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ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ ΚΑΙ ΦΥΣΙΚΩΝ ΕΠΙΣΤΗΜΩΝ

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Models and Stability of Self-Gravitating Magnetic Monopoles beyond the Standard Model

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Abstract

In this thesis we investigate the physics of magnetic monopoles and search for self-gravitating models that correspond to magnetic monopoles with structure. The main goal of this work is the construction of monopole models that predict magnetic monopoles with reduced mass, which may be detected in current or future colliders. First of all, we present the electromagnetic duality and the idea of the symmetrization of Maxwell's equations, which imply the existence of magnetic monopoles. Then, to get a sense of the interactions of magnetic monopoles with charged matter, we examine classical and quantum electromagnetic systems, which consist of an electrically charged particle and particles with magnetic charge. Additionally, we present the Dirac model of magnetic monopoles and the corresponding Dirac strings. Moreover, we examine the Dirac quantization condition, which provides an elegant explanation of the electric charge quantization. Also, we investigate the solutions of the Schrödinger, Pauli and Dirac equations considering a Coulomb-like magnetic field. Furthermore, upon considering the topological roots of the magnetic charge, we investigate some monopole models. For instance, we present the 't Hooft-Polyakov monopole solution of the Georgi-Glashow model and explicitly prove the topological origin of the conservation of the corresponding magnetic charge. The mass of the 't Hooft-Polyakov monopole must be reduced in order for the magnetic monopole to be detected in current or future colliders. Therefore, we consider self-gravitating global models, since the self-gravitational interaction and the independence of the scalar fields from gauge symmetries may reduce the mass of the monopoles. The simplest self-gravitating global monopole model is the Barriola-Vilenkin model, which predicts unstable monopoles with negative mass. Consequently, we present a string inspired magnetic monopole model derived from a self-gravitating global monopole model with Kalb-Ramond torsion. This model predicts magnetic monopoles with structure, positive mass and a very interesting mechanism, which implies the regularisation of the corresponding curvature singularity and the stability of the model. The aforementioned regularisation mechanism is a tool of high importance in the research of self-gravitating monopoles. Subsequently, we phenomenologically investigate the structured-particle solutions of the above model by determining the mass and radius of the monopole core. Finally, we impose a lower limit on the monopole mass, considering recent results from the ATLAS and MoEDAL experiments at the LHC. In the appendix of this thesis, we include comments on some papers that deal with the application of the Newman-Janis algorithm or some corresponding modified algorithms to the global monopole model.

Περίληψη

Σε αυτή τη διατριβή διερευνούμε τη φυσική των μαγνητικών μονοπόλων και αναζητούμε self-gravitating μοντέλα που αντιστοιχούν σε μαγνητικά μονόπολα με εσωτερική δομή. Ο κύριος στόχος αυτής της εργασίας είναι η κατασκευή μοντέλων μονοπόλων που προβλέπουν μαγνητικά μονόπολα με μειωμένη μάζα, τα οποία μπορούν να ανιχνευθούν σε υπάρχοντες ή μελλοντικούς επιταχυντές. Αρχικά, παρουσιάζουμε το electromagnetic duality και την ιδέα της συμμετρικοποίησης των εξισώσεων του Maxwell, που συνεπάγεται την ύπαρξη των μαγνητικών μονοπόλων. Στη συνέχεια, για να έχουμε μια αίσθηση των αλληλεπιδράσεων των μαγνητικών μονοπόλων με τη φορτισμένη ύλη, εξετάζουμε κλασσικά και κβαντικά ηλεκτρομαγνητικά συστήματα που αποτελούνται από ένα ηλεκτρικά φορτισμένο σωματίδιο και σωματίδια με μαγνητικό φορτίο. Επιπλέον, παρουσιάζουμε το μοντέλο του Dirac για τα μαγνητικά μονόπολα και τα Dirac strings. Επιπρόσθετα, εξετάζουμε τη συνθήκη κβαντισμού του Dirac, η οποία παρέχει μια κομψή εξήγηση της κβάντωσης του ηλεκτρικού φορτίου. Επίσης, διερευνούμε τις λύσεις των εξισώσεων Schrödinger, Pauli και Dirac λαμβάνοντας υπόψη ένα μαγνητικό πεδίο τύπου Coulomb. Έχοντας διερευνήσει τη βασική φυσική των μαγνητικών μονοπόλων και εξετάζοντας τις τοπολογικές ρίζες του μαγνητικού φορτίου, μελετάμε ορισμένα μοντέλα μονοπόλων. Για παράδειγμα, παρουσιάζουμε το μαγνητικό μονόπολο 't Hooft-Polyakov, το οποίο αποτελεί λύση του μοντέλου Georgi-Glashow και αποδεικνύουμε ρητά την τοπολογική προέλευση της διατήρησης του αντίστοιχου μαγνητικού φορτίου. Η μάζα του μονοπόλου 't Hooft-Polyakov πρέπει να μειωθεί προκειμένου το μαγνητικό μονόπολο να μπορεί να ανιχνευθεί στους τρέχοντες ή σε μελλοντικούς επιταχυντές. Ως εκ τούτου, επικεντρωνόμαστε σε self-gravitating global μοντέλα, καθώς η self-gravitational αλληλεπίδραση και η ανεξαρτησία των βαθμωτών πεδίων από gauge συμμετρίες μπορεί να μειώσει τη μάζα των μονοπόλων. Το απλούστερο self-gravitating global μοντέλο είναι το μοντέλο Barriola-Vilenkin, το οποίο προβλέπει ασταθή μονόπολα με αρνητική μάζα. Συνεπώς, παρουσιάζουμε ένα μοντέλο μαγνητικού μονοπόλου εμπνευσμένο από τη θεωρία χορδών, το οποίο προέρχεται από ένα self-gravitating global μοντέλο μονοπόλου με Kalb-Ramond torsion. Αυτό το μοντέλο προβλέπει μαγνητικά μονόπολα θετικής μάζας με εσωτερική δομή και έναν πολύ ενδιαφέροντα μηχανισμό, που συνεπάγεται το regularisation της αντίστοιχης curvature singularity και τη σταθερότητα του μοντέλου. Ο προαναφερθείς μηχανισμός εξομάλυνσης είναι ένα εργαλείο υψηλής σημασίας για την έρευνα self-gravitating μονοπόλων. Στη συνέχεια, διερευνούμε φαινομενολογικά τις σωματιδιακές λύσεις με δομή του παραπάνω μοντέλου προσδιορίζοντας τη μάζα και την ακτίνα του πυρήνα του μονοπόλου. Τέλος, επιβάλλουμε ένα χαμηλότερο όριο στη μάζα του μονοπόλου, λαμβάνοντας υπόψη τα πρόσφατα αποτελέσματα από τα πειράματα ATLAS και MoEDAL στον LHC. Στο παράρτημα της παρούσας διπλωματικής εργασίας, συμπεριλαμβάνουμε κάποια σχόλια πάνω σε ορισμένες εργασίες που αφορούν την εφαρμογή του αλγορίθμου Newman-Janis ή ορισμένων αντίστοιχων αλγορίθμων στο self-gravitating global μοντέλο Barriola-Vilenkin.

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Contents

Preface

In this thesis, we present the main properties of the magnetic monopoles and some theories, which involve them.

To be more precise, the hypothesis of the magnetic monopole existence yields some additional properties to some classical theories. Therefore, in the first and the second chapter, we describe the electromagnetic interaction between an electrically charged particle and a magnetic monopole, through the perspective of classical electromagnetism and quantum mechanics. Also, it is very interesting that, in many cases, the magnetic charge corresponds to the topological properties of the theory. Hence, in the third chapter, we present a brief introduction to the topology, which provides us with the necessary tools in order to find the topological roots of the magnetic charge. Moreover, using the topological properties of the monopole, we describe the construction of some monopole theories. In particular, in the fourth chapter, we search for magnetic monopole solutions of the Georgi-Glashow model, which is a gauge $SU(2)$ symmetric theory in flat space-time. The magnetic monopole, which corresponds to this theory, is named 't Hooft-Polyakov monopole. In the fifth chapter, we present the self-gravitating global monopole solutions of the Barriola-Vilenkin model. Furthermore, in the sixth chapter, we search for magnetic monopoles from self-gravitating global monopoles with Kalb-Ramond torsion. Additionally, we examine this model phenomenologically and present an elegant mechanism for the regularisation of the model's curvature singularity, from which spontaneously corresponds the stability of the monopole. This is a string-inspired theory. Finally, in the appendix, among other supplementary notes, we present a discussion on the application of the Newman-Janis algorithm to the global self-gravitating monopole model.

The resources, which mainly inspired this thesis, are the book of Yakov M. Shnir "Magnetic Monopoles" [1], the thesis of dr.Nikos Chatzifotis "Magnetic Monopoles" [2] and the original papers of Nick E. Mavromatos and Sarben Sarkar "Magnetic Monopoles from Global Monopoles in the presence of a Kalb-Ramond Field" [3] and "Regularised Kalb-Ramond Magnetic Monopole with Finite Energy" [4].

Chapter 1

Magnetic Monopoles: A classical theory approach

First of all, it is essential to describe the innovating ideas, which lead to the hypothesis of the magnetic monopole existence. The main idea is the symmetrization of Maxwell's equations. This is possible, if we assume the existence of a magnetic charge. Then, the equations are invariant under the dual transformations, which replace the electric with the magnetic charge and vice-versa. The extended Maxwell's equations lead to a new electromagnetic interaction between an electrically charged particle and a magnetic monopole.

Furthermore, as it is known, the scalar and vector electromagnetic potentials play a fundamental role in the physics of electromagnetic systems. A theory described by these potentials has four degrees of freedom. The gauge fixing and the equations of motion decrease the degrees of freedom from four to two. That is why the electromagnetic field has two degrees of freedom (two polarizations of the light) and the photon may have spin ± 1. This electromagnetic structure works well, describing numerous phenomena in the perspective of classical mechanics, quantum mechanics and quantum electrodynamics. Additionally, the scalar and vector potential form the gauge field, which corresponds to the $U(1)$ gauge symmetry of electromagnetism. Dirac's pioneering idea was to describe the magnetic induction of the magnetic monopoles using the curl of a vector field, as it is customary in classical electromagnetism. However, due to the existence of the magnetic monopoles, the vector potential has a singularity along a semi-infinite string, called the Dirac string.

1.1 Electromagnetic duality

Let us start our discussion from the hypothesis of the symmetric Maxwell's equations of the form:

$$\nabla \vec{E} = \rho_e \tag{1.1}$$

$$\nabla \vec{B} = \rho_g \tag{1.2}$$

$$\nabla \times \vec{E} = -\vec{j}_g - \frac{\partial \vec{B}}{\partial t} \tag{1.3}$$

$$\nabla \times \vec{B} = \vec{j}_e + \frac{\partial \vec{E}}{\partial t} \tag{1.4}$$

Note that the free Maxwell's equations ($\rho_e = \rho_g = 0$ and $\vec{j}_e = \vec{j}_g = \vec{0}$) are invariant under global $O(2)$ transformations [1]:

$$\begin{pmatrix} \vec{E}' \\ \vec{B}' \end{pmatrix} = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} \tag{1.5}$$

The free Maxwell's equations can be written using the electromagnetic tensor as follows:

$$\partial_\nu F^{\mu\nu} = 0 \tag{1.6}$$

$$\partial_\nu \tilde{F}^{\mu\nu} = 0 \tag{1.7}$$

where the equations (1.6) are the Euler-Lagrange equations of the system and the equations (1.7) are implied by the Bianchi identity $\partial_{[\alpha}F_{\beta\gamma]} = 0$, where the brackets [...] denote the total antisymmetrization of the respective indices. Also, the tensor $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ is the dual tensor of $F_{\mu\nu}$, where $\epsilon_{\mu\nu\rho\sigma}$ is the total antisymmetric Levi-Civita tensor defined as:

$$\epsilon_{\mu_1\mu_2...\mu_n} = \sqrt{|g|}\tilde{\epsilon}_{\mu_1\mu_2...\mu_n} \tag{1.8}$$

Similarly, the Levi-Civita tensor with upper indices reads:

$$\epsilon^{\mu_1\mu_2...\mu_n} = \frac{1}{\sqrt{|g|}}\tilde{\epsilon}^{\mu_1\mu_2...\mu_n} \tag{1.9}$$

where g is the determinant of the metric and $\tilde{\epsilon}_{\mu_1\mu_2...\mu_n}$ is the Levi-Civita tensor density, which reads:

$$\tilde{\epsilon}_{\mu_1\mu_2...\mu_n} = \begin{cases} +1 & \text{if } \mu_1\mu_2...\mu_n \text{is an even permutation of } 0123...(n-1) \\ -1 & \text{if } \mu_1\mu_2...\mu_n \text{is an odd permutation of } 0123...(n-1) \\ 0 & \text{otherwise} \end{cases} \tag{1.10}$$

and

$$\tilde{\epsilon}^{\mu_1\mu_2...\mu_n} = \begin{cases} -1 & \text{if } \mu_1\mu_2...\mu_n \text{is an even permutation of } 0123...(n-1) \\ +1 & \text{if } \mu_1\mu_2...\mu_n \text{is an odd permutation of } 0123...(n-1) \\ 0 & \text{otherwise} \end{cases} \tag{1.11}$$

Additionally, it is evident that the electromagnetic tensor satisfies the Bianchi identity (1.7) due to the absence of a magnetic charge ($\rho_g = 0$). Later on, when we work with magnetic monopole solutions, the right side of the equation (1.7) will not be zero and, instead, will be equal to the topological current, or the torsion current due to the Kalb-Ramond term in the model of chapter 6. The conservation of topological current has topological roots and yields the conservation of the magnetic charge.

The aforementioned transformation (1.5) can be expressed via the electromagnetic tensor as follows [1]:

$$\begin{pmatrix} F'_{\mu\nu} \\ \tilde{F}'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} \begin{pmatrix} F_{\mu\nu} \\ \tilde{F}_{\mu\nu} \end{pmatrix} \tag{1.12}$$

Another representation of this transformation reads:

$$\vec{F} = \vec{E} + i\vec{B} \longrightarrow \vec{F}' = e^{i\theta}(\vec{E} + i\vec{B}) \tag{1.13}$$

where θ is an arbitrary angle, which is called dual angle. To be more precise, the choice of $\theta = -\pi/2$ leads to the interchange of the electric and the magnetic field:

$$\vec{E} \longrightarrow \vec{B} \text{ and } \vec{B} \longrightarrow -\vec{E} \tag{1.14}$$

Also, there is a straightforward consequence of the aforementioned form of the transformation, which is the energy and the momentum invariance:

$$\mathcal{E} = \frac{1}{2}|\vec{F}|^2 = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \text{ and } \vec{P} = \frac{1}{2i}(\vec{F}^* \times \vec{F}) = \vec{E} \times \vec{B} \tag{1.15}$$

It is very important to examine the corresponding transformation of the Lagrangian density of the E/M field:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \longrightarrow \mathcal{L}' = -\frac{1}{4}(cos(\theta)F_{\mu\nu} - sin(\theta)\tilde{F}_{\mu\nu})^2 \Rightarrow$$

$$\mathcal{L} \longrightarrow \mathcal{L}' = -\frac{1}{4}\big(cos^2(\theta)F_{\mu\nu}F^{\mu\nu} + sin^2(\theta)\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - sin(2\theta)F_{\mu\nu}\tilde{F}^{\mu\nu}\big)$$

Considering the definition of the dual electromagnetic tensor ($\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$), we have:

$$\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\kappa\lambda}F^{\rho\sigma}F_{\kappa\lambda} = \frac{1}{4}\big[-2(\delta_\rho{}^\kappa\delta_\sigma{}^\lambda - \delta_\rho{}^\lambda\delta_\sigma{}^\kappa)\big]F^{\rho\sigma}F_{\kappa\lambda} \Rightarrow$$

$$\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = -F_{\mu\nu}F^{\mu\nu}$$

where we used the identity $\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\kappa\lambda} = -2(\delta_\rho{}^\kappa\delta_\sigma{}^\lambda - \delta_\rho{}^\lambda\delta_\sigma{}^\kappa)$. Therefore, the transformation of the Lagrangian density yields:

$$\mathcal{L} \longrightarrow \mathcal{L}' = -\frac{1}{4}\big(cos(2\theta)F_{\mu\nu}F^{\mu\nu} - sin(2\theta)F_{\mu\nu}\tilde{F}^{\mu\nu}\big) \tag{1.16}$$

It is crucial to prove that the action remains invariant under this transformation. Considering $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, which identically satisfies Bianchi identity, we obtain:

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)\tilde{F}^{\mu\nu} = 2(\partial_\mu A_\nu)\tilde{F}^{\mu\nu} = 2\partial_\mu(A_\nu\tilde{F}^{\mu\nu}) - 2A_\nu(\partial_\mu\tilde{F}^{\mu\nu}) \Rightarrow$$

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu(2A_\nu\tilde{F}^{\mu\nu}) \tag{1.17}$$

where we used the Bianchi identity (1.7). The term $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is a total derivative, as a result, it does not contribute to the dynamical equations of the fields. Consequently, the examined transformation acts like a canonical transformation on the action of the system. This confirms the invariance of the system.

Now, let us assume that we have electric and magnetic charge distributions. Then, the Maxwell's equations (1.1-1.4) can be written as:

$$\nabla(\vec{E} + i\vec{B}) = \rho_e + i\rho_g \text{ and } \nabla\times(\vec{E} + i\vec{B}) - i\frac{\partial}{\partial t}(\vec{E} + i\vec{B}) = i(\vec{j}_e + i\vec{j}_g) \tag{1.18}$$

Hence, upon considering the previously examined transformations and an equivalent interchange between electric and magnetic charges, we construct the dual transformations:

$$\vec{F} = \vec{E} + i\vec{B} \longrightarrow \vec{F}' = e^{i\theta}(\vec{E} + i\vec{B}) \text{ and } q = e + ig \longrightarrow q' = e^{i\theta}(e + ig) \tag{1.19}$$

Maxwell's equations remain invariant under the global dual transformations (1.19). The fact that the dual transformations connect the electric and the magnetic charges means that we do not have a two-charge electromagnetic theory. For instance, for $\theta = -\pi/2$ the dual transformation yields $e \longrightarrow g$ with the constraint $Q = |q| = const.$. Note that the charge Q is the experimental invariant, since it is the dual-invariant charge.

The form of the electromagnetic tensor is not trivial in the case of a magnetic monopole, where the Bianchi identity is not satisfied. Therefore, in such a case, the invariance of the electromagnetic action can be explicitly proven after the determination of the electromagnetic tensor. A dual symmetric theory that predicts magnetic monopoles, such as those we discuss later on, also predicts electric monopoles, in the case of a non-fixed dual angle.

1.2 Classical electromagnetic scattering

In this section, we present the properties of a system consisting of an electrically charged particle and a magnetic monopole (or a dyon, which has both electric and magnetic charge) [1]. In this discussion, we consider that the magnetic monopole is always located at the point $\vec{r} = \vec{0}$, hence it produces a static Coulomb-like field.

1.2.1 Classical scattering on a Magnetic Monopole

First of all, let us consider that the monopole has only magnetic charge, g. Then, the corresponding magnetic induction satisfies the equation (1.2), where the magnetic charge distribution is:

$$\rho_g = 4\pi g\delta^{(3)}(\vec{r}) \tag{1.20}$$

Demanding that the magnetic induction vanishes at infinity, the solution of equation (1.2), considering equation (1.20), is:

$$\vec{B} = g\frac{\vec{r}}{r^3} \tag{1.21}$$

This is a static Coulomb-like field. The Lorentz force, which acts on an electrically charged particle, with charge e, which moves in the external field (1.21), reads:

$$m\ddot{\vec{r}} = e\dot{\vec{r}}\times\vec{B} = \frac{eg}{r^3}(\dot{\vec{r}}\times\vec{r}) \tag{1.22}$$

where the dot denotes the total derivative with respect to time. The scalar product of equation (1.22) with the velocity $\dot{\vec{r}}$ of the charged particle yields:

$$E = \frac{1}{2}m\dot{\vec{r}}^2 = const. \text{ and } |\dot{\vec{r}}| = u = const. \tag{1.23}$$

The kinetic energy of the particle is conserved. Similarly, the scalar product of equation (1.22) with the position $\dot{\vec{r}}$ of the charge implies:

$$\vec{r} \cdot \dot{\vec{r}} = u^2 t \tag{1.24}$$

The solution of equation (1.24) reads:

$$r = \sqrt{u^2t^2 + b^2} \tag{1.25}$$

where b is the minimum distance between the charge and the monopole. From equation (1.24) we obtain the first interesting result. There are no closed orbits in the charge-monopole system, since, only at the moment $t = 0$, the position vector $\vec{r}$ of the particle becomes perpendicular to the velocity vector $\dot{\vec{r}}$ of the particle.

In the subsequent steps we determine the conserved angular momentum. The particle's angular momentum reads:

$$\tilde{\vec{L}} = \vec{r} \times m\dot{\vec{r}} \tag{1.26}$$

$$\frac{d}{dt}\tilde{\vec{L}} = \frac{d}{dt}(\vec{r} \times m\dot{\vec{r}}) = \vec{r} \times m\ddot{\vec{r}} \overset{(1.22)}{\Longrightarrow}$$

$$\frac{d}{dt}\tilde{\vec{L}} = \frac{eg}{mr^3}(\tilde{\vec{L}} \times \vec{r}) \tag{1.27}$$

It is obvious that the particle's angular momentum vector is not conserved. On the contrary, the magnitude of the angular momentum is conserved. Upon taking the scalar product of equation (1.27) with $\tilde{\vec{L}}$, we obtain:

$$\frac{d}{dt}\tilde{\vec{L}}^2 = 0 \Rightarrow$$

$$|\tilde{\vec{L}}| = \tilde{L} = const. \tag{1.28}$$

Consequently, we have:

$$\tilde{L} = \tilde{L}(t=0) = \vec{r}(t=0) \times m\dot{\vec{r}}(t=0)$$

Note that, for $t = 0$, from equation (1.24) it is yielded that $\vec{r} \perp \dot{\vec{r}}$ and from equation (1.25) it is yielded that $r(t=0) = b$. Therefore:

$$\tilde{L} = mub \tag{1.29}$$

Using the identity $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$, from equation (1.27) we obtain:

$$\frac{d}{dt}\tilde{\vec{L}} = \frac{eg}{mr^3}(r^2\dot{\vec{r}} - \vec{r}(\vec{r}\dot{\vec{r}})) = eg\left(\frac{1}{r}\dot{\vec{r}} - \frac{\vec{r}}{2r^3}\frac{dr^2}{dt}\right) = eg\left(\frac{1}{r}\dot{\vec{r}} - \frac{\vec{r}}{r^2}\dot{r}\right) \Rightarrow$$

$$\frac{d}{dt}\tilde{\vec{L}} = eg\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) \tag{1.30}$$

Defining the generalized angular momentum

$$\vec{L} = \tilde{\vec{L}} - eg\hat{r} \tag{1.31}$$

we observe that:

$$\frac{d}{dt}\vec{L} = 0 \tag{1.32}$$

Therefore, the generalized angular momentum is conserved. It is straightforward that the magnitude of $\vec{L}$ is conserved.

$$L^2 = (mub)^2 + (eg)^2 \tag{1.33}$$

It is essential to determine the source of the second term of the generalized angular momentum ($-eg\hat{r}$). In order to do so, we calculate the angular momentum of the electromagnetic field:

$$\tilde{\vec{L}}_{eg} = \frac{1}{4\pi}\int d^3r'(\vec{r}\,'\times\vec{S}) = \frac{1}{4\pi}\int d^3r'[\vec{r}\,'\times(\vec{E}\times\vec{B})] = \frac{g}{4\pi}\int d^3r'[\vec{r}\,'\times(\vec{E}\times\vec{r})]\frac{1}{r'^3} \Rightarrow$$

$$\tilde{\vec{L}}_{eg} = \frac{g}{4\pi}\int d^3r'\left[\frac{\vec{E}}{r'} - \frac{\vec{r}\,'}{r'^3}(\vec{r}\,'\vec{E})\right]$$

Note that $\nabla'\left(\frac{1}{r'}\right) = -\frac{\vec{r}\,'}{r'^3}$. Therefore:

$$\tilde{\vec{L}}_{eg} = \frac{g}{4\pi}\int d^3r'\left[\frac{\vec{E}}{r'} + \nabla'\left(\frac{1}{r'}\right)(\vec{r}\,'\vec{E})\right] \overset{\text{integration by parts}}{\Longrightarrow}$$

$$\tilde{\vec{L}}_{eg} = \frac{g}{4\pi}\int d^3r'\left[\frac{\vec{E}}{r'} - \frac{1}{r'}\nabla'(\vec{r}\,'\vec{E}) + \nabla'(\hat{r}'\vec{E})\right]$$

Using the identities $\nabla'(\vec{r}\,'\vec{E}) = \vec{r}\,'\times(\nabla'\times\vec{E}) + \vec{E}\times(\nabla'\times\vec{r}\,') + (\vec{r}\,'\nabla')\vec{E} + (\vec{E}\nabla')\vec{r}\,'$, $\nabla'\times\vec{r}\,' = 0$, the equation $\nabla'\times\vec{E} = 0$ (1.3) and the fact that $(\vec{E}\nabla')\vec{r}\,' = \vec{E}$, we obtain:

$$\tilde{\vec{L}}_{eg} = \frac{g}{4\pi}\int d^3r'(\vec{E}\nabla')\hat{r}' = \frac{g}{4\pi}\left[\hat{x}\int d^3r'(\vec{E}\nabla')\frac{x'}{r'} + \hat{y}\int d^3r'(\vec{E}\nabla')\frac{y'}{r'} + \hat{z}\int d^3r'(\vec{E}\nabla')\frac{z'}{r'}\right] =$$

$$\frac{g}{4\pi}\left[\hat{x}\int d^3r'\nabla'\left(\frac{x'\vec{E}}{r'}\right) + \hat{y}\int d^3r'\nabla'\left(\frac{y'\vec{E}}{r'}\right) + \hat{z}\int d^3r'\nabla'\left(\frac{z'\vec{E}}{r'}\right)\right] - \frac{g}{4\pi}\int d^3r'\hat{r}'(\nabla'\vec{E})$$

From Gauss's theorem we obtain: $\int d^3r'\nabla'\left(\frac{x'\vec{E}}{r'}\right) = \oint_\infty d\vec{a}'\frac{x'\vec{E}(r'\to\infty)}{r'} = 0$, where we assume $\vec{E}(r'\to\infty) = \vec{0}$. We perform similar calculations for the cases y, z. Finally, considering the equation $\nabla'\vec{E} = 4\pi e\delta^{(3)}(\vec{r} - \vec{r}')$ (1.1), the angular momentum of the E/M field reads:

$$\tilde{\vec{L}}_{eg} = -eg\hat{r} \tag{1.34}$$

Consequently, the extra term in the expression of the generalized angular momentum (1.31) is the angular momentum of the electromagnetic field. It is very interesting that the term (1.34) does not vanish even if the two particles remain stationary. Note that this case is examined above, because we ignore the magnetic induction produced by the moving electric charge.

Furthermore, it is possible to determine the trajectory of the electrically charged particle. The inner product of the generalized angular momentum (1.31) and the radial unit vector $\hat{r}$ reads:

$$\vec{L}\cdot\hat{r} = -eg \tag{1.35}$$

Hence, due to the fact that the generalized angular momentum is a constant vector, we deduce that the charged particle's trajectory lies on the surface of a cone, with an axis of symmetry along the vector $-\vec{L}$ and a cone angle θ, which can be determined as follows:

$$|\vec{L}\cdot\hat{r}| = eg \Rightarrow$$

$$cos(\theta) = \frac{eg}{L} = \frac{eg}{\sqrt{(mub)^2 + (eg)^2}} \tag{1.36}$$

Also, with straight forward calculations we obtain:

$$sin(\theta) = \frac{\tilde{L}}{L} = \frac{mub}{\sqrt{(mub)^2 + (eg)^2}} \tag{1.37}$$

$$tan(\theta) = \frac{\tilde{L}}{eg} = \frac{mub}{eg} \tag{1.38}$$

$$cot(\theta) = \frac{eg}{\tilde{L}} = \frac{eg}{mub} \tag{1.39}$$

Additionally, the inner product of the orbital and generalized angular momentum yields:

$$\vec{L} \cdot \tilde{\vec{L}} = \tilde{L}^2 = (mub)^2 = const. \tag{1.40}$$

which means that the vector of the particle's orbital angular momentum is precessing on the surface of a cone with an axis of symmetry along the constant vector $\vec{L}$ and a cone angle $\tilde{\theta}$, which reads:

$$\vec{L} \cdot \tilde{\vec{L}} = \tilde{L}^2 \Rightarrow$$

$$cos(\tilde{\theta}) = \frac{\tilde{L}}{L} = sin(\theta) = \frac{mub}{\sqrt{(mub)^2 + (eg)^2}} \quad \text{and} \quad \tilde{\theta} = \frac{\pi}{2} - \theta \tag{1.41}$$

The charged particle's velocity vector may be written as follows:

$$-\vec{r} \times \tilde{\vec{L}} = -\vec{r} \times \vec{L} = -\vec{r} \times (\vec{r} \times m\dot{\vec{r}}) = -m\vec{r}(\vec{r}\dot{\vec{r}}) + m\dot{\vec{r}}r^2 \Rightarrow$$

$$\dot{\vec{r}} = \dot{r}\hat{r} + \frac{\vec{L} \times \vec{r}}{mr^2} = \vec{\omega} \times \vec{r} + \dot{r}\hat{r} \tag{1.42}$$

where $\vec{\omega} = \dfrac{\vec{L}}{mr^2}$ is the particle's angular velocity. Also, from equation (1.25), we can calculate the radial velocity $\dot{r}$.

$$\vec{\omega}(t) = \frac{\vec{L}}{mr^2(t)} = \frac{\vec{L}}{m(u^2t^2 + b^2)} \ , \quad \dot{r}(t) = \frac{u}{\sqrt{1 + \left(\frac{b}{ut}\right)^2}} \quad \text{and} \quad r(t) = \sqrt{u^2t^2 + b^2} \tag{1.43}$$

Note that:

$$\vec{\omega}(t \to \pm\infty) = \vec{0} \ , \ \ \dot{r}(t \to \pm\infty) = u \ , \ \ r(t \to \pm\infty) \to +\infty$$

$$|\vec{\omega}|(t = 0) = \frac{\sqrt{(mub)^2 + (eg)^2}}{mb^2} \ , \ \ \dot{r}(t = 0) = 0 \ , \ \ r(t = 0) = b \tag{1.44}$$

At this point, we can make some useful observations. Let us assume that the charged particle comes from $+\infty$ and moves radially. Then, it approaches the monopole up to a minimum distance $r = b$. Subsequently, it returns to $+\infty$, moving radially towards a different direction from the initial one. This process describes the scattering of an electrically charged particle on a static magnetic monopole. The angle between the initial and the final direction of the monopole defines the scattering angle Θ, which will be calculated later on. An interesting observation is that the conditions $\vec{\omega}(t \to \pm\infty) = \vec{0}$, $\dot{r}(t \to \pm\infty) = u$ and $r_{min} = b \neq 0$ yield that the trajectory of the particle does not form a conic section. At the extreme case, where $b = 0$, we obtain $\tilde{L} = 0$, hence, the particle moves only radially towards the monopole.

Searching for the scattering angle Θ, we first calculate the magnitude of the angular velocity $\vec{\omega}$, around the axis defined by the vector $\vec{L}$.

$$\omega = \dot{\phi} = \frac{L}{mr^2} = \frac{L}{m}\frac{1}{u^2t^2 + b^2} \overset{\phi(t=0)=0}{\Longrightarrow}$$

$$\phi(t) = \frac{L}{m}\int_0^t \frac{dt'}{u^2t'^2 + b^2} = \frac{L}{mb^2}\int_0^t \frac{dt'}{\left(1 + \frac{ut'}{b}\right)^2}$$

Upon setting $tan(\xi) = \dfrac{ut}{b} \Rightarrow dt = \dfrac{b}{u}\dfrac{d\xi}{cos^2(\xi)}$ we obtain:

$$\phi(t) = \frac{L}{mbu}\int_0^{arctan\left(\frac{ut}{b}\right)} d\xi = \frac{L}{\tilde{L}} arctan\left(\frac{ut}{b}\right) \Rightarrow$$

$$\phi(t) = \frac{1}{sin(\theta)} arctan\left(\frac{ut}{b}\right) \tag{1.45}$$

The equation (1.45) explicitly proves that the trajectory is not a closed orbit. Also, we can calculate the total angle $\Delta\phi$ on the plane, which is perpendicular to the vector $\vec{L}$.

$$\Delta\phi = \phi(t \to +\infty) - \phi(t \to -\infty) = \frac{\pi}{sin(\theta)} \tag{1.46}$$

Without loss of generality we may consider that the vector $-\vec{L}$ is parallel to the unit vector $\hat{z}$. Thus, we calculate the unit vector of the velocity for $r \to \pm\infty$:

$$\vec{v}(t \to \pm\infty) = |\dot{r}|(\pm\hat{r}(t \to \pm\infty)) \Rightarrow$$

$$\hat{v}(t \to \pm\infty) = \Big(\pm\, sin(\theta)cos\Big(\frac{\Delta\phi}{2}\Big),\ sin(\theta)sin\Big(\frac{\Delta\phi}{2}\Big),\ \pm cos(\theta)\Big) \tag{1.47}$$

Consequently, the scattering angle Θ is calculated as follows:

$$cos(\Theta) = \hat{v}(+\infty)\cdot\hat{v}(-\infty) = -sin^2(\theta)cos^2\Big(\frac{\Delta\phi}{2}\Big) + sin^2(\theta)sin^2\Big(\frac{\Delta\phi}{2}\Big) - cos^2(\theta) \Rightarrow$$

$$cos(\Theta) = 2sin^2(\theta)sin^2\Big(\frac{\pi}{2sin(\theta)}\Big) - 1 \quad \text{and} \quad cos\Big(\frac{\Theta}{2}\Big) = sin(\theta)\Big|sin\Big(\frac{\pi}{2sin(\theta)}\Big)\Big| \tag{1.48}$$

Finally, the previous analysis may lead to a very useful result, which is the effective cross-section of the charged particle's scattering on the magnetic monopole. Similarly to Rutherford scattering, the effective cross-section takes the form [1]:

$$\frac{d\sigma}{d\Omega} = \Big|\frac{b\ db}{dcos(\Theta)}\Big| \tag{1.49}$$

where b is the impact parameter. We have to calculate the differential db:

$$cos(\theta) = \frac{eg}{\sqrt{(mub)^2+(eg)^2}} \Rightarrow$$

$$-sin(\theta)d\theta = -\frac{eg(mu)^2 b\ db}{\Big(\sqrt{(mub)^2+(eg)^2}\Big)^3} \Rightarrow$$

$$sin(\theta)d\theta = \frac{(mu)^2}{(eg)^2}cos^3(\theta)b\ db \Rightarrow$$

$$b\ db = \frac{(eg)^2}{(mu)^2}\frac{sin(2\theta)d\theta}{2\ cos^4(\theta)} \tag{1.50}$$

Also, taking into account the contributions from all values of the impact parameter or equivalently from all values of the cone angle ($0 \le \theta \le \pi/2$), the differential cross-section reads:

$$\frac{d\sigma}{d\Omega} = \sum_{\theta_i}\frac{(eg)^2}{(mu)^2}\frac{1}{2\ cos^4(\theta)}\Big|\frac{sin(2\theta)d\theta}{sin(\Theta)d\Theta}\Big| \tag{1.51}$$

The differential cross-section (1.51) is singular for $\Theta = \pi$ and $\frac{d\Theta}{d\theta} = 0$. In the subsequent steps we examine these spacial cases. For back scattering (glory scattering), we consider $\Theta = \pi \Rightarrow sin(\Theta) = 0$, which means that the differential cross-section is singular. Upon substituting $\Theta = \pi$ into equation (1.48), we obtain:

$$sin(\theta)sin\Big(\frac{\pi}{2sin(\theta)}\Big) = 0$$

If the cone is not degenerated ($\theta \neq 0,\ \pi$), the above equation has the following solutions:

$$sin(\theta_n) = \frac{1}{2n}\ ,\quad n \in \mathbb{N} \tag{1.52}$$

Some critical values of the cone angle (solutions of the equation (1.52)) are [1]: $\theta_1 = 0.5236,\ \theta_2 = 0.2527,\ \theta_3 = 0.1674,\ \ldots$.

In the second case, $\frac{d\Theta}{d\theta} = 0$ (rainbow scattering), we consider $\Theta \neq 0,\ \pi$ and we calculate:

$$\frac{d\Theta}{d\theta} = 0 \Rightarrow \frac{dcos(\Theta)}{d\theta} = 0 \overset{(1.48)}{\Longrightarrow}$$

$$2sin(\theta)cos(\theta)sin^2\Big(\frac{\pi}{2sin(\theta)}\Big) - 2sin^2(\theta)sin\Big(\frac{\pi}{2sin(\theta)}\Big)cos\Big(\frac{\pi}{2sin(\theta)}\Big)\frac{\pi cos(\theta)}{2sin^2(\theta)} = 0 \Rightarrow$$

$$tan\Big(\frac{\pi}{2sin(\theta_n)}\Big) = \frac{\pi}{2sin(\theta_n)} \ , \quad n \in \mathbb{N} \tag{1.53}$$

Also, some critical values of the cone angle (solutions of the equation (1.53)) are [1]: $\theta_1 = 0.3571, \ \theta_2 = 0.2048, \ \theta_3 = 0.1446, \ ...$.

Last but not least, for small scattering angles ($\Theta \to 0$), from equation (1.48) we obtain $\theta \to \pi/2$. Therefore, we can set ($\Theta \approx \pi - 2\theta$). Hence, we have:

$$\Theta \approx sin(\Theta) \approx sin(\pi - 2\theta) = sin(2\theta) = 2sin(\theta)cos(\theta) \overset{(1.36)}{\underset{(1.37)}{\Longrightarrow}}$$

$$\Theta \approx 2\frac{egmub}{(mub)^2 + (eg)^2} = 2\frac{eg}{mub}\frac{1}{1 + \Big(\frac{eg}{mub}\Big)^2}$$

Upon considering $cos(\theta \to \pi/2) \to 0$, from equation (1.36) we obtain:

$$\frac{eg}{\sqrt{(mub)^2 + (eg)^2}} \to 0 \Rightarrow \frac{1}{\sqrt{\Big(\frac{mub}{eg}\Big)^2 + 1}} \to 0 \Rightarrow$$

$$\frac{eg}{mub} \to 0$$

Summarizing, the scattering angle approximately reads:

$$\Theta \approx 2\frac{eg}{mub} \tag{1.54}$$

We have assumed that $\Theta \approx \pi - 2\theta$, hence $cos(\theta \approx \pi/2) = 0 - (\theta - \pi/2) \approx \frac{\Theta}{2}$, $sin(2\theta) \approx sin(\pi - \Theta) = sin(\Theta)$ and $\frac{d\Theta}{d\theta} = -2$. Consequently, the equation (1.51) yields:

$$\frac{d\sigma}{d\Omega} \approx \Big(\frac{2eg}{mu}\Big)^2\frac{1}{\Theta^4} \tag{1.55}$$

which is analogous to Rutherford formula for low energy scattering.

1.2.2 Classical scattering on a Dyon

In this subsection, we generalize the previous case considering that the magnetic monopole has also electric charge, i.e., it is a dyon. In particular, we examine a classical non-relativistic scattering of an electrically charged particle, with charge e, on a static dyon having both electric "Q" and magnetic "g" charges. We consider $eQ < 0$, in order for bound orbits to be obtained. The analysis is similar to the previous case.

The Lorentz force that acts on the electrically charged particle reads:

$$m\ddot{\vec{r}} = eQ\frac{\vec{r}}{r^3} - \frac{eg}{r^3}(\vec{r} \times \dot{\vec{r}}) \tag{1.56}$$

Thus, due to the fact that the extra term ($eQ\frac{\vec{r}}{r^3}$) is parallel to $\hat{r}$, the analysis which corresponds to the angular momentum is exactly the same. Hence, we summarize the main relations:

$$\frac{d\vec{L}}{dt} = \vec{0}, \ \ \vec{L} = \tilde{\vec{L}} - eg\hat{r}, \ \ \tilde{\vec{L}} = \vec{r} \times m\dot{\vec{r}}, \ \ L = \sqrt{\tilde{L}^2 + (eg)^2} \tag{1.57}$$

$$\vec{L}\cdot\hat{r} = -eg = \text{const.} < 0 \Rightarrow \begin{cases} cos(\theta) = \frac{eg}{L} = \frac{eg}{\sqrt{\tilde{L}^2 + (eg)^2}} \\ sin(\theta) = \frac{\tilde{L}}{L} = \frac{\tilde{L}}{\sqrt{\tilde{L}^2 + (eg)^2}} \end{cases} \tag{1.58}$$

$$|\tilde{L}| = \tilde{L} = \text{const.} \tag{1.59}$$

Upon taking the inner product of the particle's velocity vector with equation (1.56), we calculate the conserved energy:

$$m\dot{\vec{r}}\cdot\ddot{\vec{r}} = \frac{eQ}{r^3}\vec{r}\cdot\dot{\vec{r}} - \frac{eg}{r^3}\cancelto{0}{\dot{\vec{r}}(\vec{r}\times\dot{\vec{r}})} \ \overset{\dot{\vec{r}}(\vec{r}\times\dot{\vec{r}})=0}{=\!=\!=} \ \frac{eQ}{r^2}\dot{r} \Rightarrow$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{\vec{r}}^2 + \frac{eQ}{r}\right) = 0 \Rightarrow$$

$$E = \frac{1}{2}m\dot{\vec{r}}^2 + \frac{eQ}{r} = \frac{1}{2}m\dot{r}^2 + \frac{\tilde{L}^2}{2mr^2} + \frac{eQ}{r} = \text{const.} \tag{1.60}$$

where we recognise the extra Coulomb term. Then, we solve this equation with respect to the radial velocity:

$$\dot{r} = \pm\sqrt{\frac{2}{m}}\sqrt{E + \frac{|eQ|}{r} - \frac{\tilde{L}^2}{2mr^2}} \quad \underset{\text{bound motion}}{\overset{E<0}{\Longrightarrow}}$$

$$t = \sqrt{\frac{m}{2|E|}}\int \frac{rdr}{\sqrt{-r^2 + \frac{|eQ|r}{|E|} - \frac{\tilde{L}^2}{2m|E|}}} \tag{1.61}$$

where we considered negative total energy, due to the Coulomb term and the assumption $eQ < 0$. The condition $E < 0$ implies bound trajectories. If we set:

$$a = \frac{|eQ|}{2|E|}, \quad b = \frac{\tilde{L}}{\sqrt{2m|E|}}, \quad \epsilon = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{2|E|\tilde{L}^2}{me^2Q^2}} \tag{1.62}$$

we have:

$$t = \sqrt{\frac{m}{2|E|}}\int \frac{rdr}{\sqrt{a^2 - b^2 - (r-a)^2}} = \sqrt{\frac{m}{2|E|}}\int \frac{rdr}{\sqrt{a^2\epsilon^2 - (r-a)^2}} \tag{1.63}$$

Subsequently, we introduce the variable ξ, such that:

$$r = a\big(1 - \epsilon\ cos(\xi)\big) \Rightarrow dr = a\epsilon\ sin(\xi)d\xi \tag{1.64}$$

$$\xi = arccos\left(\frac{a-r}{a\epsilon}\right) \tag{1.65}$$

Hence, we obtain:

$$t = \sqrt{\frac{ma^2}{2|E|}}\int \frac{(1 - \epsilon cos(\xi))sin(\xi)d\xi}{\sqrt{1 - cos^2(\xi)}} \Rightarrow$$

$$t = \sqrt{\frac{ma^2}{2|E|}}[\xi - \epsilon\ sin(\xi)] = \sqrt{\frac{ma^2}{2|E|}}\left(arccos\left(\frac{a-r}{a\epsilon}\right) - \epsilon\ sin\left[arccos\left(\frac{a-r}{a\epsilon}\right)\right]\right) \tag{1.66}$$

As we did in the previous case, where $Q = 0$, we assume that $\hat{z}$ is parallel to $\vec{L}$, thus, the angular velocity reads:

$$\vec{\omega} = \frac{\vec{L}}{mr^2} \Rightarrow \omega = \frac{d\phi}{dt} = \frac{L}{mr^2} \Rightarrow \frac{d\phi\big(r(t)\big)}{dt} = \frac{L}{mr^2} \Rightarrow \frac{dr}{dt}\frac{d\phi}{dr} = \frac{L}{mr^2} \Rightarrow \frac{d\phi}{dr} = \frac{L}{mr^2}\frac{dt}{dr} \Rightarrow$$

$$\frac{d\phi}{dr} = \frac{L}{\sqrt{2m}}\frac{1}{r^2\sqrt{-|E| + \frac{|eQ|}{r} - \frac{\tilde{L}^2}{2mr^2}}} \Rightarrow$$

$$\phi(r) = \frac{L}{\sqrt{m|eQ|}}\int \frac{dr}{r^2\sqrt{\frac{-\frac{2|E|}{|eQ|}r^2 + 2r - \frac{\tilde{L}^2}{m|eQ|}}{r^2}}} \Rightarrow$$

$$\phi(r) = \frac{L\tilde{L}}{m|eQ|}\frac{1}{\epsilon}\int \frac{dr}{r^2\sqrt{\frac{\left(1 - \frac{2|E|\tilde{L}^2}{me^2Q^2}\right)r^2 - r^2 + \frac{2\tilde{L}^2}{m|eQ|}r - \left(\frac{\tilde{L}^2}{m|eQ|}\right)^2}{\epsilon^2 r^2}}} \quad \overset{\epsilon = \sqrt{1 - \frac{2|E|\tilde{L}^2}{me^2Q^2}}}{\Longrightarrow}$$

$$\phi(r) = \frac{L\tilde{L}}{m|eQ|\epsilon} \int \frac{dr}{r^2\sqrt{1 - \left(\frac{r - \frac{\tilde{L}^2}{m|eQ|}}{\epsilon r}\right)^2}} \tag{1.67}$$

To its end, we introduce the variable ξ as follows:

$$cos(\xi) = \frac{r - \frac{\tilde{L}^2}{m|eQ|}}{\epsilon r} \Rightarrow sin(\xi)d\xi = \frac{\tilde{L}^2}{\epsilon m|eQ|}\frac{dr}{r^2} \Rightarrow dr = r^2\frac{\epsilon m|eQ|}{\tilde{L}^2}sin(\xi)d\xi \tag{1.68}$$

Thus, we obtain:

$$\phi(r) = \frac{L}{\tilde{L}} \int \frac{\cancel{sin(\xi)}d\xi}{\cancel{\sqrt{1 - cos^2(\xi)}}} = \frac{L}{\tilde{L}}\xi, \quad \begin{cases} \frac{\tilde{L}}{L} = sin(\theta_c) \\ \xi = arccos\left(\frac{-r + \frac{\tilde{L}^2}{m|eQ|}}{\epsilon r}\right) \end{cases} \tag{1.69}$$

Hence, the azimuthal angle satisfies the following equation:

$$cos\big(\phi(r)sin(\theta_c)\big) = \frac{-r + \frac{\tilde{L}^2}{m|eQ|}}{\epsilon r} \tag{1.70}$$

Consequently, a charged particle in a dyon field, considering bound orbit, moves along an ellipse with semi-axes a, b and eccentricity ϵ [1]. Additionally, the elliptic trajectory lies on the surface of a cone, with axis of symmetry along $-\vec{L}$ and cone angle θ_c. It is very interesting that there is a precession of the orbit. For $\phi(r_{min}) = 0$, we have $cos\big(\phi(r_{min})sin(\theta_c)\big) = 1$, while for $\phi(r) = 2\pi$, we have $cos\big(\phi(r)sin(\theta_c)\big) \neq 1 \Rightarrow r \neq r_{min}$, since $0 < sin(\theta_c) < 1$. Therefore, we obtain $\phi(r > r_{min}) = 2\pi$, which means that after a complete term the charged particle does not return to the initial position, i.e., we have a precession angle. We calculate the precession angle as follows:

$$\phi(r_{min})sin(\theta_c) = 2\pi \Rightarrow$$

$$\phi(r_{min}) = \frac{2\pi}{sin(\theta_c)}$$

thus, the precession angle reads

$$\Delta\phi = \phi(r_{min}) - 2\pi \overset{(1.58)}{\Longrightarrow}$$

$$\Delta\phi = 2\pi\left(\frac{L}{\tilde{L}} - 1\right) \tag{1.71}$$

1.3 Dirac Magnetic Monopole

A self-consistent electromagnetic theory with magnetic monopoles, which respects the results corresponding to the classical and quantum electromagnetic theories without magnetic monopoles, must be described by the four-potential A_μ. However, a dual symmetric model may be structured by considering either the electric field or the magnetic induction as the curl of the vector potential $\vec{A}$. This becomes obvious, upon considering the dual transformation (1.12) of the electromagnetic tensor. In this section we prove that the vector potential, in presence of a magnetic monopole, is singular along a string, which is called Dirac string. The Dirac string is not observable, and hence, we need to somehow eliminate it. Interestingly, in chapter 3, trying to overcome the problematic situation due to the Dirac string, we prove that the magnetic charge seems to have topological origins. In fact, in chapter 4, where we discuss the 't Hooft-Polyakov magnetic monopole, we explicitly prove that the magnetic charge has topological roots. Additionally, if the models that predict magnetic monopoles are dual symmetric, which means that the term $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is a total derivative, these models also predict electric monopoles, and more importantly, the corresponding electric charge has topological roots. Note that the models examined in this thesis effectively provide magnetic monopole solutions, which are particles with structure. In fact, there are not dual symmetric actions that predict magnetic monopoles as elementary particles.

1.3.1 Vector potential of a Dirac Monopole

In this section, we make the usual assumption, according which we can write the magnetic induction as $\vec{B} = \nabla \times \vec{A}$.

Considering a magnetic monopole theory, the solution of equation $\nabla \vec{B}_g = \rho_g$ (1.2), where $\rho_g = 4\pi g \delta^{(3)}(\vec{r})$ reads:

$$\vec{B}_g = g\frac{\vec{r}}{r^3} \tag{1.72}$$

where g is the magnetic charge. Hence, due to the identity $\nabla(\nabla \times \vec{A}) = 0$, we cannot write $\vec{B}_g = \nabla \times \vec{A}$ for every point $\vec{r}$ in space. Instead, we may consider a field $\vec{B}$, which satisfies the equation $\nabla \vec{B} = 0$, for every $\vec{r}$, and can be written as $\vec{B} = \nabla \times \vec{A}$. Additionally, let us assume that there is a field $\vec{h}$, such that the field $\vec{B}$ reads [5]:

$$\vec{B} = \vec{B}_g + g\vec{h} = \nabla \times \vec{A} \tag{1.73}$$

Upon considering the equations $\nabla \vec{B}_g = 4\pi g \delta^{(3)}(\vec{r})$ and $\nabla \vec{B} = 0$, the field $\vec{h}$ satisfies the following equation:

$$\nabla \vec{h} = -4\pi \delta^{(3)}(\vec{r}) \tag{1.74}$$

It is evident that we can find a vector potential $\vec{A}$, which satisfies equation $\vec{B}_g = \nabla \times \vec{A}$ everywhere except a line, which extends from the origin to infinity. The vector potential $\vec{A}$ is singular along this line, which is called Dirac string (D.S.). Thus, the fields $\vec{B}$ and $\vec{h}$ are singular along the Dirac string, while $\vec{B}_g = g\frac{\vec{r}}{r^3}$ is singular only at the origin. The gradient theorem allows us to solve the equation (1.74) as follows:

$$\vec{h}(\vec{r}) = -4\pi \int_{D.S} \delta^{(3)}(\vec{r} - \vec{r}\,')d\vec{r}\,' \tag{1.75}$$

where the integration is along the Dirac string. Let us consider, without loss of generality, that the Dirac string is extended along the positive semi-axis z. Then, equation (1.75) yields:

$$\vec{h}(\vec{r}) = -4\pi \hat{z}\delta(x)\delta(y) \int_0^{+\infty} dz'\delta(z - z') \Rightarrow^{\xi = z - z'}$$

$$\vec{h}(\vec{r}) = -4\pi \hat{z}\delta(x)\delta(y) \int_{-\infty}^{z} d\xi\delta(\xi) \Rightarrow$$

$$\vec{h}(\vec{r}) = -4\pi \hat{z}\delta(x)\delta(y)\Theta(z) \tag{1.76}$$

where $\Theta(z)$ is the Heaviside step function. The field (1.76) satisfies the equation (1.74), since $\frac{d\Theta(z)}{dz} = \delta(z)$. Also, it is crucial to calculate the vector potential $\vec{A}$. The curl of equation (1.73) yields:

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \vec{A}) - \nabla^2 \vec{A} = \nabla \times \vec{B}_g + g\nabla \times \vec{h}$$

The equation (1.72) implies $\nabla \times \vec{B}_g = 0$. Upon choosing the Coulomb gauge $\nabla \vec{A} = 0$, we obtain:

$$\nabla^2 \vec{A} = -g\nabla \times \vec{h} \tag{1.77}$$

Considering that $\nabla^2 \frac{1}{|\vec{r} - \vec{r}\,'|} = -4\pi\delta^{(3)}(\vec{r} - \vec{r}\,')$, the usual solution of the equation (1.77) reads:

$$\vec{A}(\vec{r}) = \frac{g}{4\pi} \int \frac{\nabla' \times \vec{h}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3r' \tag{1.78}$$

The curl of the field (1.76) reads:

$$\nabla \times \vec{h} = -4\pi\Theta(z)(\delta(x)\frac{d\delta(y)}{dy}\hat{x} - \frac{d\delta(x)}{dx}\delta(y)\hat{y}) \tag{1.79}$$

Upon substituting equation (1.79) into equation (1.78), we obtain:

$$\vec{A}(\vec{r}) = -g\int_0^{+\infty} dz' \left[\hat{x}\int_{-\infty}^{+\infty} dy' \frac{d\delta(y)/dy}{\sqrt{x^2 + (y - y')^2 + (z - z')^2}} - \hat{y}\int_{-\infty}^{+\infty} dx' \frac{d\delta(x)/dx}{\sqrt{(x - x')^2 + y^2 + (z - z')^2}}\right] \Rightarrow$$

$$\vec{A}(\vec{r}) = -g\int_0^{+\infty} dz' \frac{-y\hat{x}+x\hat{y}}{\left[r^2-z^2+(z-z')^2\right]^{3/2}} = -g(\hat{z}\times\vec{r})\frac{z+r}{r(r^2-z^2)} \Rightarrow$$

$$\vec{A}(\vec{r}) = -\frac{g}{r}\frac{\hat{z}\times\vec{r}}{r-(\vec{r}\hat{z})} \tag{1.80}$$

More generally we can write:

$$\vec{A}(\vec{r}) = -\frac{g}{r}\frac{\hat{n}\times\vec{r}}{r-(\vec{r}\hat{n})} \tag{1.81}$$

where $\hat{n}$ is the unit vector along the Dirac string. The vector potential (1.81) is called Dirac potential and plays a fundamental role in the magnetic monopole theories. Equation (1.81) confirms that the vector potential is singular along the Dirac string. Also, we can explicitly confirm that $\vec{B}_g = \nabla\times\vec{A} = g\frac{\vec{r}}{r^3}$, away from the Dirac string, using the form (1.80) to represent the vector potential.

Later on, it will be very interesting to write the vector potential as follows [1]:

$$\vec{A}(\vec{r}) = -g(1+cos(\theta))\nabla\phi \tag{1.82}$$

where θ is the polar angle and ϕ is the azimuthal angle. The gradient of ϕ reads:

$$\nabla\phi = \Big(-\frac{sin(\phi)}{rsin(\theta)}, \frac{cos(\phi)}{rsin(\theta)}, 0\Big) = \frac{1}{rsin(\theta)}\hat{e}_\phi \tag{1.83}$$

Upon substituting equation (1.83) into equation (1.82), the equation (1.80) is confirmed. Furthermore, the equation (1.82) yields that the vector potential can be written as follows:

$$\vec{A}(\vec{r}) = (1+cos(\theta))\frac{-i}{e}U^{-1}\nabla U \ , \ \text{ where } \ U = e^{-ieg\phi} \tag{1.84}$$

This result is the first step in order to find the nature of the magnetic monopoles. This becomes more obvious in the next subsection.

Additionally, we have to clarify that the field $\vec{B} = \vec{B}_g + g\vec{h}$ does not have any physical meaning, since it is not observable. Thus, the Dirac string, which produces field $\vec{h}$, is not observable. Therefore, in section 3.2, we need to construct a non-singular vector potential using the gauge form of the vector potential in equation (1.84). Nevertheless, we will briefly examine a potential physical system which produces the field $\vec{B}$. This system consists of a solenoid along the Dirac string, at the edge of which exists a magnetic monopole. The solenoid produces the field $g\vec{h}$, while the monopole produces the field $\vec{B}_g$. The total field $\vec{B}$ (1.73) satisfies the equation $\nabla\vec{B} = 0$, which means that the total magnetic flux over a closed surface, with the monopole located at its centre, vanishes. This statement can be explicitly proven as follows:

$$\Phi_g = \oint d\vec{\sigma}\vec{B}_g = 4\pi g \tag{1.85}$$

$$\Phi_{D.S.} = g\oint d\vec{\sigma}\vec{h} = -4\pi g \tag{1.86}$$

$$\Phi_{total} = \Phi_g + \Phi_{D.S.} = 0 \tag{1.87}$$

1.3.2 Transformations of the Dirac string

As we mentioned before, the equation (1.84) plays a fundamental role in our discussion, because it is in a form of a gauge transformation $U \in U(1)$, where $U = e^{-ieg\phi}$.

Let us assume the gauge transformation $U = e^{ie\lambda(\vec{r})} \in U(1)$ that acts on the vector potential as follows:

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} - \frac{i}{e}U^{-1}\nabla U \Rightarrow$$

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \nabla\lambda(\vec{r}) \tag{1.88}$$

It is very important that away from the Dirac string, where $\vec{B}_g = \nabla\times\vec{A}$, the vector potentials $\vec{A}'$ and $\vec{A}$ correspond to the same magnetic induction, since $\nabla\times(\nabla\lambda(\vec{r})) = 0$.

Additionally, we need to determine the changes that occur in the Dirac string. Let us denote vector potential (1.80) as $\vec{A}_S$, because it is regular to the south region of the $x-y$ plane. Upon imposing a specific gauge transformation $U = e^{2ieg\phi}$ on $\vec{A}_S$, we obtain:

$$\vec{A}_S \longrightarrow \vec{A}_N = \vec{A}_S + 2g\nabla\phi = -\frac{g}{r}\frac{\hat{z}\times\vec{r}}{r-(\vec{r}\hat{z})} + \frac{2g}{rsin(\theta)}\hat{e}_\phi \Rightarrow$$

$$\vec{A}_S \longrightarrow \vec{A}_N = -\frac{g}{r}\frac{-y\hat{x}+x\hat{y}}{r-z} + \frac{2g}{rsin(\theta)}\hat{e}_\phi = \frac{g}{r}\frac{sin(\theta)sin(\phi)\hat{x} - sin(\theta)cos(\phi)\hat{y}}{1-cos(\theta)} + \frac{2g}{rsin(\theta)}\hat{e}_\phi \Rightarrow$$

$$\vec{A}_S \longrightarrow \vec{A}_N = \frac{g}{r}(1+cos(\theta))\,\frac{sin(\theta)sin(\phi)\hat{x} - sin(\theta)cos(\phi)\hat{y}}{sin^2(\theta)} + \frac{2g}{rsin(\theta)}\hat{e}_\phi \Rightarrow$$

$$\vec{A}_S \longrightarrow \vec{A}_N = \frac{g}{r}(1+cos(\theta))\,\frac{sin(\phi)\hat{x} - cos(\phi)\hat{y}}{sin(\theta)} + \frac{2g}{rsin(\theta)}\hat{e}_\phi = -\frac{g}{r}\frac{1+cos(\theta)}{sin(\theta)}\hat{e}_\phi + \frac{2g}{rsin(\theta)}\hat{e}_\phi \Rightarrow$$

$$\vec{A}_S \longrightarrow \vec{A}_N = \frac{g}{r}\frac{1-cos(\theta)}{sin(\theta)}\hat{e}_\phi \tag{1.89}$$

The vector potential $\vec{A}_N$ is singular at $\theta = \pi$, i.e., the gauge transformation has turned the singularity from the positive z semi-axis to the negative z semi-axis. Note that $\lim_{\theta\to 0}\frac{1-cos(\theta)}{sin(\theta)} = \lim_{\theta\to 0}\frac{sin(\theta)}{cos(\theta)} = 0$. Consequently, it is very interesting that the gauge transformations rotate the Dirac string. The fact that the gauge transformations leave the observable magnetic induction $\vec{B}_g$ invariant, but they change the direction of the Dirac string, proves explicitly that the Dirac string has no physical meaning. Also, this statement points out that it is crucial to avoid the Dirac strings. The formalism to do that is described in section 3.2 and leads to the Dirac quantization condition and the topological roots of the magnetic charge.

Finally, note that the gauge transformation $U = e^{2ieg\phi} \in U(1)$ acts like the parity operator P on the vector potential $\vec{A}$. For instance [1]:

$$\vec{A}_S \longrightarrow P\vec{A}_S = \frac{g}{r}\frac{1-cos(\theta)}{sin(\theta)}\hat{e}_\phi = \vec{A}_N$$

$$\vec{A}_S \longrightarrow \vec{A}_S - \frac{i}{e}U^{-1}\nabla U = \vec{A}_N \tag{1.90}$$

Note that $\vec{A}_N \neq \pm\vec{A}_S$, hence the $\vec{A}$ is not an eigenfunction of the parity operator P. We can define the generalized space reflection $\mathbb{P}$, which reads:

$$\vec{A} \longrightarrow \mathbb{P}\vec{A} = P\vec{A}_S + \frac{i}{e}U^{-1}\nabla U = \vec{A} \tag{1.91}$$

where $U = e^{2ieg\phi}$. Therefore, the vector potential $\vec{A}$ transforms as a pseudo-vector under the generalized space reflection.

1.4 Dynamical symmetries of the charge-monopole system

In this section we present the symmetries of the charge-monopole system and the corresponding conserved quantities. The Lagrangian of the model reads:

$$L = \frac{1}{2}m\dot{\vec{r}}^2 + e\dot{\vec{r}}\vec{A} \tag{1.92}$$

where $\vec{A}$ is defined by equation (1.81). The Dirac string lies along an arbitrary unit vector $\hat{n}$. The spacial variation $\vec{r}(t) \to \vec{r}(t) + \delta\vec{r}(t) \Rightarrow \dot{\vec{r}}(t) \to \dot{\vec{r}}(t) + \delta\dot{\vec{r}}(t)$, assuming that the initial and final values of the variation vanish $\delta\vec{r}(t_i) = \delta\vec{r}(t_f) = 0$, leads to the action variation of the form:

$$\delta S = \int_{t_i}^{t_f} dt\big(m\dot{\vec{r}}\delta\dot{\vec{r}} + e(\delta\dot{\vec{r}})\vec{A} + e\dot{\vec{r}}\delta\vec{A}\big) \Rightarrow$$

$$\delta S = \int_{t_i}^{t_f} dt\bigg(m\frac{d}{dt}(\dot{\vec{r}}\delta\vec{r}) - m\ddot{\vec{r}}\delta\vec{r} + e\frac{d}{dt}(\delta\vec{r}\vec{A}) - e\delta\vec{r}\frac{d\vec{A}}{dt} + e\dot{\vec{r}}(\delta\vec{r}\nabla)\vec{A}\bigg) \Rightarrow \tag{1.93}$$

$$\delta S = m\Big[\dot{\vec{r}}\delta\vec{r}\Big]_{t_i}^{t_f} + e\Big[\delta\vec{r}\vec{A}\Big]_{t_i}^{t_f} - \int_{t_i}^{t_f} dt\;\delta\vec{r}\Big(m\ddot{\vec{r}} - e[\dot{\vec{r}}\times\vec{B}]\Big) \tag{1.94}$$

In equation (1.93) we considered that $\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\dot{\vec{r}}\nabla)\vec{A} = (\dot{\vec{r}}\nabla)\vec{A}$ and that $e(\dot{\vec{r}}(\delta\vec{r}\nabla)\vec{A} - \delta\vec{r}(\dot{\vec{r}}\nabla)\vec{A}) = e(\dot{\vec{r}} \times \vec{B})$. Let us focus on the equation (1.94), where it is obvious that the term $m\big[\dot{\vec{r}}\delta\vec{r}\big]_{t_i}^{t_f}$ vanishes due to the assumption $\delta\vec{r}(t_i) = \delta\vec{r}(t_f) = 0$. If we had considered an ordinary system without magnetic monopoles, the term $e\big[\delta\vec{r}\vec{A}\big]_{t_i}^{t_f}$ would have vanished too, hence the ordinary Euler Lagrange equations $m\ddot{\vec{r}} = e[\dot{\vec{r}} \times \vec{B}]$ would have appeared. If we examine a system with a magnetic monopole, we need to be extra careful, because of the singularity of the vector potential (1.81) along the Dirac string. This is an issue, since, if one of the points $\vec{r}_i$ and $\vec{r}_f$ is located on the Dirac string, the term $e\big[\delta\vec{r}\vec{A}\big]_{t_i}^{t_f}$ does not necessarily vanish. Therefore, we need to be careful with the subsequent steps, where we are searching for conserved quantities.

Let us perform a radius-vector transformation $\delta\vec{r} = \vec{\omega} \times \vec{r} = \omega(\hat{n} \times \vec{r})$, where ω is infinitesimally small and $\hat{n}$ is the unit vector along the Dirac string. We consider this direction of the variation in order to avoid the aforementioned problematic situation. Additionally, it is evident that the direction of the Dirac string is arbitrary, as proven in subsection 1.3.2, hence, the examined transformation can be considered as general. It is obvious that the system is symmetric under this transformation. Noether's theorem yields that the corresponding conserved quantity is the following:

$$(\hat{n} \times \vec{r})\frac{\partial L}{\partial \dot{\vec{r}}} = (\hat{n} \times \vec{r})(m\dot{\vec{r}} + e\vec{A}) = const. \tag{1.95}$$

In the next few lines we perform the calculation of $(\hat{n} \times \vec{r})(m\dot{\vec{r}} + e\vec{A})$, considering equation (1.81):

$$(\hat{n} \times \vec{r})(m\dot{\vec{r}} + e\vec{A}) = \hat{n}(\vec{r} \times m\dot{\vec{r}}) - \frac{eg}{r}\frac{(\vec{r} \times \hat{n})^2}{r - (\vec{r}\hat{n})} = \hat{n}(\vec{r} \times m\dot{\vec{r}}) - \frac{eg}{r}\frac{\hat{n}([\vec{r} \times \hat{n}] \times \vec{r})}{r - (\vec{r}\hat{n})} =$$

$$\hat{n}(\vec{r} \times m\dot{\vec{r}}) + \frac{eg}{r}\frac{\vec{r}(\vec{r}\hat{n}) - \hat{n}r^2}{r - (\vec{r}\hat{n})}\hat{n} = \hat{n}(\vec{r} \times m\dot{\vec{r}}) + \frac{eg}{r}\frac{(\vec{r}\hat{n})^2 - r^2}{r - (\vec{r}\hat{n})} =$$

$$\hat{n}(\vec{r} \times m\dot{\vec{r}}) - \frac{eg}{r}(r + \vec{r}\hat{n}) = \hat{n}\Big[(\vec{r} \times m\dot{\vec{r}}) - ge\hat{r}\Big] - eg = const. \Rightarrow$$

$$\hat{n}\Big[(\vec{r} \times m\dot{\vec{r}}) - ge\hat{r}\Big] = const. \Rightarrow^{(1.31)}$$

$$\hat{n}\vec{L} = const. \tag{1.96}$$

Upon considering that the unit vector $\hat{n}$ is arbitrary, as we mentioned before, we obtain that the conserved quantity is the generalized angular momentum (1.31).

$$\vec{L} = const. \tag{1.97}$$

This result confirms the calculations in section 1.2.

Subsequently, let us perform the time translation $t \to t + \delta t \Rightarrow \vec{r} \to \vec{r} + \dot{\vec{r}}\delta t$. The Lagrangian (1.92) remains invariant under this transformation. From Noether's theorem it follows that the corresponding conserved quantity reads:

$$E = \dot{\vec{r}}\frac{\partial L}{\partial \dot{\vec{r}}} = \frac{1}{2}m\dot{\vec{r}}^2 = const. \tag{1.98}$$

The conserved quantity is the kinetic energy, as we expected from the calculations in subsection 1.2.1.

So far, the charge-monopole system seems to be completely analogous to an ordinary charge-charge system, taking into account the symmetries. Nonetheless, the system with the magnetic monopole has some additional symmetries, which form the group of dynamical symmetry. The kinetic energy of the charged particle in the monopole field is a total time derivative also under transformation of dilatation $\delta\vec{r} = \vec{v}t - \vec{r}/2$ and the special conformal transformation $\delta\vec{r} = \vec{v}t^2 - \vec{r}t$ [1][6][7]. The generators of these transformations are the following:

$$D = Ht - \frac{m}{4}[(\vec{r}\vec{v}) + (\vec{v}\vec{r})] \quad \text{and} \quad K = -Ht^2 + 2Dt + \frac{mr^2}{2} \tag{1.99}$$

where H is the Hamiltonian of the system. The generators (1.99), together with the Hamiltonian, form the algebra of the conformal group $SO(2,1)$ [1]:

$$[H, D] = iH\ , \quad [D, K] = iK\ , \quad [H, K] = 2iD \tag{1.100}$$

which is the group of dynamical symmetry of the non-relativistic charge-monopole system. Note that, the invariance of a theory with respect to a transformation involving a time dependence does not lead to the existence of a new integral of motion, but introduces some constraints on the configuration space of the classical system.

Chapter 2

Magnetic Monopoles: A quantum theory approach

The hypothesis of the existence of the magnetic monopoles leads to an elegant explanation of the quantization of the electric charge. Therefore, it is essential to present some arguments, in the perspective of quantum mechanics, for the charge quantization condition, called Dirac quantization condition. In the next chapter, we prove the Dirac quantization condition using topological arguments. In this chapter, we examine the Spin-statistics theorem in a magnetic monopole or dyon theory and we solve Schrödinger, Pauli and Dirac equations for an electrically charged particle in an external monopole field.

2.1 Charge quantization condition

The Dirac quantization condition can be derived in various ways. Let us consider a charged particle in an external monopole field. The Lagrangian of the system reads:

$$L = \frac{1}{2}m\dot{\vec{r}}^2 + e\dot{\vec{r}}\vec{A} \tag{2.1}$$

which under a gauge $U(1)$ transformation of the form $U = e^{ie\lambda(\vec{r})}$ yields:

$$\delta L = e\,\dot{\vec{r}}\,\nabla\lambda(\vec{r}) = e\frac{d\lambda(\vec{r}(t))}{dt} \tag{2.2}$$

where we used equation (1.88). Consequently, the action of the system is transformed as follows:

$$\delta S = \int_0^T dt\,\delta L = e\big[\lambda(\vec{r})\big]_0^T \tag{2.3}$$

The transition amplitude $\sim e^{iS}$ defined by the corresponding path integral must be a gauge invariant quantity [1]. Thus, the parameter of the transformation must satisfy the following equation:

$$e\delta\lambda = 2\pi n\ ,\quad n \in \mathbb{Z} \tag{2.4}$$

Let us consider the gauge transformation $U = e^{2ieg\phi}$, which rotates the Dirac string. Also, upon considering that the particle traces a closed orbit around the Dirac string, we obtain that $\phi(T) = \phi(0) + 2\pi$. Hence, the quantization condition (2.4) yields:

$$eg = \frac{n}{2}\ ,\quad n \in \mathbb{Z} \tag{2.5}$$

This is called Dirac's charge quantization condition [8].

Additionally, we may prove equation (2.5) by considering the Schrödinger equation of the aforementioned system, which reads:

$$\hat{H}\psi(\vec{r}) = E\psi(\vec{r}) \Rightarrow -\frac{1}{2m}(\nabla - ie\vec{A})^2\psi(\vec{r}) = E\psi(\vec{r}) \tag{2.6}$$

The solution of equation (2.6) reads:

$$\psi(\vec{r}) = \psi_0(\vec{r})e^{ie\int_{\vec{r}_0}^{\vec{r}} \vec{A}(\vec{r}\,')d\vec{r}\,'} \tag{2.7}$$

where $\psi_0(\vec{r})$ is the solution of the free Schrödinger equation. The wave-function $\psi(\vec{r})$ must be a single-valued function and as a result the complex phase must satisfy the following equation:

$$e\oint_\ell \vec{A}(\vec{r}\,')d\vec{r}\,' = 2\pi n\ , \quad n \in \mathbb{Z} \tag{2.8}$$

where contour ℓ is a closed loop around the Dirac string. In the case we consider the contour ℓ as an infinitesimally small loop around the Dirac string, the vector potential $\vec{A}$ describes the field $g\vec{h}(\vec{r})$ from equation (1.73), which corresponds to the solenoid along the Dirac string. Thus, we can use the Stokes' theorem in order to calculate the integral (2.8).

$$e\oint_\ell \vec{A}(\vec{r}\,')d\vec{r}\,' = e\int(\nabla\times\vec{A})d\vec{a} = eg\int \vec{h}d\vec{a} \overset{(1.76)}{=} -4\pi eg = 2\pi n \Rightarrow$$

$$eg = \frac{n}{2}\ , \quad n \in \mathbb{Z} \tag{2.9}$$

which is once again the Dirac quantization condition.

Another interesting aspect, connected with the Dirac quantization condition, is the quantization of the generalized angular momentum (1.31). If we assume that the orbital angular momentum $\tilde{L} = \vec{r}\times\vec{\pi}$ has integer eigenvalues as usual, then the equation (2.9) yields that the generalized angular momentum $\vec{L} = \tilde{\vec{L}} - eg\hat{r}$ has semi-integer eigenvalues. This can be clarified by proving that the components of the generalized angular momentum satisfy the algebra $[L_i, L_j] = i\epsilon_{ijk}L_k$. Upon considering the canonical commutation relations:

$$[x_i, x_j] = 0\ , \quad [p_i, p_j] = 0\ , \quad [x_i, p_j] = i\delta_{ij}\ , \quad i\ j = 1,\ 2,\ 3 \tag{2.10}$$

we calculate the commutations $[x_i, \pi_j]$ and $[\pi_i, \pi_j]$, where $\vec{\pi} = \vec{p} - e\vec{A}(\vec{x})$:

$$[x_i, \pi_j] = [x_i, p_j] - e[x_i, A_j(\vec{X})] = i\delta_{ij} - 0 \Rightarrow$$

$$[x_i, \pi_j] = i\delta_{ij} \tag{2.11}$$

$$[\pi_i, \pi_j] = [p_i, p_j] - e[p_i, A_j(\vec{x})] - e[A_i(\vec{x}), p_j] + e^2[A_i(\vec{x}), A_j(\vec{x})] \Rightarrow$$

$$[\pi_i, \pi_j] = ie\partial_i A_j - ie\partial_j A_i \Rightarrow$$

$$[\pi_i, \pi_j] = ie(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\partial_m A_n \Rightarrow$$

$$[\pi_i, \pi_j] = ie\ \epsilon_{ijk}\epsilon_{kmn}\partial_m A_n \Rightarrow$$

$$[\pi_i, \pi_j] = ie\ \epsilon_{ijk}\left(\nabla\times\vec{A}\right)_k \tag{2.12}$$

where we used Ehrenfest theorem:

$$[p_i, f(\vec{x})] = -i\frac{\partial f(\vec{x})}{\partial x_i} \tag{2.13}$$

and the identity:

$$\epsilon_{ijk}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \tag{2.14}$$

Subsequently, we calculate the commutation $[L_i, L_j]$:

$$[L_i, L_j] = \left[\epsilon_{iab}x_a\pi_b - eg\frac{x_i}{r}, \epsilon_{jcd}x_c\pi_d - eg\frac{x_j}{r}\right] \Rightarrow$$

$$[L_i, L_j] = \epsilon_{iab}\epsilon_{jcd}[x_a\pi_b, x_c\pi_d] - eg\ \epsilon_{iab}x_a\left[\pi_b, \frac{x_j}{r}\right] + eg\ \epsilon_{jcd}x_c\left[\pi_d, \frac{x_i}{r}\right]$$

Upon considering the relations $\left[\pi_b, \frac{x_j}{r}\right] = -i\frac{\delta_{bj}}{r} + i\frac{x_j x_b}{r^3}$, $\epsilon_{iab}x_a x_b = 0$ and the identity $[AB, CD] = A[B,C]D + [A,C]BD + CA[B,D] + C[A,D]B$, we obtain:

$$[L_i, L_j] = \epsilon_{iab}\epsilon_{jcd}(-i\delta_{cb}x_a\pi_d + i\delta_{ad}x_c\pi_b) - 2egi\ \epsilon_{ijk}\frac{x_k}{r} + \epsilon_{iab}\epsilon_{jcd}\ ie\ \epsilon_{bck}\Big(\nabla\times\vec{A}\Big)_k x_a x_c =$$

$$i\epsilon_{iab}\epsilon_{jcd}(x_a\pi_c - x_c\pi_a) - 2egi\ \epsilon_{ijk}\frac{x_k}{r} + \epsilon_{iab}\epsilon_{bck}\ ie\ \epsilon_{jcd}\Big(\nabla\times\vec{A}\Big)_k x_a x_c \overset{(2.14)}{=}$$

$$i(x_i\pi_j - x_j\pi_i) - 2egi\ \epsilon_{ijk}\frac{x_k}{r} + ie\ \epsilon_{ijc}x_c\Big(\nabla\times\vec{A}\Big)_k x_k =$$

$$i\epsilon_{ijk}\epsilon_{kab}x_a\pi_b - 2egi\ \epsilon_{ijk}\frac{x_k}{r} + ie\ \epsilon_{ijc}x_c\Big[\Big(\nabla\times\vec{A}\Big)\vec{r}\Big] \underset{(1.73)}{\overset{(1.72)}{\Longrightarrow}}$$

$$[L_i, L_j] = i\epsilon_{ijk}L_k + ieg\ \epsilon_{ijk}x_k(\vec{h}\cdot\vec{x}) \tag{2.15}$$

Note that for vanishing magnetic charge ($g = 0$), equation (2.15) corresponds to the regular commutation relation of the angular momentum. In the non-trivial case ($g \neq 0$), we observe that the field, produced by the solenoid along the Dirac string, adds an extra unnatural term to the regular commutation relation of the angular momentum. This term is unnatural, since the Dirac string is not observable. In the next few lines we calculate the commutation relation between the components of the generalized angular momentum and the Hamiltonian operator $H = \frac{\vec{\pi}}{2M}$:

$$[L_i, H] = \Big[\epsilon_{ijk}x_j\pi_k - eg\frac{x_i}{r}, \frac{\pi_a\pi_a}{2M}\Big] = \frac{\epsilon_{ijk}}{2M}[x_j\pi_k, \pi_a\pi_a] + \frac{eg}{2M}\Big[\pi_a\pi_a, \frac{x_i}{r}\Big] =$$

$$i\frac{\epsilon_{ijk}}{2M}\Big(e\ \epsilon_{kab}(\nabla\times\vec{A})_b x_j\pi_a + \pi_k\pi_a\delta_{ja} + e\ \epsilon_{kab}(\nabla\times\vec{A})_b\pi_a x_j + \pi_a\pi_k\delta_{ja}\Big) + \frac{eg}{2M}\Big(\pi_a\Big[\pi_a, \frac{x_i}{r}\Big] + \Big[\pi_a, \frac{x_i}{r}\Big]\pi_a\Big) =$$

$$\frac{ie}{2M}\Big((\vec{r}\vec{B})\pi_i - X_jB_i\pi_j + \pi_i(\vec{r}\vec{B}) - (\vec{\pi}\vec{r})B_i\Big) + i\frac{eg}{2M}\Big(-\pi_i\frac{1}{r} + \pi_a\frac{x_a x_i}{r^3} - \frac{1}{r}\pi_i + \frac{x_a x_i}{r^3}\pi_a\Big) \underset{(1.73)}{\overset{(1.72)}{\Longrightarrow}}$$

$$[L_i, H] = \frac{ieg}{2M}\Big((\vec{r}\vec{h})\pi_i - x_jh_i\pi_j + \pi_i(\vec{r}\vec{h}) - (\vec{\pi}\vec{r})h_i\Big) \tag{2.16}$$

Similarly to the previous case, for vanishing magnetic charge ($g = 0$), equation (2.16) yields the conservation of the regular angular momentum. In the non-trivial case ($g \neq 0$), the field of the solenoid refuses the conservation of the generalized angular momentum. As we mentioned before, the solenoid and the Dirac string are not observable, and therefore, we need to fix the problematic behavior of the operators $\vec{L}$ and H, which is caused by the singularity of the vector potential along the Dirac string. According to Schwinger [9], the singular operator products are correctly defined by point splitting. Thus, we may regularise the operators, avoiding their singularities. According to this method, the operators read [5] [10]:

$$H = \lim_{|\vec{\varepsilon}|\to 0}\frac{3}{m\varepsilon^2}\Big(1 - e^{-i\frac{\vec{p}\vec{\varepsilon}}{2}}Ee^{-i\frac{\vec{p}\vec{\varepsilon}}{2}}\Big) \tag{2.17}$$

$$L_i = \lim_{|\vec{\varepsilon}|\to 0}\epsilon_{ijk}x_j\frac{3\varepsilon_k}{i\varepsilon^2}\Big(1 - e^{-i\frac{\vec{p}\vec{\varepsilon}}{2}}Ee^{-i\frac{\vec{p}\vec{\varepsilon}}{2}}\Big) \tag{2.18}$$

with

$$E = exp\Big[ie\int_{\vec{r}-\frac{\vec{\varepsilon}}{2}}^{\vec{r}+\frac{\vec{\varepsilon}}{2}}\vec{A}(\vec{\xi})d\vec{\xi}\Big] \tag{2.19}$$

In appendix A we prove that, if the vector potential is regular, the average of the operators (2.17) and (2.18) over all directions of the parameter $\vec{\varepsilon}$ corresponds to the standard Hamiltonian operator $\Big(H = \frac{\vec{\pi}^2}{2M} = \frac{(\vec{p}-e\vec{A}(\vec{r}))^2}{2M}\Big)$ and the angular momentum ($\vec{L} = \vec{r}\times\vec{\pi}$), respectively. If the vector potential is singular, the equations (2.17) and (2.18) may be considered as definitions. Additionally, using the regularized operators (2.17) and (2.18), the commutation relations yield:

$$[L_i, L_j] = i\epsilon_{ijk}L_k \tag{2.20}$$

$$[L_i, H] = 0 \tag{2.21}$$

Note that the problem, caused by the singularity along the Dirac string, is fixed.

The Dirac quantization condition (2.9) is supported by many arguments, but the origin of this relation lies in the topological roots of the magnetic charge. This discussion takes place in chapter 3. Note that the quantization of the charges by the relation (2.9) has nothing in common with the standard approach to quantization, since the charges do not appear as the discrete part of the spectrum of eigenvalues of a Hermitian operator. Instead, the Dirac quantization condition has topological roots. The suggestion by Dirac was that a monopole provides a beautiful explanation to the problem of the quantization of the electric charge. It is well known that all charged particles have electric charges that are proportional to the minimal charge of an electron. Then, if there is a monopole somewhere in the universe, even one such object placed anywhere would be enough to explain the quantization of electric charges according to (2.9) [1]. Taking into account the quarks, we have to impose that all charged particles have electric charge proportional to one-third of the electron's charge. This assumption leads to some states with charges 4/3, 5/3, ... , which correspond to some exotic electrically charged structures. These representations should either be ignored or considered to describe exotic particle/distributions beyond the standard model.

2.2 Spin-statistics theorem in a Magnetic Monopole theory

The extra term of the angular momentum $\vec{T} = eg\hat{r}$, which appears in the charge-monopole system, can take both integer and half-integer values, due to the Dirac quantization condition (2.9). This result yields some very interesting properties of the magnetic monopole systems. For instance, a bound system of two dyons may follow Fermi-Dirac or Boss-Einstein statistics, depending on Dirac quantization condition.

Hence, an essential result is the generalisation of the spin-statistics theorem. Let us consider the Hamiltonian operator of an electrically charged particle in the monopole field, which reads:

$$H_e = -\frac{1}{2M_e}\left[\nabla_e - ieg\vec{A}(\vec{r}_e - \vec{r}_g)\right]^2 \tag{2.22}$$

The theory is dual invariant. Considering the dual transformation ($\theta = -\pi/2$):

$$e \to g\ , \quad g \to -e\ , \quad A_\mu \to \tilde{A}_\mu \tag{2.23}$$

the Hamiltonian yields:

$$H_e \to H_g = -\frac{1}{2M_g}\left[\nabla_g + ieg\tilde{\vec{A}}(\vec{r}_g - \vec{r}_e)\right]^2 \tag{2.24}$$

The explicit form of the dual vector potential $\tilde{\vec{A}}$ can be recovered from the conservation of the total momentum, which corresponds to the translation invariance of the theory. The particles' momentum reads:

$$M_e\vec{v}_e = \vec{p}_e - eg\vec{A}(\vec{r}_e - \vec{r}_g) \quad \text{and} \quad M_g\vec{v}_g = \vec{p}_g - eg\tilde{\vec{A}}(\vec{r}_g - \vec{r}_e) \tag{2.25}$$

The system of the particles is considered isolated. Therefore, equations (2.25) are compatible with the conservation of momentum of the whole system, $\vec{p}_e + \vec{p}_g = 0$ only if $\vec{A}(\vec{r}) = \tilde{\vec{A}}(-\vec{r})$. Thus, the vector potential and its dual potential are connected with the following gauge transformation:

$$\vec{A}(\vec{r}) \longrightarrow \vec{A}(\vec{r}) + \nabla\lambda(\vec{r}) = \vec{A}(-\vec{r}) = \tilde{\vec{A}}(\vec{r}) \tag{2.26}$$

Hence, the Hamiltonian (2.24) reads:

$$H_g = -\frac{1}{2M_g}\left[\nabla_g + ieg\vec{A}(\vec{r}_e - \vec{r}_g)\right]^2 \tag{2.27}$$

Note that the gauge transformation (2.26) corresponds to the transformation $e^{2ieg\phi}$, as we mention in section 1.3. Therefore transformation (1.88) yields:

$$\vec{A}(\vec{r}) - \vec{A}(-\vec{r}) = \frac{i}{e}U^{-1}\nabla U\ , \quad \text{where} \quad U = e^{2ieg\phi} \tag{2.28}$$

Let us consider a system of two identical dyons with electrical and magnetic charges e and g at $\vec{r}_1$ and $\vec{r}_2$, respectively. Obviously, a permutation of these particles cannot change any physical observable. Thus, the only thing that can happen is that the wave function of the system picks up an additional phase factor $e^{i\pi a}$. The effect of two consecutive interchanges is the same as that of no interchange. Thus $e^{2i\pi a} = 1$, i.e., $e^{i\pi a} = 1$ (Bose-Einstein statistics) or $e^{i\pi a} = -1$ (Fermi-Dirac statistics). Considering equations (2.22) and (2.27), the total Hamiltonian reads:

$$H = -\frac{1}{2M}\Big(\nabla_1 - ieg\vec{A}(\vec{r}_1-\vec{r}_2) + ieg\vec{A}(\vec{r}_2-\vec{r}_1)\Big)^2 - \frac{1}{2M}\Big(\nabla_2 - ieg\vec{A}(\vec{r}_2-\vec{r}_1) + ieg\vec{A}(\vec{r}_1-\vec{r}_2)\Big)^2 + V(e^2) + V(g^2) \tag{2.29}$$

$$\text{with} \quad V(e^2) = \frac{e^2}{|\vec{r}_1 - \vec{r}_2|} \quad \text{and} \quad V(g^2) = \frac{g^2}{|\vec{r}_1 - \vec{r}_2|}$$

Upon substituting transformation (2.28) into equation (2.29), we obtain:

$$H = -\frac{1}{2M}\Big(\nabla + 2ig^2 e\nabla\phi\Big)^2 - \frac{1}{2M}\Big(\nabla - 2ig^2 e\nabla\phi\Big)^2 + \frac{e^2+g^2}{|\vec{r}_1-\vec{r}_2|} \overset{\nabla^2\phi=0}{\Longrightarrow}$$

$$H = -\frac{1}{2M}(\nabla_1^2 + \nabla_2^2) + V(q^2)\ , \quad \text{where} \quad q = \sqrt{g^2+e^2} \tag{2.30}$$

Note that, as it was expected, the total Hamiltonian (2.30) of the system is dual invariant, due to the dual invariant charge q. Additionally, under the transformation (2.28) the wave-function changes as follows:

$$\psi(\vec{r}) \longrightarrow e^{2ieg\phi}\psi(\vec{r}) \tag{2.31}$$

The Hamiltonian (2.10) remains invariant under the interchange of the two dyons ($\vec{r}_1 \to \vec{r}_2 \Rightarrow \vec{r} \to -\vec{r} \Rightarrow \phi \to \phi + \pi$). Under this transformation, the wave-function change as follows:

$$e^{2ieg\phi}\psi(\vec{r}) \longrightarrow e^{2ieg\pi}e^{2ieg\phi}\psi(\vec{r})$$

The Dirac quantization condition (2.9) demands:

$$e^{2ieg\phi}\psi(\vec{r}) \longrightarrow e^{in\pi}e^{2ieg\phi}\psi(\vec{r})\ , \quad n \in \mathbb{Z} \tag{2.32}$$

The equation (2.32) yields a very interesting result:

$$\text{for } n = 1,\ 3,\ ... \quad \phi(\vec{r}) \to -\phi(\vec{r}) \quad \text{Fermi-Dirac statistics}$$

$$\text{for } n = 0,\ 2,\ ... \quad \phi(\vec{r}) \to \phi(\vec{r}) \quad \text{Boss-Einstein statistics} \tag{2.33}$$

Thus, the standard spin-statistics theorem is fulfilled, but a system of two identical dyons may satisfy Bose-Einstein as well as Fermi-Dirac statistics.

2.3 Quantum mechanical description of a charge-monopole system

In this section, we examine the system charge-monopole through the Schrödinger equation. The corresponding solutions include a very interesting generalisation of the spherical harmonics, which is explicitly defined later on.

2.3.1 The Generalized Spherical Harmonics

Let us consider once again the system of an electrically charged particle in an external monopole field. The Lagrangian of this system reads:

$$L = \frac{1}{2}m\dot{\vec{r}}^{\,2} + e\dot{\vec{r}}\vec{A} \tag{2.34}$$

The conjugate momentum is defined as follows:

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} + e\vec{A} \tag{2.35}$$

The Legendre transformation of the Lagrangian yields the Hamiltonian of the system:

$$H = \vec{p}\cdot\dot{\vec{r}} - L \Rightarrow$$

$$H = \frac{\vec{\pi}^2}{2M}\ , \quad \text{with} \quad \vec{\pi} = \vec{p} - e\vec{A} \tag{2.36}$$

Also, we consider the vector potential (1.89):

$$\vec{A}(\vec{r}) = \frac{g}{r}\frac{1-cos(\theta)}{sin(\theta)}\hat{e}_\phi \tag{2.37}$$

It is useful to write momentum $\vec{\pi}$ as follows:

$$\vec{\pi} = \vec{\pi}_r + \vec{\pi}_\perp \quad \Rightarrow \quad \vec{\pi} = \vec{p}_r + \vec{\pi}_\perp \tag{2.38}$$

where we noticed that $\vec{\pi}_r$ is identical to $\vec{p}_r$, since $\vec{A}$ is parallel to $\hat{e}_\phi$ and perpendicular to $\hat{e}_r$. Therefore, the Hamiltonian (2.36) reads:

$$H = \frac{\vec{p}_r^2}{2M} + \frac{\vec{\pi}_\perp^2}{2M} = \frac{\vec{p}_r^2}{2M} + \frac{\left(\frac{\vec{r}\times\vec{\pi}}{r}\right)^2}{2M} = \frac{\vec{p}_r^2}{2M} + \frac{\left(\vec{L}+eg\hat{r}\right)^2}{2Mr^2} \Rightarrow$$

$$H = \frac{\vec{p}_r^2}{2M} + \frac{L^2-\mu^2}{2Mr^2}\ , \quad \text{with} \quad \mu = eg \tag{2.39}$$

where the generalized angular momentum is defined as $\vec{L} = \vec{r}\times\vec{\pi} - eg\hat{r}$ and satisfies the equation (1.35) $\vec{L}\cdot\hat{r} = \hat{r}\cdot\vec{L} = -eg$. Upon using definition $\vec{p} = -i\nabla$, we define the following operators:

$$p_r^2 = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) \tag{2.40}$$

considering

$$\vec{L} = \vec{r}\times\vec{\pi} - \mu\hat{r} = \vec{r}\times\vec{p} - e\vec{r}\times\vec{A} - \mu\hat{r}$$

and

$$\vec{r}\times\vec{p} = -i\vec{r}\times\nabla = -i\hat{e}_\phi\frac{\partial}{\partial\theta} + i\frac{\hat{e}_\theta}{sin(\theta)}\frac{\partial}{\partial\phi}\ , \quad \vec{r}\times\vec{A} = -g\frac{1-cos(\theta)}{sin(\theta)}\hat{e}_\theta$$

we obtain

$$\vec{L} = \frac{\hat{e}_\theta}{sin(\theta)}\left[i\frac{\partial}{\partial\phi} + \mu(1-cos(\theta))\right] - i\hat{e}_\phi\frac{\partial}{\partial\theta} - \mu\hat{e}_r \tag{2.41}$$

and

$$L^2 = -\frac{1}{sin^2(\theta)}\left(sin(\theta)\frac{\partial}{\partial\theta}\left[sin(\theta)\frac{\partial}{\partial\theta}\right] + \left[\frac{\partial}{\partial\phi} - i\mu(1-cos(\theta))\right]^2\right) + \mu^2 \tag{2.42}$$

also considering

$$\hat{e}_r = cos(\theta)\hat{e}_z + sin(\theta)\hat{e}_\rho \quad \text{and} \quad \hat{e}_\theta = -sin(\theta)\hat{e}_z + cos(\theta)\hat{e}_\rho$$

we obtain

$$L_3 = -i\frac{\partial}{\partial\phi} - \mu \tag{2.43}$$

Note that in the ordinary case, where $\mu = 0$, the operator $\vec{L}$ corresponds to the ordinary angular momentum operator. The solutions of the Schrödinger equation $H\psi(\vec{r}) = E\psi(\vec{r})$, where the operator H is given by the equation (2.39), may be written as follows:

$$\psi(\vec{r}) = F_{k\tilde{\ell}}(r)Y_{\mu\ell m}(\theta,\phi) \tag{2.44}$$

where $Y_{\mu\ell m}(\theta,\phi)$ are the eigenfunctions of the generalized angular momentum, called generalized spherical harmonics. In the subsequent steps we determine the explicit form of the generalized spherical harmonics.

The operators L^2 and L_3 have common eigenfunctions [1], considering the Dirac quantization condition. The eigenfunctions of the operator L_3 are:

$$L_3 f(\phi) = mf(\phi) \Rightarrow -i\frac{\partial f}{\partial\phi} = (m+\mu)f(\phi) \Rightarrow$$

$$f_{\mu m}(\phi) = Ae^{i(\mu+m)\phi}\ , \quad \text{with eigenvalues } m:\ \mu+m\in\mathbb{Z} \text{ and } \mu = \frac{n}{2}\ , \quad n\in\mathbb{Z} \tag{2.45}$$

The quantization condition $\mu + m \in \mathbb{Z}$ is imposed, since function $f_{\mu m}(\phi)$ must be single-valued. The fact that the operators L^2 and L_3 have common eigenfunctions yields that the form of $Y_{\mu\ell m}(\theta,\phi)$ reads:

$$Y_{\mu\ell m}(\theta,\phi) = P_{\mu\ell m}(cos(\theta))e^{i(\mu+m)\phi} \tag{2.46}$$

Let us solve the eigenvalue equation of the L^2 operator:

$$L^2 Y_{\mu\ell m}(\theta,\phi) = \lambda Y_{\mu\ell m}(\theta,\phi) \Rightarrow$$

$$\Big(-(1-x^2)\frac{d^2}{dx^2} + 2x\frac{d}{dx} + \frac{(m+\mu x)^2}{1-x^2} + \mu^2\Big)P_{\mu\ell m}(x) = \lambda P_{\mu\ell m}(x) \tag{2.47}$$

where we have set $x = cos(\theta)$. Upon separating the singularities $x = \pm 1$, we may write the solution as follows:

$$P_{\mu\ell m}(x) = (1-x)^{-\dfrac{\mu+m}{2}}(1+x)^{-\dfrac{\mu-m}{2}}F(x) \tag{2.48}$$

Upon substituting ansatz (2.48) into equation (2.47), we obtain:

$$(1-x^2)\frac{d^2F(x)}{dx^2} + 2\big[m + (\mu-1)x\big]\frac{dF(x)}{dx} - (\mu^2 - \mu - \lambda)F(x) = 0 \tag{2.49}$$

If we introduce variable $z = \dfrac{1+x}{2}$, we obtain:

$$z(1-z)F'' + \Big[c - (a+b+1)z\Big]F' - abF = 0$$

$$\text{with} \quad c = m - \mu + 1\ , \quad ab = \mu^2 - \mu - \lambda\ , \quad a + b + 1 = 2(1-\mu) \tag{2.50}$$

This is the hypergeometric equation. Demanding finite solutions, we set $a = -n$, where $n = 0, 1, 2, \ldots$. Hence, considering the arguments for single-valued generalized spherical harmonics, presented in [11], the eigenvalues of the generalized angular momentum read:

$$\lambda = \ell(\ell+1)\ , \quad \text{with} \ \ \ell = n + |\mu|\ , \quad n = 0, 1, 2, \ldots \tag{2.51}$$

Equation (2.51) yields that ℓ and μ must be simultaneously integers or half-integers. Additionally, we have:

$$\ell = |\mu|,\ |\mu|+1,\ |\mu|+2,\ \ldots > 0 \tag{2.52}$$

Upon summarizing the solutions of the eigenvalue equation, we obtain:

$$Y_{\mu\ell m}(\theta,\phi) = N_{\mu\ell m}(1-cos(\theta))^{-\dfrac{\mu+m}{2}}(1+cos(\theta))^{-\dfrac{\mu-m}{2}}P_{\ell+m}^{-\mu-m,\,-\mu+m}(cos(\theta))e^{i(\mu+m)\phi} \tag{2.53}$$

These are the generalized spherical harmonics, where $P_n^{a,b}$ are the Jacobi polynomials:

$$P_n^{a,b}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-a}(1+x)^{-b}\frac{d^n}{dx^n}\Big((1-x)^{a+n}(1+x)^{b+n}\Big) \tag{2.54}$$

Additionally, normalisation factor $N_{\mu\ell m}$ reads:

$$N_{\mu\ell m} = 2^m\left(\frac{(2\ell+1)(\ell-m)!(\ell+m)!}{4\pi(\ell-\mu)!(\ell+\mu)!}\right)^{\frac{1}{2}} \tag{2.55}$$

Note that, in the vanishing magnetic charge case ($\mu = 0$), the generalized spherical harmonics are reduced to the standard spherical harmonics.

Additionally, we can define raising and lowering generators L_+ and L_-, respectively, according to the standard procedure:

$$L_\pm = L_1 \pm iL_2 \tag{2.56}$$

where

$$L_1 = cot(\theta)cos(\phi)\Big[i\frac{\partial}{\partial\phi} + \mu(1-cos(\theta))\Big] + i\,sin(\phi)\frac{\partial}{\partial\theta} - \mu\,sin(\theta)cos(\phi)$$

$$L_2 = cot(\theta)sin(\phi)\Big[i\frac{\partial}{\partial\phi} + \mu(1 - cos(\theta))\Big] - i\ cos(\phi)\frac{\partial}{\partial\theta} - \mu\ sin(\theta)sin(\phi) \tag{2.57}$$

thus

$$L_\pm = e^{\pm i\phi}\Big[\pm\frac{\partial}{\partial\theta} + i\ cot(\theta)\frac{\partial}{\partial\phi} - \mu\frac{sin(\theta)}{1 + cos(\theta)}\Big] \tag{2.58}$$

Upon considering algebra (2.20), we obtain:

$$L_\pm L_\mp = L^2 - L_3^2 \pm L_3 \tag{2.59}$$

Consequently, we can prove the following standard relation:

$$L_\pm Y_{\mu\ell m}(\theta,\phi) = \sqrt{\ell(\ell+1) - m(m\pm 1)}Y_{\mu\ell m\pm 1}(\theta,\phi) \tag{2.60}$$

2.3.2 Solutions of the Schrödinger equation

In order for the solution of the Schrödinger equation $H\psi(\vec{r}) = E\psi(\vec{r})$ to be completed, we need to solve the radial part of the equation. Upon substituting equations (2.39), (2.40), (2.44) and $L^2 Y_{\mu\ell m}(\theta,\phi) = \ell(\ell+1)Y_{\mu\ell m}(\theta,\phi)$ into Schrödinger equation, we obtain:

$$-\frac{1}{2M}\left[\frac{1}{r^2}\frac{\partial}{\partial r} - \frac{\ell(\ell+1) - \mu^2}{r^2}\right]F_{k\tilde{\ell}}(r) = EF_{k\tilde{\ell}}(r) \tag{2.61}$$

Introducing the parameter $k = \sqrt{2ME}$ and the variable $x = kr$, equation (2.61) yields:

$$\frac{d^2}{dx^2}F_{k\tilde{\ell}} + \frac{2}{x}\frac{d}{dx}F_{k\tilde{\ell}} + \Big(1 - \frac{\ell(\ell+1) - \mu^2}{x^2}\Big)F_{k\tilde{\ell}} = 0 \tag{2.62}$$

If we set

$$\tilde{\ell}(\tilde{\ell}+1) = \ell(\ell+1) - \mu^2 \Rightarrow$$

$$\tilde{\ell} = \sqrt{\Big(\ell + \frac{1}{2}\Big)^2 - \mu^2} - \frac{1}{2} \tag{2.63}$$

and consider finite solution at $r = 0$, the solution is given by the spherical Bessel functions:

$$F_{k\tilde{\ell}}(r) = \frac{1}{k}\sqrt{\frac{2}{\pi}}j_{\tilde{\ell}}(kr) \tag{2.64}$$

The asymptotic $r \to +\infty$ behaviour of function (2.64) reads:

$$F_{k\tilde{\ell}}(r) \approx \frac{1}{r}sin(kr - \frac{\pi\ell}{2}) \tag{2.65}$$

It is very interesting that, considering equation (2.52), we observe that the effective potential in equation (2.61) is always repulsive,

$$V_{eff}(r) = \frac{\ell(\ell+1) - \mu^2}{2Mr^2} \tag{2.66}$$

which means that there is no bound state in the spectrum of a monopole and a spinless charged particle. It is an analogous situation with the classical scattering of a charged particle on a monopole, discussed in section 1.2, where there are no closed trajectories.

Similarly to the previous case, we consider a charge-dyon system. The angular dependence of the corresponding wave-function is still given by the generalized spherical harmonics. Nevertheless, the radial part of the Schrödinger equation reads:

$$-\frac{1}{2M}\left[\frac{1}{r^2}\frac{\partial}{\partial r} - \frac{\ell(\ell+1) - \mu^2}{r^2}\right]F_{N\tilde{\ell}}(r) + \frac{eQ}{r}F_{N\tilde{\ell}}(r) = EF_{N\tilde{\ell}}(r) \tag{2.67}$$

where Q is the electric charge of the dyon and $\frac{eQ}{r}$ is an attractive Coulomb-interaction term, if we consider $eQ < 0$. Considering $E < 0$ and the ansatz

$$F_{N\tilde{\ell}}(r) = e^{-kr} r^{\tilde{\ell}} \rho_{N\tilde{\ell}}(r) \tag{2.68}$$

where $k = \sqrt{2M|E|}$, the equation (2.67) reads:

$$r\frac{d^2}{dr^2}\rho_{N\tilde{\ell}}(r)+2(\tilde{\ell}^2+1-kr)\frac{d}{dr}\rho_{N\tilde{\ell}}(r)+2(M|eQ|-k\tilde{\ell}-k)\rho_{N\tilde{\ell}}(r)+\frac{\tilde{\ell}(\tilde{\ell}+1)-\ell(\ell+1)+\mu^2}{r}\rho_{N\tilde{\ell}}(r) = 0 \tag{2.69}$$

Note that, if we consider $\tilde{\ell} = \sqrt{(\ell+\frac{1}{2})^2 - \mu^2} - \frac{1}{2}$, the solution is regular at $r = 0$. Additionally, the solution is given by the confluent hypergeometric function:

$$\rho_{N\tilde{\ell}}(r) = {}_1F_1(-N; 2\tilde{\ell}+2; 2kr) \tag{2.70}$$

where $N = 0,\ 1,\ 2,\ ...$ is a radial quantum number, which reads:

$$N = \frac{M|eQ|}{k} - \tilde{\ell} - 1\ , \quad \text{and therefore} \quad E_{N\tilde{\ell}} = -\frac{M(eQ)^2}{2(N+\tilde{\ell}+1)^2} \tag{2.71}$$

where $E_{N\tilde{\ell}}$ is the spectrum of the bound states of the system. The ground state reads $\ell = \frac{1}{2},\ N = 0,\ \mu = \frac{1}{2}$, which is doubly degenerated ($m = \pm\frac{1}{2}$). In order to conclude, the solution of the radial Schrödinger equation reads:

$$F_{N\tilde{\ell}}(r) = C\ (kr)^{\tilde{\ell}} e^{-kr}\ {}_1F_1(-N; 2\tilde{\ell}+2; 2kr) \tag{2.72}$$

where [1]:

$$C = \frac{2^{\tilde{\ell}+1}\big[MeQ\Gamma(-N; 2\tilde{\ell}+2; 2kr)\big]^{\frac{1}{2}}}{(N!)^{\frac{1}{2}}(N+\tilde{\ell}+1)\Gamma(2\tilde{\ell}+2)} \tag{2.73}$$

For $Q = 0$, equation (2.64) can be recovered. Finally, the effective potential

$$V_{eff}(r) = \frac{\ell(\ell+1)-\mu^2}{2Mr^2} - \frac{|eQ|}{r} \tag{2.74}$$

allows the existence of bound states with negative energy, as we proved explicitly. This result is analogous to that recovered in the subsection 1.2.2, where we proved that the classical charge-dyon system has bound trajectories.

2.4 Charge-Monopole system in the Pauli approximation

The Pauli equation is a low energy limit of the Dirac equation and describes fermions, whose spin interacts with an external magnetic field. In our case, which is the charge-monopole system, the external magnetic field is the monopole field. The non-relativistic Pauli equation reads:

$$H\psi(\vec{r}) = -\frac{1}{2M}\Big[\vec{\sigma}(\nabla - ie\vec{A})\Big]^2\psi(\vec{r}) = E\psi(\vec{r}) \tag{2.75}$$

Upon considering the identity $(\vec{\sigma}\vec{a})(\vec{\sigma}\vec{a}) = \vec{a}\vec{b} + i\vec{\sigma}(\vec{a}\times\vec{b})$, the Hamiltonian (2.75) yields:

$$H\psi(\vec{r}) = \left[\frac{(\vec{\sigma}\vec{\pi})^2}{2M}\right]\psi(\vec{r}) = \left[\frac{\vec{\pi}^2}{2M} + \frac{i\vec{\sigma}}{2M}(\vec{\pi}\times\vec{\pi})\right]\psi(\vec{r}) \tag{2.76}$$

Additionally, the spin operator reads $\vec{S} = \dfrac{\vec{\sigma}}{2}$, while the cross product of the momentum $\vec{\pi}$ is $\vec{\pi}\times\vec{\pi} = ie\vec{B}$. Hence, equation (2.76) implies:

$$H\psi(\vec{r}) = \left[\frac{\vec{\pi}^2}{2M} - \frac{e}{M}\vec{S}\vec{B}\right]\psi(\vec{r}) \tag{2.77}$$

where the term $-\frac{e}{M}\vec{S}\vec{B}$ corresponds to the interaction between the spin $s = \frac{1}{2}$ of the particle and the external magnetic field. Note that the gyromagnetic ratio predicted by the Pauli equation is $\gamma = 2$.

The operator of the total angular momentum is defined as follows:

$$\vec{J} = \vec{L} + \vec{S} = \vec{r}\times\vec{\pi} - eg\hat{r} + \vec{S} \tag{2.78}$$

The operator of the total angular momentum commutes with the Hamiltonian operator (2.77). Note that:

$$J^2 = L^2 + 2\vec{L}\vec{S} + \frac{3}{4} \tag{2.79}$$

The conservation of the total angular momentum (2.78) leads to some very interesting results [1]. Let us consider a minimal value of the parameter $\mu = \frac{1}{2}$, which is consistent with the charge quantization condition. Then, the ground state is a spherically symmetric s-wave with zero angular momentum $\vec{J} = \vec{0}$, since the orbital angular momentum vanishes and the extra term of the generalized angular momentum is canceled by the spin vector, i.e., $\vec{S} - \mu\hat{r} = \vec{0}$. The latter condition means that the spin angular momentum $\vec{S}$ has the same length as the extra angular momentum $-\mu\hat{r}$, and these vectors are anti-parallel. The subtle point here is that the direction of the vector $-\mu\hat{r}$ is given by the unit radial vector, with direction from the monopole to the charged particle. Therefore, if the charged particle manages to go through the core of the monopole, this component of the angular momentum must invert its direction. However, the total angular momentum is conserved, which means that the spin of the charged particle must also change its direction in order to compensate for the inversion of $-\mu\hat{r}$. However, if we consider the Dirac, or the Pauli equation, which describes a mass-less particle interacting with a monopole, the helicity is a conserved quantum number labeling the states. Therefore, the Hamiltonian of the system of a mass-less spin-1/2 charged particle and a monopole is not self-adjoint at the origin for the s-wave states. Thus, the theory becomes pathological [12][13]. We may fix this problem, if we suppose that there is an unknown mechanism inside the monopole core, according which, when an electron enters the monopole core, some process of non-electrodynamical nature leads, for example, to the conjugation of the electron charge $e \to -e$. Then, there is no need to consider the spin-flip scattering of the electron on the monopole, the amplitude of which is not well defined in the s-wave.

2.4.1 Dynamical supersymmetry of the electron-monopole system

In section 1.4 we mention the dynamical $SO(2,1)$ symmetry of the non-relativistic charge-monopole system. The addition of spin-1/2 makes the system invariant under the transformations of the dynamical conformal supergroup $OS_p(1,2)$.

The spin-1/2 degrees of freedom can be represented via the three-dimensional anti-commuting Grassmann variables ξ_i, where $\{\xi_i, \xi_j\} = \delta_{ij}$. Let us define the operator

$$S_i = -\frac{i}{2}\epsilon_{ijk}\xi_j\xi_k \tag{2.80}$$

which satisfies the same algebra with the spin operator. In the next few lines we prove this statement. The commutation of the spin operators reads:

$$[S_i, S_j] = \frac{i^2}{4}\epsilon_{iab}\epsilon_{jcd}[\xi_a\xi_b, \xi_c\xi_d]$$

We use the identity

$$[AB, CD] = A\{B, C\}D - C\{D, A\}B - AC\{B, D\} + \{A, C\}DB \tag{2.81}$$

and obtain

$$[S_i, S_j] = \frac{i^2}{4}\epsilon_{iab}\epsilon_{jcd}(\xi_a\xi_d\delta_{bc} - \xi_c\xi_b\delta_{da} - \xi_a\xi_c\delta_{bd} + \xi_d\xi_b\delta_{ac}) =$$

$$\frac{i^2}{4}\epsilon_{iab}\epsilon_{jdb}(-\xi_a\xi_d + \xi_d\xi_a - \xi_a\xi_d + \xi_d\xi_a) = \frac{i^2}{2}(\delta_{ij}\delta_{ad} - \delta_{id}\delta_{ja})(\xi_d\xi_a - \xi_a\xi_d) =$$

$$-\frac{i^2}{2}(\xi_i\xi_j - \xi_j\xi_i) = -\frac{i^2}{2}\epsilon_{ijk}\epsilon_{kab}\xi_a\xi_b \Rightarrow$$

$$[S_i, S_j] = i\,\epsilon_{ijk}S_k \tag{2.82}$$

The equation (2.82) represents the algebra of Spin operators. It is very interesting that the irreducible two-dimensional representation of Clifford's algebra of ξ_i variables is $\xi_i = \frac{\sigma_i}{\sqrt{2}}$. Hence, the representation of operator $\vec{S}$ reads:

$$S_i = -\frac{i}{2}\epsilon_{ijk}\frac{\sigma_j\sigma_k}{2}$$

using the algebra of Pauli matrices $[\sigma_i, \sigma_j] = 2i\ \epsilon_{ijk}\sigma_k$ we have

$$S_i = -\frac{i}{4}\epsilon_{ijk}\sigma_k\sigma_j + \frac{1}{2}\epsilon_{jka}\epsilon_{ijk}\sigma_a$$

and considering the identity $\epsilon_{jka}\epsilon_{ijk} = 2\delta_{ia}$, we obtain

$$S_i = \frac{\sigma_i}{2} \tag{2.83}$$

Indeed, the operator (2.80) represents the Spin operator. Note that $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$ can be considered as a vector under the rotation group. Substituting the operator (2.80) and the monopole field (1.21) into the Hamiltonian (2.77) we obtain:

$$H = \frac{\vec{\pi}^2}{2M} + i\frac{\mu}{2Mr^3}\epsilon_{ijk}r_i\xi_j\xi_k \tag{2.84}$$

The corresponding Lagrangian reads:

$$L = \frac{1}{2}M\dot{\vec{r}}^2 + \frac{1}{2}(\vec{\xi}\dot{\vec{\xi}}) + e\vec{A}\dot{\vec{r}} - i\frac{\mu}{2Mr^3}\epsilon_{ijk}r_i\xi_j\xi_k \tag{2.85}$$

Also, the total angular momentum is:

$$J_i = \epsilon_{ijk}Mr_j\dot{r}_k - \mu\hat{r} - \frac{i}{2}\epsilon_{ijk}\xi_j\xi_k \tag{2.86}$$

Note that the Lagrangian (2.85) transforms as a scalar under spatial rotations. The conjugate momentum of $\dot{\vec{r}}$ and $\dot{\vec{\xi}}$ read:

$$p_i^{(r)} = \frac{\partial L}{\partial \dot{r}_i} = M\dot{r}_i + eA_i \quad \text{and} \quad p_i^{(\xi)} = \frac{\partial L}{\partial \dot{\xi}_i} = \frac{\xi_i}{2} \tag{2.87}$$

The Hamilton's equation for ξ_i is:

$$\dot{\xi}_i = \frac{\partial H}{\partial p_i^{(\xi)}} = \frac{i\mu}{Mr^3}\epsilon_{ijk}r_j\xi_k \tag{2.88}$$

The equation (2.88) describes the classical spin precession in the monopole field [1].

Furthermore, let us consider the following supertransformations [1]:

$$Q: \quad r_i \longrightarrow r_i + \frac{i\varepsilon}{\sqrt{M}}\xi_i\ , \quad \xi_i \longrightarrow \xi_i - \varepsilon\sqrt{M}\dot{r}_i \tag{2.89}$$

$$S: \quad r_i \longrightarrow r_i + \frac{it\eta}{\sqrt{M}}\xi_i\ , \quad \xi_i \longrightarrow \xi_i - \eta\sqrt{M}(t\dot{r}_i - r_i) \tag{2.90}$$

where ε and η are anti-commuting Grassmannian transformation parameters. Transformations Q and S change the Lagrangian (2.85) by a total time derivative, hence, they form a dynamical supersymmetry. The corresponding conserved Noether supercharges reads [1][14]:

$$Q = \sqrt{M}\dot{\vec{r}}\vec{\xi} \quad \text{and} \quad S = -tQ + \sqrt{M}\vec{r}\vec{\xi} \tag{2.91}$$

Operators Q and S complete the set of generators of the conformal group (1.100). All the charges commute with the operator of the total angular momentum $\vec{J}$ [1]. Thus, the complete set of generators of the dynamical group form the following algebra:

$$\begin{array}{ccc} [H,D] = iH & [D,K] = iK & [H,K] = 2iD \\ [H,S] = -iQ & [K,Q] = iS & [K,S] = 0 \\ [H,Q] = 0 & [D,Q] = -\frac{i}{2}Q & [D,S] = \frac{i}{2}S \\ \{Q,Q\} = 2H & \{Q,S\} = -2D & \{S,S\} = 2K \end{array} \tag{2.92}$$

In Appendix B we present some proofs of the above relations. The quadratic Casimir operator of the supergroup $OSp(1,1)$ reads [14]:

$$\mathcal{J}^2 = \frac{1}{4}\big(i[Q,S] - \frac{1}{2}\big)^2 \equiv \frac{1}{4}C^2 \tag{2.93}$$

where

$$C = \vec{\sigma}(\vec{J}^2 + \mu\hat{r}) - \frac{1}{2} \tag{2.94}$$

Note that if the eigenvalues of the operator of the total angular momentum $\vec{J}^2$ are denoted by $j(j+1)$, the eigenvalues of the Casimir operator $\mathcal{J}^2$ are $\frac{\tilde{\ell}}{4}$, where $\tilde{\ell} = \sqrt{(j+\frac{1}{2})^2 - \mu^2}$. Thus, since the eigenstates of the commuting operators $\vec{J}^2$, J_3 and C transform under some irreducible representation of the supergroup $OSp(1,1)$, we can determine the spectrum of the Pauli equation in the presence of a magnetic monopole immediately [14]. However, in the next subsection we find the spectrum of the monopole-spin-1/2 particle system by solving the eigenvalue problem directly.

2.4.2 Generalized spinor harmonics

Similarly to the Schrödinger equation case, the solutions of the angular part of the Pauli equation are some generalized spherical harmonics, called generalized spinor harmonics.

The eigenvalues of the generalized angular momentum depend on the quantum number ℓ, defined in subsection 2.3.1, equation (2.52). The minimum value of ℓ is $\ell_{min} = \mu$. Equivalently, the minimum value of the quantum number of the total angular momentum $\vec{J}^2 = (\vec{L}+\vec{S})^2$ reads $j = \mu \pm \frac{1}{2}$ and depends on spin.

i) *Generalized spinor harmonics for* $j \geq \mu + \frac{1}{2}$

Upon substituting operator (2.40) into the Hamiltonian (2.77), we obtain:

$$H = -\frac{1}{2Mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{2Mr^2}\Big[L^2 - \mu^2 - \mu(\vec{\sigma}\cdot\hat{r})\Big] \tag{2.95}$$

Let us define a generalization of the parity operator of the spinors [1]:

$$K = \vec{\sigma}(\vec{r}\times\vec{\pi}) = \vec{\sigma}(\vec{L} + \mu\hat{r}) \tag{2.96}$$

In the next few lines we prove that the operator (2.96) can represent the angular part of the Hamiltonian operator:

$$K^2 = \Big[\vec{\sigma}(\vec{r}\times\vec{\pi})\Big]^2$$

Considering the identity $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{a}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}(\vec{a}\times\vec{b})$, we have

$$K^2 = \tilde{L}^2 + i\ \vec{\sigma}(\vec{r}\times\vec{\pi})\times(\vec{r}\times\vec{\pi}) = \tilde{L}^2 + i\ \sigma_i\epsilon_{ijk}\epsilon_{jab}r_a\pi_b\epsilon_{kcd}r_c\pi_d$$

Substituting the identity (2.14) into the equation, we obtain:

$$K^2 = \tilde{L}^2 + i\ \sigma_i\epsilon_{jab}r_a\pi_b r_i\pi_j$$

where we used the commutative property of r_i and the anti-commutative property of the indices of the Levi-Civita symbol. Considering the standard commutation $[r_i, \pi_j] = i\delta_{ij}$, we have:

$$K^2 = \tilde{L}^2 + i\ \vec{r}(\vec{\pi}\times\vec{\pi})(\vec{\sigma}\cdot\vec{r}) - \sigma_i\epsilon_{ijk}r_j\pi_k$$

Finally, upon substituting relations $\vec{\pi}\times\vec{\pi} = ie\vec{B}$ and $\vec{L} = \tilde{\vec{L}} - \mu\hat{r}$ into the above equation, we obtain:

$$K^2 = L^2 - \mu^2 - \mu(\vec{\sigma}\cdot\vec{r}) - \vec{\sigma}(\vec{r}\times\vec{\pi}) \Rightarrow$$

$$K^2 + K = L^2 - \mu^2 - \mu(\vec{\sigma}\cdot\hat{r}) \tag{2.97}$$

It is very important that the operator K commutes with the operators $H,\ J^2,\ J_3,\ L^2$ and $\vec{\sigma}\cdot\vec{L}$ [1].

Our goal is to determine the eigenfunctions of K operator. The common eigenfunctions of the operators $J^2 = (\vec{L}+\vec{S})^2$, J_3 and L^2 are:

$$\Phi^{(1)}_{\mu j m_j}(\theta,\phi) = \begin{pmatrix} \sqrt{\frac{j+m_j}{2j}}Y_{\mu,\,j-\frac{1}{2},\,m_j-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{j-m_j}{2j}}Y_{\mu,\,j-\frac{1}{2},\,m_j+\frac{1}{2}}(\theta,\phi) \end{pmatrix} \tag{2.98}$$

with eigenvalues

$$\begin{array}{c} J^2: \ j(j+1)\,, \quad \text{where } \ j=\ell+\frac{1}{2} \\ J_3: \ m_j \\ L^2: \ \ell(\ell+1) \end{array} \tag{2.99}$$

and

$$\Phi^{(2)}_{\mu j m_j}(\theta,\phi) = \begin{pmatrix} -\sqrt{\frac{j-m_j+1}{2j+2}}Y_{\mu,\, j+\frac{1}{2},\, m_j-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{j+m_j+1}{2j+2}}Y_{\mu,\, j+\frac{1}{2},\, m_j+\frac{1}{2}}(\theta,\phi) \end{pmatrix} \tag{2.100}$$

with eigenvalues

$$\begin{array}{c} J^2: \ j(j+1)\,, \quad \text{where } \ j=\ell-\frac{1}{2} \\ J_3: \ m_j \\ L^2: \ \ell(\ell+1) \end{array} \tag{2.101}$$

The eigenvalues of the operators L^2 and J_3 are easily verified since:

$$L^2 Y_{\mu,\ell,m_\ell}(\theta,\phi) = \ell(\ell+1)Y_{\mu,\ell,m_\ell}(\theta,\phi) \quad \text{and} \quad L_3 Y_{\mu,\ell,m_\ell}(\theta,\phi) = m_\ell Y_{\mu,\ell,m_\ell}(\theta,\phi) \tag{2.102}$$

Also, the range of the quantum number j reads:

$$\Phi^{(1)}_{\mu j m_j}(\theta,\phi): \ j-\frac{1}{2}=\ell\geq\mu \quad \text{and} \quad \Phi^{(2)}_{\mu j m_j}(\theta,\phi): \ j+\frac{1}{2}=\ell\geq\mu \tag{2.103}$$

The functions (2.98) and (2.100) are also eigenfunctions of operator $\vec{\sigma}\cdot\vec{L}$, since:

$$\vec{\sigma}\cdot\vec{L}\Phi^{(1)}_{\mu j m_j}(\theta,\phi) = \begin{pmatrix} L_3 & L_- \\ L_+ & -L_3 \end{pmatrix}\Phi^{(1)}_{\mu j m_j}(\theta,\phi)$$

Considering equations (2.56), (2.60) and (2.102), we obtain:

$$\vec{\sigma}\cdot\vec{L}\Phi^{(1)}_{\mu j m_j} = \begin{pmatrix} \left((m_j-\frac{1}{2})\sqrt{\frac{j+m_j}{2j}} + \sqrt{(j-\frac{1}{2})(j+\frac{1}{2})-(m_j+\frac{1}{2})(m_j-\frac{1}{2})}\sqrt{\frac{j-m_j}{2j}}\right)Y_{\mu,\, j-\frac{1}{2},\, m_j-\frac{1}{2}} \\ \left((m_j+\frac{1}{2})\sqrt{\frac{j-m_j}{2j}} + \sqrt{(j-\frac{1}{2})(j+\frac{1}{2})-(m_j+\frac{1}{2})(m_j-\frac{1}{2})}\sqrt{\frac{j+m_j}{2j}}\right)Y_{\mu,\, j-\frac{1}{2},\, m_j+\frac{1}{2}} \end{pmatrix} =$$

$$\vec{\sigma}\cdot\vec{L}\Phi^{(1)}_{\mu j m_j} = \begin{pmatrix} \left((m_j-\frac{1}{2})\sqrt{\frac{j+m_j}{2j}} + \sqrt{(j-m_j)(j+m_j)}\sqrt{\frac{j-m_j}{2j}}\right)Y_{\mu,\, j-\frac{1}{2},\, m_j-\frac{1}{2}} \\ \left((m_j+\frac{1}{2})\sqrt{\frac{j-m_j}{2j}} + \sqrt{(j-m_j)(j+m_j)}\sqrt{\frac{j+m_j}{2j}}\right)Y_{\mu,\, j-\frac{1}{2},\, m_j+\frac{1}{2}} \end{pmatrix} \Rightarrow$$

$$\vec{\sigma}\cdot\vec{L}\Phi^{(1)}_{\mu j m_j}(\theta,\phi) = (j-\frac{1}{2})\Phi^{(1)}_{\mu j m_j}(\theta,\phi) \tag{2.104}$$

Similarly, we can prove that:

$$\vec{\sigma}\cdot\vec{L}\Phi^{(2)}_{\mu j m_j}(\theta,\phi) = -(j+\frac{3}{2})\Phi^{(2)}_{\mu j m_j}(\theta,\phi) \tag{2.105}$$

Also, it is essential to verify the eigenvalues of the total angular momentum. The definition (2.78) yields:

$$J^2 = L^2 + 2\vec{L}\vec{S} + S^2 = L^2 + \vec{\sigma}\cdot\vec{L} + \frac{3}{4} \tag{2.106}$$

Thus, we obtain:

$$J^2\Phi^{(1)}_{\mu j m_j}(\theta,\phi) = \left(\ell(\ell+1) + j - \frac{1}{2} + \frac{3}{4}\right)\Phi^{(1)}_{\mu j m_j}(\theta,\phi) \overset{\ell=j-\frac{1}{2}}{=} \left(j^2 - \frac{1}{4} + j - \frac{1}{2} + \frac{3}{4}\right)\Phi^{(1)}_{\mu j m_j}(\theta,\phi) \Rightarrow$$

$$J^2\Phi^{(1)}_{\mu j m_j}(\theta,\phi) = j(j+1)\Phi^{(1)}_{\mu j m_j}(\theta,\phi)$$

similarly, for $\Phi^{(2)}_{\mu j m_j}(\theta,\phi)$, we have:

$$J^2\Phi^{(2)}_{\mu j m_j}(\theta,\phi) = \left(\ell(\ell+1) - j - \frac{3}{2} + \frac{3}{4}\right)\Phi^{(2)}_{\mu j m_j}(\theta,\phi) \overset{\ell=j+\frac{1}{2}}{=} \left(j^2 + 2j + \frac{3}{4} - j - \frac{3}{2} + \frac{3}{4}\right)\Phi^{(2)}_{\mu j m_j}(\theta,\phi) \Rightarrow$$

$$J^2\Phi^{(2)}_{\mu j m_j}(\theta,\phi) = j(j+1)\Phi^{(2)}_{\mu j m_j}(\theta,\phi)$$

In order to determine the eigenfunctions of operator K, we need to find how the operator $\vec{\sigma}\cdot\hat{r}$ changes spinors $\Phi^{(1)}_{\mu j m_j}$ and $\Phi^{(2)}_{\mu j m_j}$.

$$\vec{\sigma}\cdot\hat{r}\Phi^{(1)}_{\mu j m_j}(\theta,\phi) = \sqrt{\frac{4\pi}{3}}\begin{pmatrix} Y_{010} & \sqrt{2}Y_{01-1} \\ -\sqrt{2}Y_{011} & Y_{010}\end{pmatrix}\begin{pmatrix}\sqrt{\frac{j+m_j}{2j}}Y_{\mu,\, j-\frac{1}{2},\, m_j-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{j-m_j}{2j}}Y_{\mu,\, j-\frac{1}{2},\, m_j+\frac{1}{2}}(\theta,\phi)\end{pmatrix}$$

where $Y_{010}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}cos(\theta)$ and $Y_{01\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}sin(\theta)e^{\pm i\phi}$. Thus, we obtain:

$$\vec{\sigma}\cdot\hat{r}\Phi^{(1)}_{\mu j m_j}(\theta,\phi) = A\Phi^{(1)}_{\mu j m_j}(\theta,\phi) + B\Phi^{(2)}_{\mu j m_j}(\theta,\phi) \tag{2.107}$$

similarly

$$\vec{\sigma}\cdot\hat{r}\Phi^{(2)}_{\mu j m_j}(\theta,\phi) = B\Phi^{(1)}_{\mu j m_j}(\theta,\phi) - A\Phi^{(2)}_{\mu j m_j}(\theta,\phi) \tag{2.108}$$

where

$$A = -\frac{\mu}{j+\frac{1}{2}} \quad \text{and} \quad B = -\frac{\sqrt{(j+\frac{1}{2})^2-\mu^2}}{j+\frac{1}{2}} = -\sqrt{1-A^2} \tag{2.109}$$

Hence, we recognise the eigenfunctions of the operators $K = \vec{\sigma}\vec{L} + \mu\vec{\sigma}\hat{r}$, J^2 and J_3:

$$\Omega^{(1)}_{\mu j m_j}(\theta,\phi) = \frac{1}{2}\Big(\sqrt{1-A}+\sqrt{1+A}\Big)\Phi^{(1)}_{\mu j m_j}(\theta,\phi) - \frac{1}{2}\Big(\sqrt{1-A}-\sqrt{1+A}\Big)\Phi^{(2)}_{\mu j m_j}(\theta,\phi) \tag{2.110}$$

$$\Omega^{(2)}_{\mu j m_j}(\theta,\phi) = \frac{1}{2}\Big(\sqrt{1-A}-\sqrt{1+A}\Big)\Phi^{(1)}_{\mu j m_j}(\theta,\phi) + \frac{1}{2}\Big(\sqrt{1-A}+\sqrt{1+A}\Big)\Phi^{(2)}_{\mu j m_j}(\theta,\phi) \tag{2.111}$$

which are the generalized spinor harmonics. It can be proven that:

$$\vec{\sigma}\cdot\hat{r}\Omega^{(1)}_{\mu j m_j}(\theta,\phi) = -\Omega^{(2)}_{\mu j m_j}(\theta,\phi) \tag{2.112}$$

$$\vec{\sigma}\cdot\hat{r}\Omega^{(2)}_{\mu j m_j}(\theta,\phi) = -\Omega^{(1)}_{\mu j m_j}(\theta,\phi) \tag{2.113}$$

Consequently, we obtain:

$$K\Omega^{(1)}_{\mu j m_j}(\theta,\phi) = (-1+\tilde{\ell})\Omega^{(1)}_{\mu j m_j}(\theta,\phi) \tag{2.114}$$

and

$$K\Omega^{(2)}_{\mu j m_j}(\theta,\phi) = (-1-\tilde{\ell})\Omega^{(2)}_{\mu j m_j}(\theta,\phi) \tag{2.115}$$

where

$$\tilde{\ell} = \sqrt{\Big(j+\frac{1}{2}\Big)^2 - \mu^2} \tag{2.116}$$

Note that $\frac{\tilde{\ell}}{2}$ is the eigenvalue of Casimir operator (2.93).

ii) *Generalized spinor harmonics for* $j = \mu - \frac{1}{2}$

In this case, relation (2.103) yields that only one angular spinor is non-vanishing. Therefore, we have:

$$\Omega^{(3)}_{\mu,\mu-\frac{1}{2},m_j}(\theta,\phi) = \Phi^{(2)}_{\mu,\mu-\frac{1}{2},m_j}(\theta,\phi) \tag{2.117}$$

which satisfies the following equations:

$$K\Omega^{(3)}_{\mu,\mu-\frac{1}{2},m_j}(\theta,\phi) = -\Omega^{(3)}_{\mu,\mu-\frac{1}{2},m_j}(\theta,\phi) \tag{2.118}$$

and

$$(\vec{\sigma}\cdot\hat{r})\Omega^{(3)}_{\mu,\mu-\frac{1}{2},m_j}(\theta,\phi) = \Omega^{(3)}_{\mu,\mu-\frac{1}{2},m_j}(\theta,\phi) \tag{2.119}$$

2.4.3 Solutions of the radial Pauli equation

The Hamiltonian (2.95), considering equation (2.97), can be written as follows:

$$H = -\frac{1}{2Mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{K^2+K}{2Mr^2} \tag{2.120}$$

Upon considering solutions of the form:

$$\psi^{(1)}(\vec{r}) = R^{(1)}_{k\tilde{\ell}}(r)\Omega^{(1)}_{\mu j m_j}(\theta,\phi) \tag{2.121}$$

and the fact that (2.114):

$$(K^2+K)\Omega^{(1)}_{\mu j m_j}(\theta,\phi) = \tilde{\ell}(\tilde{\ell}-1)\Omega^{(1)}_{\mu j m_j}(\theta,\phi) \tag{2.122}$$

the radial part of Pauli equation reads:

$$-\frac{1}{2M}\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\tilde{\ell}(\tilde{\ell}-1)}{r^2}\right]R^{(1)}_{k\tilde{\ell}}(r) = ER^{(1)}_{k\tilde{\ell}}(r) \tag{2.123}$$

A regular solution of the above equation is given by the modified Bessel function of order $\tilde{\ell} - \frac{1}{2}$:

$$R^{(1)}_{k\tilde{\ell}}(r) = \sqrt{\frac{k}{r}}J_{\tilde{\ell}-\frac{1}{2}}(kr)\ , \quad k = \sqrt{2ME} \tag{2.124}$$

Similarly, considering a solution of the form:

$$\psi^{(2)}(\vec{r}) = R^{(2)}_{k\tilde{\ell}}(r)\Omega^{(2)}_{\mu j m_j}(\theta,\phi) \tag{2.125}$$

and the equation (2.114):

$$(K^2+K)\Omega^{(2)}_{\mu j m_j}(\theta,\phi) = \tilde{\ell}(\tilde{\ell}+1)\Omega^{(2)}_{\mu j m_j}(\theta,\phi) \tag{2.126}$$

we obtain the following radial equation:

$$-\frac{1}{2M}\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\tilde{\ell}(\tilde{\ell}+1)}{r^2}\right]R^{(2)}_{k\tilde{\ell}}(r) = ER^{(2)}_{k\tilde{\ell}}(r) \tag{2.127}$$

with regular solution:

$$R^{(2)}_{k\tilde{\ell}}(r) = \sqrt{\frac{k}{r}}J_{\tilde{\ell}+\frac{1}{2}}(kr)\ , \quad k = \sqrt{2ME} \tag{2.128}$$

Note that, close to the origin, we have $R^{(1)}(r) \sim ar^{-\tilde{\ell}} + br^{\tilde{\ell}-1}$. Supposing $j \geq \mu + \frac{1}{2} \Rightarrow \tilde{\ell} > 1$, where $R^{(1)}(r)$ is not vanishing, and demanding regular solution at $r = 0$, we obtain $R^{(1)}(0) = 0$. This result corresponds to magnetic mirror effect described in subsection 1.2.1.

Considering $j = \mu - \frac{1}{2} \Rightarrow \tilde{\ell} = 0$, the radial equation reads:

$$-\frac{1}{2M}\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right]R^{(3)}_{k,\mu-\frac{1}{2}}(r) = ER^{(3)}_{k,\mu-\frac{1}{2}}(r) \tag{2.129}$$

with solution:

$$R^{(3)}_{k,\mu-\frac{1}{2}}(r) = \frac{1}{\sqrt{\pi}}\frac{e^{\pm ikr}}{r}\ , \quad k = \sqrt{2ME} \tag{2.130}$$

The asymptotic behaviour of the above solution at the origin is $\sim \frac{1}{r}$, which means that it is not compatible with a smooth boundary condition at the origin. The reason is that the Pauli Hamiltonian is not self-adjoint over the complete space of eigenfunctions, since [1]:

$$(\psi^{(3)}, H\psi^{(3)}) \neq (H\psi^{(3)}, \psi^{(3)}) \tag{2.131}$$

This case corresponds to a charge falling down on the monopole. Note that the states 1, 2, with $j \geq \mu + \frac{1}{2}$, transform under a representation of the supergroup $OSp(1,1)$. However, the eigenvalues of the Casimir operator, corresponding to the states with $j = \mu - \frac{1}{2}$, vanish and the supercharges Q and S are not self-adjoint. Consequently, the group of dynamical symmetry consists of the group $SO(2,1)$, in the case $j = \mu - \frac{1}{2}$.

2.5 Relativistic quantum mechanical description of a charge-monopole system

In this section we represent a relativistic description of the charge-monopole system. First of all, we solve the corresponding Dirac equation:

$$H\psi(\vec{r}) = E\psi(\vec{r}) \tag{2.132}$$

where H is the Dirac Hamiltonian operator:

$$H = \vec{\alpha}\vec{\pi} + M\beta = \vec{\alpha}(\vec{p} - e\vec{A}) + M\beta = -i\vec{\alpha}(\nabla - ie\vec{A}) + M\beta \tag{2.133}$$

where

$$\alpha_i = \gamma_0\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \begin{pmatrix} M & -i\vec{\sigma}(\nabla - ie\vec{A}) \\ -i\vec{\sigma}(\nabla - ie\vec{A}) & -M \end{pmatrix} \tag{2.134}$$

Note that the above Hamiltonian commutes with the total angular momentum (2.78), hence we can immediately write the eigenfunctions of H:

$$\psi^{(1)}(\vec{r}) = \frac{1}{r}\begin{pmatrix} F^{(1)}(r)\Omega^{(1)}_{\mu j m_j}(\theta,\phi) \\ iG^{(1)}(r)\Omega^{(2)}_{\mu j m_j}(\theta,\phi) \end{pmatrix} \quad \text{and} \quad \psi^{(2)}(\vec{r}) = \frac{1}{r}\begin{pmatrix} F^{(2)}(r)\Omega^{(2)}_{\mu j m_j}(\theta,\phi) \\ iG^{(2)}(r)\Omega^{(1)}_{\mu j m_j}(\theta,\phi) \end{pmatrix}, \quad \text{for } j \geq \mu + \frac{1}{2} \tag{2.135}$$

and

$$\psi^{(3)}(\vec{r}) = \frac{1}{r}\begin{pmatrix} F^{(3)}(r)\Omega^{(3)}_{\mu j m_j}(\theta,\phi) \\ iG^{(3)}(r)\Omega^{(3)}_{\mu j m_j}(\theta,\phi) \end{pmatrix}, \quad \text{for } j = \mu - \frac{1}{2} \tag{2.136}$$

Helicity operator $\vec{\sigma}\cdot\vec{\pi} = \vec{\sigma}(\nabla - ie\vec{A})$ acts on these functions as follows [1]:

$$\begin{aligned}
(\vec{\sigma}\cdot\vec{\pi})F^{(1)}(r)\Omega^{(1)}_{\mu j m_j}(\theta,\phi) &= \Big(-\frac{d}{dr} - \frac{1}{r} + \frac{\tilde{\ell}}{r}\Big)F^{(1)}(r)\Omega^{(2)}_{\mu j m_j}(\theta,\phi) \\
(\vec{\sigma}\cdot\vec{\pi})F^{(2)}(r)\Omega^{(2)}_{\mu j m_j}(\theta,\phi) &= \Big(-\frac{d}{dr} - \frac{1}{r} - \frac{\tilde{\ell}}{r}\Big)F^{(2)}(r)\Omega^{(1)}_{\mu j m_j}(\theta,\phi) \\
(\vec{\sigma}\cdot\vec{\pi})F^{(3)}(r)\Omega^{(3)}_{\mu j m_j}(\theta,\phi) &= \Big(\frac{d}{dr} + \frac{1}{r}\Big)F^{(3)}(r)\Omega^{(3)}_{\mu j m_j}(\theta,\phi)
\end{aligned} \tag{2.137}$$

where $\tilde{\ell} = \sqrt{(j+\frac{1}{2})^2 - \mu^2}$ are the eigenvalues of Casimir operator (2.93). Also, functions $G^{(i)}$ satisfy similar equations to the above. Consequently, the radial part of the Dirac equation reads [1]:

$$\left.\begin{aligned}
\Big(\tfrac{d}{dr} - \tfrac{\tilde{\ell}}{r}\Big)F^{(1)}(r) = (M+E)G^{(1)}(r) \\
\Big(\tfrac{d}{dr} + \tfrac{\tilde{\ell}}{r}\Big)G^{(1)}(r) = (M-E)F^{(1)}(r)
\end{aligned}\right\} \Rightarrow \begin{aligned}
G^{(1)}(r) &= \sqrt{\tfrac{r}{k}}J_{\tilde{\ell}+\frac{1}{2}}(kr) \\
F^{(1)}(r) &= \tfrac{\sqrt{kr}}{E-M}J_{\tilde{\ell}-\frac{1}{2}}(kr)
\end{aligned} \tag{2.138}$$

$$\left.\begin{aligned}
\Big(\tfrac{d}{dr} + \tfrac{\tilde{\ell}}{r}\Big)F^{(2)}(r) = (M+E)G^{(2)}(r) \\
\Big(\tfrac{d}{dr} - \tfrac{\tilde{\ell}}{r}\Big)G^{(2)}(r) = (M-E)F^{(2)}(r)
\end{aligned}\right\} \Rightarrow \begin{aligned}
F^{(2)}(r) &= \sqrt{\tfrac{r}{k}}J_{\tilde{\ell}+\frac{1}{2}}(kr) \\
G^{(2)}(r) &= \tfrac{\sqrt{kr}}{E+M}J_{\tilde{\ell}-\frac{1}{2}}(kr)
\end{aligned} \tag{2.139}$$

where $k = \sqrt{E^2 - M^2} > 0$.

$$\left.\begin{aligned}
\tfrac{d}{dr}F^{(3)}(r) = -(E+M)G^{(3)}(r) \\
\tfrac{d}{dr}G^{(3)}(r) = (E-M)F^{(3)}(r)
\end{aligned}\right\} \Rightarrow$$

$$\begin{aligned}
F^{(3)}_1(r) &= \tfrac{1}{k}\sqrt{\tfrac{2}{\pi}}sin(kr+\delta) \\
G^{(3)}_1(r) &= -\tfrac{1}{E+M}\sqrt{\tfrac{2}{\pi}}cos(kr+\delta)
\end{aligned} \tag{2.140}$$

or

$$\begin{aligned} F_2^{(3)}(r) &= \tfrac{1}{E-M}\sqrt{\tfrac{2}{\pi}}cos(kr+\delta) \\ G_2^{(3)}(r) &= \tfrac{1}{k}\sqrt{\tfrac{2}{\pi}}sin(kr+\delta) \end{aligned} \tag{2.141}$$

where the phase shift "δ" is an arbitrary parameter. Summarizing the results in the third case, we obtain [1]:

$$\psi_1^{(3)}(\vec{r}) = \frac{1}{kr}\sqrt{\frac{2}{\pi}}\chi_1(r)\Omega^{(3)}_{\mu j m_j}(\theta,\phi), \quad \text{with} \quad \chi_1(r) = \begin{pmatrix} sin(kr+\delta) \\ -i\frac{k}{E+M}cos(kr+\delta) \end{pmatrix} \tag{2.142}$$

$$\psi_2^{(3)}(\vec{r}) = \frac{1}{kr}\sqrt{\frac{2}{\pi}}\chi_2(r)\Omega^{(3)}_{\mu j m_j}(\theta,\phi), \quad \text{with} \quad \chi_2(r) = \begin{pmatrix} \frac{k}{E-M}cos(kr+\delta) \\ i\ sin(kr+\delta) \end{pmatrix} \tag{2.143}$$

Note that the solutions (2.138) and (2.139), in cases one and two respectively, satisfy the boundary conditions $F^{(1,2)}(0) = G^{(1,2)}(0) = 0$. On the contrary, in case three, solutions (2.142) and (2.143) behave as $\frac{1}{r}$ at the origin, as in the case of Pauli equation.

2.5.1 Zero modes and the Witten effect

In case three, the radial part of the Hamiltonian operator that acts on spinors $\chi(r)$ can be written as follows:

$$H_0 = \begin{pmatrix} M & -i\frac{d}{dr} \\ -i\frac{d}{dr} & -M \end{pmatrix} = -i\gamma_5\frac{d}{dr} + M\beta \tag{2.144}$$

where we consider the Dirac-Pauli representation of γ_5 and β. Let us set:

$$\chi(r) = \begin{pmatrix} F(r) \\ iG(r) \end{pmatrix} \tag{2.145}$$

then the corresponding Dirac equation reads:

$$H_0\chi(r) = H_0\begin{pmatrix} F(r) \\ iG(r) \end{pmatrix} = E\chi(r) \tag{2.146}$$

Note that, as in the Pauli equation case, the Hamiltonian H_0 is not Hermitian at the origin. A self-adjoint extension of the Hamiltonian can be constructed [15] by imposing non-trivial self-consistent boundary conditions. This implies interesting physical consequences [16], for instance, the helicity operator is not Hermitian and is no longer conserved, as we explicitly prove later on.

$$(\chi, H_0\chi) - (H_0\chi, \chi) = i\ \chi^\dagger(0)\gamma_5\chi(0) = i\big[\chi_+^\dagger(0)\chi_+(0) - \chi_-^\dagger(0)\chi_-(0)\big] =$$

$$-\big[F^*(0)G(0) - G^*(0)F(0)\big] = 0 \tag{2.147}$$

where

$$\chi_\pm(r) = \frac{1}{2}(1\pm\gamma_5)\chi(r) \tag{2.148}$$

As usual, χ_+ and χ_- are eigenstates of chirality with positive and negative eigenvalues respectively. In order for the equation (2.147) to be satisfied, the boundary conditions must connect the states χ_+ and χ_-, e.g.:

$$\chi_-(0) = e^{i\theta}\chi_+(0) \tag{2.149}$$

where θ is an arbitrary angular parameter. The equation (2.147) directly yields:

$$\frac{F^*(0)}{G^*(0)} = \frac{F(0)}{G(0)} \tag{2.150}$$

which means that $\frac{F(0)}{G(0)}$ is real. Thus, the boundary condition may be written as follows [17][16]:

$$F(0) = G(0)\ tan\Big(\frac{\theta}{2} + \frac{\pi}{4}\Big) \tag{2.151}$$

Upon substituting equation (2.151) into equation (2.142), we obtain:

$$\chi(0) \propto \begin{pmatrix} sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right) \\ -i\frac{k}{E+M}cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right) \end{pmatrix} \tag{2.152}$$

Considering a massless spin-1/2 particle in the monopole field and imposing the aforementioned boundary condition on the solution $\chi(r)$, the angular parameter θ can be explained as a chiral rotation of the initial wave function (2.140)[16]:

$$\chi_\theta(r) \sim e^{i\frac{\theta}{2}\gamma_5} \begin{pmatrix} sin\left(kr+\frac{\pi}{4}\right) \\ -icos\left(kr+\frac{\pi}{4}\right) \end{pmatrix} = \begin{pmatrix} sin\left(kr+\frac{\theta}{2}+\frac{\pi}{4}\right) \\ -icos\left(kr+\frac{\theta}{2}+\frac{\pi}{4}\right) \end{pmatrix} \tag{2.153}$$

Then, we decompose spinor $\chi(r)$ into plane waves propagating in both directions

$$\chi(r) = e^{i\frac{\theta}{2}}\frac{1+i}{2\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}e^{ikr} - e^{-i\frac{\theta}{2}}\frac{1-i}{2\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}e^{-ikr} \tag{2.154}$$

and we observe that helicity changes, as we promised, when a particle passes through the origin and the phase swift is given by the term $e^{i\theta}$, as expected from equation (2.149). Note that although the Hamiltonian (2.144) commutes with the operator γ_5, its eigenfunctions depend on an arbitrary phase θ, which is a CP violating parameter [1]. Indeed, for states of the third type, the CP inversion is defined as CP : $\chi(r) \to \gamma_5\chi^*(r)$. However, in the massless case, the model is invariant under chiral rotations $\chi \to e^{ie\gamma_5\theta'}\chi$, which shift the value of this parameter as $\theta \to \theta+\theta'$, and in particular allows us just to set it to zero. Therefore the physical observables are independent on the value of θ in the absence of a chiral anomaly.

In the massive case, the eigenfunctions of the Hamiltonian, which satisfy the aforementioned boundary conditions, can be written as follows:

$$\chi_\theta(r) = \frac{k}{\sqrt{E(E-Msin(\theta))}}\left[\chi_1(r)cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)+i\,\chi_2(r)sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right]\text{ , with } E=\sqrt{k^2+M^2}>0 \tag{2.155}$$

$$\xi_\theta(r) = \frac{k}{\sqrt{|E|(|E|-Msin(\theta))}}\left[\chi_1(r)cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)+i\,\chi_2(r)sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right]\text{ , with } E=-\sqrt{k^2+M^2}<0 \tag{2.156}$$

where χ_1 and χ_2 are defined in (2.142) and (2.143). In the case $cos(\theta)<0$, there is a family of solutions [16]:

$$\chi_\theta(r) \sim \begin{pmatrix} sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right) \\ -i\,cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right) \end{pmatrix} e^{-kr} \tag{2.157}$$

where

$$E = Msin(\theta) \quad \text{and} \quad k = M|cos(\theta)| \tag{2.158}$$

From equation (2.158) arises a case with great interest. For $\theta = 0,\ \pi$, we have a state with zero energy, named zero mode of the Dirac operator whose appearance is connected with the index theorem [18][19]. This mode is part of the complete set of eigenfunctions of the Hamiltonian and cannot be neglected [1]. Note that unlike the massless case, the eigenfunctions with negative energy obviously violate the CP symmetry of the theory, since chiral rotations of the wave functions no longer leave the Hamiltonian invariant. If we still demand the theory to remain CP invariant, the value of θ must be fixed to $\theta=0$ or $\theta=\pi$. However, the physical content of these cases is different. The point is that the existence of the fermionic zero mode on the monopole background transforms a monopole into a dyon [1]. This unexpected statement must be proven explicitly.

We can obtain the wave-function (2.155), upon imposing a chiral rotation on the $\chi(r)$:

$$\chi(r) \to \chi_\theta(r) = e^{i\frac{Q_5}{2}\theta}\chi = e^{i\frac{\theta}{2}\int dr j_5^0(r)}\chi \tag{2.159}$$

where $j_5^\mu(r) = \chi^\dagger(r)\gamma_0\gamma_5\gamma^\mu\chi(r)$ is the chiral current and $Q_5 = \int dr j_5^0(r)$ is the chiral charge, i.e., the generator of the chiral rotations. Let us consider the following commutation relation [1]:

$$[j^0(x), j_5^0(x)] = i\frac{e^2}{2\pi^2}\vec{B}\nabla\delta(x-y) \tag{2.160}$$

where $j^0 = e\chi^\dagger\chi$ is the density of the electric charge of the charged particle and $\vec{B}$ is the external monopole field. It is obvious that the vacuum expectation value of density j^0 vanishes, since there are no electrically charged particles in the vacuum state. Let us act with a chiral rotation on the vacuum state:

$$e^{-i\frac{\theta}{2}Q_5} j^0(0) e^{i\frac{\theta}{2}Q_5} = j^0(0) + i\frac{\theta}{2}[j^0(0), Q_5] = j^0(0) + i\frac{\theta}{2}\int dr [j^0(0), j_5^0(r)] \Rightarrow$$

$$e^{-i\frac{\theta}{2}Q_5} j^0(0) e^{i\frac{\theta}{2}Q_5} = j^0(0) - \frac{e^2\theta}{4\pi^2}\int dr \vec{B}\nabla\delta(r) = j^0(0) + \frac{e^2\theta}{4\pi^2}\nabla\vec{B}(0) = j^0(0) + \frac{e^2 g\theta}{\pi}\delta(0) \Rightarrow$$

$$< 0|e^{-i\frac{\theta}{2}Q_5} j^0(0) e^{i\frac{\theta}{2}Q_5}|0 >= \frac{e^2 g\theta}{\pi}\delta(0) \Rightarrow$$

$$< 0|Q|0 >= \frac{e\theta}{2\pi} n \ , \quad \text{with} \quad n \in \mathbb{Z} \tag{2.161}$$

where we used the Dirac quantization condition (2.9), the equation (2.160) and

$$e^{-sA} B e^{sA} = B - s[A, B] + O(s^2) \ ,$$

$$\nabla\vec{B} = 4\pi g\delta(\vec{r})$$

The equation (2.161) yields that we observe electric charge at the origin, in absences of the electrically charged particle. Hence, a magnetic monopole interacting with a fermionic field transforms into a dyon with electric charge proportional to θ. This effect is called Witten effect.

Finally, let us suppose that there is a reason of non-electrodynamical nature that demands $\theta = 0$. In this case, we encounter the problem of the self-adjointness of the Hamiltonian, since the solutions of the equations (2.140) become regular at the origin. In the special case $E = 0$, we have [1]:

$$F(r) = G(r) = \frac{1}{\sqrt{2}} e^{-\frac{k\mu}{2Mr} - Mr} \tag{2.162}$$

2.5.2 Generalisation of the Dirac quantization condition

We can generalize the Dirac quantization condition by considering a dyon-dyon system. Assuming that the dyons' electric and magnetic charges are $(e_1, \ g_1)$ and $(e_2, \ g_2)$ respectively, the extra term of the generalized angular momentum reads:

$$\vec{L}_{dd} = (e_1 g_2 - g_1 e_2)\hat{r} \tag{2.163}$$

The quantization condition for angular momentum $\vec{L}_{dd}$ is [20][21]

$$e_1 g_2 - g_1 e_2 = n \ , \quad n \in \mathbb{Z} \tag{2.164}$$

Note that the above formula is dual invariant.

Let us discuss Dirac's statement, according to which, if there is a magnetic monopole in the universe with magnetic charge g, then it is enough to explain the quantization of the electric charge. Indeed, if we consider a single magnetic monopole, with magnetic charge g, the equation (2.164) yields:

$$e = ne_0 \ , \quad \text{with} \quad e_0 = \frac{1}{g} \quad \text{and} \quad n \in \mathbb{Z} \tag{2.165}$$

where $e_0 = \frac{1}{g}$ is the elementary electric charge.

We can also prove the quantization of the magnetic charge [22]. Let us consider two states, one with elementary electric charge $(e_0, 0)$ and a second with magnetic charge and a possible arbitrary electric charge (e, g). Then quantization condition (2.164) yields:

$$g = \frac{n}{n_0} g_0 \ , \quad \text{with} \quad g_0 = \frac{n_0}{e_0} \quad \text{and} \quad n \in \mathbb{Z} \tag{2.166}$$

where g_0 is the minimum magnetic charge and n_0 is a positive number that depends on the particular choice of the model. For instance, the Dirac quantization condition is recovered for $n_0 = \frac{1}{2}$ [1].

Chapter 3

Topological description of the magnetic charge

Probably the most interesting aspect of magnetic monopoles is their topological properties. In appendix D we describe some notions of differential geometry and topology, which are useful for the understanding of magnetic monopole physics. In particular, we define *topological space*, *homeomorphism*, *differentiable manifold*, *diffeomorphism* etc. In this chapter, we briefly discuss the connection between the *homotopic n-loops* and the topology of the monopole systems. Also, we present the formalism of magnetic monopoles without strings and the corresponding charge quantization condition.

3.1 Homotopy groups

Homotopy groups and more specifically homotopic loops play a fundamental role in our discussion, which leads to the topological roots of magnetic charge [1].

Let us consider two *topological spaces* and an "one-way" map $\phi_1 : X \to Y$, which has no inverse. If there is another map $\phi_2 : X \to Y$ and the function $\phi_1(x)$ can be continuously deformed into $\phi_2(x)$, i.e., there is a continuous family of functions $f(x,t)$, $x \in X$, $t \in [0,1]$, $f : X \times [0,1] \to Y$, where $f(x,0) = \phi_1(x)$ and $f(x,1) = \phi_2(x)$, the map ϕ_1 is homotopic to ϕ_2. The family of functions $f(x,t)$ is called a homotopy.

Note that if $\phi_1(x)$ is *homotopic* to $\phi_2(x)$ and $\phi_2(x)$ is *homotopic* to $\phi_3(x)$, then $\phi_1(x)$ is *homotopic* to $\phi_3(x)$. Thus, the space of continuous maps from X to Y is divided into equivalence classes.

Two *topological spaces* X and Y are said to be homotopically equivalent, if there exist continuous maps $f : X \to Y$ and $g : X \to Y$ that satisfy: (i) $f \circ g$ is *homotopic* to I_Y (an identity on the space Y, $I_Y(y) = y$) and (ii) $g \circ f$ is *homotopic* to I_X (an identity on the space X, $I_X(x) = x$).

We present two useful examples of homotopic spaces:

1) Let $X = \mathbb{R}^n \setminus \{0\}$ and $Y = S^{n-1}$. Also, let the maps f and g be $f : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ via $\vec{x} \to f(\vec{x}) = \hat{x} = \frac{\vec{x}}{|\vec{x}|}$, where $|\vec{x}| \neq 0$, and $g : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ via $\hat{x} \to g(\hat{x}) = \lambda\hat{x}$, where $\lambda \in \mathbb{R} \setminus \{0\}$. Then

$$f \circ g(\hat{x}) = \hat{x} = I_Y(\hat{x}) \quad \text{and} \quad g \circ f(\vec{x}) = \lambda\hat{x}$$

We need to prove that $g \circ f(\vec{x}) = \lambda\hat{x}$ is *homotopic* to I_X. Considering the *homotopy* $F : X \times [0,1] \to X$:

$$F(\vec{x},t) = t\vec{x} + (1-t)\lambda\hat{x} \Rightarrow$$

$$F(\vec{x},0) = g \circ f(\vec{x}) \quad \text{and} \quad F(\vec{x},1) = I_X(\vec{x})$$

Therefore, $g \circ f(\vec{x}) = \lambda\hat{x}$ is *homotopic* to I_X. Thus, $\mathbb{R}^n \setminus \{0\}$ and S^{n-1} are *homotopically equivalent.*

2) Let $X = S^1$ and Y be the set of unimodular complex numbers: $U_{n\delta}(\theta) = e^{i(n\theta+\delta)} \in Y$. Also, let the maps f and g be $f : X \to Y$ via $\hat{n}(\theta) \to f(\hat{n}(\theta)) = e^{i(n\theta+\delta)} = U_{n,\delta}(\theta)$, where $\theta \in [0,2\pi)$, and $g : Y \to X$ via $U_{n,\delta}(\theta) \to g(U_{n,\delta}(\theta)) = \hat{n}(-i\frac{Ln(U_{n,\delta}(\theta))}{n} - \frac{\delta}{n}) = \hat{n}(\theta)$, where $\theta \in [0,2\pi)$.
For $n = 1$ and arbitrary $\delta \in \mathbb{R}$ we obtain:

$$f \circ g(U_{1,\delta}(\theta)) = U_{1,\delta}(\theta) = I_Y(U_{1,\delta}(\theta)) \quad \text{and} \quad g \circ f(\hat{n}(\theta)) = \hat{n}(\theta) = I_X(\hat{n}(\theta))$$

Consequently, S^1 and the set of unimodular complex numbers are *homotopically equivalent.*

A very important fact is that the maps $f: X \to Y$ via $\hat{n}(\theta) \to f(\hat{n}(\theta)) = e^{i(n\theta+\delta)} = U_{n,\delta}(\theta)$, for fixed $n \in \mathbb{Z}$ and different $\delta \in \mathbb{R}$, constitute a *homotopy class*. Proof:

Let us consider the maps $f_0(\theta) = e^{i(n\theta+\delta_0)}$ and $f_1(\theta) = e^{i(n\theta+\delta_1)}$ and the continuous family of functions $F: X \times [0,1] \to Y$, such that $F(\theta,t) = e^{i(n\theta+(1-t)\delta_0+t\delta_1)}$. We observe that $F(\theta,0) = f_0(\theta)$ and $F(\theta,1) = f_1(\theta)$. Therefore, the maps f for fixed $n \in \mathbb{Z}$ and different $\delta \in \mathbb{R}$ form a *homotopy class*.

Each *homotopy class* is characterised by the integer n. Note that for $\theta \in [0,2\pi)$, using the map $f(\theta)$, we cover the unit circle in the complex plane n times. Therefore, we refer to integer n as *winding number*, which is a characteristic of the *homotopy class*.

Additionally, for a given *diffeomorphism* $f: S^1 \to S^1$ the *winding number* reads:

$$n = \frac{i}{2\pi} \int_0^{2\pi} d\theta f(\theta) \frac{\partial}{\partial\theta} f^{-1}(\theta) \ \in \mathbb{Z} \tag{3.1}$$

Note that the *winding number* plays the role of topological charge (magnetic charge) in monopole theories.

Furthermore, the notions *simply connected manifold* and *linearly connected manifold* are very important for our discussion. The *simply connected manifolds* satisfy the following properties:

i) Any two points of the given manifold can be connected by a continuous curve.
ii) Any closed curve (loop) can be shrunk continuously to a point.

The *linearly connected manifolds* satisfy only property (i). For instance, *simply connected manifold* is $\mathbb{R}^2$ and *linearly connected manifolds* are S^1 and $\mathbb{R}^2 \setminus \{0\}$. Note that a single-valued *function* f in some region of a *linearly connected manifold* can be continued to the whole space along paths connecting the points of this region with any other point. The condition for f to be single-valued requires that for any two points x_0 and x, continuations of $f(x_0)$ along any path connecting these two points must give the same result, $f(x)$. To be more precise, the continuation along any closed curve going through x_0 must lead to the initial $f(x_0)$. This is automatically true if any two paths connecting x_0 and x are *homotopic*.

We are ready to define the most important notion of our discussion, the *class of homotopic loops*. A loop in a *topological manifold* X can be defined as a continuous map (*curve*) $\gamma(t): I \to X$, such that $\gamma(0) = \gamma(1) = x_0 \in X$, where $t \in I = [0,1]$ is the parameter along the loop. The point x_0 is called a *base point* of the loop $\gamma(t)$. Two loops $\alpha(t)$ and $\beta(t)$ based at the same point are *homotopic* if the one loop can be continuously deformed into the other, i.e., there is a continuous map $H(t,s): [0,1] \times [0,1] \to X$, with $s,t \in [0,1]$, such that $H(0,s) = H(1,s) = x_0$ and $H(t,0) = \alpha(t)$, $H(t,1) = \beta(t)$. The set of loops with the same *base point*, which are connected by the *homotopy* H, is called *class of homotopic loops*.

It is very interesting that a *class of homotopic loops* equipped with a binary operation does not constitute a group. We can prove this statement explicitly. Let us consider the binary operation:

$$(\beta \circ \alpha)(t) = \begin{cases} \alpha(2t) \ , \ 0 \le t \le \frac{1}{2} \\ \beta(2t-1) \ , \ \frac{1}{2} < t \le 1 \end{cases} \tag{3.2}$$

where $\alpha(t)$ and $\beta(t)$ belong to the same *class of homotopic loops* $[\alpha]$, with $\alpha(0) = \beta(0) = \alpha(1) = \beta(1) = x_0$. The inverse element $\alpha^{-1}(t)$ is a loop with opposite direction of $\alpha(t)$ and can be defined as follows:

$$\alpha^{-1}(t) = \alpha(1-t) \tag{3.3}$$

The unit element reads:

$$I(t) = x_0 \ , \forall t \in [0,1] \tag{3.4}$$

Let us calculate the element $\alpha^{-1} \circ \alpha(t)$:

$$(\alpha^{-1} \circ \alpha)(t) = \begin{cases} \alpha(2t) \ , \ 0 \le t \le \frac{1}{2} \\ \alpha(2-2t) \ , \ \frac{1}{2} < t \le 1 \end{cases} \neq I(t) \tag{3.5}$$

The result (3.5) yields that the *class of homotopic loops* $[\alpha]$ equipped with the binary operation (3.2) does not form a group. In fact, if we consider a *class of homotopic loops* $[\alpha]$ in a *linearly connected manifold*, like $\mathbb{R}^2 \setminus \{0\}$, and additionally that loops $[\alpha]$ are located around the origin (*winding number* $n \neq 0$), then we obtain $I \notin [\alpha]$, $\alpha^{-1} \notin [\alpha]$ and $\alpha^{-1} \circ \alpha \notin [\alpha]$, since I and $\alpha^{-1} \circ \alpha$ always belong to the same *class of homotopic loops* with *winding number* $n = 0$ and α^{-1} belongs to the *class of homotopic loops* with an opposite *winding number* from $[\alpha]$. Since $I \notin [\alpha]$, $\alpha^{-1} \notin [\alpha]$ and $\alpha^{-1} \circ \alpha \notin [\alpha]$, the *class of homotopic loops* $[\alpha]$ equipped with the binary operation (3.2) does not form a group. Let us prove that the loops I and $\alpha^{-1} \circ \alpha$ always belong to the same *class of homotopic loops* with *winding number* $n = 0$. Considering the following *homotopy*:

$$H(t,s) = \begin{cases} \alpha(2ts) \ , \ 0 \le t \le \frac{1}{2} \\ \alpha\big((2-2t)s\big) \ , \ \frac{1}{2} < t \le 1 \end{cases} \tag{3.6}$$

we obtain:

$$H(t,0) = x_0 = I(t) \quad \text{and} \quad H(t,1) = (\alpha^{-1} \circ \alpha)(t) \tag{3.7}$$

and

$$H(0,s) = x_0 \quad \text{and} \quad H(1,s) = x_0 \tag{3.8}$$

Also, the *class of homotopic loops* $[I]$ has *winding number* $n = 0$, since the unit element does not perform any turning around the origin.

However, a very essential fact is that the complete set of the *classes of homotopic loops* in a *topological manifold* X equipped with the binary operation (3.2) satisfies all the group axioms. Such a group is called *first homotopy group* $\pi_1(X)$. In particular, $[\alpha^{-1}] \circ [\alpha] = [\alpha^{-1} \circ \alpha] = [I]$, with $[\alpha], [\alpha^{-1}], [I] \in \pi_1(X)$. Group axioms closure and associativity are trivially satisfied. It is obvious that, if X is a *simply connected manifold*, group $\pi_1(X)$ is trivial, since it consists of one element $[\alpha] = [I]$. Also, if X is a *non-simply connected manifold*, group $\pi_1(X)$ is non-trivial. For instance, let us consider $X = S^1$, then $\pi_1(X)$ consists of the *classes of homotopic loops* corresponding to *winding numbers* $n \in \mathbb{Z}$, usually denoted as $\pi_1(S^1) = \mathbb{Z}$.

A straightforward generalization of the 1-loops to the n-dimensional case provides the *higher rank homotopy group* $\pi_n(X)$. We define a n-loop in the *topological manifold* X as a continuous map of a sphere S^n into X. The set $[\phi_i]$ (*classes of homotopic n-loops*), where $\phi_i : \ S^n \to X$, equipped with a binary operation similar to (3.2), forms the group $\pi_n(X)$. Finally, it seems reasonable that, if we consider $X = S^n$, we obtain:

$$\pi_n(S^n) = \mathbb{Z} \tag{3.9}$$

3.2 Magnetic monopoles without strings and the quantization condition through homotopic loops

As we mention is section 1.3, the Dirac strings, along which the Dirac potential (1.80) is singular, are unobservable, i.e., they have no physical meaning. We can avoid the existence of the non-physical Dirac strings, and hence, we can prove that the Dirac quantization condition (2.5) has topological roots.

We can divide the manifold $\mathbb{R}^3 \setminus \{0\}$ (magnetic monopole at the origin) into two slightly overlapping hemispheres (north R^N and south R^S), where the vector potential reads:

$$\vec{A} = \begin{cases} \vec{A}_N = \frac{g}{r}\frac{1-cos(\theta)}{sin(\theta)}\hat{e}_\phi \ \ ,\theta \in \left[0, \frac{\pi}{2} + \varepsilon\right] \\ \vec{A}_S = -\frac{g}{r}\frac{1+cos(\theta)}{sin(\theta)}\hat{e}_\phi \ \ ,\theta \in \left[\frac{\pi}{2} - \varepsilon, \pi\right] \end{cases} \tag{3.10}$$

where we used the definitions (1.80) and (1.89). Note that the vector potential (3.10) has no Dirac strings, but it is not continuous at $\theta = \frac{\pi}{2}$. Hence, we consider the overlapping region $\theta \in [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$, $0 < \varepsilon \ll 1$, where the potential (3.10) takes either the one or the other form. Thus, the potential (3.10) corresponds to the magnetic monopole field everywhere in $\mathbb{R}^3 \setminus \{0\}$.

Note that the potentials $\vec{A}_N$ and $\vec{A}_S$ are connected with the gauge transformation $U(\phi) = e^{2ieg\phi}$, under which $\vec{A}_S \to \vec{A}_N$. Let us consider a particle moving around the overlapping region $R^N \cap R^S$. The monopole field in region $R^N \cap R^S$ is independent of the choice of the potential's form. Therefore, the wave-function $\psi_N = U(\phi)\psi_S = e^{2ieg\phi}\psi_S$ describes the system as well as wave-function ψ_S does. Also, the wave-function $\psi_N = e^{2ieg\phi}\psi_S$ must be single-valued, hence we obtain that:

$$eg = \frac{n}{2} \quad \text{with} \ \ n \in \mathbb{Z} \tag{3.11}$$

which is the Dirac quantization condition.

Let us consider the gauge transformation $U(\phi)$ as a map $U : \mathbb{R}^2 \to X$ via $U_n(\vec{r}) = e^{in\phi}$, where X is the set of unimodular complex numbers and $\mathbb{R}^2 \subset \mathbb{R}^3$, for $\theta = \frac{\pi}{2}$. Note that $U_n(\vec{r})$ is mapping a loop in $\mathbb{R}^2$ around the magnetic monopole to a loop in X, which belongs to a *homotopy class* with *winding number* n. Hence, the charge quantization and the quantum number "n" have topological roots. The loops in $\mathbb{R}^2$ need to be around the magnetic monopole in order for the gauge transformation $U_n(\vec{r})$, with $n \neq 0$, to be symmetry of the system. Therefore, the loops in $\mathbb{R}^2$ are not *homotopic* with the unit element. Thus, we conclude that manifold $\mathbb{R}^2$ is *non-simply connected* and that the magnetic monopole is a topological singularity. That is why, in the beginning of our discussion, we considered the manifold $\mathbb{R}^3 \setminus \{0\}$.

This aspect of magnetic monopole theories is the key for many models, such as the models that we discuss in chapters 4, 5 and 6. In these theories, we consider a Higgs triplet as a 2-loop from $\mathbb{R}^3 \setminus \{0\}$ to S^2. The solution of the equations of motion with *winding number* $n = 0$, corresponds to normal systems without magnetic monopoles. On the contrary, the solutions with *winding number* $n \neq 0$ correspond to models with magnetic or global monopoles. These results are compatible with the Dirac quantization condition (3.11). The global monopoles are presented in chapter 5.

Chapter 4

’t Hooft-Polyakov Monopole

As we describe in chapter 3, we may construct monopole models by considering field theories with an extra contribution of a Higgs triplet, which may be stabilized due to a self-interacting potential. In particular, we consider the Georgi-Glashow model, which is a global $SU(2)$ gauge theory, which includes a Higgs scalar triplet and an E/M field. The Higgs triplet may be considered as an n-loop on the spherical surface with its radius being the stabilized magnitude of the triplet. If the n-loop belongs to a *homotopy class* with *winding number* $n \neq 0$, the corresponding solutions have monopole properties, i.e., the solution for the E/M field is a magnetic monopole solution (Coulomb-like field). The corresponding magnetic monopole is called ’t Hooft-Polyakov monopole [23].

4.1 Georgi-Glashow model

The Georgi-Glashow model corresponds to a non-abelian gauge theory, $SU(2)$ invariant, in the adjoint representation. The Lagrangian density of the model reads:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \frac{1}{2} (\underline{\underline{D}}_\mu \underline{\underline{\Phi}})^a (\underline{\underline{D}}^\mu \underline{\underline{\Phi}})^a - V(\underline{\underline{\Phi}}^T \underline{\underline{\Phi}}) \tag{4.1}$$

where $a = 1,\ 2,\ 3$ and $\underline{\underline{\Phi}}$ a scalar Higgs triplet:

$$\underline{\underline{\Phi}} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \tag{4.2}$$

The Higgs triplet (4.2) is scalar, since the $SU(2)$ transformations in the adjoint representation are real (Appendix C). Also, $\underline{\underline{D}}_\mu$ is the covariant derivative:

$$\underline{\underline{D}}_\mu = \underline{\underline{\mathbb{I}}} \partial_\mu - ieA^a_\mu \underline{\underline{T}}^a \tag{4.3}$$

where A^a_μ are the gauge $SU(2)$ bosons and $\underline{\underline{T}}^a$ are the generators of the $SU(2)$ group. The double-line is used to declare matrices, to avoid a heavy indices notation. Additionally, the tensor $F^a_{\mu\nu}$ reads:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\epsilon_{abc} A^b_\mu A^c_\nu \tag{4.4}$$

Note that the English indices a, b, c denote vector components in a 3-dimensional euclidean space, therefore, it makes no difference whether we have upper or lower English indices. Also, ϵ_{abc} is the Levi-Civita symbol with $\epsilon_{123} = 1$. Additionally, $V(\underline{\underline{\Phi}}^T \underline{\underline{\Phi}})$ is the Higgs potential:

$$V(\underline{\underline{\Phi}}^T \underline{\underline{\Phi}}) = \frac{\lambda}{4} (\underline{\underline{\Phi}}^T \underline{\underline{\Phi}} - \eta^2)^2 = \frac{\lambda}{4} (\Phi^a \Phi^a - \eta^2)^2 \tag{4.5}$$

Due to the fact that we work in the adjoint representation of the $SU(2)$ group, we can write the Lagrangian density (4.1) in a more convenient form, in order to examine the $SU(2)$ invariance. The Lie algebra of the $SU(2)$ group reads:

$$[\underline{\underline{T}}^a, \underline{\underline{T}}^b] = i\ \epsilon_{abc} \underline{\underline{T}}^c \tag{4.6}$$

As we mention in Appendix C, the adjoint representation of the generators of a Lie group corresponds to the structure constants of the Lie algebra:

$$(T^a_{adj})^{bc} = -i\,\epsilon_{abc} \tag{4.7}$$

Thus, the covariant derivative (4.3) reads:

$$(D_\mu)^{ab} = \delta^{ab}\partial_\mu - e\epsilon_{abc}A^c_\mu \tag{4.8}$$

Then, the covariant derivative of the Higgs triplet reads:

$$(\underline{\underline{D}}_\mu\underline{\underline{\Phi}})^a = \partial_\mu\Phi^a + e\epsilon_{abc}A^b_\mu\Phi^c \Rightarrow \tag{4.9}$$

$$(\underline{\underline{D}}_\mu\underline{\underline{\Phi}})^a\underline{\underline{T}}^a = \partial_\mu\Phi^a\underline{\underline{T}}^a + e\underline{\underline{T}}^a\epsilon_{abc}A^b_\mu\Phi^c$$

The equation (4.6) yields $\epsilon_{abc}\underline{\underline{T}}^a = -i[\underline{\underline{T}}^b, \underline{\underline{T}}^c]$. Hence we have:

$$(\underline{\underline{D}}_\mu\underline{\underline{\Phi}})^a\underline{\underline{T}}^a = \partial_\mu\Phi^a\underline{\underline{T}}^a - ie[\underline{\underline{T}}^b, \underline{\underline{T}}^c]A^b_\mu\Phi^c \Rightarrow$$

$$D_\mu\Phi = \partial_\mu\Phi - ie[A_\mu, \Phi] \tag{4.10}$$

where

$$D_\mu\Phi = (\underline{\underline{D}}_\mu\underline{\underline{\Phi}})^a\underline{\underline{T}}^a\ , \quad \Phi = \Phi^a\underline{\underline{T}}^a \text{ and } A_\mu = A^a_\mu\underline{\underline{T}}^a \tag{4.11}$$

Also, we define the tensor $F_{\mu\nu}$ as follows:

$$F_{\mu\nu} = \underline{\underline{T}}^a F^a_{\mu\nu} \tag{4.12}$$

To sum up, the Lagrangian (4.1) in the $SU(2)$ adjoint representation can be written as follows:

$$\mathcal{L} = -\frac{1}{2}Tr\Big(F_{\mu\nu}F^{\mu\nu}\Big) + Tr\Big((D_\mu\Phi)(D^\mu\Phi)\Big) - V(\underline{\underline{\Phi}}^T\underline{\underline{\Phi}}) \tag{4.13}$$

In the next few lines, considering that the generators of the $SU(2)$ group, on the fundamental representation, satisfy the equation $Tr\big(\underline{\underline{T}}^a\underline{\underline{T}}^b\big) = \frac{\delta_{ab}}{2}$, we prove that the equation (4.13) yields the equation (4.1). Note that the generators in the adjoint representation satisfy $Tr\big(\underline{\underline{T}}^a\underline{\underline{T}}^b\big) = 2\delta_{ab}$. This means that we need to change some coefficients in (4.13), in order to correspond to (4.1) in the adjoint representation. However, this is not necessary, since the Lagrangian density (4.13) will be used only to examine the $SU(2)$ invariance of the model. Thus, the equation (4.13) yields:

$$\mathcal{L} = -\frac{1}{2}Tr\Big(\underline{\underline{T}}^a F^a_{\mu\nu}\underline{\underline{T}}^b F^{b\ \mu\nu}\Big) + Tr\Big((\underline{\underline{D}}_\mu\underline{\underline{\Phi}})^a\underline{\underline{T}}^a(\underline{\underline{D}}^\mu\underline{\underline{\Phi}})^b\underline{\underline{T}}^b\Big) - V(\underline{\underline{\Phi}}^T\underline{\underline{\Phi}}) =$$

$$-\frac{1}{2}F^a_{\mu\nu}F^{b\ \mu\nu}Tr\Big(\underline{\underline{T}}^a\underline{\underline{T}}^b\Big) + (\underline{\underline{D}}_\mu\underline{\underline{\Phi}})^a(\underline{\underline{D}}^\mu\underline{\underline{\Phi}})^b Tr\Big(\underline{\underline{T}}^a\underline{\underline{T}}^b\Big) - V(\underline{\underline{\Phi}}^T\underline{\underline{\Phi}}) \Rightarrow$$

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\ \mu\nu} + \frac{1}{2}(\underline{\underline{D}}_\mu\underline{\underline{\Phi}})^a(\underline{\underline{D}}^\mu\underline{\underline{\Phi}})^a - V(\underline{\underline{\Phi}}^T\underline{\underline{\Phi}})$$

which is the Lagrangian density (4.1).

4.1.1 SU(2) symmetry of the Georgi–Glashow Model

It is essential to explicitly prove the $SU(2)$ invariance of the model. It is more efficient to use the Lagrangian density (4.13).

First of all, we need to determine how the components of the model are transformed under the transformations of the $SU(2)$ adjoint representation. A transformation of the Lie group, $\underline{\underline{U}} \in SU(2)$, reads:

$$\underline{\underline{U}} = e^{ie\theta^a T^a},\ \underline{\underline{U}} = \mathbb{I} + ie\delta\theta^a\underline{\underline{T}}^a \text{ and } \underline{\underline{U}}^{-1} = \underline{\underline{U}}^\dagger = e^{-ie\theta^a T^a},\ \underline{\underline{U}}^{-1} = \underline{\underline{U}}^\dagger = \mathbb{I} - ie\delta\theta^a\underline{\underline{T}}^a \text{ for } \delta\theta^a \ll 1 \tag{4.14}$$

The Higgs triplet is transformed as follows:

$$\underline{\underline{\Phi}} \longrightarrow \underline{\underline{\Phi}}' = \underline{\underline{U}}\,\underline{\underline{\Phi}} \tag{4.15}$$

As usual we demand:

$$\underline{\underline{D}}_\mu\underline{\underline{\Phi}} \longrightarrow \underline{\underline{D}}'_\mu\underline{\underline{\Phi}}' = \underline{\underline{U}}\,\underline{\underline{D}}_\mu\underline{\underline{\Phi}} \Rightarrow$$

$$(\underline{\underline{\mathbb{I}}}\partial_\mu - ieA'^a_\mu\underline{\underline{T}}^a)\underline{\underline{U}}\,\underline{\underline{\Phi}} = \underline{\underline{U}}\,(\underline{\underline{\mathbb{I}}}\partial_\mu - ieA^a_\mu\underline{\underline{T}}^a)\underline{\underline{\Phi}} \Rightarrow$$

$$(\partial_\mu\underline{\underline{U}})\underline{\underline{\Phi}} - ieA'^a_\mu\underline{\underline{T}}^a\,\underline{\underline{U}}\,\underline{\underline{\Phi}} = -ieA^a_\mu\underline{\underline{U}}\,\underline{\underline{T}}^a\,\underline{\underline{\Phi}} \Rightarrow (\partial_\mu\underline{\underline{U}}) - ieA'^a_\mu\underline{\underline{T}}^a\,\underline{\underline{U}} = -ieA^a_\mu\underline{\underline{U}}\,\underline{\underline{T}}^a \Rightarrow$$

$$A'_\mu = A'^a_\mu\underline{\underline{T}}^a = \underline{\underline{U}}A_\mu\underline{\underline{U}}^{-1} - \frac{i}{e}(\partial_\mu\underline{\underline{U}})\underline{\underline{U}}^{-1} \tag{4.16}$$

which also means that:

$$\underline{\underline{D}}_\mu \longrightarrow \underline{\underline{D}}'_\mu = \underline{\underline{U}}\,\underline{\underline{D}}_\mu\,\underline{\underline{U}}^{-1} \tag{4.17}$$

Moreover, we can calculate the transformation of Φ (4.11):

$$\Phi \longrightarrow \Phi' = \Phi'^m\underline{\underline{T}}^m \Rightarrow$$

$$(\Phi')^{bc} = (\delta_{mn} + ie\delta\theta^a(T^a_{adj})^{mn}\Phi^n(T^m_{adj})^{bc} =$$

$$(-i\,\epsilon_{nbc} - ie\delta\theta^a\epsilon_{amn}\epsilon_{mbc})\Phi^n = \Big(-i\,\epsilon_{n\rho\sigma}\delta_{b\rho}\delta_{c\sigma} + ie\delta\theta^a(\delta_{ab}\delta_{nc} - \delta_{ac}\delta_{nb})\Big)\Phi^n =$$

$$\Big(-i\,\epsilon_{n\rho\sigma}\delta_{b\rho}\delta_{c\sigma} + ie\delta\theta^a(\delta_{ab}\delta_{nc} - \delta_{an}\delta_{bc} + \delta_{an}\delta_{bc} - \delta_{ac}\delta_{nb})\Big)\Phi^n =$$

$$\Big(-i\,\epsilon_{n\rho\sigma}\delta_{b\rho}\delta_{c\sigma} + ie\delta\theta^a\epsilon_{\sigma ac}\epsilon_{\sigma bn} + ie\delta\theta^a\epsilon_{\rho ab}\epsilon_{\rho nc}\Big)\Phi^n =$$

$$\Big(-i\,\epsilon_{n\rho\sigma}\delta_{b\rho}\delta_{c\sigma} + ie\delta\theta^a\epsilon_{a\sigma c}\epsilon_{n\rho\sigma}\delta_{\rho b} - ie\delta\theta^a\epsilon_{\rho ab}\epsilon_{n\rho\sigma}\delta_{c\sigma} + O(\delta\theta^2)\Big)\Phi^n =$$

$$(\delta_{b\rho} + e\delta\theta^a\epsilon_{ab\rho})(-i\,\epsilon_{n\rho\sigma})\Phi^n(\delta_{\sigma c} - e\delta\theta^a\epsilon_{a\sigma c}) \Rightarrow$$

$$(\Phi')^{bc} = (\underline{\underline{\mathbb{I}}} + ie\delta\theta^a\underline{\underline{T}}^a)^{b\rho}(\Phi^n\underline{\underline{T}}^n)^{\rho\sigma}(\underline{\underline{\mathbb{I}}} - ie\delta\theta^a\underline{\underline{T}}^a)^{\sigma c} \Rightarrow$$

$$\Phi \longrightarrow \Phi' = \underline{\underline{U}}\Phi\underline{\underline{U}}^{-1} \tag{4.18}$$

Considering equations (4.17) and (4.18), it is obvious that:

$$D_\mu\Phi \longrightarrow (D_\mu\Phi)' = \underline{\underline{U}}D_\mu\Phi\underline{\underline{U}}^{-1} \tag{4.19}$$

Furthermore, in order to determine the transformation of $F^a_{\mu\nu}$, we need to calculate $[\underline{\underline{D}}_\mu, \underline{\underline{D}}_\nu]\underline{\underline{\Phi}}$:

$$[\underline{\underline{D}}_\mu, \underline{\underline{D}}_\nu]\underline{\underline{\Phi}} = \underline{\underline{D}}_\mu\underline{\underline{D}}_\nu\underline{\underline{\Phi}} - \underline{\underline{D}}_\nu\underline{\underline{D}}_\mu\underline{\underline{\Phi}} =$$

$$(\underline{\underline{\mathbb{I}}}\partial_\mu - ieA^a_\mu\underline{\underline{T}}^a)(\underline{\underline{\mathbb{I}}}\partial_\nu - ieA^b_\nu\underline{\underline{T}}^b)\underline{\underline{\Phi}} - \mu \leftrightarrow \nu \;\; a \leftrightarrow b =$$

$$-ie\underline{\underline{T}}^b(\partial_\mu A^b_\nu)\underline{\underline{\Phi}} - e^2\underline{\underline{T}}^a\underline{\underline{T}}^b A^a_\mu A^b_\nu\underline{\underline{\Phi}} + ie\underline{\underline{T}}^a(\partial_\nu A^a_\mu)\underline{\underline{\Phi}} + e^2\underline{\underline{T}}^b\underline{\underline{T}}^a A^b_\nu A^a_\mu\underline{\underline{\Phi}} =$$

$$-ie(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)\underline{\underline{T}}^a\underline{\underline{\Phi}} - e^2 A^b_\mu A^c_\nu[\underline{\underline{T}}^b, \underline{\underline{T}}^c]\underline{\underline{\Phi}} = -ie(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\,\epsilon_{abc}A^b_\mu A^c_\nu)\underline{\underline{T}}^a\underline{\underline{\Phi}} \Rightarrow$$

$$F_{\mu\nu} \equiv F^a_{\mu\nu}\underline{\underline{T}}^a = \frac{i}{e}\big[\,\underline{\underline{D}}_\mu, \underline{\underline{D}}_\nu\big] \tag{4.20}$$

Consequently, equations (4.17) and (4.20) yield:

$$F_{\mu\nu} \longrightarrow F'_{\mu\nu} = \underline{\underline{U}}F_{\mu\nu}\underline{\underline{U}}^{-1} \tag{4.21}$$

In order to conclude, considering equations (4.15), (4.19) and (4.21), we examine the $SU(2)$ invariance of the Lagrangian density (4.13):

$$\mathcal{L} \longrightarrow \mathcal{L}' = -\frac{1}{2}Tr\Big(F'_{\mu\nu}F'^{\mu\nu}\Big) + Tr\Big((D_\mu\Phi)'(D^\mu\Phi)'\Big) - V(\underline{\underline{\Phi}}'^T\underline{\underline{\Phi}}') \Rightarrow$$

$$\mathcal{L}' = -\frac{1}{2}Tr\Big(\underline{\underline{U}}F_{\mu\nu}F^{\mu\nu}\underline{\underline{U}}^{-1}\Big) + Tr\Big(\underline{\underline{U}}(D_\mu\Phi)(D^\mu\Phi)\underline{\underline{U}}^{-1}\Big) - V(\underline{\underline{\Phi}}^T\underline{\underline{U}}^T\underline{\underline{U}}\,\underline{\underline{\Phi}}') \Rightarrow$$

$$\mathcal{L}' = \mathcal{L}$$

where $\underline{\underline{U}}^T = \underline{\underline{U}}^\dagger = \underline{\underline{U}}^{-1}$, since the $SU(2)$ transformations in the adjoint representation are real. Consequently, the Georgi-Glashow model is $SU(2)$ invariant.

4.1.2 Equations of motion

We calculate the Euler-Lagrange equations of motion for the fields of the Georgi-Glashow model, using the Lagrangian density (4.1). The equations of motion for the Higgs triplet read:

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi^a} - \frac{\partial\mathcal{L}}{\partial\Phi^a} = 0 \tag{4.22}$$

We calculate:

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi^a} = \partial_\mu(\underline{\underline{D}}^\mu\underline{\underline{\Phi}})^a$$

where we used equation (4.9), and

$$\frac{\partial\mathcal{L}}{\partial\Phi^a} = -e\,\epsilon_{abc}A^b_\nu(\underline{\underline{D}}^\nu\underline{\underline{\Phi}})^c - \lambda\Phi^a(\phi^b\phi^b - \eta^2)$$

Therefore, equation (4.22) yields:

$$\partial_\mu(\underline{\underline{D}}^\mu\underline{\underline{\Phi}})^a + e\,\epsilon_{abc}A^b_\nu(\underline{\underline{D}}^\nu\underline{\underline{\Phi}})^c + \lambda\Phi^a(\phi^b\phi^b - \eta^2) = 0 \Rightarrow$$

$$(\underline{\underline{D}}_\mu\underline{\underline{D}}^\mu\underline{\underline{\Phi}})^a = -\lambda\Phi^a(\phi^b\phi^b - \eta^2) \tag{4.23}$$

Subsequently, the equations of motion for the gauge fields read:

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu A^a_\nu} - \frac{\partial\mathcal{L}}{\partial A^a_\nu} = 0 \tag{4.24}$$

We calculate:

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu A^a_\nu} = -\partial_\mu F^{a\;\mu\nu}$$

and

$$\frac{\partial\mathcal{L}}{\partial A^a_\nu} = -\frac{1}{2}F^{b\;\rho\sigma}\frac{\partial}{\partial A^a_\nu}(e\,\epsilon_{bcd}A^c_\rho A^d_\sigma) + (\underline{\underline{D}}^\mu\underline{\underline{\Phi}})^b\frac{\partial}{\partial A^a_\nu}(e\,\epsilon_{bcd}A^c_\mu\Phi^d) =$$

$$-\frac{e}{2}\epsilon_{bcd}F^{b\;\rho\sigma}(\delta_{ca}\delta^\nu_\rho A^d_\sigma + \delta_{ad}\delta^\nu_\sigma A^c_\rho) + (\underline{\underline{D}}^\mu\underline{\underline{\Phi}})^b e\,\epsilon_{bcd}\delta_{ac}\delta^\nu_\mu\Phi^d =$$

$$-\frac{e}{2}(\epsilon_{bad}F^{b\;\nu\sigma}A^d_\sigma + \epsilon_{bca}F^{b\;\rho\nu}A^c_\rho) + e(\underline{\underline{D}}^\nu\underline{\underline{\Phi}})^b\epsilon_{bad}\Phi^d \Rightarrow$$

$$\frac{\partial\mathcal{L}}{\partial A^a_\nu} = -e\epsilon_{bad}F^{b\;\nu\sigma}A^d_\sigma + e(\underline{\underline{D}}^\nu\underline{\underline{\Phi}})^b\epsilon_{bad}\Phi^d$$

Therefore, equation (4.24) yields:

$$-\partial_\mu F^{a\;\mu\nu} + e\epsilon_{bad}F^{b\;\nu\sigma}A^d_\sigma - e(\underline{\underline{D}}^\nu\underline{\underline{\Phi}})^b\epsilon_{bad}\Phi^d = 0 \overset{\mu\leftrightarrow\nu}{\Longrightarrow}$$

$$\partial_\nu F^{a\;\mu\nu} + e\epsilon_{adb}A^d_\sigma F^{b\;\mu\sigma} - e(\underline{\underline{D}}^\mu\underline{\underline{\Phi}})^b\epsilon_{bad}\Phi^d = 0 \Rightarrow$$

$$D_\nu F^{a\;\mu\nu} = e\,\epsilon_{abc}\Phi^b(\underline{\underline{D}}^\mu\underline{\underline{\Phi}})^c \tag{4.25}$$

4.1.3 Energy of the system

We consider flat space-time with the convention $\eta_{\mu\nu} = diag(1, -1, -1, -1)$. Then, the energy density of the system is described by the component T^{00} of the stress–energy tensor, which reads:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \tag{4.26}$$

where $g_{\mu\nu} = \eta_{\mu\nu}$, g is the determinant of the metric tensor and S_M is the action corresponding to matter. In our case $S_M = \int d^4x\sqrt{-g}\mathcal{L}$.

$$\delta S_M = \int d^4x\delta(\sqrt{-g}\mathcal{L}) = \int d^4x\big(\delta(\sqrt{-g})\mathcal{L} + \sqrt{-g}\delta\mathcal{L}\big) \tag{4.27}$$

The variation of the determinant reads:

$$\delta\sqrt{-g} = -\sqrt{-g}\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu} \tag{4.28}$$

Also, we need to calculate the variation of the Lagrangian density (4.1)

$$\delta(F^a_{\mu\nu}F^{a\,\mu\nu}) = \delta(g^{\mu\kappa}g^{\nu\lambda}F^a_{\mu\nu}F^a_{\kappa\lambda}) = F^a_{\mu\nu}F^a_{\kappa\lambda}(\delta g^{\mu\kappa}g^{\nu\lambda} + g^{\mu\kappa}\delta g^{\nu\lambda}) = (F^a_{\mu\kappa}F^a_{\nu\lambda}g^{\kappa\lambda} + F^a_{\lambda\nu}F^a_{\kappa\mu}g^{\lambda\kappa})\delta g^{\mu\nu} \Rightarrow$$

$$\delta(F^a_{\mu\nu}F^{a\,\mu\nu}) = 2F^a_{\mu\kappa}F^a_{\nu\lambda}g^{\kappa\lambda}\delta g^{\mu\nu} \tag{4.29}$$

and

$$\delta\big((\underline{\underline{D}}_\mu\underline{\underline{\Phi}})^a(\underline{\underline{D}}^\mu\underline{\underline{\Phi}})^a\big) = (\underline{\underline{D}}_\mu\underline{\underline{\Phi}})^a(\underline{\underline{D}}_\nu\underline{\underline{\Phi}})^a\delta g^{\mu\nu} \tag{4.30}$$

Upon summarizing the equations (4.27), (4.28), (4.29) and (4.30), we obtain:

$$\delta S_M = \int d^4\,x\delta(\sqrt{-g}\mathcal{L}) = \int d^4x\,\frac{\sqrt{-g}}{2}\big(-g_{\mu\nu}\mathcal{L} - F^a_{\mu\kappa}F^a_{\nu\lambda}g^{\kappa\lambda} + (\underline{\underline{D}}_\mu\underline{\underline{\Phi}})^a(\underline{\underline{D}}_\nu\underline{\underline{\Phi}})^a\big)\delta g^{\mu\nu} \tag{4.31}$$

We substitute equation (4.31) into equation (4.26) and we obtain:

$$T_{\mu\nu} = -g_{\mu\nu}\mathcal{L} - F^a_{\mu\lambda}F^{a\ \lambda}_{\ \nu} + (\underline{\underline{D}}_\mu\underline{\underline{\Phi}})^a(\underline{\underline{D}}_\nu\underline{\underline{\Phi}})^a \tag{4.32}$$

Subsequently, we calculate the total energy of the system:

$$E = \int d^3x\;T^{00} \tag{4.33}$$

Note that $T^{00} = T_{00} = T^0_0$ for $g_{\mu\nu} = \eta_{\mu\nu} = diag(1, -1, -1, -1)$.

$$E = \int d^3x\,\big(-\mathcal{L} - F^a_{0\lambda}F^{a\ \lambda}_{\ 0} + (\underline{\underline{D}}_0\,\underline{\underline{\Phi}})^a(\underline{\underline{D}}_0\,\underline{\underline{\Phi}})^a\big) =$$

$$\int d^3x\,\Big(\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} - \frac{1}{2}(\underline{\underline{D}}_\mu\underline{\underline{\Phi}})^a(\underline{\underline{D}}^\mu\underline{\underline{\Phi}})^a + V(\underline{\underline{\Phi}}^T\underline{\underline{\Phi}}) - F^a_{0i}F^{a\ i}_{\ 0} + (\underline{\underline{D}}_0\,\underline{\underline{\Phi}})^a(\underline{\underline{D}}_0\,\underline{\underline{\Phi}})^a\Big) =$$

$$\int d^3x\,\Big(\frac{1}{4}(F^a_{0i}F^{a\,0i} + F^a_{i0}F^{a\,i0} + F^a_{ij}F^{a\,ij}) - \frac{1}{2}(\underline{\underline{D}}_0\,\underline{\underline{\Phi}})^a(\underline{\underline{D}}_0\,\underline{\underline{\Phi}})^a + \frac{1}{2}(\underline{\underline{D}}_i\,\underline{\underline{\Phi}})^a(\underline{\underline{D}}_i\,\underline{\underline{\Phi}})^a + V(\underline{\underline{\Phi}}^T\underline{\underline{\Phi}})$$

$$-F^a_{0i}F^{a\,0i} + (\underline{\underline{D}}_0\,\underline{\underline{\Phi}})^a(\underline{\underline{D}}_0\,\underline{\underline{\Phi}})^a\Big) \Rightarrow$$

$$E = \int d^3x\,\Big(-\frac{1}{2}F^a_{0i}F^{a\,0i} + \frac{1}{4}F^a_{ij}F^{a\,ij} + \frac{1}{2}(\underline{\underline{D}}_0\,\underline{\underline{\Phi}})^a(\underline{\underline{D}}_0\,\underline{\underline{\Phi}})^a + \frac{1}{2}(\underline{\underline{D}}_i\,\underline{\underline{\Phi}})^a(\underline{\underline{D}}_i\,\underline{\underline{\Phi}})^a + V(\underline{\underline{\Phi}}^T\underline{\underline{\Phi}})\Big)$$

where $i, j = 1, 2, 3$. Note that:

$$F^a_{0i}F^{a\,0i} = -\vec{E}^a\cdot\vec{E}^a \tag{4.34}$$

and

$$B_i^a = \frac{1}{2}\epsilon_{ijk}F^{a\,jk} \;\Rightarrow\; \vec{B}^a \cdot \vec{B}^a = \frac{1}{2}F_{ij}^a F^{a\,ij} \tag{4.35}$$

where $i,j = 1,2,3$. Considering equations (4.34) and (4.35), the total energy reads:

$$E = \int d^3x \left(\frac{1}{2}\vec{E}^a\vec{E}^a + \frac{1}{2}\vec{B}^a\vec{B}^a + \frac{1}{2}(\underline{\underline{D}}_0\,\underline{\underline{\Phi}})^a(\underline{\underline{D}}_0\,\underline{\underline{\Phi}})^a + \frac{1}{2}(\underline{\underline{D}}_i\,\underline{\underline{\Phi}})^a(\underline{\underline{D}}_i\,\underline{\underline{\Phi}})^a + V(\underline{\underline{\Phi}}^T\underline{\underline{\Phi}})\right) \tag{4.36}$$

Let us consider a static system $\frac{\partial}{\partial t} = 0$. The Stress-energy tensor is gauge invariant, hence the total energy is gauge invariant. Thus, we fix the gauge $A_0^a = 0$. Then, the total energy reads:

$$E = \int d^3x \left(\frac{1}{2}\vec{B}^a\vec{B}^a + \frac{1}{2}(\underline{\underline{D}}_i\,\underline{\underline{\Phi}})^a(\underline{\underline{D}}_i\,\underline{\underline{\Phi}})^a + V(\underline{\underline{\Phi}}^T\underline{\underline{\Phi}})\right) \tag{4.37}$$

Consequently, the total energy (4.37) becomes minimum when:

$$(\underline{\underline{D}}_i\,\underline{\underline{\Phi}})^a = 0, \quad F_{ij}^a = 0 \text{ and } \Phi^a\Phi^a = \eta^2 \tag{4.38}$$

4.1.4 Spontaneous symmetry breaking

The total energy becomes minimum in the case $<0|\Phi^a\Phi^a|0> = \eta^2$. Without loss of generality we can choose:

$$<0|\underline{\underline{\Phi}}|0> = \begin{pmatrix} 0 \\ 0 \\ \eta \end{pmatrix} \tag{4.39}$$

Also, we can consider the Higgs triplet as variations around the minimum (4.39):

$$\underline{\underline{\Phi}} = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \eta + \xi \end{pmatrix} \tag{4.40}$$

such that:

$$<0|\chi_1|0> = <0|\chi_2|0> = <0|\xi|0> = 0 \tag{4.41}$$

The vacuum state $|0>$ must stay invariant under the symmetry transformations, i.e. the generators of the symmetry must act on $|0>$ as $T^a|0> = 0$. The generators of the $SU(2)$ group in the adjoint representation read:

$$T_{adj}^1 = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad T_{adj}^2 = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_{adj}^3 = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.42}$$

Note that:

$$T_{adj}^{1,2} <0|\underline{\underline{\Phi}}|0> \neq 0 \quad \text{and} \quad T_{adj}^3 <0|\underline{\underline{\Phi}}|0> = 0 \tag{4.43}$$

which means that two generators break the symmetry. This is called spontaneous symmetry breaking. Also, we have a remaining $U(1)$ symmetry with generator T^3. Thus, we have the following symmetry breaking pattern:

$$SU(2) \longrightarrow U(1)$$

$$3 \text{ generators} \longrightarrow 1 \text{ generator} \tag{4.44}$$

From the Goldstone theorem it follows that there are two massless Goldstone bosons corresponding to the fields χ_1 and χ_2, since $<0|\chi_1|0> = <0|\chi_2|0> = 0$. Also, there is a massive Higgs boson corresponds to the field $\phi(x) = \eta + \xi(x)$, since $<0|\phi(x)|0> = \eta \neq 0$.

Let us count the degrees of freedom of the model. Before the spontaneous symmetry breaking, we have three massless $SU(2)$ gauge bosons with two degrees of freedom each (4 − 1 from the equations of motion − 1 from the gauge symmetry) and three real scalars with one degree of freedom each, i.e., nine degrees of freedom in total. After spontaneous symmetry breaking, the three vector fields have three degrees of freedom each, since the gauge symmetry is broken, thus we have twelve degrees of freedom in total. This inconsistency is regularized with a proper gauge fixing. We can choose a gauge such that the two Goldstone bosons are eliminated and two of the gauge bosons, which correspond to generators that break the symmetry, become massive. Hence, the degrees of freedom after the spontaneous symmetry breaking remain nine. The gauge fixing yields:

$$\underline{\underline{\Phi}} = \begin{pmatrix} 0 \\ 0 \\ \eta + \xi \end{pmatrix} \tag{4.45}$$

We can explicitly determine the masses of the fields from the square terms of the Lagrangian after the symmetry breaking.

$$\underline{\underline{D}}_{\mu}\underline{\underline{\Phi}} = -ieA^a_\mu \underline{\underline{T}}^a \begin{pmatrix} 0 \\ 0 \\ \eta \end{pmatrix} + ... = -\eta e \begin{pmatrix} 0 & A^3_\mu & -A^2_\mu \\ -A^3_\mu & 0 & A^1_\mu \\ A^2_\mu & -A^1_\mu & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + ... = \eta e \begin{pmatrix} A^2_\mu \\ -A^1_\mu \\ 0 \end{pmatrix} + ... \tag{4.46}$$

and

$$V(\underline{\underline{\Phi}}) = \frac{\lambda}{4}\Big((\eta+\xi)^2 - \eta^2\Big)^2 = \frac{\lambda}{4}\Big(2\eta\xi + \xi^2\Big)^2 + ... = \lambda\eta^2\xi^2 + ... \tag{4.47}$$

substituting equations (4.46) and (4.47) into the Lagrangian density (4.1), we obtain:

$$\mathcal{L} = \frac{1}{2}e^2\eta^2 A^1_\mu A^{1\mu} + \frac{1}{2}e^2\eta^2 A^2_\mu A^{2\mu} - \lambda\eta^2\xi^2 + ... \tag{4.48}$$

From equation (4.48) we identify the masses of the fields:

$$M_{A^1} = M_{A^2} = e\eta, \quad M_{A^3} = 0 \text{ and } M_{Higgs} = \sqrt{2\lambda}\eta \tag{4.49}$$

The gauge boson A^3_μ, corresponding to the remaining symmetry $U(1)$, remains massless, after spontaneous symmetry breaking. Therefore, the boson A^3_μ plays the role of the electromagnetic potential.

The operator of the conserved electric charge is the generator of the remaining $U(1)$ symmetry:

$$\underline{\underline{D}}_{\mu} = -ieA^3_\mu T^3 \;\Rightarrow\; Q = eT^3 \tag{4.50}$$

Note that:

$$Q\begin{pmatrix} 0 \\ 0 \\ \phi_{Higgs} \end{pmatrix} = 0 \;\Rightarrow\; Q_{Higgs} = 0 \tag{4.51}$$

It is interesting that the gauge bosons A^1_μ and A^2_μ are not eigenstates of the electric charge Q. However, a linear combination of A^1_μ and A^2_μ is eigenstate of the electric charge. To be more precise, let us set:

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \mp iA^2_\mu) \text{ and } T^\pm = \frac{1}{\sqrt{2}}(T^1 \mp iT^2) \tag{4.52}$$

where:

$$[T^3, T^\pm] = \mp T^\pm \tag{4.53}$$

Then, we obtain:

$$\underline{\underline{D}}_{\mu} = -ieW^+_\mu T^- - ieW^-_\mu T^+ - ieA^3_\mu T^3 + ... \tag{4.54}$$

Consequently, we have:

$$|A^3> = A^3_\mu T^3|0> \;\Rightarrow\; Q|A^3> = 0 \;\Rightarrow\; Q_{A^3} = 0 \tag{4.55}$$

and

$$|W^\pm> = W^\pm_\mu T^\mp|0> \;\Rightarrow\; Q|W^\pm> = \pm e|W^\pm> \;\Rightarrow\; Q_{W^\pm} = \pm e \tag{4.56}$$

where we used the equation: $T^3|0>=0$. Note that the electromagnetic field is chargeless, as it should be.

4.2 Monopole solution of the Georgi-Glashow model

The classical vacuum of the Georgi-Glashow model is degenerated, since $V(\Phi) = 0 \Rightarrow |\Phi| = \eta$, i.e., the set of vacuum values of the Higgs field forms a sphere S^2_{vac} of radius η in a 3-dimensional isotopic space. All points of the sphere are equivalent, because there is a well-defined $SU(2)$ gauge transformation that connects them.

In section 4.1 we consider the Higgs field as a map $\Phi : \mathbb{R}^3 \longrightarrow S^2_{vac}$ with *winding number* $n = 0$, since we consider $\Phi =$ *constant*. Therefore, there are no magnetic charges. In this section, we search for magnetic monopole solutions [23].

4.2.1 Topological classification of the solutions

Let us consider solutions with the property $|\Phi(r \to \infty)| = \eta$. In particular, we search for a mapping $\Phi : S^2_\infty \to S^2_{vac}$, where S^2_∞ is the boundary of the 3-dimensional euclidean space. This map is characterized by a *winding number* $\pi_2(S^2) = \mathbb{Z}$. We may consider a solution with *winding number* $n = 1$, which asymptotically behaves as follows:

$$\Phi^a(r \to \infty) \to \eta \frac{r^a}{r} \tag{4.57}$$

It is obvious that $|\Phi(r \to \infty)| = \eta$. Note that, since $\vec{\Phi}(r \to \infty) = \eta \hat{r}$, a single turn around S^2_∞ corresponds to a single close path (2-loop) in S^2_{vac}, which confirms that $n = 1$.

It is very important that the state (4.57) cannot collapse to the global minimum state $\Phi = (0, 0, \eta)$, since these two states belong to different classes, $n = 1$ and $n = 0$ respectively, and this results in the energy becoming infinite if we try to continuously deform the one into the other.

We may determine the form of the gauge bosons by considering minimum total energy, with the constraint $n = 1$, which means that $D_i\Phi = 0$ (4.38). For $r \to \infty$ we obtain:

$$D_i\Phi = 0 \Rightarrow \eta\partial_i \frac{r^a}{r} + e\eta\, \epsilon_{abc} A^b_i \frac{r^c}{r} = 0 \Rightarrow$$

$$\frac{\delta_{ia}}{r} - \frac{r^i r^a}{r^3} = -e\, \epsilon_{abc} A^b_i\, \frac{r^c}{r} \Rightarrow$$

$$(\delta_{ia}\delta_{kl} - \delta_{ik}\delta_{al}) r^k r^l = -er^2 \epsilon_{abc} A^b_i r^c \Rightarrow$$

$$\epsilon_{ilb}\epsilon_{akb} r^k r^l = er^2 \epsilon_{akb} A^b_i r^k$$

Then, a particular solution reads:

$$A^a_i = \frac{1}{e}\epsilon_{aij}\frac{r^j}{r^2}, \quad r \to +\infty \tag{4.58}$$

where $a, i, j = 1, 2, 3$. Note that the gauge is fixed $A^a_0 = 0$. Additionally, we can calculate the non-Abelian magnetic field as follows:

$$B^a_k = \frac{1}{2}\epsilon_{kli} F^a_{li} \tag{4.59}$$

where F^a_{li} is given in (4.4).

$$\epsilon_{kli}\partial_l A^a_i = \epsilon_{kli}\frac{1}{e}\epsilon_{aij}\partial_l \frac{r^j}{r^2} =^{(2.14)} \frac{1}{e}(-\delta_{ka}\delta_{jl} + \delta_{jk}\delta_{al})\Big(\frac{\delta_{ia}}{r^2} - 2\frac{r^i r^a}{r^4}\Big) \Rightarrow$$

$$\epsilon_{kli}\partial_l A^a_i = -\frac{2}{e}\frac{r^k r^a}{r^4} \tag{4.60}$$

also we calculate:

$$e\, \epsilon_{kli}\epsilon_{abc} A^b_l A^c_i = \frac{1}{e}\epsilon_{kli}\epsilon_{abc}\epsilon_{blm}\epsilon_{cin}\frac{r^m}{r^2}\frac{r^n}{r^2} =^{(2.14)} \frac{1}{e}(\delta_{kb}\delta_{im} - \delta_{mk}\delta_{ib})(\delta_{ia}\delta_{bn} - \delta_{an}\delta_{bi})\frac{r^m}{r^2}\frac{r^n}{r^2} =$$

$$\frac{1}{e}\Big(\delta_{kn}\delta_{am} + \delta_{km}\delta_{an}\Big) = \frac{1}{e}\Big(\frac{r^a r^k}{r^4} + \frac{r^k r^a}{r^4}\Big) \Rightarrow$$

$$e\ \epsilon_{kli}\epsilon_{abc}A_l^b A_i^c = \frac{2}{e}\frac{r^k r^a}{r^4} \tag{4.61}$$

Upon substituting equations (4.4), (4.60) and (4.61) into equation (4.59), we obtain:

$$B_k^a = \frac{1}{2}\epsilon_{kli}(\partial_l A_i^a - \partial_i A_l^a + e\epsilon_{abc}A_l^b A_i^c) = \epsilon_{kli}\partial_l A_i^a + \frac{e}{2}\epsilon_{kli}\epsilon_{abc}A_l^b A_i^c \Rightarrow$$

$$B_i^a = -\frac{1}{e}\frac{r^i r^a}{r^4} \tag{4.62}$$

Furthermore, in the section 4.1, if we had fixed the vacuum expectation value of the gauge triplet in a different way, e.g., $< 0|\underline{\Phi}|0 >= (\eta_1, \eta_2, \eta_3)$, with $\eta_i\eta_i = \eta^2$, the remaining $U(1)$ symmetry would have not described rotations around z-axis with generator T^3, instead, it would have described rotations with generator $\hat{\Phi}^a T^a = \frac{\Phi^a}{\eta}T^a$. Upon considering this case, the operators of the electric charge and the electromagnetic field are:

$$Q = \frac{e}{\eta}\Phi^a T^a \quad \text{and} \quad A_\mu^{EM} = \frac{1}{\eta}\Phi^a A_\mu^a \tag{4.63}$$

Considering expressions (4.63), let us express the gauge bosons A_μ^a in terms of A_μ^{EM} and the Higgs triplet. Upon minimizing the total energy, we have:

$$D_i\Phi = 0 \Rightarrow \partial_i\Phi^a + e\ \epsilon_{abc}A_i^b\Phi^c = 0 \Rightarrow \tag{4.64}$$

$$\epsilon_{akl}\partial_i\Phi^a = -e\ (\delta_{kb}\delta_{lc} - \delta_{kc}\delta_{lb})A_i^b\Phi^c \quad \Rightarrow \quad \epsilon_{akl}\Phi^k\partial_i\Phi^a = -e\ \Phi^b A_i^b\Phi^l + e\ |\Phi|^2 A_i^l$$

For $|\Phi|^2 = \eta^2$ and $A_\mu^{EM} = \frac{1}{\eta}\Phi^a A_\mu^a$ we obtain:

$$A_\mu^a = \frac{\Phi^a}{\eta}A_\mu^{EM} - \frac{1}{\eta^2 e}\epsilon_{abc}\Phi^b\partial_\mu\Phi^c \tag{4.65}$$

We define the electromagnetic tensor:

$$F_{\mu\nu} \equiv \frac{\Phi^a}{\eta}F_{\mu\nu}^a \tag{4.66}$$

and we calculate it:

$$\frac{\Phi^a}{\eta}F_{\mu\nu}^a = \frac{1}{\eta}(\Phi^a\partial_\mu A_\nu^a - \Phi^a\partial_\nu A_\mu^a + e\Phi^a\epsilon_{abc}A_\mu^b A_\nu^c) =$$

$$\partial_\mu\Big(\frac{\Phi^a}{\eta}A_\nu^a\Big) - \partial_\nu\Big(\frac{\Phi^a}{\eta}A_\mu^a\Big) - \frac{1}{\eta}A_\nu^a\partial_\mu\Phi^a + \frac{1}{\eta}A_\mu^a\partial_\nu\Phi^a + \frac{1}{\eta}e\Phi^a\epsilon_{abc}A_\mu^b A_\nu^c \underset{A_\mu^{EM}=\frac{1}{\eta}\Phi^a A_\mu^a}{\overset{(4.64)}{=}}$$

$$\partial_\mu A_\nu^{EM} - \partial_\nu A_\mu^{EM} - \frac{1}{\eta}A_\nu^a\partial_\mu\Phi^a \underset{\partial_\mu|\Phi|=0}{\overset{(4.65)}{\Longrightarrow}}$$

$$F_{\mu\nu} \equiv \frac{\Phi^a}{\eta}F_{\mu\nu}^a = \partial_\mu A_\nu^{EM} - \partial_\nu A_\mu^{EM} - \frac{1}{\eta^3 e}\epsilon_{abc}\Phi^a\partial_\mu\Phi^b\partial_\nu\Phi^c \tag{4.67}$$

Note that, for *winding number* $n = 0$, the equation (4.67) yields:

$$F_{\mu\nu}^{EM} = \partial_\mu A_\nu^{EM} - \partial_\nu A_\mu^{EM} \tag{4.68}$$

The electromagnetic strength tensor (4.68) satisfies the Bianchi identity, therefore the model does not consist of magnetic monopoles. However, for *winding number* $n \neq 0$, the electromagnetic strength tensor (4.67) does not satisfy the Bianchi identity, as we prove in the next subsection. Hence, for *winding number* $n \neq 0$ there exist magnetic monopoles.

Additionally, due to the non-Abelian gauge symmetry of the system, the electromagnetic strength tensor $F_{\mu\nu}$ is somehow arbitrary and it can be written as [3]:

$$f_{\mu\nu} = \hat{\Phi}^a F_{\mu\nu}^a - \frac{1}{e}\epsilon_{abc}\hat{\Phi}^a D_\mu\hat{\Phi}^b D_\nu\hat{\Phi}^c \tag{4.69}$$

4.2.2 Topological current

In the next few lines we explicitly prove that the electromagnetic strength tensor (4.67) does not satisfy the Bianchi identity.

$$\partial_{[a}F_{bc]} = \partial_a F_{bc} + \partial_c F_{ab} + \partial_b F_{ca} = \frac{1}{2}\epsilon_{abc\sigma}\epsilon^{\sigma\mu\nu\rho}\partial_\mu F_{\nu\rho} \tag{4.70}$$

The dual electromagnetic strength tensor is defined as:

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{4.71}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita tensor defined in (1.8).

$$\partial^\nu \tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu(\partial^\rho A^\sigma_{EM} - \partial^\sigma A^\rho_{EM} - \frac{1}{\eta^3 e}\epsilon_{abc}\Phi^a\partial^\rho\Phi^b\partial^\sigma\Phi^c) \Rightarrow$$

$$\partial^\nu \tilde{F}_{\mu\nu} = \frac{1}{2e}\epsilon_{\mu\nu\rho\sigma}\epsilon_{abc}\partial^\nu\hat{\Phi}^a\partial^\rho\hat{\Phi}^b\partial^\sigma\hat{\Phi}^c = k_\mu \tag{4.72}$$

For $k_\mu \neq 0$, the Bianchi identity is not satisfied and the theory consists of magnetic monopoles.

We define k_μ as the topological current. Note that

$$\partial^\mu k_\mu = 0 \tag{4.73}$$

due to the antisymmetric property of the Levi-Civita tensor. Consequently, the topological current is conserved. This result is very interesting, since there are no symmetries and Noether currents that yield the above conservation. The conserved topological current (4.72) has topological origins. This argument is strengthened by calculating the corresponding conserved charge:

$$k_0 = \frac{1}{2e}\epsilon_{0\nu\rho\sigma}\epsilon_{abc}\partial^\nu\hat{\Phi}^a\partial^\rho\hat{\Phi}^b\partial^\sigma\hat{\Phi}^c = \frac{1}{2e}\epsilon_{mnk}\epsilon_{abc}\partial^m\hat{\Phi}^a\partial^n\hat{\Phi}^b\partial^k\hat{\Phi}^c \Rightarrow$$

$$Q_k = \frac{1}{2e}\epsilon_{mnk}\epsilon_{abc}\int d^3x\partial^m\hat{\Phi}^a\partial^n\hat{\Phi}^b\partial^k\hat{\Phi}^c = \frac{1}{2e}\epsilon_{mnk}\epsilon_{abc}\int d^3x\partial^m\left(\hat{\Phi}^a\partial^n\hat{\Phi}^b\partial^k\hat{\Phi}^c\right) \Rightarrow$$

$$Q_k = \frac{1}{2e}\epsilon_{mnk}\epsilon_{abc}\int d^3x\partial^m\hat{\Phi}^a\partial^n\hat{\Phi}^b\partial^k\hat{\Phi}^c = \frac{1}{2e}\epsilon_{mnk}\epsilon_{abc}\int d^3x\partial^m\left(\hat{\Phi}^a\partial^n\hat{\Phi}^b\partial^k\hat{\Phi}^c\right) \Rightarrow$$

$$Q_k = \frac{1}{2e}\epsilon_{mnk}\epsilon_{abc}\int_{S^2_\infty} dS_m\hat{\Phi}^a\partial^n\hat{\Phi}^b\partial^k\hat{\Phi}^c \tag{4.74}$$

Note that $\hat{\Phi}^a(r\to\infty)\to\frac{r^a}{r}$ on the surface S^2_∞, hence the above integral does not vanish.

Let us consider a map (2-loop) $S^2 \longrightarrow S^2$, where the coordinates for the first sphere are θ,ϕ and for the second sphere are $\alpha(\theta,\phi),\beta(\theta,\phi)$. The *winding number* is given by a similar equation to (3.1), which reads [1]:

$$n = \frac{1}{4\pi}\int d^2\Omega\frac{sin(\alpha)}{sin(\theta)}\Big(\frac{\partial\alpha}{\partial\theta}\frac{\partial\beta}{\partial\phi} - \frac{\partial\beta}{\partial\theta}\frac{\partial\alpha}{\partial\phi}\Big) = \frac{1}{4\pi}\int d\theta d\phi\, sin(\alpha)\Big(\frac{\partial\alpha}{\partial\theta}\frac{\partial\beta}{\partial\phi} - \frac{\partial\beta}{\partial\theta}\frac{\partial\alpha}{\partial\phi}\Big) \tag{4.75}$$

Upon substituting the unit vectors $\hat{e}(\vec{r}) = \big(sin(\alpha)cos(\beta), sin(\alpha)sin(\beta), cos(\alpha)\big)$ into the above equation, we obtain [1]:

$$n = \frac{1}{8\pi}\epsilon_{ijk}\epsilon_{abc}\int dS_i\hat{e}^a\partial_j\hat{e}^b\partial_k\hat{e}^c \tag{4.76}$$

As we mentioned before, the Higgs triplet is a mapping $\Phi:\ S^2_\infty \longrightarrow S^2_{vac}$. Consequently, comparing the equations (4.74) and (4.76), we observe that:

$$Q_k = \frac{4\pi}{e}n, \quad n\in\mathbb{Z} \tag{4.77}$$

This wonderful result explicitly shows that the conserved topological charge is proportional to the *winding number*, which determines the topological properties of the Higgs triplet as a mapping from S^2_∞ to S^2_{vac}. Note that $4\pi n$ is the total solid angle of the covering of the S^2_{vac} during the mapping.

4.2.3 Magnetic induction of the magnetic monopole

Last but not least, we need to calculate the magnetic induction corresponding to (4.67) for $r \longrightarrow \infty$, in order to determine the magnetic charge of the magnetic monopole, whose existence is testified by equation (4.72).

$$B_k = \frac{1}{2}\epsilon_{kij}F_{ij} = \frac{1}{2}\epsilon_{kij}F^a_{ij}\frac{\Phi^a}{\eta} = B^a_k\frac{\Phi^a}{\eta} \underset{(4.62)}{\overset{(4.57)}{\Longrightarrow}}$$

$$\vec{B}(r \to \infty) = \frac{1}{e}\frac{\vec{r}}{r^3} \tag{4.78}$$

This is the magnetic field of a monopole with magnetic charge $g = \frac{1}{e}$. Note that, if we set $g = \frac{Q_k}{4\pi}$ and substitute into equation (4.77), we obtain:

$$eg = n, \quad n \in \mathbb{Z} \tag{4.79}$$

which is an analog to the Dirac quantization condition in the non-Abelian case. If we set the *winding number* $n = 1$ in the equation (4.79) we have $g = \frac{1}{e}$, which is exactly the magnetic charge of the corresponding magnetic monopole. Finally, note that the magnetic charge is conserved, since it is proportional to the conserved topological charge.

Chapter 5

Self-Gravitating Global Monopole

Some models predict monopoles, as finite self-gravitating objects with structure. These solutions may not correspond to magnetic monopoles, since there is no need for the model to include electromagnetic fields. For instance, we can consider a system of a self-gravitating scalar Higgs triplet within a self-interacting Higgs potential. In this case, we obtain monopole solutions, if we consider the Higgs triplet as a 2-loop with *winding number* $n = 1$. A model with these properties is the Barriola-Vilenkin model [24].

5.1 Barriola-Vilenkin model

The Barriola-Vilenkin model is described by the following Lagrangian density [24]:

$$\mathcal{L} = \sqrt{-g}\Big(\frac{1}{2}\partial_\mu \chi^a \partial^\mu \chi^a - \frac{\lambda}{4}(\chi^a\chi^a - \eta^2)^2 - R\Big) \tag{5.1}$$

where χ^a, $a = 1, 2, 3$, forms a scalar triplet, which parameterizes the spontaneous breaking of the global $O(3)$ symmetry of the system down to a global $O(2)$, due to the self-interacting Higgs potential $-\frac{\lambda}{4}(\chi^a\chi^a - \eta^2)^2$. Also, $g_{\mu\nu}$ is the 4-dimensional space-time metric tensor, g is the determinant of the metric and R is the Ricci scalar for the metric $g_{\mu\nu}$. Note that we use the conventions:

$$\text{Signature of the metric: } (+, -, -, -) \tag{5.2}$$

$$\text{Riemann tensor: } R^\lambda{}_{\mu\nu\sigma} = \partial_\nu \Gamma^\lambda{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\sigma}\Gamma^\lambda{}_{\rho\nu} - (\nu \leftrightarrow \sigma) \tag{5.3}$$

$$\text{Ricci tensor: } R_{\mu\sigma} = R^\lambda{}_{\mu\lambda\sigma} \tag{5.4}$$

$$\text{Ricci scalar: } R = g^{\mu\nu}R_{\mu\nu} \tag{5.5}$$

$$\text{Einstein tensor: } G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{5.6}$$

As we discuss in chapters 3 and 4, if we consider the scalar triplet as a map (2-loop) from S^2_∞ to S^2_{vac} with *winding number* $n \neq 0$, we obtain monopole solutions. In this chapter we fix $n = 1$ and we are searching for spherical symmetric solutions of the form:

$$\chi^a(\vec{r}) = \eta f(r)\frac{x^a}{r}, \text{ with } f(r \to \infty) = 1 \tag{5.7}$$

5.1.1 Equations of motion

First of all, we need to determine the equations of motion. Note that the equations of motion of the scalar triplet are satisfied by the ansatz (5.7). Additionally, the Einstein field equations read:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{5.8}$$

where

$$\boxed{\text{Stress-Energy tensor: } T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_\chi}{\delta g^{\mu\nu}}} \tag{5.9}$$

In the next few lines we determine the components of the Stress-Energy tensor. First of all, we calculate the variation of the action corresponding to the scalar triplet.

$$\delta S_\chi = \int d^4x \Big[(\delta\sqrt{-g}) \Big(\frac{1}{2}\partial_\mu \chi^a \partial^\mu \chi^a - \frac{\lambda}{4}(\chi^a\chi^a - \eta^2)^2 \Big) + \frac{\sqrt{-g}}{2}\partial_\mu\chi^a\partial_\nu\chi^a\delta g^{\mu\nu} \Big]$$

where:

$$\delta\sqrt{-g} = -\sqrt{-g}\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu} \tag{5.10}$$

Therefore, we obtain:

$$\delta S_\chi = \int d^4x \frac{\sqrt{-g}}{2}\Big[-g_{\mu\nu}\Big(\frac{\mathcal{L}}{\sqrt{-g}} + R\Big) + \partial_\mu\chi^a\partial_\nu\chi^a \Big]\delta g^{\mu\nu} \tag{5.11}$$

Upon substituting equation (5.11) into equation (5.9), we obtain:

$$\boxed{\begin{gathered} T_{\mu\nu} = -g_{\mu\nu}\Big(\frac{\mathcal{L}}{\sqrt{-g}} + R\Big) + \partial_\mu\chi^a\partial_\nu\chi^a \\ \text{or} \\ T_{\mu\nu} = -\frac{g_{\mu\nu}}{2}\partial_\rho\chi^a\partial^\rho\chi^a + g_{\mu\nu}\frac{\lambda}{4}(\chi^a\chi^a - \eta^2)^2 + \partial_\mu\chi^a\partial_\nu\chi^a \end{gathered}} \tag{5.12}$$

The most general static metric with spherical symmetry can be written as follows [24]:

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + sin^2(\theta)d\phi^2) \tag{5.13}$$

with the usual relation between the spherical coordinates, r, θ and ϕ, and the "Cartesian" coordinates x^a. The equation (5.13) yields:

$$g_{\mu\nu} \doteq \begin{pmatrix} B(r) & 0 & 0 & 0 \\ 0 & -A(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2(\theta) \end{pmatrix} \tag{5.14}$$

and

$$g^{\mu\nu} \doteq \begin{pmatrix} \frac{1}{B(r)} & 0 & 0 & 0 \\ 0 & -\frac{1}{A(r)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2 sin^2(\theta)} \end{pmatrix} \tag{5.15}$$

We need to write the components of the scalar triplet in terms of the spherical coordinates.

$$\chi^1(\vec{r}) = \eta f(r) sin(\theta)cos(\phi), \quad \chi^2(\vec{r}) = \eta f(r) sin(\theta) sin(\phi) \text{ and } \chi^3(\vec{r}) = \eta f(r) cos(\theta) \tag{5.16}$$

then we obtain:

$$\partial_\rho\chi^a\partial^\rho\chi^a = g^{\rho\sigma}\partial_\rho\chi^a\partial_\sigma\chi^a =$$

$$-\frac{1}{A(r)}\eta^2 f'^2(r) - \frac{1}{r^2}\eta^2 f^2(r)\big(cos^2(\theta)cos^2(\phi) + cos^2(\theta)sin^2(\phi) + sin^2(\theta)\big)$$

$$-\frac{1}{r^2 sin^2(\theta)}\eta^2 f^2(r)\big(sin^2(\theta)sin^2(\phi) + sin^2(\theta)cos^2(\phi)\big) \Rightarrow$$

$$\partial_\rho\chi^a\partial^\rho\chi^a = g^{\rho\sigma}\partial_\rho\chi^a\partial_\sigma\chi^a = -\frac{1}{A(r)}\eta^2 f'^2(r) - \frac{2}{r^2}\eta^2 f^2(r) \tag{5.17}$$

and

$$\chi^a(\vec{r})\chi^a(\vec{r}) = \eta^2 f^2(r) \tag{5.18}$$

Thus, we calculate the non-trivial components of the stress-energy tensor.

$$T_{tt} = \frac{B(r)}{2A(r)}\eta^2 f'^2(r) + \frac{B(r)}{r^2}\eta^2 f^2(r) + B(r)\frac{\lambda}{4}(\eta^2 f^2(r) - \eta^2)^2 \Rightarrow$$

$$\boxed{T_{tt} = \frac{B(r)}{2A(r)}\eta^2 f'^2(r) + \frac{B(r)}{r^2}\eta^2 f^2(r) + B(r)\frac{\lambda}{4}\eta^4(f^2(r) - 1)^2} \tag{5.19}$$

$$T_{rr} = -\frac{1}{2}\eta^2 f'^2(r) - \frac{A(r)}{r^2}\eta^2 f^2(r) - A(r)\frac{\lambda}{4}(\eta^2 f^2(r) - \eta^2)^2 + \eta^2 f'^2(r) \Rightarrow$$

$$\boxed{T_{rr} = \frac{1}{2}\eta^2 f'^2(r) - \frac{A(r)}{r^2}\eta^2 f^2(r) - A(r)\frac{\lambda}{4}\eta^4(f^2(r) - 1)^2} \tag{5.20}$$

$$T_{\theta\theta} = -\frac{r^2}{2A(r)}\eta^2 f'^2(r) - \eta^2 f^2(r) - r^2\frac{\lambda}{4}(\eta^2 f^2(r) - \eta^2)^2 + \eta^2 f^2(r) \Rightarrow$$

$$\boxed{T_{\theta\theta} = -\frac{r^2}{2A(r)}\eta^2 f'^2(r) - r^2\frac{\lambda}{4}\eta^4(f^2(r) - 1)^2} \tag{5.21}$$

similarly

$$\boxed{T_{\phi\phi} = T_{\theta\theta}\ sin^2(\theta)} \tag{5.22}$$

5.1.2 Asymptotic solutions of the equations of motion

Let us consider the monopole to be a finite object, which is contained within a core. In flat space-time the monopole core has size $\delta \sim \lambda^{-\frac{1}{2}}\eta^{-1}$ and its mass is $M_{core} \sim \lambda\eta^4\delta^3 \sim \lambda^{-\frac{1}{2}}\eta$ [24]. For $\eta \ll m_p$, where m_p is the Planck mass, the gravity does not change significantly the structure of the monopole at small distances, hence the flat space-time estimates of δ and M_{core} still approximately apply [24]. Outside the core, where we suppose that $r > \delta \gg$ Planck length, we obtain that $f(r) \approx 1$ and

$$T_t^{\ t} = T_r^{\ r} = \frac{\eta^2}{r^2} \quad \text{and} \quad T_\theta^{\ \theta} = T_\phi^{\ \phi} = 0 \tag{5.23}$$

The components G_{tt} and G_{rr}, considering the metric ansatz (5.14), read:

$$G_{tt} = -B(r)\frac{A(r) - A^2(r) - rA'(r)}{r^2 A^2(r)} \tag{5.24}$$

and

$$G_{rr} = \frac{B(r) - A(r)B(r) - rB'(r)}{r^2 B(r)} \tag{5.25}$$

For $B(r) = A^{-1}(r)$ and $f(r) = 1$, we obtain:

$$G_{tt} = -\frac{A(r) - A^2(r) - rA'(r)}{r^2 A^3(r)} \tag{5.26}$$

and

$$G_{rr} = \frac{A(r) - A^2(r) - rA'(r)}{r^2 A(r)} \tag{5.27}$$

Note that the components tt and rr of the Einstein field equation (5.8) are the same. Therefore, it is convenient to solve only the tt equation:

$$G_{tt} = 8\pi G T_{tt} \Rightarrow$$

$$-\frac{A(r) - A^2(r) - rA'(r)}{r^2 A^3(r)} = 8\pi G\frac{\eta^2}{A(r)r^2} \Rightarrow$$

$$B(r) = A^{-1} = 1 - 8\pi G\eta^2 - \frac{2GM}{r} \tag{5.28}$$

where M is a constant of integration. Summarizing the results, the metric reads:

$$\boxed{ds^2 = \left(1 - 8\pi G\eta^2 - \frac{2GM}{r}\right)dt^2 - \frac{dr^2}{1 - 8\pi G\eta^2 - \frac{2GM}{r}} - r^2(d\theta^2 + sin^2(\theta)d\phi^2), \quad r > \delta} \tag{5.29}$$

Note that the Schwarzschild metric is obtained in the unbroken phase $\eta \to 0$. If we introduce the rescale $t \to (1 - 8\pi G\eta^2)^{-\frac{1}{2}}t'$ and $r \to (1 - 8\pi G\eta^2)^{\frac{1}{2}}r'$, in the asymptotic limit $r \to \infty$, the metric reads:

$$ds^2 = dt'^2 - dr'^2 - (1 - 8\pi G\eta^2)r'^2(d\theta^2 + sin^2(\theta)d\phi^2), \quad r > \delta \tag{5.30}$$

which is a Minkowski metric with a conical deficit solid angle $\Delta\Omega = 32\pi^2 G\eta^2$ [3]. The metric (5.30) is not flat, since the Ricci scalar reads:

$$R = -\frac{16\pi G\eta^2}{r(1 - 8\pi G\eta^2)} \tag{5.31}$$

The corresponding deficit solid angle is predicted by many models of self-gravitating monopoles and appears to be a fundamental feature of the global monopole systems. The deficit solid angle leads to Einstein rings, which may have a cosmological impact on CMB.

5.2 The effective mass of the self-gravitating global monopole

Let us make the very generous assumption that the metric (5.29) describes space-time even inside the monopole core. We can estimate monopole mass considering a Schwarzschild-like metric [24]:

$$B(r) = 1 - \frac{2GM'}{r} \tag{5.32}$$

Also, the rest-mass M' of the monopole reads:

$$M' = \int dx^3 \sqrt{-g} T_t{}^t \tag{5.33}$$

where

$$\sqrt{-g} = r^2 sin(\theta) \quad \Rightarrow \quad d^3x\sqrt{-g} = r^2 sin(\theta) dr d\theta d\phi \tag{5.34}$$

and

$$T_t{}^t = \frac{1}{2A(r)}\eta^2 f'^2(r) + \frac{1}{r^2}\eta^2 f^2(r) + \frac{\lambda}{4}\eta^4(f^2(r) - 1)^2 \tag{5.35}$$

Thus, the equation (5.34) yields:

$$M'(r) = 4\pi\eta^2 \int_0^r dr' r'^2 \Big(\frac{1}{2A(r')} f'^2(r') + \frac{1}{r'^2} f^2(r') + \frac{\lambda}{4}\eta^2(f^2(r') - 1)^2\Big) \Rightarrow$$

$$M'(r) = 4\pi\eta^2 r + 4\pi\eta^2 \int_0^r dr' r'^2 \left(\frac{f'^2(r')}{2A(r')} + \frac{f^2(r') - 1}{r'^2} + \frac{\lambda}{4}\eta^2(f^2(r') - 1)^2\right) \tag{5.36}$$

Upon substituting equation (5.36) into equations (5.32) we obtain:

$$B(r) = 1 - 8\pi G\eta^2 - \frac{2G}{r}4\pi\eta^2 \int_0^r dr' r'^2 \left(\frac{f'^2(r')}{2A(r')} + \frac{f^2(r') - 1}{r'^2} + \frac{\lambda}{4}\eta^2(f^2(r') - 1)^2\right) \tag{5.37}$$

Then, if we compare the equations (5.28) and (5.37), we have:

$$M(r) = 4\pi\eta^2 \int_0^r dr' r'^2 \left(\frac{f'^2(r')}{2A(r')} + \frac{f^2(r') - 1}{r'^2} + \frac{\lambda}{4}\eta^2(f^2(r') - 1)^2\right) \tag{5.38}$$

Upon considering that the mass of the monopole M, as seen from outside the core, is contained in a core of radius δ and that $f(r < \delta) \approx 0$, $f'(r < \delta) \approx 0$, we obtain:

$$M = 4\pi\eta^2 \int_0^\delta dr' \Big(-1 + \frac{\lambda}{4}\eta^2 r'^2\Big) \Rightarrow$$

$$M = -4\pi\eta^2\delta + \frac{4\pi}{3}\delta^3\frac{\lambda\eta^4}{4} \tag{5.39}$$

Note that for $\delta \sim \lambda^{-\frac{1}{2}}\eta^{-1}$(flat space-time), the equation (5.39) yields:

$$M = -4\pi\eta\lambda^{-\frac{1}{2}} + \frac{\pi}{3}\lambda^{-\frac{1}{2}}\eta \Rightarrow$$

$$M = -\frac{11\pi}{3}\lambda^{-\frac{1}{2}}\eta \tag{5.40}$$

This is a very interesting result, since negative mass implies repulsive forces. Additionally, solutions with negative mass carry no horizons. Note that $T_t^{\ t} \approx \frac{\lambda\eta^4}{4}$ for $f(r < \delta) = 0$, hence the equation (5.39) yields:

$$M = \frac{4\pi}{3}\delta^3 T_t^{\ t} - 4\pi\eta^2\delta \Rightarrow$$

$$M = V_{core}T_t^{\ t} \ - \ deficit\ angle\ term \ \ < 0 \tag{5.41}$$

Also note that

$$M'(\delta) = V_{core}T_t^{\ t} = \frac{\pi}{3}\lambda^{-\frac{1}{2}}\eta \sim \lambda^{-\frac{1}{2}}\eta \sim M_{core} \tag{5.42}$$

Concluding the results, although monopole's rest-mass is positive and similar to the flat space-time solution ($M_{core} \sim \lambda^{-\frac{1}{2}}\eta$ in flat space-time), the self-gravitational interaction of the monopole implies negative ADM mass for the monopole metric. Additionally, this result can be obtained, even if we are more careful about our assumption about the inner space of the monopole core.

5.3 Repulsive gravitational effects

Let as do the approximation [25]:

$$f(r) \begin{cases} 0 & ,r < \delta \\ 1 & ,r > \delta \end{cases} \tag{5.43}$$

Then, inside the core ($r < \delta$), we have:

$$T_{\mu\nu} = g_{\mu\nu}\frac{\lambda}{4}\eta^4 \tag{5.44}$$

Therefore, the Einstein field equations read:

$$G_{\mu\nu} = g_{\mu\nu}2\pi G\lambda\eta^4 \tag{5.45}$$

The equations (5.45) correspond to a de Sitter space-time with positive cosmological constant:

$$\Lambda = 2\pi G\lambda\eta^4 > 0 \tag{5.46}$$

Thus, the metric inside the core reads:

$$ds^2 = \Big(1 - \frac{\Lambda r^2}{3}\Big)dt^2 - \frac{dr^2}{1 - \frac{\Lambda r^2}{3}} - r^2(d\theta^2 + sin^2(\theta)d\phi^2), \ \ r < \delta \tag{5.47}$$

Outside the core ($r > \delta$), the metric is given by the equation (5.29). On the monopole core bounds, the metric must satisfy the Israel conditions [26], i.e., it must be continuous and its first derivative must be continuous. These conditions yield:

$$\left.\begin{array}{c} 1 - \frac{2\pi G}{3}\lambda\eta^4\delta^2 = 1 - 8\pi G\eta^2 - \frac{2GM}{\delta} \\ -\frac{4\pi G}{3}\lambda\eta^4\delta = \frac{2GM}{\delta^2} \end{array}\right\} \Rightarrow$$

$$\delta = 2\lambda^{-\frac{1}{2}}\eta^{-1} \quad \text{and} \quad M = -\frac{16\pi}{3}\lambda^{-\frac{1}{2}}\eta < 0 \tag{5.48}$$

Note that the radius of the core and the monopole mass are similar to those calculated before. A more accurate calculation, using computational methods, without the approximation (5.43), in paper [25], yields:

$$M = -6\pi\lambda^{-\frac{1}{2}}\eta < 0 \tag{5.49}$$

which is also a similar result. The interpretation of the negative monopole mass is that the global self-gravitating monopole provides vacuum energy that induces repulsive gravitational effects. This interpretation is consistent with the monopole being an entity with complicated structure rather than an elementary particle-like excitation. Subsequently, the monopole has an expanding ($\Lambda > 0$) core surrounded by space-time with negative "Schwarzschild" mass, which has repulsive nature. Hence, it seems that the monopole, as a structured-particle, cannot be stabilized.

Chapter 6

Magnetic monopoles from self-gravitating global monopoles with Kalb-Ramond torsion

The models described in chapters 4 and 5 provide monopole solutions, which are not able to be detected in the current large hadron collider (LHC).

The Georgi-Glashow model discussed previously does not correspond to the phenomenologically correct model describing Nature [27]. However, embedding the Standard Model group to larger Grand Unified Theory (GUT) groups, which implies the existence of monopoles, we determine the mass of the monopole, which depends on the mass of the gauge boson of the GUT and the GUT gauge group coupling. The typical mass scales of such a monopole varies in the range $10^{14} - 10^{16} GeV$ [27]. Consequently, such monopoles cannot be produced in current colliders, and one can only search cosmically for them, but their density must be extremely dilute due to inflation [27].

Furthermore, some recommended solutions to the hierarchy problem that rely neither on supersymmetry nor on technicolor, propose the existence of new dimensions at a millimeter [28][29][30]. This consideration lowers the Planck scale of gravitational physics to the order of TeV and thus, in principle, gravitational effects may become observable at the LHC. In particular, micro-black holes can be produced and decay rapidly [3]. Therefore, a self-gravitational interaction may reduce the mass of the monopole, which may be observable at LHC.

Nevertheless, the global self-gravitating monopole of the Barriola-Vilenkin model has negative mass. Such a construction with negative mass would not apply to collider physics.

In this chapter, we present a model of a self-gravitating global monopole, having included additional fields, which allow for a positive mass solution. Note that the corresponding magnetic monopole has structure (core, etc). We shall call such structures "bag-models".

6.1 Description of the model

Let us consider a model of a self-gravitating global monopole with Kalb-Ramond torsion. The $O(3)$ global symmetric Lagrangian density of the model reads [3]:

$$\mathcal{L} = \sqrt{-g}\left[\frac{1}{2}\partial_\mu\chi^a\partial^\mu\chi^a - \frac{\lambda}{4}(\chi^a\chi^a-\eta^2)^2 - R + \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - V(\Phi) - \frac{1}{12}e^{-2\gamma\Phi}H_{\rho\mu\nu}H^{\rho\mu\nu} - \frac{1}{4}e^{-\gamma\Phi}f_{\mu\nu}f^{\mu\nu}\right] \quad (6.1)$$

In the following lines we describe the components of the above Lagrangian density.

1) First of all, we have a self-gravitating system, hence $g_{\mu\nu}$ is the corresponding metric, g is the determinant of the metric and R is the Ricci scalar for $g_{\mu\nu}$.

2) We seek monopole solutions. Thus, the system includes a triplet of scalar fields in the adjoint representation of the $O(3)$ group and a self-interaction Higgs potential, which allows the triplet to play the role of mapping from S^2_∞ to S^2_{vac} with *winding number* $n \neq 0$. The corresponding Higgs parameter determines the deficit angle of the monopole metric and implies a cosmological constant inside the core of the monopole, which regularises the corresponding singularity at the origin.

3) In order for the monopole to be a magnetic monopole, an E/M field strength $f_{\mu\nu}$ should contribute to the system. Since the corresponding monopole is global, there are no couplings between the particles and the E/M field. Therefore, the particles are not electrically charged. It is very interesting that the corresponding magnetic monopole will not be an elementary particle, instead it will be a combination (bag-model) of all the components of the Lagrangian density (6.1), which will effectively have a magnetic charge, whose source will be the "Kalb-Ramond torsion" charge, after the spontaneous symmetry breaking of the global $O(3)$ symmetry of the system, due to the scalar triplet.

4) The antisymmetric Kalb-Ramond tensor field strength $H_{\rho\mu\nu}$ and the field Φ are included to the model in order to succeed a positive mass solution for the magnetic monopole. The Kalb-Ramond field strength can be written as

$$H_{\rho\mu\nu} = \partial_{[\rho} B_{\mu\nu]} \tag{6.2}$$

where the brackets [...] denote total antisymmetrization of the respective indices and $B_{\mu\nu}$ is the spin-1 Kalb-Ramond gauge field (antisymmetric 2-form). In some (closed) string theories [31], the field $B_{\mu\nu}$ appears in the massless spectrum. For the bosonic gravitational part of low-energy string effective actions, the Kalb-Ramond field strength can be thought of as providing a source of torsion [32][33]. Recently [34], in string-inspired effective theories, there are considered some cosmological implications of a dual formulation of a time-dependent four-dimensional Kalb-Ramond field, in connection with the generation of matter-antimatter asymmetry in universe. In four space-time dimensions, the dual of the Kalb-Ramond field strength is a pseudoscalar axion-like field, which reads

$$H_{\mu\nu\lambda} = e^{2\Phi} \epsilon_{\mu\nu\lambda}{}^{\sigma} \partial_\sigma b \tag{6.3}$$

where $\epsilon_{\mu\nu\lambda\sigma}$ is the Levi-Civita tensor defined in (1.8). In our discussion we use the dual formulation (6.3).

5) It is also necessary to introduce the $O(3)$-singlet scalar field Φ, which is stabilised to a constant value via potential $V(\Phi)$. We can consider the model (6.1) in two ways [3]:

i) As a string-inspired theory, where Φ is the dilaton (spin-0 part of the gravitational massless string multiplet), and in principle its stabilisation could be guaranteed by an appropriate (string-loop induced) dilaton potential $V(\Phi)$. In this case $\gamma = 1$.

ii) As a theory with an ultra-heavy scalar field Φ, independent of string theory, which is stabilised by its own potential $V(\Phi)$. In this case $\gamma = 0$. Therefore, the scalar field Φ is only gravitationally coupled to the other scalar and gauge fields of the model.

Note that the Kalb-Ramond field has a "Kalb-Ramond torsion" charge, which is appeared as an axion-charge in this model, since we use the dual formulation (6.3). The existence of a field like Φ is necessary, because it plays the role of the link between the Kalb-Ramond field and the E/M field and leads to the connection between the magnetic and the "Kalb-Ramond torsion" charges.

In summary, in this chapter we present a model, in four-dimensional space-time, according to which the gravitation in the presence of Maxwell and Kalb-Ramond axion-like fields (the latter being the dual of the field strength of a spin-one antisymmetric tensor field in the massless gravitational multiplet of string theories [32][33]) can lead to a magnetic monopole with strength determined by the Kalb-Ramond charge.

6.2 Equations of motion

The first step in our analysis is the determination of the equations of motion. The Euler-Lagrange equations for the scalar triplet read:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \chi^a} - \frac{\partial \mathcal{L}}{\partial \chi^a} = 0 \Rightarrow$$

$$\partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \chi^a) + \sqrt{-g} \lambda \chi^a (\chi^b \chi^b - \eta^2) = 0 \Rightarrow$$

$$\boxed{g^{\mu\nu} \partial_\mu \partial_\nu \chi^a + \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu}) \partial_\nu \chi^a = -\lambda \chi^a (\chi^b \chi^b - \eta^2)} \tag{6.4}$$

The variation of the action with respect to $B_{\mu\nu}$ reads:

$$\delta S = -\int d^4x\, \frac{\sqrt{-g}}{6} e^{-2\gamma\Phi} H^{\rho\mu\nu} \delta H_{\rho\mu\nu}$$

where

$$\delta H_{\rho\mu\nu} = \delta\partial_{[\rho} B_{\mu\nu]} = \partial_\rho(\delta B_{\mu\nu}) + \partial_\nu(\delta B_{\rho\mu}) + \partial_\mu(\delta B_{\nu\rho})$$

note that $[\partial_\mu, \delta] = 0$, then we integrate by parts the variation δS

$$\delta S = -\int d^4x\, \partial_\rho \left[\frac{\sqrt{-g}}{6} e^{-2\gamma\Phi} H^{\rho\mu\nu} \delta B_{\mu\nu}\right] +\ \rho - \mu - \nu\ rotation\ +$$

$$+\int d^4x\, \partial_\rho \left[\frac{\sqrt{-g}}{6} e^{-2\gamma\Phi} H^{\rho\mu\nu}\right] \delta B_{\mu\nu} +\ \rho - \mu - \nu\ rotation$$

The integrals of the derivatives are equal to zero, since the fields vanish at infinity. Also, note that we have $H^{\rho\mu\nu} = H^{\nu\rho\mu} = H^{\mu\nu\rho}$ by definition. Therefore, the variation of the action yields:

$$\delta S = \int d^4x\, \partial_\rho \left[\frac{\sqrt{-g}}{2} e^{-2\gamma\Phi} H^{\rho\mu\nu}\right] \delta B_{\mu\nu}$$

Considering $\delta S = 0$ for arbitrary $\delta B_{\mu\nu}$, we obtain:

$$\partial_\rho \left[\sqrt{-g} e^{-2\gamma\Phi} H^{\rho\mu\nu}\right] = 0 \Rightarrow$$

$$\partial_\rho(e^{-2\gamma\Phi} H^{\rho\mu\nu}) + \frac{1}{\sqrt{-g}} \partial_\rho(\sqrt{-g}) e^{-2\gamma\Phi} H^{\rho\mu\nu} = 0$$

We need to calculate the derivative $\partial_\rho(\sqrt{-g})$. By the definition of the determinant we know that:

$$g = \frac{1}{4!} \tilde{\epsilon}^{abcd} \tilde{\epsilon}^{\kappa\lambda\mu\nu} g_{a\kappa} g_{b\lambda} g_{c\mu} g_{d\nu} \Rightarrow$$

$$\partial_\rho(g) = \frac{4}{4!} \tilde{\epsilon}^{abcd} \tilde{\epsilon}^{\kappa\lambda\mu\nu} g_{a\kappa} g_{b\lambda} g_{c\mu} \partial_\rho g_{d\nu} \overset{(1.9)}{\Longrightarrow}$$

$$\frac{1}{g} \partial_\rho(g) = -\frac{1}{3!} \epsilon^{abcd} \epsilon_{abc\xi} g^{\xi\nu} \partial_\rho g_{d\nu} \Rightarrow^{\epsilon^{abcd} \epsilon_{abc\xi} = -6\delta_\xi^{\ d}}$$

$$\frac{1}{g} \partial_\rho(g) = g^{\mu\nu} \partial_\rho g_{\mu\nu} \Rightarrow$$

$$\partial_\rho(\sqrt{-g}) = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\rho g_{\mu\nu} \tag{6.5}$$

Upon considering that $\Gamma^\mu_{\ \rho\mu} = \frac{1}{2} g^{\mu\nu} \partial_\rho g_{\mu\nu}$, we have

$$\partial_\rho(\sqrt{-g}) = \sqrt{-g} \Gamma^\mu_{\ \rho\mu} \tag{6.6}$$

Upon substituting equation (6.6) into equation of motion of $B_{\mu\nu}$, we obtain

$$\partial_\rho(e^{-2\gamma\Phi} H^{\rho\mu\nu}) + \Gamma^\xi_{\ \rho\xi} e^{-2\gamma\Phi} H^{\rho\mu\nu} = 0 \Rightarrow$$

$$\partial_\rho(e^{-2\gamma\Phi} H^{\rho\mu\nu}) + e^{-2\gamma\Phi} \Gamma^\rho_{\ \rho\xi} H^{\xi\mu\nu} + e^{-2\gamma\Phi} \Gamma^\mu_{\ \rho\xi} H^{\rho\xi\nu} + e^{-2\gamma\Phi} \Gamma^\nu_{\ \rho\xi} H^{\rho\mu\xi} = 0$$

since $\Gamma^\mu_{\ \rho\xi} H^{\rho\xi\nu} = \Gamma^\nu_{\ \rho\xi} H^{\rho\mu\xi} = 0$, due to the fact that $B_{\mu\nu} = -B_{\nu\mu}$ and $\Gamma^\mu_{\ \rho\xi} = \Gamma^\mu_{\ \xi\rho}$. Thus, we have

$$\boxed{\nabla_\rho(e^{-2\gamma\Phi} H^{\rho\mu\nu}) = 0} \tag{6.7}$$

This equation is identically satisfied by the axion-like particle (6.3):

$$\nabla_\rho(\epsilon^{\rho\nu\lambda\sigma} \partial_\sigma b) = \partial_\rho(\epsilon^{\rho\nu\lambda\sigma} \partial_\sigma b) + \Gamma^\rho_{\ \rho\xi} \epsilon^{\xi\nu\lambda\sigma} \partial_\sigma b + \Gamma^\nu_{\ \rho\xi} \epsilon^{\rho\xi\lambda\sigma} \partial_\sigma b + \Gamma^\lambda_{\ \rho\xi} \epsilon^{\rho\nu\xi\sigma} \partial_\sigma b \overset{(1.9)}{=}$$

$$-\frac{\sqrt{-g}}{-g} \Gamma^\xi_{\ \rho\xi} \tilde{\epsilon}^{\rho\nu\lambda\sigma} \partial_\sigma b + \epsilon^{\rho\nu\lambda\sigma} \partial_\rho \partial_\sigma b + \Gamma^\rho_{\ \rho\xi} \epsilon^{\xi\nu\lambda\sigma} \partial_\sigma b =^{(1.9)} -\Gamma^\rho_{\ \rho\xi} \epsilon^{\xi\nu\lambda\sigma} \partial_\sigma b + \Gamma^\rho_{\ \rho\xi} \epsilon^{\xi\nu\lambda\sigma} \partial_\sigma b = 0$$

Hence, the corresponding term of the Lagrangian (6.1) reads:

$$\mathcal{L} = -\frac{\sqrt{-g}}{12}e^{-2\gamma\Phi}H_{\rho\mu\nu}H^{\rho\mu\nu} + ... = -\frac{\sqrt{-g}}{12}e^{2\gamma\Phi}\epsilon_{\rho\mu\nu\lambda}\epsilon^{\rho\mu\nu\kappa}g^{\sigma\lambda}\partial_\sigma b\partial_\kappa b + ... \Rightarrow^{\epsilon_{\rho\mu\nu\lambda}\epsilon^{\rho\mu\nu\kappa}=-6\delta_\lambda^\kappa}$$

$$\mathcal{L} = \frac{\sqrt{-g}}{2}e^{2\gamma\Phi}g^{\mu\nu}\partial_\mu b\partial_\nu b + ... \tag{6.8}$$

Similarly to the previous case, the variation with respect to the E/M field yield:

$$\partial_\mu\left[\sqrt{-g}e^{-\gamma\Phi}f^{\mu\nu}\right] = 0 \Rightarrow$$

$$\partial_\mu(e^{-\gamma\Phi}f^{\mu\nu}) + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g})e^{-\gamma\Phi}f^{\mu\nu} = 0 \stackrel{(6.6)}{\Longrightarrow}$$

$$\partial_\mu(e^{-\gamma\Phi}f^{\mu\nu}) + \Gamma^\rho{}_{\mu\rho}e^{-\gamma\Phi}f^{\mu\nu} = 0 \Rightarrow$$

$$\partial_\mu(e^{-\gamma\Phi}f^{\mu\nu}) + e^{-\gamma\Phi}\Gamma^\mu{}_{\mu\rho}f^{\rho\nu} + e^{-\gamma\Phi}\Gamma^\nu{}_{\mu\rho}f^{\mu\rho} = 0$$

Note that $f^{\mu\nu}$ is antisymmetric, hence $\Gamma^\nu{}_{\mu\rho}f^{\mu\rho} = 0$. Thus, we obtain:

$$\boxed{\nabla_\mu(e^{-\gamma\Phi}f^{\mu\nu}) = 0} \tag{6.9}$$

The equations of motion for the metric tensor correspond to the Einstein field equations:

$$\boxed{G_{\mu\nu} = g_N T_{\mu\nu}} \tag{6.10}$$

where $G_{\mu\nu}$ is the Einstein tensor (5.6), $T_{\mu\nu}$ is the stress-energy tensor (5.9) and $g_N = 8\pi G$, where G is the gravitational constant. Finally, we determine the dilaton equation of motion, which is appropriate to calculate in the case $\gamma = 1$.

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi} - \frac{\partial\mathcal{L}}{\partial\Phi} = 0 \Rightarrow$$

$$\partial_\mu\partial^\mu\Phi + \frac{\partial V(\Phi)}{\partial\Phi} - e^{2\Phi}\partial_\mu b\partial^\mu b - \frac{1}{4}e^{-\Phi}f_{\mu\nu}f^{\mu\nu} = 0$$

Note that we are interested in cases in which the dilaton is stabilised to a constant value $\Phi = \Phi_0$ for which

$$\left.\frac{\partial V(\Phi)}{\partial\Phi}\right|_{\Phi=\Phi_0} = 0 \quad \text{and} \quad V(\Phi_0) = 0 \tag{6.11}$$

Then, we obtain:

$$\boxed{e^{2\Phi}\partial_\mu b\partial^\mu b + \frac{1}{4}e^{-\Phi}f_{\mu\nu}f^{\mu\nu} - \mathcal{O}\Big(\frac{\partial V(\Phi)}{\partial\Phi}\Big) - \mathcal{O}(\partial_\mu\Phi) = 0} \tag{6.12}$$

This is the dilaton equation, which relates the Kalb-Ramond axion-like field to the E/M field strength. This constraint links the Kalb-Ramond torsion charge with the magnetic charge of the magnetic monopole in the string-inspired case ($\gamma = 1$). In the non-string case, the dilaton equation is trivial and the constraint is not imposed. Nevertheless, an appropriate modification of the E/M field strength, in the spirit of the 't Hooft-Polyakov model, involving the axion-like particle and the dilaton, can be constructed. Surprisingly, this solution of the equation (6.9) also links the Kalb-Ramond torsion charge with the magnetic charge.

6.2.1 Anätze of the solutions of the equations of motion

First of all, in order to solve the equations of motion, we need to determine the stress-energy tensor. The variation of the action, ignoring the Ricci scalar term, reads:

$$\delta S^M = \int d^4x\,\delta\Bigg[\sqrt{-g}\Big(\frac{1}{2}g^{\mu\nu}\partial_\mu\chi^a\partial_\nu\chi^a - \frac{\lambda}{4}(\chi^a\chi^a - \eta^2)^2 + \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - V(\Phi) +$$

$$+\frac{1}{2}e^{2\gamma\Phi}g^{\mu\nu}\partial_\mu b\partial_\nu b - \frac{1}{4}e^{-\gamma\Phi}g^{\mu\nu}g^{\rho\sigma}f_{\mu\rho}f_{\nu\sigma}\Big)\Bigg] \Rightarrow$$

$$\delta S^M = \int d^4x \left[\delta(\sqrt{-g})\left(\frac{\mathcal{L}}{\sqrt{-g}} + R\right) + \sqrt{-g}\left(\frac{\delta g^{\mu\nu}}{2}\partial_\mu\chi^a\partial_\nu\chi^a + \frac{\delta g^{\mu\nu}}{2}\partial_\mu\Phi\partial_\nu\Phi + \right.\right.$$

$$\left.\left.\frac{\delta g^{\mu\nu}}{2}e^{2\gamma\Phi}\partial_\mu b\partial_\nu b - \frac{1}{4}e^{-\gamma\Phi}(\delta g^{\mu\nu}g^{\rho\sigma} + g^{\mu\nu}\delta g^{\rho\sigma})f_{\mu\rho}f_{\nu\sigma}\right)\right]$$

Note that from equation (6.5) we can read:

$$\delta\sqrt{-g} = \frac{\sqrt{-g}}{2}g^{\mu\nu}\delta g_{\mu\nu} \tag{6.13}$$

Also, the equation $\delta(g_{\mu\rho}g^{\rho\nu}) = 0$ yields:

$$\delta g_{\rho\sigma} = -g_{\rho\mu}g_{\sigma\nu}\delta g^{\mu\nu} \tag{6.14}$$

Thus, we have:

$$\delta\sqrt{-g} = -\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta g^{\mu\nu} \tag{6.15}$$

Upon substituting equation (6.15) into the variation of the action, we obtain:

$$\delta S^M = \int d^4x \, \frac{\sqrt{-g}}{2}\left[-g_{\mu\nu}\left(\frac{\mathcal{L}}{\sqrt{-g}} + R\right) + \partial_\mu\chi^a\partial_\nu\chi^a + \partial_\mu\Phi\partial_\nu\Phi + e^{2\gamma\Phi}\partial_\mu b\partial_\nu b - e^{-\gamma\Phi}g^{\rho\sigma}f_{\mu\rho}f_{\nu\sigma}\right]\delta g^{\mu\nu}$$

Hence, considering a stabilized dilaton field, the stress-energy tensor (5.9) reads:

$$\boxed{T_{\mu\nu} = -g_{\mu\nu}\left(\frac{\mathcal{L}}{\sqrt{-g}} + R\right) + \partial_\mu\chi^a\partial_\nu\chi^a + e^{2\gamma\Phi}\partial_\mu b\partial_\nu b - e^{-\gamma\Phi}g^{\rho\sigma}f_{\mu\rho}f_{\nu\sigma}} \tag{6.16}$$

Let us suppose that the model describes a static, non-rotating and self-gravitating monopole. In this case, the metric is static and spherical symmetric. Thus we consider the following ansatz for the metric:

$$g_{\mu\nu} \doteq \begin{pmatrix} B(r) & 0 & 0 & 0 \\ 0 & -A(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2(\theta) \end{pmatrix} \tag{6.17}$$

and

$$g^{\mu\nu} \doteq \begin{pmatrix} \frac{1}{B(r)} & 0 & 0 & 0 \\ 0 & -\frac{1}{A(r)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2 sin^2(\theta)} \end{pmatrix} \tag{6.18}$$

Moreover, we seek an E/M field strength $f_{\mu\nu}$ compatible with the solution (4.67) of the 't Hooft-Polyakov model. Note that for $r \to \infty$, the equation (4.67) yields that the non-vanishing components of $f_{\mu\nu}$ are:

$$f_{\theta\phi} = -f_{\phi\theta} \propto sin(\theta) \tag{6.19}$$

Therefore, we introduce the ansatz:

$$f_{\mu\nu} \doteq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2r sin(\theta)W(r) \\ 0 & 0 & -2r sin(\theta)W(r) & 0 \end{pmatrix} \tag{6.20}$$

and we expect $W(r \to \infty) \sim \frac{1}{r}$. Additionally, it is essential to prove that the ansatz (6.20) satisfies the equation (6.9). For a stabilized Φ we obtain:

$$\nabla_\mu f^{\mu\nu} = 0 \Rightarrow$$

$$\partial_\mu f^{\mu\nu} + \Gamma^\mu_{\ \mu\rho}f^{\rho\nu} + \Gamma^\nu_{\ \mu\rho}f^{\mu\rho} = 0 \Rightarrow$$

$$\partial_\mu f^{\mu\nu} + \Gamma^\mu_{\ \mu\rho} f^{\rho\nu} = 0$$

It is obvious that for $\nu = 0, 1$ the above equation is satisfied identically. Upon substituting the metric ansatz (6.17) into the above equation, we obtain:

for $\nu = 2$

$$\partial_\phi f^{\phi\theta} + (\Gamma^t_{\ t\phi} + \Gamma^r_{\ r\phi} + \Gamma^\theta_{\ \theta\phi} + \Gamma^\phi_{\ \phi\phi}) f^{\phi\theta} = 0$$

which is identically satisfied, since $\partial_\phi f^{\phi\theta} = 0$ and $\Gamma^t_{\ t\phi} = \Gamma^r_{\ r\phi} = \Gamma^\theta_{\ \theta\phi} = \Gamma^\phi_{\ \phi\phi} = 0$.

for $\nu = 3$

$$\partial_\theta f^{\theta\phi} + (\Gamma^t_{\ t\theta} + \Gamma^r_{\ r\theta} + \Gamma^\theta_{\ \theta\theta} + \Gamma^\phi_{\ \phi\theta}) f^{\theta\phi} = 0$$

where $\Gamma^t_{\ t\theta} = \Gamma^r_{\ r\theta} = \Gamma^\theta_{\ \theta\theta} = 0$ and $\Gamma^\phi_{\ \phi\theta} = cot(\theta)$, hence:

$$\partial_\theta f^{\theta\phi} + cot(\theta) f^{\theta\phi} = 0 \tag{6.21}$$

We need to calculate $f^{\theta\phi}$:

$$f^{\theta\phi} = g^{\theta\theta} g^{\phi\phi} f_{\theta\phi} \Rightarrow$$

$$f^{\theta\phi} = \frac{1}{r^4 sin^2(\theta)} 2r sin(\theta) W(r) \Rightarrow$$

$$f^{\theta\phi} = \frac{2W(r)}{r^3 sin(\theta)} \tag{6.22}$$

We substitute $f^{\theta\phi}$ into equation (6.21) and we have

$$-\frac{cos(\theta)}{sin^2(\theta)} \frac{2W(r)}{r^3} + \frac{cos(\theta)}{sin(\theta)} \frac{2W(r)}{r^3 sin(\theta)} = 0$$

Consequently, the ansatz (6.20) satisfies the equation (6.9). Additionally, if $W(r) \neq r^3$, the E/M field strength (6.20) does not satisfy the Bianchi identity. Thus, we have a magnetic monopole solution. The assumption $W(r) \neq r^3$ is reasonable, since a model with $f_{\mu\nu}(r \to \infty) \to \infty$ seems strange.

Then, we can calculate the corresponding magnetic induction:

$$B^a = \frac{1}{2} \frac{\eta_{abc}}{\sqrt{-g}} f_{bc} \tag{6.23}$$

where η_{abc} is the 3-dimensional Levi-Civita symbol with $\eta_{123} = 1$. Also, according the equation (6.17), we have:

$$\sqrt{-g} = \sqrt{B(r)A(r)} r^2 sin(\theta) \tag{6.24}$$

Therefore, the only non-vanishing component of the magnetic induction reads:

$$B^r = \frac{2r sin(\theta) W(r)}{\sqrt{B(r)A(r)} r^2 sin(\theta)} \Rightarrow$$

$$B^r = \frac{1}{\sqrt{B(r)A(r)}} \frac{2W(r)}{r} \tag{6.25}$$

As we emphasized earlier, a monopole solution corresponds to the existence of a mapping with *winding number* $n \neq 0$. In particular, we cause the spontaneous symmetry breaking of the global $O(3)$ symmetry, upon considering the Higgs triplet as a mapping from S^2_∞ to S^2_{vac} with *winding number* $n = 1$. The proper ansatz for the triplet reads:

$$\boxed{\chi^a = \eta f(r) \frac{x^a}{r}} \tag{6.26}$$

with the constraint $f(r \to \infty) \to 1$.

Subsequently, we suppose that the axion-like ansatz is $b(\vec{r}) = b(r)$, to be compatible with the spherically symmetric metric assumption. Thus, we substitute the anätze (6.17), (6.20) and (6.26) into the Einstein field equation (6.10) and we obtain:

For the tt component:

$$G_{tt} = \frac{B(r)}{r^2 A^2(r)}(-A(r) + A^2(r) + rA'(r)) \tag{6.27}$$

$$T_{tt} = -B(r)\left(\frac{\mathcal{L}}{\sqrt{-g}} + R\right) \tag{6.28}$$

Upon considering the stabilized dilaton $\Phi = \Phi_0 = 0$ and the equation (6.11), we have

$$\frac{\mathcal{L}}{\sqrt{-g}} + R = \frac{1}{2}g^{\mu\nu}\partial_\mu\chi^a\partial_\nu\chi^a - \frac{\lambda}{4}(\chi^a\chi^a - \eta^2)^2 + \frac{1}{2}g^{\mu\nu}\partial_\mu b\partial_\nu b - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} \tag{6.29}$$

where

$$\frac{1}{2}g^{\mu\nu}\partial_\mu b(r)\partial_\nu b(r) = -\frac{1}{2A(r)}b'^2(r) \tag{6.30}$$

also we calculate $\frac{g_{\mu\nu}}{2}\partial_\mu\chi^a\partial_\nu\chi^a$

$$\chi^1(\vec{r}) = \eta f(r)sin(\theta)cos(\phi), \quad \chi^2(\vec{r}) = \eta f(r)sin(\theta)sin(\phi) \text{ and } \chi^3(\vec{r}) = \eta f(r)cos(\theta) \tag{6.31}$$

$$\frac{g_{\mu\nu}}{2}\partial_\mu\chi^a\partial_\nu\chi^a = -\frac{1}{2}\Big(\frac{1}{A(r)}\partial_r\chi^a\partial_r\chi^a + \frac{1}{r^2}\partial_\theta\chi^a\partial_\theta\chi^a + \frac{1}{r^2 sin^2(\theta)}\partial_\phi\chi^a\partial_\phi\chi^a\Big) =$$

$$-\frac{\eta^2}{2}\left(\frac{f'^2(r)}{A(r)} + \frac{f^2(r)}{r^2} + \frac{f^2(r)}{r^2}\right) \Rightarrow$$

$$\frac{g_{\mu\nu}}{2}\partial_\mu\chi^a\partial_\nu\chi^a = -\frac{\eta^2}{2}\left(2\frac{f^2(r)}{r^2} + \frac{f'^2(r)}{A(r)}\right) \tag{6.32}$$

and

$$\chi^a(\vec{r})\chi^a(\vec{r}) = \eta^2 f^2(r) \tag{6.33}$$

finally we have to determine the last term of (6.29)

$$-\frac{1}{4}f_{\mu\nu}f^{\mu\nu} = -\frac{1}{2}f_{\theta\phi}f^{\theta\phi} = -\frac{1}{2}2r\ sin(\theta)W(r)\frac{2W(r)}{r^3 sin(\theta)} \Rightarrow$$

$$-\frac{1}{4}f_{\mu\nu}f^{\mu\nu} = -\frac{2W^2(r)}{r^2} \tag{6.34}$$

Upon summarizing the equations (6.30), (6.32), (6.33) and (6.34), the equation (6.29) yields:

$$\boxed{\frac{\mathcal{L}}{\sqrt{-g}} + R = -\frac{\eta^2}{2}\left(2\frac{f^2(r)}{r^2} + \frac{f'^2(r)}{A(r)}\right) - \frac{\lambda\eta^4}{4}(f^2(r)-1)^2 - \frac{1}{2A(r)}b'^2(r) - \frac{2W^2(r)}{r^2}} \tag{6.35}$$

Thus, the tt component of the stress-energy tensor (6.28) reads:

$$\boxed{T_{tt} = B(r)\left[\frac{2W^2(r)}{r^2} + \frac{1}{2A(r)}b'^2(r) + \frac{\eta^2}{2}\left(2\frac{f^2(r)}{r^2} + \frac{f'^2(r)}{A(r)}\right) + \frac{\lambda\eta^4}{4}(f^2(r)-1)^2\right]} \tag{6.36}$$

Consequently, the tt component of the Einstein field equations (6.10) yields:

$$\boxed{\frac{-A(r) + A^2(r) + rA'(r)}{g_N A^2(r)} = 2W^2(r) + \frac{r^2}{2A(r)}b'^2(r) + \frac{\eta^2}{2}\left(2f^2(r) + \frac{r^2 f'^2(r)}{A(r)}\right) + \frac{\lambda\eta^4}{4}(f^2(r)-1)^2 r^2} \tag{6.37}$$

For the rr component:

$$G_{rr} = \frac{B(r) - B(r)A(r) + rB'(r)}{r^2 B(r)} \tag{6.38}$$

$$T_{rr} = A(r)\bigg(\frac{\mathcal{L}}{\sqrt{-g}} + R\bigg) + \partial_r\chi^a\partial_r\chi^a + \partial_r b\partial_r b \tag{6.39}$$

where

$$\partial_r\chi^a\partial_r\chi^a = \eta^2 f'^2(r) \tag{6.40}$$

Hence, we have

$$T_{rr} = A(r)\bigg[-\frac{\eta^2}{2}\bigg(2\frac{f^2(r)}{r^2} + \frac{f'^2(r)}{A(r)}\bigg) - \frac{\lambda\eta^4}{4}(f^2(r)-1)^2 - \frac{1}{2A(r)}b'^2(r) - \frac{2W^2(r)}{r^2}\bigg] + \eta^2 f'^2(r) + b'^2(r) \Rightarrow$$

$$\boxed{T_{rr} = A(r)\bigg[-\frac{2W^2(r)}{r^2} + \frac{1}{2A(r)}b'^2(r) - \frac{\eta^2}{2}\bigg(2\frac{f^2(r)}{r^2} - \frac{f'^2(r)}{A(r)}\bigg) - \frac{\lambda\eta^4}{4}(f^2(r)-1)^2\bigg]} \tag{6.41}$$

Thus, the rr component of the Einstein field equations (6.10) yields:

$$\boxed{\frac{B(r) - B(r)A(r) + rB'(r)}{g_N A(r)B(r)} = -2W^2(r) + \frac{r^2}{2A(r)}b'^2(r) - \frac{\eta^2}{2}\bigg(2f^2(r) - \frac{r^2 f'^2(r)}{A(r)}\bigg) - \frac{\lambda\eta^4}{4}(f^2(r)-1)^2 r^2} \tag{6.42}$$

For the $\theta\theta$ component:

$$G_{\theta\theta} = -\frac{r}{4A(r)}\bigg[2\frac{A'(r)}{A(r)} + r\frac{B'^2(r)}{B^2(r)} + r\frac{A'(r)B'(r)}{A(r)B(r)} - 2\bigg(\frac{B'(r)}{B(r)} + r\frac{B''(r)}{B(r)}\bigg)\bigg] \tag{6.43}$$

$$T_{\theta\theta} = r^2\bigg(\frac{\mathcal{L}}{\sqrt{-g}} + R\bigg) + \partial_\theta\chi^a\partial_\theta\chi^a + \frac{1}{r^2 sin^2(\theta)}(f_{\theta\phi})^2 \tag{6.44}$$

where

$$\partial_\theta\chi^a\partial_\theta\chi^a = \eta^2 f^2(r) \tag{6.45}$$

and

$$\frac{1}{r^2 sin^2(\theta)}(f_{\theta\phi})^2 = \frac{1}{r^2 sin^2(\theta)}4r^2 sin^2(\theta)W^2(r) \Rightarrow$$

$$\frac{1}{r^2 sin^2(\theta)}(f_{\theta\phi})^2 = 4W^2(r) \tag{6.46}$$

Upon substituting the above equations into equation (6.44), we have:

$$T_{\theta\theta} = r^2\bigg[-\frac{\eta^2}{2}\bigg(2\frac{f^2(r)}{r^2} + \frac{f'^2(r)}{A(r)}\bigg) - \frac{\lambda\eta^4}{4}(f^2(r)-1)^2 - \frac{1}{2A(r)}b'^2(r) - \frac{2W^2(r)}{r^2}\bigg] + \eta^2 f^2(r) + 4W^2(r) \Rightarrow$$

$$\boxed{T_{\theta\theta} = 2W^2(r) - \frac{r^2}{2A(r)}b'^2(r) - \frac{\eta^2}{2}\frac{r^2 f'^2(r)}{A(r)} - \frac{\lambda\eta^4}{4}(f^2(r)-1)^2 r^2} \tag{6.47}$$

Thus, the $\theta\theta$ component of the Einstein field equations (6.10) yields:

$$\boxed{\begin{gathered}\tfrac{r}{4g_N A(r)}\bigg[2\tfrac{A'(r)}{A(r)} + r\tfrac{B'^2(r)}{B^2(r)} + r\tfrac{A'(r)B'(r)}{A(r)B(r)} - 2\bigg(\tfrac{B'(r)}{B(r)} + r\tfrac{B''(r)}{B(r)}\bigg)\bigg] = \\ -2W^2(r) + \tfrac{r^2}{2A(r)}b'^2(r) + \tfrac{\eta^2}{2}\tfrac{r^2 f'^2(r)}{A(r)} + \tfrac{\lambda\eta^4}{4}(f^2(r)-1)^2 r^2\end{gathered}} \tag{6.48}$$

Furthermore, we cannot determine the axion-like field $b(r)$ via the equation of motion (6.7), since this equation is satisfied trivially from (6.3). Nevertheless, due to the definition (6.2), the Kalb-Ramond field strength satisfies the Bianchi identity:

$$\epsilon^{\mu\nu\lambda\rho}\partial_\rho H_{\mu\nu\lambda} = \epsilon^{\mu\nu\lambda\rho}\partial_\rho\partial_{[\mu}B_{\nu\lambda]} = 0 \tag{6.49}$$

which yields:

$$\epsilon^{\mu\nu\lambda\rho}\partial_\rho H_{\mu\nu\lambda} = 0 \Rightarrow \epsilon^{\mu\nu\lambda\rho}\partial_\rho(\epsilon_{\mu\nu\lambda}{}^{\sigma}\partial_\sigma b(r)) = 0 \Rightarrow$$

$$\epsilon^{\mu\nu\lambda\rho}\partial_\rho(\sqrt{-g}\tilde{\epsilon}_{\mu\nu\lambda\sigma}g^{\sigma\alpha}\partial_\alpha b(r)) = 0 \Rightarrow$$

$$\epsilon^{\mu\nu\lambda\rho}\tilde{\epsilon}_{\mu\nu\lambda\sigma}\Big(\partial_\rho(\sqrt{-g})g^{\sigma\alpha}\partial_\alpha b(r) + \sqrt{-g}\partial_\rho(g^{\sigma\alpha})\partial_\alpha b(r) + \sqrt{-g}g^{\sigma\alpha}\partial_\rho\partial_\alpha b(r)\Big) = 0 \overset{(6.5)}{\Longrightarrow}$$

$$\epsilon^{\mu\nu\lambda\rho}\epsilon_{\mu\nu\lambda\sigma}\Big(\frac{1}{2}g^{\kappa\xi}\partial_\rho(g_{\kappa\xi})g^{\sigma\alpha}\partial_\alpha b(r) + \partial_\rho(g^{\sigma\alpha})\partial_\alpha b(r) + g^{\sigma\alpha}\partial_\rho\partial_\alpha b(r)\Big) = 0 \overset{\epsilon^{\mu\nu\lambda\rho}\epsilon_{\mu\nu\lambda\sigma}=-6\delta_\sigma^\rho}{\Longrightarrow}$$

$$\frac{1}{2}g^{\kappa\xi}\partial_\rho(g_{\kappa\xi})g^{\rho\alpha}\partial_\alpha b(r) + \partial_\rho(g^{\rho\alpha})\partial_\alpha b(r) + g^{\rho\alpha}\partial_\rho\partial_\alpha b(r) = 0 \Rightarrow$$

$$\frac{1}{2}g^{\kappa\xi}\partial_r(g_{\kappa\xi})g^{rr}\partial_r b(r) + \partial_r(g^{rr})\partial_r b(r) + g^{rr}\partial_r\partial_r b(r) = 0 \overset{(6.17)}{\Longrightarrow}$$

$$-\frac{b'(r)}{2A(r)}\left(\frac{B'(r)}{B(r)} + \frac{A'(r)}{A(r)} + \frac{4}{r}\right) + \frac{A'(r)}{A^2(r)}b'(r) - \frac{1}{A(r)}b''(r) = 0 \Rightarrow$$

$$\frac{1}{2}\left(\frac{B'(r)}{B(r)} - \frac{A'(r)}{A(r)}\right)r^2 b'(r) + 2b'(r)r + r^2 b''(r) = 0 \Rightarrow$$

$$\frac{1}{2}\sqrt{\frac{B(r)}{A(r)}}\frac{B'(r)A(r) - B(r)A'(r)}{A(r)B(r)}r^2 b'(r) + \sqrt{\frac{B(r)}{A(r)}}b'(r)(r^2)' + \sqrt{\frac{B(r)}{A(r)}}r^2 b''(r) = 0 \Rightarrow$$

$$\frac{1}{2}\sqrt{\frac{A(r)}{B(r)}}\frac{B'(r)A(r) - B(r)A'(r)}{A^2(r)}r^2 b'(r) + \sqrt{\frac{B(r)}{A(r)}}b'(r)(r^2)' + \sqrt{\frac{B(r)}{A(r)}}r^2 b''(r) = 0 \Rightarrow$$

$$\left(\sqrt{\frac{B(r)}{A(r)}}\right)' r^2 b'(r) + \sqrt{\frac{B(r)}{A(r)}}b'(r)(r^2)' + \sqrt{\frac{B(r)}{A(r)}}r^2 b''(r) = 0 \Rightarrow$$

$$\frac{d}{dr}\left(\sqrt{\frac{B(r)}{A(r)}}r^2 b'(r)\right) = 0 \Rightarrow$$

$$\boxed{b'(r) = \frac{\zeta}{r^2}\sqrt{\frac{A(r)}{B(r)}}} \tag{6.50}$$

where ζ is a constant of integration.

Finally, we substitute the ansatz (6.26) into equation (6.4), which is the equation of motion of the scalar triplet.

$$g^{\mu\nu}\partial_\mu\partial_\nu\chi^a + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu})\partial_\nu\chi^a = -\lambda\chi^a(\chi^b\chi^b - \eta^2) \overset{a=3}{\Longrightarrow}$$

$$g^{rr}\partial_r\partial_r\chi^3 + g^{\theta\theta}\partial_\theta\partial_\theta\chi^3 + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g})g^{\mu\nu}\partial_\nu\chi^3 + \partial_\mu(g^{\mu\nu})\partial_\nu\chi^a = -\lambda(\chi^b\chi^b - \eta^2)\chi^3 \underset{(6.5)}{\overset{(6.31)}{\Longrightarrow}}$$

$$-\frac{1}{A(r)}\eta f''(r)cos(\theta) + \frac{1}{r^2}\eta f(r)cos(\theta) + \frac{1}{2}g^{\rho\sigma}\partial_\mu(g_{\rho\sigma})g^{\mu\nu}\partial_\nu\chi^3 + \partial_\mu(g^{\mu\nu})\partial_\nu\chi^a = -\lambda(\chi^b\chi^b - \eta^2)\chi^3 \Rightarrow$$

$$-\frac{1}{A(r)}\eta f''(r)cos(\theta) + \frac{1}{r^2}\eta f(r)cos(\theta) + \frac{1}{2}g^{\rho\sigma}\partial_r(g_{\rho\sigma})g^{rr}\partial_r\chi^3 + \frac{1}{2}g^{\rho\sigma}\partial_\theta(g_{\rho\sigma})g^{\theta\theta}\partial_\theta\chi^3 +$$

$$+\partial_r(g^{rr})\partial_r\chi^a = -\lambda(\chi^b\chi^b - \eta^2)\chi^3 \overset{(6.33)}{\Longrightarrow}$$

$$-\frac{1}{A(r)}\eta f''(r)cos(\theta) + \frac{1}{r^2}\eta f(r)cos(\theta) - \frac{\eta}{2A(r)}f'(r)cos(\theta)\left(\frac{B'(r)}{B(r)} + \frac{A'(r)}{A(r)} + \frac{4}{r}\right) +$$

$$+\frac{\eta}{2r^2}f(r)sin(\theta)\frac{2r^2 sin(\theta)cos(\theta)}{r^2 sin^2(\theta)} + \eta\frac{A'(r)}{A^2(r)}f'(r)cos(\theta) = -\lambda\eta^3(f^2(r)-1)f(r)cos(\theta) \Rightarrow$$

$$\frac{f''(r)}{A(r)} - \frac{f(r)}{r^2} + \frac{f'(r)}{2A(r)}\left(\frac{B'(r)}{B(r)} + \frac{A'(r)}{A(r)} + \frac{4}{r}\right) - \frac{f(r)}{r^2} - \frac{A'(r)}{A^2(r)}f'(r) = \lambda\eta^2(f^2(r)-1)f(r) \Rightarrow$$

$$\boxed{\frac{f''(r)}{A(r)} - \frac{1}{2A(r)}\left(\frac{A'(r)}{A(r)} - \frac{B'(r)}{B(r)} - \frac{4}{r}\right)f'(r) - \frac{2f(r)}{r^2} = \lambda\eta^2(f^2(r)-1)f(r)} \tag{6.51}$$

Later on it will be useful to use rescaled dimensionless variables:

$$W(r) \to \frac{W(r)}{\sqrt{g_N}}, \quad r \to \sqrt{g_N}r, \quad b(r) \to \frac{b(r)}{\sqrt{g_N}} \quad \text{and} \quad \eta \to \frac{\eta}{\sqrt{g_N}} \tag{6.52}$$

The equations satisfied by these rescaled variables are the same as (6.37), (6.42) and (6.48) but with g_N replaced by 1.

6.2.2 Asymptotic solutions of the equations of motion

In this subsection, we solve the equations (6.37), (6.42), (6.48) and (6.51) in two asymptotic regions, $r \to 0$ and $r \to \infty$. The existence of the full interpolating solution is assumed and based on continuity in space. In both regions, to leading order, we require $B(r)A(r) \simeq 1$, which is compatible with the far field (Newtonian) limit. In particular, the presence of a non-trivial antisymmetric Kalb-Ramond field strength and of the scalar triplet field with non-trivial vacuum expectation value η, implies modifications in the equation $B(r)A(r) \simeq 1$, which read

$$B(r)A(r) \simeq 1 + \mathcal{O}(r^2), \quad r \to 0 \tag{6.53}$$

$$B(r)A(r) \simeq 1 + \mathcal{O}(\frac{1}{r^2}), \quad r \to \infty \tag{6.54}$$

and are crucial for the consistency of the solutions. In the region $r \to \infty$, where $B(r)A(r) \simeq 1 + O(\frac{1}{r^2})$, equation (6.50) yields:

$$b'(r) \simeq \frac{\zeta}{r^2} \Rightarrow$$

$$b(r) = -\frac{\zeta}{r} + ... \tag{6.55}$$

Consequently, we can recognize ζ as the "axion-charge", or even better as the "Kalb-Ramond torsion charge". In the string-inspired case ($\gamma = 1$), through the dilaton equation (6.12), and in the non-string case ($\gamma = 0$), through the ultra-heavy scalar, the axion-charge is connected with the magnetic charge. The corrections (6.53) and (6.54) are necessary in the case we seek magnetic monopole solutions, since if we suppose $B(r)A(r) = 1 \Rightarrow \frac{A'(r)}{A(r)} = -\frac{B'(r)}{B(r)}$ everywhere is space, the equations (6.37) and (6.42) yield:

$$\frac{r^2}{A(r)}\left[(\eta f'(r))^2 + b'^2(r)\right] = 0 \tag{6.56}$$

which means that necessarily we have $\zeta = 0$ and, as a result, we have vanishing magnetic charge. Thus, the modifications (6.53) and (6.54) contribute to a consistent magnetic monopole solution.

The leading order assumption $B(r)A(r) \simeq 1$ is used in many non-trivial black-hole solutions. For instance, for the Reissner–Nordström (R-N) black hole solution, the metric is [35]:

$$ds^2 = \Delta dt^2 - \frac{dr^2}{\Delta} - r^2 d\Omega^2 \tag{6.57}$$

where $\Delta = 1 - \frac{2GM}{r} + \frac{G\mu^2}{r^2}$, with μ the magnetic charge and M the mass of the black hole. The R-N black hole is not singular at the horizons, since the apparent singularities are coordinate artefacts. In our case, due to the corrections (6.53) and (6.54), we obtain deformed RN-type solutions, but for small monopole mass the singularity at $r = 0$ is naked. Nevertheless, the naked singularity is shielded inside the monopole core, since we consider a "bag-model". Furthermore, an extra assumption implies a positive cosmological constant observed within the core, which regularises the singularity. This aspect is presented in section 6.3.

Because of the use of scaled dimensionless variables, when r is of $\mathcal{O}(1)$ the physical r is of the order of the Planck length. At this scale the equations cannot be expected to be valid because of quantum gravity corrections; hence, in order to be able to estimate the magnetic energy of our monopole, we put the effective Planck length as a lower distance cut-off; in the estimate we will use expressions for our dependent variables which represent their leading asymptotic behaviour for $r \to 0$ and $r \to \infty$.

Considering a RN-type metric and the leading term of $B(r)A(r) \simeq 1$ in the limit $r \to 0$, we have

$$B(r) \sim \frac{p(r)}{r^2} \quad \text{and} \quad A(r) \sim \frac{r^2}{p(r)}, \quad r \to 0 \tag{6.58}$$

and

$$f(r) = f_0 r, \quad r \to 0 \tag{6.59}$$

which is compatible with a vanishing scalar triplet at the origin. Considering the equations (6.50), (6.58) and (6.59) we have

$$b'^2(r) = \frac{\zeta^2}{r^4}\frac{r^4}{p^2(r)} = \frac{\zeta^2}{p^2(r)}, \; A'(r) = \frac{2r}{p(r)} - \frac{r^2 p'(r)}{p^2(r)}, \; B'(r) = \frac{p'(r)}{r^2} - \frac{2p(r)}{r^3} \text{ and } \frac{B'(r)}{B(r)} = -\frac{A'(r)}{A(r)} \tag{6.60}$$

then the equation (6.37) yields

$$1 + \frac{r^2}{p(r)} - r\frac{p'(r)}{p(r)} = 2\frac{W^2(r)r^2}{p(r)} + \frac{\zeta^2 r^2}{2p^2(r)} + \eta^2 f_0^2 \frac{r^4}{p(r)} + \frac{\eta^2}{2} f_0^2 r^2 + \frac{\lambda\eta^4}{4}(f_0^2 r^2 - 1)^2 r^2 \frac{r^2}{p(r)} \tag{6.61}$$

and the equation (6.48) yields

$$\frac{r}{4}\left[4\frac{A'(r)}{A(r)} - 2r\frac{B''(r)}{B(r)}\right] = -2W^2(r)A(r) + \frac{r^2}{2}b'^2(r) + \frac{\eta^2}{2}r^2 f'^2(r) + A(r)\frac{\lambda\eta^4}{4}(f^2(r) - 1)^2 r^2 \Rightarrow$$

$$2 - r\frac{p'(r)}{p(r)} - \frac{r^2}{2}\frac{p''(r)}{p(r)} + 2r\frac{p'(r)}{p(r)} - 3 = -2\frac{W(r)^2}{p(r)}r^2 + \frac{\zeta^2}{2p^2(r)}r^2 + \frac{\eta^2 f_0^2}{2}r^2 + \frac{\lambda}{4}\eta^4(f_0^2 r^2 - 1)^2 r^2 \frac{r^2}{p} \Rightarrow$$

$$-1 + r\frac{p'(r)}{p(r)} - \frac{r^2}{2}\frac{p''(r)}{p(r)} = -2\frac{W(r)^2}{p(r)}r^2 + \frac{\zeta^2}{2p^2(r)}r^2 + \frac{\eta^2 f_0^2}{2}r^2 + \frac{\lambda}{4}\eta^4(f_0^2 r^2 - 1)^2 r^2 \frac{r^2}{p(r)} \tag{6.62}$$

Then, we add equation (6.62) to the equation (6.61) and we obtain:

$$\frac{r^2}{p(r)} - \frac{r^2}{2}\frac{p''(r)}{p(r)} = \frac{\zeta^2 r^2}{p^2(r)} + \eta^2 f_0^2 \frac{r^4}{p(r)} + \eta^2 f_0^2 r^2 + \frac{\lambda}{2}\eta^4(f_0^2 r^2 - 1)^2 r^2 \frac{r^2}{p(r)} \Rightarrow$$

$$1 - \frac{p''(r)}{2} = \frac{\zeta^2}{p(r)} + \eta^2 f_0^2 r^2 + \eta^2 f_0^2 p(r) + \frac{\lambda}{2}\eta^4 r^2 (f_0^2 r^2 - 1)^2 \Rightarrow$$

$$1 - \frac{p''(r)}{2} = \frac{\zeta^2}{p(r)} + \eta^2 f_0^2 (r^2 + p(r)) + \frac{\lambda}{2}\eta^4 r^2 (f_0^2 r^2 - 1)^2 \tag{6.63}$$

The above equation cannot be solved without approximation; on the right-hand side of (6.63), in the denominator of the term proportional to ζ^2, we consider $p(r)$ to be approximately a non-zero constant p_0. Then, the solution of the equation reads:

$$p(r) = c_1 cos(\sqrt{2} f_0 \eta r) + c_2 sin(\sqrt{2} f_0 \eta r) + \frac{Z(r)}{2 f_0^4 \eta^4 p_0} \tag{6.64}$$

Where:

$$Z(r) = \eta^2\big(-2\zeta^2 f_0^2 - 90 f_0^2\lambda p_0 r^2 + 12\lambda p_0 + 4f_0^2 p_0\big)$$
$$+\eta^4\big(-2f_0^4 p_0 r^2 - 12 f_0^2 p_0\lambda r^2 + 15 f_0^4 p_0 \lambda r^4 + p_0\lambda\big)$$
$$+\eta^6\big(-f_0^2 p_0\lambda r^2 + 2f_0^4 p_0\lambda r^4 - f_0^6 p_0 \lambda r^6\big) + 90\lambda p_0$$

Note that the approximation $f(r) = f_0 r$ is compatible with the equation (6.51), in the limit $r \to 0$. Additionally, since η is small (on assuming that the symmetry breaking scale is much smaller than the Planck scale), $p(r)$ is well approximated by a constant near $r = 0$. Then, we keep the expression [3]

$$B(r) \sim \frac{p_0}{r^2} \;\text{ with }\; p_0 = const. \;\text{ and }\; r \to 0 \tag{6.65}$$

Furthermore, we seek the next-to-leading order corrections in the product $A(r)B(r)$, which are induced by the antisymmetric Kalb-Ramond field strength and the non-trivial vacuum expectation value of the scalar fields. We assume that:

$$B(r)A(r) = 1 + \varepsilon(r), \;\text{ for } r \to 0 \tag{6.66}$$

where $\varepsilon(r \to 0) \to 0$. Then, the equation (6.66) implies:

$$\frac{A'(r)}{A(r)} = -\frac{B'(r)}{B(r)} + \frac{\varepsilon'(r)}{1+\varepsilon(r)} \tag{6.67}$$

Upon substituting equations (6.59), (6.65) and (6.67) into equations (6.37) and (6.42), and adding them, we obtain:

$$\frac{r}{A(r)}\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) = \frac{b'^2(r) r^2}{A(r)} + \eta^2\frac{r^2 f_0^2}{A(r)} \Rightarrow$$

$$\frac{rB(r)}{1+\varepsilon(r)}\frac{\varepsilon'(r)}{1+\varepsilon(r)} = B(r)\frac{r^2}{1+\varepsilon(r)}\frac{\zeta^2}{r^4}\frac{1+\varepsilon(r)}{B^2(r)} + B(r)\eta^2\frac{r^2 f_0^2}{1+\varepsilon(r)} \Rightarrow$$

$$\frac{\varepsilon'(r)}{(1+\varepsilon(r))^2} = \frac{\zeta^2}{p_0^2} r + \frac{\eta^2 f_0^2}{1+\varepsilon(r)} r \tag{6.68}$$

Note that $\eta^2\frac{1}{1+\varepsilon(r)} \simeq \eta^2(1-\varepsilon(r)) \simeq \eta^2$, since η is small. Therefore, we have

$$\frac{\varepsilon'(r)}{(1+\varepsilon(r))^2} = \left(\frac{\zeta^2}{p_0^2} + \eta^2 f_0^2\right) r \tag{6.69}$$

which can be solved as follows:

$$-\frac{1}{1+\varepsilon(r)} = \frac{1}{2}\left(\frac{\zeta^2}{p_0^2} + \eta^2 f_0^2\right) r^2 + C$$

We want $\varepsilon(0) = 0 \Rightarrow C = -1$. Also, for $r \to 0$ we have $-\frac{1}{1+\varepsilon(r)} \simeq -1 + \varepsilon(r)$. Hence we obtain

$$\varepsilon(r) = \left(\frac{\zeta^2}{2p_0^2} + \frac{\eta^2 f_0^2}{2}\right) r^2 \tag{6.70}$$

Subsequently, we can determine $W(r)$ in the region $r \to 0$ by considering the non-vanishing term in the equation (6.37). For the calculations we may consider the full expression of $B(r)$ function in the unit system (6.52): $B(r) = 1 - \frac{2M}{r} + \frac{p_0}{r^2}$, which is compatible with the RN-type metric. However, the mass term does not contribute to the $W(r)$ in the limit under consideration.

$$\frac{-A(r) + A^2(r) + rA'(r)}{A^2(r)} = 1 + -\frac{B(r)}{1+\varepsilon(r)} - \frac{rB'(r)}{1+\varepsilon(r)} + \frac{rB(r)\varepsilon'(r)}{(1+\varepsilon(r))^2} =$$

$$= 1 - B(r) + \varepsilon(r)B(r) - rB'(r) + r\varepsilon(r)B'(r) + rB(r)\varepsilon'(r) - 2rB(r)\varepsilon(r)\varepsilon'(r) =$$

$$= 1 - 1 + \frac{2M}{r} - \frac{p_0}{r^2} + \frac{\zeta^2}{2p_0} + \frac{\eta^2 f_0^2 p_0}{2} - \frac{2M}{r} + 2\frac{p_0}{r^2} - \frac{\zeta^2}{p_0} - \eta^2 f_0^2 p_0 + \frac{\zeta^2}{p_0} + \eta^2 f_0^2 p_0 + \mathcal{O}(r) =$$

$$= \frac{p_0}{r^2} + \frac{\zeta^2}{2p_0} + \frac{\eta^2 f_0^2 p_0}{2} \overset{(6.37)}{\Longrightarrow}$$

$$\frac{p_0}{r^2}+\frac{\zeta^2}{2p_0}+\frac{\eta^2 f_0^2 p_0}{2}=2W^2(r)+\frac{r^2}{2A(r)}b'^2(r)+\frac{\eta^2}{2}\left(2f^2(r)+\frac{r^2 f'^2(r)}{A(r)}\right)+\frac{\lambda\eta^4}{4}(f^2(r)-1)^2r^2\Rightarrow$$

$$\frac{p_0}{r^2}+\frac{\zeta^2}{2p_0}+\frac{\eta^2 f_0^2 p_0}{2}=2W^2(r)+\frac{r^2}{2A(r)}b'^2(r)+\frac{\eta^2}{2}\left(2f^2(r)+\frac{r^2 f'^2(r)}{A(r)}\right)+\frac{\lambda\eta^4}{4}(f^2(r)-1)^2r^2\Rightarrow$$

$$\frac{p_0}{r^2}+\frac{\zeta^2}{2p_0}+\frac{\eta^2 f_0^2 p_0}{2}=2W^2(r)+\frac{r^2}{2A(r)}b'^2(r)+\frac{\eta^2}{2}\frac{r^2 f_0^2}{A(r)}+\mathcal{O}(r) \tag{6.71}$$

Note that

$$\frac{r^2}{2A(r)}b'^2(r)=\frac{r^2}{2A(r)}\frac{\zeta^2}{r^4}\frac{A(r)}{B(r)}=\frac{\zeta^2}{2r^2}\frac{1}{1-\frac{2M}{r}+\frac{p_0}{r^2}}\simeq\frac{\zeta^2}{2p_0}$$

and

$$\frac{\eta^2}{2}\frac{r^2 f_0^2}{A(r)}=\frac{\eta^2 r^2 f_0^2}{2}\frac{B(r)}{1+\varepsilon(r)}=\frac{\eta^2 f_0^2 p_0}{2}+\mathcal{O}(r)$$

Consequently, the equation (6.71) implies:

$$\frac{p_0}{r^2}+\frac{\zeta^2}{2p_0}+\frac{\eta^2 f_0^2 p_0}{2}=2W^2(r)+\frac{\zeta^2}{2p_0}+\frac{\eta^2 f_0^2 p_0}{2}\Rightarrow$$

$$W^2(r)=\frac{p_0}{2r^2}+\mathcal{O}(r) \tag{6.72}$$

This is a very important result, as well as predictable, since the corresponding magnetic induction is proportional to $W(r)$, hence the magnetic charge is proportional to $\sqrt{p_0}$. Therefore, the charge-term of the RN-type metric is proportional to the square of the magnetic charge, as it should be. The normalisation factor is fixed in the string-inspired case ($\gamma = 1$). Also, $W(r) \propto \frac{1}{r}$, as we noticed earlier.

Let us focus on the non-string framework ($\gamma = 0$). In this case, the equation (6.12) does not contains the axion-like field and the E/M field strength. The correlation between the axion-charge and the magnetic charge is explicitly confirmed, if we consider an expression of the E/M field strength similar to the 't Hooft-Polyakov model (4.67). Under this consideration, the term of the E/M field strength that makes non-vanishing the dual current (4.72) consist of the fields that contribute to the magnetic monopole structure. In the 't Hooft-Polyakov model we have only the scalar triplet, while in our model we have the Kalb-Ramond field strength $H_{\rho\mu\nu}$, the ultra-heavy scalar Φ, which is singlet under the global $O(3)$ group, and the scalar triplet χ^a. Thus, we can write the term that contributes to the topological current as follows:

$$f_{\mu\nu}=-H_{\rho\sigma\kappa}\Phi^\rho\partial_\mu\Phi^\sigma\partial_\nu\Phi^\kappa \tag{6.73}$$

where

$$\Phi^\mu=(\Phi,\chi^1,\chi^2,\chi^3) \tag{6.74}$$

and

$$H_{\rho\mu\nu}=\epsilon_{\rho\mu\nu\sigma}g^{\sigma\xi}\partial_\xi b(r) \tag{6.75}$$

We can determine an explicit expression for the E/M field strength:

$$f_{\mu\nu}=-\epsilon_{\rho\sigma\kappa\lambda}g^{\lambda\xi}\partial_\xi b(r)\Phi^\rho\partial_\mu\Phi^\sigma\partial_\nu\Phi^\kappa\Rightarrow$$

$$f_{\mu\nu}=\frac{b'(r)}{A(r)}\epsilon_{\rho\sigma\kappa r}\Phi^\rho\partial_\mu\Phi^\sigma\partial_\nu\Phi^\kappa$$

We consider that the ultra-heavy scalar Φ is stabilized in a non-zero value Φ^0, due to the self-interacting potential $V(\Phi)$. Then, we obtain:

$$f_{\mu\nu}=\frac{b'(r)}{A(r)}\epsilon_{tr\sigma\kappa}\Phi^0\partial_\mu\Phi^\sigma\partial_\nu\Phi^\kappa\Rightarrow$$

$$f_{\mu\nu} = \frac{\zeta}{r^2}\frac{\sqrt{-g}}{\sqrt{B(r)A(r)}}\tilde{\epsilon}_{tr\sigma\kappa}\Phi^0\partial_\mu\Phi^\sigma\partial_\nu\Phi^\kappa \Rightarrow$$

$$f_{\mu\nu} = \zeta sin(\theta)\tilde{\epsilon}_{tr\sigma\kappa}\Phi^0\partial_\mu\Phi^\sigma\partial_\nu\Phi^\kappa$$

Finally, if we consider that the triplet Φ^a defines a S^3-spatial coordinate set $(\eta f(r), \theta, \phi)$ which maps the $SO(3)$ internal space to the three-space, the non-vanishing components of the E/M field strength read:

$$f_{\theta\phi} = -f_{\phi\theta} = \zeta\eta\Phi^0 sin(\theta) \tag{6.76}$$

Comparing with the equation (6.20), we obtain:

$$W(r) = \frac{\zeta\eta\Phi^0}{2r} \tag{6.77}$$

This equation is valid everywhere in space. The corresponding magnetic induction reads:

$$\vec{B}(\vec{r}) = \frac{\zeta\eta\Phi^0}{r^2}\hat{r} \tag{6.78}$$

The above equation is valid in the limits $r \to 0$ and $r \to \infty$, where $B(r)A(r) = 1$. The interpretation of this amazing result is that a system of an axion-like field with axion-charge $-\zeta$, an ultra-heavy scalar field stabilised at a non-zero value and a scalar triplet which acts as a map with *winding number* $n = 1$, effectively form a magnetic monopole, whose magnetic charge $g = \zeta\eta\Phi^0$ is determined by the axion-charge via the ultra-heavy particle.

In the context of the string-inspired low energy theory ($\gamma = 1$), the dilaton equation (6.12) adds a constraint to the system. Thus, without loss of generality, the value of the constant Φ^0 can be fixed at $\Phi^0 = 0$. Hence, the dilaton equation implies:

$$\partial_\mu b\partial^\mu b + \frac{1}{4}f_{\mu\nu}f^{\mu\nu} = 0 \Rightarrow$$

$$-\frac{1}{A(r)}b'^2(r) + \frac{1}{4}f_{\mu\nu}f^{\mu\nu} = 0 \Rightarrow$$

$$-\frac{\zeta^2}{r^4}\frac{1}{B(r)} + \frac{1}{4}f_{\mu\nu}f^{\mu\nu} = 0$$

Note that the spatial part of the E/M field strength yields: $\frac{1}{4}f_{ij}f^{ij} = \frac{1}{2}(B^r)^2A(r)$. Therefore, we obtain:

$$\frac{1}{2}(B^r)^2A(r) = \frac{\zeta^2}{r^4}\frac{1}{B(r)} \Rightarrow$$

$$B^r = \frac{1}{\sqrt{B(r)A(r)}}\frac{\sqrt{2}\zeta}{r^2} \tag{6.79}$$

Note that this expression is valid everywhere in space. In the extreme case, where $A(r)B(r) = 1$, we obtain:

$$\boxed{B^r = \frac{\sqrt{2}\zeta}{r^2}} \tag{6.80}$$

The magnetic charge reads:

$$\boxed{g = \sqrt{2}\zeta} \tag{6.81}$$

Additionally, the equation (6.25) implies:

$$B^r = \frac{2W(r)}{r} = \frac{\sqrt{2}\zeta}{r^2} \Rightarrow$$

$$\boxed{W(r) = \frac{\zeta}{\sqrt{2}r}} \tag{6.82}$$

Finally, upon comparing the above equation with the equation (6.72) we have

$$W(r) = \frac{\sqrt{p_0}}{\sqrt{2}r} = \frac{\zeta}{\sqrt{2}r} \Rightarrow$$

$$\boxed{p_0 = \zeta^2} \tag{6.83}$$

The equations (6.81) and (6.83) explicitly prove that the magnetic charge is determined by the axion-charge and that the charge-term of the RN-type metric is proportional to the square of the magnetic charge.

In order to finish the small-r analysis we need to prove that the ansatz $f(r) = f_0 r$ is compatible with the equation of motion (6.51) at the limit $r \to 0$, where $B(r) \sim \frac{p_0}{r^2}$:

$$\frac{f''(r)}{A(r)} - \frac{1}{2A(r)}\left(\frac{A'(r)}{A(r)} - \frac{B'(r)}{B(r)} - \frac{4}{r}\right)f'(r) - \frac{2f(r)}{r^2} = \lambda\eta^2(f^2(r)-1)f(r) \overset{\mathcal{O}(r^3)}{\Longrightarrow}$$

$$f''(r) = \frac{2-\lambda\eta^2}{p_0}r^2 \tag{6.84}$$

These equations can be solved in terms of parabolic cylinder functions which are analytic in the neighbourhood of $r = 0$ [3]. A solution exists, which (for small r) is proportional to $r + \frac{r^3}{3p_0}$. Therefore, $f(r) = f_0 r + \mathcal{O}(r^3)$, hence we have a consistent ansatz.

Additionally, let us proceed with the large-r case. We consider the ansatz

$$B(r)A(r) = 1 + \frac{\varepsilon_0}{r^2} \tag{6.85}$$

where $\varepsilon_0 = const. \in \mathbb{R}$. The above equation yields

$$\frac{A'(r)}{A(r)} = -\frac{B'(r)}{B(r)} - \frac{2\varepsilon_0}{r^3} \tag{6.86}$$

Also, in the region $r \to \infty$, a compatible expression of $B(r)$ for a monopole model implies a deficit angle:

$$B(r) \sim 1 + \beta_1 + \frac{\beta_2}{r} + \frac{\beta_3}{r^2}, \quad \text{for } r \to \infty \tag{6.87}$$

Additionally, we consider the asymptotic ansatz:

$$f(r) = 1 - \frac{\alpha_1}{r^2} + \delta(r), \quad \text{with } \alpha_1 = const. \tag{6.88}$$

The $\frac{1}{r}$-order of the equation (6.51) yields [3]:

$$\alpha_1 = \frac{1}{\lambda\eta^2} \tag{6.89}$$

Moreover, considering as leading order $\sim \frac{1}{r}$ in the equation (6.37), we obtain [3]:

$$\beta_1 = -\eta^2 < 0 \tag{6.90}$$

Hence, the term corresponding to the deficit angle is negative, as it was in the global monopole model in chapter 5. Furthermore, the equation (6.51), upon ignoring terms of $\frac{1}{r^2}$-order, implies a differential equation with respect to $\delta(r)$ [3]:

$$(1-\eta^2)\delta''(r) + \frac{2}{r}(1-\eta^2)\delta'(r) - 2\lambda\eta^2\delta(r) = 0 \tag{6.91}$$

whose solution reads:

$$\delta(r) = \frac{1}{r}e^{-\eta\sqrt{\frac{2\lambda}{1-\eta^2}}r} \tag{6.92}$$

This is a decaying solution, exponentially small and can be ignored. Additionally, upon adding the equations (6.37) and (6.42), we obtain:

$$\varepsilon_0 = -\frac{\zeta^2}{2(1-\eta^2)^2} \tag{6.93}$$

Also, similarly to the small-r case, the equation (6.37) implies:

$$W^2(r) \simeq \frac{1}{2r^2}\Big(\beta_3 + \frac{1}{\lambda}\Big) \tag{6.94}$$

which solves the equation (6.48). Upon using the string-inspired analysis, we have

$$\beta_3 = \zeta^2 - \frac{1}{\lambda} \tag{6.95}$$

Finally, we compare the metric ansatz (6.87) with the RN metric and we determine β_2

$$\beta_2 = -2M \tag{6.96}$$

where M is the ADM mass of the monopole. The next sections are devoted to the determination of the mass of the monopole.

6.3 The total rest-energy of the monopole

The total rest-energy of the magnetic monopole (rest-mass of the particle) reads:

$$\mathcal{E} = \int\int\int T_t{}^t r^2 sin(\theta) dr d\theta d\phi \tag{6.97}$$

Upon substituting equation (6.36) into the above equation, we obtain:

$$\mathcal{E}(r) = 4\pi \int_0^r dr'\, r'^2 \left[\frac{2W^2(r')}{r'^2} + \frac{1}{2A(r')} b'^2(r') + \frac{\eta^2}{2}\left(2\frac{f^2(r')}{r'^2} + \frac{f'^2(r')}{A(r')}\right) + \frac{\lambda\eta^4}{4}(f^2(r') - 1)^2\right] \tag{6.98}$$

From now on it is convenient to use the following approximation of the metric for every r:

$$B(r) = 1 - 8\pi G\eta^2 - \frac{2mG}{r} + \frac{8\pi G p_0}{r^2} \tag{6.99}$$

and

$$B(r)A(r) \approx 1 \tag{6.100}$$

Equivalently, we may use again dimensionless variables ($8\pi G = 1$) [4]:

$$B(r) = 1 - \eta^2 - \frac{2M}{r} + \frac{p_0}{r^2} \tag{6.101}$$

where

$$p_0 = \zeta^2\ , \quad M \equiv \frac{m}{8\pi} \text{ and } r \to \frac{r}{\sqrt{8\pi}} \tag{6.102}$$

In this units r, M and η are dimensionless quantities (or, equivalently, expressed in reduced Planck mass scale units, in which the Planck mass is $M_P = \ell_P^{-1} = \sqrt{8\pi}$, with ℓ_P the Planck length.

Also, note that the associated with (6.99) and (6.100) Ricci scalar reads:

$$R = -\frac{16\pi G\eta^2}{r^2} \tag{6.103}$$

which is singular at the origin. The curvature singularity at the origin corresponds to the non-vanishing deficit angle $8\pi G\eta^2$. For $\eta \ll 1$ we will prove that $M < |\zeta|$, hence there are no R-N horizons. The naked singularity is shielded inside the monopole core. Under specific circumstances, which are discussed later on, the naked singularity is spontaneously regularised, due to an effective positive cosmological constant at the origin. This cosmological constant takes place in the region where $\chi^a(r \to 0) \to 0$. Consequently, as we discuss later on, a more detailed analysis in the region ($r \to 0$), which includes the origin, corresponds to a de Sitter space-time and not to a singular RN-like space-time. Thus, the mass of the monopole must be concentrated inside a spherical shell around the de Sitter region. We may assume that this region is determined by the approximation $r \gg 1$, where $f(r) = 1 - corrections \neq 0$. Additionally, it is evident that the integral (6.98), from a radius $\delta \gg 1$ to infinity, divergences. It is an expected result, since the $SO(3)$ triplet do not vanish in infinity. Therefore, we consider a cut-off L_c and our model becomes a bag-model. Hence, the monopole mass is distributed inside a spherical shell with radii δ, L_c, where $1 \ll \delta < L_c$. We may write the inner radius as

$$\delta = aL_c \ , \quad \text{with} \ \ a < 1 \tag{6.104}$$

Considering the monopole mass in flat space-time ($M_{core} \sim \lambda^{-\frac{1}{2}}\eta$), for $\eta \ll 1$, gravity is expected not to change dramatically the structure of the monopole at small distances. Nevertheless, considering the equations (6.88) and (6.89) and the radius of the monopole core in flat space-time ($\delta_{flat} \sim \lambda^{-\frac{1}{2}}\eta^{-1}$), we obtain:

$$f(\delta_{flat}) = 1 - \frac{\lambda^{-1}\eta^{-2}}{\delta^2_{flat}} = 0 \neq 1 \tag{6.105}$$

Hence, the radius L_c of the monopole core in the examined model differs significantly from the flat space-time case, i.e. $L_c \gg \delta_{flat}$.

Let us proceed with the calculation of the rest-mass of the monopole in the case $r \to \infty$, $\eta \ll 1$, $\lambda \gg 1$, $L_c > \delta \gg 1$ and $\delta = aL_c$. We will need the equations (6.50), (6.82), (6.88), (6.89), (6.92), (6.100) and (6.101), which we summarize hear:

$$W(r) = \frac{\zeta}{\sqrt{2}r} \ , \quad A(r)B(r) \approx 1 \ , \quad A(r) \approx B(r) \approx 1$$

$$b'(r) = \frac{\zeta}{r^2} \ , \quad f(r) = 1 - \frac{1}{\lambda\eta^2 r^2} \ , \quad f^2(r) - 1 \approx -\frac{2}{\lambda\eta^2 r^2} \tag{6.106}$$

Thus, the rest-mass of the monopole reads:

$$\mathcal{E} = 4\pi \int_{aL_c}^{L_c} dr\, r^2 \left[\frac{3\zeta^2}{2r^4} + \frac{\eta^2}{2}\left(\frac{2}{r^2} - \frac{4}{\lambda\eta^2 r^4} + \frac{4}{\lambda^2\eta^4 r^6}\right) + \frac{\lambda\eta^4}{4}\frac{4}{\lambda^2\eta^4 r^4}\right] \Rightarrow$$

$$\mathcal{E} \sim 4\pi \int_{aL_c}^{L_c} dr \left[\frac{3\zeta^2}{2r^2} + \eta^2 - \frac{2}{\lambda r^2} + \frac{1}{\lambda r^2}\right] \Rightarrow$$

$$\mathcal{E} \sim 4\pi \int_{aL_c}^{L_c} dr \left[\frac{3\zeta^2}{2r^2} + \eta^2 - \frac{1}{\lambda r^2}\right] \Rightarrow$$

$$\mathcal{E} \sim \frac{1-a}{a}\left[6\pi\zeta^2 - \frac{4\pi}{\lambda}\right]\frac{1}{L_c} + 4\pi\eta^2(1-a)L_c \tag{6.107}$$

Note that due to the large λ, which ensures that the scalar fields χ^a approach their vacuum expectation values, the right-hand-side of (6.107) is practically independent of the coupling λ. In this case we have:

$$\mathcal{E} \sim \frac{1-a}{a}\frac{6\pi\zeta^2}{L_c} + 4\pi\eta^2(1-a)L_c \tag{6.108}$$

We may determine the core radius L_c by minimizing the total energy:

$$\frac{1-a}{a}\frac{6\pi\zeta^2}{L_c^2} = 4\pi\eta^2(1-a) \Rightarrow$$

$$L_c^2 = \frac{3\zeta^2}{2a\eta^2} \Rightarrow$$

$$\boxed{L_c = \sqrt{\frac{3}{2}}\frac{|\zeta|}{\sqrt{a}\eta}} \tag{6.109}$$

Thus, upon substituting equation (6.109) into equation (6.108), we obtain:

$$\mathcal{E} \sim 6\pi\sqrt{\frac{2}{3}}\frac{1-a}{\sqrt{a}}\eta|\zeta| + 4\pi\sqrt{\frac{3}{2}}\frac{1-a}{\sqrt{a}}\eta|\zeta| \Rightarrow$$

$$\boxed{\mathcal{E} \sim 4\sqrt{6}\pi\frac{1-a}{\sqrt{a}}\eta|\zeta|} \tag{6.110}$$

Finally, we may prove that there are no R-N horizons in our model:

$$1 - \frac{2M}{r} + \frac{\zeta^2}{r^2} = 0 \Rightarrow$$

$$r_{\pm} = M\left(1 \pm \sqrt{1 - \frac{\zeta^2}{M^2}}\right) \tag{6.111}$$

Note that:

$$M \leq \mathcal{E} \sim 4\sqrt{6}\pi \frac{1-a}{\sqrt{a}} \eta |\zeta| \overset{\eta \lll 1}{\Longrightarrow}$$

$$M < |\zeta| \tag{6.112}$$

Consequently, there are no R-N horizons. Note that we use symbol $\mathcal{E}$ for the rest-mass, in order to avoid the confusion with the Schwarzschild mass M, which differs from the total mass as we prove later on. In the following section we present the regularisation procedure of the naked R-N singularity.

6.4 Regularisation of the curvature singularity

It is very interesting that the rest-mass of the monopole (6.110) is not fully determined. The parameters η and ζ must be determined experimentally. Hence, we have a remaining unknown parameter, a. This parameter will be determined by a regularisation of the curvature singularity at the origin.

6.4.1 Effective cosmological constant

As we mentioned before, our analysis for the region around the origin is uncompleted. To be more precise, near the origin it is possible to be generated effectively a positive cosmological constant. Therefore, at the region of the origin we have a de Sitter space-time instead of a singular RN-like space-time. The inspiration for the subsequent discussion comes from the strongly coupled string theory, which might be expected near the origin, due to the strong gravity in this region. Thus, the string coupling reads $g_s(r \to 0) = e^{\Phi(r \to 0)} \gg 1$. In other words the dilaton field Φ, near the origin, is stabilised at a very large value, due to the self-interacting potential $V(\Phi)$. Consequently, the terms $-\frac{1}{12}e^{-2\Phi}H_{\rho\mu\nu}H^{\rho\mu\nu} - \frac{1}{4}e^{-\Phi}f_{\mu\nu}f^{\mu\nu}$ of the Lagrangian (6.1) vanish near the origin. Additionally, note that $\chi^a(r \to 0) \to 0$. Therefore, the stress-energy tensor (6.16) reads:

$$T_{\mu\nu} = g_{\mu\nu}\frac{\lambda\eta^4}{4} \tag{6.113}$$

Hence, the corresponding Einstein field equations are:

$$G_{\mu\nu} = g_{\mu\nu} 2\pi G\lambda\eta^4 \tag{6.114}$$

Thus, we recognise the positive cosmological constant:

$$\Lambda = 2\pi G\lambda\eta^4 > 0 \tag{6.115}$$

which is similar to the case of the self-gravitating global monopole described in chapter 5. Consequently, we may consider a region around the origin with radius $\delta \gg 1$ (Planck length scale), where the space-time is de Sitter with cosmological constant (6.115). Additionally, note that the region between the radii δ and L_c is the shell, which contains the most of the monopole mass. In this region, the metric has the form of a R-N space-time with an angular deficit proportional to the global $SO(3)$ spontaneous symmetry breaking scale η^2, equations (6.99) and (6.100). Outside the core of radius L_c, which is sufficiently large compared to the Planck length, the metric is approximately Minkowski-like with deficit angle $8\pi G\eta^2$. We depict the structure of the self-gravitating magnetic monopole in figure 6.1. Note that, the fact that the metric is not asymptotically Minkowski, due to the deficit angle, does not violate the Birkhoff's theorem, since the scalar $SO(3)$ triplet does not vanish in infinity.

6.4.2 Israel conditions

The cut-off radius δ, for the de Sitter space-time, can be determined by employing the Israel procedure [26] based on matching of two spherical regions in space described by different metrics. First of all, the system seems to be consistent, since the de Sitter space-time extents to the region where we approximately have $A(r)B(r) \approx 1 + \mathcal{O}(\frac{1}{r^2})$. Hence, the matching of the de Sitter and RN-like metrics is possible. Moreover, we assume that the two regions (R-N with deficit angle and de Sitter) are separated by a thin shell, on which the stress-energy tensor vanishes and, as a result, no energy flows through the boundary surface at

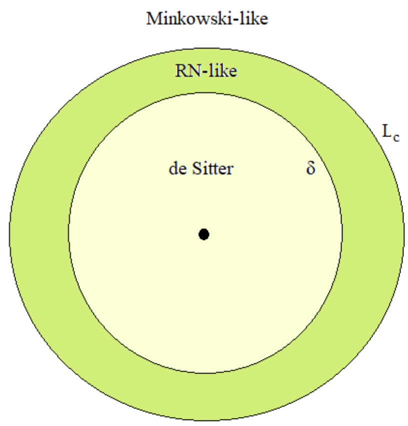

Figure 6.1: The magnetic monopole structure. Around the origin, in the region with radius $\delta \gg 1$, the space-time is de Sitter. The most of the monopole mass is concentrated inside the thin spherical shell with radii $L_c > \delta \gg 1$. In this region the space-time is R-N with deficit angle. Outside the monopole core, i.e. for $r > L_c$, the spacetime is Minkowski with deficit angle.

$r = \delta$. The Israel conditions demand the continuity of the metric and its derivatives on the thin shell. Let us apply Israel conditions to our case. The metric of our model can be written as follows:

$$ds^2_{reg} = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2\Big(d\theta^2 + sin^2(\theta)d\phi^2\Big) \tag{6.116}$$

where

$$f(r) = B_1(r)\Theta(\delta - r) + B(r)\Theta(r - \delta) \tag{6.117}$$

with

$$B_1(r) = 1 - \frac{\Lambda r^2}{3} \tag{6.118}$$

and $B(r)$ given by equation (6.101). In our static case, the Israel conditions read:

$$\begin{aligned} B_1(r = \delta) &= B(r = \delta) \\ \tfrac{d}{dr}B_1(r)\Big|_{r=\delta} &= \tfrac{d}{dr}B(r)\Big|_{r=\delta} \end{aligned} \tag{6.119}$$

Hence, we can determine the radius δ and the Schwarzschild mass M.

$$B_1(r = \delta) = B(r = \delta) \Rightarrow$$

$$\frac{\Lambda}{3}\delta^2 = \eta^2 + \frac{2M}{\delta} - \frac{\zeta^2}{\delta^2} \Rightarrow$$

$$\boxed{\delta(\Lambda) = \frac{\eta}{\sqrt{2\Lambda}}\left(1 + \sqrt{1 + \frac{4\zeta^2\Lambda}{\eta^4}}\right)^{\frac{1}{2}} \approx \Lambda^{-\frac{1}{4}}\sqrt{|\zeta|} > 0} \tag{6.120}$$

The above approximation ($\frac{4\zeta^2\Lambda}{\eta^4} \gg 1$) needs to be examined in every case separately. Also, we have

$$\frac{d}{dr}B_1(r)\Big|_{r=\delta} = \frac{d}{dr}B(r)\Big|_{r=\delta} \Rightarrow$$

$$\boxed{2M(\Lambda) = -\frac{2\Lambda}{3}\delta^3(\Lambda) + \frac{2\zeta^2}{\delta(\Lambda)}} \tag{6.121}$$

We denote the parameters as functions of the cosmological constant, since we want to emphasize that some results will be independent from Λ. Note that, if we ignore Kalb-Ramond term ($\zeta = 0$), the monopole will not be magnetic and will have negative mass. Hence, there will be repulsive forces and the monopole will not be able to be stabilised. This case is similar to that examined in chapter 5. Thus, for sufficiently large ζ^2, indeed Kalb-Ramond torsion and the mechanism that contributes to the existence of the effective magnetic charge solve the problem of the negative mass. Moreover, note that the cosmological constant (6.115) depends on the parameter η. Hence, the scalar $SO(3)$ triplet and the corresponding Higgs self-interacting potential contribute to the regularisation of the naked singularity.

Then, we substitute the cosmological constant (6.115), in units $8\pi G = 1$, into equation (6.120), while we consider the conditions $\lambda \gg 1$, $|\zeta| \gg 1$, $\lambda\zeta^2 \gg 1$, $\eta^2 \ll 1$ and $\lambda\eta^2 \ll 1$, which are consistent with the aforementioned discussion, and we obtain:

$$\delta = \sqrt{2}\lambda^{-\frac{1}{2}}\eta^{-1}\left(1+\sqrt{1+\zeta^2\lambda}\right)^{\frac{1}{2}} \Rightarrow$$

$$\delta \approx \sqrt{2}\lambda^{-\frac{1}{2}}\eta^{-1}|\zeta|^{\frac{1}{2}}\lambda^{\frac{1}{4}} \Rightarrow$$

$$\boxed{\delta \approx \sqrt{2}\lambda^{-\frac{1}{4}}\eta^{-1}|\zeta|^{\frac{1}{2}} = \sqrt{2}\frac{|\zeta|}{\eta}(\zeta^2\lambda)^{-\frac{1}{4}} \gg 1} \tag{6.122}$$

Note that $\delta \gg 1$ as we expected. Furthermore, we calculate the Schwarzschild mass M of the monopole:

$$M \approx -\frac{\lambda\eta^4}{12}2^{\frac{3}{2}}\frac{|\zeta|^3}{\eta^3}(\zeta^2\lambda)^{-\frac{3}{4}} + \zeta^2\frac{\eta}{\sqrt{2}|\zeta|}(\zeta^2\lambda)^{\frac{1}{4}} \Rightarrow$$

$$M \approx \left(\frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}}\right)|\zeta|\eta(\zeta^2\lambda)^{\frac{1}{4}} \Rightarrow$$

$$\boxed{M = \frac{m}{8\pi} \approx 0.47|\zeta|\eta(\zeta^2\lambda)^{\frac{1}{4}} > 0} \tag{6.123}$$

Note that $\frac{M}{\eta} \sim |\zeta|(\zeta^2\lambda)^{\frac{1}{4}} \gg 1$, hence $M \gg \eta$, in other words, the Schwarzschild mass M of the monopole is much greater than the $SO(3)$ spontaneous symmetry breaking scale.

Moreover, we can determine the parameter a:

$$a = \frac{\delta}{L_c} = \sqrt{2}\frac{|\zeta|}{\eta}(\zeta^2\lambda)^{-\frac{1}{4}}\sqrt{\frac{2}{3}}\frac{\sqrt{a}\eta}{|\zeta|} \Rightarrow$$

$$\boxed{a(\Lambda) = \frac{2}{3}\zeta^{-2}\eta^2\delta^2(\Lambda) = \frac{4}{3}(\zeta^2\lambda)^{-\frac{1}{2}} \ll 1} \tag{6.124}$$

Note that although $a \ll 1$, it is evident that $\delta \gg 1$, as we expected. Upon substituting equation (6.124) into equation (6.110), the rest-mass reads:

$$\boxed{\mathcal{E}(\Lambda) \sim 8\pi\frac{\sqrt{6}}{2}\frac{\eta|\zeta|}{\sqrt{a(\Lambda)}} = 8\pi\frac{3}{2}\frac{\zeta^2}{\delta(\Lambda)}}\,, \quad \text{for } a \ll 1 \tag{6.125}$$

$$\boxed{\mathcal{E} \sim 8\pi\ 1.06(\zeta^2\lambda)^{\frac{1}{4}}\eta|\zeta|} \tag{6.126}$$

Hence, we compare it with the Schwarzschild mass (6.123) of the monopole and we obtain:

$$\boxed{\frac{\mathcal{E}}{m} = 2.26} \tag{6.127}$$

Due to the angular deficit $8\pi G\eta^2$ in the asymptotic space-time and the gravitational binding energy of the system, the mass coefficient appearing in the gravitational potential is different from the total rest-energy, which in a flat space-time would be considered as the total monopole rest mass. This violation of the Weak Equivalence Principle is related with the invalidity of the Birkhoff's theorem, due to the deficit angle, as well as, it is correlated with the fact that the examined monopole is a finite structured object.

6.5 The effective mass of the monopole

In this section, we discuss the concept of the effective mass according to the work [36], which is equivalent with the calculation of the Komar integral for the Killing vector ∂_t. First of all, let us examine the case of the Reissner-Nordström space-time. In such a case, the charge, like the mass, induces curvature of space-time. In our case, the existence of the magnetic charge acts proportionally. Considering a spherical surface of radius R and centre at R-N singularity location, we may define two kind of "effective mass": the effective mass enclosed by the spherical surface (M_{Eff}^{Int}) and the effective mass associated with the exterior region (M_{Eff}^{Ext}). The M_{Eff}^{Int} can be calculated by the Whittaker's theorem [36][37]:

$$M_{Eff}^{Int} = \frac{1}{4\pi}\oint_{v_2} dv_2 V_{,i}n^i \ , \ \ n^i = (V,0,0) \tag{6.128}$$

where, in spherical polars, $dv_2 = \delta^2 sin(\theta)d\theta d\phi$ and V^2 denotes the temporal component of the metric tensor, as defined by the invariant line element $ds^2 = V^2dt^2 - g_{ij}dx^idx^j$. Particularly, in the case of the R-N metric we have:

$$V^2 = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{6.129}$$

where M is the Schwarzschild mass and Q the charge. Hence, V reads:

$$V = \Big(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\Big)^{\frac{1}{2}} \tag{6.130}$$

Thus, from equation (6.128), for $r = R$, we obtain:

$$M_{Eff}^{Int} = M - \frac{Q^2}{R} \tag{6.131}$$

Note that the result is identical with the one obtained by the corresponding Komar integral. Similarly, M_{Eff}^{Ext} is calculated in [36]:

$$M_{Eff}^{Ext} = \frac{Q^2}{R} \tag{6.132}$$

Subsequently, we may calculate the total effective mass

$$M_{Eff}^{Tot} = M_{Eff}^{Int} + M_{Eff}^{Ext} = M \tag{6.133}$$

which is the Schwarzschild mass (ADM mass) of the R-N metric. It is interesting that the effective mass M_{Eff}^{Int} appears in the expression for the radial acceleration of a neutral test particle falling into the R-N black hole [36]:

$$\frac{d^2r}{d\tau^2} = -\frac{1}{r^2}\Big(M - \frac{Q^2}{r}\Big) \tag{6.134}$$

where τ is the proper time. Note that the gravitational field varies with the distance r and becomes repulsive when the effective mass $M - \frac{Q^2}{r}$ becomes negative, i.e. when $r < \frac{Q^2}{M}$.

Considering our model, we must point out that the hypothesis of a bag-model is actually not a choice but a necessity, since the exterior total rest-energy of the monopole divergences due to the deficit angle η^2. Let us consider as M_{Eff}^{Int} the effective mass contained in the de Sitter region $r < \delta$. Then we recognise:

$$V = \Big(1 - \frac{1}{3}\Lambda r^2\Big)^{\frac{1}{2}} \tag{6.135}$$

Hence, using Whittaker's theorem [36][37] we calculate:

$$M_{Eff}^{Int}(\Lambda) = -\frac{1}{3}\Lambda\delta^3 < 0 \tag{6.136}$$

The negative value of the M_{Eff}^{Int} was expected, since the positive cosmological constant implies repulsive forces. Considering the equation (6.121) we may write:

$$\boxed{M_{Eff}^{Int}(\Lambda) = M(\Lambda) - \frac{\zeta^2}{\delta(\Lambda)} \underset{(6.120)(6.121)}{\overset{\Lambda=\frac{\lambda\eta^4}{4}}{=}} -\frac{1}{2}M(\Lambda)} \tag{6.137}$$

If we require that the sum of both exterior and interior effective masses should equals the Schwarzschild mass (6.123), i.e. the ADM mass, the M_{Eff}^{Ext} reads:

$$\boxed{M_{Eff}^{Ext}(\Lambda) = \frac{\zeta^2}{\delta(\Lambda)} = \frac{3}{2}M(\Lambda)} \tag{6.138}$$

The result $M_{Eff}^{Ext}(\Lambda) = \frac{\zeta^2}{\delta(\Lambda)}$ is a consistent result with the R-N case for any value of the cosmological constant. The total rest-energy (6.126) in the units (6.102), for $\Lambda = \frac{\lambda\eta^4}{4}$, is written as follows:

$$\mathcal{E} = 1.06(\zeta^2\lambda)^{\frac{1}{4}}\eta|\zeta| = 2.26M \tag{6.139}$$

It is very interesting that, comparing the total rest-energy (6.125) with M_{Eff}^{Ext}, for any value of Λ, we have:

$$\boxed{\mathcal{E}(\Lambda) = 1.5M_{Eff}^{Ext}(\Lambda)} \tag{6.140}$$

The last result is valid for any value of the cosmological constant with the restriction $a \ll 1$. Note that the total rest-energy of the monopole in the spherical shell is larger than the effective mass in the region exterior to the core ($\mathcal{E} > M_{Eff}^{Ext}$). This can be attributed to the non-zero contributions of the Kalb-Ramond axion-like field $b(\vec{r})$ and the electromagnetic fields to the energy functional $\mathcal{E}$, as well as the gravitational self-binding energy [4] and the non-vanishing scalar triplet at infinity. Under the assumptions that the components of the total momentum vector of the system are zero and the object is considered "quasi-static", which means that there is no significant energy present in the form of gravitational waves, then the monopole's effective mass $\tilde{m}$ can be defined as (in units of the speed of light in vacuum):

$$\tilde{m} = E_{total} + E_{binding} \tag{6.141}$$

where the binding energy satisfies the inequality: $E_{binding} < 0$, due to the repulsive nature of the self-gravitating scalar triplet, which is examined in chapter 5. In our case, for large core radius, the criterion of (approximate) space-time flatness, along with the other assumptions, is satisfied; one may thus identify $E_{total} = \mathcal{E}$ and $\tilde{m} = M_{Eff}^{Ext}$. Hence, the fact that the total rest-energy is larger than the exterior effective mass is naturally explained. We also note that, for us, the role of the infalling neutral matter is played by the Kalb-Ramond axion-like pseudoscalar field, which thus will accumulate on the surface of radius δ, since it is on this surface that the radial acceleration will vanish.

In order to conclude, in view of the negative effective mass contributions of the de Sitter regulator, the Weak Equivalence Principle, where one would equate the total rest-energy $\mathcal{E}$ with the inertial mass, fails. However, this should be expected for gravitating extended objects, as is our case, given that the Weak Equivalence Principle characterises point-like masses. Note that the total rest-energy and M_{Eff}^{Ext} are of the same order of magnitude, therefore we can conclude that our monopole moves similarly to a point particle.

6.6 Satisfaction of the Weak Equivalence Principle

The Weak Equivalence Principle can be accomplished by regulating the cosmological constant. Let us assume that the dilaton potential takes a non-zero value near the origin, as such:

$$\mathcal{E} = M_{Eff}^{Ext} \quad \text{and} \quad M_{Eff}^{Ext} = \frac{\zeta^2}{\delta} \tag{6.142}$$

In order for the equation (6.142) to be satisfied, we need to work with a cosmological constant that corresponds to $a \approx 1$, otherwise the satisfaction of the equation (6.140) is inevitable. Thus, we introduce a modified cosmological constant:

$$\Lambda(\xi) = \frac{\xi^2}{4}\lambda\eta^4 > 0\ , \quad \xi \in \mathbb{R} \tag{6.143}$$

in the units $8\pi G = 1$. Then:

$$\delta(\xi) \approx \Lambda^{-\frac{1}{4}}(\xi)\sqrt{|\zeta|} = \sqrt{2}\frac{1}{\sqrt{|\xi|}}\frac{|\zeta|}{\eta}(\zeta^2\lambda)^{-\frac{1}{4}} \gg 1 \tag{6.144}$$

$$M(\xi) = -\frac{\sqrt{2}}{6}\eta\sqrt{|\xi|}|\zeta|(\zeta^2\lambda)^{\frac{1}{4}} + \frac{|\zeta|\sqrt{|\xi|}}{\sqrt{2}}\eta(\zeta^2\lambda)^{\frac{1}{4}} \Rightarrow$$

$$m(\xi) = 8\pi\ \frac{\sqrt{2}}{3}\sqrt{\xi}\eta|\zeta|(\zeta^2\lambda)^{\frac{1}{4}} \tag{6.145}$$

Hence, the parameter a reads:

$$a(\xi) = \frac{\delta(\xi)}{L_c} = \sqrt{2}\frac{1}{\sqrt{|\xi|}}\frac{|\zeta|}{\eta}(\zeta^2\lambda)^{-\frac{1}{4}}\sqrt{\frac{2}{3}}\frac{\sqrt{a(\xi)}\eta}{|\zeta|} \Rightarrow$$

$$a(\xi) = \frac{2}{3}\frac{\eta^2}{\zeta^2}\delta^2(\xi) = \frac{4}{3|\xi|}(\zeta^2\lambda)^{-\frac{1}{2}} \tag{6.146}$$

As we proved earlier, if we suppose that $a \ll 1$, the ratio (6.140) remain invariant. In this case, the weak equivalence principle cannot me satisfied. This is a consistent result, because the assumption $a \ll 1$ corresponds to a monopole that differs significantly from a point-like particle. Hence, the parameter a needs to satisfy the condition $a \approx 1$, which implies a thin shell, which contains most of the mass of the monopole. Upon substituting equation (6.146) into equation (6.110), the rest-mass reads:

$$\mathcal{E} \sim 4\sqrt{6}\pi\frac{1-a}{\sqrt{a}}\eta|\zeta| \Rightarrow$$

$$\mathcal{E} \sim 8\pi\,\frac{3}{2}\frac{\zeta^2}{\delta(\xi)}\Big(1 - \frac{4}{3|\xi|}(\zeta^2\lambda)^{-\frac{1}{2}}\Big) \tag{6.147}$$

Note that we did not use the approximation (6.125). Thus, in the units $8\pi G = 1$, we have:

$$\mathcal{E} \sim \frac{3}{2}\Big(1 - \frac{4}{3|\xi|}(\zeta^2\lambda)^{-\frac{1}{2}}\Big)M_{Eff}^{Ext} \tag{6.148}$$

Consequently, the equation (6.142) is satisfied if we demand:

$$|\xi| = 4(\zeta^2\lambda)^{-\frac{1}{2}} \ll 1 \tag{6.149}$$

Note that $|\xi| \ll 1 \Rightarrow a = \frac{1}{3} \approx 1$. Consequently, the Weak Equivalence Principle is satisfied when the monopole mass is contained in an extremely thin shell, or, in other words, when the monopole looks like a point-particle. Additionally, we need to check the validity of the approximation (6.120). Note that $\Lambda = \frac{4\eta^4}{|\zeta|}$, hence $\frac{4\zeta^2\Lambda}{\eta^4} = 16$, which is one order of magnitude larger than the Planck scale "1". Consequently, the approximation is questionable. Nevertheless, the result makes physical sense, which can be formulated as follows: for a small enough value of the cosmological constant, the shell, which contains the most of the mass of the monopole, becomes extremely thin, hence the monopole acts like a point-like particle and the Weak Equivalence Principle is approximately satisfied.

Finally, it is essential to point out that this choice of regularisation scheme [4] is consistent with a negative dilaton potential in the de Sitter region, where the string theory (the Ultra-Violet (UV) completion of our low energy model) is strongly coupled and such a potential might be generated, for instance, through non-perturbative string-loop corrections. Outside the de Sitter region, where the gravity is weakened and, string theory is weakly coupled and our low energy model is an effective description of the dynamics, the tree level dilaton potential vanishes due to arguments based on conformal invariance [32][33]. Thus, the asymptotic solutions, which are valid outside the de Sitter region, are not affected.

6.7 Quantization of the magnetic charge and experimental constraints of the model

The quantization of the magnetic charge g, which is proportional to the torsion charge ζ, may be yielded by the standard Dirac argument [38], which considers the gauge transformations of the quantum relativistic wave-function ψ of an electron field (with electric charge e) in the presence of the Dirac-string singular vector potential $A(r)$ for the monopole magnetic field $\vec{B}$ (with $\vec{B} = \nabla\vec{A}$). Upon requiring that the wave-function is single-valued under the appropriate (singular) gauge transformations, we obtain the Dirac quantization condition:

$$g\,e = \frac{n}{2}\ ,\quad n \in \mathbb{Z} \tag{6.150}$$

Hence, the torsion charge 6.81 satisfies the following quantization rule:

$$\zeta\,e = \frac{n}{2\sqrt{2}}\ ,\quad n \in \mathbb{Z} \tag{6.151}$$

Furthermore, considering the lowest magnetic charge ($n = 1 \Rightarrow \zeta = \frac{1}{2\sqrt{2}}$) and the current bounds on the (scalar) monopole mass at the LHC, from ATLAS [39] and MoEDAL [40] experiments, we impose a lower limit on the value of the monopole rest-mass (6.126):

$$\mathcal{E} \sim 5.6\lambda^{\frac{1}{4}}\eta > 790\ GeV \tag{6.152}$$

where $\lambda \gg 1$ and $\eta \ll M_P$. Note that the mass limits for the monopoles are not very reliable, given that these experimental limits are based on Drell-Yan or photon-fusion processes, which are approximations that are not valid due to the strong magnetic coupling of a magnetic monopole. Moreover, the production cross section for magnetic monopoles might be strongly suppressed for monopoles with structure, like the ones relevant to this Thesis.

6.8 Conclusions

Searching for models that predict magnetic monopoles with reduced positive mass, which may be detected in current or future colliders, we consider a string inspired model with Kalb-Ramond torsion [3] and investigate its solutions in the spirit of the 't Hooft-Polyakov [23] and Barriola-Vilenkin [24] models. Thus, we determine a self-gravitating global solution, which corresponds to a "bag-model" with a Coulomb-like magnetic field. Consequently, we determine the magnetic charge of the corresponding magnetic monopole, which is proportional to the torsion charge. Despite the fact that the Higgs triplet, in the examined model, plays the role of a 2-loop with winding number $n \neq 0$, the quantization condition of the topological charge does not correspond to the quantization condition of the 't Hooft-Polyakov case. This is because the term that violates the Bianchi identity depends on either the axion-like field, the ultra-heavy scalar and scalar triplet, in the $\gamma = 0$ case, or the axion-like field, in the $\gamma = 1$ case. Therefore, an electromagnetic strength tensor of the form (4.67) cannot be obtained. Nevertheless, the charge quantization condition is yielded by the standard Dirac argument [38]. Additionally, the naked curvature singularity of the model is regularized [4], if we consider a de Sitter core inside the magnetic monopole. Hence, upon imposing the Israel conditions [26], we can write the radius and the ADM mass of the magnetic monopole with respect to the parameters η and λ. These parameters may be determined experimentally. For instance, we impose a lower limit on the magnetic monopole mass by considering the recent results from ATLAS [39] and MoEDAL [40] experiments.

Finally, we cannot help but mention that, in our recent paper [41], we prove that the Barriola-Vilenkin model [24] may get a positive mass if we consider it embedded in a higher curvature gravity. To be more precise, we focus on a recently developed model of Pedro G. S. Fernandes [42], which includes a conformally coupled scalar field by only requiring conformal invariance of the scalar field equation of motion and not of the action. Hence, the theory incorporates a scalar-Gauss-Bonnet sector with a coupling parameter α. In our original work, we discuss self-gravitating global $O(3)$ monopole solutions, associated with the spontaneous breaking of the $O(3)$ symmetry down to a global $O(2)$, in the above extended Gauss-Bonnet theory of gravity in $(3 + 1)$-dimensions. Thus, we determine self-gravitating monopole solutions and regularize their curvature singularity by considering a de Sitter core inside the monopole. Then, upon imposing the Israel conditions [26], we determine the radius of the de Sitter core and the ADM mass of the global monopole, which is positive. Therefore, inside the de Sitter core we have repulsive forces, while outside the core we have attractive forces. Consequently, considering these forces, the monopole seems to be stable, however a detailed stability analysis is pending.

Appendix A

Point splitting method

We may explicitly prove that the equation (2.17) for regular vector potential reduces to standard Hamiltonian, if we take its average over all directions of the parameter $\vec{\varepsilon}$. Since, $\vec{A}$ is regular and $\varepsilon \to 0$, equation (2.19) implies:

$$E = exp\Big[ie\int_{\vec{r}-\frac{\vec{\varepsilon}}{2}}^{\vec{r}+\frac{\vec{\varepsilon}}{2}} \vec{A}(\vec{\xi})d\vec{\xi}\Big] = exp\big[ie(\vec{\varepsilon}\cdot\vec{A}(\vec{r}))\big] \tag{A.1}$$

Also, we need to expand the exponentials in equation (2.17) up to the second order of $\vec{\varepsilon}$:

$$1 - e^{-i\frac{\vec{p}\vec{\varepsilon}}{2}}Ee^{-i\frac{\vec{p}\vec{\varepsilon}}{2}} \approx 1 - \Big(1 - i\frac{\vec{p}\cdot\vec{\varepsilon}}{2} - \frac{(\vec{p}\cdot\varepsilon)^2}{8}\Big)\Big(1 + ie\vec{A}\cdot\vec{\varepsilon} - \frac{e^2}{2}(\vec{A}\cdot\vec{\varepsilon})^2\Big)\Big(1 - i\frac{\vec{p}\cdot\vec{\varepsilon}}{2} - \frac{(\vec{p}\cdot\varepsilon)^2}{8}\Big) \Rightarrow$$

$$1 - e^{-i\frac{\vec{p}\vec{\varepsilon}}{2}}Ee^{-i\frac{\vec{p}\vec{\varepsilon}}{2}} \approx i\vec{\pi}\cdot\vec{\varepsilon} + \frac{(\vec{p}\cdot\vec{\varepsilon})^2}{2} - \frac{e}{2}(\vec{A}\cdot\vec{\varepsilon})(\vec{p}\cdot\vec{\varepsilon}) - \frac{e}{2}(\vec{p}\cdot\vec{\varepsilon})(\vec{A}\cdot\vec{\varepsilon}) + \frac{e^2}{2}(\vec{A}\cdot\vec{\varepsilon})^2 + ... \tag{A.2}$$

Then, in order to take the average of (2.17) over all directions of the parameter $\vec{\varepsilon}$, we integrate on the surface of a sphere with radius $|\vec{\varepsilon}|$:

$$H = \frac{3}{m\varepsilon^2}\Big[\int_0^{2\pi}\int_{-1}^{1}\varepsilon^2 dcos(\theta)d\phi\Big]^{-1}\int_0^{2\pi}\int_{-1}^{1}\Big(1 - e^{-i\vec{p}\vec{\varepsilon}}Ee^{-i\vec{p}\vec{\varepsilon}}\Big)\varepsilon^2 dcos(\theta)d\phi \tag{A.3}$$

In the subsequent steps we calculate each term of the equation (A.3) separately. Without loss of generality, we consider that $\hat{z}$ is parallel to $\vec{\pi}$ and we have:

$$\frac{3}{4\pi m\varepsilon^4}i\int_0^{2\pi}\int_{-1}^{1}(\vec{\pi}\cdot\vec{\varepsilon})\varepsilon^2 dcos(\theta)d\phi = \frac{3}{4\pi m\varepsilon^4}2\pi i\varepsilon^3|\vec{\pi}|\int_{-1}^{1}cos(\theta)dcos(\theta) = 0 \tag{A.4}$$

Additionally, considering that $\hat{z}$ is parallel to $\vec{p}$, we obtain:

$$\frac{3}{4\pi m\varepsilon^4}\int_0^{2\pi}\int_{-1}^{1}\frac{(\vec{p}\cdot\vec{\varepsilon})^2}{2}\varepsilon^2 dcos(\theta)d\phi = \frac{3}{4m\varepsilon^2}\int_{-1}^{1}p^2\varepsilon^2cos^2(\theta)dcos(\theta) \Rightarrow$$

$$\frac{3}{4\pi m\varepsilon^4}\int_0^{2\pi}\int_{-1}^{1}\frac{(\vec{p}\cdot\vec{\varepsilon})^2}{2}\varepsilon^2 dcos(\theta)d\phi = \frac{\vec{p}^2}{2m} \tag{A.5}$$

Subsequently, in order to integrate the third term of the expansion (A.2), we consider that $\hat{z}$ is parallel to $\vec{p}$ and that $\vec{A} = \frac{(\vec{A}\cdot\vec{p})\vec{p}}{p^2} + A_\perp\hat{x}$, which implies that $\hat{x}$ is parallel to $\vec{A}_\perp$. Hence, we have the equation: $\vec{A}\cdot\vec{\varepsilon} = \frac{(\vec{A}\cdot\vec{p})}{p}\varepsilon\ cos(\theta) + A_\perp\varepsilon\ sin(\theta)cos(\phi)$. Thus, we obtain:

$$-\tfrac{e}{2}\tfrac{3}{4\pi m\varepsilon^4}\int_0^{2\pi}\int_{-1}^{1}(\vec{A}\cdot\vec{\varepsilon})(\vec{p}\cdot\vec{\varepsilon})\varepsilon^2 dcos(\theta)d\phi =$$

$$-\tfrac{e}{2}\tfrac{3}{4\pi m\varepsilon^2}\Big[\int_0^{2\pi}\int_{-1}^{1}(\vec{A}\cdot\vec{p})\varepsilon^2cos^2(\theta)dcos(\theta)d\phi + \underbrace{\int_0^{2\pi}\int_{-1}^{1}A_\perp p\varepsilon^2 sin(\theta)cos(\phi)dcos(\theta)d\phi}_{\to 0}\Big] \Rightarrow$$

$$-\frac{e}{2}\frac{3}{4\pi m\varepsilon^4}\int_0^{2\pi}\int_{-1}^{1}(\vec{A}\cdot\vec{\varepsilon})(\vec{p}\cdot\vec{\varepsilon})\varepsilon^2 dcos(\theta)d\phi = -\frac{e(\vec{A}\cdot\vec{p})}{2m} \tag{A.6}$$

Similarly, the integration of the forth term of the expansion (A.2) reads:

$$-\frac{e}{2}\frac{3}{4\pi m\varepsilon^4}\int_0^{2\pi}\int_{-1}^{1}(\vec{p}\cdot\vec{\varepsilon})(\vec{A}\cdot\vec{\varepsilon})\varepsilon^2 dcos(\theta)d\phi = -\frac{e(\vec{p}\cdot\vec{A})}{2m} \tag{A.7}$$

Finally, we calculate the last term of (A.2) by considering that $\hat{z}$ is parallel to $\vec{A}$:

$$\frac{e^2}{2}\frac{3}{4\pi m\varepsilon^4}\int_0^{2\pi}\int_{-1}^{1}(\vec{A}\cdot\vec{\varepsilon})^2\varepsilon^2 dcos(\theta)d\phi = \frac{3e^2}{4m\varepsilon^4}\vec{A}^2\varepsilon^4\int_{-1}^{1}cos^2 dcos(\theta)d\phi \Rightarrow$$

$$\frac{e^2}{2}\frac{3}{4\pi m\varepsilon^4}\int_0^{2\pi}\int_{-1}^{1}(\vec{A}\cdot\vec{\varepsilon})^2\varepsilon^2 dcos(\theta)d\phi = \frac{(e\vec{A})^2}{2m} \tag{A.8}$$

Upon substituting equations (A.2), (A.4), (A.5), (A.6), (A.7) and (A.8) into equation (A.3), we obtain:

$$H = \frac{\vec{\pi}^2}{2m}\ , \quad \text{with} \quad \vec{\pi} = \vec{p} - e\vec{A} \tag{A.9}$$

The average of the regularised angular momentum (2.18), over all directions of parameter $\vec{\varepsilon}$, is calculated similarly, providing the standard angular momentum, in the case where $\vec{A}$ is regular. Let us present the calculations. Note that, in this case, the only non-vanishing term of the expansion (A.2) is $i\vec{\pi}\cdot\vec{\varepsilon}$, due to the imposition of the limit $|\vec{\varepsilon}| \to 0$. Hence, the average of the angular momentum (2.18) reads:

$$L_i = \frac{3}{4\pi i\varepsilon^4}\epsilon_{ijk}x_j\int_0^{2\pi}\int_{-1}^{1}\varepsilon_k\left(1 - e^{-i\frac{\vec{p}\vec{\varepsilon}}{2}}Ee^{-i\frac{\vec{p}\vec{\varepsilon}}{2}}\right)\varepsilon^2 dcos(\theta)d\phi \Rightarrow$$

$$L_i = \frac{3}{4\pi\varepsilon^2}\epsilon_{ijk}x_j\int_0^{2\pi}\int_{-1}^{1}\varepsilon_k(\vec{\pi}\cdot\vec{\varepsilon})dcos(\theta)d\phi \tag{A.10}$$

Note that $\varepsilon_k = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, where:

$$\begin{aligned} \varepsilon_1 &= \varepsilon\ sin(\theta)cos(\phi) \\ \varepsilon_2 &= \varepsilon\ sin(\theta)sin(\phi) \\ \varepsilon_3 &= \varepsilon\ cos(\theta) \end{aligned} \tag{A.11}$$

Also, it is useful to set

$$I_k = \int_0^{2\pi}\int_{-1}^{1}\varepsilon_k(\vec{\pi}\cdot\vec{\varepsilon})dcos(\theta)d\phi \quad \text{with} \quad k = 1,2,3 \tag{A.12}$$

and calculate the three components separately.

$$I_1 = \varepsilon^2\int_0^{2\pi}\int_{-1}^{1}sin(\theta)cos(\phi)\Big(\pi_1 sin(\theta)cos(\phi) + \pi_2 sin(\theta)sin(\phi) + \pi_3 cos(\theta)\Big)dcos(\theta)d\phi \Rightarrow$$

$$I_1 = \frac{4\pi}{3}\pi_1\varepsilon^2 \tag{A.13}$$

Similarly, the remaining two components read:

$$I_2 = \frac{4\pi}{3}\pi_2\varepsilon^2 \tag{A.14}$$

$$I_3 = \frac{4\pi}{3}\pi_3\varepsilon^2 \tag{A.15}$$

Upon substituting $I_k = \frac{4\pi}{3}\pi_k\varepsilon^2$ into equation (A.10), we obtain:

$$L_i = \frac{3}{4\pi\varepsilon^2}\epsilon_{ijk}x_j\frac{4\pi}{3}\pi_k\varepsilon^2 \Rightarrow$$

$$\vec{L} = \vec{r}\times\vec{\pi} \tag{A.16}$$

which is the desirable result, in case we assume a regular vector potential. Otherwise, if the vector potential is singular, equations (2.17) and (2.18) are considered as definitions of the respective operators, in order for the commutative relations (2.20) and (2.21) to be satisfied.

Appendix B

Superalgebra $OSp(1,1)$

Let us prove some of the relations (2.92). To be more precise, in the next few lines, we verify the anti-commutation relations between the generators of the supertransformations, which are given in equation (2.91). It would be useful to summarize the generators of the supergroup $OSp(1,1)$:

$$Q = \sqrt{M}\dot{\vec{r}} \cdot \vec{\xi}\,, \quad S = -tQ + \sqrt{M}\vec{r} \cdot \vec{\xi}$$

$$H = \tfrac{\vec{\pi}^2}{2M} - i\tfrac{\mu}{2Mr^3}\epsilon_{ijk} r_i \xi_j \xi_k \tag{B.1}$$

$$D = Ht - \tfrac{M}{4}(\vec{r} \cdot \dot{\vec{r}} + \dot{\vec{r}} \cdot \vec{r})\,, \quad K = -Ht^2 + 2Dt + \tfrac{M\vec{r}^2}{2}$$

where $\mu = eg$. Let us verify relation $\{Q, Q\} = 2H$. We will use the following identity:

$$\{AB, CD\} = A[B, C]D + AC\{B, D\} - [A, C]DB + C[D, A]B \tag{B.2}$$

Hence, we have:

$$\{Q, Q\} = M\{\dot{r}_i\xi_i, \dot{r}_j\xi_j\} \overset{(B.2)}{=} M\big(\dot{\vec{r}}^2 - \xi_j\xi_i[\dot{r}_i, \dot{r}_j]\big) \underset{\vec{\pi}=M\dot{\vec{r}}}{\overset{(2.12)}{\Longrightarrow}}$$

$$\{Q, Q\} = \frac{\vec{\pi}^2}{M} - \xi_j\xi_i\frac{ie}{M}\epsilon_{ijk}B_k \underset{\vec{B}=g\frac{\vec{r}}{r^3}}{\overset{\{\xi_i,\xi_j\}=\delta_{ij}}{\Longrightarrow}}$$

$$\{Q, Q\} = 2\Big(\frac{\vec{\pi}^2}{2M} - i\frac{\mu}{2Mr^3}\epsilon_{ijk}r_i\xi_j\xi_k\Big) \overset{(B.1)}{\Longrightarrow}$$

$$\{Q, Q\} = 2H \tag{B.3}$$

Moreover, we may verify relation $\{Q, S\} = -2D$.

$$\{Q, S\} = -t\{Q, Q\} + \{\pi_i\xi_i, r_j\xi_j\} \underset{(B.3)}{\overset{(B.2)}{\Longrightarrow}}$$

$$\{Q, S\} = -2tH + \vec{\pi}\vec{r} + i(\vec{\xi})^2 \underset{(B.3)}{\overset{(\vec{\xi})^2=(\frac{\vec{\sigma}}{\sqrt{2}})^2=\frac{3}{2}}{\Longrightarrow}}$$

$$\{Q, S\} = -2tH + \frac{1}{2}(2\vec{\pi} \cdot \vec{r} + i\delta_{ii}) \overset{[x_i,\pi_j]=i\delta_{ij}}{\Longrightarrow}$$

$$\{Q, S\} = -2\Big(tH - \frac{M}{4}(\vec{r} \cdot \dot{\vec{r}} + \dot{\vec{r}} \cdot \vec{r})\Big) \overset{(B.1)}{\Longrightarrow}$$

$$\{Q, S\} = -2D \tag{B.4}$$

Finally, we will prove that $\{S, S\} = 2K$.

$$\{S, S\} = t^2\{Q, Q\} - 2\sqrt{M}\{Q, r_j\xi_j\} + M\{r_i\xi_i, r_j\xi_j\} \Rightarrow$$

$$\{S, S\} = 2t^2H - 2t\{r_i\xi_i, \pi_j\xi_j\} + M\{r_i\xi_i, r_j\xi_j\} \overset{(B.2)}{\Longrightarrow}$$

$$\{S,S\} = 2t^2H - 2tr_i\pi_j\delta_{ij} + 2ti\xi_j\xi_i\delta_{ij} + Mr_ir_j\delta_{ij} \Rightarrow$$

$$\{S,S\} = 2t^2H - 2t\vec{r}\vec{\pi} + ti\delta_{ii} + Mr_ir_j\delta_{ij} \overset{[r_i,\pi_j]=i\delta_{ij}}{\Longrightarrow}$$

$$\{S,S\} = -2t^2H + 4t^2H - t\vec{r}\vec{\pi} - t\vec{\pi}\vec{r} + Mr_ir_j\delta_{ij} \overset{(B.1)}{\Longrightarrow}$$

$$\{S,S\} = -2t^2H + 4tD + Mr_ir_j\delta_{ij} \overset{(B.1)}{\Longrightarrow}$$

$$\{Q,S\} = 2K \tag{B.5}$$

Appendix C

Lie groups

The *Lie groups* are groups, as also differentiable manifolds, with the property that the group operations are compatible with the smooth structure. To be more precise, the group elements are smooth differentiable functions of some finite set of parameters $\theta_a \in \mathbb{R}$, and the group operation "$\odot$" depends smoothly on those parameters. A typical element of a Lie group can be represented as a "matrix valued complex phase": $g = g(\theta_1, ..., \theta_n) = e^{i\theta_a T^a} \equiv e^{i\vec{\theta}\vec{T}}$, $a = 1, ..., n$, where θ_a are continuous parameters. Note that any typical element of a Lie group is continuously connected to the identical element. The T^a are called generators of the Lie group, since any element of the Lie group, continuously connected to the identity element, can be constructed by them.

Algebra is a vector space $V = \{U^i\}$ equipped with a binary operation "$\odot$" corresponding to a linear superposition of the algebra elements: $U^i \odot U^j = f^{ij}_{\ \ k} U^k$. The coefficients $f^{ij}_{\ \ k}$ are called structure constants.

Algebra Lie is an algebra, whose elements are the generators of the Lie group and the corresponding binary operation is the commutation of the generators: $[T^a, T^b] = i f^{ab}_{\ \ c} T^c$, where the structure constant is completely antisymmetric. If the metric of the Lie group is Euclidean, it does not matter if the indices are up or down. Additionally, the *associativity* property of the group implies the Jacobi identity:

$$[[T^a, T^b], T^c] + [[T^c, T^a], T^b] + [[T^b, T^c], T^a] = 0 \tag{C.1}$$

Let us consider a Lie group with generators T^a, $a = 1, ..., n$. Then, a representation of the group in the form of $n \times n$ matrices can be the following:

$$(T^a)^{bc}_{adjoint} = -i f^{abc} \tag{C.2}$$

Consequently, it can be proven that the structure constants, i.e. $(T^a)^{bc}_{adjoint}$, satisfy the Lie algebra. Firstly, we may prove a relation between the f^{abc}, which corresponds to the Jacobi identity (C.1). Hence, the equation (C.1) implies:

$$[[T^a, T^b], T^c] + [[T^c, T^a], T^b] + [[T^b, T^c], T^a] = 0 \Rightarrow$$

$$\big(i f^{bci} i f^{aij} + i f^{cai} i f^{bij} + i f^{abi} i f^{cij}\big) T^j = 0 \overset{i \leftrightarrow c}{\Rightarrow}$$

$$f^{bic} f^{acj} + f^{iac} f^{bcj} + f^{abc} f^{icj} = 0 \tag{C.3}$$

Subsequently, we calculate the commutative relation $[T^a_{adj}, T^b_{adj}]$ as follows:

$$[T^a_{adj}, T^b_{adj}]^{ij} = (-i f^{aik})(-i f^{bkj}) - (-i f^{bik})(-i f^{akj}) \overset{(C.3)}{=} i f^{abc} (-i f^{cij}) \Rightarrow$$

$$[T^a_{adj}, T^b_{adj}]^{ij} = i f^{abc} (T^c_{adj})^{ij} \tag{C.4}$$

Note that, if the examined Lie group is the group $SU(2)$, the structure constants are represented by the standard Levi-Civita symbol. In this case, the adjoint representation of the group has purely complex generators. Hence, the elements of the group ($U(\theta_1, \theta_2, \theta_3) = e^{i\theta_a T^a}$, with $\theta_a \in \mathbb{R}$) have real components. Therefore, we have $U^\dagger = U^T$.

Appendix D

Notions of Topology

In the next few lines we present some notions of differential geometry and topology [1][35].

A *topological space* is a set X, endowed with a topology T. A *topology* is a family of open subsets $\{U_i\}_{i\in I} \subseteq X$, which obey:

i) $X \in T$ and $\emptyset \in T$
ii) if K is any finite subcollection of I, the family $\{U_k\}_{k\in K}$ satisfies $\bigcap_{k\in K} U_k \in T$
iii) if J is any (maybe infinite) subcollection of I, the family $\{U_j\}_{j\in J}$ satisfies $\bigcup_{j\in J} U_k \in T$

A topological space is called *Hausdorff*, if for every $p, q \in X$, there exist neighborhoods $U_p, U_q \in T$ such that $p \in U_p$, $q \in U_q$ and $U_p \cap U_q = \emptyset$.

A *homeomorphism* between topological spaces X and $\hat{X}$ is a map $f:\ \hat{X} \to X$, which is

i) *injective* (or one-to-one): for $p \neq q \Rightarrow f(p) \neq f(q)$
ii) *surjective* (or onto): for every $p \in X$, there is a $\hat{p} \in \hat{X}$ such that $f(\hat{p}) = p$
Functions which are both *injective* and *surjective* are called *bijective*. This ensures that they have an inverse $f^{-1}:\ X \to \hat{X}$.
iii) *bicontinuous*: Both f and its inverse f^{-1} are continuous. To define a notion of continuity, the properties of *topological space* is needed. A *homeomorphism* f is continuous if for all $U \in X$, $f^{-1}(U) \in \hat{X}$

An n-dimensional *differentiable manifold* is a *Hausdorff topological space* M equipped with a family of pairs $\{(U_i, \phi_i)\}$ under the constraint $\bigcup_i U_i = M$ such that:

i) M is locally *homeomorphic* to $\mathbb{R}^n$ by the mapping $\phi_i:\ U_i \to U_i'$, with U_i' an open subset of $\mathbb{R}^n$
ii) Given two overlapping U_i, U_j $(U_i \cap U_j \neq \emptyset)$, the map $\psi_{ij} = \phi_i \circ \phi_j^{-1}:\ \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is smooth (also called infinitely differentiable or C^∞)
The maps ϕ_i are calls *charts* and allow us to assign a coordinate system on the manifold. The map ψ_{ij} is called *transition function* and acts as a coordinate transformation.

Similarly, a *complex manifold* is locally *homeomorphic* to $\mathbb{C}^n$ and the corresponding *transition function* is *holomorphic*, i.e. is complex differentiable in a neighbourhood of each point in a domain in complex coordinate space $\mathbb{C}^n$. A complex function $f(x, y) = u(x, y) + iv(x, y)$ is *holomorphic*, if and only if, Cauchy-Riemann equations are satisfied throughout the domain we are dealing with.

Let $f:\ M \to N$ be a map from an m-dimensional manifold M to an n-dimensional manifold N. Thus, a point $p \in M$ is mapped to a point $f(p) \in N$, namely $f:\ p \to f(p)$. Taking a chart (U, ϕ) on M and (V, ψ) on N, where $p \in U$ and $f(p) \in V$, f takes the coordinate representation $\psi \circ f \circ \phi^{-1}:\ \mathbb{R}^m \to \mathbb{R}^n$. If $\psi \circ f \circ \phi^{-1}$ is C^∞ to the coordinates $\mathbb{R}^m$, then f is said to be differentiable (smooth) at $p \in M$.

Let $f:\ M \to N$ be a *homeomorphism* (f, f^{-1} are continuous) and (U, ϕ), (V, ψ) be charts on M, N respectively. If the representation of f, $\psi \circ f \circ \phi^{-1}$, is invertible, i.e. the map $\phi \circ f^{-1} \circ \psi^{-1}$ exists, and both $\psi \circ f \circ \phi^{-1}$ and $\phi \circ f^{-1} \circ \psi^{-1}$ are C^∞, then f is called *diffeomorphism* and manifold M is said to be *diffeomorphic* to manifold N.

Biholomorphism is an equivalent notion to *diffeomorphism*, between two complex manifolds X and Y. If the map $\phi : X \rightarrow Y$ and its inverse $\phi^{-1} : Y \rightarrow X$ are *holomorphic* maps, we can say that the manifolds X and Y are *isomorphic*. Such a map is called *biholomorphism*.

Note that, if the manifold X is *homeomorphic* to the manifold Y, which is *homeomorphic* to the manifold T, the composition of these two maps makes X *homeomorphic* to T. Thus, all topological manifolds can be divided into equivalence classes.

A *curve* in an m-dimensional manifold M is a map $c : (a, b) \rightarrow M$, where $(a, b) \subset \mathbb{R}$ is an open interval, such that $a < 0 < b$. Let λ be a parameter in the interval (a, b), then we can parameterise the *curve* as $c(\lambda)$. On a chart (U, ϕ), the parameter λ has the coordinate presentation $x = \phi \circ c : \mathbb{R} \rightarrow \mathbb{R}^m$.

A *function* on an m-dimensional manifold M is a smooth map $f : M \rightarrow \mathbb{R}$. The set of all smooth functions on M is denoted by $C^\infty(M)$. On a chart (U, ϕ), the coordinate presentation of f is given by $f \circ \phi^{-1} : \mathbb{R}^m \rightarrow \mathbb{R}$, which is a real-valued function of m variables.

Appendix E

Non-coordinate bases

It is often convenient to use the *tetrad* orthonormal basis [35]. A natural basis for the tangent space $T_p(\mathcal{M})$ at a point p is given by the partial derivative with respect to that point, $\hat{e}_{(\mu)} = \partial_\mu$. Similarly, a basis for the cotangent space T_p^* is given by the gradient of the coordinate function, $\hat{\theta}^{(\mu)} = dx^\mu$. However, we can introduce a set of orthonormal basis vectors $\hat{e}_{(a)}$, indexed by a Latin letter to remind that they are not related to any coordinate system. If the canonical form of the metric is written η_{ab}, which represents the Minkowski metric in a Lorentzian space-time, we demand the inner product of our vectors to be

$$g(\hat{e}_{(a)}, \hat{e}_{(b)}) = \eta_{ab} \tag{E.1}$$

We can express our old basis vectors $\hat{e}_{(\mu)} = \partial_\mu$ in terms of the new ones:

$$\partial_\mu = e_\mu{}^a \hat{e}_{(a)} \tag{E.2}$$

The components $e_\mu{}^a$ form an $n \times n$ invertible matrix. We may refer to $e_\mu{}^a$ as tetrad. The components $e^\mu{}_a$ of the inverse $\hat{e}^{(\mu)} = dx^\mu = e^\mu{}_a \hat{e}^{(a)}$ satisfy the following equation:

$$\partial_\mu dx^\nu = e_\mu{}^a e^\nu{}_b (\hat{e}_{(a)}, \hat{e}^{(b)}) = \delta_\mu{}^\nu \quad \Rightarrow \quad e_\mu{}^a e^\nu{}_b \delta_a{}^b = \delta_\mu{}^\nu \Rightarrow$$

$$e_\mu{}^a e^\nu{}_a = \delta_\mu{}^\nu \tag{E.3}$$

which also yields:

$$e^\mu{}_b e_\mu{}^a e^\nu{}_a = e^\nu{}_b \quad \Rightarrow \quad (e^\mu{}_b e_\mu{}^a) e^\nu{}_a = \delta_b{}^a e^\nu{}_a \Rightarrow$$

$$e^\mu{}_b e_\mu{}^a = \delta_b{}^a \tag{E.4}$$

Additionally, the definition of the tetrad basis implies:

$$g(\hat{e}_{(a)}, \hat{e}_{(b)}) = \eta_{ab} \quad \Rightarrow \quad g_{\mu\nu}(dx^\mu, \hat{e}_{(a)})(dx^\nu, \hat{e}_{(b)}) = \eta_{ab} \quad \Rightarrow \quad g_{\mu\nu} e^\mu{}_c e^\nu{}_d (\hat{e}^{(c)}, \hat{e}_{(a)})(\hat{e}^{(d)}, \hat{e}_{(b)}) = \eta_{ab} \Rightarrow$$

$$g_{\mu\nu} e^\mu{}_a e^\nu{}_b = \eta_{ab} \tag{E.5}$$

which means that the metric in the tetrad basis reads:

$$g = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} e^\mu{}_a \hat{e}^{(a)} e^\nu{}_b \hat{e}^{(b)} \Rightarrow$$

$$g = \eta_{ab} \hat{e}^{(a)} \hat{e}^{(b)} \tag{E.6}$$

The inverse properties imply:

$$g_{\mu\nu} e^\mu{}_a e^\nu{}_b = \eta_{ab} \Rightarrow g_{\mu\nu} e^\mu{}_a \delta_\rho{}^\nu = \eta_{ab} e_\rho{}^b \Rightarrow g_{\mu\rho} e^\mu{}_a = \eta_{ab} e_\rho{}^b \Rightarrow$$

$$\delta_\mu{}^\xi e^\mu{}_a = g^{\xi\rho} \eta_{ab} e_\rho{}^b \Rightarrow e^\xi{}_a = g^{\xi\rho} \eta_{ab} e_\rho{}^b \Rightarrow$$

$$e^\mu{}_a = g^{\mu\nu} \eta_{ab} e_\nu{}^b \tag{E.7}$$

Note that we can raise and lower the Greek indices using the metric representation $g_{\mu\nu}$ and the Latin indices using the metric representation $\eta_{\mu\nu}$. Let $V = V^\mu \partial_\mu$ be a vector. We may write the components of the vector in the tetrad basis representation:

$$V = V^\mu \partial_\mu = V^\mu e_\mu{}^a \hat{e}_{(a)} \Rightarrow$$

$$V^a = V^\mu e_\mu{}^a \quad \text{and} \quad V^\mu = e^\mu{}_a V^a \tag{E.8}$$

Similarly, let $\omega = \omega_\mu dx^\mu$ be a cotangent vector, whose components in the tetrad basis representation read:

$$\omega = \omega_\mu dx^\mu = \omega_\mu e^\mu{}_a \hat{e}^{(a)} \Rightarrow$$

$$\omega_a = \omega_\mu e^\mu{}_a \quad \text{and} \quad \omega_\mu = e_\mu{}^a \omega_a \tag{E.9}$$

Finally, we can explicitly prove that the Latin indices of vectors can be raised and lowered by the metric representation $\eta_{\mu\nu}$:

$$g(V,V) = V^a V^b \eta_{cd} (\hat{e}_{(a)}, \hat{e}^{(c)})(\hat{e}_{(b)}, \hat{e}^{(d)}) = V^a V^b \eta_{cd} \delta_a{}^c \delta_b{}^d = V^a V^b \eta_{ab} \Rightarrow$$

$$V_a = \eta_{ab} V^b \tag{E.10}$$

Appendix F

Newman-Janis algorithm

We present Newman-Janis algorithm [43] as a four-step procedure for generating new solutions of Einstein's equations from known static spherically symmetric ones [44].

Let (t, r, θ, ϕ) be a coordinate system assigned to a point p of a manifold $\mathcal{M}$ and $(\partial_t, \partial_r, \partial_\theta, \partial_\phi)$ be the basis tangent vectors of $T_p(\mathcal{M})$. A general, static and spherical symmetric metric tensor $G_0 \in \mathcal{T}^0_{2,p}(\mathcal{M})$ may be given by:

$$G_0 = G(r)dt^2 - \frac{1}{F(r)}dr^2 - H(r)d\Omega^2 \tag{F.1}$$

This is our static spherically symmetric seed line element to use as a seed for the rotating one.

Let us consider that the manifold $\mathcal{M}$ is embedded in a four-complex-dimensional manifold $\mathbb{C}\mathcal{M}$ [45], with coordinates $z^a = (t, x, z, y)$ and $\bar{z}^a = (\bar{t}, \bar{x}, \bar{y}, \bar{z})$ or equivalently $z^a = (t, r, \theta, \phi)$ and $\bar{z}^a = (\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi})$. Additionally, we introduce a Hermitian metric h on the complex manifold $\mathcal{M}$. A hermitian 2-form on a complex vector space V, which is also a real vector space (of twice the dimension), is a map $h : V \times V \to \mathbb{C}$, which is $\mathbb{C}$-linear in its first argument, and such that $h(v_1, v_2) = \overline{h(v_2, v_1)}$ for all $v_1, v_2 \in V$. It follows that h is $\mathbb{C}$-anti-linear in its second argument. Let us write the line element as the real part of the hermitian metric:

$$ds^2 = \frac{1}{2}(h_{\mu\bar{\nu}}dz^\mu d\bar{z}^{\bar{\nu}} + h_{\bar{\mu}\nu}d\bar{z}^{\bar{\mu}}dz^\nu) = g_{\mu\nu}dz^\mu d\bar{z}^\nu \tag{F.2}$$

Hence, the metric in our case is generalized as follows:

$$G_0 = G(r)dtd\bar{t} - \frac{1}{F(r)}drd\bar{r} - H(r)d\theta d\bar{\theta} - H(r)sin^2(\theta)d\phi d\bar{\phi} \tag{F.3}$$

where $G(r), F(r)$ and $H(r)$ are functions from $\mathbb{C}$ to $\mathbb{R}$. For instance, the generalization of Schwarzschild metric reads

$$ds^2 = F(r)dtd\bar{t} - \frac{1}{F(r)}drd\bar{r} - r\bar{r}d\theta d\bar{\theta} - r\bar{r}sin^2(\theta)d\phi d\bar{\phi} \tag{F.4}$$

where

$$F(r) = 1 - M\Big(\frac{1}{r} + \frac{1}{\bar{r}}\Big) = 1 - \frac{2MRe[r]}{|r|^2} \tag{F.5}$$

It is obvious that, if we consider real coordinates the solution is reduced to the standard Schwarzschild metric. Note that the complexification $\frac{2}{r} \to \frac{1}{r} + \frac{1}{\bar{r}}$ works in the case of Schwarzschild seed but it is not generally the suitable choice. For instance, if we have a metric with a deficit angle, e.g. the metric of a global self-gravitating monopole system, the complexification $\frac{2}{r} \to \frac{1}{r} + \frac{1}{\bar{r}}$ does not work, as we discuss in Appendix G. In this case, we may examine modified Newman-Janis algorithms, as the one developed by Mustapha Azreg-Aïnou in [46].

Under the consideration of a complex manifold, the Newman-Janis algorithm arises spontaneously, since we consider a complex transformation of the coordinates in order to obtain real t, r, θ, ϕ. In the following four steps, we conclude the algorithmic procedure.

Step 1. Eddington-Finkelstein coordinates

Let us consider the following coordinate transformation to outgoing Eddington-Finkelstein coordinates

$$f_1 : (t, r, \theta, \phi) \to (u, r, \theta, \phi) \tag{F.6}$$

which is defined by the relation

$$\boxed{du = dt - \frac{dr}{\sqrt{G(r)F(r)}}} \tag{F.7}$$

$$\Rightarrow \sqrt{G(r)}dt = \sqrt{G(r)}du + \frac{dr}{\sqrt{F(r)}} \Rightarrow \tag{F.8}$$

$$G(r)dtd\bar{t} = G(r)dud\bar{u} + \sqrt{\frac{G(r)}{F(r)}}d\bar{u}dr + \sqrt{\frac{G(r)}{F(r)}}dud\bar{r} + \frac{drd\bar{r}}{F(r)} \tag{F.9}$$

Hence, the metric, induced by the mapping f_1, can be written as follows:

$$G_S = G(r)dud\bar{u} + \sqrt{\frac{G(r)}{F(r)}}d\bar{u}dr + \sqrt{\frac{G(r)}{F(r)}}dud\bar{r} - H(r)d\theta d\bar{\theta} - H(r)sin^2(\theta)d\phi d\bar{\phi} \tag{F.10}$$

Step 2. Complexification of the vector space

We can turn any real vector space V into a complex vector space $V^{\mathbb{C}}$ by forming the set $V \times V$ of all pairs of (E_a, E_b), where E_a, E_b are orthonormal basis vector of V, by expressing the complex basis as $E_a + iE_b$. As such, we can construct a set of vectors forming a basis in $T_p^{\mathbb{C}}(\mathcal{M})$. In particular, we may construct a vector orthonormal tetrad basis, denoted by $e_a = \partial_a = (\partial_{\tilde{t}}, \partial_{\tilde{r}}, \partial_{\tilde{\theta}}, \partial_{\tilde{\phi}})$ and $e_{\bar{a}} = \partial_{\bar{a}} = (\partial_{\bar{\tilde{t}}}, \partial_{\bar{\tilde{r}}}, \partial_{\bar{\tilde{\theta}}}, \partial_{\bar{\tilde{\phi}}})$, using the following tetrad fields. The orthonormal one-forms can easily be found to read:

$$\begin{aligned} d\tilde{t} &= \sqrt{G(r)}dt = \sqrt{G(r)}du + \tfrac{dr}{\sqrt{F(r)}} \\ d\tilde{r} &= \tfrac{dr}{\sqrt{F(r)}} \\ d\tilde{\theta} &= \sqrt{H(r)}d\theta \\ d\tilde{\phi} &= \sqrt{H(r)}sin(\theta)d\phi \end{aligned} \tag{F.11}$$

Similarly, the conjugate one-forms are determined. Thus, the corresponding tetrad field in the Eddington-Finkelstein representation reads:

$$e^a = e^a{}_\mu dx^\mu \Rightarrow$$

$$e^a{}_\mu = \begin{pmatrix} \sqrt{G(r)} & \frac{1}{\sqrt{F(r)}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{F(r)}} & 0 & 0 \\ 0 & 0 & \sqrt{H(r)} & 0 \\ 0 & 0 & 0 & \sqrt{H(r)}sin(\theta) \end{pmatrix} \tag{F.12}$$

Then, using the relation $e^\mu{}_a = g^{\mu\nu}\eta_{ab}e_\nu{}^b$ we may find that:

$$e_a{}^\mu = \begin{pmatrix} \frac{1}{\sqrt{G(r)}} & 0 & 0 & 0 \\ -\frac{1}{\sqrt{G(r)}} & \sqrt{F(r)} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{H(r)}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{H(r)}sin(\theta)} \end{pmatrix} \tag{F.13}$$

which, by the definition $\partial_a = e_a{}^\mu \partial_\mu$ yields:

$$
\begin{aligned}
\partial_{\tilde{t}} &= \frac{1}{\sqrt{G(r)}}\partial_u \\
\partial_{\tilde{r}} &= -\frac{1}{\sqrt{G(r)}}\partial_u + \sqrt{F(r)}\partial_r \\
\partial_{\tilde{\theta}} &= \frac{1}{\sqrt{H(r)}}\partial_\theta \\
\partial_{\tilde{\phi}} &= \frac{1}{\sqrt{H(r)sin(\theta)}}\partial_\phi
\end{aligned}
\tag{F.14}
$$

Similarly, we determine the conjugate tetrad basis. Note that $G(r)$, $F(r)$, $H(r)$, ${e_a}^\mu$ and ${e^a}_\mu$ take real values. Also, it is obvious that the tetrad fields satisfy their definition: $g_{\mu\nu}{e^\mu}_a{e^\nu}_b = \eta_{ab}$. Additionally, it is obvious that ${e_\mu}^a{e^\nu}_a = {\delta_\mu}^\nu$ and ${e^\mu}_b{e_\mu}^a = {\delta_b}^a$.

Using the above results and considering that the angular coordinates θ, ϕ are real, we may represent the metric by the following linear combinations of orthonormal vectors, denoted by $(L, \bar{L}, N, \bar{N}, M, \bar{M})$:

$$
\boxed{
\begin{aligned}
L &= \frac{1}{\sqrt{F(r)}}(\partial_{\tilde{t}} + \partial_{\tilde{r}}) = \partial_r \quad \text{and} \quad \bar{L} = \partial_{\bar{r}} \\
N &= \frac{\sqrt{F(r)}}{2}(\partial_{\tilde{t}} - \partial_{\tilde{r}}) = \sqrt{\frac{F(r)}{G(r)}}\partial_u - \frac{F(r)}{2}\partial_r \quad \text{and} \quad \bar{N} = \sqrt{\frac{F(r)}{G(r)}}\partial_{\bar{u}} - \frac{F(r)}{2}\partial_{\bar{r}} \\
M &= \frac{\sqrt{2}}{2}(\partial_{\tilde{\theta}} + i\partial_{\tilde{\phi}}) = \frac{\sqrt{2}}{2\sqrt{H(r)}}\partial_\theta + i\frac{\sqrt{2}}{2sin(\theta)\sqrt{H(r)}}\partial_\phi \\
\bar{M} &= \frac{\sqrt{2}}{2}(\partial_{\tilde{\theta}} - i\partial_{\tilde{\phi}}) = \frac{\sqrt{2}}{2\sqrt{H(r)}}\partial_\theta - i\frac{\sqrt{2}}{2sin(\theta)\sqrt{H(r)}}\partial_\phi
\end{aligned}
}
\tag{F.15}
$$

where the point is that, since ∂_θ and ∂_ϕ are real, we turn vector M into a complex vector via the additional i. Note that the inverse metric tensor reads:

$$
G_S^{-1} = \sqrt{\frac{F(r)}{G(r)}}\partial_u \otimes \partial_{\bar{r}} + \sqrt{\frac{F(r)}{G(r)}}\partial_{\bar{u}} \otimes \partial_r - F(r)\partial_r \otimes \partial_{\bar{r}} - \frac{1}{H(r)}\partial_\theta \otimes \partial_\theta - \frac{1}{H(r)sin^2(\theta)}\partial_\phi \otimes \partial_\phi \tag{F.16}
$$

Hence, the inverse metric tensor may be written as follows:

$$
G_S^{-1} = L^\mu \bar{N}^\nu \partial_\mu \otimes \partial_{\bar{\nu}} + \bar{L}^\mu N^\nu \partial_{\bar{\mu}} \otimes \partial_\nu - M^\mu \bar{M}^\nu \partial_\mu \otimes \partial_{\bar{\nu}} - M^\nu \bar{M}^\mu \partial_{\bar{\mu}} \otimes \partial_\nu \tag{F.17}
$$

Using the tetrad notation introduced by Newman and Penrose

$$
Z_a^\mu = (L^\mu, N^\mu, M^\mu, \bar{M}^\mu), \quad \text{with } a = 1, 2, 3, 4
$$

we can write its components in Eddington-Finkelstein coordinates as follows:

$$
\begin{aligned}
L^\mu &= {\delta_1}^\mu \\
N^\mu &= \sqrt{\frac{F(r)}{G(r)}}{\delta_0}^\mu - \frac{F(r)}{2}{\delta_1}^\mu \\
M^\mu &= \frac{\sqrt{2}}{2\sqrt{H(r)}}{\delta_2}^\mu + i\frac{\sqrt{2}}{2\sqrt{H(r)}sin(\theta)}{\delta_3}^\mu
\end{aligned}
$$

Also, note that the components of the null tetrad vector satisfy the relations:

$$
L_\mu L^\mu = M_\mu M^\mu = N_\mu N^\mu = 0, \;\; L_\mu N^\mu = -M_\mu \bar{M}^\mu = 1, \;\; L_\mu M^\mu = N_\mu M^\mu = 0
$$

Similarly for the conjugate vectors.

Step 3. Rotated Eddington-Finkelstein coordinates

We introduce the rotated Eddington-Finkelstein coordinates by the mapping

$$
f_2 : \; (u, r, \theta, \phi) \to (u_R, r_R, \theta_R, \phi_R) \tag{F.18}
$$

through the transformation

$$\boxed{\begin{array}{c} u_R = u - ia\ cos(\theta) = \bar{u} + ia\ cos(\theta) \\ \\ r_R = r + ia\ cos(\theta) = \bar{r} - ia\ cos(\theta) \\ \\ \theta_R = \theta \\ \\ \phi_R = \phi \end{array}} \tag{F.19}$$

Let us suppose that the new coordinates $(u_R, r_R, \theta_R, \phi_R)$ are real. The system of the basis vectors $(\partial_u, \partial_r, \partial_\theta, \partial_\phi)$ transforms via f_2 as follows:

$$\begin{gathered} \partial_u = \partial_{\bar{u}} = \partial_{u_R} \\ \\ \partial_r = \partial_{\bar{r}} = \partial_{r_R} \\ \\ \partial_\theta = ia\ sin(\theta)(\partial_{u_R} - \partial_{r_R}) + \partial_{\theta_R} \\ \\ \partial_\phi = \partial_{\phi_R} \end{gathered} \tag{F.20}$$

Then, the set of basis vectors $(L, \bar{L}, N, \bar{N}, M, \bar{M})$ read:

$$\begin{gathered} L_R = \bar{L_R} = \partial_{r_R} \\ \\ N_R = \bar{N_R} = \sqrt{\tfrac{B(r_N,\theta_R)}{A(r_N,\theta_R)}}\partial_{u_R} - \tfrac{B(r_N,\theta_R)}{2}\partial_{r_R} \\ \\ M_R = \tfrac{\sqrt{2}}{2\sqrt{\Psi(r_N,\theta_R)}}\Big(ia\ sin(\theta_N)\partial_{u_R} - ia\ sin(\theta_N)\partial_{r_R} + \partial_{\theta_R} + \tfrac{i}{sin(\theta_R)}\partial_{\phi_R}\Big) \\ \\ \bar{M}_R = \tfrac{\sqrt{2}}{2\sqrt{\Psi(r_N,\theta_R)}}\Big(- ia\ sin(\theta_N)\partial_{u_R} + ia\ sin(\theta_N)\partial_{r_R} + \partial_{\theta_R} - \tfrac{i}{sin(\theta_R)}\partial_{\phi_R}\Big) \end{gathered} \tag{F.21}$$

where the fields $A(r_N, \theta_R), B(r_N, \theta_R)$ and $\Psi(r_N, \theta_R)$ are the associated metric components of the original $G(r), F(r)$ and $H(r)$ given by the relations:

$$\begin{gathered} A(r_N, \theta_R) = G(r_N - ia\ cos(\theta_N)), \quad B(r_N, \theta_R) = F(r_N - ia\ cos(\theta_N)) \quad \text{and} \\ \\ \Psi(r_N, \theta_R) = H(r_N - ia\ cos(\theta_N)) \end{gathered} \tag{F.22}$$

In this step arises a very important fact about Newman-Janis algorithm. If we know the suitable complexification of the functions of r, e.g. $\frac{1}{r}$ etc, after the transformation $r_N = r + ia\ cos(\theta)$ we know exactly the form of the functions $A(r_N, \theta_R)$, $B(r_N, \theta_R)$ and $H(r_N, \theta_R)$, otherwise we have to determine these functions solving the Einstein equations, as proposed in [46]. Continuing the process as described by Newman and Janis, the metric in contravariant form reads

$$\begin{gathered} G_R^{-1} = -\tfrac{a^2 sin^2(\theta_R)}{\Psi(r_N,\theta_R)}\partial_{u_R} \otimes \partial_{u_R} + 2\tfrac{\sqrt{B(r_N,\theta_R)}\Psi(r_N,\theta_R) + a^2\sqrt{A(r_N,\theta_R)}sin^2(\theta_R)}{\Psi(r_N,\theta_R)\sqrt{A(r_N,\theta_R)}}\partial_{u_R} \otimes \partial_{r_R} \\ \\ -2\tfrac{a}{\Psi(r_N,\theta_R)}\partial_{u_R} \otimes \partial_{\phi_R} - \tfrac{\Psi(r_N,\theta_R)B(r_N,\theta_R) + a^2 sin^2(\theta_R)}{\Psi(r_N,\theta_R)}\partial_{r_R} \otimes \partial_{r_R} + 2\tfrac{a}{\Psi(r_N,\theta_R)}\partial_{r_R} \otimes \partial_{\phi_R} \\ \\ -\tfrac{1}{\Psi(r_N,\theta_R)}\partial_{\theta_R} \otimes \partial_{\theta_R} - \tfrac{1}{\Psi(r_N,\theta_R)sin^2(\theta_R)}\partial_{\phi_R} \otimes \partial_{\phi_R} \end{gathered} \tag{F.23}$$

which yields the following line element

$$\begin{gathered} G_R = A(r_N, \theta_R)du_R^2 + 2\sqrt{\tfrac{A(r_N,\theta_R)}{B(r_N,\theta_R)}}du_R dr_R - 2a\ sin^2(\theta_R)\left(A(r_N, \theta_R) - \sqrt{\tfrac{A(r_N,\theta_R)}{B(r_N,\theta_R)}}\right)du_R d\phi_R \\ \\ -2a\ sin^2(\theta_R)\sqrt{\tfrac{A(r_N,\theta_R)}{B(r_N,\theta_R)}}dr_R d\phi_R - \Psi(r_N, \theta_R)d\theta_R^2 \\ \\ -sin^2(\theta_R)\left[\Psi(r_N, \theta_R) + a^2 sin^2(\theta_R)\left(2\sqrt{\tfrac{A(r_N,\theta_R)}{B(r_N,\theta_R)}} - A(r_N, \theta_R)\right)\right]d\phi_R^2 \end{gathered} \tag{F.24}$$

Step 4. Boyer-Lindquist coordinates

In order to bring the metric into the known Boyer-Lindquist coordinate system, we need the mapping

$$f_3 : \ (u_R, r_R, \theta_R, \phi_R) \rightarrow (T, R, \Theta, \Phi) \tag{F.25}$$

defined through

$$\boxed{\begin{aligned} du_R &= dT + Y(R)dR \\ dr_R &= dR \\ d\theta_R &= d\Theta \\ d\phi_R &= d\Phi + Z(R)dR \end{aligned}} \tag{F.26}$$

The only non-vanishing off-diagonal element of a metric of a rotating black hole is $g_{T\Phi}$. Therefore, upon requiring $g_{TR} = g_{R\Phi} = 0$, the fields $Y(R)$ and $Z(R)$ read:

$$Y(R) = -\frac{\sqrt{B(r_R, \theta_R)}\Psi(r_R, \theta_R) + a^2 sin^2(\theta_R)\sqrt{A(r_R, \theta_R)}}{\sqrt{A(r_R, \theta_R)}(B(r_R, \theta_R)\Psi(r_R, \theta_R) + a^2 sin^2(\theta_R))} \tag{F.27}$$

$$Z(R) = -\frac{a}{B(r_R, \theta_R)\Psi(r_R, \theta_R) + a^2 sin^2(\theta_R)} \tag{F.28}$$

In order for the coordinate transformation to be integrable, $Y(R)$ and $Z(R)$ must be functions of R only, for obvious reasons. Hence, the complexifications of $\frac{1}{r}$ and r^2 are consistent with the transformations f_2 and f_3 if Y, Z are functions of R only. In the Appendix G we prove that, if the metric has a deficit angle and we choose the complexifications $\frac{2}{r} \rightarrow \frac{1}{r} + \frac{1}{\bar{r}}$ and $r^2 \rightarrow r\bar{r}$, the corresponding functions are: $Y(R, \Theta)$ and $Z(R, \Theta)$. Therefore, in this case, the aforementioned complexification is not consistent. In appendix G, we modify the usual complexification and other arbitrary steps of the algorithm, in order to obtain a solution for the rotating global monopole. In the usual case, where the above complexification is valid, using the above transformation, we find that our sought after line element reads:

$$\boxed{\begin{aligned} G_{BL} = A(R,\Theta)dT^2 + 2a\ sin^2(\Theta)\left(\sqrt{\tfrac{A(R,\Theta)}{B(R,\Theta)}} - A(R,\Theta)\right)dTd\Phi - \tfrac{\Psi(R,\Theta)}{B(R,\Theta)\Psi(R,\Theta)+a^2 sin^2(\Theta)}dR^2 \\ -\Psi(R,\Theta)d\Theta^2 - sin^2(\Theta)\left[\Psi(R,\Theta) + a^2 sin^2(\Theta)\left(2\sqrt{\tfrac{A(R,\Theta)}{B(R,\Theta)}} - A(R,\Theta)\right)\right]d\Phi^2 \end{aligned}} \tag{F.29}$$

Finally, we examine the validity of the algorithm, using as seed the Schwarzschild metric:

$$ds^2 = F(r)dtd\bar{t} - \frac{1}{F(r)}drd\bar{r} - r\bar{r}d\theta d\bar{\theta} - r\bar{r}sin^2(\theta)d\phi d\bar{\phi} \tag{F.30}$$

Hence, we have

$$G(r) = F(r) = 1 - \frac{2MRe[r]}{|r|^2} \quad \text{and} \quad H(r) = r\bar{r} \tag{F.31}$$

where we considered the usual complexification. Note that $G(r)$, $F(r)$ and $H(r)$ are functions from $\mathbb{C}$ to $\mathbb{R}$, as we promised. Then, functions $A(r_N, \theta_R)$, $B(r_N, \theta_R)$ and $\Psi(r_N, \theta_R)$ read

$$A(R,\Theta) = B(R,\Theta) = 1 - \frac{2MR}{R^2 + a^2 cos^2(\Theta)} \quad \text{and} \quad \Psi(R,\Theta) = R^2 + a^2 cos^2(\Theta) \tag{F.32}$$

Thus, functions $Y(R)$ and $Z(R)$ read

$$Y(R) = -\frac{R^2 + a^2}{R^2 + a^2 - 2MR} \tag{F.33}$$

$$Z(R) = -\frac{a}{R^2 + a^2 - 2MR} \tag{F.34}$$

which are functions of R coordinate only, hence the coordinate transformation f_3 is well defined. Consequently, the rotating metric reads

$$\begin{aligned} G_{BL} = \left(1 - \tfrac{2MR}{R^2+a^2cos^2(\Theta)}\right)dT^2 + \tfrac{4MRa^2sin^2(\Theta)}{R^2+a^2cos^2(\Theta)}dTd\Phi - \tfrac{R^2+a^2cos^2(\Theta)}{R^2+a^2-2MR}dR^2 \\ -(R^2 + a^2cos^2(\Theta))d\Theta^2 - \left(R^2 + a^2 + \tfrac{2MRa^2sin^2(\Theta)}{R^2+a^2cos^2(\Theta)}\right)sin^2(\Theta)d\Phi^2 \end{aligned} \tag{F.35}$$

which is the metric of a Kerr black hole, with $a = \frac{J}{M}$, where J is the angular momentum of the rotating black hole and M is the mass of the black hole.

Appendix G

Failure of the Newman-Janis algorithm to apply to the global monopole model

The Newman-Janis algorithm [43] succeeds to provide valid solutions in the Kerr and the Kerr-Newman cases. The roots of the validity of the algorithm, in these cases, remains unknown. Nevertheless, it is reasonable to consider that applying the algorithm to other theories could be fruitful. Unfortunately, we will prove that the Newman-Janis algorithm fails to generalise metrics with deficit angle. There is a series of papers [47][48][49] that argue about the application of the Newman-Janis algorithm to the self-gravitating global monopole case. Hence, it is essential for our work to comment these papers. Since, we will prove that the regular algorithm fails to generalise the metric of the self-gravitating global monopole system, we will examine the validity of the modified Newman-Janis algorithm referred in [48] and developed by Mustapha Azreg-Aïnou in [46]. However, in the global monopole case, this method does not properly work either, as we will prove later on. Then, we will try some modifications to the aforementioned methods, without any consistent results.

First of all, the first steps described by the Newman-Janis algorithm and the process [46] are identical. Hence, let us try to generalise the metric of the non-rotating global monopole (5.29) to the case of a rotating global monopole, in the region $r > \delta \gg$ Planck length, using the modified Newman-Janis algorithm [46].

Thus, from the metric (5.29) we identify the functions

$$F(r) = G(r) = 1 - 8\pi G\eta^2 - \frac{2GM}{r} \quad \text{and} \quad H(r) = r^2 \tag{G.1}$$

Then, considering a four-complex-dimensional manifold $\mathbb{C}\mathcal{M}$ [45], with coordinates $z^a = (t, x, y, z)$ and $\bar{z}^a = (\bar{t}, \bar{x}, \bar{y}, \bar{z})$ or equivalently $z^a = (t, r, \theta, \phi)$ and $\bar{z}^a = (\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi})$, the metric is generalized into

$$G_0 = G(r)dtd\bar{t} - \frac{1}{F(r)}drd\bar{r} - H(r)d\theta d\bar{\theta} - H(r)sin^2(\theta)d\phi d\bar{\phi} \tag{G.2}$$

where $G(r), F(r)$ and $H(r)$ are functions from $\mathbb{C}$ to $\mathbb{R}$. In contrast with the standard Newman-Janis algorithm [43] we do not fix the complexification of $\frac{1}{r}$ and r^2, for reasons that will be clarified in the following steps. The only constraint imposed on $G(r), F(r)$ and $H(r)$ is that, if the coordinate r takes real values, these functions are defied as (G.1) and the metric reduces to (5.29).

Subsequently, we consider the following coordinate transformation to outgoing Eddington-Finkelstein coordinates

$$f_1 : (t, r, \theta, \phi) \to (u, r, \theta, \phi) \tag{G.3}$$

which is defined by the relation

$$du = dt - \frac{dr}{\sqrt{G(r)F(r)}} \tag{G.4}$$

Then, we calculate the transformed components of the metric:

$$\sqrt{G(r)}dt = \sqrt{G(r)}du + \frac{dr}{\sqrt{F(r)}} \Rightarrow$$

$$G(r)dtd\bar{t} = G(r)dud\bar{u} + \sqrt{\frac{G(r)}{F(r)}}d\bar{u}dr + \sqrt{\frac{G(r)}{F(r)}}dud\bar{r} + \frac{drd\bar{r}}{F(r)}$$

Hence, the metric, induced by the mapping f_1, can be written as follows:

$$G_S = G(r)dud\bar{u} + \sqrt{\frac{G(r)}{F(r)}}d\bar{u}dr + \sqrt{\frac{G(r)}{F(r)}}dud\bar{r} - H(r)d\theta d\bar{\theta} - H(r)sin^2(\theta)d\phi d\bar{\phi} \tag{G.5}$$

Furthermore, let us consider real coordinates θ, ϕ. Then, the six-basis vectors, of the corresponding complex vector space, may be written as follows:

$$L = \frac{1}{\sqrt{F(r)}}(\partial_{\tilde{t}} + \partial_{\tilde{r}}) = \partial_r \quad \text{and} \quad \bar{L} = \partial_{\bar{r}} \tag{G.6}$$

$$N = \frac{\sqrt{F(r)}}{2}(\partial_{\tilde{t}} - \partial_{\tilde{r}}) = \sqrt{\frac{F(r)}{G(r)}}\partial_u - \frac{F(r)}{2}\partial_r \quad \text{and} \quad \bar{N} = \sqrt{\frac{F(r)}{G(r)}}\partial_{\bar{u}} - \frac{F(r)}{2}\partial_{\bar{r}} \tag{G.7}$$

$$M = \frac{\sqrt{2}}{2}(\partial_{\tilde{\theta}} + i\partial_{\tilde{\phi}}) = \frac{\sqrt{2}}{2\sqrt{H(r)}}\partial_\theta + i\frac{\sqrt{2}}{2sin(\theta)\sqrt{H(r)}}\partial_\phi \tag{G.8}$$

$$\bar{M} = \frac{\sqrt{2}}{2}(\partial_{\tilde{\theta}} - i\partial_{\tilde{\phi}}) = \frac{\sqrt{2}}{2\sqrt{H(r)}}\partial_\theta - i\frac{\sqrt{2}}{2sin(\theta)\sqrt{H(r)}}\partial_\phi \tag{G.9}$$

where we used the tetrad basis (Appendix E) $\partial_{\tilde{t}}, \partial_{\tilde{r}}, \partial_{\tilde{\theta}}, \partial_{\tilde{\phi}}$ and the corresponding conjugate basis. Considering the vectors ∂_θ and ∂_ϕ as real, the inverse metric tensor reads:

$$G_S^{-1} = \sqrt{\frac{F(r)}{G(r)}}\partial_u \otimes \partial_{\bar{r}} + \sqrt{\frac{F(r)}{G(r)}}\partial_{\bar{u}} \otimes \partial_r - F(r)\partial_r \otimes \partial_{\bar{r}} - \frac{1}{H(r)}\partial_\theta \otimes \partial_\theta - \frac{1}{H(r)sin^2(\theta)}\partial_\phi \otimes \partial_\phi \tag{G.10}$$

Hence, the inverse metric tensor may be written as follows:

$$G_S^{-1} = L^\mu \bar{N}^{\bar{\nu}}\partial_\mu \otimes \partial_{\bar{\nu}} + \bar{L}^{\bar{\mu}} N^\nu \partial_{\bar{\mu}} \otimes \partial_\nu - M^\mu \bar{M}^{\bar{\nu}}\partial_\mu \otimes \partial_{\bar{\nu}} - \bar{M}^{\bar{\mu}} M^\nu \partial_{\bar{\mu}} \otimes \partial_\nu \tag{G.11}$$

Note that the metric has the form of the real part of a Hermitian tensor [45]:

$$G_S^{-1} = g^{\mu\bar{\nu}}\partial_\mu \otimes \partial_{\bar{\nu}} + g^{\bar{\mu}\nu}\partial_{\bar{\mu}} \otimes \partial_\nu \tag{G.12}$$

where $g^{\mu\bar{\nu}} = L^\mu \bar{N}^{\bar{\nu}} - M^\mu \bar{M}^{\bar{\nu}}$.

Moreover, we want to express the metric in real coordinates. Thus, we introduce the rotated Eddington-Finkelstein coordinates by the mapping

$$f_2 : \ (u, r, \theta, \phi) \rightarrow (u_R, r_R, \theta_R, \phi_R) \tag{G.13}$$

where $(u_R, r_R, \theta_R, \phi_R)$ are real. This complex transformation is arbitrary, if we do not fix the transformation f_3 and the complexification of $\frac{1}{r}$ and r^2. The transformation f_3 will be performed later on, in order to bring the metric into the known Boyer-Lindquist coordinate system. The complexification of $\frac{1}{r}$, r^2 and the transformations f_2 and f_3 must be fixed in such a way that implies a metric with the only off-diagonal component to be $g_{t\phi}$. Since, we have not determined the complexification of $\frac{1}{r}$ and r^2, we can fix the transformation f_2 through the usual relations

$$u_R = u - ia\ cos(\theta) = \bar{u} + ia\ cos(\theta) \tag{G.14}$$

$$r_R = r + ia\ cos(\theta) = \bar{r} - ia\ cos(\theta) \tag{G.15}$$

$$\theta_R = \theta \tag{G.16}$$

$$\phi_R = \phi \tag{G.17}$$

Continuing the procedure, the system of the basis vectors $(\partial_u, \partial_r, \partial_\theta, \partial_\phi)$ transforms via f_2 as

$$\partial_u = \partial_{\bar{u}} = \partial_{u_R} \tag{G.18}$$

$$\partial_r = \partial_{\bar{r}} = \partial_{r_R} \tag{G.19}$$

$$\partial_\theta = ia\ sin(\theta)(\partial_{u_R} - \partial_{r_R}) + \partial_{\theta_R} \tag{G.20}$$

$$\partial_\phi = \partial_{\phi_R} \tag{G.21}$$

Then, the set of basis vectors $(L, \bar{L}, N, \bar{N}, M, \bar{M})$ read:

$$L_R = \bar{L_R} = \partial_{r_R} \tag{G.22}$$

$$N_R = \bar{N_R} = \sqrt{\frac{B(r_N,\theta_R)}{A(r_N,\theta_R)}}\partial_{u_R} - \frac{B(r_N,\theta_R)}{2}\partial_{r_R} \tag{G.23}$$

$$M_R = \frac{\sqrt{2}}{2\sqrt{\Psi(r_N,\theta_R)}}\Big(ia\ sin(\theta_N)\partial_{u_R} - ia\ sin(\theta_N)\partial_{r_R} + \partial_{\theta_R} + \frac{i}{sin(\theta_R)}\partial_{\phi_R}\Big) \tag{G.24}$$

$$\bar{M_R} = \frac{\sqrt{2}}{2\sqrt{\Psi(r_N,\theta_R)}}\Big(- ia\ sin(\theta_N)\partial_{u_R} + ia\ sin(\theta_N)\partial_{r_R} + \partial_{\theta_R} - \frac{i}{sin(\theta_R)}\partial_{\phi_R}\Big) \tag{G.25}$$

where the fields $A(r_N,\theta_R), B(r_N,\theta_R)$ and $\Psi(r_N,\theta_R)$ are the associated metric components of the original $G(r), F(r)$ and $H(r)$ given by the relations:

$$A(r_N,\theta_R) = G(r_N - ia\ cos(\theta_N)), \quad B(r_N,\theta_R) = F(r_N - ia\ cos(\theta_N)) \quad \text{and}$$

$$\Psi(r_N,\theta_R) = H(r_N - ia\ cos(\theta_N)) \tag{G.26}$$

In this step arises the difference between the Newman-Janis algorithm [43] and the procedure described in [46] (we call it modified Newman-Janis algorithm). In Newman-Janis algorithm, where the complexification of the functions of r is fixed a priori, the functions $A(r_N,\theta_R)$, $B(r_N,\theta_R)$ and $\Psi(r_N,\theta_R)$ are known. In the modified Newman-Janis algorithm we have not yet fixed the complexification of $\frac{1}{r}$ and r^2, hence the functions $A(r_N,\theta_R)$, $B(r_N,\theta_R)$ and $\Psi(r_N,\theta_R)$ are unknown. Subsequently, the metric in the contravariant form reads

$$G_R^{-1} = -\frac{a^2 sin^2(\theta_R)}{\Psi(r_N,\theta_R)}\partial_{u_R}\otimes\partial_{u_R} + 2\frac{\sqrt{B(r_N,\theta_R)}\Psi(r_N,\theta_R) + a^2\sqrt{A(r_N,\theta_R)}sin^2(\theta_R)}{\Psi(r_N,\theta_R)\sqrt{A(r_N,\theta_R)}}\partial_{u_R}\otimes\partial_{r_R} - 2\frac{a}{\Psi(r_N,\theta_R)}\partial_{u_R}\otimes\partial_{\phi_R}$$
$$-\frac{\Psi(r_N,\theta_R)B(r_N,\theta_R) + a^2 sin^2(\theta_R)}{\Psi(r_N,\theta_R)}\partial_{r_R}\otimes\partial_{r_R} + 2\frac{a}{\Psi(r_N,\theta_R)}\partial_{r_R}\otimes\partial_{\phi_R}$$
$$-\frac{1}{\Psi(r_N,\theta_R)}\partial_{\theta_R}\otimes\partial_{\theta_R} - \frac{1}{\Psi(r_N,\theta_R)sin^2(\theta_R)}\partial_{\phi_R}\otimes\partial_{\phi_R} \tag{G.27}$$

which yields the following line element

$$G_R = A(r_N,\theta_R)du_R^2 + 2\sqrt{\frac{A(r_N,\theta_R)}{B(r_N,\theta_R)}}du_R dr_R - 2a\ sin^2(\theta_R)\left(A(r_N,\theta_R) - \sqrt{\frac{A(r_N,\theta_R)}{B(r_N,\theta_R)}}\right)du_R d\phi_R -$$
$$-2a\ sin^2(\theta_R)\sqrt{\frac{A(r_N,\theta_R)}{B(r_N,\theta_R)}}dr_R d\phi_R - \Psi(r_N,\theta_R)d\theta_R^2 -$$
$$-sin^2(\theta_R)\left[\Psi(r_N,\theta_R) + a^2 sin^2(\theta_R)\left(2\sqrt{\frac{A(r_N,\theta_R)}{B(r_N,\theta_R)}} - A(r_N,\theta_R)\right)\right]d\phi_R^2 \tag{G.28}$$

In the last step of the algorithm, we bring the metric into the known Boyer-Lindquist coordinate system via the mapping

$$f_3 :\ (u_R, r_R, \theta_R, \phi_R) \to (T, R, \Theta, \Phi) \tag{G.29}$$

defined through

$$du_R = dT + Y(R)dR \tag{G.30}$$

$$dr_R = dR \tag{G.31}$$

$$d\theta_R = d\Theta \tag{G.32}$$

$$d\phi_R = d\Phi + Z(R)dR \tag{G.33}$$

We introduce the above transformation, with $Y(R)$ and $Z(R)$ to be determined later on, in order for the only non-vanishing off-diagonal element of the metric to be $g_{T\Phi}$. This consideration ensures the axisymmetry of the metric. Note that the off-diagonal components of the form $g_{\Theta\mu}$ are identically zero. Hence, we solve the system of equations $g_{TR} = 0$ and $g_{R\Phi} = 0$ with respect to $Y(R)$ and $Z(R)$ and we have:

$$Y(R) = -\frac{\sqrt{B(r_R,\theta_R)}\Psi(r_R,\theta_R) + a^2 sin^2(\theta_R)\sqrt{A(r_R,\theta_R)}}{\sqrt{A(r_R,\theta_R)}(B(r_R,\theta_R)\Psi(r_R,\theta_R) + a^2 sin^2(\theta_R))} \tag{G.34}$$

$$Z(R) = -\frac{a}{B(r_R,\theta_R)\Psi(r_R,\theta_R) + a^2 sin^2(\theta_R)} \tag{G.35}$$

In this step arises the problem of the Newman-Janis algorithm. In order for the coordinate transformation to be integrable, $Y(R)$ and $Z(R)$ must be functions of R only. In the Newman-Janis algorithm, the complexifications $\frac{2}{r} \to \frac{1}{r} + \frac{1}{\bar{r}}$ and $r^2 \to r\bar{r}$ are fixed. Hence, equation (G.1) yields

$$F(r) = G(r) = 1 - 8\pi G\eta^2 - M\Big(\frac{1}{r} + \frac{1}{\bar{r}}\Big) = 1 - 8\pi G\eta^2 - \frac{2MRe[r]}{|r|^2} \quad \text{and} \quad H(r) = |r|^2 \tag{G.36}$$

Therefore, the functions $A(r_N,\theta_R)$, $B(r_N,\theta_R)$ and $\Psi(r_N,\theta_R)$ are fixed. Thus, considering equations (G.26), (G.31) and (G.32) they read

$$A(R,\Theta) = B(R,\Theta) = F(R - ia\ cos(\Theta)) \Rightarrow$$

$$A(R,\Theta) = B(R,\Theta) = 1 - 8\pi G\eta^2 - \frac{2GMR}{R^2 + a^2 cos^2(\Theta)} \tag{G.37}$$

and

$$\Psi(R,\Theta) = H(R - ia\ cos(\Theta)) \Rightarrow$$

$$\Psi(R,\Theta) = R^2 + a^2 cos^2(\Theta) \tag{G.38}$$

Upon substituting equations (G.37) and (G.38) into equation (G.34) and (G.35), we obtain

$$Y(R,\Theta) = -\frac{R^2 + a^2}{R^2 + a^2 - 8\pi G\eta^2 R^2 - 2GMR - 8\pi G\eta^2 a^2 cos^2(\Theta)} \tag{G.39}$$

$$Z(R,\Theta) = -\frac{a}{R^2 + a^2 - 8\pi G\eta^2 R^2 - 2GMR - 8\pi G\eta^2 a^2 cos^2(\Theta)} \tag{G.40}$$

Note that the functions $Y(R,\Theta)$ and $Z(R,\Theta)$ depend on Θ coordinate, due to the non-vanishing Higgs parameter η, which determines the deficit angle. Consequently, the transformation f_3 is not valid and the Newman-Janis algorithm cannot yield consistent results in this case, where we have a deficit angle. Consequently, the papers [47] and [49] do not present valid results. However, the work [48] propose to use the modified Newman-Janis algorithm [46] in order to determine the metric of a rotating self-gravitating global monopole. Hence, let us proceed further with these method and examine its validity.

The problem with the Newman-Janis algorithm is that it imposes the fixing of the complexification of $\frac{1}{r}$, r^2 and the transformations f_2, f_3, without a consistent way. The modified Newman-Janis algorithm [46] proposes to fix the transformations f_2, f_3, but not the complexification of $\frac{1}{r}$, r^2 a priori. Hence, up to this step, we cannot determine the functions $A(R,\Theta)$, $B(R,\Theta)$ and $\Psi(R,\Theta)$. Instead, we can require that these functions have the following form:

$$A(R,\Theta) = \frac{F(R)H(R) + a^2 cos^2(\Theta)}{(K(R) + a^2 cos^2(\Theta))^2}\Psi(R,\Theta) \quad \text{and} \quad B(R,\Theta) = \frac{F(R)H(R) + a^2 cos^2(\Theta)}{\Psi(R,\Theta)} \tag{G.41}$$

where

$$K(R) \equiv \sqrt{\frac{F(R)}{G(R)}}H(R) \tag{G.42}$$

Note that the functions $A(R,\Theta)$, $B(R,\Theta)$ are not fully determined, since they depend on $\Psi(R,\Theta)$, which is still unknown. Nevertheless, upon substituting (G.41) into equations (G.34) and (G.35) we have

$$Y(R) = -\frac{K(R)+a^2}{F(R)H(R)+a^2} \quad \text{and} \quad Z(R) = -\frac{a}{F(R)H(R)+a^2} \tag{G.43}$$

Note that $Y(R)$ (G.34) and $Z(R)$ (G.35) bring metric (G.28) into the desirable form (the only non-vanishing component is $g_{t\phi}$). Additionally, demanding the functions $A(R,\Theta)$ and $B(R,\Theta)$ to satisfy equation (G.41), the fixed transformation f_3 is valid, since $Y(R)$ and $Z(R)$ (G.43) depend only on R.

The main idea of this method is that a part of the arbitrariness of the Newman-Janis algorithm is described by $\Psi(R,\Theta)$, which will be determined by solving the Einstein equations. Note that, in order for this method to be valid, the equation (G.26) must be satisfied. This is not an issue, since we can fix the complexification of $F(r)$ and $G(r)$ even in different ways, in order for the equations (G.26) and (G.41) to be simultaneously satisfied. However, note that this method is not as general as it is considered in [46], since for any form of the function $K(R)$ the equations (G.34) and (G.35) are satisfied. The only restriction remains that the equations (G.26) and (G.41) have to be simultaneously satisfied. Hence, a part of the arbitrariness of the Newman-Janis algorithm is described by $K(R)$, which has been fixed in the modified Newman-Janis algorithm. Consequently, when we will prove that the modified Newman-Janis algorithm described in [46] fails to apply to the global monopole system, we will examine some modifications to the $K(R)$ function.

Note that the modified Newman-Janis algorithm is supposed to work, up to this step, for every metric of the form (G.2), since we have not substituted equations (G.1) yet. Subsequently, the transformation f_3 (G.29) implies the following form of the metric:

$$G_{BL} = A(R,\Theta)dT^2 + 2a\ sin^2(\Theta)\left(\sqrt{\frac{A(R,\Theta)}{B(R,\Theta)}} - A(R,\Theta)\right)dTd\Phi - \frac{\Psi(R,\Theta)}{B(R,\Theta)\Psi(R,\Theta)+a^2 sin^2(\Theta)}dR^2 -$$

$$-\Psi(R,\Theta)d\Theta^2 - sin^2(\Theta)\left[\Psi(R,\Theta) + a^2 sin^2(\Theta)\left(2\sqrt{\frac{A(R,\Theta)}{B(R,\Theta)}} - A(R,\Theta)\right)\right]d\Phi^2 \tag{G.44}$$

or equivalently, upon substituting $A(R,\Theta)$ and $B(R,\Theta)$ from (G.41), we obtain

$$\boxed{\begin{aligned} G_{BL} = \Psi(R,\Theta)\Big[&\tfrac{F(R)H(R)+a^2cos^2(\Theta)}{(K(R)+a^2cos^2(\Theta))^2}dT^2 + 2a\ sin^2(\Theta)\tfrac{K(R)-F(R)H(R)}{(K(R)+a^2cos^2(\Theta))^2}dTd\Phi - \\ &-\tfrac{1}{F(R)H(R)+a^2}dR^2 - d\Theta^2 - \left[1 + a^2 sin^2(\Theta)\tfrac{2K(R)-F(R)H(R)+a^2cos^2(\Theta)}{(K(R)+a^2cos^2(\Theta))^2}\right]sin^2(\Theta)d\Phi^2\Big]\end{aligned}} \tag{G.45}$$

The above metric can be written in a Kerr-like form:

$$\boxed{ds^2 = \frac{\Psi(r,\theta)}{\rho^2(r,\theta)}\left[\left(1-\frac{2f(r)}{\rho^2(r,\theta)}\right)dt^2 - \frac{\rho^2(r,\theta)}{\Delta(r)}dr^2 + \frac{4af(r)sin^2(\theta)}{\rho^2(r,\theta)}dtd\phi - \rho^2(r,\theta)d\theta^2 - \frac{\Sigma(r,\theta)sin^2(\theta)}{\rho^2(r,\theta)}d\phi^2\right]} \tag{G.46}$$

or

$$\boxed{ds^2 = \frac{\Psi(r,\theta)}{\rho^2(r,\theta)}\left[\frac{\Delta(r)}{\rho^2(r,\theta)}(dt - a\ sin^2(\theta)d\phi)^2 - \frac{\rho^2(r,\theta)}{\Delta(r)}dr^2 - \rho^2(r,\theta)d\theta^2 - \frac{sin^2(\theta)}{\rho^2(r,\theta)}\big(adt - (K(r)+a^2)d\phi\big)^2\right]} \tag{G.47}$$

where we have use the notation $(T,R,\Theta,\Phi) \to (t,r,\theta,\phi)$ and we have set:

$$\rho^2(r,\theta) \equiv K(r) + a^2cos^2(\theta) \tag{G.48}$$

$$f(r) \equiv \frac{K(r) - F(r)H(r)}{2} \tag{G.49}$$

$$\Delta(r) \equiv F(r)H(r) + a^2 \tag{G.50}$$

$$\Sigma(r,\theta) \equiv (K(r) + a^2)^2 - a^2\Delta(r)sin^2(\theta) \tag{G.51}$$

Let us summarize the results. Metrics (G.45), (G.46) and (G.47) are not fully determined, since the function $\Psi(r,\theta)$ remains unknown. However, the ansatz (G.45), of a rotating system with seed metric (G.2), is not the most general possible, since there are still some steps, of the modified Newman-Janis algorithm [46], fixed without a consistent way. Nevertheless, we have not fixed yet the complexification of $\frac{1}{r}$ and r^2. Hence, the arbitrariness of the aforementioned complexification leads to an arbitrary function $\Psi(r,\theta)$, which may be determined, if we suppose that the metric must satisfy Einstein field equations. Consequently, this is a more natural way to fix the complexification of $\frac{1}{r}$ and r^2, than the a priori fixing according the regular Newman-Janis algorithm. It is very interesting that the form of the metric in Boyer-Lindquist coordinates is general, i.e. it does not describe a particular model, since we have not yet substituted the components of the seed metric. Hence, the modified Newman-Janis algorithm can potentially be a useful tool in determining an ansatz of a rotating generalization of a metric.

Let us apply the modified Newman-Janis algorithm in the self-gravitating global monopole case, where we have:

$$K(r) = r^2 \tag{G.52}$$

$$\rho^2(r,\theta) = r^2 + a^2cos^2(\theta) \tag{G.53}$$

$$2f(r) = 8\pi\eta^2r^2 + 2Mr \tag{G.54}$$

$$\Delta(r) = (1 - 8\pi\eta^2)r^2 - 2Mr + a^2 \tag{G.55}$$

$$\frac{\Sigma(r,\theta)}{\rho^2(r,\theta)} = r^2 + a^2 + \frac{8\pi\eta^2r^2 + 2Mr}{\rho^2(r,\theta)}a^2sin^2(\theta) \tag{G.56}$$

Hence, the ansatz of the metric reads:

$$\begin{aligned} ds^2 = \tfrac{\Psi(r,\theta)}{r^2+a^2cos^2(\theta)}\Big[\Big(1 - \tfrac{8\pi\eta^2r^2+2Mr}{r^2+a^2cos^2(\theta)}\Big)dt^2 - \tfrac{r^2+a^2cos^2(\theta)}{(1-8\pi\eta^2)r^2-2Mr+a^2}dr^2 + \tfrac{2a\ sin^2(\theta)(8\pi\eta^2r^2+2Mr)}{r^2+a^2cos^2(\theta)}dtd\phi \\ -(r^2 + a^2cos^2(\theta))d\theta^2 - \Big(r^2 + a^2 + \tfrac{8\pi\eta^2r^2+2Mr}{r^2+a^2cos^2(\theta)}a^2sin^2(\theta)\Big)d\phi^2\Big] \end{aligned} \tag{G.57}$$

Note that inside the square parenthesis, for $\eta^2 = 0$, the metric reduces to the Kerr metric. The comment-paper [48] propose as a solution of the Einstein equation the function:

$$\Psi(r,\theta) = r^2 + a^2cos^2(\theta) \tag{G.58}$$

Also, in the same paper, it is pointed out that this solution must be checked through the Einstein equations. This is a reasonable solution, since the proportional factor of the square parenthesis equals to 1, hence for $\eta^2 = 0$ we obtain the Kerr metric. Note that, for $r > \delta \gg$ Planck-length, the $r\theta$ component of the Einstein equation reads:

$$G_{r,y} = 0 \Rightarrow$$

$$-6a^2y\Psi^2(r,y)K'(r) + 3(a^2y^2 + K(r))^2\Psi^{(0,1)}(r,y)\Psi^{(1,0)}(r,y) - 2(a^2y^2 + K(r))^2\Psi(r,y)\Psi^{(1,1)}(r,y) = 0 \tag{G.59}$$

where we considered (5.12) with $f(r) = 1$, $y = cos(\theta)$, $K'(r) = \frac{dK}{dr}$ and $\Psi^{(1,1)}(r,y) = \frac{d^2\Psi}{drdy}$. Note that the above equation, for $K(r) = r^2$, is satisfied by $\Psi(r,y) = r^2 + a^2y^2$. However, the remaining non-trivial Einstein equations do not satisfied by $\Psi(r,y) = r^2 + a^2y^2$, neither in the far-r approximation nor in the slowly rotating approximation. For convenience, we present the $t\phi$ component of the Einstein equations in the aforementioned approximations:

$$-\frac{8\pi\eta^2}{r^2}a\ sin^2(\theta) = 0 \tag{G.60}$$

Therefore, the solution (G.58) fails to satisfy the Einstein equations. However, we may try to solve the Einstein equations in order to determine the function $\Psi(r, y)$. Note that our solution has some constraints: for $\eta^2 = 0$ our metric reduces to Kerr metric, hence we suppose $\Psi(r, y) = r^2 + a^2y^2 + \eta^2 f(r, y)$, also for $a = 0$ the metric reduces to the non-rotating case, therefore we may have $\Psi(r, y) = r^2 + a^2y^2 + \eta^2a^2b(r, y)$. Additionally, since the metric must not divergence at $r \to \infty$, the function $b(r, y)$ is up to first order with respect to r. Nonetheless, the $\Psi(r, y) = r^2 + a^2y^2 + \eta^2a^2b(r, y)$ does not satisfy Einstein equations, even for the $\mathcal{O}(\eta^3)$ approximation.

Note that, the Kerr space-time asymptotically ($r \to \infty$ and $r \gg a$) is similar to the flat Minkowski space-time. It is reasonable to consider that the metric of the rotating self-gravitating global monopole may asymptotically be similar to the Minkowski space-time with deficit angle:

$$ds^2 = (1 - 8\pi\eta^2)dt^2 - \frac{dr^2}{1 - 8\pi\eta^2} - r^2(d\theta^2 + sin^2(\theta)d\phi^2) \tag{G.61}$$

Considering $\Psi(r, y) \sim r^2$ for $r \to \infty,\ r \gg a$, the ansatz (G.57) reads:

$$ds^2 = (1 - 8\pi\eta^2)dt^2 - \frac{dr^2}{1 - 8\pi\eta^2} - r^2(d\theta^2 + sin^2(\theta)d\phi^2) + 2a\ sin^2(\theta)\ 8\pi\eta^2 dtd\phi \tag{G.62}$$

Note that the off-diagonal component is non-zero. If we demand that, in this limit, the metric reads (G.61), we need to modify the procedure [46] with a consistent way. As we mentioned before, the function $K(r)$ given in (G.42) is not the most general. Hence, if we consider that $K(r)$ reads

$$K(r) \equiv \sqrt{\frac{F(r)}{G(r)}}H(r) - 8\pi\eta^2 H(r) \tag{G.63}$$

the ansatz (G.45) remains unchanged. Upon substituting the known functions into (G.45), we obtain the off-diagonal component: $g_{t\phi} = \frac{a\ sin^2(\theta)2Mr}{b^2r^2 + a^2cos^2(\theta)}$, hence for $r \to \infty$ we have $g_{t\phi} = 0$. Additionally, for $\eta^2 = 0 \Rightarrow b = 1$, we obtain Kerr solution. However, the corresponding metric is not valid, since it is not reduced to the non-rotating metric when $a = 0$. Consequently, the Newman-Janis algorithm [43], the procedure [46] and some simple modifications of [46] do not apply to the global self-gravitating monopole model.

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