# The Vidav-Palmer Theorem and C*-equivalent algebras

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# Abstract

Over the last century, the mathematical area of Operator Algebras has gained a lot of attention among mathematicians, thanks to the numerous applications on Natural Sciences. The most famous mathematical object on this area are C*-algebras. Primarily used to model the algebra of observables in quantum mechanics, a C*-algebra is a very well behaved structure with which a mathematician can approach many problems in Mathematics. This great behavior is the fact from which the purpose of this thesis arises.

Unlike the previous, the more general object called Banach algebra is not that well behaved. Therefore, the purpose of this Master's thesis is to present some conditions in order for an arbitrary Banach or *-Banach algebra to be a C*-algebra.

In the first chapter, which is an introduction to Banach algebras, we provide the essential concepts and tools in order to continue. In the second chapter we introduce the concept of an involutive Banach algebra and demonstrate the characterization of hermitian and symmetric *-Banach algebras, with the celebrated Shiralli-Ford Theorem and provide some useful results rising from Pták's Inequality. Also a brief presentation regarding Möbius Transformations takes place, necessary to prove the Russo-Dye Theorem.

In the third chapter, after the description of the concept of the Numerical Range is displayed, we find ourselves in a position to provide the first major result, the Vidav-Palmer Theorem. In the last chapter, we provide some well-known results on C*-algebras such as the Gelfand-Naimark Theorem and introduce the reader to the concept of C*-equivalent algebras. After the presentation of some early results by B. A. Barnes, we conclude with J. Cuntz's ingenious proof that local C*-equivalence implies C*-equivalence.

# Contents

# Chapter 1

# Banach Algebras

## 1.1 Preliminaries

In this first section, we will provide some basic results about the general theory of Banach algebras. Beginning with the description of the essential tools, we will come up with some early but important results in Operator Theory, on which we will continue our presentation. This chapter can also be used as an appendix for readers familiar with Banach algebras. For an excessive description of these matters, one can look in [**1**], [**2**] and the appendix of [**4**].

**Definition 1.1.1.** *A Banach algebra $\mathcal{A}$ is a complex Banach space together with an associative and distributive multiplication satisfying:*

$$\lambda(ab) = (\lambda a)b = a(\lambda b)$$

*and*

$$||ab|| \leq ||a|| ||b||$$

*for any $a, b \in \mathcal{A}$ ,$\lambda \in \mathbb{C}$. It follows immediately that the multiplication is jointly continuous. Also $\mathcal{A}$ is called commutative if $ab = ba$, for all $a, b \in \mathcal{A}$ and $\mathcal{A}$ is called unital if it possesses a unit.*

The next lemma gives a connection between unital Banach algebras and Banach algebras without a unit which is important because it simplifies the study of the second class. Lemma 1.1.2 refers to the class of unital Banach algebras giving an equivalent norm whenever $||e|| > 1$, which practically lets us assume that the norm of the unit of a Banach algebra is equal to one.

**Lemma 1.1.1.** *A Banach algebra $\mathcal{A}$ without a unit can be embedded into a unital Banach algebra $\mathcal{A}_I$ as an ideal of codimension 1.*

*Proof.* Let $\mathcal{A}_I = \mathcal{A} \oplus \mathbb{C}$ as a linear space and a multiplication in $\mathcal{A}_I$ as such: $(x, \lambda)(y, \mu) = (xy + \mu x + \lambda y, \lambda\mu)$. One can easily check that the multiplication is associative and distributive and $(0, 1)$ is the corresponding unit.
Setting $||(x, \lambda)||_{\mathcal{A}_I} = ||x|| + |\lambda|$, we have that $\mathcal{A}_I$ is a Banach space and $||(\cdot, \cdot)||_{\mathcal{A}_I}$ is an algebra-norm, for,

$$||(x, \lambda)(y, \mu)||_{\mathcal{A}_I} = ||(xy + \mu x + \lambda y, \lambda\mu)||_{\mathcal{A}_I} = ||xy + \mu x + \lambda y|| + |\lambda\mu| \leq$$
$$||x||||y|| + |\mu|||x|| + |\lambda| + |\lambda||\mu| = (||x|| + |\lambda|)(||y|| + |\mu|) = ||(x, \lambda)||_{\mathcal{A}_I}||(y, \mu)||_{\mathcal{A}_I}.$$

Thus $\mathcal{A}_I$ is a Banach algebra with unit and $\mathcal{A}$ is the ideal $\{(x, 0) : x \in \mathcal{A}\}$ in $\mathcal{A}_I$. $\square$

**Lemma 1.1.2.** *Let $\mathcal{A}$ be a Banach algebra with unit $e$. Then there is a norm $|||\cdot|||$ equivalent to the original norm, such that $(\mathcal{A}, |||\cdot|||)$ is a unital Banach algebra satisfying $||e|| = 1$*

*Proof.* Let us denote for each $x \in \mathcal{A}$ the left multiplication operator $L_x : y \mapsto xy$. It falls immediately that $L_x$ is linear, $L_x = L_{x'} \implies x = x'$ and due to the fact that $||L_x y|| = ||xy|| \leq ||x||||y|| \implies L_x$ is bounded and $||L_x|| \leq ||x||$.
Put $|||x||| = ||L_x||$, then $|||x||| \leq ||x||$ and

$$|||x||| = ||L_x|| = \sup\{||L_x y|| : ||y|| \leq 1\} = \sup\{||xy|| : ||y|| \leq 1\} \geq ||xy'|| = \frac{||x||}{||e||},$$

where $y' = \frac{e}{||e||}$ $\square$

**Definition 1.1.2.** *Let $\mathcal{A}$ be a unital Banach algebra, $x \in \mathcal{A}$ is said to be invertible if there exists an element $y \in \mathcal{A}$, such that*

$$xy = yx = e.$$

*The set of all invertible elements of $\mathcal{A}$ is denoted as $\mathcal{G}(\mathcal{A})$.*

It is easily deduced that the inverse of an element is unique and that $\mathcal{G}(\mathcal{A})$ is a group, for if $x, y \in \mathcal{G}(\mathcal{A}) \implies (xy)^{-1} = y^{-1}x^{-1}$.

**Theorem 1.1.1.** *Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$ such that $||x|| < 1$. Then $(e - x) \in \mathcal{G}(\mathcal{A})$ and $(e - x)^{-1} = \sum_{n=0}^{+\infty} x^n$. Moreover $||(e - x)^{-1}|| \leq \frac{1}{1 - ||x||}$.*

*Proof.* We have that $\sum_{n=0}^{+\infty}||x||^n$ converges as a geometrical series. Since $||\cdot||$ is an algebra-norm implying $||x^n|| \leq ||x||^n$, $\forall n \in \mathbb{N}$ we have that $\sum_{n=0}^{+\infty}||x^n||$ converges. Due to the fact that $\mathcal{A}$ is a Banach space we have that $\sum_{n=0}^{+\infty} x^n$ also converges. Thus it suffices to show that,

$$(e-x)\sum_{n=0}^{+\infty} x^n = \sum_{n=0}^{+\infty} x^n(e-x) = e.$$

We have that,
$(e-x)\sum_{k=0}^{n} x^k = \sum_{k=0}^{n} x^k - \sum_{k=0}^{n} x^{k+1} = e - x^{n+1}$. Since $\sum_{n=0}^{+\infty} x^n$ converges we have $x \xrightarrow{n} 0$.
Finally, $||(e-x)^{-1}|| = ||\sum_{n=0}^{+\infty} x^n|| \leq \sum_{n=0}^{+\infty}||x^n|| \leq \sum_{n=0}^{+\infty}||x||^n = \frac{1}{1-||x||}$. The last equality is provided by the geometric series infinite terms sum formula. □

**Proposition 1.1.1.** *Let $\mathcal{A}$ be a Banach algebra and $x_0 \in \mathcal{G}(\mathcal{A})$. If $x \in \mathcal{A}$ satisfies $||x - x_0|| < \frac{1}{||x_0^{-1}||}$, then $x \in \mathcal{G}(\mathcal{A})$.*

*Proof.* Let $x \in \mathcal{A}$ then $x$ can be written as

$$x = x_0 - (x_0 - x) = x_0(e - (e - x_0^{-1}x)) = x_0(e - x_0^{-1}(x_0 - x))$$

Since $||x - x_0|| < \frac{1}{||x_0^{-1}||}$ it is implied that

$$||x_0^{-1}(x_0 - x)|| \leq ||x_0^{-1}||||x_0 - x|| < 1.$$

By Theorem 1.1.1., $e - x_0^{-1}(x_0 - x) \in \mathcal{G}(\mathcal{A})$, thus $x_0(e - x_0^{-1}(x_0 - x)) \in \mathcal{G}(\mathcal{A}) \implies x \in \mathcal{G}(\mathcal{A})$. □

**Corollary 1.1.1.** *The group $\mathcal{G}(\mathcal{A})$ is topologically open. Conversely the set of non-invertible elements is closed.*

**Proposition 1.1.2.** *Let $\mathcal{A}$ be a unital Banach algebra. The mapping $x \mapsto x^{-1}$ is continuous in $\mathcal{G}(\mathcal{A})$.*

*Proof.* Let $x_0 \in \mathcal{G}(\mathcal{A})$ and $\{x_n\}_{n\in\mathbb{N}} \in \mathcal{G}(\mathcal{A})$ a sequence satisfying $x_n \xrightarrow{x} 0$. Since $x_n \xrightarrow{n} x_0$ there exists a $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ the following holds

$$||x_n - x_0|| < \frac{1}{||x_0^{-1}||}.$$

Furthermore, $x_n = x_0[e - x_0^{-1}(x_0 - x_n)] \in \mathcal{G}(\mathcal{A})$ implying

$$x_n^{-1} = [e - x_0^{-1}(x_0 - x_n)]^{-1} x_0^{-1}$$

and

$$||x_0^{-1}(x_0 - x_n)|| \leq ||x_0^{-1}|| ||x_0 - x_n|| < 1.$$

Finally by theorem 1.1.1. $[e - x_0^{-1}(x_0 - x_n)]^{-1} = \sum_{k=0}^{+\infty}[x_0^{-1}(x_0 - x_n)]^k \implies x_n^{-1} = \sum_{k=0}^{+\infty}[x_0^{-1}(x_0 - x_n)]^k x_0^{-1} = x_0^{-1} + \sum_{k=1}^{+\infty}[x_0^{-1}(x_0 - x_n)]^k$, implying

$$||x_n^{-1} - x_0^{-1}|| = ||\sum_{k=1}^{+\infty}[x_0^{-1}(x_0 - x_n)]^k|| \leq \sum_{k=1}^{+\infty}(||x_0^{-1}|| ||x_0 - x_n||)^k ||x_0^{-1}|| \to 0.$$

□

Next is the definition of the exponential function on a Banach algebra and some necessary properties in order to continue. We will mainly denote the exponential function with $e^x$ and $\exp(x)$ whenever there is a possibility to cause confusion between the exponential and the unit of the algebra.

**Definition 1.1.3.** *Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$. Then* $\exp(x)$ *is defined by,*

$$\exp(x) = e + \sum_{n=1}^{+\infty} \frac{1}{n!} x^n.$$

**Theorem 1.1.2.** *Let $\mathcal{A}$ be a unital Banach algebra and $a, b \in \mathcal{A}$. If $ab = ba$ then* $\exp(a + b) = \exp(a)\exp(b)$ *and* $\exp(a)\exp(-a) = e$.

*Proof.* Let:

$$x_n = e + \sum_{k=1}^{n} \frac{1}{k!} a^k, \qquad y_n = e + \sum_{k=1}^{n} \frac{1}{k!} b^k,$$

$$z_n = e + \sum_{k=1}^{n} \frac{1}{k!} (a + b)^k, \qquad \xi_n = e + \sum_{k=1}^{n} \frac{1}{k!} ||a||^k,$$

$$\eta_n = e + \sum_{k=1}^{n} \frac{1}{k!} ||b||^k, \qquad \zeta_n = e + \sum_{k=1}^{n} \frac{1}{k!} (||a|| + ||b||)^k.$$

Notice that

$$x_ny_n - z_n = \sum_{j,k=1}^{n} \lambda_{j,k} a^j b^k,$$

where $\lambda_{j,k} \geq 0$, $\forall j,k$. Hence,

$$||x_ny_n - z_n|| \leq \sum_{j,k=1}^{n} \lambda_{j,k} ||a||^j ||b||^k = \xi_n\eta_n - \zeta_n.$$

On the other hand $\xi_n\eta_n - \zeta_n \to \exp(||a||)\exp(||b||) - \exp(||a|| + b) = 0$, as $n$ approaches infinity. Thus $\exp(a+b) = \exp(a)\exp(b)$ and due to the commutativity between an element and its inverse $\exp(a)\exp(-a) = e \implies \exp(a) \in \mathcal{G}(\mathcal{A})$. $\square$

**Theorem 1.1.3.** *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. Then $\exp(a) = \lim\limits_{n\to+\infty}(e + \frac{1}{n}a)^n$.*

*Proof.* Let $x_n = e + \sum_{k=1}^{n} \frac{1}{k!} a^k$, $y_n = (e + \frac{1}{n}a)^n$, $\xi_n = e + \sum_{k=1}^{n} \frac{1}{k!}||a||^k$ and $\eta_n = (e + \frac{1}{n}||a||)^n$. Expanding $y_n$ we get $x_n - y_n = \sum_{k=2}^{n} \lambda_k a^k$, with $\lambda_k \geq 0$, hence

$$||x_n - y_n|| \leq \sum_{k=2}^{n} \lambda_k ||a||^k = \xi_n - \eta_n.$$

On the other hand $\exp(||a||) = \lim\limits_{n\to+\infty}(1 + \frac{1}{n}||a||)^n \implies \xi_n - \zeta_n \to 0$. $\square$

**Definition 1.1.4.** *Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$. The resolvent set of $x$ is the set*

$$\rho(x) = \{\lambda \in \mathbb{C} : \lambda e - x \in \mathcal{G}(\mathcal{A})\}.$$

*The resolvent function of $x$ is defined as $r_\lambda = (\lambda e - x)^{-1}$, $\forall \lambda \in \rho(x)$. The spectrum of $x$ denoted $\sigma_{\mathcal{A}}(x)$ is the complement of the resolvent set of $x$*

$$\sigma_{\mathcal{A}}(x) = \mathbb{C} \setminus \rho(x) = \{\lambda \in \mathbb{C} : \lambda e - x \in \mathcal{A} \setminus \mathcal{G}(\mathcal{A})\}.$$

**Theorem 1.1.4.** *Let $\mathcal{A}$ be a unital Banach algebra, $x \in \mathcal{A}$ and $\lambda_0 \in \rho(x)$. If $|\lambda - \lambda_0| \leq \frac{1}{||r_{\lambda_0}||}$ then $\lambda \in \rho(x)$ and $r_\lambda = \sum_{n=0}^{+\infty}(-1)^n(\lambda - \lambda_0)^n r_{\lambda_0}^{n+1}$. Moreover $||r_\lambda - r_{\lambda_0}|| \leq \frac{|\lambda-\lambda_0|}{1-||r_{\lambda_0}|||\lambda-\lambda_0|}||r_{\lambda_0}||^2$.*

*Proof.* Let $x \in \mathcal{A}$, $\lambda_0 \in \rho(x) \implies \lambda_0 e - x \in \mathcal{G}(\mathcal{A})$ and $\lambda \in \mathbb{C}$ satisfying $|\lambda - \lambda_0| < \frac{1}{||r_{\lambda_0}||}$. Notice that, $||(\lambda e - x) - (\lambda_0 e - x)|| = ||(\lambda - \lambda_0)e|| = |\lambda - \lambda_0| < \frac{1}{||r_{\lambda_0}||}$. By Proposition 1.1.1. we have that $(\lambda e - x) \in \mathcal{G}(\mathcal{A})$. For an arbitrary $x \in \mathcal{A}$ and $x_0 \in \mathcal{G}(\mathcal{A})$ we have, $x = e - (x_0^{-1}(x_0 - x))$, replacing $x$ with $r_\lambda^{-1}$ and $x_0$ with $r_{\lambda_0}^{-1}$ we have $r_\lambda^{-1} = r_{\lambda_0}^{-1}(e - r_{\lambda_0}(r_{\lambda_0}^{-1} - r_\lambda^{-1}))$. On the other hand, $r_{\lambda_0}^{-1} - r_\lambda^{-1} = (\lambda_0 - \lambda)e$. Thus,

$$r_\lambda^{-1} = r_{\lambda_0}^{-1}(e - (\lambda_0 - \lambda)r_{\lambda_0}) \implies r_\lambda = (e - (\lambda_0 - \lambda)r_{\lambda_0})^{-1} r_{\lambda_0}.$$

Finally, by Theorem 1.1.1.,

$$r_\lambda = \sum_{n=0}^{+\infty} (-1)^n (\lambda - \lambda_0)^n r_{\lambda_0}^{n+1}$$

and $||r_\lambda - r_{\lambda_0}|| \leq \frac{|\lambda - \lambda_0|}{1 - ||r_{\lambda_0}|||\lambda - \lambda_0|} ||r_{\lambda_0}||^2$. □

**Remark 1.** *From the above it is implied that the resolvent set $\rho(x)$ is always topologically open, making the spectrum $\sigma_{\mathcal{A}}(x)$ always closed. In addition the resolvent function $\lambda \mapsto r_\lambda$ is continuous.*

The following proposition is crucial not only because it plays an important role on the proof of Gelfand-Buerling's Formula for the spectral radius, but it also implies that the spectrum of an element in a Banach algebra is always bounded, a fact we are going to use shortly to prove that the spectrum is always compact.

**Proposition 1.1.3.** *Let $\mathcal{A}$ be a unital Banach algebra, $x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. If $|\lambda| > \lim\limits_{n \to +\infty} ||x^n||^{1/n}$, then $\lambda \in \rho(x)$ and $r_\lambda = \sum_{n=0}^{+\infty} \frac{x^n}{\lambda^{n+1}}$. Moreover if $|\lambda| > ||x||$ then $||r_\lambda|| \leq \frac{1}{|\lambda|} \frac{1}{1 - ||\frac{x}{\lambda}||}$.*

*Proof.* Since $\lim\limits_{n \to +\infty} ||x^n||^{1/n} < |\lambda| \implies \lim\limits_{n \to +\infty} ||\frac{x^n}{\lambda^n}||^{1/n} \leq 1 \implies \sum_{n=0}^{+\infty} \frac{x^n}{\lambda^n}$ converges. Thus

$$(\lambda e - x) \sum_{n=0}^{+\infty} \frac{x^n}{\lambda^{n+1}} = \sum_{n=0}^{+\infty} \frac{x^n}{\lambda^n} - \sum_{n=0}^{+\infty} \frac{x^{n+1}}{\lambda^{n+1}} = \lim_{k \to +\infty} (e - \frac{x^{k+1}}{\lambda^{k+1}}) = e.$$

Similarly for the left and we have $r_\lambda = \sum_{n=0}^{+\infty} \frac{x^n}{\lambda^{n+1}}$. If $|\lambda| \geq ||x||$, then

$$||r_\lambda|| = ||\sum_{n=0}^{+\infty} \frac{x^n}{\lambda^{n+1}}|| \leq \sum_{n=0}^{+\infty} ||\frac{x^n}{\lambda^{n+1}}|| = \frac{1}{|\lambda|} \sum_{n=0}^{+\infty} ||\frac{x}{\lambda}||^n = \frac{1}{|\lambda|} \frac{1}{1 - ||\frac{x}{\lambda}||}.$$

□

**Lemma 1.1.3.** *Let $\lambda,\ \mu \in \rho(x)$ then $r_\lambda - r_\mu = -(\lambda-\mu)r_\lambda r_\mu$ and $r_\lambda$ is holomorphic in $\rho(x)$.*

*Proof.* Since $\lambda,\ \mu \in \rho(x)$ we have that $r_\lambda = r_\lambda e = r_\lambda(\mu e - x)r_\mu = r_\lambda(\mu e - \lambda e + \lambda e - x)r_\mu = r_\lambda(\mu-\lambda)r_\mu + r_\lambda(\lambda e - x)r_\mu = r_\lambda(\mu-\lambda)r_\mu + r_\mu \implies r_\lambda - r_\mu = -(\lambda-\mu)r_\lambda r_\mu$.
Now let $\phi \in \mathcal{A}^*$ and $f : \rho(x) \to \mathbb{C}$ defined as $f(\lambda) = \phi(r_\lambda)$. Hence

$$\frac{f(\lambda)-f(\mu)}{\lambda-\mu} = \frac{\phi(r_\lambda)-\phi(r_\mu)}{\lambda-\mu} = \phi(\frac{r_\lambda - r_\mu}{\lambda-\mu}).$$

By the previous identity, $\phi(\frac{r_\lambda - r_\mu}{\lambda-\mu}) = \phi(\frac{-(\lambda-\mu)r_\lambda r_\mu}{\lambda-\mu}) = -\phi(r_\lambda r_\mu) \to -\phi(r_\mu^2)$ as $\lambda$ approaches $\mu$. Therefore $f(\lambda) = \phi(r_\lambda)$ is holomorphic. $\square$

**Theorem 1.1.5.** *Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$. Then $\sigma_{\mathcal{A}}(x)$ is non-empty and compact.*

*Proof.* By the remark of theorem 1.1.4. we have that $\rho(x)$ is open and $\sigma_{\mathcal{A}}(x) = \mathbb{C} \setminus \rho(x)$, thus $\sigma_{\mathcal{A}}(x)$ is closed. By proposition 1.1.3. we have that for $\lambda \in \sigma_{\mathcal{A}}(x)$ the following holds, $|\lambda| \leq \lim\limits_{n\to+\infty} ||x^n||^{1/n}$, hence $\sigma_{\mathcal{A}}(x)$ is bounded. Therefore $\sigma_{\mathcal{A}}(x)$ is compact as a closed and bounded subset of $\mathbb{C}$.
In order to prove that $\sigma_{\mathcal{A}}(x)$ is non-empty we will use the function $f(\lambda)$ defined in Lemma 1.1.3. By Theorem 1.1.4. $r_\lambda$ is continuous on $\rho(x)$ and $\phi$ is bounded implying that $f$ is bounded as a continuous function on $\overline{B(0,||x||)}$ and due to Proposition 1.1.3. $||r_\lambda|| \leq \frac{1}{|\lambda|}\frac{1}{1-||\frac{x}{\lambda}||} < M$ whenever $|\lambda| > ||x||$, implying

$$|f(\lambda)| = |\phi(r_\lambda)| \leq ||\phi||_{\mathcal{A}^*}||r_\lambda|| \leq M||\phi||_{\mathcal{A}^*}.$$

Assume that $\sigma_{\mathcal{A}}(x) = \emptyset \implies \rho(x) = \mathbb{C} \implies f$ is holomorphic and bounded on $\mathbb{C}$, thus by Liouville's Theorem $f$ is constant. On the other hand by Proposition 1.1.3. $f(\lambda) \to 0$ as $|\lambda|$ approaches infinity, making $f(\lambda) = 0$, $\forall \lambda \in \mathbb{C} \iff \phi(r_\lambda) = 0$.
Notice that $\phi$ was chosen arbitrarily thus $\forall \phi \in \mathcal{A}^*$, $\phi(r_\lambda) = 0 \implies r_\lambda = 0$, which is a contradiction because 0 cannot be the inverse of an element. $\square$

Favorably we are in the position of giving the first characterization theorem about unital Banach algebras. Gelfand-Mazur theorem partially stated by Mazur and proven by Gelfand states that every algebra $\mathcal{A}$ in the class of division algebras i.e. $\mathcal{A} = \mathcal{G}(\mathcal{A}) \cup \{0\}$ is isometrically isomorphic to $\mathbb{C}$.

**Theorem 1.1.6** (Gelfand-Mazur)**.** *Let $\mathcal{A}$ be a unital Banach algebra. If $\mathcal{G}(\mathcal{A})$ contains all non-zero elements of $\mathcal{A}$, then $\mathcal{A}$ is isometrically isomorphic to $\mathbb{C}$.*

*Proof.* Let $x \in \mathcal{A} \setminus \{0\}$ such that $\lambda e - x \neq 0$, $\forall \lambda \in \mathbb{C}$. By the hypotheses $\lambda e - x \in \mathcal{G}(\mathcal{A})$, $\forall \lambda \in \mathbb{C} \implies \rho(x) = \mathbb{C} \implies \sigma_{\mathcal{A}}(x) = \emptyset$, which is a contradiction due to Theorem 1.1.5.. Hence $\mathcal{A} = \{\lambda e : \lambda \in \mathbb{C}\}$. □

Our next step is to give the definition of the spectral radius. We will soon find out the importance of this term since it plays a vital role on all of the characterization results which will be presented in this Master's thesis.

**Definition 1.1.5.** *Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$. The spectral radius of $x$ denoted $|x|_\sigma$ is defined as*

$$|x|_\sigma = \sup\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(x)\}.$$

**Proposition 1.1.4.** *Let $\mathcal{A}$ be a unital Banach algebra. For any $x, y \in \mathcal{A}$ the following holds*

$$\sigma_{\mathcal{A}}(xy) \cup \{0\} = \sigma_{\mathcal{A}}(yx) \cup \{0\}.$$

*Proof.* Assume $x, y \in \mathcal{A}$ such that $e - xy \in \mathcal{G}(\mathcal{A})$. Put $a = (e - xy)^{-1}$ and $b = e + yax$, observe that $b$ is the inverse of $(e - xy)$, since

$$b(e-yx) = (e+yax)(e-yx) = e-yx+yax-yaxyx = e-yx+ya(e-xy)x =$$

$= e - yx + yx = e$. Now suppose $\lambda \in \mathbb{C} \setminus \{0\}$, thanks to the observation above we have that

$$\lambda e-xy \in \mathcal{G}(\mathcal{A}) \iff \lambda(e-\frac{xy}{\lambda}) \in \mathcal{G}(\mathcal{A}) \iff \lambda(e-\frac{yx}{\lambda}) \in \mathcal{G}(\mathcal{A}) \iff \lambda e-yx \in \mathcal{G}(\mathcal{A}).$$

Hence $\sigma_{\mathcal{A}}(xy) \cup \{0\} = \sigma_{\mathcal{A}}(yx) \cup \{0\}$. □

**Corollary 1.1.2.** *For any $x, y \in \mathcal{A}$, we have $|xy|_\sigma = |yx|_\sigma$.*

**Theorem 1.1.7** (Gelfand-Buerling)**.** *Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$. Then*

$$|x|_\sigma = \lim_{n \to +\infty} ||x^n||^{1/n}.$$

*Proof.* Let $x \in \mathcal{A}$, by Proposition 1.1.3. we have that if $|\lambda| > \lim_{n\to+\infty} ||x^n||^{1/n}$ then $\lambda \in \rho(x)$, thus $|x|_\sigma \leq \lim_{n\to+\infty} ||x^n||^{1/n} = R$. Now let $\mu \in \mathbb{C}$ satisfying $|x|_\sigma < |\mu| < R$ and $\phi \in \mathcal{A}^*$, setting $f(\lambda) = \phi(r_\lambda)$, $\lambda \in \rho(x)$, we have that $f$ is holomorphic and againby Proposition 1.1.3. if $|\lambda| > ||x||$ then $r_\lambda = \sum_{n=0}^{+\infty} \frac{x^n}{\lambda^{n+1}}$. Hence

$$f(\lambda) = \phi(r_\lambda) = \phi(\sum_{n=0}^{+\infty} \frac{x^n}{\lambda^{n+1}}) = \sum_{n=0}^{+\infty} \frac{\phi(x^n)}{\lambda^{n+1}}.$$

Observe that the above is the Laurent expansion of $f$ for $|\lambda| \geq ||x||$. On the other hand f is analytic whenever $|\lambda| > |x|_\sigma$ and due to the uniqueness of the Laurent expansion we have $f(\lambda) = \sum_{n=0}^{+\infty} \frac{\phi(x^n)}{\lambda^{n+1}}$, $\forall \lambda > |x|_\sigma \implies \sum_{n=0}^{+\infty} \frac{\phi(x^n)}{\lambda^{n+1}}$ converges for $\lambda = \mu$ which implies

$$\lim_{n\to+\infty} \frac{\phi(x^n)}{\mu^{n+1}} = 0 \implies |\frac{\phi(x^n)}{\mu^{n+1}}| \leq M.$$

The sequence $\{\frac{\phi(x^n)}{\mu^{n+1}}\}$ defines a sequence of operators $\{T_n\}_{n\in\mathbb{N}} : \mathcal{A} \to \mathbb{C}$ as such $T_n(\phi) = \frac{\phi(x^n)}{\mu^{n+1}}$, $\forall \phi \in \mathcal{A}^*$. As mentioned above $\{T_n(\phi)\}_{n\in\mathbb{N}}$ is bounded $\forall \phi \in \mathcal{A}^*$, thus by Banach-Steinhaus Theorem there exists a constant $c > 0$ such that

$$||T_n|| = ||\frac{x^n}{\mu^{n+1}}|| \leq c \implies ||x^n|| \leq |\mu^{n+1}|c \implies ||x^n||^{1/n} \leq |\mu||\mu|^{1/n}c.$$

Letting $n \to +\infty$ we have $R \leq |\mu| < R$, which is a contradiction. $\square$

## 1.2 Functional Calculus

In this section we will provide the Holomorphic Functional Calculus, a powerful tool in order to continue. Our prime concern here are functions between a Banach algebra $\mathcal{A}$ and complex numbers. We will start with the Spectral Mapping Theorem for polynomials and taking advantage of the property of the holomorphic functions to be represented locally as a Taylor series we will conclude with the Spectral Mapping Theorem for holomorphic functions. In this thesis the entire presentation of the Functional Calculus is not the main subject, thus some proofs will be omitted. An excessive presentation can be found in [**1**].

**Definition 1.2.1.** *Let $\mathcal{A}$ be a unital Banach algebra. Then $x \in \mathcal{A}$ is said to be right-invertible if there exists an element $y \in \mathcal{A}$ satisfying $xy = e$ and respectively left-invertible if $hx = e$.*

It follows easily that if $x$ is both left and right invertible then $x \in \mathcal{G}(\mathcal{A})$, since if $h, k$ respectively the left and right inverse we have $h = he = h(xk) = (hx)k = ek = k$.

**Lemma 1.2.1.** *Let $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{C}$ and $x \in \mathcal{A}$. Then $\prod_{i=1}^{n}(\lambda_i e - x) \in \mathcal{G}(\mathcal{A})$ if and only if $\lambda_i e - x \in \mathcal{G}(\mathcal{A})$, $\forall i = 1, ..., n$.*

*Proof.* Let $\lambda_i e - x \in \mathcal{G}(\mathcal{A})$, $\forall i = 1, ..., n$. As mentioned before $\mathcal{G}(\mathcal{A})$ is a group implying $\prod_{i=1}^{n}(\lambda_i e - x) \in \mathcal{G}(\mathcal{A})$.

For the opposite direction, let $j = 1, ..., n$ and $(\lambda_j e - x) \in \mathcal{A} \setminus \mathcal{G}(\mathcal{A})$. Assume $\prod_{i=1}^{n}(\lambda_i e - x) \in \mathcal{G}(\mathcal{A})$ and $S$ its inverse. Notice that for every $\lambda_i$, $(\lambda_i e - x)$, $(\lambda_{i+1} e - x)$ commute. Hence,

$$S \cdot \prod_{i \neq j}(\lambda_i e - x)(\lambda_j e - x) = e$$

making $S \cdot \prod_{i \neq j}(\lambda_i e - x)$ the left inverse of $(\lambda_j e - x)$. Similarly for right invertibility. Finally $\lambda_j$ was chosen arbitrarily hence $(\lambda_j e - x) \in \mathcal{G}(\mathcal{A})$, $\forall j = 1, 2, ..., n$. $\square$

**Theorem 1.2.1.** *Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$. For any polynomial $p$ the following holds*

$$p(\sigma_{\mathcal{A}}(x)) = \sigma_{\mathcal{A}}(p(x)).$$

*Proof.* Let $p$ be an n-degree polynomial and $\lambda \in \sigma_{\mathcal{A}}(x)$, then $q(z) = p(\lambda) - p(z)$ is also an n-degree polynomial and $q(\lambda) = 0$. By The Fundamental Theorem of Algebra we have,

$$p(z) = p(\lambda) - p(z) = c(\lambda - z)(\lambda_2 - z)...(\lambda_n - z),$$

implying $p(\lambda)e - p(z) = c(\lambda e - x)(\lambda_2 e - x)...(\lambda_n e - x)$. Suppose that $\lambda \in \sigma_{\mathcal{A}}(x)$, then $\lambda e - x \notin \mathcal{G}(\mathcal{A})$ and by Lemma 1.2.1 $p(\lambda)e - p(z) \notin \mathcal{G}(\mathcal{A}) \implies p(\lambda) \in \sigma_{\mathcal{A}}(p(x))$.

For the opposite inclusion let $\mu \in \sigma_{\mathcal{A}}(p(x))$. We have that $\mu - p(z) = c(\lambda_1 - z)...(\lambda_n - z) \implies \mu e - p(x) = c(\lambda_1 e - x)...(\lambda_n e - x)$. Note that $\mu \in \sigma_{\mathcal{A}}(p(x)) \implies \mu e - p(x) \notin \mathcal{G}(\mathcal{A}) \implies c(\lambda_1 e - x)...(\lambda_n e - x) \notin \mathcal{G}(\mathcal{A})$. Thus by Lemma 1.2.1. there exists $\lambda_j$ such that $(\lambda_j e - x) \notin \mathcal{G}(\mathcal{A}) \implies \lambda_j \in \sigma_{\mathcal{A}}(x)$. On the other hand,

$$\mu - p(\lambda_j) = 0 \implies \mu = p(\lambda_j) \implies \sigma_{\mathcal{A}}(p(x)) \subseteq p(\sigma_{\mathcal{A}}(x)).$$

□

**Definition 1.2.2.** *Let $\mathcal{X}$ a Banach space, $f : [a,b] \to \mathcal{X}$ and $\alpha : [a,b] \to \mathbb{C}$. If $p = \{a = t_0 < t_1 < ... < t_n = b\}$ a partition of the interval $[a,b]$ then the Riemann-Stietjes sum of $f$ on $p$ is given by*

$$S(\alpha, f, p) = \sum_{i=1}^{n} f(\xi_i)|\alpha(t_{i-1}) - \alpha(t_i)|$$

*where $\xi_i \in [t_{i-1}, t_i]$.*

*The element $y = \int_a^b f(t)d\alpha(t) \in \mathcal{X}$ is the Riemann-Stieltjes Integral of $f$ if $\forall \epsilon > 0$ there is a partition $p_\epsilon$ such that for every Riemann-Stieltjes sum and for any refinement $p$ of $p_\epsilon$ we have $||S(\alpha, f, p) - y|| < \epsilon$.*

**Definition 1.2.3.** *Let $\mathcal{X}$ a Banach space, $U$ an open subset of $\mathbb{C}$ and $c$ a rectifiable curve contained in $U$. The line integral of a continuous function $f : U \to \mathcal{X}$ over $c$ is given by*

$$\int_c f(\lambda)d\lambda = \int_a^b f(\alpha(t))d\alpha(t)$$

*where $\alpha(t)$ is a continuous function of bounded variation.*

The next theorem is a generalization of Cauchy's Integral Theorem on Banach Algebras.

**Theorem 1.2.2** (Cauchy's Integral Theorem)**.** *Let $\mathcal{A}$ be a Banach algebra, $c$ a simple closed rectifiable curve and $U \subseteq \mathbb{C}$ an open set containing $c$ and its interior. If $f : U \to \mathcal{A}$ is holomorphic then*

$$\int_c f(\lambda)d\lambda = 0$$

*and for every $\lambda$ in the interior of $c$,*

$$f(\lambda) = \frac{1}{2\pi i}\int_c \frac{f(\zeta)}{\zeta - \lambda}d\zeta.$$

**Lemma 1.2.2.** *Let $\mathcal{A}$ a unital Banach algebra and $\sum_{n=0}^{+\infty} a_n\lambda^n$ a complex power series with convergence radius $r = \frac{1}{\limsup|a_n|^{1/n}}$. If $x \in \mathcal{A}$ satisfying $|x|_\sigma < r$, Then the series $\sum_{n=0}^{+\infty} a_n x_n$ converges in $\mathcal{A}$.*

*Proof.* It follows immediately that

$$\limsup||a_n x^n||^{1/n} \leq \limsup|a_n|^{1/n} \limsup||x^n||^{1/n} = \frac{1}{r}|x|_\sigma < 1.$$

Thus $\sum_{n=0}^{+\infty} a_n x_n$ converges. □

**Proposition 1.2.1.** *Let $\mathcal{A}$ a unital Banach algebra, $x \in \mathcal{A}$ and $f$ holomorphic on $\mathcal{B}(0, r)$. If $|x|_\sigma < r$ then,*

$$f(x) = \frac{1}{2\pi i}\int_c f(\zeta) r_\zeta d\zeta.$$

*Proof.* Let $x \in \mathcal{A}$ then by Cauchy's Integral Theorem we have,

$$\frac{1}{2\pi i}\int_c f(\zeta) r_\zeta d\zeta = \frac{1}{2\pi i}\int_c \sum_{n=0}^{+\infty} a_n \zeta^n r_\zeta d\zeta = \frac{1}{2\pi i}\sum_{n=0}^{+\infty} a_n \int_c \zeta^n r_\zeta d\zeta = \sum_{n=0}^{+\infty} a_n x^n =$$

$= f(x)$. □

**Definition 1.2.4.** *Let $S$ be an open subset of $\mathbb{C}$. The set $S$ will be called a Cauchy domain if it has a finite number of connected components $S_n$ satisfying $\overline{S_i} \cap \overline{S_j} = \emptyset$, for every $i \neq j$ and the boundary of $S$, $\partial S$ is in the form $\partial S = c_1 \cup c_2 \cup ... \cup c_n$, where $c_i$ are simple, closed and rectifiable curves satisfying $c_i \cap c_j = \emptyset$, for every $i \neq j$.*

**Lemma 1.2.3.** *Let $K$ be a compact subset of $\mathbb{C}$ and $U$ an open set of $\mathbb{C}$ with $K \subset U$, then there exists a Cauchy domain $S$ satisfying $K \subseteq S \subseteq \overline{S} \subseteq U$.*

*Proof.* Since $K$ is compact and $K \subset U$, the distance $r$ between $K$ and $\mathbb{C} \setminus U$ will be strictly positive. Let $\{B_i : i \in I\}$ a cover of $K$ consisting of disks with radius $r/2$ and center an element of $K$. Due to the compactness of $K$ once again there exists a finite subcover $\{D_i : i = 1, ..., n\}$. Put $S = \cup_{i=1}^{n} D_i$, then $S$ is open, has a finite number of connected components and contains $K$. In addition the radius of $D_i$ is strictly smaller than $r$, hence $\overline{S} \subseteq U$. Finally the boundary $\partial S$ consists of circular arcs making $S$ a Cauchy domain. □

Observe that a Cauchy domain is always open, thus by the above lemma we have that for every Cauchy domain $S$ there is always another Cauchy domain $S_1$ satisfying $S_1 \subseteq S$.

**Definition 1.2.5.** *Let $\mathcal{A}$ be a unital Banach algebra, $U \subseteq \mathbb{C}$ open and $x \in \mathcal{A}$ satisfying $\sigma_{\mathcal{A}}(x) \subseteq U$. By the previous lemma there exists a Cauchy domain such that $\sigma_{\mathcal{A}}(x) \subseteq S \subseteq \overline{S} \subseteq U$. For any holomorphic $f : U \to \mathbb{C}$, the element $f(x)$ is given by,*

$$f(x) = \frac{1}{2\pi i} \int_{\partial S} f(\zeta) r_\zeta d\zeta.$$

**Theorem 1.2.3.** *Let $\mathcal{A}$ be a unital Banach algebra, $U$ an open subset of $\mathbb{C}$ and $x \in \mathcal{A}$ satisfying $\sigma_{\mathcal{A}}(x) \subseteq U$. For any holomorphic $f, g : U \to \mathbb{C}$ we have $(fg)(x) = f(x)g(x)$.*

*Proof.* Since $\sigma_{\mathcal{A}}(x)$ is compact we have by lemma 1.2.3. that there exists a Cauchy domain $S_1$ satisfying $\sigma_{\mathcal{A}}(x) \subseteq S_1 \subseteq \overline{S_1} \subseteq U$ and

$$g(x) = \frac{1}{2\pi i} \int_{\partial S_1} g(\lambda) r_\lambda d\lambda.$$

Similarly for $f$ there exists a Cauchy domain $S_0$ satisfying $\sigma_{\mathcal{A}}(x) \subseteq S_0 \subseteq S_1$ and

$$f(x) = \frac{1}{2\pi i} \int_{\partial S_0} f(\lambda) r_\lambda d\lambda.$$

Hence,

$$f(x)g(x) = \frac{1}{2\pi i} \int_{\partial S_0} f(\lambda) r_\lambda d\lambda \frac{1}{2\pi i} \int_{\partial S_1} g(\mu) r_\mu d\mu =$$

$$= \frac{1}{2\pi i} \int_{\partial S_0} f(\lambda) [\frac{1}{2\pi i} \int_{\partial S_1} g(\mu) r_\lambda r_\mu d\mu] d\lambda. \qquad (1)$$

By Lemma 1.1.3. we have that $r_\lambda - r_\mu = (\mu - \lambda) r_\lambda r_\mu \implies r_\lambda r_\mu = \frac{r_\lambda}{\mu - \lambda} + \frac{r_\mu}{\lambda - \mu}$. Considering this (1) implies that

$$f(x)g(x) = \frac{1}{2\pi i} \int_{\partial S_0} f(\lambda) r_\lambda [\frac{1}{2\pi i} \int_{\partial S_1} \frac{g(\mu)}{\mu - \lambda} r_\lambda d\mu + \frac{1}{2\pi i} \int_{\partial S_1} \frac{g(\mu)}{\mu - \lambda} r_\mu d\mu] d\lambda.$$

On the other hand $\lambda \in \partial S_0 \implies \lambda \in S_1$ and thus by Cauchy's Integral Formula

$$\frac{1}{2\pi i} \int_{\partial S_1} \frac{g(\mu)}{\mu - \lambda} d\mu = g(\lambda).$$

Since $\frac{f(\lambda)}{-\mu+\lambda}$ is holomorphic in $S_0$ and $\mu \in \mathbb{C} \setminus \overline{S_0}$, we have

$$\frac{1}{2\pi i} \int_{\partial S_0} f(\lambda)[\frac{1}{2\pi i} \int_{\partial S_1} \frac{g(\mu)}{\mu - \lambda} r_\mu d\mu] d\lambda = 0.$$

Finally:

$$f(x)g(x) = \frac{1}{2\pi i} \int_{\partial S_0} f(\lambda)g(\lambda) r_\lambda d\lambda = \frac{1}{2\pi i} \int_{\partial S_0} (fg)(\lambda) r_\lambda d\lambda = (fg)(x).$$

□

**Corollary 1.2.1.** *Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$. Then for any holomorphic $f$ satisfying $f(\lambda) \neq 0$, $\forall \lambda \in \sigma_{\mathcal{A}}(x)$ we have $f(x) \in \mathcal{G}(\mathcal{A})$ and $f^{-1}(x) = (\frac{1}{f})(x)$.*

**Theorem 1.2.4** (Spectral Mapping Theorem)**.** *Let $\mathcal{A}$ be a unital Banach algebra $U \subseteq \mathbb{C}$ and $x \in \mathcal{A}$ satisfying $\sigma_{\mathcal{A}}(x) \subseteq U$. Then for any holomorphic function $f : U \to \mathbb{C}$ the following holds,*

$$\sigma_{\mathcal{A}}(f(x)) = f(\sigma_{\mathcal{A}}(x)).$$

## 1.3 Gelfand's Theory

In this section we will focus on algebra homomorphisms, we will give some general results on Banach algebras and study the interplay between multiplicative algebra homomorphisms and maximal ideals. We shall also give the definition of Gelfand's Transformation and have a word on semi-simple Banach algebras. For additional information on semi-simple Banach algebras and automatic continuity matters which arise from some of the propositions provided, one can look in [**3**].

**Definition 1.3.1.** *Let $\mathcal{A}$ be a Banach algebra. A function $\phi : \mathcal{A} \to \mathbb{C}$ is called algebra homomorphism if for any $x, y \in \mathcal{A}$ we have $\phi(xy) = \phi(x)\phi(y)$. Moreover if $\phi \in \mathcal{A}^*$, it will be called multiplicative linear functional or character.*

**Definition 1.3.2.** *Let $\mathcal{A}$ be a Banach algebra. An ideal $J$ is said to be maximal if it is proper and not contained as a proper subset in any other proper ideal.*

**Proposition 1.3.1.** *Let $\mathcal{A}$ be a Banach algebra and $\phi$ be a character. Then $\phi(e) = 1$, $\phi(x) \neq 0$ for any $x \in \mathcal{G}(\mathcal{A})$ and $|\phi(x)| < 1$ for any $x \in \mathcal{A}$ satisfying $||x|| < 1$.*

*Proof.* Since $\phi \neq 0$ there is an element $y \in \mathcal{A}$ such that, $\phi(y) = \phi(ye) = \phi(y)\phi(e) \implies \phi(e) = 1$. Now let $x \in \mathcal{G}(\mathcal{A})$, $1 = \phi(e) = \phi(xx^{-1}) \implies \phi(x) \neq 0$.

Let $\lambda \in \mathbb{C}$ satisfying $|\lambda| \geq 1$. Since $||x|| < 1 \implies ||\frac{x}{\lambda}|| < 1$ and by Theorem 1.1.1. we have $(e - \frac{x}{\lambda}) \in \mathcal{G}(\mathcal{A})$ which implies the following

$$\phi(e - \frac{x}{\lambda}) \neq 0 \implies \phi(e) - \frac{1}{\lambda}\phi(x) \neq 0 \implies \phi(x) \neq \lambda \implies |\phi(x)| < 1.$$

The last also implies that every character is continuous. □

**Proposition 1.3.2.** *Let $\mathcal{A}$ be a Banach algebra. If $J$ is a maximal ideal then it is closed.*

*Proof.* Let $J$ be a maximal ideal, then it cannot contain any invertible elements because we would have $J = \mathcal{A}$, hence $J \subseteq \mathcal{A} \setminus \mathcal{G}(\mathcal{A})$. Since $\mathcal{G}(\mathcal{A})$ is open we have that $\mathcal{A} \setminus \mathcal{G}(\mathcal{A})$ is closed and as a result $J \subseteq \overline{J} \subseteq \mathcal{A} \setminus \mathcal{G}(\mathcal{A})$. On the other hand $J \subseteq \overline{J} \neq \mathcal{A}$ so $\overline{J}$ is a proper ideal containing $J$, thus $J = \overline{J} \implies J$ is closed. □

The following theorem is a corollary of the Gelfand-Mazur Theorem and perfectly outlines the interplay between maximal ideals and characters in a commutative Banach algebra.

**Theorem 1.3.1** (Gelfand-Mazur)**.** *Let $\mathcal{A}$ be a commutative unital Banach algebra. Then for any maximal ideal $J$ in $\mathcal{A}$ there exists a character $\phi$ such that $\ker\phi = J$ and if $\phi$ is a character then there exists a maximal ideal $J$ in $\mathcal{A}$ satisfying $\ker\phi = J$.*

*Proof.* Assume that $\phi \neq 0$ is a character then $\ker\phi = J$, we will prove that $J$ is maximal. Since $\phi \neq 0$ we have that $J \neq \mathcal{A}$. Let $a \notin J$ then any $b \in \mathcal{A}$ can be written as

$$b = a\frac{\phi(b)}{\phi(a)} + (b - a\frac{\phi(b)}{\phi(a)}).$$

Observe that $b - a\frac{\phi(b)}{\phi(a)} \in \ker\phi$ thus $\mathcal{A} = \text{span}\{a\} + J \implies J$ is a maximal ideal.

Now let $J$ be a maximal ideal by Proposition 1.3.2. $J$ is closed. We will prove that $\mathcal{A}/J$ is a division Banach algebra, assume that $a+J \in \mathcal{A}/J$ and $a+J \in \mathcal{G}(\mathcal{A}/J)$, then $J+aA$ is a proper ideal containing $J$ as a proper subset implying that $J$ is not maximal, which is a contradiction. Thus $\mathcal{A}/J$ is a division algebra and by the Gelfand-Mazur theorem there is an isomorphism $\phi : \mathcal{A}/J \to \mathbb{A}$ such that $\phi(x+J) = \lambda(e+J)$. Let $\pi : \mathcal{A} \to \mathcal{A}/J$ denote the canonical projection. Then for $a, b \in \mathcal{A}$ we have

$$\phi \circ \pi(ab) = \phi(\pi(ab)) = \phi((a+J)(b+J)) = \phi(a+J)\phi(b+J) = \phi \circ \pi(a)\phi \circ \pi(b)$$

and $\phi \circ \pi(a) = 0 \iff \pi(a) = 0 \iff a \in J$. Thus $\phi \circ \pi$ is a character with kernel equal to $J$. □

Even though the assumption of commutativity was never used in the proof it is still necessary, because it ensures us that characters exist in $\mathcal{A}$. Non-commutative Banach algebras do not necessarily posses characters.

**Definition 1.3.3.** *Let $\mathcal{A}$ be a commutative unital Banach algebra. The set of characters of $\mathcal{A}$ is called the structure space (or spectrum) of $\mathcal{A}$ and denoted as* $\operatorname{Sp}\mathcal{A}$.

**Theorem 1.3.2.** *Let $\mathcal{A}$ be a commutative unital Banach algebra. For $x \in \mathcal{A}$ and $\phi \in \operatorname{Sp}\mathcal{A}$ define $\widehat{x} : \operatorname{Sp}\mathcal{A} \to \mathbb{C}$ as*

$$\widehat{x}(\phi) = \phi(x).$$

*Then $\widehat{x}(\operatorname{Sp}\mathcal{A}) = \sigma_{\mathcal{A}}(x)$ and the Gelfand Transform $\widehat{\cdot} : \mathcal{A} \to \mathcal{C}(\operatorname{Sp}\mathcal{A})$ is a homomorphism satisfying $||\widehat{x}||_\infty \leq ||x||$.*

*Proof.* Let $x \in \mathcal{A}$ and $\phi \in \operatorname{Sp}\mathcal{A}$ then by Proposition 1.3.2. we have $\phi(x)e - x \in \ker\phi = J$ where $J$ is a maximal ideal and as a result it does not contain any invertible elements. Then $\phi(x)e - x \in \mathcal{A} \setminus \mathcal{G}(\mathcal{A}) \iff \phi(x) \in \sigma_{\mathcal{A}}(x)$ which means that $\widehat{x}(\operatorname{Sp}\mathcal{A}) \subseteq \sigma_{\mathcal{A}}(x)$.

For the opposite inclusion let $\lambda \in \sigma_{\mathcal{A}}(x) \iff \lambda e - x \in \mathcal{A} \setminus \in A$ and so it is contained in a maximal ideal $J$. By Proposition 1.3.2. we have that there exists a $\phi \in \operatorname{Sp}\mathcal{A}$ such that $\ker\phi = J$ and it is implied that $\lambda e - x \in \ker\phi \implies \phi(x) = \lambda$.

The map $\widehat{\cdot}$ is clearly a homomorphism for

$$\widehat{xy}(\phi) = \phi(xy) = \phi(x)\phi(y) = \widehat{x}(\phi)\widehat{y}(\phi),$$

for any $x, y \in \mathcal{A}$, $\phi \in \operatorname{Sp}\mathcal{A}$.

To prove that $\widehat{x} \in \mathcal{C}(\operatorname{Sp}\mathcal{A})$ it suffices to show that for any open $U \subseteq \mathbb{C}$, $\widehat{x}^{-1}(U)$ is open. Let $\phi \in \widehat{x}^{-1}(U)$, then there is a $z \in U$ such that $\widehat{x}(\phi) = z$ and since $U$ is open there exists a positive number $\epsilon$ such that $U_\epsilon = \{\lambda \in \mathbb{C} : |\lambda - z| < \epsilon\} \subseteq U$. Now let $N = N(\phi, x, \epsilon) = \psi \in \operatorname{Sp}\mathcal{A} : |\psi(x) - \phi(x)| < \epsilon$. Hence, $\psi(x) = \widehat{x}(\psi) \in U$ for any $\psi \in N$, which implies that $\widehat{x}^{-1}(U)$ is open and $\widehat{x} \in \mathcal{C}(\operatorname{Sp}\mathcal{A})$. $\square$

We can now establish a remarkable connection with a little help from the following proposition.

**Proposition 1.3.3.** *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{B}$ is a subalgebra and $x \in \mathcal{B}$, then $\sigma_{\mathcal{A}}(x) \subseteq \sigma_{\mathcal{B}}(x) \cup \{0\}$. In addition if $\mathcal{B}$ is maximal and commutative $\sigma_{\mathcal{A}}(x) \cup \{0\} = \sigma_{\mathcal{B}}(x) \cup \{0\}$.*

*Proof.* Let $x \in \mathcal{B}$ and $\lambda \notin \sigma_{\mathcal{A}}(x) \implies \lambda e - x \notin \mathcal{G}(\mathcal{A}) \implies \lambda e - x \notin \mathcal{G}(\mathcal{B})$.

Now let $\mathcal{B}$ be maximal and commutative, it suffices to show $\sigma_{\mathcal{B}}(x) \subseteq \sigma_{\mathcal{A}}(x) \cup \{0\}$. Assume $\lambda \notin \sigma_{\mathcal{A}}(x) \cup \{0\}$ then $y = \lambda e - x \in \mathcal{G}(\mathcal{A})$ which implies that there exists a $z \in \mathcal{A}$ such that $yz = zy = e$. Note that it suffices to show that $z \in \mathcal{B}$. Since $\mathcal{B}$ is commutative we have $yw = wy$ for any $w \in \mathcal{B}$. Therefore,

$$zw = zwe = zwyz = zywz = wz \implies z \in \mathcal{B}.$$

$\square$

Taking advantage of the invariance of the spectrum when it comes to restricting on a maximal commutative subalgebra and the fact that the range of the Gelfand Transform is the spectrum we can give a new definition of it as such, for any $x \in \mathcal{A}$, $\sigma_{\mathcal{A}}(x) = \sigma_{\mathcal{B}}(x) = \{\phi(x) : \phi \in \operatorname{Sp}\mathcal{B}\}$.

Next we will have a word on the radical of $\mathcal{A}$ and semi-simple Banach algebras.

**Theorem 1.3.3.** *Let $\mathcal{A}$ be a ring with unit $e$. Then the following are identical:*

*(a) The intersection of all maximal left ideals.*
*(b) The intersection of all maximal right ideals.*
*(c) the set of $x$ such that $e - zx \in \mathcal{G}(\mathcal{A})$, for any $z \in \mathcal{A}$.*
*(d) the set of $x$ such that $e - xz \in \mathcal{G}(\mathcal{A})$, for any $z \in \mathcal{A}$.*

*Proof.* We will only prove the equivalence between (a) and (c). The proof is the same for (b) $\iff$ (d), and Proposition 1.1.4. completes the circle proving (c) $\iff$ (d).

Let $x$ be contained in the intersection of all maximal left ideals of $\mathcal{A}$. If $a = e - zx \notin \mathcal{G}(\mathcal{A})$, then $\mathcal{A}a$ is a left ideal of $\mathcal{A}$ and is contained in some maximal left ideal $L_0$. Then $zx \in L_0$ and $e - zx \in L_0$, thus $e \in L_0$, which means $L_0 = \mathcal{A}$.

Now assume that $e - zx \in \mathcal{G}(\mathcal{A})$ for any $z \in \mathcal{A}$. If $x$ is not contained in the intersection of all maximal left ideals it means there exists a maximal left ideal $L_0$ such that $x \notin L_0$. As a result $L_0 + \mathcal{A}x = \mathcal{A}$ and therefore $e - zx \in L_0$ which again implies $e \in L_0$ which is a contradiction. $\square$

**Definition 1.3.4.** *Let $\mathcal{A}$ be a Banach algebra. The two-sided ideal satisfying the properties of Theorem 1.3.3., is called the radical of $\mathcal{A}$ and denoted as* $\operatorname{Rad} \mathcal{A}$. *If* $\operatorname{Rad} \mathcal{A} = \{0\}$ *we say that $\mathcal{A}$ is semi-simple.*

Remember that for a closed two-sided ideal of $\mathcal{A}$, $\mathcal{A}/I$ is a Banach algebra for the norm $||\dot{x}|| = \inf_{u \in I} ||x + u||$, where $\dot{x} = x + I$.

**Theorem 1.3.4.** *Let $\mathcal{A}$ be a unital Banach algebra. Then* $\mathcal{A}/\operatorname{Rad} \mathcal{A}$ *is semi-simple.*

*Proof.* Let $L'$ be a maximal left ideal of $\mathcal{A}/\operatorname{Rad} \mathcal{A}$. Then $L = \{x : x \in \mathcal{A}, \dot{x} \in L'\}$ is a left ideal of $\mathcal{A}$. Also $L$ is maximal because if $J$ is a left ideal such that $L \subset J$ then $L' \subset J' = \{\dot{x} : x \in J\}$ and as a result $L' = J'$ and thus $L = J$. Conversely if $L$ is a maximal left ideal of $\mathcal{A}$ then $L'$ is a maximal left ideal of $\mathcal{A}/\operatorname{Rad}$ for the same reason.
By Theorem 1.3.3. (a) we have that if $\{x : \dot{x} \in \operatorname{Rad}(\mathcal{A}/\operatorname{Rad})\}$ is contained in the intersection of all maximal left ideals of $\mathcal{A}$ then it is the radical of $\mathcal{A}$ and as a result $\mathcal{A}/\operatorname{Rad}$ is semi-simple. $\square$

**Proposition 1.3.4.** *Let $T$ be a linear operator from a commutative Banach algebra $\mathcal{A}$ into a semi-simple commutative Banach algebra $\mathcal{B}$. Then $T$ is continuous.*

*Proof.* Let us assume that $\{x_n\}_{n \in \mathbb{N}} \in \mathcal{A}$ with $x_n \to 0$ and $Tx_n \to a \in \mathcal{B}$. It suffices to show that $a = 0$. Let $\phi \in \operatorname{Sp} \mathcal{B}$, then $\phi \circ T \in \operatorname{Sp} \mathcal{A}$ and by Proposition 1.3.1. it is continuous. Hence the following is deduced

$$\phi(a) = \lim_{n \to +\infty} \phi(Tx_n) = \lim_{n \to +\infty} (\phi \circ T)(x_n) = 0.$$

Thus $a \in \operatorname{Rad} \mathcal{B} = \{0\}$. $\square$

**Corollary 1.3.1.** *On a semi-simple commutative Banach algebra $\mathcal{A}$, all Banach algebra-norms are equivalent.*

*Proof.* Let $||\cdot||_1, ||\cdot||_2$ two Banach algebra-norms on $\mathcal{A}$. Applying the previous corollary to the identity mapping from $(\mathcal{A}, ||\cdot||_1)$ to $(\mathcal{A}, ||\cdot||_2)$ the proof is complete. $\square$

Even though we have not defined the involution, we choose to provide the following result here. For the definition of the involution the reader can see Definition 2.1.1. in the next chapter.

**Corollary 1.3.2.** *Let $\mathcal{A}$ be a semi-simple commutative Banach algebra. Then every involution defined on $\mathcal{A}$ is continuous.*

*Proof.* Let us define a new norm on $\mathcal{A}$, $|||x||| = ||x^*||$. It follows immediately that $|||\cdot|||$ is sub-multiplicative and $|||e||| = 1$. Also $(\mathcal{A}, |||\cdot|||)$ is a Banach algebra because, if $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence for $|||\cdot|||$ then $\{x_n^*\}_{n\in\mathbb{N}}$ is a Cauchy sequence for $||\cdot||$ and it converges to some $a \in \mathcal{A}$. Hence

$$\lim_{n\to+\infty} |||x_n - a||| = 0$$

and consequently $|||\cdot|||$ is a Banach algebra-norm on $\mathcal{A}$ and by Corollary 1.3.1. there exists a constant $C > 0$ such that

$$|||x||| = ||x^*|| \leq C||x||,$$

making the involution continuous. $\square$

**Lemma 1.3.1.** *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{B}$ a subalgebra of $\mathcal{A}$. If $x \in \mathrm{Rad}(\mathcal{B})$ then $|x|_\sigma = 0$.*

*Proof.* Let $x \in \mathrm{Rad}(\mathcal{B})$, then we have that $\sigma_{\mathcal{A}}(x) = \{0\}$ and therefore considering $\sigma_{\mathcal{A}}(x) \subseteq \sigma_{\mathcal{B}}$ we have that $|x|_\sigma = 0$. $\square$

# Chapter 2

# Banach Algebras with involution

Continuing our presentation on Banach algebras we will introduce the concept of Banach algebras with involution. The study of this concept started to develop with the famous 1943 paper of Gelfand and Naimark [**18**]. In this paper they assumed that the involution was symmetric, meaning that every element of the form $e+x^*x$ is invertible. In 1947 I. Kaplansky first introduced the condition of hermicity, and conjectured its equivalency to symmetry. It wasn't until 1970, when S. Shirali and J. W. M. Ford proved Kaplansky's conjecture for the class of Banach *-algebras. Another powerful tool is the theory of Möbius Transforms exploited by L. A. Harris in 1972, from which we obtain a generalized result of the Russo-Dye Theorem for the class of Banach *-algebras.

## 2.1 Hermitian and Symmetric *-algebras

In this section we shall present the elegant theory V. Pták developed in 1970 for Banach *-algebras by utilizing the properties of the spectral radius and the corollaries which arise from the famous Pták inequality. After presenting Ford's square root lemma, we will complete the characterization providing the celebrated Shirali-Ford Theorem. An excessive presentation of Pták's theory with many interesting results can be found in [**5**].

**Definition 2.1.1.** *A *-algebra is an algebra $\mathcal{A}$ with a mapping ' * ': $\mathcal{A} \to \mathcal{A}$ called involution satisfying the following properties:*

*(a)* $(x+y)^* = x^* + y$
*(b)* $(\lambda x)^* = \overline{\lambda} x^*$
*(c)* $(xy)^* = y^* x^*$
*(d)* $x^{**} = x$

*Where* $x, y \in \mathcal{A}$ *and* $\lambda \in \mathbb{C}$. *If* $\mathcal{A}$ *is complete and the involution is continuous then* $\mathcal{A}$ *is called Banach* $^*$*-algebra and if the involution is not continuous* $^*$*-Banach algebra.*

**Definition 2.1.2.** *Let* $\mathcal{A}$ *be a *-algebra. A norm on* $\mathcal{A}$ *is said to satisfy the C*-property if* $||x^*x|| = ||x||^2$, *for any* $x \in \mathcal{A}$. *A C*-algebra is a Banach *-algebra whose norm satisfies the C*-property.*

More information on this class will be provided in the last chapter.

**Definition 2.1.3.** *Let* $\mathcal{A}$ *be a *-algebra. An element* $x \in A$ *is called quasi-regular if there exists an element* $y \in A$ *such that* $x \circ y \ = y \circ x \ = 0$ *where* $x \circ y = x + y - xy$.

**Definition 2.1.4.** *A *-algebra* $\mathcal{A}$ *is called symmetric if every element of the form* $-x^*x$ *is quasi-regular. If* $A$ *has a unit* $e$ *then* $A$ *is symmetric if and only if every element of the form* $e + x^*x$ *is invertible.*

In this thesis we are going to focus on the unital case, for $*$-algebras without a unit see Doran [**3**].

Before proceeding to the next definition note that for an arbitrary algebra the terms hermitian and selfadjoint do not coincide. Thus in this section whenever we call an element hermitian we mean selfadjoint. The term hermitian will be defined in the next chapter in terms of numerical range.

**Definition 2.1.5.** *A *-algebra* $\mathcal{A}$ *is called hermitian if the spectum of every self-adjoint element* $(h^* = h)$ *lies exclusively in* $\mathbb{R}$. *The set of all self-adjoint elements will be denoted as* $\mathcal{S}$.

**Proposition 2.1.1.** *Let* $\mathcal{A}$ *be a unital *-algebra. Then* $\mathcal{A}$ *is hermitian if and only if every element of the form* $-h^2$ *where* $h \in \mathcal{S}$ *is quasi-regular.*

*Proof.* Let $\mathcal{A}$ be hermitian and $h \in \mathcal{A}$ where $h^* = h$. Due to the hermicity of $h$ we have that $\sigma_{\mathcal{A}}(h) \subseteq \mathbb{R}$ and by the Spectral Mapping Theorem we have $\sigma_{\mathcal{A}}(h^2) = \sigma_{\mathcal{A}}(h)^2 \geq 0 \implies -1 \notin \sigma_{\mathcal{A}}(h^2)$ which implies that $e + h^2$ is invertible, which means that $\mathcal{A}$ is symmetric and hence $-h^2$ is quasi-regular.

Now let every element of the form $-h^2$ where $h^* = h$ be quasi-regular. Let $z = \lambda + i\mu$ be a complex number with $\mu \neq 0$ such that $z \in \sigma_{\mathcal{A}}(h)$ and the complex polynomial

$$p(t) = \frac{\lambda t^2 + (\mu^2 - \lambda^2)t}{\mu(\lambda^2 + \mu^2)}, \quad t \in \mathbb{C}$$

If $k = p(h)$ we can easily see that $k^* = k$.

$$p(\lambda + i\mu) = \frac{\lambda(\lambda + i\mu)^2 + (\mu^2 - \lambda^2)(\lambda + i\mu)}{\mu(\lambda^2 + \mu^2)} =$$

$$= \frac{\lambda(\lambda^2 + 2\lambda\mu i - \mu^2) + \mu^2\lambda + i\mu^3 - \lambda^3 - i\lambda^2\mu}{\mu(\lambda^2 + \mu^2)} =$$

$$= \frac{\lambda^3 + 2\lambda^2\mu i - \lambda\mu^2 + \mu^2\lambda + i\mu^3 - \lambda^3 - i\lambda^2\mu}{\mu(\lambda^2 + \mu^2)} =$$

$$= \frac{\lambda^2\mu i + i\mu^3}{\mu(\lambda^2 + \mu^2)} = \frac{i\mu(\lambda^2 + \mu^2)}{\mu(\lambda^2 + \mu^2)} = i$$

Now again using the Spectral Mapping Theorem we have $i \in p(\sigma_{\mathcal{A}}(h)) = \sigma_{\mathcal{A}}(p(h)) = \sigma_{\mathcal{A}}(k)$ which implies that $-1 \in \sigma_{\mathcal{A}}(k)^2 = \sigma_{\mathcal{A}}(k^2)$, therefore $e + k^2$ is not invertible $\implies -k^2$ is not quasi-regular, which is a contradiction. $\square$

**Corollary 2.1.1.** *Every symmetric *-algebra is hermitian.*

**Definition 2.1.6.** *Let $\mathcal{A}$ be a hermitian *-algebra and $x \in \mathcal{S}$. We will call $x$ positive if its spectrum is non-negative. The set of all positive elements of $\mathcal{A}$ will be denoted as $\mathcal{A}^+$. Mostly we are going to use the notation $x \geq 0$ whenever $x \in \mathcal{A}^+$.*

**Proposition 2.1.2.** *Let $\mathcal{A}$ be a *-algebra. Then $\mathcal{A}$ is symmetric if and only if $\sigma_{\mathcal{A}}(x^*x) \geq 0$ for every $x \in \mathcal{A}$.*

*Proof.* Let $\mathcal{A}$ be symmetric then by the above corollary we have that $\mathcal{A}$ is hermitian. Thus $\sigma_{\mathcal{A}}(x^*x) \subseteq \mathbb{R}$. Let $k < 0$, with $k \in \sigma_{\mathcal{A}}(x^*x) \implies \frac{x^*x}{k}$ is not quasi-regular, and $b = (-k)^{-\frac{1}{2}}$. We observe that the element $-(bx)^*(bx) = \frac{x^*x}{k}$ is not quasi-regular which is a contradiction because $\mathcal{A}$ is symmetric.
Now let $\sigma_{\mathcal{A}}(x^*x) \geq 0 \implies -1 \notin \sigma_{\mathcal{A}}(x^*x)$. Which means that every element of the form $-x^*x$ is quasi-regular. Thus $\mathcal{A}$ is symmetric.

$\square$

**Definition 2.1.7.** *Let $\mathcal{A}$ be a *-algebra. An element $k \in \mathcal{A}$ is called skew-hermitian if $k^* = -k$. Equivalently $k \in \mathcal{A}$ is skew-hermitian if there is an element $h \in \mathcal{S}$ such that $k = ih$.*

**Remark 2.** *In the class of hermitian *-Banach algebras it is immediate from the definition and the Spectral Mapping Theorem that the spectrum of a skew-hermitian element lies exclusively in $i\mathbb{R}$. For,*

$$\sigma_{\mathcal{A}}(k) = \sigma_{\mathcal{A}}(ih) = i\sigma_{\mathcal{A}}(h) \implies \sigma_{\mathcal{A}}(k) \subseteq i\mathbb{R}$$

**Definition 2.1.8.** *Let $\mathcal{A}$ be a *-algebra. An element $u \in \mathcal{A}$ will be called unitary if it satisfies the following*

$$uu^* = u^*u = e.$$

*The set of all unitary elements will be denoted as $\mathcal{U}$.*

It follows immediately due to the uniqueness of the inverse that for any $u \in \mathcal{U}$ we have $u^* = u^{-1}$.

**Lemma 2.1.1.** *Let $\mathcal{A}$ be a hermitian Banach *-algebra. Then $\sigma_{\mathcal{A}}(u) \subseteq \boldsymbol{T} = \{z \in \mathbb{C} : |z| = 1\}$, for $u \in \mathcal{U}$. In addition if the involution is continuous and $u \in \mathcal{U}$ with $\sigma_{\mathcal{A}}(u) \neq \boldsymbol{T}$, then there exists $h \in \mathcal{S}$ such that $u = e^{ih}$*

*Proof.* Since $u \in \mathcal{U}$, we have $u^*u = uu^* = e \implies u^* = u^{-1}$. Let $w \neq 0$ with $w \in \sigma_{\mathcal{A}}(u) \implies w^{-1} \in \sigma_{\mathcal{A}}(u^{-1}) = \sigma_{\mathcal{A}}(u^*)$. As follows from functional calculus $\sigma_{\mathcal{A}}(u + u^*) = \sigma_{\mathcal{A}}(u + u^{-1}) = \{\lambda + \lambda^{-1} : \lambda \in \sigma_{\mathcal{A}}(u)\}$. Hence $w+w^{-1} \in \sigma_{\mathcal{A}}(u+u^*) \subseteq \mathbb{R}$, due to the hermicity of $u+u^*$. Similarly $w-w^{-1} \in \sigma_{\mathcal{A}}(u - u^*) \subseteq i\mathbb{R}$, due to the skew-hermicity of $u - u^{-1}$. Let $w = x + iy$, then $w + w^{-1} = \frac{(x+iy)\sqrt{x^2+y^2}+x+iy}{\sqrt{x^2+y^2}} \in \mathbb{R} \implies w + w^{-1} = \frac{x(\sqrt{x^2+y^2}+1)}{\sqrt{x^2+y^2}}$ and $w - w^{-1} \in i\mathbb{R} \implies w - w^{-1} = \frac{yi(\sqrt{x^2+y^2}+1)}{\sqrt{x^2+y^2}}$
Thus $2w = w+w^{-1}+w-w^{-1} = \frac{x(\sqrt{x^2+y^2}+1)+yi(\sqrt{x^2+y^2}+1)}{\sqrt{x^2+y^2}} = \frac{(x+yi)(\sqrt{x^2+y^2}+1)}{\sqrt{x^2+y^2}} \implies$ $2(x + iy) = \frac{(x+yi)(\sqrt{x^2+y^2}+1)}{\sqrt{x^2+y^2}} \implies \sqrt{x^2 + y^2} = 1 \implies |w| = 1$

If $\sigma_{\mathcal{A}}(u) \neq \mathbf{T}$ we can take a branch of the logarithm $L$ defined on a neighborhood of $\sigma_{\mathcal{A}}(u)$ and define $h = -iL(u)$, and now because of the continuity of the involution we have $h^* = iL(u^*) = iL(u^{-1}) = -iL(u) = h$ as required. □

The next result is one of the most important in this section due to its later use. Provided by J. M. W. Ford in 1967 [**6**], the square root lemma ensures the existence of a square root for every self-adjoint element $h$ satisfying $|e-h|_\sigma < 1$ in an arbitrary Banach *-algebra.

**Lemma 2.1.2** (Ford's Square Root Lemma)**.** *Let $\mathcal{A}$ be a complex *-Banach algebra and $h \in \mathcal{S}$. If $|e-h|_\sigma < 1$, then there exists $u \in \mathcal{S}$ such that $u^2 = h$. In addition if $h \in \mathcal{A}^+$ then $u \in \mathcal{A}^+$ as well.*

*Proof.* Let $h \in \mathcal{S}$, Then there exists a maximal commutative *-subalgebra $\mathcal{B}$ containing $h$ and $\sigma_\mathcal{A}(x) = \sigma_\mathcal{B}(x)$ for every $x \in \mathcal{B}$. Observe that the function $f$ defined by,

$$f(t) = \sum_{n=0}^{+\infty} \binom{\frac{1}{2}}{n} (t-1)^n$$

converges absolutely for $|t-1| < 1$ and $(f(t))^2 = t$. Since $|e-h|_\sigma < 1$, the sequence $\{h_n\}_{n\in\mathbb{N}} \in \mathcal{B}$ defined by,

$$h_n = \sum_{k=0}^{n} \binom{\frac{1}{2}}{k} (h-e)^k$$

is a Cauchy sequence in $\mathcal{B}$ and converges to $(u+iv)$ where $u, v \in \mathcal{S} \cap \mathcal{B}$. It falls immediately that since $h = u^2 - v^2 + 2iuv$, then

$$h = u^2 - v^2 \quad and \quad uv = 0 \quad .(1)$$

Denote $\mathcal{R} = \mathrm{Rad}(\mathcal{B})$, $x_\mathcal{R}$ the coset $x + \mathcal{R}$ and $||\cdot||_\mathcal{R}$ the canonical norm on $\mathcal{B}/\mathcal{R}$. Thus,

$$||h_n - (u+iv)|| \to 0$$

and

$$||(h_n - u)_\mathcal{R} - (iv)_\mathcal{R}||_\mathcal{R} \to 0.$$

Since $\mathcal{R}$ is a closed *-ideal of $\mathcal{B}$, $\mathcal{B}/\mathcal{R}$ is a Banach *-algebra with the natural involution, $x + \mathcal{R} \to x^* + \mathcal{R}$. Note that since $\mathcal{B}/\mathcal{R}$ is commutative and semi-simple, the involution is continuous making $\mathcal{S}$ closed. As a result $(h_n - u)_\mathcal{R} \in \mathcal{S}$, for all $n \in \mathbb{N}$ and hence $(iv)_\mathcal{R} \in \mathcal{S} \implies (iv)_\mathcal{R} \in \mathcal{S} \cap i\mathcal{S} = \{0\} \implies v \in \mathcal{R}$.

Assume that $u \notin \mathcal{G}(\mathcal{B})$, then there exists $\phi \in \mathrm{Sp}\,\mathcal{A}$ such that $\phi(u) = 0$. Since $v \in \mathcal{R}$, $\phi(v) = 0$ thus $\phi(h) = \phi(u)^2 - \phi(v)^2 \implies \phi(e-h) = 1 \implies |e-h|_\sigma \geq 1$ which is a contradiction. Hence $u \in \mathcal{G}(\mathcal{A})$ and by (1) $v = u^{-1}uv = 0 \implies h = u^2$. The proof for the main result is complete.

Suppose now that $h \in \mathcal{A}^+$, it follows immediately that if $u^2 = h$ then $u \in \mathcal{B}$. Let $\theta \in \operatorname{Sp}\mathcal{B}$ then

$$\theta(u) = \lim_{n\to+\infty} \theta(h_n) = \lim_{n\to+\infty} \theta((\sum_{k=0}^{n} \binom{\frac{1}{2}}{k}(h-e)^k)) = \lim_{n\to+\infty} \sum_{k=0}^{n} \binom{\frac{1}{2}}{k}(\theta(h)-1)^k,$$

which is strictly positive due to the fact that $|e-h|_\sigma < 1$ and $\sigma_{\mathcal{A}}(h) \geq 0$. Hence $\sigma_{\mathcal{A}}(u) = \sigma_{\mathcal{B}}(u) \geq 0 \implies u \in \mathcal{A}^+$.

□

Next we give the definition of Ptak's functional, the most important concept of this chapter due to its vital role for the characterization of the class of hermitian *-algebras. It was first exploited by V. Ptak in [**5**]. Theorems like Pták's Inequality and corollaries provided later, along with many more can also be found in this paper.

**Definition 2.1.9.** *Let $\mathcal{A}$ be a unital Banach *-algebra. Ptak functional is defined on $\mathcal{A}$ by:*

$$|x|_\Sigma = |x^*x|_\sigma^{1/2}$$

*for any $x \in \mathcal{A}$.*

**Theorem 2.1.1** (Ptak's Inequality)**.** *Let $\mathcal{A}$ be a hermitian *-algebra, then the following inequality holds for every $x \in \mathcal{A}$.*

$$|x|_\sigma \leq |x|_\Sigma$$

*Proof.* In order to prove the above inequality for $x \in \mathcal{A}$ we can equivalently show that $\forall x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ with $|\lambda| > |x|_\Sigma \implies \lambda e - x \in \mathcal{G}(\mathcal{A})$. Putting $z = \lambda^{-1}x$, it suffices to show that $|z|_\Sigma < 1$ implies that $e - z \in \mathcal{G}(\mathcal{A})$.

$$|z|_\Sigma < 1 \implies |z^*z|_\sigma < 1 \implies e - z^*z \in \mathcal{G}(\mathcal{A})$$

From the above statement arises the fact that $\sigma_{\mathcal{A}}(e - z^*z) \geq 0$. In addition $e - z^*z \in \mathcal{S}$ and $|e - (e - z^*z)|_\sigma = |z^*z|_\sigma < 1$. Thus by Ford's Square Root Lemma we have that there exists a self-adjoint element $a \in \mathcal{A}$ which satisfies $a \in \mathcal{G}(\mathcal{A})$, $\sigma_{\mathcal{A}}(a) \geq 0$ and $a^2 = e - z^*z$.

$$(e + z^*)(e - z) = e + z^* - z - z^*z = a^2 + z^* - z = a(e + a^{-1}(z^* - z)a^{-1})a$$

$a \in \mathcal{S} \implies a^{-1} \in \mathcal{S}$, as a result,

$$(a^{-1}(z^* - z)a^{-1})^* = a^{-1}(z - z^*)a^{-1} = -a^{-1}(z^* - z)a^{-1}$$

which means that $a^{-1}(z^* - z)a^{-1}$ is skew-hermitian and because $\mathcal{A}$ is hermitian $\sigma_{\mathcal{A}}(a^{-1}(z^* - z)a^{-1}) \subseteq i\mathbb{R} \implies e + a^{-1}(z^* - z)a^{-1} \in \mathcal{G}(\mathcal{A})$. So $e - z$ is left invertible.
If we do the same process for $(e - z)(e + z^*)$ and consider the facts that $|\lambda| = |\bar{\lambda}|$ and $|z^*|_\Sigma = |z|_\Sigma$ we have that $e - z$ is right invertible.

□

**Definition 2.1.10.** *Let $\mathcal{A}$ be a *-algebra. An element $x \in \mathcal{A}$ will be called normal if $xx^* = x^*x$.*

**Proposition 2.1.3.** *Let $\mathcal{A}$ be a hermitian *-algebra and $h, k \in \mathcal{S}$. Then $|hk|_\sigma \leq |h|_\sigma |k|_\sigma$.*

*Proof.* Let $h, k \in \mathcal{S}$, then by induction

$$|hk|_\sigma \leq |hk|_\Sigma = |hkhk|_\sigma^{1/2} = |h^2k^2|_\sigma^{1/2} \leq |h^2k^2|_\Sigma^{1/2} = |h^4k^4|_\sigma^{1/4} \leq ... \leq |h^nk^n|_\sigma^{1/n}$$

where $n = 2^m$, $m \in \mathbb{N}$. Using the fact that the norm is always greater than the spectral radius and Gelfand-Buerling's Formula for the spectral radius we have:

$$|h^nk^n|_\sigma^{1/n} \leq ||h^nk^n||^{1/n} \leq ||h^n||^{1/n}||k^n||^{1/n},$$

letting $m \to \infty$ we have $|hk|_\sigma \leq |h|_\sigma |k|_\sigma$. □

**Corollary 2.1.2.** *The Ptak functional is sub-multiplicative on $\mathcal{A}$.*

*Proof.* Let $x, y \in \mathcal{A}$, $|xy|_\Sigma = |(xy)^*xy|_\sigma^{1/2} = |y^*x^*xy|_\sigma^{1/2} = |x^*xy^*y|_\sigma^{1/2}$
But $x^*x$ and $y^*y$ are self-adjoint, hence by the proposition above

$$|xy|_\Sigma = |x^*xy^*y|_\sigma^{1/2} \leq |x^*x|_\sigma^{1/2}|y^*y|_\sigma^{1/2} = |x|_\Sigma |y|_\Sigma.$$

□

**Proposition 2.1.4.** *Let $\mathcal{A}$ be a hermitian *-algebra and $x, y \in \mathcal{A}$. If $x, y$ are normal, then $|x|_\sigma = |x|_\Sigma$ and $|xy|_\sigma \leq |x|_\sigma |y|_\sigma$.*

*Proof.* Since $x$ is normal we have $|x|_\Sigma^2 = |x^*x|_\sigma \leq |x^*|_\sigma |x|_\sigma = |x|_\sigma^2 \leq |x|_\Sigma^2$.

By Corollary 2.1.2. we have that the Ptak functional is sub-multiplicative and by the above it is implied that the spectral radius is sub-multiplicative on normal elements. □

**Theorem 2.1.2.** *Let $\mathcal{A}$ be a hermitian *-algebra. Then $\mathcal{A}^+$ is a convex cone.*

*Proof.* It is obvious that $x \in \mathcal{S} \implies \lambda x \in \mathcal{S}$, for any $\lambda \in \mathbb{R}$ and $x + y \in \mathcal{S}$ for any $x, y \in \mathcal{S}$. Thus it suffices to prove that $\sigma_{\mathcal{A}}(x+y) \geq 0$ or equivalently $\forall \lambda < 0$,

$$\lambda \notin \sigma_{\mathcal{A}}(x+y) \iff -1 \notin \sigma_{\mathcal{A}}(\frac{x+y}{|\lambda|}).$$

It suffices to show that $1 \notin \sigma_{\mathcal{A}}(x+y) \iff e+x+y \in \mathcal{G}(\mathcal{A})$. Since $x, y \in \mathcal{A}^+$ we have that $e+x$, $e+y \in \mathcal{G}(\mathcal{A})$. Note that

$$(e+x)(e+y) = e+x+y+xy \implies e+x+y = (e+x)(e+y) - xy$$

Putting $a = (e+x)^{-1}x$, $b = (e+y)^{-1}y$, we have

$$(e+x)(e-ab)(e+y) = (e+x)[e-(e+x)^{-1}xy(e+y)^{-1}](e+y) =$$

$$= [e+x-(e+x)(e+x)^{-1}xy(e+y)^{-1}](e+y) = (e+x)(e+y) - xy$$

Thus $e+x+y = (e+x)(e-ab)(e+y)$.
Now in order to prove that $e+x+y \in A$ we have to show that $(e-ab) \in \mathcal{G}(\mathcal{A}) \implies -1 \notin \sigma_{\mathcal{A}}(ab)$.
$\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{A}}((e+x)^{-1}x)) = \{\frac{\lambda}{\lambda+1} : \lambda \in \sigma_{\mathcal{A}}(x)\} \implies \sigma_{\mathcal{A}}(a) \subseteq (1,1) \implies |a|_\sigma < 1$. Similarly for $b$, $|b|_\sigma < 1$. Also $a, b \in \mathcal{A} \implies |ab|_\sigma \leq |a|_\sigma |b|_\sigma \implies |ab|_\sigma < 1 \implies e-ab \in \mathcal{G}(\mathcal{A}) \implies e+x+y \in \mathcal{G}(\mathcal{A})$. □

**Corollary 2.1.3.** *The spectral radius is sub-additive on $\mathcal{S}$.*

*Proof.* Let $h, k \in \mathcal{S}$, then $\sigma_{\mathcal{A}}(h), \sigma_{\mathcal{A}}(k) \in \mathbb{R}$. It is also implied that $\sigma_{\mathcal{A}}(|h|_\sigma \pm h)$, $\sigma_{\mathcal{A}}(|k|_\sigma \pm k) \geq 0$. Thus by Theorem 2.1.2. we have $\sigma_{\mathcal{A}}(|h|_\sigma + |k|_\sigma \pm (h+k)) \geq 0$. By the Spectral Mapping Theorem we have,

$$\sigma_{\mathcal{A}}(|h|_\sigma + |k|_\sigma \pm (h+k)) = \{|h|_\sigma + |k|_\sigma \pm \lambda : \lambda \in \sigma_{\mathcal{A}}(h+k)\} \geq 0$$

So for all $\lambda \in \sigma_{\mathcal{A}}(h+k)$, we have,

$$|h|_\sigma + |k|_\sigma \pm \lambda \geq 0 \implies |h|_\sigma + |k|_\sigma \geq \pm\lambda \implies$$

$$|h|_\sigma + |k|_\sigma \geq |\lambda| \implies |h+k|_\sigma \leq |h|_\sigma + |k|_\sigma.$$

□

**Lemma 2.1.3.** *Let $\mathcal{A}$ be a hermitian *-algebra. Then for any $x \in \mathcal{A}$ the following holds, $|\frac{1}{2}(x^* + x)|_\sigma \leq |x|_\Sigma$.*

*Proof.* It is trivial that for any $x \in \mathcal{A}$ we have $x + x^* \in \mathcal{A}$ and $x - x^*$ is skew-hermitian which implies that $\exists k \in \mathcal{S}$ satisfying $x - x^* = ik$, also $x = (x + x^*)/2 + (x - x^*)/2 = h + ik$, where $h, k \in \mathcal{S}$.
Note that $x^*x + xx^* = 2(h^2 + k^2)$, $\sigma_{\mathcal{A}}(k^2) = \sigma_{\mathcal{A}}(k)^2 \geq 0$ and $\sigma_{\mathcal{A}}(|h^2 + k^2|_\sigma - (h^2 + k^2)) \geq 0$.
Hence by Theorem 2.1.2. $\sigma_{\mathcal{A}}(|h^2 + k^2|_\sigma - h^2) \geq 0 \implies |h^2|_\sigma \leq |h^2 + k^2|_\sigma$.
Using the above facts and Corollary 2.1.3. we have,

$$
\begin{aligned}
|\frac{1}{2}(x^* + x)|_\sigma^2 &= |h^2|_\sigma \leq |h^2 + k^2|_\sigma = \frac{1}{2}|x^*x + xx^*|_\sigma \\
&\leq \frac{1}{2}(|x^*x|_\sigma + |xx^*|_\sigma) = \frac{1}{2}(|x|_\Sigma^2 + |x|_\Sigma^2) = |x|_\Sigma^2.
\end{aligned}
$$

□

**Proposition 2.1.5.** *Let $\mathcal{A}$ a hermitian *-algebra. For any $x, y \in \mathcal{A}$ we have $|x + y|_\Sigma \leq |x|_\Sigma + |y|_\Sigma$.*

*Proof.* Let $x, y \in \mathcal{A}$ then $|x + y|_\Sigma^2 = |(x^* + y^*)(x + y)|_\sigma = |x^*x + x^*y + y^*x + y^*y|_\sigma$, and by Corollary 2.1.3. we have

$$|x + y|_\Sigma^2 \leq |x^*x|_\sigma + |x^*y + y^*x|_\sigma + |y^*y|_\sigma \qquad (1)$$

Again by Corollary 2.1.3. $|x^*y + y^*x|_\sigma \leq |y^*x|_\sigma + |x^*y|_\sigma = |y^*x|_\Sigma + |x^*y|_\Sigma = 2|y^*x|_\Sigma \leq 2|y^*|_\Sigma |x|_\Sigma = 2|x|_\Sigma |y|_\Sigma \implies$

$$|x^*y + y^*x|_\sigma \leq 2|y^*|_\Sigma |x|_\Sigma = 2|x|_\Sigma |y|_\Sigma. \qquad (2)$$

Thus (1) and (2) imply,

$$
\begin{aligned}
|x + y|_\Sigma^2 &\leq |x|_\Sigma^2 + 2|x|_\Sigma |y|_\Sigma + |y|_\Sigma^2 = (|x|_\Sigma + |y|_\Sigma)^2 \\
&\implies |x + y|_\Sigma \leq |x|_\Sigma + |y|_\Sigma.
\end{aligned}
$$

□

As mentioned before, in 1947 I. Kaplansky introduced the condition of hermicity of the involution and conjectured its equivalency to symmetry. This conjecture remained unsolved until 1970, when S. Shirali and J. W. M. Ford confirmed it for the class of Banach *-algebras in [**19**]. Note that here the hypotheses that $\mathcal{A}$ is complete is necessary due to J. Wichmann's example of a hermitian *-algebra which is not symmetric. The complete construction of this algebra can be found in [**20**].

**Theorem 2.1.3** (Shirali-Ford). *A Banach *-algebra $\mathcal{A}$ is symmetric if and only if it is hermitian.*

*Proof.* If $\mathcal{A}$ is symmetric then it is hermitian by Corollary 2.1.1.
Let's assume that $\mathcal{A}$ is hermitian, we have to show that for every element $x^*x$ in $\mathcal{A}$, we have $\sigma_{\mathcal{A}}(x^*x) \geq 0$.
Let $\delta = \sup\{-\mu : \mu \in \sigma_{\mathcal{A}}(x^*x), |x|_\Sigma \leq 1\}$. It suffices to show that $\delta \leq 0$. Assuming that $\delta > 0 \implies \exists x \in \mathcal{A}$ and $\exists \lambda \in \sigma_{\mathcal{A}}(x^*x)$, such that $-\lambda > \frac{1}{4}\delta$ and $|x|_\Sigma < 1$.
Let $y = 2x(e + x^*x)^{-1}$, then $e - y^*y = e - 4(e + x^*x)^{-1}x^*x(e + x^*x)^{-1}$. But $x^*x, (e+x^*x)^{-1}$ commute, thus $e-y^*y = e-4x^*x(e+x^*x)^{-2} = (e+x^*x)^2(e+x^*x)^{-2} - 4x^*x(e + x^*x)^{-2} = (e - x^*x)^2(e + x^*x)^{-2} \implies$

$$y^*y = e - (e - x^*x)^2(e + x^*x)^{-2}.$$

Hence by functional calculus, $\sigma_{\mathcal{A}}(y^*y) = \{1 - (\frac{1-t}{1+t})^2 : t \in \sigma_{\mathcal{A}}(x^*x)\} \subseteq (-\infty, 1)$

Now set $y = h + ik$, with $h, k \in \mathcal{S}$, then $yy^* = 2h^2 + 2k^2 - y^*y$, we also have that $\sigma_{\mathcal{A}}(e - y^*y) = \{1 - \lambda : \lambda \in \sigma_{\mathcal{A}}(y^*y)\} = (0, \infty)$. Hence by Theorem 2.1.2 $\sigma_{\mathcal{A}}(2h^2 + 2k^2 + (e - y^*y)) \geq 0 \implies \sigma_{\mathcal{A}}(yy^* + e) \geq 0$. Therefore $\forall \lambda \in \sigma_{\mathcal{A}}(yy^*) : \lambda + 1 \geq 0 \implies \lambda \geq -1 \implies \sigma_{\mathcal{A}}(yy^*) \subseteq [-1, \infty)$.
On the other hand $\sigma_{\mathcal{A}}(y^*y) \cup \{0\} = \sigma_{\mathcal{A}}(yy^*) \cup \{0\} \implies \sigma_{\mathcal{A}}(y^*y) \subset [-1, 1) \implies |y|_\Sigma \leq 1$.
Let us denote $f(\lambda) = (\frac{1-t}{1+t})^2$ with $t \in \sigma_{\mathcal{A}}(x^*x)$. Note that according to the definition of $\delta$ we have $-(1 - f(\lambda)^2) \leq \delta$, which implies $f(\lambda) \leq (1 + \delta)^{\frac{1}{2}}$ (1).
We can also observe that $f(f(t)) = t$ and $f$ is decreasing on $(-1, \infty)$. Hence by (1)

$$-\lambda = -f(f(\lambda)) \leq \frac{(1 + \delta)^{\frac{1}{2}} - 1}{(1 + \delta)^{\frac{1}{2}} + 1} \quad \leq \frac{\delta/2}{2} = \frac{\delta}{4}.$$

The last inequality holds because $\delta \geq 0 \implies (1+\delta)^{\frac{1}{2}} - 1 \geq 2$ and $(1+\delta)\frac{1}{2} - 1 \leq \frac{\delta}{2}$. Therefore the last inequality contradicts due to the selection of $\lambda$.

□

## 2.2 Möbius Transformations

In this section we will study the behavior of the set of unitary elements $\mathcal{U}$ and its convex hull. The use of Möbius Transformations and especially Potapov's Transformation outlines perfectly this behavior and provides simplified proofs of some important results in the class of Banach algebras with involution. We will also present the Russo-Dye Theorem which provides us the isometric part in the Vidav-Palmer Theorem. Most of the results in this section are provided by L. A. Harris in [**7**].

**Definition 2.2.1.** *Let $\mathcal{A}$ be a Banach *-algebra and $x \in \mathcal{A}$ with $|x^*x|_\sigma < 1$. The Potapov Mobius Transformation $T_x$ is defined by,*

$$T_x(y) = (e - xx^*)^{-1/2}(y + x)(e + x^*y)^{-1}(e - x^*x)^{1/2},$$

*for $y \in \mathcal{A}$ satisfying $|x^*y|_\sigma < 1$ and $(e - xx^*)^{-1/2}$ denotes the inverse of the square root of $(e - xx^*)$.*

**Lemma 2.2.1.** *Let $\mathcal{A}$ be a Banach *-algebra and $x \in \mathcal{A}$ satisfying $|x|_\sigma < 1$ and $|x^*x|_\sigma < 1$, and $\lambda \in \mathbb{C}$ satisfying $|\lambda| = 1$. Then,*

*(a)* $(e + \lambda x^*)^{-1}(\lambda e + x) = x + \lambda(e + \lambda x^*)^{-1}(e - x^*x)$.

*(b)* $(\lambda e + x)(e + \lambda x^*)^{-1} = x + \lambda(e - xx^*)(e + \lambda x^*)^{-1}$.

*(c)* $(e - xx^*)^{1/2}x = x(e - x^*x)^{1/2}$.

*Proof.* (a) $x + \lambda(e+\lambda x^*)^{-1}(e - x^*x) = x + \lambda(e+\lambda x^*)^{-1} - \lambda(e+\lambda x^*)^{-1}x^*x = [e-\lambda(e+\lambda x^*)^{-1}x^*]x+\lambda(e+\lambda x^*)^{-1} = [(e+\lambda x^*)^{-1}(e+\lambda x^*)-\lambda(e+\lambda x^*)^{-1}x^*]x+\lambda(e + \lambda x^*)^{-1} = (e + \lambda x^*)^{-1}x + \lambda(e + x^*)^{-1} = (e + \lambda x^*)^{-1}(\lambda e + x)$.
(b) We follow the same process as (a).
(c) As we saw in the proof of the Square Root Lemma it suffices to show that for every $n \in \mathbb{N}$, $x(\sum_{k=0}^n \binom{\frac{1}{2}}{k}(-x^*x)^k) = (\sum_{k=0}^n \binom{\frac{1}{2}}{k}(-xx^*)^k)x$
$\Leftrightarrow x(\sum_{k=0}^n \binom{\frac{1}{2}}{k}(-x^*x)^k) - (\sum_{k=0}^n \binom{\frac{1}{2}}{k}(-xx^*)^k)x = 0 \Leftrightarrow \sum_{k=0}^n \binom{\frac{1}{2}}{k}(x(-x^*x)^k - (-xx^*)^k x)$. We will now show by induction that $x(-x^*x^k) - (xx^*)^k x = 0$, $\forall k \in \mathbb{N}$. Let $k \in \mathbb{N}$, assume that $x(-x^*x)^k - (-xx^*)^k x = 0$.

$$x(-x^*x)^{k+1} - (-xx^*)^{k+1}x = x(-x^*x)(-x^*x)^k - (-xx^*)(-xx^*)^k x =$$

$$= xx^*(-xx^*)^k x - xx^*x(-x^*x)^k = xx^*((-xx^*)^k x - x(-x^*x)^k) = 0.$$

$\square$

**Lemma 2.2.2.** *Let $\mathcal{A}$ be a Banach *-algebra. If $x \in \mathcal{A}$ satisfies $|x|_\sigma < 1$ and $|x^*x|_\sigma < 1$, then the function $f(\lambda) = T_x(\lambda e)$ is holomorphic in an open neighborhood of the closed unit disk and $f(\boldsymbol{T}) \subseteq \mathcal{U}$. Moreover $f(0) = x$.*

*Proof.* Since $|x|_\sigma = |x^*|_\sigma < 1$ and $|x^*x|_\sigma = |xx^*|_\sigma < 1$, then f is defined on an open neighborhood of the closed unit disk and f is holomorphic because it is the product of the holomorphic functions $f_1(\lambda) = (e - xx^*)^{-1/2}(\lambda e + x)$ and $f_2(\lambda) = (e + \lambda x^*)^{-1}(e - x^*x)^{1/2}$.
We will now prove that if $|\lambda| = 1$ then $f(\lambda) \in \mathcal{U}$ equivalently $[f(\lambda)^{-1}]^* = f(\lambda)$.

$$[f(\lambda)^{-1}]^* = (e - xx^*)^{1/2}(\bar{\lambda}e + x^*)^{-1}(e + \bar{\lambda}x)(e - x^*x)^{-1/2} =$$
$$= (e - xx^*)^{1/2}(e + \lambda x^*)^{-1}(\lambda e + x)(e - x^*x)^{-1/2}.$$

Using the previous lemma we have

$$[f(\lambda)^{-1}]^* = (e - xx^*)^{1/2}[x + \lambda(e + \lambda x^*)^{-1}(e - x^*x)](e - x^*x)^{-1/2} =$$
$$= (e - xx^*)^{1/2}x(e - x^*x)^{-1/2} + \lambda(e - xx^*)^{1/2}(e + \lambda x^*)^{-1}(e - x^*x)^{1/2} =$$
$$= x(e - x^*x)^{1/2}(e - x^*x)^{-1/2} + \lambda(e - xx^*)^{1/2}(e + \lambda x^*)^{-1}(e - x^*x)^{1/2} =$$
$$= x + \lambda(e - xx^*)^{1/2}(e + \lambda x^*)^{-1}(e - x^*x)^{1/2} =$$
$$= (e - xx^*)^{-1/2}(e - xx^*)^{1/2}x + \lambda(e - xx^*)^{1/2}(e + \lambda x^*)^{-1}(e - x^*x)^{1/2} =$$
$$= (e - xx^*)^{-1/2}x(e - x^*x)^{1/2} + \lambda(e - xx^*)^{1/2}(e + \lambda x^*)^{-1}(e - x^*x)^{1/2} =$$
$$= [(e - xx^*)^{-1/2}x + \lambda(e - xx^*)^{1/2}(e + \lambda x^*)^{-1}](e - x^*x)^{1/2} =$$
$$= (e - xx^*)^{-1/2}(x + \lambda(e - xx^*)(e + \lambda x^*)^{-1})(e - x^*x)^{1/2} = f(\lambda).$$

Thus $f(\mathbf{T}) \subseteq \mathcal{U}$ and $f(0) = x$.

□

**Theorem 2.2.1.** *Let $\mathcal{A}$ be a Banach *-algebra. The set $\overline{\mathrm{conv}}\,\mathcal{U}$ contains all $x \in \mathcal{A}$ satisfying $|x|_\Sigma < 1$.*

*Proof.* We have $f(\lambda) = \frac{1}{2\pi i}\int_c \frac{f(\zeta)}{\zeta - \lambda}d\zeta$. Using the fact that $f(0) = x$ from the previous lemma, we have

$$x = f(0) = \frac{1}{2\pi i}\int_c \frac{f(\zeta)}{\zeta}d\zeta = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta}}ie^{i\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta$$

Now by the definition of the last integral, $x$ is equal to the limit of the Riemann-Stieltjes sum,

$$x = \lim_{n \to +\infty} \frac{1}{2\pi} \sum_{i=1}^{n} |\theta_i - \theta_{i-1}| f(e^{i\theta}).$$

We can easily observe that $\sum_{i=1}^{n} \frac{1}{2\pi} |\theta_i - \theta_{i-1}| = 1$. Hence $x$ is the limit of convex combinations of unitary elements. $\square$

**Lemma 2.2.3.** *Let $S$ be a subset of a normed linear space $\mathcal{X}$ such that* $\operatorname{conv} S$ *contains a neighborhood of* $0$. *Then* $\operatorname{conv} S$ *contains any $x \in \mathcal{X}$ satisfying $tx \in \overline{\operatorname{conv}}\, S$ for some $t > 1$.*

*Proof.* Let $\epsilon > 0$ such that: $||y|| < \epsilon \implies y \in \operatorname{conv} S$. If $tx \in \overline{\operatorname{conv}}\, S$ for some $t > 1$ then there is an $x_1 \in \operatorname{conv} S$ with $||tx - x_1|| < \epsilon(t-1)$ and also $tx - x_1 = (t-1)x_2$ for some $x_2 \in \operatorname{conv} S$. Thus $x = t^{-1}x_1 + (1 - t^{-1})x_2 \in \operatorname{conv} S$. $\square$

Now with a little help from this lemma we will come with a stronger result for the previous theorem when the involution is continuous.

**Corollary 2.2.1.** *Let $\mathcal{A}$ be a Banach *-algebra. If the involution on $\mathcal{A}$ is continuous, then* $\operatorname{conv}\mathcal{U}$ *contains all $x \in \mathcal{A}$ satisfying $|x|_\Sigma < 1$.*

*Proof.* By the previous lemma it suffices to show that $\operatorname{conv}\mathcal{U}$ contains a neighborhood of 0. Since the involution is continuous, then there is a positive number $M$ such that $||y^*|| \leq M||y||$, for all $y \in \mathcal{A}$. Let $y$ be a self-adjoint element of $\mathcal{A}$ with $||y|| < 1$, notice that the elements $u = y^2 + i(e - y^2)^{1/2}$ and $u^* = y^2 - i(e - y^2)^{1/2}$ are both hermitian and unitary. Furthermore $y = \frac{u+u^*}{2} \implies y \in \operatorname{conv}\mathcal{U}$. Now let $y$ be a skew-hermitian element of $\mathcal{A}$ with $||y|| < 1$. There exists an $h \in \mathcal{S}$ satisfying $||h|| < 1$ such that $y = ih = i\frac{u+u^*}{2} = \frac{iu+(-iu)^*}{2}$. It is implied that $y$ belongs in $\operatorname{conv}\mathcal{U}$ because $iu \in \mathcal{U}$.
Assume that $y$ is an arbitrary element of $\mathcal{A}$ with $||y|| < \frac{1}{1+M}$, therefore $y = y_1 + y_2$ where $y_1 = \frac{y+y^*}{2}$, $y_2 = \frac{y-y^*}{2}$. We have that $y + y^* \in \mathcal{S}$ and $||y + y^*|| \leq ||y|| + ||y^*|| \leq ||y|| + M||y|| < \frac{1+M}{1+M} < 1 \implies y + y^* \in \operatorname{conv}\mathcal{U}$. In addition $y - y^*$ is skew-hermitian and $||y - y^*|| \leq |||y|| - ||y^*||| < max\{||y||, ||y^*||\} < 1 \implies y - y^* \in \operatorname{conv}\mathcal{U}$. Concluding the proof we have that if $||y|| < \frac{1}{1+M}$, then $y$ is a convex combination of elements in $\operatorname{conv}\mathcal{U}$. Hence $y \in \operatorname{conv}\mathcal{U}$. $\square$

**Theorem 2.2.2.** *Let $\mathcal{A}$ be a hermitian Banach *-algebra with continuous involution and $\mathcal{E} = \{e^{ih} : h = h^*\}$. Then $\mathcal{U} \subseteq \overline{\operatorname{conv}}\,\mathcal{E}$.*

*Proof.* Let $u \in \mathcal{U}$, pick $t < 1$ and set $a = tu$ which instantly implies that $a$ is a normal element and $|a|_\sigma < 1$. We will use the Potapov Mobius Transformation once again, this time for a normal element.

$$f_a(\lambda) = T_a(\lambda e) = (e - aa^*)^{-1/2}(\lambda e + a)(e + \lambda a^*)^{-1}(e - a^*a)^{1/2}$$

We will soon come with a simplified form of $f_a(\lambda)$. For more comfortable calculations let us put $y = (e - a^*a)^{1/2} = (e - aa^*)^{1/2}$. By lemma 2.2.1.(c) we have that $ay = ya$ which implies that $(ay)^* = (ya)^* \implies ya^* = a^*y \implies (e + \lambda a^*)y = y(e + \lambda a^*) \implies (e + \lambda a^*)^{-1}y = y(e + \lambda a^*)^{-1}$, also $(\lambda e + a)y = y(\lambda e + a)$. Thus we have a simplified form for $f_a(\lambda)$

$$f_a(\lambda) = T_a(\lambda e) = (\lambda e + a)(e + \lambda a^*)^{-1}.$$

Which is defined and holomorphic in the disk $|\lambda| < t^{-1}$. In addition $f_a(\lambda) \in \mathcal{U}$ for $|\lambda| = 1$ and $f_a(0) = a$.
Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. $\lambda e + f_a(\lambda) = \lambda e + (\lambda e + a)(e + \lambda a^*)^{-1} = \lambda(e+\lambda a^*)(e+\lambda a^*)^{-1}+(\lambda e+a)(e+\lambda a^*)^{-1} = (\lambda e+\lambda^2 a^*+\lambda e+a)(e+\lambda a^*)^{-1} = \lambda(2e + \lambda a^* + \bar{\lambda} a)(e + \lambda a^*)^{-1}$. In order to prove that $\lambda e + f_a(\lambda)$ is invertible it suffices to show that $-2 \notin \sigma_{\mathcal{A}}(-\bar{\lambda} a^* - \lambda a) = \sigma_{\mathcal{A}}(-\bar{\lambda} tu^* - \lambda tu) = \sigma_{\mathcal{A}}(-\bar{\lambda} tu^{-1} - \lambda tu) = \{-(\bar{\lambda} tz^{-1} + \lambda tz) : |z| = 1\}$.

$$|-(\bar{\lambda} tz^{-1} + \lambda tz)| \leq |\bar{\lambda}||t||z^{-1}| + |\lambda||t||z| < 2 \implies |-(\bar{\lambda} a^* + \lambda a)|_\sigma < 2$$

Which implies that $-\lambda \notin \sigma_{\mathcal{A}}(f_a(\lambda))$. Hence we can use Lemma 2.1.1 and produce a self-adjoint element $h_\lambda$ such that $f_a(\lambda) = e^{ih_\lambda} \in \mathcal{E}$. As in Theorem 2.2.1.

$$a = \frac{1}{2\pi i}\int_{\mathbf{T}} f_a(\lambda)\lambda^{-1}d\lambda = \frac{1}{2\pi}\int_0^{2\pi} f_a(e^{i\theta})d\theta = \lim_{n\to+\infty}\frac{1}{2\pi}\sum_{i=1}^{n}|\theta_{i-1} - \theta i| f_a(e^{i\theta})$$

which means $a \in \overline{\operatorname{conv}}\,\mathcal{E}$. Therefore $u = \lim_{t\to 1} tu \in \overline{\operatorname{conv}}\,\mathcal{E}$. □

The biggest result of this chapter and directly connected with the Vidav-Palmer Theorem is the Russo-Dye Theorem. It was proven by B. Russo and H. A. Dye in 1966 but here we present an elementary proof given by L. T. Gardner in 1984. The original proof is contained in [**21**] and Gardner's proof can be found in [**22**].

**Theorem 2.2.3** (Russo-Dye)**.** *If $\mathcal{A}$ is a $C^*$-algebra, then the closed unit ball of $\mathcal{A}$ is the closed convex hull of its unitaries.*

*Proof.* In order to prove this theorem we will use the following mapping $T : \mathcal{A} \to \mathcal{A}$ where for $a \in \mathcal{A}$, with $||a|| < 1$, T is defined as: $T(u) = \frac{a+u}{2}$. Firstly we will prove that conv$\,\mathcal{U}$ is invariant under T, while it suffices to show that $T(\mathcal{U}) \subseteq$ conv$\,\mathcal{U}$. Let $u \in \mathcal{U}$. $\mathcal{A}$ is a $C^*$-algebra so $1 = ||e|| = ||u^{-1}u|| = ||u^*u|| = ||u||^2 \implies ||u|| = ||u^{-1}|| = 1$.

$$b = Tu = \frac{au^{-1} + e}{2}u$$

We have that $||au^{-1}|| \leq ||a|| < 1 \implies ||b|| < 1$. Thus $||bu^{-1} - e|| = ||\frac{(au^{-1}-e)}{2}|| = \frac{1}{2}||au^{-1} - e|| \leq \frac{1}{2}||e|| - ||au^{-1}|| < 1$. By Proposition 1.1.1. this means that $bu^{-1} \in \mathcal{G}(\mathcal{A})$ which implies $b \in \mathcal{G}(\mathcal{A})$. Thus $b$ can be written as $b = v(b^*b)^{1/2}$, $v \in \mathcal{U}$ and since $|b|_\sigma < 1$, we can define a $w \in \mathcal{U}$ such that $w = (b^*b)^{1/2} + i(e - b^*b)^{1/2}$ and $b = v(w + w^*)/2 \in$ conv$\,\mathcal{U}$.
We will now show that the open ball lies inside $\overline{\text{conv}}\mathcal{U}$. Let $a \in \mathcal{A}$ with $||a|| < 1$ and pick $u \in \mathcal{U}$. We will construct a sequence defined inductively $u_0 = u$ and $u_{n+1} = Tu$. Since $||u_0|| = 1$ and $||a|| < 1$ we have,

$$2 \geq ||u_0 - a|| = 2||\frac{u_0 + a - 2a}{2}|| = 2||u_1 - a|| = ... = n||u_{n-1} - a||$$

for any $n \in \mathbb{N}$. Hence,

$$||u_{n-1} - a|| \leq \frac{2}{n} \implies u_n \to a, when\ n \to +\infty$$

On the other hand, by the first section of the proof we have $u_n \in$ conv$\,\mathcal{U} \implies a = \lim\limits_{n\to+\infty} u_n \in \overline{\text{conv}}\mathcal{U}$.
We have that $B(0_A, 1) \subseteq \overline{\text{conv}}\mathcal{U}$ and $\mathcal{U} \subseteq \overline{B(0_A, 1)}$. Thus $\overline{B(0_A, 1)} = \overline{\text{conv}}\mathcal{U}$. □

It follows immediately by Theorem 2.2.2. and the fact that $\mathcal{E} \subseteq \mathcal{U}$, that $\overline{\text{conv}}\mathcal{U} = \overline{\text{conv}}\mathcal{E}$. Hence by the Russo-Dye Theorem we have that,

$$\overline{\text{conv}}\mathcal{U} = \overline{\text{conv}}\mathcal{E} = \overline{B(0_A, 1)}.$$

# Chapter 3

# The Vidav-Palmer Theorem

In this chapter we will present the main result of this thesis, the Vidav-Palmer Theorem. It was in 1956 that I. Vidav took the first step towards a deep geometrical characterization of C*-algebras. Using the concept of the numerical range he investigated the set of hermitian elements of a Banach algebra and tried to find the necessary conditions in order for a Banach algebra $\mathcal{A}$ with the natural involution to be a C*-algebra.

## 3.1 Numerical Range in Banach Algebras

In this section we will provide some basic ideas and results considering the concept of the numerical range, an essential tool for the proof. A full presentation of this matter can be found in [**8**].

**Definition 3.1.1.** *Let $\mathcal{A}$ be a unital normed algebra over a field. For an element $x \in \mathcal{A}$ satisfying $||x|| = 1$, the set $\mathcal{D}_{\mathcal{A}}(x)$ is defined as*

$$\mathcal{D}_{\mathcal{A}}(x) = \{f \in \mathcal{A}^* : f(e) = ||f|| = 1\}.$$

*Each element of $\mathcal{D}_{\mathcal{A}}(x)$ is called a support functional of the unit disk at $x$. The elements of $\mathcal{D}_{\mathcal{A}}(e)$ are called normalized states. Also by the Hahn-Banach Theorem it is guaranteed that $\mathcal{D}_{\mathcal{A}}(x)$ is non-empty for all $x$ in $\mathcal{A}$.*

**Definition 3.1.2.** *Let $\mathcal{A}$ be a unital normed algebra, $a \in \mathcal{A}$ and $x \in \mathcal{A}$ satisfying $||x|| = 1$. The numerical range of $a$ in $x$ is denoted as*

$$\mathcal{V}_{\mathcal{A}}(a, x) = \{f(ax) : f \in \mathcal{D}_{\mathcal{A}}(x), ||x|| = 1\}.$$

*The numerical range of $a$ is the set* $\mathcal{V}_{\mathcal{A}}(a) = \bigcup\limits_{||x||=1} \mathcal{V}_{\mathcal{A}}(a, x)$*, and the numerical radius* $|a|_v = sup\{|\lambda| : \lambda \in \mathcal{V}_{\mathcal{A}}(x)\}$.

**Lemma 3.1.1.** *Let $\mathcal{A}$ be a unital normed algebra then:*

$$\mathcal{V}_{\mathcal{A}}(a) = \mathcal{V}_{\mathcal{A}}(a, e).$$

*Proof.* It is obvious that $\mathcal{V}_{\mathcal{A}}(a, e) \subseteq \mathcal{V}_{\mathcal{A}}(a)$, due to the fact that the norm of the unit is 1.

For the other direction: Let $\lambda \in \mathcal{V}(A, a)$, then there is a $y \in \mathcal{A}$ with $||y|| = 1$ and $g \in \mathcal{D}(A, y)$ such that $\lambda = g(ay)$. Now let $f(x) = g(xy)$. As a result we have that $f \in \mathcal{D}(A, e)$ which follows immediately from the fact that $g \in \mathcal{D}(A, y)$. Hence:

$$\lambda = g(ay) = f(a) = f(ae) \in \mathcal{V}_{\mathcal{A}}(a, e).$$

□

**Theorem 3.1.1** (Toeplitz-Haussdorff)**.** *Let $\mathcal{A}$ be a unital normed algebra and $a \in \mathcal{A}$. Then $\mathcal{V}_{\mathcal{A}}(a)$ is compact and convex.*

*Proof.* Let $a \in \mathcal{A}$. We have that,

$$\mathcal{D}_{\mathcal{A}}(e) = \{f \in \mathcal{A}^* : ||f|| \leq 1, f(e) = 1\} =$$

$$= \{f \in \mathcal{A}^* : ||f|| \leq 1\} \cup \{f \in \mathcal{A}^* : f(e) = 1\}.$$

By the Banach-Alaoglou Theorem $\{f \in \mathcal{A}^* : ||f|| \leq 1\}$ is $w^*$-compact. Now let us prove that $\{f \in \mathcal{A}^* : f(e) = 1\}$ is $w^*$-closed. Let $f_n$ a sequence in $\mathcal{A}^*$ which converges weakly in $f$ and $f_n(e) = 1$, $\forall n \in \mathbb{N}$. Then $f_n \xrightarrow{w^*} f$ which means that $f_n(x) \to f(x), \forall x \in \mathcal{A} \implies f_n(e) \to f(e)$, but $f_n(e) = 1$ for every $n \in \mathbb{N}$, so $f(e) = 1$. Thus $\mathcal{D}_{\mathcal{A}}(e)$ is a subset of a $w^*$-compact set and $w^*$-closed $\implies \mathcal{D}_{\mathcal{A}}(e)$ is $w^*$-compact.
We will prove that $\mathcal{D}_{\mathcal{A}}(e)$ is convex. Let $f, g \in \mathcal{D}_{\mathcal{A}}(e) \implies ||f|| \leq 1, ||g|| \leq 1, f(e) = g(e) = 1$. For every $t \in [0, 1]$ we have:

$$||(1-t)g + tf|| \leq ||(1-t)g|| + ||tf|| = (1-t)||g|| + t||f|| \leq (1-t) + t \leq 1,$$

and

$$((1-t)g + tf)(e) = (1-t)g(e) + tf(e) = 1.$$

Hence $\mathcal{D}_{\mathcal{A}}(e)$ is convex and $w^*$-compact. Therefore, $\mathcal{V}_{\mathcal{A}}(a,e)$ inherits these topological properties because it is the image of $\mathcal{D}_{\mathcal{A}}(e)$ via the mapping $f \mapsto f(a)$ which is linear and continuous. Using the previous lemma the proof is completed. $\square$

**Theorem 3.1.2.** *Let $\mathcal{B}$ a subalgebra of $\mathcal{A}$ which contains the unit. Then $\forall b \in \mathcal{B}$ , $\mathcal{V}_{\mathcal{B}}(b) = \mathcal{V}_{\mathcal{A}}(b)$.*

*Proof.* Equivalently we will show that $\mathcal{V}_{\mathcal{B}}(b,e) = \mathcal{V}_{\mathcal{A}}(b,e)$. Since $\mathcal{B} \subseteq \mathcal{A}$ it follows that $\mathcal{D}_{\mathcal{A}}(e) \subseteq \mathcal{D}_{\mathcal{B}}(e) \implies \mathcal{V}_{\mathcal{A}}(b,e) \subseteq \mathcal{V}_{\mathcal{B}}(b,e)$.
Let $f \in \mathcal{D}_{\mathcal{B}}(e)$ by the Hahn-Banach theorem we have that there is a $\tilde{f} \in \mathcal{D}_{\mathcal{A}}(e)$ such that $\tilde{f}(x) = f(x), \forall x \in \mathcal{B}$ and due to the fact that the unit and b are contained in $\mathcal{B}$ the following is implied,

$$f(be) = \tilde{f}(be) \in \mathcal{V}_{\mathcal{B}}(b,e).$$

$\square$

Next we will provide some results about the connection of the numerical range and the spectrum.

**Theorem 3.1.3** (Lumer's First Numerical Range Formula)**.** *Let $\mathcal{A}$ be a normed algebra. For every $a \in \mathcal{A}$, the following holds,*

$$\max\{\operatorname{Re}\lambda : \lambda \in \mathcal{V}_{\mathcal{A}}(a)\} = \inf_{k>0} \frac{1}{k}\{||e+ka|| - 1\} = \lim_{k \to 0^+} \frac{1}{k}(||e+ka|| - 1).$$

*Proof.* Let $\mu = \max\{\operatorname{Re}\lambda : \lambda \in \mathcal{V}_{\mathcal{A}}(a)\}$. Given $f \in \mathcal{D}_{\mathcal{A}}(e)$ and $k > 0$, we have,

$$f(a) = \frac{1}{k}(f(e+ka)) \implies \operatorname{Re}(f(a)) \leq \frac{1}{k}(||e+ka|| - 1) \implies$$

$$\mu \leq \inf_{k>0} \frac{1}{k}(||e+ka|| - 1) \qquad (1)$$

Assume $a$ is not 0, otherwise the result is obvious. Let $0 < k < ||a||^{-1}$. For $x$ satisfying $||x|| = 1$ and $f \in \mathcal{D}_{\mathcal{A}}(x)$ we have, $||(e-ka)x|| \geq \operatorname{Re}(f((e-ka)x)) \geq 1 - k\mu$. Hence $||(e-ka)x|| \geq (1-k\mu)||x||, \forall x \in \mathcal{A}$.
Setting $x = e + ka$ we obtain,

$$||(e-ka)(e+ka)|| \geq (1-k\mu)||e+ka|| \implies ||e-k^2a^2|| \geq (1-k\mu)||e+ka|| \implies$$

$$||e+ka|| \leq \frac{||e-k^2a^2||}{1-k\mu} \leq \frac{1+k^2||a^2||}{1-k\mu} \implies \frac{||e+ka||}{k} - \frac{1}{k} \leq \frac{1+k^2||a^2||}{k(1-k\mu)} - \frac{1}{k}$$

$$\implies \frac{||e+ka||-1}{k} \leq \frac{1+k^2||a^2||+k\mu-1}{k(1-k\mu)} \implies \frac{||e+ka||-1}{k} \leq \frac{k||a^2||+\mu}{1-k\mu}.$$

Let $k \to 0^+$ and the proof is complete. □

Now we prove that the numerical range always contains the spectrum of an element in a Banach algebra

**Theorem 3.1.4.** *Let $\mathcal{A}$ be a complex unital Banach algebra. For any $x \in \mathcal{A}$ the following holds,*

$$\sigma_{\mathcal{A}}(x) \subseteq \mathcal{V}_{\mathcal{A}}(x).$$

*Proof.* Let $\lambda \in \sigma_{\mathcal{A}}(x)$, then $\lambda e - x \in \mathcal{A} \setminus \mathcal{G}(\mathcal{A})$. Assume that it has no left inverse and let $J = \mathcal{A}(\lambda e - x)$, it falls immediately that $J$ is a proper left ideal of $\mathcal{A}$ and since $\mathcal{A}$ is a Banach algebra and $\lambda \in \sigma_{\mathcal{A}}(x)$, we have that for any $y \in J$ the following holds,

$$||e-y|| \geq 1.$$

Thus by the Hahn-Banach theorem, there exists $f \in \mathcal{A}^*$ such that $f(e) = ||f|| = 1$ and $f(J) = \{0\}$. Hence $f \in \mathcal{D}_{\mathcal{A}}(e)$ and $f(\lambda e - x) = 0$, which means that $\lambda = f(a) \in \mathcal{V}_{\mathcal{A}}(a, e)$. Similarly for the absence of the right inverse and the proof is complete. □

**Definition 3.1.3.** *Let $\mathcal{A}$ be a Banach algebra. An algebra-norm $p$ on $\mathcal{A}$ is said to be equivalent to the given norm if there exist constants $\lambda, \kappa > 0$ such that for any $x \in \mathcal{A}$ the following holds,*

$$\lambda||x|| \leq p(x) \leq \kappa||x||.$$

*The set of all equivalent algebra-norms which satisfy $p(e) = 1$ will be denoted as $\tilde{N}$.*

Note that the numerical range is not a strictly algebraic nor analytic concept, therefore if we change the norm on $\mathcal{A}$, the numerical range of an element naturally changes. For this change let us denote the numerical range of $x \in \mathcal{A}$ with $p$ as $\mathcal{V}^p_{\mathcal{A}}(a)$.

**Lemma 3.1.2.** *Let $\mathcal{A}$ be a Banach algebra and $S$ a bounded semi-group in $\mathcal{A}$. Then there exists $p \in \tilde{N}$ such that $p(s) \leq 1$, for any $s \in S$.*

*Proof.* Without loss of generality assume that $e \in S$. Then for any $x \in \mathcal{A}$ let $q(x) = \sup\{||sx|| : s \in S\}$. Since $S$ is bounded we have that $M = \sup\{||s|| : s \in S\}$. Hence $q$ is an algebra-norm satisfying,

$$||x|| \leq q(x) \leq M||x||,$$

and

$$q(sx) \leq q(x),$$

for all $x \in \mathcal{A}$.

To conclude, take for $a \in \mathcal{A}$

$$p(a) = \sup\{q(ax) : x \in \mathcal{A}, q(x) \leq 1\}.$$

We have that $p \in \tilde{N}$ satisfying $p(s) \leq 1$. □

**Lemma 3.1.3.** *Let $\mathcal{A}$ be a Banach algebra and $a_1, a_2, ..., a_n \in \mathcal{A}$ mutually commuting elements. If $\epsilon > 0$ then there exists $p \in \tilde{N}$ such that,*

$$p(a_k) < |a_k|_\sigma + \epsilon.$$

*Proof.* Let $b_k = \frac{a_k}{(|a_k|_\sigma + \epsilon)}$ and S the multiplicative semi-group generated by $b_1, b_2, ..., b_n$, which are mutually commuting elements with spectral radius less than 1. It follows that S is bounded and by Lemma 3.1.2. there exists a $p \in \tilde{N}$ satisfying $p(b_k) \leq 1$ and thus,

$$p(a_k) \leq |a_k|_\sigma + \epsilon.$$

□

**Theorem 3.1.5.** *Let $\mathcal{A}$ be a complex Banach algebra. Then for any $x \in \mathcal{A}$, the following holds,*

$$\operatorname{conv} \sigma_{\mathcal{A}}(x) = \cap\{\mathcal{V}^p_{\mathcal{A}} : p \in \tilde{N}\}.$$

*Proof.* By Theorems 3.1.1. and 3.1.2. it is obvious that,

$$\operatorname{conv} \sigma_{\mathcal{A}}(x) \subseteq \cap\{\mathcal{V}^p_{\mathcal{A}}(x) : p \in \tilde{N}\}.$$

Since $\sigma_{\mathcal{A}}(x)$ is compact we have that $\operatorname{conv} \sigma_{\mathcal{A}}(x)$ is compact and convex and hence the intersection of the open circular disks containing $\sigma_{\mathcal{A}}(x)$. Assume that for any $\lambda \in \sigma_{\mathcal{A}}(x)$ we have

$$|\lambda - a| < r.$$

Then $|x-a|_\sigma < r$ and thus by Lemma 3.1.3. there exists $p \in \tilde{N}$ satisfying $p(x-a) < r$ for any $\lambda \in \mathcal{V}_{\mathcal{A}}^p(x)$. This means that $\cap\{\mathcal{V}_{\mathcal{A}}^p(x) : p \in \tilde{N}\}$ is contained in every open circular disk containing $\sigma_{\mathcal{A}}(x)$ implying $\cap\{\mathcal{V}_{\mathcal{A}}^p(x) : p \in \tilde{N}\} \subseteq \operatorname{conv} \sigma_{\mathcal{A}}(x)$. □

**Theorem 3.1.6** (Lumer's Second Numerical Range Formula)**.** *Let $\mathcal{A}$ be a Banach algebra. For any $a \in \mathcal{A}$ the following holds,*

$$\max\{\lambda : \lambda \in \mathcal{V}_{\mathcal{A}}(a)\} = \sup\{\frac{1}{k}\log||e^{ka}|| : k > 0\} = \lim_{k\to 0^+}\frac{1}{k}\log||e^{ka}||.$$

*Proof.* Let $\mu = \max\{\lambda : \lambda \in \mathcal{V}_{\mathcal{A}}(a)\}$ and $k \geq 0$. Recalling the proof of Lumer's first numerical range formula, for $x \in \mathcal{A}$ satisfying $||x|| = 1$ and $f \in \mathcal{D}_{\mathcal{A}}(x)$ we have,

$$||(e-ka)x|| \leq (1-k\mu)||x||. \qquad (1)$$

Suppose that $1-k\mu \geq 0$, then by induction $||(1-k\mu)^n x|| \geq (1-k\mu)^n||x||$, for any $n \in \mathbb{N}$. Observe that $1-\frac{k}{n}\mu \geq 0$ for large n. Thus replacing $k$ by $\frac{k}{n}$ and letting $n$ to infinity, we get: $||e^{-ka}x|| \geq e^{-k\mu}||x||$. Setting $x = e^{ka}$ we obtain, $||e^{ka}|| \leq e^{a\mu}$, which implies,

$$\sup\{\frac{1}{k}\log||e^{ka}|| : k > 0\} \leq \mu.$$

On the other hand we have that, $||e^{ka}|| = ||e+ka|| + \lambda a$, where for some $M > 0$, the following holds,

$$|\lambda(k)| \leq Mk^2, \qquad (2)$$

for any $k \in [0,1]$. Now using the inequality $\log(t) \geq \frac{t-1}{t}$ the following holds,

$$\frac{1}{k}\log||e^{ka}|| \geq \frac{\frac{1}{k}\{||e+ka||-1\} + \frac{1}{k}\lambda(k)}{||e+ka|| + \lambda(k)}.$$

To complete the proof use inequality (2) and see that the right side of the inequality converges to $\mu$ as $k \to 0^+$. □

The following theorem first proven by H. F. Bohnenblust and S. Karlin in [**23**], practically gives us that the numerical radius is a seminorm on $\mathcal{A}$. The elementary proof presented here is due to Bonsall and Duncan and can be found in [**8**]. An interesting fact about this theorem is that it may be false for Banach algebras over $\mathbb{R}$. Also note that here $e$ is the base of the natural logarithm.

**Theorem 3.1.7.** *Let $\mathcal{A}$ be a complex unital Banach algebra. Then for any $a \in \mathcal{A}$ the following holds,*

$$||a|| \geq |a|_u \geq \frac{||a||}{e}.$$

*Proof.* The left inequality follows immediately by the definition of the numerical range.

Let $b \in \mathcal{A}$ satisfying $|b|_u \leq \mu < 1$. For a given $x \in \mathcal{A}$ with $||x|| = 1$, there exists $f \in \mathcal{D}_{\mathcal{A}}(x)$ such that $\forall \lambda \in \mathbb{C}$ satisfying $|\lambda| \leq 1$, the following holds,

$$||(e - \lambda b)x|| \geq |f((e - \lambda b)x)| = |1 - \lambda f(b)| \geq 1 - \mu.$$

Hence $||(e - \lambda b)x|| \geq (1 - \mu)||x||$, for any $x \in \mathcal{A}$. By Theorem 3.1.4. $|b|_\sigma \leq \mu < 1 \implies e - \lambda b \in \mathcal{G}(\mathcal{A})$ and by the inequality above it is implied that:

$$||(e - \lambda b)^{-1}|| \leq (1 - \mu)^{-1}. \qquad (1)$$

Let $\omega_1, \omega_2, ..., \omega_n$ denoting the n-th roots of the unity, we have for all integers $j$,

$$\sum_{k=1}^{n} \omega_k^j = \begin{cases} 0 & j \neq 0 \mod n \\ n & j = 0 \mod n \end{cases}$$

Now let us define

$$S(r, n) = \frac{1}{n} \sum_{k=1}^{n} \omega_k^{-1} (e - \omega_k b)^{-r},$$

for $r = 1, 2, ...,$. We have,

$$\omega_k^{-1}(e - \omega_k b)^{-r} = \omega_k^{-1}\{e + r\omega_k b + \frac{r(r+1)}{2!}(\omega_k b)^2 + ...\},$$

and as a result,

$$S(r, n) = rb + \frac{r(r+1)...(r+n)}{(n+1)!} b^{n+1} + ...$$

Since $|b|_\sigma < 1$, we have: $\lim_{n\to\infty} S(r, n) = rb$. By (1) we have $||S(r, n)|| \leq (1 - \mu)^{-r}$ and as a result $r||b|| \leq (1 - \mu)^{-r}$. (2)

Now let $a \in \mathcal{A}$, $k > |a|_u$, $r \geq 2$, $\mu = \frac{1}{r}$ and $b = \frac{1}{rk} a$. Note that $b, \mu$ satisfy condition (1) and by (2) the following holds,

$$\frac{1}{k}||a|| \leq (1 - \frac{1}{r})^{-r},$$

from which the following is deduced,

$$k \geq \frac{1}{e}||a||.$$

$\square$

## 3.2 Hermitian Elements of Banach Algebras

In this section we provide the concept of hermicity in arbitrary Banach algebras. The main problem is the inability to define the involution and use the natural concept of self-adjointness to provide the characterization. Hence we take advantage of the numerical range and its convenient properties to finally come up with an equivalent description.

Let us start with a Lumer type formula which provides the maximum of the real part of the spectrum. But first, a lemma.

**Lemma 3.2.1.** *Let $f : \mathbb{N} \to \mathbb{R}$ continuous and sub-additive. Then the following holds,*

$$\lim_{t\to+\infty} \frac{1}{t} f(t) = \inf\{\frac{1}{t} f(t) : t > 0\}.$$

*Proof.* Let $k > \inf\{\frac{1}{t} f(t) : t > 0\}$ and choose $s > 0$ satisfying $\frac{1}{s} f(s) < k$. Since f is continuous, we have $\sup\{f(t) : s \leq t \leq 2s\} = m < +\infty$. Hence for any $n \in \mathbb{N}$ and $t > 0$ satisfying $(n+1)s \leq t \leq (n+2)s$ we have,

$$f(t) = f(ns + t - ns) \leq f(ns) + f(t - ns) \leq nf(s) + m,$$

and thus $\frac{1}{t} f(t) < \frac{ns}{t} k + \frac{m}{t}$. Letting $t \to +\infty$ and $n \to +\infty$ with respect to the inequality $(n+1)s \leq t \leq (n+2)s$, we obtain,

$$\lim_{t\to+\infty} \sup \frac{1}{t} f(t) \leq k.$$

$\square$

**Theorem 3.2.1.** *Let $\mathcal{A}$ be a Banach algebra. For any $a \in \mathcal{A}$ the following holds,*

$$\max\{\lambda : \lambda \in \sigma_{\mathcal{A}}(a)\} = \inf\{\frac{1}{k} \log||e^{ka}|| : k > 0\} = \lim_{k\to+\infty} \frac{1}{k} \log||e^{ka}||.$$

*Proof.* For any $a \in \mathcal{A}$ we have $|e^a|_\sigma = \lim_{n\to+\infty} ||e^{na}||^{1/n}$. Hence

$$\log|e^a|_\sigma = \lim_{n\to+\infty} \frac{1}{n} \log||e^{na}||.$$

Observe that $\log||e^{na}||$ is sub-additive. Hence by the previous lemma we obtain,

$$\lim_{k\to+\infty} \frac{1}{k} \log||e^{ka}|| = \inf\{\frac{1}{k} \log||e^{ka}|| : k > 0\}.$$

By the Spectral Mapping Theorem we have $\sigma_{\mathcal{A}}(e^a) = e^{\sigma_{\mathcal{A}}(a)}$ and as a result $|e^a|_\sigma = \max\{e^{\mathrm{Re}\,\lambda} : \lambda \in \sigma_{\mathcal{A}}(a)\} = e^{\max \mathrm{Re}\,\sigma_{\mathcal{A}}(a)}$, which implies,

$$\log|e^a|_\sigma = \log e^{\max \mathrm{Re}\,\sigma_{\mathcal{A}}(a)}$$

and by the equality obtained from the lemma above we have,

$$\max \mathrm{Re}\,\sigma_{\mathcal{A}}(a) = \log e^{\max \mathrm{Re}\,\sigma_{\mathcal{A}}(a)} = \lim_{k\to+\infty} \frac{1}{k} \log||e^{ka}||.$$

□

**Definition 3.2.1.** *Let $\mathcal{A}$ be a unital Banach algebra. An element of $h \in \mathcal{A}$ is called hermitian if $\mathcal{V}_A(h) \subseteq \mathbb{R}$. The set of all hermitian elements in $\mathcal{A}$ is denoted as $\mathcal{H}$.*

**Lemma 3.2.2.** *Let $\mathcal{A}$ be a normed algebra. Then the following are equivalent,*

*(a)* $h \in \mathcal{H}$.

*(b)* $\lim_{t\to 0} \frac{||e+ith||-1}{t} = 0$, $t \in \mathbb{R}$.

*(c)* $||\exp(ith)|| = 1$.

*Proof.* (a) $\implies$ (b)
Let $h \in \mathcal{H} \implies \mathcal{V}_{\mathcal{A}}(h) \subseteq \mathbb{R}$, by Lumer's formula for the numerical range we have,

$$\max Re\mathcal{V}_{\mathcal{A}}(ih) = \max Re\mathcal{V}_{\mathcal{A}}(-ih) = 0 \implies$$
$$\lim_{t\to 0} \frac{||e+ith||-1}{t} = \lim_{t\to 0} \frac{||e-ith||-1}{t} = 0$$
$$\implies \lim_{t\to 0} \frac{||e+ith||-1}{t} = 0.$$

(b) $\implies$ (c) By Lumer's second numerical range formula we have,
$\lim_{t\to 0} \frac{||e+ith||-1}{t} = \lim_{t\to+\infty} \frac{1}{t} \log||\exp(ith)|| = 0 \implies \lim_{t\to+\infty} \log||\exp(ith)||^{1/t} = 0 \implies ||\exp(ith)|| = 1.$ □

Condition (b) in the above lemma was Vidav's first definition of hermitian in [**24**].

Our next goal is to present Sinclair's Theorem, which for any hermitian element $h$ provides the equality between the spectral radius and the norm. Although it was proven by A. M. Sinclair in 1971, here we chose to present an elementary proof provided by F. F. Bonsall and M. J. Crabb in [**9**]. Sinclair's original proof can be found in [**25**].

**Lemma 3.2.3.** *Let $\mathcal{A}$ be a unital Banach algebra and suppose $h \in \mathcal{A}$ satisfying $\sigma_{\mathcal{A}}(h) \subseteq (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $\sigma_{\mathcal{A}}(\sin h) \subseteq \{\sin\lambda \in \mathbb{C} : |\lambda| < 1\}$ and $\arcsin(\sin h) = h$.*

*Proof.* Let $h \in \mathcal{H}$ with $\sigma_{\mathcal{A}}(h) \subseteq (-\frac{\pi}{2}, \frac{\pi}{2})$. By the Spectral Mapping Theorem $\sigma_{\mathcal{A}}(\sin h) = \{sin\lambda : \lambda \in \sigma_{\mathcal{A}}(h)\}$, which implies that $\sigma_{\mathcal{A}}(\sin h) \subseteq (-1, 1) \subseteq \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Hence,

$$\arcsin(\sin h) = \sum_{k=1}^{+\infty} \gamma_k \sin h^k.$$

Let $N$ be an open neighborhood of $\sigma_{\mathcal{A}}(h)$ such that $sin\lambda \subseteq \mathcal{B}(0_{\mathcal{A}}, 1)$, $\forall \lambda \in N$, then $\arcsin(\sin h)$ is analytic for all $\lambda$ in $N$. Moreover $\forall a \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\arcsin(\sin a) = a$. On the other hand by Functional Calculus

$$\arcsin(\sin h) = \frac{1}{2\pi i}\int_{\partial S} \arcsin(\sin\zeta) r_\zeta d\zeta = h.$$

$\square$

**Theorem 3.2.2** (Sinclair)**.** *Let $\mathcal{A}$ be a unital Banach algebra. If $h \in \mathcal{H}$ then $|h|_\sigma = |h|_v = ||h||$.*

*Proof.* We know that $|h|_\sigma \leq ||h||$ is always true, thus we only have to show the inverse. Assume that $\mathcal{A}$ is complete and $|h|_\sigma < \frac{\pi}{2}$. It suffices to show that $||h|| \leq \frac{\pi}{2}$ because if it is true then $\forall 0 < t < \frac{\pi}{2}$ we have,

$$|\frac{t}{|h|_\sigma + (\frac{\pi}{2} - t)} h|_\sigma < \frac{\pi}{2} \implies ||\frac{t}{|h|_\sigma + (\frac{\pi}{2} - t)} h|| \leq \frac{\pi}{2}$$

And if $t \to \pi/2$ we have $||h|| \leq |h|_\sigma$.
By the previous lemma we have, $h = \arcsin(\sin h) \implies ||h|| = ||\arcsin(\sin h)|| =$

$||\arcsin(\frac{1}{2i}(e^{ih} - e^{-ih}))|| = ||\sum_{k=1}^{+\infty} \gamma_k [\frac{1}{2i}(e^{ih} - e^{-ih})]^k|| \leq \sum_{k=1}^{+\infty} \gamma_k [\frac{1}{2i}||e^{ih}|| - ||e^{-ih}||]^k \leq \sum_{k=1}^{+\infty} \gamma_k \to \pi/2$.

By Theorems 3.1.4. and 3.1.7. we have that, $|h|_\sigma \leq |h|_v \leq ||h||$. Since $|h|_\sigma = ||h||$ we obtain $|h|_\sigma = |h|_v = ||h||$. □

**Proposition 3.2.1.** *Let $\mathcal{A}$ be a unital Banach algebra. If $h \in \mathcal{H}$, then $\mathcal{V}_{\mathcal{A}}(h) = \operatorname{conv} \sigma_{\mathcal{A}}(h)$.*

*Proof.* Let $h \in \mathcal{H}$, then $\operatorname{conv} \sigma_{\mathcal{A}}(h) \subseteq \mathcal{V}_{\mathcal{A}}(h) \subseteq \mathbb{R}$ and $|h-\lambda e|_\sigma \leq |h-\lambda e|_v \leq ||h - \lambda e||$. $\mathcal{V}_{\mathcal{A}}(h) \subseteq \mathbb{R}$ is compact and convex, so $\mathcal{V}_{\mathcal{A}}(h) = [a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$. Then,

$$\mathcal{V}_{\mathcal{A}}(h-ae) = [0, b-a] \implies |h-ae|_v = b-a = |h-ae|_\sigma \implies b-a \in \sigma_{\mathcal{A}}(h-ae).$$

By the Spectral Mapping Theorem we have $b \in \sigma_{\mathcal{A}}(h)$. With the same process we prove that $a \in \sigma_{\mathcal{A}}(h)$, and due to the compactness and convexity of $\operatorname{conv} \sigma_{\mathcal{A}}(h)$ the proof is complete. □

The following theorem is contained in in M. J. Crabb's Ph.D Thesis [**11**]. In the original text, the result concerns operators acting on a Banach space but since this fact is not used throughout the proof, the result holds for Banach algebras as well.

**Theorem 3.2.3.** *Let $\mathcal{A}$ be a unital Banach Algebra and $h \in \mathcal{H}$. Then $\mathcal{V}_{\mathcal{A}}(h^2)$ is contained in the right half-plane.*

*Proof.* Let $x \in \mathcal{A}$ satisfying $||x|| = 1$, $h \in \mathcal{H}$ and $t \in \mathbb{R}$. Then for some $f \in \mathcal{D}_{\mathcal{A}}(x)$ we have $||(e + ith)x|| \geq |f((e + ith)x)| = |1 + itf(h)| \geq 1$ which implies that $||(e + ith)x|| \geq ||x||$, $\forall x \in \mathcal{A}$.
Setting $x = (e + ith)y$ we have,

$$||(e + t^2h^2)y|| \geq ||y|| \implies ||e + t^2h^2|| \geq 1. \qquad (1)$$

Since $\sigma_{\mathcal{A}}(h^2) = \sigma_{\mathcal{A}}(h)^2$ and $\sigma_{\mathcal{A}}(h) \subseteq \mathcal{V}_{\mathcal{A}}(h) \subseteq \mathbb{R}$ we have $\sigma_{\mathcal{A}}(h^2) \geq 0 \implies -1 \notin \sigma_{\mathcal{A}}(h^2)$, which implies that $e+th^2 \in \mathcal{G}(\mathcal{A})$, $\forall t > 0$ and together with (1) $\implies ||(e + th^2)^{-1}|| \leq 1$. Now by the definition of the exponential function we have,

$$||\exp(-th^2)|| = ||\lim_{n\to+\infty}(e + th/n)^{-n}|| \leq 1.$$

Concluding the proof:

$$\sup\{\operatorname{Re} \mathcal{V}_A(-h^2)\} = \sup\{\frac{1}{t}\log||exp(-th^2)|| : t > 0\} \leq 0.$$

□

**Lemma 3.2.4.** *Let $\mathcal{A}$ a complex unital Banach algebra. If $h \in \mathcal{H}$ and $f \in \mathcal{D}_{\mathcal{A}}(e)$, then $f(h)^2 \leq \operatorname{Re} f(h^2)$.*

*Proof.* Let $g \in \mathcal{D}_{\mathcal{A}}(e)$, since $g(h - f(h)e) = g(h) - g(f(h)) \in \mathbb{R}$ we have $h - f(h)e \in \mathcal{H}$ and by Theorem 2.1.2. it is implied:

$$\begin{aligned} &\operatorname{Re}(f(h - f(h)e)^2) \geq 0 \implies \operatorname{Re}(f(h^2 - 2hf(h) + f(h)^2)) \geq 0 \\ &\implies \operatorname{Re}(f(h^2) - f(h)^2) \geq 0 \implies f(h^2) \leq \operatorname{Re}(f(h^2)). \end{aligned}$$

□

**Lemma 3.2.5.** *Let $\mathcal{A}$ be a unital Banach Algebra and $h, k \in \mathcal{H}$. Then $i(kh - kh) \in \mathcal{H}$.*

*Proof.* Let $h, k \in \mathcal{A}$ and $\lambda \in \mathbb{R}$. By Lemma 3.2.2. we have that for any $a \in \mathcal{A}$ and $b \in \mathcal{H}$, we have $||ae^{ib}|| = ||a||$. Hence the following holds,

$$||e^{i\lambda h} e^{i\lambda k} e^{-i\lambda h} e^{-i\lambda k}|| = 1.$$

If we expand the exponentials we get,

$$||e - \lambda^2(hk - kh)|| = 1 + O(\lambda^3),$$

and $||e + \lambda(hk - kh)|| = 1 + o(\lambda)$. As a result

$$\lim_{\lambda \to 0} \frac{1}{\lambda}\{||e + i\lambda(i(hk - kh))|| - 1\} = 0.$$

Hence again by Lemma 3.2.2. $i(hk - kh) \in \mathcal{H}$. □

## 3.3 The Proof of the Theorem

After the excessive description of the concept of hermicity in terms of numerical range we are almost ready of providing the Vidav-Palmer Theorem. In [**24**] Vidav proved the following,

**Theorem.** *Let $\mathcal{A}$ be a complex unital algebra such that,*
*(i) $\mathcal{A} = \mathcal{H} + i\mathcal{H}$,*
*(ii) if $h \in \mathcal{H}$, then $h^2$ is normal.*
*Then there exists a bicontinuous isomorphism $a \mapsto T_a$ of $\mathcal{A}$ with a $C^*$-algebra such that $T_h$ is self-adjoint and $|T_h| = ||h||$, whenever $h \in \mathcal{H}$.*

In 1966, E. Berkson in [**26**] and B. W. Glickfield in [**27**], showed by different methods that the mapping $a \mapsto T_a$ is an isometry. It was not until 1968 that T. W. Palmer in [**12**] proved that condition (ii) is not necessary and and gave a simplified proof on the isometric part.

The proof we chose to present is a modern one provided in 2013 by P. G. Spain in [**10**]. Spain's proof is a natural resumption of Ptak's characterization of Hermitian *-algebras, because as we will see later Ptak's functional is the only possible C*-norm on a Vidav algebra.

**Lemma 3.3.1.** *Let $U \subseteq \mathbb{C}$ compact. Then* $\max \operatorname{Re}\{U\} = \max \operatorname{Re}\{conv(U)\}$.

*Proof.* Let $\max \operatorname{Re}\{U\} = r$ and $z = x + yi \in \operatorname{conv}(U)$ with $r < x$. By Caratheodory's Theorem $z$ can be written as a convex combination of at most three elements of $U$ $z_j = x_j + iy_j$, $j = 1, 2, 3$. Hence $z = \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 \implies \operatorname{Re}\{z\} = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \leq (\lambda_1 + \lambda_2 + \lambda_3) r = r$ □

**Proposition 3.3.1.** *Let $\mathcal{A}$ be a unital Banach algebra. If $x = h + ik \in \mathcal{A}$ for $h, k \in \mathcal{H}$ satisfying $hk = kh$. Then* $\operatorname{conv}(\sigma_{\mathcal{A}}(x)) = \mathcal{V}_{\mathcal{A}}(x)$.

*Proof.* Let $x = h + ik$ with $h, k \in \mathcal{H}$ with $kh = hk$ and $\mathcal{B}$ the maximal commutative algebra generated by $h, k$. Then by Gelfand's Theory

$$\sigma_{\mathcal{A}}(x) = \sigma_B(x) = \{\phi(x) : \phi \in \operatorname{Sp} \mathcal{A}\} = \{\phi(h) + i\phi(k) : \phi \in \operatorname{Sp} \mathcal{A}\},$$

which automatically implies that $\max \operatorname{Re} \sigma_{\mathcal{A}}(x) = \max \sigma_{\mathcal{A}}(h)$. Since $\sigma_{\mathcal{A}}(x)$ is compact by the previous lemma we have,

$$\max \operatorname{Re}(\operatorname{conv}(\sigma_{\mathcal{A}}(x))) = \max \operatorname{conv}(\sigma_{\mathcal{A}}(h)) = \max \mathcal{V}_{\mathcal{A}}(h).$$

If we do the same process for $-x$ we get $\min \operatorname{Re}(\operatorname{conv}(\sigma_{\mathcal{A}}(x))) = \min \mathcal{V}_{\mathcal{A}}(h)$. Similarly for the imaginary part we have $\max \operatorname{Im}(\operatorname{conv}(\sigma_{\mathcal{A}}(x))) = \max \mathcal{V}_{\mathcal{A}}(k)$ and $\min \operatorname{Im}(\operatorname{conv}(\sigma_{\mathcal{A}}(x))) = \min \mathcal{V}_{\mathcal{A}}(k)$.

To conclude assume that $\exists \lambda \in \mathcal{V}_{\mathcal{A}}(x) \backslash \operatorname{conv}(\sigma_{\mathcal{A}}(x))$. Since $\lambda \notin \operatorname{conv}(\sigma_{\mathcal{A}}(x))$, then $\operatorname{Re}(\lambda) \notin \mathcal{V}_{\mathcal{A}}(h)$ and $\operatorname{Im}(\lambda) \notin \mathcal{V}_{\mathcal{A}}(k)$. On the other hand $\lambda \in \mathcal{V}_{\mathcal{A}}(x) \implies \exists f \in \mathcal{D}_{\mathcal{A}}(e)$ such that,

$$\lambda = f(x) = f(h + ik) = f(h) + if(k) \in \mathcal{V}_{\mathcal{A}}(h) + i\mathcal{V}_{\mathcal{A}}(k),$$

which is a contradiction. □

**Definition 3.3.1.** *Let $\mathcal{A}$ be a complex unital Banach algebra. Consider $\mathcal{J} = \mathcal{H}+i\mathcal{H}$ the complex linear span of $\mathcal{H}$. The natural sesquilinear involution $* : \mathcal{J} \to \mathcal{J}$ is defined as $b = h + ik \mapsto b^* = h - ik$, $h, k \in \mathcal{H}$.*

**Definition 3.3.2.** *A complex unital Banach algebra $\mathcal{A}$ is called a Vidav algebra ( abbreviated V-algebra) if $\mathcal{A} = \mathcal{H}+i\mathcal{H}$, where $\mathcal{H}$ is the set of hermitian elements in terms of numerical range.*

**Proposition 3.3.2.** *Let $\mathcal{A}$ be a V-algebra. Then the natural sesquilinear involution is a well defined and continuous involution on $\mathcal{A}$.*

*Proof.* Firstly let us prove that it is well defined. Let $b = h+ik = h'+ik' \implies h - h' = i(k - k') \in \mathcal{H} \cap i\mathcal{H} = \{0\}$.
Now let us prove that $^*$ is an algebra involution. For any $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}$, we have $(\lambda b)^* = \overline{\lambda} b^*$. In order to prove that $(ab)^* = b^* a^*$ we have to make the following process: Let $h \in \mathcal{H}$, since $\mathcal{A}$ is a V-algebra we have $h^2 = p + iq$ and thus $h(p + iq) = (p + iq)h \implies hp - ph = i(qh - hq) \in \mathcal{H} \cap i\mathcal{H} = \{0\}$. Therefore $h^2 p = ph^2 \implies qp = pq$ and by Proposition 3.2.1.

$$\mathcal{V}_{\mathcal{A}}(h^2) = \mathrm{conv}(\sigma_{\mathcal{A}}(h^2)) = \mathrm{conv}(\sigma_{\mathcal{A}}(h)^2) \subseteq \mathbb{R} \implies h^2 \in \mathcal{H}.$$

Hence for $h, k \in \mathcal{H}$ automatically $(h + k)^2 \in \mathcal{H} \implies hk + kh \in \mathcal{H}$.
Now we can finally show that $^*$ is an algebra involution on $\mathcal{A}$: For $a = h + ik$ and $b = p + iq$, $ab + b^*a^* = (h + ik)(p + iq) + (p - iq)(h - ik) = (hp + ph) + i(kp - pk) + i(hq - qh) - (kq - qk) \in \mathcal{H}$ and $i(ab - b^*a^*) = i[(hp - ph) + i(hq - qh) + i(kp - pk) + (qk - kq)] \in \mathcal{H}$. Due to the unique representation of an element in a V-algebra we have,

$$ab = \frac{ab + b^*a^*}{2} + \frac{i(ab - b^*a^*)}{2i} \implies (ab)^* = \frac{ab + b^*a^*}{2} - \frac{i(ab - b^*a^*)}{2i} = B^*a^*$$

Let $b \in \mathcal{A} \implies b = h + ik$ for some $h, k \in \mathcal{H} \implies \mathcal{V}_{\mathcal{A}}(h), \mathcal{V}_{\mathcal{A}}(k) \subseteq R$. By the definition of the numerical range we have,

$$\mathcal{V}_{A}(b) = \{f(h + ik) : f \in \mathcal{D}_{\mathcal{A}}(e)\} = \{f(h) + f(ik) : f \in \mathcal{D}_{\mathcal{A}}(e)\}$$

which implies the following,

$$\max \mathrm{Re}\, \mathcal{V}_{\mathcal{A}}(b) = \max \mathcal{V}_{\mathcal{A}}(h)$$

$$\max \mathrm{Re}\, \mathcal{V}_{\mathcal{A}}(-b) = \max \mathcal{V}_{\mathcal{A}}(-h).$$

Without loss of generality let $\max \mathcal{V}_{\mathcal{A}}(h) \geq \max \mathcal{V}_{\mathcal{A}}(-h)$ which means that $|h|_v = \max \mathcal{V}_{\mathcal{A}}(h) = \max \mathrm{Re}\, \mathcal{V}_{\mathcal{A}}(b) \leq |x|_v \leq ||h+ik||$. If we do the same process for the imaginary part of $\mathcal{V}_{\mathcal{A}}(b)$ we have $|k|_v \leq ||h+ik||$. Finally,

$$||b^*|| = ||h-ik|| \leq ||h|| + ||k|| = |h|_v + |k|_v \leq 2||h+ik||.$$

The second equality above holds due to Sinclair's Theorem.

□

**Remark 3.** *Let $\mathcal{A}$ be a Vidav algebra and $^*$ the natural involution, then $\mathcal{A}$ is a hermitian Banach *-algebra and $\mathcal{S} = \mathcal{H}$*

The following Theorem provided by P. G. Spain in [**10**] is actually the one that opens the road for an alternative and shorter proof.

**Theorem 3.3.1.** *Let $\mathcal{A}$ a complex unital Banach algebra. If $b \in \mathcal{H} + i\mathcal{H}$ and $b^*b$ is hermitian, then $|b|_v \leq |b|_\Sigma$.*

*Proof.* Let $b = h+ik$ with $h, k \in \mathcal{H}$ then $b^*b = h^2+k^2+i(hk-kh)$. Since by Lemma 3.2.5. $i(hk-kh)$ is hermitian we have $b^*b \in \mathcal{H} \iff bb^* \in \mathcal{H} \iff h^2+k^2 \in \mathcal{H}$. By Sinclair's Theorem, $|b^*b|_v = |b^*b|_\sigma = |b|_\Sigma^2 = |bb^*|_\sigma = |bb^*|_v$. Let $f \in \mathcal{D}_{\mathcal{A}}(e)$ then,
$|f(b)^2| = f(h)^2+f(k)^2 \leq \mathrm{Re}(f(h^2))+\mathrm{Re}(f(k^2)) = \mathrm{Re}(f(h^2+k^2)) = \frac{1}{2}f(b^*b+bb^*) \leq \frac{1}{2}(|b^*b|_v + |bb^*|_v) = |b|_\Sigma^2$. □

**Theorem 3.3.2** (Vidav-Palmer)**.** *Let $\mathcal{A}$ be a V-algebra. Then with the natural involution and given norm, $\mathcal{A}$ is a $C^*$-algebra.*

*Proof.* Firstly, we will prove that Ptak's functional is a $C^*$-norm on $\mathcal{A}$. As observed in Remark 3 of Proposition 3.3.2. $\mathcal{A}$ is a hermitian *-algebra, hence by Corollary 2.1.2. and Proposition 2.1.5. $|\cdot|_\Sigma$ satisfies the triangle inequality (sub-additive) and is sub-multiplicative. Concerning the first property of the norm, let $b \in \mathcal{A}$, then $|b|_\Sigma \geq 0$ and by Theorem 3.3.1. and Corollary 3.3.1.

$$|b|_\Sigma = 0 \implies |b|_v = 0 \implies b = 0.$$

Ptak's functional also has the $C^*$-property due to Corollary 1.1.2.

Moreover by Theorem 3.1.7. and Proposition 3.3.2. $|\cdot|_\Sigma$ is equivalent to the given norm,

$$e^{-1}||b|| \leq |b|_v \leq |b|_\Sigma \leq (||b^*||||b||)^{1/2} \leq (2||b||^2)^{1/2} = 2^{1/2}||b||.$$

The last inequality is due to the penultimate line in the proof of Proposition 3.3.2.

Let $a \in \mathcal{A}$ satisfying $|a|_\Sigma = 1$, applying the Russo-Dye Theorem on the C*-algebra $(\mathcal{A}, |\cdot|_\Sigma)$, we obtain that $a \in \overline{\mathrm{conv}}(\mathcal{E})$. This means that there exists a sequence $\{a_n\}_{n\in\mathbb{N}} \in \mathrm{conv}(\mathcal{E})$ such that

$$|a_n - a|_\Sigma \to 0.$$

Since $|\cdot|_\Sigma$ and $||\cdot||$ are equivalent on $\mathcal{A}$ we obtain that,

$$||a_n - a|| \to 0 \implies ||a_n|| \to ||a||.$$

On the other hand by Lemma 3.2.2. we have $||e^{ih}|| = 1$. In addition, for any $n \in \mathbb{N}$, $a_n$ is a convex combination of elements in the form of $e^{ih}$ where $h \in \mathcal{S}$. Hence

$$||a_n|| \leq 1 \implies ||a|| \leq 1 \implies ||a|| \leq |a|_\Sigma,$$

which holds for any $b \in \mathcal{A}$. Thus by Sinclair's Theorem we obtain that for any $b \in \mathcal{A}$ the following hold,

$$||b^*b|| = |b^*b|_\sigma = |b|_\Sigma^2 \geq ||b||^2 \qquad (1)$$

and

$$||b^*||||b|| \geq ||b||^2 \implies ||b^*|| \geq ||b||.$$

The latter implies that,

$$||b^*|| \geq ||b|| \implies ||(b^*)^*|| \geq ||b^*|| \implies ||b|| \geq ||b^*|| \implies ||b|| = ||b^*||. \quad (2)$$

Finally by (2) we obtain,

$$||b^*b|| \leq ||b||||b^*|| = ||b||^2 \implies ||b^*b|| = ||b||^2. \quad (3)$$

Thus by (1) and (3), $||\cdot||$ satisfies the C*-property and $(\mathcal{A}, ||\cdot||)$ is a C*-algebra. $\square$

# Chapter 4

# C*-equivalent Algebras

## 4.1 C*-algebras

The purpose of this chapter is to provide some conditions in order a Banach *-algebra $\mathcal{A}$ to be a C*-algebra. In the first section we will provide a short presentation of the well-behaved structure of C*-algebras. The main result of the first section is the Gelfand-Naimark Theorem, probably the most important characterization theorem in the field of C*-algebras provided by I. M. Gelfand and M. A. Naimark in [**18**].

**Proposition 4.1.1.** *Let $\mathcal{A}$ be a C*-algebra. Then the involution is continuous.*

*Proof.* It suffices to show that $||x|| = ||x^*||$, for some $x \in \mathcal{A}$. By the C*-property we have,

$$||x^*x|| = ||x||^2 \implies ||x||^2 \leq ||x||||x^*|| \implies ||x|| \leq ||x^*||.$$

To conclude replace $x$ with $x^*$. □

**Definition 4.1.1.** *Let $\mathcal{A}$, $\mathcal{B}$ be *-algebras. A *-homomorphism is an algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ satisfying $\phi(a^*) = \phi(a)^*$, for any $a \in \mathcal{A}$.*

**Definition 4.1.2.** *Let $\mathcal{A}$ be a C*-algebra. The intersection of the kernels of all *-homomorphisms from $\mathcal{A}$ to $\mathbb{C}$ will be called the *-radical of $\mathcal{A}$ and be denoted as* $^*\mathrm{Rad}(\mathcal{A})$. *If* $^*\mathrm{Rad}(\mathcal{A}) = \{0\}$ *then $\mathcal{A}$ will be called star-semi-simple.*

**Proposition 4.1.2.** *Let $\mathcal{A}$ be a unital C*-algebra, then $\mathcal{A}$ is hermitian.*

*Proof.* Let $h \in \mathcal{S}$, $z = h + ite \in \mathcal{A}$, with $t \in \mathbb{R}$ and $\phi \in \operatorname{Sp}\mathcal{A}$. If $\phi(h) = a + ib$ with $a, b \in \mathbb{R}$, then $\phi(z) = a + i(b+t)$ and $z^*z = (h - ite)(h + ite) = h^2 + t^2 e$. Hence, the following holds,

$$a^2 + (b+t)^2 = |\phi(z)|^2 \leq ||z||^2 = ||z^*z|| \leq ||h^2|| + t^2.$$

Moving $t^2$ we obtain: $a^2 + b^2 + 2bt \leq ||h^2||$ for all $t \in \mathbb{R}$. Thus $b = 0$ and $\phi(h) \in \mathbb{R}$. □

**Theorem 4.1.1.** *Let $\mathcal{A}$ be a C*-algebra. Then $|xx^*|_\sigma = ||x||^2$, for any $x \in \mathcal{A}$.*

*Proof.* Let $h \in \mathcal{S}$, by the C*-property we obtain $||h^2|| = ||h||^2$ and then by induction $||h^{2^n}|| = ||h||^{2^n}$. Then, using Gelfand-Buerling's Formula we have:

$$|h|_\sigma = \lim_{n\to+\infty} ||h^{2^n}||^{\frac{1}{2^n}} = ||h||.$$

Taking $h = x^*x$, we have $|x^*x|_\sigma = ||x^*x|| = ||x||^2$ □

**Corollary 4.1.1.** *For any $x \in \mathcal{A}$ normal, $|x|_\sigma = ||x||$.*

*Proof.* Since $x, x^*$ commute the spectral radius is sub-multiplicative and as a result we have:

$$||x||^2 = |x^*x|_\sigma \leq |x|_\sigma |x^*|_\sigma = |x|_\sigma^2.$$

□

The following proposition was taken by [**3**]. In the original book the proposition refers to C*-algebras, but since the only property of a C*-algebra used was that self-adjointness implies real spectrum, we decided to change the hypothesis and include hermitian Banach *-algebras as well.

**Proposition 4.1.3.** *Let $\mathcal{A}$ be a commutative hermitian *-Banach algebra and $x \in \mathcal{A}$. Then for any $\phi \in \operatorname{Sp}\mathcal{A}$ we have*

$$\phi(x^*) = \overline{\phi(x)}.$$

*Proof.* Let $x = h + ik$, with $h, k \in \mathcal{S}$. Assume $\phi \in \operatorname{Sp}\mathcal{A}$, then since $\mathcal{A}$ is hermitian we have $\phi(h) \subseteq \mathbb{R}$ and $\phi(k) \subseteq \mathbb{R}$. Thus $\phi(x^*) = \phi(h) + i\phi(k) = \overline{\phi(x)}$. □

The above proposition practically tells us that every algebra homomorphism is an algebra *-homomorphism, which implies that the radical and the *-radical of $\mathcal{A}$ are the same. Another corollary is that semi-simplicity coincides with star-semi-simplicity.

**Definition 4.1.3.** *A topological space $\mathcal{X}$ will be called locally compact, if for every $x \in \mathcal{X}$ there is a compact neighborhood of $x$.*

**Lemma 4.1.1.** *Let $\mathcal{A}$ be a C*-algebra. If $\mathcal{B} \subseteq \mathcal{A}$ is a *-subalgebra of $\mathcal{A}$ then $\mathcal{B}$ is semi-simple.*

*Proof.* Let $\mathcal{B}$ be a *-subalgebra of $\mathcal{A}$ and $x \in \mathrm{Rad}(\mathcal{B})$. Then $x^*x \in \mathrm{Rad}(\mathcal{B})$ and by Theorem 4.1.1 and lemma 1.3.1. we have $||x^*x|| = |x^*x|_\sigma = 0$ which implies that $x^*x = 0 \implies x = 0$, making $\mathcal{B}$ semi-simple. $\square$

For the next proposition we are going to need a concept from General Topology called one-point or Alexandroff compactification. Due to its incisive use, Theorem 4.1.2. will be included without a proof. For more information of this matter and the proof, the reader can look in [**13**].

**Definition 4.1.4.** *Let $\mathcal{X}$ be a non-compact Hausdorff space. Choose a point $\infty \notin \mathcal{X}$, set $\mathcal{X}_\infty = \mathcal{X} \cup \{\infty\}$ and define the topology on $\mathcal{X}_\infty$ which has as a sub-basis the collection of,*

*(i) all open subsets of $\mathcal{X}$.*

*(ii) all subsets in the form $\mathcal{X}_\infty \setminus F$ where $F$ is a compact subset of $\mathcal{X}$.*

*(iii) all subsets of $\mathcal{X}_\infty$. The space $\mathcal{X}_\infty$ is called one-point compactification of $\mathcal{X}$.*

**Theorem 4.1.2** (Alexandroff)**.** *Let $\mathcal{X}$ be a topological space. The one-point compactification $\mathcal{X}_\infty$ is compact and $\mathcal{X}$ is a subspace. The space $\mathcal{X}_\infty$ is Hausdorff if and only if $\mathcal{X}$ is locally compact and Hausdorff.*

**Proposition 4.1.4.** *Let $\mathcal{A}$ be a commutative Banach algebra, then $\mathrm{Sp}\,\mathcal{A}$ is a locally compact Hausdorff space. In addition if $\mathcal{A}$ has a unit, then $\mathrm{Sp}\,\mathcal{A}$ is compact.*

*Proof.* Let $\phi_\infty$ be the identically zero functional on $\mathcal{A}$. We have that $\mathrm{Sp}^\infty \mathcal{A} = \mathrm{Sp}\,\mathcal{A} \cup \{\phi_\infty\}$ is a subset of the closed unit ball of $\mathcal{A}^*$. Thus, in order to prove that $\mathrm{Sp}_\infty \mathcal{A}$ is compact in the weak*-topology it suffices to show that it is a weak* closed subset of $\mathcal{A}^*$. Assume that $\phi \in \mathcal{A}^* \setminus \operatorname{Sp}_\infty \mathcal{A}$. Then there exist $x, y \in \mathcal{A}$ and $\delta > 0$ such that,

$$|\phi(x)\phi(y) - \phi(xy)| = \delta.$$

Let $\epsilon = \min\{\frac{1}{2}\delta^{1/2}, \frac{1}{4}((1 + |\phi(x)| + |\phi(y)|)^{-1}\delta)\}$ and let $\psi \in \mathcal{A}^*$ with

$$|\psi(x) - \phi(x)| < \epsilon, \quad |\psi(x) - \phi(y)| < \epsilon, \quad |\psi(xy) - \phi(xy)| < \epsilon.$$

Then $|\psi(x)\psi(y) - \phi(x)\phi(y)| \leq |\psi(x) - \phi(x)||\psi(y) - \phi(y)| + |\phi(x)||\psi(y) - \phi(y)| + |\psi(x) - \phi(x)||\phi(y)| < \epsilon^2 + \epsilon(|\phi(x)| + |\phi(y)|) < (\frac{1}{2}\delta^{1/2})^2 + \frac{1}{4}\delta = \frac{1}{2}\delta$. Since also $|\psi(xy) - \phi(xy)| < \frac{1}{4}\delta$, we have $\psi(xy) \neq \psi(x)\psi(y) \implies \psi \notin \operatorname{Sp}_\infty \mathcal{A}$. Thus $\phi$ has a weak* neighborhood that does not meet $\operatorname{Sp}_\infty \mathcal{A}$, which means that $\operatorname{Sp}_\infty \mathcal{A}$ is weak* closed and compact as a subset of a compact set.

Since the weak*-topology is always Hausdorff, by Alexandroff's Theorem we have $\operatorname{Sp} \mathcal{A}$ with the weak*-topology is a locally compact and Hausdorff topological space.

If $\mathcal{A}$ possesses a unit, by Proposition 1.3.1. we have,

$$\operatorname{Sp} \mathcal{A} = \{\phi \in \operatorname{Sp}_\infty \mathcal{A} : \phi(e) = 1\}.$$

Which means that $\operatorname{Sp} \mathcal{A}$ is a closed subset of $\operatorname{Sp}_\infty \mathcal{A}$ and therefore compact. □

**Theorem 4.1.3** (Gelfand-Naimark)**.** *Let $\mathcal{A}$ be a commutative unital Banach *-algebra. Then the Gelfand Transform $\widehat{\cdot} : \mathcal{A} \mapsto \mathcal{C}(\operatorname{Sp} \mathcal{A})$ is an isometric *-isomorphism if and only if $\mathcal{A}$ is a C*-algebra.*

*Proof.* Let $x \in \mathcal{A}$, assume that $\widehat{\cdot}$ is an isometric *-isomorphism. Then,

$$||x^*x|| = ||\widehat{x^*x}||_\infty = ||\widehat{(x^*)}\widehat{x}||_\infty = ||\overline{\widehat{x}}\widehat{x}||_\infty = |||\widehat{x}|^2||_\infty = ||\widehat{x}||_\infty^2 = ||x||^2.$$

Which proves that $\mathcal{A}$ is a C*-algebra.

Conversely, assume that $\mathcal{A}$ is a C*-algebra, by proposition 4.1.2. we have that for any $h \in \mathcal{S}$, $\sigma_{\mathcal{A}}(h) \subset \mathbb{R}$, which means that $\widehat{h}(\phi) \in \mathbb{R}$, for any $\phi \in \operatorname{Sp} \mathcal{A}$. Let $x \in \mathcal{A}$, then $x = \frac{(x+x^*)}{2} + \frac{i(x-x^*)}{2i}$. Hence, we have,

$$\widehat{(x^*)}(\phi) = \phi(x^*) = \phi(\frac{(x+x^*)}{2} - \frac{i(x-x^*)}{2i}) = \phi(\frac{(x+x^*)}{2}) - i\phi(\frac{(x-x^*)}{2i}) =$$

$$\overline{\phi(\frac{(x+x^*)}{2}) + i\phi(\frac{(x-x^*)}{2i})} = \overline{\phi(x)} = \overline{\widehat{x}(\phi)}.$$

which means that $\widehat{\cdot}$ is a *-homomorphism.

To prove that $\widehat{\cdot}$ is isometric, let $h \in \mathcal{S}$, note that by Theorem 1.3.2. and Corollary 4.1.1. we have,

$$||\widehat{h}||_\infty = |h|_\sigma = ||h||.$$

Now let $x \in \mathcal{A}$, we have,

$$||\widehat{x}||_\infty^2 = ||\overline{\widehat{x}}\widehat{x}||_\infty = ||\widehat{x^*x}||_\infty = ||x^*x|| = ||x||^2.$$

The second equality comes from the fact that $\widehat{\cdot}$ is a *-homomorphism, the third from Corollary 4.1.1. and the last from the C*-property of $||\cdot||$.

To complete the proof, we have to show that $\widehat{\cdot}$ is surjective. Since $\mathcal{A}$ is complete and $\widehat{\cdot}$ is isometric, we have that the range of $\widehat{\cdot}$, $\mathrm{Ran}\widehat{\cdot}$ is closed in $\mathcal{C}(\mathrm{Sp}\,\mathcal{A})$ and since $\widehat{\cdot}$ is a *-homomorphism then $\mathrm{Ran}\widehat{\cdot}$ is a closed *-subalgebra of $\mathcal{C}(\mathrm{Sp}\,\mathcal{A})$ which contains $\widehat{e}$. Now let $\phi, \psi \in \mathrm{Sp}\,\mathcal{A}$ with $\phi \neq \psi$. By the definition of Gelfand Transform, the re is a $x \in \mathcal{A}$ such that $\phi(x) \neq \psi(x) \implies \widehat{x}(\phi) \neq \widehat{x}(\psi)$. Thus $\mathrm{Ran}\widehat{\cdot}$ separates the points of $\mathrm{Sp}\,\mathcal{A}$, and by Stone-Weierstrass Theorem we have that $\overline{\mathrm{Ran}\widehat{\cdot}} = \mathcal{C}(\mathrm{Sp}\,\mathcal{A})$, but since $\mathrm{Ran}\widehat{\cdot}$ is closed,

$$\mathrm{Ran}\widehat{\cdot} = \mathcal{C}(\mathrm{Sp}\,\mathcal{A}).$$

□

**Theorem 4.1.4.** *Let $\mathcal{A}$ be a C*-algebra without a unit. Then $\mathcal{A}$ is isometrically *-isomorphic to a C*-subalgebra of codimension 1 in a unital C*-algebra.*

*Proof.* Let $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ with the involution $x \oplus \lambda \mapsto x^* \oplus \overline{\lambda}$, multiplication $(x \oplus \lambda)(y \oplus \mu) = (xy + \lambda y + \mu x) \oplus \lambda\mu$ and the norm:

$$||x \oplus \lambda||_{\tilde{\mathcal{A}}} = \sup\{||xy + \lambda y|| : ||y|| \leq 1\}.$$

□

**Theorem 4.1.5.** *Let $\mathcal{A}$ be a commutative C*-algebra without a unit. Then there is a locally compact, non-compact, Hausdorff space $\mathcal{X}$ such that $\mathcal{A}$ is isometrically *-isomorphic to $\mathcal{C}_0(\mathcal{X})$.*

*Proof.* Let $K = \mathrm{Sp}\,\tilde{\mathcal{A}}$. Then by Proposition 4.1.4. $K$ is compact and by Theorem 4.1.3. we have $\tilde{\mathcal{A}} \simeq \mathcal{C}(K)$. Hence by the previous theorem $\mathcal{A}$ is isometrically *-isomorphic to a C*-subalgebra of $\mathcal{C}(K)$. Let $\phi \in \operatorname{Sp} \tilde{\mathcal{A}}$ defined as,

$$\phi(x) = \begin{cases} 0 & , x \in \mathcal{A} \subset \tilde{\mathcal{A}} \\ 1 & , x = e \end{cases}.$$

Thus for any $x \in \mathcal{A}$ we have $\widehat{x}(\phi) = 0$, which implies that the range of $\mathcal{A}$ under the Gelfand Transform $\widehat{\mathcal{A}} = \{f \in \mathcal{C}(K) : f \text{ vanishes at } \phi\}$.

Assume $f \in \mathcal{C}(K)$ satisfying $f(\phi) = 0$ and $x \in \tilde{\mathcal{A}}$ such that $\widehat{x} = f$. This means that $x = y + \lambda e$ for some $y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Then we have,

$$f(\phi) = 0 \implies \widehat{x}(\phi) = \phi(x) = 0 \implies \phi(x) + \mu\phi(e) = 0 \implies \mu = 0.$$

Thus $x \in \mathcal{A}$ and the Gelfand Transform on $\tilde{\mathcal{A}}$ maps $\mathcal{A}$ onto the subalgebra $\{f \in \mathcal{C}(K) : f \text{ vanishes at } \phi\}$.

Now let $\mathcal{X} = K \setminus \{\phi\}$. Then $\mathcal{X}$ is locally compact and the restriction map $f \mapsto f|_{\mathcal{X}}$ is an isometric *-isomorphism between $\{f \in \mathcal{C}(K) : f(\phi) = 0\}$ and $\mathcal{C}_0(\mathcal{X})$, which means that $\mathcal{A} \simeq \mathcal{C}_0(\mathcal{X})$.

In order to prove that $\mathcal{X}$ is not compact, let $\mathcal{X}$ be a compact then $\phi$ would be an isolated point of $K$ and by the Hahn-Banach Theorem there is an element $e \in \mathcal{A} \subset \tilde{\mathcal{A}}$ such that

$$\widehat{e}(\psi) = \begin{cases} 0 & , \psi = \phi \\ 1 & , otherwise \end{cases}.$$

Which means that $e$ is a unit for $\mathcal{A}$, which is a contradiction. Hence $\mathcal{X}$ is not compact. □

## 4.2 Locally C*-equivalent algebras

The concept of Locally C*-equivalent algebras was first exploited by B. A. Barnes in 1972 in his paper *Locally B*-equivalent algebras* [**14**]. The motivation of this investigation arises from a problem in algebraic physics. As J. Cuntz writes in [**16**] reffering to Theorem 4.3.1.: "Segal postulates that the observables in Quantum Theory correspond to the self-adjoint elements of a C*-algebra. Since the observables behave locally like continuous functions, here our theorem may serve to replace the word C*-algebra to Banach *-algebra and thus to weaken the not physically justified assumptions."

**Definition 4.2.1.** *A Banach *-algebra $\mathcal{A}$ is said to be C*-equivalent if there is an equivalent norm to the given norm making $\mathcal{A}$ a C*-algebra.*

**Theorem 4.2.1.** *Let $\mathcal{A}$ be a Banach *-algebra and $\mathcal{B}$ a C*-algebra. If $\phi : \mathcal{A} \to \mathcal{B}$ is an algebra homomorphism such that $\phi(\mathcal{A})$ is a *-subalgebra of $\mathcal{B}$ then $\phi$ is automatically continuous.*

*Proof.* Since $\mathcal{B}$ is a C*-algebra and $\phi(\mathcal{A})$ is a *-subalgebra of $\mathcal{B}$ we have that $\phi(\mathcal{A})$ is semi-simple. By Proposition 1.3.4. we have that $\phi$ is continuous. $\square$

The following theorem is actually the equivalence between the definitions of a C*-equivalent algebra.

**Theorem 4.2.2.** *A Banach *-algebra $\mathcal{A}$ is C*-equivalent if and only if there exists a *-isomorphism $\phi$ from $\mathcal{A}$ onto a C*-algebra $\mathcal{B}$.*

*Proof.* Let $(\mathcal{A}, |\cdot|)$ be C*-equivalent, then there exists an equivalent algebra-norm $||\cdot||$ such that $||a^*a|| = ||a||^2$. It follows easily that the identity operator,

$$\mathcal{I} : (\mathcal{A}, |\cdot|) \to (\mathcal{A}, ||\cdot||)$$

is a *-isomorphism from $\mathcal{A}$ onto a C*-algebra.

Let $\mathcal{B}$ be a C*-algebra and $\phi : \mathcal{A} \to \mathcal{B}$ a *-isomorphism. Since $\phi$ is surjective we have that $\phi(\mathcal{A})$ is a *-subalgebra and thus semi-simple. By the previous theorem we have that $\phi$ is continuous and hence bounded.
Notice that $||\cdot|| \circ \phi$ is a C*-norm on $\mathcal{A}$, where $||\cdot||$ is the C*-norm on $\mathcal{B}$ and since $\phi$ is bounded we have,

$$\frac{||\phi(x)||}{|x|} \leq M,$$

for some $M > 0$ which implies that $\frac{1}{M}|x| \leq ||\phi(x)|| \leq M|x|$, making $\mathcal{A}$ a C*-equivalent algebra. $\square$

At this point we will present a theorem by J. Wichmann , provided in [**28**], which will be used to provide a characterization theorem for C*-equivalence algebras which is connected to the Vidav-Palmer Theorem. We will also use it later to provide a simplified proof that local C*-equivalence implies C*-equivalence in commutative Banach *-algebras.

**Theorem 4.2.3** (Wichmann)**.** *Let $\mathcal{A}$ be a complex unital Banach *-algebra. If the set,*

$$\mathcal{E} = \{e^{irh} : h \in \mathcal{S}\}$$

*is bounded, then $\mathcal{A}$ is hermitian, $\mathcal{S}$ is closed and the involution is continuous. In addition, $\mathcal{U}$ is bounded.*

*Proof.* First we will prove that $\mathcal{A}$ is hermitian. Let $K = \sup\{||e^{ih}|| : h \in \mathcal{S}\}$, then by Buerling-Gelfand's Formula, for every $r \in \mathbb{R}$ we have:

$$|e^{irh}|_\sigma = \lim_{n\to+\infty} ||e^{inh}||^{\frac{1}{n}} \leq \lim_{n\to+\infty} K^{\frac{1}{n}} = 1.$$

By Theorem 3.2.1. we have,

$$\max\{\operatorname{Re}\lambda \in \sigma_{\mathcal{A}}(ih)\} = \lim_{n\to+\infty} \log||e^{inh}||^{\frac{1}{n}} \leq \lim_{n\to+\infty} \log K^{\frac{1}{n}} = 0.$$

Similarly $\max\{\operatorname{Re}\lambda \in \sigma_{\mathcal{A}}(-ih)\} = 0$ and thus $\mathcal{A}$ is hermitian.

In order to prove that $\mathcal{S}$ is closed, we denote for any $t \geq 0$ the following set,

$$\mathcal{S}_{(t)} = \{h \in \mathcal{S} : ||e^{irh}|| \leq t\},$$

for any $r \in \mathbb{R}$. Let $t_0 \geq 0$, since the power series of the exponential function converges absolutely we have that $\mathcal{S}_{(t_0)}$ is closed. Notice that since $\mathcal{E}$ is bounded we have that $\mathcal{S} = \mathcal{S}_{(K)}$ and thus $\mathcal{S}$ is closed.

The continuity of the involution comes from the facts that any $x \in \mathcal{A}$ can be written as $x = h + ik$ for some $h, k \in \mathcal{S}$ and that $\mathcal{S}$ is closed. Applying Theorem 2.2.2. we obtain that $\mathcal{U}$ is bounded. $\square$

The following theorem was first proven by V. Pták in 1972. Since Pták's proof seems elusive and provided before Wichmann's Theorem, we decided to include a proof provided by Spain in **[10]** which makes the following theorem an instant consequence of Wichmann's Theorem and Spain's proof for the Vidav-Palmer Theorem. It also provides the connection between the Vidav-Palmer Theorem and the last chapter of this thesis. For Pták's proof and similar theorems one can look at **[5]**.

**Theorem 4.2.4.** *Let $\mathcal{A}$ be a unital Banach *-algebra. If the set of its exponential unitaries $\mathcal{E}$ is bounded, then $\mathcal{A}$ is $C^*$-equivalent.*

*Proof.* Let $K = \sup\{||e^{ih}|| : h \in \mathcal{S}\}$. Then by Theorem 4.2.4. we have that $\mathcal{U}$ is a bounded group and thus by Lemma 3.1.2. there is an equivalent norm, let us denote it as $|\cdot|_{\mathcal{U}}$, for which every $u \in \mathcal{U}$ satisfies $||u|| = 1$.

Since for every $h \in \mathcal{S}$ and $r \in \mathbb{R}$ we have $e^{irh} \in \mathcal{U}$ and thus $|e^{irh}|_{\mathcal{U}} = 1$. By Lemma 3.2.2. we obtain that every $h \in \mathcal{S}$ is $|\cdot|_{\mathcal{U}}$-hermitian (now in terms of numerical range) and hence $(\mathcal{A}, |\cdot|_{\mathcal{U}})$ is a V-algebra. This means that by Theorem 3.3.2. $(\mathcal{A}, |\cdot|_{\Sigma})$ is a C*-algebra which is equivalent to $|\cdot|_{\mathcal{U}}$ and thus equivalent to the given norm. $\square$

As Spain remarks in [**10**] here the isometric part proven by the Russo-Dye Theorem is not necessary.

Next we will finally introduce the concept of locally C*-equivalence algebras. It was first exploited by B. A. Barnes [**14**] in 1972 and investigated broadly in [**14**] and [**15**].

**Definition 4.2.2.** *A Banach *-algebra $\mathcal{A}$ is called locally C*-equivalent if, for every $h \in \mathcal{S}$ the closed *-subalgebra $\mathcal{A}(h)$ of $\mathcal{A}$ generated by $h$ is C*-equivalent.*

Regarding the structure of $\mathcal{A}(h)$ note that the existence of a unit is not necessary. Even if there is a unit, $\mathcal{A}(h)$ does not have to contain it. The algebra generated by a self-adjoint element $h$ and $e$ will be denoted as $\mathcal{A}'(h)$.

**Proposition 4.2.1.** *Let $\mathcal{A}$ be a Banach *-algebra and $\tilde{\mathcal{A}}$ the Banach *-algebra obtained by adjoining a unit to $\mathcal{A}$. Then the following hold,*

*(i) $\mathcal{A}$ is C*-equivalent if and only if $\tilde{\mathcal{A}}$ is C*-equivalent.*

*(ii) $\mathcal{A}$ is locally C*-equivalent if and only if $\tilde{\mathcal{A}}$ is locally C*-equivalent.*

*Proof.* (i) Assume $\mathcal{A}$ is a C*-equivalent algebra, then there exists an equivalent algebra-norm $||\cdot||$ on $\mathcal{A}$ and thus the norm $||\cdot||_{\tilde{\mathcal{A}}}$ as defined in Theorem 4.1.4. is a C*-norm on $\tilde{\mathcal{A}}$.

Now let us suppose that $\tilde{\mathcal{A}}$ is C*-equivalent, by Theorem 4.1.4. $\mathcal{A}$ is *-isomorphic to a *-subalgebra of $\tilde{\mathcal{A}}$, thus by Theorem 4.2.2. $\mathcal{A}$ is C*-equivalent.

(ii) Let $\mathcal{A}$ be locally C*-equivalent, then for any $h \in \mathcal{S}$ we have $\mathcal{A}(h)$ is C*-equivalent. Since $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ it follows that for $h \in \mathcal{S}$ and $\lambda \in \mathbb{C}$ we have:

$$\mathcal{A}(h) \subseteq \mathcal{A}(h + \lambda e) \subseteq \mathcal{A}'(h),$$

and by (i) we obtain that $\mathcal{A}'(h)$ is C*-equivalent, which implies that $\tilde{\mathcal{A}}$ is locally C*-equivalent.

Let us assume that $\tilde{\mathcal{A}}$ is locally C*-equivalent, then $\mathcal{A}'(h)$ is C*-equivalent. By Theorem 4.1.4. $\mathcal{A}(h)$ is *-isomorphic to *-subalgebra of $\mathcal{A}'(h)$ and again by Theorem 4.2.2. $\mathcal{A}$ is locally C*-equivalent. □

The Proposition above allows us to assume from now on that the Banach *-algebra used possesses (or not) a unit.

**Proposition 4.2.2.** *Let $\mathcal{A}$ be a locally C*-equivalent algebra. Then the following hold,*

*(i) $\mathcal{A}$ is hermitian.*

*(ii) $\mathcal{A}$ is semi-simple.*

*(iii) $\mathcal{A}$ is *-semi-simple.*

*Proof.* (i) Let $\mathcal{A}$ be locally C*-equivalent and $h \in \mathcal{S}$, since $\mathcal{A}(h)$ is C*-equivalent and thus hermitian we have $\sigma_{\mathcal{A}(h)}(h) \subseteq \mathbb{R}$. On the other hand $\sigma_{\mathcal{A}}(h) \subseteq \sigma_{\mathcal{A}(h)}(h) \subseteq \mathbb{R}$.

(ii) Let $h \in \mathcal{S}$ such that $h \in \operatorname{Rad}\mathcal{A}$. Since $\mathcal{A}(h)$ is C*-equivalent then there is an equivalent algebra C*-norm satisfying $||h^2|| = ||h||^2$. On the other hand $h \in \operatorname{Rad}(\mathcal{A}) \cap \mathcal{A}(h) \implies h^2 \in \operatorname{Rad}(\mathcal{A}) \cap \mathcal{A}(h)$ and therefore,

$$||h||^2 = ||h^2|| = |h|_\sigma = 0 \implies h = 0.$$

Now let $x = h + ik \in \operatorname{Rad}(\mathcal{A})$ for some $h, k \in \mathcal{S}$. Since $\operatorname{Rad}(\mathcal{A})$ is a two-sided ideal we have that $h, k \in \operatorname{Rad}(\mathcal{A}) \implies h = 0$ and $k = 0 \implies x = 0$ and as a result $\mathcal{A}$ is semi-simple.

(iii) Since $\mathcal{A}$ is hermitian by Proposition 4.1.3. we have that semi-simplicity and *-semi-simplicity coincide. □

**Definition 4.2.3.** *Let $\mathcal{B}$ be a set of complex-valued functions on some compact Hausdorff space $\mathcal{X}$ and $\mathcal{F}$ a complex-valued function defined on a subset $D$ of $\mathbb{C}$. We will say that $\mathcal{F}$ operates in $\mathcal{B}$, if $\mathcal{F} \circ f \in \mathcal{B}$ whenever $f \in \mathcal{B}$ and $f(\mathcal{X}) \subset D$.*

A useful theorem for the next section is the following by Y. Katznelson. It can be used to prove that local C*-equivalence implies C*-equivalence in the class of commutative Banach *-algebras and also to derive many useful consequences for the general case. I decided not to include the proof due to its length and prerequisites from other mathematical areas. Readers can look in chapters 7 and 8 of [**17**] for the proof and similar results. French speakers can directly look at the original papers by Katznelson [**29**], [**30**].

**Theorem 4.2.5** (Katznelson)**.** *Let $\mathcal{X}$ be a compact Hausdorff space and $\mathcal{A}$ a Banach *-algebra lying in $\mathcal{C}(\mathcal{X})$ which is conjugate closed, contains the constants and separates the points of $\mathcal{X}$. If the function $\sqrt{\cdot}$ operates in $\mathcal{A}$, then $\mathcal{A} = \mathcal{C}(\mathcal{X})$.*

Barnes in his work on locally C*-equivalent algebras provided many theorems where local C*-equivalence with additional conditions implies C*-equivalence. The following theorem is one of the most important of his work and the condition needed here is commutativity. In **[15]** there are two more very important theorems which instead of the commutativity condition, the algebra $\mathcal{A}$ needs to have a dense socle (the intersection of all minimal ideals) or every self-adjoint element must have at most countable spectrum.

**Theorem 4.2.6** (Barnes)**.** *Let $\mathcal{A}$ be a locally C*-equivalent algebra. If $\mathcal{A}$ is commutative then $\mathcal{A}$ is C*-equivalent.*

*Proof.* By Proposition 4.2.1. we can assume that $\mathcal{A}$ has a unit. Let $h \in \mathcal{S}$, since $\mathcal{A}$ is locally C*-equivalent then $\mathcal{A}(h)$ possesses an equivalent C*-norm $||\cdot||_h$. By Proposition 4.2.2. we have that $\mathcal{A}$ is hermitian and semi-simple, in addition due to the commutativity of $\mathcal{A}$, by Corollary 1.3.2. we have that the involution is continuous.
Since $\mathcal{A}(h)$ admits an equivalent C*-norm it is implied that for any $k \in \mathcal{A}(h)$, $||e^{ik}|| \leq M$ for some $M \geq 0$. We will use the sets defined in Wichmann's Theorem (4.2.4.). For $n \in \mathbb{N}$. define

$$\mathcal{S}_n = \{h \in \mathcal{S} : ||e^{ih}|| \leq n\}.$$

We have that $S_n$ is closed for any $n \in \mathbb{N}$ and $\cup_{n=\in\mathbb{N}}\mathcal{S}_n = \mathcal{S}$. Thus by Baire's Category Theorem there exists $m \in \mathbb{N}$ such that the interior of $\mathcal{S}_m$ is non-empty which implies that there exists $h_0 \in \mathcal{S}_m$ and $r > 0$ such that $\mathcal{B}(h_0, r) \subseteq \mathcal{S}_m$.
Let $h \in \mathcal{S}$ satisfying $||h|| < r$, then since $\mathcal{A}$ is commutative we obtain,

$$||e^{ith}|| \leq ||e^{it(h+h_0)}||||e^{-ith_0}|| \leq m^2.$$

To conclude we have that $||e^{ith}|| \leq m^2$ for any $h \in \mathcal{S}$, thus by Theorem 4.2.5. $\mathcal{A}$ is C*-equivalent. $\square$

## 4.3 Local C*-equivalence implies C*-equivalence

The last section of this Master's Thesis will be dedicated to the presentation of J. Cuntz's proof to the conjecture of Barnes stating that local C*-equivalence implies C*-equivalence. In his work Barnes was unable to prove this theorem which remained a conjecture until 1975. Cuntz in [**16**] provides an ingenious proof introducing the concept of a K-indecomposable algebra.

Note that, since $\mathcal{A}$ is a locally C*-equivalent algebra and thus hermitian and star-semi-simple then we can easily prove that $\mathcal{A}$ possesses a C*-norm (this C*-norm is actually Pták's functional). The aim is to prove that this C*-norm is actually equivalent to the given norm. Throughout the section we will denote $||\cdot||$ as the given norm and $||\cdot||_C$ as the C*-norm. Although the proof will be stated as small lemmas let us give the theorem.

**Theorem 4.3.1.** *Let $\mathcal{A}$ be a locally C*-equivalent algebra, then $\mathcal{A}$ is C*-equivalent.*

The following formulation is equivalent.

**Theorem 4.3.2.** *Let $\mathcal{A}$ be a Banach *-algebra. If for every $h \in \mathcal{S}$ there exists a locally compact Hausdorff space $\mathcal{X}$ such that $\mathcal{A}(h)$ is *-isomorphic to the algebra $\mathcal{C}_0(\mathcal{X})$ of all continuous complex-valued functions vanishing at infinity on $\mathcal{X}$, then $\mathcal{A}$ is C*-equivalent.*

**Definition 4.3.1.** *Let $\mathcal{A}$ be a locally C*-equivalent algebra and $K > 0$ a constant. We will call $\mathcal{A}$ K-indecomposable if it has the following property. For any normal $x \in \mathcal{A}$ and any not C*-equivalent Banach *-subalgebra $\mathcal{B}$ of $\mathcal{A}$ satisfying $x \cdot \mathcal{B} = \mathcal{B} \cdot x = \{0\}$, we have $||x|| \leq K||x||_C$.*

**Lemma 4.3.1.** *Let $\mathcal{A}$ be a unital locally C*-equivalent algebra that is not C*-equivalent. Then there is a constant $K > 0$ and a Banach *-subalgebra $\mathcal{A}'$ of $\mathcal{A}$ containing the unit such that: $\mathcal{A}'$ is locally C*-equivalent, not C*-equivalent and K-indecomposable.*

*Proof.* Assume the opposite. Since $\mathcal{A}$ is not 1-indecomposable then there exists a normal $x_1 \in \mathcal{A}$ and a *-subalgebra $\mathcal{B}_1$ of $\mathcal{A}$ which is not C*-equivalent, $x_1 \cdot \mathcal{B}_1 = \mathcal{B}_1 \cdot x_1 = \{0\}$ and $||x_1|| > ||x_1||_C$. Denote as $\tilde{\mathcal{B}_1}$ the *-subalgebra obtained by adjoining a unit to $\mathcal{B}_1$. As a result we have that $x_1$ commutes with every element of $\tilde{\mathcal{B}_1}$.

Similarly, let $x_1, x_2, ..., x_{n-1} \in \mathcal{A}$ normal and commuting with each other and with a closed *-subalgebra $\tilde{\mathcal{B}}_{n-1}$, that is not C*-equivalent and has been constructed as above in order to satisfy,

$$||x_i|| > i||x_i||_C,$$

with $i \in \{1, 2, ..., n-1\}$. By the construction of these *-subalgebras we have that $\tilde{\mathcal{B}}_{n-1}$ is not n-indecomposable, which means that there exists $x_n \in \tilde{\mathcal{B}}_{n-1}$ and a closed not C*-equivalent *-subalgebra $\mathcal{B}_n$ of $\tilde{\mathcal{B}}_{n-1}$ such that $x_n \cdot \mathcal{B}_n = \mathcal{B}_n \cdot x_n = \{0\}$ and $||x_n|| > n||x_n||_C$. Also $x_1, x_2, ..., x_n$ commute with $\tilde{\mathcal{B}}_n$.

Inductively, we construct a commutative subset $\mathcal{N} = \{x_n : n \in \mathbb{N}\}$ of normal elements, which satisfies the inequality $||x_n|| > n||x_n||_C$, this is a contradiction because the *-subalgebra generated by $\mathcal{N}$ is locally C*-equivalent and commutative and hence C*-equivalent. □

From now on $\mathcal{A}$ denotes a fixed unital locally C*-equivalent but not C*-equivalent that is K-indecomposable for a fixed $K > 0$.

**Lemma 4.3.2.** *Let $x \in \mathcal{A}$ and $\mathcal{X}$ a subset of $\mathcal{A}$ such that $x\mathcal{X} = \mathcal{X}x = \{0\}$.*

*(i) If $x \in \mathcal{A}$ is normal and $\frac{||x||}{||x||_C} > K$, then $||\cdot||$ and $||\cdot||_C$ are equivalent on $\mathcal{X}$.*

*(ii) If $\frac{||x||}{||x||_C} > 2K$, then $||\cdot||$ and $||\cdot||_C$ are equivalent on $\mathcal{X}$.*

*Proof.* (i) If $\mathcal{B}$ is the closed *-subalgebra of $\mathcal{A}$ generated by $\mathcal{X}$, then $x \cdot \mathcal{B} = \mathcal{B} \cdot x = \{0\}$. Let $\mathcal{B}$ be not C*-equivalent, then due to the fact that $\mathcal{A}$ is K-indecomposable we have $||x|| \leq ||x||_C$, which is clearly a contradiction. Hence $\mathcal{B}$ is C*-equivalent.

(ii) Let $||x||_C = 1$, and $x = x_1 + ix_2$, where $x_1 = (x + x^*)/2$ and $x_2 = (x - x^*)/2i$ both self-adjoint. Then $||x_1||_C, ||x_2||_C \leq 1$ and by the triangle inequality we have $||x_1|| > K$ or $||x_2|| > K$. Thus either $x_1$ or $x_2$ satisfies the conditions of (i). □

At his point we will remind Cantor's Theorem from Functional Analysis which will be used for the proof of the next lemma.

**Theorem 4.3.3** (Cantor)**.** *Let $(\mathcal{X}, \rho)$ be a metric space. $\mathcal{X}$ is complete if and only if for every decreasing sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ of non-void closed subsets of $\mathcal{X}$ satisfying $\lim_{n\to\infty} \mathrm{diam}(\mathcal{F}_n) = 0$, there is a $x \in \mathcal{X}$, such that $\cap_{n\in\mathbb{N}}\mathcal{F}_n = \{x\}$.*

In the following lemma $e_h$ denotes the unit in $\mathcal{A}'(h)$.

**Lemma 4.3.3.** *Let $h \in \mathcal{S}$ of $\mathcal{A}$ such that $||h||_C = 1$ and $||h|| > 4K + ||e_h||$. Then the spectrum of $h$ contains two necessarily unique points $\psi_1, \psi_2$ which satisfy the following properties,*

*(i) If $f \in \mathcal{A}'(h)$ such that $f(\psi_1) = f(\psi_2) = 0$, then $||f|| \leq K||f||_C$.*

*(ii) If $f \in \mathcal{A}'(h)$ such that $0 \leq f \leq 1$ and $f(\psi_1) = 1$ and $f(\psi_2) = 0$, then $||f|| > K||f||_C$.*

*(iii) If $f \in \mathcal{A}'(h)$ such that $0 \leq f \leq 1$ and $f(\psi_1) = 0$ and $f(\psi_2) = 1$, then $||f|| > K||f||_C$.*

*Proof.* Note that $\sigma_{\mathcal{A}}(h)$ is a topological subspace of the interval $[-1, 1]$. If $g \in \mathcal{C}([-1, 1])$ then $\tilde{g} = g|_{\sigma_{\mathcal{A}}(h)} \in \mathcal{A}'(h)$.

We will inductively construct two sequences $\{g_n\}_{n\in\mathbb{N}}$ $\{g'_n\}_{n\in\mathbb{N}}$ of positive continuous functions on $[-1, 1]$ such that the supports of $g_n$ and $g'_n$ are intervals of length less than $3/n$, $\text{supp}(g_n) \subset \text{supp}(g_{n-1})$, $\text{supp}(g'_n) \subset \text{supp}(g'_{n-1})$, and such that $||\cdot||$, $||\cdot||_C$ are not equivalent on $\tilde{g}_n \cdot \mathcal{A} \cdot \tilde{g}'_n$. Define the function $e_0$ by $e_0(\lambda) = 1$ whenever $\lambda \in [-1, 1]$, such that $\tilde{e_0} = e$. We write $g_1 = g'_1 = e_0$.

Suppose that $g_1, ..., g_{n-1}$ and $g'_1, ..., g'_{n-1}$ satisfying the above conditions have been constructed. Choose $k_1, k_2, ..., k_n : [-1, 1] \to \mathbb{R}^+$ such that $k_1 + k_2 + ... + k_n = e_0$ and $l(\text{supp}(k_i)) < \frac{3}{n}$. Since $||\cdot||$, $||\cdot||_C$ are not equivalent on $\tilde{g}_{n-1} \cdot \mathcal{A} \cdot \tilde{g}'_{n-1}$ we have that for a given $r \in \mathbb{N}$ there is a $x_r \in \tilde{g}_{n-1} \cdot \mathcal{A} \cdot \tilde{g}'_{n-1}$ such that $||x_r||_C = 1$ and $||x_r|| > r$. Since $x_r = \sum_{0 \leq i,j \leq n} \tilde{k}_i x_r \tilde{k}_j$ from the triangle inequality we obtain that there are $i_r$ and $j_r$ between 1 and n, such that $||\tilde{k}_{i_r} x_r \tilde{k}_{j_r}|| > \frac{r}{n^2}$, while $||\tilde{k}_{i_r} x_r \tilde{k}_{j_r}||_C \leq ||x_r||_C = 1$. Notice that the integers between 1 and n are finitely many, while there are infinitely many $i_r$ and $j_r$, this means that there exist $i_0$ and $j_0$ such that $i_0 = i_r$ and $j_0 = j_r$ for infinitely many $r \in \mathbb{N}$. Then $||\cdot||$, $||\cdot||_C$ are not equivalent on $\tilde{k}_{i_0} \cdot \tilde{g}_{n-1} \cdot \mathcal{A} \cdot \tilde{g}'_{n-1} \cdot \tilde{k}_{j_0}$. We write $g_n = k_{i_0} g_{n-1}$ and $g'_n = k_{j_0} g'_{n-1}$.

Now let us denote the sets,

$$\mathcal{R}_n = \text{supp}(g_n + g'_n) \cap \sigma_{\mathcal{A}}(h).$$

Since the support of a continuous function is always closed, these sets form a decreasing sequence of non-empty compact sets and it follows from Cantor's Theorem, that the set $\mathcal{M} = \cap_{n=1}^{\infty} \mathcal{R}_n$ contains one or two points. One point, if the support of $g_n + g'_n$ is connected for every $n \in \mathbb{N}$ and two if there exists a $n_0 \in \mathbb{N}$ such that $\mathcal{R}_n$ has two components.

Now we will denote the set $\mathcal{I}_{\mathcal{M}} = \{f \in \mathcal{A}'(h) : f(\lambda) = 0, \forall \lambda \in \mathcal{M}\}$ and prove that if $y \in \mathcal{I}_{\mathcal{M}}$, then $||y|| \leq K||y||_C$. Let $f$ be a continuous function on $\sigma_{\mathcal{A}}(h)$ that vanishes in an open neighborhood $U$ of $\mathcal{M}$, this means that there exists a $n \in \mathbb{N}$ such that $\mathcal{R}_n \subset U$, for this particular $n$ the following holds,

$$f \cdot \tilde{g}_n \cdot \mathcal{A} \cdot \tilde{g}'_n = \tilde{g}_n \cdot \mathcal{A} \cdot \tilde{g}'_n \cdot f = f \cdot \tilde{g}'_n \cdot \mathcal{A} \cdot \tilde{g}_n = \tilde{g}'_n \cdot \mathcal{A} \cdot \tilde{g}_n \cdot f = \{0\}.$$

Since $f$ is normal and $\mathcal{A}$ is K-indecomposable it is implied that $||f|| \leq K||f||_C$. Also observe that functions like $f$ are $(||\cdot||, ||\cdot||_C)$-dense in $\mathcal{I}_{\mathcal{M}}$, which implies that

$$||y|| \leq K||y||_C,$$

for any $y \in \mathcal{I}_{\mathcal{M}}$.

Let us assume that $\mathcal{M}$ contains only one point $\mu$, then we can define $g = h - h(\mu)e_h$ where $h(\mu) \in \mathbb{R}$, $|h(\mu)| \leq 1$ and $g \in \mathcal{I}_{\mathcal{M}}$ satisfying $||g||_C \leq 2$. If this was true we would have,

$$||h|| \leq ||e_h|| + ||g|| \leq ||e_h|| + K||g||_C \leq ||e_h|| + 2K,$$

which is clearly a contradiction.

Hence $\mathcal{M}$ contains two points $\psi_1$ and $\psi_2$ and (i) has been proven. To prove (ii) and (iii) choose $f \in \mathcal{A}'(h)$ satisfying the conditions $0 \leq f \leq 1$, $f(\psi_1) = 1$ and $f(\psi_2) = 0$, then defining $g$ by:

$$h - h(\psi_1)f - h(\psi_2)(e_h - f) = g,$$

we obtain that $g \in \mathcal{I}_{\mathcal{M}}$ and since $|h(\psi_1)| \leq 1$, $|h(\psi_2)| \leq 1$ and $||h||_C = 1$, we obtain that $||g||_C \leq 2$ and

$$||h|| \leq 2||f|| + ||e_h|| + ||g|| \leq 2||f|| + ||e_h|| + 2K \implies$$

$$||f|| \geq \frac{1}{2}(||h|| - ||e_h|| - 2K) > \frac{1}{2}(4K + ||e_h|| - ||e_h|| - 2K) > K.$$

Finally (iii) can be shown in the same way. □

As Doran mentions in [**4**], the proof of the previous lemma practically allows us to choose two elements $k_1, k_2 \in \mathcal{A}'(h)$, which satisfy the conditions $0 \leq k_1, k_2 \leq 1$ and $k_1(\psi_1) = k_2(\psi_2) = 1$ and $k_1 \cdot k_2 = 0$, such that $||\cdot||$ and $||\cdot||_C$ are not equivalent on $k_1 \cdot \mathcal{A} \cdot k_2$.

The above holds because, we can choose sufficiently large $n \in \mathbb{N}$ such that the supports of $\tilde{g_n}$ and $\tilde{g_n}'$ are disjoint. Then choosing functions $k_1, k_2 \in \mathcal{A}'(h)$ with $0 \leq k_1, k_2 \leq 1$, $k_1 \cdot k_2 = 0$ and $k_1 = 1$ $k_2 = 1$ on $\mathrm{supp}(\tilde{g_n})$ and $\mathrm{supp}(\tilde{g_n}')$

respectively, we obtain that $k_1 \cdot \tilde{g_n} = \tilde{g_n}$ and $\tilde{g_n}' \cdot k_2 = \tilde{g_n}'$. Since $\mathcal{A}$ is a two-sided ideal it follows that:

$$k_1 \cdot \mathcal{A} \cdot k_2 \supseteq k_1 \cdot \tilde{g_n} \cdot \mathcal{A} \cdot \tilde{g_n}' \cdot k_2 = \tilde{g_n} \cdot \mathcal{A} \cdot \tilde{g_n}'.$$

Now the fact that $||\cdot||$ and $||\cdot||_C$ are not equivalent on $\tilde{g_n} \cdot \mathcal{A} \cdot \tilde{g_n}'$ implies that they are not equivalent on $k_1 \cdot \mathcal{A} \cdot k_2$ as well.
**Notation**: From now on we will denote by $\mathcal{K}$ the $||\cdot||$-closure of $k_1 \cdot \mathcal{A} \cdot k_2$.

**Lemma 4.3.4.** *The set $\mathcal{K}$ has the following properties,*
*(i) $||\cdot||_C$ and $||\cdot||$ are not equivalent on $\mathcal{K}$.*
*(ii) $||\cdot||_C$ and $||\cdot||$ are equivalent on $\mathcal{K} \cdot \mathcal{K}^*$ and $\mathcal{K}^* \cdot \mathcal{K}$.*
*(iii) $\mathcal{K} \cdot \mathcal{K} = \{0\}$.*
*(iv) If $u \in \mathcal{K}$, then $u \cdot \mathcal{A}(u^*u) \subset \mathcal{K}$.*

*Proof.* Properties (i), (iii), are obvious from the definition of $\mathcal{K}$.

In order to prove (iv), let $u \in k_1 \cdot \mathcal{A} \cdot k_2 \implies u = k_1 \cdot x \cdot k_2$ for some $x \in \mathcal{A}$. It follows that $u^*u = k_2^* \cdot x^* \cdot k_1^* k_1 \cdot x \cdot k_2$ and:

$$u(u^*u) = k_1 \cdot x \cdot k_2 \cdot k_2^* \cdot x^* \cdot k_1^* \cdot k_1 \cdot x \cdot k_2 \in k_1 \cdot \mathcal{A} \cdot k_2.$$

Since the above holds, we obtain that $u \cdot p(u^*u) \in k_1 \cdot \mathcal{A} \cdot k_2$, where $p$ is a polynomial satisfying $p(0) = 0$. By continuity of multiplication and the fact that $k_1 \cdot \mathcal{A} \cdot k_2$ is dense in $\mathcal{K}$, we obtain that $up(u^*u) \in \mathcal{K}$ for any $u \in \mathcal{K}$ and polynomial $p$. On the other hand by the Stone-Weierstrass theorem, we have that the expressions $p(u^*u)$ are dense in $\mathcal{A}(u^*u)$, which completes the proof.

Regarding (ii), by Lemma 4.3.3. we obtain that $||k_1|| > K||k_1||_C$ and $||k_2|| > K||k_2||_C$. Since $k_1 \cdot k_2 = 0$, we have

$$k_1 \cdot \mathcal{K}^* \cdot \mathcal{K} = \mathcal{K}^* \cdot \mathcal{K} \cdot k_1 = \{0\} = k_2 \cdot \mathcal{K} \cdot \mathcal{K}^* = \mathcal{K} \cdot \mathcal{K}^* \cdot k_2.$$

To complete the proof apply Lemma 4.3.2. □

**Lemma 4.3.5.** *The norms $||\cdot||$ and $||\cdot||_C$ are equivalent on $u \cdot \mathcal{A}'(u^*u)$, whenever $u \in \mathcal{A}$ and $u^2 = 0$.*

*Proof.* Suppose that $||\cdot||$ and $||\cdot||_C$ are not equivalent on $u \cdot \mathcal{A}'(u^*u)$ and without loss of generality that $||u||_C = 1$. Since $u^*u \geq 0$ we represent $\mathcal{A}'(u^*u)$ as the algebra of continuous functions on $\sigma_{\mathcal{A}}(u^*u) \subset [0, 1]$.

For a given $a \in (0, 1)$, we have the following linear spaces,

$$\mathcal{I}_a = \{uf : f \in \mathcal{A}'(u^*u), \mathrm{supp}(f) \subset [a, 1]\}.$$

and

$$\mathcal{J}_a = \{uf : f \in \mathcal{A}'(u^*u), \mathrm{supp}(f) \subset [0,a]\}.$$

By Ford's Square Root Lemma we obtain that $(u^*u)^{1/2}$ exists and thus we can define the mapping,

$$\Phi : uf \mapsto (u^*u)^{1/2} f.$$

Observe that $\mathcal{I}_a$ is $||\cdot||_C$-isometrically isomorphic to the space $\widehat{\mathcal{I}_a} = \{g \in \mathcal{A}'(u^*u) : \mathrm{supp}(g) \subset [a,1]\}$ through $\Phi$ since,

$$||(u^*u)^{1/2}f||_C^2 = ||[(u^*u)^{1/2}f]^*[(u^*u)^{1/2}f]||_C = ||f^*u^*uf||_C = ||(uf)^*(uf)||_C =$$
$$= ||uf||_C^2.$$

We have that $\widehat{\mathcal{I}_a}$ is $||\cdot||_C$-closed and $||\cdot||_C$-complete, which implies that $\mathcal{I}_a$ is $||\cdot||_C$-complete. This means that $\mathcal{I}_a$ is $||\cdot||$-closed and $||\cdot||$-complete. By the Open Mapping Theorem we obtain that the two norms are equivalent on $\mathcal{I}_a$.

Now let $0 < a < b < 1$. Then every $x \in u \cdot \mathcal{A}'(u^*u)$ can be written as $x = x_1 + x_2$ where $x_1 \in \mathcal{I}_a$, $x_2 \in \mathcal{J}_b$ and $||x_1||_C \le ||x||_C$, $||x_2||_C \le ||x||_C$. If $||\cdot||$ and $||\cdot||_C$ were equivalent on $\mathcal{J}_b$, we would have,

$$||x||_C \le ||x_1||_C + ||x_2||_C \le M_1||x_1|| + M_2||x_2|| \le (M_1 + M_2)||x||.$$

This means that $||\cdot||_C$ and $||\cdot||$ would be equivalent on $u \cdot \mathcal{A}'(u^*u)$, which contradicts the assumption. Thus $||\cdot||$ and $||\cdot||_C$ are not equivalent on $\mathcal{J}_b$, whenever $0 < b < 1$.

If $0 < b < a < 1$ and $uf \in \mathcal{I}_a$, $ug \in \mathcal{J}_b$, we have $(uf)(ug)^* = 0$ and $(ug)^*(uf) = g^*u^*uf = u^*ug^*f = 0$. On the other hand, since $u^2 = 0$ we have $up(u^*u)u = 0$ for any polynomial $p$. This means that $u \cdot \mathcal{A}'(u^*u)u = 0$ and as a result $(ufu)g = (ugu)f = 0$. Observe that $\mathcal{J}_b$ satisfies the hypothesis of Lemma 4.3.2. but $||\cdot||$ and $||\cdot||_C$ are not equivalent on $\mathcal{J}_b$, which implies that

$$||uf|| \le 2K||uf||_C,$$

for all $uf \in \mathcal{I}_a$.

At this point, note that the space $\mathcal{I} = \cup_{a>0}\mathcal{I}_a$ is $||\cdot||_C$-dense in $u \cdot \mathcal{A}'(u^*u)$. This follows directly from the fact that $\Phi(\mathcal{I})$ is $||\cdot||_C$-dense on $\Phi(u \cdot \mathcal{A}'(u^*u))$.

Let $\{x_n\}_{n\in\mathbb{N}} \in \mathcal{I}$ which $||\cdot||_C$-converges to $x \in u \cdot \mathcal{A}'(u^*u)$, then $\{x_n\}_{n\in\mathbb{N}}$ is a $||\cdot||$-Cauchy sequence since $||x_n - x_m|| \le 2K||x_n - x_m||_C$, therefore $\{x_n\}_{n\in\mathbb{N}}$ $||\cdot||$-converges to $x$ and as a result,

$$||x|| = \lim_{n\to\infty}||x_n|| \le \lim_{n\to\infty}||x_n||_C = 2K||x||_C,$$

which is a contradiction to the assumption. □

Now, observe that $\mathcal{A}(u^*u)$ as a subalgebra of $\mathcal{A}$ coincides with $\mathcal{A}((u^*u)^{1/2})$. This practically allows us to represent every element of the form $v^*v$ where $v \in u \cdot \mathcal{A}(u^*u)$ as a function in $\sigma_{\mathcal{A}}((u^*u)^{1/2})$. Also let us remind the reader that $\mathcal{A}(u^*u)$ consists of those functions in $\mathcal{A}'(u^*u)$ which vanish at 0.

**Lemma 4.3.6.** *Let $N > 0$ and $u \in \mathcal{K}$ satisfying $||u||_C = 1$ and $||u|| > N$. Then there exists an $\tilde{u} \in u \cdot \mathcal{A}'(u^*u)$ such that $||\tilde{u}||_C = 1$, $||\tilde{u}|| > N/2$ and $(\tilde{u}^*\tilde{u})^{1/2}$ vanishes on a neighborhood of 0 (as a function on $\sigma_{\mathcal{A}}((u^*u)^{1/2})$).*

*Proof.* By lemma 4.3.4. (iii) we have that $\mathcal{K} \cdot \mathcal{K} = \{0\}$, from which we obtain that $u^2 = 0$, implying that $u$ is not invertible. Since $u^2 = 0$ from lemma 4.3.5. follows that there exists $M > 0$ such that $||y|| \leq ||y||_C$ for any $y \in u \cdot \mathcal{A}(u^*u)$.

Now let $\epsilon > 0$ and $0 \leq f \leq 1$ a continuous function on $\sigma_{\mathcal{A}}((u^*u)^{1/2}) \subset [0, 1]$, such that the support of $f$ is contained in $[0, \epsilon]$ and $f$ is identically 1 on $[0, \epsilon/2] \cap \sigma_{\mathcal{A}}((u^*u)^{1/2})$. Since $(1 - f)$ vanishes at 0, we obtain that $1 - f \in \mathcal{A}((u^*u)^{1/2}) = \mathcal{A}(u^*u)$.

If $\epsilon < \min\{1, \frac{N}{2M}\}$, then $||u(e_{u^*u} - f)||_C = ||(u^*u)^{1/2}(e_{u^*u} - f)||_C = 1$, since $1 = ||u||_C^2 = ||u^*u||_C = ||(u^*u)^{1/2}||_C^2$, which also implies that $1 \in \sigma_{\mathcal{A}}((u^*u)^{1/2})$. Also

$$||u - u(e - f)|| = ||uf|| \leq M||uf||_C \leq M\epsilon < N/2.$$

The inequality $M||uf||_C \leq M\epsilon$ comes from the fact that $(u^*u)^{1/2}$ is the identity function in $\mathcal{C}(\sigma_{\mathcal{A}}((u^*u)^{1/2}))$ and it is less than $\epsilon$ on $\text{supp}(f)$. Hence $\tilde{u} = u(e - f)$ has the required properties.

□

**Lemma 4.3.7.** *Let $u, N, \tilde{u}$ with the properties of Lemma 4.3.6. and $N \geq 4K$. Also let $g$ be a continuous function on $\sigma_{\mathcal{A}}((u^*u)^{1/2})$, satisfying $0 \leq g \leq 1$ such that $g \equiv 1$ on a neighborhood of 0 and $(u^*u)^{1/2}g = 0$. Then the following hold,*

*(i) There exists $z \in u \cdot \mathcal{A}((u^*u)^{1/2})$ such that $g = e - z^*z$.*

*(ii) Writing $g' = e - zz^*$, we have $\tilde{u}g = 0$, $g'\tilde{u} = 0$, $g\tilde{u} = \tilde{u}$ and $\tilde{u}g' = \tilde{u}$.*

*(iii) $||\cdot||$ and $||\cdot||_C$ are equivalent on $g' \cdot \mathcal{K} \cdot g$.*

*Proof.* (i) Let the function

$$f(\lambda) = [\frac{(1 - g)(\lambda)}{(u^*u)(\lambda)}]^{1/2},$$

Since $(u^*u)^{1/2}$ is the identity function in $\mathcal{C}(\sigma_{\mathcal{A}}((u^*u)^{1/2}))$, f is continuous on $\sigma_{\mathcal{A}}((u^*u)^{1/2})$. Observe that $z = uf$ has the required properties.

(ii) By the definition of $g$ we have that,

$$0 = ||(\tilde{u}^*\tilde{u})^{1/2}g||_C^2 = ||g^*\tilde{u}^*\tilde{u}g||_C = ||(\tilde{u}g)^*(\tilde{u}g)||_C = ||\tilde{u}g||_C^2,$$

which implies that $\tilde{u}g = 0$. Since $\tilde{u} \in u\cdot\mathcal{A}(u^*u) \subseteq \mathcal{K}$ and $z \in u\cdot\mathcal{A}((u^*u)^{1/2}) = u\cdot\mathcal{A}(u^*u) \subseteq \mathcal{K}$ by Lemma 4.3.4. (iii), we obtain $z\tilde{u} = \tilde{u}z = 0$. As a result

$$g\tilde{u} = (e - z^*z)\tilde{u} = \tilde{u}$$

and

$$\tilde{u}g' = \tilde{u}(1 - zz^*) = \tilde{u}.$$

To obtain $g'\tilde{u} = 0$, let us write $\tilde{u} = u\tilde{f}$, where $f \in \mathcal{A}((u^*u)^{1/2}) = \mathcal{A}(u^*u)$ and $z = uf$ where $f$ is as in the proof of (i). Since $(\tilde{u}^*\tilde{u})^{1/2}g = 0$ and $(\tilde{u}^*\tilde{u})^{1/2} = (\tilde{f}^*(u^*u)\tilde{f})^{1/2}$, $\tilde{f}$ must vanish at all points where $g(\lambda) > 0$, hence,

$$\tilde{f}(e_{u^*u} - g) = \tilde{f}.$$

Then $g'\tilde{u} = (e_{u^*u} - uffu^*)\tilde{u} = \tilde{u} - uf^2u^*u\tilde{f} = \tilde{u} - u(e_{u^*u} - g)\tilde{f} = \tilde{u} - u\tilde{f} = 0$.

(iii) Since $\tilde{u} \in \mathcal{K}$ and $\mathcal{K}\cdot\mathcal{K} = \{0\}$ by (ii) we obtain:

$$\tilde{u}\cdot g'\cdot\mathcal{K}\cdot g = g'\cdot\mathcal{K}\cdot g\cdot\tilde{u} = \tilde{u}\cdot g\cdot\mathcal{K}^*\cdot g' = g\cdot\mathcal{K}^*\cdot g'\cdot\tilde{u} = \{0\}.$$

On the other hand, from the hypotheses we obtain,

$$||\tilde{u}|| > \frac{N}{2}||\tilde{u}||_C \geq 2K||\tilde{u}||_C.$$

Lemma 4.3.2. (ii) completes the proof. □

**Proof of Theorem 4.3.1.** Take and fix $u, g, g', z$ with the properties described in lemmas 4.3.6. and 4.3.7. By now, we have shown that $||\cdot||$ and $||\cdot||_C$ are equivalent on the set,

$$\mathcal{B} = \mathcal{K}^*\cdot\mathcal{K} \cup \mathcal{K}\cdot\mathcal{K}^* \cup g'\cdot\mathcal{K}\cdot g \cup \mathcal{A}'(u^*u).$$

This means that there exists a constant $C > 1$ such that $||y|| \leq C||y||_C$ for any $y \in \mathcal{B}$. Now every element $x \in \mathcal{K}$ can be written as,

$$x = x(e_{u^*u} - g) + (e_{u^*u} - g')xg + g'xg = xz^*z + zz^*xg + g'xg.$$

As a result, we have,

$$||x|| \leq ||xz^*||||z|| + ||z||||z^*x||||g|| + ||g'xg|| \implies$$

$$||x|| \leq C||z||||x||_C + C^2||z||||x||_C + C||x||_C \implies$$

$$||x|| \leq C^2(2||z|| + 1)||x||_C.$$

The second inequality holds since $xz^* \in \mathcal{K} \cdot \mathcal{K}^*$, $z^*x \in \mathcal{K}^* \cdot \mathcal{K}$, $g \in \mathcal{A}'(u^*u)$, $g'xg \in g' \cdot \mathcal{K} \cdot g$ and since $||xz^*||_C \leq ||x||_C$, $||z^*x||_C \leq ||x||_C$, $||g||_C \leq 1$, $||g'xg||_C \leq ||x||_C$.

From the inequality $||x|| \leq C^2(2||z|| + 1)||x||_C$ we obtain that $||\cdot||$ and $||\cdot||_C$ are equivalent in $\mathcal{K}$ which contradicts Lemma 4.3.4. (i). Therefore, every locally C*-equivalent algebra is C*-equivalent.

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