kappa\, \mathbb{E}^{\theta}\left[\delta(x - X(s;\theta))\, \frac{\partial \delta(y - \dot{X}(t;\theta))}{\partial \dot{X}(t;\theta)}\, \dot{\Xi}(t;\theta)\right].$$

The two averaged terms appearing on the right-hand side of the above expressions are readily calculated by making use of the delta projection formalism. Substitution of the reevaluated averaged terms in Eq. (6.4) results in the following ***two-time stochastic Liouville equation of*** $X(s;\theta)$ ***and*** $\dot{X}(t;\theta)$:

$$\frac{\partial f_{X(s)\dot{X}(t)}(x,\, y)}{\partial t} + \frac{\partial}{\partial y}\Big(y\, \mathbb{E}^{\theta}\big[\delta(x - X(s;\theta))\, \delta(y - \dot{X}(t;\theta)) h'(X(t;\theta))\big]\Big) =$$
$$= -\, \kappa\, \frac{\partial}{\partial y}\, \mathbb{E}^{\theta}\big[\delta(x - X(s;\theta))\, \delta(y - \dot{X}(t;\theta))\, \dot{\Xi}(t;\theta)\big], \qquad s \neq t \tag{6.5}$$

As expected, Eq. (6.5) is a differential equation with respect to time t, while time s enters as a parameter. As such, similar to the two-time response pdf, the initial condition to Eq. (6.5) will also be parametric with respect to s. More specifically, the said initial condition is written as $f_{X(s)\dot{X}(t_0)}(x,\, y)$ and its corresponding SLE will be formulated in the subsequent paragraph 6.1.2.

Transformed SLE. SLE (6.5) is non-closed due to both averaged terms appearing on each of its sides. Nevertheless, the one appearing on the right-hand side of Eq. (6.5) can be further evaluated using the appropriate extension of the Novikov-Furutsu theorem. Thus, recalling the dependence of the response over the initial value $X_0(\theta)$ and the time history of the excitation $\Xi(\bullet;\theta)$, and under the understanding that $\dot{X}(t;\theta)$ can, in turn, be regarded through equation (6.1a) as a function of the response, we can write

$$\mathbb{E}^{\theta}\big[\delta(x - X(s;\theta))\, \delta(y - \dot{X}(t;\theta))\, \dot{\Xi}(t;\theta)\big] =$$
$$= \mathbb{E}^{\theta}\left[\delta\Big(x - X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\Big)\, \delta\Big(y - X[X_0(\theta);\Xi(\bullet|_{t_0}^{s};\theta)]\Big)\, \dot{\Xi}(t;\theta)\right]. \tag{6.6}$$

Therefore, in accordance with the approach followed for the two-time response SLE, the product of random delta functions in the above averaged term can be considered as a FFℓ of initial value $X_0(\theta)$ and excitation $\Xi(\bullet;\theta)$ over the time interval $[t_0, t_1]$, with $t_1 = \max(t, s)$. Under this convenient notation, and regardless of the time ordering of t and s, the aforementioned product can always be regarded as a FFℓ like $\mathcal{F} = \mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t_1};\theta)]$. Hence, by employing Eq. (2.21) the averaged term can be calculated as follows:

$$
\begin{aligned}
&\mathbb{E}^{\theta}\left[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))\,\dot{\Xi}(t;\theta)\right] = \\
&\quad = \dot{m}_{\Xi}(t)\mathbb{E}^{\theta}\left[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))\right] + \\
&\qquad + \dot{C}_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\left[\frac{\partial\left\{\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))\right\}}{\partial X_0(\theta)}\right] + \\
&\qquad + \int_{t_0}^{t_1} C_{\dot{\Xi}\Xi}(t,u)\,\mathbb{E}^{\theta}\left[\frac{\delta\left\{\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))\right\}}{\delta\Xi(u;\theta)}\right] du\,.
\end{aligned}
\tag{6.7}
$$

Further, by utilizing the product and rules for the derivatives that appear on the right-hand side of the above equation as well as recalling the properties of the delta projection formalism, results in

$$
\begin{aligned}
&\mathbb{E}^{\theta}\left[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))\,\dot{\Xi}(t;\theta)\right] = \\
&\quad = \dot{m}_{\Xi}(t)\mathbb{E}^{\theta}\left[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))\right] - \\
&\qquad - \dot{C}_{X_0\Xi}(t)\frac{\partial}{\partial x}\mathbb{E}^{\theta}\left[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))\frac{\partial X(s;\theta)}{\partial X_0(\theta)}\right] - \\
&\qquad - \dot{C}_{X_0\Xi}(t)\frac{\partial}{\partial y}\mathbb{E}^{\theta}\left[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))\frac{\partial \dot{X}(t;\theta)}{\partial X_0(\theta)}\right] - \\
&\qquad - \frac{\partial}{\partial x}\int_{t_0}^{t_1} C_{\dot{\Xi}\Xi}(t,u)\,\mathbb{E}^{\theta}\left[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))\frac{\delta X(s;\theta)}{\delta\Xi(u;\theta)}\right] du - \\
&\qquad - \frac{\partial}{\partial y}\int_{t_0}^{t_1} C_{\dot{\Xi}\Xi}(t,u)\,\mathbb{E}^{\theta}\left[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))\frac{\delta \dot{X}(t;\theta)}{\delta\Xi(u;\theta)}\right] du\,.
\end{aligned}
\tag{6.8}
$$

Due to causality, the variational derivatives $\delta X(s;\theta)/\delta\Xi(\tau;\theta)$, $\delta\dot{X}(t;\theta)/\delta\Xi(\tau;\theta)$ appearing on the right-hand side of the above expression, are zero for $t<\tau$ and $s<\tau$ respectively, since a variation of excitation at a certain time instance cannot result in variation of response in previous time instances. Thus, the upper limits of the integrals in Eq. (6.8) must be adjusted accordingly. Moreover, in Eq. (6.8) the usual variational derivatives appear which, as described in Chapter 3, are given by

$$
V_{X_0}(s;\theta) \equiv \frac{\partial X(s;\theta)}{\partial X_0(\theta)} = \exp\left(\int_{t_0}^{s} h'\big(X(u;\theta)\big)\,du\right),
\tag{6.9a}
$$

$$
V_{\Xi(u)}(s;\theta) \equiv \frac{\delta X(s;\theta)}{\delta\Xi(u;\theta)} = \kappa\exp\left(\int_{u}^{s} h'\big(X(\tau;\theta)\big)\,d\tau\right).
\tag{6.9b}
$$

Further, the “new” variational derivatives $\partial\dot{X}(t;\theta)\big/\partial X_0(\theta)$, $\delta\dot{X}(t;\theta)\big/\delta\Xi(u;\theta)$ can be easily calculated by directly applying to Eq. (6.1a) the operators $\partial\bullet\big/\partial X_0(\theta)$ and $\delta\bullet\big/\delta\Xi(u;\theta)$, respectively. As such, we find

$$\frac{\partial \dot{X}(t;\theta)}{\partial X_0(\theta)} = h'(X(t;\theta))\, V_{X_0}(t;\theta), \tag{6.10a}$$

$$\frac{\delta \dot{X}(t;\theta)}{\delta \Xi(u;\theta)} = h'(X(t;\theta))\, V_{\Xi(u)}(t;\theta) + \kappa\, \delta(t-u). \tag{6.10b}$$

Combining, now, Eqs. (6.8)-(6.10) yields

$$\begin{aligned}
&\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))\,\dot{\Xi}(t;\theta)\Big] = \\
&= \dot{m}_{\Xi}(t)\, f_{X(s)\dot{X}(t)}(x,y) - \\
&\quad - \dot{C}_{X_0\Xi}(t)\frac{\partial}{\partial x}\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))V_{X_0}(s;\theta)\Big] - \\
&\quad - \dot{C}_{X_0\Xi}(t)\frac{\partial}{\partial y}\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))h'(X(t;\theta))V_{X_0}(s;\theta)\Big] - \\
&\quad - \frac{\partial}{\partial x}\int_{t_0}^{s} C_{\dot{\Xi}\Xi}(t,u)\,\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))V_{\Xi(u)}(s;\theta)\Big]\,du - \\
&\quad - \frac{\partial}{\partial y}\int_{t_0}^{t} C_{\dot{\Xi}\Xi}(t,u)\,\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))h'(X(t;\theta))V_{\Xi(u)}(t;\theta)\Big]\,du - \\
&\quad - \kappa\frac{\partial}{\partial y}\int_{t_0}^{t} C_{\dot{\Xi}\Xi}(t,u)\,\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))\Big]\delta(t-u)\,du.
\end{aligned} \tag{6.11}$$

Last, by employing the identity of the delta function, the averaged term is written

$$\begin{aligned}
&\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))\,\dot{\Xi}(t;\theta)\Big] = \\
&= \dot{m}_{\Xi}(t)\, f_{X(s)\dot{X}(t)}(x,y) - \kappa\, C_{\dot{\Xi}\Xi}(t,t)\frac{\partial}{\partial y}\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))\Big] - \\
&\quad - \dot{C}_{X_0\Xi}(t)\frac{\partial}{\partial x}\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))V_{X_0}(s;\theta)\Big] - \\
&\quad - \dot{C}_{X_0\Xi}(t)\frac{\partial}{\partial y}\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))h'(X(t;\theta))V_{X_0}(t;\theta)\Big] - \\
&\quad - \frac{\partial}{\partial x}\int_{t_0}^{s} C_{\dot{\Xi}\Xi}(t,u)\,\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))V_{\Xi(u)}(s;\theta)\Big]\,du - \\
&\quad - \frac{\partial}{\partial y}\int_{t_0}^{t} C_{\dot{\Xi}\Xi}(t,u)\,\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))h'(X(t;\theta))V_{\Xi(u)}(t;\theta)\Big]\,du.
\end{aligned} \tag{6.12}$$

Finally, substituting the above expression into SLE (6.5) results in the following ***transformed SLE for the two-time joint pdf of*** $X(s;\theta)$ ***and*** $\dot{X}(t;\theta)$:

$$\frac{\partial f_{X(s)\dot{X}(t)}(x,y)}{\partial t} + \frac{\partial}{\partial y}\Big(y\,\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))h'(X(t;\theta))\Big]\Big) +$$
$$+\kappa\, m_{\dot{\Xi}}(t)\,\frac{\partial f_{X(s)\dot{X}(t)}(x,y)}{\partial y} =$$
$$= \kappa^2 C_{\dot{\Xi}\Xi}(t,t)\frac{\partial^2 f_{X(s)\dot{X}(t)}(x,y)}{\partial y^2} +$$
$$+\kappa\,\dot{C}_{X_0\Xi}(t)\frac{\partial^2}{\partial x\partial y}\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))V_{X_0}(s;\theta)\Big] +$$
$$+\kappa\,\dot{C}_{X_0\Xi}(t)\frac{\partial^2}{\partial y^2}\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))h'(X(t;\theta))V_{X_0}(t;\theta)\Big] +$$
$$+\kappa\frac{\partial^2}{\partial x\partial y}\int_{t_0}^{s} C_{\dot{\Xi}\Xi}(t,u)\,\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))V_{\Xi(u)}(s;\theta)\Big]du +$$
$$+\kappa\frac{\partial^2}{\partial y^2}\int_{t_0}^{t} C_{\dot{\Xi}\Xi}(t,u)\,\mathbb{E}^{\theta}\Big[\delta(x-X(s;\theta))\,\delta(y-\dot{X}(t;\theta))h'(X(t;\theta))V_{\Xi(u)}(t;\theta)\Big]du \tag{6.13}$$

It is not surprising that Eq. (6.13) is non-closed due to a) the averaged term on its left-hand side, b) the terms multiplying the product of random delta function inside the averages on its right-hand side. Moreover, these terms are not in the appropriate form to implement the usual approximation scheme, first presented in paragraph 3.1.3, because of the occurrence of $h'(X(t;\theta))$. As in the case of the evolution equations corresponding to an RDE excited by both additive and multiplicative noise Eqs. (3.71), (4.73), the time argument of $h'(X(\bullet;\theta))$ does not match the one of the delta function of this response $\delta(x-X(s;\theta))$ and thus, the said term cannot be treated via the delta projection method. An appropriate closure technique to derive the required pdf evolution equation is proposed in paragraph 6.1.3

6.1.2 Auxiliary, initial stochastic Liouville equation corresponding to the RDE

Having obtained SLE (6.13), it is essential to also determine its initial value $f_{X(s)\dot{X}(t_0)}(x,y)$. As per usual, our starting point is the delta projection method which provides us with the following representation:

$$f_{X(t)\dot{X}(t_0)}(x,y) = \mathbb{E}^{\theta}\Big[\delta\big(x-X(t;\theta)\big)\,\delta\big(y-\dot{X}(t_0;\theta)\big)\Big], \tag{6.14}$$

in which $\dot{X}(t_0;\theta)$ is given by Eq. (6.2b). By differentiating Eq. (6.14) over time t, we obtain

$$\begin{aligned}\frac{\partial f_{X(t)\dot{X}(t_0)}(x,y)}{\partial t} &= \mathbb{E}^{\theta}\left[\frac{\partial\delta\big(x-X(t;\theta)\big)}{\partial t}\,\delta\big(y-\dot{X}(t_0;\theta)\big)\right]\\ &= \mathbb{E}^{\theta}\left[\frac{\partial\delta\big(x-X(t;\theta)\big)}{\partial X(t;\theta)}\,\dot{X}(t;\theta)\,\delta\big(y-\dot{X}(t_0;\theta)\big)\right].\end{aligned} \tag{6.15}$$

Eq. (6.15) can be further evaluated by employing RDE (6.1a) as:

$$\frac{\partial f_{X(t)\dot{X}(t_0)}(x,y)}{\partial t} = \mathbb{E}^{\theta}\left[\frac{\partial\delta\left(x-X(t;\theta)\right)}{\partial X(t;\theta)}\, h(X(t;\theta))\,\delta\Big(y-\dot{X}(t_0;\theta)\Big)\right] + \\ + \kappa\,\mathbb{E}^{\theta}\left[\frac{\partial\delta\left(x-X(t;\theta)\right)}{\partial X(t;\theta)}\,\Xi(t;\theta)\,\delta\Big(y-\dot{X}(t_0;\theta)\Big)\right]. \tag{6.16}$$

Finally, carrying out the familiar delta projection method manipulations, we acquire the following ***stochastic Liouville equation for*** $X(t;\theta)$ ***and*** $\dot{X}(t_0;\theta)$:

$$\frac{\partial f_{X(t)\dot{X}(t_0)}(x,y)}{\partial t} + \frac{\partial}{\partial x}\Big(h(x)\,f_{X(t)\dot{X}(t_0)}(x,y)\Big) = \\ = -\kappa\,\frac{\partial}{\partial x}\,\mathbb{E}^{\theta}\Big[\delta\left(x-X(t;\theta)\right)\delta\Big(y-\dot{X}(t_0;\theta)\Big)\,\Xi(t;\theta)\Big]. \tag{6.17}$$

SLE (6.17) is also supplemented with the following initial condition

$$f_{X(t_0)\dot{X}(t_0)}(x,y) = f_{X_0\dot{X}(t_0)}(x,y), \tag{6.18}$$

which is a bivariate Gaussian distribution, in accordance with its components.

Transformed SLE. At this point, let us proceed by evaluating the non-closed averaged term appearing on the right-hand side of Eq. (6.17). Thus, recalling the dependence of the response on $X_0(\theta)$ and $\Xi(\bullet|_{t_0}^{t};\theta)$ as well as Eq. (6.2b) for $\dot{X}(t_0;\theta)$, the product of random delta functions can be seen as a FFℓ like $\mathcal{F} = \mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]$. Hence, we are able to apply the extended NF theorem, Eq. (2.18):

$$\mathbb{E}^{\theta}\Big[\delta\left(x-X(t;\theta)\right)\delta\Big(y-\dot{X}(t_0;\theta)\Big)\,\Xi(t;\theta)\Big] = \\ = m_{\Xi}(t)\,\mathbb{E}^{\theta}\Big[\delta\left(x-X(t;\theta)\right)\delta\Big(y-\dot{X}(t_0;\theta)\Big)\Big] + \\ + C_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\left[\frac{\partial\Big\{\delta\left(x-X(t;\theta)\right)\delta\Big(y-\dot{X}(t_0;\theta)\Big)\Big\}}{\partial X_0(\theta)}\right] + \\ + \int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\frac{\delta\Big\{\delta\left(x-X(t;\theta)\right)\delta\Big(y-\dot{X}(t_0;\theta)\Big)\Big\}}{\delta\Xi(\tau;\theta)}\right]d\tau. \tag{6.19}$$

By employing, now, the chain and product rule for the derivatives, Eq. (6.21) is transformed into

$$\mathbb{E}^{\theta}\Big[\delta\left(x-X(t;\theta)\right)\delta\Big(y-\dot{X}(t_0;\theta)\Big)\,\Xi(t;\theta)\Big] = \\ = m_{\Xi}(t)\,\mathbb{E}^{\theta}\Big[\delta\left(x-X(t;\theta)\right)\delta\Big(y-\dot{X}(t_0;\theta)\Big)\Big] + \\ + C_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\left[\frac{\partial\delta\left(x-X(t;\theta)\right)}{\partial X(t;\theta)}\,\delta\Big(y-\dot{X}(t_0;\theta)\Big)\,V_{X_0}(t;\theta)\right] +$$

$$+ C_{X_0\Xi}(t)\, \mathbb{E}^{\theta}\left[\delta\big(x - X(t;\theta)\big)\, \frac{\partial \delta\big(y - \dot{X}(t_0;\theta)\big)}{\partial \dot{X}(t_0;\theta)}\, \frac{\partial \dot{X}(t_0;\theta)}{\partial X_0(\theta)} \right] +$$

$$+ \int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\, \mathbb{E}^{\theta}\left[\frac{\delta \delta\big(x - X(t;\theta)\big)}{\delta X(t;\theta)}\, \delta\big(y - \dot{X}(t_0;\theta)\big)\, V_{\Xi(\tau)}(t;\theta) \right] d\tau + \tag{6.20}$$

$$+ \int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\, \mathbb{E}^{\theta}\left[\delta\big(x - X(t;\theta)\big)\, \frac{\delta \delta\big(y - \dot{X}(t_0;\theta)\big)}{\delta \dot{X}(t_0;\theta)}\, \frac{\delta \dot{X}(t_0;\theta)}{\delta \Xi(\tau;\theta)} \right] d\tau\, .$$

In Eq. (6.20), the familiar variational derivatives $V_{X_0}(t;\theta)$, $V_{\Xi(\tau)}(t;\theta)$ appear, as well as the unknowns $\partial \dot{X}(t_0;\theta) \big/ \partial X_0(\theta)$ and $\delta \dot{X}(t_0;\theta) \big/ \delta \Xi(\tau;\theta)$. These are straightforwardly apprehended by applying the operators $\partial \bullet \big/ \partial X_0(\theta)$ and $\delta \bullet \big/ \delta \Xi(\tau;\theta)$ on both sides of Eq. (6.2b), resulting in

$$\frac{\partial \dot{X}(t_0;\theta)}{\partial X_0(\theta)} = h'\big(X_0(\theta)\big), \tag{6.21a}$$

$$\frac{\delta \dot{X}(t_0;\theta)}{\delta \Xi(\tau;\theta)} = \kappa\, \frac{\delta \Xi(t_0;\theta)}{\delta \Xi(\tau;\theta)} = \kappa\, \delta(t_0 - \tau)\, . \tag{6.21b}$$

Using Eqs. (6.21a,b), the delta function identity for $\delta(t_0 - \tau)$ and the delta projection method's properties, Eq. (6.20) is written as

$$\mathbb{E}^{\theta}\left[\delta\big(x - X(t;\theta)\big)\, \delta\big(y - \dot{X}(t_0;\theta)\big)\, \Xi(t;\theta) \right] =$$

$$= m_{\Xi}(t)\, \mathbb{E}^{\theta}\left[\delta\big(x - X(t;\theta)\big)\, \delta\big(y - \dot{X}(t_0;\theta)\big) \right] -$$

$$- C_{X_0\Xi}(t)\, \frac{\partial}{\partial x}\, \mathbb{E}^{\theta}\left[\delta\big(x - X(t;\theta)\big)\, \delta\big(y - \dot{X}(t_0;\theta)\big)\, V_{X_0}(t;\theta) \right] -$$

$$- C_{X_0\Xi}(t)\, \frac{\partial}{\partial y}\, \mathbb{E}^{\theta}\left[\delta\big(x - X(t;\theta)\big)\, \delta\big(y - \dot{X}(t_0;\theta)\big)\, h'\big(X_0(\theta)\big) \right] - \tag{6.22}$$

$$- \frac{\partial}{\partial x} \int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\, \mathbb{E}^{\theta}\left[\delta\big(x - X(t;\theta)\big)\, \delta\big(y - \dot{X}(t_0;\theta)\big)\, V_{\Xi(\tau)}(t;\theta) \right] d\tau -$$

$$- \kappa\, C_{\Xi\Xi}(t, t_0)\, \frac{\partial}{\partial y}\, \mathbb{E}^{\theta}\left[\delta\big(x - X(t;\theta)\big)\, \delta\big(y - \dot{X}(t_0;\theta)\big) \right].$$

Last, substituting Eq. (6.22) into SLE (6.13) results in the following, ***transformed SLE for the joint pdf of*** $X(t;\theta)$ ***and*** $\dot{X}(t_0;\theta)$:

$$\frac{\partial f_{X(t)\dot{X}(t_0)}(x,y)}{\partial t} + \frac{\partial}{\partial x}\Big[\Big(h(x) + \kappa\, m_{\Xi}(t)\Big)\, f_{X(t)\dot{X}(t_0)}(x,y)\Big] =$$

$$= \kappa\, C_{X_0\Xi}(t)\, \frac{\partial^2}{\partial x^2}\, \mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big)\, \delta\Big(y - \dot{X}(t_0;\theta)\Big) V_{X_0}(t;\theta)\Big] +$$

$$+ \kappa\, C_{X_0\Xi}(t)\, \frac{\partial^2}{\partial x \partial y}\, \mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big)\, \delta\Big(y - \dot{X}(t_0;\theta)\Big) h'\Big(X_0(\theta)\Big)\Big] +$$

$$+ \kappa\, \frac{\partial^2}{\partial x^2} \int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\, \mathbb{E}^{\theta}\Big[\delta\big(x - X(t;\theta)\big)\, \delta\Big(y - \dot{X}(t_0;\theta)\Big) V_{\Xi(\tau)}(t;\theta)\Big]\, d\tau +$$

$$+ \kappa^2\, C_{\Xi\Xi}(t,t_0) \frac{\partial^2 f_{X(t)\dot{X}(t_0)}(x,y)}{\partial x \partial y}. \tag{6.23}$$

As expected, Eq. (6.23) is non-closed due to the variational derivatives as well as the term $h'\Big(X_0(\theta)\Big)$ multiplying the product of random delta functions inside the averaged appearing on its right-hand side. The treatment of these term in a way that allows us to derive a closed pdf evolution equation from SLE (6.23) is presented in the following paragraph.

6.1.3 Novel, two-time evolution equations for the pdf of the response and its derivative

Let us, first, rewrite Eq. (6.13) by substituting the variational derivatives by Eqs. (3.24) and (3.30) and using the notation

$$\mathcal{I}_{h'}[X(\bullet|_{\tau}^{t};\theta)] = \int_{\tau}^{t} h'\big(X(u;\theta)\big)\, du\,. \tag{6.24}$$

Using the aforementioned equations, we obtain the following ***exact, non-closed evolution equation for the two-time joint pdf of the response and its derivative*** $f_{X(s)\dot{X}(t)}(x,y)$:

$$\frac{\partial f_{X(s)\dot{X}(t)}(x,y)}{\partial t} + \frac{\partial}{\partial y}\Big(y\, \mathbb{E}^{\theta}\Big[\delta(x - X(s;\theta))\, \delta(y - \dot{X}(t;\theta)) h'(X(t;\theta))\Big]\Big) +$$

$$+ \kappa\, m_{\dot{\Xi}}(t)\, \frac{\partial f_{X(s)\dot{X}(t)}(x,y)}{\partial y} = \kappa^2 C_{\dot{\Xi}\Xi}(t,t) \frac{\partial^2 f_{X(s)\dot{X}(t)}(x,y)}{\partial y^2} +$$

$$+ \kappa\, \dot{C}_{X_0\Xi}(t) \frac{\partial^2}{\partial x \partial y} \mathbb{E}^{\theta}\Big[\delta(x,y;X_s,\dot{X}_t)\, \exp\Big(\mathcal{I}_{h'}[X(\bullet|_{t_0}^{s};\theta)\Big)\Big] +$$

$$+ \kappa\, \dot{C}_{X_0\Xi}(t) \frac{\partial^2}{\partial y^2} \mathbb{E}^{\theta}\Big[\delta(x,y;X_s,\dot{X}_t)\, h'(X(t;\theta)) \exp\Big(\mathcal{I}_{h'}[X(\bullet|_{t_0}^{t};\theta)\Big)\Big] +$$

$$+ \kappa^2 \frac{\partial^2}{\partial x \partial y} \int_{t_0}^{s} C_{\dot{\Xi}\Xi}(t,u)\, \mathbb{E}^{\theta}\Big[\delta(x,y;X_s,\dot{X}_t)\, \exp\Big(\mathcal{I}_{h'}[X(\bullet|_{u}^{s};\theta)]\Big)\Big]\, du +$$

$$+ \kappa^2 \frac{\partial^2}{\partial y^2} \int_{t_0}^{t} C_{\dot{\Xi}\Xi}(t,u)\, \mathbb{E}^{\theta}\Big[\delta(x,y;X_s,\dot{X}_t)\, \exp\Big(\mathcal{I}_{h'}[X(\bullet|_{u}^{t};\theta)]\Big)\Big]\, du\,, \tag{6.25}$$

where the shorthand $\delta(x, y; X_s, \dot{X}_t) = \delta(x - X(s;\theta))\, \delta(y - \dot{X}(t;\theta))$ has also been used. Accordingly, by using the same notation for Eq. (6.23) written for $t = s$, we obtain the following, ***exact, non-closed evolution equation for the joint pdf of*** $X(s;\theta)$ ***and*** $\dot{X}(t_0;\theta)$:

$$\frac{\partial f_{X(s)\dot{X}(t_0)}(x, y)}{\partial s} + \frac{\partial}{\partial x}\Big[\Big(h(x) + \kappa\, m_{\Xi}(s)\Big) f_{X(s)\dot{X}(t_0)}(x, y)\Big] =$$

$$= \kappa\, C_{X_0 \Xi}(s) \frac{\partial^2}{\partial x^2} \mathbb{E}^{\theta}\Big[\delta(x, y; X_s, \dot{X}_{t_0}) \exp\Big(\mathcal{I}_{h'}[X(\bullet|_{t_0}^{s};\theta)\Big)\Big] +$$

$$+ \kappa\, C_{X_0 \Xi}(s) \frac{\partial^2}{\partial x \partial y} \mathbb{E}^{\theta}\Big[\delta(x, y; X_s, \dot{X}_{t_0}) h'\Big(X_0(\theta)\Big)\Big] +$$

$$+ \kappa^2 \frac{\partial^2}{\partial x^2} \int_{t_0}^{s} C_{\Xi\Xi}(s, \tau)\, \mathbb{E}^{\theta}\Big[\delta(x, y; X_s, \dot{X}_{t_0}) \exp\Big(\mathcal{I}_{h'}[X(\bullet|_{\tau}^{s};\theta)]\Big)\Big] d\tau +$$

$$+ \kappa^2 C_{\Xi\Xi}(s, t_0) \frac{\partial^2 f_{X(s)\dot{X}(t_0)}(x, y)}{\partial x \partial y}. \tag{6.26}$$

In this notation, it becomes clear that Eqs. (6.25) and (6.26) are non-closed due to the nonlocal terms multiplying the product of random delta functions. What is more, the nonlocal terms $h'(\cdots)$ resemble the ones appeared in the response and response-excitation pdf evolution equations of the RDE excited by both additive and multiplicative colored noise and thus, a similar current-time approximation can be employed for them. Further, the exponential terms are of similar form as those presented in paragraph 3.1.3 and as such, an appropriately adjusted approximation scheme, analogous to the one introduced in the aforementioned paragraph, can be implemented.

First, let us specify the approximate form of Eq. (6.26). As already discussed, this is performed by implementing the usual approximation scheme for the nonlocal terms as well as considering the Taylor expansion of $h'\Big(X_0(\theta)\Big)$ around evolution time s up to first order; the latter is carried out under the assumption that we work for small correlation times. As such, the ***closed, approximate evolution equation for the joint pdf of*** $X(s;\theta)$ ***and*** $\dot{X}(t_0;\theta)$ reads

$$\frac{\partial f_{X(s)\dot{X}(t_0)}(x, y)}{\partial s} + \frac{\partial}{\partial x}\Big[\Big(h(x) + \kappa\, m_{\Xi}(s)\Big) f_{X(s)\dot{X}(t_0)}(x, y)\Big] =$$

$$= \frac{\partial^2}{\partial x^2}\left\{\left[\sum_{m=0}^{M} \frac{1}{m!} D_m^{\text{eff}}\Big[R_{h'}(\bullet), s, s\Big]\, \varphi_{h'}^{m}\Big(x; R_{h'}(s)\Big)\right] f_{X(s)\dot{X}(t_0)}(x, y)\right\} + \tag{6.27}$$

$$+\Big\{\kappa\, C_{X_0 \Xi}(s)\Big[h'(x) + h''(x) h(x)(s - t_0)\Big] + \kappa^2 C_{\Xi\Xi}(s, t_0)\Big\} \frac{\partial^2 f_{X(s)\dot{X}(t_0)}(x, y)}{\partial x \partial y},$$

where D_m^{eff} is the generalized effective noise intensities given by Eq. (5.29). In order to derive Eq. (6.27), in the Taylor expansion of $h'\Big(X_0(\theta)\Big)$ we have disregarded the terms which explicitly contain the excitation as was also performed in the case of a both multiplicatively and additively excited RDE. Further, for the above equation to be valid the additional assumption that function $h(x)$ has continuous second derivative must be introduced.

Let us, now, continue with the treatment of Eq. (6.25). First, for the nonlocal terms $h'(X(t;\theta))$, under the assumption that we work for small correlation times, we perform a Taylor expansion around parameter time s and once more disregard the term containing the excitation. Then, we apply the familiar approximation scheme for the exponential terms and, anew, carry out current-time approximations when it is required. Thus, we obtain the following ***approximate, closed evolution equation for the two-time joint pdf of the response and its derivative*** $f_{X(s)\dot{X}(t)}(x,y)$:

$$\begin{aligned} &\frac{\partial f_{X(s)\dot{X}(t)}(x,y)}{\partial t} + \left(h'(x) - h''(x)h(x)(s-t)\right)\frac{\partial}{\partial y}\left(y\, f_{X(s)\dot{X}(t)}(x,y)\right) + \\ &\quad + \kappa\, m_{\dot{\Xi}}(t)\,\frac{\partial f_{X(s)\dot{X}(t)}(x,y)}{\partial y} = \kappa^2 C_{\dot{\Xi}\Xi}(t,t)\frac{\partial^2 f_{X(s)\dot{X}(t)}(x,y)}{\partial y^2} + \\ &\quad + \frac{\partial^2}{\partial x \partial y}\left\{\left[\sum_{m=0}^{M}\frac{1}{m!}G_m^{(1)}\left[R_{h'}(\bullet),t,s\right]\varphi_{h'}^{m}\left(x;R_{h'}(s)\right)\right] f_{X(s)\dot{X}(t)}(x,y)\right\} + \\ &\quad + \frac{\partial^2}{\partial y^2}\left\{\left[\sum_{m=0}^{M}\frac{1}{m!}G_m^{(2)}\left[R_{h'}(\bullet),t,t\right]\psi_1^{m}\left(x;R_{h'}(s)\right)\right] f_{X(s)\dot{X}(t)}(x,y)\right\}, \end{aligned} \tag{6.28}$$

where

$$\begin{aligned} G_m^{(1)}\left[R_{h'}(\bullet),t,s\right] &= \kappa \exp\left(\int_{t_0}^{s} R_{h'}(u)\,du\right)\dot{C}_{X_0\Xi}(t)\,(s-t_0)^m + \\ &\quad + \kappa^2 \int_{t_0}^{s}\partial_t C_{\Xi\Xi}(t,u)\exp\left(\int_{u}^{s} R_{h'}(\tau)\,d\tau\right)(s-u)^m\,du, \end{aligned} \tag{6.29}$$

$$G_m^{(2)}\left[R_{h'}(\bullet),t,t\right] = \left(h'(x) - h''(x)h(x)(s-t)\right)G_m^{(1)}\left[R_{h'}(\bullet),t,t\right], \tag{6.30}$$

$$\psi_1^{m}\left(x;R_{h'}(s)\right) = \varphi_{h'}^{m}\left(x;R_{h'}(s)\right) - \varphi_{h'}'^{\,m}\left(x;R_{h'}(s)\right)h(x)\,(s-t). \tag{6.31}$$

Through the D_m^{eff}, G, φ and ψ terms Eqs. (6.27), (6.28) retain a considerable amount of the nonlinearity and nonlocality (in time), thus, maintaining the non-Markovian character of the original problem. All in all, the treatment of Eqs. (6.25), (6.26) clearly showcases the intricacies associated with the derivation of closed, computable pdf evolution equations involving both the response and its derivative. In the present paragraph, a closure technique was proposed which could provide a starting point for their numerical solution.

A different, and evidently simpler, set of pdf evolution equations which involve both the response and its derivative can be derived by employing **Hänggi's ansatz** (P. Hänggi & Jung, 1995; Peter Hänggi, Mroczkowski, Moss, & McClintock, 1985) for the exponential terms of Eqs. (6.25), (6.26). This set of "new" equations constitute only a special case of the ones presented above.

According to Hänggi's ansatz, the random quantity inside the nonlocal, exponential terms is approximated by its mean value, e.g.

$$\exp\left(\int_{\tau}^{t} h'\big(X(u;\theta)\big)\, du \right) \cong \exp\left(\int_{\tau}^{t} R_{h'}(u)\, du \right), \tag{6.32}$$

where $R_{h'}(u) = \mathbb{E}^{\theta}\left[h'\big(X(u;\theta)\big) \right]$. Therefore, it can be readily seen that this scheme is a simplification of the one presented above since it can be directly obtained by disregarding the fluctuation exponentials and their subsequent treatment. However, it must be noted that the current-time approximation for the $h'(\cdots)$ terms is unavoidable. Thus, application of Hänggi's ansatz results in the following approximate, closed evolution equation for the two-time joint pdf of the response and its derivative $f_{X(s)\dot{X}(t)}(x, y)$:

$$\begin{aligned} &\frac{\partial f_{X(s)\dot{X}(t)}(x, y)}{\partial t} + \big(h'(x) - h''(x)h(x)(s-t)\big)\frac{\partial}{\partial y}\Big(y\, f_{X(s)\dot{X}(t)}(x, y)\Big) + \\ &\quad + \kappa\, m_{\dot{\Xi}}(t)\, \frac{\partial f_{X(s)\dot{X}(t)}(x, y)}{\partial y} = G^{(1)}\Big[R_{h'}(\bullet), t, s\Big] \frac{\partial^2 f_{X(s)\dot{X}(t)}(x, y)}{\partial x \partial y} + \\ &\quad + \Big\{ \kappa^2 C_{\dot{\Xi}\Xi}(t, t) + G^{(2)}\Big[R_{h'}(\bullet), t, t\Big] \Big\} \frac{\partial^2 f_{X(s)\dot{X}(t)}(x, y)}{\partial y^2}, \end{aligned} \tag{6.33}$$

where

$$\begin{aligned} G^{(1)}\Big[R_{h'}(\bullet), t, s\Big] &= \kappa \exp\left(\int_{t_0}^{s} R_{h'}(u)\, du \right) \dot{C}_{X_0 \Xi}(t) + \\ &\quad + \kappa^2 \int_{t_0}^{s} \partial_t C_{\Xi\Xi}(t, u) \exp\left(\int_{u}^{s} R_{h'}(\tau)\, d\tau \right) du, \end{aligned} \tag{6.34}$$

$$G^{(2)}\Big[R_{h'}(\bullet), t, t\Big] = \big[h'(x) - h''(x)h(x)(s-t)\big]\, G^{(1)}\Big[R_{h'}(\bullet), t, t\Big]. \tag{6.35}$$

Accordingly, the auxiliary, initial pdf evolution equation for the joint pdf of $X(s;\theta)$ and $\dot{X}(t_0;\theta)$ is specified into

$$\begin{aligned} &\frac{\partial f_{X(s)\dot{X}(t_0)}(x, y)}{\partial s} + \frac{\partial}{\partial x}\Big[\Big(h(x) + \kappa\, m_{\Xi}(s)\Big) f_{X(s)\dot{X}(t_0)}(x, y) \Big] = \\ &\quad = D^{\text{eff}}\Big[R_{h'}(\bullet), s, s\Big] \frac{\partial^2 f_{X(s)\dot{X}(t_0)}(x, y)}{\partial x^2} + \\ &\quad + \Big\{ \kappa\, C_{X_0\Xi}(s)\Big[h'(x) + h''(x)h(x)(s - t_0)\Big] + \kappa^2 C_{\Xi\Xi}(s, t_0) \Big\} \frac{\partial^2 f_{X(s)\dot{X}(t_0)}(x, y)}{\partial x \partial y}, \end{aligned} \tag{6.36}$$

where the term D^{eff} is given via the following expression:

$$D^{\mathrm{eff}}\left[R_{h'}(\bullet), t, s\right] = \kappa \exp\left(\int_{t_0}^{s} R_{h'}(u)\, du\right) C_{X_0\Xi}(t) + \\ + \kappa^2 \int_{t_0}^{s} \exp\left(\int_{\tau}^{s} R_{h'}(u)\, du\right) C_{\Xi\Xi}(t, \tau)\, d\tau . \tag{6.37}$$

Having derived Eqs. (6.33) and (6.36), it becomes apparent that they are simpler versions of their counterparts Eqs. (6.28) and (6.27), respectively. However, they are also much simpler to comprehend as well as attempt to solve numerically.

6.2 Second variant of the evolution equations for the two-time pdf of the response and its derivative

In this section, we are going to formulate another SLE for the response and its derivative at a different time instance. Thus, using the delta representation, the sought-for pdf is written as:

$$f_{X(t)\dot{X}(s)}(x, y) = \mathbb{E}^{\theta}\left[\delta(x - X(t;\theta))\, \delta(y - \dot{X}(s;\theta))\right]. \tag{6.38}$$

Note that Eq. (6.38) is not the same as Eq. (6.3); this time the time argument of the derivative of the response will be treated as a parameter in the following calculations.

6.2.1 The corresponding stochastic Liouville equation

As performed in section 6.1, we differentiate both sides of Eq. (6.38) with respect to time t and also employ RDE (6.1a), resulting in

$$\frac{\partial f_{X(t)\dot{X}(s)}(x, y)}{\partial t} = \mathbb{E}^{\theta}\left[\frac{\partial \delta(x - X(t;\theta))}{\partial X(t;\theta)}\, \delta(y - \dot{X}(s;\theta))\, h(X(t;\theta))\right] + \\ + \kappa\, \mathbb{E}^{\theta}\left[\frac{\partial \delta(x - X(t;\theta))}{\partial X(t;\theta)}\, \delta(y - \dot{X}(s;\theta))\, \Xi(t;\theta)\right]. \tag{6.39}$$

Each term on the right-hand side of Eq. (6.39) is readily evaluated using the delta projection formalism. Thus, by performing the usual manipulations the above equation is written

$$\frac{\partial f_{X(t)\dot{X}(s)}(x, y)}{\partial t} + \frac{\partial}{\partial x}\Big(h(x)\, f_{X(t)\dot{X}(s)}(x, y)\Big) = \\ = -\kappa \frac{\partial}{\partial x} \mathbb{E}^{\theta}\left[\delta(x - X(t;\theta))\, \delta(y - \dot{X}(s;\theta))\, \Xi(t;\theta)\right]. \tag{6.40}$$

The above expression is another variant for the ***stochastic Liouville equation for the joint pdf of*** $X(t;\theta)$ ***and*** $\dot{X}(s;\theta)$ which is a also a differential equation with respect to time t and time s enters as a parameter. Further, SLE (6.40) is supplemented with the following initial condition:

$$f_{X(t_0)\dot{X}(s)}(x, y) = f_{X_0\dot{X}(s)}(x, y), \tag{6.41}$$

whose SLE will be formulated via the delta projection method in the ensuing paragraph 6.2.2.

Transformed SLE. As in all the previous cases, SLE (6.40) in non-closed due to the averaged term on its right-hand side. This term is of similar form to the one appearing in Eq. (6.9) with the exception that the product of the random delta functions is multiplied by $\Xi(t;\theta)$ rather than $\dot{\Xi}(t;\theta)$. As such, again the said product can be regarded as a FFℓ like $\mathcal{F} = \mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t_1};\theta)]$ with $t_1 = \max(t,s)$. Thus, by employing Eq. (2.19), the averaged term can be written as

$$\begin{aligned}
&\mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta))\,\delta(y-\dot{X}(s;\theta))\,\Xi(t;\theta)\Big] = \\
&\quad = m_{\Xi}(t)\mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta))\,\delta(y-\dot{X}(s;\theta))\,\Xi(t;\theta)\Big] + \\
&\qquad + C_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\left[\frac{\partial\left\{\delta(x-X(t;\theta))\,\delta(y-\dot{X}(s;\theta))\,\Xi(t;\theta)\right\}}{\partial X_0(\theta)}\right] + \\
&\qquad + \int_{t_0}^{t_1} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\frac{\delta\left\{\delta(x-X(t;\theta))\,\delta(y-\dot{X}(s;\theta))\,\Xi(t;\theta)\right\}}{\delta\Xi(\tau;\theta)}\right]d\tau .
\end{aligned} \tag{6.42}$$

Eq. (6.42) is further evaluated by following the usual approach into

$$\begin{aligned}
&\mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta))\,\delta(y-\dot{X}(s;\theta))\,\Xi(t;\theta)\Big] = \\
&\quad = m_{\Xi}(t)\mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta))\,\delta(y-\dot{X}(s;\theta))\Big] - \\
&\quad - C_{X_0\Xi}(t)\,\frac{\partial}{\partial x}\,\mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta))\,\delta(y-\dot{X}(s;\theta))\,V_{X_0}(t;\theta)\Big] - \\
&\quad - C_{X_0\Xi}(t)\,\frac{\partial}{\partial y}\,\mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta))\,\delta(y-\dot{X}(s;\theta))\,h'(X(s;\theta))\,V_{X_0}(s;\theta)\Big] - \\
&\quad - \frac{\partial}{\partial x}\int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta))\,\delta(y-\dot{X}(s;\theta))V_{\Xi(\tau)}(t;\theta)\Big]d\tau - \\
&\quad - \frac{\partial}{\partial y}\int_{t_0}^{s} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta))\,\delta(y-\dot{X}(s;\theta))\,h'(X(s;\theta))\,V_{\Xi(\tau)}(s;\theta)\Big]d\tau - \\
&\quad - \kappa\,C_{\Xi\Xi}(t,s)\,\frac{\partial}{\partial y}\,\mathbb{E}^{\theta}\Big[\delta(x-X(t;\theta))\,\delta(y-\dot{X}(s;\theta))\Big].
\end{aligned} \tag{6.43}$$

Finally, combining Eqs. (6.40) and (6.43), results in the following ***transformed Liouville equation for*** $X(t;\theta)$ ***and*** $\dot{X}(s;\theta)$:

$$\frac{\partial f_{X(t)\dot{X}(s)}(x,y)}{\partial t} + \frac{\partial}{\partial x}\Big[\big(h(x) + \kappa\, m_{\Xi}(t)\big) f_{X(t)\dot{X}(s)}(x,y)\Big] =$$
$$= \kappa\, C_{X_0\Xi}(t)\,\frac{\partial^2}{\partial x^2}\,\mathbb{E}^{\theta}\Big[\delta(x - X(t;\theta))\,\delta(y - \dot{X}(s;\theta))\,V_{X_0}(t;\theta)\Big] +$$
$$+ \kappa\, C_{X_0\Xi}(t)\,\frac{\partial^2}{\partial x \partial y}\,\mathbb{E}^{\theta}\Big[\delta(x - X(t;\theta))\,\delta(y - \dot{X}(s;\theta))\,h'(X(s;\theta))\,V_{X_0}(s;\theta)\Big] +$$
$$+ \kappa\,\frac{\partial^2}{\partial x^2}\int_{t_0}^{t} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\Big[\delta(x - X(t;\theta))\,\delta(y - \dot{X}(s;\theta))\,V_{\Xi(\tau)}(t;\theta)\Big]d\tau +$$
$$+ \kappa\,\frac{\partial^2}{\partial x \partial y}\int_{t_0}^{s} C_{\Xi\Xi}(t,\tau)\,\mathbb{E}^{\theta}\Big[\delta(x - X(t;\theta))\,\delta(y - \dot{X}(s;\theta))\,h'(X(s;\theta))\,V_{\Xi(\tau)}(s;\theta)\Big]d\tau +$$
$$+ \kappa^2 C_{\Xi\Xi}(t,s)\,\frac{\partial^2 f_{X(t)\dot{X}(s)}(x,y)}{\partial x \partial y}. \tag{6.44}$$

In consistence with all the prior examined cases, SLE (6.44) is non-closed due to the terms multiplying the product of random delta functions inside the averages. These will be accordingly addressed in the ensuing paragraph 6.2.3 in order to obtain a closed alternative of SLE (6.44).

6.2.2 Auxiliary, initial stochastic Liouville equation corresponding to the RDE

In this subsection, we are going to formulate the SLE for $X_0(\theta)$ and $\dot{X}(t;\theta)$ which will serve an initial condition to Eq. (6.44). As always, our starting point is the delta projection method:

$$f_{X_0\dot{X}(t)}(x,y) = \mathbb{E}^{\theta}\Big[\delta\big(x - X_0(\theta)\big)\,\delta\big(y - \dot{X}(t;\theta)\big)\Big]. \tag{6.45}$$

By differentiating, now, both sides of the above equation and employing RDE (6.1a), we find

$$\frac{\partial f_{X_0\dot{X}(t)}(x,y)}{\partial t} = \mathbb{E}^{\theta}\left[\delta\big(x - X_0(\theta)\big)\,\frac{\partial \delta\big(y - \dot{X}(t;\theta)\big)}{\partial \dot{X}(t;\theta)}\,h'\big(X(t;\theta)\big)\,\dot{X}(t;\theta)\right] + \kappa\,\mathbb{E}^{\theta}\left[\delta\big(x - X_0(\theta)\big)\,\frac{\partial \delta\big(y - \dot{X}(t;\theta)\big)}{\partial \dot{X}(t;\theta)}\,\dot{\Xi}(t;\theta)\right]. \tag{6.46}$$

The averaged terms on the right-hand side of the above expression can be further evaluated using the familiar formalism, thus, resulting in the following ***SLE for*** $X_0(\theta)$ ***and*** $\dot{X}(t;\theta)$ in its non-closed form:

$$\frac{\partial f_{X_0\dot{X}(t)}(x,y)}{\partial t} + \frac{\partial}{\partial y}\left(y\,\mathbb{E}^{\theta}\Big[\delta\big(x - X_0(\theta)\big)\,\delta\big(y - \dot{X}(t;\theta)\big)\,h'\big(X(t;\theta)\big)\Big]\right) = - \kappa\,\frac{\partial}{\partial y}\,\mathbb{E}^{\theta}\Big[\delta\big(x - X_0(\theta)\big)\,\delta\big(y - \dot{X}(t;\theta)\big)\dot{\Xi}(t;\theta)\Big]. \tag{6.47}$$

SLE (6.47) in non-closed due to the averaged terms appearing on both of its sides. The first one will be treated in paragraph 6.2.3 where an approximation scheme is proposed that results in a closed pdf evolution equation. The one appearing on its right-hand side can easily be treated by the appropriate NF theorem, as it is subsequently presented.

Transformed SLE. Recalling Eq. (6.1a) as well as the dependence of the response on the initial value $X_0(\theta)$ and the time history of the excitation $\Xi(\bullet;\theta)$ over $[t_0, t]$, the averaged term appearing on the right-hand side of Eq. (6.47) can be written as

$$\begin{aligned} &\mathbb{E}^{\theta}\Big[\delta\big(x-X_0(\theta)\big)\,\delta\big(y-\dot{X}(t;\theta)\big)\dot{\Xi}(t;\theta)\Big] = \\ &\quad = \mathbb{E}^{\theta}\Big[\delta\big(x-X_0(\theta)\big)\,\delta\Big(y-X[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]\Big)\dot{\Xi}(t;\theta)\Big], \end{aligned} \tag{6.48}$$

which is the appropriate form for the application of the Extension III of the NF theorem, Eq. (2.21), under the understanding that the product of the random delta functions is regarded as a FFℓ like $\mathcal{F}[X_0(\theta);\Xi(\bullet|_{t_0}^{t};\theta)]$. Implementation of the said extended NF theorem yields

$$\begin{aligned} &\mathbb{E}^{\theta}\Big[\delta\big(x-X_0(\theta)\big)\,\delta\big(y-\dot{X}(t;\theta)\big)\dot{\Xi}(t;\theta)\Big] = \\ &\quad = \dot{m}_{\Xi}(t)\,\mathbb{E}^{\theta}\Big[\delta\big(x-X_0(\theta)\big)\,\delta\big(y-\dot{X}(t;\theta)\big)\Big] + \\ &\qquad + \dot{C}_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\left[\frac{\partial\Big\{\delta\big(x-X_0(\theta)\big)\,\delta\big(y-\dot{X}(t;\theta)\big)\Big\}}{\partial X_0(\theta)}\right] + \\ &\qquad + \int_{t_0}^{t} C_{\dot{\Xi}\Xi}(t,\tau)\,\mathbb{E}^{\theta}\left[\frac{\delta\Big\{\delta\big(x-X_0(\theta)\big)\,\delta\big(y-\dot{X}(t;\theta)\big)\Big\}}{\delta\Xi(\tau;\theta)}\right]d\tau\,, \end{aligned} \tag{6.49}$$

where $C_{\dot{\Xi}\Xi}(t,\tau)=\partial_t C_{\Xi\Xi}(t,\tau)$. Further, by using the product and chain rules for derivatives and substituting the variational derivatives that arise, we obtain the following final form for the averaged term:

$$\begin{aligned} &\mathbb{E}^{\theta}\Big[\delta\big(x-X_0(\theta)\big)\,\delta\big(y-\dot{X}(t;\theta)\big)\dot{\Xi}(t;\theta)\Big] = \\ &\quad = \dot{m}_{\Xi}(t)\,\mathbb{E}^{\theta}\Big[\delta\big(x-X_0(\theta)\big)\,\delta\big(y-\dot{X}(t;\theta)\big)\Big] - \\ &\quad - \dot{C}_{X_0\Xi}(t)\,\frac{\partial}{\partial x}\,\mathbb{E}^{\theta}\Big[\delta\big(x-X_0(\theta)\big)\,\delta\big(y-\dot{X}(t;\theta)\big)\Big] - \\ &\quad - \frac{\partial}{\partial y}\,\dot{C}_{X_0\Xi}(t)\,\mathbb{E}^{\theta}\Big[\delta\big(x-X_0(\theta)\big)\,\delta\big(y-\dot{X}(t;\theta)\big)\,h'(X(t;\theta))\,V_{X_0}(t;\theta)\Big] - \\ &\quad - \frac{\partial}{\partial y}\int_{t_0}^{t} C_{\dot{\Xi}\Xi}(t,\tau)\,\mathbb{E}^{\theta}\Big[\delta\big(x-X_0(\theta)\big)\,\delta\big(y-\dot{X}(t;\theta)\big)\,h'(X(t;\theta))V_{\Xi(\tau)}(t;\theta)\Big]\,d\tau - \\ &\quad - \kappa\,C_{\dot{\Xi}\Xi}(t,t)\,\frac{\partial}{\partial y}\,\mathbb{E}^{\theta}\Big[\delta\big(x-X_0(\theta)\big)\,\delta\big(y-\dot{X}(t;\theta)\big)\Big]. \end{aligned} \tag{6.50}$$

Finally, by substituting Eq. (6.50) into Eq. (6.47) gives rise to

$$\frac{\partial f_{X_0 \dot{X}(t)}(x, y)}{\partial t} + \frac{\partial}{\partial y}\left(y \mathbb{E}^\theta\left[\delta\left(x - X_0(\theta)\right)\delta\left(y - \dot{X}(t;\theta)\right)h'\left(X(t;\theta)\right)\right]\right) +$$
$$+ \kappa\, m_{\dot{\Xi}}(t)\frac{\partial f_{X_0 \dot{X}(t)}(x, y)}{\partial y} =$$
$$= \kappa\, \dot{C}_{X_0 \Xi}(t)\frac{\partial^2 f_{X_0 \dot{X}(t)}(x, y)}{\partial x \partial y} + \kappa^2 C_{\dot{\Xi}\Xi}(t, t)\frac{\partial^2 f_{X_0 \dot{X}(t)}(x, y)}{\partial y^2} + \tag{6.51}$$
$$+ \kappa\, \dot{C}_{X_0 \Xi}(t)\frac{\partial^2}{\partial y^2}\mathbb{E}^\theta\left[\delta\left(x - X_0(\theta)\right)\delta\left(y - \dot{X}(t;\theta)\right)h'(X(t;\theta))\, V_{X_0}(t;\theta)\right] +$$
$$+ \kappa^2 \frac{\partial^2}{\partial y^2}\int_{t_0}^{t} C_{\dot{\Xi}\Xi}(t, \tau)\, \mathbb{E}^\theta\left[\delta\left(x - X_0(\theta)\right)\delta\left(y - \dot{X}(t;\theta)\right)h'(X(t;\theta))V_{\Xi(\tau)}(t;\theta)\right]d\tau.$$

Eq. (6.51) is the ***transformed SLE for the joint pdf of*** $X_0(\theta)$ ***and*** $\dot{X}(t;\theta)$. Once more, it is useful to reiterate that not only the averaged term on the left-hand side of Eq. (6.51) constitute the apprehended expression non-closed, but also the additional terms multiplying the product of random delta functions.

6.2.3 Novel, two-time evolution equations for the pdf of the response and its derivative

Having formulated SLEs (6.44) and (6.51), we are able to proceed with the derivation of the second variant of the evolution equations corresponding to the pdf of the response and its derivative. However, let us first rewrite them in a notation that makes the terms requiring approximation more evident. More specifically, using the notation presented in paragraph 6.1.3, the ***exact, non-closed pdf evolution equation for the joint pdf of*** $X(t;\theta)$ ***and*** $\dot{X}(s;\theta)$ reads

$$\frac{\partial f_{X(t)\dot{X}(s)}(x, y)}{\partial t} + \frac{\partial}{\partial x}\left[\left(h(x) + \kappa\, m_\Xi(t)\right)f_{X(t)\dot{X}(s)}(x, y)\right] =$$
$$= \kappa\, C_{X_0 \Xi}(t)\,\frac{\partial^2}{\partial x^2}\,\mathbb{E}^\theta\left[\delta(x, y; X_t, \dot{X}_s)\exp\left(\mathcal{I}_{h'}[X(\bullet|_{t_0}^{t};\theta)\right)\right] +$$
$$+ \kappa\, C_{X_0 \Xi}(t)\,\frac{\partial^2}{\partial x \partial y}\,\mathbb{E}^\theta\left[\delta(x, y; X_t, \dot{X}_s)\,h'(X(s;\theta))\exp\left(\mathcal{I}_{h'}[X(\bullet|_{t_0}^{s};\theta)\right)\right] +$$
$$+ \kappa^2 \frac{\partial^2}{\partial x^2}\int_{t_0}^{t} C_{\Xi\Xi}(t, \tau)\,\mathbb{E}^\theta\left[\delta(x, y; X_t, \dot{X}_s)\exp\left(\mathcal{I}_{h'}[X(\bullet|_{\tau}^{t};\theta)]\right)\right]d\tau +$$
$$+ \kappa^2 \frac{\partial^2}{\partial x \partial y}\int_{t_0}^{s} C_{\Xi\Xi}(t, \tau)\,\mathbb{E}^\theta\left[\delta(x, y; X_t, \dot{X}_s)\,h'(X(s;\theta))\exp\left(\mathcal{I}_{h'}[X(\bullet|_{\tau}^{s};\theta)]\right)\right]d\tau +$$
$$+ \kappa^2 C_{\Xi\Xi}(t, s)\,\frac{\partial^2 f_{X(t)\dot{X}(s)}(x, y)}{\partial x \partial y}. \tag{6.52}$$

Eq. (6.52) must also be supplemented by an initial condition, which is obtained through Eq. (6.45) for $t = s$ as follows:

$$\begin{aligned}
&\frac{\partial f_{X_0\dot{X}(s)}(x,y)}{\partial s} + \frac{\partial}{\partial y}\left(y\mathbb{E}^{\theta}\left[\delta(x,y;X_{t_0},\dot{X}_s)\,h'\big(X(s;\theta)\big)\right]\right) + \\
&\qquad + \kappa\, m_{\dot{\Xi}}(s)\frac{\partial f_{X_0\dot{X}(s)}(x,y)}{\partial y} = \\
&\quad = \kappa\,\dot{C}_{X_0\Xi}(s)\frac{\partial^2 f_{X_0\dot{X}(s)}(x,y)}{\partial x\partial y} + \kappa^2 C_{\dot{\Xi}\Xi}(s,s)\frac{\partial^2 f_{X_0\dot{X}(s)}(x,y)}{\partial y^2} + \\
&+ \kappa\,\dot{C}_{X_0\Xi}(s)\frac{\partial^2}{\partial y^2}\,\mathbb{E}^{\theta}\left[\delta(x,y;X_{t_0},\dot{X}_s)h'(X(s;\theta))\exp\left(\mathcal{I}_{h'}[X(\bullet|_{t_0}^{s};\theta)\right)\right] + \\
&+ \kappa^3\frac{\partial^2}{\partial y^2}\int_{t_0}^{s} C_{\dot{\Xi}\Xi}(s,\tau)\,\mathbb{E}^{\theta}\left[\delta(x,y;X_{t_0},\dot{X}_s)h'(X(s;\theta))\exp\left(\mathcal{I}_{h'}[X(\bullet|_{\tau}^{s};\theta)]\right)\right]d\tau .
\end{aligned} \tag{6.53}$$

Eq. (6.53) is the ***exact, non-closed pdf evolution equation for the joint pdf of*** $X_0(\theta)$ ***and*** $\dot{X}(s;\theta)$. Under this notation, it is readily recognized that the nonlocal terms which constitute the above equations non-closed are similar to the ones appearing in Eqs. (6.25), (6.26) and thus, the same approximation can be implemented. As such, from Eq. (6.53) the ***following closed, approximate pdf evolution equation for the joint pdf of*** $X_0(\theta)$ ***and*** $\dot{X}(s;\theta)$:

$$\begin{aligned}
&\frac{\partial f_{X_0\dot{X}(s)}(x,y)}{\partial s} + \left[h'(x)+h''(x)h(x)(s-t_0)\right]\frac{\partial}{\partial y}\left(y\,f_{X_0\dot{X}(t)}(x,y)\right) + \\
&\qquad + \kappa\, m_{\dot{\Xi}}(s)\frac{\partial f_{X_0\dot{X}(s)}(x,y)}{\partial y} = \kappa\,\dot{C}_{X_0\Xi}(s)\frac{\partial^2 f_{X_0\dot{X}(s)}(x,y)}{\partial x\partial y} + \\
&\qquad + \kappa^2 C_{\dot{\Xi}\Xi}(s,s)\frac{\partial^2 f_{X_0\dot{X}(s)}(x,y)}{\partial y^2} + \left[h'(x)+h''(x)h(x)(s-t_0)\right]\times \\
&\qquad\qquad \times\frac{\partial^2}{\partial y^2}\left\{\left[\sum_{m=0}^{M}\frac{1}{m!}G_m^{(3)}\left[R_{h'}(\bullet),s\right]\psi_2^m\left(x;R_{h'}(s)\right)\right]f_{X_0\dot{X}(s)}(x,y)\right\}.
\end{aligned} \tag{6.54}$$

where

$$\begin{aligned}
G_m^{(3)}\left[R_{h'}(\bullet),s\right] &= \kappa\exp\left(\int_{t_0}^{s}R_{h'}(u)\,du\right)\dot{C}_{X_0\Xi}(s)\,(s-t_0)^m + \\
&\qquad + \kappa^3\int_{t_0}^{s}\partial_s C_{\Xi\Xi}(s,\tau)\exp\left(\int_{\tau}^{s}R_{h'}(u)\,du\right)(s-\tau)^m\,d\tau ,
\end{aligned} \tag{6.55}$$

$$\psi_2\left(x;R_{h'}(s)\right) = \varphi_{h'}\left(x;R_{h'}(s)\right) + \varphi'_{h'}\left(x;R_{h'}(s)\right)h(x)\,(s-t_0)\,. \tag{6.56}$$

Subsequently, by working accordingly in the case of Eq. (6.52) we obtain the following ***approximate, closed evolution equation for the two-time joint pdf of the response and its derivative*** $f_{X(t)\dot{X}(s)}(x,y)$:

$$\frac{\partial f_{X(t)\dot{X}(s)}(x,y)}{\partial t} + \frac{\partial}{\partial x}\Big[\big(h(x) + \kappa\, m_{\Xi}(t)\big) f_{X(t)\dot{X}(s)}(x,y)\Big] =$$
$$= \frac{\partial^2}{\partial x \partial y}\Big\{\Big(\big[h'(x) + h''(x)h(x)(s-t)\big]A_m^{\mathrm{eff}}[t,s] + \kappa^2 C_{\Xi\Xi}(t,s)\Big) f_{X(t)\dot{X}(s)}(x,y)\Big\} +$$
$$+ \frac{\partial^2}{\partial x^2}\left\{\sum_{m=0}^{M}\frac{1}{m!}D_m^{\mathrm{eff}}\big[R_{h'}(\bullet),t,s\big]\,\varphi_{h'}^{m}\big(x\,;R_{h'}(s)\big)\, f_{X(t)\dot{X}(s)}(x,y)\right\}, \tag{6.57}$$

where $A_m^{\mathrm{eff}}[t,s]$ is a shorthand for

$$A_m^{\mathrm{eff}}[t,s] = \left[\sum_{m=0}^{M}\frac{1}{m!}D_m^{\mathrm{eff}}\big[R_{h'}(\bullet),t,s\big]\,\psi_1^{m}\big(x\,;R_{h'}(s)\big)\right]. \tag{6.58}$$

This approximate equation seems somewhat simpler than its counterpart Eq. (6.28) because the manipulations required for Eq. (6.52) are more familiar and are confined to the terms only on its right-hand side. Nevertheless, this time, more complicacies arise in the treatment of the initial condition, Eq. (6.53).

Let us, also, consider the simpler alternatives of Eqs. (6.54) and (6.57) by taking into account only the terms which would appear if we had implemented Hänggi's ansatz. Thus, for the approximate pdf evolution equation for the joint pdf of $X_0(\theta)$ and $\dot{X}(s;\theta)$ we find

$$\begin{aligned}
&\frac{\partial f_{X_0\dot{X}(s)}(x,y)}{\partial s} + \big[h'(x) + h''(x)h(x)(s-t_0)\big]\frac{\partial}{\partial y}\Big(y\, f_{X_0\dot{X}(t)}(x,y)\Big) + \\
&\quad + \kappa\, m_{\dot{\Xi}}(s)\frac{\partial f_{X_0\dot{X}(s)}(x,y)}{\partial y} = \kappa\, \dot{C}_{X_0\Xi}(s)\frac{\partial^2 f_{X_0\dot{X}(s)}(x,y)}{\partial x \partial y} + \\
&\quad + \big[h'(x) + h''(x)h(x)(s-t_0)\big]G^{(3)}\big[R_{h'}(\bullet),s\big]\frac{\partial^2 f_{X_0\dot{X}(s)}(x,y)}{\partial y^2} + \\
&\quad + \kappa^2 C_{\dot{\Xi}\Xi}(s,s)\frac{\partial^2 f_{X_0\dot{X}(s)}(x,y)}{\partial y^2},
\end{aligned} \tag{6.59}$$

where

$$\begin{aligned}
G^{(3)}\big[R_{h'}(\bullet),s\big] &= \kappa \exp\left(\int_{t_0}^{s} R_{h'}(u)\,du\right)\dot{C}_{X_0\Xi}(s) + \\
&\quad + \kappa^3\int_{t_0}^{s}\partial_s C_{\Xi\Xi}(s,\tau)\exp\left(\int_{\tau}^{s}R_{h'}(u)\,du\right)d\tau .
\end{aligned} \tag{6.60}$$

Working in similar fashion for the case of the approximate, closed evolution equation for the two-time joint pdf of the response and its derivative $f_{X(t)\dot{X}(s)}(x,y)$, we obtain

$$\frac{\partial f_{X(t)\dot{X}(s)}(x,y)}{\partial t} + \frac{\partial}{\partial x}\Big[\Big(h(x) + \kappa\, m_{\Xi}(t)\Big) f_{X(t)\dot{X}(s)}(x,y)\Big] =$$
$$= \big[h'(x) + h''(x)h(x)(s-t)\big]\, D^{\text{eff}}\Big[R_{h'}(\bullet), t, s\Big] \frac{\partial^2 f_{X(t)\dot{X}(s)}(x,y)}{\partial x \partial y} + \tag{6.61}$$
$$+ \kappa^2 C_{\Xi\Xi}(t,s) \frac{\partial^2 f_{X(t)\dot{X}(s)}(x,y)}{\partial x \partial y} + D^{\text{eff}}\Big[R_{h'}(\bullet), t, s\Big] \frac{\partial^2 f_{X(t)\dot{X}(s)}(x,y)}{\partial x^2},$$

in which $D^{\text{eff}}\Big[R_{h'}(\bullet), t, s\Big]$ is given by Eq. (6.37). At this point, it is easily understood that Eqs. (6.59), (6.61) are similar to their alternatives (6.54), (6.57) respectively, while also being easier to comprehend and manage.

6.3 Exact pdfs for a linear, additively excited RDE

In this section, let us, for the last time, examine the results of this methodology in the linear case. By setting $h(x) = \eta x$, with $\eta < 0$ for stability reasons, in RDE (6.1a), we obtain the RIVP

$$\dot{X}(t;\theta) = \eta X(t;\theta) + \kappa\, \Xi(t;\theta), \qquad X(t_0;\theta) = X_0(\theta). \tag{6.62a,b}$$

As was previously discussed, in this case, the variational derivatives are explicitly calculated into $V_{X_0}(t;\theta) = e^{\eta(t-t_0)}$ and $V_{\Xi(\tau)}(t;\theta) = \kappa\, e^{\eta(t-\tau)}$. It is also useful to reiterate that in the linear case, the joint pdf of the response and its derivative as well as the auxiliaries, initial pdfs are expected to be Gaussian distributions whose moments have been specified in sec. B.2 by formulating and solving their corresponding moment problems.

Subsequently, we are going to determine the pdfs for the linear, additively excited RIVP (6.62a,b) by considering the two variants for the two-time joint pdf of the response and its derivative separately. This will allow us to examine the potency of both derived variants and comment on any discrepancies that may or may not arise between them.

6.3.1 First variant of the exact pdfs

In this subsection, we shall rewrite the derived SLEs for the first variant of the problem in the linear case and then, by employing an appropriate Fourier transform, the exact pdfs corresponding to the RIVP will be specified. For this, it is necessary to first consider the auxiliary, initial pdf $f_{\dot{X}(t_0)X(t)}(x,y)$.

(a) Exact initial pdf $f_{\dot{X}(t_0)X(t)}$

Using the aforementioned calculated variational derivatives, the SLE (6.23) for the joint pdf $f_{\dot{X}(t_0)X(t)}$ is specified into

$$\frac{\partial f_{\dot{X}(t_0)X(t)}(x,y)}{\partial t} + \frac{\partial}{\partial y}\left[\left(\eta y + \kappa\, m_\Xi(t)\right) f_{\dot{X}(t_0)X(t)}(x,y)\right] =$$
$$= \left(\kappa\eta C_{X_0\Xi}(t) + \kappa^2\, C_{\Xi\Xi}(t,t_0)\right)\frac{\partial^2 f_{\dot{X}(t_0)X(t)}(x,y)}{\partial x\partial y} + \tag{6.63}$$
$$+ \kappa\, D^{\text{eff}}(t)\,\frac{\partial^2 f_{\dot{X}(t_0)X(t)}(x,y)}{\partial y^2},$$

where $D^{\text{eff}}(t)$ is the generalized effective noise intensity given by Eq. (3.41). In contrast with SLE (6.23) and consistent with all the linear cases examined so far, Eq. (6.63) is closed and exact. Further. it is supplemented by the following initial condition:

$$f_{\dot{X}(t_0)X(t_0)}(x,y) = f_{\dot{X}(t_0)X_0}(x,y), \tag{6.64}$$

which is a bivariate Gaussian distribution, in accordance with its components.

Solution of Eq. (6.63) in the Fourier domain. We employ the two-dimensional Fourier transform $\varphi_{\dot{X}(t_0)X(t_0)}(\boldsymbol{u}) = \varphi_{\dot{X}(t_0)X(t)}(u_0,u_1) = \int_{\mathbb{R}^2} e^{i(u_0x+u_1y)}\, f_{\dot{X}(t_0)X(t_0)}(x,y)\,dx\,dy$, which results in the transformed Eq. (6.63):

$$\frac{\partial\varphi_{\dot{X}(t_0)X(t)}(\boldsymbol{u})}{\partial t} = \eta u_1\,\frac{\partial\varphi_{\dot{X}(t_0)X(t)}(\boldsymbol{u},t)}{\partial u_1} + \varphi_{\dot{X}(t_0)X(t)}(\boldsymbol{u})\times$$
$$\times\left[i\kappa\, m_\Xi(t)u_1 - \left(\kappa\eta C_{X_0\Xi}(t) + \kappa^2\, C_{\Xi\Xi}(t,t_0)\right)u_0u_1 - \kappa D^{\text{eff}}(t)u_1^2\right], \tag{6.65}$$

supplemented by the transformed initial condition

$$\varphi_{\dot{X}(t_0)X(t_0)}(\boldsymbol{u}) = \exp\left(i\dot{m}_{X_0}u_0 + i\,m_{X_0}u_1 - \frac{1}{2}\dot{\sigma}^2_{X_0}u_0^2 - \frac{1}{2}\sigma^2_{X_0}u_1^2 - u_0u_1C_{\dot{X}_0X_0}\right). \tag{6.66}$$

The IVP (6.65) and (6.66) is solved in the familiar manner. First, the solution to the homogeneous variant of Eq. (6.41) determines the change of variables $(u_0,u_1)\to(u_0,\upsilon)$, $\upsilon = u_1e^{\eta t}$. Under these variables, Eq. (6.65) is transformed into a linear ODE with respect to time t. By solving the said equation and then returning to the original variables $\boldsymbol{u}$, we determine the solution of IVP (6.65)-(6.66) as

$$\varphi_{\dot{X}(t_0)X(t)}(\boldsymbol{u}) = \exp\left(i\dot{m}_{X_0}u_0 - \frac{1}{2}\dot{\sigma}^2_{X_0}u_0^2\right)\times$$
$$\times\exp\left[iu_1\left(m_{X_0}\,e^{\eta(t-t_0)} + \kappa\int_{t_0}^{t} m_\Xi(\tau)\,e^{\eta(t-\tau)}\,d\tau\right)\right]\times$$
$$\times\exp\left[-\frac{1}{2}u_1^2\left(\sigma^2_{X_0}\,e^{2\eta(t-t_0)} + 2\kappa\int_{t_0}^{t} D^{\text{eff}}(\tau)\,e^{2\eta(t-\tau)}\,d\tau\right)\right]\times$$
$$\times\exp\left[-u_0u_1\,C_{\dot{X}(t_0)X(t_0)}\,e^{\eta(t-t_0)}\right]\times$$

$$\times \exp\left[-u_0 u_1 \left(\kappa\eta \int_{t_0}^{t} C_{X_0\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau + \kappa^2 \int_{t_0}^{t} C_{\Xi\Xi}(\tau, t_0)\, e^{\eta(t-\tau)}\, d\tau\right)\right]. \quad (6.67)$$

By utilizing Eq. (A.3) for $m_X(t)$, the verified Eq. (3.47) for $\sigma_X^2(t)$, as well as Eq. (B.55) for $C_{\dot{X}(t_0)X(t_0)}$, solution (6.67) is written equivalently

$$\begin{aligned}\varphi_{\dot{X}(t_0)X(t)}(\boldsymbol{u}) &= \exp\left(i\dot{m}_{X_0}u_0 - \frac{1}{2}\dot{\sigma}_{X_0}^2 u_0^2 + i m_X(t)u_1 - \frac{1}{2}\sigma_X^2(t)u_1^2\right)\times \\ &\times \exp\left[-u_0 u_1\left(\eta\, C_{X_0X_0}\, e^{\eta(t-t_0)} + \kappa\, C_{X_0\Xi}(t_0)\, e^{\eta(t-t_0)}\right)\right]\times \\ &\times \exp\left[-u_0 u_1\left(\kappa\eta\int_{t_0}^{t} C_{X_0\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau + \kappa^2\int_{t_0}^{t} C_{\Xi\Xi}(\tau, t_0)\, e^{\eta(t-\tau)}\, d\tau\right)\right].\end{aligned} \quad (6.68)$$

Last, utilizing Eq. (B.56) for $C_{\dot{X}X(t)}(t_0, t)$ we obtain

$$\begin{aligned}\varphi_{\dot{X}(t_0)X(t)}(\boldsymbol{u}) &= \exp\left(i\dot{m}_{X_0}u_0 - \frac{1}{2}\dot{\sigma}_{X_0}^2 u_0^2 + i m_X(t)u_1\right)\times \\ &\times \exp\left(-\frac{1}{2}\sigma_X^2(t)u_1^2 - C_{\dot{X}X(t)}(t_0, t)u_0 u_1\right).\end{aligned} \quad (6.69)$$

At this point, Eq. (6.69) is easily recognized as the characteristic function of the correct, bivariate Gaussian pdf.

(b) Exact two-time, joint pdf of $X(s;\theta)$ and $\dot{X}(t;\theta)$

Let us, now, examine the two-time pdf $f_{X(s)\dot{X}(t)}(x, y)$ in the linear case. Thus, substituting in SLE (6.13) $h(x) = \eta x$, and employing the aforementioned calculated variational derivative, we obtain the following, exact pdf evolution equation:

$$\begin{aligned}\frac{\partial f_{X(s)\dot{X}(t)}(x, y)}{\partial t} + \frac{\partial}{\partial y}\left[\left(\eta y + \kappa m_{\dot{\Xi}}(t)\right) f_{X(s)\dot{X}(t)}(x, y)\right] &= \\ = \left(\kappa\eta\, G(t) + \kappa^2\, C_{\dot{\Xi}\Xi}(t, t)\right)\frac{\partial^2 f_{X(s)\dot{X}(t)}(x, y)}{\partial y^2} &+ \\ + \kappa\, G(t, s)\,\frac{\partial^2 f_{X(s)\dot{X}(t)}(x, y)}{\partial x \partial y}&,\end{aligned} \quad (6.70)$$

where $G(t)$ is given by Eq. (4.36), while

$$G(t, s) = \dot{C}_{X_0\Xi}(t)\, e^{\eta(s-t_0)} + \kappa\int_{t_0}^{s} \partial_t C_{\Xi\Xi}(t, \tau)\, e^{\eta(s-\tau)}\, d\tau. \quad (6.71)$$

Note that comparison of Eq. (6.71) and Eq. (4.36) yields $G(t) = G(t, t)$. Further, through (6.70) it becomes apparent that the required initial condition for its solution is $f_{X(s)\dot{X}(t_0)}(x, y)$. Under the obvious identity $f_{X(s)\dot{X}(t_0)}(x, y) = f_{\dot{X}(t_0)X(s)}(y, x)$, it is readily seen that the required initial condition has already been specified as the solution of Eq. (6.63).

Solution of Eq. (6.70) using Fourier transform. As performed, plenty of times so far, the two-time pdf evolution equation (6.70) is solved by employing the two-dimensional Fourier transform; $\varphi_{X(s)\dot{X}(t)}(\boldsymbol{u}) = \int_{\mathbb{R}^2} e^{i(u_1 x + u_2 y)} f_{X(s)\dot{X}(t)}(x, y)\, dxdy$, resulting in the equation

$$\begin{aligned} \frac{\partial \varphi_{X(s)\dot{X}(t)}(\boldsymbol{u})}{\partial t} &= \eta u_2 \frac{\partial \varphi_{X(s)\dot{X}(t)}(\boldsymbol{u})}{\partial u_2} + \varphi_{X(t)\dot{X}(s)}(\boldsymbol{u}) \times \\ &\times \left[i\kappa\, \dot{m}_{\Xi}(t)\, u_2 - \left(\kappa\eta\, G(t) + \kappa^2\, C_{\dot{\Xi}\Xi}(t, t)\right) u_2^2 - \kappa\, G(t, s)\, u_1 u_2 \right]. \end{aligned} \tag{6.72}$$

Eq. (6.72) is also supplemented, through Eq. (6.63), by the initial condition

$$\begin{aligned} \varphi_{X(s)\dot{X}(t_0)}(\boldsymbol{u}) &= \exp\left(i m_X(s) u_1 - \frac{1}{2}\sigma_X^2(s) u_1^2 + i\dot{m}_{X_0} u_2 \right) \times \\ &\times \exp\left(-\frac{1}{2}\dot{\sigma}_{X_0}^2 u_2^2 - C_{X(s)\dot{X}(t_0)}(s)\, u_1 u_2 \right). \end{aligned} \tag{6.73}$$

The solution of IVP (6.72), (6.73) is carried out in the familiar fashion, that is the method of characteristics, resulting in

$$\begin{aligned} \varphi_{X(s)\dot{X}(t)}(\boldsymbol{u}) &= \exp\left[i m_X(s) u_1 - \frac{1}{2}\sigma_X^2(s) u_1^2 \right] \times \\ &\times \exp\left[i u_2 \left(\kappa \int_{t_0}^{t} \dot{m}_{\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau + \dot{m}_{X_0}\, e^{\eta(t-t_0)} \right) - \frac{1}{2} u_2^2\, \dot{\sigma}_{X_0}^2\, e^{2\eta(t-t_0)} \right] \times \\ &\times \exp\left[-\frac{1}{2} u_2^2 \left(2\kappa\eta \int_{t_0}^{t} G(\tau)\, e^{2\eta(t-\tau)}\, d\tau + 2\kappa^2 \int_{t_0}^{t} C_{\dot{\Xi}\Xi}(\tau, \tau)\, e^{2\eta(t-\tau)}\, d\tau \right) \right] \times \\ &\times \exp\left[-u_1 u_2 \left(C_{X(s)\dot{X}(t_0)}(s)\, e^{\eta(t-t_0)} + \kappa \int_{t_0}^{t} G(\tau, s)\, e^{\eta(t-\tau)}\, d\tau \right) \right]. \end{aligned} \tag{6.74}$$

At this point, using Eqs. (B.29b) and (B.30), the second exponential term in the above equation is identified as $\dot{m}_X(t)$. Then, using Eq. (B.57) for $t = t_0$, Eq. (4.37a) for the connection between $G(t)$ and $D^{\text{eff}}(t)$, and performing some simple calculations, the second exponential term is recognized as $\sigma_{\dot{X}}^2(t)$. Last, employing Eq. (B.56) for $C_{X(s)\dot{X}(t_0)}(s)$ as well as definition (6.71) for $G(\tau, s)$ and performing simple algebraic manipulations, the rightmost exponential term of Eq. (6.74) is identified through Eq. (B.54) as $C_{X\dot{X}}(s, t)$. Combining all of the above with Eq. (6.50) results into

$$\varphi_{X(s)\dot{X}(t)}(\boldsymbol{u}) = \exp\left(i m_X(s)u_1 - \frac{1}{2}\sigma_X^2(s)u_1^2 + i\dot{m}_X(t)u_2 \right) \times \\ \times \exp\left(-\frac{1}{2}\sigma_{\dot{X}}^2(t)u_2^2 - C_{X\dot{X}}(s,t)\,u_1 u_2 \right). \tag{6.75}$$

Finally, Eq. (6.75) is identified as the characteristic function of the expected bivariate Gaussian distribution, a finding which reaffirms the approach presented herein.

6.3.2 Second variant of the exact pdfs

Let us, now, repeat the process presented in the previous paragraph 6.3.1 for the equations derived in section 6.2.

(a) Exact initial pdf $f_{X_0\dot{X}(t)}(x,y)$

In this subsection, we are going to determine the pdf $f_{X_0\dot{X}(t)}(x,y)$ in the linear case. Thus, setting $h(x) = \eta x$ in SLE (6.51), and substituting the calculated variational derivatives $V_{X_0}(t;\theta) = e^{\eta(t-t_0)}$ and $V_{\Xi(\tau)}(t;\theta) = \kappa e^{\eta(t-\tau)}$, results in the following, exact pdf evolution equation:

$$\frac{\partial f_{X_0\dot{X}(t)}(x,y)}{\partial t} + \frac{\partial}{\partial y}\left[\left(\eta y + \kappa\,\dot{m}_\Xi(t) \right) f_{X_0\dot{X}(t)}(x,y) \right] = \\ = \kappa\,\dot{C}_{X_0\Xi}(t)\frac{\partial^2 f_{X_0\dot{X}(t)}(x,y)}{\partial x \partial y} + \left(\kappa^2 C_{\dot{\Xi}\Xi}(t,t) + \kappa\eta G(t) \right)\frac{\partial^2 f_{X_0\dot{X}(t)}(x,y)}{\partial y^2}, \tag{6.76}$$

where $G(t)$ is again given by Eq. (4.36). The initial condition for Eq. (6.76) is the same as the one for Eq. (6.70), i.e. the inverse Fourier transform of Eq. (6.69).

Solution of Eq. (6.76) using Fourier transform. Following the usual approach, the two-time pdf evolution equation (6.76) is solved by employing the two-dimensional Fourier transform; $\varphi_{X_0\dot{X}(t)}(\boldsymbol{u}) = \int_{\mathbb{R}^2} e^{i(u_0 x + u_1 y)}\, f_{X_0\dot{X}(t)}(x,y)\, dxdy$, resulting in the equation

$$\frac{\partial \varphi_{X_0\dot{X}(t)}(\boldsymbol{u})}{\partial t} = \eta u_1 \frac{\partial \varphi_{X_0\dot{X}(t)}(\boldsymbol{u})}{\partial u_1} + \varphi_{X_0\dot{X}(t)}(\boldsymbol{u}) \times \\ \times \left[i\kappa\,\dot{m}_{\dot{\Xi}}(t)u_1 - \kappa\dot{C}_{X_0\Xi}(t)u_0 u_1 - \left(\kappa^2 C_{\dot{\Xi}\Xi}(t,t) + \kappa\eta\, G(t) \right) u_1^2 \right]. \tag{6.77}$$

Eq. (6.77) is also supplemented by the transformed initial condition which, in this case, coincides with Eq. (6.69).

The solution of IVP (6.69), (6.77) commences by considering the homogenous variant of Eq. (6.77). This provides us with the change of variables $(u_0, u_1) \to (u_0, \upsilon)$, $\upsilon = u_1 e^{\eta t}$. Under these variables, Eq. (6.77) is transformed into a linear ODE with respect to time t. By solving the said equation and then returning to the original variables $\boldsymbol{u}$, we determine the solution of the aforementioned IVP as:

$$\varphi_{X_0\dot{X}(t)}(\boldsymbol{u}) = \exp\left(i m_{X_0} u_0 - \frac{1}{2}\sigma^2_{X_0} u_0^2\right)\times$$
$$\times \exp\left[i u_1 \left(\dot{m}_{X_0}\, e^{\eta(t-t_0)} + \kappa \int_{t_0}^{t} \dot{m}_{\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau \right) - \frac{1}{2} u_1^2 \sigma^2_{\dot{X}_0}\, e^{2\eta(t-t_0)} \right]\times$$
$$\times \exp\left[-\frac{1}{2} u_1^2 \left(2\kappa^2 \int_{t_0}^{t} C_{\dot{\Xi}\Xi}(\tau,\tau)\, e^{2\eta(t-\tau)}\, d\tau + 2\kappa\eta \int_{t_0}^{t} G(\tau)\, e^{2\eta(t-\tau)}\, d\tau \right)\right]\times$$
$$\times \exp\left[- u_0 u_1 \left(C_{X_0\dot{X}_0}\, e^{\eta(t-t_0)} + \kappa \int_{t_0}^{t} \dot{C}_{X_0\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau \right)\right]. \tag{6.78}$$

All of the terms appearing in the parentheses inside the exponentials are of the same form as the ones in Eq. (6.74). Thus, performing the manipulations describes in section 6.3.1, Eq. (6.78) is equivalently rewritten as:

$$\varphi_{X_0\dot{X}(t)}(\boldsymbol{u}) = \exp\left(i m_{X_0} u_0 - \frac{1}{2}\sigma^2_{X_0} u_0^2 + i \dot{m}_X(t) u_1 - \frac{1}{2}\sigma^2_{\dot{X}}(t) u_1^2 - C_{X_0\dot{X}}(t)\, u_0 u_1 \right). \tag{6.79}$$

Last, Eq. (6.79) is easily recognized as the expected Gaussian characteristic function.

(b) Exact two-time joint pdf of $X(t;\theta)$ and $\dot{X}(s;\theta)$

We will now examine the linear counterpart of SLE (6.44). Again, substituting $h(x) = \eta x$ and calculating the variational derivatives results in

$$\frac{\partial f_{X(t)\dot{X}(s)}(x,y)}{\partial t} + \frac{\partial}{\partial x}\left[\left(\eta x + \kappa\, m_{\Xi}(t)\right) f_{X(t)\dot{X}(s)}(x,y)\right] =$$
$$= D^{\text{eff}}(t) \frac{\partial^2 f_{X(t)\dot{X}(s)}(x,y)}{\partial x^2} + \left(\eta D(t,s) + \kappa^2 C_{\Xi\Xi}(t,s)\right) \frac{\partial^2 f_{X(t)\dot{X}(s)}(x,y)}{\partial x \partial y}, \tag{6.80}$$

where $D^{\text{eff}}(t)$ is defined by Eq. (3.41) and $D(t,s)$ by Eq. (5.43). Eq. (6.80) is also supplemented by the initial condition $f_{X(t)\dot{X}(t_0)}(x,y)$, which has already been specified in the previous paragraph.

Solution of Eq. (6.80) using Fourier transform. Eq. (6.80) is solved by employing the two-dimensional Fourier transform; $\varphi_{X(t)\dot{X}(s)}(\boldsymbol{u}) = \int_{\mathbb{R}^2} e^{i(u_1 x + u_2 y)}\, f_{X(t)\dot{X}(s)}(x,y)\, dxdy$, resulting in the equation

$$\frac{\partial \varphi_{X(t)\dot{X}(s)}(\boldsymbol{u})}{\partial t} = \eta u_1 \frac{\partial \varphi_{X(t)\dot{X}(s)}(\boldsymbol{u})}{\partial u_1} + \varphi_{X(t)\dot{X}(s)}(\boldsymbol{u}) \times$$
$$\times\left[i\kappa m_{\Xi}(t)\, u_1 - D^{\text{eff}}(t)\, u_1^2 - \left(\eta D(t,s) + \kappa^2 C_{\Xi\Xi}(t,s)\right) u_1 u_2 \right], \tag{6.81}$$

which, in turn, is supplemented by the initial condition Eq. (6.79) written for $s = t$.

Following the usual approach, the solution of IVP (6.79), (6.81) is attained:

$$\begin{aligned}
\varphi_{X(t)\dot{X}(s)}(\boldsymbol{u}) = {} & \exp\left[i u_1 \left(m_{X_0}\, e^{\eta(t-t_0)} + \kappa \int_{t_0}^{t} m_{\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau \right)\right] \times \\
& \times \exp\left[-\frac{1}{2} u_1^2 \left(\sigma_{X_0}^2\, e^{2\eta(t-t_0)} + 2\int_{t_0}^{t} D^{\text{eff}}(\tau)\, e^{2\eta(t-\tau)}\, d\tau \right) - u_1 u_2 C_{X_0 \dot{X}}(s)\, e^{\eta(t-t_0)} \right] \times \\
& \times \exp\left[-u_1 u_2 \left(\eta \int_{t_0}^{t} D(\tau, s)\, e^{\eta(t-\tau)}\, d\tau + \kappa^2 \int_{t_0}^{t} C_{\Xi\Xi}(\tau, s)\, e^{\eta(t-\tau)}\, d\tau \right)\right] \times \\
& \times \exp\left[i \dot{m}_X(s) u_2 - \frac{1}{2} \sigma_{\dot{X}}^2(s) u_2^2 \right].
\end{aligned} \tag{6.82}$$

At this point, using Eq. (B.3) for $m_X(t)$, verified Eq. (3.47) for $\sigma_X^2(t)$, defining Eq. (5.43) for $D(t, s)$ as well as Eq. (B.53) for $C_{X_0 \dot{X}}(s)$; Eq. (6.82) can be written as

$$\begin{aligned}
\varphi_{X(t)\dot{X}(s)}(\boldsymbol{u}) = {} & \exp\left(i m_X(t) u_1 - \frac{1}{2}\sigma_X^2(t) u_1^2 + i \dot{m}_X(s) u_2 \right) \times \\
& \times \exp\left(-\frac{1}{2}\sigma_{\dot{X}}^2(s) u_2^2 - C_{X\dot{X}}(t, s)\, u_1 u_2 \right).
\end{aligned} \tag{6.83}$$

Eq. (6.83) is readily identified as the expected, bivariate Gaussian characteristic function. Thus, even though the use of SLE (6.44) seems rather inconvenient in the nonlinear case, its validation in the linear case reaffirms its consistency with the entire methodology presented in this work.

All in all, the correspondence, in the linear case, of the results derived by solving our new equations with the ones obtained via formulating and solving the corresponding moment problems serves as a preliminary indication of the validity of our overall approach.

Chapter 7

Conclusions and Future works

In this thesis, we extended a methodology used to derive response pdf evolution equations for RDEs excited by colored Gaussian noise in order to account for more systems. More specifically, we first showcased the applicability of this methodology in another case, namely the one in which the RDE is excited by both multiplicative and additive noise. Furthermore, we outlined the manner in which this methodology can be implemented in order to derive equations governing the joint, second-order pdfs of the system depending on all responses (and/or excitations) on the same or different time instances.

To sum up, the derivation of these new equations commences with the delta projection method which was concisely described in sec. 3.1[2]. By virtue of this method, we are able to easily represent the sought-for pdf as the average of a random delta function or a product of them and thus, by executing some straightforward algebraic manipulations we readily derive a SLE corresponding to the examined RIVP. The acquired SLE is each time non-closed due to some averaged terms dependent on a time instance of the excitation, the initial value of the RDE as well as the entire time-history of the response; the occurrence of the terms was previously considered somewhat burdensome (Venturi et al., 2012a). Nevertheless, by considering the response as a functional or a FFℓ, we are able to employ novel extensions to the NF theorem – formulated and proven in Chapter 2 by following the approach presented in (Athanassoulis & Mamis, 2019) – that the SLE In the case of the scalar, nonlinear additively excited RDE this results in the appearance of the variational derivatives of the RIVP that depend on the time-history of the response, while, in the case of an RDE also excited by multiplicative noise there is also a dependence on the history of the excitation. Last, probably the most convoluted step of this approach is that in order to obtain closed, approximate pdf evolution equations, we must effectively elaborate on this nonlocal, nonlinear terms which arise from the variational derivatives. The potency of these results was also tested in the case of a linear, additively excited RDE in which they attained the correct Gaussian pdfs.

Since, the vast majority of the theoretical modelling for the presented cases has been established in this work, in the future, a significant step forward would be to conduct numerical simulations based on these equations and evaluate their efficiency. In particular, it would be very useful to first test this methodology in benchmark cases, e.g. for Ornstein-Uhlenbeck excitation, or more practical cases where the probabilistic characteristics of the noise have been specified through existent data. Another useful step forward is to rederive these equations for a multidimensional system of RDEs in order to clarify any intricacies that may arise and implement them in useful applications. Last, it would also be very useful to test the equations for the joint pdf of the response with its derivative in first-passage problems.

[2] A more thorough description of this methodology can be found in (Mamis, 2020; Mamis et al., 2019)

Appendix A

Proofs of Lemmata 2.7-2.12

The proof of Lemmata 2.7-2.12 are proven herein by expressing the appropriate $\bar{\mathcal{T}}-$ operators in series form. This is accomplished via the expansion of the exponentials appearing on the right-hand sides of Eqs. (2.47a,b,c), as follows:

$$\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_0}\bullet = \sum_{p=0}^{\infty}\frac{1}{p!}\frac{1}{2^p}\int_{t_0}^{t}\overset{(2p)}{\cdots}\int_{t_0}^{t} C_{\Xi_0\Xi_0}(\tau_1^{(1)},\tau_2^{(1)})\cdots C_{\Xi_0\Xi_0}(\tau_1^{(p)},\tau_2^{(p)})\times$$
$$\times\frac{\delta^{2p}\bullet}{\delta\upsilon(\tau_1^{(1)})\cdots\delta\upsilon(\tau_1^{(p)})\delta\upsilon(\tau_2^{(1)})\cdots\delta\upsilon(\tau_2^{(p)})}d\tau_1^{(1)}\cdots d\tau_1^{(p)}\,d\tau_2^{(1)}\cdots d\tau_2^{(p)}, \tag{A.1a}$$

$$\bar{\mathcal{T}}_{\hat{\Xi}_1\hat{\Xi}_1}\bullet = \sum_{p=0}^{\infty}\frac{1}{p!}\frac{1}{2^p}\int_{t_0}^{t}\overset{(2p)}{\cdots}\int_{t_0}^{t} C_{\Xi_1\Xi_1}(\tau_1^{(1)},\tau_2^{(1)})\cdots C_{\Xi_1\Xi_1}(\tau_1^{(p)},\tau_2^{(p)})\times$$
$$\times\frac{\delta^{2p}\bullet}{\delta u(\tau_1^{(1)})\cdots\delta u(\tau_1^{(p)})\delta u(\tau_2^{(1)})\cdots\delta u(\tau_2^{(p)})}d\tau_1^{(1)}\cdots d\tau_1^{(p)}\,d\tau_2^{(1)}\cdots d\tau_2^{(p)}, \tag{A.1b}$$

$$\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\bullet = \sum_{p=0}^{\infty}\frac{1}{p!}\int_{t_0}^{t}\overset{(2p)}{\cdots}\int_{t_0}^{t} C_{\Xi_0\Xi_1}(\tau_0^{(1)},\tau_1^{(1)})\cdots C_{\Xi_0\Xi_1}(\tau_0^{(p)},\tau_1^{(p)})\times$$
$$\times\frac{\delta^{2p}\bullet}{\delta\upsilon(\tau_0^{(1)})\cdots\delta\upsilon(\tau_0^{(p)})\delta u(\tau_1^{(1)})\cdots\delta u(\tau_1^{(p)})}d\tau_0^{(1)}\cdots d\tau_0^{(p)}\,d\tau_1^{(1)}\cdots d\tau_1^{(p)}. \tag{A.1c}$$

The proofs of the aforementioned lemmata is based on the above series expansion, in conjunction with the linearity of integrals and derivatives.

Proof of Lemma 2.7: $\bar{\mathcal{T}}$ – operators are linear. The action of operator $\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}$ on $\alpha\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]+\beta\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ is expressed via Eq. (A.1c) as

$$\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\left[\alpha\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]+\beta\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]=$$
$$=\sum_{p=0}^{\infty}\frac{1}{p!}\int_{t_0}^{t}\overset{(2p)}{\cdots}\int_{t_0}^{t} C_{\Xi_0\Xi_1}(\tau_0^{(1)},\tau_1^{(1)})\cdots C_{\Xi_0\Xi_1}(\tau_0^{(p)},\tau_1^{(p)})\times$$
$$\times\frac{\delta^{2p}\left[\alpha\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]+\beta\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\delta\upsilon(\tau_0^{(1)})\cdots\delta\upsilon(\tau_0^{(p)})\delta u(\tau_1^{(1)})\cdots\delta u(\tau_1^{(p)})}d\tau_0^{(1)}\cdots d\tau_0^{(p)}\,d\tau_1^{(1)}\cdots d\tau_1^{(p)}.$$

By taking advantage of the linearity of derivatives and integrals and assuming that a and β are independent from the differentiation arguments $\upsilon(\bullet)$ and $u(\bullet)$, each term on the right-hand side of the above equations in linearly decomposed, thus, resulting in the linearity of the $\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}$ operator; the proof of Lemma 2.7 for $\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}$ has been completed. The proof for the other two $\bar{\mathcal{T}}$ – operators is similar. ■

Proof of Lemma 2.8: $\bar{\mathcal{T}}$ – operators commute with $\upsilon(\tau)-$ and $u(\tau)-$ differentiation. We shall prove this lemma for operator $\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}$, which is also the most complicated case. By using Eq. (A.1c), we have

$$\frac{\delta}{\delta\upsilon(\tau_0)}\left[\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]=$$

$$=\frac{\delta}{\delta\upsilon(\tau_0)}\sum_{p=0}^{\infty}\frac{1}{p!}\int_{t_0}^{t}\overset{(2p)}{\cdots}\int_{t_0}^{t}C_{\Xi_0\Xi_1}(\tau_0^{(1)},\tau_1^{(1)})\cdots C_{\Xi_0\Xi_1}(\tau_0^{(p)},\tau_1^{(p)})\times$$
$$\times\frac{\delta^{2p}\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]}{\delta\upsilon(\tau_0^{(1)})\cdots\delta\upsilon(\tau_0^{(p)})\delta u(\tau_1^{(1)})\cdots\delta u(\tau_1^{(p)})}d\tau_0^{(1)}\cdots d\tau_0^{(p)}\,d\tau_1^{(1)}\cdots d\tau_1^{(p)}.$$

Employing, now, the continuity and linearity of the derivative, the above expression is rewritten as:

$$=\sum_{p=0}^{\infty}\frac{1}{p!}\int_{t_0}^{t}\overset{(2p)}{\cdots}\int_{t_0}^{t}C_{\Xi_0\Xi_1}(\tau_0^{(1)},\tau_1^{(1)})\cdots C_{\Xi_0\Xi_1}(\tau_0^{(p)},\tau_1^{(p)})\times$$
$$\times\frac{\delta^{2p+1}\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]}{\delta\upsilon(\tau_0)\delta\upsilon(\tau_0^{(1)})\cdots\delta\upsilon(\tau_0^{(p)})\delta u(\tau_1^{(1)})\cdots\delta u(\tau_1^{(p)})}d\tau_0^{(1)}\cdots d\tau_0^{(p)}\,d\tau_1^{(1)}\cdots d\tau_1^{(p)}=$$
$$=\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\left[\frac{\delta\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]}{\delta\upsilon(\tau_0)}\right]$$

Proof of the Lemma for $\frac{\delta}{\delta u(\tau_1)}\left[\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]$ and for the other two operators is similar. ■

Proof of Lemma 2.9: $\bar{\mathcal{T}}$ – operators commute with each other. For the sake of simplicity, we shall prove the commutativity of operators $\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}$ and $\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_0}$; the proof for the rest of the cases is similar. Using Eqs. (A.1a,c), we obtain

$$\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\left[\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_0}\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]=$$

$$=\sum_{p=0}^{\infty}\frac{1}{p!}\int_{t_0}^{t}\overset{(2p)}{\cdots}\int_{t_0}^{t}C_{\Xi_0\Xi_1}(\tau_0^{(1)},\tau_1^{(1)})\cdots C_{\Xi_0\Xi_1}(\tau_0^{(p)},\tau_1^{(p)})\times$$

$$\times\frac{\delta^{2p}}{\delta\upsilon(\tau_0^{(1)})\cdots\delta\upsilon(\tau_0^{(p)})\delta u(\tau_1^{(1)})\cdots\delta u(\tau_1^{(p)})}\times$$

$$\times\left[\sum_{m=0}^{\infty}\frac{1}{m!}\frac{1}{2^m}\int_{t_0}^{t}\overset{(2m)}{\cdots}\int_{t_0}^{t}C_{\Xi_0\Xi_0}(\sigma_1^{(1)},\sigma_2^{(1)})\cdots C_{\Xi_0\Xi_0}(\sigma_1^{(m)},\sigma_2^{(m)})\times\right.$$

$$\left.\times\frac{\delta^{2m}\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]}{\delta\upsilon(\sigma_1^{(1)})\cdots\delta\upsilon(\sigma_1^{(m)})\delta\upsilon(\sigma_2^{(1)})\cdots\delta\upsilon(\sigma_2^{(m)})}d\sigma_1^{(1)}\cdots d\sigma_1^{(m)}\,d\sigma_2^{(1)}\cdots d\sigma_2^{(m)}\right]\times$$

$$\times\,d\tau_0^{(1)}\cdots d\tau_0^{(p)}\,d\tau_1^{(1)}\cdots d\tau_1^{(p)}.$$

Now, by employing the linearity of derivatives and integrals and rearranging the order of summations, we obtain

$$\bar{\mathcal{T}}_{\hat{\Xi}_0(\bullet)\hat{\Xi}_1(\bullet)}\left[\bar{\mathcal{T}}_{\hat{\Xi}_0(\bullet)\hat{\Xi}_0(\bullet)}\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]=$$

$$=\sum_{m=0}^{\infty}\frac{1}{m!}\frac{1}{2^m}\int_{t_0}^{t}\overset{(2m)}{\cdots}\int_{t_0}^{t}C_{\Xi_0\Xi_0}(\sigma_1^{(1)},\sigma_2^{(1)})\cdots C_{\Xi_0\Xi_0}(\sigma_1^{(m)},\sigma_2^{(m)})\times$$

$$\times\frac{\delta^{2m}}{\delta\upsilon(\sigma_1^{(1)})\cdots\delta\upsilon(\sigma_1^{(m)})\delta\upsilon(\sigma_2^{(1)})\cdots\delta\upsilon(\sigma_2^{(m)})}\times$$

$$\times\left[\sum_{p=0}^{\infty}\frac{1}{p!}\int_{t_0}^{t}\overset{(2p)}{\cdots}\int_{t_0}^{t}C_{\Xi_0\Xi_1}(\tau_0^{(1)},\tau_1^{(1)})\cdots C_{\Xi_0\Xi_1}(\tau_0^{(p)},\tau_1^{(p)})\times\right.$$

$$\left.\times\frac{\delta^{2p}\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]}{\delta\upsilon(\tau_0^{(1)})\cdots\delta\upsilon(\tau_0^{(p)})\delta u(\tau_1^{(1)})\cdots\delta u(\tau_1^{(p)})}d\tau_0^{(1)}\cdots d\tau_0^{(p)}\,d\tau_1^{(1)}\cdots d\tau_1^{(p)}\right]\times$$

$$\times\,d\sigma_1^{(1)}\cdots d\sigma_1^{(m)}\,d\sigma_2^{(1)}\cdots d\sigma_2^{(m)}=$$

$$=\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_0}\left[\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\mathcal{G}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right].\qquad\blacksquare$$

Lemmata 2.7-2.7 are essential for the proof of both extensions to the NF theorem. Subsequently, we are going to evaluate the action of $\bar{\mathcal{T}}$ – operators on the products $u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ and $\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$. However, before we are able proceed, we must introduce, in each case, the following ***product rule for Volterra functional derivatives***:

$$\frac{\delta^k\left[u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\delta u(\tau^{(1)})\cdots\delta u(\tau^{(k)})} = u(s)\frac{\delta^k\left[\mathcal{F}[u(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\delta u(\tau^{(1)})\cdots\delta u(\tau^{(k)})} + \\ + \sum_{n=1}^{k}\delta(s-\tau^{(n)})\frac{\delta^{k-1}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\prod_{\substack{\ell=1\\ \ell\neq n}}^{k}\delta u(\tau^{(\ell)})}, \tag{A.2a}$$

$$\frac{\delta^k\left[\dot u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\delta u(\tau^{(1)})\cdots\delta u(\tau^{(k)})} = \dot u(s)\frac{\delta^k\left[\mathcal{F}[u(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\delta u(\tau^{(1)})\cdots\delta u(\tau^{(k)})} - \\ - \sum_{n=1}^{k}\delta'(s-\tau^{(n)})\frac{\delta^{k-1}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\prod_{\substack{\ell=1\\ \ell\neq n}}^{k}\delta u(\tau^{(\ell)})}. \tag{A.2b}$$

with $\prod_{\substack{\ell=1\\ \ell\neq n}}^{k}\delta u(\tau^{(\ell)}) = \delta u(\tau^{(1)})\cdots\delta u(\tau^{(n-1)})\delta u(\tau^{(n+1)})\cdots\delta u(\tau^{(k)})$ and $\delta'(s-\tau^{(n)}) = \partial\delta(s-\tau^{(n)})/\partial s$.

The first case. Eq. (A.2a) can easily be proven via mathematical induction on index k, commencing from the product rule for the first order Volterra derivative

$$\frac{\delta\left[u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\delta u(\tau)} = u(s)\frac{\delta\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\delta u(\tau)} + \\ + \frac{\delta u(s)}{\delta u(\tau)}\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})], \tag{A.3}$$

with $\delta u(s)/\delta u(\tau) = \delta(s-\tau)$.

For the second case, under the assumption that $u(\bullet)$ is $C^1\left([t_0,t]\to\mathbb{R}\right)$, like the paths of $\Xi(\bullet;\theta)$, $\dot u(s)$ can expressed as a linear functional of integral type with a singular kernel

$$\dot u(s) = \int_{t_0}^{s}\delta'(s-\tau)\,u(\tau)\,d\tau, \tag{A.4}$$

where $\delta'(s-\tau)\equiv-\partial\delta(s-\tau)/\partial\tau$. Expression (A.3) is formal, yet it makes the Volterra derivative of $\dot u(s)$ easily computable to

$$\frac{\delta\dot u(s)}{\delta u(\tau)} = \delta'(s-\tau). \tag{A.5}$$

Using, now, Eq. (A.3) the product rule for first-order Volterra derivatives of $\dot u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ can be seen through Eq. (A.3) as

$$\frac{\delta\left[\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\delta u(\tau)} = \dot{u}(s)\frac{\delta\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\delta u(\tau)} + \\ + \delta'(s-\tau)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]. \tag{A.6}$$

Thus, commencing from Eq. (A.6), Eq. (A.2b) can easily be proven via mathematical induction on index k.

Proof of Lemma 2.10: The action of operator $\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_0}$ on $u(t)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ is given by

$$\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_0}\left[u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right] = u(s)\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_0}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]. \tag{A.7}$$

Eq. (A.7) holds true due to the linearity of operator $\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_0}$, since the scalar function $u(s)$ is independent from the differentiation argument of $\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_0}$, i.e. $\upsilon(\bullet)$.

Proof of Lemma 2.11. For this lemma, we are going to present the proof for $\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ since it constitutes the most complicated case; the proof for $u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ is similar and omitted for reasons of brevity. Using the series expansion, Eq. (A.1c) and employing the product rule for Volterra derivatives, Eq. (A.2b), the action of operator $\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}$ on $\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ can be expressed as

$$\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\left[\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right] = A + B,$$

with

$$A = \sum_{p=0}^{\infty}\left[\frac{1}{p!}\int_{t_0}^{t}\overset{(2p)}{\cdots}\int_{t_0}^{t}\prod_{\ell=1}^{k} C_{\Xi_0\Xi_1}(\tau_0^{(\ell)},\tau_1^{(\ell)})\dot{u}(s)\frac{\delta^{2p}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\prod_{\ell=1}^{p}\delta\upsilon(\tau_0^{(\ell)})\delta u(\tau_1^{(\ell)})}\prod_{\ell=1}^{p} d\tau_0^{(\ell)}d\tau_1^{(\ell)}\right],$$

$$B = -\sum_{p=1}^{\infty}\left[\frac{1}{p!}\int_{t_0}^{t}\overset{(2p)}{\cdots}\int_{t_0}^{t}\prod_{\ell=1}^{p} C_{\Xi_0\Xi_1}(\tau_0^{(\ell)},\tau_1^{(\ell)}) \times \right. \\ \left. \times\sum_{n=1}^{p}\delta'(s-\tau^{(n)})\frac{\delta^{2p-1}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\prod_{\ell=1}^{p}\delta\upsilon(\tau_0^{(\ell)})\prod_{\substack{\ell=1\\ \ell\neq n}}^{p}\delta u(\tau_1^{(\ell)})}\prod_{\ell=1}^{p} d\tau_0^{(\ell)}d\tau_1^{(\ell)}\right].$$

Let us, first, elaborate on A:

$$A = \sum_{p=0}^{\infty}\left[\frac{1}{p!}\int_{t_0}^{t}\overset{(2p)}{\cdots}\int_{t_0}^{t}\prod_{\ell=1}^{k} C_{\Xi_0\Xi_1}(\tau_0^{(\ell)},\tau_1^{(\ell)})\dot{u}(s)\frac{\delta^{2p}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\prod_{\ell=1}^{p}\delta\upsilon(\tau_0^{(\ell)})\delta u(\tau_1^{(\ell)})}\prod_{\ell=1}^{p} d\tau_0^{(\ell)}d\tau_1^{(\ell)}\right] =$$

$$= \dot{u}(s)\sum_{p=0}^{\infty}\left[\frac{1}{p!}\int_{t_0}^{t}\overset{(2p)}{\cdots}\int_{t_0}^{t}\prod_{\ell=1}^{k}C_{\Xi_0\Xi_1}(\tau_0^{(\ell)},\tau_1^{(\ell)})\,\frac{\delta^{2p}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\prod_{\ell=1}^{p}\delta\upsilon(\tau_0^{(\ell)})\delta u(\tau_1^{(\ell)})}\prod_{\ell=1}^{p}d\tau_0^{(\ell)}d\tau_1^{(\ell)}\right]$$

where the sum appearing on the rightmost side of the above expression can be identified via Eq. (A.1c) as $\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]$. As such,

$$A = \dot{u}(s)\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]. \tag{A.8}$$

We shall now evaluate the second term, B:

$$B = -\sum_{p=1}^{\infty}\left[\frac{1}{p!}\int_{t_0}^{t}\overset{(2p)}{\cdots}\int_{t_0}^{t}\prod_{\ell=1}^{p}C_{\Xi_0\Xi_1}(\tau_0^{(\ell)},\tau_1^{(\ell)})\times\right.$$

$$\left.\times\sum_{n=1}^{p}\delta'(s-\tau_1^{(n)})\frac{\delta^{2p-1}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\prod_{\ell=1}^{p}\delta\upsilon(\tau_0^{(\ell)})\prod_{\substack{\ell=1\\ \ell\neq n}}^{p}\delta u(\tau_1^{(\ell)})}\prod_{\ell=1}^{p}d\tau_0^{(\ell)}d\tau_1^{(\ell)}\right] =$$

$$= \sum_{p=1}^{\infty}\left[\frac{1}{p!}p\int_{t_0}^{t}\overset{(2p)}{\cdots}\int_{t_0}^{t}\partial_s\prod_{\ell=1}^{p}C_{\Xi_0\Xi_1}(\tau_0^{(\ell)},\tau_1^{(\ell)})\times\right.$$

$$\left.\times\,\delta(s-\tau_1^{(n)})\frac{\delta^{2p-1}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\prod_{\ell=1}^{p}\delta\upsilon(\tau_0^{(\ell)})\prod_{\substack{\ell=1\\ \ell\neq n}}^{p}\delta u(\tau_1^{(\ell)})}\prod_{\ell=1}^{p}d\tau_0^{(\ell)}d\tau_1^{(\ell)}\right] =$$

$$= \sum_{p=1}^{\infty}\left[\frac{1}{(p-1)!}\int_{t_0}^{t}\overset{(2p-1)}{\cdots}\int_{t_0}^{t}\partial_s C_{\Xi_0\Xi_1}(\tau_0^{(n)},s)\prod_{\substack{\ell=1\\ \ell\neq n}}^{p}C_{\Xi_0\Xi_1}(\tau_0^{(\ell)},\tau_1^{(\ell)})\times\right.$$

$$\left.\times\frac{\delta^{2p-1}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\delta\upsilon(\tau_0^{(n)})\prod_{\substack{\ell=1\\ \ell\neq n}}^{p}\delta\upsilon(\tau_0^{(\ell)})\delta u(\tau_1^{(\ell)})}d\tau_0^{(n)}\prod_{\substack{\ell=1\\ \ell\neq n}}^{p}d\tau_0^{(\ell)}d\tau_1^{(\ell)}\right].$$

By performing, in each term of sum $\sum_{p=1}^{\infty}\cdots$, the change of integration variables $\tau_0 = \tau_0^{(n)}$ and $\tau_1^{(m)} = \tau_1^{(\ell)}$ for $\ell < n$, $s_i^{(m)} = s_i^{(\ell-1)}$ for $\ell > n$, we obtain

$$\prod_{m=1}^{p-1}C_{\Xi_0\Xi_1}(\tau_0^{(m)},\tau_1^{(m)}) = \prod_{\substack{\ell=1\\ \ell\neq n}}^{p}C_{\Xi_0\Xi_1}(\tau_0^{(\ell)},\tau_1^{(\ell)}), \tag{A.9a}$$

$$\prod_{m=1}^{p-1}\delta\upsilon(\tau_0^{(m)})\delta u(\tau_1^{(m)}) = \prod_{\substack{\ell=1\\ \ell\neq n}}^{p}\delta\upsilon(\tau_0^{(\ell)})\delta u(\tau_1^{(\ell)}), \tag{A.9b}$$

$$\prod_{m=1}^{p-1}d\tau_0^{(m)}d\tau_1^{(m)} = \prod_{\substack{\ell=1\\ \ell\neq n}}^{p}d\tau_0^{(\ell)}d\tau_1^{(\ell)}. \tag{A.9c}$$

Under this change of variables, we find

$$B = \sum_{p=1}^{\infty}\left[\frac{1}{(p-1)!}\int_{t_0}^{t} \overset{(2p-1)}{\cdots} \int_{t_0}^{t} \partial_s C_{\Xi_0\Xi_1}(\tau_0, s) \prod_{m=1}^{p-1} C_{\Xi_0\Xi_1}(\tau_0^{(m)}, \tau_1^{(m)}) \times \right.$$
$$\left. \times \frac{\delta^{2p-1}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t}); u(\bullet|_{t_0}^{t})]\right]}{\delta\upsilon(\tau_0)\prod_{m=1}^{p-1}\delta\upsilon(\tau_0^{(m)})\delta u(\tau_1^{(m)})} d\tau_0 \prod_{m=1}^{p-1} d\tau_0^{(m)} d\tau_1^{(m)}\right].$$

By performing the change of index $k = p-1$, and interchanging τ_0 integration with summation the above expression is equivalently written

$$B = \int_{t_0}^{t} \partial_s C_{\Xi_0\Xi_1}(\tau_0, s) \sum_{k=0}^{\infty}\left[\frac{1}{(k)!}\int_{t_0}^{t} \overset{(2k)}{\cdots} \int_{t_0}^{t} \partial_s C_{\Xi_0\Xi_1}(\tau_0, s) \prod_{m=1}^{k} C_{\Xi_0\Xi_1}(\tau_0^{(m)}, \tau_1^{(m)}) \times \right.$$
$$\left. \times \frac{\delta^{2k}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t}); u(\bullet|_{t_0}^{t})]\right]}{\prod_{m=1}^{k}\delta\upsilon(\tau_0^{(m)})\delta u(\tau_1^{(m)})} \frac{\delta \mathcal{F}[\upsilon(\bullet|_{t_0}^{t}); u(\bullet|_{t_0}^{t})]}{\delta\upsilon(\tau_0)} \prod_{m=1}^{k} d\tau_0^{(m)} d\tau_1^{(m)}\right] d\tau_0 .$$

The sum on the right-hand side of the above equation, is identified via Eq. (A.1c) as

$$\bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\left[\frac{\delta \mathcal{F}[\upsilon(\bullet|_{t_0}^{t}); u(\bullet|_{t_0}^{t})]}{\delta\upsilon(\tau_0)}\right].$$

Thus,

$$B = \int_{t_0}^{t} \partial_s C_{\Xi_0\Xi_1}(\tau_0, s)\, \bar{\mathcal{T}}_{\hat{\Xi}_0\hat{\Xi}_1}\left[\frac{\delta \mathcal{F}[\upsilon(\bullet|_{t_0}^{t}); u(\bullet|_{t_0}^{t})]}{\delta\upsilon(\tau_0)}\right] d\tau_0 . \qquad \text{(A.10)}$$

Finally, combining Eqs. (A.8) and (A.10), the proof of Lemma 2.11 is completed. ■

Proof of Lemma 2.12. Once more, for reasons of brevity, we are only going to present the proof of Lemma 2.12 for $\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t}); u(\bullet|_{t_0}^{t})]$. Using the series expansion Eq. (A.1b) as well as Eq. (A.2b), the action of operator $\bar{\mathcal{T}}_{\hat{\Xi}_1\hat{\Xi}_1}$ on $\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t}); u(\bullet|_{t_0}^{t})]$ is given by

$$\bar{\mathcal{T}}_{\hat{\Xi}_1\hat{\Xi}_1}\left[\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t}); u(\bullet|_{t_0}^{t})]\right] = \dot{u}(s)\Gamma + \Delta ,$$

where Γ is given by

$$\Gamma = \sum_{p=0}^{\infty}\frac{1}{p!}\frac{1}{2^p}\int_{t_0}^{t} \overset{(2p)}{\cdots} \int_{t_0}^{t} C_{\Xi_1\Xi_1}(\tau_1^{(1)}, \tau_2^{(1)}) \cdots C_{\Xi_1\Xi_1}(\tau_1^{(p)}, \tau_2^{(p)}) \times$$
$$\times \frac{\delta^{2p}\mathcal{F}[\upsilon(\bullet|_{t_0}^{t}); u(\bullet|_{t_0}^{t})]}{\prod_{\ell=1}^{p}\delta u(\tau_1^{(\ell)})\delta u(\tau_2^{(\ell)})} \prod_{\ell=1}^{p} d\tau_1^{(\ell)} d\tau_2^{(\ell)} ,$$

while Δ, after taking into account the symmetry of autocorrelation function $C_{\Xi_1\Xi_1}(\tau_1,\tau_2)$ and the properties of the delta function, can be written as follows:

$$\Delta = \sum_{p=0}^{\infty}\frac{1}{p!}\frac{1}{2^p}\int_{t_0}^{t}\overset{(2p)}{\cdots}\int_{t_0}^{t}\partial_s C_{\Xi_1\Xi_1}(s,\tau_2^{(n)})\prod_{\substack{\ell=1\\ \ell\neq n}}^{p} C_{\Xi_1\Xi_1}(\tau_1^{(\ell)},\tau_2^{(\ell)})\times$$
$$\times\frac{\delta^{2p-1}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\delta u(\tau_2^{(n)})\prod_{\substack{\ell=1\\ \ell\neq n}}^{p}\delta u(\tau_1^{(\ell)})\delta u(\tau_2^{(\ell)})}d\tau_2^{(n)}\prod_{\substack{\ell=1\\ \ell\neq n}}^{p}d\tau_1^{(\ell)}d\tau_2^{(\ell)}.$$

Through Eq. (A.1b), Γ is identified as

$$\Gamma = \bar{\mathcal{T}}_{\hat{\Xi}_1\hat{\Xi}_1}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]. \tag{A.11}$$

Let us, now, return to the treatment of Δ and perform, for the terms inside the sum, the change of variables presented in the proof of Lemma 2.11, Eqs. (A.9a,b,c). it is easy to see that all $2p$ terms in the $n-$ sum are equal, resulting in

$$\Delta = \sum_{p=1}^{\infty}\frac{1}{(p-1)!}\frac{1}{2^{p-1}}\int_{t_0}^{t}\overset{(2p-1)}{\cdots}\int_{t_0}^{t}\partial_s C_{\Xi_1\Xi_1}(s,\tau_2)\prod_{m=1}^{p-1}C_{\Xi_1\Xi_1}(\tau_1^{(m)},\tau_2^{(m)})\times$$
$$\times\frac{\delta^{2p-1}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\delta u(\tau_2)\prod_{m=1}^{p-1}\delta u(\tau_1^{(m)})\delta u(\tau_2^{(m)})}d\tau_2\prod_{m=1}^{p-1}d\tau_1^{(m)}d\tau_2^{(m)}.$$

Performing the index change $k = p-1$ and interchanging τ_2- integration with summation, results in

$$\Delta = \int_{t_0}^{t}\partial_s C_{\Xi_1\Xi_1}(s,\tau_2)\sum_{k=0}^{\infty}\frac{1}{(k)!}\frac{1}{2^k}\int_{t_0}^{t}\overset{(2k)}{\cdots}\int_{t_0}^{t}\left[\prod_{m=1}^{k}C_{\Xi_1\Xi_1}(\tau_1^{(m)},\tau_2^{(m)})\times\right.$$
$$\left.\times\frac{\delta^{2k}\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\prod_{m=1}^{k}\delta u(\tau_1^{(m)})\delta u(\tau_2^{(m)})}\left[\frac{\delta\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\delta u(s)}\right]\prod_{m=1}^{k}d\tau_1^{(m)}d\tau_2^{(m)}\right]d\tau_2$$

The sum in the right-hand side of the above equation is identified, via Eq. (19b), as

$$\Delta = \int_{t_0}^{t}\partial_s C_{\Xi_1\Xi_1}(s,\tau_2)\ \bar{\mathcal{T}}_{\hat{\Xi}_1\hat{\Xi}_1}\left[\frac{\delta\left[\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]\right]}{\delta u(\tau_2)}\right]d\tau_2. \tag{A.12}$$

Last, combining Eq. (A.11) and (A.12) Lemma 2.12 for the case of $\dot{u}(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$.

The proof in the case of $u(s)\mathcal{F}[\upsilon(\bullet|_{t_0}^{t});u(\bullet|_{t_0}^{t})]$ is similar.

Appendix B

Moment problem for a linear, additively excited RDE

In this appendix, we will derive and solve the deterministic initial value problems for the moments of the familiar linear, additively excited RIVP:

$$\dot{X}(t;\theta) = \eta X(t;\theta) + \kappa \, \Xi(t;\theta), \qquad X(t_0;\theta) = X_0(\theta). \tag{B.1a,b}$$

The derivation of these moment equations will be performed by multiplying both sides of Eqs. (B.1a,b) with the appropriate, for each problem, random function, and then by taking the average $\mathbb{E}^{\theta}[\bullet]$ of both sides of the equation. Since the original RIVP is a linear SDE, the corresponding moment problems will be linear ODEs of the general form

$$\frac{dx(t)}{dt} = a\,x(t) + b\,y(t), \qquad x(t_0) = x_0 \in \mathbb{R}, \tag{B.i}$$

whose solution is

$$x(t) = x_0\, e^{a(t-t_0)} + b\int_{t_0}^{t} y(\tau)\, e^{a(t-\tau)}\, d\tau. \tag{B.ii}$$

At this point, it is useful to discuss on which moment functions should be considered as data for the moment IVPs, and which moments should be considered as unknowns, and thus, for their determination, additional moment IVPs should be constructed. In the original IVP, Eqs. (B.1a,b), initial value and excitation are considered known, as in every IVP. Since initial value is the random variable $X_0(\theta)$ and excitation is the random function $\Xi(t;\theta)$, the aforementioned knowledge of initial value and excitation means, from a probability theory point of view, the knowledge of the pdf $f_{X_0}(x)$ and the family of pdfs $f_{\Xi(t_1)\cdots\Xi(t_N)}(\xi_1, \dots, \xi_N)$, for all $N \in \mathbb{N}$ respectively. What is more, the *simultaneous* knowledge of $X_0(\theta)$, $\Xi(t;\theta)$ implies also the knowledge of the cross-pdf initial value-excitation family $f_{X_0\Xi(t_1)\cdots\Xi(t_N)}(x, \xi_1, \dots, \xi_N)$, for all $N \in \mathbb{N}$.

B.1 Moment problem for the response

In this section, we will derive, from Eqs. (B1a,b), the corresponding IVPs for the first (mean value) and second (autocorrelation) moments of the response $X(t;\theta)$, as well as the cross-correlation functions of the response with excitation $\Xi(t;\theta)$ and initial value $X_0(\theta)$.

B.1.1 IVP for mean value of the response $m_X(t)$

We construct an equation for the mean value of $X(t;\theta)$ by averaging both sides of Eqs. (B1a,b)

$$\mathbb{E}^{\theta}\left[\dot{X}(t;\theta)\right] = \eta\,\mathbb{E}^{\theta}\left[X(t;\theta)\right] + \kappa\,\mathbb{E}^{\theta}\left[\Xi(t;\theta)\right],$$
$$\mathbb{E}^{\theta}\left[X(t_0;\theta)\right] = \mathbb{E}^{\theta}\left[X_0(\theta)\right].$$

Interchanging, now, average and differentiation operators results in

$$\frac{d}{dt}\mathbb{E}^{\theta}\left[X(t;\theta)\right] = \eta\,\mathbb{E}^{\theta}\left[X(t;\theta)\right] + \kappa\,\mathbb{E}^{\theta}\left[\Xi(t;\theta)\right],$$
$$\mathbb{E}^{\theta}\left[X(t_0;\theta)\right] = \mathbb{E}^{\theta}\left[X_0(\theta)\right],$$

which can equivalently be written as

$$\frac{d\,m_X(t)}{dt} = \eta\,m_X(t) + \kappa\,m_{\Xi}(t), \qquad m_X(t_0) = m_{X_0}. \tag{B2a,b}$$

Thus, initial value problem (B2a,b) can be solved using Eqs. (B.i), (B.ii) as

$$m_X(t) = m_{X_0}\,e^{\eta(t-t_0)} + \kappa\int_{t_0}^{t} m_{\Xi}(\tau)\,e^{\eta(t-\tau)}\,d\tau \tag{B.3}$$

B.1.2 IVP for the two times response-excitation cross-correlation $R_{X\Xi}(t,s)$

We will construct an equation for the cross-correlation function of the response $X(t;\theta)$ and the excitation $\Xi(s;\theta)$. For this, we first multiply both sides of Eq. (B.1a) with $\Xi(s;\theta)$

$$\dot{X}(t;\theta)\,\Xi(s;\theta) = \eta\,X(t;\theta)\,\Xi(s;\theta) + \kappa\,\Xi(t;\theta)\,\Xi(s;\theta),$$

and then, take the average as follows:

$$\mathbb{E}^{\theta}\left[\dot{X}(t;\theta)\,\Xi(s;\theta)\right] = \eta\,\mathbb{E}^{\theta}\left[X(t;\theta)\,\Xi(s;\theta)\right] + \kappa\,\mathbb{E}^{\theta}\left[\Xi(t;\theta)\,\Xi(s;\theta)\right].$$

Interchanging, now, average and differential operators

$$\frac{\partial}{\partial t}\,\mathbb{E}^{\theta}\left[X(t;\theta)\,\Xi(s;\theta)\right] = \eta\,\mathbb{E}^{\theta}\left[X(t;\theta)\,\Xi(s;\theta)\right] + \kappa\,\mathbb{E}^{\theta}\left[\Xi(t;\theta)\,\Xi(s;\theta)\right].$$

and introducing moments, we obtain

$$\frac{\partial R_{X\Xi}(t,s)}{\partial t} = \eta\,R_{X\Xi}(t,s) + \kappa\,R_{\Xi\Xi}(t,s). \tag{B.4a}$$

The same procedure is also applied to procure the initial condition

$$X(t_0;\theta)\,\Xi(s;\theta) = X_0(\theta)\,\Xi(s;\theta) \;\Rightarrow$$
$$\mathbb{E}^{\theta}\left[X(t_0;\theta)\,\Xi(s;\theta)\right] = \mathbb{E}^{\theta}\left[X_0(\theta)\,\Xi(s;\theta)\right] \;\Rightarrow$$
$$\mathbb{E}^{\theta}\left[X(t_0;\theta)\,\Xi(s;\theta)\right] = \mathbb{E}^{\theta}\left[X_0(\theta)\,\Xi(s;\theta)\right] \;\Rightarrow$$

$$R_{X\Xi}\left(t_0,\, s\right) \;=\; R_{X_0\Xi}(s)\,. \tag{B.4b}$$

Solution to initial value problem (B.4a,b) with respect to t using Eqs. (i), (ii) is written

$$R_{X\Xi}\left(t,\, s\right) \;=\; R_{X_0\Xi}(s)\, e^{\eta\,(t-t_0)} + \kappa \int_{t_0}^{t} R_{\Xi\Xi}\left(\tau,\, s\right) e^{\eta\,(t-\tau)}\; d\tau\,. \tag{B.5}$$

B.1.3 IVP for the initial value-response cross-correlation $R_{X X_0}(t)$

We shall, now, construct an equation for the initial value-response cross-correlation $R_{X X_0}(t)$. By multiplying Eq. (B.1a) with initial value $X_0(\theta)$

$$X_0(\theta)\; \dot{X}(t;\theta) \;=\; \eta\, X_0(\theta)\; X(t;\theta) + \kappa\; X_0(\theta)\; \Xi(t;\theta)\,,$$

and performing the following manipulations

$$\mathbb{E}^{\theta}\Big[\, X_0(\theta)\; \dot{X}(t;\theta)\Big] \;=\; \eta\,\mathbb{E}^{\theta}\Big[\, X_0(\theta)\; X(t;\theta)\Big] + \kappa\; \mathbb{E}^{\theta}\Big[\, X_0(\theta)\; \Xi(t;\theta)\Big] \;\Rightarrow$$

$$\frac{d}{dt}\; \mathbb{E}^{\theta}\Big[\, X_0(\theta)\; X(t;\theta)\Big] \;=\; \eta\,\mathbb{E}^{\theta}\Big[\, X_0(\theta)\; X(t;\theta)\Big] + \kappa\; \mathbb{E}^{\theta}\Big[\, X_0(\theta)\; \Xi(t;\theta)\Big],$$

we find

$$\frac{d\, R_{X X_0}(t)}{dt} \;=\; \eta\, R_{X X_0}(t) + \kappa\; R_{X_0\Xi}(t)\,. \tag{B.6a}$$

Accordingly, the initial condition is specified as

$$X(t_0;\theta)\; X_0(\theta) \;=\; X_0(\theta)\; X_0(\theta) \;\Rightarrow$$

$$\mathbb{E}^{\theta}\Big[\, X(t_0;\theta)\; X_0(\theta)\Big] \;=\; \mathbb{E}^{\theta}\Big[\, X_0(\theta)\; X_0(\theta)\Big] \;\Rightarrow$$

$$R_{X X_0}(t_0) \;=\; R_{X_0 X_0}\,. \tag{B.6b}$$

Thus, the solution to initial value problem (B.6a,b) using Eqs. (B.i), (B.ii) is

$$R_{X X_0}(t) \;=\; R_{X_0 X_0}\; e^{\eta\,(t-t_0)} + \kappa \int_{t_0}^{t} R_{X_0\Xi}(\tau)\, e^{\eta\,(t-\tau)}\; d\tau\,. \tag{B.7}$$

B.1.4 IVP for the two-time autocorrelation of the response $R_{XX}(s,\, t)$

Following the same procedure, we will construct an equation for the autocorrelation of the response. First, Eq. (B.1a,b) is multiplied by $X(s;\theta)$

$$X(s;\theta)\; \dot{X}(t;\theta) \;=\; \eta\, X(s;\theta)\; X(t;\theta) + \kappa\; X(s;\theta)\; \Xi(t;\theta)\,.$$

Then, by applying the average operator and carrying out the usual manipulations

$$\mathbb{E}^{\theta}\Big[\, X(s;\theta)\; \dot{X}(t;\theta)\Big] \;=\; \eta\,\mathbb{E}^{\theta}\big[X(s;\theta)\; X(t;\theta)\big] + \kappa\; \mathbb{E}^{\theta}\big[X(s;\theta)\; \Xi(t;\theta)\big] \;\Rightarrow$$

$$\frac{\partial}{\partial t}\; \mathbb{E}^{\theta}\big[X(s;\theta)\; X(t;\theta)\big] \;=\; \eta\,\mathbb{E}^{\theta}\big[X(s;\theta)\; X(t;\theta)\big] + \kappa\; \mathbb{E}^{\theta}\big[X(s;\theta)\; \Xi(t;\theta)\big],$$

results in

$$\frac{\partial R_{XX}(s,t)}{\partial t} = \eta R_{XX}(s,t) + \kappa R_{X\Xi}(s,t). \tag{B.8a}$$

The initial condition to Eq. (B.8a) is analogously obtained as

$$\begin{aligned} &X(s;\theta)\, X(t_0;\theta) = X(s;\theta)\, X_0(\theta) \Rightarrow \\ &\mathbb{E}^{\theta}\Big[X(s;\theta)\, X(t_0;\theta)\Big] = \mathbb{E}^{\theta}\Big[X(s;\theta)\, X_0(\theta)\Big] \Rightarrow \\ &R_{XX}(s,t_0) = R_{XX_0}(s). \end{aligned} \tag{B.8b}$$

Therefore, by solving initial value problem (B.8a,b) through Eqs. (B.i), (B.ii), we obtain the two-time autocorrelation of the response

$$R_{XX}(s,t) = R_{XX_0}(s)\, e^{\eta(t-t_0)} + \kappa \int_{t_0}^{t} R_{X\Xi}(s,\tau)\, e^{\eta(t-\tau)}\, d\tau. \tag{B.9}$$

In Eq. (B.9), quantities $R_{X\Xi}(s,\tau)$, $R_{XX_0}(s)$ can be determined by Eqs. (B.5) and (B.7) respectively. First, in Eq. (B.7) we substitute $t = s$; this substitution is legitimate since the right-hand side of Eq. (B.7) is continuous with respect to the two arguments s, t. Thus, we find

$$R_{XX_0}(s) = R_{X_0X_0}\, e^{\eta(s-t_0)} + \kappa \int_{t_0}^{s} R_{X_0\Xi}(\tau)\, e^{\eta(s-\tau)}\, d\tau. \tag{B.10a}$$

Then, in Eq. (B.5) we interchange the two arguments

$$R_{X\Xi}(s,t) = R_{X_0\Xi}(t)\, e^{\eta(s-t_0)} + \kappa \int_{t_0}^{s} R_{\Xi\Xi}(u,t)\, e^{\eta(s-u)}\, du,$$

and set $t = \tau$:

$$R_{X\Xi}(s,\tau) = R_{X_0\Xi}(\tau)\, e^{\eta(s-t_0)} + \kappa \int_{t_0}^{s} R_{\Xi\Xi}(u,\tau)\, e^{\eta(s-u)}\, du. \tag{B.10b}$$

Finally, substituting Eqs. (B.10a,b) into Eq. (B.9) yields

$$\begin{aligned} R_{XX}(s,t) = &\left[R_{X_0X_0}\, e^{\eta(s-t_0)} + \kappa \int_{t_0}^{s} R_{X_0\Xi}(\tau)\, e^{\eta(s-\tau)}\, d\tau \right] e^{\eta(t-t_0)} + \\ &+ \kappa \int_{t_0}^{t} \left[R_{X_0\Xi}(\tau)\, e^{\eta(s-t_0)} + \kappa \int_{t_0}^{s} R_{\Xi\Xi}(u,\tau)\, e^{\eta(s-u)}\, du \right] e^{\eta(t-\tau)}\, d\tau \Rightarrow \end{aligned}$$

$$\begin{aligned} R_{XX}(s,t) = &R_{X_0X_0}\, e^{\eta(t+s-2t_0)} + \kappa \int_{t_0}^{s} R_{X_0\Xi}(\tau)\, e^{\eta(t+s-\tau-t_0)}\, d\tau \\ &+ \kappa \int_{t_0}^{t} R_{X_0\Xi}(\tau)\, e^{\eta(t+s-\tau-t_0)}\, d\tau + \kappa^2 \int_{t_0}^{t}\int_{t_0}^{s} R_{\Xi\Xi}(u,\tau)\, e^{\eta(t+s-\tau-u)}\, du\, d\tau. \end{aligned} \tag{B.11}$$

B.1.5 IVP for the one-time autocorrelation of the response $R_{XX}(t,t)$

The one-time autocorrelation of the response can readily be obtained by substituting $s=t$ in Eq. (B.11). This is rigorously performed since the right-hand side of Eq. (B.11) is continuous with respect to both time arguments. As such, we obtain

$$R_{XX}(t,t) = R_{X_0X_0}\, e^{2\eta(t-t_0)} + 2\kappa\, e^{2\eta t}\int_{t_0}^{t} R_{X_0\Xi}(\tau)\, e^{-\eta(\tau+t_0)}\, d\tau + + \kappa^2\, e^{2\eta t}\int_{t_0}^{t}\int_{t_0}^{t} R_{\Xi\Xi}(u,\tau)\, e^{-\eta(\tau+u)}\, du\, d\tau . \tag{B.12}$$

Let us now differentiate both sides of Eq. (12) with respect to t, resulting in

$$\frac{d R_{XX}(t,t)}{dt} = 2\eta R_{X_0X_0}\, e^{2\eta(t-t_0)} + 2\kappa\, R_{X_0\Xi}(t)\, e^{\eta(t-t_0)} + + 4\eta\kappa\, e^{2\eta t}\int_{t_0}^{t} R_{X_0\Xi}(\tau)\, e^{-\eta(\tau+t_0)}\, d\tau + 2\kappa^2\int_{t_0}^{t} R_{\Xi\Xi}(u,t)\, e^{\eta(t-u)}\, du + + 2\eta\kappa^2\, e^{2\eta t}\int_{t_0}^{t}\int_{t_0}^{t} R_{\Xi\Xi}(u,\tau)\, e^{-\eta(\tau+u)}\, du\, d\tau .$$

Further, identifying via Eq. (B.12) $R_{XX}(t,t)$ on the right-hand side of the above equation, we find

$$\frac{1}{2}\frac{d R_{XX}(t,t)}{dt} = \eta R_{XX}(t,t) + \kappa R_{X_0\Xi}(t)\, e^{\eta(t-t_0)} + \kappa^2\int_{t_0}^{t} R_{\Xi\Xi}(u,t)\, e^{\eta(t-u)}\, du . \tag{B.13}$$

Eq. (B.13), along with the initial condition determined by the data of the problem as

$$R_{XX}(t_0,t_0) = R_{X_0X_0}, \tag{B.14}$$

constitutes the IVP for the *one-time* autocorrelation function $R_{XX}(t,t)$. Note that this IVP for $R_{XX}(t,t)$, Eqs. (B.13), (B.14), cannot be obtained by simply substituting $s=t$ in the IVP for $R_{XX}(s,t)$, Eqs. (B.6) under the substitution of Eq. (B.5) for $R_{X\Xi}(s,t)$

$$\frac{\partial R_{XX}(s,t)}{\partial t} = \eta R_{XX}(s,t) + \kappa\, R_{X_0\Xi}(t)\, e^{\eta(s-t_0)} + \kappa^2\int_{t_0}^{s} R_{\Xi\Xi}(u,t)\, e^{\eta(s-u)}\, du , \tag{B.15a}$$

and the initial condition

$$R_{XX}(s,t_0) = R_{XX_0}(s). \tag{B.15b}$$

We observe that the multiplying factor $1/2$ in the left-hand side of Eq. (B.13) cannot be obtained by Eq. (B.15a) for $s=t$.

Formulae for the central moments of the response

Let us, now, define the second-order central moments of the data of the problem, i.e. the initial value and excitation

initial value variance: $\sigma^2_{X_0} = C_{X_0 X_0} = R_{X_0 X_0} - m^2_{X_0}$, (B.16)

two-time excitation autocovariance: $C_{\Xi\Xi}(t, s) = R_{\Xi\Xi}(t, s) - m_{\Xi}(t) m_{\Xi}(s)$, (B.17)

initial value-excitation cross-covariance: $C_{X_0\Xi}(t) = R_{X_0\Xi}(t) - m_{X_0} m_{\Xi}(t)$. (B.18)

Also, we define the second-order central moments that include the response, i.e.

initial value-response cross-covariance: $C_{X_0 X}(t) = R_{X_0 X}(t) - m_{X_0} m_X(t)$, (B.19)

two-time response excitation cross-covariance: $C_{X\Xi}(t, s) = R_{X\Xi}(t, s) - m_X(t) m_{\Xi}(s)$, (B.20)

two-time response autocovariance: $C_{XX}(t, s) = R_{XX}(t, s) - m_X(t) m_X(s)$, (B.21)

By substituting now, the relations: (B.3) for $m_X(t)$, (B.5) for $R_{X\Xi}(t, s)$, (B.7) for $R_{X_0 X}(t)$ and (B.11) for $R_{XX}(t, s)$, into definition relations (B.19)-(B.21), and employing the relations (B.16)-(B.18), we obtain

$$C_{X\Xi}(t, s) = C_{X_0\Xi}(s)\, e^{\eta(t-t_0)} + \kappa \int_{t_0}^{t} C_{\Xi\Xi}(\tau, s)\, e^{\eta(t-\tau)}\, d\tau, \tag{B.22}$$

$$C_{X_0 X}(t) = C_{X_0 X_0}\, e^{\eta(t-t_0)} + \kappa \int_{t_0}^{t} C_{X_0\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau, \tag{B.23}$$

$$\begin{aligned} C_{XX}(t, s) = {} & C_{X_0 X_0}\, e^{\eta(t+s-2t_0)} + \kappa \int_{t_0}^{t} C_{X_0\Xi}(\tau)\, e^{\eta(t+s-\tau-t_0)}\, d\tau + \\ & + \kappa \int_{t_0}^{s} C_{X_0\Xi}(\tau)\, e^{\eta(t+s-\tau-t_0)}\, d\tau + \kappa^2 \int_{t_0}^{s}\int_{t_0}^{t} C_{\Xi\Xi}(u, \tau)\, e^{\eta(t+s-\tau-u)}\, du\, d\tau. \end{aligned} \tag{B.24}$$

Last, for the variance of the response, we set $s = t$, in Eq. (B.24)

$$\begin{aligned} \sigma^2_X(t) = C_{XX}(t, t) = {} & \sigma^2_{X_0}\, e^{2\eta(t-t_0)} + \\ & + 2\kappa\, e^{\eta(t-t_0)} \int_{t_0}^{t} C_{X_0\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau + \kappa^2 \int_{t_0}^{t}\int_{t_0}^{t} C_{\Xi\Xi}(\tau, u)\, e^{\eta(2t-\tau-u)}\, du\, d\tau. \end{aligned} \tag{B.25}$$

Since the integrand of the double integral is symmetric with respect to the two integration variables, u, τ,

$$\int_{t_0}^{t}\int_{t_0}^{t} C_{\Xi\Xi}(\tau, u)\, e^{\eta(2t-\tau-u)}\, du\, d\tau = 2\int_{t_0}^{t}\int_{t_0}^{\tau} C_{\Xi\Xi}(\tau, u)\, e^{\eta(2t-\tau-u)}\, du\, d\tau.$$

Thus, Eq. (B.25) is expressed equivalently as

$$\sigma_X^2(t) = \sigma_{X_0}^2 e^{2\eta(t-t_0)} + 2\kappa e^{\eta(t-t_0)} \int_{t_0}^{t} \left(C_{X_0\Xi}(\tau) e^{\eta(t-\tau)} + \kappa \int_{t_0}^{\tau} C_{\Xi\Xi}(\tau, u) e^{\eta(2t-\tau-u)} du \right) d\tau,$$

and after some algebraic manipulations

$$\sigma_X^2(t) = \sigma_{X_0}^2 e^{2\eta(t-t_0)} + 2\kappa \int_{t_0}^{t} \left(C_{X_0\Xi}(\tau) e^{\eta(\tau-t_0)} + \kappa \int_{t_0}^{\tau} C_{\Xi\Xi}(\tau, u) e^{\eta(\tau-u)} du \right) e^{2\eta(t-\tau)} d\tau. \tag{B.26}$$

By employing now Eq. (B.22), Eq. (B.26) reads as

$$\sigma_X^2(t) = \sigma_{X_0}^2 e^{2\eta(t-t_0)} + 2\kappa \int_{t_0}^{t} C_{X\Xi}(\tau, \tau) e^{2\eta(t-\tau)} d\tau. \tag{B.27}$$

B.2 Moment problem for the first derivative of the response

Let us, now, move towards determining the first (mean value) and second (autocorrelation) moments of the first derivative of the response $\dot{X}(t;\theta)$, the cross-correlation functions of the response with excitation $\Xi(t;\theta)$, $\dot{\Xi}(t;\theta)$ as well as initial values $X_0(\theta)$ and $\dot{X}(t_0;\theta)$. The derivation of these moment equations requires to consider the following RDE which is derived, as discussed in Ch. 7, by differentiating both sides of Eq. (B.1a) with respect to t:

$$\ddot{X}(t;\theta) = \eta \dot{X}(t;\theta) + \kappa \dot{\Xi}(t;\theta). \tag{B.28a}$$

Eq. (B.28a) is also supplemented by the initial value obtained by setting in the initial problem $t = t_0$ as follows:

$$\dot{X}(t_0;\theta) = \eta X_0(\theta) + \kappa \Xi_0(\theta), \tag{B.28b}$$

where $\Xi_0(\theta) = \Xi(t_0;\theta)$ is used in order to simplify the notation. The sought-for moments are derived in this section by utilizing the already presented methodology. More specifically, we shall take the average $\mathbb{E}^\theta[\bullet]$ of both sides of RIVP (B.28a,b) which each time is multiplied by the appropriate random function, and then solve the acquired, linear ODE. Nevertheless, the expressions presented in this section can also be straightforwardly obtained by differentiating (in the mean-square calculus sense) of the already derived moments.

Before we begin with the derivation of the aforementioned moments, we have to make some additional comments regarding what is considered data for the moment IVPs and what unknowns; the latter shall be subsequently obtained by formulating and solving their corresponding IVPs. Thus, in this section, in addition to the knowledge of the pdfs mentioned in the previous section, the knowledge of the family of pdfs $f_{\dot{\Xi}(t_1)\cdots\dot{\Xi}(t_N)}(\xi_1, \ldots, \xi_N)$ and the cross pdf family of the initial value with the derivative of the excitation $f_{X_0\dot{\Xi}(t_1)\cdots\dot{\Xi}(t_N)}(x, \xi_1, \ldots, \xi_N)$ is required and apprehended though the knowledge of the initial value and excitation.

B.2.1 IVP for mean value of the first derivative of the response $\dot{m}_X(t)$

We construct an equation for the mean value of $\dot{X}(t;\theta)$ by first averaging both sides of Eqs. (B.28a)

$$\mathbb{E}^{\theta}\left[\ddot{X}(t;\theta)\right] = \eta\,\mathbb{E}^{\theta}\left[\dot{X}(t;\theta)\right] + \kappa\,\mathbb{E}^{\theta}\left[\dot{\Xi}(t;\theta)\right].$$

Then, we interchange mean value and differentiation operators

$$\frac{d}{dt}\mathbb{E}^{\theta}\left[\dot{X}(t;\theta)\right] = \eta\,\mathbb{E}^{\theta}\left[\dot{X}(t;\theta)\right] + \kappa\,\mathbb{E}^{\theta}\left[\dot{\Xi}(t;\theta)\right],$$

and introducing moments, we obtain

$$\frac{d\,\dot{m}_X(t)}{dt} = \eta\,\dot{m}_X(t) + \kappa\,\dot{m}_{\Xi}(t). \tag{B.29a}$$

Following, now, the same procedure for Eq. (B.28b)

$$\mathbb{E}^{\theta}\left[\dot{X}(t_0;\theta)\right] = \eta\,\mathbb{E}^{\theta}\left[X_0(\theta)\right] + \kappa\,\mathbb{E}^{\theta}\left[\Xi_0(\theta)\right] \Rightarrow$$
$$\mathbb{E}^{\theta}\left[\dot{X}(t_0;\theta)\right] = \eta\,\mathbb{E}^{\theta}\left[X_0(\theta)\right] + \kappa\,\mathbb{E}^{\theta}\left[\Xi_0(\theta)\right],$$

results in the following initial condition:

$$\dot{m}_X(t_0) = \eta\, m_{X_0} + \kappa\, m_{\Xi_0}. \tag{B.29b}$$

Finally, by solving IVP (B.29a,b) using Eqs. (B.i), (B.ii), the mean value of the first derivative of the response is written as

$$\dot{m}_X(t) = \left(\eta\, m_{X_0} + \kappa\, m_{\Xi_0}\right) e^{\eta(t-t_0)} + \kappa\int_{t_0}^{t} \dot{m}_{\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau\ , \tag{B.30}$$

which can be further calculated as

$$\dot{m}_X(t) = \left(\eta\, m_{X_0} + \kappa\, m_{\Xi_0}\right) e^{\eta(t-t_0)} + \kappa\left[m_{\Xi}(t) - m_{\Xi}(t_0)\, e^{\eta(t-t_0)}\right] +$$
$$+\kappa\eta\int_{t_0}^{t} m_{\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau \Rightarrow$$

$$\dot{m}_X(t) = \eta\, m_{X_0}\, e^{\eta(t-t_0)} + \kappa m_{\Xi}(t) + \kappa\eta\int_{t_0}^{t} m_{\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau. \tag{B.31}$$

Moreover, recalling Eq. (B.3) for the mean value of the response Eq. (B.31) can be equivalently be expressed as

$$\dot{m}_X(t) = \eta\, m_X(t) + \kappa m_{\Xi}(t). \tag{B.32}$$

Eq. (B.32) can also be derived by directly applying the average operator $\mathbb{E}^{\theta}[\bullet]$ to both sides of Eq. (B.1a).

B.2.2 IVP for the two times response – excitation dot cross-correlation $R_{X\dot{\Xi}}(t,s)$

We will construct an equation for the cross-correlation function of the derivative of the response and the excitation. Thus, by multiplying both sides of Eq. (B.1a) with $\dot{\Xi}(s;\theta)$, taking the average and performing the usual treatment, we obtain

$$\begin{aligned}
&\dot{\Xi}(s;\theta)\dot{X}(t;\theta) = \eta\dot{\Xi}(s;\theta)X(t;\theta) + \kappa\,\dot{\Xi}(s;\theta)\Xi(t;\theta) \Rightarrow \\
&\mathbb{E}^{\theta}\left[\dot{\Xi}(s;\theta)\dot{X}(t;\theta)\right] = \eta\mathbb{E}^{\theta}\left[\dot{\Xi}(s;\theta)X(t;\theta)\right] + \kappa\,\mathbb{E}^{\theta}\left[\dot{\Xi}(s;\theta)\Xi(t;\theta)\right] \Rightarrow \\
&\frac{\partial}{\partial t}\mathbb{E}^{\theta}\left[\dot{\Xi}(s;\theta)X(t;\theta)\right] = \eta\mathbb{E}^{\theta}\left[\dot{\Xi}(s;\theta)X(t;\theta)\right] + \kappa\,\mathbb{E}^{\theta}\left[\dot{\Xi}(s;\theta)\Xi(t;\theta)\right],
\end{aligned}$$

which can be rewritten by introducing moments as

$$\frac{\partial}{\partial t}R_{X\dot{\Xi}}(t,s) = \eta\partial_s R_{X\dot{\Xi}}(t,s) + \kappa\,\partial_s R_{X\Xi}(t,s). \tag{B.33a}$$

The same procedure is also applied to the initial condition (B.1b)

$$\begin{aligned}
&X(t_0;\theta)\dot{\Xi}(s;\theta) = X_0(\theta)\dot{\Xi}(s;\theta) \Rightarrow \\
&\mathbb{E}^{\theta}\left[X(t_0;\theta)\dot{\Xi}(s;\theta)\right] = \mathbb{E}^{\theta}\left[X_0(\theta)\dot{\Xi}(s;\theta\right] \Rightarrow \\
&R_{X\dot{\Xi}}(t_0,s) = R_{X_0\dot{\Xi}}(t,s).
\end{aligned} \tag{B.33b}$$

Solution to initial value problem (B.33a,b) with respect to t is written

$$R_{X\dot{\Xi}}(t,s) = R_{X_0\dot{\Xi}}(s)\,e^{\eta(t-t_0)} + \kappa\int_{t_0}^{t} R_{\Xi\dot{\Xi}}(\tau,s)\,e^{\eta(t-\tau)}\,d\tau. \tag{B.34}$$

B.2.3 IVP for the two-time cross-correlation of the response with its first derivative $R_{X\dot{X}}(s,t)$

We will construct an equation for the two-time cross-correlation function $R_{X\dot{X}}(s,t)$, by multiplying both sides of Eq. (B.28a) with $X(s;\theta)$ and taking the average

$$\begin{aligned}
&X(s;\theta)\ddot{X}(t;\theta) = \eta X(s;\theta)\dot{X}(t;\theta) + \kappa\,X(s;\theta)\dot{\Xi}(t;\theta) \Rightarrow \\
&\mathbb{E}^{\theta}\left[X(s;\theta)\ddot{X}(t;\theta)\right] = \eta\mathbb{E}^{\theta}\left[X(s;\theta)\dot{X}(t;\theta)\right] + \kappa\,\mathbb{E}^{\theta}\left[X(s;\theta)\dot{\Xi}(t;\theta)\right] \Rightarrow \\
&\frac{\partial}{\partial t}\mathbb{E}^{\theta}\left[X(s;\theta)\dot{X}(t;\theta)\right] = \eta\mathbb{E}^{\theta}\left[X(s;\theta)\dot{X}(t;\theta)\right] + \kappa\,\mathbb{E}^{\theta}\left[X(s;\theta)\dot{\Xi}(t;\theta)\right],
\end{aligned}$$

we find

$$\frac{\partial}{\partial t}R_{X\dot{X}}(s,t) = \eta\,R_{X\dot{X}}(s,t) + \kappa\,R_{X\dot{\Xi}}(s,t). \tag{B.35}$$

Note that the rightmost term of Eq. (B.35) can be identified by Eq. (B.34), rewritten by interchanging the time arguments as

$$R_{X\dot{\Xi}}(s,t) = R_{X_0\dot{\Xi}}(t)\, e^{\eta(s-t_0)} + \kappa \int_{t_0}^{s} R_{\Xi\dot{\Xi}}(\tau,t)\, e^{\eta(s-\tau)}\, d\tau .$$

Now, substituting the above expression into Eq. (B.35), we find

$$\frac{\partial}{\partial t} R_{X\dot{X}}(s,t) = \eta\, R_{X\dot{X}}(s,t) + \kappa\, R_{X_0\dot{\Xi}}(t)\, e^{\eta(s-t_0)} + \\ + \kappa^2 \int_{t_0}^{s} R_{\Xi\dot{\Xi}}(\tau,t)\, e^{\eta(s-\tau)}\, d\tau . \tag{B.36a}$$

Likewise, we obtain the initial condition

$$X(s;\theta)\dot{X}(t_0;\theta) = \eta\, X(s;\theta) X_0(\theta) + \kappa\, X(s;\theta)\Xi_0(\theta) \Rightarrow$$
$$\mathbb{E}^{\theta}\left[X(s;\theta)\dot{X}(t_0;\theta)\right] = \eta\,\mathbb{E}^{\theta}\left[X(s;\theta) X_0(\theta)\right] + \kappa\, \mathbb{E}^{\theta}\left[X(s;\theta)\Xi_0(\theta)\right] \Rightarrow$$
$$R_{X\dot{X}}(s,t_0) = \eta R_{XX_0}(s) + \kappa\, R_{X\Xi_0}(s). \tag{B.36b}$$

Solution to initial value problem (B.36a,b) with respect to t is written

$$R_{X\dot{X}}(s,t) = \left(\eta R_{XX_0}(s) + \kappa\, R_{X\Xi_0}(s)\right) e^{\eta(t-t_0)} + \kappa \int_{t_0}^{t} R_{X\dot{\Xi}}(s,\tau) e^{\eta(t-\tau)}\, d\tau =$$
$$= \left(\eta R_{XX_0}(s) + \kappa\, R_{X\Xi_0}(s)\right) e^{\eta(t-t_0)} +$$
$$+ \kappa \left[R_{X\Xi}(s,t) - R_{X\Xi}(s,t_0) e^{\eta(t-t_0)}\right] + \kappa\eta \int_{t_0}^{t} R_{X\Xi}(s,\tau) e^{\eta(t-\tau)}\, d\tau \Rightarrow$$

$$R_{X\dot{X}}(s,t) = \eta R_{XX_0}(s)\, e^{\eta(t-t_0)} + \kappa\, R_{X\Xi}(s,t) + \kappa\eta \int_{t_0}^{t} R_{X\Xi}(s,\tau)\, e^{\eta(t-\tau)}\, d\tau \tag{B.37}$$

Further, the terms $R_{XX_0}(s)$ and $R_{X\Xi}(s,t)$ can be expressed through Eqs. (B.7) and (B.5) respectively as:

$$R_{XX_0}(s) = R_{X_0X_0}\, e^{\eta(s-t_0)} + \kappa \int_{t_0}^{s} R_{X_0\Xi}(\tau)\, e^{\eta(s-\tau)}\, d\tau , \tag{B.38}$$

$$R_{X\Xi}(s,t) = R_{X_0\Xi}(t)\, e^{\eta(s-t_0)} + \kappa \int_{t_0}^{s} R_{\Xi\Xi}(u,t)\, e^{\eta(s-u)}\, du . \tag{B.39}$$

As such, using Eqs. (B.38) and (B.39) the two-time cross-correlation of the response and its first derivative, Eq. (B.37), takes the following final form:

$$R_{X\dot{X}}(s,t) = \eta R_{X_0X_0}\, e^{\eta(s+t-2t_0)} + \kappa\eta \int_{t_0}^{s} R_{X_0\Xi}(\tau) e^{\eta(s+t-\tau-t_0)}\, d\tau +$$
$$+ \kappa\, R_{X_0\Xi}(t)\, e^{\eta(s-t_0)} + \kappa^2 \int_{t_0}^{s} R_{\Xi\Xi}(u,t)\, e^{\eta(s-u)}\, du + \qquad \text{(B.40)}$$
$$+ \kappa\eta \int_{t_0}^{t} R_{X_0\Xi}(\tau) e^{\eta(s+t-\tau-t_0)}\, d\tau + \kappa^2\eta \int_{t_0}^{t}\int_{t_0}^{s} R_{\Xi\Xi}(u,\tau)\, e^{\eta(s+t-\tau-u)}\, du\, d\tau$$

B.2.4 IVP for the cross-correlation of the initial value with the first derivative of the response $R_{\dot{X}X_0}(t)$

We will construct an equation for the cross-correlation function, by multiplying both sides of Eq. (B.28a) with $X_0(\theta)$, and taking the average

$$X_0(\theta)\ddot{X}(t;\theta) = \eta X_0(\theta)\dot{X}(t;\theta) + \kappa\, X_0(\theta)\dot{\Xi}(t;\theta) \Rightarrow$$
$$\mathbb{E}^\theta\left[X_0(\theta)\ddot{X}(t;\theta)\right] = \eta\mathbb{E}^\theta\left[X_0(\theta)\dot{X}(t;\theta)\right] + \kappa\,\mathbb{E}^\theta\left[X_0(\theta)\dot{\Xi}(t;\theta)\right] \Rightarrow$$
$$\frac{d}{dt}\mathbb{E}^\theta\left[X_0(\theta)\dot{X}(t;\theta)\right] = \eta\mathbb{E}^\theta\left[X_0(\theta)\dot{X}(t;\theta)\right] + \kappa\,\mathbb{E}^\theta\left[X_0(\theta)\dot{\Xi}(t;\theta)\right] \Rightarrow$$
$$\frac{d}{dt}R_{\dot{X}X_0}(t) = \eta R_{\dot{X}X_0}(t) + \kappa\, R_{\dot{\Xi}X_0}(t). \qquad \text{(B.41a)}$$

The same procedure is also applied to the initial condition

$$X_0(\theta)\dot{X}(t_0;\theta) = \eta X_0^2(\theta) + \kappa\, X_0(\theta)\Xi_0(\theta) \Rightarrow$$
$$\mathbb{E}^\theta\left[X_0(\theta)\dot{X}(t_0;\theta)\right] = \eta\mathbb{E}^\theta\left[X_0^2(\theta)\right] + \kappa\,\mathbb{E}^\theta\left[X_0(\theta)\Xi_0(\theta)\right] \Rightarrow$$
$$R_{\dot{X}X_0}(t_0) = \eta R_{X_0X_0} + \kappa\, R_{X_0\Xi_0}. \qquad \text{(B.41b)}$$

Solution to initial value problem (B.41a,b) with respect to t using Eqs. (B.i), (B.ii) is written

$$R_{\dot{X}X_0}(t) = \left(\eta R_{X_0X_0} + \kappa\, R_{X_0\Xi_0}\right)e^{\eta(t-t_0)} + \kappa\left[R_{\Xi X_0}(t) - R_{\Xi X_0}(t_0)e^{\eta(t-t_0)}\right] +$$
$$+ \kappa\eta \int_{t_0}^{t} R_{\Xi X_0}(\tau) e^{\eta(t-\tau)}\, d\tau \Rightarrow$$
$$R_{\dot{X}X_0}(t) = \eta R_{X_0X_0}\, e^{\eta(t-t_0)} + \kappa\, R_{X_0\Xi}(t) + \kappa\eta \int_{t_0}^{t} R_{X_0\Xi}(\tau) e^{\eta(t-\tau)}\, d\tau \qquad \text{(B.42)}$$

B.2.5 IVP for the cross-correlation of the first derivative of the response with the excitation $R_{\dot{X}\Xi}(t,s)$

We will construct an equation for the cross-correlation function, by multiplying both sides of Eq. (B.28a) with $\Xi(s;\theta)$, and taking the average

$$\Xi(s;\theta)\ddot{X}(t;\theta) = \eta\Xi(s;\theta)\dot{X}(t;\theta) + \kappa\,\Xi(s;\theta)\dot{\Xi}(t;\theta) \Rightarrow$$
$$\mathbb{E}^\theta\left[\Xi(s;\theta)\ddot{X}(t;\theta)\right] = \eta\mathbb{E}^\theta\left[\Xi(s;\theta)\dot{X}(t;\theta)\right] + \kappa\,\mathbb{E}^\theta\left[\Xi(s;\theta)\dot{\Xi}(t;\theta)\right] \Rightarrow$$
$$\frac{\partial}{\partial t}\mathbb{E}^\theta\left[\Xi(s;\theta)\dot{X}(t;\theta)\right] = \eta\mathbb{E}^\theta\left[\Xi(s;\theta)\dot{X}(t;\theta)\right] + \kappa\,\mathbb{E}^\theta\left[\Xi(s;\theta)\dot{\Xi}(t;\theta)\right] \Rightarrow$$
$$\frac{\partial}{\partial t}R_{\dot{X}\Xi}(t,s) = \eta R_{\dot{X}\Xi}(t,s) + \kappa\,R_{\dot{\Xi}\Xi}(t,s). \quad \text{(B.43a)}$$

The same procedure is also applied to the initial condition

$$\Xi(s;\theta)\dot{X}(t_0;\theta) = \eta\Xi(s;\theta)X_0(\theta) + \kappa\,\Xi(s;\theta)\Xi_0(\theta) \Rightarrow$$
$$\mathbb{E}^\theta\left[\Xi(s;\theta)\dot{X}(t_0;\theta)\right] = \eta\mathbb{E}^\theta\left[\Xi(s;\theta)X_0(\theta)\right] + \kappa\,\mathbb{E}^\theta\left[\Xi(s;\theta)\Xi_0(\theta)\right] \Rightarrow$$
$$R_{\dot{X}\Xi}(t_0,s) = \eta R_{X_0\Xi}(s) + \kappa\,R_{\Xi_0\Xi}(s). \quad \text{(B.43b)}$$

Solution to initial value problem (B.47a,b) with respect to t using Eqs. (B.i), (B.ii) is written

$$R_{\dot{X}\Xi}(t,s) = \left(\eta R_{X_0\Xi}(s) + \kappa\,R_{\Xi_0\Xi}(s)\right)e^{\eta(t-t_0)} + \kappa\int_{t_0}^{t} R_{\dot{\Xi}\Xi}(\tau,s)e^{\eta(t-\tau)}\,d\tau \Rightarrow$$
$$R_{\dot{X}\Xi}(t,s) = \eta R_{X_0\Xi}(s)e^{\eta(t-t_0)} + \kappa\,R_{\Xi\Xi}(t,s) + \kappa\eta\int_{t_0}^{t} R_{\Xi\Xi}(\tau,s)e^{\eta(t-\tau)}\,d\tau$$
$$\text{(B.44)}$$

B.2.6 The autocorrelation of the first derivative of the response

We will obtain the autocorrelation function, by multiplying both sides of Eq. (B.1a) with $\dot{X}(t;\theta)$, and taking the average

$$\dot{X}(t;\theta)\dot{X}(s;\theta) = \eta\dot{X}(t;\theta)X(s;\theta) + \kappa\,\dot{X}(t;\theta)\Xi(s;\theta) \Rightarrow$$
$$\mathbb{E}^\theta\left[\dot{X}(t;\theta)\dot{X}(s;\theta)\right] = \eta\mathbb{E}^\theta\left[\dot{X}(t;\theta)X(s;\theta)\right] + \kappa\,\mathbb{E}^\theta\left[\dot{X}(t;\theta)\Xi(s;\theta)\right] \Rightarrow$$
$$R_{\dot{X}\dot{X}}(t,s) = \eta R_{X\dot{X}}(s,t) + \kappa\,R_{\dot{X}\Xi}(t,s). \quad \text{(B.45)}$$

By substituting Eqs. (B.40) and (B.44) into (B.45) we obtain the two-time autocorrelation function

$$R_{\dot{X}\dot{X}}(t,s) = \eta^2 R_{X_0X_0}\,e^{\eta(s+t-2t_0)} + \kappa\eta^2\int_{t_0}^{s} R_{X_0\Xi}(\tau)e^{\eta(s+t-\tau-t_0)}\,d\tau +$$
$$+ \kappa\eta R_{X_0\Xi}(t)\,e^{\eta(s-t_0)} + \kappa^2\eta\int_{t_0}^{s} R_{\Xi\Xi}(\sigma,t)\,e^{\eta(s-\sigma)}\,d\sigma +$$
$$+ \kappa\eta^2\int_{t_0}^{t} R_{X_0\Xi}(\tau)e^{\eta(s+t-\tau-t_0)}\,d\tau + \kappa^2\eta^2\int_{t_0}^{t}\int_{t_0}^{s} R_{\Xi\Xi}(\sigma,\tau)\,e^{\eta(s+t-\tau-\sigma)}\,d\sigma\,d\tau +$$
$$+ \kappa\eta R_{X_0\Xi}(s)e^{\eta(t-t_0)} + \kappa^2\,R_{\Xi\Xi}(t,s) + \kappa^2\eta\int_{t_0}^{t} R_{\Xi\Xi}(\tau,s)e^{\eta(t-\tau)}\,d\tau. \quad \text{(B.46)}$$

In addition, recognizing that the right-hand side of the above equation is continuous for both time arguments, we can make the legitimate substitution $s = t$. Finally, by taking advantage of the symmetry property of $R_{\Xi\Xi}(t,t)$, the one-time autocorrelation function of the first temporal derivative of the response is derived.

$$\begin{aligned} R_{\dot{X}\dot{X}}(t,t) &= \eta^2 R_{X_0X_0}\, e^{2\eta(t-t_0)} + \kappa^2 R_{\Xi\Xi}(t,t) + 2\kappa\eta R_{X_0\Xi}(t)\, e^{\eta(t-t_0)} + \\ &+ 2\kappa\eta^2 \int_{t_0}^{t} R_{X_0\Xi}(\tau)\, e^{\eta(2t-\tau-t_0)}\, d\tau + 2\kappa^2\eta \int_{t_0}^{t} R_{\Xi\Xi}(\tau,t)\, e^{\eta(t-\tau)}\, d\tau + \\ &+ \kappa^2\eta^2 \int_{t_0}^{t}\int_{t_0}^{t} R_{\Xi\Xi}(\sigma,\tau)\, e^{\eta(2t-\tau-\sigma)}\, d\sigma\, d\tau \end{aligned} \tag{B.47}$$

Formulae for central moments of the derivative of the response

First, we define the second-order central moments of the data of the problem (B.28a,b),

variance of the derivative of the response at $t = t_0$:

$$\sigma^2_{\dot{X}}(t_0) = C_{\dot{X}\dot{X}}(t_0,t_0) = R_{\dot{X}\dot{X}}(t_0,t_0) - m^2_{\dot{X}}(t_0)\,, \tag{B.48}$$

two-time cross-covariance of the excitation and its derivative:

$$C_{\dot{\Xi}\Xi}(t,s) = R_{\dot{\Xi}\Xi}(t,s) - m_{\dot{\Xi}}(t)\, m_{\Xi}(s)\,, \tag{B.49}$$

cross-covariance of the initial value and the derivative of the excitation:

$$C_{X_0\dot{\Xi}}(t) = R_{X_0\dot{\Xi}}(t) - m_{X_0}\, m_{\dot{\Xi}}(t)\,. \tag{B.50}$$

Also, we define the following, useful second-order central moments that include the response and its derivative, i.e.
cross-covariance of the derivative of the response and the initial value $X_0(\theta)$:

$$C_{\dot{X}X_0}(t) = R_{\dot{X}X_0}(t) - m_{\dot{X}}(t)\, m_{X_0}\,, \tag{B.51}$$

two-time cross-covariance of the response and its derivative:

$$C_{X\dot{X}}(t,s) = R_{X\dot{X}}(t,s) - m_X(t)\, m_{\dot{X}}(s)\,, \tag{B.52}$$

At this point, using the above expressions as well as Eqs (B.40) for $R_{X\dot{X}}(t,s)$ and (B.42) for $R_{\dot{X}X_0}(t)$, we find

$$C_{\dot{X}X_0}(t) = \eta C_{X_0X_0}\, e^{\eta(t-t_0)} + \kappa\, C_{X_0\Xi}(t) + \kappa\eta \int_{t_0}^{t} C_{X_0\Xi}(\tau)\, e^{\eta(t-\tau)}\, d\tau\,, \tag{B.53}$$

$$C_{X\dot{X}}(s,t) = \eta C_{X_0X_0} e^{\eta(s+t-2t_0)} + \kappa\eta \int_{t_0}^{s} C_{X_0\Xi}(\tau) e^{\eta(s+t-\tau-t_0)} d\tau +$$
$$+ \kappa C_{X_0\Xi}(t) e^{\eta(s-t_0)} + \kappa^2 \int_{t_0}^{s} C_{\Xi\Xi}(u,t) e^{\eta(s-u)} du + \tag{B.54}$$
$$+ \kappa\eta \int_{t_0}^{t} C_{X_0\Xi}(\tau) e^{\eta(s+t-\tau-t_0)} d\tau + \kappa^2 \eta \int_{t_0}^{t}\int_{t_0}^{s} C_{\Xi\Xi}(u,\tau) e^{\eta(s+t-\tau-u)} du\, d\tau .$$

Moreover, by setting $t = t_0$ in Eqs. (B.53), (B.54), we obtain

$$C_{\dot{X}X_0}(t_0) = C_{\dot{X}_0X_0} = \eta C_{X_0X_0} + \kappa C_{X_0\Xi}(t_0), \tag{B.55}$$

$$C_{X\dot{X}}(s,t_0) = \eta C_{X_0X_0} e^{\eta(s-t_0)} + \kappa\eta \int_{t_0}^{s} C_{X_0\Xi}(\tau) e^{\eta(s-\tau)} d\tau +$$
$$+ \kappa C_{X_0\Xi}(t_0) e^{\eta(s-t_0)} + \kappa^2 \int_{t_0}^{s} C_{\Xi\Xi}(u,t_0) e^{\eta(s-u)} du . \tag{B.56}$$

The derivation of Eqs. (B.55)-(B.56) is rigorous since the right-hand side of Eqs. (B.53)-(B.54) is continuous with respect to both time arguments.

Last, the variance of the first derivative of the response, $\sigma^2_{\dot{X}}(t)$, can similarly be derived by considering $\sigma^2_{\dot{X}}(t) = \mathrm{C}_{\dot{X}\dot{X}}(t,t) = \mathrm{R}_{\dot{X}\dot{X}}(t,t) - m_{\dot{X}}(t) m_{\dot{X}}(t)$. However, it is much easier to acquire it via its definition, as

$$\begin{aligned}
\sigma^2_{\dot{X}}(t) &= \mathbb{E}^{\theta}\left[\left(\dot{X}(t;\theta) - \dot{m}_X(t)\right)^2\right] = \\
&= \mathbb{E}^{\theta}\left[\dot{X}^2(t;\theta) - 2\dot{m}_X(t)\dot{X}(t;\theta) + \dot{m}^2_X(t)\right] = \\
&= \mathbb{E}^{\theta}\left[\dot{X}^2(t;\theta)\right] - 2\dot{m}_X(t)\,\mathbb{E}^{\theta}\left[\dot{X}(t;\theta)\right] + \dot{m}^2_X(t) = \\
&= \mathbb{E}^{\theta}\left[\dot{X}^2(t;\theta)\right] - \dot{m}^2_X(t).
\end{aligned}$$

Further, using Eq. (B.1a), we find

$$\begin{aligned}
\sigma^2_{\dot{X}}(t) &= \mathbb{E}^{\theta}\left[\left(\eta X(t;\theta) + \kappa\Xi(t;\theta)\right)^2\right] - \dot{m}^2_X(t) = \\
&= \eta^2\,\mathbb{E}^{\theta}\left[X^2(t;\theta)\right] + 2\eta\kappa\,\mathbb{E}^{\theta}\left[X(t;\theta)\Xi(t;\theta)\right] + \kappa^2\,\mathbb{E}^{\theta}\left[\Xi^2(t;\theta)\right] - \dot{m}^2_X(t) \overset{\text{Eq. (B.32)}}{=} \\
&= \eta^2 R_{XX}(t,t) + 2\eta\kappa R_{X\Xi}(t,t) + \kappa^2 R_{\Xi\Xi}(t,t) - \left(\eta m_X(t) + \kappa m_{\Xi}(t)\right)^2 \Rightarrow
\end{aligned}$$

Finally, by introducing central moments we reach the expression

$$\sigma^2_{\dot{X}}(t) = \eta^2 \sigma^2_X(t) + 2\eta\kappa C_{X\Xi}(t,t) + \kappa^2 \sigma^2_{\Xi}(t). \tag{B.57}$$

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