Continuous-time Equilibrium Returns in Markets with Frictions

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Chapter 1

INTRODUCTION

The concept of optimality with regards to investment selection is an extensively studied "problem" in Mathematical Finance. The substantial variety of approaches aiming to answer the question: "*What constitutes an optimal portfolio?*" can be traced back, at least partially, to the following:

What are we trying to optimize? In other words, what would be a realistic representation of an intestor's criterion.

For this thesis, this question is the initial setting. For this, we adapt a stochastic version of the *Mean-variance* portfolio selection (MVPS) criterion. More precisely, we consider that the investors participating in the market invest in a *non self-financing strategy* aiming for maximal mean and minimal variance [1]. Moreover, after having determined the optimal strategies—according to the investors' preferences—we study their influence on the market at the *equilibrium*.

More precisely, the main goals of this thesis are:

(I) Study the MVPS criterion when the drift of the market is derived endogenously.

(II) Analyze the model and the results of [Bou+18].

(III) Introduce the price impact model of [Ant17] in continuous time without frictions.

(IV) Generalize the notion of impact for a market with frictions.

The second chapter of the thesis is introductory and it aims to define some mathematical concepts which are frequently used. This chapter is partially supplemented by Appendix A, which we use to further analyze and/or clarify some concepts present in the main text. To this end, throughout

[1] Alternatively, as expressed in [GP16], we could say that the investors have *mean-variance preferences* over the change in their wealth over time.

the thesis, there can also be found some special blocks of text called “Closer looks”, which focus on delving somewhat deeper in a few relevant notions without disrupting the main flow of the text.

In the third chapter and onwards, we work towards analyzing the main part of the thesis, that is defining and solving the portfolio optimization problem and exploring the influence of its solutions at the equilibrium, in various types of markets. Note that in Chapter 3 and 4, we essentially analyze and discuss the results of [Bou+18]. More precisely, we initially consider a financial market with 1 riskless and d risky assets, modelled by a diffusion process, where the diffusion coefficient is exogenously given, while its drift is derived endogenously by an *equilibrium condition*. There are N investors with heterogenous risk aversions that participate in the market and trade amongst themselves. The investors are endowed with a random wealth process consisting of two parts: their holdings on the financial market, which naturally depend on their strategy of choice, and a exogenously given random endowment which captures other sources of income, possibly correlated with the assets. The rest of the market's participants, called noise traders, are assumed to follow strategies that are not derived through any specific optimization criterion. Having introduced the above, we assume that the trades of risky assets in the market do not incur transaction costs, i.e. a *frictionless market*, and define an appropriate objective function, representing the mean-variance preferences over the change of an investor's wealth. Optimizing the aforementioned function over the space of potential strategies and for each investor independently gives us the optimal asset allocation in a frictionless market. Closing the third chapter, we concern ourselves with determining the drift coefficient of the process that drives the market. In fact, the equilibrium condition that is imposed in the market dictates that the sum of the optimal strategies of the investors matches the exogenous demand of the noise traders. Via this condition we derive endogenously the drift of the market, called *equilibrium returns*.

In the fourth chapter, we introduce frictions in the market. We follow [Bou+18] and assume that each trade incurs a cost in the form of a *transaction tax*, which goes to an exogenous recipient (who does not participate in the market). To derive the respective optimal strategies in a market with frictions, we have to assume that the strategies of the investors are now absolutely continuous and incur transaction costs proportional to the square of their pointwise derivatives (that is, the investors' *trading rates*). This yields a new objective function for investors' optimal allocation, which in turn is characterized by a system of coupled but linear Forward-Backward SDEs. These equations can be solved explicitly in terms of matrix power series, leading to closed-form expressions for the liquidity premia between the equilibrium returns in a market with frictions and their frictionless counterpart. Interestingly enough, under the assumption of homogenous risk aversions and without the presence of any noise traders in the market, the frictional equilibrium returns revert to their frictionless form.

In the fifth chapter we go back to the frictionless optimization problem presented in the third chapter, and we introduce a notion of *price impact*. Price impact can be defined as the effect that an investor has on the price of a risky asset as a result of her buying or selling it. In some sense, we could view the price of the assets as a function of an investor's strategy. The aforementioned give us a natural way to model the concept of "impact", through the equilibrium returns. Recall that this process drives the prices of the risky assets that exist in the market and is determined via the strategies of the investors by the equilibrium condition. Similar concepts are also studied in [Ant17] and [AK17]. Having derived a form for the price impact through the equilibrium returns, we have essentially created a new optimization problem (since the wealth of each investor depends on the assets' return process). Its solution determines the respective optimal strategies in a frictionless market with price impact (called *best-response*). When all N investors adapt the same best-response strategy, the market equilibrates at the induced fixed point, determining a *Nash-equilibrium*. We also note that under the assumptions of homogenous risk aversions and without the presence of any noise traders in the market, the Nash equilibrium reverts to the frictionless equilibrium returns of the third chapter.

Lastly, in the sixth chapter we extend the notions already introduced during the fifth chapter, but in a market with frictions. Therein, for tractability, we consider that the investors have homogenous risk aversions. By this assumption, the solution to the new objective function in a market with frictions under the price impact of a single investor is characterized by a second order, linear, non-homogenous (random) ODE. We again note that the frictional equilibrium returns under the price impact of a single investor revert to their frictionless counterpart in both of the following cases: (i)

same risk aversions and no noise traders and (*ii*) the transaction costs, which come in the form of a transaction tax, go to zero (while the investors have homogenous risk aversions).

Chapter 2

INTRODUCTORY CONCEPTS

In this chapter we shall give a brief introduction of the concepts depicted in the graph below:

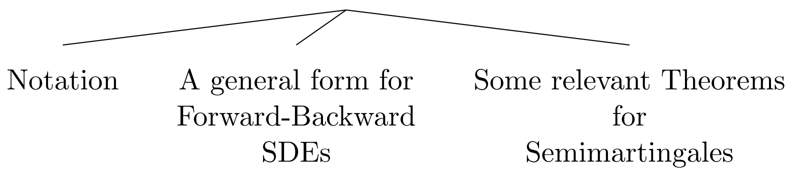

Figure 2.1: Outline of the 2^{nd} chapter

2.1 Notation

Throughout this thesis, we use the following notations for a continuous stochastic process for $t \in \mathscr{T}$, $\omega \in \Omega$ and a state space S:

(I) The mapping depicted as either X or $(X_t)_{t\in\mathscr{T}}$, or equivalently $X(\cdot,\cdot) : \mathscr{T} \times \Omega \to S$ denotes a stochastic process.

(II) By fixing t we get a random variable $X(t,\cdot)$ or equivalently $X_t : \omega \mapsto X(t,\omega)$. From this perspective the stochastic process is a collection of random variables X_t, indexed by the time variable t.

(III) By fixing ω we get a function of time $X(\cdot,\omega)$ or equivalently $X^\omega : t \mapsto X(t,\omega)$. X^ω denotes a realization/sample path of the stochastic process, which is formed by taking a single possible value of each random variable of the stochastic process. From this perspective the process is a function of time.

We fix a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ equipped with a complete filtration $(\mathscr{F}_t)_{t\in\mathscr{T}}$ [1] generated by a standard d-dimensional Brownian motion $(W_t)_{t\in\mathscr{T}}$, where $\mathscr{T} = [0, T]$ for $T \in (0, \infty)$ ("finite time horizon") or $\mathscr{T} = [0, \infty)$ for $T = \infty$ ("infinite time horizon"). Moreover, we fix a non-negative constant $r \geq 0$ (which will stand for the discounting interest rate) and let an $\mathbb{R}^l$-valued progressive process $(X_t)_{t\in\mathscr{T}}$ belonging to $\mathscr{L}_r^p$, $p \geq 1$, if $\|\cdot\|_{(p,r)} = \left(\mathbb{E}[\int_0^T e^{-rt}\|X_t\|^p dt]\right)^{1/p} < \infty$ holds [2]. That is:

For more information about progressive processes, local martingales and the space $\mathscr{L}_r^p$, refer to *A.1.*

$$\mathscr{L}_r^p = \left\{X : \Omega \times \mathscr{T} \to \mathbb{R}^l : \text{ X is progressively measurable s.t. } \|\cdot\|_{(p,r)} < \infty\right\}^3.$$

2.2 Forward Backward Stochastic Differential Equations (FBSDE)

As stated in the introduction, the equilibrium of a market with frictions will be characterized as a unique solution of a system of coupled but linear Forward-Backward Stochastic Differential Equations (FBSDE). Below, we give a general representation for such system considering an one-dimensional Brownian motion, as in [MJ07].

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbf{P})$ be a complete filtered probability space on which an one-dimensional standard Brownian motion $(W_t)_{t\geq 0}$ is defined, such that $(\mathscr{F}_t)_{t\geq 0}$ is the natural filtration generated by it. A general form of a system of coupled linear FBSDE, driven by a one-dimensional Brownian motion, is:

$$\begin{cases} dX_t = \Big(AX_t + BY_t + CZ_t + Db_t\Big)dt + \Big(A_1X_t + B_1Y_t + C_1Z_t + D_1\sigma_t\Big)dW_t, \\ dY_t = \Big(\hat{A}X_t + \hat{B}Y_t + \hat{C}Z_t + \hat{D}\hat{b}_t\Big)dt + \Big(\hat{A}_1X_t + \hat{B}_1Y_t + \hat{C}_1Z_t + \hat{D}_1\hat{\sigma}_t\Big)dW_t, \\ t \in [0, T], \\ X_0 = x, \qquad Y_T = GX_T + Fg. \end{cases}$$

[1] Let $\mathscr{N}_\mathbf{P} = \{A \subseteq \Omega : A \subseteq B \; s.t. \; \mathbf{P}(B) = 0\}$. A filtration is called complete if every $(\mathscr{F}_t)_{t\geq 0}$ contains $\mathscr{N}_\mathbf{P}$.

[2] Note that $\|\cdot\|$ could be any norm in $\mathbb{R}^l$.

[3] Note that since we are working with Lebesgue-integrable functions, most equalities are understood in an "almost everywhere sense".

In the above, $A, A_1, \hat{A}, \hat{A}_1, B, B_1, \hat{B}, \hat{B}_1$ etc, are (deterministic) matrices of suitable sizes, $b, \sigma, \hat{b}, \hat{\sigma}$ are measurable, adapted stochastic processes and g is a measurable random variable. We see that the equation is forward with regards to $X(\cdot)$, its initial point is given and backward with regards to $Y(\cdot)$. Note that the solution to the above system will be of the form (X_t, Y_t, Z_t). It is clear that the above system is coupled, since at least one differential equation (in this case two) depends on both output variables.

2.3 A primer on semimartingales and quadratic variation

Let us initially define the concept of quadratic variation as well as some relevant properties.

A Closer Look I: Quadratic variation, covariation and the notion of compensators

Following the notation of [KS91], the *variation* of a stochastic process $X : [0, t] \to \mathbb{R}$ over some partition $P = \{0 = t_0, t_1, ..., t_n = t\}$ is defined by:

$$V_1(X, P) = \sum_{i=1}^{n} |\underbrace{X(t_i) - X(t_{i-1})}_{\Delta X}|,$$

$$V_1(X, [0, t]) = \sup_P \{V_1(X, P) : \text{ for any partition } P \text{ of } [0, t]\}.$$

The above equations defines the total variation of a process. A function is of bounded variation on $[0, t]$ if $V_1(X, [0, t])$ is finite [a]. Let us now focus on the concept of quadratic variation, which plays a fundamental role in the theory of stochastic processes. To this end, we define the following:

$$V_2(X, P) = \sum_{i=1}^{n} (X(t_i) - X(t_{i-1}))^2,$$

$$[X]_t = \lim_{\|P\| \to 0} V_2(X, P),$$

where $\|P\|$ is the so-called *mesh* [b]. If the above limit converges in probability for any partition, then we say that the quadratic variation of X exists [c]. In a similar manner, the notion of *covariation* between X, Y is defined as follows (see further in [Kal02]):

$$C(X,Y,P) = \sum_{i=1}^{n} \underbrace{(X(t_i) - X(t_{i-1}))}_{\Delta X} \underbrace{(Y(t_i) - Y(t_{i-1}))}_{\Delta Y}$$

$$[X,Y]_t = \lim_{\|P\| \to 0} C(X,Y,P)$$

If the above limit converges in probability for any partition, we say that the covariation of X, Y exists. Equivalently, the above concepts in integral form are:

$$V_1(X,[0,t]) = \int_0^t |dX_s|,$$
$$[X]_t = \int_0^t (dX_s)^2,$$
$$[X,Y]_t = \int_0^t dX_s dY_s.$$

In the case of a d-dimensional process $X = (X^{(1)}, ..., X^{(d)})$, the quadratic variation $[X]$ is defined as the $d \times d$ matrix valued process:

$$[X] = \begin{bmatrix} [X^{(1)}, X^{(1)}] & [X^{(1)}, X^{(2)}] & \cdots & [X^{(1)}, X^{(d)}] \\ [X^{(2)}, X^{(1)}] & [X^{(2)}, X^{(2)}] & \cdots & [X^{(2)}, X^{(d)}] \\ \vdots & \ddots & \cdots & \vdots \\ [X^{(d)}, X^{(1)}] & [X^{(d)}, X^{(2)}] & \cdots & [X^{(d)}, X^{(d)}] \end{bmatrix},$$

the covariation process is also defined accordingly.

[a] The above concepts can be generalized for $X : [0,t] \to Y$, where Y is a normed space, as $V_1(X,[0,t] = \sup_P \left\{ \sum_{i=1}^{n} \|X(t_i) - X(t_{i-1})\|_Y : \text{ for any partition } P \text{ of } [0,t] \right\}$.

[b] The mesh of a partition can be defined as: $max\{|t_i - t_{i-1}| : i = 1, ..., n\}$ [Hij16]. Another partition Q of the given interval $[a,b]$ is defined as a refinement of the partition P, if Q contains all the points of P and possibly some other points as well. In this case Q is said to be "finer" than P.

[c] Note that in the context of Analysis, a different definition is offered. For more information, refer to [MW12].

We now give some useful definitions as stated in [Pro05], which help us better understand the modelling process of the problem at hand and also provide some insight for a few concepts discussed later on.

Definition 2.3.1 (Finite variation process) *An adapted, a.s. right continuous with left limits process*

A is a finite variation process (FV) if a.s. the paths of A are of bounded variation on each compact interval of $[0, \infty)$. ♢

Definition 2.3.2 (Predictable σ-algebra) *The predictable σ-algebra on $\mathbb{R}_+ \times \Omega$, also denoted by $\mathscr{P}$, is generated by the left continuous and adapted processes (or by the continuous adapted processes). A stochastic process said to be predictable if it is $\mathscr{P}$-measurable [4].* ♢

Definition 2.3.3 (Decomposable process) *An adapted, a.s. right continuous with left limits process X is* decomposable *if there exist processes N, A such that:*

$$X_t = X_0 + N_t + A_t,$$

with $N_0 = A_0 = 0$, N is a locally square integrable (i.e. $\mathbb{L}^2$ on all compact subsets of the domain) local martingale and A is a finite variation process (these concepts shall be discussed in more length below). ♢

Definition 2.3.4 (Classical semimartingale) *An adapted, a.s. right continuous with left limits process Y is a classical semimartingale if there exist processes M, B with $M_0 = B_0$ such that:*

$$Y_t = Y_0 + M_t + B_t,$$

where M is a local martingale and B is a finite variation process. If, furthermore, B is a predictable process, then the above decomposition is also unique. ♢

The two aforementioned Definitions are linked by the below result as proven in [Pro05], for any local martingale M.

$$M = N + C,$$

where N is a locally square integrable, local martingale and C a FV process. Therefore, the above notions can be summarized in the Theorem presented below which gives equivalent statements for a semimartingale, as shown in [Pro05].

[4] In a similar manner, we define the optional σ-algebra, also denoted by $\mathscr{O}$, which is generated by the right continuous and adapted processes.

Theorem 2.3.1 (Equivalent statements for semimartingales) *Let X be an adapted right continuous with left limits process. The following are equivalent:*

(I) X is a semimartingale.

(II) X is decomposable.

(III) X is a classical semimartingale. ◊

We are now ready to define the concept of *compensators* in the context of quadratic variation/co-variation. The following definitions are given in accordance with [Kal02]. Let A be a process of locally integrable variation. Then A can be uniquely decomposed by means of the *Rao's Theorem* [Pro05] as follows:

$$A = M + \hat{A},$$

where $\hat{A}$ is a predictable finite variation process. Alternatively, we could state that $\hat{A}$ is the unique, predictable, finite variation process such that $A - \hat{A}$ is a local martingale. The following definition stems naturally from the above.

Definition 2.3.5 (Compensator) *Let A be an adapted process with locally integrable variation. The unique, predictable finite variation process $\hat{A}$, with $\hat{A}_0 = 0$, such that $A - \hat{A}$ is a local martingale is called the* compensator *of A.* ◊

Notice that some obvious observations are:

- We have that $\mathbb{E}[A_t] = \mathbb{E}[\hat{A}_t]$ for all t, $0 \leq t \leq \infty$.
- Let A be an adapted increasing process of (locally) integrable variation and let $\mathbb{E}[\int_0^t H_s dA_s] < \infty$ a.s. for every $t > 0$ and progressively measurable H. Then, we have that:

$$\mathbb{E}\left[\int_0^t H_s dA_s\right] = \mathbb{E}\left[\int_0^t H_s d\hat{A}_s\right].$$

  The above stems from the fact that since $A - \hat{A}$ is a martingale, $\int H_s d\big(A_s - \hat{A}_s\big)$ is also and therefore the aforementioned quantity has constant expectation, equal to zero.

Now let's place the above in the context of variations.

Definition 2.3.6 (Quadratic variation's compensator) *Let X be a semimartingale with locally integrable quadratic variation. The* predictable quadratic variation $\langle X, X\rangle_t$ *is the compensator of the quadratic variation process* $[X, X]$*, that is* $\hat{A} = [X, X]$. $\diamondsuit$

Here is a relevant property as shown in [Kal02]:

- For a continuous local martingale the predictable quadratic variation process exists and coincides with the quadratic variation process a.s.

Definition 2.3.7 (Covariation's compensator) *Let X, Y be two semimartingales with locally integrable quadratic variation. The predictable covariation $\langle X, Y\rangle_t$ is the compensator of the covariation process* $[X, Y]$. $\diamondsuit$

Here is a relevant property as shown in [Kal02]:

- For continuous local martingales X, Y, the predictable covariation exists and coincides with the covariation process a.s.

Chapter 3

THE MODEL FOR THE FRICTIONLESS MARKET

In this chapter we examine the concepts stated in the graph below:

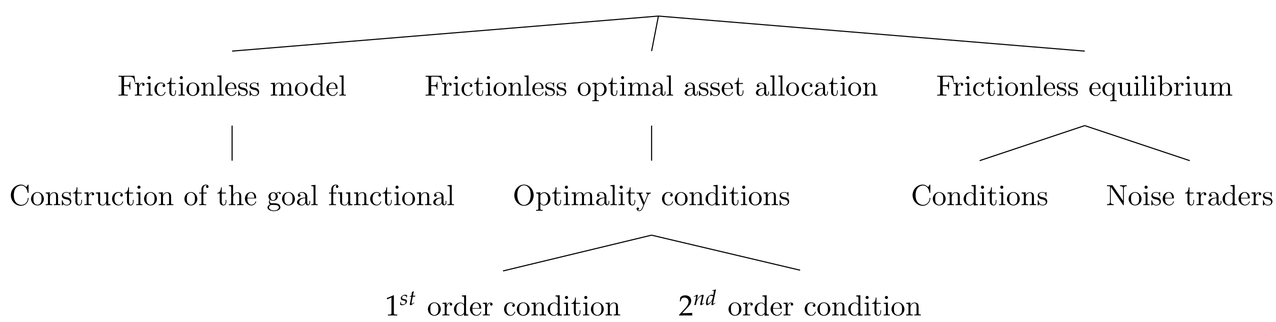

Figure 3.1: Outline of the 3^{rd} chapter

3.1 The frictionless model

In this section we are going to primarily discuss the construction of the optimization problem, i.e. the optimal allocation of assets of an investor, in the context of a market without frictions. For such a market this means one can trade continuously with no transaction costs, taxes, or other encumbrances of any kind [Dur10]. The market consists of two kinds of participants, that is:

(I) The investors, indexed by $n = 1, ..., N$ with mean-variance preferences and heterogenous risk exposure and risk aversions.

(II) The noise traders whose decisions to buy or sell an asset are not determined by any optimization criterion.

The objective is to derive a progressively measurable optimal asset allocation $(\phi_t^n)_{t\in\mathscr{T}} \in \mathscr{L}_r^2$, for each investor $n = 1, ..., N$, which maximizes the discounted expected (proportional changes) of her wealth, penalized by their quadratic variation. Equivalently we could say, as expresed in [GP16], the aforementioned objective means that the investor has mean-variance preferences over the change in her wealth for each time period.

Let us now explain a bit more thoroughly the various "building blocks" used to construct the above representation. We consider a financial market with $1+d$ assets. The first one holds no risk and its price process is exogenous and normalized to one. The remaining assets are risky, with dynamics which are driven by a d-dimensional standard Brownian motion $(W_t)_{t\in\mathscr{T}}$.

$$dS_t = \begin{bmatrix} \mu_t^{(1)} \\ \vdots \\ \mu_t^{(d)} \end{bmatrix} \cdot \, dt + \begin{bmatrix} \sigma_{1,1} & \cdots & \sigma_{1,d} \\ \vdots & & \vdots \\ \sigma_{d,1} & \cdots & \sigma_{d,d} \end{bmatrix} \cdot \, dW_t. \tag{3.1}$$

As previously stated, (3.1) defines the *dynamics of the proportional returns of the market.* The drift parameter of (3.1) is defined by the $\mathbb{R}^d$-process $(\mu_t)_{t\in\mathscr{T}}$ which depicts the (instantaneous) return process of the risky assets and will be determined endonegously by an equilibrium condition. Futhermore, the diffusion parameter of the above SDE is assumed to be constant, $\mathbb{R}^{d\times d}$-valued volatility matrix σ, given exogenously—which in turn yields the infinitesimal covariance matrix $\Sigma dt = \sigma\sigma^T dt$. Note that Σ is assumed to be positive-definite and therefore non-singular.

The previously described optimization problem can be posed through the maximization of the functional presented below:

$$\mathscr{F}^n(\phi) = \mathbb{E}\left[\int_0^T e^{-rt}\left(\underbrace{\overbrace{(\phi_t^n)^T dS_t}^{Market}\ \overbrace{+dY_t^n}^{Endowment}}_{Dynamics\ of\ Wealth} - \frac{\gamma^n}{2} d\left\langle \int_0^{\cdot} (\phi^n)^T dS_s + Y^n \right\rangle_t\right)\right] \to maximize, \tag{3.2}$$

where $\gamma^n > 0$ denotes the investor's n risk aversion. We also set without loss of generality that:

$$\gamma^N = \max(\gamma^1, ...\gamma^N).$$

Furthermore, each of the aforementioned investors receive cumulative random endowments $(Y_t^n)_{t\in\mathscr{T}}$ with dynamics:

$$dY_t^n = dA_t^n + (\zeta_t^n)^T \sigma dW_t + dM_t^{\perp,n}, \quad n = 1, ..., N. \tag{3.3}$$

We now discuss in more detail the terms that comprise (3.3) in accordance with [Bou+18].

- We define $(A_t^n)_{t\in\mathcal{T}}$ as the $\mathbb{R}$-valued finite variation process [1], such that $\mathbb{E}[\int_0^T e^{-rt}|dA_t|] < \infty$, which models both cash inflows and outflows for each investor at any moment. We can think of this specific component as the cummulative cashflow of each position that is not necessarily spanned by the assets (eg. some other obligations, salary, private placements etc.).
- We define $\zeta^n \in \mathcal{L}_r^2$ as the $\mathbb{R}^d$-valued process [2], which describes the exposure of an investor to market's shocks. This component essentially means that the endowment of an investor is correlated with the risky assets. Each investor has motive to hedge against those fluctuations of her endowment through trading in the market. Note that the strategies are not self-financing.
- We define $M^{\perp,n}$, such that $\mathbb{E}[\| \int_0^T e^{-rt} d[M]_t \|] < \infty$, as the $\mathbb{R}$-valued orthogonal martingale which models unhedgeable shocks [3]. The purpose of this process is to model forces stemming from outside the market that the investors participate or more specifically, forces that cannot be hedged against by trading in the market.

Motivated by the formal setting of the concepts presented above, we simplify (3.2) as follows:

For more information about tools used for such a simplification, refer to *A.2.*

$$\begin{aligned}
&\mathbb{E}\Bigg[\int_0^T e^{-rt}\Big((\phi_t^n)^T(\mu_t dt + \sigma dW_t) + dA_t^n + (\zeta_t^n)^T\sigma dW_t + dM_t^{\perp,n} \\
&-\frac{\gamma^n}{2}\Big\{2(\phi_t^n)^T dS_t dY_t^n + (\phi_t^n)^T\Sigma\phi_t^n dt + (\zeta_t^n)^T\Sigma\zeta_t^n dt + d\langle M^{\perp,n}\rangle_t\Big\}\Big)\Bigg] = \\
&\mathbb{E}\Bigg[\int_0^T e^{-rt}\Big((\phi_t^n)^T\mu_t dt + \overbrace{(\phi_t^n)^T\sigma dW_t}^{a} + dA_t^n + \overbrace{(\zeta_t^n)^T\sigma dW_t}^{b} + \overbrace{dM_t^{\perp,n}}^{c} \\
&-\frac{\gamma^n}{2}\Big\{\underbrace{(\phi_t^n)^T\Sigma\zeta_t^n dt + (\zeta_t^n)^T\Sigma\phi_t^n dt + (\phi_t^n)^T\Sigma\phi_t^n dt + (\zeta_t^n)^T\Sigma\zeta_t^n dt}_{(\phi_t^n+\zeta_t^n)^T\Sigma(\phi_t^n+\zeta_t^n)dt} + d\langle M^{\perp,n}\rangle_t\Big\}\Big)\Bigg].
\end{aligned} \tag{3.4}$$

[1] A_t is of size $N \times 1$.

[2] ζ_t is of size $d \times N$.

[3] $M_t^{\perp}$ is of size $N \times 1$.

Commenting on (3.4), by the linearity of expectation and the Fubini-Tonelli Theorem, the terms $a,\ b,\ c$ are equal to zero. The aforementioned is a direct result of the following theorem coupled with the basic properties of martingales (see [Kal02] for the proof):

Theorem 3.1.1 (Stochastic integral: Itô, Kunita and Watanabe) *For every continuous local martingale M and progressively measurable process V such that $(V^2 \cdot \langle M \rangle)_t = \int_0^t V_s^2 d\langle M \rangle_s < \infty$ a.s. for every $t > 0$, there exists an a.s. unique continuous local martingale $(V \cdot M)_t = \int_0^t V_s dM_s$ with $(V \cdot M)_0 = 0$.* ◊

Therefore, by properly rearranging the terms of (3.4), the goal functional becomes:

$$
\begin{aligned}
\mathscr{F}^n(\phi) = \mathbb{E}\Bigg[\int_0^T e^{-rt} \Big(\overbrace{(\phi_t^n)^T \mu_t - \frac{\gamma^n}{2}(\phi_t^n + \zeta_t^n)^T \Sigma (\phi_t^n + \zeta_t^n)}^{1^{st}\ part} \Big) dt \\
+ \int_0^T e^{-rt} \Big(\underbrace{dA_t^n - \frac{\gamma^n}{2} d\langle M^{\perp,n} \rangle_t}_{2^{nd}\ part} \Big) \Bigg].
\end{aligned}
\tag{3.5}
$$

Remark 3.1.1 *The first part of (3.5) is linked (but not the same) to the "usual" mean-variance goal functional, as expressed in the context of the MVPS problem (i.e. the problem of producing a portfolio with maximal mean and minimal variance). More precisely, a discrete version of the MVPS problem can be found by maximizing the following expression:*

$$w^T \mu - \gamma w^T \Sigma w,$$

where γ defines the risk aversion parameter, w is a vector of portfolio weights, Σ is the covariance matrix, $w^T \Sigma w$ is the variance of the portfolio and $w^T \mu$ is the expected return of the portfolio.

An appropriate generalization of the MVPS problem in the context of our continuous time model is given by:

$$\max\ \mathbb{E}[V_T(x,\theta)] - \gamma Var[V_T(x,\theta)]\ \textit{over all}\ \theta \in \Theta,$$

where, V depicts the portoflio value, with initial value x, $\gamma > 0$ is a risk aversion parameter, θ_t stands for the holdings in time t and Θ is defined as a class of progressive processes with *appropriate integrability conditions.*

By the above it should be clearer that by optimizing (3.2)*, an investor's goal in investing on the market is twofold. On the one hand, the investor invests to produce a portfolio* ϕ_t^n *that has an optimal trade-off between risk* (Σ) *and expected returns* (μ_t) *and on the other to hedge against the exposure of her endowment* (ζ_t^n) *to the fluctuations of the market. Lastly note, the optimizer for a goal functional of the type shown in* (3.2) *is not dependent on the* 2^{nd} *part of* (3.2)*.* ◊

3.2 Frictionless optimal asset allocation

In this section we discuss the problem of optimal allocation of assets for each investor and how its solution, by certain conditions, leads us to the derivation of equilibirum returns in a frictionless market. To this end, we develop some appropriate background. For more information about how the Gâteaux derivative is defined, refer to A.3.

Optimality conditions

We shall initially present the conditions for a maximum of the goal functional (3.5) and then continue with explicit calculations, in order to derive the optimal allocation of assets for each investor. Note that the goal functionals we consider throughout this thesis are (strictly) concave, therefore we need a few "tools" from convex analysis at our disposal. Namely, we present a few properties which are discussed extensively in [ET99], [GH04] and [KZ05]:

Proposition 3.2.1 *Let* X *be a Banach space and let* $\mathscr{F} : X \supseteq \mathscr{D}(\mathscr{F}) \to \mathbb{R}$ *be Gâteaux differentiable over the closed, convex set* $\mathscr{D}(\mathscr{F})$*.*

(I) $\mathscr{F}$ *being concave over* $\mathscr{D}(\mathscr{F})$ *is equivalent to the second Gâteaux differential existing and being non-positive over* $\mathscr{D}(\mathscr{F})$ *for every "direction" in the domain. In a similar manner, strict concavity is implied by the second diferential being negative* [4]*.*

[4] Note that the second differential being negative is a sufficient condition but not a necessary one for strict concavity.

(II) Assuming that $\mathscr{F}$ is as defined above and concave, if the Gâteaux differential is zero at $\bar{u} \in \mathscr{D}(\mathscr{F})$, then $\bar{u}$ is a global maximum on $\mathscr{D}(\mathscr{F})$. Moreover, strict concavity implies uniqueness. ◊

First order condition

Note that, as previously discussed, ϕ_t^n is a $d \times 1$ vector. We now recall (3.5) and get for all $\theta_t^n \in \mathscr{L}_r^2$:

$$\frac{d\mathscr{F}^n(\phi^n + \rho\theta^n)}{d\rho} = \mathbb{E}\left[\int_0^T e^{-rt}\left((\theta_t^n)^T\mu_t - \frac{\gamma^n}{2}\left(2(\phi_t^n)^T\Sigma\theta_t^n + 2(\theta_t^n)^T\Sigma\theta_t^n\rho + 2(\zeta_t^n)^T\Sigma\theta_t^n\right)\right)dt\right] \Rightarrow$$

$$\left(d\mathscr{F}^n(\phi^n), \theta^n\right) = \mathbb{E}\left[\int_0^T e^{-rt}\left(\mu_t^T\theta_t^n - \gamma^n\left((\phi_t^n)^T\Sigma\theta_t^n + (\zeta_t^n)^T\Sigma\theta_t^n\right)\right)dt\right] =$$

$$\mathbb{E}\left[\int_0^T e^{-rt}\left(\mu_t^T - \gamma^n(\phi_t^n + \zeta_t^n)^T\Sigma\right)\theta_t^n dt\right].$$

Therefore, the first order condition imposes:

$$\mathbb{E}\left[\int_0^T e^{-rt}\left(\mu_t^T - \gamma^n(\phi_t^n + \zeta_t^n)^T\Sigma\right)\theta_t^n dt\right] = 0.$$

An extended version of *fundamental theorem of calculus of variations* (see [JLJ98] among others), gives us the following:

Lemma 3.2.1 *Let f be a real function with $f \in \mathbb{L}^2(\Omega)$, and suppose that for all compactly supported and sufficiently smooth w we have that:*

$$\int_\Omega fw\,dx = 0.$$

Then, $f = 0$ (in $\mathbb{L}^2$, that is almost everywhere in $\mathscr{L}^2$ with respect to the assigned measure). ◊

Proof: Since, as shown in [JLJ98], $C_0(\Omega)$ (as well as $C_0^\infty(\Omega)$) is dense in $\mathbb{L}^2(\Omega)$ and since $w \mapsto \int_\Omega fw\,dx$ is a continuous linear functional on $\mathbb{L}^2(\Omega)$, we obtain that:

$$\int_\Omega fw\,dx = 0 \quad \textit{for all } w \in \mathbb{L}^2(\Omega).$$

Putting $w = kf$, for positive real function k yields the result, since a continuous non-negative real function with zero integral is the zero function. ■

Hence, by Lemma 3.2.1 we have that:

$$\mu_t - \gamma^n \Sigma(\phi_t^n - \zeta_t^n) = 0 \quad d\mathbf{P} \otimes dt - a.e.$$

which leads to the following frictionless optimizer for a positive definite (and hence invertible) Σ:

$$\phi_t^n = \frac{\Sigma^{-1}\mu_t}{\gamma^n} - \zeta_t^n. \tag{3.6}$$

The first term of the optimizer is similar to the usual *Merton's optimal allocation* (see [KS08] and [MS92]), while the second term deals with hedging against the exposure of the endowment to asset price shocks (see discussion in Remark 3.1.1).

Second order condition

For the second order condition in the direction of $\theta_t^n \in \mathscr{L}_r^2$, note that (s.f. [GF63], [JLJ98]) by Proposition 3.2.1 we have that the second Gâteaux differential of (3.2) is:

$$\left(d^2\mathscr{F}^n(\phi^n)\theta^n, \theta^n\right) = \mathbb{E}\left[\int_0^T e^{-rt}\left(-\gamma^n(\theta_t^n)^T\Sigma\theta_t^n\right)dt\right] < 0.$$

Therefore, for $\gamma^n > 0$ and positive-definite matrix Σ, the asset allocation of (3.6) is the optimal one under no frictions.

3.3 Frictionless Equilibrium

In this section we deal with determining the returns process $(\mu_t)_{t\in\mathscr{T}}$, i.e. the drift of the tradeable assets in the market, as expressed in (3.1). We initially study the case of the *frictionless market*. For the derivation of $(\mu_t)_{t\in\mathscr{T}}$, we introduce *the equilibrium condition* which imposes that the exogenous demand of noise traders $\psi_t \in \mathscr{L}_r^2$ matches the aggregate optimal asset allocation of the investors (a market clearing condition). In other words, we impose that the following equality holds for the optimal strategies at all times:

$$\phi_t^1 + \cdots + \phi_t^N + \psi_t = 0. \tag{3.7}$$

By the above we can deduce an explicit form for the returns process $(\mu_t)_{t\in\mathscr{T}}$, which we henceforth call *equilibrium returns*, by utilizing (3.6) as follows:

$$\begin{aligned} -\psi_t &= \Sigma^{-1}\mu_t \sum_{n=1}^{N} 1/\gamma^n - \sum_{n=1}^{N} \zeta_t^n \Rightarrow \\ \mu_t &= \frac{\Sigma(\zeta_t - \psi_t)}{\delta}, \end{aligned} \tag{3.8}$$

where $\zeta_t = \sum_{n=1}^{N} \zeta_t^n$ defines the aggregate exposure of the investors and $\delta = \sum_{n=1}^{N} 1/\gamma^n$ their aggregate risk tolerance.

Let us now give a more intuitive explanation for the terms involved in (3.8):

- Note that as the exposure of the investors increases so do the equilibrium returns. To get a better sense on why that happens, initially recall that $\zeta_t^{n,k}$ denotes the exposure of investor n's endowment to market shocks in asset k and assume, for the sake of simplicity, that there is only this asset in the market (hence σ is one-dimensional and denotes the asset's standard deviation). In this context, we loosely define a "shock" as a market influencing factor that alters either the demand or supply of k, therefore causing a subsequent change to its price. Shocks that produce a positive co-movement in the risky asset and the endowment of the investor n (i.e. $\zeta_t^{n,k} > 0$) reflect the fact that both the risky asset k and the investor's endowment are exposed to them. Shocks that drive the price of the asset and the value of the endowment in opposite directions (i.e. $\zeta_t^{n,k} < 0$), arise from asset k providing a hedge for the endowment's fluctuations. Thus, a negative exposure to asset k implies that investor n "uses" this asset to hedge against the market risk to her endowment, driving asset k's price down in the process and vice versa.

- Large values of γ^n contribute towards the increase of the absolute value of equilibrium returns. This stems from the fact that, by definition, as the risk aversity of an investor increases, she requires higher returns from her investments on the risky assets in order to be willing to undertake the same level of risk.

- Note that μ_t also increases with Σ. This stems from the fact that, in general, investments with higher risk need to be able to provide the investor with greater returns when compared with less risky ones, in order for her to be willing to undertake the excess amount of risk.

In the next chapter, we analyze the above results in a market with frictions following [Bou+18]. Chapter 5 aims to introduce the price impact model of [Ant17] in continuous time without frictions. In other words, we study a new maximization problem where the investors act optimally after considering how their transactions impact the prices of the risky assets. As we see on a later note, the equilibrium condition provides a natural way to model the concept of "impact" into our problem. Lastly, chapter 6 combines the aforementioned, generalizing the notion of price impact in a market with frictions.

Chapter 4

THE MODEL FOR A MARKET WITH FRICTIONS

In this chapter we study the concepts depicted in the graph below:

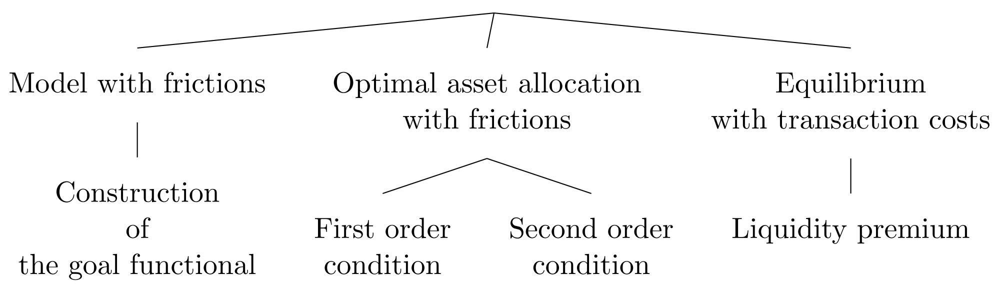

Figure 4.1: Outline of the 4^{th} chapter

4.1 The model under frictions

We are now ready to present the respective version of the goal functional, (3.5), in a market with frictions. This model is taken from [Bou+18], while similar models are also discussed in [CL13], [GP16], [KXG15], [XZ16] and [Zit09]. In this context the investors' strategies ϕ_t^n are absolutely continuous processes, given as follows:

For more information about absolutely continuous functions, refer to *A.4*.

$$\phi_t^n = \phi_0^n + \int_0^t \dot{\phi}_s^n \, ds, \quad t \in \mathscr{T}, \quad \text{for } n = 1, ..., N,$$

where $\phi_t^n, \dot{\phi}_t^n \in \mathscr{L}_r^2$ and $\dot{\phi}_t^n$ defines the *trading rate* of the investor. Moreover, we require that $\phi_0^n = 0$, and since ϕ_t^n is absolutely continuous, we get:

$$\frac{d\phi_t^n}{dt} = \dot{\phi}_t^n, \quad \phi_0^n = 0, \quad \text{for } n = 1, ..., N. \tag{4.1}$$

We furthermore assume that these costs take the form of a transaction tax, which goes to an exogenous recipient that does not trade in the market. More precisely, we define $\Lambda \in \mathbb{R}^{d \times d}$ as the diagonal matrix with elements $\lambda^m > 0,\ m = 1, ..., d$ that express the costs levied separately on each investor's order flow for a specific asset. Therefore, the goal functional becomes:

$$\begin{aligned}\mathscr{F}^{\Lambda,n}(\dot{\phi}) = \mathbb{E}\left[\int_0^T e^{-rt}\Big((\phi_t^n)^T\mu_t - \frac{\gamma^n}{2}(\phi_t^n + \zeta_t^n)^T\Sigma(\phi_t^n + \zeta_t^n) - (\dot{\phi}_t^n)^T\Lambda\dot{\phi}_t^n\Big)dt\right] \\ + \mathbb{E}\left[\int_0^T e^{-rt}\Big(dA_t^n - \frac{\gamma^n}{2}d\langle M^{\perp,n}\rangle_t\Big)\right]^1.\end{aligned} \tag{4.2}$$

Remark 4.1.1 *Following the discussion in Remark 3.1.1, a similar form to that of* (4.2)*—stemming from the objective of determining a portfolio with maximal mean and minimal variance, under the presence of transaction costs—can be represented as follows:*

$$\max_w \{w^T\mu - \gamma w^T\Sigma w - c(w)\},$$

where $c(w)$ represents the total transcation costs incurred on the portfolio. In general the above formulation constitutes a non-concave problem, however simplifying assumptions are usually imposed in the relevant literature in order to end up with problems that are easier to solve. More precisely, in our case as previously mentioned, this takes the form of quadratic costs levied on each investor's order flow.

The above reveals that the goal of an investor remains the same as the frictionless market, but under the presence of transaction costs. On the one hand, the investor invests to produce a portfolio ϕ_t^n that has an optimal trade-off between risk (Σ) and expected returns (μ_t) and to hedge against the exposure of her endowment (ζ_t^n) to the fluctuations of a market with transaction costs. ◊

4.2 Optimal asset allocation under frictions

First order condition

We now move towards the calculation of the optimal asset allocation of an investor in a market with frictions. More precisely, in the case of a finite time horizon, the initial condition of (4.1) is

[1] Note that by the absolute continuity of $\phi_t^n,\ n = 1, ..., N$, the goal functional of (4.2) can be seen as either a function of ϕ_t^n or $\dot{\phi}_t^n$. For this case, we specifically choose to view it as a function of $\dot{\phi}_t^n$.

coupled by a terminal condition for $\dot{\phi}_t^n$, while in the case of infinite time horizon, an appropriate transversality condition is used. Namely, we have:

$$\begin{aligned} &T < \infty : \dot{\phi}_T^n = 0, \\ &T = \infty : e^{-rt}\dot{\phi}_t^n \xrightarrow{\mathbf{P}-a.s.} 0 \text{ as } t \text{ goes to } \infty. \end{aligned} \tag{4.3}$$

The above tells us that for infinite time horizons, the usual terminal condition is replaced by an implicit condition. To get a better sense on what is the intuition behind $e^{-rt}\dot{\phi}_t^n \xrightarrow{\mathbf{P}-a.s.} 0$ as t goes to ∞ consider that $\dot{\phi}_t^n$ represents the rate at which an investor n either buys or sells an asset. In this context, this condition imposes that the present value of the marginal value of an additional unit of a risky asset must go to zero as t goes to infinity. Hence, the current value (at time t) of an additional unit must either be finite or grow at a rate slower than the discount rate r, so that the discount factor e^{-rt}, pushes the present value to zero. This is tied to a concept called *time value of money*, which assumes that a specific amount of money in the present (acquired by trading the asset) is worth more than the same amount in the future.

In a similar manner to that of the frictionless market, the first order condition can be expressed as:

$$\left(d\mathscr{F}^{\Lambda,n}(\dot{\phi}^n), \dot{\theta}^n\right) = 0.$$

By the above, we could construct the following lemma:

Lemma 4.2.1 *Recall that* $\phi_t^n = \frac{\Sigma^{-1}\mu_t}{\gamma^n} - \zeta_t^n$ *is the frictionless maximizer from* (3.6)*. The goal functional* (4.2) *has a unique optimizer, characterized by the following FBSDE:*

$$\begin{aligned} d\phi_t^{\Lambda,n} &= \dot{\phi}_t^{\Lambda,n} dt, \quad \phi_0^{\Lambda,n} = 0, \\ d\dot{\phi}_t^{\Lambda,n} &= dM_t^n + \frac{\gamma^n \Lambda^{-1}\Sigma}{2}(\phi_t^{\Lambda,n} - \phi_t^n)dt + r\dot{\phi}_t^{\Lambda,n} dt, \end{aligned} \tag{4.4}$$

where $\phi_t^{\Lambda,n}, \dot{\phi}_t^{\Lambda,n} \in \mathscr{L}_r^2$ *and the* $\mathbb{R}^d$*-valued square-integrable martingale* M^n *needs to be determined as part of the solution. The terminal condition is given in* (4.3). ◊

Proof: After simplifying the expression for the first order condition, for all $\theta_t^n, \dot{\theta}_t^n \in \mathscr{L}_r^2$ with $\theta_0^n = 0$ and $\theta_t^n = \int_0^t \dot{\theta}_s^n ds$ [2] and by the linearity of the integral (and the expectation) we get:

$$\frac{d\mathscr{F}^{\Lambda,n}(\dot{\phi}^n + \rho\dot{\theta}^n)}{d\rho} = \mathbb{E}\Bigg[\int_0^T e^{-rt}\Bigg(\Big(\int_0^t \dot{\theta}_s^n ds\Big)^T \mu_t - \frac{\gamma^n}{2}\Bigg(2\Big(\int_0^t \phi_s^n ds\Big)^T \Sigma\Big(\int_0^t \dot{\theta}_s^n ds\Big) +$$

$$2\Big(\int_0^t \dot{\theta}_s^n ds\Big)^T \Sigma\Big(\int_0^t \dot{\theta}_s^n ds\Big)\rho + 2(\zeta_t^n)^T \Sigma\Big(\int_0^t \dot{\theta}_s^n ds\Big)\Bigg) - 2(\dot{\theta}_t^n)^T \Lambda \dot{\phi}_t^n - 2(\dot{\theta}_t^n)^T \Lambda \dot{\theta}_t^n \rho\Bigg) dt\Bigg] \Rightarrow$$

$$\Big(d\mathscr{F}^{\Lambda,n}(\dot{\phi}^n), \dot{\theta}^n\Big) = \mathbb{E}\Bigg[\int_0^T e^{-rt}\Bigg(\mu_t^T\Big(\int_0^t \dot{\theta}_s^n ds\Big) - \gamma^n\Bigg(\Big(\int_0^t \phi_s^n ds\Big)^T \Sigma\Big(\int_0^t \dot{\theta}_s^n ds\Big) +$$

$$(\zeta_t^n)^T \Sigma\Big(\int_0^t \dot{\theta}_s^n ds\Big)\Bigg) - 2(\dot{\theta}_t^n)^T \Lambda \dot{\phi}_t^n\Bigg) dt\Bigg]$$

$$= \mathbb{E}\Bigg[\int_0^T e^{-rt}\Big(\mu_t^T - \gamma^n(\phi_t^n + \zeta_t^n)^T \Sigma\Big) \int_0^t \dot{\theta}_s^n ds dt - \int_0^T 2e^{-rt}(\dot{\theta}_t^n)^T \Lambda \dot{\phi}_t^n dt\Bigg].$$

By Fubini's Theorem the above can be written as:

$$\mathbb{E}\Bigg[\int_0^T \Bigg(\int_t^T e^{-rs}\Big(\mu_s^T - \gamma^n(\phi_s^n + \zeta_s^n)^T \Sigma\Big) ds\Bigg)\dot{\theta}_t^n dt - \int_0^T 2e^{-rt}(\dot{\theta}_t^n)^T \Lambda \dot{\phi}_t^n dt\Bigg].$$

By the law of total expectation in turn we get ($\dot{\theta}_t^n$ is $\mathscr{F}_t$-measurable, since $\dot{\theta}_t^n \in \mathscr{L}_r^2$):

$$\mathbb{E}\Bigg[\int_0^T \Bigg(\mathbb{E}\Bigg[\int_t^T e^{-rs}\Big(\mu_s^T - \gamma^n(\phi_s^n + \zeta_s^n)^T \Sigma\Big) ds \Bigg| \mathscr{F}_t\Bigg] - 2e^{-rt}(\dot{\phi}_t^n)^T \Lambda\Bigg)\dot{\theta}_t^n dt\Bigg].$$

Now by Lemma 3.2.1, for non-singular Λ and with the help of the *Fubini-Tonelli* Theorem as studied in [Sch05], we get [3]:

$$\mathbb{E}\Bigg[\int_t^T e^{-rs}\Big(\mu_s^T - \gamma^n(\phi_s^n + \zeta_s^n)^T \Sigma\Big) ds \Bigg| \mathscr{F}_t\Bigg] - 2e^{-rt}(\dot{\phi}_t^n)^T \Lambda = 0 \quad d\mathbf{P} \otimes dt - a.e.$$

Therefore, we have:

$$\dot{\phi}_t^{\Lambda,n} = \frac{\gamma^n \Lambda^{-1} \Sigma}{2} e^{rt} \mathbb{E}\Bigg[\int_t^T e^{-rs}\Big(\frac{\Sigma^{-1}\mu_s}{\gamma^n} - \zeta_s^n - \phi_s^{\Lambda,n}\Big) ds \Bigg| \mathscr{F}_t\Bigg]. \tag{4.5}$$

---

[2] In the case of a finite time horizon market we also have $\dot{\theta}_T^n = 0$, while for the case of infinite time horizon the transversality condition of (4.3) is used.

[3] Note that $\phi_t^{\Lambda,n}$ denotes the optimal asset allocation under frictions, as oposed to ϕ_t^n in the case of a frictionless market.

Consequently, in order for the first order condition of (4.2) to be satisfied, the above must also have a solution. Going back to the finite time horizon model (4.3), as previously examined, we should also have $\dot{\phi}_T^{\Lambda,n} = 0$. So, we essentially are looking for an adapted backward solution, since we have a terminal condition, that has the dynamics of (4.5). Note that $\frac{\gamma^n \Lambda^{-1}\Sigma}{2} \int_t^T e^{-rs}\Big(\frac{\Sigma^{-1}\mu_s}{\gamma^n} - \zeta_s^n - \phi_s^{\Lambda,n}\Big)ds$ is not necessarily $(\mathscr{F}_t)_{t\geq 0}$-adapted, it is $\mathscr{F}_T$-measurable. That is why we reformulated the terminal value problem by inserting the conditional expectation as presented above, so that we may allow a solution which is $(\mathscr{F}_t)_{t\geq 0}$-adapted. Let $\tilde{M}_t = \frac{\gamma^n \Lambda^{-1}\Sigma}{2}\mathbb{E}\Big[\int_0^T e^{-rs}\Big(\frac{\Sigma^{-1}\mu_s}{\gamma^n} - \zeta_s^n - \phi_s^{\Lambda,n}\Big)ds\Big|\mathscr{F}_t\Big]$, $t \in \mathscr{T}$, we could write:

$$e^{-rt}\dot{\phi}_t^{\Lambda,n} = \tilde{M}_t - \frac{\gamma^n \Lambda^{-1}\Sigma}{2}\mathbb{E}\Big[\int_0^t e^{-rs}\Big(\frac{\Sigma^{-1}\mu_s}{\gamma^n} - \zeta_s^n - \phi_s^{\Lambda,n}\Big)ds\Big|\mathscr{F}_t\Big].$$

Now if we apply *Itô's Lemma*, we get:

$$\begin{aligned}
&d(e^{-rt}\dot{\phi}_t^{\Lambda,n}) = d\tilde{M}_t - \underbrace{\frac{\gamma^n \Lambda^{-1}\Sigma}{2}e^{-rt}(\phi_t^n - \phi_t^{\Lambda,n})dt}_{\mathscr{F}_t-measurable} \Rightarrow \\
&-re^{-rt}\dot{\phi}_t^{\Lambda,n}dt + e^{-rt}d\dot{\phi}_t^{\Lambda,n} = d\tilde{M}_t - \frac{\gamma^n \Lambda^{-1}\Sigma}{2}e^{-rt}(\phi_t^n - \phi_t^{\Lambda,n})dt \Rightarrow \\
&e^{-rt}d\dot{\phi}_t^{\Lambda,n} = d\tilde{M}_t - \frac{\gamma^n \Lambda^{-1}\Sigma}{2}e^{-rt}(\phi_t^n - \phi_t^{\Lambda,n})dt + re^{-rt}\dot{\phi}_t^{\Lambda,n}dt \Rightarrow \\
&d\dot{\phi}_t^{\Lambda,n} = e^{rt}d\tilde{M}_t - \frac{\gamma^n \Lambda^{-1}\Sigma}{2}(\phi_t^n - \phi_t^{\Lambda,n})dt + r\dot{\phi}^{\Lambda,n}dt.
\end{aligned}$$

Note that $\tilde{M}_t$ shall be a part of the above BSDE's solution. Moreover, note that $\tilde{M}_t$ is square integrable, that is $\mathbb{E}[|\tilde{M}_t|^2] < \infty$, $t \in \mathscr{T}$ [4] and by Lemma A.1.2 in A.1, it is also continuous. Now note that since e^{rt} is trivially predictable (and hence progressive), $dM_t^n = e^{rt}d\tilde{M}_t$, $M_0^n = 0$ also defines a continuous local martingale with finite second moments, as shown in Theorem 3.1.1. Coupling the above with (4.1), for $T < \infty$, the first order condition of (4.2) can be characterized by the following linear, coupled FBSDE:

$$\begin{aligned}
&d\phi_t^n = \dot{\phi}_t^n dt, \quad \phi_0^n = 0, \\
&d\dot{\phi}_t^{\Lambda,n} = dM_t^n - \frac{\gamma^n \Lambda^{-1}\Sigma}{2}(\phi_t^n - \phi_t^{\Lambda,n})dt + r\dot{\phi}^{\Lambda,n}dt, \quad \dot{\phi}_T^{\Lambda,n} = 0.
\end{aligned}$$

[4]This is a direct consequence of the triangle inequality for integrals, the Cauchy-Swartz inequality and the fact that the integrand inside $\tilde{M}_t$ is in $\mathscr{L}_r^2$.

Notice that the solution of the above system of SDEs shall be of the form: $\left(\phi_t^{\Lambda,n}, \dot{\phi}_t^{\Lambda,n}, M^n\right)$. To see that the second equation of (4.4) does indeed satisfy (4.5), we initially note that:

$$e^{-rt}\dot{\phi}_t^{\Lambda,n} = \dot{\phi}_0^{\Lambda,n} + \int_0^t e^{-rs} dM_s^n + \int_0^t e^{-rs} \frac{\gamma^n \Lambda^{-1}\Sigma}{2} (\phi_s^{\Lambda,n} - \phi_s^n) ds. \tag{4.6}$$

Next, if we let:

$$\dot{\phi}_0^{\Lambda,n} = -\int_0^T e^{-rs} dM_s^n - \int_0^T e^{-rs} \frac{\gamma^n \Lambda^{-1}\Sigma}{2} (\phi_s^{\Lambda,n} - \phi_s^n) ds. \tag{4.7}$$

For finite time horizon models this holds by the terminal condition $\dot{\phi}_T^{\Lambda,n} = 0$. Conversely, in the case of $T = \infty$ we arrive at the same result through the transversality condition of $\dot{\phi}_t^{\Lambda,n}$. In other words, there exists an increasing sequence $(t_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} t_k = \infty$ along which $e^{-rt_k}\dot{\phi}_{t_k}^{\Lambda,n}$ converges to zero. For the right-hand side of (4.6) we use the *Martingale convergence* Theorem as shown in [Bou+18]. Thus, inserting (4.7) to (4.6) and taking conditional expectations we arrive at (4.5). ■

Second order condition

The concavity of (4.2) in $\dot{\phi}_t^n$ is direct, since we have a negative quadratic form.

Remark 4.2.1 *Note that as shown in detail on [Bou+18],* (4.4) *can be reduced to a first order ODE of the following form:*

$$\dot{\phi}^{\Lambda,n} = \mathbb{S}\left(\mathbb{TP} - \phi^{\Lambda,n}\right), \quad n = 1, ..., N,$$

where $\mathbb{S}, \mathbb{TP}$ *are functions which depend on the nature of time horizon. By the above equation we could say that, in the context of individually optimal strategies, the optimal trading rate of an investor prescribes to trade with speed* $\mathbb{S}$ *towards a target portfolio* $\mathbb{TP}$*. Thus, when the investor reaches her target, she stops trading (at time* t*). More precisely, as discussed in [MMKS14], as transaction costs get smaller the target portfolio tends closer to the frictionless optimizer and the current position in the market with frictions is pushed back more aggresively to that target. Hence, the frictional optimal allocation trades towards the current frictionless benchmark, rather than a different optimum.* ◊

4.3 Equilibrium with frictions

In this subsection we shall show an appropriate form for the equilibrium returns process, $(\mu_t)_{t\in\mathcal{T}}$, under the pressence of transaction costs. To this end, we work in a similar way as in the corresponding frictionless form of (3.8), using the dynamics for the individually optimal strategies, as expressed in (4.4). Specifically, we require that the optimal demands of the investors match the null net supply of the risky asset.

We initially introduce the (exogenous) dynamics for the noise traders in the market under frictions, which are modelled in a similar way to that of the investors:

$$d\psi_t = \dot{\psi}_t dt,$$
$$d\dot{\psi}_t = \mu_t^{\psi} dt + dM_t^{\psi},$$

where, $\mu^{\psi}, \psi, \dot{\psi} \in \mathscr{L}_r^2$ and M^{ψ} a continuous local martingale (more information about the exact localization is given on a later note).

In order to derive the equilibrium returns, we need the following proposition (the proof is taken from [Kal02]).

Proposition 4.3.1 *If M is a continuous local martingale of locally finite variation, then $M = M_0$ a.s.* ◊

Proof: By localization we may reduce to the case when $M_0 = 0$ and M is of bounded variation. In fact, let $V_1(M,[0,t])$ be the total variation of M and note that $V_1(M,[0,t])$ is continuous and adapted. Now introduce a sequence of the optional times [5] that makes M a local martingale with bounded variation, that is $\tau_n = \inf\{t \geq 0 : V_1(M,[0,t]) = n\}$. Note that $M^{\tau_n} - M_0$ is a continuous martingale with total variation bounded by n. Note also that $\tau_n \to \infty$ and that if $M^{\tau_n} = M_0$, *a.s.* for each n, then we have $M = M_0$, *a.s.*.

[5] For some basic information about stopping/optional times, refer to A.1.

In the reduced case (i.e. the case where the above localizing sequences ensures the properties we require), fix $t > 0$, consider a partition P, with $t_{n,k} = \frac{kt}{n}$ and conclude from the continuity of M that a.s.:

$$V_2(M,P) = \sum_{k=1}^{n}(M_{t_{n,k}} - M_{t_{n,k-1}})^2 \leq V_1(M,[0,t]) \sup_{1\leq k\leq n} |M_{t_{n,k}} - M_{t_{n,k-1}}| \to 0,$$

since M is continuous on $[0,t]$ and thus uniformly continuous on the same interval, for $\epsilon > 0$, $\exists\ a(\epsilon) : |u - v| < a$ then $|M_u - M_v| < \epsilon$ for all $u, v \in [0,t]$. Now consider $\|P\| < a$ and note that $|M_{t_{n,k}} - M_{t_{n,k-1}}| < \epsilon$. Thus, the above tends to zero as $n \to \infty$ ($\|P\| \to 0$). Moreover, we have that $V_2(M,P) \leq [M]$ ($[M]$ is increasing), which is bounded by a constant. It now follows by the martingale property and the dominated convergence theorem (which lets us bring the limit inside the expectation $\mathbb{E}[V_2(M,P)]$, since $V_2(M,P)$ is bounded by a constant) that $\mathbb{E}[M_t^2] = \mathbb{E}[V_2(M,P)] \stackrel{n\to\infty}{\longrightarrow} 0$ [6]. Thus, we get that $M_t = 0,\ a.s.$ for each $t > 0$. ■

Theorem 4.3.1 *As shown in [Bou+18], the unique frictional equilibrium returns is of the form:*

$$\mu_t^{\Lambda} = \sum_{n=1}^{N-1} \frac{(\gamma^n - \gamma^N)\Sigma}{N}\phi_t^{\Lambda,n} + \sum_{n=1}^{N} \frac{\gamma^n\Sigma}{N}\zeta_t^n - \frac{\gamma^N\Sigma}{N}\psi_t + \frac{2\Lambda}{N}(\mu_t^{\psi} - r\dot{\psi}_t), \tag{4.8}$$

◊

where for the investors $n = 1, ..., N$, $\phi^{\Lambda,1}, ..., \phi^{\Lambda,N-1}$ denotes the optimal asset allocation in the frictional market, which maximizes (4.2) and satisfies the FBSDE coupled system depicted in (4.4). Moreover, we have that $\phi^{\Lambda,N} = -\sum_{n=1}^{N-1}\phi^{\Lambda,n} - \psi$.

Below we address how (4.8) is derived.

Proof: Let $\nu \in \mathscr{L}_r^2$ be any equilibrium return and $\theta^{\Lambda} = (\theta^{\Lambda,1}, ..., \theta^{\Lambda,N})$ the corresponding optimal strategies of the investors, under frictions. Now notice that the conditions imposed at equilibrium, imply the following:

$$0 = \sum_{n=1}^{N} \theta_t^{\Lambda,n} + \psi_t, \tag{4.9}$$

[6] Note that by the martingale property we have that $\mathbb{E}[M_t^2] = \mathbb{E}[\sum(\Delta M_{t_i})^2]$ when $n \to \infty$, since for any $i > j$ we have that $Cov(\Delta M_{t_i}, \Delta M_{t_j}) = 0$. This stems from the fact that $\mathbb{E}[\Delta M_{t_i}]\mathbb{E}[\Delta M_{t_j}] = (\mathbb{E}[M_{t_i}] - \mathbb{E}[M_{t_{i-1}}])\mathbb{E}[\Delta M_{t_j}] = 0$ and $\mathbb{E}[\Delta M_{t_i}\Delta M_{t_j}] = \mathbb{E}[\mathbb{E}[\Delta M_{t_i}\Delta M_{t_j}|\mathscr{F}_{t_j}]] = \mathbb{E}[\Delta M_{t_j}\mathbb{E}[\Delta M_{t_i}|\mathscr{F}_{t_j}]] = 0$.

$$0 = \sum_{n=1}^{N} \dot{\theta}_t^{\Lambda,n} + \dot{\psi}_t. \tag{4.10}$$

Now going back to (4.4), setting $\theta_t^n = \frac{\Sigma^{-1}\nu_t}{\gamma^n} - \zeta_t^n$ and summing the trading rates for all the investors, (4.10) becomes:

$$0 = dM_t + \sum_{n=1}^{N} \frac{\Lambda^{-1}}{2} \left[\gamma^n \Sigma \theta_t^{\Lambda,n} - \underbrace{(\nu_t - \gamma^n \Sigma \zeta_t^n)}_{\gamma^n \Sigma \phi_t^n} \right] dt + \sum_{n=1}^{N} r \dot{\theta}_t^{\Lambda,n} dt + \underbrace{d\dot{\psi}_t}_{\mu_t^{\psi} dt + dM_t^{\psi}},$$

where M_t, M_t^{ψ} are continuous local martingales, which are reduced to be of finite variation, as shown in the proof for Proposition 4.3.1. Now, from (4.9), we get: $0 = \theta^{\Lambda,N} + \sum_{n=1}^{N-1} \theta^{\Lambda,n} + \psi \Rightarrow \theta^{\Lambda,N} = -\sum_{n=1}^{N-1} \theta^{\Lambda,n} - \psi$. Furthermore, through (4.10) we have $\sum_{n=1}^{N} \dot{\theta}_t^{\Lambda,n} = -\dot{\psi}_t$. Thus, by substituting the aforementioned, we arrive at:

$$\begin{aligned} 0 = dM_t + \frac{\Lambda^{-1}}{2} \left[\Sigma \sum_{n=1}^{N-1} \gamma^n \theta_t^{\Lambda,n} - \gamma^N \Sigma \underbrace{\Big(\sum_{n=1}^{N-1} \theta_t^{\Lambda,n} + \psi_t \Big)}_{\theta_t^{\Lambda,N}} - \sum_{n=1}^{N} (\nu_t - \gamma^n \Sigma \zeta_t^n) \right] dt - r\dot{\psi}_t dt + \\ \mu_t^{\psi} dt + dM_t^{\psi} \Rightarrow \\ 0 = dM_t + \frac{\Lambda^{-1}}{2} \left[\sum_{n=1}^{N-1} (\gamma^n - \gamma^N) \Sigma \theta_t^{\Lambda,n} - \sum_{n=1}^{N} (\nu_t - \gamma^n \Sigma \zeta_t^n) - \gamma^N \Sigma \psi_t \right] dt - r\dot{\psi}_t dt + \mu_t^{\psi} dt + dM_t^{\psi}. \end{aligned} \tag{4.11}$$

Now note that for properly localized continuous martingales M_t, M_t^{ψ} and by Proposition 4.3.1, dM_t, dM_t^{ψ} vanish since the aforementioned martingales are constant. Then, (4.11) becomes:

$$\begin{aligned} 0 = \frac{\Lambda^{-1}}{2} \left[\sum_{n=1}^{N-1} (\gamma^n - \gamma^N) \Sigma \theta_t^{\Lambda,n} - N\nu_t + \sum_{n=1}^{N} \gamma^n \Sigma \zeta_t^n - \gamma^N \Sigma \psi_t \right] dt - r\dot{\psi}_t dt + \mu_t^{\psi} dt \Rightarrow \\ \nu_t = \sum_{n=1}^{N-1} \frac{(\gamma^n - \gamma^N)\Sigma}{N} \theta_t^{\Lambda,n} + \sum_{n=1}^{N} \frac{\gamma^n \Sigma}{N} \zeta_t^n - \frac{\gamma^N \Sigma}{N} \psi_t + \frac{2\Lambda}{N} (\mu_t^{\psi} - r\dot{\psi}_t). \end{aligned} \tag{4.12}$$

For a detailed explanation of the uniqueness of the equilibrium returns in a market with frictions, refer to [Bou+18]. Naturally the uniqueness of μ_t^{Λ}, which as shown in [Bou+18] is given by the uniqueness of the frictional solutions, implies that $\nu_t = \mu_t^{\Lambda}$. ■

As shown in [Bou+18] the FBSDE system of (4.4), which characterizes the individually optimal asset allocation in a frictional market, is a part of linear and coupled systems of FBSDEs. A more general representation of this class is the following:

$$\begin{aligned} d\phi_t &= \dot{\phi}_t dt, \quad \phi_0 = 0, \quad t \in \mathscr{T} \\ d\dot{\phi}_t &= dM_t + B(\phi_t - \xi_t)dt + r\dot{\phi}_t dt, \quad t \in \mathscr{T} \end{aligned} \tag{4.13}$$

where $B \in \mathbb{R}^{d\times d}$ is a matrix with strictly positive eigenvalues, $r \geq 0$ and $\xi \in \mathscr{L}_r^2$ is derived through the equilibrium, as shown in Remark 4.3.1. The terminal condition of the BSDE in the system depends on the nature of the time horizon, as it was already explained for (4.4).

As is thoroughly explained in [Bou+18] the BSDE of (4.13) reverts to a known (stochastic) ODE yielding an explicit representation of an investor's optimal asset allocation in a market with frictions. Once again, the solutions depend on the nature of the time horizon and are briefly expressed below.

Finite time horizon solution

For a finite time horizon, system (4.13) is reduced to the following expression, which characterizes the individually optimal problem at equilibrium:

$$\dot{\phi}_t = \bar{\xi}_t - F_n(t)\phi_t. \tag{4.14}$$

The above ODE, as shown in [Bou+18], in turn yields the optimal asset allocation in a finite time horizon market with frictions for $n = 1, ..., N-1$:

$$\begin{aligned} \phi_t^{\Lambda,n} &= \int_0^t e^{-\int_s^t F_n(u)du} \bar{\xi}_s^n ds, \\ F_n(t) &= -\left(\Delta_n G_n(t) - \frac{r}{2}\dot{G}_n(t)\right)^{-1} B_n \dot{G}_n(t), \\ \bar{\xi}_t^n &= \left(\Delta_n G_n(t) - \frac{r}{2}\dot{G}_n(t)\right)^{-1} \mathbb{E}\left[\int_t^T \left(\Delta_n G_n(s) - \frac{r}{2}\dot{G}_n(s)\right) B_n e^{-\frac{r}{2}(s-t)} \xi_s^n ds \middle| \mathscr{F}_t\right], \\ G_n(t) &= \cosh\left(\sqrt{\Delta_n}(T-t)\right) \textit{ with } G_n(T) = 1 \textit{ and } \ddot{G}_n(t) = \Delta_n G_n(t), \\ \dot{G}_n(t) &= -\sqrt{\Delta_n} \sinh\left(\sqrt{\Delta_n}(T-t)\right) \textit{ with } \dot{G}_n(T) = 0 \end{aligned} \tag{4.15}$$

where $\Delta_n = B_n + \frac{r^2}{4} I_d$ has only positive eigenvalues, $r \geq 0$ and $\phi_t^{\Lambda,N}$ is determined by the equilibrium condition (see Remark 4.3.1).

In this context we could interpet the optimal asset allocation process in a market with frictions, $\phi_t^{\Lambda,n}$, as the discounting of $(\bar{\xi}_t^n)_{t\in\mathcal{T}}$, using an exponential kernel. Note that the latter process, is controlled by the "signal process" $(\xi_t^n)_{t\in\mathcal{T}}$, which stems from the frictionless optimizer, ϕ_t^n.

Remark 4.3.1 *Let us clarify why the optimal allocation of* (4.15) *holds for investors* $n = 1, ..., N-1$, *while investor N is determined via the equilibrium. Recall that the optimization of the goal functional in a market with frictions yielded the following system of SDEs:*

$$d\phi_t^{\Lambda,n} = \dot{\phi}_t^{\Lambda,n} dt, \quad \phi_0^{\Lambda,n} = 0,$$
$$d\dot{\phi}_t^{\Lambda,n} = dM_t^n + \frac{\gamma^n \Lambda^{-1}\Sigma}{2}\Big(\phi_t^{\Lambda,n} - \underbrace{\phi_t^n(\mu^\Lambda)}_{\star}\Big)dt + r\dot{\phi}_t^{\Lambda,n} dt,$$

where $\phi_t^n = \frac{\Sigma^{-1}\mu_t^\Lambda}{\gamma^n} - \zeta_t^n$. *Note that the term* $\star$ *is a function of* μ_t^Λ, *which as shown in* (4.8), *takes the following form:*

$$\mu_t^\Lambda = \sum_{n=1}^{N-1} \frac{(\gamma^n - \gamma^N)\Sigma}{N} \phi_t^{\Lambda,n} + \sum_{n=1}^{N} \frac{\gamma^n \Sigma}{N} \zeta_t^n - \frac{\gamma^N \Sigma}{N} \psi_t + \frac{2\Lambda}{N}(\mu_t^\psi - r\dot{\psi}_t).$$

The above expression does not directly depend on $\phi_t^{\Lambda,N}$. *In other words, the equilibrium condition was used to express the optimal asset allocation of investor N as:* $\phi_t^{\Lambda,N} = -\sum_{n=1}^{N-1} \phi_t^{\Lambda,n} - \psi_t$. *If we now substitute the above expression for* μ_t^Λ *into* $\star$, *bring it to the form of* (4.13), *reduce the system to* (4.14) *and solve it, we arrive at* (4.15) *for* $n = 1, ..., N-1$. *It should be clear by the above that* (4.15) *is directly linked to how we express* μ_t^Λ *and whether we choose to substitute it in the* $\star$ *term or not.*

Lastly note that the process ξ_t^n *of* (4.15) *is in essence the* $\star$ *term after we substitute the frictional equilibrium returns of* (4.8) *and group together all the terms but* $\phi_t^{\Lambda,n}$, *transforming* (4.4) *into* (4.13). ◊

Infinite time horizon

In a similar manner to that of the finite time horizon, as shown in [Bou+18], (4.13) is reduced to the following ODE:

$$\dot{\phi}_t^{\Lambda,n} = \bar{\xi}_t^n - \left(\sqrt{\Delta_n} - \frac{r}{2} I_d\right) \phi_t^{\Lambda,n}. \tag{4.16}$$

The above ODE, as shown in [Bou+18], in turn yields the optimal asset allocation in an infinite time horizon market with frictions for $n = 1, ..., N-1$:

$$\begin{aligned} \phi_t^{\Lambda,n} &= \int_0^t e^{-(\sqrt{\Delta_n} - \frac{r}{2} I_d)(t-s)} \bar{\xi}_s^n ds, \\ \bar{\xi}_t^n &= \left(\sqrt{\Delta_n} - \frac{r}{2} I_d\right) \mathbb{E}\left[\int_t^\infty \left(\sqrt{\Delta_n} + \frac{r}{2} I_d\right) e^{-(\sqrt{\Delta_n} + \frac{r}{2} I_d)(s-t)} \xi_s^n ds \middle| \mathscr{F}_t \right] \end{aligned} \tag{4.17}$$

where $\Delta_n = B_n + \frac{r^2}{4} I_d$ has only positive eigenvalues, $r > 0$ and $\phi_t^{\Lambda,N}$ is determined by the equilibrium condition.

Liquidity premium

A direct step, which is implied by the form of the equilibrium returns in a market with frictions in (4.8), is to discuss the so-called *liquidity premium* between μ_t^{Λ} and its frictionless counterpart in (3.8). In this context, if we recognize the presence of transactions costs as a new source of risk, one that is connected to the notion of market illiquidity and is not present in the frictionless market, we could view the liquidity premium as a form of additional return which is required by the investors to undertake the excess amount of risk. To make this more clear, we define this premium as follows:

$$LiPr_t = \mu_t^{\Lambda} - \mu_t,$$

where μ_t^{Λ} denotes the frictional equilibrium of (4.8) and μ_t its frictionless counterpart of (3.8).

Let us now derive an explicit expression for the liquidity premium. To this end, if we substitute the frictionless equilibrium of (3.8) in (3.6), we get:

$$\bar{\phi}_t^n = \frac{\overbrace{1/\gamma^n}^{\delta^n}\Big(\overbrace{\sum_{m=1}^{N}\zeta_t^m}^{\zeta_t} - \psi_t\Big)}{\underbrace{\sum_{m=1}^{N} 1/\gamma^m}_{\delta}} - \zeta_t^n, \quad n = 1, ..., N.$$

With this change in notation and while having in mind that $\mu_t = \frac{\Sigma(\zeta_t - \psi_t)}{\delta}$, the corresponding form for the frictional equilibrium returns becomes:

$$\mu_t = \sum_{n=1}^{N} \frac{\gamma^n \Sigma}{N}(\bar{\phi_t}^n + \zeta_t^n).$$

Substracting the above from (4.8), we get:

$$LiPr_t = \sum_{n=1}^{N-1} \frac{(\gamma^n - \gamma^N)\Sigma}{N}\phi_t^{\Lambda,n} - \frac{\gamma^N\Sigma}{N}\psi_t + \sum_{n=1}^{N} \frac{\gamma^n\Sigma}{N}\zeta_t^n + \frac{2\Lambda}{N}(\mu_t^\psi - r\dot{\psi}_t) - \sum_{n=1}^{N} \frac{\gamma^n\Sigma}{N}(\bar{\phi_t}^n + \zeta_t^n) \Rightarrow$$

$$LiPr_t = \frac{\Sigma}{N}\Big(\sum_{n=1}^{N-1} \gamma^n \phi_t^{\Lambda,n} \underbrace{-\gamma^N\Big(\sum_{n=1}^{N-1}\phi_t^{\Lambda,n} + \psi_t\Big)}_{\gamma^N\phi_t^{\Lambda,N}}\Big) + \sum_{n=1}^{N} \frac{\gamma^n\Sigma}{N}\zeta_t^n + \frac{2\Lambda}{N}(\mu_t^\psi - r\dot{\psi}_t) - \sum_{n=1}^{N} \frac{\gamma^n\Sigma}{N}(\bar{\phi_t}^n + \zeta_t^n) \Rightarrow$$

$$LiPr_t = \underbrace{\frac{\Sigma}{N}\left(\sum_{n=1}^{N-1} \gamma^n \phi_t^{\Lambda,n} + \gamma^N\phi_t^{\Lambda,N}\right)}_{\frac{\Sigma}{N}\sum_{n=1}^{N}\gamma^n\phi_t^{\Lambda,n}} + \sum_{n=1}^{N} \frac{\gamma^n\Sigma}{N}\zeta_t^n + \frac{2\Lambda}{N}(\mu_t^\psi - r\dot{\psi}_t) - \sum_{n=1}^{N} \frac{\gamma^n\Sigma}{N}\bar{\phi_t}^n - \sum_{n=1}^{N} \frac{\gamma^n\Sigma}{N}\zeta_t^n \Rightarrow$$

$$LiPr_t = \frac{\Sigma}{N}\sum_{n=1}^{N} \gamma^n(\phi_t^{\Lambda,n} - \bar{\phi_t}^n) + \frac{2\Lambda}{N}(\mu_t^\psi - r\dot{\psi}_t). \tag{4.18}$$

Equivalently, we express the liquidity premium as follows:

$$LiPr_t = \frac{\Sigma}{N}\Big(\sum_{n=1}^{N} \gamma^n(\phi_t^{\Lambda,n} - \bar{\phi_t}^n) - \bar{\gamma}\underbrace{\sum_{n=1}^{N}(\phi_t^{\Lambda,n} - \bar{\phi_t}^n)\Big)}_{\sum_{n=1}^{N}(\phi_t^{\Lambda,n} - \bar{\phi_t}^n) = -\psi_t + \psi_t} + \frac{2\Lambda}{N}(\mu_t^\psi - r\dot{\psi}_t) \Rightarrow$$

$$LiPr_t = \frac{\Sigma}{N}\sum_{n=1}^{N}(\gamma^n - \bar{\gamma})(\phi_t^{\Lambda,n} - \bar{\phi_t}^n) + \frac{2\Lambda}{N}(\mu_t^\psi - r\dot{\psi}_t), \tag{4.19}$$

where $\bar{\gamma} = \sum_{n=1}^{N}\gamma^n/N$ defines the average risk aversion of the investors. Let us now discuss some specific cases, taken from [Bou+18].

Remark 4.3.2 *Note that, when:*

(I) There are no noise traders in the market.

(II) The investors are assumed to have the same risk aversions $\bar{\gamma} = \gamma^1 = ... = \gamma^N$.

Then the $LiPr_t = 0$.

The absence of noise traders, implies that the market consists solely of investors. Therefore, going back to (4.18) *assuming the homegenous risk aversions, we have that:*

$$LiPr_t = \frac{\bar{\gamma}\Sigma}{N} \underbrace{\sum_{n=1}^{N} (\phi_t^{\Lambda,n} - \bar{\phi}_t{}^n)}_{\star}.$$

The liquidity premium is hence determined by the relative position of each investor between the market with frictions and its frictionless counterpart. In the absence of noise traders, the investors trade each risk asset amongst themselves. Therefore, by the equilibrium condition we have that the aggregate demand of the investors must be null, hence the $\star$ term vanishes. Note that while the equilibrium condition implies that the aggregate demand for the assets is equal between the two markets (frictionless and with frictions), this does not suggest in any form *an equality among the individual strategies $\phi_t^{\Lambda,n}$ and ϕ_t^n.* ◊

Corollary 4.3.1 *Assume that the investors have same risk aversions $\bar{\gamma} = \gamma^1 = ... = \gamma^N$. Then:*

$$LiPr_t = \frac{2\Lambda}{N}(\mu_t^{\psi} - r\dot{\psi}_t). \qquad \diamond$$

Note that, by the above result, if the noise traders *only* sell at a constant rate $\dot{\psi}_t < 0$ we have that $-\frac{2\Lambda r}{N}\dot{\psi}_t > 0$, leading to a positive liquidity premium. This serves to underline that even in this context, we could get positive liquidity premia in market expanding at a rate $\propto \dot{\psi}_t$.

Corollary 4.3.2 *Assume that there are no noise traders in the market. Then* (4.19) *reduces to:*

$$LiPr_t = \frac{\Sigma}{N} \sum_{n=1}^{N} (\gamma^n - \bar{\gamma}) \underbrace{(\phi_t^{\Lambda,n} - \bar{\phi}_t{}^n)}_{\phi_t^{R,n}}. \qquad \diamond$$

As it is also stated in [Bou+18] we view the above result as the covariance between the vector of risk aversions $(\gamma^1, ..., \gamma^N)$ and the vector of the relative positions of the investors between the two markets $\phi_t^{R,n}$. The liquidity premium becomes positive if and only if $(\gamma^n - \bar{\gamma})$ and $(\phi_t^{\Lambda,n} - \bar{\phi}_t{}^n)$

have the same sign. That is to say that in the market with frictions, the less risk averse investors tend to hold smaller positions on the risky assets compared to the frictionless case.

Chapter 5

PRICE IMPACT IN A FRICTIONLESS MARKET

In this chapter, we study the concepts in the graph below:

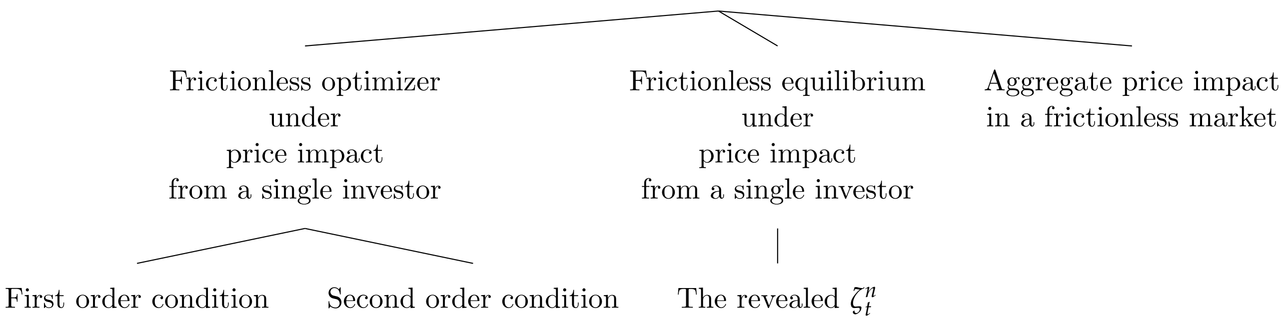

Figure 5.1: Outline of the 5^{th} chapter

5.1 Frictionless optimization under price impact of a single investor

We are now ready to consider the notion of "price impact" and how it can be applied in the context of this thesis. The so-called price impact can be defined as the effect that an investor has on the price of a risky asset as a result of her buying or selling it. In a sense, one could see the price of assets as a function of an investor's strategy. This give us a natural way to model the concept of "impact", through the equilibrium returns. Recall that this process drives the prices of the tradeable risky assets and is determined via the strategies of the investors by the equilibrium condition. Let us first consider, without loss of generality, the price impact of investor 1 and express μ_t as a function of investor her demand. The goal is now to determine how the incorporation of price impact affects the optimization problem both on the market with frictions and on its frictionless counterpart. Similar ideas are also expressed in [Ant17]. Note that the concept of price impact is closely reated to that of market liquidity and as such, it can be thought as an additional source of frictions in a market. To better understand the importance of such generalization, it suffices to consider the case when an investor's "buying strength" is significant relative to the size of the market. If such an investor for example enters a big long position on an asset, relative to the size of the market, she drives its price up in the process. Consequently, she now is required to pay an additional amount to that of the original price of the asset as a result of price impact. Naturally, price impact is a stronger force in relatively thinner markets.

In order to model the price impact of investor 1, we "extend" the original equilibrium condition of (3.7) such that it holds for every strategy ϕ_t^1, while the rest of the investors act optimally as in (3.6). Therefore, in the case of the frictionless market, we have:

$$\phi_t^1 + \sum_{n=2}^{N} \left\{ \frac{\Sigma^{-1}\mu_t}{\gamma^n} - \zeta_t^n \right\} + \psi_t = 0. \tag{5.1}$$

Solving the above for μ_t, the price impact of investor 1 on the market is modelled as follows:

$$\mu_t(\phi^1) = \frac{\Sigma(\zeta_t^{-1} - \psi_t - \phi_t^1)}{\delta_{-1}}, \tag{5.2}$$

where we define $\zeta_t^{-1} = \sum_{n=2}^{N} \zeta_t^n$ and $\delta_{-1} = \sum_{n=2}^{N} 1/\gamma^n$. Having defined (5.2) we now establish a new goal functional, which is to be optimized in order to determine investor 1's optimal strategy under price impact. To this end, we modify the frictionless goal functional of (3.5) as follows:

$$\mathscr{F}^n(\phi) = \mathbb{E}\Bigg[\int_0^T e^{-rt} \Big((\phi_t^n)^T \mu_t(\phi^1) - \frac{\gamma^n}{2} (\phi_t^n + \zeta_t^n)^T \Sigma (\phi_t^n + \zeta_t^n) \Big) dt$$
$$+ \int_0^T e^{-rt} \Big(dA_t^n - \frac{\gamma^n}{2} d\langle M^{\perp,n} \rangle_t \Big) \Bigg].$$

Thus, for investor 1 we get:

$$\mathscr{F}^1(\phi^1) = \mathbb{E}\Bigg[\int_0^T e^{-rt} \Bigg((\phi_t^1)^T \Big[\frac{\Sigma(\zeta_t^{-1} - \psi_t - \phi_t^1)}{\delta_{-1}} \Big] - \frac{\gamma^1}{2} (\phi_t^1 + \zeta_t^1)^T \Sigma (\phi_t^1 + \zeta_t^1) \Bigg) dt$$
$$+ \int_0^T e^{-rt} \Big(dA_t^1 - \frac{\gamma^1}{2} d\langle M^{\perp,1} \rangle_t \Big) \Bigg] \Rightarrow$$
$$\mathscr{F}^1(\phi^1) = \mathbb{E}\Bigg[\int_0^T e^{-rt} \Bigg(\frac{(\phi_t^1)^T \Sigma \zeta_t^{-1}}{\delta_{-1}} - \frac{(\phi_t^1)^T \Sigma \psi_t}{\delta_{-1}} - \frac{(\phi_t^1)^T \Sigma \phi_t^1}{\delta_{-1}} -$$

$$\frac{\gamma^1}{2}\Big[2(\phi_t^1)^T\Sigma\zeta_t^1+(\phi_t^1)^T\Sigma\phi_t^1+(\zeta_t^1)^T\Sigma\zeta_t^1\Big]\Big)dt+\int_0^T e^{-rt}\Big(dA_t^1-\frac{\gamma^1}{2}d\langle M^{\perp,1}\rangle_t\Big)\Big]\Rightarrow$$

$$\mathscr{F}^1(\phi^1)=\mathbb{E}\Big[\int_0^T e^{-rt}\Big(-\frac{(\phi_t^1)^T\Sigma\psi_t}{\delta_{-1}}-\underbrace{\Big(\frac{1}{\delta_{-1}}+\frac{\gamma^1}{2}\Big)}_{\frac{k_1}{2}>0}(\phi_t^1)^T\Sigma\phi_t^1-(\phi_t^1)^T\Sigma\Big(\gamma^1\zeta_t^1-\frac{\zeta_t^{-1}}{\delta_{-1}}\Big)-\frac{\gamma^1}{2}(\zeta_t^1)^T\Sigma\zeta_t^1\Big)dt$$

$$+\int_0^T e^{-rt}\Big(dA_t^1-\frac{\gamma^1}{2}d\langle M^{\perp,1}\rangle_t\Big)\Big].$$

Hence, when investor 1's price impact is taken into account, her goal functional becomes:

$$\begin{aligned}\tilde{\mathscr{F}}^1(\phi^1)=\mathbb{E}\Big[\int_0^T e^{-rt}\Big((\phi_t^1)^T\Sigma\Big(\frac{\zeta_t^{-1}}{\delta_{-1}}-\gamma^1\zeta_t^1-\frac{\psi_t}{\delta_{-1}}\Big)-\frac{k_1}{2}(\phi_t^1)^T\Sigma\phi_t^1-\frac{\gamma^1}{2}(\zeta_t^1)^T\Sigma\zeta_t^1\Big)dt\\+\int_0^T e^{-rt}\Big(dA_t^1-\frac{\gamma^1}{2}d\langle M^{\perp,1}\rangle_t\Big)\Big].\end{aligned}\tag{5.3}$$

Henceforth, we use the tilde notation to refer to quantities pertaining to single investor price impact.

First order condition

Regarding the first order condition of (5.3), for $\theta_t^1\in\mathscr{L}_r^2$ we have:

$$\frac{d\tilde{\mathscr{F}}^1(\phi^1+\rho\theta^1)}{d\rho}=\mathbb{E}\Big[\int_0^T e^{-rt}\Big(-\frac{(\theta_t^1)^T\Sigma\psi_t}{\delta_{-1}}-k_1(\phi_t^1)^T\Sigma\theta_t^1-k_1\rho(\theta_t^1)^T\Sigma\theta_t^1-\gamma^1(\theta_t^1)^T\Sigma\zeta_t^1+(\theta_t^1)^T\Sigma\frac{\zeta_t^{-1}}{\delta_{-1}}\Big)dt\Big]$$

$$\Rightarrow\Big(d\tilde{\mathscr{F}}^1(\phi^1),\theta^1\Big)=\mathbb{E}\Big[\int_0^T e^{-rt}\Big(-\frac{\psi_t^T\Sigma}{\delta_{-1}}-k_1(\phi_t^1)^T\Sigma-\gamma^1(\zeta_t^1)^T\Sigma+\frac{(\zeta_t^{-1})^T}{\delta_{-1}}\Sigma\Big)\theta_t^1dt\Big].$$

Therefore, by Lemma (3.2.1) we have:

$$-\frac{\psi_t^T\Sigma}{\delta_{-1}}-k_1(\phi_t^1)^T\Sigma-\gamma^1(\zeta_t^1)^T\Sigma+\frac{(\zeta_t^{-1})^T}{\delta_{-1}}\Sigma=0\quad d\mathbf{P}\otimes dt-a.e.,$$

which leads to the following representation of the frictionless optimizer under the price impact of investor 1:

$$\tilde{\phi}_t^1=\frac{\zeta_t^{-1}-\psi_t-\delta_{-1}\gamma^1\zeta_t^1}{\delta_{-1}k_1},\quad k_1=2\Big(\frac{1}{\delta_{-1}}+\frac{\gamma^1}{2}\Big).\tag{5.4}$$

Equivalently, defining investor 1's *risk tolerance* $\delta_1 = 1/\gamma^1$, her *risk tolerance* $\lambda_1 = \delta_1/\delta$ and $\lambda_{-1} = 1 - \lambda_1$, we have that: $\delta_{-1}k_1 = 2(1 + \delta_{-1}/2\delta_1) = (2\delta_1 + \delta_{-1})/\delta_1 = (\delta_1 + \delta)/\delta_1$. Hence, (5.4) becomes:

$$\begin{aligned}\tilde{\phi}_t^1 &= \frac{\delta_1(\zeta_t^{-1} - \psi_t) - \delta_{-1}\zeta_t^1}{\delta_1 + \delta} \\ &= \frac{\delta\Big(\frac{\delta_1}{\delta}(\zeta_t^{-1} - \psi_t) - \frac{\delta_{-1}}{\delta}\zeta_t^1\Big)}{\delta\big(\delta_1/\delta + 1\big)}.\end{aligned}$$

Therefore, we have:

$$\tilde{\phi}_t^1 = \frac{\lambda_1(\zeta_t^{-1} - \psi_t) - \lambda_{-1}\zeta_t^1}{\lambda_1 + 1}. \tag{5.5}$$

Second order condition

Similarly to the frictionless case (see Proposition 3.2.1), for $\theta_t^1 \in \mathscr{L}_r^2$ we get:

$$\Big(d^2\tilde{\mathscr{F}}^1(\phi^1)\theta^1, \theta^1\Big) = \mathbb{E}\left[\int_0^T e^{-rt}\Big(-k_1(\theta_t^1)^T\Sigma\theta_t^1\Big)dt\right] < 0,$$

for a constant $k_1 > 0$ and a positive-definite matrix Σ. Therefore, the optimizer presented in (5.4) is indeed a (global, unique) maximum.

5.2 Frictionless equilibrium under the price impact of a single investor

Motivated by the result of (5.4), we now move forward to define a new form for the equilibrium returns under the price impact of a single investor. It should be noted that for the sake of clarity, we make the following comments with regards to some of the parameters discussed above:

(I) We make a slight abuse of notation and use μ_t to represent both the frictionless equilibrium returns of (3.8) and the yet "undetermined returns" in the frictionless optimizer $\phi_t^n = \frac{\Sigma^{-1}\mu_t}{\gamma^n} - \zeta_t^n,\ n = 1, ..., N$ of (3.6). Instead, in order to ease notation, when we want to denote the former instead of the latter we state it explicitly. The same goes for any strategy in the frictionless market and the frictionless optimizer $\phi_t^n = \frac{\Sigma^{-1}\mu_t}{\gamma^n} - \zeta_t^n,\ n = 1, ..., N$ of (3.6).

(II) We shall denote the new equilibrium returns process as $(\tilde{\mu}_t)_{t\in\mathcal{T}}$, in order to distinguish it from the process $(\mu_t)_{t\in\mathcal{T}}$ in (3.6).

Proposition 5.2.1 *The frictionless equilibrium returns process under the price impact of investor* 1, *takes the following form:*

$$\tilde{\mu}_t^1 = \frac{\delta_1 \mu_{t,-1} + \delta \mu_t}{\delta_1 + \delta}, \tag{5.6}$$

where $\mu_t = \frac{\Sigma(\zeta_t - \psi_t)}{\delta}$ *from* (3.8) *and* $\mu_{t,-1} = \frac{\Sigma(\zeta_t^{-1} - \psi_t)}{\delta_{-1}}$. ◊

Proof: Going back to the equilibrium condition of (3.7) and substituting (5.4) for the optimal strategy of investor 1, we get for $\delta_{-1} k_1 = \gamma^1(1/\gamma^1 + \delta)$:

$$\begin{aligned}
\tilde{\mu}_t^1 &= \Sigma(\zeta_t^{-1} - \psi_t) \div \delta_{-1} - \overbrace{\frac{\Sigma(\zeta_t^{-1} - \psi_t - \delta_{-1}\gamma^1\zeta_t^1)}{\delta_{-1}k_1}}^{\tilde{\phi}_t^1} \div \delta_{-1} \Rightarrow \\
\tilde{\mu}_t^1 &= \mu_{t,-1} - \frac{\cancel{\delta_{-1}}\left(\frac{\Sigma(\zeta_t^{-1} - \psi_t)}{\delta_{-1}} - \gamma^1\Sigma\zeta_t^1\right)}{\cancel{\delta_{-1}}\delta_{-1}k_1} = \mu_{t,-1} - \frac{\left(\frac{\Sigma(\zeta_t^{-1} - \psi_t)}{\delta_{-1}} - \gamma^1\Sigma\zeta_t^1\right)}{\delta_{-1}\underbrace{2\left(\frac{1}{\delta_{-1}} + \frac{\gamma^1}{2}\right)}_{k_1}} \Rightarrow \\
\tilde{\mu}_t^1 &= \mu_{t,-1} + \frac{-\mu_{t,-1} + \gamma^1\Sigma\zeta_t^1}{\gamma^1(1/\gamma^1 + \delta)}.
\end{aligned} \tag{5.7}$$

Now we notice the following:

$$\begin{aligned}
\mu_t - \mu_{t,-1} &= \frac{\Sigma(\zeta_t - \psi_t)}{\delta} - \frac{\Sigma(\zeta_t^{-1} - \psi_t)}{\delta_{-1}} \Rightarrow \\
\mu_t - \mu_{t,-1} &= \frac{\Sigma\Big((\zeta_t^{-1} + \zeta_t^1) - \psi_t\Big)}{\delta_{-1} + 1/\gamma^1} - \frac{\Sigma(\zeta_t^{-1} - \psi_t)}{\delta_{-1}} \Rightarrow \\
\mu_t - \mu_{t,-1} &= \frac{\cancel{\delta_{-1}\Sigma\zeta_t^{-1}} + \delta_{-1}\Sigma\zeta_t^1 - \cancel{\delta_{-1}\Sigma\psi_t} - \cancel{\delta_{-1}\Sigma\zeta_t^{-1}} + \cancel{\delta_{-1}\Sigma\psi_t} - \frac{1}{\gamma^1}\Sigma\zeta_t^{-1} + \frac{1}{\gamma^1}\Sigma\psi_t}{\delta\delta_{-1}} \Rightarrow \\
\mu_t - \mu_{t,-1} &= \frac{-\frac{1}{\gamma^1}\Sigma(\zeta_t^{-1} - \psi_t) + \delta_{-1}\Sigma\zeta_t^1}{\delta\delta_{-1}} = \frac{-\frac{\delta_{-1}\mu_t^-}{\gamma^1} + \delta_{-1}\Sigma\zeta_t^1}{\delta\delta_{-1}}.
\end{aligned}$$

So, we get:

$$\mu_t - \mu_{t,-1} = \frac{-\mu_{t,-1} + \gamma^1\Sigma\zeta_t^1}{\gamma^1\delta}. \tag{5.8}$$

Therefore (5.7) becomes:

$$\tilde{\mu}_t^1 = \mu_{t,-1} + \frac{\gamma^1\delta(\mu_t - \mu_{t,-1})}{\gamma^1(1/\gamma^1 + \delta)} \Rightarrow$$
$$\tilde{\mu}_t^1 = \frac{\mu_{t,-1} + \cancel{\gamma^1\delta\mu_{t,-1}} + \gamma^1\delta\mu_t - \cancel{\gamma^1\delta\mu_{t,-1}}}{\gamma^1(1/\gamma^1 + \delta)}.$$

Then, by the above, we get the following form for the equilibrium returns under price impact, which is expressed as a weighted average between $\mu_{t,-1}$ and μ_t:

$$\tilde{\mu}_t^1 = \frac{\frac{1}{\gamma^1}\mu_{t,-1} + \delta\mu_t}{1/\gamma^1 + \delta}.$$

(5.6) does indeed verify (5.2), reverting to the new optimal asset allocation, as expressed in (5.4).

$$\frac{\frac{1}{\gamma^1}\mu_{t,-1} + \delta\mu_t}{\delta_{-1}k_1/\gamma^1} = \mu_{t,-1} - \frac{\Sigma\tilde{\phi}_t^1}{\delta_{-1}} \Rightarrow$$
$$\Sigma\tilde{\phi}_t^1 = \delta_{-1}\mu_{t,-1} - \frac{\mu_{t,-1} + \gamma^1\delta\mu_t}{k_1} = \frac{k\delta_{-1}\mu_{t,-1} - \mu_{t,-1} - \gamma^1\delta\left(\mu_{t,-1} + \frac{-\mu_{t,-1} + \gamma^1\Sigma\zeta_t^1}{\gamma^1\delta}\right)}{k_1} \Rightarrow$$
$$\Sigma\tilde{\phi}_t^1 = \frac{\gamma^1(1/\gamma^1 + \delta)\mu_{t,-1} - \gamma^1\delta\mu_{t,-1} - \gamma^1\Sigma\zeta_t^1}{k_1} = \frac{\mu_{t,-1} - \gamma^1\Sigma\zeta_t^1}{k_1} \Rightarrow$$
$$\tilde{\phi}_t^1 = \frac{\zeta_t^{-1} - \psi_t - \gamma^1\delta_{-1}\zeta_t^1}{\delta_{-1}k_1}.$$

■

In order to better understand the form of (5.6) we also examine the following liquidity premium:

Corollary 5.2.1 *The liquidity premium between the equilibrium returns under price impact of* (5.6) *and the frictionless equilibrium of* (3.8) *is:*

$$\tilde{\mu}_t^1 - \mu_t = \frac{\lambda_1}{\lambda_{-1}}\Sigma\phi_t^1, \tag{5.9}$$

where $\mu_t = \frac{\Sigma(\zeta_t - \psi_t)}{\delta}$ *and* $\phi_t^1 = \frac{\Sigma^{-1}\mu_t}{\gamma^1} - \zeta_t^1$*.* ◊

Proof: Going back to to (5.6), we have for $\gamma^1\delta = 1 + \gamma^1\delta_{-1}$:

$$
\begin{aligned}
\tilde{\mu}_t^1 - \mu_t &= 1/\gamma^1 \mu_{t,-1} - 1/\gamma^1 \mu_t \\
&= \frac{\Sigma(\zeta_t^{-1} - \psi_t)}{\gamma^1 \delta_{-1}} - \frac{\Sigma(\zeta_t - \psi_t)}{1 + \gamma^1 \delta_{-1}} \\
&= \frac{\Sigma(\zeta_t^{-1} - \psi_t) + \gamma^1 \delta_{-1} \Sigma(\zeta_t^{-1} - \psi_t) - \gamma^1 \delta_{-1} \Sigma(\zeta_t - \psi_t)}{\gamma^1 \delta_{-1}(1 + \gamma^1 \delta_{-1})} \\
&= \frac{\Sigma(\zeta_t^{-1} - \psi_t) - \gamma^1 \delta_{-1} \Sigma \zeta_t^1}{\gamma^1 \delta_{-1}(1 + \gamma^1 \delta_{-1})} \\
&= \frac{\Sigma(\zeta_t^{-1} - \psi_t) \pm \Sigma \zeta_t^1 - \gamma^1 \delta_{-1} \Sigma \zeta_t^1}{\gamma^1 \delta_{-1}(1 + \gamma^1 \delta_{-1})} \\
&= \frac{\Sigma(\zeta_t - \psi_t)}{\gamma^1 \delta_{-1} \gamma^1 \delta} - \frac{\Sigma \zeta_t^1}{\gamma^1 \delta_{-1}} \\
&= \frac{\Sigma}{\gamma^1 \delta_{-1}} \Big(\frac{\zeta_t - \psi_t}{\gamma^1 \delta} - \zeta_t^1 \Big) \\
&= \frac{\Sigma}{\gamma^1 \delta_{-1}} \Big(\frac{\Sigma^{-1} \mu_t}{\gamma^1} - \zeta_t^1 \Big).
\end{aligned}
$$

Now note that $\frac{1}{\gamma^1 \delta_{-1}} = \frac{1}{\frac{1}{\delta_1} \delta_{-1}} = \frac{1}{\frac{\delta}{\delta_1} \frac{\delta_{-1}}{\delta}} = \frac{\lambda_1}{\lambda_{-1}}$, which concludes the proof. ■

The price impact derived in (5.9) is endogenous and linearly dependent on the frictionless optimizer at equilibrium. More precisely, we note that the price impact of investor 1 grows with her relative risk tolerance, which is consistent with relevant literature.

The revealed form for ζ_t^n

A Closer Look II: The optimal asset allocation $\tilde{\phi}_t^1$

Let us take a closer look on investor 1's optimal asset allocation under price impact as in (5.4). To this end, to simplify the discussion, we note that:

$$
\tilde{\phi}_t^1 \propto \zeta_t^{-1} - \psi_t - \delta_{-1} \gamma^1 \zeta_t^1. \tag{CLII.1}
$$

Furthermore, in order to focus on the interaction between investor 1 and the rest, we ignore for now the pressence of noise traders. Therefore, (CLII.1) becomes:

$$
\tilde{\phi}_t^1 \propto \Big(\zeta_t^2 - \frac{\gamma^1}{\gamma^2} \zeta_t^1 \Big) + \Big(\zeta_t^3 - \frac{\gamma^1}{\gamma^3} \zeta_t^1 \Big) + \cdots + \Big(\zeta_t^N - \frac{\gamma^1}{\gamma^N} \zeta_t^1 \Big). \tag{CLII.2}
$$

The above is controlled by two parameters: on the one hand the fraction $\frac{\gamma^1}{\gamma^m}$, $m = 2, ..., N$, which can by seen as the relative risk aversion of investor 1 with respect to investor 1, and on the other the relationship between ζ_t^1 and ζ_t^m.

We can see that, in essence $\tilde{\phi}_t^1$ is expressed as the endowments' exposure of the rest of the investors, penalized by the weighted exposure of the first investor. In this context, the weights are the relative risk aversion between investors 1 and m.

We may argue that the strategic investor's (or non-price taker) goal is twofold. In other words, she uses the information she has about the other investors' exposure to set her demand, while simultaneously taking into account the "needs" of her own endowment. Moreover, this is controlled by $\frac{\gamma^1}{\gamma^m}$ in the sense that if the aforementioned fraction is greater than one, then the second term is each parenthesis of (CLII.2) is strengthened and vice versa.

In (5.4) we essentially gave a new expression for the optimal asset allocation of the strategic investor, which takes into account her influence to the equilibrium. Investor 1 stirs the returns to a different equilibrium point, as shown in (5.6). To put it differently, we could once more say that the investor invests to hedge agaist the exposure of her endowment to price shocks. The only feature that changes is the aforementionted exposure. We are interested in determining this new exposure process $\tilde{\zeta}_t^1$ that drives the market to the new equilibrium point $\tilde{\mu}_t$ (in a similar manner as in [Ant17]). Generalizing the above concepts, after taking into account that the optimization problem in a frictionless market under the price impact of a single investor is symmetrical, we give the following definition:

Definition 5.2.1 (The revealed exposure process) *Consider a frictionless market under the price impact of a single investor n. Then this investor's revealed exposure is defined to be:*

$$\tilde{\zeta}_t^n = \frac{\Sigma^{-1}\tilde{\mu}_t}{\gamma^n} - \tilde{\phi}_t^n, \quad n = 1, ..., N,$$

where $\tilde{\mu}_t$ is the "new" equilibrium point, determined by the equilibrium condition of (3.7) *and $\tilde{\phi}_t^n$ is the strategy determined by the optimization of the frictionless goal functional with price impact.* ◊

Proposition 5.2.2 *The revealed exposure under the price impact of one investor in a frictionless market can be expressed in the following equivalent forms:*

$$\tilde{\zeta}_t^n = \frac{\zeta_t^{-n} - \psi_t}{(\gamma^n\delta - 1)(\gamma^n\delta + 1)} + \frac{\gamma^n \delta \zeta_t^n}{\gamma^n\delta + 1}, \quad n = 1, ..., N, \tag{5.10}$$

where we define $\zeta_t^{-n} = \sum_{\substack{m=1 \\ m \neq n}}^{N} \zeta_t^m$ *and* $\delta_{-n} = \sum_{\substack{m=1 \\ m \neq n}}^{N} 1/\gamma^m$. ◊

Proof: Note that below we derive $\tilde{\zeta}_t^1$, which generalizes directly to the rest of the investors by their symmetry. Recalling Definition 5.2.1, substituting $\tilde{\phi}_t^1$ from (5.4) and $\tilde{\mu}_t^1$ from (5.6), we have:

$$\frac{\zeta_t^{-1} - \psi_t - \delta_{-1}\gamma^1\zeta_t^1}{\delta_{-1}k_1} = \frac{\Sigma^{-1}\overbrace{\left(\frac{\frac{1}{\gamma^1}\mu_{t,-1} + \delta\mu_t}{1/\gamma^1 + \delta}\right)}^{\tilde{\mu}_t^1}}{\gamma^1} - \tilde{\zeta}_t^1 \overset{\times\Sigma,\pm\zeta_t^1}{\underset{\times\div\delta}{\Longrightarrow}}$$

$$\frac{\delta\Sigma(\zeta_t - \cancel{\psi_t})}{\cancel{\delta\delta_{-1}k_1}} - \frac{\Sigma\zeta_t^1}{\delta_{-1}k_1} - \frac{\gamma^1\delta_{-1}\Sigma\zeta_t^1}{\delta_{-1}k_1} = \frac{\frac{1}{\gamma^1}\mu_{t,-1}}{\delta_{-1}k_1} + \frac{\delta\cancel{\mu_t}}{\cancel{\delta_{-1}}k_1} - \Sigma\tilde{\zeta}_t^1 \Rightarrow$$

$$\Sigma\tilde{\zeta}_t^1 = \frac{\frac{1}{\gamma^1}\frac{\Sigma(\zeta_t^{-1} - \psi_t)}{\delta_{-1}}}{\gamma^1(\delta + 1/\gamma^1)} + \frac{\Sigma\zeta_t^1}{\gamma^1(\delta + /\gamma^1)} + \frac{\gamma^1\delta_{-1}\Sigma\zeta_t^1}{\gamma^1(\delta + 1/\gamma^1)}.$$

By the above, we arrive at the first form of $\tilde{\zeta}_t^1$, which shall fascilitate the comparison with ζ_t^1 in a later note.

$$\tilde{\zeta}_t^1 = \frac{\frac{\zeta_t^{-1} - \psi_t}{\gamma^1\delta_{-1}} + \zeta_t^1 + \gamma^1\delta_{-1}\zeta_t^1}{\gamma^1(\delta + 1/\gamma^1)}. \tag{5.11}$$

We also derive the following equivalent form for $\tilde{\zeta}_t^1$:

$$\underset{\pm\zeta_t^1}{\Longrightarrow} \tilde{\zeta}_t^1 = \frac{\frac{\zeta_t - \psi_t}{\gamma^1\delta_{-1}} - \frac{\zeta_t^1}{\gamma^1\delta_{-1}} + \zeta_t^1 + \gamma^1\delta_{-1}\zeta_t^1}{\gamma^1\delta + 1} \Rightarrow$$

$$\tilde{\zeta}_t^1 = \frac{\zeta_t - \psi_t - \zeta_t^1 + \gamma^1\delta(\gamma^1\delta - 1)\zeta_t^1}{(\gamma^1\delta - 1)(\gamma^1\delta + 1)}.$$

Therefore, we arrive at:

$$\tilde{\zeta}_t^1 = \frac{\zeta_t^{-1} - \psi_t}{(\gamma^1\delta - 1)(\gamma^1\delta + 1)} + \frac{\gamma^1\delta\zeta_t^1}{\gamma^1\delta + 1}.$$

■

Remark 5.2.1 *Let us now give a more intuitive explanation of investor* 1*'s revealed exposure. Recall that by* (5.10)*, we have:*

$$\begin{aligned}\tilde{\zeta}_t^1 &= \underbrace{\frac{\zeta_t^{-1} - \psi_t}{(\gamma^1\delta - 1)(\gamma^1\delta + 1)}}_{\star} + \underbrace{\frac{\gamma^1\delta}{\gamma^1\delta + 1}}_{\star\star}\zeta_t^1 \\ &= \frac{\lambda_1^2}{(1-\lambda_1)(1+\lambda_1)}(\zeta_t^{-1} - \psi_t) + \frac{1}{1+\lambda_1}\zeta_t^1.\end{aligned}$$

Note that the $\star\star$ *term is positive and less than one, since* $\gamma^1\delta > 1$*. Whence the revealed exposure consists of a fraction of the true exposure, plus the* $\star$ *term. We call* $\zeta_t^{-1} - \psi_t$ *as the* residual exposure to risk*. If we now assume for the sake of the argument, that there are not any noise traders in the market then* $\star$ *becomes solely proportional to the aggregate exposure of the rest of the investors. In other words, strategic investor* 1 *on the one hand lessens the true exposure of her endowment and on the other supplements it by a fraction* $\propto \zeta_t^{-1}$*. We could think of the above concept as investor* 1 *declaring only a part of her true exposure* ($\star\star$) *to the rest of the investors, while exploiting her influence on the equilibrium by "informing" her declared exposure with a portion of the aggregate exposure of the rest of the investors* $\left((\gamma^1\delta)^2 - 1 > 0\right)$*. Moreover, it is crucial to note that the constants that affect both* ζ_t^1 *and* ζ_t^{-1} *are directly related to the relative risk aversion between the strategic investor and the rest of the investors, i.e.:* $1 + \frac{\gamma^1}{\gamma^2} + \cdots + \frac{\gamma^1}{\gamma^N}$. ◊

Comparison between ϕ_t^1 and $\tilde{\phi}_t^1$

In this subsection we will examine more closely how the following optimal strategies compare: the optimal asset allocation under no price impact, derived by the optimization of (3.5) and the optimal asset allocation under the effects of price impact, derived by the optimization of (5.3).

Proposition 5.2.3 *The following holds at equilibrium* [1]*:*

$$\tilde{\phi}_t^n = \frac{\Sigma^{-1}\mu_t}{\gamma^n} - \zeta_t^n, \quad n = 1, ..., N \quad d\mathbf{P} \otimes dt - a.e.$$

If and only if:

[1] Note that while we usually omit to underline that most equalities throughout this thesis are understood in an almost everywhere sense (opting to underline the equalities that hold for every t instead, in order to ease notation), we chose to explicitly express it for this proposition to make the comparison between the optimal strategies clearer.

$$\frac{\Sigma^{-1}\mu_t}{\gamma^n} - \zeta_t^n = 0, \quad n = 1, ..., N \quad d\mathbf{P} \otimes dt - a.e,$$

where $\mu_t = \frac{\Sigma(\zeta_t - \psi_t)}{\delta}$ $d\mathbf{P} \otimes dt$ *almost everywhere.* ◊

Proof: Below we derive the case for investor 1, which generalizes directly to the rest of the investors by their symmetry. Now note that $\tilde{\phi}_t^1 = \frac{\Sigma^{-1}\mu_t}{\gamma^1} - \zeta_t^1$ $d\mathbf{P} \otimes dt - a.e.$ implies that by the equilibrium condition of (3.7), the new equilibrium point $\tilde{\mu}_t$ reverts to its frictionless counterpart of (3.8). In turn, by the definition of the revealed exposure process in Definition 5.2.1, we have that:

$$\tilde{\zeta}_t^n = \zeta_t^n, \quad n = 1, ..., N \quad d\mathbf{P} \otimes dt - a.e.$$

More precisely, for the above assertion to hold, (5.11) implies that:

$$\frac{\zeta_t^{-1} - \psi_t}{\gamma^1 \delta_{-1}} + \zeta_t^1 + \gamma^1(\delta - 1/\gamma^1)\zeta_t^1 - \zeta_t^1 - \gamma^1 \delta \zeta_t^1 = 0 \Rightarrow$$
$$\frac{\Sigma^{-1}\mu_{t,-1}}{\gamma^1} - \zeta_t^1 = 0 \quad d\mathbf{P} \otimes dt - a.e., \tag{5.12}$$

where $\mu_{t,-1} = \frac{\Sigma(\zeta_t^{-1} - \psi_t)}{\delta_{-1}}$. Now, using (5.8), we have that $\mu_{t,-1} = \mu_t - \frac{(\gamma^1 \Sigma \zeta_t^1 - \mu_{t,-1})}{\gamma^1 \delta}$ for $\mu_t = \frac{\Sigma(\zeta_t - \psi_t)}{\delta}$. Expanding the left-hand side of (5.12), we have:

$$\begin{aligned}
\frac{\Sigma^{-1}\mu_{t,-1}}{\gamma^1} - \zeta_t^1 &= \frac{\Sigma^{-1}\mu_t}{\gamma^1} - \zeta_t^1 - \frac{\Sigma^{-1}(\gamma^1 \Sigma \zeta_t^1 - \mu_{t,-1})}{(\gamma^1)^2 \delta} \\
&= \frac{\Sigma^{-1}\mu_t}{\gamma^1} - \zeta_t^1 - \frac{\zeta_t^1}{\gamma^1 \delta} + \frac{\zeta_t^{-1} - \psi_t}{\gamma^1 \delta_{-1} \gamma^1 \delta} \\
&= \frac{\Sigma^{-1}\mu_t}{\gamma^1} - \zeta_t^1 + \frac{\zeta_t^{-1} - \psi_t - \gamma^1 \delta_{-1} \zeta_t^1}{\gamma^1 \delta_{-1} \gamma^1 \delta} \\
&= \frac{\Sigma^{-1}\mu_t}{\gamma^1} - \zeta_t^1 + \frac{\zeta_t^{-1} - \psi_t \pm \zeta_t^1 - \gamma^1 \delta_{-1} \zeta_t^1}{\gamma^1 \delta_{-1} \gamma^1 \delta} \\
&= \frac{\Sigma^{-1}\mu_t}{\gamma^1} - \zeta_t^1 + \frac{\zeta_t - \psi_t}{\gamma^1 \delta_{-1} \gamma^1 \delta} - \frac{\zeta_t^1}{\gamma^1 \delta_{-1}} \\
&= \frac{\Sigma^{-1}\mu_t}{\gamma^1} - \zeta_t^1 + \frac{1}{\gamma^1 \delta_{-1}} \underbrace{\Big(\frac{\zeta_t - \psi_t}{\gamma^1 \delta} - \zeta_t^1\Big)}_{\frac{\Sigma^{-1}\mu_t}{\gamma^1} - \zeta_t^1} \\
&= \Big(\frac{\Sigma^{-1}\mu_t}{\gamma^1} - \zeta_t^1\Big)\Big(1 + \frac{1}{\gamma^1 \delta_{-1}}\Big).
\end{aligned}$$

Therefore, by (5.12) we have:

$$\Big(\frac{\Sigma^{-1}\mu_t}{\gamma^1} - \zeta_t^1\Big)\Big(1 + \frac{1}{\gamma^1\delta_{-1}}\Big) = 0 \quad d\mathbf{P}\otimes dt - a.e.$$

In turn, by noting that $1 + \frac{1}{\gamma^1\delta_{-1}} > 0$ we have that $\frac{\Sigma^{-1}\mu_t}{\gamma^1} - \zeta_t^1 = 0 \; d\mathbf{P}\otimes dt - a.e.$

For the other direction, we have that:

$$\begin{aligned} 0 &= \frac{\Sigma^{-1}\overbrace{\frac{\Sigma(\zeta_t - \psi_t)}{\delta}}^{\mu_t}}{\gamma^1} - \zeta_t^1 \Rightarrow \\ \zeta_t - \psi_t &= \gamma^1\delta\zeta_t^1 \Rightarrow \\ (\zeta_t^{-1} - \psi_t) + \cancel{\zeta_t^1} &= \gamma^1\delta_{-1} + \cancel{\zeta_t^1} \Rightarrow \\ \zeta_t^{-1} - \psi_t - \gamma^1\delta_{-1}\zeta_t^1 &= 0. \end{aligned}$$

Now by recalling how $\tilde{\phi}_t^1$ was derived in (5.4) and using the above result, we see that the numerator of $\tilde{\phi}_t^1$ vanishes, yielding $\tilde{\phi}_t^1 = \phi_t^1 = 0 \; d\mathbf{P}\otimes dt - a.e.$ ■

5.3 The Nash equilibrium in a frictionless market

The new goal now becomes to derive the new point of equilibrium returns when all investors apply the same price impact strategy. That is, we consider a market where all the investors are aware of their impact on the risky assets. This is the so-called Nash equilibrium, where all investors apply the same strategic behavior (as investor 1 in (5.4)), and the market equilibrates at the induced fixed point. Thus, going back how the revealed exposure was defined in Definition 5.2.1, after substituting (5.4) for each investor, we have:

$$\tilde{\zeta}_t^n = \frac{\Sigma^{-1}\tilde{\mu}_t}{\gamma^n} - \Big(\frac{\tilde{\zeta}_{t,-n} - \psi_t - \gamma^n\delta_{-n}\zeta_t^n}{\delta_{-n}k_n}\Big), \quad n = 1, ..., N. \tag{5.13}$$

Note that ζ_t^{-n} became $\tilde{\zeta}_t^{-n}$, since in this setting all the investors influence the equilibrium. In other words, the aggregate revealed exposure of the investors drives the market to a new equilibrium, taking into account the price impact of each investor, which is to be determined by the equilibrium condition.

Theorem 5.3.1 *The Nash equilibrium in a frictionless market is of the following form:*

$$\tilde{\mu}_t = \frac{\Sigma}{\delta} \frac{(\zeta_t - \psi_t) - \sum_{n=1}^N \lambda_n \zeta_t^n}{1 - \sum_{n=1}^N \lambda_n^2}, \tag{5.14}$$

where $\delta_n = 1/\gamma^n$ *and* $\lambda_n = \delta_n/\delta$. ◊

Proof: Going back to (5.13), adding and substracting $\frac{\tilde{\zeta}_{t,-n}}{\delta_{-n}k_n}$ on the right-hand side, we get:

$$\begin{aligned}
\tilde{\zeta}_t^n &= \frac{\Sigma^{-1}\tilde{\mu}_t}{\gamma^n} - \frac{\tilde{\zeta}_t}{\delta_{-n}k_n} + \frac{\tilde{\zeta}_t^n}{\delta_{-n}k_n} + \frac{\psi_t}{\delta_{-n}k_n} + \frac{\gamma^n\delta_{-n}\zeta_t^n}{\delta_{-n}k_n} \Rightarrow \\
\tilde{\zeta}_t^n - \frac{\tilde{\zeta}_t^n}{\delta_{-n}k_n} &= \frac{\Sigma^{-1}\tilde{\mu}_t}{\gamma^n} - \frac{\tilde{\zeta}_t}{\delta_{-n}k_n} + \frac{\psi_t}{\delta_{-n}k_n} + \frac{\gamma^n\delta_{-n}\zeta_t^n}{\delta_{-n}k_n} \Rightarrow \\
\frac{\overbrace{(\delta_{-n}k_n - 1)}^{\gamma^n\delta}}{\delta_{-n}k_n}\tilde{\zeta}_t^n &= \frac{\Sigma^{-1}\tilde{\mu}_t}{\gamma^n} - \frac{\tilde{\zeta}_t}{\delta_{-n}k_n} + \frac{\psi_t}{\delta_{-n}k_n} + \frac{\gamma^n\delta_{-n}\zeta_t^n}{\delta_{-n}k_n} \Rightarrow \\
\tilde{\zeta}_t^n &= \frac{(1/\gamma^n + \delta)\Sigma^{-1}\tilde{\mu}_t}{\gamma^n\delta} - \frac{\tilde{\zeta}_t}{\gamma^n\delta} + \frac{\psi_t}{\gamma^n\delta} + \frac{\delta_{-n}\zeta_t^n}{\delta} \\
&= \frac{\Sigma^{-1}\tilde{\mu}_t}{(\gamma^n)^2\delta} + \frac{\Sigma^{-1}\tilde{\mu}_t}{\gamma^n} - \frac{\tilde{\zeta}_t}{\gamma^n\delta} + \frac{\psi_t}{\gamma^n\delta} + \frac{\delta_{-n}\zeta_t^n}{\delta} \Rightarrow \\
\sum_{n=1}^N \tilde{\zeta}_t^n &= \Sigma^{-1}\tilde{\mu}_t \sum_{n=1}^N \frac{1}{(\gamma^n)^2\delta} + \Sigma^{-1}\tilde{\mu}_t\delta - \tilde{\zeta}_t \sum_{n=1}^N \frac{1}{\gamma^n\delta} + \psi_t \sum_{n=1}^N \frac{1}{\gamma^n\delta} + \frac{1}{\delta}\sum_{n=1}^N \delta_{-n}\zeta_t^n \\
&= \Sigma^{-1}\tilde{\mu}_t\delta\Big(1 + \sum_{n=1}^N \frac{1}{(\gamma^n)^2\delta^2}\Big) - \tilde{\zeta}_t + \psi_t + \frac{1}{\delta}\sum_{n=1}^N \delta_{-n}\zeta_t^n.
\end{aligned}$$

Therefore, we have:

$$\tilde{\zeta}_t = \frac{\Sigma^{-1}\tilde{\mu}_t\delta\Big(1 + \sum_{n=1}^N \frac{1}{(\gamma^n)^2\delta^2}\Big) + \psi_t + \frac{1}{\delta}\sum_{n=1}^N \delta_{-n}\zeta_t^n}{2}. \tag{5.15}$$

In turn, by (5.15), the equilibrium condition of (3.7) and the linearity of summation we get:

$$\begin{aligned}
0 &= \overbrace{\sum_{n=1}^N \frac{\Sigma^{-1}\tilde{\mu}_t}{\gamma^n} - \tilde{\zeta}_t}^{\sum_{n=1}^N \tilde{\phi}_t^n} + \psi_t \overset{\times 2}{\Rightarrow} \\
0 &= \Sigma^{-1}\tilde{\mu}_t\delta - \Sigma^{-1}\tilde{\mu}_t\delta \sum_{n=1}^N \frac{1}{(\gamma^n)^2\delta^2} + \psi_t - \frac{1}{\delta}\sum_{n=1}^N \delta_{-n}\zeta_t^n \Rightarrow \\
\Sigma^{-1}\tilde{\mu}_t\delta - \Sigma^{-1}\tilde{\mu}_t\delta \sum_{n=1}^N \frac{1}{(\gamma^n)^2\delta^2} &= \zeta_t - \sum_{n=1}^N \frac{\zeta_t^n}{\gamma^n\delta} - \psi_t \Rightarrow
\end{aligned}$$

$$\tilde{\mu}_t - \tilde{\mu}_t \sum_{n=1}^{N} \frac{1}{(\gamma^n)^2\delta^2} = \frac{\Sigma(\zeta_t - \psi_t)}{\delta} - \frac{\Sigma \sum_{n=1}^{N} \zeta_t^n/\gamma^n}{\delta^2}.$$

Thus, taking into account that the matrix $\left(I_d - I_d \sum_{n=1}^{N} \frac{1}{(\gamma^n)^2\delta^2}\right) = \left(1 - \sum_{n=1}^{N} \frac{1}{(\gamma^n)^2\delta^2}\right) I_d$ is positive definite (since $\gamma^n \delta > 1$), and thus invertible, the Nash equilibrium is given by:

$$\tilde{\mu}_t = \frac{\Sigma}{\delta} \frac{(\zeta_t - \psi_t) - \sum_{n=1}^{N} \lambda_n \zeta_t^n}{1 - \sum_{n=1}^{N} \lambda_n^2}.$$

This concludes the proof. ■

Corollary 5.3.1 *Let the following conditions hold:*

(I) There are not any noise traders in the market.

(II) The investors have homogenous risk profiles, $\gamma^1 = ... = \gamma^N = \bar{\gamma}$.

Then $\tilde{\mu}_t = \mu_t$, *where* $\mu_t = \frac{\bar{\gamma}\Sigma\zeta_t}{N}$. ◊

Proof: Note that by substracting the frictionless equilibrium of (3.8) from the Nash equilibrium of (5.14) we have:

$$\begin{aligned} \tilde{\mu}_t - \mu_t &= \frac{\Sigma}{\delta} \frac{(\zeta_t - \psi_t) - \sum_{n=1}^{N} \lambda_n \zeta_t^n}{1 - \sum_{n=1}^{N} \lambda_n^2} - \frac{\Sigma(\zeta_t - \psi_t)}{\delta} \\ &= \frac{\Sigma}{\delta} \left(\frac{(\zeta_t - \psi_t) \sum_{n=1}^{N} \lambda_n^2 - \sum_{n=1}^{N} \lambda_n \zeta_t^n}{1 - \sum_{n=1}^{N} \lambda_n^2} \right). \end{aligned}$$

Assuming there are no noise traders in the market and the investors have the same risk aversions, and therefore $\lambda_n = \frac{1}{N}$, the above becomes:

$$\begin{aligned} \tilde{\mu}_t - \mu_t &= \frac{\Sigma}{\delta} \left(\frac{\zeta_t \sum_{n=1}^{N} (1/N)^2 - \sum_{n=1}^{N} (1/N) \zeta_t^n}{1 - \sum_{n=1}^{N} (1/N)^2} \right) \\ &= 0. \end{aligned}$$

This concludes the proof. ■

Chapter 6

PRICE IMPACT IN A MARKET WITH FRICTIONS

In this chapter, we study the notions depicted in the following graph:

Optimal allocation with frictions under price impact	Equilibrium with frictions under the price impact of a single investor

Figure 6.1: Outline of the 6^{th} chapter

In this chapter we extend the notion of price impact in a finite time horizon market with frictions. To this end, also recall the discussion about the solution of the FBSDE (4.4) in Chapter 4.

6.1 Optimal allocation with frictions under price impact

In this section, we consider the optimization problem in a market with frictions, under the price impact of a single investor (say investor 1). To this end, apart from the concepts already introduced in Chapter 4, we define the following which hold throughout this chapter:

(I) Assume that investor 1's trading rate is absolutely continuous, given by:

$$\dot{\varphi}^1_t = x + \int_0^t \ddot{\varphi}^1_s ds, \quad t \in [S, T],$$

where $\ddot{\varphi}^1_t \in \mathscr{L}^2_r$.

For more information about absolutely continuous functions, refer to *A.4.*

(II) The time horizon of the market is finite.

(III) Investors have the same risk aversions.

(IV) We consider the optimization problem, in a market with frictions and price impact, on a compact interval $t \in [S, T]$, where $0 \leq S < T$ (S, T are fixed). Furthermore, we define the "$\mathscr{F}_S$-restrictions" of the following processes: $\underline{\zeta}_t^n = \mathbb{E}[\zeta_t^n|\mathscr{F}_S]$, $\forall\ n \in N$, $\underline{\psi}_t = \mathbb{E}[\psi_t|\mathscr{F}_S]$, $\underline{\dot{\psi}}_t = \mathbb{E}[\dot{\psi}_t|\mathscr{F}_S]$ and $\underline{\mu}_t^{\psi} = \mathbb{E}[\mu_t^{\psi}|\mathscr{F}_S]$ [1]. Lastly, while the terminal condition remains the same to that of a finite time horizon market with frictions (i.e. $\dot{\phi}_T^1 = 0$), we now have $\phi_S^1 = 0$ (the new starting point, which of course might be $S = 0$).

(V) We define the following classes:

$$\mathscr{W}^{1,2} = \left\{X : \Omega \times [S,T] \to \mathbb{R}^l : X \text{ is absolutely continuous s.t. } \dot{X}_t \in \mathscr{L}_r^2\right\},$$
$$\mathscr{W}^{2,2} = \left\{X : \Omega \times [S,T] \to \mathbb{R}^l : X \text{ is absolutely continuous s.t. } \dot{X}_t \in \mathscr{W}^{1,2}\right\}.$$

Lemma 6.1.1 *Assume* $\gamma^1 = ... = \gamma^N = \bar{\gamma}$. *Then, the respective form of investor* 1*'s frictionless price impact of* (5.2), *in a finite time horizon market with frictions is given as follows:*

$$\mu_t(\phi^1, \dot{\phi}^1, \ddot{\phi}^1) = \frac{2\Lambda}{N-1}\ddot{\phi}_t^1 - \frac{2r\Lambda}{N-1}\dot{\phi}_t^1 - \frac{\bar{\gamma}\Sigma}{N-1}\phi_t^1 + \frac{\bar{\gamma}\Sigma}{N-1}\sum_{n=2}^{N}\underline{\zeta}_t^n - \frac{\gamma\Sigma}{N-1}\underline{\psi}_t + \frac{2\Lambda}{N-1}(\underline{\mu}_t^{\psi} - r\underline{\dot{\psi}}_t). \ \diamond$$

Proof: In a similar way to the one we followed to derive the price impact of a single investor in a frictionless market, (5.1) generalizes to the following equilibrium conditions, "extending" (4.9) and (4.10) respectively by the conditions of this chapter:

$$\phi_t^1 + \sum_{n=2}^{N}\phi_t^{\Lambda,n} + \underline{\psi}_t = 0, \tag{6.1}$$
$$\dot{\phi}_t^1 + \sum_{n=2}^{N}\dot{\phi}_t^{\Lambda,n} + \underline{\dot{\psi}}_t = 0, \tag{6.2}$$

where $\phi_t^1, \dot{\phi}_t^1$ are any strategies and trading rates of investor 1, $\phi_t^{\Lambda,n},\ n = 2, ..., N$ are the optimal strategies of the rest of the investors in a finite time horizon market with frictions, without price impact and $\dot{\phi}_t^{\Lambda,n},\ n = 2, ..., N$ their respective optimal trading rates. Equivalently, the differential form of (6.2) becomes:

[1] Note that in the case where $S = 0$, $\mathscr{F}_S$ becomes the trivial σ-algebra.

$$d\dot{\phi}_t^1 + \sum_{n=2}^{N} d\dot{\phi}_t^{\Lambda,n} + d\underline{\dot{\psi}}_t = 0. \tag{6.3}$$

Let us begin by determining $d\dot{\phi}_t^n$, $n = 2, ..., N$. The new form of (4.2) under the conditions introduced in this chapter will be:

$$\mathscr{F}^{\Lambda,n}(\dot{\phi}) = \mathbb{E}\left[\int_S^T e^{-rt}\Big((\phi_t^n)^T \mu_t - \frac{\bar{\gamma}}{2}(\phi_t^n + \underline{\zeta}_t^n)^T \Sigma (\phi_t^n + \underline{\zeta}_t^n) - (\dot{\phi}_t^n)^T \Lambda \dot{\phi}_t^n\Big) dt\right]$$
$$+ \mathbb{E}\left[\int_S^T e^{-rt}\Big(dA_t^n - \frac{\gamma^n}{2} d\langle M^{\perp,n}\rangle_t\Big)\right].$$

Following the same process as we did in Chapter 4, for (4.5) we get for $t \in [S, T]$:

$$\dot{\phi}_t^{\Lambda,n} = \frac{\bar{\gamma}\Lambda^{-1}\Sigma}{2} e^{rt} \mathbb{E}\left[\int_t^T e^{-rs}\Big(\frac{\Sigma^{-1}\mu_s}{\bar{\gamma}} - \underline{\zeta}_s^n - \phi_s^{\Lambda,n}\Big) ds \Big| \mathscr{F}_t\right], \quad n = 2, ..., N.$$

As previously shown, the above for a finite time horizon is characterized by the following system of FBSDEs, derived in Lemma 4.2.1:

$$\begin{aligned} d\phi_t^{\Lambda,n} &= \dot{\phi}_t^{\Lambda,n} dt, \quad \phi_0^{\Lambda,n} = 0, \\ d\dot{\phi}_t^{\Lambda,n} &= dM_t^n + \frac{\bar{\gamma}\Lambda^{-1}\Sigma}{2}(\phi_t^{\Lambda,n} - \phi_t^n) dt + r\dot{\phi}_t^{\Lambda,n} dt, \quad \dot{\phi}_T^{\Lambda,n} = 0, \end{aligned} \tag{6.4}$$

where now we respectively have $\phi_t^n = \frac{\Sigma^{-1}\mu_t}{\bar{\gamma}} - \underline{\zeta}_t^n$. Furthermore, coupling the absolute continuity of $\dot{\phi}_t^1$ with the terminal condition at $t = T$, we have:

$$d\dot{\phi}_t^1 = \ddot{\phi}_t^1 dt, \quad \dot{\phi}_T^1 = 0, \tag{6.5}$$

where $\ddot{\phi}_t^1 \in \mathscr{L}_r^2$.

Substituting (6.5) for investor 1 and the BSDE of (6.4) for the rest of the investors in (6.3), we have:

$$0 = \ddot{\phi}_t^1 dt + \underbrace{dM_t}_{n=2,...,N} + \frac{\Lambda^{-1}}{2} \sum_{n=2}^{N} \Big(\bar{\gamma}\Sigma\phi_t^{\Lambda,n} - (\mu_t - \bar{\gamma}\Sigma\underline{\zeta}_t^n)\Big) dt + r \sum_{n=2}^{N} \dot{\phi}_t^{\Lambda,n} dt + \underbrace{d\underline{\dot{\psi}}_t}_{\mu_t^{\psi} dt + dM_t^{\psi}},$$

where M_t, M_t^{ψ} are continuous local martingales. Moreover, (6.2) yields $r\sum_{n=2}^{N} \dot{\phi}_t^{\Lambda,n} = -r(\dot{\phi}_t^1 + \underline{\dot{\psi}}_t)$. Therefore, we have:

$$0 = \ddot{\phi}^1_t dt + dM_t + \frac{\Lambda^{-1}\Sigma}{2} \sum_{n=2}^{N} \bar{\gamma}\phi^{\Lambda,n}_t dt - \frac{\Lambda^{-1}(N-1)}{2}\mu_t dt + \frac{\Lambda^{-1}\Sigma}{2} \sum_{n=2}^{N} \bar{\gamma}\underline{\zeta}^n_t dt$$
$$- r(\dot{\phi}^1_t + \dot{\underline{\psi}}_t)dt + \mu^{\psi}_t dt + dM^{\psi}_t.$$

Now recall that, using the same arguments as in (4.8), dM_t, dM^{ψ}_t vanish, as shown in [Bou+18], yielding:

$$0 = \ddot{\phi}^1_t dt + \frac{\Lambda^{-1}\Sigma}{2} \sum_{n=2}^{N} \bar{\gamma}\phi^{\Lambda,n}_t dt - \frac{\Lambda^{-1}(N-1)}{2}\mu_t dt + \frac{\Lambda^{-1}\Sigma}{2} \sum_{n=2}^{N} \bar{\gamma}\underline{\zeta}^n_t dt - r(\dot{\phi}^1_t + \dot{\underline{\psi}}_t)dt + \mu^{\psi}_t dt.$$

By the homogenous risk aversions and (6.1) we in turn have that $\bar{\gamma}\sum_{n=2}^{N} \phi^{\Lambda,n}_t = -\bar{\gamma}\phi^1_t - \bar{\gamma}\underline{\psi}_t$, which gives us:

$$0 = \ddot{\phi}^1_t dt + \frac{\Lambda^{-1}\Sigma}{2}(-\bar{\gamma}\phi^1_t - \bar{\gamma}\underline{\psi}_t)dt - \frac{\Lambda^{-1}(N-1)}{2}\mu_t dt + \frac{\Lambda^{-1}\Sigma}{2} \sum_{n=2}^{N} \bar{\gamma}\underline{\zeta}^n_t dt - r(\dot{\phi}^1_t + \dot{\underline{\psi}}_t)dt + \mu^{\psi}_t dt.$$

Solving the above for μ_t concludes the proof. ■

Remark 6.1.1 *Let us now briefly discuss the form of $\mu_t(\phi^1, \dot{\phi}^1, \ddot{\phi}^1)$ in Lemma 6.1.1. Recall that in the case of a frictionless market under the price impact of investor 1, we have:*

$$\mu_t(\phi^1) = \frac{\Sigma(\zeta^{-1}_t - \psi_t - \phi^1_t)}{\delta_{-1}},$$

which under the homogenous risk aversion assumption, becomes:

$$\mu_t(\phi^1) = \frac{\bar{\gamma}\Sigma(\zeta^{-1}_t - \psi_t - \phi^1_t)}{N-1}.$$

Thus, by the above we have:

$$\begin{aligned}
\mu_t(\phi^1, \dot{\phi}^1, \ddot{\phi}^1) &= \frac{2\Lambda}{N-1}\ddot{\phi}^1_t - \frac{2r\Lambda}{N-1}\dot{\phi}^1_t - \frac{\bar{\gamma}\Sigma}{N-1}\phi^1_t + \frac{\gamma\Sigma}{N-1}\underline{\zeta}^{-1}_t - \frac{\gamma\Sigma}{N-1}\underline{\psi}_t + \frac{2\Lambda}{N-1}(\mu^{\psi}_t - r\dot{\underline{\psi}}_t) \\
&= \frac{2\Lambda}{N-1}(\ddot{\phi}^1_t - r\dot{\phi}^1_t + \mu^{\psi}_t - r\dot{\underline{\psi}}_t) + \frac{\bar{\gamma}\Sigma}{N-1}(\underline{\zeta}^{-1}_t - \underline{\psi}_t - \phi^1_t) \\
&= \frac{2\Lambda}{N-1}(\ddot{\phi}^1_t - r\dot{\phi}^1_t + \mu^{\psi}_t - r\dot{\underline{\psi}}_t) + \mu_t(\phi^1),
\end{aligned}$$

where $\underline{\mu}_t(\phi^1) = \frac{\bar{\gamma}\Sigma(\underline{\zeta}_t^{-1} - \underline{\psi}_t - \phi_t^1)}{N-1}$. *Note, by the last equality, that as* Λ *goes to zero, the price impact in a market with frictions goes towards* $\underline{\mu}_t(\phi^1)$. *This is to be expected since, as previously discussed in Chapter 4, the transaction costs in a market with frictions are modelled through the matrix* Λ.$\diamondsuit$

We are now ready to establish a new goal functional, in a market with frictions, which takes into account investor 1's price impact. Therefore, in a similar manner to that of the frictionless case and by Lemma 6.1.1, we have:

$$\mathscr{F}^{\Lambda,1}(\phi) = \mathbb{E}\Bigg[\int_S^T e^{-rt}\Big((\phi_t^1)^T\Big(\frac{2\Lambda}{N-1}\ddot{\phi}_t^1 - \frac{2r\Lambda}{N-1}\dot{\phi}_t^1 - \frac{\bar{\gamma}\Sigma}{N-1}\phi_t^1 + \frac{\bar{\gamma}\Sigma}{N-1}\sum_{n=2}^N \underline{\zeta}_t^n - \frac{\bar{\gamma}\Sigma}{N-1}\underline{\psi}_t$$
$$+ \frac{2\Lambda}{N-1}(\mu_t^{\psi} - r\underline{\dot{\psi}}_t)\Big) - \frac{\bar{\gamma}}{2}(\phi_t^1 + \underline{\zeta}_t^1)^T\Sigma(\phi_t^1 + \underline{\zeta}_t^1) - (\dot{\phi}_t^1)^T\Lambda\dot{\phi}_t^1\Big)dt + \int_S^T e^{-rt}\Big(dA_t^1 - \frac{\bar{\gamma}}{2}d\langle M^{\perp,1}\rangle_t\Big)\Bigg]. \tag{6.6}$$

Theorem 6.1.1 *Assume that* $\gamma^1 = ... = \gamma^N = \bar{\gamma}$. *Then the solution to* (6.6) *is characterized by the following system of linear, coupled, non-homogenous first order ODEs:*

$$\begin{aligned} \frac{d\phi_t^1}{dt} &= \dot{\phi}_t^1, \quad \phi_S^1 = 0, \\ \frac{d\dot{\phi}_t^1}{dt} &= r\dot{\phi}_t^1 + B\phi_t^1 + Q_t^1, \quad \dot{\phi}_T^1 = 0, \end{aligned} \tag{6.7}$$

where $B = \frac{\bar{\gamma}\Lambda^{-1}\Sigma}{2}$ *and* $Q_t^1 = \frac{(N-1)B\underline{\zeta}_t^1 - B\underline{\zeta}_t^{-1} - (\mu_t^{\psi} - r\underline{\dot{\psi}}_t) + B\underline{\psi}_t}{N+1}$. $\diamondsuit$

Proof: Taking the Gâteaux differential of (6.6) for $\theta_t^1 \in \mathscr{W}^{2,2}$ with $\theta_S^1 = 0$, $\dot{\theta}_T^1 = 0$ and using integration by parts, we have [2]:

$$\Big(d\mathscr{F}^{\Lambda,1}(\phi^1), \theta^1\Big) = \mathbb{E}\Bigg[\int_S^T \Big[re^{-rt}(\phi_t^1)^T\frac{2\Lambda}{N-1}\theta_t^1\Big]_S^T + 2e^{-rt}(\ddot{\phi}_t^1)^T\frac{2\Lambda}{N-1}\theta_t^1 - 2re^{-rt}(\dot{\phi}_t^1)^T\frac{2\Lambda}{N-1}\theta_t^1$$
$$+ r^2e^{-rt}(\phi_t^1)^T\frac{2\Lambda}{N-1}\theta_t^1 - \Big[re^{-rt}(\phi_t^1)^T\frac{2\Lambda}{N-1}\theta_t^1\Big]_S^T - r^2e^{-rt}(\phi_t^1)^T\frac{2\Lambda}{N-1}\theta_t^1 - 2e^{-rt}(\phi_t^1)^T\frac{\bar{\gamma}\Sigma}{N-1}\theta_t^1 +$$
$$e^{-rt}\Big(\sum_{n=2}^N \underline{\zeta}_t^n\Big)^T\frac{\bar{\gamma}\Sigma}{N-1}\theta_t^1 + e^{-rt}(\mu_t^{\psi} - r\underline{\dot{\psi}})^T\frac{2\Lambda}{N-1}\theta_t^1 - e^{-rt}\underline{\psi}_t^T\frac{\bar{\gamma}\Sigma}{N-1}\theta_t^1 - e^{-rt}\Big((\phi_t^1)^T\bar{\gamma}\Sigma\theta_t^1 + (\underline{\zeta}_t^1)^T\bar{\gamma}\Sigma\theta_t^1\Big)$$

[2] Note that in a finite time horizon market, $\ddot{\phi}_t^1 \in \mathscr{L}_r^2$ suffices in order to have that $\phi_t^1, \dot{\phi}_t^1 \in \mathscr{L}_r^2$. For more information about the integrability of $\phi_t^1, \dot{\phi}_t^1$ and (the Lebesgue) integration by parts, refer to A.5.

$$- e^{-rt}(\dot{\phi}_t^1)^T 2r\Lambda\theta_t^1 + e^{-rt}(\ddot{\phi}_t^1)^T 2\Lambda\theta_t^1 dt\Bigg]$$

$$= \mathbb{E}\Bigg[\int_S^T e^{-rt}\Bigg((\ddot{\phi}_t^1)^T\frac{4\Lambda}{N-1} + (\ddot{\phi}_t^1)^T 2\Lambda - (\dot{\phi}_t^1)^T\frac{4r\Lambda}{N-1} - (\dot{\phi}_t^1)^T 2r\Lambda - (\phi_t^1)^T\frac{2\bar{\gamma}\Sigma}{N-1} - (\phi_t^1)^T\bar{\gamma}\Sigma +$$

$$\Big(\sum_{n=2}^{N}\underline{\zeta}_t^n\Big)^T\frac{\bar{\gamma}\Sigma}{N-1} - (\underline{\zeta}_t^1)^T\bar{\gamma}\Sigma + (\mu_t^{\psi} - r\underline{\psi})^T\frac{2\Lambda}{N-1} - \underline{\psi}_t^T\frac{\bar{\gamma}\Sigma}{N-1}\Bigg)\theta_t^1 dt\Bigg].$$

Now by using Lemma 3.2.1 and multiplying by the factor $\frac{e^{rt}(N-1)\Lambda^{-1}}{2(N+1)}$, we finally arrive at the following second order linear random ODE with constant coefficients:

$$\ddot{\tilde{\phi}}_t^{\Lambda,1} - r\dot{\tilde{\phi}}_t^{\Lambda,1} - B\tilde{\phi}_t^{\Lambda,1} = \underbrace{\frac{(N-1)B\underline{\zeta}_t^1}{N+1} - \frac{B\underline{\zeta}_t^{-1}}{N+1} - \frac{(\mu_t^{\psi} - r\underline{\psi}_t)}{N+1} + \frac{B\underline{\psi}_t}{N+1}}_{Q_t^1}, \tag{6.8}$$

where $B = \frac{\bar{\gamma}\Lambda^{-1}\Sigma}{2}$ and $\underline{\zeta}_t^{-1} = \sum_{n=2}^{N}\underline{\zeta}_t^n$.

To ensure that (6.8) does indeed characterize a unique global maximum, we deal with the second order condition. Using once more integration by parts we have:

$$\mathbb{E}\Bigg[\int_S^T e^{-rt}\Bigg(-(\theta_t^1)^T\frac{2\bar{\gamma}\Sigma}{N-1}\theta_t^1 - (\dot{\theta}_t^1)^T\frac{4\Lambda}{N-1}\dot{\theta}_t^1 - (\theta_t^1)^T\bar{\gamma}\Sigma\theta_t^1 - (\dot{\theta}_t^1)^T 2\Lambda\dot{\theta}_t^1\Bigg)dt\Bigg] < 0,$$

for $\bar{\gamma}, \frac{\bar{\gamma}}{N-1} > 0$ and positive definite matrices Σ, Λ. Therefore, by Proposition 3.2.1, (6.8) characterizes the unique global maximum.

Lastly, we note that (6.8) can be reduced to the following system of linear, coupled, non-homogenous first order ODEs by considering:

$$\frac{d}{dt}\begin{bmatrix}\tilde{\phi}_t^{\Lambda,1}\\ \dot{\tilde{\phi}}_t^{\Lambda,1}\end{bmatrix} = \begin{bmatrix}0_{d,d} & I_d\\ B & rI_d\end{bmatrix}\begin{bmatrix}\tilde{\phi}_t^{\Lambda,1}\\ \dot{\tilde{\phi}}_t^{\Lambda,1}\end{bmatrix} + \begin{bmatrix}0_{d,1}\\ Q_t^1\end{bmatrix},$$

where $0_{m,n}$ denotes a $m \times n$ zero matrix. ■

Before moving forward, let us introduce a usefull theorem as shown in [Sch05]:

Theorem 6.1.2 (Conditional Fubini Theorem) *Let $(X, \mathscr{A}, \mu), (Y, \mathscr{B}, \nu)$ be σ-finite measure spaces and $\mathscr{G} \subset \mathscr{A}$ be a sub-σ-algebra containing a sequence $(G_n)_{n\in\mathbb{N}}$ such that $G_n \uparrow X$ and $\mu(G_n) < \infty$. If $u(x,y)$ satisfies $u \in \mathscr{L}^1(\mu \otimes \nu)$, then:*

$$\mathbb{E}\left[\int_Y u(\cdot,y)\nu(dy)\Big|\mathscr{G}\right] = \int_Y \mathbb{E}[u(\cdot,y)|\mathscr{G}]\nu(dy). \qquad \Diamond$$

Lemma 6.1.2 *Consider a frictional market with a single risky asset. Then the solution to (6.7), is given by:*

This result can be generalized to d risky assets through the use of matrix exponentials.

$$\tilde{\phi}_t^{\Lambda,1} = \frac{C_1^1 e^{(r/2+\sqrt{\Delta})t}(\sqrt{\Delta}-r/2)}{B} - \frac{C_2^1 e^{(r/2-\sqrt{\Delta})t}(r/2+\sqrt{\Delta})}{B} + H_t^1, \tag{6.9}$$

where:

$$C_1^1 = -\frac{(r/2+\sqrt{\Delta})e^{-2\sqrt{\Delta}S}J_T^1}{(\sqrt{\Delta}-r/2)e^{(r/2-\sqrt{\Delta})T} + (r/2+\sqrt{\Delta})e^{-2\sqrt{\Delta}S}e^{(r/2+\sqrt{\Delta})T}},$$
$$C_2^1 = \frac{(r/2-\sqrt{\Delta})J_T^1}{(\sqrt{\Delta}-r/2)e^{(r/2-\sqrt{\Delta})T+(r/2+\sqrt{\Delta})e^{-2\sqrt{\Delta}S}e^{(r/2+\sqrt{\Delta})T}}},$$
$$J_t^1 = \int_S^t e^{r/2(t-s)}\left(\cosh\left(\sqrt{\Delta}(t-s)\right) + \frac{r}{2\sqrt{\Delta}}\sinh\left(\sqrt{\Delta}(t-s)\right)\right)Q_s^1 ds,$$
$$Q_t^1 = \frac{(N-1)B\underline{\zeta}_t^1 - B\underline{\zeta}_t^{-1} - (\mu_t^{\psi} - r\underline{\dot{\psi}}_t) + B\underline{\psi}_t}{N+1},$$
$$H_t^1 = \int_S^t \frac{e^{r/2(t-s)}\sinh\left(\sqrt{\Delta}(t-s)\right)}{\sqrt{\Delta}}Q_s^1 ds.$$

with $B = \frac{\bar{\gamma}\Lambda^{-1}\Sigma}{2}$, $\Delta = r^2/4 + B$ *and* $\underline{\zeta}_t^{-1} = \sum_{n=2}^N \underline{\zeta}_t^n$. $\Diamond$

Proof: Going back to Theorem 6.1.1, we have the following system of ODEs which characterize the optimal strategy:

$$\frac{d}{dt}\begin{bmatrix}\tilde{\phi}_t^{\Lambda,1}\\ \dot{\tilde{\phi}}_t^{\Lambda,1}\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ B & r\end{bmatrix}}_{M}\begin{bmatrix}\tilde{\phi}_t^{\Lambda,1}\\ \dot{\tilde{\phi}}_t^{\Lambda,1}\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}Q_t^1. \tag{6.10}$$

Beginning with the homogenous solution, we calculate the eigenvalues and eigenvectors of matrix M such that $Mv = zv$:

$$|M - zI| = z^2 - rz - B.$$

The above quadratic in turns yields the following eigenvalues for $D = r^2 + 4B > 0 \rightarrow z_{1,2} = \frac{r \pm \sqrt{D}}{2}$. Moving forward with the eigenvectors, for $z_1 = \frac{r+\sqrt{D}}{2}$, we have:

$$\begin{bmatrix} \frac{-r-\sqrt{D}}{2} & 1 \\ B & \frac{r-\sqrt{D}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

Dividing the first row by $\frac{-r-\sqrt{D}}{2}$ and by Gaussian elimination, the above matrix becomes:

$$\begin{bmatrix} 1 & \frac{r-\sqrt{D}}{2B} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0,$$

which in turns yields the first eigenvector:

$$\begin{bmatrix} -\frac{(r-\sqrt{D})}{2B} \\ 1 \end{bmatrix}.$$

In a similar way for $z_2 = \frac{r-\sqrt{D}}{2}$, we arrive at:

$$\begin{bmatrix} -\frac{(r+\sqrt{D})}{2B} \\ 1 \end{bmatrix}.$$

Therefore, the homogenous solution (HS) of (6.10) will be of the form:

$$C_1^1 e^{\frac{r+\sqrt{D}}{2}t} \begin{bmatrix} -\frac{(r-\sqrt{D})}{2B} \\ 1 \end{bmatrix} + C_2^1 e^{\frac{r-\sqrt{D}}{2}t} \begin{bmatrix} -\frac{(r+\sqrt{D})}{2B} \\ 1 \end{bmatrix}, \tag{HS}$$

with C_1^1, C_2^1 to be determined by the initial and terminal conditions of the system.

For the particular solution, we are searching a solution in the following form:

$$u_1 \begin{bmatrix} -\frac{(r-\sqrt{D})}{2B} e^{\frac{r+\sqrt{D}}{2}t} \\ e^{\frac{r+\sqrt{D}}{2}t} \end{bmatrix} + u_2 \begin{bmatrix} -\frac{(r+\sqrt{D})}{2B} e^{\frac{r-\sqrt{D}}{2}t} \\ e^{\frac{r-\sqrt{D}}{2}t} \end{bmatrix}.$$

Using the variation of parameters method, we essentially need to solve the following system:

$$\underbrace{\begin{bmatrix} -\frac{(r-\sqrt{D})}{2B}e^{\frac{r+\sqrt{D}}{2}t} & -\frac{(r+\sqrt{D})}{2B}e^{\frac{r-\sqrt{D}}{2}t} \\ e^{\frac{r+\sqrt{D}}{2}t} & e^{\frac{r-\sqrt{D}}{2}t} \end{bmatrix}}_{K} \begin{bmatrix} u_1^{'} \\ u_2^{'} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ Q_t^1 \end{bmatrix}}_{b}.$$

The above system can be solved by Cramer's rule, yielding u_1, u_2 as follows:

$$u_i^{'} = \frac{|K_i|}{|K|}, \quad i = 1, 2,$$

where K_i is the matrix formed by replacing the i-th column of matrix K by the column vector b. Calculating the above determinants, we have that $u_1 = \int_S^t \frac{(r+\sqrt{D})e^{\frac{r-\sqrt{D}}{2}s}}{2\sqrt{D}e^{rs}} Q_s^1 ds$ and $u_2 = -\int_S^t \frac{(r-\sqrt{D})e^{\frac{r+\sqrt{D}}{2}s}}{2\sqrt{D}e^{rs}} Q_s^1 ds$. Noting that $\sqrt{D} = 2\sqrt{r^2/4 + B} = 2\sqrt{\Delta}$ (by (4.15)), we have a particular solution (PS) of (6.10) can be written in the form:

$$\begin{bmatrix} H_t^1 \\ J_t^1 \end{bmatrix} = \int_S^t \frac{(r/2+\sqrt{\Delta})e^{(r/2-\sqrt{\Delta})s}}{2\sqrt{\Delta}e^{rs}} Q_s^1 ds \begin{bmatrix} -\frac{(r/2-\sqrt{\Delta})}{B}e^{(r/2+\sqrt{\Delta})t} \\ e^{(r/2+\sqrt{\Delta})t} \end{bmatrix}$$
$$- \int_S^t \frac{(r/2-\sqrt{\Delta})e^{(r/2+\sqrt{\Delta})s}}{2\sqrt{\Delta}e^{rs}} Q_s^1 ds \begin{bmatrix} -\frac{(r/2+\sqrt{\Delta})}{B}e^{(r/2-\sqrt{\Delta})t} \\ e^{(r/2-\sqrt{\Delta})t} \end{bmatrix}.$$

Equivalently, using the definitions of hyperbolic functions, the above can be written as:

$$\begin{aligned} H_t^1 &= \int_S^t \frac{e^{r/2(t-s)} \sinh\left(\sqrt{\Delta}(t-s)\right)}{\sqrt{\Delta}} Q_s^1 ds, \\ J_t^1 &= \int_S^t e^{r/2(t-s)} \left(\cosh\left(\sqrt{\Delta}(t-s)\right) + \frac{r}{2\sqrt{\Delta}} \sinh\left(\sqrt{\Delta}(t-s)\right) \right) Q_s^1 ds. \end{aligned} \tag{PS}$$

Note that H_t^1, J_t^1 are absolutely continuous, by the definition of absolute continuity and the fact that the sum and the product of absolutely continuous functions over a compact domain is absolutely continuous. Furthermore, H_t^1, J_t^1 are $\mathscr{F}_t$-adapted, by Lemma A.1.2, since their integrands are in $\mathscr{L}_r^2$. Combining (HS) and (PS), we get:

$$\begin{aligned} \tilde{\phi}_t^{\Lambda,1} &= \frac{C_1^1 e^{(r/2+\sqrt{\Delta})t}(\sqrt{\Delta} - r/2)}{B} - \frac{C_2^1 e^{(r/2-\sqrt{\Delta})t}(r/2+\sqrt{\Delta})}{B} + H_t^1, \\ \dot{\tilde{\phi}}_t^{\Lambda,1} &= C_1^1 e^{(r/2+\sqrt{\Delta})t} + C_2^1 e^{(r/2-\sqrt{\Delta})t} + J_t^1. \end{aligned}$$

Using the initial and terminal conditions, i.e. $\tilde{\phi}_S^{\Lambda,1} = 0$ and $\dot{\tilde{\phi}}_T^{\Lambda,1} = 0$, we derive C_1^1, C_2^1, which we substitute in the above to get the following general solution (GS):

$$\tilde{\phi}_t^{\Lambda,1} = \frac{C_1^1 e^{(r/2+\sqrt{\Delta})t}(\sqrt{\Delta} - r/2)}{B} - \frac{C_2^1 e^{(r/2-\sqrt{\Delta})t}(r/2 + \sqrt{\Delta})}{B} + H_t^1,$$
$$\dot{\tilde{\phi}}_t^{\Lambda,1} = C_1^1 e^{(r/2+\sqrt{\Delta})t} + C_2^1 e^{(r/2-\sqrt{\Delta})t} + J_t^1, \tag{GS}$$

where $C_1^1 = -\frac{(r/2+\sqrt{\Delta})e^{-2\sqrt{\Delta}S}J_T}{(\sqrt{\Delta}-r/2)e^{(r/2-\sqrt{\Delta})T}+(r/2+\sqrt{\Delta})e^{-2\sqrt{\Delta}S}e^{(r/2+\sqrt{\Delta})T}}$ and $C_2^1 = \frac{(r/2-\sqrt{\Delta})J_T}{(\sqrt{\Delta}-r/2)e^{(r/2-\sqrt{\Delta})T}+(r/2+\sqrt{\Delta})e^{-2\sqrt{\Delta}S}e^{(r/2+\sqrt{\Delta})T}}$. Note that by the fact that $Q_t \in \mathscr{L}_r^2$ (the other parts of the integrands of H_t^1, J_t^1 are uniformly continuous on a compact interval and therefore bounded) and Theorem 6.1.2 we have:

$$J_T^1 = \int_S^T e^{r/2(T-s)}\left(\cosh\left(\sqrt{\Delta}(T-s)\right) + \frac{r}{2\sqrt{\Delta}}\sinh\left(\sqrt{\Delta}(T-s)\right)\right)\overbrace{\left(\frac{(N-1)B\underline{\zeta}_s^1 - B\underline{\zeta}_s^{-1} - (\underline{\mu}_s^{\psi} - r\dot{\underline{\psi}}_s) + B\underline{\psi}_s}{N+1}\right)}^{Q_s^1} ds$$
$$= \mathbb{E}\left[\int_S^T e^{r/2(T-s)}\left(\cosh\left(\sqrt{\Delta}(T-s)\right) + \frac{r}{2\sqrt{\Delta}}\sinh\left(\sqrt{\Delta}(T-s)\right)\right)\overbrace{\left(\frac{(N-1)B\zeta_s^1 - B\zeta_s^{-1} - (\mu_s^{\psi} - r\dot{\psi}_s) + B\psi_s}{N+1}\right)}^{Q_s^1} ds \middle| \mathscr{F}_S\right],$$

which makes J_T^1 $\mathscr{F}_S$-measurable and therefore by the definition and completeness of the filtration $\mathscr{F}_t$-measurable for each $t \in [S, T]$. Thus (GS) is indeed in the class of admissible strategies [3]. Lastly, we can see that (HS) is indeed a homogenous solution for (6.8). Likewise for the particular solution, we have by integration by parts for J_t^1:

$$\frac{dJ_t^1}{dt} = \frac{r}{2}e^{(r/2+\sqrt{\Delta})t}u_1 + \sqrt{\Delta}e^{(r/2+\sqrt{\Delta})t}u_1 - \frac{r}{2}e^{(r/2-\sqrt{\Delta})t}u_2 + \sqrt{\Delta}e^{(r/2-\sqrt{\Delta})t}u_2$$
$$+ \frac{(r/2+\sqrt{\Delta})Q_t^1}{2\sqrt{\Delta}} - \frac{(r/2-\sqrt{\Delta})Q_t^1}{2\sqrt{\Delta}}.$$

We also have:

$$-rJ_t^1 - BH_t^1 = -re^{(r/2+\sqrt{\Delta})t}u_1 + re^{(r/2-\sqrt{\Delta})t}u_2 + \frac{r}{2}e^{(r/2+\sqrt{\Delta})t}u_1 - \sqrt{\Delta}e^{(r/2+\sqrt{\Delta})t}u_1$$
$$- \frac{r}{2}e^{(r/2-\sqrt{\Delta})t}u_2 - \sqrt{\Delta}e^{(r/2-\sqrt{\Delta})t}u_2$$
$$= -\frac{r}{2}e^{(r/2+\sqrt{\Delta})t}u_1 + \frac{r}{2}e^{(r/2-\sqrt{\Delta})t}u_2 - \sqrt{\Delta}e^{(r/2+\sqrt{\Delta})t}u_1 - \sqrt{\Delta}e^{(r/2-\sqrt{\Delta})t}u_2.$$

Combining the above in (6.8) we finally get:

[3] The fact that $\ddot{\tilde{\phi}}_t^{\Lambda,1} \in \mathscr{L}_r^2$ can be directly derived, since $\ddot{\tilde{\phi}}_t^{\Lambda,1}$ is given in feedback form by (6.8).

$$\frac{dJ_t^1}{dt} - rJ_t^1 - BH_t^1 = \frac{(r/2+\sqrt{\Delta})Q_t^1}{2\sqrt{\Delta}} - \frac{(r/2-\sqrt{\Delta})Q_t^1}{2\sqrt{\Delta}} = Q_t^1.$$

Therefore (GS) does indeed satisfy (6.8). ■

6.2 Equilibrium with frictions under the price impact of a single investor

We are now ready to derive an explicit form for the equilibrium returns in a finite time horizon market with frictions and under the price impact of investor 1. Recall that, as shown in Remark 4.3.2, the equilibrium with frictions without price impact reverts to its frictionless counterpart if we have homogenous risk aversions and there are no noise traders in the market. As we see in the following corollary, this is not the case for the equilibrium with frictions under the price impact of a single investor.

Corollary 6.2.1 *Assume that* $\gamma^1 = ... = \gamma^N = \bar{\gamma}$. *Then, the equilibrium returns in a market with frictions and under the price impact of investor* 1, *take the following form:*

$$\tilde{\mu}_t^{\Lambda,1} = \frac{N^2}{(N-1)(N+1)}\mathbb{E}[\mu_t|\mathscr{F}_S] - \frac{\bar{\gamma}\Sigma}{(N-1)(N+1)}\underline{\zeta}_t^1 + \frac{2N\Lambda}{(N-1)(N+1)}(\mu^{\underline{\psi}} - r\dot{\underline{\psi}}_t), \tag{6.11}$$

where $\mu_t = \frac{\bar{\gamma}\Sigma(\zeta_t - \psi_t)}{N}$ *is the frictionless equilibrium.* ◊

Proof: Going back to Lemma 6.1.1 after substituting the optimal dynamics in a market with frictions and under the price impact of investor 1, as derived in Theorem 6.1.1, we have:

$$\tilde{\mu}_t^{\Lambda,1} = \frac{2\Lambda}{N-1}\Big(\ddot{\tilde{\phi}}_t^{\Lambda,1} - r\dot{\tilde{\phi}}_t^{\Lambda,1} - \underbrace{\frac{\bar{\gamma}\Lambda^{-1}\Sigma}{2}}_{B}\tilde{\phi}_t^{\Lambda,1}\Big) + \frac{\bar{\gamma}\Sigma}{N-1}\underline{\zeta}_t^{-1} - \frac{\bar{\gamma}\Sigma}{N-1}\underline{\psi}_t + \frac{2\Lambda}{N-1}(\mu^{\underline{\psi}} - r\dot{\underline{\psi}}_t).$$

By (6.8) the above becomes:

$$\tilde{\mu}_t^{\Lambda,1} = \frac{2\Lambda}{N-1}\Big(\overbrace{\frac{(N-1)B\underline{\zeta}_t^1}{N+1} - \frac{B\underline{\zeta}_t^{-1}}{N+1} - \frac{(\mu_t^{\underline{\psi}} - r\dot{\underline{\psi}}_t)}{N+1} + \frac{B\underline{\psi}_t}{N+1}}^{Q_t^1}\Big) + \frac{\bar{\gamma}\Sigma\underline{\zeta}_t^{-1}}{N-1} - \frac{\bar{\gamma}\Sigma\underline{\psi}_t}{N-1} + \frac{2\Lambda(\mu_t^{\underline{\psi}} - r\dot{\underline{\psi}}_t)}{N-1}$$

$$= \frac{\bar{\gamma}\Sigma\underline{\zeta}_t^1}{N+1} + \frac{N\bar{\gamma}\Sigma\underline{\zeta}_t^{-1}}{(N-1)(N+1)} + \frac{2N\Lambda(\mu_t^{\psi} - r\dot{\underline{\psi}}_t)}{(N-1)(N+1)} - \frac{N\bar{\gamma}\Sigma\underline{\psi}_t}{(N-1)(N+1)}.$$

Now recall that the frictionless equilibrium under no price impact is of the form $\mu_t = \frac{\Sigma(\zeta_t - \psi_t)}{\delta}$, which in turn under the homogenous risk aversions assumption becomes $\mu_t = \frac{\bar{\gamma}\Sigma(\zeta_t - \psi_t)}{N}$. Therefore, after adding and substracting $\frac{N\bar{\gamma}\Sigma\underline{\zeta}_t^1}{(N-1)(N+1)}$ in the above and by the linearity of the conditional expectation, we get:

$$\begin{aligned}\tilde{\mu}_t^{\Lambda,1} &= \frac{\bar{\gamma}\Sigma}{N+1}\underline{\zeta}_t^1 - \frac{N\bar{\gamma}\Sigma}{(N-1)(N+1)}\underline{\zeta}_t^1 + \frac{N}{(N-1)(N+1)}\bar{\gamma}\Sigma(\underline{\zeta}_t - \underline{\psi}_t) + \frac{2N\Lambda}{(N-1)(N+1)}(\mu_t^{\psi} - r\dot{\underline{\psi}}_t) \\ &= \frac{N^2}{(N-1)(N+1)}\mathbb{E}[\mu_t|\mathscr{F}_S] - \frac{\bar{\gamma}\Sigma}{(N-1)(N+1)}\underline{\zeta}_t^1 + \frac{2N\Lambda}{(N-1)(N+1)}(\mu_t^{\psi} - r\dot{\underline{\psi}}_t).\end{aligned}$$

This concludes the proof. ■

Let us now briefly discuss the results of Corollary 6.2.1 and compare them with some previous ones. Note that in order to make the comparisons more clear, besides the standing assumption of common risk aversions, we assume that there are no noise traders in the market. By Remark 4.3.2, the equilibrium with frictions from (4.8) reverts to its frictionless counterpart from (3.8), that is:

$$\mu_t^{\Lambda} = \mu_t.$$

Having said the above, let us now compare $\tilde{\mu}_t^{\Lambda,1}$ with its frictionless counterpart $\tilde{\mu}_t^1$, under the price impact of a single investor.

Corollary 6.2.2 *Assume that there are no noise traders in the market and* $\gamma^n = \bar{\gamma}$, $\forall n = 1, ..., N$. *Then:*

$$\tilde{\mu}_t^{\Lambda,1} = \mathbb{E}[\tilde{\mu}_t^1|\mathscr{F}_S], \quad t \in [S, T],$$

where $\tilde{\mu}_t^{\Lambda,1}, \tilde{\mu}_t^1$ *are the equilibrium returns in a market with and without frictions, under the price impact of investor* 1*, derived in* (6.11) *and* (5.6) *respectively.* ◊

Proof: Going back to (5.6), we have:

$$\tilde{\mu}_t^1 = \frac{\frac{1}{\bar{\gamma}}\mu_{t,-1} + \delta\mu_t}{1/\bar{\gamma} + \delta}$$

$$
\begin{aligned}
&= \frac{\frac{1}{\bar{\gamma}}\frac{\bar{\gamma}\Sigma(\zeta_t^{-1}-\psi_t)}{N-1} + \frac{N}{\bar{\gamma}}\frac{\bar{\gamma}\Sigma(\zeta_t-\psi_t)}{N}}{1/\bar{\gamma}+N/\bar{\gamma}} \\
&= \frac{\frac{\Sigma(\zeta_t^{-1}-\psi_t)}{N-1} + \Sigma(\zeta_t-\psi_t)}{(N+1)/\bar{\gamma}} \\
&= \frac{N\Sigma(\zeta_t-\psi_t) - \Sigma(\zeta_t-\psi_t) + \Sigma(\zeta_t^{-1}-\psi_t)}{\frac{N-1}{(N+1)/\bar{\gamma}}} \\
&= \frac{N\Sigma(\zeta_t-\psi_t) - \Sigma\zeta_t^1}{\frac{N-1}{(N+1)/\bar{\gamma}}} \\
&= \frac{N\bar{\gamma}\Sigma(\zeta_t-\psi_t) - \bar{\gamma}\Sigma\zeta_t^1}{N^2-1} \\
&= \frac{N^2}{N^2-1}\mu_t - \frac{\bar{\gamma}\Sigma}{N^2-1}\zeta_t^1.
\end{aligned}
$$

Corollary 6.2.1 and the linearity of conditional expectation yields the result. ■

Remark 6.2.1 *By Corollary 6.2.2 we have determined that the frictionless equilibrium returns under the price impact of a single investor, of* (5.6)*, can be equivalently written as:*

$$
\tilde{\mu}_t^1 = \frac{N^2}{N^2-1}\mu_t - \frac{\gamma\Sigma}{N^2-1}\zeta_t^1, \tag{6.12}
$$

where μ_t *is the frictionless equilibrium of* (3.8)*. Thus, the counterpart of* (6.12) *in a market with frictions, as shown in* (6.11)*, can be written as:*

$$
\tilde{\mu}_t^{\Lambda,1} = \mathbb{E}[\tilde{\mu}_t^1|\mathscr{F}_S] + \frac{2N\Lambda}{N^2-1}(\mu_t^{\psi} - r\underline{\dot{\psi}}_t).
$$

By the above, it becomes easy to see that as Λ *goes to zero,* (6.11) *reverts to its frictionless counterpart of* (5.6)*, restricted by the "information" of* $\mathscr{F}_S$*.*

Further examining the results of Corollary 6.2.2, if we drop the assumption of no noise traders in the market and compare the results with that of Corollary 4.3.1 we have:

$$
\tilde{\mu}_t^{\Lambda,1} - \tilde{\mu}_t^1 = \mathbb{E}[\tilde{\mu}_t^1|\mathscr{F}_S] - \tilde{\mu}_t^1 + \frac{2N\Lambda}{N^2-1}(\mu_t^{\psi} - r\underline{\dot{\psi}}_t), \tag{6.13}
$$

$$
\mu_t^{\Lambda} - \mu_t = \frac{2\Lambda}{N}(\mu_t^{\psi} - r\dot{\psi}_t). \tag{6.14}
$$

Note that the liquidity premium of (6.13) *now also depends on the amount of information investor* 1 *has, depicted through the* σ*-algebra* $\mathscr{F}_S$. *We also note that as* $\Lambda \to 0$, $\tilde{\mu}_t^{\Lambda,1} - \tilde{\mu}_t^1$ *goes towards* $\mathbb{E}[\tilde{\mu}_t^1|\mathscr{F}_S] - \tilde{\mu}_t^1$. ♢

Remark 6.2.2 (Investor 1's frictional best response strategy under small transaction costs) *Recall that by* (6.8) *in Theorem 6.1.1 investor* 1*'s optimal strategy in a market with frictions, under her price impact (alternatively called as her best response strategy under frictions), is characterized by the following second order ODE:*

$$\ddot{\tilde{\phi}}_t^{\Lambda,1} - r\dot{\tilde{\phi}}_t^{\Lambda,1} - B\tilde{\phi}_t^{\Lambda,1} = Q_t^1,$$

where $Q_t^1 := \frac{(N-1)B\zeta_t^1}{N+1} - \frac{B\zeta_t^{-1}}{N+1} - \frac{(\mu_t^{\psi} - r\dot{\psi}_t)}{N+1} + \frac{B\psi_t}{N+1}$ *and* $B := \frac{\gamma\Lambda^{-1}\Sigma}{2}$. *Now note that the above equation can be equivalently written as:*

$$\dot{\tilde{\phi}}_t^{\Lambda,1} = B/r\Big(\mathbb{TP}_t^1 + \mathbb{D}_t^1 - \tilde{\phi}_t^{\Lambda,1}\Big), \tag{6.15}$$

where we set $\mathbb{D}_t^1 := B^{-1}\ddot{\tilde{\phi}}_t^{\Lambda,1}$ *and* $\mathbb{TP}_t^1 := -B^{-1}Q_t^1$ *and propose that* $\mathbb{TP}_t^1$ *represents investor* 1*'s "target portfolio". In other words, the investor's best response strategy in a market with frictions trades towards the previously stated "target" and is influenced by* B, r *and* $\ddot{\tilde{\phi}}_t^{\Lambda,1}$ *in the process. Likewise,* $\mathbb{D}_t^1$ *can be thought as the distortions to the price of an asset, caused by the investor's trading in the market.*

Now recall that investor 1*'s frictionless optimal strategy under her price impact is of the following form, as shown in* (5.4)*:*

$$\begin{aligned}\tilde{\phi}_t^1 &= \frac{\zeta_t^{-1} - \psi_t - \delta_{-1}\gamma^1\zeta_t^1}{\delta_{-1}k_1}, \quad k_1 := 2\Big(\frac{1}{\delta_{-1}} + \frac{\gamma^1}{2}\Big) \\ &= \frac{\zeta_t^{-1} - \psi_t - (N-1)\zeta_t^1}{N+1},\end{aligned} \tag{6.16}$$

where the second equality stems from the homogenous risk aversions assumption. Now, by comparing $\mathbb{TP}_t^1$ *in* (6.15) *with* (6.16), *provided the individual limits exist, it should be clear that:*

$$\mathbb{TP}_t^1 \to \mathbb{E}[\tilde{\phi}_t^1|\mathscr{F}_S] \;\&\; \mathbb{D}_t^1 \to 0 \text{ as } \Lambda \to 0, \tag{6.17}$$

♢

since both the distortions $\mathbb{D}_t^1$ *and* $B^{-1}\frac{(\mu_t^{\psi}-r\dot{\psi}_t)}{N+1}$ *in* $\mathbb{TP}_t^1$ *vanish as* Λ *goes to zero. That is to say, as transaction costs get smaller, investor* 1*'s target portfolio converges to her frictionless best response, while being infuenced by the distortions she causes to the market. Hence, as stated in [MMKS14], the frictional optimal strategy under the price impact of a single investor is "myopic" in the sense that it trades towards the current frictionless maximizer (rather a projected future optimum) with a speed determined by current market and preference parameters. Similar results are discussed for relevant problems in [GP16] and [MMKS14].*

Appendix A

REMARKS AND PROOFS

A.1 Section 2.1

In this section we explore in more detail the concepts of progressive measurability, localizing sequences and the space $\mathscr{L}_r^2$. For more information, refer to [KS91] and [Kal02].

Progressive measurability and localizing sequences

Remark A.1.1 *We fix a filtration $(\mathscr{F}_t)_{t\geq 0}$ on $(\Omega, \mathscr{F}, \mathbf{P})$. Following the notation of [Gal16], a random process $(X(t,\omega))_{t\geq 0}$ is called* adapted *if, for every $t \geq 0$, $X(t,\omega)$ is $\mathscr{F}_t$-measurable. Equivalently, we could say that $X : \mathbb{R}_0^+ \times \Omega \to \mathbb{R}$ is $\mathscr{F}_s$-adapted iff $(s,\cdot) \mapsto X(s,\omega)$ defined on $[0,t]\times\Omega$ is measurable with respect to $\mathscr{F}_s$ for each fixed s. Moreover, this process is said to be* progressive *(or progressively measurable)* [1] *if, for every $t \geq 0$, the mapping:*

$$(s,\omega) \mapsto X(s,\omega).$$

defined on $[0,t]\times\Omega$ is measurable for the σ-algebra $\mathscr{B}([0,t]) \otimes \mathscr{F}_t$ [2].

From the above it is evident that progressive measurability is stronger than adaptedness and measurability. Specifically, a progressive process is both adapted and measurable. Saying that a process is measurable *is equivalent to saying that, for each $\omega \in \Omega$, the mapping $(\cdot,\omega) \mapsto X(s,\omega)$ defined on $[0,t]\times\Omega$ is $\mathscr{B}([0,\infty]) \otimes \mathscr{F}$-measurable.The above relationships can be summarized in the graph presented below* [3]*:*

[1] Note that the Lebesgue integral of a progressive process is also progressive [KS91].

[2] Note that if X is adapted with right/left continuous sample paths (i.e. for every $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is right/left continuous), then X is progressive. On the other hand if the aforementioned process is adapted and with left continuous sample paths, then it is predictable [Gal16], [KS91].

[3] Note that in discrete time optional, progressive and adapted processes coincide.

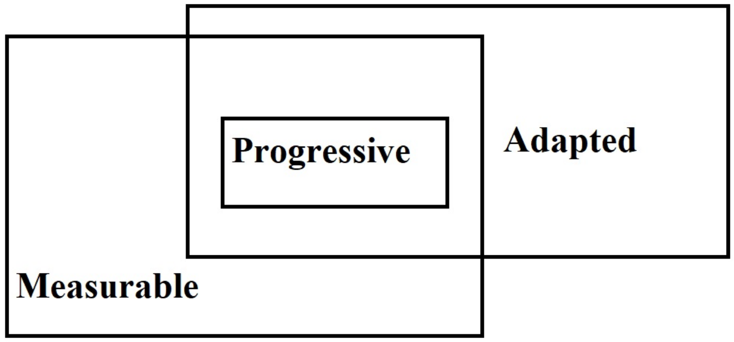

Figure A.1: The relationship between Progressiveness, Adaptedness and Measurability

We could now move to some general concepts about stopping times *in a continuous context. Specifically, given a filtered probability space* $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbf{P})$, *a stopping time* $\tau : \Omega \to [0, \infty]$ *is a random variable satisfying* $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathscr{F}_t,\ \forall t \geq 0$ [4]. *Correspondigly, a* stopped process *is a stochastic process that is forced to assume the same value after the prescribed stopping time and is denoted by* $X^{\tau}(t, \omega)$ *or* $X_{t\wedge\tau}(\omega)$. *A stopping time can be thought as a strategy that dictates the specific point of "exit". Therefore, rephrasing the above definition, in order for* τ *to be a stopping time, it is not permitted to "see into the future". Naturally, there are many more things to consider, for the above cocnepts, which are out of the scope of this thesis. For more, you could for example refer to [Kal02], [KS91].*

We shall conclude this remark by introducing the concept of a localizing sequence. We define a localizing sequence *as an increasing sequence of stopping times* $(\tau_n)_{n\geq 1}$, *such that* $\mathbf{P}(\lim_{n\to\infty} = \infty) = 1$. *Localizing sequences are generally used to ensure that a process* X_t *has certain properties we want, at least in a local manner, that would not hold universally (for the specific process). A noteworthy*

[4] An *optional time* is defined similarly, that is a random variable satisfying $\{\omega \in \Omega : \tau'(\omega) < t\} \in \mathscr{F}_t,\ \forall t \geq 0$. Note that optional and stopping times coincide for a right continuous filtration, as shown in [KS91].

example would be a local martingale, which should satify the following:

For all $n \geq 1$, the stopped process $(X_{t\wedge\tau_n})_{t\geq 0}$ should be a martingale, adapted to the respective filtration. ◊

$\mathbb{L}^p$ vs $\mathscr{L}^p$ spaces and equivalence classes

Below we give some basic information about the $\mathbb{L}^2$ and $\mathscr{L}^2$ spaces. For more information as well as proofs for the statements presented below, refer to [MW12], [Sch05]. We fix a measure space $(\Omega, \mathscr{F}, \mu)$ and define the following set of functions:

Definition A.1.1 *The set of square-integrable functions is defined to be:*

$$\mathscr{L}^p(\Omega, \mathscr{F}, \mu) = \left\{ f : \Omega \to \mathbb{R} : f \text{ is } \mathscr{F}\text{-measurable and } \int_\Omega |f|^2 d\mu < \infty \right\}.$$

Henceforth simply denoted as $\mathscr{L}^2$. ◊

For an $\mathscr{F}$-measurable function f, we define the 2-norm of f as:

$$\|f\|_2 = \left(\int_\Omega |f|^2 d\mu \right)^{1/2}.$$

Proposition A.1.1 *The $\mathscr{L}^2$ is a linear space and the 2-norm defines a seminorm on $\mathscr{L}^2$.* ◊

Recall that while a seminorm satisfies the triangle inequality and absolute homogeneity properties of a norm, it is not positive definite, i.e. $\|f\|_2 = 0$ does not imply $f = 0$ in $\mathscr{L}^2$. This becomes clearer by the following lemma:

Lemma A.1.1 *Let $f : \Omega \to \mathbb{R}$ be $\mathscr{F}$-measurable. Then $\|f\|_2 = 0$ if and only if $f = 0$ μ-a.e.* ◊

In order to ensure the positive definiteness of $\|f\|_2$, we consider the functions which are a.e. equal in $(\Omega, \mathscr{F}, \mu)$. This is done by introducing an equivalence relation in $\mathscr{L}^2$ and then taking the quotient space.

Definition A.1.2 *We say that two functions $f, g \in \mathscr{L}^2$ are equivalent if $f = g$ μ-a.e. More precisely, we write:*

$$f \sim g \Leftrightarrow f = g \ \mu - a.e. \qquad \diamond$$

The relation presented in te above definition is an equivalence relation. We are now ready to define the notion of an equivalence class in $\mathscr{L}^2$. More precisely, we write $[f]$ for the equivalence class of a function $f \in \mathscr{L}^2$. That is,

$$[f] = \{g \in \mathscr{L}^2 : f \sim g\}.$$

Definition A.1.3 *Let $\sim$ be the equivalence relation from Definition A.1.2. Define $\mathbb{L}^2(\Omega, \mathscr{F}, \mu)$ to be the quotient space $\mathscr{L}^2(\Omega, \mathscr{F}, \mu)/\sim$. That is:*

$$\mathbb{L}^2(\Omega, \mathscr{F}, \mu) = \{[f] : f \in \mathscr{L}^2(\Omega, \mathscr{F}, \mu)\}. \qquad \diamond$$

$\mathbb{L}^2$ is a linear space in which $\|[f]\|_2 = \|f\|_2 = \left(\int_\Omega |f|^2 d\mu\right)^{1/2}$ defines a norm.

Remark A.1.2 (The space $\mathscr{L}^2_r$) *In this subsection we explore in more detail the space $\mathscr{L}^2_r$, which will be referenced extensively throughout this thesis.*

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space and $X : \mathscr{T} \times \Omega \to \mathbb{R}^l$ a measurable mapping w.r.t. $\mathscr{B}([0,\infty]) \otimes \mathscr{F}$[5]. Then, as previously shown, the norm $\|\cdot\|_{(2,r)}$ is defined as [6]:

$$\|X\|^2_{(p,r)} = \mathbb{E}\left[\int_0^T e^{-rt} \sum_{i=1}^{l} |X_{t,i}|^2 dt\right] = \int_0^T \int_\Omega e^{-rt} \|X_t\|^2 \, d\mathbf{P} \, dt. \qquad \text{(CLVI.1)}$$

Note that the addition of e^{-rt} enables us to treat finite and infinite horizon models in a unified manner. More precisely, in order to deal with the infinite limit int he time integral, we essentially require that e^{-rt} goes to zero faster than $X_{t,i}$ goes to infinity. Hence, the current value (at time t) of $X_{t,i}$ must either be finite or grow at a rate slower than r so that the discount factor pushes the present value to zero. That is why in infinite horizon models we have the strict inequality $r > 0$.

[5]Note that for finite dimension spaces, such as $\mathbb{R}^l$, all norms (or seminorms) are *equivalent*, i.e. the following holds for x in a normed space and constants $C_1,\ C_2 > 0$: $C_1\|x\|_q \leq \|x\|_p \leq C_2\|x\|_q$. Equivalent norms define equivalent metrics, which produce the same topology and Cauchy sequences [Fol13].

[6]We also make use of the Tonelli Theorem, as shown in [Sch05], which lets us change the order of integration for measurable, non-negative functions.

We could also interpet e^{-rt} as a "weight" in the context of Lebesgue spaces. This stems from the fact that with the help of Radon-Nikodym Theorem we could construct a new measure $d\mu = e^{-rt}dt$ which is absolutely continuous with regards to dt [7]*. Having said the above, we could also represent* (CLVI.1) *as follows:*

$$\int \int \|X(t,\omega)\|^2 d\mathbf{P} d\mu. \tag{CLVI.2}$$

(CLVI.1) *is a part of a general class of norms, which form spaces known as "mixed norm Lebesgue spaces". Specifically as shown in [EM18], a mixed norm Lebesgue space consists of measurable multivariable functions with a norm defined in terms of potentially different, iteratively calculated p-norms. More precisely, for (Ω_i, μ_i), $i = 1, ..., n$ σ-finite measurable spaces, $p_i \geq 1$ and $(p_1, .., p_{n-1}, p_n)$ and a mapping f, we could defined the following norm:*

$$\|f\| = \Big(\int_{\Omega_1} \Big(\cdots \Big(\int_{\Omega_n} \|f\|^{p_n} d\mu_n \Big)^{p_{n-1}/p_n} \cdots \Big)^{p_1/p_2} d\mu_1 \Big)^{1/p_1}. \qquad \diamond$$

We now present a useful property, as shown in [Sch05], regarding integrals of functions over a null set. More precisely, we have:

Theorem A.1.1 *Let (X, Σ, μ) be a measure space. Let $f : X \to \mathbb{R}$ be a μ-integrable function and N a μ-null set. Then:*

$$\int_N f d\mu = 0. \qquad \diamond$$

Lastly, we present an important lemma relevant to the class $\mathscr{L}_r^2$ (see further in [KS91]). Namely:

Lemma A.1.2 *Let $X_t \in \mathscr{L}_r^2$, then the process:*

$$Y_t = \int_{s \leq t} X_s ds,$$

is continuous in t (for almost all ω) and progressively measurable. $\diamond$

[7]If we define two measures μ, ν on a measurable space $(X, \mathscr{F})$ and let $\mathscr{N}_\mu = \{A \in \mathscr{F} \mid \mu(A) = 0\}$, $\mathscr{N}_\nu = \{A \in \mathscr{F} \mid \nu(A) = 0\}$ be their respective null sets, then the measure ν is said to be absolutely continuous with respect to μ iff $\mathscr{N}_\nu \supseteq \mathscr{N}_\mu$. We denote this relationship as: $\nu \ll \mu$. In a similar manner, the two measures are called equivalent iff $\mu \ll \nu$ and $\nu \ll \mu$. It should be noted that equivalent measures share their a.s. properties [Kal17].

A.2 Section 3.1

In this section we see in detail how (3.2) transforms into (3.5). For more information about the statements and proofs presented below, refer to [KS91], [Kal02], [Pro05], [MW12] and [Ose12].

Remark A.2.1 *Let X, Y be two semimartingales taking values in $\mathbb{R}^d$. We say that X and Y are orthogonal, if for any non-negative integers i, j we have:*

$$\left[X^i, Y^j\right] = 0. \tag{A.1}$$

◊

Theorem A.2.1 *Let X, Y be semimartingales and ξ be a X-integrable process. Then, ξ is $[X, Y]$-integrable and the following hold:*

(I) $\left[\int \xi dX\right] = \int \xi^2 d[X]$.

(II) $\left[\int \xi dX, Y\right] = \int \xi d[X, Y]$. ◊

Before continuing further, let us extent the definition of stochastic integrals to a larger class of integrators as shown in [Kal02]. As previously expressed, a process X is said to be a continuous semimartingale if it can be written in the form:

$$X = M + A,$$

where M is a continuous (local) martingale and A is a continuous adapted process of locally finite variation with $A_0 = 0$. Now let $L(A)$ denote the class of progressively measurable processes V such that the process $(V \cdot A)_t = \int_0^t V dA$ exists in the Lebesgue-Stieltjes sense. For any continuous martingale X, we may write $L(X) = L(M) \cap L(A)$, where $L(M)$ denotes the class of all progressive processes V such that $(V^2 \cdot [M])_t < \infty$ a.s. for each $t > 0$. By the above, we can define the integral of the process $V \in L(X)$, as:

$$\underbrace{\int V dX}_{V \cdot X} = \underbrace{\int V dM}_{V \cdot M} + \underbrace{\int V dA}_{V \cdot A}.$$

Note that $V \cdot X$ is again a continuous semimartingale with decomposition $V \cdot X = V \cdot M + V \cdot A$.

Lemma A.2.1 *Let a function $f_t : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and set a partition $P = (t_0, t_1, ..., t_n)$ of $[a,b]$.*

$$\text{If } V_1(f,[a,b]) < \infty \Rightarrow [f]_{[a,b]} = 0. \qquad \Diamond$$

Proof: Note that the term $V_2(f,P) = \sum_{i=1}^{n} \left| f_{t_i} - f_{t_{i-1}} \right|^2$ can be decomposed as follows:

$$\sum_{i=1}^{n} \left| f_{t_i} - f_{t_{i-1}} \right|^2 = \sum_{i=1}^{n} \left| f_{t_i} - f_{t_{i-1}} \right| \left| f_{t_i} - f_{t_{i-1}} \right| \leq \sup_{1 \leq i \leq n} |f_{t_i} - f_{t_{i-1}}| V_1(f,P).$$

Now, by the continuity of f on $[a,b]$ we have:

$$\sup_{1 \leq i \leq n} |f_{t_i} - f_{t_{i-1}}| \to 0 \quad as \quad n \to \infty,$$

since f is continuous on $[a,b]$ and thus uniformly continuous on the same interval, for $\epsilon > 0$, $\exists\ c(\epsilon) : |u - v| < c$ then $|f_u - f_v| < \epsilon$ for all $u, v \in [a,b]$. Now consider $\|P\| < c$ and note that $|f_{t_i} - f_{t_{i-1}}| < \epsilon$. The result follows. *The same of course holds for the covariation $[f,g]_{[a,b]}$ between f and a continuous $g : [a,b] \to \mathbb{R}$*, since:

$$C(f,g,P) = \sum_{i=1}^{n} (f_{t_i} - f_{t_{i-1}})(g_{t_i} - g_{t_{i-1}}) \leq \sup_{1 \leq i \leq n} |g_{t_i} - g_{t_{i-1}}| V_1(f,[a,b]),$$

using the same argument as before, we conclude the proof. ∎

Lemma A.2.2 *Let f be non-decreasing function on $[a,b]$, then f has bounded variation on $[a,b]$ and $V_1(f,[a,b]) = f_b - f_a$.* $\Diamond$

Proof: Let $P = (t_0, t_1, ..., t_n)$ be a partition of $[a,b]$. Then:

$$V_1(f,P) = \sum_{i=1}^{n} |f_{t_i} - f_{t_{i-1}}| = \sum_{i=1}^{n} (f_{t_i} - f_{t_{i-1}}) = (\cancel{f_{t_1}} - f_{t_0}) + (\cancel{f_{t_2}} - \cancel{f_{t_1}}) + (f_{t_3} - \cancel{f_{t_2}}) + ... = f_b - f_a$$

The above result holds for every partition of $[a,b]$. A similar argument can be made for a non-increasing function g, with $V_1(g,[a,b]) = g_a - g_b$. ∎

Some other useful results for functions of bounded variation are presented in the following theorems:

Theorem A.2.2 *Let f, g be functions of bounded variation on $[a,b]$ and k be a constant. Then:*

(I) f is bounded on $[a,b]$.

(II) f is of bounded variation on every closed subinterval of $[a,b]$.

(III) kf is of bounded variation on $[a,b]$.

(IV) $f+g$ and $f-g$ are of bounded variation on $[a,b]$.

(V) fg is of bounded variation on $[a,b]$. ◊

Theorem A.2.3 *Let f be a function defined on $[a,b]$ and $c \in (a,b)$. If f is of bouned variation on $[a,c]$ and $[c,b]$, then it is of bounded variation on $[a,b]$ with $V_1(f,[a,b]) = V_1(f,[a,c]) + V_1(f,[c,b])$.* ◊

Theorem A.2.4 (Quadratic variation of Brownian motion) *For every $0 \leq a < b$ the quadratic variation of the Brownian motion W on $[a,b]$ is $b-a$. Moreover the following hold:*

(I) For every partition P with $\|P\| \to 0$, we have that $V_2(W,P) \xrightarrow{\mathbb{L}^2} b-a$ [8].

(II) For every partition P with $\sum_{i=1}^{\infty} \|P\| < \infty$, $V_2(W,P) \to b-a$ with probability 1 [9]. ◊

Proof: Let $P = (a = t_0, t_1, ..., t_n = b)$. We set $Y_i = (W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1}),\ i = 1,2,...,n$. Then:

$$[V_2(W,P) - (b-a)]^2 = \left[\sum_{i=1}^{n} Y_i\right]^2 = \sum_{i=1}^{n} Y_i^2 + \underbrace{2\sum_{1\leq i<j\leq n} Y_i Yj}_{\sum_{i=1}^{n}\sum_{j\neq i}^{n} Y_i Y_j}.$$

Notice that Y_i are independent and as such $\mathbb{E}[Y_i] = 0$, since $W_t - W_s \sim N(0, t-s)$, $(t \geq s)$, and $\mathbb{E}[Y_i^2] = E[(W_{t_i} - W_{t_{i-1}})^4] - 2(t_i - t_{i-1})\mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2] + (t_i - t_{i-1})^2 = (t_i - t_{i-1})^2\mathbb{E}[Z^4] - (t_i - t_{i-1})^2$ with $Z \sim N(0,1)$ [10]. Thus, $\mathbb{E}[Y_i^2] = 2(t_i - t_{i-1})^2$. Combining the above, we get:

$$\mathbb{E}\big[\big(V_2(W,P) - (b-a)\big)^2\big] = \sum_{i=1}^{n}(t_i - t_{i-1})^2 \leq (b-a)\|P\| \xrightarrow{\mathbb{L}^2} 0.$$

---

[8] A sequence X_n converges in $\mathbb{L}^p$, $p > 0$ to X if $\lim_{n\to\infty} \mathbb{E}[|X_n - X|^p] = 0$.

[9] A sequence X_n converges almost surely to X if $P\left(\lim_{n\to\infty} X_n = X\right) = 1$.

[10] Note that with the help of the characteristic function of $N(0,1)$ we can deduce that $\mathbb{E}[Z^{2k}] = \frac{(2k)!}{2^k k!}$, $\mathbb{E}[Z^{2k+1}] = 0,\ k = 0,1,....$

For the second part of the proof, we set $U_n = V_2(W, P_n) - (b-a)$ for different partitions P_n. The above result together with the linearity of the expectation give us that $\mathbb{E}\left[\sum_{n=1}^{\infty} U_n^2\right] = \sum_{n=1}^{\infty} \mathbb{E}[U_n^2] < \infty$. In other words we have, with probability 1, that $\sum_{n=1}^{\infty} U_n < \infty$ and consequently $\lim_{n\to\infty} U_n = 0$. ■

Remark A.2.2 *For one-dimensional semimartingales Z_t, X_t, Y_t with $Z_t = aX_t + bY_t$, following the notation introduced in CLII, we calculate the quadratic variation as follows:*

$$[Z,Z]_t = \lim_{\|P\|\to 0} \sum_{i=1}^{n} (a\Delta X_i + b\Delta Y_i)^2 = \lim_{\|P\|\to 0} \sum_{i=1}^{n} \left(a^2(\Delta X_i)^2 + b^2(\Delta Y_i)^2 + 2ab\Delta X_i \Delta Y_i\right)$$
$$= a^2 \underbrace{\lim_{\|P\|\to 0} \sum_{i=1}^{n} (\Delta X_i)^2}_{[X]_t} + b^2 \underbrace{\lim_{\|P\|\to 0} \sum_{i=1}^{n} (\Delta Y_i)^2}_{[Y]_t} + 2ab \underbrace{\lim_{\|P\|\to 0} \sum_{i=1}^{n} \Delta X_i \Delta Y_i}_{[X,Y]_t}.$$

Then by the compensator's properties, as expressed in Definition 2.3.5, we have:

$$\mathbb{E}\left[\int_0^T \frac{\gamma^n}{2} d\left\langle \int_0^{\cdot} (\phi^n)^T dS_s + Y^n \right\rangle_t\right] = \mathbb{E}\left[\int_0^T \frac{\gamma^n}{2} d\left[\int_0^{\cdot} (\phi^n)^T dS_s + Y^n\right]_t\right] \tag{A.2}$$
$$= \mathbb{E}\left[\int_0^T \frac{\gamma^n}{2} d\left(2\left\langle \int_0^{\cdot} (\phi^n)^T dS_s, Y^n \right\rangle_t + \left\langle \int_0^{\cdot} (\phi^n)^T dS_s \right\rangle_t + \langle Y^n \rangle_t\right)\right].$$

Calculating explicitly the above quantities, we omit the expectations in order to simplify notation. Moreover, we should have in mind that the reason behind the use of the predictable quadratic variation inside the expectation is to ensure that the process we consider is predictable (which is generally not the case for the "usual" quadratic variation). ◊

Excplicitly calculating the terms in (A.2), we finally get:

$$d\left\langle \int_0^{\cdot} (\phi^n)^T dS_s \right\rangle_t \overset{11}{=} (\phi_t^n)^T \Sigma \phi_t^n dt,$$

$$d\langle Y^n \rangle_t = \left(dA_t^n + (\zeta_t^n)^T \sigma dW_t + dM_t^{\perp,n}\right)\left(dA_t^n + (\zeta_t^n)^T \sigma dW_t + dM_t^{\perp,n}\right) \Rightarrow$$
$$d\langle Y^n \rangle_t \overset{12}{=} (\zeta_t^n)^T \Sigma \zeta_t^n dt + d\langle M^{\perp,n}\rangle_t,$$

[11] A direct result of Lemmas $A.2.1$, $A.2.2$ and Theorems $A.2.4$, $A.2.1$.

$$d\left\langle \int_0^{\cdot} (\phi^n)^T dS_s, Y^n \right\rangle_t = (\phi^n_t)\left(\mu_t dt + \sigma dW_t\right)\left(dA^n_t + (\zeta^n_t)^T \sigma dW_t + dM^{\perp,n}_t\right) \Rightarrow$$
$$d\left\langle \int_0^{\cdot} (\phi^n_t)^T dS_s, Y^n \right\rangle_t \overset{13}{=} (\phi^n_t)^T \Sigma \zeta^n_t dt.$$

A.3 Section 3.2

In this section we see in detail how the Gâteaux derivative is defined in infinite-dimensional spaces. For more information, refer to [ET99], [GF63] and [JLJ98].

Generally we could make the following distinction based on the kind of spaces a mapping operates, i.e.:

- A mapping of the form $f : \mathbb{R} \supseteq [a, b] \rightarrow X$, where X is a general normed space is denoted as an *abstract function*.
- A mapping of the form $\mathscr{F} : X \supseteq \mathscr{D}(\mathscr{F}) \rightarrow \mathbb{R}$, where X is a general normed space is denoted as a *functional*.
- A mapping of the form $\mathbf{F} : X \supseteq \mathscr{D}(\mathbf{F}) \rightarrow Y$, where X, Y are general normed spaces is denoted as an *operator*. Clearly this category includes the other two.

In this context, we could express the functional depicted in (3.2) as:

$$\mathscr{F}^n : \left(\mathscr{L}^2_r, \|\cdot\|_{(2,r)}\right) \rightarrow \left(\mathbb{R}, \|\cdot\|\right).$$

Let us now move on to define the generalization of directional derivative to locally convex topological vector spaces (e.g. Banach spaces) [14]. Specifically following the concepts expressed in [Car83], [Eva98] we have:

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[12] A direct result of Lemma $A.2.1$, Theorems $A.2.4$, $A.2.1$ and (A.1)

[13] A direct result of Lemmas $A.2.1$, $A.2.2$ and Theorems $A.2.1$, $A.2.4$, (A.1)

[14] Locally convex topological vector spaces generalize normed spaces. Following the notation of [NB11], a locally convex topological vector space is one that is induced by a family of seminorms.

Definition A.3.1 (Directional derivative - generalization) *Let X, Y be linear topological vector spaces, $\mathbf{F} : X \supseteq \mathscr{D}(\mathbf{F}) \to Y$ be an operator, x_0 an interior point of $\mathscr{D}(\mathbf{F})$* [15] *and $\theta \in X$, $\theta \neq 0_X$. Then there is an interval $I_\epsilon = (-\epsilon, \epsilon)$, $\epsilon > 0$ such that $x_0 + \rho\theta \in \mathscr{D}(\mathbf{F})$, $\forall\ \rho \in I_\epsilon$. Therefore, we could define an abstract function $g : I_\epsilon \to Y$, with $g(\rho) = \mathbf{F}(x_0 + \rho\theta)$. We shall say that the operator $\mathbf{F}$ is differentiable at x_0 and in the direction of θ, if the aforementioned abstract function g is differentiable at $\rho = 0$. Then, the directional derivative at x_0 is given by:*

$$d\mathbf{F}(x_0;\theta) = \left.\frac{d\mathbf{F}(x_0 + \rho\theta)}{d\rho}\right|_{\rho\to 0}.$$

Note that in some cases it might be meaningful to consider the one-sided limit, thus defining the appropriate right/left directional derivatives as follows:

$$d_+\mathbf{F}(x_0;\theta) = \left.\frac{d\mathbf{F}(x_0 + \rho\theta)}{d\rho}\right|_{\rho\to 0+},$$

$$d_-\mathbf{F}(x_0;\theta) = \left.\frac{d\mathbf{F}(x_0 + \rho\theta)}{d\rho}\right|_{\rho\to 0-}. \qquad \Diamond$$

In a similar manner, we can define higher order directional derivatives as follows:

$$d^n\mathbf{F}(x_0;\theta) = \left.\frac{d^n\mathbf{F}(x_0 + \rho\theta)}{d\rho^n}\right|_{\rho\to 0}.$$

Note that the above defines a homogeneous function of degree n in θ (it is not necessarily additive).

Definition A.3.2 (Gâteaux derivative) *Following the notions presented in Definition A.3.1, $\mathbf{F}$ is Gâteaux differentiable at a point $x_0 \in \mathscr{D}(\mathbf{F})$ if:*

- *$d\mathbf{F}(x_0;\theta)$ exists for all $\theta \in X$.*
- *$\theta \mapsto d\mathbf{F}(x_0;\theta)$ is a linear continuous function, that is:*

$$d\mathbf{F}(x_0;\theta) = d\mathbf{F}(x_0)\theta, \quad d\mathbf{F}(x_0) \in LC\Big(\mathscr{D}(\mathbf{F}), Y\Big).$$ [16]

We denote $d\mathbf{F}(x_0)$, $\Big(d\mathbf{F}(x_0), \theta\Big)$ as the Gâteaux derivative and differential at x_0 respectively.

[15] Note that $x_0 \in \mathscr{D}(\mathbf{F})$ is called an interior point if $\exists \epsilon > 0$ such that $B(x_0, \epsilon) \subseteq \mathscr{D}(\mathbf{F})$.

[16] The space of linear continuous mappings.

- *In a similar manner,* $\mathbf{F}$ *is twice Gâteaux differentiable at a point* $x_0 \in \mathscr{D}(\mathbf{F})$ *in the directions* $\theta, h \in X$ *if the operator* $d\mathbf{F}(x_0)\theta$ *is once Gâteaux differentiable at point* x_0 *in the direction* h. ◊

Latly, regarding all the goal functionals that are considered throughout this thesis, note the following:

- The linearity of $\theta \mapsto d\mathbf{F}(x_0;\theta)$ is easily verified when calculating the Gâteaux differential.
- The continuity of $\theta \mapsto d\mathbf{F}(x_0;\theta)$ is a direct consequence of the Cauchy-Swartz inequality (CS) and the Triangle Inequality for integrals (TI) [17].

To prove the aforementioned two points we examine the case of the Gâteaux derivative for the frictionless goal functional without price impact of (3.2). Similar results for all the other derivatives presented throughout this thesis can be derived in the same manner. Recall that the Gâteaux derivative of (3.2) is of the following form:

$$\Big(d\mathscr{F}^n(\phi^n),\theta^n\Big) = \mathbb{E}\left[\int_0^T e^{-rt}\Big(\mu_t^T - \gamma^n(\phi_t^n+\zeta_t^n)^T\Sigma\Big)\theta_t^n dt\right] = 0,$$

where $\mathscr{F}^n : \Big(\mathscr{L}_r^2, \|\cdot\|_{(2,r)}\Big) \to \Big(\mathbb{R}, \|\cdot\|_\infty\Big)$. The linearity of $\theta^n \mapsto d\mathscr{F}(\phi^n;\theta^n)$ is direct. To prove the continuity of the derivative in θ^n we use the epsilon-delta definition of continuity, where $\|\theta^n - \hat{\theta}^n\|_{(2,r)} < \delta$. Therefore, we have:

$$\begin{aligned}\left|\Big(d\mathscr{F}^n(\phi^n),\theta^n\Big) - \Big(d\mathscr{F}^n(\phi^n),\hat{\theta}^n\Big)\right| &= \left|\mathbb{E}\left[\int_0^T e^{-rt}\Big(\mu_t^T - \gamma^n(\phi_t^n+\zeta_t^n)^T\Sigma\Big)(\theta_t^n-\hat{\theta}_t^n)dt\right]\right| \\ &\underset{(TI)}{\overset{(CS)}{\leq}} \|\mu^T - \gamma^n(\phi^n+\zeta^n)^T\Sigma\|_{(2,r)}\|\theta^n-\hat{\theta}^n\|_{(2,r)} \\ &< \|\mu^T - \gamma^n(\phi^n+\zeta^n)^T\Sigma\|_{(2,r)}\delta.\end{aligned}$$

Finally, choosing $\delta = \frac{\epsilon}{\|\mu_t^T - \gamma^n(\phi_t^n+\zeta_t^n)^T\Sigma\|_{(2,r)}}$ yields the desired result.

---

[17]The functional in Chapter 6 is a more special case. Nevertheless, continuity is also achieved there by the finite $\mathscr{L}_r^2$ (semi) norms of ϕ_t^1 and its derivatives.

A.4 Section 4.1

In this section we explore in more detail the concept of absolute continuity and its relation with regards to differentiability. For more information about the following statements and their proofs, refer to [MW12]. We begin with some introductory statements about Riemann integrable functions. More precisely, recall that the *First Fundamental Theorem of Calculus* states: Suppose f is Riemann integrable on $[a, b]$. Let:

$$F(x) = \int_a^x f(t)dt, \quad a \leq x \leq b.$$

Then F is differentiable at all points which f is continuous and at such points $\frac{dF(x)}{dx} = f(x)$. In other words we have:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x),$$

at all continuity points of f. The above results can be generalized for the Lebesgue integral, leading to the following theorem:

Theorem A.4.1 (First Fundamental Theorem of Calculus - Lebesgue) *Suppose $f \in \mathscr{L}^1(\lambda)$, where λ denotes the Lebesgue measure* [18] *and set:*

$$F(x) = \int_a^x f(t)dt, \quad a \leq x \leq b.$$

Then F is differentiable almost everywhere on $[a, b]$ and $\frac{dF(x)}{dx} = f(x)$ for almost all $x \in [a, b]$. ◊

We are now ready to introduce a definition for an absolutely continuous function. That is:

Definition A.4.1 (Absolutely continuous function) *Suppose that F is defined on $[a, b]$, f exists almost everywhere and $f \in \mathscr{L}^1(\lambda)$ on $[a, b]$ and:*

$$F(x) = F(a) + \int_a^x f(t)dt, \quad a \leq x \leq b.$$

Then f is said to be absolutely continuous on $[a, b]$. ◊

[18] For more information about Lebesuge spaces, refer to A.1.

Equivalently for $F(a) = 0$, we say that f is absolutely continuous if it can represented as $F(x) = \int_a^x f(t)dt$ for all x and therefore it is differentiable almost everywhere with $\frac{dF(x)}{dx} = f(x)$ for almost all x.

Definition A.4.2 (Absolutely continuous process) *Let a probability space* $(\Omega, \mathscr{F}, \mathbf{P})$. *Then a stochastic process* X *is absolutely continuous if* $t \mapsto X(t, \omega)$ *is absolutely continuous for* $d\mathbf{P}$ *almost all* ω. ♢

Now recall that for a market with frictions, the strategy for investor n is of the following form ($\phi_0^n = 0$):

$$\phi_t^n = \int_0^t \dot{\phi}_s^n ds, \quad t \in \mathscr{T},$$

where $\phi_t^n, \dot{\phi}_t^n \in \mathscr{L}_r^2$. Therefore, by the above we can reason that ϕ_t^n is almost everywhere differentiable with $\frac{d\phi_t^n}{dt} = \dot{\phi}_t^n$ almost everywhere for (ω, t) on the product measure $d\mathbf{P} \otimes dt$ [19]. Note that the almost everywhere existence of the pointwise derivative (instead of everywhere) does not affect the optimization of the goal functional, since by Theorem A.1.1 integrals over nullsets are null. Lastly, note a relevant property for absolutely continuous functions presented in the following proposition:

Proposition A.4.1 *If* f *is absolutely continuous on* $[a, b]$, *then it is continuous and of bounded variation on* $[a, b]$. ♢

A.5 Section 6.2

Recall that in the case of a market with frictions and under the price impact of investor 1, her strategy and trading rate are respectively given as follows (taking into account the initial condition):

$$\phi_t^1 = \int_0^t \dot{\phi}_s^1 ds. \quad t \in \mathscr{T},$$

[19] This is a direct consequence of the fact that for for two σ-finite measure spaces: $(X_1, \Sigma_1, \mu_1), (X_2, \Sigma_2, \mu_2)$, the product measure is uniquely defined for every measurable E as $(\mu_1 \otimes \mu_2)(E) = \int_{X_2} \mu_1(E^y) d\mu_2(y) = \int_{X_1} \mu_2(E_x) d\mu_1(x)$, where $E_x = \{y \in X_2 : (x, y) \in E\}$ and $E^y = \{x \in X_1 : (x, y) \in E\}$. That is, almost everywhere on sections implies almost everywhere on product for measurable sets (the opposite is generally not true).

$$\dot{\phi}_t^1 = x + \int_0^t \ddot{\phi}_s^1 ds, \quad x \in \mathbb{R}^d, \quad t \in \mathscr{T}.$$

Furthermore, recall that by the discussion in A.4, ϕ_t^1 is differentiable everywhere on $t \in \mathscr{T}$ with pointwise derivative: $\frac{d\phi_t^1}{dt} = \dot{\phi}_t^1$, since $\dot{\phi}_t^1$ is continuous on $t \in \mathscr{T}$. On the other hand, $\dot{\phi}_t^1$ is differentiable almost everywhere on $t \in \mathscr{T}$ with pointwise derivative $\frac{d\dot{\phi}_t^1}{dt} = \ddot{\phi}_t^1$ and $\ddot{\phi}_t^1 \in \mathscr{L}_r^2$. Let us now examine the integrability of $\dot{\phi}_t^1$ in a finite time horizon market (the same arguments can be made for ϕ_t^1). More precisely, we examine the case where $\dot{\phi}_t^1$ is one-dimensional. Similar arguments can be made in the multi-dimensional case.

Lemma A.5.1 *Consider a finite time horizon market ($T < \infty$) with two assets (one risky and one riskless) and let:*

$$\dot{\phi}_t^1 = x + \int_0^t \ddot{\phi}_s^1 ds, \quad x \in \mathbb{R}, \quad t \in \mathscr{T},$$

where $\ddot{\phi}_t^1 \in \mathscr{L}_r^2$. Then:

$$\dot{\phi}_t^1 \in \mathscr{L}_r^2 \qquad \Diamond$$

Proof: The progressive measurability of $\dot{\phi}_t^1$ is a direct consequence of Lemma A.1.2. For the mixed integrability, we initially use the trivial inequality $|x+y|^2 \leq 2|x|^2 + 2|y|^2$ [20], getting:

$$\mathbb{E}\Big[\int_0^T e^{-rt}\Big|x + \int_0^t \ddot{\phi}_s^1 ds\Big|^2 dt\Big] \leq \mathbb{E}\Big[\int_0^T e^{-rt}\Big(2|x|^2 + 2\Big|\int_0^t \ddot{\phi}_s^1 ds\Big|^2\Big) dt\Big].$$

By the triangle inequality for intergrals, as shown in [Sch05] we have:

$$\begin{aligned}\mathbb{E}\Big[\int_0^T e^{-rt}\Big(2|x|^2 + 2\Big|\int_0^t \ddot{\phi}_s^1 ds\Big|^2\Big) dt\Big] &\leq \overbrace{2|x|^2\Big(\frac{1-e^{-rT}}{r}\Big)}^{C} + 2\mathbb{E}\Big[\int_0^T e^{-rt}\Big(\int_0^t |\ddot{\phi}_s^1| ds\Big)^2 dt\Big] \\ &= C + 2\mathbb{E}\Big[\int_0^T e^{-rt}\Big(\int_0^T \underbrace{\mathbf{1}_{\{s\leq t\}}}_{f}\, \underbrace{|\ddot{\phi}_s^1|\mathbf{1}_{\{s\leq t\}}}_{g} ds\Big)^2 dt\Big].\end{aligned}$$

Applying the Cauchy-Swartz inequality in the time domain for f, g, we have:

[20] For $x, y \in \mathbb{R}$, we have that $(x-y)^2 \geq 0 \Rightarrow |x|^2 + |y|^2 \geq 2xy$. Thus, $(x+y)^2 = |x|^2 + |y|^2 + 2xy \leq 2|x|^2 + 2|y|^2$.

$$C + 2\mathbb{E}\left[\int_0^T e^{-rt}\left(\int_0^T \mathbf{1}_{\{s\leq t\}}|\ddot{\phi}_s^1|\mathbf{1}_{\{s\leq t\}}ds\right)^2 dt\right] \leq C + 2\mathbb{E}\left[\int_0^T e^{-rt}\left(t\int_0^t |\ddot{\phi}_s^1|^2 ds\right)dt\right].$$

Applying Fubini's Theorem, we have:

$$\begin{aligned}
C + 2\mathbb{E}\left[\int_0^T e^{-rt}\left(t\int_0^t |\ddot{\phi}_s^1|^2 ds\right)dt\right] &= C + 2\mathbb{E}\left[\int_0^T\int_s^T e^{-rt}t|\ddot{\phi}_s^1|^2 dtds\right] \\
&= C + 2\mathbb{E}\left[\int_0^T \frac{(rs+1)e^{-rs}}{r^2}|\ddot{\phi}_s^1|^2 ds - \int_0^T \frac{(rT+1)e^{-rT}}{r^2}|\ddot{\phi}_s^1|^2 ds\right].
\end{aligned}$$

Noting that $\frac{(rT+1)e^{-rT}}{r^2}|\ddot{\phi}_s^1|^2 \geq 0$ and using the monotonicity and the linearity of the integral (and the expectation), we have:

$$C + 2\mathbb{E}\left[\int_0^T \frac{(rs+1)e^{-rs}}{r^2}|\ddot{\phi}_s^1|^2 ds - \int_0^T \frac{(rT+1)e^{-rT}}{r^2}|\ddot{\phi}_s^1|^2 ds\right] \leq C + 2\mathbb{E}\left[\int_0^T \frac{(rs+1)e^{-rs}}{r^2}|\ddot{\phi}_s^1|^2 ds\right].$$

Noting that $\frac{(rs+1)e^{-rs}}{r^2} \leq \frac{(rT+1)e^{-rs}}{r^2}$ and using once more the monotonicity of the integral, we have:

$$\begin{aligned}
C + 2\mathbb{E}\left[\int_0^T \frac{(rs+1)e^{-rs}}{r^2}|\ddot{\phi}_s^1|^2 ds\right] &\leq C + 2\mathbb{E}\left[\int_0^T \frac{(rT+1)e^{-rs}}{r^2}|\ddot{\phi}_s^1|^2 ds\right] \\
&= C + \frac{2T}{r}\|\ddot{\phi}^1\|^2_{(2,r)} + \frac{2}{r^2}\|\ddot{\phi}^1\|^2_{(2,r)}.
\end{aligned}$$

This concludes the proof. ■

Lastly, we present a generalized form of integration by parts used in Lebesgue spaces, for absolutely continuous functions. For more information, refer to [MW12].

Lemma A.5.2 *[Integration by parts - Lebesgue] Let* $[a,b]$ *be a finite interval,* $u, v \in \mathbb{L}^1([a,b])$ *and* U, V *absolutely continuous functions, that is:*

$$\begin{aligned}
U_t &= U_a + \int_a^t u_s ds, \quad t \in [a,b], \\
V_t &= V_a + \int_a^t v_s ds, \quad t \in [a,b].
\end{aligned}$$

Then:

$$\int_a^b u_t V_t dt = U_t V_t\Big|_a^b - \int_a^b U_t v_t dt. \qquad \diamond$$

Proof: Note that:

$$\int_s^b u_t dt = \int_a^b u_t dt - \int_a^s u_t dt = U_b - U_a - \int_a^s u_t dt = U_b - U_s. \tag{$\star$}$$

Now, using Fubini's Theorem and the absolute continuity of U_t, V_t, we have:

$$\begin{aligned}
\int_a^b u_t V_t dt &= \int_a^b u_t \Big(V_a + \int_a^t v_s ds \Big) dt \\
&= \int_a^b u_t V_a dt + \int_a^b u_t \int_a^t v_s ds dt \\
&= (U_b - U_a) V_a + \int_a^b \int_s^b u_t v_s dt ds \\
&\overset{(\star)}{=} (U_b - U_a) V_a + \int_a^b (U_b - U_s) v_s ds \\
&= (U_b - U_a) V_a + U_b (V_b - V_a) - \int_a^b U_s v_s ds \\
&= U_b V_b - U_a V_a - \int_a^b U_t v_t dt.
\end{aligned}$$

This concludes the proof. ■

Appendix B

THE MERTON'S PROBLEM

The *Merton's problem* defines an integral part of the process of understanding (3.6). Specifiacally through the aforementioned problem, which shares many similarities with the frictionless optimizer of (3.6), one can better understnd the guiding principles behind the formulation of such an optimization problem.

Note that before we delve deeper into the above, it is deemed crucial to introduce some relevant concepts of *Dynamic programming*. For more information about the concepts examined in this section of the appendix, refer to [Bel54], [Ber12], [How60], [FR12], [KS08] and [MS92].

One could argue that the main idea behind a problem in the context of Dynamic programming lies in the interaction between a set of *actions* that we can do, a set of *states* that we could be and a set of *rewards* that are linked with those actions in each of these states. In this context, our goal essentially becomes to define a specific sequence of actions that optimizes the sum of the aforementioned rewards, by "breaking" the original problem into a sequence of smaller sub-problems. Let us now approach the above more rigorously.

Basic definitions of Dynamic programming

In the context of problem approached through the lense of Dynamic programming (dynamic program), in discrete time $t = 0, ..., T$, we could make the following observations:

(I) We denote x as a specific state and $\mathscr{X}$ as the set of all possible states of the program, with $x_t \in \mathscr{X}$ being the state of the dynamic program at time t.

(II) We denote a as a specific action and $\mathscr{A}$ as the set of possible actions.

(III) We denote the reward that is linked with a specific action a in a specific state x as $r(x, a)$. Moreover, the reward for terminating in state x at time T is denoted as $r(x)$.

(IV) We define the mapping:

$$\hat{x} = f(x, a),$$

which connects an action a in a specific state x with the next state $\hat{x} \in \mathscr{X}$. The above mapping is also called *Plant equation* of a dynamic program, where $f : \mathscr{X} \times \mathscr{A} \to \mathscr{A}$.

(V) The guiding principle behind chosing a specific action on a specific time $t = 0, ..., T-1$, is called *policy* and is denoted as $\pi = (\pi_t : t = 0, ..., T-1)$.

(VI) We evaluate how good each policy is by the sum of it rewards (cumulative rewards):

$$R(x_0, \pi) = r(x_0, \pi_0) + r(x_1, \pi_1) + \cdots + r(x_T).$$

In a similar manner we could also define $R_t(x_t, \pi) = \sum_{s=T-t}^{T-1} r(x_s, \pi_s) + r(x_T)$.

(VII) For a cumulative reward function $R(x, \pi)$, we define the value function to be the maximum reward:

$$V(x_0) = \max_{\pi} R(x_0, \pi).$$

In a similar manner we could also define $V_t(x_t) = \max_{\pi} R_t(x_t, \pi)$.

Having introduced the basic "building blocks" of a dynamic program, we could now move on its specific definition.

Definition B.0.1 (Dynamic program) *Given an initial state x_0, a dynamic program is the following optimization for $t = 0, ..., T-1$:*

$$\begin{aligned} V(x_0) = \max \quad & R(x_0, \pi) \\ s.t. \quad & x_{t+1} = f(x_t, \pi_t) \\ for \quad & \pi_t \in \mathscr{A}. \end{aligned}$$

◊

As it was previously expressed, the main idea behind a dynamic program is the reduction of the original problem to a sequence of simpler optimizations. Specifically, under this condition, the optimization problem could be depicted via the *Bellman equation*, which is presented below.

Definition B.0.2 (Bellman equation) *The optimality of a dynamic program for* $V_0(x) = r_T(x)$ *and* $t = 0, ..., T-1$ *can be given by:*

$$V_t = \max_{a \in \mathscr{A}} \{r(x, a) + V_{t-1}(\hat{x})\},$$

where $x \in \mathscr{X}$ *and* $\hat{x} = f(x, a)$. ◊

Diffusion Control Problems (DCP)

Generalizing accordingly the above concepts, in this context time is continuous $\in \mathbb{R}_+$, $X_t \in \mathbb{R}^n$ defines the state of the dynamic program at time t and $a_t \in \mathscr{A}$ defines the respective action at time t.

Definition B.0.3 (Plant equation) *Given the drift* $\mu(X_t, a_t)$ *and the diffusion* $\sigma(X_t, a_t)$ *processes of* X_t*, the state of the dynamic program evolves according to the following dynamics:*

$$dX_t = \mu(X_t, a_t)dt + \sigma(X_t, a_t)dW_t,$$

where W_t *is a* d*-dimensional Brownian motion. The above SDE defines the Plant equation in the context of a DCP.* ◊

Furthermore, in a similar manner to the discrete case defined above, a policy π chooses an action at each time t (with the only technical difference being that usually in this context, we assume that π_t is predictable). Let $\mathscr{P}$ be the set of policies, the a dynamic program in the context of a DCP is reformulated as follows:

Definition B.0.4 (Diffusion control problem) *Given initial state* x_0*, a dynamic program is the following optimization:*

$$V(x_0) = \max_{\pi \in \mathscr{P}} R(x_0, \pi) = \mathbb{E}_{x_0}\left[\int_0^T e^{-at} r_t(X_t, \pi_t)dt + e^{-aT} r_T(X_T)\right].$$

The above can be restated accordingly for a minimization problem. ◊

Regarding the Bellman equation, as it was presented above, the corresponding form in this context is called *Hamilton-Jacobi-Bellman equation* (HJB) and it yileds the necessary and sufficient conditions for optimality of a DCP.

Definition B.0.5 (Hamilton-Jacobi-Bellman equation) *The optimality of a DCP can be given by:*

$$0 = \max_{a \in \mathscr{A}} \left[r_t(x, a) + \partial_t V_t(x) + \partial_x V_t(x) + \frac{1}{2} \sigma^T \sigma \partial_{xx} V_t(x) - a V_t(x) \right],$$

where ∂_t, ∂_x, ∂_{xx} denote the respective first and second order patial derivatives of V_t. ◊

Modelling for the Merton's problem

We shall now venture to examine the one-dimensional case of Merton's problem, under which an investor chooses to invest between a riskless and a risky asset. In this context, the goal is the maximization of the expected utility of the investor's long run consumption. In other words, we assume that the investor could either allocate his money in a riskless investment (e.g. bank), receiving interest r, or invest in a risky asset which is driven by the following dynamics:

$$dS_t = S_t(\mu dt + \sigma dW_t),$$

where μ, σ are constants and $(W_t)_{t \in \mathscr{T}}$ defines the standard Brownian motion. Furthermore, under the aforementioned modelling procedure, the wealth $(K_t)_{t \in \mathscr{T}}$ of the investor evolves according to the following SDE

$$dK_t = r \underbrace{(K_t - n_t S_t)}_{\star} dt + \underbrace{n_t dS_t}_{\star\star} - \underbrace{c_t dt}_{consumption}, \tag{B.1}$$

where $\star$ is the wealth allocated in the risky asset, $\star\star$ is the wealth allocated in the risky asset, c_t is the investor's rate of consumption at time t and n_t is the number of stocks in the risky asset at time t. Moreover, we define θ_t to be the wealth in the risky asset at time t.

By the above, it is clear that (B.1) defines the Plant equation in this problem. Therefore, in this context, the objective becomes:

$$V(k_0) = \max_{(n_t, c_t)_{t \geq 0} \in \mathscr{P}(k_0)} \mathbb{E} \left[\int_0^\infty e^{-\rho t} u(c_t) dt \right], \tag{B.2}$$

where $\rho > 0$ defines the discounting factor, $u(c)$ defines a concave function and $\mathscr{P}(k_0)$ defines the set of all admissible policies. Let us now move forward to the optimization of the problem at hand.

Proposition B.0.1 *The Hamilton-Jacobi-Bellman equation for the Merton's problem can be written as:*

$$0 = \max_c\{u(c) - c\partial_k V\} + \max_\theta \left\{\theta(\mu - r)\partial_k V + \frac{1}{2}\sigma^2\theta^2\partial_{kk}V\right\} - \rho V + rk\partial_k V. \tag{B.3}$$

◊

Proof: Initially note that (B.1) can be equivalently written as:

$$dK_t = \big(rK_t + (\mu - r)\theta_t - c_t\big) + \theta_t\sigma dW_t,$$

if we now apply Ito's Lemma on $V(K_t)$ we get:

$$\begin{aligned}
dV_t(K_t) &= \partial_K V(K_t)dK_t + \frac{1}{2}\partial_{KK}V(K_t)d[K]_t \Rightarrow \\
dV_t(K_t) &= \partial_K V(K_t)\Big(\big(rK_t + (\mu - r)\theta_t - c_t\big) + \theta_t\sigma dW_t\Big) + \frac{\theta_t^2\sigma^2}{2}\partial_{KK}V(K_t)dt.
\end{aligned}$$

Substituting the above into (B.3), while having in mind how the HJB is defined in the context of a DCP, we arrive at the result. ■

Proposition B.0.2 *The optimal wealth invested in the risky asset θ^*, is given by:*

$$\theta^* = -\frac{\partial_K V}{\partial_{KK}V}\sigma^{-2}(\mu - r). \tag{B.4}$$

◊

Proof: Differentiating (B.3) w.r.t. θ yields the desired result. ■

The Merton's problem for CRRA utility

Let us now derive (B.4) under a specific representation for the utility function called: *Constant Relative Risk Aversion* (CRRA) utility function, which is defined as follows:

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma}, & if \quad \gamma > 0, \gamma \neq 1 \\ lnc, & if \quad \gamma = 1 \end{cases} \tag{B.5}$$

where γ defines the relative risk aversion of the investor.

Remark B.0.1 *It should be noted that* (B.5) *takes that name since the coefficient of relative risk aversion for the aforementioned expression is constant. Specifically, we have:*

$$R(c) = -\frac{cu''(c)}{u'(c)} = \gamma.$$

Let us now see in detail how we arrive at such an expression for $R(c)$. Intuitevely, we could say that there is a certain amount x_{CE} of a good or service for which a consumer is indifferent between either taking x_{CE} or taking a chance to receive more (or less). Specifically, we could say that the above state could be depicted as: $u(x_{CE}) = \mathbb{E}[u(x)]$, where x_{CE} is denoted as Certainty Equivalent value. *We now move forward to defining the following:*

(I) The Absolute Risk premium*: $p_A = \mathbb{E}[x] - x_{CE}$.*

(II) The Relative Risk premium*: $p_R = p_A/\mathbb{E}[x]$.*

Through the use of Taylor's Theorem, *note that by expanding $u(x)$ (up to the second order term) and $u(x_{CE})$ (up to the first order term) around $\mathbb{E}[x]$, we get the following approximations:*

$$u(x) \approx u(E[x]) + u'(E[x])(x - E[x]) + \frac{1}{2}u''(E[x])(x - E[x])^2$$
$$u(x_{CE}) \approx u(E[x]) + u'(E[x])(x_{CE} - E[x]),$$

taking expectations on the first expression and by the equality $u(x_{CE}) = \mathbb{E}[u(x)]$, we then have:

$$u'(E[x])(x_{CE} - E[x]) \approx \frac{1}{2}u''(E[x])E\left[(x - E[x])^2\right] \Rightarrow$$
$$p_A = E[x] - x_{CE} \approx -\frac{1}{2}\frac{u''(E[x])}{u'(E[x])}E\left[(x - E[x])^2\right].$$

We define $A(x) = -\frac{u''(x)}{u'(x)}$ as the absolute risk aversion coefficient. In a similar manner, we have:

$$p_R = \frac{p_A}{E[x]} = -\frac{1}{2}\frac{u''(E[x])E[x]}{u'(E[x])}\frac{E\left[(x - E[x])^2\right]}{E[x]^2}.$$

We define $R(x) = -\frac{u''(x)x}{u'(x)}$ *as the relative risk aversion coefficient.*

Having defined all the above, let us now see more intuitively what the CRRA utility function essentially depicts. It is easy to see that through the choice of a utility function, we could deduce an investor's stance on undertaking risk for a given initial wealth. Therefore, if for example we have 1 *euro at our disposal (which corresponds to a specific value for a given utility function) and a potential investment arises which yields* 1 *euro with probability* p *or* -1 *euro with probability* $1-p$*, if we calculate the corresponding utility for this investment and compare it with our current utility (which stems by keeping our initial wealth, without participating in this investment), we could deduce if it is in our interest to undertake this "excess risk" or not.*

An important condition on the above "simulation", to determine if we are going to undertake the excess risk or not, constitutes our initial wealth relative to the amount we stand to "win" or "lose". This condition would not make a difference under the CRRA utility function if our relative winnings or losses remain the same. In other words, under the CRRA utility function, the decision to participte in an investment which takes us to either 2 *or* 0 *euro, when our initial wealth is* 1 *euro, will have the same answer as to that for an investment which takes us to either* 200 *or* 0 *euros, while having* 100*.* ♢

Thus, by using (B.5), (B.2) becomes:

$$V(k_0) = \max_{(n_t,c_t)_{t\geq 0}\in \mathscr{P}(k_0)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right].$$

Proposition B.0.3 *For the CRRA utility function, the following holds:*

$$V(k) = c\frac{k^{1-\gamma}}{1-\gamma}, \tag{B.6}$$

where c *is a positive constant.* ♢

Proof: Note that for a constant multiplier λ, we have:

$$V(\lambda k) = \max_{(n_t,c_t)_{t\geq 0}\in \mathscr{P}(\lambda k_0)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right]$$

$$= \max_{(n_t, c_t)_{t\geq 0} \in \mathscr{P}(k_0)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{(\lambda c_t)^{1-\gamma}}{1-\gamma} dt \right] = \lambda^{1-\gamma} V(k).$$

Thus, by setting $\lambda = k^{-1}$, we have:

$$V(k) = \frac{k^{1-\gamma}}{k^{1-\gamma}} V(k) = k^{1-\gamma} \underbrace{\frac{V(k)}{k^{1-\gamma}}}_{V(k^{-1}k)=V(1)} = \frac{k^{1-\gamma}}{1-\gamma} \underbrace{(1-\gamma)V(1)}_{c}.$$

■

Proposition B.0.4 *The optimal amount invested in the risky asset under the CRRA utility function is given by:*

$$\theta^* = \frac{k}{\gamma} \sigma^{-2} (\mu - r). \qquad \diamond$$

Proof: Differentiating (B.6) w.r.t. k, yields: $\partial_k V(k) = ck^{-\gamma}$ and $\partial_{kk} V(k) = -\gamma c k^{-\gamma-1}$, consequently:

$$\theta^* = -\frac{\partial_k V}{\partial_{kk} V} \sigma^{-2} (\mu - r) = \frac{ck^{-\gamma}}{\gamma c k^{-\gamma-1}} \sigma^{-2} (\mu - r) = \frac{k}{\gamma} \sigma^{-2} (\mu - r).$$

■

If we now want to derive the fraction of the investor's wealth which is to be invested in the risky asset, this easily emerges by dividing with k resulting in:

$$\phi^* = \frac{\mu - r}{\sigma^2 \gamma}.$$

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