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PHANTOM BLACK HOLES
Local solutions with Phantom scalar field hair

MASTER THESIS OF
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In accordance with the requirements of the degree of Interdepartmental Program of Postgraduate studies in Physics and Technological Applications., in the School of Applied Mathematical and Physical Sciences, I present the following thesis entitled,

Phantom Black Holes
Local Solutions in Phantom Fields

This work was performed under the supervision of Eleftherios Papantonopoulos. I declare that the work submitted in this thesis is my own, except as acknowledged in the text and footnotes, and has not been previously submitted for a degree at National Technical University of Athens or any other institution.

Dimitrios Giannakopoulos

To Ankor...

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I must also thank my family, my friends, and all my teachers for their unconditional support throughout these years...

Abstract

The purpose of this Master thesis is to examine the exact solutions of black holes with phantom scalar fields and investigate the nature of phantom fields as a dark matter candidate. The first chapter is a general introduction to the topic, while the second chapter deals with the general case of an n-dimensional theory in which gravity is coupled to a dilaton and Maxwell field. In Chapter 3 we investigate the nature of Phantom Black Holes and combine the dilaton and phantom cases. Chapter 4 focuses on the energy conditions and thermodynamic laws of phantom black holes. Finally, conclusions are drawn in Chapter 5.

Contents

Chapter 1

Introduction

The accelerating expansion of the universe is a fact. The origin of the matter that induces this expansion of the universe is still unknown and it is dubbed as dark energy. Although we don't know much about its nature, there are phenomenological models which can suitably describe the current expansion of the Universe. ΛCDM,(Λ-cosmological constant, cold, dark, matter) predicts an equation of state with a constant pressure to energy density ratio of -1. Meanwhile, this model takes into account dark matter's existence and is still the best fit to the observational data. However, ΛCDM has its own malfunctions. Recent findings have suggested that violations of the cosmological principle, especially of isotropy, exist. These violations have called the ΛCDM model into question, with some authors suggesting that the cosmological principle is now obsolete or that the Friedmann–Lemaître–Robertson–Walker metric breaks down in the late universe.

Physicists didn't exclude other models that could suitably describe the current acceleration as well. In fact, several models that could induce a positive acceleration have been suggested, e.g.

- Quintessence models, which are those that preserve the null energy condition, i.e. $0 \leq \rho + p$, in such a way that w (equation of state) parameter is always larger than 1.

- Phantom models where the null energy condition is violated and the w parameter can go below 1.

Surprisingly, phantom models are not excluded, but even seem to be favoured by recent observations, as Planck 2018 suggest. We are going to mention few observational data in the last chapter. Phantom Black Holes is a topic which made its appearance due to phantom scalar fields. One of the first papers trying to propose the behaviour of these fields is called "a phantom menace" and investigates the consequences of a dark energy component with supernegative equation of state [5]

In past few years there have been tremendous efforts in modeling the dark energy. They include scalar field models, some models of brane worlds and specific compactification schemes in string theory which have been shown to mimic the dark energy like behaviour.

A wide variety of scalar field models have been conjectured for this purpose including quintessence as we mentioned, K-essence, tachyonic scalar fields with the last one being originally motivated by string theory. All these models of scalar fields lead to the equation of state parameter $w = p/\rho$ greater than or equal to minus one. However, the recent observations of SN Ia (supernova of type Ia) do not seem to exclude values of this parameter less than minus one [20].

It is therefore important to look for theoretical models to describe dark energy with $w < -1$ called phantom energy. This kind of energy comes up from a theoretical built, named phantom field, and inevitably has its own spacetime anomalies such as the phantom black holes. Or else, phantom black holes are the exact solutions of black holes with phantom fields.

Phantom fields have negative kinetic energy. This constraint is enforced on the action level by considering a plus sign in front of the kinetic term of the field. So the action of phantom fields in n-dimensions is assumed to be:

$$S = \int d^n x \sqrt{-g} \left[R + \frac{4}{n-2} \partial_\mu \psi \partial^\mu \psi - V(\psi) \right], \tag{1.1}$$

where g is the metric, R the Ricci scalar, the factor in front of the kinetic term is a consequence of the compactification of Brans-Dicke theory in n-dimensions. More about this certain compactification and the origins of dilaton and phantom fields, can someone find in [23]. Finally, V is the phantom potential.

In chapter 2 we will start the analysis of the dilatonic black hole spacetime, extract physical quantities and walk through the equations of motion. Our effort will go back to the paper of Horowitz and Strominger [16].

Chapter 3 is focused on the same procedure, but we are considering the phantom case of the field. In particular, via a Wick rotation on the dilaton, one may derive the corresponding action of the phantom black hole and solve the corresponding equations of motion in a similar manner. The analogy would be straightforward. What we are going to do is to combine the results already existed in the bibliography, and show them in the most intuitive way we can. In the end, always comes physics, so we need to come up with the physical quantities of mass and charge. The results which are going to be extracted are,

$$Q^2 = \frac{(n-2)(n-3)^2}{2(n-3-\beta^2)} r_+^{n-3} r_-^{n-3} \tag{1.2}$$

$$M = \frac{r_+}{2}(n-3)\left[1-\left(\frac{r_-}{r_+}\right)^{n-3}\right]^{\frac{(n-3)^2-\beta^2}{(n-3)(n-3-\beta^2)}} + \frac{(n-2)(n-3)}{2(n-3-\beta^2)}r_-^{n-3} \quad (1.3)$$

of course those expressions are immensely simplified in 4 dimensions and in more specific cases, sush as spherical symmetry and staticity.

To continue, we just mention that some authors investigate a step further and deal with the quintom case, which is a dilaton-phantom mixture.

Thesis is ending in Chapter 4, where we examine the energy conditions and thermodynamics of phantom black holes. Next, we just mention the intense research of the community on phantom fields, including all of its extra parts. In particular, the research on dark matter and dark energy expands from gravity and quantum mechanics to thermodynamics and experimental processes such as gravitational lensing, and further to observational data analysis. My hope is that if someone, if not the author of this thesis himself, wishes to start his own research in these interesting fields, this thesis will prove itself helpful.

Chapter 2

Dilaton Black Holes

Let us first set up the stage of our work. We are considering a gravitational action with a dilaton contribution and a Maxwell field. We will perform the corresponding variations of the degrees of freedom and extract the corresponding equations of motion. Then we will follow the steps of [8] and estimate dilaton black hole's metric. In bibliography we have found the phantom black hole's metric calculation through a transformation from the dilaton's case. So our steps, is finding the dilaton's behaviour and then transform our results in those we need, i.e. the phantom ones.

2.1 Dilaton's Coupling behaviour

2.1.1 Action's Variation

In Brans–Dicke theory of gravity, Newton's constant is not presumed to be constant but instead $1/G$ is replaced by a scalar field ϕ and the associated particle is the dilaton. It should be noted that the dilaton originated from Kaluza Klein theory as a classical unified field theory of gravitation and electromagnetism built around the idea of a fifth dimension. Dilaton (or radion) was an intrinsic component of the metric tensor and since then, it stands in scalar field theories of gravity on its own.

To cut a long story short, the properties of black holes are modified when the dilaton field is present and naturally, so does when the phantom field is present as well. But let us not be hasty, we will come there later. Some authors say that dilaton changes the causal structure of the black hole and leads to the curvature singularities at finite radii.

We are going to have a more analytic flow, but anyway be guided by [8]. So we begin by considering the n-dimensional theory in which gravity is coupled to dilaton and Maxwell field with an action

$$S = \int d^n x \sqrt{-g} \left[R - \frac{4}{n-2} \partial_\mu \phi \partial^\mu \phi - V(\phi) - e^{-\frac{4a\phi}{n-2}} F^2 \right], \tag{2.1}$$

where R is the scalar curvature, $F^2 = F_{\mu\nu}F^{\mu\nu}$ is the usual Maxwell contribution, and $V(\phi)$ is a potential of dilaton ϕ. The parameter α is an arbitrary coupling constant governing the strength of the interaction between the dilaton and the Maxwell field as the authors mention. One should note that the minus sign in front of dilaton's kinetic term makes the dilaton automatically inappropriate as a field-candidate for the $w < -1$ case.

Let us now go through the variation of the action with respect to the metric, Maxwell, and dilaton field respectively, which will yield 3 independent equations. Starting from the variation of the metric we will follow the same pattern and deal with the 3 main parts of the action separately, as follows:

$$\delta S = \delta S_1 + \delta S_2 + \delta S_3 \tag{2.2}$$

where

$$S_1 = \int d^n x\sqrt{-g}R, \tag{2.3}$$

$$S_2 = -\int d^n x\sqrt{-g}\left[\frac{4}{n-2}\partial_\mu\phi\partial^\mu\phi + V(\phi)\right], \tag{2.4}$$

$$S_3 = -\int d^n x\sqrt{-g}e^{\frac{-4\alpha\phi}{n-2}}F^2 \tag{2.5}$$

(2.3) is the classical Einstein-Hilbert action, (2.4) contains the dilaton's kinetic term and dilaton's potential, while (2.5) is the Maxwell term coupled with a suitable dilaton exponential, such that the coupling vanishes at infinity.

Starting from the variation of the classical Einstein Hilbert action we have

$$\delta S_1 = \int d^n x\left[\delta(\sqrt{-g})R + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\right] \tag{2.6}$$

the last term leads to a total derivative which doesn't contribute to the equations of motion. So we have

$$\delta S_1 = \int d^n x\sqrt{-g}\left[-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu}\right]\delta g^{\mu\nu} \tag{2.7}$$

from where we will extract the trace reversed Einstein equation, which we will work on. In particular, by performing the contraction with the metric on the corresponding Einstein equations, we find that

$$(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)g^{\mu\nu} = G_{\mu\nu}g^{\mu\nu} = -R = -T \tag{2.8}$$

where T is the trace of the total stress energy tensor. This will greatly simplify the calculations. Next up is the dilaton's kinetic and potential term. We are going through the metric's variation process.

$$\delta S_2 = -\int d^n x \left[\delta(\sqrt{-g}) \frac{4}{n-2} g_{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \delta(\sqrt{-g}) V(\phi) \right. \\ \left. + \sqrt{-g} \frac{4}{n-2} \delta(g_{\mu\nu}) \partial_\mu \phi \partial_\nu \phi \right] \tag{2.9}$$

now through the following equation

$$\delta(\sqrt{-g}) = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \tag{2.10}$$

we may take

$$\delta S_2 = \int d^n x \left[\sqrt{-g} \frac{2}{n-2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{n-2} g_{\mu\nu} V(\phi) - \frac{4}{n-2} \partial_\mu \phi \partial_\nu \phi \right] \delta g_{\mu\nu},$$

$$\delta S_2 = \int d^n x \sqrt{-g} \left[+ \frac{4}{n-2} \partial_\mu \phi \partial_\nu \phi - \frac{2}{n-2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right. \\ \left. + \frac{1}{n-2} g_{\mu\nu} V(\phi) \right] \delta g_{\mu\nu} \tag{2.11}$$

The second term in the last equation produces a total derivative which doesn't contribute in the equations of motion, so we have,

$$\delta S_2 = \int d^n x \sqrt{-g} \left[\frac{4}{n-2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{n-2} g_{\mu\nu} V(\phi) \right] \delta g_{\mu\nu} \tag{2.12}$$

Finally we deal with the Maxwell term and variate with respect to the metric. We can write the S_3 term like:

$$S_3 = \int d^n x \delta \left[\sqrt{-g} e^{\frac{-4\alpha\phi}{n-2}} F_{\mu\nu} g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} \right] \tag{2.13}$$

So through variation one can get:

$$\delta S_3 = \int d^n x \delta(\sqrt{-g}) \left[e^{\frac{-4\alpha\phi}{n-2}} F_{\mu\nu} g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} \right] + \int d^n x \sqrt{-g} \left[e^{\frac{-4\alpha\phi}{n-2}} F_{\mu\nu} \delta(g^{\mu\rho}) g^{\nu\sigma} F_{\rho\sigma} \right] \\ + \int d^n x \sqrt{-g} \left[e^{\frac{-4\alpha\phi}{n-2}} F_{\mu\nu} g^{\mu\rho} \delta(g^{\nu\sigma}) F_{\rho\sigma} \right]$$

By inserting the equations (2.8), (2.12) and (2.13) with (2.2), we finaly find the trace reversed Einstein equations, which read:

$$R_{\mu\nu} = \frac{4}{n-2}\left(\partial_\mu\phi_\nu\phi + \frac{1}{4}g_{\mu\nu}V\right) + 2e^{\frac{-4\alpha\phi}{n-2}}\left(F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{2n-4}g_{\mu\nu}F^2\right) \quad (2.14)$$

Next we will deal with the variation via the Maxwell field. Only the S_3 term is the one which contributes in this variation and yields

$$\partial_\mu\left(\sqrt{-g}e^{\frac{-4\alpha\phi}{n-2}}F^{\mu\nu}\right) = 0 \quad (2.15)$$

The last equation is provided from the variation with respect to the dilaton field. So we have:

$$\delta S_1 = 0, \quad (2.16)$$

$$\delta S_2 = \int d^n x\delta\left[\sqrt{-g}\left(-\frac{4}{n-2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right)\right], \quad (2.17)$$

$$\delta S_3 = -\int d^n x\delta\left(e^{-\frac{4\alpha\phi}{n-2}}F^2\right) \quad (2.18)$$

The variation of the dilaton field and potential of (2.17) via (2.10) yields

$$\begin{aligned}
\delta S_2 &= \int d^n x\sqrt{-g}\left(\frac{4}{n-2}g^{\mu\nu}\partial_\mu(\delta\phi)\partial_\nu\phi + \frac{4}{n-2}g^{\mu\nu}\partial_\mu\phi\partial_\nu(\delta\phi) - \frac{dV(\phi)}{d\phi}\delta\phi\right) \\
&= \int d^n x\sqrt{-g}\left(\frac{8}{n-2}g^{\mu\nu}\partial_\mu(\delta\phi)\partial_\nu\phi - \frac{dV(\phi)}{d\phi}\delta\phi\right) \\
&= -\int d^n x\frac{8}{n-2}\left[\partial_\nu\left(\sqrt{-g}\partial^\nu\phi\delta\phi\right) - \partial_\nu\left(\sqrt{-g}\partial^\nu\phi\right)\delta\phi\right] - \int d^n x\sqrt{-g}\frac{dV(\phi}{d\phi}\delta\phi \\
&= -\int d^n x\sqrt{-g}\frac{8}{n-2}\left[\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\partial^\nu\phi\delta\phi\right) - \frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\partial^\nu\phi\right)\delta\phi\right] - \int d^n x\sqrt{-g}\frac{dV(\phi)}{d\phi}\delta\phi \\
&= -\int d^n x\sqrt{-g}\frac{8}{n-2}\left[\nabla_\nu\left(\partial^\nu\phi\delta\phi\right) - \nabla_\nu\left(\partial^\nu\phi\right)\delta\phi\right] - \int d^n x\sqrt{-g}\frac{dV(\phi)}{d\phi}\delta\phi \\
&= \int d^n x\sqrt{-g}\left[\frac{8}{n-2}\left(\nabla^\mu\nabla_\mu\phi\right) - \frac{dV(\phi)}{d\phi}\right]\delta\phi
\end{aligned}$$

Meanwhile, (2.18) is trivially found to be

$$\delta S_3 = \int d^n x\frac{4\alpha}{n-2}e^{\frac{4\alpha\phi}{n-2}}F^2\delta\phi \quad (2.19)$$

So finally, taking into account (2.16), (2.17) and (2.19), we have the following equation of motion:

$$\delta S = 0 => \delta S_1 + \delta S_2 + \delta S_3 = 0 => \nabla^\mu\nabla_\mu\phi = \frac{n-2}{8}\frac{\partial V}{d\phi} - \frac{\alpha}{2}e^{\frac{4\alpha\phi}{n-2}}F^2 \quad (2.20)$$

So (2.14), (2.15) and (2.20) gives as the whole picture of the variation which we are going to exploit in order to produce the metric for the spacetime we seek.

2.1.2 Dilaton's metric

We will take the most general form of the metric for the static dilaton black hole. The metric ansatz we are going to use is the following:

$$ds^2 = -U(r)dt^2 + \frac{1}{U(r)}dr^2 + f(r)^2 d\Omega_{n-2}^2 \tag{2.21}$$

Firstly, the equation (2.15) is easily solved to yield

$$F_{01} = Q\frac{e^{-\frac{4\alpha\phi}{n-2}}}{\sqrt{-g}} \tag{2.22}$$

since the F_{01} component is the only non-trivial component of our case. Q is simply an integration constant (which will assume the role of the electromagnetic charge). The denominator can be written due to (2.21) as f^{n-2}, so we will get

$$F_{01} = Q\frac{e^{-\frac{4\alpha\phi}{n-2}}}{f^{n-2}} \tag{2.23}$$

Working out the laplacian operator and taking into consideration the metric and (2.22) equation, the equations of motion (2.14, 2.15 and 2.20) become via trivial mathematical operations, the following ones:

$$\frac{1}{f^{n-2}}\frac{d}{dr}\left(f^{n-2}U\frac{d\phi}{dr}\right) = \frac{n-2}{8}\frac{\partial V}{\partial \phi} + \alpha\frac{Q^2 e^{\frac{4\alpha\phi}{n-2}}}{f^{2n-4}}, \tag{2.24}$$

$$\frac{1}{f}\frac{d^2 f}{dr^2} = -\frac{4}{(n-2)^2}\left(\frac{d\phi}{dr}\right)^2, \tag{2.25}$$

$$\frac{1}{f^{n-2}}\frac{d}{dr}\left[U\frac{d}{dr}(f^{n-2})\right] = \frac{(n-2)(n-3)}{f^2} - V - 2\frac{Q^2 e^{\frac{4\alpha\phi}{n-2}}}{f^{2n-4}}. \tag{2.26}$$

There is a problem here. We have four unknown quantites U(r), f(r), ϕ(r) and V(ϕ), with three equations. So we can't solve the system in a straightforward way. Our way out of this conundrum is to exploit the results of Strominger and Horowitz in their paper [16] where they found an explicit expression for the n-dimensional dilaton black hole with vanishing potential.

The metric in equation (11) as far as the aforementioned paper is concerned is the following one,

$$
\begin{aligned}
ds^2 = &- \left[1 - \left(\frac{r_+}{r}\right)^{n-3}\right]\left[1 - \left(\frac{r_-}{r}\right)^{n-3}\right]^{1-\gamma(n-3)} dt^2 \\
&+ \left[1 - \left(\frac{r_+}{r}\right)^{n-3}\right]^{-1}\left[1 - \left(\frac{r_-}{r}\right)^{\gamma-1}\right] dr^2 \\
&+ r^2\left[1 - \left(\frac{r_-}{r}\right)^{n-3}\right]^{\gamma} d\Omega^2_{n-2}
\end{aligned}
\tag{2.27}
$$

where r_+, r_- are the two event horizons of the black hole and γ is a constant which is related with the coupling constant α and the dimensions n of the spacetime. The authors of both papers [16, 8], agree in the value of γ which should be:

$$
\gamma = \frac{2\alpha^2}{(n-3)(n-3+\alpha^2)}, \tag{2.28}
$$

.

Strominger and Horowitz, found the value of γ via boundary conditions in their model at the horizon and spatial infinity, while Gao and Zhang agreed on the same value as a final assumption in their theory, where only by imposing this value for γ, the metric would have the correct form.

We want to express the (2.27) metric in a form similar to the Schwarzschild case, so we will have to fix the components in order to achieve, $-g_{00} = g^{11}$. This is achieved via the following coordinate transformation,

$$
r(x) = \int dr \left[1 - \left(\frac{r_-}{r}\right)^{n-3}\right]^{-\gamma(n-4)/2}, \qquad i.e. \quad \frac{\partial r}{\partial x} = r' = \left[1 - \left(\frac{r_-}{r}\right)^{n-3}\right]^{\gamma(n-4)/2},
$$

where on goes from (t,r) coordinate system to (t,x), with x variable taking the aforementioned expression, not to be confused with the 4 dimension ansatz.

Equation (2.27) takes the form we wanted

$$
\begin{aligned}
ds^2 = &- \left[1 - \left(\frac{r_+}{r(x)}\right)^{n-3}\right]\left[1 - \left(\frac{r_-}{r(x)}\right)^{n-3}\right]^{1-\gamma(n-3)} dt^2 \\
&+ \left[1 - \left(\frac{r_+}{r(x)}\right)^{n-3}\right]^{-1}\left[1 - \left(\frac{r_-}{r(x)}\right)^{n-3}\right]^{-1+\gamma(n-3)} dx^2 \\
&+ r^2\left[1 - \left(\frac{r_-}{r(x)}\right)^{n-3}\right]^{\gamma} d\Omega^2_{n-2},
\end{aligned}
\tag{2.29}
$$

where one can obtain that $g_{00} = -g^{11}$.

We wish to note that Gao and Zhang have also examined the case of the four dimensional dilaton black hole with cosmological constant [7], and extracted a metric of the form:

$$
\begin{aligned}
ds^2 = & -\left[\left(1-\frac{r_+}{x}\right)\left(1-\frac{r_-}{x}\right)^{\frac{1-a^2}{1+a^2}} - \frac{1}{3}\lambda x^2\left(1-\frac{r_-}{x}\right)^{\frac{2a^2}{1+a^2}}\right] dt^2 \\
& + \left[\left(1-\frac{r_+}{x}\right)\left(1-\frac{r_-}{x}\right)^{\frac{1-a^2}{1+a^2}} - \frac{1}{3}\lambda x^2\left(1-\frac{r_-}{x}\right)^{\frac{2a^2}{1+a^2}}\right]^{-1} dx^2 \\
& + x^2\left(1-\frac{r_-}{x}\right)^{\frac{2a^2}{1+a^2}} d\Omega_2^2,
\end{aligned}
\tag{2.30}
$$

where r_+, r_- are the two horizons of the black hole. α is an arbitrary constant governing the strength of the coupling between the dilaton and the Maxwell field in the same manner to our case.

2.2 Dilaton's charge and potential

Inspired by the two solutions, Gao and Zhang considered the following metric ansatz for the case of the n-dimensional dilatonic black hole with a dilatonic potential: ,

$$
U(r) = \left[1-\left(\frac{r_+}{r(x)}\right)^{n-3}\right]\left[1-\left(\frac{r_-}{r(x)}\right)^{n-3}\right]^{1-\gamma(n-3)} - \frac{1}{3}\lambda x^2\left[\left(1-\frac{r_-}{x}\right)^{n-3}\right]^{\gamma},
$$

$$
f(r) = r^2\left[1-\left(\frac{r_-}{r(x)}\right)^{n-3}\right]^{\gamma/2} \tag{2.31}
$$

The equations of motion for this case can easily be found to be,

$$
\frac{1}{f^{n-2}} r' \frac{d}{dr}\left(f^{n-2} U r' \frac{d\phi}{dr}\right) = \frac{n-2}{8}\frac{\partial V}{\partial \phi} + \alpha \frac{Q^2 e^{\frac{4\alpha\phi}{n-2}}}{f^{2n-4}}, \tag{2.32}
$$

$$
\frac{1}{f}\frac{d}{dr}\left(r'\frac{df}{dr}\right) = -\frac{4}{(n-2)^2}\left(\frac{d\phi}{dr}\right)^2 r', \tag{2.33}
$$

$$
\frac{1}{f^{n-2}} r' \frac{d}{dr}\left[U r' \frac{d}{dr}(f^{n-2})\right] = \frac{(n-2)(n-3)}{f^2} - V - 2\frac{Q^2 e^{\frac{4\alpha\phi}{n-2}}}{f^{2n-4}}, \tag{2.34}
$$

which can now be solved by virtue of the fact that they fixed two of the degrees of freedom to expressions that would agree with the two known solutions of the n-dimensional case with zero potential (and zero cosmological constant) and the four dimensional case with cosmological constant respectively.

Indeed, it is now easy to calculate the remaining degrees of freedom. In particular equation (2.33) can be easily integrated and yields the expression

$$e^{2\phi} = e^{2\phi_0}\left[1 - \left(\frac{r_-}{r}\right)^{n-3}\right]^{(n-2)\sqrt{\gamma}\sqrt{2+3\gamma-n\gamma}/2}, \tag{2.35}$$

where ϕ_0 is an integration constant assuming the asymptotic value of the dilaton. From equations (2.32-2.34) the "founders" of the dilaton black hole have found the important expressions of V(ϕ) and Q considering the same value of γ we previously mentioned. The electromagnetic charge Q can straightforwardly be found to be

$$Q^2 = \frac{(n-2)(n-3)^2}{2(n-3+\alpha^2)} e^{-\frac{4a\phi_0}{n-2}} r_+^{n-3} r_-^{n-3}, \tag{2.36}$$

while it can be deduced that under these reasonable assumptions, the potential for the dilaton is necessarily of the following form:

$$\begin{aligned} V(\phi) =& \frac{\lambda}{3(n-3+\alpha^2)^2}\Bigg[-\alpha^2(n-2)(n^2-n\alpha^2-6n+\alpha^2+9)e^{-\frac{4(n-3)(\phi-\phi_0)}{(n-2)\alpha}} \\ &+ (n-2)(n-3)^2(n-1-\alpha^2)e^{-\frac{4\alpha(\phi-\phi_0)}{(n-2)}} \\ &+ 4\alpha^2(n-3)(n-2)^2 e^{-\frac{4(\phi-\phi_0)(n-3-\alpha^2)}{(n-2)\alpha}}\Bigg], \end{aligned} \tag{2.37}$$

One may verify that the above expressions can indeed be reduced to the aforementioned known cases under suitable limits (λ=0 for the n-dimensional dilatonic black hole and n=4 for the four dimensional case with the cosmological constant respectively).

In the Schwarzschild coordinate system we wish to make use of, the final expression for the n-dimensional black hole with the above dilatonic potential is the following:

$$\begin{aligned} ds^2 =& -\left\{\left[1-\left(\frac{r_+}{r}\right)^{n-3}\right]\left[1-\left(\frac{r_-}{r}\right)^{n-3}\right]^{1-\gamma(n-3)} - \frac{1}{3}\lambda r^2\left[1-\left(\frac{r_-}{r}\right)^{n-3}\right]^{\gamma}\right\}dt^2 \\ &+\left\{\left[1-\left(\frac{r_+}{r}\right)^{n-3}\right]\left[1-\left(\frac{r_-}{r}\right)^{n-3}\right]^{1-\gamma(n-3)} - \frac{1}{3}\lambda r^2\left[1-\left(\frac{r_-}{r}\right)^{n-3}\right]^{\gamma}\right\}^{-1} \\ &\cdot\left[1-\left(\frac{r_-}{r}\right)^{n-3}\right]^{-\gamma(n-4)} dr^2 + r^2\left[1-\left(\frac{r_-}{r}\right)^{n-3}\right]^{\gamma} d\Omega_{n-2}^2 \end{aligned} \tag{2.38}$$

We see now that when the couling constant between the Maxwell and dilaton field is zero, then respectively, $\gamma = 0$, and then we get in 4 dimensions the Reissner-Nordstrom metric as expected. This model fits good with the string theory background, although it remains a theoretical project. The reason we got involved with the dilaton's case, is to pass through to the phantom's case via a Wick transformation.

Chapter 3

Phantom Black Holes

In this chapter we are going focus on case of the phantom field. As we said before, compared to the action of the ordinary scalar fields, the phantom field has the negative kinetic term. To reproduce this term we are going to follow [14] and make use of the mathematical trick of Wick rotation.

3.1 Via a Wick Rotation

So, with ϕ being the dilaton field, ψ the phantom field and α, β being the coulping constants in either case, the Wick rotation will be,

$$\phi \to i\psi, \qquad \alpha \to i\beta \tag{3.1}$$

which yields the following action

$$S = \int d^n x \sqrt{-g} \left[R + \frac{4}{n-2} \partial_\mu \psi \partial^\mu \phi - V(\psi) - e^{-\frac{4\beta\psi}{n-2}} F^2 \right] \tag{3.2}$$

while the potential of the phantom field via the same transformation will be, similar to (2.36),

$$\begin{aligned} V(\psi) =& \frac{\lambda}{3(n-3-\beta^2)^2} \Bigg[\beta^2(n-2)(n^2+n\beta^2-6n-\beta^2+9)e^{-\frac{4(n-3)\psi}{(n-2)\beta}} \\ &+ (n-2)(n-3)^2(n-1+\beta^2)e^{-\frac{4\beta\psi}{(n-2)}} \\ &- 4\beta^2(n-3)(n-2)^2 e^{-\frac{2\psi(n-3+\beta^2)}{(n-2)\beta}} \Bigg] \end{aligned} \tag{3.3}$$

We straightforwardly observe, how the the action reduces to the Einstein-Maxwell one when $\psi = 0$ or $\beta = 0$. Now the procedure followed in the previous chapter will be of use for the case of the metric. So taking into account the (3.1) transformation, we can write down the metric of the phantom black hols with cosmological constant in a similar fashion to the dilaton version.

$$\begin{aligned} ds^2 = &- \left\{ \left[k - \left(\frac{r_+}{r}\right)^{n-3} \right] \left[1 - \left(\frac{r_-}{r}\right)^{n-3} \right]^{1-\gamma(n-3)} - \frac{1}{3}\lambda r^2 \left[1 - \left(\frac{r_-}{r}\right)^{n-3} \right]^{\gamma} \right\} dt^2 \\ &+ \left\{ \left[k - \left(\frac{r_+}{r}\right)^{n-3} \right] \left[1 - \left(\frac{r_-}{r}\right)^{n-3} \right]^{1-\gamma(n-3)} - \frac{1}{3}\lambda r^2 \left[1 - \left(\frac{r_-}{r}\right)^{n-3} \right]^{\gamma} \right\}^{-1} \\ &\cdot \left[1 - \left(\frac{r_-}{r}\right)^{n-3} \right]^{-\gamma(n-4)} dr^2 + r^2 \left[1 - \left(\frac{r_-}{r}\right)^{n-3} \right]^{\gamma} d\Omega^2_{n-2} \end{aligned} \tag{3.4}$$

where r_+ and r_- are the two horizons of the black hole, and γ, physical mass M and electrical charge Q are respectively given by

$$\gamma = -\frac{2\beta^2}{(n-3)(n-3-\beta^2)}, \tag{3.5}$$

$$Q^2 = \frac{(n-2)(n-3)^2}{2(n-3-\beta^2)} r_+^{n-3} r_-^{n-3}, \tag{3.6}$$

$$M = \frac{r_+}{2}(n-3) \left[1 - \left(\frac{r_-}{r_+}\right)^{n-3} \right]^{\frac{(n-3)^2-\beta^2}{(n-3)(n-3-\beta^2)}} + \frac{(n-2)(n-3)}{2(n-3-\beta^2)} r_-^{n-3} \tag{3.7}$$

where $k = 0, \pm 1$ denotes the three kinds of topologies of black holes, torus, an (n-2) dimensional sphere, and hyperboloid respectively. (see [14] for more.)

3.2 Four dimensional spherical Phantom Black Holes

The most simple example to deal with is the spherical and four dimensional phantom black holes. So, setting n=4 and k=1, there occurs the case we wanted. By means of simplicity the cosmological constant is ommitted too. So the metric of the previous relationship, taking into consideration the relationship (3.5), is given by

$$\begin{aligned} ds^2 = &- \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1+\beta^2}{1-\beta^2}} dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1} \left(1 - \frac{r_-}{r}\right)^{-\frac{1+\beta^2}{1-\beta^2}} dr^2 \\ &+ r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2\beta^2}{1-\beta^2}} d\Omega^2_2 \end{aligned} \tag{3.8}$$

Now, the expressions of the physical quantities are simplified a lot, and then we have for the phantom field (in a similar way with the dilaton's case), the physical mass, and the electrical charge of the phantom black hole, the following relationships,

$$e^{-2\beta\psi} = \left(1 - \frac{r_-}{r}\right)^{\frac{1+\beta^2}{1-\beta^2}}, Q^2 = \frac{r_+ r_-}{1-\beta^2}, \qquad (3.9)$$
$$M = \frac{r_+}{2} + \frac{1+\beta^2}{1-\beta^2} \cdot \frac{r_-}{2}$$

If the coupling constant β of the phantom field with the Maxwell one, is zero then we go to the Reissner-Norström solution as it should be expected. However, if $\beta \neq 0$ there the case differs a lot. Following the analysis by Chang Jun Gao and Shuang Nan Zhang in [14] , for all β, $r = r_+$ is an event horizon. The surface $r = r_-$ is a curvature singularity except for the case $\beta = 0$ when it is a nonsingular inner horizon. Thus they describe black holes only when $r_- < r_+$."
One can find that the two horizons r_+ and r_- locate respectively at

$$\begin{aligned} r_+ &= M + \sqrt{M^2 - (1+\beta^2)Q^2}, \\ r_- &= \frac{1-\beta^2}{1+\beta^2}\left[M - \sqrt{M^2 - (1+\beta^2)Q^2}\right]. \end{aligned} \qquad (3.10)$$

What these equations tell us is that when, $\beta >> 1$, a small amount of electrical charge whould be responsible for a large change in the geometry close to the horizon. Meanwhile for $\beta \neq 0$ the extremal black hole, where $r_- = r_+$ can never be achieved. Other gravitational quantities, is the surface gravity which will be,

$$\kappa = \frac{1}{2r_+}\left(1 - \frac{r_-}{r_+}\right)^{\frac{1+\beta^2}{1-\beta^2}}. \qquad (3.11)$$

Thus the surface gravity will never approach zero except for $\beta = 0$. For all β, the surface gravity does not diverge. Since the temperature is proportional to κ, the third law of thermodynamics for black holes still holds as we are going to explain in next chapter. The transition between black holes and naked singularities occurs at $Q = M/\sqrt{1+\beta^2}$. For the phantom black holes, the extremal value corresponds to the case where the repulsive forces of electrical charge and phantom charge can exactly destroy the event horizon. In other words, dilaton field contributes an extra attractive force and phantom field contributes an extra repulsive force between black holes. So for a given M, one needs a smaller Q to destroy the event horizon. Furthermore the curvature singularity in the phantom case is present only for $0 \leq \beta < 1$.

Now lets go back to recover the Schwarzschild case from phantom's point of view. If we consider the case $\beta = 1$ then the metric, with the help of the exponential series, take the following form

$$\mathrm{ds}^2 = -\left(1 - \frac{r_+}{r}\right) e^{\frac{-r_-}{r}} dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1} e^{\frac{r_-}{r}} dr^2 + r^2 e^{\frac{r_-}{r}} d\Omega_2^2,$$

with the two horizons being now

$$r_+ = M + \sqrt{M - 2Q^2}, \qquad r_- = M - \sqrt{M - 2Q^2} \tag{3.12}$$

The authors of [14] propose the kind of charge named phantom charge which is a parameter determined only by the mass and the charge. This one results to be,

$$P = \frac{1}{4\pi} \int d^2 \Sigma^\mu \nabla_\mu \psi = \frac{1}{2} (\sqrt{M^2 - 2Q^2} - M) \tag{3.13}$$

As we can see by (3.12) the Phantom Charge goes from $-M/2$ to 0. Now the metric simplifies to

$$ds^2 = -\left(1 - \frac{2M + 2P}{r}\right) e^{\frac{2P}{r}} dt^2 + \left(1 - \frac{2M + 2P}{r}\right)^{-1} e^{-\frac{2P}{r}} dr^2 + r^2 e^{-\frac{2P}{r}} d\Omega_2^2$$

with $\psi = \frac{P}{r}$.

The metric approaches the Schwarzschild case if we set $P = 0$ as someone would expect. It is obvious when you get rid of the charge, you are leading to the basic case, of a static and spherical black hole. If we go back to the Newtonian gravitational field now, we will remember, if we set $G = 1$ using the natural system of units, that $\psi_N = -M/r$ where N denotes Newtonian. So the phantom field will be $\psi = P/r$, which denotes then that the Phantom field will have a negative charge. However someone can't make the assumption, that the phantom charge contributes a long-range, attractive force to the physical mass.

Similar to Schwarzschild case, $r = 2M + 2P$ is the regular event horizon and r = 0 is the curvature singularity. The corresponding Hawking temperature is

$$T = \frac{e^{\frac{-P}{M+P}}}{8\pi(M + P)} \tag{3.14}$$

This reveals that the corresponding Hawking temperature increases with the presence of the phantom charge. For the maximum value of electrical charge, i.e. Q = M/ 2 (That is the transition between black hole and singularity), we have the non-vanishing temperature T = e/(4M).

3.3 In understanding the Phantom Black Hole

In [14] there exists a chapter named physical realizations where we can see the exact formation of the phantom black hole, via a theoretical model of a fluid ball being soaked in the phantom field. One can their understand the process in which a phantom black hole should be constructed, similar with the Reissner-Nordström case. Alhtough we won't get busy with this situation. What we will see is a few words about the physical part of the phantom black holes, and whether they or not violate the energy conditions of general relativity.

As far as the energy conditions are concerned, in [4], we see that a phantom scalar field itself doesn't respect the usual energy conditions so nonsingular solutions would be expected in black holes physics or in cosmology. Their they propose a big variety of solutions, and end in a way where they say:

"Such solutions also lead to the idea that our Universe could be created from a phantom dominated collapse in another universe, with KS (Kantowski-Sachs) expansion and isotropization after crossing the horizon. Explicit examples of regular solutions are built and discussed. Possible generalizations include k -essence type scalar fields (with a potential) and scalar-tensor theories of gravity"

There, someone can see how the phantom fields may be a part of the possible cosmological endings. More about energy conditions will be presented in last chapter.

3.4 Both Phantom and Dilaton fields

In the beginning of this analysis we dealt with the dilaton black hole and then went into the phantom case, with a Wick rotation. However, there is no reason why these two kind of fields can't coexist in the same theory. Especially, the dilaton-phantom case is being examined in [14]. There, the authors deal with a combined action of these fields, which reads

$$S = \int d^4x\sqrt{-g}[R - \partial_\mu\phi\partial^\mu\phi + \partial_\mu\psi\partial^\mu\psi - e^{-2\alpha\phi+2\beta\psi}F^2] \tag{3.15}$$

and as we saw ϕ, ψ are the dilaton and phantom fields while α, β their coupling constants respectively. Following the same analysis, the authors vary the action with respect to the metric, Maxwell, phantom and dilaton fields. Then they took the most general form of the metric for the static spacetime and solve the equations of motion in the end. There, they found that:

$$
\begin{aligned}
e^{2\phi/\alpha} = e^{2\psi/\beta} &= \left(1 - \frac{r_-}{r}\right)^{\frac{2}{1+\alpha^2+\beta^2}} \\
M &= \frac{r_+}{2} + \frac{1-\alpha^2+\beta^2}{1+\alpha^2+\beta^2} \cdot \frac{r_-}{2}, \\
Q^2 &= \frac{r_+ r_-}{1+\alpha^2-\beta^2}.
\end{aligned} \tag{3.16}
$$

So we can see with a glimpse that when $\alpha^2 = \beta^2$, revives the Reissner-Nordström case. While the other cases are $\alpha^2 > \beta^2$ and $\alpha^2 < \beta^2$ where we get a dilaton like and a phantom like black hole respectivily. There is always a sign difference before the two coupling constants in the expressions of the metric and the physical mass and charge. It follows then that the effect of the phantom field is opposite to that of the dilaton field. There it comes to enhance this point the paper, [14] where we encounter the following words, which we have already mentioned:

"The transition between black holes and naked singularities occurs at $Q = M/\sqrt{1+\beta^2}$ rather than $Q = M/\sqrt{1-\beta 2}$ as in the dilaton case. For the electrically charged dilaton black holes, the extremal value corresponds to the case where the repulsive force of the electric charge can exactly destroy the event horizon (or the repulsive force of electric charge exactly balances the attractive forces of mass and dilaton). However, for the phantom black holes, the extremal value corresponds to the case where the repulsive forces of electrical charge and phantom charge can exactly destroy the event horizon. In other words, dilaton field contributes an extra attractive force and phantom field contributes an extra repulsive force between black holes."

There we see again the repulsive force bringing apart even the phantom black holes, by nature of the phantom fields. We encounter their in a way, the famous riveting senario of the Big Rip, bringing apart every corner of space.

3.4.1 A Quintom model for Dark Energy

As we mentioned earlier, current observations seem to mildly favor an evolving dark energy with the equation of state getting across -1 [20]. However, neither quintessence nor phantom can fulfill this transition. So the models of combination of quintessence scalar field and phantom scalar field, which is called quintom have been developed (from the words quintessence and phantom fields respectively). As we found in research papers [13], "oscillating Quintom can unify the early inflation and current acceleration of the universe, leading to oscillations of the Hubble constant and a recurring universe". Though oscillating Quintom wouldn't lead to a Big Rip as a phantom menace would expect, with the universe torned apart, neither to a big crunch, with the universes heat death. The Quintom senario suggest that "the scale factor keeps increasing from one period to another and leads naturally to a highly flat universe. The universe in this model recurs itself and we are only staying among one of the epochs, in which sense the coincidence problem is reconciled."

It would be interesting setting here just a note for this topic, because the quintom model can also be realized in the dilaton-phantom frame. So if we consider the action in the case of both phantom and dilaton fields we have:

$$S = \int d^4x\sqrt{-g}[R - p - 2\partial_\mu\phi\partial^\mu\phi + 2\partial_\mu\psi\partial^\mu\psi - V_1(\phi) - V_2(\psi)], \quad (3.17)$$

Where $V_1(\phi), V_2(\psi)$ are the potentials of dilaton and phantom fields which have been examinded in previous analysis [(see) (2.36) and (3.3)] and p the Lagrangian for dark matter. Cosmologically, in the Friedmann equations, one can see that, if we consider a flat universe described by the flat FRW metric, the equations of motion would be:

$$\begin{aligned} 3H^2 &= 8\pi(\dot{\phi}^2 - \dot{\psi}^2 + \frac{1}{2}V_1 + \frac{1}{2}V_2 + \rho_{m0}\alpha^{-3}), \\ \ddot{\phi} &= -3H\dot{\phi} - \frac{1}{4}\frac{\partial V_1}{\partial \phi}, \\ \ddot{\psi} &= -3H\dot{\psi} + \frac{1}{4}\frac{\partial V_2}{\partial \psi}, \end{aligned} \quad (3.18)$$

where we see again the sign difference between the phantom and dilaton fields, while dot denotes the derivative with respect to time and $\alpha(t)$ is the scale factor of the Universe. $H \equiv \dot{a}/a$ is the Hubble parameter and ρ_{m0} is the energy density of the dark matter today. So the equation of state for dark energy would be,

$$w = \frac{\dot{\phi}^2 - \dot{\psi}^2 - \frac{1}{2}V_1 - \frac{1}{2}V_2}{\dot{\phi}^2 - \dot{\psi}^2 + \frac{1}{2}V_1 + \frac{1}{2}V_2} \quad (3.19)$$

This equation tells us whether the difference of the kinetic energy between the dilaton field and phantom field evolves or not. So the difference is initially positive, then zero, finally negative, then w crosses 1 smoothly. This is the effect of quintom.

Chapter 4

Conditions of Phantom Black Holes

4.1 Energy Conditions

4.1.1 A glimpse to the principles

We want to remember in this part the energy conditions in the general case, and then investigate if phantom black holes discipline in these principles. So energy conditions are some rules that tells us if a physical system is truly "physical", if it obeys some natural identities of matter, energy and logic. Even if the logic is the one we have for the behaviour of natural systems nowadays. Energy conditions are coordinate-invariant restrictions on the energy-momentum tensor, which takes part in Einstein's equations. There from this tensor the theory constructs some scalar-tensor quantities by contracting it with timelike or null vectors and express some physical restrictions on the result. This is a point where the fluid like nature of the stress-energy tensor is indeed very helpful in extracting meaningful results. So we will set for reader's facility the basic energy conditions here, as they are proposed in [9].

- Weak Energy Condition (WEC): The energy density as measured by any observer with a timelike four-velocity t^μ, is non-negative, which formally can be expressed as

$$T_{\mu\nu}t^\mu t^\nu \geq 0, \qquad \forall t: \qquad t^\mu t_\mu < 0 \tag{4.1}$$

  This equivalently means that $\rho \geq 0$ and $\rho + P \geq 0$

- Null Energy Condition (NEC): expresses the requirement that the geometry has an attractive effect on null geodesics,

$$T_{\mu\nu}l^\mu l^\nu \geq 0, \qquad \forall l: \qquad l^\mu l_\mu = 0 \tag{4.2}$$

where l^μ is any null four-vector. This is a straightforward generalization of the WEC. The energy density may now be negative as long as there is a compensating positive pressure.

- Strong Energy Condition (SEC): expresses the requirement that the geometry has an attractive effect on timelike geodesics.

$$T_{\mu\nu}t^\mu t^\nu \geq \frac{1}{2}Tg_{\mu\nu}t^\mu t^\nu, \qquad \forall t: \qquad t^\mu t_\mu < 0 \tag{4.3}$$

  where T is the gravitational trace of the stress tensor $T \equiv g^{\mu\nu}T_{\mu\nu}$.

- Dominant Energy Condition (DEC): This energy condition refers to the current density: $J^\alpha = -T^\alpha_\beta t_\beta$, i.e. the energy density current as seen by an observer with 4-velocity t^μ. J^α should be causal and future directed for all timelike and future directed t^α. Since t^α is timelike and future directed, the above conditions are mathematically expressed as follows

$$J_\alpha t^\alpha \leq 0 \quad and \quad J_\alpha J^\alpha \leq 0 \tag{4.4}$$

  which yields the DEC as a set of the two requirements

$$T_{\mu\nu}t^\mu t^\nu \geq 0 \quad and \quad T_{\mu\nu}T^\mu_\alpha t^\nu t^\alpha \leq 0 \quad \forall t: \qquad t^\mu t_\mu < 0 \tag{4.5}$$

  Where the first one is the Weak Energy Condition.

The most physically reasonable condition is the Dominant Energy Condition (DEC), which is what allows one to prove that energy can't propagate faster than the speed of light. Sean Carroll then pointed out that DEC would exclude the $w < -1$ possibility. So going back to the words of its article called "Phantom Energy"[6] "people were happily ignoring $w < -1$ a priori" as they believed that this occasion was breaking causality, we suppose. Although, Robert Caldwell, had another idea on, when investigating the "phantom menace". [5].

4.1.2 Phantom's Energy Conditions

The idea of Robert Cladwell was the following: have a scalar field rolling in a potential, but give it a negative kinetic energy. That means that the field tends to roll up the hill to the top of the potential, rather than rolling down to the bottom. The energy density thus tends to increase, implying $w < -1$. Caldwell called his idea "phantom energy," both because the Phantom Menace had just come out (the first episode of the movie series Star Wars) and also because negative-kinetic-energy fields also appear in the context of quantized gauge theories, where they are called "ghost" fields. If w is less than -1 and constant, the energy density grows without bound and everything in the universe is ripped to shreds at some finite point in the future. This is the idea of the Big Rip, however noone tells us that w won't variate and change value in the nick of time, because then the Big Rip senario would go away.

DEC violation

From the above we see that in a way the phantom fields violate the Dominant Energy Condition, if someone imagines how it breaks the common sense, i.e. the causality of the energy density, for a field to run up to an uphill potential. This nature of physics would be something new, and changes things as we know them. So by then it is where the the phantom fields are a candidate in dark matter-energy and quintessence theories.

WEC violation

Even from the begining of this thesis, one could have noticed the violation of WEC. The first point where a sharp mind would stare, was in the introduction of dilaton and phantom. Both of them own their existence in string theory and Brans-Dick extensions in Einstein's work. Violation of WEC is one of the natural consequences of string theory. But lets investigate the case from "another" point of view.

If we go to the basics of the dark energy we' ll see that dark energy it is a component with a positive energy density $\rho = 0$ and a negative pressure $p < -(1/3)\rho$. This dark energy would be either of the form of a vacuum energy with $p = -\rho$ either a dynamically evolving scalar field with negative pressure. By definition, phantom energy is being proposed as $\rho + p < 0$. In this case the cosmological phantom energy density grows at large times and disrupts finally all bounded objects up to subnuclear scale, and violates by definition the weak energy condition which states that $\rho + p \geq 0$.

NEC violation

As we saw the energy density in a rolling up potential tends to increase. However, in [21] we can see that (4.2), the NEC, can be seen through the conservation of the covariant energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$. Through cosmology, and Friedman equations, we can read this equation as,

$$\frac{d\rho}{dt} = -3H(\rho + p). \tag{4.6}$$

Rubakov says, "Thus, the NEC implies that energy density always decreases in expanding Universe" as we can see from the minus sign in the above equation. Phantom fields so seem to violate the most basic energy condition, the NEC, because if this one is violated then in a "chain reaction" all the other conditions will be violated too.

Their is a wide range of papers and investigations, which propose scalar tensor-theories and ways in which the dark energy seem to violate the NEC- and approach de Sitter from below- the phantom region. In [11, 21] someone can investigate more and more scalar tensor theories, with or without extra degrees of freedom (as there is in dilaton's or phantom's case), which violate some of the energy conditions.

4.2 Black Hole Thermodynamics

Except from the energy conditions, which are a measure in the minds of the physicist, for a physical theory to be acceptable for research, there are the thermodynamic laws of black holes too, which we have to test in the phantom black hole's case.

As we did in the previous subsection we are going to mention the laws of black hole thermodynamics, and then see what happens in the phantom black hole case. Our reference for the following is [9]

- Zeroth law: The surface gravity of a stationary black hole remains constant on the horizon. A system in thermal equilibrium will have settled to a stationary state, corresponding to a stationary black hole. In a few words, the temperature of a body in thermal equilibrium is constant all over the body.

- First law: If a stationary black hole of mass M, angular momentum J (and electric charge Q) is perturbed so that it settles down to another black hole of mass $M + M\delta M$, angular momentum $J + \delta J$ (and charge $Q+$), then.
$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J (+\Phi_H \delta Q) \tag{4.7}$$
The above equation yields, in principle, the concept of conservation of energy

- Second Law: In any physical process, the area of the black hole does not decrease. However, in the general case this can be true only if spacetime does not violate the null energy conditions (NEC). In thermodynamics, the second law states in a thermodynamic process, the entropy of an isolated system cannot decrease, which indeed verifies our understanding of considering the area of the black hole as its entropy.

- Third Law: No physical process exists that allows one to reach an exactly extremal black hole. An extremal black hole have vanishing surface gravity, which implies that their temperature may reach absolute zero. Recalling the third law of thermodynamics: It is impossible to reach the absolute zero temperature state with a finite number of thermodynamic steps. We can go to zero temperature asymptotically, but we cannot reach it.

4.2.1 Phantom's thermodynamic conditions

2nd law violation

One may have already noticed that in the phrase of the second law, mentioned above, it is being proposed that this law is true only if the NEC isn't violated. Although, we have already seen that this case isn't the one which fits with the phantom black hole manner. We have seen that phantom field has important consequences on the properties of black holes. Babichev, Dokuchaev and Eroshenko [2] have found a very interesting feature, that the mass (and consequently the entropy) of a black hole decreases in such a process as an accretion of phantom fluid onto a black hole. This one is similar to the result of Gao and Shuang. This result comes in contrast with the second law of thermodynamics which states that the entropy of an isolated system in a thermodynamical process, cannot decrease. Now since the flow is outwards all the time in the expanding Universe, the authors conclude that at some times, the phantom black holes would be evaporated completely. And this is why they are part of the phantom fields senario, the Big Rip. Babichev and his partners took it a step further and found the "law" of the black hole mass evolution through time.

$$M = M_i\left(1 + \frac{M_i}{M_0\tau}\frac{t}{\tau - t}\right) \tag{4.8}$$

where τ is the time of the "doomsday", M_i is the initial mass of the black hole and M_0 is a constant depending on the accretion flux A and the coupling constant α. The diminishing of a black hole mass is caused by the violation of the dominant energy condition, which includes the WEC , $\rho + p \geq 0$, as a principal assumption of the classical black hole 'non-diminishing' theorems. The possible physical interpretation of a black hole mass diminishing is that accreting particles of phantom scalar field have a total negative energy. The similar negative energy particles are created in the Hawking radiation process and participate the Penrose black hole rotation energy extraction mechanism.

3rd law conservation

We have already mentioned about the third law of black hole mechanics. We have seen that for large coupling constant, (β in the equations of previous sections), a small amount of electrical charge would make remarkable change on the structure of spacetime. In particular, we have mentioned that, for every value of the coupling constant β, taking part in (3.11), the surface gravity does not diverge. So extremal black holes which have vanishing surface gravity can never be achieved in the phantom case, so the third law of thermodynamics for black holes is remedied. Another clue is that for $\beta \neq 0$ the extremal black hole, which exist in the case where $r_- = r_+$, can't be achieved. The case of $r_- = r_+$ necessarilly leads to $\beta = 0$ which means no coupling constant, no phantom field and thus, the original Reissner- Norström spacetime is recovered, which, as we know, the extremal black hole case, violates the concept of the third thermodynamical law, and is being regarded as a theoretical toy.

A note would be that in the dilaton, for $\beta < 1$ the surface gravity goes to zero in the extremal limit, for $\beta = 1$ it approaches a constant and for $\beta > 1$ it diverges, which propose a new investigation channel. However we won't deal with it.

4.3 About Constraints

Phantom black holes, phantom energy, phantom fields etc. are very interesting topics, with a big amount of papers concerning them. However, the nature of physics needs experimentally questionable theories, and so constraints do exist in the theories. Although, scientists did not stop at the weakness of $w < -1$ to describe physics, due to WEC violation. If they would, nothing of the above would have been existed. The present accelerated expansion of the universe seems to be an experimental fact, since 2003, when data from distant type Ia supernovae [20] have been corroborated by those from the cosmic microwave background from WMAP observations. [3]. Furthermore large scale structure (LSS) brings further cosmological data. These three experimental-cosmological ways of seeing the know-how of the cosmos, allowed the theory of phantom energy and therefore phantom black holes to exist, as so they did in many other quintessence theories. These cosmological data, set the constraints for the equation of state for dark energy and so physicists saw the possibility of $w_Q < -1$, being a real one.[15]

Recent papers, after 2016 use another way to constrain the model parameters, the Baryonic Acoustic Oscillations, coming from the Planck 2015 results. Planck 2018 seem to favor phantom models from the observational results. In addition, the cosmological perturbations have been a useful tool for cosmologists. They predict the matter distribution that can be compared with the observations. The predicted observables within the cosmological perturbations theory have been widely used to test several models of dark energy, as well as dark energy-dark matter interacting models and f(R) modified gravity.

The test for observational constraints, goes long way through the cosmological data of the aforementioned methods. Phantom black holes could exist, only if phantom fields and all about phantom models could exist. So the constraints, are the same. We mention a few papers, which investigate recent observational data, and the constraints which they contribute. [10, 1, 17]

Chapter 5

Conclusion

Concluding what we have dealt with in this review, the phantom field considered has some great features because of the negative kinetic energy. Using the phantom field, we have constructed through the dilaton's case the exact solutions of electrically charged phantom black holes with the cosmological constant. We have seen, how one can get the Reissner-Nordström and the Scwarzschild metric, going in the classical situations of black holes.

We went from dilaton black holes to phantom black holes with a Wick rotation, and examined the differences between ths kind of spacetime anomalies. We dealt with the phantom black hole's metric and special features such as surface gravity and the scenario of extremal black holes. Even if we didn't make a great analysis of this part, we mentioned the quintom case and its possible cosmological consequences.

The energy conditions of phantom black holes is an important aspect which we encountered and takes a crucial role on how we imagine those gravitational anomalies, as being a candidate for dark matter. Violation of NEC, tests our understanding of phantom's nature and makes phantom black holes much different from simple black holes. As they seem to evaporate and end their lives in the Big Rip senario, thermodynamics of phantom black holes seem to bear some differences with the classical black holes, when someone comes up with the violation of the second law of black hole thermodynamics. In the end we slightly mentioned some observational methods of cosmological data and observational constraints in phantom theories, at the availability of whoever wants to research more.

There is a huge amount of papers and analysis in this topic which goes from gravitational quasinormal modes, [18], experimental methods, as gravtitational lensing in phantom black holes [12] , and a way to another huge topic, the influence of phantom fields in Cosmology. There are many research to be made so forth, and its going on, in a theoretical level, such as investigating the thermal stability and criticality of those systems[22] or the research of wormholes in modified f(R) gravities sourced by these phantom scalar fields [19].

The good news is that more and more precise observational results are being collected from large research groups and research programms, while the theoretical background flourish every year passing. Ending this thesis, relevant to phantom black holes and the phantom nature generally, we have to mention that dark matter and dark energy still remains a mystery. here are more to be investigated in the corners of universe's future and its own acceleration, whether phantom scalar fields have to be blamed or not.

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