# NATIONAL TECHNICAL UNIVERSITY OF ATHENS SCHOOL OF NAVAL ARCHITECTURE AND MARINE ENGINEERING <br> <br> DIVISION OF MARINE STRUCTURES 

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Development of analytical models of mixed boundary value problems for the determination of the hydrodynamic loading on solid bodies, with a focus on violent slamming of free-surface flows

## Theodosis D. Tsaousis

Naval Architect and Marine Engineer, NTUA

Supervisor: I. K. Chatjigeorgiou, Professor NTUA

A thesis submitted for the degree of

> Doctor of Philosophy

Athens, September 2021

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## Dedicated to my family

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#### Abstract

Slamming phenomena are of major concern for Naval Architects and Marine Engineers, because they can cause devastating effects if ignored. The present thesis focuses on the investigation of the impact pressures and forces exerted on a rigid body, when it is hit by a mass of water, briefly and violently. Their enormous magnitude and very short duration make the nature of the impact loads completely different from those exerted by the regular gravitational waves. The arising boundary value problems (bvp) are of mixed type and are considered within the realm of analytical solutions in marine hydrodynamics. This thesis discusses the following two problems: i) the steep wave impact on a vertical, circular cylinder and ii) the breaking wave impact on a vertical impermeable wall.

The problem of a steep wave impact, i.e., a rectangular mass of water with a completely vertical wave front colliding with a cylinder violently, is considered first. This approach is taken in order to simulate the vertical wave front of a breaking wave (flip-through impact), which is considered as the worst scenario of impact. The bvp is of mixed type, given that a Neumann condition holds on the impacted part of the cylinder and a Dirichlet condition holds beyond it. The solution derived is based on the small-time approximation. This means that the impact is considered at the very early stages. The fundamental assumption of the analysis is that the instantaneous contact line is weakly dependent on the vertical coordinate. The problem is solved within the original Wagner approach in 3D. Incompressible, inviscid and irrotational flow is assumed, so that the solution sought can be determined with the aid of the velocity potential. The main results concern the instantaneous contact line and its evolution with time, the deformation of the wave front, the slamming force and the impulse exerted on the cylinder. The range of validity of the existing 2D theories of von Karman and Wagner when applied to real 3D geometries is discussed. To this end, the results derived are compared with those anticipated by the implementation of the existing 2D theories; significant differences are noticed. CFD results demonstrate the validity of the assumption of the almost vertical contact line.

The second part of this thesis deals with the breaking wave impact on a vertical impermeable wall. The scrutiny on this filed is indispensable, owing to the large energy dissipation which always accompanies the wave breaking. During the collision, a small air pocket is generated between the water and the wall. Hence, an idealized, but rational model of a slightly overturning breaking wave is assumed. The bvp is of mixed type, since a Neumann condition holds on the impacted part of the wall, while a Dirichlet condition holds on the part of the wall occupied by the air. The jet formed at the upper part of the air pocket is neglected initially. Nevertheless, it is discussed sufficiently, in the last part of Chapter 3. The analysis is performed in 2D. Once again, the small-time assumption is employed. Therefore, using the time as a small parameter, a perturbation technique is applied in order to fragment the governing hydrodynamic bvp in a sequence of subproblems. Firstly, the leading order problem is solved. The method applied, demands the solution to a system of dual trigonometrical series. Results are presented for the velocity potential and the free-surface elevation. Owing to the formation of a splash at the intersection point between the still water level and the wall,


a logarithmic singularity for the vertical velocity arises. To this end, the method of solution is enhanced with that by King and Needham (1993).

Subsequently, the higher order problem is considered. The bvp is reduced to a novel and very challenging one-dimensional Sturm-Liouville problem with mixed conditions. The boundary conditions are far more complicated when compared to the form they are encountered usually, since they involve infinite series. Relevant problems have not been investigated in the past, at least in the field of analytical hydrodynamics associated with slamming phenomena. Interesting corrections to the leading order problem were noticed.

The results of this PhD thesis highlighted the very complicated procedures occurring during water impact and enriched the existing literature of analytical solutions in slamming phenomena undoubtedly.

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#### Abstract

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## Chapter 1 Literature review

Slamming in marine applications can be encountered in several forms such as steep wave and breaking wave impact, wetdeck slamming, green water slamming, water entry and sloshing [Faltinsen et al. (2004), Greco et al. (2012), Swidan et al. (2016)]. Moreover, extreme impact conditions are encountered in dam-break flows, tsounamis and generally storm surges [Liang et al. (2016)]. In this dissertation we focus on the first two cases.

The problem of water entry, i.e., the violent penetration of the water free-surface by a body has already been studied since early 30s of the past century. The two pioneering studies, which have established the water entry theory, are those by von Karman (1929) and Wagner (1932). The first theory was motivated by the problem of the alighting of seaplanes. Calculations were made for a wedge of small and moderate deadrise angle penetrating the flat free-surface of the water. The limiting case of a flat-bottom body was also discussed. However, the deformation of the free-surface due to the impact was not taken into account by von Karman (1929). A few years later, Wagner (1932) expanded this theory by considering the pile-up of the water for a blunt body, with small deadrise angle. The main idea of his theory was the substitution of the wetted part of the body by a time expanding flat disk. Since the free-surface elevation is of the same order as the penetration depth, the free-surface boundary conditions and the body condition can be linearized and imposed on the undisturbed water surface.

Undoubtedly, the most widely examined body penetrating violently an initially calm, flat water surface is the wedge. The classical study of Dobrovol'skaya (1969) proposes a fully nonlinear solution for the self-similar flow concerning the vertical entry of a symmetric wedge, with constant speed. The free-surface elevation was calculated through the solution to a singular integral equation. Hughes (1972) provided a selfsimilar solution using the conformal mapping method. He used the Wagner function to map the entire fluid domain into an isosceles triangle. Given that the disturbance of the fluid is caused by the wetted part of the body only, he highlighted that each body can be considered as "finite", regardless its shape or dimensions outside the fluid. This fact has a direct influence in the calculation of the flow far from the body, as the body may be replaced by an equivalent body fully submerged. Watanabe (1986) obtained a 2D analytical solution for the wedge water entry problem using asymptotic techniques, by calculating an outer solution, an inner one and matching them. Howison et al. (1991) derived also a 2D asymptotic solution for the wedge entry problem with small deadrise angles. They examined how the air entrapped influences the dynamics of the impact by considering a flat-bottom wedge. It was also shown that the flow in the jet region is governed by the shallow-water equations. They questioned the relevance of their model to water exit problems too. A classical numerical study in 2D is by Zhao and Faltinsen (1993) for the water entry of a body with arbitrary cross-section, using a nonlinear boundary element method (BEM) and incorporating the jet formed in the intersection between the body and the water surface. They presented results for wedges with a wide range of deadrise angles and they derived an asymptotic solution for the small deadrise angles. Very recently, among the great number of experimental researches, the one by Duan et al. (2020) scrutinizes the influence of the most important parameters that affect the slamming pressure distribution on a wedge, i.e., the deadrise angle, the drop height
and the structural flexibility. Perhaps, the most noticeable outcome of this study is that the slamming pressure exhibits propagating features when the deadrise angle is greater than the critical value of $0.2^{\circ}$. Wang and Guedes Soares (2020) examined how the compressibility affects the dynamics of impact during the water entry problem of a wedge cylinder and they inferred that it can be ignored during the slamming stage of the water entry process.

The study of the wedge entry problems has led gradually to more complicated models over the years. For example, the entry of a perforated wedge has been studied by Molin and Korobkin (2001). The main parameters involved are the deadrise angle, the porosity ratio and a discharge coefficient. They highlighted that even a small increase in the porosity has a drastic reduction in the impact force. The problem of a porous/perforated wedge entry was also studied by Iafrati and Korobkin (2005), who considered an intermediate thin layer between the rigid, impermeable body and the freesurface which modifies the boundary conditions. They solved the bvp by introducing a modified velocity potential. Their results were based on the self-similar solution. Semenov and Wu (2016) applied an integral hodograph method to study the case of a liquid wedge hitting a wedge-shaped permeable body. They considered separately the impact on a i) porous body and ii) perforated body. In the first case the relationship between the velocity flow and the pressure is linear, whereas in the second case is quadratic. They verified that the maximum pressure decreases with the increasing permeability ratio, given that fluid particles pass through the body. This effect is more pronounced for small deadrise angles. Moreover, they concluded that the jet tip moves closer to the centerline, a fact that implies a decrease in the contact area.

Considerable effort has been put on the case of asymmetric wedges. Such a study was conducted numerically in 2D by Riccardi and Iafrati (2004). Their model was based on an initially floating wedge on a still liquid surface. Owing to the asymmetry, a singularity occurs in the apex which is removed by introducing viscous effects (as point vortices). Later, Semenov and Iafrati (2006) proposed a nonlinear model for the vertical water entry of asymmetric wedge. It is mentioned that, due to the asymmetry, a reduction in the pressure peak occurs on the side with the smaller deadrise angle. Shams et al. (2015) conducted an experimental study, examining the influence of the heel angle (asymmetric impact) on the velocity and the pressure fields, defining a critical value of $30^{\circ}$. They concluded that the impact force increases as the heel angle increases.

The research in the water entry problems was expanded further to blunt bodies penetrating the flat free-surface of the water. Faltinsen et al. (1977) studied the water entry of circular cylinders numerically and experimentally. They examined the cases of rigid and elastic cylinders. They obtained the slamming force by using Newton's second law, by time differentiation of the added mass in the vertical direction (for a constant velocity). They derived results until the full submergence of the cylinder (penetration depth equals to the diameter of the cylinder). The water entry of a sphere was studied by Miloh (1981) who derived analytical expressions for the vertical slamming force exerted on a sphere, the added mass coefficients and the wetting factor. He commented that, contrary to the water entry of a cylinder, the slamming force on a sphere rises sharply from zero, at the first contact, to a maximum value and then decreases. Meanwhile, Cointe and Armand (1987) investigated the vertical entry of a rigid cylinder. They solved the bvp by applying a perturbation method and using the matched
asymptotic expansions to obtain valid expressions for the outer and inner solutions. Miloh's (1981) study for the vertical water entry problem of a sphere was later expanded, by the same author, for an oblique entry of a sphere. His solution was based on the generalized Kirchhoff-Lagrange equations, by exploiting the kinetic energy of the fluid. He had to compute the derivatives of the horizontal and vertical added masses with respect to the submergence depth. He underlined that a horizontal component of the entry velocity tends to reduce the slamming load [Miloh (1991)]. Faltinsen and Chezhian (2005) performed a numerical study to calculate the loads acting on an idealized but more complicated structure, which consisted of a cylindrical mid-body and hemispherical ends. The results anticipated by the numerical simulation were compared to experimental results. Lately, the influence of ice and its position in relation to the impact point on a rigid body, was examined in detail by Khabakhpasheva et al. (2018).

Typically, the flow region for the water entry problems is subdivided in three subdomains: the outer/far-field solution, the jet roots and the jet regions, which are generally of different orders [Fontaine and Cointe (1997)]. One of the most classical, analytical studies concerning the solution to the water entry problems in 3D in the outer field is that due to Scolan and Korobkin (2001). They solved the inverse problem of the linearized classical Wagner approach: they assumed the kinematics of the entering body and the contact line are known and they consequently searched for the shape of the body. In addition, they proved that an elliptic paraboloid penetrating the flat freesurface of the water with constant velocity or constant acceleration has an elliptic contact line. Thereafter, they extended their study by considering the entry of almost axisymmetric bodies. To this end, they introduced a small perturbation parameter which accounts for the slight differences relative to the corresponding completely axisymmetric body [Korobkin and Scolan, (2002), (2006)]. The case of an elliptical paraboloid within the Wagner theory was also studied analytically by Korobkin (2002) by making use of the Lamé functions. He concluded that the contact area between the body and the liquid is elliptical for any law of penetration. Scolan and Korobkin (2012) showed that results for elliptic contact regions can be also obtained using Galin's theorem. The case of elliptical contact regions was studied recently by Chatjigeorgiou (2019), who focused on the impact on a prolate spheroid and on an elliptical paraboloid. It was found that the solution is detached from the angular dependence.

The normal (vertical) water entry problem has been extended to the case of oblique impact. A preliminary study was carried out by Korobkin (1988) who formed bvp of $O(t)$ for blunt bodies penetrating the flat free-surface, by applying asymptotic analysis, given that the problem is considered at the very early stages of impact. The solutions derived were valid for an outer region, describing properly the flow far from the contact points. In later years, the research on the inclined impact was extended to 3D as well. Moore et al. (2012) investigated the case of oblique impact, within the Wagner theory, for bodies with small deadrise angles, deriving a solution for the splash sheet region too. The cavity was not considered. Sun and Wu (2013) examined the oblique water entry of a cone. They concluded that the vertical force is not strongly influenced by the horizontal component of the velocity. However, for a constant ratio of horizontal to vertical velocity, a reduction in the deadrise angle leads to a substantial increase in the vertical force. The oblique water entry of an elliptic paraboloid was studied by Scolan
(2014). He showed that this configuration slightly differs from the pure vertical entry, as well as that the leading order of the vertical force remains unaffected by the horizontal kinematics. A similar study by the same author for an elliptic paraboloid penetrating obliquely the flat free-surface, discusses that under a proper combination of the velocity components, zero pressure could occur on the wetted surface, provided that the horizontal velocity is large enough [Scolan (2014)].

Generally, the water free-surface is assumed flat. The case of a curved free-surface and its influence on the impact force has been studied by Cointe (1989) who also demonstrated how the results derived can be applied to the impact of spilling breakers on marine structures. The case of water entry of an elliptic paraboloid was studied for a non-flat free-surface as well. Within the linearized Wagner theory, Scolan (2014) considered the case of impact on cylindrical waves. He proved that a proper combination of the characteristics of the elliptic paraboloid and the wave's parameters, may lead the contact region to be a circle instead of an ellipse, as generally happens. The water entry of a cone in a $5^{\text {th }}$ order Stokes wave was investigated by Sun et al. (2019) by means of a BEM. They solved the fully nonlinear problem for different wave heights and orientations of the falling cone. Wen et al. (2020) studied the case of the wedge entry problem with varying speed to a non-flat free-surface. In their model, the initial water surface is curved and has a specific angle far from the impact zone (less than $90^{\circ}$-flat case-). They firstly derived the solution for a wedge penetrating a wedgeshaped free-surface with constant velocity, based on the results for the self-similar solution for a wedge penetrating the flat free-surface of the water with constant speed. Thereafter, they extended this solution for a wedge with varying speed.

In water entry problems, it is always assumed that the body penetrates the interface between the air and the water. Nevertheless, Yakimov (1973) conducted a study to demonstrate the qualitative differences in the flow and the jets formed when the standard ratio of the density of the air to the water is changed $\left(\sim 10^{-3}\right)$. He particularly considered three cases of a cylinder penetrating the liquid free-surface surrounded by air of pressure 1 atm , air of pressure 0.13 atm and helium of pressure 1 atm . He mentioned that the jet is strongly influenced by the aerodynamic forces acting on it in each case.

Generally, in water entry problems, the gravity and the surface tension are omitted and the water is assumed incompressible, inviscid and the flow irrotational. These simplifications are mainly owing to the small-time assumption. However, if either the gravity is taken into account or the change in the entry speed is significant then non self-similar solutions have to be derived. Greenhow (1987) concluded that gravity can be ignored if $V \geq 2 g t$, where $V$ is the impact velocity, $g$ is the gravitational acceleration and $t$ is the duration of the impact. He also considered the nonlinearities of the free-surface. Tyvand (1990) discussed the range of validity of the small-time approximations sufficiently. According to Korobkin and Pukhnachov (1988), the solution of the problem for an incompressible liquid can be considered as the leading asymptotic term of the similar problem for a compressible fluid where $M \rightarrow 0$ ( $M$ is the Mach number). Faltinsen and Zhao (1997) commented that fluid accelerations involved in water impact, are typically much higher than the gravitational acceleration. They mentioned also that during the water entry of a blunt body with a small deadrise angle, hydroelastic effects and air compressibility should be taken into account. Scolan and

Korobkin (2003) conducted a study to examine the energy distribution during water entry problems. They concluded that when including the jet root and the jet regions for a blunt body with an elliptic contact line in the analysis, the energy is equally transmitted in the bulk liquid and the jets if the acceleration after impact is zero (constant velocity). If the acceleration is negative, which means a reduction in the body velocity, most of the energy is transmitted to the jets. Fairlie-Clarke and Tveitnes (2008) proposed a model in CFD code, using a finite volume method, to solve the conservation of mass and momentum equations. The interface between the air and the water was modelled using the Volume of Fluid (VoF) method. They examined how the change in the added mass, the flow momentum and the gravity, affect the entry force. For this reason, they distinguished three regions in the fluid: the far-field, which was treated as undisturbed, the second fluid field within which, the water is displaced due to the wedge entry and the third represents the expanding added mass of the fluid, which is considered as being attached to the wedge.

In the last, more or less, twenty years, many researchers have contributed to the improvement of the classical model of Wagner (1932), as it was discussed in the first paragraph of this Chapter. Zhao et al. (1997) introduced actually the so-called generalized Wagner model (GWM). In this model, the body boundary condition is imposed on the actual position of the entering part of the body, the free-surface conditions are linearized and imposed on the pile-up height -instead of the undisturbed free-surface-, which is determined as part of the solution. The hydrodynamic pressure is given by the nonlinear Bernoulli equation. Moreover, they discussed how the flow separation is correlated to the geometry of the entering body and commented that for a concave body the jet generated is able to follow the geometry of the body, contrary to a convex one. The GWM was studied later by Mei et al. (1999) using the conformal mapping method. This analytical 2D study focuses on the models of a wedge, of a circular cylinder and of Lewis-form ship sections, while the body boundary condition was imposed on the exact wetted surface. It was shown that the impact force is overestimated when is calculated using Newton's second law, through time differentiation of the infinite frequency vertical added mass. In a free-falling wedge, similarly to the sphere, the force starts from zero and increases as the impact evolves. Korobkin (2004) has worked on improving the classical Wagner model significantly. He derived some analytical expressions for bodies with small and moderate deadrise angles and discusses the differences in the calculation of the hydrodynamic pressure when using i) only the linear term of the Bernoulli equation, ii) the quadratic term and iii) the original or iv) modified Logvinovich model (MLM). The MLM was applied to ship sections too, by Korobkin and Malenica (2005). They concluded that the results obtained by the MLM are very close to those predicted by the GWM, while the computational cost of the latter is much greater. The water entry of a symmetric contour following the GWM was studied numerically by Khabakhpasheva et al. (2014). In this study, particular attention was given to the flow and the pressures close to the contact points. In order to calculate the slamming force, they took into account only the positive pressures in the contact area. Korobkin et al. (2014) showed that the slamming force (calculated using Newton's second law) can be decomposed in two components, with slight proper modifications of the OWM, MLM, GWM: the first component is proportional to the entry speed squared and the second, to the body acceleration. The
coefficients involved are functions of the penetration depth only, and they can be precomputed for a given shape of the body, even for ship sections.

The effect of flow separation and air cavity dynamics may be important in later stages, as the penetration evolves. The water entry of a blunt body with small deadrise angle, has been studied with the entrapment of a cavity between the bottom of the body and the water surface, by Korobkin (1996). He assumed that the pressure inside the cavity is a function of time only and he considered cases in which, either the cavity is filled with air or there is no gas inside it. Tassin et al. (2014) conducted an analytical 2D study for a wedge. They suggested a correction of the Logvinovich model for estimating the force acting on a wedge during the separation stage, i.e., when the jet generated, is detached from the body surface. They also applied the concept of the "fictitious body continuation". A 2D analysis for a free-fall wedge with several deadrise angles and a wide range of Froude numbers, has been conducted by Wang et al. (2015). They focused on the dynamics of impact when separation occurs and on the cavity dynamics. To this end, they defined three stages: slamming stage, transition stage and collapse stage.

Typically, in water entry problems the water is assumed of infinite depth and the body penetration is small compared to the depth. Korobkin (1995) suggested that in the case of a comparable penetration to the water depth, one more subregion should be taken into account; this is the region beneath the entering body. Korobkin (1999) expanded his previous study for a box-like structure impacting on shallow water. No cavity was considered, and the flow region was fragmented in six subregions. The jet formed was observed to be inclined towards the body and moving away from it. If the penetration is comparable to the water depth, Howison et al. (2002) introduced the socalled "Korobkin" flow where the influence of the bottom is considered as important. With the aid of asymptotic analysis, they studied the cases of convex smooth bodies or non-smooth but flat-bottomed bodies. The case of inclined impact of a blunt body was studied by Khabakhpasheva and Korobkin (2012).

In the last years, some researchers have tried to investigate the water entry of a body whose shape varies in time. Such a 2D study is carried out by Tassin et al. (2013), whose mathematical modelling was based on the MLM. For the exit stage, they modified the von Karman approach. In order to calculate the hydrodynamic pressure they applied a Taylor series expansion for the acceleration potential. Scolan and Korobkin (2015) examined the case in which a smooth body penetrating the water surface changes its shape over time, within the Wagner approach. It is noted that the leading order vertical force is independent of the horizontal kinematics. On the other hand, they highlighted that the force and the moments are strongly dependent on the angles of the body rotation and discussed the existence of negative-pressure zones in the contact line.

Conformal mapping is probably the most common technique for solving the bvp in water impact theories. Among others, Malleron et al. (2007) turned the physical body contour into a flat plate through consecutive mappings. An alternative solution for the mixed bvp in impact problems was given by Scolan and Korobkin (2012). Conformal mapping was initially used to obtain the flat disk within the Wagner theory and the solution on the disk was broken down as Fourier series of the azimuthal angle and a combination of functions of the radial coordinate.

The effect of elasticity in the water impact problems has also been studied. In order to simulate the impact on a ship shell, Khabakhpasheva (2005) studied the impact on an elastic plate, which is connected with springs to the main structure. The hydrodynamic part of the problem was solved within the Wagner theory, while the structural part is governed by the Euler's beam equation. It was found that at the beginning of the impact, the edges of the plate move towards the water, independently of the spring's stiffness. A 2D analytical study for an elastic wedge has been carried out by Khabakhpasheva and Korobkin (2013). The side of the wedges were modelled as Euler beams. The hydrodynamic part of the problem was formulated within the context of the classical Wagner theory. The structural part was governed by the typical set of differential equations for a beam subjected to bend loading. The two parts of the problem are coupled through the added mass matrix. Very recently, Khabakhpasheva and Korobkin (2020) investigated the oblique impact of an elastic plate. They examined the possible mechanisms of air entrapment in a thin liquid layer caused by the oblique impact of a deformable body. The first study which combined the MLM for the calculation of the hydrodynamic loads with a structural model of the impacted body was by Sun et al. (2021). The structural part of the problem accounted for the interaction between the elastic and the rigid motion modes. The flexible deformation of the body was represented by a modal superposition approach, while Lagrangian equations were used for the penetration depth and the general coordinates of the problem.

Studies for the second-order Wagner theory have been performed as well. Korobkin (2007) made an extensive asymptotic analysis for a parabola-like shape of water hitting a rigid flat plate with a constant velocity. He commented that despite the fact that the second-order potential is more complicated than that of the first-order, the corresponding hydrodynamic pressure has the same form as in the first-order theory. In addition, he mentioned that the second-order theory does not give a contribution to the dimension of the contact region as a function of time. Oliver (2007) found that the leading order theory overestimates the vertical force relatively to the one predicted by the second-order, which is closer to the numerical and experimental results.

An alternative aspect of the water entry problems, but substantially more complicated, is the concept of steep wave impact. This means that a rectangular mass of water, with vertical wave front moves towards a fixed body. The idea behind this approach is the attempt to simulate a breaking wave in its theoretically more hazardous configuration. This concept was introduced for the first time by Korobkin and Malenica (2007). They studied the impact of such a wave onto an elastic wall. The analysis was conducted in the field of acoustic approximations. Korobkin (2008) considered the steep wave impact on a vertical wall. He discussed the special boundary conditions which hold on the wall if it is perforated, porous or protected with a soft cover. Both studies were in 2D. A few years later, Chatjigeorgiou et al. (2016), who utilized the tool of the pressure impulse theory, considered the impact of a steep wave on blunt structures, which form time spreading rectangular contact regions. The corresponding rectangular plate was treated as a degenerate ellipse with a zero semi-minor axis. This assumption allowed the solution of the boundary value problem with the aid of the special Mathieu functions, since elliptical coordinates were used. Results were presented for a circular cylinder. This study was further expanded by Chatjigeorgiou et al. (2017) who introduced an open rectangular section on the plate, which allowed the
water to escape from the plate during impact. The bvp needed special treatment, as it involved triple trigonometrical series.

The steep wave impact on a vertical circular cylinder was initially studied by Chatjigeorgiou et al. (2016). Their analysis was performed within the linearized Wagner theory. They made the fundamental assumption that the contact line between the cylinder and the water is weakly dependent on the vertical coordinate $z$, so as the Laplace equation was not violated. This study was further expanded by Tsaousis et al. (2020) who compared the actual results within the 3D Wagner theory with those anticipated by the 2D theories of von Karman and Wagner. Moreover, they discussed the range of validity of the 2D theories when applied on real 3D blunt geometries. Ghadirian and Bredmose (2019) proposed three models within the pressure impulse theory in 3D. Specifically, they considered i) a box-shaped volume of fluid impacting a vertical wall, ii) an idealized all-directional wave hitting a cylinder and iii) a wedgeshaped volume of water hitting a cylinder.

Wave breaking is a highly nonlinear and complicated process. Peregrine et al. (1980), in their numerical study, defined three regions for a wave approaching breaking, that were supposed to be significant for its dynamics: i) a region at the top of the wave, where the velocity of the fluid particles is greater than the phase velocity of the wave, ii) a region where the particles' accelerations are greater than gravity, just under the wave crest and iii) a region with low accelerations, at the rear part of the wave. Beyond any question, the distinctive features of the breaking waves are the randomness and the strongly unsymmetrical profile between the crest and the trough, which was clearly demonstrated by Bonmarin (1989). Breaking criteria and categorization of breaker types are proposed by Oumeraci et al. (1993), based on a sequence of parameters such as the wave steepness, the ratio of the water depth at the breaking point to the still water depth and the ratio of the horizontal velocity of the breaker to the vertical velocity of the water in front of the wall. Moreover, the reflection of a wave due to the presence of a structure may also affect the breaking process. A discussion for the evolution of the breaking wave characteristics was made by Chella et al. (2016), with the aid of a numerical model. The differences between a plunging and a spilling breaker while approaching the shore, were reported. Lubin et al. (2019) focused on the instabilities that could occur during wave breaking. They originate mainly from the high-density differences between the two fluids (air and water), velocity differences, surface tension, pressure perturbations and centrifugal acceleration. They are responsible for jet formations and development (for plunging breakers), droplet generation, air entrapment and vortices. The characteristics of the velocity and acceleration field of a plunging breaking wave in shallow waters were discussed by Scolan and Guilcher (2020), for four different points of the free-surface of the breaker. They also proposed a simple, complex formula for the velocity field in the plunging jet. It was concluded that the inner shape of the breaker determines the kinematics of the free-surface dominantly and that the maximum velocity computed along the free-surface is also the maximum velocity of the whole velocity field.

There are many classical studies focusing on the breaking wave impact on walls. Bagnold (1939) highlighted the lack of repeatability in the breaking wave impact process, since he was the first to observe that the peak pressures change unpredictably from one experiment to the other. This is in contrast to the behavior of the total impulse.

These deductions led to the concept of the "kinetic mass", a mass of water of certain geometry replacing the actual wave. The issue of the shock pressures was also discussed. The pressure impulse concept is a tool which is widely exploited in impact problems. In this direction, Cooker and Peregrine (1990a) demonstrated that the presence of the wall has a negligible influence on the overturning part of the wave. On the other hand, the part of the water in front of the wall, rises up the wall quickly, since it cannot pass through it. Due to the high accelerations in the jet (even 6000g) the hydrodynamic pressure is significantly greater than the instantaneous hydrostatic pressure. Moreover, Cooker and Peregrine (1990b) conducted a study in order to discuss the issue of the pressure impulse exerted on a wall, using analytical models. They concluded that the impulse on the wall increases as the impact zone increases and is non zero even in the sea bed. The impulse on the wall is attributed to the loss of momentum from the colliding wave. Given that this loss originates from a narrow zone close to the wall, the impulse is practically the same for any shape of the impacting wave they examined (triangular or infinite strip). To this end, they also defined the term of the "momentum length". Hattori and Arami (1992) suggested an interesting model for the air pocket in the following way: they suggested that the 2D air pocket be substituted by a 3D square pillar of specific width, height and length. The compression of the air pocket was simulated via a spring of stiffness $k$. Results for the profile and pressure contours of a tanh wave, have been derived numerically for different amplitudes and breaking locations, by Peregrine and Topliss (1994). One year later, Cooker and Peregrine (1995) published an analytical study for the liquid-solid and solid-solid impact. The pressure impulse in confined spaces increases, a conclusion which is of special interest for sloshing of liquids in containers. Since the computations are for incompressible, irrotational flow, the energy loss is transmitted to the jet up the wall. The idea of a filling flow was adopted by Peregrine and Kalliadasis (1996) and was considered proper to describe sloshing phenomena in an almost full container. The pressure impulse model was extended later by Wood et al. (2000), who considered that during impact the entrapped air firstly decelerates the water and thereafter could cause the water to move backwards, a phenomenon called as "bounce back". They concluded that the results anticipated by the bounce back theory are in accord with the experimental results when the length of the air pocket is less than half of the wave height.

The wave impact on the wall is strongly affected by many parameters, such as the water depth in front of the wall, the special features of the wall (vertical or inclined, impermeable or perforated), the exact shape of the breaking wave, the aeration of the colliding part of the water and the air pocket entrapment. The role of the entrained air has been examined by Peregrine and Thais (1996) with the aid of a compressible model that used the state equation for water-air mixtures. A classical study which discusses some of these issues was conducted by Peregrine (2003). He discussed and highlighted the case of the flip-through impact, which has some special features such as the small impacted region of very high pressure and the enormous fluid particles' accelerations. Moreover, he noted that the aeration of the water generally tends to behave as a cushion to the most violent impacts. Abrahamsen and Faltinsen (2012) derived analytical expressions for the calculation of the natural frequency of a small air pocket exhibiting simple geometry. They studied cases in which the air pocket was entrapped during the
breaking wave impact on a vertical wall or in the upper corner of a high leveled tank, during sloshing phenomena. The impact on a permeable wall was studied by Cooker (2013), who derived an expression for the pressure impulse as a function of the barrier's porosity. The analytical study by Chatjigeorgiou et al. (2015) considers a simplified geometry for a breaking wave hitting a wall which entraps a small air pocket violently. Their case study involved two impacted regions of the wall and mathematically the solution of the problem demanded the solution to a system of triple trigonometrical series.

Despite the randomness of the breaking process, experimental studies permit a deeper insight of the physical phenomenon. Kirkgöz (1990) analyzed the experimental results statistically and concluded that the probability distribution of the maximum impact pressures on a vertical wall can be described by the log-normal distribution. Perhaps, the most interesting result of his investigation is that the lower and longer lasting impact pressures cause larger wall deflections instead of those with high magnitude and very short duration. Based on the deflection criterion, he proposed a range for the maximum impact pressure, which should be taken into account for design purposes. Hattori et al. (1994) studied the impact pressure on a vertical wall for various types of breaking waves. They concluded that the highest pressure, characterized by very small duration, occurs when a small amount of air pocket is entrapped, while the point of the first contact is around the vicinity of the still water level generally. In contrast, larger air pockets were found to reduce the pressure peak, while they increased the duration of the impact. The pulsation of the air pocket, owing to the compressibility of the air, was found to lead to pressure oscillations and to the presence of subatmospheric/negative pressures. Given that the aeration levels are higher in sea water than in fresh water, the differences in the measured impact pressures within experiments were demonstrated by Bullock et al. (2001). They concluded that the entrained air tended to reduce the maximum impact pressures, therefore the pressures were higher in fresh water. Furthermore, they focused on the Froude and the Cauchy scaling laws in order to examine the influence of the scale of the prototype structure in the results. A few years later, Bullock et al. (2007) conducted large scale experiments. Even though it was found that high aeration tends to reduce the pressure peak, the longer rise time along with the fact that the impact is less spatially localized, does not necessarily guarantee the reduction of the impulse associated with the impact. A major outcome is that the impact impulse on a vertical wall is greater than that on an inclined (for nominally identical circumstances of impact) and can reach up to $30 \%$ of the total impulse on the wall. On the other hand, the total impulse seems to remain independent of the wall slope. The extension of the previous study is due to Bredmose et al. (2009) who combined experimental and numerical results focusing on the influence of the air on pressure shock waves generated by the impact and their interaction with the structure and a reflective sea bed. Nevertheless, experimental procedures bear another one difficulty, which arises from the fact that the mapping of the impact pressures is deteriorated by the effect of the air entrapment. These issues have been discussed in detail in the work of Marzeddu et al. (2018). PIV (particle image velocimetry) technique was utilized in the work of Jensen (2019) in order to derive the velocity field of a flipthrough and a perfect breaker impact.

Chan and Melville (1989) deduced that the presence of the wall had a larger influence on the dynamics of impact compared to that of a cylinder. They mentioned that the impact pressures can be more than ten times greater than the non-impact pressures on a wall. In addition, they noticed that the chance of having a breaking wave impact (plunging breaker) in a severe sea state is $20 \%$. They also recommended that the scaling of the impact loads may be feasible through the impulse, given that it is much more consistent in relation to the impact pressures, provided that the structure response time is much higher than the time scale of the impulse. Zhang et al. (1996) accounted for the initial stage of impact of a plunging breaker on a vertical wall, within the theory of oblique impact of a liquid wedge on a wall, which was described with the aid of the similarity solution. Their solution was approached numerically, using a mixed Eulerian-Lagrangian boundary integral method. The entrapped air was described by a polytropic gas law. Besides, they focused on deriving appropriate scaling laws for the impact process. They also concluded that the maximum pressure and impact force were reduced for an elastic wall by $6 \%$ and $8 \%$ respectively. The high impact forces along with the short rise times, can lead to significant structural vibrations. Cuomo et al. (2010) suggested new equations for the calculation of the quasi-static and impulsive loads exerted on a structure by a breaking wave, taking into account the reflection coefficient of the structure, the significant spectral wave height, the wavelength at the toe of the structure, the water depth at the breaking location and the water depth in front of the wall.

More recently, another numerical, multifluid model was proposed by Plumerault et al. (2012), who obtained results for a constant breaking distance and different aeration levels and vice versa. Particular attention was paid to the pressure fluctuations in two different regions: the first region was close to the air pocket. These pressure fluctuations were attributed to the air pocket oscillations. The second region was below the air pocket, where the pressure was related to acoustic waves. In contrast to the general results, they found out that the breaking location had a limited influence on the hydrodynamic parameters. In order to capture the local flow characteristics in the impact zone more satisfactorily, Song and Zhang (2018) introduced a multi-scale algorithm along with a stretched coordinate system and they solved the bvp by means of BEM. They found that secondary inner jets could be formed owing to the contraction-expansion process of the air pocket. Liu et al. (2019) developed a 2D CFD, two-phase model to derive results for the breaking wave impact on a vertical wall. The air was considered as an ideal, compressible gas, while the water was treated as an incompressible liquid. For this reason, the Ghost Fluid Method was also applied which accounts for the discontinuity of the fluid properties. The cases examined were for a slightly breaking wave, a flip-through impact, a plunging breaker capturing a small and/or large air pocket and a broken wave. They concluded that the largest total forces on the wall occurred in the flip-through and in large air pocket cases. Pressure oscillations in the air pocket were noticed, owing to its compression and decompression. Similarly to the previous results, larger air pockets led to lower maximum pressure compared to small air pockets, but of higher duration. Finally, they adopted an alternative configuration of the sloping part of the bottom, since the region in front of the wall was assumed to have two slopes. Sun et al. (2019) in their numerical study, concluded that a compressible model simulates the impact process of a two-phase
flow more efficiently and is generally more realistic than an incompressible one. An enhanced and more accurate model relative to Smoothed Particle Hydrodynamics (SPH) method was developed within the Finite Particle Method (FPM) and it was examined in the work of He et al. (2020), for a solitary wave breaking over a slope. However, given that their model is single-phase, it cannot capture the air entrapment and splash-up stages very satisfactorily.

There are a few studies examining the breaking wave impact by plunging breakers in deep waters. Pressure oscillations were once again observed to exist during the breaking wave impact process by Chan and Melville (1988). These oscillations were of lower or higher frequency, depending on the amount of the entrapped air. The high randomness, which characterizes the breaking process, dictates the difficulty of deriving results from scale to prototype conditions. Given that walls and breakwaters are situated close to the coast, whereas the offshore structures are in deep waters mainly, the experimental study by Chan (1994) showed that when the wave kinematics and the impact zone are comparable in the shallow and deep water, so are the impact loads.

The flip-through impact is a special kind of the breaking wave impact process. It is undoubtedly a highly localized phenomenon. An indicative experimental study for the role of the flip-through on the impact pressure on a wall is by Lugni et al. (2006). They found that the evolution of the flow which leads to the flip-through event, consists essentially of the following three steps: i) the wave advancement, ii) the focusing and finally iii) the flip-through. The flip-through phenomenon was also studied numerically by Scolan (2010), who used a desingularized technique along with a conformal mapping method to treat the singularities originating from the Green's function. The flip-through impact on a breakwater placed over a porous rubble mound was studied numerically by Martin-Medina et al. (2018), who used a multi-phase Navier-Stokes model, without taking the turbulence into account though, along with a VoF technique for the interface evolution. They derived results for the pressure distribution, the freesurface elevation and the acceleration of the fluid particles for different locations of the breakwater, which, in substance, implies different shapes of the breaker. They concluded that the steeper the wave, the more localized the impact region is, as it was proven by the velocity profile.

The breaking wave impact on a vertical cylinder has been widely examined numerically. Such a study is by Chella et al. (2016) who used a CFD model which solves the Reynolds-Averaged Navier-Stokes (RANS) equations along with the $k-\omega$ model for the turbulence and the level set method for the free-surface. They concluded that the curve, which demonstrates the pressure distribution, becomes narrower and sharper as the distance from the still water level increases, while the contribution of the wave impact force to the total force, increases as the wave height increases. Another CFD model solving the RANS equations and the continuity equation, along with the level set method was developed by Kamath et al. (2016). They examined the effect of five different incident wave heights and five different impact scenarios, determined by the relative distance between the breaking point and the location of the cylinder. The highest force was noticed when the breaking wave impacts the cylinder just below the crest level, while the force is increased when the incoming wave height increases too. Xie et al. (2017) proposed a novel numerical model with unstructured, adaptive meshes in order to capture the complex surface formations during the breaking wave-cylinder
interaction (jets, splash-up) more satisfactorily. Tai et al. (2019) conducted an experimental study for the impact forces on a vertical cylinder and they discussed two decomposition methods that can be used to obtain the quasi-static and the dynamic components of the measured force. In order to investigate the characteristics of the irregular breaking wave forces on a monopile, through spectral analysis, Aggarwal et al. (2019) used a two-phase flow CFD model.

In the last years, the numerical research for impact problems is moving towards the Smoothed Particle Hydrodynamics method (SPH) [Monaghan and Rafiee (2012)]. For the time being, the methodology for impact problem is rather immature. It involves inherent difficulties due to the selection of the proper kernels and the fragmentation of the fluid in particles. They are developed to simulate dam-break flows hitting a column or a cylinder violently [Cummins et al. (2012), Yang et al. (2016), Peng et al. (2021)], water entry problems [Gong et al. (2009), Lind et al. (2015), Aly and Asai (2018), Shao et al. (2019)] and breaking wave impact on walls [Didier et al. (2014), Rafiee et al. (2015), Pugliese Carratelli et al. (2016)].

Except for simple vertical walls and cylinders, the breaking wave impact process has been investigated in alternate models too. Kisacik et al. (2016) examined experimentally the impact on a vertical wall with a horizontal cantilever slab just above it. Due to the first impact of the breaking wave on the wall, the generated jet escaped upwards with great accelerations, impacting the slab directly above the wall. A similar study has been conducted by Huang and Chen (2020), who detected three consecutive impacts: the first one on the upper position of the vertical wall, the second on the lower part of the vertical wall and third in the corner between the cantilever slab and the vertical wall. By trying to approximate the vertical section of a FPSO, Hu et al. (2017) investigated the impact on a truncated wall, by comparing numerical and experimental results. They noticed that the characteristics of the breaking wave impact on a truncated wall were similar to those for a full depth vertical wall. Their main outcome was that the highest pressures recorded, were associated with the case of a flip-through impact, whereas the highest impact force occurred in the impact involving large air pockets. Comparisons among numerical models, for several loading conditions, were conducted by Park et al. (2018) for a simplified box-shaped structure. The box was elevated, in a sense that void existed between its bottom and the still water level. In an attempt to model an offshore structure, Yan et al. (2019) examined the case of plunging wave impact on a box-shape structure, experimentally and numerically. Contrary to the general deduction for the vertical walls, they found that the peak pressure on the structure does not change significantly among several experiments. A prototype approach to the problem of the discussion of the pressure oscillations was through the measurement of the area of the cavity. The classical approach of a breaking wave propagating over a sloping beach and impacting a wall has been extended to the investigation of an impact on a storm wall for a dike-promenade structure, numerically by De Finis et al. (2020). They considered three different locations for the storm wall and five different wall heights.

The continuously increasing needs of renewable energy has led to a remarkable progress in the field of offshore wind technology. Wind turbines are established in many locations offshore, around the world. However, offshore structures, either floating or fixed, must survive extreme environmental loads very often, since they are
exposed to slamming loads due to breaking waves, especially from plunging breakers. The breaking wave forces exerted on a monopile supporting a wind turbine were examined by Aggarwal et al. (2017), who concluded that the peak spectral wave density is higher besides the cylinder due to shoaling and the reflected waves by the cylinder. Woo et al. (2017) proposed a numerical model for the calculation of slamming loads on a jacket structure, by discretizing it using the lumped masses method. Tu et al. (2017) mention that for the effective calculation of the slamming loads on offshore wind turbines, one should consider the following aspects: i) detection of the breaking waves, ii) calculation of the slamming loads and iii) their integration in the fully coupled analysis. Guo et al. (2020) calculated the impact loads on a semi-submersible platform experimentally, by applying the Froude scaling law.

Continuing in the field of renewable energy discussed in the previous paragraph, the attempt to exploit wave energy, has led to considerable evolution of the oscillating wave surge converters (OWSC). Given that they are not protected by the wave action, impact pressures may question their survivability. To this end, slamming loads have been calculated numerically and experimentally, along with semi-analytical models. Similarities with the water entry problem of a wedge (within the Wagner approach) have been observed, while the role of the developed jet during the impact has been discussed by Henry et al. (2015) and Dias et al. (2017). The spatial-temporal distribution showed that the maximum slamming pressure is located at the middle of the flap below the mean water level [Wei et al. (2016)]. In any case, since slamming pressures are involved in many phenomena associated with Naval and Ocean Engineering, it is strongly recommended to be taken into account, especially during the design stage of a structure [Dias and Ghidaglia (2018)]. Very recently, the concept of pressure impulse was applied, by taking also the aeration of the water into account and numerical results were derived for its distribution on the flap of a wave energy converter [Renzi et al. (2018)].

## Chapter 2 The steep wave impact on a rigid, vertical circular cylinder

### 2.1 Theoretical modelling

In this Chapter, we investigate the hydrodynamic problem of a steep wave impact on a rigid, vertical circular cylinder. Beyond any question, cylinders are the most widely used body in the field of Offshore Engineering. They can be encountered in several forms: as trusses of jacket structures, members of semi-submersibles and floating platforms generally, as Oscillating Water Column (OWC) devices for the exploitation of the wave energy and as monopiles supporting wind turbines.

Our purpose is to propose a robust mathematical model which allows us to determine with sufficient accuracy the contact area between the liquid and the body at every time instant, the 3D fluid flow induced by the impact and consequently the slamming force exerted on the cylinder. The method of solution is analytical. The bvp which arises is of mixed type. Dirichlet and Neumann boundary conditions should hold simultaneously at different portions of the same surface (wave front). The solution obtained within the context of this analysis is valid everywhere in the fluid domain except for a small vicinity near the jet root and the jet splash. Our analysis relies on the small-time assumption and particular attention is given on the initial stage of the impact where $t \rightarrow$ 0 . Once we determine the flow, we can calculate several physical quantities including the hydrodynamic pressure on the wetted area, using the Bernoulli equation, and accordingly the slamming force exerted on the cylinder, the pressure impulse on the cylinder and the displacement of the wave front.

Breaking waves may hit violently a structure either with a vertical wave front, a case known as flip-through impact, or inclined, entrapping a small or a larger amount of air. Therefore, we simulated this kind of impact in the following manner: a completely vertical wave front is assumed, bounded from above by a flat free-surface and extending to the semi-infinity. The water is of depth $h$ and the bottom is horizontal and impermeable. Without loss of generality, we may assume that $h=1$. The wave moves along the negative $x$-axis with a steady velocity $V$, towards a cylinder of radius $R$. A schematic representation of the physical problem under consideration is shown in Fig. 2.1.1. $\Omega$ is the domain of the liquid, $S_{u}$ is the upper free-surface, considered flat, $S_{B}$ denotes the bottom and finally $S_{F}$ is the vertical wave front which collides with the cylinder.


Fig. 2.1.1. 3D configuration of the steep wave impact on a vertical circular cylinder.
Apart from the modelling of a completely vertical wave front of a breaking wave, the steep wave concept can be used to model dam break flows moving against a fixed body. The major difference between the physical modelling of a steep wave impact and that anticipated when a freak or a rogue wave hit an offshore structure, is that in our case the bottom just in front of the cylinder is dry. Besides, the underlying physics is completely different in each case. Even though the study of extreme waves impact on marine structures is not examined so widely, there are some connections between our modelling and that one. It is known that a freak wave may evolve into a plunging breaker. Some studies examined the impact pressures from a freak wave taking the kinematics and the orientation of the incident wave into account. In their experimental study, Luo et al. (2020) generated a freak wave using the focused wave theory assuming constant wave amplitude for each wave component of the wave packet. The chosen wave celerity was the one prescribed for intermediate water depths, using the middle frequency of the corresponding range of the wave packet, as the characteristic one. Rudman and Cleary (2013), trying to model the notorious Draupner wave [Clauss (2002)], used the SPH methodology to investigate the impact of rogue waves in a TLP. Another numerical study by the same authors, proved that a rogue wave impact with an angle of $45^{\circ}$ was the worst case of loading [Rudman and Cleary (2016)]. In both studies, the impact velocity was chosen to be $20 \mathrm{~m} / \mathrm{s}$.

At the time instant $t=0$, the wave front hits the cylinder. For $t>0$ the cylinder penetrates the water by a horizontal distance $V t$ and the liquid/body contact area is delimited by the contact line $b(z, t)$, along the vertical $z$-axis, at each time instant. From a physical aspect, the contact line denotes the end of the portion of the body that the water hits and bounds the wetted surface of the body. Alternatively, from a mathematical point of view, the contact line can be contemplated as the boundary which divides a surface to two sub-regions, each one governed by a different boundary
condition (Dirichlet and Neumann). Under the small-time assumption, which fundamentally leads to the linearization of the bvp, the contact area can be approximated by its projection on the vertical $(y, z)$ plane, as shown in Fig. 2.1.2 [Scolan and Korobkin (2001), Korobkin and Scolan (2006)].


Fig. 2.1.2. The contact line $b(z, t)$ and the projection of the contact area in the $(y, z)$ plane.


Fig. 2.1.3. The von Karman, $b_{0}(t)$, and the Wagner, $b(z, t)$, contact lines, at a time instant $t>0$. The cylinder penetrates the liquid by a horizontal distance $V t$ and the wave front displacement is denoted by $\eta(y, z, t)$. The problem is considered in 3D using a Cartesian representation. The coordinate system moves with the wave front. The vertical $z$-axis increases towards the viewer; the disturbed wave front asymptotes to the $y$-axis.

As already discussed in the literature review, there are two established theories, both referring to 2 D wedge water entry problems, which determine the width $b(z, t)$. von Karman's theory does not account for the pile-up of the water during the collision and the contact line is explicitly calculated by a simple geometrical relation. Wagner's theory assumes that, due to the collision, a portion of the water surface, which hits the
cylinder, denoted by $\eta(y, z, t)$, deforms and exhibits an asymptotical behavior along the $y$-axis, as shown in Fig. 2.1.3. The contact line is slightly displaced compared to that described by the von Karman model. As a result, the first theory clearly underestimates the actual contact area and consequently the hydrodynamic loading, while the second one overestimates it, when considered in 3D. Herein, we deal with the 3D Wagner impact problem and we prove that the real contact line lies between those prescribed by these two theories and has to be determined along with the fluid flow after the impact.

### 2.2 The mixed boundary value problem of the hydrodynamic slamming

The mathematical formulation of the governing mixed bvp is considered in this Section, within the Wagner impact theory. We assume inviscid, incompressible fluid and irrotational flow, so the fluid flow after impact can be described by the linear potential theory. Surface tension and gravity are ignored. The governing hydrodynamic system will be derived using asymptotic analysis of the flow and the pressure distribution at the initial stage of the impact. Asymptotic analysis has been performed by Korobkin (2004), Miloh (1991) and Howison et al. (1991) among others. Here, the asymptotic analysis is implemented under the assumption that the horizontal dimension (along $y$ ) of the wetted part of the cylinder starts from zero and grows progressively but is less than $2 R$. Cylinder's penetration $d$ relative to the undeformed wave front $S_{F}$ at the initial stage of the impact is of the order of $f(R)$ and is very small compared to the horizontal dimension of the contact region and accordingly the ratio $\epsilon=f(R) / R$ can be used as the scaling factor.

The length scale is represented by the radius $R$. Further, $\epsilon R$ is the displacement scale, $\epsilon R / V$ is the time scale, $\rho V^{2} / \epsilon$ is the scale of the hydrodynamic pressure, while $R V$ is the scale of the velocity potential coined $\phi(x, y, z, t)$. Following Korobkin (2004) and using nondimensional variables, which are denoted by the symbol of the tilde, the nondimensional velocity potential $\tilde{\phi}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$ should satisfy the following bvp:

$$
\begin{gather*}
\nabla^{2} \tilde{\phi}=0, \quad(\text { in } \widetilde{\Omega}),  \tag{2.2.1}\\
\frac{\partial \tilde{\phi}}{\partial \tilde{t}}+\epsilon \frac{1}{2}|\nabla \tilde{\phi}|^{2}=0, \quad\left(\text { on } \tilde{S}_{F}, \tilde{S}_{U}\right),  \tag{2.2.2}\\
\frac{\partial \tilde{\eta}}{\partial \tilde{t}}+\epsilon \frac{\partial \tilde{\eta}}{\partial \tilde{y}} \frac{\partial \tilde{\phi}}{\partial \tilde{y}}=\frac{\partial \tilde{\phi}}{\partial \tilde{x}}, \quad[\tilde{x}=-\epsilon \tilde{\eta}(\tilde{y}, \tilde{t})],  \tag{2.2.3}\\
\frac{\partial \tilde{\phi}}{\partial \tilde{z}}=0, \quad\left(\text { on } \tilde{S}_{B}\right),  \tag{2.2.4}\\
\frac{\partial \tilde{\phi}}{\partial \tilde{x}}=\epsilon \tilde{f}^{\prime}(\tilde{y}) \frac{\partial \tilde{\phi}}{\partial \tilde{y}}+\frac{\partial \tilde{d}}{\partial \tilde{t}}, \quad\{\tilde{x}=\epsilon[\tilde{f}(\tilde{y})+\tilde{d}(\tilde{t})]\},  \tag{2.2.5}\\
\tilde{\phi} \rightarrow 0, \quad\left(\tilde{x}^{2}+\tilde{y}^{2} \rightarrow \infty\right) . \tag{2.2.6}
\end{gather*}
$$

Eq. (2.2.1) is the Laplace equation which holds in the entire fluid field, Eq. (2.2.2) is the dynamic condition on the upper free-surface and the vertical wave front, Eq. (2.2.3) is the kinematic condition on the wave front, while Eq. (2.2.4) is the zero-velocity condition on the impermeable flat and horizontal bottom. Eq. (2.2.5) is the kinematic condition on the wetted part of the cylinder and finally Eq. (2.2.6) is the far-field boundary condition, which implies that any disturbance caused by the impact should vanish at infinity. Clearly, the underlying system of Eqs. (2.2.1)-(2.2.6) assumes that no deformation of the velocity field occurs in the vertical direction, which allows us to neglect the $\partial \tilde{\phi} / \partial \tilde{z}$ terms. Under the specific assumption, the present formulation ignores the deformations of the upper free-surface, including the wave run-up on the cylinder during impact. Following the same reasoning, no free-surface kinematical condition exists on $\tilde{S}_{U}$.

We restore the dimensionalized variables, and noting that $d(t)=V t$ is the penetration depth of the cylinder to the liquid at the very early stages of the impact, we let $\epsilon \rightarrow 0$ and the system of Eqs. (2.2.1)-(2.2.6) is linearized (to the leading order) according to

$$
\begin{array}{ccc}
\nabla^{2} \phi=0, & x \geq 0, \quad 0 \leq y<+\infty, & -h \leq z \leq 0, \\
\phi=0, \quad x \geq 0, \quad 0 \leq y<+\infty, & z=0, \\
\phi=0, \quad x=0, \quad b(z, t)<y<+\infty, & -h \leq z \leq 0, \\
\frac{\partial \eta}{\partial t}=\frac{\partial \phi}{\partial x}, \quad x=0, \quad y>b(z, t), \quad-h \leq z \leq 0, \\
\frac{\partial \phi}{\partial z}=0, \quad x \geq 0, \quad 0 \leq y<+\infty, \quad z=-h, \\
\frac{\partial \phi}{\partial x}=V, \quad x=0, \quad 0 \leq y \leq b(z, t), & -h \leq z \leq 0, \\
& \phi \rightarrow 0, \quad x^{2}+y^{2} \rightarrow \infty . \tag{2.2.13}
\end{array}
$$

Given that the problem is symmetrical relative to the $z$-axis for $x=0$ (Fig. 2.1.2), the governing set can be enhanced using the artificial Neumann condition in $y$ direction

$$
\begin{equation*}
\frac{\partial \phi}{\partial y}=0, \quad x \geq 0, \quad y=0, \quad-h \leq z \leq 0 . \tag{2.2.14}
\end{equation*}
$$

Finally, the hydrodynamic system of wave impact is completed introducing the Wagner condition, which literally means that the elevated liquid meets the surface of the cylinder. By geometry (Fig. 2.1.3), the Wagner condition is written as

$$
\begin{equation*}
\eta[b(z, t), z, t]=V t-R+\sqrt{R^{2}-[b(z, t)]^{2}} . \tag{2.2.15}
\end{equation*}
$$

In the context of the classical Wagner approximation the liquid flow due to the impact is described by the set of equations (2.2.7)-(2.2.15). This theory is valid at the very early stages of the impact, when the penetration depth is much smaller than the horizontal dimension of the wetted body, which in our case means that $V t \ll b(z, t)$ [Scolan and Korobkin (2001), Korobkin (2002)]. It is important to note that despite the linearization of the boundary conditions, the bvp remains nonlinear. This follows from the fact that the contact region is unknown in advance and must be determined as a part of the solution. Clearly, the associated bvp is of mixed type, given that it involves both Neumann and Dirichlet conditions in different portions of the same surface (the wave front). In particular, the final system under consideration assumes a zero (total) velocity on the liquid/solid contact area [Eq. (2.2.12)] and zero hydrodynamic pressures, i) on the entire upper free-surface at $z=0$ and ii) beyond the contact area on the wave front [Eqs. (2.2.8) and (2.2.9)].

### 2.3 Solution of the mixed boundary value problem

### 2.3.1 The contact line and the velocity potential on the contact area

Using separable solutions of the Laplace equation (2.2.7), we write an expression for $\phi(x, y, z, t)$ as follows

$$
\begin{equation*}
\phi(x, y, z, t)=\sum_{n=1}^{\infty} \phi_{n}(x, y, t) \sin \left(\lambda_{n} z\right) \tag{2.3.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda_{n}=(n-1 / 2) \pi / h, \tag{2.3.2}
\end{equation*}
$$

which satisfies the boundary conditions (2.2.8) and (2.2.11). Substituting expression (2.3.1) in the Laplace equation (2.2.7) yields

$$
\begin{equation*}
\frac{\partial^{2} \phi_{n}}{\partial x^{2}}+\frac{\partial^{2} \phi_{n}}{\partial y^{2}}-\lambda_{n}^{2} \phi_{n}=0 \tag{2.3.3}
\end{equation*}
$$

The separable solutions of Eq. (2.3.3) should construct products of the form

$$
\begin{equation*}
\phi_{n}(x, y, t) \sim \cos (u y) e^{-x \sqrt{u^{2}+\lambda_{n}^{2}}} \tag{2.3.4}
\end{equation*}
$$

where $u$ denotes an arbitrary variable. In addition to Eq. (2.3.3), Eq. (2.3.4) satisfies both the symmetry [Eq. (2.2.14)] and the far-field [Eq. (2.2.13)] conditions. To generalize the solution for $\phi_{n}$ relative to the variable $u$, and apparently the time $t$, we write

$$
\begin{equation*}
\phi_{n}(x, y, t)=\int_{0}^{\infty} \xi_{n}(u, t) \cos (u y) e^{-x \sqrt{u^{2}+\lambda_{n}^{2}}} d u . \tag{2.3.5}
\end{equation*}
$$

Hence, it follows that the fundamental solution $\phi(x, y, z, t)$ is written in the form

$$
\begin{equation*}
\phi(x, y, z, t)=\sum_{n=1}^{\infty} \sin \left(\lambda_{n} z\right) \int_{0}^{\infty} \xi_{n}(u, t) \cos (u y) e^{-x \sqrt{u^{2}+\lambda_{n}^{2}}} d u, \tag{2.3.6}
\end{equation*}
$$

where $\xi_{n}(u, t)$ are functions to be determined. Note that the coefficients which should multiply the eigenfunction in the $z$-direction, originating from the separation of variables of the Laplace equation, are actually encompassed in the unknown functions $\xi_{n}(u, t)$.

Furthermore, we endeavor to satisfy the Dirichlet and the Neumann conditions of Eqs. (2.2.9) and (2.2.12) respectively, that define the concerned bvp as a problem of mixed type. Thus, introducing Eq. (2.3.6) into Eqs. (2.2.9) and (2.2.12), one gets

$$
\begin{gather*}
\sum_{n=1}^{\infty} \sin \left(\lambda_{n} z\right) \int_{0}^{\infty} \xi_{n}(u, t) \cos (u y) d u=0, \quad b<y,  \tag{2.3.7}\\
\sum_{n=1}^{\infty} \sin \left(\lambda_{n} z\right) \int_{0}^{\infty} \xi_{n}(u, t) \sqrt{u^{2}+\lambda_{n}^{2}} \cos (u y) d u=-V, \quad 0 \leq y \leq b . \tag{2.3.8}
\end{gather*}
$$

The cosine terms in Eqs. (2.3.7) and (2.3.8) are next recast using Abramowitz and Stegun (1970)

$$
\begin{equation*}
\cos (u y)=\sqrt{\pi y u / 2} J_{v}(u y), \quad v=-1 / 2 \tag{2.3.9}
\end{equation*}
$$

where $J_{v}$ is the Bessel function of the first kind with fractional order $v$. Subsequently, Eqs. (2.3.7) and (2.3.8) form a problem of mixed type that involves integral equations. To cope with this problem, we follow the technique suggested by Tranter (1960a) which aims to satisfy first the Dirichlet condition via proper selection of the unknown kernel $\xi_{n}(u, t)$. However, particular attention must be paid on this issue, since the existence and convergence of the infinite integrals on $u \in[0, \infty)$ is not necessarily guaranteed for a random choice of the unknown functions $\xi_{n}(u, t)$. Hence, we exploit a relation that can be found in Gradshteyn and Ryzhik (2007)

$$
\begin{equation*}
\int_{0}^{\infty} u^{v+1} \frac{J_{m}\left(b \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{m / 2}} J_{v}(u y) d u=0, \quad b<y \tag{2.3.10}
\end{equation*}
$$

The only requirement for the validity of Eq. (2.3.10) is that $m>v>-1$ [Gradshteyn and Ryzhik (2007); p. 693]. The form of Eq. (2.3.10) dictates the following assumption for the unknown functions $\xi_{n}(u, t)$

$$
\begin{equation*}
\xi_{n}(u, t)=\sum_{m=0}^{\infty} A_{m n} \frac{J_{m}\left(b \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{m / 2}} \tag{2.3.11}
\end{equation*}
$$

The series expansion of Eq. (2.3.11) relative to the order $m$ of the Bessel function, ensures the generality of the solution. As previously mentioned, the arbitrary coefficients $A_{m n}$ incorporate the coefficients, which should multiply the sine function in Eq. (2.3.6). After introducing Eqs. (2.3.9) and (2.3.11) in Eq. (2.3.7) and recalling Eq. (2.3.10), one gets

$$
\begin{equation*}
\sqrt{\frac{\pi y}{2}} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{m n} \sin \left(\lambda_{n} z\right) \int_{0}^{\infty} u^{1 / 2} \frac{J_{m}\left(b \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{m / 2}} J_{-1 / 2}(u y) d u=0 . \tag{2.3.12}
\end{equation*}
$$

Hence, using Eq. (2.3.11) in Eq. (2.3.7) yields an identity. Nonetheless, it should be highlighted that the general consideration of the 3D Wagner wave impact problem, requires that the shape of the contact line, represented herein by the width $b$, should depend on both the time $t$ and the vertical coordinate $z$, i.e., $b \equiv b(z, t)$. Consequently, although the particular selection of the kernel $\xi_{n}$ satisfies the Dirichlet condition (2.3.7), it actually violates the Laplace equation. Thus, the only mathematically acceptable option is to consider that the width of the contact area $b$ is weakly dependent on $z$, which in turn allows assuming that the Laplace equation is not being violated. The concerned hypothesis implies that both $d b(z, t) / d z$ and $d^{2} b(z, t) / d z^{2}$ should approach to zero along the whole interval $z \in[-h, 0]$, a condition that is satisfied assuming that the instantaneous contact line is "nearly" vertical and weakly dependent on $z$. That ensures that Eq. (2.3.6) still satisfies the Laplace equation. Further, the expansion coefficients $A_{m n}$ should also be functions of both $z$ and $t$, i.e., $A_{m n} \equiv$ $A_{m n}(z, t)$, while it is assumed that they are also weakly dependent on $z$.

Having satisfied one of the two mixed conditions, in the next step we proceed to the calculation of the unknown expansion coefficients $A_{m n}$ by requiring the satisfaction of the kinematic condition of Eq. (2.3.8). Introducing Eqs. (2.3.9) and (2.3.11) into Eq. (2.3.8) yields

$$
\begin{gather*}
\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{m n} \sin \left(\lambda_{n} z\right) \int_{0}^{\infty} u^{1 / 2} \frac{J_{m}\left(b \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{(m-1) / 2}} J_{-1 / 2}(u y) d u=  \tag{2.3.13}\\
-V \sqrt{\frac{2}{\pi y}}, \quad 0 \leq y \leq b .
\end{gather*}
$$

The infinite integral in Eq. (2.3.13) is divergent for $m=0$ and $m=1$. It can be verified by taking the limits of the Bessel functions for large arguments [Abramowitz and Stegun (1970)]. The asymptotic behavior of the Bessel function is like $u^{-1 / 2}$ for $u \rightarrow$ $\infty$. Therefore, we are allowed to assume artificially that $A_{0 n}=A_{1 n}=0$ for $n=1,2, \ldots$. It is noted that there is no explicit analytic formula for calculating the infinite integral in Eq. (2.3.13). All infinite integrals employed herein, are calculated using MATLAB's built-in function "integral", which allows the efficient numerical approximation of improper integrals such as the integrals included in the present study. Relying further on numerical computations, it can be easily shown that the values of the infinite integral for $m \geq 3,4, \ldots$, are rather negligible compared to the corresponding values for $m=2$. To support this statement, Figs. 2.3.1.-2.3.3 are given, which show graphically the value
of the infinite integral of Eq. (2.3.13) as a function of the mode number $n$, for a specific value of $b$ and three random values of $y$, provided that $0 \leq y \leq b$.


Fig. 2.3.1. Values of the infinite integral of Eq. (2.3.13) as a function of the mode number $n$ for $b=0.1581$ and $y=0.005$.


Fig. 2.3.2. Values of the infinite integral of Eq. (2.3.13) as a function of the mode number $n$ for $b=0.1581$ and $y=0.09$.


Fig. 2.3.3. Values of the infinite integral of Eq. (2.3.13) as a function of the mode number $n$ for $b=0.1581$ and $y=0.155$.

Therefore, it can be safely assumed that the truncation of the $m$-series does not influence the convergence of the left-hand side of Eq. (2.3.13). Using the assumption that only $A_{2 n}=A_{n}$ are nonzero, Eq. (2.3.13) is reduced to

$$
\begin{gather*}
\sum_{n=1}^{\infty} A_{n} \sin \left(\lambda_{n} z\right) \int_{0}^{\infty} u^{1 / 2} \frac{J_{\mu}\left(b \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{(\mu-1) / 2}} J_{-1 / 2}(u y) d u=-V \sqrt{\frac{2}{\pi y}},  \tag{2.3.14}\\
0 \leq y \leq b,
\end{gather*}
$$

where we assumed $\mu=2$. In order to calculate the expansion coefficients $A_{n}$, we have to make use of the orthogonality property of $\sin \left(\lambda_{n} z\right)$ in the interval $[-1,0]$. However, this approach cannot be implemented in a straightforward manner, since $b$, in the argument of the Bessel function of order $\mu$, is also a varying function of $z$. Thus, further mathematical manipulation of the Eq. (2.3.14) is required. Firstly, we average both sides over the defining interval $0 \leq y \leq b$. In particular we assume that $y=b r, 0 \leq$ $r \leq 1$. Next, both sides of Eq. (2.3.14) are multiplied by $r^{v+1}$ and are integrated over the defining interval of $r$. Making use of the result [Gradshteyn and Ryzhik (2007); p. 676]

$$
\begin{equation*}
\int_{0}^{1} r^{v+1} J_{v}(b u r) d r=\frac{1}{b u} J_{v+1}(b u) \tag{2.3.15}
\end{equation*}
$$

Eq. (2.3.14) yields

$$
\begin{gather*}
\sum_{n=1}^{\infty} A_{n} \sin \left(\lambda_{n} z\right) \int_{0}^{\infty} u^{-1 / 2} \frac{J_{2}\left(b \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{1 / 2}} J_{1 / 2}(u b) d u=-V \sqrt{\frac{2 b}{\pi}}  \tag{2.3.16}\\
0 \leq y \leq b .
\end{gather*}
$$

Nevertheless, additional elaboration on the Eq. (2.3.16) is needed, since we have not eliminated the dependence of the Bessel function $J_{2}$ on the vertical coordinate $z$ yet. To this end, it is recalled that the width of the instantaneous Wagner contact region $b$ is a function of both $z$ and $t$, but it varies weakly around the vertical so that the Laplace equation is not violated. Next, the width $b$ is assumed to be of the form

$$
\begin{equation*}
b(z, t)=b_{0}(t)[1+\epsilon(z, t)], \quad \epsilon(z, t) \ll 1, \quad-1 \leq z \leq 0 . \tag{2.3.17}
\end{equation*}
$$

The instantaneous contact line $b_{0}(t)$ denotes the von Karman approach (see Fig. 2.1.3) and is determined explicitly by the simple geometrical relation

$$
\begin{equation*}
b_{0}(t)=\sqrt{R^{2}-(R-V t)^{2}} . \tag{2.3.18}
\end{equation*}
$$

Having defined the small scaling factor $\epsilon \equiv \epsilon(z, t)$, in Section 2.2, we are allowed to expand in Taylor series, all the terms of Eq. (2.3.16), which involve the factor $b(z, t)$, around the known von Karman width $b_{0}(t)$, as follows

$$
\begin{align*}
& J_{2}\left(b \sqrt{u^{2}+\lambda_{n}^{2}}\right) \\
& =J_{2}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right) \\
& +\epsilon\left[-b_{0} \sqrt{u^{2}+\lambda_{n}^{2}} J_{3}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right)\right.  \tag{2.3.19}\\
& \left.+2 J_{2}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right)\right]+O\left(\epsilon^{2}\right) \text {, } \\
& J_{1 / 2}(b u)=J_{1 / 2}\left(b_{0} u\right) \\
& +\epsilon\left[b_{0} u J_{-1 / 2}\left(b_{0} u\right)-(-1 / 2) J_{1 / 2}\left(b_{0} u\right)-J_{1 / 2}\left(b_{0} u\right)\right]  \tag{2.3.20}\\
& +O\left(\epsilon^{2}\right) \text {, } \\
& b^{1 / 2}=b_{0}^{1 / 2}+\frac{\epsilon}{2} b_{0}^{1 / 2}+O\left(\epsilon^{2}\right) . \tag{2.3.21}
\end{align*}
$$

The last term of Eq. (2.3.16), which must be uncoupled by the $z$-dependence, is the expansion coefficients $A_{n}$, which are accordingly expanded in a perturbation series as

$$
\begin{equation*}
A_{n}(z, t)=A_{n}^{(0)}(t)+\epsilon(z, t) A_{n}^{(1)}(t)+O\left(\epsilon^{2}\right) \tag{2.3.22}
\end{equation*}
$$

Thereafter, it is remarked that the coefficients $A_{n}^{(k)}, k=0,1,2, \ldots$, are solely functions of time. Introducing Eqs. (2.3.19)-(2.3.21) in (2.3.16), we segregate like orders of $\epsilon$ and finally exclude the scaling factor from further manipulations. Following this analysis, one gets

Order $O\left(\epsilon^{0}\right)$

$$
\begin{equation*}
\sum_{n=1}^{\infty} A_{n}^{(0)} \sin \left(\lambda_{n} z\right) \int_{0}^{\infty} u^{-1 / 2} \frac{J_{2}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{1 / 2}} J_{1 / 2}\left(u b_{0}\right) d u=-V \sqrt{\frac{2 b_{0}}{\pi}} \tag{2.3.23}
\end{equation*}
$$

Order $O\left(\epsilon^{1}\right)$

$$
\begin{align*}
\sum_{n=1}^{\infty} \sin \left(\lambda_{n} z\right)\{ & b_{0} A_{n}^{(0)} \int_{0}^{\infty} u^{1 / 2} \frac{J_{2}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{1 / 2}} J_{-1 / 2}\left(u b_{0}\right) d u \\
& -(-1 / 2) A_{n}^{(0)} \int_{0}^{\infty} u^{-1 / 2} \frac{J_{2}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right.}{\left(u^{2}+\lambda_{n}^{2}\right)^{1 / 2}} J_{1 / 2}\left(u b_{0}\right) d u \\
& -A_{n}^{(0)} \int_{0}^{\infty} u^{-1 / 2} \frac{J_{2}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{1 / 2}} J_{1 / 2}\left(u b_{0}\right) d u \\
& -b_{0} A_{n}^{(0)} \int_{0}^{\infty} u^{-1 / 2} J_{3}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right) J_{1 / 2}\left(u b_{0}\right) d u  \tag{2.3.24}\\
& +2 A_{n}^{(0)} \int_{0}^{\infty} u^{-1 / 2} \frac{J_{2}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{1 / 2}} J_{1 / 2}\left(u b_{0}\right) d u \\
+ & \left.+A_{n}^{(1)} \int_{0}^{\infty} u^{-1 / 2} \frac{J_{2}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{1 / 2}} J_{1 / 2}\left(u b_{0}\right) d u\right\}=-\frac{V}{2} \sqrt{\frac{2 b_{0}}{\pi}}
\end{align*}
$$

The form of Eqs. (2.3.23) and (2.3.24) enables the immediate exploitation of the orthogonality property of the $z$-eigenfunctions. It is reminded that

$$
\begin{equation*}
\int_{-1}^{0} \sin \left(\lambda_{n} z\right) \sin \left(\lambda_{s} z\right) d z=\frac{1}{2} \delta_{n s} \tag{2.3.25}
\end{equation*}
$$

where $\delta_{n s}$ is the Kronecker's delta function. Eqs. (2.3.23) and (2.3.24) are reduced to

$$
\begin{equation*}
A_{n}^{(0)} \int_{0}^{\infty} u^{-1 / 2} \frac{J_{2}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{1 / 2}} J_{1 / 2}\left(u b_{0}\right) d u=\frac{2 V}{(n-1 / 2) \pi} \sqrt{\frac{2 b_{0}}{\pi}} \tag{2.3.26}
\end{equation*}
$$

$$
\begin{align*}
& A_{n}^{(1)} \int_{0}^{\infty} u^{-1 / 2} \frac{J_{2}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{1 / 2}} J_{1 / 2}\left(u b_{0}\right) d u \\
&=\frac{V}{(n-1 / 2) \pi} \sqrt{\frac{2 b_{0}}{\pi}} \\
&-b_{0} A_{n}^{(0)} \int_{0}^{\infty} u^{1 / 2} \frac{J_{2}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{1 / 2}} J_{-1 / 2}\left(u b_{0}\right) d u \\
&+(-1 / 2) A_{n}^{(0)} \int_{0}^{\infty} u^{-1 / 2} \frac{J_{2}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{1 / 2}} J_{1 / 2}\left(u b_{0}\right) d u  \tag{2.3.27}\\
&+A_{n}^{(0)} \int_{0}^{\infty} u^{-1 / 2} \frac{J_{2}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{1 / 2}} J_{1 / 2}\left(u b_{0}\right) d u \\
&+b_{0} A_{n}^{(0)} \int_{0}^{\infty} u^{-1 / 2} J_{3}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right) J_{1 / 2}\left(u b_{0}\right) d u \\
&-2 A_{n}^{(0)} \int_{0}^{\infty} u^{-1 / 2} \frac{J_{2}\left(b_{0} \sqrt{u^{2}+\lambda_{n}^{2}}\right)}{\left(u^{2}+\lambda_{n}^{2}\right)^{1 / 2}} J_{1 / 2}\left(u b_{0}\right) d u .
\end{align*}
$$

Hence, the coefficients of the perturbation expansion (2.3.22) are easily obtained through the simple algebraic relations of Eqs. (2.3.26) and (2.3.27). The involved infinite integrals are calculated numerically. Clearly the first-order coefficients $A_{n}^{(1)}$ necessitate the computation of the leading, zeroth order coefficients $A_{n}^{(0)}$.


Fig. 2.3.4. Leading order expansion coefficients $A_{n}^{(0)}(t)$ vs time, for the first five modes. Here, $R=0.5 \mathrm{~m}, V=3 \mathrm{~m} / \mathrm{s}$.


Fig. 2.3.5. First order expansion coefficients $A_{n}^{(1)}(t)$ vs time, for the first five modes. Here, $R=0.5 \mathrm{~m}, V=3 \mathrm{~m} / \mathrm{s}$.

Figs. 2.3.4 and 2.3.5 show the behavior of those coefficients, for the first five modes, over a short period of time $t \in(0,0.01]$. The leading order coefficients are almost constant in time. Their values have a clear declining tendency towards zero with increasing mode number. In contrast, the first order coefficients, demonstrate a more radical reduction, especially in the first moments of the impact. At later stages, they seem to obtain a constant value.

Returning back to the bvp, integration of the kinematic condition, Eq. (2.2.10), of the wave front beyond the instantaneous contact line yields

$$
\begin{equation*}
\eta(y, z, t)=\int_{0}^{t} \frac{\partial \phi(0, y, z, \tau)}{\partial x} d \tau \tag{2.3.28}
\end{equation*}
$$

Clearly, the constant of integration is zero since no disturbance exists at the time instant $t=0$ and the wave front is completely flat.

Having satisfied all boundary conditions and having calculated the unknown expansion coefficients, the velocity potential is obtained by combining Eqs. (2.3.6), (2.3.9) and (2.3.11) as

$$
\begin{align*}
\phi(x, y, z, t)= & \sqrt{\frac{\pi y}{2}} \sum_{n=1}^{\infty} A_{n}(z, t) \sin \left(\lambda_{n} z\right)  \tag{2.3.29}\\
& \times \int_{0}^{\infty} u^{1 / 2} \frac{J_{2}\left[b(z, t) \sqrt{u^{2}+\lambda_{n}^{2}}\right]}{\left(u^{2}+\lambda_{n}^{2}\right)} J_{-1 / 2}(u y) e^{-x \sqrt{u^{2}+\lambda_{n}^{2}}} d u .
\end{align*}
$$

Lastly, introducing Eq. (2.3.29) in Eq. (2.3.28), differentiating with respect to $x$ and evaluating on the instantaneous contact line, one gets

$$
\begin{align*}
& \eta\left[b_{0}(t)+\epsilon(z, t) b_{0}(t), z, t\right]=-\sqrt{\frac{\pi\left[b_{0}(t)+\epsilon(z, t) b_{0}(t)\right]}{2}} \\
& \times \sum_{n=1}^{\infty} \sin \left(\lambda_{n} z\right) \int_{0}^{t}\left[A_{n}^{(0)}(\tau)+\epsilon(z, \tau) A_{n}^{(1)}(\tau)\right]  \tag{2.3.30}\\
& \times \int_{0}^{\infty} u^{1 / 2} \frac{J_{2}\left\{\left[b_{0}(\tau)+\epsilon(z, \tau) b_{0}(\tau)\right] \sqrt{u^{2}+\lambda_{n}^{2}}\right\}}{\left(u^{2}+\lambda_{n}^{2}\right)^{1 / 2}} \\
& \times J_{-1 / 2}\left\{u\left[b_{0}(\tau)+\epsilon(z, \tau) b_{0}(\tau)\right]\right\} d u d \tau .
\end{align*}
$$

Eqs. (2.3.30) and (2.2.15) form a nonlinear algebraic equation to be solved in terms of the unknown small parameter $\epsilon(z, t)$. For this purpose, "fzero" command was used. At this point, it is necessary to provide some details about the adopted procedure. For a given time $t$, the interval $[0, t]$ was discretized into a sufficient number of nodes $(\tau)$ to secure convergence. For each of these nodes, we calculated the von Karman width $b_{0}(\tau)$ through Eq. (2.3.18). The double integration in Eq. (2.3.30) was performed numerically for every mode $n$, each time instant $\tau$, at a fixed position $z$. Specifically, the time integration was executed using the "trapz" command, while that of the Bessel functions with "integral". All the aforementioned commands are MATLAB's built-in functions. The expansion coefficients $A_{n}^{(0)}(\tau)$ and $A_{n}^{(1)}(\tau)$ are obtained using Eqs. (2.3.26) and (2.3.27). Once the parameter $\epsilon(z, t)$ is calculated, the Wagner 3D instantaneous contact line $b(z, t)$ can be explicitly determined with the aid of Eq. (2.3.17). The last step of the solution is the calculation of the velocity potential through Eq. (2.3.29), on the impacted area of the cylinder for $x=0$. Having calculated $\epsilon(z, \tau)$, the velocity potential follows from

$$
\begin{align*}
& \phi(x, y, z, t)=\sqrt{\frac{\pi y}{2}} \sum_{n=1}^{\infty}\left[A_{n}^{(0)}(t)+\epsilon(z, t) A_{n}^{(1)}(t)\right] \sin \left(\lambda_{n} z\right) \\
& \times \int_{0}^{\infty} u^{1 / 2} \frac{J_{\mu}\left\{\left[b_{0}(t)+\epsilon(z, t) b_{0}(t)\right] \sqrt{u^{2}+\lambda_{n}^{2}}\right\}}{\left(u^{2}+\lambda_{n}^{2}\right)} J_{-1 / 2}(u y) e^{-x \sqrt{u^{2}+\lambda_{n}^{2}}} d u . \tag{2.3.31}
\end{align*}
$$

In the following we provide some numerical results for the instantaneous contact line and the velocity potential on the impacted area. Two model cases are considered, defined by a fixed radius $R=0.5 \mathrm{~m}$ and two distinct velocities $V=1 \mathrm{~m} / \mathrm{s}$ and $V=$ $3 \mathrm{~m} / \mathrm{s}$.


Fig. 2.3.6. Evolution of the instantaneous contact lines during the impact of a steep wave that collides with a steady velocity $V=1 \mathrm{~m} / \mathrm{s}$ with a vertical circular cylinder with radius $R=$ 0.5 m .


Fig. 2.3.7. Evolution of the instantaneous contact lines during the impact of a steep wave that collides with a steady velocity $V=3 \mathrm{~m} / \mathrm{s}$ with a vertical circular cylinder with radius $R=$ 0.5 m .

Figs. 2.3.6 and 2.3.7 depict the motion of the instantaneous contact line between the liquid (using the 3D Wagner steep wave impact model elaborated in the present) and the vertical cylinder. These snapshots determine essentially the half contact area between the liquid and the solid. The first remark is that the contact line is actually vertical along nearly the total height of the cylinder, a fact that verifies the accuracy of the fundamental assumption employed in the present, according to which the semiwidth $b(z, t)$ is weakly dependent on $z$, a consideration which maintains the Laplace
equation intact. However, one should note the bend of the contact line towards the upper free-surface. Finally, it is remarked that the curvature is larger as the phenomenon is evolving, while the impact of the collision is more intense at the beginning of the phenomenon.

Moreover, while formulating the governing hydrodynamic bvp, in Section 2.2, we noted that the Wagner impact theory holds under the assumption that the penetration depth is much smaller than the horizontal dimension of the contact area, so that the boundary conditions can be imposed on its projection. This statement can be verified e.g., with the aid of Fig. 2.3.7. For the time instant $t=0.005 \mathrm{~s}$, the penetration depth of the cylinder is $d=V t=0.015 \mathrm{~m}$ and the width of the contact line, at a section $z=$ -0.4 , is $b=0.1567$. A straightforward comparison gives $V t / b=0.096 \ll 1$.

One of the most important queries that this study intends to answer is the range of validity of the 2D Wagner approach when applied to 3D convex geometries such as vertical circular cylinders. A credible answer can be given using the results depicted in Figs. 2.3.8 and 2.3.9, which show the normalized semi-width of the total impacted area of a vertical circular cylinder, again with fixed radius $R=0.5 \mathrm{~m}$ and subjected to a steep wave impact with constant velocities $V=1 \mathrm{~m} / \mathrm{s}$ and $V=3 \mathrm{~m} / \mathrm{s}$. The time is taken in the nondimensional form $\tau=V t / R$. Figs. 2.3.8 and 2.3.9 originate actually from Figs. 2.3.6 and 2.3.7, if one computes the evolution of a contact point by considering a section at a specific height of the cylinder. The results presented are for a section at $z=-0.7$. Three basic concepts are considered and compared: the simplified von Karman approximation [von Karman (1929)], the 2D Wagner approach [Wagner (1932), Mei et al. (1999)] and the present 3D sophisticated solution. The von Karman contact line is explicitly determined by Eq. (2.3.18). The corresponding 2D Wagner solid-liquid intersection point is obtained by [Korobkin (2004)]

$$
\begin{equation*}
\tau=1-\frac{2}{\pi} E\left[\frac{b(\tau)}{R}\right], \tag{2.3.32}
\end{equation*}
$$

where $E$ denotes the complete elliptic integral of the second kind. The 3D Wagner solution is approximated by the position of the contact line, that extends vertically almost along the complete height of the cylinder. That is, the upper bend of the liquidsolid intersection is omitted. Nevertheless, even with the employed simplification one could draw the conclusion that the von Karman approach underestimates the surface of the contact area, while the 2D Wagner approach provides overestimated results given that it leads to broader liquid-solid contact regions, which are progressively increased with time. The 2D Wagner solution of a circular cross section penetrating a liquid could be applied to some extent, in practical 3D problems, for small water impact velocities and at the very beginning of the collision. The 2D and 3D Wagner solutions are almost identical for a dimensionless time duration of $V t / R \approx 0.003$ which leads to the real time of $t=0.0015 \mathrm{~s}$, for the first impact scenario, while for the second one, where higher impact velocity in considered, this dimensional time leads to the infinitesimal time duration of $t=0.0005 \mathrm{~s}$. Both examples prove that the 2D Wagner solution has an extremely limited range of validity, when applied in realistic 3D convex geometries. In addition, higher impact velocities lead to larger discrepancies (Fig. 2.3.9), a fact that implies that the 2D Wagner solution, when employed to 3D geometries, misses
information, especially in the vicinity of the contact point, and unavoidably leads to larger hydrodynamic loads.


Fig. 2.3.8. Evolution of the position of the intersection point during the impact of a steep wave that collides with a steady velocity $V=1 \mathrm{~m} / \mathrm{s}$ with a vertical circular cylinder with radius $R=0.5 \mathrm{~m}$.


Fig. 2.3.9. Evolution of the position of the intersection point during the impact of a steep wave that collides with a steady velocity $V=3 \mathrm{~m} / \mathrm{s}$ with a vertical circular cylinder with radius $R=0.5 \mathrm{~m}$.

As the impact evolves and the cylinder keeps penetrating the liquid, the contact area, evidently, expands continuously. Nevertheless, it is definitely challenging to investigate the rate of expansion of the contact area and how this is correlated to the impulsive
nature of the phenomenon. At first, a general idea may be extracted by a careful observation of the distance between two consecutive curves, in Figs. 2.3.6 and 2.3.7. At later stages, the contact lines are closer to each other. This fact implies that the rate of expansion of the contact area decreases with time. Moreover, this conclusion can be clearly reached if one estimates the slope of the curves of Figs. 2.3.8 and 2.3.9, at several time instants. However, a more consistent answer to this question is required. To this end, Tables 2.3.1-2.3.4 and Figs. 2.3.10-2.3.11 are provided, which depict the rate of expansion of the contact area between the cylinder and the liquid, as predicted by the three theories, against time. In order to calculate these rates, the finite difference method was utilized. For the first and the last time instant, the forward and backward method of accuracy 1 was applied respectively, whereas for the intermediate the central method of accuracy 2 was used.

Table 2.3.1. The evolution of the semi-widths of the contact line according to the three theories for a section at $z=-0.7$. A steep wave collides with a steady velocity $V=1 \mathrm{~m} / \mathrm{s}$ with a vertical circular cylinder with radius $R=0.5 \mathrm{~m}$.

| $t_{i}(\mathrm{~s})$ | $b_{0}^{v K}(\mathrm{~m})$ | $b^{W 2 D}(\mathrm{~m})$ | $b^{W 3 D}(\mathrm{~m})$ |
| :---: | :---: | :---: | :---: |
| 0.001 | 0.0316 | 0.0435 | 0.0432 |
| 0.002 | 0.0447 | 0.0613 | 0.0604 |
| 0.003 | 0.0547 | 0.0745 | 0.0735 |
| 0.004 | 0.0631 | 0.0865 | 0.0845 |
| 0.005 | 0.0705 | 0.0970 | 0.0941 |
| 0.006 | 0.0772 | 0.1065 | 0.1028 |
| 0.007 | 0.0834 | 0.1153 | 0.1108 |
| 0.008 | 0.0891 | 0.1234 | 0.1181 |
| 0.009 | 0.0944 | 0.1310 | 0.1250 |
| 0.01 | 0.0995 | 0.1383 | 0.1315 |

Table 2.3.2. The rate of expansion of the contact area according to the three theories. for a section at $z=-0.7$. A steep wave collides with a steady velocity $V=1 \mathrm{~m} / \mathrm{s}$ with a vertical circular cylinder with radius $R=0.5 \mathrm{~m}$.

| $t_{i}(\mathrm{~s})$ | $\frac{d b_{0}^{v K}}{d t}(\mathrm{~m} / \mathrm{s})$ | $\frac{d b^{W 2 D}}{d t}(\mathrm{~m} / \mathrm{s})$ | $\frac{d b^{W 3 D}}{d t}(\mathrm{~m} / \mathrm{s})$ |
| :---: | :---: | :---: | :---: |
|  | 13.10 | 17.8 | 17.20 |
| 0.002 | 11.55 | 15.50 | 15.15 |
| 0.003 | 9.20 | 12.60 | 12.05 |
| 0.004 | 7.90 | 11.25 | 10.30 |
| 0.005 | 7.05 | 10.00 | 9.15 |
| 0.006 | 6.45 | 9.15 | 8.35 |
| 0.007 | 5.95 | 8.45 | 7.65 |
| 0.008 | 5.50 | 7.85 | 7.10 |
| 0.009 | 5.20 | 7.45 | 6.70 |
| 0.0095 | 5.10 | 7.30 | 6.50 |



Fig. 2.3.10. The rate of expansion of the cylinder's contact area relative to the impact velocity for a steep wave that collides with a steady velocity $V=1 \mathrm{~m} / \mathrm{s}$ with a vertical circular cylinder with radius $R=0.5 \mathrm{~m}$.

Table 2.3.3. The evolution of the semi-widths of the contact line according to the three theories for a section at $z=-0.7$. A steep wave collides with a steady velocity $V=3 \mathrm{~m} / \mathrm{s}$ with a vertical circular cylinder with radius $R=0.5 \mathrm{~m}$.

| $t_{i}(\mathrm{~s})$ | $b_{0}^{v K}(\mathrm{~m})$ | $b^{W 2 D}(\mathrm{~m})$ | $b^{W 3 D}(\mathrm{~m})$ |
| :---: | :---: | :---: | :---: |
| 0.001 | 0.0547 | 0.0815 | 0.0735 |
| 0.002 | 0.0772 | 0.1115 | 0.1028 |
| 0.003 | 0.0944 | 0.1340 | 0.1250 |
| 0.004 | 0.1089 | 0.1540 | 0.1435 |
| 0.005 | 0.1216 | 0.1730 | 0.1596 |
| 0.006 | 0.1330 | 0.1890 | 0.1740 |
| 0.007 | 0.1434 | 0.2050 | 0.1872 |
| 0.008 | 0.1530 | 0.2190 | 0.1995 |
| 0.009 | 0.1621 | 0.2315 | 0.2093 |
| 0.01 | 0.1706 | 0.2435 | 0.2184 |

Table 2.3.4. The rate of expansion of the contact area according to the three theories. for a section at $z=-0.7$. A steep wave collides with a steady velocity $V=3 \mathrm{~m} / \mathrm{s}$ with a vertical circular cylinder with radius $R=0.5 \mathrm{~m}$.

| $t_{i}(\mathrm{~s})$ | $\frac{d b_{0}^{v K}}{d t}(\mathrm{~m} / \mathrm{s})$ | $\frac{d b^{W 2 D}}{d t}(\mathrm{~m} / \mathrm{s})$ | $\frac{d b^{W 3 D}}{d t}(\mathrm{~m} / \mathrm{s})$ |
| :---: | :---: | :---: | :---: |
| 0.0015 | 22.50 | 30.00 | 29.30 |
| 0.002 | 19.85 | 26.25 | 25.75 |
| 0.003 | 15.85 | 21.25 | 20.35 |
| 0.004 | 13.60 | 19.50 | 17.30 |
| 0.005 | 12.05 | 17.50 | 15.25 |
| 0.006 | 10.90 | 16.00 | 13.80 |
| 0.007 | 10.00 | 15.00 | 12.75 |
| 0.008 | 9.35 | 13.25 | 11.05 |
| 0.009 | 8.80 | 12.25 | 9.45 |
| 0.0095 | 8.50 | 12.00 | 9.10 |



Fig. 2.3.11. The rate of expansion of the cylinder's contact area relative to the impact velocity for a steep wave that collides with a steady velocity $V=3 \mathrm{~m} / \mathrm{s}$ with a vertical circular cylinder with radius $R=0.5 \mathrm{~m}$.

From Figs. 2.3.10 and 2.3.11, the constantly decreasing rate of expansion of the contact area is indeed ascertained, ensuring the impulsive nature of the phenomenon, especially at the early stages. Literally, this rate is the corresponding velocity of expansion of the contact area. It is interesting to comment that this velocity is many times higher than the impact velocity. It is noticed that the rate of expansion of the contact area within the 3D Wagner solution is lying between those of the 2D Wagner and von Karman approaches. For smaller impact velocities, the rate obtained by the presented sophisticated method, is close enough to the one predicted by the 2D Wagner solution, while larger deviations occur at later stages, similarly to the results for the evolution of the position of the intersection point (Figs. 2.3.8 and 2.3.9). On the other
hand, for higher impact velocities, these variations are more intense and the solution given by the 3D Wagner theory tends to coincide with the von Karman's approximation at later stages.


Fig. 2.3.12. The vertical variation of the absolute, normalized velocity potential for several sections along the $y$-axis, during the impact of a steep wave that collides with a steady velocity $V=1 \mathrm{~m} / \mathrm{s}$ with a vertical circular cylinder with radius $R=0.5 \mathrm{~m}$, at the time instant

$$
t=0.001 \mathrm{~s} . \text { Here } b=0.042 \mathrm{~m}
$$



Fig. 2.3.13. The vertical variation of the absolute, normalized velocity potential for several sections along the $y$-axis, during the impact of a steep wave that collides with a steady velocity $V=3 \mathrm{~m} / \mathrm{s}$ with a vertical circular cylinder with radius $R=0.5 \mathrm{~m}$, at the time instant $t=0.001 \mathrm{~s}$. Here $b=0.073 \mathrm{~m}$.

Having calculated the contact area, we can now proceed to the computation of the velocity potential, with the aid of Eq. (2.3.31) at a specific time instant and at the projection of the contact area at $x=0$. The boundary conditions dictate that the potential must be zero i) on the upper free-surface [Eq. (2.2.8)] and ii) beyond the contact area on the wave front [Eq. (2.2.9)]. Figs. 2.3.12 and 2.3.13 show the behavior of the absolute, normalized velocity potential in the vertical direction for several sections along the horizontal $y$-axis, at the time instant $t=0.001 \mathrm{~s}$. The values of the $y$-sections are relative to the centerline of the wetted area $(y=0)$. The decaying tendency of the velocity potential towards zero is obvious, as the section approaches the limit of the contact line $[y \rightarrow b(z ; t)]$, while the maximum value is encountered in the centerline of the contact area. Moreover, a clear convergence to zero is observed, close to the free-surface.

Next, Figs. 2.3.14 and 2.3.15 demonstrate the behavior of the absolute, normalized velocity potential along the horizontal $y$-axis, for several $z$ - sections. Once again, it is verified that the potential is indeed zero at the contact line limit, whereas it obtains its maximum value at the centerline of the contact area. In addition, the velocity potential curves are almost identical as moving farther from the free-surface to the bottom of the cylinder. This proves that the velocity potential on the contact area exhibits a specific, constant configuration, while the vertical dependence seems to be rather moderate.


Fig. 2.3.14. The horizontal variation of the absolute, normalized velocity potential for several sections along the $z$-axis, during the impact of a steep wave that collides with a steady velocity $V=1 \mathrm{~m} / \mathrm{s}$ with a vertical circular cylinder with radius $R=0.5 \mathrm{~m}$, at the time instant $t=0.001 \mathrm{~s}$.


Fig. 2.3.15. The horizontal variation of the absolute, normalized velocity potential for several sections along the $z$-axis, during the impact of a steep wave that collides with a steady velocity $V=3 \mathrm{~m} / \mathrm{s}$ with a vertical circular cylinder with radius $R=0.5 \mathrm{~m}$, at the time instant

$$
t=0.001 \mathrm{~s}
$$

A rational question, which has not been answered yet, is whether the impact is of the same intensity in the whole contact area. In other words, where the potential - or analogously the hydrodynamic pressure - obtain its maximum value.


Fig. 2.3.16. The distribution of the absolute velocity potential on the contact area during the impact of a steep wave that collides with a steady velocity $V=1 \mathrm{~m} / \mathrm{s}$ with a vertical circular cylinder with radius $R=0.5 \mathrm{~m}$, at the time instant $t=0.001 \mathrm{~s}$. Here $b=0.042 \mathrm{~m}$.


Fig. 2.3.17. The distribution of the absolute velocity potential on the contact area during the impact of a steep wave that collides with a steady velocity $V=3 \mathrm{~m} / \mathrm{s}$ with a vertical circular cylinder with radius $R=0.5 \mathrm{~m}$, at the time instant $t=0.001 \mathrm{~s}$. Here $b=0.073 \mathrm{~m}$.

The results depicted in Figs. 2.3.16 and 2.3.17 can be considered of paramount importance, particularly for designing purposes. They present the absolute values of the velocity potential distribution on the contact area, accompanied with a colourbar, indicative of the intensity of the impact. Clearly, the potential obtains its higher values in a zone extending laterally from the centerline to the one third of the contact area. The potential, as expected, tends to zero in a region close to the free-surface and close to the contact line. Once again, it is concluded that in the vertical direction, no significant fluctuations occur.

### 2.3.2 The wave front displacement

The shapes of the wave front at several fixed sections along the cylinder are depicted in Figs. 2.3.18 and 2.3.19. It is shown that they do exhibit the curved shape, as expected by the Wagner theory for the water pile-up beyond the contact line. The shape of the wave front progressively decays at large distances from the solid until it is practically level. Here, the disturbances are eliminated at approximately $5 b$ from the center of the cylinder. The results presented correspond to the time instant $t=0.01 \mathrm{~s}$. It is noticed that the deformation of the wave front is not identical along the cylinder; it enlarges as moving from the free-surface to the bottom. Despite the fact that the differences are indeed slight (as suggested by the fundamental assumption for the weak vertical dependence of the contact line), and recalling in mind Fig. 2.1.3, this fact suggests that in the case of a cylinder, the contact point is not constant along its main axis of symmetry and varies sideways along the vertical direction, as the water wraps the cylinder. Therefore, this statement substantially implies that the contact line is not completely vertical, but very small fluctuations occur.


Fig. 2.3.18. The wave front displacements, normalized by the instantaneous semi-width of the contact line during 3D Wagner steep wave impact, at several positions along the cylinder;

$$
V=1 \mathrm{~m} / \mathrm{s}, R=0.5 \mathrm{~m}
$$



Fig. 2.3.19. The wave front displacements, normalized by the instantaneous semi-width of the contact line during 3D Wagner steep wave impact, at several positions along the cylinder;

$$
V=3 \mathrm{~m} / \mathrm{s}, R=0.5 \mathrm{~m}
$$

### 2.3.3 The hydrodynamic pressure and the slamming force

The hydrodynamic pressure at a specific time instant, evaluated on $x=0$, is obtained using the nonlinear Bernoulli equation

$$
\begin{equation*}
p=-\rho \frac{\partial \phi}{\partial t}-\frac{1}{2} \rho|\nabla \phi|^{2} \tag{2.3.33}
\end{equation*}
$$

where $\rho=1025 \mathrm{~kg} / \mathrm{m}^{3}$ is the water density. Accordingly, the slamming force is given by direct pressure integration over the instantaneous impacted area $S$ of the cylinder as

$$
\begin{equation*}
F_{x}(t)=\int_{S} p(0, y, z, t) d y d z=\int_{-h}^{0}\left\{\int_{0}^{b(z, t)} p(0, y, z, t) d y\right\} d z \tag{2.3.34}
\end{equation*}
$$

Only a horizontal component of the slamming force is assumed, since the impact is not oblique and no rotation of the body occurs as well. For the purposes of the present, the operations in (2.3.33) and (2.3.34) are performed numerically reusing the finite difference formulae of the time derivative of the linear term and the built-in function of MATLAB "gradient" for the nonlinear term. The hydrodynamic pressure and accordingly the load, is evaluated at $t+\Delta t / 2$. Numerical results for the associated slamming forces are shown in Figs. 2.3.20 and 2.3.21. The depicted results provide actually half of the total force. Those are normalized by $\rho R^{2} V^{2}$ and are given as functions of the normalized time $V t / R$. The calculations that refer to the present method show both the load that corresponds to the linear term of Bernoulli's equation and the effect of the quadratic term of Eq. (2.3.33). The lower and the upper envelope curves of the 2D von Karman and the 2D Wagner approaches respectively, assume only the linear hydrodynamic pressure term and are calculated on the theoretical contact area described by these two theories.


Fig. 2.3.20. The normalized slamming force during impact of a steep wave onto a vertical cylinder; $V=1 \mathrm{~m} / \mathrm{s}, R=0.5 \mathrm{~m}$.


Fig. 2.3.21. The normalized slamming force during impact of a steep wave onto a vertical cylinder; $V=3 \mathrm{~m} / \mathrm{s}, R=0.5 \mathrm{~m}$.

Regarding the linear component, it is evident that even small differences in the variable (along $z$ ) contact area lead to substantial differences in loading, generating a 3D solution, which falls between the 2D von Karman and the 2D Wagner approximations, as it was truly expected. Once again, the discrepancies between the 2D Wagner and the 3D Wagner results are more emphasized with the evolution of time. In all cases, the slamming force follows a decreasing trend with small sloping, while the 3D solution exhibits a faster decrease. The decrease rate of the von Karman approach is negligible, especially in the section $V t / R \in(0,0.02]$. Moreover, it should be remarked that the effect of the quadratic term of Bernoulli's pressure is not actually negligible. Although it does not change the variation particulars of the total slamming force significantly, it is reduced by approximately $15 \%$. The depicted results demonstrate that the effect of the quadratic term is stronger as the time and/or the velocity are increased.

The strongest impact occurs at the beginning of the collision and accordingly the force is monotonically decreased. However, this observation cannot be conceived as a general conclusion for water impact problems. For example, the slamming force exerted on a sphere, which penetrates violently and vertically the still water surface, has a zero initial value, then gradually reaches a maximum and at later stages decreases, reaches a minimum and then increases again until the spere is fully submerged [Miloh (1981)]. Finally, it should be recalled that the presented method of investigation of 3D steep wave impact refers only to the initial stage; so does the slamming force. As the impact evolves, the assumptions made have to be clearly reexamined, as effects of cavity, ventilation, viscosity and flow separation cannot be neglected, influencing the force exerted on the body.

At this point, it is useful to examine the effect of the impact velocity and the radius of the cylinder on the impact force. Let us examine the loading in the time instant $t=$ 0.006 s . For a pair of parameters $(R, V)=(0.5,1)$ we get a nondimensional time of
$V t / R=0.012$ that leads to a nondimensional force of $\tilde{F}=7$ or $F=1793.8 \mathrm{~N}$. However, this force grows very fast for the case of impact $(R, V)=(0.5,3)$ and becomes $F=12684 \mathrm{~N}$. We deduce then that an increase on the impact velocity leads to a substantial increase in the slamming force, which was actually anticipated. On the other hand, for a pair of parameters $(R, V)=(0.2,1)$ we get the considerably reduced slamming $F=241.9 \mathrm{~N}$. As a conclusion, smaller radius leads to a reduction of the slamming load exerted on the cylinder.

For validation purposes, the results for the slamming force extracted by the presented method are compared to those reported by Chatjigeorgiou et al. (2016), who calculated the linear hydrodynamic loading acting on a vertical cylinder impacted by a steep wave. The underlying theory relies on the wave impact on a rigid vertical plate with specified width but infinitesimal thickness, an approach which enables the assimilation of the plate by an elliptical vertical cylinder with zero semi-minor axis. This assumption allowed the employment of elliptical harmonics for the solution of the Laplace equation and involved the special Mathieu functions. The study was extended to approximate the hydrodynamic loading on a vertical cylinder for which, however, a specified wetted width was required. The width was determined using the 2D Wagner and the 2D von Karman models of the water entry of a circle. The "plate" approximation prespecifies a vertical contact line and it is evident therefore that the methodology adopted in the present study takes the theory a step further.


Fig. 2.3.22. Comparison of solution methodologies for the 3D Wagner steep wave impact problem; solid (middle) line: present approach; envelope lines von Karman 3D (dashed line) and Wagner 3D (chained line) approaches, taken by Chatjigeorgiou et al. (2016). The obtained results correspond to $R=0.5 \mathrm{~m}, V=3 \mathrm{~m} / \mathrm{s}$.

The solid line of Fig. 2.3.22 corresponds to the linear term of Bernoulli's hydrodynamic pressure so that there is a direct comparison with the assumptions of Chatjigeorgiou et al. (2016). Clearly, the present integrated approach, which relies on the assumption of a vertical contact line that varies weakly with $z$, yields computations that fall within the
area specified by the 2D von Karman and the 2D Wagner approaches, extended to 3D though. There is an underlying difference between the two approximations contrasted in Fig. 2.3.22. The present study assumes that the instantaneous contact line is unknown and should be obtained as part of the integrated solution sought, while the approach taken by Chatjigeorgiou et al. (2016) is relying on 2D impact models. Again, it is remarked that the 2D von Karman and the 2D Wagner models, under- and overestimate the actual 3D hydrodynamic loading as suggested by Korobkin (2004) and Mei et al. (1999).

### 2.3.4 The concept of pressure impulse

An important issue of concern, associated with hydrodynamic slamming problems, is the peak pressure exerted on the impacted solid. As firstly highlighted by Bagnold (1939), it is acknowledged that the peak pressures change unpredictably from one impact to the next (among repetitions of waves), even for eventually identical conditions of impact. However, the component that is often referred as pressure impulse is approximately constant at a fixed point on the impacted solid and accordingly, the brief collision between a volume of liquid and a rigid body is better quantified by this term. Pressure impulse has been mainly used to model flows induced instantaneously from rest, i.e., the impact of rigid body striking the surface of still water [Cointe and Armand (1987), Cointe (1989)]. Pressure impulse theory applied to the changes in the moving liquid domain that collides with a fixed structure was introduced by Cooker and Peregrine (1995). At a given time instant, pressure impulse may be defined, similarly to the widely known concept of impulse, as

$$
\begin{equation*}
P(x, y, z, t)=\int_{t}^{\mathrm{t}+\Delta t} p(x, y, z, t) d t \tag{2.3.35}
\end{equation*}
$$

where the pressure is approximated by the linear term of the Bernoulli equation. In other words, the pressure impulse is expressed by the change in the velocity potential at time $t$ during impact. Calculation of $P(x, y, z, t)$ allows the computation of the total impulse exerted on the structure as a function of time. Accordingly, denoting the total impulse by $I(t)$ it follows that

$$
\begin{equation*}
I(t)=-\rho \int_{S} \phi(0, y, z, t) d y d z=-\rho \int_{-h}^{0}\left\{\int_{0}^{b(z, t)} \phi(0, y, z, t) d y\right\} d z . \tag{2.3.36}
\end{equation*}
$$

Clearly, Eq. (2.3.36) provides half of the total impulse due to symmetry. Relevant calculations of the total impulse exerted on the vertical cylinder during impact are shown in Fig. 2.3.23.


Fig. 2.3.23. Total pressure impulse exerted on the cylinder during 3D Wagner steep wave impact.

In this Subsection, only the present 3D theory of Wagner impact was employed. The results have been normalized by the typical scale of the total impulse, i.e., $\rho V R^{3}$. Three different slamming cases are considered, defined by the pair $(R, V)$, which all show that the total impulse increases with time, in contrast to the total hydrodynamic loading. It is interesting to observe that a reduction in the cylinder's radius, while keeping the impact velocity constant, leads to an increase of the total impulse exerted on the cylinder. This conclusion opposes the one that the slamming force for a decreasing radius of a cylinder decreases as well, as already commented in Subsection 2.3.3. On the other hand, an increase of the impact velocity, leads to a higher total impulse, as it was expected.

### 2.3.5 Validation against CFD results

As already discussed, the fundamental assumption of our analysis is that the contact line varies weakly along the vertical direction. To support this assumption and to formalize the approach taken, we provide comparisons between the theoretical model, as outlined in the previous Sections, and the concerned 3D impact problem which is simulated through CFD computations, included in the study of Tsaousis et al. (2020). In order to avoid the run-up effect due to the impact, which is also ignored in the theoretical model, the CFD computations assume that the height of the cylinder is equal to the height of the wave front $(h=1 m)$. The concept of the simulation was as follows: a single test case was investigated that considers a cylinder with radius $R=0.5 \mathrm{~m}$, subjected to a steep wave impact that propagates towards the cylinder with constant velocity $V=1 \mathrm{~m} / \mathrm{s}$. The computational domain is depicted in Fig. 2.3.24 (a). The numerical results are shown in two planes: one that extends in the longitudinal direction and one that lies perpendicularly to the axis of the cylinder [Fig. 2.3.24 (b), Fig. 2.3.24 (c)]. The simulations were performed using a mesh composed by $5,328,864$ cells, with increasing density close to the contact region, as shown in Fig. 2.3.25. The
computations relied on the interDyMFoam algorithm of the OpenFOAM C++ package, which combines a dynamic mesh with a Volume of Fluid (VoF) solver.


Fig. 2.3.24. a) Geometry of the computational domain and the steep wave formulation. The simulation results are depicted on b) the longitudinal plane and c) the plane, which lies perpendicularly to the cylinder.


Fig. 2.3.25. Mesh densities close to the cylinder.
Figs. 2.3.26 and 2.3.27 provide a visual representation of the water impact on the vertical cylinder through side and top screenshots at selected time instants. From Fig. 2.3.26 it is immediately apparent that the instantaneous contact line in nearly vertical, which accordingly verifies the validity of the assumption taken in the theoretical model. In addition, it is clearly shown that the contact line exhibits a slight bend close to the top of the cylinder, which is in accord with the theoretical model (Figs. 2.3.6 and 2.3.7). With the evolution of the phenomenon, the contact line loses its cohesion and degenerates progressively. This fact validates the basis of the approach developed in
this Chapter, which is the small-time assumption. At the time instant $t=0.005 \mathrm{~s}$, the CFD model approximates the semi-width of the contact area $b$ to 0.12 m (middle screenshot in the first row of Fig. 2.3.26), while the theoretical model yields a value of $\sim 0.1 \mathrm{~m}$.


Fig. 2.3.26. Side screenshots of the steep wave impact on the vertical cylinder at selected time instants; $R=0.5 \mathrm{~m}, V=1 \mathrm{~m} / \mathrm{s}$.


Fig. 2.3.27. Top screenshots of the steep wave impact on the vertical cylinder at selected time instants; $R=0.5 \mathrm{~m}, V=1 \mathrm{~m} / \mathrm{s}$.

### 2.4 Conclusions

In this Chapter, an analytical model for the steep wave impact on a rigid, vertical circular cylinder was formulated. A breaking wave with a completely vertical wave front was assumed, in order to simulate the concept of flip-through impact, which for some researchers in the literature is considered as the worst case of loading. The governing hydrodynamic problem is simplified, in terms of the linearization of the boundary conditions, by adopting the small-time assumption, which enables the employment of the boundary condition on the projection of the real contact area. Nevertheless, the problem remains nonlinear, since the contact line is a priori unknown and has to be sought along with the fluid flow. The problem was solved within the context of 3D Wagner theory of water impact. The most important assumption for the validity of the solution is the weak dependence of the contact line on the vertical coordinate, which was indeed verified by the results. The results presented for the 3D Wagner problem are directly compared to those expected from the 2D von Karman and 2D Wagner approximations. Some results of the theoretical model were compared to those derived by CFD computations. The main outcomes of this research can be summarized as follows:
I. All the results obtained by the 3D Wagner solution related to the hydrodynamic physical quantities of the phenomenon, always lie between those from the 2 D von Karman and 2D Wagner approximations. This statement is in general wellknown in the literature. However, we commented on these variations and we answered the most important question explicitly. This question has to do with the range of validity of the 2D theories. The 2D von Karman theory clearly underestimates the contact area, even at the initial stage of the impact, and cannot be applied safely for 3D impact on convex geometries. On the other hand, the 2D Wagner solution gives comparative results to the 3D solution, in terms of the contact area, for a dimensionless time of $V t / R=0.004$. The discrepancies are more evident with the evolution of time, as well as for higher impact velocities.
II. The contact line is indeed weakly dependent on the vertical coordinate $z$, especially during the initial stage of impact, in which this study undoubtedly focuses. This fact was emphasized clearly by the CFD results. The bend of the contact line in the upper part of the cylinder was also verified by the numerical results. Nevertheless, small discrepancies do exist along the main axis of the cylinder.
III. The rate of expansion of the contact area (or in other words the velocity in which the contact area expands) is many times higher than the impact velocity.
IV. An important remark of this research is that we answered straightforwardly which portion of the cylinder undergoes the most intense impact. It has been proved that the highest impact pressure is localized in an area expanding from the centerline of the cylinder to the one third of the total contact area.
V. The wave front displacement, which actually reflects the deformation of the surface due to the impact, is larger as it moves from the free-surface to the bottom.
VI. The maximum of the slamming force is at the beginning of the impact, contrary to the impact of a sphere, and then decreases constantly with time, with an approximately constant slope. For a prescribed impact velocity and a specific time instant, we can reduce the slamming load on the cylinder by decreasing its radius.
VII. The normalized slamming force provided by all three methods seems to decrease almost linearly with time. Particularly, the force given by the von Karman theory is nearly constant, especially at the initial stage of the impact.
VIII. Although the vertical variations of the contact line are small, substantial differences in loading, from the 2D theories, occur.
IX. The total impulse exerted on the cylinder increases with time. In contrast to the slamming force, reducing the radius of the cylinder leads to a higher impulse.

### 2.5 Further research

The investigation conducted for the steep wave impact on a rigid, vertical circular cylinder has definitely filled several research gaps, while it has offered an alternative approach to the widely examined issue of the body water entry problem. However, further research should deal with the following fields:
I. It would be remarkable to provide an approximate solution for the inner flow problem, i.e., a small vicinity close to the contact point, which can be encompassed to the outer solution presented in this Chapter, through matching techniques. The jet and accordingly the splash which are formed at the contact point are reflected to a singularity for the velocity. It would be of particular interest to obtain results for the jet thickness, velocity and the pressure distribution in it.
II. The cylinder examined herein is rigid, which means no particular deformation occurs due to the impact. An approach including the elasticity of the body would be useful, in order to examine how the peculiar nature of the slamming force affects the body.
III. The model proposed suggests a constant impact velocity and a completely vertical wave front of the breaking wave, as we tried to simulate the flip-through impact. The model can be enhanced by assuming a vertical variation $V(z)$ of the velocity. This consideration will lead to a change of the breaking wave profile which hits the cylinder. This is of great significance, as the most important parameters of an impact are the location of the breaking and consequently the shape of the breaker.
IV. The cylinder was assumed vertical. It would be intriguing to investigate the impact of a steep wave to an inclined cylinder and therefore compare the slamming forces exerted on the body in each case.
V. In this analysis, the cylinder's height is supposed to be equal to the water's depth. A next step could be to examine the case of a cylinder higher than the water depth and accordingly take the front run-up of the water into account. It would be stimulating to derive results for the slamming force in this case, since the contact area changes significantly.
VI. A fundamental prerequisite for the validity of the methodology developed, is the linearization of the boundary conditions of the 3D Wagner impact problem. A more comprehensive solution would be extracted if the Dirichlet condition on the wave front was imposed on the real position of the free-surface $x=$ $-\eta(y, z ; t)$; by analogy to the Neumann condition on the impacted portion of the body.
VII. The effect of air entrapment is always important when examining water impact or more generally water entry problems. The air compressibility may affect the dynamics of the phenomenon. During the impact, escape of air is expected to occur in all possible directions.
VIII. The case of impact between an extreme wave, such as a freak/rogue wave, and a cylinder would be intriguing to be discussed, due to their nature and special wave characteristics.

At this point we need to mention that the last two suggestions may require numerical solutions which are out of the field of analytical solutions in water impact problems established in the present thesis.

# Chapter 3 Breaking wave impact on a vertical wall. Part 1. The leading order problem 

### 3.1 Theoretical modelling

Chapter 3 deals with the breaking wave impact on a vertical impermeable wall. The purpose of this study is to take the existing analytical researches one step further. To this end, a small air pocket is assumed to be entrapped during the collision and the setup of the problem takes the free-surface elevation into account. However, we assume that air entrapment does not affect the dynamics of the impact significantly. Walls (seawalls/breakwaters) are widely examined in Coastal Engineering, since they are designed to protect shores from the action of violent waves. Moreover, the approach adopted in this part of the thesis, can lead to the development of a simplified model for a barge impact loading condition.

To this end, we contemplate an idealized overturning-like wave, which is close to the concept of a plunging breaker. During the collision, a small air pocket is entrapped between the wave and the wall. The air pocket width is assumed negligible and accordingly compression effects are omitted. The dynamic pressure in the air pocket was taken equal to zero, a hypothesis which is valid if we consider that in the volume captured by the cavity there is nothing but vacuum. The problem is treated in 2D and is considered in a time instant just after impact. The main difference in the modelling between this case and the steep wave impact on a cylinder is that in the first case the impact is restricted to a part of a wall, whereas in the second the impact occurs along the total wave front. This kind of impact simulates also an impact by a focused wave generated by a wavemaker. The wave celerity affects the rate of focus of the wave definitely. Banks and Abdussamie (2017) investigated the dynamic behavior of a semisubmersible platform under the influence of extreme unidirectional waves generated by a piston type wavemaker. Fang et al. (2020) proposed a methodology to examine impact by extreme waves on coastal and offshore structures using first and second order wave maker theory.

The fluid is assumed inviscid, incompressible and the flow irrotational so that the tool of the linear potential theory can be utilized. In order to avoid satisfying the freesurface kinematic and dynamic boundary conditions on the a priori unknown freesurface elevation due to the collision, a Taylor series expansion around the still water level is employed. The hydrodynamic bvp is clearly of mixed type, owing to the Neumann and Dirichlet boundary conditions, which hold on different parts of the wall. The problem is formulated with nondimensional variables and is examined in its early stages. Accordingly, a perturbation method for the velocity potential and the freesurface elevation is applied, with the dimensionless time playing the role of the small parameter. This approach allows the fragmentation of the initial governing hydrodynamic problem in orders like of the dimensionless time parameter, a fact which actually permits the elimination of the time dependence on the solution sought. In this Chapter, we solve only for the leading order problem. The problem obtained requires a solution to a system of dual trigonometrical series. Results are presented for the distribution of the velocity potential on the wall and in the rest of the liquid domain, as well as for the free-surface elevation. The calculation of the free-surface elevation
reveals a logarithmic singularity exactly at the point where the undisturbed water level meets the wall. This singularity is removed with the aid of the methodology provided by King and Needham (1994).

The physical problem is formulated schematically in Fig. 3.1.1, which shows a highly idealized shape of the wave at a time instant immediately after the collision. The simplified breaking wave is bounded from the top by the initially undisturbed upper free-surface and propagates from right to left with constant velocity $V$. At $t=0(t$ is the time) the volume of water strikes the vertical impermeable wall that is situated at $x=0$. The free-surface runs-up the wall and elevates relative to the undisturbed freesurface by $H \equiv H(x, t)$. The undisturbed free-surface is situated at $z=0$. The water depth is constant and equal to $h$. During the brief collision a small air cavity is formed and air is entrapped between the liquid and the wall. The air pocket width is assumed to be negligible $(\delta \rightarrow 0)$ and it extends between $-h \leq z \leq-a$ in the Cartesian representation of Fig. 3.1.1. Air compression effects are omitted.


Fig. 3.1.1. Schematic representation of the breaking wave impact on a vertical impermeable wall situated at $x=0$. The air pocket extends between $-h \leq z \leq-a$ and is assumed of infinitesimal width, $\delta(z) \rightarrow 0$. The free-surface elevation, immediately after impact, is denoted by $H$; the water depth is constant and equal to $h$.

### 3.2 The governing mixed boundary value problem

We assume incompressible, inviscid fluid and irrotational flow. Hence, potential theory can be employed and the problem is treated in the Laplace domain. The problem will be considered in its nondimensional form using the following tilde variables

$$
\begin{aligned}
\tilde{x}=\frac{x}{h}, & \tilde{z}=\frac{z}{h}, & \tilde{a}=\frac{a}{h}, & \tilde{H}=\frac{H}{h}, \quad \tilde{\phi}=\frac{\phi}{V h^{\prime}} \\
T & =\frac{h}{V}, & \tilde{t}=\frac{t}{T}, & \tilde{p}=\frac{p}{\rho V^{2}},
\end{aligned}
$$

where $a$ is the upper point of the air pocket, $\phi$ denotes the velocity potential, $t$ is the time and $T$ is the time scale of the problem, $p$ is the hydrodynamic pressure and $\rho$ is the
water density. According to our approach, the impact condition is governed by the following bvp, expressed in a nondimensional form:

$$
\begin{gather*}
\nabla^{2} \tilde{\phi}=0, \quad \tilde{x} \geq 0, \quad-1 \leq \tilde{z} \leq 0,  \tag{3.2.1}\\
\frac{\partial \tilde{\phi}}{\partial \tilde{x}}=1, \quad \tilde{z} \in I_{1}, \quad-\tilde{a}<\tilde{z}<0, \quad \tilde{x}=0,  \tag{3.2.2}\\
\frac{\partial \tilde{\phi}}{\partial \tilde{t}}=0, \quad \tilde{z} \in I_{2}, \quad-1 \leq \tilde{z}<-\tilde{a}, \quad \tilde{x}=0,  \tag{3.2.3}\\
\frac{\partial \tilde{\phi}}{\partial \tilde{z}}=0, \quad \tilde{z}=-1, \quad \tilde{x} \geq 0,  \tag{3.2.4}\\
\frac{\partial \widetilde{H}}{\partial \tilde{t}}=\frac{\partial \tilde{\phi}}{\partial \tilde{z}}, \quad \tilde{z}=\tilde{H}, \quad \tilde{x} \geq 0,  \tag{3.2.5}\\
\frac{\partial \tilde{\phi}}{\partial \tilde{t}}+\frac{g h}{V^{2}} \widetilde{H}=0, \quad \tilde{z}=\tilde{H}, \quad \tilde{x} \geq 0,  \tag{3.2.6}\\
\tilde{\phi} \rightarrow 0, \quad \tilde{x} \rightarrow \infty, \quad-1 \leq \tilde{z} \leq 0 . \tag{3.2.7}
\end{gather*}
$$

Eq. (3.2.1) is the Laplace equation which holds in the entire liquid domain. Eq. (3.2.2) is the normalized Neumann condition due to the collision between the wall and the corresponding portion of the water, while Eq. (3.2.3) is the air pocket dynamic condition, imposed precisely on the wall due to the negligible width of the air pocket. The relevant condition is represented by a single term that arises from the Bernoulli's equation after omitting the quadratic velocity component and the hydrostatic term. At this point we should note the difference between the present model and the model of Eq. (2.2.9). In the first case the pressure is assumed to be zero beyond the contact line, whereas in this case the hydrodynamic pressure is zero in the air pocket. The kinematic condition in the cavity area is not considered (in this Section), which implies that no run-up is expected to occur at the limits of the air pocket. Furthermore, Eq. (3.2.4) holds on the impermeable bottom, while Eqs. (3.2.5) and (3.2.6) are the kinematic and the dynamic boundary conditions of the upper free-surface, respectively. It should be noted that Eq. (3.2.6) takes the hydrostatic term of the Bernoulli's equation into account, because of the free-surface elevation (run-up) immediately after the first impact. Again, we have omitted the quadratic terms from both the dynamic and the kinematic boundary conditions of the upper free-surface. In completing the presentation of the governing bvp, it is mentioned that Eq. (3.2.7) is the far-field boundary condition, which implies that any disturbance caused by the collision must vanish at infinity.

It is interesting to observe that the normalization process introduces the nondimensional parameter $k \sim g h / V^{2}$, which is the square of the inverse Froude number based on the depth, $F n=V / \sqrt{g h}$, i.e.,

$$
k=\frac{g h}{V^{2}}=\left(\frac{1}{F n}\right)^{2} .
$$

The parameter introduced, correlates the impact velocity with the gravity acceleration and the water depth and is involved in the solution of the higher order problem. It particularly means that gravity effects are not important in the leading order approach. In the following steps we omit the tilde and we analyze the problem in normalized form.

### 3.3 Perturbation analysis

The phenomenon is examined only at the very early stages of the impact $(t \rightarrow 0)$. Conditions (3.2.5) and (3.2.6) are imposed on the unknown boundary at $z=H$. In order to cope with this difficulty, we apply a Taylor series expansion around the still water level at $z=0$, in accord with the Stokes theory, employed in regular wave hydrodynamics. Hence, Eqs. (3.2.5) and (3.2.6) become

$$
\begin{gather*}
\frac{\partial H}{\partial t}=\frac{\partial \phi}{\partial z}+H \frac{\partial^{2} \phi}{\partial z^{2}}+\cdots, \quad z=0, \quad x \geq 0,  \tag{3.3.1}\\
\frac{\partial \phi}{\partial t}+\frac{\partial H}{\partial t} \frac{\partial \phi}{\partial z}+H \frac{\partial^{2} \phi}{\partial z \partial t}+\cdots+k H=0, \quad z=0, \quad x \geq 0 . \tag{3.3.2}
\end{gather*}
$$

Clearly, the effort put into reducing the original conditions (3.2.5) and (3.2.6) relative to the undisturbed free-surface resulted in the nonlinear expansions (3.3.1) and (3.3.2). To further simplify the associated relations and make the bvp solvable, we employ the method of the Stokes perturbations, first introduced by Stokes himself [Lamb (1932)]. In regular wave hydrodynamics, Stokes suggested the expansion of the velocity potential and the free-surface elevation in perturbation series using the wave steepness as a perturbation parameter, which is by default a very small parameter for gravity waves. Here, we employ the same approach using the time as a scaling factor, which indeed obtains very small values given that the problem is considered at the very early stages of the impact. Accordingly, following the expansion proposed by King and Needham (1994), the normalized velocity potential and the upper free-surface elevation are written as

$$
\begin{align*}
& \phi=t \phi_{1}+t^{3} \phi_{3}+\cdots  \tag{3.3.3}\\
& H=t^{2} \eta_{2}+t^{4} \eta_{4}+\cdots \tag{3.3.4}
\end{align*}
$$

Using Eqs. (3.3.3) and (3.3.4), Eqs. (3.3.1) and (3.3.2) are transformed into

$$
\begin{align*}
2 t \eta_{2}+4 t^{3} \eta_{4}+\cdots & =t \frac{\partial \phi_{1}}{\partial z}+t^{3}\left(\frac{\partial \phi_{3}}{\partial z}+\eta_{2} \frac{\partial^{2} \phi_{1}}{\partial z^{2}}\right)+\cdots,  \tag{3.3.5}\\
z & =0, \quad x \geq 0
\end{align*}
$$

$$
\begin{gather*}
\phi_{1}+t^{2}\left(3 \phi_{3}+2 \eta_{2} \frac{\partial \phi_{1}}{\partial z}+\eta_{2} \frac{\partial \phi_{1}}{\partial z}+\frac{g h}{V^{2}} \eta_{2}\right)+\cdots=0,  \tag{3.3.6}\\
z=0, \quad x \geq 0 .
\end{gather*}
$$

Eqs. (3.3.5) and (3.3.6) correspond to the kinematic and the dynamic boundary conditions of the free-surface. Equating like powers of $t$, we obtain the perturbation systems at $O(t)$ and $O\left(t^{3}\right)$ in terms of the leading- and higher-order potentials $\phi_{1}$ and $\phi_{3}$. Higher order terms are neglected. Clearly, the derivation of $\phi_{3}$ through the associated system dictates the solution of the leading order problem.

### 3.3.1 Boundary value problem for the leading order potential $\boldsymbol{\phi}_{\mathbf{1}}$

The bvp for the leading order reads

$$
\begin{gather*}
\nabla^{2} \phi_{1}=0, \quad x \geq 0, \quad-1 \leq z \leq 0,  \tag{3.3.7}\\
\frac{\partial \phi_{1}}{\partial x}=1, \quad z \in I_{1}, \quad-a<z<0, \quad x=0,  \tag{3.3.8}\\
\phi_{1}=0, \quad z \in I_{2}, \quad-1 \leq z<-a, \quad x=0,  \tag{3.3.9}\\
\frac{\partial \phi_{1}}{\partial z}=0, \quad z=-1, \quad x>0,  \tag{3.3.10}\\
\phi_{1}=0, \quad z=0, \quad x>0,  \tag{3.3.11}\\
\phi_{1} \rightarrow 0, \quad x \rightarrow \infty, \quad-1 \leq z \leq 0, \tag{3.3.12}
\end{gather*}
$$

The elevation $\eta_{2}$ is obtained from Eq. (3.3.5) after equating powers of $t$ as

$$
\begin{equation*}
\eta_{2}=\frac{1}{2} \frac{\partial \phi_{1}}{\partial z} \tag{3.3.13}
\end{equation*}
$$

### 3.3.2 Boundary value problem for the higher order velocity potential $\boldsymbol{\phi}_{\mathbf{3}}$

The resulting higher order bvp in terms of $\phi_{3}$ reads

$$
\begin{gather*}
\nabla^{2} \phi_{3}=0, \quad x \geq 0, \quad-1 \leq z \leq 0  \tag{3.3.14}\\
\frac{\partial \phi_{3}}{\partial x}=0, \quad z \in I_{1}, \quad-a<z<0, \quad x=0,  \tag{3.3.15}\\
\phi_{3}=0, \quad z \in I_{2}, \quad-1 \leq z<-a, \quad x=0,  \tag{3.3.16}\\
\frac{\partial \phi_{3}}{\partial z}=0, \quad z=-1, \quad x \geq 0  \tag{3.3.17}\\
\phi_{3}=-\eta_{2} \frac{\partial \phi_{1}}{\partial z}-\frac{k}{3} \eta_{2}, \quad z=0, \quad x \geq 0 \tag{3.3.18}
\end{gather*}
$$

$$
\begin{equation*}
\phi_{3} \rightarrow 0, \quad x \rightarrow \infty, \quad-1 \leq z \leq 0 \tag{3.3.19}
\end{equation*}
$$

The elevation $\eta_{4}$ is obtained from Eq. (3.3.5) after equating powers of $t^{3}$ as

$$
\begin{equation*}
\eta_{4}=\frac{1}{4} \frac{\partial \phi_{3}}{\partial z}+\frac{1}{4} \eta_{2} \frac{\partial^{2} \phi_{1}}{\partial z^{2}} . \tag{3.3.20}
\end{equation*}
$$

### 3.4 Solution to the leading order problem

### 3.4.1 The distribution of the velocity potential on the wall and on the rest of the liquid domain

As already mentioned in Section 3.1, the solution to the leading order problem only, will be presented in the analysis of the present Chapter 3. The solution to the higher order problem will be discussed in Chapter 4. The solution to the leading order problem satisfies Eqs. (3.3.7)-(3.3.12). Applying separable solutions for the Laplace equation (3.3.7) it can be shown that the normalized velocity potential is given by

$$
\begin{equation*}
\phi_{1}=\sum_{n=1}^{\infty} B_{n} \sin \left(\lambda_{n} z\right) e^{-\lambda_{n} x}, \quad \lambda_{n}=\left(n-\frac{1}{2}\right) \pi \tag{3.4.1}
\end{equation*}
$$

where $B_{n}$ are unknown expansion coefficients to be determined. Eq. (3.4.1) satisfies Eqs. (3.3.7) and (3.3.10)-(3.3.12). In order to calculate the expansion coefficients, we need to satisfy the boundary conditions (3.3.8) and (3.3.9), which after substitution lead to

$$
\begin{align*}
& \sum_{n=1}^{\infty} \lambda_{n} B_{n} \sin \left(\lambda_{n} z\right)=-1, \quad-a<z<0  \tag{3.4.2}\\
& \sum_{n=1}^{\infty} B_{n} \sin \left(\lambda_{n} z\right)=0, \quad-1 \leq z<-a . \tag{3.4.3}
\end{align*}
$$

Both the infinite series of Eqs. (3.4.2) and (3.4.3) are divergent at $z=-a$, owing to the fact that no kinematic condition was taken into account at that point. These dual trigonometrical series [Tranter (1959), Tranter (1960b)], reveal a singularity for the velocity, explicitly at $z=-a$. The singularity for the velocity implies the formation of a jet in the upper part of the air pocket. Letting $z=-y / \pi$ and $c=a \pi$, the system of Eqs. (3.4.2) and (3.4.3) becomes identical with the one reported and analyzed by Sneddon (1966) [p. 151-152], namely

$$
\begin{equation*}
\sum_{n=1}^{\infty}(n-1 / 2) B_{n} \sin [(n-1 / 2) y]=\tilde{F}(y) \equiv \frac{1}{\pi}, \quad 0<y<c \tag{3.4.4}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{n=1}^{\infty} B_{n} \sin [(n-1 / 2) y]=0, \quad c<y \leq \pi \tag{3.4.5}
\end{equation*}
$$

for the special case $p=1$. Note that $\tilde{F}(y)$ herein is a constant. The solution to this problem is taken directly from Sneddon (1966) [p. 158]. The solution for $B_{n}$ reads

$$
\begin{gather*}
B_{1}=\frac{1}{\sqrt{2}} \int_{0}^{c} h_{1}(\tau)\left[1-P_{1}(\cos \tau)\right] d \tau  \tag{3.4.6}\\
B_{n}=\frac{1}{\sqrt{2}} \int_{0}^{c} h_{1}(\tau)\left[P_{n-2}(\cos \tau)-P_{n}(\cos \tau)\right] d \tau, \quad n=2,3, \ldots, \tag{3.4.7}
\end{gather*}
$$

where $P_{n}$ denotes the $n$th degree Legendre Polynomial and

$$
\begin{align*}
& h_{1}(\tau)=\frac{1}{\pi} \frac{d}{d \tau} \int_{0}^{\tau} \frac{d \chi}{\sqrt{\cos \chi-\cos \tau}} \int_{0}^{\chi} \tilde{F}(u) d u  \tag{3.4.8}\\
&-\frac{\sqrt{2} \gamma_{1}}{\pi} \frac{d}{d \tau} \int_{0}^{\tau} \frac{(1-\cos \chi / 2)}{\sqrt{\cos \chi-\cos \tau}} d \chi .
\end{align*}
$$

The constant $\gamma_{1}$ is obtained from

$$
\begin{equation*}
\gamma_{1}\left\{1+\frac{\sqrt{2}}{\pi} \int_{0}^{c} \frac{(1-\cos \chi / 2)}{\sqrt{\cos \chi-\cos c}} d \chi\right\}=\frac{1}{\pi} \int_{0}^{c} \frac{d \chi}{\sqrt{\cos \chi-\cos c}} \int_{0}^{\chi} \tilde{F}(u) d u . \tag{3.4.9}
\end{equation*}
$$

Given than no analytical expressions are available for all the elliptic integrals involved in (3.4.8) and (3.4.9), the corresponding calculations are performed numerically using MATLAB's built-in function "integral", which allows the efficient approximation of improper integrals, as already discussed in the Subsection 2.3.1. The integrals involved in the expressions of the unknown coefficients $B_{n}$, are calculated using MATLAB's command "trapz", which applies the trapezoidal rule. Similarly, for the differentiations in Eqs. (3.4.8) and (3.4.9) the "gradient" command was applied.


Fig. 3.4.1. The expansion coefficients $B_{n}, n \geq 1$, for the leading order velocity potential. The air pocket was assumed to extend between $-1 \leq z \leq-0.7$.


Fig. 3.4.2. The normalized Neumann condition $\frac{\partial \phi_{1}}{\partial x}$ (Eq. 3.3.8) imposed on the wall. The air pocket was assumed to extend between $-1 \leq z \leq-0.7$. A singularity in noted at $z=-0.7$.

Fig. 3.4.1 shows that the expansion coefficients exhibit a transient, oscillatory behavior, but they decrease to zero relatively fast, which implies a quick convergence for the expression of the velocity potential. For the test cases examined herein, 200 modes were found enough to secure convergence of four significant digits for the velocity potential.

Fig. 3.4.2 depicts the Neumann condition of Eq. (3.3.8), which is actually the vertical distribution of the impact velocity on the wall boundary. The normalized Neumann condition dictates that the velocity must be equal to 1 in the impacted area. The singularity of the velocity at the upper part of the air pocket $z=-a$ is expressed i ) from the mathematical point of view, through the divergence of the series of Eq. (3.4.2) and ii) through the discontinuity of the graph. Physically, this singularity is attributed to the formation of a jet at that point.

The normalized distribution of the leading order velocity potential on the wall is shown in Fig. 3.4.3, where some remarkable results are obtained. To begin with, it is immediately noted that the potential is indeed zero i) at the part of the wall captured by the air pocket and ii) on the free-surface, as required by the Dirichlet conditions (3.3.9) and (3.3.11) respectively. Moreover, recalling Bernoulli's equation, the most interesting outcome from the results depicted in Fig. 3.4.3 is that the maximum hydrodynamic pressure is expected to occur in the middle of the impacted area, for each test case. Herein, each test case is defined by the air pocket length. Furthermore, it is evident that the presence of the air pocket affects the potential explicitly and correspondingly the hydrodynamic pressure on the wall. Large deviations are found to occur even in the same impacted areas. Particularly, it should be highlighted that smaller air pockets lead to higher impact loads in the same part of the wall. In contrast, the potential at the upper part of the wall, close to the free-surface, remains rather unaffected by the presence of the air pocket. Finally, the semi-elliptical shape of the potential verifies its sinus-like dependence.


Fig. 3.4.3. The normalized leading order velocity potential on the wall for several air pocket positions.

At this point, invoking the perturbation expansion of Eq. (3.3.3) and the leading order bvp of Eqs. (3.3.7)-(3.3.12), it should be mentioned that the results of Fig. 3.4.3
depict actually the term $t \phi_{1}$, which means that the results are the same no matter what the dimensionless time is. This happens because the time is in its first power and this is in contrast to the corresponding term for the free-surface elevation, which will be discussed in the following Subsection. In the general case, the expansion coefficients $B_{n}$ are functions of time. Nevertheless, given that the impacting velocity is constant, we are allowed to introduce a second time scale and assume that the expansion coefficients are weakly dependent on time, i.e., constants.


Fig. 3.4.4. Distribution of the magnitude of the velocity potential in a liquid domain of $1 \times 1$.
The air pocket was assumed to extend between $-1 \leq z \leq-0.5$.


Fig. 3.4.5. Distribution of the magnitude of the velocity potential in a liquid domain of $1 \times 1$. The air pocket was assumed to extend between $-1 \leq z \leq-0.7$.

The distribution of the magnitude of the normalized velocity potential in the liquid domain, restricted in an area of $1 \times 1$, is shown in the 3D graphs of Figs. 3.4.4 and 3.4.5, for different air pocket lengths. Once again, the vertical semi-elliptical configuration and the zero value for the potential in the air pocket are evident. In addition, one can easily observe the exponential decrease of the potential as we move further from the wall. The associated colourbar in both graphs, which clearly indicates the intensity of the impact, proves that as $x \rightarrow \infty$ then $\phi_{1} \rightarrow 0$, as dictated by the expression for the potential through Eq. 3.4.1.

The projection of the distribution of the velocity potential on the liquid domain of $1 \times 1$ is cited in Figs. 3.4.6 and 3.4.7. Clearly, the impact is more intense on the wall and in a small region close to the wall. Let us examine the case that the air pocket extends between $-1 \leq z \leq-0.7$ (see Fig. 3.4.7). It is interesting to observe that for distance from the wall $x>0.4$, the potential diminishes rapidly, approaching to zero. This fact implies that the disturbance caused by the collision vanishes. In our simplified approach, the impacted zone equals actually to the wave height; in the test case discussed this is 0.7 . Therefore, we can deduce that for distances greater than half of the wave height, the disturbance actually fades. The same conclusion is derived when examining the case where the air pocket extends between $-1 \leq z \leq-0.5$. This is in accord with the conclusion by Cooker and Peregrine (1995), who commented that for distances larger than half of the wave height, the wave parameters are not so important. This observation suggests that the phenomenon is generally localized.


Fig. 3.4.6. Projection of the 3D graph of Fig. 3.4.4 on the 2D liquid domain of $1 \times 1$. The air pocket was assumed to extend between $-1 \leq z \leq-0.5$.


Fig. 3.4.7. Projection of the 3D graph of Fig. 3.4.5 on the 2D liquid domain of $1 \times 1$. The air pocket was assumed to extend between $-1 \leq z \leq-0.7$.

Some contours of the velocity potential are shown in Figs. 3.4.8 and 3.4.9. The isopotential lines demonstrate once again that the phenomenon is more intense in the wave impact zone, as it was truly expected.


Fig. 3.4.8. Contours of the velocity potential. The air pocket was assumed to extend between $-1 \leq z \leq-0.5$.


Fig. 3.4.9. Contours of the velocity potential. The air pocket was assumed to extend between $-1 \leq z \leq-0.7$.

### 3.4.2 Validation against available results from other analytical studies

In order to check the accuracy and validity of the methodology adopted, it is needed to compare our results with some existing ones. It should be mentioned that straight comparison with other experimental or numerical results would not be representative for the accuracy of our method, because we cannot obtain exactly the same impact conditions, since analytical studies demand idealizations for the problem under consideration. Therefore, with the aim to secure identical conditions for the impact problem, we shall compare our results to those derived from the classical analytical study of Cooker and Peregrine (1990b). They calculated the pressure impulse exerted on a vertical wall, by a semi-infinite, rectangular strip of water.

To this end, we solve the bvp again, but this time without taking the existence of the air pocket into account. Particularly, we simply adjust the portion of the wall that is impacted, while the rest is assumed to remain untouched. Consequently, the mixed bvp is greatly simplified, since only the modified Neumann condition is valid on the wall. In detail, the new Neumann condition on the wall is transformed into

$$
\frac{\partial \phi_{1}}{\partial x}=-\sum_{n=1}^{\infty} \lambda_{n} B_{n} \sin \left(\lambda_{n} z\right)= \begin{cases}1, & -a<z \leq 0  \tag{3.4.10}\\ 0, & -1 \leq z<-a .\end{cases}
$$

The rest of the equations which compose the bvp are still the same. The expansion coefficients $B_{n}$ are now easy to be calculated, by using Fourier analysis. They are calculated with the aid of the Eq. (3.4.11)

$$
\begin{equation*}
B_{n}=2 \frac{1-\cos \left(a \lambda_{n}\right)}{\lambda_{n}^{2}} . \tag{3.4.11}
\end{equation*}
$$

The corresponding expression in the study by Cooker and Peregrine (1990b), is given by

$$
\begin{equation*}
B_{n}=\frac{-2 \rho U_{0}}{H} \frac{1-\cos \left(a \lambda_{n}\right)}{\lambda_{n}^{2}} \tag{3.4.12}
\end{equation*}
$$

where, $\rho$ is the water density, $U_{0}$ is the velocity of the fluid before impact (or equally the impact velocity) and $H$ is the water depth.

Fig. 3.4.10 depicts the normalized pressure impulse exerted on the wall, as anticipated by i) the analysis described so far and ii) the study by Cooker and Peregrine (1990b). The impact region is chosen to be $a=0.5$ in both cases. Surely, several important outcomes can be reached. First of all, it is demonstrated that the results provided by the global analysis are qualitatively accurate. The curves exhibit a similar pattern in the impacted part of the wall. Additionally, it is deduced that the presence of the air pocket reduces the pressure impulse on the wall drastically. Specifically, in case 1 , the maximum value of the pressure impulse is $\sim 0.21$, whereas in the second case it is $\sim 0.29$, which implies an increase of $38 \%$. There is also a displacement in the position of the maximum value downwards, from $z=-0.25$ to $z=-0.35$. However, in the upper part of the wall, close to free-surface, the curves are almost identical and the discrepancies are negligible.


Fig. 3.4.10. The non-dimensional pressure impulse exerted on the wall as predicted by the following two approaches: i) present study, which assumes the entrapment of a small air pocket and ii) work by Cooker and Peregrine (1990b). The air pocket was assumed to extend between $-1 \leq z \leq-0.5$.


Fig. 3.4.11. The non-dimensional pressure impulse exerted on the wall as predicted by the following two approaches: i) present study, which assumes the entrapment of a small air pocket for five different air pocket heights and ii) work by Cooker and Peregrine (1990b), where the impact was assumed to occur along the total height of the wall.

Fig. 3.4.11 shows a comparison between a group of curves which describe the pressure impulse as the air pocket height tends to zero asymptotically and the one predicted by the study of Cooker and Peregrine (1990b) where the impact occurs along the total height of the wall. At this point, it is useful to remind that the maximum impact pressure is expected to occur in the middle of the impacted region (see for example Fig. 3.4.3). However, the most noticeable feature in this case is that as $a \rightarrow 1$, the point of the wall where the maximum pressure is anticipated to occur moves downwards from the middle of the impacted region. This is in complete alignment with the conclusion of Cooker and Peregrine (1990b), who concluded that when the total height of the wall is impacted, the curve of the pressure impulse changes form and its maximum value is anticipated at the lowest part of the wall.

### 3.4.3 The free-surface elevation

Due to the violent collision between the water and the wall, a run-up along the wall is immediately observed at a time instant just after impact (Fig. 3.1.1). The parabolalike shape of the free-surface elevation $\eta_{2}(x)$, can be determined explicitly through the series of Eq. (3.3.13), in the fluid domain where $x \geq 0$. It is reminded that Eq. (3.3.13) originates from the kinematic condition of the free-surface. Nonetheless, that series is divergent when calculated exactly on the wall, i.e., $x=0$, increasing continuously as the number of modes increases to infinity. A deeper insight into the physical problem can clearly explain this mathematical observation. At the moment of collision, a sprayjet is formed exactly at the intersection point of the still water level and the wall, $(x, z)=(0,0)$, which implies infinite vertical velocity at that point. In order to tackle this mathematical difficulty, we apply the methodology suggested by King and Needham (1994), who treated the problem of a jet caused by the sudden acceleration of
a vertical plate into an initially stationary fluid analytically. A similar study for the impulsive motion of a plate moving towards an initially quiescent fluid was also conducted by Chwang (1983) who applied the small-time assumption as well, but rather focused on the hydrodynamic pressure on the plate. Roberts (1987) studied the same problem analytically as well.

The solution derived from the mathematical manipulation of the physical problem in the present investigation, dictates that the free-surface elevation is given by

$$
\begin{equation*}
t \eta_{2}(x)=\frac{1}{2} \sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right) \pi B_{n} e^{-(n-1 / 2) \pi x} . \tag{3.4.13}
\end{equation*}
$$

At this point, a brief discussion for the left-hand side of Eq. (3.4.13) is needed. It is reminded that the expansion coefficients $B_{n}$ of the normalized leading order problem, calculated using Eqs. (3.4.6) and (3.4.7), give the leading order velocity potential multiplied by the time, $t \phi_{1}$. Consequently, exploiting Eq. (3.3.13), these coefficients permit the calculation of the term $t \eta_{2}(x)$. The corresponding expression by King and Needham (1994) is

$$
\begin{equation*}
\eta_{2}(x)=\frac{4 \sigma}{\pi} \sum_{n=0}^{\infty} \frac{e^{-(n+1 / 2) \pi x}}{2 n+1}=\frac{2 \sigma}{\pi} \ln \left[\operatorname{coth} \frac{\pi x}{4}\right] \tag{3.4.14}
\end{equation*}
$$

where $\sigma$ is a parameter that expresses the ratio between the acceleration of the plate and the gravitational acceleration. The series of Eq. (3.4.14) is clearly singular as $x \rightarrow 0$. King and Needham (1994) assumed an outer region $(x>0)$ where the series expansion for the free-surface elevation is valid and an inner region $(x=0)$ where the expansion is singular. In order to raise this singularity, they reclaimed the Euler momentum equations, by retaining the terms neglected in the outer region. Thus, they actually made a correction to the leading order problem. In the inner region they applied the following asymptotic expansion

$$
\begin{equation*}
\eta_{\text {inner }}=-t^{2} \ln (t) \eta_{1}-t^{2} \tilde{\eta}_{2}, \tag{3.4.15}
\end{equation*}
$$

where $\eta_{1}$ and $\tilde{\eta}_{2}$ denote the first and the second order free-surface elevation in the inner region. The approach taken by King and Needham (1994) can be applied in the present analysis as well, provided that the sums of Eqs. (3.4.13) and (3.4.14) are subjected to the same behavior at $x=0$, for an infinite number of modes. To examine this assertion Fig. 3.4.12 is provided. Both sums grow indefinitely for increasing number of modes, owing to the logarithmic singularity of the sums for $x=0$. Although evident differences are observed, it can be easily deduced that they nearly coincide and become practically indistinguishable for large truncation of the sums. The oscillatory behavior at the first part of the curve for the sum of Eq. (3.4.13) is attributed to the numerical calculation of the coefficients $B_{n}$.


Fig. 3.4.12. Solid line: shows the behavior of the sum of Eq. (3.4.13). Dashed line: shows the behavior of the sum of Eq. (3.4.14). Both sums are calculated for $x=0$. The air pocket was assumed to extend between $-1 \leq z \leq-0.5$.

Accordingly, the mixed bvp methodology discussed in the present, is enhanced with King and Needham's (1994) asymptotic analysis formulae explicitly for $x=0$. We are therefore allowed to transform Eq. (3.4.13) in the equivalent form

$$
\begin{equation*}
t \eta_{2}(x)=\frac{1}{2} \sum_{n=0}^{\infty} \frac{e^{-(n+1 / 2) \pi x}}{2 n+1}=\frac{1}{4} \ln \left[\operatorname{coth} \frac{\pi x}{4}\right] \tag{3.4.16}
\end{equation*}
$$

Eq. (3.4.14) is written in a more convenient form, for the asymptotic procedure, as

$$
\begin{equation*}
\eta_{2}(x)=-\frac{2 \sigma}{\pi} \ln x+\frac{2 \sigma}{\pi} \ln \frac{4}{\pi}+O\left(x^{2}\right), \quad x \rightarrow 0 \tag{3.4.17}
\end{equation*}
$$

Similarly, Eq. (3.4.16) is written in the form

$$
\begin{equation*}
\operatorname{t\eta }_{2}(x)=-\frac{1}{4} \ln x+\frac{1}{4} \ln \frac{4}{\pi}+O\left(x^{2}\right), \quad x \rightarrow 0 \tag{3.4.18}
\end{equation*}
$$

Following the asymptotic expansion suggested by King and Needham (1994), one gets for the inner region

$$
\begin{equation*}
t \tilde{\eta}_{2}(x)=\frac{1}{4}\left\{\ln (-\ln t)-\ln \left(\frac{4}{\pi}\right)+\ln \left(\frac{1}{4}\right)-1-\frac{\Gamma^{\prime}\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}+x\right\} \tag{3.4.19}
\end{equation*}
$$

and

$$
\begin{equation*}
t \eta_{1}=\frac{1}{2} \tag{3.4.20}
\end{equation*}
$$

Expression (3.4.19) has no singularity neither for $x \rightarrow 0$ nor for $t \rightarrow 0$.


Fig. 3.4.13. The free-surface elevation $H(x, t)$ at three different time instants, including only the leading order free-surface elevation. The air pocket was assumed to extend between

$$
-1 \leq z \leq-0.5 .
$$



Fig. 3.4.14. The free-surface elevation $H(x, t)$ at three different time instants, including only the leading order free-surface elevation. The air pocket was assumed to extend between

$$
-1 \leq z \leq-0.7 .
$$

Figs. 3.4.13 and 3.4.14 depict the free-surface elevation given by Eq. (3.3.4) retaining only the leading order term, at three different time instants immediately after
the first impact. With time passing, the water runs-up the wall at higher heights, as it was truly expected. The results from Figs. 3.4.13 and 3.4.14 validate the aforementioned deduction, from another perspective, that for $x>0.4$ the disturbance tends to zero (see Subsection 3.4.1). In addition, it is noted that for $x>1$, the freesurface remains completely flat and horizontal, unaffected by the collision.

Fig. 3.4.15 displays the free-surface elevation given by Eq. (3.3.4) for two different air pocket lengths at three time instants and is cited to examine how the air pocket length affects the free-surface elevation. It therefore offers a straightforward comparison between the results of Figs. 3.4.13 and 3.4.14. It manifests that smaller air pockets lead to a slightly higher free-surface elevation. A reasonable hypothesis for this observation is that the phenomenon is more intense for smaller air pockets. The discrepancies are more conspicuous as the impact evolves. They are negligible on the wall, probably owing to the fact that the potential close to the free-surface is rather unaffected by the air pocket length (see Fig. 3.4.3) Nevertheless, they are more evident farther from the wall.


Fig. 3.4.15. Comparison of the free-surface elevation for two different air pocket lengths at three time instants.

### 3.4.4 The singularity on the upper point of the air pocket

The fundamental solution of the Laplace equation in 2D is

$$
\begin{equation*}
\phi_{1} \sim \log \left(\sqrt{x^{2}+z^{2}}\right) \tag{3.4.21}
\end{equation*}
$$

Therefore, the relevant expression at the point of singularity $(x, z)=(0,-a)$ reads

$$
\begin{equation*}
\phi_{1} \sim \log \left(\sqrt{x^{2}+(z+a)^{2}}\right) . \tag{3.4.22}
\end{equation*}
$$

Similarly to Eq. (3.3.13), the corresponding expression for the "inner elevation" is given by

$$
\begin{equation*}
\zeta_{2}=\frac{1}{2} \frac{\partial \phi_{1}}{\partial z}, \quad x>0, \quad z=-a . \tag{3.4.23}
\end{equation*}
$$

Combining Eqs. (3.4.22) and (3.4.23) yields

$$
\begin{equation*}
\zeta_{2} \sim \frac{z+a}{x^{2}+(z+a)^{2}} . \tag{3.4.24}
\end{equation*}
$$

It is noticed that this expression is singular exactly at the point $(x, z)=(0,-a)$, as it was literally expected. Using Eqs. (3.4.23) and (3.4.1) we obtain the expression for the "inner elevation" as

$$
\begin{equation*}
\zeta_{2}=\frac{1}{2} \sum_{n=1}^{\infty} \lambda_{n} B_{n} \cos \left(-\lambda_{n} a\right) . \tag{3.4.25}
\end{equation*}
$$

As already discussed, the series of Eq. (3.4.25) is divergent for increasing number of modes. However, the aforementioned methodology to tackle the singularity, developed in the Subsection 3.4.3, can be utilized again, provided that the sums of Eqs. (3.4.25) and (3.4.14) exhibit the same behavior for $x=0$. To examine this allegation, Fig. 3.4.16 is provided.


Fig. 3.4.16. Solid line: shows the behavior of the sum of Eq. (3.4.25). Dashed line: shows the behavior of the sum of Eq. (3.4.14). Both sums are calculated for $x=0$. The air pocket was assumed to extend between $-1 \leq z \leq-0.5$.

The solid line corresponds to the sum of Eq. (3.4.25) multiplied by a constant number, so as to make the two curves coincide. The dashed line holds for the sum of Eq. (3.4.14).

Even if there are large discrepancies for the first 100 modes, a convergence of the two curves is observed for increasing number of modes and particularly for $n>160$ they become practically identical. Therefore, the methodology described in detail in the Subsection 3.4.3 can be exploited again.

### 3.5 Solution to the leading order problem assuming linear variation of the pressure in the air pocket

In this Section, we examine the case that the pressure exhibits linear variation in the air pocket. Consequently, similarly to the set of Eqs. (3.3.7)-(3.3.12), the bvp will now read

$$
\begin{gather*}
\nabla^{2} \phi_{1}=0, \quad x \geq 0, \quad-1 \leq z \leq 0  \tag{3.5.1}\\
\frac{\partial \phi_{1}}{\partial x}=1, \quad z \in I_{1}, \quad-a<z \leq 0, \quad x=0  \tag{3.5.2}\\
\phi_{1}=f(z), \quad z \in I_{2}, \quad-1 \leq z<-a, \quad x=0,  \tag{3.5.3}\\
\frac{\partial \phi_{1}}{\partial z}=0, \quad z=-1, \quad x>0  \tag{3.5.4}\\
\phi_{1}=0, \quad z=0, \quad x>0  \tag{3.5.5}\\
\phi_{1} \rightarrow 0, \quad x \rightarrow \infty, \quad-1 \leq z \leq 0 \tag{3.5.6}
\end{gather*}
$$

where $f(z)=m_{1} z+m_{2}$. The coefficients $m_{1}$ and $m_{2}$ are constants and are taken as known. They incorporate the water density $\rho$ (through the Bernoulli's equation in the air pocket) as well as the impact velocity $V$ and the water depth $h$, owing to the nondimensionalization process.

The solution process is exactly the same as outlined in Subsection 3.4.1. This means that the normalized velocity potential has the form

$$
\begin{equation*}
\phi_{1}=\sum_{n=1}^{\infty} B_{n} \sin \left(\lambda_{n} z\right) e^{-\lambda_{n} x}, \quad \lambda_{n}=\left(n-\frac{1}{2}\right) \pi \tag{3.5.7}
\end{equation*}
$$

The only difference lies to the system of the dual trigonometrical series which has to be solved. The unknown expansion coefficients must satisfy the following system

$$
\begin{align*}
& \sum_{n=1}^{\infty} \lambda_{n} B_{n} \sin \left(\lambda_{n} z\right)=H_{1}(z) \equiv-1, \quad-a<z \leq 0,  \tag{3.5.8}\\
& \sum_{n=1}^{\infty} B_{n} \sin \left(\lambda_{n} z\right)=H_{2}(z)=f(z), \quad-1 \leq z<-a . \tag{3.5.9}
\end{align*}
$$

In order to find a proper solution, we have to decompose the problem of Eqs. (3.5.8) and (3.5.9) in the following two subproblems

$$
\begin{gather*}
\sum_{n=1}^{\infty} \lambda_{n} B_{n}^{(1)} \sin \left(\lambda_{n} z\right)=H_{1}(z), \quad-a<z \leq 0,  \tag{3.5.10}\\
\sum_{n=1}^{\infty} B_{n}^{(1)} \sin \left(\lambda_{n} z\right)=0, \quad-1 \leq z<-a, \tag{3.5.11}
\end{gather*}
$$

and

$$
\begin{gather*}
\sum_{n=1}^{\infty} \lambda_{n} B_{n}^{(2)} \sin \left(\lambda_{n} z\right)=0, \quad-a<z \leq 0,  \tag{3.5.12}\\
\sum_{n=1}^{\infty} B_{n}^{(2)} \sin \left(\lambda_{n} z\right)=H_{2}(z), \quad-1 \leq z<-a . \tag{3.5.13}
\end{gather*}
$$

In fact, this is the approach suggested by Sneddon (1966) [p. 158]. The solution to the first subproblem is similar to that of the leading order problem. Hence, letting $z=$ $-\psi / \pi$, the unknown coefficients will be given by Sneddon (1966) [p. 158]

$$
\begin{gather*}
B_{1}^{(1)}=\frac{1}{\sqrt{2}} \int_{0}^{c} h_{1}(\tau)\left[1-P_{1}(\cos \tau)\right] d \tau  \tag{3.5.14}\\
B_{n}^{(1)}=\frac{1}{\sqrt{2}} \int_{0}^{c} h_{1}(\tau)\left[P_{n-2}(\cos \tau)-P_{n}(\cos \tau)\right] d \tau, \quad n=2,3, \ldots, \tag{3.5.15}
\end{gather*}
$$

where $P_{n}$, again, denotes the $n$th degree Legendre Polynomial, while

$$
\begin{align*}
h_{1}(\tau)=\frac{1}{\pi} \frac{d}{d \tau} & \int_{0}^{\tau} \frac{d \psi}{\sqrt{\cos \psi-\cos \tau}} \int_{0}^{\psi} H_{1}(u) d u  \tag{3.5.16}\\
& \quad-\frac{\sqrt{2} \gamma_{1}}{\pi} \frac{d}{d \tau} \int_{0}^{\tau} \frac{(1-\cos \psi / 2)}{\sqrt{\cos \psi-\cos \tau}} d \psi
\end{align*}
$$

The constant $\gamma_{1}$ yields from

$$
\begin{equation*}
\gamma_{1}\left\{1+\frac{\sqrt{2}}{\pi} \int_{0}^{c} \frac{(1-\cos \psi / 2)}{\sqrt{\cos \psi-\cos c}} d \psi\right\}=\frac{1}{\pi} \int_{0}^{c} \frac{d \psi}{\sqrt{\cos \psi-\cos c}} \int_{0}^{\psi} H_{1}(u) d u \tag{3.5.17}
\end{equation*}
$$

The solution to the second subproblem is given by Sneddon (1966) [p. 159]

$$
\begin{gather*}
B_{1}^{(2)}=\frac{1}{\sqrt{2}} \int_{c}^{\pi} h_{2}(\tau)\left[1+P_{1}(\cos \tau)\right] d \tau  \tag{3.5.18}\\
B_{n}^{(2)}=\frac{1}{\sqrt{2}} \int_{c}^{\pi} h_{2}(\tau)\left[P_{n}(\cos \tau)-P_{n-2}(\cos \tau)\right] d \tau, \quad n=2,3, \ldots,  \tag{3.5.19}\\
h_{2}(\tau)=\frac{1}{\pi} \frac{d}{d \tau} \int_{\tau}^{\pi} \frac{H_{2}(\psi) d \psi}{\sqrt{\cos \tau-\cos \psi}} . \tag{3.5.20}
\end{gather*}
$$

However, this classical approach to the bvp was found to provide unreliable results, especially in the part of the wall, which is captured by the air pocket. The major problem was the numerical instabilities, which do not allow the proper convergence of the infinite series of Eq. (3.5.9). In order to tackle this difficulty an alternative approach was taken. To this end, we write the function $f(z)$ in a series-like form as

$$
\begin{equation*}
\sum_{n=1}^{\infty} w_{n} \sin \left(\lambda_{n} z\right)=f(z) \tag{3.5.21}
\end{equation*}
$$

Using orthogonality, the expansion coefficients, $w_{n}$, become

$$
\begin{equation*}
w_{n}=2 \int_{-1}^{0} f(z) \sin \left(\lambda_{n} z\right) d z \tag{3.5.22}
\end{equation*}
$$

Next, we define $D_{n}=B_{n}-w_{n}$ and accordingly, Eq. (3.5.8) becomes

$$
\begin{align*}
\sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right) D_{n} & \sin [(n-1 / 2) \psi] \\
& =\frac{1}{\pi}-\sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right) w_{n} \sin [(n-1 / 2) \psi] \tag{3.5.23}
\end{align*}
$$

while Eq. (3.5.9) is reduced to

$$
\begin{equation*}
\sum_{n=1}^{\infty} D_{n} \sin [(n-1 / 2) \psi]=0 . \tag{3.5.24}
\end{equation*}
$$

The solution given by Sneddon (1966) can now be applied properly to the bvp of Eqs. (3.5.23)-(3.5.24), in terms of the coefficients $D_{n}$. Once the coefficients $D_{n}$ have been determined, it is easy to obtain $B_{n}$.

Fig. 3.5.1 shows the potential on the wall for two models of linear variation of the pressure compared to the case of zero pressure in the air pocket. It is proved that the case of linear pressure distribution in the air pocket has a very limited influence on the pressure distribution on the wall and on the maximum pressure as well. It is observed that once again the maximum pressure is expected to occur in the middle of the impacted zone.


Fig. 3.5.1. The normalized leading order velocity potential on the wall with linear pressure variation in the air pocket relative to the zero-pressure case. The air pocket was assumed to extend between $-1 \leq z \leq-0.7$.

### 3.6 Conclusions

In this Chapter, the breaking wave impact on a vertical impermeable wall was investigated. To this end, a slightly overturning breaking wave was idealized in a logical manner, so that an analytical approach is applicable. Specifically, the wave was considered as bounded from above by the still water level, while a small air pocket was entrapped between the water and the wall. The theory developed was established mainly on the assumption that the pressure is zero in the air pocket, a case corresponding to pure vacuum. The problem was formulated using nondimensional variables. The fluid was assumed to be inviscid, incompressible and the flow irrotational. The bvp is of mixed type, due to the Neumann and Dirichlet boundary conditions, which hold on the two different parts of the wall. A perturbation expansion was employed, using the dimensionless time as the small parameter. After separating like orders of $t$, the governing bvp was broken into two subproblems; only the leading order problem was solved in the present Chapter. Moreover, a Taylor series expansion was used to cope with the boundary conditions of the free-surface. Results were derived in terms of the velocity potential distribution on the wall and the liquid field as well as the free-surface elevation. Results for the velocity potential were also presented assuming linear variation of the pressure in the air pocket. The free-surface elevation was found to be singular at the point $(x, z)=(0,0)$, owing to the splash generated due to the collision.

To this end, our model was enhanced by encompassing the mathematical solution proposed by King and Needham (1994). The main results from this Chapter can be itemized as follows:
I. Gravity is not involved in the leading order problem, but it is explicitly introduced in the higher order problem, through the Froude number.
II. The results for the leading order velocity potential are in terms of $t \phi_{1}$, which particularly means that they are actually independent of the dimensionless time. This happens because the time is in the first power. On the contrary, the results for the free-surface elevation which depend on $t^{2}$, depend clearly on the time. This approach suggests the introduction of a second time scale.
III. The maximum hydrodynamic pressure is expected to occur in the middle of the impacted zone.
IV. Judging from the 3D graph for the distribution of the magnitude of the velocity potential in the liquid domain as well as the contour results, it is remarked that the phenomenon is localized and more intense in the vicinity of the wall.
V. The length of the air pocket was found to affect the distribution of the velocity potential on the wall and correspondingly, the hydrodynamic pressure. Particularly, smaller air pockets lead to higher impact loads. However, the potential seems to remain rather unaffected by the presence of the air pocket at the upper part of the wall.
VI. The projection of the magnitude of the velocity potential in the liquid field revealed that the disturbance due to the collision practically vanishes if the distance from the wall is greater than half of the wave height.
VII. Regarding the case of linear variation of the pressure in the air pocket, it was found to have a negligible effect on the pressure distribution on the impacted area and on the maximum pressure as well.
VIII. The singularity of the free-surface elevation at the point $(x, z)=(0,0)$ was found to be logarithmic. An inner asymptotic expansion was adopted, as suggested by the study of King and Needham (1994).
IX. The run-up along the wall is higher as the phenomenon evolves, as it was expected.
X. The free-surface elevation is also affected by the length of the air pocket, even if the discrepancies are not that important. It was derived that smaller air pockets lead to greater elevations, probably owing to the fact that the impact is more intense in that case.

### 3.7 Further research

The investigation presented in this Chapter, aimed to propose an alternative aspect of the breaking wave impact and provide answers to some essential research questions. Nonetheless, some improvements of the model considered in this study, would probably offer a deeper insight into the phenomenon. Some suggestions are mentioned in the following:
I. The velocity of the breaking wave in the impact zone was considered constant. A vertical distribution $V(z)$ of the impact velocity would bring the approach a step closer to the real impact conditions.
II. An air pocket with non-negligible width would be interesting to be examined, from a mathematical point of view mainly, since it would claim the solution to two bvp: i) in the vertical direction and ii) in the horizontal direction.
III. The pressure in the air pocket, in this approach, was considered either as zero (void) or linearly dependent on the vertical coordinate. No time dependence was introduced. Obtaining some results for the pressure history record and its vertical distribution in the air pocket and introducing them to the bvp herein, would be challenging. Experiments would be useful in that respect. Perhaps an asymptotic analysis in orders of time would be necessary, in order to fragment the governing bvp to subproblems of orders of time.
IV. In order to simulate the wave's breaking process, while approaching the shore, one could take a non-flat bottom into account, by introducing a small inclination of the sea bed.
V. It is generally accepted that during impact the air pocket is compressed and then expands, leading to pressure oscillations. Escape of air occurs as well. Consequently, taking the compression effects into account would take the analysis a step further.
VI. Breakwaters are substantial for the protection of the shores or the structures located very close to them, from violent waves. Examining the case of a permeable wall and/or inclined wall would be interesting to observe whether these configurations would increase the safety level.
VII. A concept of oblique impact between the semi-infinite strip of water and the wall is definitely a special case that could be examined in the near future.

Similarly to the comment of Chapter 2, it should be mentioned again that some of the concepts proposed to be investigated could demand numerical solutions.

# Chapter 4 Breaking wave impact on a vertical wall. Part 2. The higher order problem 

### 4.1 Theoretical modelling

Chapter 4 presents the solution of the higher order bvp, for a two-dimensional breaking wave impact on a vertical wall. At this point, we shall recall that the general bvp, defined by the set of Eqs. (3.2.1)-(3.2.7), was fragmented in subproblems of orders of time. To this end, i) a Taylor series expansion around the still water level -so as to avoid satisfying the kinematic and dynamic boundary conditions of the free-surface on the initially unknown boundary of the free-surface elevation- and ii) a perturbation method for the expansion of the normalized velocity potential $\phi$, were employed. The dimensionless time $t$ was used as the small parameter in our perturbation analysis. Besides, the non-dimensionalization process introduced the square of the inverse Froude number based on depth. Clearly, the solution of the higher order problem dictates the solution of the leading order problem. It also leads to a substantially complicated, inhomogeneous Sturm-Liouville problem, with mixed, inhomogeneous boundary conditions, which involve infinite series. Consequently, it is required to be treated in a sophisticated manner. Furthermore, the results derived for the higher order velocity potential distribution on the wall reveal interesting corrections that should be made in the leading order theory, so that the results predicted by the developed mathematical model will be more accurate. To this end, they are analyzed by a mathematical as well as an engineering point of view.

### 4.2 The higher order boundary value problem

The higher order bvp is composed of Eqs. (4.2.1)-(4.2.6), as follows

$$
\begin{gather*}
\nabla^{2} \phi_{3}=0, \quad x \geq 0, \quad-1 \leq z \leq 0,  \tag{4.2.1}\\
\frac{\partial \phi_{3}}{\partial x}=0, \quad z \in I_{1}, \quad-a<z<0, \quad x=0,  \tag{4.2.2}\\
\phi_{3}=0, \quad z \in I_{2}, \quad-1 \leq z<-a, \quad x=0,  \tag{4.2.3}\\
\frac{\partial \phi_{3}}{\partial z}=0, \quad z=-1, \quad x \geq 0,  \tag{4.2.4}\\
\phi_{3}=-\eta_{2} \frac{\partial \phi_{1}}{\partial z}-\frac{k}{3} \eta_{2}, \quad z=0, \quad x \geq 0,  \tag{4.2.5}\\
\phi_{3} \rightarrow 0, \quad x \rightarrow \infty, \quad-1 \leq z \leq 0, \tag{4.2.6}
\end{gather*}
$$

where $k$ is the non-dimensional parameter used for the square of the inverse Froude number based on depth. Eq. (4.2.1) is the Laplace equation for the higher order velocity potential and Eq (4.2.2) is the normalized Neumann condition on the impacted part of
the wall, after the implementation of the perturbation method. Eq. (4.2.3) is the dynamic boundary condition in the air pocket, after omitting the quadratic and hydrostatic terms. Eq. (4.2.4) expresses the impermeability condition of the flat, horizontal bottom, while Eq (4.2.5) holds for the dynamic condition of the free-surface applied on the still water level, after the employment of the Taylor series expansion. Finally, Eq. (4.2.6) is the far-field boundary condition. The higher order free-surface elevation is obtained from

$$
\begin{equation*}
\eta_{4}=\frac{1}{4} \frac{\partial \phi_{3}}{\partial z}+\frac{1}{4} \eta_{2} \frac{\partial^{2} \phi_{1}}{\partial z^{2}} . \tag{4.2.7}
\end{equation*}
$$

### 4.3 Solution to the higher order problem

### 4.3.1 The methodology for the derivation of the expression for the distribution of the higher order velocity potential on the wall

The problem of mixed type herein, determined via Eqs. (4.2.2)-(4.2.3), is defined differently due to the zero right-hand side term in both equations. The solution sought is not trivial owing to the inhomogeneous dynamic boundary condition of the freesurface. Specifically, the right-hand side term of Eq. (4.2.5) involves the leading order free-surface elevation and the leading order vertical velocity. At this point, it is useful to recall Eqs. (3.3.13) and (3.4.13), which dictate that the inhomogeneous term is a function of the horizontal coordinate $x$, defined as $f(x)$. Consequently, in order to build up the solution process, a different strategy is assumed. Particularly, we aim to satisfy initially Eqs. (4.2.4) and (4.2.5). To this end, a proper solution that complies with these requirements can be expressed as

$$
\begin{equation*}
\phi_{3}=-f(x) g(z)+\sum_{n=1}^{\infty} F_{n}(x) \sin \left(\lambda_{n} z\right) \tag{4.3.1}
\end{equation*}
$$

where $F_{n}(x)$ and $g(z)$ are functions to be determined. It was also assumed that

$$
\begin{equation*}
f(x)=\eta_{2} \frac{\partial \phi_{1}}{\partial z}+\frac{k}{3} \eta_{2}, \tag{4.3.2}
\end{equation*}
$$

which is properly behaved at infinity. Specifically, $f(x) \rightarrow 0$ as $x \rightarrow \infty$. The latter remark is shown graphically in Fig. 4.3.1. In more detail, the term (4.3.2) converges to zero for distances from the wall equal to the water depth $(x=1)$. In addition, given that the term $f(x)$ involves the leading order free-surface elevation, it exhibits a singularity at the point $x=0$. To overcome this difficulty and obtain a numerical value exactly on the wall, we follow the process outlined in Section 3.4.3.


Fig. 4.3.1. The behavior of the term $f(x)$ (see Eq. (4.3.2)), for the free-surface boundary condition of the higher order hydrodynamic problem. It was assumed that $k=1.09(V=$ $3 \mathrm{~m} / \mathrm{s}), t=0.01$ and the air pocket extends between $-1 \leq z \leq-0.5$.

Eq. (4.3.1) satisfies both Eqs. (4.2.4) and (4.2.5) provided that $g^{\prime}(-1)=0$ and $g(0)=$ 1 , where the prime denotes differentiation with respect to the argument. Hence, a proper expression for $g(z)$ reads [Gradshteyn and Ryzhik (2007); p.46]

$$
\begin{equation*}
g(z)=\frac{4}{\pi} \sum_{i=1}^{\infty}(-1)^{i-1} \frac{\cos [(2 i-1) \pi z]}{2 i-1} . \tag{4.3.3}
\end{equation*}
$$



Fig. 4.3.2. The function $g(z)$ given by the expression (4.3.3).

The function $g(z)$ is shown graphically in Fig. 4.3.2, which demonstrates that indeed $g(0)=1$. Moreover, differentiating the relation (4.3.3) with respect to $z$, it is very simple to show that $g^{\prime}(-1)=0$. Next, the general expression (4.3.1) for the higher order potential is substituted in the Laplace equation (4.2.1), which after employing the orthogonality property for $\sin \left(\lambda_{n} z\right)$ in the vertical domain $-1 \leq z \leq 0$ yields the simple formula for $F_{n}(x)$

$$
\begin{equation*}
F_{n}^{\prime \prime}(x)-\lambda_{n}^{2} F_{n}(x)=S_{n}(x), \quad 0 \leq x<\infty, \tag{4.3.4}
\end{equation*}
$$

where

$$
\begin{equation*}
S_{n}(x)=2 \int_{-1}^{0}\left[f^{\prime \prime}(x) g(z)+f(x) g^{\prime \prime}(z)\right] \sin \left(\lambda_{n} z\right) d z \tag{4.3.5}
\end{equation*}
$$

The inhomogeneous component of the differential Eq. (4.3.4) is denoted by $S_{n}(x)$ and can be regarded as the forcing term that excites the system. The factor 2 in (4.3.5) originates from the orthogonality property for $\sin \left(\lambda_{n} z\right)$. The term $S_{n}(x)$ comprises the functions $f(x), g(z)$ as well as their second derivatives with respect to their arguments. The behavior of the term $S_{n}(x)$ is depicted in Fig. 4.3.3, for the first four modes. Clearly, it obtains much higher values than $f(x)$ but asymptotes to zero much more rapidly.


Fig. 4.3.3. The behavior of the inhomogeneous term of Eq. (4.3.4), as described by the Eq.
(4.3.5). It was assumed that $k=1.09(V=3 \mathrm{~m} / \mathrm{s}), t=0.01$ and the air pocket extends between $-1 \leq z \leq-0.5$.

Subsequently, using Eq. (4.3.1) in the mixed, homogeneous conditions (4.2.2) and (4.2.3), it follows that

$$
\begin{align*}
& \sum_{n=1}^{\infty} F_{n}^{\prime}(0) \sin \left(\lambda_{n} z\right)=f^{\prime}(0) g(z), \quad-a<z<0,  \tag{4.3.6}\\
& \sum_{n=1}^{\infty} F_{n}(0) \sin \left(\lambda_{n} z\right)=f(0) g(z), \quad-1 \leq z<-a . \tag{4.3.7}
\end{align*}
$$

Using Eq. (4.2.6) and taking into account that $f(x) \rightarrow 0$ for $x \rightarrow \infty$, the last condition that has to be considered is

$$
\begin{equation*}
F_{n}(x) \rightarrow 0, \quad x \rightarrow \infty \tag{4.3.8}
\end{equation*}
$$

Eqs. (4.3.4) and (4.3.6)-(4.3.8) determine a novel and challenging one-dimensional, boundary value, Sturm-Liouville problem with mixed, inhomogeneous conditions, which is even more complicated due to the infinite series involved. Relevant problems have not been investigated so far, at least in the field of analytical hydrodynamics associated with slamming phenomena. The solution to this problem is considered next.

### 4.3.2 An approximate approach for the one-dimensional Sturm-Liouville problem with mixed conditions

In order to solve the Sturm-Liouville problem composed of Eqs. (4.3.4) and (4.3.6)(4.3.8) and obtain numerical results, we followed several approaches. In this Subsection, we outline briefly two possible, but approximate, methods of solution. Typical Sturm-Liouville bvp are solved by means of the governing Green's function. Possible solutions are assumed to be of the form (e.g., for Eq. (4.3.4))

$$
\begin{equation*}
F_{n}(x)=\int_{0}^{\infty} S_{n}(y) G_{n}(x ; y) d y \tag{4.3.9}
\end{equation*}
$$

in which the Green's function must satisfy the homogeneous field equation (e.g., Eq. (4.3.4)), and the same boundary conditions as the unknown function to be obtained. In addition, the Green's function must be continuous at $y=x$, while the derivative of the Green's function, calculated at the same point, must be discontinuous exhibiting a jump singularity [Dettman (1988); p.230, Chatjigeorgiou (2018); p.111]. The problem herein arises mainly from the fact that the condition on the known boundary at $x=0$ is not uniform, but each condition holds on a different portion of the wall. In addition, the boundary condition is not described by a simple formula (say Neumann or Dirichlet) as it involves infinite series. Therefore, once the Green's function is derived, a first approximate solution can be obtained by substituting the general expression (4.3.9) in the boundary conditions (4.3.6) and (4.3.7). The expansion coefficients involved in the Green's function can be calculated by the dual trigonometrical series system formed. A second approximate solution can be obtained by demanding that the Green's function satisfies exactly the same boundary conditions with the unknown function $F_{n}(x)$. At this point, it should be mentioned that these approximate solutions have been discussed
briefly for the completeness of the present thesis. In the next Subsection, we proceed with the exact and robust method of solution for the Sturm-Liouville problem.

### 4.3.3 The exact solution of the one-dimensional Sturm-Liouville problem with mixed conditions

In this method of solution, we follow a rather straightforward approach, starting from the fundamental basic Eq. (4.3.4). Therefore, the solution for $F_{n}(x)$, that is composed by the homogeneous solution and the particular solution, is given by

$$
\begin{align*}
& F_{n}(x)=A_{n} e^{-\lambda_{n} x} \\
&-\frac{1}{2 \lambda_{n}}\left\{\int_{0}^{x} e^{-\lambda_{n}(x-\xi)} S_{n}(\xi) d \xi+\int_{x}^{\infty} e^{-\lambda_{n}(\xi-x)} S_{n}(\xi) d \xi\right\}, \tag{4.3.10}
\end{align*}
$$

where $A_{n}$ are unknown expansion coefficients to be calculated. Expression (4.3.10) satisfies the Laplace equation and secures convergence as $x \rightarrow \infty$. Substituting Eq. (4.3.10) into the boundary conditions (4.3.6) and (4.3.7) yields the following bvp of mixed type

$$
\begin{align*}
& \sum_{n=1}^{\infty} \lambda_{n} A_{n} \sin \left(\lambda_{n} z\right)=H_{1}(z), \quad-a<z<0,  \tag{4.3.11}\\
& \sum_{n=1}^{\infty} A_{n} \sin \left(\lambda_{n} z\right)=H_{2}(z), \quad-1 \leq z<-a \tag{4.3.12}
\end{align*}
$$

where the inhomogeneous terms are given by the expressions

$$
\begin{equation*}
H_{1}(z)=-f^{\prime}(0) g(z)-\frac{1}{2} \sum_{n=1}^{\infty} \int_{0}^{\infty} e^{-\lambda_{n} \xi} S_{n}(\xi) d \xi \sin \left(\lambda_{n} z\right) \tag{4.3.13}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{2}(z)=f(0) g(z)+\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\lambda_{n}} \int_{0}^{\infty} e^{-\lambda_{n} \xi} S_{n}(\xi) d \xi \sin \left(\lambda_{n} z\right) \tag{4.3.14}
\end{equation*}
$$

Eqs. (4.3.11) and (4.3.12) form a dual trigonometrical series system which has to be solved in terms of the unknown coefficients $A_{n}$. The necessary steps are fully developed in the analysis that follows.

## First approach

In order to find a proper solution, we have to decompose the problem of Eqs. (4.3.11), (4.3.12) into the two following subproblems

$$
\begin{gather*}
\sum_{n=1}^{\infty} \lambda_{n} A_{n}^{(1)} \sin \left(\lambda_{n} z\right)=H_{1}(z), \quad-a<z<0,  \tag{4.3.15}\\
\sum_{n=1}^{\infty} A_{n}^{(1)} \sin \left(\lambda_{n} z\right)=0, \quad-1 \leq z<-a, \tag{4.3.16}
\end{gather*}
$$

and

$$
\begin{align*}
& \sum_{n=1}^{\infty} \lambda_{n} A_{n}^{(2)} \sin \left(\lambda_{n} z\right)=0, \quad-a<z<0,  \tag{4.3.17}\\
& \sum_{n=1}^{\infty} A_{n}^{(2)} \sin \left(\lambda_{n} z\right)=H_{2}(z), \quad-1 \leq z<-a . \tag{4.3.18}
\end{align*}
$$

In fact, this is the approach suggested by Sneddon (1966) [p.158]. The solution to the first subproblem is similar to that of the leading order problem [see Eqs. (3.4.4)-(3.4.9)]. Hence, letting $z=-\psi / \pi$, the unknown coefficients will be given by (Sneddon (1966) [p. 158])

$$
\begin{gather*}
A_{1}^{(1)}=\frac{1}{\sqrt{2}} \int_{0}^{c} h_{1}(\tau)\left[1-P_{1}(\cos \tau)\right] d \tau  \tag{4.3.19}\\
A_{n}^{(1)}=\frac{1}{\sqrt{2}} \int_{0}^{c} h_{1}(\tau)\left[P_{n-2}(\cos \tau)-P_{n}(\cos \tau)\right] d \tau, \quad n=2,3, \ldots, \tag{4.3.20}
\end{gather*}
$$

where $P_{n}$, again, denotes the $n$th degree Legendre Polynomial, while

$$
\begin{align*}
h_{1}(\tau)=\frac{1}{\pi} \frac{d}{d \tau} & \int_{0}^{\tau} \frac{d \psi}{\sqrt{\cos \psi-\cos \tau}} \int_{0}^{\psi} H_{1}(u) d u  \tag{4.3.21}\\
& -\frac{\sqrt{2} \gamma_{1}}{\pi} \frac{d}{d \tau} \int_{0}^{\tau} \frac{(1-\cos \psi / 2)}{\sqrt{\cos \psi-\cos \tau}} d \psi
\end{align*}
$$

The constant $\gamma_{1}$ yields from

$$
\begin{equation*}
\gamma_{1}\left\{1+\frac{\sqrt{2}}{\pi} \int_{0}^{c} \frac{(1-\cos \psi / 2)}{\sqrt{\cos \psi-\cos c}} d \psi\right\}=\frac{1}{\pi} \int_{0}^{c} \frac{d \psi}{\sqrt{\cos \psi-\cos c}} \int_{0}^{\psi} H_{1}(u) d u \tag{4.3.22}
\end{equation*}
$$

The solution to the second subproblem is given by (Sneddon (1966) [p. 159])

$$
\begin{gather*}
A_{1}^{(2)}=\frac{1}{\sqrt{2}} \int_{c}^{\pi} h_{2}(\tau)\left[1+P_{1}(\cos \tau)\right] d \tau,  \tag{4.3.23}\\
A_{n}^{(2)}=\frac{1}{\sqrt{2}} \int_{c}^{\pi} h_{2}(\tau)\left[P_{n}(\cos \tau)-P_{n-2}(\cos \tau)\right] d \tau, \quad n=2,3, \ldots, \tag{4.3.24}
\end{gather*}
$$

while

$$
\begin{equation*}
h_{2}(\tau)=\frac{1}{\pi} \frac{d}{d \tau} \int_{\tau}^{\pi} \frac{H_{2}(\psi) d \psi}{\sqrt{\cos \tau-\cos \psi}} \tag{4.3.25}
\end{equation*}
$$

However, this classical approach to the bvp was found to provide unreliable results, especially in the part of the wall that is captured by the air pocket. This particularly means that in the region of the air pocket the series of Eq. (4.3.18) did not converge to the required value satisfactorily; the part of the velocity potential which had to be zero (in the air pocket region) did not satisfy the corresponding boundary condition and clearly deviates from this value. Particularly, this is attributed to the very complicated mathematical manipulations concerning the higher order problem. Despite the fact that the solution given by Sneddon (1966) is absolutely exact and accurate, it involves the calculation of improper, elliptic integrals for which no analytical solutions exist. Therefore, these integrals have to be treated numerically. Consequently, in order to derive accurate results, a second approach had to be taken, which actually adjusts Sneddon's (1966) suggested solution.

## Second approach

We reexamine the main problem of Eqs. (4.3.11) and (4.3.12) and we write the latter in the form

$$
\begin{equation*}
\sum_{n=1}^{\infty}\left\{A_{n}-\frac{1}{2 \lambda_{n}} \int_{0}^{\infty} e^{-\lambda_{n} \xi} S_{n}(\xi) d \xi\right\} \sin \left(\lambda_{n} z\right)=f(0) g(z) . \tag{4.3.26}
\end{equation*}
$$

Next, we define

$$
\begin{equation*}
C_{n}=A_{n}-\frac{1}{2 \lambda_{n}} \int_{0}^{\infty} e^{-\lambda_{n} \xi} S_{n}(\xi) d \xi \tag{4.3.27}
\end{equation*}
$$

and we write the right-hand side term of Eq. (4.3.26) in a series-like form

$$
\begin{equation*}
\sum_{n=1}^{\infty} b_{n} \sin \left(\lambda_{n} z\right)=f(0) g(z) \tag{4.3.28}
\end{equation*}
$$

Using orthogonality, the expansion coefficients, $b_{n}$, become

$$
\begin{equation*}
b_{n}=2 f(0) \int_{-1}^{0} g(z) \sin \left(\lambda_{n} z\right) d z \tag{4.3.29}
\end{equation*}
$$

Further, we define $D_{n}=C_{n}-b_{n}$ and accordingly, the unknown coefficients $A_{n}$ are obtained from

$$
\begin{equation*}
A_{n}=D_{n}+b_{n}+\frac{1}{2 \lambda_{n}} \int_{0}^{\infty} e^{-\lambda_{n} \xi} S_{n}(\xi) d \xi \tag{4.3.30}
\end{equation*}
$$

By substituting Eq. (4.3.30) into Eq. (4.3.11) yields

$$
\begin{align*}
& \sum_{n=1}^{\infty} \lambda_{n}\left\{D_{n}+b_{n}+\frac{1}{2 \lambda_{n}} \int_{0}^{\infty} e^{-\lambda_{n} \xi} S_{n}(\xi) d \xi\right\} \sin \left(\lambda_{n} z\right) \\
& =-f^{\prime}(0) g(z)-\frac{1}{2} \sum_{n=1}^{\infty} \int_{0}^{\infty} e^{-\lambda_{n} \xi} S_{n}(\xi) d \xi \sin \left(\lambda_{n} z\right) \text {, }  \tag{4.3.31}\\
& -a<z<0 .
\end{align*}
$$

Eq. (4.3.31) is transformed in a more convenient form according to

$$
\begin{align*}
\sum_{n=1}^{\infty} \lambda_{n} D_{n} \sin ( & \left.\lambda_{n} z\right) \\
& =-f^{\prime}(0) g(z) \\
& -\sum_{n=1}^{\infty}\left\{\int_{0}^{\infty} e^{-\lambda_{n} \xi} S_{n}(\xi) d \xi+\lambda_{n} b_{n}\right\} \sin \left(\lambda_{n} z\right),  \tag{4.3.32}\\
& -a<z<0
\end{align*}
$$

while Eq. (4.3.12) is reduced to

$$
\begin{equation*}
\sum_{n=1}^{\infty} D_{n} \sin \left(\lambda_{n} z\right)=0, \quad-1 \leq z<-a . \tag{4.3.33}
\end{equation*}
$$

The solution given by Sneddon (1966) can now be applied properly to the bvp of Eqs. (4.3.32)-(4.3.33), to obtain the coefficients $D_{n}$. Once the coefficients $D_{n}$ have been determined, it is easy to obtain $A_{n}$ using Eq. (4.3.30).


Fig. 4.3.4. The coefficients $A_{n}$ for the higher order problem and $k=1.09(V=3 \mathrm{~m} / \mathrm{s})$. The air pocket extends between $-1 \leq z \leq-0.7$.

The solution for the higher order expansion coefficients $A_{n}$, is convergent again, as manifested in Fig. 4.3.4, for the first 50 modes. The results demonstrate a small amplitude oscillatory behavior around zero, which decays rather fast, in a manner similar to the coefficients of the leading order problem, but of a greater magnitude. An analogous behavior is observed for the infinite integral of Eq. (4.3.32) as a function of the mode number. A fast convergence to zero can be easily seen as the number of modes grows to infinity.


Fig. 4.3.5. The infinite integral of Eq. (4.3.32) for $k=1.09(V=3 \mathrm{~m} / \mathrm{s})$. The air pocket extends between $-1 \leq z \leq-0.7$.

### 4.3.4 The distribution of the higher order velocity potential on the wall

This is calculated by expression (4.3.1) explicitly, substituting $x=0$. Fig. 4.3.6 depicts the higher order potential on the wall for three different air pocket heights. The first point that one notices is the change in the sign of the potential. Recalling the results of the leading order solution, it is deduced that the leading order theory slightly overestimates the hydrodynamic pressure on the impacted part of the wall (see Fig. 3.4.3). On the other hand, the corrections due to the higher order solution amplify the leading order counterpart on the part of the wall directly above the fluid-air pocket intersection point. In addition, the presence of the air pocket seems to have a negligible effect on the potential on the upper part of the wall, close to the still water level. Finally, Fig. 4.3.6 verifies that $\phi_{3}$ is non-zero on the free-surface, $z=0$, as dictated by the Dirichlet boundary condition (4.2.5).


Fig. 4.3.6. The third order velocity potential on the wall for several air pocket heights. It was assumed that $k=1.09(V=3 \mathrm{~m} / \mathrm{s})$.

Figs. 4.3.7 and 4.3.8 show the pattern of $\phi_{3}$ for several values of the parameter $k$, which correspond to impact velocities $V=1,2,3 \mathrm{~m} / \mathrm{s}$ and two different air pocket heights. The velocity potential that ignores the hydrostatic term $(k=0)$ is shown for reference as well. It is observed that the higher the impact velocity is (or equivalently, the smaller the dimensionless parameter $k$ ), the smaller the magnitude of the higher order potential becomes. Consequently, the corrections that originate from the contribution of the higher order term are more limited. Thus, the main outcome from Figs. 4.3.7 and 4.3.8 is that the leading order predicts the velocity potential distribution on the wall for higher impact velocities more accurately.


Fig. 4.3.7. The third order velocity potential on the wall for several values of the parameter $k$. The air pocket extends between $-1 \leq z \leq-0.7$.


Fig. 4.3.8. The third order velocity potential on the wall for several values of the parameter $k$. The air pocket extends between $-1 \leq z \leq-0.8$.

Nevertheless, the contribution of the higher order potential does not change the results of the leading order problem for any impact velocity substantially. To verify this allegation, Fig. 4.3.9 is provided. One can easily notice that the shape of the potential distribution on the wall, as well as its magnitude, are essentially determined by the
leading order solution. Furthermore, small discrepancies are noticed in the upper part of the wall and just above the upper point of the air pocket. In the leading order solution, the maximum value of the velocity potential is $\sim-0.3$ whereas when the correction of the higher order problem is involved, it is $\sim-0.28$, which corresponds to a reduction of $6.7 \%$. Therefore, the hydrodynamic pressure is expected to exhibit the same variation.


Fig. 4.3.9. The distribution of the total velocity potential on the wall when i) retaining the leading order term only and ii) considering the correction from the higher order problem. It was assumed that $k=1.09(V=3 \mathrm{~m} / \mathrm{s}), t=0.01$ and the air pocket extends between $-1 \leq z \leq-0.7$.

The last remarks herein concern the higher order free-surface elevation $\eta_{4}$. Combining Eqs. (3.3.20), (3.4.1) and (4.3.1) and evaluating on $z=0$ yields

$$
\begin{align*}
\eta_{4}=\frac{1}{4} \sum_{n=1}^{\infty} \lambda_{n} & \left\{A_{n} e^{-\lambda_{n} x}\right. \\
& -\frac{1}{2 \lambda_{n}}\left[\int_{0}^{x} e^{-\lambda_{n}(x-\xi)} S_{n}(\xi) d \xi\right.  \tag{4.3.34}\\
& \left.\left.+\int_{x}^{\infty} e^{-\lambda_{n}(\xi-x)} S_{n}(\xi) d \xi\right]\right\}
\end{align*}
$$

Letting $x=0$ gives the run-up on the wall due to the higher order potential, which is

$$
\begin{equation*}
\left.\eta_{4}\right)_{x=0}=\frac{1}{4} \sum_{n=1}^{\infty} \lambda_{n} A_{n}-\frac{1}{8} \sum_{n=1}^{\infty} \int_{0}^{\infty} e^{-\lambda_{n} \xi} S_{n}(\xi) d \xi . \tag{4.3.35}
\end{equation*}
$$

It can be numerically verified that $\eta_{4}$, calculated on the wall is singular and that is due to the first sum, which is divergent on $x=0$, owing to the expansion coefficients $A_{n}$. The first term exhibits an analogous oscillating behavior with that of Eq. (3.4.13), but of greater magnitude, which was totally expected. Particularly, given that the expansion coefficients for the higher order problem $A_{n}$ are negative (see Fig. 4.3.4), this sum tends to negative infinity for increasing number of modes. The second sum tends asymptotically to a constant value as $n$ grows to infinity. Consequently, a general, approximate solution can be derived from the multiplication of the value of the leading order free-surface elevation on the wall with a negative constant. The results for the higher order free-surface elevation are shown in Fig. 4.3.10. It should be noted that $\eta_{4}$ is negative, a fact which implies that the leading order slightly overestimates the freesurface elevation. However, recalling Eq. (3.3.4), $\eta_{4}$ is multiplied by the term $t^{4}$, thus its contribution for small times is literally negligible. Nonetheless, in order to obtain a more accurate solution for research, rather than for practical purposes, an asymptotic analysis at $x \rightarrow 0$ should be conducted.


Fig. 4.3.10. The higher order free-surface elevation at $t=0.005$. It was assumed that the air pocket extends between $-1 \leq z \leq-0.5$.

### 4.4 Conclusions

In this Chapter, the solution to the higher order bvp was analyzed in detail. After several mathematical manipulations, it was reduced to a complicated Sturm-Liouville problem, with mixed inhomogeneous boundary conditions, which involved dual trigonometrical, infinite series. The difficulties and the numerical instabilities encountered during the solution process were also discussed. The main conclusions drawn from this part of the research, can be summarized as follows:
I. The gravity was found to affect the results of the velocity potential when we considered the higher order bvp.
II. The most noticeable outcome of this analysis was the change in the sign of the potential of the third order solution on the impacted portion of the wall. This particularly means that the results anticipated by the leading order solution are attenuated on the upper part and amplified on the part just above the fluid-air pocket intersection point.
III. The presence of the air pocket does not affect the potential and consequently the hydrodynamic pressure on the upper part of the wall, close to the freesurface.
IV. The leading order theory predicts the impact pressures on the wall more accurately for large impact velocities.
V. The contribution of the higher order term leads to a reduction of $6.7 \%$ in the maximum value of the velocity potential on the wall.
VI. The higher order free-surface elevation $\eta_{4}$ was proved to obtain negative values, which implies that the corresponding leading order solution slightly overestimates the results. In any case, its contribution is considered as negligible.

### 4.5 Further research

The mathematical analysis we dealt with in this Chapter, prescribed the solution to a challenging Sturm-Liouville problem.
I. The methodology presented was initiated by assuming a general expression for the higher order velocity potential [see Eq. (4.3.1)]. It unavoidably demanded the selection of the function $g(z)$. Therefore, it would be interesting to examine how any different, but proper, expressions for the function $g(z)$ would change the results. Besides, it is challenging to investigate an alternate general expression for the higher order velocity potential $\phi_{3}$.
II. An asymptotic analysis as $x \rightarrow 0$ for the treatment of the higher order freesurface elevation singularity could be conducted.

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## Chapter 5 Original contributions of the PhD thesis

The research conducted within the context of this dissertation has several novel features, which aimed to give clear answers to important and unclarified scientific queries concerning analytical methods in water impact problems. The scrutiny in this field was absolutely indispensable, given that this class of problems inherently introduce difficulties in the physical modelling of the phenomenon and the mathematical solution as well. Particularly,
— Chapter 2:

- Provided a straightforward comparison between the slamming force exerted on the vertical cylinder as calculated by the solution of the 3D Wagner bvp, and those anticipated by the corresponding 2D theories.
- Provided a straightforward comparison between the results concerning the evolution with time of the real 3D contact line and those anticipated by the 2 D theories.
- Discussed in detail the range of validity of the 2D Wagner theory when applied to real 3D geometries.
- Demonstrated comparative results for the velocity of the expansion of the contact area, between the cylinder and the liquid, relative to the impact velocity.
- Chapter 3:
- Enhanced the existing analytical models utilized to simulate breaking wave impact on vertical walls, based on the concept of pressure-impulse, by taking the free-surface elevation and the entrapment of a small air pocket into account.
- Presented for the first time a solution to this type of impact problems using dual trigonometrical series system.
— Chapter 4:
- Enriched the available mathematical literature concerning SturmLiouville problems. The solution developed aimed to tackle considerable difficulties arising from the inhomogeneous boundary conditions of the inhomogeneous Sturm-Liouville problem, which even involved infinite series.
- Examined and discussed the accuracy of the leading order results thoroughly. It also discussed the rate of contribution of the higher order solution of the velocity potential on the wall.

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## APPENDIX A

## LIST OF PUBLICATIONS

## PAPERS IN INTERNATIONAL JOURNALS (PEER REVIEW)

1. Tsaousis T.D., Papadopoulos P.K., Chatjigeorgiou I.K. (2020): "A semianalytical solution for the three-dimensional Wagner steep wave impact on a vertical circular cylinder", Applied Mathematical Modelling (Impact Factor: 3.633).
2. Tsaousis T.D., Chatjigeorgiou I.K. (2020): "An analytical approach for the two-dimensional plunging breaking wave impact on a vertical wall with air entrapment", Fluids (CiteScore: 1.8).
3. Tsaousis T.D., Chatjigeorgiou I.K. (2021): "Higher order phenomena connected with the two-dimensional breaking wave impact on a vertical impermeable wall with air entrapment", European Journal of MechanicsB/Fluids (Impact Factor: 2.131).

## PAPERS IN INTERNATIONAL CONFERENCES

1. Tsaousis T.D., Chatjigeorgiou I.K. (2021): "An idealized two-dimensional breaking wave impact on a vertical wall", $31^{s t}$ ISOPE, 20-25 June 2021, Rodos, Greece (Online/ virtual).
2. Tsaousis T.D., Chatjigeorgiou I.K. (2021): "Higher order phenomena for a twodimensional breaking wave impact on a vertical wall", $36^{\text {th }}$ IWWWFB, 25-28 April 2021, Seoul, Korea, (Online/ virtual).

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## APPENDIX B

## CURRICULUM VITAE

Name: Theodosis Tsaousis
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## EDUCATION

- PhD candidate, School of Naval Architecture and

Nov. 2016 - Sept. 2021 Marine Engineering, National Technical University of Athens (NTUA)
Title of PhD thesis: Development of analytical models of mixed boundary value problems for the determination of the hydrodynamic loading on solid bodies, with a focus on violent slamming of free-surface flows.

Predoctoral courses: Wave phenomena in the sea Mar. 2017 - Jun. 2018 environment, FEM in the static and dynamic analysis of the structures, Environmental conditions and sea loads on marine structures, Non-linear waves, Hydromechanic analysis and optimal mooring design of moored floating structures

- Diploma in Naval Architecture and Marine Engineering
School of Naval Architecture and Marine Engineering, National Technical University of Athens
Diploma thesis: Steady state slug flow in risers for oil and gas exploitation


## SCIENTIFIC PUBLICATIONS

## PAPERS IN INTERNATIONAL JOURNALS (PEER REVIEW)

1. Tsaousis T.D., Papadopoulos P.K., Chatjigeorgiou I.K. (2020): "A semianalytical solution for the three-dimensional Wagner steep wave impact on a vertical circular cylinder", Applied Mathematical Modelling (Impact Factor: 3.633).
2. Tsaousis T.D., Chatjigeorgiou I.K. (2020): "An analytical approach for the two-dimensional plunging breaking wave impact on a vertical wall with air entrapment", Fluids (CiteScore: 1.8).
3. Tsaousis T.D., Chatjigeorgiou I.K. (2021): "Higher order phenomena connected with the two-dimensional breaking wave impact on a vertical impermeable wall with air entrapment", European Journal of MechanicsB/Fluids (Impact Factor: 2.131).

## PAPERS IN INTERNATIONAL CONFERENCES

1. Tsaousis T.D., Chatjigeorgiou I.K. (2021): "An idealized two-dimensional breaking wave impact on a vertical wall", $31^{s t}$ ISOPE, 20-25 June 2021, Rodos, Greece (Online/ virtual).
2. Tsaousis T.D., Chatjigeorgiou I.K. (2021): "Higher order phenomena for a twodimensional breaking wave impact on a vertical wall", $36^{\text {th }} I W W W F B, 25-28$ April 2021, Seoul, Korea, (Online/ virtual).

## WORK EXPERIENCE

## Research projects

- FHMES: Floating Hybrid Mooring Jan. 2020 - present Wind Turbine Energy System
- PICASSO: Preventing incidents Jun. 2017 - Jul. 2017 and accidents by safer ships in the oceans


## TEACHING EXPERIENCE

- Dynamics of floating structures Nov. 2016 - present
- Design of floating structures

Nov. 2016 - present

## INVITED SPEAKER

- Imperial College London, Department of Civil \& Environmental Engineering. The title of the lecture was: "Steep wave impact on a vertical circular cylinder", Online/ virtual, May 12, 2021.


## LANGUAGES

Greek (Mother tongue), English (Certificate in Advanced English, CAE, University of Cambridge), French (DELF B1)

## IT SKILLS

MATLAB, Maple, Tribon, AutoCAD, Fortran, Abaqus, MS-Office

## SEMINARS/ WORKSHOPS

- Integrated tank testing for the offshore renewable industry, $25^{\text {th }}-29^{\text {th }}$ Nov. 2019, MaREI, Cork, Ireland (with fellowship)


## FELLOWSHIPS

- By the Research Committee of the

Jun. 2018 - present National Technical University of Athens

## AWARDS

- Runner-up of the "Tuck Fellowship" for young researchers, by the international conference $36^{\text {th }} I W W W F B, 25-28$ April 2021, Seoul, Korea


## MILITARY SERVICE

Hellenic Air Forces
Nov. 2015 - Nov. 2016

## INTERESTS

Swimming, basketball, strolling, theatre, literature, travelling

