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APPLICATION OF MODE ANALYSIS METHOD FOR THE UNSTEADY LIFTING SURFACE PROBLEM

Diploma Thesis

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Introduction

The purpose of this thesis is the development of a numerical schema, based on the lifting surface theory, for the prediction of lifting or thrust forces produced by a single lifting surface (wing) or multiple surfaces arranged symmetrically (propeller).

Historically the first methods were based on the lifting line theory and were basically reversing the procedure used for the design problem (Kerwin [1959]). Although a simple and fast method, the lifting line theory could not describe the complex forms of propeller blades or new wing designs. Nevertheless this method is still being used as first approximation.

A better approximation to the problem was achieved by the use of the lifting surface theory, around 1960. The process was based on the linearization of the wing geometry, which was represented by a reference surface. Using this method the actual problem could be split into a camber and thickness problem.

Tsakonas et al [1968, 1973, 1983] developed the acceleration potential method for the steady and unsteady flow. A set of spanwise and chordwise mode functions were used to approximate the unknown loading distribution, whose amplitude could be determined.

Kerwin and Lee [1978], Van Gent [1977] and Greeley [1982] focused on the vortex lattice method. Here, the continuous distribution of vorticity is approximated by a set of line vorticies placed on the mean surface of the blade or wing (linearized lifting surface theory). The unknowns in this case are the intensities of the vorticies, which can be calculated by the implementation of the no-entrance boundary condition at specific points of the reference surface.

In the mode analysis method, developed by Cummings [1973], the vorticity distribution is approximated by six mode functions with unknonw amptitudes A_m. Each mode is the product of a trigonometric mode in the spanwise direction a two modes in the chordwise direction (the first corresponds to the flat plate loading and the second one to the loading of the camber surface). The amplitudes are determined by the no-entrance b.c at 8 spanwise and 4 chordwise positions and the system is solved with a least squares method.

In later years the non-linear problem was solved using a Boundary Elements Method (B.E.M). According to that method the real surface of the wing is approximated with a series of panels.

Hess and Valarenzo [1985] developed a B.E method, where the four-sided panels where used to approximate the upper and lower surface of the blade. Inside each panel is assumed a constant source-sink distribution and a dipole distribution which constant along the chord of the blade. The wake on the other hand is approximated like in the vortex lattice method. The contribution of non-lifting surfaces like the hub of the propeller is modeled using only a source-sink distribution. All control points are placed in the middle of the panels and implementing the no-entrance boundary condition they calculate the dipole and source-sink intensities.

The present analysis is based on the Segkos' work [1986] who studied the mode analysis method for the steady problem using a double Fourier series. The premise of this work is to generalize the theory to include the unsteady case of a freely moving body. Specifically the problem is no longer described using the moving coordinate system, x^i, but the inertial system $\underset{\bowtie}{x}^i$. This allows us, to examine the 'general' motion (high amplitudes compared to the chord) of the wing and solve the unsteady lifting surface problem. As a result, the program is divided into two separate programs. The first handles the geometry and motion of the wing and provides the input for the second program to solve the problem using the mode analysis method. At this point we mention that for simplicity we use the frozen wake model, although a free wake model could be also implemented.

The basic advantages of the mode analysis method compared to the vortex lattice method are :

- The approximation of the unknown lifting surface vorticity by a C^∞ dipole distribution function and not by C^0 dipole patches used in vortex lattice numerical approximation.
- The accurate satisfaction of the surface and vorticity and continuity equation.
- The analytical calculation of the self induction factors using the Glauert integral methodology.

The main disadvantage is the complexity in the required mathematical formalism to derive the proper expressions for all conditions entered (no entrance, pressure Kutta condition at the trailing edge and calculation of forces and momments) for the case of a Riemann surface (which mathematically is the lifting surface) embedded in a 3D Cartesian space.

The thesis consists of 8 sections :

Section 1, introduces the basic theory of tensor analysis and the tensorial notation which will be used systematically for the description of the coordiante systems.

Section 2, deals with the various coordinate systems which will be used to describe the motion and the geometry of the lifting surface. Specifically section 2.1 introduces the inertial system $\underset{\bowtie}{x}^i$ and the relative to it moving system x^i. We split the motion into transitional and rotational and describe the quantities needed to represent each one of them. Furthermore, we define the absolute and relative velocities to a point M. In section 2.2 the rest of the coordinate systems, used to describe the geometry of the lifting surface, are introduced. We examine the geometries of a propeller and a wing and give the relation connecting the coordinates of the various systems.

Section 3, examines the physical problem and make the necessary simplifying assumptions to derive the relations for the boundary conditions. It focuses on the no-entrance b.c, the continuity of vorticity and the pressure kutta condition at the trailing edge. Specifically, in section 3.2 we use the assumptions of perturbation theory to split the no entrance b.c into a thickness and a camber b.c and use the mean velocity (and velocity jump) on the reference surface to describe them. In section 3.3 we derive the exact relation for the pressure difference on the lifting surface and then linearize it. In section 3.4 we apply the Bernoulli 's equation on the vortex sheet and examine the kinematic and dynamic conditions on it. In section 3.5 we examine the kinematic conditions on the blade by applying the continuity of vorticity on the outline of the surface. Finally in section 3.6 we apply the representation theorem of velocity on the reference surface.

Section 4, applies the mode analysis method to each boundary condition examined in the previous section and derives the relations for the numerical solution of the problem. We start in section 4.1 by taking the results of the 2D lifting line theory referring to the (bound) vorticity and apply them in the 3D problem to get double Fourier series for the modified vorticity. In chapters 4.3, 4.4, 4.5 and 4.6 we substitute this series in the kinematic, dynamic and no entrance boundary conditions to get a more useful expression. It is worth noting that although we study the full series only a part of it (sinus terms) are needed to describe the problem at hand. In fact the selection of terms of the series should be examined for the general case, to ensure the speed and stability of the solution. In section 4.7 we introduce the numerical scheme for the specific case of the sinus bound vorticity series, for the blade and the wake. In section 4.8 we describe the contribution of the wake (Kutta strip and free wake) as well as the boundary of the lifting surface, using a lattice of vortexes. The contribution of the Kutta strip is split between the boundary of the lifting surface and the free wake. The first part is described using mode velocities which will be added to the mode velocities of the inner part of the lifting surface, in the no entrance b.c. The other part of the Kutta strip contribution is combined with the wake and is known at every instant. Finally in section 4.9 we create the linear system for the solution of the coefficients, A, of the modified bound vorticity.

Section 5, deals with the calculation of the pressure distribution and forces on the lifting surface. It includes the general description of the forces acting on the body, which is simplified to give the forces acting on the camber surface. In the next subsection we correct the produced force by adding the leading edge suction force, which cannot by calculated by the lifting surface theory.

Section 6, examines the Filon integration method (a modified Simpsons method) for the calculations of mode velocities. In section 6.2 we develop the general Filon method by approximating the integrand function (without the trigonometric term) using a Lagrange polynomial of any order.

Section 7, includes the the summary of the programs along with a description of their variables. There are also example of input and output files for each one of them.

Section 8, includes the results of the program for three different problems. First we compare lift coefficient as a function of angle of attack of a chordwise cross section with the 2D steady case experimental results of NACA 4412 airfoil. Next we take the theoretical results of the 2D heaving and pitching potion of a flat plate and compare the lifting coefficient and its phase with respect to the motion as functions of the reduced frequency. Finally we examine the thrust coefficient of a 3D flapping hydrofoil for various geometries and motion combinations. Those results are compared to the respective results of the UBEM program designed by G.Politis.

1 Brief review of tensor analysis

The introduction of the indicial notation and tensor algebra is necessary in order to cope with the complex relations that describe the physical problems. A formal approach to tensor analysis can be found in [1]. In this section we will mention only a few basic properties and relations of the tensors needed for the better understanding of the following analysis.

Since we use both space and surface tensors, for the distinction between the two will use the usual convention : Latin indices for the space tensors with values 1, 2, 3 and Greek indices for the surface tensors with values 1 and 2. Depending on the position of the index we distinguish two cases :

- Covariant component of a tensor : When the index is in the lower right part of the variable, as a subscript e.g. x_i, x_α.
- Contravariant component of a tensor : When the index is in the upper right part of the variable, as a supersctipt e.g. x^i, x^α

1.1 Space tensors

Supposing two oblique coordinate systems x^r and $\overline{x}^r$, and the using the summation convention the transformations between the two systems are :

$$\overline{x}^r = c_s^r \cdot x^s \tag{1.1 α}$$

$$x^r = \gamma_s^r \cdot \overline{x}^s \tag{1.2 α}$$

The general case of nonlinear transformation is given by relations :

$$\overline{x}^r = f^r(x^1, x^2, x^3) \tag{1.1 β}$$

$$x^r = g^r(\overline{x}^1, \overline{x}^1, \overline{x}^1) \tag{1.2 β}$$

Note that relations (1.2) are valid only in case the Jacobian of the direct transformation

$$c = \left| \frac{\partial \overline{x}^r}{\partial x^s} \right| \tag{1.3}$$

is non-zero. By differentiating relations (1.1α), (1.2α) or taking the differentials of relations (1.1β),(1.2β) we get:

$$c_s^r = \frac{\partial \overline{x}^r}{\partial x^s}, \qquad \gamma_s^r = \frac{\partial x^r}{\partial \overline{x}^s} \tag{1.4}$$

In case of linear transformation ((1.1α), (1.2α)), c_s^r, γ_s^r are constant. Otherwise in the general transformation case of (1.1β) and (1.2β), c_s^r, γ_s^r are functions of x^r (and thus of $\overline{x}^r$). A useful relation which connects c_s^r ,γ_s^r is :

$$c_m^r \cdot \gamma_s^m = \gamma_m^r \cdot c_s^m = \delta_s^r \tag{1.5}$$

where δ_s^r is Kronecker's delta :

$$\delta_s^r = \begin{cases} 1, & \text{if } r = s \\ 0, & \text{otherwise} \end{cases}$$

Let α be a number which in general can be a function of position. If α is constant under linear transformation

$$\overline{\alpha} = \alpha \tag{1.6}$$

then it's called a scalar or an invariant or a tensor of zero order.

A system of numbers or functions of position $\alpha^r = (\alpha^1, \alpha^2, \alpha^3)$, is called a contravariant vector, or contravariant tensor of first order if :

$$\overline{\alpha}^r = c_s^r \cdot \alpha^s \tag{1.7}$$

and a covariant tensor of first order, if:

$$\overline{\alpha}_r = \gamma_r^s \cdot \alpha_s \tag{1.8}$$

It can be proven that :

$$\alpha^r = \gamma_m^r \cdot \overline{\alpha}^m \tag{1.9}$$

$$\alpha_r = c_r^m \cdot \overline{\alpha}_m \tag{1.10}$$

Generalizing the definition of the tensor, we call the system $\alpha_{s_1 s_2 \ldots s_n}^{r_1 r_2 \ldots r_m}$, a covariant tensor of order n, with respect to the lower indices and contravariant tensor of order m, with respect to the upper indices, when :

$$\overline{\alpha}_{s_1 s_2 \ldots s_n}^{r_1 r_2 \ldots r_m} = \left(c_{p_1}^{r_1} \cdot c_{p_2}^{r_2} \ldots c_{p_m}^{r_m}\right)\left(\gamma_{s_1}^{q_1} \cdot \gamma_{s_2}^{q_2} \ldots \gamma_{s_n}^{q_n}\right) \cdot \alpha_{q_1 q_2 \ldots q_n}^{p_1 p_2 \ldots p_m} \tag{1.11}$$

Next we define the relative tensors of weight M. For the sake of simplicity we assume a mixed triple tensor, contravariant with respect to the upper index and covariant with respect to the lower indices, then the following relation should hold :

$$\overline{\alpha}_{st}^r = (\gamma)^M \cdot c_m^r \cdot \gamma_s^n \cdot \gamma_t^p \cdot \alpha_{np}^m \tag{1.12}$$

where

$$\gamma = \left|\frac{\partial x^i}{\partial \overline{x}^j}\right| \tag{1.13}$$

When M=0 we have the absolute tensors or simply tensors.

By the same logic, we define the general tensors, which are tensors under the linear transformation connecting the differentials between the initial and the final variables. In other words, general tensors are defined locally, with respect to a single point in space.

A basic concept, of the tensor analysis is that of the metric tensor. Referring to relations (1.1 α), (1.2 α), the metric tensor g_{mn}, is defined by the relation:

$$\delta^2 = g_{mn} \cdot x^m \cdot x^n \tag{1.13 α}$$

where δ, is the distance from the origin point O, to a point P = x^r, of a linear coordinate system. Otherwise in case of the general coordinate system the metric tensor is defined by the relation :

$$ds^2 = g_{mn} \cdot dx^m \cdot dx^n \tag{1.13 β}$$

where ds is the differential distance between two points.

If x^i is an orthonormal system and $\overline{x}^i$ is a general curvilinear system, then it can be shown that the metric tensor in the x^i system is given by:

$$g_{mn} = \frac{\partial x^i}{\partial \overline{x}^m} \frac{\partial x^j}{\partial \overline{x}^n} \delta_{ij}$$

which shows that the metric tensor is symmetric. Obviously for an orthonormal coordinate system the metric tensor takes the oversimplified form :

$$g_{mn} = \delta_{mn} \ , \ g^{mn} = \delta^{mn} \tag{1.14}$$

A useful type of tensor, is the e-system :

$$e^{rst} = \begin{cases} +1, & \text{if (r,s,t) is an even permutation of (1,2,3)} \\ -1, & \text{if (r,s,t) is an odd permutation of (1,2,3)} \\ 0, & \text{otherwise} \end{cases}$$

In a similar manner we define e_{rst}. It can be easily proved that the permutation symbols e_{rst}, e^{rst} are relative tensors of weight -1 and +1 respectively.

Below we list some very useful relations of δ^r_s , e^{rst} and e_{rst} :

$$\delta^r_r = 3 \tag{1.15 α}$$

$$e^{rmn} e_{rst} = \delta^m_s \delta^n_t - \delta^m_t \delta^n_s \tag{1.15 β}$$

$$e^{rmn} e_{rms} = 2! \delta^n_s \tag{1.15 γ}$$

$$e^{rmn} e_{rmn} = 3! \tag{1.15 δ}$$

$$e^{rst} \delta^s_r = 0 \tag{1.15 ϵ}$$

$$e^{rst} A_{st} = e_{rst} A^{st} \tag{1.15 ζ}$$

where A_{st} , A^{st} are symmetric tensors.

The determinant $\alpha \equiv |\alpha^m_n|$ of the second order tensor α^m_n is given by the relation :

$$\alpha = \frac{1}{3!}\ \delta^{ijk}_{rst}\ \alpha^r_i\ a^s_j\ \alpha^t_k$$

or alternatively (1.16)

$$e^{ijk}\ \alpha^r_i\ \alpha^s_j\ \alpha^t_k = \alpha\ e^{rst}$$
$$e_{ijk}\ \alpha^i_r\ \alpha^j_s\ \alpha^k_t = \alpha\ e_{rst}$$

where the generalized Kronecker's delta has been used

$$\delta^{ijk}_{rst} \equiv e^{ijk}\ e_{rst}$$

Using relation (1.16) we can calculate the determinant of the metric tensor (or metric for short):

$$g = \frac{1}{3!}\, e^{rst}\, e^{mnp}\, g_{rm}\, g_{sn}\, g_{tp} \tag{1.17}$$

It can be shown that the metric is a relative invariant of weight 2, since e^{rst}, e^{mnp} are relative tensors of weight 1. Therefore we can define the ε-systems as :

$$\varepsilon_{rst} = \sqrt{g}\ e_{rst} \quad , \quad \varepsilon^{rst} = \frac{1}{\sqrt{g}}\ e^{rst} \tag{1.18}$$

which are absolute tensors

The magnitude, A, of a contravariant vector A^r, is an invariant defined by the equation

$$A = (g_{mn}\, A^m\, A^n)^{1/2} \tag{1.19}$$

or similarly for the covariant vector B_r

$$B = (g^{mn}\, B_m\, B_n)^{1/2}$$

The angle θ between the directions defined by two unit vectors λ^m and μ^n is:

$$\cos\theta = g_{mn}\, \lambda^m\, \mu^n \tag{1.20}$$

The last two relations show the importance of the metric tensor, namely that g_{mn} provides information about the scaling of the axes and the angles between them. For the first, g_{11}, g_{22} ,g_{33} are necessary, while for the second the g_{12}, g_{23}, g_{13} are needed. Of course the opposite is also true. Knowing the scaling of the axes and the angle between them, we immediately know g_{mn}, through the following relations :

$$\varepsilon^r_{(1)} = \frac{1}{\sqrt{g_{11}}}\,\delta^r_1 \ , \quad \varepsilon^r_{(2)} = \frac{1}{\sqrt{g_{22}}}\,\delta^r_2 \ , \quad \varepsilon^r_{(3)} = \frac{1}{\sqrt{g_{33}}}\,\delta^r_3$$

and

$$\cos\theta_{12} = \frac{g_{12}}{\sqrt{g_{11}g_{22}}} \ , \quad \cos\theta_{23} = \frac{g_{23}}{\sqrt{g_{22}g_{33}}} \ , \quad \cos\theta_{13} = \frac{g_{13}}{\sqrt{g_{11}g_{33}}}$$

where $\varepsilon^r_{(m)}$ is the unit vector of the (m) axis (or their tangent in a point P in the case of a curvilinear system) and θ_{mn} is the angle between the (m) and (n) axes (or their tangents at P).

With the help of the metric tensor we can define the inner and the cross product.

Inner product of A^r, B^r : $$A\,B\cos\theta = g_{mn}\,A^m\,B^n = A^m\,B_m = A_m\,B^m \tag{1.21}$$

where relation (1.20) and the following identities have been used :

$$A_m = g_{mn}A^n \ , \ A^m = g^{mn}A_n \tag{1.22}$$

Cross Product of A^r, B^r : $$A\,B\,\sin\theta\ v^r = \varepsilon^{rmn}\,A_m\,B_n \tag{1.23}$$

where v^r is a unit vector normal to the A^m, B^m.

An other important concept is that of the physical component of a vector x^r in the direction of the unit vector q^r. It is defined as the projection of x^r in the q^r direction and it is independent of the components of x^r :

$$x_{[\phi]} = g_{mn}x^m q^n = g^{mn}x_m q_n \tag{1.24}$$

The physical component of a contravariant vector in the (i) axis is given by the relation :

$$x^{(i)}_{[\phi]} = g_{ri}\,x^r/\sqrt{g_{II}} \tag{1.25}$$

which can be written as:

$$x^{(i)}_{[\phi]} = x_i/\sqrt{g_{II}} \tag{1.26}$$

which shows that the covariant components, x_i, of a vector are the scaled physical components with respect to the contravariant axis. Similarly we get :

$$x_{[\phi]\,(i)} = g^{ri}x_r/\sqrt{g^{II}} = x^i/\sqrt{g^{II}} \tag{1.27}$$

Note that in the above relations the summation convention doesn't apply for the capital letter indicies ($i \equiv I$ in arithmetic value)

1.2 Surface Tensors

Let x^r be a cooridinate system in space. A surface, S, is defined as the geometrical set of points whose coordinates are functions of two independent variables u^1, u^2. The equations of a surface are:

$$x^r = x^r(u^1, u^2) \tag{1.28}$$

Taking the differential of the relation (1.28), on a certain point we get:

$$dx^r = \frac{\partial x^r}{\partial u^\alpha} du^\alpha$$

thus in every direction $\dfrac{du^\alpha}{ds}$ tangent to the surface in a certain point in space, corresponds the :

$$\frac{dx^r}{ds} = x^r_\alpha \frac{du^\alpha}{ds} \tag{1.29}$$

where ds is the differential physical length along du^α on the surface and

$$x^r_\alpha \equiv \frac{\partial x^r}{\partial u^\alpha} \tag{1.30}$$

is at the same time a contravariant space vector and a covariant surface vector. For every surface vector A^α (or tensor of any order), we can assign a space vector A^r through the relation :

$$A^r = x^r_\alpha A_r \Longrightarrow A_\alpha = x^r_\alpha A_r \qquad (1.31), (1.32)$$

The surface metric tensor, $\alpha_{\alpha\beta}$, is defined as:

$$ds^2 = \alpha_{\alpha\beta}\, du^\alpha\, du^\beta \tag{1.33}$$

or using relation (1.32)

$$\alpha_{\alpha\beta} = g_{mn}\, x^m_\alpha\, x^n_\beta \tag{1.34}$$

Furthermore we can define the surface tensors δ^α_β, $e^{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$:

$$\left.\begin{aligned} \delta^\alpha_\beta &= \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases} \\ e^{\alpha\beta} &= \begin{cases} +1, & \alpha = 1, \beta = 2 \\ -1, & \alpha = 2, \beta = 1 \\ 0, & \alpha = \beta \end{cases} \\ \varepsilon^{\alpha\beta} &= \frac{1}{\sqrt{\alpha}}\, e^{\alpha\beta} \end{aligned}\right\} \tag{1.35}$$

and in a similar manner the $\delta^{\alpha\beta}$, $\delta_{\alpha\beta}$, $e^{\alpha\beta}$, $\varepsilon_{\alpha\beta}$.

Let ξ^r be a unit vector normal to a point on a surface. Then it can be shown that:

$$\xi_r = \frac{1}{2}\,\varepsilon^{\alpha\beta}\,\varepsilon_{rst}\,x^s_\alpha\,x^t_\beta \tag{1.36}$$

Finally we give the relation for the divergence of :

a) space vector X^r :
$$div\,X^r \equiv X^r_{.,r} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^r}(\sqrt{g}\,X^r) \tag{1.37}$$

b) surface vector X^α :
$$div\,X^\alpha \equiv X^\alpha_{.,\alpha} = \frac{1}{\sqrt{\alpha}}\frac{\partial}{\partial x^\alpha}(\sqrt{\alpha}\,X^\alpha) \tag{1.38}$$

Note that the symbol $X^\alpha_{.,\beta}$ is defined as :

$$X^\alpha_{.,\beta} \equiv \frac{\partial X^\alpha}{\partial u^\beta} + \left\{ \begin{array}{c} \alpha \\ \sigma\ \beta \end{array} \right\} X^\sigma$$

where

$$\left\{ \begin{array}{c} \alpha \\ \beta\ \gamma \end{array} \right\} \equiv \alpha^{\alpha\delta}\ [\,\beta\,\gamma\,,\,\delta\,]$$

and

$$[\,\alpha\,\beta\,,\,\gamma\,] \equiv \frac{1}{2}\left(\frac{\partial\alpha_{\gamma\alpha}}{\partial u^\beta} + \frac{\partial\alpha_{\gamma\beta}}{\partial u^\alpha} - \frac{\partial\alpha_{\alpha\beta}}{\partial u^\gamma}\right)$$

It can be shown that :

$$\left\{ \begin{array}{c} \gamma \\ \gamma\ \beta \end{array} \right\} = \frac{1}{\sqrt{\alpha}}\,\frac{\partial\sqrt{\alpha}}{\partial u^\beta}$$

Closing this section notice that almost all of the above relations can be found with proofs in [1].

2 Preliminaries for Geometry and Motion

In this section we discuss the preliminary ideas used in the module of "Geometry and Motion Preprocessor", which provides the necessary input for the main program, "Solver".

2.1 Lifting Surface Motion - Coordinate Systems

In order to describe the unsteady problem for the lifting surface we have to define first an inertial coordinate system, which we denote by $\underset{\bowtie}{x}^i$. All velocities are measured relative to this coordinate system unless it is specified otherwise. We then define a moving coordinate system relative to $\underset{\bowtie}{x}^i$, which we denote by x^i. Both systems are considered to be orthonormal with their axis initially (t=0) coinciding.

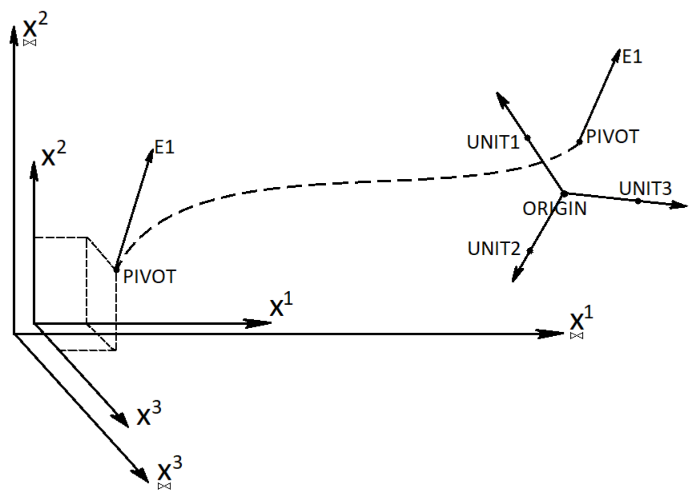


Figure 2.1.1

The moving coordinate system is used to describe the general motion of the lifting surface. Specifically we assume that the lifting surface has a fixed position relative to x^i, which is used in the next section to describe the geometry of the surface.

Suppose a point M fixed on the wing's surface (moving with x^i system), then the following velocities can be defined.

$$\vec{u}_M \longrightarrow \text{velocity of M, relative to the inertia system} \tag{2.1.1}$$

$$\vec{V}_\infty \longrightarrow \text{undisturbed fluid velocity at M, relative to the inertial system} \tag{2.1.2}$$

$$\vec{q} = \vec{V}_\infty - \vec{u}_M \longrightarrow \text{undisturbed fluid velocity relative to M} \tag{2.1.3}$$

$$\vec{v} \longrightarrow \text{perturbation velocity relative to the inertial system} \tag{2.1.4}$$

$$\vec{V}_\infty + \vec{v} \longrightarrow \text{total fluid velocity relative to the inertial system.} \tag{2.1.5}$$

$$\vec{w} = \vec{q} + \vec{v} \longrightarrow \text{total fluid velocity relative to M} \tag{2.1.6}$$

The movement of x^i system is described by a parallel transition and a rotation around a fixed point relative to x^i, named PIVOT. In other words the motion of the moving system can be described by the motion of PIVOT and the rotation vector $\theta(t) \cdot \vec{E}1$ at PIVOT, where $\theta(t)$ is the angle of rotation and $\vec{E}1$ the direction of rotation.

2.2 Lifting Surface Geometry - Coordinate Systems

Here we will describe the Lifting surface of two different bodies. The first one is the propeller which practically consists of multiple blades - lifting surfaces placed symmetrically around the axis of rotation. In this analysis we don't include the contribution of the hub and therefore we don't model it. Next we examine the case of a single wing which can be view as a special case of the propeller, as far as the description of the geometry goes (not the motion).

In order to describe the geometry of the propeller we use three space and two surface coordinate systems, all moving (i.e. non-inertia), according to the discussion of the previous section. The distinction between them is accomplished with the different underline symbols. Referring to Figure 2.2.1, of the propeller we define the following coordinate systems for the reference blade :

a) The *orthonormal* coordinate system, x^i, with the x^1 axis coinciding with the propeller axis and x^2, x^3 axes completing the right hand system.

b) The *cylindrical* coordinate system, $\underset{\wedge}{x}^i$, with $\underset{\wedge}{x}^1 \equiv x^1$ and $\underset{\wedge}{x}^2, \underset{\wedge}{x}^3$ according to figure 2.2.1, which is also a right hand system.

c)The general *curvilinear system*, $\underline{x}^i$, with the $\underline{x}^1$ axis coinciding with the blade reference line [1] and parametrized with the radial coordinate $\underset{\wedge}{x}^2$, the $\underline{x}^2$ axis coinciding with the helicoidal lines through the blade reference line with pitch angle $\phi(\underset{\wedge}{x}^2)$ and parametrized with the distance from the blade reference line expressed as a fraction of chord. Finally $\underline{x}^3$ coincides with the helicoidal lines with pitch $\phi(\underset{\wedge}{x}^2) - \pi/2$. Notice that this is a left hand system. With the previous considerations in mind the three coordinate systems are connected by the following relations :

$$x^1 = \underset{\wedge}{x}^1 \tag{2.2.1}$$

$$x^2 = \underset{\wedge}{x}^2 \cos \underset{\wedge}{x}^3 \tag{2.2.2}$$

$$x^3 = \underset{\wedge}{x}^2 \sin \underset{\wedge}{x}^3 \tag{2.2.3}$$

and

$$\underset{\wedge}{x}^1 = X(\underline{x}^1) + c(\underline{x}^1) \left[\underline{x}^2 \sin \phi(\underline{x}^1) - \underline{x}^3 \cos \phi(\underline{x}^1) \right] \tag{2.2.4}$$

$$\underset{\wedge}{x}^2 = \underline{x}^1 \tag{2.2.5}$$

$$\underset{\wedge}{x}^3 = \Theta(\underline{x}^1) + c(\underline{x}^1) \left[\underline{x}^2 \cos \phi(\underline{x}^1) + \underline{x}^3 \sin \phi(\underline{x}^1) \right] / \underline{x}^1 \tag{2.2.6}$$

[1] For the propeller performace problem we are studying, blade reference line is usually selected to be the locus of blade section midchord points. For the design problem it is selected to differ from the unknown locus of blade midchord points by a first order quantity. Other choices of blade reference line are also possible.

where $X(\underline{x}^1)$ and $\Theta(\underline{x}^1)$ are the rake and skew angle of the blade reference line respectively and $c(\underline{x}^1)$ is the chord length, at the radial position $\underline{x}^1$.

We also introduce two surface coordinate systems which describe the intrinsic geometry of the blade reference surface, according to the following relations :

a) The $u^\alpha \equiv (\underline{x}^i)_{i=\alpha}$

b) The y^α , which is defined through the relations :

$$u^1 = \frac{1}{2}(R_o + R_H) - \frac{1}{2}(R_o - R_H)\cos y^1 \tag{2.2.7}$$

$$u^2 = -\frac{1}{2}\cos y^2 \tag{2.2.8}$$

where R_H, is the radius of the hub and R_o is the radius of the propeller. The difference $R_o - R_H$, is the span of the blade.

The range of values, for the y^1, y^2 is $[0, \pi]$ where :

$$y^1 = 0 \leftrightarrow u^1 = R_H$$

$$y^1 = \pi \leftrightarrow u^1 = R_o$$

and

$$y^2 = 0 \leftrightarrow u^2 = -1/2 \text{ (leading edge)}$$

$$y^2 = \pi \leftrightarrow u^2 = 1/2 \text{ (trailing edge)}$$

The surface coordinate systems have axes, that coincide with the $\underline{x}^1$ and $\underline{x}^2$ axes of the general curvilinear system, so we define them as left hand systems. This is induced by the properties of x^i as left handed. For the sake of uniformity we introduce the $\breve{x}^i$, coordinate system as :

$$\breve{x}^\alpha \equiv y^\alpha \quad , \qquad \breve{x}^3 = \underline{x}^3$$

The introduction of three different coordinate systems in our development was necessary since each coordinate system has some special advantages over the others for some part of the subsequent analysis. Thus, the orthonormal system is necessary for the introduction of the representation theorems of potential theory, the cylindrical system is helpful in the case of the propeller since it simplifies the representation of its geometry and finally the general curvilinear coordinate system is very useful in deriving the linearized no entrance blade surface boundary condition since the equations of blade surface for this coordinate system become particularly simple. Finally the y^α surface coordinate system is necessary for the application of the mode analysis method. With the help of the equations (2.2.7) and (2.2.8), the integral equations of the perturbation velocities, with the Cauchy kernel, can take a simplified form using the familiar Glauert integrals.

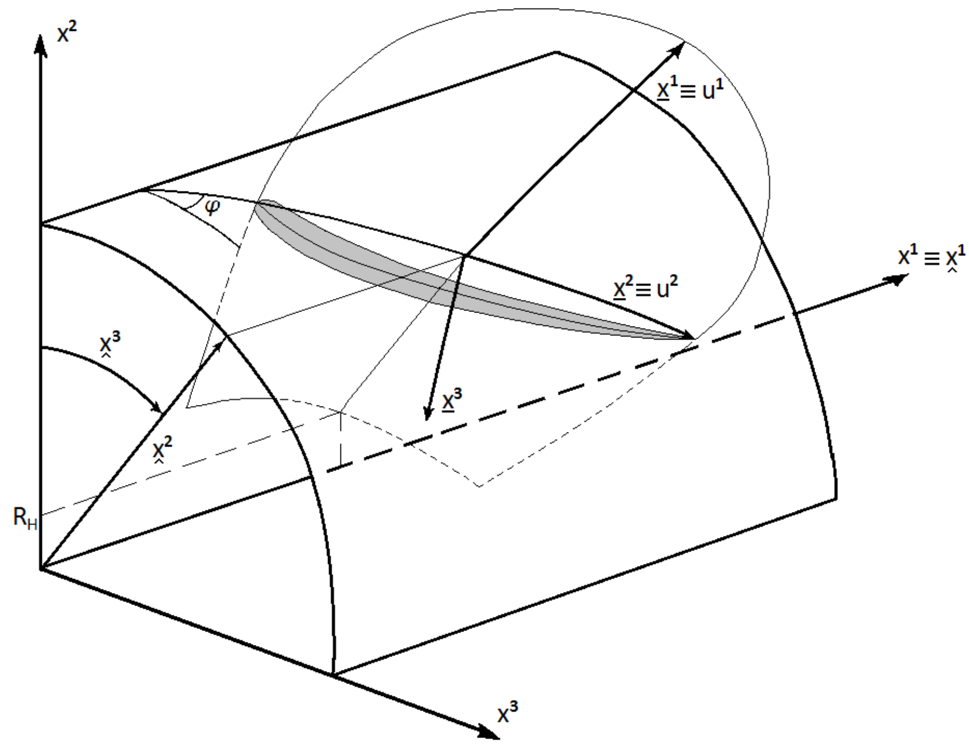


Figure 2.2.1

As an other example of lifting surface geometry is that of the a wing. The main difference between the propeller and the wing is that the blade sections of the first lie on cylindrical surfaces while the blade sections of the later are flat and therefore there is no need for the cylindrical coordinates.

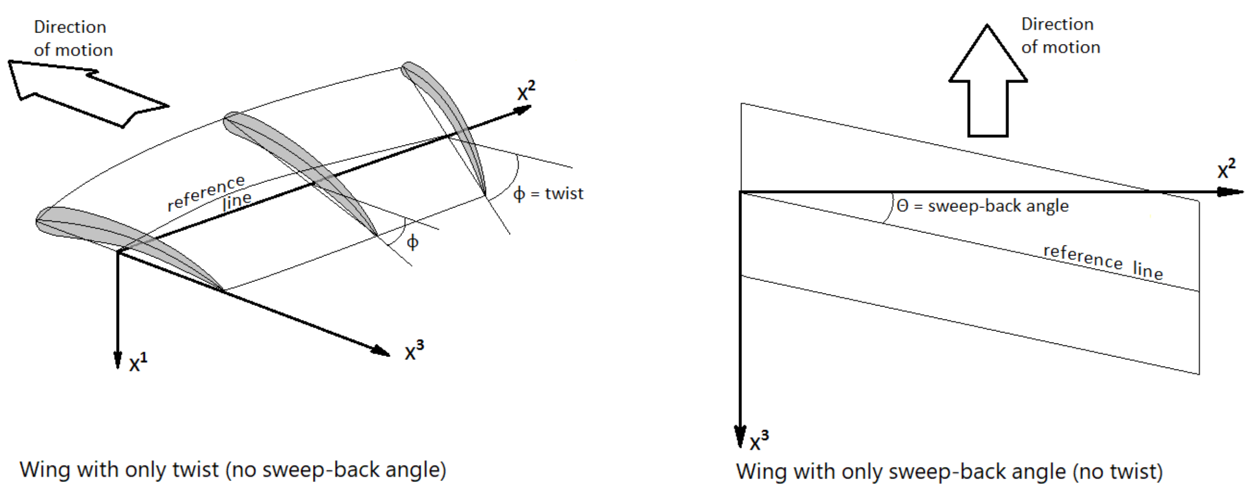


Figure 2.2.2

The parameters of skew and pitch in the case of the wing are replaced by sweep-back and twist angle respectively. Normally a wing with no twist lies on the x^2, x^3 plane. Furthermore, similarly to the propeller geometry we can add an extra degree of freedom, $X(x^2)$, denoting the distance of each cross section from the x^2, x^3 plane which is the wing skew-back. In fact changing the point of view in figure 2.2.2 we could view the wing like in figure 2.2.3. Notice that figures 2.2.1 and 2.2.3 are very similar and in fact we can use the slightly modified equations to (2.2.1)-(2.2.6) to describe the reference surface of the wing, as follows.

$$x^1 = X(\underline{x}^1) + c(\underline{x}^1)\left[\underline{x}^2 \sin\phi(\underline{x}^1) - \underline{x}^3 \cos\phi(\underline{x}^1)\right] \tag{2.2.9}$$

$$x^2 = \underline{x}^1 \tag{2.2.10}$$

$$x^3 = \underline{x}^1\ \tan(\Theta(\underline{x}^1)) + c(\underline{x}^1)\left[\underline{x}^2 \cos\phi(\underline{x}^1) + \underline{x}^3 \sin\phi(\underline{x}^1)\right] \tag{2.2.11}$$

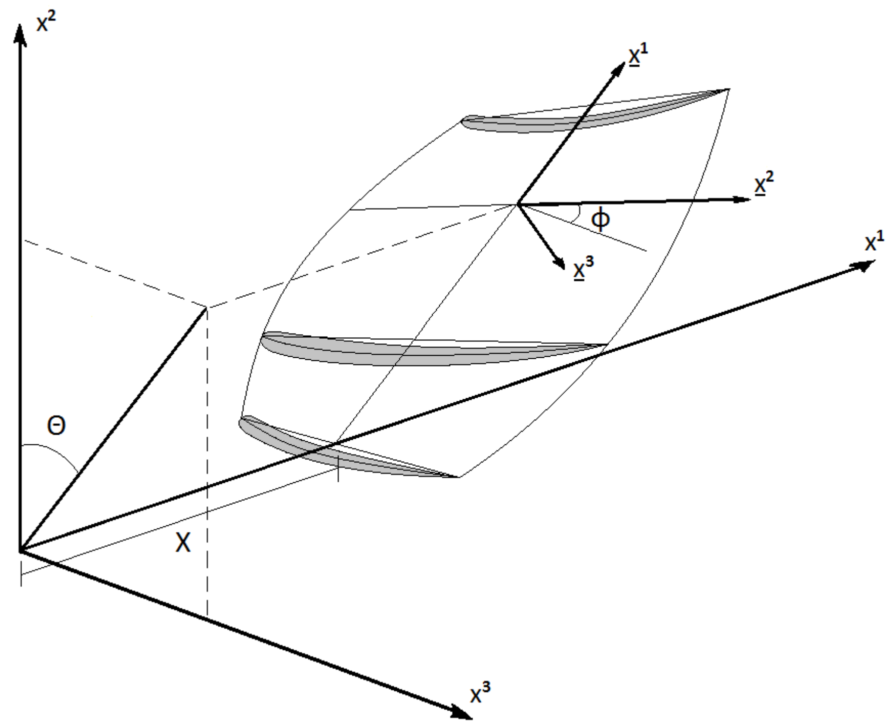


Figure 2.2.3

In the sequel we use the following conventions:

We underline each quantity with the symbol of the coordinate system to which this quantity refers, as well as double underlining for quantities that connect two coordinate systems (transformation matrices). For example :

g_{ij}, g, ε_{ijk}, the metric tensor , the metric tensor determinant and the alternating symbol respectively in the cartesian coordinate system.

$\underset{\wedge}{g}_{ij}$, $\underset{\wedge}{g}$, $\underset{\wedge}{\varepsilon}_{ijk}$, the metric tensor, the metric tensor determinant and the alternating symbol respectively in the cylindrical coordinate system.

As for the transformation matrices between two coordinate systerms e.g $\underline{x}^i$, $\underset{\wedge}{x}^i$ there are two conventions. In the first one we underline the quantities (tensors, transormation matricies etc). We use c and γ to denote the direct and inverse transformation respectively.

$$\underline{A}^m = \underset{\underline{\wedge}}{c^m_j} \; \underset{\wedge}{A}^j \quad , \qquad \underset{\wedge}{A}^j = \underset{\underline{\wedge}}{\gamma^j_m} \; \underline{A}^m \qquad \text{(transformation by contravariance)}$$

$$\underline{A}_n = \underset{\underline{\wedge}}{\gamma^m_n} \; \underset{\wedge}{A}_m \quad , \qquad \underset{\wedge}{A}_n = \underset{\underline{\wedge}}{c^m_n} \; \underline{A}_m \qquad \text{(transformation by covariance)}$$

In the second one we underline the indicies associated with each coordinate system, thus we use the same symbol for both the direct and inverse transformation.

$$A^{\underline{m}} = c^{\underline{m}}_{\underset{\wedge}{j}} \; A^{\underset{\wedge}{j}} \quad , \qquad A^{\underset{\wedge}{j}} = c^{\underset{\wedge}{j}}_{\underline{m}} \; A^{\underline{m}} \qquad \text{(transformation by contravariance)}$$

$$A_{\underline{n}} = c^{\underset{\wedge}{m}}_{\underline{n}} \; A_{\underset{\wedge}{m}} \quad , \qquad A_{\underset{\wedge}{n}} = c^{\underline{m}}_{\underset{\wedge}{n}} \; A_{\underline{m}} \qquad \text{(transformation by covariance)}$$

In the following pages we use the first convention. The trasformation matrices between the various coordinate systems are shown in the following figure. The analytic relations for the transformation matrices as well as the metrics of the coordinate systems are shown in Appendix A.

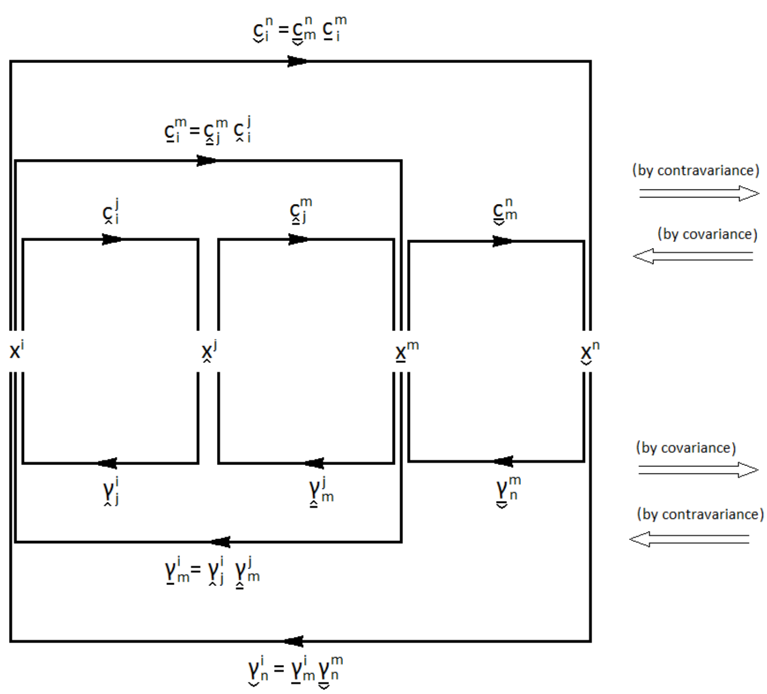


Figure 2.2.4

3 Linearised Lifting Surface Theory

Here we will present the basic parts of the linearised lifting surface theory, on which the present study is based on. In this chapter we shall present the various quantities used to formulate the problem and work out the form of the boundary conditions using basic assumptions to linearise them. In the next chapters we shall review the same boundary conditions and make further assumptions to acquire more useful expressions to use for the program.

3.1 Velocity field and the cross section geometry

The basic assumptions for the development of the theory are that the fluid surrounding the propeller extends infinitely in all directions (open region) with boudaries consisting of a finite number of smooth (Lyapunov) surfaces. Moreover the flow is considered to be frictionless and irrotational for the whole region except for the trailing vortex sheet of each blade. Thus it is possible to describe the flow using a potential (scalar function) and the representation theorem.

As seen in chapter 2.1 the undisturbed fluid velocity relative to a point M on the propeller blade is denoted by $\vec{q}$ and its projection on the cylindrical coordinate system has the simple form :

$$\underset{\wedge}{q}^i = \left\{U^x \ , \ U^R \ , \ U^t\right\} \tag{3.1.1}$$

where U^x is the axial wake velocity, U^R the radial wake velocity and U^t the tangential wake velocity, which, for compatibility with the tensor notation, has dimensions of radial velocity [rad/sec]. Note that in order for the propeller to rotate correctly (with the leading edge preceded), U^t, should be negative.

The perturbation velocity due to the existence of blades and the trailing vortex sheets emanating form each blade shall be denoted by v^i.

Total velocities relative to M are denoted by w^i and the following relation holds :

$$w^i = q^i + v^i \tag{3.1.2}$$

Let $E_c(u^1, u^2)$ be the blade camber distribution and $E_T(u^1, u^2)$ the blade thickness form nondimensionalised with the chord length. The following order of magnitude estimates shall be used repeatedly in the sequel as a part of the linearised lifting surface theory.

$$E_c = O(\varepsilon) \tag{3.1.3}$$

$$E_T = O(\varepsilon) \tag{3.1.4}$$

$$\frac{\underset{\wedge}{v}^i}{U^x} = O(\varepsilon) \tag{3.1.5}$$

$$tan\phi - \tan\beta = O(\varepsilon) \ , \quad (\text{or } \phi - \beta = O(\varepsilon)) \tag{3.1.6}$$

where $\tan\beta = U^x/(\omega \underset{\wedge}{x}^2)$ and ε is a small parameter, for example ε=camber/chord). Relations (3.1.3) to (3.1.5) express the fact that the propeller blade camber and thickness which is the cause of fluid disturbance and perturbation velocity v^i are first order quantities. Finally relation (3.1.6) expresses the fact that the blade reference surface which by assumption is selected near the real blade surface (since using the blade reference surface propeller geometry is expressed by the first order quantities E_c and E_T) differs by a first order quantity from the undisturbed flow surface.

We shall assume that the derivatives with respect to the independent variables of the previous quantities conserve the order of magnitude estimates although this is not generally true near the blade leading edge and tips.

Knowing the camber and thickness distribution of the blade we can construct the blade surface as shown in Figure 3.1. By definition the camber surface is the set of points in the middle of the thickness, measuring vertically from the camber surface. Thus the blade surface geometry in the general curvilinear coordinate system $\underline{x}^i$ can be described in parametric form by the following relations :

For the back of the blade (upper side) :

$$\begin{cases} \underline{x}_u^2 = u^2 - \frac{1}{2}E_T(u^1,u^2)\sin\psi \\ \underline{x}_u^3 = E_c(u^1,u^2) + \frac{1}{2}E_T(u^1,u^2)\cos\psi \end{cases}$$

For the face of the blade (lower side) :

$$\begin{cases} \underline{x}_L^2 = u^2 + \frac{1}{2}E_T(u^1,u^2)\sin\psi \\ \underline{x}_L^3 = E_c(u^1,u^2) - \frac{1}{2}E_T(u^1,u^2)\cos\psi \end{cases}$$

where $\psi = \frac{\partial E_c}{\partial u^2}$ is the slope of the camber surface. Using the previous assumptions we get, $\psi = O(\varepsilon)$, $\sin\psi = O(\varepsilon)$, $\cos\psi = 1 + O(\varepsilon)$.

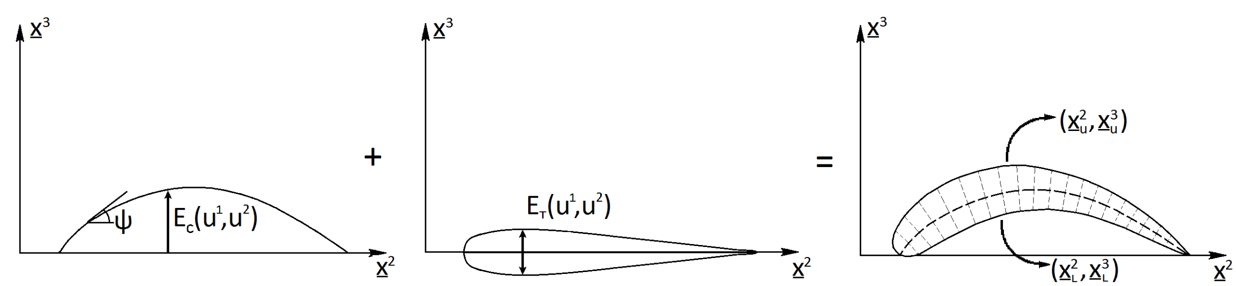


Figure 3.1

Keeping only the first order terms we end up with the simpler relations :

$$\left.\begin{aligned} \underline{x}^1 &= u^1 \\ \underline{x}^2_{u,L} &= u^2 \\ \underline{x}^3_{u,L} &= E(u^1, u^2) \end{aligned}\right\} \quad (3.1.7)$$

where $E(u^1, u^2) = E_c(u^1, u^2) \pm \frac{1}{2} E_T(u^1, u^2)$ (3.1.8)

3.2 Blade surface boundary condition

The no entrance blade boundary condition expressed in the $\underline{x}^i$ coordinate system reads :

$$(\underline{v}^i + \underline{V}^i_\infty)\,\underline{N}_i \;=\; \underline{u}^i_M\,\underline{N}_i \;\;\Leftrightarrow\;\; \underline{w}^i\underline{N}_i = 0 \tag{3.2.1}$$

where $\underline{N}_i$ is a unit vector normal to the blade surface and $\underline{w}^i$ is the corresponding total fluid velocity relative to M.

Boundary condition (3.2.1) refers to the real blade surface and in order to linearize we have to transfer it to the blade reference surface $\underline{x}^3 = 0$. Applying Taylor's expansion theorem around the blade's reference surface (the quantities of which shall be denoted with the overline symbol "$\sim$" we get :

$$\underline{w}^i = \widetilde{\underline{w}}^i \;+\; \frac{\partial \widetilde{\underline{w}}^i}{\partial \underline{x}^j}\,\delta\underline{x}^j \;+\; O(\varepsilon^2) \tag{3.2.2}$$

where $\delta\underline{x}^j = (0, 0, E)$ and $\dfrac{\partial \widetilde{\underline{w}}^i}{\partial \underline{x}^j} \equiv \lim\limits_{\underline{x}^3 \to 0} \dfrac{\partial \underline{w}^i}{\partial \underline{x}^j}$

Relation (3.1.2) expressed in the $\underline{x}^i$ coordinate system reads :

$$\widetilde{\underline{w}}^i \;=\; \widetilde{\underline{q}}^i \;+\; \widetilde{\underline{v}}^i \tag{3.2.3}$$

or taking the partial derivatives :

$$\frac{\partial \widetilde{\underline{w}}^i}{\partial \underline{x}^j} \;=\; \frac{\partial \widetilde{\underline{q}}^i}{\partial \underline{x}^j} \;+\; O(\varepsilon) \tag{3.2.4}$$

Substituting relations (3.2.3) and (3.2.4) to (3.2.2) we get :

$$\widetilde{\underline{w}}^i \;=\; \widetilde{\underline{q}}^i + \widetilde{\underline{v}}^i + \frac{\partial \widetilde{\underline{q}}^i}{\partial \underline{x}^j}\delta x^j \;+\; O(\varepsilon^2) \tag{3.2.5}$$

A unit vector normal on the blade's surface can be found using relation (1.36) :

$$\underline{N}_i = \frac{1}{2}\,\underline{\varepsilon}^{\alpha\beta}\,\underline{\varepsilon}_{ijk}\,\underline{x}^j_\alpha\,\underline{x}^k_\beta \tag{3.2.6}$$

where $\underline{x}^s_\alpha = \dfrac{\partial \underline{x}^s_{u,L}}{\partial u^\alpha}$ (3.2.7)

Since both $\underline{x}^i$, u^α are left handed coordinate systems we get $\underline{\varepsilon}^{\alpha\beta} = -e^{\alpha\beta}/\sqrt{\underline{\alpha}'}$ and $\underline{\varepsilon}_{ijk} = -e_{ijk}\sqrt{\underline{g}'}$, where $\underline{\alpha}'$ and $\underline{g}'$ refer to the real blade surface. Instead of $\underline{N}_i$ we will use the non-unit vector $\underline{n}_i$:

$$\underline{n}_i = \frac{1}{2}\, e^{\alpha\beta}\, e_{ijk}\, \underline{x}^j_\alpha\, \underline{x}^k_\beta \;=$$

$$= \left[\left(\underline{x}^2_1\, \underline{x}^3_2 \,-\, \underline{x}^3_1\, \underline{x}^2_2\right), \left(\underline{x}^3_1\, \underline{x}^1_2 \,-\, \underline{x}^1_1\, \underline{x}^3_2\right), \left(\underline{x}^1_1\, \underline{x}^2_2 \,-\, \underline{x}^2_1\, \underline{x}^1_2\right)\right] \;=$$

$$= \left(-\underline{x}^3_1 \;, -\underline{x}^3_2 \;, 1\right) \;=\; \left(-\frac{\partial E}{\partial u^1}, -\frac{\partial E}{\partial u^2} \;, 1\right) \tag{3.2.8}$$

Alternatively if $E(u^1, u^2)$ is the parametric blade surface, then:

$$dE = \frac{\partial E}{\partial u^1} du^1 \;+\; \frac{\partial E}{\partial u^2} du^2 \Rightarrow$$

$$0 = \left(-\frac{\partial E}{\partial u^1}, -\frac{\partial E}{\partial u^2} \;, 1\right)\left(du^1, du^2, dE\right) \Rightarrow \underline{n}_i \; d\underline{x}^i = 0$$

where $d\underline{x}^i$, is a differential on the blade surface. Note also that the transition from the first to the second equation is valid since $\left(-\frac{\partial E}{\partial u^1}, -\frac{\partial E}{\partial u^2} \;, 1\right)$ are the covariant components of the normal vector while $\left(du^1, du^2, dE\right)$ are the contravariant components of the differential.

Using the non-unit $\underline{n}_i$ vector has the advantage of simplifying taylor expansion for $\underline{n}_i$. Specifically $\underline{n}_i$ does not depend on the coordinate $\underline{x}^3$ in the $\underline{x}^i$ coordinate system and thus its taylor expansion has the form :

$$\underline{n}_i = \overset{\sim}{\underline{n}}_i \tag{3.2.9}$$

where $\overset{\sim}{\underline{n}}_i$ refers to points of the blade reference surface. The disadvantage of using $\underline{n}_i$ instead of $\underline{N}_i$ is that boundary condition (3.2.1) has lost its invariant character and this has to be taken into consideration in the interpretation of the final result.

Substituting relations (3.2.8) to (3.2.9) and using (3.1.8) we get :

$$\overset{\sim}{\underline{n}}{}^{+,-}_i \;=\; \pm\overset{\overset{o}{\sim}}{\underline{n}}_i \;\pm\; \overset{\overset{c}{\sim}}{\underline{n}}_i \;+\; \frac{1}{2}\,\overset{\overset{t}{\sim}}{\underline{n}}_i \tag{3.2.10}$$

where

$$\overset{\overset{o}{\sim}}{\underline{n}}_i = \overset{o}{\underline{n}}_i = \{0, 0, 1\} \tag{3.2.11}$$

$$\overset{\overset{c}{\sim}}{\underline{n}}_i = \overset{c}{\underline{n}}_i = \left\{-\frac{\partial E_c}{\partial u^1}, -\frac{\partial E_c}{\partial u^2}, 0\right\} \tag{3.2.12}$$

$$\overset{\overset{t}{\sim}}{\underline{n}}_i = \overset{t}{\underline{n}}_i = \left\{-\frac{\partial E_T}{\partial u^1}, -\frac{\partial E_T}{\partial u^2}, 0\right\} \tag{3.2.13}$$

and $\overset{\sim}{\underline{n}}{}^+_i$ denotes a vector normal to the upper blade surface transferred to the blade reference surface whereas $\overset{\sim}{\underline{n}}{}^-_i$ is a vector normal to the lower blade surface transferred to the blade reference surface.

Obviously $\tilde{\underline{n}}_i$ is not normal to the blade reference surface. On the other hand $\overset{o}{\tilde{\underline{n}}}_i$ lies on the $\underline{x}_3$ axis which is reciprocal and thus by definition perpendicular to the $\underline{x}^1$, $\underline{x}^2$ axes. Note that although $\underline{x}^2$ and $\underline{x}^3$ are perpendicular ($\underline{g}_{23} = 0$ (A78)) the reference line which coincides with the $\underline{x}^1$ axis is not perpendicular to the other two axes ($\underline{g}_{12} \neq 0$ (A75), $\underline{g}_{13} \neq 0$ (A76)).

Substituting relations (3.2.10) to the boundary condition (3.2.1) we get :

$$\underline{w}^{i+,-}\, \tilde{\underline{n}}_i^{+,-} \;=\; \left(\underline{q}^{i+,-} + \underline{v}^{i+,-}\right)\left(\pm\overset{o}{\tilde{\underline{n}}}_i \pm \overset{c}{\tilde{\underline{n}}}_i + \frac{1}{2}\overset{t}{\tilde{\underline{n}}}_i\right) \;=\; 0 \tag{3.2.14}$$

which can be written as :

$$\left(\underline{q}^{i+} + \underline{v}^{i+}\right)\left(\overset{o}{\tilde{\underline{n}}}_i + \overset{c}{\tilde{\underline{n}}}_i + \frac{1}{2}\overset{t}{\tilde{\underline{n}}}_i\right) \;=\; 0 \tag{3.2.15}$$

$$\left(\underline{q}^{i-} + \underline{v}^{i-}\right)\left(-\overset{o}{\tilde{\underline{n}}}_i - \overset{c}{\tilde{\underline{n}}}_i + \frac{1}{2}\overset{t}{\tilde{\underline{n}}}_i\right) \;=\; 0 \tag{3.2.16}$$

By adding and subtracting relations (3.2.15) and (3.2.16) we get :

$$\left[\left(\underline{q}^{i+} - \underline{q}^{i-}\right) + \left(\underline{v}^{i+} - \underline{v}^{i-}\right)\right]\left(\overset{o}{\tilde{\underline{n}}}_i + \overset{c}{\tilde{\underline{n}}}_i\right) + \left[\left(\underline{q}^{i+} + \underline{q}^{i-}\right) + \left(\underline{v}^{i+} + \underline{v}^{i-}\right)\right]\frac{1}{2}\overset{t}{\tilde{\underline{n}}}_i \;=\; 0 \tag{3.2.17}$$

$$\left[\left(\underline{q}^{i+} + \underline{q}^{i-}\right) + \left(\underline{v}^{i+} + \underline{v}^{i-}\right)\right]\left(\overset{o}{\tilde{\underline{n}}}_i + \overset{c}{\tilde{\underline{n}}}_i\right) + \left[\left(\underline{q}^{i+} - \underline{q}^{i-}\right) + \left(\underline{v}^{i+} - \underline{v}^{i-}\right)\right]\frac{1}{2}\overset{t}{\tilde{\underline{n}}}_i \;=\; 0 \tag{3.2.18}$$

Applying Taylor's expansion Theorem for the undisturbed velocities on the upper and lower surfaces of the blade we get :

$$\underline{q}^{i+} \;=\; \tilde{\underline{q}}^i + \frac{\partial\tilde{\underline{q}}^i}{\partial\underline{x}^j}\,\delta\underline{x}^j + O(\varepsilon^2) \;=\; \tilde{\underline{q}}^i + \frac{\partial\tilde{\underline{q}}^i}{\partial\underline{x}^3}\,E + O(\varepsilon^2) \tag{3.2.19}$$

$$\underline{q}^{i-} \;=\; \tilde{\underline{q}}^i + \frac{\partial\tilde{\underline{q}}^i}{\partial\underline{x}^j}\,\delta\underline{x}^j + O(\varepsilon^2) \;=\; \tilde{\underline{q}}^i - \frac{\partial\tilde{\underline{q}}^i}{\partial\underline{x}^3}\,E + O(\varepsilon^2) \tag{3.2.20}$$

where $\delta\underline{x}^j = (0, 0, \pm E)$ and $\dfrac{\partial\tilde{\underline{q}}^i}{\partial\underline{x}^j} \equiv \lim\limits_{\underline{x}^3\to 0}\dfrac{\partial\underline{q}^i}{\partial\underline{x}^j}$

By adding and subtracting relations (3.2.19) and (3.2.20) we get the jump and the mean value of the velocity between the two sides of the blade.

$$\langle\underline{q}^i\rangle \;\overset{def}{=}\; \left(\underline{q}^{i+} - \underline{q}^{i-}\right) \;=\; 2\,\frac{\partial\tilde{\underline{q}}^i}{\partial\underline{x}^3}\,E + O(\varepsilon^2) \;=\; O(\varepsilon) \tag{3.2.21}$$

$$\rangle\underline{q}^i\langle \;\overset{def}{=}\; \frac{1}{2}\left(\underline{q}^{i+} + \underline{q}^{i-}\right) \;=\rangle\tilde{\underline{q}}^i\langle + O(\varepsilon^2) \tag{3.2.22}$$

In the actual program the undisturbed velocities on the upper and lower surface are know and thus, in calculating the jump and the mean value of q^i, from relations (3.2.21) and (3.2.22) we use only their definition (first equation). Nevertheless the above relations provide us with the physical meaning that for thin blade the jump in the undisturbed velocity is negligible while the mean velocity can be approximated by the undisturbed velocity on the reference surface.

In a similar manner we can take the Taylor's expansion of the perturbation velocity and considering the assumption that the derivatives of the perturbation are also first order quantities, we get:

$$\underline{v}^i \;=\; \tilde{\underline{v}}^i \;+\; \frac{\partial \tilde{\underline{v}}^i}{\partial \underline{x}^j}\,\delta \underline{x}^j \;+\; O(\varepsilon^2) \;=\; \tilde{\underline{v}}^i \;+\; O(\varepsilon^2) \tag{3.2.23}$$

therefore

$$\langle \underline{v}^i \rangle \;\stackrel{def}{=}\; \left(\underline{v}^{i+} \;-\; \underline{v}^{i-}\right) \;=\; \langle \tilde{\underline{v}}^i \rangle \;+\; O(\varepsilon^2) \tag{3.2.24}$$

$$\rangle \underline{v}^i \langle \;\stackrel{def}{=}\; \frac{1}{2}\left(\underline{v}^{i+} \;+\; \underline{v}^{i-}\right) \;=\; \rangle \tilde{\underline{v}}^i \langle + O(\varepsilon^2) \tag{3.2.25}$$

where $\tilde{\underline{v}}^{i+}$ and $\tilde{\underline{v}}^{i-}$ denote the perturbation velocities on upper ($\underline{x}^3 \;=\; 0^+$) and lower ($\underline{x}^3 \;=\; 0^-$) faces of the blade reference surface. This means that the jump and the mean value of the perturbation velocity between the surfaces of the blade can be approximated by the respective values on the reference surface.

Substituting relations (3.2.21),(3.2.22),(3.2.24) and (3.2.25) to relations (3.2.17) and (3.2.18) we get :

$$\langle \underline{q}^i \rangle \overset{o}{\tilde{\underline{n}}}_i \;+\; \langle \underline{v}^i \rangle \overset{o}{\tilde{\underline{n}}}_i + \rangle \underline{q}^i \langle \overset{t}{\tilde{\underline{n}}}_i \;+\; O(\varepsilon^2) \;=\; 0 \tag{3.2.26}$$

$$\rangle \underline{q}^i \langle \left(\overset{o}{\tilde{\underline{n}}}_i \;+\; \overset{c}{\tilde{\underline{n}}}_i \right) \;+ \rangle \underline{v}^i \langle \overset{o}{\tilde{\underline{n}}}_i \;+\; O(\varepsilon^2) \;=\; 0 \tag{3.2.27}$$

Relations (3.2.26) and (3.2.27) are linearized forms of the blade boundary condition of the lifting surface theory. We usually say that equation (3.2.26) defines the 'thickness problem' of LST while equation (3.2.27) defines the 'camber problem' of LST.

Multiplying relation (3.2.26) by $\sqrt{\tilde{\underline{g}}} \;/\; \sqrt{\underline{\alpha}}$,where $\tilde{\underline{g}}$ is the metric tensor determinant at a point of the blade reference surface and $\underline{\alpha}$ is the surface metric tensor determinant, and introducing :

$$\overset{o}{\tilde{\underline{N}}}_i \;=\; \overset{o}{\tilde{\underline{n}}}_i \sqrt{\tilde{\underline{g}}} \;/\; \sqrt{\underline{\alpha}} \tag{3.2.28}$$

$$\overset{c}{\tilde{\underline{N}}}_i \;=\; \overset{c}{\tilde{\underline{n}}}_i \sqrt{\tilde{\underline{g}}} \;/\; \sqrt{\underline{\alpha}} \tag{3.2.29}$$

$$\overset{t}{\tilde{\underline{N}}}_i \;=\; \overset{t}{\tilde{\underline{n}}}_i \sqrt{\tilde{\underline{g}}} \;/\; \sqrt{\underline{\alpha}} \tag{3.2.30}$$

where $\underset{\sim}{\overset{o}{\tilde{N}}}_i$ is a unit vector normal to the blade reference surface, we get :

$$\langle \underline{q}^i \rangle \overset{o}{\tilde{\underline{N}}}_i \;+\; \langle \underline{v}^i \rangle \overset{o}{\tilde{\underline{N}}}_i + \rangle \underline{q}^i \langle \overset{t}{\tilde{\underline{N}}}_i \;+\; O(\varepsilon^2) \;=\; 0 \;\Rightarrow$$

$$\Rightarrow \sigma \;=\; -\rangle \underline{q}^i \langle \overset{t}{\tilde{\underline{N}}}_i \;-\; \langle \underline{q}^i \rangle \overset{o}{\tilde{\underline{N}}}_i + O(\varepsilon^2) \;\Rightarrow$$

$$\Rightarrow \sigma \;=\; \left(\rangle \underline{q}^1 \langle \frac{\partial E_T}{\partial u^1} + \rangle \underline{q}^2 \langle \frac{\partial E_T}{\partial u^2} \;-\; \langle \underline{q}^3 \rangle \right) \sqrt{\frac{\tilde{\underline{g}}}{\underline{\alpha}}} + O(\varepsilon^2) \tag{3.2.31}$$

where σ is the source intensity on the reference surface, defined as :

$$\sigma \;\overset{def}{=}\; \langle \tilde{\underline{v}}^i \rangle \overset{o}{\tilde{\underline{N}}}_i \;=\; \langle \tilde{\underline{v}}^3 \rangle \tag{3.2.32}$$

From the relation (3.2.27) of the camber problem we get:

$$\rangle \underline{v}^3 \langle \;=\; -\rangle \underline{q}^i \langle \left(\overset{o}{\tilde{\underline{n}}}_i \;+\; \overset{c}{\tilde{\underline{n}}}_i \right) \Rightarrow$$

$$\Rightarrow\; \tilde{\underline{c}}_1^3 \rangle v^1 \langle + \tilde{\underline{c}}_2^3 \rangle v^2 \langle + \tilde{\underline{c}}_3^3 \rangle v^3 \langle \;=\; \left(\rangle \underline{q}^1 \langle \frac{\partial E_c}{\partial u^1} + \rangle \underline{q}^2 \langle \frac{\partial E_c}{\partial u^2} - \rangle \underline{q}^3 \langle \right) \tag{3.2.33}$$

since from relation (3.2.25) we get:

$$\rangle \underline{v}^i \langle = \rangle \tilde{\underline{v}}^i \langle + O(\varepsilon^2) \;=\; \tilde{\underline{c}}_j^i \rangle \tilde{v}^j \langle + O(\varepsilon^2) \;=\; \tilde{\underline{c}}_j^i \rangle v^j \langle + O(\varepsilon^2) \tag{3.2.34}$$

We emphasize the fact that relations (3.2.31) and (3.2.33) hold for points away from the blades' leading edge.

3.3 Prediction of pressure distribution

For the inertial reference frame, $\underset{\bowtie}{x}^i$, since the perturbation velocity field is irrotational, we can define the

$$\Phi(\underset{\bowtie}{x}^i) \longrightarrow \text{perturbation potential}$$

where $\underset{\bowtie}{\Phi}_{,i} = \underset{\bowtie}{v_i}$ is the (absolute) perturbation velocity with regard to the inertial reference frame, projected on the inertial system.

Let S_b be the surface of a blade and M a point on S_b. Let u^α be the curvilinear (surface) coordinatization of S_b, as described in section 2.2 and t the time. Then the following analytical description of S_b is valid :

$$\vec{x}_M \;=\; \vec{x}_M(u^\alpha, t) \tag{3.3.1}$$

where $\vec{x}_M$ is a continuous function of the position of M on the surface with regard to the inertial frame.

The trace of Φ on S is denoted by the function of $\phi(u^\alpha, t)$. The total differential of trace ϕ with respect to u^α and t as a function of Φ and $\underset{\bowtie}{x}^i(u^\alpha, t)$ (i.e $\phi(u^\alpha, t) \;=\; \Phi(\underset{\bowtie}{x}^i(u^\alpha, t), t)$, is given by:

$$d\phi \;=\; \frac{\partial \Phi}{\partial \underset{\bowtie}{x}^i}\left(\frac{\partial \underset{\bowtie}{x}^i}{\partial u^\alpha} du^\alpha \;+\; \frac{\partial \underset{\bowtie}{x}^i}{\partial t} dt\right) \;+\; \frac{\partial \Phi}{\partial t} dt \tag{3.3.2}$$

For a fixed point M on S_b ($u^\alpha \;=\;$ const), with velocity $\underset{\bowtie}{u}^i_M$, relation (3.3.2) gives :

$$d\phi \;=\; \frac{\partial \Phi}{\partial \underset{\bowtie}{x}^i}\,\frac{\partial \underset{\bowtie}{x}^i_M}{\partial t}\, dt \;+\; \frac{\partial \Phi}{\partial t}\, dt \quad\Rightarrow\quad \left.\frac{d\phi}{dt}\right|_M \;=\; \underset{\bowtie}{\Phi}_{,i}\, \underset{\bowtie}{u}^i_M \;+\; \frac{\partial \Phi}{\partial t} \quad\Rightarrow$$

$$\Rightarrow \quad \frac{\partial \Phi}{\partial t} \;=\; \left.\frac{d\phi}{dt}\right|_M \;-\; \underset{\bowtie}{v_i}\, \underset{\bowtie}{u}^i_M \tag{3.3.3}$$

where $\frac{\partial}{\partial t} \rightarrow$ derivative at fixed $\underset{\bowtie}{x}^i$

$\frac{d}{dt} \rightarrow$ derivative at fixed u^α

Let $\vec{V}_\infty$ be the (absolute) backround velocity of the fluid at M, with regard to the inertial reference frame and Φ_∞ the corresponding potential, then $\underset{\bowtie}{\Phi}_{\infty,i} \;=\; \underset{\bowtie}{V}_{\infty\, i}$ and :

$$\Phi \;+\; \Phi_\infty \;\longrightarrow \text{total potential of the flow at M}$$

Applying Bernoulli's theorem at M and using the definitions of chapter 2.1, we get :

$$\frac{p}{\rho} + \frac{\partial(\Phi + \Phi_\infty)}{\partial t} + \frac{1}{2}(\underset{\bowtie}{v}^i + \underset{\bowtie}{V}^i_\infty)(\underset{\bowtie}{v}_i + \underset{\bowtie}{V}_{\infty\, i}) = const \Rightarrow$$

$$\Rightarrow \quad \frac{p}{\rho} + \frac{d(\phi + \phi_\infty)}{dt} - (\underset{\bowtie}{v}_i + \underset{\bowtie}{V}_{\infty\, i})\, \underset{\bowtie}{u}^i_M + \frac{1}{2}(\underset{\bowtie}{v}^i + \underset{\bowtie}{V}^i_\infty)(\underset{\bowtie}{v}_i + \underset{\bowtie}{V}_{\infty\, i}) = const \Rightarrow$$

$$\Rightarrow \quad \frac{p}{\rho} + \frac{d(\phi + \phi_\infty)}{dt} + \frac{1}{2}(\underset{\bowtie}{v}^i + \underset{\bowtie}{V}^i_\infty - \underset{\bowtie}{u}^i_M)(\underset{\bowtie}{v}_i + \underset{\bowtie}{V}_{\infty\, i} - \underset{\bowtie}{u}_{Mi}) - \frac{1}{2}\underset{\bowtie}{u}^i_M\, \underset{\bowtie}{u}_{Mi} = const \tag{3.3.4}$$

Since all of the terms are tensor invariants (scalar), any coordinate system can be used for their representation. Using the curvilinear coordinate system $\underline{x}^i$ (see Section 2.2), we get :

$$\frac{p}{\rho} + \frac{d(\phi + \phi_\infty)}{dt} + \frac{1}{2}\underline{w}^i\, \underline{w}_i - \frac{1}{2}\underline{u}^i_M\, \underline{u}_{Mi} = const \tag{3.3.5}$$

Applying relation (3.3.5) on both sides of blade's reference surface, we get:

$$\frac{p^+ - p^-}{\rho} + \frac{d(\phi^+ - \phi^-)}{dt} + \frac{1}{2}\tilde{\underline{w}}^{i+}\, \tilde{\underline{w}}^+_i - \frac{1}{2}\tilde{\underline{w}}^{i-}\, \tilde{\underline{w}}^-_i = 0 \tag{3.3.6}$$

since both ϕ_∞ and u^i_M, are continuous through the reference surface.

Using relations (3.1.2) and (3.2.24) we arrive at the linearized form of the Bernoulli's theorem :

$$\frac{p^+ - p^-}{\rho} + \frac{d(\phi^+ - \phi^-)}{dt} + \frac{1}{2}\tilde{\underline{v}}^{i+}\, \tilde{\underline{v}}^+_i + \tilde{\underline{q}}^i \left(\tilde{\underline{v}}^+_i - \tilde{\underline{v}}^-_i\right) - \frac{1}{2}\tilde{\underline{v}}^{i-}\, \tilde{\underline{v}}^-_i = 0 \Rightarrow$$

$$\Rightarrow \frac{p^+ - p^-}{\rho} + \frac{d(\phi^+ - \phi^-)}{dt} + \tilde{\underline{q}}^i \langle \tilde{\underline{v}}_i \rangle + O(\varepsilon^2) = 0 \tag{3.3.7}$$

We can now develop $\tilde{\underline{q}}^i$ and $\langle \tilde{\underline{v}}^i \rangle$ into their normal and tangential components to the blade reference surface :

$$\tilde{\underline{q}}^i = \overset{N}{\tilde{\underline{q}}}{}^i + \overset{T}{\tilde{\underline{q}}}{}^i \tag{3.3.8}$$

$$\langle \tilde{\underline{v}}_i \rangle = \langle \overset{N}{\tilde{\underline{v}}}_i \rangle + \langle \overset{T}{\tilde{\underline{v}}}_i \rangle \tag{3.3.9}$$

where the overscore N or T denotes the normal and tangential component respectively (see Appendix B)

We shall now try to derive an expression for each of the components of $\tilde{\underline{q}}^i \langle \tilde{\underline{v}}_i \rangle$ in terms of the surface (blade reference surface) coordinates, starting with $\overset{T}{\tilde{\underline{q}}}{}^i \langle \overset{T}{\tilde{\underline{v}}}_i \rangle$. Therefore we introduce the blade reference surface vorticity vector $\underline{\gamma}^i$:

$$\underline{\gamma}_i \overset{def}{=} \underline{\varepsilon}_{ijk} \langle \tilde{\underline{v}}^j \rangle \; \overset{o}{\tilde{\underline{N}}}{}^k \tag{3.3.10}$$

which is a space vector tangent to the blade reference surface and $\overset{\circ}{\tilde{\underline{N}}}{}^{k}$ a space vector normal to the surface under consideration. In the case of the blade's reference surface the normal vector is defined by the relations (3.2.11) and (3.2.28). The tangent vector $\langle\, \overset{T}{\tilde{\underline{v}}}{}^{i} \rangle$ in terms of $\underline{\gamma}^{i}$ is given by the relation (B13) :

$$\langle\, \overset{T}{\tilde{\underline{v}}}{}^{i} \rangle \;=\; \underline{\varepsilon}^{ijk}\, \underline{\gamma}_{j}\, \overset{\circ}{\tilde{\underline{N}}}_{k} \tag{3.3.11}$$

With the help of relation (3.3.11), we get:

$$\overset{T}{\tilde{\underline{q}}}{}^{i}\, \langle \overset{T}{\tilde{\underline{v}}}_{i} \rangle \;=\; \overset{T}{\tilde{\underline{q}}}_{i}\, \langle\, \overset{T}{\tilde{\underline{v}}}{}^{i} \rangle \;=\; \overset{T}{\tilde{\underline{q}}}_{i} \left(\underline{\varepsilon}^{ijk}\, \underline{\gamma}_{j}\, \overset{\circ}{\tilde{\underline{N}}}_{k} \right) \overset{(B4)}{=} \overset{T}{\tilde{\underline{q}}}_{i}\, \underline{\gamma}_{j} \left(\underline{x}^{i}_{\alpha}\, \underline{x}^{j}_{\beta}\, \varepsilon^{\alpha\beta} \right) \;=\; \varepsilon_{\alpha\beta}\; \overset{T}{\tilde{\underline{q}}}{}^{\alpha}\, \underline{\gamma}^{\beta} \tag{3.3.12}$$

where (using relation (B15))

$$\overset{T}{\tilde{\underline{q}}}_{\alpha} \;=\; \underline{x}^{j}_{\alpha}\, \overset{T}{\tilde{\underline{q}}}_{j} \qquad\qquad \underline{\gamma}_{\beta} \;=\; \underline{x}^{k}_{\beta}\, \underline{\gamma}_{k} \tag{3.3.13),(3.3.14}$$

are the surface components of the space vectors $\overset{T}{\tilde{\underline{q}}}_{j}$, $\underline{\gamma}_{k}$, which are tangent to the blade reference surface.

Similarly for the rest of the components :

$$\overset{N}{\tilde{\underline{q}}}{}^{i} \langle \overset{N}{\tilde{\underline{v}}}_{i} \rangle \;=\; \left[\overset{\circ}{\tilde{\underline{N}}}{}^{i} \left(\tilde{\underline{q}}^{j}\, \overset{\circ}{\tilde{\underline{N}}}_{j} \right) \right] \left[\overset{\circ}{\tilde{\underline{N}}}_{i} \left(\langle \tilde{\underline{v}}_{j} \rangle\, \overset{\circ}{\tilde{\underline{N}}}{}^{j} \right) \right] \;=\; \left(\tilde{\underline{q}}^{j}\, \overset{\circ}{\tilde{\underline{N}}}_{j} \right) \cdot \sigma \;=\; \tilde{\underline{q}}^{3} \sqrt{\frac{\tilde{\underline{\underline{g}}}}{\underline{\alpha}}}\, \sigma \tag{3.3.15}$$

$$\overset{N}{\tilde{\underline{q}}}{}^{i} \langle \overset{T}{\tilde{\underline{v}}}_{i} \rangle \;=\; \left[\overset{\circ}{\tilde{\underline{N}}}{}^{i} \left(\tilde{\underline{q}}^{j}\, \overset{\circ}{\tilde{\underline{N}}}_{j} \right) \right] \left[\underline{\varepsilon}_{ijk}\, \underline{\gamma}^{j}\, \overset{\circ}{\tilde{\underline{N}}}{}^{k} \right] \;=\; \left(\underline{\varepsilon}_{ijk}\, \overset{\circ}{\tilde{\underline{N}}}{}^{i}\, \underline{\gamma}^{j}\, \overset{\circ}{\tilde{\underline{N}}}{}^{k} \right) \left(\tilde{\underline{q}}^{j}\, \overset{\circ}{\tilde{\underline{N}}}_{j} \right) \;=\; 0 \tag{3.3.16}$$

$$\overset{T}{\tilde{\underline{q}}}{}^{i}\, \langle \overset{N}{\tilde{\underline{v}}}_{i} \rangle \;=\; \left[\underline{\varepsilon}^{ijk}\, \underline{\gamma}^{(q)}_{j}\, \overset{\circ}{\tilde{\underline{N}}}_{k} \right] \left[\overset{\circ}{\tilde{\underline{N}}}_{i} \left(\langle \tilde{\underline{v}}_{j} \rangle\, \overset{\circ}{\tilde{\underline{N}}}{}^{j} \right) \right] \;=\; \left(\underline{\varepsilon}^{ijk}\, \underline{\gamma}^{(q)}_{j}\, \overset{\circ}{\tilde{\underline{N}}}_{k}\, \overset{\circ}{\tilde{\underline{N}}}_{i} \right) \left(\langle \tilde{\underline{v}}_{j} \rangle\, \overset{\circ}{\tilde{\underline{N}}}{}^{j} \right) \;=\; 0 \tag{3.3.17}$$

where

$$\underline{\gamma}^{(q)}_{i} \;=\; \underline{\varepsilon}_{ijk}\, \tilde{\underline{q}}^{j}\, \overset{\circ}{\tilde{\underline{N}}}{}^{k} \tag{3.3.18}$$

$$\langle \tilde{\underline{v}}^{i} \rangle \;=\; \overset{\circ}{\tilde{\underline{N}}}{}^{i}\, \sigma \;+\; \underline{\varepsilon}^{ijk}\, \underline{\gamma}_{j}\, \overset{\circ}{\tilde{\underline{N}}}_{k} \tag{3.3.19}$$

with $\quad \sigma = \overset{\circ}{\tilde{\underline{N}}}_{i}\, \langle \tilde{\underline{v}}^{i} \rangle \quad , \quad \underline{\gamma}_{i} = \underline{\varepsilon}_{ijk}\, \langle \tilde{\underline{v}}^{j} \rangle\, \overset{\circ}{\tilde{\underline{N}}}{}^{k}$

Substituting relations (3.3.12),(3.3.15),(3.3.16) and (3.3.17) to the linearized bernoulli equation (3.3.7):

$$\frac{p^+ - p^-}{\rho} + \frac{d(\phi^+ - \phi^-)}{dt} + \underline{\varepsilon}_{\alpha\beta}\; \overset{T}{\tilde{\underline{q}}}{}^{\alpha}\, \underline{\gamma}^{\beta} + \tilde{\underline{q}}^{3} \sqrt{\frac{\tilde{\underline{g}}}{\underline{\alpha}}}\,\sigma + O(\varepsilon^2) = 0 \tag{3.3.20}$$

For a further simplification of relation (3.3.20) we shall try to find an order of estimate for $\overset{T}{\tilde{\underline{q}}}{}^{\alpha}$. To this respect we have :

$$\overset{T}{\tilde{\underline{q}}}{}^{\alpha} = \underline{\alpha}^{\alpha\beta}\, \overset{T}{\tilde{\underline{q}}}{}_{\beta} = \underline{\alpha}^{\alpha\beta}\, \underline{x}^{r}_{\beta}\, \overset{T}{\tilde{\underline{q}}}{}_{r} = \underline{\alpha}^{\alpha\beta}\, \underline{x}^{r}_{\beta}\, \tilde{\underline{q}}_{r} \tag{3.3.21}$$

In order to prove the last equation of (3.3.21) we note that $\underline{\alpha}^{\alpha\beta}\, \underline{x}^{r}_{\beta}\, \underline{N}_{r} = \underline{\alpha}^{\alpha\beta}\, \underline{x}^{r}_{\beta} \left(\frac{1}{2}\, \varepsilon^{\gamma\delta}\, \underline{\varepsilon}_{rst}\, \underline{x}^{s}_{\gamma}\, \underline{x}^{t}_{\delta}\right)$ while $\underline{\varepsilon}_{rst}\, \underline{x}^{r}_{\beta}\, \underline{x}^{s}_{\gamma}\, \underline{x}^{t}_{\delta} = \varepsilon_{\beta\gamma\delta} = 0$, since β, γ, δ range only from 1 to 2 and

$$\overset{T}{\tilde{\underline{q}}}{}^{\alpha=1} = \underline{\alpha}^{1\beta}\, \underline{x}^{r}_{\beta}\, (\underline{g}_{rj}\, \tilde{\underline{q}}^{j}) = (\underline{\alpha}^{1\beta}\, \underline{g}_{\beta j})\, \tilde{\underline{q}}^{j} = \delta^{1}_{j}\, \tilde{\underline{q}}^{j} \tag{3.3.22}$$

since $\underline{x}^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}$ and $\underline{x}^{3}_{\beta} = 0$

$$\underline{\alpha}^{\alpha\beta}\, \underline{g}_{\beta j} = \underline{g}^{\alpha\beta}\, \underline{g}_{\beta j} = \delta^{\alpha}_{j} \qquad \text{where } \alpha = 1,2 \qquad j = 1,2,3 \tag{3.3.23}$$

Repeating the same process for $\overset{T}{\tilde{q}}{}^{\alpha=2}$ we get the obvious results :

$$\overset{T}{\tilde{\underline{q}}}{}^{\alpha=1} = \tilde{\underline{q}}^{1}\,, \qquad \overset{T}{\tilde{\underline{q}}}{}^{\alpha=2} = \tilde{\underline{q}}^{2} \qquad (3.3.24)\ ,(3.3.25)$$

Substituting relations (3.3.24) and (3.3.25) to (3.3.20) we get :

$$\frac{p^+ - p^-}{\rho} = -\frac{d(\phi^+ - \phi^-)}{dt} - \underline{\varepsilon}_{12}\; \tilde{\underline{q}}^{1}\, \underline{\gamma}^{2} - \underline{\varepsilon}_{21}\; \tilde{\underline{q}}^{2}\, \underline{\gamma}^{1} - \tilde{\underline{q}}^{3} \sqrt{\frac{\tilde{\underline{g}}}{\underline{\alpha}}}\,\sigma + O(\varepsilon^2) \tag{3.3.26}$$

Given that $\underline{\varepsilon}_{\alpha\beta} = -\sqrt{\underline{\alpha}}\, e_{\alpha\beta}$ we introduce the following notation :

$$\underline{\Gamma}^{1} = \sqrt{\underline{\alpha}}\, \underline{\gamma}^{1} \quad \longrightarrow \quad \text{Modified bound vorticity} \tag{3.3.27}$$

$$\underline{\Gamma}^{2} = \sqrt{\underline{\alpha}}\, \underline{\gamma}^{2} \quad \longrightarrow \quad \text{Modified trailing vorticity [2]} \tag{3.3.28}$$

Thus relation (3.3.26) becomes :

$$\frac{p^+ - p^-}{\rho} = -\frac{d(\phi^+ - \phi^-)}{dt} + \tilde{\underline{q}}^{1}\, \underline{\Gamma}^{2} - \tilde{\underline{q}}^{2}\, \underline{\Gamma}^{1} - \tilde{\underline{q}}^{3} \sqrt{\frac{\tilde{\underline{g}}}{\underline{\alpha}}}\,\sigma + O(\varepsilon^2) \tag{3.3.29}$$

[2]Notice that since $\underline{\gamma}^{\alpha}$ is an absolute tensor and $\underline{\alpha}$ is a relative invariant of weight two, $\underline{\Gamma}^{\alpha}$ should be a relative tensor of weight one.

where σ is the source intensity on the surface under consideration, defined by relation (3.2.32) as $\sigma \;=\; \langle \underline{\widetilde{v}}^{\,i} \rangle \, \underline{\overset{\circ}{\tilde{N}}}_i$. In the case of the blade's reference surface, the source intensity is calculated as :

$$\sigma \;=\; \Big(\rangle \underline{q}^1 \langle \frac{\partial E_T}{\partial u^1} + \rangle \underline{q}^2 \langle \frac{\partial E_T}{\partial u^2} \;-\; \langle \underline{q}^3 \rangle \Big) \sqrt{\frac{\underline{\widetilde{g}}}{\underline{\alpha}}} \tag{3.2.31*}$$

The components of vorticity (bound vorticity $\underline{\gamma}^2$ and trailing vorticity $\underline{\gamma}^1$) are connected with the continuity of the surface vorticity distribution expressed in the intrinsic blade reference surface coordinate system (Appendix C) :

$$\underline{\gamma}^{\alpha}_{,\alpha} \;=\; \frac{\partial}{\partial u^1} \big(\sqrt{\underline{\alpha}} \, \underline{\gamma}^1 \big) \;+\; \frac{\partial}{\partial u^2} \big(\sqrt{\underline{\alpha}} \, \underline{\gamma}^2 \big) \;=\; 0 \tag{3.3.30}$$

or using relations (3.3.27) and (3.3.28)

$$\frac{\partial}{\partial u^1} \big(\underline{\Gamma}^1 \big) \;+\; \frac{\partial}{\partial u^2} \big(\underline{\Gamma}^2 \big) \;=\; 0 \tag{3.3.31}$$

3.4 Trailing vortex sheet

At the trailing edge of a lifting body, the meeting of the two different layers (one from the face and the other from the back) gives rise to a surface of fluid where a tangential discontinuity of fluid velocity exists. At the limit of high Reynolds numbers the thickness of the free shear layer becomes small and thus we arrive at the prototype of a surface of discontinuity in a flow field.

A surface of discontinuity of flow can be introduced by considering that the surface of the free shear layer S_F, is a two sided surface with sides S_F^+ and S_F^-, in each side of which a different potential exists. Both sides of S_F are parts of the fluid and thus are free to move with the fluid. Since the sides S_F^+ and S_F^- constitute always the same surface S_F , S_F^+ is allowed to slip with respect to S_F^- but S_F^+ cannot separate from S_F^- to form a different surface. Thus the normal to S_F fluid velocities on the two sides S_F^+ and S_F^- should be equal. This is termed the kinematic boundary condition of the free shear layer and reads :

$$\tilde{\underline{N}}_i(\tilde{\underline{v}}^{i+} + V_\infty^i) = \tilde{\underline{N}}_i(\tilde{\underline{v}}^{i-} + V_\infty^i) \quad \Leftrightarrow \quad \tilde{\underline{N}}_i \, \langle \tilde{\underline{v}}^i \rangle \ = \ 0 \tag{3.4.1}$$

where $\tilde{\underline{N}}_i$ is the normal vector on a point M on the free shear layer expressed in the components of the $\underline{x}_i$ curvilinear system and $\tilde{\underline{v}}^{i\pm}$ are the traces of the velocity field on the free shear layer.

According to relation (3.4.1) the source intensity of the wake is zero :

$$\sigma \ = \ \tilde{\underline{N}}_i \, \langle \tilde{\underline{v}}^i \rangle \ = \ 0 \tag{3.4.2}$$

which means that the normal component of the perturbation velocity is continuous through the free shear layer:

$$\langle \ \overset{N}{\tilde{\underline{v}}}{}^i \rangle \ = \ \sigma \ \tilde{\underline{N}}^i \ = \ (0,0,0) \tag{3.4.3}$$

Furthermore since S_F is a fluid surface, it cannot carry loading. This is usually termed dynamic loading condition of the free shear layer and it reads :

$$p^+ \ = \ p^- \tag{3.4.4}$$

Applying relation (3.3.7) to a point on the free shear layer we get:

$$\frac{p^+ - p^-}{\rho} \ + \ \frac{d(\phi^+ - \phi^-)}{dt} \ + \ \tilde{\underline{q}}^i \langle \tilde{\underline{v}}_i \rangle \, + \, \rangle \, \tilde{\underline{v}}^i \langle \ \langle \tilde{\underline{v}}_i \rangle \ = \ 0 \ \Rightarrow$$

$$\Rightarrow \ \frac{p^+ - p^-}{\rho} \ + \ \frac{d(\phi^+ - \phi^-)}{dt} \ + \ \left(\tilde{\underline{q}}^i \, + \, \rangle \, \tilde{\underline{v}}^i \langle \right) \ \langle \tilde{\underline{v}}_i \rangle \ = \ 0 \tag{3.4.5}$$

Applying both the kinematic and dynamic conditions of the free shear layer we get :

$$\frac{d(\phi^+ - \phi^-)}{dt} \ + \ \left(\tilde{\underline{q}}^i \, + \, \rangle \, \tilde{\underline{v}}^i \langle \right) \ \langle \overset{T}{\tilde{\underline{v}}}_i \rangle \ = \ 0 \tag{3.4.6}$$

Given that for the inertial system $(\underset{\bowtie}{\phi_{,i}}^{+} - \underset{\bowtie}{\phi_{,i}}^{-}) = \frac{\partial(\phi^+ - \phi^-)}{\partial \underset{\bowtie}{x^i}} = \langle \underset{\bowtie}{\overset{T}{\tilde{v}_i}} \rangle$ from relation (3.3.3) we get :

$$\frac{\partial(\phi^+ - \phi^-)}{\partial t} + \underset{\bowtie}{u^i_M} \langle \underset{\bowtie}{\overset{T}{\tilde{v}_i}} \rangle + \left(\underset{\bowtie}{\tilde{q}^i} + \rangle \underset{\bowtie}{\tilde{v}^i} \langle \right) \langle \underset{\bowtie}{\overset{T}{\tilde{v}_i}} \rangle = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial(\phi^+ - \phi^-)}{\partial t} + \left(\underset{\bowtie}{u^i_M} + \underset{\bowtie}{\tilde{q}^i} + \rangle \underset{\bowtie}{\tilde{v}^i} \langle \right) \langle \underset{\bowtie}{\overset{T}{\tilde{v}_i}} \rangle = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial \mu}{\partial t} + \left(\underset{\bowtie}{V^i_\infty} + \rangle \underset{\bowtie}{\tilde{v}^i} \langle \right) \langle \underset{\bowtie}{\overset{T}{\tilde{v}_i}} \rangle = 0 \tag{3.4.7}$$

where $\mu = \phi^+ - \phi^-$ is the dipole intensity on the free shear layer.

Comparing relations (3.3.3) and (3.4.7) we see that the dipoles on the free shear layer move with the mean total flow velocity $(\underset{\bowtie}{V^i_\infty} + \rangle \underset{\bowtie}{\tilde{v}^i} \langle)$ as seen from the inertial frame system.

Relation (3.4.7) examines the dipole intensities μ instead of the vorticity distribution γ^i that we examined in the previous paragraphs. The latter includes information about the geometry of the trailing vortex sheet, which in the general case cannot be described analytically as it contains instabilities especially away from the trailing edge. Moreover as we will show in a following paragraph the contribution of the free shear layer can be calculated separately from the contribution of the blade's vorticity distribution.

Assuming that both velocities $\rangle \underset{\bowtie}{\tilde{v}^i} \langle$ and $\langle \underset{\bowtie}{\overset{T}{\tilde{v}_i}} \rangle$ are of first order $O(\varepsilon)$, relation (3.4.7) becomes :

$$\frac{\partial \mu}{\partial t} + \langle \underset{\bowtie}{\overset{T}{\tilde{v}_i}} \rangle \underset{\bowtie}{V^i_\infty} + O(\varepsilon^2) = 0 \tag{3.4.8}$$

This means that the dipole distribution moves with the absolute undisturbed velocity of the flow, which corresponds to the frozen wake.

Assuming (functional) continuity of the field variables in passing from S_b to S_F through a point on the trailing edge, both kinematic and dynamic conditions (3.4.2) and (3.4.4) of the free shear layer can be applied on the trailing edge. Substituting the two previous relations in the Bernoulli equation (3.3.29) we get :

$$-\frac{d(\phi^+ - \phi^-)}{dt} + \underline{\tilde{q}}^1 \underline{\Gamma}^2 - \underline{\tilde{q}}^2 \underline{\Gamma}^1 + O(\varepsilon^2) = 0 \tag{3.4.9}$$

Consider a closed curve with all its points entirely into the fluid with the exception its starting and ending points which coincide with a point on the trailing edge. The direction of C is clockwise, so that it starts on the side of S^{F-} with potential ϕ^- and ends on the side of S^{F+} with potential ϕ^+, as shown in figure 3.4.1. Then the circulation Γ around the curve, with tangent vector $d\underline{l}^i$ is:

$$\Gamma \;=\; \oint_C (\underline{V}_{\infty\, i} \,+\, \underline{v}_i)\, d\underline{l}^i \;=\; \oint_C \underline{v}_i\, d\underline{l}^i \qquad (3.4.10)$$

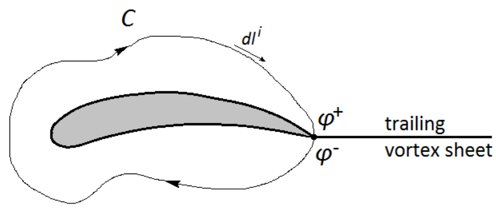


Figure 3.4.1

Since the undisturbed fluid field velocity with regard to the inertial system is supposed to be irrotational, so for every closed curve in it $\oint_C \underline{V}_{\infty\, i}\, d\underline{l}^i \,=\, 0$. Therefore :

$$\Gamma \;=\; \oint_C \underline{v}_i\, d\underline{x}^i \;=\; \oint_C \underline{\phi_{,i}}\, d\underline{x}^i \;=\; \phi^+ \,-\, \phi^- \;=\; \mu \qquad (3.4.11)$$

Note that in order to find the dipole intensity at a given point, the curve should only intersect with the reference blade surface (or the trailing vortex sheet) at that particular point, as shown in Figure 3.4.1, in the case of the trailing edge.

If the curve C belongs on the reference surface of the blade, then by definition $d\underline{l}^i$ should be tangent to the surface and thus $\overset{N}{\tilde{\underline{v}}}{}_i^{\pm}\, d\underline{l}^i \,=\, 0$. Assuming also that the path lies parallel to the $\underline{x}^2$ axis, then $d\underline{l}^i = \delta^i_2\, d\underline{x}^2$, for the part of the curve tangent to the upper surface and $d\underline{l}^i = -\delta^i_2\, d\underline{x}^2$, for the part of the curve tangent to the lower surface. Using relations (3.2.24) and (3.3.11) :

$$\phi^+ \,-\, \phi^- \;=\; \oint_C \overset{T}{\tilde{\underline{v}}}_i\, d\underline{l}^i \;=\; \int_{L.E}^{T.E} \langle \overset{T}{\tilde{\underline{v}}}_2 \rangle\, d\underline{x}^2 \;=\; \int_{L.E}^{T.E} \underline{\varepsilon}_{2jk}\, \underline{\gamma}^j \;\overset{\circ}{\tilde{\underline{N}}}{}^k\, d\underline{x}^2 \qquad (3.4.12)$$

where $\overset{\circ}{\tilde{\underline{N}}}_k \;=\; (0,0,1)\sqrt{\dfrac{\tilde{g}}{\underline{\alpha}}}$ is the normal unit vector on the blade reference surface. Since $\underline{\gamma}^3 = 0$:

$$\phi^+ \,-\, \phi^- \;=\; \int_{L.E}^{T.E} \underline{\varepsilon}_{213}\, \underline{\gamma}^1 \;\overset{\circ}{\tilde{\underline{N}}}{}^3\, d\underline{x}^2 \qquad (3.4.13)$$

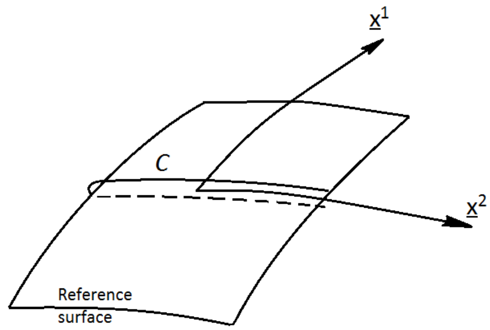


Figure 3.4.2

but

$$\underline{\varepsilon}_{213}\ \overset{\overset{o}{\sim}3}{\underline{N}} \;=\; -e_{213}\sqrt{\underline{\tilde{g}}}\left(\ \overset{\sim i3}{\underline{g}}\ \overset{\overset{o}{\sim}}{\underline{N}}_i\right) \;=\; \sqrt{\underline{\tilde{g}}}\left(\ \overset{\sim 33}{\underline{g}}\sqrt{\frac{\underline{\tilde{g}}}{\underline{\alpha}}}\right) \;=$$

$$= \frac{\overset{\sim 33}{\underline{G}}}{\sqrt{\underline{\alpha}}} \;=\; \frac{\underline{\alpha}_{11}\,\underline{\alpha}_{22}\;-\;\underline{\alpha}_{12}\,\underline{\alpha}_{21}}{\sqrt{\underline{\alpha}}} \;=\; \sqrt{\underline{\alpha}} \tag{3.4.14}$$

since $\underline{g}_{ij}\Big|_{i,j=1,2} \;=\; \underline{g}_{\alpha\beta} \;=\; \underline{\alpha}_{\alpha\beta}$

Substituting relations (3.4.14) and (3.3.27) to (3.4.13) we get :

$$\phi^+ \;-\; \phi^- \;=\; \int\limits_{L.E}^{T.E} \underline{\Gamma}^1\, d\underline{x}^2 \;=\; \int\limits_{-1/2}^{1/2} \underline{\Gamma}^1\, d\underline{x}^2 \tag{3.4.15}$$

Substituting relation (3.4.15) to (3.4.9) we get :

$$-\int\limits_{-1/2}^{1/2} \frac{d\underline{\Gamma}^1}{dt}\, d\underline{x}^2 \;+\; \underline{\tilde{q}}^{\,1}\,\underline{\Gamma}^2 \;-\; \underline{\tilde{q}}^{\,2}\,\underline{\Gamma}^1 \;+\; O(\varepsilon^2) \;=\; 0 \tag{3.4.16}$$

According to relation (A95), $\sqrt{\underline{\alpha}}$ and therefore $\underline{\Gamma}^1$ and μ are proportional to the chord length. This means that for $c \longrightarrow 0$ the inertial term (first term) of (3.4.16) tends to zero while the dynamic pressure introduced by the second and third term are invariant. Therefore equation (3.4.16) degenerates to the steady case.

The pressure distribution on any point on the reference blade is given by the relation :

$$\frac{p^+ - p^-}{\rho} \;=\; -\int\limits_{-1/2}^{\underline{x}_2} \frac{d\underline{\Gamma}^1}{dt}\, d\underline{x}^2 \;+\; \underline{\tilde{q}}^{\,1}\,\underline{\Gamma}^2 \;-\; \underline{\tilde{q}}^{\,2}\,\underline{\Gamma}^1 \;-\; \left(\underline{\tilde{q}}^{\,3}\sqrt{\frac{\underline{\tilde{g}}}{\underline{\alpha}}}\,\sigma\right) \;+\; O(\varepsilon^2) \tag{3.4.17}$$

3.5 No entrance of vorticity boundary condition

Let S be a part of the reference blade surface, and C a curve entirely in it. If λ^α and μ^α are the tangent and normal vector of C on particular point, then using Green's theorem we get :

$$\iint_S \underline{\gamma}^\alpha_{,\alpha}\, dS \;=\; -\oint_C \underline{\gamma}^\alpha\, \mu_\alpha dl \;=\; -\oint_C \varepsilon_{\alpha\beta}\gamma^\alpha\lambda^\beta\, dl = 0 \tag{3.5.1}$$

$\underline{\gamma}^\alpha_{,\alpha} = 0$ expresses the continuity of vorticity in a differential form, while $\oint_C \varepsilon_{\alpha\beta}\gamma^\alpha\lambda^\beta\, dl = 0$ does so in an integral form. Obviously the second one is a weak form of the continuity since it doesn't demand that the vorticity is a differentiable function. In our theory we have assumed the validity of $\underline{\gamma}^\alpha_{,\alpha} = 0$ for points in or out of the reference surface. At boundary points of the lifting surface (i.e. the blade outline) the differential form for γ^α has meaning only in sense of trace. Thus at those points we have to use the integral form, to manage vorticity continuity. Based on 2D flat plate results, which has been embodied in our mode function approximation for bound vorticity, the γ^1 component tends to infinity as we approach the leading edge ($\gamma^1 = A_0/\tan(y^2) + ...$). More specifically, γ^1 could be a non-continues function at the leading edge. Therefore approaching the boundary of S, relation $\underline{\gamma}^\alpha_{,\alpha} = 0$ has not meaning and we are forced to use the integral expression of the continuity of vorticity.

Suppose now a curve C with points A, B and Γ where the first two lie outside of S and the third inside. We assume that the $\widehat{AB}$ part of the curve lies outside the leading edge of the reference surface and it's parallel to it.

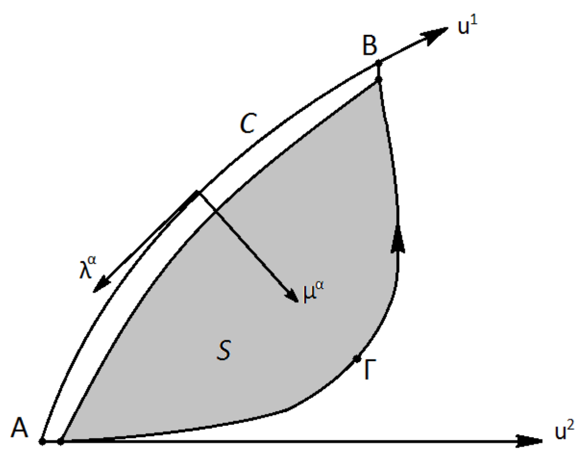


Figure 3.5.1

Given that approaching the leading edge from the outside we have continuity of velocities $\underset{\sim}{\underline{v}}^{i+} = \underset{\sim}{\underline{v}}^{i-}$, the vorticity distribution will be zero. Thus there will be no vorticity entering S from $\widehat{AB}$.

$$\oint_C \varepsilon_{\alpha\beta}\gamma^\alpha\lambda^\beta\, dl \;=\; \oint_{\widehat{AB}} \varepsilon_{\alpha\beta}\cancelto{0}{\gamma^\alpha}\lambda^\beta\, dl \;+\; \oint_{\widehat{A\Gamma B}} \varepsilon_{\alpha\beta}\gamma^\alpha\lambda^\beta\, dl \;=\; 0 \tag{3.5.2}$$

Assuming that $S \rightarrow 0$,but with the $\widehat{AB}$ part of the curve remaining the same, then $\widehat{A\Gamma B} \rightarrow \widehat{AB}$. Therefore the tangent vector λ^β on $\widehat{A\Gamma B}$ will be parallel with the tangent vector on $\widehat{AB}$, and thus :

$$\varepsilon_{\alpha\beta}\gamma^\alpha\lambda^\beta = 0 \qquad (3.5.3)$$

with λ^β tangent to $\widehat{AB}$.

Relation (3.5.3) means that the vorticity $\underline{\gamma}^\alpha$ on the blade is tangent to the L.E.

We observe that a unit vector across the leading edge $u^2 = -\frac{1}{2}$ is $\lambda^\alpha = \frac{1}{\sqrt{\underline{\alpha}_{11}}}\delta^\alpha_{(1)}$, where $\underline{\alpha}_{11}$ has to be evaluated at the leading edge points. Therefore relation (3.5.3) becomes :

$$\underline{\gamma}^2(u^1, u^2 = -\frac{1}{2}) = 0 \qquad (3.5.4)$$

In other word we have proven the continuity of $\underline{\gamma}^2$ at the leading edge. Using again relation (3.5.1) and (3.5.4) it's easy to prove that $\underline{\gamma}^1 = constant$ at the leading edge. In fact it's zero as we approach the L.E outside of S and non-zero (infinite for the linearized theory) as we approach the L.E from the inside of S.

The tip of the blade is usually considered the point where the leading edge meets the trailing edge which emits the trailing vortex sheet. In our case though we assume that the reference surface of the blade ends prior to the actual tip of the blade. In other words we cut the end of the blade so that the tip is now a line of finite length (chord>0) parallel to the $\underline{x}^2$ axis. This is essential for the numerical calculations, since according to relation (A95) the surface metric tensor goes to zero for c=0.

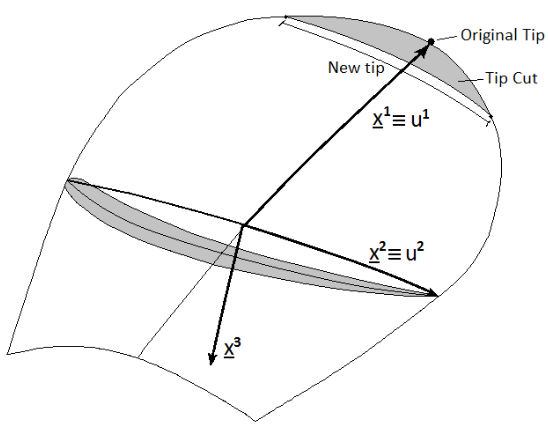


Figure 3.5.2

In order to characterize the tip as part of the leading edge or the trailing edge we examine if it emits a trailing vortex sheet. Supposing that the mean total velocity relative to a point M on the tip is pointing away from the reference surface $((\tilde{\underline{q}}^3 + \rangle\tilde{\underline{v}}^3\langle) \neq 0$ or $(\tilde{\underline{q}}^1 + \rangle\tilde{\underline{v}}^1\langle)\mu_1 < 0)$, then according to relation (3.4.7) there is a surface of discontinuity, with dipole intensity μ, which follows the total flow velocity. That makes the tip part of the trailing edge, meaning that $\tilde{\underline{v}}^{i+} \neq \tilde{\underline{v}}^{i-}$. Otherwise if the flow is point inwards or if it's parallel to the chord $((\tilde{\underline{q}}^1 + \rangle\tilde{\underline{v}}^1\langle)\,\mu_1 \geq 0$ and $(\tilde{\underline{q}}^3 + \rangle\tilde{\underline{v}}^3\langle) = 0)$, the tip is a part of the leading edge and so $\tilde{\underline{v}}^{i+} = \tilde{\underline{v}}^{i-}$ outside of the tip. In that last case, we can apply the no entrance vorticity condition (3.5.3), which reads :

$$\underline{\gamma}^1(u^1 = R_o, u^2) \;=\; 0 \tag{3.5.5}$$

In the program we assume for the sake of simplicity that the flow is parallel to the tip, $(\tilde{\underline{q}}^i + \rangle\tilde{\underline{v}}^i\langle) = (\tilde{\underline{q}}^2 + \rangle\tilde{\underline{v}}^2\langle)\delta_2^i$. Therefore relation (3.5.5) is always valid at the tip. Although this limits the range of movements of the blade, it is pretty accurate in the context of the small perturbations of linearized theory where the angle of attack and the direction of the flow have a very short range. Assuming that $(\tilde{\underline{q}}^i + \rangle\tilde{\underline{v}}^i\langle) = (\tilde{\underline{q}}^2 + \rangle\tilde{\underline{v}}^2\langle)\delta_2^i$ at the tip, then by substituting (3.4.15) to (3.3.7) :

$$\frac{p^+ - p^-}{\rho} + \frac{d(\phi^+ - \phi^-)}{dt} + \frac{1}{2}\tilde{\underline{v}}^{i+}\,\tilde{\underline{v}}_i^+ + \tilde{\underline{q}}^i\left(\tilde{\underline{v}}_i^+ - \tilde{\underline{v}}_i^-\right) - \frac{1}{2}\tilde{\underline{v}}^{i-}\,\tilde{\underline{v}}_i^- \;=\; 0 \;\Rightarrow$$

$$\Rightarrow\; \frac{p^+ - p^-}{\rho} \;=\; -\int_{-1/2}^{\underline{x}_2} \frac{d\underline{\Gamma}^1}{dt}\, d\underline{x}^2 \;-\; \left(\rangle\tilde{\underline{v}}^i\langle + \tilde{\underline{q}}^i\right)\,\langle\tilde{\underline{v}}_i\rangle \;\Rightarrow \qquad \left(\frac{d\underline{\Gamma}^1}{dt} \overset{(3.5.5)}{=} 0\right)$$

$$\Rightarrow\; \frac{p^+ - p^-}{\rho} \;=\; -\left(\rangle\tilde{\underline{v}}^2\langle + \tilde{\underline{q}}^2\right)\,\varepsilon_{2jk}\,\underline{\gamma}^j\,\underline{N}^k \overset{(3.5.5)}{=} 0 \tag{3.5.6}$$

Relation (3.5.6) of the equalization of pressure, guaranties the compatibility of the boundary conditions at the junction of the trailing edge with the tip and hub and should always be valid near the tips (or tip and hub in case of the propeller), unless $(\tilde{\underline{q}}^1 + \rangle\tilde{\underline{v}}^1\langle)\,\mu_1 > 0$. The physical interpretation of this, in the case of the hub, is that a secondary flow from the back of one blade equalizes the pressure on the face of the neighboring blade. When there is no vortex sheet emanating form the tip and flow is pointing towards the inside of the blade $(\tilde{\underline{q}}^1 + \rangle\tilde{\underline{v}}^1\langle)\,\mu_1 > 0$, then the tip is a part of the leading edge and the pressure difference there assumes infinite values in the linearized theory.

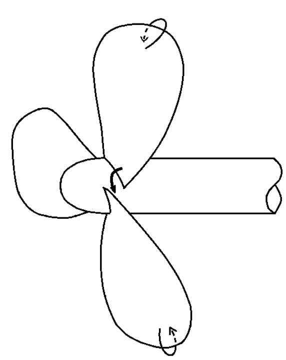


Figure 3.5.3

When the wing has a burst start, at t=0 there is no trailing vortex sheet even behind the trailing edge. Therefore, due to the continuity in velocity around the blade ($\tilde{\underline{v}}^{i+} = \tilde{\underline{v}}^{i+}$), we can apply the no entrance vorticity condition.

$$\underline{\gamma}^2(u^1, u^2 = -\frac{1}{2}) \;=\; 0 \rightarrow \text{Zero free vorticity at the leading edge} \tag{3.5.7}$$

$$\underline{\gamma}^2(u^1, u^2 = \frac{1}{2}) \;=\; 0 \rightarrow \text{Zero free vorticity at the trailing edge} \tag{3.5.8}$$

$$\underline{\gamma}^1(u^1 = R_o, u^2) \;=\; 0 \rightarrow \text{Zero bound vorticity at the (upper) tip} \tag{3.5.9}$$

$$\underline{\gamma}^1(u^1 = 0, u^2) \;=\; 0 \rightarrow \text{Zero bound vorticity at the (lower) tip} \tag{3.5.10}$$

According to relations (3.5.7) - (3.5.10), the vorticity vector, $\underline{\gamma}^\alpha = (\underline{\gamma}^1, \underline{\gamma}^2)$, is tangent to the outline of the blade, meaning that this is a vortex line. Due to the continuity of vorticity (3.5.1), the modified vorticity should be constant on the blade's outline. In the case of the lifting surface theory $\underline{\Gamma}^1$ is infinite at the leading edge and as we will see in chapter 4, $\underline{\Gamma}^2$ assumes also infinite values at the tips. This agrees with experiments where we have spikes in the pressure distribution near the L.E and strong vortices starting at the tips. The physical interpretation of relation (3.5.8) is that the starting vortex is strong in the case of burst start. In fact according to the linearized theory $\underline{\Gamma}^1 \rightarrow \infty$, for t=0, making necessary the use of a mollifier.

For $t > 0$ all of the above relations should hold with the exception of (3.5.8). Specifically, following the Kutta-Joukowski hypothesis, we assume that the bound vorticity at the trailing edge has a finite value.

$$|\underline{\gamma}^1(u^1, u^2 = \tfrac{1}{2})| < \infty \tag{3.5.11}$$

Obviously if the tip is a part of the trailing edge then a similar relation should hold. In general if at the tip or hub $\underline{\gamma}^1 \neq 0$, then they are not part of the leading edge's vortex line. Therefore :

$$\underline{\gamma}^1(u^1 = R_H, R_o, u^2) \neq 0 \;\Rightarrow\; |\underline{\gamma}^2(u^1 = R_H, R_o, u^2)| < \infty \tag{3.5.12}$$

Both relations (3.5.11),(3.5.12) are important because they allow us to apply the Bernoulli's equation at the blades outline.

3.6 Calculation of the perturbation velocity field

In the general case of the propeller, we assume that the fluid extends to infinity and we omit any contribution from the hub. In other words we model the propeller as NBL (Number of Blades) [3] wings placed symmetrically around the x^1 axis. Supposing that the flow field is potential in nature apart from the NBL trailing vortex sheets emanating from the trailing edges of the NBL blades, the representation theorem for the potential function can be proved to be :

$$-4\,\pi\,\Phi(P) \;=\; \int_{S_b} \Phi(Q)_{,i}\,N^i(Q)\frac{dS}{r} \;+\; \int_{S_b} \Phi(Q)\,\frac{x^i(Q)\;-\;x^i(P)}{r^3}\,N_i(Q)dS \;+$$

$$+\int_{S_F} \mu(Q)\,\frac{x^i(Q)\;-\;x^i(P)}{r^3}\,N_i(Q)dS, \qquad Q\in S_b\cup S_F \tag{3.6.1}$$

where $$r \;=\; \left[\left(x^i(Q)-x^i(P)\right)\left(x_i(Q)-x_i(P)\right)\right]^{1/2} \tag{3.6.2}$$

$$S_b \;=\; \cup\left[S_{u_Z}\cup S_{l_Z}\right],\ \mathrm{Z}=1\ \mathrm{to\ NBL} \tag{3.6.3}$$

$$S_F \;=\; \cup\left[S^+_{W_Z}\cup S^-_{W_Z}\right],\ \mathrm{Z}=1\ \mathrm{to\ NBL} \tag{3.6.4}$$

and S_{u_Z} ,S_{l_Z} ,$S^+_{W_Z}$,$S^-_{W_Z}$ denote the upper blade surface, lower blade surface and the two sides of the trailing sheet respectively for Z^{th} blade.

The existence of the finite distance term $x^i(Q)-x^i(P)$, which is not tensorial in character under generalized coordinate transformations avoids relation (3.6.1) to be tensorial in the generalized sense. This is not obviously the case if only Cartesian tensor transformation are allowed. Thus the validity of the representation theorem, relation (3.6.1), is limited to Cartesian only coordinate system such as the system x^i of our application.

Taking the covariant derivative of relation (3.6.1) and after some algebra we get :

$$-4\,\pi v^j(P) \;=\; \int_{S_b} v^k(Q)\,N_k(Q)\,\frac{x^j(Q)-x^j(P)}{r^3}\,dS \;+$$

$$+\int_{S_b}\varepsilon^{jil}\,\left(\varepsilon_{ipk}N^p(Q)\,v^k(Q)\right)\,\frac{x_l(Q)-x_l(P)}{r^3}\,dS \;+\; \int_{S_F}\varepsilon^{jil}\,\gamma_j\,\frac{x_l(Q)-x_l(P)}{r^3}\,dS$$

$$-\int_{L}\mu\,\varepsilon^{jil}\,\frac{x_l(Q)-x_l(P)}{r^3}\,dl^i \tag{3.6.5}$$

where $$L \;=\; \cup L_Z = \cup\partial(S_{u_Z}\cup S_{l_Z}\cup S^+_{W_Z}\cup S^-_{W_Z}),\ \mathrm{Z}=1\ \mathrm{to\ NBL} \tag{3.6.6}$$

[3] We try to make use of the same symbols as in the program.

while $v_k = \Phi_{,k}$ and Φ have been assumed continuous [4] functions of position for points on S_{u_Z}, S_{l_Z}, $S^+_{W_Z}$, $S^-_{W_Z}$. Note that no vertical velocity jump on the vortex sheet can occur. [5]

To proceed further we shall transfer the integration on S_{u_Z}, S_{l_Z}, $S^+_{W_Z}$, $S^-_{W_Z}$ to the blade and wake reference surfaces S_{R_Z} and S_{RW_Z}. Since v^k is already of first order, only zero order terms in the Taylor expansions of the quantities entered in the surface integrations should remain. Assuming that the equation of the blade surface is given by :

$$x^i(Q) \;=\; \tilde{x}^i(u^1, u^2) \;+\; \delta x^i(u^1, u^2) \tag{3.6.7}$$

where $\tilde{x}^i = \tilde{x}^i(u^1, u^2)$ is the equation of the blades' reference surface and :

$$\delta x^i \;=\; O(\varepsilon) \tag{3.6.8}$$

we get (Appendix D):

$$dS_{u_Z} \;=\; dS_{R_Z} \;+\; O(\varepsilon) \tag{3.6.9}$$

$$v^k \;=\; \tilde{v}^k \;+\; O(\varepsilon^2) \tag{3.6.10}$$

$$1/r^3 \;=\; 1/\tilde{r}^3 \;+\; O(\varepsilon) \tag{3.6.11}$$

where $$\tilde{r} \;=\; \Big[(\tilde{x}^i(Q) \;-\; x^i(P))\,(\tilde{x}_i(Q) \;-\; x_i(P))\Big]^{1/2} \tag{3.6.12}$$

and similarly for dS_{l_Z} and dS_{W_Z}. Substituting relations (3.6.7), (3.6.9), (3.6.10) and (3.6.11) to relation (3.6.5), the velocity representation theorem becomes:

$$-4\,\pi\,v^j(P) \;=\; \int\limits_{\bigcup_Z S_{R_Z}} \sigma(Q)\,\frac{x^j(Q) - x^j(P)}{r^3}\,dS \;+$$

$$+ \int\limits_{\bigcup_Z (S_{R_Z} \cup S_{RW_Z})} \varepsilon^{jil}\gamma_i(Q)\,\frac{x_l(Q) - x_l(P)}{r^3}\,dS \;-\; \int\limits_{L_Z} \mu\,\varepsilon^{jil}\;\frac{x_l(Q) - x_l(P)}{r^3}\,dl^i \tag{3.6.13}$$

where γ_i is the surface vorticity tensor and σ is the surface intensity of sources and sinks defined by the relations (3.3.10) and (3.2.32) respectively.

The wave overscore for $x^j(Q)$ and r in the representation theorem, relation (3.6.13), has been omitted since their meaning is obvious from the integration domain.

Formula (3.6.13) holds for points off blade reference surface. In the case of $P \in S_{RZ}$ the respective formula can be proved to be the same as relation (3.6.13) apart from a change of the left hand side

[4]if Φ has a line of discontinuity on S then this can proved to be equivalent to a singular vortex line coinciding with the line of potential discontinuity

[5]This is a result of the continuity of velocity field on points of the trailing edge sheets

from $4\,\pi\,v^j$ to $4\,\pi\,\rangle v^j\langle$ (Appendix D) :

$$-4\,\pi\,\rangle v^j(P)\langle = \int\limits_{\bigcup_Z S_{R_Z}} \sigma(Q)\,\frac{x^j(Q)-x^j(P)}{r^3}\,dS\,+$$

$$+\int\limits_{\bigcup_Z (S_{R_Z}\cup S_{RW_Z})} \varepsilon^{jil}\gamma_i(Q)\,\frac{x_l(Q)-x_l(P)}{r^3}\,dS\;-\int\limits_{L_Z}\mu\,\varepsilon^{jil}\;\frac{x_l(Q)-x_l(P)}{r^3}\,dl^i \qquad (3.6.14)$$

The main difference between relations (3.6.13) and (3.6.14) lays in the existence of a Cauchy type surface singularity in the Kernel of the surface integral in the right hand side of equation (3.6.14).

4 Application of "Mode Analysis Method"

4.1 Approximation of the vorticity distribution with double Fourier Series

In the theory of 2D airfoils in the steady potential flow, the (bound) vorticity, γ, on the reference line is considered to be a function of position x, with domain restricted between the leading and trailing edge ($-1/2 \leq x \leq 1/2$).According to the solution of the zero thickness problem, using complex functions the vorticity should approach infinity near the leading edge while being zero at the trailing edge. The first condition is met by assuming that γ is consisted of a regular and an irregular part, which behaves like the $1/\tan(\theta)$ function. As for the regular part, it is approximated by a sinus Fourier series.

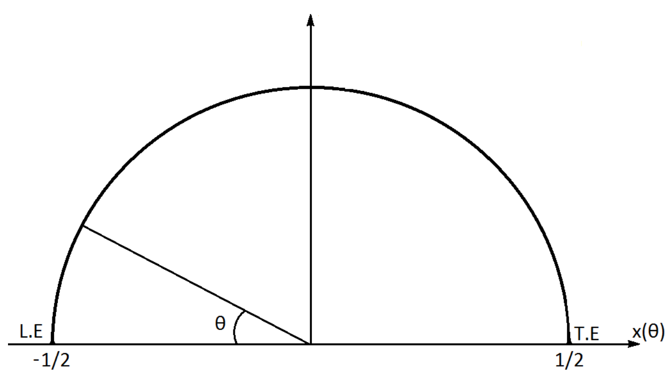


Figure 4.1.1

$$\gamma(x(\theta)) \;=\; A_0' \frac{1}{\tan(\theta/2)} + \sum_{n=0}^{\infty} (A_n \sin(n\theta)) \tag{4.1.1}$$

where $x(\theta) \;=\; -\frac{1}{2}\cos\theta$ is the non dimensional chord length and $0 \leq \theta \leq \pi$

Notice that the relation (4.1.1) satisfies implicitly both boundary conditions at the leading and trailing edge. Although the simplest, (4.1.1) is not the only representation of the solution to the problem. In fact both

$$\gamma(x(\theta)) \;=\; A_0' \frac{1}{\tan(\theta/2)} + \sum_{n=0}^{\infty} (B_n \cos(n\theta)) \tag{4.1.2}$$

and

$$\gamma(x(\theta)) \;=\; A_0' \frac{1}{\tan(\theta/2)} + \sum_{n=0}^{\infty} (A_n \sin(n\theta) + B_n \cos(n\theta)) \tag{4.1.3}$$

with $$\sum_{n=0}^{\infty} (B_n \cos(n\pi)) \;=\; 0 \tag{4.1.4}$$

are valid representations of the solution of the steady problem. Specifically since the (bound) vorticity γ is defined only in $[0, \pi]$, it's not a periodical function and thus we should expanded across all $\mathbb{R}$. In fact (4.1.1) is the sinus expansion (odd function), (4.1.2) is the co-sinus expansion (even function) and (4.1.3) is the full expansion of γ.

In the case of the unsteady problem of a 2D airfoil, the vorticity is a function of time and so the coefficients in the Fourier representation should also be function of time. Assuming the same boundary condition at the leading edge, the irregular part stays as is. However the boundary condition at the trailing edge demands a non zero value for the vorticity, much like relation (3.4.16). This could be achieved by using either relations (4.1.2) or (4.1.3) with the condition $\gamma = c(t)$, where c is a given value, instead of (4.1.4). Another interesting solution is to take the relation (4.1.1) and add the constant $c(t)$, which is effectively the same as taking (4.1.3) and setting $B_n = 0$ for $n \geq 1$.

Choosing the full expansion, (4.1.3), of γ we get :

$$\gamma(x(\theta), t) \;=\; A'_0(t)\,\frac{1}{\tan(\theta/2)} \;+\; \sum_{n=0}^{\infty} \left[A_n(t)\,\sin(n\theta) + B_n(t)\,\cos(n\theta)\right] \tag{4.1.5}$$

We emphasize the fact that by expanding the regular part of γ periodically as an odd function, there will be a discontinuity at the multiples of π. Therefore the Fourier series shows a difficulty in convergence at those points due to the Gibbs phenomenon. Adding those extra co-sinus terms is not a matter of convergence, but they are dictated by the physical problem at hand. In other words, given the real solution to the unsteady problem we could approximate it infinitely close with the sinus series, but starting with only the sinus factor solves a different problem.

Since the problem we are dealing with is the exact analog in 3D space, we assume the solution (4.1.5) for every cross section of the blades. In a particular blade section $\underline{x}^1$ of the propeller the bound vorticity is (using the symbols of chapter 2.2) :

$$\underline{\gamma}^1(\underline{x}^1, \underline{x}^2(y^2), Z, t) \;=$$

$$= \; A'_{0Z}(\underline{x}^1, t)\,\frac{1}{\tan(y^2/2)} \;+\; \sum_{n=0}^{\infty}\left[A_{nZ}(\underline{x}^1, t)\,\sin(ny^2) \;+\; B_{nZ}(\underline{x}^1, t)\,\cos(ny^2)\right] \tag{4.1.6}$$

where A_{nZ}, B_{nZ} are the coefficients of the Z^{th} blade .

Writing the coefficients A_{nz} and B_{nz} as fourier series with respect to y^1, we get:

$$A'_{0Z} \;=\; \sum_{m=0}^{\infty}\left[A^s_{m0Z}(t)\,\sin(my^1) + A^c_{m0Z}(t)\,\cos(my^1)\right] \tag{4.1.7}$$

$$A_{nZ} \;=\; \sum_{m=0}^{\infty}\left[A^{ss}_{mnZ}(t)\,\sin(my^1) + A^{cs}_{mnZ}(t)\,\cos(my^1)\right] \tag{4.1.8}$$

$$B_{nZ} \;=\; \sum_{m=0}^{\infty}\left[A^{sc}_{mnZ}(t)\,\sin(my^1) + A^{cc}_{mnZ}(t)\,\cos(my^1)\right] \tag{4.1.9}$$

Substituting relations (4.1.7),(4.1.8) and (4.1.9) to (4.1.6):

$\underline{\gamma}^1(y^1, y^2, Z, t) \; =$

$$= \sum_{m=0}^{\infty} \left[A^s_{m0Z}(t)\,\sin(my^1) \,+\, A^c_{m0Z}(t)\,\cos(my^1)\right] \frac{1}{\tan(y^2/2)} \,+$$

$$+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left[A^{ss}_{mnZ}(t)\,\sin(my^1)\sin(ny^2) \,+\, A^{cs}_{mnZ}(t)\,\cos(my^1)\sin(ny^2)\right]$$

$$+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left[A^{sc}_{mnZ}(t)\,\sin(my^1)\cos(ny^2) \,+\, A^{cc}_{mnZ}(t)\,\cos(my^1)\cos(ny^2)\right] \qquad (4.1.10)$$

According to relation (3.3.31), the continuity of vorticity using the surface coordinate system reads:

$$\frac{\partial}{\partial u^1}\left(\underline{\Gamma}^1(y^\alpha(u^\alpha), t)\right) \,+\, \frac{\partial}{\partial u^2}\left(\underline{\Gamma}^2(y^\alpha(u^\alpha), t)\right) \,=\, 0 \qquad (4.1.11)$$

Given that relation (4.1.10) expresses the bound vorticity using the parametric surface coordinate system y^α, we rewrite the continuity of vorticity using the same coordinate system.

$$\frac{\partial}{\partial y^1}\left(\underset{\smile}{\Gamma}^1(y^1, y^2, t)\right) \,+\, \frac{\partial}{\partial y^2}\left(\underset{\smile}{\Gamma}^2(y^1, y^2, t)\right) \,=\, 0 \qquad (4.1.12)$$

Since the modified vorticity is a relative tensor, using relations (A52),(A53) and (A97) we get:

$$\underset{\smile}{\Gamma}^1 \overset{def}{=} \sqrt{\underset{\smile}{\underline{\alpha}}}\,\underset{\smile}{\gamma}^1 \,=\, \left(\frac{1}{4}\,\sin y^1\,\sin y^2\,\sqrt{\underline{\alpha}}\right)\left(\frac{2}{(R_o - R_H)\,\sin y^1}\,\underline{\gamma}^1\right) \,=\, \frac{1}{2}\,\sin y^2\,\underline{\Gamma}^1 \qquad (4.1.13)$$

$$\underset{\smile}{\Gamma}^2 \overset{def}{=} \sqrt{\underset{\smile}{\underline{\alpha}}}\,\underset{\smile}{\gamma}^2 \,=\, \left(\frac{1}{4}\,\sin y^1\,\sin y^2\,\sqrt{\underline{\alpha}}\right)\left(\frac{2}{\sin y^2}\,\underline{\gamma}^2\right) \,=\, \frac{1}{2}\,(R_o - R_H)\,\sin y^2\,\underline{\Gamma}^2 \qquad (4.1.14)$$

Notice that the bound modified vorticity $\underset{\smile}{\Gamma}^\alpha$ doesn't have an irregular part and it has finite values in all of $[0, \pi]$, since :

$$\left(\frac{1}{2}\,\sin y^2\right)\frac{1}{\tan(y^2/2)} \,=\, \left(\frac{1}{2}\,\sin y^2\right)\frac{\cos y^2 + 1}{\sin y^2} \,=\, \frac{cosy^2 + 1}{2}$$

Both the T.E b.c and the continuity of vorticity equation make use of the modified vorticity $\underline{\Gamma}^\alpha = \sqrt{\underline{\alpha}}\,\underline{\gamma}^\alpha$, instead of the γ^α. Multiplying by the surface metric $\sqrt{\underline{\alpha}}$, the manipulation of the modified vorticity (e.g integration, derivation) becomes too complicated. Therefore instead of expressing $\underline{\gamma}^a$ though relation (4.1.10) we choose to represent the modified bound vorticity $\underset{\smile}{\Gamma}^\alpha$, as a double Fourier series, since according to relations (4.1.13) and (A95) both $\sqrt{\underline{\alpha}}$ and $\frac{1}{2}\,\sin y^2\,\underline{\gamma}^1$ are regular functions. Thus :

$$\underset{\smile}{\Gamma}^1(y^1,y^2,Z,t) \;=\; \sum_{m=0}^{\infty}\left[A'^{s}_{m0Z}(t)\,\sin(my^1) \;+\; A'^{c}_{m0Z}(t)\,\cos(my^1)\right] \;+$$

$$+\;\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[A^{ss}_{mnZ}(t)\,\sin(my^1)\sin(ny^2) \;+\; A^{cs}_{mnZ}(t)\,\cos(my^1)\sin(ny^2)\right.$$

$$\left.+\;A'^{sc}_{mnZ}(t)\,\sin(my^1)\cos(ny^2) \;+\; A'^{cc}_{mnZ}(t)\,\cos(my^1)\cos(ny^2)\right] \qquad (4.1.15)$$

The coefficients A of relation (4.1.15) are different from those of relation (4.1.10), but we keep the same symbols for the sake of simplicity. Furthermore we set the following relations :

$$A'^{sc}_{m1Z} \;=\; A^{sc}_{m1Z} \;+\; \frac{1}{2}\,A^{s}_{m0Z} \quad , \qquad A'^{cc}_{m1Z} \;=\; A^{cc}_{m1Z} \;+\; \frac{1}{2}\,A^{c}_{m0Z} \qquad (4.1.16)$$

$$A'^{s}_{m0Z} \;=\; \frac{1}{2}\,A^{s}_{m0Z} \quad , \qquad A'^{c}_{m0Z} \;=\; \frac{1}{2}\,A^{c}_{m0Z} \qquad (4.1.17)$$

Substituting the above relations to (4.1.15) we get:

$$\underset{\smile}{\Gamma}^1(y^1,y^2,Z,t) \;=$$

$$=\;\sum_{m=0}^{\infty}\left[A^{s}_{m0Z}(t)\,\sin(my^1) \;+\; A^{c}_{m0Z}(t)\,\cos(my^1)\right]\frac{\cos y^2+1}{2} \;+$$

$$+\;\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[A^{ss}_{mnZ}(t)\,\sin(my^1)\sin(ny^2) \;+\; A^{cs}_{mnZ}(t)\,\cos(my^1)\sin(ny^2)\right.$$

$$\left.+\;A^{sc}_{mnZ}(t)\,\sin(my^1)\cos(ny^2) \;+\; A^{cc}_{mnZ}(t)\,\cos(my^1)\cos(ny^2)\right] \qquad (4.1.18)$$

Solving (4.1.12) for $\underset{\smile}{\Gamma}^2$, we get:

$$\underset{\smile}{\Gamma}^2(y^1,y^2,Z,t) \;=\; \cancel{\underset{\smile}{\Gamma}^2(y^1,y^2=0,Z,t)}\;^{(3.5.4)} \;-\; \int_0^{y^2}\frac{\partial\underset{\smile}{\Gamma}^1}{\partial y^1}\,dy^2$$

$$=\;\sum_{m=1}^{\infty}\left[-m\,A^{s}_{m0Z}(t)\,\cos(my^1) \;+\; m\,A^{c}_{m0Z}(t)\,\sin(my^1)\right]\frac{y^2+\sin y^2}{2} \;+$$

$$+\;\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[-\frac{m}{n}\,A^{ss}_{mnZ}(t)\,\cos(my^1)(1-\cos(ny^2)) \;+\; \frac{m}{n}\,A^{cs}_{mnZ}(t)\,\sin(my^1)(1-\cos(ny^2))\right.$$

$$\left.-\frac{m}{n}\,A^{sc}_{mnZ}(t)\,\cos(my^1)\sin(ny^2) \;+\; \frac{m}{n}\,A^{cc}_{mnZ}(t)\,\sin(my^1)\sin(ny^2)\right] \qquad (4.1.19)$$

Notice that although relation (4.1.18),(4.1.19) are regular double Fourier Series we choose to separate the terms n=0, so that they resemble the original relation (4.1.10).

Solving relations (4.1.13) and (4.1.14) for $\underline{\Gamma}^1$ and $\underline{\Gamma}^2$ respectively we get :

$$\underline{\Gamma}^1(y^1, y^2, Z, t) =$$

$$= \sum_{m=0}^{\infty} \left[A^s_{m0Z}(t)\, \sin(my^1) \,+\, A^c_{m0Z}(t)\, \cos(my^1)\right] \frac{1}{\tan(y^2/2)} +$$

$$+2 \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \left[A^{ss}_{mnZ}(t)\, \sin(my^1)\frac{\sin(ny^2)}{\sin y^2} \,+\, A^{cs}_{mnZ}(t)\, \cos(my^1)\frac{\sin(ny^2)}{\sin y^2}\right.$$

$$\left. +\, A^{sc}_{mnZ}(t)\, \sin(my^1)\frac{\cos(ny^2)}{\sin y^2} \,+\, A^{cc}_{mnZ}(t)\, \cos(my^1)\frac{\cos(ny^2)}{\sin y^2}\right] \tag{4.1.20}$$

Using D' Hospital's rule $\lim_{y^2\to 0} \sin(ny^2)/\sin(y^2) = n$. Apart from the first sum with the $1/\tan(y^2/2)$ factor, the two last terms of this relation exhibit the same infinity behavior near the leading edge since $\cos(ny^2) \to 1$.

$$\underline{\Gamma}^2(y^1, y^2, Z, t) =$$

$$= \sum_{m=1}^{\infty} \left[-m\, A^s_{m0Z}(t)\, \frac{\cos(my^1)}{\sin y^1} \,+\, m\, A^c_{m0Z}(t)\, \frac{\sin(my^1)}{\sin y^1}\right] \frac{y^2 + \sin y^2}{R_o - R_H} +$$

$$+ \frac{2}{R_o - R_H} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left[-\frac{m}{n} A^{ss}_{mnZ}(t)\frac{\cos(my^1)}{\sin y^1}(1-\cos(ny^2)) + \frac{m}{n} A^{cs}_{mnZ}(t)\frac{\sin(my^1)}{\sin y^1}(1-\cos(ny^2))\right.$$

$$\left. -\frac{m}{n} A^{sc}_{mnZ}(t)\frac{\cos(my^1)}{\sin y^1}\sin(ny^2) + \frac{m}{n} A^{cc}_{mnZ}(t)\frac{\sin(my^1)}{\sin y^1}\sin(ny^2)\right] \tag{4.1.21}$$

The free bound vorticity could assume infinite value [6] at the tips of the wing (or tip and hub for the propeller) as implied by the first, third and fifth term, due to the $\frac{\cos(my^1)}{\sin y^1}$ factor. As mentioned in section 3.5 this is expected in the linearized theory.

In order to solve the problem arithmetically we are obliged to truncate the Fourier series at some harmonic. Assuming M_0 spanwise and N_0 chordwise harmonics we get :

$$m \,=\, 0, 1, 2, ..., M_0 \tag{4.1.22}$$

$$n \,=\, 0, 1, 2, ..., N_0 \tag{4.1.23}$$

$$Z \,=\, 1, 2, ...\ ,\text{NBL} \tag{4.1.24}$$

This means that we have $(2M_0+1)(2N_0+1)$ unknowns for each of the NBL blades. However under center assumptions we can reduce the number of unknowns to just $M_0(N_0+1)$ for each blade.

[6] For $\lim_{y^1\to 0,\pi} \sum_{m=0}^{M} A_m \frac{\cos(my^1)}{siny^1} < \infty$, D' Hospital's rule must be valid, so $\lim_{y^1\to 0,\pi} \sum_{m=0}^{M} A_m \cos(my^1) = 0$

4.2 Kinematic b.c at the boundary of the reference surface

In this section we will apply each of the kinematic boundary conditions (3.5.7)-(3.5.12) to relations (4.1.20) and (4.1.21) to extract the corresponding relations for the coefficients, A.

According to relation (3.5.7) the free bound vorticity should always be zero at the leading edge :

$$\underline{\gamma}^2(u^1, u^2 = -\frac{1}{2}) \;=\; 0 \;\Rightarrow\; \underline{\Gamma}^2(y^1, y^2 = 0) = 0 \tag{4.2.1}$$

Substituting $y^2 = 0$ to (4.1.21), satisfies the above relation, which is a direct consequence of the continuity of vorticity.

In the case where no free sheer layer is emitted from the tips, according to (3.5.9) and (3.5.10) the bound vorticity is zero :

$$\underline{\gamma}^1(u^1 = 0, R_o\;, u^2) \;=\; 0 \;\Rightarrow \underline{\Gamma}^1(y^1 = 0, \pi\;, y^2) \;=\; 0 \tag{4.2.2}$$

Substituting to relation (4.1.20):

$\underline{\Gamma}^1(y^1, y^2, Z, t) \;=\; 0 \Rightarrow$

$$\Rightarrow \sum_{m=0}^{\infty}\left[A^c_{m0Z}\,\frac{1}{\tan(y^2/2)} \;+\; 2\sum_{n=1}^{\infty}\left(A^{cs}_{mnZ}\,\frac{\sin(ny^2)}{\sin y^2} \;+\; A^{cc}_{mnZ}\,\frac{\cos(ny^2)}{\sin y^2}\right)\right]\cos(my^1)|_{y^1=0,\pi} \;=\; 0$$

$$\Rightarrow \sum_{m=0}^{\infty}\left[A^c_{m0Z}(\cos y^2 + 1) \;+\; 2\sum_{n=1}^{\infty}\left(A^{cs}_{mnZ}\;\sin(ny^2) \;+\; A^{cc}_{mnZ}\;\cos(ny^2)\right)\right]\cos(my^1)|_{y^1=0,\pi} \;=\; 0$$

Since the functions $1, \cos(ny^2), \sin(ny^2)$ are functionally independent, each of their factors should be zero.

$$\sum_{m=0}^{\infty} A^c_{m0Z}\,(\pm 1)^m \;=\; 0\;, \qquad n = 0 \tag{4.2.3}$$

$$\sum_{m=0}^{\infty}(\cancel{A^c_{m0Z}}^{\;\cdot(4.2.3)} + 2\,A^{cc}_{m1Z})\,(\pm 1)^m \;=\; 0\;, \qquad n = 1 \tag{4.2.4}$$

$$\sum_{m=0}^{\infty} A^{cc}_{mnZ}\,(\pm 1)^m \;=\; 0\;, \qquad n = 2, 3, ... \tag{4.2.5}$$

$$\sum_{m=0}^{\infty} A^{cs}_{mnZ}\,(\pm 1)^m \;=\; 0\;, \qquad n = 1, 2, ... \tag{4.2.6}$$

At the intersections of tip-T.E and hub-T.E relations (4.2.3)-(4.2.6) should satisfy the pressure type kutta boundary condition at the trailing edge, which according to relation (3.5.6) do only if the flow is parallel to the surface $(\underset{\sim}{\underline{q}}^i + \rangle\underset{\sim}{\underline{v}}^i\langle = (\underset{\sim}{\underline{q}}^2 + \rangle\underset{\sim}{\underline{v}}^2\langle)\,\delta^i_2)$.

For a burst start and t=0, according to relation (3.5.8) there should be zero free vorticity at the trailing edge.

$$\underline{\gamma}^2(u^1, u^2 = \frac{1}{2}) \,=\, 0 \Rightarrow \underline{\Gamma}^2(y^1, y^2 = \pi) \,=\, 0 \tag{4.2.7}$$

Substituting to relation (4.1.21) :

$\underline{\Gamma}^2(y^1, y^2, Z, t) \,=$

$$\sum_{m=1}^{\infty}\left[-mA^s_{m0Z}\frac{y^2+\sin y^2}{2} + \sum_{n=1}^{\infty}(-\frac{m}{n}A^{ss}_{mnZ}(1-\cos(ny^2)) - \frac{m}{n}A^{sc}_{mnZ}\sin(ny^2))\right]\frac{2\cos(my^1)}{(R_o-R_H)\sin y^1}+$$

$$\sum_{m=1}^{\infty}\left[mA^c_{m0Z}\frac{y^2+\sin y^2}{2} + \sum_{n=1}^{\infty}(\frac{m}{n}A^{cs}_{mnZ}(1-\cos(ny^2)) + \frac{m}{n}A^{cc}_{mnZ}\sin(ny^2))\right]\frac{2\sin(my^1)}{(R_o-R_H)\sin y^1} = 0$$

Since the trigonometric functions $\sin(my^1)$, $\cos(my^1)$ are functionally independent, each of their factors should be zero at $y^2 = \pi$.

$$A^s_{m0Z}\,\frac{\pi}{2} \,+\, \sum_{n=1}^{\infty}\left(A^{ss}_{mnZ}\,\frac{1-(-1)^n}{n}\right) \,=\, 0 \quad , \qquad \text{m} = 1,2,... \tag{4.2.8}$$

$$A^c_{m0Z}\,\frac{\pi}{2} \,+\, \sum_{n=1}^{\infty}\left(A^{cs}_{mnZ}\,\frac{1-(-1)^n}{n}\right) \,=\, 0 \quad , \qquad \text{m} = 1,2,... \tag{4.2.9}$$

For $t > 0$ the Kutta-Joukowski hypothesis, (3.5.11), assumes finite value for the bound vorticity.

$$\lim_{u^2\to 1/2}|\underline{\gamma}^1(u^1,u^2)| < \infty \;\;\Rightarrow\;\; \lim_{y^2\to\pi}|\underline{\Gamma}^1(y^1,y^2)| < \infty \tag{4.2.10}$$

For relation (4.2.10) we should be able to apply D' Hospital's rule, which means that :

$$\lim_{y^2\to\pi}\underset{\smile}{\Gamma}^1(y^1, y^2, Z, t) \,=\, 0 \;\;\overset{(4.1.18)}{\Rightarrow}$$

$$\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[A^{sc}_{mnZ}(t)\,\sin(my^1)\cos(n\pi) \,+\, A^{cc}_{mnZ}(t)\,\cos(my^1)\cos(n\pi)\right] \,=\, 0$$

Since the trigonometric functions $\sin(my^1)$, $\cos(my^1)$ are functionally independent, each of their factors should be zero at $y^2 = \pi$.

$$\sum_{n=1}^{\infty}A^{sc}_{mnZ}\,(-1)^n \,=\, 0 \quad , \qquad \text{m} = 1,2,... \tag{4.2.11}$$

$$\sum_{n=1}^{\infty}A^{cc}_{mnZ}\,(-1)^n \,=\, 0 \quad , \qquad \text{m} = 0,1,2,... \tag{4.2.12}$$

In case the hub or the tip are not part of the leading edge's vortex line then relation (3.5.12) should hold.

$$|\underline{\gamma}^2(u^1 = R_H, R_o, u^2)| < \infty \;\Rightarrow\; |\underline{\Gamma}^2(y^1 = 0, \pi, y^2)| < \infty \tag{4.2.13}$$

For relation (4.2.13) we should be able to apply D' Hospital's rule, which means that :

$$\lim_{y^1 \to 0,\pi} \underset{\smile}{\Gamma}^2(y^1, y^2, Z, t) \;=\; 0 \quad \overset{(4.1.19)}{\Rightarrow}$$

$$\Rightarrow \sum_{m=1}^{\infty} \left[-m\, A^s_{m0Z}\, \cos(my^1)\right] \frac{y^2 + \sin y^2}{2} +$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \Big[-\frac{m}{n}\, A^{ss}_{mnZ}\, \cos(my^1)(1 - \cos(ny^2)) \;-\; \frac{m}{n}\, A^{sc}_{mnZ}\, \cos(my^1)\sin(ny^2) \Big] \;=\; 0$$

Choosing the 1, $\sin(my^1)$, $\cos(my^1)$ functions as our basis, then each of their factors should be zero at $y^1 = 0, \pi$.

$$\sum_{m=1}^{\infty} \left(m\, A^s_{m0Z}\frac{1}{2} \;+\; \frac{m}{n}\, A^{sc}_{m1Z} \;+\; m\, A^s_{m0Z}\, Y^s_1 \right) (\pm 1)^m \;=\; 0 \quad , \qquad \text{n =1} \tag{4.2.14}$$

$$\sum_{m=1}^{\infty} \left(\frac{m}{n}\, A^{sc}_{mnZ} \;+\; m\, A^s_{m0Z}\, Y^s_n \right) (\pm 1)^m \;=\; 0 \quad , \qquad \text{n =2,3,...} \tag{4.2.15}$$

$$\sum_{m=1}^{\infty} \left(-\frac{m}{n}\, A^{ss}_{mnZ} \;+\; m\, A^s_{m0Z}\, Y^c_n \right) (\pm 1)^m \;=\; 0 \quad , \qquad \text{n =1,2,...} \tag{4.2.16}$$

$$\sum_{m=1}^{\infty} \Bigg[\sum_{n=1}^{\infty} \left(\frac{m}{n}\, A^{ss}_{mnZ} \right) \;+\; m\, A^s_{m0Z}\, Y^c_n \Bigg[\; (\pm 1)^m \;=\; 0 \quad , \qquad (\text{n} = 0) \tag{4.2.17}$$

where

$$y^2/2 \;=\; \sum_{n=0}^{\infty} \left(Y^c_n\, \cos(ny^2) \;+\; Y^s_n\, \sin(ny^2) \right)$$

is the Fourier series of $y^2/2$, in the $[0, \pi]$ (e.g we could expand $y^2/2$ as an odd or even function)

As already mentioned in section 3.5, in the program we make the assumption that the flow is parallel to the tips of the wing $(\underset{\sim}{q}^i + \rangle \underset{\sim}{v}^i \langle = (\underset{\sim}{q}^2 + \rangle \underset{\sim}{v}^2 \langle)\, \delta^i_2)$, meaning that relation (4.2.2) is always valid. In that case instead of using the full Fourier series in the y^1 direction, we can expand periodically $\underline{\Gamma}^1$ as an odd function. Since $\underline{\Gamma}^1 = 0$ at $y^1 = 0, \pi$ we can approximate the bound vorticity infinitely well using only the sinus part in the spanwise direction, y^1. Therefore relation (4.1.20) becomes :

$$\underline{\Gamma}^1(y^1, y^2, Z, t) \;=\; \sum_{m=1}^{\infty} \left[A^s_{m0Z}(t)\,\sin(my^1)\right] \frac{1}{\tan(y^2/2)} \,+$$

$$+\,2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left[A^{ss}_{mnZ}(t)\,\sin(my^1)\frac{\sin(ny^2)}{\sin y^2} \,+\, A^{sc}_{mnZ}(t)\,\sin(my^1)\frac{\cos(ny^2)}{\sin y^2}\right] \tag{4.2.18}$$

Notice that the terms $A^{sc}_{mnZ}(t)$ could give the infinite value at the leading edge, that the 2D theory demands. Although, since the first term, $A^s_{m0Z}(t)$, is dedicated for that singular part at the leading edge, the last term, $A^{sc}_{mnZ}(t)$, is used to model the infinity of the bound vorticity at the trailing edge for $t = 0$ (see section 3.5).Given that infinities introduce instabilities in the code, the results at the start could be unreliable. Therefore instead of (4.2.7) at $t = 0$, we use relation (4.2.10). This means that the terms in the double summation are used to model the regular part of the bound vorticity. It is obvious that relation (4.2.18) is not a Fourier series in the chordwise direction, y^2. Instead using the parametric space :

$$\underset{\smile}{\Gamma}^1(y^1, y^2, Z, t) \;=\; \sum_{m=1}^{\infty} \left[A^s_{m0Z}(t)\,\sin(my^1)\right] \frac{\cos y^2 + 1}{2} \,+$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left[A^{ss}_{mnZ}(t)\,\sin(my^1)\sin(ny^2) \,+\, A^{sc}_{mnZ}(t)\,\sin(my^1)\cos(ny^2)\right] \tag{4.2.19}$$

According to relations (4.2.10) and (4.1.13), $\underset{\smile}{\Gamma}^1(y^1, y^2 = \pi, t) = 0$. Given that any non zero values at the leading edge are given by the (dedicated for that reason) first part, the double summation of (4.2.19) should also be zero at the leading edge. Therefore, we could approximate the double summation part of (4.2.19) using only a sinus series in the y^2 direction. Thus:

$$\underset{\smile}{\Gamma}^1(y^1, y^2, Z, t) = \sum_{m=1}^{\infty} \left[A^s_{m0Z}\,\sin(my^1)\right] \frac{\cos y^2 + 1}{2} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left[A^{ss}_{mnZ}\,\sin(my^1)\sin(ny^2)\right] \tag{4.2.20}$$

or returning to the physical space:

$$\underline{\Gamma}^1(y^1, y^2, Z, t) = \sum_{m=1}^{\infty} \left[A^s_{m0Z}\,\sin(my^1)\right] \frac{1}{\tan(y^2/2)} + 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left[A^{ss}_{mnZ}\,\sin(my^1)\frac{\sin(ny^2)}{\sin y^2}\right] \tag{4.2.21}$$

$$\underline{\Gamma}^2(y^1, y^2, Z, t) = \sum_{m=1}^{\infty} \left[-m\,A^s_{m0Z}\,\cos(my^1)\right] \frac{y^2 + \sin y^2}{R_o - R_H} +$$

$$+\,\frac{2}{R_o - R_H}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left[-\frac{m}{n}\,A^{ss}_{mnZ}\,\frac{\cos(my^1)}{\sin y^1}(1 - \cos(ny^2))\right] \tag{4.2.22}$$

In summary, using only $M_0(N_0{+}1)$ coefficients we can approximate the unsteady flow after the burst start, with the assumption that $\tilde{\underline{q}}^i + \tilde{\underline{v}}^i = (\tilde{\underline{q}}^2 + \tilde{\underline{v}}^2)\,\delta^i_2$, while implicitly satisfying the boundary conditions (4.2.1), (4.2.2) and (4.2.10).

4.3 Dynamic b.c at the boundary of the reference surface

The Bernoulli equation (3.4.17) written for the tip and hub reads :

$$\frac{p^+ - p^-}{\rho} \;=\; - \int_{-1/2}^{\underline{x}^2} \frac{d\underline{\Gamma}^1}{dt}\, d\underline{x}^2 \;+\; \underset{\sim}{\tilde{q}}^{\,1}\, \underline{\Gamma}^2 \;-\; \underset{\sim}{\tilde{q}}^{\,2}\, \underline{\Gamma}^1 \;-\; \left(\underset{\sim}{\tilde{q}}^{3} \sqrt{\frac{\underset{\sim}{\tilde{g}}}{\underline{\underline{\alpha}}}}\, \sigma \right) \;+\; O(\varepsilon^2) \tag{4.3.1}$$

Applying conditions (3.4.2) and (3.4.4) we get :

$$- \int_0^{y^2} \frac{d\breve{\Gamma}^1}{dt}\, dy^2 \;+\; \underset{\sim}{\tilde{q}}^{1}\, \underline{\Gamma}^2 \;-\; \underset{\sim}{\tilde{q}}^{2}\, \underline{\Gamma}^1 \;+\; O(\varepsilon^2) \;=\; 0 \tag{4.3.2}$$

First we examine the tip and hub at $y^1 = \pi$ and $y^1 = 0$ respectively.In order to do so, we need to express relation (4.3.2) as Fourier series in the y^2 direction.Therefore we multiply the above relation with $\sin y^2$ and analyze every part as a sum of $\cos(ny^2)$ and $\sin(ny^2)$. For the proof see Appendix E.

$$\int_0^{y^2} \frac{d\breve{\Gamma}^1}{dt}\, dy^2 \;\sin y^2 \;\overset{(E12)}{=} \tag{4.3.3}$$

$$= \sum_{m=0}^{\infty} \left(\frac{dA^c_{m0Z}}{dt} \frac{1}{2} + \frac{dA^{cc}_{m1Z}}{dt} \right) \frac{(\pm 1)^m}{2} +$$

$$+ \sum_{n=1}^{\infty} \left[\sum_{m=0}^{\infty} \left(\frac{dA^c_{m0Z}}{dt} \frac{(-\delta^2_n)}{2} + \frac{1}{n+1} \frac{dA^{cc}_{m,n+1,Z}}{dt} - \frac{1-\delta^1_n}{n-1} \frac{dA^{cc}_{m,n-1,Z}}{dt} \right) \frac{(\pm 1)^m}{2} \right] \cos(ny^2) +$$

$$+ \left[\sum_{m=0}^{\infty} \left(\frac{dA^c_{m0Z}}{dt}\, \beta_n + 2\, \frac{\delta^1_n}{n} \frac{dA^{cs}_{mnZ}}{dt} - \frac{1-\delta^1_n}{n-1} \frac{dA^{cs}_{m,n-1,Z}}{dt} + \frac{1}{n+1} \frac{dA^{cs}_{m,n+1,Z}}{dt} \right) \frac{(\pm 1)^m}{2} \right] \sin(ny^2)$$

$$\underline{\Gamma}^2(y^1, y^2, Z, t)\; \sin y^2 \;\overset{(E13)}{=}$$

$$= \sum_{m=1}^{\infty} \left(A^c_{m0Z} \frac{1}{2} \;+\; A^{cc}_{m1Z} \right) \frac{m\,(\pm 1)^{m+1}}{R_o - R_H} +$$

$$+ \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \left(A^c_{m0Z} \frac{(-\delta^2_n)}{2} + \frac{1}{n+1} A^{cc}_{m,n+1,Z} - \frac{1-\delta^1_n}{n-1} A^{cc}_{m,n-1,Z} \right) \frac{m\,(\pm 1)^{m+1}}{R_o - R_H} \right] \cos(ny^2) +$$

$$+ \left[\sum_{m=1}^{\infty} \left(A^c_{m0Z}\, \beta_n + 2\, \frac{\delta^1_n}{n} A^{cs}_{mnZ} - \frac{1-\delta^1_n}{n-1} A^{cs}_{m,n-1,Z} + \frac{1}{n+1} A^{cs}_{m,n+1,Z} \right) \frac{m\,(\pm 1)^{m+1}}{R_o - R_H} \right] \sin(ny^2) \tag{4.3.4}$$

$$\underline{\Gamma}^1(y^1=0,\pi,y^2,Z,t)\,\sin y^2 \overset{(E14)}{=}$$

$$= \sum_{m=0}^{\infty} A^c_{m0Z}(\pm 1)^m + \sum_{n=1}^{\infty}\left[\sum_{m=0}^{\infty}\left(2A^{cc}_{mnZ}+A^c_{m0Z}\delta^1_n\right)(\pm 1)^m\right]\cos(ny^2)+$$

$$+\left[\sum_{m=0}^{\infty} 2A^{cs}_{mnZ}(\pm 1)^m\right]\sin(ny^2) \qquad (4.3.5)$$

Substituting (4.3.3), (4.3.4) and (4.3.5) to (4.3.2), for every y^2 on the tip and hub, the coefficients of the trigonometric functions $\cos(ny^2)$, $\sin(ny^2)$ should be zero.

For $n=0$, $(\cos(0y^2))$:

$$-\left[\sum_{m=0}^{\infty}\left(\frac{dA^c_{m0Z}}{dt}\frac{1}{2}+\frac{dA^{cc}_{m1Z}}{dt}\right)\frac{(\pm 1)^m}{2}\right]+$$

$$+\underset{\sim}{\underline{q}}^1\left[\sum_{m=1}^{\infty}\left(A^c_{m0Z}\frac{1}{2}+A^{cc}_{m1Z}\right)\frac{m\,(\pm 1)^{m+1}}{R_o-R_H}\right]-\underset{\sim}{\underline{q}}^2\left[\sum_{m=0}^{\infty}A^c_{m0Z}(\pm 1)^m\right]=0 \qquad (4.3.6)$$

For $n=1,2,...,$ $(\cos(ny^2))$:

$$-\left[\sum_{m=0}^{\infty}\left(\frac{dA^c_{m0Z}}{dt}\frac{(-\delta^2_n)}{2}+\frac{1}{n+1}\frac{dA^{cc}_{m,n+1,Z}}{dt}-\frac{1-\delta^1_n}{n-1}\frac{dA^{cc}_{m,n-1,Z}}{dt}\right)\frac{(\pm 1)^m}{2}\right]+$$

$$+\underset{\sim}{\underline{q}}^1\left[\sum_{m=1}^{\infty}\left(A^c_{m0Z}\frac{(-\delta^2_n)}{2}+\frac{1}{n+1}A^{cc}_{m,n+1,Z}-\frac{1-\delta^1_n}{n-1}A^{cc}_{m,n-1,Z}\right)\frac{m\,(\pm 1)^{m+1}}{R_o-R_H}\right]-$$

$$-\underset{\sim}{\underline{q}}^2\left[\sum_{m=0}^{\infty}\left(2A^{cc}_{mnZ}+A^c_{m0Z}\delta^1_n\right)(\pm 1)^m\right]=0 \qquad (4.3.7)$$

For $n=1,2,...,$ $(\sin(ny^2))$:

$$-\left[\sum_{m=0}^{\infty}\left(\frac{dA^c_{m0Z}}{dt}\beta_n+2\frac{\delta^1_n}{n}\frac{dA^{cs}_{mnZ}}{dt}-\frac{1-\delta^1_n}{n-1}\frac{dA^{cs}_{m,n-1,Z}}{dt}+\frac{1}{n+1}\frac{dA^{cs}_{m,n+1,Z}}{dt}\right)\frac{(\pm 1)^m}{2}\right]$$

$$+\underset{\sim}{\underline{q}}^1\left[\sum_{m=1}^{\infty}\left(A^c_{m0Z}\beta_n+2\frac{\delta^1_n}{n}A^{cs}_{mnZ}-\frac{1-\delta^1_n}{n-1}A^{cs}_{m,n-1,Z}+\frac{1}{n+1}A^{cs}_{m,n+1,Z}\right)\frac{m\,(\pm 1)^{m+1}}{R_o-R_H}\right]-$$

$$-\underset{\sim}{\underline{q}}^2\left[\sum_{m=0}^{\infty}2A^{cs}_{mnZ}(\pm 1)^m\right]=0 \qquad (4.3.8)$$

Substituting $y^2 = \pi$ to relation (4.3.2) we get the Bernoulli equation at the trailing edge.

$$-\int_0^{\pi} \frac{d\underset{\smile}{\Gamma}^1}{dt}\, dy^2 \;+\; \underline{\tilde{q}}^1\, \underline{\Gamma}^2 \;-\; \underline{\tilde{q}}^2\, \underline{\Gamma}^1 \;+\; O(\varepsilon^2) \;=\; 0 \tag{4.3.9}$$

Repeating the same process, we multiply (4.3.9) by $\sin y^1$ in order to express it as a Fourier series. Analyzing each part as a sum of $\sin(my^1)$ and $\cos(my^1)$ we get (see Appendix E):

$$\int_0^{\pi} \frac{d\underset{\smile}{\Gamma}^1}{dt}\, dy^2 \,\sin y^1 \overset{(E18)}{=}$$

$$\begin{aligned}
+ \sum_{m=0}^{\infty} \Bigg\{ & \left[\frac{dA^s_{m+1,0,Z}}{dt}\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{dA^{ss}_{m+1,n,Z}}{dt}\frac{1-(-1)^n}{2n} \right) \right] - \\
& - \left[\frac{dA^s_{m-1,n,Z}}{dt}\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{dA^{ss}_{m-1,n,Z}}{dt}\frac{1-(-1)^n}{2n} \right) \right](1-\delta^0_m) + \\
& + \left[\frac{dA^s_{00Z}}{dt}\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{dA^{ss}_{0nZ}}{dt}\frac{1-(-1)^n}{2n} \right) \right] \delta^1_m \Bigg\} \cos(my^1) + \\
+ \Bigg\{ & \left[\frac{dA^c_{m-1,0,Z}}{dt}\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{dA^{cs}_{m-1,n,Z}}{dt}\frac{1-(-1)^n}{2n} \right) \right](1-\delta^0_m) - \\
& - \left[\frac{dA^c_{m+1,n,Z}}{dt}\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{dA^{cs}_{m+1,n,Z}}{dt}\frac{1-(-1)^n}{2n} \right) \right] + \\
& + \left[\frac{dA^c_{00Z}}{dt}\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{dA^{cs}_{0nZ}}{dt}\frac{1-(-1)^n}{2n} \right) \right] \delta^1_m \Bigg\} \sin(my^1)
\end{aligned} \tag{4.3.10}$$

$$\underline{\Gamma}^2(y^1, y^2 = \pi, Z)\sin y^1 \overset{(E19)}{=}$$

$$\begin{aligned}
= \sum_{m=1}^{\infty} & \left[-mA^s_{m0Z}\frac{y^2+\sin y^2}{R_o - R_H} + \sum_{n=1}^{\infty}\left(-\frac{m}{n}A^{ss}_{mnZ}(1-\cos(ny^2) - \frac{m}{n}A^{sc}_{mnZ}\sin(ny^2) \right) \right] \cos(my^1) \\
& \left[mA^c_{m0Z}\frac{y^2+\sin y^2}{R_o - R_H} + \sum_{n=1}^{\infty}\left(\frac{m}{n}A^{cs}_{mnZ}(1-\cos(ny^2)) - \frac{m}{n}A^{cc}_{mnZ}\sin(ny^2) \right) \right] \sin(my^1)
\end{aligned} \tag{4.3.11}$$

$$\underline{\Gamma}^1(y^1, y^2=\pi, Z, t)\,\sin y^1 \stackrel{(E17)}{=}$$

$$= \sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty}\left(A^{ss}_{m+1,n,Z} \;-\; A^{ss}_{m-1,n,Z}(1-\delta^0_m) \;+\; A^{ss}_{0nZ}\,\delta^1_m\right)(\pm 1)^m\right]\cos(my^1) +$$

$$+\left[\sum_{n=1}^{\infty}\left(A^{cs}_{m-1,n,Z}(1-\delta^0_m) \;-\; A^{cs}_{m+1,n,Z} \;+\; A^{cs}_{0nZ}\delta^1_m\right)(\pm 1)^m\right]\sin(my^1) \tag{4.3.12}$$

Substituting (4.3.10),(4.3.11) and (4.3.12) to (4.3.9), for every y^1 on the trailing edge, the coefficients of the trigonometric functions $\cos(my^1)$, $\sin(my^1)$ should be zero.

For m = 0,1,... ($\cos(my^1)$) :

$$-\left\{\left[\frac{dA^s_{m+1,0,Z}}{dt}\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{dA^{ss}_{m+1,n,Z}}{dt}\frac{1-(-1)^n}{2n}\right)\right] - \right.$$

$$- \left[\frac{dA^s_{m-1,n,Z}}{dt}\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{dA^{ss}_{m-1,n,Z}}{dt}\frac{1-(-1)^n}{2n}\right)\right](1-\delta^0_m) +$$

$$\left. + \left[\frac{dA^s_{00Z}}{dt}\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{dA^{ss}_{0nZ}}{dt}\frac{1-(-1)^n}{2n}\right)\right]\delta^1_m\right\}\cos(my^1) +$$

$$+\underline{\tilde{q}}^1\left[-mA^s_{m0Z}\frac{y^2+\sin y^2}{R_o-R_H} + \sum_{n=1}^{\infty}\left(-\frac{m}{n}A^{ss}_{mnZ}(1-\cos(ny^2) \;-\; \frac{m}{n}A^{sc}_{mnZ}\sin(ny^2)\right)\right] -$$

$$-\underline{\tilde{q}}^2\left[\sum_{n=1}^{\infty}\left(A^{ss}_{m+1,n,Z} \;-\; A^{ss}_{m-1,n,Z}(1-\delta^0_m) \;+\; A^{ss}_{0nZ}\,\delta^1_m\right)(\pm 1)^m\right] \;=\; 0 \tag{4.3.13}$$

For m = 1,2,... ($\sin(my^1)$)

$$-\left\{\left[\frac{dA^c_{m-1,0,Z}}{dt}\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{dA^{cs}_{m-1,n,Z}}{dt}\frac{1-(-1)^n}{2n}\right)\right](1-\delta^0_m) - \right.$$

$$\left. - \left[\frac{dA^c_{m+1,n,Z}}{dt}\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{dA^{cs}_{m+1,n,Z}}{dt}\frac{1-(-1)^n}{2n}\right)\right] \;+\; \left[\frac{dA^c_{00Z}}{dt}\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{dA^{cs}_{0nZ}}{dt}\frac{1-(-1)^n}{2n}\right)\right]\delta^1_m\right\}$$

$$+\underline{\tilde{q}}^1\left[\;mA^c_{m0Z}\frac{y^2+\sin y^2}{R_o-R_H} + \sum_{n=1}^{\infty}\left(\frac{m}{n}A^{cs}_{mnZ}(1-\cos(ny^2)) \;+\; \frac{m}{n}A^{cc}_{mnZ}\sin(ny^2)\right)\right] -$$

$$-\underline{\tilde{q}}^2\left[\sum_{n=1}^{\infty}\left(A^{cs}_{m-1,n,Z}(1-\delta^0_m) \;-\; A^{cs}_{m+1,n,Z} \;+\; A^{cs}_{0nZ}\delta^1_m\right)(\pm 1)^m\right] \tag{4.3.14}$$

In the program we assume that $\underset{\sim}{\tilde{q}}^i + \underset{\sim}{\tilde{v}}^i = (\underset{\sim}{\tilde{q}}^2 + \underset{\sim}{\tilde{v}}^2)\,\delta^i_2$, therefore instead of relations (4.3.6),(4.3.7) and (4.3.8) for the tip/hub we use relation (4.2.2). Moreover instead of relations (4.3.13) and (4.3.14) we can a take far simpler relation for the trailing edge. Specifically the Bernoulli 's equation (4.3.9) reads :

$$-\int_0^{y^2} \frac{d\breve{\Gamma}^1}{dt}\, dy^2 \;+\; \underset{\sim}{\tilde{q}}^1 \underline{\Gamma}^2 \;-\; \underset{\sim}{\tilde{q}}^2\, \underline{\Gamma}^1 \;+\; O(\varepsilon^2) \;=\; 0 \;\Rightarrow$$

$$\Rightarrow -\left[\sum_{m=0}^{\infty}\left(\frac{dA^s_{m0Z}}{dt}\sin(my^1) \;+\; \frac{dA^c_{m0Z}}{dt}\cos(my^1)\right)\frac{y^2+\sin y^2}{2} \;+\right.$$

$$+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(\frac{dA^{ss}_{mnZ}}{dt}\sin(my^1)\frac{1-\cos(ny^2)}{n} \;+\; \frac{dA^{cs}_{mnZ}}{dt}\cos(my^1)\frac{1-\cos(ny^2)}{n} \;+\right.$$

$$\left.\left. +\; \frac{dA^{sc}_{mnZ}}{dt}\sin(my^1)\frac{\sin(ny^2)}{n} \;+\; \frac{dA^{cc}_{mnZ}}{dt}\cos(my^1)\frac{\sin(ny^2)}{n}\right)\right] -$$

$$-\underset{\sim}{\tilde{q}}^2\left[\sum_{m=0}^{\infty}\left(A^s_{m0Z}\sin(my^1) \;+\; A^c_{m0Z}\cos(my^1)\right)\frac{1}{\tan(y^2/2)} \;+\right.$$

$$+2\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(A^{ss}_{mnZ}\sin(my^1)\frac{\sin(ny^2)}{\sin y^2} \;+\; A^{cs}_{mnZ}\cos(my^1)\frac{\sin(ny^2)}{\sin y^2} \;+\right.$$

$$\left.\left. +\; A^{sc}_{mnZ}\sin(my^1)\frac{\cos(ny^2)}{\sin y^2} \;+\; A^{cc}_{mnZ}\cos(my^1)\frac{\cos(ny^2)}{\sin y^2}\right)\right] + O(\varepsilon^2) \;=\; 0 \;\Rightarrow$$

Using relations (4.2.11), (4.2.12) and applying D' Hospital rule as $y^2 \to \pi$ we get :

$$\Rightarrow \sum_{m=0}^{\infty}\left\{\left[\frac{dA^s_{m0Z}}{dt}\frac{\pi}{2} \;+\; \sum_{n=1}^{\infty}\left(\frac{dA^{ss}_{mnZ}}{dt}\frac{1-(-1)^n}{n}\right)\right] \;+\; \left[\underset{\sim}{\tilde{q}}^2\sum_{n=1}^{\infty} 2\,A^{ss}_{mnZ}\,n\,(-1)^{n+1}\right]\right\}\sin(my^1)$$

$$\left\{\left[\frac{dA^c_{m0Z}}{dt}\frac{\pi}{2} \;+\; \sum_{n=1}^{\infty}\left(\frac{dA^{cs}_{mnZ}}{dt}\frac{1-(-1)^n}{n}\right)\right] \;+\; \left[\underset{\sim}{\tilde{q}}^2\sum_{n=1}^{\infty} 2\,A^{cs}_{mnZ}\,n\,(-1)^{n+1}\right]\right\}\cos(my^1) +$$

$$+\,O(\varepsilon^2) \;=\; 0 \qquad (4.3.15)$$

In order to implement the boundary condition (4.3.15) in a numerical scheme, we discretize the time period of the simulation to small time steps, Δt. The time, t, at the in-th instant is :

$$t = (in - 1)\,\Delta t \;, \qquad in = 1, 2, ..., Moments \tag{4.3.16}$$

where the variable $Moments$ is the maximum number of instants used in the program

For a burst (step) start, at in=0 the blades are at their starting position with zero velocity while at in=1 (t=0) the blades are also at their starting position but with "maximum" velocity.

We can now calculate numerically the time derivatives assuming Lagrange interpolation (see Appendix F) of the variables in question. Therefore the time derivative of $A(t)$ at t_{in} is :

First order interpolation :

$$\frac{dA_{in}}{dt} \approx \frac{A_{in} - A_{in-1}}{\Delta t} \;, \qquad in \geq 2 \tag{4.3.17}$$

Second order interpolation :

$$\frac{dA_{in}}{dt} \approx \frac{3A_{in} - 4A_{in-1} + A_{in-2}}{2\Delta t} \;, \qquad in \geq 3 \tag{4.3.18}$$

where $A_{in} = A(t_{in})$

It's obvious that for the calculation of the derivative (4.3.18), A(t) should be known for three consecutive instants t_{in}, t_{in-1}, t_{in-2}. This is not possible at the beginning of the calculations, which is why the first order approximation, (4.3.17), is needed. Introducing the symbols

$$P = \begin{cases} \frac{1}{\Delta t}, & \text{1st order interpolation} \\ \frac{3}{2\Delta t} & \text{2nd order interpolation} \end{cases} \tag{4.3.19}$$

$$Q = \begin{cases} -\frac{A_{in-1}}{\Delta t}, & \text{1st order interpolation} \\ \frac{-4A_{in-1}+A_{in-2}}{2\Delta t} & \text{2nd order interpolation} \end{cases} \tag{4.3.20}$$

relations (4.3.17) and (4.3.18) become :

$$\frac{dA_{in}}{dt} = P A_{in} + Q \tag{4.3.21}$$

Given that relation (4.3.15) is valid of every y^1, since the trigonometric functions are functionally independent each of their coefficients should be zero. After substituting relation (4.3.21) we get :

For m = 0,1,... $(\cos(my^1))$:

$$P\,A^{c}_{m0Z,in}\frac{\pi}{2}\;+\;\sum_{n=1}^{\infty}\left(P\,A^{cs}_{mnZ,in}\frac{1-(-1)^n}{n}\right)\;+\;\underline{\tilde{q}}^2\sum_{n=1}^{\infty}2\,A^{cs}_{mnZ,in}\,n\,(-1)^{n+1}\;=$$

$$=\;-\left[Q^{c}_{m0Z,in}\frac{\pi}{2}\;+\;\sum_{n=1}^{\infty}Q^{cs}_{mnZ,in}\frac{1-(-1)^n}{n}\right] \tag{4.3.22}$$

For m = 1,2,... $(\sin(my^1))$:

$$P\,A^{s}_{m0Z,in}\frac{\pi}{2}\;+\;\sum_{n=1}^{\infty}\left(P\,A^{ss}_{mnZ,in}\frac{1-(-1)^n}{n}\right)\;+\;\underline{\tilde{q}}^2\sum_{n=1}^{\infty}2\,A^{ss}_{mnZ,in}\,n\,(-1)^{n+1}\;=$$

$$=\;-\left[Q^{s}_{m0Z,in}\frac{\pi}{2}\;+\;\sum_{n=1}^{\infty}Q^{ss}_{mnZ,in}\frac{1-(-1)^n}{n}\right] \tag{4.3.23}$$

Using the simplified relation (4.2.21) for the bound vorticity we don't need relation (4.3.22) and the boundary condition at the trailing edge is only the (4.3.23).

4.4 Calculation of the mode velocities fields

According to relation (3.6.14) the mean velocity of the disturbance is :

$$-4\,\pi\,\rangle v^j(P)\langle\; = \int\limits_{\bigcup_Z S_{R_Z}} \sigma(Q)\,\frac{x^j(Q)-x^j(P)}{r^3}\,dS\; +$$

$$+ \int\limits_{\bigcup_Z (S_{R_Z}\cup S_{RW_Z})} \varepsilon^{jil}\gamma_i(Q)\,\frac{x_l(Q)-x_l(P)}{r^3}\,dS\; - \int\limits_L \mu\,\varepsilon^{jil}\;\frac{x_l(Q)-x_l(P)}{r^3}\,dl^i \qquad (4.4.1)$$

For the components of the above equation that refer to the Cartesian coordinate system x^i, the following relations hold.

$$\gamma_i\;=\;\gamma^i\,, \qquad x_i\;=\;x^i\,, \qquad \varepsilon^{jil}\;=\;e^{jil}\,, \qquad (4.4.2)$$

Furthermore since

$$dS\;=\;\sqrt{\underset{\smile}{\alpha}}dy^1\,dy^2 \qquad (4.4.3)$$

and

$$\gamma_i(Q)\;=\;\gamma^i(Q)\;=\;\underset{\smile}{\tilde{\gamma}}{}^i_{\alpha}\,\underset{\smile}{\gamma}{}^{\alpha}(Q) \qquad (4.4.4)$$

Relation (4.4.1) can be written as :

$$-4\,\pi\,\rangle v^j(P)\langle\;=$$

$$= \int\limits_{\bigcup_Z S_{R_Z}} \sigma(Q)\sqrt{\underset{\smile}{\alpha}}\,\frac{x^j(Q)-x^j(P)}{r^3}\,dy^1\,dy^2\;+$$

$$+ \int\limits_{\bigcup_Z (S_{R_Z}\cup S_{RW_Z})} e^{jil}\underset{\smile}{\tilde{\gamma}}{}^i_{\alpha}\,\underset{\smile}{\gamma}{}^{\alpha}(Q)\sqrt{\underset{\smile}{\alpha}}\,\frac{x^l(Q)-x^l(P)}{r^3}\,dy^1dy^2\;-\int\limits_L \mu\,e^{jil}\;\frac{x^l(Q)-x^l(P)}{r^3}\,dl^i\;=$$

$$= \int\limits_{\bigcup_Z S_{R_Z}} S(Q)\,\frac{x^j(Q)-x^j(P)}{r^3}\,dy^1\,dy^2\;+$$

$$+ \int\limits_{\bigcup_Z (S_{R_Z}\cup S_{RW_Z})} e^{jil}\left(\underset{\smile}{\tilde{\gamma}}{}^i_1\,\underset{\smile}{\Gamma}{}^1+\underset{\smile}{\tilde{\gamma}}{}^i_2\,\underset{\smile}{\Gamma}{}^2\right)\,\frac{x^l(Q)-x^l(P)}{r^3}\,dy^1dy^2-\int\limits_L \mu\,e^{jil}\;\frac{x^l(Q)-x^l(P)}{r^3}\,dl^i \qquad (4.4.5)$$

where relation (4.1.13) and (4.1.14) have been used, as well as the :

$$S(Q) \stackrel{def}{=} \sigma(Q) \sqrt{\underset{\smile}{\alpha}} \tag{4.4.6}$$

The coordinates $\tilde{x}^i \equiv x^i(Q)$, $x^i(P)$ for the reference surface $\underline{x}^3 = 0$ of the propeller are given by relations (2.2.1) - (2.2.6):

$$\tilde{x}^1 = \underset{\wedge}{x}^1(\underline{x}^3 = 0) = X(u^1) + c(u^1)\, u^2 \sin\phi(u^1) \tag{4.4.7}$$

$$\tilde{x}^2 = \underset{\wedge}{x}^2 \cos \underset{\wedge}{\tilde{x}}^3 = u^1 \cos \underset{\wedge}{\tilde{x}}^3 \tag{4.4.8}$$

$$\tilde{x}^3 = \underset{\wedge}{x}^2 \sin \underset{\wedge}{\tilde{x}}^3 = u^1 \sin \underset{\wedge}{\tilde{x}}^3 \tag{4.4.9}$$

where

$$\underset{\wedge}{\tilde{x}}^3 = \underset{\wedge}{x}^3(\underline{x}^3 = 0) = \Theta(u^1) + c(u^1)\, \frac{u^2}{u^1} \cos\phi(u^1) + \delta_Z \tag{4.4.10}$$

and

$$\delta_Z = \frac{2\pi}{NBL}(Z - 1) \tag{4.4.11}$$

is the angle of rotation of the x^i system when Q belongs on the Z-th blade. Obviously for the reference blade [7] $\delta_1 = 0$.

The coordinates of the reference surface of the wing, are given by the relations (2.2.9) - (2.2.11):

$$\tilde{x}^1 = X(u^1) + c(u^1)\, u^2 \sin\phi(u^1) \tag{4.4.12}$$

$$\tilde{x}^2 = u^1 \tag{4.4.13}$$

$$\tilde{x}^3 = u^1 \tan(\Theta(u^1)) + c(u^1)\, \underline{u}^2 \cos\phi(u^1) \tag{4.4.14}$$

[7] For a symmetrical body like the propeller the mode velocities are calculated only for the reference surface.

4.5 Introduction of Induction and Self Induction Factors of Vorticity

The contribution of the vorticity of the reference surface is given by the second term of relations (4.4.5). Specifically the contribution of a single blade is :

$$-4\ \pi\rangle v^j_{\Gamma}(P)\langle_{S_{R_Z}} = \iint_{S_{R_Z}} e^{jil} \left(\frac{1}{2} \left(R_o - R_H\right) \sin y^1 \, \underset{\smile}{\Gamma}^1 \, \underline{\tilde{\gamma}}^i_1 \; + \; \frac{1}{2} \sin y^2 \, \underset{\smile}{\Gamma}^2 \, \underline{\tilde{\gamma}}^i_2 \right) \frac{\delta x^l}{r^3} \, dy^2 \, dy^1 \tag{4.5.1}$$

Substituting relations (4.1.18) and (4.1.19) :

$$= \iint_{s_{R_Z}} \frac{1}{2} \left(R_o - R_H\right) \sin y^1 \left(\sum_{m=0}^{\infty} A^s_{m0Z} \sin(m y^1) \, \frac{\cos y^2 + 1}{2} \right) e^{jil} \underline{\tilde{\gamma}}^i_1 \, \frac{\delta x^l}{r^3} \, dy^2 \, dy^1$$

$$+ \iint_{s_{R_Z}} \frac{1}{2} \left(R_o - R_H\right) \sin y^1 \left(\sum_{m=0}^{\infty} A^c_{m0Z} \cos(m y^1) \, \frac{\cos y^2 + 1}{2} \right) e^{jil} \underline{\tilde{\gamma}}^i_1 \, \frac{\delta x^l}{r^3} \, dy^2 \, dy^1$$

$$+ \iint_{s_{R_Z}} \frac{1}{2} \left(R_o - R_H\right) \sin y^1 \left(\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A^{ss}_{mnZ} \sin(m y^1) \, \sin(n y^2) \right) e^{jil} \underline{\tilde{\gamma}}^i_1 \, \frac{\delta x^l}{r^3} \, dy^2 \, dy^1$$

$$+ \iint_{s_{R_Z}} \frac{1}{2} \left(R_o - R_H\right) \sin y^1 \left(\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A^{cs}_{mnZ} \cos(m y^1) \, \sin(n y^2) \right) e^{jil} \underline{\tilde{\gamma}}^i_1 \, \frac{\delta x^l}{r^3} \, dy^2 \, dy^1$$

$$+ \iint_{s_{R_Z}} \frac{1}{2} \left(R_o - R_H\right) \sin y^1 \left(\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A^{sc}_{mnZ} \sin(m y^1) \, \cos(n y^2) \right) e^{jil} \underline{\tilde{\gamma}}^i_1 \, \frac{\delta x^l}{r^3} \, dy^2 \, dy^1$$

$$+ \iint_{s_{R_Z}} \frac{1}{2} \left(R_o - R_H\right) \sin y^1 \left(\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A^{cc}_{mnZ} \cos(m y^1) \, \cos(n y^2) \right) e^{jil} \underline{\tilde{\gamma}}^i_1 \, \frac{\delta x^l}{r^3} \, dy^2 \, dy^1$$

$$+ \iint_{s_{R_Z}} \frac{1}{2} \sin y^2 \left(\sum_{m=1}^{\infty} -m \, A^s_{m0Z} \cos(m y^1) \, \frac{y^2 + \sin y^2}{2} \right) e^{jil} \underline{\tilde{\gamma}}^i_2 \, \frac{\delta x^l}{r^3} \, dy^1 \, dy^2$$

$$+ \iint_{s_{R_Z}} \frac{1}{2} \sin y^2 \left(\sum_{m=1}^{\infty} m \, A^c_{m0Z} \sin(m y^1) \, \frac{y^2 + \sin y^2}{2} \right) e^{jil} \underline{\tilde{\gamma}}^i_2 \, \frac{\delta x^l}{r^3} \, dy^1 \, dy^2$$

$$+ \iint_{s_{R_Z}} \frac{1}{2} \sin y^2 \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -\frac{m}{n} \, A^{ss}_{mnZ} \cos(m y^1) \left(1 - \cos(n y^2)\right) \right) e^{jil} \underline{\tilde{\gamma}}^i_2 \, \frac{\delta x^l}{r^3} \, dy^1 \, dy^2$$

$$+\iint\limits_{s_{R_Z}} \frac{1}{2}\,\sin y^2 \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m}{n}\,A^{cs}_{mnZ}\sin(my^1)\,(1-\cos(ny^2))\right) e^{jil}\underset{\sim}{\gamma}{}^{i}_{2}\,\frac{\delta x^l}{r^3}\,dy^1\,dy^2$$

$$+\iint\limits_{s_{R_Z}} \frac{1}{2}\,\sin y^2 \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} -\frac{m}{n}\,A^{sc}_{mnZ}\cos(my^1)\,\sin(ny^2)\right) e^{jil}\underset{\sim}{\gamma}{}^{i}_{2}\,\frac{\delta x^l}{r^3}\,dy^1\,dy^2$$

$$+\iint\limits_{s_{R_Z}} \frac{1}{2}\,\sin y^2 \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m}{n}\,A^{cc}_{mnZ}\sin(my^1)\,\sin(ny^2)\right) e^{jil}\underset{\sim}{\gamma}{}^{i}_{2}\,\frac{\delta x^l}{r^3}\,dy^1\,dy^2 \tag{4.5.2}$$

where S_{R_Z} is the reference surface of the Z-th blade. We also use the symbol δx^l instead of the difference $x^l(Q) - x^l(P)$.

We define the following quantities :

$$B^{sj}_{m0Z} = \int_0^{\pi}\int_0^{\pi} \left(\sin(my^1)\frac{1+\cos y^2}{2}\right)\sin y^1\, e^{jil}\,\underset{\sim}{\gamma}{}^{i}_{1}\frac{\delta x^l}{r^3}\,dy^1\,dy^2 \tag{4.5.3}$$

$$B^{cj}_{m0Z} = \int_0^{\pi}\int_0^{\pi} \left(\cos(my^1)\frac{1+\cos y^2}{2}\right)\sin y^1\, e^{jil}\,\underset{\sim}{\gamma}{}^{i}_{1}\frac{\delta x^l}{r^3}\,dy^1\,dy^2 \tag{4.5.4}$$

$$B^{ssj}_{mnZ} = \int_0^{\pi}\int_0^{\pi} \left(\sin(my^1)\sin(ny^2)\right)\sin y^1\, e^{jil}\,\underset{\sim}{\gamma}{}^{i}_{1}\frac{\delta x^l}{r^3}\,dy^1\,dy^2 \tag{4.5.5}$$

$$B^{csj}_{mnZ} = \int_0^{\pi}\int_0^{\pi} \left(\cos(my^1)\sin(ny^2)\right)\sin y^1\, e^{jil}\,\underset{\sim}{\gamma}{}^{i}_{1}\frac{\delta x^l}{r^3}\,dy^1\,dy^2 \tag{4.5.6}$$

$$B^{scj}_{mnZ} = \int_0^{\pi}\int_0^{\pi} \left(\sin(my^1)\cos(ny^2)\right)\sin y^1\, e^{jil}\,\underset{\sim}{\gamma}{}^{i}_{1}\frac{\delta x^l}{r^3}\,dy^1\,dy^2 \tag{4.5.7}$$

$$B^{ccj}_{mnZ} = \int_0^{\pi}\int_0^{\pi} \left(\cos(my^1)\cos(ny^2)\right)\sin y^1\, e^{jil}\,\underset{\sim}{\gamma}{}^{i}_{1}\frac{\delta x^l}{r^3}\,dy^1\,dy^2 \tag{4.5.8}$$

$$C^{sj}_{m0Z} = \int_0^\pi \int_0^\pi \left(\cos(my^1)\frac{y^2+\sin y^2}{2}\right) \sin y^2 \, e^{jil} \, \underset{\sim}{\gamma}{}^i_2 \frac{\delta x^l}{r^3} \, dy^1 \, dy^2 \tag{4.5.9}$$

$$C^{cj}_{m0Z} = \int_0^\pi \int_0^\pi \left(\sin(my^1)\frac{y^2+\sin y^2}{2}\right) \sin y^2 \, e^{jil} \, \underset{\sim}{\gamma}{}^i_2 \frac{\delta x^l}{r^3} \, dy^1 \, dy^2 \tag{4.5.10}$$

$$C^{ssj}_{mnZ} = \int_0^\pi \int_0^\pi \big(\cos(my^1)(1-\cos(ny^2))\big) \sin y^2 \, e^{jil} \, \underset{\sim}{\gamma}{}^i_2 \frac{\delta x^l}{r^3} \, dy^1 \, dy^2 \tag{4.5.11}$$

$$C^{csj}_{mnZ} = \int_0^\pi \int_0^\pi \big(\sin(my^1)(1-\cos(ny^2))\big) \sin y^2 \, e^{jil} \, \underset{\sim}{\gamma}{}^i_2 \frac{\delta x^l}{r^3} \, dy^1 \, dy^2 \tag{4.5.12}$$

$$C^{scj}_{mnZ} = \int_0^\pi \int_0^\pi \big(\cos(my^1)\sin(ny^2)\big) \sin y^2 \, e^{jil} \, \underset{\sim}{\gamma}{}^i_2 \frac{\delta x^l}{r^3} \, dy^1 \, dy^2 \tag{4.5.13}$$

$$C^{ccj}_{mnZ} = \int_0^\pi \int_0^\pi \big(\sin(my^1)\sin(ny^2)\big) \sin y^2 \, e^{jil} \, \underset{\sim}{\gamma}{}^i_2 \frac{\delta x^l}{r^3} \, dy^1 \, dy^2 \tag{4.5.14}$$

Notice that the dependence of B and C from the blade (index Z) comes from the difference δx^l.

Introducing the induction factors :

$$B^{sj}_m \overset{def}{=} \left(cosy^2 - cosy^2_P\right) \int_0^\pi \sin(my^1) \, \sin y^1 \, e^{jil} \, \underset{\sim}{\gamma}{}^i_1 \, \frac{\delta x^l}{r^3} \, dy^1 \tag{4.5.15}$$

$$B^{cj}_m \overset{def}{=} \left(cosy^2 - cosy^2_P\right) \int_0^\pi \cos(my^1) \, \sin y^1 \, e^{jil} \, \underset{\sim}{\gamma}{}^i_1 \, \frac{\delta x^l}{r^3} \, dy^1 \tag{4.5.16}$$

$$C^{j}_0 \overset{def}{=} \left(cosy^1 - cosy^1_P\right) \int_0^\pi \frac{y^2+\sin y^2}{2} \, \sin y^2 \, e^{jil} \, \underset{\sim}{\gamma}{}^i_2 \, \frac{\delta x^l}{r^3} \, dy^2 \tag{4.5.17}$$

$$C^{sj}_n \overset{def}{=} \left(cosy^1 - cosy^1_P\right) \int_0^\pi \big(1-\cos(ny^2)\big) \, \sin y^2 \, e^{jil} \, \underset{\sim}{\gamma}{}^i_2 \, \frac{\delta x^l}{r^3} \, dy^2 \tag{4.5.18}$$

$$C^{cj}_n \overset{def}{=} \left(cosy^1 - cosy^1_P\right) \int_0^\pi \sin(ny^2) \, \sin y^2 \, e^{jil} \, \underset{\sim}{\gamma}{}^i_2 \, \frac{\delta x^l}{r^3} \, dy^2 \tag{4.5.19}$$

and the corresponding self-induction factors

$$B^{sj}_{m_{self}} \stackrel{def}{=} \lim_{y^2 \to y^2_P} B^{sj}_m \quad , \qquad B^{cj}_{m_{self}} \stackrel{def}{=} \lim_{y^2 \to y^2_P} B^{cj}_m \tag{4.5.20}$$

$$C^j_{0_{self}} \stackrel{def}{=} \lim_{y^1 \to y^1_P} C^j_0 \quad , \qquad C^{sj}_{n_{self}} \stackrel{def}{=} \lim_{y^1 \to y^1_P} C^{sj}_n \quad , \qquad C^{cj}_{n_{self}} \stackrel{def}{=} \lim_{y^1 \to y^1_P} C^{cj}_n \tag{4.5.21}$$

with $y^1_P = y^1(P)$ and $y^2_P = y^2(P)$

and substituting to relations (4.5.3) - (4.5.14) we get:

$$B^{sj}_{m0Z} = \int_0^\pi \frac{1+\cos y^2}{2} \frac{B^{sj}_m - B^{sj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 + B^{sj}_{m_{self}} \fint_0^\pi \frac{\frac{1}{2} + \frac{1}{2}\cos y^2}{\cos y^2 - \cos y^2_P}\, dy^2 \tag{4.5.22}$$

$$B^{cj}_{m0Z} = \int_0^\pi \frac{1+\cos y^2}{2} \frac{B^{cj}_m - B^{cj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 + B^{cj}_{m_{self}} \fint_0^\pi \frac{\frac{1}{2} + \frac{1}{2}\cos y^2}{\cos y^2 - \cos y^2_P}\, dy^2 \tag{4.5.23}$$

$$B^{ssj}_{mnZ} = \int_0^\pi \sin(ny^2) \sin y^2 \frac{\frac{B^{sj}_m}{\sin y^2} - \frac{B^{sj}_{m_{self}}}{\sin y^2_P}}{\cos y^2 - \cos y^2_P}\, dy^2 + \frac{B^{sj}_{m_{self}}}{\sin y^2_P} \fint_0^\pi \frac{\sin(ny^2)\, \sin y^2}{\cos y^2 - \cos y^2_P}\, dy^2 \tag{4.5.24}$$

$$B^{csj}_{mnZ} = \int_0^\pi \sin(ny^2) \sin y^2 \frac{\frac{B^{cj}_m}{\sin y^2} - \frac{B^{cj}_{m_{self}}}{\sin y^2_P}}{\cos y^2 - \cos y^2_P}\, dy^2 + \frac{B^{cj}_{m_{self}}}{\sin y^2_P} \fint_0^\pi \frac{\sin(ny^2)\, \sin y^2}{\cos y^2 - \cos y^2_P}\, dy^2 \tag{4.5.25}$$

$$B^{scj}_{mnZ} = \int_0^\pi \cos(ny^2) \frac{B^{sj}_m - B^{sj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 + B^{sj}_{m_{self}} \fint_0^\pi \frac{\cos(ny^2)}{\cos y^2 - \cos y^2_P}\, dy^2 \tag{4.5.26}$$

$$B^{ccj}_{mnZ} = \int_0^\pi \cos(ny^2) \frac{B^{cj}_m - B^{cj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 + B^{cj}_{m_{self}} \fint_0^\pi \frac{\cos(ny^2)}{\cos y^2 - \cos y^2_P}\, dy^2 \tag{4.5.27}$$

$$C^{sj}_{m0Z} = \int_0^\pi \cos(my^1) \frac{C^j_0 - C^j_{0_{self}}}{\cos y^1 - \cos y^1_P}\, dy^1 + C^j_{0_{self}} \fint_0^\pi \frac{\cos(my^1)}{\cos y^1 - \cos y^1_P}\, dy^1 \tag{4.5.28}$$

$$C^{cj}_{m0Z} = \int_0^\pi \sin(my^1) \sin y^1 \frac{\frac{C^j_0}{\sin y^1} - \frac{C^j_{0_{self}}}{\sin y^1_P}}{\cos y^1 - \cos y^1_P}\, dy^1 + \frac{C^j_{0_{self}}}{\sin y^1_P} \fint_0^\pi \frac{\sin(my^1) \sin y^1}{\cos y^1 - \cos y^1_P}\, dy^1 \tag{4.5.29}$$

$$C^{ssj}_{mnZ} \;=\; \int_0^{\pi} \cos(my^1)\frac{C^{sj}_n - C^{sj}_{n_{self}}}{\cos y^1 - \cos y^1_P}\, dy^1 \;+\; C^{sj}_{n_{self}} \fint_0^{\pi} \frac{\cos(my^1)}{\cos y^1 - \cos y^1_P}\, dy^1 \tag{4.5.30}$$

$$C^{csj}_{mnZ} \;=\; \int_0^{\pi} \sin(my^1)\sin y^1 \frac{\frac{C^{sj}_n}{\sin y^1} - \frac{C^{sj}_{n_{self}}}{\sin y^1_P}}{\cos y^1 - \cos y^1_P}\, dy^1 \;+\; \frac{C^{sj}_{n_{self}}}{\sin y^1_P} \fint_0^{\pi} \frac{\sin(my^1)\sin y^1}{\cos y^1 - \cos y^1_P}\, dy^1 \tag{4.5.31}$$

$$C^{scj}_{mnZ} \;=\; \int_0^{\pi} \cos(my^1)\frac{C^{cj}_n - C^{cj}_{n_{self}}}{\cos y^1 - \cos y^1_P}\, dy^1 \;+\; C^{cj}_{n_{self}} \fint_0^{\pi} \frac{\cos(my^1)}{\cos y^1 - \cos y^1_P}\, dy^1 \tag{4.5.32}$$

$$C^{ccj}_{mnZ} \;=\; \int_0^{\pi} \sin(my^1)\sin y^1 \frac{\frac{C^{cj}_n}{\sin y^1} - \frac{C^{cj}_{n_{self}}}{\sin y^1_P}}{\cos y^1 - \cos y^1_P}\, dy^1 \;+\; \frac{C^{cj}_{n_{self}}}{\sin y^1_P} \fint_0^{\pi} \frac{\sin(my^1)\sin y^1}{\cos y^1 - \cos y^1_P}\, dy^1 \tag{4.5.33}$$

where the second terms of the right hand side of relations (4.5.22) - (4.5.33) are Cauchy Principal Value integrals. These terms are identical or can be transformed to the known Glauert integral.

$$\fint_0^{\pi} \frac{\cos(n\theta)}{\cos\theta - \cos\theta_0} \;=\; \pi\,\frac{\sin(n\theta_0)}{\sin\theta_0} \tag{4.5.34}$$

Therefore :

$$\fint_0^{\pi} \frac{\frac{1}{2} + \frac{1}{2}\cos y^2}{\cos y^2 - \cos y^2_P}\, dy^2 \;=\; \frac{\pi}{2}\,\frac{\sin(0\cdot y^2_P)}{\sin y^2_P} \;+\; \frac{\pi}{2}\,\frac{\sin(y^2_P)}{\sin y^2_P} \;=\; \frac{\pi}{2} \tag{4.5.35}$$

$$\fint_0^{\pi} \frac{\sin(ny^2)\,\sin y^2}{\cos y^2 - \cos y^2_P}\, dy^2 \;=\; \frac{1}{2}\fint_0^{\pi} \frac{\cos((n-1)y^2) - \cos((n+1)y^2)}{\cos y^2 - \cos y^2_P}\, dy^2$$

$$= \frac{\pi}{2\,\sin y^2_P}\left(\sin((n-1)y^2_P) \;-\; \sin((n+1)y^2_P)\right) \;=\; -\pi\,\cos(ny^2_P) \tag{4.5.36}$$

Similarly,

$$\fint_0^{\pi} \frac{\cos(my^1)}{\cos y^1 - \cos y^1_P}\, dy^1 \;=\; \pi\,\frac{\sin(my^1)}{\sin(y^1_P)} \tag{4.5.37}$$

$$\fint_0^{\pi} \frac{\sin(my^1)\sin y^1}{\cos y^1 - \cos y^1_P}\, dy^1 \;=\; -\pi\,\cos(my^1_P) \tag{4.5.38}$$

Substituting relations (4.5.35) - (4.5.38) to (4.5.22) - (4.5.33) we get :

$$B^{sj}_{m0Z} \;=\; \int_0^{\pi} \frac{1+\cos y^2}{2}\,\frac{B^{sj}_m - B^{sj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\,dy^2 \;+\; \frac{\pi}{2}\,B^{sj}_{m_{self}} \tag{4.5.39}$$

$$B^{cj}_{m0Z} \;=\; \int_0^{\pi} \frac{1+\cos y^2}{2}\,\frac{B^{cj}_m - B^{cj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\,dy^2 \;+\; \frac{\pi}{2}\,B^{cj}_{m_{self}} \tag{4.5.40}$$

$$B^{ssj}_{mnZ} \;=\; \int_0^{\pi} \sin(ny^2)\sin y^2\,\frac{\frac{B^{sj}_m}{\sin y^2} - \frac{B^{sj}_{m_{self}}}{\sin y^2_P}}{\cos y^2 - \cos y^2_P}\,dy^2 \;-\; \pi\,\frac{\cos(ny^2_P)}{\sin y^2_P}\,B^{sj}_{m_{self}} \tag{4.5.41}$$

$$B^{csj}_{mnZ} \;=\; \int_0^{\pi} \sin(ny^2)\sin y^2\,\frac{\frac{B^{cj}_m}{\sin y^2} - \frac{B^{cj}_{m_{self}}}{\sin y^2_P}}{\cos y^2 - \cos y^2_P}\,dy^2 \;-\; \pi\,\frac{\cos(ny^2_P)}{\sin y^2_P}\,B^{cj}_{m_{self}} \tag{4.5.42}$$

$$B^{scj}_{mnZ} \;=\; \int_0^{\pi} \cos(ny^2)\frac{B^{sj}_m - B^{sj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\,dy^2 \;+\; \pi\,\frac{\sin(ny^2_P)}{\sin y^2_P}B^{sj}_{m_{self}} \tag{4.5.43}$$

$$B^{ccj}_{mnZ} \;=\; \int_0^{\pi} \cos(ny^2)\frac{B^{cj}_m - B^{cj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\,dy^2 \;+\; \pi\,\frac{\sin(ny^2_P)}{\sin y^2_P}B^{cj}_{m_{self}} \tag{4.5.44}$$

$$C^{sj}_{m0Z} \;=\; \int_0^{\pi} \cos(my^1)\frac{C^j_0 - C^j_{0_{self}}}{\cos y^1 - \cos y^1_P}\,dy^1 \;+\; \pi\,\frac{\sin(my^1_P)}{\sin y^1_P}C^j_{0_{self}} \tag{4.5.45}$$

$$C^{cj}_{m0Z} \;=\; \int_0^{\pi} \sin(my^1)\sin y^1\,\frac{\frac{C^j_0}{\sin y^1} - \frac{C^j_{0_{self}}}{\sin y^1_P}}{\cos y^1 - \cos y^1_P}\,dy^1 \;-\; \pi\,\frac{\cos(my^1_P)}{\sin y^1_P}\,C^j_{0_{self}} \tag{4.5.46}$$

$$C^{ssj}_{mnZ} \;=\; \int_0^{\pi} \cos(my^1)\frac{C^{sj}_n - C^{sj}_{n_{self}}}{\cos y^1 - \cos y^1_P}\,dy^1 \;+\; \pi\,\frac{\sin(my^1_P)}{\sin y^1_P}C^{sj}_{n_{self}} \tag{4.5.47}$$

$$C^{csj}_{mnZ} \;=\; \int_0^{\pi} \sin(my^1)\sin y^1\,\frac{\frac{C^{sj}_n}{\sin y^1} - \frac{C^{sj}_{n_{self}}}{\sin y^1_P}}{\cos y^1 - \cos y^1_P}\,dy^1 \;-\; \pi\,\frac{\cos(my^1_P)}{\sin y^1_P}\,C^{sj}_{n_{self}} \tag{4.5.48}$$

$$C^{scj}_{mnZ} \;=\; \int\limits_0^{\pi} \cos(my^1) \frac{C^{cj}_n - C^{cj}_{n_{self}}}{\cos y^1 - \cos y^1_P}\, dy^1 \;+\; \pi\, \frac{\sin(my^1_P)}{\sin y^1_P} C^{cj}_{n_{self}} \tag{4.5.49}$$

$$C^{ccj}_{mnZ} \;=\; \int\limits_0^{\pi} \sin(my^1) \sin y^1 \frac{\frac{C^{cj}_n}{\sin y^1} - \frac{C^{cj}_{n_{self}}}{\sin y^1_P}}{\cos y^1 - \cos y^1_P}\, dy^1 \;-\; \pi\, \frac{\cos(my^1_P)}{\sin y^1_P}\, C^{cj}_{n_{self}} \tag{4.5.50}$$

We can now define the mode perturbation velocities :

$$T^{sj}_{m0Z} \;=\; -\frac{1}{4\pi}\left[\frac{1}{2}\,(R_o - R_H)\, B^{sj}_{m0Z} \;-\; \frac{1}{2}\, m\, C^{sj}_{m0Z}\right] \tag{4.5.51}$$

$$T^{cj}_{m0Z} \;=\; -\frac{1}{4\pi}\left[\frac{1}{2}\,(R_o - R_H)\, B^{cj}_{m0Z} \;+\; \frac{1}{2}\, m\, C^{cj}_{m0Z}\right] \tag{4.5.52}$$

$$T^{ssj}_{mnZ} \;=\; -\frac{1}{4\pi}\left[\frac{1}{2}\,(R_o - R_H)\, B^{ssj}_{mnZ} \;-\; \frac{1}{2}\,\frac{m}{n}\, C^{ssj}_{mnZ}\right] \tag{4.5.53}$$

$$T^{csj}_{mnZ} \;=\; -\frac{1}{4\pi}\left[\frac{1}{2}\,(R_o - R_H)\, B^{csj}_{mnZ} \;+\; \frac{1}{2}\,\frac{m}{n}\, C^{csj}_{mnZ}\right] \tag{4.5.54}$$

$$T^{scj}_{mnZ} \;=\; -\frac{1}{4\pi}\left[\frac{1}{2}\,(R_o - R_H)\, B^{scj}_{mnZ} \;-\; \frac{1}{2}\,\frac{m}{n}\, C^{scj}_{mnZ}\right] \tag{4.5.55}$$

$$T^{ccj}_{mnZ} \;=\; -\frac{1}{4\pi}\left[\frac{1}{2}\,(R_o - R_H)\, B^{ccj}_{mnZ} \;+\; \frac{1}{2}\,\frac{m}{n}\, C^{ccj}_{mnZ}\right] \tag{4.5.56}$$

Thus relation (4.5.1) can be written as :

$$\rangle v^j_{\Gamma}(P)\langle = \sum_{Z=1}^{NBL}\left[\sum_{m=0}^{\infty}\left(T^{sj}_{m0Z}\, A^{s}_{m0Z} \;+\; T^{cj}_{m0Z}\, A^{c}_{m0Z}\right) \;+\right.$$

$$\left. +\; \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(T^{ssj}_{mnZ}\, A^{ss}_{mnZ} \;+\; T^{csj}_{mnZ}\, A^{cs}_{mnZ} \;+\; T^{scj}_{mnZ}\, A^{sc}_{mnZ} \;+\; T^{ccj}_{mnZ}\, A^{cc}_{mnZ}\right)\right] \tag{4.5.57}$$

The analytic expressions of the self induction factors can be proven to be (see Appendix G):

$$B^{sj}_{m_{self}} \;=\; -\frac{8}{R_o - R_H}\, e^{jil}\, \underset{\sim}{\gamma}{}^i_1\, \underset{\sim}{\gamma}{}^l_2\, \sin(my^1_P)\, \frac{\sqrt{\underset{\sim}{g}_{11}}}{\underline{\alpha}} \tag{4.5.58}$$

$$B^{cj}_{m_{self}} \;=\; -\frac{8}{R_o - R_H}\, e^{jil}\, \underset{\sim}{\gamma}{}^i_1\, \underset{\sim}{\gamma}{}^l_2\, \cos(my^1_P)\, \frac{\sqrt{\underset{\sim}{g}_{11}}}{\underline{\alpha}} \tag{4.5.59}$$

$$C^{j}_{0_{self}} \;=\; \frac{4}{R_o - R_H}\, e^{jil}\, \underset{\sim}{\gamma}^{i}_{1}\, \underset{\sim}{\gamma}^{l}_{2}\, (y^2_P + \sin y^2_P)\, \frac{\sqrt{\widetilde{\underline{g}}_{22}}}{\underline{\alpha}} \tag{4.5.60}$$

$$C^{sj}_{n_{self}} \;=\; \frac{8}{R_o - R_H}\, e^{jil}\, \underset{\sim}{\gamma}^{i}_{1}\, \underset{\sim}{\gamma}^{l}_{2}\, (1 - \cos(n y^2_P))\, \frac{\sqrt{\widetilde{\underline{g}}_{22}}}{\underline{\alpha}} \tag{4.5.61}$$

$$C^{cj}_{n_{self}} \;=\; \frac{8}{R_o - R_H}\, e^{jil}\, \underset{\sim}{\gamma}^{i}_{1}\, \underset{\sim}{\gamma}^{l}_{2}\, \sin y^2_P\, \frac{\sqrt{\widetilde{\underline{g}}_{22}}}{\underline{\alpha}} \tag{4.5.62}$$

Alternatively by changing the integration order the induction factors become :

$$B^{j}_{0} \stackrel{def}{=} \left(\cos y^1 - \cos y^1_P\right) \int\limits_0^{\pi} \frac{1 + \cos y^2}{2}\, e^{jil}\, \underset{\sim}{\gamma}^{i}_{1}\, \frac{\delta x^l}{r^3}\, dy^2 \tag{4.5.63}$$

$$B^{sj}_{n} \stackrel{def}{=} \left(\cos y^1 - \cos y^1_P\right) \int\limits_0^{\pi} \sin(n y^2)\, e^{jil}\, \underset{\sim}{\gamma}^{i}_{1}\, \frac{\delta x^l}{r^3}\, dy^2 \tag{4.5.64}$$

$$B^{cj}_{n} \stackrel{def}{=} \left(\cos y^1 - \cos y^1_P\right) \int\limits_0^{\pi} \cos(n y^2)\, e^{jil}\, \underset{\sim}{\gamma}^{i}_{1}\, \frac{\delta x^l}{r^3}\, dy^2 \tag{4.5.65}$$

$$C^{sj}_{m} \stackrel{def}{=} \left(\cos y^2 - \cos y^2_P\right) \int\limits_0^{\pi} \cos(m y^1)\, e^{jil}\, \underset{\sim}{\gamma}^{i}_{2}\, \frac{\delta x^l}{r^3}\, dy^1 \tag{4.5.66}$$

$$C^{cj}_{m} \stackrel{def}{=} \left(\cos y^2 - \cos y^2_P\right) \int\limits_0^{\pi} \sin(m y^1)\, e^{jil}\, \underset{\sim}{\gamma}^{i}_{2}\, \frac{\delta x^l}{r^3}\, dy^1 \tag{4.5.67}$$

and the corresponding self-induction factors :

$$B^{j}_{0_{self}} \stackrel{def}{=} \lim_{y^1 \to y^1_P} B^{j}_{0} \;\;,\qquad B^{sj}_{n_{self}} \stackrel{def}{=} \lim_{y^1 \to y^1_P} B^{sj}_{n} \;\;,\qquad B^{cj}_{n_{self}} \stackrel{def}{=} \lim_{y^1 \to y^1_P} B^{cj}_{n} \tag{4.5.68}$$

$$C^{sj}_{m_{self}} \stackrel{def}{=} \lim_{y^2 \to y^2_P} C^{sj}_{m} \;\;,\qquad C^{cj}_{m_{self}} \stackrel{def}{=} \lim_{y^2 \to y^2_P} C^{cj}_{m} \tag{4.5.69}$$

Therefore relations (4.5.3) - (4.5.14) can be written as :

$$B^{sj}_{m0Z} \;=\; \int\limits_0^\pi \sin(my^1)\sin y^1 \,\frac{B^j_0 - B^j_{0_{self}}}{\cos y^1 - \cos y^1_P}\, dy^1 \;+\; B^j_{0_{self}} \fint\limits_0^\pi \frac{\sin(my^1)\sin y^1}{\cos y^1 - \cos y^1_P}\, dy^1 \tag{4.5.70}$$

$$B^{cj}_{m0Z} \;=\; \int\limits_0^\pi \cos(my^1)\frac{\sin y^1\, B^j_0 - \sin y^1_P\, B^j_{0_{self}}}{\cos y^1 - \cos y^1_P}\, dy^1 \;+\; \sin y^1_P\, B^j_{0_{self}} \fint\limits_0^\pi \frac{\cos(my^1)}{\cos y^1 - \cos y^1_P}\, dy^1 \tag{4.5.71}$$

$$B^{ssj}_{mnZ} \;=\; \int\limits_0^\pi \sin(my^1)\sin y^1 \,\frac{B^{sj}_n - B^{sj}_{n_{self}}}{\cos y^1 - \cos y^1_P}\, dy^1 \;+\; B^{sj}_{n_{self}} \fint\limits_0^\pi \frac{\sin(my^1)\sin y^1}{\cos y^1 - \cos y^1_P}\, dy^1 \tag{4.5.72}$$

$$B^{csj}_{mnZ} \;=\; \int\limits_0^\pi \cos(my^1)\frac{\sin y^1\, B^{sj}_n - \sin y^1_P\, B^{sj}_{n_{self}}}{\cos y^1 - \cos y^1_P}\, dy^1 \;+\; \sin y^1_P\, B^{sj}_{n_{self}} \fint\limits_0^\pi \frac{\cos(my^1)}{\cos y^1 - \cos y^1_P}\, dy^1 \tag{4.5.73}$$

$$B^{scj}_{mnZ} \;=\; \int\limits_0^\pi \sin(my^1)\sin y^1 \,\frac{B^{cj}_n - B^{cj}_{n_{self}}}{\cos y^1 - \cos y^1_P}\, dy^1 \;+\; B^{cj}_{n_{self}} \fint\limits_0^\pi \frac{\sin(my^1)\sin y^1}{\cos y^1 - \cos y^1_P}\, dy^1 \tag{4.5.74}$$

$$B^{ccj}_{mnZ} \;=\; \int\limits_0^\pi \cos(my^1)\frac{\sin y^1\, B^{cj}_n - \sin y^1_P\, B^{cj}_{n_{self}}}{\cos y^1 - \cos y^1_P}\, dy^1 \;+\; \sin y^1_P\, B^{cj}_{n_{self}} \fint\limits_0^\pi \frac{\cos(my^1)}{\cos y^1 - \cos y^1_P}\, dy^1 \tag{4.5.75}$$

$$\begin{aligned} C^{sj}_{m0Z} \;=\; & \int\limits_0^\pi (siny^2)^2 \frac{\frac{y^2+\sin y^2}{2\sin y^2} C^{sj}_m - \frac{y^2_P+\sin y^2_P}{2\sin y^2_P}\, C^{sj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 + \\ & + \frac{y^2_P+\sin y^2_P}{2\sin y^2_P}\, C^{sj}_{m_{self}} \fint\limits_0^\pi \frac{(\sin y^2)^2}{\cos y^2 - \cos y^2_P}\, dy^2 \end{aligned} \tag{4.5.76}$$

$$\begin{aligned} C^{cj}_{m0Z} \;=\; & \int\limits_0^\pi (siny^2)^2 \frac{\frac{y^2+\sin y^2}{2\sin y^2} C^{cj}_m - \frac{y^2_P+\sin y^2_P}{2\sin y^2_P}\, C^{cj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 + \\ & + \frac{y^2_P+\sin y^2_P}{2\sin y^2_P}\, C^{cj}_{m_{self}} \fint\limits_0^\pi \frac{(\sin y^2)^2}{\cos y^2 - \cos y^2_P}\, dy^2 \end{aligned} \tag{4.5.77}$$

$$\begin{aligned} C^{ssj}_{mnZ} \;=\; & \int\limits_0^\pi (1-\cos(ny^2))\frac{\sin y^2\, C^{sj}_m - \sin y^2_P\, C^{sj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 + \\ & + \sin y^2_P\, C^{sj}_{m_{self}} \fint\limits_0^\pi \frac{1-\cos(ny^2)}{\cos y^2 - \cos y^2_P}\, dy^2 \end{aligned} \tag{4.5.78}$$

$$C^{csj}_{mnZ} \;=\; \int_0^{\pi} (1-\cos(ny^2))\,\frac{\sin y^2\, C^{cj}_{m} - \sin y^2_P\, C^{cj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 \,+$$

$$+\, \sin y^2_P\, C^{cj}_{m_{self}} \fint_0^{\pi} \frac{1-\cos(ny^2)}{\cos y^2 - \cos y^2_P}\, dy^2 \qquad (4.5.79)$$

$$C^{scj}_{mnZ} \;=\; \int_0^{\pi} \sin(ny^2)\sin y^2\, \frac{C^{sj}_{m} - C^{sj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 \;+\; C^{sj}_{m_{self}} \fint_0^{\pi} \frac{\sin(ny^2)\sin y^2}{\cos y^2 - \cos y^2_P}\, dy^2 \qquad (4.5.80)$$

$$C^{ccj}_{mnZ} \;=\; \int_0^{\pi} \sin(ny^2)\sin y^2\, \frac{C^{cj}_{m} - C^{cj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 \;+\; C^{cj}_{m_{self}} \fint_0^{\pi} \frac{\sin(ny^2)\sin y^2}{\cos y^2 - \cos y^2_P}\, dy^2 \qquad (4.5.81)$$

where the second terms of the right hand side of relations (4.5.70) - (4.5.81) are Cauchy Principal Value integrals. These terms are identical or can be transformed to the known Glauert integral.

$$\fint_0^{\pi} \frac{\sin(my^1)\sin y^1}{\cos y^1 - \cos y^1_P}\, dy^1 \;=\; \frac{1}{2}\fint_0^{\pi} \frac{\cos((m-1)y^1) - \cos((m+1)y^1)}{\cos y^1 - \cos y^1_P}\, dy^1$$

$$=\; \frac{\pi}{2\sin y^1_P}\left(\sin((m-1)y^1_P) - \sin((m+1)y^1_P)\right) \;=\; -\pi\, \cos(my^1_P) \qquad (4.5.82)$$

$$\fint_0^{\pi} \frac{\cos(my^1)}{\cos y^1 - \cos y^1_P}\, dy^1 \;=\; \pi\, \frac{\sin(my^1_P)}{\sin y^1_P} \qquad (4.5.83)$$

$$\fint_0^{\pi} \frac{\sin y^2 \sin y^2}{\cos y^2 - \cos y^2_P}\, dy^2 \;=\; -\pi\, \cos y^2_P \qquad (4.5.84)$$

$$\fint_0^{\pi} \frac{1-\cos(ny^2_P)}{\cos y^2 - \cos y^2_P}\, dy^2 \;=\; \pi\, \frac{\sin(0\cdot y^2_P)}{\sin y^1_P} \;-\; \pi\, \frac{\sin(ny^2_P)}{\sin y^2_P} \;=\; -\pi\, \frac{\sin(ny^2_P)}{\sin y^2_P} \qquad (4.5.85)$$

$$\fint_0^{\pi} \frac{\sin(ny^2)\sin y^2}{\cos y^2 - \cos y^2_P}\, dy^2 \;=\; -\pi\, \cos(ny^2_P) \qquad (4.5.86)$$

Therefore relations (4.5.70) - (4.5.81) can be written as :

$$B^{sj}_{m0Z} \;=\; \int_0^\pi \sin(my^1)\sin y^1 \frac{B^j_0 - B^j_{0_{self}}}{\cos y^1 - \cos y^1_P}\, dy^1 \;-\; \pi\, \cos(my^1_P)\, B^j_{0_{self}} \tag{4.5.87}$$

$$B^{cj}_{m0Z} \;=\; \int_0^\pi \cos(my^1)\frac{\sin y^1 B^j_0 - \sin y^1_P B^j_{0_{self}}}{\cos y^1 - \cos y^1_P}\, dy^1 \;+\; \pi\, \sin(my^1_P)\, B^j_{0_{self}} \tag{4.5.88}$$

$$B^{ssj}_{mnZ} \;=\; \int_0^\pi \sin(my^1)\sin y^1 \frac{B^{sj}_n - B^{sj}_{n_{self}}}{\cos y^1 - \cos y^1_P}\, dy^1 \;-\; \pi\, \cos(my^1_P)\, B^{sj}_{n_{self}} \tag{4.5.89}$$

$$B^{csj}_{mnZ} \;=\; \int_0^\pi \cos(my^1)\frac{\sin y^1 B^{sj}_n - \sin y^1_P B^{sj}_{n_{self}}}{\cos y^1 - \cos y^1_P}\, dy^1 \;+\; \pi\, \sin(my^1_P)\, B^{sj}_{n_{self}} \tag{4.5.90}$$

$$B^{scj}_{mnZ} \;=\; \int_0^\pi \sin(my^1)\sin y^1 \frac{B^{cj}_n - B^{cj}_{n_{self}}}{\cos y^1 - \cos y^1_P}\, dy^1 \;-\; \pi\, \cos(my^1_P)\, B^{cj}_{n_{self}} \tag{4.5.91}$$

$$B^{ccj}_{mnZ} \;=\; \int_0^\pi \cos(my^1)\frac{\sin y^1 B^{cj}_n - \sin y^1_P B^{cj}_{n_{self}}}{\cos y^1 - \cos y^1_P}\, dy^1 \;+\; \pi\, \sin(my^1_P)\, B^{cj}_{n_{self}} \tag{4.5.92}$$

$$C^{sj}_{m0Z} \;=\; \int_0^\pi (siny^2)^2 \frac{\frac{y^2+\sin y^2}{2\sin y^2_P} C^{sj}_m - \frac{y^2_P+\sin y^2_P}{2\sin y^2_P}\, C^{sj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 \;-\; \pi\, \frac{y^2_P + \sin y^2_P}{2}\, \frac{\cos y^2_P}{\sin y^2_P}\, C^{sj}_{m_{self}} \tag{4.5.93}$$

$$C^{cj}_{m0Z} \;=\; \int_0^\pi (siny^2)^2 \frac{\frac{y^2+\sin y^2}{2\sin y^2} C^{cj}_m - \frac{y^2_P+\sin y^2_P}{2\sin y^2_P}\, C^{cj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 \;-\; \pi\, \frac{y^2_P + \sin y^2_P}{2}\, \frac{\cos y^2_P}{\sin y^2_P}\, C^{cj}_{m_{self}} \tag{4.5.94}$$

$$C^{ssj}_{mnZ} \;=\; \int_0^\pi (1-\cos(ny^2))\frac{\sin y^2\, C^{sj}_m - \sin y^2_P\, C^{sj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 \;-\; \pi\, \sin(ny^2_P)\, C^{sj}_{m_{self}} \tag{4.5.95}$$

$$C^{csj}_{mnZ} \;=\; \int_0^\pi (1-\cos(ny^2))\frac{\sin y^2\, C^{cj}_m - \sin y^2_P\, C^{cj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 \;-\; \pi\, \sin(ny^2_P)\, C^{cj}_{m_{self}} \tag{4.5.96}$$

$$C^{scj}_{mnZ} \;=\; \int_0^\pi \sin(ny^2)\sin y^2 \frac{C^{sj}_m - C^{sj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 \;-\; \pi\, \cos(ny^2_P)\, C^{sj}_{m_{self}} \tag{4.5.97}$$

$$C^{ccj}_{mnZ} \;=\; \int_0^\pi \sin(ny^2)\sin y^2 \frac{C^{cj}_m - C^{cj}_{m_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 \;-\; \pi\, \cos(ny^2_P) C^{cj}_{m_{self}} \tag{4.5.98}$$

The analytic expressions of the self induction factors can be proven to be (see Appendix G) :

$$B^{j}_{0_{self}} = \frac{4}{R_o - R_H} e^{jil} \underset{\sim}{\gamma}^{i}_{1} \underset{\sim}{\gamma}^{l}_{2} \frac{1 + \cos y^2_P}{\sin y^2_P} \frac{\widetilde{\underline{g}}_{12}}{\sqrt{\widetilde{\underline{g}}_{22}}\,\underline{\alpha}} \tag{4.5.99}$$

$$B^{sj}_{n_{self}} = \frac{8}{R_o - R_H} e^{jil} \underset{\sim}{\gamma}^{i}_{1} \underset{\sim}{\gamma}^{l}_{2} \frac{\sin(n y^2_P)}{\sin y^2_P} \frac{\widetilde{\underline{g}}_{12}}{\sqrt{\widetilde{\underline{g}}_{22}}\,\underline{\alpha}} \tag{4.5.100}$$

$$B^{cj}_{n_{self}} = \frac{8}{R_o - R_H} e^{jil} \underset{\sim}{\gamma}^{i}_{1} \underset{\sim}{\gamma}^{l}_{2} \frac{\cos(n y^2_P)}{\sin y^2_P} \frac{\widetilde{\underline{g}}_{12}}{\sqrt{\widetilde{\underline{g}}_{22}}\,\underline{\alpha}} \tag{4.5.101}$$

$$C^{sj}_{m_{self}} = -\frac{8}{R_o - R_H} e^{jil} \underset{\sim}{\gamma}^{i}_{1} \underset{\sim}{\gamma}^{l}_{2} \frac{\cos(m y^1_P)}{\sin y^1_P} \frac{\widetilde{\underline{g}}_{12}}{\sqrt{\widetilde{\underline{g}}_{11}}\,\underline{\alpha}} \tag{4.5.102}$$

$$C^{cj}_{m_{self}} = -\frac{8}{R_o - R_H} e^{jil} \underset{\sim}{\gamma}^{i}_{1} \underset{\sim}{\gamma}^{l}_{2} \frac{\sin(m y^1_P)}{\sin y^1_P} \frac{\widetilde{\underline{g}}_{12}}{\sqrt{\widetilde{\underline{g}}_{11}}\,\underline{\alpha}} \tag{4.5.103}$$

4.6 Introduction of Induction and Self Induction Factors of Sources

The contribution of the surface source intensity of the reference surface is given by the first term of relations (4.4.5). Specifically the contribution of a single blade is :

$$-4\ \pi\rangle \underset{S_{R_Z}}{v_S^j}(P)\langle = \int\limits_{S_{R_Z}} S(Q)\, \frac{x^j(Q)-x^j(P)}{r^3}\, dy^1\, dy^2 \tag{4.6.1}$$

where according to relations (4.4.6), (3.2.31):

$$S\ =\ \sigma(Q)\ \sqrt{\underset{\smile}{\alpha}}\ \overset{(A97)}{=}$$

$$=\ \left(\frac{1}{4}\,(R_o-R_H)\,\sin y^1\,\sin y^2\right)\sqrt{\underline{\alpha}}\ \left(\rangle \underline{q}^1\langle\,\frac{\partial E_T}{\partial \underline{x}^1}+\rangle \underline{q}^2\langle\,\frac{\partial E_T}{\partial \underline{x}^2}\ -\ \langle q^3\rangle\right)\ \sqrt{\frac{\underset{=}{\tilde{g}}}{\underline{\alpha}}}\ \overset{(A83)}{=}$$

$$=\ \left(\frac{1}{4}\,(R_o-R_H)\,\sin y^1\,\sin y^2\right)\ c^2\, f_s(Q) \tag{4.6.2}$$

with

$$f_s(Q)\ =\ \left(\rangle \underline{q}^1\langle\,\frac{\partial E_T}{\partial \underline{x}^1}+\rangle \underline{q}^2\langle\,\frac{\partial E_T}{\partial \underline{x}^2}\ -\ \langle q^3\rangle\right) \tag{4.6.3}$$

Substituting (4.6.2) to (4.6.1) we get :

$$-4\ \pi\rangle \underset{S_{R_Z}}{v_S^j}(P)\langle = \int\limits_0^\pi\int\limits_0^\pi S(Q)\,\frac{x^j(Q)-x^j(P)}{r^3}\,dy^1 dy^2\ =$$

$$=\ \frac{c^2}{4}\,(R_o-R_H)\int\limits_0^\pi\int\limits_0^\pi f_s(Q)\,\sin y^1\,\sin y^2\,\frac{\delta x^j}{r^3}\,dy^1 dy^2 \tag{4.6.4}$$

We define the induction factor :

$$D_Z^j\ \overset{def}{=}\ (\cos y^2-\cos y_P^2)\int\limits_0^\pi f_S(Q)\,\sin y^1\,\sin y^2\,\frac{\delta x^j}{r^3}\,dy^1 \tag{4.6.5}$$

and the corresponding self-induction factor :

$$D_{Z_{self}}^j\ =\ \lim_{y^2\to y_P^2} D_Z^j \tag{4.6.6}$$

Therefore relation (4.6.4) becomes :

$$-4\,\pi\rangle v^j_{\substack{S\\S_{R_Z}}}(P)\langle = \left[\int_0^\pi \frac{D^j_Z - D^j_{Z_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2 \;+\; D^j_{Z_{self}} \fint_0^\pi \frac{1}{\cos y^2 - \cos y^2_P}\, dy^2\right] \frac{c^2}{4}\,(R_o - R_H) \tag{4.6.7}$$

where the second integral of the right hand side is the known Glauert integral.

$$\fint_0^\pi \frac{1}{\cos y^2 - \cos y^2_P}\, dy^2 \;=\; \fint_0^\pi \frac{\cos(0\cdot y^2)}{\cos y^2 - \cos y^2_P}\, dy^2 \;=\; \pi\,\frac{\sin(0\cdot y^2_P)}{\sin y^2_P} \;=\; 0 \tag{4.6.8}$$

Therefore :

$$\rangle v^j_S(P)\langle = -\frac{1}{4\pi}\sum_{Z=1}^{NBL}\left[\frac{c^2}{4}(R_o - R_H)\int_0^\pi \frac{D^j_Z - D^j_{Z_{self}}}{\cos y^2 - \cos y^2_P}\, dy^2\right] \tag{4.6.9}$$

4.7 Numerical Scheme

Before we examine the contribution of the wake we should first determine the numerical scheme, since the surface of the wake cannot be described analytically. In that end we choose the boundary conditions which will be used in the program. In particular given that $\underline{\tilde{q}}^i + \rangle \underline{\tilde{v}}^i \langle = (\underline{\tilde{q}}^2 + \rangle \underline{\tilde{v}}^2 \langle) \, \delta_2^i$ we have :

- Zero bound vorticity at the tip and the hub
- Zero pressure difference at the trailing edge
- No-entrance condition at the inner points of the blade

As mentioned in chapter 4.2, relation (4.2.21) of the bound vorticity

$$\underline{\Gamma}^1(y^1, y^2, Z, t) = \sum_{m=1}^{\infty} \left[A^s_{m0Z} \sin(my^1)\right] \frac{1}{\tan(y^2/2)} + 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[A^{ss}_{mnZ} \sin(my^1) \frac{\sin(ny^2)}{\sin y^2}\right] \quad (4.7.1)$$

can fully describe this particular problem while satisfying implicitly the first condition at the tip and hub. The unknown variables are the coefficients, A, of relation (4.7.1). Assuming that all the spanwise harmonics up to the M_0-th and all the chordwise harmonics up to the N_0-th satisfy the linearized problem we have :

- A^s_{m0Z} , m = 1,2,...,M_0
- A^{ss}_{mnZ} , m = 1,2,...,M_0, n = 1,2,...,N_0

Therefore we have a total number of $M_0(N_0 + 1)$ unknowns and need an equivalent number of equations. Using the y^α coordinate system of the blade we satisfy the above boundary conditions in a set (lattice) of points consisting of $M_0 + 2$ rows and $N_0 + 1$ columns equally spaced as seen in the figure below. These control points (C.P) span from the hub ($y^1 = 0$) to the tip ($y^1 = \pi$) and from a point forward of the leading edge to the trailing edge ($y^2 = \pi$).

Notice that there are no points on the leading edge because the determinant of the surface metric tensor of the real surface goes to infty, when parametrized by u1,u2 ($\alpha' \to \infty$), thus the normal vector looses meaning in u1,u2 parametrization and cannot be approximated by relations (3.2.28) - (3.2.30). This means that points near the leading edge cannot be used for the no-entrance boundary condition, when solving in the parametrized space, however the zero-free vorticity boundary condition (4.2.1) is implicitly satisfied.

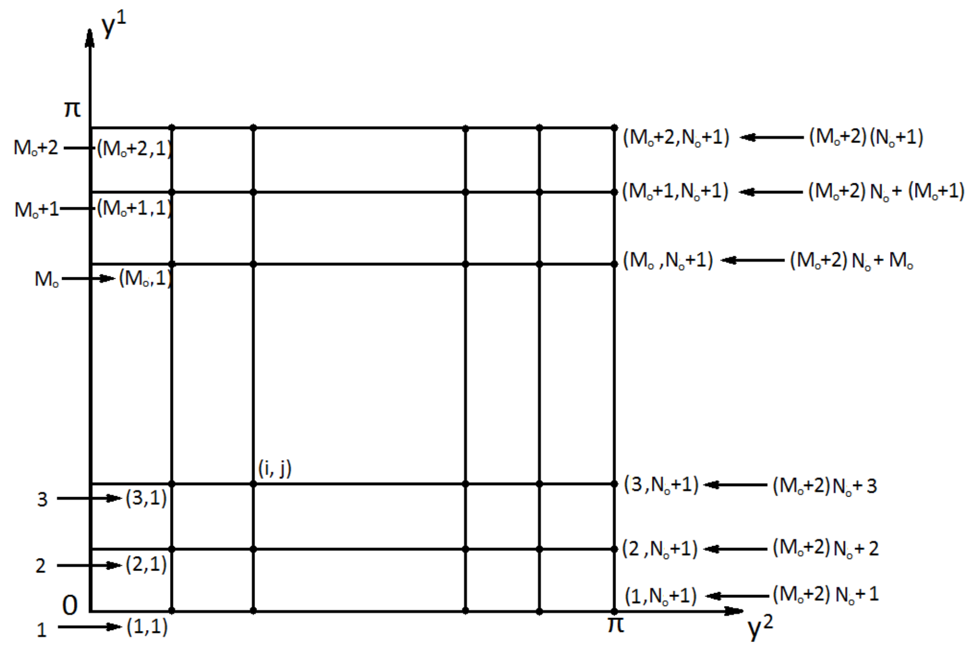


Figure 4.7.1

In the figure above, every point can be described by two integer numbers (i,j) which indicate the number of row and column respectively. Starting form point (1,1) we move up the column to (M_0+2,1) and continue the same process form the bottom ($y^1 = 0$) of the second column. Instead of two variables we can use only one to describe the position of the C.P at each blade :

$$k = i + (j-1)(M_0+2) + (Z-1)(M_0+2)(N_0+1) \tag{4.7.2}$$

where

i = 1,2,...,(M_0+2), j = 1,2,...,(N_0+1)

Z = 1,2,...,NBL , k = 1,2,...,NBL(M_0+2)(N_0+1)

For the points on hub : MOD(k-1, M_0+2) = 0, i=1 (4.7.3)

For the points on tip : MOD(k, M_0+2) = 0, i=M_0+2 (4.7.4)

For the points on the T.E : j = N_0+1 (4.7.5)

Following the same logic, instead of using variables m,n,Z to describe the coefficients, A, we can use the following variable :

$$l = m + n\,M_0 + (Z-1)\,M_0\,(N_0+1) \tag{4.7.6}$$

where

$$\text{m} = 1{,}2{,}...{,}M_0\ , \qquad \text{n} = 0{,}1{,}2{,}...{,}N_0$$

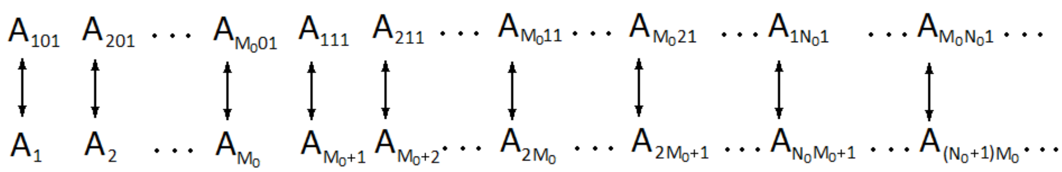


Figure 4.7.2

4.8 Contribution of Wake to a control point

The contribution of the wake of each blade is given by the second and third terms of relation (4.4.5).

$$-4\,\pi\,\rangle v^j_W(P)\langle = \int_{\bigcup_Z S_{RW_Z}} e^{jil}\,\gamma^i\,\frac{x^l(Q)-x^l(P)}{r^3}\,dS - \int_{\bigcup_Z L_Z} \mu\,e^{jil}\,\frac{x^l(Q)-x^l(P)}{r^3}\,dl^i \qquad (4.4.5)^*$$

where L_Z is the boundary line of the Z-th blade's and its wake, $L_Z = \partial(S_{R_Z} \cup S_{RW_Z})$ (Z=1,...,NBL).

In our case it's useful to define a "Kutta" strip, S_{K_Z}, behind each blade. This strip is narrow piece of the wake connecting the trailing edge and the free part of the wake (see figure below). The relative position of the Kutta strip to the blade's real surface is examined in Chapter D - Section 3 of [3] but in the case of the lifting surface theory we assume it to be tangent to the reference surface of the blade.

Notice that with the introduction of S_{K_Z}, the boundary L_Z can be written as $L_Z = \partial(S_{R_Z} \cup S_{K_Z} \cup S_{RW_Z})$, so the above relation can be written as :

$$-4\,\pi\,\rangle v^j_W(P)\langle = \int_{\bigcup_Z (S_{RW_Z} \cup S_{K_Z})} e^{jil}\,\gamma^i\,\frac{x^l(Q)-x^l(P)}{r^3}\,dy^1 dy^2$$

$$- \left(\int_{\bigcup_Z \partial(S_{RW_Z} \cup S_{K_Z})} \mu\,e^{jil}\,\frac{x^l(Q)-x^l(P)}{r^3}\,dl^i + \int_{\bigcup_Z \partial S_{R_Z}} \mu\,e^{jil}\,\frac{x^l(Q)-x^l(P)}{r^3}\,dl^i \right) \qquad (4.8.1)$$

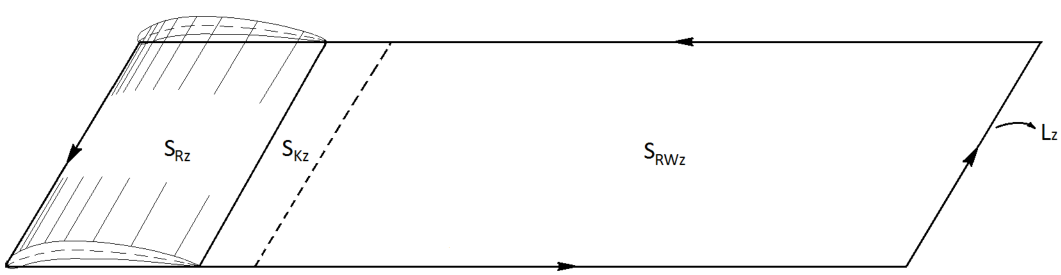


Figure 4.8.1

For the wake of a single blade we take $M_0 + 2$ control points in the y^1 - spanwise direction, which are as many as the control points in the same direction on the reference surface. Along the wake (and the Kutta strip) we have in-1 control points excluding those on the trailing edge, where "in" is the current instant of the simulation (in=1,2,...,Moments).

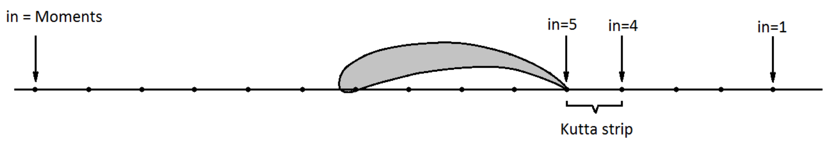


Figure 4.8.2

On each of the $(M_0+2)(in-1)$ C.P on the wake we know the dipole intensity μ from relations (3.4.7), (3.4.11). For every four nearby nodes on the wake, a surface boundary element (B.E) is defined, with a closed curve, L_{MZT}, as its boundary. We approximate this B.E using bi linear polynomials, meaning that L_{MZT} is consisted by four line segments connecting the four nodes. Similarly we approximate the boundary L_Z at the tip and hub as well as the T.E with linear splines in between the nodes $N = 1, 2, ..., N_0 + 1$ which we name L^{hub}_{NZ}, L^{tip}_{NZ} and $L^{T.E}_{MZ}$.

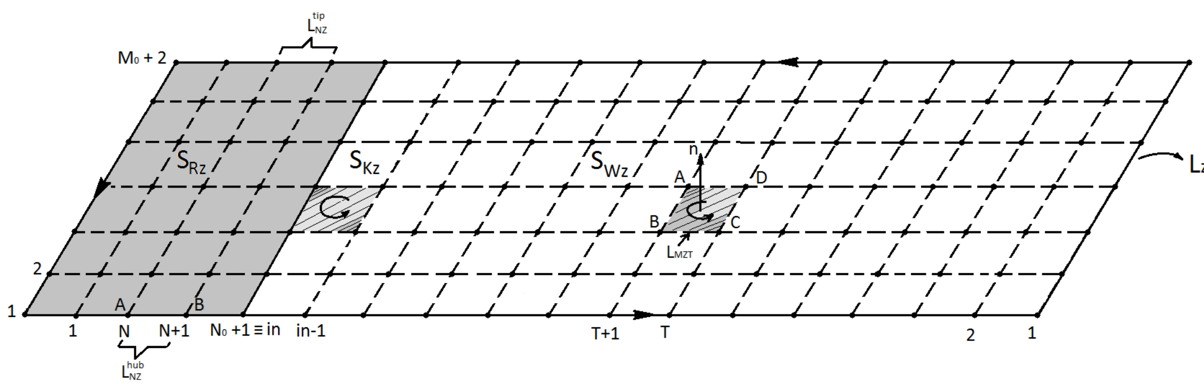


Figure 4.8.3

Using the modified Stokes theorem (see Chapter B - Section 7 of [3]) and assuming constant dipole intensity within each B.E, the integration on the wake and kutta strip (two first terms of (4.8.1)) can be written as :

$$-\frac{1}{4\pi}\int\limits_{\bigcup_Z (S_{RW_Z}\cup S_{K_Z})} e^{jil}\,\gamma^i(Q)\,\frac{x^l(Q)-x^l(P)}{r^3}\,dS + \frac{1}{4\pi}\int\limits_{\bigcup_Z \partial(S_{RW_Z}\cup S_{K_Z})} \mu\, e^{jil}\,\frac{x^l(Q)-x^l(P)}{r^3}\,dl^i =$$

$$= \sum_{M=1}^{M_0+1}\sum_{Z=1}^{NBL}\sum_{T=1}^{in-1}\frac{\overline{\mu}_{MZT}}{4\pi}\oint\limits_{L_{MZT}} e^{jil}\,dl^i\,\frac{x^l(Q)-x^l(P)}{r^3} \qquad (4.8.2)$$

As for the third term on the right hand side of (4.8.1), it obviously includes the leading edge, which doesn't contribute to the perturbation velocity since according to (3.4.15) :

$$\mu(\underline{x}^2) = \int_{-1/2}^{\underline{x}^2} \underline{\Gamma}^1 \, d\underline{x}^2 = \int_0^{y^2} \underset{\smile}{\Gamma}^1 \, dy^2 =$$

$$= \sum_{m=0}^{\infty} \left(A^s_{m0Z} \sin(my^1) + A^c_{m0Z} \cos(my^1) \right) \frac{\sin y^2 + y^2}{2} +$$

$$+ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \Bigg(\frac{1}{n} A^{ss}_{mnZ} \sin(my^1) \, (1 - \cos(ny^2)) + \frac{1}{n} A^{cs}_{mnZ} \cos(my^1) \, (1 - \cos(ny^2))$$

$$+ \frac{1}{n} A^{sc}_{mnZ} \sin(my^1) \sin(ny^2) + \frac{1}{n} A^{cc}_{mnZ} \cos(my^1) \sin(ny^2) \Bigg) \tag{4.8.3}$$

the dipole intensity is zero at the leading edge $\mu(y^2 = 0)$. Therefore we are left only with the L^{hub}_{NZ}, L^{tip}_{NZ} and L^{TE}_{MZ}. Assuming a constant dipole intensity on each line segment, the third term of (4.8.1) can be written as :

$$\frac{1}{4\pi} \int_{\underset{Z}{\cup} S_{R_Z}} \mu \, e^{jil} \, dl^i \, \frac{x^l(Q) - x^l(P)}{r^3} = \sum_{M=1}^{M0+1} \sum_{Z=1}^{NBL} \frac{\overline{\mu}^{T.E}_{MZ}}{4\,\pi} \oint_{L^{TE}_{NZ}} e^{jil} \, dl^i \, \frac{x^l(Q) - x^l(P)}{r^3} +$$

$$+ \sum_{N=0}^{N_0} \sum_{Z=1}^{NBL} \left[\frac{\overline{\mu}^{tip}_{NZ}}{4\pi} \oint_{L^{tip}_{NZ}} e^{jil} \, dl^i \, \frac{x^l(Q) - x^l(P)}{r^3} + \frac{\overline{\mu}^{hub}_{NZ}}{4\pi} \oint_{L^{hub}_{NZ}} e^{jil} \, dl^i \, \frac{x^l(Q) - x^l(P)}{r^3} \right] \tag{4.8.4}$$

Note: The direction of the tangent vector dl^i on the boundary depends on the direction of the normal vector on the B.E according to the right hand rule.

In the case of a line vortex extending between points A and B, with a unit dipole intensity, the induced velocity at a point P is given by the relation :

$$\vec{D}^{AB} \stackrel{def}{=} \frac{1}{4\pi} \int_{AB} \frac{d\vec{l} \times (\vec{x}(Q) - \vec{x}(P))}{r^3} = -\frac{1}{4\pi h} (\cos\alpha + \cos\beta) \frac{\vec{PA} \times \vec{PB}}{|\vec{PA} \times \vec{PB}|} \tag{4.8.5}$$

where $h \neq 0$ is the normal distance from control point P to the line AB. For $h = 0$, but $P \notin AB$ it can be proved by the definition of (4.8.5) that $\vec{D}^{AB} = \vec{0}$.

In the case of the wake B.E we get ($M \leq M0 + 1$, $T \leq in - 1$) :

$$\vec{D}_{MZT} = \vec{D}^{AB}_{MZT} + \vec{D}^{BC}_{MZT} + \vec{D}^{CD}_{MZT} + \vec{D}^{DA}_{MZT} \tag{4.8.6}$$

while for the tip, hub and T.E ($M \leq M0 + 1$, $N \leq N0$) :

$$\vec{D}_{NZ} = \vec{D}^{AB}_{NZ}, \qquad \vec{D}_{MZ,IN} = -\vec{D}^{AB}_{MZ,IN} = \vec{D}^{CD}_{M,Z,IN} \tag{4.8.7}$$

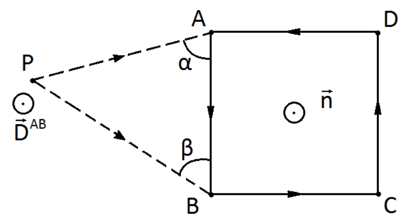


Figure 4.8.4

The constant value of the dipole intensity in the ABCD B.E is given by the mean value of dipole intensity on each one of the four points :

$$\overline{\mu}_{MZT} = \frac{1}{4}\left(\mu^{A}_{MZT} + \mu^{B}_{MZT} + \mu^{C}_{MZT} + \mu^{D}_{MZT}\right), \qquad T \le IN-1, \;\; M \le M0+1 \tag{4.8.8}$$

Similarly the constant value of dipole intensity on each line segment at the tip and hub is given as the mean value of the dipole intensity at points A and B.

$$\overline{\mu}_{NZ} = \frac{1}{2}\left(\mu^{A}_{NZ} + \mu^{B}_{NZ}\right), \qquad N \le N0 \tag{4.8.9}$$

For the T.E :

$$\overline{\mu}_{MZ,IN} = \frac{1}{2}\left(\mu^{A}_{MZ,IN} + \mu^{B}_{MZ,IN}\right), \qquad M \le M0+1 \tag{4.8.10}$$

Since we are using boundary conditions (3.5.9) and (3.5.10), according to (4.8.3), $\mu = 0$ at the tip and hub. Nevertheless we include these extra terms for the sake of completeness.

Notice that, although we know analytically the dipole intensity,μ, at each point of the blade from relation (4.8.3), we still choose to approximate it with constant values on the tip and hub. The main reason for this is because the $\vec{dl} \times (\vec{x}(Q) - \vec{x}(P))$ depends on the shape of the blade, which although has an analytic expression, makes the integration in (4.8.5) much more difficult. Furthermore, since the difference $(\vec{x}(Q) - \vec{x}(P))$ is finite the expression is not tensorial in the generalized sense.

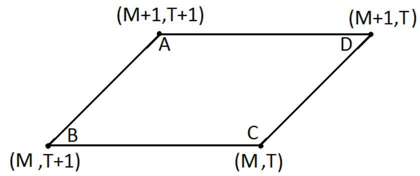
For the wake B.E the following expressions are valid :

$$\mu^{A}_{MZT} = \mu_{(M+1),Z,(T+1)} \;, \qquad \mu^{D}_{MZT} = \mu_{(M+1),Z,T} \tag{4.8.11}$$

$$\mu^{B}_{MZT} = \mu_{M,Z,(T+1)} \;, \qquad \mu^{C}_{MZT} = \mu_{M,Z,T} \tag{4.8.12}$$

so

$$\overline{\mu}_{MZT} = \frac{1}{4}\left(\mu_{M+1,Z,T+1} + \mu_{M,Z,T+1} + \mu_{M,Z,T} + \mu_{M+1,Z,T}\right) \tag{4.8.13}$$



$Figure\ \ 4.8.5$

For the line segments on the tip, hub and T.E we have :

$$\mu^{A}_{NZ}\big|_{tip,hub} = \mu_{M,N,Z}\ , \qquad \mu^{B}_{NZ}\big|_{tip,hub} = \mu_{M,N+1,Z}\ , \qquad M = 1, M0+2 \tag{4.8.14}$$

$$\mu^{A}_{MZ,IN}\big|_{T.E} = \mu_{M,Z,IN}\ , \qquad \mu^{B}_{MZ,IN}\big|_{T.E} = \mu_{M+1,Z,IN}\ , \qquad N = N0+1 \tag{4.8.15}$$

so

$$\overline{\mu}_{MNZ}|_{tip,hub} = \frac{1}{2}\left(\mu_{M,N,Z} + \mu_{M,N+1,Z}\right) \tag{4.8.16}$$

$$\overline{\mu}_{MZ,IN}\big|_{T.E} = \frac{1}{2}\left(\mu_{M,Z,IN} + \mu_{(M+1),Z,IN}\right) \tag{4.8.17}$$

Setting :

$$\mu^{s}_{m0} = \sin(my^1)\,\frac{\sin y^2 + y^2}{2}\ , \qquad \mu^{c}_{m0} = \cos(my^1)\,\frac{\sin y^2 + y^2}{2} \tag{4.8.18}$$

$$\mu^{ss}_{mn} = \frac{1}{n}\,\sin(my^1)\,(1 - cos(ny^2))\ , \qquad \mu^{cs}_{mn} = \frac{1}{n}\,\cos(my^1)\,(1 - cos(ny^2)) \tag{4.8.19}$$

$$\mu^{sc}_{mn} = \frac{1}{n}\,\sin(my^1)\,\sin(ny^2)\ , \qquad \mu^{cc}_{mn} = \frac{1}{n}\,\cos(my^1)\,\sin(ny^2) \tag{4.8.20}$$

relation (4.8.3) becomes :

$$\mu_{MNZT} = \sum_{m=0}^{\infty}\left(A^{s}_{m0Z}\,\mu^{s}_{m0}|_{MNT} + A^{c}_{m0Z}\,\mu^{c}_{m0}|_{MNT}\right) +$$

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(A^{ss}_{mnZ}\,\mu^{ss}_{mn}|_{MNT} + A^{cs}_{mnZ}\,\mu^{cs}_{mn}|_{MNT} + A^{sc}_{mnZ}\,\mu^{sc}_{mn}|_{MNT} + A^{cc}_{mnZ}\,\mu^{cc}_{mn}|_{MNT}\right) \tag{4.8.21}$$

For $T \geq IN$, the coefficients A are unknown.

For the B.E of the wake the dipole intensity is :

$$\mu_{MZT} = \sum_{m=0}^{\infty}\left(A^{s}_{m0Z}\,\overline{\mu}^{s}_{m0}|_{MT} + A^{c}_{m0Z}\,\overline{\mu}^{c}_{m0}|_{MT}\right) +$$

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(A^{ss}_{mnZ}\,\overline{\mu}^{ss}_{mn}|_{MT} + A^{cs}_{mnZ}\,\overline{\mu}^{cs}_{mn}|_{MT} + A^{sc}_{mnZ}\,\overline{\mu}^{sc}_{mn}|_{MT} + A^{cc}_{mnZ}\,\overline{\mu}^{cc}_{mn}|_{MT}\right) \tag{4.8.22}$$

with $M = 1, ..., M_0 + 1 \quad , \qquad T = 1, ..., IN - 1 \quad , \qquad Z = 1, ..., NBL$

Similarly for the tip and the hub:

$$\mu_{NZ}|_{tip,hub} \;=\; \sum_{m=0}^{\infty} (A^s_{m0Z}\,\overline{\mu}^s_{m0}|_{MN} \;+\; A^c_{m0Z}\,\overline{\mu}^c_{m0}|_{MN}) \;+$$

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} (A^{ss}_{mnZ}\,\overline{\mu}^{ss}_{mn}|_{MN} \;+\; A^{cs}_{mnZ}\,\overline{\mu}^{cs}_{mn}|_{MN} \;+\; A^{sc}_{mnZ}\,\overline{\mu}^{sc}_{mn}|_{MN} \;+\; A^{cc}_{mnZ}\,\overline{\mu}^{cc}_{mn}|_{MN}) \tag{4.8.23}$$

with $M = 1, M_0 + 2 \quad , \qquad N = 1, ..., N_0 \quad , \qquad Z = 1, ..., NBL$

For the T.E :

$$\mu_{MZ,IN}|_{T.E} \;=\; \sum_{m=0}^{\infty} (A^s_{m0Z}\,\overline{\mu}^s_{m0}|_{M,IN} \;+\; A^c_{m0Z}\,\overline{\mu}^c_{m0}|_{M,IN}) \;+$$

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} (A^{ss}_{mnZ}\,\overline{\mu}^{ss}_{mn}|_{M,IN} + A^{cs}_{mnZ}\,\overline{\mu}^{cs}_{mn}|_{M,IN} + A^{sc}_{mnZ}\,\overline{\mu}^{sc}_{mn}|_{M,IN} + A^{cc}_{mnZ}\,\overline{\mu}^{cc}_{mn}|_{M,IN}) \tag{4.8.24}$$

with $M = 1, ..., M_0 + 1 \quad , \qquad N = 1, ..., N_0 \quad , \qquad Z = 1, ..., NBL$

Notice that for the B.E of the wake, the dipole intensity $\overline{\mu}_{MZT}$, with T=IN-1, refers to the B.E of Kutta strip. Specifically in relation (4.8.13), the dipoles $\overline{\mu}^A_{MZT} = \mu_{M+1,Z,IN}$, $\overline{\mu}^B_{MZT} = \mu_{M,Z,IN}$ are on the trailing edge, so their value is unknown. Therefore we write:

$$-\frac{1}{4\pi}\int_{\bigcup_Z (S_{RW_Z}\cup S_{K_Z})} e^{jil}\,\gamma^i(Q)\,\frac{x^l(Q)-x^l(P)}{r^3}\,dS \;+\; \frac{1}{4\pi}\int_{\bigcup_Z \partial(S_{RW_Z}\cup S_{K_Z})} \mu\,e^{jil}\;\frac{x^l(Q)-x^l(P)}{r^3}\,dl^i \;=$$

$$= \sum_{M=1}^{M_0+1}\sum_{Z=1}^{NBL}\sum_{T=1}^{in-1} \frac{\overline{\mu}_{MZT}}{4\,\pi}\oint_{L_{MZT}} e^{jil}\,dl^i\,\frac{x^l(Q)-x^l(P)}{r^3} \;=$$

$$= \sum_{M=1}^{M_0+1}\sum_{Z=1}^{NBL} \frac{\overline{\mu}_{MZ,in-1}}{4\,\pi}\oint_{L_{MZ,in-1}} e^{jil}\,dl^i\,\frac{x^l(Q)-x^l(P)}{r^3} +$$

$$+ \sum_{M=1}^{M_0+1}\sum_{Z=1}^{NBL}\sum_{T=1}^{in-2} \frac{\overline{\mu}_{MZT}}{4\,\pi}\oint_{L_{MZT}} e^{jil}\,dl^i\,\frac{x^l(Q)-x^l(P)}{r^3} \;=$$

$$= \sum_{M=1}^{M_0+1}\sum_{Z=1}^{NBL} \widetilde{\mu}_{MZ,in-1}\,\vec{D}_{MZ,in-1} \;+\; \sum_{M=1}^{M_0+1}\sum_{Z=1}^{NBL}\sum_{T=1}^{in-1} \overline{\mu}_{MZT}\,\vec{D}_{MZT} \tag{4.8.25}$$

where

$$\widetilde{\mu}_{MZ,in-1} \overset{def}{=} \frac{1}{4}\left(\mu_{M+1,Z,in} + \mu_{MZ,in}\right) \overset{(4.8.10)}{=} \frac{1}{2}\,\overline{\mu}_{MZ,IN} \tag{4.8.26}$$

$$\overline{\mu}_{MZ,in-1} \overset{def}{=} \frac{1}{4}\left(\mu_{M+1,Z,in-1} + \mu_{MZ,in-1}\right) \tag{4.8.27}$$

In the program, due to limitations in the symbols, we set for $T < in-1$:

$$\vec{W}^{AB}_{MZT} = \vec{D}^{AB}_{MZT} \quad , \quad \vec{W}^{BC}_{MZT} = \vec{D}^{BC}_{MZT} \quad , \quad \vec{W}^{CD}_{MZT} = \vec{D}^{CD}_{MZT} \quad , \quad \vec{W}^{DA}_{MZT} = \vec{D}^{DA}_{MZT} \tag{4.8.28}$$

and

$$\vec{W}^{P} = \sum_{M=1}^{M_0+1}\sum_{Z=1}^{NBL}\sum_{T=1}^{in-1} \overline{\mu}_{MZT}\,\vec{D}_{MZT} \tag{4.8.29}$$

For $T = in-1$ (for the unknown dipole intensity):

$$\vec{E}^{AB}_{MZT} = \vec{D}^{AB}_{MZT} \quad , \quad \vec{E}^{BC}_{MZT} = \vec{D}^{BC}_{MZT} \quad , \quad \vec{E}^{CD}_{MZT} = \vec{D}^{CD}_{MZT} \quad , \quad \vec{E}^{DA}_{MZT} = \vec{D}^{DA}_{MZT} \tag{4.8.30}$$

and

$$\vec{E}^{P}_{MZT} = \vec{E}^{AB}_{MZT} + \vec{E}^{BC}_{MZT} + \vec{E}^{CD}_{MZT} + \vec{E}^{DA}_{MZT} \tag{4.8.31}$$

where P = Control Point

Similarly to (4.8.25) the expression for the blade's outline (tip, hub, T.E, L.E) is :

$$\frac{1}{4\pi}\int\limits_{\bigcup_Z \partial S_{R_Z}} \mu\, e^{jil}\,\frac{x^l(Q)-x^l(P)}{r^3} =$$

$$= \sum_{N=0}^{N_0}\sum_{Z=1}^{NBL}\left[\frac{\overline{\mu}^{tip}_{NZ}}{4\pi}\int\limits_{L^{tip}_{NZ}} e^{jil}\,dl^i\,\frac{x^l(Q)-x^l(P)}{r^3} + \frac{\overline{\mu}^{hub}_{NZ}}{4\pi}\int\limits_{L^{hub}_{NZ}} e^{jil}\,dl^i\,\frac{x^l(Q)-x^l(P)}{r^3}\right] =$$

$$\sum_{M=1}^{M_0+1}\sum_{Z=1}^{NBL}\left[+\frac{\overline{\mu}^{T.E}_{MZ,IN}}{4\pi}\int\limits_{L^{T.E}_{MZ,IN}} e^{jil}\,dl^i\,\frac{x^l(Q)-x^l(P)}{r^3}\right] =$$

$$= \sum_{M=1,M0+2}\sum_{N=0}^{N_0}\sum_{Z=1}^{NBL}\overline{\mu}_{MNZ}\,\vec{D}_{MNZ} + \sum_{M=1}^{M_0+1}\sum_{Z=1}^{NBL}\overline{\mu}_{MZ,IN}\,\vec{D}_{MZ,IN} \tag{4.8.32}$$

In the program we set :

$$\vec{E}^{P}_{MNZ} = \vec{D}^{AB}_{MNZ}\,, \qquad \vec{E}^{P}_{MZ,IN} = -\vec{D}^{AB}_{MZ,IN} \tag{4.8.33}$$

Substituting (4.8.25) and (4.8.32) to relation (4.8.1) becomes :

$$\rangle v^j_W(P)\langle = -\frac{1}{4\pi} \int\limits_{\bigcup_Z \partial(S_{RW_Z} \cup S_{K_Z})} e^{jil}\, \gamma^i \, \frac{x^l(Q) - x^l(P)}{r^3}\, dy^1 dy^2$$

$$+ \frac{1}{4\pi} \int\limits_{\bigcup_Z \partial(S_{RW_Z} \cup S_{K_Z})} \mu\, e^{jil}\; \frac{x^l(Q) - x^l(P)}{r^3}\, dl^i \,+\, \frac{1}{4\pi} \int\limits_{\bigcup_Z \partial S_{R_Z}} \mu\, e^{jil}\; \frac{x^l(Q) - x^l(P)}{r^3}\, dl^i \;=$$

$$= \sum_{M=1,M0+2} \sum_{N=0}^{N_0} \sum_{Z=1}^{NBL} \overline{\mu}_{MNZ}\, \vec{E}^P_{MNZ} \,+\, \sum_{M=1}^{M_0+1} \sum_{Z=1}^{NBL} \widetilde{\mu}_{M,N_0+1,Z}\, (\vec{E}^P_{M,N_0+1,Z} - 2\vec{D}_{MZ,IN}) \,+ \vec{W}^P \quad (4.8.34)$$

where $$\widetilde{\mu}_{M,N_0+1,Z} \,=\, \widetilde{\mu}_{MZ,in-1}\,, \qquad \overline{\mu}_{M,N_0+1,Z} \,=\, \overline{\mu}_{MZ,in-1} \,=\, 2\,\widetilde{\mu}_{MZ,in-1} \quad (4.8.35)$$

Analyzing the $\overline{\mu}_{MNZ}$ unknowns with respect to their harmonics, using relations (4.8.18) - (4.8.20) we get:

$$T\vec{H}T^s_{m0Z} \;=\; \sum_{M=1,M_0+2} \sum_{N=1}^{N0} \overline{\mu}^s_{m0}|_{MN}\, \vec{E}^P_{MNZ} \,+\, \sum_{M=1}^{M_0+1} \widetilde{\mu}^s_{m0}|_{M,N_0+1}\, \vec{E'}^P_{M,N_0+1,Z} \quad (4.8.36)$$

$$T\vec{H}T^c_{m0Z} \;=\; \sum_{M=1,M_0+2} \sum_{N=1}^{N0} \overline{\mu}^c_{m0}|_{MN}\, \vec{E}^P_{MNZ} \,+\, \sum_{M=1}^{M_0+1} \widetilde{\mu}^c_{m0}|_{M,N_0+1}\, \vec{E'}^P_{M,N_0+1,Z} \quad (4.8.37)$$

$$T\vec{H}T^{ss}_{mnZ} \;=\; \sum_{M=1,M_0+2} \sum_{N=1}^{N0} \overline{\mu}^{ss}_{mn}|_{MN}\, \vec{E}^P_{MNZ} \,+\, \sum_{M=1}^{M_0+1} \widetilde{\mu}^{ss}_{mn}|_{M,N_0+1}\, \vec{E'}^P_{M,N_0+1,Z} \quad (4.8.38)$$

$$T\vec{H}T^{cs}_{mnZ} \;=\; \sum_{M=1,M_0+2} \sum_{N=1}^{N0} \overline{\mu}^{cs}_{mn}|_{MN}\, \vec{E}^P_{MNZ} \,+\, \sum_{M=1}^{M_0+1} \widetilde{\mu}^{cs}_{mn}|_{M,N_0+1}\, \vec{E'}^P_{M,N_0+1,Z} \quad (4.8.39)$$

$$T\vec{H}T^{sc}_{mnZ} \;=\; \sum_{M=1,M_0+2} \sum_{N=1}^{N0} \overline{\mu}^{sc}_{mn}|_{MN}\, \vec{E}^P_{MNZ} \,+\, \sum_{M=1}^{M_0+1} \widetilde{\mu}^{sc}_{mn}|_{M,N_0+1}\, \vec{E'}^P_{M,N_0+1,Z} \quad (4.8.40)$$

$$T\vec{H}T^{cc}_{mnZ} \;=\; \sum_{M=1,M_0+2} \sum_{N=1}^{N0} \overline{\mu}^{cc}_{mn}|_{MN}\, \vec{E}^P_{MNZ} \,+\, \sum_{M=1}^{M_0+1} \widetilde{\mu}^{cc}_{mn}|_{M,N_0+1}\, \vec{E'}^P_{M,N_0+1,Z} \quad (4.8.41)$$

where

$$\vec{E'}^P_{M,N_0,Z} \,=\, \vec{E}^P_{M,N_0,Z} \,-\, 2\,\vec{D}_{MZ,IN} \,=\, -\vec{E}^{AB}_{MZT} \,+\, \vec{E}^{BC}_{MZT} \,+\, \vec{E}^{CD}_{MZT} \,+\, \vec{E}^{DA}_{MZT} \quad (4.8.42)$$

Note that the Z in $T\vec{H}T^{cc}_{mnZ}$ refers to the blade of the integration point Q and not the blade of the control point P.

Relations (4.8.32) - (4.8.36) give the contribution of the blade's boundary (tip,hub,TE,LE) of the Z-th blade to the perturbation at P. Substituting to (4.8.34)-(4.8.42) we get :

$$\rangle v^j_W(P)\langle = -\frac{1}{4\pi} \int\limits_{\bigcup_Z S_{RW_Z}} e^{jil}\,\gamma^i\,\frac{x^l(Q)-x^l(P)}{r^3}\,dy^1dy^2 \;+\; \frac{1}{4\pi}\int\limits_{\bigcup_Z L_Z} \mu\,e^{jil}\;\frac{x^l(Q)-x^l(P)}{r^3}\,dl^i \;=$$

$$= \sum_{Z=1}^{NBL}\left[\sum_{m=0}^{\infty}\left(A^s_{m0Z}\,THT^{js}_{m0Z} \,+\, A^c_{m0Z}\,THT^{jc}_{m0Z}\right) \,+\right.$$

$$+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(A^{ss}_{mnZ}\,THT^{jss}_{mnZ} \,+\, A^{cs}_{mnZ}\,THT^{jcs}_{mnZ} \,+\right.$$

$$\left.\left. +\; A^{sc}_{mnZ}\,THT^{jsc}_{mnZ} \,+\, A^{cc}_{mnZ}\,THT^{jcc}_{mnZ}\right)\right] \,+\, W^{jP} \qquad (4.8.43)$$

The above relation refers to the contribution of the wake and the blades' boundary to the perturbation velocity at a single C.P. Notice that each blade contributes with each own coefficients A_{mnZ}.

4.9 Calculation of Coefficients A

As we've seen in the previous chapters the total perturbation velocity is the sum of perturbation velocities generated by the vorticity and sources on the reference surfaces of the blades as well as the wake vorticity.

$$\rangle v^j(P)\langle = \rangle v^j_\Gamma(P)\langle + \rangle v^j_S(P)\langle + \rangle v^j_W(P)\langle \tag{4.9.1}$$

Using relations (4.5.57),(4.6.9) and (4.8.43) :

$$\rangle v^j_\Gamma(P)\langle = \sum_{z=1}^{NBL}\left[\sum_{m=0}^{\infty}\left(T^{sj}_{m0z}\,A^s_{m0z} + T^{sj}_{m0z}\,A^s_{m0z}\right) + \right.$$

$$\left. + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(T^{ssj}_{mnz}\,A^{ss}_{mnz} + T^{csj}_{mnz}\,A^{cs}_{mnz} + T^{scj}_{mnz}\,A^{sc}_{mnz} + T^{ccj}_{mnz}\,A^{cc}_{mnz}\right)\right] \tag{4.9.2}$$

$$\rangle v^j_S(P)\langle = -\frac{1}{4\pi}\sum_{z=1}^{NBL}\left[\frac{c^2}{4}(R_o - R_H)\int_0^\pi \frac{D^j_z - D^j_{self}}{\cos y^2 - \cos y^2_P}\,dy^2\right] \tag{4.9.2}$$

$$\rangle v^j_W(P)\langle = THT^{jP} + W^{jP} \tag{4.9.3}$$

where

$$THT^{jP} = \sum_{Z=1}^{NBL}\left[\sum_{m=0}^{\infty}\left(A^s_{m0z}\,THT^{js}_{m0z} + A^c_{m0z}\,THT^{jc}_{m0z}\right) + \right.$$

$$\left. + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(A^{ss}_{mnz}\,THT^{jss}_{mnz} + A^{cs}_{mnz}\,THT^{jcs}_{mnz} + A^{sc}_{mnz}\,THT^{jsc}_{mnz} + A^{cc}_{mnz}\,THT^{jcc}_{mnz}\right)\right] \tag{4.9.4}$$

The no-entrance boundary condition for the camber problem, (3.2.33), becomes :

$$\underset{\sim}{c}^3_1\,\rangle v^1\langle + \underset{\sim}{c}^3_2\,\rangle v^2\langle + \underset{\sim}{c}^3_3\,\rangle v^3\langle = \left(\rangle \underline{q}^1\langle \frac{\partial E_c}{\partial u^1} + \rangle \underline{q}^2\langle \frac{\partial E_c}{\partial u^2} - \rangle \underline{q}^3\langle\right) \Rightarrow$$

$$\Rightarrow \underset{\sim}{c}^3_1\left(\rangle v^1_\Gamma\langle + THT^{1P}\right) + \underset{\sim}{c}^3_2\left(\rangle v^2_\Gamma\langle + THT^{2P}\right) + \underset{\sim}{c}^3_3\left(\rangle v^3_\Gamma\langle + THT^{3P}\right) =$$

$$= -\left[\underset{\sim}{c}^3_1\left(\rangle v^1_S\langle + W^{1P}\right) + \underset{\sim}{c}^3_2\left(\rangle v^2_S\langle + W^{2P}\right) + \underset{\sim}{c}^3_3\left(\rangle v^3_S\langle + W^{3P}\right)\right] + \rangle \underline{q}^1\langle \frac{\partial E_c}{\partial u^1} + \rangle \underline{q}^2\langle \frac{\partial E_c}{\partial u^2} - \rangle \underline{q}^3\langle \Rightarrow$$

$$\Rightarrow \sum_{Z=1}^{NBL} \Bigg[\sum_{m=0}^{\infty} (VT^{s}_{m0Z} A^{s}_{m0Z} + VT^{c}_{m0Z} A^{c}_{m0Z}) +$$

$$+ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (VT^{cs}_{mnZ} A^{ss}_{mnZ} + VT^{cs}_{mnZ} A^{cs}_{mnZ} + VT^{sc}_{mnZ} A^{sc}_{mnZ} + VT^{cc}_{mnZ} A^{cc}_{mnZ}) \Bigg] = B \qquad (4.9.5)$$

where

$$VT^{s}_{m0Z} = \overset{\sim}{\underline{c}}{}^{3}_{1} (T^{s1}_{m0Z} + THT^{s1}_{m0Z}) + \overset{\sim}{\underline{c}}{}^{3}_{2} (T^{s2}_{m0Z} + THT^{s2}_{m0Z}) + \overset{\sim}{\underline{c}}{}^{3}_{3} (T^{s3}_{m0Z} + THT^{s3}_{m0Z}) \qquad (4.9.6)$$

$$VT^{c}_{m0Z} = \overset{\sim}{\underline{c}}{}^{3}_{1} (T^{c1}_{m0Z} + THT^{c1}_{m0Z}) + \overset{\sim}{\underline{c}}{}^{3}_{2} (T^{c2}_{m0Z} + THT^{c2}_{m0Z}) + \overset{\sim}{\underline{c}}{}^{3}_{3} (T^{c3}_{m0Z} + THT^{c3}_{m0Z}) \qquad (4.9.7)$$

$$VT^{ss}_{mnZ} = \overset{\sim}{\underline{c}}{}^{3}_{1} (T^{ss1}_{mnZ} + THT^{ss1}_{mnZ}) + \overset{\sim}{\underline{c}}{}^{3}_{2} (T^{ss2}_{mnZ} + THT^{ss2}_{mnZ}) + \overset{\sim}{\underline{c}}{}^{3}_{3} (T^{ss3}_{mnZ} + THT^{ss3}_{mnZ}) \qquad (4.9.8)$$

$$VT^{cs}_{mnZ} = \overset{\sim}{\underline{c}}{}^{3}_{1} (T^{cs1}_{mnZ} + THT^{cs1}_{mnZ}) + \overset{\sim}{\underline{c}}{}^{3}_{2} (T^{cs2}_{mnZ} + THT^{cs2}_{mnZ}) + \overset{\sim}{\underline{c}}{}^{3}_{3} (T^{cs3}_{mnZ} + THT^{cs3}_{mnZ}) \qquad (4.9.9)$$

$$VT^{sc}_{mnZ} = \overset{\sim}{\underline{c}}{}^{3}_{1} (T^{sc1}_{mnZ} + THT^{sc1}_{mnZ}) + \overset{\sim}{\underline{c}}{}^{3}_{2} (T^{sc2}_{mnZ} + THT^{sc2}_{mnZ}) + \overset{\sim}{\underline{c}}{}^{3}_{3} (T^{sc3}_{mnZ} + THT^{sc3}_{mnZ}) \qquad (4.9.10)$$

$$VT^{cc}_{mnZ} = \overset{\sim}{\underline{c}}{}^{3}_{1} (T^{cc1}_{mnZ} + THT^{cc1}_{mnZ}) + \overset{\sim}{\underline{c}}{}^{3}_{2} (T^{cc2}_{mnZ} + THT^{cc2}_{mnZ}) + \overset{\sim}{\underline{c}}{}^{3}_{3} (T^{cc3}_{mnZ} + THT^{cc3}_{mnZ}) \qquad (4.9.11)$$

and

$$B = -\Big[\overset{\sim}{\underline{c}}{}^{3}_{1} \left(\rangle v^{1}_{S} \langle + W^{1P} \right) + \overset{\sim}{\underline{c}}{}^{3}_{2} \left(\rangle v^{2}_{S} \langle + W^{2P} \right) + \overset{\sim}{\underline{c}}{}^{3}_{3} \left(\rangle v^{3}_{S} \langle + W^{3P} \right) \Big] +$$

$$+ \rangle \underline{q}^{1} \langle \frac{\partial E_c}{\partial u^1} + \rangle \underline{q}^{2} \langle \frac{\partial E_c}{\partial u^2} - \rangle \underline{q}^{3} \langle \qquad (4.9.12)$$

Notice that the VT factors of relations (4.9.6) - (4.9.11) depend only on the geometry of the body which in this case is the reference surface and its boundary. If the geometry doesn't change with time the relative position between the reference surfaces and their form (e.g the metric) stays the same, which is the case of the propeller and the wing. This means that we only have to calculate them one time. In case of biomimetic movements like the flapping of a bird's wings and movement of a jellyfish, these factors change at each time step, however they might be periodic with time.

Relation (4.9.5) refers to a particular point on a blade and the contribution of all the blades (as well as the wakes) to it. Having calculated the factors VT for each control point of the reference blade, when the C.P moves to a different blade, we don't have to repeat the same process. Specifically due to the symmetry in the propeller's blades around the rotation axis, C.P at similar positions (same y^1,y^2) but on different blades we have the same VT factors.

For the C.P on the reference surface (Z=1), we calculate the factors VT with the integration point Q on all KZ=1,...,NBL blades. [8] For the C.P on the Z-th blade the following relation is true :

[8] In the program the variable Z is reserved for the C.P while KZ is for the integration point,Q. However in relations (4.9.5)-(4.9.11), Z refers to the integration point

$$VT_{mn,KZ}|_{C.P\ at\ Z} = VT_{mn,KZZ}|_{C.P\ at\ Z=1} \tag{4.9.13}$$

where
$$KZZ = KZ + NBL - (Z - 1) \tag{4.9.14}$$

with Z,KZ = 1,..., NBL

If (KZZ > NBL) then we set KZZ = KZZ - NBL [9] (4.9.15)

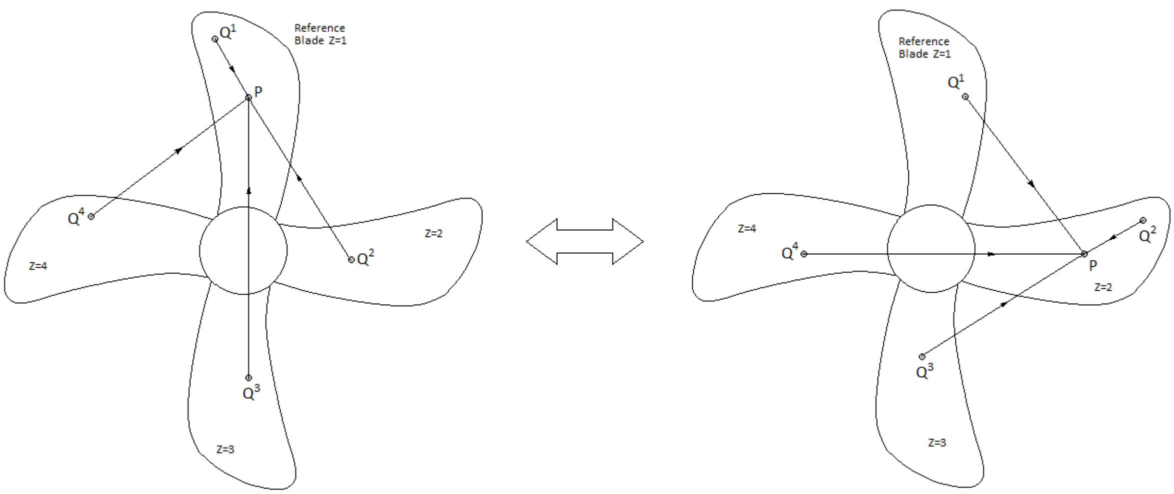


Figure 4.9.1

In the program we assume that $\underline{q}^i + \rangle \underline{v}^i \langle = (\underline{q}^2 + \rangle \underline{v}^2 \langle)\, \delta_2^i$, so using the relation (4.2.21) for the bound vorticity, relation (4.9.5) becomes :

$$\sum_{Z=1}^{NBL} \left[\sum_{m=1}^{\infty} (VT^{s}_{m0Z} A^{s}_{m0Z}) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (VT^{cs}_{mnZ} A^{ss}_{mnZ}) \right] = B \tag{4.9.16}$$

where

$$B = - \left[\overset{\sim 3}{\underline{c}_1} \left(\rangle v_S^1 \langle + W^{1P} \right) + \overset{\sim 3}{\underline{c}_2} \left(\rangle v_S^2 \langle + W^{2P} \right) + \overset{\sim 3}{\underline{c}_3} \left(\rangle v_S^3 \langle + W^{3P} \right) \right] +$$

$$+ \rangle \underline{q}^1 \langle \frac{\partial E_c}{\partial u^1} + \rangle \underline{q}^2 \langle \frac{\partial E_c}{\partial u^2} - \rangle \underline{q}^3 \langle \tag{4.9.17}$$

[9] $KZZ = MOD(KZ + NBL - (Z - 1), NBL)$

The satisfaction of the dynamic condition at the trailing edge is given by relation (4.3.23):

For m = 1,2,... ($\sin(my^1)$) :

$$P\,A^{s}_{m0Z,IN}\frac{\pi}{2} + \sum_{n=1}^{\infty}\left(P\,A^{ss}_{mnZ,IN}\frac{1-(-1)^n}{n}\right) + \underset{\sim}{\tilde{q}}^2\sum_{n=1}^{\infty}2\,A^{ss}_{mnZ,IN}\,n\,(-1)^{n+1} =$$

$$= -\left[Q^{s}_{m0Z,IN}\frac{\pi}{2} + \sum_{n=1}^{\infty}Q^{ss}_{mnZ,IN}\frac{1-(-1)^n}{n}\right] \tag{4.9.18}$$

For the points on the trailing edge we set :

$$VT^{s}_{m0Z} = P\,\frac{\pi}{2} \qquad\qquad VT^{ss}_{mnZ} = P\,\frac{1-(-1)^n}{n} + 2\,\underset{\sim}{\tilde{q}}^2\,n(-1)^{n+1} \tag{4.9.19}$$

$$B = -\left[Q^{s}_{m0Z,IN}\,\frac{\pi}{2} + \sum_{n=1}^{\infty}Q^{ss}_{mnZ,IN}\,\frac{1-(-1)^n}{n}\right] \tag{4.9.20}$$

so relation (4.9.18) becomes :

$$VT^{s}_{m0Z}A^{s}_{m0Z} + \sum_{n=1}^{\infty}(VT^{ss}_{mnZ}A^{ss}_{mnZ}) = B \quad , \qquad m = 1,2,... \tag{4.9.21}$$

Assuming that $m = 1, 2, ..., M_0$, $n = 1, 2, ..., N_0$ and using relations (4.7.2) and (4.7.6), relation (4.9.16) becomes:

$$\sum_{l=1}^{MNT} VT(k,l)\,A^{s}_{l} = B(k) \quad , \qquad MNT = NBL \cdot M_0 \cdot (N_0+1) \tag{4.9.22}$$

with

$$k = 1, 2, ..., NBL(M_0+2)(N_0+1), \qquad\qquad j \neq N_0+1 \tag{4.9.23}$$

and $VT(k,l)$, $B(k)$ defined by relations (4.9.6) - (4.9.12).

Similarly relation (4.9.21) for the trailing edge becomes :

$$\sum_{n=1}^{N_0} VT(k,l(n))\,A^{s}_{l(n)} = B(k) \tag{4.9.24}$$

where $VT(k,l)$, $B(k)$ are defined by relations (4.9.19),(4.9.20) when the control point is on the trailing edge (not the tip and hub). $l(n)$ is a function of the chordwize harmonic and it's calculated by relation (4.7.6) by setting for $m = i-1$, where i= spanwise position of the C.P in relation (4.7.2). For $m \neq i-1$ on the trailing edge, $VT(k,l) = 0$.

5 Calculation of Forces

5.1 Forces on the camber surface

Let A be the surface of a body and V the volume enclosed in it. Let $\vec{n}$ be the normal vector on the surface pointing outwards, then for constant pressure p_∞, the net force acting on it is :

$$\vec{L} \,=\, -\int\limits_A p_\infty \,\vec{n}\, dA \,=\, -\int\limits_V \nabla p_\infty \, dV \,=\, 0 \tag{5.1.1}$$

In the case of the wing where the upper and lower surface of the are near the reference surface we get :

$$\vec{L} \,=\, -\int\limits_A p\vec{n}\, dA \,=\, -\int\limits_{A^+} (p^+ - p_\infty)\,\vec{n}^+\, dA \,-\, \int\limits_{A^-} (p^- - p_\infty)\,\vec{n}^-\, dA \,=$$

$$=\, -\int\limits_{A_R} (p^+ - p_\infty)\,\vec{n}^+ \,+\, (p^- - p_\infty)\,\vec{n}^-\, dA \,+\, O(\varepsilon^2) \tag{5.1.2}$$

Setting $\tilde{\underline{N}}_i$ the normal vector on the real surface as described in section 3.2, the force acting on the reference surface per unit of area is :

$$d\underline{L}_i \,=\, -\,p^+\, \tilde{\underline{N}}_i^{+} \,-\, p^-\, \tilde{\underline{N}}_i^{-} \,=$$

$$=\, -\,(p^+ \,-\, p^-)\big(\, \overset{o}{\tilde{\underline{N}}}{}_i^{+} \,+\, \overset{c}{\tilde{\underline{N}}}{}_i^{-}\,\big) \,-\, \big((p^+ \,-\, p_\infty) \,+\, (p^- \,-\, p_\infty)\big)\, \overset{t}{\tilde{\underline{N}}}{}_i^{+}/2 \tag{5.1.3}$$

where

$$\frac{p - p_\infty}{\rho} \,+\, \frac{\partial \Phi}{\partial t} \,+\, \frac{1}{2}(\vec{v} + \vec{V}_\infty)^2 \,-\, \frac{1}{2}\vec{V}_\infty^2 \,=\, 0 \Rightarrow p \,-\, p_\infty \,=\, O(\varepsilon) \tag{5.1.4}$$

Since we don't calculate the the absolute values of pressure but only their differences we omit the last term of (5.1.2) as a second order term. This term has significant contribution to the total force only near the leading edge, where the linearized theory fails to give accurate results. Moreover the description of $\overset{t}{\tilde{\underline{N}}}{}_i^{+}$ in section 3.2 is not valid near the leading edge due to the infinite slope of thickness at this region. Therefore, the contribution of the leading edge will be calculated directly from the results of complex analysis of 2D airfoils and will be added as a correction in the final result.

The approximation of the normal vector on the camber surface in the $\underline{x}^i$ system is :

$$\big(\, \overset{o}{\tilde{\underline{N}}}{}_i^{+} \,+\, \overset{c}{\tilde{\underline{N}}}{}_i^{-}\,\big) \,=\, \left\{-\frac{\partial E_c}{\partial u^1}, -\frac{\partial E_c}{\partial u^1}, 1\right\} \sqrt{\frac{\tilde{\underline{g}}}{\underline{\alpha}}} \tag{5.1.5}$$

the same vector in the cartesian system is $\underset{\sim}{c}^{i}_{j}\,\underline{\tilde{N}}_i$. The net force in the Cartesian system is :

$$L_i = -\sum_{Z=1}^{NBL} \iint_{S_{R_Z}} (p^+ - p^-)\, \tilde{N}_i\, dS = -\sum_{Z=1}^{NBL} \iint_{S_{R_Z}} (p^+ - p^-)\, (\tilde{N}_i \sqrt{\underset{\smile}{\alpha}})\, dy^1\, dy^2$$

$$= -\sum_{Z=1}^{NBL} \int_0^{\pi}\int_0^{\pi} (p^+ - p^-)\, (\tilde{n}_i \sqrt{\underline{\tilde{g}}})\, \sqrt{\underset{\smile}{\underline{\alpha}}}\, dy^1\, dy^2 \tag{5.1.6}$$

For the orthogonal Cartesian system $L^i = L_i = L_i|_{physical}$.

In order to calculate the pressure difference on the reference surface we use the Bernoulli 's equation (3.4.17) :

$$\frac{p^+ - p^-}{\rho} = -\int_{-1/2}^{\underline{x}^2} \frac{d\underline{\Gamma}^1}{dt}\, d\underline{x}^2 - \underline{\tilde{q}}^2\, \underline{\Gamma}^1 + \underline{\tilde{q}}^1\, \underline{\Gamma}^2 - \left(\underline{\tilde{q}}^3 \sqrt{\frac{\underline{\tilde{g}}}{\underline{\alpha}}}\sigma \right) + O(\varepsilon^2) =$$

$$= \sum_{m=0}^{\infty} (A^s_{m0Z}\, VT^s_{m0Z} + A^c_{m0Z}\, VT^c_{m0Z}) +$$

$$+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} (A^{ss}_{mnZ}\, VT^{ss}_{mnZ} + A^{cs}_{mnZ}\, VT^{cs}_{mnZ} + A^{sc}_{mnZ}\, VT^{sc}_{mnZ} + A^{cc}_{mnZ}\, VT^{cc}_{mnZ}) - B \tag{5.1.7}$$

where $VT^s_{mnZ}, ..., VT^{cc}_{mnZ}, B$ are the coefficients defined in section (4.9) for the trailing edge.

5.2 Correction for the leading edge suction force

In the case of a flat plate (zero camber and thickness) the force calculated by relation (5.1.6) is always perpendicular to the surface. However in the case of a 2D airfoil (span = ∞) according to Blasius' theorem, the force is normal to the undisturbed flow velocity. This happens because apart from $\vec{L}$, there is a force parallel to the plate, so that the total force is normal to the flow at infinity.

This extra component is called leading edge suction force and cannot be calculated by the lifting surface (or lifting line) theory, since its a result of infinite pressure and infinitesimal surface area at the leading edge. In the case of a 2D airfoil the leading edge suction force is calculated using complex functions and a proof can be found in [7] :

$$LE_{SF}^{2D} \;=\; -\rho\,(Q_\infty \sin\alpha)\Gamma \tag{5.2.1}$$

where Q_∞ is the velocity at infinity, α the angle of attack and Γ the circulation around the airfoil. Using the symbols in this work we get :

$$LE_{SF}^{2D} \;=\; -\rho\,\underset{\sim}{\underline{q}}{}^{3}_{[\phi]}\,\Gamma \tag{5.2.2}$$

The above relations are valid for a uniform flow around the body which is not always the case for our problem. Since in the proof the velocities around the leading edge are used, we set $\underline{q}^3_{[\phi]}$ to be the velocity at the leading edge of the chordwise section.

The total leading edge suction force is :

$$LE_{SF}^{i} \;=\; \sum_{Z=1}^{NBL} \int_0^{\pi} LE_{SF}^{2D}\,\varepsilon^{i}_{(\underline{2})}\left(\frac{1}{2}\,span\,\sin y^1\right) dy^1 \tag{5.2.3}$$

where $\varepsilon^{i}_{(\underline{2})}$ is the unit vector of the $\underline{x}^2$ axis with coordinates in the x^i system.

$$\varepsilon^{i}_{(\underline{2})} \;=\; \underline{\varepsilon}^{j}_{(2)}\,\tilde{\underline{\gamma}}{}^{i}_{j} \;=\; \left(\delta^j_2\,\frac{1}{\sqrt{\tilde{\underline{g}}_{22}}}\right)\tilde{\underline{\gamma}}{}^{i}_{j} \;=\; \tilde{\underline{\gamma}}{}^{i}_{2}\,/\,\sqrt{\tilde{\underline{g}}_{22}} \tag{5.2.4}$$

and

$$\tilde{\underline{q}}{}^{(3)}_{[\phi]} \;=\; \tilde{\underline{g}}_{r3}\,\tilde{\underline{q}}{}^{r}\,/\,\sqrt{\tilde{\underline{g}}_{33}} \;=\; \tilde{\underline{g}}_{11}\,\tilde{\underline{q}}{}^{1}\,/\,\sqrt{\tilde{\underline{g}}_{33}} \;+\; \sqrt{\tilde{\underline{g}}_{33}}\,\tilde{\underline{q}}{}^{r} \tag{5.2.5}$$

since $\tilde{\underline{g}}_{23} = 0$

6 Numerical Integration

6.1 Filon Method for integration f(x)cos(px), f(x)sin(px), p=0,1,... .[11]

The type of integrals that appear repeatedly throughout the calculations are :

$$I = \int_a^b f(x)\cos(px)dx\ , \qquad p = 0,1,2,... \tag{6.1.1}$$

$$J = \int_a^b f(x)\sin(px)dx\ , \qquad p = 1,2,... \tag{6.1.2}$$

The numerical calculation of these kind of integrals with the usual Simpson's rule or other similar methods is quite difficult, especially when p is not small. This happens due to the rapid oscillations of the cos(px) and sin(px) terms, meaning that a great number of integration points is required in order to achieve a decent precision.

Filon's method uses the simple idea that the function f(x) (without the trigonometric factor) can be approximated by second order polynomials. Therefore the number of integration points needed for the Filon method are the same as those needed for the integration of f(x). Essentially, Filon's method is a modified version of the Simpson's rule, which takes into account the trigonometric factors cos(px), sin(px) in its coefficients. The details of the method can be found in [11], while a generalization of it using higher order Lagrange polynomials is discussed in the following section.

In the program we use Filon's method for a second and first order polynomial interpolation of f(x). Specifically, if n (n is assumed an odd number) the number of integration points in the interval $[a, b]$,

$$h = \frac{b-a}{n-1} \tag{6.1.3}$$

integration length and

$$\theta = p\,h \tag{6.1.4}$$

then, quantities I,J are given by the relations

$$I = h\ [\alpha\,(f_n\sin(pb) - f_1\sin(pa))\ +\ \beta C_{2k-1}\ +\ \gamma C_{2k}] \tag{6.1.5}$$

$$J = h\ [-\alpha\,(f_n\cos(pb) - f_1\cos(pa))\ +\ \beta S_{2k-1}\ +\ \gamma S_{2k}] \tag{6.1.6}$$ [10]

where

[10] Don't confuse the greek coefficients α, β, γ with the latin boundary values a,b of the interval $[a, b]$.

$$C_{2k-1} = \sum_{i=3(2)}^{n-2} f_i \cos(px_i) + \frac{1}{2} (f_1 \cos(pa) + f_n \cos(pb)) \qquad (6.1.7)$$ [11]

$$C_{2k} = \sum_{i=2(2)}^{n-1} f_i \cos(px_i) \qquad (6.1.8)$$

$$S_{2k-1} = \sum_{i=3(2)}^{n-2} f_i \sin(px_i) + \frac{1}{2} (f_1 \sin(pa) + f_n \sin(pb)) \qquad (6.1.9)$$

$$S_{2k} = \sum_{i=2(2)}^{n-1} f_i \sin(px_i) \qquad (6.1.10)$$

Furthermore we use the following notation :

$$x_i \equiv a + (i-1)\, h\,, \qquad i = 1, ...n$$

$$f_i \equiv f(x_i)$$

Obviously $x_1 = a$, $x_n = b$. The coefficients a, b, γ are given by the relations :

$$a = \left(\theta^2 + \theta \sin\theta \cos\theta - 2 \sin^2\theta\right) / \theta^3 \qquad (6.1.11)$$

$$b = 2\left[\theta\left(1 + \cos^2\theta\right) - 2 \sin\theta \cos\theta\right] / \theta^3 \qquad (6.1.12)$$

$$\gamma = 4\,(\sin\theta - \theta\cos\theta) / \theta^3 \qquad (6.1.13)$$

For the small values of θ we need to use the Taylor's expansion for $\sin\theta$, $\cos\theta$. Therefore (6.1.11)-(6.1.13) can be written as:

$$a = \frac{2\theta^3}{45} - \frac{2\theta^5}{315} + \frac{2\theta^7}{4725} - \frac{8\theta^9}{467775} + ... \qquad (6.1.14)$$

$$b = \frac{2}{3} + \frac{2\theta^2}{15} - \frac{4\theta^4}{105} + \frac{2\theta^6}{567} - ... \qquad (6.1.15)$$

$$\gamma = \frac{4}{3} - \frac{2\theta^2}{15} + \frac{\theta^4}{210} - \frac{\theta^6}{11340} + ... \qquad (6.1.16)$$

For $p = 0$ and thus for $\theta = 0$, we get $\alpha = 0$, $\beta = 2/3$, $\gamma = 4/3$ and relation (6.1.5) becomes :

$$I = \frac{h}{3}\left[2\, C_{2i-1} + 4\, C_{2i}\right] \qquad (6.1.17)$$

which is the known Simplon's rule.

[11] $\sum_{i=3(2)}^{n-1}$ is the summation for the index i with initial value 3, step 2 and final value n-1

6.2 General Filon Method

Here we will examine only the integral

$$I = \int_a^b f(x)\, \cos(px) dx \tag{6.2.1}$$

the same are valid for $J = \int_a^b f(x)\, \sin(px) dx$

Suppose we approximate $f(x)$ using series of Lagrange polynomials of order m $(m = 1, 2, 3, ..)$. We divide the integration interval $[a, b]$ in n equal parts of length :

$$h = \frac{b-a}{n} \tag{6.2.2}$$

where $n = k \cdot m$, $k = 1, 2, ...$

Then relation (6.2.1) becomes : $$I = \sum_{i=m(m)}^{n} I_i \tag{6.2.3}$$

with $$I_i = \int_{x_{i-m}}^{x_i} f(x) \cos(px)\, dx \quad , \qquad i = m, 2m, ..., n \tag{6.2.4}$$

The approximation of $f(x)$ in $[x_{i-m}, x_i]$ with a Lagrange polynomial of order m, is :

$$f(x) = \sum_{j=i-m}^{i} f(x_j)\, l_j(x) \tag{6.2.5}$$

where $$l_j(x) = \frac{(x - x_{i-m})...(x - x_{j-1})(x - x_{j+1})...(x - x_i)}{(x_j - x_{i-m})...(x_j - x_{j-1})(x_j - x_{j+1})...(x_j - x_i)} \tag{6.2.6}$$

Substituting relation (6.2.5) to (6.2.4), we get :

$$I_i = \int_{x_{i-m}}^{x_i} \sum_{j=i-m}^{i} f(x_j)\, l_j(x)\, \cos(px)\, dx \equiv \sum_{j=i-m}^{i} J_j \tag{6.2.7}$$

We should therefore calculate the integrals :

$$J_j = \int_{x_{i-m}}^{x_i} f_j\, l_j(x) \cos(px)\, dx \quad , \qquad j = i-m,\, i-m+1, ..., i \tag{6.2.8}$$

where

$$f_j \equiv f(x_j)\,, \qquad x_j \;=\; a + j\,h \qquad (j = i-m, ..., i \quad \text{and} \quad i = m, 2m, ..., k \cdot n)$$

and for i=m, j=i-m : $x_0 \equiv a$, i=n, j=i : $x_n \equiv b$

Integrating by parts multiple time, relation (6.2.8) becomes (for simplicity we set $i_1 \equiv i-1$, $i_{m1} \equiv i - m + 1$) :

$$J_i = p^{-1}\, f_i\, \sin px_i +$$

$$+p^{-2}\frac{f_i}{D_i}\left\{h^{m-1}\left[m!\sum_{k=0}^{m-1}\frac{1}{m-k}\cos px_i \,-\, (-1)^{m-1}(m-1)!\cos px_{i-m}\right] \,+\, ... +\right.$$

$$+\,l\,h^{m-l}\left[\frac{m!}{l}\left(\sum_{k_1=0}^{m-1}\frac{1}{m-k_1}\sum_{k_2=0*}^{m-1}\frac{1}{m-k_2}...\sum_{k_l=0*}^{m-1}\frac{1}{m-k_l}\right)\cos^{[l-1]} px_i - \right.$$

$$\left. -(-1)^{m-l}(m-1)!\left(\sum_{k_1=0}^{m-2}\frac{1}{m-k_1-1}\sum_{k_2=0*}^{m-2}\frac{1}{m-k_2-1}...\sum_{k_{l-1}=0*}^{m-2}\frac{1}{m-k_{l-1}-1}\right)\cos^{[l-1]} px_{i-m}\right]$$

$$\left. +\, ... \,+\, m!\left[\cos^{[m-1]} px_i \,-\, \cos^{[m-1]} px_{i-m}\right]\right\} \tag{6.2.9}$$

$$J_{i-m} = -p^{-1}\, f_{i-m}\, \sin px_{i-m} +$$

$$+p^{-2}\frac{f_{i-m}}{D_{i-m}}\left\{h^{m-1}\left[(m-1)!\,\cos px_i \,-\, (-1)^{m-1}\sum_{k=0}^{m-1}\frac{1}{m-k}\cos px_{i-m}\right] \,+\, ... +\right.$$

$$+\,l\,h^{m-l}\left[(m-1)!\left(\sum_{k_1=0}^{m-2}\frac{1}{m-k_1-1}\sum_{k_2=0*}^{m-2}\frac{1}{m-k_2-1}...\sum_{k_l=0*}^{m-2}\frac{1}{m-k_{l-1}-1}\right)\cos^{[l-1]} px_i - \right.$$

$$\left. -(-1)^{m-l}\,\frac{m!}{l}\left(\sum_{k_1=0}^{m-1}\frac{1}{m-k_1}\sum_{k_2=0*}^{m-1}\frac{1}{m-k_2}...\sum_{k_{l-1}=0*}^{m-1}\frac{1}{m-k_l}\right)\cos^{[l-1]} px_{i-m}\right]$$

$$\left. +\, ... \,+\, m!\left[\cos^{[m-1]} px_i \,-\, \cos^{[m-1]} px_{i-m}\right]\right\} \tag{6.2.10}$$

In the above relations the summation symbol $\sum_{k_i=0*}^{m-1}$ is the same as the $\sum_{k_i=0}^{m-1}$ but with the added conditions that $k_i \neq k_{i-1}$, $k_i \neq k_{i-2}$, ..., $k_i \neq k_1$

$$
\begin{aligned}
J_j \;=\; p^{-2}\,\frac{f_j}{D_j}\,\Bigg\{ & h^{m-1}\left[\frac{m!}{i-j}\cos px_i \;-\; (-1)^{m-1}\frac{m!}{j-(i-m)}\;\cos px_{i-m}\right] + \\
& + 2\,h^{m-2}\left[\frac{m!}{i-j}\left(\sum_{k_1=0}^{m-1}\frac{1}{m-k_1} \;-\; \frac{1}{i-j}\right)\cos^{[1]} px_i - \right. \\
& \left. -\;(-1)^{m-2}\frac{m!}{j-(i-m)}\left(\sum_{k_1=0}^{m-1}\frac{1}{m-k_1} \;-\; \frac{1}{j-(i-m)}\right)\cos^{[1]} px_{i-m}\right] \;+\;...+ \\
& + l\,h^{m-l}\left[\frac{m!}{i-j}\left(\sum_{k_1=0***}^{m-1}\frac{1}{m-k_1}\sum_{k_2=0***}^{m-1}\frac{1}{m-k_2}...\sum_{k_{l-1}=0***}^{m-1}\frac{1}{m-k_{l-1}}\right)\cos^{[l-1]} px_i \right. \\
& \left. -\,(-1)^{m-l}\frac{m!}{j-(i-m)}\left(\sum_{k_1=0**}^{m-1}\frac{1}{m-k_1}\sum_{k_2=0**}^{m-1}\frac{1}{m-k_2}...\sum_{k_{l-1}=0**}^{m-1}\frac{1}{m-k_{l-1}}\right)\cos^{[l-1]} px_{i-m}\right] + \\
& + ... \;+\; m!\left[\cos^{[m-1]} px_i \;-\; \cos^{[m-1]} px_{i-m}\right]\Bigg\}
\end{aligned}
\tag{6.2.11}
$$

In relation (6.2.11) the summation symbol $\sum_{k_p=0**}^{m-1}$ is the same as the $\sum_{k_p=0}^{m-1}$ but with the added conditions that $k_p \neq k_{p-1}$, $k_p \neq k_{p-2}$, ..., $k_p \neq k_1$, $m-k_p \neq j-(i-m)$

Similarly the summation symbol $\sum_{k_p=0***}^{m-1}$ is the same as the $\sum_{k_p=0}^{m-1}$ but with the added conditions that $k_p \neq k_{p-1}$, $k_p \neq k_{p-2}$, ..., $k_p \neq k_1$, $m-k_p \neq i-j$

In relations (6.2.9)-(6.2.11) the index, l, represents the $l-st$ term of the summation ($l = 2, ..., m-1$) and

$$D_j \;=\; (m-(i-j))!\;\;(i-j)!\;\;h^m\,(-1)^{i-j} \quad , \qquad \text{for } j = i-m, ..., i \tag{6.2.12}$$

$\cos^{[k]} px$: k-th derivative

Knowing J_j in every interval, $[x_{i-m}, x_i]$, we can, by using relations (6.2.3) and (6.2.7) calculate I. However, the algorithm for the calculation of I as given by relations (6.2.9)-(6.2.11) is not practical. Specifying the order of polynomials, m, after some calculations we can produce some more useful formulas for I. The procedure is as follows :

We set :

$$\theta \equiv p \cdot h \tag{6.2.13}$$

For every integral J_j, we express px_i, px_{i-m} as a function of px_j, θ :

$$px_i \;=\; px_j \;+\; (i-j)\,\theta \tag{6.2.14}$$

$$px_{i-m} \;=\; px_j \;-\; (j-(i-m))\,\theta$$

and using the following trigonometric identities :

$$\cos^{[n]}(x \pm y) \;=\; \cos^{[n]} x \cdot \cos y \mp sin^{[n]} x \cdot \sin y \tag{6.2.15}$$

we calculate the integral in the form of :

$$J_j \;=\; h\, f_j\,(A_j\, \sin px_j \;+\; B_j\, \cos px_j) \tag{6.2.16}$$

where A_j and B_j are functions of θ (j=i-m,...,i). The number of functions A_j and B_j at every interval, $[x_{i-m}, x_i]$, is equal to the number of integration points, meaning we have (m+1) A_j and (m+1) B_j functions. Due to the symmetry of relations (6.2.14) and (6.2.15), A_j and B_j have the same absolute value for integration points, j, symmetrically to the middle of $[x_{i-m}, x_i]$ (see Figure 6.2.1). Therefore the actual total number of A_j, B_j is (m+1).

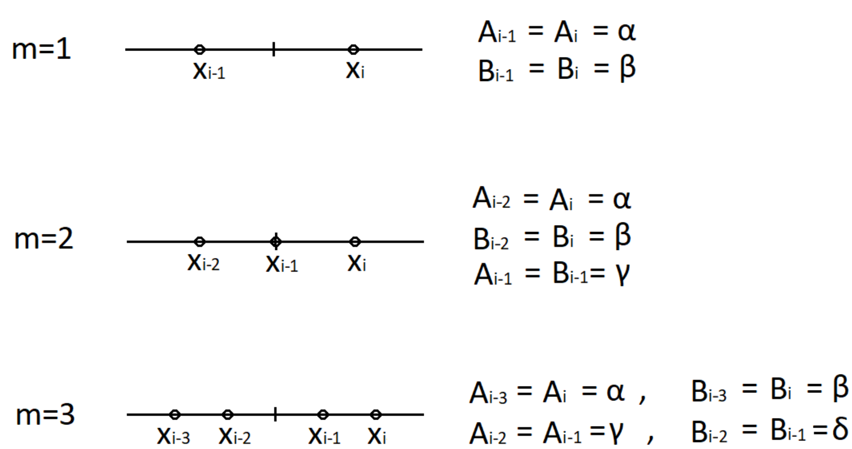


Figure 6.2.1

Note that when the integration point coincides with the middle of the interval, like the case of $m = 2$ in the Figure, we have the extra symmetry $A_j = B_j$.

Finally the calculation of (6.2.3) and (6.2.7) is quite simple and the formula depends on the order, m, of the Lagrange polynomial. Below we present the first three cases for m=1,2,3.

<u>The trapezoidal rule (m=1)</u>

Applying relations (6.2.9)-(6.2.12) we get :

$$J_{i-1} = -p^{-1} f_{i-1} \sin px_{i-1} + p^2 \frac{f_{i-1}}{-h}(\cos px_i - \cos px_{i-1}) =$$

$$= f_{i-1}(-\theta \sin px_{i-1} - \cos px_i + \cos px_{i-1})/p\,\theta \overset{(6.2.14)}{\underset{(6.2.15)}{=}}$$

$$= f_{i-1}(-\theta \sin px_{i-1} - \cos px_{i-1} \cos\theta + \sin px_{i-1} \sin\theta + \cos px_{i-1})/p\,\theta \tag{6.2.17}$$

Similarly :

$$J_i = f_i(\theta \sin px_i + \cos px_i - \cos\theta \cos px_i - \sin\theta \sin px_i)/p\,\theta \tag{6.2.18}$$

Relations (6.2.17), (6.2.18) can be written as :

$$J_{i-1} = h\, f_{i-1} \frac{1}{\theta^2}(-(\theta - \sin\theta)\sin px_{i-1} + (1-\cos\theta)\cos px_{i-1}) \tag{6.2.19}$$

$$J_i = h\, f_i \frac{1}{\theta^2}((\theta - \sin\theta)\sin px_i + (1-\cos\theta)\cos px_i) \tag{6.2.20}$$

Setting

$$\alpha = \theta^{-1} - \theta^{-2}\sin\theta \qquad\qquad \beta = 2\,\theta^{-2}(1-\cos\theta) \tag{6.2.21}$$

we get

$$I_i = h\left[\alpha(f_i \sin px_i - f_{i-1}\sin px_{i-1}) + \frac{1}{2}\beta(f_{i-1}\cos px_{i-1} + f_i \cos px_i)\right] \tag{6.2.22}$$

Substituting (6.2.22) to (6.2.3) we get :

$$I = h\,[\alpha(f(b)\sin pb - f(a)\sin pa) + \beta\, C_k] \tag{6.2.23}$$

where

$$C_k = \sum_{i=2}^{n-1} f_i \cos px_i + \frac{1}{2}(f_1 \cos pa + f_n \cos pb) \tag{6.2.24}$$

Following the same procedure for the integral (6.1.2), we get :

$$J \;=\; \int\limits_a^b \sin px\, dx \;=\; h\;[-\alpha(f(b)\cos pb \;-\; f(a)\cos pa) \;+\; \beta\, S_k] \qquad (6.2.25)$$

where

$$S_k \;=\; \sum_{i=2}^{n-1} f_i\, \sin px_i \;+\; \frac{1}{2}(f_1 \sin pa \;+\; f_n \sin pb) \qquad (6.2.26)$$

Note that for $\theta \longrightarrow 0$, coeffients α, β are :

$$\alpha \;=\; \frac{\theta}{6} \;-\; \frac{\theta^3}{120} \;+\; \frac{\theta^5}{5040} \;-\; ... \longrightarrow 0 \qquad (6.2.27)$$

$$\beta \;=\; 1 \;-\; \frac{\theta^2}{12} \;+\; \frac{\theta^4}{360} \;-\; \frac{\theta^6}{20160} \;+\; ... \longrightarrow 1 \qquad (6.2.28)$$

and thus relation (6.2.23) for $p = 0$ becomes :

$$I \;=\; h\;\left[\frac{1}{2}\, f_0 \;+\; f_1 \;+\; f_2 \;+\; ... \;+\; f_{n-1} \;+\; \frac{1}{2}\, f_n\right] \qquad (6.2.29)$$

which is the know trapezoidal rule.

Simpson's rule (m=2)

This case has been analyzed in section 6.1

Simpson's 3/8 rule (m=3)

Following the same procedure, we calculate the following coefficients :

$$\alpha \;=\; \theta^{-1} \;-\; \frac{1}{3}\,\theta^{-2}\sin 3\theta \;-\; 2\,\theta^{-3} \;-\; \theta^{-3}\cos 3\theta \;+\; \theta^{-4}\sin 3\theta \qquad (6.2.30)$$

$$\beta \;=\; 2\;\left(\frac{11}{6}\,\theta^{-2} \;-\; \frac{1}{3}\,\theta^{-2}\cos 3\theta \;+\; \theta^{-3}\sin 3\,\theta \;-\; \theta^{-4} \;+\; \theta^{-4}\cos 3\theta\right) \qquad (6.2.31)$$

$$\gamma = 3\theta^{-2}\,\sin\theta + \frac{3}{2}\,\theta^{-2}\,\sin 2\theta + 5\,\theta^{-3}\cos\theta + 4\,\theta^{-3}\cos 2\theta - 3\theta^{-4}\sin\theta - 3\theta^{-4}\sin 2\theta \qquad (6.2.32)$$

$$\delta \;=\; -3\,\theta^{-2}\cos\theta + \frac{3}{2}\theta^{-2}\cos 2\theta + 5\theta^{-3}\sin\theta - 4\theta^{-3}\sin 2\theta + 3\theta^{-4}\cos\theta - 3\theta^{-4}\cos 2\theta \qquad (6.2.33)$$

then

$$I \; = \; h \; [a\,(f(b)\sin pb \; - \; f(a)\sin pa) \; + \; \beta\, C_{3k} \; + \; \gamma(S_{1k} + S_{2k}) \; + \; \delta C_{12k}] \tag{6.2.34}$$

where

$$C_{3k} \; = \sum_{i=3(3)}^{n-3} f_i \cos px_i \; + \; \frac{1}{2}(f(a)\cos pa + f(b)\cos pb) \tag{6.2.35}$$

$$S_{1k} \; = \sum_{i=2(3)}^{n-1} f_i \sin px_i \tag{6.2.36}$$

$$S_{2k} \; = \; - \sum_{i=1(3)}^{n-2} f_i \sin px_i \tag{6.2.37}$$

$$S_{12k} \; = \sum_{i=1}^{n} f_i \cos px_i \quad , \qquad i \neq m, 2m, ..., k \cdot m = n \tag{6.2.38}$$

At the limit, $\theta \longrightarrow 0$, (p=0), it can be proven that :

$$\alpha \longrightarrow 0 \,, \qquad \beta \longrightarrow \frac{3}{4} \,, \qquad \gamma \longrightarrow 0 \,, \qquad \delta \longrightarrow \frac{9}{8} \tag{6.2.39}$$

Therefore relation (6.2.34) becomes Simpson's 3/8 rule :

$$I_i \; = \; h \left[\frac{3}{8}(f_{i-3} + f_i) \; + \; \frac{9}{8}(f_{i-2} \; + \; f_{i-1}) \right] \tag{6.2.40}$$

7 Programs Wing, Solver

7.1 Wing - Variables and Subroutines Summary

In this thesis we are trying try solve the lifting surface problem of a single wing. In order to do so, we use two separate programs. The first one is called 'Wing' and consist of two basic subroutines, which provide the geometric properties of the wing at number of points specified by the user and the motion of the wing, meaning the velocities and position of the wing relative to the inertial system. The program has a .txt file input named 'Input_Wing' containing the geometry of the wing. We assume that every cross section is geometrically similar to a specific NACA 2D airfoil, minimizing that way the degrees of freedom for the description of the wing. The current version doesn't take the motion of the wing as an input, instead the user must define the functions of path, rotation, and direction of rotation inside the code. The output is a .txt file named 'Input_for_solver' and can be read directly by the second program ('Solver'), without any modification .

<u>MODULE NRTYPE</u>

The program consist of two modules. The first one is called 'NRTYPE', and in that are defined the basic constants like π, euler's number etc. as well as the parameters used to describe the type of variables (single precision, double precision, logical etc.) used in the subroutines.

<u>MODULE GEOMETRY AND MOTION PRE-PROCESSOR</u>

This module is main part of the program and contains all the necessary subroutines to describe the geometry and motion of the wing. The global variables are :

NR = Number of spanwise positions of ouput data
NCH = Number of chordwise positions of ouput data
NBL = Number of blades (for the propeller problem)
MOMENTS = Number of (time) instants - (Size of varible T)
T = Array with the time at each instant
DT = Time step
SPAN = The span of the wing
TH = Angle of rotation

A = Parallel transition path Point (see chapter 2.1)
UNIT1 = Point on the x^1 axis with coordinates in $\underset{\bowtie}{x}^i$ system
UNIT2 = Point on the x^2 axis with coordinates in $\underset{\bowtie}{x}^i$ system
UNIT3 = Point on the x^3 axis with coordinates in $\underset{\bowtie}{x}^i$ system
ORIGIN= Origin point of the x^i system with coordinates in the $\underset{\bowtie}{x}^i$ system

U1 = Spanwise position
U2 = Chordwise position (non dimesionalized with the chord) form -0.5 to 0.5
XS = Reference line distance from the origin point in the x^1 direction (rake for the propeller)
SKEW = Sweep back angle (for the wing) or Skew (for the propeller)
FI = Twist angel (for the wing) or Pitch angle (for the propeller)
CHORD = Chord length at U1
DXS, DSKEW, DFI, DCHORD are the derivatives in the U1 direction
ECMAX = Maximum camber divided by chord in position U1
ETMAX = Maximum thickness divided by chord in position U1
CH = Chordwise position (non dimesionalized with the chord) form 0 to 1
ECCH = Camber at (U1,U2) divided by ECMAX at U1
ETCH = Thickness at (U1,U2) divided by ETMAX at U1

The above variables (U1, ..., ETCH) are calculated at the NR spanwise and NCH chordwise positions using a least square approximation of the data. The names of the respective input data variables are the same with an added '1' in front (e.g XS1, SKEW1, CH1, U21). The input variable of U1 is R1 and is non dimensionalized with the span. Variables used for reading the data are allocatable.

FACE = Position vector of a point on the face of the wing relative to the inertial system
BACK = Position vector of a point on the back of the wing relative to the inertial system
VFACE = Velocity of point a point on the face of the wing relative to the inertial system
VBACK = Velocity of point a point on the back of the wing relative to the inertial system

SUBROUTINE MAIN

This subroutine calls 'WING_GEOMETRY' to calculate variables FACE and BACK and those along with some other variables are written in the output file by calling 'OUTPUT GEOMETRY'. After initiallizing variables UNITj, ORIGIN we calculate at every instant the velocities on the face and back by calling 'VELOCITIES'. The position of the wing and the velocities are written in the output file using 'OUTPUT_POSITION_VELOCITY'. The new position of the wing is calculated by 'MOVEMENT' and the process repeats for every instant.

SUBROUTINE VELOCITIES

In order to calculate the velocities on the wing we take as input the current position of the wing and using the subroutine 'MOVEMENT' we calculate the position of the wing at DDT seconds later. The distance between each point divided by DDT is approximately the current velocity. The greater the acceleration the smaller DDT should be. Variables for FACE, BACK, UNITj, and ORIGIN are defined as locals in this subroutine because 'MOVEMENT' changes their values.

SUBROUTINE MOVEMENT

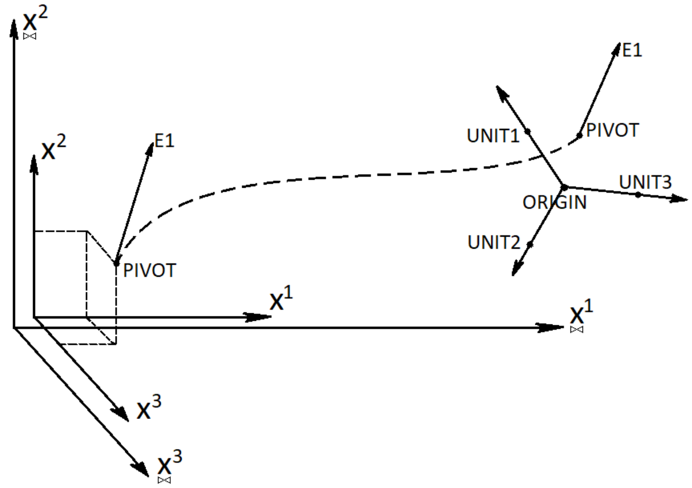


Figure 7.1.1

In order to describe the position of the moving coordinate system relative to the inertial, we use four point on the x^i axes whose coordinates are relative to $\underset{\bowtie}{x}^i$. In particular, we define points UNIT1, UNIT2, UNIT3, which lie on each of the three axes x^1, x^2, x^3 respectively, as well ORIGIN at their intersection. If $\vec{x}_{UNITj}$ (j=1,2,3) and $\vec{x}_{ORIGIN}$ are the position vectors of these points with regard to the inertial system, then we demand that $|\vec{x}_{UNITj} - \vec{x}_{ORIGIN}| = 1$. In other words we have defined the unit base of the moving system and expressed it in the $\underset{\bowtie}{x}^i$ coordinate system. Obviously at t=0, $\underset{\bowtie}{x}^i_{ORIGIN} = 0$ and $\underset{\bowtie}{x}^i_{UNITj} = \delta^i_j$.

The motion of the x^i system, between two instants t_1, t_2 can be broken down to a parallel transition and a rotation around a certain axis. Assuming that a point, denoted as PIVOT, makes only parallel transitions, then then whole x^i system should revolve around that point. Let $\vec{R(t)}$ be the parallel transition path, which corresponds to the position vector of a specific point (not the position vector of PIVOT) with regard to the inertial system for a non rotating motion. Then the parallel transition of any point is defined as $\vec{DR} = \vec{R}(t_2) - \vec{R}(t_1)$. Notice that by using the parallel transition vector $\vec{DR}$, there is no need for $\vec{R(t)}$ to point at the PIVOT.

For the rotation of the coordinate system x^i we must define the axis and direction of rotation Equivalently we can define a point of rotation (PIVOT) and a rotation vector, $\vec{\delta\theta}$, according to the right hand rule. The direction of the rotation vector is defined by the unit vector $\vec{E1}$, with coordinates in the $\underset{\bowtie}{x}^i$ system, which may vary with time. Therefore, if $\dot{\theta}(t)$ is the magnitude of the rotational speed, the rotation vector between t_1 and t_2 for $\vec{E1} = constant$ is :

$$\delta\vec{\theta} = \int_{t_1}^{t_2} \dot{\theta}(t)\,\vec{E}1\,dt \;=\; \vec{E}1 \int_{t_1}^{t_2} \dot{\theta}(t)\,dt \;=\; \vec{E}1\,(\theta(t_2)\;-\;\theta(t_1))$$

First we define the angle of rotation TH between each time step. The direction of rotation, E1, and the parallel path, A, are defined by subroutines 'DIRECTION' and 'PATH'. All three variables are defined as functions of time by the user inside the code (no input data are given in this version). Each point of the face and back as well as UNITj and ORIGIN are rotated around the PIVOT point by calling 'ROTATION'. Then we add the parallel transition of every point using 'PATH' for the next instant.

SUBROUTINE WING_GEOMETRY

PSI = Angle of camber line derivative in the U2 direction
ET = Half thickness of wing section at (U1,U2)
EC = Camber of wing section at (U1,U2)
X_FACE = Horizontal position of a point on the face at U2 position on the reference surface (see section 3.1)
Y_FACE = Vertical position of a point on the face at U2 position on the reference surface (see section 3.1)
X_BACK = Horizontal position of a point on the back at U2 position on the reference surface (see section 3.1)
Y_BACK = Vertical position of a point on the back at U2 position on the reference surface (see section 3.1)

Variables with the letter P at the end (e.g U1_P, CHORD_P) refer to the properties at the pivot point.

Subroutine 'WING_GEOMETRY', calls 'INPUT_DATA' to read the data from the file 'Input _Data.txt'. Using the subroutine 'INTERPOLATION' we calculate the values of those variables at specific points on the wing, and these are the values that we will use to make any other calculation. We then dimensionalize the variables and calculate the derivatives. Using the relations of section 3.2 we calculate X_FACE, X_BACK, Y_FACE and Y_BACK. Similarly, using the relations of section 2.2 we calculate variables FACE and BACK. Finally we repeat the calculations for the PIVOT point which is set to be in the mid-span and mid-chord.

SUBROUTINE LSA

X_I = x-coordinate of known point
Y_I = y-coordinate of known point
X = x-coordinate of desired point

LSAP= y-coordinate of desired point
FLAG= Variable that determines the type of funtions used by LSA M = degree of polynomial

Subroutine 'LSA' takes as input a number of points (X_I, Y_I) and performs a least square approximation. The type of function approximating this set of points is determined by the value of FLAG and can be 1,2,3 or 4.

FLAG = 1 (for simple approximations)
$f(x) = k_0 + k_1 x + k_2 x^2 + ... + k_m x^m$

FLAG = 2 (for zero values at x=0,1)
$f(x) = x(1-x)(k_0 + k_1 x + k_2 x^2 + ... + k_m x^{m-2})$

FLAG = 3 (for zero values at x=0,1 and infite derivative at x=0)
$f(x) = \sqrt{x}(1-x)(k_0 + k_1 x + k_2 x^2 + ... + k_m x^{m-2})$

FLAG = 4 (for zero values at x=0,1 and infite derivative at x=0,1)
$f(x) = \sqrt{x(1-x)}(k_0 + k_1 x + k_2 x^2 + ... + k_m x^{m-2})$

Coefficients, k, are determined by solving a linear system using 'DGAUSS'. Knowing k_n (n = 0, 1, ..., m), we can calculate the values at any given point x.

SUBROUTINE DGAUSS

A = System Matrix
B = Right-hand side vector (Input) - System solution (Output)
D = Determinant of A
L = Vector of pivoting indicies
IOPT = 0, Decomposition of A and solution of the system
. = 1, Decomposition of A only
. = 2, Solution of the system (skips decomposition)
IREG = 0, singular matrix A
. = 1, regular matrix A

Subroutine 'DGAUSS' uses the Gauss elimination method to determine the solution of a linear system. Depeding on the value of 'IOPT' the output can be the decomposed matrix A or the solution of the system. The subroutine calculates also the determinant of the matrix A and checks if the matrix is singular.

7.2 Wing - Input file example

The input file for wing is called 'Input_Wing.txt' and contains the necessary parameters and values to describe the geometry of a wing with cross-sections of NACA airfoils.

The first line of the file has the span of the wing (real), the desired number of spanwise NR (integer) and chordwise NCH (integer) positions. Number NR and NCH do not refer to the spanwise and chordwise positions of the input data, rather to the number of intermediate positions which will be used for the description of the wing in the output file 'Input_for_Solver'. The number of spanwise and chordwise positions of the input file 'Input_Wing.txt' are not determined.

Under the first line, there is an array with seven columns. The first column is the non-dimensional spanwise position (form 0 to 1). The rest of the columns refer to variables CHORD1, XS1, SKEW1, FI1, ETMAX1, ECMAX1 in order. Each row of this array refers to the geometric values of the wing at the non-dimensional spanwise position of the first column.

Under that array, there is a second array with three columns. The first column is the non- dimensional chordwise position (from 0 to 1). The rest of the columns refer to variables ETCH1, ECCH1 in order. Each row of this array refers to the geometric values of the wing at the non- dimensional chordwise position of the first column.

An exaple of input data for wing is given in the next page. Note that the names of the variables are not part of the file and are only included here as a description.

SPAN	NR	NCH
6.0000	13	17

R1	CHORD1	XS1	SKEW1	FI1	ETMAX1	ECMAX1
0.0000	1.0000	0.0000	0.0000	0.0000	0.1200	0.0400
0.0500	1.0000	0.0000	0.0000	0.0000	0.1200	0.0400
0.1000	1.0000	0.0000	0.0000	0.0000	0.1200	0.0400
0.2000	1.0000	0.0000	0.0000	0.0000	0.1200	0.0400
0.3000	1.0000	0.0000	0.0000	0.0000	0.1200	0.0400
0.4000	1.0000	0.0000	0.0000	0.0000	0.1200	0.0400
0.5000	1.0000	0.0000	0.0000	0.0000	0.1200	0.0400
0.6000	1.0000	0.0000	0.0000	0.0000	0.1200	0.0400
0.7000	1.0000	0.0000	0.0000	0.0000	0.1200	0.0400
0.8000	1.0000	0.0000	0.0000	0.0000	0.1200	0.0400
0.9000	1.0000	0.0000	0.0000	0.0000	0.1200	0.0400
0.9500	1.0000	0.0000	0.0000	0.0000	0.1200	0.0400
1.0000	1.0000	0.0000	0.0000	0.0000	0.1200	0.0400

CH1	ETCH1	ECCH1
0.0000	0.0000	0.0000
0.0100	0.1420	0.0494
0.0250	0.2179	0.1211
0.0500	0.2962	0.2344
0.1000	0.3901	0.4375
0.2000	0.4777	0.7500
0.3000	0.4992	0.9375
0.4000	0.4819	1.0000
0.5000	0.4385	0.9722
0.6000	0.3765	0.8889
0.7000	0.3002	0.7500
0.8000	0.2119	0.5556
0.9000	0.1121	0.3056
0.9500	0.0577	0.1597
0.9750	0.0293	0.0816
0.9900	0.0118	0.0330
1.0000	0.0000	0.0000

7.3 Wing - Output file example

The output file of wing is called 'Input_for_solver.txt' and contains the necessary parameters and values to describe the geometry and motion of the wing. The two subroutines writing in that file are 'OUTPUT_GEOMETRY' and 'OUTPUT_POSITION_VELOCITY'

<u>'OUTPUT GEOMETRY'</u>

The first line has the basic parameters. The first two numbers are the spanwise position of the tips. We set the right tip a u1=span and the left tip at u1=0. The third number is the number of wing or blades, which in our case is always 1. The forth and the fifth variables are the number of spanwise and chordwise position where the geometric values of wing are known. The last variable is the number of instants (MOMENTS).

After the first line follows an array with NR rows. The rows from left to right refer to the variables U1, CHORD, XS, SKEW, FI, DCHORD, DXS, DSKEW, DFI. The variables in each row refer to the spanwise position of the first column.

Below that, is another array with NCH columns and NR sets of 5 rows. Each of the five rows refers to U2, DEC_DU1, DEC_DU2, DET_DU1 and DET_DU2, which are the chordwise position and the derivative of camber and thickness in the u1 and u2 directions. Each row refers to the chordwise position U2.

Below those two arrays there is a label saying 'MOMENTS' followed by a list of all the instants (in seconds) where the position and velocities of the wing are known.

<u>'OUTPUT_POSITION_VELOCITY'</u>

Next there is a label saying 'POSITION VECTOR FACE' and a list of all the position vectors of the points on the face of the wing. The first columns is the $\underset{\bowtie}{x}^1$ coordinate, the second is the $\underset{\bowtie}{x}^2$ coordinate and the third is the $\underset{\bowtie}{x}^3$ coordinate. The same list but with the position vector of the points on the back follows.

Next there are the labels 'POSITION VECTOR PATH', 'UNIT VECTOR 1', 'UNIT VECTOR 2' and 'UNIT VECTOR 3' and under each one the respective vector coordinates in the inertial system.

Finally we have the labels 'VELOCITIES FACE', 'VELOCITIES BACK' and under each one a list of the velocities on every point.

The output of subroutine 'OUTPUT POSITION VELOCITY' repeats for every instant.

```
SPAN                NBL   NR      NCH     MOMENTS
6.0        0.0      1     100     17      200
```

```
U1       CHORD  XS       SKEW    FI       DCHORD   DXS      DSKEW   DFI
0.000    0.357  0.000    0.000   0.000     2.234   0.000    0.000   0.000
1.510    0.360  0.000    0.000   0.000     2.226   0.000    0.000   0.000
.......................
6.000    0.301  0.000    0.000   0.000    -2.556   0.000    0.000   0.000
```

```
U2(1,J)   ,    J=1,NCH
-0.50 -0.49 -0.48 -0.45 -0.42 -0.38 -0.33 -0.27 -0.20 -0.13 -0.05 0.02 0.11 0.20 0.30 0.40 0.50
DEC_DU1(1,J)   ,    J=1,NCH
0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00 0.00 0.00 0.00 0.00 0.00 0.00
DEC_DU2(1,J)   ,   J=1,NCH
0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00 0.00 0.00 0.00 0.00 0.00 0.00
DET_DU1(1,J)   ,    J=1,NCH
0.00  0.02  0.05  0.07  0.09  0.11  0.12  0.13  0.13  0.13  0.12 0.11 0.10 0.08 0.05 0.03 0.00
DET_DU2(1,J)   ,    J=1,NCH
0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00 0.00 0.00 0.00 0.00 0.00 0.00
.............................
U2(NR,J)   ,    J=1,NCH
-0.50 -0.49 -0.48 -0.45 -0.42 -0.38 -0.33 -0.27 -0.20 -0.13 -0.05 0.02 0.11 0.20 0.30 0.40 0.50
DEC_DU1(NR,J)   ,    J=1,NCH
0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00 0.00 0.00 0.00 0.00 0.00 0.00
DEC_DU2(NR,J)   ,    J=1,NCH
0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00 0.00 0.00 0.00 0.00 0.00 0.00
DET_DU1(NR,J)   ,    J=1,NCH
0.00  -0.02 -0.04 -0.06 -0.08 -0.09 -0.10 -0.11 -0.11 -0.11 -0.10 -0.09 -0.08 -0.06 -0.04 -0.02 0.00
DET_DU2(NR,J)   ,    J=1,NCH
0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00 0.00 0.00 0.00 0.00 0.00 0.00
```

```
MOMENTS
.    0.0000
.    0.0300
.    ....
.    5.9700
```

POSITION VECTOR FACE
. 0.00000 0.00000 -0.17861
. ...
. 0.00000 6.00000 0.12131
. 0.00000 6.00000 0.15088
POSITION VECTOR BACK
. 0.00000 0.00000 -0.17861
. ...
. 0.00000 6.00000 0.12131
. 0.00000 6.00000 0.15088

POSITION VECTOR PATH
. 0.00000 0.00000 0.00000
UNIT VECTOR 1
. 1.00000 0.00000 0.00000
UNIT VECTOR 2
. 0.00000 1.00000 0.00000
UNIT VECTOR 3
. 0.00000 0.00000 1.00000

VELOCITIES FACE
. 0.07565 0.00000 -2.00000
. ...
. 0.18405 0.00000 -2.00000
VELOCITIES BACK
. 0.07565 0.00000 -2.00000
. ...
. 0.18405 0.00000 -2.00000

7.4 Solver - Variables and Subroutines Summary'

The second program is called 'Solver' and interacts with 'Wing' through the 'Input_for_Solver'. Given the geometry, position and motion of the lifting surface, in our case a wing, it calculates the pressure distribution, the lift of every 2D - cross section and the thrust (or total lift) of the whole surface at every instant. The output of the program is chosen by the user in the code and its written in a .txt file named 'Output_of_solver'. For example the output of the program could be the total thrust of the wing at every instant. Furthermore, there is an extra output file called 'LS_motion' and contains the position of the lifting surface and its wake at every instant. The results of 'LS_motion' are processed by a MATLAB programzz named 'Mat_Fort_Wing_Graph' to produce a video of the wing's motion.

MODULE SINUS SOLVER

This module is the main part of the program and contains all the necessary subroutines to solve the lifting surface problem, using the modified vorticity of relation (4.2.21). Below we describe some of the global variables used by the subroutines.

M0 = Maximum spanwise harmonic (e.g SIN(II*Y1) , II=1,...,M0)
N0 = Maximum chordwise harmonic (e.g SIN(JJ*Y2) , JJ=1,...,N0)
P0 = Extra chordwise control points (usually P0=0)
CPB= M0*(N0+1+P0)*NBL_MAX , Number of C.P on wing (blades)
CPW= X*(M0+2-1)+1, Number of spanwise C.P on wake (we choose X=10)

WAKEP = Position Vector of a C.P on wake relative to the x^i system, with coordinate in the inertial system.
WAKE_I = Position Vector of a C.P on wake relative to the inertial system, with coordinate in the inertial system.
M_WAKE= Dipole intensity of a C.P on wake.

IE1 = Number of integration points of outer integral for interval 1
IE0 = Number of integration points of outer integral for interval 0
IE2 = Number of integration points of outer integral for interval 2
AB1 = Length, in radians, of interval 1
F0 = Length, in radians, of interval 0
AB2 = Length, in radians, of interval 2
SS_S01 = Double integral for A^s_{m0Z} of interval 1
SS_S00 = Double integral for A^s_{m0Z} of interval 0
SS_S02 = Double integral for A^s_{m0Z} of interval 2
SS_S0 = Double integral for A^s_{m0Z} of interval $[0, \pi]$ (only used for multiple blades).
SS_SN1 = Double integral for A^{ss}_{mnZ} of interval 1
SS_SN0 = Double integral for A^{ss}_{mnZ} of interval 0
SS_SN2 = Double integral for A^{ss}_{mnZ} of interval 2
SS_SN = Double integral for A^{ss}_{mnZ} of interval $[0, \pi]$ (only used for multiple blades).

EBS = $B^{sj}_{m_{self}}$, Self nduction factor for $A^{s}_{mn(0)Z}$, due to bound vorticity
EC0 = $C^{sj}_{0_{self}}$, Self nduction factor for A^{s}_{m0Z}, due to free vorticity
ECS = $C^{sj}_{n_{self}}$, Self nduction factor for A^{s}_{mnZ}, due to free vorticity
BSM0 = B^{s}_{m0Z} , Induction factor for A^{s}_{m0Z}, due to bound vorticity
CSM0 = C^{s}_{m0Z} , Induction factor for A^{s}_{m0Z}, due to free vorticity
BSSMN = B^{ss}_{mnZ} , Induction factor for A^{ss}_{mnZ}, due to bound vorticity
CSSMN = C^{ss}_{mnZ} , Induction factor for A^{ss}_{mnZ}, due to free vorticity
TS_S0N = T^{sj}_{m0Z} , Mode velocity for A^{s}_{mnZ}
TS_SMN = T^{sj}_{m0Z} , Mode velocity for A^{ss}_{mnZ}
THTP = THT^{j} , Contribution of tip, hub and trailing edge to the mode velocities

X1,X2,X3 = coordinates in the x^i system
U1,U2 = coordinates in the u^i system
Y1,Y2 = coordinates in the y^i system
GIJ = $\underset{\sim}{\underline{\gamma}}^{i}_{j}$, transformation by covariance between the x^i and $\underline{x}^i$ coordiante systems
G = $|\underset{\sim}{\underline{\gamma}}^{i}_{j}|$, determinant of transformation $\underset{\sim}{\underline{\gamma}}^{i}_{j}$
CIJ = $\underset{\sim}{\underline{c}}^{i}_{j}$, transformation by contravariance between the x^i and $\underline{x}^i$ coordiante systems
GGIJ= $\underset{\sim}{\underline{g}}^{i}_{j}$, metric tensor of the $\underline{x}^i$ coordiante systems
GP = $\underline{g}$, metric tensor determinant of $\underset{\sim}{\underline{g}}^{i}_{j}$ at C.P.
A12 = $\underline{\alpha}_{12}$, component surface metic tensor of the u^{α} coordinate system
ALPHA = $\underline{\alpha}$, determinant of surface metric tensor $\underline{\alpha}_{\alpha\beta}$
E2 = Unit vector of $\underline{x}^2$ axis on the leading edge
NP = Modified normal vector of camber surface, on the reference surface, with coordinates x^i

VFACEP = Undisturbed fluid velocity relative to a point on the face with coordinates in the inertial system
VBACKP = Undisturbed fluid velocity relative to a point on the back with coordinates in the inertial system
WFACEP = Undisturbed fluid velocity relative to a point on the face with coordinates in the x^i system
WBACKP = Undisturbed fluid velocity relative to a point on the back with coordinates in the x^i system
Q(1) = $\rangle \underline{q}^1 \langle$ Mean undisturbed fluid velocity in the $\underline{x}^1$ direction relative to the C.P on the reference surface
Q(2) = $\rangle \underline{q}^2 \langle$ Mean undisturbed fluid velocity in the $\underline{x}^2$ direction relative to the C.P on the reference surface
Q(3) = $\rangle \underline{q}^3 \langle$ Mean undisturbed fluid velocity in the $\underline{x}^3$ direction relative to the C.P on the reference surface
Q(4) = $\langle \underline{q}^3 \rangle$ Undisturbed fluid velocity jump in the $\underline{x}^3$ direction relative to the C.P on the reference surface
S = σ , Source intensity at control point

VT = Array of linear system
A = Coefficients of bound vorticity (unknowns)
B = Right hand side of the linear system

DPRES = $(p^+ - p^-)/\rho * \sqrt{\underset{\smile}{\underline{\alpha}}}$, modified pressure difference on the reference surface

DL = 2D-force of a chordwise section in the NP^j direction
LIFT = Force of z-th blade in the x^j direction
THRUST = Force of all blades in the x^j direction

SUBROUTINE MAIN

Subroutine 'MAIN' is the main part of the program, which calls all the other subroutines in order to solve a linear system and define the coefficients of the modified bound vorticity at every instant. Initially, it calls 'INPUT_GEOMETRY' to read the geometric data at the NR spanwise and NCH chordwise positions. These values will be used to calculate (approximate) the geometric features of the wing at specific control points.

For every instant it calls 'INPUT_POSITION_VELOCITY' to read the position and velocity of every point on the face and the back of the wing. Then, uses 'WAKE_POSITION', 'WAKE_INERTIAL' and 'WAKE_DIPOLE_INTENSITY' to calculate the new position of the wake relative to the x^i and the inertial system, as well as the intensity of the dipoles on the new spanwise wake strip based on the coefficients A of the previous instant.

It then calculates the geometric and kinematic properties at all the control points using 'PROPERTIES_AT_CP', which will be used by the 'LINEAR_SYSTEM_FORMATION' to calculate the elements of the matrix for the no-entrance and pressure Kutta conditions.

Once we have done that for every C.P, we can solve the linear system either by the Gauss elimination method or the Least square approximation method, which correspond to subroutines 'DGAUSS' and 'GLSQ'. In case we use more C.P than boundary conditions ($P0 > 0$) we can't use 'DGAUSS'.

Having solved for the coefficients A, we call 'PRESSURE_DISTRIBUTION' to calculate the modified pressure distribution on the wing as well as the 2D-leading edge suction force on every chordwise section. These values are then used by 'LIFT_THRUST' to calculate the forces acting on the lifting surface and the process repeats for the next instant.

SUBROUTINE WAKE_POSITION

The wake is split into two parts. The free part of the wake and the Kutta strip that connects the trailing edge and the free wake. The Kutta strip is assumed to be tangent to the reference surface with chordwise position at u2 = 0.5 + 0.1 and therefore the length of the Kutta strip is 10% of the chord length at u1. Normally this length should be a function of the time stem and number of

C.P on the wing. Since the positions of the strip is constant relative to the trailing edge, the new Kutta strip assumes the position of the old one :

WAKEP(M,Z,IN,J)=WAKEP(M,Z,IN-1,J)
WAKEP(M,Z,IN-1,J)=WAKEP(M,Z,IN-2,J)

SUBROUTINE WAKE_INERTIAL

Assuming a frozen wake model the position of the free wake doesn't change relative to the inertial reference frame, meaning that the variable WAKEP_I doesn't change either. We first calculate the position of the new Kutta strip in the inertial frame by projecting the x^i coordinates of WAKEP in the inertial system and adding the ORIGING vector. Then we calculate the new position of the free wake relative to the x^i system by reversing the process.

SUBROUTINE PROPERTIES_AT_CP

Given the surface coordinates of the C.P, we calculate the transformation by covariance $\underline{\tilde{\gamma}}^i_j$ and the x^i coordinates by calling 'GEOM'. From there we calculate the transformation by contravariance $\underline{\tilde{c}}^i_j$, the metric tensor $\underline{\tilde{g}}_{ij}$, the surface metric tensor $\underline{\alpha}_{ij}$ and the respective determinants.

Using 'FILLIN' we approximate the values of camber and thickness derivatives, as well as the undisturbed fluid velocities VFACE, VBACK for the C.P. In the process we use some auxiliary variables named AAJ and BBJ. We can then calculate the mean flow velocity relative to C.P, the source intensity and the modified normal vector, NP, of the camber surface.

SUBROUTINE LINEAR_SYSTEM_FORMATION

For the inner points of the wing subroutine 'LINEAR_SYSTEM_FORMATION' calls 'MODE_VELOCITIES' which calculates the induction factors and mode velocities to implement on the no-entrance boundary condition. For the right hand side of the system, it calls 'WAKE_CONTRIBUTION', which returns the wake induced velocity at C.P.

For the points on the trailing edge we use the pressure Kutta condition. The time derivative in the Bernoulli equation is approximated by relations (4.3.19)-(4.3.21).

SUBROUTINE MODE_VELOCITIES

The induction factors are calculated in three steps. First we calculate the self induction factors from relations (4.5.58), (4.5.60) and (4.5.61), then the inner integral from relations (4.5.15), (4.5.17) and (4.5.18) for every spanwise or chordwise section and finally the outer integral from relations (4.5.22), (4.5.24), (4.5.28) and (4.5.30). We split the outer integral into tree parts and we denote those integration interval as 1, 0, 2. Inside of interval 0, is the C.P for which the induction factor is calculated. In order to reduce the calculation time, the number of integration points IE1, IE0 and IE2 is varies according to the position of C.P on the blade. The number of integration points of the inner integral, ND, depends on the position of C.P on the blase, the dimensions of the blade and the distance of integration points from th C.P.

Specifically, subroutine 'MODE_VELOCITIES' initially calculates the self-induction factors and then the number of integration points IE1, IE0 and IE2 of the outer integral. The induction factors of the bound and free vorticity are calculated by subroutines 'BIND' and 'CIND' respectively. Those induction factors are combined to get the mode velocity induced by the inner points of the wing. The contribution of tip, hub and trailing edge is calculated in the form of mode velocities by subroutine 'TIP_HUB_TE'.

SUBROUTINE BIND

The calculation of the induction factors due to the bound vorticity is split into tree parts. In the first one we calculate the minimum number of integration points, ND, for the inner integral. Assuming equally spaced integration points in the y1, y2 directions we approximate the geometric properties at each integration point. Finally using 'FILLON_METHOD' we calculate first the inner and then the outer integral.

The same process is followed in the subroutine 'CIND'.

SUBROUTINE TIP_HUB_TE

For the calculation of the mode velocities induced by the boundary of the wing and partially by the Kutta strip, we first find the dipole intensity at the points of the boundary. Since the coefficients A are unknown we calculate only the modes (trigonometric terms) of the dipole intensity. The dipole intensity of the line segments at the tips is the mean value of the intensity at the ends of the segment. Instead at the trailing edge the dipole intensity is half of the mean value. For every given line segment on the boundary, we calculate the components of relation (4.8.5) using the following subroutines :

'POINT_TO_LINE_DISTANCE' = Finds the distance between the C.P and the line segment
'CROSS' = Performs the cross product of two vectors
'VECTOR_ANGLE' = Finds the angle between two vectors
'NORM' = Finds the magnitude of a vector

For the points on the trailing edge, we include the contribution of the Kutta strip as described in section 4.8

Finally by combining the unit velocity of each line segment with the respective mode of the dipole intensity, we get the mode velocity induced by the boundary and partially by the Kutta strip.

SUBROUTINE WAKE_CONTRIBUTION

The calculation of the velocity at C.P, induced by the wake is similar to that of 'TIP_HUB_TE'. In this case, however, the dipole intensity at every point on the wake is known and saved in the variable M_WAKE. The dipole intensity of every boundary element is the mean value of intensities at its corner. In the case of Kutta strip the mean value is calculated by considering only the corners of the B.E which don't lie on the trailing edge (and thus the dipole intensities are known). The intensity of each of the Kutta B.E is half of that mean value.

SUBROUTINE PRESSURE_DISTRIBUTION

Having solved the linear system, the coefficients A for this instant are known. Multiplying relation (5.1.7) by $\sqrt{\underset{\smile}{\underline{\alpha}}} = \frac{1}{4} span \, \sin(y^1) \, \sin(y^2)$ we get the modified pressure difference on the lifting surface. Note that the modified pressure difference is finite at the leading edge and zero at tips and trailing edge.

In this subroutine we also calculate the leading edge suction force for every chordwise section of the wing.

SUBROUTINE LIFT_THRUST

Here we first calculate the lift of every chordwise section of the wing by integrating the modified pressure, DPRE, by the modified normal vector of the camber surface. The result is corrected by the leading edge suction force and integrated across the span of the wing. In case of multiple wings or blades we add the lifting force of all the bodied to get the total force, THRUST. The components of the resulting vector are on the x^i system, so using the unit vectors UNITj we get the total force expressed in the inertial system.

In this subroutine we write on the 'Output_of_Solver'. The output may change depending on the users preferences. For example we could write the 2D-Lift of a section or the total force at every instant.

7.5 Solver - LS_motion, output file example

The program creates a .txt file named 'LS_motion' which contains the position of the wing and it's wake relative to the inertial system. Specifically the first line has six integer number referring to the maximum spanwise and chordwise harmonics, the number of blades, the number of spanwise and chordwise positions where the geometric values are known and the number of instants.

Under that line there are three columns with real number, representing the coordinates of position vector in the inertial system. The NCH*NR*NBL first rows refer to the points of the face and the next NCH*NR*NBL rows to the points of the back. The next (10*(M0+2-1)+1)*NBL*IN rows with IN=1,2,...MOMENTS, refer to the points of the wake. The data for the face, back and wake repeat for all the instants.

M0	N0	NBL	NR	NCH	MOMENTS
9	7	1	100	17	200

FACE - $\underset{\bowtie}{x}^1$	FACE - $\underset{\bowtie}{x}^2$	FACE - $\underset{\bowtie}{x}^3$
0.00000	0.00101	-0.18029
0.00004	0.00101	-0.17856
0.00007	0.00101	-0.17336
....		
0.00010	0.00101	-0.16477
0.00013	0.00101	-0.15284
0.00015	0.00101	-0.13772
BACK - $\underset{\bowtie}{x}^1$	BACK - $\underset{\bowtie}{x}^2$	BACK - $\underset{\bowtie}{x}^3$
0.00017	0.00101	-0.11952
0.00018	0.00101	-0.09844
0.00018	0.00101	-0.07468
....		
0.00018	0.00101	-0.04846
0.00017	0.00101	-0.02004
0.00015	0.00101	0.01031
WAKEP_I - $\underset{\bowtie}{x}^1$	WAKEP_I - $\underset{\bowtie}{x}^2$	WAKEP_I - $\underset{\bowtie}{x}^3$
0.00013	0.00101	0.04230
0.00011	0.00101	0.07562
0.00008	0.00101	0.10995
....		
0.00004	0.00101	0.14495
0.00000	0.00101	0.18029
0.00000	0.00403	-0.18531

....

8 Results

8.1 Steady Case - High aspect ratio, results at midspan

We first examine the results of the program at midspan for the steady case of a high aspect ratio wing. Specifically we compare those results with the 2D-equivalent experimental results of NACA 4412 airfoil for the highest Reynolds number. We compare the 2D-lift coefficient, which is defined by the relation $C_L = Lift/(\frac{1}{2}\rho U^2\, chord)$, for set of angles of attack in the linear region.

The maximum harmonic in the spanwise and chordwise direction is $M0 = 9$ and $N0 = 7$ respectively. We use a rectangular wing with chord length $c = 1m$ and span to chord ration 30. The lift is calculated for a chordwise section in the midspan.

A typical evolution of the lifting coefficient with time can be seen in the figure below.

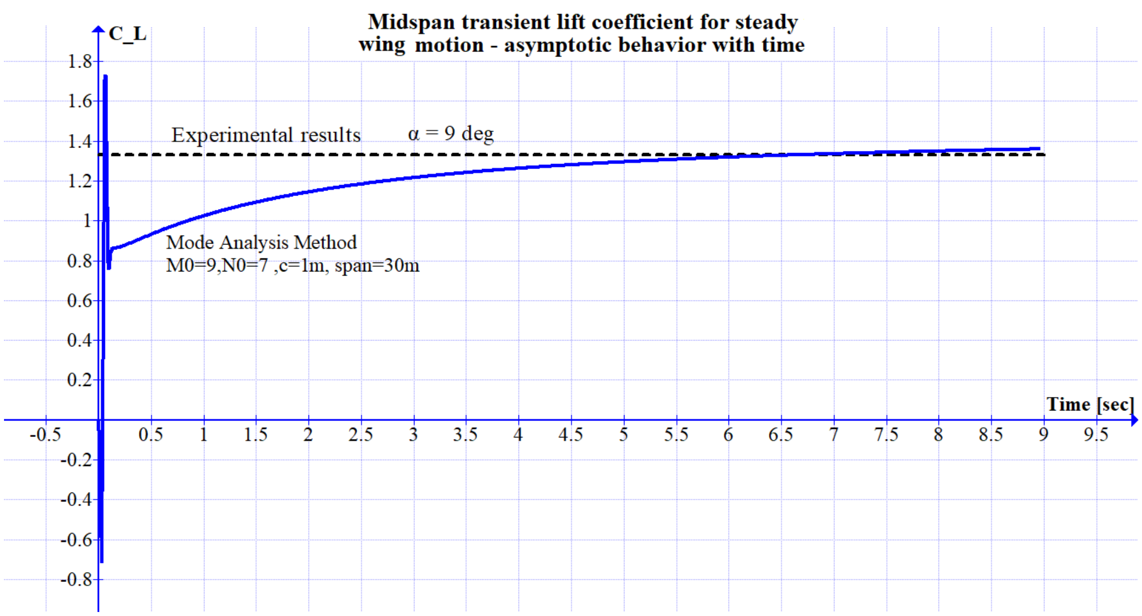


Figure 8.1.1

It can be seen that the first 3 instants there are spikes in the lift coefficient, due to the bursting start of the wing. After those instants, the lift quickly stabilizes and tends to a value around 1.4 for great values of t.

In the following figure the experimental results correspond to the solid (continues). The results of the program are five points for -9, -5, 0, 5, and 9 degrees. The slope of the dotted line connecting the points is smaller than the theoritical slope, 2π, by approximately 8%.

The difference in the results could by due to the following reasons :

- Linearization of the problem - 3D Effects that reduce the total pressure - In the program we don't calculate the effect of sources - The position of the wake.

Specifically for the last reason, the wake exits tangent to the reference surface and is frozen relative to the inertial system. In other word the plane of the wake is parallel to the motion of the wing, with the exception of the Kutta strip, which is tangent to the reference surface. This is obviously wrong since according to the lifting line theory, the downwash created by the wake changes the position of the wake.

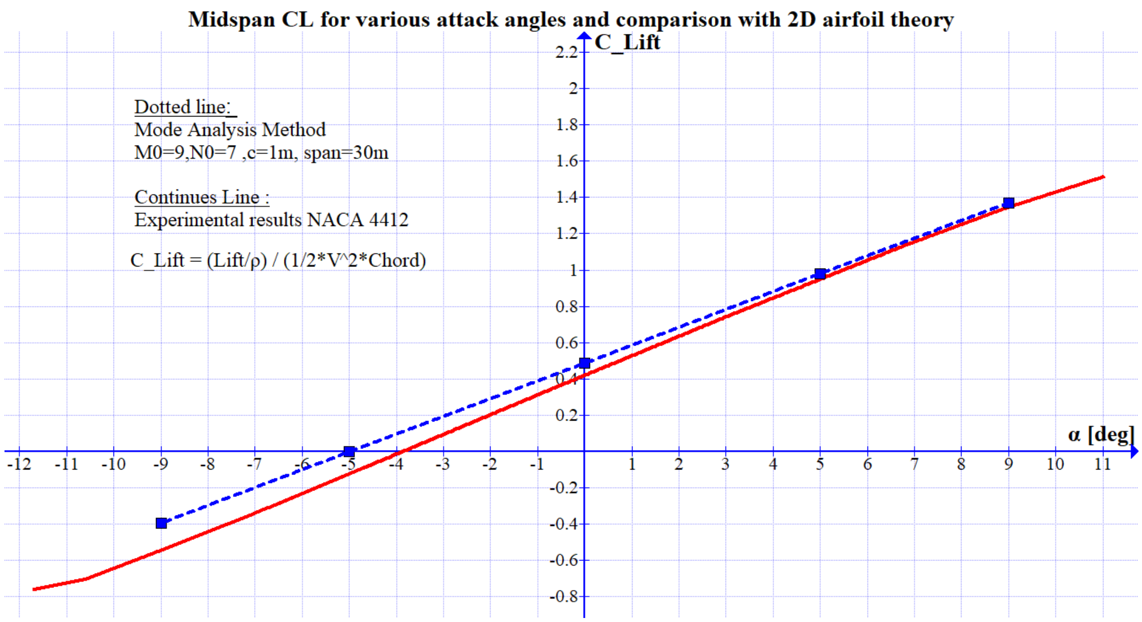


Figure 8.1.2

8.2 2D-Flat plate - Analytical solution for the unsteady case

Here examine the 2D unsteady case of a heaving and pitching flat plate. An extensive analysis on these problems can be found in [9]. In particular examine the case of wing moving with a mean (transitional) speed U, while performing one of two harmonic oscillations (in th x,y coordinate system):

$$y = y_0\, b\, e^{i\omega t} \qquad \text{(heaving motion - vertical oscillation)}$$

$$y = -\alpha_0\, x\, e^{i\omega t} \qquad \text{(pitching motion - rotational oscillation)}$$

where
b = semichord of the airfoil, taken as 1
ω = circular frequency
y_0 = the ratio of the amplitude of the vertical motion to the semichord b
α_0 = maximum angle of rotation

Instead of the circular frequency, the results are presented in the nondimesional reduced frequency k, which is defined as :

$$k = \frac{\omega\, b}{U} = \frac{\omega}{U}$$

For the sake of comparison with the results of section 8.3, the relation of Strouhal number and reduced frequency for the heaving motion is (for $y_0 = 0.5$ and $b = 1m$) $Str = \frac{k}{2\pi}$

It can be proven that the lift coefficient as defined in the previous section has the following form for each one of the movements :

$$C_L = \pi\, y_0\, k^2 \left[1 - \frac{2i}{k} C(k)\right] e^{iUkt} \qquad \text{(heaving motion)}$$

$$C_L = \pi\, k\, a_0 \left[i + (i + \frac{2}{k}) C(k)\right] e^{iUkt} \qquad \text{(pitching motion)}$$

The function C(k) is often referred to as Theodorsen's function and is defined by the relation :

$$C(k) = \frac{K_1(ik)}{k_1(ik) + K_0(ik)} = F(k) + iG(k)$$

where K_0, K_1 are the Bessel functions of second kind of order zero and one respectively. Values for F and G can be found in page 214 of [9].

Using the above relation for the lift coefficient we can calculate the amplitude and phase of the lift. The theoretical results can be seen in figures 8.2.2 - 8.2.5 in the continues lines with the reduced frequency as a free variable.

In order to calculate the lift and its phase we use the same rectangular wing as in section with chord 1 m, span 30 m and $M0 = 9$, $N0 = 7$. A typical output of the program for the case of the heaving motion ca be seen in figure 8.2.1.

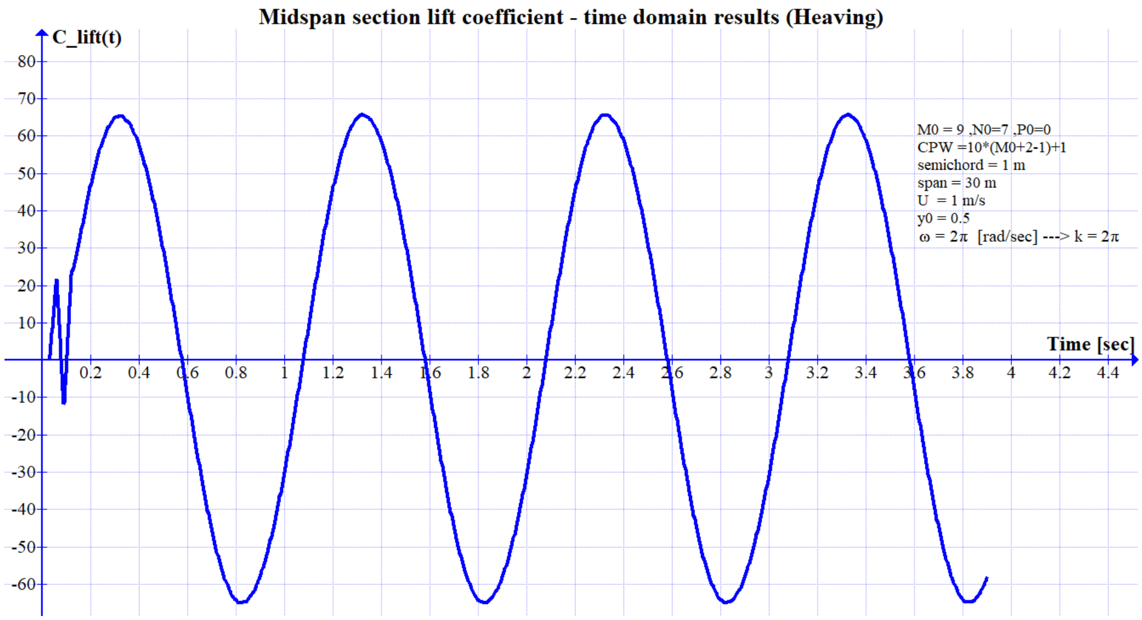


Figure 8.2.1

In Figures 8.2.2, 8.2.3 we examine the amplitude and phase of the heaving motion and in figures 8.2.4, 8.2.5 the same results for the pitching motion. It's interesting to notice that the amplitude of the heave and the phase of the pitch approximate better the theoretical results.

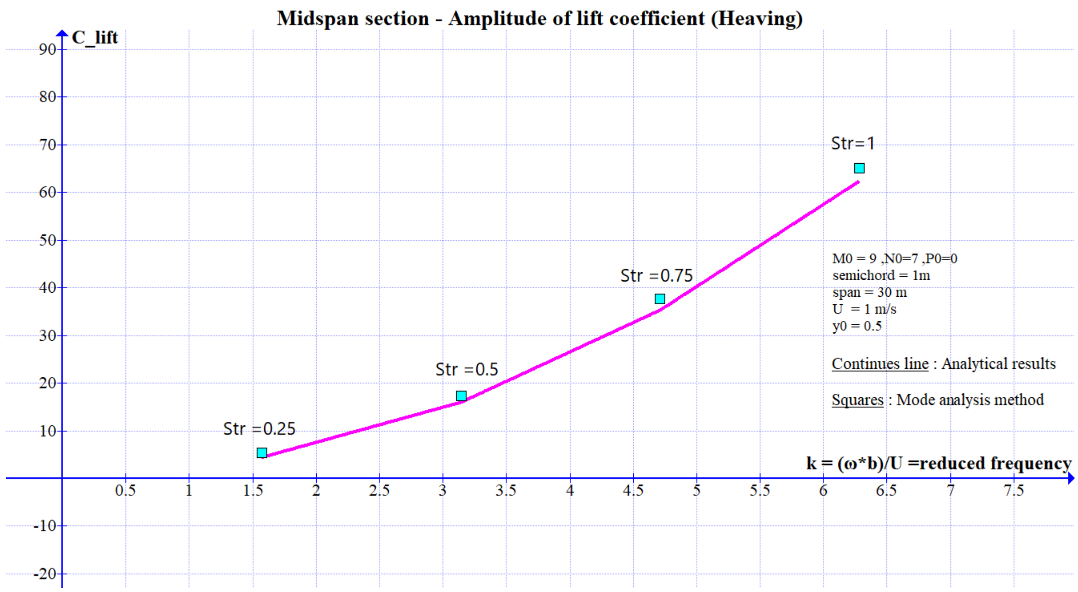


Figure 8.2.2

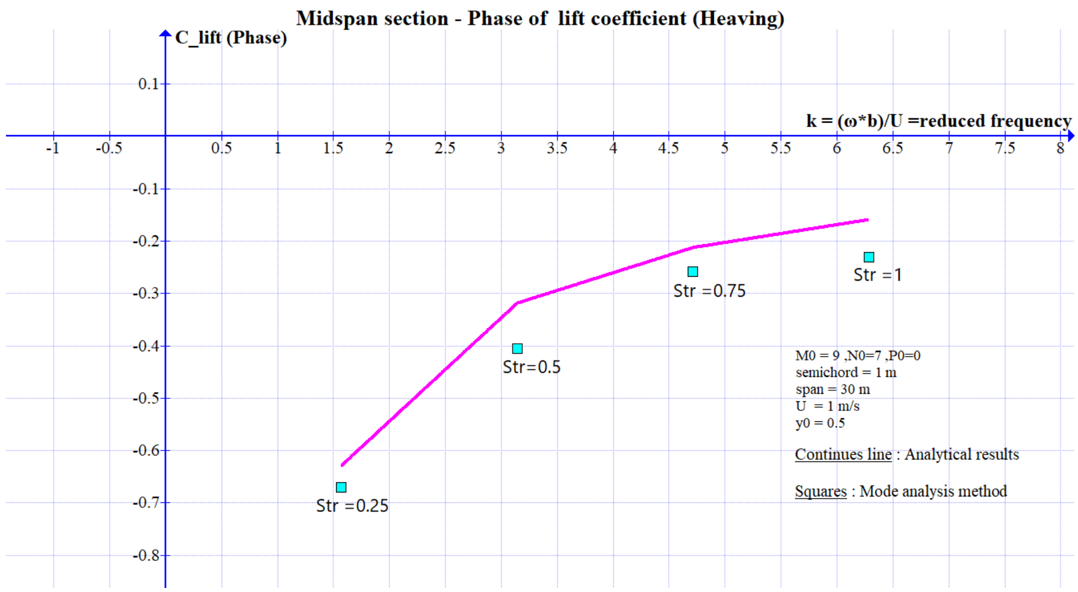


Figure 8.2.3

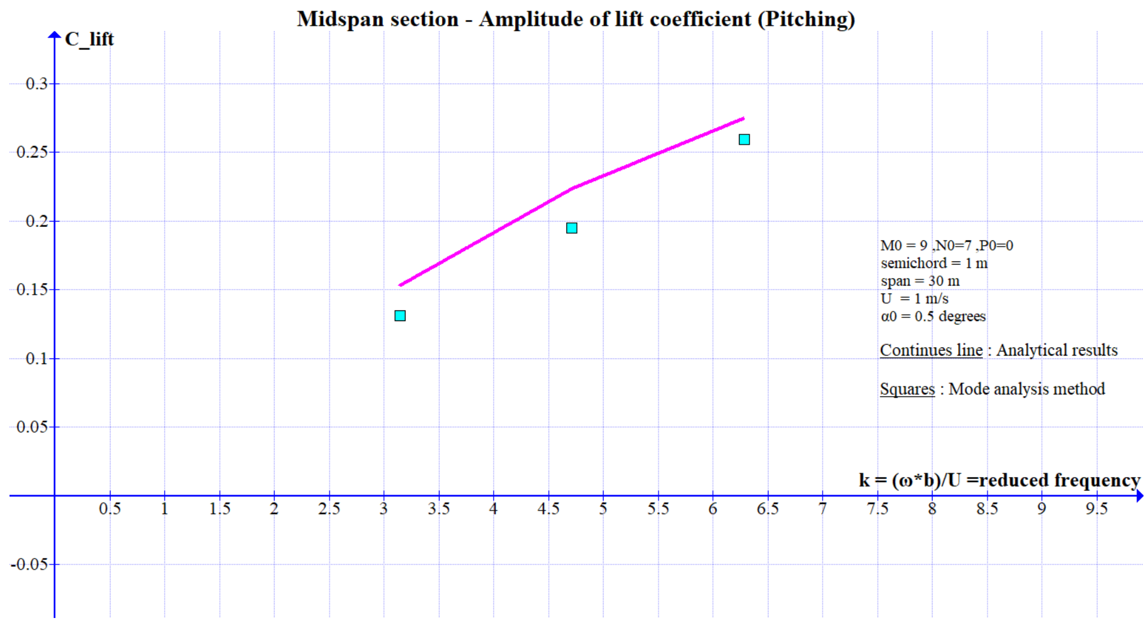


Figure 8.2.4

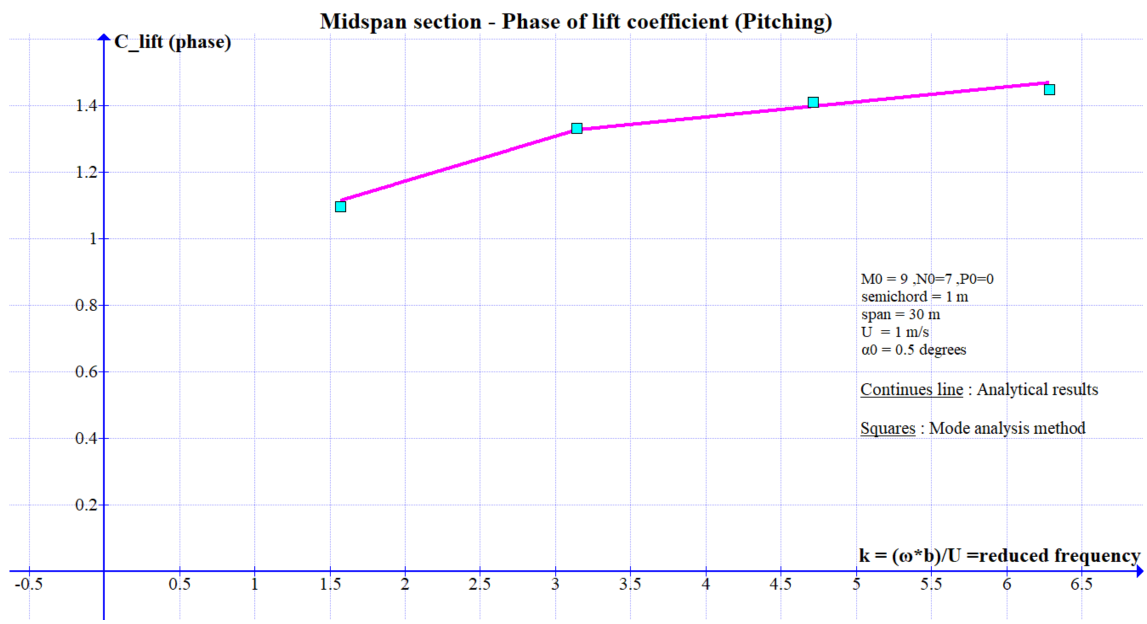


Figure 8.2.5

8.3 Comparisons with panel methods

In order to examine the 3D unsteady case we choose to model a flapping wing. Specifically we compare the thrust parallel to the transitional motion of the wing with the results given in [10], produced by UBEM. We examine 2 cases of wing geometry, 3 cases of oscillation amplitude and up to 4 cases of motion (Strouhal number).

The flapping of the wing is a combination of three motions. The first is a parallel transition with constant velocity U. To that we add a sinusoidal heaving and pitching motion separated by a phase $\psi = 90^o$. The frequency, n, is the same for both heave and pitch and the respective amplitudes are h_0 and θ_0.

$$h(t) \;=\; h_0 \,\sin(2\pi\, n\, t) \quad , \quad \text{(heaving motion)}$$

$$\theta(t) \;=\; \theta_0 \,\sin(2\pi\, n\, t \,+\, \psi) \quad , \quad \text{(pitching motion)}$$

The rotation is performed around a point (PIVOT) at distance b from the leading edge of the midspan chord. The angle of attack at that point is :

$$\alpha(t) \;=\; \theta(t) \,-\, \tan^{-1}(\dot{h}(t)/U) \quad , \quad \text{(angle of attack at the pivot point)}$$

The non dimensional parameter characterizing the motion is the Strouhal, defined by :

$$Str \;=\; \frac{n\, h}{U} \;, \quad h \;=\; 2\, h_0$$

The wing outline is characterized by zero skewback and twist. The chord changes from c/4 (at tip) to c (at c/2 from the tip). The interpolation schema for the chord, for spanwise positions between tip and and c/2, is a cubic spline with end conditions of $d^2c/ds^2 = -5$ at tip and $dc/ds = 0$ at c/2. A NACA 0012 thickness form has been selected spanwisely. We choose a standard chord length $c = 1m$.

The non-dimensional parameters have been chosen as follows:

- Wing with span to chord ratio : $s/c = 4,\, 6$

- Heaving amplitude : $h_0/c = 0.5,\, 1,\, 2$

- Position of pitch axis (from leading edge) : b/c = 0.1

- Strouhal number : Str = 0.1, 0.2, 0.3 (and 0.15 for $h_0/c = 0.5$)

The the thrust coefficient is defined as $C_T(t) \;=\; T(t)/(\frac{1}{2}\,\rho\, U^2\, S)$, where $T(t)$ denotes the time dependent wing thrust, ρ denotes fluid density, U denotes the transitional velocity of the wing and S denotes the wing swept surface ($S \;=\; s \cdot h$). In figures 8.3.2 - 8.3.7 we take the mean value of C_T after a few periods.

A typical graph of the thrust coefficient, C_T, as a function of time can be seen in Figure 8.3.1. In the vertical axis we have the thrust and on the horizontal axis the time in secondes. From this figure it can be concluded that, for our family of wings and motions, the transient phenomenon is limited to the few initial time steps after the burst start. Thus it is safe to use the 2nd period of simulation, to calculate the mean thrust.

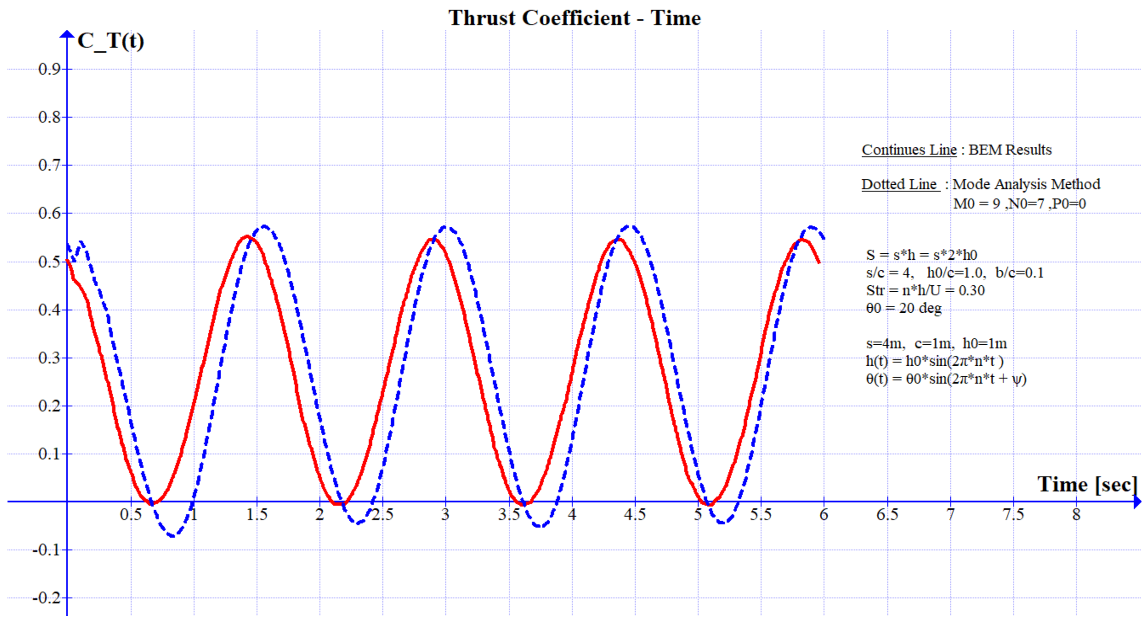


Figure 8.3.1

The Lifting surface program doesn't calculate added resistance due to friction, instead UBEM takes friction into consideration using a constant speed $U = 2.3m/s$ for all cases. The simplest way to estimate it is by using the ITTC curve for the friction of a flat plate and non-dimesionalize it by $\frac{1}{2}\rho U^2 S$, where S is the swept surface. A more systematic way to estimate the friction is discussed in [13]. Here we use the first method to calculate the friction and the correction varies form 1% up to 5%.

It can be seen in the figure that for small Strouhal number the results of the program come very close to those of UBEM. For higher Strouhal number, especially for $h_0/c = 0.5$ there is a great deviation of the two methods. For a constant speed, the lower the amplitude ratio h_0/c is, the greater the frequency, n, should be to get the same Strouhal number. It is therefore possible that for high frequency movements our method divert from the results of UBEM.

Its also worth noticing that for all the figure and Strouhal numbers the results of our method tends to diverge from those of UBEM at small pitching angles θ. For those angles, the vertical forces on the wing (lift), have a negligible contribution to the force in the direction of motion (thrust). The primary force contributing to the thrust is the leading edge suction force, which in our program is calculated simplistically using the 2D theory. In contrast, UBEM, calculates exactly the forces developed on the panels of upper lower surface near the leading edg, which might explain the difference.

Apart from the reasons mentioned above we should also include the following ones :

- The non-linearity of the phenomenon

- The non-free wake

- Errors in the code

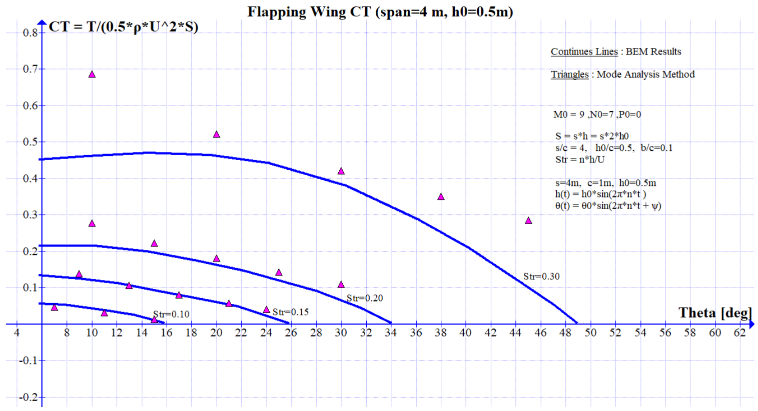


Figure 8.3.2

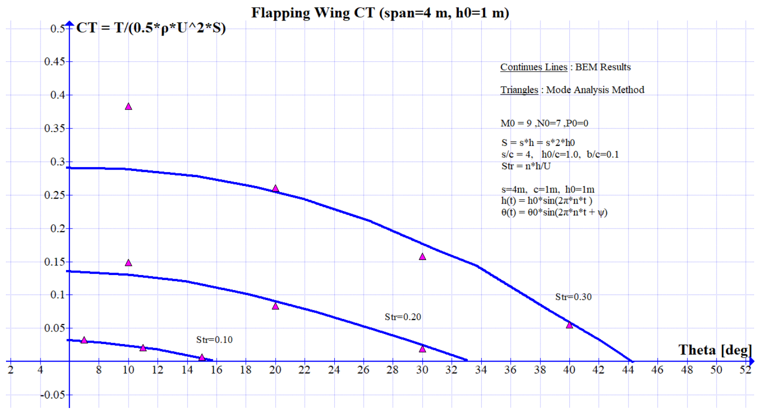


Figure 8.3.3

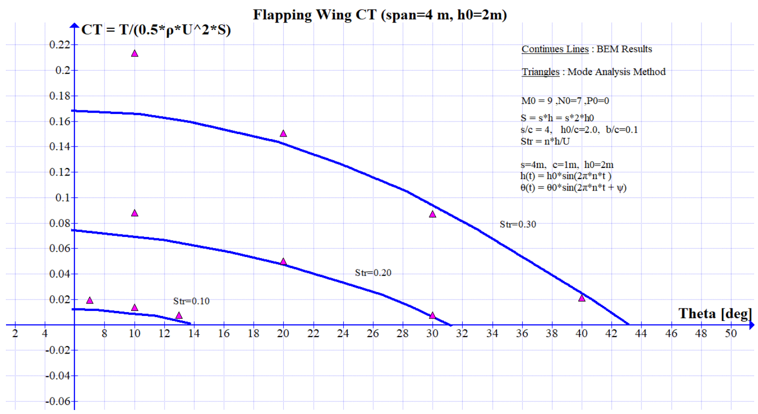


Figure 8.3.4

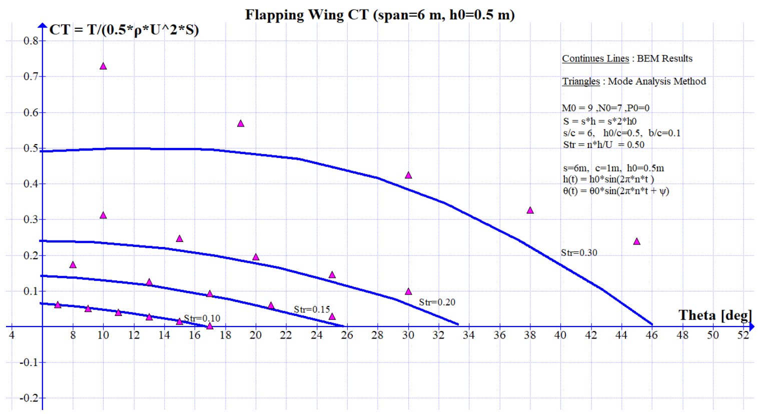


Figure 8.3.5

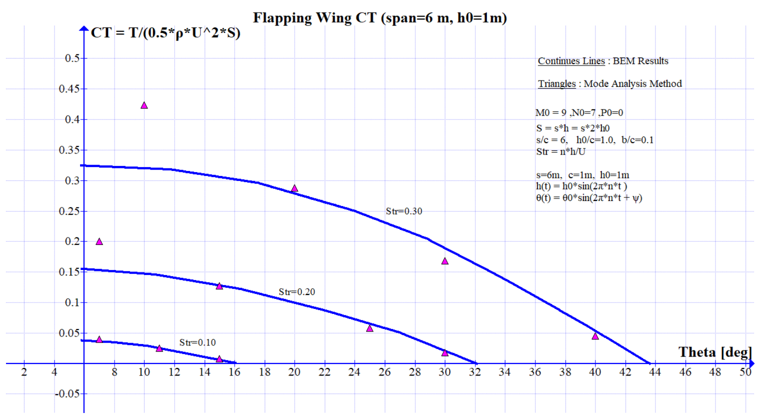


Figure 8.3.6

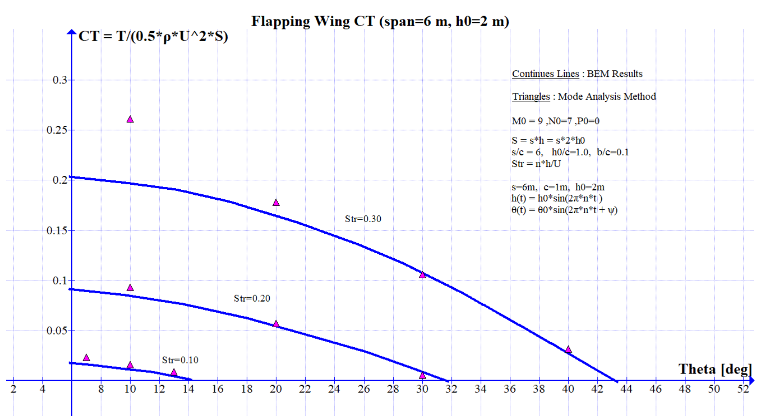


Figure 8.3.7

9 Conclusion-Future work

According to the results of the previous section (Fig 8.3.2 - 8.3.7), it's obvious that the program in its current state produce acceptable results for the light loading and low frequency conditions, while failing to make the correct predictions for the high loading, angles of attack and frequencies. Although this is expected from a linear theory the results of sections 8.1 and 8.2 (steady case and linear unsteady case) suggest that there is still room for improvement in our current version.

It should be noted that the current version of the program solves only a simplified version of the problem using specific boundary conditions at the tips ($\gamma^1 = 0$) and a frozen wake model. In other word it should be seen as very basic solver to test capabilities of the mode analysis method. There is certainly a lot of work that could be done expanding and refining the program, but a few basic suggestions are shown below :

- Improve the processing time using a better numerical scheme for the integrals of induction factors, or making a better distribution of the integration points. This is something that is examined thoroughly in [2].
- Calculate the induced velocities on the wake for a free wake model.
- Make use of the proper Fourier series and implement the correct boundary conditions of (either $p^+ = p^-$, or $\gamma^1 = 0$) at the tips.
- Include the case of multiple surfaces for the case of the propeller.

Appendix A

A.1 Transformation matrices

The following relations can also be found in either [2] or [4].

The matrix of transformation by contravariance $\underset{\triangle}{c}{}^i_r$ between the cylindrical $\underset{\wedge}{x}{}^i$ and the curvilinear $\underline{x}^i$ coordinate system is given by the relation :

$$\underset{\triangle}{c}{}^i_r = \frac{1}{2!}\delta^{ijk}_{rst}\ \underset{\triangle}{\gamma}{}^s_j\ \underset{\triangle}{\gamma}{}^t_k\ /\ \underset{\triangle}{\Gamma} \tag{A1}$$

where :

$$\underset{\triangle}{\Gamma} = \frac{1}{3!}\delta^{ijk}_{rst}\ \underset{\triangle}{\gamma}{}^r_i\ \underset{\triangle}{\gamma}{}^s_j\ \underset{\triangle}{\gamma}{}^t_k \tag{A2}$$

and $\underset{\triangle}{\gamma}{}^i_j$ are the matrices of transformation by covariance which can be calculated directly from relations (2.2.4),(2.2.5) and (2.2.6) :

$$\underset{\triangle}{\gamma}{}^i_j = \frac{\partial \underset{\wedge}{x}{}^i}{\partial \underline{x}^j} \tag{A3}$$

Applying this procedure we get :

$$\underset{\triangle}{\gamma}{}^1_1 = \frac{dX}{du^1} + \frac{dc}{du^1}\left(u^2\sin\phi - \underline{x}^3\cos\phi\right) + c\left(u^2\cos\phi + \underline{x}^3\sin\phi\right)\frac{d\phi}{du^1} \tag{A4}$$

$$\underset{\triangle}{\gamma}{}^1_2 = c\sin\phi \tag{A5}$$

$$\underset{\triangle}{\gamma}{}^1_3 = -c\cos\phi \tag{A6}$$

$$\underset{\triangle}{\gamma}{}^2_1 = 1\ , \qquad \underset{\triangle}{\gamma}{}^2_2 = 0\ , \qquad \underset{\triangle}{\gamma}{}^2_3 = 0 \tag{A7}$$

$$\underset{\triangle}{\gamma}{}^3_1 = \frac{d\Theta}{du^1} + \frac{dc}{du^1}\left(\frac{u^2\cos\phi + \underline{x}^3\sin\phi}{u^1}\right) + c\left(\frac{-u^2\sin\phi + \underline{x}^3\cos\phi}{u^1}\right)\frac{d\phi}{du^1} - \frac{c}{u^1}\left(\frac{u^2\cos\phi + \underline{x}^3\sin\phi}{u^1}\right) \tag{A8}$$

$$\underset{\triangle}{\gamma}{}^3_2 = \frac{c}{u^1}\cos\phi\ , \qquad \underset{\triangle}{\gamma}{}^3_3 = \frac{c}{u^1}\sin\phi \tag{A9}$$

The determinant of the covariant transformation matrix is

$$\underset{\triangle}{\Gamma} = -\frac{c^2}{u^1} \tag{A10}$$ [12]

so using relation (A1) we calculate the components of the contravariant transformation matrix.

$$\underset{\triangle}{c}{}_1^1 = 0\ , \quad \underset{\triangle}{c}{}_2^1 = 1\ , \quad \underset{\triangle}{c}{}_3^1 = 0 \tag{A11}$$

$$\underset{\triangle}{c}{}_1^2 = \sin\phi / c \tag{A12}$$

$$\begin{aligned}\underset{\triangle}{c}{}_2^2 = &-\frac{\sin\phi}{c}\left[\frac{dX}{du^1} + \frac{dc}{du^1}\left(u^2\sin\phi - \underline{x}^3\cos\phi\right) + c\left(u^2\cos\phi + \underline{x}^3\sin\phi\right)\frac{d\phi}{du^1}\right]\\ &-\frac{\cos\phi}{c}\left[u^1\frac{d\Theta}{du^1} + \frac{dc}{du^1}\left(u^2\cos\phi + \underline{x}^3 sin\phi\right) + c\left(-u^2\sin\phi + \underline{x}^3\cos\phi\right)\frac{d\phi}{du^1}\right.\\ &\left.-\frac{c}{u^1}\left(u^2\cos\phi + \underline{x}^3\sin\phi\right)\right]\end{aligned} \tag{A13}$$

$$\underset{\triangle}{c}{}_3^2 = \frac{u^1}{c}\cos\phi \tag{A14}$$

$$\underset{\triangle}{c}{}_1^3 = -\cos\phi / c \tag{A15}$$

$$\begin{aligned}\underset{\triangle}{c}{}_2^3 = &\frac{\cos\phi}{c}\left[\frac{dX}{du^1} + \frac{dc}{du^1}\left(u^2\sin\phi - \underline{x}^3 cos\phi\right) + c\left(u^2\cos\phi + \underline{x}^3\sin\phi\right)\frac{d\phi}{du^1}\right]\\ &-\frac{\sin\phi}{c}\left[u^1\frac{d\Theta}{du^1} + \frac{dc}{du^1}\left(u^2\cos\phi + \underline{x}^3\sin\phi\right) + c\left(-u^2\sin\phi + \underline{x}^3\cos\phi\right)\frac{d\phi}{du^1}\right.\\ &\left.-\frac{c}{u^1}\left(u^2\cos\phi + \underline{x}^3\sin\phi\right)\right]\end{aligned} \tag{A16}$$

$$\underset{\triangle}{c}{}_3^3 = u^1 sin\phi / c \tag{A17}$$

At points on the blade reference surface, $\underline{x}^3 = 0$, formulas (A4) to (A17) degenerate to :

$$\underset{\triangle}{\tilde{\gamma}}{}_1^1 = \frac{dX}{du^1} + \frac{dc}{du^1}u^2\sin\phi + c\ u^2\cos\phi\frac{d\phi}{du^1} \tag{A18}$$

$$\underset{\triangle}{\tilde{\gamma}}{}_2^1 = c\sin\phi \tag{A19}$$

$$\underset{\triangle}{\tilde{\gamma}}{}_3^1 = -c\cos\phi \tag{A20}$$

[12] in c^2, 2 is not a tensor index but a power

$$\underset{\triangle}{\tilde{\gamma}}{}^2_1 = 1 \ , \qquad \underset{\triangle}{\tilde{\gamma}}{}^2_2 = 0 \ , \qquad \underset{\triangle}{\tilde{\gamma}}{}^2_3 = 0 \tag{A21}$$

$$\underset{\triangle}{\tilde{\gamma}}{}^3_1 = \frac{d\Theta}{du^1} + \frac{dc}{du^1}\,\frac{u^2}{u^1}\cos\phi - \frac{c}{u^1}u^2\sin\phi\,\frac{d\phi}{du^1} - \frac{c\,u^2}{u^1u^1}\cos\phi \tag{A22}$$

$$\underset{\triangle}{\tilde{\gamma}}{}^3_2 = \frac{c}{u^1}\cos\phi \tag{A23}$$

$$\underset{\triangle}{\tilde{\gamma}}{}^3_3 = \frac{c}{u^1}\sin\phi \tag{A24}$$

$$\underset{\triangle}{\tilde{c}}{}^1_1 = 0 \ , \qquad \underset{\triangle}{\tilde{c}}{}^1_2 = 1 \ , \qquad \underset{\triangle}{\tilde{c}}{}^1_3 = 0 \tag{A25}$$

$$\underset{\triangle}{\tilde{c}}{}^2_1 = \sin\phi/c \tag{A26}$$

$$\underset{\triangle}{\tilde{c}}{}^2_2 = -\frac{\sin\phi}{c}\,\frac{dX}{du^1} - \frac{d\Theta}{du^1}\,\frac{cos\phi}{c}\,u^1 - \frac{u^2}{c}\,\frac{dc}{du^1} + \frac{u^2}{u^1}\cos^2\phi \tag{A27}$$

$$\underset{\triangle}{\tilde{c}}{}^2_3 = \frac{u^1}{c}\cos\phi \tag{A28}$$

$$\underset{\triangle}{\tilde{c}}{}^3_1 = -\cos\phi/c \tag{A29}$$

$$\underset{\triangle}{\tilde{c}}{}^3_2 = +\frac{\cos\phi}{c}\,\frac{dX}{du^1} - \frac{\sin\phi}{c}\,u^1\,\frac{d\Theta}{du^1} + u^2\,\frac{d\phi}{du^1} + \frac{u^2}{u^1}\,\sin\phi\cos\phi \tag{A30}$$

$$\underset{\triangle}{\tilde{c}}{}^3_3 = u^1 sin\phi/c \tag{A31}$$

Similarly using relations (2.2.1),(2.2.2) and (2.2.3) the matrices of transformation $\underset{\wedge}{c}{}^i_j$ and $\underset{\wedge}{\gamma}{}^i_j$ between the cartesian x^i and the cylindrical $\underset{\wedge}{x}{}^i$ coordinate systems are given by the relations :

$$\underset{\wedge}{\gamma}{}^1_1 = 1 \qquad \underset{\wedge}{\gamma}{}^1_2 = 0 \qquad \underset{\wedge}{\gamma}{}^1_3 = 0 \tag{A32}$$

$$\underset{\wedge}{\gamma}{}^2_1 = 0 \qquad \underset{\wedge}{\gamma}{}^2_2 = \cos\underset{\wedge}{x}{}^3 \qquad \underset{\wedge}{\gamma}{}^2_3 = -\underset{\wedge}{x}{}^2\sin\underset{\wedge}{x}{}^3 \tag{A33}$$

$$\underset{\wedge}{\gamma}{}^3_1 = 0 \qquad \underset{\wedge}{\gamma}{}^3_2 = \sin\underset{\wedge}{x}{}^3 \qquad \underset{\wedge}{\gamma}{}^3_3 = \underset{\wedge}{x}{}^2\cos\underset{\wedge}{x}{}^3 \tag{A34}$$

$$\underset{\wedge}{c}{}^1_1 = 1 \qquad \underset{\wedge}{c}{}^1_2 = 0 \qquad \underset{\wedge}{c}{}^1_3 = 0 \tag{A35}$$

$$\underset{\wedge}{c}{}_1^2 = 0 \qquad \underset{\wedge}{c}{}_2^2 = \cos \underset{\wedge}{x}{}^3 \qquad \underset{\wedge}{c}{}_3^2 = \sin \underset{\wedge}{x}{}^3 \tag{A36}$$

$$\underset{\wedge}{c}{}_1^3 = 0 \qquad \underset{\wedge}{c}{}_2^3 = -\frac{\sin \underset{\wedge}{x}{}^3}{\underset{\wedge}{x}{}^2} \qquad \underset{\wedge}{c}{}_3^3 = \frac{\cos \underset{\wedge}{x}{}^3}{\underset{\wedge}{x}{}^2} \tag{A37}$$

The transformation matrices for the reference surface are produced by setting $\underset{\wedge}{\tilde{x}}{}^3 = \underset{\wedge}{x}{}^3(\underline{x}^3 = 0)$.

The transformation matrices $\underline{\gamma}{}_j^i$, $\underline{c}{}_j^i$ between the orthonormal, x^i, and the general curvilinear system, $\underline{x}^i$, are given by the relations :

$$\underline{c}{}_j^i = \underset{\underline{\wedge}}{c}{}_k^i \; \underset{\wedge}{c}{}_j^k \tag{A38}$$

$$\underline{\gamma}{}_j^i = \underset{\wedge}{\gamma}{}_k^i \; \underset{\underline{\wedge}}{\gamma}{}_j^k \tag{A39}$$

Specifically for the reference surface we have the reletions :

$$\underline{\tilde{c}}{}_j^i = \underset{\underline{\wedge}}{\tilde{c}}{}_k^i \; \underline{\tilde{c}}{}_j^k \tag{A40}$$

$$\underline{\tilde{\gamma}}{}_j^i = \underset{\wedge}{\tilde{\gamma}}{}_k^i \; \underset{\underline{\wedge}}{\tilde{\gamma}}{}_j^k \tag{A41}$$

Using relation (A49) we calculate the values of $\underline{\tilde{\gamma}}{}_j^i$ since they will be used later on :

$$\underline{\tilde{\gamma}}{}_1^1 = \frac{dX}{du^1} + \frac{dc}{du^1} u^2 \sin\phi + c\, u^2 \cos\phi \, \frac{d\phi}{du^1} \tag{A42}$$

$$\underline{\tilde{\gamma}}{}_1^2 = \cos \underset{\wedge}{\tilde{x}}{}^3 - u^1 \; \underset{\underline{\wedge}}{\tilde{\gamma}}{}_1^3 \; \sin \underset{\wedge}{\tilde{x}}{}^3 \tag{A43}$$

$$\underline{\tilde{\gamma}}{}_1^3 = \sin \underset{\wedge}{\tilde{x}}{}^3 + u^1 \; \underset{\underline{\wedge}}{\tilde{\gamma}}{}_1^3 \; \cos \underset{\wedge}{\tilde{x}}{}^3 \tag{A44}$$

$$\underline{\tilde{\gamma}}{}_2^1 = c \;\sin\phi \quad , \qquad \underline{\tilde{\gamma}}{}_3^1 = -c \cdot \cos\phi \tag{A45}$$

$$\underline{\tilde{\gamma}}{}_2^2 = -c \;\cos\phi \;\sin \underset{\wedge}{\tilde{x}}{}^3 \quad , \qquad \underline{\tilde{\gamma}}{}_3^2 = -c \cdot \sin\phi \;\sin \underset{\wedge}{\tilde{x}}{}^3 \tag{A46}$$

$$\underline{\tilde{\gamma}}{}_2^3 = c \;\cos\phi \;\cos \underset{\wedge}{\tilde{x}}{}^3 \quad , \qquad \underline{\tilde{\gamma}}{}_3^3 = c \cdot \sin\phi \;\cos \underset{\wedge}{\tilde{x}}{}^3 \tag{A47}$$

Below we calculate the matrices of transformation $\underset{\underline{\smile}}{\gamma}{}_j^i$, $\underset{\underline{\smile}}{c}{}_j^i$ between the curvilinear systems, $\underline{x}^i$, $\underset{\smile}{x}{}^i$ of the reference surface :

$$\underset{\smile}{\gamma^{1}_{\underline{1}}} = \frac{1}{2}\,(R_o - R_H)\,\sin y^1 \qquad \underset{\smile}{\gamma^{1}_{\underline{2}}} = 0 \qquad \underset{\smile}{\gamma^{1}_{\underline{3}}} = 0 \tag{A48}$$

$$\underset{\smile}{\gamma^{2}_{\underline{1}}} = 0 \qquad \underset{\smile}{\gamma^{2}_{\underline{2}}} = \frac{1}{2}\,\sin y^2 \qquad \underset{\smile}{\gamma^{2}_{\underline{3}}} = 0 \tag{A49}$$

$$\underset{\smile}{\gamma^{3}_{\underline{1}}} = 0 \qquad \underset{\smile}{\gamma^{3}_{\underline{2}}} = 0 \qquad \underset{\smile}{\gamma^{3}_{\underline{3}}} = 1 \tag{A50}$$

The determinant of the covariant transformation is : $\underset{\smile}{\underline{\Gamma}} = \frac{1}{4}\,(R_o - R_H)\sin y^1 \sin y^2$ (A51)

$$\underset{\smile}{c^{1}_{\underline{1}}} = \frac{2}{(R_o - R_H)\,\sin y^1} \qquad \underset{\smile}{c^{1}_{\underline{2}}} = 0 \qquad \underset{\smile}{c^{1}_{\underline{3}}} = 0 \tag{A52}$$

$$\underset{\smile}{c^{2}_{\underline{1}}} = 0 \qquad \underset{\smile}{c^{2}_{\underline{2}}} = \frac{2}{\sin y^2} \qquad \underset{\smile}{c^{2}_{\underline{3}}} = 0 \tag{A53}$$

$$\underset{\smile}{c^{3}_{\underline{1}}} = 0 \qquad \underset{\smile}{c^{3}_{\underline{2}}} = 0 \qquad \underset{\smile}{c^{3}_{\underline{3}}} = 1 \tag{A54}$$

Finally the matrices of transformation $\underset{\smile}{\gamma^i_j}$, $\underset{\smile}{c^i_j}$, between the orthornormal, x^i, and the curvilinear, $\underset{\smile}{x^i}$ system are :

$$\underset{\smile}{\gamma^i_j} = \underline{\gamma}^i_k \; \underset{\smile}{\gamma^k_{\underline{j}}} \,, \qquad \underset{\smile}{c^i_j} = \underset{\smile}{c^i_{\underline{k}}} \; \underline{c}^k_j \tag{A55),(A56}$$

Specifically for the reference surface we get :

$$\underset{\smile}{\tilde{\gamma}^i_j} = \tilde{\underline{\gamma}}^i_k \; \underset{\smile}{\gamma^k_{\underline{j}}} \,, \qquad \underset{\smile}{\tilde{c}^i_j} = \underset{\smile}{c^i_{\underline{k}}} \; \tilde{\underline{c}}^k_j \tag{A58),(A57}$$

since $\underset{\smile}{\tilde{\gamma}^k_{\underline{j}}} = \underset{\smile}{\gamma^k_{\underline{j}}}$, $\underset{\smile}{\tilde{c}^i_{\underline{k}}} = \underset{\smile}{c^i_{\underline{k}}}$

Note that : $\underset{\smile}{x^r_\alpha} \equiv \underset{\smile}{\gamma^r_\alpha} = \dfrac{\partial x^r}{\partial \underset{\smile}{y^\alpha}}$ (A59)

Using the last relation we get :

$$\underset{\smile}{\tilde{\gamma}^r_1} = \; \tilde{\underline{\gamma}}^r_1 \; \frac{1}{2}\,(R_o - R_H)\,\sin y^1 \tag{A60}$$

$$\underset{\smile}{\tilde{\gamma}^r_2} = \; \tilde{\underline{\gamma}}^r_2 \; \frac{1}{2}\,\sin y^2 \tag{A61}$$

A.2 Metric Tensors

Having calculate the transformation matrices by covariant and contravariant between the various coordinate systems we can easily calculate the corresponding metric tensors $\underset{\wedge}{g}^{ij}$, $\underline{g}^{ij}$, from the existing metric tensor of the orthonormal system $g_{ij} = \delta_{ij}$, $g^{ij} = \delta^{ij}$.

For the cylindrical system the covariant components of the metric tensor are :

$$\underset{\wedge}{g}_{ij} = g_{rs}\ \underset{\wedge}{\gamma}_i^r \underset{\wedge}{\gamma}_j^s \tag{A62}$$

or

$$\underset{\wedge}{g}_{11} = 1 \qquad \underset{\wedge}{g}_{12} = 0 \qquad \underset{\wedge}{g}_{13} = 0 \tag{A63}$$

$$\underset{\wedge}{g}_{21} = 0 \qquad \underset{\wedge}{g}_{22} = 1 \qquad \underset{\wedge}{g}_{23} = 0 \tag{A64}$$

$$\underset{\wedge}{g}_{31} = 0 \qquad \underset{\wedge}{g}_{32} = 0 \qquad \underset{\wedge}{g}_{33} = \underset{\wedge}{x}^2 \underset{\wedge}{x}^2 \tag{A65}$$

For the contravariant components we have :

$$\underset{\wedge}{g}^{ij} = g^{rs}\ \underset{\wedge}{c}_r^i \underset{\wedge}{c}_s^j \tag{A66}$$

or

$$\underset{\wedge}{g}^{11} = 1 \qquad \underset{\wedge}{g}^{12} = 0 \qquad \underset{\wedge}{g}^{13} = 0 \tag{A67}$$

$$\underset{\wedge}{g}^{21} = 0 \qquad \underset{\wedge}{g}^{22} = 1 \qquad \underset{\wedge}{g}^{23} = 0 \tag{A68}$$

$$\underset{\wedge}{g}^{31} = 0 \qquad \underset{\wedge}{g}^{32} = 0 \qquad \underset{\wedge}{g}^{33} = \frac{1}{\underset{\wedge}{x}^2 \underset{\wedge}{x}^2} \tag{A69}$$

Using relations similar to (A62) and (A66) we can calculate $\underline{g}_{ij}$, $\underline{g}^{ij}$. Since we are interested in the metric tensor of the reference surface (system u^α) :

$$\underline{\alpha}_{\alpha\beta} = \left(\underline{g}_{ij}\right)_{i=\alpha, j=\beta} = \delta_{ij}\ \overset{\sim i}{\underline{\gamma}}_\alpha\ \overset{\sim j}{\underline{\gamma}}_\beta \tag{A70}$$

Expanding the above relation we have :

$$\underline{\alpha}_{11} = \overset{\sim 1}{\underline{\gamma}}_1 \overset{\sim 1}{\underline{\gamma}}_1 + \overset{\sim 2}{\underline{\gamma}}_1 \overset{\sim 2}{\underline{\gamma}}_1 + \overset{\sim 3}{\underline{\gamma}}_1 \overset{\sim 3}{\underline{\gamma}}_1 \tag{A71}$$

$$\underline{\alpha}_{22} = \overset{\sim 1}{\underline{\gamma}}_2 \overset{\sim 1}{\underline{\gamma}}_2 + \overset{\sim 2}{\underline{\gamma}}_2 \overset{\sim 2}{\underline{\gamma}}_2 + \overset{\sim 3}{\underline{\gamma}}_2 \overset{\sim 3}{\underline{\gamma}}_2 \tag{A72}$$

$$\underline{\alpha}_{12} = \underline{\alpha}_{21} = \overset{\sim 1}{\underline{\gamma}}_1 \overset{\sim 1}{\underline{\gamma}}_2 + \overset{\sim 2}{\underline{\gamma}}_1 \overset{\sim 2}{\underline{\gamma}}_2 + \overset{\sim 3}{\underline{\gamma}}_1 \overset{\sim 3}{\underline{\gamma}}_2 \tag{A73}$$

where the $\underline{\tilde{\gamma}}^{r}_{\alpha}$ are given by the relation (A42) to (A47).

Finally the metric tensor $\underset{\smile}{g}_{ij}$ for the $\underset{\smile}{x}^{i}$ coordinate system is :

$$\underset{\smile}{g}_{ij} = \underline{g}_{rs}\ \underset{\smile}{\underline{\gamma}}{}^{r}_{i}\ \underset{\smile}{\underline{\gamma}}{}^{s}_{j}$$

$$\underset{\smile}{g}_{11} = \underline{g}_{11}\ \frac{1}{4}\,(R_o - R_H)^2\ \sin^2 y^1 \tag{A74}$$

$$\underset{\smile}{g}_{12} = \underset{\smile}{g}_{21} = \underline{g}_{12}\ \frac{1}{4}\,(R_o - R_H)\ \sin y^1 \sin y^2 \tag{A75}$$

$$\underset{\smile}{g}_{13} = \underset{\smile}{g}_{31} = \underline{g}_{13}\ \frac{1}{2}\,(R_o - R_H)\ \sin y^1 \tag{A76}$$

$$\underset{\smile}{g}_{22} = \underline{g}_{22}\ \frac{1}{4}\ \sin^2 y^2 \tag{A77}$$

$$\underset{\smile}{g}_{23} = \underset{\smile}{g}_{32} = 0 \qquad (\text{since } \underline{g}_{23} = 0) \tag{A78}$$

$$\underset{\smile}{g}_{33} = \underline{g}_{33} \tag{A79}$$

A.3 Metric Determinants

We can derive the space metric tensor determinant using relation (1.16) and as for the surface determinants we use the following relations :

$$\underline{\alpha} = \underline{\alpha}_{11}\ \underline{\alpha}_{22} - \underline{\alpha}_{12}\ \underline{\alpha}_{21} \tag{A80}$$

$$\underset{\smile}{\alpha} = \underset{\smile}{\alpha}_{11}\ \underset{\smile}{\alpha}_{22} - \underset{\smile}{\alpha}_{12}\ \underset{\smile}{\alpha}_{21} \tag{A81}$$

Because of the simple relation connecting the systems $\underline{x}^{i}$, $\underset{\smile}{x}^{i}$ we can easily calculate the determinants $\underset{\smile}{g}$, $\underset{\smile}{\alpha}$ given the $\underline{g}$, $\underline{\alpha}$. We shall try to derive the space and surface metric determinants for the $\underline{x}^{i}$ and u^{α} coordinate systems, not by calculating their components but instead by using their property to be relative tensors of weight two. In particular, by choosing a coordinate system in which the calculation of the metric tensor is simple, the previous determinant is calculated from relation (1.12) of the definition of relative tensors.

For the $\underset{\wedge}{x}^{i}$ system from relations (A63) to (A65) we get :

$$\underset{\wedge}{g} = \underset{\wedge}{x}^{2}\underset{\wedge}{x}^{2} \tag{A82}$$

and from relation (A10) of the determinant of the covariant transformation, $\underset{\triangle}{\Gamma}$, between the, $\underset{\wedge}{x}^i$ and $\underline{x}^i$ we get :

$$\underline{g} \;=\; \underset{\wedge}{g}\,(\underset{\triangle}{\Gamma})^2 = (u^1)^2 \left[-\left(\frac{c^2}{u^1}\right)\right]^2 = c^4 \tag{A83}$$ [13]

Note that since the chord length, c, depends on our position on the reference line, $\underline{g}$ will also be a function of u^1. Now in order to calculate the surface determinant $\underline{\alpha}$ we choose a new coordinate system according to the relations :

$$\eta^1 = \underset{\wedge}{x}^2 \qquad\qquad \eta^2 = \underset{\wedge}{x}^3$$

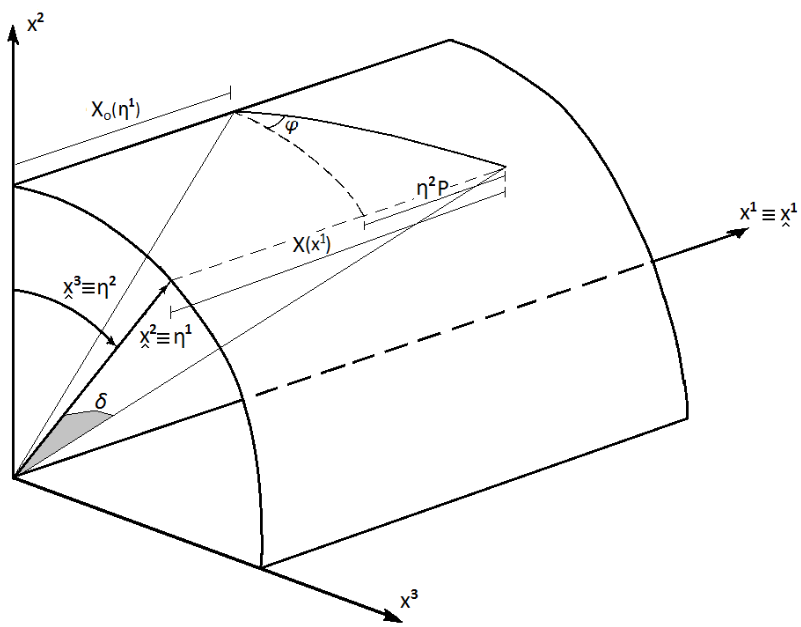
so the parametric equations of the reference surface become :

$$\underset{\wedge}{x}^1 = X_o(\eta^1) \,+\, \eta^1\,\eta^2\,\tan\phi(\eta^1) \tag{A84}$$

$$\underset{\wedge}{x}^2 = \eta^1 \tag{A85}$$

$$\underset{\wedge}{x}^3 = \eta^2 \tag{A86}$$

Note that, $X(u^1) = X_o(\eta^1) + \Theta(\eta^1)\,u^1\,\tan\phi$, where X_o is the rake of the reference line and θ the skew-angle.



FigureA.1

[13]in c^4, 4 is not a tensor index but a power

We can calculate the reference surface metric tensor to be :

$$\underset{\cdot}{\alpha_{\alpha\beta}} = \underset{\wedge}{g_{ij}} \; \frac{\partial \underset{\wedge}{x^i}}{\partial \eta^\alpha} \; \frac{\partial \underset{\wedge}{x^j}}{\partial \eta^\beta} \tag{A87}$$

where $\underset{\wedge}{g_{ij}}$ is given by the relations (A63),(A64) and (A65).

$$\underset{\cdot}{\alpha_{11}} = \tan^2 \delta + 1 \tag{A88}$$

$$\underset{\cdot}{\alpha_{12}} = P \;\; \tan \delta \tag{A89}$$

$$\underset{\cdot}{\alpha_{22}} = P^2 + \underset{\wedge}{x^2} \, \underset{\wedge}{x^2} \tag{A90}$$

where

$$P = \eta^1 \;\; \tan \phi(u^1) \tag{A91}$$

$$\tan \delta = \frac{\partial \underset{\wedge}{x^1}}{\partial \eta^1} = \frac{dX_o}{d\eta^1} + \eta^2 \; \frac{\partial P}{\partial \eta^1} \tag{A92}$$

Also

$$\underset{\cdot}{\alpha} \;=\; \underset{\cdot}{\alpha_{11}} \, \underset{\cdot}{\alpha_{22}} \;-\; \underset{\cdot}{\alpha_{12}} \, \underset{\cdot}{\alpha_{21}} \;=\; \tan^2 \delta \; (\underset{\wedge}{x^2} \underset{\wedge}{x^2}) + P^2 + \underset{\wedge}{x^2} \underset{\wedge}{x^2} \tag{A93}$$

Surface coordinates η^1, η^2 and u^1, u^2 are related by :

$$\eta^1 = u^1$$

$$\eta^2 = u^2 \, c \, \cos \phi(u^1)/u^1 + \Theta(u^1)$$

The determinant of the transformation by covariance between η^α and u^α is given by :

$$\gamma = \frac{\partial \eta^1}{\partial u^1} \frac{\partial \eta^2}{\partial u^2} - \frac{\partial \eta^1}{\partial u^2} \frac{\partial \eta^2}{\partial u^1} = \frac{c \, \cos \phi(u^1)}{u^1} \tag{A94}$$

Since the surface metric tensor is a relative tensor of weight two we have :

$$\begin{aligned} \underline{\alpha} \;&=\; \underset{\cdot}{a} \, \gamma^2 \\ &=\; (\tan^2 \delta \; (x^2 x^2) + P^2 + x^2 x^2) \;\; (c \, \cos \phi / \eta^1)^2 \\ &=\; c^2 \; (\tan^2 \delta \cos^2 \phi + \frac{P^2 \cos^2 \phi}{\eta^1 \eta^1} + \cos^2 \phi) \\ &=\; c^2 \; (\tan^2 \delta \cos^2 \phi + 1) \end{aligned} \tag{A95}$$

The determinant $\underset{\smile}{g}$, is given by the relation $\underset{\smile}{g} = e^{ijk}\, \underset{\smile}{g}_{i1}\, \underset{\smile}{g}_{j2}\, \underset{\smile}{g}_{k3}$ or using relations (A74) to (A79)

$$\underset{\smile}{g} = \left[\frac{1}{16} \left(R_o - R_H \right)^2 \;\sin^2 y^1 \sin^2 y^2 \right] \; \underline{g} \tag{A96}$$

Similarly

$$\underset{\smile}{\alpha} = \left[\frac{1}{16} \left(R_o - R_H \right)^2 \;\sin^2 y^1 \sin^2 y^2 \right] \; \underline{\alpha} \tag{A97}$$

A.4 Wing Transformation matrices

The matrix transformation $\underline{\gamma}^i_j$ between the orthonormal and the curvilinear system in the case of the wing can be calculated directly from relations (2.2.9), (2.2.10) and (2.2.11).

$$\underline{\gamma}^1_1 = \frac{dX}{du^1} + \frac{dc}{du^1} \left[u^2 \sin\phi - \underline{x}^3 \cos\phi \right] + c \left[u^2 \cos\phi + \underline{x}^3 \sin\phi \right] \frac{d\phi}{du^1} \tag{A98}$$

$$\underline{\gamma}^1_2 = c \sin\phi \tag{A99}$$

$$\underline{\gamma}^1_3 = -c \cos\phi \tag{A100}$$

$$\underline{\gamma}^2_1 = 1 \; , \qquad \underline{\gamma}^2_2 = 0 \; , \qquad \underline{\gamma}^2_3 = 0 \tag{A101}$$

$$\underline{\gamma}^3_1 = \tan\Theta + \frac{u^1}{\cos^2\Theta} \frac{d\Theta}{du^1} + \frac{dc}{du^1} \; \left[u^2 \cos\phi + \underline{x}^3 \sin\phi \right] \; + \; c \left[-u^2 \sin\phi + \underline{x}^3 \cos\phi \right] \frac{d\phi}{du^1} \tag{A102}$$

$$\underline{\gamma}^3_2 = c \cos\phi \; , \qquad \underline{\gamma}^3_3 = c \sin\phi \tag{A103}$$

For the points on the reference surface we set $\underline{x}^3 = 0$:

$$\underline{\tilde{\gamma}}^1_1 = \frac{dX}{du^1} + \frac{dc}{du^1} u^2 \sin\phi + c\, u^2 \cos\phi \, \frac{d\phi}{du^1} \tag{A104}$$

$$\underline{\tilde{\gamma}}^1_2 = c \sin\phi \tag{A105}$$

$$\underline{\tilde{\gamma}}^1_3 = -c \cos\phi \tag{A106}$$

$$\underline{\tilde{\gamma}}^2_1 = 1 \; , \qquad \underline{\tilde{\gamma}}^2_2 = 0 \; , \qquad \underline{\tilde{\gamma}}^2_3 = 0 \tag{A107}$$

$$\underline{\tilde{\gamma}}^3_1 = \tan\Theta + \frac{u^1}{\cos^2\Theta} \frac{d\Theta}{du^1} + \frac{dc}{du^1} \; u^2 \cos\phi \; - \; c\, u^2 \sin\phi \, \frac{d\phi}{du^1} \tag{A108}$$

$$\underline{\tilde{\gamma}}^3_2 = c \cos\phi \; , \qquad \underline{\tilde{\gamma}}^3_3 = c \sin\phi \tag{A109}$$

Notice the similarities between relations (A104) to (A109) with the corresponding relations (A42) to (A47). In fact setting $\underset{\wedge}{x}^3 = 0$ while $u^1 \rightarrow \infty$, they coincide.

The determinant of the transformation is : $$\underline{\Gamma} = -c^2 \tag{A110}$$

Thus the metric of the system is : $$\underline{g}_{ij} = \delta_{kl}\, \underline{\gamma}_i^k\, \underline{\gamma}_j^l \;\Rightarrow\; \underline{g} = \underline{\Gamma}^2 = c^4 \tag{A111}$$

Appendix B: Normal and Tangential projection of a vector on a given surface

The following relations can also be found in [4] - Part I.

Let u^a an intrinsic to a given surface coordinate system and $x^i(u^1, u^2)$ the parametric equations of the surface.

The projection of a vector q^i normal to the $x^i(u^1, u^2)$ surface, with N_i the normal unit vector is:

$$\overset{Ni}{q} = N^i \left(q^j N_j\right) \tag{B1}$$

while the projection of q^i tangential to the given surface is :

$$\overset{Ti}{q} = \varepsilon^{ijk} \left(\varepsilon_{jmn} N^m q^n\right) N_k \tag{B2}$$

Another useful form of relation (B2) can be obtained as follows : Raise the indices j,m,n in ε_{jmn} and lower the indices of N^m and q^n in relation (B2) to obtain:

$$\overset{Ti}{q} = \varepsilon^{ijk} \left(\varepsilon^{tmn} N_m q_n g_{jt}\right) N_k \tag{B3}$$

Using relation (McConnell, page 197):

$$\varepsilon^{\alpha\beta} x^r_\alpha x^s_\beta = \varepsilon^{mrs} N_m \tag{B4}$$

relation (B3) becomes:

$$\overset{Ti}{q} = \left(\varepsilon^{\alpha\beta} x^i_\alpha x^j_\beta\right) \left(\varepsilon^{\gamma\delta} x^n_\gamma x^t_\delta\right) g_{jt} q_n = (\varepsilon^{\alpha\beta} x^i_\alpha) (\varepsilon^{\gamma\delta} x^n_\gamma) \alpha_{\beta\delta} q_n \tag{B5}$$

and since (McConnel, page 166):

$$\frac{1}{\alpha} e^{\alpha\beta} e^{\gamma\delta} \alpha_{\beta\delta} = \alpha^{\alpha\gamma} \tag{B6}$$

relation (B5) becomes :

$$\overset{Ti}{q} = x^i_\alpha \left(x^n_\gamma \alpha^{\alpha\gamma} q_n\right) \tag{B7}$$

Form (B7) has the advantage of providing the tangential projection using only the surface metric tensor and the partial derivatives:

$$x^i_\alpha = \frac{\partial x^i}{\partial u^\alpha} \tag{B8}$$

of the parametric equations of the surface. Using (B7) the tensor as a function of his projection is given by the equation :

$$q^i \;=\; \overset{Ti}{q} + \overset{Ni}{q} \;=\; x^i_\alpha \left(x^n_\gamma \, \alpha^{\alpha\gamma} \, q_n \right) \,+\, N^i \left(q^j \, N_j \right) \tag{B9}$$

In another representation, the vector q^i can be characterized by the auxiliary parameters σ and γ_j defined by the relations :

$$\sigma \;\overset{def}{=}\; q^j \, N_j \tag{B10}$$

$$\gamma_j \;\overset{def}{=}\; \varepsilon_{jmn} \, N^m \, q^n \tag{B11}$$

Using these parameters relations (B1) and (B2) become :

$$\overset{Ni}{q} \;=\; N^i \, \sigma \tag{B12}$$

$$\overset{Ti}{q} \;=\; \varepsilon^{ijk} \, \gamma_j \, N_k \tag{B13}$$

Thus q^i as a function of σ and γ_j becomes

$$q^i \;=\; \varepsilon^{ijk} \, \gamma_j \, N_k \,+\, N^i \, \sigma \tag{B14}$$

In relation (B11) γ_j is by definition tangent to the given surface, therefore exists a surface vector γ_α, corresponding to the space vector γ_j there given by (McConnell, page 196):

$$\gamma_\alpha \;=\; x^j_\alpha \, \gamma_j \tag{B15}$$

We shall now try to express γ_α as a function of q_n, x^j_α and the surface metric tensor $\alpha_{\alpha\beta}$. Substituting relation (B11) to relation (B15) we get :

$$\gamma_\alpha \;=\; x^j_\alpha \, \left(\varepsilon_{jmn} \, N^m \, q^n \right) \;=\; x^j_\alpha \, \left(\varepsilon^{tmn} \, N_m \, q_n \right) g_{tj} \tag{B16}$$

Raising the index α in relation (B16) and using relation (B4) we get:

$$\gamma^\delta \;=\; x^j_\alpha \, \alpha^{\alpha\delta} \left[\left(\varepsilon^{\beta\gamma} \, x^n_\beta \, x^t_\gamma \right) \, q_n \right] \, g_{tj} \;=\; \left(\alpha^{\alpha\delta} \, \alpha_{\alpha\,\gamma} \right) \varepsilon^{\beta\gamma} \, x^n_\beta \, q_n \;=\; \varepsilon^{\beta\delta} \, x^n_\beta \, q_n \tag{B17}$$

which is the required relation.

Appendix C: Equation of continuity for the surface distribution of vorticity

The following relations can also be found in [4] - Part I.

For the development of this appendix, n_i is a unit vector normal to the blade reference surface. Therefore by definition :

$$\gamma^i \;=\; \varepsilon^{ilm}\, n_l\, \mu_{,m} \tag{C1}$$

where $\mu_{,m}$ is the covariant derivative of the scalar potential difference on blade reference surface : $\mu \;=\; \Phi^+ \;-\; \Phi^-$

Since γ^i is a vector tangent to the blade reference surface ($\gamma^i\, n_i \;=\; 0$), we transform it to its surface coordinates, using relation (1.32) :

$$\gamma_\alpha \;=\; \gamma_j\, x^j_\alpha \;=\; \gamma^i\, g_{ij}\, x^j_\alpha \tag{C2}$$

where $x^i_\alpha = \frac{\partial x^j}{\partial u^\alpha}$ and $x^j = f^j(u^1, u^2)$ are the parametric equations of the blade reference surface in the x^j coordinate system. Substituting (C1) to (C2) we get :

$$\gamma_\alpha \;=\; \left(\varepsilon^{ilm}\, n_l\, \mu_{,m}\right)\, g_{ij}\, x^j_\alpha \tag{C3}$$

Expressing the normal n_l to the blade reference surface as a function of the parametric equations of the surface :

$$n_l \;=\; \frac{1}{2}\, \varepsilon^{\beta\gamma}\, \varepsilon_{lst}\, x^s_\beta\, x^t_\gamma \tag{C4}$$

and substituting to relation (C3) we get :

$$\begin{aligned}
\gamma_\alpha \;&=\; \left(\varepsilon^{ilm}\, n_l\, \mu_{,m}\right)\, g_{ij}\, x^j_\alpha \;= \\
&=\; \left[\varepsilon^{ilm}\, \left(\frac{1}{2}\, \varepsilon^{\beta\gamma}\, \varepsilon_{lst}\, x^s_\beta\, x^t_\gamma\right)\, \mu_{,m}\right]\, g_{ij}\, x^j_\alpha \;= \\
&=\; \left(-\varepsilon^{lim}\, \varepsilon_{lst}\right)\, \frac{1}{2}\, \varepsilon^{\beta\gamma}\, x^s_\beta\, x^t_\gamma\, \mu_{,m}\, g_{ij}\, x^j_\alpha \;= \\
&=\; \left(\delta^i_t\, \delta^m_s \;-\; \delta^i_s\, \delta^m_t\right)\, \frac{1}{2}\, \varepsilon^{\beta\gamma}\, x^s_\beta\, x^t_\gamma\, \mu_{,m}\, g_{ij}\, x^j_\alpha \quad \overset{(*)}{=} \quad \varepsilon^{\beta\gamma}\, x^s_\beta\, \mu_{,s}\, \alpha_{\gamma\alpha}
\end{aligned} \tag{C5}$$

The transition of the last relation (*) should be obvious, though we give it below for the sake of completeness.

$$\stackrel{(*)}{=} \left(\delta^i_t\,\delta^m_s \;-\; \delta^i_s\,\delta^m_t\right)\,\frac{1}{2}\,\varepsilon^{\beta\gamma}\,x^s_\beta\,x^t_\gamma\,\mu_{,m}\,g_{ij}\,x^j_\alpha \;=$$

$$= \;\frac{1}{2}\varepsilon^{\beta\gamma}\,x^t_\gamma\,\left(x^s_\beta\,\mu_{,s}\right)\,g_{tj}\,x^j_\alpha \;-\; \frac{1}{2}\varepsilon^{\beta\gamma}\,x^s_\beta\,\left(x^t_\gamma\,\mu_{,t}\right)\,g_{sj}\,x^j_\alpha \;=$$

$$= \;\frac{1}{2}\,\varepsilon^{\beta\gamma}\left(x^s_\beta\,\mu_{,s}\right)\,\alpha_{\gamma\alpha} \;-\; \frac{1}{2}\,\varepsilon^{\beta\gamma}\left(x^t_\gamma\,\mu_{,t}\right)\alpha_{\beta\alpha} \;=$$

$$= \;\varepsilon^{\beta\gamma}\left(x^s_\beta\,\mu_{,s}\right)\,\alpha_{\gamma\alpha}$$

Raising the index α in relation (C5) by multiplying by $\alpha^{\alpha\delta}$ in both sides, we get :

$$\gamma^\delta \;=\; \varepsilon^{\beta\delta}\,\mu_{,s}\,x^s_\beta \tag{C6}$$

Take now the covariant derivative of relation (C6) and contract the contravariant with the covariant indicies of the surface vorticity to get :

$$\gamma^\delta_{,\delta} \;=\; \varepsilon^{\beta\delta}\,\mu_{,s\,\delta}\,x^s_\beta \;+\; \varepsilon^{\beta\,\delta}\,\mu_{,s}\,x^s_{\beta\,,\delta} \;+\; \varepsilon^{\beta\,\delta}_{\;\;,\delta}\,\mu_{,s}\,x^s_\beta \tag{C7}$$

But

$$\varepsilon^{\beta\,\delta}_{\;\;,\delta} \;=\; 0 \tag{C8}$$

$$x^s_{\delta\,,\beta} \;=\; x^s_{\beta\,,\delta} \tag{C9}$$

$$\mu_{,s\delta}\,x^s_\beta \;=\; \mu_{,s\,\beta}\,x^s_\delta \tag{C10}$$

where (C8) is a well known property of the altenating tensor (McConnel page 200) and the symmetry properties (C9) and (C10) can be prooved by direct calculations. Using furthermore the property of the alternating tensor $\varepsilon^{\alpha\beta}\,A_{\alpha\beta} \;=\; 0$ when $A_{\alpha\beta}$ is symmetric with respect to its indices, relation (C7) becomes :

$$\gamma^\delta_{,\delta} \;=\; 0 \tag{C11}$$

which is the required surface continuity equation. For flat surfaces coordinatized by rectilinear coordinates the determinant of the surface metric tensor becomes $\alpha = \textit{constant}$ and relation (C11) degenerates to the known form :

$$\frac{\partial\gamma^1}{\partial u^1} \;+\; \frac{\partial\gamma^2}{\partial u^2} \;=\; 0 \tag{C12}$$

Appendix D: Supplement to paragraph 3.6

The following relations can also be found in [4] - Part I.
Using relation (3.6.7) we get for the surface metric tensor of the real blade surface S_u (or S_l)

$$\underline{\alpha}'_{\alpha\beta} = \underline{g}_{ij}\, \underline{x}^i_\alpha\, \underline{x}^j_\beta = \underline{g}_{ij}\underline{\tilde{x}}^i_\alpha\underline{\tilde{x}}^j_\beta + \underline{g}_{ij}\underline{\tilde{x}}^i_\alpha\delta\underline{\tilde{x}}^j_\beta + \underline{g}_{ij}\delta\underline{\tilde{x}}^i_\alpha\underline{\tilde{x}}^j_\beta + \underline{g}_{ij}\delta\underline{\tilde{x}}^i_\alpha\delta\underline{\tilde{x}}^j_\beta \tag{D1}$$

where $\underline{\alpha}'_{\alpha\beta}$ refers to the real blade surface and $\underline{\alpha}_{\alpha\beta}$ to the reference surface and $\underline{g}_{ij} = \tilde{\underline{g}}_{ij} + O(\varepsilon)$.

$$\underline{\alpha}' = e^{\alpha\beta}\underline{\alpha}'_{\alpha 1}\underline{\alpha}'_{\beta 2} \overset{(D1)}{=} e^{\alpha\beta}\underline{\alpha}_{\alpha 1}\underline{\alpha}_{\beta 2} + O(\varepsilon) = \underline{\alpha} + O(\varepsilon) \tag{D2}$$

Using relation (D2) we get :

$$dS_{u_Z} = \sqrt{\underline{\alpha}'}\, du^1\, du^2 = \sqrt{\underline{\alpha} + O(\varepsilon)}\, du^1\, du^2 = \sqrt{\underline{\alpha}}\, du^1\, du^2 + O(\varepsilon) = dS_{R_Z} + O(\varepsilon) \tag{D3}$$

Thus relation (3.6.9) has been proved.

To prove relation (3.6.10) expand v^k in a Taylor series around $\tilde{v}^k$ to get:

$$v^k = \tilde{v}^k + \frac{\partial \tilde{v}^k}{\partial x^j}\,\delta x^j + O(\varepsilon^2) = \tilde{v}^k + O(\varepsilon^2) \tag{D4}$$

since both $\dfrac{\partial \tilde{v}^k}{\partial x^j}$ and δx^j are of first order.

To prove relation (3.6.11) we use a coordinate system originating from point P. For this system :

$$r = \left(x^i(Q)\, x_i(Q)\right)^{1/2} \tag{D5}$$

Using relation (3.6.7) we get :

$$r^2 = \tilde{r}^2 + 2\,\tilde{x}^i\,\delta x_i = \tilde{r}^2 + O(\varepsilon) \tag{D6}$$

where $\tilde{r}$ is defined by relation (3.6.12). Then

$$\frac{1}{r^3} = \frac{1}{\left[\tilde{r}^2 + O(\varepsilon)\right]^{3/2}} = \frac{1}{\tilde{r}^3}\left[1 + \frac{O(\varepsilon)}{\tilde{r}^2}\right]^{3/2} = \frac{1}{\tilde{r}^3}\left[1 - \frac{3}{2}\frac{O(\varepsilon)}{\tilde{r}^2}\right] + O(\varepsilon^2) = \frac{1}{\tilde{r}^3} + O(\varepsilon) \tag{D7}$$

Thus relation (3.6.11) has been proved.

We shall now proceed to prove relation (3.6.14). Relation (3.6.5) for $P \in S_{u_Z}$ remains unchanged apart from a change of factor multiplying its left hand side from 4π to 2π. The same holds for the case of $P \in S_{l_Z}$. Repeating afterwards the linearization procedure we shall arrive at a relation similar to (3.6.14) but with his left hand side changed from $4\pi\,\underline{\tilde{v}}^j$ to $2\pi\left[\underline{\tilde{v}}^{j+} + \underline{\tilde{v}}^{j-}\right]$. Substituting $4\pi\,\rangle\underline{\tilde{v}}^j\langle$ instead of $\left[\underline{\tilde{v}}^{j+} + \underline{\tilde{v}}^{j-}\right]$ from relation (3.2.25) we get the required relation (3.6.14).

Appendix E: Supplement to paragraph 4.2

The following trigonometric identities are true :

$$\sin y^2 \cos(ny^2) \;=\; \frac{1}{2}\left[\sin((n+1)y^2) - \sin((n-1)y^2)\right] \tag{E1}$$

$$\sin y^2 \sin(ny^2) \;=\; \frac{1}{2}\left[\cos((n-1)y^2) - \cos((n+1)y^2)\right] \tag{E2}$$

We expand the term $y^2 \sin y^2$ in $[0,\pi]$ in a Fourier Series. Assuming that the desired function is odd and periodical with period 2π we have :

$$f(y^2) = \begin{cases} y^2\,\sin y^2 \quad , \quad y^2 \in [0,\pi] \\ -(y^2-2\pi)\,\sin(y^2-2\pi) \quad , \quad y^2 \in [\pi, 2\pi] \end{cases}$$

For $n \neq 1$:

$$\int_0^{2\pi} f(y^2)\sin(ny^2)dy^2 \;=$$

$$= \int_0^{\pi} y^2 \sin y^2 \sin(ny^2)dy^2 \;-\; \int_{\pi}^{2\pi} (y^2-2\pi)\,\sin(y^2-2\pi)\,\sin(ny^2)dy^2 \;=$$

$$= \int_0^{\pi} y^2 \sin y^2 \sin(ny^2)dy^2 \;-\; \int_{\pi}^{2\pi} y^2 \sin y^2 \sin(ny^2)dy^2 \;+\; \cancelto{0}{2\pi \int_{\pi}^{2\pi} \sin y^2 \sin(ny^2)dy^2} \overset{(E2)}{=}$$

$$= \frac{1}{2(n-1)}\left\{\cancelto{0}{\left[y^2\sin((n-1)y^2)\right]_0^{\pi}} - \int_0^{\pi} \sin((n-1)y^2)\,dy^2\right\} -$$

$$- \frac{1}{2(n+1)}\left\{\cancelto{0}{\left[y^2\sin((n+1)y^2)\right]_0^{\pi}} - \int_0^{\pi} \sin((n+1)y^2)\,dy^2\right\} -$$

$$- \frac{1}{2(n-1)}\left\{\cancelto{0}{\left[y^2\sin((n-1)y^2)\right]_{\pi}^{2\pi}} - \int_{\pi}^{2\pi} \sin((n-1)y^2)\,dy^2\right\} +$$

$$+ \frac{1}{2(n+1)}\left\{\cancelto{0}{\left[y^2\sin((n+1)y^2)\right]_{\pi}^{2\pi}} - \int_{\pi}^{2\pi} \sin((n+1)y^2)\,dy^2\right\} =$$

$$= \frac{1}{2(n-1)^2}\left[\cos((n-1)y^2)\right]_0^{\pi} - \frac{1}{2(n+1)^2}\left[\cos((n+1)y^2)\right]_0^{\pi} -$$

$$- \frac{1}{2(n-1)^2}\left[\cos((n-1)y^2)\right]_{\pi}^{2\pi} + \frac{1}{2(n+1)^2}\left[\cos((n+1)y^2)\right]_{\pi}^{2\pi} =$$

$$= \frac{(-1)^{n-1}-1}{(n-1)^2} - \frac{(-1)^{n+1}-1}{(n+1)^2} = -\left((-1)^n+1\right)\frac{4n}{(n^2-1)^2} \tag{E3}$$

For $n = 1$ we get

$$\int_0^{2\pi} f(y^2)\sin y^2 dy^2 =$$

$$= \int_0^{\pi} y^2 \sin y^2 \sin y^2 dy^2 - \int_{\pi}^{2\pi} (y^2 - 2\pi)\ \sin(y^2 - 2\pi)\ \sin y^2 dy^2 =$$

$$= \int_0^{\pi} y^2 \sin y^2 \sin y^2 dy^2 - \int_{\pi}^{2\pi} y^2 \sin y^2 \sin y^2 dy^2 + 2\pi \int_{\pi}^{2\pi} \sin y^2\ \sin y^2 dy^2 \overset{(E2)}{=}$$

$$= \frac{1}{2}\left[\frac{(y^2)^2}{2}\right]_0^{\pi} - \frac{1}{4}\left\{\cancelto{0}{\left[y^2\sin(2y^2)\right]_0^{\pi}} - \int_0^{\pi}\sin(2y^2)\,dy^2\right\} -$$

$$- \frac{1}{2}\left[\frac{(y^2)^2}{2}\right]_{\pi}^{2\pi} + \frac{1}{4}\left\{\cancelto{0}{\left[y^2\sin(2y^2)\right]_{\pi}^{2\pi}} - \int_{\pi}^{2\pi}\sin(2y^2)\,dy^2\right\} + \pi^2 =$$

$$= \frac{\pi^2}{4} - \frac{1}{8}\cancelto{0}{\left[\cos(2y^2)\right]_0^{\pi}} - \frac{3\pi^2}{4} - \frac{1}{8}\cancelto{0}{\left[\cos(2y^2)\right]_{\pi}^{2\pi}} + \pi^2 = \frac{\pi^2}{2} \tag{E4}$$

Since $f(y^2)$ is odd :

$$\int_0^{2\pi} f(y^2)\ \cos(ny^2)\,dy^2 = 0 \tag{E5}$$

Given that $\int_0^{2\pi} \sin^2(ny^2)dy^2 = \pi$, (n=1,2..) :

$$y^2 \sin y^2 = \frac{\pi}{2}\sin y^2 + \sum_{n=2}^{\infty} -\left((-1)^n+1\right)\frac{4n}{(n^2-1)^2\,\pi}\sin(ny^2) \tag{E6}$$

Setting

$$\beta_1 \;=\; \pi/2\;, \qquad\qquad \beta_n \;=\; -\left((-1)^n+1\right)\;\tfrac{4n}{(n^2-1)^2\,\pi} \tag{E7}$$

we get :

$$y^2\,\sin y^2 \;=\; \sum_{n=1}^{\infty}\beta_n\,\sin(ny^2) \tag{E8}$$

Therefore the following are true :

$$\frac{\sin y^2+y^2}{2}\,\sin y^2 \;=\; \frac{1}{4}\,(1-\cos(2y^2)) \;+\; \frac{1}{2}\sum_{n=1}^{\infty}\beta_n\,\sin(ny^2) \tag{E9}$$

$$(1\,-\,\cos(ny^2))\,\sin y^2 \;=\; \sin y^2\;-\;\frac{1}{2}\left[\sin((n+1)y^2))\;-\;\sin((n-1)y^2)\right] \tag{E10}$$

$$\sin(ny^2)\,\sin y^2 \;=\; \frac{1}{2}\left[\cos((n-1)y^2)\;-\;\cos((n+1)y^2)\right] \tag{E11}$$

After multiplying the Bernoulli's (at $y^1=0,\pi$) equation with $\sin y^2$ each term is analyzed as :

$$\int_0^{y^2}\frac{d\breve{\Gamma}^1}{dt}\,dy^2\,\sin y^2 \;=$$

$$=\sum_{m=0}^{\infty}\frac{dA^c_{m0Z}}{dt}\,\cos(my^1)\,\frac{\sin y^2+y^2}{2}\,\sin y^2+$$

$$+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{n}\,\frac{dA^{cs}_{mnZ}}{dt}\,\cos(my^1)\,(1-\cos(ny^2))\;+\;\frac{1}{n}\,\frac{dA^{cc}_{mnZ}}{dt}\,\cos(my^1)\,\sin(ny^2)\right)\sin y^2$$

$$=\sum_{m=0}^{\infty}\left(\frac{dA^c_{m0Z}}{dt}\cos(my^1)\right)\,\frac{1}{2}\left(\frac{1-\cos(2y^2)}{2}\;+\;\sum_{n=1}^{\infty}\beta_n\,\sin(ny^2)\right)\;+$$

$$+\sum_{n=1}^{\infty}\left[\sum_{m=0}^{\infty}\left(\frac{1}{n}\frac{dA^{cs}_{mnZ}}{dt}\,\cos(my^1)\right)\right]\,\frac{1}{2}\left(2\,\sin y^2\;-\;(\sin((n+1)y^2)\;-\;\sin((n-1)y^2))\right)\;+$$

$$+\left[\sum_{m=0}^{\infty}\left(\frac{1}{n}\frac{dA^{cc}_{mnZ}}{dt}\,\cos(my^1)\right)\right]\,\frac{1}{2}\left(\cos((n-1)y^2)\;-\;\cos((n+1)y^2))\right)\;=$$

$$= \sum_{m=0}^{\infty} \left(\frac{dA^{c}_{m0Z}}{dt} \frac{1}{2} + \frac{dA^{cc}_{m1Z}}{dt} \right) \frac{(\pm 1)^m}{2} +$$

$$+ \sum_{n=1}^{\infty} \left[\sum_{m=0}^{\infty} \left(\frac{dA^{c}_{m0Z}}{dt} \frac{(-\delta^2_n)}{2} + \frac{1}{n+1} \frac{dA^{cc}_{m,n+1,Z}}{dt} - \frac{1-\delta^1_n}{n-1} \frac{dA^{cc}_{m,n-1,Z}}{dt} \right) \frac{(\pm 1)^m}{2} \right] \cos(ny^2) +$$

$$+ \left[\sum_{m=0}^{\infty} \left(\frac{dA^{c}_{m0Z}}{dt} \beta_n + 2 \frac{\delta^1_n}{n} \frac{dA^{cs}_{mnZ}}{dt} - \frac{1-\delta^1_n}{n-1} \frac{dA^{cs}_{m,n-1,Z}}{dt} + \frac{1}{n+1} \frac{dA^{cs}_{m,n+1,Z}}{dt} \right) \frac{(\pm 1)^m}{2} \right] \sin(ny^2) \quad \text{(E12)}$$

Since we are applying the dynamic boundary condition at the tip (or the hub) the relation (4.2.13) should hold. Therefore using relations (4.2.14), (4.2.15) and (4.2.16) we get :

$$\underline{\Gamma}^2(y^1 = 0, \pi, y^2, z, t) \sin y^2 \; =$$

$$\sum_{m=1}^{\infty} \left(m\, A^{c}_{m0Z} \frac{\sin(my^1)}{\sin y^1} \right) \frac{y^2 + \sin y^2}{R_o - R_H} \sin y^2 +$$

$$+ \frac{2}{R_o - R_H} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{m}{n} A^{cs}_{mnZ} (1 - \cos(ny^2)) \frac{\sin(my^1)}{\sin y^1} + \frac{m}{n} A^{cc}_{mnZ} \sin(ny^2) \frac{\sin(my^1)}{\sin y^1} \right) \sin y^2$$

$$= \sum_{m=1}^{\infty} \left(m\, A^{c}_{m0Z} \frac{\sin(my^1)}{\sin y^1} \right) \frac{1}{Ro - R_H} \left(\frac{1 - \cos(2y^2)}{2} + \sum_{n=0}^{\infty} \beta_n \sin(ny^2) \right) +$$

$$+ \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \left(\frac{m}{n} A^{cs}_{mnZ} \frac{\sin(my^1)}{\sin y^1} \right) \right] \frac{1}{R_o - R_H} \left(2 \sin y^2 - (\sin((n+1)y^2) - \sin((n-1)y^2)) \right) +$$

$$\left[\sum_{m=1}^{\infty} \left(\frac{m}{n} A^{cc}_{mnZ} \frac{\sin(my^1)}{\sin y^1} \right) \right] \frac{1}{R_o - R_H} \left(\cos((n-1)y^2) - \cos((n+1)y^2) \right) =$$

$$= \sum_{m=1}^{\infty} \left(A^{c}_{m0Z} \frac{1}{2} + A^{cc}_{m1Z} \right) \frac{m\,(\pm 1)^{m+1}}{R_o - R_H} +$$

$$+ \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \left(A^{c}_{m0Z} \frac{(-\delta^2_n)}{2} + \frac{1}{n+1} A^{cc}_{m,n+1,Z} - \frac{1-\delta^1_n}{n-1} A^{cc}_{m,n-1,Z} \right) \frac{m\,(\pm 1)^{m+1}}{R_o - R_H} \right] \cos(ny^2) +$$

$$+ \left[\sum_{m=1}^{\infty} \left(A^{c}_{m0Z}\, \beta_n + 2 \frac{\delta^1_n}{n} A^{cs}_{mnZ} - \frac{1-\delta^1_n}{n-1} A^{cs}_{m,n-1,Z} + \frac{1}{n+1} A^{cs}_{m,n+1,Z} \right) \frac{m\,(\pm 1)^{m+1}}{R_o - R_H} \right] \sin(ny^2) \quad \text{(E13)}$$

Notice that $A^{c}_{m0Z}\frac{1}{2} + A^{cc}_{m1Z} = A'^{cc}_{m1Z}$ by relation (4.1.16)

$$\underline{\Gamma}^1(y^1 = 0, \pi, y^2, Z, t)\,\sin y^2 =$$

$$= \sum_{m=0}^{\infty} A^{c}_{m0Z}\,\cos(my^1)\,\left(\cos y^2 + 1\right) +$$

$$2\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(A^{cs}_{mnZ}\,\cos(my^1)\sin(ny^2) + A^{cc}_{mnZ}\,\cos(my^1)\cos(ny^2)\right)$$

$$= \sum_{m=0}^{\infty} A^{c}_{m0Z}(\pm 1)^m + \sum_{n=1}^{\infty}\left[\sum_{m=0}^{\infty}\left(2A^{cc}_{mnZ} + A^{c}_{m0Z}\delta^1_n\right)(\pm 1)^m\right]\cos(ny^2) +$$

$$+ \left[\sum_{m=0}^{\infty} 2A^{cs}_{mnZ}(\pm 1)^m\right]\sin(ny^2) \qquad \text{(E14)}$$

The following trigonometric identities are true :

$$\sin y^1 \, \cos(my^1) \;=\; \frac{1}{2}\left[\sin((m+1)y^1) - \sin((m-1)y^1)\right] \tag{E15}$$

$$\sin y^1 \, \sin(my^1) \;=\; \frac{1}{2}\left[\cos((m-1)y^1) - \cos((m+1)y^1)\right] \tag{E16}$$

Therefore:

$\underline{\Gamma}^1(y^1, y^2, Z, t)\, \sin y^1 \;=$

$$= \sum_{m=0}^{\infty}\left[A^{s}_{m0Z}\frac{1}{\tan(y^2/2)} \;+\; 2\sum_{n=1}^{\infty}\left(A^{ss}_{mnZ}\frac{\sin(ny^2)}{\sin y^2} \;+\; A^{sc}_{mnZ}\frac{\cos(ny^2)}{\sin y^2}\right)\right]\sin(my^1)\,\sin y^1$$

$$+\left[A^{c}_{m0Z}\frac{1}{\tan(y^2/2)} \;+\; 2\sum_{n=1}^{\infty}\left(A^{cs}_{mnZ}\frac{\sin(ny^2)}{\sin y^2} \;+\; A^{cc}_{mnZ}\frac{\cos(ny^2)}{\sin y^2}\right)\right]\cos(my^1)\,\sin y^1$$

For $t > 0$ the boundary condition (4.2.10) should be valid, so using relations (4.2.11),(4.2.12) we get:

$\underline{\Gamma}^1(y^1, y^2 = \pi, Z, t)\, \sin y^1 \;=$

$$= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(2\,A^{ss}_{mnZ}(\pm 1)^{n+1}\right)\left(\cos((m-1)y^1) - \cos((m+1)y^1)\right) \;+$$

$$+\sum_{n=1}^{\infty}\left(2\,A^{cs}_{mnZ}(\pm 1)^{n+1}\right)\left(\sin((m+1)y^1) - \sin((m-1)y^1)\right) \;=$$

$$=\;2\sum_{n=1}^{\infty}\left(A^{ss}_{0nZ}\cos((0-1)y^1) \;-\; A^{cs}_{0nZ}\sin((0-1)y^1)\right)(\pm 1)^{n+1} +$$

$$+2\sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty}\left(A^{ss}_{m+1,n,Z} \;-\; A^{ss}_{m-1,n,Z}(1-\delta^0_m)\right)(\pm 1)^{n+1}\right]\cos(my^1) +$$

$$+\left[\sum_{n=1}^{\infty}\left(A^{cs}_{m-1,n,Z}(1-\delta^0_m) \;-\; A^{cs}_{m+1,n,Z}\right)(\pm 1)^{n+1}\right]\sin(my^1) \;=$$

$$=2\sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty}\left(A^{ss}_{m+1,n,Z} \;-\; A^{ss}_{m-1,n,Z}(1-\delta^0_m) \;+\; A^{ss}_{0nZ}\,\delta^1_m\right)(\pm 1)^{n+1}\right]\cos(my^1) +$$

$$+\left[\sum_{n=1}^{\infty}\left(A^{cs}_{m-1,n,Z}(1-\delta^0_m) \;-\; A^{cs}_{m+1,n,Z} \;+\; A^{cs}_{0nZ}\delta^1_m\right)(\pm 1)^{n+1}\right]\sin(my^1) \tag{E17}$$

$$\int_0^{y^2} \frac{d\breve{\Gamma}}{dt}\, dy^2 \, \sin y^1 \; =$$

$$= \sum_{m=0}^{\infty} \left[\frac{dA^s_{m0Z}}{dt} \frac{\sin y^2 + y^2}{2} \; + \sum_{n=1}^{\infty} \left(\frac{dA^{ss}_{mnZ}}{dt} \frac{1-\cos(ny^2)}{n} \; + \; \frac{dA^{sc}_{mnZ}}{dt} \frac{\sin(ny^2)}{n} \right) \right] \sin(my^1) \sin y^1$$

$$+ \left[\frac{dA^c_{m0Z}}{dt} \frac{\sin y^2 + y^2}{2} \; + \sum_{n=1}^{\infty} \left(\frac{dA^{cs}_{mnZ}}{dt} \frac{1-\cos(ny^2)}{n} \; + \; \frac{dA^{cc}_{mnZ}}{dt} \frac{\sin(ny^2)}{n} \right) \right] \cos(my^1) \sin y^1$$

For $y^2 = \pi$ and using relations (E15) and (E16) :

$$= \sum_{m=0}^{\infty} \left[\frac{dA^s_{m0Z}}{dt} \frac{\pi}{2} \; + \sum_{n=1}^{\infty} \left(\frac{dA^{ss}_{mnZ}}{dt} \frac{1-(-1)^n}{n} \right) \right] \frac{1}{2} \left(\cos((m-1)y^1) - \cos((m+1)y^1) \right)$$

$$+ \left[\frac{dA^c_{m0Z}}{dt} \frac{\pi}{2} \; + \sum_{n=1}^{\infty} \left(\frac{dA^{cs}_{mnZ}}{dt} \frac{1-(-1)^n}{n} \right) \right] \frac{1}{2} \left(\sin((m+1)y^1) - \sin((m-1)y^1) \right)$$

$$= \left[\frac{dA^s_{00Z}}{dt} \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{dA^{ss}_{0,n,Z}}{dt} \frac{1-(-1)^n}{2n} \right) \right] \cos y^1 + \left[\frac{dA^c_{00Z}}{dt} \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{dA^{cs}_{0,n,Z}}{dt} \frac{1-(-1)^n}{2n} \right) \right] \sin y^1$$

$$+ \sum_{m=0}^{\infty} \left\{ \left[\frac{dA^s_{m+1,0,Z}}{dt} \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{dA^{ss}_{m+1,n,Z}}{dt} \frac{1-(-1)^n}{2n} \right) \right] - \right.$$

$$\left. - \left[\frac{dA^s_{m-1,n,Z}}{dt} \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{dA^{ss}_{m-1,n,Z}}{dt} \frac{1-(-1)^n}{2n} \right) \right] (1-\delta^0_m) \right\} \cos(my^1) +$$

$$+ \left\{ \left[\frac{dA^c_{m-1,0,Z}}{dt} \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{dA^{cs}_{m-1,n,Z}}{dt} \frac{1-(-1)^n}{2n} \right) \right] (1-\delta^0_m) - \right.$$

$$\left. - \left[\frac{dA^c_{m+1,n,Z}}{dt} \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{dA^{cs}_{m+1,n,Z}}{dt} \frac{1-(-1)^n}{2n} \right) \right] \right\} \sin(my^1) \qquad \text{(E18)}$$

Finally :

$$\underline{\Gamma}^2(y^1, y^2=\pi, Z)\sin y^1 \ =$$

$$= \sum_{m=1}^{\infty}\left[-mA^s_{m0Z}\frac{y^2+\sin y^2}{R_o-R_H} + \sum_{n=1}^{\infty}\left(-\frac{m}{n}A^{ss}_{mnZ}(1-\cos(ny^2)) - \frac{m}{n}A^{sc}_{mnZ}\sin(ny^2)\right)\right]\cos(my^1)$$

$$\left[mA^c_{m0Z}\frac{y^2+\sin y^2}{R_o-R_H} + \sum_{n=1}^{\infty}\left(\frac{m}{n}A^{cs}_{mnZ}(1-\cos(ny^2)) + \frac{m}{n}A^{cc}_{mnZ}\sin(ny^2)\right)\right]\sin(my^1) \quad \text{(E19)}$$

Appendix F: Second order Time derivative

Suppose the set of three points $(t_1, A(t_1)) = (0, A_1)$, $(t_2, A(t_2)) = (\Delta t, A_2)$ and $(t_3, A(t_3)) = (2\,\Delta t, A_3)$. Then the Lagrange polynomial through those points is :

$$A(t) \;=\; A_1\,\frac{(t-\Delta t)(t-2\,\Delta t)}{2\,\Delta t^2} \;+\; A_2\,\frac{t\,(t-2\,\Delta t)}{-\Delta t^2} \;+\; A_3\,\frac{t(t-\Delta t)}{2\,\Delta t^2} \tag{F1}$$

Taking the derivative :

$$\frac{dA(t)}{dt} \;=\; A_1\,\frac{2\,t-3\,\Delta t}{2\,\Delta t^2} \;+\; A_2\,\frac{2\,t-2\,\Delta t}{-\Delta t^2} \;+\; A_3\,\frac{2\,t-\Delta t}{2\,\Delta t^2} \tag{F2}$$

Setting $t = t_3 = 2\,\Delta t$, we get :

$$\left.\frac{dA(t)}{dt}\right|_{t=t_3} \;=\; \frac{A_1 - 4\,A_2 + 3\,A_3}{2\,\Delta t} \tag{F3}$$

Appendix G: Calculation of self-induction factors

Here we will prove the analytic expressions (4.5.58)-(4.5.62) and (4.5.99)-(4.5.103) of the self induction factors. Similar calculations can be found in [2].

A. $B^{sj}_{m_{self}}$ $\quad (m = 1, 2, ...)$

According to the definition (4.5.15),(4.5.20) we get :

$$B^{sj}_{m_{self}} = \lim_{y^2 \to y^2_P} (\cos y^2 - \cos y^2_P) \int_0^{\pi} \sin(my^1) \sin y^1 \, e^{jil} \underset{\sim}{\gamma}{}^i_1 \, \frac{x^l(Q) - x^l(P)}{r^3} \, dy^1 =$$

$$= \lim_{y^2 \to y^2_P} (\cos y^2 - \cos y^2_P) \left(\int_0^{y^1-\varepsilon} + \int_{y^1-\varepsilon}^{y^1+\varepsilon} + \int_{y^1+\varepsilon}^{\pi} \right) \tag{G1}$$

where ε is a very small positive number.

It is obvious that the first and the third integrals don't exhibit any irregularity, so after being multiplied with $\lim_{y^2 \to y^2_P} (\cos y^2 - \cos y^2_P) = 0$, have zero contribution to the self induction factor. Therefore the only thing left to calculate is the second integral :

$$B^{sj}_{m_{self}} = \lim_{\substack{y^2 \to y^2_P \\ \varepsilon \to 0}} (\cos y^2 - \cos y^2_P) \int_{y^1-\varepsilon}^{y^1+\varepsilon} \sin(my^1) \sin y^1 \, e^{jil} \underset{\sim}{\gamma}{}^i_1 \frac{x^l(Q) - x^l(P)}{r^3} \, dy^1 \tag{G2}$$

In the neighborhood of (y^1_P, y^2_P) the expression inside the integral has tensorial character for the generalized coordinate transformation. Moreover using the Taylor expansion for $\cos y^2$ around y^2_P $(\cos y^2 - \cos y^2_P = -(y^2 - y^2_P) \sin y^2_P + O(\varepsilon^2))$:

$$B^{sj}_{m_{self}} = -(y^2 - y^2_P) \sin y^2_P \int \sin(my^1) \, \sin y^1 \, e^{jil} \underset{\sim}{\gamma}{}^i_1 \, \frac{\underset{\smile}{\tilde{\gamma}}{}^l_\alpha \, (y^\alpha - y^\alpha_P)}{\left\{ (y^\beta - y^\beta_P)(y_\beta - y_{P\,\beta}) \right\}^{3/2}} \, dy^1 =$$

$$= -(y^2 - y^2_P) \sin y^2_P \int \sin(my^1) \, \sin y^1 \, e^{jil} \underset{\sim}{\gamma}{}^i_1 \underset{\sim}{\gamma}{}^l_1 \, \frac{R_o - R_H}{2} \, \sin y^1 \, \frac{(y^1 - y^1_P)}{\left\{ (y^\beta - y^\beta_P)(y_\beta - y_{P\,\beta}) \right\}^{3/2}} \, dy^1$$

$$-(y^2 - y^2_P) \sin y^2_P \int \sin(my^1) \, \sin y^1 \, e^{jil} \underset{\sim}{\gamma}{}^i_1 \underset{\sim}{\gamma}{}^l_2 \, \frac{1}{2} \, \sin y^2 \frac{(y^2 - y^2_P)}{\left\{ (y^\beta - y^\beta_P)(y_\beta - y_{P\,\beta}) \right\}^{3/2}} \, dy^1 =$$

Due to symmetry $e^{jil} \underset{\sim}{\gamma}{}^i_1 \underset{\sim}{\gamma}{}^l_1 = 0$, we are left only with the second integral :

$$= -\frac{1}{2}\,\sin y_P^2\,\sin y_P^2\,\sin(my_P^1)\,\sin y_P^1\, e^{jil}\,\underset{\sim}{\gamma}{}_1^{\,i}\,\underset{\sim}{\gamma}{}_2^{\,l}\,\left(y^2-y_P^2\right)^2 \int \frac{dy^1}{\left\{(y^\beta-y_P^\beta)(y_\beta-y_{P\,\beta})\right\}^{3/2}} =$$

$$= -\frac{1}{2}\,(\sin y_P^2)^2\,\sin(my_P^1)\,\sin y_P^1\; e^{jil}\,\underset{\sim}{\gamma}{}_1^{\,i}\,\underset{\sim}{\gamma}{}_2^{\,l}\,\frac{2\sqrt{\underset{\smile}{\tilde{g}}_{11}}}{\underset{\smile}{\alpha}} =$$

$$\overset{(A74)}{=} -(\sin y_P^2)^2\,\sin(my_P^1)\,\sin y_P^1\; e^{jil}\,\underset{\sim}{\gamma}{}_1^{\,i}\,\underset{\sim}{\gamma}{}_2^{\,l}\,\frac{\frac{1}{2}\,(R_o-R_H)\,\sin y_P^1}{\frac{1}{16}\,(R_o-R_H)^2\,\sin^2 y_P^1\,\sin^2 y_P^2}\frac{\sqrt{\underline{\tilde{g}_{11}}}}{\underline{\alpha}} \Rightarrow$$

Therefore

$$\Rightarrow B^{sj}_{m_{self}} = -\frac{8}{R_o-R_H}\; e^{jil}\,\underset{\sim}{\gamma}{}_1^{\,i}\,\underset{\sim}{\gamma}{}_2^{\,l}\,\sin(my_P^1)\frac{\sqrt{\underline{\tilde{g}}_{11}}}{\underline{\alpha}}\,, \qquad m=1,2,... \tag{G3}$$

In the above relations we omitted the limit for simplicity. Furthermore we assumed that the $\sin y^1$, $\sin y^2$, $\underset{\sim}{\gamma}{}_1^{\,i}$, $\underset{\sim}{\gamma}{}_2^{\,i}$ are approximately constant in the neighborhood of (y_P^1, y_P^2) and equal to their value at (y_P^1, y_P^2).

$B.\ \ B^{cj}_{m_{self}}$ $\qquad (m\ =\ 0,1,2,...)$

According to the definitions (4.5.16), (4.5.20) we get :

$$B^{cj}_{m_{self}} \overset{def}{=} \lim_{y^2\to y_P^2}\left(cosy^2-cosy_P^2\right)\int_0^\pi \cos(my^1)\,\sin y^1\,e^{jil}\,\underset{\sim}{\gamma}{}_1^{\,i}\,\frac{\delta x^l}{r^3}\,dy^1 \Rightarrow \tag{G4}$$

The relation for $B^{cj}_{m_{self}}$ is similar to (G3) but instead of $\sin(my_P^1)$ we have $\cos(my_P^1)$:

$$\Rightarrow B^{cj}_{m_{self}} = -\frac{8}{R_o-R_H}\; e^{jil}\,\underset{\sim}{\gamma}{}_1^{\,i}\,\underset{\sim}{\gamma}{}_2^{\,l}\,\cos(my_P^1)\frac{\sqrt{\underline{\tilde{g}}_{11}}}{\underline{\alpha}}\,, \qquad m=0,1,2,... \tag{G5}$$

$C.\ \ C^{sj}_{n_{self}}$ $\qquad (n\ =\ 1,2,...)$

According to the definitions (4.5.18), (4.5.21) we get :

$$C^{sj}_{n_{self}} = \lim_{y^1\to y_P^1}(\cos y^1-\cos y_P^1)\int_0^\pi (1-\cos(ny^2))\,\sin y^2\,e^{jil}\,\underset{\sim}{\gamma}{}_2^{\,i}\,\frac{x^l(Q)-x^l(P)}{r^3}\,dy^2$$

$$= \lim_{y^1 \to y^1_P} (\cos y^1 - \cos y^1_P) \left(\int_0^{y^2_P - \varepsilon} + \int_{y^2_P - \varepsilon}^{y^2_P + \varepsilon} + \int_{y^2_P - \varepsilon}^{\pi} \right) =$$

Following the same process we calculate only the second integral :

$$= -(y^1 - y^1_P) \sin y^1_P \int (1 - \cos(ny^2)) \sin y^2 \, e^{jil} \; \underset{\sim}{\overset{\sim}{\gamma}}{}^i_2 \frac{\underset{\smile}{\overset{\sim}{\gamma}}{}^l_\alpha (y^\alpha - y^\alpha_P)}{\left\{ (y^\beta - y^\beta_P)(y_\beta - y_{P\,\beta}) \right\}^{3/2}} \, dy^2 \; =$$

$$= -\sin y^1_P \, (1 - \cos(ny^2_P)) \sin y^2_P e^{jil} \underset{\sim}{\overset{\sim}{\gamma}}{}^i_2 \underset{\sim}{\overset{\sim}{\gamma}}{}^l_1 \frac{R_o - R_H}{2} \sin y^1_P \, (y^1 - y^1_P) \int \frac{y^1 - y^1_P}{\left\{ (y^\beta - y^\beta_P)(y_\beta - y_{\beta P}) \right\}^{3/2}} dy^2$$

$$= -\sin y^1_P (1 - \cos(ny^2_P)) \sin y^2_P e^{jil} \underset{\sim}{\overset{\sim}{\gamma}}{}^i_2 \underset{\sim}{\overset{\sim}{\gamma}}{}^l_1 \frac{R_o - R_H}{2} \sin y^1_P \, \frac{\frac{1}{2} \sin y^2_P}{\frac{1}{16} (R_o - R_H)^2 \sin^2 y^1_P \sin^2 y^2_P} \frac{2 \sqrt{\tilde{g}_{22}}}{\underset{\sim}{\alpha}} \Rightarrow$$

$$\Rightarrow C^{sj}_{n_{self}} \; = \; \frac{8}{R_o - R_H} \, e^{jil} \; \underset{\sim}{\overset{\sim}{\gamma}}{}^i_1 \underset{\sim}{\overset{\sim}{\gamma}}{}^l_2 (1 - \cos(ny^2_P)) \frac{\sqrt{\tilde{g}_{22}}}{\underset{\sim}{\alpha}} \qquad n = 1, 2, ... \tag{G6}$$

$\underline{D. \; C^{cj}_{n_{self}}}$ $\qquad (n \; = \; 1, 2, ...)$

The relation for $C^{cj}_{n_{self}}$ is similar to (G6) but instead of $(1 - \cos(ny^2_P))$ we have $\sin(ny^2_P)$:

$$C^{cj}_{n_{self}} \; = \; \frac{8}{R_o - R_H} \, e^{jil} \; \underset{\sim}{\overset{\sim}{\gamma}}{}^i_1 \underset{\sim}{\overset{\sim}{\gamma}}{}^l_2 \sin(ny^2_P) \frac{\sqrt{\tilde{g}_{22}}}{\underset{\sim}{\alpha}} \tag{G7}$$

$\underline{E. \; C^{j}_{0_{self}}}$

The relation for $C^{j}_{0_{self}}$ is similar to (G6) but instead of $(1 - \cos(ny^2_P))$ we have $(y^2_P + \sin y^2_P)/2$:

$$C^{j}_{0_{self}} \; = \; \frac{4}{R_o - R_H} \, e^{jil} \; \underset{\sim}{\overset{\sim}{\gamma}}{}^i_1 \underset{\sim}{\overset{\sim}{\gamma}}{}^l_2 (y^2_P + \sin y^2_P) \frac{\sqrt{\tilde{g}_{22}}}{\underset{\sim}{\alpha}} \tag{G8}$$

We now prove the relations (4.5.99) - (4.5.103) of the alternative self-induction factors :

F. $B^j_{0_{self}}$

$$B^j_{0_{self}} = -(y^1 - y^1_P)\,\sin y^1_P \int \frac{1+\cos y^2}{2}\, e^{jil}\, \underset{\sim}{\gamma}{}^i_1 \frac{\underset{\smile}{\tilde{\gamma}}{}^l_\alpha (y^\alpha - y^\alpha_P)}{\left\{(y^\beta - y^\beta_P)(y_\beta - y_{P\,\beta})\right\}^{3/2}}\, dy^2 =$$

$$= -\sin y^1_P \frac{1+\cos y^2_P}{2}\, e^{jil} \underset{\sim}{\gamma}{}^i_1\, \underset{\sim}{\gamma}{}^l_2\, \frac{1}{2}\, \sin y^2_P\, (y^1 - y^1_P) \int \frac{(y^2 - y^2_P)}{\left\{(y^\beta - y^\beta_P)(y_\beta - y_{P\,\beta})\right\}^{3/2}}\, dy^2$$

$$= -\frac{1}{2}\, e^{jil}\, \underset{\sim}{\gamma}{}^i_1\, \underset{\sim}{\gamma}{}^l_2\, \frac{1+\cos y^2_P}{2}\, \sin y^1_P\, \sin y^2_P \frac{-2\,\underset{\smile}{\tilde{g}}{}_{12}}{\sqrt{\underset{\smile}{\tilde{g}}{}_{22}}\,\underline{\alpha}} =$$

$$= e^{jil}\, \underset{\sim}{\gamma}{}^i_1\, \underset{\sim}{\gamma}{}^l_2\, \frac{1+\cos y^2_P}{4}\, \sin y^1_P\, \sin y^2_P \frac{\frac{1}{4}(R_o - R_H)\sin y^1_P\, \sin y^2_P}{\frac{1}{2}\, \sin y^2_P\, \frac{1}{16}(R_o - R_H)^2 \sin^2 y^1_P \sin^2 y^2_P} \frac{2\underline{\tilde{g}}_{12}}{\sqrt{\underline{\tilde{g}}_{22}}\,\underline{\alpha}} \Rightarrow$$

$$\Rightarrow B^j_{0_{self}} = \frac{4}{R_o - R_H}\, e^{jil}\, \underset{\sim}{\gamma}{}^i_1\, \underset{\sim}{\gamma}{}^l_2\, \frac{1+\cos y^2_P}{\sin y^2_P}\, \frac{\underline{\tilde{g}}_{12}}{\sqrt{\underline{\tilde{g}}_{22}}\,\underline{\alpha}} \tag{G9}$$

G. $B^{sj}_{n_{self}}$ $\quad (n = 1, 2, ...)$

Setting $sin(ny^2_P)$ to (G9) instead of $(1 + \cos y^2_P)/2$ we get :

$$B^{sj}_{n_{self}} = \frac{8}{R_o - R_H}\, e^{jil}\, \underset{\sim}{\gamma}{}^i_1\, \underset{\sim}{\gamma}{}^l_2\, \frac{\sin(ny^2_P)}{\sin y^2_P}\, \frac{\underline{\tilde{g}}_{12}}{\sqrt{\underline{\tilde{g}}_{22}}\,\underline{\alpha}} \tag{G10}$$

H. $B^{cj}_{n_{self}}$ $\quad (n = 0, 1, ...)$

Setting $cos(ny^2_P)$ to (G9) instead of $(1 + \cos y^2_P)/2$ we get :

$$B^{cj}_{n_{self}} = \frac{8}{R_o - R_H}\, e^{jil}\, \underset{\sim}{\gamma}{}^i_1\, \underset{\sim}{\gamma}{}^l_2\, \frac{\cos(ny^2_P)}{\sin y^2_P}\, \frac{\underline{\tilde{g}}_{12}}{\sqrt{\underline{\tilde{g}}_{22}}\,\underline{\alpha}} \tag{G11}$$

I. $\underline{C^{sj}_{m_{self}}}$ $\quad (m = 1, 2, ...)$

Setting $\cos(my^1_P)$ to (G9) instead of $(1 + \cos y^2_P)/2$ and swapping the indices the of the transformation matrices $\underset{\sim}{\tilde{\gamma}}$ (changing the sing of the relation), we get :

$$C^{sj}_{m_{self}} = -\frac{8}{R_o - R_H}\, e^{jil}\, \underset{-}{\tilde{\gamma}}{}^i_1\, \underset{-}{\tilde{\gamma}}{}^l_2\, \frac{\cos(my^1_P)}{\sin y^1_P}\, \frac{\underline{\tilde{g}}_{12}}{\sqrt{\underline{\tilde{g}}_{11}}\,\underline{\alpha}} \tag{G12}$$

J. $\underline{C^{cj}_{m_{self}}}$ $\quad (m = 0, 1, ...)$

Setting $\sin(my^1_P)$ to (G9) instead of $(1+\cos y^2_P)/2$ and swapping the indices the of the transformation matrices $\underset{\sim}{\tilde{\gamma}}$ (changing the sing of the relation), we get :

$$C^{cj}_{m_{self}} = -\frac{8}{R_o - R_H}\, e^{jil}\, \underset{-}{\tilde{\gamma}}{}^i_1\, \underset{-}{\tilde{\gamma}}{}^l_2\, \frac{\sin(my^1_P)}{\sin y^1_P}\, \frac{\underline{\tilde{g}}_{12}}{\sqrt{\underline{\tilde{g}}_{11}}\,\underline{\alpha}} \tag{G13}$$

K. $\underline{D^j_{self}}$

Finally we calculate the self induction of the source-sink term. Specifically, according to definitions (4.6.5), (4.6.6) :

$$D^j_{Z_{self}} = -(y^2 - y^2_P)\, \sin y^2_P \int f_S(Q)\, \sin y^1\, \sin y^2 \frac{\underset{\smile}{\tilde{\gamma}}{}^j_\alpha\, (y^\alpha - y^\alpha_P)}{\Big\{(y^\beta - y^\beta_P)(y_\beta - y_{\beta\, P})\Big\}^{3/2}}\, dy^1 =$$

$$= -(y^2 - y^2_P)\, \sin y^2_P \int f_S(Q)\, \sin y^1\, \sin y^2 \frac{\underset{\smile}{\tilde{\gamma}}{}^j_1\, (y^1 - y^1_P)}{\Big\{(y^\beta - y^\beta_P)(y_\beta - y_{\beta\, P})\Big\}^{3/2}}\, dy^1$$

$$-(y^2 - y^2_P)\, \sin y^2_P \int f_S(Q)\, \sin y^1\, \sin y^2 \frac{\underset{\smile}{\tilde{\gamma}}{}^j_2 (y^2 - y^2_P)}{\Big\{(y^\beta - y^\beta_P)(y_\beta - y_{\beta\, P})\Big\}^{3/2}}\, dy^1$$

$$= -\sin y_P^1\,(\sin y_P^2)^2\; f_S(P)\;\left(\frac{1}{2}(R_o - R_H)\,\sin y_P^1\,\underline{\tilde{\gamma}}_1^j\right)(y^2 - y_P^2)\int \frac{y^1 - y_P^1}{\left\{(y^\beta - y_P^\beta)(y_\beta - y_{\beta\,P})\right\}^{3/2}}\,dy^1$$

$$-\sin y_P^1\,(\sin y_P^2)^2\; f_S(P)\;\left(\frac{1}{2}\,\sin y_P^2\,\underline{\tilde{\gamma}}_2^j\right)(y^2 - y_P^2)\int \frac{y^2 - y_P^2}{\left\{(y^\beta - y_P^\beta)(y_\beta - y_{\beta\,P})\right\}^{3/2}}\,dy^1$$

$$= -\frac{1}{2}\,(R_o - R_H)\,(\sin y_P^1)^2\,(\sin y_P^2)^2\, f_S(P)\underline{\tilde{\gamma}}_1^j\frac{-2\,\underset{\smile}{\tilde{g}}_{12}}{\sqrt{\underset{\smile}{\tilde{g}}_{11}\underset{\smile}{\alpha}}}\;-\;\frac{1}{2}\,(\sin y_P^2)^3\,\sin y_P^1\; f_S(P)\;\underline{\tilde{\gamma}}_2^j\,\frac{2\sqrt{\underset{\smile}{\tilde{g}}_{11}}}{\underset{\smile}{\alpha}}$$

$$= (R_o - R_H)(\sin y_P^1)^2(\sin y_P^2)^2 f_S(P)\underline{\tilde{\gamma}}_1^j\frac{\frac{1}{4}(R_o - R_H)\sin y_P^1\sin y_P^2}{\frac{1}{2}(R_o - R_H)\sin y_P^1\frac{1}{16}(R_o - R_H)^2(\sin y_P^1)^2(\sin y_P^2)^2}\frac{\underline{\tilde{g}}_{12}}{\sqrt{\underline{\tilde{g}}_{11}\underline{\alpha}}}$$

$$-\sin y_P^1\,(\sin y_P^2)^3\, f_S(P)\,\underline{\tilde{\gamma}}_2^j\frac{\frac{1}{2}(R_o - R_H)\,\sin y_P^1}{\frac{1}{16}(R_o - R_H)^2\,(\sin y_P^1)^2\,(\sin y_P^2)^2}\frac{\sqrt{\underline{\tilde{g}}_{11}}}{\underline{\alpha}}\Rightarrow$$

$$\Rightarrow D^j_{Z_{self}} \;=\; \frac{8}{R_o - R_H}\;\sin y_P^2\, f_S(P)\;\left(\underline{\tilde{\gamma}}_1^j\,\underline{\tilde{g}}_{12}\;-\;\underline{\tilde{\gamma}}_2^j\underline{\tilde{g}}_{11}\right)\frac{1}{\sqrt{\underline{\tilde{g}}_{11}\underline{\alpha}}} \tag{G14}$$

In the proof of the above relations we used the following integral:

$$D^{\alpha\,\beta\,\gamma} \;\equiv\; y^\alpha \int_{-\varepsilon_\gamma}^{\varepsilon^\gamma} \sqrt{\underset{\smile}{\alpha}}\,\frac{y^\beta}{(y^\beta\, y_\beta)^{3/2}}\,dy^\gamma \tag{G15}$$

with $(\alpha, \beta, \gamma) \in \{(2,1,1), (1,1,2), (2,2,1), (1,2,2)\}$ and

$$D^{\alpha\,\beta\,\gamma}_{self} \;\equiv\; \lim_{y^\alpha \to 0} D^{\alpha\,\beta\,\gamma} \tag{G16}$$

which is proven to be :

$$D^{211}_{self} \;=\; -\frac{2\,\underset{\smile}{\tilde{g}}_{12}}{\sqrt{\underset{\smile}{\tilde{g}}_{11}}\sqrt{\underset{\smile}{\alpha}}} \qquad\qquad D^{112}_{self} \;=\; \frac{2\sqrt{\underset{\smile}{\tilde{g}}_{22}}}{\sqrt{\underset{\smile}{\alpha}}} \tag{G17}$$

$$D^{221}_{self} \;=\; \frac{2\sqrt{\underset{\smile}{\tilde{g}}_{11}}}{\sqrt{\underset{\smile}{\alpha}}} \qquad\qquad D^{122}_{self} \;=\; -\frac{2\,\underset{\smile}{\tilde{g}}_{12}}{\sqrt{\underset{\smile}{\tilde{g}}_{22}}\sqrt{\underset{\smile}{\alpha}}} \tag{G18}$$

Since relations (G17) and (G18) are symmetrical, we only need to prove the first relation of each.

<u>proof :</u>

In the following realtions we omit the underscores for simplicity :

$$\begin{aligned}
y^{\delta}\,y_{\delta} \;&=\; \alpha_{11}\,y^1\,y^1 \,+\, 2\,\alpha_{12}\,y^1\,y^2 \,+\, \alpha_{22}\,y^2\,y^2 \;= \\
&= (\sqrt{\alpha_{11}}\,y^1)^2 \,+\, 2\,\frac{\alpha_{12}}{\sqrt{\alpha_{11}}\sqrt{\alpha_{22}}}(\sqrt{\alpha_{11}}\,y^1)\,(\sqrt{\alpha_{22}}\,y^2) \,+\, (\sqrt{\alpha_{22}}\,y^2)^2 \;= \\
&= \left[(\sqrt{\alpha_{11}}\,y^1) \,+\, \frac{\alpha_{12}}{\sqrt{\alpha_{11}}\sqrt{\alpha_{22}}}\,(\sqrt{\alpha_{22}}\,y^2)\right]^2 \,+\, \left[\frac{\sqrt{\alpha}}{\sqrt{\alpha_{11}}\sqrt{\alpha_{22}}}\,(\sqrt{\alpha_{22}}\,y^2)\right]^2 \;= \\
&= (s^1 \,+\, \cos\phi\,s^2)^2 \,+\, (\sin\phi\,s^2)^2 \;= \\
&= x^2 \,+\, y^2 \;=\; r^2
\end{aligned} \tag{G19}$$

where

$$s^1 \;=\; \sqrt{\alpha_{11}}\,y^1, \qquad\qquad s^2 \;=\; \sqrt{\alpha_{22}}\,y^2 \tag{G20}$$

$$\cos\phi \;=\; \frac{\alpha_{12}}{\sqrt{\alpha_{11}}\sqrt{\alpha_{22}}}, \qquad\qquad \sin\phi \;=\; \frac{\sqrt{\alpha}}{\sqrt{\alpha_{11}}\sqrt{\alpha_{22}}}, \tag{G21}$$

$$x \;=\; s^1 \,+\, \cos\phi\,s^2 \qquad\qquad y \;=\; \sin\phi\,s^2 \tag{G22}$$

$$\begin{aligned}
&\lim_{y^2\to 0} y^2 \int \frac{\sqrt{\alpha}\,y^2}{(y^{\delta}\,y_{\delta})^{3/2}}\,dy^1 \;= \\
&= \lim_{y^2\to 0} y^2 \int \frac{\frac{\sqrt{\alpha}}{\sqrt{\alpha_{11}}\sqrt{\alpha_{22}}}(\sqrt{\alpha_{22}}\,y^2)}{(y^{\delta}\,y_{\delta})^{3/2}}\,d\left(\sqrt{\alpha_{11}}\,y^1 \,+\, \frac{\alpha_{12}}{\sqrt{\alpha_{11}}\sqrt{\alpha_{22}}}(\sqrt{\alpha_{22}}\,y^2)\right) \;= \\
&= \lim_{y\to 0}\left(\frac{\sqrt{\alpha_{11}}}{\sqrt{\alpha}}\,y\right)\int \frac{y}{(x^2 \,+\, y^2)^{3/2}}\,dx \;= \\
&= \lim_{y\to 0}\left(\frac{\sqrt{\alpha_{11}}}{\sqrt{\alpha}}\,y\right)\left[\frac{x}{y\sqrt{x^2 \,+\, y^2}}\right]_{-x}^{x} \;=\; 2\,\frac{\sqrt{\alpha_{11}}}{\sqrt{\alpha}}
\end{aligned} \tag{G23}$$

For orthogonal coordinate systems $\sqrt{\alpha_{11}}/\sqrt{\alpha} \;=\; 1/\sqrt{\alpha_{22}}$. If the system is also Cartesian then

$\sqrt{\alpha_{11}}/\sqrt{\alpha} = 1$. The factor 2, in relation (G23) is due to the symmetry around zero. If the boundary was a circle the corresponding factor for the Cauchy integral would be 2π.

$$\lim_{y^2\to 0} y^2 \int \frac{\sqrt{\alpha}\, y^1}{(y^\delta\, y_\delta)^{3/2}}\, dy^1 =$$

$$= \lim_{y^2\to 0} y^2 \frac{\sqrt{\alpha}}{\alpha_{11}} \int_{-x}^{x} \frac{\left[(\sqrt{\alpha_{11}}y^1) + \frac{\alpha_{12}}{\sqrt{\alpha_{11}\alpha_{22}}}(\sqrt{\alpha_{22}}y^2)\right] - \frac{\alpha_{12}}{\sqrt{\alpha_{11}\,\alpha_{22}}}(\sqrt{\alpha_{22}}y^2)}{(y^\delta y_\delta)^{3/2}}$$

$$d\left(\sqrt{\alpha_{11}}y^1 + \frac{\alpha_{12}}{\sqrt{\alpha_{11}}\sqrt{\alpha_{22}}}(\sqrt{\alpha_{22}}y^2)\right)$$

$$= \lim_{y\to 0}\left(\frac{\sqrt{\alpha_{11}}}{\sqrt{\alpha}}\, y\right)\left(\frac{\sqrt{\alpha}}{\alpha_{11}}\right)\int_{-x}^{x} \frac{x - y/\tan\phi}{\left(\sqrt{x^2+y^2}\right)^3}\, dx =$$

$$= \lim_{y\to 0}\left(\frac{\sqrt{\alpha_{11}}}{\sqrt{\alpha}}\, y\right)\left(\frac{\sqrt{\alpha}}{\alpha_{11}}\right)\left[-\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{\tan\phi}\,\frac{x}{y\,\sqrt{x^2+y^2}}\right]_{-x}^{x} =$$

$$= \lim_{y\to 0}\left(\frac{\sqrt{\alpha_{11}}}{\sqrt{\alpha}}\, y\right)\left(\frac{\sqrt{\alpha}}{\alpha_{11}}\right)\left(-\frac{1}{\tan\phi}\,\frac{2\,x}{y\,\sqrt{x^2+y^2}}\right) =$$

$$= \left(\frac{\sqrt{\alpha_{11}}}{\sqrt{\alpha}}\, y\right)\left(\frac{\sqrt{\alpha}}{\alpha_{11}}\right)\left(-2\,\frac{\alpha_{12}}{\sqrt{\alpha}}\right) = -\frac{2\,\alpha_{12}}{\sqrt{\alpha_{11}\,\alpha}} \qquad \text{(G24)}$$

According to relation (A70) $\alpha_{11} = g_{11}$, $\alpha_{22} = g_{22}$ and $\alpha_{12} = g_{12}$, so substituting to (G23), (G24) we get (G17) and (G18)

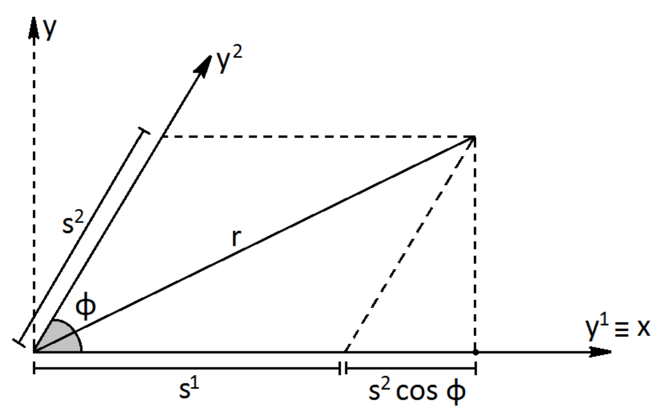


Figure G.1

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