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ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ ΚΑΙ ΦΥΣΙΚΩΝ ΕΠΙΣΤΗΜΩΝ

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Black Holes with Axionic Clouds in Chern-Simons Modified Gravity
Μαύρες οπές με Αξιονικά Νέφη σε τροποποιημένη βαρύτητα Chern-Simons

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Black Holes with Axionic Clouds in Chern-Simons Modified Gravity

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Abstract

The first formulation of the no-hair conjecture, first posed by Werner Israel and reformulated by Bekenstein, states that black holes can be completely described only by three parameters, the angular momentum, the mass and the electric charge, quantities that can be measured asymptotically and are subject to a Gauss law. No other physical quantities, to which "hair" gives some kind of a metaphor, should exist. Black holes are uniquely determined by these three parameters, and other information about the matter that formed the black hole "disappears" behind the black hole's event horizon, being permanently inaccessible to external observers. An intuitive perspective is given when we consider, in this context, the impact of the energy conditions. These energy conditions imply "reasonable impacts" on the underlined geometry from the presence of matter and energy, in agreement with the gravitational equations of motion. There is a close relation between the violation of the energy conditions and the hair of a black hole. Hairy black holes solutions have been found in multiple cases. The introduction of a negative cosmological constant or the coupling of a (pseudo-)scalar field with higher order curvature terms are just some examples of cases where hair does exist. In all of these cases, the energy conditions are violated, which gives us the physical intuition that the evasion of the no-hair theorems is achived by the violation of these conditions. Modified theories of gravity provide a great area to seek for "hairy" solutions. In this thesis, we examine the Chern Simons modified gravity, defined as the sum of the Einstein-Hilbert action and a new, higher order correction. If a (pseudo-)scalar field is directly coupled to curvature invariants, "hairy" black holes can be generated. The scalar hair is present due to the interaction with the underlined geometry. We seek for a slowly rotating black hole solution dressed with axionic hair and we examine how such a system violates the Null Energy Condition in the region outside the horizon. We study the angular momentum of the axion-black hole system and the behaviour of timelike geodesics around it, leading to a completely new behaviour of the effective potential, which presents a repulsive nature for the counter rotating geodesics near the horizon.

Contents

Chapter 1

Introduction

After the gravitational collapse of a large enough star in the presence of any type of matter and energy, the most general outcome is the Kerr-Newman black hole, completely described only by three parameters, the angular momentum, the mass and the electric charge, quantities that can be measured asymptotically and are subject to a Gauss law. No other physical quantities, to which "hair" gives some kind of a metaphor, should exist. This is a first formulation of the no-hair conjecture, first posed by Werner Israel [1] and reformulated by Bekenstein's considerations made in [2]. However, being the final state of gravitational collapse, two black holes with the same mass, angular momentum and charge, are indistinguishable in any way the one from the other, no matter what was the original composition of the collapsed matter field. There is, consequently, an information of the initial matter field hidden inside the black hole. So, a question arises: can this information be extracted?

An intuitive perspective is given when we consider, in this context, the impact of the energy conditions on the underlined geometry. These energy conditions [3] are formed in such a way in order to capture the idea of the "positiveness of the energy", or in other words, we demand "reasonable impacts" on the underlined geometry from the presence of matter and energy, in agreement with the gravitational equations of motion. There is a close relation between the energy conditions and the hair of a black hole. It seems that by challenging the validity of these conditions in specific cases, the no-hair theorems are also called into question.

However, throughout the years, stationary black holes solutions with new global charges *(primary hair)* or new non-trivial fields, determined by the standard global charges *(secondary hair)* have been found ([4–10] etc.). These solutions are known as "*hairy black holes*". Hairy black holes solutions have been found in multiple cases. The introduction of a negative cosmological constant or the coupling of a (pseudo-)scalar field with higher order curvature terms, as is the cases of the Gauss-Bonnet [8] or the Chern-Simons topological terms [5], are just some examples of various other cases where hair does exist. In all of these cases, the energy conditions are violated, which gives us the physical intuition that the evasion of the no-hair theorems is achived by the violation of these conditions.

In particular, modified theories of gravity provide a great area to seek for "hairy" solutions. One of the features of these theories is the rôle of the existence of a (pseudo)scalar field in a black hole background. In such theories, the scalar field backreacts on the geometry and "dresses" the black hole with hair. An example of a gravity theory that contains higher order curvature terms is the Chern-Simons (CS) theory [11]. The action of the Chern Simons modified gravity is defined as the sum of the Einstein-Hilbert action and a new, parity violating correction. If a (pseudo-)scalar field is directly coupled to curvature invariants, "hairy" black holes can be generated. The scalar hair is present due to the interaction with the underlined geometry. As mentioned before, there is a strong relation between the violation of the energy conditions and the formation of scalar hair in the region outside of the horizon of a black hole.

This thesis is orginised as follows: in Section 2 we see specific examples of no-hair theorems, starting from Bekenstein's no-scalar-hair conjecture and expanding to other cases, following the same reasoning. In Section 2.2 and 2.2.1, we try to make the considarations much more general, based on the violation of the energy conditions, formulating a model-independant approach, relying only on the properties of an effective energy momentum tensor and argue about the existence of hair or not. In Section 3, we present the Chern Simons modified theory of gravity, seeking for a slowly rotating Kerr-type black hole solution, dressed with axionic hair. In Section 3.3 we see how such a system violates the Null Energy Condition in the region outside the horizon, due to the existence of the axionic hair. We continue by studying the angular momentum of the axion-black hole system in Section 3.4, and finally, in Section 3.5, we examine the behaviour of timelike geodesics around the axionic black hole, leading to a completely new behaviour of the effective potential, which presents a repulsive nature for the counter rotating geodesics near the horizon.

Chapter 2

No-hair theorems and Black hole mechanics

2.1 No-hair theorems

Multiple no-hair theorems have been studied with interactions of black holes with matter taken into consideration. The main attention was firstly turned to scalar fields, as the most realistic candidates for dressing a black hole with hair. The no-hair theorems excluded for a long time scalar fields, vector fields, spinors and abelian Higgs hair from stationary black holes' exterior region.

Bekenstein [2] was the first one to propose a no-(scalar)-hair theorem, ruling out a large variety of coupled scalar fields as candidates, under some basic assumptions. The statement that black holes have no-hair means that they can be dressed only by fields that respect a Gauss-like law, like the electromagnetic field. ”Hair”, therefore, would represent a new parameter required to describe the black hole, other than the mass, the anglular momentum and the electric charge.

As we will refer extensively in what follows, hair, this new charge, could be characterized as *primary*, if it induces a new independent parameter, or *secondary*, if it depends on the already existing ones. In the second case, no-new charge is introduced, but the alterations of the black hole's spacetime are of utmost importance.

There are different approaches to prove a no-hair theorem for different cases. Mainly, the arguments made concern the symmetries that the spacetime admits, the finitiness of the energy momentum tensor and/or the agreement with the energy conditions. We will present some basic cases and see how specific examples can be excluded (or not), considering the existence of hair in the outside region of the black hole. The metric convention throughout this thesis (unless noted otherwise) is (- + + +).

2.1.1 Scalar-vacuum vs Electro-vacuum

We start by wondering what would be a distinction between an electric field and a scalar field on a black hole spacetime? We start with the two actions

$$\mathcal{S} = \frac{1}{4\pi}\int d^4x\sqrt{-g}\left(\frac{R}{4} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \tag{2.1}$$

for the electro-vacuum, and also the scalar-vacuum, for a massless and real scalar field, which is described by the action

$$\mathcal{S} = \frac{1}{4\pi}\int d^4x\sqrt{-g}\left(\frac{R}{4} - \frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi\right) \tag{2.2}$$

In both cases, the Schwarzschild metric of vacuum general relativity is a solution, with $F_{\mu\nu} = 0$, $\nabla_\mu\Phi = 0$, respectively. In Schwarzschild coordinates it reads (see appendix 1.1):

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.3}$$

Let's now consider on the Schwarzschild background a test, spherically symmetric: $i)$ electric field, described by the potential $A = \phi_E(r)dt$ and the corresponding Maxwell tensor $F = dA = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $ii)$ a scalar field which is described as $\Phi = \Phi(r)$. From the source-free Klein-Gordon and Maxwell equations, we get:

$$\nabla_\mu F^{\mu\nu} = 0 \;\Rightarrow\; \partial_r\phi_E(r) = \frac{Q_E}{r^2} \;\Rightarrow\; \phi_E(r) = -\frac{Q_E}{r} \tag{2.4}$$

and for the scalar field:

$$\Box\Phi(r) = 0 \;\Rightarrow\; \partial_r\Phi(r) = \frac{Q_S}{r^2}\left(1 - \frac{2M}{r}\right)^{-1} \;\Rightarrow\; \Phi(r) = \frac{Q_S}{2M}\ln\left(\frac{2M}{r} - 1\right) \tag{2.5}$$

where Q_E, Q_S are constants from integration.

Electric field: In the case of the electric field we have a regular solution on the horizon and also outside the horizon. Also, the field sources an energy-momentum on and outside the horizon given by

$$T^E_{\mu\nu} = F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \tag{2.6}$$

with its non-trivial components given as:

$$(T^E)^t{}_t = (T^E)^r{}_r = -\frac{Q_E^2}{2r^4} = -(T^E)^\theta{}_\theta = -(T^E)^\phi{}_\phi \;. \tag{2.7}$$

Considering the backreaction of the electric field on the metric and solving the Einstein equations $G_{\mu\nu} = 2T^E_{\mu\nu}$, we would get as we know the Reissner-Nordström black hole solution, with the metric given by (see appendix 1.2):

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q_E^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - 2M/r + Q_E^2/r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.8}$$

$$A = -\frac{Q_E}{r} dt \tag{2.9}$$

The Reissner-Nordström black hole solution connects continuously with the Schwarzschild solution and its horizon is regular as long as $|Q_E| < M$. We can compute this as the electric flux on a closed 2-surface $\partial\Sigma$:

$$Q_E = \frac{1}{8\pi} \oint_{\partial\Sigma} F^{\mu\nu} dS_{\mu\nu} \tag{2.10}$$

where $dS_{\mu\nu}$ is the area element. We can choose $\partial\Sigma$ as a r, t=constant surface at any r outside the black hole due to the spherical symmetry of our case, and conclude afterall that Q_E is the electric charge, obeying a Gauss law.

Scalar field: In the second case, we can check from eq.(2.5) that there is a divergence as $r \to r_H$ for the gradient of the scalar field, with $r_H = 2M$ the horizon, which means that the scalar field Φ diverges logarithmically there. What is more, the energy-momentum tensor given by

$$T^S_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \Phi \partial^\alpha \Phi \ , \tag{2.11}$$

also diverges at the horizon :

$$(T^S)^r{}_r = \frac{Q_S^2}{2r^4}\left(1 - \frac{r_H}{r}\right)^{-1} = -(T^S)^t{}_t = -(T^S)^\theta{}_\theta = -(T^S)^\phi{}_\phi \ . \tag{2.12}$$

The test field approximation we did for the electric field fails for the scalar field near the horizon, no matter how small Q_S is. This is what provides a first indication that a regular (on and outside the horizon), static and spherically symmetric solution of a black hole with scalar hair does not exist, which connects in a continuous way to the Schwarzschild solution.

One physical interpretation for the difference between the scalar and the electric fields is related to the Gauss's law which exists for the electric field and fails for the scalar field. Thus, a regular electric field on the horizon and outside of it can be sourced by charges that have fallen into the black hole. The existence, on the other hand, of a non trivial scalar field outside the horizon, no matter how small it becomes near it, implies an infinite pile up on the area of the horizon. This can be understood because any finite scalar field outside the black hole area whould disperse to infinity or will inevitably fall into the black hole, leaving no trace outside the black hole, because of the absence of a Gauss's law, as opposed to the case of the electric field

2.1.2 Bekenstein's no-scalar-hair theorem

In his 1995 paper [2], Bekenstein rules out a large class of scalar fields, as valid candidates for dressing a black hole. He proved the no-hair theorem for black holes dressed with a multicomponent scalar field. To sketch the proof, it starts in the simple form of an action

$$S_{\psi} = -\frac{1}{2}\int \left[\nabla_{\alpha}\psi\nabla^{\alpha}\psi + V(\psi^2)\right]\sqrt{-g}d^4x \tag{2.13}$$

for a static scalar field in a static black hole background. From Euler–Lagrange equations, we get

$$\begin{gathered}\frac{\partial L}{\partial \psi} = \partial_{\mu}\frac{\partial L}{\partial(\partial_{\mu}\psi)} => \\ g^{ab}\nabla_{\alpha}\nabla_{\beta}\psi - \psi V'(\psi^2) = 0\end{gathered} \tag{2.14}$$

and integrating (2.14) over the exterior of the black hole at a given time t, and multiply it by ψ, we get

$$\int \left[g^{ab}\psi\nabla_{\alpha}\nabla_{\beta}\psi - \psi^2 V'(\psi^2)\right]\sqrt{-g}d^3x = 0 \tag{2.15}$$

which with partial integration and vanishing $[\psi, \nabla\psi]$ boundary terms, we get

$$\int \left[g^{ab}\nabla_{\alpha}\psi\nabla_{\beta}\psi + \psi^2 V'(\psi^2)\right]\sqrt{-g}d^3x = 0 \tag{2.16}$$

where a, b are running over the space coordinates.

If $V'(\psi^2) \geq 0$ everywhere and vanishes only at some discrete values ψ_j, then the ψ-field must be constant at the outside region of the black hole, with ψ taking values from the interval $[0, \psi_j]$, so this rules out hair for the case of action (2.13). Objections were made to Bekenstein's argument. In particular, exponentially decaying scalar hair could be attached to a spherical, static black hole. However, in some regions the condition $V'(\psi^2) \geq 0$ for the potential would be violated. We want to mention that the theorem fails for any field violating the above condition.

Bekenstein challenged back these objections with the simple demonstration that a positive-definite field energy density is enough to rule out black hole solutions with scalar hair. The first step considered, is an action and a multiplet of scalar fields $(\psi, \chi, ...)$ as

$$S_{\psi,\chi,...} = -\int \mathcal{E}(\mathcal{K}, \mathcal{J}, \mathcal{M}, ..., \psi, \chi, ...)\sqrt{-g}d^4x \tag{2.17}$$

where $\mathcal{K} = g^{\alpha\beta}\nabla_{\alpha}\chi\nabla_{\beta}\psi$, $\mathcal{J} = g^{\alpha\beta}\nabla_{\alpha}\psi\nabla_{\beta}\psi$, $\mathcal{M} = g^{\alpha\beta}\nabla_{\alpha}\chi\nabla_{\beta}\chi$. Despite the generality of the above action, **we assume minimally coupled scalar fields**.

We will consider only two fields, χ and ψ, since the generalization to more fields is obvious. We now may assume that the energy density that the scalar field carries, is non-negative. We start from $T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}$ and we end up with

$$T_{\alpha}{}^{\beta} = -\mathcal{E}\delta_{\alpha}{}^{\beta} + 2\left(\frac{\partial\mathcal{E}}{\partial\mathcal{J}}\right)\nabla_{\alpha}\psi\nabla^{\beta}\psi + 2\left(\frac{\partial\mathcal{E}}{\partial\mathcal{M}}\right)\nabla_{\alpha}\chi\nabla^{\beta}\chi + \left(\frac{\partial\mathcal{E}}{\partial\mathcal{K}}\right)(\nabla_{\alpha}\chi\nabla^{\beta}\psi + \nabla_{\alpha}\psi\nabla^{\beta}\chi) \tag{2.18}$$

and the energy density for an observer with a 4-velocity U^{μ} is given by ($U^{\mu}U_{\mu} = -1$):

$$\rho = T_{\alpha\beta}U^{\alpha}U^{\beta} = \mathcal{E} + 2\left(\frac{\partial\mathcal{E}}{\partial\mathcal{J}}\right)(\nabla_{\alpha}\psi U^{\alpha})^2 + 2\left(\frac{\partial\mathcal{E}}{\partial\mathcal{M}}\right)(\nabla_{\alpha}\chi U^{\alpha})^2 + 2\left(\frac{\partial\mathcal{E}}{\partial\mathcal{K}}\right)(\nabla_{\alpha}\chi U^{\alpha}\nabla_{\beta}\psi U^{\beta}) \tag{2.19}$$

Now, suppose that the field admits a timelike Killing vector, as it should, for the case of static black hole with scalar hair. For an observer moving along a Killing-vector, $\nabla_{\alpha}\chi U^{\alpha} = 0$, $\nabla_{\alpha}\psi U^{\alpha} = 0$, so we see from (2.19) that $\rho = \mathcal{E}$ and with the assumption of positive energy density

$$\rho = \mathcal{E} \geq 0 \tag{2.20}$$

Now, let's consider a second observer moving with a relative 3-velocity to the Killing observer. In a freely falling frame, co-moving momentarily with the first observer, we have $U^0 = \frac{1}{(1-v^2)^{1/2}}U$, while $U = \frac{v}{(1-v^2)^{1/2}}$. In the relativistic limit, the terms involving derivatives in eq. (2.19) dominate $\mathcal{E}$.

The conditions in this case for the positivity of the result of energy density yield

$$\frac{\partial\mathcal{E}}{\partial\mathcal{J}} \geq 0, \quad \frac{\partial\mathcal{E}}{\partial\mathcal{M}} \geq 0$$
$$\left(\frac{\partial\mathcal{E}}{\partial\mathcal{K}}\right)^2 \leq 4\left(\frac{\partial\mathcal{E}}{\partial\mathcal{J}}\right)\left(\frac{\partial\mathcal{E}}{\partial\mathcal{M}}\right) \tag{2.21}$$

We may now proceed to the theorem. As we said previously, the theorem concerns an arbitrary number of coupled scalar fields, regarding the properties of the corresponding energy-momentum tensor. The assumptions we are going to consider are:

- an asymptotically flat solution of the Einstein and scalar field equations with characteristics of a static, spherically symmetric black hole. The metric outside the horizon may be taken as $ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta} = -e^{v}dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$
- Assymptotic flatness: $v = v(r), \lambda = \lambda(r)$ and $\lambda, v \sim O(1/r)$ as $r \to \infty$
- Non-trivial scalar field: $\psi = \psi(r)$ and $\chi = \chi(r)$
- Event horizon at $r = r_h$ where $e^{v(r_h)} = e^{-\lambda(r_h)} = 0$
- $T_{\mu\nu}$ should be finite at $r = r_h$ and at $r \to \infty$

The conservation of the energy-momentum tensor is expressed as

$$\nabla_\nu T_\mu{}^\nu = \frac{1}{\sqrt{-g}}\partial_\lambda(\sqrt{-g}T_\mu{}^\lambda) - \frac{1}{2}(\partial_\mu g_{\alpha\beta})T^{\alpha\beta} = 0 \tag{2.22}$$

and so $T_r{}^r$ component is given by

$$[\sqrt{-g}T_r{}^r]' - \sqrt{-g}\frac{1}{2}(\partial_r g_{\alpha\beta})T^{\alpha\beta} = 0 \tag{2.23}$$

where prime denotes $\partial_r = \frac{\partial}{\partial r}$. Another consequence of spherical symmetry is the fact that $T_\mu{}^\nu$ should be diagonal and $T_\theta{}^\theta = T_\phi{}^\phi$. These conditions combined with the metric determinant $\sqrt{-g} = e^{\frac{\lambda+v}{2}}r^2 sin\theta$ lead to

$$\left(e^{\frac{\lambda+v}{2}}r^2 T_r{}^r\right)' - \frac{1}{2}e^{\frac{\lambda+v}{2}}r^2\left[v'T_t{}^t + \lambda' T_r{}^r + \frac{4}{r}T_\theta{}^\theta\right] = 0 \tag{2.24}$$

The λ' terms cancel out and we get

$$\left(e^{\frac{v}{2}}r^2 T_r{}^r\right)' - \frac{1}{2}e^{\frac{v}{2}}r^2\left[v'T_t{}^t + \frac{4}{r}T_\theta{}^\theta\right] = 0 \tag{2.25}$$

Now, looking at equation 2.18 and taking into account the symmetries, with $\psi = \psi(r)$ and $\chi = \chi(r)$, we get $T_\theta{}^\theta = T_\phi{}^\phi = T_t{}^t = -\mathcal{E}$. Now, substituting in (2.25) $\frac{1}{2}e^{\frac{v}{2}}r^2\left[v'T_t{}^t + \frac{4}{r}T_\theta{}^\theta\right] = -\left(e^{\frac{v}{2}}r^2\right)'\mathcal{E}$ so

$$\left(e^{\frac{v}{2}}r^2 T_r{}^r\right)' = -\left(e^{\frac{v}{2}}r^2\right)'\mathcal{E} \tag{2.26}$$

Now, with integration at the exterior of black hole of eq.(2.26) from r_h to r we find

$$T_r{}^r(r) = -\frac{e^{-\frac{v}{2}}}{r^2}\int_{r_h}^{r}(e^{\frac{v}{2}}r^2)'\mathcal{E}dr \tag{2.27}$$

We reached eq.(2.27) with the assumption that the boundary term at the horizon vanishes because we have $e^{v(r_h)} = 0$ and $T_{\mu\nu}$ is finite there. Now, after the differentiation of eq.(2.25) we find

$$(T_r{}^r)' = -e^{-\frac{v}{2}}r^{-2}(e^{\frac{v}{2}}r^2)'(\mathcal{E} + T_r{}^r) \tag{2.28}$$

What is next, is the study of the behaviour of $(T_r{}^r)'$ and $T_r{}^r$ near the horizon and asymptotically.

Near the horizon

At the horizon $e^{v(r_h)} = 0$ and for $r = r_h + \epsilon$, $\epsilon << 1$, we see that $r^2 e^{v(r_h)}$ must grow with r sufficiently for $r = r_h + \epsilon$, so $(e^{\frac{v}{2}}r^2)'$ is positive near the horizon, due to the staticity of the spacetime and our chosen signature. So, taken into account the positivity of $\mathcal{E}$, we see from (2.27) that $T_r{}^r < 0$ sufficiently near the horizon.

Now lets see what happens for $(T_r{}^r)'$ near the horizon. We start with eq.(2.18) and find that under the previous assumptions becomes

$$\mathcal{E} + T_r{}^r = 2e^{-\lambda}\left[\left(\frac{\partial\mathcal{E}}{\partial\mathcal{J}}\right)\nabla_r\psi\nabla_r\psi + \left(\frac{\partial\mathcal{E}}{\partial\mathcal{M}}\right)\nabla_r\chi\nabla_r\chi + \left(\frac{\partial\mathcal{E}}{\partial\mathcal{K}}\right)(\nabla_r\chi\nabla_r\psi)\right] \tag{2.29}$$

and from the conditions of eq.(2.21) we ensure that $\mathcal{E} + T_r{}^r \geq 0$ at the exterior region of our black hole, everywhere. So, from eq.(2.28) and from our discusion about $(e^{\frac{v}{2}}r^2)'$ and $e^{\frac{v}{2}}r^2$ being positive near and outside the horizon, we coclude that $(T_r{}^r)' < 0$ sufficiently near the horizon, just like $T_r{}^r$. **So, as a sum, sufficiently near and outside the horizon, both $(T_r{}^r)'$ and $T_r{}^r$ should be negative, following our physical assumptions.**

Asymptotically

Asymptotically, we have that $e^{\frac{v(r)}{2}} \to 1$, since $v = v(r) = O(1/r)$ and $\lambda = \lambda(r) = O(1/r)$ as $r \to \infty$. When we put that in eq.(2.28) we easily conclude that, assymptotically, $(T_r{}^r)' < 0$. Now, for $T_r{}^r$ we will look at eq.(2.27). As we will see below, $\mathcal{E}$ must behave as $O(r^{-3})$ asymptotically, which is obtained from Einstein's equation (see eq.(2.30) and the explanation that follows). So, with $e^{\frac{v(r)}{2}} \to 1$ for $r \to \infty$, the integral in eq.(2.27) converges and the quantity under integration yields a negative result, so $T_r{}^r$, should be positive asymptotically, $T_r{}^r > 0$. **Summing up again, asymptotically,** $T_r{}^r > 0$ **and** $(T_r{}^r)' < 0$.

Moving on, let's analyse these results. We see that $T_r{}^r > 0$ but also $T_r{}^r$ decreasing asymptotically, since $(T_r{}^r)' < 0$, for large r. But, sufficiently near the horizon, we argued that $T_r{}^r < 0$. What it means is that at some intermediate interval $[r_a, r_b]$, $(T_r{}^r)'$ should change its sign and also $T_r{}^r$ should become positive at some r_c, $r_a < r_c < r_b$ and stay positive at the corresponding interval $[r_c, r_b]$. We will show now that this result is a source of contradiction, that ensures the no-hair theorem in the case of this analysis.

We continue with the (rr) and (tt) component of Einstein's field equations

$$e^{-\lambda}\left(r^{-2} - r^{-1}\lambda'\right) - r^{-2} = 8\pi G T_t{}^t = -8\pi G\mathcal{E} \tag{2.30}$$

$$e^{-\lambda}\left(r^{-1}v' + r^{-2}\right) - r^{-2} = 8\pi G T_r{}^r \tag{2.31}$$

The assyptotic flatness we have considered, requires $\lambda = \lambda(r) = O(1/r)$, $\lambda' = O(1/r^2)$ and hence $\mathcal{E} = O(r^{-3})$, for large r.

From the first equation, by the substitution of $(e^{-\lambda}r)' = e^{-\lambda}(1 - r\lambda')$ and integration at the exterior of the horizon, with $e^{-\lambda(r_h)} = 0$, we get

$$e^{-\lambda} = 1 - 8\pi G r^{-1}\int_{r_h}^{r}\mathcal{E}r^2 dr - \frac{r_h}{r} \tag{2.32}$$

From eq.(2.32) it follows that $e^{\lambda} \geq 1$ throughout the black hole exterior region and doesn't change sign to respect the metric signature and the black hole behavior. From

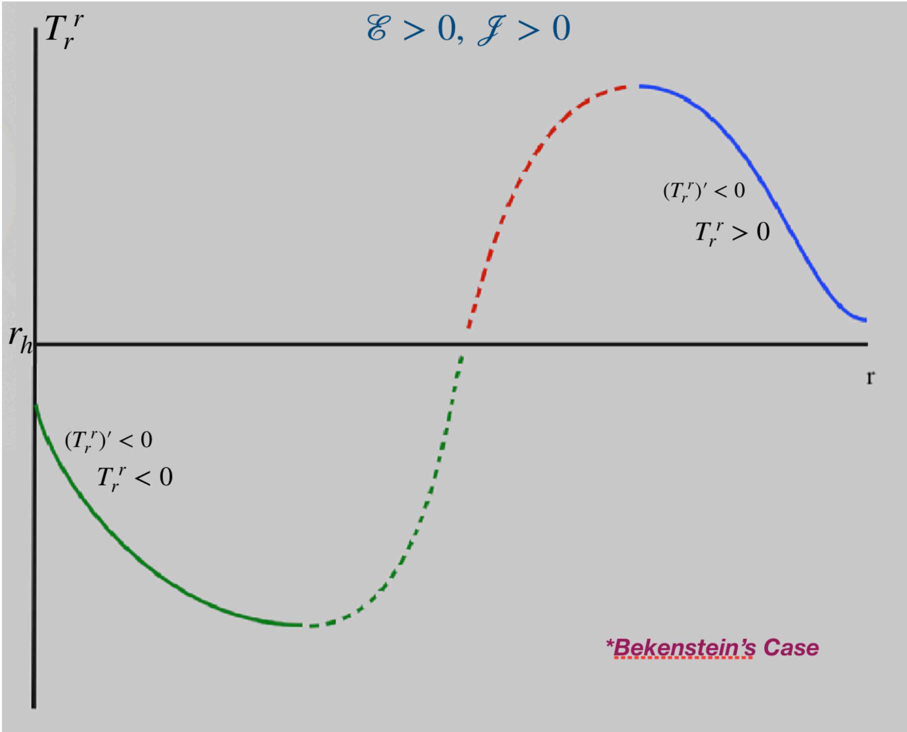

Figure 2.1: This is the Bekenstein no-hair theorem (red dotted lines imply the forbidden interval). This is the *minimum behaviour* of T_r^r (there can exist multiple such forbidden intervals).

the second, (rr), Einstein's equation, we may rewrite it as

$$\frac{e^{-\frac{v}{2}}}{r^2}(e^{\frac{v}{2}}r^2)' = 4\pi G r T_r^{\ r} e^{\lambda} + \frac{e^{\lambda}+3}{2r} > 4\pi G r T_r^{\ r} e^{\lambda} + \frac{2}{r} \tag{2.33}$$

From our previous discussion about the behaviour of $T_r^{\ r}$ and $(T_r^{\ r})'$ near the horizon and asymptotically, we argued that in an interval $[r_c, r_b]$, $T_r^{\ r} > 0$, and so $\frac{e^{-\frac{v}{2}}}{r^2}(e^{\frac{v}{2}}r^2)' > 0$ there. But, looking back to eq. (2.28) for $(T_r^{\ r})'$ we see that it would mean that $(T_r^{\ r})' < 0$ in this interval. But we found that this is the interval where $(T_r^{\ r})'$ has changed its sign and remains positive in $[r_a, r_b]$. So, here is the contradiction we mentionned earlier, as shown in Figure(2.1). $(T_r^{\ r})'$ and $T_r^{\ r}$ can't be both positive in order to respect the Einstein's equations. The only way that we can overcome this contradiction, is to consider our fields $\psi, \chi, ...$ as constants everywhere in the black hole exterior region and the values they should take, must make all components of $T_\mu^{\ \nu}$ to vanish identically, such as $\mathcal{E}(0, 0, 0, ..., \psi, \chi, ...) = 0$. Such values should exist for a trivial solution of the scalar equation be possible in free space. This is the solution which serves as an asymptotic boundary condition in our case. The solution of black hole then must be the Schwarzschild solution. Were the black hole magnetically and/or electrically charged, the solution would be identically the Reissner-Norström black hole.

2.1.3 No-hair for scalar tensor theories

As an extra step, let's abandon the assumption of minimal coupling and consider a scalar field non-minimally coupled to the geometry. The field equations, then, will involve the curvature and the scalar field would be part of gravitational interactions in this case. We will follow the arguments made by Saa in [12]. The purpose is to exclude finite scalar hair for any static, asymptotically flat and spherically symmetric black hole background for a system described by the action

$$S[g,\phi] = \int d^4x\sqrt{-g}\left\{f(\phi)R - h(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right\}, \tag{2.34}$$

and $f(\phi), h(\phi) > 0$. We will present a covariant method to generate solutions for the system described by eq.(2.34). Such a method will be the central point in order to formulate of the no-hair theorem in this case.

We will start from the minimally coupled case and use conformal transformations to seek a relation between them. The minimally coupled case is described by the action

$$\bar{S}[\bar{g},\bar{\phi}] = \int d^4x\sqrt{-\bar{g}}\left\{\bar{R} - \bar{g}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi}\right\}. \tag{2.35}$$

The equation of motion of eq.(2.34) are given as

$$f(\phi)R_{\mu\nu} - h(\phi)\partial_\mu\phi\partial_\nu\phi - \nabla_\mu\nabla_\nu f(\phi) - \frac{1}{2}g_{\mu\nu}\Box f(\phi) = 0 \tag{2.36}$$

$$2h(\phi)\Box\phi + h'(\phi)g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + f'(\phi)R = 0 \tag{2.37}$$

where the prime denotes derivation with respect to ϕ (see appendix B for analytical derivation of 2.36,2.37). Equations derived from (2.35) are

$$\bar{R}_{\mu\nu} - \partial_\mu\bar{\phi}\partial_\nu\bar{\phi} = 0 \tag{2.38}$$

$$\bar{\Box}\,\bar{\phi} = 0 \tag{2.39}$$

Now, we continue with the conformal transformation we mentioned earlier, in order to see how the two sets of solutions above are related. We consider a conformal transformation $g_{\mu\nu} = \Omega^2\overline{g}_{\mu\nu}$. Under such a conformal transformation, the curvature-scalar will transform as[1]

$$R(\Omega^2\overline{g}_{\mu\nu}) = \Omega^{-2}\bar{R} - 6\Omega^{-3}\,\bar{\Box}\,\Omega \tag{2.40}$$

[1]The result $R(\Omega^2\overline{g}_{\mu\nu}) = \Omega^{-2}\bar{R} - 6\Omega^{-3}\,\bar{\Box}\,\Omega$ comes under the assumption of 4-dimensional spacetime and starting from the conformal transformation of Christoffel symbols as $\overline{g_{ab}} \to \Omega^2 g_{ab}$ so $\overline{g^{ab}} \to \Omega^{-2}g^{ab}$ we find that the Christoffel symbols become $\overline{\Gamma^a_{bc}} = \frac{1}{2}\Omega^{-2}g^{ad}\left[\partial_b(\Omega^2 g_{cd}) + \partial_c(\Omega^2 g_{bd}) - \partial_d(\Omega^2 g_{bc})\right] = \Gamma^a_{bc} + \Omega^{-1}\left[\delta^a{}_c\partial_b(\Omega) + \delta^a{}_b\partial_c(\Omega) - g^{ad}g_{bc}\delta^a{}_c\partial_d(\Omega)\right]$ and from $R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho{}_{\nu\sigma} - \partial_\nu\Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}$ we find that $\overline{R^\rho{}_{\sigma\mu\nu}} = ...$ and continue with same steps for the curvature scalar $R = g^{\sigma\nu}R^\rho{}_{\sigma\rho\nu}$ to get relation (2.40).

and with the choice of $f(\phi) = \Omega^{-2}$ we get $\sqrt{-g} = \Omega^4\sqrt{-\bar{g}} \equiv [f(\phi)]^{-2}\sqrt{-\bar{g}}$ and that $f(\phi)R(\Omega^2\bar{g}_{\mu\nu}) = [f(\phi)]^2\bar{R} - 6f(\phi)^{5/2}\,\bar{\Box}\,[f(\phi)]^{-1/2}$. Substituting them to eq.(2.34) yields

$$
\begin{aligned}
S[\Omega^2\bar{g},\phi] = \int d^4x\sqrt{-\bar{g}}f(\phi)]^{-2}\left[f(\phi)]^2\bar{R} - 6f(\phi)^{5/2}\,\bar{\Box}\,[f(\phi)]^{-1/2} - h(\phi)f(\phi)\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] &\Rightarrow \\
= \int d^4x\sqrt{-\bar{g}}\left[\bar{R} - 6f(\phi)^{1/2}\,\bar{\Box}\,[f(\phi)]^{-1/2} - \frac{h(\phi)}{f(\phi)}\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] &\Rightarrow \\
\int d^4x\sqrt{-\bar{g}}\left[\bar{R} - 6f(\phi)^{1/2}\bar{g}^{\mu\nu}\bar{\nabla}_\nu[\bar{\nabla}_\mu(f(\phi))^{-1/2} - \frac{h(\phi)}{f(\phi)}\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] &\Rightarrow \\
\int d^4x\sqrt{-\bar{g}}\left[\bar{R} + 3f(\phi)^{1/2}\bar{g}^{\mu\nu}\bar{\nabla}_\nu[(f(\phi)^{-3/2}f'(\phi)\partial_\mu\phi] - \frac{h(\phi)}{f(\phi)}\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] &\Rightarrow \\
= \int d^4x\sqrt{-\bar{g}}\left[\bar{R} - \frac{3}{2}\left(\frac{f'(\phi)}{f(\phi)}\frac{f'(\phi)}{f(\phi)}\right)\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{h(\phi)}{f(\phi)}\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]&
\end{aligned}
\tag{2.41}
$$

where, from line four to line five we used partial integration for the integral term $3f(\phi)^{1/2}\bar{g}^{\mu\nu}\bar{\nabla}_\nu[(f(\phi)^{-3/2}f'(\phi)\partial_\mu\phi]$ with $\phi, \partial\phi = 0$ at the boundary. Now, with $\frac{d}{d\phi}lnf(\phi) = \frac{f'(\phi)}{f(\phi)}$ we finally reach equation (2.42):

$$
S[\Omega^2\bar{g},\phi] = \int d^4x\sqrt{-\bar{g}}\left\{\bar{R} - \left(\frac{3}{2}\left(\frac{d}{d\phi}\ln f(\phi)\right)^2 + \frac{h(\phi)}{f(\phi)}\right)\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right\} \tag{2.42}
$$

Now, defining the field $\bar{\phi}(\phi)$ to be

$$
\bar{\phi}(\phi) = \int_a^\phi d\xi\sqrt{\frac{3}{2}\left(\frac{d}{d\xi}\ln f(\xi)\right)^2 + \frac{h(\xi)}{f(\xi)}} \tag{2.43}
$$

with a being arbitrary, we get that $S[\Omega^2\bar{g},\phi(\bar{\phi})] \equiv S[g,\phi] = \bar{S}[\bar{g},\bar{\phi}]$. With the positive-definiteness of $f(\phi), h(\phi)$, we can see that the right hand side of eq.(2.43) is a monotonically increasing function, as ϕ increases. The result of all the above is that finally we see that the transformation we choose earlier maps a solution of eq.(2.36,2.37) to a unique solution of eq.(2.38,2.39). We can conclude. finally, keeping also in mind that the aforementioned transformation is a symmetry preserving one, that if we know all solutions $(\bar{g}_{\mu\nu}, \bar{\phi})$ with a given symmetry we automatically know all $(g_{\mu\nu}, \phi)$ with the same symmetry.

The general static, asymptotically flat and spherically symmetric solution $(\bar{g}_{\mu\nu}, \bar{\phi})$ for the minimally coupled case has been found in [13] and reads

$$
\bar{\phi} = \sqrt{2(1-\lambda^2)}\ln\mathcal{R} \tag{2.44}
$$

$$
ds^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = -\mathcal{R}^{2\lambda}dt^2 + \left(1 - \frac{r_0^2}{r^2}\right)^2\mathcal{R}^{-2\lambda}\left(dr^2 + r^2d\Omega^2\right) \tag{2.45}
$$

where the above equations represent a two-parameter (λ, r_0) family of solutions with $\mathcal{R} = \frac{r-r_0}{r+r_0}$, with the values of λ being in the interval $[-1, 1]$.

Now let's analyse this result. The negative range of the values of λ, as shown in [13], are negleted, because the solution would have a negative ADM mass, which can be checked, keeping in mind that asymptitically the spacetime coincides with the Schwarzschild solution, in the sense that (i.e. for $\lambda = -1$):

$$\begin{aligned} g_{tt} &= -\left(1 - \frac{2M}{r}\right) + \mathcal{O}(r^2) \;\Rightarrow \\ \frac{2M}{r} &= 1 - \left(\frac{r-r_0}{r+r_0}\right)^2 + \mathcal{O}(r^2) \;\Rightarrow \\ M &= -\frac{2r^2 r_0}{r-r_0} + \mathcal{O}(r^2) \;, which\ is\ negative\ for\ r > r_0 \end{aligned} \tag{2.46}$$

For $\lambda = 1$, the solution becomes **the exterior vacuum Schwarzschild solution** with the horizon located at $r_0' = 4r_0$, which can be seen using the transformation $r' = r\left(1 + \frac{r_0}{r}\right)^2$, since

$$\begin{aligned} r'^2 d\Omega^2 = \left(1 - \frac{r_0^2}{r^2}\right)^2 \mathcal{R}^{-2} r^2 d\Omega^2 \Rightarrow \\ r' = \frac{r+r_0}{r-r_0}\frac{r^2-r_0^2}{r} \Rightarrow \\ r' = \frac{(r+r_0)^2}{r} = r(1+\frac{r_0}{r})^2 \end{aligned} \tag{2.47}$$

and we get the usual exterior vacuum Schwarzschild solution with the horizon at $r_0' = 4r_0$.

For $0 \leq \lambda < 1$, the above solution is not a black-hole due to fact that the $r = r_0$ surface is not a horizon but **represents a naked singularity**. This can be verified by calculating the scalar of curvature

$$\bar{R} = \frac{8r_0^2 r^4}{(r+r_0)^{2(2+\lambda)}} \times \frac{1-\lambda^2}{(r-r_0)^{2(2-\lambda)}} \tag{2.48}$$

and observing that $r = r_0$ is a curvature singularity, whilst the sign of $g_{tt} = -\mathcal{R}^{2\lambda}$ never changes for $r > r_0$, and so in this case, no-black hole solution exist, since no-horizon is formed around the singularity.

Consequently, in total accordance with the original no-scalar-hair theorem, we see the only solution representing a black-hole of eq.(2.44,2.45) is the one for which $\lambda = 1$ and consequently $\bar{\phi} = 0$ from eq.(2.44), our usual Schwarzschild solution.

The properties of the conformal transformations we used for $\phi(r)$ lead us to the conclusion that a solution with finite ϕ in the $r = r_0$ surface is the one for which ϕ

should be constant for $r > r_0$. In this case, looking at 2.43, we see that $\bar{\phi} = 0 \Rightarrow \phi = \alpha$, for the integration to yield zero. So, the only valid solution is the $(g_{\mu\nu}, \phi = \alpha)$ and is the known Schwarzschild solution.

To sum up, we conclude that *the only asymptotically flat, static, and spherically symmetric exterior solution of the system governed by the action*

$$S = \int d^4x\sqrt{-g}\left\{f(\phi)R - h(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right\}$$

with $f(\phi), h(\phi) > 0$ and ϕ *finite everywhere is the Schwarzschild solution.*

A specific example of all the above is *the Brans-Dicke theory*, for which $f(\phi) = \phi$ and $h(\phi) = \frac{\omega}{\phi}$ which is described, at the original Jordan frame, by the action:

$$\mathcal{S}^J_{\text{BD}} = \int d^4x\sqrt{-\hat{g}}\left[\frac{1}{16\pi}\left(\varphi\hat{R} - \frac{\omega_0}{\varphi}\hat{\nabla}_\mu\varphi\hat{\nabla}^\mu\varphi\right)\right] \tag{2.49}$$

for which we can show that the only black-hole solution of with finite ϕ is the Schwarzschild one, a theory that has been generilised also to the case of $\omega = \omega(\phi)$ [6].

2.1.4 Time-dependent scalar fields

Following the arguments made by Graham and Jha in [14], let's consider a stationary, asymptotically flat, four-dimensional spacetime. We will also assume that the spacetime contains a scalar field and that the spacetime must also be axisymmetric. The metric takes the following form

$$ds^2 = -e^{\mu(r,\theta)}dt^2 + 2\rho(r,\theta)dtd\phi + e^{\nu(r,\theta)}d\phi^2 + e^{A(r,\theta)}dr^2 + e^{B(r,\theta)}d\theta^2 \tag{2.50}$$

Now, considering the existence of a scalar field, let's take the action

$$\mathcal{S} = \frac{1}{4\pi}\int d^4x\sqrt{-g}\left(\frac{R}{4} + P(\Phi, X)\right) \tag{2.51}$$

,where the action depends only on scalar field's first derivates and not to higher order ones, but it can contain a kinetic term which is non-canonical, with $X = -\frac{1}{2}\nabla_\alpha\Phi\nabla^\alpha\Phi$. Taking only the scalar field part of the action, $\mathcal{S}_\phi = \int d^4x\sqrt{-g}P(\Phi, X)$ and taking the equations of motion, we get

$$\begin{gathered}\frac{\partial P}{\partial\Phi} = \nabla_\alpha\frac{\partial P}{\partial X}\frac{\partial X}{\partial(\partial_\alpha\Phi)} \Rightarrow \\ \nabla_\alpha\left[\frac{\partial P}{\partial X}\nabla^\alpha\Phi\right] + \frac{\partial P}{\partial\Phi} = 0\end{gathered} \tag{2.52}$$

assuming that $\frac{\partial P}{\partial X} \neq 0$. The variation with respect to the metric for the scalar field part of the action yields for the energy momentum tensor that

$$
\begin{aligned}
T_{\alpha b} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = -\frac{2}{\sqrt{-g}}\frac{\delta[\sqrt{-g}\mathcal{L}]}{\delta g^{\mu\nu}} \\
= -\frac{2}{\sqrt{-g}}\frac{\delta[\sqrt{-g}P(\Phi,X)]}{\delta g^{\mu\nu}} = \\
-\frac{2}{\sqrt{-g}}\left(-\frac{1}{2}g_{\alpha b}\sqrt{-g}\frac{\delta g^{\mu\nu}}{\delta g^{\mu\nu}}P(\Phi,X)\right) - \frac{2}{\sqrt{-g}}\sqrt{-g}\frac{\delta P(\Phi,X)}{\delta g^{\mu\nu}} = \\
g_{\alpha b}P(\Phi,X) - 2\frac{\partial P(\Phi,X)}{\partial X}\frac{\delta X}{\delta g^{\mu\nu}} \Rightarrow \\
T_{\alpha b} = g_{\alpha b}P(\Phi,X) + \frac{\partial P(\Phi,X)}{\partial X}\nabla_\alpha\Phi\nabla_b\Phi
\end{aligned}
\tag{2.53}
$$

For the metric in eq.(2.50), we can compute that the components $(tr),(t\theta),(r\phi),(\theta\phi)$ of the Ricci tensor vanish

$$
R_{tr} = R_{t\theta} = R_{r\phi} = R_{\theta\phi} = 0 \tag{2.54}
$$

Now, with the above result. let's move to the Einstein's field equations $R_{ab} - \frac{1}{2}g_{ab}R = 8\pi T_{ab} \Rightarrow R_{ab} = 8\pi\left(T_{ab} - \frac{1}{2}g_{ab}T\right)$ which implies for the $(tr),(t\theta)$ components, that

$$
\begin{aligned}
R_{tr} = 8\pi T_{tr} - \frac{1}{2}g_{tr}T = 0 \Rightarrow T_{tr} = 0 \\
\Rightarrow g_{tr}P(\Phi,X) + \frac{\partial P(\Phi,X)}{\partial X}\nabla_t\Phi\nabla_r\Phi = 0 \Rightarrow \nabla_t\Phi\nabla_r\Phi = 0 \\
\text{and also } R_{t\theta} = 0 \Rightarrow \nabla_t\Phi\nabla_\theta\Phi = 0
\end{aligned}
\tag{2.55}
$$

It's clear from the above, keeping in mind $\frac{\partial P}{\partial X} \neq 0$, that considering Φ to be time dependent, then it cannot depend upon the coordinates r and θ, so $\Phi = \Phi(t,\phi)$. The following step is to see form eq.(2.52) that P can't depend on Φ after all, that $P = P(X)$. Also, from eq.(2.53) we argue that Φ should have a linear dependence of time t and that's because being otherwise, we would have components of energy-momentum tensor $T_{\alpha b}$ with explicit dependence on time. These assumption are important because if either of them was false, then some components of the energy-momentum tensor will have explicit time-dependence, which would alter the spacetime geometry from being stationary.

What is more, we can rule out the ϕ dependence of scalar field Φ. The axisymmetry of our spacetime implies that $T_{\alpha b}$ can only be consistent with spacetime axisymmetry if Φ has a linear dependence on ϕ, but ϕ being a periodic coordinate clearly would make Φ-field to be a no-continuous, no-single-valued function, since $\Phi = \alpha\phi + \beta$ would make $\Phi_{\phi=2\pi} \neq \Phi_{\phi=0}$. So we end up with the consideration of $\Phi = \Phi(t)$, scalar field only depends on time. We can now write the field as

$$
\Phi = \alpha t + \beta \tag{2.56}
$$

with α,β constants. We want now to show that the only compatible case with our assumptions is the one for which $\alpha = 0$. To do so, we return to our metric in eq.(2.50) and consider asymptotic flatness, $\mu(r,\theta), \nu(r,\theta) \to O(\frac{1}{r})$ as $r \to \infty$, so $g^{tt} \to -1$ as $r \to \infty$ and consequently, with the field given from (2.56) we have $X \to \frac{\alpha^2}{2}$. So, the $(tt), (rr)$ energy-momentum components behave asymptotically as

$$T_{tt} \to\sim \alpha^2, \quad T_{rr} \to\sim \alpha^2 \tag{2.57}$$

For our spacetime to be asymptotically flat, these must tend to zero, which is satisfied, for non-pathological solutions, when $\alpha = 0$, since for $\alpha \neq 0$ and with the assumption $\frac{\partial P}{\partial X} \neq 0$ we have no-solution and the black cannot support scalar, time dependent hair. For $\alpha = 0$, the scalar field Φ is constant, something that ensures that the only case for the field to exist in the black hole background, is when $\Phi = \beta$, the field being constant and the black hole possesses no scalar hair in this case.

We want to emphasize that this theorem does not apply to more than one scalar fields, or to one or more complex scalar fields. This can be seen by considering Φ to be complex, the scalar field part of the action in (2.51) and equations in (2.55) would be replaced by

$$\begin{gathered} \mathcal{S} = \int d^4x\sqrt{-g}P(|\Phi|^2, Y) \\ \partial_{(t}\Phi^*\partial_{r)}\Phi = 0, \quad \partial_{(t}\Phi^*\partial_{\theta)}\Phi = 0 \end{gathered} \tag{2.58}$$

where $|\Phi|^2 = \Phi^*\Phi$ and $Y = -\nabla^\alpha\Phi^*\nabla_\alpha\Phi$ where eq.(2.58) no longer implies $\partial_r\Phi = 0$ and $\partial_\theta\Phi = 0$ if we have $\partial_t\Phi \neq 0$.

2.1.5 Examples of hairy black hole solutions

We may now continue with explicit examples of how, in multiple cases, hairy solutions may occur and discuss how they would behave, depending on some physical assumptions we have to reconsider. We will sketch some examples for the cases where scalar hair may occur, based on the violations of some previous assumptions, following the arguments and the discussion made in [4].

Scalar field's Potential not strictly positive:

As a simple example, let's consider the electrostatic potential of eq.(2.5) $\Phi(r) = -\frac{Q_S}{r}$ and considering the action of the form

$$\mathcal{S} = \frac{1}{4\pi}\int d^4x\sqrt{-g}\left(\frac{R}{4} - \frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi - V(\Phi)\right) \tag{2.59}$$

we have from the Klein-Gordon equation that

$$\nabla_\mu\nabla^\mu\Phi - V'(\Phi) = 0 \tag{2.60}$$

where prime denotes differentiation with respect to Φ. As we can verify, the above is a solution for a quintic potential of the form

$$V(\Phi) = -\lambda\Phi^5 < 0 \tag{2.61}$$

We have considered a test scalar field in the Schwarzschild black hole background, so we see that the verification of eq.(2.60) yields

$$Q_S = \left(\frac{2M}{5\lambda}\right)^{1/3} < 0 \tag{2.62}$$

where for $Q_S \to \infty$, $\lambda \to 0$. In particular, what we did now is that we constructed a smooth configuration which, as can be checked, gives an energy-momentum tensor which is regular at the horizon, and this happens because of the form of the potential in eq.(2.61).

Conformal scalar vacuum:

We are interested in possible black hole solutions which support scalar hair, that can continuously connect to the Schwarzschild solution. A simple question to pose is whether there could be a hairy black hole that cannot connect to the Schwarzschild one, with a non-diverging energy momentum tensor at the horizon. We consider the conformal scalar vaccum action given by

$$\mathcal{S} = \frac{1}{4\pi}\int d^4x\sqrt{-g}\left(\frac{R}{4} - \frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi - \frac{1}{12}R\Phi^2\right) \tag{2.63}$$

The field equation for the scalar field is given by $\nabla_\mu\nabla^\mu\Phi - \Phi R/6 = 0$ and is invariant under a local-conformal transformation, $g_{\mu\nu} \to \hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ and $\Phi \to \hat{\Phi} = \Phi/\Omega$. It is a special case of the theories we discussed in the subsection (2.1.3) for scalar-tensor theories. A solution for this case has been found and has been widely discussed since. It's the Bocharova-Bronnikov-Melnikov-Bekenstein (BBMB) solution [9, 15], and it's given by a one-parameter family of solutions

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \frac{dr^2}{(1 - M/r)^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.64}$$

$$\Phi = \frac{\sqrt{3}M}{r - M} \tag{2.65}$$

The parameter is obvisouly the mass M of the black hole. For $M = 0$ we recover the Minkowski spacetime. For any other value of M we get the extremal Reissner-Nordström black hole (see 1.2), where $|Q_E| = M$, with a regular horizon at $r = M$. This is a hairy solution which does not connect to Schwarzschild one. Moreover, the scalar field diverges at the horizon, even though the geometry is regular therein.

- This kind of hair is called *secondary hair*, and it has to do with the fact that there is no new independant parameter, since our scalar field Φ depends again to one of the initial black hole parameters, the mass.
- The scalar hair would be called *primary hair* if it introduced a new, independent parameter in order to describe the black hole solution.
- Secondary hair still have really important effects to the resulting geometry, due to its backreaction. Secondary hair, therefore, has physical consequences since it can induce a black hole geometry different from those of general relativity's black holes.

Scalar fields coupled to terms of higher order curvature:

A general theory of gravity, including all quadratic curvature invariants coupled to a single scalar field would be written in the form

$$\mathcal{S} = \int d^4x\sqrt{-g}\Big(f_1(\Phi)R^2 + f_2(\Phi)R_{\mu\nu}R^{\mu\nu} + f_3(\Phi)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + f_4(\Phi)R_{\mu\nu\rho\sigma}\,{}^*R^{\mu\nu\rho\sigma}\Big) \tag{2.66}$$

where ${}^*R_{\mu\nu\rho\sigma}$ is the dual Riemann tensor, as we will see again in Section (3). Such theories are motivated from very fundamental physics, such as in low-energy expansions of string theory etc. Something intresting in these cases is that the modified Einstein equations give us an effective energy momentum tensor that involves an extra contribution due to the form of the action in eq.(2.66). This effective energy momentum tensor often violates some energy condition, as we will discuss later. The violation of these energy conditions is probably an important indication for the existence or not of hairy solutions. In other words, the existence of these couplings with the higher order curvature terms acts as a source in the resulting Klein-Gordon equation from the field's equation of motion, which frequently gives birth to no-hair theorems.

The general form of the action (2.66) reduces to known theories, choosing special values for the parameters. For example, for $f_1 = \alpha e^{\gamma\Phi}$, $f_2 = -4f_1$ and $f_3 = f_1$ we get the the *Einstein-Gauss-Bonnet-dilaton model*, while for $f_1 = f_2 = f_2 = 0$ and $f_4 = \alpha\Phi$ we recover the *Chern-Simons gravity*, which would be the main focus of this thesis later on.

2.2 Energy Conditions and Hairy Black Holes

As we said previously, the original form of the the Bekenstein's work [2] concerns an arbitrary number of coupled scalar fields, minimally coupled to gravity, and at most coupled the one with the other. However, now we will refer only to the properties of the corresponding effective energy-momentum tensor, without reference to the Lagrangian density that produces such a $T_{\mu\nu}^{eff}$. This subtraction aims to make our considerations quite more general, since any Lagrangian density that gives an energy-momentum tensor with some kind of properties, will be excluded. We proceed by presenting the basic features of energy conditions, and their physical intuition. We will also discuss the relation between the violations of these energy conditions in multiple cases and the existence or not of hair. There is a close relation between the energy conditions and the existence of hairy black holes. Every hairy solution that has been proposed so far, violates these energy contitions, something that challenges us to examine what's the underlying physics of such a behaviour.

Energy conditions

With the term of "energy condition", we roughly speak about a relation we demand for the energy-momentum tensor of matter to satisfy in order to capture the idea of "positive energy". We demand, in other words, "reasonable" effects on the underlined geometry due to the existance of matter and energy. Such a simple and almost trivial idea has great effect on our understanding of the structure of spacetimes. Sadly or not, we have not a clear or profound understanding of the nature of such conditions, what fundamental physics are involved or when they should be satisfied and when they should not. It's important to realize that the study of energy conditions and their violations in general relativity physics can be proved very fruitful in multiple cases, regarding some conceptual apsects of our theory. We move on with the energy conditions, following [3], with some proofs about the origin of the arguments we present.

We consider an energy-momentum tensor $T_{\mu\nu}$ and also the gravitational field equations

$$R_{\mu\nu} = \kappa^2 \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \tag{2.67}$$

The energy conditions are the following:

- **Weak Energy Condition (WEC):** The energy density measured by any observer with a timelike four-velocity t^μ, is non-negative, which formally can be expressed as

$$T_{\mu\nu} t^\mu t^\nu \geq 0, \ \forall \ t \ : \ t^\mu t_\mu < 0 \tag{2.68}$$

- **Null Energy Condition (NEC):** Expresses the requirement that the geometry has a focusing (attractive) effect on null geodesics,

$$T_{\mu\nu} l^\mu l^\nu \geq 0, \ \forall \ l \ : \ l^\mu l_\mu = 0 \tag{2.69}$$

  with l^μ any null four-vector. This is a generalization of the weak energy condition. The energy density may now be negative as long as there is a compensating positive pressure.

- **Strong Energy Condition (SEC):** Expresses the requirement that the geometry has a focusing (attractive) effect on timelike geodesics.

$$T_{\mu\nu}t^{\mu}t^{\nu} \geq \frac{1}{2}g_{\mu\nu}Tt^{\mu}t^{\nu}, \ \forall \ t \ : \ t^{\mu}t_{\mu} < 0 \tag{2.70}$$

or, from Einstein equation, $R_{\mu\nu}t^{\mu}t^{\nu} \geq 0$.

- **Dominant Energy Condition (DEC):** This energy condition refers to the current density $J^{\alpha} = -T^{\alpha}_{\ \beta}t^{\beta}$, (the energy-momentum current seen by an observer with 4-velocity t^{μ}). This energy condition expresses essentially the statement that the speed of the flow of energy cannot exceed that of light. So, J^{α} should remain causal for all future directed timelike vectors t^{μ}.

$$T_{\mu\nu}t^{\mu}t^{\nu} \geq 0, \ and \ T_{\mu\nu}T^{\mu}_{\ \alpha}t^{\nu}t^{\alpha} \leq 0, \ \forall \ t \ : \ t^{\mu}t_{\mu} < 0 \tag{2.71}$$

Note that, the first restriction is just the weak energy condition, while the second one ensures the causal structure of J_{α}.

We can see that, the above conditions satisfy some specific relations among them, meaning that the one entails the other. With simle arguments about continuity to the limiting values of the above inequalities, we get that :

- *DEC* $\longrightarrow$ *WEC* $\longrightarrow$ *NEC*
- *SEC* $\longrightarrow$ *NEC*

while, usinh the same arguments, we find that the violation of the Null Energy Condition implies the violation of all energy conditions:

- *NEC violation* $\longrightarrow$ *DEC,SEC,WEC violation*

Raychadhuri equations: Now, let's make a step back, and briefly discuss about the origin of these conditions, especially for the NEC and SEC [16]. We start with a *one-parameter family of geodesics*, defining U which are denoted as **the tangent vectors** of the geodesics and S, a vector *tangent to the curves of constant parameter* λ, which is called **the deviation vector**. We can easily imply that the Lie bracket $[U, S] = 0$ which is translated as $U^{\beta}\nabla_{\beta}S^{\alpha} = S^{\beta}\nabla_{\beta}U^{\alpha}$. Defining now a new tensor, $B_{\alpha\beta} = \nabla_{\beta}U_{\alpha}$, which describes the evolution and the deformation of the deviation vector along the geodesic, measuring in other words the failure of S to be parallelly transported along the geodesic, we can find from the geodesic equation (with U being affinely parametrized) that:

$$\begin{aligned} U^{\beta}\nabla_{\beta}U^{\alpha} = 0 \Rightarrow U^{\beta}B^{\alpha}_{\ \beta} = 0 \quad and \\ \frac{1}{2}\nabla_{\beta}(U^{\alpha}U_{\alpha}) = U^{\alpha}B_{\alpha\beta} = 0 \end{aligned} \tag{2.72}$$

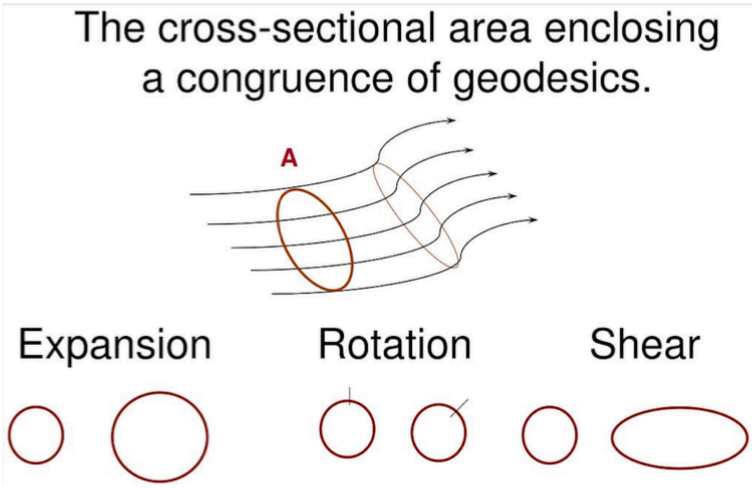

Figure 2.2: Pictorial representation of a geodesic congruence and of shear, expansion and rotation.

The relative acceleration of the geodesics is given by

$$\begin{aligned} A^{\alpha} &= U^{\rho}\nabla_{\rho}(U^{\sigma}\nabla_{\sigma}S^{\alpha}) \; where \; after \; some \; algebra \\ A^{\alpha} &= U^{\rho}(\nabla_{\rho}\nabla_{\sigma}U^{\alpha} - \nabla_{\sigma}\nabla_{\rho}U^{\alpha})S^{\sigma} \rightarrow \\ A^{\alpha} &= R^{\alpha}{}_{\beta\rho\sigma}U^{\beta}U^{\rho}S^{\sigma} \end{aligned} \tag{2.73}$$

where the last line gives the geodesic deviation equation: the relative acceleration between two neighboring geodesics should be proportional to the curvature. We continue by considering a timelike geodesic congruence ($U^2 = -1$), and expand our tensor $B_{\mu\nu}$ as

$$B_{\mu\nu} = \frac{\theta}{3}P_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} \tag{2.74}$$

where $P_{\mu\nu}$ is the projection tensor, $\sigma_{\mu\nu} = B_{(\mu\nu)} - \frac{\theta}{3}P_{\mu\nu}$ and $\omega_{\mu\nu} = B_{[\mu\nu]}$. The trace of $B_{\mu\nu}$ is given by $B_{\mu\nu}g^{\mu\nu} = \theta$. The scalar θ is defined as the divergence of U, or as is known, *the expansion of the congruence*. A positive θ value yields that the congruence is expanding, while a negative one says that the congruence is contracting. $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ give the *shear* and the *twist* of the congruence respectively, as shown in Figuer 2.2. To obtain the Raychadhuri equation for the timelike case we are studying, we want to calculate the evolution of $B_{\mu\nu}$ with respect to an affine parameter λ, which yields the generilized form of the Raychadhuri equations:

$$\frac{d}{d\lambda}B_{\mu\nu} = -B^{\rho}{}_{\nu}B_{\mu\rho} - R_{\mu\lambda\nu\rho}U^{\lambda}U^{\rho} \tag{2.75}$$

where, taking the trace of eq.(2.75), we get

$$\frac{d}{d\lambda}\theta = -\frac{\theta^2}{3} - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu} - R_{\mu\nu}U^{\mu}U^{\nu} \tag{2.76}$$

In the above equation we have the vanishing of the twist $\omega_{\mu\nu}$ using the Forbenius theorem $U_{[\alpha}\nabla_{\beta}U_{\gamma]} = 0$ and also that the shear $\sigma_{\mu\nu}$ has a spatial nature and so $\sigma_{\mu\nu}\sigma^{\mu\nu} \geq$

0. The gravitation acts as an attractive force if the strong energy condition holds, and the geodesics get focused as a result of this attraction ($\frac{d}{d\lambda}\theta = -R_{\mu\nu}U^{\mu}U^{\nu}... \leq 0$). This can be translated to $R_{\mu\nu}U^{\mu}U^{\nu} \geq 0$ **which is the geometric expression of the Strong Energy Condition**. The same procedure can be done for null geodesics, and for a focusing effect on these null geodesics we will have $R_{\mu\nu}l^{\mu}l^{\nu} \geq 0$, where $l^{\mu}l_{\mu} = 0$, ***yielding the geometric expression of the Null Energy Condition***.

Moving on, as we said at the start of this chapter, we aim to make our considerations more general, studying only the properties of the energy-momentum tensor $T_{\mu\nu}$ and not considering some specific action. The assumptions of Bekenstein in [2], which we presented analytically in (2.1.2), rule out a large class of coupled scalar fields, minimally coupled to gravity. The assumptions in [2] by Bekenstein, assume that the scalar fields construct a conserved energy-momentum tensor, which defines a non-negative energy density as measured by any observer of any timelike four-velocity, in other words the *Weak Energy Condition (WEC) is satisfied.* We want to check what this means for the components of $T_{\mu\nu}$ without refering to a specific Lagrangian density at all. The assumptions we are going to consider are:

- an asymptotically flat solution of the Einstein and scalar field equations with characteristics of a static, spherically symmetric black hole. The metric outside the horizon may be taken as
$$ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta} = -e^{v}dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{2.77}$$
- Assymptotic flatness: $v = v(r), \lambda = \lambda(r) = O(1/r)$ as $r \to \infty$
- Nontrivial scalar field: $\psi = \psi(r)$ and $\chi = \chi(r)$
- Event horizon at $r = r_h$ where $e^{v(r_h)} = e^{-\lambda(r_h)} = 0$
- $T_{\mu\nu}$ should be finite at $r = r_h$ and at $r \to \infty$
- From WEC validation : $-T^t_t = \mathcal{E} \geq 0$
- From NEC validation : $\mathcal{J} = \mathcal{E} + T^r{}_r \geq 0$ and $\mathcal{G} = \mathcal{E} + T^{\theta}{}_{\theta} = \mathcal{E} + T^{\phi}{}_{\phi} = 0$

We consider a special case of non-violation of the NEC according to the tangential direction, as we assume $\mathcal{G} = 0$ and not the general case of $\mathcal{G} \geq 0$ (we will talk analytically in Section (2.2.1) about when and how $\mathcal{G} \neq 0$). Like before, let us write the conservation of an effective energy momentum tensor as
$$\nabla_{\nu}T_{\mu}{}^{\nu} = \frac{1}{\sqrt{-g}}\partial_{\lambda}(\sqrt{-g}T_{\mu}{}^{\lambda}) - \frac{1}{2}(\partial_{\mu}g_{\alpha\beta})T^{\alpha\beta} = 0 \tag{2.78}$$
Looking back to eqs.(2.27,2.28), we have
$$T_r{}^r(r) = -\frac{e^{-\frac{v}{2}}}{r^2}\int_{r_h}^{r}(e^{\frac{v}{2}}r^2)'\mathcal{E}dr$$
$$(T_r{}^r)' = -e^{-\frac{v}{2}}r^{-2}(e^{\frac{v}{2}}r^2)'(\mathcal{E} + T_r{}^r) = -e^{-\frac{v}{2}}r^{-2}(e^{\frac{v}{2}}r^2)'\mathcal{J}$$

Studying again the sign of the above expressions near the horizon $r = r_h$ and asymptotically, we find that:

<u>Near the horizon</u>: From the arguments about asymptotic flatness and finiteness of $T_{\mu\nu}$ we have:

$$\begin{aligned} T_r{}^r \leq 0, \quad if \quad \mathcal{E} \geq 0, \quad T_r{}^r \geq 0, \quad if \quad \mathcal{E} \leq 0 \\ (T_r{}^r)' \leq 0, \quad if \quad \mathcal{J} \geq 0, \quad (T_r{}^r)' \geq 0, \quad if \quad \mathcal{J} \leq 0 \end{aligned} \tag{2.79}$$

We see that for $\mathcal{J} \leq 0$ we have a violation of NEC, while for $\mathcal{J} \geq 0$ the NEC still holds.

<u>Asymptotically</u>: Again, from asymptotic flatness we get that, $(T_r{}^r)'$ asympototically behaves as

$$(T_r{}^r)' \approx -\frac{2}{r}\mathcal{J} \tag{2.80}$$

so, asymptotically

$$\begin{aligned} T_r{}^r \geq 0, \quad if \quad \mathcal{J} \geq 0, \quad T_r{}^r \leq 0, \quad if \quad \mathcal{J} \leq 0 \\ (T_r{}^r)' \leq 0, \quad if \quad \mathcal{J} \geq 0, \quad (T_r{}^r)' \geq 0, \quad if \quad \mathcal{J} \leq 0 \end{aligned} \tag{2.81}$$

where, similarly, for $\mathcal{J} \leq 0$ we have a violation of NEC, while for $\mathcal{J} \geq 0$ the NEC still holds.

In the subsection (2.1.2) we found that the (rr) component of *Einstein's field equations* is given by

$$\frac{e^{-\frac{v}{2}}}{r^2}(e^{\frac{v}{2}}r^2)' = 4\pi G r T_r{}^r e^{\lambda} + \frac{e^{\lambda}+3}{2r} \tag{2.82}$$

Substituting to the left hand side of the above relation the equation for $(T_r{}^r)'$ which yields $(T_r{}^r)' = -e^{-\frac{v}{2}}r^{-2}(e^{\frac{v}{2}}r^2)'\mathcal{J}$, we get

$$-\frac{(T_r{}^r)'}{\mathcal{J}} = 4\pi G r T_r{}^r e^{\lambda} + \frac{e^{\lambda}+3}{2r} \tag{2.83}$$

We mention here that the quantity $\mathcal{G} = \mathcal{E} + T_\theta{}^\theta = 0$, something that has to do with the radial nature of the fields and the minimally coupling with the gravity (we will see later on how this assumption can be relaxed). Let's see what are the possible configurations of this analysis, looking closer to the behaviour of eq.(2.83).

⇒*<u>No NEC violation:</u>* $\mathcal{J} \geq 0 \longrightarrow$ We have two cases for the sign of $(T_r{}^r)'$

1. $(T_r{}^r)' > 0 \rightarrow T_r{}^r < -e^{-\lambda}\frac{e^{\lambda}+3}{8\pi G r^2} \rightarrow T_r{}^r$ cannot have positive values
2. $(T_r{}^r)' < 0 \rightarrow T_r{}^r > -e^{-\lambda}\frac{e^{\lambda}+3}{8\pi G r^2} \rightarrow T_r{}^r$ can have every value.

⇒<u>*NEC violation:*</u> $\mathcal{J} \leq 0 \longrightarrow$ Again we have two cases for the sign of $({T_r}^r)'$

1. $({T_r}^r)' > 0 \rightarrow {T_r}^r > -e^{-\lambda}\frac{e^{\lambda}+3}{8\pi G r^2} \rightarrow {T_r}^r$ can have every value
2. $({T_r}^r)' < 0 \rightarrow {T_r}^r > -e^{-\lambda}\frac{e^{\lambda}+3}{8\pi G r^2} \rightarrow {T_r}^r$ cannot have positive values

Now, let's see the four combinations of the sings of $\mathcal{E}, \mathcal{J}$:

- *First case:* $\mathcal{E} > 0, \quad \mathcal{J} > 0$, which translates to

$$\begin{aligned} Near\ the\ horizon &: ({T_r}^r)' \leq 0\ and\ {T_r}^r \leq 0 \\ Asymptotically &: ({T_r}^r)' \leq 0\ and\ {T_r}^r \geq 0 \end{aligned} \tag{2.84}$$

  which is exactly the case we talked about in subsection 2.1.2 for Bekenstein's approach, which rules out a large class of scalar hair. So, in this case, hair does not exist.

- *Second case:* $\mathcal{E} > 0, \quad \mathcal{J} < 0$, which translates to

$$\begin{aligned} Near\ the\ horizon &: ({T_r}^r)' \geq 0\ and\ {T_r}^r \leq 0 \\ Asymptotically &: ({T_r}^r)' \geq 0\ and\ {T_r}^r \leq 0 \end{aligned} \tag{2.85}$$

  In this case, there is no forbidden region in order for $T^r{}_r$ to be smoothly connected, which means that this case allows hair to exist.

- *Third case:* $\mathcal{E} < 0, \quad \mathcal{J} > 0$, which translates to

$$\begin{aligned} Near\ the\ horizon &: ({T_r}^r)' \leq 0\ and\ {T_r}^r \geq 0 \\ Asymptotically &: ({T_r}^r)' \leq 0\ and\ {T_r}^r \geq 0 \end{aligned} \tag{2.86}$$

  In this case, there is no NEC violation, which means that again there is no restriction on the sign of $T^r{}_r$ as $(T^r{}_r)' \leq 0$ everywhere. This case also allows hair to exist.

- *Fourth case:* $\mathcal{E} < 0, \quad \mathcal{J} < 0$, which translates to

$$\begin{aligned} Near\ the\ horizon &: ({T_r}^r)' \geq 0\ and\ {T_r}^r \geq 0 \\ Asymptotically &: ({T_r}^r)' \geq 0\ and\ {T_r}^r \leq 0 \end{aligned} \tag{2.87}$$

  In this case, despite the NEC violation, for a smooth connection of $T^r{}_r$ we must have at least on region where $T^r{}_r > 0$ and $(T^r{}_r)' < 0$, (similar arguments as in Bekenstein's novel no-scalar-hair theorem, see (2.1.2)). So, in this case, hair does not exist.

Let's sum up the results (see Figures 2.3,2.4,2.5,2.6):

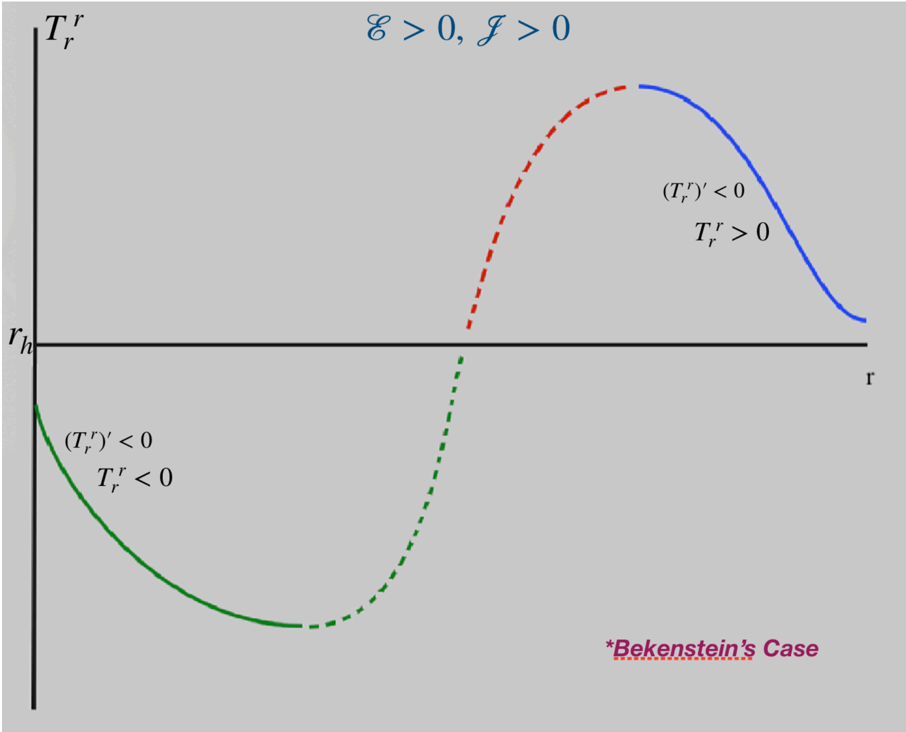

Figure 2.3: $\mathcal{E} > 0$ and $\mathcal{J} > 0$: This is just the Bekenstein case (red dotted lines→forbidden interval).

$\mathcal{E} > 0$ and $\mathcal{J} > 0$	Hair not allowed
$\mathcal{E} > 0$ and $\mathcal{J} < 0$	Hair could be allowed
$\mathcal{E} < 0$ and $\mathcal{J} > 0$	Hair could be allowed
$\mathcal{E} < 0$ and $\mathcal{J} < 0$	Hair not Allowed

The *first case*, is just Bekenstein's approach and no hair is allowed. *In the second case*, there is NEC violation, and consequently WEC is also violated. However, $\mathcal{E} \geq 0$, which means that the energy density is non-negative for the static observer, but *it is negative for sure for fastly moving observers, by means of continuity, since the NEC is violated.* In this case, hair can be allowed. *For the third case*, there is no NEC violation, but as $\mathcal{E} \leq 0$, there is a WEC violation. In this case, hair could be allowed. *For the last case*, NEC and WEC is violated, with negative energy density for every observer, so hair is not allowed. It seems that hair can be supported in cases where the static and the fastly moving observers have a disagreement about the sign of the energy-density $\mathcal{E}$.

Let us remind that for all the above, we have two basic assumption: the finiteness of the energy-momentum tensor $T_{\mu\nu}$ and also that the angular part of the NEC is fixed to be equal to zero ($\mathcal{G} = 0$) and it does not violate the energy condition. We could be much more thorough and consider the cases where $\mathcal{G}$ is not fixed or aboandon the

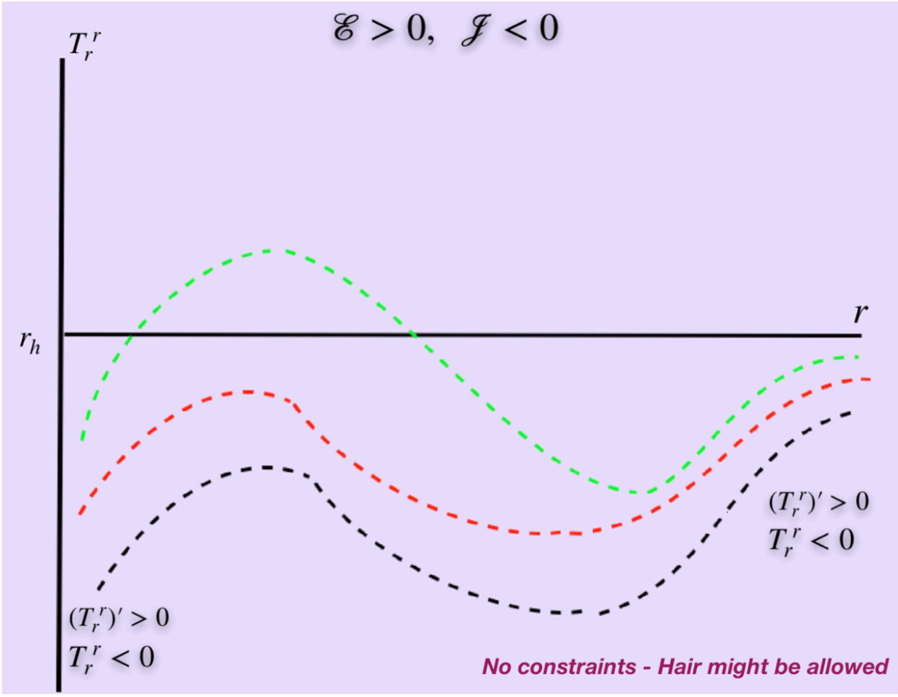

Figure 2.4: $\mathcal{E} > 0$ and $\mathcal{J} < 0$: This is a case where hair might be allowed since no constraints for the sign of T_r^r occured.

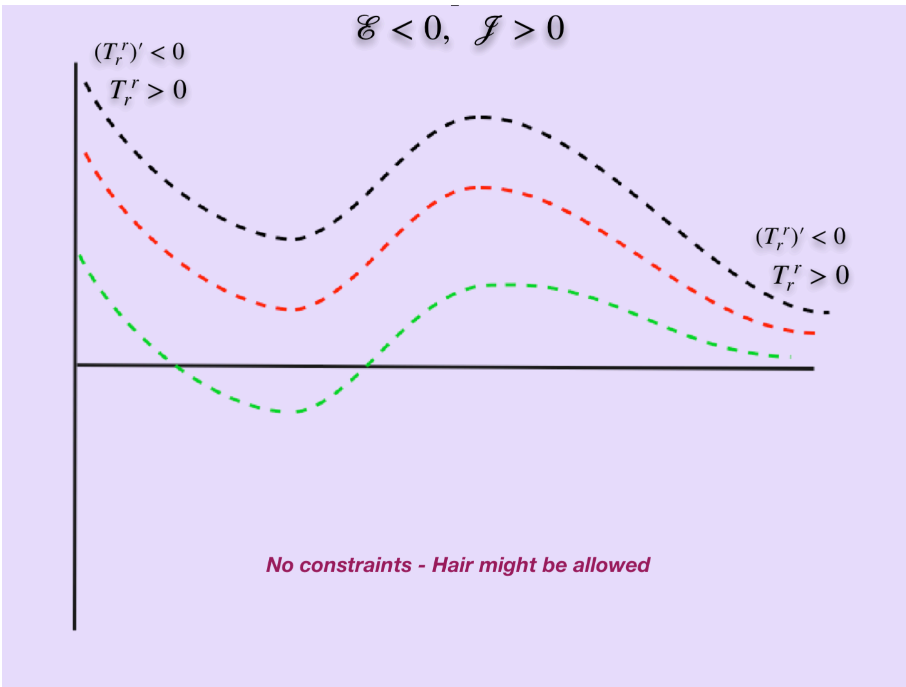

Figure 2.5: $\mathcal{E} < 0$ and $\mathcal{J} > 0$: This is a case where hair might be allowed since no constraints for the sign of T_r^r occured.

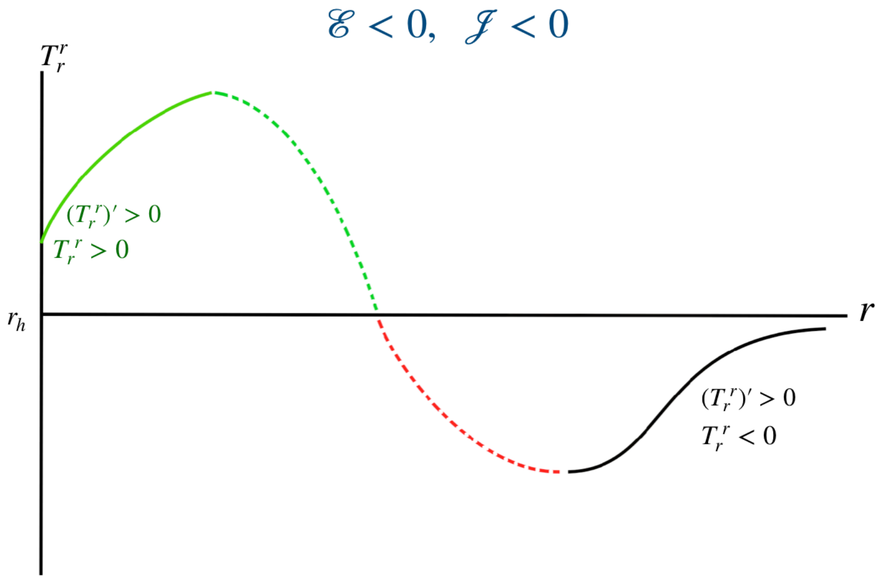

Figure 2.6: $\mathcal{E} < 0$ and $\mathcal{J} < 0$: This is a case where hair is not allowed, in the same way as the Bekenstein's case (red dotted lines→forbidden interval).

assumptions about spherical symmetry and/or asympotitical flatness.

2.2.1 Field coupled to higher order curvature terms

Let's continue now with the same procedure, **but now considering that** $\mathcal{G} = \mathcal{E} + \mathbf{T}_\theta{}^\theta \neq \mathbf{0}$. The case of $\mathcal{G} = 0$ in Bekenstein's case is based on the fact that the fields in this case are completely radial, where by a simple check of equation 2.18, we see that $\mathcal{G} = 0$. This is not the case if the field has an angular dependance, for example if we have a θ- dependance of the field, or if we have to deal with a (pseudo-)scalar field coupled to higher order curvature terms, where the effective energy momentum tensor does not respect the equality $\mathcal{G} = 0$. In the analysis below, since we assume spherical symmetry, the $\theta-$ dependance would require a stationary and axisymmetric solution, so we interpret this analysis for cases where we have higher order corrections, with our fields coupled to higher order curvature terms.

The case of $\mathcal{G} \neq 0$ is an extra relaxation of the previous analysis and we start again in the same way as in Bekenstein's analysis, from the covariant divergence of an effective EMT which now satisfies $\mathcal{G} \neq 0$.

Starting from eq.(2.25), we have that

$$\begin{aligned}
\left(e^{\frac{v}{2}}r^2 T_r{}^r\right)' - \frac{1}{2}e^{\frac{v}{2}}r^2\left[v'T_t{}^t + \frac{4}{r}T_\theta{}^\theta\right] = 0 \Rightarrow \\
\left(e^{\frac{v}{2}}r^2 T_r{}^r\right)' = \frac{1}{2}e^{\frac{v}{2}}r^2\left[v'T_t{}^t + \frac{4}{r}T_t{}^t - \frac{4}{r}T_t{}^t + \frac{4}{r}T_\theta{}^\theta\right] \Rightarrow \\
\left(e^{\frac{v}{2}}r^2 T_r{}^r\right)' = \frac{1}{2}e^{\frac{v}{2}}r^2\left[-\mathcal{E}(v' + \frac{4}{r}) + \frac{4}{r}(\mathcal{E} + T_\theta{}^\theta)\right] \Rightarrow \\
\left(e^{\frac{v}{2}}r^2 T_r{}^r\right)' = -\left[e^{\frac{v}{2}}r^2\right]'\mathcal{E} + 2e^{\frac{v}{2}}r\mathcal{G} \Rightarrow \\
e^{\frac{v}{2}}r^2 (T_r{}^r)' = -\left[e^{\frac{v}{2}}r^2\right]'(\mathcal{E} + T_r{}^r) + 2e^{\frac{v}{2}}r\mathcal{G} \Rightarrow \\
(T_r{}^r)' = -\frac{e^{-\frac{v}{2}}}{r^2}\left[e^{\frac{v}{2}}r^2\right]'(\mathcal{E} + T_r{}^r) + \frac{2}{r}\mathcal{G} \Rightarrow \\
(T_r{}^r)' = -\left(\frac{2}{r} + \frac{v'}{2}\right)\mathcal{J} + \frac{2}{r}\mathcal{G}
\end{aligned} \tag{2.88}$$

Integrating now at the exterior region, and remembering that $e^{v(r_h)} = 0$, we get:

$$\begin{aligned}
\left(e^{\frac{v}{2}}r^2 T_r{}^r\right)' = -\left[e^{\frac{v}{2}}r^2\right]'\mathcal{E} + 2e^{\frac{v}{2}}r\mathcal{G} \Rightarrow \\
T_r{}^r = -\frac{e^{-\frac{v}{2}}}{r^2}\int_{r_h}^{r}\left[\left[e^{\frac{v}{2}}r^2\right]'\mathcal{E} - 2e^{\frac{v}{2}}r\mathcal{G}\right]dr
\end{aligned} \tag{2.89}$$

Before moving on, let's write down the $(tt), (rr)$ and $(\theta\theta)$ components of the Einstein's field equations, where the metric is obviously given as before, by $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -e^v dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$. So we have

$$\begin{aligned}
(tt)\ component\ &: e^{-\lambda}\left(r^{-2} - r^{-1}\lambda'\right) - r^{-2} = 8\pi G T_t{}^t = -8\pi G\mathcal{E} \\
(rr)\ component\ &: e^{-\lambda}\left(r^{-1}v' + r^{-2}\right) - r^{-2} = 8\pi G T_r{}^r \\
(\theta\theta)\ component\ &: \frac{e^{-\lambda}\left(-2\lambda' + 2rv'' + (2 - r\lambda')v' + rv'^2\right)}{4r} = 8\pi G T_\theta{}^\theta
\end{aligned} \tag{2.90}$$

As we have already mentioned, **for the asymptotic behaviour** of $\mathcal{E}, T_r{}^r, T_\theta{}^\theta$, given that asymptotically $v, \lambda = O(\frac{1}{r})$, we can see from the above equations (2.90) that

$$\begin{aligned}
\mathcal{E} &\sim O(r^{-3}) \\
T_r{}^r &\sim O(r^{-3}) \\
T_\theta{}^\theta &\sim O(r^{-3})
\end{aligned} \tag{2.91}$$

and consequantly

$$\begin{aligned}
\mathcal{E} &\sim O(r^{-3}) \\
\mathcal{J} &\sim O(r^{-3}) \\
\mathcal{G} &\sim O(r^{-3})
\end{aligned} \tag{2.92}$$

Before moving on, it's also important to mention that, **near the horizon**, the function $v(r)$ has a divergent derivative for $r = r_h$, as we can check, since near the r_h, $v'(r) \sim \frac{1}{r-r_h}$ which means that $v' >> 1$ for $r \to r_h$.

Near the horizon

From eq.(2.88), we get for $(T_r{}^r)'$, remembering that $v'(r) >>$ for $r \to r_h$, that

$$\begin{aligned}
(T_r{}^r)' &= -\frac{e^{-\frac{v}{2}}}{r^2}\left[e^{\frac{v}{2}}r^2\right]'(\mathcal{E} + T_r{}^r) + \frac{2}{r}\mathcal{G} \Rightarrow \\
(T_r{}^r)' &= -\left(\frac{2}{r} + \frac{v'}{2}\right)\mathcal{J} + \frac{2}{r}\mathcal{G} \Rightarrow \\
(T_r{}^r)' &\simeq -\frac{v'}{2}\mathcal{J} \Rightarrow \\
(T_r{}^r)' &= -\frac{v'}{2}(\mathcal{E} + T_r{}^r) \Rightarrow \\
(T_r{}^r)' &= \frac{v'}{2}\left(T_t{}^t - T_r{}^r\right)
\end{aligned} \tag{2.93}$$

and also for $T_r{}^r$ we find from eq.(2.89) that (remembering $e^{v(r)} \to 0$ for $r \to r_h$)

$$\begin{aligned}
T_r{}^r &= -\frac{e^{-\frac{v}{2}}}{r^2}\int_{r_h}^{r}\left[\left[e^{\frac{v}{2}}r^2\right]'\mathcal{E} - 2e^{\frac{v}{2}}r\mathcal{G}\right]dr \Rightarrow \\
T_r{}^r &\simeq -\frac{e^{-\frac{v}{2}}}{r^2}\int_{r_h}^{r}\frac{v'r^2}{2}e^{\frac{v}{2}}\mathcal{E}dr \Rightarrow \\
T_r{}^r &\simeq \frac{e^{-\frac{v}{2}}}{r^2}\int_{r_h}^{r}\frac{v'r^2}{2}e^{\frac{v}{2}}T_t{}^t dr
\end{aligned} \tag{2.94}$$

Asymptotically

Again, making use of the asymptotic behaviour of our components, looking at equations (2.91, 2.92), we have for $(T_r{}^r)'$ that:

$$\begin{aligned}
(T_r{}^r)' &= -\frac{e^{-\frac{v}{2}}}{r^2}\left[e^{\frac{v}{2}}r^2\right]'\mathcal{J} + \frac{2}{r}\mathcal{G} \Rightarrow \\
(T_r{}^r)' &= -\left(\frac{2}{r} + \frac{v'}{2}\right)\mathcal{J} + \frac{2}{r}\mathcal{G} \Rightarrow \\
(T_r{}^r)' &\simeq \frac{2}{r}(\mathcal{G} - \mathcal{J}) \Rightarrow \\
(T_r{}^r)' &\simeq \frac{2}{r}\left(T_\theta{}^\theta - T_r{}^r\right)
\end{aligned} \tag{2.95}$$

since, asymptotically, $v' \sim O(r^{-2})$. For ${T_r}^r$, we get

$$
\begin{aligned}
{T_r}^r &= -\frac{e^{-\frac{v}{2}}}{r^2}\int_{r_h}^{r}\left[\left(\frac{2}{r}+\frac{v'}{2}\right)\mathcal{E}-\frac{2}{r}\mathcal{G}\right]dr \Rightarrow \\
{T_r}^r &= \frac{1}{r^2}\int_{r_h}^{r}\frac{2}{r}\left(\mathcal{G}-\mathcal{E}\right)dr \Rightarrow \\
{T_r}^r &= \frac{1}{r^2}\int_{r_h}^{r}\frac{2}{r}{T_\theta}^\theta dr
\end{aligned}
\tag{2.96}
$$

The (rr)-component of the Einstein's equations, eq.(2.90), can be written as (see Section 2.1.2, eq.(2.33)):

$$
\frac{e^{-\frac{v}{2}}}{r^2}(e^{\frac{v}{2}}r^2)' = 4\pi G r {T_r}^r e^\lambda + \frac{e^\lambda+3}{2r} > 4\pi G r {T_r}^r e^\lambda + \frac{2}{r}
\tag{2.97}
$$

and looking back at eq.(2.88), we can express the above left hand side $\frac{e^{-\frac{v}{2}}}{r^2}(e^{\frac{v}{2}}r^2)'$ as

$$
\frac{e^{-\frac{v}{2}}}{r^2}(e^{\frac{v}{2}}r^2)' = -\frac{({T_r}^r)'}{\mathcal{J}} + \frac{2}{r}\frac{\mathcal{G}}{\mathcal{J}} = \frac{v'}{2}+\frac{2}{r}
\tag{2.98}
$$

and finally get that

$$
\begin{aligned}
-\frac{({T_r}^r)'}{\mathcal{J}} + \frac{2}{r}\frac{\mathcal{G}}{\mathcal{J}} &> 4\pi G r {T_r}^r e^\lambda + \frac{2}{r} \Rightarrow \\
\frac{({T_r}^r)'}{{T_t}^t - {T_r}^r} &> 4\pi G r {T_r}^r e^\lambda + \frac{2}{r}\left[1 - \frac{{T_t}^t - {T_\theta}^\theta}{{T_t}^t - {T_r}^r}\right] \Rightarrow \\
\frac{({T_r}^r)'}{{T_t}^t - {T_r}^r} &> 4\pi G r {T_r}^r e^\lambda + \frac{2}{r}\left[\frac{{T_\theta}^\theta - {T_r}^r}{{T_t}^t - {T_r}^r}\right]
\end{aligned}
\tag{2.99}
$$

or, for ${T_r}^r$ and returning to the $\mathcal{G},\mathcal{J}$-symbolism, we get :

$$
\begin{aligned}
{T_r}^r &< \frac{e^{-\lambda}}{4\pi G r}\left[\frac{({T_r}^r)'}{{T_t}^t - {T_r}^r} - \frac{2}{r}\frac{({T_\theta}^\theta - {T_r}^r)}{({T_t}^t - {T_r}^r)}\right] \Rightarrow \\
T_r^r &< \frac{e^{-\lambda}}{4\pi G r}\left[\frac{-({T_r}^r)'}{\mathcal{J}} + \frac{2}{r}\frac{\mathcal{G}-\mathcal{J}}{\mathcal{J}}\right] \Rightarrow \\
T_r^r &< \frac{e^{-\lambda}}{4\pi G r}\left[-\frac{({T_r}^r)'}{\mathcal{J}} + \frac{2}{r}\frac{\mathcal{G}}{\mathcal{J}}\right]
\end{aligned}
\tag{2.100}
$$

Now, let's see what eq.(2.100) tells us. We start, assuming **no-violation at all**, so $\mathcal{J} > 0$, and $\mathcal{G} > 0$. Looking at eq.(2.100), we get:

- For $({T_r}^r)' > 0$, we see that there is **no restriction** for the values of T_r^r, while the same happens even if $({T_r}^r)' < 0$. This is something intresting, since with *no-violation of the energy conditions* ($\mathcal{J},\mathcal{G} > 0$), there is the possibility of the existence of hair, in the sense that T_r^r is not restricted in a specific range of values.

Assuming now the violation as $\mathcal{J} < 0$, we see that:

- Again, as long as $\mathcal{G}/\mathcal{J} > 0$, we get *no constraint at all for the values of* T_r^r, *no matter the sign of* $(T_r^r)'$, leaving again an open window regarding the existence of hair

Something different happens when we assume that $\mathcal{G}/\mathcal{J} < 0$. For $\mathcal{J} > 0$ and $(T_r^r)' > 0$, we see that

$$T_r^r < \frac{e^{-\lambda}}{4\pi G r}\left[-\frac{(T_r{}^r)'}{\mathcal{J}}\right] \tag{2.101}$$

which means that the positive values of T_r^r **are restricted, which is the case of Bekenstein no-hair**. The same happens as long as $\mathcal{J} < 0$ and $(T_r^r)' < 0$, reducing this case too to Bekenstein's no-hair theorem too.

To sum up the above, we have that:

- For $\mathcal{J} > 0$ we have two cases:
  1. $(T_r^r)' > 0$, where we get:
  $$\begin{aligned} &\frac{\mathcal{G}}{\mathcal{J}} > 0 \longrightarrow \ no\ constraint\ for\ the\ values\ of\ T_r^r \\ &\frac{\mathcal{G}}{\mathcal{J}} \leq 0 \ \longrightarrow Bekenstein's\ case. \end{aligned} \tag{2.102}$$
  2. $(T_r^r)' < 0$, where in this case we have *no constraint for the values of* T_r^r no matter what the sing of $\mathcal{G}/\mathcal{J}$ is.
- For $\mathcal{J} < 0$ we have again two cases:
  1. $(T_r^r)' > 0$, where we have *no constraint for the values of* T_r^r no matter what the sing of $\mathcal{G}/\mathcal{J}$ is.
  2. $(T_r^r)' < 0$, where we get:
  $$\begin{aligned} &\frac{\mathcal{G}}{\mathcal{J}} > 0 \longrightarrow \ no\ constraint\ for\ the\ values\ of\ T_r^r \\ &\frac{\mathcal{G}}{\mathcal{J}} \leq 0 \ \longrightarrow Bekenstein's\ case. \end{aligned} \tag{2.103}$$

As a discussion, it's intresting that in each case, as long as the sing of $\mathcal{G}$ and $\mathcal{J}$ is the same, we allow the possibility of hair to exist, in the sense that no constraints occured regarding the sign of $T_r{}^r$.

We can also check what the no-constraint case means for the behaviour of the g_{tt} component, based on equation (2.98) in order to reinforce or alter the Bekenstein's theorem in cases where we have to deal with higher order corrections in modified gravity theories.

Chapter 3

Chern-Simons modified gravity

3.1 Properties of Chern-Simons gravitational theory

Chern-Simons modified gravity is a 4-dimensional deformation of general relativity, postulated by Jackiw and Pi [11]. The gravitational Chern-Simons term CS(Γ) is a three-dimensional quantity

$$CS(\Gamma) = \frac{1}{4\pi^2}\int d^3x\ \varepsilon^{ijk}\ (\frac{1}{2}\ {}^3\Gamma^p_{iq}\,\partial_j{}^3\Gamma^q_{kp} + \frac{1}{3}{}^3\Gamma^p_{iq}\ {}^3\Gamma^q_{jr}\ {}^3\Gamma^r_{kp}). \tag{3.1}$$

where the the superscript "3" denotes three-dimensional quantities. The three-dimensional Christoffel connection is constructed in the usual way via the metric tensor, which is our fundamental dynamical variable.

A related four-dimensional quantity is the *Chern-Simons topological current*

$$K^\mu = 2\varepsilon^{\mu\alpha\beta\gamma}\left[\frac{1}{2}\ \Gamma^\sigma_{\alpha\tau}\,\partial_\beta\ \Gamma^\tau_{\gamma\sigma} + \frac{1}{3}\ \Gamma^\sigma_{\alpha\tau}\,\Gamma^\tau_{\beta\eta}\Gamma^\eta_{\gamma\sigma}\right], \tag{3.2}$$

which is proved to satisfy (see Appendix 3.3)

$$\partial_\mu K^\mu = \frac{1}{2}\tilde{R}^\sigma{}_\tau{}^{\mu\nu}\ R^\tau{}_{\sigma\mu\nu} \tag{3.3}$$

where $R^\tau{}_{\sigma\mu\nu}$ is the four-dimensional Riemann tensor

$$R^\tau{}_{\sigma\mu\nu} = \ \partial_\nu\ \Gamma^\tau_{\mu\sigma} - \partial_\mu\ \Gamma^\tau_{\nu\sigma} + \Gamma^\tau_{\nu\eta}\Gamma^\eta_{\mu\sigma} - \Gamma^\tau_{\mu\eta}\Gamma^\eta_{\nu\sigma}, \tag{3.4}$$

and $\tilde{R}^\tau{}_\sigma{}^{\mu\nu}$ is its dual

$$\tilde{R}^\tau{}_\sigma{}^{\mu\nu} = \frac{1}{2}\ \varepsilon^{\mu\nu\alpha\beta}R^\tau{}_{\sigma\alpha\beta}. \tag{3.5}$$

with $\varepsilon^{\mu\nu\alpha\beta} = \frac{\hat{\epsilon}^{\mu\nu\alpha\beta}}{\sqrt{-g}}$ the covariant Levi-Civita tensor with upper indices, under the convention that the symbol $\hat{\epsilon}_{0123} = 1$, or $\hat{\epsilon}^{0123} = -1$ *etc.*

Let's mention here that the term $\tilde{R}^{\sigma}{}_{\tau}{}^{\mu\nu} R^{\tau}{}_{\sigma\mu\nu}$, also known as **the Pontryagin density, is a parity violating term**. Later on, we will consider the field $b(x)$ as an axion field, and that's because, being a pseudo-scalar and transforming as $b \to -b$ under parity transformation, the coupling term $b\tilde{R}R$ does not violate parity.

The action we choose as an extension to Einstein-Hilbert's one is given by

$$\mathrm{I} = \frac{1}{16\pi G} \int d^4x \left(\sqrt{-g}R + \frac{1}{4} b\tilde{R}R \right) = \frac{1}{16\pi G} \int d^4x \left(\sqrt{-g}R - \frac{1}{2}(\nabla_\mu b) K^\mu \right) \tag{3.6}$$

where the second part of the equality comes from eq.(3.3) and from partial integration, respecting the boundary conditions of vanishing $b, \nabla b$ at the boundary. The variation of the CS term gives a traceless symmetric, second-rank tensor, which we name the four-dimensional Cotton tensor $C^{\mu\nu}$ (see analytical derivation at Appendix 3.1,3.4):

$$\delta I_{CS} = \delta \frac{1}{4} \int d^4x \, b\tilde{R}R \equiv \int d^4x \sqrt{-g} C^{\mu\nu} \delta g_{\mu\nu} = - \int d^4x \sqrt{-g} C_{\mu\nu} \delta g^{\mu\nu} \tag{3.7}$$

where, the Cotton tensor $C^{\mu\nu}$ is given by

$$C^{\mu\nu} = -\frac{1}{2\sqrt{-g}} \left[\nabla_\sigma b \left(\varepsilon^{\sigma\mu\alpha\beta} \nabla_\alpha R^\nu_\beta + \varepsilon^{\sigma\nu\alpha\beta} \nabla_\alpha R^\mu_\beta \right) + \nabla_\sigma \nabla_\tau b \left(\tilde{R}^{\tau\mu\sigma\nu} + \tilde{R}^{\tau\nu\sigma\mu} \right) \right] \tag{3.8}$$

or, we can prove (see Appendix 3.1) that

$$C^{\mu\nu} = -\nabla_\tau \frac{\nabla_\sigma b}{2\sqrt{-g}} \left(\tilde{R}^{\tau\mu\sigma\nu} + \tilde{R}^{\tau\nu\sigma\mu} \right) \tag{3.9}$$

The deformed of Einstein's field equations read

$$G^{\mu\nu} + C^{\mu\nu} = -8\pi G T^{\mu\nu} \tag{3.10}$$

The Bianchi identity forces $\nabla_\mu G^{\mu\nu} = 0$, while diffeomorphism invariance implies that the energy-momentum tensor $T^{\mu\nu}$ similarly satisfies $\nabla_\mu T^{\mu\nu} = 0$. But, we can check that the covariant divergence for the four-dimensional Cotton tensor $C^{\mu\nu}$ is not zero. The divergence is analytically calculated in Appendix 3.2 and gives:

$$\nabla_\mu C^{\mu\nu} = \frac{\nabla^\nu b}{8\sqrt{-g}} \tilde{R}R \tag{3.11}$$

So, we can always construct an effective EMT which satisfies $\nabla_\mu T^{\mu\nu}_{eff} = 0$, by means of the Bianchi identity.

Now, let's make a quick discussion about the confinement of the space of solutions of this theory, since it will be proven very usefull for what follows. We follow the arguments made in [17].

The quantity $b(x)$ of the previous relations is the Chern-Simons coupling field, which is generally not constant, but a function of spacetime, thus it serves as a "deformation". If b =constant, Chern-Simons gravity reduces to General Relativity, which can be easily seen, getting away of boundary terms and the vanishing of ∇b, for $b =$ constant.

When $b(x)$ becomes finite, Chern-Simons gravity becomes significantly different from GR. The quantity ∇b acts as an embedding coordinate, embedding a standard 3-dimensional theory to a 4-dimensional spacetime. The equation of motion for the quantity $b(x)$, which comes from the vanishing of the variation of action (3.6), yields:

$$\Box b(x) \sim \tilde{R}R \tag{3.12}$$

where $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ is the D'Alembertian operator. We recognize eq.(3.12) as the Klein-Gordon equation, without a potential, in the presence of sourcing term, the Pontryagin density term. We see now that the evolution of the $b(x)$ coupling is not determined only by its stress-energy tensor given below at eq,(3.13), but also by the spacetime curvature. The stress-energy tensor is given by:

$$T^b_{\mu\nu} = \nabla_\mu b(x)\nabla_\nu b(x) - \frac{1}{2}g_{\mu\nu}(\nabla b(x))^2 \tag{3.13}$$

From the field equation (3.10), we get

$$\nabla_\mu G^{\mu\nu} + \nabla_\mu C^{\mu\nu} = -8\pi G\nabla_\mu T^{\mu\nu} \tag{3.14}$$

where $\nabla_\mu G^{\mu\nu}$ vanishes identically by means of the Bianchi identity. What's left is that

$$\nabla_\mu T^{\mu\nu} \sim \nabla_\mu C^{\mu\nu} \sim \tilde{R}R \tag{3.15}$$

where, the above equation, implies some kind of "energy exchange" between $b(x)$ and the gravitational anomaly.

In the *non-dynamical framework*, where $b = constant$, the theory becomes constrained, because every solution must satisfy the condition (also known the Pontryagin constraint) $\tilde{R}R = 0$. In the *dynamical framework*, the Pontryagin constraint is replaced by the equation for the evolution of the field $b(x)$ (eq. 3.12), which doesn't impose a direct constraint on the space of solutions, while it couples the evolution of the $b(x)$ to the field equations.

Another important aspect we shall mention, is that of **vacuum solutions** of CS theory. Finding exact solutions, without any approximation technique, is always crucial, and that's why we will refer to two specific cases: i) Spherically Symmetric Spacetimes, and ii) Kerr metric

Spherically Symmetric Spacetimes: We start by considering the most general, spherically-symmetric spacetime. The line element of such a spacetime always leads to the vanishing of the Pontryagin density $\rightarrow \tilde{R}R = 0$. This actually decouples the field equations $G^{\mu\nu} + C^{\mu\nu} = -8\pi G T^{\mu\nu}$, and gives $R_{\mu\nu} = 0$ and $C_{\mu\nu} = 0$.

For the *non-dynamical framework*, spherically symmetric line elements reduce our theory to GR, cause of $C_{\mu\nu} = 0$. In the *dynamical framework*, one must also keep in mind the evolution equation for $b(x)$, which reduces to a wave-like equation without a source and without a potential, $\Box b(x) = 0$. For decaying, well-defined solutions of the above equation that will lead to a finite energy-momentum tensor and a good boundary behaviour, *the scalar field is forced to become constant*, which again reduces CS gravity to GR. Consequently, the Schwarzschild solution still holds by means of the validity of the Birkhoff's theorem in CS-gravity (*the most general and spherically symmetric solution for the vacuum Einstein field equations is the Schwarzschild metric*).

Kerr metric: Considering the Kerr solution line element, we can check that the Pontryagin constraint is not satisfied, since it leads to $\tilde{R}R \neq 0$. Just because the Kerr line element does not solve Chern-Simons gravity, it should not disappoint us since a rotating black hole solution still can be found. It is actually an open question, which specific modification of the Kerr solution can solve the deformed equations of Chern-Simons modified gravity.

3.2 The search for local solutions: slowly rotating Kerr-type axionic black holes

The action we will consider so on, that's giving the coupling of the axion field b to the $\tilde{R}R = R_{CS}$ is the following:

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} - \frac{1}{2}(\partial_\mu b)(\partial^\mu b) - A\, b\, R_{CS} \right] \tag{3.16}$$

where R_{CS} is the Chern-Simons term $R_{CS} = \frac{1}{2} R^{\mu}{}_{\nu\rho\sigma} \widetilde{R}^{\nu}{}_{\mu}{}^{\rho\sigma}$, where $\tilde{R}^{\tau}{}_{\sigma}{}^{\mu\nu} = \frac{1}{2}\, \varepsilon^{\mu\nu\alpha\beta} R^{\tau}{}_{\sigma\alpha\beta}$ is the dual of the Riemann tensor, $\varepsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita tensor with upper indices, $\kappa = M_{\rm Pl}^{-1}$ is the inverse of the reduced Planck mass, b is a pseudoscalar translated as the axion matter field and A is a parameter for the coupling of $b(x)$ axion field to the CS term with dimension of length (see Appendix D for the origin of such actions [18]).

The CS terms are accompanied by the axion field b, which, as we shall discuss below, is in the weak approximation for slowly-rotating black holes of large mass $\mathcal{M}$ compared to the Planck scale,

$$\mathcal{M}\, \kappa \gg 1\,, \tag{3.17}$$

which we will assume so on. Hence, in the limit of small angular momentum, our slow-rotation approximations in which we keep terms of up to linear order in the black hole angular momentum would be valid.

In the spirit of the previous section, we derive the equations of motion from the variation of the action (3.16) which reads (the variation of the Chern-Simons term is derived analytically in Appendix 3.1):

$$\begin{aligned} \delta\mathcal{S} = \delta\mathcal{S}_{EH} + \delta\mathcal{S}_b + \delta\mathcal{S}_{CS} = 0 \Rightarrow \\ G_{\mu\nu} = \kappa^2 T^b_{\mu\nu} + 4\kappa^2 A C_{\mu\nu} \ , \\ \Box b = A\, R_{CS} \ , \end{aligned} \tag{3.18}$$

where $T^b_{\mu\nu}$ is the stress energy-momentum tensor associated with the kinetic term of the axion matter field,

$$T^b_{\mu\nu} = \nabla_\mu b \nabla_\nu b - \frac{1}{2} g_{\mu\nu} (\nabla b)^2 \ . \tag{3.19}$$

The Cotton tensor $C_{\mu\nu}$ is also derived in Appendix (3.1) and is given by

$$C_{\mu\nu} = -\frac{1}{2}\nabla^\alpha \left[(\nabla^\beta b)\widetilde{R}_{\alpha\mu\beta\nu} + (\nabla^\beta b)\widetilde{R}_{\alpha\nu\beta\mu} \right] \ . \tag{3.20}$$

and the covariant divergence of $C^{\mu\nu}$, derived in Appendix (3.2), is given by

$$\nabla_\mu C^{\mu\nu} = -\frac{1}{4}(\nabla^\nu b) R_{CS} \ . \tag{3.21}$$

The above result can be also recovered from the equations of motion. From the Bianchi identity, $\nabla^\mu G_{\mu\nu} = 0$, where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R$ is the Einstein tensor, and with $\nabla^\mu T_{\mu\nu} = \Box b \nabla^\nu b$, we get that

$$0 = \nabla^\mu G_{\mu\nu} = \kappa^2 \nabla^\mu T^b_{\mu\nu} + 4\kappa^2 A \nabla^\mu C_{\mu\nu} \, , \tag{3.22}$$

and consequently

$$\begin{aligned} 0 = \kappa^2 \Box b \nabla^\nu b + 4\kappa^2 A\, \nabla^\mu\, C_{\mu\nu} \Longrightarrow A\ R_{CS} \nabla^\nu b = -4A\, \nabla^\mu\, C_{\mu\nu} \Rightarrow \\ \nabla^\mu\, C_{\mu\nu} = -\frac{1}{4} (\nabla^\nu b)\, R_{CS} \end{aligned} \tag{3.23}$$

We see that the covariant-conservation of energy-momentum tensor of the axion, $T^b_{\mu\nu}$, is violated, since $\nabla^\mu T^b_{\mu\nu}$ does not vanish. Actually, we find that

$$\nabla^\mu\, T^b_{\mu\nu} = -4\, A\, \nabla^\mu\, C_{\mu\nu} = A\, \frac{1}{4}\, (\nabla^\nu b)\, R_{CS} \ . \tag{3.24}$$

where we see that $\nabla^\mu T^b_{\mu\nu} \sim R_{CS}$. The latter implies an exchange of energy between the axion matter field b and the gravitational spacetime background.

The goal from now on, is to find a slowly rotating Kerr-type black hole solution of CS modified gravity, using perturbative arguments. Since the axion matter field $b(x)$ is a pseudo-scalar, it enforces an axial symmetry on the underlying spacetime. As we said in the previous section, the R_{CS} term identically vanishes for spherical symmetry and so the axion dynamics are forced to vanish and CS gravity reduces to GR. The presence of the dynamical axion field and of axisymmetry **motivates the search of rotating solutions**. We keep our analysis to the slowly rotating case, in leading order for the angular momentum parameter a of our slowly-rotating spacetime.

In General Relativity, the geometry of a spinning massive black hole is described by the Kerr metric, which is not the case of Chern-Simons modified gravity, as we mentioned before. Spinning black hole solutions have been found, using slow-rotation and small-coupling approximation [5, 7, 19, 20]. To understand how we will proceed, we will sketch out how to get an approximate solution for a rotating black hole in CS-modified gravity.

We discuss the slowly rotating case by considering the perturbation of a spherically symmetric and static spacetime, given in general form by the ansatz

$$ds^2 = -G(r)dt^2 + F(r)dr^2 - 2r^2\alpha \sin^2\theta W(r)dtd\phi + r^2 d\Omega^2 \ . \tag{3.25}$$

where $G(r), F(r), W(r)$ are functions of r. To make this more clear, starting from a spherically symmetric and static spacetime, with ”(0)” denoting the zeroth order of the pertubation, and with $b^{(0)} = 0$ as a vacuum solution, we have

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu}, \tag{3.26}$$

$$b = b^{(1)} \tag{3.27}$$

where $g^{(1)}_{\mu\nu}$ and $b^{(1)}$ are at least first-order quantities of a the small parameter α of the ansatz in eq.(3.25), which is the *angular momentum parameter of our slow-rotation case*. This means that $g^{(1)}_{\mu\nu}, b^{(1)}$ are quantities of order $\sim \mathcal{O}(\alpha)$, that's why we consider the ansatz of eq.(3.25) . The axion field, actually, is excited by the curvature of the spacetime, by means of the equations of motion for the evolution of b in eq.(3.18).

While for spherically symmetric spacetimes, the Chern-Simons term R_{CS} vanishes identically, this is not the case considering the pertubation and keeping terms up to first order in the angular momentum $\mathcal{O}(\alpha)$. Looking at eq.(3.18), from $\Box b = AR_{CS}$, we see that the order of the axion filed b should be $b \sim \mathcal{O}(\alpha)$, since the we keep only terms of order $\mathcal{O}(\alpha)$ for the Chern-Simons term R_{CS}.

Looking at the Einstein equations (3.18) again, we see that the tt and rr components of the gravitational equations of motion are satisfied in vacuum, and that's because the corrections of these components would be of order $\mathcal{O}(\alpha^2)$, which are ignored in our approximation. This naturally means that the black hole background we consider is similar to the Schwarzschild background for the (rr) and (tt) components by means of the Birkhoff's theorem, yielding that

$$G(r) = \frac{1}{F(r)} = 1 - \frac{2M}{r} \ , \quad M \equiv G\mathcal{M} \tag{3.28}$$

with $G = 8\pi\kappa^{-2}$ is the Newton's gravitational constant. Therefore, any backreaction on the background, due to the presence of the axion, will be encoded in the **off-diagonal metric components**. We define

$$W(r) = \frac{2M}{r^3} + w(r) \tag{3.29}$$

and try to extract the corrections $w(r)$ on our slowly rotating spacetime (3.25), corrections that are caused by the axionic backreaction on our background spacetime. We express $W(r)$ in this way in eq.(3.29) in order to recover the Kerr metric in the limit of slow rotation parameter α, asymptotically.

We mention here, that, from $G_{\mu\nu} = \kappa^2 T^b_{\mu\nu} + 4\kappa^2 A C_{\mu\nu}$, we see that the CS correction to the Einstein's field equation in eq.(3.18) $AC_{\mu\nu}$ should also be of order $C_{\mu\nu} \sim \mathcal{O}(\alpha)$. Therefore, when we take the specific black hole background, the CS correction to the rotation of the black hole should be of order $\mathcal{O}(\alpha)$. We mention here that we haven't considered, at any point so far, a small-coupling approximation, aiming to find a solution for all the powers of the coupling parameter A.

We continue by presenting the steps for **a solution of the D.E.** that have occured, also mentioning some important physical assumptions we have to consider along the way. Using separation of variables we deduce that the axion field b may be written as

$$b = aAu(r)P_1(\cos\theta) \ , \tag{3.30}$$

where the $P_1(\cos\theta)$ comes from the observation that $R\tilde{R} \sim cos\theta$ in order $\mathcal{O}(\alpha)$, and P_1 is the Legendre polynomial for the first order, which cancels the angular dependence on the axionic equation of motion. We consider the $t\phi$ component for the equations of motion (3.18). We can obtain differential equations for (tt), (rr), $(r\theta)$, $(\theta\theta)$, $(\phi\phi)$, $(r\phi)$, $(\theta\phi)$, (tr) and $(t\theta)$ components of equations of motion (3.18) that give homogeneous differential equations. For those components, it can be checked that we have $C^{\mu\nu} \sim \mathcal{O}\left(\alpha^2\right)$. Finally, the only component yielding equations of order $\mathcal{O}(\alpha)$ is the $(t\phi)$ component. For this component:

$$G_{t\phi} = \kappa^2 T_{t\phi} + 4A\kappa^2 C_{t\phi} \ ,$$

where $C_{t\phi}$ is calculated as

$$4A\kappa^2 C_{t\phi} = -\frac{12A^2\kappa^2 M(r-2M)(ru'-u)}{r^5} a\sin^2\theta + \mathcal{O}(a^2) \ . \tag{3.31}$$

and $G_{t\phi}$ is found to be

$$G_{t\phi} = \frac{1}{2}\alpha\sin^2(\theta)(r-2M)\left(rw''(r) + 4w'(r)\right) + \mathcal{O}\left(\alpha^2\right) \tag{3.32}$$

For $T_{t\phi}$, with a simple check of eq.(3.19), we can verify that it gives

$$T_{t\phi} \sim \mathcal{O}(\alpha^3) \tag{3.33}$$

So, from the equations of motion, we ignore the contribution of $T_{t\phi}$ since we only keep terms of order $\mathcal{O}(\alpha)$ and get that:

$$
\begin{gathered}
G_{t\phi} = \kappa^2 T_{t\phi} + 4A\kappa^2 C_{t\phi} \Rightarrow \\
\frac{1}{2}\alpha \sin^2(\theta)(r-2M)\left(rw''(r)+4w'(r)\right) = -\frac{12A^2\kappa^2 M(r-2M)(ru'-u)}{r^5} a \sin^2\theta \Rightarrow \\
\left(r^4 w''(r) + 4r^3 w'(r)\right) = -\frac{24A^2\kappa^2 M(ru'-u)}{r^2}
\end{gathered} \tag{3.34}
$$

and so, with $\left(r^4 w''(r) + 4r^3 w'(r)\right) = (r^4 w')'$, and with $(ru'-u)/r^2 = (u/r)'$, we substitute and finally get that:

$$
\left(\frac{u}{r}\right)' = -\frac{1}{24A^2\kappa^2 M}(r^4 w')' \ .
$$

Integration of the above expression, and minimizing the integration constant for a non-divergent axion, we get

$$
u(r) = -\frac{r^5 w'}{24A^2\kappa^2 M} \ . \tag{3.35}
$$

Going back to the axionic equation $\Box b = A\, R_{CS}$ and plugging our results from eq.(3.35), we end up with the following differential equation

$$
\begin{gathered}
\frac{\alpha r^2 \cos\theta\left((28r-50M)w'(r) + r^2(r-2M)w'''(r) + r(12r-22M)w''(r)\right)}{24A\kappa^2 M} = \\
\frac{24A\alpha M \sin\theta\cos\theta\left(r^4 w'(r) - 6M\right)}{r^7 \sin\theta}
\end{gathered} \tag{3.36}
$$

which, after simple algebra, leads to the D.E. :

$$
\begin{aligned}
r^{11}(r-2M)w''' + 2r^{10}(6r-11M)w'' + (28r^{10} - 50Mr^9 - 576A^2\kappa^2M^2r^4)w' \\
+3456A^2\kappa^2M^3 = 0 \ .
\end{aligned} \tag{3.37}
$$

To solve the above equation we consider a series expansion for the function $w(r)$. For the radial component of the axion field to asymptotically vanish, $w'(r)$ needs to be at least of order $\mathcal{O}(r^{-5})$ by means of eq.(3.35), which means that $w(r)$ needs to be at least of order $\mathcal{O}(r^{-4})$. This implies that, for the $g_{t\phi}$ component, the spacetime asymptotically coincides with the one of the slow-rotation limit of the Kerr metric looking at eq.(3.29), which will be of our use later on. We define $w(r)$ in a non-closed form as

$$
w(r) = \sum_{n=4}^{\infty} \frac{d_n M^{n-2}}{r^n} \ , \tag{3.38}
$$

where we introduce M^{n-2} to maintain d_n-coefficients dimensionless. So, we now have to determine the coefficients d_n *(let us note that the series expansion of eq.(3.38) converges for any value of $r > 2M$ and any value of the dimensionless parameter γ given by eq.(3.47)), as is analytically shown in Appendix* F). As expected, the next step is to replace w with the above eq.(3.38) in eq.(3.37), and get that (see analytical derivation at Appendix 5.1):

$$3456A^2\kappa^2M^3 - 162M^7d_9 + 256M^7d_8 + 8M^2d_4r^5 +$$
$$\sum_{n=-4}^{-1}\frac{M^{n+7}\left[-(n+3)(n+6)(n+9)d_{n+9} + 2(n+4)^2(n+8)d_{n+8}\right]}{r^n} +$$
$$\sum_{n=1}^{\infty}\frac{M^{n+7}\left[-(n+3)(n+6)(n+9)d_{n+9} + 2(n+4)^2(n+8)d_{n+8}\right] + 576A^2\kappa^2M^{n+3}(n+3)d_{n+3}}{r^n} = 0 \ . \tag{3.39}$$

In order for the left hand side of this equation to be zero for any r, all the coefficients of the powers r^n have to vanish. This leads us to the following equations

$$\begin{aligned}
&d_4 = 0 \ , \\
&256d_8 - 162d_9 = -3456\frac{A^2\kappa^2}{M^4} \\
&-(n+3)(n+6)(n+9)d_{n+9} + 2(n+4)^2(n+8)d_{n+8} = 0 \ , \ \text{ for } n = -1,-2,-3,-4 \\
&-(n+3)(n+6)(n+9)d_{n+9} + 2(n+4)^2(n+8)d_{n+8} + 576\frac{A^2\kappa^2}{M^4}(n+3)d_{n+3} = 0 \text{ for } n \geq 1 \ .
\end{aligned} \tag{3.40}$$

From the last line of the above equation, we can conclude that

$$d_n = \frac{2(n-5)^2(n-1)}{n(n-6)(n-3)}d_{n-1} + \frac{576A^2\kappa^2}{n(n-3)M^4}d_{n-6}, \ \text{ for } n \geq 10 \ . \tag{3.41}$$

which comes from the shift of $n \to n-9$. From eq.(3.41), we see that $d_4, d_5, d_6, d_7, d_8, d_9$ must be known for the calculatation of the other coefficients. From eq.(3.40), we can find that

$$\begin{aligned}
d_4 = d_5 &= 0 \ , \\
-28d_7 + 48d_6 &= 0 \ , \\
-80d_8 + 126d_7 &= 0 \ , \\
256d_8 - 162d_9 &= -3456\frac{A^2\kappa^2}{M^4} \ .
\end{aligned} \tag{3.42}$$

We end up with four unknown coefficients, but with only three equations. **So, we need one more constraint, which we can extract from the the weak field limit, which is valid asymptotically**. Then, our metric background is the slow-rotation limit of the Kerr spacetime, in order $\mathcal{O}(\alpha)$. We can proceed in this way, since, by looking at the correction function $W(r) = 2M/r^3 + w(r)$ and keeping in mind that $w(r) \sim r^{-4}$ for large r, the term $2M/r^3$ dominates asymtotically and the background coincides with the slow-rotation limit of the Kerr spacetime. Plugging such a *background to the axionic* *equation of motion* (3.18) $\Box b = AR_{CS}$, *we calculate*:

$$\begin{aligned}\Box b &= \frac{A\alpha\cos\theta\,[r(r-2M)u''(r)+2(r-M)u'(r)-2u(r)]}{r^2}+\mathcal{O}\left(\alpha^2\right), \quad \textit{and} \\ AR_{CS} &= \frac{72A\alpha M^2\sin 2\theta}{r^7 sin\theta}+\mathcal{O}\left(\alpha^2\right)=\frac{144A\alpha M^2\cos\theta}{r^7}+\mathcal{O}\left(\alpha^2\right)\end{aligned} \tag{3.43}$$

and so, from $\Box b = AR_{CS}$ and after some simple algebra, we get a differential equation for $u(r)$, which reads:

$$-2u(r)+2(r-M)u'(r)+(r^2-2Mr)u''(r)=\frac{144M^2}{r^5}\ , \tag{3.44}$$

By solving eq.(3.44), we find that (see analytical derivation in Appendix 5.2):

$$u(r)=-\frac{5}{4Mr^2}-\frac{5}{2r^3}-\frac{9M}{2r^4}\ . \tag{3.45}$$

which gives the behaviour of the function $u(r)$, and consequently of the axionic field b (3.30). From eq.(3.35) and making use of eq.(3.38), we can expand the sum to its first terms, up to r^{-4} terms, to get:

$$u(r)=\frac{1}{24A^2\kappa^2M}\left[4d_4M^2+\frac{5d_5M^3}{r}+\frac{6d_6M^4}{r^2}+\frac{7d_7M^5}{r^3}+\frac{8d_8M^6}{r^4}\right]+\mathcal{O}(1/r^5)\ . \tag{3.46}$$

and with a simple match of the coefficients of eq.(3.46) and eq.(3.45), we find that

$$\begin{aligned} d_4=d_5=0\ ,\ d_6=-5\gamma^2\ ,\ d_7=-\frac{60\gamma^2}{7}\ ,\ d_8=-\frac{27\gamma^2}{2}\ ,\ d_9=0, \\ \text{where}\quad \gamma^2=\frac{A^2\kappa^2}{M^4}\ . \end{aligned} \tag{3.47}$$

Now, making use again of eq.(3.41), we can check that for $n \geq 10$, we get

$$d_{10}=d_{11}=0\ ,\ d_{12},\, d_{13},\, d_{14}\ \sim\ -\gamma^4\ ,\ \text{....and so on for even powers of } \gamma \tag{3.48}$$

We write only temrs of order γ^2, since the γ^{2n} terms can be calculated via the recurrence relation. This means that the correction function $w(r)$ will be of order $\mathcal{O}(A^2)$ and, again, the terms of order $\mathcal{O}(A^{2n})$ can be calculated via eq.(3.41).

Then, our slowly rotating Kerr-like metric can be expressed as:

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\frac{dr^2}{1-\frac{2M}{r}}+r^2\,d\Omega^2-2\,r^2\,a\,\sin^2\theta\,W(r)\,dt\,d\phi\ , \tag{3.49}$$

where the off-diagonal correction term reads

$$W(r)=\frac{2M}{r^3}-\frac{A^2\kappa^2(189M^2+120Mr+70r^2)}{14r^8}+\mathcal{O}(A^{2n}),\quad \text{with } n\geq 2\,, \tag{3.50}$$

where the $\mathcal{O}(A^{2n})$-terms can be found from eq.(3.41).

The axion can now be calculated by eq.(3.30) and eq.(3.35) in the same order of A as

$$b = aA\cos\theta\left(-\frac{5}{4Mr^2} - \frac{5}{2r^3} - \frac{9M}{2r^4}\right) + \mathcal{O}(A^m), \quad \text{for } m = 2n+1,\ n \in \mathbb{Z}^+, \tag{3.51}$$

with $\mathbb{Z}^+$ the positive integers. The Cotton tensor contribution, in order $\mathcal{O}(\alpha)$, can now be found to be from eq.(3.31):

$$4\kappa^2 A\, C_{t\phi} = a\frac{r-2M}{2}\sum_{n=4}^{\infty}\frac{n(n-3)d_n M^{n-2}}{r^{n+1}}\sin^2(\theta)\ , \tag{3.52}$$

This contribution is negative for every $r > 2M$, because, as can be checked, the coefficients $d_n \leq 0$ for all n. The axionic hair of the black-hole, *constitutes secondary hair for our black hole solution*, since it depends on the existing black hole parameters, the mass M and the angular momentum α.

Let us mention here, that, throughout the above analysis, we have not considered any kind of small coupling approximation. This means that we didn't regard the coupling A as a perturbative parameter, keeping only the first order quantities, as we did for the angular momentum parameter. So, *the solution can be expanded to every power of the coupling A. This allows us to get arbitrary close to the black hole horizon.*

3.3 Axionic Hair and the violation of energy conditions

As we presented in detail in Section 2.2, there is a close relation between the existence or not of hairy black holes and with the violation of the energy conditions. In what follows, we demostrate the violation of the Null Energy Condition for an effective energy momentum tensor $T^{eff}_{\mu\nu}$ for which $\nabla^\mu T^{eff}_{\mu\nu} = 0$. For our case, considering the action of eq.(3.16), where, in order $\mathcal{O}(\alpha)$, the only contribution comes from the $(t\phi)$-component, where $T_{t\phi} \sim \mathcal{O}(\alpha^3)$. So, the above relation reduces to

$$T^{eff}_{t\phi} = 4\kappa^2 A C_{t\phi}\ , \quad \textit{in order } \mathcal{O}(\alpha) \tag{3.53}$$

which is calculated in eq.(3.52) and yields

$$T^{eff}_{t\phi} = 4\kappa^2 A\, C_{t\phi} = \alpha\frac{r-2M}{2}\sum_{n=4}^{\infty}\frac{n(n-3)d_n M^{n-2}}{r^{n+1}}\sin^2(\theta)\ . \tag{3.54}$$

The $C_{t\phi}$ component contributes to the energy momentum tensor, and this contribution is clearly negative for $r > 2M$, since $d_n < 0$. To demonstrate the violation of the Null

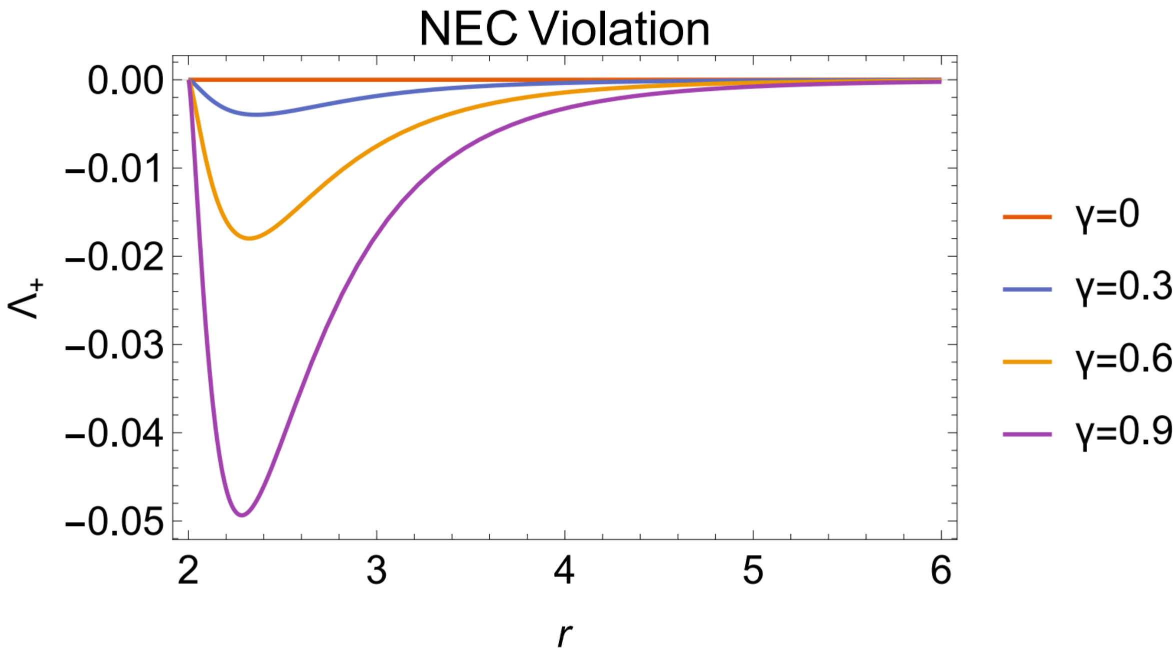

Figure 3.1: Behavior of $\Lambda_+ = T^{eff}_{\mu\nu} l^\mu_+ l^\nu_+ / \alpha$ at the equitorial plane $\theta = \pi/2$ up to $\mathcal{O}(A^2)$ with respect to r.

Energy Condition in the way we argued in Section (2.2), we define the future directed null vector

$$l^\mu_\pm = \left(1, 0, 0, -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}} \right) . \tag{3.55}$$

where, contracting with $T^{eff}_{\mu\nu}$, we get

$$T^{eff}_{\mu\nu} l^\mu_\pm l^\nu_\pm = \pm \frac{\alpha (r - 2M)^{3/2}}{2} \sin\theta \sum_{n=4}^{\infty} \frac{n(n-3) d_n M^{n-2}}{r^{n+5/2}} + \mathcal{O}(\alpha^2) . \tag{3.56}$$

Equation (3.56) implies that $T^{eff}_{\mu\nu} l^\mu_+ l^\nu_+ \leq 0$ and $T^{eff}_{\mu\nu} l^\mu_- l^\nu_- \geq 0$, where "+" is for co-rotating and " $-$ " for the counter rotating case, whilst the equality holds for $r = 2M$. So, it is clear that outside the horizon ($r > 2M$), the Null Energy Condition is violated, expect for the poles, $\theta = 0$ and $\theta = \pi$, where the contraction identically vanishes.

The violation of NEC has its origin to the axion coupling with the Chern-Simons term R_{CS}, and the Cotton tensor correction to the field equations of motion. For larger values of this coupling, the deformation of the spacetime becomes more and more important and hence the violation of NEC gets stronger, as is shown in Figure(3.1), where for larger values of γ, the NEC violation becomes more and more apparent. Stronger γ means stronger coupling, and/or smaller mass M for the black hole. We also see that the NEC reach its maximum value near the horizon, which is something we expected, since the axion "lives" mostly near the horizon, and vanishes asymptotically. Black holes with hair provide us an insightful understanding on which way the spacetime deformation is achieved. The deformation occurs in such a way as to be permitted for the axionic field to exist in the region outside the horizon.

3.4 Angular momentum of the Axionic Black Hole

In this chapter, we aim to derive the angular momentum of the axionic black hole. Making use of the antisymmetries in the first and last two indices and the cyclic property of the Riemann tensor $R_{\mu[\nu\alpha\beta]}$, we get:

$$\begin{aligned} R_{\mu[\nu\alpha\beta]} = \frac{1}{3!}\left(R_{\mu\nu\alpha\beta} - R_{\mu\alpha\nu\beta} + R_{\mu\alpha\beta\nu} - R_{\mu\beta\alpha\nu} + R_{\mu\beta\nu\alpha} - R_{\mu\nu\beta\alpha}\right) = \\ \frac{2}{3!}\left(R_{\mu\nu\alpha\beta} + R_{\mu\alpha\beta\nu} + R_{\mu\beta\nu\alpha}\right) = 0 \end{aligned} \tag{3.57}$$

Contracting the above relation with the Killing vector that corresponds to the polar isometry of our spacetime, $\xi = \partial_\phi$, and using the symmetry of the Riemannn tensor in the change of the first two with the last two indices ($R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$), we have that:

$$\begin{aligned} R_{\mu\nu\alpha\beta}\xi^\nu + R_{\mu\alpha\beta\nu}\xi^\nu + R_{\mu\beta\nu\alpha}\xi^\nu = 0 \Rightarrow \\ R_{\mu\nu\alpha\beta}\xi^\nu + R_{\beta\nu\mu\alpha}\xi^\nu + R_{\nu\alpha\mu\beta}\xi^\nu = 0 \Rightarrow \\ -[\nabla_\beta, \nabla_\alpha]\xi_\mu - [\nabla_\alpha, \nabla_\mu]\xi_\beta - [\nabla_\mu, \nabla_\beta]\xi_\alpha = 0 \Rightarrow \\ \nabla_\beta\nabla_\alpha\xi_\mu - \nabla_\alpha\nabla_\beta\xi_\mu + \nabla_\alpha\nabla_\mu\xi_\beta - \nabla_\mu\nabla_\alpha\xi_\beta + \nabla_\mu\nabla_\beta\xi_\alpha - \nabla_\beta\nabla_\mu\xi_\alpha = 0 \Rightarrow \\ \nabla_\beta\left(\nabla_\alpha\xi_\mu - \nabla_\mu\xi_\alpha\right) - \nabla_\alpha\left(\nabla_\beta\xi_\mu - \nabla_\mu\xi_\beta\right) - \nabla_\mu\left(\nabla_\alpha\xi_\beta - \nabla_\beta\xi_\alpha\right) = 0 \Rightarrow \\ 2\nabla_\beta\nabla_\alpha\xi_\mu - 2\nabla_\alpha\nabla_\beta\xi_\mu - 2\nabla_\mu\nabla_\alpha\xi_\beta = 0 \\ [\nabla_\beta, \nabla_\alpha]\xi_\mu = \nabla_\mu\nabla_\alpha\xi_\beta \Rightarrow \\ -R_{\mu\nu\alpha\beta}\xi^\nu = -\nabla_\mu\nabla_\beta\xi_\alpha \Rightarrow \\ R_{\mu\nu\alpha\beta}\xi^\nu = \nabla_\mu\nabla_\beta\xi_\alpha \end{aligned} \tag{3.58}$$

where, at the 6th line we used the antisymmetry coming from the Killing equation $\nabla_\mu\xi_\nu = -\nabla_\nu\xi_\mu$ for the Killing vector ξ. Contracting the result of the above calculations with $g^{\mu\alpha}$ and using again the antisymmetry of the Killing equation, we get:

$$R_{\nu\beta}\xi^\nu = \nabla_\mu\nabla_\beta\xi^\mu = -\nabla_\mu\nabla^\mu\xi_\beta \tag{3.59}$$

or, as we will use it from now on, we have:

$$\nabla_\beta\nabla^\beta\xi^\alpha = -R^\alpha_\beta\xi^\beta \ , \tag{3.60}$$

where R^α_β is the Ricci tensor. Integrating the above equation, we get

$$\int_S d\Sigma_\alpha \nabla_\beta\nabla^\beta\xi^\alpha = -\int_S d\Sigma_\alpha R^\alpha{}_\beta\xi^\beta \ . \tag{3.61}$$

where S is an hypersurface.

Now, we will make use of the Stokes' theorem, and that's because we will take advantage of the antisymmetry of the tensor $B^{\alpha\beta} = \nabla^\beta \xi^\alpha$, by means of the Killing equation $\nabla_\mu \xi_\nu = -\nabla_\nu \xi_\mu$. What Stokes' theorem really does, is to relate the integral of an n-form ω, in our case a 2-form, over a boundary, to the integral of exterior derivative of this 2-form over the enclosed submanifold. It translates like this:

$$\int_{\partial\Sigma} \omega = \int_{\Sigma} \mathbf{d}\omega \tag{3.62}$$

or, in our case:

$$\int_S d\Sigma_\alpha \nabla_\beta \nabla^\beta \xi^\alpha = \int_{\partial S} d\Sigma_{\alpha\beta} \nabla^\alpha \xi^\beta = -\int_S d\Sigma_\alpha R^\alpha{}_\beta \xi^\beta. \tag{3.63}$$

S is a 3-hypersurface, whith $d\Sigma_\alpha$ representing the 3-dimensional directed surface element on S, whilst $d\Sigma_{\alpha\beta}$ is the 2-dimensional directed surface element of the boundary ∂S, with ∂S being a 2-surface.

Now, let us proceed to the calculation of the angular momentum of our system [20]. We will use the Komar integrals, and to do so, we start with the conserved current $J^\mu = R^{\mu\nu}\xi_\nu$, which gives

$$\nabla_\mu J^\mu = R^{\mu\nu}\nabla_\mu \xi^\nu + \xi_\nu \nabla_\mu R^{\mu\nu} = 0 + \xi_\nu \nabla_\mu \left(G^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R \right) = \xi_\nu \frac{1}{2} g^{\mu\nu} R = 0 \tag{3.64}$$

where $R^{\mu\nu}\nabla_\mu \xi^\nu = 0$ due to the symmetry of Ricci tensor in $[\mu, \nu]$ and the antisymmetry cause of the Killing equation of $\nabla_\mu \xi^\nu = -\nabla_\nu \xi^\mu$, $\nabla_\mu G^{\mu\nu} = 0$ from the Bianchi identity and also $\xi_\nu \frac{1}{2} g^{\mu\nu} R = 0$, which can be easily proven, a relation that expresses that the curvature remains constant along the Killing vector. The conservation of the current J^μ implies the existence of a conserved charge corresponding to the Killing vector of polar isometries, $\xi = \partial\phi$. This charge, is the angular momentum J of the black hole, given by

$$J = \frac{1}{8\pi} \int_{\partial S_\infty} d\Sigma_{\alpha\beta} \nabla^\alpha \xi^\beta \tag{3.65}$$

where ∂S_∞ is the two-sphere at infinity. The hypersurface S represents the exterior of the black hole and we may express the boundary consisting of a 2-surface at spatial infinity ∂S_∞ and the event horizon $\mathcal{H}$. So, we can write

$$\int_{\partial S} d\Sigma_{\alpha\beta} \nabla^\alpha \xi^\beta = \int_{\partial S_\infty} d\Sigma_{\alpha\beta} \nabla^\alpha \xi^\beta + \int_{\mathcal{H}} d\Sigma_{\alpha\beta} \nabla^\alpha \xi^\beta \tag{3.66}$$

and so, with the help of eq.(3.65) and eq.(3.63), we get

$$\begin{aligned} J = \frac{1}{8\pi} \int_{\partial S_\infty} d\Sigma_{\alpha\beta} \nabla^\alpha \xi^\beta = \frac{1}{8\pi} \int_{\partial S} d\Sigma_{\alpha\beta} \nabla^\alpha \xi^\beta - \frac{1}{8\pi} \int_{\mathcal{H}} d\Sigma_{\alpha\beta} \nabla^\alpha \xi^\beta \Rightarrow \\ J = -\frac{1}{8\pi} \int_S d\Sigma_\alpha R^\alpha{}_\beta \xi^\beta - \frac{1}{8\pi} \int_{\mathcal{H}} d\Sigma_{\alpha\beta} \nabla^\alpha \xi^\beta \end{aligned} \tag{3.67}$$

where in the second line we used the relation of eq.(3.63). Using now the gravitational equation of motion, and taking its trace, we can write the Ricci tensor with respect to the effective energy momentum tensor, which we will write for simplicity as $T^{eff}_{\mu\nu} = T_{\mu\nu}$. So, we have

$$R^{\alpha}{}_{\beta} = T^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}T \tag{3.68}$$

and so, eq.(3.67) becomes

$$J = J_{Matter} + J_{Horizon} = -\frac{1}{8\pi}\int_S d\Sigma_\alpha \left(T^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}T\right)\xi^\beta - \frac{1}{8\pi}\int_{\mathcal{H}} d\Sigma_{\alpha\beta}\nabla^\alpha\xi^\beta \tag{3.69}$$

In the total absence of matter, we have $T_{\mu\nu} = 0$ and the total angular momentum is $J = J_{Horizon}$, which comes only from the black hole. This would be the Kerr vacuum solution. Now, as is shown in eq.(3.69), we have two contribution to the total angular momentum, as it would be measured at infinity. The term

$$J_{Horizon} = -\frac{1}{8\pi}\int_{\mathcal{H}} d\Sigma_{\alpha\beta}\nabla^\alpha\xi^\beta \tag{3.70}$$

represents the contribution from *the rotating black hole*, while the term

$$J_{Matter} = -\frac{1}{8\pi}\int_S d\Sigma_\alpha \left(T^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}T\right)\xi^\beta \tag{3.71}$$

is representing *the angular momentum contribution from the axionic matter field.*

The slowly rotating Kerr like metric of our problem is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2M}{r}\right)} + r^2 d\Omega^2 + 2g_{t\phi}(r,\theta)dtd\phi\ , \tag{3.72}$$

where

$$\begin{aligned} g_{t\phi} &= -r^2\alpha\sin^2\theta W(r) = -r^2\left(\frac{2M}{r^3} + w(r)\right)\alpha\sin^2(\theta) \Rightarrow \\ g_{t\phi} &= -\alpha sin^2\theta\left(\frac{2M}{r} + r^2 w(r)\right) \equiv -\alpha sin^2\theta\left(\frac{2M}{r} + y(r)\right) \end{aligned} \tag{3.73}$$

where we redefined $r^2 w(r) \equiv y(r)$.

Moving on, we have to perform the integrations, and to do so, we need to define how the directed surface elements $d\Sigma_{\alpha\beta}$ and $d\Sigma_\alpha$ should be expressed. We construct the boundary ∂S by $t, r = const$ and we will take the appropriate limits at the end of the calculations, which should be valid by means of continuity. The two normal one-forms are given by $n_\mu = c_1\partial_\mu t$ and $\sigma_\mu = c_2\partial_\mu r$, where c_1, c_2 are normalization constants. The normalization is given by $n_\mu n^\mu = -\sigma_\mu\sigma^\mu = -1$. At order $\mathcal{O}(a)$, we have

$c_1 = 1/c_2 = \pm\sqrt{1-2M/r}$. We choose $c_1 < 0$ and $c_2 > 0$, in order to have future and outgoing orientation for the 2-surfaces. So, we get:

$$
\begin{aligned}
n_\mu &= -\sqrt{1-\frac{2M}{r}}(\partial_t)_\mu \ , \\
\sigma_\mu &= \frac{1}{\sqrt{1-\frac{2M}{r}}}(\partial_r)_\mu \ .
\end{aligned}
\tag{3.74}
$$

The sing of $c_1 < 0$ comes from our claim that the vector n^μ should have the same, positive (future) time direction as $(\partial_t) = (1,0,0,0)^T$, and so, since $g_{tt} < 0$, for $n_\mu = c_1\partial_\mu t$, we get that c_1 must have the negative sing. The same arguments for σ^μ, but now with the positive g_{rr} we cunclude that $c_2 > 0$.

Consequently, the surface element can be expressed as $d\Sigma_{\alpha\beta} = \sqrt{g^{(2)}}n_\alpha\sigma_\beta d^2x$, where $g^{(2)}$ is the determinant of the induced 2-metric, and is found to be $\sqrt{g^{(2)}} = \sqrt{r^4\sin^2\theta} = r^2\sin\theta$. Also, $\nabla^\alpha\xi^\beta = g^{\alpha\mu}\nabla_\mu\xi^\beta = g^{\alpha\mu}\partial_\mu\xi^\beta + g^{\alpha\mu}\Gamma^\beta_{\mu\nu}\xi^\nu = g^{\alpha\mu}\Gamma^\beta_{\mu\phi}\xi^\phi = g^{\alpha\mu}\Gamma^\beta_{\mu\phi}$. So, we have for $J_{Horizon}$ that

$$
J_{Horizon} = -\frac{1}{8\pi}\int_{\mathcal{H}} d\Sigma_{\alpha\beta}\nabla^\alpha\xi^\beta \ = -\frac{1}{8\pi}\int_0^\pi d\theta\int_0^{2\pi} d\phi r^2\sin\theta n_\mu\sigma_\nu g^{\mu\alpha}\Gamma^\nu_{\alpha\phi} \ . \tag{3.75}
$$

For our Kerr-like black hole of relation (3.25), we find

$$
n_\mu\sigma_\nu g^{\mu\alpha}\Gamma^\nu_{\alpha\phi} = \frac{\sin^2\theta(-6M-2ry+r^2y')a}{2r^2} \ . \tag{3.76}
$$

Substituting, we have

$$
\begin{aligned}
J &= -\frac{1}{8\pi}\int_0^\pi d\theta\int_0^{2\pi} d\phi r^2\sin\theta\frac{\sin^2\theta(-6M-2ry+r^2y')a}{2r^2} \ \Rightarrow \\
J &= -\frac{(-6M-2ry+r^2y')a}{16\pi}\int_0^\pi \sin^3\theta d\theta\int_0^{2\pi} d\phi \ \Rightarrow \\
J &= \frac{(6M+2ry-r^2y')a}{8}\int_0^\pi \sin^3\theta d\theta \Rightarrow \\
J &= \frac{(6M+2ry-r^2y')a}{6} \Rightarrow \\
J &= \left[1+\frac{2ry-r^2y'}{6M}\right]M\alpha
\end{aligned}
\tag{3.77}
$$

For $r \to +\infty$, we obviously reproduce the result for the Kerr black hole, which is $J_{r\to\infty} = M\alpha$, since both y, y' fall faster than r and r^{-2}, respectively. For $r = 2M$, we get the black hole's angular momentum

$$
J_{Horizon} = \left[1+\frac{2ry(r)-r^2y'(r)}{6M}\right]_{r=2M} M\alpha \ . \tag{3.78}
$$

For the axionic field, we get with similar steps that:

$$J_{Matter} = -\frac{1}{8\pi}\int_S d\Sigma_\alpha \left(T^\alpha_\beta - \frac{1}{2}\delta^\alpha_\beta T\right)\xi^\beta = -\frac{1}{8\pi}\int d^3x n_0 \sqrt{g^{(3)}} T^t_\phi \ , \tag{3.79}$$

where $g^{(3)}$ is the determinant of the induced 3-metric at the $t = const$ hypersurface and is given by $\sqrt{g^{(3)}} = \sqrt{r^4 \sin^2\theta}/\sqrt{(1-2M/r)} = r^2 \sin\theta/\sqrt{1-2M/r}$ and with $n_0 = c_1 = -\sqrt{1-2M/r}$ So, we have:

$$\begin{aligned} J_{Matter} &= \frac{1}{8\pi}\int_0^{2\pi} d\phi \int_0^\pi d\theta \int_{2M}^\infty dr r^2 \sin\theta T^t_\phi \Rightarrow \\ J_{Matter} &= \frac{1}{4}\int_0^\pi \sin\theta d\theta \int_{2M}^\infty dr r^2 T^t_\phi \end{aligned} \tag{3.80}$$

where we calculate that T^t_ϕ up tp $\mathcal{O}(a)$ is given by

$$T^t_\phi = \frac{1}{2}\sin^2\theta\left(\frac{2y}{r^2} - y''\right)\alpha \tag{3.81}$$

Substituting the above, we get

$$\begin{aligned} J_{Matter} &= \frac{1}{8}\alpha\int_0^\pi \sin^3\theta d\theta \int_{2M}^\infty \left(2y - r^2 y''\right) dr \Rightarrow \\ J_{Matter} &= -\frac{1}{6}\alpha\int_{2M}^\infty \frac{d}{dr}\left(r^2 y' - 2ry\right) dr \Rightarrow \\ J_{Matter} &= -\frac{\alpha}{6}\left[r^2 y' - 2ry\right]_{2M}^\infty \end{aligned} \tag{3.82}$$

and since $r^2 y(r)' \to 0$ and $ry(r) \to 0$, as $r \to \infty$, we finally end up with:

$$J_{Matter} = -\left[\frac{2ry(r) - r^2 y(r)'}{6M}\right]_{r=2M} M\alpha \ . \tag{3.83}$$

Thus, adding togenther $J_{Horizon}$ and J_{Matter}, we find that *the total angular momentum yields*:

$$J = J_{Horizon} + J_{Matter} = M\alpha \tag{3.84}$$

This is an intresting result. We see that in accordance with the Kerr case, the total angular momentum is equal to $J = Ma$, but now there is an important differentiation, and it has to do with the **non-trivial inner structure, consisting of the axionic and of the black hole contribution to the angular momentum**. The existence of the dynamical axionic field implies the rotation of the background spacetime. So, the axionic field causes in a way the rotation of the black hole. The γ parameter appearing in eq.(3.47), given by $\gamma^2 = A^2\kappa^2/M^4$, **measures the strength of the backreaction on the background geometry**, in the sense that the backreaction is stronger for an increasing γ. *The axion cloud around the black hole acquires an angular momentum in such a way that the total angular momentum remains constant and equal to* Ma, as shown in Figure (3.2). This is a demonstration of the exchange of energy between the axionic and the gravitational field.

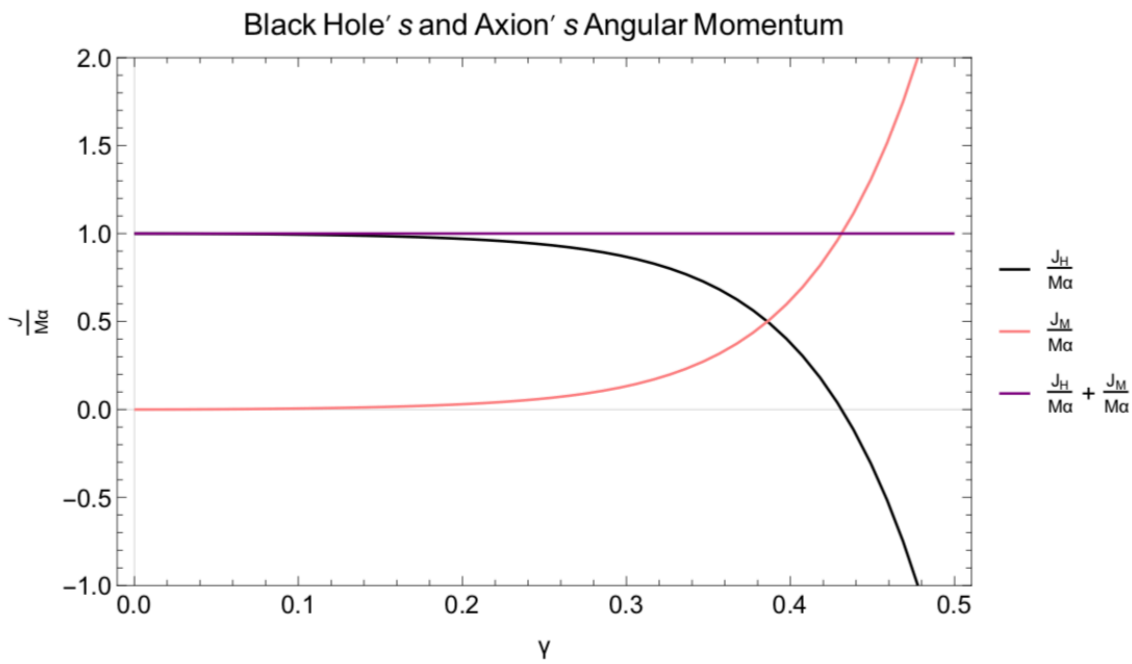

Figure 3.2: Total angular momentum for the system of the black hole and the axionic matter field.

3.5 Geodesics in slow-rotating Black Holes

In the following analysis, we study the effects of the perturbation parameter γ we mentioned before, given by $\gamma = A\kappa/M^2$, and study its influence on the background spacetime. It is this dimensionless parameter that appeared in the correction function of the Kerr-like black hole solution we presented above, and captures the strength of the axionic field backreaction on the geometry. We showed that the total angular momentum still equals to Ma, as in the Kerr case, with an important differantiation: *the existence of an internal structure consisting of the axionic and the black hole contribution.*

We will demonstrate that for increasing γ, the contribution to the total angular momentum coming from the black hole decreases, reaching a critical value for $\gamma = \gamma_{crit.}$, after which, the axionic black hole starts to counter-rotate. Hence, the angular momentum of the black hole may reach large values in magnitude, as long as the J_{total} remains constant and equals to $M\alpha$. The slowly rotation approximation is still valid cause of the aforementionned internal structure and the "interaction" of the two opossite systems, the black hole and the axionic field.

3.5.1 Geodesics in a Slowly Rotating Kerr Black Hole

We start by studying the motion of a particle around the Kerr black hole in the slow-rotation limit, which is given by the line element:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2 + 2g_{t\phi}dtd\phi \ . \tag{3.85}$$

where

$$g_{t\phi} = -\frac{2M\alpha}{r}\sin^2\theta \tag{3.86}$$

The parameter α, defines the *angular momentum of the black hole per unit mass* derived from the Komar integral. An axisymmetric and stationary spacetime, independent of t, ϕ, admits the corresponding Killing vectors:

$$k = \partial_t \quad and \quad \xi = \partial_\phi \ , \tag{3.87}$$

with k being timelike and ξ being spacelike outside the black hole horizon. According to these Killing vectors, there are three constants of motion, namely

$$\begin{aligned} &E = -k^\mu u_\mu \to g_{tt}\dot{t} + g_{t\phi}\dot{\phi} = -E \ , \\ &L_z = \xi^\mu u_\mu \to g_{t\phi}\dot{t} + g_{\phi\phi}\dot{\phi} = L_z \ , \\ &g_{\mu\nu}u^\mu u^\nu = \epsilon \ , \end{aligned} \tag{3.88}$$

where E is the energy, L_z the z-component of its angular momentum, assuming the rotation is around the z-axis, and u^μ is the tangent vector to the geodesic, with $\epsilon = -1$

for massive particles and $\epsilon = 0$ for massless particles. From the first two constants of motion, we have:

$$\begin{aligned} \dot{t} &= \frac{g_{t\phi}L_z + g_{\phi\phi}E}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}} \ , \\ \dot{\phi} &= -\frac{g_{t\phi}E + g_{tt}L_z}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}} \ , \end{aligned} \tag{3.89}$$

while, substitution to the third one results in the equation

$$\begin{aligned} & g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{tt}\dot{t}^2 + g_{\phi\phi}\dot{\phi}^2 + 2g_{t\phi}\dot{t}\dot{\phi} - \epsilon = 0 \Rightarrow \\ & g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + \frac{L_z^2 g_{tt} + E^2 g_{\phi\phi} + 2EL_z g_{t\phi}}{g_{tt}g_{\phi\phi} - g_{t\phi}^2} - \epsilon = 0 \ , \\ & where \ \ V^E(r,\theta) = \frac{L_z^2 g_{tt} + E^2 g_{\phi\phi} + 2EL_z g_{t\phi}}{g_{tt}g_{\phi\phi} - g_{t\phi}^2} - \epsilon \ . \end{aligned} \tag{3.90}$$

In order to define the effective potential, we set $\dot{r} = \dot{\theta} = 0$ and solve the equation $V^E(r,\theta) = 0$ with respect to E, where the corresponding function can be interpreted as our effective potential. We may rewrite $V^E(r,\theta)$ as

$$V^E(r,\theta) = -\frac{g_{\phi\phi}E^2 + 2L_z g_{t\phi}E + L_z^2 g_{tt} + \epsilon\tilde{\Delta}}{\tilde{\Delta}} \ , \tag{3.91}$$

where $\tilde{\Delta} = g_{t\phi}^2 - g_{tt}g_{\phi\phi}$. The metric determinant is given by $g = -g_{rr}g_{\theta\theta}\left[g_{t\phi}^2 - g_{tt}g_{\phi\phi}\right] = -g_{rr}g_{\theta\theta}\tilde{\Delta}$. Since the metric determinant is negative, and $g_{rr}g_{\theta\theta} > 0$, we get $\tilde{\Delta} > 0$. Setting now $V^E = 0$ we extract a quadratic equation with respect to E, with the solution given by:

$$V^{(\pm)}(r,\theta) = -\frac{L_z g_{t\phi}}{g_{\phi\phi}} \pm \frac{\sqrt{\tilde{\Delta}}}{g_{\phi\phi}}\sqrt{L_z^2 - \epsilon g_{\phi\phi}} \ . \tag{3.92}$$

and so eq.(3.90) becomes

$$g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 = \frac{(E - V^{(+)})(E - V^{(-)})}{\tilde{\Delta}} \ . \tag{3.93}$$

The left hand side of the above equation is non-negative and vanishes at the turning points of the phase space at which $E = V^{(+)}$ or $E = V^{(-)}$.

For the slowly rotating case that we are considering here, **the ergosphere in the exterior region vanishes, coinciding with the event horizon at** $g_{tt} = 0$. This implies that the energy of the particle is $E = -k^\mu u_\mu$ and is positive definite outside the horizon, since the Killing vector remains timelike in the exterior region. It can easily be checked, since, considering the ergosphere, the Killing vector k becomes spacelike outside the event horizon (since it's already spacelike at the event horizon and is timelike at infinity) at: $r^2 - 2GMr + \alpha^2 cos^2\theta = 0 \Rightarrow r = GM + \sqrt{(GM)^2 - \alpha^2 cos^2\theta}$. In the slow rotation limit for small α, this implies the vanishing of the ergosphere.

Focusing on the potential of eq.(3.92) we may conclude that $V^{(-)}$ can be eliminated as a candidate, making the following arguments:

1. $L_z g_{t\phi} < 0$: Since $g_{t\phi}$ is negative, this case corresponds to particles *co-rotating with the black hole geometry*. In this case, $V^{(+)} > 0$ for all the exterior region, by a simple check of eq.(3.92). Assuming $V^{(-)} \geq 0$ and setting $\epsilon = -1$ for the timelike case, we find that:

$$\begin{aligned}
&V^{(-)}(r,\theta) \geq 0 \Rightarrow \\
&-\frac{L_z g_{t\phi}}{g_{\phi\phi}} - \frac{\sqrt{\tilde{\Delta}}}{g_{\phi\phi}}\sqrt{L_z^2 + g_{\phi\phi}} > 0 \Rightarrow \\
&-L_z g_{t\phi} > \sqrt{\tilde{\Delta}}\sqrt{L_z^2 + g_{\phi\phi}} \\
&L_z^2 g_{t\phi}^2 > \tilde{\Delta} L_z^2 + \tilde{\Delta} g_{\phi\phi} \Rightarrow \\
&g_{tt} L_z^2 > g_{t\phi}^2 - g_{tt} g_{\phi\phi} \Rightarrow \\
&L_z^2 < \frac{g_{t\phi}^2}{g_{tt}} - g_{\phi\phi} \ , \quad Not \ \ possible.
\end{aligned} \tag{3.94}$$

where the last line can't be valid, and that's because we consider $g_{tt} < 0$, $g_{t\phi} < 0$ and $g_{\phi\phi} > 0$ from our metric in eq.(3.85). The above tell us that $V^{(-)} < 0$ and so $(E - V^{(-)}) > 0$, $\forall E$. Thus, the effective potential cannot be given in this case by $V^{(-)}$ and should be given by $V^{(+)}$.

2. $L_z g_{t\phi} > 0$: This case corresponds to *counter rotating particles*. Obviously, in this case $V^{(-)} < 0$ from eq.(3.92) and consequently $(E - V^{(-)}) > 0$, $\forall E$. Making now the assumtion that $V^{(+)} < 0$, we get, like before, that: $-g_{tt}L_z^2 < -\tilde{\Delta}$, for $\epsilon = -1$. Since, $\tilde{\Delta} > 0$ outside the event horizon, we conclude that $V^{(+)}$ becomes negative in a region defined by $g_{tt} > 0$, which is in the inside region of the black hole. So, we can interpret $V^{(+)}$ as the effective potential since the regions of motion are determined by $V^{(+)} \leq E$.

We argued that $V^{(+)}$ must be the effective potential, given by:

$$V_{eff}(r,\theta) = -\frac{L_z g_{t\phi}}{g_{\phi\phi}} + \frac{\sqrt{\tilde{\Delta}}}{g_{\phi\phi}}\sqrt{L_z^2 - \epsilon g_{\phi\phi}} \ , \tag{3.95}$$

with the allowed motion confined in regions where $E \geq V_{eff}(r,\theta)$. Up to $\mathcal{O}(a)$ for the line element of eq.(3.85), we have:

$$V_{eff}(r,\theta) = \frac{\sqrt{r(r-2M)(L_z^2 + r^2\sin^2(\theta))}}{r^2\sin(\theta)} + \frac{2L_z M a}{r^3} + \mathcal{O}(a^2) \ . \tag{3.96}$$

which is plotted in the above Figures (3.3,3.4) taken from [20]:

A local maximum and a local minimum do appear, characterising bound orbits. Our effective potential is θ-dependent, implying that the motion isn't constrained on a plane. We can draw the turning points at the θ–r with $V_{effective} = E$ and E some

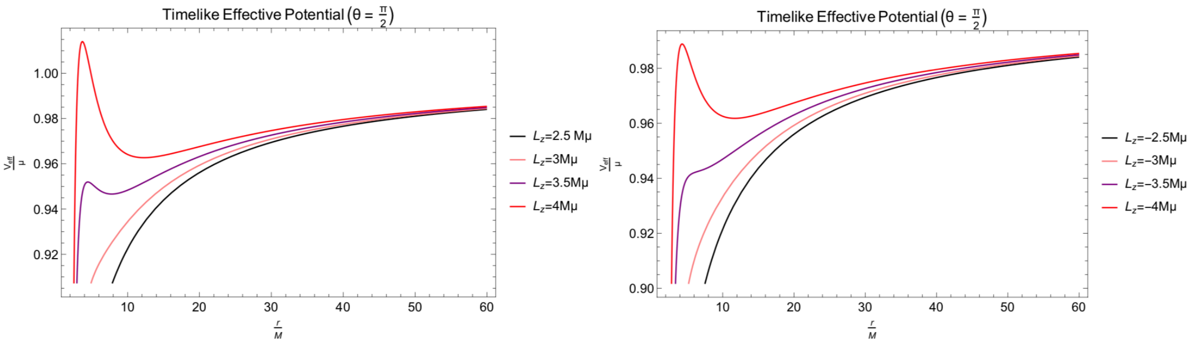

Figure 3.3: Effective potential at the equatorial plane for timelike geodesics for the slowly rotating Kerr metric. The parameter α is fixed to $\alpha = 0.1$

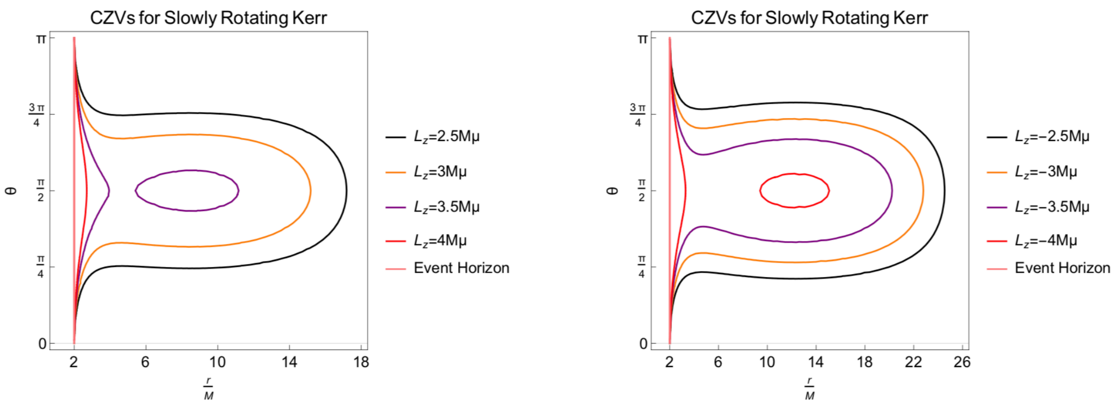

Figure 3.4: Curves of Zero Velocity (CZV) for timelike geodesics. The plots are evaluated for $M = 1$, $\alpha = 0.1$ and $E = 0.95$ (for the left hand site) and $E = 0.9633$ (for the right hand site).

energy and then plot the corresponding contour plot, as we do in Fig. 3.3. The phase space points of $V_{eff} = E$ defines curves of zero velocity (CZV), since the left hand side of the equation enforces $\dot{r} = \dot{\theta} = 0$. The graphs in the CZV figures were obtained for a particle's energy $E = 0.95$ (in units of its rest mass μ). Such particles can't escape to infinity. We mention here that whether a particle falls into the black hole or orbits around it, depends on the initial conditions. There is a case in the graphs that yields a region enclosed by an ellipse, in the exterior region, for which massive particles are trapped there and they cannot escape to infinity nor fall into the black hole. For counter-rotating trajectories ($L_z < 0$) the quality of the structure is the same.

3.5.2 Kerr-like Rotating Black Hole with Axionic-hair

Let's continue with the same procedure, considering now the action (3.16):

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} - \frac{1}{2}(\partial_\mu b)(\partial^\mu b) - A\, b\, R_{CS} \right] \tag{3.97}$$

Following the results of our previous analysis, we have the equations of motion given by eq.(3.18):

$$\begin{aligned} \delta\mathcal{S} = \delta\mathcal{S}_{EH} + \delta\mathcal{S}_b + \delta\mathcal{S}_{CS} = 0 \Rightarrow \\ G_{\mu\nu} = \kappa^2 T^b_{\mu\nu} + 4\kappa^2 A C_{\mu\nu} \ , \\ \Box b = A\, R_{CS} \ , \end{aligned} \tag{3.98}$$

The solution we discussed in Section 3.2, reads:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2 + 2 g_{t\phi}(r,\theta) dt d\phi \ , \tag{3.99}$$

with

$$g_{t\phi} = -\left(\frac{2M}{r} + y(r)\right) \alpha\ \sin^2(\theta) \ , \tag{3.100}$$

and

$$y(r) \equiv r^2 w(r) = \sum_{n=4}^{\infty} \frac{d_n M^{n-2}}{r^{n-2}} \ , \tag{3.101}$$

The coefficients d_n are given by:

$$d_n = \frac{2(n-5)^2(n-1)}{n(n-6)(n-3)} d_{n-1} + \frac{576 A^2 \kappa^2}{n(n-3)M^4} d_{n-6}, \ \text{ for } n \geq 10 \ , \tag{3.102}$$

where, as we explained:

$$d_4 = d_5 = 0 \ , \ d_6 = -5\gamma^2 \ , \ d_7 = -\frac{60\gamma^2}{7} \ , \ d_8 = -\frac{27\gamma^2}{2} \ , \ d_9 = 0, \tag{3.103}$$

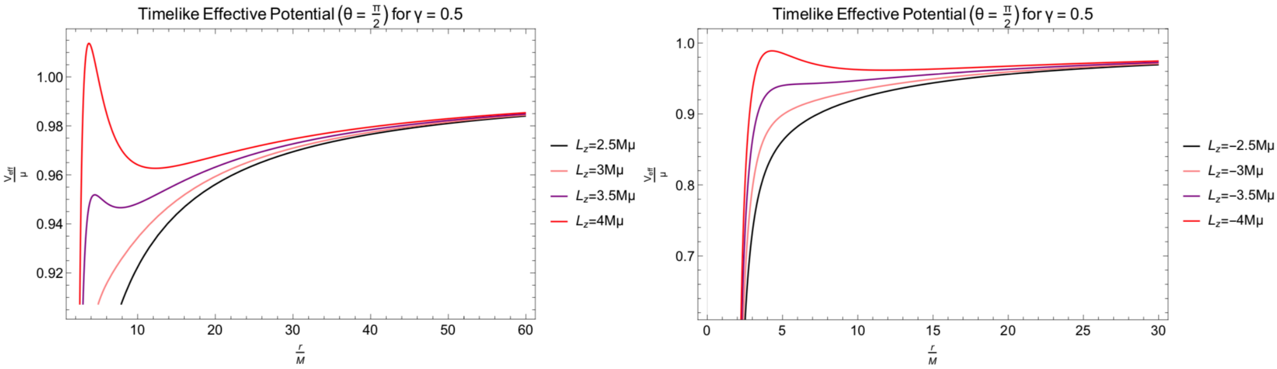

Figure 3.5: Effective potential at the equatorial plane for timelike geodesics for the slowly rotating deformed Kerr metric. ($\gamma = 0.5, \alpha = 0.1$)

with γ the dimensionless parameter

$$\gamma^2 = \frac{A^2 \kappa^2}{M^4} \,, \tag{3.104}$$

measuring how strong the axionic backreaction becomes on our geometry. The **axion field b** reads:

$$b(r,\theta) = -\frac{r^5 (y/r^2)'}{24\gamma^2 M^5} \alpha A \cos(\theta) = \alpha A \cos\theta \left(-\frac{5}{4Mr^2} - \frac{5}{2r^3} - \frac{9M}{2r^4} \right) + \mathcal{O}(A^m) \quad \text{for } \; m = 2n+1, \; n \in \mathbb{Z}^+ \tag{3.105}$$

Since the function $y(r)$ (and consequently $w(r)$) is expressed in terms of power series of γ, the Kerr metric deformation depends both on the coupling constant A and on the black hole mass M.

The function $y(r) = r^2 w(r)$ exists due to the Kerr metric deformation; it results from the backreacting axion field to the background spacetime. Looking at the deformed metric now, given by eq.(3.99) and $g_{t\phi}$ given by eq.(3.100), the effective potential of the geodesics is calculated to be:

$$V_{eff} = V_{eff}^{Kerr} + \frac{L_z \alpha y(r)}{r^2} \tag{3.106}$$

Because of the alteration of the effective potential due to the $y(r)$-term, we expect to see some kind of different behaviour concerning the orbits of particles (timelike in our case) around the black hole. Plotting again the effective potential and the zero velocity curves for increasing values of γ and for positive and negative angular momentum, we get the Figures (3.5,3.6,3.7,3.8).

For $\gamma = 2$ we see something that differentiates. The effective potential starts to behave in a repulsive way for the counter-rotating case, close to the horizon. What

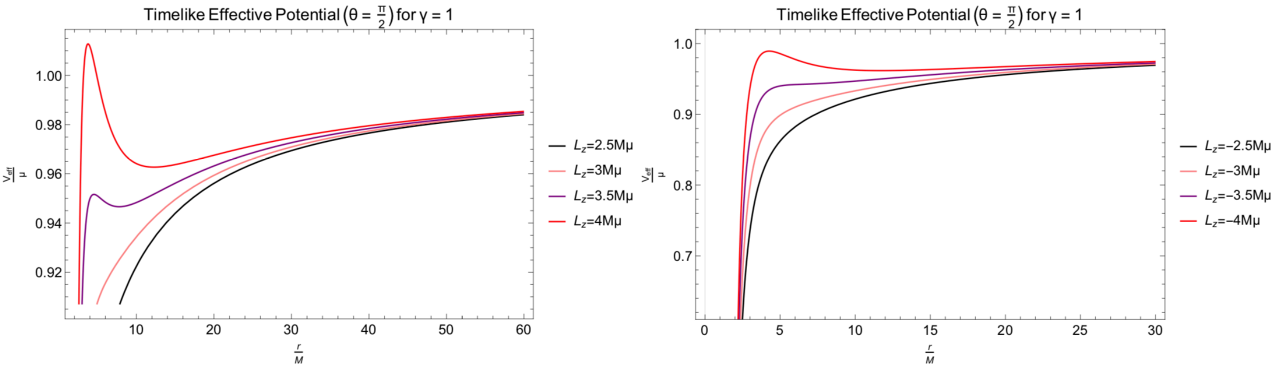

Figure 3.6: Effective potential at the equatorial plane for timelike geodesics for the slowly rotating deformed Kerr metric. $(\gamma = 1, \alpha = 0.1)$

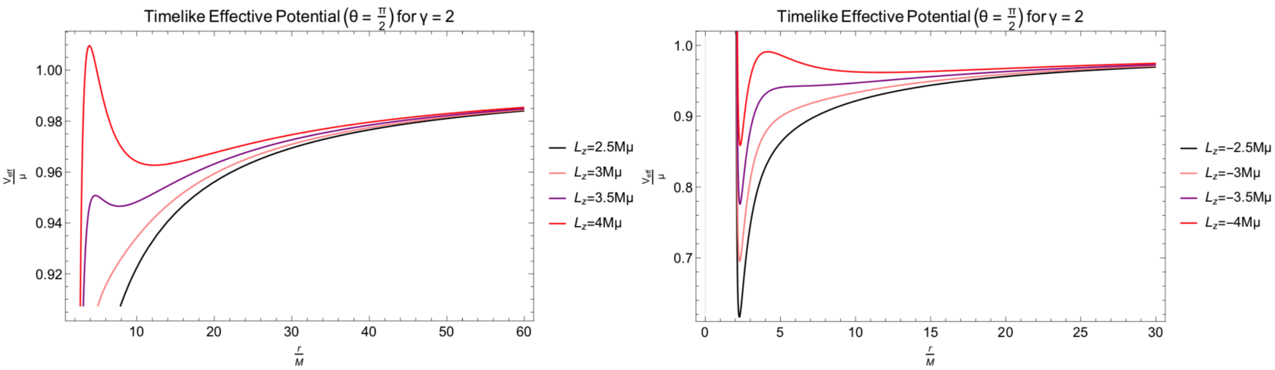

Figure 3.7: Effective potential at the equatorial plane for timelike geodesics for the slowly rotating deformed Kerr metric. $(\gamma = 2, \alpha = 0.1))$

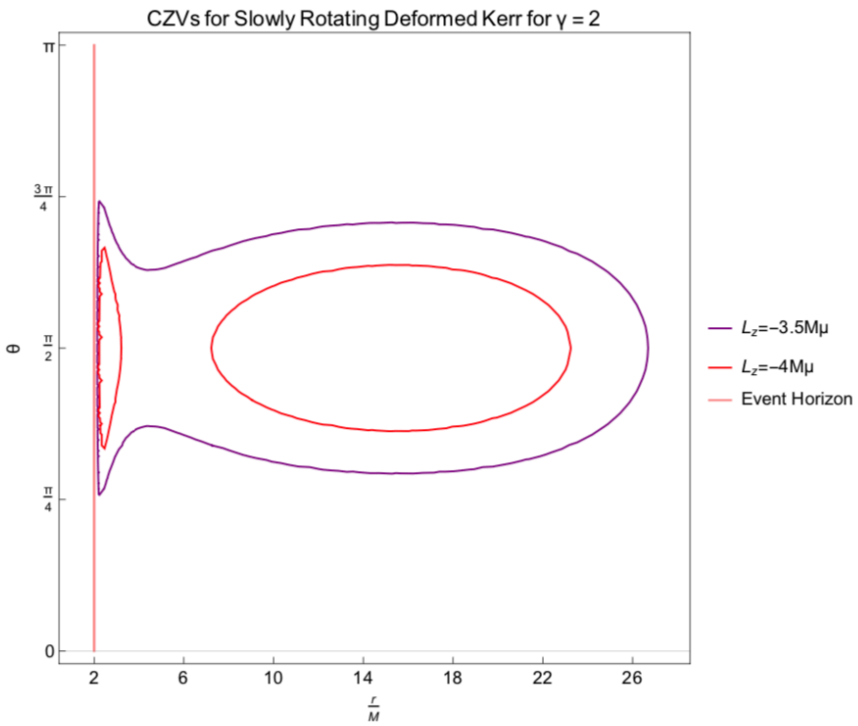

Figure 3.8: Curves of Zero Velocity for rotating deformed Kerr metric of our case. $(\gamma = 2, \alpha = 0.1)$

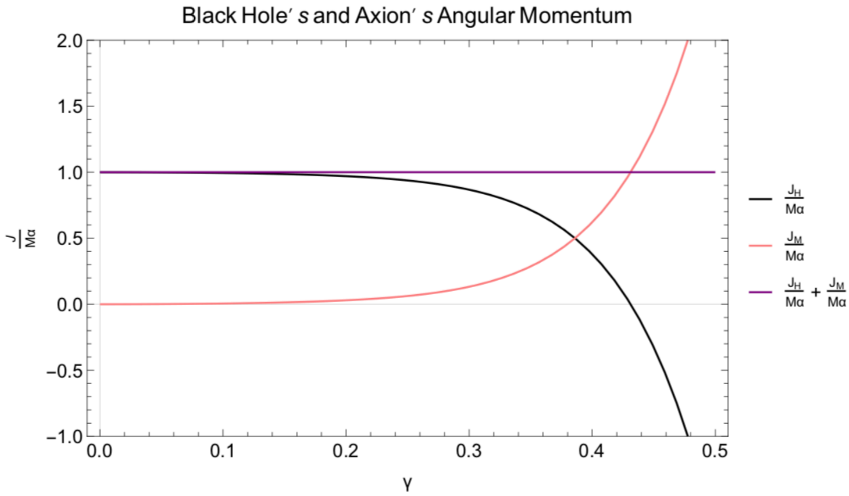

Figure 3.9: Total angular momentum for the system of the Black Hole and the axionic matter field.

is more, there seems to appear new local minimum, implying bound orbits, close to the horizon. Going too close to the horizon, the behaviour of the potential signifies scattering for such particles, and they cannot fall into the black hole region.

In Figure 3.8 for the zero velocity curves, we can see this effect, where, far away from the horizon, the behaviour is similar to the slowly rotating Kerr metric, but turning points close to the horizon start to appear, preventing counter rotating particles to fall into the black hole.

As we discussed in Section 3.4, the total angular momentum in the deformed Kerr case equals to Ma, like the Kerr case, with an internal structure consisting of two competitive systems, the axionic field and the black hole. After the calculations performed, we see that for increasing γ, the black hole's angular momentum starts to decrease, where, for some critical value of $\gamma = \gamma_{crit.}$, the black hole's angular momentum vanishes. For $\gamma > \gamma_{crit.}$, the black hole starts to rotate in the opposite direction. Thus, the two systems can obtain large angular momentum values in magnitude, as long as the total angular momentum measured at infinity remains constant and equal to $M\alpha$, like in the slowly rotating case. This is qualitatively illustrated in Figure 3.9, where we see that because of the energy exchange of the two systems, the axionic field "tends" to rotate the black hole in the other way, as γ becomes stronger.

Chapter 4

Outlook

The subject of this thesis was an analytical review, both physical and mathematical, about the nature and the validity of the no-hair theorems and the examination of a system consisting of a slowly rotating axionic black hole. We started by excluding the possibility of hair for specific theories, following the reasonings made by Bekenstein's no-scalar-hair conjecture. We seeked to generilize these ideas and formulate a model independant analysis, looking only at the properties of an effective energy momentum tensor. The motivation of such a formulation originates from the insight that the violation of the energy conditions is strongly related to the evasion of the no-hair conjecture.

For Bekenstein's basic assumption, which is the case of no-violation of the energy conditions at all, no hair is allowed. We argued that hair might be allowed only for the cases where WEC or NEC is violated, but not both. This means, that a black hole might have hair, not just by the violation of the NEC or WEC. It seems that hair is allowed in cases where the static and the fastly moving observers have a disagreement about the sign of the energy-density.

The above calculations where based on the fact that $\mathcal{G} = \mathcal{E} + T_\theta{}^\theta = 0$. We can be more general and consider the cases where $\mathcal{G} \neq 0$, which is exaclty the case for an effective energy momentum tensor if the field has a θ-dependance and/or is coupled to higher order curvature terms. We argued that, as long as $\mathcal{G}/\mathcal{J} \geq 0$, there is a possibility for hair to exist.

In the second part of this thesis, we constructed a slowing rotating black hole, considering an axion field coupled to the Chern-Simons geometric term. We seeked an exact slow rotating Kerr-type black hole solution, "dressed" with axionic hair. The solution is expressed in analogy to the inverse powers of the radial distance from the centre of the black hole. Such a behaviour allows us to go arbitrarily close to the horizon of the axionic black hole.

We constructed an effective energy momentum tensor and confirmed that the NEC is violated in the outside region of the axionic black hole. NEC is violated in such a way that the spacetime deformation around the horizon allows the axionic field to exist. The violation is stronger near the horizon, which comes from the fact that the axion "lives" mostly near the horizon, as it vanishes asymptotically. The violation becomes more important as the coupling becomes stronger and stronger.

We continued by studying the angular momentum of the axion-black hole system. The axionic matter in the outside region of the horizon acquires an angular momentum in a way that the total angular momentum of the axionic black hole remains constant and equals to Ma. The system is characterised by a dimensionless parameter $\gamma \sim A/M^2$. We found that, as γ increases, the black hole's angular momentum decreases more and more, reaching a critical value $\gamma_{crit.}$ beyond which it starts to counter-rotate, reaching larger and larger values in magnitude. This is an effect of two competing systems, the Kerr-like black hole and the axionic-matter rotating outside the horizon. As the coupling gets stronger, the energy exchange between the axionic field and the gravitational field increases.

Finally, we looked at the behaviour of the timelike geodesics around the axionic black hole. Because of the alteration of the Kerr effective potential due to the $y(r)$-term (the $t\phi$ correction of our metric), we expected to witness a different behaviour concerning the orbits of timelike particles around the black hole. Plotting the effective potential and the zero velocity curves for increasing values of γ for positive and negative angular momentum, we found that for $\gamma = 2$ the effective potential starts to behave in a repulsive way for the counter-rotating case, close to the horizon. What is more, there seems to appear new local minimum, implying bound orbits, close to the horizon.

This is a behaviour that we also find in the extreme case of the Kerr metric, but this is not a right analogy. That's because, in the extreme Kerr, we have to deal with a highly rotating horizon in vacuum. representing a highly rotating effect. In our case, we have a spacetime that remains slowly rotating ($\mathcal{O}(\alpha)$), but we have two competitive systems that can reach larger and larger values of angular momentum, as long as the total angular momentum of the spacetime remains constant. The violation of the NEC, which is stronger near the horizon, is repsonsible for the repulsive nature of the geometry concerning the counter rotating geodesics, something that motivates a more thorough and in-depth investigation regarding these "puzzling" energy conditions and their relation to the existance of hairy black holes.

It's important to study more thorough the case of $\mathcal{G} = \mathcal{E} + T_\theta{}^\theta > 0$, in the spirit of Bekenstein's work in [2], and see if we can argue about the existence of hair in cases of higher order corrections.

It would also be intresting, since in this thesis our study was restricted to a slowly-rotating black hole ($\mathcal{O}(\alpha)$), to extend this work for fastly rotating black hole, at least of order $\mathcal{O}(\alpha^2)$ in the angular momentum parameter α, and see if this behaviour for the counter rotating geodesics would still be valid.

Appendices

Appendix A

Black Hole Solutions

1.1 Schwarzschild Solution

We begin with the simplest case, the Schwarzschild solution, describing the spacetime under the influence of a non-rotating, massive and spherically symmetric object. The assumption we are going to make are the following:

1. The system under consideration is spherically symmetric.
2. We assume vacuum conditions ($T_{\mu\nu} = 0$), so the Einstein field equations become $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$, or $R_{\mu\nu} = 0$.
3. A static spacetime; every metric component does not depend on time. The system is static and invariant under time reversal $t \to -t$.
4. Metric signature used here is (+,-,-,-).
5. $\hbar = c = 1$
6. No vacuum energy.

We begin by diagonalising the metric, where we can easily see from symmetry assumptions that every metric component $g_{\mu\nu} = 0$ for $\mu \neq \nu$, leading to a metric of the form $ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2$.

Moving on, we can go further and see that for a constant t, θ, ϕ hypersurface, $g_{00} = B(r)$, a function of only one r-variable, which we can also show for $g_{11} = -A(r)$.

For t, r =constant, we recover the 2-sphere metric $dl^2 = r^2(d\theta^2 + sin^2\theta d\phi^2)$, so the metric in of the form:

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{A.1}$$

What we have to do now, is to compute each Christoffel symbol from

$$\Gamma^i{}_{kl} = \frac{1}{2}g^{im}(\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl}) \tag{A.2}$$

in order to compute the Riemann curvature tensor

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma} \tag{A.3}$$

We start with a first simplification of the calculations, by noticing that

1. every derivative with respect to t is zero
2. $g_{\mu\nu} = 0$ and $g^{\mu\nu} = 0$ for $\mu \neq \nu$
3. the lower indices symmetry of the Christoffel symbols $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$

The only non-vanishing Christoffel symbols are the following (where prime U', V' denotes $\partial_r U, \partial_r V$):

$$\begin{aligned}
\Gamma^0_{01} = \Gamma^0_{10} &= \frac{B'}{2B} \\
\Gamma^1_{00} &= \frac{B'}{2A} \\
\Gamma^1_{11} &= \frac{A'}{2A} \\
\Gamma^1_{22} &= -\frac{r}{A} \\
\Gamma^1_{33} &= -\frac{r}{A}sin^2\theta \\
\Gamma^2_{12} = \Gamma^2_{21} &= \frac{1}{r} \\
\Gamma^2_{33} &= -cos\theta sin\theta \\
\Gamma^3_{31} = \Gamma^3_{13} &= \frac{1}{r} \\
\Gamma^3_{32} = \Gamma^3_{23} &= \frac{cos\theta}{sin\theta}
\end{aligned}$$

Now, we move on by calculating the Ricci tensor, noticing that $R_{\mu\nu} = 0$ for $\mu \neq \nu$

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu} = \partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\rho \Gamma^\rho_{\mu\nu} + \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\sigma\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} \tag{A.4}$$

and also the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} \tag{A.5}$$

and substituting them into the Einstein field equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$. We find that

$$R_{\mu\nu} = \begin{pmatrix} (-\frac{B''}{2A} + \frac{B'}{2A}(\frac{A'}{A} + \frac{B'}{B}) - \frac{B'}{rA}) & 0 & 0 & 0 \\ 0 & (\frac{B''}{2B} - \frac{B'}{4B}(\frac{A'}{A} + \frac{B'}{B}) - \frac{A'}{rA}) & 0 & 0 \\ 0 & 0 & (-1 - \frac{r}{2A}(\frac{A'}{A} - \frac{B'}{B}) + \frac{1}{A)} & 0 \\ 0 & 0 & 0 & sin^2\theta R_{22} \end{pmatrix} \tag{A.6}$$

and for the Ricci scalar we have

$$R = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} = \frac{1}{B}R_{00} - \frac{1}{A}R_{11} - \frac{1}{r^2}R_{22} - \frac{1}{r^2 sin^2\theta}R_{22} \tag{A.7}$$

and we get

$$R = R^{\mu}_{\mu} = -\frac{B''}{AB} + \frac{A'B'}{2BA^2} + \frac{(B')^2}{2B^2A} - \frac{2B'}{rAB} + \frac{2A'}{rA^2} + \frac{2}{r^2}(1 - \frac{1}{A}) \tag{A.8}$$

and so we can proceed with the componets of the field equations as follows:

$$\begin{aligned} R_{00} - \frac{1}{2}g_{00}R &= 0 \\ R_{11} - \frac{1}{2}g_{11}R &= 0 \\ R_{22} - \frac{1}{2}g_{22}R &= 0 \\ R_{33} - \frac{1}{2}g_{33}R &= 0 \end{aligned}$$

THe above equations lead to

$$\begin{aligned} \frac{A'}{rA^2} + \frac{1}{r^2}(1 - \frac{1}{A}) &= 0 \\ -\frac{B'}{rAB} + \frac{1}{r^2}(1 - \frac{1}{A}) &= 0 \\ -\frac{B'}{B} + \frac{A'}{A} - \frac{rB''}{B} + \frac{rA'B'}{2AB} + \frac{r(B')^2}{2B^2} &= 0 \\ R_{22} + \frac{r^2}{2}R &= 0 \end{aligned}$$

Noticing that the first from the above equation is only in terms of A, we solve and get

$$A(r) = \frac{1}{1 - \frac{C}{r}} \tag{A.9}$$

and inserting this result to $R_{11} - \frac{1}{2}g_{11}R = 0$ we find

$$B(r) = 1 - \frac{C}{r} \tag{A.10}$$

Finally, identifying $C = 2GM$ from the weak field approximation, we get the Schwarzschild solution to be

$$ds^2 = (1 - \frac{2GM}{r})dt^2 - (1 - \frac{2GM}{r})^{-1}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{A.11}$$

1.2 Reissner-Nordström solution

The Reissner–Nordström metric is a static solution of the Einstein–Maxwell field equations, corresponding to the gravitational field of a non-rotating, charged and spherically symmetric body of mass M. Considering the Einstein-Maxwell action $S = \int d^4x\sqrt{|g|}(\frac{R}{16\pi G} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu})$, we start with the field equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 8\pi G T_{\mu\nu}$, and the assumptions in this case are the following:

1. Charged point singularity in an otherwise empty space
2. Spherical symmetry and staticity
3. No vacuum energy
4. Metric signature used here is (+,-,-,-)

which gives us:

1. Varying $S = \int d^4x\sqrt{|g|}(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu})$ with respect to the metric we get for the electromagnetic tensor $T_{\mu\nu} = -\frac{2}{\sqrt{|g|}}\frac{\delta S_{EM}}{\delta g^{\mu\nu}} = F_{\alpha\mu}F^{\alpha\beta}g_{\nu\beta} - \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}$
2. $\Lambda = 0$
3. $g_{\mu\nu} =$ diag$(B(r), -A(r), -r^2, -r^2 sin^2\theta)$
4. $A_\mu = (\frac{Q}{4\pi r}, 0, 0, 0)$

Claclulating Christoffel symbols, we get from $\Gamma^i{}_{kl} = \frac{1}{2}g^{im}(\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl})$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{B'}{2B}$$
$$\Gamma^1_{00} = \frac{B'}{2A}$$
$$\Gamma^1_{11} = \frac{A'}{2A}$$
$$\Gamma^1_{22} = -\frac{r}{A}$$
$$\Gamma^1_{33} = -\frac{r}{A} sin^2\theta$$
$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$$
$$\Gamma^2_{33} = -cos\theta sin\theta$$
$$\Gamma^3_{31} = \Gamma^3_{13} = \frac{1}{r}$$
$$\Gamma^3_{32} = \Gamma^3_{23} = \frac{cos\theta}{sin\theta}$$

like the Schwarzschild case, and now, calculating Ricci tensor

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = \partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\rho \Gamma^\rho_{\mu\nu} + \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\sigma\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} \tag{A.12}$$

and we find

$$R_{\mu\nu} = \begin{pmatrix} (-\frac{B''}{2A} + \frac{B'}{2A}(\frac{A'}{A} + \frac{B'}{B}) - \frac{B'}{rA}) & 0 & 0 & 0 \\ 0 & (\frac{B''}{2B} - \frac{B'}{4B}(\frac{A'}{A} + \frac{B'}{B}) - \frac{A'}{rA}) & 0 & 0 \\ 0 & 0 & (-1 - \frac{r}{2A}(\frac{A'}{A} - \frac{B'}{B}) + \frac{1}{A}) & 0 \\ 0 & 0 & 0 & sin^2\theta R_{22} \end{pmatrix} \tag{A.13}$$

Now, before checking the Einstein equations, we calculate the electromagnetic tensor $F_{\mu\nu}$ and $F^{\mu\nu}$:

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{A.14}$$

where the latter equality comes from the anti-symmetry of the elecromagnetic tensor and the symmetry of the lower indices of the Christoffel symbols. So, we go on and find

$$F_{01} = -F_{10} = -\partial_1 A_0 = -\partial_r \frac{Q}{4\pi r} = \frac{Q}{4\pi r^2} \tag{A.15}$$

and the contravariant components are

$$F^{01} = g^{00}g^{11}F_{01} = -\frac{Q}{4\pi r^2 AB}, \quad F^{10} = g^{11}g^{00}F_{10} = \frac{Q}{4\pi r^2 AB} \tag{A.16}$$

and we can now write $F_{\mu\nu}$ and $F^{\mu\nu}$ in the better form

$$F_{\mu\nu} = \begin{pmatrix} 0 & \frac{Q}{4\pi r^2} & 0 & 0 \\ -\frac{Q}{4\pi r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.17}$$

and

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{Q}{4\pi r^2 AB} & 0 & 0 \\ \frac{Q}{4\pi r^2 AB} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.18}$$

What follows, is the computation of the stress energy momentum tensor $T_{\mu\nu}$, so we start by calculating

$$F^{\alpha\beta}F_{\alpha\beta} = F^{01}F_{01} + F^{10}F_{10} = -\frac{Q^2}{8\pi^2 r^4 AB} \tag{A.19}$$

and from $T_{\mu\nu} = F_{\alpha\mu}F^{\alpha\beta}g_{\nu\beta} - \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}$ we have:

$$\begin{aligned} T_{00} = F_{\alpha 0}F^{\alpha\beta}g_{0\beta} - \frac{1}{4}g_{00}F^{\alpha\beta}F_{\alpha\beta} &= -\frac{Q^2}{32\pi^2 r^4 A} \\ T_{11} = F_{\alpha 1}F^{\alpha\beta}g_{1\beta} - \frac{1}{4}g_{11}F^{\alpha\beta}F_{\alpha\beta} &= \frac{Q^2}{32\pi^2 r^4 B} \\ T_{22} = F_{\alpha 2}F^{\alpha\beta}g_{2\beta} - \frac{1}{4}g_{22}F^{\alpha\beta}F_{\alpha\beta} &= -\frac{Q^2}{32\pi^2 r^2 AB} \\ T_{33} = F_{\alpha 3}F^{\alpha\beta}g_{3\beta} - \frac{1}{4}g_{33}F^{\alpha\beta}F_{\alpha\beta} &= -\frac{sin^2\theta Q^2}{32\pi^2 r^2 AB} \end{aligned}$$

or, in the same sense:

$$T_{\mu\nu} = \begin{pmatrix} -\frac{Q^2}{32\pi^2 r^4 A} & 0 & 0 & 0 \\ 0 & \frac{Q^2}{32\pi^2 r^4 B} & 0 & 0 \\ 0 & 0 & -\frac{Q^2}{32\pi^2 r^2 AB} & 0 \\ 0 & 0 & 0 & -\frac{sin^2\theta Q^2}{32\pi^2 r^2 AB} \end{pmatrix} \tag{A.20}$$

and now we are ready to construct the field equations for the unknown metric functions in order to find every component of our metric. Substituting the above into the Einstein field equations, and working with the trace inverted equations (because stress energy momentum tensor is traceless we get:

$$R_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{A.21}$$

from which we find

$$\begin{aligned}(-\frac{B''}{2A}+\frac{B'}{2A}(\frac{A'}{A}+\frac{B'}{B})-\frac{B'}{rA}) &= -8\pi G\frac{Q^2}{32\pi^2r^4A}\\ (\frac{B''}{2B}-\frac{B'}{4B}(\frac{A'}{A}+\frac{B'}{B})-\frac{A'}{rA}) &= 8\pi G\frac{Q^2}{32\pi^2r^4B}\\ (-1-\frac{r}{2A}(\frac{A'}{A}-\frac{B'}{B})+\frac{1}{A}) &= -8\pi G\frac{Q^2}{32\pi^2r^2AB}\\ (-1-\frac{r}{2A}(\frac{A'}{A}-\frac{B'}{B})+\frac{1}{A}) &= -8\pi G\frac{Q^2}{32\pi^2r^2AB}\end{aligned}$$

From the first two of the above equations, dividing the first by B and the second by A and adding them together, we can easily show that A and B are multiplicative inverses $\rightarrow A = B^{-1}$. Now, inserting this to the third equation, we get

$$1 - rB' - B = \frac{GQ^2}{4\pi r^2} \tag{A.22}$$

and, demanding a solution of the form $B(r) = 1 - \frac{2MG}{r} + f(Q,r)$, because we want to recover the Schwarzschild Solution as $f(Q,r) \rightarrow 0$ for $Q \rightarrow 0$, we get

$$-rf'(Q,r) - f(Q,r) = \frac{GQ^2}{4\pi r^2} \tag{A.23}$$

and finally we get

$$f(Q,r) = \frac{GQ^2}{4\pi r^2} \tag{A.24}$$

and we reached our solution, since we now can find

$$B(r) = 1 - \frac{2MG}{r} + \frac{GQ^2}{4\pi r^2}, \;\; A(r) = \frac{1}{1 - \frac{2MG}{r} + \frac{GQ^2}{4\pi r^2}} \tag{A.25}$$

and, afterall,

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{2MG}{r}+\frac{GQ^2}{4\pi r^2} & 0 & 0 & 0\\ 0 & -\frac{1}{1-\frac{2MG}{r}+\frac{GQ^2}{4\pi r^2}} & 0 & 0\\ 0 & 0 & -r^2 & 0\\ 0 & 0 & 0 & -r^2 sin^2\theta \end{pmatrix} \tag{A.26}$$

and the line element takes its final form:

$$ds^2 = 1 - \frac{2MG}{r} + \frac{GQ^2}{4\pi r^2}dt^2 - \frac{1}{1-\frac{2MG}{r}+\frac{GQ^2}{4\pi r^2}}dr^2 - r^2d\theta^2 - r^2sin^2\theta d\phi^2 \tag{A.27}$$

1.3 Kerr solution

We consider a uniformly rotating black hole, described by a stationary and axisymmetric spacetime. Stationarity is translated to a timelike Killing vector K for our spacetime, while axisymmetry to a spacelike Killing vector R. Actually,

$$K = \partial_t, \qquad R = \partial_\phi \tag{A.28}$$

Enforcing these Killing vectors on our spacetime and consequently get a metric independent of the t and ϕ coordinates. Another implication of stationarity and axisymmetry is that the metric possesses the isometry of $t \to -t$, $\phi \to -\phi$. This implies that the $g_{t\phi}$ component of the metric is not trivial, while the $g_{tr}, g_{t\theta}, g_{r\phi}, g_{\theta\phi}$ components of the metric need to be vanished. The general form of the metric with the above assumptions reads

$$g_{\mu\nu} = \begin{pmatrix} g_{tt}(r,\theta) & 0 & 0 & g_{t\phi}(r,\theta) \\ 0 & g_{rr}(r,\theta) & g_{r\theta}(r,\theta) & 0 \\ 0 & g_{r\theta}(r,\theta) & g_{\theta\theta}(r,\theta) & 0 \\ g_{t\phi}(r,\theta) & 0 & 0 & g_{\phi\phi}(r,\theta) \end{pmatrix} \tag{A.29}$$

We may use a coordinate transformation to enforce that the $g_{r\theta}$ components of the metric vanish, by diagonalising the spacelike three-dimensional submatrix of (A.29). So, the metric ansatz becomes:

$$ds^2 = g_{tt}(r,\theta)dt^2 + 2g_{t\phi}(r,\theta)dtd\phi + g_{rr}(r,\theta)dr^2 + g_{\theta\theta}(r,\theta)d\theta^2 + g_{\phi\phi}(r,\theta)d\phi^2 \tag{A.30}$$

We continue with two physical assumptions :

1. The metric should be reduced to the Schwarzschild one when rotation goes to zero.
2. Rotating bodies change in shape in the equatorial plane, away from the rotation-axis. This suggest that we may start using ellipsoidal coordinates to describe our case.
3. Metric signature used here is (-,+,+,+)

So, we have, for a parameter α, that:

$$x = \sqrt{r^2 + \alpha^2} sin\theta cos\phi, \quad y = \sqrt{r^2 + \alpha^2} sin\theta sin\phi, \quad z = rcos\theta \tag{A.31}$$

and the flat metric is of the form

$$ds^2 = -dt^2 + \frac{r^2 + \alpha^2 cos^2\theta}{r^2 + \alpha^2} dr^2 + \left(r^2 + \alpha^2 cos^2\theta\right) d\theta^2 + (r^2 + \alpha^2) sin^2\theta d\phi^2 \tag{A.32}$$

The next step, is to find a coordinate system (T, r, θ, ϕ), such that g_{TT} is inverse proportional to g_{rr} for the Schwarzschild metric limit to be visible in a clear way. Noticing that $g_{rr} = \rho^2 = r^2 + \alpha^2 cos^2\theta$, we may re-write:

$$-dt^2 = -\frac{\rho^2}{\rho^2}dt^2 = -\left(\frac{r^2+\alpha^2}{\rho^2}\right)dt^2 + \left(\frac{\alpha^2 sin^2\theta}{\rho^2}\right)dt^2 \tag{A.33}$$

while for the $g_{\phi\phi}$ component we have

$$\begin{aligned} g_{\phi\phi}d\phi^2 = (r^2+\alpha^2)sin^2\theta d\phi^2 = \frac{\rho^2}{\rho^2}(r^2+\alpha^2)sin^2\theta d\phi^2 = \\ \frac{(r^2+\alpha^2)^2 sin^2\theta}{\rho^2}d\phi^2 - \frac{(r^2+\alpha^2)\alpha^2 sin^4\theta}{\rho^2}d\phi^2 \end{aligned} \tag{A.34}$$

So, rewritting the flat metric A.32 and after some algebra, we have

$$\begin{aligned} ds^2 = -\left(\frac{r^2+\alpha^2}{\rho^2}\right)\left[dt^2 + \alpha^2 sin^4\theta d\phi^2\right] + \left(\frac{\rho^2}{r^2+\alpha^2}\right)dr^2 + \rho^2 d\theta^2 + \\ \frac{sin^2\theta}{\rho^2}\left[(r^2+\alpha^2)^2 d\phi^2 + \alpha^2 dt^2\right] \end{aligned} \tag{A.35}$$

Now, by adding and subtracting the term $2\alpha sin^2\theta\frac{r^2+\alpha^2}{\rho^2}dtd\phi$, we end up with

$$ds^2 = -\left(\frac{r^2+\alpha^2}{\rho^2}\right)dT^2 + \left(\frac{\rho^2}{r^2+\alpha^2}\right)dr^2 + \rho^2 d\theta^2 + \frac{sin^2\theta}{\rho^2}d\Phi^2 \tag{A.36}$$

where

$$dT = dt - \alpha sin^2\theta d\phi, \qquad d\Phi = (r^2+\alpha^2)d\phi - \alpha dt \tag{A.37}$$

and finally, for the case of a rotating black hole we have the metric

$$\begin{aligned} ds^2 = -\left(\frac{r^2+k(r)+\alpha^2}{\rho^2}\right)\left[dt - \alpha sin^2\theta d\phi\right]^2 + \left(\frac{\rho^2}{r^2+j(r)+\alpha^2}\right)dr^2 + \rho^2 d\theta^2 + \\ \frac{sin^2\theta}{\rho^2}\left[(r^2+\alpha^2)d\phi - \alpha dt\right]^2 \end{aligned} \tag{A.38}$$

where let us explain the "extra" terms $k(r), j(r)$. The function $k(r)$ at the g_{TT} component breaks staticity, while the mass terms contained in Schwarzschild metric enforce $k(r)$ and $j(r)$ functions to be inserted as well. The existence of these two functions break spherical symmetry down to axisymmetry.

The constraint that the above metric reduces to Schwarzschild in the limit of zero rotation yields that

$$k(r) = j(r) = -2GMr \tag{A.39}$$

Finally, after simple calculations, we have extracted **the metric for Kerr black hole in the Boyer-Lindquist coordinates**, which is given by

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 - \frac{4GMr}{\rho^2}\alpha sin^2\theta dtd\phi + \frac{sin^2\theta}{\rho^2}\left[(r^2+\alpha^2)^2 - \alpha^2 sin^2\theta\Delta\right]d\phi^2 \tag{A.40}$$

where, for J the angular momentum of the black hole:

$$\Delta = r^2 - 2GMr + \alpha^2, \quad \rho^2 = r^2 + \alpha^2 cos^2\theta, \quad \alpha = \frac{J}{M} \tag{A.41}$$

*Let's mention here that the Kerr metric can be genererilised to the **Kerr-Newman black hole**, which contains also the electromagnetic charge, with $\Delta = r^2 - 2GMr + \alpha^2 + G(Q^2 + P^2)$ with Q and P are the electric and magnetic charges.

Appendix B

Derivation of equations of motion 2.36,2.37

We start with the action (2.34)

$$S[g,\phi] = \int d^4x\sqrt{-g}\left[f(\phi)R - h(\phi)g^{\mu\nu}\nabla_\mu\phi\nabla\nu\phi\right] \tag{B.1}$$

Now let us to concentrate in the variation gravitational part of the action . Variation with respect to the metric, assuming that the ϕ-field is fixed, reads:

$$\delta_g S[g,\phi] = \int d^4x\left[\delta\sqrt{|g|}\left[f(\phi)R - h(\phi)(\nabla\phi)^2\right] + \sqrt{|g|}\left[f(\phi)\delta R - h(\phi)\nabla_\mu\phi\nabla_\nu\phi\delta g^{\mu\nu}\right]\right] \tag{B.2}$$

we have, also, that

$$\begin{aligned} \delta\sqrt{|g|} &= -\frac{1}{2}\sqrt{|g|}g_{\mu\nu}\delta g^{\mu\nu}, \\ \delta R &= R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}. \end{aligned} \tag{B.3}$$

Collecting terms together, we end up with

$$\delta_g S[g,\phi] = \int d^4x\sqrt{|g|}\delta g^{\mu\nu}\left[f(\phi)G_{\mu\nu} - h(\phi)\left[\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2\right]\right] + \delta_g\bar{S} \tag{B.4}$$

and

$$\delta_g\bar{S} = \int d^4x\sqrt{|g|}\,f(\phi)g^{\mu\nu}\delta R_{\mu\nu} \tag{B.5}$$

We know from Palatini equation that

$$\delta R_{\mu\nu} = \nabla_\lambda\left(\delta\Gamma^\lambda_{\mu\nu}\right) - \nabla_\nu\left(\delta\Gamma^\lambda_{\mu\lambda}\right) \tag{B.6}$$

with $\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\lambda}\left(\partial_\nu g_{\mu\lambda} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\mu\nu}\right)$ the Christoffel symbols, and the components of the Ricci tensor $R_{\mu\nu} = \partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\mu\nu}\Gamma^\tau_{\lambda\tau} - \Gamma^\tau_{\mu\lambda}\Gamma^\lambda_{\tau\nu}$.

The variation of eq.(B.5) can then be written as

$$\delta_g \bar{S} = \int d^4x \sqrt{|g|}\, f(\phi) g^{\mu\nu} \left[\nabla_\lambda \left(\delta\Gamma^\lambda_{\mu\nu}\right) - \nabla_\nu \left(\delta\Gamma^\lambda_{\mu\lambda}\right)\right] \tag{B.7}$$

Moving on, we have that

$$\begin{aligned} \nabla_\lambda \left(f(\phi) g^{\mu\nu} \delta\Gamma^\lambda_{\mu\nu}\right) &= \nabla_\lambda f(\phi) g^{\mu\nu} \delta\Gamma^\lambda_{\mu\nu} + \underline{f(\phi) g^{\mu\nu} \nabla_\lambda \left(\delta\Gamma^\lambda_{\mu\nu}\right)} \\ \nabla_\nu \left(f(\phi) g^{\mu\nu} \delta\Gamma^\lambda_{\mu\lambda}\right) &= \nabla_\nu f(\phi) g^{\mu\nu} \delta\Gamma^\lambda_{\mu\lambda} + \underline{f(\phi) g^{\mu\nu} \nabla_\nu \left(\delta\Gamma^\lambda_{\mu\lambda}\right)} \end{aligned} \tag{B.8}$$

The full divergence terms in the left-hand-side of the above equationss are boundary terms since they contribute an integral over the boundary of a volume under integration, by virtue of the Gauss' theorem. This means that the boundary terms may be safely omitted since we require the vanishing of the surface integrals due to the stationary action principle (vanishing of metric variations and of its first derivatives on the boundary of the integration). So, we end up with

$$\begin{aligned} \delta_g \bar{S} = \int d^4x \sqrt{|g|} \left[\nabla_\nu f(\phi) g^{\mu\nu} \delta\Gamma^\lambda_{\mu\lambda} - \nabla_\lambda f(\phi) g^{\mu\nu} \delta\Gamma^\lambda_{\mu\nu}\right] = \\ \int d^4x \sqrt{|g|}\, \nabla_\lambda f(\phi) \left[g^{\mu\lambda} \delta\Gamma^\nu_{\mu\nu} - g^{\mu\nu} \delta\Gamma^\lambda_{\mu\nu}\right] \end{aligned} \tag{B.9}$$

where, we know that

$$\delta\Gamma^\lambda_{\mu\nu} = -\frac{1}{2}\left[g_{\nu\sigma} \nabla_\mu (\delta g^{\lambda\sigma}) + g_{\mu\sigma} \nabla_\nu (\delta g^{\lambda\sigma}) - g_{\mu\sigma} g_{\nu\tau} \nabla^\lambda (\delta g^{\sigma\tau})\right]$$

and also that

$$\delta\Gamma^\lambda_{\mu\lambda} = -\frac{1}{2} g_{\lambda\sigma} \nabla_\mu (\delta g^{\lambda\sigma}),$$

So, for the terms of eq. (B.9) we have

$$\nabla_\lambda f(\phi) \left[g^{\mu\lambda} \delta\Gamma^\nu_{\mu\nu} - g^{\mu\nu} \delta\Gamma^\lambda_{\mu\nu}\right] = \nabla_\lambda f(\phi) \left[\nabla_\mu (\delta g^{\mu\lambda}) - g_{\mu\nu} \nabla^\lambda (\delta g^{\mu\nu})\right] \tag{B.10}$$

and so

$$\delta_g \bar{S} = \int d^4x \sqrt{|g|} \left[\nabla_\nu f(\phi) \nabla_\mu (\delta g^{\mu\nu}) - \nabla_\lambda f(\phi) g_{\mu\nu} \nabla^\lambda (\delta g^{\mu\nu})\right] \tag{B.11}$$

where, folloing the same logic as before about the boundary behaviour of metric, we have, after some algebra:

$$\begin{aligned} \nabla_\mu \left(\nabla_\nu f(\phi) \delta g^{\mu\nu}\right) &= \nabla_\mu \nabla_\nu f(\phi) \delta g^{\mu\nu} + \underline{\nabla_\nu f(\phi) \nabla_\mu (\delta g^{\mu\nu})}, \\ \nabla^\lambda \left(\nabla_\lambda f(\phi) g_{\mu\nu} \delta g^{\mu\nu}\right) &= \Box f(\phi) g_{\mu\nu} \delta g^{\mu\nu} + \underline{\nabla_\lambda f(\phi) g_{\mu\nu} \nabla^\lambda (\delta g^{\mu\nu})}, \end{aligned}$$

where the left hand side is a full divergence, whose integral vanishes by virtue of the Gauss theorem, keeping in mind the requirement of stationary action principle. So, we end up with

$$\delta_g \bar{S} = \int d^4x \sqrt{|g|} \delta g^{\mu\nu} (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu) f(\phi) \tag{B.12}$$

So, going back to eq.(B.4) we have

$$\begin{gathered}\delta_g S[g,\phi] = \int d^4x \sqrt{|g|} \delta g^{\mu\nu} \left[f(\phi) G_{\mu\nu} - h(\phi) \left[\nabla_\mu \phi \nabla_\nu f(\phi) - \frac{1}{2} g_{\mu\nu} (\nabla\phi)^2 \right] \right] + \delta_g \bar{S} = \\ \int d^4x \sqrt{|g|} \delta g^{\mu\nu} \left[f(\phi) G_{\mu\nu} - h(\phi) \left[\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla\phi)^2 \right] + g_{\mu\nu} \Box f(\phi) - \nabla_\mu \nabla_\nu f(\phi) \right]\end{gathered} \tag{B.13}$$

which, setting $\delta_g S[g,\phi] = 0$ yields

$$\begin{aligned} f(\phi) G_{\mu\nu} - h(\phi) \left[\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla\phi)^2 \right] + g_{\mu\nu} \Box f(\phi) - \nabla_\mu \nabla_\nu f(\phi) = 0 \Rightarrow \\ g^{\mu\nu} f(\phi) G_{\mu\nu} = -h(\phi) g^{\mu\nu} \nabla_\mu \phi \nabla \nu \phi + g^{\mu\nu} [g_{\mu\nu} \Box f(\phi) - \nabla_\mu \nabla_\nu f(\phi)] = 0 \Rightarrow \\ -f(\phi) R = -h(\phi) (\nabla\phi)^2 + [\Box f(\phi) - 4\Box\phi] \Rightarrow \\ f(\phi) R = h(\phi) (\nabla\phi)^2 + 3\Box f(\phi) \end{aligned} \tag{B.14}$$

so, from $f(\phi) G_{\mu\nu} = f(\phi) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(\phi) R$ we find

$$\begin{aligned} f(\phi) G_{\mu\nu} - h(\phi) \left[\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla\phi)^2 \right] + g_{\mu\nu} \Box f(\phi) - \nabla_\mu \nabla_\nu f(\phi) = 0 \Rightarrow \\ f(\phi) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h(\phi) (\nabla\phi)^2 - \frac{3}{2} g_{\mu\nu} \Box f(\phi) = \frac{1}{2} g_{\mu\nu} h(\phi) (\nabla\phi)^2 + \nabla_\mu \nabla_\nu f(\phi) - g_{\mu\nu} \Box f(\phi) \Rightarrow \\ f(\phi) R_{\mu\nu} - h(\phi) \partial_\mu \phi \partial_\nu \phi - \nabla_\mu \nabla_\nu f(\phi) - \frac{1}{2} g_{\mu\nu} \Box f(\phi) = 0 \end{aligned} \tag{B.15}$$

where the last line is exactly the eq. of motion (2.36).

For eq.(2.37), starting from action (2.34)

$$S[g,\phi] = \int d^4x \sqrt{-g} \left[f(\phi) R - h(\phi) g^{\mu\nu} \nabla_\mu \phi \nabla \nu \phi \right] \tag{B.16}$$

we have the Euler-Lagrange equations of motion

$$\frac{\partial L}{\partial \phi} = \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi)} \right) \tag{B.17}$$

from which we can easily derive eq. (2.37)

$$2h(\phi)\Box\phi + h'(\phi) g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + f'(\phi) R = 0 \tag{B.18}$$

Appendix C

Mathematical properties of CS action and Cotton tensor

3.1 Variation of Chern-Simons term

We use metric signature with one negative eigenvalue $(-,+,+,+)$ and the Riemann tensor is given as:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}{}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma} \tag{C.1}$$

and its dual

$$\tilde{R}^{\tau}{}_{\sigma}{}^{\mu\nu} = \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}R^{\tau}{}_{\sigma\alpha\beta} \tag{C.2}$$

The topological current is expressed as

$$K^{\mu} = 2\varepsilon^{\mu\alpha\beta\gamma}\left[\frac{1}{2}\,\Gamma^{\sigma}_{\alpha\tau}\,\partial_{\beta}\,\Gamma^{\tau}_{\gamma\sigma} + \frac{1}{3}\,\Gamma^{\sigma}_{\alpha\tau}\,\Gamma^{\tau}_{\beta\eta}\Gamma^{\eta}_{\gamma\sigma}\right], \tag{C.3}$$

and the action reads:

$$\mathrm{I} = \frac{1}{16\pi G}\int d^4x\left(\sqrt{-g}R + \frac{1}{4}b\tilde{R}R\right) = \frac{1}{16\pi G}\int d^4x\left(\sqrt{-g}R - \frac{1}{2}(\nabla_{\mu}b)K^{\mu}\right) \tag{C.4}$$

We calculate the variation of the Chern-Simons term

$$I_{CS} = \frac{1}{4}\int b\tilde{R}Rd^4x = \frac{1}{2}\int b\nabla_{\mu}K^{\mu}d^4x = -\frac{1}{2}\int d^4x(\nabla_{\mu}b)K^{\mu} \tag{C.5}$$

$$\begin{aligned}\delta I_{CS} &= -\delta\int(\nabla_{\mu}b)\varepsilon^{\mu\alpha\beta\gamma}[\frac{1}{2}\,\Gamma^{\sigma}_{\alpha\tau}\partial_{\beta}\,\Gamma^{\tau}_{\gamma\sigma} + \frac{1}{3}\,\Gamma^{\sigma}_{\alpha\tau}\Gamma^{\tau}_{\beta\eta}\Gamma^{\eta}_{\gamma\sigma}]\\ &= -\int(\nabla_{\mu}b)\varepsilon^{\mu\alpha\beta\gamma}[\frac{1}{2}\,\delta(\Gamma^{\sigma}_{\alpha\tau})\partial_{\beta}\,\Gamma^{\tau}_{\gamma\sigma} + \frac{1}{2}\,\Gamma^{\sigma}_{\alpha\tau}\partial_{\beta}\,(\delta\Gamma^{\tau}_{\gamma\sigma}) + \frac{1}{3}\,(\delta\Gamma^{\sigma}_{\alpha\tau})\Gamma^{\tau}_{\beta\eta}\Gamma^{\eta}_{\gamma\sigma}\\ &\qquad + \frac{1}{3}\,\Gamma^{\sigma}_{\alpha\tau}(\delta\Gamma^{\tau}_{\beta\eta})\Gamma^{\eta}_{\gamma\sigma} + \frac{1}{3}\,\Gamma^{\sigma}_{\alpha\tau}\Gamma^{\tau}_{\beta\eta}(\delta\Gamma^{\eta}_{\gamma\sigma})]d^4x\end{aligned}$$

Rearranging the indices and using the antisymmetric properties of $\varepsilon^{\mu\alpha\beta\gamma}$ and the symmetry at the lower indices of Christoffel symbols $\Gamma^{\tau}_{\beta\sigma} = \Gamma^{\tau}_{\sigma\beta}$ we can write the 5 terms of the above result as:

1. $\varepsilon^{\mu\alpha\beta\gamma}\frac{1}{2}\ \delta(\Gamma^{\sigma}_{\alpha\tau})\partial_{\beta}\ \Gamma^{\tau}_{\gamma\sigma}$
2. For the term→ $\varepsilon^{\mu\alpha\beta\gamma}\frac{1}{2}\ \Gamma^{\sigma}_{\alpha\tau}\partial_{\beta}\ (\delta\Gamma^{\tau}_{\gamma\sigma})$, we can get with partial integrtion, keeping in mind the boundary conditions and hence that the first derivative of our field b vanishes at infinity, we get that
$\varepsilon^{\mu\alpha\beta\gamma}\frac{1}{2}\ \Gamma^{\sigma}_{\alpha\tau}\partial_{\beta}\ (\delta\Gamma^{\tau}_{\gamma\sigma}) = -\frac{1}{2}\partial_{\beta}\Gamma^{\sigma}_{\alpha\tau}(\delta\Gamma^{\tau}_{\gamma\sigma})\varepsilon^{\mu\alpha\beta\gamma} = -\frac{1}{2}\partial_{\gamma}\Gamma^{\tau}_{\beta\sigma}\delta(\Gamma^{\sigma}_{\alpha\tau})\varepsilon^{\mu\alpha\beta\gamma}$,
where in the last line we used the antisymmetric properties of $\varepsilon^{\mu\alpha\beta\gamma}$ and the symmetry at the lower indices of Christoffel symbols $\Gamma^{\tau}_{\beta\sigma} = \Gamma^{\tau}_{\sigma\beta}$
3. $\varepsilon^{\mu\alpha\beta\gamma}\frac{1}{3}\ (\delta\Gamma^{\sigma}_{\alpha\tau})\Gamma^{\tau}_{\beta\eta}\Gamma^{\eta}_{\gamma\sigma}$
4. For the term $\varepsilon^{\mu\alpha\beta\gamma}\frac{1}{3}\ \Gamma^{\sigma}_{\alpha\tau}(\delta\Gamma^{\tau}_{\beta\eta})\Gamma^{\eta}_{\gamma\sigma}$ we use same arguments as the second one, where with partial inegration, index manipulation and symmtric-antisymmetric properties we can re-write it as $\frac{1}{3}\ \varepsilon^{\mu\alpha\beta\gamma}\Gamma^{\sigma}_{\alpha\tau}(\delta\Gamma^{\tau}_{\beta\eta})\Gamma^{\eta}_{\gamma\sigma} = -\varepsilon^{\mu\alpha\beta\gamma}\frac{1}{3}\ \Gamma^{\eta}_{\beta\sigma}(\delta\Gamma^{\sigma}_{\alpha\tau})\Gamma^{\tau}_{\gamma\eta}$
5. Similarly, $\varepsilon^{\mu\alpha\beta\gamma}\frac{1}{3}\ \Gamma^{\sigma}_{\alpha\tau}\Gamma^{\tau}_{\beta\eta}(\delta\Gamma^{\eta}_{\gamma\sigma}) = -\varepsilon^{\mu\alpha\beta\gamma}\frac{1}{3}\ \Gamma^{\tau}_{\gamma\eta}\Gamma^{\eta}_{\beta\sigma}(\delta\Gamma^{\sigma}_{\alpha\tau})$

So, the relation for the variation of (C.5) can be written as:

$$\delta I_{CS} = -\int(\nabla_{\mu}b)\varepsilon^{\mu\alpha\beta\gamma}(\delta\Gamma^{\sigma}_{\alpha\tau})[\frac{1}{2}\partial_{\beta}\ \Gamma^{\tau}_{\gamma\sigma} - \frac{1}{2}\partial_{\gamma}\Gamma^{\tau}_{\beta\sigma} + \frac{1}{3}\Gamma^{\tau}_{\beta\eta}\Gamma^{\eta}_{\gamma\sigma} - \frac{1}{3}\Gamma^{\tau}_{\gamma\eta}\Gamma^{\eta}_{\beta\sigma} - \frac{1}{3}\Gamma^{\tau}_{\gamma\eta}\Gamma^{\eta}_{\beta\sigma}]d^4x \tag{C.6}$$

Now, we add and remove to relation (C.6) the term $\frac{1}{2}(\Gamma^{\tau}_{\beta\eta}\Gamma^{\eta}_{\gamma\sigma} - \Gamma^{\tau}_{\gamma\eta}\Gamma^{\eta}_{\beta\sigma})$ where we do that in order to create the Riemman tensor of (C.1) and have an expression of the following form:

$$\delta I_{CS} = -\int(\nabla_{\mu}b)\varepsilon^{\mu\alpha\beta\gamma}(\delta\Gamma^{\sigma}_{\alpha\tau})(\frac{1}{2}R^{\tau}{}_{\sigma\gamma\beta} + terms) \tag{C.7}$$

where with "terms", we mean the rest, which is:

$$\begin{aligned}\varepsilon^{\mu\alpha\beta\gamma}[terms] &= \varepsilon^{\mu\alpha\beta\gamma}[\frac{1}{3}\Gamma^{\tau}_{\beta\eta}\Gamma^{\eta}_{\gamma\sigma} - \frac{1}{3}\Gamma^{\tau}_{\gamma\eta}\Gamma^{\eta}_{\beta\sigma} - \frac{1}{3}\Gamma^{\tau}_{\gamma\eta}\Gamma^{\eta}_{\beta\sigma} - \frac{1}{2}(\Gamma^{\tau}_{\beta\eta}\Gamma^{\eta}_{\gamma\sigma} - \Gamma^{\tau}_{\gamma\eta}\Gamma^{\eta}_{\beta\sigma})] \\ &= \varepsilon^{\mu\alpha\beta\gamma}(\frac{1}{6}\Gamma^{\tau}_{\beta\eta}\Gamma^{\eta}_{\gamma\sigma} + \frac{1}{6}\Gamma^{\tau}_{\gamma\eta}\Gamma^{\eta}_{\beta\sigma}) = \varepsilon^{\mu\alpha\beta\gamma}(\frac{1}{6}\Gamma^{\tau}_{\beta\eta}\Gamma^{\eta}_{\gamma\sigma} - \frac{1}{6}\Gamma^{\tau}_{\beta\eta}\Gamma^{\eta}_{\gamma\sigma}) = 0\end{aligned}$$

Where in the last line with $\beta \Longleftrightarrow \gamma$ and using the antisymmetric properties of $\varepsilon^{\mu\alpha\beta\gamma}$ we arrived at the conclusion that the contribution of this part of the variation vanishes, and so we are left with:

$$\delta I_{CS} = -\frac{1}{2}\int(\nabla_{\mu}b)\varepsilon^{\mu\alpha\beta\gamma}(\delta\Gamma^{\sigma}_{\alpha\tau})R^{\tau}{}_{\sigma\gamma\beta}d^4x \tag{C.8}$$

The variation of the connection with respect to the metric tensor.

$$\delta\Gamma^{\sigma}_{\alpha\tau} = \frac{g^{\sigma\nu}}{2}(\nabla_{\alpha}\delta g_{\nu\tau} + \nabla_{\tau}\delta g_{\nu\alpha} - \nabla_{\nu}g_{\alpha\tau}) \tag{C.9}$$

From (C.8,C.9) we get:

$$\delta I_{CS} = -\frac{1}{4}\int (\nabla_\mu b)\varepsilon^{\mu\alpha\beta\gamma} R^{\tau}{}_{\sigma\gamma\beta} g^{\sigma\nu}(\nabla_\alpha \delta g_{\nu\tau} + \nabla_\tau \delta g_{\nu\alpha} - \nabla_\nu g_{\alpha\tau}) \tag{C.10}$$

an raising σ in the above relation we reach

$$\delta I_{CS} = -\frac{1}{4}\int (\nabla_\mu b)\varepsilon^{\mu\alpha\beta\gamma} R^{\tau\nu}{}_{\gamma\beta}(\nabla_\alpha \delta g_{\nu\tau} + \nabla_\tau \delta g_{\nu\alpha} - \nabla_\nu g_{\alpha\tau}) d^4x \tag{C.11}$$

In (C.11), because of $R^{\tau\nu}{}_{\gamma\beta}$ antisymmetry in $[\tau,\nu]$, the first term in parentheses does'n contribute, and the last one can be written as $-R^{\tau\nu}{}_{\gamma\beta}\nabla_\nu g_{\alpha\tau} = R^{\tau\nu}{}_{\gamma\beta}\nabla_\tau \delta g_{\nu\alpha}$ for the same reason. So the last two terms do combine, and (C.11) becomes:

$$\delta I_{CS} = -\frac{1}{2}\int (\nabla_\mu b)\varepsilon^{\mu\alpha\beta\gamma} R^{\tau\nu}{}_{\gamma\beta}(\nabla_\tau \delta g_{\nu\alpha}) d^4x \tag{C.12}$$

Now, moving on with partial integration and getting away of $[b, \nabla_\mu b]$ boundary terms, (C.12) becomes:

$$\delta I_{CS} = \frac{1}{2}\int \nabla_\tau[(\nabla_\mu b)\varepsilon^{\mu\alpha\beta\gamma} R^{\tau\nu}{}_{\gamma\beta}]\delta g_{\nu\alpha}) d^4x \tag{C.13}$$

and because b is a scalar, $\nabla_\mu\nabla_\tau b = \nabla_\tau\nabla_\mu b$ and (C.13) becomes

$$\delta I_{CS} = \frac{1}{2}\int (\nabla_\mu\nabla_\tau b)\varepsilon^{\mu\alpha\beta\gamma} R^{\tau\nu}{}_{\gamma\beta} + (\nabla_\mu b)\varepsilon^{\mu\alpha\beta\gamma}\nabla_\tau R^{\tau\nu}{}_{\gamma\beta}]\delta g_{\nu\alpha}) d^4x \tag{C.14}$$

In the first term we use the Bianchi identity $\nabla_\tau R^{\tau\nu}{}_{\gamma\beta} = \nabla_\gamma R^\nu_\beta - \nabla_\beta R^\nu_\gamma$, while the second integral can be written in terms of the dual Riemann tensor of (C.2). Thus, with $\beta \Longleftrightarrow \gamma$ we have $\varepsilon^{\mu\alpha\beta\gamma}(\nabla_\gamma R^\nu_\beta - \nabla_\beta R^\nu_\gamma) = 2\varepsilon^{\mu\alpha\beta\gamma}\nabla_\gamma R^\nu_\beta$ and with $\varepsilon^{\mu\alpha\beta\gamma} R^{\tau\nu}{}_{\gamma\beta} = -\varepsilon^{\mu\alpha\beta\gamma} R^{\tau\nu}{}_{\mu\alpha} = -2\tilde{R}^{\tau\nu\mu\alpha}$, equation C.14 becomes

$$\delta I_{CS} = \int [(\nabla_\mu b)\varepsilon^{\mu\alpha\beta\gamma}\nabla_\gamma R^\nu_\beta - (\nabla_\mu\nabla_\tau b)\tilde{R}^{\tau\nu\mu\alpha}]\delta g_{\nu\alpha} d^4x \tag{C.15}$$

or, we can start from equation C.13 and with partial integration and set equal to zero $[b, \nabla b]$ terms, we get

$$\delta I_{CS} = -\frac{1}{2}\int [(\nabla_\mu b)\varepsilon^{\mu\alpha\beta\gamma}\nabla_\tau R^{\tau\nu}{}_{\gamma\beta}]\delta g_{\nu\alpha}) d^4x \tag{C.16}$$

and because of the symmetry in $[\nu, \alpha]$ of the metric tensor, only the symmetric part of Riemann dual tensor in these incices survives, hence:

$$\begin{aligned}\delta I_{CS} &= -\frac{1}{2}\int [(\nabla_\mu b)\varepsilon^{\mu\alpha\beta\gamma}\nabla_\tau R^{\tau\nu}{}_{\gamma\beta}]\delta g_{\nu\alpha}) d^4x \\ &= -\int [(\nabla_\mu b)\nabla_\tau \tilde{R}^{\tau\nu\mu\alpha}]\delta g_{\nu\alpha}) d^4x \\ &= -\frac{1}{2}\int [(\nabla_\mu b)\nabla_\tau(\tilde{R}^{\tau\nu\mu\alpha} + \tilde{R}^{\tau\alpha\mu\nu})]\delta g_{\nu\alpha}) d^4x\end{aligned} \tag{C.17}$$

and finally we have the two relation we need for the Cotton tensor, in the form above:

$$\delta I_{CS} = \delta\frac{1}{4}\int d^4x\, b\tilde{R}R \equiv \int d^4x\sqrt{-g}C^{\mu\nu}\delta g_{\mu\nu} = -\int d^4x\sqrt{-g}C_{\mu\nu}\delta g^{\mu\nu} \tag{C.18}$$

where, with the same argument as before, because of the symmetry of the metric tensor, only the symmetric part survives and hence:

$$C^{\mu\nu} = -\frac{1}{2\sqrt{-g}}\left[\nabla_\sigma b\left(\varepsilon^{\sigma\mu\alpha\beta}\nabla_\alpha R^\nu_\beta + \varepsilon^{\sigma\nu\alpha\beta}\nabla_\alpha R^\mu_\beta\right) + \nabla_\sigma\nabla_\tau b\left(\tilde{R}^{\tau\mu\sigma\nu} + \tilde{R}^{\tau\nu\sigma\mu}\right)\right] \tag{C.19}$$

or, straightforward from C.13 we have

$$C^{\mu\nu} = -\nabla_\tau\frac{\nabla_\sigma b}{2\sqrt{-g}}\left(\tilde{R}^{\tau\mu\sigma\nu} + \tilde{R}^{\tau\nu\sigma\mu}\right) \tag{C.20}$$

3.2 Covariant divergence of the Cotton tensor

We calculate the divergence of Cotton tensor $C^{\mu\nu}$, and we will start from the relation C.20

$$C^{\mu\nu} = -\nabla_\tau\frac{\nabla_\sigma b}{2\sqrt{-g}}\left(\tilde{R}^{\tau\mu\sigma\nu} + \tilde{R}^{\tau\nu\sigma\mu}\right)$$

Using the antisymmetry in $[\tau,\mu]$ of $\tilde{R}^{\tau\mu\sigma\nu}$, we present $\nabla_\mu C^{\mu\nu}$ as

$$\nabla_\mu C^{\mu\nu} = -\nabla_\tau\nabla_\mu\frac{\nabla_\sigma b}{2\sqrt{-g}}\tilde{R}^{\tau\nu\sigma\mu} + [\nabla_\tau,\nabla_\mu]\frac{\nabla_\sigma}{2\sqrt{-g}}(\tilde{R}^{\tau\nu\sigma\mu} + \frac{1}{2}\tilde{R}^{\tau\mu\sigma\nu}) \tag{C.21}$$

The first contribution to $\nabla_\mu C^{\mu\nu}$ vanishes, by noting that there occurs

$$\nabla_\mu\frac{\nabla_\sigma b}{2\sqrt{-g}}\tilde{R}^{\tau\nu\sigma\mu} = \frac{\nabla_\mu\nabla_\sigma b}{2\sqrt{-g}}\tilde{R}^{\tau\nu\sigma\mu} + \frac{\nabla_\sigma b}{2\sqrt{-g}}\varepsilon^{\sigma\mu\alpha\beta}\nabla_\mu R^{\tau\nu}{}_{\alpha\beta}.$$

Since $\nabla_\mu\nabla_\sigma$ is symmetric and $\tilde{R}^{\tau\nu\sigma\mu}$ is antisymmetric in $[\sigma,\mu]$, the first term on the right is zero. The second one is also zero, owing to the Bianchi identity for the Riemann tensor. The remainder of (C.21) involves the commutator of covariant derivatives, where from

$$[\nabla_\lambda,\nabla_\nu]T^{\alpha\beta\gamma\delta} = R_{\lambda\nu}{}^\alpha{}_\kappa T^{\kappa\beta\gamma\delta} + R_{\lambda\nu}{}^\beta{}_\kappa T^{\alpha\kappa\gamma\delta} + R_{\lambda\nu}{}^\gamma{}_\kappa T^{\alpha\beta\kappa\delta} + R_{\lambda\nu}{}^\delta{}_\kappa T^{\alpha\beta\gamma\kappa} \tag{C.22}$$

calculating the quantities $[\nabla_\tau,\nabla_\mu]\tilde{R}^{\tau\nu\sigma\mu}$ and $[\nabla_\tau,\nabla_\mu]\tilde{R}^{\tau\mu\sigma\nu}$ we get

$$\begin{aligned}\nabla_\mu C^{\mu\nu} = {} & \frac{\nabla_\sigma b}{2\sqrt{-g}}\Bigg[\left(\tilde{R}^{\lambda\nu\sigma\mu} + \frac{1}{2}\tilde{R}^{\lambda\mu\sigma\nu}\right)R^\tau{}_{\lambda\mu\tau} + \tilde{R}^{\tau\lambda\sigma\mu}R^\nu{}_{\lambda\mu\tau} + \frac{1}{2}\tilde{R}^{\tau\lambda\sigma\nu}R^\mu{}_{\lambda\mu\tau} \\ & + \tilde{R}^{\tau\nu\sigma\lambda}R^\mu{}_{\lambda\mu\tau} + \frac{1}{2}\tilde{R}^{\tau\mu\sigma\lambda}R^\nu{}_{\lambda\mu\tau}\Bigg] \\ = {} & \frac{\nabla_\sigma b}{2\sqrt{-g}}\Bigg[-\left(\tilde{R}^{\lambda\nu\sigma\mu} + \frac{1}{2}\tilde{R}^{\lambda\mu\sigma\nu}\right)R_{\lambda\mu} + \left(\tilde{R}^{\tau\nu\sigma\lambda} + \frac{1}{2}\tilde{R}^{\tau\lambda\sigma\nu}\right)R_{\lambda\tau} \\ & + \left(\tilde{R}^{\tau\lambda\sigma\mu} + \frac{1}{2}\tilde{R}^{\tau\mu\sigma\lambda}\right)R^\nu{}_{\lambda\mu\tau}\Bigg].\end{aligned} \tag{C.23}$$

The quantities involving the Ricci tensor vanish owing to its symmetry. The last term in brackets is expanded by using the antisymmetry of $\tilde{R}^{\tau\lambda\sigma\mu}$ in $[\tau, \lambda]$. Thus we are left with

$$\begin{aligned}
\nabla_\mu C^{\mu\nu} &= \frac{\nabla_\sigma b}{4\sqrt{-g}}\left[\tilde{R}^{\tau\lambda\sigma\mu}\left(R^\nu{}_{\lambda\mu\tau} - R^\nu_{\tau\mu\lambda}\right) + \tilde{R}^{\tau\mu\sigma\lambda}R^\nu{}_{\lambda\mu\tau}\right] \\
&= \frac{\nabla_\sigma b}{4\sqrt{-g}}\left[\tilde{R}^{\tau\lambda\sigma\mu}R^\nu{}_{\mu\lambda\tau} + \tilde{R}^{\tau\mu\sigma\lambda}R^\nu{}_{\lambda\mu\tau}\right] \\
&= \frac{\nabla_\sigma b}{2\sqrt{-g}}\tilde{R}^{\tau\lambda\sigma\mu}R_{\lambda\tau}{}^\nu{}_\mu. \qquad (C.24)
\end{aligned}$$

Cyclic properties of the Riemann tensor allow passage from one expression to the next in (B.3b). Finally we use the identity

$$\tilde{R}^\tau{}_\lambda{}^{\sigma\mu}R^\lambda{}_{\tau\nu\mu} = \frac{1}{4}\delta^\sigma_\nu{}^*RR, \qquad (C.25)$$

to conclude that

$$\nabla_\mu C^{\mu\nu} = \frac{\nabla^\nu b}{8\sqrt{-g}}{}^*RR \qquad (C.26)$$

3.3 Proof for the Covariant divergence of the topological current

The topological current is expressed as

$$K^\alpha = \varepsilon^{\alpha\beta\gamma\delta}\left[\ \Gamma^\kappa_{\beta\lambda}\,\partial_\gamma\ \Gamma^\lambda_{\delta\kappa} + \frac{2}{3}\ \Gamma^\kappa_{\beta\lambda}\,\Gamma^\lambda_{\gamma\mu}\Gamma^\mu_{\delta\kappa}\right] \qquad (C.27)$$

so, we calculate the covariant divergence:

$$\begin{aligned}
\nabla_\alpha K^\alpha &= \nabla_\alpha[\varepsilon^{\alpha\beta\gamma\delta}[\ \Gamma^\kappa_{\beta\lambda}\partial_\gamma\ \Gamma^\lambda_{\delta\kappa} + \frac{2}{3}\ \Gamma^\kappa_{\beta\lambda}\Gamma^\lambda_{\gamma\mu}\Gamma^\mu_{\delta\kappa}] \\
&= \varepsilon^{\alpha\beta\gamma\delta}[\partial_\alpha\Gamma^\kappa_{\beta\lambda}\partial_\gamma\ \Gamma^\lambda_{\delta\kappa} + \Gamma^\kappa_{\beta\lambda}\partial_\alpha\partial_\gamma\Gamma^\lambda_{\delta\kappa} + \frac{2}{3}\partial_\alpha\Gamma^\kappa_{\beta\lambda}\Gamma^\lambda_{\gamma\mu}\Gamma^\mu_{\delta\kappa} + \\
&\quad \frac{2}{3}\Gamma^\kappa_{\beta\lambda}\partial_\alpha\Gamma^\lambda_{\gamma\mu}\Gamma^\mu_{\delta\kappa} + \frac{2}{3}\Gamma^\kappa_{\beta\lambda}\Gamma^\lambda_{\gamma\mu}\partial_\alpha\Gamma^\mu_{\delta\kappa}]
\end{aligned} \qquad (C.28)$$

The term $\varepsilon^{\alpha\beta\gamma\delta}\Gamma^\kappa_{\beta\lambda}\partial_\alpha\partial_\gamma\Gamma^\lambda_{\delta\kappa}$ vanishes cause $\partial_\alpha\partial_\gamma$ is symmetric in $[\alpha, \gamma]$ while $\varepsilon^{\alpha\beta\gamma\delta}$ is antisymmetric in these two indices. so, C.28 becomes

$$\nabla_\alpha K^\alpha = \varepsilon^{\alpha\beta\gamma\delta}[\partial_\alpha\Gamma^\kappa_{\beta\lambda}\partial_\gamma\ \Gamma^\lambda_{\delta\kappa} + \frac{2}{3}\partial_\alpha\Gamma^\kappa_{\beta\lambda}\Gamma^\lambda_{\gamma\mu}\Gamma^\mu_{\delta\kappa} + \frac{2}{3}\Gamma^\kappa_{\beta\lambda}\partial_\alpha\Gamma^\lambda_{\gamma\mu}\Gamma^\mu_{\delta\kappa} + \frac{2}{3}\Gamma^\kappa_{\beta\lambda}\Gamma^\lambda_{\gamma\mu}\partial_\alpha\Gamma^\mu_{\delta\kappa}] \qquad (C.29)$$

Following the same logic as for the variation of CS term above, we move one and rearrange the 4 terms using index manipulation and symmetry-antisymmetry of connection and Levi-Civita symbol, we can easily conclude that:

1. $\varepsilon^{\alpha\beta\gamma\delta}\partial_\alpha\Gamma^\kappa_{\beta\lambda}\partial_\gamma\ \Gamma^\lambda_{\delta\kappa}$

2. $\varepsilon^{\alpha\beta\gamma\delta}\frac{2}{3}\partial_\alpha\Gamma^\kappa_{\beta\lambda}\Gamma^\lambda_{\gamma\mu}\Gamma^\mu_{\delta\kappa}$

3. $\varepsilon^{\alpha\beta\gamma\delta}\frac{2}{3}\Gamma^\kappa_{\beta\lambda}\partial_\alpha\Gamma^\lambda_{\gamma\mu}\Gamma^\mu_{\delta\kappa} = \varepsilon^{\alpha\beta\gamma\delta}\frac{2}{3}\partial_\alpha\Gamma^\kappa_{\beta\lambda}\Gamma^\lambda_{\gamma\mu}\Gamma^\mu_{\delta\kappa}$

4. $\varepsilon^{\alpha\beta\gamma\delta}\frac{2}{3}\Gamma^\kappa_{\beta\lambda}\Gamma^\lambda_{\gamma\mu}\partial_\alpha\Gamma^\mu_{\delta\kappa} = \varepsilon^{\alpha\beta\gamma\delta}\frac{2}{3}\partial_\alpha\Gamma^\kappa_{\beta\lambda}\Gamma^\lambda_{\gamma\mu}\Gamma^\mu_{\delta\kappa}$

So, C.29 becomes

$$\begin{aligned}\nabla_\alpha K^\alpha &= \varepsilon^{\alpha\beta\gamma\delta}[\partial_\alpha\Gamma^\kappa_{\beta\lambda}\partial_\gamma\ \Gamma^\lambda_{\delta\kappa} + 2\partial_\alpha\Gamma^\kappa_{\beta\lambda}\Gamma^\lambda_{\gamma\mu}\Gamma^\mu_{\delta\kappa}] \\ &= \varepsilon^{\alpha\beta\gamma\delta}[\partial_\alpha\Gamma^\kappa_{\beta\lambda}\partial_\gamma\ \Gamma^\lambda_{\delta\kappa} + \partial_\alpha\Gamma^\kappa_{\beta\lambda}\Gamma^\lambda_{\gamma\mu}\Gamma^\mu_{\delta\kappa} + \partial_\gamma\Gamma^\kappa_{\delta\lambda}\Gamma^\lambda_{\alpha\mu}\Gamma^\mu_{\beta\kappa}]\end{aligned} \tag{C.30}$$

where the last two terms come from the symmetry of $\varepsilon^{\alpha\beta\gamma\delta}\partial_\alpha\Gamma^\kappa_{\beta\lambda}\Gamma^\lambda_{\gamma\mu}\Gamma^\mu_{\delta\kappa}$ in $\alpha\beta \iff \gamma\delta$. Expanding each term using its antisymmetry to the relevant indices, we have

$$\begin{aligned}\nabla_\alpha K^\alpha = \varepsilon^{\alpha\beta\gamma\delta}[\partial_\alpha\Gamma^\kappa_{\beta\lambda}\partial_\gamma\ \Gamma^\lambda_{\delta\kappa} &+ \partial_\alpha\Gamma^\kappa_{\beta\lambda}\Gamma^\lambda_{\gamma\mu}\Gamma^\mu_{\delta\kappa} + \partial_\gamma\Gamma^\kappa_{\delta\lambda}\Gamma^\lambda_{\alpha\mu}\Gamma^\mu_{\beta\kappa}] \\ = \frac{1}{4}\varepsilon^{\alpha\beta\gamma\delta}&[(\partial_\alpha\Gamma^\kappa_{\beta\lambda} - \partial_\beta\Gamma^\kappa_{\alpha\lambda})(\partial_\gamma\ \Gamma^\lambda_{\delta\kappa} - \partial_\delta\ \Gamma^\lambda_{\gamma\kappa}) + \\ &(\partial_\alpha\Gamma^\kappa_{\beta\lambda} - \partial_\beta\Gamma^\kappa_{\alpha\lambda})(\Gamma^\lambda_{\gamma\mu}\Gamma^\mu_{\delta\kappa} - \Gamma^\lambda_{\delta\mu}\Gamma^\mu_{\gamma\kappa}) + \\ &(\partial_\gamma\Gamma^\kappa_{\delta\lambda} - \partial_\delta\Gamma^\kappa_{\gamma\lambda})(\Gamma^\lambda_{\alpha\mu}\Gamma^\mu_{\beta\kappa} - \Gamma^\lambda_{\beta\mu}\Gamma^\mu_{\alpha\kappa})]\end{aligned} \tag{C.31}$$

where we used antisymmetry of the terms in the parenthesis combined with $\varepsilon^{\alpha\beta\gamma\delta}$ in $\alpha \iff \beta$ and $\gamma \iff \delta$ corespondingly. The next step, is to add the vanishing term $\varepsilon^{\alpha\beta\gamma\delta}\Gamma^\kappa_{\xi\alpha}\Gamma^\xi_{\lambda\beta}\Gamma^\lambda_{\sigma\gamma}\Gamma^\sigma_{\kappa\delta}$, since it's identically equal to zero, and expanding it, as well, in antisymmetric properties of $\alpha \iff \beta$ and $\gamma \iff \delta$, we may write equation C.31 as

$$\begin{aligned}\nabla_\alpha K^\alpha = \frac{1}{4}\varepsilon^{\alpha\beta\gamma\delta}[&(\partial_\alpha\Gamma^\kappa_{\beta\lambda} - \partial_\beta\Gamma^\kappa_{\alpha\lambda})(\partial_\gamma\ \Gamma^\lambda_{\delta\kappa} - \partial_\delta\ \Gamma^\lambda_{\gamma\kappa}) + \\ &(\partial_\alpha\Gamma^\kappa_{\beta\lambda} - \partial_\beta\Gamma^\kappa_{\alpha\lambda})(\Gamma^\lambda_{\gamma\mu}\Gamma^\mu_{\delta\kappa} - \Gamma^\lambda_{\delta\mu}\Gamma^\mu_{\gamma\kappa}) + \\ &(\partial_\gamma\Gamma^\kappa_{\delta\lambda} - \partial_\delta\Gamma^\kappa_{\gamma\lambda})(\Gamma^\lambda_{\alpha\mu}\Gamma^\mu_{\beta\kappa} - \Gamma^\lambda_{\beta\mu}\Gamma^\mu_{\alpha\kappa}) + \\ &(\Gamma^\kappa_{\xi\alpha}\Gamma^\xi_{\lambda\beta} - \Gamma^\kappa_{\xi\beta}\Gamma^\xi_{\lambda\alpha})(\Gamma^\lambda_{\sigma\gamma}\Gamma^\sigma_{\kappa\delta} - \Gamma^\lambda_{\sigma\delta}\Gamma^\sigma_{\kappa\gamma})]\end{aligned} \tag{C.32}$$

And after some algebra, C.32 becomes

$$\begin{aligned}\nabla_\alpha K^\alpha = \frac{1}{4}\varepsilon^{\alpha\beta\gamma\delta}[&(\partial_\alpha\Gamma^\kappa_{\beta\lambda} + \Gamma^\kappa_{\xi\alpha}\Gamma^\xi_{\lambda\beta} - \partial_\beta\Gamma^\kappa_{\alpha\lambda} - \Gamma^\kappa_{\xi\beta}\Gamma^\xi_{\lambda\alpha})\cdot \\ &(\partial_\gamma\ \Gamma^\lambda_{\delta\kappa} + \Gamma^\lambda_{\sigma\gamma}\Gamma^\sigma_{\kappa\delta} - \partial_\delta\ \Gamma^\lambda_{\gamma\kappa} - \Gamma^\lambda_{\sigma\delta}\Gamma^\sigma_{\kappa\gamma})]\end{aligned} \tag{C.33}$$

which is written with the help of the Riemann tensor as

$$\nabla_\alpha K^\alpha = \frac{1}{4}\varepsilon^{\alpha\beta\gamma\delta}R^\kappa_{\lambda\alpha\beta}R^\lambda_{\kappa\gamma\delta} \tag{C.34}$$

or, remembering that $\tilde{R}^{\tau}{}_{\sigma}{}^{\mu\nu} = \frac{1}{2}\ \varepsilon^{\mu\nu\alpha\beta}R^\tau{}_{\sigma\alpha\beta}$, we arrive the desired result:

$$\nabla_\alpha K^\alpha = \frac{1}{2}R^\kappa_{\lambda\alpha\beta}\tilde{R}^{\lambda}{}_{\kappa}{}^{\alpha\beta} \tag{C.35}$$

3.4 Traceless Cotton Tensor

As we found in equation C.20, the cotton tensor is

$$C^{\mu\nu} = -\nabla_\tau \frac{\nabla_\sigma b}{2\sqrt{-g}}\left(\tilde{R}^{\tau\mu\sigma\nu} + \tilde{R}^{\tau\nu\sigma\mu}\right) \tag{C.36}$$

and we want to calculate $g_{\mu\nu}C^{\mu\nu}$ which yields

$$\begin{aligned} g_{\mu\nu}C^{\mu\nu} &= -\nabla_\tau \frac{\nabla_\sigma b}{2\sqrt{-g}} g_{\mu\nu}\left(\tilde{R}^{\tau\mu\sigma\nu} + \tilde{R}^{\tau\nu\sigma\mu}\right) \\ &= -\nabla_\tau \frac{\nabla_\sigma b}{4\sqrt{-g}}\left(g_{\mu\nu}R^{\tau\mu}{}_{\alpha\beta}\varepsilon^{\alpha\beta\sigma\nu} + g_{\mu\nu}R^{\tau\nu}{}_{\alpha\beta}\varepsilon^{\alpha\beta\sigma\mu}\right) \\ &= -\nabla_\tau \frac{\nabla_\sigma b}{2\sqrt{-g}}\left(R^{\tau}{}_{\mu\alpha\beta}\varepsilon^{\alpha\beta\sigma\mu}\right) \\ &= \nabla_\tau \frac{\nabla_\sigma b}{2\sqrt{-g}}\left(R^{\tau}{}_{\mu\alpha\beta}\varepsilon^{\mu\alpha\beta\sigma}\right) = 0 \end{aligned} \tag{C.37}$$

where the last equality comes from Riemann tensor symmetries, in particular $R^{\tau}{}_{[\mu\alpha\beta]} = 0$. where [...] denotes complete antisymmetrization of the corresponding indices, which immediately implies that $R^{\tau}_{\mu\alpha\beta}\varepsilon^{\mu\alpha\beta\sigma} = 0$.

→**Proof of right hand of C.18**

$$\int d^4x\sqrt{-g}C^{\mu\nu}\delta g_{\mu\nu} = -\int d^4x\sqrt{-g}C_{\mu\nu}\delta g^{\mu\nu} \tag{C.38}$$

We have that

$$\delta g_{\rho\lambda} = -g_{\nu\lambda}g_{\mu\rho}\delta g^{\mu\nu} \tag{C.39}$$

so it is straightforward to show that

$$\int d^4x\sqrt{-g}C^{\mu\nu}\delta g_{\mu\nu} = -\int d^4x\sqrt{-g}C_{\mu\nu}\delta g^{\mu\nu} \tag{C.40}$$

with simple subtitution of C.39.

Appendix D

Origin of action of eq.(3.16)

Actions like eq.(3.16) are inspired from bosonic string theory, where the lowest energy ground state consists of a spin-0 scalar dilaton field, a symmetric traceless graviton tensor field, $g_{\mu\nu} = g_{\nu\mu}$ and a spin-1 antisymmetric tensor field $B_{\mu\nu} = B_{\nu\mu}$, where $\mu, \nu = 0....D$–1 and $D = 4$ the spacetime dimension of the string after compactification. In 4 dimensions, $B_{\mu\nu}$ has the $U(1)$ symmetry:

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \theta_\nu - \partial_\nu \theta_\mu \ , \quad \mu, \nu = 0, 1, 2, 3 \tag{D.1}$$

where θ_μ are gauge parameters. Thus, the action will respect the same symmetry, depending only on the field strength tensor of $B_{\mu\nu}$, which is expressed by:

$$H_{\nu\mu\rho} = \partial_{[\mu} B_{\nu\rho]} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} \tag{D.2}$$

and also satisfies the Bianchi identity

$$\partial_{[\sigma} H_{\mu\nu\rho]} = 0 \tag{D.3}$$

where with [...] we mean antisymmetrisation of the respected indices. The effective 4-dimensional action becomes:

$$\mathcal{S} = \int d^4x \sqrt{-g} \left(\frac{R}{2\kappa^2} - \frac{1}{6} H_{\nu\mu\rho} H^{\nu\mu\rho} + ... \right) \tag{D.4}$$

where ... denote higher-derivative terms. Calculating the Euler-Lagrange equations of motion we have:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial B_{\alpha\beta}} &= \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu B_{\alpha\beta})}\right] \Rightarrow \\
0 &= \partial_\mu \left[-\frac{1}{3}H^{\kappa\lambda\rho}\frac{\partial H_{\kappa\lambda\rho}}{\partial(\partial_\mu B_{\alpha\beta})}\right] \Rightarrow \\
0 &= \partial_\mu \left[H^{\kappa\lambda\rho}\frac{\partial_\kappa B_{\lambda\rho} + \partial_\rho B_{\kappa\lambda} + \partial_\lambda B_{\rho\kappa}}{\partial(\partial_\mu B_{\alpha\beta})}\right] \Rightarrow \\
0 &= \partial_\mu \left[H^{\kappa\lambda\rho}(\delta^\mu_\kappa\delta^\alpha_\lambda\delta^\beta_\rho + \delta^\mu_\lambda\delta^\alpha_\rho\delta^\beta_\kappa + \delta^\mu_\rho\delta^\alpha_\kappa\delta^\beta_\lambda)\right] \Rightarrow \\
0 &= \partial_\mu \left(H^{\mu\alpha\beta} + H^{\beta\mu\alpha} + H^{\alpha\beta\mu}\right)
\end{aligned} \tag{D.5}$$

There is a duality of the field strength and an (axion-like) pseudoscalar field $b(x)$, Thus, we can express the field strength tensor as $H_{\nu\mu\rho} = N\epsilon_{\mu\nu\rho\sigma}\partial^\sigma b(x)$, where $\epsilon_{\mu\nu\rho\sigma}$ is the Minkowski space-time totally antisymmetric Levi-Civita symbol. The Bianchi identity becomes

$$\epsilon^{\mu\nu\rho\sigma}\partial_\sigma H_{\mu\nu\rho} = 0 \tag{D.6}$$

We proceed by considering the above relation, the Bianchi identity, as a constraint, with the help of a Lagrange multiplier, which is our field $b(x)$, and we write again the action of equation D.4 as:

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2} - \frac{1}{6}H_{\nu\mu\rho}H^{\nu\mu\rho} + b(x)\epsilon^{\mu\nu\rho\sigma}\partial_\sigma H_{\mu\nu\rho}...\right) \tag{D.7}$$

where, with partial integration of the last part of the above expression including the field $b(x)$, and keeping in mind the boundary conditions and therefore that the field and its first derivative vanish at spatial infinity, we get:

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2} - \frac{1}{6}H_{\nu\mu\rho}H^{\nu\mu\rho} - \epsilon^{\mu\nu\rho\sigma}\partial_\sigma b(x)H_{\mu\nu\rho}...\right) \tag{D.8}$$

Expressing now the field strength tensor as $H_{\nu\mu\rho} = N\epsilon_{\mu\nu\rho\lambda}\partial^\lambda b(x)$, we have that:

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}\partial_\sigma b(x)H_{\mu\nu\rho} &= \\
\epsilon^{\mu\nu\rho\sigma}\partial_\sigma b(x)N\epsilon_{\mu\nu\rho\lambda}\partial^\lambda b(x) &= \\
6N\delta^\sigma{}_\lambda\partial_\sigma b(x)\partial^\lambda b(x) &= \\
6N\partial_\sigma b(x)\partial^\sigma b(x)&
\end{aligned} \tag{D.9}$$

where, in 3rd line, we used the indentity $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\lambda} = 6\delta^\sigma{}_\lambda$. So, the action now becomes:

$$\begin{aligned}
\mathcal{S} &= \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2} - \frac{1}{6}H_{\nu\mu\rho}H^{\nu\mu\rho} - 6N\partial_\sigma b(x)\partial^\sigma b(x)...\right) \Rightarrow \\
\mathcal{S} &= \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2} - 6N\partial_\sigma b(x)\partial^\sigma b(x)...\right)
\end{aligned} \tag{D.10}$$

However, in string theory, in the extra dimensional space, there are gauge and gravitational anomalies. These anomalies are expressed through the modification of the field strength $H_{\nu\mu\rho}$ by adding the **Chern-Simons three-forms**:

$$H = \mathbf{d}B + c_1\Omega^L - c_2\Omega^Y \tag{D.11}$$

where c is a positive constant, $\mathbf{d}$ denotes partial differentiation with antisymmetrisation and Ω^L and Ω^Y are the gravitational ("Lorentz", L) and gauge (Y) anomalous counterterms given as:

$$\Omega^L = \left(\omega_\alpha\partial_\gamma\omega_\beta + \frac{2}{3}\omega_\alpha\omega_\beta\omega_\gamma\right)\left(dx^\alpha \wedge dx^\beta \wedge dx^\gamma\right) \tag{D.12}$$

$$\Omega^Y = \left(A_\alpha\partial_\gamma A_\beta + \frac{2}{3}A_\alpha A_\beta A_\gamma\right)\left(dx^\alpha \wedge dx^\beta \wedge dx^\gamma\right) \tag{D.13}$$

where $\wedge$ is the wedge product, A denotes the Yang-Mills gauge field and ω_α is the one form spin-connection.

This modification leads to the altered Bianchi identity, which takes the form:

$$\mathbf{d}H = Tr\left[c_1 R \wedge R + c_2 F \wedge F\right] \tag{D.14}$$

or, in the component form:

$$\epsilon^{\mu\nu\rho\sigma}\nabla_\sigma H_{\mu\nu\rho} = c_1 R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + c_2 F_{\mu\nu\rho\sigma}\tilde{F}^{\mu\nu\rho\sigma} \tag{D.15}$$

where $F = dA + A \wedge A$ is the Yang-Mills field strength tensor, $R = d\omega + \omega \wedge \omega$ is the curvature 2-form and $\epsilon^{\mu\nu\rho\sigma}$ the Levi-Civita symbol. Assuming that $A = 0$ and consequently $F = 0$, we shall perform the same procedure as before, adding now to the action the quantity $b(x)\left[\epsilon^{\mu\nu\rho\sigma}\nabla_\sigma H_{\mu\nu\rho} - c_1 R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right]$ and the action (D.4) now takes the following form:

$$\begin{aligned}\mathcal{S} &= \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2} - b(x)\left[\epsilon^{\mu\nu\rho\sigma}\nabla_\sigma H_{\mu\nu\rho} - c_1 R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right]...\right) \Rightarrow \\ \mathcal{S} &= \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2} - 6N\partial_\sigma b(x)\partial^\sigma b(x) + c_1 b(x) R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}...\right)\end{aligned} \tag{D.16}$$

where in the second line we used the result derived before adding the correction terms, that $\epsilon^{\mu\nu\rho\sigma}\partial_\sigma b(x) H_{\mu\nu\rho} = 6N\partial_\sigma b(x)\partial^\sigma b(x)$. Fixing the normalization constant N to have a canonical kinetic term for the axionic field $b(x)$ and ignoring the higher derivatives terms denoted in the above action by (...), we reach:

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\sigma b(x)\partial^\sigma b(x) + c_1 b(x) R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right) \tag{D.17}$$

which is general form of the action we considered in eq.(3.16).

Appendix E

Derivation and Solution of the Differential Equation

5.1 Derivation of eq.(3.39)

We start from eq.(3.37):

$$r^{11}(r-2M)w''' + 2r^{10}(6r-11M)w'' + (28r^{10} - 50Mr^9 - 576A^2\kappa^2M^2r^4)w' \\ +3456A^2\kappa^2M^3 = 0 \ . \tag{E.1}$$

where, for $w(r)$ and its derivatives we have:

- $$w(r) = \sum_{n=4}^{\infty} \frac{d_n M^{n-2}}{r^n} \ , \tag{E.2}$$

- $$w'(r) = -\sum_{n=4}^{\infty} \frac{n d_n M^{n-2}}{r^{n+1}} \ , \tag{E.3}$$

- $$w''(r) = \sum_{n=4}^{\infty} \frac{n(n+1) d_n M^{n-2}}{r^{n+2}} \ , \tag{E.4}$$

- $$w'''(r) = -\sum_{n=4}^{\infty} \frac{n(n+1)(n+2) d_n M^{n-2}}{r^{n+3}} \ , \tag{E.5}$$

Putting the above and decompoising the terms of eq.(E.1), we have:

▶ The first term $r^{11}(r-2M)w'''$ becomes:

$$r^{11}(r-2M)w''' = -\sum_{n=4}^{\infty}\frac{n(n+1)(n+2)d_nM^{n-2}}{r^{n-9}} + \sum_{n=4}^{\infty}\frac{2n(n+1)(n+2)d_nM^{n-1}}{r^{n-8}} \tag{E.6}$$

▶ The second term $2r^{10}(6r-11M)w''$ becomes:

$$2r^{10}(6r-11M)w'' = \sum_{n=4}^{\infty}\frac{12n(n+1)d_nM^{n-2}}{r^{n-9}} - \sum_{n=4}^{\infty}\frac{22n(n+1)d_nM^{n-1}}{r^{n-8}} \tag{E.7}$$

▶ The third term $(28r^{10}-50Mr^9-576A^2\kappa^2M^2r^4)w'$ becomes:

$$(28r^{10}-50Mr^9-576A^2\kappa^2M^2r^4)w' = -\sum_{n=4}^{\infty}\frac{28nd_nM^{n-2}}{r^{n-9}} + \sum_{n=4}^{\infty}\frac{50nd_nM^{n-1}}{r^{n-8}} + \sum_{n=4}^{\infty}\frac{576A^2\kappa^2M^2nd_nM^{n}}{r^{n-3}} \tag{E.8}$$

Performing now the shift $n \to m+9$, the above relations become:

▶ The first term $r^{11}(r-2M)w'''$ becomes:

$$r^{11}(r-2M)w''' = -\sum_{m=-5}^{\infty}\frac{(m+9)(m+10)(m+11)d_{m+9}M^{m+7}}{r^{m}} + \sum_{m=-5}^{\infty}\frac{2(m+9)(m+10)(m+11)d_{m+9}M^{m+8}}{r^{m+1}} \tag{E.9}$$

▶ The second term $2r^{10}(6r-11M)w''$ becomes:

$$2r^{10}(6r-11M)w'' = \sum_{m=-5}^{\infty}\frac{12(m+9)(m+10)d_{m+9}M^{m+7}}{r^{m}} - \sum_{m=-5}^{\infty}\frac{22(m+9)(m+10)d_{m+9}M^{m+8}}{r^{m+1}} \tag{E.10}$$

▶ The third term $(28r^{10}-50Mr^9-576A^2\kappa^2M^2r^4)w'$ becomes:

$$(28r^{10}-50Mr^9-576A^2\kappa^2M^2r^4)w' = -\sum_{m=-5}^{\infty}\frac{28(m+9)d_{m+9}M^{m+7}}{r^{m}} + + \sum_{m=-5}^{\infty}\frac{50(m+9)d_{m+9}M^{m+8}}{r^{m+1}} + \sum_{m=-5}^{\infty}\frac{576A^2\kappa^2M^2(m+9)d_{m+9}M^{m+9}}{r^{m+6}} \tag{E.11}$$

The next step is to bring all the above sums to the form of $\sim r^{-m}$. So, making the relative shifts, we get :

- The first term $r^{11}(r-2M)w'''$ becomes:

$$\begin{aligned} r^{11}(r-2M)w''' = -\sum_{m=-5}^{\infty}\frac{(m+9)(m+10)(m+11)d_{m+9}M^{m+7}}{r^m} + \\ \sum_{m=-4}^{\infty}\frac{2(m+8)(m+9)(m+10)d_{m+8}M^{m+7}}{r^m} \end{aligned} \tag{E.12}$$

- The second term $2r^{10}(6r-11M)w''$ becomes:

$$\begin{aligned} 2r^{10}(6r-11M)w'' = \sum_{m=-5}^{\infty}\frac{12(m+9)(m+10)d_{m+9}M^{m+7}}{r^m} \\ -\sum_{m=-4}^{\infty}\frac{22(m+8)(m+9)d_{m+8}M^{m+7}}{r^m} \end{aligned} \tag{E.13}$$

- The third term $(28r^{10}-50Mr^9-576A^2\kappa^2M^2r^4)w'$ becomes:

$$\begin{aligned} (28r^{10}-50Mr^9-576A^2\kappa^2M^2r^4)w' = -\sum_{m=-5}^{\infty}\frac{28(m+9)d_{m+9}M^{m+7}}{r^m} + \\ +\sum_{m=-4}^{\infty}\frac{50(m+8)d_{m+8}M^{m+7}}{r^m} + \\ \sum_{m=1}^{\infty}\frac{576A^2\kappa^2M^2(m+3)d_{m+3}M^{m+3}}{r^m} \end{aligned} \tag{E.14}$$

Now, extracting from the above sums the terms for $m=-5$ and for $m=0$, we have for these terms that:

$$\begin{aligned} -120d_4M^2r^5 - 990d_9M^7 + 240d_4M^2r^5 + 1080d_9M^7 - 112d_4M^2r^5 - 252d_9M^7 + \\ 1440M^7d_8 - 1584d_8M^7 + 400d_8M^7 \;=\; 8M^2d_4r^5 - 162d_9M^7 + 256d_8M^7 \end{aligned} \tag{E.15}$$

and the rest is the sums from $m = -4$ to $m = -1$ and from $m = 1$ to $m \to \infty$. So, eq.(E.1) will be:

$$
\begin{aligned}
0 = 8M^2 d_4 r^5 - 162 d_9 M^7 + 256 d_8 M^7 + 3456 A^2 \kappa^2 M^3 - \\
\sum_{m=-4}^{-1} \frac{(m+9)(m+10)(m+11) d_{m+9} M^{m+7}}{r^m} + \\
\sum_{m=-4}^{-1} \frac{12(m+9)(m+10) d_{m+9} M^{m+7}}{r^m} - \\
\sum_{m=-4}^{-1} \frac{28(m+9) d_{m+9} M^{m+7}}{r^m} + \\
\sum_{m=-4}^{-1} \frac{2(m+8)(m+9)(m+10) d_{m+8} M^{m+7}}{r^m} - \\
\sum_{m=-4}^{-1} \frac{22(m+8)(m+9) d_{m+8} M^{m+7}}{r^m} + \\
\sum_{m=-4}^{-1} \frac{50(m+8) d_{m+8} M^{m+7}}{r^m} - \\
-(\textit{...The same sums but with } m \textit{ from } m = 1 \textit{ to } m \to \infty\textit{...}) + \\
+ \sum_{m=1}^{\infty} \frac{576 A^2 \kappa^2 (m+3) d_{m+3} M^{m+3}}{r^m}
\end{aligned}
\tag{E.16}
$$

Putting the sums from $m = -4 \to -1$ and from $m = 1 \to \infty$ together, the above relation becomes:

$$
\begin{aligned}
&0 = 8M^2 d_4 r^5 - 162 d_9 M^7 + 256 d_8 M^7 + 3456 A^2 \kappa^2 M^3 + \\
&\sum_{m=-4}^{-1} \frac{M^{m+7}}{r^m} [d_{m+9}(m+9)\left(-28 + 12(m+10) - (m+10)(m+11)\right) \\
&+ d_{m+8}(m+8)\left(50 - 22(m+9) + 2(m+9)(m+10)\right)] \\
&+ \sum_{m=1}^{\infty} \frac{M^{m+7}}{r^m} [d_{m+9}(m+9)\left(-28 + 12(m+10) - (m+10)(m+11)\right) \\
&+ d_{m+8}(m+8)\left(50 - 22(m+9) + 2(m+9)(m+10)\right)] \\
&+ \sum_{m=1}^{\infty} \frac{A^2 \kappa^2 576 (m+3) d_{m+3} M^{m+3}}{r^m}
\end{aligned}
\tag{E.17}
$$

where, it's a simple task to show that

$$(-28 + 12(m+10) - (m+10)(m+11)) = -m^2 - 9m - 18 = -(m+3)(m+6)$$

and that

$$(50 - 22(m+9) + 2(m+9)(m+10)) = 2m^2 + 16m + 32 = 2(m+4)^2$$

So, we finally reach the form of the relation of eq.(3.39), which is given by:

$$3456A^2\kappa^2M^3 - 162M^7d_9 + 256M^7d_8 + 8M^2d_4r^5 +$$
$$\sum_{m=-4}^{-1}\frac{M^{m+7}\left[-(m+3)(m+6)(m+9)d_{m+9} + 2(m+4)^2(m+8)d_{m+8}\right]}{r^m} +$$
$$\sum_{m=1}^{\infty}\frac{M^{m+7}\left[-(m+3)(m+6)(m+9)d_{m+9} + 2(m+4)^2(m+8)d_{m+8}\right] + 576A^2\kappa^2M^{m+3}(m+3)d_{m+3}}{r^m} = 0 \ . \tag{E.18}$$

5.2 Solution of DE eq.(3.44)

In this Appendix, we solve eq.(3.44) given by:

$$-2u(r) + 2(r-M)u'(r) + (r^2 - 2Mr)u''(r) = \frac{144M^2}{r^5} \ , \tag{E.19}$$

The first step is to solve the homogeneous ODE,

$$-2u(r) + 2(r-M)u'(r) + (r^2 - 2Mr)u''(r) = 0 \ . \tag{E.20}$$

We note that a particular solution is $u_1 = c_1\left(\frac{r-M}{M}\right)$. We may use this solution to simplify the ODE via $u = u_1(r)z(r)$. Then, our simplified expression reads

$$\frac{1}{M}\left[2(M^2 - 4Mr + 2r^2)z' + r(2M^2 - 3Mr + r^2)z''\right] = 0 \ , \tag{E.21}$$

which implies that

$$\begin{aligned}
(\ln z') &= \int \frac{8Mr - 2M^2 - 4r^2}{(r-2M)(r-M)r}dr = -\ln[r(r-2M)(r-M)^2] + c_2 \\
\implies z' &= \frac{c_2}{r(r-2M)(r-M)^2} \\
z &= \frac{c_2}{(r-M)M^2} + \frac{c_2}{2M^3}\ln\left[1 - \frac{2M}{r}\right] + c_3 \ ,
\end{aligned}$$

which means that the homogeneous solution to the ODE is

$$u_h = c_1u_1(r) + c_2u_2(r) \ , \tag{E.22}$$

where

$$u_1(r) = \left(\frac{r-M}{M}\right) \ , \qquad u_2(r) = 1 + \frac{1}{2M}(r-M)\ln\left[1 - \frac{2M}{r}\right] \ . \tag{E.23}$$

In order to find the complete solution, we will make use of the method of variation of parameters. We consider that the general solution of the differential equation is expressed as

$$u(r) = C_1(r)u_1(r) + C_2(r)u_2(r) \ . \tag{E.24}$$

Then, $C_1(r), C_2(r)$ can be solved by the system

$$C_1'(r)u_1(r) + C_2'(r)u_2(r) = 0 \ , \tag{E.25}$$

$$C_1'(r)u_1'(r) + C_2'(r)u_2'(r) = \frac{144M^2}{r^5(r^2-2Mr)} \ . \tag{E.26}$$

Therefore,

$$\begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \frac{1}{\mathcal{W}} \begin{bmatrix} u_2' & -u_2 \\ -u_1' & u_1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{144M^2}{r^5(r^2-2Mr)} \end{bmatrix} \ , \tag{E.27}$$

where $\mathcal{W}$ denotes the Wronskian of our solutions. The system can be easily solved to yield that

$$C_1(r) = \frac{81M}{2r^4} - \frac{5}{r^3} - \frac{15}{4Mr^2} - \frac{15}{4M^2r} + \ln\left(1-\frac{2M}{r}\right)\frac{6(4r-3M)}{r^4} - \frac{15}{8M^3}\ln\left(1-\frac{2M}{r}\right) + c_1 \ , \tag{E.28}$$

$$C_2(r) = \frac{36M}{r^4} - \frac{48}{r^3} + c_2 \ . \tag{E.29}$$

Making use of (E.22), we find that the complete solution reads

$$\begin{gathered} u(r) = -c_1 + c_2 - \frac{15}{4M^3} - \frac{9M}{2r^4} - \frac{5}{2r^3} - \frac{5}{4Mr^2} + c_1\frac{r}{M} + \\ \ln\left(1-\frac{2M}{r}\right)\left[\frac{15}{8M^3} - \frac{c_2}{2}\right] + \ln\left(1-\frac{2M}{r}\right)\left[\frac{c_2 r}{2M} - \frac{15r}{8M^4}\right] \end{gathered} \tag{E.30}$$

In order to cancel the divergent terms, we fix the integration constants to $c_1 = 0$ and $c_2 = \frac{15}{4M^3}$ and we find that the asymptotic solution of the axion reads

$$u(r) = -\frac{9M}{2r^4} - \frac{5}{2r^3} - \frac{5}{4Mr^2} \ . \tag{E.31}$$

Appendix F

Proof of Convergence of the series 3.38

We have for the $t\phi$-component that:

$$g_{t\phi} = r^2 \left(-\frac{2M}{r^3} - w(r)\right) a \ \sin^2(\theta) \equiv \left(-\frac{2M}{r} - \widetilde{w}(r)\right) a \ \sin^2(\theta), \quad \widetilde{w}(r) = \sum_{n=4}^{\infty} \frac{d_n M^{n-2}}{r^{n-2}}\,, \tag{F.1}$$

where d_n is determined by the following recurrence relation:

$$d_n = \frac{2(n-5)^2(n-1)}{n(n-6)(n-3)} d_{n-1} + \frac{576\gamma^2}{n(n-3)} d_{n-6}, \quad \text{for } n \geq 10\ , \tag{F.2}$$

and initial conditions:

$$d_4 = d_5 = 0 \ \ , \ \ d_6 = -5\gamma^2 \ \ , \ \ d_7 = -\frac{60\gamma^2}{7} \ \ , \ \ d_8 = -\frac{27\gamma^2}{2} \ \ , \ \ d_9 = 0 \tag{F.3}$$

with $\gamma \in \mathbb{R}$. We restrict ourselves in the exterior to horizon region $r \geq 2M$.

For notational convenience and brevity we redefine $r \to r/M$, hence

$$\widetilde{w}(r) = \sum_{n=4}^{\infty} \frac{d_n}{r^{n-2}}. \tag{F.4}$$

Below we shall prove the convergence of this sum for all $r \geq 2$ for all $\gamma \in \mathbb{R}$.

To this end, we first note that $d_n \leq 0$, $\forall n$. Thus, we define the sequence $c_n = -d_n$, for which $c_n \geq 0$, $\forall n$. Then, in terms of c_n, we have:

$$\widetilde{w}(r) = -\sum_{n=4}^{\infty} \frac{c_n}{r^{n-2}} \tag{F.5}$$

where

$$c_n = a_n c_{n-1} + b_n c_{n-6}, \quad \text{for } n \geq 10 \ , \tag{F.6}$$

with

$$a_n = \frac{2(n-5)^2(n-1)}{n(n-6)(n-3)} \quad b_n = \frac{576\gamma^2}{n(n-3)}, \tag{F.7}$$

and initial conditions:

$$c_4 = c_5 = 0 \ , \ c_6 = 5\gamma^2 \ , \ c_7 = \frac{60\gamma^2}{7} \ , \ c_8 = \frac{27\gamma^2}{2} \ , \ c_9 = 0 \tag{F.8}$$

Let us define the sequence:

$$\tilde{\Sigma}_N = \sum_{n=4}^{N} \frac{c_n}{r^{n-2}}$$

As c_n/r^{n-2} are non-negative, $\tilde{\Sigma}_N$ is an increasing sequence, meaning $\tilde{\Sigma}_{N+1} \geq \tilde{\Sigma}_N$, $\forall N \in \mathbb{N}$.

For increasing sequences, the following theorem exists:

Theorem: *An increasing sequence tends either to a finite limit or to $+\infty$.*

Hence, a necessary and sufficient condition for the convergence of $\tilde{\Sigma}$ is the demonstration that it is bounded, *i.e.* that there exists a finite, positive number $\mathcal{N}$, such that:

$$\tilde{\Sigma} = \sum_{n=4}^{\infty} \frac{c_n}{r^{n-2}} \leq \mathcal{N}$$

Statement 1: *If $\sum_{n=4}^{\infty} \frac{c_n}{2^{n-2}}$ converges, then $\sum_{n=4}^{\infty} \frac{c_n}{r^{n-2}}$ converges $\forall r \geq 2$.*

Suppose that $\sum_{n=4}^{\infty} \frac{c_n}{2^{n-2}} \leq \mathcal{K}$, where $\mathcal{K}$ finite. For $r > 2 \rightarrow 1/r^{n-2} < 1/2^{n-2} \rightarrow c_n/r^{n-2} \leq c_n/2^{n-2}$ $\forall n \geq 4$, where the equality holds in the case of $c_n = 0$. Thus,

$$\sum_{n=4}^{\infty} \frac{c_n}{r^{n-2}} \leq \sum_{n=4}^{\infty} \frac{c_n}{2^{n-2}} \quad \Rightarrow \quad \sum_{n=4}^{\infty} \frac{c_n}{r^{n-2}} \leq \mathcal{K}$$

which means that $\sum_{n=4}^{\infty} \frac{c_n}{r^{n-2}}$ converges $\forall r > 2$.

Hence, according to the above statement 1, we should establish the convergence of the infinite series

$$\Sigma = \sum_{n=4}^{\infty} \frac{c_n}{2^{n-2}}$$

For a_n and b_n, we have:

$$1)\ a_n \to 2,\ \text{as}\ n \to +\infty$$
$$2)\ b_n \to 0,\ \text{as}\ n \to +\infty$$

i.e. a_n and b_n both converge.

Below, we use the mathematical result *that a sequence, say s_n, is convergent, i.e. tends to a finite limit s as $n \to \infty$, implies that s_n is bounded* This implies that, since a_n, b_n are convergent, in view of the above result, they are bounded, meaning that there exist k_1, k_2, such that

$$|a_n| \leq k_1 \ \ and \ \ |b_n| \leq k_2 \ \ \forall n \in \mathbb{N} \tag{F.9}$$

<u>**Statement 2**</u>: *The sequence c_n is bounded by induction.*

Suppose that there exists a subsequence $c_{N-6}, ...c_{N-1}$, for some N, that is bounded, i.e. there exists N', such that

$$|c_{N-6}|, |c_{N-5}|, ..., |c_{N-1}| \leq N'$$

.
Then, using the triangle inequality

$$|c_N| \leq |a_N||c_{N-1}| + |b_N||c_{N-6}| \Rightarrow \quad |c_N| \leq (k_1 + k_2)N' \Rightarrow |c_N| \leq \widetilde{\mathcal{M}}\,,$$

where $\widetilde{\mathcal{M}} = (k_1 + k_2)\, N'$ finite.

Thus, as the corresponding subsequence exists for $N = 10$, as $c_4, ..., c_9$ are finite, by induction, there exists finite $\mathcal{D}$, such that

$$|c_n| \leq \mathcal{D},\ \forall n \geq 10$$

concluding that c_n is bounded $\forall n \geq 4$.

Thus, as c_n is bounded and non-negative, there exists $\mathcal{D} > 0$, such that:

$$0 \leq c_n \leq \mathcal{D}\ ,\forall\ n \geq 4$$

Thence,

$$\Sigma = \sum_{n=4}^{\infty} \frac{c_n}{2^{n-2}} \leq \mathcal{D} \sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^{n-2}$$

or

$$\Sigma \leq \mathcal{D} \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^{n}$$

For the geometric series, we know that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad \text{for } -1 < x < 1$$

Thus, for $x = 1/2$, it is easy to show that:

$$\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2}$$

which implies

$$\Sigma = \sum_{n=4}^{\infty} \frac{c_n}{2^{n-2}} \leq \frac{\mathcal{D}}{2},$$

i.e., the infinite series Σ is bounded.

This proves the required result, that the sum $\widetilde{w}(r)$, and thus $w(r)$ ((3.38)), converges $\forall r \geq 2M$ and $\forall \gamma \in \mathbb{R}$.

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List of figures