NATIONAL TECHNICAL UNIVERSITY OF ATHENS

SCHOOL OF APPLIED MATHEMATICAL AND PHYSICAL SCIENCES

DEPARTMENT OF PHYSICS

NCSR ”DEMOKRITOS”

INSTITUTE OF NANOSCIENCE AND NANOTECHNOLOGY

INSTITUTE OF NUCLEAR AND PARTICLE PHYSICS

Interdepartmental Program of Postgraduate Studies

« Physics and Technological Applications »

Inflation Without Inflaton Fields in a Cosmological Model with Gravitational Anomalies

Thesis for the Master’s Degree of

Chiotellis Vyron-Alexandros

Thesis Supervisor: Prof. Mavromatos Nikolaos

25/2/2022

Athens

Acknowledgments

Foremost, I would like to express my sincere gratitude to my thesis supervisor Prof. Nikolaos Mavromatos for his continuous support and guidance throughout my studying and writing of this thesis, for his enthusiasm, patience, immense knowledge and the opportunity he provided by introducing me to this subject.

Besides my supervisor, I would like to thank the rest of my thesis committee: Prof. Georgios Koutsoubas and Prof. Eleftherios Papantonopoulos for their insightful comments during my presentation and the tools and knowledge they bestowed to me during their courses.

My many thanks also go to PhD student Nick Chatzifotis, whose help in matters of general relativity was insightful, and my friend and fellow student Katerina Filopoulou for her comments on the thesis and support.

Last but not least, I'd like to express my deepest gratitude to my mother, who without her encouragement and sacrifices I could not have completed my postgraduate studies.

Abstract

Cosmic Inflation is a physical process of vital importance in cosmology, whose occurrence determine the properties of the universe and most predictions have been phenomenologically verified. Nonetheless, there is no coherent theoretical explanation about the physical process that caused inflation within standard cosmology, which creates a gap of understanding in the physics of the early universe. With this motivation, in this master's thesis an extensive analysis of a cosmological model with gravitational anomalies, inspired by string theory, is being made, which produce a self-consistent inflationary epoch. Gravitational anomalies are an emergent property of quantum gravity and, in combination with primordial gravitational waves , can provide a positive, de-Sitter type, vacuum energy density capable to drive inflation. Additionally, the total energy density of the model is of Running-Vacuum type, which is capable of connecting the inflationary era with today's cosmology, thus providing a phenomenologically testable argument compatible with Quantum Field Theory (QFT) and Renormalization Group (RG).

In this thesis, a full review of the aforementioned model, both with physical and mathematical argumentation, is given. The first section summarizes the status-quo of modern (ΛCDM) cosmology, the current cosmological tensions and the main elements of inflation. Then, it provides an extensive analysis of the Running Vacuum Models (RVM). The second section introduce the reader with the string-inspired cosmological model used throughout this thesis and its properties, focusing on the KR axion quantum field and the emergence of the gravitational anomalies. The main body of this thesis is section three, with the objective of studying the gravitational anomaly and its non-zero vacuum expectation value, induced by primordial gravitational waves. Additionally, in the same chapter, the RV total energy density is derived, containing the positive de-Sitter type contribution, necessary for inflation to occur. In the last section, a summary of the work is given with some additional information and supplementary thoughts on this model. Furthermore, two Appendices are included, with the first covering the necessary notation and formulas of General Relativity used throughout this thesis, while the second provides the analytical calculations done in various part not given in the main body of the thesis.

Περίληψη

Ο Κοσμικός Πληθωρισμός είναι μία φυσική διαδικασία ύψιστης σημασίας για τη κοσμολογία, με την ύπαρξη του να καθορίζει της ιδιότητες του σύμπαντος και τις περισσότερες προβλέψεις του να έχουν επιβεβαιωθεί φαινομενολογικά. Παρόλα αυτά, δεν υπάρχει κάποια συναφή θεωρητική εξήγηση για την φυσική διεργασία που προκάλεσε τον πληθωρισμό, αφήνοντας έτσι ένα κενό κατανόησης στη φυσική του πρώιμου σύμπαντος. Με αυτό το κίνητρο, σε αυτή η μεταπτυχιακή διπλωματική εργασία γίνεται μία εκτενής ανάλυση ενός κοσμολογικού μοντέλου με βαρυτικές ανωμαλίες, εμπνευσμένο από την θεωρεία χορδών, το οποίο παράγει ένα, συνεπή με το μοντέλο, πληθωρισμό. Οι βαρυτικές ανωμαλίες είναι μία αναδυόμενη ιδιότητα της κβαντικής βαρύτητας και, σε συνδυασμό με αρχαία βαρυτικά κύματα, μπορεί να διαθέσει μια θετική, de-sitter τύπου, πυκνότητα κενού ικανή για παράγει τον πληθωρισμό. Επιπλέον, η ολική πυκνότητα του μοντέλου είναι τύπου ”Τρεχούμενου-Κενού”, με την δυνατότητα να συνδέσει την πληθωριστική εποχή με την σημερινή κοσμολογία, και συνεπώς να παρέχει ένα φαινομενολογικά ελέγξιμο επιχείρημα συμβατό με την Κβαντική Θεωρεία Πεδίου (ΚΘΠ) και την Ομάδα Επανακανονικοποίησης (ΟΕ).

Σε αυτή τη διπλωματική γίνεται μία πλήρη ανασκόπηση του προαναφερθέντος μοντέλου, τόσο με φυσική όσο και με μαθηματική επιχειρηματολογία. Το πρώτο κεφάλαιο συνοψίζει τη παρούσα κατάσταση στη σύγχρονη (ΛCDM) κοσμολογία, τις τωρινές κοσμολογικές εντάσεις και τα βασικά στοιχεία του πληθωρισμού. Έπειτα, παρέχει μία εκτενή ανάλυση των μοντέλων ”Τρεχούμενου-Κενού”. Το δεύτερο κεφάλαιο εισάγει τον αναγνώστη στο κοσμολογικό μοντέλο εμπνευσμένο από την θεωρεία χορδών και της ιδιότητες του, εστιάζοντας στο KR αξιονικό κβαντικό πεδίο και την ανάδυση των βαρυτικών ανωμαλιών. Το κύριο σώμα της διατριβής είναι το κεφάλαιο τρία, με σκοπό την μελέτη της βαρυτικής ανωμαλίας και την μη-μηδενική αναμενόμενη τιμή κενού της, προκαλούμενη από αρχαία βαρυτικά κύματα. Επιπλέον, στο ίδιο κεφάλαιο, υπολογίζεται η ολική πυκνότητα τύπου ”Τρεχούμενου-Κενού”, η οποία περιέχει την θετική, de-Sitter, συνεισφορά αναγκαία για να λάβει χώρα ο πληθωρισμός. Στο τελευταίο κεφάλαιο δύναται μία σύνοψη της εργασίας με κάποιες επιπλέον πληροφορίες και επιπρόσθετες σκέψεις όσον αφορά το μοντέλο. Επιπλέον, περιλαμβάνονται δυο παραρτήματα, με το πρώτο να παρέχει τον απαραίτητο φορμαλισμό και τους μαθηματικούς τύπους της Γενικής Σχετικότητας που χρησιμοποιούνται σε όλη αυτή τη διπλωματική, ενώ το πρώτο παρέχει τους αναλυτικούς υπολογισμούς που έγιναν σε ποικίλα σημεία οι οποίοι δεν δίνονται στο κυρίως σώμα της εργασίας.

Contents

1. Introduction: The Running Vacuum

1.1. Cosmology of Today

One of the most discussed and controversial topics in modern cosmology is that of cosmic inflation. This theory, even thought it hasn't achieved a theoretical validation, solves numerous questions about the observable universe, making it impossible for current cosmology to stand without some kind of inflationary epoch in the early universe. Thus, a big effort of today's theoretical research is involved, directly or indirectly, with developing a theoretical model of cosmic inflation that could accommodate with current observational data.

In the previous century, the development of General Relativity by Albert Einstein gave the mathematical tools necessary to explain a plethora of phenomena about the nature and dynamics of the universe. All that progress have converged in today's Cosmological- Constant-Cold-Dark-Matter (ΛCDM) model of cosmology, which introduces to the Friedmann equations the cosmological constant (CC) and its associated vacuum energy density, $\rho_\Lambda = \Lambda/(8\pi G)$, to accommodate the role of Dark Energy (DE) and the accelerated expansion of the universe. This addition to the Einstein Equations was necessary for accurately explaining the global properties of the observable universe. On top of that, in order to explain various properties of today's universe, mainly the cosmological principle, the concept of inflationary expansion of the early universe was introduced. This expansion, similar (but much stronger) to DE, was of de-Sitter nature and exponential expanded the universe, very fast and in many orders of magnitude. Yet, it's cause, and connection with today's universe and CC is an unresolved theoretical problem.

This is thought to be because the introduction of the CC in the Einstein Equations is not theory emergent, but a fine-tuning forced into the Friedmann equations [24]. Thus, even though it can perfectly describe the recent accelerated expansion of the universe, it fails to to explain the dynamical properties of the early universe, most importantly, in and after the inflationary epoch, and it even comes with huge disagreement with Quantum Field Theory about the nature of the CC and the vacuum energy density, with a discrepancy as high as 120 orders of magnitude [20]. Moreover, conflict between various experimental data has recently emerged. The main pillar of this conflict is known as the H_0 tension and it emerges though discrepancies in the data of the Hubble constant [3, 4, 5, 6]. Specifically, new data collected by the Plank Telescope, using mainly measurements of the Cosmic Microwave Background (CMB) anisotropies [2] and fixed by constrains of the ΛCDM, have constructed a value

(67.4 ± 0.5 km/s/Mpc) for the Hubble parameter which is many standard deviations away from the accepted value (73.5 ± 1.6 km/s/Mpc) emerging from local (i.e. 1A supernova/Cepheid variables) measurements. Additionally, the aforementioned tension is reinforced by the so-called σ_8 tension [7, 8], which sees discrepancy between the data of the Plank Collaboration and the Redshift Space Distortion (RDS) measurements for the cosmological structure growth (σ_8, $f\sigma_8$). Despite the fact that many inconsistencies in the data could be given a conventional explanation (i.e. statistical uncertainty, better cosmic distance measurement methods) both the H_0 and σ_8 tensions have prompted researches to find an explanation by seeking new physics model beyond the ΛCDM.

Many, if not all, of the aforementioned problems can be resolved with the simple assumption that the parameter Λ is not a constant, but a function of time $\Lambda \equiv \Lambda(t)$ [12], within the so-called Running Vacuum Model [12, 13, 14], which has also recently managed to alleviate simultaneously both type of tensions [11]. This model requires the cosmic vacuum to have a de Sitter-type equation of state $p(t) = -\rho(t) = -\Lambda(t)$, despite the vacuum energy being dependent on cosmic time t. This idea was actually considered even before the discovery of the accelerating expansion of the universe and it has appealed to many scientists since it can resolve most of the CC problems and be used as an alternative to the ΛCDM model and dark energy.

An interesting approach in finding an effective form of $\Lambda(t)$ is through QFT within the context of renormalization group. These Running Vacuum Models define the vacuum energy as a well-motivated function of the Hubble rate $\Lambda = \Lambda(H(t))$, which contains only even powers of H. The success of these models is that they can describe the whole history of the evolution of the universe, from inflation till the present era without the requirement of the additional inflaton field, while also inflation comes naturally, driven by a H^4 term. They also come in agreement with many observational data while they are devoid of mathematical inconsistencies, such as the initial singularity in the big bang model.

1.2. Modern Inflation

The inflation mechanism is a necessary ingredient for realizing the cosmological principle, which states that the distribution of matter in the universe is homogeneous and isotropic, something fundamental for the ΛCDM model. In addition, it accounts for various observational facts about the universe, such as the observed flatness of the universe and the absence of magnetic monopoles [25]. All of the above make modern cosmology impossible to stand without some kind of inflationary era.

In all inflationary models, inflation occurs soon after the Big Bang and, in a tiny fraction of a second, it multiplied the size of the universe in a factor much larger than in the following 13.8 years of regular expansion. Then, after the inflationary period, the universe stops its exponential acceleration and enters a reheating phase. A defining characteristic of inflation is that the scale factor grew exponentially, which is expressed by

$$\alpha(t) \propto e^{Ht} \tag{1.2.1}$$

while accelerated expansion is only possible if

$$P < -\frac{1}{3}\rho \tag{1.2.2}$$

where P is the pressure and ρ is the energy density of the substance driving inflation.

The increase of the scale factor between time t_i of the beginning of inflation and time t_f when inflation ended. Is given by

$$\frac{\alpha(t_f)}{\alpha(t_i)} = e^{H(t_f - t_i)} = e^N \tag{1.2.3}$$

with N being the number of e-foldings of inflation.

In most models of inflation the driving force behind inflation is one (or more) scalar field φ, called the inflaton. The physical nature of such field is yet to be determined and many hypotheses have been proposed, but, for now, we are going to just use it to parameterize the inflation mechanism. To do so we start with the action describing the dynamics of the field φ

$$S_{R+\varphi} = \int dx^4 \sqrt{-g} \left[\frac{1}{2\kappa^2} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right] \tag{1.2.4}$$

Then, the energy momentum tensor of the scalar field is given by

$$T^{\varphi}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_\varphi}{\delta g^{\mu\nu}} = \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \left(\frac{1}{2} \partial_\alpha \varphi \partial^\alpha \varphi + V(\varphi) \right) \tag{1.2.5}$$

assuming φ can be described as a perfect fluid we get

$$\rho_\varphi = \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \tag{1.2.6}$$

$$P_\varphi = \frac{1}{2} \dot{\varphi}^2 - V(\varphi) \tag{1.2.7}$$

where ρ_φ is the energy density and P_φ the pressure of the field. It is obvious that (1.2.2) is satisfied only if $V(\varphi) > \frac{1}{2}\dot{\varphi}^2$. Thus, for the time period in which inflation occurs, the potential energy of the field must dominate over the kinetic energy. The proposed model is that, initially the potential has a big flat region, also known as a false vacuum, where $V(\varphi)$ is dominates over $\frac{1}{2}\dot{\varphi}^2$ and we get inflation. Then, there is a local minimum which when the fields enters, the kinetic term dominates and inflation stops. This mechanism is called slow-roll inflation.

By substituting the above expressions into the Friedmann equations (shown in Appendix A, but we omitted the terms containing Λ) leads to

$$H^2 = \frac{\kappa^2}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right) \tag{1.2.8}$$

$$\frac{\ddot{\alpha}}{\alpha} = -\frac{\kappa^2}{3}\left(\dot{\varphi}^2 - V(\varphi)\right) \tag{1.2.9}$$

where H is the Hubble parameter and G newtons constant. Expanding (1.2.9) gives

$$\frac{\ddot{\alpha}}{\alpha} = -\frac{\kappa^2}{2}\dot{\varphi}^2 + H^2 = H^2\left(1 - \kappa^2\frac{\dot{\varphi}^2}{2H^2}\right) \tag{1.2.10}$$

$$\Rightarrow \qquad \dot{H} = -\frac{\kappa^2}{2}\dot{\varphi}^2 \tag{1.2.11}$$

from (1.2.10) we define the slow-roll parameter ϵ_φ as:

$$\frac{\ddot{\alpha}}{\alpha} = H^2\left(1 - \epsilon_\varphi\right) \Rightarrow \epsilon_\varphi = \kappa^2\frac{\dot{\varphi}^2}{2H^2} \tag{1.2.12}$$

thus, solving (1.2.12) for $\dot{\varphi}^2$ and inserting to (1.2.11) gives

$$\epsilon_\varphi = -\frac{\dot{H}}{H^2} \ll 1 \tag{1.2.13}$$

Th slow-roll parameter is very important for the phenomenological verification of any model discussing inflation since observable quantities, like the scalar and tensor power spectrum, are commonly expressed in terms of it [25].

1.3. The Running Vacuum Model

As discussed previously, in current physical cosmological models, for describing a dynamical expansion rate of the universe is necessary to insert an extra de-Sitter type term Λ is the Einstein equations. For a FLRW universe containing normal matter, the Einstein equations are:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa^2 T_{\mu\nu} - g_{\mu\nu}\Lambda \tag{1.3.1}$$

with $T_{\mu\nu}$ is the matter stress-energy tensor. Assuming that it describes Λ a fluid like substance, we denote with ρ_Λ the vacuum energy density, where $\rho_\Lambda = \Lambda/\kappa^2$. Thus we may define the stress energy tensor of system.

$$T_{\mu\nu}^{total} = T_{\mu\nu} - g_{\mu\nu}\rho_\Lambda \tag{1.3.2}$$

Then, in an Isotropic and homogeneous universe, we define the total energy density ρ_τ and the total pressure P_τ as

$$T_{00}^{total} = \rho_\tau \tag{1.3.3}$$

$$T_{ii}^{total} = -g_{ii}P_\tau \tag{1.3.4}$$

thus, the matter equation of state reads

$$P = \omega\rho. \tag{1.3.5}$$

with ω being the equation of state constant, ρ the matter energy density and P the matter pressure. Additionally, the vacuum equation of state is

$$P_\Lambda = -\rho_\Lambda \tag{1.3.6}$$

with P_Λ being the vacuum energy pressure. Finally, the corresponding Einstein field equations, for the FLRW metric (see Appendix A), of this system are

$$\kappa^2\rho_\tau \equiv \kappa^2\rho + \Lambda = 3H^2 \tag{1.3.7}$$

$$\kappa^2 P_\tau \equiv \kappa^2 P - \Lambda = -2\dot{H} - 3H^2 \tag{1.3.8}$$

where, $H \equiv \dot{a}/a$ is the Hubble parameter, $a = a(t)$ is the scale factor and the overdot denotes derivative with respect to the comic time t.

The Running Vacuum Models make the assumption of a dynamic Λ. Hence, we understand that the vacuum energy density and pressure need to be functions of time, i.e. $\rho_\Lambda = \rho_\Lambda(t)$ and $P_\Lambda(t) = -\rho_\Lambda(t)$ [12, 24].

By taking the zero component of the conservation of energy equation of the system described by (1.3.7) and (1.3.8) with a dynamic vacuum, we derive the following conservation law (details in Appendix A).

$$\nabla^\mu T_{\mu 0}^{total} = \dot{\rho}_\Lambda + \dot{\rho} + 3(1+\omega)\rho H = 0 \tag{1.3.9}$$

From a phenomenological point of view, this equation reveals the first huge difference between the running vacuum models compered with ΛCDM cosmology where, in the later, with $\rho_\Lambda = \text{const.}$, the above conservation law reduces to the typical equation $\dot{\rho} + 3(1+\omega)\rho H = 0$. The non-zero value of $\dot{\rho}_\Lambda$ states an energy exchange between matter and vacuum and thus, it is clear that running vacuum model could describe a universe with different dynamical properties of that of standard cosmology, which may have evolved very differently from our current understanding.

Let's now introduce the following parameter:

$$\beta(t) = \frac{\rho_\Lambda - \rho_{\Lambda 0}}{\rho_\tau} \tag{1.3.10}$$

where $\rho_{\Lambda 0}$ it's the vacuum energy density of current time according to the ΛCDM. Using the above relation and some algebra is trivial to show that the following equation holds

$$\rho_\Lambda = \rho_{\Lambda 0} + \beta(t)\rho_\tau \tag{1.3.11}$$

and with the help of (1.3.7) we can also show that

$$\Lambda(t) = \Lambda_0 + 3\beta(t)H^2 \tag{1.3.12}$$

Therefore, we showed with some simple phenomenological arguments that it is possible to relate the time depended vacuum energy with the Hubble parameter.

A more fundamental theoretical approach would be to use the tools given by RG and QFT to study the behavior of ρ_Λ and subsequently $\Lambda(t)$. Assuming that we can use the RG method to study the behavior of the QFT structure of the vacuum energy, then the vacuum energy density becomes a running parameter. Therefore, there should exist a corresponding β-function of the form

$$\beta(\rho_\Lambda) = \frac{d\rho_\Lambda}{d\ln\mu} \tag{1.3.13}$$

which describers the quantum contributions to vacuum energy density. The quantity ρ_Λ is a μ-dependent renormalized parameter of the QFT structure of the vacuum energy and μ is the renormalization parameter of the effective action. In standard QFT the gauge coupling constants, which determine the strength of the corresponding interaction, are related with the parameter μ which is associated with the energy scale of the physical process in question. Similarly, the quantum effects of the matter fields described by the running of ρ_Λ can be associated with the change in spacetime curvature, and hence, with the change of the energy of the external gravitational field. Therefore we may associate μ^2 with the Ricci scalar R, where

$$R = 12H^2 + 6\dot{H} \quad \text{(details in Appendix A)} \tag{1.3.14}$$

Assuming the acceleration of the universe is constant (and thus $\dot{H} \sim H^2$) and since $\mu^2 \sim R$ we have $\mu^2 \sim H^2$ [21]. Within the RG approach we choose $\mu = H$, thus, the rate of change of ρ_Λ satisfies the equation

$$(4\pi)^2 \frac{d\rho_\Lambda}{d\ln\mu^2} = \sum_{m=1,2,...} A_{2m}\mu^{2m} = A_2\mu^2 + A_4\mu^4 + ... \tag{1.3.15}$$

The above expression is the definition of the running vacuum model [12, 13] and, also, of the β-function for the RG running of ρ_Λ. The coefficients A_{2m} receive the loop contributions from the matter fields with masses M_i. We notice that only even powers of are involved. This is due to our need for $\mu^2 \sim H^2 \sim R$ and also for the effective action of QFT to be a covariant quantity. The dimentionality of $d\rho_\Lambda/d\ln\mu^2$ restrict every term $A_{2m}\mu^{2m}$ to be of order $\sim [M]^4$. This is also the reason we have omitted the A_0 which should be of order M_i^4 and would have overtake the running of ρ_Λ because $M_i > \mu$ for all known particles. Since we know that μ and H have dimensions of mass we may rewrite the coefficients A_{2m} in a way which every term of the β-function will appropriately be expressed in $[M]^4$ dimensions. Thus, we can rewrite (1.3.15) in the form

$$\frac{d\rho_\Lambda}{d\ln H^2} = \frac{1}{(4\pi)^2} \sum_i \left[a_i M_i^2 H^2 + b_i H^4 + c_i \frac{H^6}{M_i^2} + ...\right] \tag{1.3.16}$$

by integrating the above equation and using the relation $\rho_\Lambda(H) = \Lambda(H)/(8\pi G)$ we get (the details of the calculation are in Appendix)

$$\Lambda(H) = c_0 + \frac{\kappa^2}{(4\pi)^2} \sum_i \left[a_i M_i^2 H^2 + \frac{b_i}{2} H^4 + \frac{c_i}{3} \frac{H^6}{M_i^2} + ...\right] \tag{1.3.17}$$

As we can see, the RG formulation has provided a fundamental expression of the running vacuum withing the limits of QFT in curved spacetime which also owns the general covariance of the effective action. It is obvious that for low energies, where H is much smaller than the mass of any known particle, the expression (1.3.17) converges very fast and only the loop contributions of the matter fields, via the term $a_i M_i^2 H^2$, are significant. On the other hand, if we are to study the inflationary epoch, where H is large enough (but still smaller than M_i) the term H^4 will be much more significant, where the terms H^6/M_i^2 and above are less and less important.

Taking the expression (1.3.17) while only considering the terms H^2 and H^4 and doing some re-arrangement we get the more compact form of $\Lambda(H)$

$$\Lambda(H) = c_0 + 3\left(\nu H^2 + a\frac{H^4}{H_I^2}\right) \tag{1.3.18}$$

where

$$\nu = \frac{1}{6\pi}\sum_i a_i \frac{M_i^2}{M_{pl}^2} \tag{1.3.19}$$

and

$$a = \frac{1}{12\pi}\sum_i b_i \frac{H_I^2}{M_{pl}^2} \tag{1.3.20}$$

Let's explain the reasoning behind this formulation. Firstly, we re-arranged the expression of $\Lambda(H)$ in such a way that it behaves as (1.3.12) in low energies, where the H^4 term vanishes (i.e. when $H^4 << H_I^2$). We also may describe our current ΛCDM universe when $H \sim H_0 << H_I$ with the H^2 term representing small corrections to the dominant c_0 term of present time. The second term comes in relevance at great energies, such as the early universe, where the Hubble parameter is large and thus $H^4 \sim H_I^2$. H_I represents a physical quantity and, specifically, equates with the value of the Hubble parameter during pure inflation, where $H \sim H_I = const >> c_0$ [14]. One can easily show that replacing H with H_I in (1.3.18) we get that $\Lambda \propto H_I^2$, which describes a de Sitter era of the universe. The coefficients ν and α act as the dimensionless β-functions for the RG running of ρ_Λ at low and high energies, respectively, which receive the loop contributions from all matter particles. Both are considered to be small since $M_i << M_{pl}$, (with M_{pl} being the plank mass where $M_{pl} = \sqrt{1/G}$ for $c = \hbar = 1$), for every particle (even in the GUT scale). Interestingly, the coefficient ν has already be experimentally verified with a value of $|\nu| \lesssim \mathcal{O}(10^{-3})$ [22], which is also in agreement with the GUT scale estimate [12, 13, 16], and make possible for the alleviation of the cosmological tensions.

Using (1.3.7), (1.3.9) and (1.3.18) we obtain the following differential equation for the evolution of the Hubble parameter (details in Appendix):

$$\dot{H} + \frac{3}{2}(1+\omega)H^2\left[1 - \nu - \frac{c_0}{3H^2} - a\left(\frac{H}{H_I}\right)^2\right] = 0 \tag{1.3.21}$$

for the early universe, where $c_0 << H^2$ and thus can be neglected, the above equations takes the solution:

$$H(\alpha) = \sqrt{\frac{1-\nu}{a}}\frac{H_I}{\sqrt{D\alpha^{3(1-\nu)(1+\omega)} + 1}} \tag{1.3.22}$$

where $D > 0$ is a constant. For the very early universe, where the scale factor can be thought to be small, and thus $D\alpha^{3(1-\nu)(1+\omega)} << 1$, equation (1.3.22) reduces to $H = \sqrt{(1-\nu)/a}H_I \sim H_I$. Which shows that the quantity $H_I \simeq 10^{-5}M_{pl}$ [14] dominates and we get a de sitter era of the universe. After inflation, where the universe has grown in size, the solution shows that

$H \sim \alpha^{-3/2(1-\nu)(1+\omega)}$. For $\nu << 1$ and $\omega = 1/3$ the relation equates to $H \sim \alpha^{-4}$ which signals a radiation dominated universe.

Remarkably, the RV model has given a solution that can perfectly describe a universe that enters in an inflationary de sitter era, at early times, and later concludes with a radiation dominated era. Additionally, in an older universe, the term c_0/H^2 of (1.3.21) becomes dominant while H^2/H_I^2 gets insignificant. This results in a universe dominated by c_0 and thus (with appropriate constant fixing) H_0, i.e. the Hubble constant [16]. Furthermore, the theoretical values of the parameters of the model come in agreement with the data gathered via the numerous experiments done over the years and solve many inconsistencies between the data and the ΛCDM, while also providing predictions that are yet to be verified [12, 13, 16, 22]. All in all, the RV model is a perfect candidate for describing the whole history of the evolution of the universe and unifying under one umbrella the otherwise fractured physics of the different cosmic eras.

All of the above comes down to the fact that the cosmological constant can be described by the sum of even powers of the Hubble parameter, something that results from the RG approach of the vacuum energy density. Yet there was no discussion about the nature of this energy density and the physics behind the quantum field that would make the RG approach of the above discussion possible.

In the next chapter, a stringy toy-model is going to be discussed in which gravitational wave anomalies will give birth to the inflationary phase without the need of any extra inflationary field. This is going to be achieved via the coupling of the KR axion field of the string multiplet and gravity, which will result in a gravitational ”condensate”. This will translate in the H^4 term in the total energy density, which is needed to make the connection with the discussion of this chapter.

2. String-Inspired Cosmology

2.1. The Stringy Action

In bosonic string theory the lowest energy ground state consists of a scalar (spin 0) dilaton field, a symmetric traceless graviton tensor (spin 2) field, $g_{\mu\nu} = g_{\nu\mu}$ and the spin-1 antisymmetric tensor Kabl-Ramond (KR) field $B_{\mu\nu} = -B_{\nu\mu}$, where μ,$\nu = 0....D-1$ and $D = 4$ the space-time dimension of the string after compactification [15, 16, 26]. In 3+1 dimensions the tensor field $B_{\mu\nu}$ has the following U(1) symmetry:

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \theta_\nu - \partial_\nu \theta_\mu \quad \mu, \nu = 0, ..., 3 \tag{2.1.1}$$

where θ_μ are the gauge parameters. Thus, the effective action will be invariant under the same U(1) symmetry and will only depend on the field strength of $B_{\mu\nu}$ which takes the following expression:

$$H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \tag{2.1.2}$$

and satisfies the bianchi identity

$$\partial_{[\sigma} H_{\mu\nu\rho]} = 0 \tag{2.1.3}$$

where [....] denotes antisymmetrisation of the respected indices. Hence, the low-energy effective 4-dimensional action corresponding to the bosonic massless string multiplet, in which we kept only the lowest order of the derivative expansion, takes the form [15, 16]:

$$S(g, H) = \int dx^4 \sqrt{-g} \left(\frac{1}{2\kappa^2} R - \frac{1}{6} H_{\mu\nu\rho} H^{\mu\nu\rho} + ... \right) \tag{2.1.4}$$

In this particular case we fixed the dillaton field at an appropriate constant value ($\Phi =$ const.), corresponding to the minimization of the potential, in order to give the string coupling $g_s =$ exp(Φ) values which agree with phenomenological data [26]. Also, the corresponding path integral of the effective action is

$$\mathbb{Z} = \int DgDH \exp\left[S(g, H)\right] \tag{2.1.5}$$

where D denotes pth integration over all fields. Naturally, the Lagrangian density $\mathcal{L} = R - (1/6) H_{\mu\nu\rho} H^{\mu\nu\rho}$ of (2.1.4) must satisfy the Euler-Lagrange equations. By expanding those we get:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial B_{\alpha\beta}} &= \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu B_{\alpha\beta})}\right] \\
\Rightarrow 0 &= \partial_\mu \left[-\frac{1}{3}H^{\kappa\lambda\rho}\frac{\partial H_{\kappa\lambda\rho}}{\partial(\partial_\mu B_{\alpha\beta})}\right] \\
&= \partial_\mu \left[H^{\kappa\lambda\rho}\frac{\partial(\partial_\kappa B_{\lambda\rho} + \partial_\lambda B_{\rho\kappa} + \partial_\rho B_{\kappa\lambda})}{\partial(\partial_\mu B_{\alpha\beta})}\right] \\
&= \partial_\mu [H^{\kappa\lambda\rho}(\delta^\mu_\kappa \delta^\alpha_\lambda \delta^\beta_\rho + \delta^\mu_\lambda \delta^\alpha_\rho \delta^\beta_\kappa + \delta^\mu_\rho \delta^\alpha_\kappa \delta^\beta_\lambda)] \\
&\Rightarrow \partial_\mu (H^{\mu\alpha\beta} + H^{\beta\mu\alpha} + H^{\alpha\beta\mu}) = 0
\end{aligned} \tag{2.1.6}$$

One may see that the above expression is always satisfied by expressing the field strength as $H_{\mu\nu\rho} = N\epsilon_{\mu\nu\rho\sigma}\partial^\sigma b(x)$, where $\epsilon_{\mu\nu\rho\sigma}$ the Minkowski-space-time Levi-Civita totally antisymmetric symbol. By doing so there is an apparent duality between $H_{\mu\nu\rho}$ and the axion-like pseudoscalar field $b(x)$. Also, since $H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$, it is trivial to prove that the bianchi identity (2.1.3) becomes

$$\epsilon^{\mu\nu\rho\sigma}\partial_\sigma H_{\mu\nu\rho} = 0 \tag{2.1.7}$$

using the bianchi identity as a constrain in (2.1.5) and with the help of the Lagrange multiplier formula of b(x):

$$\delta\left(\epsilon^{\mu\nu\rho\sigma}\partial_\sigma H_{\mu\nu\rho}\right) = \int Db(x)\,\mathrm{exp}\left[i\int dx^4\sqrt{-g}\left(\epsilon^{\mu\nu\rho\sigma}\partial_\sigma H_{\mu\nu\rho}\right)b(x)\right] \tag{2.1.8}$$

we can rewrite the path integral as shown bellow

$$\mathbb{Z} = \int DgDHDb(x)\,\mathrm{exp}\left[i\int dx^4\sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{6}H_{\mu\nu\rho}H^{\mu\nu\rho} + b(x)\epsilon^{\mu\nu\rho\sigma}\partial_\sigma H_{\mu\nu\rho} + ...\right)\right] \tag{2.1.9}$$

taking into account that the fields and their first derivatives vanish at space-time infinities (boundary conditions), the partial integration of the integral in the exponential can give us

$$\int dx^4\sqrt{-g}b(x)\epsilon^{\mu\nu\rho\sigma}\partial_\sigma H_{\mu\nu\rho} = 0 - \int dx^4\sqrt{-g}\left(H_{\mu\nu\rho}\partial^\sigma b(x)\right)e^{\mu\nu\rho\sigma} \tag{2.1.10}$$

and thus we get

$$\mathbb{Z} = \int DgDHDb(x)\,\mathrm{exp}\left[i\int dx^4\sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{6}H_{\mu\nu\rho}H^{\mu\nu\rho} - \epsilon^{\mu\nu\rho\sigma}\partial_\sigma b(x)H_{\mu\nu\rho} + ...\right)\right] \tag{2.1.11}$$

by replacing the equation $H_{\mu\nu\rho} = N\epsilon_{\mu\nu\rho\lambda}\partial^\lambda b(x)$ in the exponential of the above integral and using the Levi-Civita property $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\lambda} = 6\delta^{\sigma}{}_{\lambda}$, we have

$$\begin{aligned}
&-\frac{1}{6}H_{\mu\nu\rho}H^{\mu\nu\rho} - \epsilon^{\mu\nu\rho\sigma}\partial_\sigma b(x)H_{\mu\nu\rho} = \\
&= -\frac{1}{6}H_{\mu\nu\rho}H^{\mu\nu\rho} - N\epsilon^{\mu\nu\rho\sigma}\partial_\sigma b(x)\epsilon_{\mu\nu\rho\lambda}\partial^\lambda b(x) = \\
&= -\frac{1}{6}H_{\mu\nu\rho}H^{\mu\nu\rho} - 6N\delta^{\sigma}{}_{\lambda}\partial_\sigma b(x)\partial^\lambda b(x) = \\
&= -\frac{1}{6}H_{\mu\nu\rho}H^{\mu\nu\rho} - 6N\partial_\sigma b(x)\partial^\sigma b(x)
\end{aligned}$$

then (2.1.11) becomes

$$\begin{aligned}
Z &= \int DgDHDb(x)\,\text{exp}\left[i\int dx^4\sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{6}H_{\mu\nu\rho}H^{\mu\nu\rho} - 6N\partial_\sigma b(x)\partial^\sigma b(x) + ...\right)\right] \\
&= \int DgDb(x)C\,\text{exp}\left[i\int dx^4\sqrt{-g}\left(\frac{1}{2\kappa^2}R - 6N\partial^\sigma b(x)\partial_\sigma b(x) + ...\right)\right]
\end{aligned}$$

where

$$C = \int DH\,\text{exp}\left[i\int dx^4\sqrt{-g}\left(-\frac{1}{6}H_{\mu\nu\rho}H^{\mu\nu\rho}\right)\right] = \text{const.}$$

by properly fixing the normalization constant N to ensure canonical kinetic terms we get the final form of the path integral

$$\mathbb{Z} = \int DgDb(x)\,\text{exp}\left[i\int dx^4\sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{2}\partial^\sigma b(x)\partial_\sigma b(x) + ...\right)\right] \tag{2.1.12}$$

which translates in the action:

$$S = \int dx^4\sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{2}\partial^\sigma b(x)\partial_\sigma b(x) + ...\right) \tag{2.1.13}$$

However, in string-inspired cosmology, there exist gauge and gravitational anomalies in the extra dimensional space. These anomalies are expressed through the modification of the field strength $H_{\mu\nu\rho}$ by adding the Chern-Simons three-forms [27]:

$$H = \mathbf{d}B + c_1\Omega^L - c_2\Omega^Y \tag{2.1.14}$$

where c_1 and c_2 are positive constants, **d** denotes partial differentiation with antisymmetrisation and Ω^L and Ω^Y are the gravitational ("Lorentz", L) and gauge (Y) anomalous terms given by the expressions bellow

$$\Omega^L = \left(\omega_\alpha\partial_\gamma\omega_\beta + \frac{2}{3}\omega_\alpha\omega_\beta\omega_\gamma\right)\left(dx^\alpha \wedge dx^\beta \wedge dx^\gamma\right) \tag{2.1.15}$$

$$\Omega^Y = \left(A_\alpha\partial_\gamma A_\beta + \frac{2}{3}A_\alpha A_\beta A_\gamma\right)\left(dx^\alpha \wedge dx^\beta \wedge dx^\gamma\right) \tag{2.1.16}$$

where $\wedge$ denotes the wedge product, A denotes the Yang-Mills gauge field and ω_α is the spin connection one form. This modification of the field strength leads to the Bianchi identity bellow.

$$\mathbf{d}H = \text{Tr}\left[c_1 R \wedge R + c_2 F \wedge F\right] \tag{2.1.17}$$

where $F = dA + A \wedge A$ is the Yang-Mills field strength and $R = d\omega + \omega \wedge \omega$ the curvature two form. In component form, the above identity becomes

$$\varepsilon^{\mu\nu\rho\sigma}\nabla_\sigma H_{\mu\nu\rho} = \sqrt{-g}\left(c_1 R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma} + c_2 F_{\mu\nu}\widetilde{F}^{\mu\nu}\right) \tag{2.1.18}$$

where $\varepsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\epsilon_{\mu\nu\rho\sigma}$ is the covariant Levi-Civita tensor and ∇_μ the covariant derivative. The symbol $\sim$ over the curvature and gauge field strength tensors denotes the corresponding duals, defined as:

$$\widetilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}R^{\alpha\beta}{}_{\rho\sigma} \tag{2.1.19}$$

$$\widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{2.1.20}$$

This none zero quantity in the right hand side of the (2.1.17) is the mixed gauge and gravitational quantum anomalies. These extra terms turn up in the action (2.1.4) and usually are canceled with the introduction of the Green-Schwarz counterterms, making possible the previous analysis. Also, we shall assume that in the action of the early, before-inflation, universe appeared only external bosonic fields and that fermionic and gauge matter was only created after inflation. With the above assumptions we are able to set $A = 0$ and rewrite the bianchi identity in the form of

$$\varepsilon^{\mu\nu\rho\sigma}\nabla_\sigma H_{\mu\nu\rho} = c_1\sqrt{-g}R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma} \tag{2.1.21}$$

Using this equation instead of (2.1.7) and following the previous procedure we arrive on the effective action which describes the early epoch of a string-inspired Universe

$$S = \int dx^4\sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{2}\partial^\sigma b(x)\partial_\sigma b(x) - c_1 b(x)R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma} + ...\right) \tag{2.1.22}$$

the extra terms (+....) not shown in the expression denote higher derivative terms and also other axion fields arising from the compactification of the string theory [26, 27]. Those terms are subdominant and we shall not focus on them in this study.

One may see that the action now has a new anomalous CP violation term: $b(x)R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma}$ of the coupling between the axion field $b(x)$ and the gravitational anomalies. Taking the variation of the anomalous terms gives (the calculation can be seen in Appendix)

$$\delta\left[\int d^4x\sqrt{-g}b(x)R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma}\right] = -4\int d^4x\sqrt{-g}\mathcal{C}_{\mu\nu}\delta g^{\mu\nu} \tag{2.1.23}$$

where

$$\mathcal{C}^{\mu\nu} = -\frac{1}{2}\left[\partial_\sigma b\left(\varepsilon^{\sigma\mu\alpha\beta}\nabla_\alpha R^\nu_{\ \beta} + \varepsilon^{\sigma\nu\alpha\beta}\nabla_\alpha R^\mu_{\ \beta}\right) + \partial_\tau\partial_\sigma b\left(\widetilde{R}^{\tau\mu\sigma\nu} + \widetilde{R}^{\tau\nu\sigma\mu}\right)\right] \tag{2.1.24}$$

is the traceless [15, 16] Cotton tensor.

The action (2.1.22) can be decomposed in $S = S^R + S^b + S^{bR}$, with S^R being the Einstein-Hilbert action, S^b being the matter action of the $b(x)$ field and S^{bR} being the gravitational anomaly term. Thus, the variation of the whole action, which we demand to be zero, is trivially expressed as:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \kappa^2(T_b^{\mu\nu} + 8c_1\mathcal{C}^{\mu\nu}) \tag{2.1.25}$$

which are the corresponding Einstein equations, with $T_b^{\mu\nu}$ being the mass-energy tensor associated with the pseudoscalar field b(x) (i.e. axion matter) and $\mathcal{C}^{\mu\nu}$ being the cotton tensor which comes from the gravitational anomalies. Taking the derivative of the left side of (2.1.25) gives

$$\nabla_\mu R^{\mu\nu} - \frac{1}{2}\nabla_\mu R = 0 \tag{2.1.26}$$

and thus, we have the following conservation law for our toy-cosmology.

$$\nabla_\mu T_b^{\mu\nu} + 8c_1\nabla_\mu\mathcal{C}^{\mu\nu} = 0 \tag{2.1.27}$$

Usually, for acquiring FRLW geometries, the term $b(x)R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma}$ of the action (2.1.22) must vanish. This may happen with the introduction of Green-Schwarz counterterms in the action, in order to cancel the gravitation anomalies [16, 26, 27], or by choosing specific gravitational backgrounds [28], where $R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma} = 0$. In those cases the Cotton tensor is equal to zero and the matter stress-energy tensor is conserved. Instead, in our case, the presence of gravitational anomalies and the Cotton term breaks the classic conservation law $\nabla_\mu T_b^{\mu\nu} = 0$. This is actually something common when evaluating higher-order effective actions and is emended with defining a new conserved tensor to play the role of the energy-momentum tensor. Thus, in the same principle, and looking in (2.1.27), we may define the modified energy tensor:

$$T_{b+C}^{\mu\nu} = T_b^{\mu\nu} + 8c_1\mathcal{C}^{\mu\nu} \quad\Rightarrow\quad \nabla_\mu T_{b+C}^{\mu\nu} = 0 \tag{2.1.28}$$

Equation (2.1.28) indicates that, when gravitational anomalies ares present, there is an exchange in energy between the stress-energy tensor and the Cotton tensor, which means there is an exchange in energy between the field $b(x)$ and the gravitational field [15, 16, 17].

In latter sections it is going to be discussed how primordial gravitational waves in the early times of an FLRW universe can create CP-violating anomalous gravitational terms, which can be translated to ”gravitational condensates”, and how those terms can provide to the FLRW universe a positive cosmological constant, which will drive inflation.

2.2. The Axion Matter of Early Universe

In the case of a FLRW early universe, where the interaction between $b(x)$ and the gravitational anomalies is zero, the only term contributing to the stress-energy tensor is the kinematic term of $b(x)$. Using the definition of the stress-energy tensor from General Relativity [24]:

$$T^b_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S^b}{\delta g^{\mu\nu}} \tag{2.2.1}$$

for

$$S^b = -\frac{1}{2}\sqrt{-g}g^{\alpha\beta}\partial_\alpha b\partial_\beta b \tag{2.2.2}$$

we get

$$\begin{aligned} T^b_{\mu\nu} &= \frac{1}{\sqrt{-g}}\frac{1}{\delta g^{\mu\nu}}\left[\delta(\sqrt{-g})g^{\alpha\beta}\partial_\alpha b\partial_\beta b + \sqrt{-g}\delta(g^{\alpha\beta})\partial_\alpha b\partial_\beta b\right] \\ &= \frac{1}{\cancel{\sqrt{-g}}}\frac{1}{\cancel{\delta g^{\mu\nu}}}\left[-\cancel{\sqrt{-g}}\frac{1}{2}g_{\mu\nu}\cancel{\delta g^{\mu\nu}}g^{\alpha\beta}\partial_\alpha b\partial_\beta b + \cancel{\sqrt{-g}}\,\cancel{\delta g^{\mu\nu}}\partial_\mu b\partial_\nu b\right] \\ \Rightarrow \qquad T^b_{\mu\nu} &= \partial_\mu b\partial_\nu b - \frac{1}{2}g_{\mu\nu}\partial_\alpha b\partial^\alpha b \end{aligned} \tag{2.2.3}$$

Also, assuming $b(x)$ is a perfect fluid and a homogeneous and isotropic universe, the stress-energy tensor for an observer moving with a four-velocity u_μ is [24]

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu - g_{\mu\nu}P \tag{2.2.4}$$

where ρ is the energy density and P is the pressure of said fluid. Then, the system of (2.2.3) and (2.2.4) for an inertial observer ($u_\mu = (-1, 0, 0, 0)$) for $T^b_{00} = \rho^b$ and T^b_{ii} is:

$$T^b_{ii} = -g_{ii}P^b = -\frac{1}{2}g_{ii}\dot{b}^2 \tag{2.2.5}$$

$$T^b_{00} = \rho^b + P^b - g_{00}P^b = \dot{b}^2 - \frac{1}{2}g_{00}\dot{b}^2 \tag{2.2.6}$$

which corresponds to the following equation of state

$$\rho^b = P^b = \frac{1}{2}\dot{b}^2 \tag{2.2.7}$$

hence, without CP violating terms, the equation of state parameter is $\omega = 1$, which hold no properties of a running vacuum fluid of (1.3.6) but instead that of stiff matter [17]. Also, one may see that there isn't any exponential growth by calculating the scaling of the energy density of stiff matter.

$$\rho^b \propto \alpha^{-3(1+w)} = \alpha^{-6}, \text{ for } \omega = 1 \tag{2.2.8}$$

Lets now compute the equations of motion of b(x) via the Euler-Lagrange equations without specifically choosing if the gravitational anomalies exist or not. To do so we will need the Lagrangian densities in (2.1.22) where $b(x)$ is involved, specifically those of corresponding to S^b and S^{bR}. Thus, for

$$\mathcal{L}^{b+bR} = \sqrt{-g}\left[-\frac{1}{2}\partial^\mu b(x)\partial_\mu b(x) - c_1 b(x) R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma}\right] \tag{2.2.9}$$

the Euler-Lagrange equations are

$$\begin{aligned}
&\frac{\partial\mathcal{L}}{\partial b(x)} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu b(x))}\right] \\
\Rightarrow\quad & -\sqrt{-g}c_1 R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma} = \sqrt{-g}\partial_\mu\left[-\partial^\mu b(x)\frac{\partial(\partial_\mu b(x))}{\partial(\partial_\mu b(x))}\right] \\
\Rightarrow\quad & \sqrt{-g}c_1 R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma} = -\sqrt{-g}\partial_\mu\partial^\mu b(x)
\end{aligned}$$

hence, the equation of motion for $b(x)$ is

$$\sqrt{-g}\left[\partial_\mu\partial^\mu b(x) - c_1 R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma}\right] = 0 \tag{2.2.10}$$

here, we may define the quantity:

$$\sqrt{-g}\nabla_\mu\mathcal{K}^\mu = \partial_\mu\left(\sqrt{-g}\mathcal{K}^\mu\right) = \sqrt{-g}R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma} \tag{2.2.11}$$

this quantity $\sqrt{-g}\mathcal{K}^\mu$ may be seen as the axial current density, as its covariant derivative is related to the gravitational anomaly. Using (2.2.11) we may rewrite (2.2.10) as

$$\partial_\mu\left[\sqrt{-g}\left(\partial^\mu b(x) - c_1\mathcal{K}^\mu\right)\right] = 0 \tag{2.2.12}$$

and by integration we get

$$\sqrt{-g}\left(\partial^\mu b(x) - c_1\mathcal{K}^\mu\right) - C_0 = 0 \tag{2.2.13}$$

Again, in a homogeneous and isotropic early universe, the field must only be time depended. Thus, only the component $\mu = 0$ of $\mathcal{K}^\mu$ is possible to have non-zero value. Also, the constant C_0 can be set to zero without any loss of generality, since it will be far smaller than any inflationary dominant term. With those assumptions we get a solution of (2.2.13) in the form of

$$\dot{b} = c_1 \mathcal{K}^0 \tag{2.2.14}$$

while $\mathcal{K}^0$ takes the form of

$$\frac{d}{dt}\left(\sqrt{-g}\mathcal{K}^0\right) = \langle \sqrt{-g} R_{\mu\nu\rho\sigma} \widetilde{R}^{\mu\nu\rho\sigma} \rangle \tag{2.2.15}$$

where $\langle ... \rangle$ denote average over spacetime fluctuations.

Equations (2.2.14) and (2.2.15) are self consistent for both the existence or not of gravitational anomalies. In the first case, the non-zero value of the right hand side of (2.2.15) will imply a different stress-energy tensor, derived using S^{b+bR} instead of S^b, which will lead to extra terms in energy density. In the second case, the right hand side of (2.2.15) is zero, which leads to

$$\mathcal{K}^0 = \frac{1}{\sqrt{-g}} = \alpha^{-3} \tag{2.2.16}$$

and implies $\dot{b}^2 \propto \alpha^{-6}$, a result which is in agreement with (2.2.7) and (2.2.8).

Hence, we have showed that, via (2.2.15) and (2.2.14), the dynamical properties of the KR axion field are related with the expectation value of the gravitational anomalies, if they exist. Then, (2.2.7) shows that the same anomalies will give a contribution to the density of the field, and thus, its energy momentum tensor. This is due to the exchange in energy between gravity and the field b, as stated in the previous subsection, which has an important role in understanding the dynamics of the universe during inflation. Thus the study of the gravitational anomalous term in (2.1.22) is of big importance, and this shall be the way we will continue.

3. String-Inspired Running Vacuum

3.1. The Gravitational Anomaly

In the previous section we discussed about the physical properties of the low-energy 4-dimensional effective action in a string inspired universe (2.1.22), mainly the $b(x)$ pseudoscalar action field and the emergence of the gravitational anomalous term. As noted before, the gravitational anomalous term is CP violating and does not contribute in a FLRW universe. Nevertheless, we shall show bellow that this is not the case in the presence of metric perturbations (i.e gravitational waves).

Let's consider the most general form of weakly perturbed spatially-flat FLRW metric, with scale factor $\alpha = \alpha(t)$:

$$ds^2 = (1+2\varphi)dt^2 - w_i dt dx^i - \alpha^2 \left\{ \left[(1+2\psi)\delta_{ij} + h_{ij} \right] dx^i dx^j \right\} \tag{3.1.1}$$

where φ and ψ, w_i and h_{ij} are the scalar, vector and tensor perturbations of the metric, respectively. Since concern is with the term $R\widetilde{R}$ where only the tensor perturbations can contribute we may omit φ, ψ and w_i without the loss of generality. This leads to

$$ds^2 = dt^2 - \alpha^2 \left[(\delta_{ij} + h_{ij}) dx^i dx^j \right] \tag{3.1.2}$$

Assuming that the gravitational waves are propagating along the z spatial direction, the z axes must be unperturbed. Additionally, the metric tensor is symmetric and it's trace must be invariant. These statements imply that the values h_{ij} for $i, j = 3$ must be zero, that $h_{12} = h_{21} = h_2$ and h_{ij} must be traceless, thus $h_{11} = -h_{22} = h_1$. Hence, the metric reads

$$ds^2 = dt^2 - \alpha^2 \left[(1+h_1)dx^2 + (1-h_1)dy^2 + 2h_2 dx dy + dz^2 \right] \tag{3.1.3}$$

and the corresponding metric tensor is

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\alpha^2 & 0 & 0 \\ 0 & 0 & -\alpha^2 & 0 \\ 0 & 0 & 0 & -\alpha^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\alpha^2 h_1 & -\alpha^2 h_2 & 0 \\ 0 & -\alpha^2 h_2 & \alpha^2 h_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.1.4}$$

where the scale factor satisfies (1.2.1) and h_1, h_2 are functions of t, z. The CP violation is more explicit if one uses the helicity basis:

$$h_{R,L} = \frac{1}{\sqrt{2}} \left(h_1 \pm i h_2 \right) \tag{3.1.5}$$

With h_R and h_L being complex conjugate scalar fields build from wave-functions of left-handed gravitons [18, 19, 26].

We may now calculate the gravitational anomaly. Firstly, expanding $R\widetilde{R}$ gives

$$R\widetilde{R} = \frac{1}{2} R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\alpha\beta} \varepsilon^{\alpha\beta\rho\sigma} = -\frac{1}{2} R^{\mu}{}_{\nu\rho\sigma} R^{\nu}{}_{\mu\alpha\beta} \varepsilon^{\alpha\beta\rho\sigma} \tag{3.1.6}$$

and replacing with the corresponding Christoffel symbols we get

$$\begin{aligned} R\widetilde{R} = & -\frac{1}{2} \left[\partial_\rho \Gamma^{\mu}_{\sigma\nu} - \partial_\sigma \Gamma^{\mu}_{\rho\nu} + \Gamma^{\mu}_{\rho\lambda} \Gamma^{\lambda}_{\sigma\nu} - \Gamma^{\mu}_{\sigma\lambda} \Gamma^{\lambda}_{\rho\nu} \right] \\ & \times \left[\partial_\alpha \Gamma^{\nu}_{\beta\mu} - \partial_\beta \Gamma^{\nu}_{\alpha\mu} + \Gamma^{\nu}_{\alpha\lambda} \Gamma^{\lambda}_{\beta\mu} - \Gamma^{\nu}_{\beta\lambda} \Gamma^{\lambda}_{\alpha\mu} \right] \varepsilon^{\alpha\beta\rho\sigma} \end{aligned} \tag{3.1.7}$$

using the antisymmetric properties of $\varepsilon^{\alpha\beta\rho\sigma}$ and some index manipulation we simplify to

$$R\widetilde{R} = -2 \left(\partial_\rho \Gamma^{\mu}_{\sigma\nu} + \Gamma^{\mu}_{\rho\lambda} \Gamma^{\lambda}_{\sigma\nu} \right) \left(\partial_\alpha \Gamma^{\nu}_{\beta\mu} + \Gamma^{\nu}_{\alpha\lambda} \Gamma^{\lambda}_{\beta\mu} \right) \varepsilon^{\alpha\beta\rho\sigma} \tag{3.1.8}$$

For (3.1.4), the non-zero Christoffel symbols are

$$\begin{aligned} \Gamma^{0}_{ij} &= -\frac{1}{2} \partial_t h_{ij} - \frac{1}{2} \partial_t (\alpha^2) \delta_{ij} \\ \Gamma^{i}_{0j} &= \frac{1}{2\alpha^2} \partial_t (\alpha^2) \delta^{i}_{j} - \frac{1}{2\alpha^2} h_{jk} \delta^{ik} + \frac{1}{2} \partial_t (\alpha^2) \delta_{ik} - \frac{1}{2} h^{i\sigma} \partial_t h_{j\sigma} \\ \Gamma^{k}_{ij} &= -\frac{1}{2\alpha^2} \left(\partial_i h_{jk} + \partial_j h_{ik} - \partial_k h_{ij} \right) - \frac{1}{2} h^{k\sigma} \left(\partial_i h_{j\sigma} + \partial_j h_{i\sigma} - \partial_\sigma h_{ij} \right) \end{aligned} \tag{3.1.9}$$

Using (1.2.1) for the scale factor during inflation, with $H \sim const.$, the helicity basis (3.1.5) and substituting (3.1.9) into (3.1.8) while keeping the lowest order of h_L, h_R we get the contribution of tensor perturbations in $R\widetilde{R}$.

$$\begin{aligned} R\widetilde{R} = \frac{4i}{\alpha^3} \Bigg[& \left(\partial_z^2 h_R \partial_z \partial_t h_L - \partial_z^2 h_L \partial_z \partial_t h_R \right) + \alpha^2 \left(\partial_t^2 h_R \partial_z \partial_t h_L - \partial_t^2 h_L \partial_z \partial_t h_R \right) \\ & + H\alpha^2 \left(\partial_t h_R \partial_z \partial_t h_L - \partial_t h_L \partial_z \partial_t h_R \right) \Bigg] \end{aligned} \tag{3.1.10}$$

We observe that (3.1.10) has three terms with two terms within, these two terms are equivalent and differ only in the exchange of h_R and h_L. So, if the fields (which are complex conjugates) have the same dispersion relations the terms in (3.1.10) will cancel each other and $R\widetilde{R}$ will vanish. Thus, for $R\widetilde{R}$ to not be zero and the existence of CP-violating interactions imply that h_R and h_L behave differently. These phenomenon is called *cosmological birefringence* [15, 18] and states the possibility that the plane of linear polarization of quantum fields rotates while the fields are traveling over cosmological distances. To see gravitational birefringence, we need to find and solve the equations of motion for h_L and h_R, something that will also help with the evaluation of the expectation value of $R\widetilde{R}$, which is necessary for the continuation of our analysis.

The Lagrangian density of the gravitational waves and the perturbation fields h_R, h_L will correspond with the one of the effective action (2.1.22) without the kinematic terms of the pseudoscalar field $b(x)$.

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{2\kappa^2}R - c_1 b R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma}\right) \tag{3.1.11}$$

Furthermore, we can calculate the Ricci scalar using the Christoffel symbols (3.1.9), this computation yields:

$$\begin{aligned} R = 4\Bigg(&\frac{1}{2\alpha^2}h_L\partial_z^2 h_R - \frac{1}{2}h_L\partial_t^2 h_R - \frac{3}{2}Hh_L\partial_t h_R \\ &+\frac{1}{2\alpha^2}h_R\partial_z^2 h_L - \frac{1}{2}h_R\partial_t^2 h_L - \frac{3}{2}Hh_R\partial_t h_L\Bigg) \end{aligned} \tag{3.1.12}$$

and by defining

$$\Box = \partial_t^2 + 3H\partial_t - \alpha^{-2}\partial_z^2 \tag{3.1.13}$$

we get

$$R = -2\left(h_L\Box h_R + h_R\Box H_L\right) \tag{3.1.14}$$

by replacing (3.1.10) and (3.1.14) into(3.1.11) and with $\sqrt{-g} \sim \alpha^3$ the Lagrangian density takes the form:

$$\begin{aligned} \mathcal{L} = &-\frac{\alpha^3}{\kappa^2}\left(h_L\Box h_R + h_R\Box h_L\right) \\ &+ c_1 b 16 i\Big[\left(\partial_z^2 h_R\partial_z\partial_t h_L - \partial_z^2 h_L\partial_z\partial_t h_R\right) \\ &+ \alpha^2\left(\partial_t^2 h_R\partial_z\partial_t h_L - \partial_t^2 h_L\partial_z\partial_t h_R\right) \\ &+ H\alpha^2\left(\partial_t h_R\partial_z\partial_t h_L - \partial_t h_L\partial_z\partial_t h_R\right)\Big] \end{aligned} \tag{3.1.15}$$

it is trivial to show that only the first and last term will contribute to the Euler-Lagrange equations. Specifically, by observing (3.1.15) and since the equations are

$$\frac{\partial\mathcal{L}}{\partial h_{R,L}} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu h_{R,L})}\right] \tag{3.1.16}$$

we note that only the first term of (3.1.15) will contribute to the left-hand side of (3.1.16) and the last one will contribute to the right-hand side, since we need to keep only second order derivatives. Additionally, only derivatives for $\mu = 0, 3$ will contribute. Hence we have

$$\begin{aligned} \partial\mathcal{L} = &-\frac{\alpha^3}{\kappa^2}\left[\partial(h_L)\Box h_R + \partial(h_R)\Box h_L\right] \\ &+ c_1 b 16 i H\big[\partial(\partial_t h_R)\partial_z\partial_t h_L - \partial(\partial_t h_L)\partial_z\partial_t h_R\big] \\ &+ \text{not-contributing terms} \end{aligned} \tag{3.1.17}$$

and (3.1.16) become:

$$\Box h_L = -\frac{c_1 16 i\kappa^2}{\alpha^3}\partial_t\left(H\alpha^2 b\partial_z\partial_t h_L\right) \quad \Box h_R = \frac{c_1 16 i\kappa^2}{\alpha^3}\partial_t\left(H\alpha^2 b\partial_z\partial_t h_R\right) \tag{3.1.18}$$

for which we solved (3.1.16) for h_R and h_L respectively. Computing the time derivative gives the following equations of motion

$$\Box h_L = -2i\frac{\Theta}{\alpha}\partial_z\partial_t h_L \quad \Box h_R = 2i\frac{\Theta}{\alpha}\partial_z\partial_t h_R \tag{3.1.19}$$

where we omitted third-order derivatives of h_L and h_R, approximate $H\dot{\alpha}^2 = \ddot{\alpha}^2 \sim 0$ and

$$\Theta = 8c_1\kappa^2 H\dot{b} \tag{3.1.20}$$

is a dimensionless quantity associated with the anomalous term in (2.1.22).

Lets focus into one of the equations, for example the one for h_L. For convenience, we also introduce the conformal time during inflation

$$d\eta = \frac{dt}{\alpha} \rightarrow \eta = -\frac{1}{\alpha H} \tag{3.1.21}$$

Thus, expanding the equation of motion of h_L gives

$$\begin{aligned}
&\left[\frac{d^2}{dt^2} + 3H\frac{d}{dt} - \frac{1}{\alpha^2}\frac{d^2}{dz^2}\right] h_L = -2i\frac{\Theta}{\alpha}\frac{d^2}{dzdt}h_L \\
\Rightarrow &\left[\frac{d}{\alpha d\eta}\left(\frac{d}{\alpha d\eta}\right) + 3H\frac{d}{\alpha d\eta} - \frac{1}{\alpha^2}\frac{d^2}{dz^2}\right] h_L = -2i\frac{\Theta}{\alpha}\frac{d^2}{dz\alpha d\eta}h_L \\
\Rightarrow &\left[\frac{d^2}{d\eta^2} - \frac{2}{\eta}\frac{d}{d\eta} - \frac{d^2}{dz^2}\right] h_L = -2i\Theta\frac{d^2}{dzd\eta}h_L
\end{aligned} \tag{3.1.22}$$

To find the solutions, lets firstly assume that $\Theta = 0$ and $h_L \sim e^{ikz}$. By doing so the differential equation becomes the equation of a spherical Bessel function

$$\frac{d^2}{d\eta^2}h_L - \frac{2}{\eta}\frac{d}{d\eta}h_L + k^2 h_L = -0 \tag{3.1.23}$$

which accepts the solutions

$$h_L^{\pm} = e^{\pm ik(\eta+z)}(1 \mp ik\eta) \tag{3.1.24}$$

with $h_L^{\pm}$ being the positive and negative frequency solutions and are complex conjugates. Now, taking Θ into account, (3.1.22) takes the solutions

$$h_L = e^{ikx}(-ik\eta)e^{k\Theta\eta}g(\eta) \tag{3.1.25}$$

where $g(\eta)$ is a Coulomb wave function [18, 19], satisfying the differential equation

$$\frac{d^2}{d\eta^2}g + \left[k^2(1-\Theta^2) - \frac{2}{\eta^2} - \frac{2k\Theta}{\eta}\right] g = 0 \tag{3.1.26}$$

with the solution

$$g(\eta) = \exp\left[ik\sqrt{1-\Theta^2}\eta(1+\alpha(\eta))\right] \tag{3.1.27}$$

with $\alpha(\eta) \sim log(\eta)/\eta$. Knowing the expression (3.1.25) of the solution of the h_L field, we can apply it to find the expectation value of $R\widetilde{R}$. For this to be done, we will follow the standard QFT procedure for the scalar field h_L. To this end, and by definition, we will use the relation between Green's function and the expectation value of the time ordered product between h_L and is complex conjugate h_R.

$$G(x-x';t-t') = \left\langle h_L(x,t)h_R(x',t')\right\rangle \tag{3.1.28}$$

which can be related with the conformal time using the following Fourier transformation

$$G(x-x';t-t') = \int \frac{d^3k}{(2\pi)^3} G_k(\eta-\eta')e^{ik(x-x')} \tag{3.1.29}$$

Then, for k parallel to z, we get

$$\frac{d^2}{dz^2}h_L = -k^2 h_L \;\; \text{and} \;\; \frac{d}{d\eta}\left(\frac{d}{dz}h_L\right) = -ik\frac{d}{d\eta}h_L \tag{3.1.30}$$

thus (3.1.22), with the denominator α^2 factor reappearing, becomes

$$\frac{1}{\alpha^2}\left[\frac{d^2}{d\eta^2} - 2\left(\frac{1}{\eta}+k\Theta\right)\frac{d}{d\eta} + k^2\right]h_L = 0 \tag{3.1.31}$$

which means that $G_k(\eta-\eta')$ must satisfy the source equation

$$\left[\frac{d^2}{d\eta^2} - 2\left(\frac{1}{\eta}+k\Theta\right)\frac{d}{d\eta} + k^2\right]G_k(\eta-\eta') = i(H\eta)^2\delta(\eta-\eta') \tag{3.1.32}$$

where we took note of (3.1.21). Taking $\Theta = 0$, (3.1.32) has the following solution

$$G_k^{(\Theta=0)}(\eta-\eta') = \begin{cases} \frac{H^2}{2k^3}h_L^+(k,\eta)h_R^-(-k,\eta') & \text{for} \quad \eta<\eta' \\ \frac{H^2}{2k^3}h_L^-(k,\eta)h_R^+(-k,\eta') & \text{for} \quad \eta>\eta' \end{cases} \tag{3.1.33}$$

where $h_L^\pm$ are (3.1.24) and $h_R^\pm$ the corresponding solutions of the h_R equation. Even though (3.1.33) are for $\Theta = 0$, the above structure shall be preserved. To do this, we may check that for $\Theta \neq 0$ (3.1.32) is satisfied for

$$G_k(\eta-\eta') = e^{k\Theta(\eta'-\eta)}G_k^{(\Theta=0)}(\eta-\eta') \tag{3.1.34}$$

using this Green's function we may now calculate the expectation value of $R\widetilde{R}$ with the use of (3.1.28), (3.1.29) and (3.1.34) in (3.1.10). The calculation, peaking only the leading order $k\eta \gg 1$, yields [18, 19]:

$$\langle R\widetilde{R}\rangle = \frac{16}{\alpha^4}\int\frac{d^3k}{2(2\pi)^3}H^2k\Theta + \mathcal{O}(\Theta^3) \tag{3.1.35}$$

Hence, we have a non-zero expectation value for the gravitational anomalous term induced by primordial gravitational waves during inflation. This result is a consequence of the breaking of CP-symmetry, and thus, the different physical behavior of h_R, h_L, due to inflation (1.2.1), (3.1.21) and its effect on the equations of motion (3.1.22). Prior to inflation, this could not be the case and we would have calculated nonzero amplitudes. But, inflationary expansion freezes and collapses the wave functions locally giving specific values and break CP-symmetry.

Additionally. the non-zero value of (3.1.35) leads to the non-zero value of the Cotton tensor (2.1.23), which implies that the stress-energy tensor tensor is not conserved. This may be the hint that we have gone to a point where the classical Einstein Equations may not hold due to the usage of graviton quantum fluctuations in the aforementioned calculations.

To be consistent with low energy gravity and GR in this approach, one need to calculate (3.1.35) up to the Ultra-Violate (UV) μ on the physical momentum k/α. This will protect from inconsistencies in the calculations from graviton contributions and is defined as [16, 18]

$$k\eta < \mu/H \tag{3.1.36}$$

Furthermore, we assume slow-roll for b,

$$\dot{b} \ll H/\kappa \tag{3.1.37}$$

which means that $|\Theta| \ll 1$ and thus be able to ignore the $\mathcal{O}(\Theta^3)$ terms in (3.1.35). For slow-roll to hold one must must choose $c_1 = c_1'/M_s$, with M_s being the string mass scale, and thus $c_1 H^2 \ll 1$ during inflation. A natural choice of M_s is to assume that it is near the four dimensional reduced Planck mass scale M_{pl}. Then $c_1 = c_1' M_{pl}^{-1}$. Additionally, the relation $\kappa^2 = M_{pl}^{-2}$ also holds. Phenomenological data [2] support a Hubble scale of during inflation $H\kappa \ll 10^{-4}$, which give a value of

$$\frac{H}{M_{pl}} \sim 10^{-4} \tag{3.1.38}$$

thus the slow-roll condition (3.1.37) we assumed before are well consistent.

Taking into account all of the above, the integration of (3.1.35) up to leading order of the CP violating quantity Θ will give

$$\langle R\widetilde{R}\rangle = \frac{16}{\alpha^4}\int^{\mu} \frac{d^3k}{2(2\pi)^3}\left(\frac{H}{M_pl}\right)^2 k\Theta + \mathcal{O}(\Theta^3) = \frac{1}{\pi^2}\left(\frac{H}{M_pl}\right)^2 \mu^4\Theta \tag{3.1.39}$$

and substituting (3.1.20) and then (2.2.14) leaves us with

$$\begin{aligned}\langle R\widetilde{R}\rangle &= \frac{8c_1\kappa^2}{\pi^2}\left(\frac{H}{M_pl}\right)^2 H\mu^4\dot{b} = \frac{8c_1^2\kappa^2}{\pi^2}\left(\frac{H}{M_pl}\right)^2 H\mu^4\mathcal{K}^0 \\ &= cM_{pl}\left(\frac{H}{M_{pl}}\right)^3\left(\frac{\mu}{M_{pl}}\right)^4\mathcal{K}^0\end{aligned} \tag{3.1.40}$$

with $c = 8c_1'^2/\pi^2$ being the collection of proportionality constants. From this result, and with the help of (2.2.15)

$$\begin{aligned}\frac{d}{dt}\left(\sqrt{-g}\mathcal{K}^0\right) &= \langle\sqrt{-g}R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma}\rangle \simeq \sqrt{-g}\langle R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma}\rangle \\ &\simeq cM_{pl}\left(\frac{H}{M_{pl}}\right)^3\left(\frac{\mu}{M_{pl}}\right)^4(\sqrt{-g}\mathcal{K}^0)\end{aligned} \tag{3.1.41}$$

we may use (3.1.21) to integrate the above expression,but before we do so we must note that we set the beginning of inflation at $t = 0$, which equates to $\eta = H^{-1}$. Knowing that during inflation $H \ll M_{pl} \to H^{-1} \gg M_{pl}^{-1}$ and that the end of inflation occurs for $t \gg M_{pl}^{-1}$ we may set with good approximation the end of inflation at $t \to \infty$, and thus, $\eta = 0$. Then the duration of inflation can be approximated with $d\eta \sim H^{-1}$. In conformal coordinates (3.1.41) becomes

$$\frac{d}{dt}\left(\sqrt{-g}\mathcal{K}^0(t(\eta))\right) = -\eta H\frac{d}{d\eta}\left(\sqrt{-g}\mathcal{K}^0\right) \simeq cM_{pl}\left(\frac{H}{M_{pl}}\right)^3\left(\frac{\mu}{M_{pl}}\right)^4\left(\sqrt{-g}\mathcal{K}^0(t(\eta))\right)$$

$$\Rightarrow$$

$$\frac{d}{d\eta}\left(\sqrt{-g}\mathcal{K}^0\right) \simeq -\frac{1}{\eta H}cM_{pl}\left(\frac{H}{M_{pl}}\right)^3\left(\frac{\mu}{M_{pl}}\right)^4\left(\sqrt{-g}\mathcal{K}^0(t(\eta))\right) \tag{3.1.42}$$

by noting that

$$\frac{d}{dx}Ce^{-A\ln Bx} = -\frac{A}{Bx}Ce^{-A\ln Bx} \tag{3.1.43}$$

we get

$$\sqrt{-g}\mathcal{K}^0(t(\eta)) = \mathcal{K}_b\exp\left[cM_{pl}\left(\frac{H}{M_{pl}}\right)^3\left(\frac{\mu}{M_{pl}}\right)^4\ln(H\eta)\right] \tag{3.1.44}$$

additionally, from (3.1.21) and $\sqrt{-g} \sim \alpha^3$, we have $1/\sqrt{-g} = e^{-3Ht}$ and $\ln(H\eta) = -Ht$. Thus, we conclude with

$$\begin{aligned}\mathcal{K}^0(t(\eta)) &= \frac{\mathcal{K}_b}{\sqrt{-g}}\exp\left[c\left(\frac{H}{M_{pl}}\right)^2\left(\frac{\mu}{M_{pl}}\right)^4\ln(H\eta)\right] \\ &\sim \mathcal{K}_b\exp\left[-3Ht\left(1-\frac{c}{3}\left(\frac{H}{M_{pl}}\right)^2\left(\frac{\mu}{M_{pl}}\right)^4\right)\right] \\ &= \mathcal{K}_b\exp[-3Ht\mathcal{A}]\end{aligned} \tag{3.1.45}$$

The value $\mathcal{K}_b$ corresponds with the value of $\mathcal{K}^0(t(\eta))$ at the beginning of inflation $t = 0$ and is a boundary condition that can be determined phenomenologically. Additionally, one should expect that the gravitational wave effects must wash out, or completely vanish, at the end of inflation and (3.1.45) and should then approximate (2.2.16). This is actually true since the second term of the factor $\mathcal{A}$ in the exponent

$$\mathcal{A} = 1 - \frac{c}{3}\left(\frac{H}{M_{pl}}\right)^2\left(\frac{\mu}{M_{pl}}\right)^4 \tag{3.1.46}$$

is $\ll 1$ due the the approximately constant nature of the Hubble parameter during inflation where $H \ll M_{pl}$ and the natural range of the UV cutoff μ being $\mu \lesssim M_{pl}$ [16], leaving us with $\mathcal{A} \sim 1$. Moreover, it is interesting to see that for $\mathcal{A} = 0$, (3.1.46) leads to

$$\frac{H}{M_{pl}} = \sqrt{\frac{3}{c}}\left(\frac{M_{pl}}{\mu}\right)^2 \tag{3.1.47}$$

which, if one assumes that (3.1.38) still holds, translates to

$$\mu^2 \sim \sqrt{\frac{3}{c}}10^4 M_{pl}^2 \tag{3.1.48}$$

This value of μ is of great phenomenological argumentation, since is much larger than the Planck scale, which means that, if (3.1.47) holds, we would require the involvement of *transplanckian modes* [15, 16]. This would initially indicate the breakdown of an effective QFT and the week gravity conjecture, meaning that the effective field theory used is inconsistent with a theory of quantum gravity. Yet, an interesting explanation of the above result is that it is the first hint that the gravitational waves which created the condensate are indeed of quantum origin generated at the transplankian region, while they appear to us as classical gravitational waves below the Planck scale. Additionally, since the calculations are done in modes bellow the Planck scale, the above low-energy effective field theory of inflation is still perfectly valid.

The above argumentation allows for an approximately constant $\mathcal{K}^0(t(\eta))$:

$$\mathcal{K}^0(t(\eta)) \sim \mathcal{K}_b \tag{3.1.49}$$

and via (2.2.14) we get the necessary condition for $\dot{b}$ to also be approximately constant throughout the duration of inflationary period.

$$\dot{b} = c_1\mathcal{K}_b \sim const. \tag{3.1.50}$$

additionally, we can use (1.2.12) to connect $\dot{b}$ with the slow-roll parameter. By doing so, considering CMB observations [2] and noting that in standard GR $8\pi G = \kappa^2 = M_{pl}^{-2}$, we have

$$\epsilon_\varphi = \frac{1}{2}8\pi G\frac{\dot{b}}{H^2} = \frac{\dot{b}}{2H^2M_{pl}^2} \sim 10^{-2} \tag{3.1.51}$$

and thus

$$\dot{b} \sim \sqrt{2\epsilon_\varphi} M_{pl} H \sim 0.14 M_{pl} H \tag{3.1.52}$$

and time integration of this equation gives

$$b(t) = b(0) + 0.14 M_{pl} H t \tag{3.1.53}$$

with $b(0)$ being initial value of the KR axion field, at the beginning of inflation ($t = 0$).

Using (3.1.50) and (3.1.52) we may also fix the initial value of $\mathcal{K}^0(t(\eta))$ as

$$\mathcal{K}_b \sim \frac{0.14}{c_1} M_{pl} H \tag{3.1.54}$$

3.2. Anomaly Induced Inflation

In the previous subsection we discussed about the non-zero CP violating value of the condensate $\langle R\widetilde{R} \rangle$ created because by primordial gravitational waves during inflation. We also noted how those gravitational waves are hinted as of quantum origin and how they are physically connected with KR axion field $b(x)$. Yet, we have not showed what drives inflation, which throughout the whole discussion was preset and necessary for the followed procedure.

As discussed in the first section of this thesis, for inflation to occur the necessary condition of (1.2.2) for the total energy density must be satisfied. Thus we ought to find if such a condition is present. Our starting point will of course be the effective action (2.1.22), but due to the gravitational fluctuations during inflation an extra term, induced by (3.1.39), in the form of $\langle b(x) R\widetilde{R} \rangle$ must be taken into account. This shall be the vacuum expectation value of the effective action during inflation because of the CP-violating primordial gravitational waves and is viewed as a condensate of graviton fluctuations. Such an effect is analogous to the symmetry breaking in condensed matter physics, and thus the name gravitational condensate throughout the thesis. We must note that since the calculations are done up to the physical cut-off defined by (3.1.36), this expectation value is formed in the context of a UV complete theory of quantum gravity, such as string theory [15, 16]. However, the purpose of this thesis is to explain the phenomenological argumentation of such an effect with the presuming existence of such condensate in early inflationary epochs using the effective action framework.

Such a condensate can be expressed by expanding the S^{bR} gravitational anomaly term in the effective action (2.1.22) over quantum fluctuations about it [16]. Then the action, excluding the term S^R of gravity, shall be

$$S^{b+bR+condensate} = S^b + S^{bR} + S^{condensate} =$$
$$= -\int dx^4\sqrt{-g}\left(\frac{1}{2}\partial^\sigma b(x)\partial_\sigma b(x) + c_1\mathcal{T}[b(x)R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma}] + c_1\langle b(x)R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma}\rangle\right) \quad (3.2.1)$$

with $\mathcal{T}[.....]$ denoting proper quantum ordering of field operators. This quantum ordered term will give rise to the quantum ordered Cotton tensor as of (2.1.23) via its variation with respect to the gravitational field, which is traceless.

The first term of (3.2.1) corresponds the the matter of the early universe, i.e the axion field $b(x)$ whose analysis is given in 2.2. However, having now an explicit form of $\dot{b}$ in the context of inflation (3.1.52), we may rewrite (2.2.7) as

$$\rho^b = P^b \simeq 0,9895\epsilon_\varphi M_{pl}^2 H^2 \quad (3.2.2)$$

which of course satisfies the stiff matter properties, as explained in 2.2.

The second term of (3.2.1) corresponds to the cotton tensor defined by (2.1.23). Assuming a homogeneous and isotropic temporal component of the Cotton tensor $\mathcal{C}^{00}$, we may define the corresponding energy density

$$\rho^{bR} = 8c_1\mathcal{C}^{00} \quad (3.2.3)$$

It is useful the calculate the covariant derivative of the Cotton tensor, which can be expressed as (calculation in Appendix B):

$$\nabla_\mu\mathcal{C}^{\mu\nu} = -\frac{1}{8}\partial^\nu b(x)R_{\alpha\beta\gamma\delta}\widetilde{R}^{\alpha\beta\gamma\delta} \quad (3.2.4)$$

with the explicit form of the covariant derivative being

$$\nabla_\mu\mathcal{C}^{\mu\nu} = \partial_\mu\mathcal{C}^{\mu\nu} + \Gamma^\mu_{\mu\beta}\mathcal{C}^{\beta\nu} + \Gamma^\nu_{\mu\beta}\mathcal{C}^{\mu\beta} \quad (3.2.5)$$

Considering a flat FLRW background metric and a non-zero vacuum expectation value of the anomalous term, due to gravitation waves during inflation, The zero component of (3.2.4) yields

$$\nabla_\mu\mathcal{C}^{\mu 0} = \partial_\mu\mathcal{C}^{\mu 0} + \Gamma^\mu_{\mu\beta}\mathcal{C}^{\beta 0} + \Gamma^0_{\mu\beta}\mathcal{C}^{\mu\beta} = -\frac{1}{8}\dot{b}\langle R_{\alpha\beta\gamma\delta}\widetilde{R}^{\alpha\beta\gamma\delta}\rangle \quad (3.2.6)$$

Since we assumed that the Cotton tensor is homogeneous and isotropic in its temporal component, $\mathcal{C}^{i0} = 0$ for $i = 1,2,3$. Additionally, the christofel symbols on the flat FLRW spacetime read

$$\Gamma^\mu_{\mu 0} = \frac{\dot{\alpha}}{\alpha}, \quad \Gamma^0_{\mu\beta} = \alpha\dot{\alpha}\delta_{\mu\beta} \ \text{ for } \ \mu,\beta = 1,2,3 \quad (3.2.7)$$

Thus, left-hand side (3.2.6) becomes

$$\nabla_\mu \mathcal{C}^{\mu 0} = \partial_0 \mathcal{C}^{00} + \frac{\dot{\alpha}}{\alpha}\mathcal{C}^{00} + \alpha\dot{\alpha}\left(\mathcal{C}^{11} + \mathcal{C}^{22} + \mathcal{C}^{33}\right) \tag{3.2.8}$$

On top of that, the Cotton tensor is traceless [15, 16], in flat FLRW this translates into

$$g_{\mu\nu}\mathcal{C}^{\mu\nu} = \mathcal{C}^{00} - \alpha^2\left(\mathcal{C}^{11} + \mathcal{C}^{22} + \mathcal{C}^{33}\right) = 0 \tag{3.2.9}$$

taking into account this, and the fact that during inflation the scale factor satisfies (1.2.1) and the condensate (3.1.40). We have

$$\frac{d}{dt}\mathcal{C}^{00} + 4H\mathcal{C}^{00} \simeq -\frac{c_1\kappa^2}{\pi^2}\left(\frac{H}{M_pl}\right)^2 H\mu^4\dot{b}^2 \tag{3.2.10}$$

Assuming $\mathcal{C}^{00}$ is approximately constant with time, to be aligned with the argumentation of $\dot{b}$, whose expression is give by (3.1.52), we derive the solution for the Cotton tensor:

$$\mathcal{C}^{00} \simeq -\epsilon_\varphi \frac{c_1\kappa^2}{4\pi^2}\mu^4 H^4 \tag{3.2.11}$$

which we use, via (3.2.3), to derive the energy density of vacuum contributions:

$$\rho^{bR} \simeq -\epsilon_\varphi \frac{2c_1'^2}{\pi^2}\left(\frac{\mu}{M_{pl}}\right)^4 H^4 < 0 \tag{3.2.12}$$

where, as before, we used the relations $c_1 = c_1'/M_{pl}$ and $\kappa^2 = 1/M_{pl}^2$. By using (3.1.47), we then obtain

$$\rho^{bR} \simeq -\mathcal{N}\epsilon_\varphi M_{pl}^2 H^2 < 0 \tag{3.2.13}$$

with $\mathcal{N} = 9c_1'^2/32\pi^2c^2$ being the product of positive constants in the expression.

Furthermore, we may find the relation between (3.2.2) and (3.2.13), using the modified stress-energy tensor $T^{\mu\nu}_{b+\mathcal{C}}$ (2.1.28). Hence, we have

$$\nabla_\mu T^{\mu\nu}_{b+\mathcal{C}} = \nabla_\mu T^{\mu\nu} + \nabla_\mu(8c_1\mathcal{C}^{\mu\nu}) = 0 \tag{3.2.14}$$

then the zero component of the axion matter stress-energy tensor is

$$\nabla_\mu T^{\mu 0} = \partial_0 T^{00} + 3HT^{00} + H\alpha^2 T^{ii} = \frac{d}{dt}\rho^b + 6H\rho^b \tag{3.2.15}$$

while also

$$\nabla_\mu(8c_1\mathcal{C}^{\mu 0}) = \partial_0(8c_1\mathcal{C}^{00}) + 4H(8c_1\mathcal{C}^{00}) = \frac{d}{dt}\rho^{bR} + 4H\rho^{bR} \tag{3.2.16}$$

for which the relations (2.2.5), (2.2.7) and (3.2.8). Thus (3.2.14) becomes

$$\frac{d}{dt}\rho^b + 6H\rho^b + \frac{d}{dt}\rho^{bR} + 4H\rho^{bR} = \frac{d}{dt}(\rho^b + \rho^{bR}) + 3H(2\rho^b + \frac{4}{3}\rho^{bR}) = 0 \tag{3.2.17}$$

and if $\frac{d}{dt}(\rho^b + \rho^{bR}) = 0$ we have

$$\rho^b = -\frac{2}{3}\rho^{bR} \tag{3.2.18}$$

This result explains the negative nature of (3.2.13), since, if this was not the case, ρ^b would be negative and thus would clash with (3.2.2). On the other hand, one may use (3.2.2) to phenomenologically derive $\mathcal{N} = 1.484$ and thus, the constant c_1. Moreover, we observe that

$$\rho^b + \rho^{bR} = -\frac{2}{3}\rho^{bR} + \rho^{bR} = \frac{1}{3}\rho^{bR} = -0.496\epsilon_\varphi M_{pl}^2 H^2 < 0 \tag{3.2.19}$$

The first statement of this equation is that the contribution of the modified stress-energy tensor to the vacuum energy density of our universe will be of order H^2. But, more importantly states that the gravitational anomalies, due to their negative contribution to the energy density of the system, would make the fluid behave like exotic matter, with negative energy density and positive pressure. Something like this would be catastrophic, since it seems to violate the weak energy condition. Nevertheless, this is not the total density of our toy-universe, given that we have not touched on the third term of (3.2.1), that of the gravitational condensate.

By isolating the condensate term we get

$$S^{condensate} = -c_1 \int dx^4 \sqrt{-g} \langle b R_{\mu\nu\rho\sigma} \widetilde{R} \rangle \tag{3.2.20}$$

with be the axion background field satisfying (3.1.53). Since this is only in function of time, its average would be proportional to the duration of the whole inflationary epoch, which by the definition of (1.2.3)

$$H\Delta t = Ht_{end} = N \tag{3.2.21}$$

with $N = 60 - 70$ [2] being the number of e-foldings of inflation. Hence total b reads as

$$b(t) = b(0) + \sqrt{2\epsilon_\varphi} M_{pl} N \tag{3.2.22}$$

then, using this result in parallel with (3.1.40), (3.1.52) and the usual definitions of $c_1 = c_1' M_{pl}^{-1}$, $\kappa^2 = M_{pl}^{-2}$, we get

$$S^{condensate} = -\int dx^4 \sqrt{-g} \left[\frac{8c_1'^2}{\pi^2} \sqrt{2\epsilon_\varphi} \left(\frac{\mu}{M_{pl}} \right)^4 \left(\frac{|b(0)|}{M_{pl}} + \sqrt{2\epsilon_\varphi} N \right) H^4 \right] \tag{3.2.23}$$

We notice that the value inside the [....] does not depend on the metric tensor, and can be collectively be written as

$$S^{condensate} = -\int dx^4 \sqrt{-g} \frac{\Lambda}{\kappa^2} \tag{3.2.24}$$

and the variation would then lead to

$$\delta S^{condensate} = -\int dx^4 \delta\sqrt{-g} \frac{\Lambda}{\kappa^2} = -\int dx^4 \sqrt{-g} \frac{g_{\mu\nu}\Lambda}{2\kappa^2} \delta g^{\mu\nu} \tag{3.2.25}$$

which would add an $g_{\mu\nu}\Lambda$ term in the Einstein equations. Of course this a de-Sitter Dark Energy type term, which by definition satisfies the equation of state

$$\rho^{condensate} = -P^{condensate} \tag{3.2.26}$$

This term can not be classically derived, and thus does not exist in the Einstein equations (2.1.25). Nonetheless, quantum graviton fluctuations during inflation can induce naturally such a term in the Einstein Equations, via the condensate (3.2.20), and acts as an effective induced contribution to the vacuum energy density (3.2.19) [16]. Thus, we managed to derive a positive quantity $\Lambda > 0$ approximately constant ($H \sim const.$) that behaves as a cosmological de-Sitter constant term, which is responsible for driving inflation, as illustrated by the H^4 term inside (3.2.23).

This is a profound result, in which by imposing the existence of gravitational anomalies (such as those derived by string theory) we come with a mechanism that allows for natural inflation to occur, without the addition of any external field. Let us remind that the b axion field is not the drive of inflation, but the matter of our early toy-string-universe, which couples with gravity creating the gravitational anomalies.

Let's now show the contribution of (3.2.23) in the energy density of our model. In this case, the total energy density would read

$$\rho^{total} = \rho^b + \rho^{bR} + \rho^{condensate} \tag{3.2.27}$$

with $\rho^b + \rho^{bR}$ being given by (3.2.19) and

$$\rho^{condensate} = \left[\frac{8c_1'^2}{\pi^2}\sqrt{2\epsilon_\varphi}\mu^4\left(\frac{|b(0)|}{M_{pl}} + \sqrt{2\epsilon_\varphi}N\right)\left(\frac{H}{M_{pl}}\right)^4\right] \tag{3.2.28}$$

the M_{pl}^{-2} factor came from the definition of in (3.2.24). Considering at the beginning of inflation $|b(0)| > \sqrt{2\epsilon_\varphi}M_{pl}N$ (in accordance to the trans-Planckian effects of inflation), that $\epsilon_\varphi \sim 10^{-2}$ and (3.1.48), we get the total energy density:

$$\rho^{total} = 3M_{pl}^4\left[-1.65\times 10^{-3}\left(\frac{H}{M_{pl}}\right)^2 + \frac{\sqrt{2}c_1'^2}{3c}\times 2.43\times 10^7\frac{|b(0)|}{M_{pl}}\left(\frac{H}{M_{pl}}\right)^4\right] \tag{3.2.29}$$

This result solve the "exotic fluid" issue of (3.2.19), since the positive contribution of $\rho^{condensate}$ its much larger than that of $\rho^b + \rho^{bR}$ resulting in an overall positive total energy density. Furthermore, we make the important observation between the similarities of (3.2.29) and (1.3.16), hinting of a running vacuum type of energy density.

$$\rho^{RVM} = \frac{\Lambda(H)}{\kappa^2} = \frac{c_0}{\kappa^2} + \frac{3}{\kappa^2}\left(\nu H^2 + a\frac{H^4}{H_I^2}\right) \tag{3.2.30}$$

To support this argument, we need to show that

$$\rho^{total}(H) = -P^{total}(H) \tag{3.2.31}$$

to do that, we use the definition of the pressure P^{bR}:

$$C^{ii} = -g^{ii} P^{bR} \tag{3.2.32}$$

and combining this with traceless property of the time ordered Cotton tensor (3.2.9), in the flat FLRW backgroud, we have

$$\rho^{bR} = 3P^{bR} \tag{3.2.33}$$

thus

$$P^b + P^{bR} = \rho^b + \frac{1}{3}\rho^{bR} = -\frac{2}{3}\rho^{bR} + \frac{1}{3}\rho^{bR} = -\frac{1}{3}\rho^{bR} \tag{3.2.34}$$

meaning that

$$P^b + P^{bR} = -\left(\rho^b + \rho^{bR}\right) \tag{3.2.35}$$

and with the addition of (3.2.26) we prove that (3.2.31) holds. Thus, we see that (3.2.29) is of conventional RV type, as explained in subsection 1.3. However, the calculation of the condensate was done assuming an approximately constant H during inflation, hence a constant (3.2.29). But having shown that the primordial fluid of our model is of de-Sitter nature, the temporal evolution of the fluid must satisfy the solution (1.3.21) of the evolution equation of H (1.3.22), which for convenience we show bellow.

$$H(\alpha) = \sqrt{\frac{1-\nu}{a}} \frac{H_I}{\sqrt{D\alpha^{3(1-\nu)(1+\omega)} + 1}} \tag{3.2.36}$$

For consistency, we apply the Stringy model in (3.2.36). Thus, $\omega_m = 1$, since the matter is our case the $b(x)$ field, and $\nu = -1.65 \times 10^{-3}$ due to the comparison between (3.2.29) and (3.2.30). With a positive exponent of α in (3.2.36) and due to the fact that in the early universe $\alpha \ll 1$, we get an unstable de-Sitter phase, characterized by an approximate constant H:

$$H = \sqrt{\frac{(1-\nu)}{a}} H_I \tag{3.2.37}$$

we can also adjust the constant a by properly fixing the boundary value $|b(0)|$, which then will lead to $H \simeq H_I$, as required for consistency, and set $c_0 = 0$ in (3.2.30), since there is no evidence of a positive cosmological constant in the early eras of the universe [2, 15].

This profound results states that the the gravitational anomalies in the effective stringy-action (2.1.22), due to primordial gravitational perturbations of quantum origin, give a non-zero vacuum expectation value term, which is equivalent to a de-Sitter like term (3.2.24) in the action. Such a term, identified by the H^4 contribution, is able to exponentially accelerate an early universe, meaning the birth and drive of an inflationary era with an approximately constant Hubble parameter (3.2.37) and total energy density (3.2.29). Additionally, the dynamic property of the Hubble parameter and thus, the de-Sitter term (3.2.28) allows, due to the Running Vacuum nature of this model, for a graceful unbiased exit of inflation, when the Hubble parameter becomes small enough (equivalently the scale factor big enough) and the H^4 terms insignificant. This will allow for the proper decay of the running vacuum term and the creation of matter and radiation [15, 16, 25], thus reheating and the creation of fermions, important elements of the radiation era. The fermionic and gauge degrees of freedom who will appear after inflation will also contribute in matter vacuum energy density, giving a positive constant ν instead of the negative $\nu = -1.65 \times 10^{-3}$ during inflation, thus coming in total agreement with RVM. On top of that, this graceful exist of inflation, due to the Running Vacuum nature of (3.2.29), allows for a coherent connection between pre and post inflationary eras (and even connection with modern eras), thus, allow for phenomenological, theoretical and experimental test-ability of the model. Lastly, one other important aspect of this model is that it doesn't require any external modifications to be self-sustained, since we didn't invoke any additional field for the drive of inflation. Here, only gravity and the KR axion field (early matter) b are involved and the latter doesn't drive inflation itself. Inflation is purely geometrical of origin due to the interaction and exchange in energy between spacetime (i.e. the graviton field) and the b axion field, via the addition of the Cherm-Simons term in (2.1.22).

4. Conclusions

The subject of this thesis was an analytical review, both physical and mathematical, about a string-inspired theoretical model in which, an inflationary expansion of the early universe may naturally occur, due to the important vacuum energy density contributions of the CP-violating gravitational anomalies emergent in string theory. Those anomalies, translated in the effective action via the Cherm-Simons anomalous term, are induced by primordial gravitational waves of quantum origin during inflation, in the presence of CP-violating backgrounds. Additionally, the vacuum energy density of this model is of Running Vacuum type, implying a Cosmological-Constant-like term during inflation which nonetheless is time dependent and, thus, the ability of a coherent connection between early and later eras of the universe.

An important subject of study for this thesis was the gravitational anomalous term $R\widetilde{R}$ whose vacuum expectation value, normally being zero in a FLRW background, had a none-zero value consequent of the primordial gravitational waves, which in turn resulted in the gravitational condensate that drove inflation. This process is the result of energy exchange between the KR axion field and gravity and is apparent by the non conservative nature of the matter stress-energy tensor of the KR axion field due to the existence of the second rank Cotton tensor, which holds the information about the energy contribution of the gravitational anomalies. The trigger of inflation is embodied in the dominant H^4 contribution in the energy density of the system, with H being the Hubble parameter, followed by a negative subdominant H^2 term. Interestingly, was not for the condensate contribution in the total energy density of the system, the early fluid of the universe would behave like exotic matter, something that makes the existence of the gravitational condensate even more essential for satisfying the weak energy condition. On top of that, the RVM nature of the total energy density allows for a graceful end of inflation, due the exponential enlargement of the scale factor consequent of inflation, meaning the necessary mechanisms for both the beginning and ending of inflation are given, along with the ability to physical connect the post inflationary era with the universe before inflation. A substantial aspect of this model is that is self-sustained, meaning there is no need of external fields, such as the inflaton field, to drive inflation. Inflation occurs naturally within the boundaries of this model making no extra additions to justify its results.

Another fundamental part of this thesis is the understanding of the Running Vacuum Models. The motivation of development of such models is becoming more apparent by inconsistencies between a plethora of new collected data the ΛCDM model of cosmology, mainly evident in H_0 and σ_8 tensions. Additionally, the ability of Running Vacuum Models, even withing the semi-classical QFT, to explaining many fundamental and phenomenological questions about the universe, such as the emergence of inflation, and also be consistent throughout cosmic time, makes them excellent candidates for answering and providing a deeper understanding about the nature of the universe. The main argumentation of RVM is the non-constant nature of the Cosmological Constant, but instead its dependence of time, which also results in the non constant nature of the Hubble parameter, and finally, using an RG approach, its evolution equation. Such a statement makes the next logical leap in trying to understand the nature of the vacuum energy density, dark energy, and inflation. Additionally RVM are build using QFT argumentation, thus, their results should be perfectly applicable within other quantum theories.

In addition to inflation, the string-inspired model reviewed by this thesis is capable of providing possible answers to many phenomena, outside of the context of this thesis [15, 16, 17]. One of those is that it predicts matter-antimatter asymmetry post inflation where the CP-violating KR axion background remains undiluted post inflation and, in models involving heavy right-handed sterile neutrinos, leads to baryogenesis via leptogenesis. Additionally, uncompensated gauge chiral anomalies of the fermionic axial current also contribute an H^2 term in the running vacuum. In the late era of the universe, those contributions involve an additive constant term that was absent in the early universe and takes a value close to the cosmological constant of ΛCDM model, thus Dark Energy. Also, because of the same chiral anomalies, large-scale cosmic magnetic fields are able to from which allow for the KR axion field to act as Dark Matter.

Therefore, models with gravitational anomalies provide a powerful tool in understanding, not only inflation, but various cosmological questions, such as leptrogenesis, dark matter and dark energy. Hence, more theoretical development is needed in parallel with phenomenological consideration. Furthermore, these are few of the subjects which can me experimentally tested via observation, and thus clarify or falsify the theories in question. Thus, additional research in the context of string theory must be developed, while also the study of gravitational anomalies within the context of other quantum gravity theories (ex. loop-quantum gravity) should also be pursued. Finally, data collection is important, especially with the launch of James Webb Space Telescope which, in hopes of giving definitive answers to the cosmological tensions and guide future theoretical research.

Appendix A - Gravity and FLRW Cosmology

The purpose of this Appendix is to give the definitions and conventions of important quantities used throughout this Thesis in the context of General Relativity.

- **Gravitation**

  In General Relativity, the effect of gravity is caused by the curvature of spacetime which is a Riemannian manifold. The dynamics of this manifold is described (for $c = 1$) by the Einstein Field Equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

  where $T^{\mu}{}_{\nu}$ is the energy-momentum tensor, $R_{\mu\nu} = g^{\lambda\sigma} R_{\sigma\mu\lambda\nu}$ is the Ricci tensor and $R = g^{\mu\nu} R_{\mu\nu}$ the Ricci scalar, both defined as contractions of the Riemann Tensor $R_{\mu\nu\rho\sigma}$, and $g_{\mu\nu}$ is the metric tensor.

  The Riemann Tensor is identified by the expression

$$R^{\sigma}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\sigma}_{\nu\mu} - \partial_\nu \Gamma^{\sigma}_{\lambda\mu} + \Gamma^{\sigma}_{\lambda\rho} \Gamma^{\rho}_{\nu\mu} - \Gamma^{\sigma}_{\nu\rho} \Gamma^{\rho}_{\lambda\mu}$$

  and satisfies the Bianchi identity

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$$

  where ∇ is the covariant derivative, defined as

$$\nabla_\mu = \partial_\mu + \Gamma^{\nu}_{\mu\lambda}$$

  and $\Gamma^{\mu}_{\lambda\nu}$ are the Christoffel symbols, given by

$$\Gamma^{\mu}_{\lambda\nu} = \frac{1}{2} g^{\mu\rho} \left(\partial_\lambda g_{\nu\rho} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\lambda\nu} \right)$$

- **FLRW cosmology**

  A dynamic, isotropic and homogeneous universe is described by the Friedmann–Lemaître–Robertson–Walker (FLRW) metric which is given, in its most general form, as:

$$ds^2 = dt^2 - \alpha(t)^2 dX^2$$

  where X implies a three dimensional spatial metric of uniform curvature and $\alpha(t)$ is the scale factor, which describes how the size of the Universe changes with time. In the case of a flat ($\kappa = 0$) universe the metric take the form

$$ds^2 = dt^2 - \alpha(t)^2 \left(dr^2 + r^2 d\Omega \right)$$

the Christoffel symbols for the above metric are

$$\begin{aligned}
&\Gamma^0_{11} = \alpha\dot{\alpha} \quad \Gamma^0_{22} = \alpha\dot{\alpha}r^2 \quad \Gamma^0_{33} = \alpha\dot{\alpha}r^2\sin^2\theta \\
&\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{\alpha}}{\alpha} \\
&\Gamma^1_{22} = -r \quad \Gamma^1_{33} = -r\sin^2\theta \\
&\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \\
&\Gamma^2_{33} = -\sin\theta\cos \quad \Gamma^3_{23} = \Gamma^3_{32} = -\cot\theta
\end{aligned}$$

and using them one can calculate the components of the Ricci tensor

$$\begin{aligned}
&R_{00} = -3\frac{\ddot{\alpha}}{\alpha} \\
&R_{11} = \ddot{\alpha}\alpha + 2\dot{\alpha}^2 \\
&R_{22} = r^2(\ddot{\alpha}\alpha + 2\dot{\alpha}^2) \\
&R_{22} = r^2\sin^2\theta(\ddot{\alpha}\alpha + 2\dot{\alpha}^2)
\end{aligned}$$

and the others being zero. The Ricci scalar then is

$$R = 6\left(\frac{\ddot{\alpha}}{\alpha} + \frac{\dot{\alpha}^2}{\alpha^2}\right)$$

Supposing that the universe contains a fluid of total energy density $\rho_\tau(t)$ and pressure $P_\tau(t)$, then we define the energy-momentum tensor as

$$T^\mu{}_\nu = \begin{pmatrix} \rho_\tau & 0 & 0 & 0 \\ 0 & -P_\tau & 0 & 0 \\ 0 & 0 & -P_\tau & 0 \\ 0 & 0 & 0 & -P_\tau \end{pmatrix}$$

and by taking the covariant derivative of its zeroth component we get the following conservation law

$$\begin{aligned}
&\nabla_\mu T^\mu{}_0 = 0 && \rightarrow \\
&\partial_\mu T^\mu{}_0 + \Gamma^\mu_{\mu 0}T^0{}_0 - \Gamma^\lambda_{\mu 0}T^\mu{}_\lambda = 0 && \rightarrow \\
&\dot{\rho}_\tau + 3\frac{\dot{\alpha}}{\alpha}\rho_\tau + 3\frac{\dot{\alpha}}{\alpha}P_\tau = 0 && \rightarrow \\
&\dot{\rho}_\tau + 3\frac{\dot{\alpha}}{\alpha}\left(\rho_\tau + P_\tau\right) = 0
\end{aligned}$$

With the use of the scale factor we may define the Hubble parameter

$$H(t) \equiv \frac{\dot{\alpha}}{\alpha}$$

where $\dot{\alpha} \equiv d\alpha/dt$, and using the fact that

$$\dot{H} = \frac{\ddot{\alpha}\alpha - \dot{\alpha}\dot{\alpha}}{\alpha^2} = \frac{\ddot{\alpha}}{\alpha} - H^2$$

we may rewrite the Ricci Scalar and the conservation law in terms of the Hubble parameter

$$R = 12H^2 + 6\dot{H}$$
$$\dot{\rho}_\tau + 3H(\rho_\tau + P_\tau) = 0$$

Also, solving Einstein's equations for $\mu\nu = 00$ and $\mu\nu = ij$ respectively, gets us to the Friedmann equations

$$H^2 = \frac{8\pi G}{3}\rho_\tau + \frac{\Lambda}{3}$$
$$\frac{\ddot{\alpha}}{\alpha} = -\frac{4\pi G}{3}\left(\rho_\tau + 3P_\tau\right) + \frac{\Lambda}{3}$$

Finally, given the equations of state for matter $P = \omega\rho$ and vacuum energy $P_\Lambda = -\rho_\Lambda$ we can rewrite the continuity equation in terms of the scale factor

$$\frac{d\ln\rho_\tau}{d\ln a} = -3(1+w)$$

and by integration we get

$$\rho_\tau \propto \alpha^{-3(1+w)}$$

We may also define the critical energy density

$$\rho_c \equiv \frac{3H^2}{8\pi G}$$

the density parameter

$$\Omega \equiv \frac{\rho_\tau}{\rho_c} = \frac{8\pi G}{3H^2}\rho_\tau$$

and the deceleration parameter

$$q \equiv -\frac{\ddot{\alpha}\alpha}{\dot{\alpha}^2} = -1 - \frac{\dot{H}}{H^2}$$

from which we get the relation

$$\dot{H} = -(1+q)H^2$$

Appendix B - Mathematical Calculations

- Derivation of (1.3.17)

Using the fact that

$$d\ln H^2 = \frac{dH^2}{H^2}$$

equation (1.3.16) becomes

$$\begin{aligned} d\rho_\Lambda &= \left\{ \frac{1}{(4\pi)^2} \sum_i \left[a_i M_i^2 H^2 + b_i H^4 + c_i \frac{H^6}{M_i^2} + ... \right] \right\} \frac{dH^2}{H^2} \\ &= \frac{1}{(4\pi)^2} \sum_i \left[a_i M_i^2 + b_i H^4 + c_i \frac{H^4}{M_i^2} + ... \right] dH^2 \end{aligned}$$

The integration gives

$$\begin{aligned} \int d\rho_\Lambda = \rho_\Lambda = \int \frac{1}{(4\pi)^2} \sum_i \left[a_i M_i^2 + b_i H^2 + c_i \frac{H^4}{M_i^2} + ... \right] dH^2 \\ = const + \frac{1}{(4\pi)^2} \sum_i \left[\int a_i M_i^2 dH^2 + \int b_i H^2 dH^2 + \int c_i \frac{H^4}{M_i^2} dH^2 + ... \right] \\ = const + \frac{1}{(4\pi)^2} \sum_i \left[a_i M_i^2 H^2 + \frac{b_i}{2} H^4 + \frac{c_i}{3} \frac{H^6}{M_i^2} + ... \right] \end{aligned}$$

and having the relation $\rho_\Lambda(H) = \Lambda(H)/(8\pi G)$ gives

$$\begin{aligned} (8\pi G)\rho_\Lambda = \Lambda(H) = (8\pi G) \left\{ const + \frac{1}{(4\pi)^2} \sum_i \left[a_i M_i^2 H^2 + \frac{b_i}{2} H^4 + \frac{c_i}{3} \frac{H^6}{M_i^2} + ... \right] \right\} \\ = c_0 + \frac{8\pi G}{(4\pi)^2} \sum_i \left[a_i M_i^2 H^2 + \frac{b_i}{2} H^4 + \frac{c_i}{3} \frac{H^6}{M_i^2} + ... \right] \end{aligned}$$

- Derivation of (1.3.21)

Using (1.3.7) and dividing by H^2 we have

$$\begin{aligned} &\frac{8\pi G\rho + \Lambda}{H^2} = 3 \\ &\Rightarrow \frac{8\pi G\rho}{H^2} = 3 - \frac{\Lambda}{H^2} \\ &\Rightarrow \rho = \frac{H^2}{8\pi G} \left(3 - \frac{\Lambda}{H^2} \right) \end{aligned}$$

replacing Λ with (1.3.18) we get

$$\begin{aligned}\rho &= \frac{H^2}{8\pi G}\left[3 - \frac{1}{H^2}\left(c_0 + 3\left(\nu H^2 + a\frac{H^4}{H_I^2}\right)\right)\right] \\ &= \frac{3H^2}{8\pi G}\left[1 - \frac{c_0}{3H^2} - \nu - a\left(\frac{H}{H_I}\right)^2\right]\end{aligned}$$

Also, the first derivative of (1.3.7) with respect of time gives

$$\begin{aligned}&8\pi G\rho + \Lambda = 3H^2 \\ &\Rightarrow 8\pi G(\rho + \rho_\Lambda) = 3H^2 \\ &\Rightarrow 8\pi G(\dot{\rho} + \dot{\rho}_\Lambda) = 6H\dot{H} \\ &\Rightarrow \dot{\rho} + \dot{\rho}_\Lambda = \frac{3H\dot{H}}{4\pi G}\end{aligned}$$

Finally, by replacing the two expressions above into (1.3.9) we get

$$\begin{aligned}&\dot{\rho}_\Lambda + \dot{\rho} + 3(1+\omega)\rho H = 0 \\ &\Rightarrow \frac{3H\dot{H}}{4\pi G} + 3(1+\omega)H\frac{3H^2}{8\pi G}\left[1 - \frac{c_0}{3H^2} - \nu - a\left(\frac{H}{H_I}\right)^2\right] = 0 \\ &\Rightarrow \frac{3H}{4\pi G}\left\{\dot{H} + \frac{3}{2}(1+\omega)H^2\left[1 - \frac{c_0}{3H^2} - \nu - a\left(\frac{H}{H_I}\right)^2\right]\right\} = 0 \\ &\Rightarrow \dot{H} + \frac{3}{2}(1+\omega)H^2\left[1 - \frac{c_0}{3H^2} - \nu - a\left(\frac{H}{H_I}\right)^2\right] = 0\end{aligned}$$

- Derivation of (1.3.22)

With $\dot{H} = dH/dt = (dH/d\alpha)(d\alpha/dt)$ and setting $c_0 = 0$, equation (1.3.21) becomes

$$\begin{aligned}&\dot{\alpha}\frac{dH}{d\alpha} + \frac{3}{2}(1+\omega)H^2\left[1 - \nu - a\left(\frac{H}{H_I}\right)^2\right] = 0 \\ &\Rightarrow \frac{\alpha}{H}\frac{dH}{d\alpha} + \frac{3}{2}(1+\omega)\left[1 - \nu - a\left(\frac{H}{H_I}\right)^2\right] = 0 \\ &\Rightarrow \frac{\alpha}{d\alpha}\frac{dH}{H} + \frac{3}{2}(1+\omega)(1-\nu)\left[1 - \frac{a}{1-\nu}\left(\frac{H}{H_I}\right)^2\right] = 0\end{aligned}$$

by setting $H/H_I = \tilde{H}$ we get

$$\begin{aligned}&\frac{\alpha}{d\alpha}\frac{d\tilde{H}}{\tilde{H}} + \frac{3}{2}(1+\omega)(1-\nu)\left[1 - \left(\frac{a}{1-\nu}\right)\tilde{H}^2\right] = 0 \\ &\Rightarrow \frac{d\tilde{H}}{\left[1 - \left(\frac{a}{1-\nu}\right)\tilde{H}^2\right]\tilde{H}} = -\frac{3}{2}(1+\omega)(1-\nu)\frac{d\alpha}{\alpha}\end{aligned}$$

lets simplify the expression by setting $\beta = a/(1-\nu)$ and $\xi = -3(1+\omega)(1-\nu)/2$, the above expression then becomes

$$\frac{d\tilde{H}}{\left(1-\beta\tilde{H}^2\right)\tilde{H}} = \xi\frac{d\alpha}{\alpha}$$
$$\Rightarrow \frac{d\tilde{H}}{\tilde{H}} + \beta\tilde{H}\frac{d\tilde{H}}{1-\beta\tilde{H}^2} = \xi\frac{d\alpha}{\alpha}$$
$$\Rightarrow \frac{d(\beta\tilde{H})}{\beta\tilde{H}} + \frac{1}{2}\frac{d(\beta\tilde{H}^2)}{1-\beta\tilde{H}^2} = \xi\frac{d\alpha}{\alpha}$$

Integration of the above gives

$$\int\frac{d(\beta\tilde{H})}{\beta\tilde{H}} + \frac{1}{2}\int\frac{d(\beta\tilde{H}^2)}{1-\beta\tilde{H}^2} = \xi\int\frac{d\alpha}{\alpha}$$
$$\Rightarrow \ln\left(\beta\tilde{H}\right) - \frac{1}{2}\ln\left(1-\beta\tilde{H}^2\right) = \xi\ln\alpha + D$$
$$\Rightarrow \ln\left(\beta\tilde{H}\right) - \ln\left(1-\beta\tilde{H}^2\right)^{1/2} = \ln\left(D\alpha^{\xi}\right)$$
$$\Rightarrow \ln\left(\frac{\beta\tilde{H}}{\sqrt{1-\beta\tilde{H}^2}}\right) = \ln\left(D\alpha^{\xi}\right)$$
$$\Rightarrow -\ln\left(\frac{\sqrt{1-\beta\tilde{H}^2}}{\beta\tilde{H}}\right) = \ln\left(D\alpha^{\xi}\right)$$
$$\Rightarrow \ln\left(\sqrt{\frac{1}{\beta\tilde{H}^2}-1}\right) = \ln\left(D\alpha^{-\xi}\right)$$
$$\Rightarrow \frac{1}{\beta\tilde{H}^2} = D\alpha^{-2\xi} + 1$$
$$\Rightarrow \frac{1}{\beta}\frac{1}{D\alpha^{-2\xi}+1} = \tilde{H}^2$$

Finally, replacing $\beta = a/(1-\nu)$, $\xi = -3(1+\omega)(1-\nu)/2$ and $\tilde{H} = H/H_I$ we have

$$\frac{H^2}{H_I^2} = \frac{1-\nu}{a}\frac{1}{D\alpha^{-3(1+\omega)(1-\nu)}+1}$$
$$\Rightarrow H = \sqrt{\frac{1-\nu}{a}}\frac{H_I}{\sqrt{D\alpha^{-3(1+\omega)(1-\nu)}+1}}$$

- Derivation of (2.1.23)

Starting with the variation of the anomalous term of (2.1.22) we have

$$\delta CS = \left[\int d^4x\sqrt{-g}bR_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma}\right] =$$
$$= \int d^4x\sqrt{-g}b\delta(R_{\mu\nu\rho\sigma})\widetilde{R}^{\mu\nu\rho\sigma} + \int d^4x\sqrt{-g}bR_{\mu\nu\rho\sigma}\delta(\widetilde{R}^{\mu\nu\rho\sigma})$$

and using (2.1.19) gives

$$\delta CS = \frac{1}{2}\int d^4x\sqrt{-g}b\delta(R_{\mu\nu\rho\sigma})R^{\mu\nu}{}_{\alpha\beta}\varepsilon^{\alpha\beta\rho\sigma}$$
$$+ \frac{1}{2}\int d^4x\sqrt{-g}bR_{\mu\nu\rho\sigma}\delta(R^{\mu\nu}{}_{\alpha\beta})\varepsilon^{\alpha\beta\rho\sigma}$$

The variation of the Riemann Tensor reads as

$$\delta(R^{\mu}{}_{\nu\rho\sigma}) = \nabla_\rho(\delta\Gamma^{\mu}_{\sigma\nu}) - \nabla_\sigma(\delta\Gamma^{\mu}_{\rho\nu})$$

thus, the variation of the anomalous term becomes

$$\delta CS = \frac{1}{2}\int d^4x\sqrt{-g}b\nabla_\rho(\delta\Gamma_{\mu\sigma\nu})R^{\mu\nu}{}_{\alpha\beta}\varepsilon^{\alpha\beta\rho\sigma}$$
$$- \frac{1}{2}\int d^4x\sqrt{-g}b\nabla_\sigma(\delta\Gamma_{\mu\rho\nu})R^{\mu\nu}{}_{\alpha\beta}\varepsilon^{\alpha\beta\rho\sigma}$$
$$+ \frac{1}{2}\int d^4x\sqrt{-g}bR_{\mu\nu\rho\sigma}\nabla_\alpha(\delta\Gamma^{\mu\nu}_{\beta})\varepsilon^{\alpha\beta\rho\sigma}$$
$$- \frac{1}{2}\int d^4x\sqrt{-g}bR_{\mu\nu\rho\sigma}\nabla_\beta(\delta\Gamma^{\mu\nu}_{\alpha})\varepsilon^{\alpha\beta\rho\sigma}$$

Index manipulation of the second and forth term as $\sigma \leftrightarrow \rho$ and $\alpha \leftrightarrow \beta$, respectively, gives

$$\delta CS = \int d^4x\sqrt{-g}b\nabla_\rho(\delta\Gamma_{\mu\sigma\nu})R^{\mu\nu}{}_{\alpha\beta}\varepsilon^{\alpha\beta\rho\sigma} + \int d^4x\sqrt{-g}bR_{\mu\nu\rho\sigma}\nabla_\alpha(\delta\Gamma^{\mu\nu}_{\beta})\varepsilon^{\alpha\beta\rho\sigma}$$

Continuing with partial integration (and respecting the boundary conditions) we get

$$\delta CS = -\int d^4x\sqrt{-g}\left\{\nabla_\rho(b)\delta(\Gamma_{\mu\sigma\nu})R^{\mu\nu}{}_{\alpha\beta}\varepsilon^{\alpha\beta\rho\sigma} + b\delta(\Gamma_{\mu\sigma\nu})\nabla_\rho(R^{\mu\nu}{}_{\alpha\beta})\varepsilon^{\alpha\beta\rho\sigma}\right.$$
$$\left. + \nabla_\alpha(b)R_{\mu\nu\rho\sigma}\delta(\Gamma^{\mu\nu}_{\beta})\varepsilon^{\alpha\beta\rho\sigma} + b\nabla_\alpha(R_{\mu\nu\rho\sigma})\delta(\Gamma^{\mu\nu}_{\beta})\varepsilon^{\alpha\beta\rho\sigma}\right\}$$

Again, interchanging indices as $\sigma \leftrightarrow \beta$ and $\rho \leftrightarrow \alpha$ in both the second and forth terms of the relations above yields

$$\delta CS = -2\int d^4x\sqrt{-g}\left[b\nabla_\rho(R^{\mu\nu}{}_{\alpha\beta})\varepsilon^{\alpha\beta\rho\sigma} + \nabla_\rho(b)R^{\mu\nu}{}_{\alpha\beta}\varepsilon^{\alpha\beta\rho\sigma}\right]\delta(\Gamma_{\mu\sigma\nu})$$

where

$$\delta(\Gamma_{\mu\sigma\nu}) = \frac{1}{2}g_{\mu}{}^{\lambda}\left[\nabla_\sigma(\delta g_{\lambda\nu}) + \nabla_\nu(\delta g_{\lambda\sigma}) - \nabla_\lambda(\delta g_{\sigma\nu})\right]$$

Inspecting both equations we see that with the use of the metric ${g_{\mu}}^{\lambda}$ we have ${R^{\mu\nu}}_{\alpha\beta}{g_{\mu}}^{\lambda} = {R^{\lambda\nu}}_{\alpha\beta}$, yet, because ${R^{\lambda\nu}}_{\alpha\beta}$ is antisymmetric in $[\lambda, \nu]$ the first term in the parenthesis of $\delta(\Gamma_{\mu\sigma\nu})$ does not contribute. Also, the remaining two terms can be combined with an exchange of indices $\lambda \leftrightarrow \nu$ in one of the two terms. Additionally, we see that the second term of δCS is the bianchi identity:

$$\nabla_{[\alpha}{R_{\rho\sigma]}}^{\mu\nu} = \epsilon^{\beta\alpha\rho\sigma}\nabla_{\alpha}{R_{\rho\sigma}}^{\mu\nu} = 0 \tag{4.0.1}$$

Thus, δCS takes the form

$$\delta CS = -2\int d^4x\sqrt{-g}\left[\nabla_{\rho}(b){R^{\lambda\nu}}_{\alpha\beta}\varepsilon^{\alpha\beta\rho\sigma}\right]\nabla_{\nu}(\delta g_{\lambda\sigma})$$

Partial integration yields

$$\begin{aligned}\delta CS = -2\int d^4x\sqrt{-g}\,\Big[&\nabla_{\rho}(b)\nabla_{\nu}({R^{\lambda\nu}}_{\alpha\beta})\varepsilon^{\alpha\beta\rho\sigma}\\ &+\nabla_{\nu}\nabla_{\rho}(b){R^{\lambda\nu}}_{\alpha\beta}\varepsilon^{\alpha\beta\rho\sigma}\Big](\delta g_{\lambda\sigma})\end{aligned}$$

The third term can be expressed as:

$$\nabla_{\nu}\nabla_{\rho}(b){R^{\lambda\nu}}_{\alpha\beta}\varepsilon^{\alpha\beta\rho\sigma} = 2\nabla_{\nu}\nabla_{\rho}(b)\widetilde{R}^{\lambda\nu\rho\sigma}$$

Also, we expand the bianchi identity $\nabla_{[\lambda}R_{\rho\sigma]\mu\nu}$ as bellow

$$\begin{aligned}0 &= \nabla_{\lambda}R_{\rho\sigma\mu\nu} + \nabla_{\rho}R_{\sigma\lambda\mu\nu} + \nabla_{\sigma}R_{\lambda\rho\mu\nu}\\ &= g^{\sigma\nu}\left(\nabla_{\lambda}R_{\rho\sigma\mu\nu} + \nabla_{\rho}R_{\sigma\lambda\mu\nu} + \nabla_{\sigma}R_{\lambda\rho\mu\nu}\right)\\ &= \nabla_{\lambda}{R_{\rho}}^{\nu}{}_{\mu\nu} + \nabla_{\rho}{R^{\nu}}_{\lambda\mu\nu} + \nabla^{\nu}R_{\lambda\rho\mu\nu}\\ &= \nabla_{\lambda}{R^{\nu}}_{\rho\nu\mu} - \nabla_{\rho}{R^{\nu}}_{\lambda\nu\mu} + \nabla^{\nu}R_{\lambda\rho\mu\nu}\\ &\Rightarrow \nabla^{\nu}R_{\nu\mu\lambda\rho} = \nabla_{\lambda}R_{\rho\mu} - \nabla_{\rho}R_{\lambda\mu}\end{aligned}$$

by combining all of the above in $\delta(\Gamma_{\mu\sigma\nu})$ we get

$$\begin{aligned}\delta CS = -2\int d^4x\sqrt{-g}\,\Big\{&\nabla_{\rho}(b)\left[\nabla_{\alpha}{R^{\lambda}}_{\beta} - \nabla_{\beta}{R^{\lambda}}_{\alpha}\right]\varepsilon^{\alpha\beta\rho\sigma}\\ &+2\nabla_{\nu}\nabla_{\rho}(b)\widetilde{R}^{\lambda\nu\rho\sigma}\Big\}(\delta g_{\lambda\sigma})\end{aligned}$$

using the antisymmetric property of $\varepsilon^{\alpha\beta\rho\sigma}$ between $[\alpha, \beta]$ gives

$$\delta CS = -4\int d^4x\sqrt{-g}\left\{\nabla_{\rho}(b)\nabla_{\alpha}{R^{\lambda}}_{\beta}\varepsilon^{\alpha\beta\rho\sigma} + \nabla_{\nu}\nabla_{\rho}(b)\widetilde{R}^{\lambda\nu\rho\sigma}\right\}(\delta g_{\lambda\sigma})$$

finally, the symmetry of the metric $g_{\lambda\sigma}$ between $[\lambda, \sigma]$ help as rewrite the expression into

$$\delta CS = -4 \int d^4x \sqrt{-g}.\frac{1}{2} \left\{ \nabla_\nu(b) \left[\varepsilon^{\nu\lambda\alpha\beta} \nabla_\alpha R^\sigma{}_\beta + \varepsilon^{\nu\sigma\alpha\beta} \nabla_\alpha R^\lambda{}_\beta \right] - \nabla_\nu \nabla_\rho(b) \left[\widetilde{R}^{\nu\sigma\rho\lambda} + \widetilde{R}^{\nu\lambda\rho\sigma} \right] \right\} (\delta g_{\lambda\sigma})$$

Thus, by comparison with (2.1.23) we conclude to

$$\delta \left[\int d^4x \sqrt{-g} b(x) R_{\mu\nu\rho\sigma} \widetilde{R}^{\mu\nu\rho\sigma} \right] = 4 \int d^4x \sqrt{-g} \mathcal{C}^{\mu\nu} \delta g_{\mu\nu} = -4 \int d^4x \sqrt{-g} \mathcal{C}_{\mu\nu} \delta g^{\mu\nu}$$

where

$$\mathcal{C}^{\mu\nu} = -\frac{1}{2} \left\{ \nabla_\sigma(b) \left[\varepsilon^{\sigma\mu\alpha\beta} \nabla_\alpha R^\nu{}_\beta + \varepsilon^{\sigma\nu\alpha\beta} \nabla_\alpha R^\mu{}_\beta \right] + \nabla_\sigma \nabla_\tau(b) \left[\widetilde{R}^{\tau\mu\sigma\nu} + \widetilde{R}^{\tau\nu\sigma\mu} \right] \right\}$$

- Derivation of (3.2.4)

  Taking into account the following the equations, derived previously:

  - $\delta CS = -2 \int d^4x \sqrt{-g} \left[\nabla_\rho(b) \nabla_\nu (R^{\lambda\nu}{}_{\alpha\beta}) \varepsilon^{\alpha\beta\rho\sigma} + \nabla_\nu \nabla_\rho(b) R^{\lambda\nu}{}_{\alpha\beta} \varepsilon^{\alpha\beta\rho\sigma} \right] (\delta g_{\lambda\sigma})$
  - $\delta \left[\int d^4x \sqrt{-g} b(x) R_{\mu\nu\rho\sigma} \widetilde{R}^{\mu\nu\rho\sigma} \right] = -4 \int d^4x \sqrt{-g} \mathcal{C}_{\mu\nu} \delta g^{\mu\nu}$

  and (2.1.19), we may rewrite the cotton tensor as

$$\mathcal{C}^{\mu\nu} = \frac{1}{2} \left[\nabla_\rho(b) \nabla_\nu (R^{\lambda\nu}{}_{\alpha\beta}) \varepsilon^{\alpha\beta\rho\sigma} + \nabla_\nu \nabla_\rho(b) R^{\lambda\nu}{}_{\alpha\beta} \varepsilon^{\alpha\beta\rho\sigma} \right]$$

  and stating that the covariant derivative of $\sqrt{-g}$ is zero, that $\varepsilon^{\alpha\beta\rho\sigma} = \frac{sgn(g)}{\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma}$ we may write the above expression as

$$\mathcal{C}^{\lambda\sigma} = \frac{1}{2} \left[\nabla_\rho(b) \nabla_\nu (\widetilde{R}^{\lambda\nu\rho\sigma}) + \nabla_\nu \nabla_\rho(b) \widetilde{R}^{\lambda\nu\rho\sigma} \right]$$

  partial integration of the first term gives

$$\nabla_\rho(b) \nabla_\nu (\widetilde{R}^{\lambda\nu\rho\sigma}) = \left(\nabla_\rho(b) (\widetilde{R}^{\lambda\nu\rho\sigma}) \right) - \nabla_\nu \nabla_\rho(b) \widetilde{R}^{\lambda\nu\rho\sigma}$$

  combining the above, doing some index manipulation and using the asymmetric property of $\widetilde{R}^{\lambda\nu\rho\sigma}$, we get

$$\mathcal{C}^{\lambda\sigma} = -\frac{1}{2} \nabla_\nu \left[\nabla_\rho(b) (\widetilde{R}^{\lambda\nu\rho\sigma} + \widetilde{R}^{\lambda\sigma\rho\nu}) \right]$$

then, taking the covariant derivative of the cotton tensor gives:

$$\begin{aligned}\nabla_\lambda \mathcal{C}^{\lambda\sigma} &= -\frac{1}{2}\nabla_\lambda\nabla_\nu\left[\nabla_\rho(b)(\widetilde{R}^{\lambda\nu\rho\sigma}+\widetilde{R}^{\lambda\sigma\rho\nu})\right] \\ &= \frac{1}{2}\left[\nabla_\lambda,\nabla_\nu\right]\nabla_\rho(b)(\widetilde{R}^{\lambda\nu\rho\sigma}+\frac{1}{2}\widetilde{R}^{\lambda\sigma\rho\nu}) - \frac{1}{2}\nabla_\lambda\nabla_\nu\left[\nabla_\rho(b)\widetilde{R}^{\lambda\nu\rho\sigma}\right]\end{aligned}$$

with $[\nabla_\lambda,\nabla_\nu] = \nabla_\lambda\nabla_\nu - \nabla_\nu,\nabla_\lambda$. The second term of the above expression vanishes, since $\nabla_\lambda\nabla_\nu$ is symmetric and $\widetilde{R}^{\lambda\nu\rho\sigma}$ antisymmetric in $[\lambda,\nu]$. We also take notice that the derivatives of $\nabla_\rho(b)$ cancel out, hence, without the loss of generality

$$\nabla_\lambda \mathcal{C}^{\lambda\sigma} = \frac{1}{2}\nabla_\rho(b)[\nabla_\lambda,\nabla_\nu]\left(\widetilde{R}^{\lambda\sigma\rho\nu}+\frac{1}{2}\widetilde{R}^{\lambda\nu\rho\sigma}\right)$$

using the Ricci identity for any tensor T

$$[\nabla_\lambda,\nabla_\nu]T^{\alpha\beta\gamma\delta} = R_{\lambda\nu}{}^{\alpha}{}_{\kappa}T^{\kappa\beta\gamma\delta} + R_{\lambda\nu}{}^{\beta}{}_{\kappa}T^{\beta\kappa\gamma\delta} + R_{\lambda\nu}{}^{\gamma}{}_{\kappa}T^{\alpha\beta\kappa\delta} + R_{\lambda\nu}{}^{\delta}{}_{\kappa}T^{\alpha\beta\gamma\kappa}$$

we get

$$[\nabla_\lambda,\nabla_\nu]\widetilde{R}^{\lambda\sigma\rho\nu} = R^{\lambda}{}_{\kappa\lambda\nu}\widetilde{R}^{\kappa\sigma\rho\nu} + R^{\sigma}{}_{\kappa\lambda\nu}\widetilde{R}^{\lambda\kappa\rho\nu} + R^{\rho}{}_{\kappa\lambda\nu}\widetilde{R}^{\lambda\sigma\kappa\nu} + R^{\nu}{}_{\kappa\lambda\nu}\widetilde{R}^{\lambda\sigma\rho\kappa}$$

and

$$[\nabla_\lambda,\nabla_\nu]\widetilde{R}^{\lambda\nu\rho\sigma} = R^{\lambda}{}_{\kappa\lambda\nu}\widetilde{R}^{\kappa\nu\rho\sigma} + R^{\nu}{}_{\kappa\lambda\nu}\widetilde{R}^{\lambda\kappa\rho\sigma} + R^{\rho}{}_{\kappa\lambda\nu}\widetilde{R}^{\lambda\nu\kappa\sigma} + R^{\sigma}{}_{\kappa\lambda\nu}\widetilde{R}^{\lambda\nu\rho\kappa}$$

hence, we have

$$\begin{aligned}\nabla_\lambda \mathcal{C}^{\lambda\sigma} = \frac{1}{2}\nabla_\rho(b)\Bigg[&\left(\widetilde{R}^{\kappa\sigma\rho\nu}+\frac{1}{2}\widetilde{R}^{\kappa\nu\rho\sigma}\right)R^{\lambda}{}_{\kappa\lambda\nu} + \left(\widetilde{R}^{\lambda\sigma\rho\kappa}+\frac{1}{2}\widetilde{R}^{\lambda\kappa\rho\sigma}\right)R^{\nu}{}_{\kappa\lambda\nu} \\ &+ \left(\widetilde{R}^{\lambda\sigma\kappa\nu}+\frac{1}{2}\widetilde{R}^{\lambda\nu\kappa\sigma}\right)R^{\rho}{}_{\kappa\lambda\nu} + \left(\widetilde{R}^{\lambda\kappa\rho\nu}+\frac{1}{2}\widetilde{R}^{\lambda\nu\rho\kappa}\right)R^{\sigma}{}_{\kappa\lambda\nu}\Bigg]\end{aligned}$$

The first two terms of the expression vanish because of the symmetry of the Ricci tensor. The two last terms can be added together using index manipulation in $[\rho,\sigma]$ and can be expanded using the antisymmetry of $R^{\lambda\kappa\rho\nu}$. After all of the above, we are left with.

$$\nabla_\lambda \mathcal{C}^{\lambda\sigma} = \frac{1}{4}\nabla_\rho(b)\left[\widetilde{R}^{\lambda\kappa\rho\nu}\left(R^{\sigma}{}_{\kappa\nu\lambda} - R^{\sigma}{}_{\lambda\nu\kappa}\right) + \widetilde{R}^{\lambda\nu\rho\kappa}R^{\sigma}{}_{\kappa\lambda\nu}\right]$$

and with the cyclic property of the bianchi identity

$$R_{\alpha\beta\gamma\delta} + R_{\beta\gamma\alpha\delta} + R_{\gamma\alpha\beta\delta} = 0$$

we get

$$R_{\sigma\lambda\nu\kappa} = -R_{\sigma\nu\kappa\lambda} - R_{\sigma\kappa\lambda\nu}$$

which leave us with

$$\nabla_\lambda \mathcal{C}^{\lambda\sigma} = \frac{1}{4}\nabla_\rho(b)\left[\widetilde{R}^{\lambda\kappa\rho\nu}R^{\sigma}{}_{\nu\kappa\lambda} + \widetilde{R}^{\lambda\nu\rho\kappa}R^{\sigma}{}_{\kappa\lambda\nu}\right]$$

using the same property we conclude to

$$\nabla_\lambda \mathcal{C}^{\lambda\sigma} = \frac{1}{2}\nabla_\rho(b)\widetilde{R}^{\lambda\kappa\rho\nu}R^{\sigma}{}_{\nu\kappa\lambda}$$

finally, using the identity

$$\widetilde{R}^{\lambda\kappa\rho\nu}R^{\sigma}{}_{\nu\kappa\lambda} = -\frac{1}{4}\delta^{\rho}_{\sigma}\widetilde{R}R$$

we reach to

$$\nabla_\lambda \mathcal{C}^{\lambda\sigma} = -\frac{1}{8}\nabla^{\sigma}(b)\widetilde{R}R$$

References

[1] J. Martin, C. Ringeval and V. Vennin, "*Encyclopædia Inflationaris*," *Phys. Dark Univ.* **5-6** (2014) 75-235 [arXiv:1303.3787].

[2] Planck collaboration, *Planck 2018 results. VI. Cosmological parameters*, *Astron. Astrophys.* **641** (2020) A6 [arXiv:1807.06209].

[3] L. Verde, T. Treu and A.G. Riess, *Tensions between the Early and the Late Universe*, *Nature Astron.* **3** (2019) 891 [arXiv:1907.10625].

[4] A.G. Riess et al., *A 2.4% Determination of the Local Value of the Hubble Constant*, *Astrophys. J.* **826** (2016) 56 [arXiv:1604.01424].

[5] A.G. Riess et al., *Milky Way Cepheid Standards for Measuring Cosmic Distances and Application to Gaia DR2: Implications for the Hubble Constant*, *Astrophys. J.* **861** (2018) 126 [arXiv:1804.10655].

[6] V. Bonvin et al., *H0LiCOW – V. New COSMOGRAIL time delays of HE 0435−1223: H_0 to 3.8 per cent precision from strong lensing in a flat ΛCDM model*, *Mon. Not. Roy. Astron. Soc.* **465** (2017) 4914 [arXiv:1607.0179].

[7] E. Macaulay, I.K. Wehus and H.K. Eriksen, *Lower Growth Rate from Recent Redshift Space Distortion Measurements than Expected from Planck*, *Phys. Rev. Lett.***111** (2013) 161301 [arXiv:1303.6583].

[8] E. Di Valentino et al., *Cosmology intertwined III: $f\sigma_8$ and S_8*, *Astropart. Phys.* **131** (2021) 102604 [arXiv:2008.11285].

[9] W.L. Freedman, *Cosmology at a Crossroads*, *Nature Astron.***1** (2017) 0121 [arXiv:1706.02739].

[10] E. Di Valentino, O. Mena, S. Pan, L. Visinelli, W. Yang, A. Melchiorri et al., *In the realm of the Hubble tension—a review of solutions*, *Class. Quant. Grav.* **38** (2021) 153001 [arXiv:2103.01183].

[11] J. Solà Peracaula, A. Gómez-Valent, J. de Cruz Perez and C. Moreno-Pulido, *Running vacuum against the H_0 and σ_8 tensions*, *EPL* **134** (2021) 19001 [arXiv:2102.12758].

[12] E. L. D. Perico, J. A. S. Lima, S. Basilakos and J Solà, *Complete cosmic history with a dynamical $\Lambda = \Lambda(t)$ term*, *Phys. Rev. D* **88** (2013) 063531 [arXiv:1306.0591]

[13] J. A. S. Lima, S. Basilakos and J. Solà, *From inflation to dark energy through a dynamical* Λ*: an attempt at alleviating fundamental cosmic puzzles*, *Int. J. Mod. Phys. D* **22** (2013) 1342008 [arXiv:1307.6251]

[14] J. Solà and A. Gomez-Valent, *THE ΛCDM COSMOLOGY: FROM INFLATION TO DARK ENERGY THROUGH RUNNING Λ*, *Int. J. Mod. Phys. D* **24** (2015) 1541003 [arXiv:1501.03832]

[15] N. E. Mavromatos, *Geometrical origins of the Universe dark sector: string-inspired torsion and anomalies as seeds for inflation and dark matter* [arXiv:2108.02152] (2021)

[16] S. Basilakos, N. E. Mavromatos and J. Solà, *Gravitational and Chiral Anomalies in the Running Vacuum Universe and Matter-Antimatter Asymmetry*, *Phys. Rev. D* **101** (2020) 045001 [arXiv:1907.04890]

[17] N. E. Mavromatos and J. Solà, *Stringy-Running-Vacuum-Model Inflation: from primordial Gravitational Waves and stiff Axion Matter to Dynamical Dark Energy*, *Eur.Phys.J.ST* **230** (2021) 2077–2110 [arXiv:2012.07971]

[18] S. H.S Alexander, M. Peskin and M. M. Sheikh-Jabbari, *Leptogenesis from Gravity Waves in Models of Inflation*, *Phys. Rev. Lett* **96** (2006) 081301 [arXiv:hep-th/0403069]

[19] S. H.S Alexander, M. Peskin and M. M. Sheikh-Jabbari, *Gravi-Leptogenesis: Leptogenesis from Gravity Waves in Pseudo-scalar Driven Inflation Models*, [arXiv:hep-ph/0701139] (2006)

[20] S. Weinberg, *The Cosmological Constant Problems* [arXiv:astro-ph/0005265] (2000)

[21] I. L. Shapiro and J. Solà, : *The Scaling Evolution of the Cosmological Constant*, *JHEP* **0202:006** (2002) [arXiv:0012227]

[22] S. Basilakos, M. Plionis and J. Solà, *Hubble expansion and structure formation in time varying vacuum models*, *Phys. Rev. D* **80** (2009) 083511 [arXiv:0907.4555]

[23] R. Bluhm, *Explicit versus Spontaneous Diffeomorphism Breaking in Gravity Phys. Rev. D* **91** (2015) 065034 [arXiv:1401.4515]

[24] S. M. Carroll, *Lecture Notes on General Relativity*, University of California (1997)

[25] D. Baumann, *TASI Lectures on Inflation*, Harvard University (2012)

[26] J. Polchinski, *String Theory. Volume 1, An Introduction to the Bosonic String*, Cambridge University Press (1998)

[27] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory. Vols.1: Introduction, Vols. 2: Loop Amplitudes, Anomalies And Phenomenology*, Cambridge University (1987)

[28] D. Baumann, *TASI Lectures on Inflation*, [`arXiv:0907.5424`] (2012)