ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ

ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ ΚΑΙ ΦΥΣΙΚΩΝ ΕΠΙΣΤΗΜΩΝ

ΣΧΟΛΗ ΜΗΧΑΝΟΛΟΓΩΝ ΜΗΧΑΝΙΚΩΝ

ΕΚΕΦΕ «ΔΗΜΟΚΡΙΤΟΣ»

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Διατμηματικό Πρόγραμμα Μεταπτυχιακών Σπουδών

«Φυσική και Τεχνολογικές Εφαρμογές»

A string-inspired gravitational-field theory with torsion and anomalies, and potential relevance to inflation

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

του Ανδρέα Παπατριανταφύλλου

Επιβλέπων: Νίκος Μαυρόματος

Αθήνα, Φεβρουάριος, 2022

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Antreas Papatriantafyllou

National Technical University of Athens,
Physics Department,
Zografou 157 80, Athens, Greece

Thesis submitted for the Master's Degree
of the Interdepartmental Programme of
Postgraduate Studies
« Physics and Technological Applications»
National Tech. University of Athens and NCSR « Demokritos»
Supervisor: Prof. N.E. Mavromatos

February 2022

To my family,friends and supervising professor

Abstract

In this Thesis I shall attempt to present the main theoretical characteristics of a (3+1)-dimensional string-inspired low-energy effective gravitational theory with Chern-Simons anomalies and an axion field, called

Kalb-Ramond (KR) or string-model-independent axion, which in four dimensions is the dual of the three form associated with the field strength of the spin-one antisymmetric tensor field of the massless gravitational multiplet of the underlying string theory. This field strength plays the role of a totally antisymmetric torsion component. This this theory contains both anomalies and torsion, whose presence is manifested through the emergence of the pseudoscalar KR field, This theoretical framework allows for the quantum fluctuations of the axion field to affect the gravitational field and vise versa. We will construct the effective action of the graviton and axion fields obtained in the low-energy limit of string theories, and explain how the Chern-Simons couplings between the gravity and the axion lead to the aforementioned phenomenon. Then we will try to use this model to show that quantum axion perturbations lead to time-dependent vacuum energy densities for this string-inspired cosmology, which is essential in driving the Universe evolution in a way characterising the so-called "running-vacuum-model" framework of cosmology. We shall extract the form of this running (with the cosmic time) vacuum energy density, and show that it contains terms proportional to H^2 and H^4, where H is the Hubble parameter. We will show that, at the inflationary era, the H^4 term is dominant. Most importantly, we shall demonstrate that the H^4 terms arise from the gravitational anomalies that characterize the theory, as a consequence of the formation of condensates due to primordial gravitational wave perturbations of the space-time. This means that this term lead to inflation in a dynamical scenario whereby the role of the inflaton field is played be an effective intrinsic scalar field (vacuumon), associated with the non-linearities due to the H^4 term of the vacuum energy density.

Θεωρία βαρύτητας με στρέψη καί ανωμαλίες εμπνευσμένη από τη θεωρία χορδών, και πιθανή της σημασία για το φαινόμενο του κοσμικού πληθωρισμού

Ανδρέας Παπατριανταφύλλου

Εθνικό Μετσόβιο Πολυτεχνείο (Ε.Μ.Π.), Ζωγράφου 157 80, Αθήνα

Διατριβή υποβληθείσα για το πυχίο Master
του διατμηματικού προγράμματος
μεταπτυχιακών σπουδών
« Φυσική και Τεχνολογικές Εφαρμογές»
Ε.Μ.Π. και Εθνικό Κέντρο Έρευνας Φυσικών (Ε.Κ.Ε.Φ.Ε.) Επιστημώω
« Δημόκριτος»

Επιβλέπων: Καθ. Ν.Ε. Μαυρόματος

Φεβρουάριος 2022

Στην οικογένειά μου, τους φίλους και τον επιβλέποντα καθηγητή μου

Περίληψη

Σε αυτή τη διατριβή θα προσπαθήσω να παρουσιάσω τα κύρια θεωρητικά χαρακτηριστικά μιας ενεργής (3+1)-διάστατης βαρυτικής θεωρίας χαμηλής ενέργειας βαρύτητας με ανωμαλίες Chern-Simons και ένα πεδίο άξιονίου, που ονομάζεται Kalb-Ramond (KR) ή αξιόνιο ανεξάρτητο από το μοντέλο χορδής, το οποίο σε τέσσερις διαστάσεις είναι δυϊκό της έντασης πεδίου του αντισυμμετρικού τανυστικού πεδίου με spin 1 που υπάρχει στην άμαζη βαρυτική πολλαπλότητα της υποκείμενης θεωρίας χορδών. Αυτή η ένταση πεδίου παίζει το ρόλο ενός πλήρως αντισυμμετρικού στοιχείου στρέψης του χωρόχρονου. Αυτή η θεωρία περιέχει τόσο ανωμαλίες όσο και στρέψη, η παρουσία των οποίων εκδηλώνεται μέσω της εμφάνισης του ψευδοβαθμωτού πεδίου KR. Αυτό το θεωρητικό πλαίσιο επιτρέπει στις κβαντικές διακυμάνσεις του πεδίου του αξιονίου να επηρεάζουν το βαρυτικό πεδίο και το αντίστροφο. Θα κατασκευάσουμε την ενεργή δράση των πεδίων βαρυτονίου και αξιονίου που λαμβάνονται στο όριο χαμηλής

ενέργειας των θεωριών χορδών και θα εξηγήσουμε πώς οι συζεύξεις Chern-Simons μεταξύ της βαρύτητας και του αξιονίου οδηγούν στο προαναφερθέν φαινόμενο. Στη συνέχεια, θα προσπαθήσουμε να χρησιμοποιήσουμε αυτό το μοντέλο για να δείξουμε ότι οι διαταραχές των κβαντικών αξιονίων οδηγούν σε εξαρτώμενες από το χρόνο πυκνότητες ενέργειας κενού για αυτήν την εμπνευσμένη από χορδές κοσμολογία, η οποία είναι απαραίτητη για την οδήγηση της εξέλιξης του Σύμπαντος με τρόπο που χαρακτηρίζει το κοσμολογικό πλαίσιο του λεγόμενου "μοντέλου του τρεχούμενου κενού". Θα εξαγάγουμε τη μορφή αυτής της τρέχουσας (με τον κοσμικό χρόνο) πυκνότητας ενέργειας κενού και θα δείξουμε ότι περιέχει όρους ανάλογους των H^2 και H^4, όπου η H είναι η παράμετρος Hubble. Θα δείξουμε ότι, στην εποχή του πληθωρισμού, ο όρος H^4 είναι κυρίαρχος. Το πιο σημαντικό, όπως θα δείξουμε, είναι ότι οι όροι H^4 προκύπτουν από τις βαρυτικές ανωμαλίες που χαρακτηρίζουν τη θεωρία, ως συνέπεια του σχηματισμού συμπυκνωμάτων λόγω των αρχέγονων διαταραχών των βαρυτικών κυμάτων του χωροχρόνου. Αυτό σημαίνει ότι αυτός ο όρος οδηγεί σε πληθωρισμό σε ένα δυναμικό σενάριο όπου ο ρόλος του πεδίου inflaton παίζεται από ένα ενεργό και εγγενές βαθμωτό πεδίο (vacuumon), που σχετίζεται με τις μη γραμμικότητες λόγω του όρου H^4 της ενεργειακής πυκνότητας κενού.

Contents

1 Introduction and Summary

Over the last two decades a plethora of cosmological observations [1] have changed our perception of the universe.We know now that in our current epoch the universe mostly consists of an unknown form of energy (dark energy) (69% of the current energy budget) whose equation of state is close to that of a spacetime characterized by a positive cosmological constant ($w \simeq -1$).In addition 26% of the Universe's current energy budget consists of dark matter. Ordinary matter constitutes only 5% of the observed Cosmos.The dominance of dark energy results in the acceleration of the universe in later eras while due to the equation of state above leads at a new de Sitter phase (the first having occurred during inflation).

All these cosmological data have been interpreted using the ΛCDM model (cosmological constant λ plus cold dark matter (CDM)).In this model data is fitted to the flat six-parameter canonical version of the ΛCDM model the so-called base ΛCDM [1]. The ΛCDM paradigm is not yet understood at a microscopic level, in that we do not know the microscopic origin of the cosmological constant.

The simplicity of this model, however, and the excellent fit to the plethora of the cosmological data, prompt physicists to accept it and attempt to develop microscopic models that could approximate the λCDM paradigm in the late Universe. However, recently there appear also to be tensions between the Planck Collaboration [1] data concerning the Value of the current-era Hubble parameter $H_0 = 67.27 \pm 0.60$ km/s/Mpc based on cosmic-microwave-background (CMB) observations and fits with the ΛCDM model, and the value $H_0 = 73.24 \pm 1.74$ km/s/Mpc obtained from direct local measurements (e.g. cepheid galaxy measurements [2]). Although such tensions may accept more mundane astrophysical and/or statistical-analysis explanations, nontheless several physics were prompted to look for viable alternatives of the ΛCDM paradigm [3].

So the main question is whether the de sitter phase can be described by this model or we need a time dependent vacuum energy density which can describe better this phase and probably offers a resolution to the observed tensions.

The vacuum energy is probably a result of quantum gravity effects and so the understanding of its microscopic nature will have to wait,until a better theory of quantum gravity is available.Nonetheless,it might be an effective theory (a theory that does not care about the internal interaction of the building blocks of the system but only for the external effects which can be verified by observational data). Such an attempt is the running vacuum model (RVM) of cosmology [4], which are going to analyze in this work. Apart from a smooth cosmological evolution of a Universe with a time-dependent vacuum energy [5], this framework also provides a correct phenomenology in the current era of the Universe [6] and a potential resolution of the cosmic tensions [7]. In [8] a derivation of a RVM from a string-inspired cosmology was given. The model is characterised by gravitational anomalies and torsion, the latter corresponding to a pseudoscalar (axion-like) degree of freedom which couple to the anomaly terms in a non trivial way. This anomalous coupling is crucial, under certain circumstances we shall discuss in detail below, in providing the non-linear terms necessary for the RVM framework to induce inflation in early stages, without inflaton fields. This is the main topic of this dissertation.

The structure of the thesis is as follows. In the next section 2, we review the basic features of the RVM model. In section 3 we discuss the string-inspired anomalous gravitational theory that will be the basic model which will lead to an effective RVM cosmology, In section 4 we discuss some important mathematical properties of the anomalous gravitational theory, related to the anomalous coupling of the axion "matter" to the Chern-Simons gravitational anomaly term. In section 5 we discuss the effects of gravitational waves on the gravitational anomaly Chern-Simons terms, which become non trivial in the presence of the former, while in section 6 we present an alternative method to calculate those terms, using the covariant derivative of the Cotton tensor. In section 7 we compute the condensate of the anomalous terms, which will be crucial for our analysis in the following section 8, where we demonstrate that the above gravitational anomalous system, leads to a running vacuum model cosmology and inflation due to the associated quartic terms in the Hubble parameter, which are dominant in the early Universe and lead to inflation without inflaton fields. Final, section 9 contains our conclusions and outlook.

2 Brief description of the Running Vacuum Model Cosmology

An important feature of this model is its departure from the classical phenomenological framework of cosmology, the LCDM model in which the cosmological constant Λ is truly constant and, as a result, the vacuum energy density is also constant

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}. \tag{1}$$

In contrast, our cosmological theory will be of the type provided in the so-cxalled the running vacuum model (RVM) of the Universe [4, 5], in which Λ and ρ are both time dependent.

The reason is that running vacuum is gravitationaly induced and as we shall show below gravitational anomalies in our theory create this dependence [8]. The main object of this theory is the running vaccuum energy density which, as the name hints, expresses the time dependent energy modification that gravitational anomalies creates with the vaccuum being the mediator.The ρ_{RVM} is given by the renormalised equation as a function of Hubble rate [4] $H = \frac{\dot{a}}{a}$

$$\frac{d\rho_{RVM}}{dlnH^2} = \frac{1}{(4\pi)^2}\sum_i [a_i M_i^2 H^2 + b_i H^4 + c_i \frac{H^6}{M_i^2}] \tag{2}$$

where the right-hand side consists only of even powers of H^2, on account of general covariance (that is, it should depend on terms involving positive integer powers of the Riemann tensor and its contractions, the latter being proportional to H^2). In general there could be dependence on $\dot{H}$ as well, however this quantity can be expressed in terms of the cosmic acceleration $q = -\frac{\ddot{a}\,a}{(\dot{a})^2} = -\frac{\ddot{a}}{a}\,H^{-2}$ during each era

$$\dot{H} = -(1+q^2)\,H^2 \tag{3}$$

Making the approximation, sufficient for the phenomenology in our case, that in each era q is approximately constant, we can then express $\dot{H}$ in terms of H^2 as above. In this way, by integrating (2) we have

$$\rho_{RVM}(H,\dot{H}) = a_0 + a_1\dot{H} + a_2H^2 + a_3\dot{H}^2 + a_4H^4 + a_5\dot{H}H^2 ... \tag{4}$$

In our situation (4) we have $a_1 = a_3 = a_5 = 0$ and so the final equation is of the form [4]

$$\rho_{RVM} = \frac{3}{\kappa^2}(c_0 + vH^2 + \alpha\frac{H^4}{H_I^2}) + \dots \tag{5}$$

By considering a spatially flat FLRW space the main cosmological evolution equation for ρ_{RVM} is provided by [5]

$$\dot{H} + \frac{3}{2}(1+\omega)H(1 - v - \frac{c_0}{H^2} - a\frac{H^2}{H_I^2}) \tag{6}$$

in which $\omega = \frac{\rho_m}{p_m}$ denotes the equation of state of matter and/or radiation.

For all practical purposes, the expansion in power of the Hubble parameter can be truncated to order H^4, as this suffices to describe the cosmology of the entire universe [4, 8]. A solution of (6) is given by

$$H(a) = \left(\frac{1-\nu}{\alpha}\right)^{1/2} \frac{H_I}{\sqrt{D\, a^{3(1-\nu)(1+\omega)} + 1}}\,, \tag{7}$$

where $D > 0$ is an integration constant. It is easy to check from (7) that for $Da^{4(1-\nu)} \ll 1$ the universe starts from an unstable de Sitter era $H^2 = (1-\nu)H_I^2/\alpha$ which is powered by the H^4 term in (5).

The H^4 term is dominant in the early Universe, and as we we will see in the next chapters of this thesis is associated with the coupling between the KR axion field which is the generalization of the matter field and the gravitational anomalies which are expressed in the low energy sting framework as an effective Chern-Simons term. As we shall discuss later on, there is a condensate of anomalies, powered by primordial gravitational-wave perturbations, and the this leads to violation of Lorentz symmetry, via an axion background satisfying $\dot{b} =$ constant. As discussed in [8], such constant axion backgrounds couple to axial fermion currents, and result in CPT violation, which may produce a lepton asymmetry during the radiation era. The H^4 term, due to the anomalies, will lead to inflation in the early de Sitter eralaton, but this inflation will not be induced by an external inflaton field, but through these non-linear quartic terms in the Hubble parameter, which organically hide an effective scalar mode. The situation is reminiscent of Starobinski inflation [9], but it is different from it, as it involves here higher curvature terms that are due to gravitational anomalies. This will be the main point of research in this report.

3 Anomalous string effective action with gravitational anomalies

In the low energy string theory which we use there is three fundimental vibrational modes of a string,a traceless, symmetric ,dimensionless ,spin-2 field $g_{\mu\nu}$ which is the gravitational field a spin-2 scalar gravitational field the dilaton Φ with $g_s = e^{\Phi}$ and the spin-1 antisymmetric tensor field the KR axion field with $B_{\mu\nu} = -B_{\nu\mu}$.the dilaton we will see that plays the central role in the effective theory because it binds the gravitational and the axion field together in an extra dimensional manifold with metric $\tilde{g}_{\mu\nu}$.Spesifically,it alters the gravitational field equation to ($\tilde{g}_{\mu\nu}$,Φ) so the GR field is now expressed in $4+d$ dimensions.Also it appears naturally in the action of the matter axion field.Finally,the dilaton characterizes the size of the new manifold.

Now we are going to extract the effective action.in order to do do that we are going to asume te form of the action of the bosonic part and we are going to calculate the path integrnal integrating over H.Then we are going to reveal a δ functional constraint which is a product of the Bianchi identity.

The action of the bosonic part is:

$$S_B = -\int d^4x\sqrt{-g}(\frac{1}{2\kappa^2}R + \frac{1}{6}H_{\lambda\mu\nu}H^{\lambda\mu\nu}...) \tag{8}$$

It is known [10] that the KR field strength terms $\mathcal{H}^2$ in (8) can be absorbed (up to an irrelevant total divergence) into a contorted generalised curvature $\overline{R}(\overline{\Gamma})$, with a “torsional connection” [11] $\overline{\Gamma}$, corresponding to a contorsion tensor proportional to $\mathcal{H}^{\rho}_{\mu\nu}$ field strength,

$$\overline{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + \frac{\kappa}{\sqrt{3}}\mathcal{H}^{\rho}_{\mu\nu} \neq \overline{\Gamma}^{\rho}_{\nu\mu} \ , \tag{9}$$

where $\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu}$ is the torsion-free Christoffel symbol. As we shall demonstrate below, there is an axion degree of freedom associated via a duality transformation to the torsion [12]. This is the so-called KR axion, or string-model-independent axion [13].

Now I calculate the path integral over H:

$$\int dH \exp iS \tag{10}$$

using the δ functional constraint associated with the Bianchi identity:

$$\Pi_x \delta(\varepsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma}(\chi)_{;\mu} - G(\omega, A)) =$$
$$\int Db \exp[i \int d^4x \sqrt{-g} \frac{1}{\sqrt{3}} b(x)(\varepsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma}(\chi)_{;\mu} - G(\omega, A))] \tag{11}$$

Integrating by parts we obtain:

$$\Pi_\chi \delta(...) = \int Db \exp[-i \int d^4x \sqrt{-g} (\partial^\mu b(x) \frac{1}{\sqrt{3}} \varepsilon_{\mu\nu\rho\sigma} H^{\nu\rho\sigma}$$
$$+ \frac{b(x)}{\sqrt{3}} G(\omega, A)] \tag{12}$$

So the path integral becomes:

$$\int DbdH \exp[-i \int d^4x \sqrt{-g} (\frac{1}{2\kappa^2} R + \frac{1}{6} H_{\lambda\mu\nu} H^{\lambda\mu\nu}$$
$$+ \partial^\mu b(x) \frac{1}{\sqrt{3}} \varepsilon_{\mu\nu\rho\sigma} H^{\nu\rho\sigma} + \frac{b(x)}{\sqrt{3}} G(\omega, A)] \tag{13}$$

We observe that we can complete the square in the integral by adding and subtructing the term $\frac{1}{2}\partial_\mu b \partial^\mu b$ So the path integral becomes:

$$\int DbdH \exp[-i \int d^4x \sqrt{-g} (\frac{1}{2\kappa^2} R + (\frac{1}{\sqrt{6}} \varepsilon_{\mu\nu\rho\sigma} H^{\mu\nu\sigma}$$
$$+ \frac{1}{\sqrt{2}} \partial_\mu b)^2 - \frac{1}{2} \partial_\mu b \partial^\mu b + \frac{b(x)}{\sqrt{3}} G(\omega, A)] \tag{14}$$

The integration with the dH gives a functional Gaussian integral which is not part of the action. Now we are going to replace the tensor notation of the Bianchi constraint, which is:

$$G(\omega, A) = \frac{\alpha'}{32\kappa} (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - F_{\mu\nu} F^{\mu\nu}) \tag{15}$$

In the last equation by changing the constant in order the action to be renormalisable we have that the effective action is:

$$S_B^{eff} = \int d^4x\sqrt{-g}[-\frac{1}{2\kappa^2}R + \sqrt{\frac{2}{3}}\frac{\alpha'}{96\kappa}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - F_{\mu\nu}F^{\mu\nu}) + ...] \quad (16)$$

Were dots denote to gauge and higher derivative terms appearing in the action.And so we see that the KR axion field couples to the gravitational and gauge field.A very important term in this action is the Hirzebruch-Pontryagin term:

$$\sqrt{-g}(\frac{\alpha'}{96\kappa}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - F_{\mu\nu}F^{\mu\nu}) \quad (17)$$

In the model of [8], which we adopt for our purposes here, it is assumed that only fields from the massless gravitational bosonic string multiplet appear as external fields in the early Universe, that is only gravitons, dilatons and antisymmetric tensor KR axions. The dilatons are self consistently assumed to be constant [8], and as such are set to zero, without loss of generality. Hence, from now on we ignore gauge fields in our discussion.

4 Analysis of the properties of Cotton tensor

In this section we are going to extract the form and the basic property of cotton tensor.this is a traceless antisymmetric tensor which is ensensial for the expration of pertubations at a macroscopic level Now we can write the effective action as a sum of 3 actions its of which express the contributions of the field modes that we have studied before.

$$S_B^{eff} = S^{grav} + S^b + S^{b-grav} \quad (18)$$

The S^{grav} express the pure gravity scalar action,the S^b express the action of the KR axion matter field $b(x)$ (ignoring its anomalous gravitational coupling) and

$$S^{b-grav} = -\sqrt{\frac{2}{3}}\frac{\alpha'}{96\kappa}\int d^4x\sqrt{-g}(\partial_\mu b(x)K^\mu)$$
$$= \sqrt{\frac{2}{3}}\frac{\alpha'}{96\kappa}\int d^4x\sqrt{-g}bR_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$$

describes the interactions between the KR axion gravitational anomaly term. Now by calculating the variation of the KR axion term we can extract the Cotton tensor.

$$\delta[\int d^4x\sqrt{-g}bR_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}] = 4\int d^4x\sqrt{-g}C^{\mu\nu}\delta g_{\mu\nu}$$
$$= -4\int d^4x\sqrt{-g}C_{\mu\nu}\delta g^{\mu\nu} \tag{19}$$

Now we have:

$$\delta[\int d^4x\sqrt{-g}bR_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}] = \delta[\int d^4x\sqrt{-g}(\partial_\mu b(x)K^\mu)]$$
$$= \int d^4x(\partial_\mu b(x)\delta(\sqrt{-g})K^\mu)) \tag{20}$$

The $(\sqrt{-g})K^\mu$ term is called Chern-Simons anomaly current term and is equal to:

$$\sqrt{-g}K^\mu = 2\varepsilon^{\mu\alpha\beta\gamma}[\frac{1}{2}\Gamma^\sigma_{\alpha\tau}\partial_\beta\Gamma^\tau_{\gamma\sigma} + \frac{1}{3}\Gamma^\sigma_{\alpha\tau}\Gamma^\tau_{\beta\eta}\Gamma^\eta_{\gamma\sigma}] \tag{21}$$

The variation of this term is:

$$\delta(\sqrt{-g}K^\mu) = 2\varepsilon^{\mu\alpha\beta\gamma}[\delta(\Gamma^\sigma_{\alpha\tau})\partial_\beta\Gamma^\tau_{\gamma\sigma} + \Gamma^\sigma_{\alpha\tau}\delta(\partial_\beta\Gamma^\tau_{\gamma\sigma})$$
$$+ \delta(\Gamma^\sigma_{\alpha\tau})\Gamma^\tau_{\beta\eta}\Gamma^\eta_{\gamma\sigma} + \Gamma^\sigma_{\alpha\tau}\delta(\Gamma^\tau_{\beta\eta})\Gamma^\eta_{\gamma\sigma} + \Gamma^\sigma_{\alpha\tau}\Gamma^\tau_{\beta\eta}\delta(\Gamma^\eta_{\gamma\sigma})] \tag{22}$$

We write the second term as a derivative:

$$\Gamma^\sigma_{\alpha\tau}\delta(\partial_\beta\Gamma^\tau_{\gamma\sigma}) = \partial_\beta(\Gamma^\sigma_{\alpha\tau}\delta(\Gamma^\tau_{\gamma\sigma})) - \delta(\Gamma^\tau_{\gamma\sigma})\partial_\beta(\Gamma^\sigma_{\alpha\tau}).$$

The first term inside the integral gives 0. For the second tern at first we change the silent markers by $\tau \leftrightarrow \sigma$.the non silent markers obeys the rule of cyclic translations so $\gamma \leftrightarrow \alpha$, $\beta \leftrightarrow \gamma$ so the term becomes: $\delta(\Gamma^{\sigma}_{\alpha\tau})\partial_{\gamma}\Gamma^{\tau}_{\beta\sigma}$

For the term $\delta(\Gamma^{\sigma}_{\alpha\tau}\Gamma^{\tau}_{\beta\eta}\Gamma^{\eta}_{\gamma\sigma})$ we keep the first of the terms that we found as it is and in the other two we do the necessary translations in order to extract the variation term as common. Due to the symmetry of those terms we need to multiply with $\frac{1}{2}$ after we do the translations. Hence:

$$\Gamma^{\sigma}_{\alpha\tau}\delta(\Gamma^{\tau}_{\beta\eta})\Gamma^{\eta}_{\gamma\sigma} = -\frac{1}{2}\delta(\Gamma^{\sigma}_{\alpha\tau})\Gamma^{\eta}_{\beta\sigma}\Gamma^{\tau}_{\gamma\eta}$$

in which we did the translations: $\tau \rightarrow \sigma, \sigma \rightarrow \eta, \eta \rightarrow \tau$ for the silent markers and $\alpha \leftrightarrow \beta$ for the non silent.And:

$$\Gamma^{\sigma}_{\alpha\tau}\Gamma^{\tau}_{\beta\eta}\delta(\Gamma^{\eta}_{\gamma\sigma}) = -\frac{1}{2}\delta(\Gamma^{\sigma}_{\alpha\tau})\Gamma^{\eta}_{\beta\sigma}\Gamma^{\tau}_{\gamma\eta}$$

in which we did the translations: $\eta \rightarrow \sigma$, $\sigma \rightarrow \tau$, $\tau \rightarrow \eta$ for the silent markers and $\alpha \leftrightarrow \gamma$ for the non silent. in conclusion the vatiation of the Chern Shimons term becomes equal to the riemman tensor multiply by $\delta(\Gamma^{\sigma}_{\alpha\tau})$.So the initial integral (20) becomes:

$$2\int d^4x \varepsilon^{\mu\alpha\beta\gamma}\partial_{\mu}bR^{\tau}_{\sigma\gamma\beta}\delta(\Gamma^{\sigma}_{\alpha\tau}) \tag{23}$$

The variation of the Christoffel symbol is given as:

$$\delta\Gamma^{\sigma}_{\alpha\tau} = \frac{g^{\sigma\nu}}{2}(D_{\alpha}\delta g_{\nu\tau} + D_{\tau}\delta g_{\nu\alpha} - D_{\nu}\delta g_{\alpha\tau})$$

so the integral becomes:

$$\int d^4\varepsilon^{\mu\alpha\beta\gamma}\partial_{\mu}bR^{\tau\nu}_{\gamma\beta}(D_{\alpha}\delta g_{\nu\tau} + D_{\tau}\delta g_{\nu\alpha} - D_{\nu}\delta g_{\alpha\tau})$$

The Riemann tensor is antisymmetric in $[\tau,\nu]$ and because $\delta g_{\nu\tau}$ is symmetric their tensor product gives 0.for the last 2 term we do the necessary translations (we do $\tau \leftrightarrow \nu$ on the Riemann and the third term and then because Riemann is antisymmetric gives a "-" and also g is symmetric) and we combine them.

$$\int d^4x\sqrt{-g}\varepsilon^{\mu\alpha\beta\gamma}\partial_\mu b R^{\tau\nu}_{\gamma\beta} D_\tau \delta g_{\nu\alpha}$$

Now we integrate by parts to obtain:

$$\int d^4x(\varepsilon^{\mu\alpha\beta\gamma}\partial_\mu b D_\tau R^{\tau\nu}_{\gamma\beta} + \partial_{\mu\tau} b\varepsilon^{\mu\alpha\beta\gamma} R^{\tau\nu}_{\gamma\beta})\delta g_{\nu\alpha}$$

Finally by using the Bianchi identity $D_\tau R^{\tau\nu}_{\gamma\beta} = D_\gamma R^\nu_\beta - D_\beta R^\nu_\gamma$ and the definition of the dual Riemann tensor$\tilde{R}^{\tau\mu\nu\alpha} = \frac{1}{2}\varepsilon^{\mu\alpha\beta\gamma}R^{\tau\nu}_{\gamma\beta}$ and by doing the necessary translation after the combination with ε (we do β↔γ and then we have an odd translation on ε so it gives a "-" and the two terms becomes the same) so we end up with:

$$-2\int d^4x(\varepsilon^{\mu\alpha\beta\gamma}\partial_\mu b D_\gamma R^\nu_\beta + \partial_{\mu\tau} b\tilde{R}^{\tau\mu\nu\alpha})\delta g_{\nu\alpha}$$

because we want the cotton tensor to be symmetric under [μ,ν] we do the same process but instead of μ from the beginning we have ν and then we simply add all the terms. So the final form of the cotton tensor is:

$$C^{\mu\nu} = -\frac{1}{2}[u_\sigma(\varepsilon^{\sigma\mu\alpha\beta}R^\nu_{\beta;\alpha} + \varepsilon^{\sigma\nu\alpha\beta}R^\mu_{\beta;\alpha} + u_{\sigma\tau}(\tilde{R}^{\tau\mu\sigma\nu} + \tilde{R}^{\tau\nu\sigma\mu}] \tag{24}$$

Another important property we need to discuss is the apparent breaking of diffeomorfism invariance in the presence of gravitational anomalies, which will manifest itself as a failure of the covariant conservation law for the stress tensor of matter in spacetime backgrounds characterised by a non trivial Cotton tensor (such as the ones with gravitational waves).

In order to do that we have to take the covariant derivative of cotton tensor in the form:

$$C^{\mu\nu} = -\frac{1}{2}[D_\lambda(u_\sigma\tilde{R}^{\lambda\mu\sigma\nu}) + D_\lambda(u_\sigma\tilde{R}^{\lambda\nu\sigma\mu})] \tag{25}$$

And the derivative is:

$$D_\mu C^{\mu\nu} = -\frac{1}{2}(u_{\mu\sigma\lambda}\tilde{R}^{\lambda\mu\sigma\nu} + u_{\mu\sigma\lambda}\tilde{R}^{\lambda\nu\sigma\mu} +$$
$$u_{\sigma\lambda}D_\mu\tilde{R}^{\lambda\mu\sigma\nu} + u_{\sigma\lambda}D_\mu\tilde{R}^{\lambda\nu\sigma\mu} + u_\sigma D_{\mu\lambda}\tilde{R}^{\lambda\mu\sigma\nu} + u_\sigma D_{\mu\lambda}\tilde{R}^{\lambda\nu\sigma\mu} +$$
$$u_{\mu\sigma}D_\lambda\tilde{R}^{\lambda\mu\sigma\nu} + u_{\mu\sigma}D_\lambda\tilde{R}^{\lambda\nu\sigma\mu})$$

Now by regrouping the derivatives ans by considering that [λ,μ] antisymmetric so that $[D_\lambda, D_\mu] = 2D_{\mu\lambda}(u_\sigma\tilde{R}^{\lambda\mu\sigma\nu})$ and the other is $D_\mu D_\lambda(u_\sigma\tilde{R}^{\lambda\nu\sigma\mu}) = [D_\lambda, D_\mu]u_\sigma\tilde{R}^{\lambda\nu\sigma\mu} - D_\lambda D_\mu(u_\sigma\tilde{R}^{\lambda\nu\sigma\mu})$ So the derivative of cotton ends up as follows:

$$D_\mu C^{\mu\nu} = \frac{1}{2}(-D_\lambda D_\mu u_\sigma\tilde{R}^{\lambda\nu\sigma\mu} + [D_\lambda, D_\mu](u_\sigma\tilde{R}^{\lambda\nu\sigma\mu} + \frac{1}{2}u_\sigma\tilde{R}^{\lambda\mu\sigma\nu})) \quad (26)$$

The first term becomes:

$$D_\mu u_\sigma\tilde{R}^{\lambda\nu\sigma\mu} = u_{\sigma\mu}\tilde{R}^{\tau\nu\sigma\mu} + u_\sigma\varepsilon^{\sigma\alpha\beta\gamma}D_\tau R^{\tau\nu}_{\alpha\beta}$$

For this the first term is 0 because is the product of a symmetric and antisymmetric tensor and the other is also 0 due to the bianchi indentity. So the remainder of the cotton derivative becomes by using the formula of the communicator of covariant derivatives of the riemman tensor which is:

$$[D_\lambda, D_\mu]\tilde{R}^{\lambda\nu\sigma\mu} = R^\lambda_{\tau\lambda\mu}\tilde{R}^{\tau\nu\sigma\mu} + R^\nu_{\tau\lambda\mu}\tilde{R}^{\lambda\tau\sigma\mu} + R^\mu_{\tau\lambda\mu}\tilde{R}^{\lambda\nu\sigma\tau}$$

And after we do the same job for the other we have:

$$D_\mu C^{\mu\nu} = u_\sigma[(\tilde{R}^{\tau\nu\sigma\mu} + \frac{1}{2}\tilde{R}^{\tau\mu\sigma\nu})R^\lambda_{\tau\lambda\mu} + R^\nu_{\tau\lambda\mu}\tilde{R}^{\lambda\tau\sigma\mu} +$$
$$R^\mu_{\tau\lambda\mu}\tilde{R}^{\lambda\nu\sigma\tau} + \frac{1}{2}R^\mu_{\tau\mu\lambda}\tilde{R}^{\lambda\mu\sigma\nu} + \frac{1}{2}R^\nu_{\tau\mu\lambda}\tilde{R}^{\lambda\mu\sigma\tau}$$

Now we do the nesesary contractions in order to reveal the Ricci tensor so:

$$D_\mu C^{\mu\nu} = \frac{1}{2}u_\sigma[-(\tilde{R}^{\tau\nu\sigma\mu} + \frac{1}{2}\tilde{R}^{\tau\mu\sigma\nu})R_{\tau\mu} + (\tilde{R}^{\lambda\nu\sigma\tau}$$
$$+ \frac{1}{2}\tilde{R}^{\lambda\tau\sigma\nu})R_{\tau\lambda} + (\tilde{R}^{\lambda\tau\sigma\mu} + \frac{1}{2}\tilde{R}^{\lambda\mu\sigma\tau})R^\nu_{\tau\mu\lambda}]$$

because the Ricci tensor is symmetric and Riemmann antisymmetric the product gives 0 and the two first terms vanishes.for the last term the first Riemmann is antisymmetric on the $[\lambda, \tau]$ so we can write that $\tilde{R}^{\lambda\tau\sigma\mu} = \frac{1}{2}(\tilde{R}^{\lambda\tau\sigma\mu} - \tilde{R}^{\tau\lambda\sigma\mu})$ So we have:

$$D_\mu C^{\mu\nu} = \frac{1}{4}u_\sigma[\tilde{R}^{\lambda\tau\sigma\mu}R^\nu_{\tau\mu\lambda} - \tilde{R}^{\tau\lambda\sigma\mu}R^\nu_{\tau\mu\lambda} + \tilde{R}^{\lambda\mu\sigma\tau}R^\nu_{\tau\mu\lambda}]$$

In the second term we do the translation $\tau <-> \lambda$ so we have:

$$D_\mu C^{\mu\nu} = \frac{1}{4}u_\sigma[\tilde{R}^{\lambda\tau\sigma\mu}(R^\nu_{\tau\mu\lambda} - R^\nu_{\lambda\mu\tau}) + \tilde{R}^{\lambda\mu\sigma\tau}R^\nu_{\tau\mu\lambda}]$$

Now by using the Bianchi identity $R^\nu_{\tau\mu\lambda} - R^\nu_{\lambda\mu\tau} - R^\nu_{\mu\lambda\tau} = 0$ we have that:

$$D_\mu C^{\mu\nu} = \frac{1}{4}u_\sigma[\tilde{R}^{\lambda\tau\sigma\mu}R^\nu_{\mu\lambda\tau} + \tilde{R}^{\lambda\mu\sigma\tau}R^\nu_{\tau\mu\lambda}]$$

Now by using the identity $\tilde{R}^{\lambda\tau\sigma\mu}R^\nu_{\mu\lambda\tau} = \frac{1}{4}\delta^\sigma_\nu\tilde{R}R$ we end up with:

$$D_\mu C^{\mu\nu} = \frac{1}{8}u_\nu\tilde{R}R\,, \tag{27}$$

where we remind the reader $u_\mu = \partial_\mu b$, is the derivative of the axion field.

This shows us that the Cotton tensor is not covariantly conserved, and, as a result, affects the diffeomorphism invariance, since as we shall demonstrate below the property (27) implies non conservation of the matter stress-energy tensor, something that should not happened in GR.

One would think that, in order to solve that problem, we should work only in specific gravitational backgrounds (specific manifolds) like the FLRW space-time with axion field b(t) in which, in the non perturbed case, the Cotton tensor vanishes. However we can in general make sense of gravitational anomalies,such as, for instance, the cosmology with gravitational waves, as follows: in order for the Einstein equation to be diffeomorphism invariant we need to modify the stress energy tensor of the axion field by including the Cotton tensor, as expressing an exchange of energy between the matter (axion) and the gravitational sector. So:$T^{\mu\nu} \Rightarrow T^{\mu\nu}_{b+\Lambda+gCS}$ and

$$\kappa^2 \tilde{T}^{\mu\nu}_{b+\Lambda+gCS} = \sqrt{\frac{2}{3}} \frac{\alpha\kappa}{12} C^{\mu\nu} + \kappa^2 T^{\mu\nu}_b + \Lambda g^{\mu\nu}. \tag{28}$$

The Einstein equations in the presence of gravitational anomalies (that is in the presence of a non trivial $C_{\mu\nu} \neq 0$, are then given

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 \tilde{T}^{\mu\nu}_{b+\Lambda+gCS} \tag{29}$$

This is consistent with the Bianchi identity of the Einstein tensor, $\Big(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\Big)_{;\nu} = 0$, from which we obtain the generalised conservation

$$\tilde{T}^{\mu\nu}_{b+\Lambda+gCS;\mu} = 0\,. \tag{30}$$

In this sense, the apparent failure of non conservation of the matter stress tensor in the presence of gravitational anomalies does not signal any fundamental breaking of diffeomorphism invariance, nor requires the selection of only specific metric backgrounds without this anomalies. As mentioned previously, the presence of anomalies signifies the exchange of energy between axion matter and the gravitational environment, and we can absorb this into the modified stress-energy tensor (28) in order to leave the gravity field invariant.As we are going to see later if we have quantum anomalies which stems from the axion field those modify the gravitational field and not the stress-energy tensor and so the invariance breaks and so they effect in in the specific cosmological era which they act(inflationary era).

5 Calculation of the effect of Chiral anomalies in the FLRW spacetime due to gravitational waves

In this section we will explain the mechanism that creates inflation due to the effect of gravitational waves of primordial black holes. this gravitational waves affects the axion field and especially the chirality which creates anomalies at the initial CP violation through the polarization of GW. The backround spacetime is the FRW with metric $dS^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$ and the metric of the GW is

$$ds^2 = dt^2 - a^2[(1 - h_+(t,z))dx^2 + (1 + h_+(t,z))dy^2 + \\ 2h_x(t,z)dxdy + dz^2 \tag{31}$$

The total metric is the linear comdination of this two metrics thus $g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$ and $g^{\mu\nu} = g^{\mu\nu}_0 - h^{\mu\nu}$.Now we want to calculate the term $R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$.this is the term of the effective action that express the Hirzebruch-Pontryagin density of the axion field which as we will see it going to birth the inflation term.for the total riemman tensor we have:

$$R_{\mu\nu\rho\sigma} = R^0_{\mu\nu\rho\sigma} + R^{gw}_{\mu\nu\rho\sigma}$$

so the $R\tilde{R}$ becomes:

$$R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} = R^0_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}_0 + R^{gw}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}_0 \\ + R^0_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}_{gw} + R^{gw}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}_{gw} \tag{32}$$

the first term is 0 because it express purely the FRW and this is unpertubed.For the rest terms by using the expression for the dual Riemmann and by doing the usual transformations with the metric in order to bring the Riemmann tensor components in the form that we are going to calculate them we have:

$$R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} = \frac{1}{2\sqrt{-g}}[R^0_{\mu\nu\rho\sigma}\varepsilon^{\rho\sigma\pi\lambda}h^{\kappa\nu}R^{\mu}_{gm_{\kappa\pi\lambda}}$$
$$+ R^0_{i\kappa\pi\lambda}g_0^{\kappa\nu}g_0^{\mu i}\varepsilon^{\rho\sigma\pi\lambda}h_{\mu\zeta}R^{\zeta}_{gm_{\nu\rho\sigma}}$$
$$+ h_{\mu\zeta}R^{\zeta}_{gm_{\nu\rho\sigma}}\varepsilon^{\rho\sigma\pi\lambda}h^{\kappa\nu}R^{\mu}_{gm_{\kappa\pi\lambda}}] \tag{33}$$

The term $\frac{1}{2\sqrt{-g}}$ comes from the need the ε be covariant. Now we need to calculate the Christoffel symbols.for the FRW spacetime we can take them directly from the bibliography so: $\Gamma^t_{ij} = \alpha^2 H\delta_{ij}$,$\Gamma^i_{jt} = \Gamma^i_{tj} = H\delta^i_j$,$\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}$ and the Riemman tensors are: $Ritjt = -(\dot{H} + H^2)\alpha^2\delta_{ij}$,$R_{ijkl} = \alpha^4 H^2(\delta_{ik}\delta_{lj} - \delta_{il}\delta_{kj})$ were $i, j, k, l = x, y, z$

Now for the GW metric the cristoffel symbols are more complicated.so we have: $\Gamma^t_{tt} = \Gamma^t_{ti} = 0$,$\Gamma^t_{xx} = \frac{1}{2}\partial_t a^2 - \frac{1}{2}a^2\partial_t h_+ - \frac{1}{2}\partial_t a^2 h_+$,$\Gamma^t_{yy} = \frac{1}{2}\partial_t a^2 + \frac{1}{2}a^2\partial_t h_+ + \frac{1}{2}\partial_t a^2 h_+$,$\Gamma^t_{xy} = \Gamma^t_{yx} = \partial_t a^2 h_x + a^2\partial_t h_x$,$\Gamma^t_{zz} = \frac{1}{2}\partial_t a^2$.

For Γ^{χ}_{ij} we have:$\Gamma^{\chi}_{\chi\chi} = \Gamma^{\chi}_{xy} = \Gamma^{\chi}_{yy} = \Gamma^{\chi}_{zz} = \Gamma^{\chi}_{tt} = 0$.

For Γ^{χ}_{ti} we have:$\Gamma^{\chi}_{tx} = -\frac{1}{2}a^2\partial_t a^2 h_+ + \frac{1}{2}a^2\partial_t a^2 + \frac{1}{2}a^4\partial_t h_+ - 2a^2 h_x^2\partial_t a^2 - a^4\partial_t h_x^2 + \frac{1}{2}a^2\partial_t a^2 h_+^2 - a^4\partial_t\partial_t h_+^2$.

$\Gamma^{\chi}_{ty} = -2a^2\partial_t a^2 h_x - a^4\partial_t h_x + a^4 h_+\partial_t h_x - a^2 h_x\partial_t h+$.

For Γ^x_{iz} we have

$\Gamma^x_{yz} = -a^4\partial_z h_x + a^4 h_+\partial_z h_x + a^4 H_x\partial_z H_+$
$\Gamma^x_{xz} = \frac{1}{2}a^4\partial_z h+ - \frac{1}{2}a^4 h_+\partial_z h_+ - 2a^4 h_x\partial_z h_x$

For Γ^y_{ij} we have:$\Gamma^y_{\chi\chi} = \Gamma^y_{xy} = \Gamma^y_{yy} = \Gamma^y_{zz} = \Gamma^y_{tt} = 0$

For Γ^y_{ti} we have:$\Gamma^y_{yt} = -\frac{1}{2}a^2\partial_t a^2 h_+ - \frac{1}{2}a^2\partial_t a^2 - \frac{1}{2}a^4\partial_t h_+ - \frac{1}{2}a^2\partial_t a^2 h_t^2 - -\frac{1}{2}a^4 h_+\partial_t h+ - 2a^2\partial_t a^2 h_x^2 - a^4\partial_t h_x^2$

$\Gamma^{y}_{xt} = -2a^2\partial_t a^2 h_x - a^4\partial_t h_x - a^4 h_+\partial_t h_x + a^4 h_x\partial_t h_+.$

$\Gamma^{y}_{yz} = -\frac{1}{2}a^4\partial_z h_+ - \frac{1}{2}a^4 h_+\partial_z h_+ + 2a^4 h_x\partial_z h_x.$

$\Gamma^{y}_{xz} = -a^4\partial_z h_x - a^4 h_+\partial_z h_x - a^4 h_x\partial_z h_+.$

$\Gamma^{z}_{ij} = 0$ for $i, j = t, x, y, z$ accept $\Gamma^{z}_{zt} = -\frac{1}{2}a^2\partial_t a^2.$

Now by expressing the polarizasions of gw in the chiral graviton basis $h_+ = \frac{1}{\sqrt{2}}(h_L + h_R)$, $h_+ = \frac{1}{\sqrt{2}}i(h_L - h_R)$ in the symbols and substitute those to the mixed riemmanian tensor in the equation(30) we have an expression with first order and higher orde terms for h_L andh_R.We keep only the first order terms because those are the ones that contribute more so the final expration of equation (32) is:

$$R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} = 4i\frac{1}{a^3}[\partial_z^2 h_R\partial_t\partial_z h_L + a^2\partial_t^2 h_R\partial_t\partial_z h_L + \frac{1}{2}\partial_t^2(a^2)\partial_t h_R\partial_t\partial_z h_L - \partial_z^2 h_L\partial_t\partial_z h_R - a^2\partial_t^2 h_L\partial_t\partial_z h_R - \frac{1}{2}\partial_t^2(a^2)\partial_t h_L\partial_t\partial_z h_R \tag{34}$$

which we shall make use of later on when we evaluate the gravitational-wave-induced condensate.

6 Conformal method of extaction of $R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$

Another way to extract the above result is by using the equation of GR and exploit the fact that field perturbation can be expressed inside it through the Cotton tensor. Someone could say that this is not possible because this is a macroscopic equation of motion and we add quantum fluctuations of the axion field. The reason that this method is valid, hoever, is that, even if we used the action at first in order to calculate the Cotton tensor, the latter is not a field term. It is a product of the variation of the Chern-Simons term, which is a string theory term. As a result, the Cotton tensor can be expressed with the same ease in terms of both gravitational and quantum fluctuations due to the unification in the string theory level. Now we come to the GR equation and substitute the total Ricci tensor $R^{\mu\nu} = R_0^{\mu\nu} + R_{gw}^{\mu\nu}$ and the Ricci scalar by

$$R = g_{\mu\nu}R^{\mu\nu} = g^0_{\mu\nu}R_0^{\mu\nu} + g^0_{\mu\nu}R_{gw}^{\mu\nu} + h_{\mu\nu}R_0^{\mu\nu} + h_{\mu\nu}R_{gw}^{\mu\nu}$$

so the equation of GR is:

$$\begin{aligned} R_0^{\mu\nu} + R_{gw}^{\mu\nu} + (g_0^{\mu\nu} - h^{\mu\nu})(g^0_{\mu\nu}R_0^{\mu\nu} + g^0_{\mu\nu}R_{gw}^{\mu\nu} + h_{\mu\nu}R_0^{\mu\nu} + \\ h_{\mu\nu}R_{gw}^{\mu\nu}) = \Lambda g^{\mu\nu} + \sqrt{\frac{2}{3}}\frac{a'\kappa}{12}C^{\mu\nu} + \kappa^2 T^{\mu}_{matter} \end{aligned} \tag{35}$$

and from that we find the covariant derivative. Here instead of modifying the stress tensor we leave it as is, and though the covariant derivative the terms that express purely the FLRW space-time background give zero, while the rest are used equation (26). The $b(t)$ term is extracted by the Chern Simons term and expresses the graviton quantum fluctuation in the axion field.The equation of this term is given below (*cf.* (36), but we are going to extract it property in later sections. The variation of this term is the energy density of the axion field.

In order to calculate the

$$\dot{b}(t) = \sqrt{\frac{2}{3}}\frac{a'}{96\kappa}K^0 \tag{36}$$

we use the expression of the Chern-Simons term (21). We need to be careful that we do not want h-dependent terms in that quantity, because it is a macroscopic one and h are quantum, which exhibit fully space-time dependence, and we only need t dependence for a description of isotropic and homogeneous quantities.

This method in my opinion is much more suitable to show the special characteristics of the effective field duo to the breaking of the differential invariance.the macroscopic quantities that the quantum fluctuations induced (which part of it is inflation)are characteristics of this and end only this cosmological era because we cannot do any conformal transformation in order to go to another.

7 Calculation of $< R\tilde{R} >$

In order to calculate $< R\tilde{R} >$ we need to find the eq of motion of the of the chiral fluctuations. This can be done by solving the Einstein action for the total metric that we have.this given by the equations:

$$\Delta h_L = -2i\frac{\Theta}{\alpha}\dot{h_L}', \qquad \Delta h_R = 2i\frac{\Theta}{\alpha}\dot{h_R}'.$$

The Θ parameter is associated with the anomalous interactions and we are going to see is the reason we the mean value isn't 0 something that would have happened if the perturbations was symmetric.

We transform the t to the proper time trough the relation $\eta = \frac{\exp -Ht}{H}$ $,\frac{1}{dt} = a(t)\frac{1}{d\eta}$.We can see that proper time and cosmic time t have oposite sings so when the cosmic time increase the proper decrease.this means that now we have actualy a countdown which becomes 0 when t is near the infinite future and express the end of the action of the $< R\tilde{R} >$.

Now we have for the h_L(similar for the other) that:

$$\frac{d^2}{d\eta^2}h_L - 2\frac{1}{\eta}\frac{d}{d\eta}h_L - \frac{d^2}{dz^2}h_L = -2i\Theta\frac{d^2}{d\eta dz}h_L \tag{37}$$

for $\Theta = 0$(something we are going to need later) we have that the positive frequency solution:

$$h_L^+(k,\eta) = \exp ik(\eta + z)(1 - ik\eta) \tag{38}$$

and the negative is the complex conjugate of the positive.For the general solution we have that:

$$h_L = \exp ikz(-ik\eta)\exp(k\Theta\eta)g(\eta) \tag{39}$$

The function $g(\eta)$ satisfies the equation

$$\frac{d^2}{d\eta^2}g + [k^2(1-\Theta^2) - \frac{2}{\eta^2} - \frac{2k\Theta}{\eta}]g = 0$$

this gives:

$$g(\eta) = exp[ik(1-\Theta^2)\eta(1+\alpha(\eta))] \tag{40}$$

The h_R relations are the complex cojugate of h_L so

$$h_R = \exp -ikz(ik\eta)\exp(k\Theta\eta)g^*(\eta) \tag{41}$$

$h_L^+ = h_R^-$, $h_L^- = h_R^+$ Now we are going to insert those values in $R\tilde{R}$ after we do the proper time transformation so:

$$\begin{aligned}
&R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} = 4i\frac{1}{a^3}[a\partial_z^2 h_R\partial_\eta\partial_z h_L + a^5\partial_\eta^2 h_R\partial_\eta\partial_z h_L + \\
&a^4\frac{1}{2}\partial_\eta^2(a^2)\partial_\eta h_R\partial_\eta\partial_z h_L - a\partial_z^2 h_L\partial_\eta\partial_z h_R - a^5\partial_\eta^2 h_L\partial_\eta\partial_z h_R - \\
&a^4\frac{1}{2}\partial_\eta^2(a^2)\partial_\eta h_L\partial_\eta\partial_z h_R]
\end{aligned} \tag{42}$$

After the substitution (38),(39),(40) we have:

$$\begin{aligned}
R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} &= 4i\frac{1}{a^3}[[-ik^3 a(F^*(\eta) - F(\eta)) + ika^5((|F(\eta)|^2)' \\
&+ (F^*(\eta) - F(\eta))|F(\eta)|^2) + ika^4(\dot{a}^2 + a\ddot{a})|F(\eta)|^2]h_R h_L]
\end{aligned} \tag{43}$$

where $F(\eta) = (\frac{1}{\eta} + k\Theta + ik(1-\Theta^2)(1+a+\dot{a})$ and $F(\eta)' = (-\frac{1}{\eta^2} + ik(1-\Theta^2)(\dot{a}+\ddot{a})$ and the complex conjugates respectively.

Now as we can see the computation of the mean value is about the $< h_L h_R >$. in this problem we actually want to compute the mean value in relation of the spacial coordinates and live the time coordinate free.this because we want the general macroscopic effect of the anomalous spacial part of the perturbation and at the same time to be able to see how this term will evolve through time.Mathematically the reason that we can separate the spacial from the time part is that the coordinates are not entangled .As a result we can see the term as the product of a spacial and a time function. For the calculation of the term we use the Green's function and after we do a Fourier transformation. We thus have:

$$G(x,t;x't') =< h_L(x,t)h_R(x',t') >$$
$$= \int \frac{d^3k}{(2\pi)^3} e^{ik(x-x')} G_k(\eta,\eta') \tag{44}$$

The Green's function (44) satisfies the equation:

$$[\frac{d^2}{d\eta^2} - 2(\frac{1}{\eta} + k\Theta)\frac{d}{d\eta} - k^2]G_k(\eta,\eta') = -i\frac{(H\eta)^2}{M_{PI}^2}\delta(\eta-\eta')$$

The general solution has an exponential dependence over Θ so :

$$G_k = e^{-k\Theta\eta} G_{k0} e^{k\Theta\eta} \tag{45}$$

the G_{k0} is the solution for $\Theta = 0$ which is:

$$G_{k0}(\eta,\eta') = (\frac{H^2}{2k^3M_{PI}^2})h_L^+(k,\eta)h_R^-(-k,\eta') \text{ for } \eta < \eta'$$
$$G_{k0}(\eta,\eta') = (\frac{H^2}{2k^3M_{PI}^2})h_L^-(k,\eta)h_R^+(-k,\eta') \text{ for } \eta' < \eta \tag{46}$$

Were $h_L^+(k,\eta) = h_R^-(k,\eta)$ and $h_R^+(k,\eta)+ = h_L^-(k,\eta)$ because the h_R^+ need to have positive frequency and the frequency of the h_R equation has opposite sign so the initial positive solution due to $-k$ becomes negative and vise versa.This means $h_L^+(k,\eta) = h_R^+(-k,\eta)$ and $h_L^-(k,\eta) = h_R^-(-k,\eta)$ even if they are conjugate relation between them

So the solution of (45) is:

$$G_{k0}(\eta,\eta') = (\frac{H^2}{2k^3M_{PI}^2})e^{-ik(\eta'-\eta)}(1-ik\eta)(1+ik\eta') \text{ for } \eta < \eta' \tag{47}$$

$$G_{k0}(\eta,\eta') = (\frac{H^2}{2k^3M_{PI}^2})e^{-ik(\eta-\eta')}(1+ik\eta)(1-ik\eta') \text{ for } \eta' < \eta. \tag{48}$$

Now we put the the G_k in the Green's function (43) and then in equation (42) and after taking the $\langle \dots \rangle$ and doing the calculation using mathematica, we find a relation with a first order term in Θ as well as higher-order (cubic and higher) terms of Θ. We keep the first order, as we asre dealing with weak gravitational-wave perturbations, so we finally have:

$$< R\tilde{R} > \frac{16}{a^4}\int \frac{d^3k}{(2\pi)^3}\frac{H^2}{2k^3M_{PI}^2}k^4\Theta(\eta) + O(\Theta^3) \tag{49}$$

Were Θ is:

$$\Theta = \frac{2}{3}\frac{a'\kappa}{12}H\dot{\tilde{b}}(\eta) \tag{50}$$

We shall use now the above result to extra a running vacuum model behaviour for this Universe, in the phase where there exist condensation of gravitational waves, that in turn produce the condensate (49) and induce inflation without the need for inflaton fields.

8 Extraction of the running vacuum energy density and inflation coefficients

The connection of the above approach to the running vacuum model can be achieved by demonstrating that the following total energy vacuum density :

$$\rho_{RVM} = \rho^b + \rho^{gCS} + \rho^{\Lambda} \tag{51}$$

acquires a running vacuum model form, that is the form (5).

In (51), the ρ^b is the energy density of the axion field ,the ρ^{gCS} is the density due to the KR axion gravitational term which expresses the coupling of the axion field with the gravitational anomalous terms and ρ^{Λ} denotes the pure gravitational contribution associated with the condensate (48). This latter term is the dominant de-Sitter contribution which induces and sets the scale of inflation. Let us see how we achoeve this.

Our starting point is the effective action for the KR axion and graviton fields in the presence of a condensate:

$$\begin{aligned} S_B^{\rm eff} = & \int d^4x \sqrt{-g}\Big[-\frac{1}{2\kappa^2}\, R + \frac{1}{2}\, \partial_\mu b\, \partial^\mu b \\ & + \sqrt{\frac{2}{3}}\, \frac{\alpha^\prime}{96\, \kappa}\, b(x)\, R_{\mu\nu\rho\sigma}\, \widetilde{R}^{\mu\nu\rho\sigma} + \dots \Big] \\ = & \int d^4x \sqrt{-g}\Big[-\frac{1}{2\kappa^2}\, R + \frac{1}{2}\, \partial_\mu b\, \partial^\mu b \\ & - \sqrt{\frac{2}{3}}\, \frac{\alpha^\prime}{96\, \kappa}\, K^\mu(\omega)\, \partial_\mu b(x) + \dots \Big], \end{aligned} \tag{52}$$

where in the second line we used the anomaly equation, which expresses the Chern-Simons anomaly as the gravitational covariant derivative of the anomaly current K (21).

The axion equation of motion obtained from (52) yields

$$\frac{1}{\sqrt{-g}}\, \partial_\mu\Big(\sqrt{-g}\, \Big[\partial^\mu b - \sqrt{\frac{2}{3}}\frac{\alpha^\prime}{96\kappa}\, K^\mu\Big]\Big) = 0 \tag{53}$$

admits as an isotropic and homogeneous solution (which had been prennounced in (36))

$$\dot{b}(t) = \sqrt{\frac{2}{3}}\, \frac{\alpha^\prime}{96\kappa} K^0, \tag{54}$$

assuming that the dominant components of the anomaly current in a FLRW background are the temporal ones.

Writing the anomaly equation (21) for the condensate under the assumption of homogeneity and isotropy of the FLRW Universe, as

$$\frac{d}{dt} K^0 + 3H\, K^0 =< R_{\mu\nu\rho\sigma}\, \widetilde{R}^{\mu\nu\rho\sigma} >= \text{constant}, \tag{55}$$

which will thus play the rôle of a slow-roll parameter.

Finally, contrary to the conventional wisdom, it is not the axion field b that drives inflation, but the non-linearities of the total energy density of the vacuum (51) in terms of the Hubble parameter, in particular its quartic term H^4 due to the gravitational anomalies condensate, which we next proceed to demonstrate.

Let us start by calculating the density of the axion field

$$\rho^b = \frac{1}{2}(\dot{\tilde{b}})^2 \tag{56}$$

. this satisfying a stiff equation of state

$$p^b = \rho^b \tag{57}$$

To this end, we need to calculate the Chern Simons term for time dependence. This is worked out in the paper [], and below we shall sketch the most basic features and results for the analysis that will be crucial in our approach.

To leading order in the small parameter Θ, the computation of the condensate, using an UV cutoff for the momentum modes μ yields:

$$\begin{aligned}\langle R_{\mu\nu\rho\sigma}\, \widetilde{R}^{\mu\nu\rho\sigma}\rangle &= \frac{1}{\pi^2}\Big(\frac{H}{M_{\rm Pl}}\Big)^2\, \mu^4\, \Theta \\ &= \frac{2}{3\pi^2}\frac{1}{96\times 12}\,\Big(\frac{H}{M_{\rm Pl}}\Big)^3\,\Big(\frac{\mu}{M_s}\Big)^4\, M_{\rm Pl}\,\times\; K^0(t).\end{aligned} \tag{58}$$

In which we use the eq (48) for Θ. Above, $M_s = 1/\sqrt{\alpha'}$ is the string scale. We the need large string mass scales, near the Planck scale, such that $\alpha' H^2 << 1$ during inflation in this model.

Using this result in the anomaly-current evolution equation (55), we easily obtain:

$$\begin{aligned}&\frac{d}{dt}\Big(\sqrt{-g}\, K^0(t(\eta))\Big) = -(\eta H)\,\frac{d}{d\eta}\Big(\sqrt{-g}\, K^0(t(\eta))\Big) \\ &= \Big[\frac{1}{3\pi^2\times 6\times 96}\,\Big(\frac{H}{M_{\rm Pl}}\Big)^3\,\Big(\frac{\mu}{M_s}\Big)^4\, M_{\rm Pl}\Big]\,\times\,\Big(\sqrt{-g}\, K^0(t(\eta))\Big).\end{aligned} \tag{59}$$

Taking into account that H remains approximately constant during the inflation period, (59) can then be integrated over the entire duration of inflation, to yield

$$K^0(t) \sim K^0_{\rm begin}(t=0)\ \times$$
$$\exp\Big[-3H\,t\,\Big(1 - \tfrac{1}{3\pi^2\times 18\times 96}\,\Big(\tfrac{H}{M_{\rm Pl}}\Big)^2\,\Big(\tfrac{\mu}{M_s}\Big)^4\Big)\Big], \tag{60}$$

where we assume the beginning of inflation at $t=0$.

$$K^0(t(\eta)) = K^0_{begin}(t(\eta = H^{-1}))exp[-2Ht(\eta)A] \tag{61}$$

with A being:

$$A = \Big(1 - \frac{1}{3\pi^2\times 18\times 96}\,\Big(\frac{H}{M_{\rm Pl}}\Big)^2\,\Big(\frac{\mu}{M_s}\Big)^4\Big) \tag{62}$$

The value $K^0_{\rm begin}(t=0)$ corresponds, on account of (54), to an initial condition for the cosmic time derivative of the KR axion, $\dot{\bar{b}}(0)$, and thus is a boundary condition to be determined phenomenologically.

The presence of gravitational waves during the inflationary phase may lead to a decrease in general, or even complete elimination, of the exponential washing out effects of inflation as $t \to +\infty$.So due to the slow running of H during inflation A is approximately constant and we may assume

$$A = 0 \Rightarrow \frac{H}{M_{PI}} = (15.06)(\frac{M_s}{\mu})^2 \tag{63}$$

by fixing the string scale. And so K^0 in eq. (60) is constant during the inflationary era. On using the eq (53) we then have that

$$\dot{\bar{b}} = \text{constant} \tag{64}$$

which violates spontaneously Lorentz symmetry.

Since phenomenologically Planck Collaboration results [1] have indicated $H/M_{\rm Pl} \sim 10^{-5}$, one obtains from (63) that

$$\mu \sim 5\times 10^3\, M_s. \tag{65}$$

The UV cutoff μ can be taken as high as the Planck scale, in agreement with the transplanckian conjecture. This provides a self-consistent and necessary condition for $\dot{b}$ to be approximately constant during inflation,

Now because of (64), the slow roll parameter for the single field inflation ε is linked to $\dot{b}$. We therefore have from observations [1] that

$$\varepsilon \sim \frac{1}{2}\frac{1}{(HM_{PI})^2}(\dot{\tilde{b}})^2 \sim >(\dot{\tilde{b}}) = \sqrt{2\varepsilon}M_{PI}H \tag{66}$$

and so $K^0_{begin}(t(\eta = H^{-1})) \sim HM^2_{PI}$ and $K^0(t(\eta)) \sim K^0_{begin}(t(\eta = H^{-1}))$ from the relations(54),(56) and,(66) we have that

$$\rho^b = vM^4_{PI}(\frac{H}{M_{PI}})^2 \tag{67}$$

For the ρ_Λ as we say the Λ parameter express the gravitational anomaly condensate which satisfy a de sitter eq of state $p_{cond} = -\rho_{cond}$. This can be seen by the action:

$$\begin{aligned} S_\Lambda = \int d^4x|\rho^\Lambda| = \int d^4\sqrt{-g}(5.86 \times 10^7\sqrt{2\varepsilon}[\frac{\tilde{b}(0)}{M_{PI}} \\ + \sqrt{2\varepsilon}N]H^4) = -\int d^4x\sqrt{-g}\frac{\Lambda}{\kappa^2} \end{aligned} \tag{68}$$

Were the N is the number of e-fold and for ε 10^{-2} we have that the N term is negligible so

$$\rho^\Lambda = 5.86 \times 10^7\sqrt{2\varepsilon}\frac{|\tilde{b}(0)|}{M_{PI}}H^4 \tag{69}$$

This term cannot arise by classical general relativistic treatment and so is an essential component of the total density.

Now he have to calculate the density due to the coupling.The equation of state that the ρ^{gCS} obeys is radiation like so :

$$p_{gCS} = \frac{1}{3}\rho_{gCS} \tag{70}$$

in which the pressure is asosiated with the diagonal spacial components of the cotton tensor and the density with the diagonal temporal one. now because we work with weak pertubation due to their string nature.implies that the metric component which the product with the cotton tensor gives the physical quantities above is the metric of the FRLW.this is very important in order to find the relation between the ρ^{gCS} and ρ_b also the sign of those. Now in order to do that we are going to use the second methodology in which we extract the $R\tilde{R}$.the reason is that the equation of the density is

$$\rho^{gCS} = \frac{2}{3}\frac{a'}{96\kappa}C^{00} \tag{71}$$

the main difference with the axion density is that we wanted the pure expression of the quantum fluctuations but now we want the effect of those on the gravitational field something that is expressed with the cotton tensor.Because we want the density to be purely time dependent we have to calculate the $C^{00}_{;0}$ term which:

$$C^{00}{}_{;0} = \frac{d}{dt}C^{00} + 4HC^{00} = -\frac{1}{8}u_\nu < R\tilde{R} > \tag{72}$$

We calculate the $< R\tilde{R} >$ in the eq (58) and we had:

$$< R\tilde{R} >= \frac{1}{\pi^2}(\frac{H}{M_{PI}})^2\mu^4\Theta$$

And by substituting Θ (50) we have:

$$C^{00}{}_{;0} = -\frac{1}{8}\frac{2}{3}\frac{a'\kappa}{12}H\frac{1}{\pi^2}(\frac{H}{M_{PI}})^2\mu^4(\dot{\tilde{b}}(\eta))^2 \tag{73}$$

so by using the $\dot{\tilde{b}}$ we found in (54) and the assumtion that C^{00} is constant in time we have:

$$C^{00} = -\varepsilon\frac{2}{3}\frac{a'\kappa}{192}\frac{1}{\pi^2}\mu^4 H^4 \tag{74}$$

Therefore, the density is

$$\rho_{gCS} = -2.932\chi 10^{-5}\varepsilon(\frac{\mu}{M_s})H^4 \tag{75}$$

and from the relation for the factor A (61):
$\frac{H}{M_{PI}} (\frac{M_{PI}}{\mu})^2 > (\frac{\mu}{M_{PI}})^2 \frac{M_{PI}}{H}$
we found before gives the order of magnitute:

$$\rho_{gCS} = -1.484 e M_{PI} H^2 \tag{76}$$

By substitute the total time dependent cotton tensor(73) in the modified einstein (28) and taking into account that we have to use the FRLW metric we have that:

$$\frac{d}{dt}(\rho^b + \rho_{gCS}) + 3H((1 + w_b)\rho^b + \frac{4}{3}\rho_{gCS}) = 0$$
$$=> \rho^b = -\frac{2}{3}\rho_{gCS} \tag{77}$$

the last result holds for $\frac{d}{dt}(\rho^b + \rho_{gCS}) = 0$ which implies that the contribution of the anomalies to the density is constant.Also because we have pure b fluid $w_b = 1$

Now fom the eqs (77),(78) we have that:

$$\rho^b = -\frac{2}{3}\rho_{gCS} => \rho^b + \rho_{gCS} = \frac{1}{3}\rho_{gCS} = -0.496 e M_{PI} H^2 \tag{78}$$

From the (76) we see that $\rho_{gCS} < 0$ and so from eq.(70) $p_{gCS} < 0$ so we have negative pressure and from (77) we have: $\rho^b + \rho_{gCS} < 0$ and by using the (57),(70) we also have:

$$p^b + p_{gCS} = \rho^b + \frac{1}{3}\rho_{gCS} = -\frac{1}{3}\rho_{gCS} > 0 \tag{79}$$

this is very weird because the energy is negative and presure positive.This leads to exotic state but as we will see the contribution of this term to the total ρ_{RVM} is very small in relation to the condensate

By substituting the necessary coefficients that stem from the above analysis into(69)-(78) we have therefore for the total vacuum energy density :

$$\rho_{RVM} = 3M_{PI}^4[-1.7x10^{-3}(\frac{H}{M_{PI}})^2 + \frac{\sqrt{2}}{3}5.86 \times 10^6 \frac{|\tilde{b}(0)|}{M_{PI}}(\frac{H}{M_{PI}})^4] \tag{80}$$

which is of the form of the RVM energy density (5), but with $c_0 = 0$ and the coefficient of the H^2 term being negative. This latter feature is due to the negative contributions of the Chern-Simons gravitational anomaly terms to the stress-energy tensor [8]. The evolution equation of the RVM model (6), then, implies the early de Sitter era (7), in a self consistent way with our analysis so far, where we assumed a constant Hubble parameter to evaluate the condensate of the gravitational anomalies.

As we can see in our string inspired model $c_0 = 0$ in the inflationary era. Also we see that the contributions of the Cotton tensor is actually very small compared to those of the actual quantum fluctuations of the anomalies. The reason is that the Cotton tensor does not express the effect of those flux in the gravitational field but the effect of the GWs themselves.That sound very strange at first because we said that the Cotton tensor express the effect of those in the field.The reason of this is that the the resonant amplification of the quantum fluctuations happened at the radiation era before the inflation one by preexisting GWs and they transfer all the way through the inflation era due to a mechanism that called two field theory which is not the subject of this paper. Also as we said the contribution of the anomalous density terms $-1.7 * 10^{-3}$ is negative and this could cause problems due to the exotic nature of this state. Lucky for us,the coefficient of the condensate-induced term ρ^{Λ} is positive $a = \frac{\sqrt{2}}{3} \times 5.86 \times 10^6 \frac{|\tilde{b}(0)|}{M_{PI}}(\frac{H}{M_{PI}})^2 = 2.8 \times 10^{-2} \frac{|\tilde{b}(0)|}{M_{PI}}$ and bigger and by using the necessary GUT-like potential we have that $\rho_{total} \simeq \rho^{\Lambda}$ which means that the inflation happened due to the Λ de sitter contribution.

Finally I am going to say some thing about the nature of this Λ

condensate.This is actually a composite scalar field consisting by the superposition of quantum b-axion and graviton modes.this means that the effect bosonic string modes that express the gravitons and fermionic sting modes of the axion binds together on the background that the dilaton creates ,that is why is a scalar field,and through that affect the gravitational field by creating the inflation.the mapping of the H^4 in this scalar field is called vacuumon [14].To do that mapping of the RVM in the vacuumon picture we follow the following corespodence between the total density and pressure:

$$\rho_{tot} = \rho_\varphi = \frac{\dot{\varphi}^2}{2} + V(\varphi), p_{tot} = p_\varphi = \frac{\dot{\varphi}^2}{2} - V(\varphi) \tag{81}$$

with

$$\dot{\varphi}^2 = -\frac{2}{\kappa^2}\dot{H} \tag{82}$$

and the vacuumon potential is [14]:

$$V = \frac{3H^2}{\kappa^2}(1 + \frac{\dot{H}}{3H^2}) = \frac{3H^2}{\kappa^2}(1 + \frac{\alpha}{6H^2}\frac{dH^2}{d\alpha}). \tag{83}$$

The scalar field φ is a classical field which can be used to described the temporal evolution of RVM. Using the eq. (83) we can compute the potential assosiated to the RVM density:

$$U(\varphi) = \frac{H_I^2}{\alpha\kappa^2}\frac{2 + cosh^2(\kappa\varphi)}{cosh^4(\kappa\varphi)} \tag{84}$$

This potential in our scenario would have been used in the calculation of the density but there is a problem. The potential (84) is classical and, thus, it cannot express the quantum fluctuation inside the condensate which needs the proper string theory framework for their calculation.As a result, the true vacuumon-field effective potential stemming from these fluctuations might be very different from (83), (84).

9 Conclusions and Outlook

In this paper we tried to explain the emergence of the running vacuum model of cosmology RVM as a low-energy limit of string cosmologies with anomalies and totally antisymmetric space-time torsion, which are based on the effective gravitational theory of the fields belonging to the massless gravitational multiplet of the string. The fields that enter our (3+1)-dimensional construction, after appropriate string compactification, which we did not discuss here, are the graviton and the KR axion. The four dimensional KR axion field is dual to the field strength of the spin-1 antisymmetric tensor Kalb-Ramond field of the massless string gravitational multiplet, which also provides a totally antisymmetric component of space-time torsion in this theory.

The dilaton was assumed constant throughout our analysis. The full dynamics of the dilaton, whose exponential determines the coupling of the string interactions, is non trivial and belongs to the realm of the fully quantum string theory. In the present dissertation, as already mentioned, we assumed the dilaton as stabilised during inflation, somehow due to its dynamics, via minimisation of an appropriate potential, induced by string loops(like virtual masses). Further brief discussion on future projects on the potential effects of a non trivial dilaton on thd current cosmological model is given in the outlook paragraphs at the end of the concluding section.

By considering condensation of primordial gravitational waves, we have argued in favour of the emergence of condensates of the gravitational Chern-Simons (CP-violating) anomaly terms, which couple to the axion field, which can then drive an early de Sitter era of RVM type, leading to inflation. It is worthy of remarking at this point that the existence of the condensate of the Chern-Simons anomaly term, leads, through its coupling with the KR axion, to an effective linear axion potential. This leads to a slow-roll evolution of the KR axion, during inflation. The slow-roll parameter in this inflationary scenario is provided by the time derivative of the KR axion, which is proportional to the (approximately constant) condensate (vacuum expectation value) of the temporal component of the gravitational anomaly current. However, it is not the axion that drives inflation, but rather the non-linear H^4 terms in the resulting RVM energy density of the cosmological vacuum, which owe their existence to the anomaly condensates and dominate at early epochs.

On exploiting properties of the Cotton tensor that characterises the gravitational variation of the anomaly terms, we have shown that the equation of state of this cosmological fluid, in the epoch of domination of the gravitational-wave condensate, is that of the RVM vacuum. Furthermore,we have described the basic characteristics of the Cotton tensor (19),(27) and shown that the presence of gravitational anomalies lead, through it, to the modification of the stress-energy tensor in order for the diffeomorphism invariance to be preserved. In this way, one does not have to restrict themselves only to anomaly-free specific metric backgrounds. We were able to calculate the gravitational-wave-induced Hirzebruch-Pontryagin term $R\tilde{R}$ (17), sourced by the KR axion field (49). After that, we have managed to demonstrate that the vacuum energy density is of a running vacuum model (RVM) type, (5), and we showed that the dominant H^4 term at early epochs, due to the condensate, leads to an almost de-Sitter-like dark-energy gravitational term. Finally we described briefly the condensate itself through an effective, non linear scalar field, the vacuumon, and discussed how it yields a classical picture of the RVM potential which is expected to be different though from the (yet uknown) one, due to the (full) quantum fluctuations of the condensate.

In this work we did not discuss the potential rôle of world-sheet instantons (non-perturbative stringy effects), which may lead to the emergence of periodic potentials for the KR axion [8, 12], as well as the other axion fields that exist in string theory, as a result of compactification. Such instanton-induced periodic potentials may affect the densities of the primordial black holes in such cosmologies, and eventually the profile of the gravitational waves during the inflationary epoch. This latter phenomenon may have phenomenological implications for the radiation-era profile of the gravitational waves, which we did not explore in this thesis, but is a feature of the model worthy of future exploration.

Another issue to be looked at in detail in the future is the rôle of potentially non-trivial dilaton fields that might exist at the end of the

RVM inflationary era. In the picture of [8], adopted here, the dilaton is assumed stabilised by an appropriate (yet unspecified in our effective approach here) potential, which is assumed to be generated at a full quantum string theory level. Incorporation of time dependent dilatons at the end of the RVM inflationary era might be possible, as a way of providing an explicit description of the decay of the running vacuum, first to a brief phase with just gravitons, KR axions and slow-moving dilatons, still in the presence of a (now slowly-decreasing in value with the cosmic time) anomaly condensate. This brief phase is then succeeded by the generation of massless chiral fermionic matter and radiation, also due to the decay of the running vacuum. After this second stage, the chiral fermions generate their own gravitational anomalies, which can cancel the primordial ones, paving the way for the passage to the anomaly-free post-inflationary radiation and matter dominated eras, as explained in [8].

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