ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ

ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ ΚΑΙ ΦΥΣΙΚΩΝ ΕΠΙΣΤΗΜΩΝ

ΣΧΟΛΗ ΜΗΧΑΝΟΛΟΓΩΝ ΜΗΧΑΝΙΚΩΝ

ΕΚΕΦΕ «ΔΗΜΟΚΡΙΤΟΣ»

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Διατμηματικό Πρόγραμμα Μεταπτυχιακών Σπουδών

«Φυσική και Τεχνολογικές Εφαρμογές»

<u>Exploring the landscape of anomaly-free 6D supergravities</u>

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

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Αθήνα, Ιούλιος, 2023

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1 Introduction

Given a general Quantum Field Theory, a fundamental consistency check is that of anomaly cancellation. In dimensions $D = 4k + 2$ one finds gravitational, gauge and mixed anomalies, and the requirement that these cancel places severe constraints on the space of available theories. In $D = 10$, anomaly-free theories only admit a handful of gauge groups, as shown in [2]. Similarly in $D = 6$, anomaly cancellation provides a strong constraint on the possible gauge groups and particle spectrum of a theory. In fact, it was shown in [3], that if we additionally require that the gauge-kinetic terms are positive and that the number of tensor multiplets is less than 9, there is only a finite number of anomaly-free models. On a similar note, it was shown in [5] that anomaly cancellation together with other conjectured Swampland constraints leaves again only a finite number of possible models.

Knowing that there is a finite number of possible configurations of gauge groups and matter representations (an estimate of under a billion was given in [4]), a logical next step would be an attempt to enumerate these models. In [1], such an enumeration was performed for a specific set of allowed gauge groups. In particular, and in order to make the search feasible, the following restrictions were placed on the solution space:

1. The number of tensor multiplets was required to be $T = 1$.

2. The semisimple gauge group was required to be a product of up to two simple groups.

3. The rank of classical groups had to lie within specific ranges (e.g. for SO(N) it was required that $10 \leq N \leq 64$).

In this thesis we aim to expand upon that search by relaxing some of the above conditions. Specifically:

1. We allow the semisimple gauge group to be a product of any number of simple groups.

2. We only require a lower bound of $D \geq 10$ for the considered representations, i.e. the rank of the classical groups is not a priori bounded from above.

In order to achieve this, we will need to use more efficient searching algorithms as well as theoretical bounds on the amount of hypermultiplets and vector multiplets of a configuration when only part of it is fixed, which will be the main subject of this work.

2 Anomaly cancellation in six dimensions

We begin with a brief outline of the necessary material, namely the content of $\mathcal{N} = 1$, $D = 6$ supergravity and the cancellation of anomalies through the Green-Schwarz mechanism.

2.1 Particle content

The massless representations of $\mathcal{N} = 1$ supergravity in 6 dimensions are as follows:

- Supergravity multiplet: $(g_{\mu\nu}, B^+_{\mu\nu}, \psi^{i-}_{\mu\nu})$
- Tensor multiplet: $(B^-_{\mu\nu}, \phi, \chi^{i+})$
- Vector multiplet: (A_μ, λ^{i-})
- Hypermultiplet: $(4\phi, 2\psi^+)$

We denote by T, V and H the number of tensor multiplets, vector multiplets and hypermultiplets of the theory respectively. V is given by the sum of the dimensions of the constituent simple gauge groups, while H is given by the sum of the dimensions of the hypermultiplets multiplied by their multiplicities, plus the number of singlets (which we will not be writing down explicitly). The combination of these with one supergravity multiplet gives rise to a generic $D = 6$, $\mathcal{N} = 1$ supergravity with spectrum

$$(g_{\mu\nu}, B^+_{\mu\nu}, \psi^{i-}_{\mu\nu}) + T(B^-_{\mu\nu}, \phi, \chi^{i+}) + V(A_\mu, \lambda^{i-}) + H(4\phi, 2\psi^+) \tag{1}$$

This spectrum may be contain local or global anomalies. In this thesis we will focus only on the local anomalies. Next we describe the mechanism with which these are cancelled.

2.2 Cancellation of local anomalies

The anomaly polynomial includes contributions from gravity, gauge and mixed anomalies. Its full form is

$$
\begin{aligned}
I_8 = \frac{H-V+29T-273}{360}\,\mathrm{tr}\,R^4 + \frac{H-V-7T+51}{288}(\mathrm{tr}\,R^2)^2 \\
+\frac{1}{6}\,\mathrm{tr}\,R^2\sum_i A_i\,\mathrm{tr}\,F_i^2 \\
-\frac{2}{3}\sum_i B^i\,\mathrm{tr}\,F_i^4 - \frac{2}{3}\sum_i C^i(\mathrm{tr}\,F_i^2)^2 + 4\sum_{i<j} A^{ij}\,\mathrm{tr}\,F_i^2\,\mathrm{tr}\,F_j^2
\end{aligned} \tag{2}
$$

where

$$A^i = a^i_{\mathrm{adj}} - \sum_r n^i_r a^i_r \tag{3}$$

$$B^i = b^i_{\mathrm{adj}} - \sum_r n^i_r b^i_r \tag{4}$$

$$C^i = c^i_{\mathrm{adj}} - \sum_r n^i_r c^i_r \tag{5}$$

$$A^{ij} = \sum_{r,s} n^{ij}_{rs} a^i_r a^j_s \tag{6}$$

In the above, n^i_r is the number of hypermultiplets transforming under the representation r and n^{ij}_{rs} is the number of hypermultiplets transforming simultaneously under r and s.

This polynomial must factorize for the anomaly to cancel through the Green-Schwarz mechanism. As a result we must have

$$H - V = 273 - 29T \tag{7}$$

and

$$B_i = 0 \tag{8}$$

for groups with fourth-order invariants. The polynomial can then be written more concisely as

$$I_8 = \frac{1}{2}(\mathrm{tr}\,X_4)^2 = \frac{1}{2}\Omega_{\alpha\beta}\,\mathrm{tr}\,X_4^\alpha\,\mathrm{tr}\,X_4^\beta \tag{9}$$

where

$$X_4^\alpha = \frac{1}{2}a^\alpha\,\mathrm{tr}\,R^2 + \sum_i b_i^\alpha\,\mathrm{tr}\,F_i^2 \tag{10}$$

By expanding (9) and matching the coefficients we arrive at the anomaly cancellation conditions.

$$a \cdot a = 9 - T \tag{11}$$

$$a \cdot b_i = \frac{1}{6} A_i \tag{12}$$

$$b_i \cdot b_i = -\frac{1}{3} C_i \tag{13}$$

$$b_i \cdot b_j = A_{ij} \tag{14}$$

$$B_i = 0, \qquad \text{for groups with fourth-order invariants} \tag{15}$$

In the above, a and b_i are vectors in $\mathbb{R}^{1,T}$ and $x \cdot y$ denotes the $\mathrm{SO}(1,T)$-invariant product $\Omega_{\alpha\beta} x^\alpha y^\beta$. a^i_r, b^i_r, c^i_r are group theoretic constants which can be found in [1].

When $T = 1$, the first three equations can be solved explicitly. We first take

$$\Omega_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{16}$$

By using the scaling invariance of the anomaly polynomial [3], we can fix $a = (-2, -2)$ and we also set $b_i = \frac{1}{2}(a_i, \tilde{a}_i)$. This way, equations (12, 13) take the form

$$a_i + \tilde{a}_i = -\frac{1}{6} A^i \tag{17}$$

$$a_i \tilde{a}_i = -\frac{2}{3} C^i \tag{18}$$

with solutions

$$a_i, \tilde{a}_i = \frac{-A^i \pm \sqrt{D^i}}{12} \tag{19}$$

where $D^i = (A^i)^2 + 96C^i$ and the values can be assigned to the $a_i, \tilde{a}_i$ in both orders. Equation (6) becomes

$$a_i \tilde{a}_j + \tilde{a}_i a_j = 4A^{ij} \tag{20}$$

Using (19) this can be written more explicitly as

$$A^i A^j \mp \sqrt{D^i D^j} = 288 A^{ij} \tag{21}$$

where the $-$ corresponds to assigning the same sign to a_i and a_j in (19), while the $+$ corresponds to assigning the same sign to a_i and $\tilde{a}_j$.

As discussed in [4], the coefficients of the gauge kinetic terms, $a_i e^\phi + \tilde{a}_i e^{-\phi}$, must all be positive for some value of the dilaton ϕ:

$$a_i e^\phi + \tilde{a}_i e^{-\phi} > 0, \text{ for all } i \tag{22}$$

This prevents infinite families of models such as the SU(N):$2N$ $\langle N \rangle$ from coupling to each other. Condition (22) places some restrictions on the relative signs of the $a_i, \tilde{a}_i$ and on the possible values of $e^{2\phi}$, as follows:

- If a_i and $\tilde{a}_i$ are both positive, (22) is always satisfied.
- a_i and $\tilde{a}_i$ cannot both be negative.
- If one of the $a_i, \tilde{a}_i$ is zero, the other one must be positive, in which case (22) is always satisfied.
- If $a_i > 0$ and $\tilde{a}_i < 0$, then $e^{2\phi} > -\frac{\tilde{a}_i}{a_i}$.
- If $a_i < 0$ and $\tilde{a}_i > 0$, then $e^{2\phi} < -\frac{\tilde{a}_i}{a_i}$.

Note that in the last two cases, the quantity $-\frac{\tilde{a}_i}{a_i}$ is positive.

To summarize, we can calculate $a_i, \tilde{a}_i$ using (19) and then check (20) for each pair of groups. In addition, depending on the values a_i, $\tilde{a}_i$, (22) might be impossible to satisfy, it might always be satisfied or it might restrict the values of $e^{2\phi}$ to a certain range. If multiple groups fall into the latter category, all the inequalities must be satisfied simultaneously, i.e. the allowed ranges must have a non-vanishing intersection.

Finally, the solution as a whole must satisfy

$$H - V \leq 273 - 29T \tag{23}$$

3 Scanning the solution space

With the above in mind, we could in principle find all the solutions with the following procedure:

1. We first find all the solutions that have $H - V \leq M$, where M is a constant the value of which will be specified in later sections. We can represent each such solution as a node in a graph.
2. We then find which of these solutions can be combined according to (20). When two solutions can be combined we draw an edge between the respective nodes in the graph.
3. The solutions containing more than two factors will then be the subgraphs of this graph that are fully connected, i.e. there exists an edge between each pair of nodes. Such subgraphs are known as complete subgraphs or "cliques" in graph theory. The cliques of a graph can easily be found programmatically.

An example can be seen in figure 1 where we have drawn part of the graph for $T = 1$.

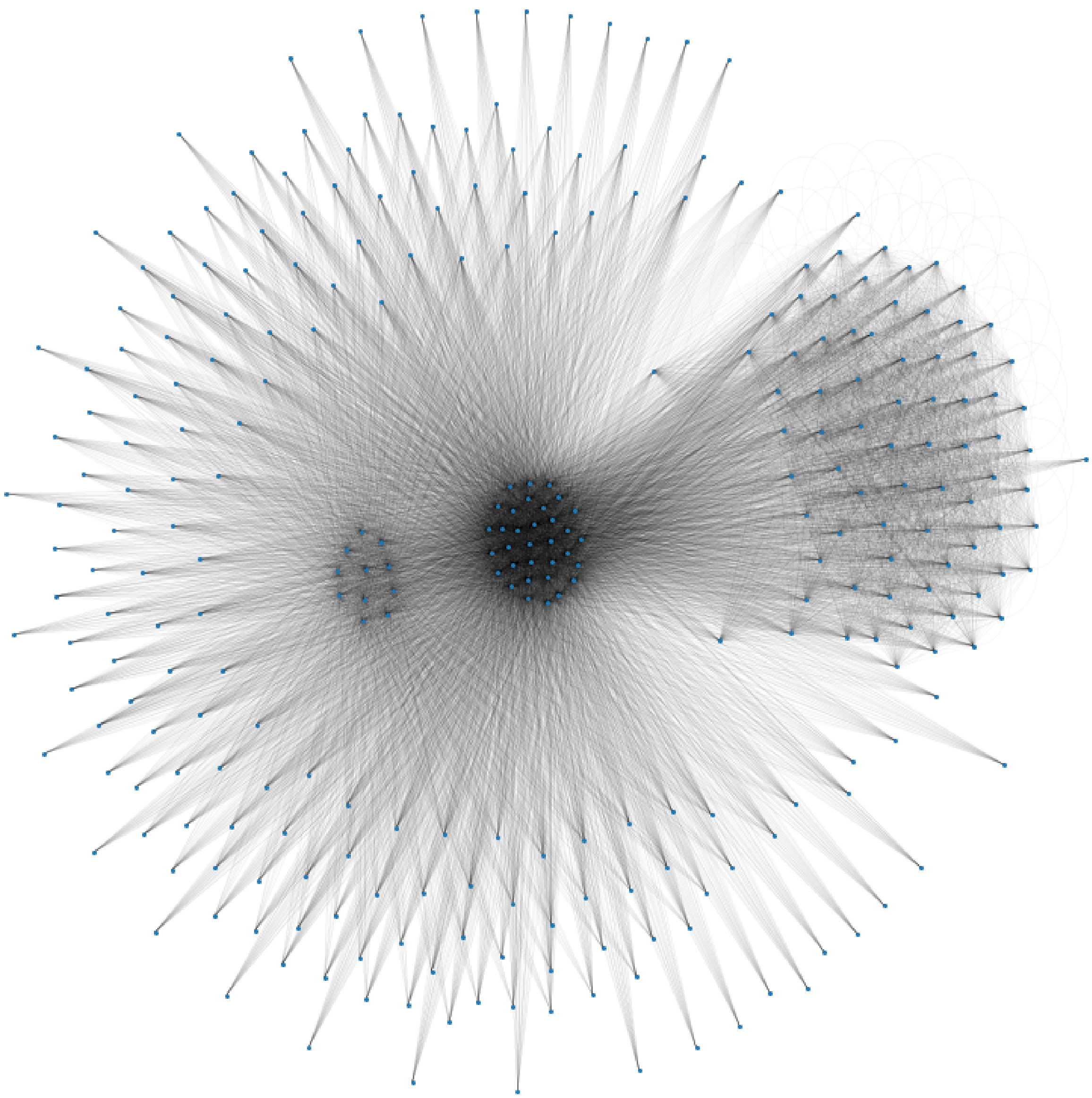

Figure 1: Part of the graph for $T = 1$.

We constrain our search to $T = 1$, and consider all the exceptional groups except for G_2 and all the classical groups SU(N), SO(N), Sp(N), with rank $N \geq 10$. Additionally for the classical groups, we will only consider the fundamental, the adjoint, the symmetric and the antisymmetric representations, and not higher dimensional representations such as the spinorial. The above restrictions are placed in order to facilitate an exhaustive search, given that the number of possible nodes is significantly larger for representations with dimension $D < 10$ and especially for the groups SU(2), SU(3) and G_2. That being said, using the tools that we present here along with some improvements on the developed algorithms and with the use of better hardware, it might be feasible to achieve a complete enumeration of all the possible models.

4 Bounding the contribution to $H - V$

4.1 General scheme

We will begin by establishing some notation. Let

$$M = G : \sum_r n_r \langle D_r \rangle \tag{24}$$

be a model containing n_r hypermultiplets transforming under the representation D_r of the simple gauge group G. For example, taking $G = \mathrm{SU}(N)$ we may write

$$M = \mathrm{SU}(N) : (N-8)\langle N \rangle + \langle \frac{N(N+1)}{2} \rangle \tag{25}$$

We may combine two such models according to

$$M = M_1 \oplus M_2 = G_1 \times G_2 : \sum_{r,s} n_{rs} \langle D_r, D_s \rangle \tag{26}$$

where n_{rs} is the number of hypermultiplets transforming under both D_r and D_s (note that the sum includes the trivial representations, with $D_r = 1$ or $D_s = 1$). As an example, we can take

$$M = \mathrm{SU}(N) \times \mathrm{SU}(N) : 2\langle N, N \rangle \tag{27}$$

In a similar manner, we can fix g models of the form (24), i.e. we can pick some groups G_i and the multiplicities n_r of their representations, and combine them according to

$$M = \bigoplus_{i=1}^{g} M_i = \prod_{i=1}^{g} G_i : \sum_{r_1,\ldots,r_g} N_{r_1,\ldots,r_g} \langle D_{r_1}, \ldots, D_{r_g} \rangle \tag{28}$$

Note that, since we fixed the multiplicities n_r of each constituent model, these must be the same in the composite model. In other words, we must have

$$n_r = \sum_{r \notin r_1,\ldots,r_k} N_{r_1 \ldots r_k} D_{r_1} \cdots D_{r_k}, \text{ for every } r \tag{29}$$

The contribution of (28) to $H - V$ is

$$C_M = \sum_{r_1,\ldots,r_g} N_{r_1,\ldots,r_g} D_{r_1} \cdots D_{r_g} - \sum_{i=1}^{g} A_i \tag{30}$$

where A_i are the dimensions of the adjoint representations of the groups G_i.

Our first objective will be to find a way of defining an individual contribution $C_{r_1,\ldots,r_g}$ to $H - V$ for each term in (28), so that

$$C_M = \sum_{r_1,\ldots,r_g} C_{r_1,\ldots,r_g} \tag{31}$$

Obviously, each term has a positive contribution $N_{r_1,\ldots,r_g} D_{r_1} \cdots D_{r_g}$. Note however, that there may be multiple terms transforming under different representations of a single group G_i with adjoint dimension A_i. This adjoint dimension will appear only once in (31), so we need to define a way of distributing it to the corresponding terms. To make this clearer, let's take (25) as an example. This has

$$C_M = (N-8)N + \frac{N(N+1)}{2} - (N^2 - 1) \tag{32}$$

We want to rewrite this in such a way so that the negative contribution $-(N^2 - 1)$ is distributed to the first two terms, corresponding to the two involved groups. For example, we may choose to distribute it equally as follows:

$$C_M = [(N-8)N - \frac{1}{2}(N^2 - 1)] + [\frac{N(N+1)}{2} - \frac{1}{2}(N^2 - 1)] \tag{33}$$

The distribution could become more "fair" if we assigned to each term a fraction of the adjoint, proportional to the contribution of this term to the fourth-order Casimir cancellation condition. More precisely, we assign to the term

$$N_{r_1,\ldots,r_g} \langle D_{r_1}, \ldots, D_{r_g} \rangle \tag{34}$$

with $r_i \in r_1, \ldots, r_g$, a fraction $f_{r_i} A_i$ of the negative contribution of G_i to $C_{r_1,\ldots,r_g}$. $0 \leq f_{r_i} \leq 1$ is the relative contribution of r_i to the left-hand side of the constraint $\sum_r n_r B_r = B_i$, so that

$$f_{r_i} = \frac{n_{r_i} B_{r_i}}{B_i} \tag{35}$$

In this way, we assign to the term (34) a fraction of the adjoint A_i for each of the involved representations. We finally have

$$C_{r_1,\ldots,r_g} = N_{r_1,\ldots,r_g} \prod_{i=1}^{g} D_{r_i} - \sum_{i=1}^{g} f_{r_i} A_i \tag{36}$$

This definition allows us to assign a fraction of the adjoint to one term even if we don't know what other terms involving G_i appear in the sum (31). Terms involving a representation from an exceptional group will get the full negative contribution of the adjoint representation. We will make a few more comments about the effect of the choice of distribution in the following, after we have established our end goal.

Now let's assume that $\lambda + 1$ out of the g representations of (34) are non-trivial, i.e. their dimensions are not equal to 1. To simplify notation we will denote these by $D, D_i, i = 1, \ldots, \lambda$. Similarly we denote by n, n_i the multiplicities of the representations in the original constituent models and by A, A_i the dimensions of the adjoint representations multiplied by the respective fractions as defined in (35). Finally, we will denote the multiplicity of the hypermultiplet by $N_{r_1,\ldots,r_g} \equiv m$ and its total contribution to H by $N_{r_1,\ldots,r_g} D_{r_1} \cdots D_{r_g} = mDD_1 \cdots D_\lambda \equiv S$. With these definitions we have

$$nD = n_1 D_1 = \ldots = n_\lambda D_\lambda = S \tag{37}$$

and we will also denote

$$C \equiv C_{r_1,\ldots,r_g} = S - A - \sum_i A_i \tag{38}$$

The question that we want to answer is the following: if we are given the multiplicity n and the dimension D of the first representation in the above term, how should we choose λ other dimensions and their multiplicities so that the individual contribution of the term to $H - V$, in the sense of (36), has its minimum possible value?

Before we try to answer this question, let's recall that in order to talk about the individual contribution of each term, we had to find a way of distributing the negative contribution from the dimension of the adjoint to the various representations. As we mentioned, there are many ways to do this and we chose one that would allow us to examine each representation separately. Will this choice affect the minimum value that we find for C in (38)? The answer is that it will affect the minimum that we arrive at, but that will always be less than or equal to the actual value $C_{\min}$ that exists for a given pair n, D. This is because we could always arrive at that minimum by putting together the different pieces (i.e. terms in (31)) of the actual configuration that has the contribution $C_{\min}$. In other words, by allowing the pieces to be put together in any possible configuration (potentially including ones that are not allowed), we will arrive at a less conservative bound. Of course the choice of method for the distribution is still important, since it could lead us to underestimate the minimum so much that it diverges to $-\infty$. This would happen, for example, if we assigned all of the negative contribution to the fundamental representation, without taking into account its multiplicity.

Now let's look for a way to bound C. We first rewrite (38) as

$$C = S - A - \lambda \overline{A_i} \tag{39}$$

where the overline denotes the average value. We also have

$$
\begin{aligned}
S = mDD_1 \cdots D_\lambda \Rightarrow D_1 \cdots D_\lambda = \frac{S}{mD} = \frac{n}{m} \\
\Rightarrow \sum_i \log D_i = \log \frac{n}{m} \Rightarrow \lambda \overline{\log D_i} = \log \frac{n}{m} \\
\Rightarrow \lambda = \frac{\log \frac{n}{m}}{\overline{\log D_i}}
\end{aligned}
\tag{40}
$$

Substituting this into (39) we get

$$
C = S - A - \log \frac{n}{m} \frac{\overline{A_i}}{\overline{\log D_i}} \tag{41}
$$

Note that $D_i \geq 2$ for all i, so that all the $\log D_i$ are positive, regardless of the base of the logarithm. It is then straightforward to see that

$$
\frac{\overline{A_i}}{\overline{\log D_i}} = \frac{\sum_i A_i}{\sum_i \log D_i} \leq \max_i \frac{A_i}{\log D_i} \tag{42}
$$

so that

$$
C \geq S - A - \log \frac{n}{m} \max_i \frac{A_i}{\log D_i} \tag{43}
$$

In other words, to minimize C, we should pick all the representations to be the ones for which the quantity $\frac{A_i}{\log D_i}$ is maximized. If we denote these optimal dimensions by $\tilde{D}$ and $\tilde{A}$, the number of groups involved becomes

$$
\lambda = \log_{\tilde{D}} \frac{n}{m} \tag{44}
$$

so that we can rewrite (43) as

$$
C \geq nD - A - \tilde{A} \log_{\tilde{D}} \frac{n}{m} \tag{45}
$$

In addition, we must have

$$
\lambda \geq 1 \Rightarrow \frac{n}{m} \geq \tilde{D} \tag{46}
$$

It is obvious that the value of the minimum increases with the multiplicity $m \geq 1$ of the term so from now on we fix $m = 1$. This way we get $\tilde{D} \leq n$, and from $\tilde{n}\tilde{D} = nD$ we also get $D \leq \tilde{n}$.

To reiterate, let's assume that we are given a representation of dimension D with multiplicity n. We find the $\tilde{D}$ and $\tilde{A}$ that correspond to the representation for which the fraction $\frac{\tilde{A}}{\log \tilde{D}}$ has the maximum value, subject to the constraints

$$
\tilde{n}\tilde{D} = nD \text{ and } \tilde{D} \leq n, D \leq \tilde{n} \tag{47}
$$

Then, the minimum individual contribution of a term involving the $n\langle D\rangle$ will be

$$C_{n\langle D\rangle} = nD - A - \tilde{A}\log_{\tilde{D}} n \tag{48}$$

We can follow a very similar approach to obtain the minimum contribution of a term without being given the values n, D and A, but only the value the positive contribution

$$S = mD_1 \cdots D_\lambda \tag{49}$$

As before, we have

$$C \geq S - \log\frac{S}{m}\max_i \frac{A_i}{\log D_i} \tag{50}$$

with the equality being obtained when

$$D_1 = \ldots = D_\lambda = \tilde{D} \tag{51}$$

Taking again $m = 1$, the minimum can be written as

$$C_S = \tilde{D}^\lambda - \lambda\tilde{A} \tag{52}$$

where $\lambda \geq 1$ is now a free parameter and

$$\tilde{D} = S^{\frac{1}{\lambda}} \tag{53}$$

Depending on the values of $\tilde{D}$ and $\tilde{A}$, the optimal value for λ could be either 1 or 2. For the purposes of the present analysis it will always be 1, since we are only considering dimensions $D \geq 10$.

In the following sections, we will go through each individual type of multiplet $n_i\langle D_i\rangle$, subject to either (47) or (53), and obtain the minimum values.

4.2 Exceptional groups

We begin with the exceptional groups. For a given A_i we want to keep the dimension D_i as small as possible, so we will take D_i to be the smallest dimension for each group. To get a more general picture, let's assume that $A_i = aD_i$. For example, for F_4 we have $a = 2$. The fraction then takes the value

$$\frac{A_i}{\log D_i} = \frac{aD_i}{\log D_i} \leq \frac{an}{\log n} \tag{54}$$

where we made use of (47). By looking at each of the exceptional groups we see that $a < 3$ in all cases. Taking $a = 3$ for the sake of simplicity sake we arrive at

$$\frac{A_i}{\log D_i} \leq \frac{3n}{\log n} \tag{55}$$

We can then use (48) to get

$$C_{n\langle D\rangle} = n(D-3) - A \tag{56}$$

Similarly, we find

$$C_S = D^{\lambda} - 3\lambda D = S - 3\lambda S^{\frac{1}{\lambda}} \tag{57}$$

4.3 Finite rank classical groups

It is well known that for large values of the rank N, there are only a few possible configurations containing representations of the classical groups SU(N), SO(N), Sp(N), that satisfy the fourth-order Casimir cancellation condition. These belong to a set of families, listed in table 1, which we will examine in the next subsection. Here we will look into the configurations that don't belong in this list, by keeping the rank bounded. We can easily obtain the maximal rank for which we can have models other than the infinite ones. For SU(N), we must have

$$n_0 + 2Nn_1 + (N+8)n_2 + (N-8)n_3 = 2N \tag{58}$$

This equation has only the solutions of table 1, unless

$$\frac{2N}{N-8} \geq 3 \Rightarrow N \leq 24 \tag{59}$$

The groups SO(N) and Sp(N) can be viewed as special cases of SU(N) in the context of the present analysis. In particular, we can view SO(N) as SU(N) with $n_1 = 0, n_2 = 1$ and Sp(N) as SU($2N$) with $n_1 = 0, n_3 \geq 1$. For this reason we will focus only on SU(N) in the following.

It is then straightforward to obtain the functions $C_{n\langle D\rangle}$ and C_S for each possible configuration as we did for the case of the exceptional groups. These are plotted in figures 2 and 3.

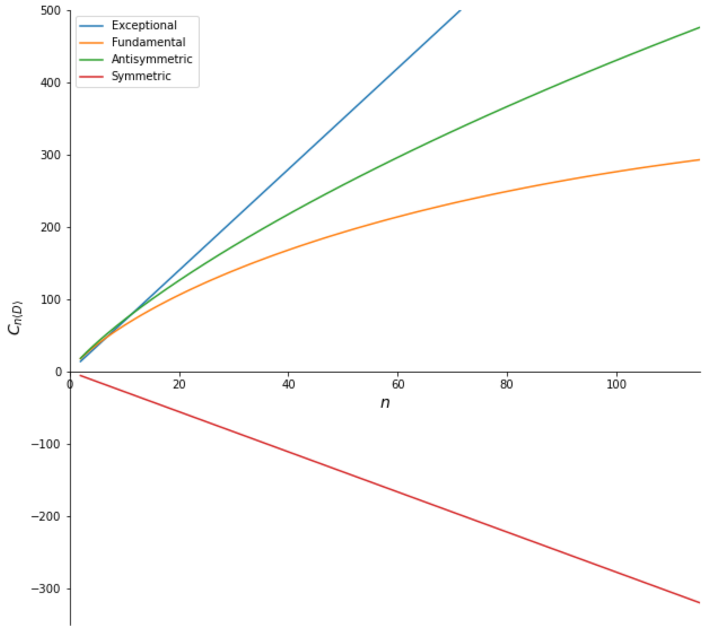

Figure 2: $C_{n\langle D\rangle}$ for $D = 10$.

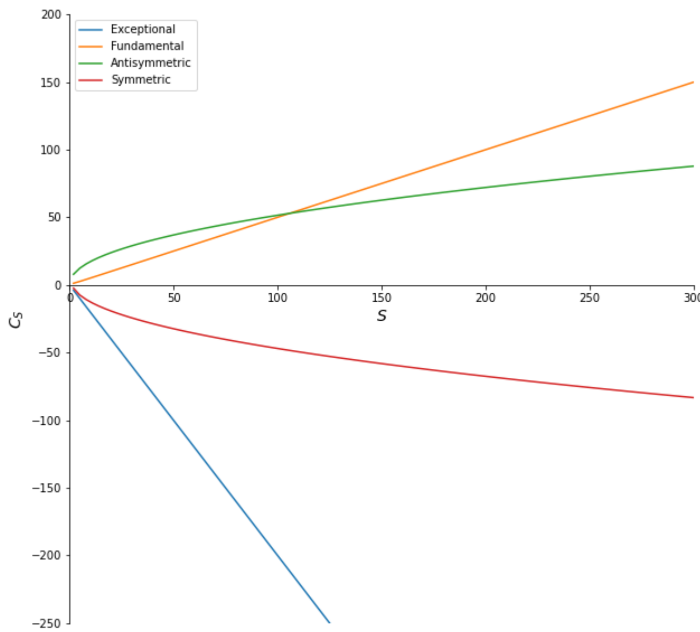

Figure 3: C_S for $\lambda = 1$.

Figure 2 in particular has been plotted for $n \leq \frac{2N_0^2}{D}$ where $N_0 = 24$. For larger n, we will need to break $n\langle D\rangle$ down to multiple terms, each of which will couple to one finite group. We will see this in more detail in section 4.5.

Looking at $C_{n\langle D\rangle}$, we note that the symmetric representation is the one with the smallest (and even negative) value for a range of values of n. Importantly, this representation can appear only once in any configuration involving the representations that we are considering. To see this, note that for $N \geq 8$, equation (58) has either $n_2 = 0$ or $n_2 = 1$. The solutions with $n_2 = 1$ correspond to the families of table 1, which cannot coexist in the same model.

Out of the rest of the curves, we can see that the optimal one is that of the fundamental for most values of n. For small values, the optimal curve is the one corresponding to the exceptional groups.

4.4 Infinite rank classical groups

We finally move on to the infinite families. This case is distinct from the ones examined so far, in that each model can include up to one of these configurations, given that in most cases two of them cannot coexist in the same model [4]. Moreover, in this case we will only need to calculate C_S because the final result will not depend on the dimension D but only on the total positive contribution S. The results for each family are summarized in table 1.

Matter	C_S
$2N\,\langle N\rangle$	$\frac{S}{2}+1$
$\langle\frac{N(N+1)}{2}\rangle + \langle\frac{N(N-1)}{2}\rangle$	S
$(N+8)\,\langle N\rangle + \langle\frac{N(N-1)}{2}\rangle$	$\frac{S}{2}+\frac{7}{2}\sqrt{S+16}-14$
$(N-8)\,\langle N\rangle + \langle\frac{N(N+1)}{2}\rangle$	$\frac{S}{2}-\frac{7}{2}\sqrt{S+16}-14$
$16\,\langle N\rangle + 2\,\langle\frac{N(N-1)}{2}\rangle$	$\frac{15}{16}S+1$

Table 1: Infinite families of SU(N) along with the contribution C_S after the coupling to $n\,\langle D\rangle$.

Note that these models can couple to multiple terms whose total contribution is S, and the final contribution will always be that given in the second column, regardless of the way in which the S is split. To see this, let's denote by $S(N), A(N), n(N)$ the contributions of a family and the multiplicity of its fundamental respectively. Additionally, let's assume that the i-th term, with positive contribution S_i and individual negative contribution A_i, couples to a fraction f_i of the $n(N)$. It is then obvious that $S_i = f_i S(N)$ and $A_i = f_i A(N)$, so that the contribution of the final configuration to $H-V$ is

$$C_S = S - A - \sum_i A_i = S - A - \sum_i f_i A(N) = S - A - A(N) \tag{60}$$

In addition, from table 1 we can see that the reduction of the total contribution as a result of the coupling to an infinite family is increasing with the positive contribution S. This means that the minimum contribution is obtained by coupling the infinite model to every other term. Furthermore, the final contribution, i.e. the expression in the second column, is also an increasing function of S, which means that we can't further decrease the contribution of our model by adding extra terms which are coupled to the unbounded rank model.

As we mentioned earlier, in most cases, two models from table 1 cannot coexist. The exception to this is the model of the last row, since two models of this type can coexist in a composite model, though without being coupled to each other. This means that they need to be involved in different terms of the sum (28). As a result, if we have k such models each being involved in one term with contribution S_i, $i = 1, \ldots, k$, so that $S = \sum_{i=1}^{k} S_i$, the total contribution will be $C_S = \sum_{i=1}^{k} (\frac{15}{16} S_i + 1) = \frac{15}{16} S + k \geq \frac{15}{16} S + 1$. In other words, the minimum value is obtained by coupling all the terms to a single model, in a manner similar to what was described above (60).

4.5 Optimal configuration

Let's summarize our results so far. We have calculated:

1. the minimal contribution of each individual term when we fix the representations involved and their multiplicities,
2. the same contribution when we only know the total positive contribution S to $H - V$,
3. the effect of coupling a model containing any combination of the above terms, to a model that belongs to one of the infinite families of table 1.

Now let's assume that we fix one of the group factors and the multiplicities of its representations, i.e. we fix

$$M_0 = G_0 : \sum_r n_r \langle D_r \rangle \tag{61}$$

What is the minimal possible contribution to $H - V$ of a configuration that contains (61)? Using the results we have so far we can answer this question, which will allow us to specify the upper bound M that was mentioned around (23). More specifically, M will be the maximal value of $\sum_r n_r D_r$ for which the minimal contribution is at most $273 - 29T$.

As we argued, we can only have one group factor of unbounded rank, and the minimal contribution is obtained when the term with unbounded multiplicity is coupled to all the other terms. This may contribute up to

$$-A_{\text{unbounded}} = -\frac{1}{2}\sum_r n_r D_r - \frac{7}{2}\sqrt{\sum_r n_r D_r + 16} - 14 \tag{62}$$

There can also be terms that are not coupled to (61). The total contribution of the final model is decreased as a result of these terms, when their individual contributions C_S are negative. As we can see in figure 3, this is the case for a few models with exceptional groups and for the antisymmetric representation. The exceptional group models with negative contributions (blue line in figure 3) are the ones that have $n_0 \leq \lfloor \frac{A}{D_0} \rfloor$, where $\langle D_0 \rangle$ is the representation with the smallest dimension, and $n_i = 0$ for $i \neq 0$. It is easy to check that these cannot coexist, so that we can only pick one of them for each model. The one with the minimal C_S is obviously the model of E_8 without matter content, which has $C_S = -248$. Coincidentally, none of these models can coexist with the antisymmetric representation, as we can check explicitly. As a consequence, if we pick the E_8 contribution, we must also have $-A_{\text{unbounded}} = -\frac{1}{2}\sum_r n_r D_r$, which corresponds to the first row of table 1, instead of (62). The total contribution of the unbounded group factor and of the terms that are not coupled to (61), can be written as

$$-A_{\text{unbounded}} - A_{\text{not coupled}} = -\frac{1}{2}\sum_r n_r D_r - \max\{248, \frac{7}{2}\sqrt{\sum_r n_r D_r + 16} + 14\} \tag{63}$$

Finally, we can have additional terms coupled to (61). These can either be terms from exceptional groups or terms that transform in the fundamental of classical groups. We thus get

$$-A_{\text{coupled}} = -\sum_r \max\{A_{\text{exceptional}}, A_{\text{fundamental}}\} \tag{64}$$

where

$$A_{\text{exceptional}} = 3n_r \tag{65}$$

and

$$A_{\text{fundamental}} = \log_{N_0} n_r \cdot f_r(N_0^2 - 1) \tag{66}$$

with $f_r = \frac{n_r D_r}{2N_0^2}$. For $n_r D_r > 2N_0^2$, it is obvious that one $2N_0 \langle N_0 \rangle$ is not enough to fully couple to the term $n_r \langle D_r \rangle$ and we need to divide the n_r into parts, so that

$$n_r = q\frac{2N_0^2}{D_r} + r \tag{67}$$

where $q = \lfloor \frac{n_r D_r}{2N_0^2} \rfloor$ and $r = n_r - \frac{qD_r}{2N_0^2}$ are respectively the quotient and the remainder of the Euclidean division of n_r by $\frac{2N_0^2}{D_r}$. We then have

$$A_{\text{fundamental}} = q \log_{N_0}(2N_0^2) \cdot (N_0^2 - 1) + \log_{N_0} r \cdot \frac{rD_r}{2N_0^2}(N_0^2 - 1) \tag{68}$$

Putting everything together, we have

$$C_{\min} = \frac{1}{2}\sum_r n_r D_r - A - \sum_r \max\{A_{\text{exceptional}}, A_{\text{fundamental}}\} - \max\{248, \frac{7}{2}\sqrt{\sum_r n_r D_r + 16} + 14\} \tag{69}$$

We can use the above methodology to calculate an upper bound for the rank of the classical groups. As an example, for the family in the fourth row of table 1 we have

$$C_{\min}(N) = (N-8)N - \frac{N(N-1)}{2} + 1 - \max\{A_{\text{exceptional}}, A_{\text{fundamental}}\}|_{n_r = N-8} - 248 \tag{70}$$

Requiring that $C_{\min}(N) \leq 244$, we find $N < 50$. In a similar manner, we can find the maximal rank for the rest of the families.

5 Results

We present here some notable results. The scan took a few hours on a standard CPU, with a total of 1295 models being found. The largest number of groups in a single model was 4, realized by the following 5 models:

- $\mathrm{SU}(10) \times \mathrm{SU}(10) \times \mathrm{SU}(11) \times \mathrm{SU}(16) : 2\,\langle 45, 1, 1, 1\rangle + 2\,\langle 1, 45, 1, 1\rangle + 2\,\langle 1, 1, 55, 1\rangle + \langle 1, 1, 1, 16\rangle + \langle 10, 1, 1, 16\rangle + \langle 1, 10, 1, 16\rangle + \langle 1, 1, 11, 16\rangle$
- $\mathrm{SU}(10) \times \mathrm{SU}(10) \times \mathrm{SU}(10) \times \mathrm{SU}(16) : 2\,\langle 45, 1, 1, 1\rangle + 2\,\langle 1, 45, 1, 1\rangle + 2\,\langle 1, 1, 45, 1\rangle + 2\,\langle 1, 1, 1, 16\rangle + \langle 10, 1, 1, 16\rangle + \langle 1, 10, 1, 16\rangle + \langle 1, 1, 10, 16\rangle$
- $\mathrm{SU}(10) \times \mathrm{SU}(10) \times \mathrm{SU}(12) \times \mathrm{SU}(16) : 2\,\langle 45, 1, 1, 1\rangle + 2\,\langle 1, 45, 1, 1\rangle + 2\,\langle 1, 1, 66, 1\rangle + \langle 10, 1, 1, 16\rangle + \langle 1, 10, 1, 16\rangle + \langle 1, 1, 12, 16\rangle$
- $\mathrm{SU}(10) \times \mathrm{SU}(10) \times \mathrm{SU}(10) \times \mathrm{SU}(15) : \langle 10, 1, 1, 1\rangle + 2\,\langle 45, 1, 1, 1\rangle + \langle 1, 10, 1, 1\rangle + 2\,\langle 1, 45, 1, 1\rangle + \langle 1, 1, 10, 1\rangle + 2\,\langle 1, 1, 45, 1\rangle + \langle 10, 1, 1, 15\rangle + \langle 1, 10, 1, 15\rangle + \langle 1, 1, 10, 15\rangle$
- $\mathrm{SU}(10) \times \mathrm{SU}(11) \times \mathrm{SU}(11) \times \mathrm{SU}(16) : 2\,\langle 45, 1, 1, 1\rangle + 2\,\langle 1, 55, 1, 1\rangle + 2\,\langle 1, 1, 55, 1\rangle + \langle 10, 1, 1, 16\rangle + \langle 1, 11, 1, 16\rangle + \langle 1, 1, 11, 16\rangle$

268 models with 3 group factors were found. The maximum number of massless modes found was 1504, realized by the model

$$\mathrm{SU}(16) \times \mathrm{SU}(32) : 2\,\langle 1, 496\rangle + \langle 16, 32\rangle \tag{71}$$

Finally, the following models which contain no singlets were found:

- $\mathrm{SU}(18)$: $6\,\langle 18\rangle + 3\,\langle 153\rangle$
- $\mathrm{E}_6 \times \mathrm{F}_4$: $10\,\langle 27, 1\rangle + 4\,\langle 1, 26\rangle$
- $\mathrm{E}_6 \times \mathrm{SU}(15)$: $9\,\langle 27, 1\rangle + \langle 78, 1\rangle + 7\,\langle 1, 15\rangle + \langle 1, 120\rangle$
- $\mathrm{E}_7 \times \mathrm{SO}(11)$: $3\,\langle 133, 1\rangle + 3\,\langle 1, 11\rangle$
- $\mathrm{F}_4 \times \mathrm{SU}(11)$: $5\,\langle 26, 1\rangle + 16\,\langle 1, 11\rangle + 2\,\langle 1, 55\rangle$
- $\mathrm{SO}(14) \times \mathrm{SU}(10)$: $6\,\langle 14, 1\rangle + 8\,\langle 1, 10\rangle + 6\,\langle 1, 45\rangle$
- $\mathrm{SO}(14) \times \mathrm{SU}(20)$: $6\,\langle 14, 1\rangle + 4\,\langle 1, 20\rangle + 3\,\langle 1, 190\rangle$
- $\mathrm{SO}(14) \times \mathrm{Sp}(11)$: $6\,\langle 14, 1\rangle + 2\,\langle 1, 22\rangle + 2\,\langle 1, 230\rangle$
- $\mathrm{SO}(15) \times \mathrm{SU}(18)$: $7\,\langle 15, 1\rangle + 6\,\langle 1, 18\rangle + 3\,\langle 1, 153\rangle$
- $\mathrm{SU}(14) \times \mathrm{SU}(16)$: $2\,\langle 91, 1\rangle + 18\,\langle 1, 16\rangle + \langle 14, 16\rangle$
- $\mathrm{SU}(14) \times \mathrm{Sp}(12)$: $6\,\langle 14, 1\rangle + \langle 105, 1\rangle + 2\,\langle 1, 275\rangle$
- $\mathrm{SU}(15) \times \mathrm{SU}(17)$: $9\,\langle 15, 1\rangle + 3\,\langle 105, 1\rangle + 9\,\langle 1, 17\rangle + \langle 1, 153\rangle$

The full list of solutions is provided as an ancillary file.

6 Conclusions

In this thesis we performed an extended search for anomaly-free $\mathcal{N} = 1$ supergravities in $D = 6$ with one tensor multiplet, by leaving the number of group factors constituting the semisimple gauge group as well as the rank of the classical groups unbounded. To do so, we obtained lower bounds on the contribution of a given model to $H - V$, which made the search space finite, and employed computational techniques to ensure the feasibility of the scan.

It would be interesting to further extend these results by relaxing the lower bound on the dimensions of the representations, as well as taking into account additional representations such as the spinorial of $\mathrm{SO}(N)$. Although this would require generalizing the results of section 4, it could be an important step towards the complete enumeration of the 6D supergravity landscape.

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