NATIONAL TECHNICAL UNIVERSITY OF ATHENS

# Analysis, optimal management and metastability of stochastic systems in environmental economics

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*Doctor of Philosophy*

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I dedicate this to my father.

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# Contents

# Chapter 1

# Introduction

The shallow lake problem is a well-known problem of environmental economics with a great mathematical interest. It is observed that the heavy use of fertilizers on the surrounding land and the increased inflow of waste water from industries and human settlements release phosphorus into the lakes and this causes an intense growth of phytoplankton. As a result, the shallow lakes flip from a clear state (oligotrophic state) to a turbid state (euthrophic state) with a greenish look. Limnologists have shown a great interest in this natural phenomenon and have proposed a model to quantify the evolution of the concentration of phosphorus into the lake. More specifically, the amount of phosphorus in algae is usually modeled by the non-linear stochastic differential equation:

$$\begin{cases} dx(t) = (u(t) - bx(t) + r(x(t)))\, dt + \sigma x(t) dW_t, \\ x(0) = x \geq 0. \end{cases} \tag{1.0.1}$$

The first term, $u: [0,\infty) \to (0,\infty)$, in the drift part of the dynamics, represents the exterior load of phosphorus as a result of human activities. The second term is the rate of loss $bx(t)$, which is due to sedimentation, outflow and sequestration in other biomass. The third term, $r(x(t))$, is the rate of recycling of phosphorus on the bed of the lake. This term is assumed to be a sigmoid function (see [15]) and the typical choice in the literature is the function $x \mapsto x^2/(x^2+1)$. An uncertainty in the rate of loss is assumed and is introduced in the model through a linear multiplicative

Gaussian white noise with intensity $\sigma$.

The economics of the lake arise from the conflicting services it offers to the community. On the one hand, a clear lake is a recourse for ecological services. On the other hand, the lake serves as a waste sink for the agricultural and industrial activities. When the users of the lake cooperate, the loading strategy, $u \in \mathfrak{U}_x$ (to be determined in the next section), can be used as a control to maximize the total benefit from the lake. Assuming an infinite horizon, this benefit is typically defined as

$$J(x;u) = \mathbb{E}_x \left[ \int_0^\infty e^{-\rho t} \big( \ln u(t) - cx^2(t) \big) \, dt \right], \tag{1.0.2}$$

where $\rho > 0$ is the discount rate and $x(\cdot)$ is the solution to (1.0.1), for a given exterior loading (control) $u(\cdot)$, and initial state $x(0) = x$. The total benefit of the lake increases with the increase of loading of phosphorus as $\ln u$, but at the same time decreases with the existing amount of phosphorus inside the lake as $-cx^2$, due to the implied decline in quality of its ecological services. The positive parameter $c$ reflects the relative weight of this component.

For the optimal management of the lake, we need to maximize the total benefit over all admissible controls $u \in \mathfrak{U}_x$. The set of admissible control $\mathfrak{U}_x$ will be specified in the next chapter. In this way, the value function of the problem is defined as

$$V(x) = \sup_{u \in \mathfrak{U}_x} J(x;u), \tag{1.0.3}$$

Therefore, the shallow lake problem becomes a problem of control theory or a differential game in the case where we have competitive users of the lake [13, 15, 50]. As a control problem (see [23], Section III.7), the value function $V$ given by (1.0.3) is expected to satisfy the HJB equation

$$\rho V - H(x, V_x) - \frac{1}{2}\sigma^2 x^2 V_{xx} = 0 \tag{1.0.4}$$

where the Hamiltonian $H$ is defined by

$$H(x,p) = \sup_{u>0} \left[ (u - bx + r(x))\, p + \ln u - cx^2 \right]. \tag{1.0.5}$$

Assuming that $V_x < 0$, (1.0.4) reduces to:

$$\rho V - (r(x) - bx)\, V_x + \ln(-V_x) + cx^2 + 1 - \frac{1}{2}\sigma^2 x^2 V_{xx} = 0 \qquad (1.0.6)$$

The shallow lake problem has been extensively studied in the literature, especially its deterministic version. When $\sigma = 0$, the case where the optimally controlled lake has two equilibria and a Skiba point or indifference point [47, 49] is of particular interest. The leftmost (oligotrophic) equilibrium point of the system of the lake corresponds to a lake with low concentration of phosphorus, while the rightmost one (eutrophic) corresponds to a lake with high concentration of phosphorus. At the Skiba point, there are two different optimal strategies, each one driving the system to a different equilibrium and the value function is not differentiable thereat. Therefore, the value function, $V$, cannot be a classical solution to the HJB eq. (1.0.4). Actually, the correct mathematical framework to work with, especially when the value function does not a priori possess the regularity of a classical solution, is that of viscosity solutions, as it was developed by Crandall and Lions [17]. The connection of control theory problems with HJB equations has been extensively studied, see e.g [3, 23, 22, 37, 38].

In order to uncover the range of parameters for which Skiba points appear, an extensive exploration of the parameter space and the qualitative differences of the Pontryagin system of the shallow lake (bifurcation analysis) has been conducted [31, 49]. Properties of the value function of the deterministic shallow lake problem have been proved in [32].

A basic question that, to the best of our knowledge, has not been completely answered so far is that of the existence of optimal control. The existence of optimal control is usually taken as a hypothesis and the optimal dynamics of the lake is studied mostly through the necessary conditions, which are determined by the Pontryagin Maximum Principle, and the equilibrium points of the corresponding dynamical system [49, 50]. A rigorous answer to this question was given by Bartaloni in [7, 8], albeit under restrictions that do not fully cover the range of the parameters for which Skiba points are present.

In addition, there has been increasing interest recently in the stochastic version of the problem ($\sigma \neq 0$). Specifically, deterministic systems with two equilibrium points and one Skiba point have a fundamentally different behaviour from their stochastic counterparts. In particular, in the presence of noise, random fluctuations lead the system from the one equilibrium point to the other one (metastability). In the case of a shallow lake, variations in the rate of loss drive the lake from the oligotrophic to the eutrophic state and vice versa, a phenomenon which is naturally observed. The interest in the study of metastable systems firstly arose from phenomena in the field of chemistry. Arrhenius [2] in 1889 physically justified an expression for the mean transition time of the system to go from the one local equilibrium to the other. Later, H. Eyring and H. A. Kramers [20, 35] with the well-known Eyring-Kramers law refined the Arrhenius law by specifying the prefactor term in Arrhenius' expression. In the sequel, M.I.Freindlin and A.D.Wentzell [24] introduced the theory of large deviations to explain and understand the metastable behaviour of various dynamical systems. Even though, metastable systems have been extensively studied ever since (see e.g. [12, 9]), the majority of results concern dynamical systems for which the drift function is not a function of the noise intensity. However, in the context of (stochastic) control theory, one naturally expects that in metastable systems, like the shallow lake system in the case of Skiba points, the drift term of the optimally controlled system will depend on the noise via the presence of the value function, which in turn depends on the noise via the Hamilton-Jacobi-Bellman correction. The phenomenon of metastability in the shallow lake problem is studied numerically in [26], where the value function of the shallow lake problem is approximated for small $\sigma$, based on heuristic methods of perturbation analysis.

Furthermore, thorough examination of the stochastic version of the shallow lake problem is conducted by Kossioris, Loulakis and Souganidis [33] who analytically derive properties of the value function and characterise it as the unique (in a suitable class) state-constraint viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation (1.0.4).

The shallow lake problem has some nonstandard features and, hence, it requires some special analysis. First of all, the problem is formulated as a state constraint one on a semi-infinite domain. Viscosity solutions with state constraint boundary conditions were introduced for first order equations by [48, 14]. For second order equations one should consult [30, 36, 1]. In addition, a priori knowledge of the properties of the solution is necessary to guarantee that the Hamiltonian is well-defined, due to the logarithmic term in the cost functional, which leads to a logarithm of the derivative of the value function in (1.0.6). Then, in the case of the stochastic shallow lake problem, ellipticity of (1.0.6) degenerates at the boundary, $x = 0$. Finally, the control space is open and unbounded, so the usual assumptions made to prove existence in control problems with infinite horizon (see e.g. [10, 19, 44]) are not satisfied here.

The first main contribution of this work is the rigorous proof of existence of optimal control in both the stochastic and deterministic shallow lake problem without any restrictions in the parameter space. In the presence of noise, the proof follows the general lines of a verification principle (see e.g. [23]) with appropriate modifications to address the loss of ellipticity at the boundary and a possible blow-up of the benefit for small controls. This approach is not always feasible in the deterministic problem, since the value function may fail to be differentiable. In [7] and [8] the existence of optimal control is established by proving uniform localization lemmas followed by diagonal arguments. This approach is successfully carried out under the assumption that either the parameter $b$ or the discount parameter $\rho$ are greater than $3\sqrt{3}/8$. Our approach here is entirely different and does not require any restrictions on the parameter space. More specifically, we prove that both the value function and the total benefit achieved when the system is driven by the candidate optimal control suggested by the Pontryagin Maximum Principle are viscosity solutions to the same well-posed problem. In this way, it is proved that the optimal total benefit, that is the value function $V$, is attained by an admissible control and in this way this control is characterised as optimal.

The second main contribution of this work is the analysis of the metastable behaviour of the shallow lake problem, carried out in the more general framework of stochastic control problems which exhibit Skiba points. In more detail, we study the expected value of the transition time from the one well to the other for a process in a noise-dependent double-well potential and we prove a generalization of the Arrhenius law. To do this, we firstly exploit the fact that the mean transition time solves a Poisson problem, the solution of which is given in an explicit integral form in the one-dimensional case. In the following, we prove the locally uniform convergence of the noisy integrand to a noiseless one, by proving the convergence of the derivatives of the noisy value function to the derivatives of the noiseless value function. To prove this, we adapt a methodology introduced by Fleming and Souganidis in [21], which is based on a semiconvexity argument.

This Thesis is organised as follows. Chapter 2 is preliminary and contains some necessary definitions and results that will be useful in the following. In Chapter 3, we generalize the results in [33] to include sigmoid recycling rates that are more general than the standard choice, $x^2/(x^2+1)$, as well as the penalty parameter, $c$, which cannot be scaled away with a suitable change of variables. In Chapter 4, we prove existence of optimal control in both the deterministic and stochastic case. In Chapter 5, we study the asymptotic behaviour of the value function, $V$, at $+\infty$ as well as the tails of the invariant density of the optimally controlled process, $x^*(t)$, that is the optimally controlled concentration of phosphorus into the lake. In Chapter 6, we implement a numerical scheme constructed based on Barles and Souganidis scheme [4] and study numerically the paths and the properties of the optimally controlled lake. Finally, in Chapter 7, we study the metastable behaviour of stochastic control problems which exhibit Skiba points and prove the generalization of Arrhenius law for the case of noise-dependent double-well potential.

# Chapter 2

# Preliminaries

In this Chapter, we present the main notation and definitions we will use in the following.

## 2.1 Viscosity Solutions

Following the notation and definitions in [16], we consider ordinary differential equations of the form $F(x, u(x), Du(x), D^2u(x)) = 0$, where $F : \mathcal{O} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Here, $u$ is a real-valued function defined in a subset $\mathcal{O}$ of $\mathbb{R}$ and $Du$, $D^2u$ correspond to the first and second derivative of $u$, respectively. We require $F$ to be a continuous function which satisfies the following monotonicity conditions:

$$F(x, r, p, X) \leq F(x, s, p, X) \text{ whenever } r \leq s \tag{2.1.1}$$

$$F(x, r, p, X) \leq F(x, r, p, Y) \text{ whenever } X \geq Y \tag{2.1.2}$$

A function $F$ which satisfies condition 2.1.2 is called *degenerate elliptic* and a function $F$ which satisfies both conditions 2.1.1 and 2.1.2 is called *proper*.

We now proceed with the definition of viscosity solution.

**Definition 2.1.1.** *Let $u : O \to \mathbb{R}$ be continuous function.*

*(i) $u$ is a viscosity subsolution of $F = 0$ on $\mathcal{O}$ if*

$$F(x, u(x), D\phi(x), D^2\phi(x)) \leq 0,$$

*for all* $x \in \mathcal{O}$ *and for all* $\phi = \phi_x \in C^2(\mathcal{O})$ *such that* $x$ *is a maximum point of* $u - \phi$*.*

*(ii)* $u$ *is a viscosity supersolution of* $F = 0$ *on* $\mathcal{O}$ *if*

$$F(x, u(x), D\phi(x), D^2\phi(x)) \geq 0,$$

*for all* $x \in \mathcal{O}$ *and for all* $\phi = \phi_x \in C^2(\mathcal{O})$ *such that* $x$ *is a minimum point of* $u - \phi$*.*

*(iii) We say that* $u$ *is a viscosity solution of* $F = 0$ *on* $\mathcal{O}$ *if it is both a subsolution and supersolution of* $F = 0$*.*

In the following, we will give an equivalent definition of the viscosity solutions, based on the notion of second-order semijet of the function $u : \mathcal{O} \to \mathbb{R}$.

**Definition 2.1.2.** *If* $u : \mathcal{O} \to \mathbb{R}$, $\hat{x} \in \mathcal{O}$

*(i) we say that* $(p, X) \in J_{\mathcal{O}}^{2,+}u(\hat{x})$ *(the second-order "superjet" of* $u$ *at* $x$*) if*

$$u(x) \leq u(\hat{x}) + p(x - \hat{x}) + \frac{1}{2}X(x - \hat{x})^2 + o(|x - \hat{x}|^2) \text{ as } \mathcal{O} \ni x \to \hat{x} \tag{2.1.3}$$

*(ii) we say that* $(p, X) \in J_{\mathcal{O}}^{2,-}u(\hat{x})$ *(the second-order "subjet" of* $u$ *at* $x$*) if*

$$u(x) \geq u(\hat{x}) + p(x - \hat{x}) + \frac{1}{2}X(x - \hat{x})^2 + o(|x - \hat{x}|^2) \text{ as } \mathcal{O} \ni x \to \hat{x} \tag{2.1.4}$$

Based on the above definition, we have the following equivalent definition of viscosity solutions.

**Definition 2.1.3.** *Let* $u : O \to \mathbb{R}$ *be continuous function.*

*(i)* $u$ *is a viscosity subsolution of* $F = 0$ *on* $\mathcal{O}$ *if*

$$F(x, u(x), p, X) \leq 0 \text{ for all } x \in \mathcal{O} \text{ and } (p, X) \in J_{\mathcal{O}}^{2,+}u(x).$$

*(ii)* $u$ *is a viscosity supersolution of* $F = 0$ *on* $\mathcal{O}$ *if*

$$F(x, u(x), p, X) \geq 0 \text{ for all } x \in \mathcal{O} \text{ and } (p, X) \in J_{\mathcal{O}}^{2,-}u(x).$$

*(iii) We say that $u$ is a viscosity solution of $F = 0$ on $\mathcal{O}$ if it is both a subsolution and supersolution of $F = 0$.*

**Remark 2.1.1.** *For first-order differential equations of the form*

$$F(x, u(x), Du(x)) = 0,$$

*we define the viscosity solutions accordingly. In this case, the first-order "superjet" and "subjet", denoted by $D^+u(x)$ and $D^-u(x)$, are defined as in Definition 2.1.3 considering a first-order approximation of $u$ and are called the superdifferential and subdifferential of $u$ at $x$ (see e.g. [29], page 139).*

Next, we recall the notion of constrained viscosity solutions (see e.g. [48], [14], [30], [51]).

**Definition 2.1.4.** *Let $u : O \to \mathbb{R}$ be continuous function. We say that $u$ is a constrained viscosity solution to $F = 0$ if*

*(i) $u$ is a viscosity subsolution of $F = 0$ on $\bar{\mathcal{O}}$ and*

*(ii) $u$ is a viscosity supersolution of $F = 0$ on $\mathcal{O}$*

## 2.2 Pontryagin Maximum Principle and Skiba points

In this section, we will state the Pontryagin Maximum Principle (PMP) which provides a general set of necessary conditions for existence of optimal control in a control theory problem. In this direction, we will firstly present the general setting of a maximization problem in an infinite time horizon in order to comply with the setting of the shallow lake problem.

We consider the (controlled) ordinary differential equation:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \quad t \geq 0 \\ x(0) = x_0 \end{cases} \tag{2.2.1}$$

where $u \in \mathcal{U}$, the set of admissible controls, and the cost functional:

$$J(x; u) = \int_0^\infty e^{-\rho t} g(x(t), u(t)) dt$$

where $x(t)$ is the solution to (2.2.1) with control $u$, which starts at $x(0) = x$. Control problem:

$$\text{maximise } J(x;u) \text{ over all } u \in \mathcal{U}$$

If there exists a control $u^* \in \mathcal{U}$ that maximizes the cost functional $J$, then the control $u^*$ is called *optimal*.

If $u^*$ is an optimal control and $x^*$ is the associated optimal trajectory, then Pontryagin's necessary conditions state that there exists a function $p^*$, called the co-state, such that if

$$\bar{H}(x,u,p) = g(x,u) + pf(x,u) \tag{2.2.2}$$

then

1. $x^*, p^*$ are solutions to the system:

$$\begin{cases} \dot{x} = \frac{\theta \bar{H}}{\theta p} \\ \dot{p} = -\frac{\theta \bar{H}}{\theta x} + \rho p \end{cases} \tag{2.2.3}$$

2. $u^*(t)$ maximizes the Hamiltonian $\bar{H}$ i.e.

$$\max_u \bar{H}(x^*(t), u, p^*(t)) = \bar{H}(x^*(t), u^*(t), p^*(t)) \tag{2.2.4}$$

3. $p^*(t)$ satisfies the transversality condition

$$\lim_{t \to \infty} e^{-\rho t} p^*(t) = 0 \tag{2.2.5}$$

Following [49] and assuming that $f_u \neq 0$ and $H_{uu} < 0$, it is proved that relation (2.2.4) implies a one-to-one correspondence between $(x,p)$ and $(x,u)$ representations of an optimal trajectory. Therefore, one can derive the system (2.2.3) of the problem expressed in the state-control space. Then, according to PMP, one should look for candidate optimal trajectories among the phase curves of this system which satisfy also the transversality condition (2.2.5). In the one-dimensional case with infinite time horizon, the admissible curves for optimality are usually situated on the stable manifolds of a steady state of the state-control system (see e.g. [31]).

When for an initial state $x_0$, there exist two candidate optimal trajectories $(x_1(\cdot), u_1(\cdot))$ and $(x_2(\cdot), u_2(\cdot))$ with $J(x_0; u_1) = J(x_0; u_2)$, the point $x_0$ is called an indifference point or a Skiba point (see [47]). For a proof of the existence of such points see e.g. [49] and in the case of the shallow lake problem see e.g. the appendix of [50].

### 2.2.1 Application to the shallow lake problem

if $u^* : [0, \infty) \to (0, \infty)$ is an optimal control and $x^*(t)$ is the associated optimal trajectory, then there exists a function $p^*(t)$ such that if

$$\bar{H}(x, u, p) = \ln u - cx^2 + p\,(u - bx + r(x)) \tag{2.2.6}$$

then

1. $x^*, p^*$ are solutions of the system:

$$\begin{cases} \dot{x} = \frac{\theta \bar{H}}{\theta p} = u - bx + r(x) \\ \dot{p} = -\frac{\theta \bar{H}}{\theta x} + \rho p = (\rho + b - r'(x))\,p + 2cx \end{cases} \tag{2.2.7}$$

2. $u^*(t)$ maximizes the Hamiltonian $\bar{H}$ i.e.

$$\max_u \bar{H}(x^*(t), u, p^*(t)) = \bar{H}(x^*(t), u^*(t), p^*(t)) \;\Rightarrow\; u^*(t) = -\frac{1}{p^*(t)} \tag{2.2.8}$$

3. $p^*(t)$ satisfies the transversality condition

$$\lim_{t\to\infty} e^{-\rho t} p^*(t) = 0 \tag{2.2.9}$$

Due to relation (2.2.8), there is a one-to-one correspondence between the control $u^*$ and the co-state $p^*$ and the system (2.2.7) can be rewritten in the state-control form:

$$\begin{cases} \dot{x} = u - bx + r(x) =: f(u, x) \\ \dot{u} = -\,(\rho + b - r'(x))\,u + 2cxu^2 =: g(u, x) = 2cxu\,(u - g_1(x)) \end{cases} \tag{2.2.10}$$

The autonomous system (2.2.10) is called the *shallow lake system* and its phase curves correspond to potential optimal trajectories of the shallow lake problem. This system may either have one or multiple equilibria (see [49]). In the former case, the equilibrium is a saddle point, while in the latter there are always two saddle points. The leftmost one is characterised as the *oligotrophic* steady state of the lake and the rightmost one is called the *eutrophic* steady state. In the Appendix of [49], it is proved that the only admissible solution curves for optimality are on the stable manifolds of the saddle points and three different cases are distinguished.

- The lake moves towards the oligotrophic steady state regardless of its initial pollution level, $x_0$.
- The lake moves towards the eutrophic steady state regardless of its initial pollution level, $x_0$.
- There exists a threshold value, $x_*$, of the initial pollution level: if $x_0 < x_*$, then the lake moves towards the oligotrophic steady state, whereas if $x_0 > x_*$, the lake moves towards the eutrophic steady state. The point $x_*$ can either be a repeller or an indifference point (Skiba point). When $x_*$ is a repeller, it is itself a steady state and the resulting policy is everywhere single-valued. On the other hand, indifference points are initial states for which there are two distinct controls corresponding to the same total benefit. One of these controls leads to the oligotrophic steady-state while the other one leads to the eutrophic steady-state. In this case, the resulting policy is everywhere single-valued except for the indifference point, at which it may take two values, see Figure 2.1.

Based on this analysis, the candidate optimal path $(x^*, u^*)$ suggests a candidate total benefit, called $J_P$, which, following [32], is constructed as follows:

Let $(x_0, u_0)$ be a saddle point of (2.2.10). By definition of the saddle points, following [32], the total benefit $J_P$, which corresponds to system (2.2.10) computed at $x_0$ is given by:
$J_P(x_0) = \int\limits_0^\infty e^{-\rho t} \left(\ln u(t) - cx^2(t)\right) dt = \int\limits_0^\infty e^{-\rho t} \left(\ln u_0 - cx_0^2\right) dt = \frac{\ln u_0 - cx_0^2}{\rho}$

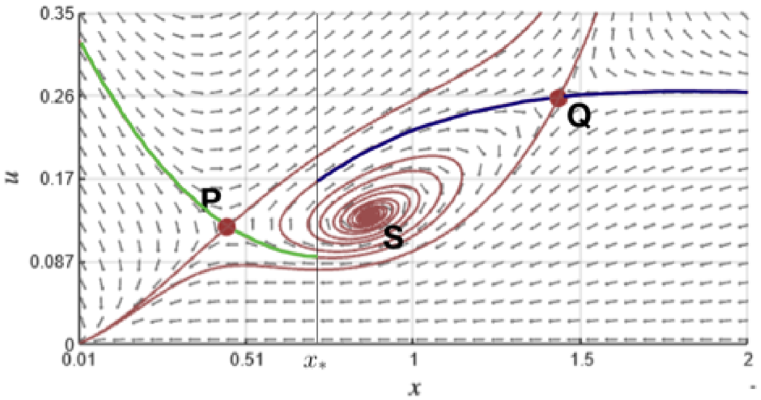



Figure 2.1: Part of the phase plane of the system (2.2.10). The points P, Q are the saddle steady-states, the point S is a vortex and $x_*$ is the Skiba point. The green and the blue curve form the optimal solution, which is everywhere single-valued except for the Skiba point whereat the optimal control may take two values.

Then the total benefit at any point x can be found via the stable manifold of the corresponding saddle point (see the three cases above) through integration, as follows:

$$J_P(x) = J_P(x_0) + \int_{x_0}^{x} \frac{dJ_P}{dk}(k)dk = J_P(x_0) + \int_{x_0}^{x} p(k)dk = J_P(x_0) - \int_{x_0}^{x} \frac{1}{u(k)}dk \tag{2.2.11}$$

For the second equality, we used that $\dfrac{dJ_P}{dx} = p$ along the trajectories of (2.2.10) (for a proof, see the Appendix of [50]), while for the third one, we used (2.2.8). In Chapter 4, the function $J_P$ will serve as a *candidate value function*, and this is how we will refer to it.

# Chapter 3

# Regularity Estimates

The stochastic version of the shallow lake problem was thoroughly studied by Kossioris, Loulakis and Souganidis [33]. In their paper, they analytically derived properties of the value function and characterised it as the unique (in a suitable class) state-constraint viscosity solution of a Hamilton-Jacobi-Bellman (HJB) equation. In this section, we review their results [33] and we extend them to include 1) sigmoid recycling rates that are more general than the standard choice $x^2/(x^2+1)$ and 2) all positive values of the weight parameter $c$. Notice that some of the proofs are simple modifications of the ones presented in [33], but we include them here for completeness purposes. Results that do not demand any adaptation will be recorded here without their proofs.

## 3.1 The value function and the HJB

We assume that there exists a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions, and a Brownian motion $W = \{W_t,\ t \geq 0\}$ defined on that space. An admissible control $u(\cdot) \in \mathfrak{U}_x$ is an $\mathcal{F}_t$-adapted, $\mathbb{P}$-a.s. locally integrable process with values in $U = (0, \infty)$, satisfying

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \ln u(t) dt\right] < \infty, \tag{3.1.1}$$

such that the problem (1.0.1) has a unique strong solution $x(\cdot)$. Furthermore, throughout the paper, the recycling rate function $r$ is a sigmoid function satisfying Assumption 1.

**Assumption 1.** *The rate of recycling $r(x)$ satisfies the following:*

1. $r \in C^1([0,\infty))$ *and nondecreasing*
2. $r(0) = 0$ *and* $r(x) \leq (b+\rho)x$ *close to* $0$.
3. $a := \lim_{x\to\infty} r(x) < \infty$
4. *The limit* $\lim_{x\to\infty}(a - r(x))x =: C$ *exists and is a finite, necessarily nonnegative, real number.*
5. $\lim_{x\to\infty} r'(x) = 0$.

One main difficulty in the study of this problem is related to the fact that the control functions $u$ take values in the open unbounded set $(0,\infty)$ so that supremum in (1.0.5) might take infinite values. Indeed, when $U = (0,\infty)$, we find

$$H(x,p) = \begin{cases} (r(x) - bx)\,p - (\ln(-p) + cx^2 + 1) & if\ p < 0, \\ +\infty & if\ p \geq 0. \end{cases} \tag{3.1.2}$$

One naturally expects that since the shallow lake looses its value with a higher concentration of phosphorus, the value function is a decreasing function of the initial state of phosphorus. Assuming that $V_x < 0$, (1.0.4) becomes (1.0.6).

Since the problem is set on $(0,\infty)$, it is necessary to introduce boundary conditions to guarantee the well-posedness of the corresponding boundary value problem. Given the possible degeneracies of Hamilton-Jacobi-Bellman equations at $x = 0$, the right framework is that of continuous viscosity solutions in which boundary conditions are considered in the weak sense. Since at the boundary point $x = 0$

$$\inf_{u\in U}\{-u + bx - r(x)\} < 0, \tag{3.1.3}$$

that is, there always exists a control that can drive the system inside $(0,\infty)$, the problem should be considered as a state constraint one on the interval

$[0, \infty)$, meaning that $V$ is a subsolution in $[0, \infty)$ and a supersolution in $(0, \infty)$.

Now we present the main results of the Chapter and their proofs are given in section 3.4.

Theorem 1 characterises the value function of the stochastic shallow lake problem as a state-constraint viscosity solution to the Hamilton-Jacobi-Bellman (HJB) equation (1.0.4)

**Theorem 1.** *If $0 < \sigma^2 < \rho + 2b$, the value function $V$ is a continuous constrained viscosity solution to equation* (1.0.4) *in* $[0, \infty)$.

In particular, the value function $V$ is characterised as the unique constrained viscosity solution of (1.0.6) because of the following comparison principle (Theorem 2). In section 3.3, it is proved that $V$ satisfies the conditions of Theorem 2.

**Theorem 2.** *If $0 < \sigma^2 < \rho + 2b$, assume that $u \in C([0, \infty))$ is a strictly decreasing subsolution to* (1.0.4) *in $[0, \infty)$ and $v \in C([0, \infty))$ is a strictly decreasing supersolution to* (1.0.4) *in $(0, \infty)$ such that $v \geq -c_1(1 + x^q)$, where $q$ can be any real number strictly smaller than $|k(\sigma)|$, where $k(\sigma)$ is the negative root of* (3.4.5) *and $Du \leq -\frac{1}{c_2}$ in the viscosity sense, for $c_1$, $c_2$ positive constants. Then $u \leq v$ in $[0, \infty)$.*

**Remark 3.1.1.** *Theorem 2 was stated in [33] (see Theorem 2.2) with the parameter $q$ being equal to 2. Actually, it follows from* (3.4.5) *that when $\sigma^2 \in [0, \rho + 2b)$, we have $|k(\sigma)| > 2$.*

Notice that both of these results were proven in [33] only for the stochastic case ($\sigma > 0$.)
The proofs of Theorems 1-2 are presented in detail in section 3.4.

## 3.2 Properties of the dynamics

In order to prove Theorem 1, we need to establish first some key properties of the dynamics of the lake and of the value function of the problem. In this section, we state and prove some basic properties which refer to the lake's dynamics. In more detail, it is proved that i) the concentration of phosphorus, $x_t$, remains non-negative when the process starts from a non-negative quantity, $x$, ii) the set of admissible controls is independent of the initial state, $x$, and iii) the higher the loading of phosphorus, $u$, the higher the resulting concentration of phosphorus, $x(\cdot)$. These results apply to both the stochastic and deterministic case.

Let

$$Z_t = e^{\sigma W_t - \left(b+\sigma^2/2\right)t} \quad \text{and} \quad M_t(u) = \int\limits_0^t \frac{Z_t}{Z_s} u(s)ds \tag{3.2.1}$$

**Proposition 3.2.1.**

(i) *If $x \geq 0$, $u \in \mathfrak{U}_x$, and $x(\cdot)$ is the solution to (1.0.1), then $\mathbb{P}\big[x(t) \geq 0,\ \forall t \geq 0\big] = 1$. In particular, $\mathbb{P}\big[x(t) \geq M_t(u),\ \forall t \geq 0\big] = 1$.*

(ii) *For all $x, y \geq 0$, $\mathfrak{U}_x = \mathfrak{U}_y =: \mathfrak{U}$.*

(iii) *Suppose $x_1(\cdot)$, $x_2(\cdot)$ satisfy (1.0.1) with controls $u_1$, $u_2 \in \mathfrak{U}$, respectively, and $x_1(0) = x_1$, $x_2(0) = x_2$. If $x_1 \leq x_2$ and $\mathbb{P}\big[u_1(t) \leq u_2(t),\ \forall t \geq 0\big] = 1$, then*

$$\mathbb{P}\big[x_2(t) - x_1(t) \geq (x_2 - x_1)Z_t,\ \forall t \geq 0\big] = 1.$$

*Proof.* (i) Let $u \in \mathfrak{U}_x$ be an arbitrary admissible control. We define $y(t) = e^{-\sigma W(t)}x(t)$ and apply Itô's rule to get

$$\begin{aligned} dy(t) &= e^{-\sigma W(t)}dx(t) - \sigma y(t)dW(t) + \frac{\sigma^2}{2}y(t)dt + d\langle e^{-\sigma W(\cdot)}, x(\cdot)\rangle_t \\ &= \left\{ e^{-\sigma W(t)}\big(u(t) + r\left(x(t)\right)\big) - (b + \frac{\sigma^2}{2})y(t) \right\} \ dt, \end{aligned}$$

that is $y(\cdot)$ satisfies a regular ODE with random coefficients. By variation of parameters we can get the following pathwise implicit representation for $x(\cdot)$

$$x(t) = xZ_t + \int_0^t \frac{Z_t}{Z_s}\left(u_s + r(x_s)\right)ds. \tag{3.2.2}$$

The claim is now obvious, since $u$ and $r$ are non-negative. □

(ii) Fix $u \in \mathfrak{U}_x$ and $x \in [0, \infty)$ and let $x(\cdot)$ the unique strong solution to (1.0.1) with $x(0) = x$, and, for any $y \geq 0$, consider the sde

$$\begin{cases} dw(t) = \Big\{ - bw(t) - r\left(x(t) - w(t)\right) + r\left(x(t)\right) \Big\} dt + \sigma w(t) dW(t) \\ w(0) = x - y, \end{cases} \tag{3.2.3}$$

and note that, based on the assumptions made on $r$, the coefficients are Lipschitz and grow at most linearly. Therefore, the eq. (3.2.3) has a unique strong solution defined for all $t \geq 0$. It is easy to see now that the process $y(t) = x(t) - w(t)$ satisfies (1.0.1) with $y(0) = y$. Moreover, the uniqueness of $y(\cdot)$ follows from that of $x(\cdot)$, so $u \in \mathfrak{U}_y$.
□

(iii) Since the processes $x_1(\cdot)$, $x_2(\cdot)$ are the unique strong solutions to (1.0.1), the conditions $x_1 \leq x_2$ and $\mathbb{P}\big[u_1(t) \leq u_2(t),\ \forall t \geq 0\big] = 1$ imply that $\mathbb{P}\big[x_1(t) \leq x_2(t),\ \forall t \geq 0\big] = 1$ based on the Comparison Theorem of Ikeda and Watanabe [27]. Furthermore, based on (3.2.2), the process $w(t) = x_2(t) - x_1(t)$ satisfies

$$w(t) = (x_2 - x_1) Z_t + \int_0^t \frac{Z_t}{Z_s} \big(u_2(s) - u_1(s) + r(x_2(s)) - r(x_1(s))\big)\, ds.$$

The claim is now immediate, since $r$ is nondecreasing.

□

Proposition 3.2.1(ii) indicates that the set of admissible controls is independent of the initial state $x$. Therefore, in the following, we will denote the set of admissible controls by $\mathfrak{U}$, regardless of the starting point $x$ in (1.0.1).

## 3.3 Properties of the value function

Propositions 3.3.1, 3.3.2, 3.3.3 refer to properties of the value function $V$ in (1.0.3). Note that these properties are derived directly from the definition of $V$ in (1.0.3), so they are not a consequence of any differential equation, such as (1.0.6), that $V$ may satisfy. On the contrary, they are used as a crucial input in the characterization of the value function as a constrained viscosity solution to (1.0.4), as they ensure that the associated Hamiltonian of the control problem is finite, and they outline a class of functions among which there is uniqueness of solutions to (1.0.4). Notice also that the results of Proposition 3.3.3 assume that $\sigma > 0$, while the other two also apply to the deterministic case.

**Remark 3.3.1.** *In the following, we will assume that*

$$\sigma^2 < \rho + 2b,$$

*because otherwise the value function $V$ is not finite. Indeed, if we consider an admissible control $u \in \mathfrak{U}$ and $x(\cdot)$ the unique strong solution to (1.0.1) with $x(0) = x$, from Proposition 3.2.1(i), we have that*

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} x^2(t) dt\right] \geq \mathbb{E}\left[\int_0^\infty e^{-\rho t} M_t^2(u) dt\right].$$

*After simple computations (see Appendix Lemma A.1(iii) ), it follows that if $\sigma^2 \geq \rho + 2b$, $\mathbb{E}\left[\int_0^\infty e^{-\rho t} M_t^2(u) dt\right] = \infty$ and so $V \equiv \infty$.*

Let

$$A = \frac{c}{\rho + 2b - \sigma^2} \tag{3.3.1}$$

**Proposition 3.3.1.** *Suppose* $0 \leq \sigma^2 < \rho + 2b$

(i) *The function $x \mapsto V(x) + Ax^2$, where $A$ is defined in* (3.3.1), *is decreasing on $[0, +\infty)$.*

(ii) *The value function at zero satisfies $V(0) \leq \frac{1}{\rho} \ln\left(\frac{b+\rho}{\sqrt{2ec}}\right)$.*

(iii) *Fix $x_1, x_2 \in [0, \infty)$ with $x_1 < x_2$, and, for $u \in \mathfrak{U}$, let $x(\cdot)$ be the solution to (1.0.1) with control $u$ and $x(0) = x_1$. If $\tau_u$ is the hitting time of $x(\cdot)$ on $[x_2, +\infty)$, that is, $\tau_u = \inf\{t \geq 0 : x(t) \geq x_2\}$, then*

$$V(x_1) = \sup_{u \in \mathfrak{U}} \mathbb{E}\left[\int_0^{\tau_u} e^{-\rho t}\big(\ln u(t) - cx^2(t)\big)\, dt + e^{-\rho\tau_u} V(x_2)\right]. \tag{3.3.2}$$

*Proof.* (i) Fix $x_1, x_2 \geq 0$ with $x_1 \leq x_2$. It suffices to show that, for $J$ as in (1.0.2) and for any control $u \in \mathfrak{U}$,

$$J(x_2; u) + Ax_2^2 \leq J(x_1; u) + Ax_1^2,$$

Since this holds trivially if $J(x_2; u) = -\infty$, we may assume that $J(x_2; u) > -\infty$.

Consider now the solutions $x_1(\cdot), x_2(\cdot)$ to (1.0.1) with initial conditions $x_1, x_2$ and a common control $u \in \mathfrak{U}$. Relation (3.2.2) and Proposition 3.2.1(iii) implies that, $\mathbb{P}$-a.s. and for all $t \geq 0$,

$$x_1(t) + x_2(t) \geq (x_1 + x_2)Z_t, \qquad \text{and} \qquad x_2(t) - x_1(t) \geq (x_2 - x_1)Z_t.$$

Note that since $u \in \mathfrak{U}$ and $J(x_2; u) > -\infty$,

$$\int_0^\infty e^{-\rho t} x_2^2(t)\, dt < +\infty \Rightarrow \int_0^\infty e^{-\rho t} x_1^2(t)\, dt < +\infty$$

$$\Rightarrow J(x_1; u) > -\infty.$$

In particular,

$$\begin{aligned} J(x_2; u) - J(x_1; u) &= \mathbb{E}\Bigg[\int_0^\infty e^{-\rho t}\big(\ln u(t) - c x_2^2(t)\big) dt \\ &\qquad - \int_0^\infty e^{-\rho t}\big(\ln u(t) - c x_1^2(t)\big) dt\Bigg] \\ &= -\mathbb{E}\left[\int_0^\infty e^{-\rho t} c\big(x_2(t) - x_1(t)\big)\big(x_2(t) + x_1(t)\big) dt\right] \\ &\leq -c(x_2^2 - x_1^2)\int_0^\infty e^{-\rho t}\mathbb{E}\big[Z_t^2\big]\, dt = -A(x_2^2 - x_1^2). \end{aligned}$$

□

(ii) Using Prop. 3.2.1(iii), Jensen's inequality and part (i) of Lemma A.1, we find

$$\frac{lnc}{2\rho} + \mathbb{E}\left[\int_0^\infty e^{-\rho t} \ln u(t)\ dt\right] \le \frac{1}{\rho} \ln \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} \sqrt{c} u(t)\ dt\right]$$
$$= \frac{1}{\rho} \ln \mathbb{E}\left[\int_0^\infty \rho(\rho+b) e^{-\rho t} \sqrt{c} M_t(u)\ dt\right]$$
$$\le \frac{\ln(b+\rho)}{\rho} + \frac{1}{\rho} \ln \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} \sqrt{c} x(t)\ dt\right]$$
$$\le \frac{\ln(b+\rho)}{\rho} + \frac{1}{2\rho} \ln \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} c x^2(t)\ dt\right].$$

In view of (3.1.1), we need only consider $u \in \mathfrak{U}$ such that

$$D := \mathbb{E}\left[\int_0^\infty e^{-\rho t} c x^2(t)\ dt\right] < \infty.$$

Then

$$\mathbb{E}\left\{\int_0^\infty e^{-\rho t}\big[\ln u(t) - cx^2(t)\big] dt\right\} \le \frac{\ln(b+\rho)}{\rho} + \frac{\ln(\rho D)}{2\rho} - D - \frac{lnc}{2\rho}$$
$$\le \frac{1}{\rho} \ln\left(\frac{b+\rho}{\sqrt{2ec}}\right),$$

and the assertion holds. □

(iii) We have

$$J(x_1; u) = \mathbb{E}\left[\int_0^{\tau_u} e^{-\rho t}\big(\ln u(t) - cx^2(t)\big)\ dt\right]$$
$$+ \mathbb{E}\left[\int_{\tau_u}^\infty e^{-\rho t}\big(\ln u(t) - cx^2(t)\big)\ dt;\ \tau_u < +\infty\right].$$

Conditioning on the $\sigma$-field $\mathcal{F}_{\tau_u}$, and applying the strong Markov property, the rightmost term becomes

$$\mathbb{E}\left[e^{-\rho\tau_u}\mathbb{E}\left[\int_{\tau_u}^\infty e^{-\rho(t-\tau_u)}\big(\ln u(t) - cx^2(t)\big)\ dt\ \Big|\mathcal{F}_{\tau_u}\right];\ \tau_u < +\infty\right]$$
$$\le \mathbb{E}\left[e^{-\rho\tau_u}\right] V(x_2),$$

since on the event $\{\tau_u < +\infty\}$, $x(\tau_u + \cdot)$ satisfies (1.0.1) with initial condition $x(\tau_u) = x_2$ and control $u(\tau_u + \cdot)$. Taking the supremum over $u \in \mathfrak{U}$ we see that the left hand side of (3.3.2) is less than or equal the right hand one.

For the reverse inequality, take any $u \in \mathfrak{U}$ and consider (1.0.1) driven by the Brownian motion $B(t) = W(\tau_u + t) - W(\tau_u)$, and, for $\varepsilon > 0$, choose a control $u_\varepsilon$ such that

$$V(x_2) < J(x_2; u_\varepsilon) + \varepsilon.$$

Define now the new control $u_* \in \mathfrak{U}$ as

$$u_*(t) = \begin{cases} u(t) & \text{for } t \le \tau_u \\ u_\varepsilon(t - \tau_u) & \text{for } t > \tau_u. \end{cases}$$

Just as in the proof of the upper bound we get

$$\begin{aligned} V(x_1) \ge J(x_1; u_*) &= \mathbb{E}\left[\int_0^{\tau_u} e^{-\rho t}\big(\ln u(t) - cx^2(t)\big)\,dt + e^{-\rho\tau_u} J(x_2; u_\varepsilon)\right] \\ &> \mathbb{E}\left[\int_0^{\tau_u} e^{-\rho t}\big(\ln u(t) - cx^2(t)\big)\,dt + e^{-\rho\tau_u} V(x_2)\right] - \varepsilon, \end{aligned}$$

which concludes the proof. □

□

Based on Prop. 3.3.1(i) and 3.3.1(ii), it follows that $V < \infty$ in $[0, \infty)$ when $\sigma^2 < \rho + 2b$. Furthermore, Prop. 3.3.1(iii) is a special case of the dynamic programming principle. In the next two Propositions (3.3.2, 3.3.3), we prove the key properties of the value function $V$ which guarantee that $V$ satisfies the assumptions of the comparison Theorem 2.

In particular, Proposition 3.3.2, states that $V$ does not go to minus infinity more quickly than $-Cx^2$. Furthermore, it shows that $V$ is strictly decreasing and that $V$ satisfies $DV \le -C < 0$ in the viscosity sense.

**Proposition 3.3.2.** *Suppose* $0 \le \sigma^2 < \rho + 2b$.

(i) *There exist constants* $K_1, K_2 > 0$*, such that, for any* $x \ge 0$*, we have*

$$K_1 \;\le\; V(x) + A\left(x + \frac{a}{b+\rho}\right)^2 + \frac{1}{\rho}\ln\left(x + \frac{a}{b+\rho}\right) \;\le\; K_2. \quad (3.3.3)$$

(ii) *There exist constant* $C_1 > 0$ *and function* $c : [0, +\infty) \to (0, \infty)$ *with* $\lim_{x \to 0} c(x) = e^{-(\rho V(0)+1)}$ *such that, for any* $x_1, x_2 \in [0, +\infty)$ *with* $x_1 < x_2$,

$$\frac{V(x_2) - V(x_1)}{x_2 - x_1} \le -c(x_2) \le -C_1 < 0. \tag{3.3.4}$$

*Proof.* (i) *Proof lower bound:* The claim will follow by choosing the control $u(t) = \frac{1 \vee x(t)}{1+x^2(t)} + a - r(x(t))$, which is clearly admissible. Then, (3.2.2) gives

$$\begin{aligned} x(t) &= xZ_t + \int_0^t \frac{Z_t}{Z_s} \Big(a + \frac{1 \vee x(t)}{1 + x^2(t)})\Big) ds \\ &= xZ_t + aM_t(1) + \int_0^t \frac{Z_t}{Z_s} \frac{1 \vee x(s)}{1 + x^2(s)} ds \end{aligned} \tag{3.3.5}$$

and, hence,

$$\begin{aligned} x^2(t) = x^2 Z_t^2 + 2axZ_tM_t(1) + a^2 M_t^2(1) + \Big(\int_0^t \frac{Z_t}{Z_s} \frac{1 \vee x(s)}{1 + x^2(s)} ds\Big)^2 \\ + 2aM_t(1) \int_0^t \frac{Z_t}{Z_s} \frac{1 \vee x(s)}{1 + x^2(s)} ds + \int_0^t \frac{Z_t^2}{Z_s} \frac{2x \cdot (1 \vee x(s))}{1 + x^2(s)} ds. \end{aligned}$$

To estimate the rightmost term from above, note that, in view of (3.3.5), $x \le x(s) Z_s^{-1}$, while for the third and fourth terms of the sum we use that $\frac{1 \vee x(s)}{1+x^2(s)} \le 1$. It follows that

$$cx^2(t) \le cx^2 Z_t^2 + 2acxZ_tM_t(1) + c(a+1)^2 M_t^2(1) + 2c \int_0^t \frac{Z_t^2}{Z_s^2} ds, \tag{3.3.6}$$

It is easy to see that

$$\begin{aligned} \mathbb{E}\left[\int_0^\infty e^{-\rho t} cx^2 Z_t^2 dt\right] &= \int_0^\infty e^{-\rho t} cx^2 \mathbb{E}\left[Z_t^2\right] dt \\ &= cx^2 \int_0^\infty e^{-(\rho + 2b - \sigma^2)t} dt = Ax^2, \end{aligned} \tag{3.3.7}$$

Lemma A.1 (ii) gives

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} 2acxZ_tM_t(1) dt\right] = 2aAx \int_0^\infty e^{-\rho t} \mathbb{E}\big[Z_t\big] dt = \frac{2aAx}{(\rho + b)}, \tag{3.3.8}$$

while Lemma A.1 (i) and Lemma A.1 (iii) yield

$$\int_0^\infty e^{-\rho t} c\mathbb{E}\big[M_t^2(1)\big]\; dt = 2A\int_0^\infty e^{-\rho t}\mathbb{E}\big[M_t(1)\big]\; dt = \frac{2A}{\rho(\rho+b)}. \tag{3.3.9}$$

We also have

$$\int_0^\infty ce^{-\rho t}\int_0^t \mathbb{E}\left[\frac{Z_t^2}{Z_s^2}\right] ds\; dt = c\int_0^\infty e^{-\rho t}\int_0^t e^{(\sigma^2-2b)(t-s)} ds\; dt = \frac{A}{\rho}. \tag{3.3.10}$$

Using the last four observations in (3.3.6), we find for some constant $B$,

$$\int_0^\infty e^{-\rho t}\mathbb{E}\big[cx^2(t)\big]\; dt \le A\left[\big(x+\frac{a}{\rho+b}\big)^2 + B\right]. \tag{3.3.11}$$

On the other hand, using that, for all $x \ge 0$, $\dfrac{1\vee x}{1+x^2} \ge \dfrac{1}{1+x}$, and Jensen's inequality, we find

$$\begin{aligned}\int_0^\infty e^{-\rho t}\mathbb{E}\big[\ln u(t)\big]\; dt &\ge -\int_0^\infty e^{-\rho t}\mathbb{E}\big[\ln\big(1+x(t)\big)\big]\; dt\\ &\ge -\frac{1}{\rho}\ln\left(\int_0^\infty \rho e^{-\rho t}\Big(1+\mathbb{E}\big[x(t)\big]\Big)dt\right)\\ &= -\frac{1}{\rho}\ln\left(1+\rho\int_0^\infty e^{-\rho t}\mathbb{E}\big[x(t)\big]dt\right).\end{aligned}$$

By (3.3.5) it follows that $\mathbb{E}\big[x(t)\big] \le x\mathbb{E}\big[Z_t\big] + (a+1)\mathbb{E}\big[M_t(1)\big] = xe^{-bt} + (a+1)\mathbb{E}\big[M_t(1)\big]$.

Hence, using Lemma A.1 (i), we obtain

$$\int_0^\infty e^{-\rho t}\mathbb{E}\big[\ln u(t)\big]\; dt \ge -\frac{1}{\rho}\ln\left(1+\frac{\rho x}{\rho+b}+\frac{a+1}{\rho+b}\right).$$

The preceding estimate and (3.3.11) together imply that, for some suitable constant $K_1$,

$$\begin{aligned}V(x) \ge J(x;u) &= \mathbb{E}\big[\int_0^\infty e^{-\rho t}\big(\ln u(t) - cx^2(t)\big)\; dt\big]\\ &\ge -A\left(x+\frac{a}{b+\rho}\right)^2 - \frac{1}{\rho}\ln\left(x+\frac{a}{b+\rho}\right) + K_1.\end{aligned}$$

*Proof of the upper bound:* In view of Prop. 3.3.1(i) and 3.3.1(ii), it suffices to find $K_2 > 0$, such that the asserted inequality holds for $x \geq 1$.

Fix $u \in \mathfrak{U}$. Then

$$\begin{aligned} x^2(t) &\geq x^2 Z_t^2 + 2xZ_t^2 \int_0^t \frac{1}{Z_s}\left(u(s) + r(x(s))\right) ds \\ &= x^2 Z_t^2 + 2xZ_t M_t(a+u) - \int_0^t \frac{Z_t^2}{Z_s} 2x(a - r(x(s))ds \\ &\geq x^2 Z_t^2 + 2axZ_t M_t(1) + 2xZ_t M_t(u) \\ &\qquad - \int_0^t \frac{Z_t^2}{Z_s^2} 2x(s)\left(a - r(x(s))\right) ds \end{aligned}$$

Since $\lim_{x\to\infty} x(a-r(x)) = C \in \mathbb{R} \Rightarrow x(a-r(x)) \leq K$ for some suitable positive constant $K$, we can further estimate $x^2(t)$ from below by

$$x^2(t) \geq x^2 Z_t^2 + 2axZ_t M_t(1) + 2xZ_t M_t(u) - 2K \int_0^t \frac{Z_t^2}{Z_s^2} ds. \quad (3.3.12)$$

Using the elementary inequality $\ln a \leq ab - \ln b - 1$, which holds for all $a, b > 0$, and Lemma A.1 (ii), we obtain, for some B,

$$\begin{aligned} \int_0^\infty e^{-\rho t}\mathbb{E}\left[\ln u(t)\right] dt &\leq \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left\{2AxZ_t u(t) - \ln\left(2AxZ_t\right)\right\} dt\right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-\rho t} 2cxZ_t M_t(u)\, dt\right] - \frac{\ln(2Ax)}{\rho} + \frac{2b+\sigma^2}{2\rho^2} \\ &\leq c\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(x^2(t) - x^2 Z_t^2 - 2axZ_t M_t(1) + 2K\int_0^t \frac{Z_t^2}{Z_s^2} ds\right) dt\right] \\ &\qquad - \frac{\ln x + B}{\rho}, \end{aligned}$$

where in the final step we have used (3.3.12).

In view of (3.3.7), (3.3.8) and (3.3.10), for every $u \in \mathfrak{U}$ there exists $K_2 > 0$ such that

$$J(x;u) \leq -A\left(x + \frac{a}{b+\rho}\right)^2 - \frac{1}{\rho}\ln\left(x + \frac{a}{\rho+b}\right) + K_2.$$

The assertion now follows by taking the supremum over $u \in \mathfrak{U}$.

(ii) Let $\eta > 0$ such that $r(x) < (b+\rho)x \ \forall x \in \ (0,\eta)$. In view of Proposition 3.3.1(i), it suffices to assume that $x_2 \leq \eta$, since otherwise we have

$$V(x_2) - V(x_1) \leq -A(x_2^2 - x_1^2) < -A\eta(x_2 - x_1).$$

For a positive constant $d$, choose a control $u_d \in \mathfrak{U}$ that is constant and equal to $d$ up to time $\tau_d = \tau_{u_d}$. Then, Proposition 3.3.1(iii) yields

$$V(x_1) \geq \frac{\ln d - cx_2^2}{\rho}(1 - \mathbb{E}\big[e^{-\rho\tau_d}\big]) + \mathbb{E}\big[e^{-\rho\tau_d}\big]V(x_2),$$

or equivalently,

$$\big(V(x_2) - V(x_1)\big)\mathbb{E}\big[e^{-\rho\tau_d}\big] \leq -\big(\ln d - \rho V(x_1) - cx_2^2\big)\ \mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t}\, dt\right]. \tag{3.3.13}$$

Consider now the solution $x_d(\cdot)$ to (1.0.1) with $x(0) = x_1$ and control $u_d$. Applying Itô's formula to $e^{-\rho t}x_d(t)$, followed by the optional stopping theorem for the bounded stopping time $\tau_N = \tau_d \wedge N$, we get

$$\mathbb{E}\big[e^{-\rho\tau_N}x_d(\tau_N)\big] - x_1 = \mathbb{E}\left[\int_0^{\tau_N} e^{-\rho t}\big(d - (b+\rho)x_d(t) + r(x_d)\big)\, dt\right]. \tag{3.3.14}$$

The leftmost term of (3.3.14) is equal to

$$x_2\mathbb{E}\big[e^{-\rho\tau_d}; \tau_d \leq N\big] + e^{-\rho N}\mathbb{E}\big[x_d(\tau_N); \tau_d > N\big].$$

On the other hand, since we have assumed that $x_2 \leq \eta$, we have $x_d(t) \leq \eta$ up to time $\tau_d$. Thus, the right hand side of (3.3.14) is bounded by $\mathbb{E}\left[\int_0^{\tau_N} e^{-\rho t} d\ dt\right]$.

Letting $N \to \infty$ in (3.3.14), by the monotone convergence theorem, we have

$$x_2\mathbb{E}\big[e^{-\rho\tau_d}\big] - x_1 \leq d\,\mathbb{E}\big[\int_0^{\tau_d} e^{-\rho t} dt\big]$$

$$\Leftrightarrow (x_2 - x_1)\mathbb{E}\big[e^{-\rho\tau_d}\big] \leq (d + \rho x_1)\ \mathbb{E}\big[\int_0^{\tau_d} e^{-\rho t} dt\big].$$

Substituting this in (3.3.13) and choosing $\ln d = \rho V(x_1) + 1 + cx_2^2$, we find

$$V(x_2) - V(x_1) \leq -(x_2 - x_1)\left(e^{\rho V(x_1)+1+cx_2^2} + \rho x_1\right)^{-1}. \qquad (3.3.15)$$

The assertion now follows setting

$$c(x_2) = A\eta \mathbf{1}\{x_2 > \eta\} + \left(e^{\rho V(0)+1+cx_2^2} + \rho x_2\right)^{-1} \mathbf{1}\{x_2 \leq \eta\}$$

and

$$C_1 = A\eta \wedge \left(e^{\rho V(0)+1+c\eta^2} + \rho\eta\right)^{-1} > 0.$$

□

The next assertion together with (3.3.4) states the locally Lipschitz continuity of the value function in the stochastic case. Moreover, relation (3.3.17) gives us the appropriate boundary condition for the HJB eq. (1.0.4) that guarantees the well-posedness of the corresponding boundary value problem.

**Proposition 3.3.3.** *Suppose* $0 < \sigma^2 < \rho + 2b$.

*(i) There exists an increasing function* $L_\sigma : [0, \infty) \to \mathbb{R}$ *with* $\lim_{x\to 0} L_\sigma(x) = e^{-(\rho V(0)+1)}$ *such that, for any* $x_1, x_2 \in [0, \infty)$ *with* $x_1 < x_2$,

$$\frac{V(x_2) - V(x_1)}{x_2 - x_1} \geq -L_\sigma(x_2) \qquad (3.3.16)$$

*(ii)* $V$ *is differentiable at zero and*

$$\ln\big(-V'(0)\big) + \rho V(0) + 1 = 0. \qquad (3.3.17)$$

*Proof.* (i) Fix $x_1, x_2$ as in the statement. It follows from Proposition 3.3.1(iii) that for any $\epsilon > 0$ there exists a control $u_\varepsilon \in \mathfrak{U}$ such that

$$V(x_1) \leq \mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\rho t} \ln u_\varepsilon(t)\, dt\right] + \mathbb{E}\big[e^{-\rho\tau_\varepsilon}\big]V(x_2) + \varepsilon c(x_2^2 - x_1^2), \qquad (3.3.18)$$

where $\tau_\varepsilon$ is the hitting time on $[x_2, +\infty)$ of the solution $x_\varepsilon(\cdot)$ to (1.0.1) with $x(0) = x_1$ and control $u_\varepsilon$.

Using the elementary inequality

$$\ln u_\varepsilon(t) \le \ln \kappa + \frac{u_\varepsilon(t)}{\kappa} - 1, \quad \text{with } \kappa = e^{\rho V(x_2)+1},$$

we find

$$V(x_1) - V(x_2) \le \frac{1}{\kappa}\mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\rho t} u_\varepsilon(t)\ dt\right] + \varepsilon c(x_2^2 - x_1^2). \tag{3.3.19}$$

To conclude it suffices to show that

$$\frac{1}{\kappa}\mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\rho t} u_\varepsilon(t)\ dt\right] \le L_\sigma(x_2)(x_2 - x_1). \tag{3.3.20}$$

To do this, we apply Itô's rule to the semimartingale $Y_t = e^{-\rho t + \gamma x_\varepsilon(t)}$, where $\gamma > 0$ is a constant to be determined, and find

$$\begin{aligned} Y_t - e^{\gamma x_1} &= \int_0^t Y_s\big( -\rho\, ds + \gamma\, dx_\varepsilon(s) + \frac{\gamma^2}{2}\, d\langle x_\varepsilon\rangle_s\big) \\ &= \int_0^t Y_s\Big( -\rho + \gamma\big(u_\varepsilon(s) - bx_\varepsilon(s) + r(x_\varepsilon(s))\big) \\ &\qquad\qquad + \frac{\gamma^2\sigma^2 x_\varepsilon^2(s)}{2}\Big) ds + M_t, \end{aligned}$$

where $M_t$ stands for the martingale $\gamma\sigma \int_0^t Y_s x_\varepsilon(s)\, dW(s)$.

Next, we apply the optional stopping theorem for the bounded stopping time $\tau_N = \min\{\tau_\varepsilon, N\}$, with $N \in \mathbb{N}$, to find

$$\begin{aligned} \mathbb{E}\big[Y_{\tau_N}\big] - e^{\gamma x_1} = \mathbb{E}\Bigg[\int_0^{\tau_N} Y_s\Big( -\rho + \gamma\big(u_\varepsilon(s) - bx_\varepsilon(s) + r(x_\varepsilon(s))\big) \\ + \frac{\gamma^2\sigma^2 x_\varepsilon^2(s)}{2}\Big) ds\Bigg]. \end{aligned}$$

Since $0 \le x_\varepsilon(s) \le x_2$ in $[0, \tau_\varepsilon]$ and $\rho \ge 0$,

$$\begin{aligned} e^{\gamma x_2}\mathbb{E}\big[e^{-\rho\tau_N}\big] - e^{\gamma x_1} \ge \mathbb{E}\Bigg[\int_0^{\tau_N} Y_t\Big( -\rho + \gamma\big(u_\varepsilon(t) - bx_\varepsilon(t)\big) \\ + \frac{\gamma^2\sigma^2 x_\varepsilon^2(t)}{2}\Big) dt\Bigg], \end{aligned}$$

and since $0 \leq Y_t \leq e^{-\rho t} e^{\gamma x_2}$ in $[0, \tau_\varepsilon]$,

$$e^{\gamma x_2} - e^{\gamma x_1} \geq \gamma \mathbb{E}\left[\int_0^{\tau_N} e^{-\rho t} u_\varepsilon(t)\, dt\right] + \mathbb{E}\Bigg[\int_0^{\tau_N} Y_t \Big( -b\gamma x_\varepsilon(t) + \frac{\gamma^2 \sigma^2 x_\varepsilon^2(t)}{2} \Big) dt \Bigg].$$

Note that the term in the parenthesis above is nonnegative if $\gamma x_\varepsilon(t) \geq 2b/\sigma^2$, and greater than or equal to $-b^2/2\sigma^2$ in any case. Hence,

$$e^{\gamma x_2} - e^{\gamma x_1} \geq \gamma \mathbb{E}\left[\int_0^{\tau_N} e^{-\rho t} u_\varepsilon(t)\, dt\right] - \frac{b^2 e^{\frac{2b}{\sigma^2}}}{2\sigma^2} \mathbb{E}\left[\int_0^{\tau_N} e^{-\rho t} dt\right].$$

Letting $N \to \infty$ we get

$$e^{\gamma x_2} - e^{\gamma x_1} \geq \gamma \mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\rho t} u_\varepsilon(t)\, dt\right] - \frac{b^2 e^{\frac{2b}{\sigma^2}}}{2\sigma^2} \mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\rho t} dt\right]. \quad (3.3.21)$$

To show (3.3.20), it suffices to control the term $\mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\rho t} dt\right]$ by $\mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\rho t} u_\varepsilon(t)\, dt\right]$.

Without loss of generality we can now assume that $\varepsilon c < A$. Then, Proposition 3.3.1(i) and eq. (3.3.18) give

$$\begin{aligned} 0 &\leq V(x_1) - V(x_2) - \varepsilon c (x_2^2 - x_1^2) \\ &\leq \mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\rho t} \ln u_\varepsilon(t)\ dt\right] - \rho V(x_2) \mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\rho t}\ dt\right]. \end{aligned}$$

Jensen's inequality then implies that

$$\mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\rho t} u_\varepsilon(t)\, dt\right] \geq e^{\rho V(x_2)}\ \mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\rho t}\ dt\right],$$

and (3.3.21) gives,

$$\gamma e^{\gamma x_2}(x_2 - x_1) \geq e^{\gamma x_2} - e^{\gamma x_1} \geq (\gamma - C(x_2))\ \mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\rho t} u_\varepsilon(t)\, dt\right]. \quad (3.3.22)$$

with $C(x_2) = \frac{b^2}{2\sigma^2} e^{\frac{2b}{\sigma^2} - \rho V(x_2)}$.

We will now choose $\gamma$ appropriately in order to optimize the preceding inequality. Choosing $\gamma = q(x_2)/x_2$ in (3.3.22), where

$$q(x_2) = \frac{x_2 C(x_2)}{2} + \sqrt{\left(\frac{x_2 C(x_2)}{2}\right)^2 + x_2 C(x_2)}, \tag{3.3.23}$$

we obtain

$$\mathbb{E}\left[\int_0^{\tau_\varepsilon} e^{-\rho t} u_\varepsilon(t)\, dt\right] \leq (x_2 - x_1)\big(1 + q(x_2)\big)e^{q(x_2)}.$$

Substituting this in (3.3.19) and letting $\varepsilon \to 0$ yield

$$\frac{V(x_2) - V(x_1)}{x_2 - x_1} \geq -\big(1 + q(x_2)\big)e^{q(x_2)-1-\rho V(x_2)}. \tag{3.3.24}$$

The claim now follows. □

(ii) It follows from (3.3.15) that, for any $x \in (0, \eta]$,

$$\frac{V(x) - V(0)}{x} \leq -e^{-\rho V(0)-1-cx^2}.$$

Letting $x \to 0$ we get

$$\limsup_{x\to 0} \frac{V(x) - V(0)}{x} \leq -e^{-\rho V(0)-1},$$

while (3.3.24) gives

$$\frac{V(x) - V(0)}{x} \geq -\big(1 + q(x)\big)e^{q(x)-1-\rho V(x)}.$$

Letting $x \to 0$ and noting that $q(x) \to 0$ and $V(x) \to V(0)$, we have

$$\liminf_{x\to 0} \frac{V(x) - V(0)}{x} \geq -e^{-\rho V(0)-1},$$

which proves the claim.

□

## 3.4 Proof of the main results

In this section, we prove Theorems 1 and 2.

**Proof of Theorem 1:** Since the Hamiltonian (3.1.2) can take infinite values, we have a singular stochastic control problem and the value function (1.0.3) should satisfy the following variational inequality (see [41], Section 4):

$$\min\Big[\rho V - H(x, V_x) - \frac{1}{2}\sigma^2 x^2 V_{xx}, -V_x\Big] = 0, \qquad \text{in } [0,\infty). \tag{3.4.1}$$

That $V$ is a viscosity solution in $(0,\infty)$ follows as in [41], so we omit the details. It remains to prove the subsolution property at $x = 0$. Let $\phi$ be a test function such that $V - \phi$ has a maximum at $x = 0$ with $V(0) - \phi(0) = 0$. It suffices to assume that $-\phi'(0) > 0$, otherwise the result is immediate. Given that $-\phi'(0) > 0$, we have:

$$\begin{aligned} \rho\phi(0) - H(0, \phi'(0)) &= \rho\phi(0) + 1 + \ln\big(-\phi'(0)\big) \\ &\leq \rho V(0) + 1 + \ln\big(-V'(0)\big) = 0 \end{aligned}$$

where we used that $V - \phi$ has a maximum at 0 and relation (3.3.17). Now, it follows that (1.0.3) is a continuous constrained viscosity solution of the equation (1.0.6) since inequality (3.3.4) implies that $p \leq -C$ for any $p \in D^{\pm}V(x)$, with $x \in (0,\infty)$. The regularity of $V$ in $(0,\infty)$ follows from the classical results for uniformly elliptic operators. □

Next we prove the comparison Theorem 2. The proof is based on the strategy in [28] (see also [18], [51]). Based on Lemma 4.2 [33], the difference $u - v$ of the functions $u$, $v$ satisfying the assumptions of Theorem 2 is a subsolution of the corresponding linearized equation.

**Lemma 3.4.1** (Lemma 4.2 [33]). *Suppose $u$, $v$ satisfy the assumptions of Theorem 2. Then $\psi = u - v$ is a subsolution of*

$$\rho\psi + bxD\psi - (a + c^*)\,|D\psi| - \frac{1}{2}\sigma^2 x^2 D^2\psi = 0 \quad \textit{in} \quad [0,\infty). \tag{3.4.2}$$

Then, we conclude to our main result by comparing $u - v$ with the appropriate supersolution of the linearized equation; see also [18] and [51]. The difference with the existing results is that, due to the presence of the logarithmic term, the commonly used functions of simple polynomials do not yield a supersolution of the equation.

**Proof of Theorem 2:** The main step is the construction of a solution of the linearized equation. For this, we consider the ode

$$\rho w + \big(bx - (a + c^*)\big)w' - \frac{1}{2}\sigma^2 x^2 w'' = 0, \tag{3.4.3}$$

which has a solution of the form

$$w(x) = x^{-k}\mathcal{J}(\frac{2a + 2c^*}{\sigma^2 x}), \tag{3.4.4}$$

where $k$ is a root of

$$k^2 + \Big(1 + \frac{2b}{\sigma^2}\Big)k - \frac{2\rho}{\sigma^2} = 0 \tag{3.4.5}$$

and $\mathcal{J}$ a solution of the degenerate hypergeometric equation

$$xy'' + (\tilde{b} - x)y' - \tilde{a}y = 0 \tag{3.4.6}$$

with $\tilde{a} = k$ and $\tilde{b} = 2(k + 1 + b/\sigma^2)$.

Since we are looking for a solution of (3.4.3) with superquadratic growth at $+\infty$, we choose $k$ to be the negative root of (3.4.5).

We further choose $\mathcal{J}$ to be the Tricomi solution of (3.4.6) which satisfies

$$\mathcal{J}(0) > 0 \qquad \text{and} \qquad \mathcal{J}(x) = x^{-k}\big(1 + \frac{2\rho}{\sigma^2 x} + o(x^{-1})\big) \quad \text{as } x \to \infty.$$

With this choice, the function $w$ defined in (3.4.4) for $x > 0$ and by continuity at $x = 0$, satisfies $w(0), w'(0) > 0$ and $w(x) \sim \mathcal{J}(0)x^{-k}$, as $x \to \infty$.

Note that $w$ is increasing in $[0, \infty)$ since it would otherwise have a positive local maximum and this is impossible by (3.4.3). In particular, $w$ satisfies (3.4.2).

Set now $\psi = u - v$ and consider $\epsilon > 0$. Since

$$\begin{aligned}\lim_{x\to\infty} (\psi(x) - \epsilon w(x)) &\le \lim_{x\to\infty} \big(u(0) + c_1(1 + x^q) - \epsilon J(0)x^{-k}\big) \\ &= \lim_{x\to\infty} -\epsilon J(0)x^{-k} = -\infty,\end{aligned}$$

there exists $x^\epsilon \in [0,\infty)$ such that

$$\max_{x\geq 0}\big(\psi(x)-\epsilon w(x)\big)=\psi(x^\epsilon)-\epsilon w(x^\epsilon).$$

By Lemma 3.4.1, $\psi$ is a subsolution of (3.4.2). We now use $\epsilon w$ as a test function to find that

$$\begin{aligned}0 &\geq \rho\psi(x^\epsilon)+\epsilon b x^\epsilon w(x^\epsilon)-\epsilon\,(a+c^*)\,|w'(x^\epsilon))|-\frac{1}{2}\epsilon\sigma^2(x^\epsilon)^2 w''(x^\epsilon)\\ &= \rho(\psi(x^\epsilon)-\epsilon w(x^\epsilon)).\end{aligned}$$

Hence, $\psi(x)\leq \epsilon w(x)$ for all $x\in[0,\infty)$. Since $\epsilon$ is arbitrary, this proves the claim. □

# Chapter 4

# Existence of Optimal Control

A basic question that, to the best of our knowledge, has not been completely answered so far is that of existence of optimal control. In the presence of noise, the elliptic regularity of the value function permits the adoption of the usual methodology. On the other hand, this approach is not always feasible in the deterministic case because the value function is not expected in general to be smooth. Indeed, when the system of the lake has a Skiba point, the value function is not differentiable thereat.

## 4.1 Deterministic shallow lake problem

In this section, we prove the existence of optimal control in the absence of noise. All the results of this section assume that $\sigma = 0$. To prove that the optimal value is attainable, we establish a Comparison Principle (Theorem 3) and we use it to dominate the value function, $V$, by the candidate value function, $J_P$ of section 2.2.1, which is constructed based on the Pontryagin Maximum Principle.

Therefore, the first step in our methodology is to derive a Comparison Principle, analogous to Theorem 2. The proof of Theorem 3 is a modification of the proof of Theorem 2.

**Theorem 3.** *Assume that*

- $u \in C([0, \infty))$ *is a strictly decreasing subsolution of* (1.0.6) *(with*

*$\sigma = 0$) in $[0, \infty)$, with $Du \leq -\frac{1}{c^*}$, in the viscosity sense, for some positive constant $c^*$.*

- *$v \in C([0, \infty))$ is a strictly decreasing supersolution of* (1.0.6) *(with $\sigma = 0$) in $(0, \infty)$, such that $v \geq -c_2(1 + x^q)$, where $c_2$ can be any positive constant and $q$ can be any real number.*

*Then, $u \leq v$ in $[0, \infty)$.*

*Proof.* Let $\eta > 0$ sufficiently small. We consider the ordinary differential equation

$$\rho w + \big(bx - (a + c^*)\big)w' - \frac{1}{2}\eta x^2 w'' = 0, \tag{4.1.1}$$

which has a solution of the form

$$w(x) = x^{-k}\mathcal{J}\Big(\frac{2a + 2c^*}{\eta x}\Big), \tag{4.1.2}$$

where $k$ is a root of

$$k^2 + \Big(1 + \frac{2b}{\eta}\Big)k - \frac{2\rho}{\eta} = 0 \tag{4.1.3}$$

and $\mathcal{J}$ a solution of the degenerate hypergeometric equation

$$xy'' + (\tilde{b} - x)y' - \tilde{a}y = 0 \tag{4.1.4}$$

with $\tilde{a} = k$ and $\tilde{b} = 2(k + 1 + b/\eta)$.

We choose $k$ to be the negative root of (4.1.3). We further choose $\mathcal{J}$ to be the Tricomi solution of (4.1.4) which satisfies

$$\mathcal{J}(0) > 0 \qquad \text{and} \qquad \mathcal{J}(x) = x^{-k}\big(1 + \frac{2\rho}{\eta x} + o(x^{-1})\big) \quad \text{as } x \to \infty.$$

With this choice, the function $w$ defined in (4.1.2) for $x > 0$ and by continuity at $x = 0$, satisfies $w(0), w'(0) > 0$ and $w(x) \sim \mathcal{J}(0)x^{-k}$, as $x \to \infty$. We choose $\eta > 0$ sufficiently small so that $k < -q$.

Note that $w$ is increasing in $[0, \infty)$ since it would otherwise have a positive local maximum and this is impossible by (4.1.1).

Set now $\psi = u - v$ and consider $\epsilon > 0$. Since $\psi - \epsilon w < 0$ in a neighborhood of infinity, there exists $x^\epsilon \in [0, \infty)$ such that

$$\max_{x \geq 0} \big(\psi(x) - \epsilon w(x)\big) = \psi(x^\epsilon) - \epsilon w(x^\epsilon).$$

By Lemma 3.4.2 (with $\sigma = 0$), $\psi$ is a subsolution of

$$\rho\psi + bxD\psi - (a + c^*)|D\psi| = 0 \ \in [0, \infty)$$

We now use $\epsilon w$ as a test function to find that

$$0 \geq \rho\psi(x^\epsilon) + \epsilon b x^\epsilon w'(x^\epsilon) - \epsilon(a + c^*)|w'(x^\epsilon))|$$
$$\stackrel{(4.1.1)}{=} \rho\left(\psi(x^\varepsilon) - \varepsilon w(x^\varepsilon)\right) + \frac{\eta}{2}(x^\varepsilon)^2 \varepsilon w''(x^\varepsilon)$$
$$\rho\left(\psi(x^\varepsilon) - \varepsilon w(x^\varepsilon)\right) \leq -\frac{1}{2}\eta(x^\varepsilon)^2 \varepsilon w''(x^\varepsilon)$$

Note that the function $g(x) = \frac{1}{2}\eta x^2 w''(x)$ is bounded from below because from (4.1.1) for $x > (a+c^*)/b$ we have $g(x) = \rho w(x) + (bx - (a+c^*))w'(x) \geq \rho w(0)$, since $w$ is increasing in $[0, \infty)$. Therefore,

$$\rho\left(\psi(x) - \varepsilon w(x)\right) \leq -\varepsilon \inf_{[0,\infty)} g \ \text{ for all } x \in [0, \infty)$$

Since $\varepsilon$ is arbitrary, this proves the claim. □

Notice that based on Proposition 3.3.2(ii) the value function $V$ is strictly decreasing and $DV \leq -\frac{1}{c}$, in the viscosity sense, for some positive constant $c$. We proceed now be proving that the value function $V$ is a constrained viscosity solution of eq. (1.0.6) in $[0, \infty)$ when $\sigma = 0$.

**Theorem 4.** *The value function $V$ is a continuous constrained viscosity solution of eq.* (1.0.6) *on* $[0, \infty)$. *Particularly, $V$ satisfies* (1.0.6) *at $x = 0$ in the classical sense.*

*Proof.* Following exactly the steps of the proof of Theorem 1, presented in Chapter 3 for the stochastic case, it suffices to prove that the value function $V$ satisfies (3.3.17) when $\sigma = 0$. Therefore, we will show that $V$ is differentiable at 0 and

$$\ln(-V'(0)) + \rho V(0) + 1 = 0 \tag{4.1.5}$$

We will first prove that $\liminf_{h \to 0^+} \dfrac{V(h) - V(0)}{h} \geq -e^{-(\rho V(0)+1)}$.

**Claim 1.** *There exists $h > 0$ such that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ if $u^\varepsilon$ is an $\varepsilon-$optimal control, then $\tau^h_{u^\varepsilon} := \inf\{t \geq 0 : x_\varepsilon(t) \geq h\} < \infty$, where $x_\varepsilon(\cdot)$ is the solution of the* (1.0.1) *with control $u^\varepsilon$ and $x_\varepsilon(0) = 0$.*

*Proof.* By contradiction, let us assume that for any $h > 0$ there exists $\{\varepsilon_n\}_{n\in\mathbb{N}}$ ($\varepsilon_n \downarrow 0$) for which there exists $u^n$, $\varepsilon_n-$optimal control with $\tau^h_{u^n} = \infty$. Then $\forall\ n \in \mathbb{N}$, if $h_n = 1/n$, $\exists\ \varepsilon_n > 0$ ($\varepsilon_n \downarrow 0$) and $u^n$ $\varepsilon_n-$optimal control such that $\tau^h_n := \tau^h_{u^n} = \infty$. If $x_n = x_{\varepsilon_n}$, then $x_n(t) \leq 1/n\ \forall t \geq 0$.
Using the elementary inequality $\ln u^n \leq \ln A_n + \dfrac{u^n}{A_n} - 1$ with $A_n = 1/n$, we find

$$J(0; u^n) = \int_0^\infty e^{-\rho t}(\ln u^n(t) - cx_n^2(t))dt \leq \frac{\ln(1/n)}{\rho} + n\int_0^\infty e^{-\rho t}u^n(t)dt$$

$$\begin{aligned}\int_0^\infty e^{-\rho t}u^n(t)dt &= \int_0^\infty e^{-\rho t}\dot{x}_n(t)dt + \int_0^\infty e^{-\rho t}\left(bx_n(t) - r\left(x_n(t)\right)\right)dt \\ &= \rho\int_0^\infty e^{-\rho t}x_n(t)dt + \int_0^\infty e^{-\rho t}\left(bx_n(t) - r\left(x_n(t)\right)\right)dt \\ &\leq \frac{\rho + b}{\rho n}\end{aligned}$$

Therefore

$$J(0; u^n) \leq \frac{\ln(1/n) + \rho + b}{\rho} \ \Rightarrow\ V(0) = \lim_{n\to\infty} J(0; u^n) = -\infty$$

which is a contradiction.

□

**Claim 2.** $c(h) = \inf\limits_{0<\varepsilon<\varepsilon_0}\{\tau^h_{u^\varepsilon}\} > 0$, *where* $h$ *and* $\{\tau^h_{u^\varepsilon}\}_{0<\varepsilon<\varepsilon_0}$ *are defined in Claim 1.*

*Proof.* By contradiction, let us assume that $\inf\limits_{0<\varepsilon<\varepsilon_0}\{\tau^h_{u^\varepsilon}\} = 0$. Then there exists a subsequence of stopping times $\{\tau^h_{u^n}\}_{n\in\mathbb{N}}$ corresponding to a sequence in $(0, \varepsilon_0)$ $\varepsilon_n \downarrow 0$ such that $\tau^h_{u^n} \to 0$.
Then, since for all $t \in \left(0, \tau^h_{u^n}\right)$, $n \in \mathbb{N}$,

$$h = \int_0^{\tau^h_{u^n}} \left(u^n(t) - bx_n(t) + r\left(x_n(t)\right)\right)dt$$

and

$$0 \leq x_n(t) \leq h$$

we have

$$\lim_{n\to\infty}\int_0^{\tau^h_{u^n}} u^n(t)dt = h \tag{4.1.6}$$

Then for $\phi_n = \dfrac{1-e^{-\rho\tau^h_{u^n}}}{\rho}$, Jensen's inequality gives:

$$\begin{aligned} J(0;u^n) &\le \int_0^{\tau^h_{u^n}} e^{-\rho t}(\ln u^n(t) - c{x^2}_n(t))dt + e^{-\rho\tau^h_{u^n}}V(h) \\ &\le \phi_n \ln\left(\frac{1}{\phi_n}\int_0^{\tau^h_{u^n}} e^{-\rho t}u^n(t)dt\right) + V(h) \\ &\le \phi_n \ln\left(\frac{1}{\phi_n}\right) + \phi_n \ln\left(\int_0^{\tau^h_{u^n}} u^n(t)dt\right) + V(h) \end{aligned} \tag{4.1.7}$$

Since $\lim_{n\to\infty}\phi_n = 0$, by (4.1.6), we get:

$$V(0) = \lim_{n\to\infty} J(0;u^n) \le V(h)$$

which is a contradiction since $V$ is (strictly) decreasing. □

We consider now $h$, $u^\varepsilon, \tau^h_{u^\varepsilon}$ and $x_\varepsilon$ as in Claim 1. Then for all $\varepsilon \in (0,\varepsilon_0)$

$$V(0) - \varepsilon < J(0;u^\varepsilon) < \int_0^{\tau^h_{u^\varepsilon}} e^{-\rho t}(\ln u^\varepsilon(t) - cx^2_\varepsilon(t))dt + e^{-\rho\tau^h_{u^\varepsilon}}V(h)$$

Using the elementary inequality $\ln u^\varepsilon \le \ln A + \dfrac{u^\varepsilon}{A} - 1$, for $A > 0$ we find that

$$V(0) - \varepsilon < \frac{\ln A - 1}{\rho}\left(1 - e^{-\rho\tau^h_{u^\varepsilon}}\right) + \frac{1}{A}\int_0^{\tau^h_{u^\varepsilon}} e^{-\rho t}u^\varepsilon(t)dt + e^{-\rho\tau^h_{u^\varepsilon}}V(h)$$

Moreover,

$$\begin{aligned} e^{-\rho\tau^h_{u^\varepsilon}}h &= \int_0^{\tau^h_{u^\varepsilon}} e^{-\rho t}\left(u^\varepsilon(t) - (b+\rho)x_\varepsilon(t) + r\left(x_\varepsilon(t)\right)\right)dt \\ &\ge \int_0^{\tau^h_{u^\varepsilon}} e^{-\rho t}u^\varepsilon(t)dt - \frac{1-e^{-\rho\tau^h_{u^\varepsilon}}}{\rho}(b+\rho)h \end{aligned}$$

$$V(0) - \varepsilon < \left(\ln A - 1 + \frac{(b+\rho)h}{A}\right) \frac{1 - e^{-\rho\tau^h_{u^\varepsilon}}}{\rho} + e^{-\rho\tau^h_{u^\varepsilon}} V(h) + \frac{he^{-\rho\tau^h_{u^\varepsilon}}}{A}$$

$$V(0) - V(h) \leq \frac{1 - e^{-\rho\tau^h_{u^\varepsilon}}}{\rho} \left(\ln A - 1 + \frac{(b+\rho)h}{A} - \frac{\rho h}{A} - \rho V(h)\right) + \varepsilon + \frac{h}{A}$$

Choosing $A = \dfrac{h}{V(0) - V(h)}$, we find

$$-\ln\left(-\frac{V(h) - V(0)}{h}\right) - 1 + b(V(0) - V(h)) - \rho V(0) \geq -\frac{\rho\varepsilon}{1 - e^{-\rho\tau^h_{u^\varepsilon}}}$$

$$\frac{V(h) - V(0)}{h} \geq -\exp\left(\left(-1 - \rho V(0) + b(V(0) - V(h) + \frac{\rho\varepsilon}{1 - e^{-\rho c(h)}}\right)\right)$$

where the last inequality follows from Claim 2. Letting now $\varepsilon \to 0^+$ and then $h \to 0^+$, we find:

$$\liminf_{h \to 0^+} \frac{V(h) - V(0)}{h} \geq -e^{(-1-\rho V(0))}$$

Moreover, from Proposition 3.3.2(ii), we have that

$$\limsup_{h \to 0^+} \frac{V(h) - V(0)}{h} \leq -e^{(-1-\rho V(0))}$$

and this gives (4.1.5) and concludes the proof (see also proof of Theorem 1).

□

We now show that our candidate value function, $J_P$ satisfies the assumptions made for the supersolution $v$ in Theorem 3.

**Lemma 4.1.1.** *Let $J_P$ be the candidate value function of* (2.2.11)*. Then,*

*i. $J_P$ is decreasing.*

*ii. $J_P$ is a viscosity solution to* (1.0.6) *with $\sigma = 0$, in $(0, \infty)$.*

*iii. There exists $c_2 > 0$, such that $J_P(x) \geq -c_2(1 + x^2)$, for all $x \geq 0$.*

*Proof.* i. Along the stable manifold $\dfrac{dJ_P}{dx} = -\dfrac{1}{u} < 0$. Therefore, $J_P$ is decreasing. Moreover, $\dot{u} = g(x, u) < 0$ for $x$ close to zero, which implies that $J_P(0) = \lim_{x \to 0} J_P(x) < \infty$.

ii. The total benefit $J_P(x)$ of (2.2.11), for $x$ different from the Skiba point (if there exists such a point), is the classical solution to eq. (1.0.6) constructed by the method of characteristics. In the case of a Skiba point, it was proved in [32] that $J_P$ is also a viscosity solution to (1.0.6) at the Skiba point.

iii. Let $x > \frac{a+1}{b}$. Since $J_P$ is a classical solution to (1.0.6) at $x$ and $J_P' < 0$, we have that

$$\begin{aligned}\rho J_P(x) &= \sup_{u>0} \left\{ (u - bx + r(x))J_P'(x) + \ln u - x^2 \right\} \\ &\geq \ln\left(bx - r(x)\right) - x^2 \geq -x^2\end{aligned}$$

By continuity of $J_p$ in $[0, \infty)$, we conclude the proof.

□

We have now collected all the key ingredients to establish our main existence result.

**Corollary 4.1.1.** *The deterministic shallow lake problem admits an optimal control.*

*Proof.* It is a direct consequence of Theorems 3 and 4 , Lemma 4.1.1 and Proposition 3.3.2(ii) that $V \leq J_P$. Hence, the control suggested by the Pontryagin Maximun Principle is indeed optimal and it is given in the feedback form as $u^*(t) = u(x(t)) = -\frac{1}{V'(x(t))}$. □

## 4.2 Stochastic shallow lake problem

In this section, we prove existence of optimal control in the presence of noise. Based on Theorem 1, $V$ is a constrained viscosity solution to (1.0.4) in $[0, \infty)$ and from classical results for uniformly elliptic operators, it follows that $V$ is actually a classical solution to (1.0.6) in $(0, \infty)$. In fact, it can be proved that $V$ is actually two times differentiable at $x = 0$ and in this way, from Proposition 3.3.3(ii), it follows that $V$ is a classical solution to (1.0.6) in $[0, \infty)$. This result is stated in Proposition 4.2.1.

**Proposition 4.2.1.** *If* $0 < \sigma^2 < \rho + 2b$, *the value function* $V$ *is a classical solution to the equation* (1.0.6) *on* $[0, \infty)$ *and*

$$V''(0) = -(\rho + b - r'(0))\,(V'(0))^2$$

*Proof.* What remains to be shown is that $V$ satisfies (1.0.6) at $x = 0$ in the classical sense. It suffices to show that $V$ is two times differentiable at $x = 0$ and $C^1([0, \infty))$. We will first show that $V \in C^1([0, \infty))$. Since $V$ is $C^2(0, \infty)$, we will prove the regularity of $V$ (close) to zero.
Let $0 < x < b$. From Propositions 3.3.2(ii) and 3.3.3(i), we have that:

$$-\Phi_\sigma(x) \leq V'(x) \leq -c(x)$$

Taking $x \to 0$, it follows that $\lim_{x \to 0} V'(x) = e^{-\rho V(0) - 1} = V'(0)$.
Therefore, $V \in C^1([0, \infty))$.
Now, we proceed with the second derivative of $V$.
From eq. (1.0.6), we have that for $x > 0$:

$$V''(x) = \frac{2}{\sigma^2}\left[\frac{\rho V(x) + (bx - r(x))\,V'(x) + \ln(-V'(x)) + 1}{x^2}\right] + \frac{2}{\sigma^2}c \quad (4.2.1)$$

Setting $Q(x) = \rho V(x) + (bx - r(x))V'(x) + \ln(-V'(x)) + 1$, we have that

$$Q'(x) = (\rho + b - r'(x))V'(x) + (bx - r(x))V''(x) + \frac{V''(x)}{V'(x)} \;\Rightarrow$$

$$Q'(x) = \frac{2}{\sigma^2}\left(\frac{b}{x} + \frac{1}{x^2 V'(x)} - \frac{r(x)}{x^2}\right) Q(x) + (\rho + b - r'(x))V'(x)$$
$$+ \frac{2}{\sigma^2}c\left(bx - r(x) + \frac{1}{V'(x)}\right) \Rightarrow$$

$$Q'(x) = \alpha(x)Q(x) + \beta(x)$$

Let $\varepsilon > 0$. Then

$$\left(Q(x)e^{-\int_\varepsilon^x \alpha(s)ds}\right)' = \beta(x)e^{-\int_\varepsilon^x \alpha(s)ds}$$

$$Q(x) = Q(\varepsilon)e^{\int\limits_\varepsilon^x \alpha(s)ds} + \int_\varepsilon^x \beta(s)e^{\int_s^x \alpha(t)dt}ds$$

Since $\lim\limits_{\varepsilon\to 0^+} \int\limits_\varepsilon^x a(s)ds = -\infty$ and $\lim\limits_{\varepsilon\to 0} Q(\varepsilon) = 0$,

$$\frac{Q(x)}{x^2} = \frac{1}{x^2}\int\limits_0^x \beta(s)e^{\int\limits_s^x \alpha(t)dt} ds$$

Let $0 < \eta < \min\{-\beta(0), -1/V'(0)\}$. There exists $\varepsilon > 0$ such that $\forall s \in [0,\varepsilon]$

$$\begin{cases} \beta(0) - \eta < \beta(s) < \beta(0) + \eta \\ \frac{2b}{\sigma^2}\frac{1}{s} + \left(\frac{1}{V'(0)} - \eta\right)\frac{2}{\sigma^2 s^2} < \alpha(s) < \frac{2b}{\sigma^2}\frac{1}{s} + \left(\frac{1}{V'(0)} + \eta\right)\frac{2}{\sigma^2 s^2} \end{cases}$$

Then

$$\begin{aligned} \frac{Q(x)}{x^2} &\leq \frac{(\beta(0)+\eta)\int\limits_0^x e^{\int\limits_s^x \left(\frac{2b}{\sigma^2}\frac{1}{t} + \left(\frac{1}{V'(0)}-\eta\right)\frac{2}{\sigma^2 t^2}\right)dt} ds}{x^2} \\ &\leq \frac{(\beta(0)+\eta)\int\limits_0^x s^{-2b/\sigma^2} e^{\left(\frac{1}{V'(0)}-\eta\right)\frac{2}{\sigma^2 s}} ds}{x^{(2-2b/\sigma^2)} e^{(1/V'(0)-\eta)\frac{2}{\sigma^2 x}}} \end{aligned}$$

$$\limsup_{x\to 0} \frac{Q(x)}{x^2} \leq -\beta(0)V'(0)\sigma^2/2 \tag{4.2.2}$$

Similarly,

$$\liminf_{x\to 0} \frac{Q(x)}{x^2} \geq -\beta(0)V'(0)\sigma^2/2 \tag{4.2.3}$$

Therefore, from (4.2.1), (4.2.2), (4.2.3), we have

$$\lim_{x\to 0} V''(x) = -(\rho + b - r'(0))(V'(0))^2$$

Since $V \in \mathbb{C}^1[0,\infty) \cap \mathbb{C}^2(0,\infty)$, the assertion follows. □

The elliptic regularity of the Value function in the presence of noise permits the adoption of the usual methodology in order to prove the existence of the optimal control. In this direction, we follow the steps described in [23], with appropriate modifications to address the loss of ellipticity at the boundary and a possible blow-up of the benefit for small controls, due to the presence of the logarithmic term.

**Theorem 5.** *The stochastic shallow lake problem admits an optimal (feedback) control, which satisfies:*

$$u^*(t) = -\frac{1}{V'(x(t))},\ t \geq 0 \tag{4.2.4}$$

*where* $x(\cdot)$ *is the solution of* (1.0.1) *corresponding to this control.*

*Proof.* Let $x(t)$ be the solution of the sde

$$\begin{cases} dx(t) = f(x(t), -\frac{1}{V'(x(t))})dt + \sigma x(t)dW_t \\ x(0) = x \end{cases}$$

and $u(t) = -\frac{1}{V'(x(t))}$ the corresponding control.
We apply Itô's Rule to the stochastic process $g(t, x(t)) = e^{-\rho t}V(x(t))$ and we find for $t \geq 0$:

$$\begin{aligned} V(x) &= e^{-\rho t}V(x(t)) \\ &\quad + \int_0^t e^{-\rho s}\left(\rho V(x(s)) - (r(x(s))) - bx(s)))V'(x(s))) + 1\right) ds \\ &\quad - \int_0^t \frac{1}{2}V''(x(s)))\sigma^2 x^2(s)ds - \int_0^t e^{-\rho s}\sigma x(s))V'(x(s)))dW_s \\ &\overset{(1.0.6)}{=} e^{-\rho t}V(x(t)) + \int_0^t e^{-\rho s}(\ln(u(s)) - cx^2(s))ds \\ &\quad - \int_0^t e^{-\rho s}\sigma x(s)V'(x(s))dW_s \end{aligned}$$

We also consider the stopping times $\theta_n = \inf\{t \geq 0 : \; |x(t) - x| \geq n\} \wedge n$ and taking expected values in the above relation, we get that, for all $n \in \mathbb{N}$:

$$V(x) = \mathbb{E}\left[e^{-\rho\theta_n} V(x(\theta_n))\right] + \mathbb{E}\left[\int_0^{\theta_n} e^{-\rho s}(\ln(u(s)) - cx^2(s))ds\right]$$

- Since $V$ is decreasing, we have that

$$\limsup_{n\to\infty} \mathbb{E}\left[e^{-\rho\theta_n} V(x(\theta_n))\right] \leq \lim_{n\to\infty} \mathbb{E}\left[e^{-\rho\theta_n}\right] V(0) = 0$$

- Since $V'(x)$ is bounded from above by a negative quantity, we have that $u(s)$ is also bounded from above by a constant, say $C$. Thus, we have that:

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^{\theta_n} e^{-\rho s}(C - \ln(u(s)))ds\right] = \mathbb{E}\left[\int_0^{\infty} e^{-\rho s}(C - \ln(u(s)))ds\right]$$

  by monotone convergence theorem. Therefore,

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^{\theta_n} e^{-\rho s}\ln(u(s))ds\right] = \mathbb{E}\left[\int_0^{\infty} e^{-\rho s}\ln(u(s))ds\right]$$

- Regarding the last term, we also have that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^{\theta_n} e^{-\rho s}x^2(s)ds\right] = \mathbb{E}\left[\int_0^{\infty} e^{-\rho s}x^2(s)ds\right]$$

  by monotone convergence theorem.

Thus, we have that

$$V(x) \leq \mathbb{E}\left[\int_0^{\infty} e^{-\rho s}(\ln(u(s)) - cx^2(s))ds\right] = J(x;u) \leq V(x)$$

and this concludes the proof.

□

**Remark 4.2.1.** *The optimal control, written in feedback form,* $u^*(x) = -\frac{1}{V'(x)}$*, for* $\sigma > 0$*, is obviously a bounded and locally Lipschitz function. Therefore, the problem* (1.0.1) *has a unique strong solution* $x(\cdot)$ *(see Theorem 3.4 [39]) and the optimal control* $u^*$ *is admissible. Furthermore, the admissibility of the optimal control in the deterministic case is an immediate consequence, from the way it was constructed, since it is located on the stable manifold of the Pontryagin system of the lake* (2.2.10)*.*

# Chapter 5

# Asymptotic Behaviour

In this chapter, we study the asymptotic behaviour of the system as $x \to +\infty$ and we present two main results.

The first, Theorem 6, describes the exact asymptotic behavior of the value function $V$ at $+\infty$. In Chapter 6, we present and implement a monotone numerical scheme approximating (1.0.3). Relation (5.0.1) is crucial for the accurate computation of $V$ in this setting, because it suggests the boundary condition on the right end of the computational domain. The proof of Theorem 6 follows the lines of the proof of Theorem 2.3 in [33].

**Theorem 6.** *As $x \to \infty$,*

$$V(x) = -A\left(x + \frac{a}{b+\rho}\right)^2 - \frac{1}{\rho}\ln\left[2A(x + \frac{a}{b+\rho})\right] + K + o(1). \tag{5.0.1}$$

*where*

$$K = \frac{1}{\rho}\left(\frac{2b+\sigma^2}{2\rho} - \frac{Aa^2(\rho+2b)}{(b+\rho)^2} - 1 + 2AC\right) \tag{5.0.2}$$

*Proof.* We write $V$ as

$$V(x) = -A\left(x + \frac{a}{b+\rho}\right)^2 - \frac{1}{\rho}\ln\left(2A(x + \frac{a}{b+\rho})\right) + K + v(x).$$

Straightforward calculations yield that $v$ is a viscosity solution in $(0, \infty)$

of the equation

$$\rho v + (bx - r(x))\, v' + \ln\left(1 + \frac{1 - \rho\big(x + \frac{a}{b+\rho}\big)v'}{2A\rho\left(x + \frac{a}{b+\rho}\right)^2}\right) - \frac{1}{2}\sigma^2 x^2 v'' + f = 0, \tag{5.0.3}$$

where

$$\begin{aligned} f(x) = {} & \frac{a(b + \frac{\sigma^2}{2}) + (b+\rho)r(x)}{\rho\big(a + x(b+\rho)\big)} + \frac{\sigma^2 x(b+\rho)}{2\rho\big(a + x(b+\rho)\big)^2} \\ & - \frac{2A}{b+\rho}(a - r(x))(a + x(b+\rho)) + 2AC. \end{aligned}$$

Note that $f$ is smooth on $[0, \infty)$ and vanishes as $x \to \infty$ (see Assumption 1).

Let $v_\lambda(y) = v(\frac{y}{\lambda})$ and observe that, if $v_\lambda(1) \to 0$ as $\lambda \to 0$, then $v(x) \to 0$ as $x \to \infty$. It turns out that $v_\lambda$ solves

$$\begin{aligned} \rho v_\lambda + \Big(bx - \lambda r(\frac{x}{\lambda})\Big)\, v_\lambda' + \ln\left(1 + \frac{\lambda^2\big(1 - \rho\big(x + \frac{\lambda a}{b+\rho}\big)v_\lambda'\big)}{2A\rho\left(x + \frac{\lambda a}{b+\rho}\right)^2}\right) & \\ - \frac{1}{2}\sigma^2 x^2 v_\lambda'' + f\big(\frac{x}{\lambda}\big) = 0. & \end{aligned}$$

Since, by (3.3.3) $v_\lambda$ is uniformly bounded, we consider the half-relaxed limits $v^*(y) = \limsup_{x \to y, \lambda \to 0} v_\lambda(x)$ and $v_*(y) = \liminf_{x \to y, \lambda \to 0} v_\lambda(x)$ in $(0, \infty)$, which are (see [6]) respectively sub- and super-solutions of

$$\rho w + byw' - \frac{1}{2}\sigma^2 y^2 w'' = 0. \tag{5.0.4}$$

It is easy to check that for any $y > 0$ we have $v^*(y) = \limsup_{x \to \infty} v(x)$ and $v_*(y) = \liminf_{x \to \infty} v(x)$.

The subsolution property of $v^*$ and the supersolution property of $v_*$ give

$$\limsup_{x \to \infty} v(x) \le 0 \le \liminf_{x \to \infty} v(x) \le \limsup_{x \to \infty} v(x).$$

□

Next, we study the stationary distribution of the optimally controlled lake. Since the optimal control, as it is indicated by Theorem 5, is a feedback control, we can substitute it back in the dynamics of the lake and derive the stochastic differential equation of the optimally controlled system.

Indeed, the optimal dynamics for the shallow lake problem are described by

$$\begin{cases} dx^*(t) = \left(-\frac{1}{V'(x^*(t))} - bx^*(t) + r(x^*(t))\right) dt + \sigma x^*(t) dW(t) \\ x^*(0) = x \end{cases} \tag{5.0.5}$$

The last Proposition describes the behaviour of the tails of the stationary distribution of the amount of phosphorous in the optimally controlled lake.

**Proposition 5.0.1.** *The density, $f$, of the stationary distribution of the optimal dynamics* (5.0.5) *is*

$$f(x) = \frac{1}{Z} x^{-2\left(1+\frac{b}{\sigma^2}\right)} e^{-\frac{2}{\sigma^2}\Phi_\sigma(x)} \tag{5.0.6}$$

*where $Z$ is a normalising constant and*

$$\Phi_\sigma(x) = \int_x^\infty \Big( - \frac{1}{V_\sigma'(u)} + r(u) \Big) \frac{du}{u^2}, \qquad x > 0.$$

*In particular,*

$$\lim_{x \to 0} x\Phi_\sigma(x) = \frac{1}{|V_\sigma'(0)|} \qquad \textit{and} \qquad \lim_{x \to \infty} \Phi_\sigma(x) = 0. \tag{5.0.7}$$

*Proof.* In order to calculate the stationary distribution $f$ of a process $y_t$, we need to solve the stationary Fokker-Planck equation:

$$\mathcal{L}_y^*(f) = 0,\ f \geq 0,\ \int f = 1 \tag{5.0.8}$$

where $\mathcal{L}_y^*$ is the adjoint of the generator $\mathcal{L}_y$ of the process. Equation (5.0.8) takes a convenient form, when the corresponding process $y(t)$ has a constant diffusion coefficient. Particularly, if $dy(t) = g(y(t))dt + \sigma dw_t$, eq. (5.0.8) becomes

$$\frac{\sigma^2}{2}\frac{d^2 f}{dy^2} = \frac{d(gf)}{dy},\ f \geq 0,\ \int f = 1 \tag{5.0.9}$$

from where we deduce that

$$f(y) = Cexp\left(\frac{2}{\sigma^2}\int g(y)dy\right) \tag{5.0.10}$$

where $C$ is a normalization constant. Therefore, regarding the dynamics of the optimally controlled stochastic lake, it is reasonable to apply the transformation $y(t) = \log(x^*(t))$, in order to conclude to a constant diffusion process. Indeed, applying Itô's rule to the process $y(t) = \log x^*(t)$, we find

$$dy(t) = \left(e^{-y(t)}h(e^{y(t)}) - \frac{\sigma^2}{2}\right)dt + \sigma dW_t \tag{5.0.11}$$

where $h(x) = -\frac{1}{V'(x)} - bx + r(x)$ is the drift of the optimally controlled lake $x^*$. If we denote by $g$ the drift of the process $y$ in (5.0.11), then using (5.0.10) we deduce that the stationary distribution of $x^*$ is

$$f(x) = \frac{f_Y(logx)}{x} = \frac{1}{x}Cexp\left(-\frac{2}{\sigma^2}k(x)\right) \tag{5.0.12}$$

where for the function $k$ we have:

$$k'(x) = \frac{g(logx)}{x} = \frac{1}{x^2V'(x)} + \left(b + \frac{\sigma^2}{2}\right)\frac{1}{x} - \frac{r(x)}{x^2}$$

Setting

$$\Phi_\sigma(x) = \int_x^\infty \frac{1}{V'(u)u^2}du - \int_x^\infty \frac{r(u)}{u^2}du,$$

then for a suitable normalization constant $Z$, we conclude to relation (5.0.6) From this representation, we can deduce the asymptotic behaviour of the function $\Phi_\sigma$ at zero and at infinity.

□

# Chapter 6

# Numerical Approximation

The aim of this Chapter is the numerical investigation of the shallow problem. In particular, we want to study the behaviour of the optimally controlled lake and how it depends on the various parameters of the problem. As it was proven in Chapter 4, the optimal control is given in the feedback form, $u(x) = -\frac{1}{V'(x)}$. Therefore, we need first to compute numerically the value function of the problem. To do this, we will solve numerically the Hamilton-Jacobi-Bellman equation (1.0.6) based on the methodology proposed in [33].

## 6.1 Barles–Souganidis Scheme

Barles and Souganidis in [4], developed a general argument to establish convergence of approximation schemes to the viscosity solutions of fully nonlinear second-order elliptic or parabolic, possibly degenerate, partial differential equations. Their methodology doesn't make any convexity or concavity assumptions and has been extensively applied for the numerical approximation of solutions to first-order equations, see e.g. [45, 46, 34, 43], and for various nonlinear second-order equations, see e.g. [40, 5, 11, 25].

Next, following [32] which considered the deterministic problem, we present and implement the monotone finite difference scheme constructed in [33] to approximate numerically the value function of the deterministic and stochastic shallow lake problem and recover numerically the dynamics of optimally controlled lake.

Let $\Delta x$ denote the step size of a uniform partition $0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = l$ of $[0, l]$ for $l > 0$ sufficiently large. Having in mind (3.3.4), if $V_i$ is the approximation of $V$ at $x_i$, we employ a backward finite difference discretization to approximate the first derivative in the linear term of (1.0.6), a forward finite difference discretization for the derivative in the logarithmic term and a central finite difference scheme to approximate the second derivative.

Therefore, we have the following finite-difference, for $i = 1, \ldots, N-1$:

$$V_i - \frac{1}{\rho}\Big(r(x_i) - bx_i\Big)\frac{V_i - V_{i-1}}{\Delta x} + \frac{1}{\rho}\left[cx_i^2 + 1 + \ln\left(-\frac{V_{i+1} - V_i}{\Delta x}\right)\right] - \frac{\sigma^2}{2\rho}x_i^2\frac{V_{i+1} + V_{i-1} - 2V_i}{(\Delta x)^2} = 0. \quad (6.1.1)$$

Setting

$$g(x, w, z, d) = \Big[(\Delta x)^2 - \frac{1}{\rho}\Big(r(x) - bx\Big)\Delta x + \frac{\sigma^2}{\rho}x^2\Big]w + \frac{1}{\rho}(cx^2+1)(\Delta x)^2 + \frac{1}{\rho}(\Delta x)^2\ln\left(-\frac{z - w}{\Delta x}\right) + \frac{1}{\rho}\Delta x\Big(r(x) - bx\Big)d - \frac{\sigma^2}{2\rho}x^2(z+d), \quad (6.1.2)$$

the numerical approximation of $V$ satisfies

$$g(x_i, V_i, V_{i+1}, V_{i-1}) = 0, \text{ for } i = 1, \ldots, N-1 . \quad (6.1.3)$$

Following [4], the numerical approximation scheme defined by (6.1.3) can be written in the form

$$S(r, x, v^r(x), v^r) = 0 \text{ in } [0, \infty), \quad (6.1.4)$$

where $S : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times M(\mathbb{R}^+) \to \mathbb{R}$, $r = \Delta x$ and $v^r$ is defined by $v^r(y) = V_i$ for $y \in [x_i, x_{i+1})$. Here, $M(\mathbb{R}^+)$ is the space of locally bounded functions defined in $\mathbb{R}^+$. For the numerical scheme 6.1.4, it was proven in [33] the following convergence result

**Proposition 6.1.1** (Proposition 2, [33]). *A numerical scheme defined by (6.1.4), with $v^r(x_{i+1}) < v^r(x_i)$, is consistent and monotone, provided that*

$$\Delta x(r(x) - bx) \leq \frac{\sigma^2}{2}x^2. \quad (6.1.5)$$

*In addition, it is stable and* $v^r$, $r \to 0$, *converges locally uniformly to the unique constrained viscosity solution of the equation* (1.0.6).

For the numerical computations, since in (6.1.1) we have $N-1$ equations with $N+1$ unknowns. we exploit the boundary condition (3.3.17) at $x=0$ and we also estimate the value of $V$ at the right endpoint $x=l$ based on the formula (5.0.1) of the asymptotic behaviour of the value function $V$ as $x \to +\infty$.

The set of equations

$$\begin{cases} V_i - \frac{1}{\rho}\Big(r(x_i) - bx_i\Big)\frac{V_i - V_{i-1}}{\Delta x} + \frac{1}{\rho}\left[cx_i^2 + 1 + \ln\left(-\frac{V_{i+1}-V_i}{\Delta x}\right)\right] \\ \qquad\qquad -\frac{\sigma^2}{2\rho}x_i^2\frac{V_{i+1}+V_{i-1}-2V_i}{(\Delta x)^2} = 0 \;\text{ for }\; i = 1,2,...,N-1 \\ V_0 + \frac{1}{\rho}\left[1 + \ln\left(-\frac{V_1 - V_0}{\Delta x}\right)\right] = 0. \end{cases} \tag{6.1.6}$$

together with the boundary condition

$$V_N = -A\left(l + \frac{a}{b+\rho}\right)^2 - \frac{1}{\rho}\ln\left(2A(l + \frac{a}{b+\rho})\right) + K \tag{6.1.7}$$

form a $N \times N$ system of nonlinear equations. In this work, we approximate the solution to this system using the Newton-Raphson method. As an initial estimation of the solution, in order to run the Newton-Rapshon algorithm, we considered a quadratic function $V^0$, such that $V^0(x_N) = V_N$, $V^0(x_0) = V^0(0) = \frac{1}{\rho}\ln\left(\frac{b+\rho}{\sqrt{2ec}}\right)$ (see Proposition 3.3.1(ii)) and $V^0(x_1) = V^0(0) - \Delta x e^{-(\rho V^0(0)+1)}$, so that the second equation of (6.1.6) is (initially) satisfied.

The significance of our methodology for the computation of the constrained viscosity solution $V$ is that we are free to choose any value of the parameter $\sigma$ we want, as long as the condition $\sigma^2 < \rho + 2b$ is met (see Remark 3.3.1) and the condition 6.1.5 is satisfied. In this way, we are not restricted only to small values of the noise parameter $\sigma$, as in the small-noise asymptotics approximation of [26]. Notice also that in the case of the usual choice of the recycling function $r(x) = x^2/(x^2+1)$, condition 6.1.5 is satisfied if e.g. $b \geq 0.5$, independently of the step size $\Delta x$.

In the first part of our numerical investigation, we study the problem with the typical choice of the function $r$, i.e. $r(x) = x^2/(x^2+1)$, while in the second part we study the properties of the value function $V$, which corresponds to a hyperbolic tangent function.

## 6.2 The value function V and the optimal policy

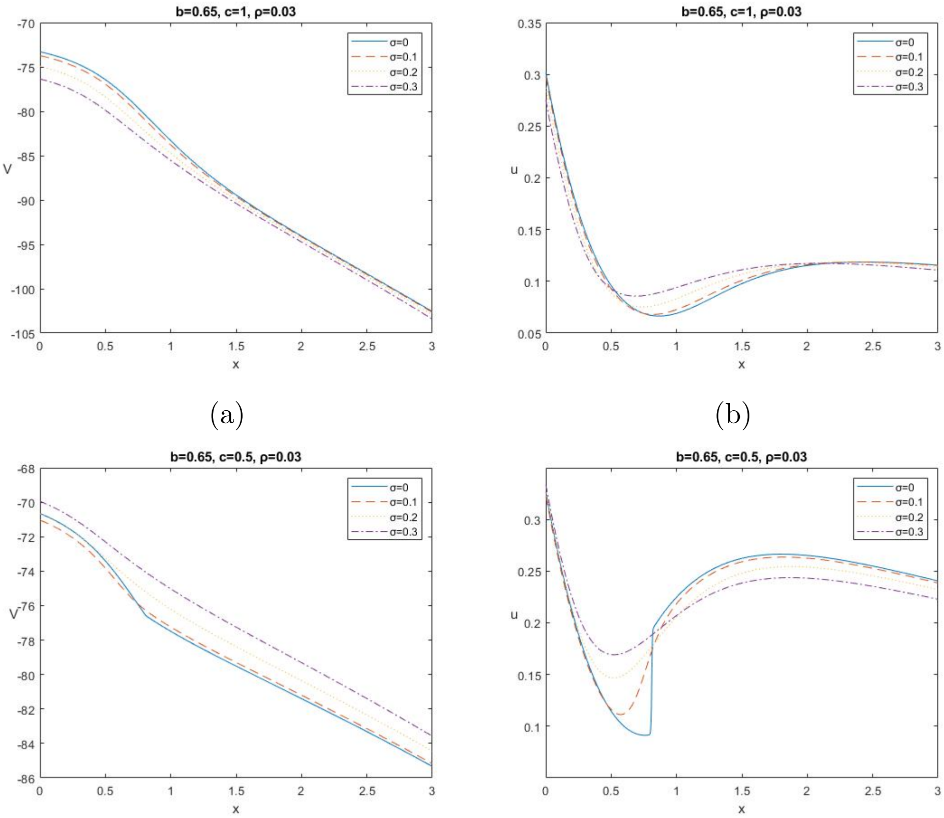
In order to gain some first insight into the problem, we begin by exploring the properties of the value function $V$ for various values of the parameters $b, c, \rho, \sigma$. In our analysis, our choice of parameters is based on the bifurcation analysis made in [49].

Figures 6.1a and 6.1c show the graph of the value function for the fixed parameters $(b, c, \rho) = (0.65, 1, 0.03)$ and $(b, c, \rho) = (0.65, 0.5, 0.03)$ respectively, with the noise $\sigma$ varying. Notice that these graphs also depict the value function in the deterministic case ($\sigma = 0$). Since the optimal policy is given in a feedback form as $u^*(x) = -1/V'(x)$, we can also illustrate the optimal loading of phosphorus as a function of the current amount of phosphorus. In Figures 6.1b and 6.1d, the optimal management policies that correspond to the value functions of Figures 6.1a and 6.1c are shown. For the choice of parameters $(b, c, \rho) = (0.65, 1, 0.03)$, the optimal policies are smooth functions. On the other hand, when $(b, c, \rho) = (0.65, 0.5, 0.03)$, the system exhibits a Skiba point and the noiseless optimal policy is discontinuous at this point. This can be seen as a jump of the optimal policy at the Skiba point.

## 6.3 Invariant distribution

In this section, we numerically investigate the properties of the equilibrium distribution of the optimally controlled lake for different combinations of the parameters of the problem.

From Proposition 5.0.1, we have an exact expression for the stationary distribution $f$ of the optimally controlled lake. For the usual choice of the recycling rate $r$, that is the function $x^2/(x^2 + 1)$, formula (5.0.6) takes the




(a) (b)

(c) (d)

Figure 6.1: The value function $V$ (left) and the optimal policy (right) for different values of noise including the deterministic case ($\sigma = 0$). Parameters: Fig 6.1a- 6.1b: $(b, c, \rho) = (0.65, 1, 0.03)$ and Fig 6.1c-6.1d: $(b, c, \rho) = (0.65, 0.5, 0.03)$

.

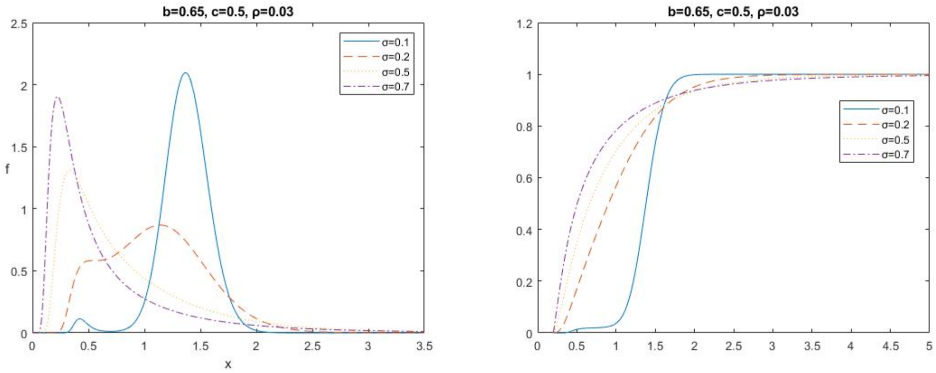
form:

$$f(x) = \frac{1}{Z} x^{-2\left(1+\frac{b}{\sigma^2}\right)} \exp\left(-\frac{2}{\sigma^2}\left(\arctan(x) - \pi/2 + \int_x^{\infty} \frac{1}{V_\sigma'(u)u^2} du\right)\right)$$

where $Z$ is a normalization constant. Therefore, since we have the approximation of the value function $V$, we can also compute the invariant density $f$.

Apart from the invariant density, $f$, and cumulative distribution, $F$, of the optimally controlled lake, we also present some bifurcation diagrams based on its transformation invariant function, $I = \sigma x f$. The utility of this function as a basis of bifurcation theory has been already highlighted (see e.g. [52] and [49]). The main advantage of the transformation invariant function is that it is invariant under transformations of the coordinate systems and gives us insight into the inherent properties of the dynamics of the optimally controlled (stochastic) lake. Following the definitions introduced in [26], the local maximisers of the transformation invariant $I$ are called *stochastic attractors* of the process, while the local minimiser of $I$ is called the *regime switching threshold*. The stochastic attractors are the natural analogue of the attracting steady states of the deterministic problem and the regime switching threshold is the analogue of the indifference point (the Skiba point).

For the fixed parameters $(b, c, \rho) = (0.65, 0.5, 0.03)$ Figure 6.2 shows the invariant density and cumulative distribution functions for several values of the noise parameter $\sigma$. For this set of parameters, the deterministic problem exhibits a Skiba point. In the presence of small noise, the lake spends most of the time in the eutrophic state, while we notice that as noise increases, the lake tends to be "clean" most of the time. Nevertheless, the invariant density function fails to be that concentrated around a particular point as in the case of small noise. In particular, the polynomial tails of the density function get fatter as $\sigma$ increases (see Prop. 5.0.6). A more detailed presentation of this shift from a bimodal distribution (with a peak at the eutrophic state) to a unimodal one with a peak to the oligotrophic (clean) state due to the increase of noise is depicted in Figure 6.3. In this diagram, the locations of the modes and antimodes of the transformation invariant $I$ with respect to $\sigma$ are illustrated. In the case of the fixed param-




(a) (b)

Figure 6.2: Invariant density and cumulative distribution function for different values of the noise parameter $\sigma$. The choice of parameters is $(b, c, \rho) = (0.65, 0.5, 0.03)$.

eters $(b, c, \rho) = (0.8, 0.5, 0.03)$, the deterministic problem exhibits a unique equilibrium in the eutrophic state (see [49]). Therefore, we have qualitatively different dynamics comparing to the previous case. In the presence of small noise, the invariant is unimodal with a peak at the eutrophic state, but with the introduction of more and more noise the location of the mode is moved to a cleaner state. These results are summarized in Figures 6.4 and 6.5. The same behaviour for large values of noise, as in the previous cases, is also present for combinations of parameters for which the deterministic problem exhibits a unique equilibrium in the oligotrophic state. Based on the above observations, we could say that the introduction of more noise seems to eventually "clean" the lake. Notice that if we were limited to small values of noise, e.g. $\sigma < 0.2$ see Figures 6.3 and 6.5, we could not observe this behaviour.

Figure 6.6a illustrates a bifurcation diagram for the fixed parameters $(b, \rho) = (0.65, 0.03)$ and noise $\sigma = 0.1$ with respect to the cost of pollution $c$. As it was expected, according to the definition of the total benefit (1.0.2), large values of $c$ give more weight to the ecological services of the lake and thus "clean" the lake. This is not the case, for the bifurcation diagram with respect to the discount factor $\rho$. In Figure 6.6b the bifurcation diagram

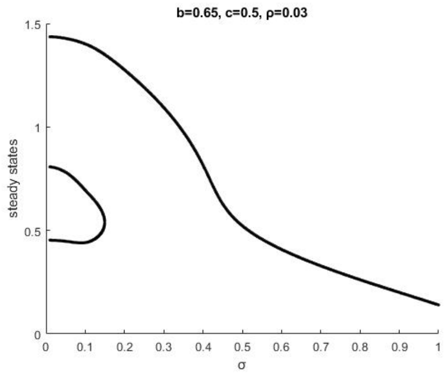
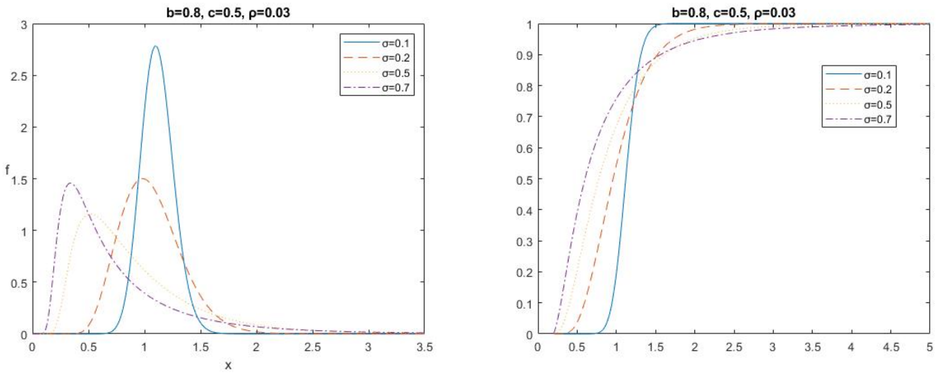



Figure 6.3: Bifurcation diagrams for the extrema of the transformation invariant function with respect to the noise parameter $\sigma$ when $(b, c, \rho) = (0.65, 0.5, 0.03)$. The vertical axis corresponds to the location of the extrema of the transformation invariant function.

with respect to $\rho$ for the fixed parameters $(b, c) = (0.65, 0.8)$ and noise $\sigma = 0.1$ is depicted. In this diagram, we observe that as the discount factor $\rho$ increases, the stochastic attractors of the system move towards the eutrophic states. This may indicate that as we increase $\rho$, that is we care less for the future generations and more for the current value of the lake, we conclude to load more phosphorus into the lake.

## 6.4 The rate of recycling

In this section, we present some numerical results, when a hyperbolic tangent function is used as the rate of recycling. Initially, we consider as rate of recycling the function $r(x) = \tanh(x - 3) + \tanh(3)$. In Figure 6.7, we present the value function for different combinations of the parameters $(b, c, \rho)$ and different values of noise. In Figure 6.8 the invariant density functions and the optimal policies, which correspond to the preceding value functions, are shown. We observe that the lake exhibits two attractors for small values of noise, when $(b, c, \rho) = (0.8, 0.06, 0.5)$, while it has only one when $(b, c, \rho) = (0.65, 0.5, 0.03)$ and $(b, c, \rho) = (0.5, 0.5, 0.01)$. Nevertheless, in all three cases, the increase of noise shifts the mass to cleaner states of the lake. Afterwards, in Figure 6.10 we illustrate the changes in the value




(a) (b)

Figure 6.4: Invariant density and cumulative distribution function for different values of the noise parameter $\sigma$. The choice of parameters is $(b, c, \rho) = (0.8, 0.5, 0.03)$.

function, which are induced by small changes in the rate of recycling $r$. Particularly, we numerically approximate the value functions $V$ that correspond to the rate of recycling $r(x) = \frac{1}{2}(\tanh(a(x-3)) + \tanh(3a))$ for various values of the parameter $a$ and the step function $\mathbf{1}\{x > 3\}$ in the deterministic and stochastic ($\sigma = 0.1$) case.

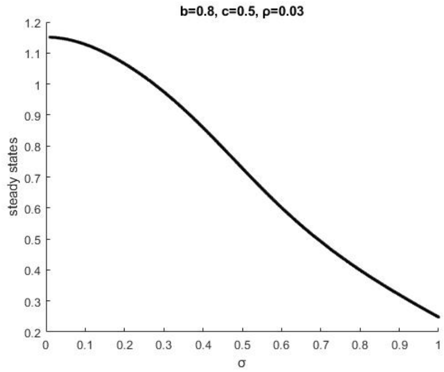
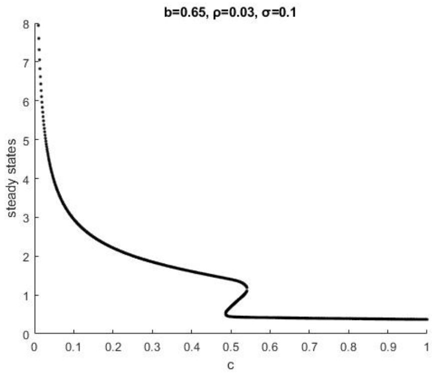
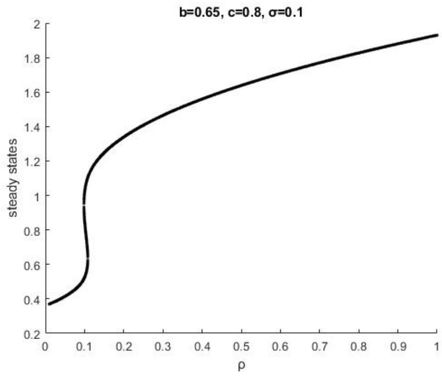



Figure 6.5: Bifurcation diagram for the extrema of the Transformation Invariant function with respect to the noise parameter $\sigma$ when $(b, c, \rho) = (0.8, 0.5, 0.03)$. The vertical axis stands for the location of the critical points of the Transformation Invariant function.

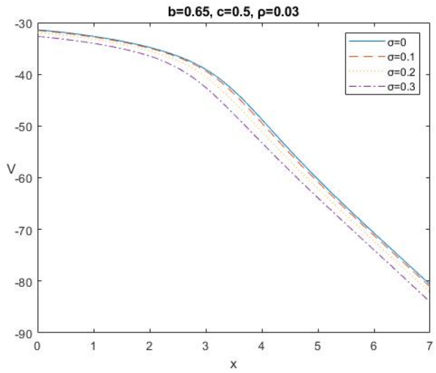



(a)

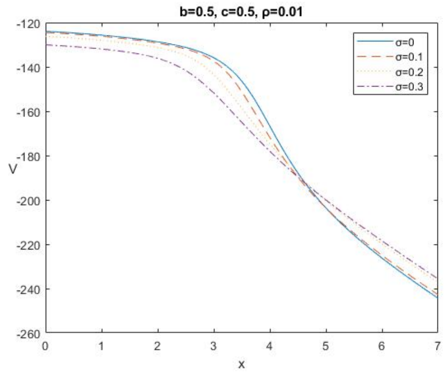



(b)

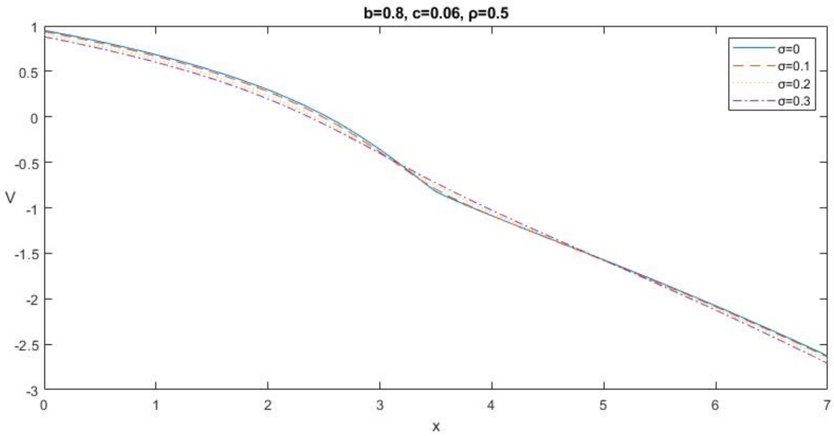
Figure 6.6: Left: Bifurcation diagram for the transformation invariant function with respect to the cost of pollution $c$ when $(b, \rho, \sigma) = (0.65, 0.03, 0.1)$. Right: Bifurcation diagram for the transformation invariant distribution with respect to the discount factor $\rho$ when $(b, c, \sigma) = (0.65, 0.8, 0.1)$. The vertical axes correspond to the location of the extrema of the transformation invariant function.




(a)




(b)




(c)

Figure 6.7: The value function $V$ for different values of noise. Fig. 6.7a $(b, c, \rho) = (0.65, 0.5, 0.03)$, Fig. 6.7b $(b, c, \rho) = (0.5, 0.5, 0.01)$, Fig. 6.7c $(b, c, \rho) = (0.8, 0.06, 0.5)$

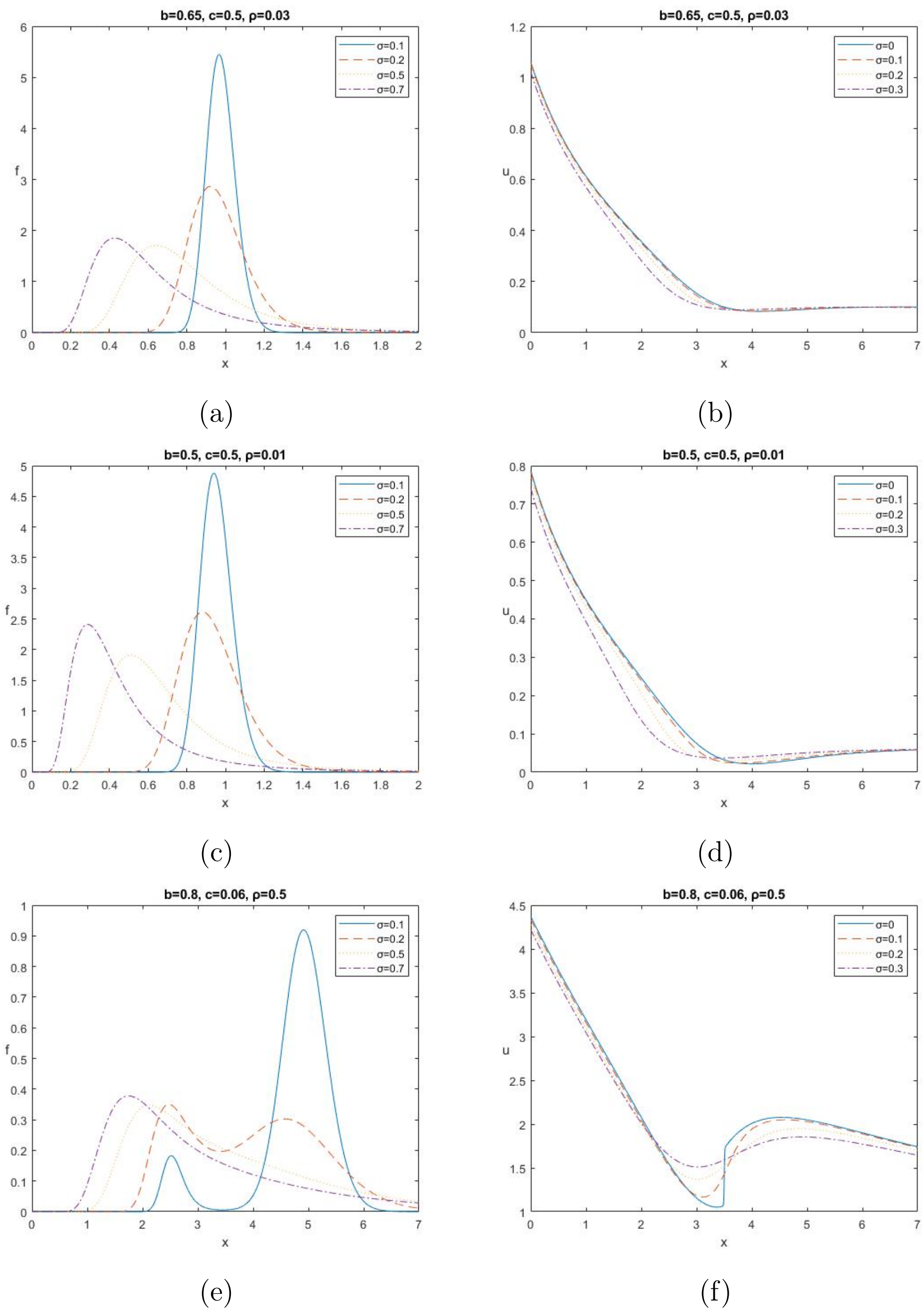



Figure 6.8: The equilibrium distribution (left) and optimal policy (right) for different values of noise.

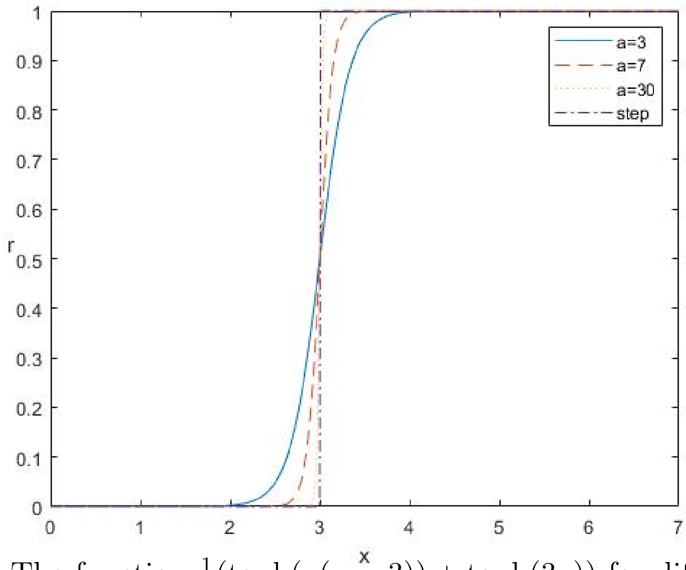
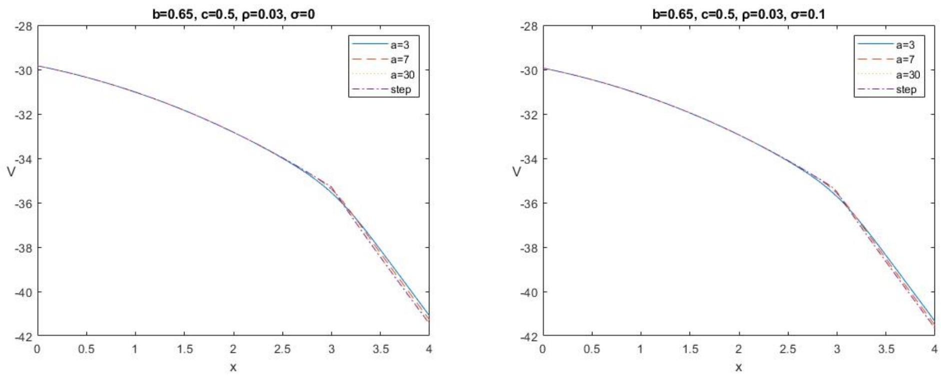



Figure 6.9: The function $\frac{1}{2}(\tanh(a(x-3)) + \tanh(3a))$ for different values of the parameter $a$ and the step function $\mathbf{1}\{x > 3\}$ .




(a) (b)

Figure 6.10: The Value function $V$ for rate of recycling of the form $\frac{1}{2}(\tanh(a(x-3)) + \tanh(3a))$ for different values of the parameter $a$ and the step function $\mathbf{1}\{x > 3\}$ in the deterministic (left) and stochastic (right) case.

# Chapter 7

# Metastability

In this Chapter, we present the last main contribution of this work which is an analysis of the metastable behaviour of stochastic control problems which exhibit Skiba points. In more detail, we study the expected value of the transition time from the one well to the other for a process in a noise-dependent double-well potential and we prove a generalization of the Arrhenius law. In section 7.3, we consider the shallow lake problem as an application to the described methodology.

## 7.1 Main results

We assume that there exists a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions, and a Brownian motion $W(\cdot)$ defined on that space. Let us also assume that the state dynamics is now described by the following autonomous stochastic differential equation:

$$\begin{cases} dx^\varepsilon(t) = f(x^\varepsilon(t), u(t))dt + \sqrt{2\varepsilon}dW_t \quad t \geq 0 \\ x^\varepsilon(0) = x \geq 0 \end{cases} \tag{7.1.1}$$

where the function $f \in C^1(\mathbb{R} \times U)$ and it satisfies:

$$\begin{cases} |f_x| \leq C \\ |f(x,u)| \leq C(1 + |x| + |u|) \end{cases} \tag{7.1.2}$$

Let $G$ be continuous on $\mathbb{R} \times U$, $G(x, \cdot) \in C^1(U)$ and $\rho > 0$. We consider the value function of the problem

$$V_\varepsilon(x) = \sup_{u \in \mathfrak{U}} \mathbb{E}_x \left[ \int_0^\infty e^{-\rho s} G(x^\varepsilon(s), u(s)) ds \right] \tag{7.1.3}$$

where $\mathfrak{U}$ is the set of $\mathcal{F}_t-$adapted, $\mathbb{P}-$a.s. locally integrable processes with values in $U$ satisfying

$$\mathbb{E}_x \left[ \int_0^\infty e^{-\rho s} G(x^\varepsilon(s), u(s)) ds \right] < \infty$$

such that the stochastic differential equation (7.1.1) has a unique strong solution $x(\cdot)$.
Henceforward, we make the following assumptions:

**Assumption 2.** *(i) The value function $V_\varepsilon$ is a (classical) solution to the associated HJB equation:*

$$-\varepsilon V_\varepsilon'' - H(x, V_\varepsilon') + \rho V_\varepsilon = 0 \tag{7.1.4}$$

*where* $H(x,p) = \sup_{u \in U} \{ f(x,u)p + G(x,u) \}$.

*(ii) There exists $\varepsilon_0 > 0$ such that $V_\varepsilon$ and $V_\varepsilon'$ are uniformly bounded with respect to $\varepsilon < \varepsilon_0$ on every compact subset of $\mathbb{R}$*

*(iii) $H(x,p)$ is $C^2$ and $H_{pp} > 0$*

*(iv) There exists an optimal stationary Markov control policy of the form $u^*(s) = g\left(x^*(s), V_\varepsilon'(x^*(s))\right) =: \bar{u}^*(x^*(s))$ such that*

$$f_u(x, \bar{u}^*(x)) V_\varepsilon'(x) + G_u(x, \bar{u}^*(x)) = 0$$

*and $g$ is a continuous function.*

Under Assumption 2, the optimally controlled system (7.1.1) takes the form:

$$\begin{cases} dx^\varepsilon(t) = -F_\varepsilon'(x^\varepsilon(t))dt + \sqrt{2\varepsilon} dW_t & t \geq 0 \\ x^\varepsilon(0) = x \geq 0 \end{cases} \tag{7.1.5}$$

where $F'_\varepsilon(x) = -f(x, g(x, V'_\varepsilon(x))$. Based on Assumption 2 (i)-(ii) and stability property of viscosity solutions,

$$V_\varepsilon \overset{\varepsilon \to 0}{\to} V_0 \text{ locally uniformly} \tag{7.1.6}$$

where $V_0$ is a viscosity solution to the HJB equation (7.1.4) for $\varepsilon = 0$. We make the following assumption on function $V_0$.

**Assumption 3.** *The function $V_0$ of* (7.1.6) *is almost everywhere differentiable.*

**Lemma 7.1.1.** *Assuming 2 and 3, let $\Omega$ be a compact subset of $\mathbb{R}$. Then:*

*(i) There exists $C = C(\Omega)$ such that $V''_\varepsilon(x) \geq C$ for all $x \in \Omega$ and $\varepsilon < \varepsilon_0$.*

*(ii) The family of functions $\{F_\varepsilon\}_{\varepsilon>0}$ converges locally uniformly to a function $F_0$, which is almost everywhere differentiable with*

$$F'_0(x) = -f(x, g(x, V'_0(x))).$$

We are now ready to state our main result:

**Theorem 7.** *We assume that the function $F_0$ of Lemma 7.1.1 forms a double well potential with local minima $x_\pm$ and local maximum $x_*$, with $x_- < x_* < x_+$ and that $F_0$ is $C^1$ close to $x_\pm, x_*$. Furthermore, we assume that there exist $a, q, M > 0$ and $b \in \mathbb{R}$ (independent of $\varepsilon$) such that $F_\varepsilon(x) \geq ax^q + b$, for all $x > M$. Let $\tau^\varepsilon_{x_-} = \inf\{t \geq 0 : \; x^\varepsilon(t) \leq x_-\}$ be the first hitting time of $x_-$ of the stochastic process $x^\varepsilon$ of* (7.1.5). *Then the expectation of $\tau^\varepsilon_{x_-}$, when $x^\varepsilon$ starts at $x_+$, satisfies*

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_{x_+} \left[\tau^\varepsilon_{x_-}\right] = F_0(x_*) - F_0(x_+) \tag{7.1.7}$$

We will now state a result which is of general interest and stands on its own. We begin by stating the assumptions of the Lemma.

**Assumption 4.** *We assume that the drift $f$ is linear with respect to $u$, i.e. it is of the form $f(x, u) = a(x)u + b(x)$ and the function $G(x, u)$ is a concave function of $u$ for all $x \in \mathbb{R}$. We further assume that the noiseless value function $V_0$ is a classical solution to the HJB eq.* (7.1.4) *for $\varepsilon = 0$ everywhere except for a finite number of points $x_0$ whereat it is not differentiable, but there exist the side derivatives $V'_0(x_0(-))$, $V'_0(x_0(+))$.*

**Lemma 7.1.2.** *Under Assumption 4, if there exists the optimal control, then it is given in the feedback form* $u^*(x) = h_x(-a(x)V_0'(x))$*, where* $h_x$ *is the inverse of the function* $G_u(x,\cdot)$*, and for the drift of the optimally controlled system at* $x_0$ *we have:*

$$f(x_0, u^*(x_0(-))) \cdot f(x_0, u^*(x_0(+))) \neq 0$$

*Proof.* Since the optimal control exists, it should satisfy the following relation:

$$a(x)V_0'(x) + G_u(x,u) = 0$$

which gives $u^*(x) = h_x(-a(x)V_0'(x))$. By contradiction, let us assume that

$$f(x_0, u^*(x_0(-))) = a(x_0)h_{x_0}\left(-a(x_0)V_0'(x_0(-))\right) + b(x_0) = 0, \qquad (7.1.8)$$

and without loss of generality we assume that $V_0'(x_0(-)) < V_0'(x_0(+))$. Taking $x \to x_0(-)$ in HJB, by (7.1.8), we find:

$$\rho V(x_0) = G\left(x_0, h_{x_0}(-a(x_0)V_0'(x_0(-)))\right) \qquad (7.1.9)$$

Taking now $x \to x_0(+)$ in HJB and substituting (7.1.9) and (7.1.8), we find:

$$\begin{aligned} G\left(x_0, h_{x_0}(-a(x_0)V_0'(x_0(-)))\right) - G\left(x_0, h_{x_0}(-a(x_0)V_0'(x_0(+)))\right) = \\ -\, a(x_0)\Big(h_{x_0}(-a(x_0)V_0'(x_0(-))) - h_{x_0}(-a(x_0)V_0'(x_0(+)))\Big)V'(x_0(+)) \end{aligned}$$

which by Mean Value Theorem and the monotonicity of $h_{x_0}$, gives

$$G_u(x_0, h_{x_0}(-a(x_0)y) = -a(x_0)V'(x_0(+))) \qquad (7.1.10)$$

for some $y \in (V'(x_0(-)), V'(x_0(+)))$, which is a contradiction, since relation (7.1.10) implies that $y = V'(x_0(+))$. □

An interesting application of this result is presented in Lemma B.1(iv) in Appendix B in the case of the shallow lake problem. The fact that the drift of the optimally controlled lake does not disappear at the Skiba point, from the left and from the right, guarantees the existence of the side limits of the second derivative of $V$ at the Skiba point.

# 7.2 Proofs

## 7.2.1 Proof of Lemma 7.1.1

(i) We will prove this claim following the lines of proof of Lemma 3.1 in [21]. Let $m = \inf\{\Omega\} > -\infty$, $M = \sup\{\Omega\} < \infty$. If $l = \max\{|m|, |M|\}$, we consider a function $\phi \in C^\infty(\mathbb{R})$ such that

$$\begin{aligned} \phi(x) &= 1 \quad \text{if } x \in \Omega \\ \phi(x) &= 0 \quad \text{if } x < -2l \text{ or } x > 2l \\ \tfrac{(\phi'(x))^2}{\phi(x)} &\leq C_l \quad \text{on supp } \phi \end{aligned}$$

For brevity, we write $u = V_\varepsilon$, $u_1 = V'_\varepsilon$ and $u_{11} = V''_\varepsilon$. Differentiating twice eq. (7.1.4) with respect to $x$, we obtain:

$$-\varepsilon u''_{11} - H_{xx} - 2H_{px}u_{11} - H_{pp}\,(u_{11})^2 - H_p u'_{11} + \rho u_{11} = 0$$

Next we define $w = \phi u_{11}$ and compute (on $\phi > 0$):
$-\varepsilon w'' - H_p(x, u_1)w' + \rho w + 2\varepsilon \frac{\phi'}{\phi} w'$

$$= \phi H_{pp}(x, u_1)\,(u_{11})^2 + \phi H_{xx} + 2\phi H_{px} u_{11} + \varepsilon \left(\frac{2(\phi')^2)}{\phi} - \phi''\right) u_{11} + H_p \phi' u_{11}$$

Let $x_0$ be a point in $(\phi > 0)$ at which $w$ attains a negative minimum. If for some value of $\varepsilon$, $w$ is non-negative on $(\phi > 0)$, we can conclude that $V''_\varepsilon(x) \geq 0$ $\forall x \in \Omega$. Then $w'(x_0) = 0$, $w(x_0) \leq 0$ and $w''(x_0) \geq 0$. Moreover, by Assumption 2(iii) there exists $\eta_0 > 0$ such that $H_{pp}(x, u_1) > \eta_0 > 0$ for all $x \in \Omega$.
Therefore, we have at $x = x_0$

$$\eta_0\,(w(x_0))^2 \leq -\phi^2 H_{xx} + (-2\phi H_{px} - \varepsilon C_l + \varepsilon\phi'' - H_p\phi')\,w(x_0)$$

$$\Rightarrow \eta_0 w^2(x_0) \leq A + Bw(x_0) \;\Rightarrow\; w(x_0) \geq \tilde{C}$$

where $\tilde{C}$ constant independent of $\varepsilon$ (and $x_0$). Here, we used the fact that since $u_1 = V'_\varepsilon$ is bounded on supp $\phi$ by Assumption 2(ii) (uniformly with respect to $\varepsilon$), each partial derivative of $H$ is bounded at $(x, u_1(x))$ for $x \in$ supp $\phi$.
Therefore $V''_\varepsilon(x) \geq w(x_0) \geq \tilde{C}$, $\forall x \in \Omega,\ \varepsilon \leq \varepsilon_0$. □

(ii) Since $V_\varepsilon''$ is locally uniformly bounded with respect to $\varepsilon$ and $V_0$ is almost everywhere differentiable, it follows (see Theorem 3.2 (i) [21]) that $V_\varepsilon' \to V_0'$ almost everywhere. Therefore, the continuity of $g$ and the boundedness of $V_\varepsilon'$ on compact sets imply, by bounded convergence theorem, that $F_\varepsilon$ converges locally uniformly to a function $F_0$, which is almost everywhere differentiable with

$$F_0'(x) = -f(x, g(x, V_0'(x))).$$

$\square$

### 7.2.2 Proof of Theorem 7

For every $\varepsilon > 0$, we know that the function $h(x) = \mathbb{E}_x\left[\tau_{x-}^\varepsilon\right]$ solves the Poisson problem:

$$\begin{cases} \mathcal{L}h(x) = -1, & x > x_- \\ h(x) = 0, & x \le x_- \end{cases}$$

where $\mathcal{L}$ is the generator of the process $x^\varepsilon$ in (7.1.5). That is

$$\begin{cases} (\varepsilon h''(x) - F_\varepsilon'(x)h'(x)) = -1, & x > x1 \\ h(x) = 0, & x \le x_- \end{cases}$$

This problem is solved explicitly and for $x = x_+$ takes the form:

$$\mathbb{E}_{x_+}\left[\tau_{x-}^\varepsilon\right] = \frac{1}{\varepsilon}\int_{x_-}^{x_+}\int_{z}^{\infty} \exp\left(\frac{F_\varepsilon(z) - F_\varepsilon(y)}{\varepsilon}\right) dydz$$

We denote by $D$ the area of integration. Notice that in $D$, the function $(z, y) \mapsto F_0(z) - F_0(y)$ attains its maximum at $(x_*, x_+)$ and for $A > x_+$, let us consider the compact set $D_1 = D \cap ((-\infty, A] \times \mathbb{R})$ which contains the point $(x_*, x_+)$ and the unbounded set $D_2 = D \setminus D_1$. Moreover, denote by $I_1(\varepsilon), I_2(\varepsilon)$ the integral of $\exp\left(\frac{F_\varepsilon(z)-F_\varepsilon(y)}{\varepsilon}\right)$ over $D_1$ and $D_2$, respectively, to get

$$\iint_D e^{\frac{1}{\varepsilon}(F_\varepsilon(z)-F_\varepsilon(y))}dydz = I_1(\varepsilon) + I_2(\varepsilon) \tag{7.2.1}$$

Since $D_1$ is compact and $F_\varepsilon$ converges uniformly on compact sets to $F_0$, there exists a non-negative function $\lambda(\varepsilon)$, with $\lim_{\varepsilon\to 0}\lambda(\varepsilon) = 0$, such that

$$|(F_\varepsilon(z) - F_\varepsilon(y)) - (F_0(z) - F_0(y))| \le \lambda(\varepsilon)\ \forall (y,z) \in D_1.$$

Therefore,

$$e^{-\frac{\lambda(\varepsilon)}{\varepsilon}} \iint_{D_1} e^{\frac{1}{\varepsilon}(F_0(z)-F_0(y))} dydz \le I_1(\varepsilon) \le \iint_{D_1} e^{\frac{1}{\varepsilon}(F_0(z)-F_0(y))} dydz e^{\frac{\lambda(\varepsilon)}{\varepsilon}}$$

Which gives from standard Laplace asymptotics that,

$$\lim_{\varepsilon\to 0} \varepsilon \log I_1(\varepsilon) = F_0(x_*) - F_0(x_+) \tag{7.2.2}$$

Since $F_\varepsilon$ converges uniformly to $F_0$ on $[x_-, x_+]$, there exists $\varepsilon_1 > 0$ such that $F_\varepsilon(z) < F_0(z) + 1 \le F_0(x_*) + 1$, $\forall z \in [x_-, x_+]$, $\forall \varepsilon < \varepsilon_0$. Thus,

$$I_2(\varepsilon) = \int_{x_-}^{x_+}\int_{A}^{\infty} e^{\frac{1}{\varepsilon}(F_\varepsilon(z)-F_\varepsilon(y))} dydz \le e^{\frac{1}{\varepsilon}(F_0(x_*)+1)}(x_+ - x_-)\int_{A}^{\infty} e^{-\frac{1}{\varepsilon}(ay^q+b)} dy$$

which based on bounds of the upper incomplete Gamma function (see [42]), gives

$$\limsup_{\sigma\to 0} \varepsilon \log I_2(\varepsilon) \le F_0(x_*) + 1 - b - aA^q \tag{7.2.3}$$

By choosing $A$ sufficiently large (such that $1 - b - aA^q < -F_0(x_+)$), from relations (7.2.1), (7.2.2), (7.2.3) we find that

$$\lim_{\varepsilon\to 0} \varepsilon \log \iint_D \exp\left(\frac{F_\varepsilon(z) - F_\varepsilon(y)}{\varepsilon}\right) dydz = F_0(x_*) - F_0(x_+)$$

which concludes the proof. □

## 7.3 Application to the shallow lake problem

The metastable behaviour of shallow lakes is naturally observed and in mathematical terms corresponds to a system with two equilibrium points and a Skiba point. In the presence of noise, the system moves from the oligotrophic state to the eutrophic state and vice versa, see Figure 7.1.

Furthermore, the markovian nature of the optimal control leads to a system with a noise-dependent drift function. Therefore, the shallow lake problem is offered as a suitable application of our result, Theorem 7. In Figure 7.2, the double-well potential of the deterministic shallow lake problem is depicted indicating the difference term appearing in relation (7.1.7). Notice that this quantity is exactly identified as the height of the barrier that the process has to overcome in order to get to the first well. In the following, we assume that the parameters $b, c, \rho$ are such that the deterministic shallow lake problem possesses two equilibrium points and one Skiba point.

If we apply the transformation $y(t) = \log x(t)$ to the process $x(t)$ of the stochastic shallow lake problem (1.0.1), we find by It$\hat{o}$'s rule:

$$\begin{cases} dy(t) = \left(e^{-y(t)}u(t) - b + r\left(e^{y(t)}\right)e^{-y(t)} - \frac{\sigma^2}{2}\right)dt + \sigma dW_t \\ y(0) = y \end{cases} \tag{7.3.1}$$

Therefore, the dynamics of the shallow lake problem is described in terms of (7.1.1). We can now consider the value function of the shallow lake problem in terms of the process $y(t)$, i.e.

$$\tilde{V}(y) = \sup_{u\in\mathfrak{U}} \mathbb{E}_y \left[\int_0^\infty e^{-\rho t}\left(\ln u(t) - ce^{2y(t)}\right)dt\right] = V(e^y)$$

From Proposition 4.2.1, it follows that $\tilde{V}$ is a classical solution in $\mathbb{R}$ to equation

$$-\frac{1}{2}\sigma^2\tilde{V}'' - \tilde{H}(x, \tilde{V}') + \rho\tilde{V} = 0 \tag{7.3.2}$$

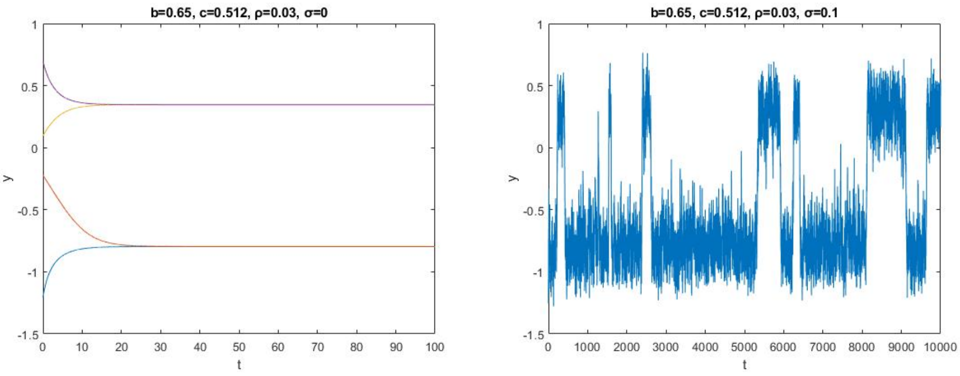
where $\tilde{H}(x,p) = \left(r(e^x)e^{-x} - b - \frac{\sigma^2}{2}\right)p - \ln(-p) + x - 1 - ce^{2x}$ and the optimally controlled system (7.3.1) takes the form:

$$\begin{cases} dy^\sigma(t) = -F'_\sigma(y^\sigma(t))dt + \sigma dW_t \\ y^\sigma(0) = y \end{cases} \tag{7.3.3}$$

where

$$F'_\sigma(y) = \frac{1}{\tilde{V}'_\sigma(y)} + b - r\left(e^y\right)e^{-y} + \frac{\sigma^2}{2}$$

Obviously, our verification argument for the deterministic shallow lake problem implies that $F_0$ is $C^1\left(\mathbb{R}\setminus\{x_*\}\right)$, where $x_*$ is the logarithm of




(a) (b)

Figure 7.1: 7.1a: The paths of the optimally controlled lake (deterministic case) for different initial positions. 7.1b: One simulated path of the optimally controlled lake (stochastic case) with two stochastic attractors.

the Skiba point. Proposition 7.3.1 and Lemma 7.3.1 show that the shallow lake problem, under the assumption of two saddle equilibrium points and one Skiba point, satisfies the hypothesis of Theorem 7. Therefore, the Arrhenius Law is stable under our model.

**Proposition 7.3.1.** *We assume that the recycling rate function $r$, in addition to Assumption 1, satisfies also that $r(x) < bx \;\forall\; x > 0$. Let $\Omega \subset \mathbb{R}$ compact and $\sigma_0 < \sqrt{\rho}$. Then there exists $C = C(\Omega, \sigma_0) > 0$ such that $|\tilde{V}_\sigma'(x)| \leq C$ for all $x \in \Omega$, $\sigma \leq \sigma_0$.*

*Proof.* Since $\tilde{V}_\sigma'(x) = V'(e^x)e^x$, Proposition 3.3.2(ii) implies that there exists $C > 0$ (independent of $\sigma$) such that $\tilde{V}'(x) \leq -C < 0$ for all $x \in \Omega$. Therefore, it suffices to show that there exists a continuous function $\Phi : (0, \infty) \to (0, \infty)$ independent of $\sigma$ such that $V_\sigma'(x) \geq -\Phi(x)$ for all $x > 0$.

Let $0 < x_1 < x$ and for $d > 0$ consider a control $u \in \mathfrak{U}$ which equals $d$ up to time $\tau_d = \inf\{t \geq 0 : \; x(t) \leq x_1\}$ where $x(\cdot)$ is the solution of (1.0.1) with control $u$ and $x(0) = x$. Then

$$V_\sigma(x) \geq \mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t}(\ln u(t) - cx^2(t))dt\right] + \mathbb{E}\left[e^{-\rho\tau_d}\right] V_\sigma(x_1)$$

$$= \ln d \cdot \mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t} dt\right] - c\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t} x^2(t) dt\right] + \mathbb{E}\left[e^{-\rho\tau_d}\right] V_\sigma(x_1)$$

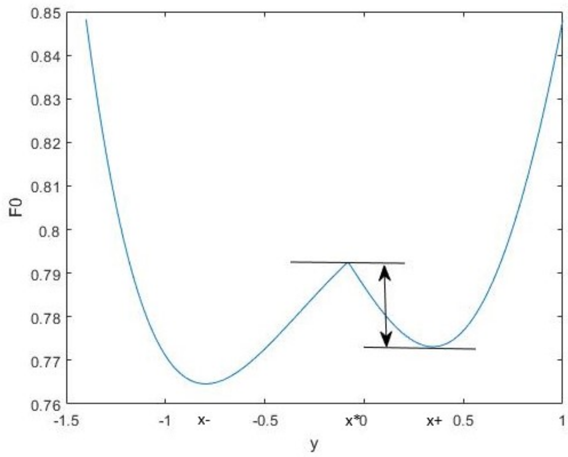



Figure 7.2: The potential $F_0$ of the deterministic optimally controlled shallow lake when $(b, c, \rho, \sigma) = (0.65, 0.512, 0.03, 0)$

Based on Proposition 3.3.1(ii) and 3.3.2(ii), $V_\sigma(x) \leq \frac{1}{\rho}\ln\left(\frac{b+\rho}{\sqrt{2ec}}\right) =: D$ for all $\sigma \leq \sigma_0$, $x \geq 0$. Thus, we find:

$$V_\sigma(x) - V_\sigma(x_1) \geq (\ln d - \rho D)\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t} dt\right] - \mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t} c x^2(t) dt\right] \quad (7.3.4)$$

Applying Itô's rule to the semimartingale $Y_t = e^{-\rho t}x^2(t)$, then the optional stopping theorem for the bounded stopping time $\tau_N = \tau_c \wedge \inf\{t \geq 0 : x(t) \geq N\} \wedge N$, and letting $N \to \infty$, we find

$$x_1^2 \mathbb{E}\left[e^{-\rho\tau_d}\right] - x^2 \leq \mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t}\Big( -\rho x^2(t) + 2x(t)(d - bx(t) + r(x(t))) + \sigma^2 x^2(t)\Big) dt\right]$$

If $m = m(x_1) = -\sup_{x \geq x_1}\{-bx + r(x)\}$, then $m > 0$. Choosing $d \leq m/2$, we have

$$x_1^2 - x^2 \leq -(\rho - \sigma_0^2)\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t} x^2(t) dt\right] + \rho x_1^2 \mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t} dt\right] \quad (7.3.5)$$

Combining (7.3.4) and (7.3.5), we find

$$V_\sigma(x) - V_\sigma(x_1) \geq \left(\ln d - \rho D - \frac{c\rho x_1^2}{\rho - \sigma_0^2}\right)\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t} dt\right] + \frac{c}{\rho - \sigma_0^2}(x_1^2 - x^2) \quad (7.3.6)$$

In order to control the term $\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t}dt\right]$, we now apply Itô's rule to the semimartingale $Z_t = e^{-\rho t}x(t)$, following the same steps as above, to find:

$$x_1\mathbb{E}\left[e^{-\rho\tau_d}\right] - x \leq \mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t}\left(-\rho x(t) + d - bx(t) + r(x(t))\right)dt\right]$$
$$\leq x_1\mathbb{E}\left[e^{-\rho\tau_d}\right] - x_1 - \frac{m}{2}\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t}dt\right]$$

Thus

$$x_1 - x \leq -\frac{m}{2}\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t}dt\right] \tag{7.3.7}$$

If we finally choose $d = d(x_1) = \min\left[\exp\left(\rho D + \frac{c\rho x_1^2}{\rho - \sigma_0^2}\right), m(x_1)\right]/2$, we find

$$V_\sigma(x) - V_\sigma(x_1) \geq \frac{2}{m(x_1)}\left(\ln d(x_1) - \rho D - \frac{c\rho x_1^2}{\rho - \sigma_0^2}\right)(x - x_1) - \frac{c}{\rho - \sigma_0^2}2x(x - x_1) \tag{7.3.8}$$

Dividing by $x - x_1$ and taking the limit $x \to x_1$, we conclude that

$$V_\sigma'(x) \geq \frac{2}{m(x)}\left(\ln d(x) - \rho D - \frac{c\rho x^2}{\rho - \sigma_0^2}\right) - \frac{2cx}{\rho - \sigma_0^2} =: \Phi(x)$$

Therefore, $\tilde{V}'(x) \geq \Phi(e^x)e^x$ for all $x \in \mathbb{R}$.

□

**Lemma 7.3.1.** *$F_\sigma(x) \geq bx + C$ for all $x > 1$, where $C$ constant independent of $\sigma$.*

*Proof.* Let $x > 1$. Since $V_\sigma'(x) \leq -C < 0$, where $C$ is a constant independent of $\sigma$ and $r(x) \leq a$ for all $x \geq 0$, we have that

$$F_\sigma(x) = \int_1^x \left(\frac{1}{V_\sigma'(e^y)} - r(e^y)\right)e^{-y}dy + (b + \frac{\sigma^2}{2})x > bx + Const$$

□

# Appendix A

# Some useful identities

**Lemma A.1.** *Assume that $f$ is a positive $\mathbb{P}$-a.s. locally integrable $\mathcal{F}_t$ and let $M_t(f)$ be defined as in (3.2.1). Then,*

*(i)* $\mathbb{E}\left[\int_0^\infty e^{-\rho t} M_t(f)\ dt\right] = \dfrac{1}{\rho+b}\ \mathbb{E}\left[\int_0^\infty e^{-\rho t} f(t)\ dt\right].$

*(ii)*

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} Z_t M_t(f)\ dt\right] = \begin{cases} Ac^{-1}\mathbb{E}\left[\int_0^\infty e^{-\rho t} Z_t f(t)\ dt\right] & \text{if } \sigma^2 < \rho + 2b, \\ \infty & \text{if } \sigma^2 \geq \rho + 2b. \end{cases}$$

*(iii)*

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} M_t^2(f)\ dt\right] = \begin{cases} 2Ac^{-1}\mathbb{E}\left[\int_0^\infty e^{-\rho t} f(t) M_t(f)\ dt\right] & \text{if } \sigma^2 < \rho + 2b, \\ \infty & \text{if } \sigma^2 \geq \rho + 2b. \end{cases}$$

*Proof:* (i) Since $f$ is $\mathcal{F}_t$-adapted we have

$$\mathbb{E}\big[M_t(f)\big] = \mathbb{E}\left[\int_0^t \mathbb{E}\big[Z_t \big| \mathcal{F}_s\big] \frac{f(s)}{Z_s}\ ds\right] = \int_0^t e^{-b(t-s)}\mathbb{E}\big[f(s)\big]\ ds.$$

Therefore,

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} M_t(f)\ dt\right] = \int_0^\infty e^{bs}\mathbb{E}\big[f(s)\big] \int_s^\infty e^{-(\rho+b)t} dt\ ds = \frac{1}{\rho+b}\ \mathbb{E}\left[\int_0^\infty e^{-\rho t} f(t)\ dt\right].$$

(ii) Conditioning first on $\mathcal{F}_s$ we have

$$\mathbb{E}\big[Z_t M_t(f)\big] = \mathbb{E}\left[\int_0^t \mathbb{E}\big[Z_t^2\big|\mathcal{F}_s\big]\frac{f(s)}{Z_s}\ ds\right] = \int_0^t e^{(\sigma^2-2b)(t-s)}\mathbb{E}\left[Z_s f(s)\right]\ ds,$$

and, hence,

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty e^{-\rho t} Z_t M_t(f)\ dt\right] &= \int_0^\infty e^{(\sigma^2-2b)s}\mathbb{E}\left[Z_s f(s)\right]\int_s^\infty e^{-(\rho+2b-\sigma^2)t}\ dt\ ds \\ &= \begin{cases} Ac^{-1}\mathbb{E}\left[\int_0^\infty e^{-\rho t} Z_t f(t)\ dt\right] & \text{if } \sigma^2 < \rho+2b \\ \infty & \text{if } \sigma^2 \geq \rho+2b. \end{cases}\end{aligned}$$

(iii) By Fubini's theorem we have

$$\begin{aligned}\mathbb{E}\big[M_t^2(f)\big] &= 2\ \mathbb{E}\left[\int_0^t\int_s^t \frac{Z_t^2}{Z_s Z_q} f(s)f(q)\ dq\ ds\right] \\ &= 2\ \mathbb{E}\left[\int_0^t\int_s^t \mathbb{E}\big[Z_t^2\big|\mathcal{F}_q\big]\frac{1}{Z_s Z_q} f(s)f(q)\ dq\ ds\right] \\ &= 2\int_0^t\int_s^t e^{(\sigma^2-2b)(t-q)}\mathbb{E}\left[\frac{Z_q}{Z_s} f(s)f(q)\right]\ dq\ ds \\ &= 2\int_0^t e^{(\sigma^2-2b)(t-q)}\mathbb{E}\left[f(q)M_q(f)\right]\ dq,\end{aligned}$$

and, therefore,

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty e^{-\rho t} M_t^2(f)\ dt\right] &= \int_0^\infty e^{(2b-\sigma^2)q}\mathbb{E}\big[f(q)M_q(f)\big]\int_q^\infty e^{-(\rho+2b-\sigma^2)t} dt\ dq \\ &= \begin{cases} 2Ac^{-1}\mathbb{E}\left[\int_0^\infty e^{-\rho t} f(t)M_t(f)\ dt\right] & \text{if } \sigma^2 < \rho+2b, \\ \infty & \text{if } \sigma^2 \geq \rho+2b. \ \square \end{cases}\end{aligned}$$

# Appendix B

# Further estimates on V

We begin by proving a result on the asymptotic behaviour of the derivative of the value function at $+\infty$. We should highlight that this result is true for the value function of both the stochastic and the deterministic shallow lake problem.

**Proposition B.0.1.** *For $0 \leq \sigma^2 < \rho + 2b$, there exist constants $M, B > 0$, $C \in \mathbb{R}$ such that*

$$V'(x) \geq -Bx + C \;\; \forall x > M.$$

*Proof.* Let $0 < x_1 < x_2$ and for $d > 0$ consider a control $u \in \mathfrak{U}$ which equals $d$ up to time $\tau_d = \inf\{t \geq 0 : \; x(t) \leq x_1\}$ where $x(\cdot)$ is the solution of (1.0.1) with control $u$ and $x(0) = x_2$. Then

$$V(x_2) \geq \mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t}(\ln u(t) - cx^2(t))dt\right] + \mathbb{E}\left[e^{-\rho\tau_d}\right] V(x_1)$$

$$(V(x_2) - V(x_1))\mathbb{E}[e^{-\rho\tau_d}] \geq (\ln d - \rho V(x_2))\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t}dt\right] - c\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t}x^2(t)dt\right] \quad \text{(B.0.1)}$$

Applying Itô's rule to the semimartingale $Y_t = e^{-\rho t}x^2(t)$, we find

$$Y_t - x_2^2 = \int_0^t e^{-\rho s}\left[(\sigma^2 - 2b - \rho)x^2(s) + 2x(s)\left(u(s) + r(x(s))\right)\right] ds + M_1(t)$$

where $M_1(t) = 2\sigma \int_0^t e^{-\rho s}x^2(s)dW_s$

Applying now the optional stopping theorem for the bounded stopping time $\tau_N = \tau_d \wedge \inf\{t \geq 0 : \ x(t) \geq N\} \wedge N$, we have

$$\mathbb{E}[Y_{\tau_N}] - x_2^2 = \mathbb{E}\left[\int_0^{\tau_N} e^{-\rho s}\left((\sigma^2 - 2b - \rho)x^2(s) + 2x(s)\left(d + r(x(s))\right)\right) ds\right]$$

Since $x(s) \geq x_1$ on $[0, \tau_N]$, we have

$$x_1^2\mathbb{E}[e^{-\rho\tau_N}] - x_2^2 \leq -\frac{c}{A}\mathbb{E}\left[\int_0^{\tau_N} e^{-\rho s}x^2(s)ds\right] + 2(d+a)\mathbb{E}\left[\int_0^{\tau_N} e^{-\rho s}x(s)ds\right]$$

Letting $N \to \infty$, we get

$$x_1^2\mathbb{E}[e^{-\rho\tau_d}] - x_2^2 \leq -\frac{c}{A}\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s}x_s^2 ds\right] + 2(d+a)\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s}x_s ds\right]$$

$$(x_1^2 - x_2^2)\mathbb{E}[e^{-\rho\tau_d}] \leq -\frac{c}{A}\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s}x^2(s)ds\right] + 2(d+a)\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s}x(s)ds\right] + \rho x_2^2\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s}ds\right]$$

$$-c\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s}x^2(s)ds\right] \geq A(x_1^2 - x_2^2)\mathbb{E}[e^{-\rho\tau_d}] - 2A(d+a)\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s}x(s)ds\right] - A\rho x_2^2\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s}ds\right] \quad \text{(B.0.2)}$$

Now (B.0.1) gives

$$(V(x_2) - V(x_1))\mathbb{E}[e^{-\rho\tau_d}] \geq (\ln d - \rho V(x_2) - A\rho x_2^2)\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s}ds\right] + A(x_1^2 - x_2^2)\mathbb{E}[e^{-\rho\tau_d}] - 2A(d+a)\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s}x(s)ds\right] \quad \text{(B.0.3)}$$

In order to control the last term in relation (B.0.3), we apply It$\hat{o}$'s rule to the semimartingale $\tilde{Y}_t = e^{-\rho t}x(t)$.

$$\tilde{Y}_t - x_2 = \int_0^t e^{-\rho s}\left(u(s) - (b+\rho)x(s) + r(x(s))\right) ds + M_2(t)$$

where $M_2(t) = \sigma\int_0^t e^{-\rho s}x(s)dW_s$. Applying again the optional stopping theorem for the bounded stopping time $\tau_N$, we have

$$\mathbb{E}[\tilde{Y}_{\tau_N}] - x_2 = \mathbb{E}\left[\int_0^{\tau_N} e^{-\rho s}\left(d - (b+\rho)x(s) + r(x(s))\right) ds\right]$$

Since $x(s) \geq x_1$ on $[0, \tau_N]$, letting $N \to \infty$, we have

$$x_1\mathbb{E}[e^{-\rho\tau_d}] - x_2 \leq \mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s}\left(d - (b+\rho)x(s) + r(x(s))\right) ds\right]$$

$$(x_1 - x_2)\mathbb{E}[e^{-\rho\tau_d}] \leq (d + \rho x_2)\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s} ds\right] + \mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s}\left(-(b+\rho)x(s) + r(x(s))\right) ds\right] \quad \text{(B.0.4)}$$

$$(x_1 - x_2)\mathbb{E}[e^{-\rho\tau_d}] \leq (d + \rho x_2 + a)\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s} ds\right] - (b+\rho)\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s} x(s) ds\right] \quad \text{(B.0.5)}$$

$$-\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s} x(s) ds\right] \geq \frac{1}{b+\rho}(x_1 - x_2)\mathbb{E}[e^{-\rho\tau_d}] - \frac{d + \rho x_2 + a}{b+\rho}\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s} ds\right] \quad \text{(B.0.6)}$$

Relation (B.0.3) based on (B.0.6) becomes:

$$(V(x_2) - V(x_1))\mathbb{E}[e^{-\rho\tau_d}] \geq \left(\ln d - \rho V(x_2) - A\rho x_2^2 - \frac{2A(d+a)(d+\rho x_2 + a)}{b+\rho}\right)\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t} dt\right] + A(x_1^2 - x_2^2)\mathbb{E}[e^{-\rho\tau_d}] + \frac{2A(d+a)}{b+\rho}(x_1 - x_2)\mathbb{E}\left[e^{-\rho\tau_d}\right] \quad \text{(B.0.7)}$$

If we denote by $g(d; x_2)$ the coefficient of $\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho t} dt\right]$, we can consider the following two possible scenarios.

- Case 1: If $\max_{d>0} g(d; x_2) \geq 0$, we choose $d = d(x_2) = \arg\max_{d>0} g(d; x_2)$ and relation (B.0.7) gives:

$$\begin{aligned}\frac{V(x_2) - V(x_1)}{x_2 - x_1} &\geq -A(x_1 + x_2) - \frac{2A(d(x_2) + a)}{b+\rho} \\ &\geq -2Ax_2 - \frac{2A(d(x_2) + a)}{b+\rho}\end{aligned}$$

Notice that $d(x_2)$ is bounded from above, since $d(x_2)$ is such that $g'(d; x_2) = 0$, which gives that

$$d(x_2) = \frac{-2A(\rho x_2 + 2a) + \sqrt{4A^2(\rho x_2 + 2a)^2 + 16A(b+\rho)}}{8A} \leq \sqrt{\frac{b+\rho}{4A}} =: d_1$$

Thus,

$$\frac{V(x_2) - V(x_1)}{x_2 - x_1} \geq -2Ax_2 - \frac{2A(d_1 + a)}{b+\rho} \tag{B.0.8}$$

- Case 2: If $g(d; x_2) < 0 \ \ \forall d > 0$, then focusing on relation (B.0.5) and choosing $\frac{2(\rho+a)}{b} < x_1 < x_2 < x_1 + 1$, we have that for all $x \geq x_1$:

$$d + \rho x_2 + a - (b+\rho)x \leq d + \rho(x_1 + 1) + a - (b+\rho)x_1 < d - (\rho + a)$$

So if we choose $d = \rho$, relation (B.0.5) becomes:

$$(x_1 - x_2)\mathbb{E}[e^{-\rho\tau_d}] \leq -a\mathbb{E}\left[\int_0^{\tau_d} e^{-\rho s} ds\right] \tag{B.0.9}$$

$$\text{(B.0.7)} \overset{\text{(B.0.9)}}{\Rightarrow} \frac{V(x_2) - V(x_1)}{x_2 - x_1} \geq \frac{g(\rho; x_2)}{a} - \frac{2A(\rho+a)}{b+\rho} - 2Ax_2 \tag{B.0.10}$$

Since $V(x) + Ax^2$ is decreasing by Proposition 3.3.1(i), we have:

$$g(\rho; x_2) \geq \ln\rho - \rho V(0) - \frac{2A(\rho+a)^2}{b+\rho} - \frac{2A\rho(\rho+a)}{b+\rho} x_2 \tag{B.0.11}$$

Relation (B.0.10) based on (B.0.11) becomes

$$\frac{V(x_2) - V(x_1)}{x_2 - x_1} \geq \frac{1}{a}(\ln\rho - \rho V(0)) - \frac{2A(\rho+a)(\rho+2a)}{a(b+\rho)} - \left(\frac{2A\rho(\rho+a)}{a(b+\rho)} + 2A\right) x_2 \tag{B.0.12}$$

The assertion now follows by taking $B = \frac{2A\rho(\rho+a)}{a(b+\rho)} + 2A$, $M = \frac{2(\rho \vee d_1 + a)}{b}$ and $C = \min\{\frac{1}{a}(\ln\rho - \rho V(0)) - \frac{2A(\rho+a)(\rho+2a)}{a(b+\rho)}, -\frac{2A(d_1+a)}{b+\rho}\}$. □

The fact that $V'$ does not go to minus infinity more quickly than linearly along with its upper bound is enough to establish the boundedness of the second derivative of $V$ at $+\infty$ as it is stated in the following corollary.

**Corollary B.0.1.** *For the value function, $V$, of the stochastic shallow lake problem, we have*

$$-\infty < \liminf_{x\to\infty} V''(x) \leq \limsup_{x\to\infty} V''(x) < \infty$$

*Proof.* Since $V$ is a classical solution to (1.0.6), we have that

$$V''(x) = \frac{2}{\sigma^2}\left(\rho\frac{V(x)}{x^2} - (\frac{r(x)}{x} - b)\frac{V'(x)}{x} + \frac{\ln(-V'(x))}{x^2} + c + \frac{1}{x^2}\right) \tag{B.0.13}$$

Based on Proposition 3.3.3, $\lim_{x\to\infty} \frac{V(x)}{x^2} = -A$. Furthermore, based on Proposition B.0.1 and 3.3.4, for $x >> 1$, $-Bx + C \leq V'(x) \leq -C_1$. Therefore, relation (B.0.13) gives

$$\frac{2}{\sigma^2}(-\rho A - bB + c) \leq \liminf_{x\to\infty} V''(x) \leq \limsup_{x\to\infty} V''(x) \leq \frac{2}{\sigma^2}(-\rho A + c)$$

□

We now proceed by proving properties of the noiseless value function. Proposition B.0.2 refers to the regularity of $V$.

**Proposition B.0.2.** *The value function, $V$, of the deterministic shallow lake problem is $C^2((0,\infty)\setminus\{x_*\})$ where $x_*$ is the Skiba point.*

*Proof.* According to the analysis of section 4.1, it follows that the value function $V$ is identical to the function $J_P$ constructed based on the Pontryagin Maximum Principle. Along the optimal solution of the system of the lake (2.2.10), we have that $f(x,u) \neq 0$ for all $x \neq x_-, x_+, x_*$. Therefore, based on the Implicit Function Theorem and the HJB eq. (1.0.6), it follows that the value function, $V$, is $C^2\left([0,\infty)\setminus\{x_-, x_+, x_*\}\right)$. Since along the optimal trajectories $\frac{dV}{dx} = -\frac{1}{u}$ and $\frac{du}{dx} = \frac{g(u,x)}{f(u,x)}$, it remains to prove that there exists the limit $\lim_{(x,u)\to(x_0,u_0)} \frac{g(u,x)}{f(u,x)}$ and it is finite, where for simplicity reasons we denote by $(x_0, u_0)$ the saddle steady states of the system (2.2.10). The point $(x_0, u_0)$ satisfies:

$$\begin{cases} u_0 = bx_0 - r(x_0) \\ u_0 = \frac{b+\rho}{2cx_0} - \frac{r'(x_0)}{2cx_0} \end{cases} \tag{B.0.14}$$

The linear approximation around $(x_0, u_0)$ gives:

$$d\begin{pmatrix} x-x_0 \\ u-u_0 \end{pmatrix} = \underbrace{\begin{pmatrix} -b+r'(x_0) & 1 \\ 2u_0^2+\frac{2\left(1-3x_0^2\right)}{\left(x_0^2+1\right)^3}u_0 & -(b+\rho)+\frac{2x_0}{\left(x_0^2+1\right)^2}+4x_0u_0 \end{pmatrix}}_{A}\begin{pmatrix} x-x_0 \\ u-u_0 \end{pmatrix} \tag{B.0.15}$$

In order to determine the direction of the optimal trajectory close to $(x_0, u_0)$, we need to compute the eigenvector which corresponds to the negative eigenvalue of $A$. If we denote by $\lambda^* < 0$, the negative eigenvalue of $A$, then the corresponding eigenvector is

$$v^* = \begin{pmatrix} 1 \\ b-r'(x_0)+\lambda^* \end{pmatrix} \tag{B.0.16}$$

Therefore, close to $(x_0, u_0)$ along the optimal trajectory we have that

$$u-u_0 = \underbrace{(b-r'(x_0)+\lambda^*)}_{k(u_0,x_0)}(x-x_0) \tag{B.0.17}$$

By (B.0.17), we can now compute the following limit along the optimal trajectory:

$$\begin{aligned}
\lim_{(x,u)\to(x_0,u_0)}\frac{g(u,x)}{f(u,x)} &= \lim_{x\to x_0}\frac{g(u_0+k(u_0,x_0)(x-x_0),x)}{f(u_0+k(u_0,x_0)(x-x_0),x)} \\
&= \lim_{x\to x_0}\Big[\frac{-\left((b+\rho)-r'(x)\right)\left(u_0+k(u_0,x_0)(x-x_0)\right)}{u_0+k(u_0,x_0)(x-x_0)-bx+r(x)} \\
&\qquad + \frac{2cx\left(u_0+k(u_0,x_0)(x-x_0)\right)^2}{u_0+k(u_0,x_0)(x-x_0)-bx+r(x)}\Big] \\
&\underset{DLH}{\overset{\frac{0}{0}}{=}} \lim_{x\to x_0}\frac{h(x,x_0,u_0)}{k(x_0,u_0)-b+r'(x)} = \frac{h(x_0,x_0,u_0)}{k(x_0,u_0)-b+r'(x_0)} \\
&= \frac{h(x_0,x_0,u_0)}{\lambda^*(\neq 0)}
\end{aligned}$$

□

Now, Lemma B.1 collects all the key properties of $V$ to establish the boundedness of its second derivative, whereat the second derivative exists.

**Lemma B.1.** *For the value function, $V$, of the deterministic shallow lake problem we have that* *(i)* $\lim_{x\to\infty} \frac{V'(x)}{x} = \frac{c}{b}(\frac{\rho}{\rho+2b} - 1)$

*(ii)* $V''(0) = -(\rho + b - r'(0))\,(V'(0))^2$

*(iii) the limit* $\lim_{x\to\infty} V''(x)$ *exists and is finite*

*(iv) the limits* $\lim_{x\to x_*-} V''(x)$ *and* $\lim_{x\to x_*+} V''(x)$ *exist and are finite.*

*Proof.* (i) Since $V$ is a classical solution to the HJB eq. (1.0.6) for $x > x_*$, we have

$$\begin{aligned}\lim_{x\to\infty} \frac{V'(x)}{x} &= \lim_{x\to\infty} \frac{x}{bx - r(x)} \left( -\frac{\rho V(x)}{x^2} - \frac{\ln(-V'(x))}{x^2} - c - \frac{1}{x^2} \right) \\ &= \frac{1}{b}(A\rho - c) = \frac{c}{b}\left(\frac{\rho}{\rho + 2b} - 1\right)\end{aligned}$$

where in the last line we used Propositions 3.3.3, B.0.1 and 3.3.4.

(ii) Differentiating once the HJB eq. (1.0.6) for $x$ close to 0, we find

$$V''(x) = \frac{(r'(x) - (\rho + b))\,V'(x) - 2cx}{-r(x) + bx + \frac{1}{V'(x)}} \tag{B.0.18}$$

Taking $x \to 0$, the assertion follows since $V$ is $C^1([0,\infty) \setminus x_*)$ and $C^2((0,\infty) \setminus x_*)$.

(iii) By relation (B.0.18), we find

$$\begin{aligned}\lim_{x\to\infty} V''(x) &= \lim_{x\to\infty} \frac{(r'(x) - (\rho + b))\,\frac{V'(x)}{x} - 2c}{-\frac{r(x)}{x} + b + \frac{1}{xV'(x)}} \\ &= -(\rho + b)\frac{c}{b^2}\left(\frac{\rho}{\rho + 2b} - 1\right) - \frac{2c}{b}\end{aligned}$$

(iv) Because of (B.0.18) and the fact that the side derivatives $V_P'(x_*(-))$, $V_P'(x_*(+))$ exist (see Proposition 3.8 [32]), we just need to show that the limit of the denominator of (B.0.18) as $x \to x_*(\pm)$ is not zero. This is an immediate consequence of Lemma 7.1.2.

□

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# ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ

# Ανάλυση, βέλτιστη διαχείριση και μελέτη της μεταευστάθειας στοχαστικών συστημάτων από την περιβαλλοντική οικονομία

Διατριβή η οποία υποβλήθηκε προς απόκτηση
*Διδακτορικού Τίτλου*

από την
Αγγελική Κουτσιμπέλα

Τομέας Μαθηματικών
Σχολή Εφαρμοσμένων Μαθηματικών και Φυσικών Επιστημών
Εθνικό Μετσόβιο Πολυτεχνείο

Σεπτέμβριος, 2023

Η υλοποίηση της διδακτορικής διατριβής συγχρηματοδοτήθηκε από την Ελλάδα και την Ευρωπαϊκή Ένωση (Ευρωπαϊκό Κοινωνικό Ταμείο) μέσω του Επιχειρησιακού Προγράμματος «Ανάπτυξη Ανθρώπινου Δυναμικού, Εκπαίδευση και Δια Βίου Μάθηση», 2014-2020, στο πλαίσιο της Πράξης «Ενίσχυση του ανθρώπινου δυναμικού μέσω της υλοποίησης διδακτορικής έρευνας Υποδράση 2: Πρόγραμμα χορήγησης υποτροφιών ΙΚΥ σε υποψηφίους διδάκτορες των ΑΕΙ της Ελλάδας.

Ευρωπαϊκή Ένωση
Ευρωπαϊκό Κοινωνικό Ταμείο

Επιχειρησιακό Πρόγραμμα
Ανάπτυξη Ανθρώπινου Δυναμικού,
Εκπαίδευση και Διά Βίου Μάθηση

Με τη συγχρηματοδότηση της Ελλάδας και της Ευρωπαϊκής Ένωσης

ΕΣΠΑ 2014-2020
ανάπτυξη - εργασία - αλληλεγγύη

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# Detailed abstract in Greek

Το πρόβλημα της ρηχής λίμνης είναι ένα γνωστό πρόβλημα της περιβαλλοντικής οικονομίας με μεγάλο μαθηματικό ενδιαφέρον. Παρατηρείται ότι η μεγάλη χρήση λιπασμάτων στα περιβάλλοντα εδάφη και η αυξημένη εισροή λυμάτων από τις βιομηχανίες και τους οικισμούς απελευθερώνουν φώσφορο στις λίμνες και αυτό προκαλεί την έντονη ανάπτυξη του φυτοπλαγκτού. Ως αποτέλεσμα, οι ρηχές λίμνες μεταπίπτουν από μια καθαρή κατάσταση (ολιγοτροφική κατάσταση) σε μια θολή κατάσταση (ευτροφική κατάσταση) με πρασινωπή όψη. Οι λιμνολόγοι έχουν δείξει έντονο ενδιαφέρον για αυτό το φυσικό φαινόμενο και έχουν προτείνει ένα μοντέλο για την ποσοτικοποίηση της χρονικής εξέλιξης της συγκέντρωσης του φωσφόρου μέσα στη λίμνη. Πιο συγκεκριμένα, η ποσότητα του φωσφόρου στη λίμνη μοντελοποιείται συνήθως με τη μη γραμμική στοχαστική διαφορική εξίσωση:

$$\begin{cases} dx(t) = (u(t) - bx(t) + r(x(t)))\, dt + \sigma x(t) dW_t, \\ x(0) = x \geq 0. \end{cases} \tag{B'.0.19}$$

Ο πρώτος όρος, $u : [0, \infty) \to (0, \infty)$, στον συντελεστή ολίσθησης (drift), αντιπροσωπεύει το εξωτερικό φορτίο φωσφόρου ως αποτέλεσμα των ανθρώπινων δραστηριοτήτων. Ο δεύτερος όρος είναι ο ρυθμός απώλειας $bx(t)$, ο οποίος οφείλεται σε φυσικές διεργασίες, όπως την καθίζηση, την εκροή και τη δέσμευση σε άλλη βιομάζα. Ο τρίτος όρος, $r(x(t))$, είναι ο ρυθμός ανακύκλωσης του φωσφόρου στον πυθμένα της λίμνης. Αυτός ο όρος θεωρείται ότι είναι μια σιγμοειδής συνάρτηση (βλέπε [15]) και η τυπική επιλογή στη βιβλιογραφία είναι η συνάρτηση $x \mapsto x^2/(x^2+1)$. Υποθέτουμε την ύπαρξη αβεβαιότητας στον ρυθμό απώλειας και η αβεβαιότητα αυτή εισάγεται στο μοντέλο μέσω ενός γραμμικού πολλαπλασιαστικού γκαουσιανού λευκού θορύβου με ένταση $\sigma$.

Η οικονομική διάσταση του προβλήματος της ρηχής λίμνης προκύπτει από τις αντικρουόμενες υπηρεσίες που προσφέρει στην κοινότητα. Από τη μία πλευρά, μια καθαρή λίμνη προσφέρει οικολογικές υπηρεσίες. Από την άλλη πλευρά, η λίμνη χρησιμεύει ως αποδέκτης αποβλήτων για τις γεωργικές και βιομηχανικές δραστηριότητες. Όταν οι χρήστες της λίμνης συνεργάζονται, η στρατηγική φόρτωσης, $u \in \mathfrak{U}_x$, μπορεί να χρησιμοποιηθεί ως έλεγχος για τη μεγιστοποίηση του συνολικού οφέλους από τη λίμνη. Υποθέτοντας άπειρο χρονικό ορίζοντα, το όφελος αυτό ορίζεται συνήθως ως εξής

$$J(x;u) = \mathbb{E}_x \left[ \int_0^\infty e^{-\rho t} \big( \ln u(t) - c x^2(t) \big)\, dt \right], \tag{B'.0.20}$$

όπου $\rho > 0$ είναι το προεξοφλητικό επιτόκιο και $x(\cdot)$ είναι η λύση της (B′.0.19), για δεδομένο εξωτερικό φορτίο (έλεγχος) $u(\cdot)$, και $x(0) = x$. Το συνολικό όφελος της λίμνης αυξάνεται με την αύξηση του φορτίου φωσφόρου ως $\ln u$, αλλά ταυτόχρονα μειώνεται με την τρέχουσα ποσότητα φωσφόρου στο εσωτερικό της λίμνης ως $-cx^2$, λόγω της συνεπαγόμενης πτώσης της ποιότητας των οικολογικών υπηρεσιών της λίμνης. Η θετική παράμετρος $c$ αντικατοπτρίζει τη σχετική βαρύτητα αυτής της συνιστώσας. Για τη βέλτιστη διαχείριση της λίμνης, πρέπει να μεγιστοποιήσουμε το συνολικό όφελος ως προς όλους τους αποδεκτούς ελέγχους $u \in \mathfrak{U}_x$. Με αυτόν τον τρόπο, η συνάρτηση αξίας του προβλήματος ορίζεται ως εξής

$$V(x) = \sup_{u \in \mathfrak{U}_x} J(x;u), \tag{B'.0.21}$$

Επομένως, το πρόβλημα της ρηχής λίμνης μετατρέπεται σε πρόβλημα της θεωρίας ελέγχου ή σε διαφορικό παίγνιο στην περίπτωση που έχουμε ανταγωνιστικούς χρήστες της λίμνης [13, 15, 50]. Ως πρόβλημα ελέγχου (βλ. [23], Κεφάλαιο III.7), η συνάρτηση αξίας $V$ της σχ. (1.0.3) αναμένεται να ικανοποιεί την εξίσωση HJB

$$\rho V - H(x, V_x) - \frac{1}{2}\sigma^2 x^2 V_{xx} = 0 \tag{B'.0.22}$$

όπου η συνάρτηση $H$ ορίζεται ως

$$H(x,p) = \sup_{u>0} \left[ (u - bx + r(x))\, p + \ln u - cx^2 \right]. \tag{B'.0.23}$$

Υποθέτοντας ότι $V_x < 0$, η εξ. (1.0.4) καταλήγει στη:

$$\rho V - (r(x) - bx)\, V_x + \ln(-V_x) + cx^2 + 1 - \frac{1}{2}\sigma^2 x^2 V_{xx} = 0 \qquad (B'.0.24)$$

Το πρόβλημα της ρηχής λίμνης έχει μελετηθεί εκτενώς στη βιβλιογραφία, ιδίως η ντετερμινιστική του εκδοχή. Όταν $\sigma = 0$, η περίπτωση όπου η βέλτιστα ελεγχόμενη λίμνη έχει δύο σημεία ισορροπίας και ένα σημείο Skiba, ή σημείο αδιαφορίας (indifference point) [47, 49] παρουσιάζει ιδιαίτερο ενδιαφέρον. Το αριστερότερο (ολιγοτροφικό) σημείο ισορροπίας του συστήματος της λίμνης αντιστοιχεί σε μια λίμνη με χαμηλή συγκέντρωση φωσφόρου, ενώ το δεξιότερο (ευτροφικό) αντιστοιχεί σε μια λίμνη με υψηλή συγκέντρωση φωσφόρου. Στο σημείο Skiba υπάρχουν δύο διαφορετικές βέλτιστες στρατηγικές, καθεμία από τις οποίες οδηγεί το σύστημα σε διαφορετική κατάσταση ισορροπίας και η συνάρτηση αξίας δεν είναι παραγωγίσιμη εκεί. Επομένως, η συνάρτηση αξίας, $V$, δεν μπορεί να είναι μια κλασική λύση της εξίσωσης HJB (1.0.4). Στην πραγματικότητα, το σωστό μαθηματικό πλαίσιο με το οποίο μπορούμε να εργαστούμε, ειδικά όταν η συνάρτηση αξίας δεν έχει εκ των προτέρων την ομαλότητα μιας κλασικής λύσης, είναι αυτό των λύσεων ιξώδους (viscosity solutions), όπως αναπτύχθηκε από τους Crandall και Lions [17]. Η σύνδεση των προβλημάτων της θεωρίας ελέγχου με τις εξισώσεις HJB έχει μελετηθεί εκτενώς, βλέπε π.χ. [3, 23, 22, 37, 38].

Προκειμένου να προσδιοριστεί το εύρος των παραμέτρων για τις οποίες εμφανίζονται τα σημεία Skiba, διεξήχθη εκτεταμένη διερεύνηση του χώρου των παραμέτρων και των ποιοτικών διαφορών του συστήματος Pontryagin της ρηχής λίμνης (ανάλυση διακλάδωσης) [31, 49]. Ιδιότητες της συνάρτησης αξίας του ντετερμινιστικού προβλήματος της ρηχής λίμνης έχουν αποδειχθεί στο [32].

Ένα βασικό ερώτημα που, εξ' όσων γνωρίζουμε, δεν έχει απαντηθεί πλήρως μέχρι σήμερα είναι αυτό της ύπαρξης βέλτιστου ελέγχου. Η ύπαρξη βέλτιστου ελέγχου λαμβάνεται συνήθως ως υπόθεση και η βέλτιστη δυναμική της λίμνης μελετάται κυρίως μέσω των αναγκαίων συνθηκών, οι οποίες προσδιορίζονται από την αρχή μεγίστου του Pontryagin, και των σημείων ισορροπίας του αντίστοιχου δυναμικού συστήματος [49, 50]. Μια αυστηρή απάντηση στο ερώτημα αυτό δόθηκε από τον Bartaloni στα [7, 8], αν και υπό περιορισμούς που δεν καλύπτουν πλήρως το εύρος των παραμέτρων για τις οποίες υπάρχουν σημεία Skiba.

Επιπλέον, πρόσφατα αναπτύχθηκε έντονο ενδιαφέρον για τη στοχαστική εκδοχή του προβλήματος ($\sigma \neq 0$). Συγκεκριμένα, τα ντετερμινιστικά συστήματα με δύο σημεία ισορροπίας και ένα σημείο Skiba έχουν θεμελιωδώς διαφορετική συμπεριφορά από τα αντίστοιχα στοχαστικά συστήματα. Συγκεκριμένα, παρουσία θορύβου, οι τυχαίες διακυμάνσεις οδηγούν το σύστημα από το ένα σημείο ισορροπίας στο άλλο (μεταευστάθεια). Στην περίπτωση της ρηχής λίμνης, οι διακυμάνσεις του ρυθμού απώλειας οδηγούν τη λίμνη από την ολιγοτροφική στην ευτροφική κατάσταση και αντίστροφα, φαινόμενο που παρατηρείται συχνά στη φύση. Το ενδιαφέρον για τη μελέτη των μεταευσταθών συστημάτων προέκυψε αρχικά από φαινόμενα στον τομέα της Χημείας. Ο Arrhenius [2] το 1889 δικαιολόγησε μια έκφραση για τον μέσο χρόνο μετάβασης του συστήματος από το ένα τοπικό ελάχιστο στο άλλο. Αργότερα, οι H. Eyring και H. A. Kramers [20], [35] με τον γνωστό νόμο Eyring-Kramers συμπλήρωσαν τον νόμο Arrhenius προσδιορίζοντας τον προ-παράγοντα στην έκφραση του Arrhenius. Στη συνέχεια, οι M.I.Freindlin και A.D.Wentzell [24] εισήγαγαν τη θεωρία των Μεγάλων Αποκλίσεων για την εξήγηση και κατανόηση της μεταευσταθούς συμπεριφοράς διαφόρων δυναμικών συστημάτων. Παρόλο που τα μεταευσταθή συστήματα έχουν μελετηθεί έκτοτε εκτενώς (βλ. π.χ. [12], [9]), η πλειονότητα των αποτελεσμάτων αφορά δυναμικά συστήματα για τα οποία η συνάρτηση ολίσθησης δεν είναι συνάρτηση της έντασης του θορύβου. Ωστόσο, στο πλαίσιο της (στοχαστικής) θεωρίας ελέγχου, είναι φυσικό να αναμένει κανείς ότι σε μεταευσταθή συστήματα, όπως το σύστημα της ρηχής λίμνης στην περίπτωση των σημείων Skiba, ο όρος ολίσθησης του βέλτιστα ελεγχόμενου συστήματος θα εξαρτάται από τον θόρυβο μέσω της παρουσίας της συνάρτησης αξίας, η οποία με τη σειρά της εξαρτάται από το θόρυβο μέσω της διόρθωσης Hamilton-Jacobi-Bellman. Το φαινόμενο της μεταευστάθειας στο πρόβλημα της ρηχής λίμνης μελετάται αριθμητικά στο [26], όπου η συνάρτηση αξίας του προβλήματος της ρηχής λίμνης προσεγγίζεται για μικρές τιμές του $\sigma$, με βάση ευρετικές μεθόδους ανάλυσης διαταραχών.

Επιπλέον, διεξοδική εξέταση της στοχαστικής εκδοχής του προβλήματος της ρηχής λίμνης διεξάγεται από τους Kossioris, Loulakis και Souganidis [33] οι οποίοι εξάγουν αναλυτικά ιδιότητες της συνάρτησης αξίας και τη χαρακτηρίζουν ως τη μοναδική (σε μια κατάλληλη κλάση) λύση ιξώδους με περιορισμούς κατάστασης (state-constraint) της εξίσωσης Hamilton-Jacobi-Bellman (HJB) (1.0.4).

Το πρόβλημα της ρηχής λίμνης έχει ορισμένα μη τυπικά χαρακτηριστικά και, ως εκ τούτου, απαιτεί ειδική ανάλυση. Πρώτα απ' όλα, το πρόβλημα διατυπώνεται ως πρόβλημα περιορισμού κατάστασης σε ένα ημι-άπειρο πεδίο. Λύσεις ιξώδους με συνοριακές συνθήκες περιορισμού κατάστασης εισήχθησαν για εξισώσεις πρώτης τάξης από [48] και [14]. Για εξισώσεις δεύτερης τάξης θα πρέπει να συμβουλευτεί κανείς τα [30], [36] και [1]. Επιπλέον, η εκ των προτέρων γνώση των ιδιοτήτων της λύσης είναι απαραίτητη για να εξασφαλιστεί ότι η Χαμιλτονιανή είναι καλά ορισμένη, λόγω του λογαριθμικού όρου στη συνάρτηση κόστους, ο οποίος οδηγεί σε λογάριθμο της παραγώγου της συνάρτησης αξίας στο (1.0.6). Στη συνέχεια, στην περίπτωση του στοχαστικού προβλήματος ρηχής λίμνης, η ελλειπτικότητα του (1.0.6) εκφυλίζεται στο άκρο, $x = 0$. Τέλος, ο χώρος ελέγχου είναι ανοικτός και άπειρος, οπότε οι συνήθεις υποθέσεις που γίνονται για την απόδειξη της ύπαρξης σε προβλήματα ελέγχου με άπειρο ορίζοντα (βλ. π.χ. [10, 19, 44]) δεν ικανοποιούνται εδώ.

Η πρώτη κύρια συνεισφορά αυτής της εργασίας είναι η αυστηρή απόδειξη της ύπαρξης βέλτιστου ελέγχου τόσο στο στοχαστικό όσο και στο ντετερμινιστικό πρόβλημα της ρηχής λίμνης χωρίς περιορισμούς στο χώρο των παραμέτρων. Με την παρουσία θορύβου, η απόδειξη ακολουθεί τις γενικές γραμμές μιας αρχής επαλήθευσης (verification principle) (βλ. π.χ. [23]) με κατάλληλες τροποποιήσεις για την αντιμετώπιση της απώλειας της ελλειπτικότητας στο σύνορο και μιας πιθανής πολύ αρνητικής τιμής ($\sim -\infty$) του οφέλους για μικρούς ελέγχους. Αυτή η προσέγγιση δεν είναι πάντα εφικτή στο ντετερμινιστικό πρόβλημα, καθώς η συνάρτηση αξίας μπορεί να μην είναι παραγωγίσιμη. Στα [7] και [8] η ύπαρξη βέλτιστου ελέγχου τεκμηριώνεται με την απόδειξη λημμάτων που αφορούν ομοιόμορφα φραγμένους ελέγχους ακολουθούμενων από διαγώνια επιχειρήματα. Η προσέγγιση αυτή πραγματοποιείται με επιτυχία υπό την προϋπόθεση ότι είτε η παράμετρος $b$ είτε η παράμετρος προεξόφλησης $\rho$ είναι μεγαλύτερες από $3\sqrt{3}/8$. Η προσέγγισή μας εδώ είναι εντελώς διαφορετική και δεν απαιτεί κανέναν περιορισμό στο χώρο των παραμέτρων. Πιο συγκεκριμένα, αποδεικνύουμε ότι τόσο η συνάρτηση αξίας όσο και το συνολικό όφελος που επιτυγχάνεται όταν το σύστημα οδηγείται από τον υποψήφιο βέλτιστο έλεγχο που προτείνεται από την Αρχή Μεγίστου του Pontryagin είναι λύσεις ιξώδους στο ίδιο καλά τεθειμένο πρόβλημα. Με αυτόν τον τρόπο αποδεικνύεται ότι το βέλτιστο συνολικό όφελος, δηλαδή η συνάρτηση αξίας $V$, επιτυγχάνεται από έναν αποδεκτό έλεγχο και με αυτόν τον τρόπο ο έλεγχος αυτός χαρακτηρίζεται ως βέλτιστος.

Η δεύτερη κύρια συνεισφορά της παρούσας εργασίας είναι η ανάλυση της μεταευσταθούς συμπεριφοράς του προβλήματος της ρηχής λίμνης, η οποία πραγματοποιείται στο γενικότερο πλαίσιο των προβλημάτων στοχαστικού ελέγχου που παρουσιάζουν σημεία Skiba. Πιο αναλυτικά, μελετάμε την αναμενόμενη τιμή του χρόνου μετάβασης από το ένα πηγάδι στο άλλο για μια διαδικασία σε ένα θορυβοεξαρτώμενο δυναμικό διπλού πηγαδιού και αποδεικνύουμε μια γενίκευση του νόμου Arrhenius. Για να το κάνουμε αυτό, εκμεταλλευόμαστε κατ΄ αρχάς το γεγονός ότι ο μέσος χρόνος μετάβασης επιλύει ένα πρόβλημα Poisson, η λύση του οποίου δίνεται επακριβώς σε ολοκληρωτική μορφή στη μονοδιάστατη περίπτωση. Στη συνέχεια, αποδεικνύουμε την τοπικά ομοιόμορφη σύγκλιση του θορυβώδους ολοκληρώματος σε ένα αθόρυβο, αποδεικνύοντας τη σύγκλιση των παραγώγων της θορυβώδους συνάρτησης αξίας στις παραγώγους της αθόρυβης συνάρτησης αξίας. Για να το αποδείξουμε αυτό, προσαρμόζουμε μια μεθοδολογία που εισήγαγαν οι Fleming και Souganidis στο [21], η οποία βασίζεται σε ένα επιχείρημα ημικυρτότητας.

## Β΄.1 Κεφάλαιο 2

Το Κεφάλαιο 2 είναι προκαταρκτικό και περιέχει ορισμένους απαραίτητους ορισμούς και αποτελέσματα που είναι χρήσιμα για τα επόμενα κεφάλαια. Συγκεκριμένα, παρουσιάζονται οι ορισμοί της λύσης ιξώδους (viscosity solution) και της λύσης ιξώδους με περιορισμό κατάστασης (state-constraint). Επιπλέον, διατυπώνεται η Αρχή του Pontryagin, η μορφή που παίρνει στην περίπτωση του Προβλήματος της ρηχής λίμνης και με βάση αυτή κατασκευάζεται η υποψήφια συνάρτηση αξίας, $J_P$ που θα χρησιμοποιήσουμε στο Κεφάλαιο 4 για την απόδειξη ύπαρξης βέλτιστου ελέγχου στην περίπτωση του ντετερμινιστικού προβλήματος.

## Β′.2 Κεφάλαιο 3

Στο Κεφάλαιο 3, γενικεύουμε τα αποτελέσματα του [33] για να συμπεριλάβουμε σιγμοειδείς ρυθμούς ανακύκλωσης που είναι πιο γενικοί από την τυπική επιλογή, $x \mapsto x^2/(x^2+1)$ καθώς και την παράμετρο βάρους, $c$, που δεν μπορεί να απορροφηθεί μετά από κατάλληλη αλλαγή μεταβλητών. Συγκεκριμένα, υποθέτουμε ότι η συνάρτηση ρυθμού ανακύκλωσης $r$ είναι μια σιγμοειδής συνάρτηση που ικανοποιεί την Υπόθεση Β′.1.

**Υπόθεση Β′.1.** *Ο ρυθμός ανακύκλωσης $r(x)$ ικανοποιεί τα ακόλουθα:*

1. $r \in C^1([0,\infty))$ *και αύξουσα*
2. $r(0) = 0$ *και* $r(x) \leq (b+\rho)x$ *κοντά στο* $0$.
3. $a := \lim_{x\to\infty} r(x) < \infty$
4. *Το όριο* $\lim_{x\to\infty}(a - r(x))x =: C$ *υπάρχει και είναι ένας πεπερασμένος, αναγκαστικά μη αρνητικός, πραγματικός αριθμός*
5. $\lim_{x\to\infty} r'(x) = 0$.

Τα κύρια αποτελέσματα του Κεφαλαίου αφορούν τον χαρακτηρισμό της συνάρτησης αξίας ως τη λύση ιξώδους περιορισμού-κατάστασης ενός καλώς τεθειμένου προβλήματος. Οι αποδείξεις τους δίνονται στην ενότητα 3.4.

**Θεώρημα Β′.1.** *Εάν $0 < \sigma^2 < \rho + 2b$, η συνάρτηση αξίας $V$ είναι μια συνεχής λύση ιξώδους με περιορισμούς κατάστασης (state-constraint) στην εξίσωση* (Β′.0.22) *στο* $[0,\infty)$.

Συγκεκριμένα, η συνάρτηση αξίας $V$ χαρακτηρίζεται ως η μοναδική λύση ιξώδους περιορισμού κατάστασης της (Β′.0.24) λόγω της ακόλουθης αρχής σύγκρισης (Θεώρημα Β′.2).

**Θεώρημα B′.2.** *Αν* $0 < \sigma^2 < \rho + 2b$, *υποθέτοντας ότι* $u \in C([0,\infty))$ *είναι μια γνησίως φθίνουσα υπολύση της* (B′.0.22) *στο* $[0,\infty)$ *και* $v \in C([0,\infty))$ *είναι μια γνησίως φθίνουσα υπερλύση της* (B′.0.22) *στο* $(0,\infty)$ *τέτοια ώστε* $v \geq -c_1(1+x^q)$, *όπου* $q$ *μπορεί να είναι οποιοσδήποτε πραγματικός αριθμός αυστηρά μικρότερος από* $|k(\sigma)|$, *όπου* $k(\sigma)$ *είναι η αρνητική ρίζα του* (3.4.5) *και* $Du \leq -\frac{1}{c_2}$ *με την έννοια του ιξώδους, για* $c_1$, $c_2$ *θετικές σταθερές. Τότε* $u \leq v$ *στο* $[0,\infty)$.

Στη συνέχεια, διατυπώνονται, και αποδεικνύονται στις ενότητες 3.2 και 3.3, ιδιότητες της δυναμικής του προβλήματος και της συνάρτησης αξίας. Συγκεκριμένα, στην Πρόταση B′.1, αποδεικνύεται ότι i) η συγκέντρωση φωσφόρου, $x_t$, παραμένει μη αρνητική όταν η διαδικασία ξεκινά από μια μη αρνητική ποσότητα, $x$, ii) το σύνολο των επιτρεπτών ελέγχων είναι ανεξάρτητο από την αρχική κατάσταση, $x$, και iii) όσο μεγαλύτερη είναι η φόρτιση φωσφόρου, $u$, τόσο μεγαλύτερη είναι η προκύπτουσα συγκέντρωση φωσφόρου, $x(\cdot)$.
Έστω

$$Z_t = e^{\sigma W_t - \left(b+\sigma^2/2\right)t} \text{ ανδ } M_t(u) = \int_0^t \frac{Z_t}{Z_s} u(s)ds \qquad (\text{B}'.2.1)$$

**Πρόταση B′.1.**

(i) *Αν* $x \geq 0$, $u \in \mathfrak{U}_x$, *και* $x(\cdot)$ *είναι η λύση της (B′.0.19), τότε* $\mathbb{P}\big[x(t) \geq 0,\ \forall t \geq 0\big] = 1$. *Συγκεκριμένα,* $\mathbb{P}\big[x(t) \geq M_t(u),\ \forall t \geq 0\big] = 1$.

(ii) *Για κάθε* $x, y \geq 0$, $\mathfrak{U}_x = \mathfrak{U}_y =: \mathfrak{U}$.

(iii) *Υποθέτουμε ότι* $x_1(\cdot)$, $x_2(\cdot)$ *ικανοποιούν την εξ. (B′.0.19) με ελέγχους* $u_1$, $u_2 \in \mathfrak{U}$, *αντίστοιχα, και* $x_1(0) = x_1$, $x_2(0) = x_2$. *Αν* $x_1 \leq x_2$ *και* $\mathbb{P}\big[u_1(t) \leq u_2(t),\ \forall t \geq 0\big] = 1$, *τότε*

$$\mathbb{P}\big[x_2(t) - x_1(t) \geq (x_2 - x_1)Z_t,\ \forall t \geq 0\big] = 1.$$

Οι Προτάσεις B′.2, B′.3, B′.4 αναφέρονται σε ιδιότητες της συνάρτησης αξίας $V$ της (B′.0.21). Σημειώστε ότι αυτές οι ιδιότητες προκύπτουν απευθείας από τον ορισμό της $V$ στο (B′.0.21), οπότε δεν είναι συνέπεια οποιασδήποτε διαφορικής εξίσωσης, όπως η (B′.0.24), που μπορεί να ικανοποιεί η $V$. Αντίθετα, χρησιμοποιούνται ως καθοριστικός παράγοντας στον χαρακτηρισμό της συνάρτησης αξίας ως μοναδικής λύσης ιξώδους περιορισμού κατάστασης της (B′.0.22), καθώς εξασφαλίζουν ότι η σχετική Χαμιλτονιανή του προβλήματος ελέγχου είναι πεπερασμένη και περιγράφουν μια κλάση συναρτήσεων μεταξύ των οποίων υπάρχει μοναδικότητα λύσεων της (B′.0.22). Σημειώστε επίσης ότι τα αποτελέσματα της Πρότασης B′.4 υποθέτουν ότι $\sigma > 0$, ενώ τα άλλα δύο ισχύουν και για την ντετερμινιστική περίπτωση. Στη συνέχεια, υποθέτουμε ότι

$$\sigma^2 < \rho + 2b,$$

επειδή διαφορετικά η συνάρτηση αξίας $V$ δεν είναι πεπερασμένη (βλ. Σχόλιο 3.3.1).

Έστω

$$A = \frac{c}{\rho + 2b - \sigma^2} \tag{B′.2.2}$$

**Πρόταση B′.2.** *Υποθέτουμε ότι* $0 \leq \sigma^2 < \rho + 2b$.

(i) *Η συνάρτηση* $x \mapsto V(x) + Ax^2$*, όπου* $A$ *ορίζεται στη* (B′.2.2)*, είναι φθίνουσα στο* $[0, +\infty)$.

(ii) *Η συνάρτηση αξίας στο μηδέν ικανοποιεί* $V(0) \leq \frac{1}{\rho} \ln\left(\frac{b+\rho}{\sqrt{2ec}}\right)$.

(iii) *Έστω* $x_1, x_2 \in [0, \infty)$ *με* $x_1 < x_2$*, και, για* $u \in \mathfrak{U}$*, έστω* $x(\cdot)$ *η λύση στην εξ. (1.0.1) με έλεγχο* $u$ *και* $x(0) = x_1$*. Αν* $\tau_u$ *είναι ο χρόνος κρούσης της* $x(\cdot)$ *στο* $[x_2, +\infty)$*, δηλαδή,* $\tau_u = \inf\{t \geq 0 : x(t) \geq x_2\}$*, τότε*

$$V(x_1) = \sup_{u \in \mathfrak{U}} \mathbb{E}\left[\int_0^{\tau_u} e^{-\rho t}\big(\ln u(t) - cx^2(t)\big)\, dt + e^{-\rho \tau_u} V(x_2)\right]. \tag{B′.2.3}$$

Στις επόμενες δύο Προτάσεις (B′.3, B′.4), αναφέρουμε τις βασικές ιδιότητες της συνάρτησης αξίας $V$ που εγγυώνται ότι η $V$ ικανοποιεί τις υποθέσεις του θεωρήματος σύγκρισης B′.2.

Ειδικότερα, η Πρόταση B′.3, δηλώνει ότι η $V$ δεν πηγαίνει στο μείον άπειρο πιο γρήγορα από το $-Cx^2$. Επιπλέον, δείχνει ότι η $V$ είναι γνησίως φθίνουσα και ότι ικανοποιεί την $DV \leq -C < 0$ με την έννοια του ιξώδους.

**Πρόταση B′.3.** *Ας υποθέσουμε ότι* $0 \leq \sigma^2 < \rho + 2b$.

(i) *Υπάρχουν σταθερές* $K_1, K_2 > 0$, *τέτοιες ώστε, για κάθε* $x \geq 0$, *έχουμε*

$$K_1 \leq V(x) + A\left(x + \frac{a}{b+\rho}\right)^2 + \frac{1}{\rho}\ln\left(x + \frac{a}{b+\rho}\right) \leq K_2. \quad \text{(B}'\text{.2.4)}$$

(ii) *Υπάρχει σταθερά* $C_1 > 0$ *και συνάρτηση* $c : [0, +\infty) \to (0, \infty)$ *με* $\lim_{x\to 0} c(x) = e^{-(\rho V(0)+1)}$ *τέτοια ώστε, για κάθε* $x_1, x_2 \in [0, +\infty)$ *με* $x_1 < x_2$,

$$\frac{V(x_2) - V(x_1)}{x_2 - x_1} \leq -c(x_2) \leq -C_1 < 0. \quad \text{(B}'\text{.2.5)}$$

Ο επόμενος ισχυρισμός μαζί με το (B′.2.5) δηλώνει την τοπικά Lipschitz ιδιότητα της συνάρτησης αξίας στη στοχαστική περίπτωση. Επιπλέον, η σχέση (B′.2.7) μας δίνει την κατάλληλη συνοριακή συνθήκη για την εξίσωση HJB, (B′.0.22) που εγγυάται ότι το αντίστοιχο πρόβλημα συνοριακών τιμών είναι καλά τεθειμένο.

**Πρόταση B′.4.** *Ας υποθέσουμε ότι* $0 < \sigma^2 < \rho + 2b$.

*(i) Υπάρχει μια αύξουσα συνάρτηση* $L_\sigma : [0, \infty) \to \mathbb{R}$ *με* $\lim_{x\to 0} L_\sigma(x) = e^{-(\rho V(0)+1)}$ *τέτοια ώστε, για κάθε* $x_1, x_2 \in [0, \infty)$ *με* $x_1 < x_2$,

$$\frac{V(x_2) - V(x_1)}{x_2 - x_1} \geq -L_\sigma(x_2) \quad \text{(B}'\text{.2.6)}$$

*(ii) Η συνάρτηση* $V$ *είναι παραγωγίσιμη στο μηδέν και*

$$\ln\big(-V'(0)\big) + \rho V(0) + 1 = 0. \quad \text{(B}'\text{.2.7)}$$

## B′.3 Κεφάλαιο 4

Στο Κεφάλαιο 4, αποδεικνύουμε την ύπαρξη βέλτιστου ελέγχου τόσο στην ντετερμινιστική όσο και στη στοχαστική περίπτωση. Παρουσία θορύβου, η ελλειπτική ομαλότητα της συνάρτησης αξίας επιτρέπει την υιοθέτηση της συνήθους μεθοδολογίας. Από την άλλη πλευρά, η προσέγγιση αυτή δεν είναι πάντα εφικτή στη ντετερμινιστική περίπτωση, επειδή η συνάρτηση αξίας δεν αναμένεται γενικά να είναι ομαλή. Πράγματι, όταν το σύστημα της λίμνης έχει ένα σημείο Skiba, η συνάρτηση αξίας δεν είναι διαφορίσιμη εκεί.

Στη ντετερμινιστική περίπτωση, προκειμένου να αποδείξουμε ότι η βέλτιστη αξία είναι εφικτή, αποδεικνύουμε μια Αρχή Σύγκρισης (Θεώρημα B'.3) και τη χρησιμοποιούμε για να κυριαρχήσουμε τη συνάρτηση αξίας, $V$, από την υποψήφια συνάρτηση αξίας, $J_P$ της ενότητας 2.2.1, η οποία κατασκευάζεται με βάση την Αρχή Μέγιστου Pontryagin.

**Θεώρημα B'.3.** *Ας υποθέσουμε ότι*

- $u \in C([0,\infty))$ *είναι μια γνησίως φθίνουσα υπολύση της* (B'.0.24) *(με* $\sigma = 0$*) στο* $[0,\infty)$*, με* $Du \leq -\frac{1}{c_1}$*, με την έννοια του ιξώδους, για κάποια θετική σταθερά* $c_1$.
- $v \in C([0,\infty))$ *είναι μια γνησίως φθίνουσα υπερλύση της* (B'.0.24) *(με* $\sigma = 0$*) στο* $(0,\infty)$*, τέτοια ώστε* $v \geq -c_2(1+x^q)$*, όπου* $c_2$ *μπορεί να είναι οποιαδήποτε θετική σταθερά και* $q$ *μπορεί να είναι οποιοσδήποτε πραγματικός αριθμός.*

*Τότε,* $u \leq v$ *στο* $[0,\infty)$.

Σημειώστε ότι με βάση την Πρόταση B'.2.0(ii) η συνάρτηση αξίας $V$ είναι γνησίως φθίνουσα και $DV \leq -\frac{1}{c}$, με την έννοια του ιξώδους, για κάποια θετική σταθερά $c$. Συνεχίζουμε έπειτα αποδεικνύοντας ότι η συνάρτηση αξίας $V$ είναι μια λύση ιξώδους περιορισμού-κατάστασης της εξίσωσης (1.0.6) στο $[0,\infty)$ όταν $\sigma = 0$.

**Θεώρημα B'.4.** *Η συνάρτηση αξίας* $V$ *είναι μια συνεχής λύση ιξώδους περιορισμού κατάστασης της εξίσωσης* (B'.0.24) *στο* $[0,\infty)$*. Ειδικότερα, η* $V$ *ικανοποιεί την* (B'.0.24) *στο* $x = 0$ *με την κλασική έννοια.*

Έπειτα δείχνουμε ότι η υποψήφια συνάρτηση αξίας μας, $J_P$, ικανοποιεί τις υποθέσεις που έγιναν για την υπερλύση $v$ στο Θεώρημα B'.3.

**Λήμμα B'.1.** *Έστω ότι* $J_P$ *είναι η υποψήφια συνάρτηση αξίας της* (2.2.11)*. Τότε,*

*i.* $J_P$ *είναι φθίνουσα.*

*ii.* $J_P$ *είναι λύση ιξώδους στο* (1.0.6) *με* $\sigma = 0$*, στο* $(0,\infty)$*.*

*iii. Υπάρχει* $c_2 > 0$*, τέτοιο ώστε* $J_P(x) \geq -c_2(1 + x^2)$*, για κάθε* $x \geq 0$.

Έτσι, καταλήγουμε στο βασικό συμπέρασμα

**Πόρισμα Β′.1.** *Το ντετερμινιστικό πρόβλημα της ρηχής λίμνης επιδέχεται βέλτιστο έλεγχο.*

Έπειτα, αποδεικνύουμε την ύπαρξη βέλτιστου ελέγχου παρουσία θορύβου ($\sigma > 0$). Με βάση το Θεώρημα Β′.1, η $V$ είναι μια λύση ιξώδους της εξ. (Β′.0.22) στο $[0, \infty)$ και από κλασικά αποτελέσματα για ομοιόμορφα ελλειπτικούς τελεστές, προκύπτει ότι η $V$ είναι στην πραγματικότητα μια κλασική λύση της (Β′.0.24) στο $(0, \infty)$. Εν τέλει, αποδεικνύεται ότι η $V$ είναι δύο φορές παραγωγίσιμη στο $x = 0$ και με αυτόν τον τρόπο, από την Πρόταση Β′.4(ii), προκύπτει ότι η $V$ είναι μια κλασική λύση της εξ. (Β′.0.24) στο $[0, \infty)$. Το αποτέλεσμα αυτό αναφέρεται στην Πρόταση Β′.5.

**Πρόταση Β′.5.** *Αν* $0 < \sigma^2 < \rho + 2b$*, η συνάρτηση αξίας* $V$ *είναι μια κλασική λύση της εξίσωσης* (Β′.0.24) *στο* $[0, \infty)$ *και*

$$V''(0) = -(\rho + b - r'(0)) \left(V'(0)\right)^2$$

Η ελλειπτική ομαλότητα της συνάρτησης αξίας παρουσία θορύβου επιτρέπει την υιοθέτηση της συνήθους μεθοδολογίας για την απόδειξη της ύπαρξης του βέλτιστου ελέγχου. Προς αυτή την κατεύθυνση, ακολουθούμε τα βήματα που περιγράφονται στο [23], με κατάλληλες τροποποιήσεις για την αντιμετώπιση της απώλειας της ελλειπτικότητας στο όριο και μιας έκρηξης του οφέλους για μικρούς ελέγχους, λόγω της παρουσίας του λογαριθμικού όρου.

**Θεώρημα Β′.5.** *Το στοχαστικό πρόβλημα της ρηχής λίμνης δέχεται έναν βέλτιστο (ανατροφοδοτούμενο) έλεγχο, ο οποίος είναι της μορφής:*

$$u(t) = -\frac{1}{V'(x(t))},\ t \geq 0 \qquad \text{(Β′.3.1)}$$

*όπου* $x(\cdot)$ *είναι η λύση της* (Β′.0.19) *που αντιστοιχεί σε αυτόν τον έλεγχο.*

## Β′.4 Κεφάλαιο 5

Στο Κεφάλαιο 5, μελετάμε την ασυμπτωτική συμπεριφορά της συνάρτησης αξίας, $V$, στο $+\infty$ καθώς και τις ουρές της αναλλοίωτης κατανομής της βέλτιστα ελεγχόμενης διαδικασίας, $x^*(t)$, που είναι η βέλτιστα ελεγχόμενη συγκέντρωση φωσφόρου στη λίμνη.

Το Θεώρημα Β′.6 περιγράφει την ακριβή ασυμπτωτική συμπεριφορά της συνάρτησης αξίας $V$ στο $+\infty$. Στο κεφάλαιο 6, παρουσιάζουμε και υλοποιούμε ένα μονότονο αριθμητικό σχήμα που προσεγγίζει την (Β′.0.21). Η σχέση (Β′.4.1) είναι καθοριστικής σημασίας για τον ακριβή υπολογισμό της $V$ σε αυτό το πλαίσιο, καθώς προτείνει την οριακή συνθήκη στο δεξιό άκρο του υπολογιστικού πεδίου.

**Θεώρημα Β′.6.** *Καθώς $x \to \infty$,*

$$V(x) = -A\left(x + \frac{a}{b+\rho}\right)^2 - \frac{1}{\rho}\ln\left[2A(x + \frac{a}{b+\rho})\right] + K + o(1). \quad (\text{B}'.4.1)$$

*΄όπου*

$$K = \frac{1}{\rho}\left(\frac{2b+\sigma^2}{2\rho} - \frac{Aa^2(\rho+2b)}{(b+\rho)^2} - 1 + 2AC\right) \quad (\text{B}'.4.2)$$

Στη συνέχεια, μελετάμε την αναλλοίωτη κατανομή της βέλτιστα ελεγχόμενης λίμνης. Δεδομένου ότι ο βέλτιστος έλεγχος, όπως υποδεικνύεται από το Θεώρημα Β′.5, είναι ένας έλεγχος ανατροφοδότησης, μπορούμε να τον αντικαταστήσουμε ξανά στη δυναμική της λίμνης και να εξάγουμε τη στοχαστική διαφορική εξίσωση του βέλτιστα ελεγχόμενου συστήματος:

$$\begin{cases} dx^*(t) = \left(-\frac{1}{V'(x^*(t))} - bx^*(t) + r(x^*(t))\right)dt + \sigma x^*(t)dW(t) \\ x^*(0) = x \end{cases} \quad (\text{B}'.4.3)$$

Η τελευταία Πρόταση περιγράφει τη συμπεριφορά των ουρών της αναλλοίωτης κατανομής της ποσότητας φωσφόρου στη βέλτιστα ελεγχόμενη λίμνη.

**Πρόταση Β′.6.** *Η πυκνότητα, $f$, της αναλλοίωτης κατανομής της βέλτιστα ελεγχόμενης διαδικασίας* (Β′.4.3) *είναι:*

$$f(x) = \frac{1}{Z}x^{-2\left(1+\frac{b}{\sigma^2}\right)}e^{-\frac{2}{\sigma^2}\Phi_\sigma(x)} \quad (\text{B}'.4.4)$$

*όπου $Z$ σταθερά κανονικοποίησης και*

$$\Phi_\sigma(x) = \int_x^\infty \Big(-\frac{1}{V_\sigma'(u)} + r(u)\Big)\frac{du}{u^2}, \qquad x > 0.$$

*Συγκεκριμένα,*

$$\lim_{x\to 0} x\Phi_\sigma(x) = \frac{1}{|V_\sigma'(0)|} \qquad \textit{και} \qquad \lim_{x\to\infty}\Phi_\sigma(x) = 0. \quad (\text{B}'.4.5)$$

## Β′.5 Κεφάλαιο 6

Στο Κεφάλαιο 6, υλοποιούμε ένα αριθμητικό σχήμα που κατασκευάστηκε με βάση το σχήμα [4] των Barles και Souganidis και μελετάμε αριθμητικά τις τροχιές και τις ιδιότητες της βέλτιστα ελεγχόμενης λίμνης.

Έστω $\Delta x$ το βήμα της ομοιόμορφης διαμέρισης $0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = l$ του $[0, l]$ για $l > 0$ αρκετά μεγάλο. Χρησιμοποιούμε μια προς τα πίσω διακριτοποίηση πεπερασμένων διαφορών για την προσέγγιση της πρώτης παραγώγου στον γραμμικό όρο του (Β′.0.24), μια προς τα εμπρός διακριτοποίηση πεπερασμένων διαφορών για την παράγωγο στον λογαριθμικό όρο και ένα κεντρικό σχήμα πεπερασμένων διαφορών για την προσέγγιση της δεύτερης παραγώγου.

Επομένως, έχουμε το παρακάτω σχήμα πεπερασμένων διαφορών, για $i = 1, \ldots, N-1$:

$$V_i - \frac{1}{\rho}\Big(r(x_i) - bx_i\Big)\frac{V_i - V_{i-1}}{\Delta x} + \frac{1}{\rho}\left[cx_i^2 + 1 + \ln\left(-\frac{V_{i+1} - V_i}{\Delta x}\right)\right] - \frac{\sigma^2}{2\rho}x_i^2\frac{V_{i+1} + V_{i-1} - 2V_i}{(\Delta x)^2} = 0. \quad (\text{Β}'.5.1)$$

Ακολουθώντας το [4], αποδεικνύεται ότι το παραπάνω σχήμα είναι συνεπές και μονότονο δεδομένου ότι

$$\Delta x(r(x) - bx) \leq \frac{\sigma^2}{2}x^2. \quad (\text{Β}'.5.2)$$

Επιπλέον είναι ευσταθές και συγκλίνει ομοιόμορφα στα συμπαγή στη μοναδική λύση ιξώδους περιορισμού-καταστάσεων της HJB.

Για τους αριθμητικούς υπολογισμούς, δεδομένου ότι στο (Β′.5.1) έχουμε $N - 1$ εξισώσεις με $N + 1$ αγνώστους. εκμεταλλευόμαστε την συνοριακή συνθήκη (Β′.2.7) στο $x = 0$ και εκτιμούμε επίσης την τιμή του $V$ στο δεξιό άκρο $x = l$ με βάση τον τύπο (Β′.4.1) της ασυμπτωτικής συμπεριφοράς της συνάρτησης αξίας $V$ καθώς $x \to +\infty$.

Το σύστημα εξισώσεων

$$\begin{cases} V_i - \frac{1}{\rho}\Big(r(x_i) - bx_i\Big)\frac{V_i - V_{i-1}}{\Delta x} + \frac{1}{\rho}\left[cx_i^2 + 1 + \ln\left(-\frac{V_{i+1} - V_i}{\Delta x}\right)\right] \\ \qquad - \frac{\sigma^2}{2\rho}x_i^2\frac{V_{i+1} + V_{i-1} - 2V_i}{(\Delta x)^2} = 0 \ \ \text{φορ} \ \ i = 1, 2, ..., N-1 \\ V_0 + \frac{1}{\rho}\left[1 + \ln\left(-\frac{V_1 - V_0}{\Delta x}\right)\right] = 0. \end{cases} \quad (\text{Β}'.5.3)$$

μαζί με τη συνοριακή συνθήκη

$$V_N = -A\left(l + \frac{a}{b+\rho}\right)^2 - \frac{1}{\rho}\ln\left(2A(l + \frac{a}{b+\rho})\right) + K \qquad (\text{B}'.5.4)$$

αποτελούν ένα σύστημα $N \times N$ μη γραμμικών εξισώσεων. Στην παρούσα εργασία, προσεγγίζουμε τη λύση αυτού του συστήματος χρησιμοποιώντας τη μέθοδο Newton-Raphson.

Η σημαντικότητα της μεθοδολογίας μας για τον υπολογισμό της λύσης ιξώδους $V$ έγκειται στο γεγονός ότι είμαστε ελεύθεροι να επιλέξουμε οποιαδήποτε τιμή της παραμέτρου $\sigma$ θέλουμε, εφόσον ικανοποιείται η συνθήκη $\sigma^2 < \rho + 2b$ (βλέπε Σχόλιο 3.3.1) και η συνθήκη Β′.5.2. Με αυτό τον τρόπο δεν περιοριζόμαστε μόνο σε μικρές τιμές του θορύβου, $\sigma$, όπως στο [26].

Στη συνέχεια, υλοποιούμε το παραπάνω σχήμα και μελετάμε τις ιδιότητες της συνάρτησης αξίας (βλ. εικόνες 6.1a και 6.1c) και της βέλτισης στρατηγικής εναπόθεσης φωσφόρου (βλ. εικόνες 6.1b και 6.1d) καθώς αλλάζει η τιμή του θορύβου, $\sigma$ για τη συνήθη επιλογή της σιγμοειδούς συνάρτησης, $r = x^2/(x^2+1)$. Η επιλογή των παραμέτρων καλύπτει και την περίπτωση μοναδικού σημείου ισορροπίας και την περίπτωση ύπαρξης σημείου Skiba. Στην τελευταία περίπτωση, παρατηρούμε ένα άλμα (ασυνέχεια) του βέλτιστου ελέγχου στο σημείο Skiba.

Έπειτα, μελετάμε τις ποιοτικές αλλαγές της συμπεριφοράς της αναλλοίωτης κατανομής του βέλτιστα ελεγχόμενου συστήματος σε σχέση με τις διάφορες παραμέτρους. Τόσο στην περίπτωση του μοναδικού σημείου ισορροπίας (είτε ολιγοτροφικό είτε ευτροφικό), όσο και στην περίπτωση του σημείου Skiba, παρατηρούμε ότι το σημείο μεγίστου της κατανομής μετακινείται προς μικρότερες τιμές του φωσφόρου καθώς αυξάνεται ο θόρυβος, ενώ οι ουρές της κατανομής στο άπειρο παχαίνουν. Τα αποτελέσματα αυτά συνοψίζονται στις εικόνες 6.2 και 6.4, ενώ αναδεικνύονται πιο λεπτομερώς στα γραφήματα διακλάδωσης (bifurcation diagrams) 6.3 και 6.5, όπου απεικονίζονται οι θέσεις των τοπικών μεγίστων και ελαχίστων της αναλλοίωτης κατανομής σαν συνάρτηση του θορύβου, $\sigma$.

Οι εικ. 6.6a και 6.6b δείχνουν την αλλαγή της συμπεριφοράς της αναλλοίωτης κατανομής σε σχέση με το κόστος μόλυνσης, $c$, και το προεξοφλητικό παράγοντα, $\rho$, αντίστοιχα.

Τέλος, παρουσιάζουμε γραφήματα της συνάρτησης αξίας, του βέλτιστου ελέγχου και της αναλλοίωτης κατανομής, όταν ο ρυθμός ανακύκλωσης είναι η συνάρτηση υπερβολικής εφαπτομένης (βλ. εικόνες 6.7, 6.8 και 6.10).

## Β′.6 Κεφάλαιο 7

Στο Κεφάλαιο 7, μελετάμε τη μεταευσταθή συμπεριφορά προβλημάτων στοχαστικού ελέγχου που εμφανίζουν σημεία Skiba και αποδεικνύουμε τη γενίκευση του νόμου Arrhenius για την περίπτωση του θορυβοεξαρτώμενου δυναμικού διπλού πηγαδιού.

Θεωρούμε την αυτόνομη στοχαστική διαφορική εξίσωση:

$$\begin{cases} dx^\varepsilon(t) = f(x^\varepsilon(t), u(t))dt + \sqrt{2\varepsilon}dW_t \quad t \geq 0 \\ x^\varepsilon(0) = x \geq 0 \end{cases} \tag{Β′.6.1}$$

όπου η συνάρτηση $f \in C^1(\mathbb{R} \times U)$ ικανοποιεί:

$$\begin{cases} |f_x| \leq C \\ |f(x,u)| \leq C(1 + |x| + |u|) \end{cases} \tag{Β′.6.2}$$

Έστω $G$ συνεχής συνάρτηση στο $\mathbb{R} \times U$, $G(x,\cdot) \in C^1(U)$ και $\rho > 0$. Θεωρούμε τη συνάρτηση αξίας του προβλήματος:

$$V_\varepsilon(x) = \sup_{u \in \mathfrak{U}} \mathbb{E}_x \left[ \int_0^\infty e^{-\rho s} G(x^\varepsilon(s), u(s)) ds \right] \tag{Β′.6.3}$$

όπου $\mathfrak{U}$ είναι το σύνολο $\mathcal{F}_t-$προσαρμοσμένων, $\mathbb{P}-$σ.β. τοπικά ολοκληρώσιμων διαδικασιών με τιμές στο $U$ που ικανοποιούν:

$$\mathbb{E}_x \left[ \int_0^\infty e^{-\rho s} G(x^\varepsilon(s), u(s)) ds \right] < \infty$$

έτσι ώστε η στοχαστική διαφορική εξίσωση (Β′.6.1) να έχει ισχυρή λύση $x(\cdot)$. Στη συνέχεια, κάνουμε τις ακόλουθες υποθέσεις:

**Υπόθεση Β′.2.** *(i) Η συνάρτηση αξίας $V_\varepsilon$ είναι (κλασική) λύση της σχετικής εξίσωσης HJB:*

$$-\varepsilon V_\varepsilon'' - H(x, V_\varepsilon') + \rho V_\varepsilon = 0 \tag{Β′.6.4}$$

*όπου $H(x,p) = \sup_{u \in U}\{f(x,u)p + G(x,u)\}$.*

*(ii)* *Υπάρχει* $\varepsilon_0 > 0$ *τέτοιο ώστε οι συναρτήσεις* $V_\varepsilon$ *και* $V'_\varepsilon$ *να είναι ομοιόμορφα φραγμένες ως προς* $\varepsilon < \varepsilon_0$ *σε κάθε συμπαγές υποσύνολο του* $\mathbb{R}$

*(iii)* $H(x,p)$ *είναι* $C^2$ *και* $H_{pp} > 0$

*(iv)* *Υπάρχει μια βέλτιστη στρατηγική ελέγχου Markov της μορφής* $u^*(s) = g\left(x^*(s), V'_\varepsilon(x^*(s))\right) =: \bar{u}^*(x^*(s))$ *τέτοια ώστε*

$$f_u(x,\bar{u}^*(x))V'_\varepsilon(x) + G_u(x,\bar{u}^*(x)) = 0$$

*και η* $g$ *είναι συνεχής συνάρτηση.*

Υπό την υπόθεση Β΄.2, το βέλτιστα ελεγχόμενο σύστημα (Β΄.6.1) παίρνει τη μορφή:

$$\begin{cases} dx^\varepsilon(t) = -F'_\varepsilon(x^\varepsilon(t))dt + \sqrt{2\varepsilon}dW_t \quad t \geq 0 \\ x^\varepsilon(0) = x \geq 0 \end{cases} \tag{Β΄.6.5}$$

όπου $F'_\varepsilon(x) = -f(x, g(x, V'_\varepsilon(x))$. Με βάση την Υπόθεση 2 (i)-(ii) και την ιδιότητα ευστάθειας των λύσεων ιξώδους,

$$V_\varepsilon \overset{\varepsilon\to 0}{\to} V_0 \text{ τοπικά ομοιόμορφα} \tag{Β΄.6.6}$$

όπου $V_0$ είναι η λύση ιξώδους στην εξίσωση (7.1.4) για $\varepsilon = 0$. Κάνουμε την ακόλουθη υπόθεση για τη συνάρτηση $V_0$.

**Υπόθεση Β΄.3.** *Η συνάρτηση* $V_0$ *της σχ. (7.1.6) είναι σχεδόν παντού παραγωγίσιμη.*

**Λήμμα Β΄.2.** *Υπό τις υποθέσεις Β΄.2 και Β΄.3, έστω* $\Omega$ *συμπαγές υποσύνολο του* $\mathbb{R}$*. Τότε:*

*(i)* *Υπάρχει* $C = C(\Omega)$ *τέτοιο ώστε* $V''_\varepsilon(x) \geq C$ *φορ αλλ* $x \in \Omega$ *και* $\varepsilon < \varepsilon_0$.

*(ii)* *Η οικογένεια συναρτήσεων* $\{F_\varepsilon\}_{\varepsilon>0}$ *συγκλίνει ομοιόμορφα στα συμπαγή στη συνάρτηση* $F_0$.

Το βασικό μας αποτέλεσμα είναι το ακόλουθο:

**Θεώρημα Β′.7.** *Υποθέτουμε ότι η συνάρτηση* $F_0$ *του Λήμματος Β′.2 σχηματίζει ένα δυναμικό διπλού πηγαδιού με τοπικά ελάχιστα* $x_\pm$ *και τοπικό μέγιστο* $x_*$, *με* $x_- < x_* < x_+$ *και ότι η* $F_0$ *είναι* $C^1$ *κοντά στα* $x_\pm, x_*$. *Επιπλέον, υποθέτουμε ότι υπάρχουν* $a, q, M > 0$ *και* $b \in \mathbb{R}$ *(ανεξάρτητα του* $\varepsilon$*) τέτοια ώστε* $F_\varepsilon(x) \geq ax^q + b$, *για κάθε* $x > M$. *Έστω* $\tau^\varepsilon_{x_-} = \inf\{t \geq 0 : x^\varepsilon(t) \leq x_-\}$ *ο χρόνος πρώτης κρούσης του* $x_-$ *από τη διαδικασία* $x^\varepsilon$ *της εξ.* (Β′.6.5). *Τότε η αναμενόμενη τιμή του χρόνου* $\tau^\varepsilon_{x_-}$, *όταν η διαδικασία* $x^\varepsilon$ *ξεκινάει στο* $x_+$, *ικανοποιεί*

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_{x_+} \left[\tau^\varepsilon_{x_-}\right] = F_0(x_*) - F_0(x_+) \qquad (\text{Β}'.6.7)$$

Η μεταευσταθής συμπεριφορά των ρηχών λιμνών παρατηρείται στη φύση και σε μαθηματικούς όρους αντιστοιχεί σε ένα σύστημα με δύο σημεία ισορροπίας και ένα σημείο Skiba. Στην παρουσία θορύβου, το σύστημα κινείται από την ολιγοτροφική κατάσταση στην ευτροφική και αντίστροφα, βλ. εικόνα 7.1. Επιπλέον, η μαρκοβιανή φύση του βέλτιστου ελέγχου οδηγεί σε ένα σύστημα με θορυβοεξαρτώμενη συνάρτηση ολίσθησης. Επομένως, το πρόβλημα της ρηχής λίμνης προσφέρεται ως κατάλληλη εφαρμογή του αποτελέσματός μας, Θεώρημα Β′.7. Υποθέτουμε ότι οι παράμετροι $b, c, \rho$ είναι τέτοιες ώστε το ντετερμινιστικό πρόβλημα της ρηχής λίμνης να διαθέτει δύο σημεία ισορροπίας και ένα σημείο Skiba.

Εφαρμόζοντας τον μετασχηματισμό $y(t) = \log x(t)$ στη διαδικασία $x(t)$ του στοχαστικού προβλήματος της ρηχής λίμνης (Β′.0.19), βρίσκουμε :

$$\begin{cases} dy(t) = \left(e^{-y(t)}u(t) - b + r\left(e^{y(t)}\right)e^{-y(t)} - \frac{\sigma^2}{2}\right)dt + \sigma dW_t \\ y(0) = y \end{cases} \qquad (\text{Β}'.6.8)$$

Ως εκ τούτου, η δυναμική του προβλήματος της ρηχής λίμνης περιγράφεται όπως στην εξ. (Β′.6.1) και μπορούμε να εξετάσουμε τη συνάρτηση αξίας του προβλήματος της ρηχής λίμνης ως προς τη διαδικασία $y(t)$, δηλαδή

$$\tilde{V}(y) = \sup_{u \in \mathfrak{U}} \mathbb{E}_y \left[\int_0^\infty e^{-\rho t}\left(\ln u(t) - ce^{2y(t)}\right)dt\right] = V(e^y)$$

Από την Πρόταση Β′.5, προκύπτει ότι η συνάρτηση $\tilde{V}$ είναι κλασσική λύση στο $\mathbb{R}$ της εξίσωσης

$$-\frac{1}{2}\sigma^2 \tilde{V}'' - \tilde{H}(x, \tilde{V}') + \rho\tilde{V} = 0 \qquad (\text{Β}'.6.9)$$

όπου $\tilde{H}(x,p) = \left(r(e^x)e^{-x} - b - \frac{\sigma^2}{2}\right) p - \ln(-p) + x - 1 - ce^{2x}$ και το βέλτιστα ελεγχόμενο σύστημα (B′.6.8) παίρνει τη μορφή:

$$\begin{cases} dy^\sigma(t) = -F'_\sigma(y^\sigma(t))dt + \sigma dW_t \\ y^\sigma(0) = y \end{cases} \tag{B′.6.10}$$

όπου

$$F'_\sigma(y) = \frac{1}{\tilde{V}'_\sigma(y)} + b - r\left(e^y\right)e^{-y} + \frac{\sigma^2}{2}$$

Για το μοντέλο αυτό αποδεικνύουμε τα δύο τελευταία αποτελέσματά μας.

**Πρόταση B′.7.** *Υποθέτουμε ότι η συνάρτηση ρυθμού ανακύκλωσης $r$, εκτός από την Υπόθεση B′.1, ικανοποιεί επίσης και $r(x) < bx \; \forall \; x > 0$. Έστω $\Omega \subset \mathbb{R}$ συμπαγές και $\sigma_0 < \sqrt{\rho}$. Τότε υπάρχει $C = C(\Omega, \sigma_0) > 0$ τέτοιο ώστε $|\tilde{V}'_\sigma(x)| \leq C$ για κάθε $x \in \Omega$, $\sigma \leq \sigma_0$.*

**Λήμμα B′.3.** *$F_\sigma(x) \geq bx + C$ για κάθε $x > 1$, όπου $C$ σταθερά ανεξάρτητη του $\sigma$.*

Το επιχείρημα επαλήθευσής μας για το ντετερμινιστικό πρόβλημα της ρηχής λίμνης συνεπάγεται ότι η $F_0$ είναι $C^1\left(\mathbb{R} \setminus \{x_*\}\right)$, όπου $x_*$ είναι ο λογάριθμος του σημείου Skiba. Η Πρόταση B′.7 και το Λήμμα B′.3 δείχνουν ότι το πρόβλημα της ρηχής λίμνης, υπό την υπόθεση ύπαρξης δύο σημείων ισορροπίας και ενός σημείου Skiba, ικανοποιεί τις υποθέσεις του Θεωρήματος B′.7. Επομένως, ο νόμος Arrhenius είναι ευσταθής για το μοντέλο μας.