IRREVERSIBLE ENERGY TRANSFER IN THE FORM OF DISCRETE BREATHERS IN NONLINEAR LATTICES WITH ASYMMETRY

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Preface

Applied mathematics have played a highly significant role in the development of the science. One part of the applied mathematics that are of a high interest is transfer of energy. Over the years many scientists have tried to unlock the mysteries of the energy of a specific system. In this thesis, we shall see how energy transfers through a certain type of "energy carrier", called discrete breather.

In Chapter 1, we present the general idea of solitons and their behavior. We study three important equations that have solitons as solutions. We also, study some special solitonic structures, which are called breathers. The importance of the breathers is going to be shown in chapter 2 and 4.

In Chapter 2, we show how solitons have a direct connection with nonlinear lattices and how moving breathers make their appearance in nonlinear lattices. The first part is will be shown through the FPU problem and how the lattice they were studying has a deep connection with the KdV equation. The second part will be shown numerically.

In Chapter 3, we present the complexification averaging method. This is done by first presenting the multiple scale method through an example. After that, by using a simple example again we show how passing to complex variables is useful to solving the problem.

Finally, in Chapter 4, we study how energy transfers in a nonlinear lattice through travelling breathers. In particular, we study numerically what conditions have to be satisfied in order to achieve irreversible energy transfer.

Special thanks to my teacher Vasileios Rothos and my family for all their help and guidance through this last year...

1 Solitons

1.1 An Introduction to Solitons and the KdV Equation

In 1834, John Scott Russell (1808–1882) while conducting some experiments in the Union Canal in Scotland, he discovered a phenomenon that he described as the wave of translation. A solitary wave is a localized wave of translation that arises from a balance between nonlinear and dispersive effects. In most types of solitary waves, the pulse width depends on the amplitude. A soliton is a solitary wave that behaves like a "particle", in the sense that it satisfies the conditions given in the definition below. The term soliton was given by Norman Zabusky (1929-2018) and Martin David Kruskal (1925-2006) in 1965.

Definition 1.1.1. *A soliton is a solution to a nonlinear P.D.E. which:*

1.) is localised

2.) moves with constant shape and velocity in isolation

3.) is preserved under collisions with other solitons

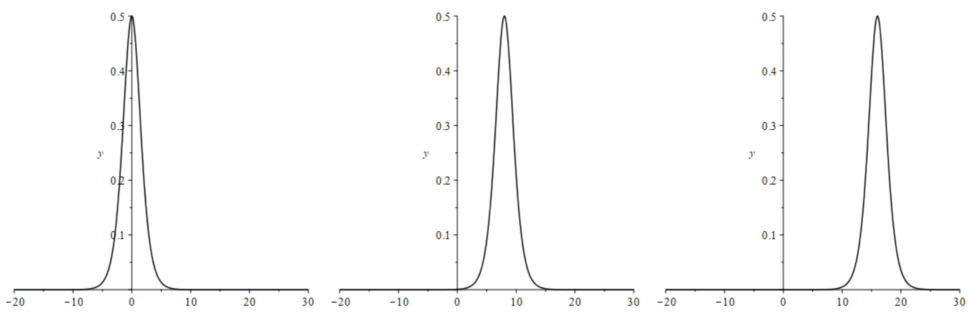

Figure 1.1.1: A soliton which moves in one direction, invariantly.

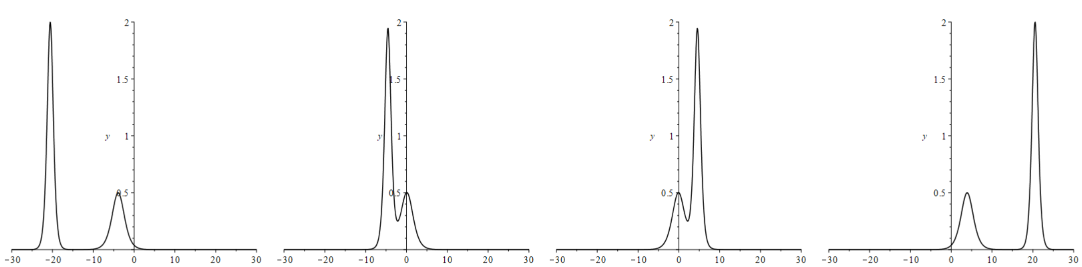

Figure 1.1.2: Two solitons interacting with each other and then emerging from the collision with the same shapes and velocities.

Remark 1.1.2. As we saw in Definition 1.1.1., a soliton emerges unchanged from a collision with another soliton. However, its only characteristic that can possibly change is its phase (phase shift). With the term phase shift, we mean where the soliton would have been if the other soliton had not been there.

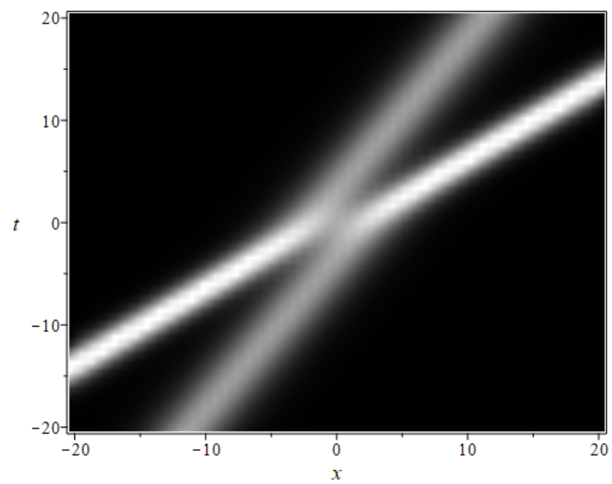

Figure 1.1.3: A space-time diagram of the collision of two solitons. Space is on the horizontal axis and time is on the vertical axis. Observe the phase shift that occurs when the two solitons collide.

One of the most known and classic P.D.E. which has solitons as solutions is the Korteweg-de Vries equation (KdV). This is one of the equations that we are going to study in order to understand what exactly is a soliton and how it behaves. The dutch mathematicians Diederik Johannes Korteweg (1848-1941) and Gustav de Vries (1866-1934), continuing the work of the french mathematician Joseph Valentin Boussinesq (1842-1929), derived a nonlinear equation governing long one dimensional, small amplitude, surface gravity waves propagating in a shallow channel of water[2]

$$\frac{\partial \eta}{\partial \tau} = \frac{3}{2}\sqrt{\frac{g}{h}} \cdot \frac{\partial}{\partial \xi}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma \cdot \frac{\partial^2 \eta}{\partial \xi^2}\right), \quad \sigma = \frac{1}{3}h^3 - \frac{Th}{\rho g} \tag{1.1.1}$$

where

η : the surface elevation of the wave above the equilibrium level h
a : a small arbitrary constant related to the uniform motion of the liquid
g : the gravitational constant
T : the surface tension
ρ : the density

The terms "long" and "small" are meant in comparison to the depth of the channel. Now, by making the following transformations in equation (1.1.1)

$$t = \frac{1}{2}\sqrt{\frac{g}{h\sigma}} \cdot \tau, \qquad x = -\sigma^{-\frac{1}{2}}\xi, \qquad u = \frac{1}{2}\eta + \frac{1}{3}a$$

we obtain the most common form of the KdV equation

$$\frac{\partial u(x,t)}{\partial t} + 6 \cdot u(x,t) \cdot \frac{\partial u(x,t)}{\partial x} + \frac{\partial^3 u(x,t)}{\partial x^3}$$

or

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1.1.2}$$

where now

$u = u(x,t)$: height of the wave at position x and time t

The presence of the term u_{xxx} in the relation (1.1.2) is the one responsible for the dispersive phenomena and the term $u \cdot u_x$ describes the nonlinearity of the propagating mean. The coexistence of these two terms in balance, leads to the emergence of a stable, solitary pulse which is the soliton.

In order to understand better the nature of the soliton we examine what happens if we remove certain terms from the KdV equation. We examine two distinct cases. First, we remove the nonlinear term and we have

$$u_t + u_{xxx} = 0$$

The result is dispersion i.e. waves spread and energy disperses.

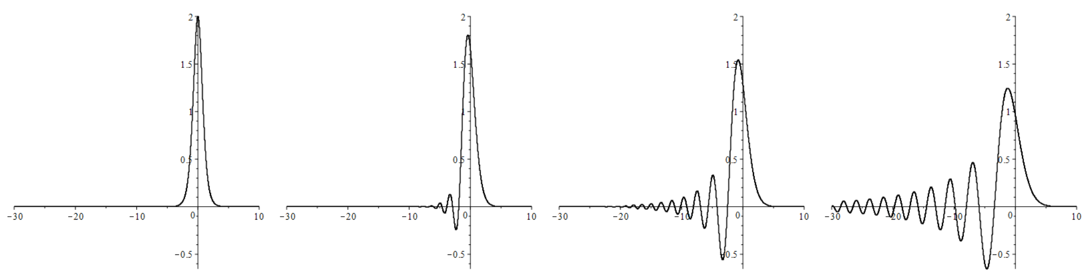

Figure 1.1.4: The initial localised wave disperses. In particular it spreads out so there are these waves which move towards x equals minus infinity.

The next option is to remove the dispersive term leaving only the nonlinear term and then obtain

$$u_t + 6uu_x = 0$$

Now, the result is breaking i.e. waves energy concentrates until it becomes singular

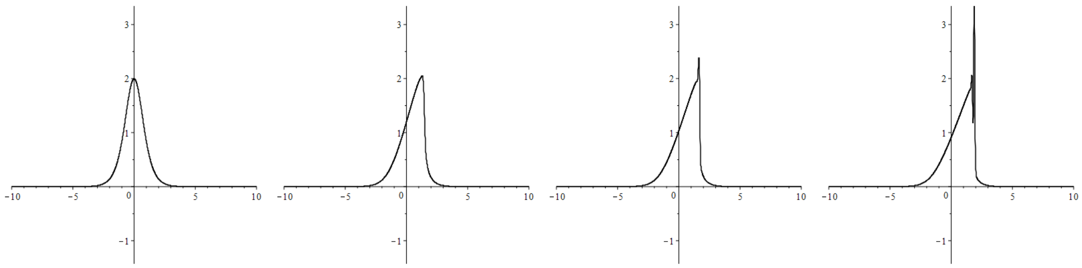

Figure 1.1.5: The wave remains localised. However, it piles up towards the right and after some finite time it breaks down.

The reason we presented these two cases and the graphs above, is to show the importance of the balance of the dispersive term and the nonlinear term in the KdV equation. Even if these two terms exist in the relation (1.1.2) at the same time, if they do not coexist in balance the appearance of a soliton shall not happen.

The following step is to solve the KdV equation, explicitly, which is given again below

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1.1.3}$$

We know that the soliton is a propagating wave so we consider the trial solution

$$u(x, t) = v(x - ct)$$

where now v is the unknown function and c is the velocity of the wave. Now, we consider the variable ξ as

$$\xi = x - ct$$

and we have the relation

$$u(x, t) = v(x - ct) \equiv v(\xi) \tag{1.1.4}$$

Furthermore, we have the relations

$$\frac{\partial u}{\partial t} = \frac{dv}{d\xi} \cdot \frac{\partial \xi}{\partial t} = -cv_\xi = -cv' \tag{1.1.5}$$

and

$$\frac{\partial u}{\partial x} = \frac{dv}{d\xi} \cdot \frac{\partial \xi}{\partial x} = v_\xi = v', \qquad \frac{\partial^3 u}{\partial x^3} = v''' \tag{1.1.6}$$

We substitute the relations (1.1.4), (1.1.5), (1.1.6) in the equation (1.1.3) and we obtain the ordinary differential equation

$$-cv' + 6vv' + v''' = 0$$

By integrating we get

$$-cv + 3v^2 + v'' = A, \quad A \in \mathbb{R} \tag{1.1.7}$$

We multiply by v' and we have

$$-cvv' + 3v^2v' + v'v'' = Av'$$

We integrate again and we have

$$-\frac{c}{2}v^2 + v^3 + \frac{1}{2}(v')^2 = Av + B, \quad A, B \in \mathbb{R} \tag{1.1.8}$$

According to the definition of the soliton as a *localised wave which moves in isolation*, we get the boundary conditions for the equation (1.1.8)

$$\text{as } \xi \to \pm\infty \;\text{ then }\; v, v', v'' \to 0$$

By combining the previous BCs and the equations (1.1.8) and (1.1.7) we have that $A = B = 0$. Hence, the relation (1.1.8) becomes

$$-\frac{c}{2}v^2 + v^3 + \frac{1}{2}(v')^2 = 0 \implies$$

$$-cv^2 + 2v^3 + (v')^2 = 0 \implies$$

$$(v')^2 = cv^2 - 2v^3 = (c - 2v) \cdot v^2 \implies$$

$$v' = \pm v \cdot \sqrt{c - 2v} \implies$$

$$\frac{dv}{d\xi} = \pm v \cdot \sqrt{c - 2v} \implies$$

$$\pm \frac{dv}{v \cdot \sqrt{c - 2v}} = d\xi$$

We integrate and we obtain

$$\pm \int^{v} \frac{dy}{y \cdot \sqrt{c - 2y}} = \xi - x_0 \tag{1.1.9}$$

In order to compute the integral of the left-hand side of the above equation, we are going to need the following identities for the hyperbolic functions

$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \operatorname{sech} x = \frac{1}{\cosh x}, \qquad \operatorname{sech}^2 x = 1 - \tanh^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = -\tanh x \cdot \operatorname{sech} x, \qquad \operatorname{sech}(-x) = \operatorname{sech} x$$

Now, we are solving the integral of the equation (1.1.9). Initially, we apply the transformation[3]

$$y = \frac{1}{2}c \cdot \operatorname{sech}^2 z \tag{1.1.10}$$

Moreover, we get the following relations

$$c - 2y = c - \frac{1}{2} \cdot 2c \cdot \operatorname{sech}^2 z =$$

$$c - c \cdot \operatorname{sech}^2 z =$$

$$c(1 - sech^2 z) \implies$$

$$c - 2y = c \cdot \tanh^2 z \tag{1.1.11}$$

and

$$\frac{dy}{dz} = \frac{1}{2}c \cdot 2 \operatorname{sech} z \cdot (\operatorname{sech} z)' =$$

$$c \cdot \operatorname{sech} z \cdot (-\tanh z \cdot \operatorname{sech} z) =$$

$$-c \cdot \tanh z \cdot \operatorname{sech}^2 z =$$

$$-c \cdot \frac{\sinh z}{\cosh z} \cdot \frac{1}{\cosh z} =$$

$$-c \cdot \frac{\sinh z}{\cosh^3 z} \implies$$

$$dy = -c \cdot \frac{\sinh z}{\cosh^3 z} \quad dz \tag{1.1.12}$$

Also, the upper integration limit of the integral in (1.1.9) becomes

$$v = \frac{1}{2}c \cdot \operatorname{sech}^2 z \implies$$

$$\frac{2v}{c} = \operatorname{sech}^2 z \implies$$

$$z = \operatorname{sech}^{-1} \sqrt{\frac{2v}{c}} := w \tag{1.1.13}$$

We substitute the relations (1.1.10), (1.1.11), (1.1.12), (1.1.13) in the equation (1.1.9) and we derive

$$\xi - x_0 = \pm \int^{w} \frac{-c \cdot \frac{\sinh z}{\cosh^3 z}}{\frac{1}{2}c \cdot \operatorname{sech}^2 z \cdot \sqrt{c \cdot \tanh^2 z}} \, dz =$$

$$\pm\frac{2}{\sqrt{c}}\int^{w}\frac{\sinh z}{\operatorname{sech}^2 z\cdot\tanh z\cdot\cosh^3 z}\,dz=$$

$$\pm\frac{2}{\sqrt{c}}\int^{w}\sinh z\cdot\cosh^2 z\cdot\frac{\cosh z}{\sinh z}\cdot\frac{1}{\cosh^3 z}\,dz=$$

$$\pm\frac{2}{\sqrt{c}}\int^{w}dz$$

Therefore, we have that

$$\xi-x_0=\pm\frac{2}{\sqrt{c}}\cdot w\overset{(1.1.13)}{\Longrightarrow}$$

$$\xi-x_0=\pm\frac{2}{\sqrt{c}}\cdot\operatorname{sech}^{-1}\sqrt{\frac{2v}{c}}\implies$$

$$\operatorname{sech}\left[\frac{\sqrt{c}}{2}(\xi-x_0)\right]=\sqrt{\frac{2v}{c}}\implies$$

$$\frac{2v}{c}=\operatorname{sech}^2\left[\frac{\sqrt{c}}{2}(\xi-x_0)\right]\implies$$

$$v(\xi)=\frac{c}{2}\cdot\operatorname{sech}^2\left[\frac{\sqrt{c}}{2}(\xi-x_0)\right]\overset{\xi=x-ct}{\Longrightarrow}$$

$$u(x,t)=\frac{c}{2}\cdot\operatorname{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct-x_0)\right]$$

Now, we are presenting two graphs of the solution above.

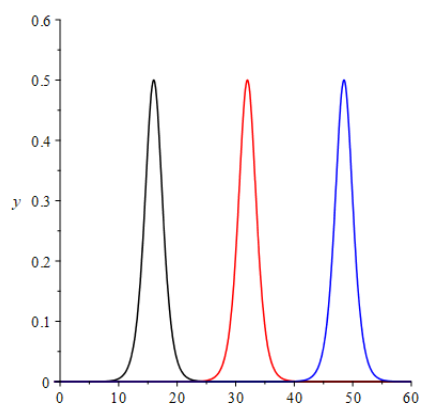

Figure 1.1.6: The soliton of the KdV equation at different times which moves in one direction and it remains unchanged (same shape and same velocity).

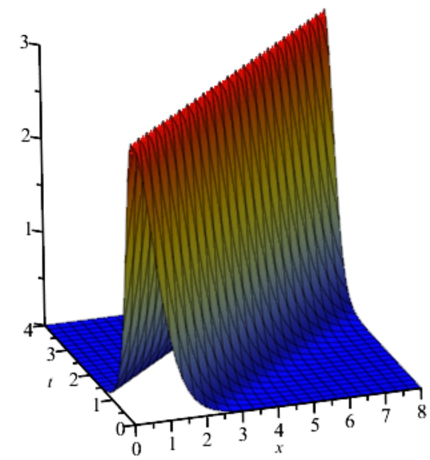

Figure 1.1.7: The soliton of the KdV equation on the three dimensions.

Let us now construct a KdV solution with two soliton components. The first such solution, published by Zabusky in 1968, has the form[4]

$$u(x,t) = 12\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2}$$

This is an exact solution of the equation (1.1.3), describing the collision of two solitons near the origin of the (x,t)-plane.

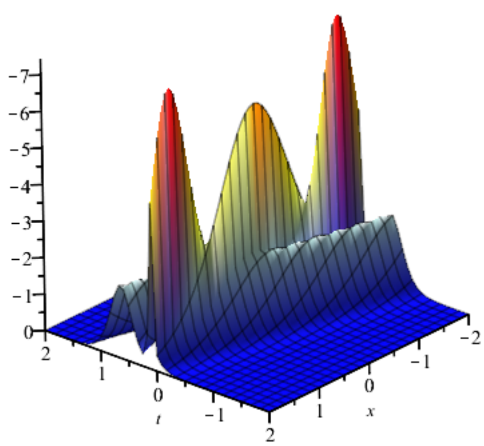

Figure 1.1.8: Two solitons of the KdV equation interacting with each other.

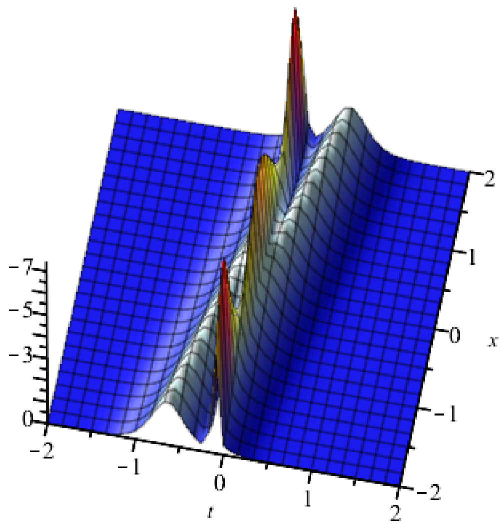

Figure 1.1.9: A different angle of the previous graph. After their interaction, the two solitons continue their motion with the same shape and velocity. There is only a slightly change in their phase.

1.2 The sine-Gordon Equation, Kinks-Antikinks and Breathers

The sine-Gordon equation is a partial differential equation which appears in differential geometry and relativistic field theory. Its name is a wordplay on its similar form to the Klein-Gordon equation. The equation, as well as several solution techniques, were known in the 19th century, but the equation grew greatly in importance when it was realized that it led to solutions ("kink" and "antikink") with the collisional properties of solitons. The sine-Gordon equation also appears in a number of other physical applications including the propagation of fluxons in Josephson junctions (a junction between two superconductors), the motion of rigid pendula attached to a stretched wire, and dislocations in crystals.[11]

The sine-Gordon equation which is a real-valued, hyperbolic, nonlinear wave equation is the following

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u(x,t)}{\partial t^2} = \sin\big(u(x,t)\big)$$

or

$$u_{xx} - u_{tt} = \sin u \tag{1.2.1}$$

We try to find soliton solutions for the SG equation. As we did with the KdV equation, we consider the trial solution

$$u(x,t) = v(x-ct)$$

We, also, consider the variable ξ as

$$\xi = x - ct$$

and then we have

$$u(x,t) = v(x-ct) \equiv v(\xi) \tag{1.2.2}$$

Additionally, we get the following relations

$$\frac{\partial u}{\partial x} = \frac{dv}{d\xi}\cdot\frac{\partial \xi}{\partial x} = v_\xi = v', \qquad \frac{\partial^2 u}{\partial x^2} = v'' \tag{1.2.3}$$

and

$$\frac{\partial u}{\partial t} = \frac{dv}{dt}\cdot\frac{\partial \xi}{\partial t} = -cv_\xi = -cv', \qquad \frac{\partial^2 u}{\partial t^2} = c^2 v'' \tag{1.2.4}$$

By substituting the relations (1.2.2), (1.2.3) and (1.2.4) in the equation (1.2.1) we obtain

$$v'' - c^2 v'' = \sin v \implies$$

$$(1 - c^2)v'' = \sin v$$

We multiply by v'

$$(1-c^2)v'v'' = v' \sin v$$

and we integrate with respect to ξ

$$\int (1-c^2)v'v''\ d\xi = \int v' \sin v\ d\xi \implies$$

$$\frac{1-c^2}{2}(v')^2 + A = -\cos v, \quad A \in \mathbb{R} \tag{1.2.5}$$

We consider the BCs of the soliton

$$\text{as } \xi \to \pm\infty \ \text{ then } \ v, v' \to 0$$

and from the previous equation we get

$$A = -\cos 0 \implies A = -1$$

So, the relation (1.2.5) becomes

$$\frac{(1-c^2)}{2}(v')^2 = 1 - \cos v \implies$$

$$v' = \pm\sqrt{2} \cdot \sqrt{\frac{1-\cos v}{1-c^2}} \implies$$

$$\frac{1}{\sqrt{1-\cos v}}\ dv = \pm\sqrt{2}\frac{1}{\sqrt{1-c^2}}\ d\xi$$

We integrate and we obtain

$$I := \int^v \frac{1}{\sqrt{1-\cos y}}\ dy = \pm\sqrt{2}\frac{1}{\sqrt{1-c^2}}(\xi - x_0)$$

Now, we are solving the integral of the left hand side of the above equation. By using trigonometric identities we derive

$$I = \int^v \frac{1}{\sqrt{2}\sin(\frac{y}{2})}\ dy$$

We make the substitution $z = \dfrac{y}{2}$ and thus we get

$$I = \int^{\frac{v}{2}} \frac{\sqrt{2}}{\sin z}\ dz =$$

$$\sqrt{2}\int^{\frac{v}{2}} \frac{1}{\sin z}\ dz =$$

$$\sqrt{2}\int^{\frac{v}{2}} \csc z \, dz =$$

$$\sqrt{2}\cdot \ln\left(\csc\frac{v}{2} - \cot\frac{v}{2}\right)$$

By using the trigonometric identity $\tan\left(\frac{x}{2}\right) = \csc x - \cot x$ we obtain

$$I = \sqrt{2}\cdot \ln\left[\tan\left(\frac{v}{4}\right)\right]$$

Therefore,

$$\sqrt{2}\cdot \ln\left[\tan\left(\frac{v}{4}\right)\right] = \pm\sqrt{2}\frac{1}{\sqrt{1-c^2}}(\xi - x_0) \implies$$

$$\ln\left[\tan\left(\frac{v}{4}\right)\right] = \pm\frac{1}{\sqrt{1-c^2}}(\xi - x_0) \implies$$

$$\tan\frac{v}{4} = \exp\left[\pm\frac{1}{\sqrt{1-c^2}}(\xi - x_0)\right] \implies$$

$$v(\xi) = 4\arctan\left[\exp\left(\pm\frac{\xi - x_0}{\sqrt{1-c^2}}\right)\right] \overset{\xi = x-ct}{\implies}$$

$$u(x,t) = 4\arctan\left[\exp\left(\pm\frac{x - ct - x_0}{\sqrt{1-c^2}}\right)\right]$$

The solution above describes a soliton moving with velocity $0 \leq c < 1$ and changing phase from 0 to 2π (*kink*, the case of $+$ sign) or from 2π to 0 (*anti-kink*, the case of $-$ sign). Next, we present some graphs of kinks and anti-kinks.

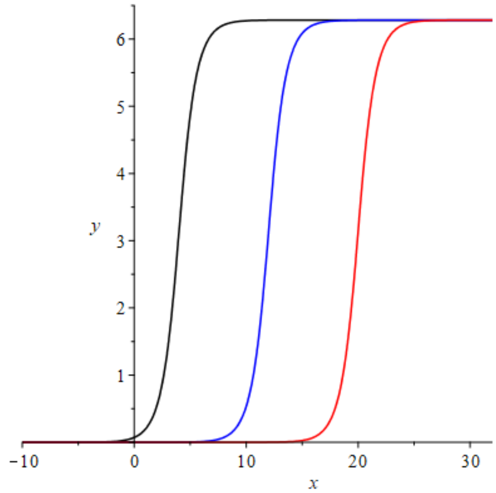

Figure 1.2.1: A kink of the sine-Gordon equation at different times.

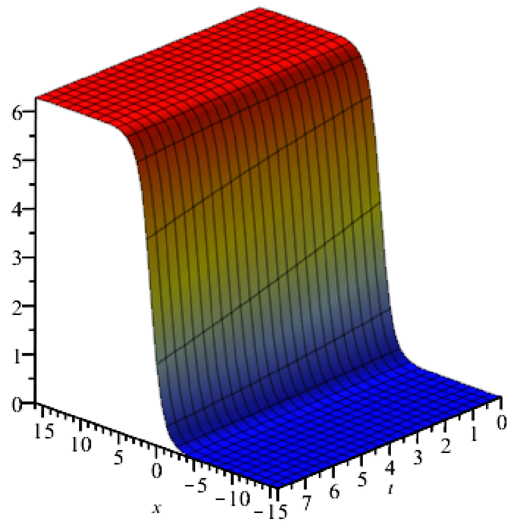

Figure 1.2.2: A kink of the sine-Gordon equation on the three dimensions.

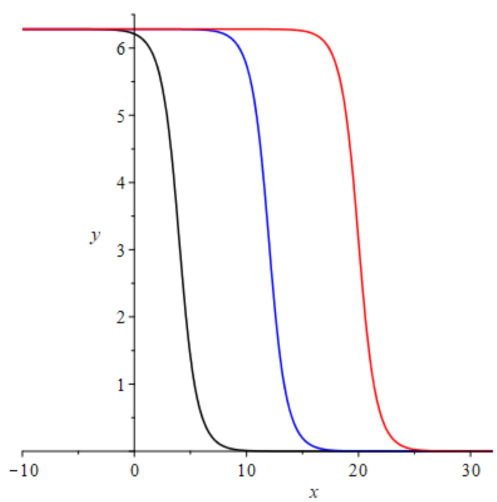

Figure 1.2.3: An anti-kink of the sine-Gordon equation at different times.

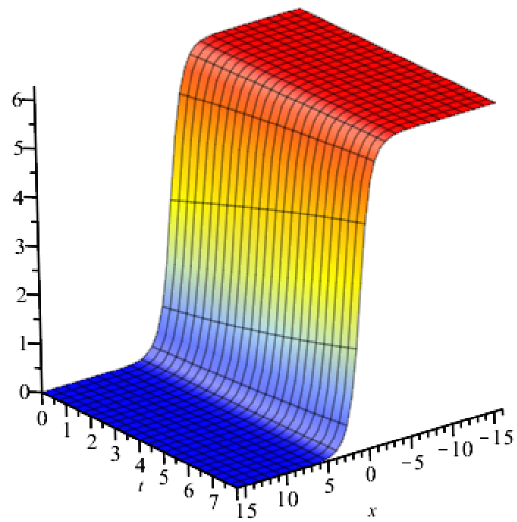

Figure 1.2.4: An anti-kink of the sine-Gordon equation on the three dimensions.

Remark 1.2.1. The sine-Gordon equation does not express a "classic" pulse which moves to one specific direction. Here, a pulse is considered to be the change of the value of u from 0 to 2π, or vice versa, as ξ goes from $-\infty$ to $+\infty$. Thus, if for some time $\tilde{t}$ a fixed area of the propagating mean has value $u = 0$, then after the pulse travels through this area, the area will have value $u = 2\pi$.

As we saw in the example of the KdV equation, we can construct a solution of the sine-Gordon equation consisting of two solitons. One such solution which describes a kink of velocity c, colliding with a kink of velocity $-c$ is given below[1]

$$u(x,t) = \arctan\left[\frac{c \cdot \sinh\left(\frac{x}{\sqrt{1-c^2}}\right)}{\cosh\left(\frac{ct}{\sqrt{1-c^2}}\right)}\right] \tag{1.2.6}$$

In the same way, the interaction of a kink with an anti-kink is given by the following relation[1]

$$u(x,t) = \arctan\left[\frac{\sinh\left(\frac{ct}{\sqrt{1-c^2}}\right)}{c \cdot \cosh\left(\frac{x}{\sqrt{1-c^2}}\right)}\right] \tag{1.2.7}$$

Now, we present two graphs of the above solutions.

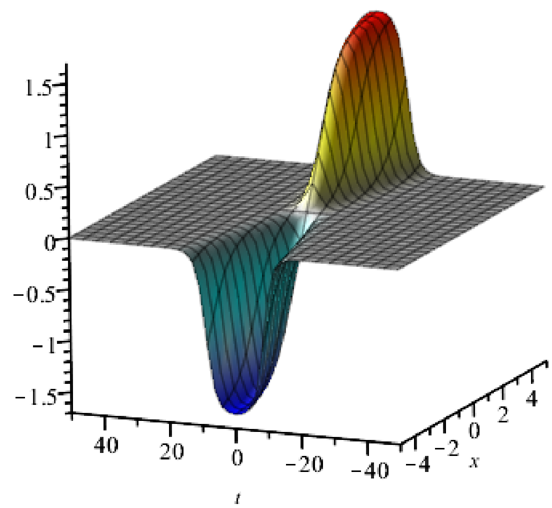

Figure 1.2.5: Two kinks with opposite directions and same velocities, interacting with each other according to the relation (1.2.6).

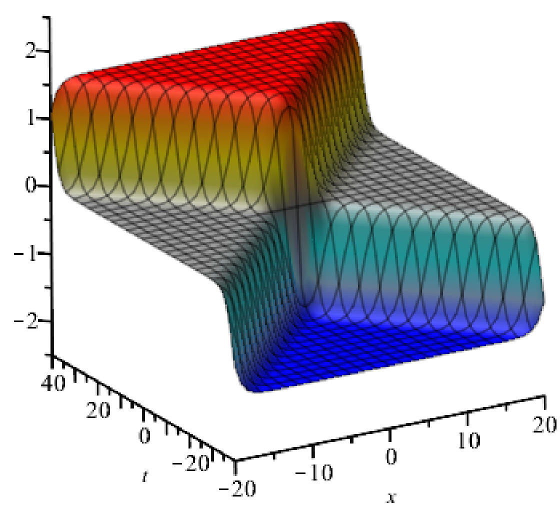

Figure 1.2.6: A kink and an anti-kink colliding with each other at the origin according to the relation (1.2.7).

Except the solutions presented previously, some other solutions of the sine-Gordon equation which have particular interest are these called *breathers*.

Definition 1.2.2. *A breather is a localized periodic solution of either continuous media equations or discrete lattice equations. Breathers are solitonic structures and there are two types of breathers: standing or traveling ones.*

In order to find such solutions that verify the sine-Gordon equation, the inverse scattering transformation has to be used. A breather oscillates, changing its amplitude between its two extreme values. That is, the value of the amplitude of a breather goes from $u = 0$ to $u = u_{\max}$ to $u = 0$ to $u = u_{\min}$ to $u = 0$ etc. The following equation[1]

$$u(x,t) = \arctan\left[\frac{\beta \cdot \sin(\omega t)}{\omega \cdot \cosh\left(\beta(x - x_0)\right)}\right] \tag{1.2.8}$$

describes a standing breather, where $\omega < 1$ is the period of the breather's oscillation and β is a coefficient with the property $\beta^2 + \omega^2 = 1$.

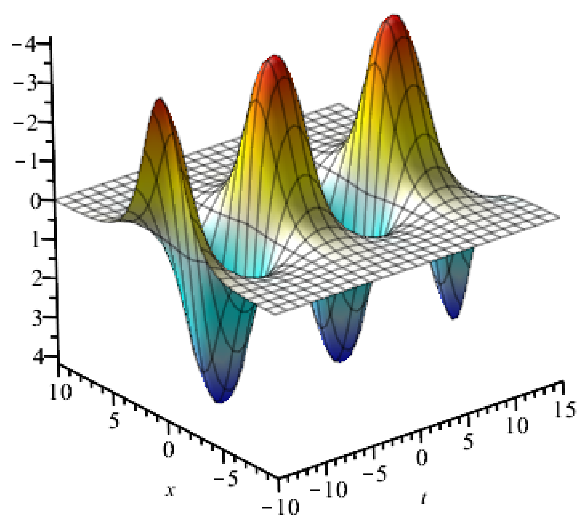

Figure 1.2.7: A stationary breather which is described by equation (1.2.8).

Next, we construct an equation[1] which describes a moving breather of frequency ω and velocity c

$$u(x,t) = 4\arctan\left(\frac{\sqrt{1-\omega^2}}{\omega}\sin\left[\frac{\omega(t-cx)}{\sqrt{1-c^2}}\right]\operatorname{sech}\left[\frac{\sqrt{1-\omega^2}(x-ct)}{\sqrt{1-c^2}}\right]\right) \quad (1.2.9)$$

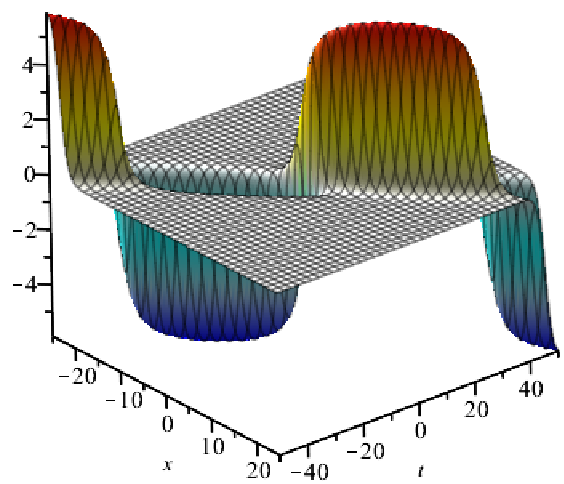

Figure 1.2.8: A moving breather which is described by equation (1.2.9). As the breather travels through the propagating media, it oscillates, at the same time, between its maximum and minimum amplitude.

1.3 The Nonlinear Schrödinger Equation

In theoretical physics, the (one-dimensional) nonlinear Schrödinger (NLS) equation is a nonlinear variation of the Schrödinger equation. It is a classical field equation whose principal applications are to the propagation of light in nonlinear optical fibers and planar waveguides and to Bose–Einstein condensates confined to highly anisotropic cigar-shaped traps, in the mean-field regime. Additionally, the equation appears in the studies of small-amplitude gravity waves on the surface of deep inviscid (zero-viscosity) water; the Langmuir waves in hot plasmas; the propagation of plane-diffracted wave beams in the focusing regions of the ionosphere; the propagation of Davydov's alpha-helix solitons, which are responsible for energy transport along molecular chains; and many others.[14][15][16][17]

When we study nonlinear wave packets, it is often convenient to write the solution of a linear wave problem in the form[4]

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}(k)\cdot \mathrm{e}^{i(kx-\omega t)}\; dk \tag{1.3.1}$$

where $\mathcal{F}(k)$ is the Fourier transform of $u(x,0)$ and ω is a function of k i.e. $\omega = \omega(k)$. In case that $\omega \neq k$, then each component in equation (1.3.1) travels at a different speed $\dfrac{\omega}{k}$ and the wave disperses. Therefore, the relation $\omega = \omega(k)$ is called dispersion relation.

A wave packet is a special form of the equation (1.3.1) with the Fourier components lying close to some propagation number k_0 and its corresponding frequency ω_0. That is, $\mathcal{F}(k)$ achieves its maximum value at $k = k_0$, falling rapidly as $|k-k_0|$ increases. This phenomenon, allows us to expand the dispersion relation as a power series around k_0 and to obtain

$$\omega = \omega_0 + b_1(k-k_0) + b_2(k-k_0)^2 + ... \tag{1.3.2}$$

Now, using relation (1.3.2) up until the second order term, the equation (1.3.1) becomes

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}(k)\cdot \mathrm{e}^{\mathrm{i}[kx-\{\omega_0+b_1(k-k_0)+b_2(k-k_0)^2\}\cdot t]}\; dk =$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}(k)\cdot \mathrm{e}^{\mathrm{i}[kx-\omega_0 t-b_1(k-k_0)t-b_2(k-k_0)^2 t]}\; dk =$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathcal{F}(k)\cdot \mathrm{e}^{\mathrm{i}[kx+k_0x-k_0x-\omega_0t-b_1(k-k_0)t-b_2(k-k_0)^2t]}\;dk =$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathcal{F}(k)\cdot \mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)}\cdot \mathrm{e}^{\mathrm{i}[(k-k_0)x-b_1(k-k_0)t-b_2(k-k_0)^2t]}\;dk \implies$$

$$u(x,t) = \mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathcal{F}(k)\cdot \mathrm{e}^{\mathrm{i}[(k-k_0)x-b_1(k-k_0)t-b_2(k-k_0)^2t]}\;dk$$

where the factor $\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)}$ is a carrier wave with velocity $u_c = \dfrac{\omega_0}{k_0}$ and riding over the carrier is an envelope wave

$$\phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathcal{F}(k)\cdot \mathrm{e}^{\mathrm{i}[(k-k_0)x-b_1(k-k_0)t-b_2(k-k_0)^2t]}\;dk \tag{1.3.3}$$

Now, by changing the variable of integration from k to $\kappa \equiv k-k_0$, relation (1.3.3) takes the form

$$\phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathcal{F}(\kappa+k_0)\cdot \mathrm{e}^{\mathrm{i}[\kappa x-b_1\kappa t-b_2\kappa^2t]}\;d\kappa$$

First, we take the partial derivative of $\phi(x,t)$ with respect to t and we derive

$$\frac{\partial\phi}{\partial t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{i}(-b_1\kappa-b_2\kappa^2)\cdot\mathcal{F}(\kappa+k_0)\cdot \mathrm{e}^{\mathrm{i}[\kappa x-b_1\kappa t-b_2\kappa^2t]}\;d\kappa =$$

$$-b_1\cdot\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{i}\kappa\cdot\mathcal{F}(\kappa+k_0)\cdot \mathrm{e}^{\mathrm{i}[\kappa x-b_1\kappa t-b_2\kappa^2t]}\;d\kappa\;+$$

$$+\;\mathrm{i}b_2\cdot\frac{1}{2\pi}\int_{-\infty}^{+\infty}-\kappa^2\cdot\mathcal{F}(\kappa+k_0)\cdot \mathrm{e}^{\mathrm{i}[\kappa x-b_1\kappa t-b_2\kappa^2t]}\;d\kappa$$

Subsequently, we take the partial derivative of $\phi(x,t)$ with respect to x and we obtain the following equations

$$\frac{\partial\phi}{\partial x} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{i}\kappa\cdot\mathcal{F}(\kappa+k_0)\cdot \mathrm{e}^{\mathrm{i}[\kappa x-b_1\kappa t-b_2\kappa^2t]}\;d\kappa$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}-\kappa^2\cdot\mathcal{F}(\kappa+k_0)\cdot \mathrm{e}^{\mathrm{i}[\kappa x-b_1\kappa t-b_2\kappa^2t]}\;d\kappa$$

Now, by combing the three previous equations we can obtain the following P.D.E. which governs time evolution of the envelope

$$\frac{\partial \phi}{\partial t} = -b_1 \frac{\partial \phi}{\partial x} + \mathrm{i} b_2 \frac{\partial^2 \phi}{\partial x^2} \implies$$

$$\frac{\partial \phi}{\partial t} + b_1 \frac{\partial \phi}{\partial x} - \mathrm{i} b_2 \frac{\partial^2 \phi}{\partial x^2} = 0 \implies$$

$$\mathrm{i}\left(\frac{\partial \phi}{\partial t} + b_1 \frac{\partial \phi}{\partial x}\right) + b_2 \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{1.3.4}$$

Now, we shall consider how a small amount of nonlinearity will affect relation (1.3.4) If a is the local amplitude of the wave envelope, the lowest order contribution to the dispersion relation will be proportional to a^2. Also, the equation $a^2 = |\phi|^2$ holds. From equation (1.3.4) the dispersion relation of the envelope is

$$\omega = b_1 \kappa + b_2 \kappa^2$$

If the equation (1.3.4) is converted to the nonlinear P.D.E.

$$\mathrm{i}\left(\frac{\partial \phi}{\partial t} + b_1 \frac{\partial \phi}{\partial x}\right) + b_2 \frac{\partial^2 \phi}{\partial x^2} + \lambda |\phi|^2 \phi = 0, \ \lambda \in \mathbb{R} \tag{1.3.5}$$

then the nonlinear dispersion relation becomes

$$\omega = b_1 \kappa + b_2 \kappa^2 - \lambda a^2$$

Equation (1.3.5) is the nonlinear Schrödinger equation, which has the three following properties:

1. NLS equation describes the propagation of an envelope wave, riding over a carrier.

2. If the envelope varies sufficiently slowly with x and is of small enough amplitude, then the last two terms on the left-hand side of NLS equation can be neglected and the modulation travels with velocity b_1.

3. The term $b_2 \frac{\partial^2 \phi}{\partial x^2}$ introduces wave dispersion at the lowest level of approximation. On the other hand, the term $\lambda |\phi|^2 \phi$ introduces nonlinearity at the lowest level of approximation. Hence, the NLS equation is generic, arising whenever one wishes to consider the lowest order effects of dispersion and nonlinearity on a wave packet.

By making a proper transformation of the variables, one can obtain the normalised form of the NLS equation

$$\mathrm{i}\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+2|u|^2u=0 \tag{1.3.6}$$

on the infinite domain $-\infty<x<+\infty$ and $-\infty<t<+\infty$. Now we shall try to solve the NLS equation (1.3.6). However, we can not use the same technique as we did with KdV and sine-Gordon equations. We can not consider the trial solution

$$u(x,t)=v(x-ct)$$

because we would require the velocity to be imaginary. Therefore, we should consider solutions of the more general form

$$u(x,t)=\phi(x,t)\cdot \mathrm{e}^{\mathrm{i}\theta(x,t)} \tag{1.3.7}$$

where ϕ and θ are real valued functions. First, we take the derivative of u with respect to t and we have that

$$\frac{\partial u}{\partial t}=\frac{\partial \phi}{\partial t}\cdot \mathrm{e}^{\mathrm{i}\theta}+\mathrm{i}\phi\cdot\frac{\partial \theta}{\partial t}\cdot \mathrm{e}^{\mathrm{i}\theta} \tag{1.3.8}$$

Next, we take the derivative of u with respect to x and we obtain the relations

$$\frac{\partial u}{\partial x}=\frac{\partial \phi}{\partial x}\cdot \mathrm{e}^{\mathrm{i}\theta}+\mathrm{i}\phi\cdot\frac{\partial \theta}{\partial x}\cdot \mathrm{e}^{\mathrm{i}\theta}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=\frac{\partial^2 \phi}{\partial x^2}\cdot \mathrm{e}^{\mathrm{i}\theta}+\mathrm{i}\cdot\frac{\partial \phi}{\partial x}\cdot\frac{\partial \theta}{\partial x}\cdot \mathrm{e}^{\mathrm{i}\theta}+\mathrm{i}\cdot\frac{\partial \phi}{\partial x}\cdot\frac{\partial \theta}{\partial x}\cdot \mathrm{e}^{\mathrm{i}\theta}+\\&+\mathrm{i}\phi\cdot\frac{\partial^2 \theta}{\partial x^2}\cdot \mathrm{e}^{\mathrm{i}\theta}-\phi\cdot\frac{\partial \theta}{\partial x}\cdot\frac{\partial \theta}{\partial x}\cdot \mathrm{e}^{\mathrm{i}\theta}\implies\\ \frac{\partial^2 u}{\partial x^2}&=\frac{\partial^2 \phi}{\partial x^2}\cdot \mathrm{e}^{\mathrm{i}\theta}+\mathrm{i}\cdot\frac{\partial \phi}{\partial x}\cdot\frac{\partial \theta}{\partial x}\cdot \mathrm{e}^{\mathrm{i}\theta}+\mathrm{i}\cdot\frac{\partial \phi}{\partial x}\cdot\frac{\partial \theta}{\partial x}\cdot \mathrm{e}^{\mathrm{i}\theta}+\\&+\mathrm{i}\phi\cdot\frac{\partial^2 \theta}{\partial x^2}\cdot \mathrm{e}^{\mathrm{i}\theta}-\phi\cdot\left(\frac{\partial \theta}{\partial x}\right)^2\cdot \mathrm{e}^{\mathrm{i}\theta}\end{aligned} \tag{1.3.9}$$

Now, using relations (1.3.7), (1.3.8) and (1.3.9), equation (1.3.6) becomes

$$\begin{aligned}&\mathrm{i}\cdot\frac{\partial \phi}{\partial t}\cdot \mathrm{e}^{\mathrm{i}\theta}-\phi\cdot\frac{\partial \theta}{\partial t}\cdot \mathrm{e}^{\mathrm{i}\theta}+\frac{\partial^2 \phi}{\partial x^2}\cdot \mathrm{e}^{\mathrm{i}\theta}+\mathrm{i}\cdot\frac{\partial \phi}{\partial x}\cdot\frac{\partial \theta}{\partial x}\cdot \mathrm{e}^{\mathrm{i}\theta}+\mathrm{i}\cdot\frac{\partial \phi}{\partial x}\cdot\frac{\partial \theta}{\partial x}\cdot \mathrm{e}^{\mathrm{i}\theta}+\\&+\mathrm{i}\phi\cdot\frac{\partial^2 \theta}{\partial x^2}\cdot \mathrm{e}^{\mathrm{i}\theta}-\phi\cdot\left(\frac{\partial \theta}{\partial x}\right)^2\cdot \mathrm{e}^{\mathrm{i}\theta}+2\phi^2\cdot\left|\mathrm{e}^{\mathrm{i}\theta}\right|^2\cdot\phi\cdot \mathrm{e}^{\mathrm{i}\theta}=0\overset{\mathrm{e}^{\mathrm{i}\theta}\neq 0}{\implies}\end{aligned}$$

$$\mathrm{i}\cdot\frac{\partial\phi}{\partial t}-\phi\cdot\frac{\partial\theta}{\partial t}+\frac{\partial^2\phi}{\partial x^2}+\mathrm{i}\cdot\frac{\partial\phi}{\partial x}\cdot\frac{\partial\theta}{\partial x}+\mathrm{i}\cdot\frac{\partial\phi}{\partial x}\cdot\frac{\partial\theta}{\partial x}+\mathrm{i}\phi\cdot\frac{\partial^2\theta}{\partial x^2}-\phi\cdot\left(\frac{\partial\theta}{\partial x}\right)^2+2\phi^3=0 \implies$$

$$-\phi\cdot\frac{\partial\theta}{\partial t}+\frac{\partial^2\phi}{\partial x^2}-\phi\cdot\left(\frac{\partial\theta}{\partial x}\right)^2+2\phi^3+\mathrm{i}\left(\frac{\partial\phi}{\partial t}+2\cdot\frac{\partial\phi}{\partial x}\cdot\frac{\partial\theta}{\partial x}+\phi\cdot\frac{\partial^2\theta}{\partial x^2}\right)=0$$

Now, by equating real and imaginary parts we get the system

$$\begin{cases}-\phi\cdot\dfrac{\partial\theta}{\partial t}+\dfrac{\partial^2\phi}{\partial x^2}-\phi\cdot\left(\dfrac{\partial\theta}{\partial x}\right)^2+2\phi^3=0\\[2ex] \dfrac{\partial\phi}{\partial t}+2\cdot\dfrac{\partial\phi}{\partial x}\cdot\dfrac{\partial\theta}{\partial x}+\phi\cdot\dfrac{\partial^2\theta}{\partial x^2}=0\end{cases}$$

or with a more simply notation

$$\begin{cases}-\phi\theta_t+\phi_{xx}-\phi\theta_x^2+2\phi^3=0\\ \phi_t+2\phi_x\theta_x+\phi\theta_{xx}=0\end{cases}$$

Now, we can try the usual wave solutions for the above system, therefore we can write

$$\theta(x,t)=\tilde{\theta}(x-c_\theta t)\equiv\tilde{\theta}(\xi_\theta)$$
$$\phi(x,t)=\tilde{\phi}(x-c_\phi t)\equiv\tilde{\phi}(\xi_\phi)$$

So, our system becomes

$$\begin{cases}c_\theta\cdot\tilde{\phi}\cdot\tilde{\theta}_{\xi_\theta}+\tilde{\phi}_{\xi_\phi\xi_\phi}-\tilde{\phi}\cdot\tilde{\theta}_{\xi_\theta}^2+2\tilde{\phi}^3=0\\ -c_\phi\cdot\tilde{\phi}_{\xi_\phi}+2\cdot\tilde{\phi}_{\xi_\phi}\cdot\tilde{\theta}_{\xi_\theta}+\tilde{\phi}\cdot\tilde{\theta}_{\xi_\theta\xi_\theta}=0\end{cases}\tag{1.3.10}$$

We multiply the second equation by $2\tilde{\phi}$ and we get

$$-2c_\phi\cdot\tilde{\phi}\cdot\tilde{\phi}_{\xi_\phi}+4\cdot\tilde{\phi}\cdot\tilde{\phi}_{\xi_\phi}\cdot\tilde{\theta}_{\xi_\theta}+2\cdot\tilde{\phi}^2\cdot\tilde{\theta}_{\xi_\theta\xi_\theta}=0 \implies$$

$$4\cdot\tilde{\phi}\cdot\tilde{\phi}_{\xi_\phi}\cdot\tilde{\theta}_{\xi_\theta}+2\cdot\tilde{\phi}^2\cdot\tilde{\theta}_{\xi_\theta\xi_\theta}=2c_\phi\cdot\tilde{\phi}\cdot\tilde{\phi}_{\xi_\phi} \implies$$

$$\frac{\partial}{\partial x}\left(2\cdot\tilde{\phi}^2\cdot\tilde{\theta}_{\xi_\theta}\right)=c_\phi\cdot\frac{\partial}{\partial x}\left(\tilde{\phi}^2\right)$$

We integrate with respect to x and we have

$$2\cdot\tilde{\phi}^2\cdot\tilde{\theta}_{\xi_\theta}=\tilde{\phi}^2+A,\ A\in\mathbb{R} \implies$$

$$\tilde{\phi}^2\left(2\cdot\tilde{\theta}_{\xi_\theta}-c_\phi\right)=A,\ A\in\mathbb{R}$$

Without loss of generality, A can be assumed to be equal to zero and so we obtain

$$\tilde{\phi}^2\big(2 \cdot \tilde{\theta}_{\xi_\theta} - c_\phi\big) = 0 \implies$$

$$2 \cdot \tilde{\theta}_{\xi_\theta} - c_\phi = 0 \implies$$

$$\tilde{\theta}_{\xi_\theta} = \frac{c_\phi}{2}$$

We substitute this expression in the first equation of system (1.3.10) and we get

$$c_\theta \cdot \tilde{\phi} \cdot \frac{c_\phi}{2} + \tilde{\phi}_{\xi_\phi \xi_\phi} - \tilde{\phi} \cdot \frac{c_\phi^2}{4} + 2\tilde{\phi}^3 = 0 \implies$$

$$\frac{d^2\tilde{\phi}}{d\xi_\phi^2} = \frac{c_\phi^2 - 2c_\phi c_\theta}{4} \cdot \tilde{\phi} - 2\tilde{\phi}^3 \implies$$

$$\frac{1}{\frac{c_\phi^2 - 2c_\phi c_\theta}{4} \cdot \tilde{\phi} - 2\tilde{\phi}^3}\, d^2\tilde{\phi} = d\xi_\phi^2 \implies$$

$$\iint \frac{1}{\frac{c_\phi^2 - 2c_\phi c_\theta}{4} \cdot \tilde{\phi} - 2\tilde{\phi}^3}\, d^2\tilde{\phi} = \int d\xi_\phi^2 \implies$$

$$\iint \frac{1}{\sqrt{\frac{c_\phi^2 - 2c_\phi c_\theta}{4} \cdot \tilde{\phi} - 2\tilde{\phi}^3}}\, d^2\tilde{\phi} = \pm\xi_\phi \overset{\xi_\phi = x - c_\phi t}{\implies}$$

$$\int^{\tilde{\phi}} \frac{dy}{\sqrt{P(y)}} = \pm(x - c_\phi t) \tag{1.3.11}$$

where

$$P(y) = B + \frac{c_\phi^2 - 2c_\phi c_\theta}{4} \cdot y^2 - y^4$$

In general, if $B \neq 0$ then the integral of the left-hand side of equation (1.3.11) is an elliptic integral and it can be calculated using elliptic functions. However, if $B = 0$, then we get the solution

$$\tilde{\phi} = a \operatorname{sech}\big[a(x - c_\phi t)\big] \tag{1.3.12}$$

where a is the wave amplitude and satisfies the following relation

$$a^2 = \frac{c_\phi^2 - 2c_\phi c_\theta}{4} \tag{1.3.13}$$

Now, we solve

$$\tilde{\theta}_{\xi_\theta} = \frac{c_\phi}{2} \implies$$

$$\tilde{\theta} = \frac{c_\phi}{2}\xi_\theta = \frac{c_\phi}{2}(x - c_\theta t) \implies$$

$$\tilde{\theta} = \frac{c_\phi}{2}x - \frac{c_\phi c_\theta}{2}t \overset{(1.3.13)}{\implies}$$

$$\tilde{\theta} = \frac{c_\phi}{2}x + \left(a^2 - \frac{c_\phi^2}{4}\right)t \tag{1.3.14}$$

Finally, using relations (1.2.12) and (1.3.14) we obtain an exact solution for the NLS equation and relation (1.3.7) becomes

$$u(x,t) = a\,\mathrm{sech}\left[a(x - c_\phi t - x_0)\right] \cdot \exp\left[\mathrm{i}\frac{c_\phi}{2}x + \mathrm{i}\left(a^2 - \frac{c_\phi^2}{4}\right)t\right] \tag{1.3.15}$$

Now, if we set $c_\phi = 0$, we obtain the stationary breather which is localised around the point $x = x_0$, oscillates at a frequency of a^2 and is given by the following equation

$$u(x,t) = a\,\mathrm{sech}\left[a(x - x_0)\right] \cdot \mathrm{e}^{\mathrm{i}a^2 t} \tag{1.3.16}$$

Next, we present a graph of the above solution.

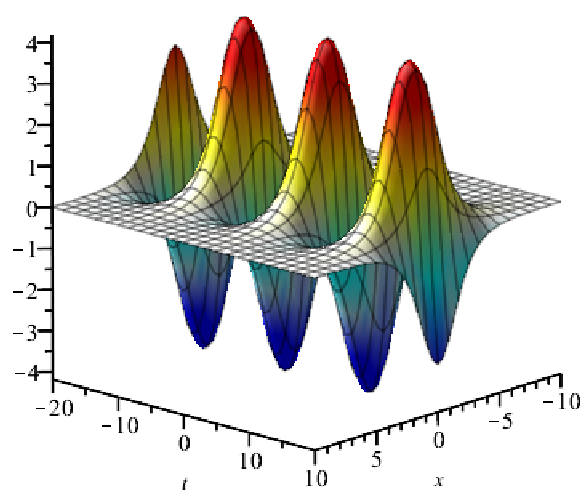

Figure 1.2.7: A stationary breather of the NLS equation which is described by relation (1.3.16).

Now, we can choose the constant B in equation (1.3.11) to be negative and then we can write $P(y)$ as

$$P(y) = (a^2 - y^2)(y^2 - b^2)$$

and it holds that $P(y) \in \mathbb{R}$ for $b \leq y \leq a$. So, the equation

$$\int^{\tilde{\phi}} \frac{dy}{\sqrt{P(y)}} = \pm(x - c_\phi t)$$

gives that

$$\tilde{\phi} = a \cdot \mathrm{dn}\,[a(x - c_\phi t); k]$$

where $k^2 = \dfrac{1 - b^2}{a^2}$ and dn is one of the Jacobi elliptic functions. The quantity $\tilde{\phi}$ oscillates between a maximum value of a and a minimum value of b with period of $T = \dfrac{2}{a} \cdot K(k)$ where $4(a^2 + b^2) = c_\phi^2 - 2c_\phi c_\theta$. Therefore, the solution of the NLS equation takes the form

$$u(x,t) = \mathrm{dn}[a(x - c_\phi t - x_0); k] \cdot a \exp\left[\mathrm{i}\frac{c_\phi}{2}x + \mathrm{i}\left(a^2 + b^2 - \frac{c_\phi^2}{4}\right)t\right]$$

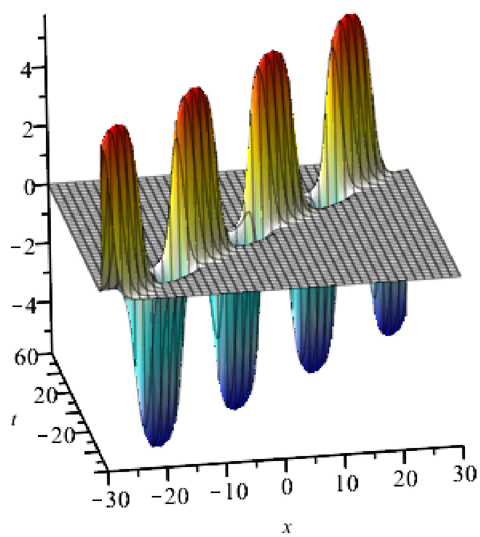

Figure 1.2.7: A moving breather of the NLS equation which is described by the previous relation.

2 Nonlinear Lattices

2.1 Nonlinear Normal Modes

Before we introduce the concept of nonlinear lattices, we shall make a brief presentation of nonlinear normal modes (NNMs). We will focus on NNMs in discrete systems. For nonlinear systems, NNMs can be regarded as generalizations of the linear normal modes of classical vibration theory. While NNMs lack many of the useful mathematical properties of linear normal modes, such as superposition and invariance, they do provide a valuable tool for understanding nonlinear systems. The concept of a NNMs was first introduced by Rosenberg in the 1960s and it was defined as the following.[27]

Definition 2.1.1. *A nonlinear normal mode is any vibration-in-unison of a conservative nonlinear system—i.e. where the coordinates of the system pass through the equilibrium and reach their extrema simultaneously.*

Remark 2.1.2. The motion of all coordinates is periodic and of the same period. Furthermore, at any given time, oscillations of all coordinates can be parameterized by any one coordinate i.e. the coordinates are related by functional relations of the form

$$\mathbf{x}_i = \hat{\mathbf{x}}_i(\mathbf{x}_m), \quad i = 1, ..., n \text{ and } m \text{ is fixed, } m \leq n$$

where n is the number of the coordinates of the system.

When a discrete system vibrates in a NNM, the corresponding oscillation is represented by a line in its configuration space, which is termed modal line. A modal line represents the synchronous oscillation of the system in the configuration space during a NNM motion. Linear systems possess straight modal lines since their coordinates are related linearly during a normal mode oscillation. In nonlinear systems, the modal lines can be either straight or curved. The later cases are generic in nonlinear discrete systems, since straight nonlinear modal lines reflect symmetries of the system.[26]

Now, we are going to make a schematic demonstration of how a 2-degree of freedom system with cubic stiffness behaves and we will compare it with its corresponding linear system. Let us consider our system (S1) which is governed by the equations of motion

$$\begin{cases} m\ddot{x}_1 = -2kx_1 + kx_2 - Kx_1^3 \\ m\ddot{x}_2 = -2kx_2 + kx_1 \end{cases}$$

where m is the mass of the two oscillators, k is the spring constant, K is a cubic stiffness coefficient and x_1 and x_2 are the displacements of the oscillators.

As we mentioned, its corresponding linear system (S2) is the one given by the equations below

$$\begin{cases} m\ddot{x}_1 = -2kx_1 + kx_2 \\ m\ddot{x}_2 = -2kx_2 + kx_1 \end{cases}$$

We shall present two schemes of the oscillations of the linear system (S2) where they move in-phase and out-of-phase. The reason we present the corresponding linear edition of the system (S1) is to become easier to visualize the normal mode motion.

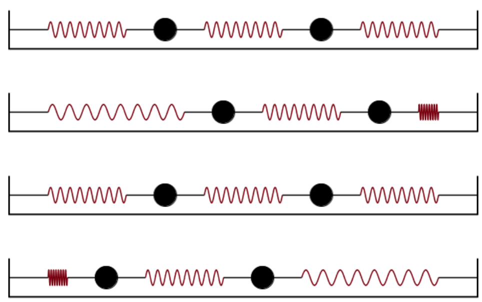

Figure 2.1.1: The in-phase motion of the two coupled oscillators. At the first scheme they are at equilibrium position. Then, they move to the right with the same initial velocity v_0. This is the in-phase normal mode.

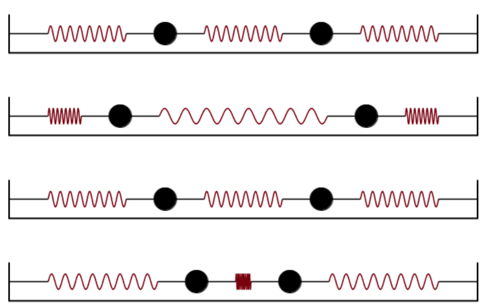

Figure 2.1.2: The out-of-phase motion of the two coupled oscillators. At the first scheme they are at equilibrium position. Then, the first oscillator moves to the left with a negative velocity v_0 and the second oscillator moves to the right with a positive velocity v_1, where $|v_0| = |v_1|$. This is the out-of-phase normal mode.

The in-phase and out-of-phase normal mode motions of the linear system (S2) and then of the nonlinear system (S1) are depicted in the following graphs.

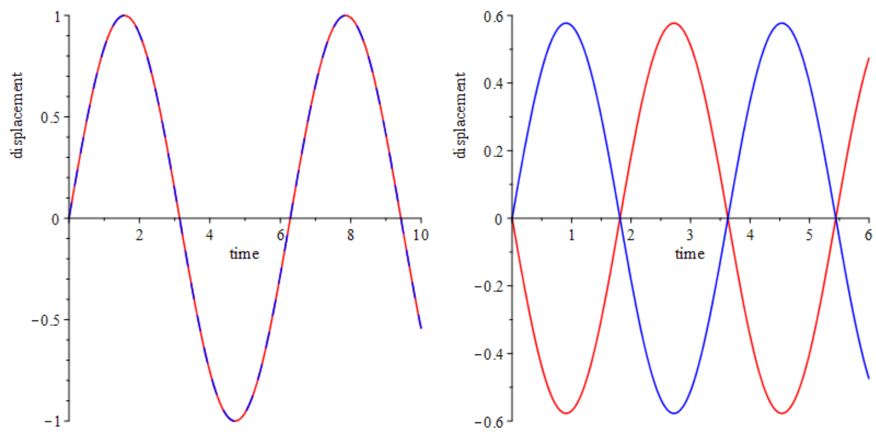

Figure 2.1.3: The in-phase LNM (left) and the out-of-phase LNM (right). Observe that the two oscillators reach their maximum amplitude at the same time.

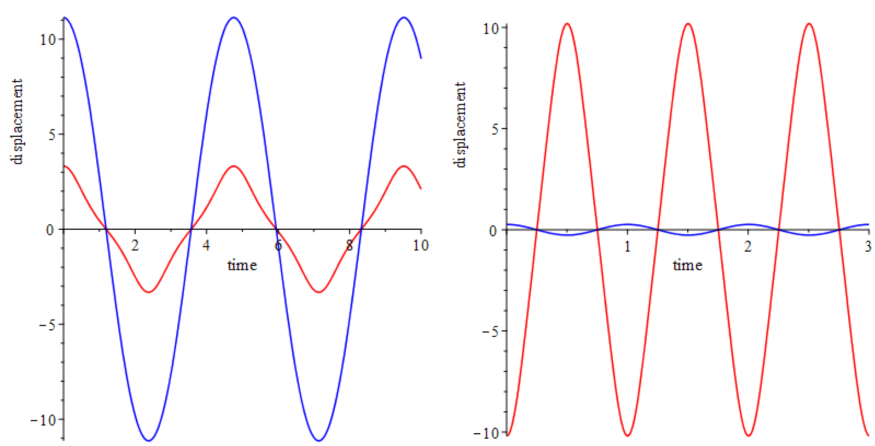

Figure 2.1.4: The in-phase NNM (left) and the out-of-phase NNM (right).

2.2 The Fermi-Pasta-Ulam Problem

In the early 1950s MANIAC-I had just been completed and the time had come for scientists to tackle some very important problems. Physicist and Nobel prize winner Enrico Fermi, computer expert and physicist John Pasta and mathematician Stan Ulam saw the chance and decided to grasp it. But the question was, which problem was of great interest? Fermi suggested to study (numerically) the one-dimensional analogue of atoms in a crystal: a long chain of particles linked by springs that obey Hooke's law (a linear interaction), but with a weak nonlinear correction. The aim of this numerical experiment was to investigate the statistical properties of the chain, and in particular the question how fast a many particle system reaches thermal equilibrium. Fermi, Pasta and Ulam thought that, due to the nonlinear correction, the energy introduced into the lowest frequency mode k=1 should have slowly drifted to the other modes, until the equipartition of energy, a consequence of ergodicity, would have been reached. The beginning of the calculation indeed suggested that this was the case. Modes k=2, k=3,. . . , were successively excited, reaching a state close to equipartition, as shown in figure 2.2.1 below. However, by accident, one day, they let the program run longer. When they realized their oversight and came back to the computer room, they noticed that the system, after remaining in the near equipartition state for a while, had then departed from it. To their great surprise, after 157 periods of the mode k=1, almost all the energy was back to this mode. [32][33]

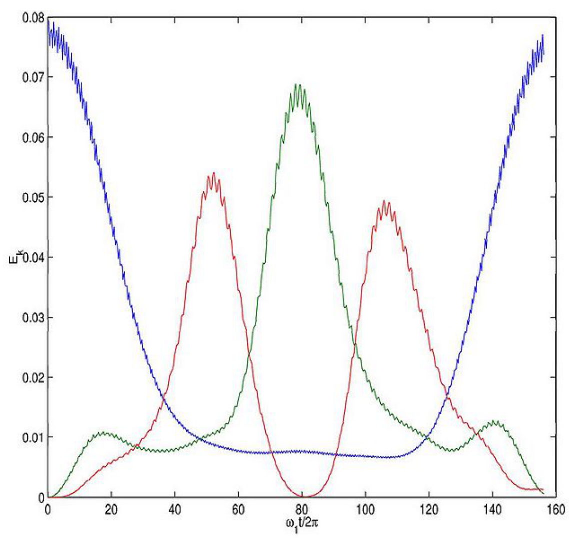

Figure 2.2.1: The plot shows the time evolution of the energy of each of the three lowest normal modes, related to the displacements. Initially, only mode k=1 (blue) is excited. After flowing to other modes, k=2 (green), k=3 (red), etc., the energy almost fully returns to mode k=1! [33]

Now, as we described above, let us consider a one-dimensional lattice of particles with nearest neighbor interaction. Let m denote the mass of every particle, l the length of the string and N the total number of particles. The N oscillators have equilibrium positions

$$p_i = ih, \ \ i = 0, ..., N-1$$

where $h = \dfrac{l}{N-1}$ is the lattice spacing i.e. the space between two neighbor particles. Their positions at time t are

$$X_i(t) = p_i + x_i(t)$$

where x_i denotes the displacement of the oscillators from equilibrium. Now, the force attracting any oscillator to one of its neighbors is taken as

$$F = k(\delta + a\delta^2)$$

where δ is the deviation of the distance separating these two oscillators from their equilibrium position h. The force acting on the $i-$th oscillator because of its right neighbor is

$$F_i(x)^+ = k\big[(x_{i+1} - x_i) + a(x_{i+1} - x_i)^2\big]$$

while the force acting on the $i-$th oscillator because of its left neighbor is

$$F_i(x)^- = k\big[(x_{i-1} - x_i) - a(x_{i-1} - x_i)^2\big]$$

Therefore, the total force on the $i-$th particle is given by

$$F_i(x) = F_i(x)^+ + F_i(x)^- =$$

$$k\big[(x_{i+1} - x_i) + a(x_{i+1} - x_i)^2\big] + k\big[(x_{i-1} - x_i) - a(x_{i-1} - x_i)^2\big] =$$

$$k(x_{i+1} - x_i) + ka(x_{i+1} - x_i)^2 + k(x_{i-1} - x_i) - ka(x_{i-1} - x_i)^2 =$$

$$k(x_{i+1} - 2x_i + x_{i-1}) + ka(x_{i+1}^2 - 2x_ix_{i+1} + x_i^2 - x_{i-1}^2 + 2x_ix_{i-1} - x_i^2) =$$

$$k(x_{i+1} - 2x_i + x_{i-1}) + ka(x_{i+1}^2 - 2x_ix_{i+1} - x_{i-1}^2 + 2x_ix_{i-1}) =$$

$$k(x_{i+1} - 2x_i + x_{i-1}) + ka(x_{i+1}^2 - 2x_ix_{i+1} + x_{i+1}x_{i-1} - x_{i-1}^2 + 2x_ix_{i-1} - x_{i+1}x_{i-1}) =$$

$$k(x_{i+1} - 2x_i + x_{i-1}) + ka\big[x_{i+1}(x_{i+1} - 2x_i + x_{i-1}) - x_{i-1}(x_{i-1} - 2x_i + x_{i+1})\big] =$$

$$k(x_{i+1} - 2x_i + x_{i-1}) + ka(x_{i+1} - 2x_i + x_{i-1})(x_{i+1} - x_{i-1}) =$$

$$k(x_{i+1} - 2x_i + x_{i-1})\big[1 + a(x_{i+1} - x_{i-1})\big]$$

Thus, by Newton's equation of motion we derive

$$m\ddot{x}_i = k(x_{i+1} - 2x_i + x_{i-1})\big[1 + a(x_{i+1} - x_{i-1})\big] \tag{2.2.1}$$

The BCs are $x_0(t) = x_{N-1}(t) = 0$, since the first and the last particle of the lattice can not move. Furthermore, the lattice starts its motion from rest so we have the initial conditions $\dot{x}_i(0) = 0$, for $i = 0, ..., N-1$.

Now we shall rewrite equation (2.2.1) in a more convenient way. If we denote ρ the density of the string, then $m = \rho h$. Also, if κ denotes the spring constant for a piece of unit length, then $k = \dfrac{\kappa}{h}$ will be the spring constant for a piece of length h. Now, let us define $c = \sqrt{\dfrac{\kappa}{\rho}}$. So equation (2.2.1) becomes

$$m\ddot{x}_i = k(x_{i+1} - 2x_i + x_{i-1})\big[1 + a(x_{i+1} - x_{i-1})\big] \implies$$

$$\ddot{x}_i = \frac{k}{m}(x_{i+1} - 2x_i + x_{i-1})\big[1 + a(x_{i+1} - x_{i-1})\big] \implies$$

$$\ddot{x}_i = \frac{\kappa}{h} \cdot \frac{1}{\rho h}(x_{i+1} - 2x_i + x_{i-1})\big[1 + a(x_{i+1} - x_{i-1})\big] \implies$$

$$\ddot{x}_i = \frac{\kappa}{\rho} \cdot \frac{1}{h^2}(x_{i+1} - 2x_i + x_{i-1})\big[1 + a(x_{i+1} - x_{i-1})\big] \implies$$

$$\ddot{x}_i = c^2 \left(\frac{x_{i+1} - 2x_i + x_{i-1}}{h^2}\right)\big[1 + a(x_{i+1} - x_{i-1})\big] \tag{2.2.2}$$

Now, we shall try to find a good continuum limit for the nonlinear FPU lattice. Let us denote $u(x,t)$ the function which describes the displacement of the particle of string with equilibrium position x, at a given time t. Thus, if $x = p_i$, then $x_i(t) = u(x,t)$, $x_{i+1}(t) = u(x+h,t)$ and $x_{i-1}(t) = u(x-h,t)$ and finally $\ddot{x}_i(t) = u_{tt}(x,t)$. So, equation (2.2.2) takes the form

$$u_{tt}(x,t) = c^2 \left(\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}\right)\big[1 + a\big(u(x+h,t) - u(x-h,t)\big)\big] \tag{2.2.3}$$

Now, according to Taylor's theorem we can derive that

$$u(x+h,t) =$$

$$u(x,t) + hu_x(x,t) + \frac{h^2}{2!}u_{xx}(x,t) + \frac{h^3}{3!}u_{xxx}(x,t) + \frac{h^4}{4!}u_{xxxx}(x,t) + O(h^5)$$

and

$$u(x-h,t) =$$

$$u(x,t) - hu_x(x,t) + \frac{h^2}{2!}u_{xx}(x,t) - \frac{h^3}{3!}u_{xxx}(x,t) + \frac{h^4}{4!}u_{xxxx}(x,t) + O(h^5)$$

By adding the two previous equations we get that

$$u(x+h,t)+u(x-h,t)=2u(x,t)+h^4u_{xx}(x,t)+\frac{h^2}{12}u_{xxxx}(x,t)+O(h^6) \implies$$

$$u(x+h,t)+u(x-h,t)-2u(x,t)=h^2u_{xx}(x,t)+\frac{h^4}{12}u_{xxxx}(x,t)+O(h^6) \implies$$

$$\frac{u(x+h,t)+u(x-h,t)-2u(x,t)}{h^2}=u_{xx}(x,t)+\frac{h^2}{12}u_{xxxx}(x,t)+O(h^4)$$

Now, by subtracting the two previous equations we get that

$$u(x+h,t)-u(x-h,t)=2hu_x(x,t)+\frac{h^3}{3}u_{xxx}(x,t)+O(h^5) \implies$$

$$a\big(u(x+h,t)-u(x-h,t)\big)=2ahu_x(x,t)+\frac{ah^3}{3}u_{xxx}(x,t)+O(h^5)$$

So, after the replacements, equation (2.2.3) becomes

$$\frac{1}{c^2}\cdot u_{tt}=\left(u_{xx}+\frac{h^2}{12}u_{xxxx}+O(h^4)\right)\cdot\left(1+2ahu_x+\frac{ah^3}{3}u_{xxx}+O(h^5)\right)=$$

$$u_{xx}+\frac{h^2}{12}u_{xxxx}+O(h^4)+\left(u_{xx}+\frac{h^2}{12}u_{xxxx}+O(h^4)\right)\cdot\left(2ahu_x+\frac{ah^3}{3}u_{xxx}+O(h^5)\right)=$$

$$u_{xx}+\frac{h^2}{12}u_{xxxx}+2ahu_xu_{xx}+O(h^4) \implies$$

$$\frac{1}{c^2}\cdot u_{tt}-u_{xx}=\frac{h^2}{12}u_{xxxx}+2ahu_xu_{xx}+O(h^4)$$

Now, we can remove the term $O(h^4)$ as negligible and the above equation takes the form

$$\frac{1}{c^2}\cdot u_{tt}-u_{xx}=\frac{h^2}{12}u_{xxxx}+2ahu_xu_{xx} \tag{2.2.4}$$

Now, if we differentiate this equation with respect to x and make the substitution $v=u_x$, we see that the equation takes the more familiar form

$$\frac{1}{c^2}\cdot v_{tt}=v_{xx}+\frac{h^2}{12}v_{xxxx}+ah\frac{\partial(v^2)}{\partial x^2}$$

which is known as Boussinesq equation.

Since, $h \neq 0$ equation (2.2.4) cannot be considered a true continuum limit of the FPU lattice. It should rather be regarded as an asymptotic approximation to the lattice model that works for small lattice spacing h (and hence large N). If a and h are small enough, solutions of (2.2.4) should behave qualitatively like solutions of the linear wave equation $u_{tt} = c^2 u_{xx}$. The general solution of the linear wave equation is $u(x,t) = f(x+ct)+g(x-ct)$, i.e., the sum of an arbitrary left moving traveling wave and an arbitrary right moving traveling wave, both moving with speed c.

Now, suppose that $y(\xi, \tau)$ is a smooth function of two real variables such that the map

$$\tau \to y(\cdot, \tau)$$

is uniformly continuous from $\mathbb{R}$ into the bounded functions on $\mathbb{R}$ with the sup norm. That is, for given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|\tau - \tau_0| < \delta \implies |y(\xi, \tau) - y(\xi, \tau_0)| < \varepsilon$$

Then for $|\tau - \tau_0| < T = \dfrac{\delta}{ahc}$ we have

$$|ahct - ahct_0| < \delta$$

so

$$|y(x-ct, ahct) - y(x-ct, ahct_0)| < \varepsilon$$

In other words, the function $u(x,t) = y(x-ct, ahct)$ is uniformly approximated by the traveling wave $u^0(x,t) = y(x-ct, ahct_0)$ on the interval $|t-t_0| < T$ (and of course $T \to \infty$ as a and h go to 0). If $y(\xi, \tau)$ is periodic or almost periodic in τ, the gradually changing shape of the approximate traveling wave will also be periodic or almost periodic.

Now, we define the new variables $\xi = x - ct$ and $\tau = (ah)ct$. Then, by the chain rule we obtain

$$\frac{\partial^k}{\partial x^k} = \frac{\partial^k}{\partial x^k}$$

$$\frac{\partial}{\partial t} = -c\left(\frac{\partial}{\partial \xi} - ah\frac{\partial}{\partial \tau}\right)$$

$$\frac{\partial^2}{\partial t^2} = c^2\left(\frac{\partial^2}{\partial \xi^2} - 2ah\frac{\partial^2}{\partial \tau \partial \xi} + (ah)^2\frac{\partial^2}{\partial \tau^2}\right)$$

Therefore, in these new coordinates we have that

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = -2ah\frac{\partial^2}{\partial\xi\partial\tau} + (ah)^2\frac{\partial^2}{\partial\tau^2}$$

By doing the substitution $u(x,t) = y(\xi,\tau)$, equation (2.2.4) becomes

$$y_{\xi\tau} - \frac{ah}{2}y_{\tau\tau} = -y_\xi y_{\xi\xi} - \frac{h}{24a}y_{\xi\xi\xi\xi}$$

and now, we can pass to the continuum limit. We assume that a and h tend to zero at the same rate, i.e., that as h tends to zero, $\frac{h}{a}$ tends to a positive limit, and we define

$$\delta = \lim_{h\to 0}\sqrt{\frac{h}{24a}}$$

Then, $ah = O(h^2)$, so letting $h \to 0$ we get that $y_{\xi\tau} + y_\xi y_{\xi\xi} + \delta^2 y_{\xi\xi\xi\xi} = 0$. If we set $v = y_\xi$ then we have that

$$v_\tau + vv_\xi + \delta^2 v_{\xi\xi\xi} = 0$$

Now, to summarize the relationship between the FPU lattice and the KdV equation. Given a solution $x_i(t)$ of the FPU lattice we get a function $u(x,t)$ by interpolation, that is, $u(ih,t) = x_i(t), i = 0, ..., N$. For small lattice spacing h and nonlinearity parameter a there will be solutions $x_i(t)$ so that the corresponding $u(x,t)$ will be an approximate right moving traveling wave with slowly varying shape, i.e., it will be of the form $u(x,t) = y(x - ct, ahct)$ for some smooth function $y(\xi,\tau)$, and the function $v(\xi,\tau) = y_\xi(\xi,\tau)$ will satisfy the KdV equation $v_\tau + vv_\xi + \delta^2 v_{\xi\xi\xi} = 0$, where $\delta = \frac{h}{24a}$. Having found this relationship between the FPU lattice and the KdV equation, Kruskal and Zabusky made some numerical experiments, solving the KdV initial value problem for various initial data.[35]

3 Complexification Averaging Method

3.1 The Method of Multiple Scales

The method of multiple scales comprises techniques used to construct uniformly valid approximations to the solutions of perturbation problems in which the solutions depend simultaneously on widely different scales. This is done by introducing fast-scale and slow-scale variables for an independent variable, and subsequently treating these variables, fast and slow, as if they are independent. Let, us consider the linear damped mass-spring system with no external forces. The equation for the displacement $y(\tau)$ is

$$m\ddot{y} + c\dot{y} + ky = 0 \tag{3.1.1}$$

If initially the mass is released from a positive displacement y_i with no initial velocity, we have the following initial conditions

$$y(0) = y_i, \quad \dot{y}(0) = 0 \tag{3.1.2}$$

We assume that $c \ll m, k$. Choosing y_i and $\sqrt{\dfrac{m}{k}}$ as the characteristic distance and characteristic time respectively, we define the following dimensionless variables

$$x = \frac{y}{y_i}, \quad t = \frac{\tau}{\sqrt{\frac{m}{k}}} \tag{3.1.3}$$

By substituting equations (3.1.3) in the equation (3.1.1) and (3.1.2) we get that

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0 \tag{3.1.4}$$

$$x(0) = 1, \quad \dot{x}(0) = 0 \tag{3.1.5}$$

where

$$\varepsilon = \frac{c}{2\sqrt{mk}} \ll 1$$

is a dimensionless parameter. This equation corresponds to a linear oscillator with weak damping, where the time variable has been scaled by the period of the undamped system. The analytical solution of (3.1.4)-(3.1.5) is the following

$$x(t) = \mathrm{e}^{-\varepsilon t}\left(\cos(\sqrt{1-\varepsilon^2}\cdot t) + \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\sin(\sqrt{1-\varepsilon^2}\cdot t)\right) \tag{3.1.6}$$

Observe, that if the oscillation is undamped, i.e. if $\varepsilon = 0$, then the exact solution is

$$x(t) = \cos t$$

where both amplitude and phase of the oscillation remain constant. However, with the presence of damping, (3.1.6) shows that both amplitude and phase change with time. In fact, the amplitude changes on the time scale ε^{-1}, while the phase changes on the longer time scale ε^{-2}.

If we look at the relation (3.1.6) we see that

$$x(t) = \cos t + O(\varepsilon), \quad \text{for } t = O(1) \tag{3.1.7}$$

is uniformly true, but is not uniformly valid for $t = O\left(\frac{1}{\varepsilon}\right)$. If we are interested in times which are $O\left(\frac{1}{\varepsilon}\right)$ then the combination εt must be preserved in the exponential function. Then it is uniformly valid to state that

$$x(t) = \mathrm{e}^{-\varepsilon t} \cos t + O(\varepsilon), \quad \text{for } t = O\left(\frac{1}{\varepsilon}\right) \tag{3.1.8}$$

If we are interested in values of t which are $O(\frac{1}{\varepsilon^2})$ then (3.1.8) is no longer valid. In this case terms of the form $\varepsilon^2 t$ must be preserved in the cosine function appearing in (3.1.6). Using binomial expansion, we have

$$\sqrt{1-\varepsilon^2} = 1 - \frac{\varepsilon^2}{2} - \frac{\varepsilon^4}{8} - \frac{\varepsilon^6}{16} - ...$$

So,

$$x(t) = \mathrm{e}^{-\varepsilon t} \cos\left(1 - \frac{\varepsilon^2}{2}\right) t + O(\varepsilon), \quad \text{for } t = O\left(\frac{1}{\varepsilon^2}\right) \tag{3.1.9}$$

That is (3.1.9) is uniformly valid for $t = O(\frac{1}{\varepsilon^2})$.

Now, let us see how multiple scales method is applied. Any asymptotic expansion of (3.1.6) must simultaneously depict both the decaying and oscillatory behaviors of the solution in order to be uniformly valid in $t = O(\frac{1}{\varepsilon^k})$. The method of multiple scales is a more general approach that involve two key tricks. The first is the idea of introducing scaled space and time coordinates to capture the slow modulation of the pattern, and treating these as separate variables in addition to the original variables that must be retained to describe the pattern state itself. This is essentially the idea of multiple scales. The second is the use of what are known as solvability conditions in the formal derivation. We note from analytical solution (3.1.6) that the functional dependence of x on t and ε is not disjoint because x depends on the combination of εt as well as on the individual t and ε. Hence, in place of $x = x(t;\varepsilon)$, we write $x = \hat{x}(t, \varepsilon t; \varepsilon)$. The oscillator has three processes acting on their own time scales. Fist, there is the basic oscillation on the time scale of 1 from the inertia causing the restoring force to overshoot the equilibrium position. Then there is a small drift in the amplitude on the time scale of ε^{-1} and finally a very small drift in the phase on the time scale of ε^{-2} due to the small friction. We introduce three time variables.

$$\begin{aligned}
&T_0 = t, \ \text{ the fast time of the oscillation} \\
&T_1 = \varepsilon t, \ \text{ the slow time of the amplitude change} \\
&T_2 = \varepsilon^2 t, \ \text{ the slower time of the phase change}
\end{aligned}$$

So, now we look for a solution in the form of

$$x(t;\varepsilon) = x(T_0, T_1, T_2; \varepsilon)$$

Generally, if we choose n time scales for the expansion we look for a solution of the form

$$x(t;\varepsilon) = x(T_0, T_1, T_2, ..., T_n; \varepsilon)$$

where

$$\begin{aligned} T_0 &= t \\ T_1 &= \varepsilon t \\ &\vdots \\ T_n &= \varepsilon^n t \end{aligned}$$

Therefore, instead of determining x as a function of t, we determine x as a function of $T_0, T_1, ..., T_n$. Note that as real time t increases, the fast time T_0 increases at the same rate, while the slower time $T_j,\ 1 \le j \le n$, increases slowly. Using the chain rule we have

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial}{\partial T_2}\frac{\partial T_2}{\partial t} + ... = \\ &= \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2} + ... \end{aligned} \tag{3.1.10}$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon\frac{\partial^2}{\partial T_0 \partial T_1} + \varepsilon^2\left(\frac{\partial^2}{\partial T_0 \partial T_2} + \frac{\partial^2}{\partial T_1^2}\right) + ... \tag{3.1.11}$$

So, (3.1.4)-(3.1.5) become

$$\frac{\partial^2 x}{\partial T_0^2} + 2\varepsilon\frac{\partial^2 x}{\partial T_0 \partial T_1} + \varepsilon^2\left(\frac{\partial^2 x}{\partial T_0 \partial T_2} + \frac{\partial^2 x}{\partial T_1^2}\right) + 2\varepsilon\left(\frac{\partial x}{\partial T_0} + \varepsilon\frac{\partial x}{\partial T_1} + \varepsilon^2\frac{\partial x}{\partial T_2}\right) + x + ... = 0 \tag{3.1.12}$$

$$x(\mathbf{0}) = 1,\quad \frac{\partial x(\mathbf{0})}{\partial T_0} + \varepsilon\frac{\partial x(\mathbf{0})}{\partial T_1} + \varepsilon^2\frac{\partial x(\mathbf{0})}{\partial T_2} + ... = 0,\quad \text{for}\quad T_0 = T_1 = ... = T_n = 0 \tag{3.1.13}$$

We note that when $t = 0$, all $T_0, T_1, ..., T_n$ are zero. We now search an asymptotic approximation for x of the form

$$x(t) \equiv x(T_0, T_1, ..., T_n; \varepsilon) \sim x_0(T_0, T_1, ..., T_n) + \varepsilon x_1(T_0, T_1, ..., T_n) + \varepsilon^2 x_2(T_0, T_1, ..., T_n) + ... \tag{3.1.14}$$

Here, we must realize that there are only two independent variables, t and ε in relation (3.1.14) - $T_j,\ 0 \le j \le n$ are functions of these two and so they are not independent. The basic steps are to find coefficients x_n as though $T_0, ..., T_n$ and ε were independent variables. Now, we will assume that there are only two time scales involved in our problem. The scales are defined as

$$T_0 = t \quad \text{and } T_1 = \varepsilon t$$

So, instead of determining x as a function of t, we determine x as a function of T_0 and T_1. Therefore, the differential equation and initial conditions given in (3.1.12)-(3.1.13) become

$$\frac{\partial^2 x}{\partial T_0^2} + 2\varepsilon \frac{\partial^2 x}{\partial T_0 \partial T_1} + \varepsilon^2 \frac{\partial^2 x}{\partial T_1^2} + 2\varepsilon \left(\frac{\partial x}{\partial T_0} + \varepsilon \frac{\partial x}{\partial T_1} \right) + x + ... = 0 \tag{3.1.15}$$

$$x(\mathbf{0}) = 1, \quad \frac{\partial x(\mathbf{0})}{\partial T_0} + \varepsilon \frac{\partial x(\mathbf{0})}{\partial T_1} = 0, \quad \text{for} \quad T_0 = T_1 = 0 \tag{3.1.16}$$

We seek an asymptotic approximation for x of the form

$$x(t) \equiv x(T_0, T_1; \varepsilon) \sim x_0(T_0, T_1) + \varepsilon x_1(T_0, T_1) \tag{3.1.17}$$

Substituting (3.1.17) into (3.1.15) we derive

$$\frac{\partial^2 x_0}{\partial T_0^2} + \varepsilon \frac{\partial^2 x_1}{\partial T_0^2} + 2\varepsilon \frac{\partial^2 x_0}{\partial T_0 \partial T_1} + 2\varepsilon \frac{\partial x_0}{\partial T_0} + x_0 + \varepsilon x_1 + ... = 0 \implies$$

$$\frac{\partial^2 x_0}{\partial T_0^2} + x_0 + \varepsilon \left(\frac{\partial^2 x_1}{\partial T_0^2} + 2 \frac{\partial^2 x_0}{\partial T_0 \partial T_1} + 2 \frac{\partial x_0}{\partial T_0} + x_1 \right) = 0$$

Equating coefficients of like powers of ε to 0, gives the following sequence of linear partial differential equations

$$O(1): \ \frac{\partial^2 x_0}{\partial T_0^2} + x_0 = 0 \tag{3.1.18}$$

$$O(\varepsilon): \ \frac{\partial^2 x_1}{\partial T_0^2} + x_1 = -2 \frac{\partial^2 x_0}{\partial T_0 \partial T_1} - 2 \frac{\partial x_0}{\partial T_0} \tag{3.1.19}$$

The respective initial conditions for (3.1.18) and (3.1.19) are given by

$$x_0 = 1, \quad \frac{\partial x_0}{\partial T_0} = 0, \quad \text{for} \quad T_0 = T_1 = 0 \tag{3.1.20}$$

$$x_1 = 0, \quad \frac{\partial x_1}{\partial T_0} = -\frac{\partial x_0}{\partial T_1}, \quad \text{for} \quad T_0 = T_1 = 0 \tag{3.1.21}$$

Since T_0 and T_1 are being treated as independent (for now), the differential equation (3.1.18) is actually a partial differential equation for a function x_0 of two variables T_0 and T_1. However, since no derivatives with respect to T_1 appear in (3.1.18), it may be regarded instead as an ordinary differential equation for a function of T_0 regarding T_1 as merely an auxiliary parameter. Therefore the general solution of (3.1.18) may be obtained from the general solution of the corresponding ordinary differential equation just by letting the arbitrary constants become arbitrary functions of T_1. Thus the general solution of (3.1.18) can be given in the form

$$x_0 = A_0(T_1)\cos T_0 + B_0(T_1)\sin T_0 \tag{3.1.22}$$

in which A_0 and B_0 are constant as far as the fast T_0 variations are concerned, but are allowed to vary over the slow T_1 time. The initial conditions are

$$A_0(0) = 1 \quad \text{and} \quad B_0(0) = 0 \tag{3.1.23}$$

Now, in order to compute the functions A_0 and B_0, we must consider the next order of approximation, i.e. $O(\varepsilon)$. From (3.1.22) we have that

$$\frac{\partial x_0}{\partial T_0} = -A_0(T_1)\sin T_0 + B_0(T_1)\cos T_0$$

$$\frac{\partial^2}{\partial T_1 \partial T_0} = \frac{\partial}{\partial T_1}\left(\frac{\partial x_0}{\partial T_0}\right) = -\sin T_0 \frac{\partial A_0}{\partial T_1} + \cos T_0 \frac{\partial B_0}{\partial T_1}$$

By combining, equations (3.1.21) and (3.1.22) we get

$$\frac{\partial^2 x_1}{\partial T_0^2} + x_1 = 2\left(\frac{\partial A_0}{\partial T_1} + A_0\right)\sin T_0 - 2\left(\frac{\partial B_0}{\partial T_1} + B_0\right)\cos T_0 \tag{3.1.24}$$

Since both the right-hand side of (3.1.24) and the complementary function of this equation contain terms proportional to $\sin T_0$ and $\cos T_0$, the particular solution of x_1 will have secular terms in it. Thus, to obtain a uniform expansion, each of the coefficients of $\sin T_0$ and $\cos T_0$ must independently vanish. The vanishing of these coefficients yields the condition for the determination of A_0 and B_0. So

$$\frac{\partial A_0}{\partial T_1} + A_0 = 0 \implies A_0 = a_0 \mathrm{e}^{-T_1} \tag{3.1.25}$$

$$\frac{\partial B_0}{\partial T_1} + B_0 = 0 \implies B_0 = b_0 \mathrm{e}^{-T_1} \tag{3.1.26}$$

where a_0 and b_0 are constants of integration. We substitute (3.1.25) and (3.1.26) in (3.1.22) and we obtain

$$x_0 = a_0 \mathrm{e}^{-T_1}\cos T_0 + b_0 \mathrm{e}^{-T_1}\sin T_0 \tag{3.1.27}$$

Now, using the initial conditions (3.1.21) we conclude that

$$a_0 = 1 \quad \text{and} \quad b_0 = 0$$

therefore, the solution is

$$x_0 = \mathrm{e}^{-T_1} \cos T_0$$

We ensure that secular terms are avoided so that we may write

$$x_0 = \mathrm{e}^{-T_1} \cos T_0 + O(\varepsilon)$$

In terms of the original variables, x becomes

$$x = \mathrm{e}^{-\varepsilon t} \cos t + O(\varepsilon)$$

which is uniformly valid for $t = O(\frac{1}{\varepsilon})$ and in agreement with exact solution (3.1.6) to $O(\varepsilon)$.

3.2 Complex Equations of Motion and Solution by the Multiple Scales Method

The main purpose of this chapter is the introduction of the asymptotic approach based on the complex representation of equations of motion. Let us consider a nonlinear single-degree-of-freedom system described by the equation

$$\ddot{x}(t) + x = \varepsilon f(x, \dot{x}) \tag{3.2.1}$$

where, ε is as small parameter, all the derivatives are with respect to time t and f is a piecewise-differentiable function. Let us write the equation of motion (3.2.1) as a system of two first order equations

$$\begin{aligned} \dot{y} + x &= \varepsilon f(x, y) \\ \dot{x} &= y \end{aligned} \tag{3.2.2}$$

Now, we introduce the complex variables

$$\psi = y + \mathrm{i}x$$
$$\overline{\psi} = y - \mathrm{i}x$$

where $\overline{\psi}$ is the complex conjugate of the complex number ψ. By utilizing the two previous equations one can derive

$$\begin{aligned} y &= \frac{1}{2}(\psi + \overline{\psi}) \\ x &= \frac{1}{2\mathrm{i}}(\psi - \overline{\psi}) \end{aligned} \tag{3.2.3}$$

Multiplying the second equation (3.2.2) by imaginary unit i and adding it to the first one of (3.2.2) we obtain

$$\frac{d}{dt}\psi - \mathrm{i}\psi = \varepsilon f\left[\frac{1}{2\mathrm{i}}(\psi - \overline{\psi}), \frac{1}{2}(\psi + \overline{\psi})\right] \tag{3.2.4}$$

Thus instead of the second order equation (3.2.1) we deal with the first order complex equation (3.2.4). The solution of this equation will be obtained by the multiple scale method as we presented in the previous paragraph. Let us introduce times of various scales $\tau_n = \varepsilon^n t,\ n = 0, 1, ...$ and consider the required complex function as a function of variables $\tau_0 (= t), \tau_1, ...$ Using the differentiation rule for compound functions one can obtain the operator of differentiation by t in the form of expansion

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + ... \tag{3.2.5}$$

where $D_k = \dfrac{\partial}{\partial \tau^k}$.

Then, the function $\psi(t)$ is expanded in the asymptotic series by the small parameter ε as it follows

$$\psi(t) = \psi_0(t) + \varepsilon\psi_1(t) + \varepsilon^2\psi_2(t) + ... \tag{3.2.6}$$

In order the functions $\psi_k(t)$ to be uniquely defined, some additional conditions should be imposed on them, similarly to those for the averaging method in real variables. These are the orthogonality conditions

$$\int_0^{2\pi} \psi_0(t) \cdot \overline{\psi}_1(t)\ dt = 0$$

$$\int_0^{2\pi} \psi_0(t) \cdot \overline{\psi}_2(t)\ dt = 0$$

$$\int_0^{2\pi} \psi_1(t) \cdot \overline{\psi}_2(t)\ dt = 0$$

Furthermore, initial conditions are applied on the function $\psi_0(t)$. All the other functions $\psi_k(t),\ k = 1, 2, 3, ...$ satisfy the initial conditions $\psi_k(0) = 0,\ k = 1, 2, ...$ Now, if we substitute relations (3.2.5) and (3.2.6) on the equation (3.2.4) we derive

$$\begin{gathered} D_0(\psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + ...) + \varepsilon D_1(\psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + ...) + \\ \varepsilon^2 D_2(\psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + ...) + \mathrm{i}(\psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + ...) = \\ \varepsilon F(\psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + ..., \overline{\psi}_0 + \varepsilon\overline{\psi}_0 + \varepsilon^2\overline{\psi}_0 + ...) \end{gathered}$$

where

$$F(\psi, \overline{\psi}) \equiv f\left[\tfrac{1}{2\mathrm{i}}(\psi - \overline{\psi}), \tfrac{1}{2}(\psi + \overline{\psi})\right]$$

Then we expand function F in series in parameter ε

$$\begin{gathered} F(\psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + ..., \overline{\psi}_0 + \varepsilon\overline{\psi}_0 + \varepsilon^2\overline{\psi}_0 + ...) = \\ F(\psi_0, \overline{\psi}_0) + \varepsilon\big[F_\psi(\psi_0, \overline{\psi}_0)\psi_1 + F_{\overline{\psi}}(\psi_0, \overline{\psi}_0)\overline{\psi}_1\big] + ... \end{gathered}$$

where F_ψ denotes the derivative of F with respect to ψ. Equating coefficients at increasing powers of ε to zero, one can obtain the following relations

$$D_0\psi_0 - \mathrm{i}\psi_0 = 0 \tag{3.2.7}$$

$$D_0\psi_1 - \mathrm{i}\psi_1 = -D_1\psi_0 + F(\psi_0, \overline{\psi}_0) \tag{3.2.8}$$

$$D_0\psi_2 - \mathrm{i}\psi_2 = -D_1\psi_1 - D_2\psi_0 + F_\psi(\psi_0, \overline{\psi}_0)\psi_1 + F_{\overline{\psi}}(\psi_0, \overline{\psi}_0)\overline{\psi}_1 \tag{3.2.9}$$

Now, it follows from equation (3.2.7) that

$$\psi_0 = A\mathrm{e}^{\mathrm{i}t} \tag{3.2.10}$$

where A depends on the slow time i.e. $A = A(\tau_1, \tau_2, ...)$. Now, the obtained expansion (3.2.6) is a uniformly suitable first order solution, if we retain only the first term in the form (3.2.10) with A being determined from a procedure which is not being analyzed in this thesis. By working with this method we can have second, third,... uniformly suitable solutions in case we add more terms. This method is full analytically described in [41]. The steps described above will be used in a similar way in the next chapter.

4 Energy Transfer in Nonlinear Lattices

4.1 Irreversible Energy Transfer and Localization in the Lattice Network

A semi-infinite network of two coupled semi-infinite nonlinear lattices is studied. In particular, we are going to study passive irreversible energy transfer (redirection) from an "excited lattice" (forced by an impulse) to an "absorbing lattice" (that is not directly forced). As we presented at the previous chapter, in certain one-dimensional nonlinear lattices – e.g, Klein–Gordon nonlinear lattices, we can observe the existence of traveling discrete breathers. What we shall study in this chapter is how energy transfers through these breathers in a specific lattice network. To be more precise, we study if it is possible to achieve irreversible energy transfer in such coupled lattices. First, we make an approach at the symmetric lattice network whose scheme is presented below. [39]

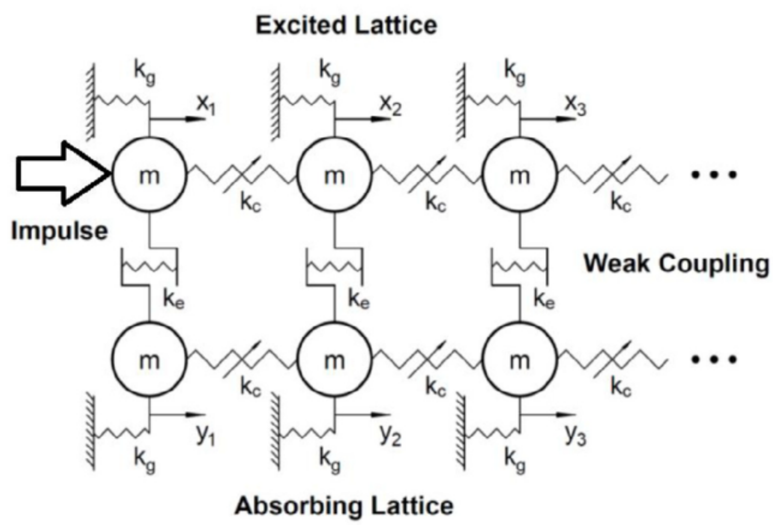

Figure 4.1.1: Scheme of the symmetric lattice network. [39]

In Figure 4.1.1 we present the semi-infinite lattice network, composed of two identical 1D lattices of linearly grounded, undamped oscillators coupled to their next neighbors through essentially nonlinear cubic stiffness. The two lattices are weakly coupled through linear stiffness connecting the corresponding oscillators of each lattice. An impulsive excitation is applied at $t_0 = 0$ to the leading oscillator of one of the lattices, i.e. the "excited lattice", while the other, "absorbing lattice", is not directly forced. This is equivalent to an initial velocity of the excited oscillator at $t_0 = 0$ with all other oscillations assumed at rest. We are interested only in primary wave transmission, and not be concerned by reflections at the boundaries. The equations of motion of the symmetric lattice network are the following

$$
\begin{aligned}
m\ddot{x}_1 &= -k_g x_1 - k_c(x_1 - x_2)^3 - k_e(x_1 - y_1) \\
m\ddot{y}_1 &= -k_g y_1 - k_c(y_1 - y_2)^3 - k_e(y_1 - x_1) \\
m\ddot{x}_2 &= -k_g x_2 - k_c\big[(x_2 - x_1)^3 + (x_2 - x_3)^3\big] - k_e(x_2 - y_2) \\
m\ddot{y}_2 &= -k_g y_2 - k_c\big[(y_2 - y_1)^3 + (y_2 - y_3)^3\big] - k_e(y_2 - x_2) \\
&\vdots \\
m\ddot{x}_n &= -k_g x_n - k_c\big[(x_n - x_{n-1})^3 + (x_n - x_{n+1})^3\big] - k_e(x_n - y_n),\ n \geq 2 \\
m\ddot{y}_n &= -k_g y_n - k_c\big[(y_n - y_{n-1})^3 + (y_n - y_{n+1})^3\big] - k_e(y_n - x_n),\ n \geq 2
\end{aligned}
\tag{4.1.1}
$$

In the above equations, x_n and y_n are the displacements of the nth oscillators of the excited and absorbing lattices, respectively, m and k_g their mass and grounding stiffness, k_c the coefficient of the cubic coupling stiffness and $k_e \ll k_g A^2$ the coefficient of the weak linear coupling stiffness, with A being a characteristic displacement.

Now, the initial conditions of the system above - let us denote it (S1) - are $\dot{x}_1(0) = v_0$ and zero otherwise. Each lattice is composed of 50 oscillators and the system parameters are listed below with $v_0 = 0.06\frac{m}{s}$.

m(kg)	k$_g(\frac{N}{m})$	k$_c(\frac{N}{m^3})$	k$_e(\frac{N}{m})$
0.022	1467.27	2.48E9	92.21

In the next figure, we show the spatio-temporal evolution of the normalized (with respect to the impulsive energy) energies of the two lattices. As we see, at time $t_0 = 0$ the total energy of the system i.e. the impulsive energy, is at the oscillator number one. After some time, the energy is transferred to the oscillator number two of the absorbing lattice. And as the time goes on, we observe a recurrent energy exchange between the two lattices in the form of propagating breathers. This can be understood by the contour lines of the following figure. So there is this periodic state of how the oscillators are moving. Each breather is an oscillatory wavepacket with a “fast frequency” that is modulated by a “slow-varying” localized envelope. The recurrent energy exchanges are due to the symmetry and weak coupling of the lattice network.

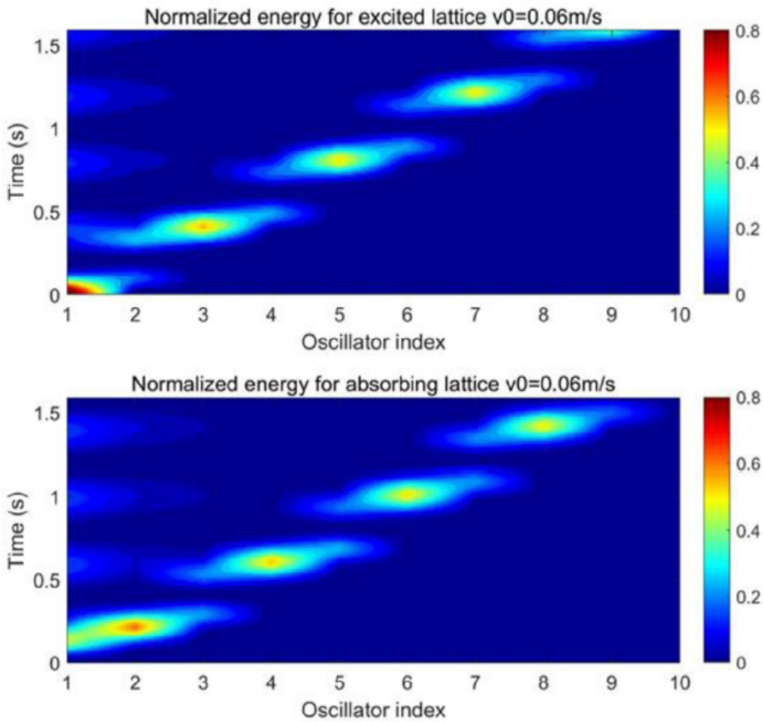

Figure 4.1.2: Spatio-temporal evolution of the normalized energy of the lattices of the symmetric network showing recurrent energy exchanges between lattices ($v_0 = 0.06\frac{m}{s}$). [39]

Now, we follow a different path. We will break the symmetry of the lattice network while retaining weak coupling. And as we will see, the results vary from the previous situation. The asymmetric lattice network (ALN) is identical to the network of Figure 4.1.1 but for a symmetry-breaking spatial variation of the linear grounding stiffness of the oscillators of the excited lattice, i.e., except for the leading oscillator all the other oscillators of that lattice have uniform softer grounding stiffness.

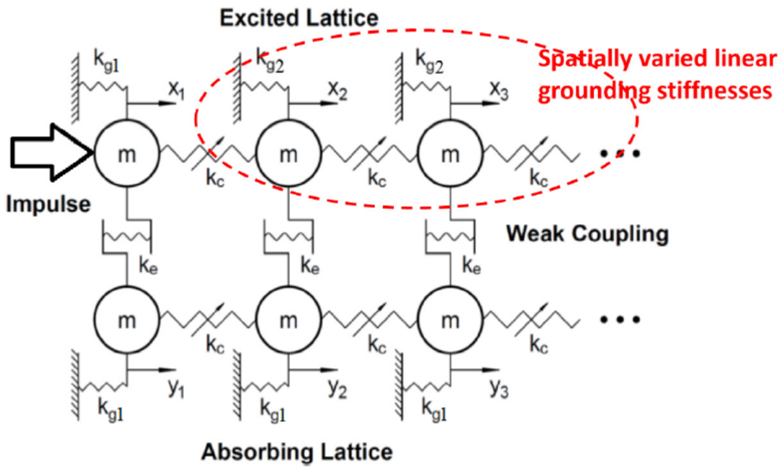

Figure 4.1.3: Scheme of the asymmetric lattice network. [39]

The equations of motion of (ALN) are

$$
\begin{aligned}
m\ddot{x}_1 &= -k_{g_1}x_1 - k_c(x_1-x_2)^3 - k_e(x_1-y_1)\\
m\ddot{y}_1 &= -k_{g_1}y_1 - k_c(y_1-y_2)^3 - k_e(y_1-x_1)\\
m\ddot{x}_2 &= -k_{g_2}x_2 - k_c\big[(x_2-x_1)^3+(x_2-x_3)^3\big] - k_e(x_2-y_2)\\
m\ddot{y}_2 &= -k_{g_1}y_2 - k_c\big[(y_2-y_1)^3+(y_2-y_3)^3\big] - k_e(y_2-x_2)\\
&\vdots\\
m\ddot{x}_n &= -k_{g_2}x_n - k_c\big[(x_n-x_{n-1})^3+(x_n-x_{n+1})^3\big] - k_e(x_n-y_n),\ n\geq 2\\
m\ddot{y}_n &= -k_{g_1}y_n - k_c\big[(y_n-y_{n-1})^3+(y_n-y_{n+1})^3\big] - k_e(y_n-x_n),\ n\geq 2
\end{aligned}
\tag{4.1.2}
$$

The parameters of the (ALN) are given in the table below. They are the same as the previous ones, except the new parameter k_{g_2}, and the initial velocity is again $v_0 = 0.06\frac{m}{s}$.

m(kg)	k$_{g_1}(\frac{N}{m})$	k$_{g_2}(\frac{N}{m})$	k$_c(\frac{N}{m^3})$	k$_e(\frac{N}{m})$
0.022	1467.27	687.53	2.48E9	92.21

As we depict in the next graph, almost immediately, the (total) energy (i.e. the energy due to the initial velocity) of the oscillator number one of the excited lattice is transferred to the oscillator number one of the absorbing lattice. And what we observe after that, is that the energy continues to being transferred only in the absorbing lattice in the form of travelling, discrete breathers. So, by breaking the symmetry of the lattice network, we see that no energy remains on the excited lattice after some time t_1 where t_1 is exceptionally close to zero. On the other hand, in the absorbing lattice we notice that the propagating breathers progress periodically and now all the oscillators of the absorbing lattice will move as time goes on (while in the symmetric lattice only oscillators two, four, six,... were having energy).

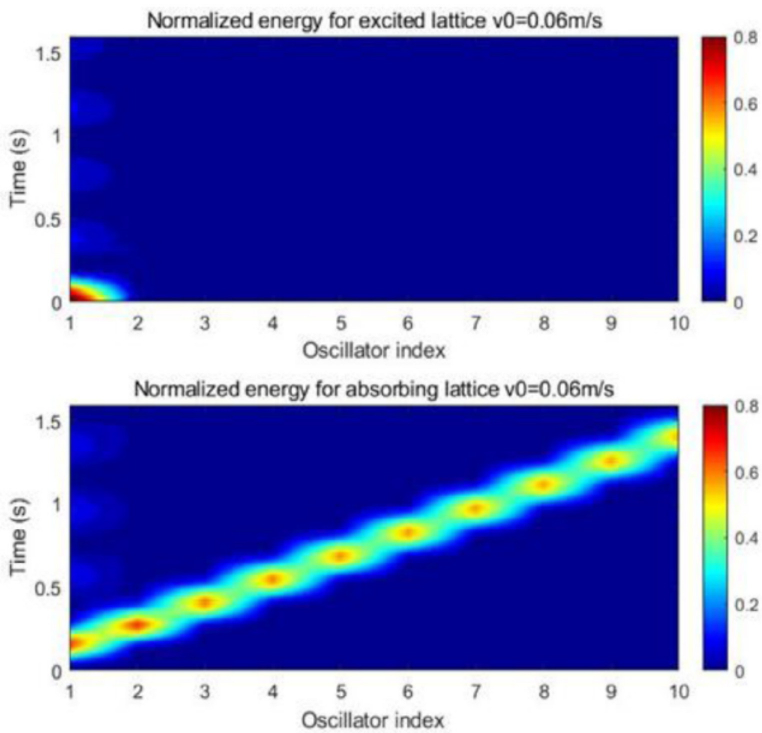

Figure 4.1.4: Spatio-temporal evolution of the energy of the lattices of the asymmetric network showing irreversible energy transfer between lattices ($v_0 = 0.06\frac{m}{s}$). [39]

Furthermore, since the (ALN) is a strongly nonlinear system, its acoustics depend strongly on the input energy. This is presented in the next graph where the (ALN) is studied for various initial velocities. We depict the contour plots of the maximum instantaneous normalized energy (with respect to input energy) over a certain time period for each of the leading 10 oscillators at a given velocity v_0. The nonlinear acoustics are categorized into four Regions, labeled as I–IV.

Region I corresponds to low intensity impulses with the energy being localized in the leading oscillators of the excited and absorbing lattices. For intermediate impulses Region II is realized irreversible energy redirection from the excited to the absorbing lattice, and breather propagation in the absorbing lattice (as we saw on the previous pages for the initial velocity $v_0 = 0.06\frac{m}{s}$). By increasing further the intensity of the impulse Region III is realized with energy localized in the leading oscillator of the excited lattice. For very high intensity impulses as we see in the Region IV, there is still no energy transfer from the excited to the absorbing lattice, but there exist propagating breathers in the excited lattice.

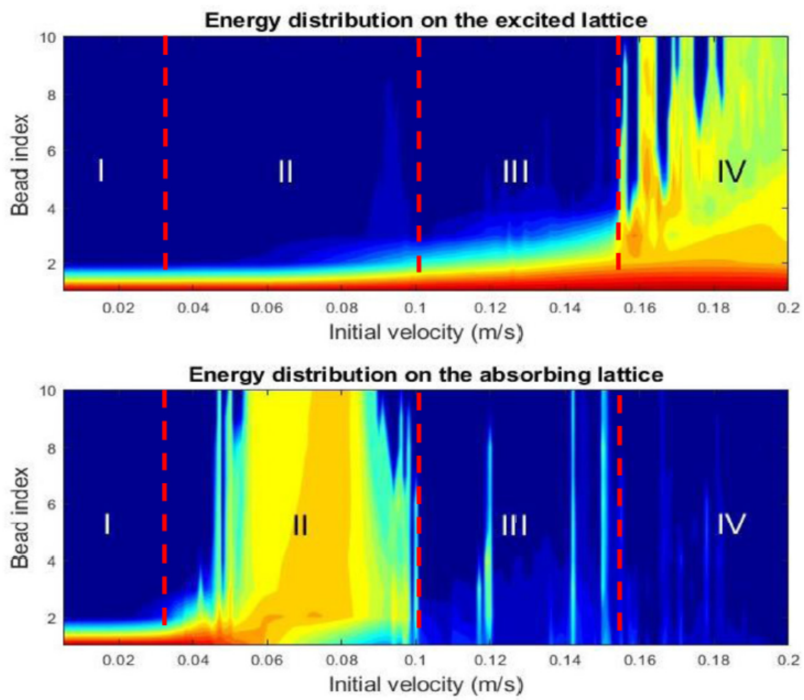

Figure 4.1.5: Normalized energy in the excited and absorbing lattices of the (ALN) for different impulse intensity. [39]

4.2 Irreversible Energy Transfer in the ALN in Region II

From the ALN in Figure (4.1.3) we denote γ as the the reduction ratio of the grounding stiffness

$$0 \leq \gamma \equiv \frac{k_{g_1} - k_{g_2}}{k_{g_1}} < 1$$

When $\gamma = 0$ energy is reversibly exchanged between the two lattices in the form of propagating breathers. However, when $\gamma > 0$, irreversible energy transfer from the excited to the absorbing lattice is possible. Now, let us consider the new normalized time τ,

$$\tau = \left(\frac{k_{g_1} + k_e}{m}\right)^{\frac{1}{2}} t$$

In order to have all the derivatives with respect to time τ we will have to apply the chain rule. So, for the oscillator x_1 of the (ALN)

$$\frac{d}{dt}x_1 = \frac{dx_1}{d\tau} \cdot \frac{d\tau}{dt} = \left(\frac{k_{g_1} + k_e}{m}\right)^{\frac{1}{2}} \dot{x}_1(\tau)$$

Also,

$$\frac{d^2}{dt^2}x_1 = \frac{d}{dt}\left[\left(\frac{k_{g_1} + k_e}{m}\right)^{\frac{1}{2}} \dot{x}_1(\tau)\right] = \left(\frac{k_{g_1} + k_e}{m}\right)^{\frac{1}{2}} \dot{x}_1(\tau) \cdot \frac{d\tau}{dt} =$$

$$\left(\frac{k_{g_1} + k_e}{m}\right)^{\frac{1}{2}} \ddot{x}_1(\tau) \cdot \left(\frac{k_{g_1} + k_e}{m}\right)^{\frac{1}{2}} \implies$$

$$\ddot{x}_1(t) = \frac{k_{g_1} + k_e}{m} \cdot \ddot{x}_1(\tau)$$

So, the equation of x_1 of the system (4.1.2) with respect to the normalized time τ, becomes

$$m \cdot \frac{k_{g_1} + k_e}{m} \cdot \ddot{x}_1(\tau) = -k_{g_1}x_1(\tau) - k_c\big(x_1(\tau) - x_2(\tau)\big)^3 - k_e\big(x_1(\tau) - y_1(\tau)\big) \implies$$

$$(k_{g_1} + k_e)\ddot{x}_1 + k_{g_1}x_1 + k_e x_1 = -k_c(x_1 - x_2)^3 + k_e y_1 \implies$$

$$(k_{g_1} + k_e)\ddot{x}_1 + (k_{g_1} + k_e)x_1 = -k_c(x_1 - x_2)^3 + k_e y_1 \implies$$

$$\ddot{x}_1 + x_1 = \frac{-k_c}{k_{g_1} + k_e}(x_1 - x_2)^3 + \frac{k_e}{k_{g_1} + k_e}y_1 \implies$$

$$\ddot{x}_1 + x_1 = -\varepsilon a(x_1 - x_2)^3 + 2\varepsilon\beta y_1$$

where $0 < \varepsilon \ll 1$ is a small scaling parameter denoting the smallness of the aforementioned parameters, $\varepsilon a \equiv \dfrac{k_c}{k_{g_1}}$ is the normalized cubic stiffness coefficient and $\varepsilon\beta \equiv \dfrac{k_e}{2(k_{g_1} + k_e)}$ is the normalized coupling stiffness coefficient.

Following the same reasoning, the equation of y_1 of the system (4.1.2) with respect to the normalized time τ, becomes

$$\ddot{y}_1 + y_1 = -\varepsilon a(y_1 - y_2)^3 + 2\varepsilon\beta x_1$$

Now, we are going to see how equations of x_2 and y_2 are transformed. We have

$$m \cdot \frac{k_{g_1} + k_e}{m} \cdot \ddot{x}_2 = -k_{g_2}x_2 - k_c\big[(x_2 - x_1)^3 + (x_2 - x_3)^3\big] - k_e(x_2 - y_2) \implies$$

$$(k_{g_1} + k_e)\ddot{x}_2 + k_{g_2}x_2 + k_e x_2 = -k_c\big[(x_2 - x_1)^3 + (x_2 - x_3)^3\big] + k_e y_2 \implies$$

$$(k_{g_1} + k_e)\ddot{x}_2 + (k_{g_2} + k_e)x_2 = -k_c\big[(x_2 - x_1)^3 + (x_2 - x_3)^3\big] + k_e y_2 \implies$$

$$\ddot{x}_2 + \frac{k_{g_2} + k_e}{k_{g_1} + k_e}x_2 = \frac{-k_c}{k_{g_1} + k_e}\big[(x_2 - x_1)^3 + (x_2 - x_3)^3\big] + \frac{k_e}{k_{g_1} + k_e}y_2 \implies$$

$$\ddot{x}_2 + (1 - \varepsilon\gamma)x_2 = -\varepsilon a\big[(x_2 - x_1)^3 + (x_2 - x_3)^3\big] + 2\varepsilon\beta y_2$$

where

$$\frac{k_{g_2} + k_e}{k_{g_1} + k_e} = \frac{k_{g_1} + k_e - k_{g_1} + k_{g_2}}{k_{g_1} + k_e} = 1 - \frac{k_{g_1} - k_{g_2}}{k_{g_1} + k_e} = 1 - \varepsilon\gamma$$

Similarly, we can derive

$$(k_{g_1} + k_e)\ddot{y}_2 + k_{g_1}y_2 + k_e y_2 = -k_c\big[(y_2 - y_1)^3 + (y_2 - y_3)^3\big] + k_e x_2 \implies$$

$$(k_{g_1} + k_e)\ddot{y}_2 + (k_{g_1} + k_e)y_2 = -k_c\big[(y_2 - y_1)^3 + (y_2 - y_3)^3\big] + k_e x_2 \implies$$

$$\ddot{y}_2 + y_2 = \frac{-k_c}{k_{g_1} + k_e}\big[(y_2 - y_1)^3 + (y_2 - y_3)^3\big] + \frac{k_e}{k_{g_1} + k_e}x_2 \implies$$

$$\ddot{y}_2 + y_2 = -\varepsilon a\big[(y_2 - y_1)^3 + (y_2 - y_3)^3\big] + 2\varepsilon\beta x_2$$

Therefore, the complete transformed system with respect to the new normalized time τ becomes

$$\begin{aligned}
\ddot{x}_1 + x_1 &= -\varepsilon a(x_1 - x_2)^3 + 2\varepsilon\beta y_1 \\
\ddot{y}_1 + y_1 &= -\varepsilon a(y_1 - y_2)^3 + 2\varepsilon\beta x_1 \\
\ddot{x}_n + (1 - \varepsilon\gamma)x_n &= -\varepsilon a\big[(x_n - x_{n-1})^3 + (x_n - x_{n+1})^3\big] + 2\varepsilon\beta y_n,\ n \geq 2 \\
\ddot{y}_n + y_n &= -\varepsilon a\big[(y_n - y_{n-1})^3 + (y_n - y_{n+1})^3\big] + 2\varepsilon\beta x_n,\ n \geq 2
\end{aligned} \tag{4.2.3}$$

Applying the condition of 1:1 resonance, we follow the complexification-averaging method (CX-A) approach[44],[45], and introduce the complex variables (with $\mathrm{i} = \sqrt{-1}$)

$$\psi_n^x = \dot{x}_n + \mathrm{i}x_n \quad \text{and} \quad \psi_n^y = \dot{y}_n + \mathrm{i}y_n, \; n \geq 1 \tag{4.2.4}$$

with all oscillators possessing a normalized fast frequency equal to unity. Denoting the complex conjugation by $\overline{\psi}$, for x_n of the system (4.2.3) we have that

$$\psi_n^x = \dot{x}_n + \mathrm{i}x_n, \quad \overline{\psi}_n^x = \dot{x}_n - \mathrm{i}x_n \tag{4.2.5}$$

$$\dot{x}_n = \frac{1}{2}(\psi_n^x + \overline{\psi}_n^x), \quad x_n = \frac{1}{2\mathrm{i}}(\psi_n^x - \overline{\psi}_n^x) \tag{4.2.6}$$

for $n \geq 1$. Similarly, for y_n of the system (4.1.3) we write

$$\psi_n^y = \dot{y}_n + \mathrm{i}y_n, \quad \overline{\psi}_n^y = \dot{y}_n - \mathrm{i}y_n \tag{4.2.7}$$

$$\dot{y}_n = \frac{1}{2}(\psi_n^y + \overline{\psi}_n^y), \quad y_n = \frac{1}{2\mathrm{i}}(\psi_n^y - \overline{\psi}_n^y) \tag{4.2.8}$$

for $n \geq 1$. Now, by using equations (4.2.5)-(4.2.8), the equation of x_1 of the system (4.2.3) becomes

$$\ddot{x}_1 + x_1 = -\varepsilon a(x_1 - x_2)^3 + 2\varepsilon\beta y_1 \implies$$

$$\ddot{x}_1 + \mathrm{i}\dot{x}_1 - \mathrm{i}\dot{x}_1 + x_1 = -\varepsilon a(x_1 - x_2)^3 + 2\varepsilon\beta y_1 \implies$$

$$\frac{d}{d\tau}(\dot{x}_1 + \mathrm{i}x_1) - \mathrm{i}(\dot{x}_1 - \mathrm{i}x_1) = -\varepsilon a(x_1 - x_2)^3 + 2\varepsilon\beta y_1 \implies$$

$$\frac{d}{d\tau}\psi_1^x - \mathrm{i}\psi_1^x = -\varepsilon a\left[\frac{\psi_1^x - \overline{\psi}_1^x}{2\mathrm{i}} - \frac{\psi_2^x - \overline{\psi}_2^x}{2\mathrm{i}}\right]^3 - \mathrm{i}\varepsilon\beta(\psi_1^y - \overline{\psi}_1^y) \tag{4.2.9}$$

Likewise, the equation of y_1 of the system (4.1.3) becomes

$$\frac{d}{d\tau}\psi_1^y - \mathrm{i}\psi_1^y = -\varepsilon a\left[\frac{\psi_1^y - \overline{\psi}_1^y}{2\mathrm{i}} - \frac{\psi_2^y - \overline{\psi}_2^y}{2\mathrm{i}}\right]^3 - \mathrm{i}\varepsilon\beta(\psi_1^x - \overline{\psi}_1^x) \tag{4.2.10}$$

Now, let us see how the equation of $x_n, \; n \geq 2$ of the system (4.2.3) becomes. We have

$$\ddot{x}_n + (1 - \varepsilon\gamma)x_n = -\varepsilon a\big[(x_n - x_{n-1})^3 + (x_n - x_{n+1})^3\big] + 2\varepsilon\beta y_n \implies$$

$$\ddot{x}_n + \mathrm{i}\dot{x}_n - \mathrm{i}\dot{x}_n + (1 - \varepsilon\gamma)x_n = -\varepsilon a\big[(x_n - x_{n-1})^3 + (x_n - x_{n+1})^3\big] + 2\varepsilon\beta y_n \implies$$

$$\frac{d}{d\tau}\psi_n^x - \mathrm{i}\psi_n^x =$$

$$-\varepsilon a\left\{\left[\frac{\psi_n^x-\overline{\psi}_n^x}{2\mathrm{i}}-\frac{\psi_{n-1}^x-\overline{\psi}_{n-1}^x}{2\mathrm{i}}\right]^3+\left[\frac{\psi_n^x-\overline{\psi}_n^x}{2\mathrm{i}}-\frac{\psi_{n+1}^x-\overline{\psi}_{n+1}^x}{2\mathrm{i}}\right]^3\right\}$$

$$-\mathrm{i}\varepsilon\beta(\psi_n^y-\overline{\psi}_n^y)-\frac{\mathrm{i}\gamma}{2}\varepsilon(\psi_n^x-\overline{\psi}_n^x),\ n\geq 2 \tag{4.2.11}$$

In a similar way, one can obtain

$$\frac{d}{d\tau}\psi_n^y - \mathrm{i}\psi_n^y =$$

$$-\varepsilon a\left\{\left[\frac{\psi_n^y-\overline{\psi}_n^y}{2\mathrm{i}}-\frac{\psi_{n-1}^y-\overline{\psi}_{n-1}^y}{2\mathrm{i}}\right]^3+\left[\frac{\psi_n^y-\overline{\psi}_n^y}{2\mathrm{i}}-\frac{\psi_{n+1}^y-\overline{\psi}_{n+1}^y}{2\mathrm{i}}\right]^3\right\}$$

$$-\mathrm{i}\varepsilon\beta(\psi_n^x-\overline{\psi}_n^x),\ n\geq 2 \tag{4.2.12}$$

Anticipating slow and fast time scales in the solution, we asymptotically approximate the irreversible energy transfer employing the method of multiple scales, introducing the fast time scale $\tau_0=\tau$ and the slow time scale $\tau_1=\epsilon\tau$, and expressing the solutions in the series forms

$$\psi_n^x=\psi_{n_0}^x+\epsilon\psi_{n_1}^x+O(\epsilon^2),\quad \psi_n^y=\psi_{n_0}^y+\epsilon\psi_{n_1}^y+O(\epsilon^2),\ n\geq 1 \tag{4.2.13}$$

Substituting (4.2.13) into (4.2.9)-(4.2.12), expressing the time derivatives in terms of the new time scales, and separating terms of different orders of the small parameter we obtain a group of linear subproblems. The $O(1)$ subproblem reads (for $n\geq 1$),

$$\frac{\partial}{\partial\tau_0}\psi_{n_0}^x-\mathrm{i}\psi_{n_0}^x=0 \implies \psi_{n_0}^x=\phi_{n_0}^x(\tau_1)\cdot \mathrm{e}^{\mathrm{i}t_0} \tag{4.2.14}$$

$$\frac{\partial}{\partial\tau_0}\psi_{n_0}^y-\mathrm{i}\psi_{n_0}^y=0 \implies \psi_{n_0}^y=\phi_{n_0}^y(\tau_1)\cdot \mathrm{e}^{\mathrm{i}t_0} \tag{4.2.15}$$

indicating that, to leading order, the responses of the asymmetric lattice network can be expressed in terms of fast oscillations that are modulated by the slow-varying complex envelopes $\phi_{n_0}^{x,y}(\tau_1)$. These are approximated by eliminating the secular terms from the $O(\epsilon)$ subproblems, yielding the following modulation equations or slow flow[39]

$$\begin{gathered}\frac{\partial}{\partial\tau_1}\phi_{10}^x=\frac{3}{8}a\mathrm{i}|\delta_1^x|^2\delta_1^x-\mathrm{i}\beta\phi_{10}^y\\ \frac{\partial}{\partial\tau_1}\phi_{10}^y=\frac{3}{8}a\mathrm{i}|\delta_1^y|^2\delta_1^y-\mathrm{i}\beta\phi_{10}^x\\ \frac{\partial}{\partial\tau_1}\phi_{n_0}^x=\frac{3}{8}a\left[-\mathrm{i}|\delta_{n-1}^x|^2\delta_{n-1}^x-(-\mathrm{i})|\delta_n^x|^2\delta_n^x\right]-\frac{\mathrm{i}\gamma}{2}\phi_{n0}^x-\mathrm{i}\beta\phi_{n0}^y,\ n\geq 2\\ \frac{\partial}{\partial\tau_1}\phi_{n_0}^y=\frac{3}{8}a\left[-\mathrm{i}|\delta_{n-1}^y|^2\delta_{n-1}^y-(-\mathrm{i})|\delta_n^y|^2\delta_n^y\right]-\mathrm{i}\beta\phi_{n0}^x,\ n\geq 2\end{gathered} \tag{4.2.16}$$

where $\delta_n^{x,y} \equiv \phi_{n0}^{x,y} - \phi_{(n+1)0}^{x,y}$ are the differences between the complex envelopes of neighboring oscillators in the two lattices.

In the following scheme, we can see that the slow flow predicts irreversible energy transfer from the excited to the absorbing lattice. We observe that rapid and complete energy transfer is initiated immediately after the impulse is applied, through 1:1 resonance interaction between the leading oscillators of the two lattices. We note that 1:1 resonance causes the initial intense energy transfer from the leading oscillator of the excited lattice to the leading oscillator of the absorbing lattice, but once this initial energy transfer is completed, the 1:1 resonance is "broken" so that the energy cannot transfer back to the excited lattice.

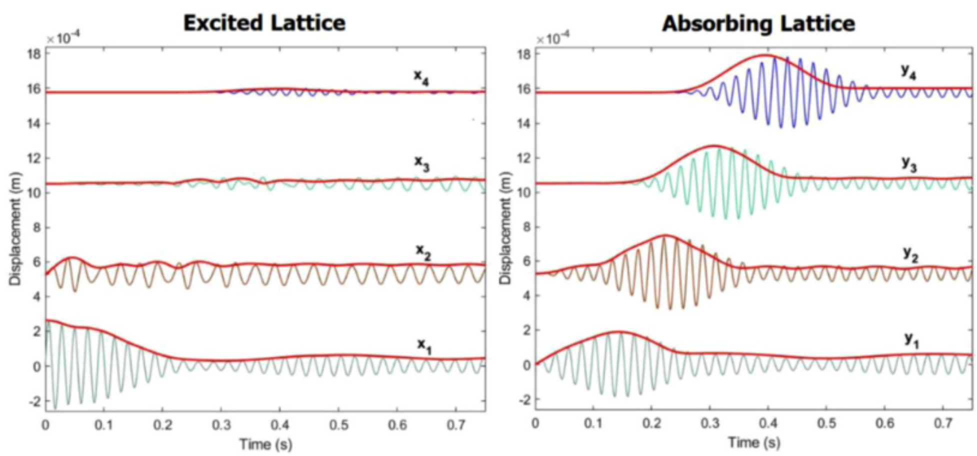

Figure 4.2.1: Comparison between the exact time series of the four leading oscillators of each lattice obtained by numerical integration of system (4.2.3), and the envelopes predicted by the slow flow (4.2.16) (red solid lines). [39]

We now develop a reduced-order model (ROM) to better understand the nonlinear mechanism governing irreversible energy transfer in the asymmetric lattice network under condition of 1:1 resonance. We assume that energy propagates along the two lattices in the form of breathers with constant speed under condition of 1:1 resonance, so that the energy is localized at one oscillator at a given time. Therefore the excited and absorbing lattices can be replaced by two oscillators. As the absorbing lattice has spatially uniform grounding stiffness, the grounding stiffness of the "absorbing oscillator" is constant (time-invariant); however, since the grounding stiffness of the excited lattice varies as the breather propagates, the "excited oscillator" possesses time-varying grounding stiffness. To construct the ROM we introduce the new variables,

$$u = \sum_{n=1}^{+\infty} x_n, \quad v = \sum_{n=1}^{+\infty} y_n \tag{4.2.17}$$

Adding the x_1 and x_n equations of (4.2.3) we get

$$\ddot{x_1} + x_1 + \ddot{x_n} + x_n - \varepsilon\gamma x_n = 2\varepsilon\beta y_1 + \varepsilon\beta y_n \implies$$

$$\ddot{x_1} + x_1 + \ddot{x_n} + x_n - \varepsilon\gamma x_n - \varepsilon\gamma x_1 + \varepsilon\gamma x_1 = 2\varepsilon\beta y_1 + 2\varepsilon\beta y_n \implies$$

$$\ddot{x_1} + \ddot{x_n} + x_1 + x_n - \varepsilon\gamma(x_1 + x_n) + \varepsilon\gamma x_1 = 2\varepsilon\beta y_1 + 2\varepsilon\beta y_n \implies$$

$$\ddot{u} + (1 - \varepsilon\gamma)u + \varepsilon\gamma x_1 = 2\varepsilon\beta v$$

Doing the same process, adding the y_1 and y_n equations of (4.2.3) we get

$$\ddot{v} + v = 2\varepsilon\beta u$$

So, we obtain the system of equations

$$\begin{gathered} \ddot{u} + (1 - \varepsilon\gamma)u + \varepsilon\gamma x_1 = 2\varepsilon\beta v \\ \ddot{v} + v = 2\varepsilon\beta u \end{gathered} \tag{4.2.18}$$

where the nonlinear terms have canceled by the summation. Equations of system (4.2.18) are exact up to this point, but to derive the (ROM) in its final form it is necessary to express $x_1(\tau)$ in terms of the variables (4.2.17). This can only be accomplished approximately, e.g., by introducing the relationship

$$x_1(\tau) = \xi(\tau)u(\tau)$$

Then, assuming that the impulsive energy is irreversibly transferred completely from the excited to the absorbing oscillator, the function $\xi(\tau)$ can be approximated following a three-stage approach. First, the energy is assumed to be localized entirely in the leading oscillators of the excited and absorbing lattices so that all other oscillators have zero response and

$$u(\tau) \approx x_1(\tau) \implies \xi(\tau) \approx 1$$

Following that, there is initiation of the propagating breather in the absorbing lattice, i.e., of nearly complete energy transfer from the leading oscillator of that lattice to its neighboring oscillator, so $\xi(\tau)$ can be assumed to linearly decrease in the range

$$0 \leq \xi(\tau) \leq 1$$

Finally, after the propagating breather has been initiated, energy propagates in the absorbing lattice, and $\xi(\tau) \approx 0$. Hence, we arrive to the final form of the (ROM), in the form of the linear, time-varying system

$$\begin{gathered} \ddot{u} + (1 - \varepsilon\gamma)u + \varepsilon\gamma\xi u = 2\varepsilon\beta v \\ \ddot{v} + v = 2\varepsilon\beta u \end{gathered} \tag{4.2.19}$$

where $\xi(\tau)$ is the $O(1)$ piecewise linear function

$$\xi(\tau) = \begin{cases} 1, & \tau \leq c_1 \\ \dfrac{c_1 + c_2 - \tau}{c_2}, & c_1 < \tau \leq c_1 + c_2 \\ 0, & c_1 + c_2 < \tau \end{cases} \tag{4.2.20}$$

In (4.2.19) the u-oscillator is the excited oscillator and the v-oscillator is the absorbing one. In (4.2.20) c_1 is the time required for the energy to completely transfer from the leading oscillator of the excited lattice to the leading oscillator of the absorbing one and c_2 is the time required for the energy to be completely transferred from the leading oscillator of the absorbing lattice to its neighboring oscillator in the same lattice. The analytical study of the ROM is carried out similarly to the original network (4.2.3) by the (CX-A) method. We introduce the complex variables

$$\psi_u = \dot{u} + \mathrm{i}u \quad \text{and} \quad \psi_v = \dot{v} + \mathrm{i}v$$

under condition of 1:1 resonance between the two oscillators of the (ROM). As before we introduce the fast and slow time scales, τ_0 and τ_1, respectively, and expand the complex variables as

$$\psi_u = \psi_{u_0} + \varepsilon\psi_{u_1} + O(\varepsilon^2) \quad \text{and} \quad \psi_v = \psi_{v_0} + \varepsilon\psi_{v_1} + O(\varepsilon^2)$$

Then, the leading order approximation of the solution is expressed as

$$\begin{aligned} \psi_u &= \phi_{u_0}(\tau_1)\mathrm{e}^{\mathrm{i}\tau_0} + O(\varepsilon) \\ \psi_v &= \phi_{v_0}(\tau_1)\mathrm{e}^{\mathrm{i}\tau_0} + O(\varepsilon) \end{aligned} \tag{4.2.21}$$

where the slowly varying complex envelopes are governed by the following reduced slow flow

$$\begin{aligned} 2\mathrm{i}\frac{\partial}{\partial\tau_1}\phi_{u_0} &= 2\beta\phi_{v_0} + \gamma\eta(\tau_1)\phi_{u_0} \\ \mathrm{i}\frac{\partial}{\partial\tau_1}\phi_{v_0} &= \beta\phi_{u_0} \end{aligned} \tag{4.2.22}$$

and the parametric, slowly varying term is expressed as

$$\eta(\tau_1) = 1 - \xi(\tau_1 - \varepsilon c_1) = \begin{cases} 0, & \tau_1 \leq \varepsilon c_1 \\ \dfrac{\tau_1}{\varepsilon c_2}, & \varepsilon c_1 \leq \tau_1 \leq \varepsilon c_2 \\ 1, & \tau_1 > \varepsilon c_2 \end{cases} \tag{4.2.23}$$

We see that the (ROM) system (4.2.21)-(4.2.22) describes the system after the energy has been transferred from the leading oscillator of the excited lattice to the leading oscillator of the absorbing lattice in the time interval $\tau_1 \leq \varepsilon c_1$. During the time interval $[0, \varepsilon c_1]$ the energy is completely transferred from the excited oscillator to the absorbing one, under condition of 1:1 resonance. At time $\tau_1 = \varepsilon c_1$ energy transfer has been completed to the leading oscillator of the absorbing lattice, so the initial conditions for (4.2.22) are

$$\phi_{u_0} = 0 \quad \text{and} \quad \phi_{v_0} = 1$$

In the time interval $[\varepsilon c_1, \varepsilon c_2]$ energy gets transferred from the leading oscillator to the other oscillators of the absorbing lattice and for $\tau_1 > \varepsilon c_2$ the breather in the absorbing lattice has been initiated. The (ROM) (4.2.21)-(4.2.22) can be explicitly solved by defining the ratio $a = \frac{\gamma}{\varepsilon c_2}$ and combing the complex equations (4.2.22) into a linear complex equation in terms of the response of the absorbing oscillator

$$\frac{d^2}{d\tau_1^2}\phi_{v_0} + \mathrm{i}\frac{\tau_1 a}{2} \cdot \frac{d}{d\tau_1}\phi_{v_0} + \beta^2 \phi_{v_0} = 0 \tag{4.2.24}$$

with initial conditions $\phi_{v_0}(0) = 1$ and $\frac{d}{d\tau_1}\phi_{v_0}(0) = 0$. Through the transformations,

$$\Phi = \phi_{v_0}(\tau_1)\mathrm{e}^{\mathrm{i}a\frac{\tau_1^2}{8}}, \quad z = \tau_1 \left(\frac{a}{2}\right)^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}, \quad \nu = 2\mathrm{i}\frac{\beta^2}{a}$$

equation (4.2.24) becomes

$$\frac{d^2}{dz^2}\Phi + \left(\nu + \frac{1}{2} - \frac{z^2}{4}\right)\Phi = 0 \tag{4.2.25}$$

with initial conditions $\Phi(0) = 1$ and $\Phi'(0) = 0$. Equation (4.2.25) is the normal form of Weber's equation[39] which admits an explicit asymptotic solution in terms of tabulated functions. As it is a linear, homogeneous, second-order differential equation, its general solution is expressed as a linear combination of two fundamental solutions

$$\Phi(z) = C_1 D_{-\nu-1}(\mathrm{i}z) + C_2 D_{-\nu-1}(-\mathrm{i}z) \tag{4.2.26}$$

where $D_{-\nu-1}(\pm\mathrm{i}z)$ are Weber's functions with well-defined asymptotic behaviors[39] as $z \to \infty$

$$\begin{aligned} &\lim_{z\to\infty} D_{-\nu-1}(\mathrm{i}z) = 0 \\ &\lim_{z\to\infty} D_{-\nu-1}(-\mathrm{i}z) = \frac{\sqrt{2\pi}}{G(\nu+1)}\mathrm{e}^{\frac{\pi\nu\mathrm{i}}{4}}\mathrm{e}^{\frac{\mathrm{i}|z|^2}{4}}|z|^{\nu} \end{aligned} \tag{4.2.27}$$

Imposing the previous initial conditions, we may evaluate the two unknown constants C_1 and C_2 as follows

$$
\begin{aligned}
C_1 &= \frac{2^{\frac{\nu}{2}}\sqrt{2\pi}}{\pi\Gamma(\frac{-\nu}{2})}\cos\frac{\pi(\nu+1)}{2} - \Gamma(-\nu)\Gamma(\nu+1)\sin\frac{\pi(\nu+1)}{2}\sin\pi\left(\nu+\frac{1}{2}\right) \\
C_2 &= \frac{2^{\frac{\nu}{2}}\sqrt{2\pi}}{\pi\Gamma(\frac{-\nu}{2})}\Gamma(-\nu)\Gamma(\nu+1)\sin\frac{\pi(\nu+1)}{2}
\end{aligned}
\tag{4.2.28}
$$

Relations (4.2.26)-(4.2.28) are an exact analytic expression for the leading-order response of the absorbing oscillator $\phi_{v_0}(\tau_1)$ and through the first of relations (4.2.22), also of the corresponding response of the excited oscillator, $\phi_{u_0}(\tau_1)$. Now, we consider the leading-order approximations of the energies of the two oscillators of the (ROM) as $\tau_1 \to +\infty$. These are analytically evaluated by the squares of ϕ_{v_0} and ϕ_{u_0}, yielding the following asymptotic expression of the energy that is irreversibly transferred to the absorbing oscillator

$$
\lim_{\tau_1\to\infty} |\phi_{v_0}(\tau_1)|^2 = \lim_{\tau_1\to\infty} |\Phi(z)|^2 = \frac{1+\mathrm{e}^{-\frac{2\pi\beta^2}{a}}}{2} = \frac{1+\mathrm{e}^{-\frac{2\pi\beta^2\varepsilon c_2}{\gamma}}}{2} \tag{4.2.29}
$$

where εc_2 is the time required for the generation of the propagating breather in the absorbing lattice, i.e., the time required for energy to be completely transferred from the leading oscillator of the absorbing lattice to its neighboring oscillator in the same lattice. According to [39] let us consider

$$
\varepsilon aTA^2 = \frac{0.7975}{\frac{3}{8}}
$$

where A is the amplitude of the breather, T the peak-to-peak time delay and a the normalized coefficient of the cubic nonlinearity. Substituting into (4.2.29) we obtain the final expression

$$
\lim_{\tau_1\to\infty} |\phi_{v_0}(\tau_1)|^2 = \frac{1+\mathrm{e}^{-\frac{2\pi\beta^2\cdot 2.13}{\gamma a A^2}}}{2} \tag{4.2.30}
$$

In (4.2.30), the normalized coefficient denoting the strength of the cubic nonlinearity, aA^2, and the normalized coupling stiffness coefficient, β, must be of the same order, or the energy will localize in the leading oscillators of the two lattices. If, in addition, the stiffness reduction ratio γ is sufficiently larger than the normalized coupling coefficient β, i.e. $\gamma \gg \beta$, then

$$
\lim_{\tau_1\to\infty} |\phi_{v_0}(\tau_1)|^2 \approx 1
$$

and nearly all of the impulsive energy becomes irreversibly localized to the absorbing oscillator. Otherwise, if $\gamma \ll \beta$, then

$$
\lim_{\tau_1\to\infty} |\phi_{v_0}(\tau_1)|^2 \approx \frac{1}{2}
$$

and the energy is reversibly and recursively exchanged between the two oscillators in a nonlinear beat phenomenon. As a final step we wish to validate the approximate (ROM) (4.2.19)-(4.2.20) based on numerical simulations of the exact (ALN). In the exact network the grounding stiffness of the excited lattice is spatially varying, whereas in the (ROM) the grounding stiffness of the excited oscillator is time varying. So, we compute the weighted-averaged grounding stiffness over

$$\overline{k}_g(t) = \frac{\sum_{j=1}^{n} k_{g_j} E_j(t)}{E} \tag{4.2.31}$$

where, E is the total energy provided by the impulse, $E_j(t)$ the instantaneous energy of the j-th oscillator pair of the excited and absorbing lattices and k_{g_j} the grounding stiffness of the j-th oscillator of the excited lattice. Now, we compare the result with the analytical approximation (4.2.20). According to the following scheme we note that the difference between the approximate and numerical results is small, validating the (ROM).

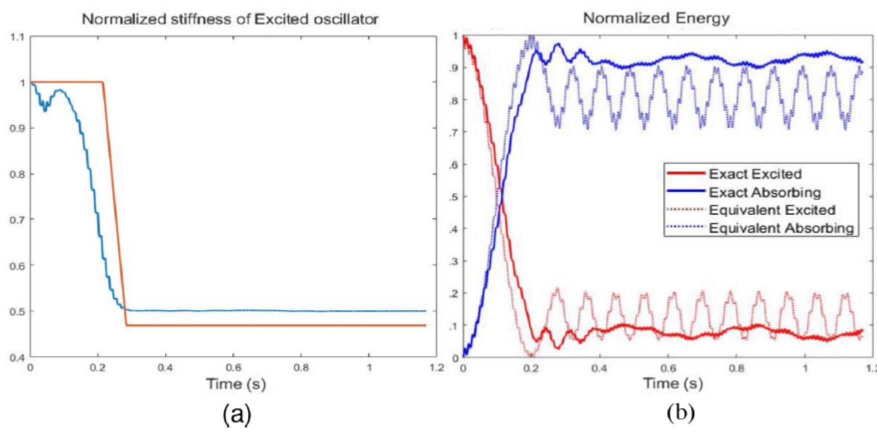

Figure 4.2.2: The (ROM) (4.2.19), (4.2.20) compared to the (ALN) (3). (a) Weighted-averaged grounded stiffness (4.2.31) – blue line, versus the approximation (4.2.20) – red line; (b) instantaneous energies of the excited and absorbing lattices – solid lines, versus the analytical (ROM) predictions for the excited and absorbing oscillators – dashed lines.[39]

Finally, in Figure (4.2.2) we validate the analytical prediction of the (ROM) regarding irreversible energy transfer by comparing to the results of direct numerical simulations. We note good agreement, especially in the critical, early regime of the response during which irreversible energy transfer from the leading oscillator of the excited lattice to the corresponding oscillator of the absorbing lattice. The small oscillations in the energy values at later times are due to the simplifying assumptions of the (ROM), and its inability to more accurately capture the time-varying stiffness of the excited oscillator.

5 Appendix

5.1 An Introduction to Reduced Order Model Theory

In many fields of mathematics and physics, we come across hard problems that are describable by a differential equation and many times by systems of differential equations. The study of the behavior of these systems is, often, a tough task. One way to deal with such problems is the Reduced Order Model (ROM) theory. How does ROM work? Basically, it substitutes the original large scale system with a much smaller one, but the new ROM has the same qualitative behavior as the original, to a very high accuracy. Thus, by simulating only the ROM, one can study the original system doing much easier work. Let us present a simple example to demonstrate how ROM works. We consider the following system

$$\dot{\underline{\mathbf{x}}} = \mathbf{A}\underline{\mathbf{x}} + \underline{\mathbf{b}} \cdot f(t) \tag{5.1.1}$$

$$y = \underline{\mathbf{c}}^T \underline{\mathbf{x}} \tag{5.1.2}$$

where $\underline{\mathbf{x}} = \underline{\mathbf{x}}(t)$ is an N-dimensional vector, $\mathbf{A}$ is an NxN constant matrix and $\underline{\mathbf{b}}$, $\underline{\mathbf{c}}$ are N-dimensional constant vectors. For each function $f(t)$ we plug at (5.1.1), a new function $y(t)$ is generated by equation (5.1.2). Therefore, let us denote function $f(t)$ as the input-function (IF) and function $y(t)$ as the output-function (OF). So, ROM theory asks the question: can we find another system

$$\dot{\underline{\mathbf{z}}} = \mathbf{A_1}\underline{\mathbf{z}} + \underline{\mathbf{b}}_1 \cdot f(t) \tag{5.1.3}$$

$$y_1 = \underline{\mathbf{c}}_1^T \underline{\mathbf{z}} \tag{5.1.4}$$

where $\underline{\mathbf{z}} = \underline{\mathbf{z}}(t)$ is an n-dimensional vector, $\mathbf{A_1}$ is an nxn constant matrix, $\underline{\mathbf{b}}_1$, $\underline{\mathbf{c}}_1$ are n-dimensional constant vectors and n is much smaller than N? For instance, N could be thousands and n less than 10. And the ROM system has similar behavior with the original system i.e. for any (IF) $f(t)$ the ROM system (5.1.3) - (5.1.4) shall generate an (OF) $y_1(t)$ such that $y_1(t)$ is very close to $y(t)$. Therefore, once we have the ROM, we can study the behavior of the original system to a good precision by studying and simulating the ROM instead.

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