ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ

ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ ΚΑΙ ΦΥΣΙΚΩΝ ΕΠΙΣΤΗΜΩΝ

ΣΧΟΛΗ ΜΗΧΑΝΟΛΟΓΩΝ ΜΗΧΑΝΙΚΩΝ

ΕΚΕΦΕ «ΔΗΜΟΚΡΙΤΟΣ»

ΙΝΣΤΙΤΟΥΤΟ ΝΑΝΟΕΠΙΣΤΗΜΗΣ ΚΑΙ ΝΑΝΟΤΕΧΝΟΛΟΓΙΑΣ

ΙΝΣΤΙΤΟΥΤΟ ΠΥΡΗΝΙΚΗΣ ΚΑΙ ΣΩΜΑΤΙΔΙΑΚΗΣ ΦΥΣΙΚΗΣ

Διατμηματικό Πρόγραμμα Μεταπτυχιακών Σπουδών
«Φυσική και Τεχνολογικές Εφαρμογές»

Θεωρήματα Μονοτονίας στην Ροή Ομάδας Επανακανονικοποίσης

Monotonicity Theorems on Renormalization Group flow

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

Του Λεωνίδα Αναστάσιου Καραγεώργου

Επιβλέπων: Νίκος Ήργες

Αθήνα, Ιούνιος, 2023

National and Technical University of Athens
School of Applied Mathematics and Physical Sciences, Physics Department

Monotonicity Theorems on Renormalization Group flow

Author
Karageorgos Leonidas Anastasios

Supervisor
Irges Nikos

June 18, 2023

Abstract

In Chapter 1, a comprehensive introduction to conformal symmetry is presented, starting with an overview of d-dimensional space-time conformal transformations and the structure of the conformal group. The extension of conformal symmetry to classical field theory is then discussed, including the proof of the traceless energy-momentum tensor for conformal invariant theories. The constraints imposed by conformal invariance on quantum field theories are also examined.

Chapter 2 focuses on fundamental concepts of renormalization and the renormalization group (RG) flow, employing the Wilsonian approach. The renormalization process of pseudoscalar Yukawa theory is presented as an illustrative example.

Chapter 3 provides a detailed analysis of the emergence of the trace anomaly in conformal field theories within curved space.

Chapter 4 centers around the reproduction of Zomolochikov's C-theorem, which establishes the irreversibility of the RG flow for two-dimensional renormalizable field theories. Additionally, the application of this theorem to the massive Thiring Model is discussed.

Finally, Chapter 5 demonstrates that Zomolochikov's approach is not applicable to theories in dimensions other than two ($d \neq 2$). By incorporating Cardy's conjecture regarding the one-point function of the trace, the proof of the a-theorem is reconstructed by considering the RG flow as a manifestation of spontaneously broken conformal symmetry as Komargodski and Schwimmer originally did.

Ευχαριστίες

Αυτό αποτελεί το πιο δύσκολο κεφάλαιο . Δεν υπάρχουν αρκετές λέξεις για να γράψω για όσους μου στάθηκαν καθ' όλη την διάρκεια αυτού του μεταπτυχιακού.

Ξεκινώντας από τους καθηγητές μου, θα ήθελα πρώτα από όλα να ευχαριστήσω τον ακαδημαϊκό (και όχι μόνο) επιβλέπων μου, Νίκο Ήργες που μου έδωσε την ευκαιρία να καταπιαστώ με ένα τόσο απαιτητικό θέμα. Ο κύριος Ήργες ήταν δίπλα μου καθ' όλη την διάρκεια της διπλωματικής, δίνοντάς μου στήριξη σε ζητήματα που έχουν να κάνουν με την διπλωματική, αλλά δεν περιορίστηκε μόνο εκεί, καθώς μου έδωσε πολύτιμες συμβουλές για την ακαδημαϊκή πορεία που καλούμαι να χαράξω. Ελπίζω η συνεργασία μας αυτή να συνεχιστεί και να καρποφορήσει.

Επίσης θα ήθελα να ευχαριστήσω τον κύριο Νίκο Μαυρόματο, ο οποίος αποτελεί έναν από τους ανθρώπους που με ενέπνευσαν να ασχοληθώ με την Κβαντική Θεωρία Πεδίου. Επίσης ο κύριος Μαυρόματος με τον αυθόρμητο χαρακτήρα του, με έκανε να νιώσω αρκετά έξυπνος (ίσως και πιο έξυπνος από ότι πραγματικά είμαι) ώστε να συνεχίσω την πορεία μου στο μονοπάτι της Θεωρητικής Φυσικής. Η έλευση του κύριου Μαυρόματου στη ΣΕΜΦΕ είναι ένα δώρο για όλους εμάς.

Επιπλέον θα ήθελα να ευχαριστήσω τον κύριο Γεώργιο Κουτσούμπα, που μου πρόσφερε τις πρώτες μου διαλέξεις στη Κβαντική Θεωρία Πεδίου και θα μπορούσα να πω ότι εξ' αιτίας του πλέον "έχω μπει σε μπελάδες".

Δεν θα μπορούσα να παραλείψω τον κύριο Λευτέρη Παπαντωνόπουλο, που μόνο η παρουσία του δίνει δύναμη σε όλους εμάς του τρίτου ορόφου να συνεχίζουμε. Αποτελεί παράδειγμα ανθρώπου που αγαπά την Φυσική, αλλά και τους φοιτητές του.

Περνώντας στην αμέσως κατώτερη ακαδημαϊκή βαθμίδα, θα ήθελα να ευχαριστήσω τα παιδιά του τρίτου ορόφου, τους γνωστούς και ως "3rd Floor United". Πάνο, Νίκο, Θανάση, Σωτήρη, Χαράλαμπε, σας ευχαριστώ για όλα, χωρίς εσάς θα ήταν πολύ πιο δύσκολο.

Αστείρευτη πηγή δύναμης και αντοχής αποτελούν τα άτομα που μοιραζόμαστε κοινές αξίες και ιδανικά. Είναι οι άνθρωποι που κάθε μέρα παλεύουμε από κοινού για να γίνουμε η σπίθα που θα κάνει τα σκοτάδια φως. Τα άτομα αυτά είναι τόσο πολλά που δεν θα έφτανε το χαρτί για να τους απαριθμήσω.

Μέσα σε αυτά τα άτομα όμως, ανήκει και ένα ξεχωριστό πρόσωπο, που έκανε τα δικά μου σκοτάδια φως και που συνεχίζει να κάνει πιο όμορφη τη ζωή μου. Είναι η κοπέλα μου Φανή, που παρόλο που δεν καταλαβαίνει ούτε λέξη από το κείμενο της διπλωματικής που θα ακολουθήσει, στέκεται δίπλα μου και δείχνει εμπιστοσύνη στις ικανότητές μου.

Ειδική αναφορά πρέπει να κάνω στην Χριστίνα, που μέσα σε αυτόν τον χρόνο είχε σημαντική συνεισφορά στο να βελτιώσω τα Αγγλικά μου, ώστε να γράψω αυτή την διπλωματική. Το ακόμα πιο σημαντικό όμως, είναι ότι με βοήθησε να διαχειριστώ το παιδί που έχω μέσα μου και να καταφέρω να κάνω το επόμενο βήμα.

Κλείνοντας, γνωρίζω καλά ότι δεν θα μπορούσα να καταφέρω τίποτα χωρίς να έχω δίπλα την οικογένειά μου. Ευχαριστώ τους γονείς μου, Μαριάννα και Λεωνίδα που ξέρω ότι η αγάπη τους είναι αρκετή για να σηκώσει οποιοδήποτε βάρος και να εμφανιστεί, την αδερφή μου Μαλβίνα και την υπέροχη οικογένεια της που φροντίζουν καθημερινά ό,τι αγαπώ και τον αδερφό μου Γιάννη που κάθε μέρα μου δίνει δύναμη να περπατώ τα βήματα για δύο ανθρώπους

Contents

Chapter 1

Conformal symmetry

In this chapter, our focus will be on the examination of conformal symmetry at a classical level. Initially, we will introduce the concept of conformal transformations within D-dimensional spacetime. Subsequently, we will proceed to explore the generalization of conformal symmetry for classical fields, necessitating the application of classical field theory in curved space. Moreover, we will demonstrate that conformally invariant theories possess the characteristic of a traceless energy-momentum tensor. Additionally, we will provide a brief introduction to specific aspects of two-dimensional conformal theories. Finally, we will elucidate the constraints that arise from the presence of conformal symmetry in quantum theories.

1.1 Conformal Transformations in D-dimensional spacetime

A conformal transformation is a transformation that changes the scale locally. This transformation can be expressed, for the metric, in this way:

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \tag{1.1}$$

Where $\Omega(x)$ is a positive function and $g_{\mu\nu}$ is the metric of a d-dimensional manifold.

The laws of physics are diffeomorphism invariant. So they do not change under a transformation:

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \tag{1.2}$$

For simplicity the metric $g_{\mu\nu}$ is considered flat (Euclidean or Minkowski). By applying an infinitesimal transformation ($|\epsilon(x)| \ll 1$), we get:

$$\begin{aligned} g'_{\mu\nu} &= g_{\rho\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \\ &= g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \end{aligned} \tag{1.3}$$

From eq. 1.1 it is easy to convince someone that the second term of the last equation is proportional to the metric. So we can write down the above differential equation ;

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = f(x) g_{\mu\nu} \tag{1.4}$$

The $f(x)$ is not any arbitrary function. There are some constraints that should be satisfied. By multiplying the last equation with $g^{\mu\nu}$ from both sides:

$$\partial \cdot \epsilon = \frac{d}{2} f(x) \tag{1.5}$$

By acting with ∂^ν to the 1.4, while the metric is flat ($\partial_\kappa g_{\mu\nu} = 0$), it gives:

$$\begin{aligned} \partial_\mu (\partial \cdot \epsilon) + \partial^2 \epsilon_\mu = \partial_\mu f(x) \ , \quad \partial \cdot \epsilon = \frac{d}{2} f(x) \\ \partial^2 \epsilon_\mu = \frac{2-d}{2} \partial_\mu f(x) \end{aligned} \tag{1.6}$$

Acting once again with ∂^μ to the last expression, it gives:

$$\begin{aligned} \partial^2 (\partial \cdot \epsilon) &= \frac{2-d}{2} \partial^2 f(x) \\ (1-d) \partial^2 f(x) &= 0 \end{aligned} \tag{1.7}$$

This result gives a constraint for $f(x)$, which is that this function is at most linear in x for all the dimensions $d > 1$.

By acting with ∂_ν to 1.6 gives:

$$\begin{aligned} \partial^2 \partial_\nu \epsilon_\mu &= \frac{2-d}{2} \partial_\mu \partial_\nu f(x) \Rightarrow \\ g_{\mu\nu} \partial^2 f(x) - \partial^2 (\partial_\mu \epsilon_\nu) &= \frac{2-d}{2} \partial_\mu \partial_\nu f(x) \Rightarrow \\ g_{\mu\nu} \partial^2 f(x) - \frac{2-d}{2} \partial_\mu \partial_\nu f(x) &= \frac{2-d}{2} \partial_\mu \partial_\nu f(x) \Rightarrow \\ (2-d) \partial_\mu \partial_\nu f(x) &= g_{\mu\nu} \partial^2 f(x) \end{aligned} \tag{1.8}$$

As we can see, the additional condition for $d > 2$ is that

$$\partial_\mu \partial_\nu f(x) = 0 \tag{1.9}$$

From the derived condition, we can conclude that the third derivatives of ϵ vanish. So conformal Killing vectors are at most quadratic to x. Going back to function $f(x)$ the general solution that satisfies the previous conditions is:

$$f(x) = \lambda + 2b \cdot x \tag{1.10}$$

Now we have everything needed to solve the Conformal Killing equation 1.4, which is a non-homogeneous partial differential equation. So we can separate the solution into two parts, a homogeneous one and a non-homogeneous one, $\epsilon = \epsilon_\mu^{(0)} + \epsilon_\mu^{(n)}$, where:

$$\partial_\mu \epsilon_\nu^{(0)} + \partial_\nu \epsilon_\mu^{(0)} = 0 \tag{1.11}$$

The general solution of this equation is:

$$\epsilon_\mu^{(0)} = a^\mu + \omega_\nu^\mu x^\nu \tag{1.12}$$

Where $\omega_{\mu\nu}$ is an anti-symmetric tensor and a^μ is constant. This solution generates the Poincare Group (Space-time translations + Lorentz transformations), which as we will see is a subgroup of the Conformal Group. Here it should be noted that the Poincare Group is the fundamental symmetry of space-time underlying all QFTs. So we can suppose that Conformal symmetry brings a "more symmetric QFT'.

Coming back to the general solution of 1.4, the corresponding value of ϵ^μ is:

$$e^\mu = \epsilon^\mu_{(0)} + \lambda x^\mu + 2(b \cdot x) x^\mu - x^2 b^\mu \tag{1.13}$$

Now there have been added two extra types of infinitesimal transformations. The first one is called dilation (scale symmetry):

$$x'^\mu = (1 + \lambda) x^\mu \tag{1.14}$$

Dilatation seems not to be a good symmetry of physical systems, as there does not exist a fundamental energy scale on which all observers must agree.

The second one is called special conformal transformation:

$$x'^\mu = x^\mu + 2(b \cdot x) x^\mu - x^2 b^\mu \tag{1.15}$$

All these infinitesimal transformations do not commute each other, but they form a group, which is called the "conformal group". The generators of this group are:

$$P^\mu = i\partial^\mu \tag{1.16}$$

$$M^{\mu\nu} = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) \tag{1.17}$$

$$D = i x^\mu \partial_\mu \tag{1.18}$$

$$K^\mu = i\left(2x^\mu x^\nu \partial_\nu - x^2 \partial^\mu\right) \tag{1.19}$$

Where P^μ stands for translations, $M^{\mu\nu}$ for Lorentz transformations, D for scale transformations and K^μ for special conformal transformations. The algebra of this group is given above:

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\left(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho}\right) \tag{1.20}$$

$$[M^{\mu\nu}, P^{\rho}] = -i\left(g^{\mu\rho}P^{\nu} - g^{\nu\rho}P^{\mu}\right) \tag{1.21}$$

$$[M^{\mu\nu}, K^{\rho}] = -i\left(g^{\mu\rho}K^{\nu} - g^{\nu\rho}K^{\mu}\right) \tag{1.22}$$

$$[D, P^{\mu}] = -iP^{\mu} \tag{1.23}$$

$$[D, K^{\mu}] = iK^{\mu} \tag{1.24}$$

$$[P^{\mu}, K^{\nu}] = 2i\left(g^{\mu\nu}D - M^{\mu\nu}\right) \tag{1.25}$$

$$[M^{\mu\nu}, D] = [P^{\mu}, P^{\nu}] = [K^{\mu}, K^{\nu}] = 0 \tag{1.26}$$

The conformal algebra is isomorphic to $SO(D{+}1,1)$ [1]. We consider a $(D{+}2)$-dimensional Minkowski space, with coordinates $X^1, X^2, ...X^{D+2}$, where the X^{D+2} is the timelike direction. Working on the light cone coordinates then we have:

$$ds^2 = \sum_{i=1}^{D}\left(dX^i\right)^2 - dX^+dX^- \tag{1.27}$$

And the conformal generators will be identified as:

$$J_{\mu\nu} = M_{\mu\nu}, \tag{1.28}$$

$$J_{\mu+} = P_{\mu}, \tag{1.29}$$

$$J_{\mu-} = K_{\mu} \tag{1.30}$$

$$J_{+-} = D, \tag{1.31}$$

And the algebra of these generators obeys the $SO(D+1,1)$ algebra.

Now we can count down the number of generators. From Lorentz transformations, we get $\frac{d(d-1)}{2}$ since $M^{\mu\nu}$ is a $d \times d$ antisymmetric matrix, one from scale transformation, d from translations and d from special conformal transformations. So the total number of generators in a conformally symmetric d-dimensional spacetime is:

$$Number\ of\ generators = \frac{(d+1)(d+2)}{2}\ ,\ d > 2 \tag{1.32}$$

The case where $d = 2$ is a special one. There are infinitely many more conformal transformations in d = 2. As we will see later, there is no remarkable difference between Euclidean and Minkowski conformal transformations in d = 2, as the transformation acts essentially on the two light-cone(for the Minkowski) /holomorphic (for the Eucledean) coordinates independently.

1.2 Conformal Field Theory for classical fields

By now we have seen how conformal transformations act on coordinates. Now we have to consider conformally symmetric field theories. Conformal symmetry plays an important role in classical field theory. It is a type of symmetry that allows for the transformation of a field into a new field that is conformally related to the original field. This allows for the study of a field under different conditions without necessarily changing the form of the equations that govern it.

In classical field theory, conformal symmetry is a special case of the more general notion of invariance under a group of transformations. A conformal transformation is one that preserves angles and magnitudes of distances. This means that the same field equation can be used to describe the same field under a different set of conditions.
Conformal symmetry is particularly important in quantum field theory, where it is used to describe the behavior of particles and forces at very small distances. We postpone this discussion to the next section.

In classical field theory, conformal symmetry is related to the notion of scale invariance. A scale-invariant system is one in which the equations of motion do not change when the system is scaled up or down. This is an essential property in a number of physical systems. Conformal symmetry is a generalization of scale invariance and allows for studying a field under a wider range of transformations. Finally, it is important to note that these transformations are associated with the shape of spacetime and can be used to describe the behavior of physical systems in the presence of gravity.

1.2.1 Classical Field Theory in curved space

A conformal transformation changes the geometry of spacetime, something that has an impact on the dynamics of the field. So in order to study conformal symmetry for classical fields, first of all, we should introduce the classical field theory in curved space.

We begin with the action that describes the system. The general form of a generally covariant action in curved spacetime is:

$$S = \int d^d x \sqrt{-g} \mathcal{L}(g_{\mu\nu}, \Phi_i, \partial_\mu \Phi_i, ...) \tag{1.33}$$

There are some requirements for the action [2]:

1. The action should be real-valued
2. The action is a local functional of fields and their derivatives
3. The equations of motion of the fields contain at most second-order derivatives
4. For a flat background spacetime the action should be Poincare invariant, something that means that there is not an explicit dependence on $\mathbf{x}$ and t

5. For an arbitrarily curved spacetime the action has a general covariant form
6. If the system has internal symmetries (e.g. gauge symmetries), the action should be invariant under these transformations.

For non-trivial geometries (existence of gravity), the field that describes gravity is the metric tensor, so due to the term $\sqrt{-g}$, there is a coupling between the other types of fields and gravity. The simplest type of field theories in curved space are those, known as minimal coupled. The action (for a scalar field) is given by:

$$S = \int d^d x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} (\partial_\mu \Phi)(\partial_\nu \Phi) - V(\Phi) \right] \tag{1.34}$$

The equations of motion for the scalar field in curve space are given by:

$$\partial_\nu \left(\sqrt{-g} g^{\mu\nu} \partial_\mu \Phi \right) - \sqrt{-g} \frac{dV(\Phi)}{d\Phi} = 0 \tag{1.35}$$

So we see that curved background impacts the equations of motion for the scalar field. The last equation can be written in a covariant form:

$$\nabla^\mu \nabla_\mu \Phi - \frac{dV(\Phi)}{d\Phi} = 0 \tag{1.36}$$

Simple example of Field Theories in FRW Universe

We are going to discuss a simple example of field theories with dynamical geometries. This one will be the scalar field in an expanding flat Universe. This example is very interesting for Cosmology.

First we consider the FRW(0) spacetime:

$$ds^2 = -dt^2 + a^2(t) d\mathbf{x}^2 \tag{1.37}$$

This metric is conformally flat, so we can define the conformal time $\eta(t)$:

$$\eta(t) = \int_0^t \frac{dt}{a(t)} \tag{1.38}$$

So the line element is given as:

$$ds^2 = a^2(\eta) \left[-d\eta^2 + d\mathbf{x}^2 \right] = a^2(\eta) \eta_{\mu\nu} dx^\mu dx^\nu \tag{1.39}$$

For this metric we have that $\sqrt{-g} = a^4(\eta)$, $g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$ and $g^{\mu\nu} = a^{-2}(\eta)\eta_{\mu\nu}$. The simplest model that we can study is the free massive scalar field, where we substitute $V(\Phi) = \frac{1}{2}m^2\Phi^2$. It is a straightforward calculation to find the equations of motion, which are ;

$$\Phi''(\eta, \mathbf{x}) + 2\frac{a'(\eta)}{a(\eta)}\Phi'(\eta, \mathbf{x}) - \nabla^2_{\mathbf{x}}\Phi + a^2(\eta)m^2\Phi = 0 \tag{1.40}$$

Now we can define an auxiliary field $\chi(x) = a(\eta)\Phi(x)$. Under straightforward calculation, the equations of motion in terms of the auxiliary field are:

$$\chi'' - \nabla^2\chi + \left(m^2a^2 - \frac{a''}{a}\right)\chi = 0 \tag{1.41}$$

We can identify the last equation as the Klein-Gordon equation for the field χ with time-dependent mass, $m^2_{eff}(\eta) = m^2a^2 - \frac{a''}{a}$ and the equations of motion are:

$$\chi'' - \nabla^2\chi + m^2_{eff}\chi = 0 \tag{1.42}$$

In analogy the action in χ-field terms is written as:

$$\begin{aligned} S &= \int d^3\vec{x}d\eta a^4\left[\frac{1}{2}a^{-2}\left(\phi'^2 - (\nabla\phi)^2\right) - \frac{1}{2}m\phi^2\right] \\ &= \int d^3\vec{x}d\eta\left[\frac{1}{2}\left(\chi'^2 - (\nabla\chi)^2\right) - \frac{1}{2}ma^2\chi^2 - \frac{2\chi\chi'a'}{a} + \frac{\chi^2a'^2}{a^2}\right] \\ &= \int d^3\vec{x}d\eta\frac{1}{2}\left[\chi'^2 - (\nabla\chi)^2 - m^2_{eff}\chi^2 + \frac{1}{2}\left(\frac{\chi^2a'}{a}\right)'\right] \end{aligned}$$

The last term is a total time derivative, so after an integration vanishes. We conclude, the action in terms of the auxiliary field is:

$$\int d^3\vec{x}d\eta\frac{1}{2}\left[\chi'^2 - (\nabla\chi)^2 - m^2_{eff}\chi^2\right] \tag{1.43}$$

Which is the action for the free massive[1] scalar field in flat spacetime. The dynamics of the scalar field Φ in FRW spacetime are mathematically equivalent to the dynamics of the auxiliary field χ in Minkowski spacetime, with time-dependent mass.

[1] With time-dependent mass

Dynamics of the metric- Gravity

By now we have introduced the metric as the field of gravity, but we were considering this field as a background field. Now we are going to study its dynamics. The simplest scalar that we can construct by the curvature tensor is the Ricci scalar. The action of pure gravity (without matter) is:

$$S_{GR} = \int d^d x \sqrt{-g} R \tag{1.44}$$

The variation with respect to the metric, gives the equations of motion for gravity, the well-known Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{1.45}$$

Now we can take a further step, and study the field equations of gravity interacting with matter. The action of this system is:

$$S = \frac{1}{16\pi G} S_{GR} + S_{Matter} \tag{1.46}$$

So the equations of motion now will take the well-known form:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{1.47}$$

Where $T_{\mu\nu}$ is the energy momentum tensor and is given by:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{Matter}}{\delta g^{\mu\nu}} \tag{1.48}$$

1.2.2 Conformally coupled fields with gravity

From now on we will think of the metric tensor as a field and assume that conformal transformation is a transformation of fields. This way of thinking requires studying the theory in curved space. So the action that describes the free massless scalar field is [3]:

$$S = \int d^d x \sqrt{-g} \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \alpha R \phi^2 \right] \tag{1.49}$$

Where R is the Ricci scalar, and α is a dimensionless constant. There is a specific value of α for which the previous action is invariant under infinitesimal conformal transformations.

The equations of motion of this action are:

$$(\nabla^\mu \nabla_\mu - 2\alpha R)\phi = 0 \tag{1.50}$$

So now we have to define the transformation of fields as a scale transformation of the scalar field combined with a Weyl transformation of the metric. So we get:

$$\delta g_{\mu\nu} = -\omega(x) g_{\mu\nu} \rightarrow \delta g^{\mu\nu} = \omega(x) g^{\mu\nu} \tag{1.51}$$
$$\delta\phi = \frac{d-2}{4}\omega(x)\phi \tag{1.52}$$

By doing the variation of the previous action we get:

$$\begin{aligned}\delta S = &\int d^d x \left[-\frac{1}{2}\delta(\sqrt{-g}g^{\mu\nu})\partial_\mu\phi\partial_\nu\phi - \sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu(\delta\phi)\right] \rightarrow \delta S_1 \\ &+ \int d^d x \left[\alpha\delta(\sqrt{-g}R)\phi^2\right] \rightarrow \delta S_2 \\ &+ \int d^d x \left[2\alpha\sqrt{-g}R\phi\delta\phi\right] \rightarrow \delta S_3 \end{aligned} \tag{1.53}$$

So now we have to do straightforward calculations.

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta(g^{\mu\nu}) = -\frac{d}{2}\omega\sqrt{-g} \tag{1.54}$$
$$\delta(\sqrt{-g}g^{\mu\nu}) = -\frac{d-2}{2}\omega\sqrt{-g}g^{\mu\nu} \tag{1.55}$$
$$\delta(R_{\mu\nu}) = \nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu}) = \frac{d-2}{2}\nabla_\mu\nabla_\nu\omega + \frac{1}{2}g_{\mu\nu}\nabla^\lambda\nabla_\lambda\omega \tag{1.56}$$
$$\delta R = \delta(g^{\mu\nu})R_{\mu nu} + g^{\mu\nu}\delta R_{\mu\nu} = \omega R + (d-1)\nabla^\lambda\nabla_\lambda\omega \tag{1.57}$$

Now we have all the tools needed to continue the calculation of the specific value of α.

$$\delta S_1 = \int d^d x\sqrt{-g}\left[\frac{d-2}{4}\omega\partial_\mu\phi\partial^\mu\phi - \frac{d-2}{4}\partial^\mu\phi\partial_\mu(\omega\phi)\right] \tag{1.58}$$
$$\delta S_2 = \int d^d x\sqrt{-g}\alpha\left[\omega\frac{2-d}{2}R + (d-1)\nabla^\lambda\nabla_\lambda\omega\right]\phi^2 = \int d^d x\sqrt{-g}\alpha\omega\left[\frac{2-d}{2}R + (d-1)\nabla^\lambda\nabla_\lambda\right]\phi^2 \tag{1.59}$$
$$\delta S_3 = \int d^d x\sqrt{-g}\alpha\omega\frac{d-2}{2}R\phi^2 \tag{1.60}$$

We see that the first term of δS_2 cancels with δS_3. For the second term of δS_2 we get ;

$$\nabla^\lambda \nabla_\lambda \phi^2 = \nabla^\lambda (2\phi \nabla_\lambda \phi) = 2\partial_\mu \phi \partial^\mu \phi + 2\phi \nabla^\lambda \nabla_\lambda \phi$$
$$\Rightarrow \delta S_2 + \delta S_3 = \int d^d x \sqrt{-g} 2\alpha\omega (d-1)(\partial_\mu \phi \partial^\mu \phi + \phi \nabla^\lambda \nabla_\lambda \phi) \tag{1.61}$$

By integrating by parts the second term of δS_1 we get:

$$\delta S_1 = \int d^d x \sqrt{-g} \frac{d-2}{4} \omega (\partial_\mu \phi \partial^\mu \phi + \phi \nabla^\lambda \nabla_\lambda \phi) \tag{1.62}$$
$$\Rightarrow \delta S = \int d^d x \sqrt{-g} \omega \left[\left(\frac{d-2}{4} + 2\alpha(d-1) \right) (\partial_\mu \phi \partial^\mu \phi + \phi \nabla^\lambda \nabla_\lambda \phi) \right]$$
$$\alpha = -\frac{1}{2} \frac{d-2}{4(d-1)} \tag{1.63}$$

For this value, conformal transformations are the symmetry of the system. We see again that $d = 2$ is a special case, where $\alpha = 0$.

An important fact about classical conformal field theories is the tracelessness of the energy-momentum tensor. Under an arbitrary coordinate transformation, the change of the action is:

$$\delta S = \int d^d x T^{\mu\nu} \partial_\mu \epsilon_\nu$$

The energy-momentum tensor is assumed to be symmetric. So we can write:

$$\delta S = \frac{1}{2} \int d^d x T^{\mu\nu} (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)$$

recalling (1.4) we get:

$$\delta S = \frac{1}{d} \int d^d x T^\mu_\mu \partial \cdot \epsilon \Rightarrow T^\mu_\mu = 0 \tag{1.64}$$

So we conclude that the tracelessness of the stress-energy tensor implies the invariance of the system under conformal transformation. It is important to note that this is valid even if the equations of motion are unsatisfied. The converse is not true, since ϵ^μ is not arbitrary

1.3 Two-dimensional Conformal Field Theory

We are going to study the special case of $d = 2$, where we suppose a Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$. In this case the eq.(1.4) gives as a result:

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2 \ , \ \partial_1 \epsilon_2 = -\partial_2 \epsilon_1 \tag{1.65}$$

We can identify these two equations as the Cauchy-Riemann equations, so it is useful to turn on the complex plane with $z, \bar{z} = x^1 \pm i x^2$, where we can define the holomorphic and anti-holomorphic functions $f(z) = \epsilon_1 + i\epsilon_2$ and $\bar{f}(\bar{z}) = \epsilon_1 - i\epsilon_2$.
The metric of the complex plane becomes:

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \ , \ g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{1.66}$$

So the conformal coordinate transformations in the complex plane are:

$$z \to z + f(z) \tag{1.67}$$
$$\bar{z} \to \bar{z} + \bar{f}(\bar{z}) \tag{1.68}$$

Since functions $f, \bar{f}$ are holomorphic and anti-holomorphic, we can expand them as :

$$f(z) = \sum_n \epsilon_n z^{n+1} \tag{1.69}$$
$$\bar{f}(\bar{z}) = \sum_n \bar{\epsilon}_n \bar{z}^{n+1} \tag{1.70}$$

So we can conclude that there exists an infinite number of generators for the infinitesimal conformal transformations, which are:

$$l_n = -z^{n+1}\partial \tag{1.71}$$
$$\bar{l}_n = -\bar{z}^{n+1}\bar{\partial} \tag{1.72}$$

The algebra of these generators is known as Witt algebra:

$$[l_m, l_n] = (m-n) l_{m+n} \tag{1.73}$$
$$[\bar{l}_m, \bar{l}_n] = (m-n) \bar{l}_{m+n} \tag{1.74}$$
$$[l_m, \bar{l}_n] = 0 \tag{1.75}$$

This algebra is satisfied only for the classical conformal field theories. For a quantum theory symmetry transformations act projectively on states. Projective representations of an algebra are equivalent to representations of a centrally extended algebra. The central extension of Witt algebra is also known as Virasoro algebra.

Here occurs a tricky question about two-dimensional conformal symmetry. From the analysis for a d-dimensional CFT we get that the number of generators of the symmetry is $\frac{(d+1)(d+2)}{2}$, so there should be 6 generators, instead of an infinite number.

In order to give an answer to this paradox, we consider the vector that generates the conformal transformations:

$$v(z) = \sum_n a_n l_n \tag{1.76}$$

This vector field should be regular throughout the whole Riemann sphere $(C \cup \{\infty\})$. Looking at the vicinity of 0 we get that $a_n \neq 0$ for $n \geqslant -1$. By using the transformation $z = \frac{1}{w}$ we can study the behavior in a neighborhood of the point infinity:

$$v(z) = \sum_n a_n \left(\frac{1}{w}\right)^{n-1} \partial_w \tag{1.77}$$

Under the requirement that the vector field is regular at infinity, we have that $a_n \neq 0$ for $n \leqslant 1$. So we can conclude that conformal transformations in two dimensions globally defined on the Riemann sphere correspond to $n = 0, \pm 1$. The same result can be obtained for an anti-analytic vector field. However, the number of parameters does not double since the generators of the two algebras that preserve the real section of C^2 are expressed by the linear combinations. So in two dimensions, we must distinguish between the global and the local conformal transformations. This distinction exists only in two-dimensional CFTs. The global conformal transformations are those that are uniquely invertible and well-defined on all of the complex plane plus infinity. So strictly speaking, the only conformal group in two dimensions is the global conformal group.

By choosing to work on complex coordinates we get some interesting properties of the stress-energy tensor of a conformally symmetric classical theory. Firstly from the tracelessness property, we get:

$$T^{\mu\nu} g_{\mu\nu} = T^{z\bar{z}} g_{z\bar{z}} + T^{\bar{z}z} g_{\bar{z}z} = T^{\bar{z}z} = 0 \tag{1.78}$$

From conservation law we get ;

$$\begin{aligned} \partial_\alpha T^{\alpha\beta} &= \partial_z T^{z\beta} + \partial_{\bar{z}} T^{\bar{z}\beta} \\ \partial_\alpha T^{\alpha z} = \partial_z T^{zz} + \partial_{\bar{z}} T^{\bar{z}z} &= \partial_z T^{zz} = 0 \\ \partial_\alpha T^{\alpha\bar{z}} = \partial_z T^{z\bar{z}} + \partial_{\bar{z}} T^{\bar{z}\bar{z}} &= \partial_{\bar{z}} T^{\bar{z}\bar{z}} = 0 \end{aligned} \tag{1.79}$$

By lowering the indices we get that:

$$\bar{\partial} T = 0 \ , \ T = T_{zz} \tag{1.80}$$
$$\partial \bar{T} = 0 \ , \ \bar{T} = T_{\bar{z}\bar{z}} \tag{1.81}$$

so T and $\bar{T}$ are analytic and anti-analytic functions respectively.

1.4 Constraints of conformal invariance for a quantum theory

All the information of a quantum system can be encoded in the N-point functions. In order to have a theory with conformal invariance the above properties should be satisfied[4]:

- There is a set of fields-operators $\{f_i\}$, which in general is infinite and contains the derivatives of all the fields f_i
- There exists a subset of fields $\{\phi_i\}$ with dimension Δ_i, called "quasi-primary", that under global conformal transformations transform :

$$\phi_i(x) \to \left|\frac{\partial x'}{\partial x}\right|^{\Delta_i/d} \phi_i(x') \tag{1.82}$$

  where $\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{\sqrt{g'}} = \Omega^{-d/2}$ is the jacobian of the transformation. The N-point correlation functions are covariant under this transformation, in the sense that :

$$\langle \phi_1(x_1)...\phi_n(x_n) \rangle = \prod_{i=1}^{n} \left|\frac{\partial x'}{\partial x}\right|_{x=x_i}^{\Delta_i/d} \langle \phi_1(x'_1)...\phi_n(x'_n) \rangle \tag{1.83}$$

- All the other fields can be written as linear combinations of the quasi-primary fields and their derivatives.
- There is a vacuum state $|0\rangle$, which is invariant under the global conformal group. This vacuum state is included in a Hilbert space. In order to define this Hilbert space, space-time is foliated into surfaces of equal time, and to each time slice we associate a Hilbert space of quantum states. For example for a scale-invariant vacuum state we have:

$$D\,|0\rangle = 0$$

From these properties, we can obtain severe restrictions about the 2- and 3- point functions of quasi-primary fields. For simplicity, we consider spinless fields.
We will study the case of 2-point functions. We can perform the same analysis for the

3-point correlation functions. From translation and rotation invariance it is required that the 2-point function is a function of $|x_1 - x_2|$

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1 - x_2|)$$

From scale invariance we get:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2} \langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle$$

the last two symmetry restrictions give a result:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{1.84}$$

By demanding special-conformal invariance we get:

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \frac{(\gamma_1\gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{1.85}$$

with $\gamma_i = 1 - 2bx_i + b^2c_i^2$.

This constraint is satisfied only if $\Delta_1 = \Delta_2$. So for the quasi-primary fields, we get that the 2-point function is:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \begin{cases} \frac{C_{12}}{|x_1-x_2|^{2\Delta}} & \Delta_1 = \Delta_2 = \Delta \\ 0 & \Delta_1 \neq \Delta_2 \end{cases} \tag{1.86}$$

We can follow the same procedure in order to define the three-point function. A recent work about the derivation of the correlators of N operators is presented at [5]

1.4.1 Ward Identities

At the quantum level, the main objects of study are the correlation functions. As we saw before symmetry leads to constraints relating to different correlation functions. The consequences of symmetry can be expressed via the Ward identities. Considering an infinitesimal transformation, generated by a set of generators $\{G_a\}$, we can write [6]:

$$\Phi'(x) = (1 - ie_aG_a)\Phi(x) \tag{1.87}$$

where $\{e_a\}$ is a collection of infinitesimal parameters.

Under local infinitesimal transformations, the action is not invariant and its variation is given by:

$$\delta S = \int d^d x \partial_\mu j^\mu_a e_a(x) \tag{1.88}$$

where j^μ_a is the current associated with the infinitesimal transformation. We set $X = \Phi(x_1)\Phi(x_2)...\Phi(x_n)$, we can write:

$$\langle X \rangle = \int [d\Phi] (X + \delta X) e^{-S[\Phi] - \int d^d x \partial_\mu j^\mu_a e_a(x)} \tag{1.89}$$

By expanding to first order in $e_a(x)$ we get that:

$$\langle \delta X \rangle = \int d^d x \partial_\mu \langle j^\mu_a X \rangle e_a(x) \tag{1.90}$$

The variation δX is given explicitly:

$$\langle \delta X \rangle = -i \sum_{i=1}^{n} e_a(x_i) \langle \Phi(x_1)...G_a\Phi(x_i)...\Phi(x_n) \rangle \tag{1.91}$$

It is straightforward to show that for any infinitesimal $e_a(x)$ we can write the following identity, also known as Ward Identity:

$$\partial_\mu \langle j^\mu_a(x)\Phi(x_1)...\Phi(x_n) \rangle = -i \sum_{i=1}^{n} \delta(x - x_i) \langle \Phi(x_1)...G_a\Phi(x_i)...\Phi(x_n) \rangle \tag{1.92}$$

By integrating the last identity, and supposing that the points x_i are included in the region, then we get:

$$\int d\Sigma_\mu \langle j^\mu X \rangle = -i e_a \sum_{i=1}^{N} \langle \Phi(x_1)...G_a\Phi(x_i)...\Phi(x_n) \rangle \tag{1.93}$$

The r.h.s. of the last equation is the variation of the correlation function $\delta_e \langle X \rangle$. Looking at the l.h.s. we can assume that the integral vanishes since on the hypersurface of the region the $\langle j^\mu X \rangle$ goes to zero, by hypothesis. So we conclude that the variation of the N-point correlation function vanishes under infinitesimal transformations.

Ward Identities for Conformal invariance

Now we are going to find the Ward identities implied by conformal invariance. First, we begin with translations. The conserved current of translation symmetry is the energy-momentum tensor and the generator is $P^{\mu} = -i\partial^{\mu}$. So the first Ward Identity is:

$$\partial_{\mu} \langle T^{\mu}{}_{\nu} X \rangle = -\sum_{i} \delta(x - x_i) \partial_{\nu} \langle X \rangle \tag{1.94}$$

For the Lorentz symmetry, the generator is $M^{\mu\nu} = i\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right) + S^{\mu\nu}$ and the associated current is $j^{\mu\nu\rho} = T^{\mu\nu}x^{\rho} - T^{\mu\rho}x^{\mu}$. So the Ward identity takes the form:

$$\partial_{\mu} \left\langle \left(T^{\mu\nu}x^{\rho} - T^{\mu\rho}x^{\mu}\right) X \right\rangle = \sum_{i} \delta(x - x_i) \left[\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right) \langle X \rangle - iS_i^{\mu\nu} \langle X \rangle\right] \tag{1.95}$$

Taking into account that the derivative acts both on the stress-energy tensor and on the coordinates, and by using the Ward identity for the translation symmetry we reduce the above to

$$\left\langle \left(T^{\mu\nu} - T^{\nu\mu}\right) X \right\rangle = -i \sum_{i} \delta(x - x_i) S_i^{\mu\nu} \langle X \rangle \tag{1.96}$$

Finally, the current associated with the scale symmetry is $j_D^{\mu} = x_{\nu}T^{\mu\nu}$ and the generator is $-i\left(x^{\mu}\partial_{\mu} + \Delta\right)$. So the Ward identity reduces to:

$$\langle T^{\mu}{}_{\mu} X \rangle = -\sum_{i} \delta(x - x_i) \Delta_i \tag{1.97}$$

These identities will be useful in order to prove the monotonicity theorem for two-dimensional field theories (the C-theorem)

Chapter 2

Aspects on Renormalization and Renormalization Group Flow

In this chapter, the quantization of field theories, will not be introduced. Both canonical and path integral quantization procedures are very well introduced in several introductory textbooks for QFT as[7],[8], but it will not be discussed in this thesis. The path integral formulation will not be presented, but it is very well introduced by the "father" of this formulation R.P. Feynman in his textbook [9]. The Wilsonian approach will be displayed and we will see how the renormalization group occurs. Moreover, in this chapter, the Callan-Symanzik equation will be presented and we will define the well-known β and γ functions, which encode all the information about the flow in the space of coupling constants. At the end of this chapter, the example of Yukawa pseudoscalar theory will be displayed, in order to show how actually the flow in the space of coupling constants works. Two different versions of the Renormalization Group (RG) are used in QFT, the continuum RG, and the Wilsonian approach. [7] This discussion will be focused on the Wilsonian RG.

2.1 Wilsonian approach to Renormalization Group

Wilson's analysis states that every quantum field theory fundamentally has a cutoff Λ. This fundamental scale has a physical significance, for example, in fundamental particle theories, the cutoff should be proportional to scales that there is no need for a quantum theory of gravity. With this assumption, then all the loops are finite and the theory is well-defined. By using the functional integral formulation, it can be proven that by changing the cutoff, the physical system is described by an effective Lagrangian, where the coupling constants have been transformed. The set of transformations of coupling constants is called the Renormalization Group. This idea is based on the fact that the description of a physical system at energy scales smaller than μ should be done by the appropriate set of variables that are defined on this scale. So the underlying principle of this picture is that all the parameters of a field theory can usefully be thought of as scale-dependent entities.

It is known that all the information about a physical system is encoded into the partition function $Z[J]$, which is given by:

$$Z[J] = \int \mathcal{D}\phi e^{i\int d^n x(\mathcal{L}+J\phi)} \tag{2.1}$$

For simplicity, only scalar fields are considered in the theory

With respect to Wilson's analysis, there is a sharp UV cutoff to the theory. To control the large values of momenta we should apply a Wick rotation and impose the UV cutoff in Euclidean space (where there are no null directions, where the components of k are extremely large, but k^2 is still zero). As a result, the momenta values in the Fourier expansion, are restricted $|k| \leqslant \Lambda$, where k is Euclidean. So the partition Function is more appropriate to be written as:

$$Z[J] = \int [\mathcal{D}\phi]_\Lambda e^{-\int d^n x(\mathcal{L}_E+J\phi)} \tag{2.2}$$

The next step is to find a way to integrate the high momentum degrees of freedom. By introducing a parameter $b < 1$, we are trying to write the partition function as:

$$Z[J] = \int [\mathcal{D}\phi]_{b\Lambda} e^{-\int d^n x\left(\mathcal{L}_{eff}+J\phi\right)} \tag{2.3}$$

To achieve this, we can schematically separate the momentum space into two regions:

$$\phi \to \hat{\phi} + \phi \tag{2.4}$$

$$\hat{\phi} = \int_{b\Lambda}^{\Lambda} \frac{d^n k}{(2\pi)^n} \tilde{\hat{\phi}}(k) e^{ikx} \tag{2.5}$$

$$\phi = \int_0^{b\Lambda} \frac{d^n k}{(2\pi)^n} \tilde{\phi}(k) e^{ikx} \tag{2.6}$$

So we can rewrite the partition function as follows:

$$Z = \int \mathcal{D}\phi \mathcal{D}\hat{\phi} e^{-S[\phi+\hat{\phi}]} \tag{2.7}$$

The action $S[\phi + \hat{\phi}]$ reproduces the initial action but with a smaller cutoff $b\Lambda$. So it is not false to write:

$$Z = \int [\mathcal{D}\phi]_{b\Lambda} e^{-S[\phi]} \int \mathcal{D}\hat{\phi} e^{-S_{int}[\phi,\hat{\phi}]} \tag{2.8}$$

So by calculating the second path integral, the terms with interactions of the ϕ field will add some extra terms to the action, so, as a result, there will be an effective action that describes the system:

$$Z = \int [\mathcal{D}\phi]_{b\Lambda} e^{-S_{eff}[\phi]} \tag{2.9}$$

The previous conversation gives the general idea behind Wilson's approach. Now, a more convenient way will be introduced. The underlying principle of the Renormalization Group is that physics at energy scales $E \ll \Lambda$ is independent of the precise value of Λ, so the region of interest is the "low energy" limit. [1] For simplicity, we will rescale the scalar ϕ^4 theory. With the term "rescale" we mean that we change the energies and the distances that we study.

$$k' \to k/b \tag{2.10}$$

$$x' \to bx \tag{2.11}$$

where $b < 1$

This rescale will have an impact on the term of the action.

$$S_{eff} = \int d^n x \left[\frac{1}{2}(1+\Delta Z)(\partial_\mu \phi)^2 + \frac{1}{2}(m^2 + \Delta m^2)\phi^2 \frac{1}{4}(\lambda + \Delta\lambda)\phi^4 + \Delta C(\partial_\mu \phi)^4 + \Delta D \phi^6 + ... \right] =$$
$$= \int d^n x' b^{-n} \left[\frac{b^2}{2}(1+\Delta Z)(\partial'_\mu \phi)^2 + \frac{1}{2}(m^2 + \Delta m^2)\phi^2 \frac{1}{4}(\lambda + \Delta\lambda)\phi^4 + \Delta C b^4(\partial'_\mu \phi)^4 + \Delta D \phi^6 + ... \right]$$

We set the $\phi' = \left[(1+\Delta Z)b^{2-n}\right]^{\frac{1}{2}} \phi$. So now we can rewrite the action ;

$$S_{eff} = \int d^n x' \left[\frac{1}{2}(\partial'_\mu \phi')^2 + \frac{1}{2} m' \phi'^2 \frac{1}{4}\lambda' \phi'^4 + C'(\partial'_\mu \phi')^4 + D' \phi'^6 + ... \right] \tag{2.12}$$

The new coupling constants of the Lagrangian, are connected with the initials with the above set of transformations:

$$m'^2 = (m^2 + \Delta m^2)(1+\Delta Z)^{-1} b^{-2} \tag{2.13}$$

$$\lambda' = (\lambda + \Delta\lambda)(1+\delta Z) b^{n-4} \tag{2.14}$$

$$C' = \Delta C (1+\Delta Z)^{-2} b^n \tag{2.15}$$

$$D' = \Delta D (1+\Delta Z)^{-3} b^{2n-6} \tag{2.16}$$

This set of transformations is called the Renormalization Group. Continuing this procedure we can integrate over another shell of momentum space and transform the Lagrangian further. By taking b close to 1 (but still smaller than 1), the shells become infinitesimally

[1] Actually we do not consider the low energy physics limit (ex. non-relativistic), but the energies are very small relative to the cutoff

thin, so the transformation becomes a continuous one.

As far as $b < 1$ the operators that are multiplied with positive powers of b after applying several times the RG transformations tend to vanish. On the other hand, multiplied operators with negative powers of b tend to be more and more important. These operators have to do with the renormalizability of the theory. As we can see, the non-renormalizable terms tend to vanish and the "super-renormalizable" ones become more important. Finally, there are the renormalizable terms (that have zero mass dimension), that have a zero coefficient, something which means that the constant stays unchanged in the first order. The operators can be classified as:

- Relevant operators are the operators that diverge throw the RG flow to the IR and are analogous to the super-renormalizable theories
- Irrelevant operators are the operators that tend to vanish throw the RG flow to the IR and are analogous to the non-renormalizable theories
- Marginals are the operators that are multiplied by b^0. These operators are analogous to renormalizable theories.

Following Wilson's approach we can think of the renormalization as a trajectory or flow in the space of all possible Lagrangians. This picture gives us a deep understanding of why Nature should be describable in terms of renormalizable QFTs.

It is essential to note that, by integrating out degrees of freedom from UV to IR, it seems that the trajectory in the space of coupling constants, the RG flow, is irreversible, by the mean that through this flow, the information about the UV degrees of freedom gets lost. We will prove later this for two-dimensional field theories, as Zomolodchikov originally did.[10]

2.2 The Callan-Symanzik Equation

Another way to obtain information on the RG flows is from the renormalized Green's functions. But now, we do not have to think about the cutoff of the theory, since it has been sent to infinity. The parameters of a renormalizable theory, are defined by a set of renormalization conditions, which are applied to a momentum scale, known as the renormalization scale. This momentum scale M is arbitrary, so it is possible to define the same theory at a different scale. As a result, the bare Green's function $G_0^{(n)}(x_1, x_2, ...x_n; g_{i(0)}, \Lambda)$ should be the same, where $g_{i(0)}$, $i \in N$, are the bare coupling constants. The renormalized Green's function is equal to the bare, up to the rescaling powers of the field strength renormalization:

$$G^{(n)}(x_1, x_2, ...x_n; g_i, M) = Z^{-n/2} G_0^{(n)}(x_1, x_2, ...x_n; g_{i(0)}, \Lambda) \tag{2.17}$$

If we shift the renormalization scale by δM there should be a corresponding shift in the coupling constants, in order to keep fixed the bare Green's function.

$$
\begin{aligned}
M &\to M + \delta M \\
g_i &\to g_i + \delta g_i \\
\phi &\to (1 + \delta\eta)\phi
\end{aligned}
$$

Since Green's function is the time-order product of n fields $G^{(n)} = \langle 0| T\phi(x_1)\phi(x_2)...\phi(x_1) |0\rangle$, then it is simply shifted by a term $n\delta\eta$

$$
\begin{aligned}
G^{(n)} &\to (1 + n\delta\eta)G^{(n)} \\
\Rightarrow dG^{(n)} &= n\delta\eta G^{(n)}
\end{aligned} \tag{2.18}
$$

As we said before, $G^{(n)}$ is computed on a certain renormalization scale, for a specific coupling constant g_i. So we can think of $G^{(n)}$ as a function of M and g_i. So the shift of Greens's function can be written as;

$$
dG^{(n)} = \frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial g_i}\delta g_i \tag{2.19}
$$

From the last two equations we obtain the differential equation:

$$
\left[M\frac{\partial}{\partial M} + \beta(g_i)\frac{\partial}{\partial g_i} + n\gamma(g_i)\right] G^{(n)}\left(\{x_i\}; M, g_i\right) = 0 \tag{2.20}
$$

Where we have defined the dimensionless functions β and γ as;

$$
\beta \equiv \frac{M}{\delta M}\delta g_i \tag{2.21}
$$

$$
\gamma \equiv -\frac{M}{\delta M}\delta\eta \tag{2.22}
$$

These two functions are universal, by the mean that they are the same for every n. Also $G^{(n)}$ is renormalized, so β and γ cannot depend on the cutoff. From dimensional analysis, these functions cannot depend on M. So they are functions only of the coupling constant g_i

Now we are going to take a closer look at these two functions In theories with dimensionless coupling constants the dependence of the Green's function to the renormalization scale M, comes from the counterterms. As a result, the β and γ functions can be computed directly from the counterterms. For simplicity, we consider massless scalar field theory and we begin with the two-point function, which in a general theory has the form:

$$G^{(2)}(p) = \frac{i}{p^2} + \frac{i}{p^2}(A \ln \frac{\Lambda^2}{-p^2} + loop\ finite\ terms) + \frac{i}{p^2}(ip^2\delta_Z)\frac{i}{p^2} + (higher\ order\ loops) \tag{2.23}$$

The dependence to the renormalization scale M comes from the term δ_Z. By applying the Callan-Symanzik equation and neglecting the term which is proportional to β we get that :

$$\begin{aligned} \gamma &= \frac{1}{2} M \frac{\partial}{\partial M} \delta_Z \\ \gamma &= -A \end{aligned} \tag{2.24}$$

Now we will follow the same procedure, for the n-point function. In analogy with the two-point function we have, the n-point function for a theory with a dimensionless coupling constant g is:

$$G^{(n)} \sim -ig - iB \ln \frac{\Lambda^2}{-p^2} - i\delta_g - ig \sum_j \left(A_j \ln \frac{\Lambda^2}{-p_j^2} - \delta_{Z_j} \right) \tag{2.25}$$

Again the dependence from the renormalization scale comes from δ_g and δ_{Z_j}. So by applying the Callan-Symanzik equation, we get:

$$\begin{aligned} \beta(g) &= M \frac{\partial}{\partial M} \left(-\delta_g + \frac{1}{2} \sum_j \delta_{Z_j} \right) \\ \beta(g) &= -2B - g \sum_j A_j \end{aligned} \tag{2.26}$$

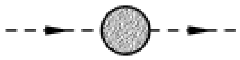
2.3 Renormalization of pseudoscalar Yukawa theory and beta-functions

The Lagrangian of the renormalized pseudoscalar Yukawa theory is :

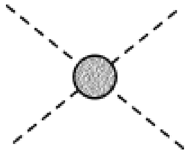
$$\begin{aligned} \mathcal{L} = &\frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m_\phi^2 \phi^2 + \bar{\psi} \left(i \partial\!\!\!/ - m_e \right) \psi - ig \bar{\psi} \gamma^5 \psi \phi - \frac{\lambda}{4!} \phi^4 \\ &+ \frac{1}{2} \delta_\phi (\partial_\mu \phi)^2 - \frac{1}{2} \delta_{m_\phi} \phi^2 + \bar{\psi} \left(i \delta_2 \partial\!\!\!/ - \delta_{m_e} \right) \psi - ig \delta_1 \bar{\psi} \gamma^5 \psi \phi - \frac{\delta_\lambda}{4!} \phi^4 \end{aligned} \tag{2.27}$$

In figure 2.1 we see the renormalization conditions that we are going to use.

We begin with the computation of the pseudo-scalar self-energy diagrams to the one-loop order, keeping only the divergent pieces.



$= \frac{i}{p^2 - m_\phi{}^2 + i\epsilon}$ with pole $= 1$.



$= -i\lambda$ at $s = 4m^2, t = u = 0$.

$$\Sigma(\not p = m) = 0.$$
$$\left.\frac{d\Sigma(\not p)}{d\not p}\right|_{\not p = m} = 0.$$
$$g\Gamma^5(q = 0) = g\gamma^5.$$

Figure 2.1: Renormalization conditions for the Yukawa theory

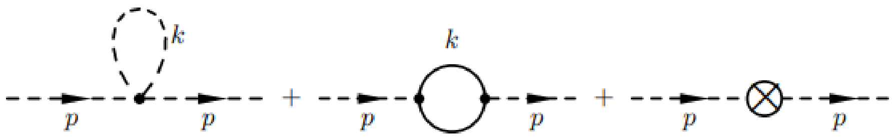
$-iM^2(p^2) =$ 

Figure 2.2: Pseudo-scalar self-energy one loop diagrams

$$\begin{aligned}
-iM^2\left(p^2\right) &= -i\frac{\lambda}{2}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2 - m_\phi^2} - g^2\int \frac{d^dk}{(2\pi)^d}\,\mathrm{Tr}\left[\frac{\gamma^5 i\left(k + \not p + m_e\right) i\gamma^5\left(\not k + m_e\right)}{\left((k+p)^2 - m_e^2\right)\left(k^2 - m_e^2\right)}\right] + i\left(p^2\delta_\phi - \delta_{m_e}\right)\\
&= -i\frac{\lambda}{2}\frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(1 - \frac{d}{2}\right)}{\left(m_\phi^2\right)^{1-d/2}} - 4g^2\int_0^1 dx\int \frac{d^dk}{(2\pi)^d}\frac{\ell^2 - x(1-x)p^2 - m_e^2}{\left(\ell^2 - \Delta\right)^2} + i\left(p^2\delta_\phi - \delta_{m_2}\right)\\
&\sim i\frac{\lambda m_\phi^2}{32\pi^2}\frac{2}{\epsilon} - 8g^2\frac{i}{(4\pi)^2}\frac{2}{\epsilon}\int_0^1 dx\left(m_e^2 - x(1-x)p^2\right) + 4g^2\frac{i}{(4\pi)^2}\frac{2}{\epsilon}\int_0^1 dx\left(m_e^2 + x(1-x)p^2\right)\\
&\quad + i\left(p^2\delta_\phi - \delta_{m_2}\right)\\
&= i\frac{\lambda m_\phi^2}{16\pi^2}\frac{1}{\epsilon} + i\frac{g^2}{4\pi^2}\frac{2}{\epsilon}\left(-2m_e^2 + \frac{2}{6}p^2 + \frac{1}{6}p^2 + m_e^2\right) + i\left(p^2\delta_\phi - \delta_{m_2}\right)\\
&= i\left(\frac{\lambda m_\phi^2}{16\pi^2} + \frac{g^2p^2}{4\pi^2} - \frac{g^2m_e^2}{2\pi^2}\right)\frac{1}{\epsilon} + i\left(p^2\delta_\phi - \delta_{m_2}\right)
\end{aligned} \tag{2.28}$$

So from renormalization conditions, we get

$$\delta_{m_\phi} = \left(\frac{\lambda m_\phi^2}{16\pi^2} - \frac{g^2m_e^2}{2\pi^2}\right)\frac{1}{\epsilon} \tag{2.29}$$

$$\delta_\phi = -\left(\frac{g^2}{4\pi^2}\right)\frac{1}{\epsilon} \tag{2.30}$$

We follow the same procedure for the self-energy of the fermion.

$$
\begin{aligned}
-i\Sigma(\not{p}) &= g^2 \int \frac{d^dk}{(2\pi)^d} \left[\gamma^5 \frac{i}{\left((p-k)^2 - m_\phi^2\right)} \frac{i\left(\not{k} + m_e\right)}{(k^2 - m_e^2)} \gamma^5 \right] + i\left(\not{p}\delta_2 - \delta_{m_e}\right), \\
&= -g^2 \int \frac{d^dk}{(2\pi)^d} \frac{\not{k} - m_e}{(k^2 - m_e^2)\left((p-k)^2 - m_\phi^2\right)} + i\left(\not{p}\delta_2 - \delta_{m_e}\right), \\
&= -g^2 \int_0^1 dz \int \frac{d^d\ell}{(2\pi)^d} \frac{\not{p}z - m_e}{\left(\ell^2 - \Delta\right)^2} + i\left(\not{p}\delta_2 - \delta_{m_2}\right), \\
&\sim -i\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\int_0^1 dz\left(\not{p}z - m_e\right) + i\left(\not{p}\delta_2 - \delta_{m_e}\right), \\
&= i\left(\frac{g^2\not{p}}{16\pi^2} - \frac{g^2 m_e}{8\pi^2}\right)\frac{1}{\epsilon} + i\not{p}\delta_2 - i\delta_{m_e}.
\end{aligned}
\tag{2.31}
$$

Therefore, by applying the renormalization conditions we find:

$$
\delta_{m_e} = -\left(\frac{g^2 m_e}{8\pi^2}\right)\frac{1}{\epsilon} \tag{2.32}
$$

$$
\delta_2 = -\frac{g^2}{16\pi^2}\frac{1}{\epsilon} \tag{2.33}
$$

For the computation of the δ_λ, there are five contributing diagrams given above. The first three are the standard of the single scalar ϕ^4 theory and they contribute with the value $\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}$. A detailed introduction to the computation of loop diagrams and vertices can be found at [11] and the calculation for the one loop contribution of the ϕ^4-theory is given explicitly at [8]

Figure 2.3: Contributing diagrams for the ϕ^4 counterterms in Yukawa pseudoscalar theory

So all the job remaining is up to the fourth diagram. It should be noted that there exists a symmetry factor of six. Also, we consider the momenta inside the loop to be much bigger than the momenta of the external legs. So we get :

$$
\begin{aligned}
i\mathcal{M} &\sim i\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon} - 6g^4 \int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{Tr}\left[\gamma^5\not{k}\gamma^5\not{k}\gamma^5\not{k}\gamma^5\not{k}\right]}{\left(k^2 - m_e^2\right)^4} - i\delta_\lambda, \\
&= i\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon} - 6g^4 \int \frac{d^dk}{(2\pi)^d} \frac{4k^4}{\left(k^2 - m_e^2\right)^4} - i\delta_\lambda, \\
&= i\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon} - 24g^4 \frac{i}{(4\pi)^{d/2}} \frac{d(d+2)}{4} \frac{\Gamma\left(2 - \frac{d}{2}\right)}{6\Delta^{2-d/2}} - i\delta_\lambda, \\
&= i\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon} - i\frac{3g^4}{\pi^2}\frac{1}{\epsilon} - i\delta_\lambda.
\end{aligned}
\tag{2.34}
$$

From the renormalization conditions, we get the result :

$$\delta_\lambda = \frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon} - \frac{3g^4}{\pi^2}\frac{1}{\epsilon} \tag{2.35}$$

Finally, the last counterterm δ_1 is given by the above diagrams:

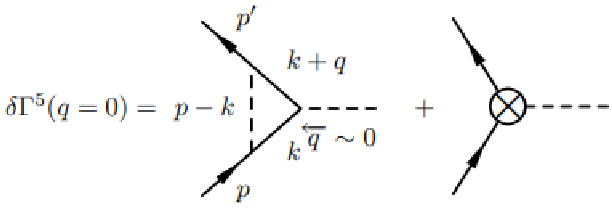


Figure 2.4

$$\begin{aligned}
\delta\Gamma^5(q=0) &= -ig^2 \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^5 (\not{k} + m_e)\, \gamma^5 (\not{k} + m_e)\, \gamma^5}{\left((p-k)^2 - m_\phi^2\right)(k^2 - m_e^2)(k^2 - m^2)} + \delta_1\gamma^5 \\
&= ig^2\gamma^5 \int \frac{d^dk}{(2\pi)^d} \frac{(\not{k} + m_e)(\not{k} - m_e)}{\left((p-k)^2 - m_\phi^2\right)(k^2 - m_e^2)(k^2 - m^2)} + \delta_1\gamma^5 \\
&= ig^2\gamma^5 \int_0^1 dz \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2 + (z^2-1)\, m_e^2}{(\ell^2 - \Delta)^3} + \delta_1\gamma^5 \\
&= ig^2\gamma^5 \int_0^1 dz(1-z) \left[\frac{i}{(4\pi)^2}\frac{d}{2}\frac{2}{\epsilon}\right] + \delta_1\gamma^5 \\
&= -\gamma^5 \frac{g^2}{8\pi^2}\frac{1}{\epsilon} + \delta_1\gamma^5
\end{aligned} \tag{2.36}$$

So we conclude that:

$$\delta_1 = \frac{g^2}{8\pi^2}\frac{1}{\epsilon} \tag{2.37}$$

Now the next step is to find the β functions for the coupling constants. For simplicity we will think the case of the massless Yukawa pseudoscalar theory. By substituting in the (2.35) we get:

$$\beta_g = \frac{5g^3}{16\pi^2} \tag{2.38}$$

$$\beta_\lambda = \frac{3\lambda^2 + 8\lambda g^2 - 48g^4}{16\pi^2} \tag{2.39}$$

Now we can find the flow of the coupling constants. By definition, the β function is the rate of change of the renormalized coupling constant.

$$\frac{d\bar{g}}{d\log p/M} = \beta_g = \frac{5g^3}{16\pi^2} \tag{2.40}$$

Solving this first-order equation we get:

$$\bar{g}^2(p) = -\frac{8\pi^2}{5\log p/M + C_g} \tag{2.41}$$

Where C_g is a constant fixed by the value of g at the renormalization scale. We see that there exists a Landau pole at the scale $p = M\exp\left\{-\frac{C_g}{5}\right\}$. The expression for the running λ is given by:

$$\bar{\lambda} = \frac{g^2}{3}\left(1 + \sqrt{145}\frac{C_\lambda + g^{2\sqrt{145/5}}}{C_\lambda - g^{2\sqrt{145/5}}}\right) \tag{2.42}$$

Where C_λ is a constant fixed by the value of λ at the renormalization scale. From this expression occur different Landau poles, that restrict the energy scales through which the theory is reliable.

Chapter 3

Quantum Field Theory in Curved Space and Conformal Anomaly

Now we are going to study the effects of curvature on quantum systems. There are two approaches. The first one is the canonical quantization process, where we promote fields to operators and built a Hilbert space of quantum states. In this approach, one should be careful about how the vacuum states are defined. The second approach is based on the path integral formulation and we are going to analyze it in this chapter.

3.1 The impact of a classical external force on quantum states

Actually, the path integral formulation is based on the action and proves that all the possible trajectories, not only those that minimize the action, contribute to the amplitude of a system. We will begin our study from the driven harmonic oscillator in order to understand the concept of the background field. Then this formulation will be extended to fields and the metric will take the role of the background field.

3.1.1 Driven Harmonic Oscillator

Classic Driven Harmonic Oscillator

The Lagrangian that describes the driven harmonic oscillator from an external force $J(t)$ is :

$$L = \frac{1}{2}\dot{q}^2 - \frac{\omega}{2}q^2 + J(t)q \tag{3.1}$$

and the equations of motion are :

$$\ddot{q} = -\omega^2 q + J \tag{3.2}$$

we introduce two new dynamical variables, a and $a^\dagger$ as:

$$
\begin{aligned}
a &= \sqrt{\frac{\omega}{2}} \left(q + \frac{i}{\omega} p \right) \\
a^{\dagger} &= \sqrt{\frac{\omega}{2}} \left(q - \frac{i}{\omega} p \right)
\end{aligned}
$$

From the Hamiltonian equations of motion, we get the time evolution of these two variables

$$
\begin{aligned}
\dot{a} + \dot{a}^{\dagger} &= i\omega \left(a^{\dagger} - a \right) \\
\dot{a} - \dot{a}^{\dagger} &= -i\omega \left(a + a^{\dagger} \right) + \frac{i}{\sqrt{2\omega}} J(t)
\end{aligned}
$$

So the equation of time evolution for a is a first-order non-homogeneous differential equation

$$
\frac{da}{dt} - -i\omega a + \frac{i}{\sqrt{2\omega}} J(t)
$$

and the solution is:

$$
a(t) = a_{in} e^{-i\omega t} + \frac{i}{\sqrt{2\omega}} \int_0^t dx J(x) e^{i\omega(x-t)} \tag{3.3}
$$

For the solution, we assumed that before the external force is "turned on" through the time window $t \geqslant 0$, and as we can see this external force has an impact printed on the new dynamical variables, something that will have consequences on the vacuum state at the quantum level.

Quantization

As far as we have solved the classical system, we can quantize it with the standard procedure. We promote the dynamic variables to operators. The equal time commutation relations are :

$$
\begin{aligned}
[a, a^{\dagger}] &= 1 \\
[a, a] &= 0 \\
[a^{\dagger}, a^{\dagger}] &= 0
\end{aligned}
$$

The Hamiltonian of the system is given by:

$$\hat{H} = \omega(\hat{a}^\dagger \hat{a} + \frac{1}{2}) - \frac{\hat{a}^\dagger + \hat{a}}{\sqrt{2\omega}} J(t)$$

We assume that the time window when the external force is turned on is $0 \leqslant t \leqslant T$. So there are two different "free" regions, the "in" region and the "out" region. The annihilation operators are defined as

$$\hat{a}_{in}(t) = e^{-i\omega t} \hat{a}_{in}$$
$$\hat{a}_{out}(t) = e^{-i\omega t} \hat{a}_{out}$$

By using the classical solution for a_{out} we get :

$$\hat{a}_{out}(t) = \hat{a}_{in}(t) + \frac{i}{\sqrt{2\omega}} \int_0^T dx J(x) e^{i\omega x} = \hat{a}_{in}(t) + J_0$$

The Hamiltonian is written as:

$$\hat{H} = \begin{cases} \omega(\hat{a}_{in}^\dagger \hat{a}_{in} + 1/2) \ , \ t \leqslant 0 \\ \omega(\hat{a}_{out}^\dagger \hat{a}_{out} + 1/2) \ , \ t \geqslant T \end{cases} \quad (3.4)$$

It is easy to imagine, that since the annihilation operators are different in each region, then the vacuum states will also be different. So we have to define two different vacuum states:

$$\hat{a}_{in} |0\rangle_{in} = 0$$
$$\hat{a}_{out} |0\rangle_{out} = 0$$

It is remarkable to notice that the $|0\rangle_{in}$ are coherent states of the "out region.

$$\hat{a}_{out} |0\rangle_{in} = J_0 |0\rangle_{in} \neq 0$$

Since $|0\rangle_{in}$ are coherent states, then we can find a relation between these two vacuum states :

$$|0\rangle_{in} = \exp\left[-\frac{|J_0|^2}{2}\right] \sum_{n=0}^{\infty} \frac{J_0^n}{\sqrt{n!}} |n\rangle_{out} = \exp\left[-\frac{|J_0|^2}{2} + J_0 \hat{a}_{out}\right] |0\rangle_{out}$$

From this analysis, we can see that the classical external force has left its signature on the vacuum state of the system, although it has been turned off.

3.1.2 Path integral and background fields

Path integral as a partition function

As we know the propagator of a quantum mechanical system is given by the path integral over all the possible trajectories between the final and the initial point. So we can write :

$$K(q_f; q_i, t_f, t_i) = \int \mathcal{D}q e^{iS[q:;t_f,t_i]} \equiv e^{i\Gamma_L}$$

In the path integral the classic solution can be thought as a fixed number so it is convenient to write

$$\mathcal{D}q = \mathcal{D}(\tilde{q} + q_{cl}) = \mathcal{D}\tilde{q}$$

The action of the driven oscillator is:

$$S[q] = \int dt \left[\frac{1}{2}\dot{q}^2 - \frac{\omega}{2}q^2 - Jq\right]$$

By preforming a Wick rotation ($t \to i\tau$) then the path integral becomes:

$$\int \mathcal{D}\tilde{q} e^{-S_E[q]}$$

and the Euclidean action is:

$$S_E\left[q_{cl} + \tilde{q}\right] = \frac{1}{2}\int d\tau(\dot{\tilde{q}}^2 + \omega^2\tilde{q}^2) - \frac{1}{2}\int d\tau J q_{cl}$$

For the form of the Euclidean action we have used the equations of motion of q in Euclidean space :

$$\frac{d^2 q_{cl}}{d\tau^2} - \omega^2 q_{cl} = J$$

Now we can evaluate the path integral

$$\int \mathcal{D}\tilde{q} e^{-S_E[q]} = \exp\left[\frac{1}{2}\int d\tau J q_{cl}\right]\int \mathcal{D}\tilde{q} e^{-\frac{1}{2}\int d\tau(\dot{\tilde{q}}^2 + \omega^2\tilde{q}^2)} =$$
$$= \mathcal{N}\exp\left[\frac{1}{2}\int d\tau J q_{cl}\right] =$$
$$= \mathcal{N}\exp\left[\frac{1}{2}\int d\tau d\xi J(\tau) G_E(\tau,\xi) J(\xi)\right]$$

where $G_E(\tau, \xi)$ is the Green's function in the Euclidean space. Going back to the Lorentzian framework the action is of the form

$$S = S[\tilde{q}] + \frac{1}{2} \int dt d\xi J(t) G_F(t, \xi) J(\xi)$$

The path integral as its own cal play the role of the partition function, with

$$Z[J] = \int \mathcal{D}q \exp\left[iS[q] + i \int dt J(t) q(t) \right]$$

and the correlation functions as :

$$\langle 0| T q_1..q_n |0\rangle = \frac{1}{Z[J]} \frac{1}{i} \frac{\delta}{\delta J(t_1)} ... \frac{1}{i} \frac{\delta}{\delta J(t_n)} Z[J] = e^{-i\Gamma_L} \frac{1}{i} \frac{\delta}{\delta J(t_1)} ... \frac{1}{i} \frac{\delta}{\delta J(t_n)} e^{i\Gamma_L}$$

3.2 Quantum fields and background field

By now we have studied the path integrals in quantum mechanics. Now we are going to extend this formalism to quantum fields. Now the integration of the path integral takes place over all the possible values of the field.

$$\int \mathcal{D}\phi \exp\left[iS[\phi, g_{\mu\nu}] \right] \equiv \exp[i\Gamma_L]$$

The background field will replace the role of the external force. In a relativistic situation, the background field has its own dynamics which is described by an action $S_B[J]$

We will think of the example of the scalar field, coupled with gravity, where the metric $g_{\mu\nu}$ takes the role of the background field. The action of this system is:

$$S = S^{(GR)}[g_{\mu\nu}] + S^{(M)}[\Phi, g_{\mu\nu}] \tag{3.5}$$

The equations of motion are given by

$$\frac{\delta S}{\delta g^{\mu\nu}} + \frac{\delta \Gamma_L}{\delta g^{\mu\nu}} = 0 \tag{3.6}$$

But we have to note that :

$$\begin{aligned} \langle T_{\mu\nu}(x) \rangle &= \frac{\int \mathcal{D}\Phi T_{\mu\nu} e^{iS}}{\int \mathcal{D}\Phi e^{iS}} \\ &= \frac{2}{\sqrt{-g}} \frac{\delta \Gamma_L}{\delta g^{\mu\nu}} \end{aligned} \tag{3.7}$$

So we conclude with the above relation:

$$R_{\mu\nu} - \frac{1}{2}R = 8\pi G \langle T_{\mu\nu} \rangle \tag{3.8}$$

The last expression is known as the semi-classical Einstein Equation. The vacuum expectation value of the energy-momentum tensor gets modified by the classical background field. This modification is known as vacuum polarization. On the other hand from the measure of the VEV we can find the backreaction of the quantum fields on the metric. This equation is applicable only to weak gravitational fields (weakly curved space) since a strong curvature requires a quantum theory for gravity.

3.2.1 Path integral in curved space

The main thing someone has to do in QFT in curved space is to calculate the vacuum expectation value of the fields and find out the difference between the values that we get in Minkowski spacetime. The main tool in order to calculate expectation values is the path integral formalism, but we have to be careful about how the measure $\mathcal{D}\Phi$ is defined. Generally, $\Phi(x)$ depends on coordinates, so the measure can not be defined as in the case of the harmonic oscillator.

We think the action of the field :

$$\begin{aligned} S[\Phi, g_{\mu\nu}] &= \frac{1}{2}\int d^{2\omega}x\sqrt{-g}\left[-g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - V(x)\Phi^2\right] = \\ &= \frac{1}{2}\int d^{2\omega}x\sqrt{-g}\left[\Phi(\Box_g^{(x)} - V(x))\Phi\right] \end{aligned} \tag{3.9}$$

It is useful to work on Euclidean space (where the Euclidean metric is $\gamma_{\mu\nu}$), so we have to wick rotate the action by $t \to -i\tau$. The Euclidean action is given by :

$$S_E[\Phi, \gamma_{\mu\nu}] = \int d^{2\omega}x_E \frac{1}{2}\left[\Phi(-\Box_\gamma^{(x)} + V(x))\Phi\right] \tag{3.10}$$

in order to well define the path integral we will follow a tricky path. First, we consider the eigenvalue problem:

$$(-\Box_\gamma^{(x)} + V(x))\varphi_n = \lambda_n\varphi \tag{3.11}$$

Where $\{\varphi_i\}$ consist an orthonormal basis

$$\int d^{2\omega}x\sqrt{\gamma}\varphi_\mu\varphi_\nu = \delta_{\mu\nu} \tag{3.12}$$

We can expand the field on this basis:

$$\Phi = \sum_n c_n \varphi_n(x) \tag{3.13}$$

So the action, by the use of the orthogonality between the eigenfunctions, can be written :

$$S_E[\Phi, \gamma_{\mu\nu}] = \frac{1}{2} \sum_n c_n^2 \lambda_n \tag{3.14}$$

It is easy to see that $\{c_n\}$ are independent of the spacetime coordinates, so the path integral measure can be defined through these quantities.

$$\mathcal{D}\Phi = \prod_n \frac{dc_n}{\sqrt{2\pi}} \tag{3.15}$$

So the effective action in Euclidean space is given :

$$\exp\{-\Gamma_E\} \equiv \int \prod_n \frac{dc_n}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\sum_n c_n^2 \lambda_n\right\} = \left[\prod_n \lambda_n\right]^{-1/2} \tag{3.16}$$

$$\Gamma_E = \frac{1}{2} \ln \det(-\Box_\gamma + V(x)) \tag{3.17}$$

Now the problem goes back to the calculation of the functional determinant. In order to achieve this we have to reformulate the problem in terms of a linear operator in an auxiliary Hilbert space $\mathcal{H}$. We define a Hermitian operation $\hat{M}$ which acts on the basis vectors $\{|\psi_n\rangle\}$ and has eigenvalues $\{\lambda_n\}$, that are the same as the eigenvalues of the differential operator.

$$\hat{M}\,|\psi_n\rangle = \lambda_n\,|\psi_n\rangle \tag{3.18}$$

We postulate a non-countable orthonormal basis $\{|x\rangle\}$ running all over the 2ω-dimensional space.

$$\langle x'|x\rangle = \delta(x' - x) \tag{3.19}$$

$$\hat{\mathbf{1}} = \int d^{2\omega}x\,|x\rangle\langle x| \tag{3.20}$$

So as in the standard quantum mechanical procedure, we can expand the $|\psi\rangle$ states on the uncountable basis

$$|\psi\rangle = \int d^{2\omega}x \; \psi(x) \, |x\rangle \;\; , \;\; \psi(x) = \langle x|\psi\rangle \tag{3.21}$$

$$\langle\psi_1|\psi_2\rangle = \int d^{2\omega}x \; \psi_1(x)\psi_2(x) \tag{3.22}$$

We see that the inner product has a non-covariant integration, so we suggest a one-to-one correspondence between the fields $\Phi(x)$ and the states $|\psi\rangle$.

$$\psi(x) = \langle x|\psi\rangle \equiv \gamma^{1/4}\Phi(x) \tag{3.23}$$

In this mapping, we have a correspondence between the linear operator $\hat{M}$ and the differential operator

$$\hat{M}\,|\psi\rangle \rightleftarrows (-\Box_\gamma + V)\Phi(x) \tag{3.24}$$

$$\Rightarrow \langle x|\,\hat{M}\,|\psi\rangle = \gamma^{1/4}(-\Box_\gamma + V)\gamma^{-1/4}\psi(x) \tag{3.25}$$

3.2.2 Zeta functions, Heat Kernels and Quantum Action

Now we are going to introduce a method that can be used to calculate the renormalized determinants of operators. We begin with the function $\zeta_M(s)$ of the operator $\hat{M}$

$$\zeta_M(s) \equiv \sum_n \left(\frac{1}{\lambda_n}\right)^s \tag{3.26}$$

It is straightforward that the derivative ζ-function with respect to s gin=ve the natural logarithm of the determinant of the operator $\hat{M}$

$$\ln det\hat{M} = -\frac{d\zeta_M(s)}{ds}|_{s=0} \tag{3.27}$$

This definition requires that we know all the eigenvalues of the operator $\hat{M}$ and this makes the calculation almost impossible. It is more convenient to compute the ζ-function using another mathematical construction which is called heat kernel. The heat Kernel is divided as :

$$\hat{K}(\tau) = \sum_n \exp\{-\lambda_n\tau\}\,|\psi_n\rangle\langle\psi_n| \tag{3.28}$$

$$\Rightarrow \mathrm{Tr}\Big\{\hat{K}(\tau)\Big\} = \sum_n \exp\{-\lambda_n\tau\} \tag{3.29}$$

By using the Euler's Γ-function, we can redefine the ζ-function as :

$$\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \left[\mathrm{Tr}\Big\{ \hat{K_M(\tau)} \Big\} \right] \tau^{s-1} \tag{3.30}$$

So now the problem goes onto the calculation of the trace of the heat kernel. It seems that we still need to know the eigenvalues, but it can be noted that:

$$\begin{aligned} \frac{d\hat{K}_M(\tau)}{d\tau} &= -\hat{M}\hat{K}_M(\tau) \\ \Rightarrow \hat{K}_M(\tau) &= \exp\Big\{-\hat{M}\tau\Big\} \end{aligned} \tag{3.31}$$

Recalling that $\hat{M} = \gamma^{1/4}(-\Box_\gamma + V)\gamma^{-1/4}$, and that the exponential of an operator is written as:

$$\exp\Big\{-\hat{M}\tau\Big\} = \sum_{n=0}^{\infty} \frac{(-\hat{M}\tau)^n}{n!} \tag{3.32}$$

the trace of heat kernel can be calculated as:

$$\mathrm{Tr}\Big\{\hat{K}_M(\tau)\Big\} = \int d^{2\omega}x \; \langle x| \exp\Big\{-\hat{M}\tau\Big\} |x\rangle \tag{3.33}$$

For the general metric and external force it is difficult to calculate the heat kernel. The assumptions that we can make, in order to be able to proceed with the calculations is to think of the case of weakly curved space and a very small external force $|V| \ll 1$. Since the space is weakly curved the metric can be written as a perturbation of the flat metric

$$\gamma_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu} \tag{3.34}$$

Since the external force is also considered to be small enough, we can perturbatively expand the heat kernel, around the flat space kernel $\hat{K}_0(\tau)$

$$\hat{K}_M(\tau) = \hat{K}_0(\tau) + \hat{K}_1(\tau) + ... \tag{3.35}$$

We begin with the flat space Kernel, where $\hat{M} = -\Box^{flat}_{(x)}$.

$$\langle x|\,\hat{K}_0(\tau)\,|x'\rangle = \exp\Big\{\tau\Box^{flat}_{(x)}\Big\}\delta(x-x') = \int \frac{d^{2\omega}k}{(2\pi)^\omega}\sum_{n=0}^{\infty}\frac{\left(\tau\Box^{flat}_{(x)}\right)^n}{n!}e^{ik(x-x')} = $$
$$= \frac{1}{(4\pi\tau)^\omega}\exp\Big\{-\frac{(x-x')^2}{4\tau}\Big\} \tag{3.36}$$

We continue the procedure up to first order. The linear operator can be written as:

$$-\hat{M} = \Box^{flat} + \hat{s}\,[h_{\mu\nu}, V] \tag{3.37}$$

We have the $\hat{K}_M = \hat{K}_0 + \hat{K}_1$ and taking into account that we work up to first order, the terms $\hat{s}\hat{K}_1$ can be neglected. So we get :

$$\frac{d\hat{K}_1}{d\tau} = \Box^{flat}\hat{K}_1 + \hat{s}\hat{K}_0\ ,\ \hat{K}_1(0) = 0 \tag{3.38}$$

We let $\hat{K}_1(\tau) = \hat{K}_0(\tau)\hat{C}(\tau)$. So we get:

$$\hat{K}_0\frac{d\hat{C}}{d\tau} = \hat{s}\hat{K}_0(\tau) \tag{3.39}$$

We recall that $\hat{K}_0^{-1}(\tau) = \hat{K}_0(-\tau)$ and $\hat{K}_0(\tau)\hat{K}_0(-\tau') = \hat{K}_0(\tau-\tau')$, the first correction is given by :

$$\hat{K}_1(\tau) = \int_0^\tau d\tau'\hat{K}_0(\tau-\tau')\hat{s}\hat{K}_0(\tau') \tag{3.40}$$

The next step is to find the expression for the operator $\hat{s}$. Combining the (3.24) with the metric $\gamma_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$, after some painful algebra we get:

$$\begin{aligned}\langle x|\,\hat{M}\,|x'\rangle = & \\ & -\Big[\Box^{flat}_{(x)} + h^{\mu\nu}\partial_\mu\partial_\nu + \partial_\nu h^{\mu\nu}\partial_\mu\Big]\,\delta(x-x') \\ & -\Big[-\frac{1}{4}\gamma^{\mu\nu}\gamma^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} - \frac{1}{4}\gamma^{\mu\nu}\partial_\nu h^{\alpha\beta}\partial_\mu h_{\alpha\beta}\Big]\,\delta(x-x') \\ & -\Big[-\frac{1}{4}\gamma^{\alpha\beta}\partial_\nu h^{\mu\nu}\partial_\mu h_{\alpha\beta} - \frac{1}{16}\gamma^{\mu\nu}\gamma^{\alpha\beta}\gamma^{\rho\sigma}\partial_\nu h_{\alpha\beta}\partial_\mu h_{\rho\sigma} - V\Big]\,\delta(x-x')\end{aligned} \tag{3.41}$$

From the last expression, we can identify the operator $\hat{s} = \hat{h} + \hat{\Gamma} + \hat{P}$

$$\langle x|\,\hat{h}\,|x'\rangle = h^{\mu\nu}\partial_\mu\partial_\nu\delta(x-x') \tag{3.42}$$

$$\langle x|\,\hat{\Gamma}\,|x'\rangle = \partial_\nu h^{\mu\nu}\partial_\mu\delta(x-x') \tag{3.43}$$

$$\langle x|\,\hat{P}\,|x'\rangle = P(x)\delta(x-x') \tag{3.44}$$

$P(x)$ is given by the last two lines of the previous equality. The first correction can be expanded as

$$\hat{K}_1(\tau) = \hat{K}_1^h(\tau) + \hat{K}_1^\Gamma(\tau) + \hat{K}_1^P(\tau)$$

Now we can calculate the trace of the first order correction. It is easier to do it term by term, where we begin with $\hat{K}_1^P$.

$$\begin{aligned}\langle x|\,\hat{K}_1^P\,|x\rangle &= \int d\tau'\,\langle x|\,\hat{K}_0(\tau-\tau')\hat{P}\hat{K}_0(\tau')\,|x\rangle \\ &= \int d\tau' \int d^{2\omega}y\;\langle x|\,\hat{K}_0(\tau-\tau')\,|y\rangle\langle y|\,\hat{K}_0(\tau')\,|x\rangle\,P(y)\end{aligned}$$

Now we use a mathematical trick in order to calculate this trace. We perform a Fourier expansion to the function $P(y)$ and then there exists a Gaussian integral with respect to y. After this procedure, we get:

$$\begin{aligned}\langle x|\,\hat{K}_1^P\,|x\rangle &= \frac{1}{(4\pi\tau)^\omega}\int d\tau' \int \frac{d^{2\omega}k}{(2\pi)^\omega}\exp\Bigg\{-\frac{\tau'(\tau-\tau')}{\tau}k^2 + i\vec{k}\cdot\vec{x}\Bigg\}\tilde{P}(k) \\ &= \frac{1}{(4\pi\tau)^\omega}\int_0^\tau d\tau'\exp\Bigg\{\frac{\tau'(\tau-\tau')}{\tau}\Box^{flat}_{(x)}\Bigg\}P(x)\end{aligned} \tag{3.45}$$

This result will be used in order to calculate the remaining terms. For the next steps, first, we will consider the non-diagonal terms and then we will take the limit $y \to x$ in which we find the trace.

$$\begin{aligned}\langle x|\,\hat{K}_1^\Gamma\,|y\rangle &= \int d\tau' \int d^{2\omega}z\;\langle x|\,\hat{K}_0(\tau-\tau')\,|z\rangle\,\partial_\nu h^{\mu\nu}(z)\frac{\partial}{\partial z^\mu}\langle z|\,\hat{K}_0(\tau')\,|y\rangle \\ &= -\frac{\partial}{\partial y^\mu}\int d\tau' \int d^{2\omega}z\;\langle x|\,\hat{K}_0(\tau-\tau')\,|z\rangle\,\partial_\nu h^{\mu\nu}(z)\,\langle z|\,\hat{K}_0(\tau')\,|y\rangle = \\ &= -\,\frac{\partial}{\partial y^\mu}\langle x|\,\hat{K}_1^P\,|y\rangle\Big|_{P(x)=\partial_\nu h^{\mu\nu}}\end{aligned} \tag{3.46}$$

so the trace is given by:

$$\langle x|\,\hat{K}_1^{\Gamma}\,|x\rangle = -\lim_{y\to x}\frac{\partial}{\partial y^\mu}\,\langle x|\,\hat{K}_1^{P}\,|y\rangle\Big|_{P(x)=\partial_\nu h^{\mu\nu}} =$$

$$= -\lim_{y\to x}\frac{\partial}{\partial y^\mu}\frac{1}{(4\pi\tau)^\omega}\int d\tau'\int\frac{d^{2\omega}k}{(2\pi)^\omega}\exp\Big\{-\frac{\tau'(\tau-\tau'}{\tau}k^2+\frac{i}{\tau}\vec{k}\cdot(\tau'\vec{x}+(\tau-\tau')\vec{y}\Big\}\tilde{P}(k)\Big|_{P(x)=\partial_\nu h^{\mu\nu}}$$

$$- = \frac{1}{(4\pi\tau)^\omega}\int d\tau'\exp\Big\{-\frac{\tau'(\tau-\tau'}{\tau}\Box^{flat}_{(x)}\Big\}\frac{\tau'(\tau-\tau'}{\tau}\partial_\mu\partial_\nu h^{\mu\nu}(x) \tag{3.47}$$

Following the same steps we can find that $\langle x|\,\hat{K}_1^{h}\,|x\rangle = \lim_{y\to x}\frac{\partial^2}{\partial y^\mu\partial y^\nu}\langle x|\,\hat{K}_1^{P}\,|y\rangle\,|_{P(x)=h^{\mu\nu}}$, which after some straightforward algebra gives:

$$\langle x|\,\hat{K}_1^{h}\,|x\rangle = \frac{1}{(4\pi\tau)^\omega}\int d\tau'\exp\Big\{-\frac{\tau'(\tau-\tau'}{\tau}\Box^{flat}_{(x)}\Big\}\left(\frac{\delta_{\mu\nu}h^{\mu\nu}}{2\tau}+\left(\frac{\tau-\tau'}{\tau}\right)^2\partial_\mu\partial_\nu h^{\mu\nu}\right) \tag{3.48}$$

By substituting and doing the summation between the last three results we get:

$$\langle x|\,\hat{K}_1(\tau)\,|x\rangle = \frac{1}{(4\pi\tau)^\omega}\Big\{P(x)\tau-\frac{1}{2}\delta_{\mu\nu}h^{\mu\nu}(x)-\frac{\tau}{6}\partial_\mu\partial_\nu h^{\mu\nu}(x)$$
$$+\frac{\tau}{6}\Box^{flat}_x P-\frac{\tau}{12}\delta_{\mu\nu}\Box^{flat}_x h^{\mu\nu}(x)-\frac{\tau}{30}\partial_\mu\partial_\nu\Box h^{\mu\nu}(x)+\Box^2(\ldots)\Big\} \tag{3.49}$$

The orders $\Box^2(\ldots)$ comes from the exponential expansion. For the $P(x)$ we select the terms that are of first order with respect to $h^{\mu\nu}$

$$P(x) = \frac{1}{4}\delta_{\mu\nu}\Box^{flat}h^{\mu\nu}-V(x)+\mathcal{O}(h^2) \tag{3.50}$$

We see that the value of the trace of the heat kernel depends on geometrical objects. We recall that the metric is $\gamma_{\mu\nu}=\delta_{\mu\nu}+h_{\mu\nu}$ and by using the identities for the determinant of the metric and the curvature scalar [12] we get that:

$$\mathrm{Tr}\,\hat{K} = \frac{1}{(4\pi\tau)^\omega}\int d2\omega\sqrt{\gamma}\left[1+\left(\frac{1}{6}R-V\right)\tau+\mathcal{O}(h^2)\right] \tag{3.51}$$

The last expression provides the first two terms as an expansion of the heat kernel in the curvature. Another type of expression of the heat kernel in powers of τ is the known Seeley-DeWitt expansion.

$$\langle x|\,\hat{K}(\tau)\,|x\rangle = \frac{\sqrt{\gamma}}{(4\pi\tau)^\omega}\left[1+a_1(x)\tau+a_2(x)\tau^2+O\left(\tau^3\right)\right] \tag{3.52}$$

This expansion is derived without the assumption that the space is weakly curved [2]. The coefficients a_i are scalar functions of the external force and the curvature. It should be noted that this expansion can not be used in order to calculate the $\zeta_M(s)$ since it is not valid for large values of τ. This means that this kind of expansion cannot describe the IR region of the system. The second-order terms were calculated in the paper [13] of Barvinsky and Vilkovisky.

$$
\begin{aligned}
\mathrm{Tr}\,\hat{K}(\tau) = \int \frac{d^{2\omega}x\sqrt{\gamma}}{(4\pi\tau)^{\omega}} \Big\{ 1 + \tau \left[\frac{R}{6} - V \right] \\
+ \frac{\tau^2}{2} \left[V - \frac{R}{6} \right] f_1(-\tau\Box) V + \tau^2 V f_2(-\tau\Box) R \qquad (3.53) \\
+ \tau^2 R f_3(-\tau\Box) R + \tau^2 R_{\mu\nu} f_4(-\tau\Box) R^{\mu\nu} + O\left(R^3, V^3, \ldots\right) \Big\} \qquad (3.54)
\end{aligned}
$$

where the auxiliary functions are defined as:

$$
\begin{aligned}
f_1(x) = \int_0^1 dy e^{-xy(1-y)}, \quad f_2(x) = -\frac{f_1(x)}{6} - \frac{f_1(x) - 1}{2x} \\
f_4(x) = \frac{a_(x) - 1 + \frac{x}{6}}{x^2}, \quad f_3(x) = \frac{f1(x)}{32} + \frac{f_1(x) - 1}{8x} - \frac{f_4(x)}{8}
\end{aligned}
\tag{3.55}
$$

The Seeley-DeWitt coefficients can be reproduced by expanding these terms in τ up to total derivative terms which vanish under the integration over all x. Neglecting the higher order terms[2] :

$$
\begin{aligned}
\mathrm{Tr}\,\hat{K}(\tau) = \int \frac{d^{2\omega}x\sqrt{\gamma}}{(4\pi\tau)^{\omega}} \Big\{ 1 + \tau \left[\frac{R}{6} - V \right] \\
+ \tau^2 \left[\frac{1}{2} V^2 - \frac{1}{6} V R + \frac{1}{120} R^2 + \frac{1}{60} R_{\mu\nu} R^{\mu\nu} \right] + O\left(\tau^3, R^3, V^3, \ldots\right) \Big\}
\end{aligned}
\tag{3.56}
$$

With this result, we can calculate the quantum action. We have just to recall that:

$$
\Gamma_E[\gamma_{\mu\nu}] = -\frac{1}{2} \frac{d\zeta(s)}{ds} \bigg|_{s=0} \tag{3.57}
$$

We will think of the simple case where $V = 0$ and $\omega = 2$. The last expression is not valid for big values of τ, since we have expanded in powers of τ. In order to cure this problem we set an IR cut off τ_1 (big values of τ respond to small values of energy, since τ has dimensions x^2). Moreover, from the eq.(3.30), one can find out some UV ($\tau = 0$) divergent terms in the limit $s \to 0$. In order to understand the type of divergence we set also a UV cut-off τ_o. So the form of ζ-function will be:

$$
\begin{aligned}
\zeta(s) = \frac{1}{(4\pi)^2 \Gamma(s)} \int d^4x \sqrt{\gamma} \Big[\int_{\tau_0}^{\tau_1} \tau^{s-3} d\tau + \frac{R}{6} \int_{\tau_0}^{\tau_1} \tau^{s-2} d\tau \\
+ \left(\frac{1}{120} R^2 + \frac{1}{60} R_{\mu\nu} R^{\mu\nu} \right) \int_{\tau_0}^{\tau_1} \tau^{s-1} d\tau \Big] + \ (finite\ terms)
\end{aligned}
\tag{3.58}
$$

Using the expansion of $\frac{1}{\Gamma(s)}$ around $s = o$ we get that:

$$\Gamma_E\left[\gamma_{\mu\nu}\right] = -\int \frac{d^4x\sqrt{\gamma}}{32\pi^2}\left[\frac{1}{2\tau_0^2} + \frac{1}{6\tau_0}R + \left(\frac{1}{120}R^2 + \frac{1}{60}R_{\mu\nu}R^{\mu\nu}\right)|\ln\tau_0| + (finite\ terms)\right] \tag{3.59}$$

This action is regularized and can be used in order to renormalize the coupling constants, by adding terms that absorb infinities. An example is the classical GR with quadratic curvature terms. It should be noted that in the general case of 2ω dimensional space,there would be $\omega + 1$ divergent terms $\tau_0^{-\omega}\ldots|\ln\tau_0|$

3.3 Conformal anomaly

The formulation of heat kernels can be used in order to calculate the vacuum expectation values of several operators. Recalling the result of Chapter 2 conformal symmetry implies that the trace of the stress-energy tensor vanishes for classic field theories. This result can be expanded to the quantum theory, under the assumption that the space is flat. If the space is curved, then the trace is polarized, by the mean that it gets a non-zero vacuum expectation value. This property is crucial for the proof of the four-dimensional α-theorem. Actually, this is the intuitive idea behind the monotonicity theorems in even dimensions. We will see that the two-dimensional c-theorem can also be proved with the use of the one-point function of the trace in curved space, although gravity in two dimensions is trivial.

The vacuum expectation value of the trace for a conformal symmetry $g_{\mu\nu} \to \Omega^2(x)g_{\mu\nu}$ can be calculated with the use of the quantum action.

$$\delta_g\Gamma_L = \int d^{2\omega}x\frac{\delta\Gamma_L}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = \int d^{2\omega}x\sqrt{-g}\left\langle T^\mu_\mu\right\rangle\delta\Omega(x) \tag{3.60}$$

where Γ_L is given straightforwardly with analytic continuations of the Euclidean quantum action.

$$\begin{aligned}\hat{M}_{\Omega^2\gamma} &= \Omega^{-1}\hat{M}_\gamma\Omega^{-1} = \hat{M}_\gamma - \left(\delta\Omega\hat{M}_\gamma + \hat{M}_\gamma\delta\omega\right) + \mathcal{O}(\delta\Omega^2)\\ \Rightarrow \zeta_{\hat{M}_{\Omega^2\gamma}}(s) &= \operatorname{Tr}\hat{M}_\gamma^{-s} + 2s\operatorname{Tr}\left[\delta\Omega\hat{M}_\gamma^{-s}\right]\end{aligned} \tag{3.61}$$

The quantum action in Euclidean space is given :

$$\begin{aligned}\Gamma_E[\gamma+\delta\gamma] = -\frac{1}{2}\frac{d\zeta_M}{ds}\bigg|_{s=o} &= \Gamma_E[\gamma_{\mu\nu}] - \lim_{s\to 0}\operatorname{Tr}\left[\delta\Omega\hat{M}_\gamma^{-s}\right]\\ \delta_\gamma\Gamma_E &= -\lim_{s\to 0}\operatorname{Tr}\left[\delta\Omega\hat{M}_\gamma^{-s}\right]\end{aligned} \tag{3.62}$$

The trace for general 2ω-dimensional space is given by :

$$\begin{aligned}\mathrm{Tr}\left[\delta\Omega\hat{M}_\gamma^{-s}\right] &= \int d^{2\omega}x\delta\Omega(x)\left\langle x\right|\hat{M}_\gamma^{-s}\left|x\right\rangle = \\ &= \int d^{2\omega}x\delta\Omega(x)\frac{\sqrt{\gamma}}{4\pi\Gamma(s)}\int_0^{+\infty}d\tau\tau^{s-1-\omega}\left[1+a_1(x)\tau+a_2(x)\tau^2+O\left(\tau^3\right)\right]\end{aligned} \tag{3.63}$$

where we have used the Seeley-DeWitt expansion. Again there exists a problem with the big values of τ since the Seeley-DeWitt expansion is not valid for them. Although this expansion does not provide all the information, most of the contribution comes from small values of τ, so it is enough in order to take a result.

An alternative way to decline the "fault" contributions from large valued τ is to diminish them in a smooth way by multiplying the integrand with the $e^{-\xi\tau}$ ($\xi > 0$) and after computing the limit $s \to 0$ then take the limit $\xi \to 0$. It is notable that with this "mathematical trick" the divergences at $\tau = 0$ are also cured. Following this procedure, the above integrals will appear:

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^\infty d\tau e^{-\xi\tau}\tau^s &= \xi^{-s-1}\frac{\Gamma(s+1)}{\Gamma(s)} \\ \frac{1}{\Gamma(s)}\int_0^\infty d\tau e^{-\xi\tau}\tau^{s-1-\omega} &= \xi^{-s+\omega}\Gamma(s-\omega) \\ \int_0^\infty d\tau e^{-\xi\tau}\tau^{s-\omega} &= \xi^{-s+\omega-1}\frac{\Gamma(s-\omega+1)}{\Gamma(s)} \\ \frac{1}{\Gamma(s)}\int_0^\infty d\tau e^{-\xi\tau}\tau^{s+1-\omega} &= \xi^{-s+\omega-2}\frac{\Gamma(s-\omega+2)}{\Gamma(s)}\end{aligned} \tag{3.64}$$

For $\omega = 1$ the only non-vanishing term, after taking the limits, is proportional to $a_1(x)$

$$\lim_{\alpha\to+0}\left(\lim_{s\to+0}\left\langle x\left|\hat{M}^{-s}\right|x\right\rangle\right) = \frac{1}{4\pi}\sqrt{\gamma}a_1(x) = \frac{\sqrt{\gamma}}{24\pi}R \tag{3.65}$$

$$\tag{3.66}$$

So for two-dimensional scalar field theories, after performing analytic continuation, the trace of energy-momentum tensor is given by :

$$\begin{aligned}\delta\Gamma_L &= -\frac{1}{24\pi}\int d^2x\sqrt{-g}\delta\Omega(x)R(x) \\ \left\langle T_\mu^\mu\right\rangle_{2d}^{(scalar)} &= -\frac{R}{24\pi}\end{aligned} \tag{3.67}$$

In the case of generic field theories, the trace in two dimensions is proportional to the charge of the Virasoro algebra

$$\left\langle T^{\mu}_{\mu} \right\rangle_{2d} = -\frac{cR}{24\pi} \tag{3.68}$$

For $\omega = 2$ the only non-vanishing term, after taking the limits, is proportional to $a_2(x)$ and the trace for generic theories (not only scalar fields) is given by :

$$\left\langle T^{\mu}_{\mu} \right\rangle_{4d} = \alpha E_4 - cW^2_{\mu\nu\rho\sigma} \tag{3.69}$$

where E_4 is the Euler density and $W_{\mu\nu\rho\sigma}$ is the Weyl tensor [14] :

$$E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2 \tag{3.70}$$

$$W^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2 \tag{3.71}$$

The values of the constants α and c for generic theories are calculated in [15]

Chapter 4

Monotonicity theorem for 2D Field theories

4.1 Zomolochikov's C-theorem

Now we are going to reproduce the proof of the irreversibility of RG flow in two dimensions. The main idea is to prove that in two dimensions there exists a function that depends on the coupling constants of the theory and has the following properties [10]:

- Is a monotonically decreasing function with respect to scale.

$$\dot{c} = \beta^i \partial_i c(g) \leqslant 0 \tag{4.1}$$

  where β is the well known β-function. The equality of the last expression stands for the fixed point $g = g^*$ where the β-function vanishes.

- At fixed points where we have $\beta(g) = 0$, it implies that $\partial_i c = 0$. At these points, we have conformal symmetry.

- In its fixed point the function $c(g^*) = c$, is equal with the central charge encountered in CFT.

4.1.1 Correlators and Differential equations for charges

As we described before, in 2d Euclidean space it is convenient to work on the complex plane. So we describe the components of the energy-momentum tensor:

$$T = T_{zz} \tag{4.2}$$
$$\Theta = -T_{\bar{z}z} \tag{4.3}$$
$$\bar{T} = T_{\bar{z}\bar{z}} \tag{4.4}$$

Moreover, we will use the connection between the tracelessness of the stress-energy tensor and the renormalizability of the theory. We can expand Θ as:

$$\Theta = \beta^i \Phi_i \tag{4.5}$$

where β^i is the β-function of the coupling constant $g_i \in \{g\}$ (which vanishes at fixed points and there we have a CFT) and Φ_i is a scalar field operator with mass dimension two that comes from the decomposition of the action density $\Phi_i = \partial_i \mathcal{L}$, with $\partial_i \equiv \frac{\partial}{\partial g^i}$ and $\mathcal{L} \equiv g^i \Phi_i$. As it is known the RG flow is the flow in the space of all the possible theories starting from a UV theory and ending to an IR. The scaling is defined as $x \equiv e^t \alpha$, with α the UV cut-off and $t > 0$ some constant that we use to rescale. For $t = \ln(z\bar{z})$ we define the correlators:

$$\langle T(x)T(0)\rangle = \frac{C(t)}{2z^4} \tag{4.6}$$

$$\langle T(x)\Theta(0)\rangle = \frac{H(t)}{z^3\bar{z}} \tag{4.7}$$

$$\langle \Theta(x)\Theta(0)\rangle = \frac{G(t)}{z^2\bar{z}^2} \tag{4.8}$$

From the conservation of energy-momentum tensor the Ward Identity becomes:

$$\partial^\mu \langle T_{\mu\nu} X\rangle = 0 \Rightarrow \partial_z \langle T_{\bar{z}\nu} X\rangle + \partial_{\bar{z}} \langle T_{z\nu} X\rangle = 0 \tag{4.9}$$

From the last equation, we can find the differential equations of the invariant under rotations amplitudes (or charges). Plugging in (4.9) $X = T(0)$ and $\nu = z$ we get:

$$\begin{aligned} -\partial_z \langle \Theta(x)T(0)\rangle + \partial_{\bar{z}} \langle T(x)T(0)\rangle &= 0 \\ \partial_z \left[-\frac{H(t)}{z^3\bar{z}}\right] + \partial_{\bar{z}} \left[\frac{C(t)}{2z^4}\right] &= 0 \\ \frac{3H(t)}{z^4\bar{z}} - \frac{\dot{H}(t)}{z^3\bar{z}}\frac{\partial t}{\partial z} + \frac{\dot{C}(t)}{2z^4}\frac{\partial t}{\partial \bar{z}} &= 0 \\ \frac{1}{2}\dot{C} + 3H - \dot{H} &= 0 \end{aligned} \tag{4.10}$$

The derivative with respect to the rescaling parameter can be expanded on the base of coupling constants, $\dot{C} = \beta^i \partial_i C$. Thanks to (4.5) we can also expand the invariant amplitude $H(t)$ at the same base as $H(t) = \beta^i H_i$ So the last equation is written as:

$$\beta^i \partial_i C = 2\beta^i \partial_i(\beta^j H_j) - 6\beta^i H_i \tag{4.11}$$

We repeat the same process for $X = \Theta(0)$ and $\nu = z$

$$\begin{aligned} \partial_z \langle \Theta(x)\Theta(0)\rangle - \partial_{\bar{z}} \langle T(x)\Theta(0)\rangle &= 0 \\ \partial_z \left[\frac{G(t)}{z^2\bar{z}^2}\right] - \partial_{\bar{z}} \left[\frac{H(t)}{z^3\bar{z}}\right] &= 0 \\ \beta^k \partial_k \left(\beta^i H_i\right) - \beta^i H_i = \beta^k \partial_k \left(\beta^i \beta^j G_{ij}\right) &- 2\beta^i \beta^j G_{ij} \end{aligned} \tag{4.12}$$

Where G_{ij} is the metric in the space of fields. From unitarity, G_{ij} is positively defined since it can be thought of as the norm of the state. Choosing an appropriate basis, the metric G_{ij} is diagonal

4.1.2 Zomolochickov's c-function

In order to define the c-function, we should make some crucial notes. First of all, as far as c is associated with the charge of the theory, it should be a linear combination of the invariant charges, which we have already defined. So the general form of c-function is:

$$c(g) = AC(g) + B\beta^k H_k + \Gamma\beta^i\beta^j G_{ij} \tag{4.13}$$

From the assertion that this function is equal to the central charge of the Virassoro algebra in its fixed point, we get that $A = 1$. This comes from the fact that $\beta^i(g^*) = 0$, plus the OPE of the energy-momentum [6] which leads to:

$$\langle T(x)T(0)\rangle = \frac{c}{2z^4} \tag{4.14}$$

From the assertion that c is monotonically decreasing we can evaluate the two other constants B, Γ. We have that $\beta^i\partial_i c(g) \leqslant 0$

$$\begin{aligned}\beta^i\partial_i c(g) &= \beta^i\partial_i C + B\beta^i\partial_i(\beta^k H_k) + \Gamma\beta^i\partial_i(\beta^j\beta^k G_{jk}) = \\ &= (B+2)\beta^i\partial_i(\beta^k H_k) - 6\beta^k H_k + \Gamma\left[2\beta^k\beta^j G_{kj} + \beta^i\partial_i(\beta^k H_k) - \beta^k H_k\right] = \\ &= (B+2+\Gamma)\,\beta^i\partial_i(\beta^k H_k) - (\Gamma+6)\,\beta^k H_k + 2\Gamma\beta^k\beta^j G_{kj}\end{aligned} \tag{4.15}$$

Since β-function and the values of its derivatives can be positive, negative, or zero, the last expression's first two terms should vanish. The last term is always positive or zero, so the constrain about the value of coefficient Γ is $\Gamma < 0$. So we conclude that:

$$\begin{cases}(B+2+\Gamma) & = 0\\ (\Gamma+6) & = 0\end{cases} \Rightarrow \begin{cases}B & = 4\\ \Gamma & = -6\end{cases} \tag{4.16}$$

So the c-function is:

$$c(g) = C(g) + 4\beta^k H_k - 6\beta^i\beta^j G_{ij} \tag{4.17}$$

with $\dot{c} = -12\beta^i\beta^j G_{ij} \leqslant 0$.

Finally, it is a straightforward calculation to prove that in fixed point $\partial_i c(g^*) = 0$

$$\begin{aligned}
\beta^i \partial_i c &= -12\beta^i \beta^j G_{ij} \\
\partial_i c &= -12\beta^j G_{ij} \ , \ \beta^j|_{g*} = 0 \Rightarrow \\
\partial_i c|_{g*} &= 0
\end{aligned} \tag{4.18}$$

4.1.3 An application to C-theorem

We are going to apply the C-theorem to the massive Thirring Model. There is a special aspect to 2-d fermionic models. In two dimensions a fermionic model can be described in terms of a bosonic one. This procedure is called bosonization. There is a duality between the Thirring model and the Sine-Gordon scalar theory. This aspect will not be analyzed in this thesis, but there exists a pretty good introduction to the bosonization at [16].

The Lagrangian of the massive Thirring model is:

$$S_{Th}[\psi, \bar{\psi}] = \int d^2x \left[\bar{\psi} i \partial\!\!\!/ \psi - m\bar{\psi}\psi - \frac{g}{2}\left(\bar{\psi}\gamma_\mu\psi\right)\left(\bar{\psi}\gamma^\mu\psi\right)\right] \tag{4.19}$$

where m is the mass and g the dimensionless coupling constant. The beta functions of this model in perturbation theory are given [17][18]:

$$\beta_g \equiv \mu\frac{dg}{d\mu} = -64\pi\frac{m^2}{\Lambda^2}, \ \beta_m \equiv \mu\frac{dm}{d\mu} = \frac{-2\left(g + \frac{\pi}{2}\right)}{g+\pi}m - \frac{256\pi^3}{(g+\pi)^2\Lambda^2}m^3 \tag{4.20}$$

The massless Thiring model is a Conformal Field Theory, so we can think of the massless theory as a UV fixed point that gets perturbed by the mass terms and then the RG flow begins.
For this model, the two-point function of the trace is given by [16]:

$$\langle\Theta(0)\Theta(r)\rangle = (m^2/(2\pi))^2\left[K_1^2(mr) - K_0^2(Mr)\right] \tag{4.21}$$

Taking the result from the C-theorem, we have to solve a first-order differential equation.

$$\begin{aligned}
&\dot{c} = -12G(t) \\
&\Rightarrow c_{UV} - c_{IR} = 12\int dt\,\langle\Theta(0)\Theta(r)\rangle\,(z\bar{z})^2 \ , \ t = \ln(z\bar{z}) \\
&c_{UV} - c_{IR} = 24\int_0^\infty dr r^3\,\langle\Theta(0)\Theta(r)\rangle = \frac{1}{2} > 0
\end{aligned} \tag{4.22}$$

Chapter 5

Monotonicity Theorem for 4D Field Theories

5.1 Zomolochikov's approach for c-theorem in $d \neq 2$

After proving the monotonicity theorem of the RG flow for two-dimensional quantum field theories, we can think about the generalization of this theorem, for four-dimensional theories. The first step is to follow the same path as in two dimensions. The difference here is that we can not use a 4-dimensional complex plane analogy, something that makes this process more difficult.

We begin the study of the two-point functions for the energy-momentum tensor. With respect to rotation invariance and parity fix, the two-point function must have the form [19]:

$$\begin{aligned}\langle T_{\mu\nu}(r)T_{\lambda\sigma}(0)\rangle = & \left(A/r^{2d+4}\right) r_\mu r_\nu r_\lambda r_\sigma \\ & + \left(B/r^{2d+2}\right)\left(r_\mu r_\nu \delta_{\lambda\sigma} + r_\lambda r_\sigma \delta_{\mu\nu}\right) \\ & + \left(C/r^{2d+2}\right)\left(r_\mu r_\lambda \delta_{\nu\sigma} + r_\nu r_\lambda \delta_{\mu\sigma} + r_\mu r_\sigma \delta_{\nu\lambda} + r_\nu r_\sigma \delta_{\mu\lambda}\right) \\ & + \left(D/r^{2d}\right)\delta_{\mu\nu}\delta_{\lambda\sigma} + \left(E/r^{2d}\right)\left(\delta_{\mu\lambda}\delta_{\nu\sigma} + \delta_{\nu\lambda}\delta_{\mu\sigma}\right)\end{aligned} \tag{5.1}$$

Considering the conservation of energy-momentum tensor we will take various relations for the invariant amplitudes A, B, C, D, E. As an analogy, to the 2-dimensional case, we must find a linear combination proportional to the correlation function $\langle\Theta\Theta\rangle$. According to John L. Cardy, the best result that can be taken is:

$$\dot{c} = -4\frac{d+1}{d-1}\langle\Theta\Theta\rangle - 2(d-2)B \tag{5.2}$$

where

$$\tilde{c} = -\frac{4}{d-1}\left[A + \frac{1}{2}\left(d^2+d+2\right)B + (d+3)C + \frac{1}{2}d(d+1)D + (d+1)E\right] \tag{5.3}$$

The eq.(5.2) implies that in a conformally invariant point, the derivative of the c-function does not vanish for general $d \neq 2$. This means that this version of the C-theorem can be proved only for two-dimensional field theories.

5.2 The four-dimensional α-theorem

5.2.1 Cardy's conjecture for another c-function

The main proposal for the generalization of the irreversibility of RG flow to 4D- field theories, is based on Cardy's conjecture, that the monotonical function is proportional to the one-point function of the trace of the energy-momentum tensor. As we proved in Chapter 3 when the theory is placed in a curved space, then the vacuum expectation value of the stress-energy tensor is proportional to terms that depend on the geometry of space. First, the two-dimensional case will be checked, and we know that the RG flow is irreversible. The candidate function is the trace integrated over the 2-sphere.

$$\tilde{C} = A_2 \int_{S^2} \langle T^\mu_\mu \rangle \tag{5.4}$$

with A_2 a normalization constant. The curvature scalar of a two-dimensional sphere is $R = \frac{2}{r^2}$. Plugging in eq.(3.68) we get:

$$\begin{aligned} \tilde{C} &= -A_2 \frac{c}{24\pi} \int d\theta d\varphi \, r^2 \sin^2\theta \frac{2}{r^2} = \\ &= -A_2 \frac{c}{6} \end{aligned} \tag{5.5}$$

By choosing $A_2 = -6$ we get that $\tilde{C} = c$. This function satisfies the third property of Zomolochikov's c-function. This is an indication that this proposal can be extended to 4-dimensional field theories.

5.2.2 RG flow as a spontaneously broken conformal symmetry

Conformal field theories are the theories that do not have a cut-off by the mean that the theory is described by a fixed point in the parametric space of coupling constants. As a result, the theory stays unchanged under varying the cut-off, something which means that there is no RG flow. However, consider a UV fixed point, described by a CFT$_{UV}$, which is perturbed by a set of relevant (or marginal) operators $\{M^{4-\Delta}\mathcal{O}\}$. This breaks the conformal symmetry and triggers an RG flow to some IR physics, which may be described by a non-trivial CFT$_{IR}$, as shown in figure 5.1

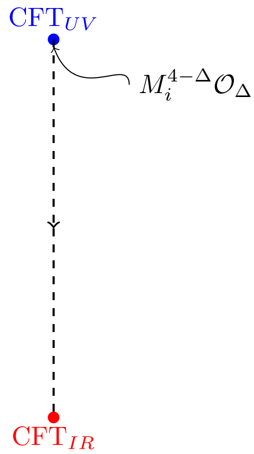


Figure 5.1: Starting from some CFT at high energies, we add a relevant operator $\mathcal{O}_\Delta$ and flow to a new CFT in the deep infrared.

The theory of dilaton

With respect to the Nambu-Goldstone theorem, due to the spontaneous symmetry braking, there exists a massless particle, the dilaton. The new action is an effective action and will have both invariant and anomalous terms. We begin with the invariant term, where we demand the action to be invariant under diff$\times$Weyl transformations. The Weyl transformation acts as:

$$g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu} \ , \ \tau \to \tau + \sigma \tag{5.6}$$

where τ is the dilaton field. The most general action up to two derivatives is given by: [1]

$$S_{kinetic} = f^2 \int d^4x \sqrt{-\hat{g}} \frac{1}{6} \hat{R} \tag{5.7}$$

where $\hat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu}$ and $\hat{R} = \hat{g}^{\mu\nu}\hat{R}_{\mu\nu}[\hat{g}]$. The term f^2 is the decay constant of the broken conformal symmetry. There is no cosmological constant term since the cosmological constant in vacua that break the conformal symmetry spontaneously is zero.

For general d-dimensions the curvature scalar is written as [20]:

$$\hat{R} = \hat{g}^{\mu\nu}\hat{R}_{\mu\nu} = e^{2\tau}\left(R + 2(d-1)\Box\tau - (d-1)(d-2)\partial_\mu\tau\partial^\mu\tau\right) \tag{5.8}$$

Working on 4-dimensional conformally flat spacetime ($g_{\mu\nu} = \eta_{\mu\nu}$) we get that :

[1]For this proof the signature used for the metric is the mostly minus $\eta_{\mu\nu} = (+,-,-,-)$

$$\hat{R} = 6e^{2\tau}\left[\Box\tau - (\partial\tau)^2\right] \tag{5.9}$$

We can evaluate the kinetic term, by using integration by parts and recalling that for the conformally flat case $\sqrt{-\hat{g}} = e^{-4\tau}$:

$$\begin{aligned} S_{kinetic} &= f^2 \int e^{-2\tau}\left[\Box\tau - (\partial\tau)^2\right] = \\ &= f^2 \int e^{-2\tau}\left[2(\partial\tau)^2 - (\partial\tau)^2\right] = f^2 \int e^{-2\tau}(\partial\tau)^2 \end{aligned} \tag{5.10}$$

and the equations of motion are given by:

$$\Box\tau = (\partial\tau)^2 \tag{5.11}$$

The most general effective action with four derivative terms, under the appropriate parameterization [2].

$$\int d^4x\sqrt{-\hat{g}}\left(\xi_1\widehat{R}^2 + \xi_2\widehat{E}_4 + \xi_3\widehat{W}^2_{\mu\nu\rho\sigma}\right) \tag{5.12}$$

The term which is proportional to ξ_1 vanishes at the flat space limit, with the use of equations of motion of τ.

$$\int d^4x\sqrt{-\hat{g}}\widehat{R}^2\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}} = 36\int d^4x\left(\Box\tau - (\partial\tau)^2\right)^2 \overset{e.o.m.}{=} 0 \tag{5.13}$$

For the second term, we have that is a total derivative that can be neglected since:

$$\begin{aligned} \sqrt{-\hat{g}}\hat{E}_4 &= \sqrt{-g}E_4 + 4\sqrt{-g}\nabla^\mu\left(R\partial_\mu\tau - 2R^\nu_\mu\partial_\nu\tau - \nabla_\mu\left(\partial_\nu\tau\partial^\nu\tau\right) + 2\partial_\mu\tau\Box\tau - 2\partial_\nu\tau\partial^\nu\tau\partial_\mu\tau\right)\right) \\ &\overset{g_{\mu\nu}=\eta_{\mu\nu}}{\longrightarrow} \partial^\mu\left(\partial_\mu\left(\partial_\nu\tau\partial^\nu\tau\right) + 2\partial_\mu\tau\Box\tau - 2\partial_\nu\tau\partial^\nu\tau\partial_\mu\tau\right) \end{aligned} \tag{5.14}$$

Finally, the third term is Weyl invariant and so will not give any contribution as far as the dilaton interactions are concerned in flat space.

The next step is to evaluate the anomalous functional. Combining eq.(3.69) with the definition of the energy-momentum tensor, we come up with:

[2] The ordinary way to write the action with four derivatives is $\int d^4x\sqrt{-\widehat{g}}\left(\kappa_1\widehat{R}^2 + \kappa_2\widehat{R}^2_{\mu\nu} + \kappa_3\widehat{R}^2_{\mu\nu\rho\sigma}\right)$, but this parametrization "hides" the conformally symmetric terms.

$$\frac{\delta S_{anomly}}{\delta \tau} = \int d^4x\sqrt{-g}\tau(cW^2_{\mu\nu\rho\sigma} - \alpha E_4) \tag{5.15}$$

The most general anomalous variation requires an extra term proportional to $b'\sigma\Box R$. We neglect this term since has to do only with g and not with tau. This is a consequence of the fact that $\delta_\sigma \int R^2 \sim \int \sigma \Box R$. As a result, if we reintroduce this term to the variation of the anomalous functional, we have just to add a term proportional to $\sqrt{-g}R^2$. The solution to this problem is well discussed in [21]. The final expression for the anomalous functional is given by [14]:

$$\begin{aligned} S_{\text{anomaly}} =& - a\int d^4x\sqrt{-g}\left(\tau E_4 + 4G^{\mu\nu}\partial_\mu\tau\partial_\nu\tau - 4(\partial\tau)^2\Box\tau + 2(\partial\tau)^4\right) \\ &+ c\int d^4x\sqrt{-g}\tau W^2_{\mu\nu\rho\sigma}. \end{aligned} \tag{5.16}$$

where $G^{\mu\nu}$ is the well know Einstein tensor. As we see, we have not added any term proportional to c- anomaly, since Weyl tensor squared is invariant on its own. A very important fact about this functional is that when the theory is projected to flat metric $g_{\mu\nu} = \eta_{\mu\nu}$, the self-interactions of dilaton, still survive. Thanks to this survival at the flat space limit we get a four derivative contribution to α-anomaly. On the other hand, c-anomaly is trivial at the flat limit. Using the leading order equations of motion of the dilaton (5.13) the anomalous functional, projected to flat space is :

$$S^{flat}_{anomaly} = 2\alpha\int d^4x(\partial\tau)^4 \tag{5.17}$$

This four-derivative scattering will give all the information needed in order to define the monotonically decreasing α-function.

5.2.3 Proof of the α-theorem

The proof of the irreversibility of RG flow for 4D field theories is based on the idea that the flow comes from the spontaneous conformal symmetry break as shown in Fig. 5.1. The flow comes from a matter theory, which is described by an action (coupled with a background metric):

$$S_{matter} = S_{matter}[\Phi_i, M_i, g_{\mu\nu}] \tag{5.18}$$

Thanks to the conservation of the stress-energy tensor the partition function is guaranteed to be diff-invariant. The violation of conformal symmetry of the partition function comes from two kinds of anomalies. The first kinds of anomalies that violate the Weyl invariance are the very well-discussed α and c anomalies.The second kind, known as operatorial anomaly, is due to the explicit mass parameters. Both of these kinds give a non-zero vacuum expectation value to the trace. The operatorial anomalies have to be distinguished from α and c anomalies which do not appear in flat space. So in a flat space, the operation equation for the previous theory is:

$$\langle T^{\mu}_{\mu} \rangle \neq 0 \tag{5.19}$$

This is the reason why we can not match straightforwardly the anomalies of the UV fixed point (α_{UV}, c_{UV}) with the anomalies of the IR fixed point (α_{IR}, c_{IR}). In order to remove the operatorial anomaly we use the dilaton as a conformal compensator. We set $\Omega = e^{-\tau}$ and the coupling with the matter theory, comes by replacing every mass scale as $M_i \to M_i\Omega$. Recalling that the kinetic term of the dilaton is multiplied by a dimensionful coefficient f^2, we can conclude that the physical dilaton fluctuations couple to the matter fields by f^{-1}. This is very important for the study of the RG flow and the matching between the anomalies. After adding the kinetic term the theory becomes:

$$S = S_{matter}[\Phi_i, M_i\Omega] + f^2 \int d^4x (\partial\Omega)^2 \tag{5.20}$$

This theory is anomaly-free and the below ward identity is satisfied :

$$\langle T^{\mu}_{\mu} \rangle = 0 \tag{5.21}$$

We can take the weakly coupled limit $M_i \ll f$. When the conformal compensator takes the vacuum expectation value $\langle \Omega \rangle = 1$ then the original matter theory is restored and the flow begins. This flow is the flow shown in Fig.5.1 perturbed only by the weak coupling to the dilaton field. As a result, the deep IR theory consists of a Conformal Field Theory sector plus the decoupled dilaton field.

We can also state that the UV theory consists of CFT_{UV} plus a decoupled dilaton. We can introduce a cutoff $\Lambda_{UV} \gg M_i$ and all momenta are restricted by $p^2 \ll \Lambda^2_{UV}$. At high energies $M_i^2 \ll p^2 \ll \Lambda^2_{UV}$ the theory consists of CFT_{UV} weakly coupled with dilaton. Recalling the limit that we are working on $M_i \ll f$ the marginal operators that describe the physical interaction of the dilaton with operators, are suppressed by a factor M_i/f. Should there exist not exactly marginal operators, plays no role in leading order since there are logarithms that are higher order in M_i/f. So it is consistent to think of the UV theory as CFT_{UV} plus a decoupled dilaton[14]. The flow between the high energies and the deep IR stays unperturbed by the dilaton. The Ward identity (5.21) allows us to match the UV and IR anomalies.

Since we assume $M_i \ll f$ we are only interested in the leading terms in 1/f. To leading order in this expansion, it is sufficient to integrate out the matter fields while the dilaton sits on external lines. The type of diagram someone has to compute, in order to find the coupling of dilaton in low energies is given below:

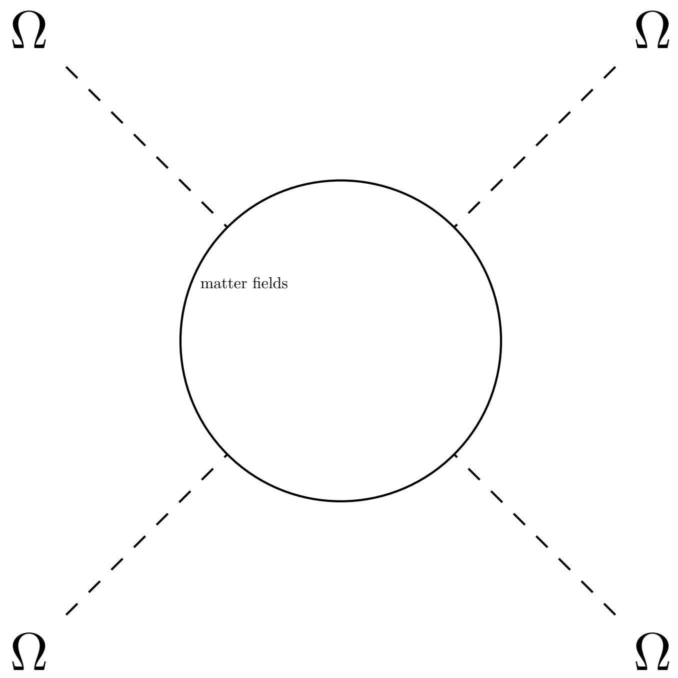


Figure 5.2: Leading order diagrams

The most general effective action for the IR theory, up to four derivatives acting on dilaton is:

$$\begin{aligned}S_{IR}\left[g_{\mu\nu}\right] &= \mathrm{CFT}_{IR}\left[g_{\mu\nu}\right] + \frac{1}{6}f^2\int d^4x\sqrt{-\widehat{g}}\widehat{R} + \frac{\kappa}{36}\int d^4x\sqrt{-\widehat{g}}\widehat{R}^2 + \kappa'\int d^4x\sqrt{-\widehat{g}}\widehat{W}^2_{\mu\nu\rho\sigma} \\ &- (a_{UV} - a_{IR})\int d^4x\sqrt{-g}\left(\tau E_4 + 4\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\partial_\mu\tau\partial_\nu\tau - 4(\partial\tau)^2\Box\tau + 2(\partial\tau)^4\right) \\ &+ (c_{UV} - c_{IR})\int d^4x\sqrt{-g}\tau W^2_{\mu\nu\rho\sigma}\end{aligned} \tag{5.22}$$

The difference between the α-anomalies is isolated by the $2 \to 2$ scattering and the leading contribution to the amplitude is:

$$\mathcal{A}(s,t,u) = \frac{\alpha_{UV} - \alpha_{IR}}{f^4}(s^2 + t^2 + u^2) + (higher\ order\ terms) \tag{5.23}$$

We consider the scattering of four dilatons such that they are all on-shell ($p_i^2 = 0$, since dilaton is massless). Working on the limit $t = 0$ and using the usual relation $s + t + u = 0$, the above amplitude becomes:

$$\mathcal{A}(s) = \frac{2(\alpha_{UV} - \alpha_{IR})}{f^4}s^2 + \mathcal{O}(s^4) \tag{5.24}$$

As a next step, we consider the amplitude $\mathcal{A}/s^3$ and by the use of dispersion relation[3] in

[3] We give more details in Appendix A

order to calculate the difference $(\alpha_{UV} - \alpha_{IR})$ we get that :

$$\alpha_{UV} - \alpha_{IR} = \frac{f^4}{\pi} \int_{s>0} ds' \frac{\mathrm{Im}\{\mathcal{A}(s')\}}{s'^3} \tag{5.25}$$

where $\mathrm{Im}\{\mathcal{A}(s')\}$ is the imaginary part of the amplitude. From the unitarity of the S-matrix, we can prove that $\mathrm{Im}\{\mathcal{A}(s)\} = s\sigma(s)$. This makes the r.h.s. of the last expression to be positively defined and thus $\alpha_{UV} > \alpha_{IR}$. We can construct a monotonically decreasing function by defining a scale dependent α-anomaly.

$$\alpha(\mu) = \alpha_{UV} - \frac{f^4}{\pi} \int_{s'>\mu} ds' \frac{\sigma(s')}{s^2} \tag{5.26}$$

This is the proof of the α-theorem.

Appendix A

$(\tau + \tau \rightarrow \tau + \tau)$ Scattering and Positivity Bound

This appendix is based on [22].

We consider the amplitude probabilities for the transition between two particles states:

$$S_{\alpha\beta} = \left\langle \Psi_\alpha^{out} \middle| \Psi_\beta^{in} \right\rangle$$

where $\left|\Psi_\beta^{in(out)}\right\rangle$ is the state defined in the far past (future) $t \rightarrow -\infty(+\infty)$. We also think of an complete and orthonormal basis:

$$\langle \Psi_\alpha | \Psi_\beta \rangle = N_\alpha \delta(\alpha - \beta) \hat{\mathbf{1}} = \int \frac{d\alpha}{N_\alpha} |\Psi_\alpha\rangle\langle\Psi_\alpha|$$

[1] It is more convenient to work with the free particles' states $|\Phi\rangle$:

$$\hat{H}_0 |\Phi_\alpha\rangle = E_{0,\alpha} |\Phi_\alpha\rangle \ , \ \langle \Phi_\alpha | \Phi_\beta \rangle = N_\alpha \delta(\alpha - \beta)$$

with N_α, the normalization constant, and $\delta(\alpha - \beta)$ stands for products of delta functions and Kronecker deltas . The interacting states can be written as :

$$|\Psi_\alpha(t)\rangle = \Omega(t) |\Phi_\alpha\rangle \ , \ \Omega(t) = e^{iHt} e^{-iH_0 t}$$

Then the amplitude can be written in terms of the free particles' states :

$$S_{\alpha\beta} = \langle \Phi_\alpha | \Omega^\dagger(-\infty)\Omega(+\infty) |\Phi_\beta\rangle \equiv \langle \Phi_\alpha | S |\Phi_\beta\rangle$$

[1] The sum over all the states is $\int d\alpha = \sum_{\sigma_1 n_1, \sigma_2 n_2, \ldots} \int d^3p_1 d^3p_2 \cdots$

Recalling that for the free theory $S = \hat{\mathbf{1}}$, we can define a scattering amplitude operator $\hat{\mathcal{M}}$

$$S = \hat{\mathbf{1}} + (2\pi)^4 \delta^{(4)} \left(\sum_i p_i \right) i\mathcal{M}$$

The S-matrix has to be unitary :

$$S^\dagger S = S S^\dagger = \hat{\mathbf{1}}$$

we define the $\mathcal{T}$ matrix such that :

$$S = \hat{\mathbf{1}} + i\mathcal{T}$$

The unitarity of the S-matrix has as a consequence the relation used for the proof of α-theorem, between the imaginary part of the amplitude and the total cross-section. Using the definition of $\mathcal{T}$ matrix we get:

$$i\left(\mathcal{T}^\dagger - \mathcal{T}\right) = \mathcal{T}^\dagger \mathcal{T}$$

By calculating the matrix elements of both r.h.s. and l.h.s using that $\langle \Phi_\alpha | \mathcal{T} | \Phi_\beta \rangle = (2\pi)^4 \delta^4(p_\beta - p_\alpha) \mathcal{M}_{\alpha\beta}$ and concerning that we have an elastic scattering (as in the case of two dilaton scattering) we conclude the relation:

$$2 \operatorname{Im} \mathcal{M}_{\alpha\alpha} = \int \frac{d\gamma}{N_\gamma} (2\pi)^4 \delta^4 \left(p_\alpha - p_\gamma\right) \left|M_{\gamma\alpha}\right|^2$$

From the above relation, we get that $\operatorname{Im} \mathcal{M}_{\alpha\alpha} \geqslant 0$, with the equality to be valid for the case of the free theory. The r.h.s. of the expression above is analogous with the total cross-section in the center of mass frame:

$$\sigma(2 \rightarrow \text{ anything }) = \frac{1}{4E_{cm} \left|\mathbf{p}_i\right|} \int \frac{d\gamma}{N_\gamma} (2\pi)^4 \delta^4 \left(p_\alpha - p_\gamma\right) \left|M_{\gamma\alpha}\right|^2$$

where we conclude that :

$$\operatorname{Im} \mathcal{M}_{2\rightarrow 2}(s)\big|_{\text{elastic forward}} = \sqrt{\left(s - m_1^2 - m_2^2\right)^2 - 4m_1^2 m_2^2} \cdot \sigma^{\text{tot}}_{2\rightarrow \text{ anything}}(s)$$

this is also known as the optical theorem. In the case of dilaton scattering, for the proof of the α-theorem we have that the masses are $m_1 = m_2 = 0$

$$\text{Im}\,\mathcal{M}_{\tau\tau\to\tau\tau}(s) = s\sigma(s)$$

Going back to the IR action (5.22) the Lagrangian of the system has the form below:

$$\mathcal{L} = (\partial\tau)^2 + \frac{\alpha_{UV} - \alpha_{IR}}{f^4}(\partial\tau)^4 + ...$$

The scattering amplitude for the four-dilaton interaction is :

$$\mathcal{M}(s) = \frac{\alpha_{UV} - \alpha_{IR}}{f^4}s^2 + \mathcal{O}(s^4) \tag{A.1}$$

This amplitude violates the unitarity of S-matrix at the UV region $s \gg f^2$, so the theory needs to be UV completed. So we can consider the previous theory as an effective theory of a linear sigma model:

$$\mathcal{L} = \partial^\mu\Phi^\dagger\partial_\mu\Phi - \lambda\left(|\Phi|^2 - v^2\right)^2$$

Giving an vacuum expectation value $\Phi = (v+h)e^{i\frac{\tau}{v}}$ The Lagrangian now is written as:

$$\mathcal{L} = \left(1 + \frac{h}{v}\right)^2(\partial\tau)^2 + (\partial h)^2 - m_H^2 h^2 + \cdots$$

Using the equations of motions for the Higgs' field in order to integrate out the h. As a result we take an interacting term $\frac{\lambda}{m_h^2}(\partial\tau)^4$. Thanks to the Higgs field's mass we have two poles for the amplitude.[2] at $s = \pm m_H^2$, due to the symmetry between the s and u channel. In order to calculate the coefficient $(\alpha_{UV} - \alpha_{IR})$ we will use the integral below:

$$I = \oint_\gamma \frac{ds}{2\pi i}\frac{\mathcal{M}(s)}{s^3}$$

The modified amplitude for the scattering is:

[2] We recall that we are working limit $t = 0$ and so $u = -s$

$$\mathcal{M}(s) = \frac{\alpha_{UV} - \alpha_{IR}}{f^4} s^2 \left[\frac{1}{s - m_h^2} - \frac{1}{s + m_h^2} \right]$$

Taking into account that for $|s| \to \infty$ the amplitude $\mathcal{M}(s) < |s|^2$ the integral I has to vanish.

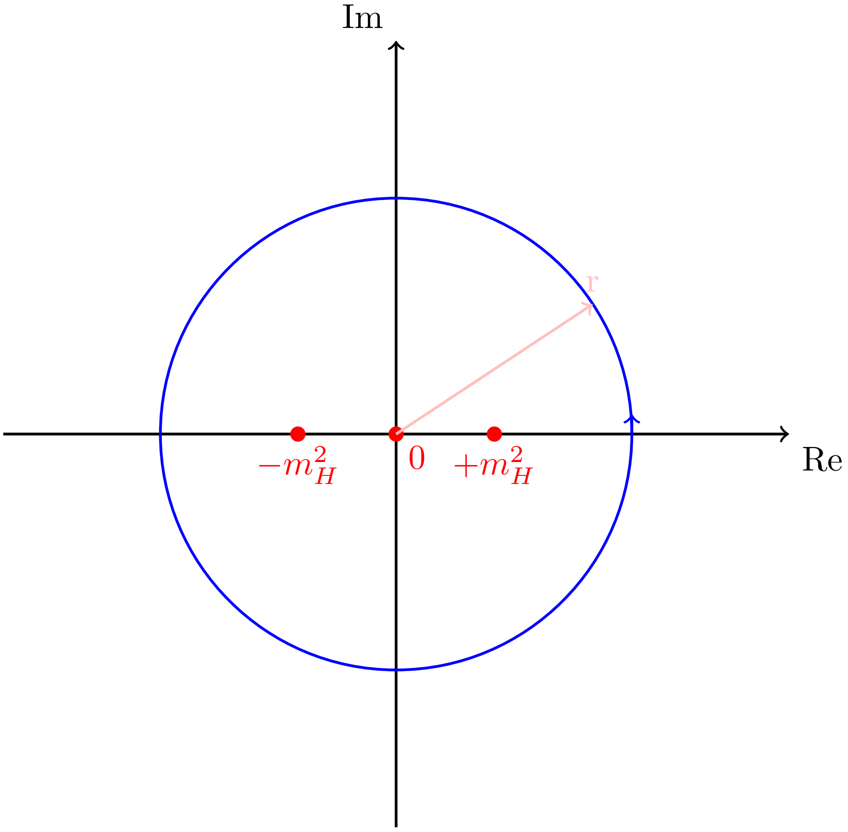


Figure A.1: Countor path with the three poles in the s-plane, with $r = |s|$

With respect to the Residues theorem, we have that:

$$\text{Res}\left[\frac{\mathcal{M}(s)}{s^3}\right]_{s=0} + \text{Res}\left[\frac{\mathcal{M}(s)}{s^3}\right]_{s=m_H^2} + \text{Res}\left[\frac{\mathcal{M}(s)}{s^3}\right]_{s=-m_H^2} = 0$$

Using the symmetry $\mathcal{M}(s) = \mathcal{M}(-s)$, we conclude:

$$\text{Res}\left[\frac{\mathcal{M}(s)}{s^3}\right]_{s=0} + 2\,\text{Res}\left[\frac{\mathcal{M}(s)}{s^3}\right]_{s=m_H^2} = 0$$

We can see that for the low energy limit $s \to 0$ we neglect the poles coming from the Higgs and though:

$$\frac{\mathcal{M}(s)}{s^3} \rightarrow \frac{\alpha_{UV}-\alpha_{IR}}{f^4}\frac{1}{s}$$
$$\Rightarrow \operatorname{Res}\left[\frac{\mathcal{M}(s)}{s^3}\right]_{s=0} = \frac{\alpha_{UV}-\alpha_{IR}}{f^4}$$

For the other two poles, we have to use the UV theory amplitude.

$$\operatorname{Res}\left[\frac{\mathcal{M}(s)}{s^3}\right]_{s=m_H^2} = \frac{\operatorname{Res}[\mathcal{M}(s)]_{s=m_H^2}}{(m_H^2)^3}$$

We have to evaluate the numerator.

$$\begin{aligned}
\operatorname{Res}[\mathcal{M}(s)]_{s=m_H^2} &= \lim_{\epsilon\to 0}\lim_{s\to m_H^2}\left(s-m_H^2+i\epsilon\right)\mathcal{M}(s) \Rightarrow \\
\mathcal{M}\left(s\to m_H^2\right) &= \frac{\operatorname{Res}[\mathcal{M}(s)]_{s=m_H^2}}{s-m_H^2+i\epsilon} = \frac{s-m_H^2 i\epsilon}{\left(s-m_H^2\right)^2+\epsilon^2}\operatorname{Res}[\mathcal{M}(s)]_{s=m_H^2} \Rightarrow \\
\operatorname{Im}\mathcal{M}\left(s\to m_H^2\right) &= -\frac{\epsilon}{\left(s-m_H^2\right)^2+\epsilon^2}\operatorname{Res}[\mathcal{M}(s)]_{s=m_H^2}.
\end{aligned}$$

Using the definition of delta function as $\pi\delta(x) = \lim_{\epsilon\to 0}\frac{\epsilon}{x^2+\epsilon^2}$ we get:

$$\operatorname{Im}\mathcal{M}\left(s\to m_H^2\right) = -\pi\delta(s-m_H^2)\operatorname{Res}[\mathcal{M}(s)]_{s=m_H^2}$$

So now we have to use this expression in order to find the initial Residue:

$$2\operatorname{Res}\left[\frac{\mathcal{M}(s)}{s^3}\right]_{s=m_H^2} = \int ds\pi\delta(s-m_H^2)\frac{\operatorname{Res}[\mathcal{M}(s)]}{s^3} = -\frac{2}{\pi}\int ds\frac{\operatorname{Im}\mathcal{M}(s)}{s^3}$$

Recalling the optical theorem $\operatorname{Im}\mathcal{M}(s) = s\sigma(s)$ we get that :

$$\frac{\alpha_{UV}-\alpha_{IR}}{f^4} = \frac{1}{\pi}\int ds\frac{\sigma(s)}{s^2} \geqslant 0$$

Bibliography

[1] Slava Rychkov. *EPFL Lectures on Conformal Field Theory in $D \geqslant 3$ Dimensions*. Springer International Publishing, 2017. DOI: 10.1007/978-3-319-43626-5. URL: https://doi.org/10.1007%2F978-3-319-43626-5 (page 4).

[2] Viatcheslav Mukhanov and Sergei Winitzki. *Introduction to quantum effects in gravity*. Cambridge University Press, June 2007. ISBN: 978-0-521-86834-1, 978-1-139-78594-5 (pages 5, 39).

[3] Marc Gillioz. *Conformal Field Theory for Particle Physicists*. Springer Nature Switzerland, 2023. DOI: 10.1007/978-3-031-27086-4. URL: https://doi.org/10.1007%2F978-3-031-27086-4 (page 8).

[4] Paul Ginsparg. *Applied Conformal Field Theory*. 1988. arXiv: hep-th/9108028 [hep-th] (page 13).

[5] Nikos Irges, Fotis Koutroulis, and Dimosthenis Theofilopoulos. «The conformal N-point scalar correlator in coordinate space». In: (Jan. 2020). arXiv: 2001.07171 [hep-th] (page 14).

[6] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. New York: Springer-Verlag, 1997. ISBN: 978-0-387-94785-3, 978-1-4612-7475-9. DOI: 10.1007/978-1-4612-2256-9 (pages 14, 45).

[7] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, 2013. DOI: 10.1017/9781139540940 (page 17).

[8] Michael Edward Peskin and Daniel V. Schroeder. *An Introduction to Quantum Field Theory*. Reading, USA: Addison-Wesley (1995) 842 p. Westview Press, 1995 (pages 17, 24).

[9] R.P. Feynman, A.R. Hibbs, and D.F. Styer. *Quantum Mechanics and Path Integrals*. Dover Books on Physics. Dover Publications, 2010. ISBN: 9780486477220. URL: https://books.google.gr/books?id=JkMuDAAAQBAJ (page 17).

[10] A. B. Zamolodchikov. «Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory». In: *JETP Lett.* 43 (1986), pp. 730–732 (pages 20, 43).

[11] M. Srednicki. *Quantum field theory*. Cambridge University Press, Jan. 2007. ISBN: 978-0-521-86449-7, 978-0-511-26720-8 (page 24).

[12] Sean Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. Benjamin Cummings, 2003. ISBN: 0805387323. URL: http://www.amazon.com/Spacetime-Geometry-Introduction-General-Relativity/dp/0805387323 (page 38).

[13] A.O. Barvinsky and G.A. Vilkovisky. «Covariant perturbation theory (II). Second order in the curvature. General algorithms». In: *Nuclear Physics B* 333.2 (1990), pp. 471–511. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(90)90047-H. URL: https://www.sciencedirect.com/science/article/pii/055032139090047H (page 39).

[14] Zohar Komargodski and Adam Schwimmer. «On Renormalization Group Flows in Four Dimensions». In: *JHEP* 12 (2011), p. 099. DOI: 10.1007/JHEP12(2011)099. arXiv: 1107.3987 [hep-th] (pages 42, 51, 52).

[15] N. D. Birrell and P. C. W. Davies. *Quantum Fields in Curved Space*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1982. DOI: 10.1017/CBO9780511622632 (page 42).

[16] Giuseppe Mussardo. *Statistical field theory: an introduction to exactly solved models in statistical physics*. New York, NY: Oxford Univ. Press, 2010 (page 46).

[17] C.-J. David Lin, Mari-Carmen Banuls, Krzysztof Cichy, Ying-Jer Kao, Yu-Ping Lin, and Tao-Lin Tan. «Investigation of the 1+1 dimensional Thirring model using the method of matrix product states». In: *Proceedings of The 36th Annual International Symposium on Lattice Field Theory — PoS(LATTICE2018)*. Sissa Medialab, May 2019. DOI: 10.22323/1.334.0229. URL: https://doi.org/10.22323%2F1.334.0229 (page 46).

[18] J. Zinn-Justin. *Quantum Field Theory at Finite Temperature: An Introduction*. 2000. arXiv: hep-ph/0005272 [hep-ph] (page 46).

[19] John L. Cardy. «Is There a c Theorem in Four-Dimensions?» In: *Phys. Lett. B* 215 (1988), pp. 749–752. DOI: 10.1016/0370-2693(88)90054-8 (page 47).

[20] Yu Nakayama. *Scale invariance vs conformal invariance*. 2014. arXiv: 1302.0884 [hep-th] (page 49).

[21] A. Schwimmer and S. Theisen. «Spontaneous breaking of conformal invariance and trace anomaly matching». In: *Nuclear Physics B* 847.3 (June 2011), pp. 590–611. DOI: 10.1016/j.nuclphysb.2011.02.003. URL: https://doi.org/10.1016%2Fj.nuclphysb.2011.02.003 (page 51).

[22] Francesco Sgarlata. «Unitarity and Causality constraints on Effective Field Theories». PhD thesis. SISSA, SISSA, Trieste, 2020 (page 55).