Stability estimates for the nonlinear Schnakenberg Reaction-Diffusion model
with Neumann boundary conditions

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Overview

1. Stability estimates under minimal regularity

2. Stability for the discrete solution

The Schnakenberg Reaction-Diffusion Model

In a bounded open Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ and a time interval $(0, T]$ consider

$$\begin{cases} u_t + \nabla(\mathbf{A}u) - D_u \Delta u & = \gamma\alpha - \gamma u + \gamma u^2 v \quad \text{in } (0, T] \times \Omega \\ v_t + \nabla(\mathbf{A}v) - D_v \Delta v & = \gamma b - \gamma u^2 v \quad \text{in } (0, T] \times \Omega \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} & = 0 \quad \text{on } (0, T] \times \partial\Omega \\ u(0, x) & = g(x) \quad \text{in } \{t = 0\} \times \Omega \\ v(0, x) & = h(x) \quad \text{in } \{t = 0\} \times \Omega \end{cases} \tag{1}$$

Here :

- $u, v : (0, T] \times \Omega \to \mathbb{R}$ — Two concentrations
- $g, h : \Omega \to \mathbb{R} \in L^2(\Omega)$ — Initial Conditions
- D_u, D_v — Constant diffusion parameters
- $\mathbf{A} = ((A^i(x, t))) \in L^\infty(\Omega_T)$ — Field flow velocity
- $\nabla \cdot \mathbf{A} \in L^\infty(\Omega_T)$ — The Gradient of the field flow velocity
- γ — Scaling parameter
- α, b — Fixed positive parameters

The Schnakenberg Reaction-Diffusion Model

Models : In biology and bio-medicine reaction–diffusion systems are used frequently to model the emergence of pattern formation, wound healing, cancer and angiogenesis. Our PDE also relates to problems involving growth and shape-changes. [J.Schnackenberg, 1979]

Domain growth has been observed experimentally to be a crucial factor in developmental biology.
Time-stepping schemes for moving grid finite elements on Growing domains can be used.
[Madzvamuse, Chung, 2006], [Elliott, Stinner, Venkataraman, 2012],
[Lakkis, Madzvamuse, Venkataraman, 2013]

To analyze our problem we first study the given PDE and find where our solutions lie.[L.Evans, 1998],[James C. Robinson , 2001.]
Thereafter, for the discretization, we are going to use standard conforming finite elements in space and the backward Euler scheme in time. [S. Brenner and L. Scott, 1996],[Suli, 2012]

The Schnakenberg Reaction-Diffusion Model

Weak formulations

Suppose that $g, h \in L^2(\Omega)$, then $u, v \in L^2[0,T;H^1(\Omega)]$, with $u', v' \in L^2[0,T;(H^1(\Omega))^*] + L^{4/3}[0,T;L^{4/3}(\Omega)]$ for $n = 2$ dimensions **or** $u, v \in L^2[0,T;H^1(\Omega)]$, with $u', v' \in L^2[0,T;(H^1(\Omega))^*] + L^{6/5}[0,T;L^{6/5}(\Omega)]$ for $n = 3$ dimensions are weak solutions of the system (1), if

$$\begin{cases} \langle u', w\rangle + (\nabla(\mathbf{A}u), w) + (D_u \nabla u, \nabla w) & = (\gamma\alpha - \gamma u, w) + \langle \gamma u^2 v, w\rangle & \\ \langle v', w\rangle + (\nabla(\mathbf{A}v), w) + (D_v \nabla v, \nabla w) & = (\gamma b, w) - \langle \gamma u^2 v, w\rangle & \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} & = 0 & \text{on} \;\; (0,T] \times \partial\Omega \\ u(0,x) & = g(x) & \text{in} \;\; \{t=0\} \times \Omega \\ v(0,x) & = h(x) & \text{in} \;\; \{t=0\} \times \Omega \end{cases} \tag{2}$$

for a.e. $0 \le t \le T$ and for every $w \in H^1(\Omega)$

- How are we going to prove the well-posedness of these weak formulations?
- Faedo-Galerkin Method [L.Evans, 1998]

Consider the eigenvalue-eigenfunction problem for the operator $\mathcal{L} = -\Delta + I$,
domain($\mathcal{L}$):$= \{u \in H^1(\Omega) \mid \ \frac{\partial u}{\partial n} = 0 \ on \ \ \partial\Omega\}$,

$$\begin{cases} \mathcal{L} w_i = \lambda_i w_i \quad \forall i \\ \\ \frac{\partial w_i}{\partial n} = 0 \quad on \ \partial\Omega \end{cases}$$

Then extract eigenfunctions $\{w_k\}_{k=1}^{\infty}$ that compose an orthogonal basis of $H^1(\Omega)$ and an orthonormal basis of $L^2(\Omega)$.
Construct a sequence of finite dimensional subspaces

$$W_m = span\{w_1, ..., w_m\} \subset H^1(\Omega)$$

where

$$W_m \subset W_{m+1} \qquad \overline{\cup W_m} = H^1(\Omega)$$

Seek approximate solutions $v_m : [0, T] \to W_m,\ u_m : [0, T] \to W_m$

$$\begin{cases} u_m(t) = \sum_{k=1}^{m} c_m^k(t) w_k \\ \\ v_m(t) = \sum_{k=1}^{m} d_m^k(t) w_k \end{cases} \tag{3}$$

that satisfies the projection of problem (1) onto the finite dimensional subspace spanned by $\{w_k\}_{k=1}^{m}$.

$$\begin{cases} (u_m', w_k) + (\nabla(\mathbf{A}u_m), w_k) + (D_u \nabla u_m, \nabla w_k) & = (\gamma\alpha - \gamma u_m + \gamma u_m^2 v_m, w_k) & k = 1, 2, ..., m \\ (v_m', w_k) + (\nabla(\mathbf{A}v_m), w_k) + (D_v \nabla v_m, \nabla w_k) & = (\gamma b - \gamma u_m^2 v_m, w_k) & k = 1, 2, ..., m \\ \dfrac{\partial u_m}{\partial n} = \dfrac{\partial v_m}{\partial n} & = 0 & \text{on } \ (0, T] \times \partial\Omega \\ u_m(0, x) & = \sum_{k=1}^{m} (g, w_k) w_k = u_{m0} & \text{in } \ \{t = 0\} \times \Omega \\ v_m(0, x) & = \sum_{k=1}^{m} (h, w_k) w_k = v_{m0} & \text{in } \ \{t = 0\} \times \Omega \end{cases} \tag{4}$$

for a.e $0 \le t \le T$, where u_{m0}, v_{m0} are the orthogonal projections onto W_m of $u(0, x) = g(x), v(0, x) = h(x)$.

Faedo-Galerkin Method

STEPS :

- Prove that such constuction exists and is well defined
- Prove energy estimates
- Passing the limits
- Uniqueness of the weak solutions

Construction exists

Our construction is well defined

For every integer m=1,2,... there exist functions u_m, v_m of the form (3) that satisfy the problem (4).

First Note : $F_1(u,v) = \alpha - u + u^2 v$ and $F_2(u,v) = b - u^2 v$ are locally Lipschitz in the sense

$$\begin{aligned} |F_1(u,v) - F_1(\overline{u},\overline{v})| &\leq L_{F_1}(\mu)(|u-\overline{u}| + |v-\overline{v}|) \\ |F_2(u,v) - F_2(\overline{u},\overline{v})| &\leq L_{F_2}(\mu)(|u-\overline{u}| + |v-\overline{v}|) \end{aligned} \tag{5}$$

where $\max\{|u|,|\overline{u}|,|v|,|\overline{v}|\} \leq \mu$ for some $\mu > 0$.

Substituting (3) into (4) we get the 'composite' form of the PDE, where we use the **Banach Fixed Point Theorem** for a suitable Banach space $(C([0,T_m], L^2(\Omega;\mathbb{R}^2)))$ and a suitable norm $\|\mathbf{v}\| = \max_{0\leq t\leq T_m} \|\mathbf{v}(t)\|_{L^2(\Omega;\mathbb{R}^2)}$, where $T_m \in (0,T]$ that is selected appropriately.

$$\begin{cases} u_m' - D_u \Delta u_m + P^m(\nabla\cdot\mathbf{A})u_m + P^m\mathbf{A}\cdot\nabla u_m & = \gamma P^m F_1(u_m, v_m) \\ v_m' - D_v \Delta v_m + P^m(\nabla\cdot\mathbf{A})v_m + P^m\mathbf{A}\cdot\nabla v_m & = \gamma P^m F_2(u_m, v_m) \\ \dfrac{\partial u_m}{\partial n} = \dfrac{\partial v_m}{\partial n} & = 0 \quad \text{on } (0,T_m]\times\partial\Omega \\ u_m(0,x) & = P^m g \quad \text{in } \{t=0\}\times\Omega \\ v_m(0,x) & = P^m h \quad \text{in } \{t=0\}\times\Omega \end{cases} \tag{6}$$

Constuction exists

$P^m : L^2(\Omega) \to W_m$ is the orthogonal projection operator onto W_m

$$(P^m u, u_m) = (u, u_m), \quad \forall u_m \in W_m, \;\; u \in L^2(\Omega)$$

Now T_m is chosen so that we can prove that the operator A that we need to define

$$A : C([0, T_m], L^2(\Omega; \mathbb{R}^2)) \to C([0, T_m], L^2(\Omega; \mathbb{R}^2))$$

$$A[\mathbf{q}] = \mathbf{r}$$

is a strict contraction. Where $\mathbf{q}$ can be taken to be the solution of our composite system, $\mathbf{q}= (u_m, v_m)$ and $\mathbf{r}$ the solution of the auxiliary parabolic problem with source term, the projection of $F_1(u_m, v_m), F_2(u_m, v_m)$, where now we consider them as $\mathcal{F}(t) = (\mathcal{F}_1(t), \mathcal{F}_2(t)) := (P^m F_1(\mathbf{q}(t)), P^m F_2(\mathbf{q}(t)))$

$$\begin{cases} x_m' - D_u \Delta x_m + P^m(\nabla \cdot \mathbf{A}) x_m + P^m \mathbf{A} \cdot \nabla x_m & = \gamma \mathcal{F}_1(t) \quad \text{in} \;\; \Omega_{T_m} \\ y_m' - D_v \Delta y_m + P^m(\nabla \cdot \mathbf{A}) y_m + P^m \mathbf{A} \cdot \nabla y_m & = \gamma \mathcal{F}_2(t) \quad \text{in} \;\; \Omega_{T_m} \\ \dfrac{\partial x_m}{\partial n} = \dfrac{\partial y_m}{\partial n} & = 0 \qquad\quad \text{on} \;\; (0, T_m] \times \partial\Omega \\ x_m(0, x) & = P^m g \quad\; \text{in} \;\; \{t = 0\} \times \Omega \\ y_m(0, x) & = P^m h \quad\; \text{in} \;\; \{t = 0\} \times \Omega \end{cases} \tag{7}$$

Constuction exists

$$\mathcal{F} \in L^2[0, T_m; L^2(\Omega; \mathbb{R}^2)]$$

After choosing $\mathbf{q}, \overline{\mathbf{q}} \in C([0, T_m], L^2(\Omega; \mathbb{R}^2))$ and defining $\mathbf{r} = A[\mathbf{q}], \bar{\mathbf{r}} = A[\overline{\mathbf{q}}]$ $\rightarrow$ consider the weak formulation of the differences of the auxilary problem, choose the correct test function $\rightarrow$ we can reach to a point where

$$\|A[\mathbf{q}] - A[\overline{\mathbf{q}}]\| \leq (C' T_m)^{1/2} \|\mathbf{q} - \overline{\mathbf{q}}\|$$

where $C' = C'(L_{F_1}(\mu), L_{F_2}(\mu))$.
Banach Fixed Point Theorem $\rightarrow$ Fixed Point $A[\mathbf{q}_0] = \mathbf{q}_0$
Need only to expand the time T_m. This is made possible by the following energy estimate (the uniform bounds) that we calculate.

Energy Estimates

Energy Estimates $\rightarrow$ Uniform Bounds $\rightarrow$ Banach-Alaoglu + Eberlein-Smulian Theorem $\rightarrow$ extract a subsequence. We have the following results

Energy Estimates

There exists a constant C that depends only on $\Omega, T, D_v, D_u, \mathbf{A} \quad \nabla \cdot \mathbf{A}, \gamma$ such that,

$$\max_{0 \leq t \leq T} \|v_m\|^2_{L^2(\Omega)} + D_v \|\nabla v_m\|^2_{L^2[0,T;L^2(\Omega)]} + 2\gamma \|u_m v_m\|^2_{L^2[0,T;L^2(\Omega)]} \quad \leq C\Big(\|h(x)\|^2_{L^2(\Omega)} + (\gamma b)^2 \Big) \tag{8}$$

$$\begin{aligned} &\max_{0 \leq t \leq T} \|u_m + v_m\|^2_{L^2(\Omega)} + D_u \|\nabla (u_m + v_m)\|^2_{L^2[0,T;L^2(\Omega)]} \\ &\qquad \leq C\Big(\|g(x)\|^2_{L^2(\Omega)} + \|h(x)\|^2_{L^2(\Omega)} + (\gamma \alpha)^2 + (\gamma b)^2 \Big) \end{aligned} \tag{9}$$

$$\max_{0 \leq t \leq T} \|u_m\|^2_{L^2(\Omega)} + D_u \|\nabla u_m\|^2_{L^2[0,T;L^2(\Omega)]} \leq C\Big(\|g(x)\|^2_{L^2(\Omega)} + \|h(x)\|^2_{L^2(\Omega)} + (\gamma \alpha)^2 + (\gamma b)^2 \Big) \tag{10}$$

Energy Estimates

- How do we prove such estimates?

We take the weak formulation of the projected problem (4) with the correct test function, doing calculation and typical Kickback arguements to achieve an inequality of the form

$$\frac{d}{dt}\left(\frac{1}{2}\left\|v_m\right\|_{L^2(\Omega)}^2\right)+D_v\left\|\nabla v_m\right\|_{L^2(\Omega)}^2+\gamma\left\|u_m v_m\right\|_{L^2(\Omega)}^2$$

$$\leq \frac{|\Omega|}{2}(\gamma b)^2+\left(\frac{1}{2}+\frac{\ell^2}{4D_v}+\left\|\nabla\cdot\mathbf{A}\right\|_{L^\infty(\Omega_T)}\right)\left\|v_m\right\|_{L^2(\Omega)}^2+D_v\left\|\nabla v_m\right\|_{L^2(\Omega)}^2$$

$$\frac{d}{dt}\left(\frac{1}{2}\left\|v_m\right\|_{L^2(\Omega)}^2\right)\leq\frac{|\Omega|}{4}(\gamma b)^2+\left(1+\frac{\ell^2}{2D_v}+2\left\|\nabla\cdot\mathbf{A}\right\|_{L^\infty(\Omega_T)}\right)\frac{1}{2}\left\|v_m\right\|_{L^2(\Omega)}^2$$

where $\ell=\sum_{i=1}^n\left\|A^i\right\|_{L^\infty(\Omega_T)}$

Now we use Gronwall's inequality

Integrating with respect to time, and maximizing the LHS we get the desired result.

- Difficulty to find estimates for $u_m \rightarrow$ Sign of the non-linear term $+\ u_m^2 v_m$
- Find estimates for the sum of the two functions $u_m + v_m \rightarrow$ Removed the non-linearity
- New difficulty $\rightarrow D_u > D_v$ or $D_u < D_v$?

$$\begin{aligned}
\big((v_m+u_m)', v_m+u_m\big) + (D_u \nabla u_m, \nabla(u_m+v_m)) + (D_v \nabla v_m, \nabla(u_m+v_m)) \\
+\big((\nabla\cdot\mathbf{A}+\gamma)(u_m+v_m), u_m+v_m\big) + \big(\nabla(u_m+v_m)\cdot\mathbf{A}, u_m+v_m\big) \\
= \int_\Omega (\gamma\alpha+\gamma b)(u_m+v_m)dx + (\gamma v_m, u_m+v_m)
\end{aligned}$$

Rewrite the term in case $D_u \le D_v$, as

$$\begin{aligned}
(D_v \nabla v_m, \nabla(u_m+v_m)) &= \big((D_u + (D_v - D_u))\nabla v_m, \nabla(u_m+v_m)\big) \\
&= (D_u \nabla v_m, \nabla(u_m+v_m)) + \big((D_v - D_u)\nabla v_m, \nabla(u_m+v_m)\big)
\end{aligned}$$

in case $D_u > D_v$, as

$$\begin{aligned}
(D_u \nabla u_m, \nabla(u_m+v_m)) &= \big((D_v + (D_u - D_v))\nabla u_m, \nabla(u_m+v_m)\big) \\
&= (D_v \nabla u_m, \nabla(u_m+v_m)) + \big((D_u - D_v)\nabla u_m, \nabla(u_m+v_m)\big)
\end{aligned}$$

As a corollary we get,

Energy Estimates

There exists a constant C that depends only on Ω, $T, D_v, D_u, \mathbf{A} \;\; \nabla \cdot \mathbf{A}, \gamma$ such that,

$$\|u_m\|^2_{L^2[0,T,H^1(\Omega)]} + \|v_m\|^2_{L^2[0,T,H^1(\Omega)]} \leq C\Big(\|g(x)\|^2_{L^2(\Omega)} + \|h(x)\|^2_{L^2(\Omega)} + (\gamma\alpha)^2 + (\gamma b)^2\Big) \tag{11}$$

for $m = 1, 2, \ldots$

Now we derive the estimates for the derivatives in time.

Energy Estimates

There exists a constant C_* which depends only on Ω, $T, D_v, D_u, \mathbf{A}\ \ \nabla\cdot\mathbf{A}, \gamma$ and the initial conditions such that,

$$\begin{cases} \|u'_m\|_{L^{4/3}[0,T;(H^1)^*]} \leq C_* \\ \|v'_m\|_{L^{4/3}[0,T;(H^1)^*]} \leq C_* \end{cases} \quad \textit{for} \quad n=2 \tag{12}$$

$$\begin{cases} \|u'_m\|_{L^{6/5}[0,T;(H^1)^*]} \leq C_* \\ \|v'_m\|_{L^{6/5}[0,T;(H^1)^*]} \leq C_* \end{cases} \quad \textit{for} \quad n=3 \tag{13}$$

for $m = 1, 2, \cdots$

For the proof we only need

- For a function $z \in H^1(\Omega)$ can be written as $z = z^1 + z^2$, where $z^1 \in W_m$ and $(z^2, w_k) = 0$ $(k = 1, \cdots, m)$ such that $\left\|z^1\right\|_{H^1(\Omega)} \leq \|z\|_{H^1(\Omega)} \leq 1$
- $$\langle u_m', z\rangle = (u_m', z) = (u_m', z^1) = -((\nabla\cdot\mathbf{A})u_m, z^1) - (\nabla u_m \cdot \mathbf{A}, z^1) - (D_u \nabla u_m, \nabla z^1) + (\gamma\alpha - \gamma u_m + \gamma u_m^2 v_m, z^1)$$
  $$\langle v_m', z\rangle = (v_m', z) = (v_m', z^1) = -((\nabla\cdot\mathbf{A})v_m, z^1) - (\nabla v_m \cdot \mathbf{A}, z^1) - (D_v \nabla v_m, \nabla z^1) + (\gamma b - \gamma u_m^2 v_m, z^1)$$
- $$\left\|u_m'\right\|_{(H^1(\Omega))^*} = \sup_{z\in H^1(\Omega), \|z\|_{H^1(\Omega)}\leq 1} \left\{\left|\langle u_m', z\rangle\right|\right\}$$
  $$\left\|v_m'\right\|_{(H^1(\Omega))^*} = \sup_{z\in H^1(\Omega), \|z\|_{H^1(\Omega)}\leq 1} \left\{\left|\langle v_m', z\rangle\right|\right\}$$

Energy Estimates

- We only need to carefully bound the nonlinear terms in both dimensions. This is done by the following remark

$$f(u_m, v_m) = u_m^2 v_m \in L^{4/3}[0, T; L^{4/3}(\Omega)] \qquad n = 2$$

$$f(u_m, v_m) = u_m^2 v_m \in L^{6/5}[0, T; L^{6/5}(\Omega)] \qquad n = 3$$

  This is done by Standard calculations using Young type inequalities + Landyzeshkayka-Gagliardo-Nirenberg inequalities.
  $\|u\|_{L^4(\Omega)} \leq C \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^1(\Omega)}^{1/2}$ for n=2 dimensions, $\|u\|_{L^3(\Omega)} \leq C \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^1(\Omega)}^{1/2}$ for n=3 dimensions
- From the above-mentioned remark, using Hölder type inequalities, correctly, we get the following inequality

$$|\langle u_m', z\rangle| \leq C\left[D_u + \ell + \|\nabla \cdot \mathbf{A}\|_{L^\infty(\Omega)} + \gamma + \gamma\alpha|\Omega|^{1/2}\right] \|u_m\|_{H^1(\Omega)} + C\gamma \left\|u_m^2 v_m\right\|_{L^p(\Omega)}$$

$$|\langle v_m', z\rangle| \leq C\left[D_u + \ell + \|\nabla \cdot \mathbf{A}\|_{L^\infty(\Omega)} + \gamma b|\Omega|^{1/2}\right] \|v_m\|_{H^1(\Omega)} + C\gamma \left\|u_m^2 v_m\right\|_{L^p(\Omega)}$$

  $p = 4/3$ or $p = 6/5$.
- Maximizing the LHS, doing basic algebra and integrating in time, we get the desired result.

Passing the Limits

- To pass the limits the only remark that we need is that the uniform bounds we constructed are in Hilbert spaces.
- Eberlein-Smulian Theorem $+$ Banach-Alaoglu Theorem $\rightarrow$ extract a weakly and weakly star converging subsequences.

Limits

Let the system (1) and $h(x), g(x) \in L^2(\Omega)$ the initial boundary conditions then we have the following limits :

$$u_m \xrightarrow{w} u \ in \ L^2[0,T;H^1(\Omega)], \qquad u_m \xrightarrow{w*} u \ in \ L^\infty[0,T;L^2(\Omega)], \qquad \begin{array}{ll} u'_m \xrightarrow{w} u' \ in \ L^{4/3}[0,T;(H^1(\Omega))^*], & n=2 \\ u'_m \xrightarrow{w} u' \ in \ L^{6/5}[0,T;(H^1(\Omega))^*], & n=3 \end{array}$$

$$v_m \xrightarrow{w} v \ in \ L^2[0,T;H^1(\Omega)], \qquad v_m \xrightarrow{w*} v \ in \ L^\infty[0,T;L^2(\Omega)], \qquad \begin{array}{ll} v'_m \xrightarrow{w} v' \ in \ L^{4/3}[0,T;(H^1(\Omega))^*], & n=2 \\ v'_m \xrightarrow{w} v' \ in \ L^{6/5}[0,T;(H^1(\Omega))^*], & n=3 \end{array}$$

$$u_m \longrightarrow u \ strongly \ in \ L^2[0,T;L^2(\Omega)], \qquad v_m \longrightarrow v \ strongly \ in \ L^2[0,T;L^2(\Omega)]$$

$$f(u_m,v_m) \xrightarrow{w} f(u,v) \ in \ L^{4/3}[0,T;L^{4/3}(\Omega)], \quad n=2$$
$$f(u_m,v_m) \xrightarrow{w} f(u,v) \ in \ L^{6/5}[0,T;L^{6/5}(\Omega)], \quad n=3$$

Passing the Limits

- For the strong convergence we are going to use compactness arguements
- Aubins - Lions Theorem for the spaces

$$W = \left\{ u \in L^2[0,T;H^1(\Omega)] \middle| \; u_t = \frac{du}{dt} \in L^{4/3}[0,T;(H^1)^*] \right\} \quad for\ n = 2$$

$$W = \left\{ u \in L^2[0,T;H^1(\Omega)] \middle| \; u_t = \frac{du}{dt} \in L^{6/5}[0,T;(H^1)^*] \right\} \quad for\ n = 3$$

then for both cases we get that

$$W \hookrightarrow\hookrightarrow L^2[0,T;L^2(\Omega)]$$

- Stronly convergence in $L^2[0,T;L^2(\Omega)] \implies$ pointwise convergence
- $f(u_m(x,t), v_m(x,t)) = u_m^2(x,t)v_m(x,t) \longrightarrow u^2(x,t)v(x,t) = f(u,v)$ *'pointwise' a.e. in* Ω_T*, for* $n = 2,3$
- By Aubins -Lion Lemma we get the desired weakly convergence for the non-linear term. [J.-L. Lions, 1969] or [James C. Robinson , 2001.]

Passing the Limits

We return to the weak formulation of the projection

- Now using density arguments $\rightarrow$ choose test function $w \in C^1([0, T]; H^1(\Omega))$
- pass the limits correctly using the above weak convergence
- It stand for every function $w \in L^4[0, T; H^1(\Omega)]$ for n=2 dimensions and $w \in L^6[0, T; H^1(\Omega)]$ for n=3 dimensions $\rightarrow$ $C^1([0, T]; H^1(\Omega))$ dense in both.

Uniqueness of the weak solution

There weak solution of the Schnakenberg system as in (2) is unique.

Uniqueness

- The proof for uniqueness is very technical
- It stands from the fact that our now solutions lie in

$$u, v \in L^4[0, T; L^4(\Omega)] \quad \textit{for } n = 2, \qquad u, v \in L^{8/3}[0, T; L^4(\Omega)] \quad \textit{for } n = 3$$

- We consider 2 solutions u, v and $\overline{u}, \overline{v} \to \eta = v - \overline{v}$, $\theta = u - \overline{u}$ and proceed to show they are the same.

From the weak formulations we calculate the differences and after choosing the correct test function we obtain

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{1}{2}\|\theta\|^2_{L^2(\Omega)}\right) + ((\nabla \cdot \mathbf{A})\theta, \theta) + (\mathbf{A} \cdot \nabla\theta, \theta) + D_u \|\nabla\theta\|^2_{L^2(\Omega)} & = -\gamma\|\theta\|^2_{L^2(\Omega)} + \gamma(u^2 v - \overline{u}^2\overline{v}, \theta) \\ \dfrac{d}{dt}\left(\dfrac{1}{2}\|\eta\|^2_{L^2(\Omega)}\right) + ((\nabla \cdot \mathbf{A})\eta, \eta) + (\mathbf{A} \cdot \nabla\eta, \eta) + D_v \|\nabla\eta\|^2_{L^2(\Omega)} & = -\gamma(u^2 v - \overline{u}^2\overline{v}, \eta) \\ \dfrac{\partial\theta}{\partial n} = \dfrac{\partial\eta}{\partial n} & = 0 \qquad \text{on } (0, T] \times \partial\Omega \\ \theta(0, x) & = 0 \qquad \text{in } \{t = 0\} \times \Omega \\ \eta(0, x) & = 0 \qquad \text{in } \{t = 0\} \times \Omega \end{cases} \tag{14}$$

- Need to handle the terms on the RHS $\to u^2 v - \overline{u}^2\overline{v} = u^2(v - \overline{v}) + (u^2 - \overline{u}^2)\overline{v} = u^2\eta + \theta(u + \overline{u})\overline{v}$

Uniqueness

- Moving the correct terms to the RHS, using general type Young and Landyzeshkayka-Gagliardo-Nirenberg inequalities we get

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|\theta\|^2_{L^2(\Omega)} + \frac{1}{2}\|\eta\|^2_{L^2(\Omega)}\right) + D_u\|\nabla\theta\|^2_{L^2(\Omega)} + D_v\|\nabla\eta\|^2_{L^2(\Omega)} + \gamma\|\theta\|^2_{L^2(\Omega)} \\ &\leq \|\nabla\mathbf{A}\|_{L^\infty(\Omega)}\left(\|\theta\|^2_{L^2(\Omega)} + \|\eta\|^2_{L^2(\Omega)}\right) + \ell\epsilon_1\|\nabla\theta\|^2_{L^2(\Omega)} + \ell\epsilon_1'\|\nabla\eta\|^2_{L^2(\Omega)} + \frac{\ell}{4\epsilon_1}\|\theta\|^2_{L^2(\Omega)} \\ &+ \frac{\ell}{4\epsilon_1'}\|\eta\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega u^2\eta^2 dx + \frac{3}{2}\int_\Omega u^2\theta^2 dx + \int_\Omega \theta^2(u+\overline{u})^2 dx + \frac{1}{2}\int_\Omega \theta^2\overline{v}^2 dx + \frac{1}{2}\int_\Omega \eta^2\overline{v}^2 dx \end{aligned} \tag{15}$$

- Depending on the dimensions of the problem we have different techniques.
- For 2 dimensions we just need to bound the last 5 terms in terms of the $L^2(\Omega)$ norm.
  $\rightarrow$ Do a kickback argument with some of the $L^2(\Omega)$ terms.
  $\rightarrow$ Use Gronwall inequality with zeroed initial conditions.
- For 3 dimension we bound the last terms in terms of the $H^1(\Omega)$ and $L^2(\Omega)$ norms, after we create terms of η and θ in, again, $H^1(\Omega)$ norm.
  $\rightarrow$ Do a kickback argument with the $H^1(\Omega)$ terms.
  $\rightarrow$ Use Gronwall inequality with zeroed initial conditions.

Space -Discretization

For the discrete part, we are going to use the Finite Element Method, which is based on the weak formulation of our PDE.
We use :

- Standard conforming elements in space
- Implicit-Backward Euler time stepping scheme

Let $\mathcal{T}^h = \{\tau\}$ be firstly shape regular triangulations of the domain Ω into disjoint open simplices $\{\tau\}$ with $h_\tau := diam \tau$ and $h := \max_{\tau \in \mathcal{T}^h} h_\tau$. It is considered a shape regular triangulation if there exists a σ s.t. $\sigma = \frac{h_\tau}{\rho_\tau}$ for every τ.

We need, now, to approximate the problem on an appropriate grid. So we need to generate a mesh in order to discretize the problem resulting to a family of finite element spaces, where an ordinary differential system in time is going to occur.

We may consider the following finite element space

$$V_h := \{v \in C(\Omega) : v_\tau \in \mathcal{P}_r \ \ \forall \tau \in \mathcal{T}^h\} \subset H^1(\Omega)$$

consisting of piecewise continuous function.

Space-Discretization

We can further impose the triangulation to be quasi-uniform by demaning that such σ is uniform, so that the inverse estimate holds.

$$\|v_h\|_{H^1(\Omega)} \leq Ch^{-1} \|v_h\|_{L^2(\Omega)} \qquad \forall v_h \in V_h$$

Let $\{P_j\}_{j=1}^{N_h}$ be the vertices of $\mathcal{T}^h$ which determines $\mathcal{P}_r$, thus creating a nodal base $\{\phi_1(x), ..., \phi_{N_h}(x)\}$.
[S. Brenner and L. Scott, 1996]

Then every function-solution $v_h, u_h \in V_h$ in this space is of the form

$$\begin{aligned} v_h(x,t) &= \sum_{j=1}^{N_h} V_j(t)\phi_j(x) \\ u_h(x,t) &= \sum_{j=1}^{N_h} U_j(t)\phi_j(x) \end{aligned} \tag{16}$$

Space-Discretization

Then for the finite element space $V_h \subset H^1(\Omega)$ we formulate the semi-discrete problem as :
$\forall$ time t find $u_h, v_h \in V_h$:

$$\begin{cases} \langle u_h', w_h \rangle + (\nabla(\mathbf{A} u_h), w_h) + (D_u \nabla u_h, \nabla w_h) & = (\gamma\alpha - \gamma u_h, w_h) + \gamma(u_h^2 v_h, w_h) \\ \langle v_h', w_h \rangle + (\nabla(\mathbf{A} v_h), w_h) + (D_v \nabla v_h, \nabla w_h) & = (\gamma b - \gamma u_h^2 v_h, w_h) \\ \dfrac{\partial u_h}{\partial n} = \dfrac{\partial v_h}{\partial n} & = 0 \\ u_h(0) & = u_h^0 \\ v_h(0) & = v_h^0 \end{cases} \tag{17}$$

$\forall w_h \in V_h,\ u_h^0, v_h^0$ are appropriate approximations of the initial conditions.

Where now, by substituting (16), the approximate solution is the solution of an h-dependent finite system consisting of ordinary differential equations in time. [V. Thomee, 1997]

We, now, need to discretize the system in time in order to get the fully discrete scheme.

Time-Discretization

- Fully-discrete scheme we discretize the time derivative with the Implicit-Backward Euler time stepping scheme.
- Partition is quasi-uniform in time $\rightarrow \Delta t = \frac{T}{N}, \quad N > 0$

So we get the mesh

$$Q_h^{\Delta t} := \{(P_j, t^n) : P_j \in \mathcal{T}^h,\ t^n = n\Delta t,\ 0 \leq n \leq N\}$$

Denote $U_h^n, V_h^n \in V^h$ the approximation of $u(t^n), v(t^n)$.

We approximate our initial problem by the finite element method, known as Implicit-Backward Euler scheme

$\forall$ time t find $U_h^n, V_h^n \in V_h$:

$$\begin{cases} \langle \frac{U_h^n - U_h^{n-1}}{\Delta t}, w_h \rangle + (\nabla(\mathbf{A}^n U_h^n), w_h) + (D_u \nabla U_h^n, \nabla w_h) & = (\gamma\alpha - \gamma U_h^n, w_h) + \gamma((U_h^n)^2 V_h^n, w_h) \\ \langle \frac{V_h^n - V_h^{n-1}}{\Delta t}, w_h \rangle + (\nabla(\mathbf{A}^n V_h^n), w_h) + (D_v \nabla V_h^n, \nabla w_h) & = (\gamma b - \gamma (U_h^n)^2 V_h^n, w_h) \\ \frac{\partial U_h^n}{\partial n} = \frac{\partial V_h^n}{\partial n} & = 0, \quad n = 1, .., N \\ U_h^0 & = u_h^0 \\ V_h^0 & = v_h^0 \end{cases} \tag{18}$$

$\forall w_h \in V_h$, $1 \leq n \leq N$ and $\mathbf{A}^n = \mathbf{A}(x, t^n)$

Stability Estimates

Let $U_h^0, V_h^0 \in L^2(\Omega)$, then for $\Delta t < \min\left\{\frac{1}{\Lambda(\epsilon_1)}, \frac{1}{M(\epsilon_1')}\right\}$ we get the following stability estimates

$$\max_{1\le n\le N} \|V_h^n\|^2_{L^2(\Omega)} + \sum_{n=1}^{N} \left\|V_h^n - V_h^{n-1}\right\|^2_{L^2(\Omega)} + D_v \Delta t \sum_{n=1}^{N} \|\nabla V_h^n\|^2_{L^2(\Omega)} + 2\gamma\Delta t \sum_{n=1}^{N} \|U_h^n V_h^n\|^2_{L^2(\Omega)}$$
$$\le e^{T\frac{\Lambda(\epsilon_1)}{1-\Delta t\Lambda(\epsilon_1)}} \left(\left\|V_h^0\right\|^2_{L^2(\Omega)} + \frac{|\Omega|(\gamma b)^2}{2\epsilon_1} \right)$$

$$\max_{1\le n\le N} \|U_h^n + V_h^n\|^2_{L^2(\Omega)} + \sum_{n=1}^{N} \left\|(U_h^n + V_h^n) - (U_h^{n-1} + V_h^{n-1})\right\|^2_{L^2(\Omega)} + D_u \Delta t \sum_{n=1}^{N} \|\nabla(U_h^n + V_h^n)\|^2_{L^2(\Omega)}$$
$$\le e^{T\frac{M(\epsilon_1')}{1-\Delta t M(\epsilon_1')}} C(\Lambda(\epsilon_1), \gamma, T, D_u, D_v) \left(\left\|U_h^0\right\|^2_{L^2(\Omega)} + \left\|V_h^0\right\|^2_{L^2(\Omega)} + \frac{|\Omega|(\gamma b)^2}{2\epsilon_1} + T\frac{|\Omega|(\gamma\alpha + \gamma b)^2}{2\epsilon_1'} \right)$$

$$\max_{1\le n\le N} \|U_h^n\|^2_{L^2(\Omega)} + \sum_{n=1}^{N} \left\|U_h^n - U_h^{n-1}\right\|^2_{L^2(\Omega)} + D_u \Delta t \sum_{n=1}^{N} \|\nabla U_h^n\|^2_{L^2(\Omega)}$$
$$\le C(\Lambda(\epsilon_1), M(\epsilon_1'), \gamma, T, D_u, D_v) \left(\left\|U_h^0\right\|^2_{L^2(\Omega)} + \left\|V_h^0\right\|^2_{L^2(\Omega)} + \frac{|\Omega|(\gamma b)^2}{2\epsilon_1} + T\frac{|\Omega|(\gamma\alpha + \gamma b)^2}{2\epsilon_1'} \right)$$

where ϵ_1, ϵ_1' are fixed positive numbers, $\|\nabla \cdot \mathbf{A}\|_{L^\infty(\Omega)} = \max_{1\le n\le N} \|\nabla \cdot \mathbf{A}^n\|_{L^\infty(\Omega)}$ and $\ell = \max_{1\le n\le N} \ell^n = \max_{1\le n\le N} \sum_{i=1}^{d} \left\|A^{n,i}\right\|_{L^\infty(\Omega)}$ (d dimensions of the problem), and $\Lambda(\epsilon_1) = 2\big(\|\nabla \cdot \mathbf{A}\|_{L^\infty(\Omega)} + \frac{\ell^2}{2D_v} + \epsilon_1 \big)$, $M(\epsilon_1') = 2\big(\|\nabla \cdot \mathbf{A}\|_{L^\infty(\Omega)} + \frac{3}{2}\gamma + \frac{\ell^2}{D_u} + \epsilon_1' \big)$

- For the proof, using the same techniques as in the formulation of the energy estimates we can get a similar stability for the discrete solution. [K. Chrysafinos, E. N. Karatzas, and D. Kostas, 2019]
- We also need the use of the Discrete Gronwall's Lemma as in [Yunzhang Zhang, Yanren Hou, and Jianping Zhaoe, 2014],
- For integer $n \ge 0$, let $\Delta t, H$ and a_n, b_n, c_n, d_n non-negative numbers satisfying the following

$$
\begin{aligned}
&a_\ell + \Delta t \sum_{n=1}^{\ell} b_n \le \Delta t \sum_{n=1}^{\ell} d_n a_n + \Delta t \sum_{n=1}^{\ell} c_n + H \quad \textit{for} \quad \ell > 0,\\
&a_0 \le H\\
&\Delta t d_n < 1
\end{aligned}
$$

then

$$
a_\ell + \Delta t \sum_{n=1}^{\ell} b_n \le e^{\Delta t \sum_{n=1}^{\ell} \frac{d_n}{1-\Delta t d_n}} \left(\Delta t \sum_{n=1}^{\ell} c_n + H \right) \tag{19}
$$

Stability

- After considering (18), the right test function and basic algebra, with the use of kickback arguements, we formulate the inequality,

$$
\begin{aligned}
\sum_{n=1}^{k}\left\|V_h^n - V_h^{n-1}\right\|_{L^2(\Omega)}^2 + \left\|V_h^k\right\|_{L^2(\Omega)}^2 + D_v\Delta t\sum_{n=1}^{k}\|\nabla V_h^n\|_{L^2(\Omega)}^2 + 2\gamma\Delta t\sum_{n=1}^{k}\|U_h^n V_h^n\|_{L^2(\Omega)}^2 \\
\leq \Lambda(\epsilon_1)\Delta t\sum_{n=1}^{k}\|V_h^n\|_{L^2(\Omega)}^2 + \left\|V_h^0\right\|_{L^2(\Omega)}^2 + \Delta t\sum_{n=1}^{k}\frac{|\Omega|(\gamma b)^2}{2\epsilon_1}
\end{aligned}
\tag{20}
$$

- After using the above Discrete Gronwall lemma, bounding the RHS and maximizing the LHS we get the desired result.
- Notice, even though we introduced the Backward-Implicit Euler scheme in time, it is in fact conditionally stable as seem in the theorem.

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