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MASTER'S THESIS

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INTERACTIONS BETWEEN DESCRIPTIVE SET THEORY AND GAME THEORY

IN THE SCIENTIFIC FIELD OF:

DESCRIPTIVE SET THEORY

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. . . .

Vasiliki Kyriakou

Στον μπαμπά μου,
που με εισήγαγε στον κόσμο των μαθηματικών.

Abstract

The present thesis aims to apply the fundamental principles of Descriptive Set Theory to the discipline of Game Theory. The discipline of Descriptive Set Theory is an active research area of modern Set Theory and specializes in the study of the structure of definable sets and functions in Polish spaces. It dates back to the work of Borel, Baire, and Lebesgue at the beginning of the 20th century.

In the first chapter, we begin by setting out the preliminaries of Set Theory and Theory of Metric and Topological Spaces, intending to introduce the concept of Polish space in the next chapter. The Polish spaces are essentially separable completely metrizable spaces. The most fundamental Polish spaces that have been studied are the Baire space and the Cantor space. Several times in this thesis, these spaces will be an important tool invoked to extract our results.

In the third chapter of the paper, we define some of the fundamental set operators on subsets of Polish spaces. We also consider the notion of the pointclass and we list a special category of pointclasses, the Borel pointclasses of finite order in Polish spaces. We introduce the Hierarchy of Borel sets of finite order and we refer to their closure properties. We proceed with the study of the more complex Projective pointclasses of finite order and we describe their hierarchy.

In chapter four, we incorporate the concept of a tree into our study. We will define the space of trees and we will prove that it is a compact Polish space. In addition, we will refer to a property that many subsets of Polish spaces have, the "Perfect Set Property". We will confirm the validity of this property, both for close and for analytic subsets of Polish spaces, proving the Cantor-Bendixson and Perfect Set Theorems, respectively. For the proof of the Perfect Set Theorem, we will need to define a new category of trees, the trees of pairs.

In the following chapter, we will consider some primary concepts of Measure Theory that are important preliminaries for our study. We will begin by mentioning the measure function, its Uniqueness and Completion. In addition, we will define Borel sets, the Borel measure, and Borel measurables functions. In turn, we will prove the "Schröder-Bernstein Theorem for good Borel monomorphisms". Our goal is, having the Theorem above as a starting point, to prove the "Borel Isomorphism Theorem" and finally to conclude with the proof that two uncountable Polish spaces are Borel isomorphic.

In chapter six, we will carry out an extensive study in Game Theory, utilizing knowledge from the previous chapters. The games we will engage in are called "Gale-Stewart". These are infinite games of perfect information between two players - player I and player II. We will thoroughly describe the rules of the games, the strategies that the players follow, as well as the way in which the winner is determined. Next, we will introduce the term "Determinacy" of a game. We will study some properties related to Determinacy and prove that the closed, $\underset{\sim}{\mathbf{\Sigma}}^0_2$ and $\underset{\sim}{\mathbf{\Pi}}^0_2$ subsets of $X^{\mathbb{N}}$ are determined. Then, we describe the G^*-Games, which are a special category of topological games. We will focus on G^*-Games that unfold on subsets of Baire and Cantor spaces, to assume the determinacy of games within sets belonging to the pointclasses $\underset{\sim}{\mathbf{\Sigma}}^1_n$, with $n \geqslant 1$, to show that these sets have a non-empty perfect subset. Ultimately, we study the Covering Games $G^\mu(A, \varepsilon)$ associated with a subset of the Cantor space and a σ-finite Borel measure μ on it. A concluding remark for our work on the games mentioned above is the fact that assuming the determinacy of a game evolving in a set belonging to the pointclasses $\underset{\sim}{\mathbf{\Sigma}}^1_n$, with $n \geqslant 1$, this set is measurable in terms of the measure μ with which the game is associated.

Keywords. Polish space, Baire space, Cantor space, Borel pointclasses, Projective pointclasses, tree, perfect set, infinite game, Gale-Stewart, strategy, winning strategy, Determinacy, G^*-Game, Covering Game, σ-finite Borel measure

Περίληψη

Η παρούσα μεταπτυχιακή εργασία αποσκοπεί στην εφαρμογή θεμελιωδών αρχών της Περιγραφικής Θεωρίας Συνόλων στον επιστημονικό κλάδο της Θεωρίας Παιγνίων. Η Περιγραφική Θεωρία Συνόλων αποτελεί ενεργή ερευνητική περιοχή της σύγχρονης Θεωρίας Συνόλων, με αντικείμενο μελέτης τη δομή ορίσιμων συνόλων και συναρτήσεων σε Πολωνικούς χώρους. Οι ρίζες της ανάγονται στο έργο των Borel, Baire και Lebesgue, στις αρχές του 20ου αιώνα.

Στην εργασία αυτή ξεκινάμε παραθέτοντας στο πρώτο κεφάλαιο κάποια βασικά προαπαιτούμενα της Θεωρίας Συνόλων, αλλά και της Θεωρίας Μετρικών και Τοπολογικών χώρων, σκοπεύοντας να εισαγάγουμε στο δεύτερο κεφάλαιο την έννοια του Πολωνικού χώρου. Οι Πολωνικοί χώροι είναι τοπολογικοί χώροι, πλήρεις και διαχωρίσιμοι ως προς τη μετρική που επάγει την τοπολογία τους. Δύο από τους σημαντικότερους Πολωνικούς χώρους που έχουν μελετηθεί είναι ο χώρος του Baire και ο χώρος του Cantor. Αρκετές φορές στην εργασία οι χώροι αυτοί θα αποτελέσουν σημαντικό εργαλείο και θα τους επικαλεστούμε για να εξαγάγουμε τα αποτελέσματά μας.

Στο τρίτο κεφάλαιο της εργασίας, θα ορίσουμε μερικούς από τους θεμελιώδεις τελεστές συνόλων, οι οποίοι έχουν ως πεδίο εφαρμογής υποσύνολα Πολωνικών χώρων. Θα αναφερθούμε επίσης στην έννοια της κλάσης συνόλων και θα παραθέσουμε μία ειδική κατηγορία κλάσεων συνόλων, τις κλάσεις Borel συνόλων πεπερασμένης τάξης σε Πολωνικούς χώρους. Θα παρουσιάσουμε την Ιεραρχία των Borel συνόλων πεπερασμένης τάξης και θα αναφερθούμε στις ιδιότητες κλειστότητάς τους. Συνεχίζοντας, θα προχωρήσουμε στη μελέτη πιο σύνθετων κλάσεων από αυτές των Borel συνόλων, τις Προβολικές κλάσεις συνόλων πεπερασμένης τάξης και θα περιγράψουμε την ιεραρχία τους.

Στο τέταρτο κεφάλαιο, εντάσσουμε στη μελέτη μας την έννοια του δένδρου. Θα ορίσουμε τον Χώρο των δένδρων και θα δείξουμε ότι αποτελεί έναν συμπαγή Πολωνικό χώρο. Επιπλέον, θα αναφερθούμε σε μία ιδιότητα που έχουν πολλά υποσύνολα των Πολωνικών χώρων, την "Ιδιότητα του Τέλειου Συνόλου". Θα επιβεβαιώσουμε την ισχύ της ιδιότητας αυτής, τόσο για τα κλειστά, όσο και για τα αναλυτικά υποσύνολα Πολωνικών χώρων, αποδεικνύοντας τα Θεωρήματα "Cantor-Bendixson" και "Τέλειου Συνόλου", αντίστοιχα. Για την απόδειξη του Θεωρήματος του Τέλειου Συνόλου θα χρειαστεί να ορίσουμε μία νέα κατηγορία δένδρου, το δένδρο ζευγών.

Προχωρώντας στο επόμενο κεφάλαιο, θα συμπεριλάβουμε στην εργασία αυτή μερικές πρωταρχικές έννοιες της Θεωρίας Μέτρου που αποτελούν ισχυρό προαπαιτούμενο για τη μελέτη μας. Θα ξεκινήσουμε κάνοντας αναφορά στη συνάρτηση του μέτρου, στη Μοναδικότητα και στην Πλήρωσή του. Επιπλέον, θα ορίσουμε τα σύνολα Borel, το μέτρο Borel, αλλά και τις Borel μετρήσιμες συναρτήσεις. Στη συνέχεια, θα αποδείξουμε το "Θεώρημα Schröder-Bernstein για καλούς Borel μονομορφισμούς". Στόχος μας είναι έχοντας ως εφόδιο το Θεώρημα που μόλις αναφέραμε να αποδείξουμε το "Θεώρημα του Borel ισομορφισμού" και τελικά βασισμένοι στα παραπάνω, να καταλήξουμε στο συμπέρασμα ότι δύο υπεραριθμήσιμοι Πολωνικοί χώροι είναι Borel ισομορφικοί.

Στο έκτο κεφάλαιο, θα πραγματοποιήσουμε μία εκτενή μελέτη στη Θεωρία Παιγνίων, αξιοποιώντας τις γνώσεις από τα κεφάλαια που προηγήθηκαν. Τα παίγνια που θα μας απασχολήσουν καλούνται "Gale-Stewart". Πρόκειται για άπειρα παίγνια τέλειας πληροφόρησης μεταξύ δύο παικτών - του παίκτη I και της παίκτριας II. Θα περιγράψουμε τους κανόνες διεξαγωγής των παιγνίων, τις στρατηγικές που ακολουθούν οι παίκτες, καθώς και τον τρόπο με τον οποίο καθορίζεται ο νικητής. Στη συνέχεια, θα εισαγάγουμε τον όρο της "Προσδιοριστότητας" ενός παιγνίου. Θα μελετήσουμε κάποιες ιδιότητες που σχετίζονται με την Προσδιοριστότητα και θα αποδείξουμε ότι τα κλειστά, τα $\underset{\sim}{\mathbf{\Sigma}}^0_2$ και τα $\underset{\sim}{\mathbf{\Pi}}^0_2$ υποσύνολα του $X^{\mathbb{N}}$ είναι προσδιορισμένα. Προχωρώντας, θα περιγράψουμε τα G^*-Παίγνια, τα οποία αποτελούν μία ειδική κατηγορία τοπολογικών παιγνίων. Θα εστιάσουμε το ενδιαφέρον μας σε G^*-Παίγνια που εκτυλίσσονται σε υποσύνολα των χώρων των Baire και Cantor, με στόχο υποθέτοντας την Προσδιοριστότητα των παιγνίων μέσα σε σύνολα που ανήκουν στις κλάσεις συνόλων $\underset{\sim}{\mathbf{\Sigma}}^1_n$, με $n \geqslant 1$, να δείξουμε ότι τα σύνολα αυτά έχουν ένα μη κενό τέλειο υποσύνολο. Τελευταία αφήνουμε να μελετηθούν τα Παίγνια Κάλυψης $G^{\mu}(A, \varepsilon)$. Τα παίγνια αυτά σχετίζονται με ένα υποσύνολο του χώρου του Cantor και ένα σ-πεπερασμένο Borel μέτρο μ μέσα στον χώρο αυτό. Συμπέρασμα της ενασχόλησής μας με τα προαναφερθέντα παίγνια αποτελεί το γεγονός ότι υποθέτοντας την Προσδιοριστότητα ενός παιγνίου που εξελίσσεται σε σύνολο που ανήκει στις κλάσεις συνόλων $\underset{\sim}{\mathbf{\Sigma}}^1_n$, με $n \geqslant 1$, το σύνολο αυτό είναι μετρήσιμο ως προς το μέτρο μ με το οποίο σχετίζεται το παίγνιο.

Λέξεις Κλειδιά. Πολωνικός χώρος, χώρος του Baire, χώρος του Cantor, κλάσεις Borel συνόλων, Προβολικές κλάσεις συνόλων, δένδρο, τέλειο σύνολο, άπειρο παίγνιο, Gale-Stewart, στρατηγική, νικητήρια στρατηγική, Προσδιοριστότητα, G^*-Παίγνιο, Παίγνιο Κάλυψης, σ-πεπερασμένο Borel μέτρο

Contents

CHAPTER 1

Introduction

1.1. Basic Notions

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ be the sets of natural, integer, explicit, and real numbers respectively, where $\mathbb{N}$ includes 0 so that $\mathbb{N} = \{0, 1, 2, \dots\}$. The main focus of Descriptive Set Theory is the study of $\mathbb{N}$, $\mathbb{R}$ and their subsets, with particular emphasis on the definable sets of integers and reals.

Given two non-empty sets X and Y, we set Y^X as the **set of all functions** from X to Y. If $(X_i)_{i \in I}$ is a family of non-empty sets, we set $\prod_{i \in I} X_i$ as the family of all functions

$$f : I \to \bigcup_{i \in I} X_i$$

with $f(i) \in X_i$, for each $i \in I$. In the case $X_i = X$ for each $i \in I$, then the set $\prod_{i \in I} X_i$ is obviously X^I. We will be particularly interested in sets of the form $X^{\mathbb{N}}$.

Definition 1.1.1. One-to-one functions are called **monomorphisms**, while onto functions are called **surjections** and we denote them, respectively, as follows:

$$f : X \rightarrowtail Y \iff f \text{ is a monomorphism}$$
$$f : X \twoheadrightarrow Y \iff f \text{ is a surjection.}$$

Definition 1.1.2. By **isomorphism** or **correspondence** we mean a one-to-one function and onto (bijection). We also denote

$$f : X \rightarrowtail\!\!\!\!\!\rightarrow Y \iff f \text{ is an isomorphism.}$$

Definition 1.1.3. If $f \in Y^X$, $A \subseteq X$ and $B \subseteq Y$ we denote by $f[A]$ the **image** of A under f and by $f^{-1}[B]$ the **inverse image** of B under f, i.e.

$$f[A] = \{y \in Y : \exists x \in X \;\; f(x) = y\}$$
$$f^{-1}[B] = \{x \in X : f(x) \in B\}.$$

Definition 1.1.4. The **restriction** of a function $f : X \to Y$ on the set $A \subseteq X$ is denoted by $f|A$. In functions $f : X \to Y$ we include cases where X, and Y are the empty sets. If $X = \varnothing$ or $Y = \varnothing$ then we will say that $f : X \to Y$ is the **empty function**.

Remark 1.1.5. We assume that the empty function is always a monomorphism and that $f : \varnothing \to \varnothing$ is an isomorphism.

Definition 1.1.6. We define the logical operators of **disjunction** "$\vee$", **conjunction** "&", **negation** "$\neg$", and logical **implication** "$\longrightarrow$", as follows:

$$P(X) \vee Q(Y) \iff x \text{ has the property } P \text{ or } y \text{ has the property } Q$$
$$P(X) \;\&\; Q(Y) \iff x \text{ has the property } P \text{ and } y \text{ has the property } Q$$
$$\neg\, P(X) \iff x \text{ does not have the property } P$$
$$P(X) \longrightarrow Q(Y) \iff \text{if } x \text{ has the property } P \text{ then } y \text{ has the property } Q.$$

1.2. Preliminaries of Set Theory

Two sets A, B are called **equinumerous** if there is a one-to-one and onto function$f : A \rightarrowtail\!\!\!\!\!\rightarrow B$. In this case, we write $A =_c B$. We realize the relation "$=_c$" as a mathematical expression of the intuitive notion "A has the same number of elements as B". Our notion that the empty function $f : \varnothing \to \varnothing$ is one-to-one and onto expresses the basic principle that the empty set has the same number of elements as itself, namely no elements at all.

We will say that the set A has a **cardinality less than or equal to** B, and we will write $A \leqslant_c B$ if there exists a one-to-one function $f : A \rightarrowtail B$. We take "$\leqslant_c$" to be a mathematical expression of the intuition "A has a smaller or equal number of elements than B". Our assumption that the empty function $f : \varnothing \to B$ is always one-to-one expresses the basic principle that the empty set has always fewer elements than any non-empty set.
The above definitions would not be so important if they did not verify the following fundamental requirement:

If A has a *smaller or equal number of elements* than B, and
B has a *smaller or equal number of elements* than A, then
A and B have the *same number of elements.*

This is satisfied for the above definitions by the following result.

Theorem 1.2.1 ((Schröder-Bernstein),[**7**])**.** *For all sets A and B, if $A \leqslant_c B$ and $B \leqslant_c A$, then $A =_c B$.*

Continuum Hypothesis (CH).
For every infinite $A \subseteq \{0,1\}^{\mathbb{N}}$, either $A =_c \mathbb{N}$ or $A =_c \{0,1\}^{\mathbb{N}}$.

Cantor believed the Continuum Hypothesis to be true and he had been trying to prove it for many years, but to no avail. Kurt Gödel proved in 1940 that the negation of the Continuum Hypothesis, i.e., the existence of a set with intermediate cardinality, could not be proved in standard set theory. The second half of the independence of the Continuum Hypothesis, i.e., unprovability of the nonexistence of an intermediate-sized set, was proved in 1963 by Paul Cohen. The Continuum Hypothesis is independent of the usual axioms of mathematics. That is, it can neither be proved nor disproved with the tools of the usual axioms we accept in mathematics. On the other hand, it quickly became known that every closed subset of $\mathbb{R}$ satisfies the Continuum Hypothesis. We conclude that the addition of a topological condition answers the problem positively.

Axiom of Choice (AC).
For any set A, B, P with $P \subseteq A \times B$, if for each $x \in A$ there exists $y \in B$, with

$$(x, y) \in P$$

then there exists a *function* $f : A \to B$, with

$$(x, f(x)) \in P, \text{ for all } x \in A.$$

The above function f is called **choice function**. The Axiom of choice is the last of Zermelo's axioms. It is an axiom of set theory equivalent to the statement that a "Cartesian product of a collection of non-empty sets is non-empty". Informally put, the Axiom of choice says that given any collection of sets, each containing at least one element, it is possible to construct a new set by arbitrarily choosing one element from each set, even if the collection is infinite.

1.3. Metric and Topological Spaces

Definition 1.3.1. Metric in a non-empty set X is a function $d : X \times X \to \mathbb{R}$ with the following properties

$$\begin{aligned}
&d(x, y) \geqslant 0 \text{ and } d(x, y) = 0 \iff x = y \\
&d(x, y) = d(y, x) \\
&d(x, y) \leqslant d(x, z) + d(z, x)
\end{aligned}$$

where $x, y \in X$. The pair (X, d) is called **metric space**.

Definition 1.3.2. The **discrete metric** d on X is defined by

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y, \end{cases}$$

for any $x, y \in X$.

In the following definitions, we assume that we are given a metric space (X, d). For each non-empty $Y \subseteq X$ we can take the restriction d_Y of the metric d on the set Y , i.e.

$$d_Y(x, y) = d(x, y) \text{ for every } x, y \in Y.$$

We will call (Y, d_Y) **subspace** of (X, d).

Definition 1.3.3. If $x \in X$ and $r > 0$ we call **open ball** of centre x and radius r on (X, d) the set

$$B_d^X(x, r) \equiv B_d(x, r) \equiv B(x, r) = \{y \in X : d(x, y) < r\}.$$

The point x is an **interior point** of $A \subseteq X$ if there is $r > 0$ with $B_d^X(x, r) \subseteq A$. The set A is **open** if every $x \in A$ is an interior point of A.

Definition 1.3.4. The **interior** A° of A is the set of all interior points of A, equivalently

$$A^\circ = \bigcup\{V \subseteq X : V \text{ is open and } V \subseteq X\}$$

Definition 1.3.5. A sequence $(x_n)_{n\in\mathbb{N}}$ in X **converges** at $x \in X$ with respect to the metric d if

$$\lim_{n\to\infty} d(x_n, x) \to 0.$$

We write $x_n \xrightarrow{d} x$ or more simply $x_n \to x$, to state that $(x_n)_{n\in\mathbb{N}}$ converges at x. We say that the sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $A \subseteq X$ is **convergent** at A, if there exists $x \in A$ with $x_n \to x$.

Definition 1.3.6. We call $x \in X$ a **boundary point** of $A \subseteq X$ if there is a sequence $(x_n)_{n\in\mathbb{N}}$ in A with $x_n \to x$ with respect to d. We allow $x_n = x$ for every $n \in \mathbb{N}$ such that every element of A is also a boundary point of A. Also, we call x a **limit point** of A if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in A with $x_n \to x$ and $x_n \neq x$ for every $n \in \mathbb{N}$.

Definition 1.3.7. The point $x \in A$ is an **isolated point** of $A \subseteq X$ if there exists $r > 0$ with

$$B(x, r) \cap A = \{x\}.$$

Remark 1.3.8. The boundary points of A are just the isolated points along with its limit points.

Definition 1.3.9. The set A is **closed** if it contains all of its boundary points (equivalently it contains all their limit points) while A is **perfect** if it is closed and has no isolated points.

Definition 1.3.10. The **closure** $\overline{A}$ of A is the set of all its boundary points, equivalently

$$\overline{A} = \bigcap\{F \subseteq X : F \text{ is closed and } F \supseteq A\}$$

Remark 1.3.11. Obviously, $A^\circ \subseteq A \subseteq \overline{A}$, A is open if and only if $A = A^\circ$, and A is closed if and only if $A = \overline{A}$. As is already known, A is open if and only if its complement is closed.

Definition 1.3.12. A set $D \subseteq X$ is **dense** in (X, d) if for every $x \in X$ and every $r > 0$ holds

$$B_d(x, r) \bigcap D \neq \varnothing.$$

If D is a dense subset of (X, d) then every element of X is the limit of a sequence of D.

Definition 1.3.13. We call a metric space (X, d) **separable** if it has a countable dense subset.

Definition 1.3.14. A set $K \subseteq X$ is **compact** if for every family of $(V_i)_{i\in I}$ open subsets of X with

$$K \subseteq \bigcup_{i\in I} V_i$$

there are $i_1, \ldots, i_n$ with

$$K \subseteq \bigcup_{t=1}^{n} V_{i_t},$$

i.e. for every open cover of K, there is a finite subcover.

Definition 1.3.15. A function $f : (X, d) \to (Y, \rho)$, where (X, d) and (Y, ρ) are metric spaces, is **continuous** at the point $x \in X$ if for every $r > 0$ there exists $\delta > 0$ such that

$$f[\,B_d(x, \delta)] \subseteq B_\rho(f(x), r),$$

equivalently, for every $V \subseteq Y$ that is ρ-open there exists $W \subseteq X$ that is d-open such that $f[\,W] \subseteq Y$.

Remark 1.3.16. According to the ***Transfer Principle***, f is continuous in x if and only if for any sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \xrightarrow{d} x$ holds $f(x_n) \xrightarrow{\rho} f(x)$.

Definition 1.3.17. A function f is **continuous** if it is continuous on every $x \in X$ or equivalently for every open $V \subseteq Y$ the inverse image $f^{-1}[\,V]$ is an open subset of X. Finally, we will say that f is a **topological isomorphism**.

Definition 1.3.18. Two **metrics** d_1 and d_2 on the set X are **equivalent**, symbolically $d_1 \sim d_2$, if they produce the same topology, i.e. for any $V \subseteq X$, V is d_1-open if and only if V is d_2-open. Equivalently, $d_1 \sim d_2$ if and only if the *identity function*

$$\mathrm{id} : (X, d_1) \rightarrowtail\!\!\!\!\!\rightarrow (X, d_2)$$

is a topological isomorphism, equivalently, for any sequence $(x_n)_{n\in\mathbb{N}}$ in X and each $x \in X$ holds

$$x_n \xrightarrow{d_1} x \iff x_n \xrightarrow{d_2} x.$$

Remark 1.3.19. For each metric space (X, d_1) the functions

$$d_2 = \min\{d_1, 1\} \quad \text{and} \quad d_3 = \frac{d_1}{1 + d_1}$$

are metrics equivalent to d_1. Note that $d_2, d_3 \leqslant 1$, so it is common to assume that the metric takes values less than or equal to unity.

Definition 1.3.20. A family $\mathcal{T}$ of subsets of X is called a **topology** on X if it satisfies the following properties:

i) $\varnothing, X \in \mathcal{T}$,
ii) $\forall\, (A_i)_{i\in I} \in \mathcal{T}$ we have that $\bigcup_{i\in I} A_i \in \mathcal{T}$,
iii) $\forall\, A_1, \ldots, A_n \in \mathcal{T}$ we have that $\bigcap_{k=1}^{n} A_k \in \mathcal{T}$

Otherwise, the family of open sets of a metric space (X, d) is a topology. This family will be called the topology of (X, d).

Definition 1.3.21. A pair $(X, \mathcal{T})$ is called a **topological space** if $\mathcal{T}$ is a topology on X. The elements of $\mathcal{T}$ are called **open** sets of the topological space. The **closed** subsets of a topological space are the complements of open ones. A topological space $(X, \mathcal{T})$ is **metrizable** or, more simply, $\mathcal{T}$ is metrizable if there exists a metric d on X such that $\mathcal{T}$ is the topology of (X, d), i.e. the family of d-open subsets of X.

Definition 1.3.22. A family $\mathcal{V}$ of subsets of a topological space $(X, \mathcal{T})$ is the **basis of topology** of X or, more simply, a **basis** of X if $\mathcal{V} \subseteq \mathcal{T}$ and each $U \in \mathcal{T}$ is equal to a union (finite or infinite) of elements of $\mathcal{V}$, i.e. there exists a family of $(B_i)_{i\in I}$ elements of $\mathcal{V}$ with $U = \bigcup_{i\in I} B_i$. In metric spaces, this means that each element of $\mathcal{V}$ is an open set and each open set is written as a union of elements of $\mathcal{V}$.

Definition 1.3.23. A sequence $(x_n)_{n\in\mathbb{N}}$ in (X, d) is called **Cauchy** or **basic** if for every $r > 0$ there exists $n_0 \in \mathbb{N}$ such that for each $n, m \geqslant n_0$ holds $d(x_n, x_m) < r$. The metric space (X, d) is **complete** if every Cauchy sequence is convergent in (X, d).

Definition 1.3.24. If we have a topological space $(X, \mathcal{T})$ and G is a non-empty subset of X then we can consider G as a topological space with the **relevant topology** of X, i.e., the topology $\mathcal{T}_G$ defined as follows

$$\mathcal{T}_G = \{V \cap G : V \in \mathcal{T}\}.$$

The pair $(G, \mathcal{T}_G)$ is called the **subspace** of $(X, \mathcal{T})$. In the case where $(X, \mathcal{T})$ is metricated by d then $(G, \mathcal{T}_G)$ is metricated by the constraint $d|(G \times G)$ of d in G.

Definition 1.3.25. The **direct sum** of two metric spaces (X, d_X) and (Y, d_Y) is the metric space (Z, d) with
$$Z = (\{0\} \times X) \bigcup (\{1\} \times Y)$$
and metric
$$d((i,x),(j,y)) = \begin{cases} d_X(x,y), & \text{if } i=j=0, \\ d_Y(x,y), & \text{if } i=j=1, \\ 1, & \text{if } i \neq j. \end{cases}$$

In other words, we consider two foreign copies of X and Y and place them at a positive distance from each other. We will denote the direct sum of (X, d_X) and (Y, d_Y) by $X \oplus Y$ and we will always denote it by the above metric d.

Definition 1.3.26. The **Cartesian product** of the spaces (X, d_X) and (Y, d_Y) is the set $X \times Y$ with the metric
$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, y_1) + d_Y(x_2, y_2).$$
We will consider $X \times Y$ with the above metric dSimilarlyway we define the **finite Cartesian product** $X_1 \times \cdots \times X_n$ of metric spaces. In particular, in $\mathbb{R}^n$ we will consider the metric
$$d(\vec{x}, \vec{y}) = \sum_{k=1}^{n} |x_k - y_k|,$$
where $\vec{x} = (x_1, \ldots, x_n), \vec{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

Product Space and Product Topology.

If we have a sequence of metric spaces $((X_n, d_n))_{n\in\mathbb{N}}$ then consider the **product space** $\prod_{n\in\mathbb{N}} X_n$ with the metric
$$d(x,y) = \sum_{n=0}^{\infty} 2^{-n} \min\{d_n(x(n), y(n)), 1\},$$
where $x = (x(n))_{n\in\mathbb{N}}$ and $y = (y(n))_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$.

If $\mathcal{V}_n$ is a basis for the topology of X_n, $n \in \mathbb{N}$, then a basis for the topology of $\prod_{n\in\mathbb{N}} X_n$ is the family $\mathcal{V}$ of all sets of the form
$$V_0 \times \cdots \times V_n \times X_{n+1} \times X_{n+2} \times \ldots,$$
where $V_i \in \mathcal{V}_i$ for each $i = 0, \ldots, n$ and $n \in \mathbb{N}$.

Remark 1.3.27. (Convergence in Product Space). If $(x_i)_{i\in\mathbb{N}}$ is a sequence of elements of $\prod_{n\in\mathbb{N}} X_n$ with $x_i = (x_i(n))_{n\in\mathbb{N}}, i \in \mathbb{N}$, and $x = (x(n))_n \in \prod_{n\in\mathbb{N}} X_n$, then
$$x_i \to x \text{ in } \prod_{n\in\mathbb{N}} X_n \iff \text{for each } n,\ x_i(n) \xrightarrow{i\to\infty} x(n) \text{ in } X_n.$$

The previous constructions of metric spaces respect *completeness* and *separability*. That is, the direct sum, finitely and infinitely countable product of complete and separable metric spaces is a complete and separable metric space. Also, the previous constructions extend to topological spaces in a way that respects metrizability.

Definition 1.3.28. The infinite product is defined as described above. If we have topological spaces $(X_n, \mathcal{T}_n)$, $n \in \mathbb{N}$, then we define $\mathcal{V}_\infty$ as the family of all sets of the form
$$V_0 \times \cdots \times V_n \times X_{n+1} \times X_{n+2} \times \ldots, \tag{1.1}$$
where $V_i \in \mathcal{T}_i$ for each $i = 0, \ldots, n$ and $n \in \mathbb{N}$. Consider the family $\mathcal{T}_\infty$ of all unions of elements of $\mathcal{V}_\infty$. Then $\mathcal{T}_\infty$ is a topology on the set $\prod_{n\in\mathbb{N}} X_n$, also known as **product topology**, with the family $\mathcal{V}_\infty$ as its basis. If X_n, $n \in \mathbb{N}$, are metrizable then $(\prod_{n\in\mathbb{N}} X_n, \mathcal{T}_\infty)$ is metrizable with the metric that we mentioned above.

CHAPTER 2

Polish Spaces

2.1. Definition of Polish Spaces

In the present chapter, we will introduce the notion of Polish spaces in our study. Polish spaces are named Polish because they were first extensively studied by Polish topologists and logicians. Some of them were W. Sierpiński, K. Kuratowski and A. Tarski. However, Polish spaces are mostly studied today because they are the primary setting for Descriptive Set Theory. Common examples of Polish spaces are the real line, the Cantor space, and the Baire space. The last two will be discussed in more detail below.

Definition 2.1.1. A topological space $(X, \mathcal{T})$ is **completely metrizable** if it admits a compatible metric d such that (X, d) is complete. A separable completely metrizable space is called **Polish space**. A metric d as above will be called a **compatible** or **suitable metric** for X.

Remark 2.1.2. We will usually denote Polish spaces by $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$. Some simple examples of Polish spaces are $\mathbb{R}$ and $\mathbb{C}$ and their closed subsets, all with the standard topology. A trivial but useful example of a Polish space is the set of natural numbers $\mathbb{N}$ with the standard metric, which is equivalent to discrete, and every subset of $\mathbb{N}$ is open.

Proposition 2.1.3. *Given a Polish space $\mathcal{X}$ and a topological space Y. If $\mathcal{X}$, Y are topologically isomorphic, then Y is also a Polish space.*

Proof. We consider a compatible metric d for $\mathcal{X}$ and a topological isomorphism $f : Y \rightarrowtail\!\!\!\!\twoheadrightarrow (\mathcal{X}, d)$. We define the function $\rho : Y \times Y \to \mathbb{R}$ with

$$\rho(y_1, y_2) = d(f(x_1), f(x_2)).$$

The function ρ is metric on Y. By definition

$$f : (Y, \rho) \rightarrowtail\!\!\!\!\twoheadrightarrow (\mathcal{X}, d)$$

is isometry (preserves the distances, and therefore the completeness, by essentially transferring the Cauchy sequences) and onto. It follows that (Y, ρ) is a complete and separable metric space. Finally, we show that ρ generates the topology of Y. For every $A \subseteq Y$ that is open in the topology of Y the set $(f^{-1})^{-1}[\,A\,] = f[\,A\,]$ is d-open because

$$f^{-1} : (\mathcal{X}, d) \rightarrowtail\!\!\!\!\twoheadrightarrow Y$$

is continuous. Therefore, the set $A = f^{-1}[\,f[\,A\,]\,]$ is ρ-open because $f : (Y, \rho) \rightarrowtail\!\!\!\!\twoheadrightarrow (\mathcal{X}, d)$ is continuous.
Conversely, we assume that $B \subseteq Y$ is ρ-open. Because $f : (\mathcal{X}, d) \rightarrowtail\!\!\!\!\twoheadrightarrow (Y, \rho)$ is continuous the set $(f^{-1})^{-1}[\,B\,] = f[\,B\,]$ is d-open and since $f : Y \rightarrowtail\!\!\!\!\twoheadrightarrow (\mathcal{X}, d)$ is continuous $B = f^{-1}[\,f[\,B\,]\,]$ is open in the topology of Y. □

2.2. F_σ and G_δ Subsets of Polish Spaces

Definition 2.2.1. Let X be a topological space and $A \subseteq X$. The set A is an $\boldsymbol{F_\sigma}$ **subset** of X, if there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of *closed* subsets of X, with

$$A = \bigcup_{n\in\mathbb{N}} F_n.$$

Definition 2.2.2. Let X be a topological space and $A \subseteq X$. The set A is a $\boldsymbol{G_\delta}$ **subset** of X, if there exists a sequence $(U_n)_{n\in\mathbb{N}}$ of *open* subsets of X, with

$$A = \bigcap_{n\in\mathbb{N}} U_n.$$

It is clear that a set $A \subseteq X$ is F_σ exactly when its complement $X \backslash A$ is G_δ. Moreover, it is clear that every closed set F is F_σ as we can get $F_n = F$, $n \in \mathbb{N}$. Equivalently, every open set is G_δ. Finally, we note that the countable union of F_σ sets is also F_σ set, because if $A_n = \bigcup_{i \in \mathbb{N}} F_n^i$, where $n \in \mathbb{N}$ and each F_n^i is closed, then $\mathcal{A} = \{F_n^i \mid i, n \in \mathbb{N}\}$ is a countable family of closed sets and

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup \mathcal{A}.$$

It follows that the countable intersection of G_δ sets is also G_δ set.

Definition 2.2.3. Given a metric space (X, d) and a non-empty $A \subseteq X$, we define the **function of the distance from the set** A,

$$f : X \to \mathbb{R} : f(x) = d(x, A) = \inf \{d(x, z) : z \in A\} \geqslant 0. \tag{2.1}$$

As is well known, f is a continuous function, for precision we have

$$|d(x, A) - d(y, A)| \leqslant d(x, y), \;\; x, y \in A.$$

Proposition 2.2.4. *If X is a metrizable topological space then every closed set of X, except F_σ, is also G_δ. Equivalently, every open subset of X is G_δ and F_σ.*

Proof. Consider a closed $F \subseteq X$, a metric d that generates the topology of X, and the distance function $f = (x \mapsto d(x, F))$ from F, as defined in (2.1) above. (We assume that $F \neq \varnothing$ otherwise the conclusion is obvious.) For any $n \in \mathbb{N}$ we define

$$U_n = \{x \in X : d(x, F) < 2^{-n}\}.$$

Then $U_n = f^{-1}[(-1, 2^{-n})]$ and since f is continuous, the set U_n is an open subset of X for every $n \in \mathbb{N}$. Since $d(x, F) = 0$ for every $x \in F$ it is clear that $F \subset \cap_{n \in \mathbb{N}} U_n$. Assume that $x \in U_n$ for every $n \in \mathbb{N}$. Then there exists a sequence $(x_n)_{n \in \mathbb{N}}$ through F with

$$d(x, x_n) < 2^{-n}, \; \forall n.$$

Therefore $x_n \xrightarrow{d} x$ and since F is closed we have $x \in F$. We conclude that $F = \cap_{n \in \mathbb{N}} U_n$ and hence F is a G_δ set. □

2.3. Finite Sequences

Consider a non-empty set X. By the term **finite sequence** in X we describe a function $u : \{i \in \mathbb{N} : i < n\} \to X$ for some $n \in \mathbb{N}$. We symbolize such a u with $(u(0), \ldots, u(n-1))$. In the finite sequences, we also include the **empty sequence**, which we symbolize with Λ. This follows from the previous definition for $n = 0$ where the domain $\{i \in \mathbb{N} : i < 0\}$ of u is the empty set. We symbolize with $X^{<\mathbb{N}}$ the set of all finite sequences in X. The preceding n in the definition of a finite sequence is unique.

Definition 2.3.1. The **length** of a finite sequence $u : \{i \in \mathbb{N} : i < n\} \to X$ is exactly that unique n and is symbolized with $|u|$. Thus we have

$$|u| = 0 \iff u = \Lambda \;\; \text{and}$$
$$u = (u(0), \ldots, u(|u| - 1)), \text{ for all } u \in X^{<\mathbb{N}}.$$

Definition 2.3.2. The **concatenation** of $u \in X^{<\mathbb{N}}$ with $v \in X^{<\mathbb{N}}$ is the sequence

$$u * v = (u(0), \ldots, u(n-1), v(0), \ldots, v(|v| - 1)).$$

Remark 2.3.3. Clearly, $u * \Lambda = \Lambda * u = u$ for every $u \in X^{<\mathbb{N}}$.

Definition 2.3.4. We define the **binary relation** $\sqsubseteq$ on the set $X^{<\mathbb{N}}$ as follows,

$$u \sqsubseteq v \iff |u| \leqslant |v| \text{ and } \forall i < |u| : \; (u(i) = v(i)).$$

Remark 2.3.5. It holds $\Lambda \sqsubseteq u$ for all $u \in X^{<\mathbb{N}}$. $\sqsubseteq$ satisfies the three properties of the order:

$$u \sqsubseteq u,$$
$$(u \sqsubseteq v \;\&\; v \sqsubseteq u) \longrightarrow u = v,$$
$$(u \sqsubseteq v \;\&\; v \sqsubseteq w) \longrightarrow u = w,$$

for each $u, v, w \in X^{<\mathbb{N}}$.

Definition 2.3.6. The **strict part** of $\sqsubseteq$ is the relation $\subsetneq$ with

$$u \subsetneq v \iff u \sqsubseteq v \;\&\; u \neq v.$$

It is clear that $u \subsetneq v$ if and only if $u \sqsubseteq v$ and $|u| < |v|$. We will say that $u \in X^{<\mathbb{N}}$ is an **initial part** of $v \in X^{<\mathbb{N}}$ or that v is an **extension** of u if $u \sqsubseteq v$. We will also say that u is a **strict initial part** of v or that v is a **strict extension** of u if $u \subsetneq v$. Also,v is a **direct extension** of u if $v = u * (x)$ for some $x \in X$. Two finite sequences u, v are said to be **compatible**, symbolically $u \| v$ if $u \sqsubseteq v$ or $v \sqsubseteq u$.

Definition 2.3.7. We define the **coding function**

$$\langle \cdot \rangle : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N} : u = (u(0), \ldots, u(n-1)) \mapsto \langle u(0), \ldots, u(n-1) \rangle$$

as follows:

$$\langle u(0), \ldots, u(n-1) \rangle = \begin{cases} p_0^{u(0)+1} \cdots\ p_{n-1}^{u(n-1)+1}, & \text{if } n \geqslant 1, \\ 1, & \text{if } n = 0. \end{cases}$$

Definition 2.3.8. We symbolize with Seq the set of all values of $\langle \cdot \rangle$,

$$\mathrm{Seq} = \{s \in \mathbb{N} : \exists u \in \mathbb{N}^{<\mathbb{N}}\ \ s = \langle u(0), \ldots, u(n-1) \rangle\}.$$

If $s = \langle u(0), \ldots, u(n-1) \rangle$ we say that s is a **code** for u.

Definition 2.3.9. We define the **natural enumeration** $(u_s)_{s \in \mathbb{N}}$ of $\mathbb{N}^{<\mathbb{N}}$ in terms of the coding $\langle \cdot \rangle$ as follows:

$$u_s = \begin{cases} (k_0, \ldots, k_{n-1}), & \text{if } s = \langle u(0), \ldots, u(n-1) \rangle \in \mathrm{Seq}, \\ \Lambda, & \text{if otherwise.} \end{cases}$$

Remark 2.3.10. We mention that we will consider the set $\mathbb{N}^{<\mathbb{N}}$ with the topology generated by the discrete metric, where each subset of $\mathbb{N}^{<\mathbb{N}}$ is open. Since $\mathbb{N}^{<\mathbb{N}}$ is a countable set it follows that it is a Polish space.

2.4. Baire Space and Cantor Space

The Baire Space

Definition 2.4.1. (Baire's space). Consider the set $\mathbb{N}^{\mathbb{N}}$ of all functions from $\mathbb{N}$ to $\mathbb{N}$ (infinite sequences). We denote the elements of $\mathbb{N}^{\mathbb{N}}$ with $\alpha, \beta, \gamma, \ldots$ and the set $\mathbb{N}^{\mathbb{N}}$ with $\mathcal{N}$. For $\alpha, \beta \in \mathcal{N}$ with $\alpha \neq \beta$ we set

$$n(\alpha, \beta) = \text{the least } n \in \mathbb{N} \text{ with } \alpha(n) \neq \beta(n).$$

We define the function $d_{\mathcal{N}} : \mathcal{N} \times \mathcal{N} \to \mathbb{R}$,

$$d_{\mathcal{N}}(\alpha, \beta) = \begin{cases} 2^{-n(\alpha,\beta)}, & \text{if } \alpha \neq \beta, \\ 0, & \text{if } \alpha = \beta, \end{cases}.$$

Then the function $d_{\mathcal{N}}$ is metric in $\mathcal{N}$. The metric space $(\mathcal{N}, d_{\mathcal{N}})$ is called **Baire space**.

Definition 2.4.2. For each $u \in \mathbb{N}^{<\mathbb{N}}$ we define the **basic region** of Baire's space,

$$\begin{aligned} \mathcal{N}_u &= \{\alpha \in \mathcal{N} : u \sqsubseteq \alpha\} \\ &= \{u(0)\} \times \cdots \times \{u(|u|-1)\} \times \mathbb{N} \times \mathbb{N} \times \ldots \end{aligned} \tag{2.2}$$

Remark 2.4.3. Note that for any $r > 0$ if n is the minimal natural with $2^{-n} < r$ then for each $\alpha, \beta \in \mathcal{N}$

$$d_{\mathcal{N}}(\alpha, \beta) < r \iff \forall i < n\ \alpha(i) = \beta(i).$$

Hence the $d_{\mathcal{N}}$-open ball of center $\alpha \in \mathcal{N}$ and radius $r > 0$ is the set of all $\beta \in \mathcal{N}$ that agree with α up to $n-1$, i.e.

$$\begin{aligned} B_{d_{\mathcal{N}}}(\alpha, r) &= \mathcal{N}_{\alpha|n} \\ &= \{\alpha(0)\} \times \{\alpha(1)\} \times \cdots \times \{\alpha(n-1)\} \times \mathbb{N} \times \mathbb{N} \times \ldots \end{aligned} \tag{2.3}$$

In the case where $n = 0$ then $\mathcal{N}_{\alpha|0} = \mathcal{N}_{\Lambda} = \mathcal{N}$.

We observe that every open ball in $(\mathcal{N}, d_{\mathcal{N}})$ is a set of the form $\mathcal{N}_u$ with $u \in \mathbb{N}^{<\mathbb{N}}$, and reverse, every set of the form $\mathcal{N}_u$ is an open ball, namely $B_{d_{\mathcal{N}}}(u * (0,0,0,\dots), 2^{-(|u|-1)})$.

The following Proposition is a direct consequence of (2.3).

Proposition 2.4.4. *For every sequence $(\alpha_i)_{i \in \mathbb{N}}$ in $\mathcal{N}$ and every $\alpha \in \mathcal{N}$ we have*

$$\begin{aligned} \alpha_i \xrightarrow{d_{\mathcal{N}}} \alpha &\iff \forall n \ \lim_{i \to \infty} \alpha_i(n) = \alpha(n) \\ &\iff \forall n \, \exists i_n \, \forall i \geqslant i_n \ \ \alpha_i(n) = \alpha(n). \end{aligned} \tag{2.4}$$

Hence, the convergence in product topology is the pointwise convergence.

Proof. Note that the last equivalence holds because $\alpha_i(n)$, $\alpha(n)$ are natural numbers. Therefore, we will show the first equivalence.
For the straight direction we consider $\alpha_i \xrightarrow{d_{\mathcal{N}}} \alpha$ and we get $n \in \mathbb{N}$. From Remark 2.4.3 for $r = 2^{-n}$ the ball $B_{d_{\mathcal{N}}}(\alpha, 2^{-n})$ is equal to the set $\mathcal{N}_{\alpha|(n+1)}$. Since the sequence $(\alpha_i)_{i \in \mathbb{N}}$ converges at $\alpha(n)$ we have

$$\alpha_i \in B_{d_{\mathcal{N}}}(\alpha, 2^{-n}) = \mathcal{N}_{\alpha|(n+1)}, \text{ for all large } i.$$

In particular, $\alpha_i(n) = \alpha(n)$ for all large i.
Conversely, we assume that for every $n \in \mathbb{N}$ we have

$$\lim_{i \to \infty} \alpha_i(n) = \alpha(n).$$

We consider $r > 0$ and N the minimal natural with $2^{-N} < r$. By Remark 2.4.3 we have

$$B_{d_{\mathcal{N}}}(\alpha, r) = \mathcal{N}_{\alpha|N}.$$

For every $n \leqslant N$ we have

$$\lim_{i \to \infty} \alpha_i(n) = \alpha(n)$$

and as we mentioned there exists $i_n \in \mathbb{N}$ such that for every $i \geqslant i_n$, it holds that $\alpha_i(n) = \alpha(n)$. We set

$$i_0 = \max\{i_n : \ n \leqslant N\},$$

then for every $i \geqslant i_0$ and every $n < N$ we have $\alpha_i(n) = \alpha(n)$. Therefore for each $i \geqslant i_0$ we have $\alpha_i|n = \alpha|N$, i.e.

$$\alpha_i \in \mathcal{N}_{\alpha|N} = B_{d_{\mathcal{N}}}(\alpha, r).$$

$\square$

Proposition 2.4.5. *The topology of $(\mathcal{N}, d_{\mathcal{N}})$ is the product topology on $\mathbb{N}^{\mathbb{N}}$.*

Proof. Since the open balls in a metric space are the basis for the metric space's topology, it follows from Remark 2.4.3 that the sets $\mathcal{N}_u$, with $u \in \mathbb{N}^{<\mathbb{N}}$, are $d_{\mathcal{N}}$-open and furthermore are the basis for the topology of $(\mathcal{N}, d_{\mathcal{N}})$.

Moreover, since the singletons in $\mathbb{N}$ are the basis for the topology of $\mathbb{N}$, we have from 1.1, that $\mathcal{N}_u$ are the basis for the product topology on $\mathbb{N}^{\mathbb{N}}$.

Hence, the topology of $(\mathcal{N}, d_{\mathcal{N}})$ and the product topology of $\mathbb{N}^{\mathbb{N}}$ have a common basis and are, subsequently, equal. $\square$

Proposition 2.4.6. *The space $(\mathcal{N}, d_{\mathcal{N}})$ is a complete and separable metric space. Hence, the Baire space $\mathcal{N}$ is a Polish space.*

Proof. The final null sequences

$$\alpha_u = u * (0,0,0,\dots) = (u(0), \dots, u(|u|-1), 0, 0, 0, \dots), \ u \in \mathbb{N}^{<\mathbb{N}}$$

are a countable and dense set of $(\mathcal{N}, d_{\mathcal{N}})$, because $\alpha_u \in \mathcal{N}_u$ for each $u \in \mathbb{N}^{<\mathbb{N}}$. Therefore, the space $(\mathcal{N}, d_{\mathcal{N}})$ is *separable.*
To prove the completeness, we consider a $d_{\mathcal{N}}$-Cauchy sequence $(\alpha_i)_{i \in \mathbb{N}}$.
We stabilize an $n \in \mathbb{N}$. Then there exists an $i_n \in \mathbb{N}$ such that for every $i, j \geqslant i_n$ we have

$$d_{\mathcal{N}}(\alpha_i, \alpha_j) < 2^{-n}.$$

It follows that

(2.5) $$\alpha_i(k) = \alpha_j(k), \text{ for every } k = 0, \ldots, n \text{ and each } i, j \geqslant i_n.$$

In particular, the sequence of natural numbers $(\alpha_i(n))_{i\in\mathbb{N}}$ is finally constant and equal to the number $\alpha_{i_n}(n)$. Therefore, we define

$$\alpha : \mathbb{N} \to \mathbb{N} : \alpha(n) = \lim_{i\to\infty} \alpha_i(n) = \alpha_{i_n}(n)$$

where i_n is as above. It is clear from (2.5) that

$$\alpha_i(k) = \alpha_{i_n}(k) = \alpha(k)$$

for every $i, j \geqslant i_n$ and every $k = 0, \ldots, n$. From (2.4) it follows that $\alpha_i \xrightarrow{d_{\mathcal{N}}} \alpha$. Hence, the space $(\mathcal{N}, d_{\mathcal{N}})$ is *complete* and so it is a Polish space. □

Theorem 2.4.7 ([**2**]). *For each Polish space $\mathcal{X}$ there is a continuous surjection $\pi : \mathcal{N} \twoheadrightarrow \mathcal{X}$.*

Proof. Consider a suitable metric d on $\mathcal{X}$ and a set

$$D = \{r_n : n \in \mathbb{N}\}$$

which is a countable and dense subset of $\mathcal{X}$. Each $x \in \mathcal{X}$ is the limit of a sequence $(r_{k_n})_{n\in\mathbb{N}}$ of $(r_n)_{n\in\mathbb{N}}$. The idea is to take $\alpha = (k_n)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and then we will have

$$x = \lim_{n\to\infty} r_{\alpha(n)}.$$

The last limit will be the value of π in α. A problem that arises is that the sequence $(r_{\alpha(n)})_{n\in\mathbb{N}}$ may not converge for every $\alpha \in \mathbb{N}$. To fix this, we will replace the sequence $(r_{\alpha(n)})_{n\in\mathbb{N}}$ with another sequence, let us denote it by $(x_n^{\alpha})_{n\in\mathbb{N}}$, which converges for every α and the function

$$a \mapsto \lim_{n\to\infty} x_n^{\alpha}$$

is continuous. Moreover, for a sufficiently large collection of $\alpha \in \mathbb{N}$, the $(x_n^{\alpha})_{n\in\mathbb{N}}$ is not substantially different from $(r_{\alpha(n)})_{n\in\mathbb{N}}$ and therefore each $x \in \mathcal{X}$ will be taken as the limit of a sequence of the form $(x_n^{\alpha})_{n\in\mathbb{N}}$, i.e. we will have a surjection.

We now proceed to the construction of π. We first define a family $(x_u)_{u\in\mathbb{N}^{<\mathbb{N}}\setminus\{\Lambda\}}$ of elements of $\mathcal{X}$ with induction on the length $|u|$ of $u \in \mathbb{N}^{<\mathbb{N}}\setminus\{\Lambda\}$.
For $|u| = 1$ with $u = (k_0)$, we define $x_u = x_{(k_0)} = r_{k_0}$. We assume that for some $n > 1$ have been defined x_w for each $w \in \mathbb{N}^{<\mathbb{N}}$, with $1 \leqslant |w| < n$.

Consider $u \in \mathbb{N}^{<\mathbb{N}}$ with $|u| = n$. We temporarily set $w = (u(0), \ldots, u(n-2))$ and $k = u(|u|-1)$ such that $u = w * (k)$. Obviously, $|w| = n-1$ and, by the Inductive Hypothesis, x_w is defined. We define

$$x_u = \begin{cases} r_k, & \text{if } d(x_w, r_k) < 2^{-n}, \\ x_w, & \text{otherwise.} \end{cases}$$

Thus the family $(x_u)_{u\in\mathbb{N}^{<\mathbb{N}}\setminus\{\Lambda\}}$. It is clear from the above definition that $d(x_w, r_k) < 2^{-|u|}$ where $u = w * u(|u|-1)$. It follows that for any $w \subsetneq u$ with $u = w * (k_0, \ldots, k_m)$,

$$\begin{aligned} d(x_w, x_u) &\leqslant d(x_w, x_{w*(k_0)}) + \cdots + d(x_{w*(k_0,\ldots,k_{m-1})}, x_u) \\ &< 2^{-(|w|+1)} + \cdots + 2^{-|u|} \\ &< \sum_{k=1}^{\infty} 2^{-(|w|+k)} = 2^{-|w|}. \end{aligned}$$

Therefore for each $\alpha \in \mathcal{N}$ and each $1 \leqslant n \leqslant m$ we have

(2.6) $$d(x_{\alpha|n}, x_{\alpha|m}) < 2^{-n}.$$

Hence for every $\alpha \in \mathcal{N}$ the sequence $(x_{\alpha|n})_{n\in\mathbb{N}}$ is d-Cauchy. Since $(\mathcal{X}, d)$ is complete, we can define the function

$$\pi : \mathcal{N} \to \mathcal{X} : \pi(\alpha) = \lim_{n\to\infty} x_{\alpha|n}.$$

Taking limit $m \to \infty$ in (2.6) we have

$$d(x_{\alpha|n}, \pi(\alpha)) \leqslant 2^{-n}, \ \forall\, \alpha \in \mathcal{N}, \ \forall\, n \in \mathbb{N}.$$

Therefore, if $\beta \in \mathcal{N}_{\alpha|n}$, i.e. if $\beta|n = \alpha|n$, then

$$d(\pi(\alpha), \pi(\beta)) \leqslant d(\pi(\alpha), x_{\alpha|n}) + d(x_{\alpha|n}, \pi(\beta)) \leqslant 2^{-n+1},$$

for each $\alpha, \beta \in \mathcal{N}$ and each $n \in \mathbb{N}$. It is straightforward that π is continuous (and indeed uniformly continuous). Finally, we show that π is an epimorphism. If $x \in \mathcal{X}$, we define $\alpha : \mathbb{N} \to \mathbb{N}$ as follows:

$$\alpha(n) = \text{the least } k \in \mathbb{N} \text{ with } d(r_k, x) < 2^{-(n+3)}. \tag{2.7}$$

(For every $n \in \mathbb{N}$ there exists such a k because the set $D = \{r_k : k \in \mathbb{N}\}$ is dense.) Then,

$$d(r_{\alpha(n)}, x) < 2^{-(n+3)}$$

and

$$d(r_{\alpha(n)}, r_{\alpha(n+1)}) \leqslant d(r_{\alpha(n)}, x) + d(x, r_{\alpha(n+1)}) < 2^{-(n+3)} + 2^{-(n+4)} < 2^{-(n+2)},$$

for each $n \in \mathbb{N}$. It follows by induction that

$$x_{\alpha|(n+1)} = r_{\alpha(n)}, \text{ for every } n \in \mathbb{N}.$$

(That is, for this α the first case of the definition of $x_{\alpha|n}$ always occurs.) Therefore,

$$\pi(\alpha) = \lim_{n\to\infty} x_{\alpha|n} = \lim_{n\to\infty} x_{\alpha|(n+1)} = \lim_{n\to\infty} r\alpha(n) = x$$

and π is a surjection. □

Remark 2.4.8. The function $\pi : \mathcal{N} \twoheadrightarrow \mathcal{X}$ of the previous Theorem admits an inverse function. That is, there exists a monomorphism $\tau : \mathcal{X} \rightarrowtail \mathcal{N}$ with

$$\pi((\tau(x)) = x, \text{ for every } x \in \mathcal{X} \text{ and } \tau(\pi(\alpha)) = \alpha, \text{ for every} \alpha \in \tau[\mathcal{X}].$$

If we set $\alpha = \tau(x)$ then as we have shown $\pi(\alpha) = x$, i.e. $\pi(\tau(x)) = x$, for every $x \in \mathcal{X}$. From this it follows that τ is a monomorphism,

$$\tau(x_1) = \tau(x_2) \implies \pi(\tau(x_1)) = \pi(\tau(x_2)) \implies x_1 = x_2.$$

Moreover, for each $\alpha = \tau(x)$ we have

$$\tau(\pi(\alpha)) = \tau(\pi(\tau(x))) = \tau(x) = \alpha.$$

Corollary 2.4.9. *Every Polish space has a cardinal number less than or equal to the cardinality of the continuum.*

Proof. For every Polish space $\mathcal{X}$, by Remark 2.4.8 there exists a monomorphism $f : \mathcal{X} \rightarrowtail \mathcal{N}$. Hence, $\mathcal{X} \leqslant_c \mathcal{N} =_c \mathbb{R}$. □

The Cantor Space

Definition 2.4.10. The set of all binary (infinite) sequences is the $\{0,1\}^{\mathbb{N}}$ which is also symbolized with $2^{\mathbb{N}}$. This is a subset of $\mathcal{N}$ and we consider on it the metric

$$d_{2^{\mathbb{N}}} = d_{\mathcal{N}} | (2^{\mathbb{N}} \times 2^{\mathbb{N}}).$$

The metric space $(2^{\mathbb{N}}, d_{2^{\mathbb{N}}})$ is called **Cantor's space**.

Remark 2.4.11. A basis for Cantor topology consists of all sets of the form $\mathcal{N}_u \cup 2^{\mathbb{N}}$, where $u \in \mathbb{N}^{<\mathbb{N}}$, precisely because $\mathcal{N}_u$, $u \in \mathbb{N}^{<\mathbb{N}}$ form the basis for topology of $\mathcal{N}$. Obviously, $\mathcal{N}_u \cup 2^{\mathbb{N}} = \varnothing$ when there exists $i < |u|$ with $u(i) > 1$, hence we can restrict to $u \in \{0,1\}^{<\mathbb{N}}$. We, therefore, conclude that a **basis for the topology of Cantor space** consists of all sets of the form

$$\mathcal{N}_u^{2^{\mathbb{N}}} = \{\alpha \in 2^{\mathbb{N}} : u \sqsubseteq \alpha\}, \ u \in \{0,1\}^{<\mathbb{N}}.$$

The topology of Cantor space is the relative topology of the Baire space, which is the product topology product on $\mathbb{N}^{\mathbb{N}}$. It follows that the topology of Cantor space is the product topology on the set $\{0,1\}^{<\mathbb{N}}$.

Remark 2.4.12. The **convergence of sequences on** $2^{\mathbb{N}}$ is characterized as in the case of the Baire space $\mathcal{N}$. That is, for each sequence $(\alpha_i)_{i\in\mathbb{N}}$ in $2^{\mathbb{N}}$ and every $\alpha \in 2^{\mathbb{N}}$ we have

$$\begin{aligned} \alpha_i \xrightarrow{d_{2^{\mathbb{N}}}} \alpha &\iff \forall n \ \lim_{i\to\infty} \alpha_i(n) = \alpha(n) \\ &\iff \forall n \, \exists i_n \, \forall i \geqslant i_n \ \alpha_i(n) = \alpha(n). \end{aligned} \tag{2.8}$$

Proposition 2.4.13. *The space* $(2^{\mathbb{N}}, d_{2^{\mathbb{N}}})$ *is a compact metric space, and therefore the corresponding topological space is a Polish space.*

Proof. If we show that $(2^{\mathbb{N}}, d_{2^{\mathbb{N}}})$ is a compact metric space, equivalently that $2^{\mathbb{N}}$ is a compact subset of $\mathcal{N}$, we will also have that $2^{\mathbb{N}}$ is closed in $\mathcal{N}$. Therefore, we will have that $2^{\mathbb{N}}$ is a Polish space. To prove the compactness, we observe that

$$2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}} = \prod_{n \in \mathbb{N}} \{0,1\}.$$

We know that the Cartesian product of compact sets is a compact set (for a random product of topological spaces we need the *Tychonoff Theorem* but here we have only a countable product). Since $\{0,1\}$ is a compact subset of $\mathcal{N}$ we have that $\prod_{n \in \mathbb{N}} \{0,1\}$ is a compact subset of $\mathbb{N}^{\mathbb{N}}$ with the product topology. By Proposition 2.4.5, this is the topology of the Baire space, so $2^{\mathbb{N}}$ is a compact subset of $\mathcal{N}$. Hence, $2^{\mathbb{N}}$ is a Polish space. $\square$

Theorem 2.4.14 ([**2**])**.** *For every perfect Polish space* $\mathcal{X}$ *there exists a continuous monomorphism* $\tau : 2^{\mathbb{N}} \rightarrowtail \mathcal{X}$.

Proof. We set $I = \{0,1\}^{<\mathbb{N}}$ and fix a compatible metric d to $\mathcal{X}$. We recursively construct a family $(V_u)_{u \in I}$ from d-open balls of $\mathcal{X}$ with the following properties:

$$\text{radius}(V_u) \leqslant 2^{-|u|},$$
$$\overline{V_{u*(i)}} \subseteq \overline{V_u},\ i = 0,1 \quad \text{and} \quad \overline{V_{u*(0)}} \cap \overline{V_{u*(1)}} = \varnothing, \text{ for each } u \in I. \tag{2.9}$$

In the basic step we choose an $x_0 \in \mathcal{X}$ and obtain $V_\Lambda = B_d(x_0, 1)$. We assume that for some $n \geqslant 1$ we have defined V_u as above for all $w \in I$ with $1 \leqslant |w| < n$.

We define $V_{u*(i)}$, $i = 0,1$, for all $w \in I$ with $|w| = n-1$. Consider such a w. The ball V_w cannot contain only its center, otherwise this would be a single point of $\mathcal{X}$. Therefore there are elements

$$x_0^w, x_1^w \text{ of } V_w, \text{ with } x_0^w \neq x_1^w.$$

Since V_w is an open set, there are open balls B_0^w and B_1^w with centres at x_0^w, x_1^w respectively, with radii less than or equal to 2^{-n}, also satisfying

$$\overline{B_0^w} \cap \overline{B_1^w} = \varnothing \quad \text{and} \quad \overline{B_i^w} \subseteq V_w \subseteq \overline{V_w},\ i = 0,1.$$

So we define $V_{w*(i)} = B_i^w$, for $i = 0,1$. It is clear that the properties of (2.9) are satisfied. This completes the construction.

For each $\alpha \in 2^{\mathbb{N}}$, $(\overline{V_{\alpha|n}})_{n \in \mathbb{N}}$ is a decreasing sequence of non-empty closed sets whose diameter converges to 0. Since $(\mathcal{X}, d)$ is complete, from the *Cantor's Intersection Theorem*, the intersection

$$\bigcap_{n \in \mathbb{N}} \overline{V_{\alpha|n}}$$

is a singleton. We define

$$\tau : 2^{\mathbb{N}} \to \mathcal{X} : \{\tau(\alpha)\} = \bigcap_{n \in \mathbb{N}} \overline{V_{\alpha|n}}.$$

If $\alpha \neq \beta$ are elements of $2^{\mathbb{N}}$ and n is the least k with $\alpha(k) \neq \beta(k)$ then

$$\alpha|n = \beta|n \quad \text{and} \quad \alpha(n) \neq \beta(n).$$

Without loss of generality, we assume $\alpha(n) = 0$ and $\beta(n) = 1$. We also set $w = \alpha|n = \beta|n$. Then

$$\tau(\alpha) \in V_{\alpha|(n+1)},\ \tau(\beta) \in V_{\beta|(n+1)}, \text{ and}$$
$$V_{\alpha|(n+1)} \cap V_{\beta|(n+1)} = \overline{V_{w*(0)}} \cap \overline{V_{w*(1)}} = \varnothing.$$

Therefore $\tau(\alpha) \neq \tau(\beta)$.

Finally, we will show that τ is continuous. We set

$$x_u = \text{ the center of the open ball } V_u.$$

Let $\alpha \in 2^{\mathbb{N}}$. Then for every n we have $\tau(\alpha) \in \overline{V_{\alpha|n}}$ and therefore

$$d(\tau(\alpha), x_{\alpha|n}) \leqslant \text{radius}(V_{\alpha|n}) \leqslant 2^{-n}.$$

Therefore, if $\beta \in \mathcal{N}_{\alpha|n}$, i.e. if $\beta|n = \alpha|n$, then

$$d(\tau(\alpha), \tau(\beta)) \leqslant d(\tau(\alpha), x\alpha|n) + d(x\alpha|n, \tau(\beta)) \leqslant 2 \cdot 2^{-n} = 2^{-n+1}.$$

It follows from the above that τ is continuous. $\square$

Proposition 2.4.15. *Every non-empty perfect subset of a Polish space has the cardinality of the continuum.*

Proof. If $\mathcal{X}$ is a Polish space and P is its non-empty perfect subset, then P with the relevant topology is a perfect Polish space. By Theorem 2.4.14, there exists a continuous monomorphism $\tau : 2^{\mathbb{N}} \rightarrowtail P$ and in particular $2^{\mathbb{N}} \leqslant_c P$. Hence,

$$\mathbb{R} =_c \{0,1\}^{\mathbb{N}} = 2^{\mathbb{N}} \leqslant_c P \leqslant_c \mathbb{R},$$

where in the last relation $\leqslant_c$ we have used the Corollary 2.4.9. By Theorem 1.2.1 (Schröder-Bernstein), we conclude that $P =_c \mathbb{R}$. $\square$

CHAPTER 3

Pointclasses

3.1. Fundamental Operators

In this chapter, we will study several set operators, restricting the scope of these operators to subsets of Polish spaces. Then, we will define the notion of pointclass and we will focus on some closure properties. In the last two sections of the chapter, we will present the definitions of the Borel Pointclasses and the Projective Pointclasses of Finite Order.

Definition 3.1.1. Using the term **set operator** we mean any operation between sets. The operators we will study are the following:

i) The **operator of the disjunction** $\vee$.
If we have $P, Q \subseteq \mathcal{X}$ we define the set of the disjunction $P \vee Q \subseteq \mathcal{X}$ by

$$x \in P \vee Q \iff x \in P \ \text{ or } \ x \in Q.$$

Here we are slightly abusing symbolism. As we have mentioned, we will use the symbol $\vee$ to denote the operator of the disjunction within logical sentences. At this point, we use the same symbol to symbolize an operation between sets. The obvious relation holds

$$x \in P \vee Q \iff x \in P \ \vee \ x \in Q.$$

ii) The **operator of the conjunction** $\&$.
If we have $P, Q \subseteq \mathcal{X}$ we define the set of coupling $P \,\&\, Q \subseteq \mathcal{X}$ by

$$x \in P \,\&\, Q \iff x \in P \ \text{ and } \ x \in Q.$$

Note that $P \,\&\, Q$ is the set-theoretic intersection $P \cap Q$, so we will also call the conjunction $P \,\&\, Q$ the intersection of P, Q. Similar observations hold with those of the operator of the disjunction.

iii) The **complement operator** c.
If $P \subseteq \mathcal{X}$ we define the complement $\mathrm{c}_{\mathcal{X}} P$ of P with respect to $\mathcal{X}$ as the set $\mathcal{X} \setminus P$. Clearly,

$$x \in \mathrm{c}_{\mathcal{X}} P \iff \neg(x \in P).$$

iv) The **operator of the infinite countable disjunction or union** $\bigvee_{\mathbb{N}}$.
If we have a sequence of sets $P_n \in \mathcal{X}$, $n \in \mathbb{N}$, we define the infinite disjunction $\bigvee_{\mathbb{N}}(P_n)_{n\in\mathbb{N}}$ as follows,

$$x \in \bigvee\nolimits_{\mathbb{N}}(P_n)_{n\in\mathbb{N}} \iff \exists n \in \mathbb{N}\, x \in P_n.$$

In other words, the infinite disjunction $\bigvee_{\mathbb{N}}(P_n)_{n\in\mathbb{N}}$ is the union $\bigcup_{n\in\mathbb{N}} P_n$ of subsets of the same Polish space. As before, we will call the infinite disjunction a union.

v) The **operator of the infinite numerical conjunction or intersection** $\bigwedge_{\mathbb{N}}$.
If we have a sequence of sets $P_n \in \mathcal{X}$, $n \in \mathbb{N}$, we define the infinite conjunction $\bigwedge_{\mathbb{N}}(P_n)_{n\in\mathbb{N}}$ as follows,

$$x \in \bigwedge\nolimits_{\mathbb{N}}(P_n)_{n\in\mathbb{N}} \iff \forall n \in \mathbb{N}\, x \in P_n.$$

The infinite conjunction $\bigwedge_{\mathbb{N}}(P_n)_{n\in\mathbb{N}}$ is the intersection $\bigcap_{n\in\mathbb{N}} P_n$ of subsets of the same Polish space, and we will call it an intersection.

vi) The **operator of the existential quantifier** $\exists^{\mathcal{Y}} P$ over $\mathcal{Y}$.
If we have a $P \subseteq \mathcal{X} \times \mathcal{Y}$ we define the set

$$\exists^{\mathcal{Y}} P = \{x \in \mathcal{X} : \exists y \ \ (x, y) \in P\}.$$

Then, $\exists^{\mathcal{Y}} P$ is exactly the *projection of P over $\mathcal{Y}$*. In other words

$$\exists^{\mathcal{Y}} P = \mathrm{pr}[\,P\,], \ \text{ where}$$
$$\mathrm{pr} : \mathcal{X} \times \mathcal{Y} \to \mathcal{X} : \mathrm{pr}(x, y) = x.$$

vii) The **operator of the universal quantifier** $\forall^{\mathcal{Y}} P$ over $\mathcal{Y}$.
If we have a $P \subseteq \mathcal{X} \times \mathcal{Y}$ we define the set

$$\forall^{\mathcal{Y}} P = \{x \in \mathcal{X} : \forall y \ \ (x, y) \in P\}.$$

It is evident that

$$\forall^{\mathcal{Y}} P = \mathrm{c}_{\mathcal{X}}(\exists^{\mathcal{Y}}(\mathrm{c}_{(\mathcal{X}\times\mathcal{Y})} P)).$$

viii) The **operator of the bounded existential quantifier** $\exists^{\leqslant}$.
Given $P \subseteq \mathcal{X} \times \mathbb{N}$ we define the set $\exists^{\leqslant} P \subseteq \mathcal{X} \times \mathbb{N}$ as follows

$$(x, n) \in \exists^{\leqslant} P \iff \exists m \leqslant n \ \ (x, m) \in P$$

Note that the set $\exists^{\leqslant} P$ remains a subset of $\mathcal{X} \times \mathbb{N}$.

ix) The **operator of the bounded universal quantifier** $\forall^{\leqslant}$.
Given $P \subseteq \mathcal{X} \times \mathbb{N}$ we define the set $\forall^{\leqslant} P \subseteq \mathcal{X} \times \mathbb{N}$ as follows

$$(x, n) \in \forall^{\leqslant} P \iff \forall m \leqslant n \ \ (x, m) \in P$$

Note that the set $\forall^{\leqslant} P$ remains a subset of $\mathcal{X} \times \mathbb{N}$.

x) The **operator of the finite union** $\bigvee_{\leqslant}$.
Given finite $P_0, \ldots, P_n \subseteq \mathcal{X}$, we define the set

$$\bigvee\nolimits_{\leqslant}(P_0, \ldots, P_n) = P_0 \cup \cdots \cup P_n.$$

In other words, $\bigvee_{\leqslant}(P_0, \ldots, P_n)$ is the finite union of the sets $P_0, \ldots, P_n$. We clarify that n is a random natural number. That is, the field of the operator $\bigvee_{\leqslant}$ is all the *non-empty finite sequences of subsets of the same space*.

xi) The **operator of the finite intersection** $\bigwedge_{\leqslant}$.
Given finite $P_0, \ldots, P_n \subseteq \mathcal{X}$, we define the set

$$\bigwedge\nolimits_{\leqslant}(P_0, \ldots, P_n) = P_0 \cap \cdots \cap P_n.$$

That is, $\bigwedge_{\leqslant}(P_0, \ldots, P_n)$ is the finite intersection of the sets $P_0, \ldots, P_n$. We clarify that n is a random natural number. That is, the field of the operator $\bigwedge_{\leqslant}$ is all the *non-empty finite sequences of subsets of the same space*.

3.2. Closure of Pointclasses

Definition 3.2.1. We define as **pointclass** the collection of all sets in metric spaces characterized by a particular property. For example, we will refer to the class of open sets. The pointlasses will usually be denoted by Γ. Unless otherwise stated the classes of sets will refer to subsets of Polish spaces.

Remark 3.2.2. For each Polish space $\mathcal{X}$ and each pointclass Γ we set

$$\Gamma(\mathcal{X}) = \{A \subseteq \mathcal{X} : A \text{ belongs to the pointclass } \Gamma\}$$

We will say that an $A \subseteq \mathcal{X}$ is a Γ-**subset** of $\mathcal{X}$ if $A \in \Gamma(\mathcal{X})$.

Remark 3.2.3. If Φ is one of the operators defined earlier, we denote by $\Phi\Gamma$ the **class resulting from all sets of the form** ΦP where P belongs to Γ and falls within the scope of Φ.

Definition 3.2.4. We will say that the class Γ is **closed under the operator** Φ if the result of the action of Φ to the sets of Γ falling within its scope is a set belonging to Γ, equivalently

$$\Phi\Gamma \subseteq \Gamma.$$

We will also refer to a **class of functions**. This term means a collection of functions among metric spaces (usually Polish spaces) characterized by a certain property. For example, we have the class of continuous functions.

Definition 3.2.5. A class Γ is **closed under continuous substitution** if for every continuous function

$$f : \mathcal{X} \to \mathcal{Y}$$

and every $Q \in \Gamma(\mathcal{Y})$ we have $f^{-1}[Q] \in \Gamma(\mathcal{X})$, equivalently the set $P \subseteq \mathcal{X}$ defined as follows

$$x \in P \iff f(x) \in Q$$

belongs to Γ.

Definition 3.2.6. More precisely, if we have a class of functions Γ' we will say that Γ is **closed under Γ'-substitution** if for every

$$f : \mathcal{X} \to \mathcal{Y}$$

belonging to Γ' and for every $Q \in \Gamma(\mathcal{Y})$ we have $f^{-1}[Q] \in \Gamma(\mathcal{X})$. Hence, the closure under continuous substitution is closure under Γ'-substitution, where Γ' is the class of all continuous functions.

3.3. The Borel Pointclasses of Finite Order

Definition 3.3.1. We define the **Borel pointclasses of finite order** (for subsets of Polish spaces) by recursion for $n \geqslant 1$ as follows:

$$\begin{aligned} \utilde{\boldsymbol{\Sigma}}^0_1 &= \text{the pointclass of all open sets,} \\ \utilde{\boldsymbol{\Pi}}^0_1 &= \mathrm{c}\utilde{\boldsymbol{\Sigma}}^0_1 = \text{the pointclass of all closed sets,} \end{aligned}$$

and

$$\begin{aligned} \utilde{\boldsymbol{\Sigma}}^0_{n+1} &= \bigvee_{\mathbb{N}} \utilde{\boldsymbol{\Pi}}^0_n = \text{the countable unions of sets of } \utilde{\boldsymbol{\Pi}}^0_n, \\ \utilde{\boldsymbol{\Pi}}^0_{n+1} &= \mathrm{c}\utilde{\boldsymbol{\Sigma}}^0_{n+1} = \text{the complements of sets of } \utilde{\boldsymbol{\Sigma}}^0_{n+1}. \end{aligned}$$

Finally, we set

$$\utilde{\boldsymbol{\Delta}}^0_n = \utilde{\boldsymbol{\Sigma}}^0_n \cap \utilde{\boldsymbol{\Pi}}^0_n.$$

The pointclasses $\utilde{\boldsymbol{\Sigma}}^0_n$ are **the Borel pointclasses of finite order**, while $\utilde{\boldsymbol{\Pi}}^0_n$ and $\utilde{\boldsymbol{\Delta}}^0_n$ are the **dual** and **ambiguous Borel pointclasses of finite order**, respectively. The collection of the aforementioned pointclasses is called the **Borel Hierarchy of sets of finite order** and visualized in the Diagram 3.1, below.

$$\begin{tikzcd}[column sep=small] & \utilde{\boldsymbol{\Sigma}}^0_1(\mathcal{X}) \arrow[dr, phantom, sloped, "\subseteq"] & & \utilde{\boldsymbol{\Sigma}}^0_2(\mathcal{X}) \arrow[dr, phantom, sloped, "\subseteq"] & & & \utilde{\boldsymbol{\Sigma}}^0_n(\mathcal{X}) \arrow[dr, phantom, sloped, "\subseteq"] & \\ \utilde{\boldsymbol{\Delta}}^0_1(\mathcal{X}) \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \utilde{\boldsymbol{\Delta}}^0_2(\mathcal{X}) \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \cdots & \utilde{\boldsymbol{\Delta}}^0_n(\mathcal{X}) \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \cdots \\ & \utilde{\boldsymbol{\Pi}}^0_1(\mathcal{X}) \arrow[ur, phantom, sloped, "\subseteq"] & & \utilde{\boldsymbol{\Pi}}^0_2(\mathcal{X}) \arrow[ur, phantom, sloped, "\subseteq"] & & & \utilde{\boldsymbol{\Pi}}^0_n(\mathcal{X}) \arrow[ur, phantom, sloped, "\subseteq"] & \end{tikzcd}$$

Diagram 3.1. The Borel Hierarchy of subsets of $\mathcal{X}$ of finite order.

Remark 3.3.2. The pointclass $\utilde{\boldsymbol{\Sigma}}^0_2$ consists exactly of the F_σ sets and therefore $\utilde{\boldsymbol{\Pi}}^0_2$ consists exactly of the G_δ sets.

Proposition 3.3.3. *For every Polish space $\mathcal{X}$ and every $n \geqslant 1$ we have*

$$\utilde{\boldsymbol{\Sigma}}^0_n(\mathcal{X}) \subseteq \utilde{\boldsymbol{\Delta}}^0_{n+1}(\mathcal{X}) \quad \textit{and also} \quad \utilde{\boldsymbol{\Pi}}^0_n(\mathcal{X}) \subseteq \utilde{\boldsymbol{\Delta}}^0_{n+1}(\mathcal{X}).$$

Proof. We show by induction on $n \geqslant 1$ that

$$\utilde{\mathbf{\Sigma}}^0_n(\mathcal{X}) \subseteq \utilde{\mathbf{\Sigma}}^0_{n+1}(\mathcal{X}) \quad \text{and} \quad \utilde{\mathbf{\Sigma}}^0_n(\mathcal{X}) \subseteq \utilde{\mathbf{\Pi}}^0_{n+1}(\mathcal{X}). \tag{3.1}$$

Obviously from the above we obtain the inclusions $\utilde{\mathbf{\Sigma}}^0_n(\mathcal{X}) \subseteq \utilde{\mathbf{\Delta}}^0_{n+1}(\mathcal{X})$. For $n = 1$, by Proposition 2.2.4 every open subset of $\mathcal{X}$ is F_σ and G_δ, so that by Remark 3.3.2 we have

$$\utilde{\mathbf{\Sigma}}^0_1(\mathcal{X}) \subseteq \utilde{\mathbf{\Sigma}}^0_2(\mathcal{X}) \quad \text{and} \quad \utilde{\mathbf{\Pi}}^0_1(\mathcal{X}) \subseteq \utilde{\mathbf{\Pi}}^0_2(\mathcal{X}).$$

We assume that for some $n \geqslant 1$, (3.1) holds and we will prove this, respectively, for $n+1$. Let $A \in \utilde{\mathbf{\Sigma}}^0_{n+1}(\mathcal{X})$. Then by definition there is a sequence $(B_i)_{i\in\mathbb{N}}$ of $\utilde{\mathbf{\Pi}}^0_1$ subsets of $\mathcal{X}$ with $A = \cup_{i\in\mathbb{N}}(B_i)$. We have

$$X\backslash B_i \in \utilde{\mathbf{\Sigma}}^0_n(\mathcal{X}) \subseteq \utilde{\mathbf{\Sigma}}^0_{n+1}(\mathcal{X})$$

where in the last inclusion we used the Inductive Hypothesis. So,

$$B \in \utilde{\mathbf{\Pi}}^0_{n+1}(\mathcal{X}), \ \forall i \in \mathbb{N}$$

and

$$A = \cup_{i\in\mathbb{N}}(B_i) \in \bigvee_{\mathbb{N}} \utilde{\mathbf{\Pi}}^0_{n+1}(\mathcal{X}) = \utilde{\mathbf{\Sigma}}^0_{n+2}(\mathcal{X}).$$

Moreover, if we take $A_i = A$ for each $i \in \mathbb{N}$ we have that $A = \cap_{i\in\mathbb{N}}(A_i)$ and $A_i \in \utilde{\mathbf{\Sigma}}^0_{n+1}(\mathcal{X})$ for each $i \in \mathbb{N}$. Therefore

$$\mathcal{X}\backslash A = \cup_{i\in\mathbb{N}}(\mathcal{X}\backslash A_i) \in \bigvee_{\mathbb{N}} \utilde{\mathbf{\Pi}}^0_{n+1}(\mathcal{X}) = \utilde{\mathbf{\Sigma}}^0_{n+2}(\mathcal{X})$$

and therefore $A \in \utilde{\mathbf{\Pi}}^0_{n+2}(\mathcal{X})$. Thus we have shown (3.1) for $n+1$. It is also clear that the inclusions $\utilde{\mathbf{\Pi}}^0_n(\mathcal{X}) \subseteq \utilde{\mathbf{\Delta}}^0_1(\mathcal{X})$ are direct from those of (3.1) by taking the complements. ☐

Remark 3.3.4. For any natural number $n \geqslant 1$, any Polish space $\mathcal{X}$ and each $A \in \utilde{\mathbf{\Sigma}}^0_n(\mathcal{X} \times \mathcal{X})$ the sets $A_i \subseteq \mathcal{X}$, $i \in \mathbb{N}$, defined as follows

$$x \in A_i \iff (x, i) \in A$$

also belong to the pointclass $\utilde{\mathbf{\Sigma}}^0_n$.

Lemma 3.3.5. *Consider a natural number $n \geqslant 1$, a Polish space $\mathcal{X}$, and a sequence of sets $(B_i)_{i\in\mathbb{N}}$ belonging to the family $\utilde{\mathbf{\Sigma}}^0_n(\mathcal{X})$. Then the set $B \subseteq \mathcal{X} \times \mathbb{N}$ defined as follows*

$$(x, i) \in B \iff x \in B_i$$

belongs to the pointclass $\utilde{\mathbf{\Sigma}}^0_n$.

The above also holds if we replace the class $\utilde{\mathbf{\Sigma}}^0_n$ with $\utilde{\mathbf{\Pi}}^0_n$.

Proposition 3.3.6. *(Equivalent Definition of $\utilde{\mathbf{\Sigma}}^0_n$ Pointclasses) For each $n \geqslant 1$ we have*

$$\utilde{\mathbf{\Sigma}}^0_{n+1} = \exists^{\mathbb{N}} \utilde{\mathbf{\Pi}}^0_n \quad \textit{and hence} \quad \utilde{\mathbf{\Pi}}^0_{n+1} = \forall^{\mathbb{N}} \utilde{\mathbf{\Sigma}}^0_n.$$

Proof. We temporarily define the classes

$$\begin{aligned} \utilde{\mathbf{\Sigma}}^*_1 &= \text{the pointclass of all open sets} &= \utilde{\mathbf{\Sigma}}^0_1 \\ \utilde{\mathbf{\Pi}}^*_1 &= \text{the pointclass of all closed sets} &= \utilde{\mathbf{\Pi}}^0_1 \end{aligned}$$

and

$$\begin{aligned} \utilde{\mathbf{\Sigma}}^*_{n+1} &= \exists^{\mathbb{N}} \utilde{\mathbf{\Pi}}^*_n \\ \utilde{\mathbf{\Pi}}^*_{n+1} &= \mathrm{c}\utilde{\mathbf{\Sigma}}^*_{n+1}. \end{aligned}$$

We will show by induction on $n \geqslant 1$ that

$$\utilde{\mathbf{\Sigma}}^0_n(\mathcal{X}) = \utilde{\mathbf{\Sigma}}^*_n(\mathcal{X}) \quad \text{and} \quad \utilde{\mathbf{\Pi}}^0_n(\mathcal{X}) = \utilde{\mathbf{\Pi}}^*_n(\mathcal{X}).$$

The latter equality is derived from the former equality by taking the complements. For $n = 1$ the requirement is straightforward by definition. We assume that for some $n \geqslant 1$ we have the requirement and show the same for $n+1$.

Let $P \subseteq \mathcal{X}$ in $\undertilde{\mathbf{\Sigma}}^*_{n+1}$. Then there exists $A \in \undertilde{\mathbf{\Pi}}^*_n(\mathcal{X} \times \mathbb{N})$ with $P = \exists^{\mathbb{N}} A$. By the Inductive Hypothesis, we have that $A \in \undertilde{\mathbf{\Pi}}^0_n(\mathcal{X} \times \mathbb{N})$. We apply Remark 3.3.4 for the pointclass $\undertilde{\mathbf{\Pi}}^0_n$ and we have that for each $i \in \mathbb{N}$ the set

$$A_i = \{x : (x,i) \in A\}$$

belongs to $\undertilde{\mathbf{\Pi}}^0_n$. Moreover

$$\begin{aligned} x \in P &\iff \exists i \ (x,i) \in A \\ &\iff \exists i \ x \in A_i, \end{aligned}$$

therefore

$$P = \bigcup_{i \in \mathbb{N}} A_i.$$

It follows that P belongs to $\undertilde{\mathbf{\Sigma}}^0_{n+1}(\mathcal{X})$. Conversely, consider a $P \subseteq \mathcal{X}$ belonging to $\undertilde{\mathbf{\Sigma}}^0_{n+1}$ and $(B_i)_{i \in \mathbb{N}}$ a sequence of elements of $\undertilde{\mathbf{\Pi}}^0_n(\mathcal{X})$ such that

$$P = \bigcup_{i \in \mathbb{N}} B_i.$$

By Lemma 3.3.5 the set

$$B = \{(x,i) : x \in B_i\}$$

also belongs to $\undertilde{\mathbf{\Pi}}^0_n$. By the Inductive Hypothesis, $B \in \undertilde{\mathbf{\Pi}}^*_n(\mathcal{X} \times \mathbb{N})$. Moreover

$$\begin{aligned} x \in P &\iff \exists i \ x \in B_i \\ &\iff \exists i \ (x,i) \in B, \end{aligned}$$

and therefore $P = \exists^{\mathbb{N}} B \in \undertilde{\mathbf{\Sigma}}^*_{n+1}(\mathcal{X})$. $\square$

Theorem 3.3.7 ([**6**], [**8**]). *(The Fundamental Closure Properties of Borel classes of finite order). The pointclasses $\undertilde{\mathbf{\Sigma}}^0_n$, $\undertilde{\mathbf{\Pi}}^0_n$ and $\undertilde{\mathbf{\Delta}}^0_n$ where $n \geqslant 1$, are closed under continuous substitution as well as under the operators $\vee$, &, $\exists^{\leqslant}$, $\forall^{\leqslant}$, $\bigvee_{\leqslant}$, $\bigwedge_{\leqslant}$.*

- *The classes $\undertilde{\mathbf{\Sigma}}^0_n$ are additionally closed under the operators $\bigvee_{\mathbb{N}}$, $\exists^{\mathbb{N}}$ and more generally $\exists^Y$, where Y is a countable Polish space.*
- *The classes $\undertilde{\mathbf{\Pi}}^0_n$ are additionally closed under the operators $\bigwedge_{\mathbb{N}}$, $\forall^{\mathbb{N}}$ and more generally $\forall^Y$, where Y is a countable Polish space.*
- *The classes $\undertilde{\mathbf{\Delta}}^0_n$ are additionally closed under the complement operator* c.

3.4. The Projective Pointclasses of Finite Order

Definition 3.4.1. We define the **Projective pointclasses of finite order** (for subsets of Polish spaces) by recursion for $n \geqslant 1$ as follows:

$$\begin{aligned} \undertilde{\mathbf{\Sigma}}^1_1 &= \exists^{\mathcal{N}} \undertilde{\mathbf{\Pi}}^0_1 \\ &= \text{the projections of closed sets } F \subseteq \mathcal{X} \times \mathcal{N} \text{ over } \mathcal{X}, \text{where } \mathcal{X} \text{ is a Polish space,} \\ \undertilde{\mathbf{\Pi}}^1_1 &= \mathrm{c}\undertilde{\mathbf{\Sigma}}^1_1 \\ &= \text{the complements of sets of } \undertilde{\mathbf{\Sigma}}^1_1, \end{aligned}$$

and

$$\begin{aligned} \undertilde{\mathbf{\Sigma}}^1_{n+1} &= \exists^{\mathcal{N}} \undertilde{\mathbf{\Pi}}^1_n \\ &= \text{the projections of } \undertilde{\mathbf{\Pi}}^1_n \text{ sets } F \subseteq \mathcal{X} \times \mathcal{N} \text{ over } \mathcal{X}, \text{where } \mathcal{X} \text{ is a Polish space,} \\ \undertilde{\mathbf{\Pi}}^1_{n+1} &= \mathrm{c}\undertilde{\mathbf{\Sigma}}^1_{n+1} \\ &= \text{the complements of sets of } \undertilde{\mathbf{\Sigma}}^1_{n+1}. \end{aligned}$$

Remark 3.4.2. It is quite convenient to set $\undertilde{\mathbf{\Pi}}^0_1 = \undertilde{\mathbf{\Pi}}^1_0$ and $\undertilde{\mathbf{\Sigma}}^1_0 = \undertilde{\mathbf{\Sigma}}^0_1$ so that

$$\undertilde{\mathbf{\Sigma}}^1_n = \exists^{\mathcal{N}} \undertilde{\mathbf{\Pi}}^1_{n-1} \quad \text{and} \quad \undertilde{\mathbf{\Pi}}^1_n = \forall^{\mathcal{N}} \undertilde{\mathbf{\Sigma}}^1_{n-1}, \text{ for each } n \geqslant 1.$$

Finally, we set

$$\undertilde{\mathbf{\Delta}}^1_n = \undertilde{\mathbf{\Sigma}}^1_n \cap \undertilde{\mathbf{\Pi}}^1_n.$$

It follows that a $P \subseteq \mathcal{X}$ belongs to $\utilde{\boldsymbol{\Delta}}^1_n$ if and only if the sets P and $\mathcal{X} \setminus P$ belong to $\utilde{\boldsymbol{\Sigma}}^1_n$. The pointclasses $\utilde{\boldsymbol{\Sigma}}^1_n$ are called **projective** or otherwise **pointclasses of Lusin** while the $\utilde{\boldsymbol{\Pi}}^1_n$ and $\utilde{\boldsymbol{\Delta}}^1_n$ are the **dual** and **ambiguous projective pointclasses**, respectively. The collection of the preceding pointclasses is called the **Hierarchy of projective sets** and is visualized in the Diagram 3.2, below. The sets of pointclass $\utilde{\boldsymbol{\Sigma}}^1_1$ are called **analytic** and those of the pointclass $\utilde{\boldsymbol{\Pi}}^1_1$ are called **coanalytic**. Also, the sets of the pointclass $\utilde{\boldsymbol{\Delta}}^1_1$ are called **bi-analytic**.

$$
\begin{tikzcd}
& \utilde{\boldsymbol{\Sigma}}^1_1(\mathcal{X}) \arrow[dr, phantom, sloped, "\subseteq"] & & \utilde{\boldsymbol{\Sigma}}^1_2(\mathcal{X}) \arrow[dr, phantom, sloped, "\subseteq"] & & \utilde{\boldsymbol{\Sigma}}^1_n(\mathcal{X}) \arrow[dr, phantom, sloped, "\subseteq"] & \\
\utilde{\boldsymbol{\Delta}}^1_1(\mathcal{X}) \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \utilde{\boldsymbol{\Delta}}^1_2(\mathcal{X}) \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \cdots \; \utilde{\boldsymbol{\Delta}}^1_n(\mathcal{X}) \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \cdots \\
& \utilde{\boldsymbol{\Pi}}^1_1(\mathcal{X}) \arrow[ur, phantom, sloped, "\subseteq"] & & \utilde{\boldsymbol{\Pi}}^1_2(\mathcal{X}) \arrow[ur, phantom, sloped, "\subseteq"] & & \utilde{\boldsymbol{\Pi}}^1_n(\mathcal{X}) \arrow[ur, phantom, sloped, "\subseteq"] &
\end{tikzcd}
$$

Diagram 3.2. The Hierarchy of projective subsets of $\mathcal{X}$ of finite order.

Proposition 3.4.3. *For every Polish space $\mathcal{X}$ and every $n \geqslant 1$ we have*

$$\utilde{\boldsymbol{\Sigma}}^1_n(\mathcal{X}) \subseteq \utilde{\boldsymbol{\Delta}}^1_{n+1}(\mathcal{X}) \quad \textit{and also} \quad \utilde{\boldsymbol{\Pi}}^1_n(\mathcal{X}) \subseteq \utilde{\boldsymbol{\Delta}}^1_{n+1}(\mathcal{X}).$$

Proof. The second inclusion follows from the first by taking the complement of the set. For the first inclusion, we will show that for each Polish space $\mathcal{X}$ every closed subset F of $\mathcal{X}$ is $\utilde{\boldsymbol{\Delta}}^1_1$. We first show the inclusion $\utilde{\boldsymbol{\Sigma}}^1_n(\mathcal{X}) \subseteq \utilde{\boldsymbol{\Pi}}^1_{n+1}(\mathcal{X})$. Consider $P \subseteq \mathcal{X}$ belonging to $\utilde{\boldsymbol{\Sigma}}^1_n$ and $Q \subseteq \mathcal{X} \times \mathcal{N}$ belonging to $\utilde{\boldsymbol{\Pi}}^1_{n-1}$, such that $P = \exists^{\mathcal{N}} Q$. We set $R = Q \times \mathcal{N}$ and we claim that R is also $\utilde{\boldsymbol{\Pi}}^1_{n-1}$ set. For each $(x, \alpha, \beta) \in \mathcal{X} \times \mathcal{N} \times \mathcal{N}$ we have

$$(x, \alpha, \beta) \in R \iff (x, \alpha) \in Q$$

and R belongs to $\utilde{\boldsymbol{\Pi}}^1_{n-1}$ due to the closure under continuous substitution. Moreover, for each $x \in \mathcal{X}$ we have

$$
\begin{aligned}
x \in P &\iff \exists \alpha \ (x, \alpha) \in Q \\
&\iff \forall \beta \, \exists \alpha \ (x, \alpha, \beta) \in R.
\end{aligned}
$$

Thus P belongs to the pointclass $\forall^{\mathcal{N}} \exists^{\mathcal{N}} \utilde{\boldsymbol{\Pi}}^1_{n-1} = \forall^{\mathcal{N}} \utilde{\boldsymbol{\Sigma}}^1_n = \utilde{\boldsymbol{\Pi}}^1_{n+1}$.

We then show that each closed set is $\utilde{\boldsymbol{\Pi}}^1_1$ subset of the same space. Let $\mathcal{X}$ be a polynomial space and $F \subseteq \mathcal{X}$ closed. The set F, as closed, is also a G_δ therefore there is a sequence $(V_n)_{n \in \mathbb{N}}$ of open subsets of $\mathcal{X}$ with

$$F = \bigcap_{n \in \mathbb{N}} V_n.$$

Consider the set

$$V = \{(x, \alpha) \in \mathcal{X} \times \mathcal{N} \ : \ x \in V_{\alpha(0)}\}$$

It is easy to see that V is an open subset of $\mathcal{X} \times \mathcal{N}$. Moreover for every $x \in \mathcal{X}$,

$$
\begin{aligned}
x \in F &\iff \forall n \ x \in V_n \\
&\iff \forall \alpha \ x \in V_{\alpha(0)} \\
&\iff \forall \alpha \ (x, \alpha) \in V.
\end{aligned}
$$

Therefore F is a $\forall^{\mathcal{N}} \utilde{\boldsymbol{\Sigma}}^0_1 = \utilde{\boldsymbol{\Pi}}^1_1$. Then we show by induction on $n \geqslant 1$ that for every Polish space $\mathcal{X}$ we have

$$\utilde{\boldsymbol{\Sigma}}^1_n(\mathcal{X}) \subseteq \utilde{\boldsymbol{\Sigma}}^1_{n+1}(\mathcal{X}).$$

For $n = 1$, consider a Polish space $\mathcal{X}$ and $P \subseteq \mathcal{X}$ belonging to $\utilde{\boldsymbol{\Sigma}}^1_1$. Then there exists a closed set $F \subseteq \mathcal{X} \times \mathcal{N}$, such that $P = \exists^{\mathcal{N}} F$. We have shown that F as a closed set, is a $\utilde{\boldsymbol{\Pi}}^1_1$ subset of $\mathcal{X} \times \mathcal{N}$. Therefore P is an $\exists^{\mathcal{N}} \utilde{\boldsymbol{\Pi}}^1_1 = \utilde{\boldsymbol{\Sigma}}^1_2$ subset of $\mathcal{X}$. Assume that for some $n \geqslant 1$ and every Polish space $\mathcal{X}$ we have

$$\utilde{\boldsymbol{\Sigma}}^1_n(\mathcal{X}) \subseteq \utilde{\boldsymbol{\Sigma}}^1_{n+1}(\mathcal{X}), \quad \text{equivalently} \quad \utilde{\boldsymbol{\Pi}}^1_n(\mathcal{X}) \subseteq \utilde{\boldsymbol{\Pi}}^1_{n+1}(\mathcal{X}).$$

Let $\mathcal{X}$ be a Polish space and a set $P \subseteq \mathcal{X} \in \utilde{\boldsymbol{\Sigma}}^1_{n+1}$. We will show that P is $\utilde{\boldsymbol{\Sigma}}^1_{n+2}$ subset of $\mathcal{X}$. By definition, there exists a $\utilde{\boldsymbol{\Pi}}^1_n$ set $Q \subseteq \mathcal{X} \times \mathcal{N}$ with $P = \exists^{\mathcal{N}} Q$.

By the Inductive Hypothesis applied to the space $\mathcal{X} \times \mathcal{N}$, the set Q is a $\underset{\sim}{\mathbf{\Pi}}^1_{n+1}$. Therefore, P is a $\exists^{\mathcal{N}} \underset{\sim}{\mathbf{\Pi}}^1_{n+1} = \underset{\sim}{\mathbf{\Sigma}}^1_{n+2}$ subset of $\mathcal{X}$. □

Theorem 3.4.4 ([**11**], [**8**]). *(The Fundamental Closure Properties of Lusin's Classes). The pointclasses $\underset{\sim}{\mathbf{\Sigma}}^1_n$, $\underset{\sim}{\mathbf{\Pi}}^1_n$ and $\underset{\sim}{\mathbf{\Delta}}^1_n$, where $n \geqslant 1$, are closed under continuous substitution and the operators* $\vee$, &, $\exists^{\leqslant}$, $\forall^{\leqslant}$, $\bigvee_{\leqslant}$, $\bigwedge_{\leqslant}$, $\exists^{\mathbb{N}}$, $\forall^{\mathbb{N}}$, $\bigvee_{\mathbb{N}}$, $\bigwedge_{\mathbb{N}}$.

- *The pointclasses $\underset{\sim}{\mathbf{\Sigma}}^1_n$ are further closed under the operator $\exists^Y$, where Y is a Polish space.*
- *The pointclasses $\underset{\sim}{\mathbf{\Pi}}^1_n$ are additionally closed under the operator $\forall^Y$, where Y is a Polish space.*
- *The pointclasses $\underset{\sim}{\mathbf{\Delta}}^1_n$ are additionally closed under the complement operator* c.

Corollary 3.4.5. *(Equivalent Definition of $\underset{\sim}{\mathbf{\Sigma}}^1_n$ Pointclasses). For each $n \geqslant 1$ the pointclass $\underset{\sim}{\mathbf{\Sigma}}^1_n$ consists exactly of the continuous images of $\underset{\sim}{\mathbf{\Pi}}^1_{n-1}$ sets.*

Proof. Each $\underset{\sim}{\mathbf{\Sigma}}^1_n$ set is a projection (and therefore a continuous image) of a $\underset{\sim}{\mathbf{\Pi}}^1_{n-1}$ set, so we need to show the converse. Let

$$f : \mathcal{X} \to \mathcal{Y}$$

continuous and $Q \subseteq \mathcal{X}$, which is a $\underset{\sim}{\mathbf{\Pi}}^1_{n-1}$. We need to show that the image $f[Q]$ is a $\underset{\sim}{\mathbf{\Sigma}}^1_n$ set. Indeed, for each $y \in \mathcal{Y}$ we have

$$\begin{aligned} y \in f[Q] &\iff \exists x \ (x \in Q \ \& \ f(x) = y) \\ &\iff \exists x \ (x \in Q \ \& \ (x, y) \in \text{Graph}(f)). \end{aligned}$$

The graph of f is a closed set and therefore by Proposition 3.4.3 (and its proof) is a $\underset{\sim}{\mathbf{\Pi}}^1_{n-1}$ set. By Theorem 3.4.4, the pointclass $\underset{\sim}{\mathbf{\Pi}}^1_{n-1}$ is closed under the operator &. In case $n = 1$, the previous holds from the closure properties of closed sets. Based on the last of the above equivalences we conclude that

$$f[Q] = \exists^{\mathcal{X}} R,$$

where R is a $\underset{\sim}{\mathbf{\Pi}}^1_{n-1}$ subset of $\mathcal{Y} \times \mathcal{X}$. Therefore, $f[Q]$ is a $\underset{\sim}{\mathbf{\Sigma}}^1_n$ subset of $\mathcal{Y}$. □

Corollary 3.4.6. *The continuous image $\underset{\sim}{\mathbf{\Sigma}}^1_n$ set is a $\underset{\sim}{\mathbf{\Sigma}}^1_n$ set.*

Proof. By Corollary 3.4.5 we know that the given $\underset{\sim}{\mathbf{\Sigma}}^1_n$ set is a continuous image of a $\underset{\sim}{\mathbf{\Pi}}^1_{n-1}$ set. Since the composition of continuous functions is a continuous function, it follows that the continuous image of the given $\underset{\sim}{\mathbf{\Sigma}}^1_n$ set is a continuous image $\underset{\sim}{\mathbf{\Pi}}^1_{n-1}$ set, and again by Corollary 3.4.5 is a $\underset{\sim}{\mathbf{\Sigma}}^1_n$ set. □

CHAPTER 4

Trees and The Perfect Set Theorem

In Descriptive Set Theory, a tree is a way of thinking about sets and collections of sets. Each set is like a tree, with the main "trunk" being the starting set. Then, each branch of the trunk leads to more sets, which branch off into even more sets. This tree of sets can get very big, and it can help us understand how different sets relate to each other. Overall, a tree is a way of organizing sets into a structure that shows how they are all related to each other. In the present chapter, we will describe the notions of trees and trees of pairs and we will provide relevant terminology. In addition, we will refer to two important theorems, the "Cantor-Bendixon Theorem" and the "Perfect Set Theorem".

4.1. Trees

For our purposes, a tree on a (non-empty) set X is a set T of finite sequences of members of X, such that if $u \in T$ and w is an initial segment of u, then $w \in T$.

Definition 4.1.1. Let X be a non-empty set. A $T \subseteq X^{<\mathbb{N}}$ is a **tree** in X if it is non-empty and closed downwards concerning the order $\sqsubseteq$, i.e.

$$\text{if } w \sqsubseteq u \text{ and } u \in T, \text{ then } w \in T.$$

For example the sets $X^{<\mathbb{N}}$ and $\{\Lambda\}$ are trees in X. Another example is

$$T = \{\Lambda, (a), (a,b), (a,c), (d)\},$$

for some $a, b, c, d \in X$.

Remark 4.1.2. Note that the empty sequence Λ belongs to every tree because $\Lambda \sqsubseteq u$ for every $u \in T \neq \varnothing$. Also, we often call the members of T *nodes* or *finite paths*.

In the following definitions, consider that we have a tree T:

Definition 4.1.3. The elements of T are called **nodes** or **leaves** of T.The empty sequence Λ is a node of every non-empty tree, so we call Λ **root** of T.

Definition 4.1.4. A node u of T is called **terminal** if it has no strict extension w within T, i.e.

$$\text{for every } w \in T \text{ with } u \sqsubseteq w, \text{ we have } u = w.$$

A tree T is called **pruned** if it has no terminal nodes.

Definition 4.1.5. We call an **infinite branch** of T a function $f : \mathbb{N} \to X$ with the property

$$(f(0), \dots, f(n)) \in T, \text{ for every } n \in \mathbb{N}.$$

The set of all infinite branches of T is called **body** of T and denoted by $[T]$. A tree T is called a **well-founded** if $[T] = \varnothing$ and **ill-founded** if $[T] \neq \varnothing$.

Definition 4.1.6. A tree S in X is called a **subtree** of T if $S \subseteq T$. For every $u \in X^{<\mathbb{N}}$ we define the **subtree T_u of the sequences that are compatible with** u, as follows

$$T_u = \{w \in T \;:\; u \| w\}.$$

In other words $w \in T_u$ if and only if $w \in T$ and

$$\text{either } w \sqsubseteq u \text{ or } u \sqsubseteq w.$$

The above definition is more interesting when $u \in T$, otherwise, T_u consists only of the strict initial parts of u belonging to T, so we may have $T_u = \{\Lambda\}$.

Trees and Topology.

It is clear that the body of a tree T in X is a subset of $X^{\mathbb{N}}$. If we consider in X the discrete topology then $X^{\mathbb{N}}$ is a metric space. As with Baire space a basis for the topology of $X^{\mathbb{N}}$ is the family of all sets of the form

$$\{x_0\} \times \cdots \times \{x_{n-1}\} \times X \times X \times \ldots$$

where $x_0, \ldots, x_{n-1} \in X$.

Remark 4.1.7. A sequence $(f_i)_{i\in\mathbb{N}}$ in $X^{\mathbb{N}}$ **converges to** $f \in X^{\mathbb{N}}$ if and only if for every $n \in \mathbb{N}$ the sequence $(f_i(n))_{i\in\mathbb{N}}$ converges to $f(n)$ in X. Equivalently, for every $n \in \mathbb{N}$ we have $f_i(n) = f(n)$, for all large i. The bodies of trees in a set X characterize the closed sets of $X^{\mathbb{N}}$.

Proposition 4.1.8. *Let X be a non-empty set with the discrete metric and $F \subseteq X^{\mathbb{N}}$. Then F is closed under the product topology of $X^{\mathbb{N}}$, if and only if, there is a tree T in X with $F = [T]$.*

Proof. For the straight direction, we assume that F is a closed subset of $X^{\mathbb{N}}$. If F is the empty set, then we choose for T any finite branching tree with an empty body, e.g. Λ. Therefore, we assume that $F \neq \varnothing$. If $u \in X^{\mathbb{N}}$ and $f \in X^{\mathbb{N}}$, we write

$$u \sqsubseteq f, \text{ when } f(k) = u(k) \text{ for every } k < |u|.$$

We define $T \subseteq X^{\mathbb{N}}$ as follows:

$$u \in T \iff \exists f \in F \ : \ u \sqsubseteq f.$$

Since $F \neq \varnothing$ we have $\Lambda \in T$ and therefore $T \neq \varnothing$. It is straightforward by definition that T is a tree.

We show that $F = [T]$. Let $g \in F$ and $n \in \mathbb{N}$. It is obvious that there exists $f \in F$ with $(g(0), \ldots, g(n)) \sqsubseteq f$, in particular we can get $f = g$. So $(g(0), \ldots, g(n)) \in T$ for every $n \in \mathbb{N}$ and $g \in [T]$.

Conversely, if $g \in [T]$ then for each $n \in \mathbb{N}$ we have $(g(0), \ldots, g(n)) \in T$ and hence there exists $f_n \in F$ with $(g(0), \ldots, g(n)) \in f_n$.

The sequence $(f_n)_{n\in\mathbb{N}}$ converges to the element g. Indeed, for every $k \in \mathbb{N}$, we get $n_0 = k$ and for every $n \geqslant n_0$ we have $f_n(k) = g(k)$ because $(g(0), \ldots, g(n)) \sqsubseteq f_n$. Hence, $f_n(k) \xrightarrow{X} g(k)$ for each $k \in \mathbb{N}$ and therefore $f_n \xrightarrow{X^{\mathbb{N}}} g$. Since $f_n \in F$ for each n and F is closed it follows that $g \in F$. Therefore $F = [T]$ and we have proved the straight direction.

For the reverse direction we consider a sequence of $(f_i)_{i\in\mathbb{N}}$ elements of $[T]$ which converges to $f \in X^{\mathbb{N}}$. We will show that $f \in T$. For every n, there exists i_0 such that for each $i \geqslant i_0$ and each $k \leqslant n$ we have $f_i(k) = f(k)$. In particular, $(f(0), \ldots, f(n)) \sqsubseteq f_{i_0}$. By definition, $(f(0), \ldots, f(n)) \in T$. It follows that $f \in [T]$. $\square$

Definition 4.1.9. Let X be a non-empty set and T a tree in X. T is **finite branching** if for every $u \in T$ there are up to finite $w \in T$ which are direct extensions of u, i.e. for each $u \in T$ there are $x_0, \ldots, x_{n-1} \in X$ such that

$$\forall x \ \ (u * (x) \in T \iff x \in \{x_0, \ldots, x_{n-1}\}).$$

The preceding n can take any large values. We can even have $n = 0$, in which case the above equivalence means that the node $u \in T$ is terminal.

Remark 4.1.10. A classical example of a finite branching tree is the set $\{0,1\}^{<\mathbb{N}}$ of all **finite binary sequences**. Obviously the body of $\{0,1\}^{<\mathbb{N}}$ is the Cantor space $2^{\mathbb{N}}$.

Trees and Continuous Functions.

Continuous functions between bodies can be approximated by functions between the corresponding tree bodies.

Definition 4.1.11. Given two trees S and T on a non-empty set X and a function $\phi : S \to T$. The function ϕ is called **monotone** if for every $u, v \in S$ with $u \sqsubseteq v$ we have $\phi(u) \sqsubseteq \phi(v)$.

We say that ϕ is **proper** if for every $f \in [S]$ the lengths of $\phi(f|n)$ acquire any length, i.e. for every $M \in \mathbb{N}$ there is $n \in \mathbb{N}$ with
$$|\phi(f|n)| \geqslant M.$$
Clearly, if we have a proper monotone function $\phi : S \to T$, then for every $f \in [S]$ we have
$$\phi((f(0))) \sqsubseteq \phi((f(0), f(1))) \sqsubseteq \cdots \sqsubseteq \phi((f(0), f(1)\ldots, f(n)))) \sqsubseteq \ldots$$
and that the union of all these branches produces an infinite branch of T. Therefore, we define the function
$$\phi^* : [S] \to [T] : \phi^*(f) = \bigcup_{n\in\mathbb{N}} \phi(f|n).$$
In other words, ϕ^* satisfies
$$\phi^*(f)(m) = x \iff \exists n\, (|(f|n)| > m \;\&\; (f|n)(m) = x),$$
for every $f \in [T]$, $m \in \mathbb{N}$, and $x \in X$.

Proposition 4.1.12. *Consider a non-empty set X with the discrete topology and two trees S and T on X, with S pruned. Then a function $\Phi : [S] \to [T]$ is continuous if and only if there exists a proper monotone $\phi : S \to T$ with $\Phi = \phi^*$.*

Proof. In the proof we denote $V_u = \{g \in X^{\mathbb{N}} : u \sqsubseteq g\}$, where $u \in X^{<\mathbb{N}}$. These V_u, $u \in X^{<\mathbb{N}}$, are the basis for $X^{\mathbb{N}}$.

We first show the reverse direction. Consider a proper monotone $\phi : S \to T$ and we show that $\phi^* : [S] \to [T]$ is continuous. For each $u \in X^{<\mathbb{N}}$ and each $f \in [S]$ we have
$$\phi^*(f) \in V_u \iff u \sqsubseteq \phi^*(f) \iff \exists n\, u \sqsubseteq \phi(f|n).$$
Therefore,
$$(\phi^*)^{-1}[V_u] = \left(\bigcup_{n\in\mathbb{N}} A_n\right) \cap [S],$$
where
$$A_n = \{f \in X^{\mathbb{N}} : u \sqsubseteq \phi(f|n)\}$$
for every $n \in \mathbb{N}$. If $f \in A_n$ and $h \in X^{\mathbb{N}}$ with $h|n = f|n$, then $h \in A_n$, hence A_n is an open subset of $X^{\mathbb{N}}$ for every $n \in \mathbb{N}$. Hence $(\phi^*)^{-1}[V_u]$ is an open set in $[S]$ and ϕ^* is a continuous function. (Note that we did not assume that S is pruned.)
Conversely, consider a continuous function $\Phi : [S] \to [T]$. The idea is to define ϕ such that we have
$$\phi[V_u \cap [S]] \subseteq V_{\phi(u)}, \text{ for every } u \in S.$$
First, observe that for each $u \in S$ and $w_1, w_2 \in T$ if
$$\Phi[V_u \cap [S]] \subseteq V_{w_1} \cap V_{w_2},$$
then $w_1 \parallel w_2$. This is the case because S is pruned and hence for each $u \in S$ we have $V_u \cap [S] \neq \varnothing$, hence $V_{w_1} \cap V_{w_2} \neq \varnothing$, which only happens when $w_1 \parallel w_2$. Moreover, for each $u \in T$ there is $w \in S$ with $\Phi[V_u \cap [S]] \subseteq V_w$, in particular $w = \Lambda$. Hence for every $u \in S$ and every $N \in \mathbb{N}$ there exists the "largest" sequence $w \in S$ with length $\leqslant N$ and $\Phi[V_u \cap [S]] \subseteq V_w$, i.e. w satisfies the last two properties, and for each $w' \in S$ satisfying the last two properties, $w' \sqsubseteq w$.

We define
$$\phi : S \to T : \phi(u) = \text{ the longest sequence } w \in S \text{ with } |w| \leqslant |u| \text{ and } \Phi[V_u \cap [S]] \subseteq V_w.$$
If $u_1 \sqsubseteq u_2 \in S$, then $|w_1| \leqslant |u_1| \leqslant |u_2|$, moreover
$$\Phi[V_{u_2} \cap [S]] \subseteq \Phi[V_{u_1} \cap [S]] \subseteq V_{\phi(u_1)}.$$
Therefore, $\phi(u_1) \in T$ is a w' satisfying $|w'| \leqslant |u_2|$ and $\Phi[V_{u_2} \cap [S]] \subseteq V_{w'}$. By the definition of $\phi(u_2)$ we have that $\phi(u_1) \sqsubseteq \phi(u_2)$.

To show the remaining properties for ϕ we consider an $f \in [S]$ and $m \in \mathbb{N}$. Since Φ is continuous on f there exists $n \in \mathbb{N}$ with
$$\Phi[V_{f|n} \cap [S]] \subseteq V_{\Phi(f)|m}.$$

We can assume that $n \geqslant M$ and therefore

$$|\Phi(f)|m| \;=\; m \;\leqslant\; n \;=\; |f|n|.$$

Therefore, by definition of $\phi(f|n)$ it holds that $\Phi(f)|m \sqsubseteq \phi(f|n)$. In particular, $\phi(f|n)$ has length at least $|\Phi(f)|m| = m$.

The above shows that ϕ is suitable and furthermore that $\phi^* = \Phi$. To grasp the latter, note that for every m there is n as above, i.e. $n \geqslant m$ and $\Phi(f)|m \sqsubseteq \phi(f|n)$. Since $\phi(f|n) \sqsubseteq \phi^*(f)$ it follows that $\Phi(f)|m \sqsubseteq \phi^*(f)$ for each m and hence $\Phi(f) = \phi^*(f)$.

□

The Space of Trees.

From now on, we will deal with trees in natural numbers, and we will be interested in the special case where $X = \mathbb{N}$.

Definition 4.1.13. (The Space of Trees Tr). Consider the set Tr of all trees in $\mathbb{N}$, as follows

$$\mathrm{Tr} = \{T \subseteq \mathbb{N}^{\mathbb{N}} : T \text{ is a tree in } \mathbb{N}\}.$$

We also consider an enumeration of $\mathbb{N}^{\mathbb{N}}$, for example the natural enumeration $(u_s)_{s\in\mathbb{N}}$, which we have stabilized in Definition 2.3.9.

Then to each tree T in $\mathbb{N}$ corresponds $\alpha_T \in 2^{\mathbb{N}}$ with

$$\alpha_T(s) = 1 \iff u_s \in T.$$

The function

$$F : \mathrm{Tr} \to 2^{\mathbb{N}} : F(T) = \alpha_T$$

is a monomorphism. We consider Tr with the *topology* obtained from F, i.e. an open $V \subseteq \mathrm{Tr}$ if and only if there exists an open $W \subseteq 2^{\mathbb{N}}$ with $V = F^{-1}[W]$. This is the minimum topology in Tr under which F is continuous.

A *compatible metric* on Tr is

$$d(T, S) = d_{\mathcal{N}}(F(T), F(S)) = d_{\mathcal{N}}(\alpha_T, \alpha_S).$$

The set Tr with the previous topology is the **Space of Trees** in $\mathbb{N}$.

Proposition 4.1.14. *The Space of Trees* Tr *is a compact Polish space.*

Proof. Consider the representation

$$F = (T \in \mathrm{Tr} \mapsto \alpha_T \in 2^{\mathbb{N}})$$

and show that the set $F[\mathrm{Tr}]$ is a closed subset of $2^{\mathbb{N}}$. Since $2^{\mathbb{N}}$ is compact, it follows that that $F[\mathrm{Tr}]$ is a compact set, equivalently $F[\mathrm{Tr}]$ with the relevant topology is compact topological space. The space $2^{\mathbb{N}}$ is topologically isomorphic to Tr via F, hence Tr is compact. We then show that the complement of $F[\mathrm{Tr}]$ is an open subset of $2^{\mathbb{N}}$. Let $\alpha \in 2^{\mathbb{N}}$ with

$$\alpha \notin \{\alpha_T : T \in \mathrm{Tr}\}$$

and $s_0 \in \mathbb{N}$ with $u_{s_0} = \Lambda$. If $\alpha(s_0) = 0$, then for any $\beta \in T$ with

$$\beta(s_0) = \alpha(s_0)$$

we have

$$\beta \notin \{\alpha_T : T \in \mathrm{Tr}\}.$$

(Otherwise $\beta = \alpha_T$ and the empty sequence would not belong to T.) Therefore we assume that $\alpha(s_0) = 1$. If for every s with $\alpha(s) = 1$ and every t with $u_t \sqsubseteq u_s$ it holds that $\alpha(t) = 1$, then $\alpha = \alpha_T$, where T is the tree defined as follows:

$$T = \{u \in \mathbb{N}^{<\mathbb{N}} : \exists s\, (u = u_s \;\&\; \alpha(s) = 1)\}.$$

But this is a contradiction because we have assumed that

$$\alpha \notin \{\alpha_T : T \in \mathrm{Tr}\}.$$

Therefore, there exist s and t with $\alpha(s) = 1$, $u_t \sqsubseteq u_s$ and $\alpha(t) = 0$. Then for every $\beta \in 2^{\mathbb{N}}$ with

$$\beta(i) = \alpha(i)$$

for each $i \leqslant \max\{t, s\}$ we have

$$\beta \notin \{\alpha_T : T \in \mathrm{Tr}\}.$$

In each case the complement

$$2^{\mathbb{N}} \setminus \{\alpha_T : T \in \mathrm{Tr}\}$$

is an open subset of $2^{\mathbb{N}}$. □

Definition 4.1.15. We define the sets

$$\mathrm{WF} = \{T \in \mathrm{Tr} : [T] = \varnothing\}$$

and

$$\mathrm{IF} = \{T \in \mathrm{Tr} : [T] \neq \varnothing\}$$

of the **well-founded** and **ill-founded** trees, respectively, in $\mathbb{N}$.

We note that the sets WF and IF are the most fundamental $\underset{\sim}{\mathbf{\Pi}}^1_1$ (i.e. coanalytic) and $\underset{\sim}{\mathbf{\Sigma}}^1_1$ (i.e. analytic) subsets of Tr, respectively.

4.2. The Cantor-Bendixson Theorem

Proposition 4.2.1. *If (X, d) is a metric space and $P \subseteq X$ is perfect, then for every open V the set $\overline{V \cap P}$ is perfect (possibly empty).*

Proof. It is obvious that the set $\overline{V \cap P}$ is closed. We will show that it has no individual points. Let $y \in \overline{V \cap P}$ and $r > 0$. We will find an element y' of $\overline{V \cap P}$ with $y' \in B(y, r)$ and $y' \neq y$. Since $y \in \overline{V \cap P}$ we have

$$B_d(y, r) \cap (V \cap P) = (B_d(y, r) \cap V) \cap P \neq \varnothing.$$

Therefore, there exists $z \in P$ with $z \in U = B_d(y, r) \cap V$. Since P is perfect and U is open, there is $w \in U \cap P$ with $w \neq z$. Then one of w, z is different from y. The required y' is one of w, z depending on which one is different from y. □

It is known that a subset of a Polish space has the "Perfect Set Property" if it is countable or if it has a nonempty perfect subset. The following Theorem, known as "Cantor-Bendixson Theorem" proves this property for the closed subsets of a Polish space. Also, this Theorem gives the relationship between closed and perfect subsets of the Polish space. (We note that having the perfect set property is not the same as being a perfect set.)

Theorem 4.2.2. *(Cantor-Bendixson). For every closed subset C of a Polish space $\mathcal{X}$, there are two sets $P, S \subseteq C$, with P perfect (possibly empty), S countable, $P \cap S = \varnothing$ and $C = P \cup S$. Indeed, the above decomposition is unique, i.e. if P', S' are two disjoint subsets of C with P' perfect, S' countable and $P' \cup S' = C$ then $P' = P$ and $S' = S$.*

Proof. We consider the closed set $C \subseteq \mathcal{X}$ and a countable basis $(V_n)_{n \in \mathbb{N}}$ for the topology of $\mathcal{X}$. We define

$$P = \{x \in C : \forall n \text{ with } x \in V_n \text{ the set } V_n \cap C \text{ is uncountable}\}$$

and

$$S = C \backslash P.$$

It is clear that $P \cap S = \varnothing$ and $C = P \cup S$.

• We will show that S is a *countable* set. For each $x \in S$ we have $x \notin P$ and hence there exists $n \in \mathbb{N}$ such that $x \in V_n$ and the set $V_n \cap C$ is countable.

We define $n(x)$ to be the least such n and

$$I = \{n(x) \in \mathbb{N} : x \in S\}.$$

Then I is countable, as a subset of $\mathbb{N}$. Also

$$S \subseteq \bigcup_{n \in I} (V_n \cap C)$$

because $x \in V_{n(x)}$, for every $x \in S \subseteq C$. Since $V_n \cap C$ is countable for every $n \in I$, we have that the set S is contained in a countable union of countable sets. Therefore, S is countable.

• We then show that P is *perfect*, i.e. it is a closed set and it has no individual points.

Firstly, we will prove that P is a closed set. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in P that converges to $x \in \mathcal{X}$. Since $P \subseteq C$ and C is closed we have that $x \in C$. Moreover, for every $n \in \mathbb{N}$ with $x \in V_n$ there exists $i \in \mathbb{N}$ with $x_i \in V_n$. Since $x_i \in P$, the set $V_n \cap C$ is uncountable. Hence, $x \in P$ and P is closed.

Then, we will prove that it has no individual points. Consider $x \in P$ and $n \in \mathbb{N}$ with $x \in V_n$. The set

$$V_n \cap C = (V_n \cap P) \cup (V_n \cap S)$$

is uncountable while $V_n \cap S$ is countable as a subset of S. Therefore, $V_n \cap P$ is uncountable and in particular there is $y \in V_n \cap P$ with $y \neq x$. Hence, x is not an isolated point of P.

• Finally, we will show the *uniqueness*. We consider a perfect set P' and a countable S' with $P' \cap S' = \varnothing$ and

$$P' \cup S' = P \cup S.$$

We consider $x \in P'$, since $P' \subseteq P \cup S = C$ we have that $x \in C$. If we have $x \in V_n$ then, using a suitably small open ball of $\mathcal{X}$, we can find m such that $x \in V_m \subseteq \overline{V_m} \subseteq V_n$. Then $\overline{V_m \cap P'}$ is a non-empty closed subset of

$$\overline{V_m} \cap P' \subseteq V_n \cap C.$$

On the other hand, by 4.2.1, $\overline{V_m \cap P'}$ has no isolated points, so it is a non-empty perfect set. More specifically, it is uncountable (by Corollary 2.4.15,) and hence the superset $V_n \cap C$ is uncountable. Therefore, $x \in P$ and $P' \subseteq P$.

If $x \in S'$ then $x \notin P'$ and since the last set is closed there is an n with $x \in V_n$ and $V_m \cap P' = \varnothing$. Therefore,

$$V_m \cap C = V_m \cap S' \subseteq S'$$

and the set $V_m \cap C$ is countable. It follows that $x \in S$ and therefore $S' \subseteq S$. Equivalently, $P \subseteq P'$.

We conclude that $P = P'$ and $S = S'$. □

Definition 4.2.3. Let $\mathcal{X}$ be a Polish space and $C \subseteq \mathcal{X}$ closed. We consider the sets P, S as in the Cantor-Bendixson Theorem. That is, P is perfect, S is countable,

$$P \cap S = \varnothing \text{ and } P \cup S = C.$$

The unique P is the **perfect kernel** of C and the unique S is the **scattered part** of C.

Corollary 4.2.4. *For every uncountable Polish space $\mathcal{X}$ there exists a continuous monomorphism $\tau : 2^{\mathbb{N}} \rightarrowtail \mathcal{X}$.*

Proof. By the Cantor-Bendixson Theorem, the closed set $\mathcal{X}$ is decomposed into its perfect kernel P and its scattered part S. If we had $P = \varnothing$, then $\mathcal{X} = S$ would be a countable set, which is a contradiction. So, $P \neq \varnothing$. Since P is closed, it follows that it is a perfect Polish space. By Theorem 2.4.14, there is a continuous monomorphism

$$\tau : 2^{\mathbb{N}} \rightarrowtail P \subseteq \mathcal{X}.$$

So, τ is the required function. □

4.3. Trees of Pairs

Definition 4.3.1. Given a non-empty set X and a $T \subseteq (X \times X)^{<\mathbb{N}}$. We call T a **tree of pairs** in X if it is a tree in $X \times X$.

Let us present some notations below. There is an obvious identification between the elements

$$w \in (X \times X)^{<\mathbb{N}} \text{ and } (u, v) \in X^{<\mathbb{N}} \times X^{<\mathbb{N}}, \text{ with } |u| = |v|.$$

In particular, each $w \in T$ has the form

$$((u_0, v_0), \ldots, (u_{n-1}, v_{n-1})), \text{ where } u_i, v_i \in X, \text{ for each } i < n,$$

so if set $u = (u_0, \dots, u_{n-1})$ and $v = (v_0, \dots, v_{n-1})$ we can identify w with (u, v).

Conversely, any pair (u, v) with $u = (u_0, \dots, u_{n-1})$ and $v = (v_0, \dots, v_{n-1})$ can be identified with the finite sequence $w = ((u_0, v_0), \dots, (u_{n-1}, v_{n-1}))$.

Notation 1. We will denote the *elements* of a tree of pairs T by (u, v) where $u, v \in X^{<\mathbb{N}}$ with $|u| = |v|$. The empty sequence Λ is identified with the pair (Λ, Λ).

The previous identification extends to infinite sequences. There is an obvious *topological isomorphism* between $(X \times X)^{<\mathbb{N}}$ and $X^{<\mathbb{N}} \times X^{<\mathbb{N}}$, namely

$$(f, g) \in X^{<\mathbb{N}} \times X^{<\mathbb{N}} \mapsto ((f(0), g(0)), \dots, ((f(n), g(n)), \dots) \in X^{\mathbb{N} \times \mathbb{N}}.$$

So we can identify every infinite branch $h \in (X \times X)^{<\mathbb{N}}$ with a pair of infinite sequences of X namely (h_1, h_2), where

$$h = ((h_1(0), h_2(0)), \dots, (h_1(n), h_2(n)), \dots).$$

Notation 2. We denote the *infinite branches* of a pair tree by pairs (f, g).

The obvious relation holds:

$$(f, g) \in [T] \iff \forall n\, ((f(0), \dots, f(n)), (g(0), \dots, g(n)))) \in T,$$

for each tree of pairs T in X, where $f, g \in X^{\mathbb{N}}$.

Finally, we observe that this particular identification respects the relation of the initial part, i.e. for each

$$w = ((u_0, v_0), \dots, (u_{n-1}, v_{n-1})) \equiv (u, v)$$

and each

$$w' = ((u'_0, v'_0), \dots, (u'_{n-1}, v'_{n-1})) \equiv (u', v')$$

we have that

$$w' \sqsubseteq w \iff u' \sqsubseteq u \ \&\ v' \sqsubseteq v.$$

Notation 3. The pair trees in X are identified with the non-empty sets $R \subseteq X^{<\mathbb{N}} \times X^{<\mathbb{N}}$ satisfying

$$(u, v) \in R \ \&\ u' \sqsubseteq u \ \&\ v' \sqsubseteq v \implies (u', v') \in R,$$

for each $u, v, u', v' \in X^{<\mathbb{N}}$.

We will be concerned with the case where $X = \mathbb{N}$. If T is a tree of pairs in $\mathbb{N}$, the elements of the body $[T]$ are of the form (α, β), where $\alpha, \beta \in \mathcal{N}$.

Lemma 4.3.2. *An $F \subseteq \mathcal{N} \times \mathcal{N}$ is closed if and only if there is a tree of pairs T in $\mathbb{N}$ with $F = [T]$, where the body of a pair tree is understood by the above identification.*

Proof. We consider the set

$$\tilde{F} = \{\gamma \in (\mathbb{N} \times \mathbb{N})^{\mathbb{N}} \ :\ (\gamma_1, \gamma_2) \in F\}$$

where

$$\gamma = ((\gamma_1(0), \gamma_2(0)), \dots, (\gamma_1(n), \gamma_2(n)), \dots).$$

The sets F and $\tilde{F}$ are topological isomorphic, therefore F is closed if and only if $\tilde{F}$ is closed.

We apply Proposition 4.1.8 for $X = \mathbb{N} \times \mathbb{N}$ and obtain that F is closed if and only if there exists a tree of pairs T in $\mathbb{N}$ with

$$\gamma \in \tilde{F} \iff \forall n\ \gamma|n \in T.$$

Therefore,

$$\begin{aligned}
(\alpha, \beta) \in F &\iff ((\alpha(0), \beta(0)), \dots, (\alpha(n), \beta(n)), \dots) \in \tilde{F} \\
&\iff \forall n\ \ ((\alpha(0), \beta(0)), \dots, (\alpha(n), \beta(n))) \in T \\
&\iff \forall n\ \ (\alpha|n, \beta|n) \in T \ \text{ (with the previous matching)} \\
&\iff (\alpha, \beta) \in [T].
\end{aligned}$$

Thus, we have proved the claim. □

Proposition 4.3.3. *For each $P \subseteq \mathcal{N}$ the following is equivalent:*

i) *P is an analytic set, i.e. $\underset{\sim}{\mathbf{\Sigma}}^1_1$*

ii) *There exists a tree of pairs T with $P = \mathrm{pr}[\,[T]\,]$.*

By pr *we mean the function of the projection in the first coordinate $(\alpha, \beta) \in \mathcal{N} \times \mathcal{N} \mapsto \alpha \in \mathcal{N}$.*

Proof. **i)** $\Longrightarrow$ **ii)** There exists a closed $F \subseteq \mathcal{N} \times \mathcal{N}$ with $P = \exists^{\mathcal{N}} F$. By Lemma 4.3.2 there exists a tree of pairs T in $\mathbb{N}$ with $F = [T]$. Therefore, for each $\alpha \in \mathcal{N}$ we have

$$\begin{aligned} \alpha \in P &\iff \exists \beta (\alpha, \beta) \in F \\ &\iff \exists \beta (\alpha, \beta) \in [T] \\ &\iff \alpha \in \mathrm{pr}[\,[T]\,]. \end{aligned}$$

ii) $\Longrightarrow$ **i)** It is obvious, as $[T]$ is a closed subset of $\mathcal{N} \times \mathcal{N}$ by Lemma 4.3.2 and $P = \mathrm{pr}[\,[T]\,] = \exists^{\mathcal{N}} P$. $\square$

4.4. The Perfect Set Theorem

Recall that a set $P \subseteq \mathcal{X}$ is **analytic** if it is $\underset{\sim}{\mathbf{\Sigma}}^1_1$, i.e there exists a closed set $F \subseteq \mathcal{X} \times \mathcal{N}$ such that for every $x \in \mathcal{X}$,

$$x \in P \iff \exists \alpha\, (x, \alpha) \in F.$$

Lemma 4.4.1. *For each Polish space $\mathcal{X}$ there is a continuous surjection $\pi : \mathcal{N} \twoheadrightarrow \mathcal{X}$ and a G_δ set $P \subseteq \mathcal{N}$ such that $\pi[P] = \mathcal{X}$ and the restriction $\pi|P$ is one-to-one.*

Therefore, every Polish space is the continuous one-to-one image of a G_δ subset of Baire space.

Theorem 4.4.2. *(The Perfect Set Theorem). For each Polish space $\mathcal{X}$ and every uncountable analytic (i.e. $\underset{\sim}{\mathbf{\Sigma}}^1_1$) $A \subseteq \mathcal{X}$ there exists a continuous monomorphism $f : 2^{\mathbb{N}} \rightarrowtail \mathcal{X}$ with $f[2^{\mathbb{N}}] \subseteq A$.*

Proof. The idea is to find a kind of "perfect kernel" for A as in the proof of the Cantor-Bendixson Theorem. The $(V_n)_{n \in \mathbb{N}}$ basis of $\mathcal{X}$, as used in the proof of the latter result, is not sufficient for our purposes, as A is not necessarily closed. Instead, we will use the analytic set representation of A based on trees of pairs (Proposition 4.3.3).

To do this, however, we need to know that $\mathcal{X} = \mathcal{N}$. We will explain why we can assume this: Suppose that we have proved the result for all uncountable analytic subsets of $\mathcal{N}$ and consider an arbitrary Polish space $\mathcal{X}$. Then, by Lemma 4.4.1, there exists a continuous surjection $\pi : \mathcal{N} \twoheadrightarrow \mathcal{X}$ and and a G_δ set $B \subseteq \mathcal{N}$ such that $\pi[B] = \mathcal{X}$ and the restriction $\pi|B$ is one-to-one.

If we have an uncountable analytic set $A \subseteq \mathcal{X}$ then by the closure of the class $\underset{\sim}{\mathbf{\Sigma}}^1_1$ under continuous substitution and since B is Borel, $B \cap \pi^{-1}[A]$ is also an analytic set. Since $\pi[B] = \mathcal{X}$ and π is a surjection, we have that

$$\begin{aligned} \pi[B \cap \pi^{-1}[A]\,] &= \pi[B] \cap \pi[\pi^{-1}[A]]] \\ &= \pi[B] \cap A \\ &= \mathcal{X} \cap A \\ &= A, \end{aligned}$$

so if $B \cap \pi^{-1}[A]$ was countable, then A would also be countable as an image of a countable set via a function (we don't even need that $\pi|B$ is one-to-one). But this is a contradiction. So $B \cap \pi^{-1}[A]$ is an uncountable analytic subset of $\mathcal{N}$ and by our hypothesis there is continuous monomorphism $f : 2^{\mathbb{N}} \rightarrowtail \mathcal{X}$ with

$$f[2^{\mathbb{N}}] \subseteq B \cap \pi^{-1}[A].$$

Then the composition

$$g = \pi \circ f : 2^{\mathbb{N}} \to \mathcal{X} : g(\gamma) = \pi(f(\gamma))$$

is continuous, one-to-one since f takes values in B and $\pi|B$ is one-to-one, and furthermore

$$g[2^{\mathbb{N}}]\,] = \pi[f[2^{\mathbb{N}}]\,]\,] \subseteq \pi[B \cap \pi^{-1}[A]\,] = A.$$

So g is a continuous monomorphism from $2^{\mathbb{N}}$ to $\mathcal{X}$, with $f[2^{\mathbb{N}}] \subseteq A$. Therefore, we can then assume that $\mathcal{X} = \mathcal{N}$.

By Proposition 4.3.3, there exists a tree of pairs T in $\mathbb{N}$ with

$$\alpha \in A \iff \exists\beta\,(\alpha, \beta) \in [T]\,. \tag{4.1}$$

For each $(u, v) \in T$ we define the sets

$$W_{(u,v)} = \{\alpha \in \mathcal{N} : u \sqsubseteq \alpha \;\; \& \;\; \exists\beta\,(v \sqsubseteq \beta \,\&\, (\alpha, \beta) \in [T]\,)\}. \tag{4.2}$$

Note that

$$W_\Lambda = \{\alpha \in \mathcal{N} : \exists\beta\,(\alpha, \beta) \in [T]\,\} = A$$

and that

$$W_{(u,v)} = \bigcup_{(u*(i),v*(j))\in T} W_{(u*(i),v*(j))}, \text{ where } (u, v) \in T.$$

In particular, we have $W_{(u,v)} \subseteq A$ for each $(u, v) \in T$. (These $W_{(u,v)}$ are used in place of the elements V_n of the basis of $\mathcal{X}$ as presented in the proof of the Cantor-Bendixson Theorem.)

In correspondence with the perfect kernel and the scattered part of a closed set, we define

$$P = \{\alpha \in A \;:\; \forall(u, v) \in T \;\; (\text{if } \alpha \in W_{(u,v)} \text{ then } W_{(u,v)} \text{ is uncountable})\}$$

and

$$\begin{aligned} S &= \{\alpha \in A \;:\; \exists(u, v) \in T \;\; (\alpha \in W_{(u,v)} \text{ and } W_{(u,v)} \text{ is countable})\} \\ &= A \setminus P. \end{aligned}$$

If we set $I = \{(u, v) \in T : W_{(u,v)} \text{ is countable}\}$, then by the definition of S we have

$$S = \bigcup_{(u,v)\in I} W_{(u,v)}.$$

The latter union is a countable union of countable sets, hence S is countable. Since $A = P \cup S$ and A is uncountable, we conclude that P is an uncountable set.

We then define the set $G \subseteq T$ as follows:

$$(u, v) \in G \iff P \cap W_{(u,v)} \neq \varnothing, \text{ with } (u, v) \in T.$$

Note that the empty sequence (which we identify with the pair (Λ, Λ)) belongs to G because

$$P \cap W_\Lambda = P \cap A = P \neq \varnothing.$$

In order to proceed, we say that two finite sequences (u_1, v_1), (u_2, v_2) are ***incompatible in the first variable*** if u_1, u_2 are incompatible. The next claim is the key point in the proof.

<u>Claim.</u> For every $(u, v) \in G$ there exist (u_1, v_1), (u_2, v_2) belonging to T, extend (u, v), are incompatible in the first variable and belong to G. (It follows that these extensions are strict, otherwise they would be compatible.)

<u>Proof of the Claim.</u> Since $(u, v) \in G$ there exists $\alpha \in P \cap W_{(u,v)}$. Therefore, the set $W_{(u,v)}$ is uncountable. Since $W_{(u,v)} \subseteq A$ and $A = P \cup S$ we have that

$$W_{(u,v)} = (W_{(u,v)} \cap P) \cup (W_{(u,v)} \cap S).$$

Since S and hence $W_{(u,v)} \cap S$ is a countable set, while $W_{(u,v)}$ is uncountable, we have from the above equality that $W_{(u,v)} \cap P$ is an uncountable set. In particular, there exist

$$\alpha_1, \alpha_2 \in W_{(u,v)} \cap P, \text{ with } \alpha_1 \neq \alpha_2.$$

We obtain the least n with $\alpha_1(n) \neq \alpha_2(n)$ and set

$$u_i = (\alpha_i(0), \ldots, \alpha_i(n)), \text{ for } i = 1, 2,$$

so that u_1, u_2 are incompatible.

Furthermore, we have $u \sqsubseteq \alpha_1, \alpha_2$ so α_1 and α_2 agree at $i < |u|$, therefore

$$u \;\sqsubseteq\; \alpha_i | n \;=\; u_i, \text{ for } i = 1, 2.$$

That is, u_1, u_2 are extensions of u. Since $\alpha_1, \alpha_2 \in W_{(u,v)}$ there exist β_1, β_2 with $v \sqsubseteq \beta_i$ such that $(\alpha_i, \beta_i) \in [T]$, $i = 1, 2$. We set

$$v_i = (\beta_i(0), \ldots, \beta_i(n)), \text{ for } i = 1, 2.$$

Since

$$v \sqsubseteq \beta_i, \; v_i = \beta_i|(n+1) \; \text{ and } \; |v| = |u| \leqslant n,$$

we have $v \sqsubseteq v_i$, for $i = 1, 2$. Moreover

$$(u_i, v_i) = ((\alpha_i(0), \ldots, \alpha_i(n)), (\beta_i(0), \ldots, \beta_i(n))) \in T$$

because $(\alpha_i, \beta_i) \in [T]$, for $i = 1, 2$.

Therefore, (u_1, v_1), (u_2, v_2) belong to T, are extensions of (u, v) and are incompatible in the first variable.

It remains to show that they belong to G. For each $i = 1, 2$, by the definition of (u_i, v_i) we have $u_i \sqsubseteq \alpha_i$, $v_i \sqsubseteq \beta_i$, and since $(\alpha_i, \beta_i) \in [T]$ we have from (4.2) that $\alpha_i \in W_{(u_i, v_i)}$. Moreover, $\alpha_i \in P$ hence $P \cap W_{(u_i, v_i)} \neq \varnothing$. From the definition of G we have $(u_i, v_i) \in G$. This proves the claim.

$\dashv$

Back to the proof, one defines by recursion a function

$$\phi : \{0,1\}^{<\mathbb{N}} \to G \subseteq T, \; \text{with} \; \phi(\Lambda) = \Lambda \equiv (\Lambda, \Lambda)$$

and $\phi(w * (0))$, $\phi(w * (1))$ are extensions of $\phi(w)$ that are incompatible in the first variable. (Because T is a countable set, we do not need the Axiom of Choice in the definition of ϕ, we just select each time the appropriate extensions incompatible in the first variable that has the least index based on an enumeration of T.)

As mentioned above, $\phi(w * (0))$, $\phi(w * (1))$ are extensions of $\phi(w)$ hence ϕ is a *proper monotone* by Definition 4.1.11 Therefore we define the function

$$\phi^* : [\{0,1\}^{<\mathbb{N}}] = 2^{\mathbb{N}} \to [T] : \phi^*(\gamma) = \bigcup_n \phi(\gamma|n),$$

which by Proposition 4.1.12 is continuous. (The continuity of ϕ^* is established from the proof of the last proposition. The basic open subsets of $[T]$ have the form

$$B_{(u,v)} = \{(\alpha, \beta) \in [T] : u \sqsubseteq \alpha \; \& \; v \sqsubseteq \beta\},$$

where $(u, v) \in T$. One can easily see that for any $\gamma \in 2^{\mathbb{N}}$ we have $\phi^*(\gamma) \in B_{(u,v)}$ if and only if there exists n with $(u, v) \sqsubseteq \phi(\gamma|n)$. It follows that $(\phi^*)^{-1}[B_{(u,v)}]$ is an open set.)

Finally, we consider the composition

$$f : 2^{\mathbb{N}} \to \mathcal{N} : f(\gamma) = (\mathrm{pr}_1 \circ \phi^*)(\gamma).$$

where pr_1 is the projection on the first coordinate $(\alpha, \beta) \mapsto \alpha$. The function f is continuous, as a composition of continuous functions. Note that f takes values in A because if we take $\gamma \in 2^{\mathbb{N}}$ then

$$\phi^*(\gamma) = (\alpha, \beta) \in [T], \text{ for some } \alpha, \beta \in \mathcal{N}.$$

Hence, $f(\gamma) = \alpha$ and from (4.2) we have that $\alpha = f(\gamma)$ belongs to A.

It remains to show that f is a monomorphism. This is due to the incompatibility given by ϕ in the first coordinate. In particular, if we have $\gamma, \gamma' \in 2^{\mathbb{N}}$ with $\gamma|n = \gamma'|n$ and $\gamma(n) \neq \gamma'(n)$ then the sequences of pairs $\phi(\gamma|(n+1)$, $\phi(\gamma'|(n+1)$ are extensions of $\phi(\gamma|n)$ that are incompatible in the first coordinate, say that they differ in $t \in \mathbb{N}$. By the definition of ϕ^* we have

$$\phi(\gamma|(n+1)) \sqsubseteq \phi^*(\gamma) \; \text{ and } \; \phi(\gamma'|(n+1)) \sqsubseteq \phi^*(\gamma'),$$

therefore

$$(\mathrm{pr}_1 \circ \phi^*)(\gamma)(t) = \mathrm{pr}_1(\phi(\gamma|(n+1)))(t) \neq \mathrm{pr}_1(\phi(\gamma'|(n+1)))(t) = (\mathrm{pr}_1 \circ \phi^*)(\gamma')(t)$$

and therefore

$$(\mathrm{pr}_1 \circ \phi^*)(\gamma)(t) \neq (\mathrm{pr}_1 \circ \phi^*)(\gamma')(t),$$

i.e. $f(\gamma) \neq f(\gamma')$. This completes the proof. $\square$

Proving the above Theorem leads us to the following definition:

Definition 4.4.3. A pointclass Γ has the "**Perfect Set Property**" if every uncountable subset A that belongs in Γ contains a non-empty perfect subset.

CHAPTER 5

Measure Theory

In this chapter, we will introduce specific notions of measure theory (based on [**3**], [**5**] and [**10**]) emphasizing the Lebesgue measure and the Borel sets and Borel measurable functions. These data will constitute necessary tools in the last chapter of this thesis, and in particular in the games associated with a measure.

5.1. Measures

Definition 5.1.1. Consider a nonempty set X and a family $\mathcal{A}$ of subsets of X. We call $\mathcal{A}$ a σ-**algebra** on X if it satisfies the following properties:

i) $\varnothing, X \in \mathcal{A}$
ii) If $A \in \mathcal{A}$, then $X \backslash A \in \mathcal{A}$.
iii) If $A_n \in \mathcal{A}$, $\forall n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

Remark 5.1.2. Some trivial examples of σ-algebras on X are the power set $\mathcal{P}(X)$ and the $(\varnothing, X)$. As is known if we have a non-empty set $\mathcal{F}$ of σ-algebras on the same set X then the intersection

$$\bigcap \mathcal{F} = \{A \subseteq X : \forall \mathcal{A} \in \mathcal{F} \;\; A \in \mathcal{A}\}$$

is also a σ-algebra on X.

Definition 5.1.3. Let X be a set and $\mathcal{A}$ a σ-algebra on X. A function $\mu : \mathcal{A} \to [0, \infty]$ is called **measure** if

i) $\mu(\varnothing) = 0$ and
ii) μ is *countably additive* (or *σ-additive*), i.e. if $(A_n)_{n\in\mathbb{N}}$ is a sequence of disjoint sets in two's in $\mathcal{A}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Remark 5.1.4. By the latter property, such a measure is often referred to as a **countably additive** (or σ-**additive**) **measure**. Also, the pair $(X, \mathcal{A})$ is called a **measurable space**, the triad $(X, \mathcal{A}, \mu)$ is called **measure space** and we say that μ is a measure on $(X, \mathcal{A})$ or simply on X. The elements of $\mathcal{A}$ are also called $\mathcal{A}$-**measurable sets**.

Definition 5.1.5. Let X be a set and $\mathcal{A}$ a σ-algebra on X. A function $\mu : \mathcal{A} \to [0, \infty]$ is called a **finitely additive measure** if

i) $\mu(\varnothing) = 0$ and
ii) μ is *finitely additive*, i.e. if $(A_j)_{j=1}^{n}$ is a finite sequence of disjoint sets in two's in $\mathcal{A}$, then

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j).$$

Every measure is also a finitely additive measure.

Proposition 5.1.6. *Let $(X, \mathcal{A}, \mu)$ be a measure space. The following holds:*

i) *The measure μ is monotone, i.e. if for $A, B \in \mathcal{A}$ it holds that $A \subseteq B$, then $\mu(A) \leqslant \mu(B)$.*
ii) *Moreover, if $\mu(A) < \infty$, then $\mu(B \backslash A) = \mu(B) - \mu(A)$.*

Proof. We write

$$B = A \cup (B \backslash A)$$

and note that A and $B\backslash A$ are disjoint sets. Thus, by the additivity of measure μ, it follows

$$\mu(B) = \mu(A) + \mu(B\backslash A).$$

Hence, $\mu(B) \geqslant \mu(A)$ and if in addition $\mu(A) < \infty$, we have

$$\mu(B\backslash A) = \mu(B) - \mu(A).$$

□

Proposition 5.1.7. *Let $(X, \mathcal{A}, \mu)$ be a measure space. The measure μ is countably subadditive (or σ-subadditive), i.e. if $(A_n)_{n\in\mathbb{N}}$ is a random sequence of elements of $\mathcal{A}$, then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} \mu(A_n).$$

Proof. We put

$$B_n = A_n \backslash \bigcup_{j=1}^{n-1} A_j, \;\; n = 1, 2, \ldots$$

Then every $B_n \in \mathcal{A}$, B_n are disjoint sets in two's, $B_n \subseteq A_n$ holds and indeed

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n.$$

Consequently,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leqslant \sum_{n=1}^{\infty} \mu(A_n)$$

due to the finitely additive of μ and monotony. □

Proposition 5.1.8. *Let $(X, \mathcal{A}, \mu)$ be a measure space. The measure μ is "continuous" in the following two senses:*

i) *If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence of elements of $\mathcal{A}$, then*

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \lim_{n\to\infty} \mu(A_n).$$

ii) *If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of elements of $\mathcal{A}$ and furthermore $\mu(A_1) < \infty$, then*

$$\mu\left(\bigcap_{j=1}^{n} A_j\right) = \lim_{n\to\infty} \mu(A_n).$$

Proof. **i)** We consider the sets

$$B_n = A_n \backslash A_{n-1}, \;\; n = 1, 2, \ldots$$

(where we have set $A_0 = \varnothing$) which are disjoint in two's and observe that for each n we have

$$A_n = \bigcap_{j=1}^{n} A_j = \bigcap_{j=1}^{n} B_j.$$

Therefore,

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \\
&= \lim_{n\to\infty} \sum_{j=1}^{n} \mu(B_j) = \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} B_j\right) \\
&= \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} A_j\right) = \lim_{n\to\infty} \mu(A_n).
\end{aligned}$$

ii) Consider the sets

$$C_n = A_1 \backslash A_n, \text{ for } n = 1, 2, \ldots$$

Then, $(C_n)_{n\in\mathbb{N}}$ is an increasing sequence of $\mathcal{A}$ with

$$\bigcup_{n=1}^{\infty} C_n = A_1 \backslash \bigcap_{n=1}^{\infty} A_n.$$

By **i)**, it follows that

$$\mu\left(\bigcup_{n=1}^{\infty} C_n\right) = \lim_{n\to\infty} \mu(C_n), \text{ i.e. } \mu\left(A_1\backslash \bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_1\backslash A_n).$$

Thus, by Proposition 5.1.6, we have

$$\mu(A_1) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n)$$

and since $\mu(A_1) < \infty$, we have what is required. □

Definition 5.1.9. Let $(X, \mathcal{A}, \mu)$ be a measure space. The measure μ is called:

i) **finite**, if $\mu(X) < \infty$,

ii) **probability measure**, if $\mu(X) = 1$ and

iii) σ-**finite**, if there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{A}$ with

$$X = \bigcup_{n=1}^{\infty} A_n$$

and

$$\mu(A_n) < \infty, \text{ for each } n = 1, 2, \ldots$$

Remark 5.1.10. Respectively, we say that the measure space $(X, \mathcal{A}, \mu)$ is **finite**, a **probability space** or an σ-**finite measure space**.

5.2. Uniqueness and Completion

Two measures μ and ν in a countable space $(X, \mathcal{A})$ are equal if for any set $A \in \mathcal{A}$, it holds that

$$\mu(A) = \nu(A).$$

Therefore, this condition is, in general, difficult to check. So it is natural to wonder, if μ and ν are identical in a "large" subfamily of $\mathcal{A}$, whether we can infer that they are identical everywhere. The answer to this, by quite good measures, is given by the following proposition:

Proposition 5.2.1. *(Uniqueness Theorem). Let $(X, \mathcal{A})$ be a countable space and Δ is a family of subsets of X closed under finite intersections, for which $\sigma(\Delta) = \mathcal{A}$. If μ and ν are two measures on $(X, \mathcal{A})$ such that*

$$\mu(\Delta) = \nu(\Delta), \text{ for every } D \in \Delta$$

and one of the following conditions holds, then $\mu = \nu$:

i) *The measures μ and ν are finite and $\mu(X) = \nu(X)$.*

ii) *The measures μ and ν are σ-finite and in particular there is an increasing sequence $(D_n)_{n\in\mathbb{N}}$ in Δ such that*

$$X = \bigcup_{n=1}^{\infty} D_n$$

and

$$\mu(D_n) = \nu(D_n) < \infty, \text{ for each } n.$$

Suppose now that in a measure space $(X, \mathcal{A}, \mu)$ we have fixed an $A \in \mathcal{A}$ with $\mu(A) = 0$. If N is any subset of $\mathcal{A}$ it is not certain that $N \in \mathcal{A}$. This depends on the choice of σ-algebra. Nevertheless, if it holds $N \in \mathcal{A}$ then surely $\mu(N) = 0$. Then this question arises:

"Can we extend the σ-algebra $\mathcal{A}$ to contain all these negligible sets?"

We will show in the following that the answer is affirmative.

Definition 5.2.2. Let $(X, \mathcal{A}, \mu)$ be a measure space and $N \subseteq X$. The set N is called μ-**null** if there exists an $A \in \mathcal{A}$ with $N \subseteq A$ and $\mu(A) = 0$. The $(X, \mathcal{A}, \mu)$ is called **complete space** (and the μ **complete measure**) if every μ-null set N belongs to $\mathcal{A}$.

Definition 5.2.3. Let $(X, \mathcal{A}, \mu)$ be a measure space. We define

i) The family
$$\mathcal{A}_\mu = \{A \subseteq X : \text{ there exist } E, F \in \mathcal{A}, \text{ with } E \subseteq A \subseteq F \text{ and } \mu(F \backslash E) = 0\}.$$
(Note that it will be $\mu(E) = \mu(F)$.)

ii) The function $\overline{\mu} : \mathcal{A}_\mu \to [0, \infty]$ defined by $\overline{\mu}(A) = \mu(E)$, where E as above. (Note that for $B \in \mathcal{A}$ with $B \subseteq A$ it is $\mu(B) \leqslant \mu(F) = \mu(E)$ and hence
$$\overline{\mu}(A) = \mu(E) = \sup\{\mu(B) : B \in \mathcal{A},\ B \subseteq A\}.$$
Thus, μ is a *well-defined* function, i.e. independent of the choice of E.)

The family $\mathcal{A}_\mu$ is called **completion** of $\mathcal{A}$, the function $\overline{\mu}$ is called **completion** of μ, and the triad $(X, \mathcal{A}_\mu, \overline{\mu})$ is a **completion** of $(X, \mathcal{A}, \mu)$.

Definition 5.2.4. We define the **symmetric difference** $\triangle$ of two sets X and Y as
$$X \triangle Y = (X \backslash Y) \cup (Y \backslash X) = (X \cup Y) \backslash (X \cap Y).$$

Definition 5.2.5. An $A \subseteq X$ is called μ-**measurable**, if there are $B \in \mathcal{A}$ and N μ-null, with
$$A = B \triangle N = (B \backslash N) \cup (N \backslash B).$$
So, the family $\mathcal{A}_\mu$ can be written as
$$\mathcal{A}_\mu = \{B \triangle N : \ B \in \mathcal{A} \text{ and } N \text{ is a } \mu\text{-null set}\}.$$

Remark 5.2.6. The elements of $\mathcal{A}_\mu$ are called μ-measurable sets. It is a direct consequence of the above definition that every μ-null set is also μ-measurable. Intuitively, the elements of $\mathcal{A}_\mu$ are those subsets of X that are "μ-negligible distance" (i.e., one μ-null set away) from the elements of $\mathcal{A}$.

Remark 5.2.7. The relation $\overline{\mu}(B \triangle N) = \mu(B)$, holds and is well defined. That is because if we have
$$B \triangle N = B' \triangle N',$$
with $B, B' \in \mathcal{A}$ and N, N' are μ-null sets, then $\mu(B) = \mu(B')$.

5.3. Lebesgue Measure

We will now define the Lebesgue outer measure λ^* in $\mathbb{R}$.

Definition 5.3.1. The **Lebesgue outer measure** $\lambda^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ **in** $\mathbb{R}$ is defined as follows:
$$\lambda^*(A) = \inf\left\{\sum_{n=1}^{\infty}(b_n - a_n) : a_n, b_n \in \mathbb{R}, \text{ and } A \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n)\right\},$$
for each $A \subseteq \mathbb{R}$.

Definition 5.3.2. The **Lebesgue outer measure** $\lambda_k^* : \mathcal{P}(\mathbb{R}^k) \to [0, \infty]$ **in** $\mathbb{R}^k$ is defined as follows:
$$\lambda_k^*(A) = \inf\left\{\sum_{n=1}^{\infty}\nu(I_n) : I_n \subseteq \mathbb{R}^k \text{ open bounded interval, and } A \subseteq \bigcup_{n=1}^{\infty} I_n\right\},$$
for each $A \subseteq \mathbb{R}^k$. By definition, an open bounded interval I of $\mathbb{R}^k$ is a set of the following form:
$$I = \prod_{j=1}^{k}(a_j, b_j) = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_k, b_k),$$
with $a_i < b_i \in \mathbb{R}$, and
$$\nu(I) = (b_1 - a_1)(b_2 - a_2) \ldots (b_k - a_k),$$
where the quantity $\nu(I)$ is the **volume of the interval** I.

Remark 5.3.3. It is clear from the definition that $\lambda_1^* = \lambda^*$. Sometimes, for simplicity, we write $\lambda_k^* = \lambda^*$.

Definition 5.3.4. Let $\phi : \mathcal{P}(X) \to [0, \infty]$ be an outer measure on the set X. A $B \subseteq X$ is called **ϕ-measurable** if

$$\phi(A) = \phi(A \cap B) + \phi(A \backslash B),$$

for every $A \subseteq X$. We denote by $\mathcal{M}_\phi$ the **family of all ϕ-measurable subsets of** X.

Theorem 5.3.5. *(Caratheodori). Let $\phi : \mathcal{P}(X) \to [0, \infty]$ be an outer measure on the set X. Then $\mathcal{M}_\phi$ is a σ-algebra on X and the restriction $\phi|_{M\phi}$ of ϕ on $\mathcal{M}_\phi$ is a complete measure.*

Definition 5.3.6. The elements of the σ-algebra $\mathcal{M}_{\lambda^*}$ are called **Lebesgue countable sets**.

Definition 5.3.7. The restriction of the Lebesgue outer measure λ_k^* on the σ-algebra $\mathcal{M}_{\lambda^*}$ is called the **Lebesgue measure** and is denoted by λ_k or simply λ.

Remark 5.3.8. According to the above, λ is a complete measure. Sometimes, and the restriction of λ_k^* on $\mathcal{B}(\mathbb{R}^k)$ will be called a Lebesgue measure.

Definition 5.3.9. Let (X, d) be a metric space, $\mathcal{A}$ a σ-algebra on X such that $\mathcal{A} \supseteq \mathcal{B}(X)$ and μ is a measure in the measurable space $(X, \mathcal{A})$. The measure μ is called **regular measure** if

i) $\mu(K) < \infty$, for every $K \subseteq X$ compact.

ii) μ satisfies the *Outer Regularity Condition*, i.e.

$$\mu(A) = \inf\{\mu(G) : G \text{ open in } X \text{ and } G \supseteq A\}, \text{ for every } A \in \mathcal{A}.$$

iii) μ satisfies the *Inner Regularity Condition*, i.e.

$$\mu(G) = \sup\{\mu(K) : K \text{ compact and } K \supseteq G\}, \text{ for every } G \subseteq X \text{ open}.$$

5.4. Borel Sets, Measures and Measurable Functions

Definition 5.4.1. Consider a metric space X and the set $\mathcal{F}$ of all σ-algebras on X containing the open subsets of X, i.e.

$$\mathcal{F} = \{\mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra on } X \text{ and } \forall \text{ open } V \subseteq X \text{ it holds } V \in \mathcal{A}\}$$

We observe that $\mathcal{P}(X) \in \mathcal{F}$ and therefore $\mathcal{F} \neq \varnothing$. The **Borel σ-algebra $\mathcal{B}(X)$ of the subsets of** X is the family

$$\mathcal{B}(X) = \bigcap \mathcal{F}.$$

A subset of X is called **Borel** if it belongs to the σ-algebra $\mathcal{B}(X)$.

Proposition 5.4.2. *Every Borel subset of $\mathbb{R}^k$ is also Lebesgue countable, i.e. $\mathcal{B}(\mathbb{R}^k) \subseteq \mathcal{M}_{\lambda^*}$.*

Remark 5.4.3. It is clear that the family $\mathcal{B}(X)$ has the following properties:

i) The $\mathcal{B}(X)$ is a σ-algebra and every open subset of X is contained in $\mathcal{B}(X)$, i.e. $\mathcal{B}(X)$ is an element of the above $\mathcal{F}$.

ii) If $\mathcal{A}$ is the σ-algebra on X containing all open subsets of X, then $\mathcal{B}(X) \subseteq \mathcal{A}$.

In other words, $\mathcal{B}(X)$ is the **minima**l σ-algebra on X that **contains the open sets**. We denote by $\mathcal{B}$ the **class** of all Borel sets in metric spaces. Some examples of Borel sets are all open sets and their complements, i.e. the closed sets. The F_σ sets are Borel as countable unions of Borel sets and also G_δ are Borel as complements of F_σ sets.

Proposition 5.4.4. *For every Polish space $\mathcal{X}$ and every $n \geqslant 1$ holds that*

$$\underset{\sim}{\boldsymbol{\Sigma}}{}^0_n(\mathcal{X}) \subseteq \mathcal{B}(\mathcal{X}),$$

i.e. every subset of $\mathcal{X}$ is also a Borel subset of $\mathcal{X}$.

Lemma 5.4.5. *The pointclass of Borel sets is closed under continuous substitution.*

Proof. We consider metric spaces X, Y, and a continuous function $f : X \to Y$. We will show that for any Borel $A \subseteq Y$ the set $f^{-1}[A]$ is a Borel subset of X. We consider the family

$$\mathcal{A} = \{A \subseteq Y : f^{-1}[A] \in \mathcal{B}(X)\}$$

and we will show that it is a σ-algebra containing all open subsets of Y. Since $\mathcal{B}(X)$ is the minimal σ-algebra containing the open subsets of Y, it follows that $\mathcal{B}(X) \subseteq A$ which is what is required. Consider an open $V \subseteq Y$. Since f is a continuous function, the inverse image $f^{-1}[V]$ is an open subset of X and, therefore, is a Borel set. Hence, $V \in \mathcal{A}$ and $\mathcal{A}$ contains all open subsets of X.

Continuing, we observe that $\varnothing, Y \in \mathcal{A}$ because the sets $\varnothing$ and Y are open. If $A \in \mathcal{A}$, then

$$f^{-1}[Y \backslash A] = X \backslash f^{-1}[A] \in \mathcal{B}(X),$$

where we used that $f^{-1}[A] \in \mathcal{B}(X)$ and that the family of Borel subsets of X is closed under complementation in X. □

Theorem 5.4.6. *(The Fundamental Closure Properties of the Pointclass of Borel Sets.) The pointclass $\mathcal{B}$ of Borel sets is closed under*

i) *continuous substitution*

ii) *the operators* $\vee$, &, $\exists^{\leqslant}$, $\forall^{\leqslant}$, $\bigvee_{\leqslant}$, $\bigwedge_{\leqslant}$, c_X, *where X is a metric space, and*

iii) *the operators* $\bigvee_{\mathbb{N}}$, $\bigwedge_{\mathbb{N}}$, $\exists^{\mathbb{N}}$, $\forall^{\mathbb{N}}$, $\exists^{Y}$, $\forall^{Y}$, *where Y is a countable Polish space.*

Remark 5.4.7. By the equivalent definition of the analytic sets, it is known that every analytic (i.e. $\underset{\sim}{\Sigma}{}^1_1$) subset of a Polish space is a continuous image of a Borel set. Therefore, as a consequence of Theorem 4.4.2, the following effect occurs:

"Every uncountable Borel set has a non-empty perfect subset."

Definition 5.4.8. Let X be a topological space and $(X, \mathcal{A}, \mu)$ be a measure space. The measure μ is called a **Borel measure** on X if

$$\mathcal{B}(X) \subseteq \mathcal{A},$$

i.e. if all Borel sets in X are in $\mathcal{A}$.

Definition 5.4.9. Let X be a topological space and μ a Borel measure on X. Then μ is called **regular** if the following are true for every Borel set E in X:

i) $\mu(E) = \inf\{\mu(U) : U \text{ open } \supseteq E\}$,

ii) $\mu(E) = \sup\{\mu(K) : K \text{ compact } \subseteq E\}$.

Definition 5.4.10. Suppose μ is a **σ-finite Borel measure** on X, i.e., a countably additive function on the Borel subsets of X with values real numbers $\geqslant 0$ or ∞ and such that we can write

$$X = \bigcup_{n=1}^{\infty} A_n$$

with $A_n \in \mathcal{B}(X)$, $\mu(A_n) < \infty$ for each n.

Let Z_μ be **the collection of null sets** or **sets of measure** 0 (in the completed measure), i.e.,

$$A \in Z_\mu \iff \text{there exists a Borel set } B \text{ such that } A \subseteq B \text{ and } \mu(B) = 0.$$

Again it is clear that Z_μ is a σ-ideal.

Definition 5.4.11. Let $\mathcal{X}, \mathcal{Y}$ be two metric spaces. A **function** $f : X \to Y$ **is Borel-measurable**, if it inverts the open subsets of Y to Borel (equivalently, open or closed) subsets of X, i.e. for every open $A \subseteq Y$ we have

$$f^{-1}[A] \in \mathcal{B}(X).$$

In other words, the inverse image of a Borel set is a Borel set, too.

Remark 5.4.12. Obviously, every continuous function $f : X \to Y$ is Borel-measurable because for for every open $A \subseteq Y$ the set $f^{-1}[U]$ is open and therefore Borel.

Definition 5.4.13. We will say that a pointclass Γ is **closed under Borel substitution** if for every Borel-measurable function $f : \mathcal{X} \to \mathcal{Y}$, where X, Y are Polish spaces, and for each $A \in \Gamma(\mathcal{Y})$ the set $f^{-1}[A]$ belongs to $\Gamma(\mathcal{X})$.

Definition 5.4.14. An isomorphism $f : X \rightarrowtail\!\!\!\!\rightarrow Y$ is a **Borel isomorphism** if the functions

$$f : X \to Y \quad \text{and} \quad f^{-1} : Y \to X$$

are Borel-measurable. The metric spaces X and Y are **Borel isomorphic** if there is a Borel isomorphism $f : X \rightarrowtail\!\!\!\!\rightarrow Y$.

Proposition 5.4.15. *Consider two metric spaces X, Y and a function $f : X \to Y$. The following are equivalent:*

- **i)** *The function f inverts open subsets of Y to Borel subsets of X, i.e., f is Borel-measurable.*
- **ii)** *The function f inverts Borel subsets of Y to Borel subsets of X, i.e. for every Borel set $B \subseteq Y$, the set $f^{-1}[B]$ is a Borel set. (Sometimes the definition of Borel-measurable functions is given by this condition.)*

Proof. The direction **ii)** $\implies$ **i)** is direct because every open subset of Y is a Borel subset of X. For direction **i)** $\implies$ **ii)**, we consider the family

$$\mathcal{A} = \{A \subseteq Y : f^{-1}[A] \in \mathcal{B}(X)\}.$$

By hypothesis, every open subset of Y belongs to $\mathcal{A}$. We will show that $\mathcal{A}$ is a σ-algebra on X. Indeed we have

$$\varnothing = f^{-1}[\varnothing] \quad \text{and} \quad X = f^{-1}[Y]$$

such that $\varnothing, Y \in \mathcal{A}$. Moreover, we have

$$f^{-1}[Y \backslash A] = X \backslash f^{-1}[A] \quad \text{and} \quad f^{-1}[\cup_{n\in\mathbb{N}} A_n] = \cup_{n\in\mathbb{N}} f^{-1}[A_n],$$

for any sequence of $(A_n)_{n\in\mathbb{N}}$ subsets of Y and any $A \subseteq Y$. Since $\mathcal{B}(X)$ is σ-algebra on X, it follows from the previous two equations that $\mathcal{A}$ is closed under the complement to Y and under the countable union of subsets of Y. We conclude that $\mathcal{A}$ is a σ-algebra. Since $\mathcal{A}$ is a σ-algebra on Y, which contains the open subsets of Y, we have that $\mathcal{B}(Y) \subseteq \mathcal{A}$, which is exactly what is required. ☐

Definition 5.4.16. Consider two Polish spaces $\mathcal{X}$, $\mathcal{Y}$ and a monomorphism $f : \mathcal{X} \rightarrowtail \mathcal{Y}$. The function f is called a **good Borel monomorphism** if

- **i)** it is Borel-measurable,
- **ii)** the image $f[\mathcal{X}]$ is a Borel subset of $\mathcal{Y}$, and
- **iii)** the inverse function $f^{-1} : f[\mathcal{X}] \to \mathcal{X}$ is Borel measurable. (We consider $f[\mathcal{X}]$ as a metric subspace of $\mathcal{Y}$.)

Remark 5.4.17. For every good Borel monomorphism $f : \mathcal{X} \rightarrowtail \mathcal{Y}$ and every Borel set $B \subseteq \mathcal{X}$ the set $f[B]$ is a Borel subset of $\mathcal{Y}$.

Remark 5.4.18. The composition of good Borel monomorphisms is a Borel monomorphism.

Lemma 5.4.19. *For every uncountable Polish space $\mathcal{X}$ there is a good Borel monomorphism $\tau : 2^{\mathbb{N}} \rightarrowtail \mathcal{X}$.*

Proof. By Corollary 4.2.4 there exists a continuous monomorphism

$$\tau : 2^{\mathbb{N}} \rightarrowtail \mathcal{X}.$$

We will show that τ is also a good Borel monomorphism. Since τ is continuous, it is also a Borel-measurable function. Moreover, the set $\tau[2^{\mathbb{N}}]$ is compact (as a continuous image of a compact set) and hence it is a closed subset of $\mathcal{X}$. In particular, $\tau[2^{\mathbb{N}}]$ is a Borel subset of $\mathcal{X}$. Finally, the inverse function

$$\tau^{-1} : \tau[2^{\mathbb{N}}] \to 2^{\mathbb{N}}$$

is continuous and hence Borel-measurable. ☐

Lemma 5.4.20. *There is a continuous monomorphism $\rho : \mathcal{N} \rightarrowtail 2^{\mathbb{N}}$ such that the function $\rho^{-1} : \rho[\mathcal{N}] \rightarrowtail 2^{\mathbb{N}}$ is continuous and the set $\rho[\mathcal{N}]$ is a $\underset{\sim}{\mathbf{\Pi}}^0_2$ subset of $2^{\mathbb{N}}$.*

Lemma 5.4.21. *For every uncountable Polish space $\mathcal{X}$ there is a good Borel monomorphism $f : \mathcal{N} \rightarrowtail \mathcal{X}$.*

Proof. Consider the function $\rho : \mathcal{N} \rightarrowtail 2^{\mathbb{N}}$, as in Lemma 5.4.20. Because every continuous function is Borel-measurable and $\underset{\sim}{\mathbf{\Pi}}^0_2$ sets are Borel, it follows that ρ is a good Borel monomorphism. By Lemma 5.4.20 there is a good Borel monomorphism $\tau : 2^{\mathbb{N}} \rightarrowtail \mathcal{X}$ and because the composition of good Borel monomorphisms is also a good Borel monomorphism, we have

$$f = \tau \circ \rho : \mathcal{N} \rightarrowtail \mathcal{X}$$

is a good Borel monomorphism. □

Lemma 5.4.22. *For every Polish space $\mathcal{X}$ there is a good Borel monomorphism $\tau : \mathcal{X} \rightarrowtail \mathcal{N}$.*

Proof. We consider a suitable metric d on $\mathcal{X}$ and a countable $D = \{r_n : n \in \mathbb{N}\}$ and a dense subset of $\mathcal{X}$. We consider the function

$$u \in \mathbb{N}^{<\mathbb{N}} \backslash \{\Lambda\} \ \mapsto \ x_u \in \mathcal{X}$$

defined as follows:

$$x_{(k)} = r_k \quad \text{and} \quad x_{u*(k)} = \begin{cases} r_k, & \text{if } d(r_k, x_u) < 2^{-(|u|+1)}, \\ x_u, & \text{if } d(r_k, x_u) \geqslant 2^{-(|u|+1)}. \end{cases}$$

As we have seen in the proof of Theorem 2.4.7 as well as in the Remark 2.4.8 the function

$$\pi : \mathcal{N} \to \mathcal{X} \ : \ \pi(\alpha) = \lim_{n\to\infty} x_{\alpha|n}$$

is a continuous surjection and the function

$$\tau : \mathcal{X} \to \mathcal{N} : \tau(x)(n) = \text{ the least k with } d(r_k, x) < 2^{-(n+2)}$$

is a monomorphism satisfying

$$\pi(\tau(x)) = x, \ x \in \mathcal{X} \quad \text{and} \quad \tau(\pi(\alpha)) = \alpha, \ \alpha \in \tau[\mathcal{X}]\,.$$

We will show that τ is a good Borel monomorphism. Firstly, we consider

$$\tau^{-1} : \tau[\mathcal{X}] \to \mathcal{X}$$

and we observe that for each $\alpha = \tau(x) \in \tau[\mathcal{X}]$, we have $\tau^{-1}(\alpha) = x = \pi(\alpha)$, i.e. $\tau^{-1} = \pi|\tau[\mathcal{X}]$. Since π is continuous it follows that τ^{-1} is continuous.

We then show that τ inverts the open sets of $\mathcal{N}$ to $\underset{\sim}{\mathbf{\Sigma}}^0_2$. Since the $\underset{\sim}{\mathbf{\Sigma}}^0_2$ sets are Borel sets it follows that τ is Borel-measurable. For each $u \in \mathbb{N}^{<\mathbb{N}}$, we have

$$\begin{aligned} \tau(x) \in \mathcal{N}_u &\iff u \sqsubseteq \tau(x) \\ &\iff \forall n < |u| \ u(n) = \tau(x)(n) \\ &\iff \forall n < |u| \ u(n) = \text{ the least } k \text{ with } d(r_k, x) < 2^{-(n+2)} \\ &\iff \forall n < |u| \ (d(r_{u(n)}, x) < 2^{-(n+2)} \ \& \ \forall t < u(n) \, d(r_t, x) \geqslant 2^{-(n+2)}). \end{aligned}$$

It follows that the set $\tau^{-1}[\mathcal{N}_u]$ is a $\underset{\sim}{\mathbf{\Delta}}^0_2$ and a $\underset{\sim}{\mathbf{\Sigma}}^0_2$ subset of $\mathcal{X}$. Since every open subset of $\mathcal{N}$ is a union of sets $\mathcal{N}_u$, for some $u \in \mathbb{N}^{<\mathbb{N}}$, we have from the closure of $\underset{\sim}{\mathbf{\Delta}}^0_2$ under the operator of countable union that τ inverts open sets to $\underset{\sim}{\mathbf{\Delta}}^0_2$.

Finally, we identify the set $\tau[\mathcal{X}]$. As we saw above, $\pi(\alpha)$ is the unique $x \in \mathcal{X}$ with $\alpha = \tau(x)$, for every $\alpha \in \tau[\mathcal{X}]$. Therefore,

$$\begin{aligned} \alpha \in \tau[\mathcal{X}] &\iff \alpha = \tau(\pi(\alpha)) \\ &\iff \forall n \ \alpha(n) = \tau(\pi(\alpha))(n) \\ &\iff \forall n \ \alpha(n) = \text{ the least } k \text{ with } d(r_k, \pi(\alpha)) < 2^{-(n+2)} \\ &\iff \forall n \ (d(\pi(\alpha), r_{\alpha(n)}) < 2^{-(n+2)} \ \& \ \forall t < \alpha(n) \, d(\pi(\alpha), r_t) \geqslant 2^{-(n+2)}). \end{aligned}$$

Therefore, $\tau[\mathcal{X}]$ is $\underset{\sim}{\mathbf{\Pi}}^0_2$ subset of $\mathcal{N}$. □

Theorem 5.4.23. *(Schröder-Bernstein for good Borel monomorphisms). For each two Polish spaces $\mathcal{X}$ and $\mathcal{Y}$, if there are good Borel monomorphisms*

$$f : \mathcal{X} \rightarrowtail \mathcal{Y} \quad \textit{and} \quad g : \mathcal{Y} \rightarrowtail \mathcal{X}$$

*then there exists a **Borel isomorphism***

$$h : \mathcal{X} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathcal{Y}.$$

Proof. By Remark 5.4.18, the composition

$$\phi : \mathcal{Y} \rightarrowtail f[\mathcal{X}] : \phi(y) = f(g(y))$$

is a good Borel monomorphism. We define the sequence $(C_n)_{n\in\mathbb{N}}$ of subsets of $\mathcal{Y}$ as follows:

$$C_0 = \mathcal{Y}\backslash f[\mathcal{X}], \ \ C_{n+1} = f[C_n].$$

Since the functions f and ϕ are good Borel monomorphisms, by applying the Remark 5.4.17, it follows inductively that every C_n is a Borel subset of $\mathcal{Y}$. We also define

$$D = \bigcup_{n\in\mathbb{N}} C_n.$$

Then D is a countable union of Borel sets and therefore it is a Borel subset of $\mathcal{Y}$. We further note that

$$\phi[D] = \phi\left[\bigcup_{n\in\mathbb{N}} C_n\right] = \bigcup_{n\in\mathbb{N}} \phi[C_n] = \bigcup_{n\geqslant 1} C_n \subseteq D.$$

Finally, we define

$$\tau : \mathcal{Y} \to \mathcal{Y} \ : \ \tau(y) = \begin{cases} \phi(y), & \text{if } y \in D, \\ y, & \text{if } y \notin D. \end{cases}$$

The function τ is Borel-measurable.

• We will show that τ takes values in $f[\mathcal{X}]$. Consider that $y \in \mathcal{Y}$. If $y \in D$ then $y \notin C_0 = \mathcal{Y}\backslash f[\mathcal{X}]$ so $y \in f[\mathcal{X}]$. Moreover, since $y \notin D$ we have $\tau(y) = y \in f[\mathcal{X}]$. If $y \in D$, then there exists $n \in \mathbb{N}$ with $y \in C_n$ and

$$\tau(y) = \phi(y) \in \phi[C_n] \subseteq f[\mathcal{X}],$$

where in the last inclusion we used that ϕ takes values in $f[\mathcal{X}]$.

• Continuing, we will show that τ is onto $f[\mathcal{X}]$. Consider $y \in f[\mathcal{X}]$. If $y \notin D$, then $\tau(y) = y$. If $y \in D$ then there exists $n \in \mathbb{N}$ with $y \in C_n$ and since $y \in f[\mathcal{X}]$, $n = 0$ cannot be true. So $n \geqslant 1$ and $y \in C_n = f[C_{n-1}]$. Therefore, there is $y' \in C_{n-1}$ with $y = \phi(y')$. Then we have $\tau(y') = \phi(y') = y$, where in the first equality we used that $y' \in C_{n-1} \subseteq D$. In each case there is $y' \in \mathcal{Y}$ with $\tau(y') = y$.

• We will then prove that τ is a monomorphism. Consider $y_1, y_2 \in \mathcal{Y}$ with $\tau(y_1) = \tau(y_2)$. If one of y_1, y_2 belongs to D (let us suppose y_1) and y_2 does not belong to D then $\tau(y_1) \in \phi[D] \subseteq D$ while $\tau(y_2) = y_2 \notin D$. Therefore, we have $\tau(y_1) \neq \tau(y_2)$ and this is a contradiction. So, either $y_1, y_2 \notin D$, in which case

$$y_1 = \tau(y_1) = \tau(y_2) = y_2,$$

or $y_1, y_2 \in D$. In the second case,

$$\phi(y_1) = \tau(y_1) = \tau(y_2) = \phi(y_2)$$

and since ϕ is a monomorphism we have $y_1 = y_2$.

• Additionally, we wiil prove that the inverse $\tau^{-1} : f[\mathcal{X}] \to \mathcal{X}$ is Borel-measurable. According to the preceding, τ^{-1} is given by

$$\tau^{-1}(y) = \begin{cases} y, & \text{if } y \notin D, \\ \phi^{-1}(y), & \text{if } y \in D. \end{cases}$$

Because $\phi : \mathcal{Y} \rightarrowtail f[\mathcal{X}]$ is a good Borel monomorphism, the inverse function $\phi^{-1} : f[\mathcal{X}] \to \mathcal{Y}$ is Borel-measurable. Hence, τ^{-1} is Borel-measurable.

It follows that τ is a good Borel monomorphism. Finally, we define

$$h : \mathcal{Y} \to \mathcal{X} \ : \ h(y) = f^{-1}(\tau(y)).$$

The function h is well-defined because τ takes values in $f[\mathcal{X}]$ and is Borel-measurable as a composition of Borel-measurable functions. Moreover, it is a monomorphism as a composition of monomorphisms and a surjection because $f^{-1} : f[\mathcal{X}] \to \mathcal{X}$ is onto $\mathcal{X}$ and τ is onto $f[\mathcal{X}]$. Finally, the inverse function $h^{-1} = \tau^{-1} \circ f$ is Borel-measurable as a composition Borel-measurable functions. $\square$

Theorem 5.4.24. *(Borel Isomorphism Theorem). Every uncountable Polish space is Borel isomorphic to Baire space.*

Proof. We consider an uncountable Polish space $\mathcal{X}$. By Lemma 5.4.21 and Lemma 5.4.22 there exist good Borel monomorphisms $f : \mathcal{N} \rightarrowtail \mathcal{X}$ and $\tau : \mathcal{X} \rightarrowtail \mathcal{N}$, respectively. Therefore, by Theorem 5.4.23 there exists a Borel isomorphism $h : \mathcal{N} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathcal{X}$. $\square$

Continuing, we will present a fact that we shall need in proving the Corollary that follows. We consider the following to be known:

"The composition of two Borel isomorphisms is a Borel isomorphism."

Corollary 5.4.25. *If $\mathcal{X}, \mathcal{Y}$ are two uncountable Polish spaces, there is $f : \mathcal{X} \to \mathcal{Y}$ that is a Borel isomorphism.*

Proof. If $\mathcal{X}, \mathcal{Y}$ are two uncountable Polish spaces then by Theorem 5.4.24 there exist Borel isomorphisms

$$f : \mathcal{X} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathcal{N} \quad \text{and} \quad g : \mathcal{N} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathcal{Y}$$

By using the composition $h = g \circ f : \mathcal{X} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathcal{Y}$ and having regard to the above fact, we conclude that h is a Borel isomorphism, too. $\square$

CHAPTER 6

Playing Games

In this chapter, having analyzed all the necessary tools in the previous chapters, we deal with the actual games. In our study, a run of the game is an infinite sequence of elements, created by the moves chosen to be made by two people, the players of the game. These elements vary depending on the space in which the game is played. Both players can follow certain rules as the game progresses. In 1953, D. Gale and J. D. Stewart introduced the notion of an infinite two-player game of perfect information and began a systematic study of these games. The games that we describe fall into this category. We will start by giving some important definitions of games and then we will deal with the concept of determinacy. In addition, we study extensively some topological games, as well as games associated with measures.

6.1. Gale-Stewart Games

Here we describe how a Gale-Stewart game is played. Let X be a fixed non-empty set and $A \subseteq X^{\mathbb{N}}$ be a set of infinite sequences from X. For two players, I (we refer to him as "He") and II (we refer to her as "She"), with each set A we associate a two-person game $G = G_X(A)$ as follows. Players I and II alternatively choose members of X ad infinitum. The game can be visualized through the Diagram 6.1, moving from left to right, with the moves of player I above and the moves of player II below (player I always starts the game by playing first).

$$\begin{array}{lcccccccccccc} \text{I} & a_0 & & & & a_2 & & & & a_4 & & & \cdot \\ & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow & & \searrow \\ \text{II} & & & a_1 & & & & a_3 & & & \cdot & & & \ldots \end{array}$$

Diagram 6.1. Playing the game $G_X(A)$.

The play continues without ending so that a single play of the game determines an infinite sequence

$$f = (a_0, a_1, a_2, a_3, \dots) \in X^{\mathbb{N}}.$$

Player I **wins** if $f \in A$, otherwise, if $f \notin A$ player II **wins**. Player I is allowed to see all $a_0, a_1, \dots, a_{n-1}$ which have preceded before he chooses a_n, for n even. Similarly, player II is allowed to see all $a_0, a_1, \dots, a_{n-1}$ for n odd. Thus, we call this the **Game of Perfect Information**.

This way, we have described a **run** (or **play**) f of the game $G = G_X(A)$. The set A is the **payoff** (or else winning) **set** for $G_X(A)$, but we will often identify A with $G_X(A)$ and when talking about the game A. Let us now explain what a strategy and a winning strategy mean through the following definitions.

Definition 6.1.1. A **strategy for player** I is any function σ with the domain being all finite sequences from X of *even length* (including the empty sequence Λ) and values in X, i.e.

$$\sigma : \{u \in X^{<\mathbb{N}} \ : \ |u| \text{ is even}\} \to X.$$

We say that player I follows (or plays) σ in a run $f = (a_0, a_1, a_2, a_3, \dots)$ of the game $G_X(A)$, if

$$\begin{aligned} a_0 &= \sigma(\Lambda), \\ a_2 &= \sigma(a_0, a_1), \\ &\vdots \\ a_n &= \sigma(a_0, a_1, \dots, a_{n-1}), \quad \text{for } n \text{ even}. \end{aligned}$$

Definition 6.1.2. A **strategy for player** II is any function τ with the domain being all finite sequences from X of *odd length* and values in X, i.e.

$$\tau : \{u \in X^{<\mathbb{N}} \ : \ |u| \text{ is odd}\} \to X.$$

Remark 6.1.3. If players I and II play with strategies σ and τ respectively, then exactly one run is produced, which we denote by $\sigma * \tau$. The run $\sigma * \tau$ is shown in the Diagram 6.2.

$$\begin{array}{llllllllllllll}
\mathrm{I} & \sigma(\Lambda) & & & & \sigma(a_0,a_1) & & & & \sigma(a_0,a_1,a_2,a_3) & & & & \cdots \\
 & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\mathrm{II} & & & \tau(a_0) & & & & \tau(a_0,a_1,a_2) & & & & \cdots & &
\end{array}$$

Diagram 6.2. When player I plays σ against player II's τ.

It is obvious that $\sigma * \tau = (a_0, a_1, a_2, \dots)$ and that the player I plays $a_n = \sigma(a_0, \cdots, a_{n-1})$, for n even, on his n^{th} move, and player II plays $a_n = \tau(a_0, \dots, a_{n-1})$, for n odd, on her n^{th} move.

Definition 6.1.4. We call σ a **winning strategy for player** I, if for every II's strategy τ,

$$\sigma * \tau \in A,$$

i.e. whatever II is playing, player I always wins when he plays σ.

Definition 6.1.5. We call τ a **winning strategy for player** II, if for every I's strategy σ,

$$\sigma * \tau \notin A.$$

6.2. Determinacy

Definition 6.2.1. We say that **the game** $G = G_X(A)$ **is determined** if either player I or player II has a winning strategy, i.e. one of them wins the game G, as we will say from now on.

Remark 6.2.2. Since we have already identified the game $G = G_X(A)$ with A, we will call the set A determined, too. It is important to note that there are games that are undetermined.

Remark 6.2.3. For an intuitive understanding, note that determinacy can be rendered as an infinite sequence of alternating quantifiers. For example, for games of infinite length, we have:

• Player I having a winning strategy in $G_X(A)$ (i.e. I wins $G_X(A)$) is equivalent to

$$\exists a_1 \, \forall a_2 \, \exists a_3 \, \forall a_4 \, \exists a_5 \, \forall a_6 \, \dots \, \exists a_n \, \forall a_{n+1} \, \dots \quad (\alpha \in A) \tag{6.1}$$

• Player II having a winning strategy in $G_X(A)$ (i.e. II wins $G_X(A)$) is equivalent to

$$\forall a_1 \, \exists a_2 \, \forall a_3 \, \exists a_4 \, \forall a_5 \, \exists a_6 \, \dots \, \forall a_n \, \exists a_{n+1} \, \dots \quad (\alpha \notin A) \tag{6.2}$$

where a_n, for n odd are I's moves and a_n, for n even are II's moves.

The game $G_X(A)$ is determined, meaning intuitively that the negation of the expression (6.1) is the expression (6.2).

Definition 6.2.4. If $A \subseteq X^{\mathbb{N}}$ and $u = (a_0, \dots, a_{n-1})$ is a sequence of even length, the **subgame** of A at u is

$$A(u) = \{f \in X^{\mathbb{N}} : (a_0, a_1, \dots, a_{n-1}, f(0), f(1), \dots) \in A\}.$$

Lemma 6.2.5 ((**AC**), [**8**]). *Let $A \subseteq X^{\mathbb{N}}$ and suppose $u = (a_0, a_1, \dots, a_{n-1})$ is a finite sequence from X of even length. If player* II *does not win the game $A(u)$, then there is some a such that for all b, player* II *does not win $A(u * (a, b))$.*

Proof. Towards a contradiction, suppose player II does not win the game $A(u)$, but that for each a, there is some b and a strategy τ which is winning for II in $A(u * (a, b))$. By using the Axiom of Choice, there is a function of choice

$$a \mapsto (b^a, \tau^a)$$

sending each a to some b^a and τ^a with these properties. From this function we conclude that if the player I starts by playing a_0, then the player II can answer by playing b^{a_0} and then following τ^{a_0} as if

she were playing in the game $A(u * (a_0, b^{a_0}))$.
To visualize the game we are describing, we present the Diagrams 6.3 and 6.4 - we set II's move b^{a_0} as a_1.

$$\begin{array}{cccccccccccccccc}
\mathrm{I} & \boldsymbol{u_0} & & & & \boldsymbol{u_2} & & & & \boldsymbol{\dots} & & & & a_0 & & \\
 & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
\mathrm{II} & & & \boldsymbol{u_1} & & & & \boldsymbol{\dots} & & & & \boldsymbol{u_{n-1}} & & & & a_1 = b^{a_0}
\end{array}$$

Diagram 6.3. Playing the game $A(u * (a_0, b^{a_0}))$ (the subgame $A(u)$ appears in bold).

$$\begin{array}{cccccccccccccccccc}
\mathrm{I} & \boldsymbol{u_0} & & & & \boldsymbol{\dots} & & & & a_0 & & & & a_2 & & & & \dots \\
 & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\mathrm{II} & & & \boldsymbol{\dots} & & & & \boldsymbol{u_{n-1}} & & & & a_1 = b^{a_0} & & & & a_3 & &
\end{array}$$

Diagram 6.4. Playing the game $A(u * (a, b))$ (the subgame $A(u)$ appears in bold).

We will now define player II's winning strategy as follows:

$$\begin{aligned}
&\tau(a_0) = b^{a_0} \\
&\tau(a_0, \dots, a_n) = \tau^{a_0}(a_2, \dots, a_n), \quad n > 1, \quad n \text{ even.}
\end{aligned}$$

By using strategy τ, we notice that for player II's moves $a_1, a_3, a_5, \dots$, applies

$$a_1 = \tau(a_0) = b^{a_0}, \quad a_3 = \tau^{a_0}(a_2), \quad a_5 = \tau^{a_0}(a_2, a_3, a_4), \quad \dots$$

So the game we described takes the following form, in Diagram 6.5.

$$\begin{array}{cccccccccccccccccc}
\mathrm{I} & \boldsymbol{u_0} & & & & \boldsymbol{\dots} & & & & a_0 & & & & a_2 & & & & \dots \\
 & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\mathrm{II} & & & \boldsymbol{\dots} & & & & \boldsymbol{u_{n-1}} & & & & b^{a_0} & & & & \tau^{a_0}(a_2) & &
\end{array}$$

Diagram 6.5. Playing the game $A(u * (a, b))$ when τ is II's winning strategy.

The sequence

$$(a_0, a_1, a_2, a_3, \dots) = (a_0, b^{a_0}, a_2, \tau^{a_0}(a_2), \dots)$$

produces a run of the game $A(u)$. Denoting this run by f, we have

$$f(0) = a_0, \ \ f(1) = a_1, \ \ \dots, \ \ f(n) = a_n, \ \ \dots$$

Suppose that I plays $f = (f(0), f(1), \dots, f(n), \dots)$ in $A(u)$ while II responds by τ, then

$$(f(2), f(3), \dots, f(n), \dots) \notin A(u * (a_0, b^{a_0})) \iff (a_2, a_3, \dots, a_n, \dots) \notin A(u * (a_0, a_1))$$

since II has been following τ^{a_0} after the first two moves. Hence,

$$\begin{aligned}
(a_0, a_1, \dots, a_n, \dots) \notin A(u) &\iff (f(0), f(1), \dots, f(n), \dots) \notin A(u) \\
&\iff f \notin A(u).
\end{aligned}$$

Therefore, player II has won this run of $A(u)$ and this is a contradiction to our hypothesis. $\square$

Theorem 6.2.6 ((**AC**), [**1**]). *(Gale-Stewart) For each $X \neq \varnothing$, every closed subset of $X^{\mathbb{N}}$ is determined.*

Proof. We use the product topology on $X^{\mathbb{N}}$ (with X discrete). Suppose that $A \subseteq X^{\mathbb{N}}$ and player II does not have a winning strategy in A. We will describe how I can play to win:

By Lemma 6.2.5, there is some a_0 such that for every b, she cannot win the subgame $A(a_0, b)$ of A. Let player I start the game by playing some a_0 with this property and let II answer by some a_1. Now player II cannot win $A(a_0, a_1)$.

Applying again Lemma 6.2.5, this time in subgame $A(a_0, a_1)$. Since II does not win in $A(a_0, a_1)$, there is some a_2 such that for every b, II cannot win the subgame $A(a_0, a_1, a_2, b)$.

Let player I play one such a_2 and continue in the same way. At the end of this run of the game, we have a play:

$$f = (a_0, a_1, a_2, \dots).$$

For every even n, II cannot win $A(a_0, a_1, a_2, \dots, a_{n-1})$. This implies that there is some $f_n \in X^{\mathbb{N}}$ with

$$f_n(0) = a_0,\ f_1 = a_1, \dots,\ f_n(n-1) = a_{n-1}, \quad \text{and} \quad f_n \in A.$$

Otherwise, player II could win $A(a_0, a_1, a_2, \dots, a_{n-1})$ by making random movements. We note that

$$\lim_{n\to\infty} f_n = f$$

and because A is a closed subset of $X^{\mathbb{N}}$, it follows that

$$f \in A$$

($A \subseteq X^{\mathbb{N}}$ is closed if and only if every convergent sequence in $X^{\mathbb{N}}$ completely contained in A has its limit in A). Therefore, player I wins A. $\square$

Theorem 6.2.7 ((**AC**), Wolfe [**12**]). *For each $X \neq \varnothing$, every $\utilde{\mathbf{\Sigma}}^0_2$ subset of $X^{\mathbb{N}}$ is determined.*

Proof. Let $A \subseteq X^{\mathbb{N}}$ and $A \in \utilde{\mathbf{\Sigma}}^0_2$. Because of $A \in \utilde{\mathbf{\Sigma}}^0_\xi$, for $\xi = 2$, it can be written as follows:

$$A = \bigcup_{i\in\mathbb{N}} F_i,$$

for suitable $F_0, F_1, \dots$ where each F_i is in $\utilde{\mathbf{\Pi}}^0_1$, therefore is closed.

We also choose trees T^i on X such that

$$F_i = [T^i] = \{f \in X^{\mathbb{N}} : \forall k\, (\, f(0), f(1), \dots, f(k-1)) \in T^i\,\}. \tag{6.3}$$

To prove that $A \in \utilde{\mathbf{\Sigma}}^0_2$ is determined, we will prove that either player I or player II wins in A.

• For player I, we will define a set of "Sure Winning Positions" in A, i.e. a set W such that it is I's turn to play and

$$u \in W \implies \text{I wins } A(u).$$

• For player II, we will show that if $\Lambda \notin W$, then she wins in A. In this way, we will have established the determinacy of A.

First of all, let us put

$$u \in W^0 \iff \exists i\,[\text{player I wins } F^i(u)]\,,$$

i.e. if $u \in W^0$, then I wins $A(u)$ trivially, by playing to get into a specific closed set F_i. So, the positions in W^0 have a simple strategy: just choose an i and play to get into F_i. (Furthermore, all even finite segments are in W^0.)

Suppose now that W^η has been defined for each $\eta < \xi$ and for $i \in \mathbb{N}$ put

$$f \in H^{\xi,i} \iff \forall \text{ even } k \left[(f(0), \dots, f(k-1)) \in \bigcup_{\eta<\xi} W^\eta \cup T^i \right].$$

Let $f_n \in H^{\xi,i}$, $n \in \mathbb{N}$ and $f_n \longrightarrow f$. We show that $f \in H^{\xi,i}$. Let $k \in \mathbb{N}$ be even. Then for all large $n \in \mathbb{N}$, we have $f_n|k = f|k$. Let one such $n \in \mathbb{N}$. Since $f_n \in H^{\xi,i}$, we have

$$f_n|k \in \bigcup_{\eta<\xi} W^\eta \cup T^i,$$

therefore

$$f|k \in \bigcup_{\eta<\xi} W^{\eta} \cup T^{i}.$$

So for all even k,

$$f|k \in \bigcup_{\eta<\xi} W^{\eta} \cup T^{i}$$

and hence $f \in H^{\xi,i}$. Therefore, $H^{\xi,i}$ is a closed set. Let

$$u \in W^{\xi} \iff \exists i\,[\text{player I wins the game } H^{\xi,i}(u)]$$

and

$$W = \bigcup_{\xi} W^{\xi},$$

where ξ is an ordinal. These W^{ξ} are finite partial plays (i.e. the sets of "Positions" to which we have already touched upon) in which player I is next to play and has a winning strategy, so these are sets of Sure Winning for I. Therefore, as we can see from the last two equations if I has a strategy to get into one of the $H^{\xi,i}$, then he can win overall.

We will prove by Transfinite Induction on ξ that

$$u \in W^{\xi} \implies \text{player I wins } A(u), \tag{6.4}$$

for all u of even length.

<u>Inductive hypothesis:</u> Suppose that (6.4) is true for all ordinals $\eta < \xi$.

Zero case: We will prove that (6.4) is true for $\xi = 0$.

Case $\xi > 0$: Let $u \in W^{\xi}$. Player I wins $A(u)$ as follows:

Choose i so that I wins $H^{\xi,i}(u)$ (u of even length) and let σ be I's winning strategy in $H^{\xi,i}(u)$.

> If $u \in T^{i}$, then $a_0 = \sigma(\Lambda)$ and player I keeps on playing the winning strategy σ.
>
> If $u * (a_0, a_1) \in T^{i}$, then I keeps on playing σ.
>
> If $u * (a_0, a_1, a_2, a_3) \in T^{i}$, then I keeps on playing σ.
>
> $\vdots$

Continuing in the same way, the run $f = (a_0, a_1, \dots, a_n, \dots)$ is produced, with

$$u * (a_0, \dots, a_{k-1}) \in T^{i}, \text{ for all even } k.$$

The Diagram 6.6 below, shows the illustration of the game we are describing.

$$\begin{array}{lccccccccccccc}
\text{I} & \boldsymbol{u_0} & & & & \boldsymbol{\dots} & & & & a_0 = \sigma(\Lambda) & & & & a_2 = \sigma(a_0, a_1) & & \\
& & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
\text{II} & & & \boldsymbol{\dots} & & & & \boldsymbol{u_{n-1}} & & & & a_1 & & & & \dots
\end{array}$$

Diagram 6.6. Playing the game $H^{\xi,i}(u)$ (the subgame $A(u)$, $u \in W^{\xi}$, appears in bold).

Thus,

$$u * f \in [T^{i}] = F^{i} \subseteq A \implies u * f \in A \implies f \in A(u).$$

So, player I wins $A(u)$. If now, as the run progresses, $u * (a_0, \dots, a_{k-1}) \in T^{i}$ doesn't happen, for all even k, then there is even k, so that

$$u * (a_0, \dots, a_{k-1}) \notin T^{i}.$$

We choose the least such even k. We continue the run by letting II play a fixed x_0 and I following σ. This produces a run

$$u * (a_0, \ldots, a_{k-1}) * g.$$

Since I wins $H^{\xi,i}$, we have

$$u * (a_0, \ldots, a_{k-1}) \in \bigcup_{\eta<\xi} W^\eta \cup T^i.$$

But,

$$u * (a_0, \ldots, a_{k-1}) \notin T^i,$$

so

$$u * (a_0, \ldots, a_{k-1}) \in \bigcup_{\eta<\xi} W^\eta.$$

He then chooses $\eta < \xi$, such that

$$u * (a_0, \ldots, a_{k-1}) \in W^\eta.$$

Therefore, by inductive hypothesis, I wins $A(u * (a_0, \ldots, a_{k-1}))$.
Let σ' be a winning strategy for player I, and I continues with σ'. Therefore, the following run is produced:

$$(a_0, \ldots, a_{k-1}, g(0), \ldots, g(n), \ldots),$$

where $g(0), \ldots, g(n), \ldots$ are the movements resulting from following σ'. Then

$$(g(0), \ldots, g(n), \ldots) \in A(u * (a_0, \ldots, a_{k-1})),$$

or equivalently

$$(a_0, \ldots, a_{k-1}) * g \in A(u).$$

This shows how I wins $A(u)$. In particular, (6.4) implies that

$$\Lambda \in W \implies \text{player I wins } A.$$

We will prove that

$$\Lambda \notin W \implies \text{player II wins } A.$$

We notice that $\forall i$, it holds

$$\eta \leqslant \xi \implies H^{\eta,i} \subseteq H^{\xi,i}.$$

This is true only because the underlying set involves a union of more sets, i.e. in particular

$$\bigcup_{\lambda<\eta} W^\lambda \cup T^i \subseteq \bigcup_{\lambda<\xi} W^\lambda \cup T^i,$$

for all $\eta < \xi$. Hence,

$$\eta \leqslant \xi \implies W^\eta \subseteq W^\xi.$$

Since they can't keep growing forever, there is some ordinal κ such that

$$W^{\kappa+1} = W^\kappa = W.$$

Suppose now that $\Lambda \notin W^{\kappa+1}$. We will describe how player II can play to win A:
By the definition of $W^{\kappa+1}$ and the determinacy of each closed game $H^{\kappa+1,i}$, player II can actually win every $H^{\kappa,i}$, since

$$u \in W^\kappa \iff \exists i \,[\text{I wins in } H^{\kappa,i}(u)]\,.$$

So,

$$u \notin W^\kappa \iff \forall i \,[\text{II wins in } H^{\kappa,i}(u)]\,.$$

Let her start by playing to win $H^{\kappa,0}$, for $i = 0$. After a while (k-moves), a finite sequence $(c_0, \ldots, c_{k-1})$ has been played and

$$(c_0, \ldots, c_{k-1}) \notin W^\kappa \;\&\; (c_0, \ldots, c_{k-1}) \notin T^0.$$

No matter how the game continues, we know at this stage that the final play will not be in F^0:

$$u \in W^0 \iff \exists i \,[\text{I wins in } F^i(u)]$$

and

$$A(u) = \bigcup_{i\in\mathbb{N}} F^i(u).$$

Therefore, player I wins $A(u)$ by playing to get into a certain closed set F^i (specifically, F^0, for $i = 0$). Thus, since II wins A, the final game will not be on F^0. Let k_0 be the first k in which this happens and using $W^\kappa = W^{\kappa+1}$, let II switch to a strategy so she can win $H^{\kappa+1,1}(c_0, \dots, c_{k_0-1})$. Again, some $k > k_0$ is reached so that

$$(c_0, \dots, c_{k_0-1}, \dots, c_{k-1}) \notin W^\kappa \ \ \& \ \ (c_0, \dots, c_{k-1}) \notin T^1.$$

At this point we have ensured that the final play will not be in F^1. Player II can continue to play in this way and ensure that the final play will not be in any of the sets $F^0, F^1, F^2, \dots$ thereby winning A. □

Remark 6.2.8. From now on we will refer to the *determinacy of a set A in pointclass* Γ, through the following equivalence. If Γ is a collection of sets, put

$$\mathrm{Det}_X(\Gamma) \iff \text{for every set } A \subseteq X^{\mathbb{N}} \text{ in } \Gamma, \text{ the game } G_X(A) \text{ is determined.}$$

We will be particularly interested in the hypotheses $\mathrm{Det}_{\mathbb{N}}(\Gamma)$ and $\mathrm{Det}_2(\Gamma)$, where $2 = \{0, 1\}$, with Γ being one of the pointclasses to which we have referred.

Theorem 6.2.9 ([**8**]). *Suppose Γ is a collection of subsets of some $X^{\mathbb{N}}$ which is closed under continuous substitution. Then*

$$\mathrm{Det}_X(\Gamma) \iff \mathrm{Det}_X(\neg\Gamma).$$

Proof. Given $A \subseteq X^{\mathbb{N}}$ in $\neg\Gamma$, let

$$B = \{(x, f(0), f(1), \dots) : x \in X, f \notin A\}.$$

It is enough to show that

$$\mathrm{Det}_X(\Gamma) \implies \mathrm{Det}_X(\neg\Gamma),$$

since we can then replace Γ by $\neg\Gamma$ (which is closed under continuous substitution) and get

$$\mathrm{Det}_X(\neg\Gamma) \implies \mathrm{Det}_X(\neg\neg\Gamma) \equiv \mathrm{Det}_X(\Gamma).$$

Assume $\mathrm{Det}_X(\Gamma)$ and $A \subseteq X^{\mathbb{N}}$ be in $\neg\Gamma$. Then the preceding B is in Γ and hence it is determined. We show that

$$\text{I wins } G_X(B) \implies \text{II wins } G_X(A) \tag{6.5}$$

and also

$$\text{II wins } G_X(B) \implies \text{I wins } G_X(A). \tag{6.6}$$

If we show (6.5) and (6.6) we have that $G_X(B)$ is determined.

For (6.5) suppose that I wins $G_X(B)$ and let σ be a winning strategy for I in $G_X(B)$. We describe how II can win $X^{\mathbb{N}} \backslash A$. Let $x_0 = \sigma(\Lambda)$. Let I play a_0 in $G_X(A)$. Player II copies a_0 in $G_X(B)$ and registers the answer of I in $G_X(B)$ according to σ, i.e

$$a_1 = \sigma(x_0, a_0).$$

This a_1 is the answer of II in $G_X(A)$. Diagrams 6.7 and 6.8 below show games $G_X(A)$ and $G_X(B)$, respectively (the copies of player I's moves in $G_X(A)$ from player II in $G_X(B)$ appears in bold).

$$\begin{array}{lccccccccc}
\text{I} & \boldsymbol{a_0} & & & & \boldsymbol{a_2} & & & \cdots & \\
 & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
\text{II} & & & a_1 & & & & a_3 & & & & \cdots
\end{array}$$

Diagram 6.7. Playing the game $G_X(A)$.

$$\begin{array}{lccccccccc}
\text{I} & x_0 = \sigma(\Lambda) & & & & a_1 = \sigma(x_0, a_0) & & & & a_3 = \sigma(x_0, a_0, a_1, a_2) & & \\
 & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
\text{II} & & & \boldsymbol{a_0} & & & & \boldsymbol{a_2} & & & & \cdots
\end{array}$$

Diagram 6.8. Playing the game $G_X(B)$, where σ is I's winning strategy.

In a similar fashion, if I plays a_2 in $G_X(A)$, we take

$$a_3 = \sigma(x_0, a_0, a_1, a_2)$$

and II plays a_3 in $G_X(A)$. If

$$(a_0, a_1, a_2, a_3, \dots)$$

is a run in $G_X(A)$ where II has played as above, since σ is a winning strategy for I in $G_X(B)$, we have

$$(x_0, a_0, a_1, a_2, a_3, \dots) \in B$$

hence

$$(a_0, a_1, a_2, a_3, \dots) \notin A.$$

So II wins the run $(a_0, a_1, a_2, a_3, \dots)$ in $G_X(A)$.
The implication (6.6) is proved similarly. □

Thus far we have shown $\mathrm{Det}_X(\mathbf{\Sigma}^0_2)$, for every X. The determinacy of the dual class $\mathbf{\Pi}^0_2$ follows from the previous Theorem.

Corollary 6.2.10. *For each X, every $\mathbf{\Pi}^0_2$ subset of $X^{\mathbb{N}}$ is determined, i.e.* $\mathrm{Det}_X(\mathbf{\Pi}^0_2)$.

Proof. From the definition of Borel pointclasses of finite order, we know that $\mathbf{\Pi}^0_2 = \mathrm{c}\mathbf{\Sigma}^0_2$. Therefore, if $A \in \mathbf{\Sigma}^0_2$, then $\mathrm{c}A \in \mathbf{\Pi}^0_2$. From Theorem (6.2.7) we know that A is determined. By applying the previous Theorem, it transpires that

$$\mathrm{Det}_X(\mathbf{\Sigma}^0_2) \implies \mathrm{Det}_X(\mathrm{c}\mathbf{\Sigma}^0_2) = \mathrm{Det}_X(\mathbf{\Pi}^0_2).$$

So $\mathrm{c}A \in \mathbf{\Pi}^0_2$ is determined. □

Proposition 6.2.11. *If Γ is a pointclass closed under continuous substitution, then*

$$\mathrm{Det}_{\mathbb{N}}(\Gamma) \implies \mathrm{Det}_2(\Gamma).$$

Proof. Given $A \subseteq 2^{\mathbb{N}}$ in Γ, define $g : \mathcal{N} \to 2^{\mathbb{N}}$ by

$$g(\alpha)(n) = \begin{cases} 0, & \text{if } \alpha(n) = 0 \\ 1, & \text{if } \alpha(n) > 0, \end{cases}$$

and let

(6.7) $$\alpha \in B \iff g(\alpha) \in A.$$

Firstly, we will show that the set B is in Γ. We have that $g : \mathcal{N} \to 2^{\mathbb{N}}$ and $B \subseteq \mathcal{N}$, so since A is in Γ, the pointclass Γ is closed under continuous substitution and (6.7) holds, it follows that B is in Γ, too. We will prove that the player who wins $G_{\mathbb{N}}(B)$ also wins $G_2(A)$. Diagrams 6.9 and 6.10 show the illustrations of the games $G_{\mathbb{N}}(B)$ and $G_2(A)$, respectively.

$$\begin{array}{llccccccc}
\text{I} & a_0(n=0) & & & & a_2(n=2) & & & \cdots \\
 & & \searrow & & \nearrow & & \searrow & \nearrow & \\
\text{II} & & & a_1(n=1) & & & \dots & &
\end{array}$$

Diagram 6.9. Playing the game $G_{\mathbb{N}}(B) \subseteq \mathcal{N}$.

$$\begin{array}{llccccccc}
\text{I} & g(\alpha)(0)(n=0) & & & & g(\alpha)(2)(n=2) & & & \cdots \\
 & & \searrow & & \nearrow & & \searrow & \nearrow & \\
\text{II} & & & g(\alpha)(1)(n=1) & & & \dots & &
\end{array}$$

Diagram 6.10. Playing the game $G_2(A) \subseteq 2^{\mathbb{N}}$.

Suppose that the game $G_{\mathbb{N}}(B)$ is determined. If player I wins B, let a run α, where I has followed his strategy

$$\alpha \in B \implies g(\alpha) \in A \implies \text{I wins } A.$$

In the case player II wins B, let a run α, where II has followed her strategy

$$\alpha \notin B \implies g(\alpha) \notin A \implies \text{II wins } A.$$

So we have shown that if $G_{\mathbb{N}}(B)$ is determined, the game $G_2(A)$ is determined, too. Therefore, the implication is proved. $\square$

6.3. The G^*-Games

Here we introduce in our study a special category of games of special topological significance. These games will be denoted by G^*_X. Given $A \subseteq X^{\mathbb{N}}$, the game $G^*_X(A)$ is played as follows:

In this game, player I chooses a finite (non-empty) sequence from X, then player II chooses a single member from X, then I chooses a finite (non-empty) sequence from X, etc. ad infinitum. So, player I is favored since he is allowed to play more than one point from X if he wishes. Diagram 6.11 shows the illustration of the game $G^*_X(A)$. Player I **wins** $G^*_X(A)$ if the play $f = (a_0, a_1, \dots)$ is in A, i.e., $f \in A$. Otherwise, player II **wins**. In particular, if I wins $G_X(A)$, he obviously wins $G^*_X(A)$ too. Strategies and winning strategies for these games are defined in an obvious way. In addition, the following equivalence applies

$$\text{Det}^*_X(\Gamma) \iff \text{for each } A \subseteq X^{\mathbb{N}} \text{ in } \Gamma, \text{ either I or II wins the game } G^*_X(A).$$

$$\begin{array}{llllllllll}
\text{I} & a_0, \dots, a_{k_0-1} & \searrow & & \nearrow & a_{k_0+1}, \dots, a_{k_1-1} & \searrow & & \nearrow & a_{k_1+1}, \dots, a_{k_2-1} \quad \searrow \\
\text{II} & & & a_{k_0} & & & & a_{k_1} & & \qquad\qquad \dots
\end{array}$$

Diagram 6.11. Playing the game $G^*_X(A)$.

Proposition 6.3.1. *If Γ is a pointclass closed under continuous substitution, containing the closed sets, then*

$$\text{Det}_{\mathbb{N}}(\Gamma) \implies \text{Det}^*_{\mathbb{N}}(\Gamma) \implies \text{Det}^*_2(\Gamma).$$

Proof. For the first implication, we define

$$f : \mathbb{N} \to \mathbb{N}^{<\mathbb{N}} : f(s) = \begin{cases} ((s)_0, \dots, (s)_{\text{lh}(s)-1}), & \text{if } s \in \text{Seq} \\ \qquad\Lambda, & \text{if } s \notin \text{Seq}, \end{cases}$$

where $s = \langle (s)_0, \dots, (s)_{\text{lh}(s)-1} \rangle$. Also, we define the continuous function $g : \mathcal{N} \to \mathcal{N}$, with

$$g(\alpha) = f(\alpha(0)) * \alpha(1) * f(\alpha(2)) * \alpha(3) * \dots$$

and

$$\text{Seq}^* = \text{Seq} \setminus \{\Lambda\}.$$

So,

$$\text{Seq}^* = \{s \in \text{Seq} \mid s > 1\}.$$

Given $A \subseteq \mathbb{N}^{\mathbb{N}} = \mathcal{N}$ in Γ, we will show that $G^*_{\mathbb{N}}(A)$ is determined. Let us define the set

$$B = g^{-1}[A] \cap \{\alpha \mid \forall n \ \alpha(2n) \in \text{Seq}^*\},$$

where we identify $\alpha(2n)$ with the finite sequence it codes (e.g. $\alpha(0) = \langle u_0, u_1, \dots, u_{k_0-1} \rangle$, $k_0 \geqslant 1$) and let

$$\alpha \in B \iff g(\alpha) \in A \ \ \& \ \ \forall n \ \alpha(2n) \in \text{Seq}^*. \tag{6.8}$$

We notice that B is in Γ because $g^{-1}[A] \in \Gamma$ (because A is in Γ and Γ is closed under continuous substitution) and $\{\alpha \mid \forall n \ \alpha(2n) \in \text{Seq}^*\} \in \Gamma$ (because it is a closed set), thus their intersection is in pointclass Γ.

• To prove the first implication, we will show that the player who wins $G_{\mathbb{N}}(B)$ also wins $G^*_{\mathbb{N}}(A)$. Firstly, let us suppose that player I wins in $G_{\mathbb{N}}(B)$. We will show that player I wins in $G^*_{\mathbb{N}}(A)$, too.

We fix a winning strategy σ for player I in $G_{\mathbb{N}}(B)$, therefore, the moves of player I are described as follows

$$a_0 = \sigma(\Lambda),\ \ a_2 = \sigma(a_0, n_0),\ \ a_4 = \sigma(a_0, n_0, n_1),\ \ \ldots$$

The games $G_{\mathbb{N}}(B)$ and $G^*_{\mathbb{N}}(A)$ are visualized in Diagrams 6.12 and 6.13 below, respectively.

$$\begin{array}{lccccccccc}
\mathrm{I} & \sigma(\Lambda) & & & & \sigma(a_0,n_0) & & & & \sigma(a_0,n_0,n_1) & \\
& & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
\mathrm{II} & & & n_0 = a_1 & & & & n_1 = a_3 & & & & \ldots
\end{array}$$

Diagram 6.12. Playing the run α of $G_{\mathbb{N}}(B)$, when player I has a winning strategy σ.

$$\begin{array}{lccccccccc}
\mathrm{I} & f(a_0) & & & & f(a_2) & & & & f(a_4) & \\
& & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
\mathrm{II} & & & a_1 = u_{k_0} \in \mathbb{N} & & & & a_3 = u_{k_1} \in \mathbb{N} & & & & \ldots
\end{array}$$

Diagram 6.13. Playing the run $g(\alpha)$ of the game $G^*_{\mathbb{N}}(A)$.

In the game $G^*_{\mathbb{N}}(A)$, the moves of player I code some finite sequences in the following way:

$$f(a_0) = f(\langle u_0, u_1, \ldots, u_{k_0-1}\rangle),\ \ f(a_2) = f(\langle u_{k_0+1}, \ldots, u_{k_1-1}\rangle),\ \ \ldots$$

So, if player I wins in $G_{\mathbb{N}}(B)$, we have

$$\alpha \in B \implies \sigma(\Lambda) * n_0 * \sigma(a_0, n_0) * n_1 * \sigma(a_0, n_0, n_1) * \ldots \in B.$$

Then, by (6.8) it follows that

$$\alpha \in B \implies g(\alpha) \in A \ \&\ \forall n\ \alpha(2n) \in \mathrm{Seq}^*.$$

Therefore, player I also wins in $G^*_{\mathbb{N}}(A)$.

In the same way, we now assume that player II wins in $G_{\mathbb{N}}(B)$. We will show that player II wins in $G^*_{\mathbb{N}}(A)$, too. Let us fix a winning strategy τ for player II in $G_{\mathbb{N}}(B)$, therefore, her moves are described as follows

$$n_0 = \tau(\langle u_0, \ldots, u_{\mathrm{lh}(u_0)-1}\rangle),\ \ n_1 = \tau(n_0, \langle u_1, \ldots, u_{\mathrm{lh}(u_1)-1}\rangle),\ \ \ldots$$

The moves of player I in $G_{\mathbb{N}}(B)$ are codes of finite sequences:

$$a_0 = \langle u_0, \ldots, u_{\mathrm{lh}(u_0)-1}\rangle,\ \ a_2 = \langle u_1, \ldots, u_{\mathrm{lh}(u_1)-1}\rangle,\ \ \ldots$$

Additionally, for player I's moves in $G^*_{\mathbb{N}}(A)$ we have

$$f(a_0) = f(\langle u_0, u_1, \ldots, u_{k_0-1}\rangle),\ \ f(a_2) = f(\langle u_{k_0+1}, \ldots, u_{k_1-1}\rangle),\ \ \ldots$$

The games $G_{\mathbb{N}}(B)$ and $G^*_{\mathbb{N}}(A)$ are visualised in Diagrams 6.14 and 6.15 below, respectively.

$$\begin{array}{lccccccccc}
\mathrm{I} & a_0 & & & & a_2 & & & & \ldots & \\
& & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
\mathrm{II} & & & \tau(\langle u_0, \ldots, u_{\mathrm{lh}(u_0)-1}\rangle) & & & & \tau(n_0, \langle u_1, \ldots, u_{\mathrm{lh}(u_1)-1}\rangle) & & & & \ldots
\end{array}$$

Diagram 6.14. Playing the run α_0 of $G_{\mathbb{N}}(B)$, when player II has a winning strategy τ.

$$\begin{array}{lcccccccc}
\mathrm{I} & f(\langle u_0, u_1, \ldots, u_{k_0-1}\rangle) & & & & f(\langle u_{k_0+1}, \ldots, u_{k_1-1}\rangle) & & & \ldots \\
& & \searrow & & \nearrow & & \searrow & & \nearrow \\
\mathrm{II} & & & a_1 = u_{k_0} \in \mathbb{N} & & & & \ldots &
\end{array}$$

Diagram 6.15. Playing the run $g(\alpha)$ of the game $G^*_{\mathbb{N}}(A)$.

If player II wins in $G_{\mathbb{N}}(B)$, we have

$$\alpha \notin B \implies \langle u_0, \ldots, u_{\mathrm{lh}(u_0)-1}\rangle * n_0 * \langle u_1, \ldots, u_{\mathrm{lh}(u_1)-1}\rangle * n_1 * \ldots \notin B.$$

Then, by (6.8) we have

$$\alpha \notin B \implies g(\alpha) \notin A \ \vee \ \exists n \ \alpha(2n) \notin \mathrm{Seq}^*$$

and since the case $\exists n \ \alpha(2n) \notin \mathrm{Seq}^*$ doesn't occur from the definition of α, player II wins $G^*_{\mathbb{N}}(A)$, too.

• For the second implication, we will apply the method of Proposition 6.2.11:

Given $D \subseteq 2^{\mathbb{N}}$ in Γ, define $h : \mathcal{N} \to 2^{\mathbb{N}}$ by

$$h(\alpha)(n) = \begin{cases} 0, & \text{if } \alpha(n) = 0 \\ 1, & \text{if } \alpha(n) > 0, \end{cases}$$

and let

$$g(\alpha) \in C \ \ \& \ \ \forall n \ \alpha(2n) \in \mathrm{Seq}^* \iff h(g(\alpha)) \in D, \tag{6.9}$$

where $C \subseteq \mathcal{N}$. Equivalently,

$$h(g(\alpha)) \in D \iff h(g(\alpha)) = h(f(\alpha(0))) * h(\alpha(1)) * h(f(\alpha(2))) * h(\alpha(3)) * \ldots$$

We will prove that the player who wins the game $G^*_{\mathbb{N}}(C)$, also wins the $G^*_2(D)$. We fix a winning strategy σ' for player I in $G^*_{\mathbb{N}}(C)$, so, his moves are described as follows

$$\sigma'(\Lambda) = f(\langle u_0, u_1, \ldots, u_{k_0-1}\rangle), \ \ \sigma'(f(a_0, a_1)) = f(\langle u_{k_0+1}, \ldots, u_{k_1-1}\rangle), \ \ \ldots,$$

while, his moves in $G^*_2(D)$ are described, respectively, as follows

$$h(f(a_0)) = h(f(\langle u_0, u_1, \ldots, u_{k_0-1}\rangle)), \ \ h(f(a_2)) = h(f(\langle u_{k_0+1}, \ldots, u_{k_1-1}\rangle)), \ \ \ldots$$

Diagrams 6.16 and 6.17 show the illustrations of the games $G^*_{\mathbb{N}}(C)$ and $G^*_2(D)$, respectively.

$$\begin{array}{lcccccccc} \mathrm{I} & \sigma'(\Lambda) & & & \sigma'(f(a_0, a_1)) & & & \cdots & \\ & & \searrow & \nearrow & & \searrow & \nearrow & & \searrow \\ \mathrm{II} & & a_1 = u_{k_0} & & & a_3 = u_{k_1} & & & \ldots \end{array}$$

Diagram 6.16. Playing the run $g(\alpha)$ of $G^*_{\mathbb{N}}(C)$,when player I has a winning strategy σ'.

$$\begin{array}{lcccccccc} \mathrm{I} & h(f(a_0)) & & & h(f(a_2)) & & & \cdots & \\ & & \searrow & \nearrow & & \searrow & \nearrow & & \searrow \\ \mathrm{II} & & h(a_1) = h(u_{k_0}) & & & h(a_3) = h(u_{k_1}) & & & \ldots \end{array}$$

Diagram 6.17. Playing the run $h(g(\alpha))$ of the game $G^*_2(D)$.

So if player I wins $G^*_{\mathbb{N}}(C)$ we have

$$g(\alpha) \in C \ \ \& \ \ \forall n \ \alpha(2n) \in \mathrm{Seq}^*$$

and thus by (6.9), we have that $h(g(\alpha)) \in D$. So player I wins the game $G^*_2(D)$, too.

In the same way, we now assume that player II wins in $G^*_{\mathbb{N}}(C)$ and we will show that player II also wins in $G^*_2(D)$. Let us fix a winning strategy τ' for player II in $G^*_{\mathbb{N}}(C)$, so her moves are described as follows

$$\tau'(f(a_0)) = a_1, \ \ \tau'(a_1, f(a_2)) = a_3, \ \ \ldots,$$

in fact, player I's moves in $G^*_2(D)$ are described in the same way as before. The games $G^*_{\mathbb{N}}(C)$ and $G^*_{\mathbb{N}}(D)$ are visualised in Diagrams 6.18 and 6.19 below, respectively.

If player II wins in $G^*_{\mathbb{N}}(C)$, we have

$$g(\alpha) \notin A \ \vee \ \exists n \ \alpha(2n) \notin \mathrm{Seq}^*$$

$$\begin{array}{lccccccc}
\mathrm{I} & f(a_0) & & & f(a_2) & & & \cdots & & \\
& & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
\mathrm{II} & & & \tau'(f(a_0)) = a_1 & & & & \tau'(a_1, f(a_2)) = \alpha_3 & & & & \cdots
\end{array}$$

Diagram 6.18. Playing the run $g(\alpha)$ of $G^*_{\mathbb{N}}(C)$, when player II has a winning strategy τ'.

$$\begin{array}{lccccccc}
\mathrm{I} & h(f(a_0)) & & & h(f(a_2)) & & & \cdots & & \\
& & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
\mathrm{II} & & & h(a_1) = h(u_{k_0}) & & & & h(a_3) = h(u_{k_1}) & & & & \cdots
\end{array}$$

Diagram 6.19. Playing the run $h(g(\alpha))$ of the game $G^*_2(D)$.

and so by (6.9), it follows that $h(g(\alpha)) \notin D$. Therefore, player II wins $G^*_2(D)$, too. Thus, the game $G^*_2(D)$ is determined and we have proved the implication. $\square$

Proposition 6.3.2. *Player* I *has a winning strategy in $G^*_2(A)$ if and only if $A \subseteq 2^{\mathbb{N}}$ has a non-empty, perfect subset.*

Proof. Let us suppose that $A \subseteq 2^{\mathbb{N}}$ and σ is a winning strategy for player I. Consider the set

$$B = \{\alpha \in 2^{\mathbb{N}} : \alpha \text{ is the play in some run of } G^*_2(A), \text{where I plays by } \sigma\}.$$

If we assume that player I plays based on the winning strategy of σ, and in the play α player II responds to every move of I by playing some $\beta(k)$, $k \in \mathbb{N}$, the set B can be written in the following equivalent form:

$$B = \{\sigma * \beta : \text{is the run where I follows } \sigma \text{ and II plays } \beta \in 2^{\mathbb{N}}\}.$$

Hence, player I's moves following σ have the following form

$$\sigma(\Lambda) = u^\beta_0,\ \ \sigma(u^\beta_0, \beta(0)) = u^\beta_1,\ \ \ldots,\ \ \sigma(u^\beta_0, \beta(0), \ldots, u^\beta_n, \beta(n)) = u^\beta_{n+1}\ \ \cdots$$

In the Diagram 6.20 below we can see how the play evolves within the set B.

$$\begin{array}{lcccccccccccccc}
\mathrm{I} & u^\beta_0 & & & & u^\beta_1 & & & & \cdots & & & & u^\beta_{n+1} & & & & \cdots \\
& & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\mathrm{II} & & & \beta(0) & & & & \beta(1) & & & & \cdots & & & & \beta(n+1) & &
\end{array}$$

Diagram 6.20. Playing some run α of $G^*_2(A)$, within the set B (where I follows the winning strategy σ).

Then we have

$$\begin{aligned}
\alpha &= \sigma * \beta \\
&= \sigma(\Lambda) * \beta(0) * \sigma(u^\beta_0, \beta(0)) * \beta(1) * \ \ldots\ * \sigma(u^\beta_0, \beta(0), \ldots, u^\beta_n, \beta(n)) * \beta(n+1) * \ \ldots \\
&= u^\beta_0 * \beta(0) * u^\beta_1 * \beta(1) * \ \ldots\ * u^\beta_{n+1} * \beta(n+1) * \ \ldots
\end{aligned}$$

Consider now a convergent sequence $(\alpha_i)_{i \in \mathbb{N}} \subseteq B$, with $\alpha_i \to \alpha$, where $\alpha_i = \sigma * \beta_i$, we have

$$u^{\beta_i}_0 = \sigma(\Lambda) = (\alpha_i(0), \ldots, \alpha_i(n_0)) = u_0$$

and since

$$\alpha_i(n_0 + 1) = \beta_i(0) \ \longrightarrow\ \alpha(n_0 + 1)$$

it follows that

$$\exists i_0\ \forall i \geqslant i_0 \ \text{ such that } \ \alpha(n_0 + 1) = \beta_i(0).$$

If we set $b_0 = \alpha(n_0+1)$, we have

$$\begin{aligned} u_1^{\beta_i} &= \sigma((\alpha_i(0),\ldots,\alpha_i(n_0)),\alpha_i(n_0+1)) \\ &= \sigma(u_0^{\beta_i},\beta_i(0)) \\ &= \sigma(u_0^{\beta_i},\alpha(n_0+1)) \\ &= \sigma(u_0^{\beta_i},b_0) \\ &= u_1. \end{aligned}$$

and since

$$\alpha_i(n_1+1) = \beta_i(1) \longrightarrow \alpha(n_1+1),$$

where

$$n_1 = \mathrm{lh}(u_0) + \mathrm{lh}(u_1) + 1,$$

it follows that

$$\exists i_1\ \forall i \geqslant i_1 \text{ such that } \alpha(n_1+1) = \beta_i(1).$$

Continuing the same way, we see that $(\beta_i)_{i\in\mathbb{N}}$ converges to some $\beta \in 2^{\mathbb{N}}$. Hence,

$$\alpha_i = \sigma * \beta_i \longrightarrow \sigma * \beta$$

and

$$\alpha = \sigma * \beta \in B.$$

Therefore, B is closed. Also, it is easy to see that B has no isolated points and thus it is a perfect set.

Conversely, let C be a perfect subset of A (i.e. C is closed and it has no isolated points) and choose a pruned tree T on $2^{\mathbb{N}}$ such that

$$C = [T] = \{\alpha : \forall n\, (a(0),\ldots,a(n-1)) \in T\}.$$

First, we remark that for all $u \in T$ there is a proper extension v of u, such that

$$v * (0) \in T \quad\text{and}\quad v * (1) \in T.$$

To see this let $u \in T$ since T is pruned, we have that $[T_u] \neq \varnothing$. So let $\alpha \in [T_u]$, i.e.,

$$\alpha \in [T] \quad\text{and}\quad u \sqsubseteq \alpha.$$

We have that α is not an isolated point of $C = [T]$, so there is $\beta \in \mathcal{N}_u$, with $\beta \neq \alpha$. Let n be the least such that

$$\beta(n) \neq \alpha(n).$$

So, $\beta|n = \alpha|n$ and $u \sqsubseteq \alpha,\beta$. Put $v = \alpha|n = \beta|n$. Then v extends u properly. Since $\alpha(n),\beta(n) \in \{0,1\}$ and $\alpha(n) \neq \beta(n)$, one of them is 0 and the other is 1. So,

$$\{v*(0), v*(1)\} = \{\alpha|(n+1) = \beta|(n+1)\}$$

and $\alpha|(n+1),\beta|(n+1) \in T$ because $\alpha,\beta \in [T_u]$. Hence, $v*(0), v*(1) \in T$.

Next, we describe how I can play to win $G_2^*(A)$. Player I starts with a non-empty sequence $(a_0,\ldots,a_{n-1})$ such that

$$(a_0,\ldots,a_{n-1},i) \in T, \text{ for } i = 0,1.$$

This is possible from the remark above taking $u = \Lambda \in T$. Suppose that II plays $a_n \in \{0,1\}$. From the choice of $(a_0,\ldots,a_{n-1})$ we have that

$$u = (a_0,\ldots,a_{n-1}) \in T.$$

We apply the preceding remark to this $u \in T$ and get a proper extension v of u,

$$v = (a_0,\ldots,a_{n-1},a_n,a_{n+1},\ldots,a_{k-1}),$$

such that $v*(i) \in T$, $i = \{0,1\}$. Then, I plays the non-empty sequence $(a_{n+1},\ldots,a_{k-1})$. We continue similarly. For every run

$$((a_0,\ldots,a_{n-1}),a_n,(a_{n+1},\ldots,a_{k-1}),\ldots),$$

where I has played as above, has the property

$$(a_0,\ldots,a_{n-1},a_n,a_{n+1},\ldots,a_{k-1},\ldots) \in [T] = C \subseteq A$$

and so I wins this run.

Therefore, we have described a winning strategy for I in $G_2^*(A)$. □

Proposition 6.3.3. *Player* II *has a winning strategy in $G_2^*(A)$ if and only if A is countable.*

Proof. Let us start the proof from the converse direction. If A is countable, then II has a winning strategy: she simply plays in her n^{th} turn to make the play different from α_n, where $A = \{\alpha_0, \alpha_1, \dots\}$. For the forward direction, suppose now that player II wins by playing τ and let α be a fixed binary sequence. We define the **sequence**

$$s_0, k_0, s_1, k_1, \dots, s_{l-1}, k_{l-1}.$$

We call this sequence **good for $\boldsymbol{\tau}$ and $\boldsymbol{\alpha}$**, if the following hold:

i) each s_i is a non-empty, finite binary sequence,

ii) each k_i is 0 or 1,

iii) the sequence $z = s_0 * k_0 * s_1 * k_1 * \cdots * s_{l-1} * k_{l-1}$ is an initial segment of α (i.e. $z \sqsubseteq \alpha$),

iv) and $s_0, k_0, s_1, k_1, \dots, s_{l-1}, k_{l-1}$ is the beginning of a run of $G_2^*(A)$ played according to τ, i.e., for $j < l$,

$$k_j = \tau(s_0, k_0, \dots, s_j).$$

In particular, k_j is player II's answer to player I's moves, by the time he's made his s_j move. More specifically, we have that

$$k_0 = \tau(s_0),\;\; k_1 = \tau(s_0, k_0, s_1),\;\; k_2 = \tau(s_0, k_0, s_1, k_1, s_2),\;\; \dots$$

The game $G_2^*(A)$ is illustrated in the Diagram 6.21, below.

$$\begin{array}{lccccccccc}
\mathrm{I} & s_0 & & & s_1 & & & s_2 & & \cdots \\
 & & \searrow & & \nearrow \;\; \searrow & & \nearrow \;\; \searrow & & \nearrow & \\
\mathrm{II} & & & \tau(s_0) & & \tau(s_0, k_0, s_1) & & & \tau(s_0, k_0, s_1, k_1, s_2) &
\end{array}$$

Diagram 6.21. Playing the game $G_2^*(A)$, where τ is II's winning strategy.

The empty sequence (for $l = 0$) is automatically good. If every good sequence has a good proper extension, then α is the play in a run of $G_2^*(A)$ where player II has followed the winning strategy τ, and hence $\alpha \notin A$.

Therefore, if $\alpha \in A$, player II has stopped following her winning strategy τ. So there must exist some

$$s_0, k_0, s_1, k_1, \dots, s_{l-1}, k_{l-1}$$

(possibly the empty sequence) which is good for τ and α and *has no proper good extension*.

We have

$$s_0 * k_0 * s_1 * k_1 * \cdots * s_{l-1} * k_{l-1} = (\alpha(0), \alpha(1), \dots, \alpha(n-1)).$$

We show that α is *uniquely determined* by $s_0, k_0, \dots, s_{l-1}, k_{l-1}$ and $\alpha(n)$.

More specifically, if

$$s_0 * k_0 * \cdots * s_{l-1} * k_{l-1} * \alpha(n) \sqsubseteq \beta,$$

then $\alpha = \beta$. To see this we claim that for all $i > n$, we have

$$\alpha(i) = 1 - \tau(s_0, k_0, \dots, s_{l-1}, k_{l-1}, (\alpha(n), \dots, \alpha(i-n))). \tag{6.10}$$

If we show (6.10), then by a simple induction on $i > n$ we induct have that for all β with

$$s_0 * k_0 * \cdots * s_{l-1} * k_{l-1} * \alpha(n) \sqsubseteq \beta,$$

it holds $\alpha = \beta$. To show (6.10), let $i > n$ and consider in $G_2^*(A)$ that

$$s_0, k_0, \dots, s_{l-1}, k_{l-1}$$

have been played. Let I play next $(\alpha(n), \dots, \alpha(i-n))$. Since $s_0 * k_0 * \cdots * s_{l-1} * k_{l-1}$ has no good extension, then II must have played against her strategy τ, so

$$\alpha(i) = 1 - \tau(s_0, k_0, \dots, s_{l-1}, k_{l-1}, (\alpha(n), \dots, \alpha(i-n)))$$

and (6.10) is proved.

To visualize the progress of a run in game $G_2^*(A)$, we present Diagrams 6.22 and 6.23.

$$
\begin{array}{llllllllll}
\mathrm{I} & s_0 = (\alpha(0), \ldots, \alpha(t_0)) & & & \cdots & & s_{l-1} & \\
& & \searrow & \nearrow & & \searrow \quad \nearrow & & \searrow \\
\mathrm{II} & & \tau(s_0) = \alpha(t_0 + 1) & & & \cdots & & \boldsymbol{k_{l-1}}
\end{array}
$$

Diagram 6.22. During a run in $G_2^*(A)$, where II follows her winning strategy τ, until her $(l-1)$-move (that appears in bold).

$$
\begin{array}{llllllll}
\mathrm{I} & \cdots & & s_l = (\alpha(n), \ldots, \alpha(i-1)) & & & \cdots \\
& \searrow & \nearrow & & \searrow & \nearrow & \\
\mathrm{II} & & \cdots & & \boldsymbol{\alpha(i)} = 1 - \tau(s_0, \ldots, k_{l-1}, s_l) & &
\end{array}
$$

Diagram 6.23. During a run in $G_2^*(A)$, where II starts playing against τ, for $i > n$ (that appears in bold).

Thus, α is completely determined by the value $\alpha(n)$ and the maximal good sequence

$$s_0, k_0, s_1, k_1, \ldots, s_{l-1}, k_{l-1}.$$

We define

$$
\begin{aligned}
C = \{\alpha \, : \, \exists l \; \exists \, s_0, k_0, \ldots, s_{l-1}, k_{l-1}, \; \exists j \in \{0, 1\}, \text{ for which} \\
s_0 * k_0 * \cdots * s_{l-1} * k_{l-1} * (j) \sqsubseteq \alpha \\
\text{is good for } \tau \text{ and } \alpha \text{ and has no proper good extension}\}.
\end{aligned}
$$

The set C can be written

$$C = \bigcup_l C_l,$$

where l is fixed. The sets C_l are finite and hence C is countable, as the countable union of finite sets (for fixed l, there are only finitely many $s_0, k_0, \ldots, s_{l-1}, k_{l-1}, j$). Therefore, $A \subseteq C$, so A is countable. $\square$

Theorem 6.3.4. *Let Γ be any of the pointclasses $\underset{\sim}{\mathbf{\Sigma}}{}^1_n$, $\underset{\sim}{\mathbf{\Pi}}{}^1_n$ and $\underset{\sim}{\mathbf{\Delta}}{}^1_n$, for $n \geqslant 1$. Then*

$$\mathrm{Det}_2^*(\Gamma) \iff \textit{every uncountable set in } \Gamma \textit{ has a non-empty perfect subset.}$$

and hence

$$\mathrm{Det}_{\mathbb{N}}^*(\Gamma) \implies \textit{every uncountable set in } \Gamma \textit{ has a non-empty perfect subset.}$$

Proof. • Let us start by proving the implication:

$$\mathrm{Det}_2^*(\Gamma) \implies \text{every uncountable set in } \Gamma \text{ has a non-empty perfect subset.}$$

We assume that $\mathrm{Det}_2^*(\Gamma)$ is true, where Γ is one of the pointclasses $\underset{\sim}{\mathbf{\Sigma}}{}^1_n$, $\underset{\sim}{\mathbf{\Pi}}{}^1_n$, $\underset{\sim}{\mathbf{\Delta}}{}^1_n$, for $n \geqslant 1$. We have to prove that if $\mathcal{X}$ is a Polish space, every uncountable set $A \subseteq \mathcal{X}$ in Γ has a non-empty perfect subset.

Let $A \subseteq \mathcal{X}$ be uncountable in Γ. Then, $\mathcal{X}$ must be uncountable, too. By Theorem 5.4.25, there is a Borel isomorphism

$$\pi : \mathcal{X} \to 2^{\mathbb{N}}.$$

Continuing, we consider $B = \pi[A] \subseteq 2^{\mathbb{N}}$. The set B is uncountable because A is also uncountable and π is a Borel isomorphism (so, π is a bijection). It is true that $B = (\pi^{-1})^{-1}[A]$. Indeed,

$$
\begin{aligned}
a \in B &\iff \exists x \in A : \; a = \pi(x) \\
&\iff \pi^{-1}(a) \in A \\
&\iff f(a) \in A, \; f = \pi^{-1} \\
&\iff a \in f^{-1}[A] \\
&\iff a \in (\pi^{-1})^{-1}[A].
\end{aligned}
$$

Since, π^{-1} is a Borel measurable and Γ is closed under Borel substitution, we have that B is in Γ. By hypothesis, $\mathrm{Det}_2^*(B)$ it is true. Thus, the game $G_2^*(B)$ is determined, so one of the two players wins.

We assume that player II has a winning strategy in $G_2^*(B)$. Then, by Proposition 6.3.3, it follows that B is countable, but this is a contradiction. Therefore, player I has a winning strategy in $G_2^*(B)$, and by Proposition 6.3.2, we have that $B \subseteq 2^{\mathbb{N}}$ has a non-empty perfect subset. Let us call this non-empty perfect subset P. Then,

$$P \subseteq B = \pi[A].$$

Knowing that π is a Borel measurable function and P is a closed set, it follows that $\pi^{-1}[P]$ is a Borel subset of $\mathcal{X}$. Additionally, P is uncountable and the sets P and $\pi^{-1}[P]$ have the same cardinality, so $\pi^{-1}[P]$ is an uncountable Borel set. Therefore, by Remark 5.4.7 there is $Q \subseteq \pi^{-1}[P]$, that is non-empty and perfect. Finally, we have that

$$Q \subseteq \pi^{-1}[P] \subseteq \pi^{-1}[B] = \pi^{-1}[\pi[A]] = A.$$

• Proceeding, we will prove the following implication:

$$\text{every uncountable set in } \Gamma \text{ has a non-empty perfect subset} \implies \mathrm{Det}_2^*(\Gamma).$$

Let $A \subseteq \mathcal{X}$ in Γ and let us suppose that $\mathcal{X} = 2^{\mathbb{N}}$. If A is countable, then from hypothesis, by Proposition 6.3.3, player II wins $G_2^*(A)$. If A is uncountable, then from hypothesis, there is $C \neq \varnothing$ perfect subset of A. By Proposition 6.3.2, player I wins $G_2^*(A)$. Therefore, the game $G_2^*(A)$ is determined and thus $\mathrm{Det}_2^*(\Gamma)$ is true.

• Last, we will prove the implication:

$$\mathrm{Det}_{\mathbb{N}}(\Gamma) \implies \text{every uncountable set in } \Gamma \text{ has a non-empty perfect subset.}$$

By Proposition 6.3.1, we have that

$$\mathrm{Det}_{\mathbb{N}}(\Gamma) \implies \mathrm{Det}_{\mathbb{N}}^*(\Gamma) \implies \mathrm{Det}_2^*(\Gamma),$$

and we also proved that

$$\mathrm{Det}_2^*(\Gamma) \implies \text{every uncountable set in } \Gamma \text{ has a non-empty perfect subset.}$$

Consequently, we have reached our goal. □

6.4. The Covering Games $G^\mu(A, \varepsilon)$

In this section, we will describe the covering game $G^\mu(A, \varepsilon)$, for $\varepsilon > 0$, associated with the usual σ-finite Borel measure μ on the space $2^{\mathbb{N}}$ and each set $A \subseteq 2^{\mathbb{N}}$. This is a game on $\mathbb{N}$, invented by L. Harrington ([**4**]). To describe the game $G^\mu(A, \varepsilon)$ we will use the **base** of $2^{\mathbb{N}}$:

$$N_u,\ u \in \{0,1\}^{<\mathbb{N}},\ \text{base of } 2^{\mathbb{N}},$$

$$N_k,\ k \in \mathbb{N},\ \text{base of } 2^{\mathbb{N}} \text{ and } t = \langle k_0, \ldots, k_{m-1} \rangle,\ m \geqslant 1.$$

So, player I plays integers $s_0, s_1, s_2, \ldots$, with each $s_i = 0$ or $s_i = 1$. In the end, he determines a binary sequence $\alpha \in 2^{\mathbb{N}}$.
Player II plays integers $t_0, t_1, t_2, \ldots$ where

$$t_0 = \langle k_0^0, \ldots, k_{m_0-1}^0 \rangle,\ \ t_1 = \langle k_0^1, \ldots, k_{m_1-1}^1 \rangle,\ \ t_2 = \langle k_0^2, \ldots, k_{m_2-1}^2 \rangle,\ \ \ldots$$

where $m_n > 0$, $n \in \mathbb{N}$ (so the sequences coded by $t_0, t_1, t_2, \ldots$ are non-empty).

Each t_n codes a finite union of basic open sets G_n, such that

$$\begin{aligned}
\mu(G_0) &= \mu(N_{k_0^0} \cup \ldots \cup N_{k_{m_0-1}^0}) \leqslant \frac{\varepsilon}{2^{2\cdot 0+2}} \\
\mu(G_1) &= \mu(N_{k_0^1} \cup \ldots \cup N_{k_{m_1-1}^1}) \leqslant \frac{\varepsilon}{2^{2\cdot 1+2}} \\
&\ \ \vdots \\
\mu(G_n) &= \mu(N_{k_0^n} \cup \ldots \cup N_{k_{m_n-1}^n}) \leqslant \frac{\varepsilon}{2^{2\cdot n+2}}.
\end{aligned}$$

To visualize the game we present the Diagram 6.24.

• Provided that the rules are followed, player I **wins** the run

$$((s_0, \ldots, s_n, \ldots),\ (t_0, \ldots, t_n, \ldots)) = (\alpha, t),$$

$$\begin{array}{lccccccccccc}
\text{I} & s_0 & & & & s_1 & & & & s_2 & & \cdots \\
& & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\text{II} & & & \langle k_0^0, \ldots, k_{m_0-1}^0 \rangle & & & & \langle k_0^1, \ldots, k_{m_1-1}^1 \rangle & & & & \cdots
\end{array}$$

Diagram 6.24. Playing the game $G^\mu(A, \varepsilon)$.

if and only if,

$$\alpha \in A \backslash G,$$

where

$$G = \bigcup_{n \in \mathbb{N}} G_n$$

and

$$G_n = N_{k_0^n} \cup \ \ldots \ \cup N_{k_{m_n-1}^n}.$$

Otherwise, player II **wins** the run

$$((s_0, \ldots, s_n, \ldots),\ (t_0, \ldots, t_n, \ldots)) = (\alpha, t),$$

if and only if,

$$\alpha \notin A \backslash G.$$

• By adapting the rules to the payoff set, player I **wins** the run

$$((s_0, \ldots, s_n, \ldots),\ (t_0, \ldots, t_n, \ldots)) = (\alpha, t),$$

where $s_n = 0, 1$ for all n, and if

either $\exists n$ such that t_n does not code a finite non-empty sequence,

or

$$\forall n \ t_n = \langle k_0^n, \ldots, k_{m_n-1}^n \rangle, \ m_n \geqslant 1, \text{ and}$$
$$\exists n \ \mu(N_{k_0^n} \cup \ \ldots \ \cup N_{k_{m_n-1}^n}) > \frac{\varepsilon}{2^{2 \cdot n+2}},$$

or

$$\forall n \ t_n = \langle k_0^n, \ldots, k_{m_n-1}^n \rangle, \ m_n \geqslant 1, \text{ and}$$
$$\forall n \ \mu(N_{k_0^n} \cup \ \ldots \ \cup N_{k_{m_n-1}^n}) \leqslant \frac{\varepsilon}{2^{2 \cdot n+2}}, \text{ and}$$
$$\alpha \in A \backslash G.$$

Player II **wins** the run (α, t), if and only if player I does not win the run (α, t). This implies that

$$\exists i \ \alpha(i) > 1,$$

or

$$\forall n \ t_n = \langle k_0^n, \ldots, k_{m_n-1}^n \rangle, \ m_n \geqslant 1, \text{ and}$$
$$\forall n \ \mu(N_{k_0^n} \cup \ \ldots \ \cup N_{k_{m_n-1}^n}) \leqslant \frac{\varepsilon}{2^{2 \cdot n+2}}, \text{ and}$$
$$\alpha \notin A \backslash G.$$

Theorem 6.4.1 ([**8**], 2H.8)**.** *For every Polish space $\mathcal{X}$ and every σ-finite Borel measure μ in $\mathcal{X}$, it holds that every $\utilde{\mathbf{\Sigma}}^1_1$-subset of $\mathcal{X}$ is μ-measurable.*

Proposition 6.4.2. *Suppose μ is a σ-finite Borel measure on $2^{\mathbb{N}}$, $A \subseteq 2^{\mathbb{N}}$ has no Borel subsets of μ-measure > 0 and for each $\varepsilon > 0$ the game $G^\mu(A, \varepsilon)$ is determined. Then, A is μ-null.*

Proof. Let us suppose by contradiction that player I, for some $\varepsilon > 0$, wins $G^\mu(A, \varepsilon)$ by playing with a strategy σ and let

$$B = \{\sigma * \tau : \ \tau \text{ is a strategy for player II}\}.$$

Since I wins $G^\mu(A, \varepsilon)$, it follows that $\sigma * \tau \in A$, for all τ which is a strategy for II, so $B \subseteq A$. As we know, τ is a function with domain all finite sequences from $\mathbb{N}$ of odd length and values in $\mathbb{N}$, i.e.

$\tau : \mathbb{N}^{\text{odd}} \to \mathbb{N}$. By enumerating $\mathbb{N}^{\text{odd}}$ we can view τ as a member of the Baire space $\mathcal{N}$, so we can think of "$\exists\, \tau$" as "$\exists^{\mathcal{N}}$". Therefore, B is a $\underset{\sim}{\mathbf{\Sigma}}^1_1$ subset of A. By Theorem 6.4.1, B is μ-measurable.

By hypothesis, A has no Borel subsets of μ-measure > 0, so $\mu(B) = 0$. Now we can find a set G which consists of all finite unions of basic neighborhoods G_n (as previously described), i.e.,

$$G = \bigcup_{n \in \mathbb{N}} G_n,$$

where

$$G_n = N_{k_0^n} \cup \ldots \cup N_{k^n_{m_n-1}}, \;\; t_n = \langle k_0^n, \ldots, k^n_{m_n-1} \rangle, \;\; m_n > 0,$$

and

$$\mu(G_n) \; = \; \mu(N_{k_0^n} \cup \ldots \cup N_{k^n_{m_n-1}}) \; \leqslant \; \frac{\varepsilon}{2^{2 \cdot n + 2}}, \;\; \text{for all} \;\; n \in \mathbb{N},$$

and

$$B \subseteq G = \bigcup_{n \in \mathbb{N}} G_n.$$

This gives a strategy τ for player II. Let $\alpha = \sigma * \tau$. Since σ is a winning strategy for I, we have $\alpha \in A \backslash G$. But $\alpha \in B \subseteq G$ is a contradiction. Therefore, for all $\varepsilon > 0$, player I cannot win $G^{\mu}(A, \varepsilon)$ and since the latter game is determined wins $G^{\mu}(A, \varepsilon)$ for all $\varepsilon > 0$.

Fix $\varepsilon > 0$, we show that $\mu(A) \leqslant \varepsilon$. Let τ be a winning strategy for II in $G^{\mu}(A, \varepsilon)$. So, if we let $(s_0, \ldots, s_n)$ be the finite binary sequence of player I's moves and $G(s_0, \ldots, s_n)$ be the finite union of basic neighborhoods coded by player II's move t_n (playing by τ) when player I plays $s_0, \ldots, s_n$, we can set

$$G = \bigcup G(s_0, \ldots, s_n).$$

Since G is a finite union of basic neighborhoods, we conclude that it is an open set. We have that

$$\begin{aligned}
\mu(G) &\leqslant \sum \{\mu(G(s_0, \ldots, s_n)) : (s_0, \ldots, s_n) \text{ is a binary sequence}\} \\
&= \sum_n \sum \{\mu(G(s_0, \ldots, s_n)) : (s_0, \ldots, s_n) \text{ a binary sequence of length } n+1\} \\
&\leqslant \sum_n \left(\sum_{0 < t \leqslant 2^{n+1}} \frac{\varepsilon}{2^{2n+2}} \right) \text{ (by the rules of the game)} \\
&= \sum_n \frac{2^{n+1} \varepsilon}{2^{2n+2}} \; = \; \varepsilon.
\end{aligned}$$

So for all $\varepsilon > 0$ there is an open G such that

$$A \subseteq G \;\; \text{and} \;\; \mu(G) < \varepsilon.$$

This proves that A is μ-null. □

Remark 6.4.3 ([**8**], 2H.7)**.** Let $\mathcal{X}$ be a Polish space and $\mu : \mathcal{B}(\mathcal{X}) \to [0, \infty]$ is a σ-finite measure on $\mathcal{X}$. Then for each $A \subseteq \mathcal{X}$ there is a Borel set B, with $A \subseteq B$, and for each $C \subseteq B \backslash A$, with $C \in \mathcal{B}(\mathcal{X})$, it holds that

$$\mu(C) = 0.$$

Theorem 6.4.4 ([**9**])**.** *Suppose Γ is any of the pointclasses $\underset{\sim}{\mathbf{\Sigma}}^1_n$, $\underset{\sim}{\mathbf{\Pi}}^1_n$ and $\underset{\sim}{\mathbf{\Delta}}^1_n$, for $n \geqslant 1$ and let μ be a σ-finite Borel measure on some Polish space $\mathcal{X}$. It holds that*

$$\mathrm{Det}_{\mathbb{N}}(\Gamma) \implies \textit{every } A \subseteq \mathcal{X} \textit{ in } \Gamma \textit{ is } \mu\textit{-measurable}.$$

Proof. Suppose first $\mathcal{X} = 2^{\mathbb{N}}$, and let $A \subseteq \mathcal{X}$ in Γ. By Remark 6.4.3, there is a Borel set B such that $A \subseteq B$ and $B \backslash A$ contains no Borel set of μ-measure > 0. We have that $B \backslash A = B \cap (2^{\mathbb{N}} \backslash A)$ is in $\neg\Gamma$, since Γ contains the Borel sets and is closed under the operator &. Further from the closure properties of Γ, for all $\varepsilon > 0$, the payoff set of the game $G^{\mu}(B \backslash A, \varepsilon)$ is in $\neg\Gamma$.

From our hypothesis, we have $\mathrm{Det}_{\mathbb{N}}(\Gamma)$ equivalently $\mathrm{Det}_{\mathbb{N}}(\neg\Gamma)$. Hence, for all $\varepsilon > 0$, $G^{\mu}(B \backslash A, \varepsilon)$ is determined. So Proposition 6.4.2 is applicable to $B \backslash A$. From the latter, we conclude that $B \backslash A$ is μ-null. Therefore, A differs from the Borel set B by a μ-null set. This shows that A is μ-measurable.

By Corollary 5.4.25, every uncountable space $\mathcal{X}$ is Borel isomorphic with $2^{\mathbb{N}}$ and we can establish the result for $\mathcal{X}$ by carrying to $2^{\mathbb{N}}$ any given measure on $\mathcal{X}$.

We have that for every σ-finite Borel measure $\mu : \mathcal{B}(2^{\mathbb{N}}) \to [0, \infty]$ on $2^{\mathbb{N}}$, every $A \subseteq 2^{\mathbb{N}}$ in Γ is μ-measurable (under $\mathrm{Det}_{\mathbb{N}}(\Gamma)$).

We want to show the same for an uncountable Polish space $\mathcal{X}$ instead of $2^{\mathbb{N}}$. Let $\mathcal{X}$ be uncountable Polish space and $\mu_{\mathcal{X}} : \mathcal{B}(\mathcal{X}) \to [0, \infty]$ be a σ-finite Borel measure on $\mathcal{X}$. Let $B \subseteq \mathcal{X}$ in Γ, too. We need to show that B is $\mu_{\mathcal{X}}$-measurable.

Let $f : 2^{\mathbb{N}} \rightarrowtail\!\!\!\twoheadrightarrow \mathcal{X}$ that is a Borel isomorphism. Then for every $A \subseteq 2^{\mathbb{N}}$, it holds that

$$A \in \mathcal{B}(2^{\mathbb{N}}) \iff f[A] \in \mathcal{B}(\mathcal{X}).$$

Define $\mu : \mathcal{B}(2^{\mathbb{N}}) \to [0, \infty]$, so

$$\mu(A) = \mu_{\mathcal{X}}(f[A]).$$

Then μ is a σ-finite Borel measure. The set $A = f^{-1}[B]$ is in Γ, because Γ is closed under Borel substitution. So, from the preceding ($\mathcal{X} = 2^{\mathbb{N}}$, under $\mathrm{Det}_{\mathbb{N}}(\Gamma)$), A is μ-measurable. Hence, there is $A_0 \in \mathcal{B}(2^{\mathbb{N}})$, such that the set

$$N = A \triangle A_0$$

is μ-null. Let $N_0 \in \mathcal{B}(2^{\mathbb{N}})$, with $N \subseteq N_0$ and $\mu(N_0) = 0$. Put $B_0 = f[A_0]$. Then B_0 is Borel and we have

$$M = B \triangle B_0 \quad \text{and} \quad M_0 = f[N_0] \in \mathcal{B}(\mathcal{X}).$$

We have that $N \subseteq N_0$, so

$$f[N] \subseteq f[N_0] = M_0 \subseteq \mathcal{X}$$

and

$$f[N] = f[A \triangle A_0] = f[A] \triangle f[A_0] = B \triangle B_0.$$

Hence,

$$B \triangle B_0 \subseteq M_0.$$

Furthermore, we have that

$$\mu_{\mathcal{X}}(M_0) = \mu_{\mathcal{X}}(f[N_0]) = \mu_{\mathcal{X}}(M_0) = 0.$$

Consequently, the set B is $\mu_{\mathcal{X}}$-null. $\square$

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