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On Logarithmic stability estimate for an inverse wave problem and on the investigation of the radial symmetric problem.

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March 22, 2024

Acknowledgements

I would like to express to my Professor Antonios Charalampopoulos, gratitude for your invaluable guidance and support throughout the process of completing my thesis. Your expertise, encouragement, and commitment to excellence have been instrumental in shaping the outcome of this work.

To the members of my thesis committee, I extend my sincere appreciation for your time, expertise, and valuable feedback during the review of my thesis. Your thoughtful contributions have significantly enriched the quality of my research, and I am grateful for the collective guidance you provided.

To my family, thank you for your constant support, understanding, and encouragement throughout my academic journey.

To my friends from Physics Department AUTh, thank you for the blast memories we had in Thessaloniki and for the emotional support through tough times. I am grateful that i met you.

To my friends from Athens, thank you for the constant support and encouragement though my journey of obtaining my masters degree.

Abstract

In chapter 1, we briefly discuss about inverse problems, the differences with direct problems and how one may approach an inverse problem in order to obtain a desired estimate. They arise in various real world problems and applications.

In chapter 2, we consider the work of Bellassoued-Choulli-Yamamoto (2009) for finding a stability estimate for an inverse problem of the wave equation and a multidimentional Borg-Levinson theorem from their analytical procedure to produce the log-type stability as it follows in theorems 14, 16 and 17. Their work is mostly based on the properties of the solution of the wave equation and providing a stability estimate for hyperbolic equation for a relatively open subset of the boundary and using generalized X-ray and Fourier transformations.

In chapter 3, we present a semi analytical-numerical procedure for verifying theorem 17 for the simplest case where we have the source of the wave equation depending only on the radius, hence we have radial symmetry, and our domain is the circle with radius 1.1. For the analytic part we solve a eigenvalue differential equation with radial symmetry considering the solutions arise from Bessel's functions. We apply Poincare-Linstedt method to have the perturbed solution familiar with Bessel coefficients, hence we got a matrix with 2 arbitrary constants. For the numerical procedure we used Wolfram Mathematica where we had to find in the graph at least 6 eigenvalues for the two sources q_1 and q_2. After finding the eigenvalues we could find the eigenfunctions and we applied the Sobolev norm to construct the terms that arise in theorem 17 and we concluded to the verification of the estimate.

In chapter 4, we provide an appendix for the basic definitions that are not covered in the previous chapters, and provide some calculations that arise in chapters 2 and 3.

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Chapter 1

Introduction to Inverse Problems

In this first chapter, we are going to briefly discuss about Inverse Problems in Partial Differential Equations. Usually in the branch of science we construct the equations that determine the behaviour of a system we want to study and find a solution. This is usually called a direct problem. In inverse problems, we are given a solution, and we have to find the equation [29] or the properties of a source that gives us the measurements in our solution. We will give the general mathematical formulation and give some examples to clarify the difference between direct and inverse problems.

1.1 What is an Inverse Problem

Three essential ingredients define an inverse problem in this book. The central element is the Measurement Operator (MO), which maps objects of interest, called parameters, to information collected about these objects, called measurements or data. The main objective of inverse problem theory is to analyze such a MO, primarily its injectivity and stability properties. Injectivity of the MO means that acquired data uniquely characterize the parameters. Often, the inversion of the MO amplifies errors in the measurements, which we refer to as noise. Stability estimates characterize this amplification [4].

When the amplification is considered “too large” by the user, which is a subjective notion, then the inverse problem needs to be modified. How this should be done depends on the structure of noise. The second essential ingredient of an inverse problem is thus a noise model, for instance a statement about its size in a given metric, or, if available, its statistical properties.

Once a MO and a noise model are available, the “too large” effect of noise on the reconstruction is mitigated by imposing additional constraints on the parameters that render the inversion well-posed. These constraints take the form of a prior model, for instance assuming that the parameters live in a finite dimensional space, or that parameters are sparsely represented in an appropriate frame.

The definition of an inverse problem (IP) starts with that of a mapping between objects of interest, which we call parameters, and acquired information about these objects, which we call data or measurements. The mapping, or forward problem, is called the measurement operator (MO). We denote it by $\mathfrak{A}$. Let $\mathcal{X}$ be a functional space for the parameters and $\mathcal{Y}$ the space of data, then we write

$$y = \mathfrak{A}(x), \quad \text{for } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}. \tag{1.1.1}$$

The MO maps the parameters to the data. The spaces $\mathcal{X}$ and $\mathcal{Y}$ are typically *Banach* or *Hilbert* spaces. Solving the inverse problem amounts to finding points $x \in \mathcal{X}$ from knowledge of the data $y \in \mathcal{Y}$ such that (1.1.1) or an approximation of (1.1.1) holds.

1.1.1 Properties we seek for the Measurement Operator

Injectivity. We want to uniquely reconstruct the parameters from the data. To do that we need to know whether the MO is *injective*. In other words:

$$\mathfrak{A}(x_1) = \mathfrak{A}(x_2) \Rightarrow x_1 = x_2, \quad \forall x_1,\ x_2 \in \mathcal{X} \tag{1.1.2}$$

In real problems the MO most of the times wont be injective yet continuous, most of the time we approximate the practical MO to a MO that is injective. When A is injective then A^{-1} can be defined which maps the known data $\mathcal{Y}$ to the parameters $\mathcal{X}$.

Stability Estimates. The goal of every inverse problem are the *Stability Estimates* which give us information on how errors in the available measurements translate into errors in the reconstructions. A general form of a stability estimate is:

$$\|x_1 - x_2\|_{\mathcal{X}} \leq \omega\left(\|A(x_1) - A(x_2)\|_{\mathcal{Y}}\right), \tag{1.1.3}$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function such that $\omega(0) = 0$. This function gives an estimate of the reconstruction error $\|x_1 - x_2\|_{\mathcal{X}}$ based on what we believe is the error in the data acquisition $\|A(x_1) - A(x_2)\|_{\mathcal{Y}}$.

1.1.2 Well-posed and ill-posed inverse problems

When noise is not amplified too drastically so that the error on the reconstructed parameters is acceptable, then we say that the inverse problem is *well-posed*. When noise is strongly amplified and the reconstruction is contaminated by too large a noisy component, then we say that the inverse problem is *ill-posed* [4].

In general, Direct Problems, in suitable function spaces and solution concepts, are well-posed; they satisfy existence, uniqueness, and continuous dependence on data. Inverse Problems are ill-posed in general because the solution does not depend continuously on data [41].

1.2 Examples of the Measurement Operator

Example 1. Integral operator. Let $\mathcal{X} = \mathcal{C}([0,1]) = \mathcal{Y}$ and define

$$A(f)(x) = \int_0^x f(y)dy.$$

Statement: The operator A is injective since the equality of data gives us equality of parameters

Proof.

$$\begin{aligned}
&A(f) = A(g) \\
&\int_0^x f(y)dy = \int_0^x g(y)dy \\
&\frac{d}{dy}\int_0^x f(y)dy = \frac{d}{dy}\int_0^x g(y)dy \\
&f(x) = g(x),\ \forall x \in \mathbb{R} \Rightarrow f = g,\ \forall f, g \in \mathcal{X}
\end{aligned}$$

□

Example 2. Derivative operator. Let $\mathcal{X} = \mathcal{C}_0^1([0,1])$ and $\mathcal{Y} = \mathcal{C}([0,1])$ define

$$A(f)(x) = f'(x)$$

We consider from the *fundamental theory of calculus* [74] that

$$f(x) = f(0) + \int_0^x f'(y)dy = \int_0^x f'(y)dy = \int_0^x A(f)(y)dy$$

where now as in *example 1* for $A(f) = A(g) \Rightarrow f = g$.

Example 3. 2-D Radon Transform. Let $\mathcal{X} = \mathcal{C}_c(\mathbb{R}^2)$, $\mathcal{Y} = \mathcal{C}(\mathbb{R} \times (0, 2\pi))$. Define $l(s,\theta)$, where $s \in \mathbb{R}$ and $\theta \in (0, 2\pi)$ as the line with direction perpendicular to $u = (\cos\theta, \sin\theta)$ and at a distance $|s|$ from the origin $(0,0)$. Let $u^\perp = (-\sin\theta, \cos\theta)$ the rotation of u by $\frac{\pi}{2}$, then

$$l(s,\theta) = \{x \in \mathbb{R}^2 : x = su + tu^\perp, \text{ for } t \in \mathbb{R}\}.$$

Define

$$A(f)(s,\theta) = \int_{l(s,\theta)} f(x)\ dl = \int_{\mathbb{R}} f(su + tu^\perp)\ dt,$$

which maps a function to the value of its integrals along any line and is called *two-dimensional Radon Transform* [75, 34, 21] with many applications in Computer Tomography (CT) [16].

Example 4. The Calderon problem. We introduce the following elliptic PDE [66]:

$$\begin{aligned} -\nabla \cdot \gamma(x)\nabla u(x) = 0, &\quad x \in X \\ u(x) = f(x) &\quad x \in \partial X, \end{aligned} \tag{1.2.1}$$

where $X \subset \mathbb{R}^n$ is smooth, bounded, open with ∂X boundary, $\gamma(x)$ is a smooth coefficient in X bounded above and below by positive constants and $f(x)$ is the Dirichlet data. We introduce the outgoing current function [36]

$$j(x) = \gamma(x)\frac{\partial u}{\partial \nu}(x),$$

with ν the outward unit normal. $j(x)$ is a well defined function, and we can define the Dirichlet-to-Neumann map, DN for short, it is defined as

$$\begin{aligned} \Lambda_\gamma : H^{1/2}(\partial X) &\to H^{-1/2}(\partial X) \\ f(x) &\mapsto \Lambda_\gamma[f](x) = j(x) = \gamma(x)\frac{\partial u}{\partial \nu}(x). \end{aligned} \tag{1.2.2}$$

we discuss more about DN map and its applications on *section 2.2*.
Let $\mathcal{X} = \mathcal{C}^2(\overline{X})$ and $\mathcal{Y} = \mathcal{L}(H^{1/2}(\partial X), H^{-1/2}(\partial X))$. We define the measurement operator

$$A(\gamma) = \Lambda_\gamma \in \mathcal{Y},\ \gamma \in \mathcal{X} \tag{1.2.3}$$

The measurement operator maps the unknown conductivity γ to the DN operator. The Calderon problem finds important applications in Electrical Impedance Tomography and Optical Tomography [77, 40, 25].

1.3 Examples of Inverse Problems

We introduce some examples from [41, 8] where we demonstrate the difference between direct and inverse problems.

1.3.1 Direct vs Inverse problems

Example 5. One-dimensional heat equation.

$$\partial_t u(x,t) = \partial_x^2 u(x,t),\ (x,t) \in (0,\pi) \times \mathbb{R}_+ \tag{1.3.1}$$

given with boundary and initial conditions

$$u(0,t) = u(\pi,t) = 0,\ t \geq 0, \quad u(x,0) = u_0(x),\ 0 \leq x \leq \pi \tag{1.3.2}$$

From this we have the solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin(nx) \tag{1.3.3}$$

with

$$a_n = \frac{2}{\pi} \int_0^\pi u_0(y) \sin(ny) dy$$

Direct Problem: Given u_0 and $T > 0$, determine $u(\cdot, T)$.
Inverse Problem: Measure $u(\cdot, T)$ and determine $u(\cdot, \tau)$ for given $\tau < T$.

We have

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \int_0^\pi u(y,\tau) \sin(ny) dy e^{-n^2(T-\tau)} \sin(nx) \tag{1.3.4}$$

and set $v = u(\cdot, \tau)$ from

$$u(x,T) = \int_0^\pi k(x,y) v(y) dy,\ 0 \leq x \leq \pi \tag{1.3.5}$$

where

$$k(x,y) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2(T-\tau)} \sin(nx) \sin(ny). \tag{1.3.6}$$

The inverse problem leads to solving a *Fredholm integral equation of the first kind* [73, 46]

Example 6. Computer Tomography (CT). Consider a fixed plane through a human body with $\rho(x_1, x_2)$ being the change of density at (x_1, x_2) which we would like to determine from measurements of intensities $l = l(L)$ of X-rays along lines L in the plain.

Parametrization of $L = L_{s,\delta}$:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s \begin{pmatrix} \cos\delta \\ \sin\delta \end{pmatrix} + t \begin{pmatrix} -\sin\delta \\ \cos\delta \end{pmatrix} \in \mathbb{R}^2,\ t \in \mathbb{R}. \tag{1.3.7}$$

The attenuation of the intensity I is approximately described by $dI = -\gamma\rho I dt$ with some constant γ. Then we integrate along the ray and we have

$$\ln I_{s,\delta} = -\gamma \int_{\mathbb{R}} \rho(s\cos\delta - t\sin\delta, s\sin\delta + t\cos\delta) dt. \tag{1.3.8}$$

Direct Problem: Given ρ (with compact support), compute line integrals.
Inverse Problem: Determine $\rho(x_1, x_2)$ from Radon transform

$$(R\rho)(s,\delta) = \int_{\mathbb{R}} \rho(s\cos\delta - t\sin\delta, s\sin\delta + t\cos\delta) dt,\ (s,\delta) \in \mathbb{R} \times [0,\pi). \tag{1.3.9}$$

Example 7. Impedance Tomography. Let $D \subset \mathbb{R}^2$ cross-section through body and $\gamma = \gamma(x_1, x_2)$ conductivity. Apply current distribution f on boundary ∂D. The potential u satisfies

$$\nabla(\gamma \nabla u) = 0 \text{ in } D, \quad \gamma \partial_\nu u = f \text{ on } \partial D. \tag{1.3.10}$$

Direct Problem: Given γ and f, solve the BVP for u.
Inverse Problem: Measure u on ∂D for many fluxes f and determine γ in D.

Example 8. Direct Scattering Problem: Given $n \in L^\infty(\mathbb{R}^3)$ such $D = \text{supp}(n-1)$ is bounded, wave number $k > 0$, and the incident field $u^{inc}(x) = e^{ik\hat{\theta}\cdot x}$ with $\hat{\theta} \in \mathbb{S}^2$, find the total field $u = u(x)$ with $\Delta u + k^2 nu = 0$ in $\mathbb{R}^3$ such that $u^s = u - u^{inc}$ satisfies a radiation condition for $|x| \to \infty$.
Inverse Scattering Problem: Given u for $|x| \to \infty$ for all directions $\hat{\theta} \in \mathbb{S}^2$. Find n or at least the shape of $D = \text{supp}(n-1)$.

1.3.2 Stability estimate with numerical example for an inverse problem

In the last example we find the stability estimate on a matrix, which is on of the most simplest inverse problems and follow up with a numerical example to visualise this.

Example 9. Matrix Inversion. Let $u \in \mathbb{R}^n$, $f \in \mathbb{R}^n$ be $n-$dimensional vectors and $K \in \mathbb{R}^{n\times n}$ be a symmetric, positive definite matrix where we have the equation

$$Ku = f. \tag{1.3.11}$$

From spectral theory [67], K has positive real eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$$

with eigenvalues $k_j \in \mathbb{R}^n$, $j \in \{1, \ldots, n\}$, therefore we can write K in the form:

$$K = \sum_{j=1}^{n} \lambda_j k_j k_j^\top \tag{1.3.12}$$

and set $\kappa = \frac{\lambda_1}{\lambda_n}$ the condition number.

Assume that we measure f^δ instead of f which is a disturbed f, then

$$\|f - f^\delta\|_2 \leq \delta\|K\| = \delta\lambda_1 \tag{1.3.13}$$

the operator norm of K is equal to its largest eigenvalue because K is symmetric. We denote as u^δ the solution to the equation

$$Ku^\delta = f^\delta \tag{1.3.14}$$

then we have by subtracting the two equations and taking applying K^{-1} from the left, we have

$$u - u^\delta = \sum_{j=1}^{n} \lambda_j^{-1} k_j k_j^\top (f - f^\delta) \tag{1.3.15}$$

$$\Rightarrow \|u - u^\delta\|_2^2 = \sum_{j=1}^{n} \lambda_j^{-2} \|k_j\|_2^2 \left|k_j^\top (f - f^\delta)\right|^2 \leq \lambda_n^{-2} \|f - f^\delta\|_2^2. \tag{1.3.16}$$

The last comes from the orthonormality of eigenvectors, Cauchy-Schwartz inequality and $\lambda_n \leq \lambda_j \Rightarrow \lambda_j^{-1} \leq \lambda_n^{-1}$, $\forall j \in \{1, \ldots, n\}$.

$$\Rightarrow \|u - u^\delta\|_2 \leq \lambda_n^{-1} \|f - f^\delta\|_2 \leq \kappa\delta. \tag{1.3.17}$$

Notice that, in worst case scenario an error δ from data is amplified by the condition number κ of $K \in \mathbb{R}^{n\times n}$. A matrix with large κ is called *ill-conditioned* [57].

Let's assume a numerical example, where

$$K = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1001}{1000} \end{pmatrix}$$

which has the eigenvalues

$$\lambda_j = 1 + \frac{1}{2000} \pm \sqrt{1 + \frac{1}{2000^2}}$$

then $\kappa \approx 4002 \gg 1$ and $\|K\| \approx 2$. For known $f = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ the solution of $Ku = f$ is given by the vector $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We assume a perturbation $f^\delta = \begin{pmatrix} 99/100 \\ 101/100 \end{pmatrix}$. The solution to the equation $Ku^\delta = f^\delta$ is given by the vector $u^\delta = \begin{pmatrix} -19.01 \\ 20 \end{pmatrix}$. We have

$$\delta = \frac{\|f - f^\delta\|}{\|K\|} \approx \frac{\sqrt{2}}{200}$$

and

$$\|u - u^\delta\| \approx 20\sqrt{2}$$

then we observe that

$$\frac{\|u - u^\delta\|}{\delta} \approx 4000 \approx \kappa$$

1.3.3 Inverse problems before it was cool

The effects to discover the causes has concerned scientists for centuries. Yet, that didn't stop them from evolving theories by observations and come up with mathematical models that are used up to this day. A historical example is the calculations of Adams and Le Verrier which led to the discovery of Neptune from the perturbed trajectory of Uranus via Newton's laws [59, 37]. However, a formal study of inverse problems was not initiated until the 20^{th} century.

One of the earliest examples of a solution to an inverse problem was discovered by Hermann Weyl and published in 1911, describing the asymptotic behavior of eigenvalues of the Laplace–Beltrami operator [82]. Today known as Weyl's law, it is perhaps most easily understood as an answer to the question of whether it is possible to hear the shape of a drum. Weyl conjectured that the eigenfrequencies of a drum would be related to the area and perimeter of the drum by a particular equation, a result improved upon by later mathematicians.

1.4 Conclusion of Chapter 1

As we seen from the examples, the inverse problems demand the knowledge of a solution or its properties via measurements for one can assert their cause. From the solutions, it is possible to have measurements with error where we have to construct our models in such a way to describe the reality, even by using approximating methods.

In the next chapter, we introduce the work of Bellassoued-Choulli-Yamamoto on finding a stability estimate for an inverse problem of the wave equation and a multidimentional Borg-Levinson theorem and how they come up with the log-type stability.

Chapter 2

Stability estimate for an inverse wave equation

2.1 Preliminaries

2.1.1 The problem

We consider the stability in an inverse problem of determining the potential q entering the wave equation in a bounded smooth domain of $\mathbb{R}^d$ from boundary observations [7]. We want to prove a log-type stability estimate in determining q from a partial Dirichlet to Neumann map where q is known in a neighbourhood of the boundary of the spatial domain with an additional condition. Let $u = u(t,x) \in H^2(Q)$, where $Q = (0,T) \times \Omega$, $T > 0$ is fixed, $\Omega \subset \mathbb{R}^d$ a bounded domain with smooth boundary Γ. The function u satisfies

$$u(0,\cdot) = \partial_t u(0,\cdot) = 0 \text{ in } \Omega,\ u(t,\cdot) = 0 \text{ on } \Sigma \tag{2.1.1}$$

where $\Sigma = (0,T) \times \Gamma$. Let $v = v(t,x) \in H^2(Q)$ be the solution to the following backward wave equation

$$\begin{cases} \partial_t^2 v - \Delta v + q(x)v = 0 & \text{in } Q, \\ v(T,\cdot) = 0, \quad \partial_t v(T,\cdot) = 0 & \text{in } \Omega, \\ v = h & \text{on } \Sigma. \end{cases} \tag{2.1.2}$$

We prove the following identity

Lemma 1. Let $v = v(t,x) \in H^2(Q)$ be the solution of (2.1.2) and $u \in H^2(Q)$ that satisfy (2.1.1), then it is true that

$$\int_Q (\partial_t^2 - \Delta + q)uv\ dxdt = -\int_\Sigma v\partial_\nu u\ dSdt. \tag{2.1.3}$$

Proof. To prove (2.1.3) we use integration by parts and Green's formula with respect to variable x. We break our original integral to three integrals, that is:

$$\begin{aligned}
\int_Q \partial_t^2 uv\ dtdx &= \int_\Omega dx \int_0^T (\partial_t^2 u)v\ dt \\
&= \int_\Omega dx \left([\partial_t uv]_0^T - \int_0^T \partial_t u \partial_t v\ dt \right) \\
&= -\int_\Omega dx \left(\int_0^T \partial_t u \partial_t v\ dt \right) \\
&= -\int_\Omega dx\, [u\partial_t v]_0^T + \int_\Omega dx \int_0^T u\partial_t^2 v\ dt \\
&= \int_Q u\partial_t^2 v\ dtdx.
\end{aligned}$$

The second integral:

$$
\begin{aligned}
-\int_Q \Delta uv\ dxdt &= -\int_0^T dt \int_\Omega \Delta uv\ dx \\
&= -\int_0^T dt \int_\Omega (\nabla\cdot\nabla u)v\ dx \\
&= -\int_0^T dt \int_\Omega \nabla\cdot(\nabla uv)\ dx + \int_0^T dt \int_\Omega \nabla u\cdot\nabla v\ dx \\
&= -\int_0^T dt \int_{\partial\Omega} \frac{\partial u}{\partial\nu} v\ dS + \int_0^T dt \int_\Omega (\nabla\cdot(u\nabla v) - u\Delta v)\ dx \\
&= -\int_0^T dt \int_{\partial\Omega} \left(\frac{\partial u}{\partial\nu}v - \frac{\partial v}{\partial\nu}u\right)\ dS - \int_Q u\Delta v\ dxdt \\
&= -\int_\Sigma (v\partial_\nu u - u\partial_\nu v)\ dSdt - \int_Q u\Delta v\ dxdt.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_Q (\partial_t^2 - \Delta + q)uv\ dxdt &= \int_Q \partial_t^2 uv\ dxdt - \int_Q \Delta uv\ dxdt + \int_Q quv\ dxdt \\
&= \int_Q u\partial_t^2 v\ dtdx - \int_\Sigma (v\partial_\nu u - u\partial_\nu v)\ dSdt - \int_Q u\Delta v\ dxdt + \int_Q uqv\ dxdt \\
&= \int_Q u\left(\partial_t^2 - \Delta + q\right) v\ dxdt - \int_\Sigma (v\partial_\nu u - u\partial_\nu v)\ dSdt.
\end{aligned}
$$

From (2.1.2), we have $\int_Q u\left(\partial_t^2 - \Delta + q\right) v\ dxdt = 0$ and

$$
-\int_\Sigma (v\partial_\nu u - u\partial_\nu v)\ dSdt = \int_\Sigma u\partial_\nu v\ dSdt - \int_\Sigma v\partial_\nu u\ dSdt
$$

where, from (2.1.1), $\int_\Sigma u\partial_\nu v\ dSdt = 0$. Finally, we conclude that

$$
\begin{aligned}
\int_Q (\partial_t^2 - \Delta + q)uv\ dxdt &= \int_Q u\left(\partial_t^2 - \Delta + q\right) v\ dxdt - \int_\Sigma (v\partial_\nu u - u\partial_\nu v)\ dSdt \\
&= -\int_\Sigma v\partial_\nu u\ dSdt.
\end{aligned}
$$

□

2.1.2 Solution for an inverse problem for the wave equation by Rakesh and Symes

From now on we consider as $\mathcal{C}^{0,\mu}(\overline{\Omega})$ be the usual Hölder space with $0 < \mu < 1$ and fix $q_0 \in \mathcal{C}^{0,\mu}(\overline{\Omega})$ and consider the set

$$
\mathfrak{X}(M,\omega) = \{q \in \mathcal{C}^{0,\mu}(\overline{\Omega});\ \|q\|_{L^\infty(\Omega)} \leq M,\ q(x) = q_0(x) \text{ in } \omega\}, \tag{2.1.4}
$$

where $\omega \subset \Omega$ is an arbitrary neighbourhood of Γ and M is a given constant. We refer a result on the existence of geometric optics solutions which is noted in [61].

Lemma 2. Let $\Phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $\theta \in \mathbb{S}^{d-1} = \{x \in \mathbb{R}^d; |x| = 1\}$, $\sigma > 0$ be arbitrarily given. Then the equation

$$
\partial_t^2 u - \Delta u + q(x)u = 0 \ \text{ in } Q
$$

has a solution $u \in H^2(Q)$ of the form

$$u(t,x) = \Phi(x+t\theta)e^{i\sigma(x\theta+t)} + \Psi_q(t,x;\sigma) \tag{2.1.5}$$

where $\Psi_q(t,x;\sigma)$ satisfies

$$\Psi_q(t,x;\sigma) = 0, \quad (t,x) \in \Sigma,$$
$$\Psi_q(s,x;\sigma) = \partial_t \Psi_q(s,x;\sigma) = 0, \quad x \in \Omega, \ s = 0 \text{ or } T,$$

and

$$\sigma\|\Psi_q(\cdot,\cdot;\sigma)\|_{L^2(Q)} + \|\nabla\Psi_q(\cdot,\cdot;\sigma)\|_{L^2(Q)} \leq C\|\Phi\|_{H^3(\mathbb{R}^d)}, \tag{2.1.6}$$

where $C = C(T,\Omega,M)$ is a constant and M is the essential boundary from (2.1.4).

Proof. We recall the work that is mentioned in this paper, [61]. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We consider $u = u(x,t)$ is a solution of the problem

$$\begin{array}{rcc} \partial_t^2 u - \Delta_x u + q(x)u = 0, & & \text{in } \overline{\Omega} \times [0,T] \\ u(x,0) = \phi(x), & \partial_t u(x,0) = \psi(x), & \text{if } x \in \Omega \\ \partial_\nu u(x,t) = f(x,t), & & \text{on } \partial\Omega \times [0,T] \end{array} \tag{2.1.7}$$

We also consider that the function $v = v(x,t)$ solves

$$\begin{array}{rc} \partial_t^2 v - \Delta_x v + q(x)v = 0, & \text{in } \Omega \times [0,T] \\ v(x,T) = \partial_t v(x,T) = 0, & x \in \Omega \\ \partial_\nu v(x,t) = g(x,t), & \text{on } \partial\Omega \times [0,T] \end{array} \tag{2.1.8}$$

Le us define the Neumann-to-Dirichlet map:

$$\begin{aligned} &\Lambda_q : \mathcal{C}^\infty(\partial\Omega \times [0,T]) \to D'(\partial\Omega \times [0,T]) \\ &f(x,t) \mapsto u_{|_{\partial\Omega\times[0,T]}} \end{aligned}$$

We can use another notation for u as $u = u_{\phi,\psi}$ that solves (2.1.7). We consider $u_{0,0}$ that solves (2.1.7) with $\phi(x) = \psi(x) = 0$ and $\tilde{u}_{\phi,\psi}$ that solves (2.1.7) with $f(x,t) = 0$. We claim that

$$\Lambda_{q,0,0}(f) = \Lambda_{q,\phi,\psi}(f) - \Lambda_{q,\phi,\psi}(0)$$

Proof. It is true that

$$\begin{aligned} \Lambda_{q,\phi,\psi}(f) &= u_{\phi,\psi}|_{\partial\Omega\times[0,T]}(x,t) \\ \Lambda_{q,0,0}(f) &= u_{0,0}|_{\partial\Omega\times[0,T]}(x,t) \\ \Lambda_{q,\phi,\psi}(0) &= \tilde{u}_{\phi,\psi}|_{\partial\Omega\times[0,T]}(x,t) \end{aligned}$$

We notice that

$$\partial_\nu u_{\phi,\psi} - \partial_\nu \tilde{u}_{\phi,\psi} = f(x,t) - 0 = f(x,t) = \partial_\nu u_{0,0}$$

that is

$$\partial_\nu \left(\Lambda_{q,\phi,\psi}(f) - \Lambda_{q,\phi,\psi}(0)\right) = \partial_\nu \Lambda_{q,0,0}(f)$$

which proves our claim. □

As in [61], it states that we reformulate our problem determing $q(x)$ knowing the bilinear form B_q defined by

$$B_q(f,g) = \int_{\partial\Omega\times[0,T]} dS_{x,t}\ (fv + gu) \tag{2.1.9}$$

It is

$$\int_{\partial\Omega\times[0,T]} dS\ (fv+gu) = \int_{\partial\Omega\times[0,T]} dS\ \left(\frac{\partial u}{\partial\nu}v + \frac{\partial v}{\partial\nu}u\right)$$

We apply Green's identity [32]:

$$\int_{\partial\Omega} dS\ v\frac{\partial u}{\partial\nu} = \int_\Omega \nabla u\cdot\nabla v\ dx + \int_\Omega v\Delta u\ dx \tag{2.1.10}$$

and we obtain

$$\begin{aligned}\int_{\partial\Omega\times[0,T]} dS\ \left(\frac{\partial u}{\partial\nu}v + \frac{\partial v}{\partial\nu}u\right) &= \int_{\Omega\times[0,T]} \nabla v\cdot\nabla u\ dxdt + \int_{\Omega\times[0,T]} u\Delta v\ dxdt \\ &\quad + \int_{\Omega\times[0,T]} \nabla v\cdot\nabla u\ dxdt + \int_{\Omega\times[0,T]} v\Delta u\ dxdt \\ &= \int_{\Omega\times[0,T]} (2\nabla v\cdot\nabla u + u\Delta v + v\Delta u)\ \ dxdt\end{aligned}$$

From (2.1.7) and (2.1.8) we have

$$\begin{aligned}\Delta u &= \partial_t^2 u + qu \quad \text{and} \\ \Delta v &= \partial_t^2 v + qv\end{aligned}$$

Therefore,

$$\begin{aligned}B_q &= \int_{\omega\times[0,T]} \left(2\nabla u\cdot\nabla v + u\partial_t^2 v + quv + v\partial_t^2 u + quv\right) \\ &= 2\int_0^T dt \int_\Omega \left(\nabla u\cdot\nabla v + quv + \frac{1}{2}\left(u\partial_t^2 v + v\partial_t^2 u\right)\right)\end{aligned}$$

The integral

$$\int_{\Omega\times[0,T]} \frac{1}{2}\left(u\partial_t^2 v + v\partial_t^2 u\right)$$

can be rewritten as

$$\frac{1}{2}\int_\Omega dx \int_0^T dt\ u\partial_t^2 v + \frac{1}{2}\int_\Omega dx \int_0^T dt\ v\partial_t^2 u$$

where we use integration by parts in these terms

$$\int_0^T dt\ u\partial_t^2 v = \underbrace{[u\partial_t v]_0^T}_{=0} - \int_0^T dt\ \partial_t u \partial_t v = -\int_0^T dt\ \partial_t u\partial_t v$$

Overall

$$\frac{1}{2}\int_{\Omega\times[0,T]} \left(u\partial_t^2 v + v\partial_t^2 u\right)\ dxdt = -\int_{\Omega\times[0,T]} \partial_t u\cdot\partial_t v\ dxdt$$

and we obtain the bilinear form

$$B_q(f,g) = 2\int_{\Omega\times[0,T]} (\nabla u\cdot\nabla v + quv - \partial_t u\cdot\partial_t v)\ \ dxdt. \tag{2.1.11}$$

Given q_0 and q_1 with $B_{q_0} = B_{q_1}$ and define $q(x,s) = sq_1 + (1-s)q_0$, for $s\in[0,1]$ and add an additional property

$$\begin{aligned}\partial_\nu u(\cdot,\cdot,s_1) &= \partial_\nu u(\cdot,\cdot,s_2) \quad \text{on} \quad \partial\Omega\times[0,T] \\ \partial_\nu v(\cdot,\cdot,s_1) &= \partial_\nu v(\cdot,\cdot,s_2) \quad \text{on} \quad \partial\Omega\times[0,T]\end{aligned} \tag{2.1.12}$$

$\forall s_1, s_2 \in [0,1]$. According with [61], with the same tools as we used to prove (2.1.11) starting with $B_{q_0} = B_{q_1}$, that

$$0 = B_{q_1}\left(\partial_\nu u(s=1)_{|\partial\Omega\times[0,T]}, \partial_\nu v(s=1)_{|\partial\Omega\times[0,T]}\right) - B_{q_0}\left(\partial_\nu u(s=0)_{|\partial\Omega\times[0,T]}, \partial_\nu v(s=0)_{|\partial\Omega\times[0,T]}\right)$$
$$= \int_0^1 ds \frac{d}{ds} B_q\left(\partial_\nu u(s)_{|\partial\Omega\times[0,T]}, \partial_\nu v(s)_{|\partial\Omega\times[0,T]}\right)$$

We apply (2.1.11) and we obtain

$$0 = \int_0^1 ds \frac{d}{ds} \int_{\Omega\times[0,T]} dxdt \; (\nabla u \cdot \nabla v - \partial_t u \cdot \partial_t v + quv)$$
$$= \int_0^1 ds \int_{\Omega\times[0,T]} dxdt \; (\nabla \dot u \cdot \nabla v + \nabla u \cdot \nabla \dot v - \partial_t \dot u \cdot \partial_t v - \partial_t u \cdot \partial_t \dot v + \dot q uv + q\dot u v + qu\dot v)$$

where $\frac{d}{ds}(f) = \dot f$

$$0 = \int_0^1 ds \int_{\Omega\times[0,T]} \dot q uv$$
$$+ \int_0^1 ds \int_{\Omega\times[0,T]} dxdt \; (q\dot u v + qu\dot v + \nabla \dot u \cdot \nabla v + \nabla u \cdot \nabla \dot v - \partial_t \dot u \cdot \partial_t v - \partial_t u \cdot \partial_t \dot v)$$

We note, from [83], that $\Box = \partial_t^2 - \Delta$ is the d'Alembert operator. We apply to the previous equation Green's identity (2.1.10) to the terms that have the operators ∇ and integration by parts to the terms that have the operator ∂_t. We have

- $\int_0^1 ds \int_{\Omega\times[0,T]} dxdt \; (\nabla \dot u \cdot \nabla v) = \int_0^1 ds \int_{\partial\Omega\times[0,T]} dS_\Omega \, (v\partial_\nu \dot u) - \int_0^1 ds \int_{\Omega\times[0,T]} dxdt \; v\Delta \dot u$
- $\int_0^1 ds \int_{\Omega\times[0,T]} dxdt \; (\nabla u \cdot \nabla \dot v) = \int_0^1 ds \int_{\partial\Omega\times[0,T]} dS_\Omega \, (u\partial_\nu \dot v) - \int_0^1 ds \int_{\Omega\times[0,T]} dxdt \; u\Delta \dot v$
- $-\int_0^1 ds \int_\Omega \int_0^T dt \; \partial_t \dot u \cdot \partial_t v = -\int_0^1 ds \int_\Omega dx \underbrace{[\partial_t \dot u \cdot v]_0^T}_{=0} + \int_0^1 ds \int_{\Omega\times[0,T]} dxdt \; v\partial_t^2 \dot u$

$$= \int_0^1 ds \int_{\Omega\times[0,T]} dxdt \; v\partial_t^2 \dot u$$

- $-\int_0^1 ds \int_\Omega \int_0^T dt \; \partial_t u \cdot \partial_t \dot v = \int_0^1 ds \int_{\Omega\times[0,T]} dxdt \; u\partial_t^2 \dot v.$

We combine our four identities with our previous equation and we use d'Alembert operator to have a more compact result

$$0 = \int_0^1 ds \int_{\Omega\times[0,T]} \dot q uv + \int_0^1 ds \int_{\Omega\times[0,T]} dxdt \; (u(\Box + q)\dot v + v(\Box + q)\dot u)$$
$$+ \int_0^1 ds \int_{\partial\Omega\times[0,T]} dS_\Omega \, (u\partial_\nu \dot v + v\partial_\nu \dot u) .$$

From (2.1.7) and (2.1.8) we have

$$\frac{d}{ds}(\Box + q)\dot u = 0 \Rightarrow (\Box + q)\dot u = -\dot q u$$
$$\frac{d}{ds}(\Box + q)\dot v = 0 \Rightarrow (\Box + q)\dot v = -\dot q v.$$

From (2.1.12) we have that $\frac{\partial u}{\partial \nu}$ and $\frac{\partial v}{\partial \nu}$ have the same value for every $s \in [0,1]$, therefore

$$\frac{\partial \dot{u}}{\partial \nu} = \frac{\partial \dot{v}}{\partial \nu} = 0$$

Therefore we have acquired the result

$$0 = \int_0^1 ds \int_{\Omega \times [0,T]} \dot{q} uv \tag{2.1.13}$$

where we can also extend q_0 and q_1 outside Ω as

$$\tilde{q}_j = \begin{cases} q_j & x \in \Omega \\ 0 & x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

then we can extend the equation (2.1.13) as

$$0 = \int_0^1 ds \int_{\mathbb{R}^n \times [0,T]} \dot{q} uv \tag{2.1.14}$$

with $\dot{q} = 0$ outside Ω and $\dot{q} = q_1 - q_0$. In [61] it is proved that $\dot{q} = 0$ in $\mathbb{R}^n$ and that is done by considering $\theta \in \mathbb{R}^n$, $|\theta| = 1$, $\sigma > 0$, $\Phi \in \mathcal{C}_0^\infty$ and a solution u of the form

$$u = u(t,x) = \Phi(x + t\theta) e^{i\sigma(x\theta + t)} + \Psi_q(t,x;\sigma)$$

where combining it with (2.1.7), also $\Box e^{i\sigma(x\theta+t)} = (i^2\sigma^2 - i^2\sigma^2|\theta|^2) e^{i\sigma(x\theta+t)} = 0$, we have

$$(\Box + q)\Psi_q = -(\Box + q)\left(\Phi(x + t\theta) e^{i\sigma(x\theta+t)}\right) = -e^{i\sigma(x\theta+t)}(\Box + q)\Phi(x + t\theta)$$

From the solution u we have that $\Psi_q = 0$ on Σ and $\Psi_q = \partial_t \Psi_q = 0$ for $x \in \Omega$ and $s = 0$ or T. Also we have

$$\|\Psi_q\|_{L^2(\Omega \times [0,T])} \leq C/\sigma \tag{2.1.15}$$

where C depends on T, $\|\Phi\|_{\mathcal{C}^3(\mathbb{R}^n)}$ and $\mathrm{Vol}(\Omega)$. Where also one can derive the H^2 regularity of Ψ_q and obtain

$$\sigma \|\Psi\|_{L^2(Q)} + \|\nabla \Psi_q\|_{L^2(Q)} \leq C' \|\Phi\|_{H^3(\mathbb{R}^n)} \tag{2.1.16}$$

Which concludes the proof of the lemma. □

2.1.3 Stability estimate for a hyperbolic equation for a relatively open subset of Γ

We choose $\rho > 0$ to be some arbitrary distance, such that

$$\omega(\rho) = \{x \in \Omega;\ \mathrm{dist}(x, \Gamma) \leq \rho\} \subset \omega \tag{2.1.17}$$

and, for $\tau > 0$, we set

$$\omega_\tau = (0,\tau) \times \omega, \quad \omega_\tau(\rho) = (0,\tau) \times \omega(\rho) \tag{2.1.18}$$

The following lemma shows a stability estimate in the continuation of the solutions of a hyperbolic equation from lateral boundary data on an arbitrary non-empty relatively open subset Γ_0 of Γ. I.e. The set Γ_0 is a part of the boundary Γ.

Lemma 3. Let $q_1 \in \mathfrak{X}(M,\omega)$ and T be sufficiently large such that $T/3 > \mathrm{Diam}(\Omega)$. Let $w \in H^2(Q)$ be a solution of the following boundary value problem

$$\begin{cases} (\partial_t^2 - \Delta + q_1(x))w = F & \text{in } Q, \\ w = 0 & \text{on } \Sigma. \end{cases} \tag{2.1.19}$$

where $F \in L^2(Q)$. Then there exists positive constants C, $T_1 > T/3$, μ and γ_0 such that the following estimate holds

$$\|w\|_{H^1(\omega_{T_1}(2\rho))} \leq \frac{C}{\sqrt{\gamma}}\|w\|_{H^2(Q)} + e^{\mu\gamma}\left(\|F\|_{L^2(\omega_T)} + \|\partial_\nu w\|_{L^2(\Sigma_0)}\right) \tag{2.1.20}$$

for any $\gamma > \gamma_0$ where $\Sigma_0 = (0,T) \times \Gamma_0$ and $C = C(\Omega, \omega, T, M)$.

Proof. We do a substitution in T by $2T$ and shift the time variable by T, we are reduced to the case $Q = (-T,T) \times \Omega$, $\Sigma = (-T,T) \times \Gamma$ and

$$\omega_\tau = (-\tau, \tau) \times \omega, \quad \omega_\tau(\rho) = (-\tau,\tau) \times \omega(\rho).$$

and assume that $w \in H^2(Q)$ is a solution to (2.1.9), where $F \in L^2(Q)$ that:

$$\begin{array}{ll} \partial_t^2 w - \Delta w + q_1(x)w = F(t,x) & \text{in } Q \\ w(x,t) = 0 & \text{on } \Sigma \end{array} \tag{2.1.21}$$

We will prove (2.1.20) for the solutions of (2.1.21). We set

$$\omega(\rho_1,\rho_2) = \{x \in \Omega; \rho_1 \leq \operatorname{dist}(x,\Gamma) \leq \rho_2\} \subset \Omega, \quad \rho_1 < \rho_2 < 8\rho,$$

where the the distance 8ρ is defined in (2.1.17) where instead of ρ we have 8ρ and, for $r > 0$,

$$\begin{array}{ll} \Omega_r = (-r,r) \times \Omega, & \omega_r(\rho, 3\rho) = (-r,r) \times \omega(\rho, 3\rho) \\ \Gamma_r = (-r,r) \times \Gamma, & \Sigma_{0,r} = (-r,r) \times \Gamma_0. \end{array} \tag{2.1.22}$$

Let $\theta \in \mathcal{C}_0^\infty(\mathbb{R})$ be a cut-off function defined by

$$\theta(t) = \left\{ \begin{array}{ll} 1, & |t| \leq T-2, \\ 0 & |t| \geq T-1. \end{array} \right. \tag{2.1.23}$$

We introduce, from [65], the partial Fourier-Bros-Iagolnitzer transformation that it is defined for $u \in \mathcal{S}(\mathbb{R}^{n'})$ with $\mathbb{R}^{n'} = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} = \mathbb{R} \times \mathbb{R}^n$, $\mathcal{S}$ is the space of rapidly decreasing functions , by

$$Tu(z_a, x_b, \lambda) = K(\lambda) \int e^{-\frac{\lambda}{2}(z_a - y_a)^2} u(y_a, x_b)\, dy_a = Tu(t+is, x, \lambda) \tag{2.1.24}$$

where $z_a \in \mathbb{C}^{n_a}$, $x_b \in \mathbb{R}^{n_b}$, $\lambda \geq 1$, $K(\lambda) = 2^{-\frac{n_a}{2}}\left(\frac{\lambda}{\pi}\right)^{\frac{n_a}{2}}$ and $z_a^2 = \sum_{j=1}^{n_a} z_{aj}^2$. In our case, it is $n_a = 1$ and $n_b = n$ with $n' = n_a + n_b = 1 + n \geq 1$. Therefore, $z_a \in \mathbb{C}^{n_a} = \mathbb{C}$, $x_b = x \in \mathbb{R}^{n_b} = \mathbb{R}^n$. Assume that $z = t + is \in \mathbb{C}$ that is reffed to our paper, $y_a = y$ and $\lambda = \gamma$. $K(\gamma) = 2^{-\frac{1}{2}}\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} = \sqrt{\frac{\gamma}{2\pi}}$. We define

$$\Phi(z_a) = \frac{1}{2}\left(\operatorname{Im} z_a\right)^2, \quad z_a \in \mathbb{C}^{n_a} \tag{2.1.25}$$

In our case it is $\Phi(z) = \frac{s^2}{2}$. For our case we have an improved version of the estimate, that is

$$|D_x^a Tu(t+is,x,\gamma)| \leq CK(\gamma)\langle x\rangle^{-M}\langle z\rangle^{-N} e^{\gamma\Phi(z) - \frac{\gamma}{2}[\operatorname{dist}(\operatorname{Re} z, \operatorname{supp} u)]^2} \cdot \sup_x \|D_x^a u(\cdot,x)\|_{H^N(\mathbb{R}^{n_a})} \tag{2.1.26}$$

where $\langle\cdot\rangle = (1 + |\cdot|^2)^{1/2}$, $a \in \mathbb{N}^{n_b}$. We apply (2.1.24) to the function $\tilde{u} = \theta \cdot u$ and integrate over all $\mathbb{R}$ over the time variable $t \to y$ and obtain

$$u_{\gamma,t}(s,x) = T\tilde{u}(z,x) = \sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\gamma}{2}(z-y)^2}\theta(y)u(y,x)dy \tag{2.1.27}$$

For our (2.1.26) estimate, we fix $M, N = 0$ then the $H^N(\mathbb{R}^{n_a})$ norm becomes $L^2(\mathbb{R})$. Therefore we have

$$|D_x^a T\tilde{u}(z,x)| \leq C\sqrt{\frac{\gamma}{2\pi}} e^{\lambda s^2/2} e^{-\frac{\gamma}{2}(\operatorname{dist}(t, \operatorname{supp}(\theta u)))^2} \sup_{x \in \mathbb{R}^n} \|D_x^a u(\cdot,x)\|_{L^2(\mathbb{R})}. \tag{2.1.28}$$

We assume that T is sufficiently large, $s \in [-3r, 3r]$, $t \in [-\frac{T}{2}, \frac{T}{2}]$ and choose a cut-off function χ with $0 \leq \chi \leq 1$, $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$, and

$$\chi(x) = \begin{cases} 1 & \text{if } x \in \omega(6\rho) \\ 0 & \text{if } x \in \Omega \setminus \omega(7\rho) \end{cases} \tag{2.1.29}$$

we assume that $w = w(t,x)$ is a solution to (2.1.21) and we set $u(t,x) = \chi(x)w(t,x)$. For $(x,t) \in Q = (-T,T) \times \Omega$ we have

$$\partial_t^2 u - \Delta u + q_1 u = \partial_t^2 \chi w - \Delta \chi w + q_1 \chi w$$

where $\Delta \chi w = [\Delta, \chi]w + \chi \Delta w$. Hence

$$\partial_t^2 u - \Delta u + q_1 u = \chi[\partial_t^2 - \Delta + q_1]w + [\Delta, \chi]w$$

Applying (2.1.21) for $(t,x) \in Q$, we obtain

$$\begin{cases} \partial_t^2 u - \Delta u + q_1 u = [\Delta, \chi]w + \chi(x)F(t,x) & \text{in } Q = (-T,T) \times \Omega \\ u(t,x) = 0 & \text{in } \Sigma = (-T,T) \times \Gamma \end{cases} \tag{2.1.30}$$

Let us define the elliptic operator by

$$P(x, D_{x,s}) = \partial_s^2 + \Delta_x = q_1(x) \tag{2.1.31}$$

from $z = t + is \Rightarrow \frac{\partial z}{\partial s} = i$, therefore

$$\begin{aligned} \partial_s \int_{\mathbb{R}} e^{-\frac{\gamma}{2}(z-y)^2} \theta(y)u(y,x)\, dy &= \int_{\mathbb{R}} e^{-\frac{\gamma}{2}(z-y)^2} (-\frac{\gamma}{2} 2(z-y)i)\theta(y)u(y,x)\, dy \\ &= -i \int_{\mathbb{R}} \left(e^{-\frac{\gamma}{2}(z-y)^2} \right)_y \theta(y)u(y,x)\, dy \\ &= i \int_{\mathbb{R}} e^{-\frac{\gamma}{2}(z-y)^2} \partial_y(\theta(y)u(y,x))\, dy \end{aligned}$$

We have

$$\partial_s^2 u_{\gamma,t} = -\sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\gamma}{2}(z-y)^2} \partial_y^2(\theta(y)u(y,x))\, dy$$

where

$$\partial_y^2(\theta(y)u(y,x)) = \theta''(y)u(y,x) + 2\theta'(y)\partial_t u(y,x) + \theta(y)\partial_y^2 u(y,x)$$

Therefore,

$$\partial_s^2 u_{\gamma,t} = -\sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\gamma}{2}(z-y)^2} (\theta''(y)u(y,x) + 2\theta'(y)\partial_t u(y,x))\, dy - \sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\gamma}{2}(z-y)^2} (\theta(y)\partial_y^2 u(y,x))\, dy$$

and

$$(\Delta - q_1)u_{\gamma,t} = \sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\gamma}{2}(z-y)^2} \theta(y)(\Delta - q_1)u(y,x)\, dy$$

Therefore

$$\begin{aligned} Pu_{\gamma,t} &= -\sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\gamma}{2}(z-y)^2} (\theta''(y)u(y,x) + 2\theta'(y)\partial_t u(y,x))\, dy \\ &\quad - \sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\gamma}{2}(z-y)^2} \theta(y)(\partial_t^2 - \Delta + q_1)u\, dy \\ &= -\sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\gamma}{2}(z-y)^2} (\theta''(y)u(y,x) + 2\theta'(y)\partial_t u(y,x))\, dy \\ &\quad - \sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\gamma}{2}(z-y)^2} \theta(y)([\Delta, \chi]w(y,x) + \chi(x)F(y,x))\, dy \end{aligned}$$

We set as

$$R_{\gamma,t}(s,x) = -\sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\gamma}{2}(z-y)^2} (\theta''(y)u(y,x) + 2\theta'(y)\partial_t u(y,x))\, dy \tag{2.1.32}$$

and

$$G_{\gamma,t}(s,x) = \sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\gamma}{2}(z-y)^2} \theta(y)([\Delta,\chi]w(y,x) + \chi(x)F(y,x))\, dy \tag{2.1.33}$$

and we obtain

$$\begin{aligned} Pu_{\gamma,t}(s,x) = R_{\gamma,t}(s,x) + G_{\gamma,t}(s,x), \quad & (s,x) \in \Omega_{3r} \\ u_{\gamma,t}(s,x) = 0, \quad & (s,x) \in \Sigma_{3r} \end{aligned} \tag{2.1.34}$$

Therefore, there exists $\eta > 0$, independent of T, such that

$$\|R_{\gamma,t}\|_{L^2(\Omega_{3r})} \leq Ce^{-\eta\gamma T}\|u\|_{H^1(Q)}, \quad \forall t \in \left[-\frac{T}{2}, \frac{T}{2}\right]. \tag{2.1.35}$$

Furthermore, there exists $\alpha > 0$, independent of T, such that

$$\|u_{\gamma,t}\|_{H^1(\Omega_{3r})} \leq Ce^{\alpha\gamma}\|u\|_{H^1(Q)} \quad \forall t \in \left[-\frac{T}{2}, \frac{T}{2}\right]. \tag{2.1.36}$$

For $(s,x) \in \omega_{3r}(6\rho)$ we have $\chi(x) = 1$, therefore

$$G_{\gamma,t} = \sqrt{\frac{\gamma}{2\pi}} \int e^{-\frac{\gamma}{2}(z-t)^2} \theta(y)F(y,x)\, dy = F_{\gamma,t}(s,x)$$

$$\|G_{\gamma,t}\|_{L^2(\omega_{3r}(6\rho))} = \|F_{\gamma,t}\|_{L^2(\omega_{3r}(6\rho))}$$

applying (2.1.36) we have

$$\|G_{\gamma,t}\|_{L^2(\omega_{3r}(6\rho))} = \|F_{\gamma,t}\|_{L^2(\omega_{3r}(6\rho))} \leq Ce^{\alpha\gamma}\|F\|_{L^2(\omega_T(7\rho))}. \tag{2.1.37}$$

We consider K a compact subset in $(-3r, 3r) \times \overline{\Omega}$ and $\psi \in \mathcal{C}^1$ with $\nabla_{s,x}\psi(s,x) \neq 0$ on K. We assume a function $\phi = \phi(s,x)$ such that

$$\phi(s,x) = e^{-\beta\psi(s,x)}, \quad \beta > 0 \text{ is sufficiently large} \tag{2.1.38}$$

We refer to [48] for the proof of the following estimate:

$$\begin{aligned} &\exists \tau_0 > 0 : \\ &\qquad C\tau\|e^{\tau\phi}u\|^2_{H^1_\tau(\Omega_{3r})} \leq \|e^{\tau\phi}Pu\|^2_{L^2(\Omega_{3r})} + \tau\|e^{\tau\phi}\|^2_{H^1_\tau} \end{aligned} \tag{2.1.39}$$

where $u \in \mathcal{C}_0^\infty$ and $\tau > \tau_0$.
We set

$$\|u\|^2_{H^1_\tau(\Omega_{3r})} = \|\nabla_{s,x}u\|^2_{L^2(\Omega_{3r})} + \tau^2\|u\|^2_{L^2(\Omega_{3r})} \tag{2.1.40}$$

and

$$\|u\|^2_{H^1_\tau(\Sigma_{3r})} = \|u\|^2_{H^1(\Sigma_{3r})} + \tau^2\|u\|^2_{L^2(\Sigma_{3r})} \tag{2.1.41}$$

Also we introduce the cut-off function $\mathcal{X}$ where $0 \leq \mathcal{X} \leq 1$, $\mathcal{X} \in \mathcal{C}^\infty(\mathbb{R})$, and

$$\mathcal{X}(\rho) = \begin{cases} 0 & \text{if } \rho \leq \frac{1}{4},\ \rho \geq 9 \\ 1 & \text{if } \rho \in [\frac{1}{2}, 8] \end{cases} \tag{2.1.42}$$

We now give some estimations that were proven in [7] without their proof near the boundary Γ_0 and $\omega_r(\rho, 3\rho)$. These estimations where necessary for proving (2.1.20) and to provide us with some estimations near the boundary.

Estimation near Γ_0

Let us estimate $u_{\gamma,t}$ in a ball $B_1 = B(x^{(1)}, r) = \{x \in \mathbb{R}^n; |x - x^{(1)}| < r\}$ over a small interval $(-r, r)$ in the given part $\Sigma_{0,3r} = (-3r, 3r) \times \Gamma_0 \subset \Sigma_{3r}$.

Lemma 4. Let $u_{\gamma,t}$ be a solution to (2.1.34). Then $\exists B_1^* \equiv (-r, r) \times B_1 \subset \Omega_{3r}$ and $\nu_0 \in (0, 1)$ such that

$$\|u_{\gamma,t}\|_{H^1(B_1^*)} \leq C \left(\|R_{\gamma,t}\|_{L^2(\Omega_{3r}} + \|F_{\gamma,t}\|_{L^2(\omega_{3r}(7\rho))} + \|\partial_\nu u_{\lambda,t}\|_{L^2(\Sigma_{0,3r})}\right)^{\nu_0} \left(\|u_{\lambda,t}\|_{H^1(\Omega_{3r})}\right)^{1-\nu_0} \tag{2.1.43}$$

for some $C > 0$.

Estimation in $\omega_r(\rho, 3\rho)$

Extending B_1^* to $\omega_r(\rho, 3\rho)$. We assume that $x^{(j)}$ satisfies $\text{dist}(x^{(j)}, \Gamma) \geq 4r$ and $B(x^{(j+1)}, r) \subset B(x^{(j)}, 2r)$. We set

$$B_j^* = (-r, r) \times B(x^{(j)}, r), \quad 2 \leq j \leq N$$

Lemma 5. Let $u_{\gamma,t}$ be a solution to (2.1.34). Then exists constants $\nu \in (0, 1)$ and $C > 0$ such that

$$\|u_{\gamma,t}\|_{H^1(B_{k+1}^*)} \leq \left(\|R_{\gamma,t}\|_{L^2(\Omega_{3r})} + \|F_{\gamma,t}\|^2_{L^2(\omega_{3r}(7\rho))} + \|u_{\gamma,t}\|_{H^1(B_k^*)}\right)^{\nu} \left(\|u_{\gamma,t}\|_{H^1(\Omega_{3r})}\right)^{1-\nu} \tag{2.1.44}$$

$\forall\ k \geq 1$.

The estimate (2.1.44) is an improved version of (2.1.43). By applying *lemma 4* from [48] we have for $\alpha_k \leq B^{1-\nu}(\alpha_{k-1} + A)^{\nu}$, $\forall \mu = (0, \nu^n]$

$$\alpha_n \leq 2^{\frac{1}{1-\nu}} B^{1-\mu}(\alpha_1 + A)^{\nu}$$

where $a_k = \|u_{\gamma,t}\|_{H^1(B_k^*)}$, $A = \|R_{\gamma,t}\|_{L^2(\Omega_{3r})} + \|\chi F_{\gamma,t}\|_{L^2(\Omega_{3r})}$, $B = \|u_{\lambda,t}\|_{H^1(\Omega_{3r})}$, one can derive our next lemma

Lemma 6. Let $u_{\gamma,t}$ be a solution to (2.1.34). There exists a constant $C > 0$ and $\mu = \nu^N$ such that

$$\|u_{\gamma,t}\|_{H^1(\omega_r(\rho,3\rho))} \leq C \left(\|R_{\gamma,t}\|_{L^2(\Omega_{3r})} + \|\chi F_{\gamma,t}\|_{L^2(\Omega_{3r})} + \|u_{\gamma,t}\|_{H^1(B_1^*)}\right)^{\mu} \left(\|u_{\gamma,t}\|_{H^1(\Omega_{3r})}\right)^{1-\mu} \tag{2.1.45}$$

where $\nu \in (0, 1)$ is the constant given in *Lemma 5.*

We introduce the Carleman estimate for $v \in H^2$

$$\|v\|_{H^1(Y)} \leq C\|v\|^{\nu}_{H^1(X)} \left[\|(\partial_t^2 + \Delta)v\|_{L^2(X)} + \|v\|_{H^1(U)}\right]$$

By applying that to the function $\tilde{u}_{\gamma,t}(s, x) = \chi_0(x)u_{\gamma,t}(s, x)$, where χ_0 is a cut-off function, with $0 \leq \chi_0 \leq 1$, $\chi_0 \in \mathcal{C}^{\infty}(\mathbb{R}^n)$ and

$$\chi_0(x) = \begin{cases} 1 & \text{if } x \in \omega(2\rho) \\ 0 & \text{if } x \in \Omega \setminus \omega(3\rho) \end{cases}$$

we have the following estimate

Lemma 7. Let $u_{\gamma,t}$ be a solution to (2.1.34) and $r_0 = r/2$. Then exist constants $C > 0$ and $\kappa \in (0, 1)$ such that

$$\|u_{\gamma,t}\|_{H^1(\omega_{r_0}(2\rho))} \leq C \left(\|R_{\gamma,t}\|_{L^2(\Omega_r)} + \|F_{\gamma,t}\|_{L^2(\omega_r(7\rho))} + \|u_{\gamma,t}\|_{H^1(\omega_r(2\rho,3\rho))}\right)^{\kappa} \left(\|u_{\gamma,t}\|_{H^1(\Omega_{3r})}\right)^{1-\kappa}. \tag{2.1.46}$$

$\forall t \in (-T/2, T/2)$.

The next lemma can be obtained by applying *Young's inequality* [20, 26]

Proposition 8. Let $\alpha, b > 0$ and $\nu \in [0, 1]$ then

$$\alpha^{1-\nu} b^{\nu} \leq (1-\nu)\alpha + \nu b$$

where equality holds if and only if $a = b$.

We apply Young's inequality to *lemma 6* and *lemma 7* and applying estimates (2.1.35) and (2.1.36) we get a better estimation that contains terms $\sim e^{\gamma}$.

Lemma 9. Let $u_{\gamma,t}$ be a solution to (2.1.34). Then there exists $C > 0$, $\alpha > 0$ and sufficiently large $T > 0$ such that

$$C\|u_{\gamma,t}\|^2_{H^1(\omega_{r_0}(2\rho))} \leq e^{-\alpha\gamma}\|u\|^2_{H^1(Q)} + e^{C\gamma}\left(\|\partial_\nu u\|^2_{L^2(\Sigma_0)} + \|F\|^2_{L^2(\Omega_T(7\rho))}\right) \tag{2.1.47}$$

$\forall t \in [-\frac{T}{2}, \frac{T}{2}]$.

Finally, we can apply *lemma 9* to our solution $u \in H^2(Q)$ of (2.1.30) and we set $u_\gamma(t, x) = u_{\gamma,t}(0, x)$. From (2.1.27) we have

$$u_\gamma(t, x) = \sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}^n} e^{-\frac{\gamma}{2}(t-y)^2}\theta(y)u(y, x)\, dy = (K_\gamma * \theta u)(t, x)$$

where

$$K_\gamma(t) = \sqrt{\frac{\gamma}{2\pi}} e^{-\frac{\gamma}{2}t^2}.$$

We introduce the *Fourier transform* [80, 69]

Definition 10. The Fourier Transform of a function u=u(x) is defined by

$$\widehat{u}(\eta) = \mathcal{F}[u(x)](\eta) = \int_{\mathbb{R}} u(x)e^{-2\pi i\eta x}dx \tag{2.1.48}$$

We denote by $\widehat{u}(\eta, x)$ the Fourier transform of $u(t, x)$ with respect to t. By the *convolution theorem* [71, 85], we have

$$\widehat{\theta u}(\eta, x) - \widehat{u_\gamma}(\eta, x) = \widehat{\theta u}(\eta, x) - \widehat{K_\gamma}\widehat{\theta u} = (1 - \widehat{K_\gamma})\widehat{\theta u}(\eta, x)$$

where

$$1 - \widehat{K_\gamma}(\eta) = 1 - \sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\gamma}{2}t^2} e^{-2\pi i\eta t}\, dt$$

To find $\widehat{K_\gamma}(\eta)$ we use *Gauss Error Function* [19, 39] (see Appendix section 4.3), hence

$$\widehat{K_\gamma}(\eta) = e^{-\frac{2\pi^2\eta^2}{\gamma}}$$

From the inequality $e^x \geq x + 1$, $\forall x \in \mathbb{R}$ we set $x = -2\pi^2\eta^2/\gamma$ we obtain

$$\frac{1 - e^{-2\pi^2\eta^2/\gamma}}{2} \leq \pi^2\eta^2/\gamma$$

and apply $|a - b| \leq |a| + |b|$ we have

$$|1 - e^{-2\pi^2\eta^2/\gamma}| \leq |e^0| + |e^{-2\pi^2\eta^2/\gamma}| = 1 - e^{-2\pi^2\eta^2/\gamma}$$

Therefore

$$|1 - e^{-2\pi^2\eta^2/\gamma}| \leq 1 + \pi^2\eta^2/\gamma \leq \eta^2/\gamma$$

since $\pi^2/\gamma \leq 1$. Therefore, we have shown that

$$|1 - \widehat{K_\gamma}(\eta)| \leq \frac{\eta^2}{\gamma}$$

else

$$\left|1 - \frac{\widehat{u_\gamma}(\eta, x)}{\widehat{\theta u}(\eta, x)}\right| \leq \frac{\eta^2}{\gamma} \Rightarrow \left|\widehat{\theta u}(\eta, x) - \widehat{u_\gamma}(\eta, x)\right| \leq \frac{\eta^2}{\gamma}|\widehat{\theta u}(\eta, x)|$$

We set $T_1 = T/2 - r_0$ which is for $\theta = 1$, therefore

$$\|u - u_\gamma\|_{L^2(\omega_{T_1}(2\rho))} \leq \frac{C}{\sqrt{\gamma}}\|u\|_{H^1(Q)}$$

and similarly

$$\|u - u_\gamma\|_{H^1(\omega_{T_1}(2\rho))} \leq \frac{C}{\sqrt{\gamma}}\|u\|_{H^2(Q)}$$

Therefore,

$$\begin{aligned}\|u\|_{H^1(\omega_{T_1}(2\rho))} &\leq C\left[\|u - u_\gamma\|_{H^1(\omega_{T_1}(2\rho))} + \|u_\gamma\|_{H^1(\omega_{T_1}(2\rho))}\right] \\ &\leq C\left[\frac{1}{\sqrt{\gamma}}\|u\|_{H^2(Q)} + \|u_\gamma\|_{H^1(\omega_{T_1}(2\rho))}\right]\end{aligned}$$

By applying Cauchy's formula, as it has been done in [6], we obtain

$$\|u_\gamma\|_{H^1(\omega_{T_1}(2\rho))} \leq e^{\mu\gamma}\|u\|^2_{H^1(Q)} + e^{\mu'\gamma}(\|u\|_{L^2(\Sigma_0)} + \|F\|_{L^2(\omega_T(7\rho))})$$

which concludes the proof to lemma 3. □

2.2 Dirichlet to Neumann map (DN map)

2.2.1 Initial Boundary Value Problem (IBVP) for the Wave Equation

In this section our aim is to prove that the IBVP (2.1.2) has a unique regular solution in order to insure that the operator

$$\begin{aligned}\Lambda_q : H^{1,1}(\Sigma) &\to L^2(\Sigma_0) \\ f &\mapsto \Lambda_q(f) = \partial_\nu u_q\end{aligned} \tag{2.2.1}$$

where $H^{1,1}(\Sigma) = L^2(0, T; H^1(\Gamma)) \cap H^1(0, T; L^2(\Gamma))$, is bounded.
This operator is called **Dirichlet to Neumann map** (DN map) [72] and has many applications. In Acoustics is used to model the behavior of sound waves in a variety of media, including air, water, and solids [27, 24]. In Electromagnetism is used to study the propagation of electromagnetic waves in various materials, such as metals, dielectrics, and plasmas [49, 50, 22]. In Medical imaging is used to reconstruct images of the interior of the body from measurements made on its surface, such as in magnetic resonance imaging (MRI) and computed tomography (CT) scans [3, 84].

To prove that (2.2.1) is bounded we consider the following IBVP

$$\begin{array}{lll}\partial_t^2 u - \Delta u = F & & \text{in } Q \\ u(0, \cdot) = u_0, & \partial_t u(0, \cdot) = u_1 & \text{in } \Omega \\ u = f & & \text{on } \Sigma.\end{array} \tag{2.2.2}$$

IBVP (2.2.2) is just an IBVP of the wave equation having $q(x) = 0$, $f \in H^{1,1}(\Sigma)$, $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega)$. We recall the theorem that it is stated in [47].

Theorem 11. Let $F \in L^1(0,T;L^2(\Omega))$, $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $f \in H^{1,1}(\Sigma)$. We assume the condition $g(0,\cdot) = u_0|_\Gamma$. The IBVP (2.2.2) has a unique solution $u \in \mathcal{C}([0,T];H^1(\Omega))$ such that $\partial_t u \in \mathcal{C}([0,T];L^2(\Omega))$ and $\partial_\nu u \in L^2(\Sigma)$. Furthermore, there exists a constant $C = C(T)$, such that

$$\|u\|_{\mathcal{C}([0,T];H^1(\Omega))}+\|\partial_t u\|_{\mathcal{C}([0,T];L^2(\Omega))}+\|\partial_\nu u\|_{L^2(\Sigma)} \leq C\left(\|F\|_{L^1(0,T;L^2(\Omega))} + \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{H^{1,1}(\Sigma)}\right). \tag{2.2.3}$$

Next, we introduce the following IBVP

$$\begin{array}{rcl} \partial_t^2 u - \Delta u + q(x)u = F & & \text{in } Q \\ u(0,\cdot) = 0, & \partial_t u(0,\cdot) = 0 & \text{in } \Omega \\ u = 0 & & \text{on } \Sigma. \end{array} \tag{2.2.4}$$

and have a similar result with the previous IBVP but this time for $q \in L^\infty(\Omega)$ and $u_0 = u_1 = 0 = f$.

Theorem 12. Let $F \in L^2(Q)$ and $q \in L^\infty(\Omega)$. Then the IBVP (2.2.4) has a unique solution $u \in \mathcal{C}([0,T];H^1(\Omega))$ such that $\partial_t u \in \mathcal{C}([0,T];L^2(\Omega))$ and $\partial_\nu u \in L^2(\Sigma)$. Furthermore, if $\|q\|_{L^\infty(\Omega)} \leq M$, M>0, there exists a constant $C = C(T,M)$, such that

$$\|u\|_{\mathcal{C}([0,T];H^1(\Omega))} + \|\partial_t u\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|\partial_\nu u\|_{L^2(\Sigma)} \leq C\|F\|_{L^1(0,T;L^2(\Omega))}. \tag{2.2.5}$$

A combination of the previous theorems will let us deal with the following IBVP and get another version of *theorem 11*.

Consider a more general IBVP

$$\begin{array}{rcl} \partial_t^2 u - \Delta u + q(x)u = F & & \text{in } Q \\ u(0,\cdot) = u_0, & \partial_t u(0,\cdot) = u_1 & \text{in } \Omega \\ u = f & & \text{on } \Sigma. \end{array} \tag{2.2.6}$$

where $q \in L^\infty(\Omega)$, $f \in H^{1,1}(\Sigma)$, $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega)$.

Theorem 13. Let $q \in L^\infty(\Omega)$ and $\|q\|_{L^\infty(\Omega)} \leq M$, M>0. if $F \in L^1(0,T;L^2(\Omega))$, $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$, $f \in H^{1,1}(\Sigma)$ and $g(0,\cdot) = u_0|_\Gamma$ then the IBVP (2.2.6) has a unique solution $u \in \mathcal{C}([0,T];H^1(\Omega))$ such that $\partial_t u \in \mathcal{C}([0,T];L^2(\Omega))$ and $\partial_\nu u \in L^2(\Sigma)$. Furthermore, there exists a constant $C = C(T,M)$ such that

$$\|u\|_{\mathcal{C}([0,T];H^1(\Omega))}+\|\partial_t u\|_{\mathcal{C}([0,T];L^2(\Omega))}+\|\partial_\nu u\|_{L^2(\Sigma)} \leq C\left(\|F\|_{L^1(0,T;L^2(\Omega))} + \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{H^{1,1}(\Sigma)}\right). \tag{2.2.7}$$

Proof. Assume that v is a solution to (2.2.2) then the following estimate holds

$$\|v\|_{\mathcal{C}([0,T];H^1(\Omega))}+\|\partial_t v\|_{\mathcal{C}([0,T];L^2(\Omega))}+\|\partial_\nu v\|_{L^2(\Sigma)} \leq C\left(\|F\|_{L^1(0,T;L^2(\Omega))} + \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{H^{1,1}(\Sigma)}\right)$$

where $C = C(T)$. Consider the IBVP

$$\begin{array}{rcl} \partial_t^2 w - \Delta w + q(x)w = F = -q(x)v & & \text{in } Q \\ w(0,x) = 0, & \partial_t w(0,x) = 0 & \text{in } \Omega \\ w = 0 & & \text{on } \Sigma. \end{array}$$

where from theorem 12 the IBVP has a unique solution $w \in \mathcal{C}([0,T];H^1(\Omega))$ where the following estimate holds

$$\|w\|_{\mathcal{C}([0,T];H^1(\Omega))} + \|\partial_t w\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|\partial_\nu w\|_{L^2(\Sigma)} \leq C\|F\|_{L^1(0,T;L^2(\Omega))} = C\|qv\|_{L^1(0,T;L^2(\Omega))}$$

We recall that $\|q\|_{L^\infty(\Omega)} \leq M$ hence, by applying Hölder's inequality we obtain

$$C\|qv\|_{L^1(0,T;L^2(\Omega))} \leq C\|q\|_{L^\infty(0,T;L^2(\Omega))}\|v\|_{L^1(0,T;L^2(\Omega))} = C\|q\|_{L^\infty(\Omega)}\left(\int_0^T dt\right)\|v\|_{L^1(0,T;L^2(\Omega))}$$

therefore

$$C\|qv\|_{L^1(0,T;L^2(\Omega))} \leq C \cdot M \cdot T \cdot \|v\|_{L^1(0,T;L^2(\Omega))}$$

hence we obtain the estimate

$$\begin{aligned}&\|w\|_{\mathcal{C}([0,T];H^1(\Omega))} + \|\partial_t w\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|\partial_\nu w\|_{L^2(\Sigma)} \leq C \cdot M \cdot T \cdot \|v\|_{L^1(0,T;L^2(\Omega))}\\ &\leq C \cdot M \cdot T \cdot \left(\|v\|_{\mathcal{C}([0,T];H^1(\Omega))} + \|\partial_t v\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|\partial_\nu v\|_{L^2(\Sigma)}\right)\end{aligned}$$

Hence the solution $u = v + w \in \mathcal{C}([0,T];H^1(\Omega))$ with $\partial_t u \in \mathcal{C}([0,T];L^2(\Omega))$, $\partial_\nu u \in L^2(\Sigma)$ is a unique solution to the IBVP (2.2.6). For the estimate of the solution $u = w + v$, it is true that

$$\begin{aligned}\|u\|_{\mathcal{C}([0,T];H^1(\Omega))} + \|\partial_t u\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|\partial_\nu u\|_{L^2(\Sigma)} &\leq \|v\|_{\mathcal{C}([0,T];H^1(\Omega))} + \|\partial_t v\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|\partial_\nu v\|_{L^2(\Sigma)}\\ &+ \|w\|_{\mathcal{C}([0,T];H^1(\Omega))} + \|\partial_t w\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|\partial_\nu w\|_{L^2(\Sigma)}\end{aligned}$$

For simplicity, we set as

$$\begin{aligned}&E_u(T) = \|u\|_{\mathcal{C}([0,T];H^1(\Omega))} + \|\partial_t u\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|\partial_\nu u\|_{L^2(\Sigma)},\\ &E_V(T) = \|v\|_{\mathcal{C}([0,T];H^1(\Omega))} + \|\partial_t v\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|\partial_\nu v\|_{L^2(\Sigma)},\\ &E_w(T) = \|w\|_{\mathcal{C}([0,T];H^1(\Omega))} + \|\partial_t w\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|\partial_\nu w\|_{L^2(\Sigma)} \text{ and}\\ &\|(F, u_0, u_1, f)\| = \|F\|_{L^1(0,T;L^2(\Omega))} + \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{H^{1,1}(\Sigma)}\end{aligned}$$

We combine the following estimates that we obtained

$$\begin{aligned}&E_u \leq E_v + E_w,\\ &E_w \leq C \cdot M \cdot T \cdot E_v \text{ and}\\ &E_v \leq C\|(F, u_0, u_1, f)\|\end{aligned}$$

We obtain

$$E_u \leq (1 + CMT)E_v \leq \tilde{C}\|(F, u_0, u_1, f)\|$$

where the constant $\tilde{C}$ is the same as $C = C(T, M)$, thus we conclude our proof. □

2.2.2 Partial DN map

Let us consider Γ_0 an arbitrary non-empty relatively open subset of Γ and set $\Sigma_0 = (0,T) \times \Gamma_0$, the partial DN map is defined as

$$\begin{aligned}&\Lambda_q^\sharp : H^{1,1}(\Sigma) \to L^2(\Sigma_0)\\ &f \mapsto \Lambda_q^\sharp(f) = \partial_\nu u_q|_{\Sigma_0}.\end{aligned} \tag{2.2.8}$$

where $\Lambda_q^\sharp$ is bounded because Λ_q is also bounded.

The results in the following section provide a log-type stability for the inverse problem determining the potential q from the partial DN map $\Lambda_q^\sharp$.

2.3 Theorems that provide a log-type stability

In this section we provide three main results from [7], which gives us log-type estimates for the inverse problem.

Theorem 14. There exists $C > 0$, $\delta \in (0,1)$ and sufficiently large T such that

$$\|q_1 - q_2\|_{H^{-1/2}(\Omega)} \leq C \left(\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|^\delta + |\log\left(\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|\right)|^{-\delta} \right) \tag{2.3.1}$$

for any $q_1, q_2 \in \mathfrak{X}(M, \omega)$.

If $q_1, q_2 \in H^s(\Omega)$ for $s > d/2$ and $\|q_j\|_{H^s(\Omega)} \leq M$, $j = 1, 2$, then there exists $\delta' \in (0,1)$ such that

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leq C \left(\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|^{\delta'} + |\log\left(\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|\right)|^{-\delta'} \right). \tag{2.3.2}$$

The consequence of *theorem 14* provide us with the following uniqueness result

Corollary 15. Let $d \geq 2$ and $g_j \in \mathcal{C}^{0,\mu}(\overline{\Omega})$, $j = 1, 2$. Assume $q_1 = q_2$ in a neighbourhood of Γ. Then there exists sufficiently large T such that $\Lambda^\sharp_{q_1} = \Lambda^\sharp_{q_2}$ implies $q_1 = q_2$ in Ω.

Let us restrict the operator $\Lambda^\sharp_q$ to the subspace

$$\mathcal{H}_1 = \{h \in H^{2d+4}(0, T; H^{3/2}(\Gamma));\ \partial_t^j h(0, \cdot) = 0,\ 0 \leq j \leq 2d + 3\}.$$

By $\tilde{\Lambda}^\sharp_q$ we denote the restriction of $\Lambda^\sharp_q$, which defines a bounded operator from $\mathcal{H}_1$ into $\mathcal{H}_2 = L^2(0, T; H^s(\Gamma_0))$, $\forall s \in [0, 1/2]$. We denote as $\|\cdot\|_s$ the norm in $\mathcal{B}(\mathcal{H}_1; \mathcal{H}_2)$.

Theorem 16. There exists $C > 0$, $\delta \in (0,1)$ and sufficiently large T such that

$$\|q_1 - q_2\|_{H^{-1/2}(\Omega)} \leq C \left(\|\tilde{\Lambda}^\sharp_{q_1} - \tilde{\Lambda}^\sharp_{q_2}\|_s^\delta + |\log\left(\|\tilde{\Lambda}^\sharp_{q_1} - \tilde{\Lambda}^\sharp_{q_2}\|_s\right)|^{-\delta} \right) \tag{2.3.3}$$

$\forall q_1, q_2 \in \mathcal{X}(M, \omega)$. If $q_1, q_2 \in H^\alpha(\Omega)$, for $\alpha > d/2$ and $\|q_j\|_{H^\alpha(\Omega)} \leq M$, $j = 1, 2$, then there exists $\delta' \in (0,1)$ such that

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leq C \left(\|\tilde{\Lambda}^\sharp_{q_1} - \tilde{\Lambda}^\sharp_{q_2}\|_s^{\delta'} + |\log\left(\|\tilde{\Lambda}^\sharp_{q_1} - \tilde{\Lambda}^\sharp_{q_2}\|_s\right)|^{-\delta'} \right) \tag{2.3.4}$$

Let $D(A_q) = H^1_0(\Omega) \cap H^2(\Omega)$ be the domain of the operator $A_q = -\Delta + q$, $q \in L^\infty(\Omega)$. Let the eigenvalues of A_q satisfy

$$0 \leq \lambda_{1,q} \leq \lambda_{2,q} \leq \ldots \leq \lambda_{k,q} \to +\infty$$

and denote the corresponding sequence of eigenfunctions by $(\phi_{k,q})$ and assume this sequence forms an orthonormal basis of $L^2(\Omega)$, that is $\|\phi_{k,q}\|_{L^2(\Omega)} = 1$. Eigenfunction $\phi_{k,q}$ is the solution to the following BVP

$$\begin{aligned} (-\Delta + q)\phi = \lambda_{k,q}\phi \quad & \text{in } \Omega, \\ \phi = 0 \quad & \text{on } \Gamma. \end{aligned}$$

it follows from theory of Elliptic PDEs [35]

$$\|\phi_{k,q}\|_{H^2(\Omega)} \leq C\lambda_{k,q}\|\phi_{k,q}\|_{L^2(\Omega)} = C\lambda_{k,q} \tag{2.3.5}$$

and

$$\|\partial_\nu \phi_{k,q}\|_{H^{1/2}(\Gamma)} \leq C\lambda_{k,q}$$

From Weyl's law [33, 28], there exists a positive constant $K \geq 1$ such that

$$K^{-1}k^{2/d} \leq \lambda_{k,q} \leq Kk^{2/d}, \quad \forall q \text{ where } 0 \leq q(x) \leq M \text{ and } x \in \Omega. \tag{2.3.6}$$

As a consequence of the trace theorem [18, 10], we have

$$\|\partial_\nu \phi_{k,q}\|_{H^{1/2}(\Gamma_0)} \leq C\|\partial_\nu \phi_{k,q}\|_{H^{1/2}(\Gamma)} \leq Ck^{2/d}.$$

We fix ζ such that $d/2+1 < \zeta \leq d+1$ and denote the sequence $a_k = k^{-2\zeta/d}\|\partial_\nu \phi_{k,q}\|_{H^{1/2}(\Gamma_0)}$ and we consider the the series:

$$\sum_{k\in\mathbb{N}} |a_k| = \sum_{k\in\mathbb{N}} k^{-2\zeta/d}\|\partial_\nu \phi_{k,q}\|_{H^{1/2}(\Gamma_0)} \leq C\sum_{k=1}^{\infty} k^{2/d}\cdot k^{-2\zeta/d}$$
$$= C\sum_{k=1}^{\infty} k^{2(1-\zeta)/d}$$

For the sum to converge, it must satisfy $\frac{d}{2(1-\zeta)} > 1 \Rightarrow \zeta > 1+\frac{d}{2} > 1-\frac{d}{2}$. Thus, the sum converges and by that we have proved that

$$\left(k^{-2\zeta/d}\|\partial_\nu \phi_{k,q}\|_{H^{1/2}(\Gamma_0)}\right) \in \ell^1.$$

Let $\boldsymbol{r} = (r_k)$ be the sequence $r_k = k^{-\zeta/d},\ \forall k \geq 1$. We consider the Banach space

$$\ell^1\left(H^{1/2}(\Gamma_0);\boldsymbol{r}\right) = \{g = (g_k);\ g_k \in H^{1/2}(\Gamma_0),\ k \geq 1,\ \left(r_k\|g_k\|_{H^{1/2}(\Gamma_0)}\right) \in \ell^1\}$$

with the natural norm

$$\|g\|_{\ell^1(H^{1/2}(\Gamma_0;\boldsymbol{r}))} = \sum_{k\in\mathbb{N}} r_k\|g_k\|_{H^{1/2}(\Gamma_0)}.$$

Consider $\mu = (\mu_k) = (\lambda_{k,0})$ the sequence of eigenvalues of A_0, where $A_0 = -\Delta$, therefore, it is true that

$$\begin{aligned} -\Delta\phi = \mu_k\phi \quad & \text{in } \Omega \\ \phi = 0 \quad & \text{on } \Gamma \end{aligned}$$

where one can obtain

$$|\lambda_{k,q} - \mu_k| = |q| \leq \|q\|_{L^\infty(\Omega)},\ k \in \mathbb{N}.$$

From this result we have that $\lambda_q = (\lambda_{k,q}) \in \tilde{\ell}^\infty = \mu + \ell^\infty$ equipped with the distance

$$d_\infty(\lambda_1 - \lambda_2) = \|(\lambda_1 - \mu) - (\lambda_2 - \mu)\|_{\ell^\infty} = \|\lambda_1 - \lambda_2\|_{\ell^\infty},\quad \text{for } \lambda_j \in \tilde{\ell}^\infty,\ j \in \{1,2\}.$$

By applying *theorem 16* we prove:

Theorem 17. $\exists C > 0,\ \mu_0 \in (0,1):$

$$\|q_1 - q_2\|_{H^{-1/2}(\Omega)} \leq C\left(|\log(|\log\eta|)|\right)^{-\mu_0} \tag{2.3.7}$$

$\forall q_1, q_2 \in \mathfrak{X}(M,\omega)$, where $\eta = d_\infty(\lambda_{q_1},\lambda_{q_2}) + \|\partial_\nu\phi_{q_1} - \partial_\nu\phi_{q_2}\|_{\ell^1(H^{1/2}(\Gamma_0);\boldsymbol{r})}$ is small and $\partial_\nu\phi_{q_j} = (\partial_\nu\phi_{k,q_j}),\ j \in \{1,2\}$

If $q_1, q_2 \in H^s(\Omega),\ s > d/2$ and $\|q_j\|_{H^s(\Omega)} \leq M,\ j \in \{1,2\}, \exists\mu_0' \in (0,1):$

$$\|q_1 - q_2\|_{L^\infty} \leq C\left(|\log(|\log\eta|)|\right)^{-\mu_0'} \tag{2.3.8}$$

In the continuation of the chapter, we provide the proof of these estimates in *theorems 15, 16 and 17.* Last, we demonstrate an example with specific potential q and solution u and we are going to show that the estimates on *theorem 17* is satisfied.

2.3.1 Geometric optics solutions and X-ray transform

Geometric optics solutions

In partial differential equations (PDE's), the term "geometric optics solutions" refers to a particular type of solution that arises in the study of high-frequency waves. Geometric optics solutions are an asymptotic approximation of the exact solutions to certain PDE's, particularly those that describe wave propagation phenomena.

In optical phenomena, not only is the wavelength short but the wave trains are long. The study of structures which have short wavelength and are in addition very short, say a short pulse, also yields a geometric theory. Long wavetrains have a longer time to allow nonlinear interactions which makes nonlinear effects more important. Long propagation distances also increase the importance of nonlinear effects. An extreme example is the propagation of light across the ocean in optical fibers. The nonlinear effects are very weak, but over 5000 kilometers, the cumulative effects can be large. To control signal degradation in such fibers the signal is treated about every 30 kilometers. Still, there is free propagation for 30 kilometers which needs to be understood. This poses serious analytic, computational, and engineering challenges. [63]

X-ray transform

In 1963, A.M. Cormack introduced a powerful diagnostic tool in radiology, computerized tomography, which is based on the mathematical properties of the X-ray transform in the Euclidean plane [14]. For a compactly supported continuous function f, its X-ray transform Xf is a function defined on the family of all straight lines l in $\mathbb{R}^2$ as follows: let the unit vector θ represent the direction of l and let p be its signed distance to the origin, so that l is represented by the pair (θ, p), then

$$Xf(l) = Xf(\theta, p) = \int_{\mathbb{R}} f(x + t\theta)\ dt$$

The X-ray transform is a mathematical operation that is used in medical imaging, computed tomography (CT), and other applications to reconstruct an image or object from its X-ray measurements. It is a fundamental concept in the field of X-ray tomography.

The X-ray transform is based on the principle that X-rays attenuate as they pass through different materials. When an X-ray beam passes through an object, the intensity of the X-rays is reduced based on the density and composition of the materials it encounters. The X-ray transform mathematically models this attenuation process.

Stability estimate of recovering a function from its X-ray transform

For starters, we consider the following assumptions:

- $0 \in \Omega$
- $T/3 > \mathrm{Diam}(\Omega)$
- let $\epsilon > 0,\ T_1 > 0 :\ T_1 > \dfrac{T}{3}$ and $T_1 - 2\epsilon > \mathrm{Diam}(\Omega)$
- $\Omega_\epsilon = \{x \in \mathbb{R}^d \setminus \overline{\Omega};\ \mathrm{dist}(x, \Omega) < \epsilon\}$
- For $\Phi \in \mathcal{C}_0^\infty(\Omega_\epsilon)$ and $\theta \in \mathbb{S}^{d-1}$ we associate $\tilde{\Phi}_\theta$ with

$\tilde{\Phi}_\theta = \Phi(x + t\theta),\ x \in \mathbb{R}^d,\ t \in \mathbb{R},\ \Phi$ is extended by 0 outside Ω_ϵ.

We will prove the following lemma:

Lemma 18. Let $q_1, q_2 \in \mathfrak{X}(M,\omega)$, $q = q_1 - q_2$ extended by 0 outside Ω. $\exists T_1 > 0,\ A > 0,\ C > 0:$ $\forall \theta \in \mathbb{S}^{d-1},\ \Phi \in \mathcal{C}_0^\infty(\Omega_\epsilon)$:

$$\left| \int_0^{T_1} \int_\Omega \Phi^2(x) q(x - s\theta)\ dxds \right| \leq C \left(\frac{1}{\gamma^{1/4}} + e^{A\gamma} \|\Lambda_{q_1}^\sharp - \Lambda_{q_2}^\sharp\| \right) \|\Phi\|_{H^3(\mathbb{R}^d)}^2 \tag{2.3.9}$$

for sufficiently large $\gamma > 0$.

Proof. We apply *lemma 2* and it follows that the initial value problem

$$\begin{array}{cc} (\partial_t^2 - \Delta + q_2(x))u = 0, & \text{in } Q = (0,T_1) \times \Omega \\ u(0,\cdot) = \partial_t u(0,\cdot) = 0 & \text{in } \Omega \end{array}$$

has a solution u_2 of the form

$$u_2(t,x) = \Phi(x + t\theta) e^{i\sigma(x\theta + t)} + \Psi_{q_2}(t,x;\sigma), \tag{2.3.10}$$

where

$$\begin{array}{cc} \Psi_{q_2}(0,x;\sigma) = \partial_t \Psi_{q_2}(0,x;\sigma) = 0, & \text{while } x \in \Omega \\ \Psi_{q_2}(t,x;\sigma) = 0, & \text{on } \Sigma_1 = (0,T_1) \times \Gamma \end{array} \tag{2.3.11}$$

and

$$\sigma \|\Psi_{q_2}(\cdot,\cdot;\sigma)\|_{L^2(Q_1)} + \|\nabla \Psi_{q_2}(\cdot,\cdot;\sigma)\|_{L^2(Q_1)} \leq C \|\Phi\|_{H^3(\mathbb{R}^d)}. \tag{2.3.12}$$

Assume $f_\sigma = u_{2|_{\Sigma_1}} = \Phi(x + t\theta) e^{i\sigma(x\theta + t)}$ and u_1 be the solution of the IBVP:

$$\left\{ \begin{array}{cc} (\partial_t^2 - \Delta + q_1)u_1 = 0 & \text{in } Q_1, \\ u_1(0,x) = \partial_t u_1(0,x) = 0 & \text{in } \Omega, \\ u_1 = u_2 = f_\sigma & \text{on } \Sigma_1. \end{array} \right.$$

By subtracting by parts the two differential equations with q_1 and q_2 and assume that $w = u_1 - u_2$ and $q(x) = q_2 - q_1$, one can get

$$\left\{ \begin{array}{cc} (\partial_t^2 - \Delta + q_1)w = qu_2 & \text{in } Q_1, \\ w(0,x) = \partial_t w(0,x) = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \Sigma_1. \end{array} \right.$$

We insert the following cut-off function $\chi \in \mathcal{C}^\infty(\mathbb{R}^d)$, $0 \leq \chi \leq 1$ and

$$\chi(x) = \left\{ \begin{array}{cc} 0 & \text{in } \omega(\rho), \\ 1 & \text{in } \Omega \setminus \omega(2\rho), \end{array} \right.$$

By multiplying the previous differential equation with the cut-off function and by considering that $[\Delta, \chi]w = \Delta w_0 - \chi \Delta w$, where $w_0 = \chi w$, we get

$$\left\{ \begin{array}{cc} (\partial_t^2 - \Delta + q_1)w_0 = qu_2 - [\Delta, \chi]w & \text{in } Q_1, \\ w_0(0,x) = \partial_t w_0(0,x) = 0 & \text{in } \Omega, \\ w_0 = 0 & \text{on } \Sigma_1. \end{array} \right.$$

Notice that we have written $\chi q u_2 = q u_2$, that is because $q = 0$ in $\omega \supset \omega(2\rho)$, therefore the term is not zero for $x \in \Omega \setminus \omega \subset \Omega \setminus \omega(2\rho)$ which gives us $\chi = 1$.

We apply again *lemma 2* for the wave equation $(\partial_t^2 - \Delta + q_1(x))v = 0$ in Q_1 and have solution of the form

$$v(t,x) = \Phi(x + t\theta) e^{-i\sigma(x\theta + t)} + \Psi_{q_1}(t,x;\sigma) \tag{2.3.13}$$

where we now write the solution with the minus sign on the exponential which still works since the roots of the solution are complex and therefore they satisfy the conjugation property. Ψ_{q_1} satisfies the followings

$$\Psi_{q_1}(t,x;\sigma)=0 \qquad (t,x)\in\Sigma_1,$$
$$\Psi(T_1,x;\sigma)=\partial_t\Psi_{q_1}(T_1,x;\sigma)=0 \qquad x\in\Omega,$$

and

$$\sigma\|\Psi_{q_1}(\cdot,\cdot;\sigma)\|_{L^2(Q_1)}+\|\nabla\Psi_{q_1}(\cdot,\cdot;\sigma)\|_{L^2(Q_1)}\leq C\|\Phi\|_{H^3(\mathbb{R}^d)}. \tag{2.3.14}$$

Due to $T_1>\text{diam}(\Omega)+2\epsilon$ and $\Phi\in C_0^\infty(\Omega_\epsilon)$ we have $\Phi(x+T_1\theta)=|\nabla\Phi(x+T_1\theta)|=0$ in Ω. Therefore

$$v(T_1,\cdot)=\partial_t v(T_1,\cdot)=0 \quad \text{in } \Omega.$$

We multiply the differential equation of w_0 with v and apply Green's formula

$$\begin{aligned}\int_{Q_1} q(x)u_2(t,x)v\ dxdt-\int_{Q_1}[\Delta,\chi]wv\ dxdt&=\int_{Q_1}((\partial_t^2-\Delta+q_1(x))w_0)v\ dxdt\\&=\int_{Q_1} w_0(\partial_t^2-\Delta+q_1(x))v\ dxdt=0\end{aligned} \tag{2.3.15}$$

From (2.3.10), (2.3.13) and (2.3.15) we have

$$\int_{Q_1} q(x)\Phi^2(x+t\theta)\ dxdt+\int_{Q_1} q(x)\Phi(x+t\theta)\left(\Psi_{q_1}e^{i\sigma(x\theta+t)}+\Psi_{q_2}e^{-i\sigma(x\theta+t)}\right)\ dxdt$$
$$\int_{Q_1} q(x)\Psi_{q_1}\Psi_{q_2}\ dxdt=\int_{Q_1}[\Delta,\chi]w(t,x)v(t,x)\ dxdt \tag{2.3.16}$$

From (2.3.12) and (2.3.14) we have

$$\left|\int_{Q_1} q(x)\Phi(x+t\theta)\left(\Psi_{q_1}e^{i\sigma(x\theta+t)+\Psi_{q_2}e^{-i\sigma(x\theta+t)}}\right)\ dxdt\right|\leq\frac{C}{|\sigma|}\|\Phi\|^2_{H^3(\mathbb{R}^d)}$$

and

$$\left|\int_{Q_1} q(x)\Psi_{q_1}\Psi_{q_2}\ dxdt\right|\leq\frac{C}{\sigma^2}\|\Phi\|^2_{H^3(\mathbb{R}^d)}$$

Also

$$\begin{aligned}\left|\int_{Q_1}[\Delta,\chi]w(t,x)v(t,x)\ dxdt\right|&\leq C\|w\|_{H^1(\omega_{T_1}(2\rho))}\|v\|_{L^2(Q_1)}\\&\leq C\|\Phi\|_{H^3(\mathbb{R}^d)}\|w\|_{H^1(\omega_{T_1}(2\rho))}\end{aligned}$$

By (2.3.16), we have

$$\left|\int_{Q_1} q(x)\Phi^2(x+t\theta)\ dxdt\right|\leq\frac{C}{\sigma}\|\Phi\|^2_{H^3(\mathbb{R}^d)}+C\|w\|_{H^1(\omega_{T_1}(2\rho))}\|\Phi\|_{H^3(\mathbb{R}^d)}$$

and apply *lemma 3* and we have

$$\left|\int_{Q_1} q(x)\Phi^2(x+t\theta)\ dxdt\right|\leq\frac{C}{\sigma}\|\Phi\|^2_{H^3(\mathbb{R}^d)}+C\left(\frac{\|w\|_{H^2(Q)}}{\sqrt{\gamma}}+e^{A\gamma}\|\partial_\nu w\|_{L^2(\Sigma_0)}\right)\|\Phi\|_{H^3(\mathbb{R}^d)}, \tag{2.3.17}$$

where $F=q(x)u_2=0$ in ω. By (2.3.12) in w and f_σ, we have the energy estimate

$$\|w\|_{H^2(Q)}\leq C\sigma\|\Phi\|_{H^3(\mathbb{R}^d)} \tag{2.3.18}$$

and

$$\|\partial_\nu w\|_{L^2(\Sigma_0)} = \|\Lambda^\sharp_{q_1}(f_\sigma) - \Lambda^\sharp_{q_2}(f_\sigma)\|_{L^2(\Sigma_0)} \le C\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|\|f_\sigma\|_{H^{1,1}(\Sigma)}$$
$$C\sigma^2\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|\|\Phi\|_{H^3(\mathbb{R}^d)}. \tag{2.3.19}$$

We combine (2.3.17), (2.3.18) and (2.3.19) and get

$$\left|\int_{Q_1} q(x)\Phi^2(x+t\theta)\ dxdt\right| \le C\left(\frac{1}{\sigma} + \frac{\sigma}{\sqrt{\gamma}} + \sigma^2 e^{A\gamma}\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|\right)\|\Phi\|^2_{H^3(\mathbb{R}^d)}. \tag{2.3.20}$$

by setting $\sigma = \gamma^{1/4}$ and substituting $x \to x + s\theta$ we obtain our estimate. □

Lemma 19. $\exists\ C > 0,\ A > 0,\ \delta > 0$ and $\gamma_0 > 0:$

$$\left|\int_{\mathbb{R}} q(y+s\theta)\ ds\right| \le \frac{C}{\gamma^\delta} + Ce^{A\gamma}\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|$$

$y \in \mathbb{R}^d\ \forall\gamma \ge \gamma_0\ \forall\theta \in \mathbb{S}^{d-1}$.

Proof. Let $\theta \in \mathbb{S}^{d-1}$ and $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ with $\|\phi\|_{L^2(\mathbb{R}^d)} = 1$. We define

$$\Phi_\kappa(x) = \kappa^{-d/2}\phi\left(\frac{x-y}{\kappa}\right)$$

where $y \in \Omega_\epsilon$ and a small positive κ.

If

$$h(x,\theta) = \int_0^{T_1} q(x - t\theta)\ dt$$

then

$$|h(y,\theta)| = \left|\int_{\mathbb{R}^d}\Phi^2_\kappa(x)h(y,\theta)\ dx\right| \le \left|\int_{\mathbb{R}^d}\Phi^2_\kappa(x)h(x,\theta)\ dx\right| + \left|\int_{\mathbb{R}^d}\Phi^2_\kappa(x)(h(y,\theta) - h(x,\theta))\ dx\right|.$$

Since

$$|h(y,\theta) - h(x,\theta)| \le \begin{cases} C|x-y|^\mu, & \text{if } q_j \in \mathcal{C}^{0,\mu}(\mathbb{R}^d) \\ C|x-y|^{\mu'}, & \text{if } q_j \in H^s(\mathbb{R}^d) \end{cases}$$

where $\mu' = s - d/2 < 1$ and $0 \le \mu' < 1$. If $s - d/2 \ge 1$ and $\kappa > 0$ is sufficiently small, we apply *lemma 18* and we have

$$|h(y,\theta)| \le C\left(\frac{1}{\gamma^{1/4}} + e^{\mu\gamma}\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|\right)\|\Phi_\kappa\|^2_{H^3(\mathbb{R}^d)} + C\int_{\mathbb{R}^d}(|x-y|^\mu + |x-y|^{\mu'})\Phi^2_\kappa(x)\ dx.$$

Also we have

$$\|\Phi_\kappa\|_{L^2(\mathbb{R}^d)} = 1, \quad \|\Phi_\kappa\|_{H^3(\mathbb{R}^d)} \le C\kappa^{-3}$$

for $\mu_0 = \min(\mu,\mu')$ it is

$$\int_{\mathbb{R}^d}(|x-y|^\mu + |x-y|^{\mu'})\Phi^2_\kappa(x)\ dx \le C\kappa^{\mu_0}.$$

$\forall\theta \in \mathbb{S}^{d-1}$ we apply *lemma 18*

$$\left|\int_0^{T_1} q(y - t\theta)\ dt\right| \le \frac{C}{\gamma^{1/4}}\kappa^{-6} + C\kappa^{-6}e^{\mu\gamma}\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\| + C\kappa^{\mu_0} \tag{2.3.21}$$

we set κ such that

$$\kappa^{\mu_0} = \frac{1}{\gamma^{1/4}}\kappa^{-6}$$

From (2.3.21) $\exists\delta > 0,\ B > 0$:

$$\left|\int_{-T_1}^{T_1} q(y+t\theta)\ dt\right| \leq \frac{C}{\gamma^\delta} + Ce^{B\gamma}\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|.$$

For $T_1 > \text{Diam}(\Omega)$ and $\text{supp}(q) \subset \Omega$ the integration can be done over $\mathbb{R}$ instead of $[-T_1, T_1] \subset \mathbb{R}$. □

2.3.2 Proof of theorems 14 and 16

We set

$$\mathcal{P}(q)(\theta, x) = \int_{\mathbb{R}} q(x+t\theta)\ dt,\ \ x \in \mathbb{R}^d,\ \ \theta \in \mathbb{S}^{d-1}.$$

we have from *lemma 19* that

$$|\mathcal{P}(q)(\theta, x)| \leq C\left(\frac{1}{\gamma^\delta} + e^{\mu\gamma}\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|\right).$$

We choose $R > 0$ such that $\Omega \subset B(0, R)$. Then we have

$$\begin{aligned}\|\mathcal{P}(q)\|^2_{L^2(\mathcal{T})} &:= \int_{\mathcal{S}^{d-1}}\int_{\theta^\perp} |\mathcal{P}(q)(\theta, y)|^2\ dyd\theta \\ &= \int_{\mathcal{S}^{d-1}}\int_{\theta^\perp\cap B(0,R)} |\mathcal{P}(q)(\theta, y)|^2\ dyd\theta \\ &\leq C\left(\frac{1}{\gamma^\delta} + e^{\mu\gamma}\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|\right),\end{aligned}$$

where we set as $\mathcal{T} = \{(\theta, y);\ \theta \in \mathbb{S}^{d-1},\ y \in \theta^\perp\}$ the tangent bundle.

We give a known estimate for the X ray transform [55]:

$$\|q\|_{H^{-1/2}(\Omega)} \leq C\|\mathcal{P}(q)\|^2_{L^2(\mathcal{T})}.$$

Combining the two estimates, we get:

$$\|q\|_{H^{-1/2}(\Omega)} \leq C\left(\frac{1}{\gamma^\delta} + e^{\mu\gamma}\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|\right) \tag{2.3.22}$$

which is valid for $\gamma \geq \gamma_0$.

$\exists\ \epsilon_0 > 0$ small enough such that if $\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\| \leq \epsilon_0$ and

$$\gamma = \frac{1-\delta}{\mu}\left|\log\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|\right|$$

because we guarantee we have $\gamma \geq \gamma_0$. Therefore, we can rewrite (2.3.22) as

$$\|q\|_{H^{-1/2}(\Omega)} \leq C\left(\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|^\delta + C'\left|\log\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|\right|^{-\delta}\right) \tag{2.3.23}$$

for $\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\| \le \epsilon_0$. On the other hand, for $\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\| \ge \epsilon_0$ we have

$$\|q\|_{H^{-1/2}(\Omega)} \le C\|q\|_{L^\infty(\Omega)} \le \frac{2CM}{\epsilon_0^\delta}\epsilon_0^\delta \le C'\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|^\delta.$$

Hence, this completes our proof for *theorem 14*.

To prove the second estimate, it is immediately deduced from *Sobolev imbedding theorem* [30] and an *interpolation inequality* [11].

Assume $\eta > 0$ such that $s = d/2 + 2\eta$, then

$$\begin{aligned}\|q\|_{L^\infty(\Omega)} &\le C\|q\|_{H^{s-\eta}(\Omega)}\\ &\le C\|q\|^\alpha_{H^{-1/2}(\Omega)}\|q\|^{1-\alpha}_{H^s(\Omega)}\\ &\le C\|q\|^\alpha_{H^{-1/2}(\Omega)},\end{aligned}$$

where $\alpha = \frac{s-d/2}{2s+1} < 1$. Then applying (2.3.23) it deduces the second estimate of *theorem 14*.

For *theorem 16* we assume the same notations as in *18*, then

$$\begin{aligned}\|\partial_\nu w\|_{L^2(\Sigma_0)} &= \|\tilde\Lambda^\sharp_{q_1}(f_\sigma) - \tilde\Lambda^\sharp_{q_2}(f_\sigma)\|_{L^2(\Sigma_0)}\\ &\le C\|\tilde\Lambda^\sharp_{q_1}(f_\sigma) - \tilde\Lambda^\sharp_{q_2}(f_\sigma)\|_{L^2(0,T;H^{3/2}(\Gamma_0))}\\ &\le C\|\tilde\Lambda^\sharp_{q_1} - \tilde\Lambda^\sharp_{q_2}\|_s\|f_\sigma\|_{H^{2d+4}(0,T;H^{3/2}(\Gamma))}.\end{aligned}$$

where

$$\|f_\sigma\|_{H^{2d+4}(0,T;H^{3/2}(\Gamma))} \le C\sigma^{2d+4}\|\Phi\|_{H^{6+2d}(\mathbb{R}^d)}$$

We combine the two estimates and we have

$$\|\partial_\nu w\|_{L^2(\Sigma_0)} \le C\sigma^{2d+4}\|\tilde\Lambda^\sharp_{q_1} - \tilde\Lambda^\sharp_{q_2}\|_s.$$

In this case, we have

$$\left|\int_Q q(x)\Phi^2(x+t\theta)\ dxdt\right| \le C\left(\frac{1}{\sigma} + \frac{\sigma}{\sqrt{\gamma}} + \sigma^{2d+4}e^{\mu\gamma}\|\tilde\Lambda^\sharp_{q_1} - \tilde\Lambda^\sharp_{q_2}\|_s\right)\|\Phi\|^2_{H^{6+2d}(\mathbb{R}^d)}.$$

By setting $\sigma = \gamma^{1/4}$ and following the same concepts as *theorem 14*, we prove *theorem 16*.

2.3.3 Proof of theorem 17

Let $q \in L^\infty(\Omega)$, $\sigma(A_q) = \{\lambda_{k,q}\}$ the spectrum of A_q and its resolvent by $\rho(A_q) = \mathbb{C} \setminus \sigma(A_q)$.

$\forall\lambda \in \rho(A_q)$, $f \in H^{3/2}(\Gamma)$ the BVP:

$$\begin{cases} -\Delta u + qu - \lambda u = 0 & \text{in } \Omega\\ u = f & \text{on } \Gamma\end{cases}$$

$\exists!\ u_{q,f} \in H^2(\Omega)$ and

$$\Pi^\sharp_q(\lambda) : f \to \partial_\nu u_{q,f}|_{\Gamma_0}$$

is bounded from $H^{3/2}(\Gamma)$ into $H^{1/2}(\Gamma_0)$.

We let

$$\mathfrak{R}^\sharp_q h = \sum_{k\ge1}\frac{1}{\lambda^{d+2}_{q,k}}(\partial_\nu\phi_{q,k})_{|\Gamma_0}\int_0^t\frac{\sin\left(\sqrt{\lambda_{q,k}}(t-s)\right)}{\sqrt{\lambda_{q,k}}}\langle-\partial_s^{2(d+2)}h(\cdot,s),\partial_\nu\phi_{q,k}\rangle\ ds,$$

where $\langle\cdot,\cdot\rangle$ is the L^2-scalar product. $\mathfrak{R}_q^\sharp$ defines a bounded operator from $\mathcal{H}_1$ into $\mathcal{H}_2$, and fix $s\in[0,1/2]$.

We need the following three lemmas which have been proven in [1, 13]

Lemma 20. Let $q\in L^\infty(\Omega)$. $\forall m>d/2$, $f\in H^{3/2}(\Gamma)$ and $\lambda\in\rho(A_q)$, we have

$$\frac{d^m}{d\lambda^m}\Pi_q^\sharp(\lambda)f=-m!\sum_{k\geq1}\frac{1}{(\lambda_{k,q}-\lambda)^{m+1}}\langle f,\partial_\nu\phi_{k,q}\rangle\partial_\nu\phi_{k,q}|_{\Gamma_0}.$$

Lemma 21. Let $N\in\mathbb{N}^*$, $q_1,q_2\in L^\infty(\Omega)$ with $0\leq q_1,q_2\leq M$ for some constant M. $\exists\ C>0$, such that

$$\left\|\frac{d^p}{d\lambda^p}[\Pi_{q_1}^\sharp-\Pi_{q_2}^\sharp]\right\|_s\leq\frac{C}{|\lambda|^{p+\frac{1-2s}{4}}},\quad \lambda\leq0 \text{ and } 0\leq p\leq N,$$

where $\|\cdot\|_s$ is the norm in $\mathcal{L}(H^{3/2}(\Gamma);H^s(\Gamma_0))$.

Lemma 22. $\forall h\in\mathcal{H}_1$

$$\tilde{\Lambda}_q^\sharp h=\sum_{j=0}^{d+1}\left[\frac{d^j}{d\lambda^j}\Pi_q^\sharp(\lambda)\right]_{\lambda=0}(-\partial_t^2h)+\mathfrak{R}_q^\sharp h,$$

where for $q\in L^\infty(\Omega)$, $\tilde{\Lambda}_q^\sharp$ is bounded from $\mathcal{H}_1$ into $\mathcal{H}_2$.

We prove the following estimate

Lemma 23. $q_1,q_2\in\mathfrak{X}(M,\omega)$, $\exists C>0$, $\delta\in(0,1)$, $\gamma_0>0$:

$$C\|\partial_\nu(\phi_{k,q_1}-\phi_{k,q_2})\|_{L^2(\Gamma)}\leq(\lambda_{k,q_1}+\lambda_{k,q_2})\gamma^{-\delta}+e^{A\gamma}(\|\partial_\nu(\phi_{k,q_1}-\phi_{k,q_2})\|_{L^2(\Gamma_0)}+|\lambda_{k,q_1}-\lambda_{k,q_2}|)\tag{2.3.24}$$

$\forall\gamma\geq\gamma_0$, $\forall k\geq1$.

Proof. Consider $\psi_k(x)=(\phi_{k,q_1}-\phi_{k,q_2})(x)$ which satisfies the BVP:

$$\begin{cases}(-\Delta+q_1-\lambda_{k,q_1})\psi_k=(q_2-q_1)\phi_{k,q_2}+(\lambda_{k,q_1}-\lambda_{k,q_2})\phi_{k,q_2} & \text{in } \Omega,\\ \psi_k=0 & \text{on } \Gamma.\end{cases}$$

Let $T>0$ large enough and

$$w_k(t,x)=e^{it\sqrt{\lambda_{k,q_1}}}\psi_k(x),\quad t\in(0,T).$$

which solves the IVP:

$$\begin{cases}(\partial_t^2-\Delta+q_1-q_1(x))w_k=F_k & \text{in } Q,\\ w_k(t,x)=0 & \text{on } \Sigma.\end{cases}$$

where

$$F_k=e^{it\sqrt{\lambda_{k,q_1}}}\left((q_2-q_1)\phi_{k,q_2+(\lambda_{k,q_1}-\lambda_{k,q_2})\phi_{k,q_2}}\right).$$

From $q_1-q_2=0$ in ω, we have

$$\|F_k\|_{L^2(\omega_T)}\leq C|\lambda_{k,q_1}-\lambda_{k,q_2}|.$$

We combine this estimate with (2.1.20) and we obtain

$$\|w_k\|_{H^1(\omega_{T_1}(2\rho))}\leq\frac{C}{\sqrt{\gamma}}\|w_k\|_{H^2(Q)}+e^{\mu\gamma}(|\lambda_{k,q_1}-\lambda_{k,q_2}|+\|\partial_\nu w_k\|_{L^2(\Sigma_0)})\tag{2.3.25}$$

and using the fact that $w_k(t,x)=e^{it\sqrt{\lambda_{k,q_1}}}\psi_k(x)$ with $\psi_k(x)=(\phi_{k,q_1}-\phi_{k,q_2})(x)$, we obtain

$$\|\phi_{k,q_1}-\phi_{k,q_2}\|_{H^1(\omega(2\rho))}\leq\frac{C}{\sqrt{\gamma}}(\lambda_{k,q_1}+\lambda_{k,q_2})+e^{\mu\gamma}(|\lambda_{k,q_1}-\lambda_{k,q_2}|+\|\partial_\nu()\|)\tag{2.3.26}$$

We use apply to (2.3.5) an interpolation inequality and we have

$$\begin{aligned}\|\partial_\nu(\phi_{k,q_1}-\phi_{k,q_2})\|_{L^2(\Gamma)} &\le C\|\phi_{k,q_1}-\phi_{k,q_2}\|_{H^{3/2}(\omega(2\rho))}\\ &\le C\|\phi_{k,q_1}-\phi_{k,q_2}\|^{1/2}_{H^1(\omega(2\rho))}\cdot\|\phi_{k,q_1}-\phi_{k,q_2}\|^{1/2}_{H^2(\omega(2\rho))}\\ &\le \gamma^{1/4}\|\phi_{k,q_1}-\phi_{k,q_2}\|_{H^1(\omega(2\rho))}+\gamma^{-1/4}(\lambda_{k,q_1}+\lambda_{k,q_2}).\end{aligned} \tag{2.3.27}$$

From (2.3.26) and (2.3.27) we acquire (2.3.24). □

Setting $Z(\lambda)=(\Pi^\sharp_{q_1}(\lambda)-\Pi^\sharp_{q_2}(\lambda))$ and applying Taylor's formula, then

$$Z^{(j)}(0)=\sum_{p=j}^{d}\frac{(-\lambda)^{p-j}}{(p-j)!}Z^{(p)}(\lambda)+\int_\lambda^0\frac{(-\tau)^{d-j}}{(d-j)!}Z^{(d+1)}(\tau)\ d\tau.$$

We prove the following lemma

Lemma 24. $\exists\ C>0,\ \mu_1\in(0,1):$

$$\|Z^{(d+1)(\lambda)}\|_s\le C|\log\eta|^{-\mu_1},\quad \forall\lambda\le 0 \tag{2.3.28}$$

where $\|\cdot\|_s$ denotes the norm in $\mathcal{L}(H^{3/2}(\Gamma);H^s(\Gamma_0))$.

Proof. We assume that $f\in H^{3/2}(\Gamma)$ to take advantage of *lemma 20*, then

$$\begin{aligned}Z^{(d+1)}(\lambda)f=&-(d+1)!\sum_{k=1}^{\infty}\frac{1}{(\lambda_{k,q_1}-\lambda)^{d+2}}\langle f,\partial_\nu\phi_{k,q_1}\rangle\partial_\nu\phi_{k,q_1}|_{\Gamma_0}\\ &+(d+1)!\sum_{k=1}^{\infty}\frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}}\langle f,\partial_\nu\phi_{k,q_2}\rangle\partial_\nu\phi_{k,q_2}|_{\Gamma_0}\end{aligned}$$

We assume that

$$\begin{aligned}I_1(\lambda)&=-(d+1)!\sum_{k=1}^{\infty}\left[\frac{1}{(\lambda_{k,q_1}-\lambda)^{d+2}}-\frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}}\right]\langle f,\partial_\nu\phi_{k,q_1}\rangle\partial_\nu\phi_{k,q_1}|_{\Gamma_0}\\ I_2(\lambda)&=-(d+1)!\sum_{k=1}^{\infty}\frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}}\langle f,\partial_\nu\phi_{k,q_1}-\partial_\nu\phi_{k,q_2}\rangle\phi_{k,q_1}|_{\Gamma_0}\\ I_3(\lambda)&=-(d+1)!\sum_{k=1}^{\infty}\frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}}\langle f,\partial_\nu\phi_{k,q_2}\rangle[\partial_\nu\phi_{k,q_1}-\partial_\nu\phi_{k,q_2}]|\Gamma_0,\end{aligned}$$

where $Z^{(d+1)}=I_1+I_2+I_3$. We will find the estimates of I_1,I_2 and I_3 in the $H^{1/2}$ sense in order to prove our desired estimate.

- For I_1:

$$\|I_1(\lambda)\|_{H^{1/2}(\Gamma)}\le(d+1)!\|f\|_{L^2(\Gamma)}\sum_{k=1}^{\infty}\left|\frac{1}{(\lambda_{k,q_1}-\lambda)^{d+2}}-\frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}}\right|\|\partial_\nu\phi_{k,q_2}\|^2_{H^{1/2}(\Gamma)}.$$

For $\lambda\le 0,\ \lambda_{k,q_j}\ge 0,\ j=1,2,$ we have

$$\left|\frac{1}{(\lambda_{k,q_1}-\lambda)^{d+2}}-\frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}}\right|\le C\max\left(\frac{1}{\lambda^{d+3}_{k,q_1}},\frac{1}{\lambda^{d+3}_{k,q_2}}\right)|\lambda_{k,q_1}-\lambda_{k,q_2}|\le\frac{C}{k^{2(d+3)/d}}|\lambda_{k,q_1}-\lambda_{k,q_2}|$$

Also, we have

$$\|\partial_\nu \phi_{k,q_2}\|^2_{H^{1/2}(\Gamma)} \leq C k^{4/d},$$

by combining these two estimates, we have

$$\begin{aligned}\|I_1(\lambda)\|_{H^{1/2}(\Gamma)} &\leq C\|f\|_{L^2(\Gamma)} d_\infty(\lambda_{q_1}, \lambda_{q_2}) \sum_{k=1}^{\infty} \frac{1}{k^{2(d+1)/d}} \\ &\leq C\eta \|f\|_{L^2(\Gamma)} \end{aligned} \tag{2.3.29}$$

- For I_2:

$$\|I_2(\lambda)\|_{H^{1/2}(\Gamma)} \leq C\|f\|_{L^2(\Gamma)} \sum_{k=1}^{\infty} \frac{\lambda_{k,q_1}}{(\lambda_{k,q_2}-\lambda)^{d+2}} \|\partial_\nu(\phi_{k,q_1} - \phi_{k,q_2})\|_{L^2(\Gamma)}.$$

We apply *lemma 23* and obtain

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\lambda_{k,q_1}}{(\lambda_{k,q_2}-\lambda)^{d+2}} \|\partial_\nu(\phi_{k,q_1} - \phi_{k,q_2})\|_{L^2(\Gamma)} &\leq C\gamma^{-\delta} \sum_{k=1}^{\infty} \frac{\lambda_{k,q_1}(\lambda_{k,q_1}+\lambda_{k,q_2})}{(\lambda_{k,q_2}-\lambda)^{d+2}} \\ &\quad + e^{A\gamma} \sum_{k=1}^{\infty} \frac{\lambda_{k,q_1}}{(\lambda_{k,q_2}-\lambda)^{d+2}} \|\partial_\nu(\phi_{k,q_1} - \phi_{k,q_2})\|_{L^2(\Gamma_0)} \\ &\quad + e^{A\gamma} \sum_{k=1}^{\infty} \frac{\lambda_{k,q_1}}{(\lambda_{k,q_2}-\lambda)^{d+2}} |\lambda_{k,q_1} - \lambda_{k,q_2}|.\end{aligned}$$

This now is

$$\sum_{k=1}^{\infty} \frac{\lambda_{k,q_1}}{(\lambda_{k,q_2}-\lambda)^{d+2}} \|\partial_\nu(\phi_{k,q_1} - \phi_{k,q_2})\|_{L^2(\Gamma)} \leq C\gamma^{-\delta} + e^{A\gamma}\eta$$

and by minimizing with respect to γ, we get

$$\|I_2(\lambda)\|_{H^{1/2}(\Gamma)} \leq C\|f\|_{L^2(\Gamma)} |\log \eta|^{-\mu_1}, \quad \mu_1 \in (0,1). \tag{2.3.30}$$

- For I_3:

$$\begin{aligned}\|I_3(\lambda)\|_{H^{1/2}(\Gamma)} &\leq C\|f\|_{L^2(\Gamma)} \sum_{k=1}^{\infty} \frac{1}{\lambda_{k,q_2}^{d+1}} \|\partial_\nu \phi_{k,q_1} - \partial_\nu \phi_{k,q_2}\|_{H^{1/2}(\Gamma_0)} \\ &\leq C\|f\|_{L^2(\Gamma)} \sum_{k=1}^{\infty} \frac{1}{k^{2\zeta/d}} \|\partial_\nu \phi_{k,q_1} - \partial_\nu \phi_{k,q_2}\|_{H^{1/2}(\Gamma_0)}.\end{aligned}$$

That gives us

$$\|I_3\|_{H^{1/2}(\Gamma)} \leq C\eta \|f\|_{L^2(\Gamma)}. \tag{2.3.31}$$

From a combination of the equations (2.3.29), (2.3.30) and (2.3.31), we derive (2.3.28) □

Using now Taylor's formula on $Z^{(j)}(0)$ and applying *lemma 21*, we have

$$\begin{aligned}\|Z^{(j)}(0)\|_s &\leq C(|\lambda|^{-j-\frac{1-2s}{4}} + |\lambda|^{d-j+1} |\log \eta|^{-\mu_1}) \\ &\leq C(|\lambda|^{-\frac{1-2s}{4}} + |\lambda|^{d+1} |\log \eta|^{-\mu_1}), \quad |\lambda| \geq 1 \\ &\leq C \min_{\rho \geq 1} (\rho^{-\frac{1-2s}{4}} + \rho^{d+1} |\log \eta|^{-\mu_1}) = C |\log \eta|^{-\mu_2}, \quad \mu_2 \in (0,1).\end{aligned}$$

We can obtain the following estimate, by proceeding with the same way as in the proof of *lemma 24*:

$$\|\mathfrak{R}^\sharp_{q_1} - \mathfrak{R}^\sharp_{q_2}\|_s \leq C |\log \eta|^{-\mu_3}. \tag{2.3.32}$$

From the estimate of *lemma 22* combined with (2.3.31) and (2.3.32), we obtain

$$\|\tilde{\Lambda}^{\sharp}_{q_1} - \tilde{\Lambda}^{\sharp}_{q_2}\|_s \leq C|\log \eta|^{-\mu_4},$$

where if we combine it with *theorem 16*, it provides us with the estimates of *theorem 17*.

2.4 Conclusion of Chapter 2

As we have seen in this chapter, the whole construction for the logarithmic estimation has also been based from other estimates and theorems from external papers, where we have mentioned a few. It requires a lot of work to even just prove only one stability estimate for an inverse problem. The procedure in this chapter was mostly a theory based scheme, where we provide in the next chapter a more applied scheme to the same problem.

Chapter 3

Confirmation of the log-type stability estimate in a problem of radial symmetry using Mathematica

In this chapter we want to verify *theorem 17* from *chapter 2* for the special case of $d = 2$ having a radial symmetry on the differential equation. We do this by introducing Bessel's functions first and second kind which lie from a specific ordinary differential equation, known as *the Bessel equation* , that has its presence on various topics in mathematical physics with cylinder or spherical symmetry [42, 76, 38].

In these case, we deal with a perturbed source $q \in H^1$, providing us with perturbed solutions of the Bessel functions. We apply Poincare-Linstedt method to our perturbed differential equation to achieve getting a solution which consists of the Bessel function J_0 and a function that gives us the perturbation for $0 < \epsilon \ll 1$. For numerical results we use Mathematica in our favour providing us a with the eigenvalues and eigenfunctions of the problem.

3.1 Introduction to Bessel's functions and Poincare-Linstedt method

3.1.1 Recurrence relation to Bessel's equation

We start by introducing *Bessel's differential equation*:

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0, \tag{3.1.1}$$

where ν is a parameter. We note that the point $x_0 = 0$ is a regular singular point and by *Frobenius method* [12, 31] we know that the power series

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

must be a solution to (3.1.1). The coefficients c_n can be found by substituting the power series to (3.1.1). That is

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)c_n x^n + \sum_{n=0}^{\infty}(r+n)c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} - \nu^2 \sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=0}^{\infty}[(r+n)(r+n-1) + (r+n) - \nu^2]c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} = 0$$

We do a substitution by setting $n+2=n'$ and then $n'\to n$ and because

$$(r+n)(r+n-1)+(r+n)-\nu^2=(r+\nu+n)(r-\nu+n)$$

we can rewrite the last equation as

$$\sum_{n=0}^{\infty}(r+\nu+n)(r-\nu+n)c_n x^n+\sum_{n=2}^{\infty}c_{n-2}x^n=0$$

or

$$(r+\nu)(r-\nu)c_0+(r+\nu+1)(r-\nu+1)c_1x+\sum_{n=2}^{\infty}[(r+\nu+n)(r-\nu+n)c_n+c_{n-2}]x^n=0.$$

From the last equation, we get that all coefficients of all powers of x must be zero. That is

$$\textbf{i.}\ (r+\nu)(r-\nu)c_0=0 \quad \text{for } c_0\neq 0\Rightarrow (r+\nu)(r-\nu)=0 \tag{3.1.2}$$

$$\textbf{ii.}\ (r+\nu+1)(r-\nu+1)c_1=0 \tag{3.1.3}$$

$$\textbf{iii.}\ (r+\nu+n)(r-\nu+n)c_n+c_{n-2}=0. \tag{3.1.4}$$

Equation (3.1.2) is the characteristic equation of Bessel's equation and its roots are

$$r_1=\nu \quad\text{and}\quad r_2=-\nu \tag{3.1.5}$$

Without loss of generality we assume that $\nu\geq 0$, and we set $r=r_1=\nu$ to (3.1.3), we have

$$(2\nu+1)c_1=0\Rightarrow c_1=0$$

and for $r=r_2=-\nu$ we have

$$(-2\nu+1)c_1=0\Rightarrow \begin{cases} c_1=0 & \text{when } \nu\neq\frac{1}{2}\\ c_1 \text{ arbitrary} & \text{when } \nu=\frac{1}{2}\end{cases}$$

From the recurrence relation (3.1.4) we notice that the term

$$R_n(r)=(r+\nu+n)(r-\nu+n),\quad n\geq 2$$

is not zero when $r=\nu$ but when $r=-\nu$ we can rewrite the term as

$$R_n(-\nu)=n(n-2\nu),\quad n\geq 2$$

where it is

$$R_n(r=-\nu)\neq 0 \ \text{ when }\ \nu\neq\frac{n}{2}=\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},\ldots$$

Therefore, when ν is not an integer or a semi-integer, we have $R_n(r)\neq 0$ for $r=\pm\nu$. Therefore, for every $\nu\geq 0$ it is possible to find a solution to the equation (3.1.1), while the second solution depends on the parameter ν.

3.1.2 Solutions to Bessel's equation for $\nu\neq 0,\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},\ldots$

When the parameter ν is not an integer or semi-integer or zero, we can find the two solutions of (3.1.1) by setting to (3.1.4) $r=r_1=\nu$ and $r=r_2=-\nu$, then we have for $n=2k$ and $n=2k+1$

$$c_{2k}=\frac{-1}{(r+\nu+2k)(r-\nu+2k)}c_{2k-2},\quad k\geq 1$$

$$c_{2k+1}=\frac{-1}{(r+\nu+2k+1)(r-\nu+2k+1)}c_{2k-1},\quad k\geq 1$$

from knowing that $c_1 = 0$ this lies from the second relation that

$$c_{2k+1} = 0, \quad \forall k \geq 0$$

From the fist relation one can try for $k = 1, 2, \ldots, k$ and then multiply by parts all the relations. That will give us

$$\begin{aligned} c_{2k} &= \frac{(-1)^k c_0}{(r+\nu+2)(r+\nu+4)\ldots(r+\nu+2k)(r-\nu+2)(r-\nu+4)\ldots(r-\nu+2k)} \\ &= \frac{(-1)^k c_0}{2^k(\frac{r+\nu}{2}+1)_k 2^k(\frac{r-\nu}{2}+1)_k} \\ &= \frac{(-1)^k \Gamma(\frac{r+\nu}{2}+1)\Gamma(\frac{r-\nu}{2}+1)c_0}{2^{2k}\Gamma(\frac{r+\nu}{2}+1+k)\Gamma(\frac{r-\nu}{2}+1+k)} \end{aligned}$$

For $r = r_1 = \nu$ and noting that $\Gamma(k+1) = k!$ we have

$$c_{2k}(r=\nu) = \frac{(-1)^k \Gamma(\nu+1)}{2^{2k}\Gamma(\nu+1+k)} c_0$$

Therefore, one solution for (3.1.1) is

$$y_1(x) = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\nu+1)}{2^{2k} k! \Gamma(\nu+1+k)} x^{2k+\nu}$$

or

$$y_1(x) = c_0 2^{\nu} \Gamma(\nu+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+1+k)} \left(\frac{x}{2}\right)^{2k+\nu}$$

by setting $c_0 2^{\nu}\Gamma(\nu+1) = 1$, we have

$$y_1(x) = J_{\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+1+k)} \left(\frac{x}{2}\right)^{2k+\nu} \tag{3.1.6}$$

in the same way, the second solution with $r = r_2 = -\nu$ is

$$y_2(x) = J_{-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(-\nu+1+k)} \left(\frac{x}{2}\right)^{2k-\nu} \tag{3.1.7}$$

Therefore, the general solution to (3.1.1) is

$$y(x) = AJ_{\nu}(x) + BJ_{-\nu}(x)$$

with A, B arbitrary constants. The function $J_{\nu}(x)$ is called *Bessel function first kind.*

3.1.3 Solutions to Bessel's equation for $\nu = m + \frac{1}{2}, \ m \in \mathbb{N}^*$

In this case, the first solution of (3.1.1) if we set $\nu = m + \frac{1}{2}$ is given by (3.1.4) as

$$J_{m+\frac{1}{2}}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(m+\frac{3}{2}+k)} \left(\frac{x}{2}\right)^{2k+m+\frac{1}{2}} \tag{3.1.8}$$

For the second solution, we notice that $r_1 - r_2 = 2\nu = 2m+1$ an odd number. In this case we have

$$R_n(r = r_2 = -m - \frac{1}{2}) = n(n-2m-1) = 0, \quad \text{for } n = 2m+1$$

but if we set $n = 2k$ and $r = r_2 = -m - \frac{1}{2}$, the term $R_n(r)$ vanishes, and we find the second solution with the same approach as the first solution, and it is

$$J_{-m-\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(-m + \frac{1}{2} + k)} \left(\frac{x}{2}\right)^{2k-m-\frac{1}{2}} \tag{3.1.9}$$

3.1.4 Solutions to Bessel's equation for $\nu = m \in \mathbb{N}$

From (3.1.6) we set $\nu = m \in \mathbb{N}$ and is

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(m + 1 + k)} \left(\frac{x}{2}\right)^{2k+m} \tag{3.1.10}$$

The second solution can't be found by Frobenius method because we have for $r = r_2 = -\nu = -m$ that

$$R_n(r = -m) = n(n - 2m) = 0, \quad \text{for } n = 2m$$

therefore, we can find the second solution by defining the following function

$$N_\nu(x) = \frac{J_\nu(x) \cos \nu\pi - J_{-\nu}(x)}{\sin \nu\pi} \tag{3.1.11}$$

The function $N_\nu(x)$ is called *Neumann function* or *Bessel function of second kind*, its a linear combination of the functions $J_\nu(x)$ and $J_{-\nu}(x)$ and it's a solution to (3.1.1). Due to the fact that $J_m(x) = (-1)^m J_{-m}(x)$, $N_\nu(x)$ becomes undetermined for $\nu = m \in \mathbb{N}$. We apply L'Hospital's rule, then

$$N_m(x) = \frac{\partial_\nu[J_\nu(x) \cos \nu\pi - J_{-\nu}(x)]}{\partial_\nu \sin \nu\pi}|_{\nu=m} = \frac{1}{\pi} \left[\partial_\nu J_\nu(x) - (-1)^m \partial_\nu J_{-\nu}(x)\right]_{\nu=m} \tag{3.1.12}$$

3.1.5 Solutions to Bessel's equation for $r_1 = r_2 = 2\nu = 0$

For the first solution we set $\nu = 0$ to (3.1.6) and we have

$$y_1(x) = c_0 J_0 = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k} \tag{3.1.13}$$

and for the second solution we can follow from Frobenius method that

$$y_2(x) = y_1(x) \ln x + \sum_{k=0}^{\infty} \partial_r c_{2k}(r)|_{r=0} x^{2k}, \quad c_{2k+1} = 0.$$

For $n = 2k$ it is

$$c_{2k} = -\frac{1}{(2k + r)^2} c_{2k-2}, \quad k \geq 1$$

therefore

$$c_{2k}(r) = \frac{(-1)^k}{(2 + r)^2(4 + r)^2 \ldots (2k + r)^2} c_0 = c_0 \prod_{p=1}^{k} \frac{(-1)^k}{(2p + r)^2}$$

by differentiating with respect to r, we have

$$\partial_r c_{2k}|_{r=0} = -c_0 \frac{(-1)^k}{2^{2k}(k!)^2} \sum_{r=1}^{k} \frac{1}{r}$$

therefore, the second solution is given by the expression

$$y_2(x) = c_0 \left[J_0(x) \ln x - \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2} \sum_{r=1}^{k} \frac{1}{r}\right] \tag{3.1.14}$$

3.1.6 Useful identities of Bessel's functions

In the previous subsection, we proved the general form of Bessel's function and we present some useful identities which one can verify from the general formula and also by introducing the generating function of the Bessel functions [79, 45]

$$w(x,t) = e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x)t^n.$$

We remind that

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}$$

are the Bessel functions of first kind and

$$Y_n(x) = \frac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi}$$

are the Bessel functions of second kind. We provide the following table of formulas that one can prove from the generating function and the Bessel functions:

$$\begin{aligned}
\boldsymbol{J_{\nu+1}(x)} &= \boldsymbol{\frac{2\nu}{x}J_\nu(x) - J_{\nu-1}(x)} & \boldsymbol{Y_{\nu+1}(x)} &= \boldsymbol{\frac{2\nu}{x}Y_\nu(x) - Y_{\nu-1}(x)} \\
\boldsymbol{J'_{\nu+1}(x)} &= \boldsymbol{\frac{1}{2}\left[J_{\nu-1}(x) - J_{\nu+1}(x)\right]} & \boldsymbol{Y'_{\nu+1}(x)} &= \boldsymbol{\frac{1}{2}\left[Y_{\nu-1}(x) - Y_{\nu+1}(x)\right]} \\
\boldsymbol{J'_\nu(x)} &= \boldsymbol{J_{\nu-1}(x) - \frac{\nu}{x}J_\nu(x)} & \boldsymbol{Y'_\nu(x)} &= \boldsymbol{Y_{\nu-1}(x) - \frac{\nu}{x}Y_\nu(x)} \\
\boldsymbol{J'_\nu(x)} &= \boldsymbol{\frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x)} & \boldsymbol{Y'_\nu(x)} &= \boldsymbol{\frac{\nu}{x}Y_\nu(x) - Y_{\nu+1}(x)} \\
\boldsymbol{\frac{d}{dx}\left[x^\nu J_\nu(x)\right]} &= \boldsymbol{x^\nu J_{\nu-1}(x)} & \boldsymbol{\frac{d}{dx}\left[x^\nu Y_\nu(x)\right]} &= \boldsymbol{x^\nu Y_{\nu-1}(x)} \\
\boldsymbol{\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right]} &= \boldsymbol{-x^{-\nu} J_{\nu+1}(x)} & \boldsymbol{\frac{d}{dx}\left[x^{-\nu} Y_\nu(x)\right]} &= \boldsymbol{-x^{-\nu} Y_{\nu+1}(x)}
\end{aligned}$$

Figure 3.1: Recurrence Formulas of the Bessel functions

A combination of the above identities from *Figure 3.1* gives us

$$\frac{d}{dx}J_0(x) = -J_1(x)$$
$$\frac{d}{dx}Y_0(x) = -Y_1(x)$$

which would be pretty useful when dealing with the numerical part using Mathematica.

3.1.7 Poincare-Linstedt method

We were motivated by [51, 56, 53] consider to use this method that arrises in perturbation theory [54, 86]. We will apply the method for terms up to ϵ^1 considering that $\epsilon^k \approx 0$, $\forall k \geq 2$. By Poincare-Linstedt method, we consider the solution

$$\phi(x) = \sum_{k=0}^{\infty} C_{2k}x^{2k} + \sum_{k=0}^{\infty} C_{2k+1}x^{2k+1} = \phi_0(x) + \epsilon\phi_1(x)$$

where

$$\phi_0(x) = J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k} = C_{2k}^{(0)} x^{2k}$$

and

$$\phi_1(x) = \sum_{k=0}^{\infty} C_k^{(1)} x^k = \sum_{k=0}^{\infty} C_{2k}^{(1)} x^{2k} + \sum_{k=0}^{\infty} C_{2k+1}^{(1)} x^{2k+1}$$

where we obtain the following recurrence relations

$$\begin{cases} C_{2k} = C_{2k}^{(0)} + \epsilon C_{2k}^{(1)} = \frac{(-1)^k}{2^{2k}(k!)^2} C_0 + \epsilon C_{2k}^{(1)} \\ C_{2k+1} = \epsilon C_{2k+1}^{(1)} \end{cases}$$

The first relation will come in handy to our solution with power series in our differential equation with source $q(r) \neq 0$.

3.2 Case in $\mathbb{R}^2$

We consider $d = 2$, therefore $q_1, q_2 \in H^s(\Omega) = H^{1+\epsilon}(\Omega)$, $0 < \epsilon \ll 1$ and $\|q_j\|_{H^{1+\epsilon}(\Omega)} \leq M$. We want to verify that

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leq C\left(|\log\left(|\log \eta|\right)|\right)^{-\mu_0'}$$

where

$$\eta = \max_k |\lambda_{k,q_1} - \lambda_{k,q_2}| + \sum_{k \geq 1} k^{-\zeta} \|\partial_\nu \phi_{k,q_1} - \partial_\nu \phi_{k,q_2}\|_{H^{1/2}(\Gamma_0)}, \quad \zeta \in (2, 3]$$

$\mu_0' \in (0, 1)$ and $C > 0$ depends from Ω, Γ_0, ω and M.
For more simplicity, the eigenvalue problems will be solved in polar coordinates with radial symmetry, that means we have the following Laplacian operator

$$-\Delta + q = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta} + q(r)$$

3.2.1 Eigenvalue problem with $q(r) = 0$ and radial symmetry $\phi = \phi(r)$

We assume that $a > 0$, $\Omega = \{x \in \mathbb{R} | x < a\}$ and consider the following eigenvalue problem

$$\begin{aligned} -\Delta\phi(r) &= \lambda\phi(r), \quad r \in [0, a] \\ \Rightarrow -\phi''(r) - \frac{1}{r}\phi'(r) &= \lambda\phi(r), \quad \frac{\partial^2 \phi}{\partial \theta^2} = 0 \\ \Rightarrow \phi''(r) + \frac{1}{r}\phi'(r) + \lambda\phi(r) &= 0 \\ \Rightarrow r^2\phi''(r) + r\phi'(r) + \lambda r^2 \phi(r) &= 0. \end{aligned} \tag{3.2.1}$$

Equation (3.2.1) reminds us of the Bessel's differential equation (3.1.1). If we do the following transformation

$$g(\xi) = \phi(r) \tag{3.2.2}$$

$$\xi = \sqrt{\lambda} r \Rightarrow \xi^2 = \lambda r^2 \tag{3.2.3}$$

we obtain

$$\phi'(r) = \frac{d}{dr}g(\xi) = g'(\xi)\frac{d\xi}{dr} = g'(\xi)\sqrt{\lambda}$$
$$\phi''(r) = \frac{d^2}{dr^2}g(\xi) = \lambda g''(\xi)$$

equation (3.2.1) takes the following form

$$\xi^2 g''(\xi) + \xi g'(\xi) + \xi^2 g(\xi) = 0 \tag{3.2.4}$$

where (3.2.4) is a Bessel differential equation with variable ξ and $\nu = 0$. Therefore, from section 3.1, we have the solution

$$g(\xi) = J_0(\xi) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+1)} \left(\frac{\xi}{2}\right)^{2m}$$

and the eigenvalues are

$$\xi_k = \sqrt{\lambda_{k,0}}a \Rightarrow \mu_k = \lambda_{k,0} = \frac{\xi_k^2}{a^2} \tag{3.2.5}$$

where a is the radius of the disk Ω. Therefore, we find that the eigenfunctions of the eigenvalue problem are

$$\phi_{k,0} = A_k J_0(\frac{\xi_k r}{a})$$

with the normalisation

$$|A_k|^2 \left\| J_0\left(\frac{\xi_k r}{a}\right)\right\|^2_{L^2(\Omega)} = 1$$

3.2.2 Eigenvalue problem with $q(r) \neq 0$ and radial symmetry $\phi = \phi(r)$

Consider now the problem

$$(-\Delta + q)\phi(r) = \lambda\phi(r), \quad r \in [0, a]$$
$$\Rightarrow -\phi''(r) - \frac{1}{r}\phi'(r) + q(r)\phi(r) = \lambda\phi(r)$$
$$\Rightarrow -r^2\phi''(r) + r\phi'(r) + r^2(\lambda - q(r))\phi(r) = 0$$

For our purpose, we choose our source $q(r)$ to be a function of the form

$$q(r) = \begin{cases} \epsilon(r^4 + \alpha r^2 + \gamma) & \text{for } 0 \leq r \leq r_0 \\ 0 & \text{for } r_0 < r \leq a \end{cases}$$

where ϵ would be a source disturbance parameter, ideally we want this to be small, α, γ are polynomial parameters and r_0 is the radius at which the source becomes identically zero outside, typically $q(x) = 0$ iff $x \in \omega = \{x \in \Omega | x \geq r_0\}$.

A selection of a more simple functions, like for example, $q_1 = E_1$ and $q_2 = E_2 = E_1 + \epsilon$ would not fit to our conditions in chapter 2, because q_1 and q_2 are distinct near the boundary and also q has to be smooth at least C^1 at $r = r_0$. Therefore $q(r)$ at $r = r_0$ satisfies the following conditions

$$r_0^4 + \alpha r_0^2 + \gamma = 0$$
$$4r_0^3 + 2\alpha r_0 = 0$$

where we find

$$\alpha = -2r_0^2 \text{ and } \gamma = r_0^4$$

Then we would have the following source function

$$q(r) = \begin{cases} \epsilon(r^2 - r_0^2)^2 & \text{for } 0 \leq r \leq r_0 \\ 0 & \text{for } r_0 < r \leq a \end{cases}$$

- For $r \in (r_0, a]$ we would have the same differential equation just like in section 3.3.1:

$$r^2\phi''(r) + r\phi'(r) + \lambda r^2\phi(r) = 0$$

therefore the solution would be a linear combination of Bessel's first and second kind functions with $\nu = 0$:

$$\phi(r) = AJ_0(\sqrt{\lambda}r) + BY_0(\sqrt{\lambda}r)$$

- For $r \in [0, r_0]$ we obtain the following diffential equation

$$r^2\phi''(r) + r\phi'(r) + [\lambda r^2 - \epsilon(r^2 - r_0^2)^2 r^2]\phi(r) = 0, \tag{3.2.6}$$

we set $g(\xi) = \phi(r)$, $\xi = \sqrt{\lambda}r$, $\tilde{\epsilon} = \frac{\epsilon}{\lambda^3}$ and $\xi_0 = \sqrt{\lambda}r_0$. Equation (3.2.6) becomes

$$\xi^2 g''(\xi) + \xi g'(\xi) + [\xi^2 - \tilde{\epsilon}(\xi^2 - \xi_0^2)^2\xi^2]g(\xi) = 0 \tag{3.2.7}$$

we note that the point $\xi = \tilde{\xi} = 0$ is a normal irregular point for (3.2.7), thus we will apply Frobenius method to solve (3.2.7).

We assume that the solution is in the form of a power series

$$g(\xi) = \sum_{n=0}^{\infty} c_n \xi^{n+\tau}, \quad c_0 \neq 0, \ \tau \in \mathbb{R}$$
$$g'(\xi) = \sum_{n=0}^{\infty} (n+\tau)c_n \xi^{n+\tau-1}$$
$$g''(\xi) = \sum_{n=0}^{\infty} (n+\tau)(n+\tau-1)c_n \xi^{n+\tau-2}$$

we substitute $g(\xi), g'(\xi)$ and $g''(\xi)$ to (3.2.7) and simplifying the terms ξ^τ, we have

$$\sum_{n=0}^{\infty} c_n(n+\tau)^2\xi^n + \sum_{n=0}^{\infty} c_n(1 - \tilde{\epsilon}\xi_0^4)\xi^{n+2} + 2\tilde{\epsilon}\sum_{n=0}^{\infty} c_n\xi_0^4\xi^{n+4} - \tilde{\epsilon}\sum_{n=0}^{\infty} c_n\xi^{n+6} = 0.$$

From the last, we get from the indicial polynomial that $\tau_1 = \tau_2 = 0$, $c_0 \neq 0$ arbitrary and $\forall k \in \mathbb{N}$ we have $c_{2k+1} = 0$. For the coefficients with even index, we have

$$c_2 \cdot 4 + c_0(1 - \tilde{\epsilon}\xi_0^4) = 0 \rightarrow \text{finding } c_2$$
$$c_2(1 - \tilde{\epsilon}\xi_0^4) + 2\tilde{\epsilon}\xi_0^2 c_0 + 4^2 \cdot c_4 = 0 \rightarrow \text{finding } c_4$$
$$c_{n+6}(n+6)^2 = -c_{n+4} + \tilde{\epsilon}[\xi_0^4 c_{n+4} - 2\xi_0^2 c_{n+2} + c_n] \rightarrow \text{finding every other } c_{2k} \text{ with } k \geq 3 \tag{3.2.8}$$

We observe that in the recurrence relation (3.2.8), every time we calculate a new term it sums up to the term with the perturbation $\tilde{\epsilon}$ accompanied by the two previous terms to give us the next coefficient. Moreover, we can generalise (3.2.8), by observing that finding the coefficients c_2 and c_4 can be done from (3.2.8) if we add the condition

$$\forall k \in \mathbb{N} \text{ we have } c_{-k} = 0.$$

We set $n+6=2k$ to (3.2.8), then

$$c_{2k}(2k)^2 = -c_{2k-2} + \tilde{\epsilon}[\xi_0^4 c_{2k-2} - 2\xi_0^2 c_{2k-4} + c_{2k-6}]$$
$$\Rightarrow c_{2k} = -\frac{c_{2k-2}}{(2k)^2} + \frac{\tilde{\epsilon}[\xi_0^4 c_{2k-2} - 2\xi_0^2 c_{2k-4} + c_{2k-6}]}{(2k)^2}.$$

The term $-\frac{c_{2k-2}}{(2k)^2}$ reminds us the coefficients $\frac{(-1)^k}{2^{2k}(k!)^2}c_0$ from Bessel's function. Therefore by *Poincare-Linstedt method*, we set

$$c_{2k} = \frac{(-1)^k}{2^{2k}(k!)^2}c_0 + \tilde{\epsilon}c_{2k}^{(1)} = c_{2k}^{(0)} + \tilde{\epsilon}c_{2k}^{(1)}$$

where we substitute that in our recurrence relation and ignoring the $\mathcal{O}(\tilde{\epsilon}^2)$ terms, we have

$$\frac{(-1)^k}{2^{2k}(k!)^2}c_0 + \tilde{\epsilon}c_{2k}^{(1)} = -\frac{1}{(2k)^2}\left[\frac{(-1)^{k-1}}{2^{2k}((k-1)!)^2}c_0 + \tilde{\epsilon}c_{2k-2}^{(1)}\right]$$
$$+ \frac{\tilde{\epsilon}c_0}{(2k)^2}\left[\frac{\xi_0^4(-1)^{k-1}}{2^{2k-2}((k-1)!)^2} - 2\xi_0^2\frac{(-1)^{k-2}}{2^{2k-4}((k-2)!)^2} + \frac{(-1)^{k-3}}{2^{2k-6}((k-3)!)^2}\right]$$

where we obtain a relation for the $c_{2k}^{(1)}$ coefficients

$$c_{2k}^{(1)} = -\frac{c_{2k-2}^{(1)}}{(2k)^2} + \frac{1}{(2k)^2}\left[\xi_0^4 c_{2k-2}^{(0)} - 2\xi_0^2 c_{2k-4}^{(0)} + c_{2k-6}^{(0)}\right],$$

where $c_{2k}^{(0)} = \frac{(-1)^k}{2^{2k}(k!)^2}c_0$. We have the coefficients

$$c_{2k} = \frac{(-1)^k}{2^{2k}(k!)^2}c_0 + \tilde{\epsilon}c_{2k}^{(1)}. \tag{3.2.9}$$

where for $k=0$ we have $c_0^{(1)} = 0$.

The recurrence relation (3.2.9) indicates that the solution to the equation (3.2.6) will be a Bessel function perturbed by a factor $\tilde{\epsilon}$. Hence, the solution will be

$$g(\xi) = c_0 J_0(\xi) + \tilde{\epsilon}\sum_{k=1}^{\infty} c_{2k}^{(1)}\xi^{2k}$$
$$= c_0\left[J_0(\xi) + \tilde{\epsilon}\sum_{k=1}^{\infty}\frac{c_{2k}^{(1)}}{c_0}\xi^{2k}\right]$$
$$= c_0\left[J_0(\xi) + \tilde{\epsilon}\sum_{k=1}^{\infty}\gamma_{2k}\xi^{2k}\right]$$
$$\Rightarrow \phi(r) = c_0\left[J_0(\sqrt{\lambda}r) + \frac{\epsilon}{\lambda^3}\sum_{k=1}^{\infty}\gamma_{2k}\lambda^k r^{2k}\right], \quad r \in [0, r_0]$$

where

$$\gamma_{2k} = \frac{c_{2k}^{(1)}}{c_0}$$

We found that the solution to the problem we started with is

$$\phi(r) = \begin{cases} c_0\left[J_0(\sqrt{\lambda}r) + \frac{\epsilon}{\lambda^3}\sum_{k=1}^{\infty}\gamma_{2k}\lambda^k r^{2k}\right], & r \in [0, r_0] \\ AJ_0(\sqrt{\lambda}r) + BY_0(\sqrt{\lambda}r), & r \in (r_0, a] \end{cases} \tag{3.2.10}$$

By the conditions we pointed earlier, our solution (3.2.10) should be at least a C^1 function at the branch point $r = r_0$ and be identically zero when it reaches $r = a$ due to the compact support. Hence, we have the following conditions

$$\begin{aligned}
&\phi(r_0^-) = \phi(r_0^+)\\
&\phi'(r_0^-) = \phi'(r_0^+)\\
&AJ_0(\sqrt{\lambda}a) + BY_0(\sqrt{\lambda}a) = 0
\end{aligned}$$

and we make the following matrix equation

$$\mathcal{A} \cdot \overrightarrow{C} = \overrightarrow{0} \Rightarrow$$
$$\begin{bmatrix} J_0(\sqrt{\lambda}a) & Y_0(\sqrt{\lambda}a) & 0 \\ J_0(\sqrt{\lambda}r_0) & Y_0(\sqrt{\lambda}r_0) & -J_0(\sqrt{\lambda}r_0) - \frac{\epsilon}{\lambda^3}\sum_{k=1}^{\infty}\gamma_{2k}\lambda^k r_0^{2k} \\ \sqrt{\lambda}J_0'(\sqrt{\lambda}r_0) & \sqrt{\lambda}Y_0'(\sqrt{\lambda}r_0) & -\sqrt{\lambda}J_0'(\sqrt{\lambda}r_0) - \frac{\epsilon}{\lambda^3}\sum_{k=1}^{\infty}2k\gamma_{2k}\lambda^k r_0^{2k-1} \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ c_0 \end{bmatrix} = \overrightarrow{0} \tag{3.2.11}$$

where we can find the eigenvalues λ_i from the equation

$$F(\lambda) = \det \mathcal{A}(\lambda) = 0 \tag{3.2.12}$$

3.3 Numerical Procedure using Mathematica in the radial symmetric eigenvalue problem with $q(r) \neq 0$ for the $\mathbb{R}^2$ case

In this section we are going to set two different sources $q(r)$ and try to get numerical results using Mathematica. For simplicity, we set $c_0 = 1$ and $r_0 = 1$, therefore we have

$$q(r)|_{r_0=1} = \begin{cases} \epsilon(r^2-1)^2 & \text{for } 0 \leq r \leq 1 \\ 0 & \text{for } 1 < r \leq a \end{cases} \tag{3.3.1}$$

$$\phi(r)|_{c_0=1,r_0=1} = \begin{cases} J_0(\sqrt{\lambda}r) + \frac{\epsilon}{\lambda^3}\sum_{k=1}^{\infty}\gamma_{2k}\lambda^k r^{2k} & r \in [0,1] \\ AJ_0(\sqrt{\lambda}r) + BY_0(\sqrt{\lambda}r), & r \in (1,a] \end{cases} \tag{3.3.2}$$

where

$$\gamma_{2k} = \gamma_{2k}(\lambda)|_{c_0=1} = \left[-\frac{c_{2k-2}^{(1)}}{(2k)^2} + \frac{1}{(2k)^2}\left(\lambda^2 c_{2k-2}^{(0)} - 2\lambda c_{2k-4}^{(0)} + c_{2k-6}^{(0)}\right)\right] \tag{3.3.3}$$

and

$$c_{2k}(\lambda)|_{c_0=1} = \frac{(-1)^k}{2^{2k}(k!)^2} + \frac{\epsilon}{\lambda^3}\gamma_{2k}(\lambda)|_{c_0=1}, \tag{3.3.4}$$

combining (3.3.3) and (3.3.4) we have

$$c_{2k}(\lambda)|_{c_0=1} = \frac{(-1)^k}{2^{2k}(k!)^2} + \frac{\epsilon}{\lambda^3}\left[-\frac{c_{2k-2}^{(1)}}{(2k)^2} + \frac{1}{(2k)^2}\left(\lambda^2 c_{2k-2}^{(0)} - 2\lambda c_{2k-4}^{(0)} + c_{2k-6}^{(0)}\right)\right] \tag{3.3.5}$$

with $c_0 = 1$ and $c_{-k} = 0 = c_{2k+1}$, $\forall k \in \mathbb{N}$. From (3.3.5), one may notice that the difference, of the consecutive terms c_{2n} and $c_{2(n+1)}$, is declining as $n \to \infty$.

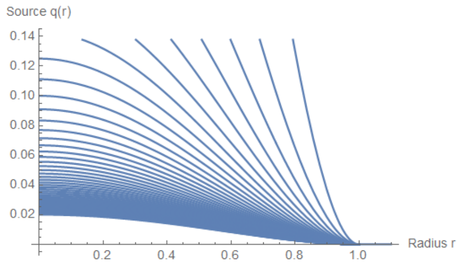


Figure 3.2: The graph of $q_n(r) = \left(\frac{1}{n}\right)(r^2-1)^2,\ r \in [0,1]$ and $q_n(r) = 0,\ r \in [1, 1.1]$.

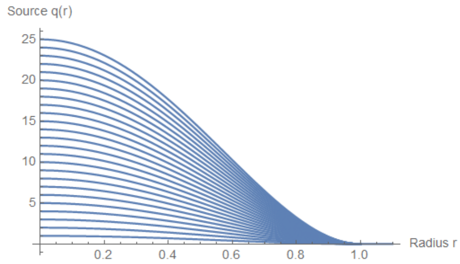


Figure 3.3: The graph of $q_n(r) = n(r^2-1)^2,\ r \in [0,1]$ and $q_n(r) = 0,\ r \in [1, 1.1]$.

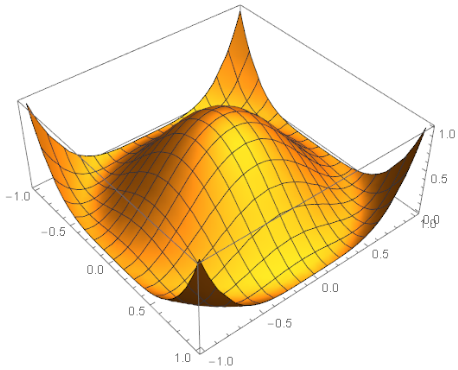


Figure 3.4: The 3D graph of $q(r) = q(x,y) = (r^2-1)^2 = (x^2+y^2-1)$, $x, y \in [-1,1]$.

3.3.1 Numerical procedure for finding eigenvalues $\lambda_{i,j}$

For this example, we set the radius of the disk Ω as $a = 1 + 10^{-1} = 1.1 \approx r_0 = 1$, we see in the next figures the behaviour of our source $q(r)$ where $r \in [0, 1.1]$

For source q_1, we choose $\epsilon = 10^{-3} \ll 1 \to q_1(r) = 10^{-3}(r^2-1)^2$, then

$$c_{2k}(\lambda)|_{c_0=1,\epsilon=10^{-3}} = \frac{(-1)^k}{2^{2k}(k!)^2} + \frac{10^{-3}}{\lambda^3}\left[-\frac{c^{(1)}_{2k-2}}{(2k)^2} + \frac{1}{(2k)^2}\left(\lambda^2 c^{(0)}_{2k-2} - 2\lambda c^{(0)}_{2k-4} + c^{(0)}_{2k-6}\right)\right].$$

From the above relation, we wish to find a few γ_{2k} terms to approximate the infinite sum we have in equation (3.2.11), knowing that the terms c_{2k} decay fast. In the following table, after calculating the term $c^{(0)}_{2k} = c^{(0)}_{2k}$ we can always find $\gamma_{2k} = \gamma_{2k}(\lambda)$:

	$c^{(0)}_{2k}$	$\gamma_{2k}(\lambda) = c^{(1)}_{2k}$
$k = 1$	-0.25	$0.25\lambda^2$
$k = 2$	0.015625	$-0.125\lambda - 0.078125\lambda^2$
$k = 3$	-0.000434028	$0.0277778 + 0.0347222\lambda + 0.0134549\lambda^2$
$k = 4$	$6.78168 \cdot 10^{-6}$	$-0.00737847 - 0.00482856\lambda - 0.00168864\lambda^2$
$k = 5$	$-6.78168 \cdot 10^{-8}$	$0.000894097 + 0.000491536\lambda + 0.000168932\lambda^2$
$k = 6$	$4.7095 \cdot 10^{-10}$	$-0.0000775222 - 0.0000410556\lambda - 0.0000140781\lambda^2$
$k = 7$	$-2.40281 \cdot 10^{-12}$	$5.5719 \cdot 10^{-6} + 2.93323 \cdot 10^{-6}\lambda + 1.00558 \cdot 10^{-6}\lambda^2$
$k = 8$	≈ 0	≈ 0

We determine our function of interest

$$f_N(\lambda) = -J_0(\sqrt{\lambda}) - \frac{10^{-3}}{\lambda^3}\sum_{k=0}^{N}\gamma_{2k}(\lambda)\lambda^k$$

which plays a crucial role of finding the eigenvalues of (3.2.12) as it points out the behaviour of $\det\mathcal{A}(\lambda)$. By the behaviour of our function of interest it is clear that we need enough terms that are far from $N = 1$ and $N = 2$, therefore we choose at most $N = 7$.

We will find at most 6 eigenvalues from each source q, we use mathematica to find the roots of (3.2.11) with the graphical method, because we can't solve this analytically. In the following table we present the eigenvalues of the eigenvalue problem with $q = q_1$ (also check *Figure 3.5*):

First 6 eigenvalues for $q = q_1$
$\lambda_{1,1} = 4.77983$
$\lambda_{1,2} = 21.3994$
$\lambda_{1,3} = 467.702$
$\lambda_{1,4} = 2494.89$
$\lambda_{1,5} = 6452.08$
$\lambda_{1,6} = 12376.8$

For source q_2, we choose $\epsilon = 10^{-4} \ll 1 \to q_2(r) = 10^{-4}(r^2-1)^2$, then

$$c_{2k}(\lambda)|_{c_0=1,\epsilon=10^{-4}} = \frac{(-1)^k}{2^{2k}(k!)^2} + \frac{10^{-4}}{\lambda^3}\left[-\frac{c_{2k-2}^{(1)}}{(2k)^2} + \frac{1}{(2k)^2}\left(\lambda^2 c_{2k-2}^{(0)} - 2\lambda c_{2k-4}^{(0)} + c_{2k-6}^{(0)}\right)\right].$$

We already proved that the coefficients $c_{2k}^{(1)}$ are independent from ϵ, therefore they remain the same and we will use the same algorithms as we did for q_1.In the following table we present the eigenvalues of the eigenvalue problem with $q = q_2$ (also check *Figure 3.6*):

First 6 eigenvalues for $q = q_2$
$\lambda_{2,1} = 4.77953$
$\lambda_{2,2} = 24.320$
$\lambda_{2,3} = 467.702$
$\lambda_{2,4} = 2494.89$
$\lambda_{2,5} = 6452.08$
$\lambda_{2,6} = 12376.8$

3.3.2 Analytical and numerical procedure for finding eigenfunctions $\phi_{i,j}$

In the previous section, we found numerically the eigenvalues of the $\mathbb{R}^2$ problem and now we are going to find every eigenfunction $\phi_{i,j}$ corresponding to a eigenvalue $\lambda_{i,j}$, with $i \in \{1,2\}$, $j \in \{1,2,2,3,4,5,6\}$. For every eigenfunction ϕ t is true that

$$\phi(r) = \begin{cases} J_0(\sqrt{\lambda}r) + \frac{\epsilon}{\lambda^3}\sum_{k\geq 1} c_{2k}^{(1)}\lambda^k r^{2k}, & r \in [0, r_0] \\ AJ_0(\sqrt{\lambda}r) + BY_0(\sqrt{\lambda}r), & r \in (r_0, \alpha] \end{cases}$$

limit condition for $r = r_0 = 1$

It is true that

$$J_0(\sqrt{\lambda}) + \frac{\epsilon}{\lambda^3}\sum_{k\geq 1} c_{2k}^{(1)}\lambda^k = AJ_0(\sqrt{\lambda}) + BY_0(\sqrt{\lambda}),$$

where we set

$$\mathcal{M}(\lambda) = \sum_{k\geq 1} c_{2k}^{(1)}\lambda^k = c_2^{(1)}\lambda + c_4^{(1)}\lambda^2 + c_6^{(1)}\lambda^3 + c_8^{(1)}\lambda^4 + c_{10}^{(1)}\lambda^5 + c_{12}^{(1)}\lambda^6 + c_{14}^{(1)}\lambda^7 + \mathcal{O}(\lambda^8) \quad (3.3.6)$$

which we are going to calculate it with the help of mathematica. Therefore, in the first limit case, we have

$$J_0(\sqrt{\lambda}) + \frac{\epsilon}{\lambda^3}\mathcal{M}(\lambda) = AJ_0(\sqrt{\lambda}) + BY_0(\sqrt{\lambda}) \quad (3.3.7)$$

limit condition for $r = \alpha = 1.1$

In this limit case, it is true that

$$AJ_0(\sqrt{\lambda}1.1) + BY_0(\sqrt{\lambda}1.1) = 0$$

where we solve for A, and we obtain

$$A = -B\frac{Y_0(1.1\sqrt{\lambda})}{J_0(1.1\sqrt{\lambda})} \quad (3.3.8)$$

If we combine (3.3.7) and (3.3.8), we get

$$B = -\frac{J_0(\sqrt{\lambda}) + \frac{\epsilon}{\lambda^3}\mathcal{M}(\lambda)}{\frac{Y_0(1.1\sqrt{\lambda})}{J_0(1.1\sqrt{\lambda})} - Y_0(\sqrt{\lambda})} \quad (3.3.9)$$

For every λ we can find the coefficients A and B from (3.3.8) and (3.3.9) with the use of mathematica.

The following table represents the numerical procedure of finding the function $\mathcal{M}(\lambda)$ from (3.3.6) using mathematica

for $q = q_1$	λ	$\mathcal{M}(\lambda)$
	4.77983	4.60963
	21.3994	$6.41606 \cdot 10^5$
	467.702	$1.05194 \cdot 10^{18}$
	2494.89	$3.74935 \cdot 10^{24}$
	6452.08	$1.94523 \cdot 10^{28}$
	12376.8	$6.84706 \cdot 10^{30}$
for $q = q_2$		
	4.77953	4.60886
	24.320	$2.10219 \cdot 10^6$
	467.702	$1.05194 \cdot 10^{18}$
	2494.89	$3.74935 \cdot 10^{24}$
	6452.08	$1.94523 \cdot 10^{28}$
	12376.8	$6.84706 \cdot 10^{30}$

The following table represents the numerical procedure of finding the coefficients from known eigenvalues.

for $q = q_1$	λ	$B = B_1 = B_{\epsilon=10^{-3}}$	$A = A_1 = A_{B_1}$
1	4.77983	0.0000102025	0.118039
2	21.3994	0.0936267	-0.202349
3	467.702	$-7.21256 \cdot 10^6$	$1.14144 \cdot 10^7$
4	2494.89	$-2.34268 \cdot 10^{11}$	$2.17812 \cdot 10^{11}$
5	6452.08	$2.12371 \cdot 10^{14}$	$8.83114 \cdot 10^{13}$
6	12376.8	$2.75546 \cdot 10^{15}$	$3.71297 \cdot 10^{15}$
for $q = q_2$	λ	$B = B_2 = B_{\epsilon=10^{-4}}$	$A = A_2 = A_{B_2}$
1	4.77953	$1.13284 \cdot 10^{-6}$	0.118044
2	24.320	0.0172908	-0.179985
3	467.702	$-7.21256 \cdot 10^5$	$1.14144 \cdot 10^6$
4	2494.89	$-2.34268 \cdot 10^{10}$	$2.17812 \cdot 10^{10}$
5	6452.08	$2.12371 \cdot 10^{13}$	$8.83114 \cdot 10^{12}$
6	12376.8	$2.75546 \cdot 10^{14}$	$3.71297 \cdot 10^{14}$

we notice that for $\lambda_{q_1,j} = \lambda_{q_2,j}$, with $j \geq 3$, the pairs A_1, A_2 and B_1, B_2 differ by a factor of 10.

3.3.3 Analytical and numerical calculation of the $\|\cdot\|_{H^{1/2}(\Gamma_0)}$ norm

Let us have the eigenfunctions

$$\phi_{q,i}(r) = \begin{cases} J_0(\sqrt{\lambda_{q,i}}r) + \frac{\epsilon}{\lambda_{q,i}^3} \sum_{k\geq 1}^{7} c_{2k}^{(1)} \lambda_{q,i}^k r^{2k}, & r \in [0, r_0] \\ A_{q,i} J_0(\sqrt{\lambda_{q,i}}r) + B_{q,i} Y_0(\sqrt{\lambda_{q,i}}r), & r \in (r_0, \alpha] \end{cases}$$

The normal derivative on the eigenfunction $\phi_{q,i}$, in our case, is the derivative with respect to r. Therefore

$$\partial_\nu \phi_{q,i} = \partial_r \phi_{q,i}(r) = \begin{cases} -\sqrt{\lambda_{q,i}} J_1(\sqrt{\lambda_{q,i}}r) + \frac{\epsilon}{\lambda_{q,i}^3} \sum_{k\geq 1}^{7} 2k c_{2k}^{(1)} \lambda_{q,i}^k r^{2k-1}, & r \in [0, r_0] \\ -A_{q,i}\sqrt{\lambda_{q,i}} J_1(\sqrt{\lambda_{q,i}}r) - B_{q,i}\sqrt{\lambda_{q,i}} Y_1(\sqrt{\lambda_{q,i}}r), & r \in (r_0, \alpha] \end{cases} \quad (3.3.10)$$

We now construct the difference

$$\partial_\nu \phi_{q_1,i} - \partial_\nu \phi_{q_2,i} = \partial_r \phi_{q_1,i}(r) - \partial_r \phi_{q_2,i}(r)$$
$$= \begin{cases} \sqrt{\lambda_{q_2,i}} J_1(\sqrt{\lambda_{q_2,i}}r) - \sqrt{\lambda_{q_1,i}} J_1(\sqrt{\lambda_{q_1,i}}r) + 10^{-4} \sum_{k\geq 1}^{7} 2k \left[10\lambda_{q_1,i}^{k-3} c_{2k}^{(1)}(\lambda_{q_1,i}) - \lambda_{q_2,i}^{k-3} c_{2k}^{(1)}(\lambda_{q_2,i}) \right] r^{2k-1} \\ A_{q_2,i}\sqrt{\lambda_{q_2,i}} J_1(\sqrt{\lambda_{q_2,i}}r) + B_{q_2,i}\sqrt{\lambda_{q_2,i}} Y_1(\sqrt{\lambda_{q_2,i}}r) - A_{q_1,i}\sqrt{\lambda_{q_1,i}} J_1(\sqrt{\lambda_{q_1,i}}r) - B_{q_1,i}\sqrt{\lambda_{q_1,i}} Y_1(\sqrt{\lambda_{q_1,i}}r) \end{cases}$$

We borrow the formulation for calculating fractional norms from [52, 17]. We have that $H^s(\Gamma_0) = W^{s,2}(\Gamma_0)$, where $s = 1/2$. For an arbitrarily open subset $\Omega \subset \mathbb{R}^n$, we have the general formula

$$\|u\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |u|^p dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx\, dy \right)^{1/p}. \quad (3.3.11)$$

In (3.3.11), we set $p = 2$ for Hilbert space $W^{s,2} = H^s$, $s = 1/2$ for $H^{1/2}$ and $n = 1$ for our case at the boundary. In *chapter 2* we mentioned that Γ_0 is a relatively open subset of Γ. We have, in first case, that $\Gamma = \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$. Therefore Γ_0 is a part of the unit circle Γ. If we switch to polar coordinates we have fixed $r = 1$, let $\theta \in I = [\theta_0, \theta_1] \subset [0, 2\pi]$ and $x = \cos\theta\hat{i} + \sin\theta\hat{j}$, equation (3.3.11) becomes

$$\|u\|_{H^{1/2}(\Gamma_0)} = \left(\int_{\theta_0}^{\theta_1} |u(1,\theta)|^2 d\theta + \int_{\theta_0}^{\theta_1} \int_{\theta_0}^{\theta_1} \frac{|u(1,\theta) - u(1,\theta')|^2}{|(\cos\theta - \cos\theta')^2 + (\sin\theta - \sin\theta')^2|} d\theta d\theta' \right)^{1/2}$$

where in our case $u(1,\theta) = \partial_r\phi_{q_1}(1,\theta) - \partial_r\phi_{q_2}(1,\theta) = \partial_r\phi_{q_1}(1) - \partial_r\phi_{q_2}(1)$, where in this case the second integral from the norm should be zero because we have independence from θ. In the following table we have calculated with Mathematica the differences of $\partial_\nu\phi_{q_1} - \partial_\nu\phi_{q_2}$ for $r = r_0 = 1$ and $r = \alpha = 1.1$

k	$\partial_\nu\phi_{q_1}(1) - \partial_\nu\phi_{q_2}(1)$	$\partial_\nu\phi_{q_1}(1.1) - \partial_\nu\phi_{q_2}(1.1)$
1	0.0189663	$3.63706 \cdot 10^{-6}$
2	1.65627	-0.0595614
3	$0.23531 \cdot 10^5$	$4.29963 \cdot 10^7$
4	$3.5797 \cdot 10^6$	$1.54791 \cdot 10^{12}$
5	$6.19338 \cdot 10^7$	$1.41138 \cdot 10^{15}$
6	$4.37214 \cdot 10^8$	$-3.33911 \cdot 10^{16}$

Therefore, for fixed $\tilde{r} \in \{r_0 = 1, \alpha = 1.1\}$, the $H^{1/2}$ norm simplifies to

$$\begin{aligned}\|\partial_r\phi_{q_1}(\tilde{r}) - \partial_r\phi_{q_2}(\tilde{r})\|_{H^{1/2}(\Gamma_0)} &= \left(\int_{\theta_0}^{\theta_1} |\partial_r\phi_{q_1}(\tilde{r}) - \partial_r\phi_{q_2}(\tilde{r})|^2 d\theta\right)^{1/2} \\ &= \left(|\partial_r\phi_{q_1}(\tilde{r}) - \partial_r\phi_{q_2}(\tilde{r})|^2(\theta_1 - \theta_2)\right)^{1/2} \\ &= |\partial_r\phi_{q_1}(\tilde{r}) - \partial_r\phi_{q_2}(\tilde{r})| \cdot \sqrt{\theta_1 - \theta_0},\end{aligned}$$

where $\sqrt{\theta_1 - \theta_0} \in (0, \sqrt{2\pi}]$. Hence, for $\zeta \in (2,3]$, we calculate the sum S using the calculations from the previous table:

$$\begin{aligned}S &= \sum_{k=1}^{6} k^{-\zeta}\|\partial_r\phi_{q_1,k}(\tilde{r}) - \partial_r\phi_{q_2,k}(\tilde{r})\|_{H^{1/2}(\Gamma_0)} \\ &= \sqrt{\theta_1 - \theta_0}\sum_{k=1}^{6} k^{-\zeta}|\partial_r\phi_{q_1,k}(\tilde{r}) - \partial_r\phi_{q_2,k}(\tilde{r})|\end{aligned}$$

- For $\tilde{r} = \alpha = 1.1$:

$$\begin{aligned}S_\alpha &= \sqrt{\theta_1 - \theta_0}\sum_{k=1}^{6} k^{-\zeta}|\partial_r\phi_{q_1,k}(1.1) - \partial_r\phi_{q_2,k}(1.1)| \\ &= \sqrt{\theta_1 - \theta_0}(3.63706 \cdot 10^{-6} + 2^{-\zeta} \cdot 0.0595614 + 3^{-\zeta} \cdot 4.29963 \cdot 10^7 \\ &+ 4^{-\zeta} \cdot 1.54791 \cdot 10^{12} + 5^{-\zeta} \cdot 1.41138 \cdot 10^{15} + 6^{-\zeta} \cdot 3.33911 \cdot 10^{16}) \geq 1.65904 \cdot 10^{14} \cdot \sqrt{\theta_1 - \theta_0}\end{aligned}$$

- For $\tilde{r} = r_0 = 1$:

$$\begin{aligned}S_{r_0} &= \sqrt{\theta_1 - \theta_0}\sum_{k=1}^{6} k^{-\zeta}|\partial_r\phi_{q_1,k}(1) - \partial_r\phi_{q_2,k}(1)| \\ &= \sqrt{\theta_1 - \theta_0}(0.0189663 + 2^{-\zeta} \cdot 1.65627 + 3^{-\zeta} \cdot 0.23531 \cdot 10^5 \\ &+ 4^{-\zeta} \cdot 3.5797 \cdot 10^6 + 5^{-\zeta} \cdot 6.19338 \cdot 10^7 + 6^{-\zeta} \cdot 4.37214 \cdot 10^8) \geq 2.57641 \cdot 10^6 \cdot \sqrt{\theta_1 - \theta_0}\end{aligned}$$

3.3.4 Verifying theorem 17 from chapter 2

Using the tables where we found the eigenvalues $\lambda_{i,j}$, it is clear that

$$\|\lambda_{q_1} - \lambda_{q_2}\|_{\ell^\infty} = \max_k |\lambda_{q_1,k} - \lambda_{q_2,k}| = \lambda_{q_2,2} - \lambda_{q_1,2} = 24.32 - 21.3994 = 2.9206.$$

Also we have that

$$\|q_1 - q_2\|_{L^\infty(\Omega)} = \inf\{C : |q_1(r) - q_2(r)| \leq C \text{ a.e. in } \Omega\}$$

where

$$|q_1(r) - q_2(r)| = |10^{-3}(r^2-1)^2 - 10^{-4}(r^2-1)^2| = 9 \cdot 10^{-4}(r^2-1)^2.$$

We know that q_1, q_2 are defined at $\Omega \setminus \omega$ with the extension of zero for all Ω. Therefore,

$$r \in [0,1] \Rightarrow (r^2-1)^2 \in [0,1],$$

hence

$$|q_1(r) - q_2(r)| \leq 9 \cdot 10^{-4} \Rightarrow \|q_1 - q_2\|_{\ell^\infty} = 9 \cdot 10^{-4}.$$

For the term $\eta = \|\lambda_{q_1} - \lambda_{q_2}\|_{\ell^\infty} + \sum_{k=1}^{6} k^{-\zeta}\|\partial_r \phi_{q_1,k}(\tilde{r}) - \partial_r \phi_{q_2,k}(\tilde{r})\|_{H^{1/2}(\Gamma_0)}$,which is supposed to be small, we assume that $\tilde{r} = \alpha$, then we have that

$$\eta \geq 2.9206 + 1.65904 \cdot 10^{14} \cdot \sqrt{\theta_1 - \theta_0} > 2.9206$$

where it is approximate 87.2238% smaller than the mean value of $\lambda_{q_1,2}$ and $\lambda_{q_2,2}$ for $\theta_1 = \theta_0$. Furthermore

$$\begin{aligned}
2.9206 \leq & \eta \leq 4.1586 \cdot 10^{14} \\
\Rightarrow 0.332106 \leq & |\log(|\log(\eta)|)| \leq 1.16492 \\
\Rightarrow 0.332106^{\mu_0'} \leq & (|\log(|\log(\eta)|)|)^{\mu_0'} \leq 1.16492^{\mu_0'}, \ \mu_0' \in (0,1) \\
\Rightarrow 9 \cdot 10^{-4} < 9 \cdot 10^{-4} \cdot 0.332106^{\mu_0'} \leq & \|q_1 - q_2\|_{L^\infty(\Omega)} \cdot (|\log(|\log(\eta)|)|)^{\mu_0'} \leq 9 \cdot 10^{-4} \cdot 1.16492^{\mu_0'} < 0.00104843
\end{aligned}$$

Therefore $\|q_1 - q_2\|_{L^\infty(\Omega)} \cdot (|\log(|\log(\eta)|)|)^{\mu_0'}$ is bounded for $\mu_0' \in (0,1)$. Hence, we can say that there exists $C > 0$ such that

$$\begin{aligned}
& \|q_1 - q_2\|_{L^\infty(\Omega)} \cdot (|\log(|\log(\eta)|)|)^{\mu_0'} \leq C \\
& \Rightarrow \|q_1 - q_2\|_{L^\infty(\Omega)} \leq C(|\log(|\log(\eta)|)|)^{-\mu_0'},
\end{aligned}$$

where the constant C depends from the circle Ω, its boundary Γ_0, $M = 9 \cdot 10^{-4}$ and $\omega \subset \Omega$ near the boundary. We control θ_0 and θ_1 and it changes the parameter η. That concludes our verification for the radial symmetry of the $\mathbb{R}^2$ case.

3.4 Conclusion of chapter 3

We have concluded to the result that was discussed in chapter 2 having radial symmetry in our problem. One can follow the same approach to other types of sources and domains and can obtain the same result with finding another $C > 0$ that can be controlled by the geometry of the domain. In general it is hard to find the eigenvalues of these problems and we only make it work for the simplest case. Any other case may follow our procedure.

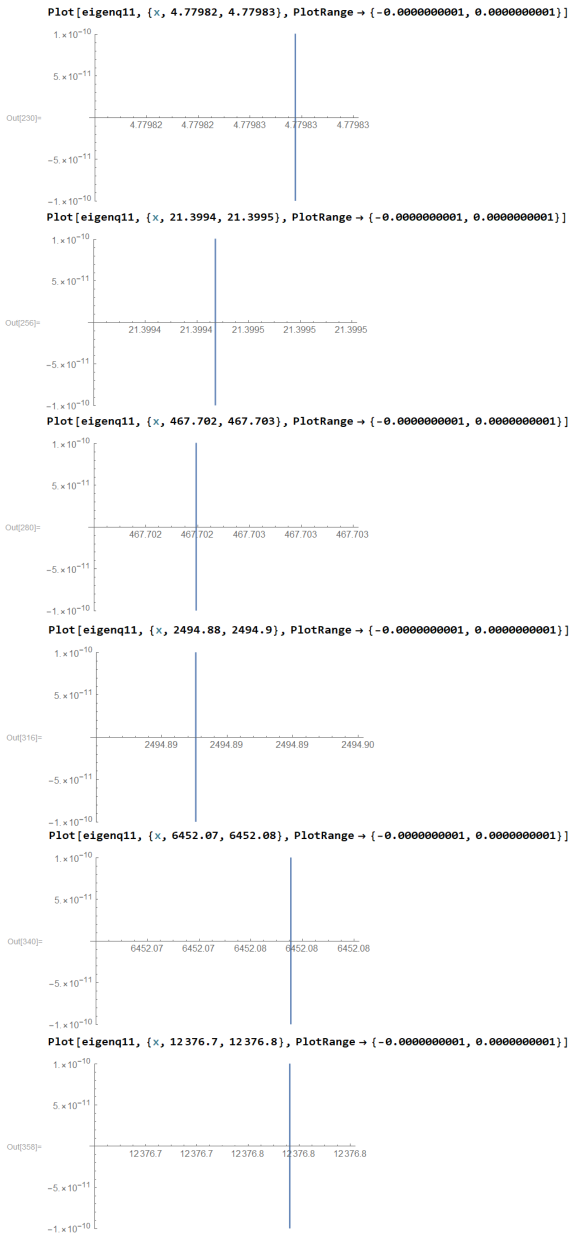


Figure 3.5: Eigenvalues $\lambda_{1,1}$ to $\lambda_{1,6}$ with the graph method for source $q = q_1$. A modern Bolzano method [62].

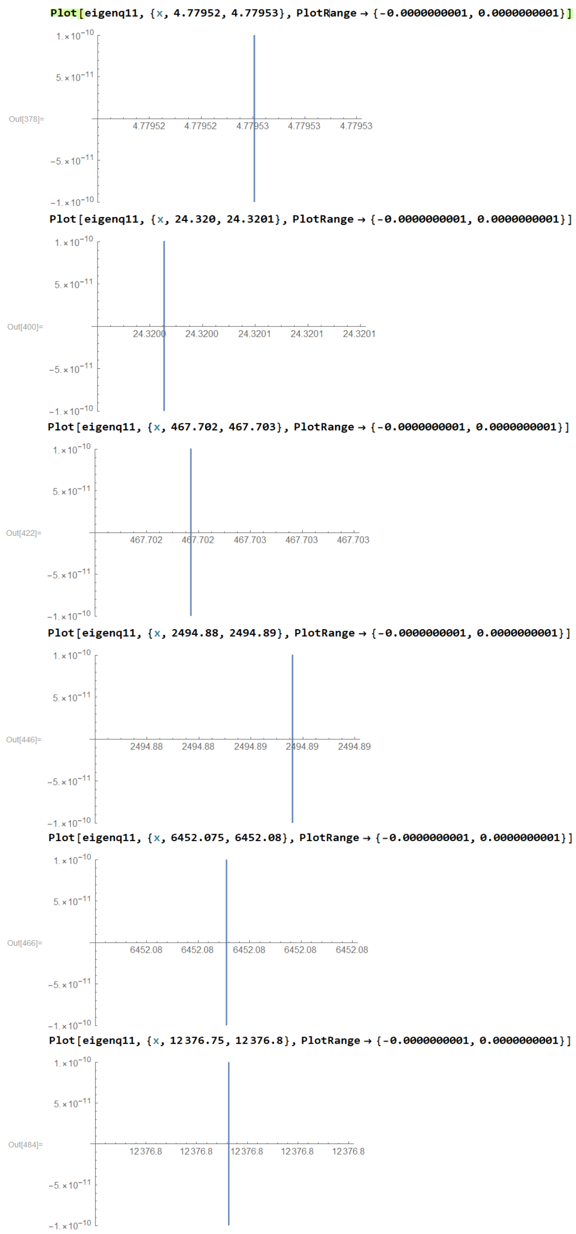


Figure 3.6: Eigenvalues $\lambda_{2,1}$ to $\lambda_{2,6}$ with the graph method for source $q = q_2$. A modern Bolzano method [62].

Chapter 4

Appendix

This chapter is devoted to the preliminaries of the main concepts that we followed in [7]. We give the general definitions and theorems that have not been defined formally. At the end of this chapter, the reader should be able to understand the concepts and the theorems which will be presented in the following chapters. For this purpose, we follow the theory that is written in the books [9, 35, 23, 78, 64], as well as we note some definitions from external notes or papers that are not mentioned in the books.

4.1 Elements of Functional Analysis and Topology

4.1.1 Euclidean space

Definition 25. Let $\mathbb{R}^n$ be n-dimensional Euclidean space. We denote the Euclidean norm of a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ by

$$|x| = \left(x_1^2 + x_2^2 + \ldots + x_n^2\right)^{1/2} = \left(\sum_{k=1}^{n} x_k^2\right)^{1/2} \tag{4.1.1}$$

Definition 26. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. We define the inner product of the vectors x and y by

$$x \cdot y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = \sum_{k=1}^{n} x_k y_k \tag{4.1.2}$$

If we compare the two definitions, one can notice that, the square of the Euclidean norm $|\cdot|^2$ is an inner product of the same vectors. Hence, $|x|^2 = x \cdot x = \sum_{k=1}^{n} x_k^2$.

We assume that $\Omega \subset \mathbb{R}^n$, we denote the complement by $\Omega^c = \mathbb{R}^n \setminus \Omega$, the closure by $\overline{\Omega}$, the interior by Ω^o and the boundary by $\Gamma = \partial\Omega = \overline{\Omega} \setminus \Omega^o$.

Definition 27. Assume that $\chi_\Omega : \mathbb{R}^n \to \mathbb{R}$ be the characteristic function which is defined by

$$\chi_\Omega(x) = \begin{cases} 1 & \text{if} \quad x \in \Omega \\ 0 & \text{if} \quad x \notin \Omega \end{cases} \tag{4.1.3}$$

We borrow the definition of a compact set from [68]:

Definition 28. Let Ω be a set. A collection $\mathscr{C} \subset 2^\Omega$ is a covering of Ω if $\bigcup_{C \in \mathscr{C}} C = \Omega$. If each $C \in \mathscr{C}$ is an open set, then $\mathscr{C}$ is called an open covering of Ω. Ω is called compact if every open covering has a finite sub covering; that is, if for every open covering $\mathscr{C}$, there exists a finite number of sets, say $C_1, \ldots, C_n \in \mathscr{C}$ for some $n \in \mathbb{N}$, such that $\Omega = \bigcup_{k=1}^{n} V_k$.

Definition 29. A set Ω' is compactly contained in an open set Ω, if and only if, $\overline{\Omega'} \subset \Omega$ and $\overline{\Omega'}$ is compact and we write $\overline{\Omega'} \Subset \Omega$.

If $\Omega' \Subset \Omega$, then

$$\operatorname{dist}(\Omega', \partial\Omega) = \inf\{|x-y| : x \in \Omega', y \in \partial\Omega\} > 0.$$

4.1.2 The Spaces C^k, C^∞ and C_0

We denote by $C\,[a,b]$ as the set that contains all the **continuous functions** that defined on the closed interval $[a,b]$. Formally, we write

$$C[a,b] = \{u : [a,b] \to \mathbb{R} : u \text{ is continuous}\}$$

In general, for an abstract set $\Omega \subset \mathbb{R}^n$, we write

$$C(\Omega) = C(\Omega; \mathbb{R}) = \{u : \Omega \to \mathbb{R} : u \text{ is continuous}\}.$$

This space is provided with the norm $\|\cdot\|_\infty$, see also [15], where

$$\forall u \in C(\Omega),\ \|u\|_\infty = \sup\{|u(x)| : x \in \Omega\}$$

$C(\Omega)$ is a metric space with the metric $\rho_\infty(u,v) = \|u-v\|_\infty$.

Let us consider a set that contains all the **functions with continuous partial derivatives** in $\Omega \subset \mathbb{R}^n$ of order less than or equal to $k \in \mathbb{N}$. We denote this set as $C^k(\Omega)$. Also, the space of **functions with continuous derivatives of all orders** by $C^\infty(\Omega)$. The space $C^k(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{C^k(\Omega)} = \sum_{|\alpha|\le k} \sup_\Omega |\partial^\alpha u| \tag{4.1.4}$$

We denote the **support of a continuous function** $u : \Omega \to \mathbb{R}^n$ by supp $u = \overline{\{\mathrm{x} \in \Omega : \mathrm{u(x)} \neq 0\}}$, where Ω is a bounded open set in $\mathbb{R}^n$.

We denote by $C_c(\Omega)$ the space of **continuous functions whose support is compactly contained in** Ω, and by $C_c^\infty(\Omega)$ the space of functions with **continuous derivatives of all orders and compact support in** Ω. The functions that lie on the space $C_c^\infty(\Omega)$ are referred as **test functions**.

4.1.3 Hölder spaces

The spaces we saw in Subsection 2.1.2 do not give us an estimation on how quickly the values of $u(x)$ of a function approach the values of $u(y)$ as $x \to y$.

We generalise the definition of continuity by adding a power, say $\alpha \in (0,1]$, to the term $|x-y|$ and define the following

Definition 30. Let $\Omega \subset \mathbb{R}^n$ and $0 < \alpha \le 1$. A function $u : \Omega \to \mathbb{R}$ is uniformly Hölder continuous with exponent α in Ω if

$$[u]_{\alpha,\Omega} = \sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty \tag{4.1.5}$$

The quantity $[u]_{\alpha,\Omega}$ is a semi-norm.

We note that, a function $u : \Omega \to \mathbb{R}$ is locally uniformly Hölder continuous with exponent α in Ω if $[u]_{\alpha,\Omega'}$ is finite for every $\Omega' \Subset \Omega$.

We denote by $C^{0,\alpha}(\Omega)$ the space of **locally uniformly Hölder continuous functions with exponent α in Ω**. The space $C^{0,\alpha}(\overline{\Omega})$, where Ω is bounded, is a Banach space with respect to the norm

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} = \sup_{\Omega} |u| + [u]_{\alpha,\Omega}. \tag{4.1.6}$$

If u is Hölder continuous with $\alpha = 1$, then we say that u is Lipschitz continuous.

We extend the definition of Hölder space to the space with functions with continuous partial derivatives.

Definition 31. Let $\Omega \subset \mathbb{R}^n$, $k \in \mathbb{N}$ and $0 < \alpha \leq 1$, then we define the space $C^{k,\alpha}(\Omega)$ which consists of all functions $u : \Omega \to \mathbb{R}$ with continuous partial derivatives in Ω of order less than or equal to k whose k^{th} derivatives are locally uniformly Hölder continuous with exponent α in Ω. If the set Ω is bounded, then the space $C^{k,\alpha}(\overline{\Omega})$ is a Banach space with respect to the norm

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} = \sum_{|\beta| \leq k} \sup_{\Omega} |\partial^\beta u| + \sum_{|\beta| = k} [\partial^\beta u]_{\alpha,\Omega} \tag{4.1.7}$$

4.1.4 L^p spaces

We introduce a more general function spaces, where we consider that Ω is a Lebesgue-measurable set in $\mathbb{R}^n$. A Lebesgue-measurable set is a set that can be divided into small pieces that can be measured, and the sum of these measures gives the measure of the whole set. Lebesgue-measurable sets are important because they allow us to define integrals and other mathematical operations that can be used to study the behavior of functions on these sets. A function $u : X \to Y$, where X and Y are measurable spaces, is said to be Lebesgue-measurable if and only if for every Borel set $B \subset Y$, the set $\{x \in X : u(x) \in B\}$ is Lebesgue-measurable in X [5].

Definition 32. For $1 \leq p < \infty$, the space $L^p(\Omega)$ consists of the Lebesgue-measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_{\Omega} |u|^p dx < \infty \tag{4.1.8}$$

and $L^\infty(\Omega)$ consists of the essentially bounded functions. Intuitively, a function $u \in L^\infty(\Omega)$ may exceed a certain bound on a set of points with zero measure. In this case, u is considered bounded.

L^p spaces are Banach spaces with respect to the norm

$$\|u\|_{L^p(\Omega)} = \left(\int_{\Omega} |u|^p dx \right)^{1/p} \tag{4.1.9}$$

and for L^∞

$$\|u\|_{L^\infty(\Omega)} = \sup_{\Omega} |u| = \inf \{M \in \mathbb{R} : u \leq M \text{ a.e. in } \Omega\}. \tag{4.1.10}$$

4.1.5 Sobolev spaces

Sobolev spaces are a class of function spaces used in the field of functional analysis and partial differential equations. They were introduced by the Russian mathematician Sergei Sobolev in the early 1930s. Sobolev spaces provide a framework for studying the regularity of functions, particularly those involved in solutions to partial differential equations.

The idea behind Sobolev spaces is to introduce a notion of "weak" derivatives for functions that may not have classical derivatives. In classical calculus, the derivative of a function measures how it changes at a given point. However, for functions that are not sufficiently smooth, classical derivatives may not exist. Sobolev spaces address this issue by considering weak derivatives, which are defined in a distributional sense.

Weak derivatives

Definition 33. Let Ω be an open subset of $\mathbb{R}^n$ and $f \in L^1_{loc}(\Omega)$. We say that f is weakly differentiable with respect to x_i if there exists $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega f \partial_i \phi dx = -\int_\Omega g_i \phi dx, \quad \forall \phi \in C_c^\infty(\Omega),$$

where g_i is considered to be the weak i^{th} derivative of f.

We can generalise the definition of the weak derivative by introducing the multi-index $\alpha \in \mathbb{N}_0^n$, then by applying integration by parts α times, we have

$$\int_\Omega (\partial^\alpha f)\phi dx = (-1)^{|a|} \int_\Omega f(\partial^\alpha \phi) dx, \quad \forall \phi \in C_c^\infty(\Omega).$$

The idea is to "move the derivatives from f to ϕ".

Example 10. Let us consider the function $f \in C(\mathbb{R})$ with

$$f(x) = \begin{cases} x, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

We consider the following integral for $\phi \in C_c^\infty(\mathbb{R})$

$$\begin{aligned} \int_\mathbb{R} f(x)\phi' dx &= \int_0^\infty x \cdot \phi' dx + \int_{-\infty}^0 0 \cdot \phi' dx \\ &= -\int_0^\infty 1 \cdot \phi dx - \int_{-\infty}^0 0 \cdot \phi dx \\ &= -\int_\mathbb{R} \chi(x)\phi dx \Leftrightarrow f'(x) = \chi(x) \end{aligned}$$

where $\chi(x)$ is the step function

$$\chi(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Distributions

In Sobolev spaces, distributions refer to generalized functions or functionals that act on a space of test functions.

Definition 34. A sequence $\{\phi_n \in C_c^\infty(\Omega) : n \in \mathbb{N}\}$ converges to $\phi \in C_c^\infty(\Omega)$ in the sence of thest functions if:
a) $\exists \Omega' \Subset \Omega : \text{supp}\phi_n \subset \Omega', \forall n \in \mathbb{N}$,
b) $\partial^\alpha \phi_n \to \partial^\alpha \phi$ as $n \to \infty$ uniformly on Ω for every $\alpha \in \mathbb{N}_0^n$
The topological vector space $\mathcal{D}(\Omega)$ consists of $C_c^\infty(\Omega)$ equipped with the topology that corresponds to convergence in the sense of test functions.

Definition 35. A distribution on Ω is a continuous linear functional

$$T : \mathcal{D}(\Omega) \to \mathbb{R}.$$

A sequence $\{T_n : n \in \mathbb{N}\}$ of distributions weakly converges to a distribution T, if $\langle T_n, \phi \rangle \to \langle T, \phi \rangle$, $\forall \phi \in \mathcal{D}(\Omega)$. The topological vector space $\mathcal{D}'(\Omega)$ consists of the distributions on Ω equipped with the topology corresponding to this notion of convergence.

Corollary 36. The space of distributions is the topological dual of the space of test functions.

Example 11. The delta-function supported at $\alpha \in \Omega$ is the distribution

$$\delta_\alpha : \mathcal{D}(\Omega) \to \mathbb{R}$$

defined by evaluation of a test function at α:

$$\langle \delta_\alpha, \phi \rangle = \phi(\alpha)$$

Example 12. Any function $f \in L^1_{loc}(\Omega)$ defines a $T_f \in \mathcal{D}'(\Omega)$ by

$$\langle T_f, \phi \rangle = \int_\Omega f \phi dx.$$

The spaces $W^{k,p}$ and $W^{s,p}$

Sobolev spaces consist of functions whose weak derivatives belong to L^p.

Definition 37. Let Ω be an open subset of $\mathbb{R}^n$, $k \in \mathbb{N}$ and $1 \le p \le \infty$. The Sobolev space $W^{k,p}(\Omega)$ consists of all locally integrable functions $f : \Omega \to \mathbb{R}$ such that $\partial^\alpha f \in L^p(\Omega)$. The sobolev space $W^{k,p}(\Omega)$ is a Banach space with the norm

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha f|^p dx \right)^{1/p}, \quad 1 \le p < \infty$$

$$\|f\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha| \le k} \sup_\Omega |\partial^\alpha f|, \quad p = \infty$$

The space $H^k(\Omega) = W^{k,2}(\Omega)$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \sum_{|\alpha| \le k} \int_\Omega (\partial^\alpha f)(\partial^\alpha g) dx$$

Generalization for $k \to s \in \mathbb{R}$:

$$\|f\|_{W^{s,p}(\Omega)} = \left(\|f\|^p_{W^{k,p}(\Omega)} + \sum_{|\alpha| = k} \int_\Omega \int_\Omega \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|^p}{|x - y|^{n + p\mu}} dx dy \right)^{1/p},$$

where $s = k + \mu$, $k \in \mathbb{N}$ and $\mu \in (0, 1)$.

4.2 Borg-Levinson theorem

This section deals with the elements of multidimensional Borg-Levinson inverse theory. Its main purpose is to establish that the Dirichlet eigenvalues and Neumann boundary data of the operator $-\Delta + q$, acting in a bounded domain of $\mathbb{R}^d$ with $d \ge 2$, uniquely determine the real-valued bounded potential q [70].

Borg (1946) and Levinson (1949) have provided us with the following result

Theorem 38. For $\lambda \in \mathbb{R}$ and for $q_j \in L^\infty(0,1;\mathbb{R})$, $j = 1, 2$, let $u_j(\cdot, \lambda)$ be the $H^2(0,1)-$solution to the initial values problem

$$\begin{array}{cc} (-\Delta + q_j(x))u_j(x,\lambda) = \lambda u_j(x,\lambda) & x \in (0,1) \\ u_j(0,\lambda) = 0, & u_j'(0,\lambda) = 1. \end{array} \tag{4.2.1}$$

Denote by $\{\lambda_{j,n},\ n \in \mathbb{N}\}$ the non-decreasing sequence of the Dirichlet eigenvalues associated with $A_q = -\Delta + q$, obtained by imposing $u_j(1, \lambda_{j,n}) = 0,\ n \in \mathbb{N}$.Then, we have the implication:

$$\big(\lambda_{1,n} = \lambda_{2,n} \text{ and } \|u_1(\cdot, \lambda_{1,n})\|_{L^2(0,1)} = \|u_2(\cdot, \lambda_{2,n})\|_{L^2(0,1)},\ n \in \mathbb{N}\big) \Longrightarrow (q_1 = q_2 \text{ in } (0,1)).$$

where the uniqueness can also be achieved if we replace $\|u_j(\cdot, \lambda_{j,n})\|_{L^2(0,1)}$, $j = 1, 2$ with $u_j'(1, \lambda)$ (I. M. Gel'fand and B. M. Levitan).

Theorem 39. Suppose that conditions of *Theorem 24* are satisfied, then

$$\big(\lambda_{1,n} = \lambda_{2,n} \text{ and } u_j'(1, \lambda_{1,n}) = u_j'(1, \lambda_{2,n}),\ n \in \mathbb{N}\big) \Longrightarrow (q_1 = q_2 \text{ in } (0,1)).$$

4.3 Application of Gauss Error Function on $\widehat{K_\gamma}(\eta)$

In section 2.1.3 we had to calculate the Fourier transform of

$$\widehat{K_\gamma}(\eta) = \sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\gamma}{2}t^2} e^{-2\pi i \eta t}\, dt$$

where we will apply Gauss error function over $\mathbb{R}$ [58]. To do this first let us focus on the integral

$$I = \int e^{-\gamma t^2/2 - 2i\pi\eta t}\, dt$$

by completing the square, one can obtain

$$I = \int e^{\left(\frac{\sqrt{-\gamma}t}{\sqrt{2}} - \frac{-2i\pi\eta}{\sqrt{-\gamma}}\right)^2 - \frac{2\pi^2\eta^2}{\gamma}}\, dt$$

and make the substitution $u = \frac{-\gamma t - 2i\pi\eta}{\sqrt{2\gamma}} \to du = -\sqrt{\frac{\gamma}{2}}dt$, we have

$$I = -\sqrt{\frac{\pi}{2\gamma}} e^{-\frac{2\pi^2\eta^2}{\gamma}} \int \frac{2e^{-u^2}}{\sqrt{\pi}}\, du$$

We can see that the indefinite integral

$$I_1 = \int \frac{2e^{-u^2}}{\sqrt{\pi}}\, du$$

is the Gauss error function $\operatorname{erf}(u)$. Therefore. Therefore,

$$I = -\sqrt{\frac{\pi}{2\gamma}} e^{-\frac{2\pi^2\eta^2}{\gamma}} \operatorname{erf}(u)$$

or

$$I = -\sqrt{\frac{\pi}{2\gamma}} e^{-\frac{2\pi^2\eta^2}{\gamma}} \operatorname{erf}\left(\frac{-\gamma t - 2i\pi\eta}{\sqrt{2\gamma}}\right)$$

Hence,

$$\widehat{K_\gamma}(\eta) = -\frac{1}{2} e^{-\frac{2\pi^2\eta^2}{\gamma}} \left[\operatorname{erf}\left(\frac{-\gamma t - 2i\pi\eta}{\sqrt{2\gamma}}\right)\right]_{\mathbb{R}} = e^{-\frac{2\pi^2\eta^2}{\gamma}}$$

4.4 Bolzano's theorem with an example for finding numerically a root for a continuous function - bisection method

If a continuous function defined on an interval is sometimes positive and sometimes negative, it must be 0 at some point [81].

Bolzano (1817) proved the theorem, which effectively proves the general case of intermediate value theorem [2], using techniques which were considered especially rigorous for his time, but which are regarded as nonrigorous in modern times (Grabiner 1983).

If f is continuous on a closed interval $[a, b]$, and c is any number between $f(a)$ and $f(b)$ inclusive, then there is at least one number x in the closed interval such that $f(x) = c$. The theorem is proven by observing that $f([a, b])$ is connected because the image of a connected set under a continuous function is connected, where $f([a, b])$ denotes the image of the interval $[a, b]$ under the function f. Since c is between $f(a)$ and $f(b)$, it must be in this connected set.

Suppose now that we have guaranteed that there exists a root x_0 for a function $f = f(x)$ on the interval I such that $f(x_0) = 0$. Where exactly is $x_0 \in I$? We cannot always find precise the root x_0 for every function f, but we can use an approximation method by applying consecutive Bolzano's method on smaller intervals that are contained in I such that $I_k \subset I_{k-1} \subset \ldots \subset I_1 \subset I$, where k is the k^{th} bolzano theorem application on a smaller interval than the previous application. Then by having smaller and smaller intervals we can approximate the value $x_0 \in I_k \subset I$.

This method, also called bisection method, is used in chapter 3 for finding the eigenvalues to our differential equation for a continuous function, however it has a slow convergence to x_0. In every step the error $\epsilon = |x_0 - x_n|$ to this method is cut in half of the previous step [60]:

$$\epsilon_{n+1} = \frac{\epsilon_n}{2} = \frac{\epsilon_{n-1}}{2^2} = \ldots = \frac{\epsilon_0}{2^n}$$

where x_n is the approximation of x_0 in the n^{th} step. Suppose that we want to achieve a specific accuracy, say E, of x_0, then the number of steps n that we need is given by the formula:

$$n = \log_2 \frac{\epsilon_0}{E}$$

The slow convergence didn't bother us where we had to find the eigenvalues in chapter 3, because we didn't applied the method exactly as it it stated, but we borrowed the idea to use it for the graph that we plotted from mathematica. We found the eigenvalues by zooming in the interval I of our interest and we continued to zoom in until mathematica didn't allow us to zoom even further due to the lack of memory.

Let us give an example where we apply bisection method exactly as it is stated

Example 13. Let us consider the function $f(x) = x^3 - x - 1$, we are going to use bisection method to find a root for the function.

1^{st} iteration:

Take $I_1 = [1, 2]$, then $f(1) = -1 < 0$ and $f(2) = 5 > 0$. By Bolzano's theorem the root lies between 1 and 2 and we consider

$$x_1 = \frac{1 + 2}{2} = 1.5$$

then

$$f(x_1) = 0.875 > 0$$

2^{nd} iteration:

Take $I_2 = [1, 1.5]$, then $f(1) = -1 < 0$ and $f(1.5) = 0.875 > 0$. By Bolzano's theorem the root lies between 1 and 1.5 and we consider

$$x_2 = \frac{1 + 1.5}{2} = 1.25$$

then

$$f(x_2) = -0.29688 < 0$$

3^{rd} iteration:

Take $I_3 = [1.25, 1.5]$, then $f(1.25) = -0.29688 < 0$ and $f(1.5) = 0.875 > 0$. By Bolzano's theorem the root lies between 1.25 and 1.5 and we consider

$$x_3 = \frac{1.25 + 1.5}{2} = 1.375$$

then

$$f(x_3) = 0.22461 > 0$$

4^{th} iteration:

Take $I_4 = [1.25, 1.375]$, then $f(1.25) = -0.29688 < 0$ and $f(1.375) = 0.22461 > 0$. By Bolzano's theorem the root lies between 1.25 and 1.375 and we consider

$$x_4 = \frac{1.25 + 1.375}{2} = 1.3125$$

then

$$f(x_4) = -0.05151 < 0$$

5^{th} iteration:

Take $I_5 = [1.3125, 1.375]$, then $f(1.3125) = -0.05151 < 0$ and $f(1.375) = 0.22461 > 0$. By Bolzano's theorem the root lies between 1.3125 and 1.375 and we consider

$$x_5 = \frac{1.3125 + 1.375}{2} = 1.34375$$

then

$$f(x_5) = 0.08261 > 0$$

6^{th} iteration:

Take $I_6 = [1.3125, 1.34375]$, then $f(1.3125) = -0.05151 < 0$ and $f(1.34375) = 0.08261 > 0$. By Bolzano's theorem the root lies between 1.3125 and 1.34375 and we consider

$$x_6 = \frac{1.3125 + 1.34375}{2} = 1.32812$$

then

$$f(x_6) = 0.01458 > 0$$

7^{th} iteration:

Take $I_7 = [1.3125, 1.32812]$, then $f(1.3125) = -0.05151 < 0$ and $f(1.32812) = 0.01458 > 0$. By Bolzano's theorem the root lies between 1.3125 and 1.32812 and we consider

$$x_7 = \frac{1.3125 + 1.32812}{2} = 1.32031$$

then

$$f(x_7) = -0.01871 < 0$$

8^{th} iteration:

Take $I_8 = [1.32031, 1.32812]$, then $f(1.32031) = -0.01871 < 0$ and $f(1.32812) = 0.01458 > 0$. By Bolzano's theorem the root lies between 1.32031 and 1.32812 and we consider

$$x_8 = \frac{1.32031 + 1.32812}{2} = 1.32422$$

then

$$f(x_8) = -0.00213 < 0$$

9^{th} iteration:

Take $I_9 = [1.32422, 1.32812]$, then $f(1.32422) = -0.00213 < 0$ and $f(1.32812) = 0.01458 > 0$. By Bolzano's theorem the root lies between 1.32422 and 1.32812 and we consider

$$x_9 = \frac{1.32422 + 1.32812}{2} = 1.32617$$

then

$$f(x_9) = 0.00621 > 0$$

10^{th} iteration:

Take $I_{10} = [1.32422, 1.32617]$, then $f(1.32422) = -0.00213 < 0$ and $f(1.32617) = 0.00621 > 0$. By Bolzano's theorem the root lies between 1.32422 and 1.32617 and we consider

$$x_{10} = \frac{1.32422 + 1.32617}{2} = 1.3252$$

then

$$f(x_{10}) = 0.00204 > 0$$

11^{th} iteration:

Take $I_{11} = [1.32422, 1.3252]$, then $f(1.32422) = -0.00213 < 0$ and $f(1.3252) = 0.00204 > 0$. By Bolzano's theorem the root lies between 1.32422 and 1.3252 and we consider

$$x_{11} = \frac{1.32422 + 1.3252}{2} = 1.32471$$

then

$$f(x_{11}) = -0.00005 < 0$$

Approximate root of the equation $x^3 - x - 1 = 0$ using Bisection method is $x_{11} = 1.32471$ with an accuracy of 4 decimals.

n	a	$f(a)$	b	$f(b)$	$c = \frac{a+b}{2}$	$f(c)$	Update
1	1	-1	2	5	1.5	0.875	$b = c$
2	1	-1	1.5	0.875	1.25	-0.29688	$a = c$
3	1.25	-0.29688	1.5	0.875	1.375	0.22461	$b = c$
4	1.25	-0.29688	1.375	0.22461	1.3125	-0.05151	$a = c$
5	1.3125	-0.05151	1.375	0.22461	1.34375	0.08261	$b = c$
6	1.3125	-0.05151	1.34375	0.08261	1.32812	0.01458	$b = c$
7	1.3125	-0.05151	1.32812	0.01458	1.32031	-0.01871	$a = c$
8	1.32031	-0.01871	1.32812	0.01458	1.32422	-0.00213	$a = c$
9	1.32422	-0.00213	1.32812	0.01458	1.32617	0.00621	$b = c$
10	1.32422	-0.00213	1.32617	0.00621	1.3252	0.00204	$b = c$
11	1.32422	-0.00213	1.3252	0.00204	1.32471	-0.00005	$a = c$

4.5 An example on the Poincare-Linstedt method

In perturbation theory, the Poincare-Linstedt method is mostly used to find periodic solutions to a differential equation or in system of equations. The method is based on the assumption that the solution for the equation

$$\partial_t x = f(x, t; \epsilon), \quad 0 \leq |\epsilon| \ll 1$$

is expressed as

$$x(t) = \sum_{k=0}^{\infty} \epsilon^k x_k(t) = x_0(t) + \epsilon x_1(t) + \ldots + \epsilon^n x_n(t) + \ldots$$

where the first term x_0 is the solution to the known problem without ϵ.

Example 14. Let us consider a problem from classical mechanics, the equation of the non-harmonic oscillator, also known as the Duffing equation [43]

$$m\partial_t^2 x + k(x - \alpha x^3) = 0, \ x \in \mathbb{R}$$

where k is the spring constant, m is the mass of the oscillator and α is a coefficient that determines the non-linearity of the spring, with $|a| \ll 1$. By setting as $\omega_0^2 = k/m$ and $\epsilon = \omega_0^2 \alpha \ll 1$ we obtain

$$\partial_t^2 x + \omega_0^2 x = \epsilon x^3 \tag{4.5.1}$$

We assume that the solution to the equation (4.5.1) is given by

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \ldots$$

and by substituting the terms ϵ^k, we obtain the following system of equations

$$\begin{array}{ll} \epsilon^0 : & \partial_t^2 x_0 + \omega_0^2 x_0 = 0 \\ \epsilon^1 : & \partial_t^2 x_1 + \omega_0^2 x_1 = x_0^3 \\ \epsilon^2 : & \partial_t^2 x_2 + \omega_0^2 x_2 = 3x_0^2 x_1 \\ \epsilon^3 : & \partial_t^2 x_3 + \omega_0^2 x_3 = 3x_0^2 x_2 + 3x_0 x_1^2 \\ \ldots & \end{array}$$

By solving the first equation we have

$$x_0(t) = A_0 \cos \omega_0 t + B_0 \sin \omega_0 t$$

where for our oscillator, without loss of generality, we may consider the initial conditions $x(0) = x_0(0) = A$ and $\partial_t x(0) = \partial_t x_0(0) = 0$, therefore

$$x_0(t) = A\cos\omega_0 t$$

We apply a Fourier transformation on $x_0(t)$ and substituting it to the next equation we have

$$\partial_t^2 x_1 + \omega_0^2 x_1 = \frac{A^3}{4}(3\cos\omega_0 t + \cos 3\omega_0 t)$$

and we obtain the solution

$$x_1(t) = A_1\cos\omega_0 t + B_1\sin\omega_0 t + \frac{3A^3}{8\omega_0} t\sin\omega_0 t - \frac{A^3}{32\omega_0^2}\cos 3\omega_0 t$$

where we have to address the fact we have the term $t\sin\omega_0 t$ that breaks our periodicity. These terms are called secular terms [44] and are the terms that gives us different results than what we expected to obtain. The presence of secular terms tells us that if the series converges, it will be only for a finite period of time

$$0 \leq t \ll 1/\epsilon$$

To obtain convergence for every $t \in \mathbb{R}$, one must to eliminate terms that are proportional to $\cos\omega_0 t$ or $\sin\omega_0 t$. Poincare and Linstedt proposed that we have to assume that the frequency ω is a unknown parameter and expand it to a series of powers of ϵ.

Let us consider

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots$$

Therefore we write (4.5.1) as

$$\partial_t^2 x + \omega^2 x = (\omega^2 - \omega_0^2)x + \epsilon x^3$$

Then we obtain the following system of equations

$$\begin{aligned}
&\partial_t^2 x_0 + \omega_0^2 x_0 = 0\\
&\partial_t^2 x_1 + \omega_0^2 x_1 = (2\omega_0\omega_1)x_0 + x_0^3\\
&\partial_t^2 x_2 + \omega_0^2 x_2 = (2\omega_0\omega_2 + \omega_1^2)x_0 + 2\omega_0\omega_1 x_1 + 2x_0^2 x_1\\
&\dots
\end{aligned}$$

By solving every differential equation we obtain the solution

$$x(t) = A\cos\omega t + \epsilon\frac{A^3}{32\omega^2}(\cos\omega t - \cos 3\omega t) + \dots$$

where

$$\omega = \omega_0 - \frac{3A^2}{8\omega_0}\epsilon + \dots$$

where the secular terms were cancelled out and we have provided periodicity of the solution $x(t)$ with frequency ω that depends on the initial values.

4.6 Mathematica codes

```
In[29]:= X = (-1)^n / (2^(2 n) (n!)^2) + (10^(-3) / λ) (λ^2 A - 2 λ B + C) / (2 n)^2;
         Y = (-1)^(n + 1) / (2^(2 (n + 1)) ((n + 1)!)^2) + (10^(-3) / λ) (λ^2 X - 2 λ A + B) / (2 (n + 1))^2;
         Y - X
```

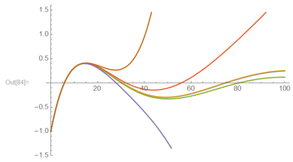
Out[31]= $$-\frac{C-2B\lambda+A\lambda^2}{4000\,n^2\,\lambda}+\frac{B-2A\lambda+\lambda^2\left(\frac{C-2B\lambda+A\lambda^2}{4000\,n^2\,\lambda}+\frac{(-1)^n\,2^{-2n}}{(n!)^2}\right)}{4000\,(1+n)^2\,\lambda}-\frac{(-1)^n\,2^{-2n}}{(n!)^2}+\frac{(-1)^{1+n}\,2^{-2(1+n)}}{((1+n)!)^2}$$

In[33]:= $$\text{Limit}\left[\frac{C-2B\lambda+A\lambda^2}{4000\,n^2\,\lambda}+\frac{B-2A\lambda+\lambda^2\left(\frac{C-2B\lambda+A\lambda^2}{4000\,n^2\,\lambda}+\frac{(-1)^n\,2^{-2n}}{(n!)^2}\right)}{4000\,(1+n)^2\,\lambda}-\frac{(-1)^n\,2^{-2n}}{(n!)^2}+\frac{(-1)^{1+n}\,2^{-2(1+n)}}{((1+n)!)^2},\ n\to\text{Infinity}\right]$$

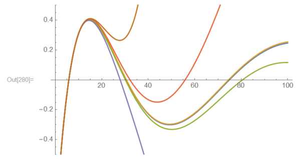
Out[33]= 0

Figure 4.1: Numerical investigation for the difference of consecutive terms c_{2n} and $c_{2(n+1)}$ using Mathematica.

```
f1 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1);
f2 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1 + x^2 d2);
f3 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1 + x^2 d2 + x^3 d3);
f4 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1 + x^2 d2 + x^3 d3 + x^4 d4);
f5 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1 + x^2 d2 + x^3 d3 + x^4 d4 + x^5 d5);
f6 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1 + x^2 d2 + x^3 d3 + x^4 d4 + x^5 d5 + x^6 d6);
Plot[{f1, f2, f3, f4, f5, f6}, {x, 0, 100}]
```



```
f1 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1);
f2 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1 + x^2 d2);
f3 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1 + x^2 d2 + x^3 d3);
f4 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1 + x^2 d2 + x^3 d3 + x^4 d4);
f5 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1 + x^2 d2 + x^3 d3 + x^4 d4 + x^5 d5);
f6 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1 + x^2 d2 + x^3 d3 + x^4 d4 + x^5 d5 + x^6 d6);
Plot[{f1, f2, f3, f4, f5, f6}, {x, 0, 100}, PlotRange → {-0.5, 0.5}]
```



```
f1 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1);
f2 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1 + x^2 d2);
f3 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1 + x^2 d2 + x^3 d3);
f4 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1 + x^2 d2 + x^3 d3 + x^4 d4);
f5 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1 + x^2 d2 + x^3 d3 + x^4 d4 + x^5 d5);
f6 = -BesselJ[0, Sqrt[x]] - 10^(-3) / x^3 (x d1 + x^2 d2 + x^3 d3 + x^4 d4 + x^5 d5 + x^6 d6);
Plot[{f1, f2, f3, f4, f5, f6}, {x, 0, 10000}, PlotRange → {-1, 1}]
```

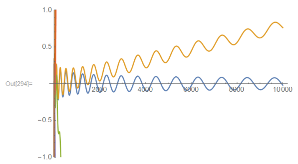


Figure 4.2: Plot of function of interest for different intervals of λ. We observe that there are two functions that behave differently than the other functions and these are the functions of interest with $N = 1$ and $N = 2$, because the term $1/\lambda^3$ outweighs the polynomial terms which are at most degree 2.

```
In[34]:= c1 = (-1) ^1 / (2^ (2 * 1) (1 !) ^2) // N
         c2 = (-1) ^2 / (2^ (2 * 2) (2 !) ^2) // N
         c3 = (-1) ^3 / (2^ (2 * 3) (3 !) ^2) // N
         c4 = (-1) ^4 / (2^ (2 * 4) (4 !) ^2) // N
         c5 = (-1) ^5 / (2^ (2 * 5) (5 !) ^2) // N
         c6 = (-1) ^6 / (2^ (2 * 6) (6 !) ^2) // N
Out[34]= -0.25
Out[35]= 0.015625
Out[36]= -0.000434028
Out[37]= 6.78168 × 10^-6
Out[38]= -6.78168 × 10^-8
Out[39]= 4.7095 × 10^-10
```

```
In[7]:= d2 = 0.25 x^2 // N
        d4 = Expand[-d2 / (2 * 2) + 1 / (2 * 2) ^2 (x^2 c2 - 2 x)  // N]
        d6 = Expand[-d4 / (2 * 3) + 1 / (2 * 3) ^2 (x^2 c4 - 2 x c2 + 1)  // N]
        d8 = Expand[-d6 / (2 * 4) + 1 / (2 * 4) ^2 (x^2 c6 - 2 x c4 + c2)  // N]
        d10 = Expand[-d8 / (2 * 5) + 1 / (2 * 5) ^2 (x^2 c8 - 2 x c6 + c4)  // N]
        d12 = Expand[-d10 / (2 * 6) + 1 / (2 * 6) ^2 (x^2 c10 - 2 x c8 + c6)  // N]
        d14 = Expand[-d12 / (2 * 7) + 1 / (2 * 7) ^2 (x^2 c12 - 2 x c10 + c8)  // N]
Out[7]= 0.25 x^2
Out[8]= -0.125 x - 0.078125 x^2
Out[9]= 0.0277778 + 0.0347222 x + 0.0134549 x^2
Out[10]= -0.00737847 - 0.00482856 x - 0.00168864 x^2
Out[11]= 0.000894097 + 0.000491536 x + 0.000168932 x^2
Out[12]= -0.0000775222 - 0.0000410556 x - 0.0000140781 x^2
Out[13]= 5.5719 × 10^-6 + 2.93323 × 10^-6 x + 1.00558 × 10^-6 x^2

In[14]:= Plot[{d2, d4, d6, d8, d10, d12}, {x, 0, 1}]
Out[14]=
```

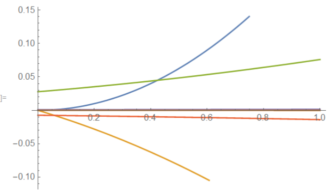


Figure 4.3: Numerical determination of $c_{2k}^{(0)}$ and $\gamma_{2k}(\lambda) = c_{2k}^{(1)}$, $k \in \{1, 2, 3, 4, 5, 6\}$

```
In[41]:= functionM[x_] := (0.25 x^2) x + (-0.125` x - 0.078125` x^2) x^2 + (0.02777777777777776` + 0.03472222222222224` x + 0.0134548611111111` x^2) x^3 +
    (-0.00737847222222222` - 0.004828559027777778` x - 0.0016886393229166665` x^2) x^4 + (0.0008940972222222222` + 0.0004915364583333333` x + 0.00016893174913194445` x^2) x^5 +
    (-0.0000775221836419753` - 0.00004105556158371913` x - 0.000014078116711275077` x^2) x^6 + (5.571899260282816` * 10^(-6) + 2.933232121697058` * 10^(-6) x + 1.0055821678986265` * 10^(-6) x^2) x^7;
  functionM[4.77983] // N
  functionM[21.3994] // N
  functionM[467.702] // N
  functionM[2494.89] // N
  functionM[6452.08] // N
  functionM[12376.8] // N
  functionM[4.77953] // N
  functionM[24.320] // N
  functionM[467.702] // N
  functionM[2494.89] // N
  functionM[6452.08] // N
  functionM[12376.8] // N
Out[42]= 4.60963
Out[43]= 641606.
Out[44]= 1.05194 × 10^18
Out[45]= 3.74935 × 10^24
Out[46]= 1.94523 × 10^28
Out[47]= 6.84706 × 10^30
Out[48]= 4.60886
Out[49]= 2.10219 × 10^6
Out[50]= 1.05194 × 10^18
Out[51]= 3.74935 × 10^24
Out[52]= 1.94523 × 10^28
Out[53]= 6.84706 × 10^30
```

Figure 4.4: Numerical calculation for the function $\mathcal{M}(\lambda)$ using Mathematica.

```
In[102]:= B1[x_] := -(BesselJ[0, Sqrt[x]] + 10^(-3) / x^3 * functionM[x]) / ((BesselY[0, 1.1 Sqrt[x]]) / (BesselJ[0, 1.1 Sqrt[x]]) - BesselY[0, Sqrt[x]])
          B1[4.77983] // N
          B1[21.3994] // N
          B1[467.702] // N
          B1[2494.89] // N
          B1[6452.08] // N
          B1[12376.8] // N

Out[103]= 0.0000102025
Out[104]= 0.0936267
Out[105]= -7.21256 × 10^6
Out[106]= -2.34268 × 10^11
Out[107]= 2.12371 × 10^14
Out[108]= 2.75546 × 10^15

In[109]:= B2[x_] := -(BesselJ[0, Sqrt[x]] + 10^(-4) / x^3 * functionM[x]) / ((BesselY[0, 1.1 Sqrt[x]]) / (BesselJ[0, 1.1 Sqrt[x]]) - BesselY[0, Sqrt[x]])
          B2[4.77953] // N
          B2[24.320] // N
          B2[467.702] // N
          B2[2494.89] // N
          B2[6452.08] // N
          B2[12376.8] // N
Out[110]= 1.13284 × 10^-6
Out[111]= 0.0172908
Out[112]= -721256.
Out[113]= -2.34268 × 10^10
Out[114]= 2.12371 × 10^13
```

Figure 4.5: Numerical calculation for the coefficient B using Mathematica.

```
In[116]:= A1[x_] := -B1[x] (BesselY[0, 1.1 Sqrt[x]]) / (BesselJ[0, 1.1 Sqrt[x]])
          A1[4.77983] // N
          A1[21.3994] // N
          A1[467.702] // N
          A1[2494.89] // N
          A1[6452.08] // N
          A1[12 376.8] // N
          A2[x_] := -B2[x] (BesselY[0, 1.1 Sqrt[x]]) / (BesselJ[0, 1.1 Sqrt[x]])
          A2[4.77953] // N
          A2[24.320] // N
          A2[467.702] // N
          A2[2494.89] // N
          A2[6452.08] // N
          A2[12 376.8] // N
```

Out[117]= 0.118039

Out[118]= -0.202349

Out[119]= 1.14144×10^{7}

Out[120]= 2.17812×10^{11}

Out[121]= 8.83114×10^{13}

Out[122]= 3.71297×10^{15}

Out[124]= 0.118044

Out[125]= -0.179985

Out[126]= 1.14144×10^{6}

Out[127]= 2.17812×10^{10}

Out[128]= 8.83114×10^{12}

Out[129]= 3.71297×10^{14}

Figure 4.6: Numerical calculation for the coefficient A using Mathematica.

```
In[1]:= difphi1[x_, y_] := Sqrt[y] BesselJ[1, Sqrt[y] r] - Sqrt[x] BesselJ[1, Sqrt[x] r] + 10^(-4) (2 (10*x (0.25 x^2) - y (0.25 y^2)) r^1
          + 4 (10*x (-0.125 x - 0.078125 x^2) - y (-0.125` y - 0.078125` y^2)) r^3
          + 6 (10*x (0.027777777777777776` + 0.034722222222222224` x + 0.013454861111111111` x^2) - y (0.027777777777777776` + 0.034722222222222224` y + 0.013454861111111111` y^2)) r^5
          + 8 (10*x (-0.007378472222222222` - 0.004828559027777778` x - 0.0016886393229166665` x^2) - y (-0.007378472222222222` - 0.004828559027777778` y - 0.0016886393229166665` y^2)) r^7
          + 10 (10*x (0.0008940972222222222` + 0.0004915364583333333` x + 0.00016893174913194445` x^2) - y (0.0008940972222222222` + 0.0004915364583333333` y + 0.00016893174913194445` y^2)) r^9
          + 12 (10*x (-0.0000775221836419753` - 0.00004105556158371913` x - 0.000014078116711275077` x^2) - y (-0.0000775221836419753` - 0.00004105556158371913` y - 0.000014078116711275077` y^2)) r^11
          + 14 (10*x (5.571899260282816`*^-6 + 2.933232121697058`*^-6 x + 1.0055821678986265`*^-6 x^2) - y (5.571899260282816`*^-6 + 2.933232121697058`*^-6 y + 1.0055821678986265`*^-6 y^2)) r^13)
      dif11[r_] := Expand[difphi1[4.77983, 4.77953] // N]
      dif12[r_] := Expand[difphi1[21.3994, 24.320] // N]
      dif13[r_] := Expand[difphi1[467.702, 467.702] // N]
      dif14[r_] := Expand[difphi1[2494.89, 2494.89] // N]
      dif15[r_] := Expand[difphi1[6452.08, 6452.08] // N]
      dif16[r_] := Expand[difphi1[12376.8, 12376.8] // N]

In[9]:= dif11[r_]
        dif12[r_]
        dif13[r_]
        dif14[r_]
        dif15[r_]
        dif16[r_]
Out[9]= 0.0491427 r - 0.0409954 r^3 + 0.0129353 r^5 - 0.00237597 r^7 + 0.00030557 r^9 - 0.0000307364 r^11 + 2.56365×10^-6 r^13 + 2.18621 BesselJ[1., 2.18621 r] - 2.18628 BesselJ[1., 2.18628 r]
Out[10]= 4.18054 r - 2.81223 r^3 + 0.761225 r^5 - 0.129475 r^7 + 0.0162543 r^9 - 0.00162661 r^11 + 0.000135564 r^13 - 4.62595 BesselJ[1., 4.62595 r] + 4.93153 BesselJ[1., 4.93153 r]
Out[11]= 0. + 46038.4 r - 28872.4 r^3 + 7474.37 r^5 - 1251.51 r^7 + 156.518 r^9 - 15.6526 r^11 + 1.30439 r^13
Out[12]= 0. + 6.98822×10^6 r - 4.37044×10^6 r^3 + 1.12947×10^6 r^5 - 189026. r^7 + 23638.2 r^9 - 2363.91 r^11 + 196.993 r^13
Out[13]= 0. + 1.20868×10^8 r - 7.55613×10^7 r^3 + 1.9523×10^7 r^5 - 3.26709×10^6 r^7 + 408553. r^9 - 40856.8 r^11 + 3404.74 r^13
Out[14]= 0. + 8.53174×10^8 r - 5.33303×10^8 r^3 + 1.37781×10^8 r^5 - 2.30566×10^7 r^7 + 2.88324×10^6 r^9 - 288334. r^11 + 24027.9 r^13

In[15]:= difphi2[x_, y_, a1_, b1_, a2_, b2_] := a2 Sqrt[y] BesselJ[1, Sqrt[y] r] + b2 Sqrt[y] BesselY[1, Sqrt[y] r] - a1 Sqrt[x] BesselJ[1, Sqrt[x] r] - b1 Sqrt[x] BesselY[1, Sqrt[x] r]
         dif21[r_] := Expand[difphi2[4.77983, 4.77953, 0.118039, 0.0000102025, 0.118044, 1.13284*10^(-6)] // N]
         dif22[r_] := Expand[difphi2[21.3994, 24.320, -0.202349, 0.0936267, -0.179985, 0.0172908] // N]
         dif23[r_] := Expand[difphi2[467.702, 467.702, 1.14144*10^7, -7.21256*10^6, 1.14144*10^6, -7.21256*10^5] // N]
         dif24[r_] := Expand[difphi2[2494.89, 2494.89, 2.17812*10^11, -2.34268*10^11, 2.17812*10^10, -2.34268*10^10] // N]
         dif25[r_] := Expand[difphi2[6452.08, 6452.08, 8.83114*10^13, 2.12371*10^14, 8.83114*10^12, 2.12371*10^13] // N]
         dif26[r_] := Expand[difphi2[12376.8, 12376.8, 3.71297*10^15, 2.75546*10^15, 3.71297*10^14, 2.75546*10^14] // N]

In[22]:= dif21[r_]
         dif22[r_]
         dif23[r_]
         dif24[r_]
         dif25[r_]
         dif26[r_]
Out[22]= 0.258069 BesselJ[1., 2.18621 r] - 0.258067 BesselJ[1., 2.18628 r] + 2.47663×10^-6 BesselY[1., 2.18621 r] - 0.0000223055 BesselY[1., 2.18628 r]
Out[23]= 0.936056 BesselJ[1., 4.62595 r] - 0.887602 BesselJ[1., 4.93153 r] - 0.433112 BesselY[1., 4.62595 r] + 0.0852701 BesselY[1., 4.93153 r]
Out[24]= -2.22167×10^8 BesselJ[1., 21.6264 r] + 1.40384×10^8 BesselY[1., 21.6264 r]
Out[25]= -9.79152×10^12 BesselJ[1., 49.9489 r] + 1.05313×10^13 BesselY[1., 49.9489 r]
Out[26]= -6.38424×10^15 BesselJ[1., 80.3248 r] - 1.53528×10^16 BesselY[1., 80.3248 r]
Out[27]= -3.71765×10^17 BesselJ[1., 111.251 r] - 2.75893×10^17 BesselY[1., 111.251 r]
```

Figure 4.7: Numerical evaluation of the differences $\partial_\nu \phi_{q_1} - \partial_\nu \phi_{q_2}$, for $r \in [0, 1]$ and $r \in [1, 1.1]$.

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