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Seismic response of asymmetric rocking museum exhibits

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Thesis

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Abstract

This thesis investigates the dynamic behavior of two rigid bodies which are simply placed on top of each other. The work is focusing on museum artefacts that can be addressed as a rocking bust on top of a solid pedestal. Pedestals are commonly adopted in order to bring up the artefacts on the eye-level. Therefore, it is important to examine the case in which the upper-body is asymmetric, while the pedestal/base is always symmetric. The system of two-bodies may rock, jump, slide or overturn under a seismic excitation. However, the current work investigated the pure rocking motion under the assumptions of rigid foundation, large friction and plastic impact. Initially, the problem can be solved using an equivalent single-block, assuming either that the pedestal is stocky enough and only the artefact will rock, or that the pedestal and the artefact have the same rotation angle. This is the simplest case, usually adopted in practice, which allows to solve the problem using the equation of motion proposed by Housner (1963)[2] or using equivalent solutions that have been proposed in the literature (e.g. Diamantopoulos and Fragiadakis, 2019)[12]. The problem is mainly studied as a two-block system, based on the same approach first presented by Psycharis (1990)[4]. However, this work (a) does not consider small rotations and (b) generalizes the equations of motion which can be used for different systems - the symmetric upper body is the simplest case. The purpose of this study is to provide the appropriate complicated equations in order to accurately predict the seismic response of stacked systems.

Extended Summary (in Greek)

Στην παρούσα διπλωματική εργασία μελετήθηκε η απόκριση μη-συμμετρικών λικνιζόμενων μουσειακών εκθεμάτων υπό σεισμική διέγερση. Το πρόβλημα που προκύπτει σχετίζεται με την έκθεση των τεχνουργημάτων, καθώς απαιτείται να τοποθετούνται χωρίς σύνδεση πάνω σε μια βάση ώστε να μην αλλοιώνονται αλλά και να θαυμάζονται με ευκολία από τον επισκέπτη. Τα συνηθέστερα εκθέματα είναι οι προτομές αγαλμάτων, οι οποίες παρουσιάζουν ιδιαίτερο ενδιαφέρον εξαιτίας της ασυμμετρίας τους. Έτσι θεωρήθηκε αναγκαία η δυναμική ανάλυση των δύο λικνιζόμενων σωμάτων θεωρώντας ασύμμετρο το άνω σώμα. Σκοπός της εργασίας είναι η μελέτη του παραπάνω συστήματος ως σύστημα δύο, ελεύθερα εδραζομένων και άκαμπτων, σωμάτων πάνω σε μια άκαμπτη επιφάνεια και στόχος είναι η δημιουργία μοντέλου για την προσέγγιση της συμπεριφοράς του κάτω από σεισμικές διεγέρσεις.

Στο κεφάλαιο 1 παρουσιάζεται η περιγραφή του προσομοιώματος (ασύμμετρο πάνω σε συμμετρικό σώμα) και η δυναμική ανάλυσή του, διατυπώνοντας τις εξισώσεις κίνησης του συστήματος.

Στο κεφάλαιο 2 επεκτείνεται η ανάλυση, παρέχοντας τις συνθήκες που οφείλουν να ισχύουν, ώστε να πραγματοποιηθούν οι μεταβάσεις ανάμεσα στα στιγμιότυπα της κίνησης του συστήματος, χωρίς να συμβεί οποιοδήποτε είδος κρούσης.

Στο κεφάλαιο 3 αναλύονται οι περιπτώσεις κατά τις οποίες συμβαίνουν οι μεταβάσεις ανάμεσα στα στιγμιότυπα εξαιτίας της κρούσης και υπολογίζονται οι σχέσεις που ορίζουν τις ταχύτητες αμέσως μετά την κρούση.

Στο κεφάλαιο 4 περιγράφεται η δομή του ηλεκτρονικού προγράμματος που αναπτύχθηκε για την επίλυση του προβλήματος. Επίσης παρουσιάζονται τα αποτελέσματα που προέκυψαν και γίνονται συγκρίσεις όχι μόνο μεταξύ συμμετρικού και ασύμμετρου άνω σώματος, αλλά και μεταξύ πειραματικών δεδομένων.

Τέλος στο κεφάλαιο 5 καταγράφονται τα συμπεράσματα που εξάχθηκαν από τα αποτελέσματα.

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Chapter 1

Dynamic analysis of asymmetric body on a rectangular body

1.1 Analytical Model

The system under consideration is presented schematically in Figure 1.1. It consists of two rigid blocks; the upper one (i=2) represents an asymmetric body which in this case is a statue, and is symmetrically placed on top of the lower symmetric body (i=1) which represents a pedestal. The lower body is freely placed on the ground.

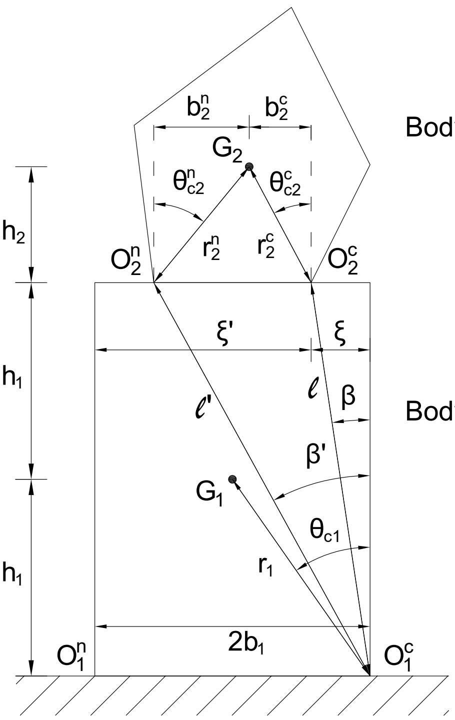

Figure 1.1: Geometric model

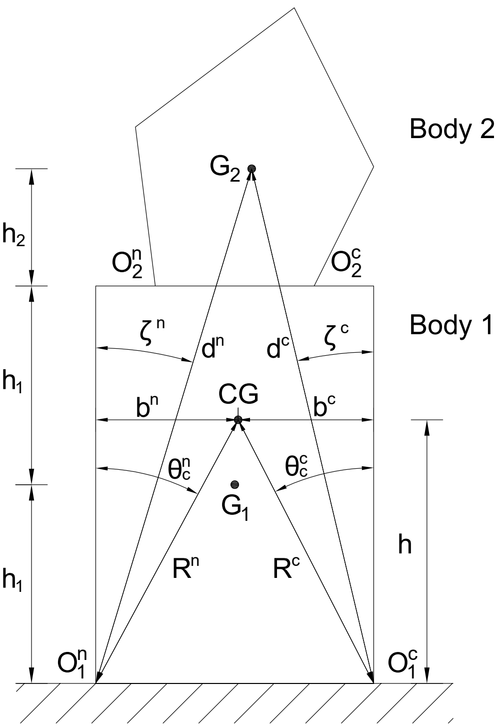

Figure 1.2: Additional parameters

In Figure 1.2 are presented some additional parameters which appear in the following analysis. Each mass is denoted by m_i and the centroid moment of inertia by I_{Gi}. The parameters that differ because of the problem's asymmetry, are superscripted with the indices **c** for critical side (right) and **n** for non-critical side (left). Their formulation follows below.

$$M = m_1 + m_2$$

$$h = \frac{m_1 h_1 + m_2(2h_1 + h_2)}{M} \quad , \quad b^{c\|n} = \frac{m_1 b_1 + m_2(\xi + b_2^{c\|n})}{M}$$

$$\theta_c^{c\|n} = \arctan(\frac{b^{c\|n}}{h}) \quad , \quad R^{c\|n} = \sqrt{(b^{c\|n})^2 + h^2}$$

$$\zeta^{c\|n} = \arctan(\frac{b_2^{c\|n} + \xi}{2h_1 + h_2}) \quad , \quad d^{c\|n} = \sqrt{(b_2^{c\|n} + \xi)^2 + (2h_1 + h_2)^2}$$

$$I_O^{c\|n} = I_{O1} + I_{G2} + m_2(d^{c\|n})^2 \quad , \quad I_{O2}^{c\|n} = I_{G2} + m_2(r_2^{c\|n})^2$$

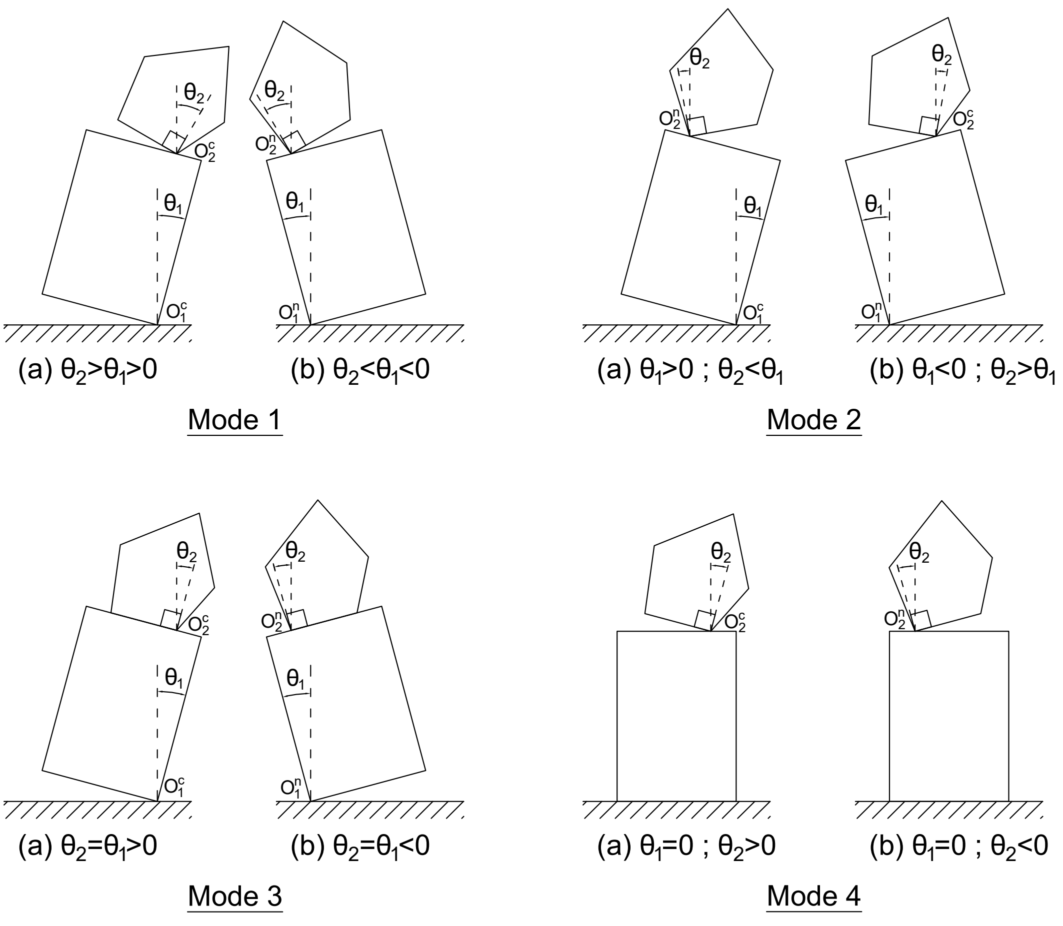

Figure 1.3: Classification of rocking modes for a system of two stacked rigid bodies with respect to the angles of rotation following Psycharis (1990)[4].

The system possesses two degrees of freedom, namely, θ_1 and θ_2, denoting the angles of rotation of the two blocks with respect to the vertical. When subjected to a base excitation, the system may exhibit four possible modes of rocking motion. Figure 1.3 illustrates the classification of the four possible modes with respect to the angles of rotation θ_1 and θ_2. Modes 1 and 2 involve a two degrees of freedom system response, and reflect rotations of the two blocks in the same or opposite direction. Modes 3 and 4 reflect a single degree of freedom system response; in particular, mode 3 describes the motion of the system rocking as one rigid structure, and mode 4 concerns the case where only the top block experiences rotation. Furthermore, each of the modes is subdivided into two subcases that account for opposite angle signs.

1.2 Equations of Motion

The equations of motion of the system are derived by applying Newton's second law to each body separately. This method was preferred over Lagrange's, in order to verify the results which were concluded from Pol D. Spanos et al(2001)[8].

1.2.1 Mode 1a

Body 1

$$\Sigma M_{O1} = I_{O1}\ddot{\theta}_1$$

$$\boxed{1} \rightarrow \quad -m_1\ddot{x}_g r_1\cos(\theta_{c1}-\theta_1) - m_1 g r_1 \sin(\theta_{c1}-\theta_1) + \boldsymbol{F_1 l\cos(\theta_1-\beta)} + \boldsymbol{F_2 l\sin(\theta_1-\beta)} = I_{O1}\ddot{\theta}_1$$

$$\boxed{\text{I}} \rightarrow \quad F_1 l\cos(\theta_1-\beta)$$

$$\boxed{\text{II}} \rightarrow \quad F_2 l\sin(\theta_1-\beta)$$

Body 2

$$\Sigma F_x = m_2 a^x_{G2} \quad\Rightarrow\quad -m_2\ddot{x}_g - F_1 = m_2 a^x_{G2} \quad\Rightarrow\quad F_1 = -m_2\ddot{x}_g - m_2 a^x_{G2}$$

$$\Sigma F_y = m_2 a^y_{G2} \quad\Rightarrow\quad F_2 - m_2 g = m_2 a^y_{G2} \quad\Rightarrow\quad F_2 = m_2 g + m_2 a^y_{G2}$$

For center of mass of Body 2

$$x_{G2} = l\sin(\theta_1-\beta) - r^c_2\sin(\theta^c_{c2}-\theta_2) \quad\Rightarrow\quad v^x_{G2} = \dot{\theta}_1 l\cos(\theta_1-\beta) + \dot{\theta}_2 r^c_2\cos(\theta^c_{c2}-\theta_2)$$

$$a^x_{G2} = \ddot{\theta}_1 l\cos(\theta_1-\beta) - \dot{\theta}_1^{\,2} l\sin(\theta_1-\beta) + \ddot{\theta}_2 r^c_2\cos(\theta^c_{c2}-\theta_2) + \dot{\theta}_2^{\,2} r^c_2\sin(\theta^c_{c2}-\theta_2)$$

$$y_{G2} = l\cos(\theta_1-\beta) + r^c_2\cos(\theta^c_{c2}-\theta_2) \quad\Rightarrow\quad v^y_{G2} = -\dot{\theta}_1 l\sin(\theta_1-\beta) + \dot{\theta}_2 r^c_2\sin(\theta^c_{c2}-\theta_2)$$

$$a^y_{G2} = -\ddot{\theta}_1 l\sin(\theta_1-\beta) - \dot{\theta}_1^{\,2} l\cos(\theta_1-\beta) + \ddot{\theta}_2 r^c_2\sin(\theta^c_{c2}-\theta_2) - \dot{\theta}_2^{\,2} r^c_2\cos(\theta^c_{c2}-\theta_2)$$

Solution for Body 1

$$\boxed{\text{I}} \Rightarrow \quad -m_2\ddot{x}_g l\cos(\theta_1-\beta) - m_2 l\boldsymbol{a^x_{G2}\cos(\theta_1-\beta)}$$

where

$$a^x_{G2}\cos(\theta_1-\beta) = \ddot{\theta}_1 l\cos^2(\theta_1-\beta) - \dot{\theta}_1^{\,2} l\sin(\theta_1-\beta)\cos(\theta_1-\beta) + \ddot{\theta}_2 r^c_2\cos(\theta^c_{c2}-\theta_2)\cos(\theta_1-\beta) + \dot{\theta}_2^{\,2} r^c_2\sin(\theta^c_{c2}-\theta_2)\cos(\theta_1-\beta)$$

$$\boxed{\text{II}} \Rightarrow \quad m_2 g l \sin(\theta_1 - \beta) + m_2 l \boldsymbol{a_{G2}^y \sin(\theta_1 - \beta)}$$

where

$$\begin{aligned} a_{G2}^y \sin(\theta_1 - \beta) = &-\ddot{\theta}_1 l \sin^2(\theta_1 - \beta) - \dot{\theta_1}^2 l \sin(\theta_1 - \beta)\cos(\theta_1 - \beta) \\ &+ \ddot{\theta}_2 r_2^c \sin(\theta_{c2}^c - \theta_2)\sin(\theta_1 - \beta) - \dot{\theta_2}^2 r_2^c \cos(\theta_{c2}^c - \theta_2)\sin(\theta_1 - \beta) \end{aligned}$$

$$\boxed{\text{I}} + \boxed{\text{II}} \Rightarrow \quad -m_2 \ddot{x}_g l \cos(\theta_1 - \beta) + m_2 g l \sin(\theta_1 - \beta) + ...$$

$$\begin{gathered} +m_2 l[-\ddot{\theta}_1 l \cos^2(\theta_1 - \beta) - \ddot{\theta}_1 l \sin^2(\theta_1 - \beta) \\ +\dot{\theta_1}^2 l \sin(\theta_1 - \beta)\cos(\theta_1 - \beta) - \dot{\theta_1}^2 l \sin(\theta_1 - \beta)\cos(\theta_1 - \beta) \\ -\ddot{\theta}_2 r_2^c \cos(\theta_{c2}^c - \theta_2)\cos(\theta_1 - \beta) + \ddot{\theta}_2 r_2^c \sin(\theta_{c2}^c - \theta_2)\sin(\theta_1 - \beta) \\ -\dot{\theta_2}^2 r_2^c \sin(\theta_{c2}^c - \theta_2)\cos(\theta_1 - \beta) - \dot{\theta_2}^2 r_2^c \cos(\theta_{c2}^c - \theta_2)\sin(\theta_1 - \beta)] \end{gathered}$$

$$\begin{gathered} \sin(A)\cos(B) \pm \cos(A)\sin(B) = \sin(A \pm B) \\ \cos(A)\cos(B) \mp \sin(A)\sin(B) = \cos(A \pm B) \end{gathered}$$

$$\begin{gathered} ... \Rightarrow \quad +m_2 l[-\ddot{\theta}_1 l 1 - \ddot{\theta}_2 r_2^c \cos(\gamma_{1a}) - \dot{\theta_2}^2 r_2^c \sin(\gamma_{1a})] \\ \gamma_{1a} = \theta_1 - \theta_2 + \theta_{c2}^c - \beta \end{gathered}$$

$$\begin{aligned} \boxed{1} \Rightarrow \quad &-m_1 \ddot{x}_g r_1 \cos(\theta_{c1} - \theta_1) - m_1 g r_1 \sin(\theta_{c1} - \theta_1) - m_2 \ddot{x}_g l \cos(\theta_1 - \beta) + m_2 g l \sin(\theta_1 - \beta) \\ &- m_2 l^2 \ddot{\theta}_1 - m_2 l r_2^c \cos(\gamma_{1a})\ddot{\theta}_2 - m_2 l r_2^c \sin(\gamma_{1a})\dot{\theta_2}^2 = I_{O1}\ddot{\theta}_1 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad &\boldsymbol{(I_{O1} + m_2 l^2)\ddot{\theta}_1 + m_2 l r_2^c \cos(\gamma_{1a})\ddot{\theta}_2 + m_2 l r_2^c \sin(\gamma_{1a})\dot{\theta_2}^2 - m_1 g r_1 \sin(\theta_1 - \theta_{c1})} \\ &\boldsymbol{- m_2 g l \sin(\theta_1 - \beta) = -[m_1 r_1 \cos(\theta_1 - \theta_{c1}) + m_2 l \cos(\theta_1 - \beta)]\ddot{x}_g} \end{aligned}$$

Solution for Body 2

$$\Sigma M_{G2} = I_{G2}\ddot{\theta}_2$$

$$\boxed{2} \rightarrow \quad F_1 r_2^c \cos(\theta_{c2}^c - \theta_2) - F_2 r_2^c \sin(\theta_{c2}^c - \theta_2) = I_{G2}\ddot{\theta}_2$$

$$\boxed{\text{III}} \rightarrow \quad F_1 r_2^c \cos(\theta_{c2}^c - \theta_2) = -m_2 r_2^c \ddot{x}_g \cos(\theta_{c2}^c - \theta_2) - m_2 r_2^c \boldsymbol{a_{G2}^x \cos(\theta_{c2} - \theta_2)}$$

where

$$a^x_{G2}\cos(\theta^c_{c2}-\theta_2)=\ddot{\theta}_1 l\cos(\theta_1-\beta)\cos(\theta^c_{c2}-\theta_2)-\dot{\theta}_1^{\,2} l\sin(\theta_1-\beta)\cos(\theta^c_{c2}-\theta_2)$$
$$+\ddot{\theta}_2 r^c_2\cos^2(\theta^c_{c2}-\theta_2)+\dot{\theta}_2^{\,2} r^c_2\sin(\theta^c_{c2}-\theta_2)\cos(\theta^c_{c2}-\theta_2)$$

$$\boxed{\text{IV}}\rightarrow\quad F_2r^c_2\sin(\theta^c_{c2}-\theta_2)=m_2r^c_2g\sin(\theta^c_{c2}-\theta_2)+m_2r^c_2\boldsymbol{a^y_{G2}\sin(\theta^c_{c2}-\theta_2)}$$

where

$$a^y_{G2}\sin(\theta^c_{c2}-\theta_2)=-\ddot{\theta}_1 l\sin(\theta_1-\beta)\sin(\theta^c_{c2}-\theta_2)-\dot{\theta}_1^{\,2} l\cos(\theta_1-\beta)\sin(\theta^c_{c2}-\theta_2)$$
$$+\ddot{\theta}_2 r^c_2\sin^2(\theta^c_{c2}-\theta_2)-\dot{\theta}_2^{\,2} r^c_2\cos(\theta^c_{c2}-\theta_2)\sin(\theta^c_{c2}-\theta_2)$$

$$\boxed{\text{III}}-\boxed{\text{IV}}\Rightarrow\quad -m_2r^c_2\ddot{x}_g\cos(\theta^c_{c2}-\theta_2)-m_2r^c_2g\sin(\theta^c_{c2}-\theta_2)-...$$

$$-m_2r^c_2[\ddot{\theta}_1 l(\cos(\theta_1-\beta)\cos(\theta^c_{c2}-\theta_2)-\sin(\theta_1-\beta)\sin(\theta^c_{c2}-\theta_2))$$
$$-\dot{\theta}_1^{\,2} l(\sin(\theta_1-\beta)\cos(\theta^c_{c2}-\theta_2)+\cos(\theta_1-\beta)\sin(\theta^c_{c2}-\theta_2))$$
$$+\ddot{\theta}_2 r^c_2 1+0]$$

$$\sin(A)\cos(B)\pm\cos(A)\sin(B)=\sin(A\pm B)$$
$$\cos(A)\cos(B)\mp\sin(A)\sin(B)=\cos(A\pm B)$$

$$...\Rightarrow\quad -m_2r^c_2[\ddot{\theta}_1 l\cos(\gamma_{1a})-\dot{\theta}_2^{\,2} l\sin(\gamma_{1a})+\ddot{\theta}_2 r^c_2]$$
$$\gamma_{1a}=\theta_1-\theta_2+\theta^c_{c2}-\beta$$

$$\boxed{2}\Rightarrow\quad -m_2r^c_2\ddot{x}_g\cos(\theta^c_{c2}-\theta_2)-m_2r^c_2g\sin(\theta^c_{c2}-\theta_2)-m_2lr^c_2\cos(\gamma_{1a})\ddot{\theta}_1$$
$$+m_2lr^c_2\sin(\gamma_{1a})\dot{\theta}_1^{\,2}-m_2(r^c_2)^2\ddot{\theta}_2=I_{G2}\ddot{\theta}_2$$

$$\Rightarrow\quad \boldsymbol{m_2lr^c_2\cos(\gamma_{1a})\ddot{\theta}_1+I^c_{O2}\ddot{\theta}_2-m_2lr^c_2\sin(\gamma_{1a})\dot{\theta}_1^{\,2}}$$
$$\boldsymbol{-m_2r^c_2g\sin(\theta_2-\theta^c_{c2})=-m_2r^c_2\ddot{x}_g\cos(\theta_2-\theta^c_{c2})}$$

1.2.2 Mode 1b

Body 1

$$\Sigma M_{O1} = I_{O1}\ddot{\theta}_1$$

$$\boxed{1} \rightarrow \quad m_1\ddot{x}_g r_1 \cos(\theta_{c1}+\theta_1) - m_1 g r_1 \sin(\theta_{c1}+\theta_1) + \boldsymbol{F_1 l \cos(\theta_1+\beta) - F_2 l \sin(\theta_1+\beta)} = -I_{O1}\ddot{\theta}_1$$

$$\boxed{\text{I}} \rightarrow \quad F_1 l \cos(\theta_1+\beta)$$
$$\boxed{\text{II}} \rightarrow \quad F_2 l \sin(\theta_1+\beta)$$

Body 2

$$\Sigma F_x = m_2 a^x_{G2} \quad \Rightarrow \quad m_2\ddot{x}_g - F_1 = -m_2 a^x_{G2} \quad \Rightarrow \quad F_1 = m_2\ddot{x}_g + m_2 a^x_{G2}$$
$$\Sigma F_y = m_2 a^y_{G2} \quad \Rightarrow \quad F_2 - m_2 g = m_2 a^y_{G2} \quad \Rightarrow \quad F_2 = m_2 g + m_2 a^y_{G2}$$

For center of mass of Body 2

$$x_{G2} = l\sin(\theta_1+\beta) + r^n_2 \sin(\theta^n_{c2}+\theta_2) \quad \Rightarrow \quad v^x_{G2} = \dot{\theta}_1 l \cos(\theta_1+\beta) + \dot{\theta}_2 r^n_2 \cos(\theta^n_{c2}+\theta_2)$$
$$a^x_{G2} = \ddot{\theta}_1 l\cos(\theta_1+\beta) - \dot{\theta}_1^2 l \sin(\theta_1+\beta) + \ddot{\theta}_2 r^n_2 \cos(\theta^n_{c2}+\theta_2) - \dot{\theta}_2^2 r^n_2 \sin(\theta^n_{c2}+\theta_2)$$

$$y_{G2} = l\cos(\theta_1+\beta) + r^n_2 \cos(\theta^n_{c2}+\theta_2) \quad \Rightarrow \quad v^y_{G2} = -\dot{\theta}_1 l \sin(\theta_1+\beta) - \dot{\theta}_2 r^n_2 \sin(\theta^n_{c2}+\theta_2)$$
$$a^y_{G2} = -\ddot{\theta}_1 l\sin(\theta_1+\beta) - \dot{\theta}_1^2 l \cos(\theta_1+\beta) - \ddot{\theta}_2 r^n_2 \sin(\theta^n_{c2}+\theta_2) - \dot{\theta}_2^2 r^n_2 \cos(\theta^n_{c2}+\theta_2)$$

Solution for Body 1

$$\boxed{\text{I}} \Rightarrow \quad m_2\ddot{x}_g l\cos(\theta_1+\beta) + m_2 l \boldsymbol{a^x_{G2}\cos(\theta_1+\beta)}$$

where

$$\begin{aligned} a^x_{G2}\cos(\theta_1+\beta) = {} & \ddot{\theta}_1 l \cos^2(\theta_1+\beta) - \dot{\theta}_1^2 l \sin(\theta_1+\beta)\cos(\theta_1+\beta) \\ & + \ddot{\theta}_2 r^n_2 \cos(\theta^n_{c2}+\theta_2)\cos(\theta_1+\beta) - \dot{\theta}_2^2 r^n_2 \sin(\theta^n_{c2}+\theta_2)\cos(\theta_1+\beta) \end{aligned}$$

$$\boxed{\text{II}} \Rightarrow \quad m_2 g l\sin(\theta_1+\beta) + m_2 l \boldsymbol{a^y_{G2}\sin(\theta_1+\beta)}$$

where

$$\begin{aligned} a^y_{G2}\sin(\theta_1+\beta) = {} & -\ddot{\theta}_1 l \sin^2(\theta_1+\beta) - \dot{\theta}_1^2 l \sin(\theta_1+\beta)\cos(\theta_1+\beta) \\ & - \ddot{\theta}_2 r^n_2 \sin(\theta^n_{c2}+\theta_2)\sin(\theta_1+\beta) - \dot{\theta}_2^2 r^n_2 \cos(\theta^n_{c2}+\theta_2)\sin(\theta_1+\beta) \end{aligned}$$

$$\boxed{\text{I}} - \boxed{\text{II}} \Rightarrow \quad m_2\ddot{x}_g l\cos(\theta_1+\beta) - m_2 g l \sin(\theta_1+\beta) + ...$$

$$+m_2 l[\ddot{\theta}_1 l\cos^2(\theta_1+\beta)+\ddot{\theta}_1 l\sin^2(\theta_1+\beta)$$
$$-\dot{\theta}_1^2 l\sin(\theta_1+\beta)\cos(\theta_1+\beta)+\dot{\theta}_1^2 l\sin(\theta_1+\beta)\cos(\theta_1+\beta)$$
$$+\ddot{\theta}_2 r_2^n\cos(\theta_{c2}^n+\theta_2)\cos(\theta_1+\beta)+\ddot{\theta}_2 r_2^n\sin(\theta_{c2}^n+\theta_2)\sin(\theta_1+\beta)$$
$$-\dot{\theta}_2^2 r_2^n\sin(\theta_{c2}^n+\theta_2)\cos(\theta_1+\beta)+\dot{\theta}_2^2 r_2^n\cos(\theta_{c2}^n+\theta_2)\sin(\theta_1+\beta)]$$

$$\sin(A)\cos(B)\pm\cos(A)\sin(B)=\sin(A\pm B)$$
$$\cos(A)\cos(B)\mp\sin(A)\sin(B)=\cos(A\pm B)$$

$$...\Rightarrow \quad +m_2 l[\ddot{\theta}_1 l1+\ddot{\theta}_2 r_2^n\cos(\gamma_{1b})+\dot{\theta}_2^2 r_2^n\sin(\gamma_{1b})]$$
$$\gamma_{1b}=\theta_1-\theta_2-\theta_{c2}^n+\beta$$

$$\boxed{1}\Rightarrow \quad m_1\ddot{x}_g r_1\cos(\theta_{c1}+\theta_1)-m_1 g r_1\sin(\theta_{c1}+\theta_1)+m_2\ddot{x}_g l\cos(\theta_1+\beta)-m_2 g l\sin(\theta_1+\beta)$$
$$+m_2 l^2\ddot{\theta}_1+m_2 l r_2^n\cos(\gamma_{1b})\ddot{\theta}_2+m_2 l r_2^n\sin(\gamma_{1b})\dot{\theta}_2^2=-I_{O1}\ddot{\theta}_1$$

$$\boldsymbol{\Rightarrow \quad (I_{O1}+m_2 l^2)\ddot{\theta}_1+m_2 l r_2^n\cos(\gamma_{1b})\ddot{\theta}_2+m_2 l r_2^n\sin(\gamma_{1b})\dot{\theta}_2^2-m_1 g r_1\sin(\theta_1+\theta_{c1})}$$
$$\boldsymbol{-m_2 g l\sin(\theta_1+\beta)=-[m_1 r_1\cos(\theta_1+\theta_{c1})+m_2 l\cos(\theta_1+\beta)]\ddot{x}_g}$$

Solution for Body 2

$$\Sigma M_{G2}=I_{G2}\ddot{\theta}_2$$
$$\boxed{2}\rightarrow \quad F_1 r_2^n\cos(\theta_{c2}^n+\theta_2)-F_2 r_2^n\sin(\theta_{c2}^n+\theta_2)=-I_{G2}\ddot{\theta}_2$$

$$\boxed{\text{III}}\rightarrow \quad F_1 r_2^n\cos(\theta_{c2}^n+\theta_2)=m_2 r_2^n\ddot{x}_g\cos(\theta_{c2}^n+\theta_2)+m_2 r_2^n\boldsymbol{a_{G2}^x\cos(\theta_{c2}^n+\theta_2)}$$

where

$$a_{G2}^x\cos(\theta_{c2}^n+\theta_2)=\ddot{\theta}_1 l\cos(\theta_1+\beta)\cos(\theta_{c2}^n+\theta_2)-\dot{\theta}_1^2 l\sin(\theta_1+\beta)\cos(\theta_{c2}^n+\theta_2)$$
$$+\ddot{\theta}_2 r_2^n\cos^2(\theta_{c2}^n+\theta_2)-\dot{\theta}_2^2 r_2^n\sin(\theta_{c2}^n+\theta_2)\cos(\theta_{c2}^n+\theta_2)$$

$$\boxed{\text{IV}}\rightarrow \quad F_2 r_2^n\sin(\theta_{c2}^n+\theta_2)=m_2 r_2^n g\sin(\theta_{c2}^n+\theta_2)+m_2 r_2^n\boldsymbol{a_{G2}^y\sin(\theta_{c2}^n+\theta_2)}$$

where

$$a_{G2}^y\sin(\theta_{c2}^n+\theta_2)=-\ddot{\theta}_1 l\sin(\theta_1+\beta)\sin(\theta_{c2}^n+\theta_2)-\dot{\theta}_1^2 l\cos(\theta_1+\beta)\sin(\theta_{c2}^n+\theta_2)$$
$$-\ddot{\theta}_2 r_2^n\sin^2(\theta_{c2}^n+\theta_2)-\dot{\theta}_2^2 r_2^n\cos(\theta_{c2}^n+\theta_2)\sin(\theta_{c2}^n+\theta_2)$$

$$\boxed{\text{III}} - \boxed{\text{IV}} \Rightarrow \quad m_2 r_2^n \ddot{x}_g \cos(\theta_{c2}^n + \theta_2) - m_2 r_2^n g \sin(\theta_{c2}^n + \theta_2) - ...$$

$$\begin{gathered} +m_2 r_2^n [\ddot{\theta}_1 l(\cos(\theta_1 + \beta)\cos(\theta_{c2}^n + \theta_2) + \sin(\theta_1 + \beta)\sin(\theta_{c2}^n + \theta_2)) \\ -\dot{\theta}_1^{\,2} l(\sin(\theta_1 + \beta)\cos(\theta_{c2}^n + \theta_2) - \cos(\theta_1 + \beta)\sin(\theta_{c2}^n + \theta_2)) \\ +\ddot{\theta}_2 r_2^n 1 + 0] \end{gathered}$$

$$\begin{gathered} \sin(A)\cos(B) \pm \cos(A)\sin(B) = \sin(A \pm B) \\ \cos(A)\cos(B) \mp \sin(A)\sin(B) = \cos(A \pm B) \end{gathered}$$

$$\begin{gathered} ... \Rightarrow \quad +m_2 r_2^n [\ddot{\theta}_1 l \cos(\gamma_{1b}) - \dot{\theta}_2^{\,2} l \sin(\gamma_{1b}) + \ddot{\theta}_2 r_2^n] \\ \gamma_{1b} = \theta_1 - \theta_2 - \theta_{c2}^n + \beta \end{gathered}$$

$$\boxed{2} \Rightarrow \quad m_2 r_2^n \ddot{x}_g \cos(\theta_{c2}^n + \theta_2) - m_2 r_2^n g \sin(\theta_{c2}^n + \theta_2) + m_2 l r_2^n \cos(\gamma_{1b})\ddot{\theta}_1 \\ - m_2 l r_2^n \sin(\gamma_{1b})\dot{\theta}_1^{\,2} + m_2 (r_2^n)^2 \ddot{\theta}_2 = -I_{G2}\ddot{\theta}_2$$

$$\boldsymbol{\Rightarrow \quad m_2 l r_2^n \cos(\gamma_{1b})\ddot{\theta}_1 + I_{O2}^n \ddot{\theta}_2 - m_2 l r_2^n \sin(\gamma_{1b})\dot{\theta}_1^{\,2} \\ - m_2 r_2^n g \sin(\theta_2 + \theta_{c2}^n) = -m_2 r_2^n \ddot{x}_g \cos(\theta_2 + \theta_{c2}^n)}$$

1.2.3 Mode 2a

Body 1

$$\Sigma M_{O1} = I_{O1}\ddot{\theta}_1$$

$$\boxed{1} \rightarrow \quad -m_1\ddot{x}_g r_1\cos(\theta_{c1}-\theta_1)-m_1 g r_1\sin(\theta_{c1}-\theta_1)+\boldsymbol{F_1 l'\cos(\beta'-\theta_1) - F_2 l'\sin(\beta'-\theta_1)} = I_{O1}\ddot{\theta}_1$$

$$\boxed{\text{I}} \rightarrow \quad F_1 l'\cos(\beta'-\theta_1)$$

$$\boxed{\text{II}} \rightarrow \quad F_2 l'\sin(\beta'-\theta_1)$$

Body 2

$$\Sigma F_x = m_2 a^x_{G2} \quad \Rightarrow \quad -m_2\ddot{x}_g - F_1 = m_2 a^x_{G2} \quad \Rightarrow \quad F_1 = -m_2\ddot{x}_g - m_2 a^x_{G2}$$
$$\Sigma F_y = m_2 a^y_{G2} \quad \Rightarrow \quad F_2 - m_2 g = m_2 a^y_{G2} \quad \Rightarrow \quad F_2 = m_2 g + m_2 a^y_{G2}$$

For center of mass of Body 2

$$x_{G2} = -l'\sin(\beta'-\theta_1) + r^n_2\sin(\theta^n_{c2}+\theta_2) \quad \Rightarrow \quad v^x_{G2} = \dot{\theta}_1 l'\cos(\beta'-\theta_1) + \dot{\theta}_2 r^n_2\cos(\theta^n_{c2}+\theta_2)$$
$$a^x_{G2} = \ddot{\theta}_1 l'\cos(\beta'-\theta_1) + \dot{\theta}_1^2 l'\sin(\beta'-\theta_1) + \ddot{\theta}_2 r^n_2\cos(\theta^n_{c2}+\theta_2) - \dot{\theta}_2^2 r^n_2\sin(\theta^n_{c2}+\theta_2)$$

$$y_{G2} = l'\cos(\beta'-\theta_1) + r^n_2\cos(\theta^n_{c2}+\theta_2) \quad \Rightarrow \quad v^y_{G2} = \dot{\theta}_1 l'\sin(\beta'-\theta_1) - \dot{\theta}_2 r^n_2\sin(\theta^n_{c2}+\theta_2)$$
$$a^y_{G2} = \ddot{\theta}_1 l'\sin(\beta'-\theta_1) - \dot{\theta}_1^2 l'\cos(\beta'-\theta_1) - \ddot{\theta}_2 r^n_2\sin(\theta^n_{c2}+\theta_2) - \dot{\theta}_2^2 r^n_2\cos(\theta^n_{c2}+\theta_2)$$

Solution for Body 1

$$\boxed{\text{I}} \Rightarrow \quad -m_2\ddot{x}_g l'\cos(\beta'-\theta_1) - m_2 l'\boldsymbol{a^x_{G2}\cos(\beta'-\theta_1)}$$

where

$$a^x_{G2}\cos(\beta'-\theta_1) = \ddot{\theta}_1 l'\cos^2(\beta'-\theta_1) + \dot{\theta}_1^2 l\sin(\beta'-\theta_1)\cos(\beta'-\theta_1)$$
$$+ \ddot{\theta}_2 r^n_2\cos(\theta^n_{c2}+\theta_2)\cos(\beta'-\theta_1) - \dot{\theta}_2^2 r^n_2\sin(\theta^n_{c2}+\theta_2)\cos(\beta'-\theta_1)$$

$$\boxed{\text{II}} \Rightarrow \quad m_2 g l'\sin(\beta'-\theta_1) + m_2 l'\boldsymbol{a^y_{G2}\sin(\theta_1-\beta)}$$

where

$$a^y_{G2}\sin(\beta'-\theta_1) = \ddot{\theta}_1 l'\sin^2(\beta'-\theta_1) - \dot{\theta}_1^2 l'\sin(\beta'-\theta_1)\cos(\beta'-\theta_1)$$
$$- \ddot{\theta}_2 r^n_2\sin(\theta^n_{c2}+\theta_2)\sin(\beta'-\theta_1) - \dot{\theta}_2^2 r^n_2\cos(\theta^n_{c2}+\theta_2)\sin(\beta'-\theta_1)$$

$$\boxed{\text{I}} - \boxed{\text{II}} \Rightarrow \quad -m_2\ddot{x}_g l'\cos(\beta'-\theta_1) - m_2 g l'\sin(\beta'-\theta_1) + ...$$

$$+m_2 l'[-\ddot{\theta}_1 l' \cos^2(\beta' - \theta_1) - \ddot{\theta}_1 l' \sin^2(\beta' - \theta_1)$$
$$-\dot{\theta}_1^2 l' \sin(\beta' - \theta_1)\cos(\beta' - \theta_1) + \dot{\theta}_1^2 l' \sin(\beta' - \theta_1)\cos(\beta' - \theta_1)$$
$$-\ddot{\theta}_2 r_2^n \cos(\theta_{c2}^n + \theta_2)\cos(\beta' - \theta_1) + \ddot{\theta}_2 r_2^n \sin(\theta_{c2}^n + \theta_2)\sin(\beta' - \theta_1)$$
$$+\dot{\theta}_2^2 r_2^n \sin(\theta_{c2}^n + \theta_2)\cos(\beta' - \theta_1) + \dot{\theta}_2^2 r_2^n \cos(\theta_{c2}^n + \theta_2)\sin(\beta' - \theta_1)]$$

$$\sin(A)\cos(B) \pm \cos(A)\sin(B) = \sin(A \pm B)$$
$$\cos(A)\cos(B) \mp \sin(A)\sin(B) = \cos(A \pm B)$$

$$... \Rightarrow \quad +m_2 l'[-\ddot{\theta}_1 l' 1 - \ddot{\theta}_2 r_2^n \cos(\gamma_{2a}) - \dot{\theta}_2^2 r_2^n \sin(\gamma_{2a})]$$
$$\gamma_{2a} = \theta_1 - \theta_2 - \theta_{c2}^n - \beta'$$

$$\boxed{1} \Rightarrow \quad -m_1 \ddot{x}_g r_1 \cos(\theta_{c1} - \theta_1) - m_1 g r_1 \sin(\theta_{c1} - \theta_1) - m_2 \ddot{x}_g l' \cos(\beta' - \theta_1) - m_2 g l' \sin(\beta' - \theta_1)$$
$$- m_2 l'^2 \ddot{\theta}_1 - m_2 l' r_2^n \cos(\gamma_{2a})\ddot{\theta}_2 - m_2 l' r_2^n \sin(\gamma_{2a})\dot{\theta}_2^2 = I_{O1}\ddot{\theta}_1$$

$$\Rightarrow \quad \boldsymbol{(I_{O1} + m_2 l'^2)\ddot{\theta}_1 + m_2 l' r_2^n \cos(\gamma_{2a})\ddot{\theta}_2 + m_2 l' r_2^n \sin(\gamma_{2a})\dot{\theta}_2^2 - m_1 g r_1 \sin(\theta_1 - \theta_{c1})}$$
$$\boldsymbol{- m_2 g l' \sin(\theta_1 - \beta') = -[m_1 r_1 \cos(\theta_1 - \theta_{c1}) + m_2 l' \cos(\theta_1 - \beta')]\ddot{x}_g}$$

Solution for Body 2

$$\Sigma M_{G2} = I_{G2}\ddot{\theta}_2$$
$$\boxed{2} \rightarrow \quad F_1 r_2^n \cos(\theta_{c2}^n + \theta_2) + F_2 r_2^n \sin(\theta_{c2}^n + \theta_2) = I_{G2}\ddot{\theta}_2$$

$$\boxed{\text{III}} \rightarrow \quad F_1 r_2^n \cos(\theta_{c2}^n + \theta_2) = -m_2 r_2^n \ddot{x}_g \cos(\theta_{c2}^n + \theta_2) - m_2 r_2^n \boldsymbol{a_{G2}^x \cos(\theta_{c2}^n + \theta_2)}$$

where

$$a_{G2}^x \cos(\theta_{c2}^n + \theta_2) = \ddot{\theta}_1 l' \cos(\beta' - \theta_1)\cos(\theta_{c2}^n + \theta_2) + \dot{\theta}_1^2 l' \sin(\beta' - \theta_1)\cos(\theta_{c2}^n + \theta_2)$$
$$+ \ddot{\theta}_2 r_2^n \cos^2(\theta_{c2}^n + \theta_2) - \dot{\theta}_2^2 r_2^n \sin(\theta_{c2}^n + \theta_2)\cos(\theta_{c2}^n + \theta_2)$$

$$\boxed{\text{IV}} \rightarrow \quad F_2 r_2^n \sin(\theta_{c2}^n + \theta_2) = m_2 r_2^n g \sin(\theta_{c2}^n + \theta_2) + m_2 r_2^n \boldsymbol{a_{G2}^y \sin(\theta_{c2}^n + \theta_2)}$$

where

$$a_{G2}^y \sin(\theta_{c2}^n + \theta_2) = \ddot{\theta}_1 l' \sin(\beta' - \theta_1)\sin(\theta_{c2}^n + \theta_2) - \dot{\theta}_1^2 l' \cos(\beta' - \theta_1)\sin(\theta_{c2}^n + \theta_2)$$
$$- \ddot{\theta}_2 r_2^n \sin^2(\theta_{c2}^n + \theta_2) - \dot{\theta}_2^2 r_2^n \cos(\theta_{c2}^n + \theta_2)\sin(\theta_{c2}^n + \theta_2)$$

$$\boxed{\text{III}} + \boxed{\text{IV}} \Rightarrow \quad -m_2 r_2^n \ddot{x}_g \cos(\theta_{c2}^n + \theta_2) + m_2 r_2^n g \sin(\theta_{c2}^n + \theta_2) + ...$$

$$+m_2 r_2^n [\ddot{\theta}_1 l'(-\cos(\beta' - \theta_1)\cos(\theta_{c2}^n + \theta_2) + \sin(\beta' - \theta_1)\sin(\theta_{c2}^n + \theta_2))$$
$$-\dot{\theta}_1^{\,2} l'(\sin(\beta' - \theta_1)\cos(\theta_{c2}^n + \theta_2) + \cos(\beta' - \theta_1)\sin(\theta_{c2}^n + \theta_2))$$
$$-\ddot{\theta}_2 r_2^n 1 + 0]$$

$$\sin(A)\cos(B) \pm \cos(A)\sin(B) = \sin(A \pm B)$$
$$\cos(A)\cos(B) \mp \sin(A)\sin(B) = \cos(A \pm B)$$

$$... \Rightarrow \quad +m_2 r_2^n [-\ddot{\theta}_1 l' \cos(\gamma_{2a}) + \dot{\theta}_2^{\,2} l' \sin(\gamma_{2a}) - \ddot{\theta}_2 r_2^n]$$
$$\gamma_{2a} = \theta_1 - \theta_2 - \theta_{c2}^n - \beta'$$

$$\boxed{2} \Rightarrow \quad -m_2 r_2^n \ddot{x}_g \cos(\theta_{c2}^n + \theta_2) + m_2 r_2^n g \sin(\theta_{c2}^n + \theta_2) - m_2 l' r_2^n \cos(\gamma_{2a})\ddot{\theta}_1$$
$$+ m_2 l' r_2^n \sin(\gamma_{2a})\dot{\theta}_1^{\,2} - m_2 (r_2^n)^2 \ddot{\theta}_2 = I_{G2}\ddot{\theta}_2$$

$$\boldsymbol{\Rightarrow \quad m_2 l' r_2^n \cos(\gamma_{2a})\ddot{\theta}_1 + I_{O2}^n \ddot{\theta}_2 - m_2 l' r_2^n \sin(\gamma_{2a})\dot{\theta}_1^{\,2}}$$
$$\boldsymbol{- m_2 r_2^n g \sin(\theta_2 + \theta_{c2}^n) = -m_2 r_2^n \ddot{x}_g \cos(\theta_2 + \theta_{c2}^n)}$$

1.2.4 Mode 2b

Body 1

$$\Sigma M_{O1} = I_{O1}\ddot{\theta}_1$$

$$\boxed{1} \rightarrow \quad m_1\ddot{x}_g r_1 \cos(\theta_{c1}+\theta_1) - m_1 g r_1 \sin(\theta_{c1}+\theta_1) + \boldsymbol{F_1 l' \cos(\beta' + \theta_1) - F_2 l' \sin(\beta' + \theta_1)} = -I_{O1}\ddot{\theta}_1$$

$$\boxed{\text{I}} \rightarrow \quad F_1 l' \cos(\beta' + \theta_1)$$

$$\boxed{\text{II}} \rightarrow \quad F_2 l' \sin(\beta' + \theta_1)$$

Body 2

$$\Sigma F_x = m_2 a^x_{G2} \quad \Rightarrow \quad m_2\ddot{x}_g - F_1 = -m_2 a^x_{G2} \quad \Rightarrow \quad F_1 = m_2\ddot{x}_g + m_2 a^x_{G2}$$

$$\Sigma F_y = m_2 a^y_{G2} \quad \Rightarrow \quad F_2 - m_2 g = m_2 a^y_{G2} \quad \Rightarrow \quad F_2 = m_2 g + m_2 a^y_{G2}$$

For center of mass of Body 2

$$x_{G2} = l' \sin(\beta' + \theta_1) - r^c_2 \sin(\theta^c_{c2} - \theta_2) \quad \Rightarrow \quad v^x_{G2} = \dot{\theta}_1 l' \cos(\beta' + \theta_1) + \dot{\theta}_2 r^c_2 \cos(\theta^c_{c2} - \theta_2)$$

$$a^x_{G2} = \ddot{\theta}_1 l' \cos(\beta' + \theta_1) - \dot{\theta}_1^{\,2} l' \sin(\beta' + \theta_1) + \ddot{\theta}_2 r^c_2 \cos(\theta^c_{c2} - \theta_2) + \dot{\theta}_2^{\,2} r^c_2 \sin(\theta^c_{c2} - \theta_2)$$

$$y_{G2} = l' \cos(\beta' + \theta_1) + r^c_2 \cos(\theta^c_{c2} - \theta_2) \quad \Rightarrow \quad v^y_{G2} = -\dot{\theta}_1 l' \sin(\beta' + \theta_1) + \dot{\theta}_2 r^c_2 \sin(\theta^c_{c2} - \theta_2)$$

$$a^y_{G2} = -\ddot{\theta}_1 l' \sin(\beta' + \theta_1) - \dot{\theta}_1^{\,2} l' \cos(\beta' + \theta_1) + \ddot{\theta}_2 r^c_2 \sin(\theta^c_{c2} - \theta_2) - \dot{\theta}_2^{\,2} r^c_2 \cos(\theta^c_{c2} - \theta_2)$$

Solution for Body 1

$$\boxed{\text{I}} \Rightarrow \quad m_2\ddot{x}_g l' \cos(\beta' + \theta_1) + m_2 l' \boldsymbol{a^x_{G2} \cos(\beta' + \theta_1)}$$

where

$$a^x_{G2} \cos(\beta' + \theta_1) = \ddot{\theta}_1 l' \cos^2(\beta' + \theta_1) - \dot{\theta}_1^{\,2} l \sin(\beta' + \theta_1) \cos(\beta' + \theta_1)$$
$$+ \ddot{\theta}_2 r^c_2 \cos(\theta^c_{c2} - \theta_2) \cos(\beta' + \theta_1) + \dot{\theta}_2^{\,2} r^c_2 \sin(\theta^c_{c2} - \theta_2) \cos(\beta' + \theta_1)$$

$$\boxed{\text{II}} \Rightarrow \quad m_2 g l' \sin(\beta' + \theta_1) + m_2 l' \boldsymbol{a^y_{G2} \sin(\beta' + \theta_1)}$$

where

$$a^y_{G2} \sin(\beta' + \theta_1) = -\ddot{\theta}_1 l' \sin^2(\beta' + \theta_1) - \dot{\theta}_1^{\,2} l' \sin(\beta' + \theta_1) \cos(\beta' + \theta_1)$$
$$+ \ddot{\theta}_2 r^c_2 \sin(\theta^c_{c2} - \theta_2) \sin(\beta' + \theta_1) - \dot{\theta}_2^{\,2} r^c_2 \cos(\theta^c_{c2} - \theta_2) \sin(\beta' + \theta_1)$$

$$\boxed{\text{I}} - \boxed{\text{II}} \Rightarrow \quad m_2\ddot{x}_g l' \cos(\beta' + \theta_1) - m_2 g l' \sin(\beta' + \theta_1) + ...$$

$$
\begin{aligned}
&+m_2 l'[\ddot{\theta}_1 l' \cos^2(\beta' + \theta_1) + \ddot{\theta}_1 l' \sin^2(\beta' + \theta_1) \\
&-\dot{\theta}_1^{\,2} l' \sin(\beta' + \theta_1) \cos(\beta' + \theta_1) + \dot{\theta}_1^{\,2} l' \sin(\beta' + \theta_1) \cos(\beta' + \theta_1) \\
&+\ddot{\theta}_2 r_2^c \cos(\theta_{c2}^c - \theta_2) \cos(\beta' + \theta_1) - \ddot{\theta}_2 r_2^c \sin(\theta_{c2}^c - \theta_2) \sin(\beta' + \theta_1) \\
&+\dot{\theta}_2^{\,2} r_2^c \sin(\theta_{c2}^c - \theta_2) \cos(\beta' + \theta_1) + \dot{\theta}_2^{\,2} r_2^c \cos(\theta_{c2}^c - \theta_2) \sin(\beta' + \theta_1)]
\end{aligned}
$$

$$
\begin{aligned}
\sin(A)\cos(B) \pm \cos(A)\sin(B) &= \sin(A \pm B) \\
\cos(A)\cos(B) \mp \sin(A)\sin(B) &= \cos(A \pm B)
\end{aligned}
$$

$$
\begin{aligned}
... \Rightarrow \quad &+m_2 l'[\ddot{\theta}_1 l' 1 + \ddot{\theta}_2 r_2^c \cos(\gamma_{2a}) + \dot{\theta}_2^{\,2} r_2^c \sin(\gamma_{2a})] \\
&\gamma_{2b} = \theta_1 - \theta_2 + \theta_{c2}^c + \beta'
\end{aligned}
$$

$$
\begin{aligned}
\boxed{1} \Rightarrow \quad & m_1 \ddot{x}_g r_1 \cos(\theta_{c1} + \theta_1) - m_1 g r_1 \sin(\theta_{c1} + \theta_1) + m_2 \ddot{x}_g l' \cos(\beta' + \theta_1) - m_2 g l' \sin(\beta' + \theta_1) \\
& + m_2 l'^2 \ddot{\theta}_1 + m_2 l' r_2^c \cos(\gamma_{2b}) \ddot{\theta}_2 + m_2 l' r_2^c \sin(\gamma_{2b}) \dot{\theta}_2^{\,2} = -I_{O1} \ddot{\theta}_1
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad & \boldsymbol{(I_{O1} + m_2 l'^2)\ddot{\theta}_1 + m_2 l' r_2^c \cos(\gamma_{2b}) \ddot{\theta}_2 + m_2 l' r_2^c \sin(\gamma_{2b}) \dot{\theta}_2^{\,2} - m_1 g r_1 \sin(\theta_1 + \theta_{c1})} \\
& \boldsymbol{- m_2 g l' \sin(\theta_1 + \beta') = -[m_1 r_1 \cos(\theta_1 + \theta_{c1}) + m_2 l' \cos(\theta_1 + \beta')] \ddot{x}_g}
\end{aligned}
$$

Solution for Body 2

$$
\begin{aligned}
&\Sigma M_{G2} = I_{G2} \ddot{\theta}_2 \\
\boxed{2} \rightarrow \quad & F_1 r_2^c \cos(\theta_{c2}^c - \theta_2) + F_2 r_2^c \sin(\theta_{c2}^c - \theta_2) = -I_{G2} \ddot{\theta}_2
\end{aligned}
$$

$$
\boxed{\text{III}} \rightarrow \quad F_1 r_2^c \cos(\theta_{c2}^c - \theta_2) = m_2 r_2^c \ddot{x}_g \cos(\theta_{c2}^c - \theta_2) + m_2 r_2^c \boldsymbol{a_{G2}^x \cos(\theta_{c2}^c - \theta_2)}
$$

where

$$
\begin{aligned}
a_{G2}^x \cos(\theta_{c2}^c - \theta_2) = {} & \ddot{\theta}_1 l' \cos(\beta' + \theta_1) \cos(\theta_{c2}^c - \theta_2) - \dot{\theta}_1^{\,2} l' \sin(\beta' + \theta_1) \cos(\theta_{c2}^c + \theta_2) \\
& + \ddot{\theta}_2 r_2^c \cos^2(\theta_{c2}^c - \theta_2) + \dot{\theta}_2^{\,2} r_2^c \sin(\theta_{c2}^c - \theta_2) \cos(\theta_{c2}^c - \theta_2)
\end{aligned}
$$

$$
\boxed{\text{IV}} \rightarrow \quad F_2 r_2^c \sin(\theta_{c2}^c - \theta_2) = m_2 r_2^c g \sin(\theta_{c2}^c - \theta_2) + m_2 r_2^c \boldsymbol{a_{G2}^y \sin(\theta_{c2}^c - \theta_2)}
$$

where

$$
\begin{aligned}
a_{G2}^y \sin(\theta_{c2}^c - \theta_2) = {} & -\ddot{\theta}_1 l' \sin(\beta' + \theta_1) \sin(\theta_{c2}^c - \theta_2) - \dot{\theta}_1^{\,2} l' \cos(\beta' + \theta_1) \sin(\theta_{c2}^c - \theta_2) \\
& + \ddot{\theta}_2 r_2^c \sin^2(\theta_{c2}^c - \theta_2) - \dot{\theta}_2^{\,2} r_2^c \cos(\theta_{c2}^c - \theta_2) \sin(\theta_{c2}^c - \theta_2)
\end{aligned}
$$

$$\boxed{\text{III}} + \boxed{\text{IV}} \Rightarrow \quad m_2 r_2^c \ddot{x}_g \cos(\theta_{c2}^c - \theta_2) + m_2 r_2^c g \sin(\theta_{c2}^c - \theta_2) + ...$$

$$+m_2 r_2^c[\ddot{\theta}_1 l'(\cos(\beta' + \theta_1)\cos(\theta_{c2}^c - \theta_2) - \sin(\beta' + \theta_1)\sin(\theta_{c2}^c - \theta_2))$$
$$-\dot{\theta}_1^{\,2} l'(\sin(\beta' + \theta_1)\cos(\theta_{c2}^c - \theta_2) + \cos(\beta' + \theta_1)\sin(\theta_{c2}^c - \theta_2))$$
$$+\ddot{\theta}_2 r_2^c 1 + 0]$$

$$\sin(A)\cos(B) \pm \cos(A)\sin(B) = \sin(A \pm B)$$
$$\cos(A)\cos(B) \mp \sin(A)\sin(B) = \cos(A \pm B)$$

$$... \Rightarrow \quad +m_2 r_2^c[\ddot{\theta}_1 l' \cos(\gamma_{2b}) - \dot{\theta}_2^{\,2} l' \sin(\gamma_{2b}) + \ddot{\theta}_2 r_2^c]$$
$$\gamma_{2b} = \theta_1 - \theta_2 + \theta_{c2}^c + \beta'$$

$$\boxed{2} \Rightarrow \quad m_2 r_2^c \ddot{x}_g \cos(\theta_{c2}^c - \theta_2) + m_2 r_2^c g \sin(\theta_{c2}^c - \theta_2) + m_2 l' r_2^c \cos(\gamma_{2b})\ddot{\theta}_1$$
$$- m_2 l' r_2^c \sin(\gamma_{2b})\dot{\theta}_1^{\,2} + m_2 (r_2^c)^2 \ddot{\theta}_2 = -I_{G2}\ddot{\theta}_2$$

$$\Rightarrow \quad \boldsymbol{m_2 l' r_2^c \cos(\gamma_{2b})\ddot{\theta}_1 + I_{O2}^c \ddot{\theta}_2 - m_2 l' r_2^c \sin(\gamma_{2b})\dot{\theta}_1^{\,2}}$$
$$\boldsymbol{- m_2 r_2^c g \sin(\theta_2 - \theta_{c2}^c) = -m_2 r_2^c \ddot{x}_g \cos(\theta_2 - \theta_{c2}^c)}$$

1.2.5 Mode 3a

System

$$\Sigma M_{O1} = I_O^c \ddot{\theta}_1$$
$$-M\ddot{x}_g R^c \cos(\theta_c^c - \theta_1) - MgR^c \sin(\theta_c^c - \theta_1) = I_O^c \ddot{\theta}_1$$
$$\Rightarrow \quad \boldsymbol{I_O^c \ddot{\theta}_1 - MgR^c \sin(\theta_1 - \theta_c^c) = -M\ddot{x}_g R^c \cos(\theta_1 - \theta_c^c)}$$

1.2.6 Mode 3b

System

$$\Sigma M_{O1} = I_O^n \ddot{\theta}_1$$
$$M\ddot{x}_g R^n \cos(\theta_c^n + \theta_1) - MgR^n \sin(\theta_c^n + \theta_1) = -I_O^n \ddot{\theta}_1$$
$$\Rightarrow \quad \boldsymbol{I_O^n \ddot{\theta}_1 - MgR^n \sin(\theta_1 + \theta_c^n) = -M\ddot{x}_g R^n \cos(\theta_1 + \theta_c^n)}$$

1.2.7 Mode 4a

Body 2

Similarly to mode 3a

$$\Sigma M_{O2} = I_{O2}^c \ddot{\theta}_2$$
$$\boldsymbol{I_{O2}^c \ddot{\theta}_2 - m_2 g r_2^c \sin(\theta_2 - \theta_{c2}^c) = -m_2 \ddot{x}_g r_2^c \cos(\theta_2 - \theta_{c2}^c)}$$

1.2.8 Mode 4b

Body 2

Similarly to mode 3b

$$\Sigma M_{O2} = I_{O2}^n \ddot{\theta}_2$$
$$\boldsymbol{I_{O2}^n \ddot{\theta}_2 - m_2 g r_2^n \sin(\theta_2 + \theta_{c2}^n) = -m_2 \ddot{x}_g r_2^n \cos(\theta_2 + \theta_{c2}^n)}$$

Chapter 2

Transitions between modes without impact

This type of transition can occur first of all, when the system is set to motion from rest. Afterwards, it occurs as the system continuously switches from modes 3 and 4 to other modes. To figure out the conditions at which that happens, the comparison between overturning and restoring moments should be evaluated. That is done by comparing the angular accelerations (which are solved for, from the respective equation of motion) before and after transition.

2.1 To mode 1a

Mode 3a → 1a

$$\ddot{\theta}_2^{\,1a} > \ddot{\theta}_2^{\,3a} = \ddot{\theta}_1^{\,3a} = \ddot{\theta}_1^{\,1a}$$

From equation of motion of body 2 and for $\theta_1 = \theta_2$

$$\frac{m_2 r_2^c}{I_{O2}^c}(-\ddot{x}_g \cos(\theta_1 - \theta_{c2}^c) + g \sin(\theta_1 - \theta_{c2}^c) + l \sin(\gamma_{1a})\dot{\theta}_1^{\,2} - l \cos(\gamma_{1a})\ddot{\theta}_1) - \ddot{\theta}_1 > 0$$

$$\gamma_{1a} = \theta_{c2}^c - \beta$$

$$\boldsymbol{-\ddot{x}_g \cos(\theta_1 - \theta_{c2}^c) + g \sin(\theta_1 - \theta_{c2}^c) + l \sin(\gamma_{1a})\dot{\theta}_1^{\,2} - (l \cos(\gamma_{1a}) + \frac{I_{O2}^c}{m_2 r_2^c})\ddot{\theta}_1 > 0}$$

Mode 4a → 1a

$$\ddot{\theta}_1^{\,1a} > 0$$

From equation of motion of body 1 and for $\theta_1 = 0$

$$\begin{aligned} &- (m_1 r_1 \cos(-\theta_{c1}) + m_2 l \cos(-\beta))\ddot{x}_g + m_2 g l \sin(-\beta) + m_1 g r_1 \sin(-\theta_{c1}) \\ &\qquad - m_2 l r_2^c \sin(\gamma_{1a})\dot{\theta}_2^{\,2} - m_2 l r_2^c \cos(\gamma_{1a})\ddot{\theta}_2 > 0 \end{aligned}$$

$$\gamma_{1a} = -\theta_2 + \theta_{c2}^c - \beta$$

$$\boldsymbol{-(m_1 + 2m_2)h_1\ddot{x}_g - m_2 g \xi - m_1 g b_1 - m_2 l r_2^c(\sin(\gamma_{1a})\dot{\theta}_2^{\,2} + \cos(\gamma_{1a})\ddot{\theta}_2) > 0}$$

2.2 To mode 1b

Mode 3b → 1b

$$\ddot{\theta}_2^{1b} < \ddot{\theta}_2^{3b} = \ddot{\theta}_1^{3b} = \ddot{\theta}_1^{1b}$$

From equation of motion of body 2 and for $\theta_1 = \theta_2$

$$\frac{m_2 r_2^n}{I_{O2}^n}(\ddot{x}_g \cos(\theta_1 + \theta_{c2}^n) - g \sin(\theta_1 + \theta_{c2}^n) - l \sin(\gamma_{1b})\dot{\theta}_1^{\,2} + l \cos(\gamma_{1b})\ddot{\theta}_1) + \ddot{\theta}_1 > 0$$

$$\gamma_{1b} = -\theta_{c2}^n + \beta$$

$$\boldsymbol{\ddot{x}_g \cos(\theta_1 + \theta_{c2}^n) - g \sin(\theta_1 + \theta_{c2}^n) - l \sin(\gamma_{1b})\dot{\theta}_1^{\,2} + (l \cos(\gamma_{1b}) + \frac{I_{O2}^n}{m_2 r_2^n})\ddot{\theta}_1 > 0}$$

Mode 4b → 1b

$$\ddot{\theta}_1^{1b} < 0$$

From equation of motion of body 1 and for $\theta_1 = 0$

$$-(m_1 r_1 \cos(\theta_{c1}) + m_2 l \cos(\beta))\ddot{x}_g + m_2 g l \sin(\beta) + m_1 g r_1 \sin(\theta_{c1}) - m_2 l r_2^n \sin(\gamma_{1b})\dot{\theta}_2^{\,2} - m_2 l r_2^n \cos(\gamma_{1b})\ddot{\theta}_2 < 0$$

$$\gamma_{1b} = -\theta_2 - \theta_{c2}^n + \beta$$

$$\boldsymbol{(m_1 + 2m_2)h_1\ddot{x}_g - m_2 g\xi - m_1 g b_1 + m_2 l r_2^n(\sin(\gamma_{1b})\dot{\theta}_2^{\,2} + \cos(\gamma_{1b})\ddot{\theta}_2) > 0}$$

2.3 To mode 2a

Mode 3a → 2a

$$\ddot{\theta}_2^{2a} < \ddot{\theta}_2^{3a} = \ddot{\theta}_1^{3a} = \ddot{\theta}_1^{2a}$$

From equation of motion of body 2 and for $\theta_1 = \theta_2$

$$\frac{m_2 r_2^n}{I_{O2}^n}(-\ddot{x}_g \cos(\theta_1 + \theta_{c2}^n) + g \sin(\theta_1 + \theta_{c2}^n) + l' \sin(\gamma_{2a})\dot{\theta}_1^{\,2} - l' \cos(\gamma_{2a})\ddot{\theta}_1) - \ddot{\theta}_1 < 0$$

$$\gamma_{2a} = -\theta_{c2}^n - \beta'$$

$$\boldsymbol{\ddot{x}_g \cos(\theta_1 + \theta_{c2}^n) - g \sin(\theta_1 + \theta_{c2}^n) - l' \sin(\gamma_{2a})\dot{\theta}_1^{\,2} + (l' \cos(\gamma_{2a}) + \frac{I_{O2}^n}{m_2 r_2^n})\ddot{\theta}_1 > 0}$$

Mode 4b → 2a

$$\ddot{\theta}_1^{2a} > 0$$

From equation of motion of body 1 and for $\theta_1 = 0$

$$-(m_1 r_1 \cos(-\theta_{c1}) + m_2 l' \cos(-\beta'))\ddot{x}_g + m_2 g l' \sin(-\beta') + m_1 g r_1 \sin(-\theta_{c1}) \\ - m_2 l' r_2^n \sin(\gamma_{2a})\dot{\theta}_2^2 - m_2 l' r_2^n \cos(\gamma_{2a})\ddot{\theta}_2 > 0$$

$$\gamma_{2a} = -\theta_2 - \theta_{c2}^n - \beta'$$

$$\boldsymbol{-(m_1 + 2m_2)h_1\ddot{x}_g - m_2 g\xi' - m_1 g b_1 - m_2 l' r_2^n(\sin(\gamma_{2a})\dot{\theta}_2^2 + \cos(\gamma_{2a})\ddot{\theta}_2) > 0}$$

2.4 To mode 2b

Mode 3b → 2b

$$\ddot{\theta}_2^{2b} > \ddot{\theta}_2^{3b} = \ddot{\theta}_1^{3b} = \ddot{\theta}_1^{2b}$$

From equation of motion of body 2 and for $\theta_1 = \theta_2$

$$\frac{m_2 r_2^c}{I_{O2}^c}(-\ddot{x}_g \cos(\theta_1 - \theta_{c2}^c) + g \sin(\theta_1 - \theta_{c2}^c) + l' \sin(\gamma_{2b})\dot{\theta}_1^2 - l' \cos(\gamma_{2b})\ddot{\theta}_1) - \ddot{\theta}_1 > 0$$

$$\gamma_{2b} = \theta_{c2}^c + \beta'$$

$$\boldsymbol{-\ddot{x}_g \cos(\theta_1 - \theta_{c2}^c) + g \sin(\theta_1 - \theta_{c2}^c) + l' \sin(\gamma_{2b})\dot{\theta}_1^2 - (l' \cos(\gamma_{2b}) + \frac{I_{O2}^c}{m_2 r_2^c})\ddot{\theta}_1 > 0}$$

Mode 4a → 2b

$$\ddot{\theta}_1^{2b} < 0$$

From equation of motion of body 1 and for $\theta_1 = 0$

$$-(m_1 r_1 \cos(\theta_{c1}) + m_2 l' \cos(\beta'))\ddot{x}_g + m_2 g l' \sin(\beta') + m_1 g r_1 \sin(\theta_{c1}) \\ - m_2 l' r_2^c \sin(\gamma_{2b})\dot{\theta}_2^2 - m_2 l' r_2^c \cos(\gamma_{2b})\ddot{\theta}_2 < 0$$

$$\gamma_{2b} = -\theta_2 + \theta_{c2}^c + \beta'$$

$$\boldsymbol{(m_1 + 2m_2)h_1\ddot{x}_g - m_2 g\xi' - m_1 g b_1 + m_2 l' r_2^c(\sin(\gamma_{2b})\dot{\theta}_2^2 + \cos(\gamma_{2b})\ddot{\theta}_2) > 0}$$

2.5 To mode 3a

Rest → 3a

$$\ddot{\theta}_1^{3a} > 0$$

From equation of motion and for $\theta_1 = 0$

$$-MR^c \cos(-\theta_c^c)\ddot{x}_g + MgR^c \sin(-\theta_c^c) > 0$$
$$\boldsymbol{-(h\ddot{x}_g + b^c g) > 0}$$

2.6 To mode 3b

Rest → 3b

$$\ddot{\theta}_1^{3b} < 0$$

From equation of motion and for $\theta_1 = 0$

$$-MR^n \cos(\theta_c^n)\ddot{x}_g + MgR^n \sin(\theta_c^n) < 0$$
$$\boldsymbol{h\ddot{x}_g - b^n g > 0}$$

2.7 To mode 4a

Rest → 4a

$$\ddot{\theta}_2^{4a} > 0$$

From equation of motion and for $\theta_2 = 0$

$$\boldsymbol{-(h_2\ddot{x}_g + b_2^c g) > 0}$$

2.8 To mode 4b

Rest → 4b

$$\ddot{\theta}_2^{4b} < 0$$

From equation of motion and for $\theta_2 = 0$

$$\boldsymbol{h_2\ddot{x}_g - b_2^n g > 0}$$

Chapter 3

Transitions between modes with impact

When the system continuously switches from one mode to another, in most cases impact has to occur. The impact can be between the two bodies or between the body 1 and the ground. Such as the bodies, the ground is considered rigid and the friction is large enough so that there will be no sliding in none of the impacts. Additionally, it is assumed that there is no uplift and the duration of the impact is negligible. Finally, the analysis proceeds to use the principle of conservation of angular momentum in order to evaluate the angular velocities of each body, immediately after the impact.

3.1 From mode 1a

Mode 1a → 2b

About $O1^n$ before impact (1a)

The angular momentum of the system:

$$(h^-_{O1^n})_{system} = (h^-_{O1^n})_{body1} + (h^-_{O1^n})_{body2}$$

Body 1

$$(h^-_{O1^n})_{body1} = I_{G1}\dot{\theta}^-_1 + m_1 v^-_{G1} d^-_1$$

$$v^-_{G1} = \dot{\theta}^-_1 r_1$$

$$\frac{r_1}{e} = \frac{b_1}{r_1} \Rightarrow \quad e = \frac{r_1^2}{b_1} \quad , \quad \frac{d^-_1}{r_1} = \frac{e - 2b_1}{e} \Rightarrow \quad d^-_1 = r_1 \frac{\frac{r_1^2}{b_1} - 2b_1}{\frac{r_1^2}{b_1}} \Rightarrow \quad d^-_1 = \frac{r_1^2 - 2b_1^2}{r_1}$$

$$\boldsymbol{(h^-_{O1^n})_{body1} = (I_{G1} + m_1 r_1 \frac{r_1^2 - 2b_1^2}{r_1})\dot{\theta}^-_1 = (I_{O1} - 2m_1 b_1^2)\dot{\theta}^-_1}$$

Body 2

$$(h^-_{O1^n})_{body2} = I_{G2}\dot{\theta}^-_2 + m_2(v^{x-}_{G2} d^y_2 - v^{y-}_{G2} d^x_2)$$

For $\theta_1 = 0$:

$$\begin{aligned} v_{G2}^{x-} &= \dot{\theta}_1^- l\cos(\beta) + \dot{\theta}_2^- r_2^c\cos(\theta_{c2}^c - \theta_2) \\ v_{G2}^{y-} &= \dot{\theta}_1^- l\sin(\beta) + \dot{\theta}_2^- r_2^c\sin(\theta_{c2}^c - \theta_2) \end{aligned}$$

$$\begin{aligned} d_2^y &= 2h_1 + r_2^c\cos(\theta_{c2}^c - \theta_2) \\ d_2^x &= \xi' - r_2^c\sin(\theta_{c2}^c - \theta_2) \end{aligned}$$

$$\begin{aligned} v_{G2}^{x-}d_2^y &= 2h_1\dot{\theta}_1^- l\cos(\beta) + r_2^c\dot{\theta}_1^- l\cos(\beta)\cos(\theta_{c2}^c - \theta_2) + 2h_1\dot{\theta}_2^- r_2^c\cos(\theta_{c2}^c - \theta_2) + \dot{\theta}_2^-(r_2^c)^2\cos^2(\theta_{c2}^c - \theta_2) \\ v_{G2}^{y-}d_2^x &= \xi'\dot{\theta}_1^- l\sin(\beta) - r_2^c\dot{\theta}_1^- l\sin(\beta)\sin(\theta_{c2}^c - \theta_2) + \xi'\dot{\theta}_2^- r_2^c\sin(\theta_{c2}^c - \theta_2) - \dot{\theta}_2^-(r_2^c)^2\sin^2(\theta_{c2}^c - \theta_2) \end{aligned}$$

$$\begin{aligned} \boldsymbol{(h_{O1^n}^-)_{body2}} &\boldsymbol{= I_{G2}\dot{\theta}_2^- + m_2 l[2h_1\cos(\beta) + r_2^c\cos(\beta - \theta_{c2}^c + \theta_2) - \xi'\sin(\beta)]\dot{\theta}_1^-} \\ &\boldsymbol{+ m_2 r_2^c[2h_1\cos(\theta_2 - \theta_{c2}^c) + \xi'\sin(\theta_2 - \theta_{c2}^c) + r_2^c]\dot{\theta}_2^-} \end{aligned}$$

System

$$\begin{aligned} \boldsymbol{(h_{O1^n}^-)_{system}} &\boldsymbol{= \{I_{O1} - 2m_1b_1^2 + m_2 l[r_2^c\cos(\theta_2 - \theta_{c2}^c + \beta) + 2h_1\cos(\beta) - \xi'\sin(\beta)]\}\dot{\theta}_1^-} \\ &\boldsymbol{+ \{I_{O2}^c + m_2 r_2^c[2h_1\cos(\theta_2 - \theta_{c2}^c) + \xi'\sin(\theta_2 - \theta_{c2}^c)]\}\dot{\theta}_2^-} \end{aligned}$$

About $O1^n$ after impact (2b)

The angular momentum of the system:

$$(h^+_{O1^n})_{system} = (h^+_{O1^n})_{body1} + (h^+_{O1^n})_{body2}$$

Body 1

$$(h^+_{O1^n})_{body1} = I_{G1}\dot{\theta}^+_1 + m_1 v^+_{G1} d^+_1$$
$$v^+_{G1} = \dot{\theta}^+_1 r_1 \quad , \quad d^+_1 = r_1$$
$$\boldsymbol{(h^+_{O1^n})_{body1} = I_{O1}\dot{\theta}^+_1}$$

Body 2

$$(h^+_{O1^n})_{body2} = I_{G2}\dot{\theta}^+_2 + m_2(v^{x+}_{G2} d^y_2 - v^{y+}_{G2} d^x_2)$$

For $\theta_1 = 0$:

$$v^{x+}_{G2} = \dot{\theta}^+_1 l' \cos(\beta') + \dot{\theta}^+_2 r^c_2 \cos(\theta^c_{c2} - \theta_2)$$
$$v^{y+}_{G2} = \dot{\theta}^+_1 l' \sin(\beta') + \dot{\theta}^+_2 r^c_2 \sin(\theta^c_{c2} - \theta_2)$$

$d^y_2 \quad , \quad d^x_2$ are the same as before impact

$$v^{x+}_{G2} d^y_2 = 2h_1\dot{\theta}^+_1 l' \cos(\beta') + r^c_2\dot{\theta}^+_1 l' \cos(\beta')\cos(\theta^c_{c2} - \theta_2) + 2h_1\dot{\theta}^+_2 r^c_2 \cos(\theta^c_{c2} - \theta_2) + \dot{\theta}^+_2 (r^c_2)^2 \cos^2(\theta^c_{c2} - \theta_2)$$
$$v^{y+}_{G2} d^x_2 = -\xi'\dot{\theta}^+_1 l' \sin(\beta') + r^c_2\dot{\theta}^+_1 l' \sin(\beta')\sin(\theta^c_{c2} - \theta_2) + \xi'\dot{\theta}^+_2 r^c_2 \sin(\theta^c_{c2} - \theta_2) - \dot{\theta}^+_2 (r^c_2)^2 \sin^2(\theta^c_{c2} - \theta_2)$$

$$\boldsymbol{(h^+_{O1^n})_{body2} = I_{G2}\dot{\theta}^+_2 + m_2 l'[2h_1\cos(\beta') + r^c_2\cos(-\theta_2 + \theta^c_{c2} + \beta') + \xi'\sin(\beta')]\dot{\theta}^+_1 + m_2 r^c_2[2h_1\cos(\theta_2 - \theta^c_{c2}) + \xi'\sin(\theta_2 - \theta^c_{c2}) + r^c_2]\dot{\theta}^+_2}$$

System

$$\boldsymbol{(h^+_{O1^n})_{system} = \{I_{O1} + m_2 l'[r^c_2\cos(\theta_2 - \theta^c_{c2} - \beta') + 2h_1\cos(\beta') + \xi'\sin(\beta')]\}\dot{\theta}^+_1 + \{I^c_{O2} + m_2 r^c_2[2h_1\cos(\theta_2 - \theta^c_{c2}) + \xi'\sin(\theta_2 - \theta^c_{c2})]\}\dot{\theta}^+_2}$$

About $O2^c$ before impact (1a)

Body 2

$$(h^-_{O2^c})_{body2} = I_{G2}\dot{\theta}^-_2 + m_2(v^{x-}_{G2}d^y_2 + v^{y-}_{G2}d^x_2)$$

v^{x-}_{G2} , v^{y-}_{G2} are the same as previously

$$d^y_2 = r^c_2 \cos(\theta^c_{c2} - \theta_2)$$
$$d^x_2 = r^c_2 \sin(\theta^c_{c2} - \theta_2)$$

$$v^{x-}_{G2}d^y_2 = \dot{\theta}^-_1 l r^c_2 \cos(\beta)\cos(\theta^c_{c2} - \theta_2) + \dot{\theta}^-_2 (r^c_2)^2 \cos^2(\theta^c_{c2} - \theta_2)$$
$$v^{y-}_{G2}d^x_2 = \dot{\theta}^-_1 l r^c_2 \sin(\beta)\sin(\theta^c_{c2} - \theta_2) + \dot{\theta}^-_2 (r^c_2)^2 \sin^2(\theta^c_{c2} - \theta_2)$$

$$\boldsymbol{(h^-_{O2^c})_{body2} = [m_2 r^c_2 l \cos(\theta_2 - \theta^c_{c2} + \beta)]\dot{\theta}^-_1 + I^c_{O2}\dot{\theta}^-_2}$$

About $O2^c$ after impact (2b)

Body 2

$$(h^+_{O2^c})_{body2} = I_{G2}\dot{\theta}^+_2 + m_2(v^{x+}_{G2}d^y_2 + v^{y+}_{G2}d^x_2)$$

v^{x+}_{G2} , v^{y+}_{G2} are the same as previously
d^y_2 , d^x_2 are the same as before impact

$$v^{x+}_{G2}d^y_2 = \dot{\theta}^+_1 l' r^c_2 \cos(\beta')\cos(\theta^c_{c2} - \theta_2) + \dot{\theta}^+_2 (r^c_2)^2 \cos^2(\theta^c_{c2} - \theta_2)$$
$$v^{y+}_{G2}d^x_2 = -\dot{\theta}^+_1 l' r^c_2 \sin(\beta')\sin(\theta^c_{c2} - \theta_2) + \dot{\theta}^+_2 (r^c_2)^2 \sin^2(\theta^c_{c2} - \theta_2)$$

$$\boldsymbol{(h^+_{O2^c})_{body2} = [m_2 r^c_2 l' \cos(\theta_2 - \theta^c_{c2} - \beta')]\dot{\theta}^+_1 + I^c_{O2}\dot{\theta}^+_2}$$

Solution

From the system of equations

$$(h^-_{O1^n})_{system} = (h^+_{O1^n})_{system} \quad | \iff | \quad B_1\dot{\theta}^-_1 + A_2\dot{\theta}^-_2 = A_1\dot{\theta}^+_1 + A_2\dot{\theta}^+_2 \quad \div A_2$$
$$(h^-_{O2^c})_{body2} = (h^+_{O2^c})_{body2} \quad | \iff | \quad B_2\dot{\theta}^-_1 + A_4\dot{\theta}^-_2 = A_3\dot{\theta}^+_1 + A_4\dot{\theta}^+_2 \quad \div A_4$$

And by subtraction:

$$(\frac{B_1}{A_2} - \frac{B_2}{A_4})\dot{\theta}^-_1 = (\frac{A_1}{A_2} - \frac{A_3}{A_4})\dot{\theta}^+_1 \quad \iff \quad \dot{\theta}^+_1 = \frac{A_4B_1 - A_2B_2}{A_1A_4 - A_2A_3}\dot{\theta}^-_1 \quad \Rightarrow \quad \boldsymbol{\dot{\theta}^+_1 = A\dot{\theta}^-_1}$$

From first equation

$$\frac{B_1}{A_2}\dot{\theta}^-_1 + \dot{\theta}^-_2 = \frac{A_1}{A_2}A\dot{\theta}^-_1 + \dot{\theta}^+_2 \quad \iff \quad \dot{\theta}^+_2 = (\frac{B_1}{A_2} - A\frac{A_1}{A_2})\dot{\theta}^-_1 + \dot{\theta}^-_2 \quad \Rightarrow \quad \boldsymbol{\dot{\theta}^+_2 = B\dot{\theta}^-_1 + \dot{\theta}^-_2}$$

in which

$$A = \frac{A_4B_1 - A_2B_2}{A_1A_4 - A_2A_3} \quad , \quad B = \frac{B_1}{A_2} - A\frac{A_1}{A_2}$$

where

$$A_1 = I_{O1} + m_2 l'[r_2^c \cos(\theta_2 - \theta_{c2}^c - \beta') + 2h_1 \cos(\beta') + \xi' \sin(\beta')]$$

$$A_2 = I_{O2}^c + m_2 r_2^c [2h_1 \cos(\theta_2 - \theta_{c2}^c) + \xi' \sin(\theta_2 - \theta_{c2}^c)]$$

$$A_3 = m_2 r_2^c l' \cos(\theta_2 - \theta_{c2}^c - \beta')$$

$$A_4 = I_{O2}^c$$

$$B_1 = I_{O1} - 2m_1 b_1^2 + m_2 l[r_2^c \cos(\theta_2 - \theta_{c2}^c + \beta) + 2h_1 \cos(\beta) - \xi' \sin(\beta)]$$

$$B_2 = m_2 r_2^c l \cos(\theta_2 - \theta_{c2}^c + \beta)$$

Mode 1a → 2a

About $O1^c$ before impact (1a)

The angular momentum of the system:

$$(h_{O1^c}^-)_{system} = (h_{O1^c}^-)_{body1} + (h_{O1^c}^-)_{body2}$$

Body 1

$$(h_{O1^c}^-)_{body1} = I_{G1}\dot{\theta}_1^- + m_1 v_{G1}^- d_1^-$$
$$v_{G1}^- = \dot{\theta}_1^- r_1 \quad , \quad d_1^- = r_1$$
$$\boldsymbol{(h_{O1^c}^-)_{body1} = I_{O1}\dot{\theta}_1^-}$$

Body 2

$$(h_{O1^c}^-)_{body2} = I_{G2}\dot{\theta}_2^- + m_2(v_{G2}^{x-} d_2^y + v_{G2}^{y-} d_2^x)$$

For $\theta_1 = \theta_2$:

$$v_{G2}^{x-} = \dot{\theta}_1^- l \cos(\theta_1 - \beta) + \dot{\theta}_2^- r_2^c \cos(\theta_{c2}^c - \theta_2)$$
$$v_{G2}^{y-} = -\dot{\theta}_1^- l \sin(\theta_1 - \beta) + \dot{\theta}_2^- r_2^c \sin(\theta_{c2}^c - \theta_2)$$

$$d_2^y = d^c \cos(\zeta^c - \theta_1)$$
$$d_2^x = d^c \sin(\zeta^c - \theta_1)$$

$$v_{G2}^{x-} d_2^y = \dot{\theta}_1^- l d^c \cos(\theta_1 - \beta)\cos(\zeta^c - \theta_1) + \dot{\theta}_2^- r_2^c d^c \cos(\theta_{c2}^c - \theta_2)\cos(\zeta^c - \theta_1)$$
$$v_{G2}^{y-} d_2^x = -\dot{\theta}_1^- l d^c \sin(\theta_1 - \beta)\sin(\zeta^c - \theta_1) + \dot{\theta}_2^- r_2^c d^c \sin(\theta_{c2}^c - \theta_2)\sin(\zeta^c - \theta_1)$$

$$\boldsymbol{(h_{O1^c}^-)_{body2} = I_{G2}\dot{\theta}_2^- + m_2 l d^c \cos(\theta_1 - \beta + \zeta^c - \theta_1)\dot{\theta}_1^- + m_2 r_2^c d^c \cos(\theta_{c2}^c - \theta_2 - \zeta^c + \theta_1)\dot{\theta}_2^-}$$

System

$$\boldsymbol{(h^-_{O1^c})_{system} = \{I_{O1} + m_2 l d^c \cos(\zeta^c - \beta)\}\dot{\theta}^-_1 + \{I_{G2} + m_2 r^c_2 d^c \cos(\zeta^c - \theta^c_{c2})\}\dot{\theta}^-_2}$$

About $O1^c$ after impact (2a)

The angular momentum of the system:

$$(h^+_{O1^c})_{system} = (h^+_{O1^c})_{body1} + (h^+_{O1^c})_{body2}$$

Body 1

$$(h^+_{O1^c})_{body1} = I_{G1}\dot{\theta}^+_1 + m_1 v^+_{G1} d^+_1$$
$$v^+_{G1} = \dot{\theta}^+_1 r_1 \quad , \quad d^+_1 = r_1$$
$$\boldsymbol{(h^+_{O1^n})_{body1} = I_{O1}\dot{\theta}^+_1}$$

Body 2

$$(h^+_{O1^n})_{body2} = I_{G2}\dot{\theta}^+_2 + m_2(v^{x+}_{G2} d^y_2 + v^{y+}_{G2} d^x_2)$$

For $\theta_1 = \theta_2$:

$$v^{x+}_{G2} = \dot{\theta}^+_1 l' \cos(\beta' - \theta_1) + \dot{\theta}^+_2 r^n_2 \cos(\theta^n_{c2} + \theta_2)$$
$$v^{y+}_{G2} = \dot{\theta}^+_1 l' \sin(\beta' - \theta_1) - \dot{\theta}^+_2 r^n_2 \sin(\theta^n_{c2} + \theta_2)$$

$d^y_2 \quad , \quad d^x_2$ are the same as before impact

$$v^{x+}_{G2} d^y_2 = \dot{\theta}^+_1 l' d^c \cos(\beta' - \theta_1)\cos(\zeta^c - \theta_1) + \dot{\theta}^+_2 r^n_2 d^c \cos(\theta^n_{c2} + \theta_2)\cos(\zeta^c - \theta_1)$$
$$v^{y+}_{G2} d^x_2 = \dot{\theta}^+_1 l' d^c \sin(\beta' - \theta_1)\sin(\zeta^c - \theta_1) - \dot{\theta}^+_2 r^n_2 d^c \sin(\theta^n_{c2} + \theta_2)\sin(\zeta^c - \theta_1)$$

$$\boldsymbol{(h^+_{O1^c})_{body2} = I_{G2}\dot{\theta}^+_2 + m_2 l' d^c \cos(\theta_1 - \beta' - \zeta^c - \theta_1)\dot{\theta}^+_1 + m_2 r^n_2 d^c \cos(\theta^n_{c2} + \theta_2 + \zeta^c - \theta_1)\dot{\theta}^+_2}$$

System

$$\boldsymbol{(h^+_{O1^c})_{system} = \{I_{O1} + m_2 l' d^c \cos(\zeta^c - \beta')\}\dot{\theta}^+_1 + \{I_{G2} + m_2 r^n_2 d^c \cos(\theta^n_{c2} + \zeta^c)\}\dot{\theta}^+_2}$$

About $O2^n$ before impact (1a)

Body 2

$$(h^-_{O2^n})_{body2} = I_{G2}\dot{\theta}^-_2 + m_2(v^{x-}_{G2}d^y_2 - v^{y-}_{G2}d^x_2)$$

v^{x-}_{G2} , v^{y-}_{G2} are the same as previously

$$d^y_2 = r^n_2 \cos(\theta^n_{c2} + \theta_2)$$
$$d^x_2 = r^n_2 \sin(\theta^n_{c2} + \theta_2)$$

$$v^{x-}_{G2}d^y_2 = \dot{\theta}^-_1 l r^n_2 \cos(\theta_1 - \beta)\cos(\theta^n_{c2} + \theta_2) + \dot{\theta}^-_2 r^c_2 r^n_2 \cos(\theta^c_{c2} - \theta_2)\cos(\theta^n_{c2} + \theta_2)$$
$$v^{y-}_{G2}d^x_2 = -\dot{\theta}^-_1 l r^n_2 \sin(\theta_1 - \beta)\sin(\theta^n_{c2} + \theta_2) + \dot{\theta}^-_2 r^c_2 r^n_2 \sin(\theta^c_{c2} - \theta_2)\sin(\theta^n_{c2} + \theta_2)$$

$$\boldsymbol{(h^-_{O2^n})_{body2} = m_2 r^n_2 l \cos(\theta^n_{c2} + \beta)\dot{\theta}^-_1 + \{I_{G2} + m_2 r^c_2 r^n_2 \cos(\theta^c_{c2} + \theta^n_{c2})\}\dot{\theta}^-_2}$$

About $O2^n$ after impact (2a)

Body 2

$$(h^+_{O2^n})_{body2} = I_{G2}\dot{\theta}^+_2 + m_2(v^{x+}_{G2}d^y_2 - v^{y+}_{G2}d^x_2)$$

v^{x+}_{G2} , v^{y+}_{G2} are the same as previously
d^y_2 , d^x_2 are the same as before impact

$$v^{x+}_{G2}d^y_2 = \dot{\theta}^+_1 l' r^n_2 \cos(\beta' - \theta_1)\cos(\theta^n_{c2} + \theta_2) + \dot{\theta}^+_2 (r^n_2)^2 \cos^2(\theta^n_{c2} + \theta_2)$$
$$v^{y+}_{G2}d^x_2 = \dot{\theta}^+_1 l' r^n_2 \sin(\beta' - \theta_1)\sin(\theta^n_{c2} + \theta_2) - \dot{\theta}^+_2 (r^n_2)^2 \sin^2(\theta^n_{c2} + \theta_2)$$

$$\boldsymbol{(h^+_{O2^n})_{body2} = m_2 r^n_2 l' \cos(\theta^n_{c2} + \beta')\dot{\theta}^+_1 + I^n_{O2}\dot{\theta}^+_2}$$

Solution

From the system of equations

$$(h^-_{O1^c})_{system} = (h^+_{O1^c})_{system} \quad |\Longleftrightarrow| \quad B_1\dot{\theta}^-_1 + B_2\dot{\theta}^-_2 = A_1\dot{\theta}^+_1 + A_2\dot{\theta}^+_2 \quad \div A_2$$
$$(h^-_{O2^n})_{body2} = (h^+_{O2^n})_{body2} \quad |\Longleftrightarrow| \quad B_3\dot{\theta}^-_1 + B_4\dot{\theta}^-_2 = A_3\dot{\theta}^+_1 + A_4\dot{\theta}^+_2 \quad \div A_4$$

And by subtraction:

$$(\frac{B_1}{A_2} - \frac{B_3}{A_4})\dot{\theta}^-_1 + (\frac{B_2}{A_2} - \frac{B_4}{A_4})\dot{\theta}^-_2 = (\frac{A_1}{A_2} - \frac{A_3}{A_4})\dot{\theta}^+_1 \quad \Longleftrightarrow \quad \dot{\theta}^+_1 = \frac{A_4B_1 - A_2B_3}{A_1A_4 - A_2A_3}\dot{\theta}^-_1 + \frac{A_4B_2 - A_2B_4}{A_1A_4 - A_2A_3}\dot{\theta}^-_2$$

$$\Rightarrow \quad \boldsymbol{\dot{\theta}^+_1 = A\dot{\theta}^-_1 + B\dot{\theta}^-_2}$$

Similarly but the 2 equations are divided by A_1 and A_3 respectively:

$$\dot{\theta}^+_2 = \frac{A_3B_1 - A_1B_3}{A_2A_3 - A_1A_4}\dot{\theta}^-_1 + \frac{A_3B_2 - A_1B_4}{A_2A_3 - A_1A_4}\dot{\theta}^-_2 \quad \Rightarrow \quad \boldsymbol{\dot{\theta}^+_2 = C\dot{\theta}^-_1 + D\dot{\theta}^-_2}$$

in which

$$A = \frac{A_4B_1 - A_2B_3}{A_1A_4 - A_2A_3} \quad , \quad B = \frac{A_4B_2 - A_2B_4}{A_1A_4 - A_2A_3}$$

$$C = \frac{A_3B_1 - A_1B_3}{A_2A_3 - A_1A_4} \quad , \quad D = \frac{A_3B_2 - A_1B_4}{A_2A_3 - A_1A_4}$$

where

$$A_1 = I_{O1} + m_2 l' d^c \cos(\zeta^c - \beta') \quad , \qquad B_1 = I_{O1} + m_2 l d^c \cos(\zeta^c - \beta)$$

$$A_2 = I_{G2} + m_2 r_2^n d^c \cos(\theta_{c2}^n + \zeta^c) \quad , \qquad B_2 = I_{G2} + m_2 r_2^c d^c \cos(\zeta^c - \theta_{c2}^c)$$

$$A_3 = m_2 r_2^n l' \cos(\theta_{c2}^n + \beta') \quad , \qquad B_3 = m_2 r_2^n l \cos(\theta_{c2}^n + \beta)$$

$$A_4 = I_{O2}^n \quad , \qquad B_4 = I_{G2} + m_2 r_2^c r_2^n \cos(\theta_{c2}^c + \theta_{c2}^n)$$

If $\dot{\theta}_2^+ > \dot{\theta}_1^+$ then transition to 3a occurs with $\dot{\theta}_2^+ = \dot{\theta}_1^+$

Mode 1a → 3a

About $O1^c$ before impact (1a)

The angular momentum of the system is the same as previously:

System

$$\boldsymbol{(h_{O1^c}^-)_{system} = \{I_{O1} + m_2 l d^c \cos(\zeta^c - \beta)\}\dot{\theta}_1^- + \{I_{G2} + m_2 r_2^c d^c \cos(\zeta^c - \theta_{c2}^c)\}\dot{\theta}_2^-}$$

About $O1^c$ after impact (3a)

System

$$\boldsymbol{(h_{O1^c}^+)_{system} = I_O^c \dot{\theta}_1^+}$$

Solution

From equation

$$(h_{O1^c}^-)_{system} = (h_{O1^c}^+)_{system} \quad \Longleftrightarrow \quad B_1\dot{\theta}_1^- + B_2\dot{\theta}_2^- = I_O^c\dot{\theta}_1^+$$

$$\dot{\theta}_1^+ = \dot{\theta}_2^+ = \frac{I_{O1} + m_2 l d^c \cos(\zeta^c - \beta)}{I_O^c}\dot{\theta}_1^- + \frac{I_{G2} + m_2 r_2^c d^c \cos(\zeta^c - \theta_{c2}^c)}{I_O^c}\dot{\theta}_2^-$$

$$\Rightarrow \quad \boldsymbol{\dot{\theta}_1^+ = \dot{\theta}_2^+ = \frac{B_1}{I_O^c}\dot{\theta}_1^- + \frac{B_2}{I_O^c}\dot{\theta}_2^-}$$

3.2 From mode 1b

Mode 1b → 2a

From 'symmetry' of modes, the same analysis (1a to 2b) also holds with the opposite sign for angle θ_2 and opposite c,n attributes.

$$\boldsymbol{\dot{\theta}_1^+ = A\dot{\theta}_1^-}$$
$$\boldsymbol{\dot{\theta}_2^+ = B\dot{\theta}_1^- + \dot{\theta}_2^-}$$

in which

$$A = \frac{A_4B_1 - A_2B_2}{A_1A_4 - A_2A_3} \quad , \quad B = \frac{B_1}{A_2} - A\frac{A_1}{A_2}$$

where

$$A_1 = I_{O1} + m_2l'[r_2^n\cos(-\theta_2 - \theta_{c2}^n - \beta') + 2h_1\cos(\beta') + \xi'\sin(\beta')]$$

$$A_2 = I_{O2}^n + m_2r_2^n[2h_1\cos(-\theta_2 - \theta_{c2}^n) + \xi'\sin(-\theta_2 - \theta_{c2}^n)]$$

$$A_3 = m_2r_2^nl'\cos(-\theta_2 - \theta_{c2}^n - \beta')$$

$$A_4 = I_{O2}^n$$

$$B_1 = I_{O1} - 2m_1b_1^2 + m_2l[r_2^n\cos(-\theta_2 - \theta_{c2}^n + \beta) + 2h_1\cos(\beta) - \xi'\sin(\beta)]$$

$$B_2 = m_2r_2^nl\cos(-\theta_2 - \theta_{c2}^n + \beta)$$

Mode 1b → 2b

From 'symmetry' of modes, the same analysis (1a to 2a) also holds with the opposite sign for angle θ_2 and opposite c,n attributes.

$$\boldsymbol{\dot{\theta}_1^+ = A\dot{\theta}_1^- + B\dot{\theta}_2^-}$$
$$\boldsymbol{\dot{\theta}_2^+ = C\dot{\theta}_1^- + D\dot{\theta}_2^-}$$

in which

$$A = \frac{A_4B_1 - A_2B_3}{A_1A_4 - A_2A_3} \quad , \quad B = \frac{A_4B_2 - A_2B_4}{A_1A_4 - A_2A_3}$$

$$C = \frac{A_3B_1 - A_1B_3}{A_2A_3 - A_1A_4} \quad , \quad D = \frac{A_3B_2 - A_1B_4}{A_2A_3 - A_1A_4}$$

where

$$A_1 = I_{O1} + m_2 l' d^n \cos(\zeta^n - \beta') \quad , \qquad B_1 = I_{O1} + m_2 l d^n \cos(\zeta^n - \beta)$$

$$A_2 = I_{G2} + m_2 r_2^c d^n \cos(\theta_{c2}^c + \zeta^n) \quad , \qquad B_2 = I_{G2} + m_2 r_2^n d^n \cos(\zeta^n - \theta_{c2}^n)$$

$$A_3 = m_2 r_2^c l' \cos(\theta_{c2}^c + \beta') \quad , \qquad B_3 = m_2 r_2^c l \cos(\theta_{c2}^c + \beta)$$

$$A_4 = I_{O2}^c = I_{G2} + m_2 (r_2^c)^2 \quad , \qquad B_4 = I_{G2} + m_2 r_2^c r_2^n \cos(\theta_{c2}^c + \theta_{c2}^n)$$

If $\underline{\dot{\theta}_2^+ < \dot{\theta}_1^+}$ then transition to 3b occurs with $\dot{\theta}_2^+ = \dot{\theta}_1^+$

Mode 1b → 3b

$$\boldsymbol{\dot{\theta}_1^+ = \dot{\theta}_2^+ = \frac{B_1}{I_O^n}\dot{\theta}_1^- + \frac{B_2}{I_O^n}\dot{\theta}_2^-}$$

3.3 From mode 2a

Mode 2a → 1b

About $O1^n$ before impact (2a)

The angular momentum of the system:

$$(h_{O1^n}^-)_{system} = (h_{O1^n}^-)_{body1} + (h_{O1^n}^-)_{body2}$$

Body 1

$$(h_{O1^n}^-)_{body1} = I_{G1}\dot{\theta}_1^- + m_1 v_{G1}^- d_1^-$$
$$v_{G1}^- = \dot{\theta}_1^- r_1$$

$$\frac{r_1}{e} = \frac{b_1}{r_1} \Rightarrow \quad e = \frac{r_1^2}{b_1} \quad , \quad \frac{d_1^-}{r_1} = \frac{e - 2b_1}{e} \Rightarrow \quad d_1^- = r_1 \frac{\frac{r_1^2}{b_1} - 2b_1}{\frac{r_1^2}{b_1}} \Rightarrow \quad d_1^- = \frac{r_1^2 - 2b_1^2}{r_1}$$

$$\boldsymbol{(h_{O1^n}^-)_{body1} = (I_{G1} + m_1 r_1 \frac{r_1^2 - 2b_1^2}{r_1})\dot{\theta}_1^- = (I_{O1} - 2m_1 b_1^2)\dot{\theta}_1^-}$$

Body 2

$$(h_{O1^n}^-)_{body2} = I_{G2}\dot{\theta}_2^- + m_2(v_{G2}^{x-} d_2^y - v_{G2}^{y-} d_2^x)$$

For $\theta_1 = 0$:

$$v_{G2}^{x-} = \dot{\theta}_1^- l' \cos(\beta') + \dot{\theta}_2^- r_2^n \cos(\theta_{c2}^n + \theta_2)$$
$$v_{G2}^{y-} = \dot{\theta}_1^- l' \sin(\beta') - \dot{\theta}_2^- r_2^n \sin(\theta_{c2}^n + \theta_2)$$

$$d_2^y = 2h_1 + r_2^n \cos(\theta_{c2}^n + \theta_2)$$
$$d_2^x = \xi + r_2^n \sin(\theta_{c2}^n + \theta_2)$$

$$v_{G2}^{x-} d_2^y = 2h_1 \dot{\theta}_1^- l' \cos(\beta') + r_2^n \dot{\theta}_1^- l' \cos(\beta') \cos(\theta_{c2}^n + \theta_2) + 2h_1 \dot{\theta}_2^- r_2^n \cos(\theta_{c2}^n + \theta_2) + \dot{\theta}_2^- (r_2^n)^2 \cos^2(\theta_{c2}^n + \theta_2)$$
$$v_{G2}^{y-} d_2^x = \xi \dot{\theta}_1^- l' \sin(\beta') + r_2^n \dot{\theta}_1^- l' \sin(\beta') \sin(\theta_{c2}^n + \theta_2) - \xi \dot{\theta}_2^- r_2^n \sin(\theta_{c2}^n + \theta_2) - \dot{\theta}_2^- (r_2^n)^2 \sin^2(\theta_{c2}^n + \theta_2)$$

$$\boldsymbol{(h_{O1^n}^-)_{body2} = I_{G2}\dot{\theta}_2^- + m_2 l'[2h_1 \cos(\beta') + r_2^n \cos(\beta' + \theta_{c2}^n + \theta_2) - \xi \sin(\beta')]\dot{\theta}_1^-}$$
$$\boldsymbol{+ m_2 r_2^n [2h_1 \cos(\theta_2 + \theta_{c2}^n) + \xi \sin(\theta_2 + \theta_{c2}^n) + r_2^n]\dot{\theta}_2^-}$$

System

$$\boldsymbol{(h_{O1^n}^-)_{system} = \{I_{O1} - 2m_1 b_1^2 + m_2 l'[r_2^n \cos(\theta_2 + \theta_{c2}^n + \beta') + 2h_1 \cos(\beta') - \xi \sin(\beta')]\}\dot{\theta}_1^-}$$
$$\boldsymbol{+ \{I_{O2}^n + m_2 r_2^n [2h_1 \cos(\theta_2 + \theta_{c2}^n) + \xi \sin(\theta_2 + \theta_{c2}^n)]\}\dot{\theta}_2^-}$$

About $O1^n$ after impact (1b)

The angular momentum of the system:

$$(h^+_{O1^n})_{system} = (h^+_{O1^n})_{body1} + (h^+_{O1^n})_{body2}$$

Body 1

$$(h^+_{O1^n})_{body1} = I_{G1}\dot{\theta}^+_1 + m_1 v^+_{G1} d^+_1$$
$$v^+_{G1} = \dot{\theta}^+_1 r_1 \quad , \quad d^+_1 = r_1$$
$$\boldsymbol{(h^+_{O1^n})_{body1} = I_{O1}\dot{\theta}^+_1}$$

Body 2

$$(h^+_{O1^n})_{body2} = I_{G2}\dot{\theta}^+_2 + m_2(v^{x+}_{G2} d^y_2 - v^{y+}_{G2} d^x_2)$$

For $\theta_1 = 0$:

$$v^{x+}_{G2} = \dot{\theta}^+_1 l\cos(\beta) + \dot{\theta}^+_2 r^n_2 \cos(\theta^n_{c2} + \theta_2)$$
$$v^{y+}_{G2} = -\dot{\theta}^+_1 l\sin(\beta) - \dot{\theta}^+_2 r^n_2 \sin(\theta^n_{c2} + \theta_2)$$

$d^y_2 \quad , \quad d^x_2$ are the same as before impact

$$v^{x+}_{G2} d^y_2 = 2h_1\dot{\theta}^+_1 l\cos(\beta) + r^n_2\dot{\theta}^+_1 l\cos(\beta)\cos(\theta^n_{c2} + \theta_2) + 2h_1\dot{\theta}^+_2 r^n_2\cos(\theta^n_{c2} + \theta_2) + \dot{\theta}^+_2 (r^n_2)^2\cos^2(\theta^n_{c2} + \theta_2)$$
$$v^{y+}_{G2} d^x_2 = -\xi\dot{\theta}^+_1 l\sin(\beta) - r^n_2\dot{\theta}^+_1 l\sin(\beta)\sin(\theta^n_{c2} + \theta_2) - \xi\dot{\theta}^+_2 r^n_2\sin(\theta^n_{c2} + \theta_2) - \dot{\theta}^+_2 (r^n_2)^2\sin^2(\theta^n_{c2} + \theta_2)$$

$$\boldsymbol{(h^+_{O1^n})_{body2} = I_{G2}\dot{\theta}^+_2 + m_2 l[2h_1\cos(\beta) + r^n_2\cos(-\theta_2 - \theta^n_{c2} + \beta) + \xi\sin(\beta)]\dot{\theta}^+_1}$$
$$\boldsymbol{+ m_2 r^n_2[2h_1\cos(\theta_2 + \theta^n_{c2}) + \xi\sin(\theta_2 + \theta^n_{c2}) + r^n_2]\dot{\theta}^+_2}$$

System

$$\boldsymbol{(h^+_{O1^n})_{system} = \{I_{O1} + m_2 l[r^n_2\cos(\theta_2 + \theta^n_{c2} - \beta) + 2h_1\cos(\beta) + \xi\sin(\beta)]\}\dot{\theta}^+_1}$$
$$\boldsymbol{+ \{I^n_{O2} + m_2 r^n_2[2h_1\cos(\theta_2 + \theta^n_{c2}) + \xi\sin(\theta_2 + \theta^n_{c2})]\}\dot{\theta}^+_2}$$

About $O2^n$ before impact (2a)

Body 2

$$(h^-_{O2^n})_{body2} = I_{G2}\dot{\theta}^-_2 + m_2(v^{x-}_{G2}d^y_2 - v^{y-}_{G2}d^x_2)$$

v^{x-}_{G2} , v^{y-}_{G2} are the same as previously

$$d^y_2 = r^n_2 \cos(\theta^n_{c2} + \theta_2)$$
$$d^x_2 = r^n_2 \sin(\theta^n_{c2} + \theta_2)$$

$$v^{x-}_{G2}d^y_2 = \dot{\theta}^-_1 l' r^n_2 \cos(\beta')\cos(\theta^n_{c2} + \theta_2) + \dot{\theta}^-_2 (r^n_2)^2 \cos^2(\theta^n_{c2} + \theta_2)$$
$$v^{y-}_{G2}d^x_2 = \dot{\theta}^-_1 l' r^n_2 \sin(\beta')\sin(\theta^n_{c2} + \theta_2) - \dot{\theta}^-_2 (r^n_2)^2 \sin^2(\theta^n_{c2} + \theta_2)$$

$$\boldsymbol{(h^-_{O2^n})_{body2} = [m_2 r^n_2 l' \cos(\theta_2 + \theta^n_{c2} + \beta')]\dot{\theta}^-_1 + I^c_{O2}\dot{\theta}^-_2}$$

About $O2^n$ after impact (1b)

Body 2

$$(h^+_{O2^n})_{body2} = I_{G2}\dot{\theta}^+_2 + m_2(v^{x+}_{G2}d^y_2 - v^{y+}_{G2}d^x_2)$$

v^{x+}_{G2} , v^{y+}_{G2} are the same as previously
d^y_2 , d^x_2 are the same as before impact

$$v^{x+}_{G2}d^y_2 = \dot{\theta}^+_1 l r^n_2 \cos(\beta)\cos(\theta^n_{c2} + \theta_2) + \dot{\theta}^+_2 (r^n_2)^2 \cos^2(\theta^n_{c2} + \theta_2)$$
$$v^{y+}_{G2}d^x_2 = -\dot{\theta}^+_1 l r^n_2 \sin(\beta)\sin(\theta^n_{c2} + \theta_2) - \dot{\theta}^+_2 (r^n_2)^2 \sin^2(\theta^n_{c2} + \theta_2)$$

$$\boldsymbol{(h^+_{O2^n})_{body2} = [m_2 r^n_2 l \cos(\theta_2 + \theta^n_{c2} - \beta)]\dot{\theta}^+_1 + I^n_{O2}\dot{\theta}^+_2}$$

Solution

From the system of equations

$$(h^-_{O1^n})_{system} = (h^+_{O1^n})_{system} \quad | \iff | \quad B_1\dot{\theta}^-_1 + A_2\dot{\theta}^-_2 = A_1\dot{\theta}^+_1 + A_2\dot{\theta}^+_2 \quad \div A_2$$
$$(h^-_{O2^n})_{body2} = (h^+_{O2^n})_{body2} \quad | \iff | \quad B_2\dot{\theta}^-_1 + A_4\dot{\theta}^-_2 = A_3\dot{\theta}^+_1 + A_4\dot{\theta}^+_2 \quad \div A_4$$

And by subtraction:

$$(\frac{B_1}{A_2} - \frac{B_2}{A_4})\dot{\theta}^-_1 = (\frac{A_1}{A_2} - \frac{A_3}{A_4})\dot{\theta}^+_1 \quad \iff \quad \dot{\theta}^+_1 = \frac{A_4B_1 - A_2B_2}{A_1A_4 - A_2A_3}\dot{\theta}^-_1 \quad \Rightarrow \quad \boldsymbol{\dot{\theta}^+_1 = A\dot{\theta}^-_1}$$

From first equation

$$\frac{B_1}{A_2}\dot{\theta}^-_1 + \dot{\theta}^-_2 = \frac{A_1}{A_2}A\dot{\theta}^-_1 + \dot{\theta}^+_2 \quad \iff \quad \dot{\theta}^+_2 = (\frac{B_1}{A_2} - A\frac{A_1}{A_2})\dot{\theta}^-_1 + \dot{\theta}^-_2 \quad \Rightarrow \quad \boldsymbol{\dot{\theta}^+_2 = B\dot{\theta}^-_1 + \dot{\theta}^-_2}$$

in which

$$A = \frac{A_4B_1 - A_2B_2}{A_1A_4 - A_2A_3} \quad , \quad B = \frac{B_1}{A_2} - A\frac{A_1}{A_2}$$

where

$$A_1 = I_{O1} + m_2 l[r_2^n \cos(\theta_2 + \theta_{c2}^n - \beta) + 2h_1 \cos(\beta) + \xi \sin(\beta)]$$

$$A_2 = I_{O2}^n + m_2 r_2^n [2h_1 \cos(\theta_2 + \theta_{c2}^n) + \xi \sin(\theta_2 + \theta_{c2}^n)]$$

$$A_3 = m_2 r_2^n l \cos(\theta_2 + \theta_{c2}^n - \beta)$$

$$A_4 = I_{O2}^n$$

$$B_1 = I_{O1} - 2m_1 b_1^2 + m_2 l'[r_2^n \cos(\theta_2 + \theta_{c2}^n + \beta') + 2h_1 \cos(\beta') - \xi \sin(\beta')]$$

$$B_2 = m_2 r_2^n l' \cos(\theta_2 + \theta_{c2}^n + \beta')$$

Mode 2a → 1a

About $O1^c$ before impact (2a)

The angular momentum of the system:

$$(h_{O1^c}^-)_{system} = (h_{O1^c}^-)_{body1} + (h_{O1^c}^-)_{body2}$$

Body 1

$$(h_{O1^c}^-)_{body1} = I_{G1}\dot{\theta}_1^- + m_1 v_{G1}^- d_1^-$$

$$v_{G1}^- = \dot{\theta}_1^- r_1 \quad , \quad d_1^- = r_1$$

$$\boldsymbol{(h_{O1^c}^-)_{body1} = I_{O1}\dot{\theta}_1^-}$$

Body 2

$$(h_{O1^c}^-)_{body2} = I_{G2}\dot{\theta}_2^- + m_2(v_{G2}^{x-} d_2^y + v_{G2}^{y-} d_2^x)$$

For $\theta_1 = \theta_2$:

$$v_{G2}^{x-} = \dot{\theta}_1^- l' \cos(\beta' - \theta_1) + \dot{\theta}_2^- r_2^n \cos(\theta_{c2}^n + \theta_2)$$

$$v_{G2}^{y-} = \dot{\theta}_1^- l' \sin(\beta' - \theta_1) - \dot{\theta}_2^- r_2^n \sin(\theta_{c2}^n + \theta_2)$$

$$d_2^y = d^c \cos(\zeta^c - \theta_1)$$

$$d_2^x = d^c \sin(\zeta^c - \theta_1)$$

$$v_{G2}^{x-} d_2^y = \dot{\theta}_1^- l' d^c \cos(\beta' - \theta_1)\cos(\zeta^c - \theta_1) + \dot{\theta}_2^- r_2^n d^c \cos(\theta_{c2}^n + \theta_2)\cos(\zeta^c - \theta_1)$$

$$v_{G2}^{y-} d_2^x = \dot{\theta}_1^- l d^c \sin(\beta' - \theta_1)\sin(\zeta^c - \theta_1) - \dot{\theta}_2^- r_2^n d^c \sin(\theta_{c2}^n + \theta_2)\sin(\zeta^c - \theta_1)$$

$$\boldsymbol{(h_{O1^c}^-)_{body2} = I_{G2}\dot{\theta}_2^- + m_2 l' d^c \cos(\beta' - \theta_1 - \zeta^c + \theta_1)\dot{\theta}_1^- + m_2 r_2^n d^c \cos(\theta_{c2}^n + \theta_2 + \zeta^c - \theta_1)\dot{\theta}_2^-}$$

System

$$\boldsymbol{(h^-_{O1^c})_{system} = \{I_{O1} + m_2 l' d^c \cos(\zeta^c - \beta')\}\dot{\theta}^-_1 + \{I_{G2} + m_2 r^n_2 d^c \cos(\zeta^c + \theta^n_{c2})\}\dot{\theta}^-_2}$$

About $O1^c$ after impact (1a)

The angular momentum of the system:

$$(h^+_{O1^c})_{system} = (h^+_{O1^c})_{body1} + (h^+_{O1^c})_{body2}$$

Body 1

$$(h^+_{O1^c})_{body1} = I_{G1}\dot{\theta}^+_1 + m_1 v^+_{G1} d^+_1$$
$$v^+_{G1} = \dot{\theta}^+_1 r_1 \quad , \quad d^+_1 = r_1$$
$$\boldsymbol{(h^+_{O1^n})_{body1} = I_{O1}\dot{\theta}^+_1}$$

Body 2

$$(h^+_{O1^n})_{body2} = I_{G2}\dot{\theta}^+_2 + m_2(v^{x+}_{G2} d^y_2 + v^{y+}_{G2} d^x_2)$$

For $\theta_1 = \theta_2$:

$$v^{x+}_{G2} = \dot{\theta}^+_1 l \cos(\theta_1 - \beta) + \dot{\theta}^+_2 r^c_2 \cos(\theta^c_{c2} - \theta_2)$$
$$v^{y+}_{G2} = -\dot{\theta}^+_1 l \sin(\theta_1 - \beta) + \dot{\theta}^+_2 r^c_2 \sin(\theta^c_{c2} - \theta_2)$$

$d^y_2 \quad , \quad d^x_2$ are the same as before impact

$$v^{x+}_{G2} d^y_2 = \dot{\theta}^+_1 l d^c \cos(\theta_1 - \beta)\cos(\zeta^c - \theta_1) + \dot{\theta}^+_2 r^c_2 d^c \cos(\theta^c_{c2} - \theta_2)\cos(\zeta^c - \theta_1)$$
$$v^{y+}_{G2} d^x_2 = -\dot{\theta}^+_1 l d^c \sin(\theta_1 - \beta)\sin(\zeta^c - \theta_1) + \dot{\theta}^+_2 r^c_2 d^c \sin(\theta^c_{c2} - \theta_2)\sin(\zeta^c - \theta_1)$$

$$\boldsymbol{(h^+_{O1^c})_{body2} = I_{G2}\dot{\theta}^+_2 + m_2 l d^c \cos(\theta_1 - \beta + \zeta^c - \theta_1)\dot{\theta}^+_1 + m_2 r^c_2 d^c \cos(\theta^c_{c2} - \theta_2 - \zeta^c + \theta_1)\dot{\theta}^+_2}$$

System

$$\boldsymbol{(h^+_{O1^c})_{system} = \{I_{O1} + m_2 l d^c \cos(\zeta^c - \beta)\}\dot{\theta}^+_1 + \{I_{G2} + m_2 r^c_2 d^c \cos(\zeta^c - \theta^c_{c2})\}\dot{\theta}^+_2}$$

About $O2^c$ before impact (2a)

Body 2

$$(h^-_{O2^c})_{body2} = I_{G2}\dot{\theta}^-_2 + m_2(v^{x-}_{G2}d^y_2 + v^{y-}_{G2}d^x_2)$$

v^{x-}_{G2} , v^{y-}_{G2} are the same as previously

$$d^y_2 = r^c_2\cos(\theta^c_{c2} - \theta_2)$$
$$d^x_2 = r^c_2\sin(\theta^c_{c2} - \theta_2)$$

$$v^{x-}_{G2}d^y_2 = \dot{\theta}^-_1 l' r^c_2\cos(\beta' - \theta_1)\cos(\theta^c_{c2} - \theta_2) + \dot{\theta}^-_2 r^c_2 r^n_2\cos(\theta^n_{c2} + \theta_2)\cos(\theta^c_{c2} - \theta_2)$$
$$v^{y-}_{G2}d^x_2 = \dot{\theta}^-_1 l' r^c_2\sin(\beta' - \theta_1)\sin(\theta^c_{c2} - \theta_2) - \dot{\theta}^-_2 r^c_2 r^n_2\sin(\theta^n_{c2} + \theta_2)\sin(\theta^c_{c2} - \theta_2)$$

$$\boldsymbol{(h^-_{O2^c})_{body2} = m_2 r^c_2 l'\cos(\theta^c_{c2} - \beta')\dot{\theta}^-_1 + \{I_{G2} + m_2 r^c_2 r^n_2\cos(\theta^c_{c2} + \theta^n_{c2})\}\dot{\theta}^-_2}$$

About $O2^c$ after impact (1a)

Body 2

$$(h^+_{O2^c})_{body2} = I_{G2}\dot{\theta}^+_2 + m_2(v^{x+}_{G2}d^y_2 + v^{y+}_{G2}d^x_2)$$

v^{x+}_{G2} , v^{y+}_{G2} are the same as previously
d^y_2 , d^x_2 are the same as before impact

$$v^{x+}_{G2}d^y_2 = \dot{\theta}^+_1 l r^c_2\cos(\theta_1 - \beta)\cos(\theta^c_{c2} - \theta_2) + \dot{\theta}^+_2 (r^c_2)^2\cos^2(\theta^c_{c2} - \theta_2)$$
$$v^{y+}_{G2}d^x_2 = -\dot{\theta}^+_1 l r^c_2\sin(\theta_1 - \beta)\sin(\theta^c_{c2} - \theta_2) + \dot{\theta}^+_2 (r^c_2)^2\sin^2(\theta^c_{c2} - \theta_2)$$

$$\boldsymbol{(h^+_{O2^c})_{body2} = m_2 r^c_2 l\cos(\theta^c_{c2} - \beta)\dot{\theta}^+_1 + I^c_{O2}\dot{\theta}^+_2}$$

Solution

From the system of equations

$$(h^-_{O1^c})_{system} = (h^+_{O1^c})_{system} \quad |\iff| \quad B_1\dot{\theta}^-_1 + B_2\dot{\theta}^-_2 = A_1\dot{\theta}^+_1 + A_2\dot{\theta}^+_2 \quad \div A_2$$
$$(h^-_{O2^c})_{body2} = (h^+_{O2^c})_{body2} \quad |\iff| \quad B_3\dot{\theta}^-_1 + B_4\dot{\theta}^-_2 = A_3\dot{\theta}^+_1 + A_4\dot{\theta}^+_2 \quad \div A_4$$

And by subtraction:

$$(\frac{B_1}{A_2} - \frac{B_3}{A_4})\dot{\theta}^-_1 + (\frac{B_2}{A_2} - \frac{B_4}{A_4})\dot{\theta}^-_2 = (\frac{A_1}{A_2} - \frac{A_3}{A_4})\dot{\theta}^+_1 \iff \dot{\theta}^+_1 = \frac{A_4B_1 - A_2B_3}{A_1A_4 - A_2A_3}\dot{\theta}^-_1 + \frac{A_4B_2 - A_2B_4}{A_1A_4 - A_2A_3}\dot{\theta}^-_2$$

$$\Rightarrow \quad \boldsymbol{\dot{\theta}^+_1 = A\dot{\theta}^-_1 + B\dot{\theta}^-_2}$$

Similarly but the 2 equations are divided by A_1 and A_3 respectively:

$$\dot{\theta}^+_2 = \frac{A_3B_1 - A_1B_3}{A_2A_3 - A_1A_4}\dot{\theta}^-_1 + \frac{A_3B_2 - A_1B_4}{A_2A_3 - A_1A_4}\dot{\theta}^-_2 \quad \Rightarrow \quad \boldsymbol{\dot{\theta}^+_2 = C\dot{\theta}^-_1 + D\dot{\theta}^-_2}$$

in which

$$A = \frac{A_4B_1 - A_2B_3}{A_1A_4 - A_2A_3} \quad , \quad B = \frac{A_4B_2 - A_2B_4}{A_1A_4 - A_2A_3}$$

$$C = \frac{A_3B_1 - A_1B_3}{A_2A_3 - A_1A_4} \quad , \quad D = \frac{A_3B_2 - A_1B_4}{A_2A_3 - A_1A_4}$$

where

$$A_1 = I_{O1} + m_2 l d^c \cos(\zeta^c - \beta) \quad , \qquad B_1 = I_{O1} + m_2 l' d^c \cos(\zeta^c - \beta')$$

$$A_2 = I_{G2} + m_2 r_2^c d^c \cos(\zeta^c - \theta_{c2}^c) \quad , \qquad B_2 = I_{G2} + m_2 r_2^n d^c \cos(\zeta^c + \theta_{c2}^n)$$

$$A_3 = m_2 r_2^c l \cos(\theta_{c2}^c - \beta) \quad , \qquad B_3 = m_2 r_2^c l' \cos(\theta_{c2}^c - \beta')$$

$$A_4 = I_{O2}^c \quad , \qquad B_4 = I_{G2} + m_2 r_2^c r_2^n \cos(\theta_{c2}^c + \theta_{c2}^n)$$

If $\dot{\theta}_2^+ < \dot{\theta}_1^+$ then transition to 3a occurs with $\dot{\theta}_2^+ = \dot{\theta}_1^+$

Mode 2a → 3a

About $O1^c$ before impact (2a)

The angular momentum of the system is the same as previously:

System

$$\boldsymbol{(h_{O1^c}^-)_{system} = \{I_{O1} + m_2 l' d^c \cos(\zeta^c - \beta')\}\dot{\theta}_1^- + \{I_{G2} + m_2 r_2^n d^c \cos(\zeta^c + \theta_{c2}^n)\}\dot{\theta}_2^-}$$

About $O1^c$ after impact (3a)

System

$$\boldsymbol{(h_{O1^c}^+)_{system} = I_O^c \dot{\theta}_1^+}$$

Solution

From equation

$$(h_{O1^c}^-)_{system} = (h_{O1^c}^+)_{system} \iff B_1\dot{\theta}_1^- + B_2\dot{\theta}_2^- = I_O^c\dot{\theta}_1^+$$

$$\dot{\theta}_1^+ = \dot{\theta}_2^+ = \frac{I_{O1} + m_2 l' d^c \cos(\zeta^c - \beta')}{I_O^c}\dot{\theta}_1^- + \frac{I_{G2} + m_2 r_2^n d^c \cos(\zeta^c + \theta_{c2}^n)}{I_O^c}\dot{\theta}_2^-$$

$$\Rightarrow \quad \boldsymbol{\dot{\theta}_1^+ = \dot{\theta}_2^+ = \frac{B_1}{I_O^c}\dot{\theta}_1^- + \frac{B_2}{I_O^c}\dot{\theta}_2^-}$$

3.4 From mode 2b

Mode 2b → 1a

From 'symmetry' of modes, the same analysis (2a to 1b) also holds with the opposite sign for angle θ_2 and opposite c,n attributes.

$$\dot{\boldsymbol{\theta}}_{\mathbf{1}}^{+} = \boldsymbol{A}\dot{\boldsymbol{\theta}}_{\mathbf{1}}^{-}$$
$$\dot{\boldsymbol{\theta}}_{\mathbf{2}}^{+} = \boldsymbol{B}\dot{\boldsymbol{\theta}}_{\mathbf{1}}^{-} + \dot{\boldsymbol{\theta}}_{\mathbf{2}}^{-}$$

in which

$$A = \frac{A_4 B_1 - A_2 B_2}{A_1 A_4 - A_2 A_3} \quad , \quad B = \frac{B_1}{A_2} - A\frac{A_1}{A_2}$$

where

$$A_1 = I_{O1} + m_2 l[r_2^c \cos(-\theta_2 + \theta_{c2}^c - \beta) + 2h_1 \cos(\beta) + \xi \sin(\beta)]$$

$$A_2 = I_{O2}^c + m_2 r_2^c[2h_1 \cos(-\theta_2 + \theta_{c2}^c) + \xi \sin(-\theta_2 + \theta_{c2}^c)]$$

$$A_3 = m_2 r_2^c l \cos(-\theta_2 + \theta_{c2}^c - \beta)$$

$$A_4 = I_{O2}^c$$

$$B_1 = I_{O1} - 2m_1 b_1^2 + m_2 l'[r_2^c \cos(-\theta_2 + \theta_{c2}^c + \beta') + 2h_1 \cos(\beta') - \xi \sin(\beta')]$$

$$B_2 = m_2 r_2^c l' \cos(-\theta_2 + \theta_{c2}^c + \beta')$$

Mode 2b → 1b

From 'symmetry' of modes, the same analysis (2a to 1a) also holds with the opposite sign for angle θ_2 and opposite c,n attributes.

$$\dot{\boldsymbol{\theta}}_{\mathbf{1}}^{+} = \boldsymbol{A}\dot{\boldsymbol{\theta}}_{\mathbf{1}}^{-} + \boldsymbol{B}\dot{\boldsymbol{\theta}}_{\mathbf{2}}^{-}$$
$$\dot{\boldsymbol{\theta}}_{\mathbf{2}}^{+} = \boldsymbol{C}\dot{\boldsymbol{\theta}}_{\mathbf{1}}^{-} + \boldsymbol{D}\dot{\boldsymbol{\theta}}_{\mathbf{2}}^{-}$$

in which

$$A = \frac{A_4 B_1 - A_2 B_3}{A_1 A_4 - A_2 A_3} \quad , \quad B = \frac{A_4 B_2 - A_2 B_4}{A_1 A_4 - A_2 A_3}$$

$$C = \frac{A_3 B_1 - A_1 B_3}{A_2 A_3 - A_1 A_4} \quad , \quad D = \frac{A_3 B_2 - A_1 B_4}{A_2 A_3 - A_1 A_4}$$

where

$$A_1 = I_{O1} + m_2 l d^n \cos(\zeta^n - \beta) \quad , \qquad B_1 = I_{O1} + m_2 l' d^n \cos(\zeta^n - \beta')$$

$$A_2 = I_{G2} + m_2 r_2^n d^n \cos(\zeta^n - \theta_{c2}^n) \quad , \qquad B_2 = I_{G2} + m_2 r_2^c d^n \cos(\zeta^n + \theta_{c2}^c)$$

$$A_3 = m_2 r_2^n l \cos(\theta_{c2}^n - \beta) \quad , \qquad B_3 = m_2 r_2^n l' \cos(\theta_{c2}^n - \beta')$$

$$A_4 = I_{O2}^n \quad , \qquad B_4 = I_{G2} + m_2 r_2^c r_2^n \cos(\theta_{c2}^c + \theta_{c2}^n)$$

If $\underline{\dot{\theta}_2^+ > \dot{\theta}_1^+}$ then transition to 3b occurs with $\dot{\theta}_2^+ = \dot{\theta}_1^+$

Mode 2b → 3b

$$\boldsymbol{\dot{\theta}_1^+ = \dot{\theta}_2^+ = \frac{B_1}{I_O^n}\dot{\theta}_1^- + \frac{B_2}{I_O^n}\dot{\theta}_2^-}$$

3.5 From mode 3a

Mode 3a → 1b

About $O1^n$ before impact (3a)

The angular momentum of the system is the same as the one from transition 1a to 2b with $\theta_2 = 0$ and $\dot{\theta}_1^- = \dot{\theta}_2^-$

$$(h_{O1^n}^-)_{system} = \{I_{O1} - 2m_1 b_1^2 + m_2 l[r_2^c \cos(-\theta_{c2}^c + \beta) + 2h_1 \cos(\beta) - \xi' \sin(\beta)] + I_{O2}^c + m_2 r_2^c [2h_1 \cos(-\theta_{c2}^c) + \xi' \sin(-\theta_{c2}^c)]\}\dot{\theta}_1^-$$

$$\boldsymbol{(h_{O1^n}^-)_{system} = (B_1 + A_2)^{1a\to 2b}\dot{\theta}_1^- \quad \to \quad B_1 = (B_1 + A_2)^{1a\to 2b}}$$

About $O1^n$ after impact (1b)

The angular momentum of the system is the same as the one from transition 2a to 1b with $\theta_2 = 0$

$$(h_{O1^n}^+)_{system} = \{I_{O1} + m_2 l[r_2^n \cos(\theta_{c2}^n - \beta) + 2h_1 \cos(\beta) + \xi \sin(\beta)]\}\dot{\theta}_1^+ + \{I_{O2}^n + m_2 r_2^n [2h_1 \cos(\theta_{c2}^n) + \xi \sin(\theta_{c2}^n)]\}\dot{\theta}_2^+$$

$$\boldsymbol{(h_{O1^n}^+)_{system} = A_1^{2a\to 1b}\dot{\theta}_1^+ + A_2^{2a\to 1b}\dot{\theta}_2^+ \quad \to \quad A_1 = A_1^{2a\to 1b} \quad , \quad A_2 = A_2^{2a\to 1b}}$$

About $O2^n$ before impact (3a)

Body 2

The angular momentum of body2 is the same as the one from transition 1a to 2a with $\dot{\theta}_1^- = \dot{\theta}_2^-$

$$(h_{O2^n}^-)_{body2} = \{m_2 r_2^n l \cos(\theta_{c2}^n + \beta) + I_{G2} + m_2 r_2^c r_2^n \cos(theta_{c2}^c + theta_{c2}^n)\}\dot{\theta}_2^-$$

$$\boldsymbol{(h_{O2^n}^-)_{system} = (B_3 + B_4)^{1a\to 2a}\dot{\theta}_1^- \quad \to \quad B_2 = (B_3 + B_4)^{1a\to 2a}}$$

About $O2^n$ after impact (1b)

Body 2

The angular momentum of body2 is the same as the one from transition 2a to 1b with $\theta_2 = 0$

$$(h_{O2^n}^+)_{body2} = [m_2 r_2^c l \cos(\theta_{c2}^c - \beta)]\dot{\theta}_1^+ + I_{O2}^c \dot{\theta}_2^+$$

$$\boldsymbol{(h_{O2^n}^+)_{system} = A_3^{2a\to 1b}\dot{\theta}_1^+ + A_4^{2a\to 1b}\dot{\theta}_2^+ \quad \to \quad A_3 = A_3^{2a\to 1b} \quad , \quad A_4 = A_4^{2a\to 1b}}$$

Solution

From the system of equations

$$\begin{aligned} (h_{O1^n}^-)_{system} = (h_{O1^n}^+)_{system} \quad &|\iff| \quad B_1\dot{\theta}_1^- = A_1\dot{\theta}_1^+ + A_2\dot{\theta}_2^+ \quad \div A_2 \\ (h_{O2^n}^-)_{body2} = (h_{O2^n}^+)_{body2} \quad &|\iff| \quad B_2\dot{\theta}_1^- = A_3\dot{\theta}_1^+ + A_4\dot{\theta}_2^+ \quad \div A_4 \end{aligned}$$

And by subtraction:

$$(\frac{B_1}{A_2} - \frac{B_2}{A_4})\dot{\theta}_1^- = (\frac{A_1}{A_2} - \frac{A_3}{A_4})\dot{\theta}_1^+ \iff \dot{\theta}_1^+ = \frac{A_4 B_1 - A_2 B_2}{A_1 A_4 - A_2 A_3}\dot{\theta}_1^- \quad \Rightarrow \quad \boldsymbol{\dot{\theta}_1^+ = A\dot{\theta}_1^-}$$

From first equation

$$\frac{B_1}{A_2}\dot{\theta}_1^- = \frac{A_1}{A_2}A\dot{\theta}_1^- + \dot{\theta}_2^+ \iff \dot{\theta}_2^+ = (\frac{B_1}{A_2} - A\frac{A_1}{A_2})\dot{\theta}_1^- \quad \Rightarrow \quad \boldsymbol{\dot{\theta}_2^+ = B\dot{\theta}_1^-}$$

in which

$$A = \frac{A_4 B_1 - A_2 B_2}{A_1 A_4 - A_2 A_3} \quad , \quad B = \frac{B_1}{A_2} - A\frac{A_1}{A_2}$$

If $\dot{\theta}_2^+ > \dot{\theta}_1^+$ then transition to 3b occurs with $\dot{\theta}_2^+ = \dot{\theta}_1^+$

Mode 3a → 3b

About $O1^n$ before impact (3a)

The angular momentum of the system is the same as previously:

System

$$\boldsymbol{(h_{O1^c}^-)_{system} = B_1\dot{\theta}_1^-}$$

About $O1^n$ after impact (3b)

System

$$\boldsymbol{(h_{O1^c}^+)_{system} = I_O^n\dot{\theta}_1^+}$$

Solution

From equation

$$(h_{O1^c}^-)_{system} = (h_{O1^c}^+)_{system} \iff B_1\dot{\theta}_1^- = I_O^n\dot{\theta}_1^+$$

$$\Rightarrow \quad \boldsymbol{\dot{\theta}_1^+ = \dot{\theta}_2^+ = \frac{B_1}{I_O^n}\dot{\theta}_1^-}$$

3.6 From mode 3b

Mode 3b → 1a

From 'symmetry' of modes, the same analysis (3a to 1b) also holds with the opposite c,n attributes.

$$\dot{\theta}_1^+ = A\dot{\theta}_1^-$$
$$\dot{\theta}_2^+ = B\dot{\theta}_1^-$$

where:

$$A_1 = A_1^{2b \to 1a} \quad , \quad A_2 = A_2^{2b \to 1a}$$
$$A_3 = A_3^{2b \to 1a} \quad , \quad A_4 = A_4^{2b \to 1a}$$
$$B_1 = (B_1 + A_2)^{1b \to 2a}$$
$$B_3 = (B_3 + B_4)^{1b \to 2b}$$

If $\dot{\theta}_2^+ < \dot{\theta}_1^+$ then transition to 3a occurs with $\dot{\theta}_2^+ = \dot{\theta}_1^+$

Mode 3b → 3a

From 'symmetry' of modes, the same analysis (3a to 3b) also holds with the opposite c,n attributes.

$$\dot{\theta}_1^+ = \dot{\theta}_2^+ = \frac{B_1}{I_O^c}\dot{\theta}_1^-$$

3.7 From mode 4a

Mode 4a → 1b

About $O1^n$ before impact (4a)

The angular momentum of the system is the same as the one from transition 1a to 2b with $\theta_2 = 0$ and $\dot{\theta}_1^- = 0$

$$(h_{O1^n}^-)_{system} = \{I_{O2}^c + m_2 r_2^c[2h_1 \cos(-\theta_{c2}^c) + \xi' \sin(-\theta_{c2}^c)]\}\dot{\theta}_2^-$$

$$\boldsymbol{(h_{O1^n}^-)_{system} = A_2^{1a\to 2b}\dot{\theta}_2^- \quad \to \quad B_1 = A_2^{1a\to 2b}}$$

About $O1^n$ after impact (1b)

The angular momentum of the system is the same as the one from transition 2a to 1b with $\theta_2 = 0$

$$(h_{O1^n}^+)_{system} = \{I_{O1} + m_2 l[r_2^n \cos(\theta_{c2}^n - \beta) + 2h_1 \cos(\beta) + \xi \sin(\beta)]\}\dot{\theta}_1^+ + \{I_{O2}^n + m_2 r_2^n[2h_1 \cos(\theta_{c2}^n) + \xi \sin(\theta_{c2}^n)]\}\dot{\theta}_2^+$$

$$\boldsymbol{(h_{O1^n}^+)_{system} = A_1^{2a\to 1b}\dot{\theta}_1^+ + A_2^{2a\to 1b}\dot{\theta}_2^+ \quad \to \quad A_1 = A_1^{2a\to 1b} \quad , \quad A_2 = A_2^{2a\to 1b}}$$

About $O2^n$ before impact (4a)

Body 2

The angular momentum of body2 is the same as the one from transition 1a to 2a with $\dot{\theta}_1^- = 0$

$$(h_{O2^n}^-)_{body2} = \{I_{G2} + m_2 r_2^c r_2^n \cos(theta_{c2}^c + theta_{c2}^n)\}\dot{\theta}_2^-$$

$$\boldsymbol{(h_{O2^n}^-)_{body2} = B_4^{1a\to 2a}\dot{\theta}_2^- \quad \to \quad B_2 = B_4^{1a\to 2a}}$$

About $O2^n$ after impact (1b)

Body 2

The angular momentum of body2 is the same as the one from transition 2a to 1b with $\theta_2 = 0$

$$(h_{O2^n}^+)_{body2} = [m_2 r_2^n l \cos(\theta_{c2}^n - \beta)]\dot{\theta}_1^+ + I_{O2}^n\dot{\theta}_2^+$$

$$\boldsymbol{(h_{O2^n}^+)_{body2} = A_3^{2a\to 1b}\dot{\theta}_1^+ + A_4^{2a\to 1b}\dot{\theta}_2^+ \quad \to \quad A_3 = A_3^{2a\to 1b} \quad , \quad A_4 = A_4^{2a\to 1b}}$$

Solution

From the system of equations

$$\begin{aligned}(h_{O1^n}^-)_{system} = (h_{O1^n}^+)_{system} \quad &|\iff| \quad B_1\dot{\theta}_2^- = A_1\dot{\theta}_1^+ + A_2\dot{\theta}_2^+ \quad \div A_2 \\ (h_{O2^n}^-)_{body2} = (h_{O2^n}^+)_{body2} \quad &|\iff| \quad B_2\dot{\theta}_2^- = A_3\dot{\theta}_1^+ + A_4\dot{\theta}_2^+ \quad \div A_4\end{aligned}$$

And by subtraction:

$$(\frac{B_1}{A_2} - \frac{B_2}{A_4})\dot{\theta}_2^- = (\frac{A_1}{A_2} - \frac{A_3}{A_4})\dot{\theta}_1^+ \iff \dot{\theta}_1^+ = \frac{A_4 B_1 - A_2 B_2}{A_1 A_4 - A_2 A_3}\dot{\theta}_2^- \Rightarrow \boldsymbol{\dot{\theta}_1^+ = A\dot{\theta}_2^-}$$

From first equation

$$\frac{B_1}{A_2}\dot{\theta}_2^- = \frac{A_1}{A_2}A\dot{\theta}_2^- + \dot{\theta}_2^+ \iff \dot{\theta}_2^+ = (\frac{B_1}{A_2} - A\frac{A_1}{A_2})\dot{\theta}_2^- \Rightarrow \boldsymbol{\dot{\theta}_2^+ = B\dot{\theta}_2^-}$$

in which

$$A = \frac{A_4 B_1 - A_2 B_2}{A_1 A_4 - A_2 A_3} \quad , \quad B = \frac{B_1}{A_2} - A\frac{A_1}{A_2}$$

If $\dot{\theta}_1^+ > 0$ then transition to 4b occurs

Mode 4a → 4b

About $O2^n$ before impact (4a)
The angular momentum of body 2 is the same as previously:
Body 2

$$\boldsymbol{(h_{O2^n}^-)_{body2} = B_2\dot{\theta}_2^-}$$

About $O2^n$ after impact (4b)
System

$$\boldsymbol{(h_{O2^n}^+)_{body2} = I_{O2}^n\dot{\theta}_2^+}$$

Solution
From equation

$$(h_{O2^n}^-)_{body2} = (h_{O2^n}^+)_{body2} \iff B_2\dot{\theta}_2^- = I_{O2}^n\dot{\theta}_2^+$$

$$\Rightarrow \boldsymbol{\dot{\theta}_2^+ = \frac{B_2}{I_{O2}^n}\dot{\theta}_2^-}$$

3.8 From mode 4b

Mode 4b → 1a

From 'symmetry' of modes, the same analysis (4a to 1b) also holds with the opposite c,n attributes.

$$\dot{\theta}_1^+ = A\dot{\theta}_2^-$$
$$\dot{\theta}_2^+ = B\dot{\theta}_2^-$$

where:

$$A_1 = A_1^{2b\to 1a} \quad , \quad A_2 = A_2^{2b\to 1a}$$
$$A_3 = A_3^{2b\to 1a} \quad , \quad A_4 = A_4^{2b\to 1a}$$
$$B_1 = A_2^{1b\to 2a}$$
$$B_2 = B_4^{1b\to 2b}$$

If $\dot{\theta}_1^+ < 0$ then transition to 4a occurs

Mode 4b → 4a

From 'symmetry' of modes, the same analysis (4a to 4b) also holds with the opposite c,n attributes.

$$\dot{\theta}_2^+ = \frac{B_2}{I_{O2}^c}\dot{\theta}_2^-$$

Chapter 4

Numerical Solution

4.1 Structure of the Program

The equations of motion for all modes, which were derived in chapter 1, are nonlinear and in order to approximate the solution, a numerical method should be applied. For that purpose, a computer program was developed in Matlab. The program takes account large angles of rotation, impact and any kind of base excitation.

Integration of the equations of motion was achieved using standard ODE solver which is provided by Matlab. Specifically, ODE45 was selected for its ability to solve differential equations which involve a variable mass matrix dependent on time $M(t, y)y' = f(t, y)$. In addition, it can locate where the impact occurs with accuracy of $\pm 10^{-18} rad$. The time step is variable and defined by the solver, so as to not lose any data from the record/excitation, a maximum step size is defined by the user. In this analysis a max step size of dt/2 is used (dt=step size of the record). The solver is based on an explicit Runge-Kutta (4,5) formula, the Dormand-Prince pair. It is a single-step solver – in computing $y(t_n)$, it needs only the solution at the immediately preceding time point, $y(t_{n-1})$ [15].

Since the problem is solved numerically, some tolerances have to be accepted. For the angles of rotation, the tolerance, for $\theta_1 = 0$ or for $\theta_2 - \theta_1 = 0$ is, as mentioned, $\pm 10^{-18} rad$. For the angular velocities the tolerance is $\pm 10^{-8} rad/sec$ ($\dot{\theta}_1 = 0, \dot{\theta}_2 - \dot{\theta}_1 = 0$) and applies after every impact.

4.2 Applications

4.2.1 System Configuration

In the following examples, a system of a statue and a pedestal was considered.

The statue (figure 4.1) is defined as:

- symmetric with $b_2^c = b_2^n = 0.109m$
- asymmetric with $b_2^c = 0.09255m \quad and \quad b_2^n = 0.12545m$

In both cases $h_2 = 0.38604m, \quad m_2 = 0.11803Mgr, \quad I_{G2} = 0.0059383Mgrm^2$.

The pedestal is defined with $2b_1 = 0.45m, \quad 2h_1 = 1.01m$.

The bodies were assumed homogeneous of equal material densities $\rho_1 = \rho_2 = 2.5Mgr/m^3$.

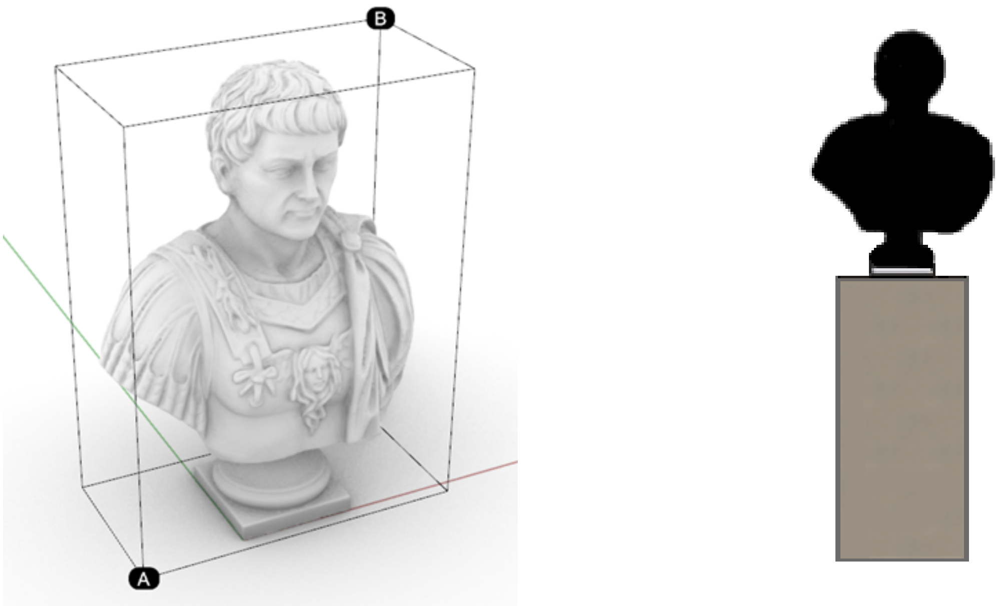

Figure 4.1: Bust of Roman Emperor Traiano.

4.2.2 Free Vibration Response

Free "vibration" or free rocking of the system is initiated by rotating the system through an initial angular displacement, for example $\theta_{01} = \theta_{02} = \theta_0$, releasing it and letting the system rock back and forth about alternative corners until the motion decays to rest.

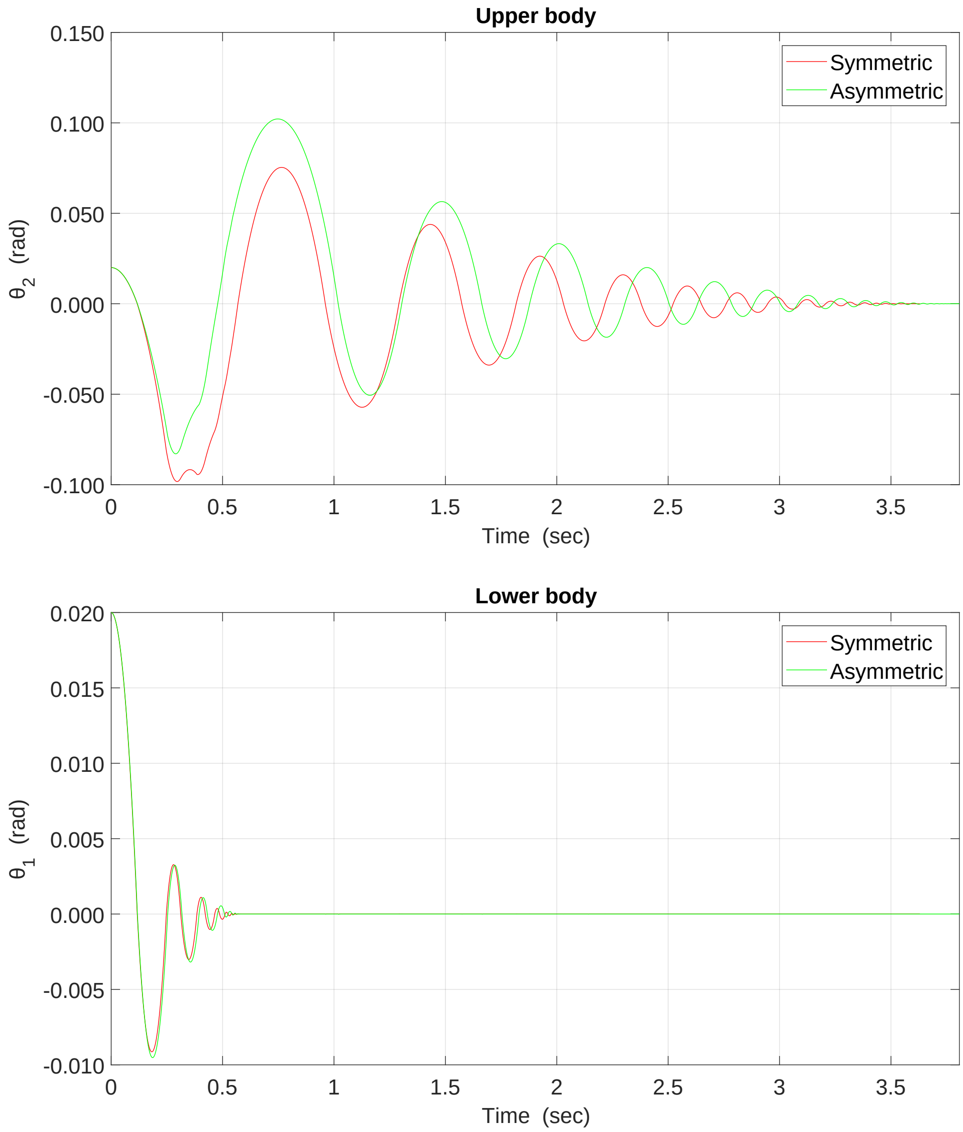

Figure 4.2: Comparison between symmetric and asymmetric upper body with initial tilt equal to 0.02 rad.

Presented in Figure 4.2 are results for $\theta_{01} = \theta_{02} = 0.02rad$ and zero initial angular velocities. It can be observed that the angles of rotation of the two bodies are decaying in an oscillatory manner with that of lower body decreasing at faster rate. The maximum angle θ_2 exceeds the initial one by a large margin and notably for the asymmetric case.

4.2.3 Response to Pulses

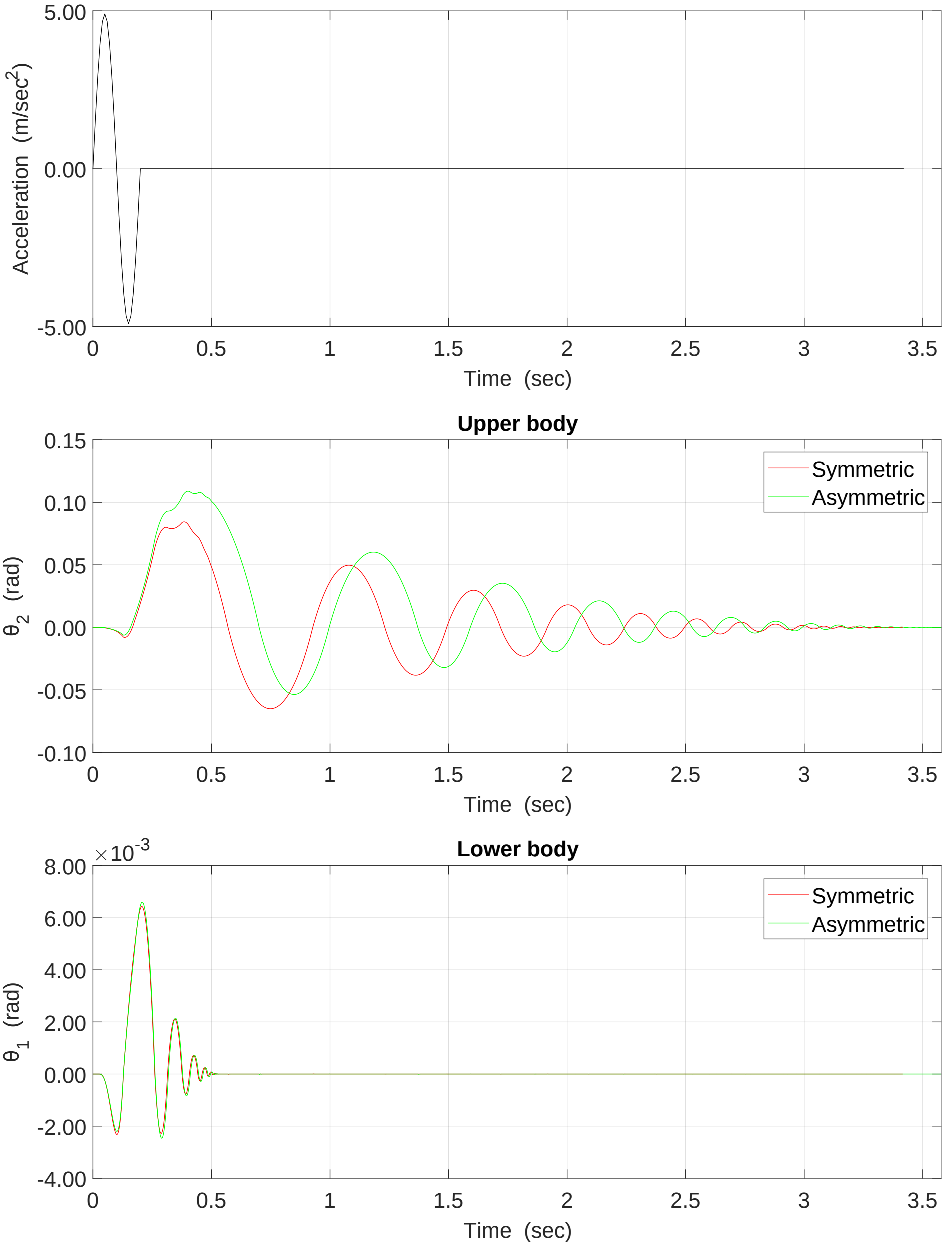

Figure 4.3: Sine pulse with amplitude 0.5g and period 0.2sec.

In Figure 4.3 a sine pulse is applied on the system with one period T=0.2sec and amplitude A=0.5g. As it can be seen, the lower body is rocking in smaller angles and there is not a significant difference between symmetric and asymmetric case, relatively to the upper one. Again the asymmetric one seems to rocking in larger positive angles by the right side.

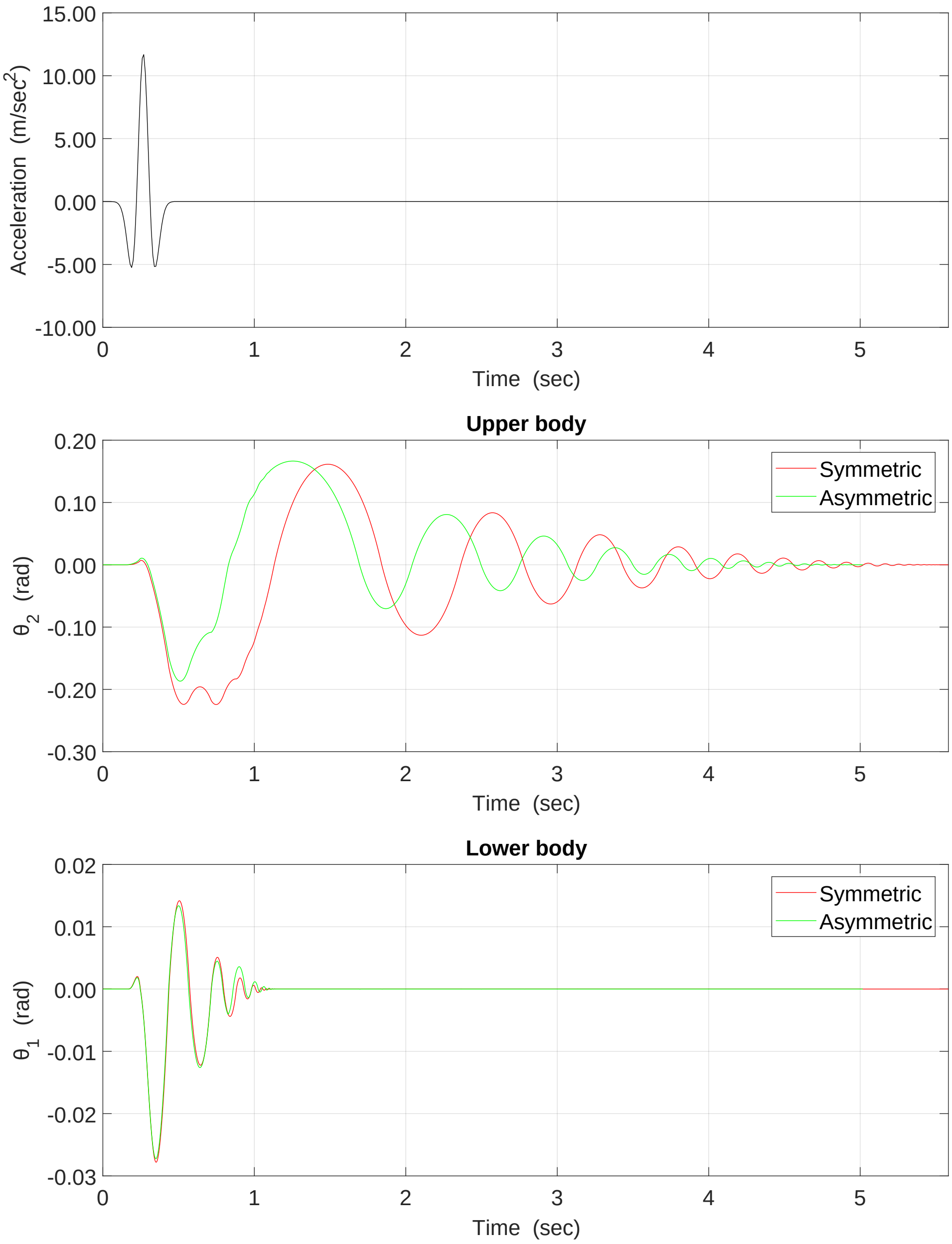

Figure 4.4: Ricker pulse with amplitude 1.2g and peak frequency 5Hz.

In Figure 4.4 a ricker pulse is applied on the system with a peak frequency of f=5Hz and amplitude A=1.2g. Unlike the sine pulse, the symmetric one seems to rocking in larger overall angles and for a little longer. That can be concluded due to the larger positive acceleration which forces the system to rock in the left side for the most part ($b^n_{\mathbf{2symmetric}} < b^n_{\mathbf{2asymmetric}}$). Again not significant difference for the lower one.

4.2.4 Response to Seismic Excitation

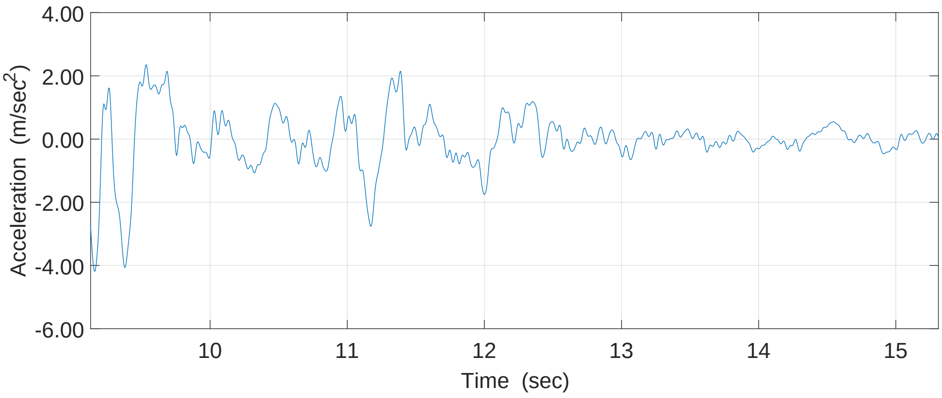

Figure 4.5: Accelerogram from the Syntagma, PGA=4.19m/s^2, earthquake.

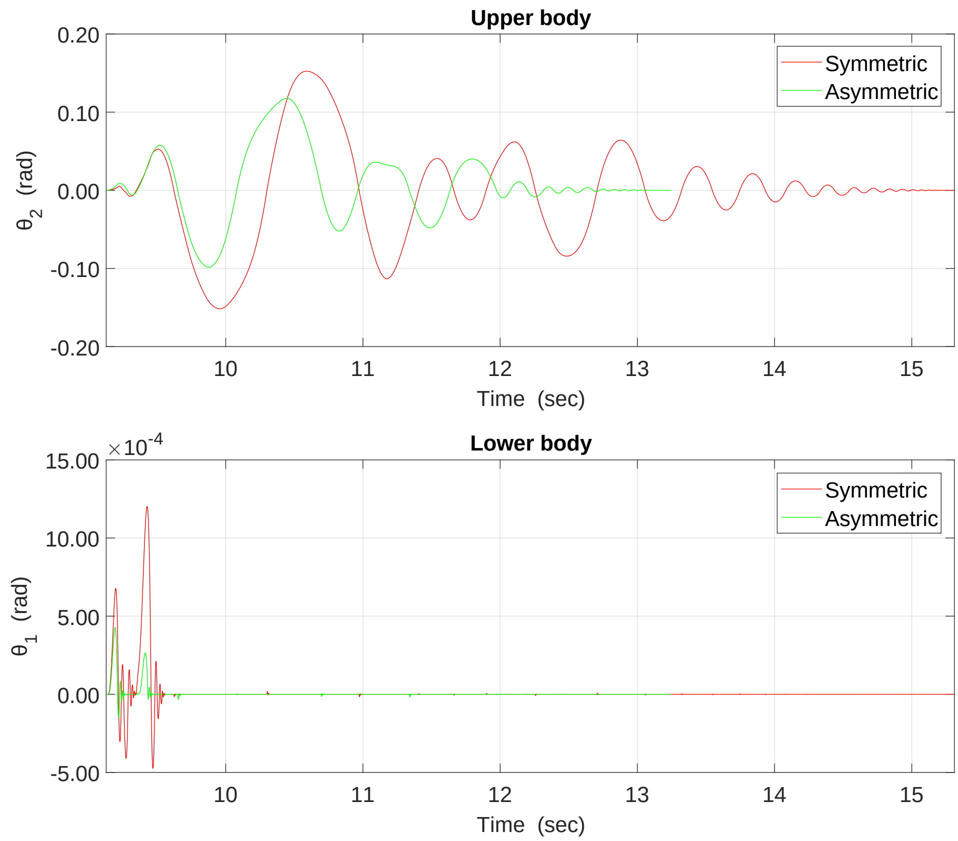

Figure 4.6: Response of the system to the Syntagma seismic record.

In Figure 4.6 an earthquake Figure 4.5 is applied on the system and is focused when motion initiates and ends. In that response, it is noticed that the symmetric approach is more unfavorable in both bodies, as the angles are larger and the rocking duration is at least 2 seconds greater. In contrast with the ricker pulse, the complexity of the seismic excitation cannot let the quantitative results to be predicted. Finally, the system comes to rest before the end of the earthquake record.

4.2.5 Comparison with Experiments

The experimental results were acquired by placing a pedestal on the seismic table and a statue on top of the pedestal with the exact same parameters (asymmetric case) that are examined here and then a seismic excitation was applied on the table.

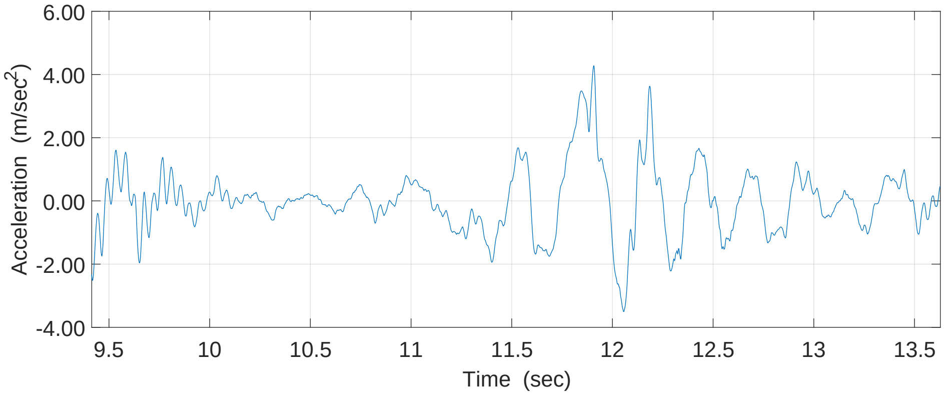

Figure 4.7: Accelerogram from the Emilia, PGA=4.28m/s^2, earthquake.

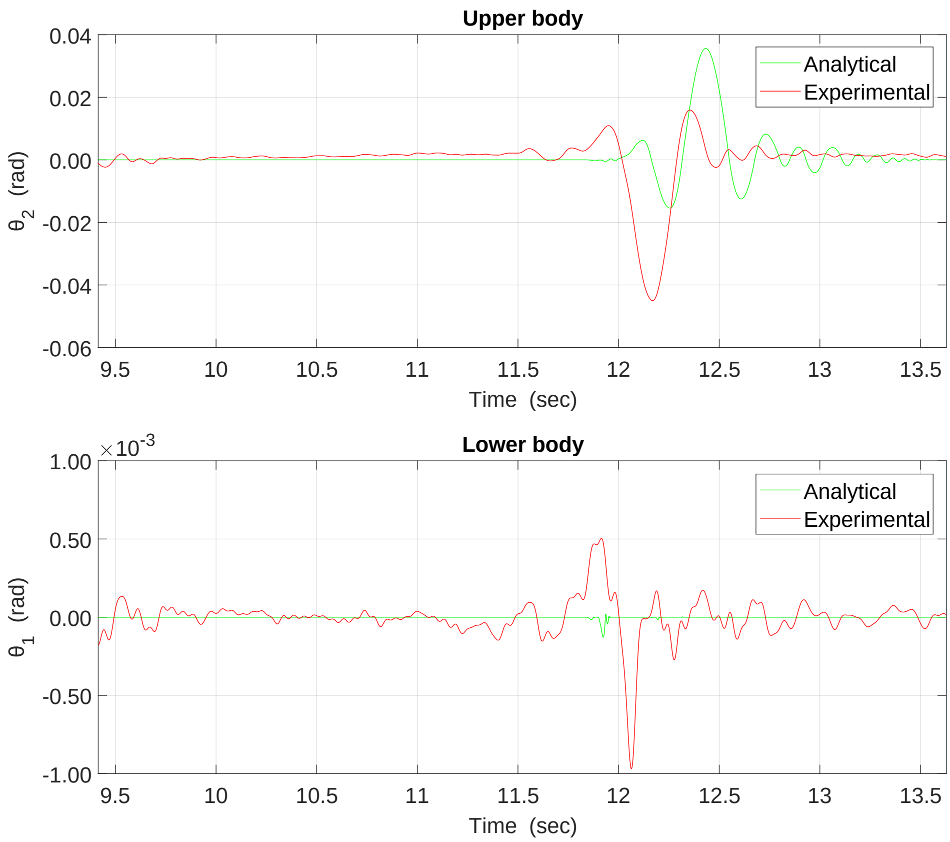

Figure 4.8: Response to the Emilia record (PGA=4.28m/s^2) compared with experimental results.

In Figure 4.8, an earthquake Figure 4.7 is applied on the system and is focused when motion initiates and ends. It can be observed that the analytical against the experimental results, do not differ significantly, given that the angles are relatively small and the analytical results are based on a 2D approach which does not take into account nor the sliding and uplift, neither the proper restitution factor during the impact.

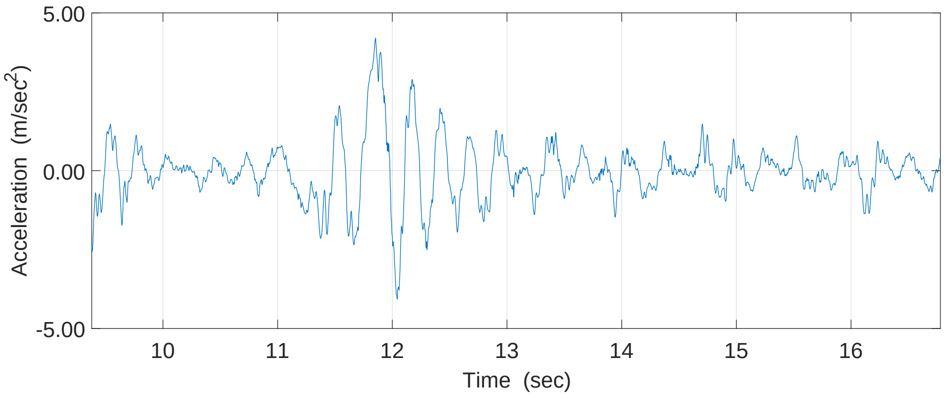

Figure 4.9: Accelerogram from the Emilia, PGA=4.20m/s^2, earthquake.

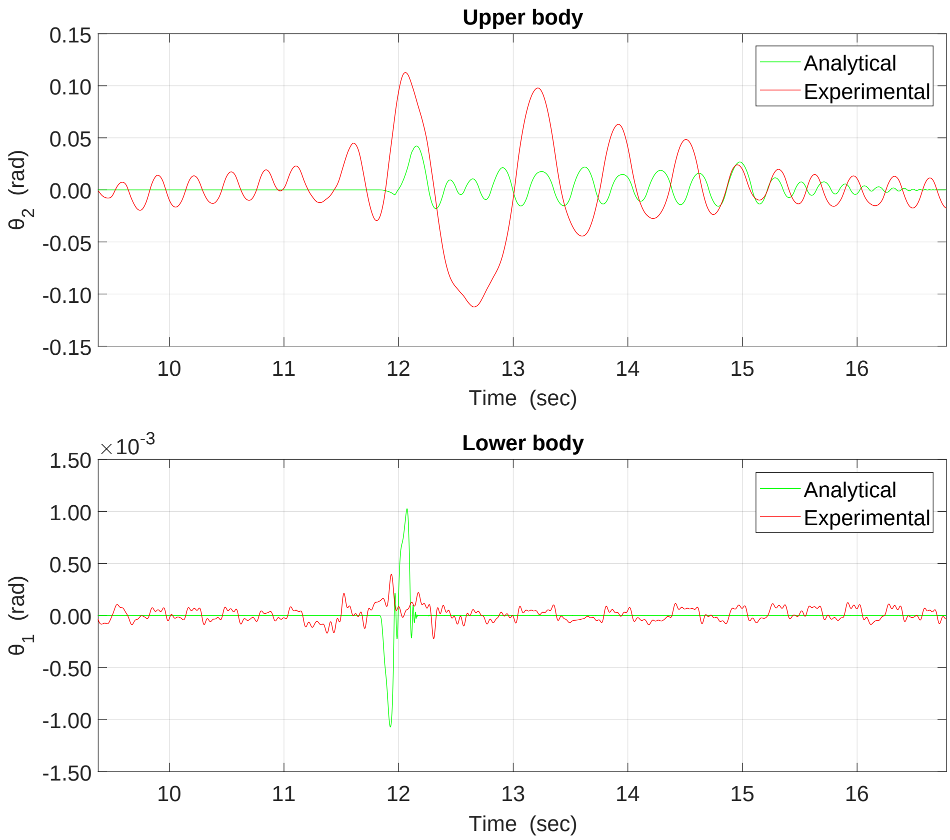

Figure 4.10: Response to the Emilia record (PGA=4.20m/s^2) compared with experimental results.

In Figure 4.10 the same record with a slightly smaller PGA, Figure 4.9, was applied on the system and is focused when motion initiates and ends. Counterintuitively, compared with Figure 4.8, it is noticed that the upper-body's response, consists of greater angles and duration in both cases (analytical/experimental). The same holds for the lower one but only in analytical results. Additionally, for larger overall values, it seems that the difference between analytical and experimental results expands, as more nonlinear phenomena take place.

Chapter 5

Conclusions

The dynamic behavior of systems consisting of a rigid pedestal and an artefact, has been discussed. Being placed on the top of the pedestal, the artefact was considered either symmetric or asymmetric. Sliding and possible jumps of the bodies are neglected and four possible modes of pure rocking motion are assumed. Particularly for the asymmetric upper body, each mode is divided into two subcases, according to the sign of the angles of rotation and according to the critical or non-critical geometrical parameters, resulting to $2X8$ equations of motions.

This analysis aims to examine the performance of equivalent solutions reducing the required computational cost, compared to a detailed Finite Element solution. Furthermore, concerning the energy dissipation and considering that it takes place when impacts occur, it is more efficient and safer to introduce it using "home-made" codes than viscous/continuous dampers that FE software use.

It should be emphasized that the behavior of the system is highly nonlinear and small changes in input or geometry may yield large changes in system response. Therefore, the quantitative results obtained in the application examples cannot be generalized.

Due to energy loss, sudden changes in angular velocities are observed in the results, while for the adequacy of the results, energy redistribution should be further examined in detail by calculating the kinetic energy of each block before and after impact.

In case of asymmetric systems, the sign of the record is also critical. Therefore, the two-block problem is multiparametric and further investigation is required.

Even if the rocking equations are taking into account large rotations, in this case more phenomena such as sliding, are likely to take place and should also be evaluated as for example in Kounadis (2018)[9] and Andreaus & Casini (1998)[6].

Nevertheless, the analysis is accurate enough to determine the possibility of overturn as long as the bodies and foundation are rigid and the contact between bodies and ground provides adequate friction. As it was proved, the effort to include the asymmetric case in the two-block problem was necessary in order to get more accurate results for a real problem which is about the preservation of archaeological remains.

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