Master of Science: 'Physics and Technological Applications'

National Technical University of Athens (NTUA) in collaboration with the National Center for scientific Research 'Demokritos'

Master Thesis

Magnetic Monopoles

Alexandros Zarafonitis
Supervisor: Prof. Nick Mavromatos

Abstract

In this master thesis we present the idea of magnetic monopoles in physics. Motivation of such a concept is given mainly by arguments of electromangetic duality. First, we explore Dirac monopole, which is done by explicitly introducing magnetic monopole configuration to Maxwell's electromagnetic theory. Then we discuss in a rigorous way 't Hooft-Polyakov monopoles, which in contrast to Dirac monopoles, arise naturally in grand unified theories. In addition to this we generalise 't Hooft-Polyakov monopoles in curved spacetime. We continue then with the main text by introducing the prospect of monopoles in electroweak theory. Such idea comes with some experimental and theoretical controversies. However, the core of this idea is useful as we will examine extensions of the standard model with Born Infeld terms.

Contents

Conventions

Throughout this text we are working with the Minkowski metric:

$$\eta_{\mu\nu} = (+, -, -, -)$$

And with physical units where:

$$\hbar = 1 \qquad c = 1$$

The Levi-Civita symbol is denotes as $\epsilon_{a_1 a_2 ....a_n}$:

$$\epsilon_{a_1 a_2 ...a_n} = \begin{cases} 1, & even \quad permutations \\ -1, & odd \quad permutations \end{cases}$$

1. INTRODUCTION

The history of magnetic monopoles is quite old, as physicists have been troubled by their absence from the early days of Maxwell's theory. In section 2, we will address this old speculation by covering the argument of electromagnetic duality. Another early introduction to magnetic monopoles was made by Henri Poincaré in 1896 when he attempted to explain the results of the Birkeland experiment. In this experiment, cathodic beams focus in a Crookes tube in the presence of a magnet. Poincaré described that this effect could be due to the forces of a magnetic pole at rest on a moving electric charge. Later Thompson in 1904 showcase that the angular momentum of such system is:

$$\vec{L} = m\vec{r} \times \frac{d\vec{r}}{dt} + eg\hat{r} \tag{1.1}$$

Where $\vec{r}$ is the position of the particle with a charge e and g is the magnetic charge of the monopole. Therefore at rest this system has angular momentum equal to eg, which is due to the interaction of the electric field of the charge particle with the monopole. Since angular momentum of the charged particle at rest is just its spin, it suggests that eg must be quantized. In the case of electron which has spin $1/2$, it means that:

$$eg = \frac{1}{2} \tag{1.2}$$

This remarkable result was derived by Dirac in 1931 when quantum mechanics have already formulated. Section 3 rigorously discusses the Dirac monopole. Such results explain why electric charge is quantized, a fact not covered by any theoretical description of particles. Dirac successfully explained why the charges of subatomic particles are quantized by explicitly introducing sources of magnetic fields. It is rather disappointing that no magnetic monopoles have ever been detected, an experimental fact that was key for the development of Maxwell's equations in the first place. It makes more sense that magnetic fields should have no sources since that is the only thing we observe. Why bother explicitly introducing them if we have never observed them? While Dirac's approach was to try to force the theoretical existence of magnetic monopoles in Maxwell's theory, in other models, this is not the case.

In 2012 Higgs boson has been detected, a remarkable observation, which solidified Higgs mechanism in the standard model of particle physics. Through this mechanism the Higgs field obtains the vacuum expectation value, which breaks symmetry $SU_c(3) \times SU_L(2) \times U_Y(1)$ of the standard model to $SU_c(3) \times U_{em}(1)$ and the main result of this procedure is that the subatomic particles obtain masses. This symmetry breaking pattern can viewed simply as $SU_L(2) \times U_Y(1)$ to $U_{em}(1)$, suggesting that above energy scale of 246 GeV, where this procedure takes place, weak interactions are unified with the electromagnetic ones. Furthermore it is widely believed that at even higher energy scales of 10^{16} GeV strong interactions unify with electroweak interactions. This known as the grand unified theory (GUT) scale.

In 1974 Georgi-Glashow develop a GUT based on the symmetry group $SU(5)$ and since the GUT scale is unexplored, models vary[1]. In this text we will focus on Georgi-Glashow theory based on $SU(2)$, which is rather a toy model than a realistic GUT theory. Despite the unrealistic nature of this model, it provides a pedagogical approach to understand topological objects in GUTs. This kind of objects enables such models to have magnetic monopole configurations, which arise naturally inside the model. We will discuss this idea rigorously in section 4.

---

[1] For example there are GUT's based on $SO(10)$, or even on other frameworks such supersymmetry and string theory.

Generalization from flat space-times to curved ones, provide a more complete picture for monopole configurations arising from GUT's like the $SU(2)$ Georgi-Glashow model. When these kind of monopoles are the source of spacetime curvature, we obtain a different view of them. Their mass becomes lighter throughout the space-time and when the Higg's vacuum expectation value is compared to Planck energy scale, event horizons form, resulting to the identification of the monopole with a charged black hole. We cover this generalization in section 5.

Standard model on the other hand, such topological objects seem to not be possible and thus magnetic monopoles cannot arise as field configurations. Cho and Maison in 1997 gave a different perspective on this matter as we will see in section 6. From this perspective standard model could have the needed topological objects such that monopole configuration arise naturally. However there is one main theoretical challenge that this concept faces, the mass of monopole is singular and thus making it unphysical. We will describe general schemes that could make the monopole mass finite, but without any specified dynamics, which does not resolve the problem completely.

Even if the energy of this so called electroweak monopole is infinity, the idea of Cho and Maison could be used for further extensions of the standard model. As we will see in section 7 Born-Infeld extension of electroweak model predicts finite energy monopoles. This type of extension could arise from non linear behavior of gauge fields. Monopole mass is predicted to be around 14 TeV, which is above capabilities of the LHC. Therefore this kind of monopoles have cosmological interest, since such energies as observed in cosmic rays.

2. ELECTROMAGNETIC DUALITY

Magnetic monopoles is a subject that fascinates physicists over a century. Initial motivations can be traced back to very existence of Maxwell equations of electromagnetism. In particular, equations in vacuum are taking the form:

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{2.1}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.2}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{2.3}$$

$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 0 \tag{2.4}$$

Notice that the equations are symmetric under the transformation $(\vec{E}, \vec{B}) \to (\vec{B}, -\vec{E})$. This kind of transformation is a particular case of $O(2)$ transformations for $\theta = -\pi/2$.

$$G = \begin{cases} E \to Ecos(\theta) - Bsin(\theta) \\ B \to Esin(\theta) + Bcos(\theta) \end{cases} \tag{2.5}$$

The electromagnetic field tensor and its a dual form are given by:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.6}$$

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = \epsilon_{\mu\nu\rho\sigma}\partial^\rho A^\sigma \tag{2.7}$$

Where $\mu, \nu = 0, 1, 2, 3$ with $F_{0i} = -\frac{1}{2}\epsilon_{0ijk}\tilde{F}^{jk} = E_i$ and $\tilde{F}_{0i} = -\frac{1}{2}\epsilon_{0ijk}F^{jk} = B_i$. Then Maxwell equations in vacuum are written as:

$$\partial_\mu F^{\nu\mu} = 0 \tag{2.8}$$

$$\partial_\mu \tilde{F}^{\nu\mu} = 0 \tag{2.9}$$

It straight forward to show that the field tensor and its dual are transforming under $O(2)$ as:

$$G = \begin{cases} F_{\mu\nu} \to F_{\mu\nu}cos(\theta) - \tilde{F}_{\mu\nu}sin(\theta) \\ \tilde{F_{\mu\nu}} \to F_{\mu\nu}sin(\theta) + \tilde{F_{\mu\nu}}\cos(\theta) \end{cases} \tag{2.10}$$

Maxwell equations can be obtained from the langragian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.11}$$

And it transforms under $O(2)$ as:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}cos(\theta) - \tilde{F_{\mu\nu}}sin(\theta))(F^{\mu\nu}cos(\theta) - \tilde{F^{\mu\nu}}sin(\theta)) =$$

$$\frac{-1}{4}[F_{\mu\nu}F^{\mu\nu}cos^2(\theta) + \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}sin^2(\theta) - 2sin(\theta)cos(\theta)F^{\mu\nu}\tilde{F}_{\mu\nu}] =$$

$$-\frac{1}{4}[F_{\mu\nu}F^{\mu\nu}cos^2(\theta) + \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu mn}F_{mn}F^{\rho\sigma}sin^2(\theta) - sin(2\theta)F^{\mu\nu}\tilde{F}_{\mu\nu}] =$$

$$-\frac{1}{4}[F_{\mu\nu}F^{\mu\nu}cos^2(\theta) + \frac{1}{2}(\delta^m_\rho\delta^n_\sigma - \delta^m_\sigma\delta^n_\rho)F_{mn}F^{\rho\sigma}sin^2(\theta) - sin(2\theta)F^{\mu\nu}\tilde{F}_{\mu\nu}] =$$

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}sin(2\theta)F^{\mu\nu}\tilde{F_{\mu\nu}}$$

Now for the term $F_{\mu\nu}\tilde{F}^{\mu\nu}$ we have:

$$\epsilon_{\mu\nu\rho\sigma}\partial^\rho A^\sigma F^{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\partial^\rho A^\sigma\partial^\mu A^\nu - \epsilon_{\mu\nu\rho\sigma}\partial^\rho A^\sigma\partial^\nu A^\mu = 2\epsilon_{\mu\nu\rho\sigma}\partial^\rho A^\sigma\partial^\mu A^\nu = 2\epsilon_{\mu\nu\rho\sigma}\partial^\rho[A^\sigma\partial^\mu A^\nu]$$

Thus this term is a total derivative and for vector fields that vanish asymptotically, we can drop it.

The Lagrangian of the source-free electromagnetism is invariant under $O(2)$. This symmetry will disappear if we introduce sources, since original Maxwell equations don't contain magnetic monopole sources. We can maintain this symmetry if we introduce both magnetic and electric sources. Such field configurations can be used to describe hypothetical particles with both electric and magnetic charges. We call these **dyons**. Maxwell equations become:

$$\vec{\nabla}\cdot\vec{E} = \rho_e \tag{2.12}$$

$$\vec{\nabla}\cdot\vec{B} = \rho_g \tag{2.13}$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = \vec{j}_e \tag{2.14}$$

$$\vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{j}_g \tag{2.15}$$

Then by letting $\vec{F} = \vec{E} + i\vec{B}$, $\rho_q = \rho_e + i\rho_g$ and $\vec{j}_q = \vec{j}_e + i\vec{j}_q$, we write the equations above as

$$\vec{\nabla}\cdot\vec{F} = \rho_q \tag{2.16}$$

$$\vec{\nabla}\times\vec{F} - i\frac{\partial\vec{F}}{\partial t} = \vec{j}_q \tag{2.17}$$

The $O(2)$ transformations are equivalent with $U(1)$ and the quantities above transform as

$$\vec{F} \to e^{i\theta}\vec{F} \tag{2.18}$$

$$\rho_q \to e^{i\theta}\rho_q \tag{2.19}$$

$$\vec{j_q} \rightarrow e^{i\theta}\vec{j_q} \tag{2.20}$$

Maxwell equations with both electric and magnetic sources are invariant under $O(2)$. In such theory electric and magnetic charges are not separated, but are a part of one observable unified charge:

$$q = \sqrt{e^2 + g^2} \tag{2.21}$$

This holds since electric and magnetic fields in this case are equivalent and there is no difference between them. Such a motivation of symmetry in the laws of electromagnetism is one of the reasons magnetic monopoles are a fascinating subject of fundamental physics. A valid counter argument of this, is that nature itself does not need to be described by beautiful mathematics to function. Nature is just nature, our perception of beauty should not be the only indication of how it works. However, history of physics seems to reward those who are driven by such indications. For example, Paul Dirac theoretically discover anti-matter, years before it was experimentally verified. In this case his drive was mathematical beauty, which at the end paid off. In the next section, we will see how the same mind theoretically describe magnetic monopoles, driven by motivations from electromagnetic duality.

3. DIRAC MONOPOLE

In this section we are going to investigate the theoretical description of magnetic monopoles by explicitly introducing them as a source of magnetic field. This work was first done by Paul Dirac, which results to some theoretical challenges that eventually lead to the quantization of electric and magnetic charges.

3.1. Dirac String

Our first attempt to describe magnetic monopoles is by considering a point-like magnetic source g. Such a source produces a static Coulomb like magnetic field:

$$\vec{B} = g\frac{\vec{r}}{r^3} \tag{3.1}$$

The vector potential of the magnetic monopole field satisfies:

$$\vec{B} = g\frac{\vec{r}}{r^3} = \vec{\nabla} \times \vec{A} \tag{3.2}$$

Since g is a point magnetic sources it holds $\vec{\nabla} \cdot \vec{B} = 4\pi g\delta^3(\vec{r})$, but from vector calculus we know that $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$. Let's analyse the problem. Given that $\vec{B}$ is spherically symmetric, we can write $\vec{A}$ as:

$$\vec{A}(r) = A(\theta)\vec{\nabla}\phi \tag{3.3}$$

Where θ is the polar angle and ϕ is the azimuth angle. By using the gradient in spherical coordinates $\nabla\phi = \frac{1}{rsin\theta}\frac{\partial}{\partial\phi}(\phi)\hat{\phi} = \frac{1}{rsin\theta}\hat{\phi}$ and picking $A(\theta) = -g(1 + cos\theta)$, we obtain:

$$\vec{A}(r) = -g\frac{1 + cos\theta}{rsin\theta}\hat{\phi} \tag{3.4}$$

This expression can written in a covariant form:

$$\vec{A}(r) = -g\frac{1 + cos\theta}{rsin\theta}\hat{\phi} = - = g\frac{sin\theta(1 + cos\theta)}{sin^2\theta}\hat{\phi} = -g\frac{1 + cos\theta}{rsin\theta}\hat{\phi}$$

$$= -g\frac{sin\theta(1 + cos\theta)}{r(1 - cos\theta)(1 + cos\theta)}\hat{\phi} = -\frac{g}{r}\frac{rsin\theta}{r - rcos\theta}\hat{\phi} = \frac{g}{r}\frac{r\hat{r} \times (cos\theta\hat{r} - sin\theta\hat{\theta})}{r - z}$$

Thus since $\hat{n} = \hat{z} = cos\theta\hat{r} - sin\theta\hat{\theta}$, we obtain the **Dirac Pontetial**

$$\vec{A}(r) = \frac{g}{r}\frac{\vec{r} \times \hat{n}}{r - \vec{r} \cdot \hat{n}} \tag{3.5}$$

A quick calculation show that the Dirac potential indeed generates (3.1), for example the x-axis component of $\vec{B}$ is:

$$B_x = -\frac{\partial A_y}{\partial z} = \frac{\partial}{\partial z}(\frac{g}{r}\frac{x}{r-z}) = -\frac{gx}{r^2(r-z)^2}\frac{\partial}{\partial z}(r^2 - zr) = -\frac{gx}{r^2(r-z)^2}(2r\frac{\partial r}{\partial z} - r - z\frac{\partial r}{\partial z})$$

$$= \frac{gx}{r^3(r-z)^2}(z^2 - 2zr - r^2) = \frac{gx}{r^3}$$

From (3.4) we see that the vector potential is singular at $\theta = 0$ and regular at $\theta = \pi$. Therefore, the static Coulomb like magnetic field in (3.1) is ill-defined along the semi infinity line of singularity. The region $\theta = 0$ where the vector potential is singular is called **Dirac String**

To avoid the singular part of the potential we write $\vec{A}$ in a regular form:

$$\vec{A_R}(r,\epsilon) = \frac{g}{R}\frac{\vec{r}\times\hat{n}}{R - \vec{r}\cdot\hat{n}} \tag{3.6}$$

Where $\epsilon \to 0$ and $R = \sqrt{x^2 + y^2 + z^2 + \epsilon^2}$. To calculate the regularized magnetic field $B_R(\vec{r},\epsilon) = \vec{\nabla}\times\vec{A_R}$, we write it in tensor form:

$$B_{R,a} = \epsilon_{abc}\partial_b A_{R,c} = \epsilon_{abc}\epsilon_{mnc} g\partial_b(\frac{r_m n_n}{R(R - \vec{r}\cdot\hat{n})}) = (\delta_{am}\delta_{bn} - \delta_{an}\delta_{mb}) g\partial_b(\frac{r_m n_n}{R(R - \vec{r}\cdot\hat{n})})$$

$$= g\partial_b(\frac{r_a n_b}{R(R - \vec{r}\cdot\hat{n})}) - g\partial_b(\frac{r_b n_a}{R(R - \vec{r}\cdot\hat{n})}) = g\frac{n_a}{R(R - \vec{r}\cdot\hat{n})} - g\frac{r_a n_b}{R^2(R - \vec{r}\cdot\hat{n})}(2R\frac{r_b}{R} - \frac{r_b}{R}\vec{r}\cdot\hat{n} - \frac{R^2}{R}n_b)$$

$$-3g\frac{n_a}{R(R - \vec{r}\cdot\hat{n})} + g\frac{n_a}{R^3(R - \vec{r}\cdot\hat{n})^2}(2Rr^2 - \vec{r}\cdot\hat{n}r^2 - \vec{r}\cdot\hat{n}R^2) = g\frac{n_a}{R(R - \vec{r}\cdot\hat{n})} - 3g\frac{n_a}{R(R - \vec{r}\cdot\hat{n})}$$

$$+2gr^2\frac{n_a}{R^3(R - \vec{r}\cdot\hat{n}}g - g\epsilon^2\frac{n_a}{R^3(R - \vec{r}\cdot n)^2}\vec{r}\cdot\hat{n} + g\frac{r_a}{R^3} = \frac{r_a}{R^3} - 2g\frac{n_a}{R^3(R - \vec{r}\cdot\hat{n})}(R^2 - r^2)$$

$$-g\epsilon^2\frac{n_a}{R^3(R - \vec{r}\cdot n)^2}\vec{r}\cdot\hat{n} = g\frac{r_a}{R^3} - g\epsilon^2 n_a(\frac{1}{R^3(R - \vec{r}\cdot\hat{n})} + \frac{R}{R^3(R - \vec{r}\cdot\hat{n})^2})$$

In vector form this is written as:

$$\vec{B_R}(r,\epsilon) = g\frac{\vec{r}}{R^3} - g\epsilon^2\hat{n}[\frac{1}{R^3(R - \vec{r}\cdot\hat{n})} + \frac{1}{R^2(R - \vec{r}\cdot\hat{n})^2}] \tag{3.7}$$

At the limit $\epsilon \to 0$, (3.7) becomes

$$\vec{B}_R(r,\epsilon) \approx g\frac{\vec{r}}{R^3} - 2g\epsilon^2\hat{n}\theta(z)[\frac{1}{r^2(x^2 + y^2 + \epsilon^2)} + \frac{2}{(x^2 + y^2 + \epsilon^2)^2}] \tag{3.8}$$

From the expression it is the second is the singular term, which is non zero on the positive infinite semi axis. We can rewrite this term by noting $\rho^2 = x^2 + y^2$:

$$\vec{B}_{sing} = -2g\hat{n}\theta(z)\epsilon^2[\frac{1}{(\rho^2 + z^2)(\rho^2 + \epsilon^2)} + \frac{2}{(\rho^2 + \epsilon^2)^2}] = -2g\hat{n}\theta(z)f(\rho)$$

Now integrate the function $f(\rho)$:

$$\epsilon^2\int_0^{2\pi} d\phi\int_0^{\infty} d\rho\rho[\frac{1}{(\rho^2 + z^2)(\rho^2 + \epsilon^2)} + \frac{2}{(\rho^2 + \epsilon^2)^2}] = \frac{\epsilon^2\pi}{z^2 - \epsilon^2}\int_0^{\infty} dw[\frac{\epsilon^2}{w + \epsilon^2} - \frac{1}{w + z^2}]$$

$$+2\pi \int_{-\infty}^{\infty} dw \frac{\epsilon^2}{(w^2+\epsilon^2)^2} = \frac{\epsilon^2 \pi}{z^2-\epsilon^2}(ln1 - ln\frac{\epsilon^2}{z^2}) + 2\pi \int_{-\infty}^{\infty} dw \frac{\epsilon^2}{(w^2+\epsilon^2)^2}$$

The first is zero when $\epsilon^2 \to 0$, while the second term behaves like 2 dimensional delta function. We obtain then:

$$\vec{B}_{sing} = -2g\hat{n}\theta(z)\delta(x)\delta(y) \tag{3.9}$$

The total magnetic flux through the closed surface with a magnetic monopole inside is:

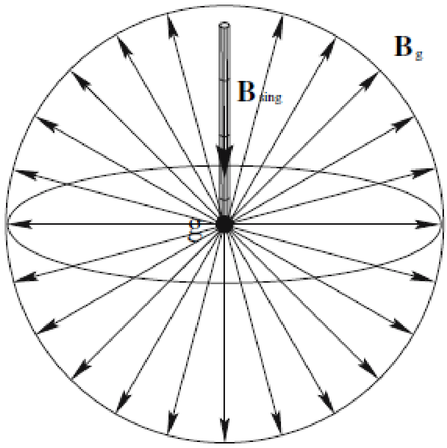

FIG. 1: Magnetic field of singular Dirac potential.

$$\Phi = \oint d\vec{S}\cdot\vec{B} = g\oint \frac{d\vec{S}\cdot\vec{r}}{r^3} - 4\pi g \oint d\vec{S}\cdot\hat{n}\theta(z)\delta(x)\delta(y) =$$

$$g\int d\Omega - 4\pi g \int\int dxdy\delta(x)\delta(y) = 4\pi g - 4\pi g = 0$$

Thus since the magnetic flux is zero the initial contradiction is resolved by making the vector potential regular, but with the cost of the Dirac string.

Dirac string must be unphysical and to see this consider the U(1) gauge transfromation of the vector potential:

$$\vec{A'} = \vec{A} - \frac{i}{e}U^{-1}\vec{\nabla}U = \vec{A} + \vec{\nabla}\lambda(r) \tag{3.10}$$

Where $U = e^{ie\lambda(r)} \in U(1)$. $\vec{A'}$ and $\vec{A}$ produce the same magnetic field. By computing the difference between magnetic flux:

$$\Delta\Phi = \oint d\vec{S}\cdot(\vec{B'} - \vec{B}) = \oint d\vec{S}\cdot\vec{\nabla}\times(\vec{A'} - \vec{A}) = \oint d\vec{S}\cdot\vec{\nabla}\times(\vec{\nabla}\lambda) = \oint dl\cdot\vec{\nabla}\lambda$$

We see that it must hold $\lambda(\phi+2\pi) = \lambda(\phi)$, otherwise the difference between the magnetic fluxes is not zero.

This leads to the gauge transformation:

$$U(\phi) = e^{2ieg\phi} \tag{3.11}$$

Where $\phi' = \phi + 2\pi$. Then the vector pontetial we consder at (3.4) transforms as:

$$\vec{A}^N = \vec{A}^S - \frac{i}{e}U^{-1}\vec{\nabla}U = -g\frac{1+cos\theta}{rsin\theta}\hat{\phi} + \frac{2g}{rsin\theta}\hat{\phi} = g\frac{1-cos\theta}{rsin\theta}\hat{\phi} \tag{3.12}$$

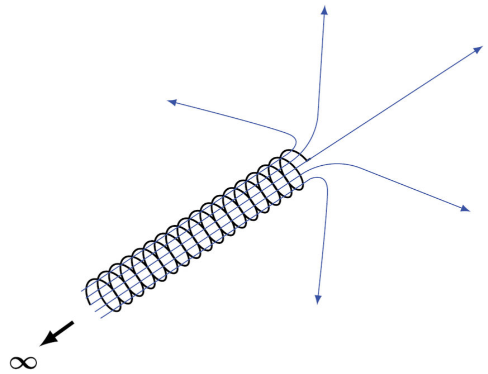

FIG. 2: Magnetic line of solenoid of infinite length.

Notice that the new vector field is singular at $\theta = \pi$ and regular at $\theta = 0$. This suggests that Dirac string has been rotated from positive semi-axis to the negative one. Therefore, the gauge transformation acts like a rotation to the Dirac string and this means that it does not have a physical significance.

This description can be summed up in the following thought experiment. We can imagine the magnetic field of a solenoid of infinite length as we see in figure 2. If the magnetic lines along the solenoid length do not have a physical significance, then the resulting magnetic field configuration is that of a magnetic monopole. These magnetic lines along the solenoid is what we describe above as the Dirac string. Combine this with the quantum description of particles interacting with the string as we will see that it results to the quantization of electric and magnetic charges. Such an idea makes sense, but at the end of day the dependence from unphysical Dirac strings for the description of magnetic monolopes to make sense seems unsatisfactory. It follows there is an equivalent description developed by Wu and Yang for the monopole without the string, which leads to the realisation that monopoles have a topological origins.

3.2. Topological Roots of the Abelian Monopole

The mathematical contradictions in the definition of (3.1) have lead us to the introduction of the Dirac string, where we have showcase that is unphysical. Now we are going to approach this problem in different manner, by giving up by the usual parameterization of $\mathbb{R}^3$ surrounding the monopole. First, we divide $\mathbb{R}^3/\{0\}$ into two sightly overlapping hemispheres, the north $\mathbb{R}^N$ and the south $\mathbb{R}^S$. The vector potential in the north hemisphere is different than the one

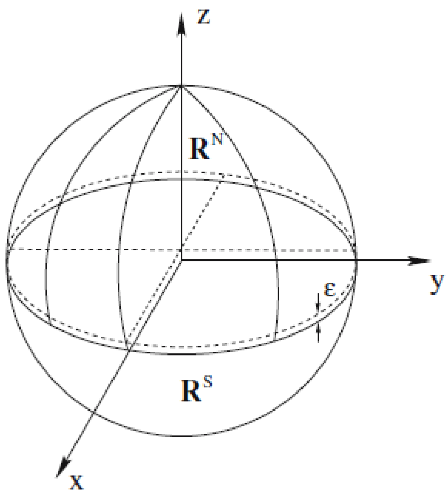

FIG. 3: Division of $\mathbb{R}^3/\{0\}$ into 2 sightly overlapping hemispheres.

in the south and these two are connected by a gauge transformation $U = e^{2ieg}$ as we saw previously.

$$\vec{A} = \begin{cases} \vec{A}^N = g\frac{1-cos\theta}{rsin\theta}\hat{\phi} & ,0 \leq \theta \leq \frac{\pi}{2} + \frac{\epsilon}{2} \\ \vec{A}^S = -g\frac{1+cos\theta}{rsin\theta}\hat{\phi} & ,-\frac{\epsilon}{2} + \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$$

This description is known as Wu-Yang monopole [23]. Consider a close loop l around the overlap region $\mathbb{R}^N \cap \mathbb{R}^S$. Then if a charged particle passes along l, then the wavefunction of the particle will pick a phase just like in a Bohm-Aharonov experiment. Recall that in quantum mechanics an electric charge interaction with electromangnetic field (classical field) can be described by the action of the system

$$S = \int_0^T dtL = \int_0^T dt(\frac{m}{2}\dot{r}^2 + e\dot{\vec{r}} \cdot \vec{A}) \tag{3.13}$$

For a close path $r(0) = r(T)$ the phase is given by $e\oint_l \vec{dr} \cdot \vec{A}$. We can calculate this phase factor with either $\vec{A}^N$ or $\vec{A}^S$ in the overlap region.

$$e\oint_l \vec{dr} \cdot \vec{A}^N = e\int_{\mathbf{R}^N} \vec{dS} \cdot [\vec{\nabla} \times \vec{A}^N] = e\int_{\mathbf{R}^N} \vec{dS} \cdot \vec{B}$$

$$e\oint_l \vec{dr} \cdot \vec{A}^S = -e\int_{\mathbf{R}^S} \vec{dS} \cdot [\vec{\nabla} \times \vec{A}^S] = -e\int_{\mathbf{R}^S} \vec{dS} \cdot \vec{B}$$

The action is then defined up to a term:

$$\Delta S = e\int_{\mathbf{R}^N \cup \mathbf{R}^S} \vec{dS} \cdot \vec{B} = e\int_V dV\vec{\nabla} \cdot \vec{B} = e4\pi g$$

This term must not have any physical significance. Therefore by recalling that the path integral is $Z \sim e^{iS}$ this term must me equal to $2\pi n$ for $n \in \mathbb{Z}$. Thus we obtain

$$eg = \frac{n}{2} \qquad n \in \mathbb{Z} \tag{3.14}$$

This is a remarkable result, since the fundamental theories of physics can not explain why the electric charge is quantized. The existence of one magnetic monopole can explain why all the electric charges are quantized. This groundbreaking idea was firstly introduced by Dirac [1] in 1948. Note that this condition isn't like the quantized physical observables from quantum mechanics, which are eingvalues of hermitean operators. Instead we will see that the quantization of electric charge has topological origin.

The corresponding wavefunctions in the overlap region are connected via the gauge transformation:

$$\psi^S = U\psi^N = e^{2eig\phi}\psi^N = e^{in\phi}\psi^N \tag{3.15}$$

Which showcase that (3.14) is essential for wavefunctions to be single valued at each region. Indeed as the azimuthal angle ϕ increases from 0 to 2π we have

$$\phi^S(0) = \phi^N(0) \qquad \phi^S(2\pi) = e^{i4\pi eg}\phi^N(2\pi)$$

Now the the integer n responsibly for the quantization condition is simply a winding number. That is how many times it circles the whole Abelian group $U(1)_{em}$. To understand view $U(1)_{em}$ as a manifold, since $U(1)_{em} \simeq S^1$. The winding number in this case is simply how many times it wraps around S^1. Thus the electric charge is quantized, because the winding number is an integer.

Winding number is a topological quantity and the fact is an integer can be traced back to the existence of magnetic monopole at the origin. We conclude then that the magnetic monopole is a topological defect and charge quantization relation has topological origin. Although such description is quite elegant, it does not seem to be fundamental. This is because we started by introducing the magnetic charges in Maxwell's theory. For instance electric charges arise naturally in electrodynamics as a result of $U(1)$ symmetry. We could try to formulate generalised electrodynamics based on $U(1) \times U(1)$, but such model does not capture the mass of the monopole. This is not the case with GUT monopoles as we will see in the next section.

4. 'T HOOFT-POLYAKOV MONOPOLES

In this section, we are going study magnetic monopoles in the context of unified theories. Such theories can have monopole solutions without explicitly introduce them. This is very useful if we are looking for a fundamental theory of magnetic monopoles. We are starting from reviewing subjects, like non-Abelian gauge theories and Higgs mechanism, that are very important to understand the concept of monopoles in unified theories. Then we will study mangnetic monopole and dyon solutions in $SU(2)$ Georgi-Glashow GUT for simplicity.

4.1. Non Abelian Gauge Theories

To formulate a description of monopoles in GUT's, it is important to review non-Abelian gauge theories. Non Abelian Gauge theories are very important in the standard model since they describe electroweak and strong interactions. There are also important for GUT's, since further unification of forces requires considering non-Abelian gauge theories based on higher symmetry groups, with the standard model symmetry group being a subgroup in such case.

Let's consider a field theory which is invariant under a local non-Abelian gauge group G. For example, strong interactions are described by $G = SU_c(3)$ and electroweak-interactions by $G = SU_L(2) \times U_Y(1)$. The effect of making the theory invariant under such a group is by shifting regular derivatives to covariant derivatives:

$$\partial_\mu \to D_\mu = \partial_\mu + ie\vec{A}_\mu \cdot \vec{T} \tag{4.1}$$

Where $\vec{T}$ is a $d = dim(G)$ dimensional vector with its components being the generators of the group G and $\vec{A}_\mu$ is the gauge potential. Consider $U = e^{ie\vec{\theta}\cdot\vec{T}} \in G$ where $\vec{\theta}$ is the continuous parameter which G depends from. Then the covariant derivative transforms as

$$D'_\mu = UD_\mu U^{-1} \tag{4.2}$$

By acting with D_μ on scalar fields ϕ_a $a = 1, ., d$, we can find how $\vec{A}_\mu \cdot \vec{T}$ transforms under G. Since ϕ_a transforms as $\phi'_a = U^b_a \phi_b$ we get

$$D'_\mu \phi'_a = U^b_a D_\mu \phi_b \Rightarrow \partial_\mu \phi'_a + ie\vec{A}'_\mu \cdot (\vec{T})^b_a \phi'_b = UD_\mu \phi_b \Rightarrow$$

$$(\partial_\mu U^b_a)\phi_b + U^b_a \partial_\mu \phi_b + ie\vec{A}'_\mu \cdot (\vec{T})^b_a U^c_b \phi_c$$

$$U[U^{-1}(\partial_\mu U) + \partial_\mu]) + ieU^{-1}(\vec{A}'_\mu \cdot \vec{T})U]\phi = U(\partial_\mu + ie\vec{A}_\mu \cdot \vec{T})\phi \Rightarrow$$

$$\vec{A}_\mu \cdot \vec{T} = -\frac{i}{e}U^{-1}\partial_\mu U + U^{-1}(\vec{A}'_\mu \cdot \vec{T})U \Rightarrow$$

$$\vec{A}'_\mu \cdot \vec{T} = \frac{i}{e}U^{-1}\partial_\mu U + U(\vec{A}_\mu \cdot \vec{T})U^{-1} \tag{4.3}$$

In the Abelian case it's straight forward to calculate the commutator of covariant derivatives:

$$[D_\mu, D_\nu]\phi = [ieA_\mu, \partial_\nu]\phi + [\partial_\mu, ieA_\nu]\phi = ieA_\mu\partial_\nu\phi - ie\partial_\nu(A_\mu\phi) + ie\partial_\mu(A_\nu\phi) - ieA_\nu\partial_\mu\phi =$$

$$ieA_\mu\partial_\nu\phi - ieA_\nu\partial_\mu\phi - ie(\partial_\nu A_\mu)\phi - ieA_\mu\partial_\nu\phi + ie(\partial_\mu A_\nu)\phi + ieA_\nu\partial_\mu\phi = ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\phi = ieF_{\mu\nu}\phi$$

$$\frac{1}{ie}[D_\mu, D_\nu] = F_{\mu\nu} \tag{4.4}$$

The commutator of covariant derivatives is equal to the electromagnetic field tensor. This suggests that the field tensor behaves like the Riemann curvature tensor $[D_\mu, D_\nu]V^a = V^b R^a{}_{b\mu\nu}$ in the field space. We can generalise this result and obtain the field tensor for the non-Abelian case by considering (4.1) and that generator T^i satisfies:

$$[T^i, T^j] = if^{ijk}T^k \qquad Tr[T^iT^j] = \frac{\delta^{ij}}{2} \tag{4.5}$$

And the field tensor can be obtain as

$$[D_\mu, D_\nu] = ie\vec{F}_{\mu\nu} \cdot \vec{T} \Rightarrow$$

$$ie(F^i_{\mu\nu}T^i)^a{}_c = [\partial_\mu \delta^a_b + ieA^i_\mu (T^i)^a{}_b, \partial_\nu \delta^b_c + ieA^i_\nu (T^i)^b{}_c] \Rightarrow$$

$$ie(F^i_{\mu\nu}T^i)^a{}_c = ie[\partial_\mu \delta^a_b, A^i_\nu (T^i)^b{}_c] + ie[A^i_\mu (T^i)^a{}_b, \partial_\nu \delta^b_c] + (ie)^2[(T^i)^a{}_b, (T^j)^b{}_c]A^i_\mu A^j_\nu \Rightarrow$$

$$ie(F^i_{\mu\nu}T^i)^a{}_c = ie(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu)(T^i)^a{}_c - ie^2 f^{ijk} A^j_\mu A^k_\nu (T^i)^a{}_c \Rightarrow$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - ef^{ijk}A^j_\mu A^k_\nu \tag{4.6}$$

And just like the Abelian case the free non-Abelian Lagrangian is:

$$\mathcal{L}_{gauge} = -\frac{1}{4}\vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu} \tag{4.7}$$

By using the trace formula in (4.5) we can write the Lagragian as

$$\mathcal{L}_{gauge} = -\frac{1}{2}Tr[(\vec{F}_{\mu\nu} \cdot \vec{T})(\vec{F}^{\mu\nu} \cdot \vec{T})] = -\frac{1}{2}Tr[F^2] \tag{4.8}$$

In this form it is clear that L is invariant under G when $F_{\mu\nu}$ transforms covariantly:

$$\vec{F}'_{\mu\nu} \cdot \vec{T} = U\vec{F}_{\mu\nu} \cdot \vec{T}U^{-1} \tag{4.9}$$

We can generalise the Bianchi indentity (2.9) to the non-Abelian case. Just act with Jacobi identity to a test function ϕ :

$$\epsilon^{\mu\nu\lambda}[D_\mu, [D_\nu, D_\lambda]]\phi = 0 \Rightarrow$$

$$\epsilon^{\mu\nu\lambda}[D_\mu, \vec{F}_{\nu\lambda} \cdot \vec{T}]\phi = 0 \Rightarrow \epsilon^{\mu\nu\lambda}D_\mu(\vec{F}_{\nu\lambda} \cdot \vec{T}\phi) - \epsilon^{\mu\nu\lambda}\vec{F}_{\nu\lambda} \cdot \vec{T}D_\mu\phi = 0 \Rightarrow \epsilon^{\mu\nu\lambda}D_\mu(\vec{F}_{\nu\lambda} \cdot \vec{T}) = 0 \Rightarrow$$

$$\epsilon^{\mu\nu\lambda}D_\mu F^a_{\nu\lambda} = 0 \tag{4.10}$$

We can obtain some useful formula from this identity. Set only spatial indices and we get:

$$\epsilon^{ijk}D_i F^a_{jk} = 0 \Rightarrow D_i(\frac{1}{2}\epsilon^{ijk}F^a_{jk}) = 0 \Rightarrow$$

$$D_i B^a_i = 0 \tag{4.11}$$

4.2. Higg's Mechanism

Higg's mechanism is an essential feature of the standard model and key for obtaining monopole configurations in GUT's Particles in nature obtain their mass when are interacting with the Higgs field. This is possible because of spontaneous symmetry breaking by the Higgs mechanism. To understand this idea briefly, consider the complex scalar Lagrangian:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi^\dagger \phi) \tag{4.12}$$

Where ϕ is a complex doublet transforming under $G = SU(2) \times U(1)$, with transformation generating by $[\frac{\sigma^1}{2}, \frac{\sigma^2}{2}, \frac{1+\sigma^3}{2}, \frac{1-\sigma^3}{2}]$

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathbb{C}^2 \tag{4.13}$$

Where $\phi_1, \phi_2 \in \mathbb{C}$. The potential can be chosen as:

$$V(\phi^\dagger \phi) = -\frac{\lambda}{2}(\phi^\dagger \phi - u^2)^2 \tag{4.14}$$

Where $\lambda > 0$. This kind of set up of the Higgs field is what we have in the standard model. The idea of symmetry breaking is that our theory may be invariant under a symmetry group G, but the vacuum expectation value $\langle 0| \phi |0\rangle$ of the scalar field is not invariant under a subgroup G/H of G. The subgroup H leaves the vacuum expectation value invariant and we write G as $G = H \times G/H$.

Since we have a doublet ϕ, the vacuum expectation value has components:

$$\langle 0| \phi_a |0\rangle = F_a \tag{4.15}$$

The vacuum expectation value can be obtain by minimizing the potential (4.14):

$$\frac{\partial V}{\partial \phi_i^\dagger} = -\lambda(\phi^\dagger \phi - u^2)\phi_i = 0 \Rightarrow$$

$$|\phi|^2 = u^2 \tag{4.16}$$

So a choice of (4.15) is:

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ u \end{pmatrix} \tag{4.17}$$

It's straight forward to show that this vector satisfies

$$\frac{(1+\sigma^3)}{2} \langle \phi \rangle = 0 \qquad \frac{(1-\sigma^3)}{2} \langle \phi \rangle \neq 0$$

$$\frac{\sigma^1}{2} \langle \phi \rangle \neq 0 \qquad \frac{\sigma^2}{2} \neq 0$$

This means that three generators brake the vacuum expectation value and one leaves it invariant. So three of these generators generate elements of G/H and remaining one elements of H. This suggests the symmetry breaking pattern:

$$SU(2) \times U(1) \xrightarrow{SSB} U(1) \tag{4.18}$$

From Goldstone theorem follows then that there are three massless bosons and one massive one. The massive one is the Higgs boson. We can write the doublet ϕ around the minimum value (4.17)

$$\phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} = \langle 0| \phi |0\rangle + \phi' = \begin{pmatrix} 0 \\ u \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_2' + i\phi_1' \\ H - i\phi_3' \end{pmatrix} \tag{4.19}$$

And the Lagrangian (4.12) becomes:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1' \partial^\mu \phi_1' \partial_\mu + \phi_2' \partial^\mu \phi_2' + \partial_\mu \phi_3' \partial^\mu \phi_3' + \partial_\mu H \partial^\mu H) - \lambda u^2 H + interaction + self - interactions \tag{4.20}$$

From the term H^2 we can obtain the mass of the Higgs boson

$$m_H = \sqrt{2\lambda} u \tag{4.21}$$

Working with Higg's mechanism and non-Abelian gauge theories, we can construct models of fundamental interactions. These can be model's of unification as it follows with Georgi-Glashow model.

4.3. Georgi-Glashow Model

The idea is to take $U(1)_{em}$ Abelian subgroup and embedded into a higher rank non-Abelian gauge group. This means to consider electromagnetism as part of a unified theory. Then any model of unification with an electromagnetic $U_{em}(1)$ subgroup embedded into a higher rank non-abelian group, which after spontaneous symmetry broken by the Higgs mechanism, possesses monopole-like solutions. To explore this idea let's consider a simply non-Abelian field theory, the $SU(2)$ Yang Mills[2] coupled with the Higgs field triplet ϕ_a. Modern unified theories are described for example by models based on higher symmetry groups like $SU(5)$ and $SO(10)$, but this model will provide us a simply introduction for the unified models. The Lagrangian of this theory is:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D_\mu\phi_a D^\mu\phi_a - V(\phi) \tag{4.22}$$

We are working with the adjoint representation where the three generators of $SU(2)$ are given by:

$$(T^i)_{jk} = -i\epsilon_{ijk} \tag{4.23}$$

For the field strength tensor, since $[T^i, T^j] = i\epsilon^{ijk}T^k$ we have:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e\epsilon^{ajk}A^j_\mu A^k_\nu \tag{4.24}$$

The potential of the scalar field is chosen to be:

$$V(\phi^a\phi^a) = \frac{\lambda}{4}(\phi^a\phi^a - u^2)^2 \tag{4.25}$$

Where u^2 corresponds to the minimal value of $V(\phi^a\phi^a)$. The equations of motion are given by Euler-Lagrange equations:

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^a_\nu)} = \frac{\partial\mathcal{L}}{\partial A^a_\nu} \qquad \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} = \frac{\partial\mathcal{L}}{\partial\phi_a} \Rightarrow$$

$$\partial_\mu F^{a\nu\mu} - e\epsilon_{abc}A^b_\mu F^{c\nu\mu} = -e_{abc}\phi^b D^\nu\phi^c \qquad \partial_\mu D^\mu\phi_a = eA^b_\mu\epsilon_{bma}D^\mu\phi_m - \lambda\phi_a(\phi^m\phi^m - u^2) \Rightarrow$$

$$D_\nu F^{a\mu\nu} = -e\epsilon_{abc}\phi^b D^\mu\phi^c \qquad D_\mu D^\mu\phi^a = -\lambda\phi^a(\phi^a\phi^a - u^2) \tag{4.26}$$

To calculate the symmetric energy momentum tensor, we write the action as:

$$S = \int dx^4\sqrt{-g}[-\frac{1}{4}g_{\mu\rho}g_{\nu\sigma}F^{a\mu\nu}F^{a\rho\sigma} + \frac{1}{2}g_{\mu\rho}D^\rho\phi_a D^\mu\phi_a - V(\phi)] \tag{4.27}$$

Where $g_{\mu\nu}$ is the Minkowski metric and $g = detg_{\mu\nu}$. Then the energy momentum tensor is given by:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} \tag{4.28}$$

And we get:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}[-\frac{1}{2}\sqrt{-g}g^{\mu\nu}\mathcal{L} + \sqrt{-g}(-\frac{1}{2}g_{\rho\sigma}F^{a\rho\mu}F^{a\sigma\nu} + \frac{1}{2}D^\mu\phi^a D^\nu\phi^a)] \Rightarrow$$

$$T^{\mu\nu} = -F^{a\ \mu}_{\ \sigma}F^{a\sigma\nu} + D^\mu\phi^a D^\nu\phi^a - g^{\mu\nu}\mathcal{L} \tag{4.29}$$

The component T^{00} is the energy density of the system[3]:

$$T^{00} = -\frac{1}{2}F^a_{0i}F^{a0i} + \frac{1}{4}F^a_{ij}F^{aij} + \frac{1}{2}(D^0\phi^a D^0\phi^a + D^i\phi^a D^i\phi^a) + \frac{\lambda}{4}(\phi^a\phi^a - u^2)^2 \Rightarrow$$

---

[2] Yang-Mills are field theories invariant under a Lie group G

[3] The indices i and j are spatial.

$$T^{00} = \frac{1}{2}(E_i^a E_i^a + B_i^a B_i^a) + \frac{1}{2}(D^0\phi^a D^0\phi^a + D^i\phi^a D^i\phi^a) + \frac{\lambda}{4}(\phi^a\phi^a - u^2)^2$$

Where $E_i^a = F_{0i}^a$ and $B_i^a = -\frac{1}{2}\epsilon_{ijk}F^{ajk}$ are the 'color' electric and magnetic fields. Because of gauge invariance we can pick $A_0^a = 0$. We are looking for the minimal energy, so we set static field configurations for ϕ^a. The energy is then:

$$E = \int d^3x [\frac{1}{2}(E_i^a E_i^a + B_i^a B_i^a + D_i\phi^a D_i\phi^a) + \frac{\lambda}{4}(\phi^a\phi^a - u^2)^2] \tag{4.30}$$

We observe that the minimal energy is obtained if:

$$F_{\mu\nu}^a = 0 \qquad D_i\phi^a = 0 \qquad \phi^a\phi^a = u^2 \tag{4.31}$$

These conditions define the Higgs vacuum of the system and the vacuum energy is zero. Now consider fluctuations H of ϕ around the vacuum $\phi^a\phi^a = u^2$, where the only non-zero component of the triplet is the third component.

$$\phi = \begin{pmatrix} 0 \\ 0 \\ H + u \end{pmatrix} \tag{4.32}$$

By rewriting the gauge potentials as $A_i^1 = \frac{A_i^+ + A_i^-}{\sqrt{2}}$ and $A_i^2 = \frac{A_i^+ - A_i^-}{\sqrt{2}}$ and by applying (4.32) we find:

$$\frac{1}{2}D^\mu\phi D_\mu\phi \ni \frac{u^2e^2}{2}(A_i^+ A_i^+ + A_i^- A_i^-)$$

$$V(\phi) = \frac{\lambda}{4}(\phi^a\phi^a - u^2)^2 \ni u^2\lambda H^2$$

Suggesting that the particle spectrum consists of 2 massive weak force bosons A_μ^+, A_μ^- with mass $m_W = ue$, a Higgs boson H with mass $m_H = \sqrt{2\lambda}u$ and a photon A_μ^3.

The unbroken subgroup of $SU(2)$ leaves $\phi = u(0,0,1)$ invariant. The generator associated with this subgroup is $\frac{\phi^a T^a}{u} = T^3$, which means that this subgroup is isomorphic to $U_{em}(1)$. We identify this generator as the electric charge operator

$$Q = e\frac{\phi^a T^a}{u} = eT^3 \tag{4.33}$$

And electromagnetic gauge potential is written as:

$$A_\mu^{em} = \frac{1}{u}\phi^a A_\mu^a = T^3 A_\mu^3 \tag{4.34}$$

Allowing us to write the covariant derivative as:

$$D_\mu = \partial_\mu + iQA_\mu^{em} \tag{4.35}$$

4.4. Topological Classification of the Solutions

Solutions of Georgi–Glashow model is much richer than one would naively expect. There are static solutions of (4.26) which are soliton-like. These type of stable solutions have finite energy density and at the asymptotic spatial they obtain the vacuum expectation value $\phi^a\phi^a = u^2$. The soliton theory[4] tell us that these solutions are stable, if they can not be deformed continuously to the trivial solution. The mathematical context, which describes this idea is homotopy theory[5].

---

[4] We describe briefly some aspects of soliton theory in appendix B
[5] An introduction for homotopy theory can be found in appendix A

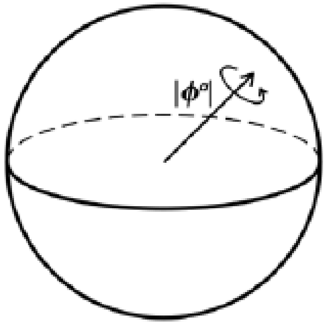

FIG. 4: Trivial Boundary.

In the Georgi-Glashow model the vacuum conditions $\phi^a\phi^a = u^2 \Rightarrow \phi^1\phi^1 + \phi^2\phi^2 + \phi^3\phi^3 = u^2$ tell us that the boundary manifold is sphere of radius u and we label it as S^2_∞. After the symmetry breaking $SU(2) \to U(1)$ the trivial boundary is associated with the Higgs field picking a particular direction, for example $\vec{\phi} = u\hat{z}$. The vacuum manifold on the other hand is the space of the broken symmetry $SU(2)/U(1) \approx S^2_{vac}$. Thus in order for soliton solutions to exist, it must exist a non-trivial map between S^2_∞ and S^2_{vac}:

$$\phi^a : \quad S^2_\infty \to S^2_{vac} \tag{4.36}$$

These maps form the second homotopy group of S^2_{vac}:

$$\pi_2(SU(2)/U(1)) = \pi_2(S^2) = Z \tag{4.37}$$

Such map/solution was proposed by Polyakov and is called hedgehog solution:

$$\phi^a = u\frac{r^a}{r}, \quad r \to \infty \tag{4.38}$$

Indeed in figure 5 we see due to the topological defect at $r = 0$, the two maps can not be deformed continuously into each other. In the following section we will see that this configuration indeed satisfies the properties we saw above.

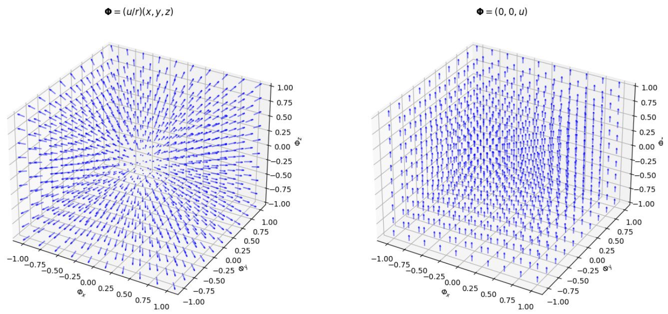

FIG. 5: Hedgehog solution vs the trivial map. The first map cannot be continuously transformed to the second one.

To sum up, the first step is to look if manifold of the solutions at the asymptotic spatial is connected non-trivially with the vacuum manifold. If this holds then soliton solutions are possible and then we solve the equation of motion with the appropriate asymptotic conditions.

4.5. Magnetic Charge

The hedgehog solution must satisfy the conditions (4.31), so that it corresponds to the vacuum energy. The third conditions gives for (4.38) :

$$(D_i)^a_b\phi_b = \partial_i(\frac{ur_a}{r}) + ie(A^c_iT^c)^a_b(\frac{ur_b}{r}) = 0 \Rightarrow$$

$$\frac{\delta_{ai}r^2 - r_i r_a}{r^3} + eA_i^c \epsilon_{cab}\frac{r_b}{r} = 0 \Rightarrow$$

$$\frac{r_b r_d}{r^3}(\delta_{ai}\delta_{bd} - \delta_{ad}\delta_{ib}) = -eA_i^c \epsilon_{cab}\frac{r_b}{r} \Rightarrow$$

$$\frac{r_b r_d}{r^3}\epsilon_{cab}\epsilon_{cid} = -eA_i^c \epsilon_{cab}\frac{r_b}{r} \Rightarrow$$

$$A_i^c = -\epsilon_{cid}\frac{r_d}{er^2} \Rightarrow$$

$$A_k^a(r) = \epsilon_{abk}\frac{r^b}{er^2}, \quad r \to \infty \tag{4.39}$$

This corresponds to the non-Abelian magnetic field :

$$B_a^i = (\nabla \times A^i)_a \Rightarrow$$

$$B_a^i = \epsilon_{abc}\epsilon_{inc}\partial_b(\frac{r^n}{er^2}) \Rightarrow$$

$$B_a^i = \frac{\delta_{ai}\delta_{bn} - \delta_{an}\delta_{bi}}{er^4}(\delta_{nb}r^2 - r^n r^b) \Rightarrow$$

$$B_k^a(r) = \frac{r^a r^k}{er^4} \tag{4.40}$$

This magnetic field falls as $1/r^2$, something that we expect from coulomb like magnetic field. This suggests to identify it as a magnetic field of the monopole, since the electric part of the field tensor is zero, $F_{0i}^a = 0$. We can construct a general solution of $D_\mu \phi^a = 0$ for $\phi^a \phi^a = u^2$

$$D_\mu \phi^a = 0 \Rightarrow$$

$$\partial_\mu \phi^a = -e\epsilon_{cab}A_\mu^c \phi^b \Rightarrow$$

$$\epsilon_{dma}\phi^m \partial_\mu \phi^a = e\epsilon_{dma}\epsilon_{cba}A_\mu^c \phi^b \phi^m \Rightarrow$$

$$\epsilon_{dma}\phi^m \partial_\mu \phi^a = e(\delta_{dc}\delta_{mb} - \delta_{db}\delta_{mc})A_\mu^c \phi^b \phi^m \Rightarrow$$

$$\epsilon_{dma}\frac{1}{e}\phi^m \partial_\mu \phi^a = A_\mu^d \phi^m \phi^m - A_\mu^m \phi^m \phi^d \Rightarrow$$

$$\frac{1}{e}\epsilon^{dma}\phi^m \partial_\mu \phi^a = u^2 A_\mu^d - uA_\mu^{em}\phi^d \Rightarrow$$

$$A_\mu^d = \frac{1}{u}A_\mu^{em}\phi^d + \frac{1}{eu^2}e_{dma}\phi^m \partial_\mu \phi^a \tag{4.41}$$

Where we have used the projection (4.34) of A_μ^a to the unbroken subgroup $U(1)$. Using this formula now we can calculate the field tensor (4.24):

$$F_{\mu\nu}^a = \partial_\mu[\frac{1}{u}A_\nu^{em}\phi^a + \frac{1}{eu^2}\epsilon_{amb}\phi^m \partial_\nu \phi^b] - \partial_\nu[\frac{1}{u}A_\mu^{em}\phi^a + \frac{1}{eu^2}\epsilon_{amb}\phi^m \partial_\mu \phi^b]$$

$$-e\epsilon_{ajk}[\frac{1}{u}A_\mu^{em}\phi^j + \frac{1}{eu^2}\epsilon_{jmb}\phi^m\partial_\mu\phi^b][\frac{1}{u}A_\nu^{em}\phi^k + \frac{1}{eu^2}\epsilon_{knc}\phi^n\partial_\nu\phi^c] =$$

$$\frac{\phi^a}{u}(\partial_\mu A_\nu^{em} - \partial_\nu A_\mu^{em}) - \frac{1}{u}(A_\mu^{em}\partial_\nu\phi^a - A_\nu^{em}\partial_\mu\phi^a) + \frac{1}{eu^2}\epsilon_{amb}[\partial_\nu\phi^b\partial_\mu\phi^m - \partial_\mu\phi^b\partial_\nu\phi^m]$$

$$-\frac{1}{u^3}[A_\mu^{em}\phi^j\partial_\nu\phi^b\phi^m\epsilon_{ajk}\epsilon_{kmb} + A_\nu^{em}\epsilon_{ajk}\epsilon_{jmb}\phi^m\partial_\mu\phi^b\phi^k] - \frac{1}{eu^4}\epsilon_{ajk}\epsilon_{jmb}\epsilon_{knc}(\phi^m\partial_\mu\phi^b\phi^n\partial_\nu\phi^c)$$

Note that:

$$\frac{1}{eu^2}\epsilon_{amb}[\partial_\nu\phi^b\partial_\mu\phi^m - \partial_\mu\phi^b\partial_\nu\phi^m] = \frac{2}{eu^2}\epsilon_{amb}\partial_\mu\phi^m\partial_\nu\phi^b$$

Focus on the term $-\frac{1}{u^3}[A_\mu^{em}\phi^j\partial_\nu\phi^b\phi^m\epsilon_{ajk}\epsilon_{kmb} + A_\nu^{em}\epsilon_{ajk}\epsilon_{jmb}\phi^m\partial_\mu\phi^b\phi^k]$ and use the formula $\epsilon_{abc}\epsilon_{amn} = (\delta_{bm}\delta_{cn} - \delta_{bn}\delta_{cm})$:

$$\frac{1}{u^3}[-\phi^a\phi^b\partial_\mu\phi^a A_\nu^{em} + u^2\partial_\mu\phi^b A_\nu^{em} + \phi^a\phi^b\partial_\nu\phi^b A_\mu^{em} - u^2\partial_\nu\phi^a A_\mu^{em}]$$

The second and the fourth term cancel the $1/u$ terms in the full expression. For the first and the third, we write for example the first

$$\phi^b\phi^a\partial_\mu\phi^a = \frac{\phi^b}{2}\partial_\mu(\frac{\phi^a\phi^a}{2}) = \frac{\phi^b}{2}\partial_\mu(\frac{u^2}{2}) = 0$$

Thus these terms are equal to zero. As for the term $-\frac{1}{eu^4}\epsilon_{ajk}\epsilon_{jmb}\epsilon_{knc}(\phi^m\partial_\mu\phi^b\phi^n\partial_\nu\phi^c)$ we write:

$$-\frac{1}{eu^4}\epsilon_{ajk}\epsilon_{jmb}\epsilon_{knc}(\phi^m\partial_\mu\phi^b\phi^n\partial_\nu\phi^c) = \frac{1}{eu^4}\epsilon_{knc}(\delta_{am}\delta_{kb} - \delta_{ab}\delta_{km})(\phi^m\phi^n\partial_\mu\phi^b\partial_\nu\phi^c)$$

$$\frac{1}{eu^4}\epsilon_{knc}(\phi^a\phi^n\partial_\mu\phi^k\partial_\nu\phi^c - \phi^k\phi^n\partial_\mu\phi^a\partial_\nu\phi^c) = -\frac{1}{eu^4}\epsilon_{cnk}\phi^a\phi^n\partial_\mu\phi^k\partial_\nu\phi^c$$

The field tensor is given then by

$$F_{\mu\nu}^a = \frac{\phi^a}{u}(\partial_\mu A_\nu^{em} - \partial_\nu A_\mu^{em}) + \frac{2}{eu^2}\epsilon_{amb}\partial_\mu\phi^m\partial_\nu\phi^b - \frac{1}{eu^4}\epsilon_{cnk}\phi^a\phi^n\partial_\mu\phi^k\partial_\nu\phi^c$$

$$F_{\mu\nu}^a = \frac{\phi^a}{u}(\partial_\mu A_\nu^{em} - \partial_\nu A_\mu^{em}) + \frac{2}{eu^2}\epsilon_{amb}\partial_\mu\phi^m\partial_\nu\phi^b - \frac{1}{eu^4}\epsilon_{cnk}\phi^a\phi^n\partial_\mu\phi^k\partial_\nu\phi^c$$

$$\phi^a F_{\mu\nu}^a = u(\partial_\mu A_\nu^{em} - \partial_\nu A_\mu^{em}) + \epsilon_{amb}\frac{2}{eu^2}\phi^a\partial_\mu\phi^m\partial_\nu\phi^b - \frac{1}{eu^2}\epsilon_{cnk}\phi^n\partial_\mu\phi^k\partial_\nu\phi^c$$

$$\frac{\phi^a F_{\mu\nu}^a}{u} = (\partial_\mu A_\nu^{em} - \partial_\nu A_\mu^{em}) + \frac{1}{eu^3}(\epsilon_{ijk}\phi^i\partial_\mu\phi^j\partial_\nu\phi^k)$$

$$F_{\mu\nu} = \frac{\phi^a F_{\mu\nu}^a}{u} = (\partial_\mu A_\nu^{em} - \partial_\nu A_\mu^{em}) + \frac{1}{eu^3}\epsilon_{ijk}\phi^i\partial_\mu\phi^j\partial_\nu\phi^k \tag{4.42}$$

In the topolagically trivial sector where $\phi^a = u(0,0,0)$ we get from (4.42):

$$F_{\mu\nu} = \partial_\mu A_\nu^{em} - \partial_\nu A_\mu^{em}$$

This expression coincides with field tensor of Maxwell's theory and satisfies the Bianchi identity. Recall that Bianchi identity corresponds to the homogeneous Maxwell equations, thus in the topologically trivial sector monopole does not exist.

In the case of non-trivial map like hedgehog solution Bianchi identity doesn't hold anymore and we get:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \Rightarrow$$

$$\epsilon^{a\lambda\mu\nu}\partial_\lambda F_{\mu\nu} = 0 \Rightarrow$$

$$\epsilon^{a\lambda\mu\nu}\partial_\lambda[(\partial_\mu A^{em}_\nu - \partial_\nu A^{em}_\mu) + \frac{1}{eu^3}\epsilon_{ijk}\phi^i\partial_\mu\phi^j\partial_\nu\phi^k = 0 \Rightarrow$$

$$\partial_\lambda \tilde{F}_{\lambda a} = \frac{1}{2eu^3}\epsilon_{ijk}\epsilon_{a\lambda\mu\nu}\partial_\lambda\phi^i\partial_\mu\phi^j\partial_\nu\phi^k \Rightarrow$$

$$\partial_\lambda \tilde{F}^{a\lambda} = k^a \tag{4.43}$$

$$k^a = \frac{1}{2eu^3}\epsilon^{a\lambda\mu\nu}\epsilon_{ijk}\partial_\lambda\phi^i\partial_\mu\phi^j\partial_\nu\phi^k \tag{4.44}$$

Where $\tilde{F}^{a\lambda} = \frac{1}{2}\epsilon^{a\lambda\mu\nu}(\partial_\mu A^{em}_\nu - \partial_\nu A^{em}_\mu)$ is the dual electromagnetic tensor and we identify k^a as a magnetic current. This current is conserved:

$$\partial_a k^a = 0 \tag{4.45}$$

Since the terms after the action of ∂_a are antisymmetric-symmetric products. We should note that this is not a Noether current, because it does not originate from a continuous symmetry. The magnetic charge that composes this current is called **t'Hooft-Polyakov monopole**.

We can calculate the magnetic charge by taking the zeroth component of the magnetic current :

$$g = \int d^3x k^0 = \frac{1}{2eu^3}\int d^3x \epsilon^{\lambda\mu\nu}\epsilon_{ijk}\partial_\lambda\phi^i\partial_\mu\phi^j\partial_\nu\phi^k = \frac{1}{2eu^3}\int d^3x\epsilon^{\lambda\mu\nu}\epsilon_{ijk}\partial_\lambda[\phi^i\partial_\mu\phi^j\partial_\nu\phi^k] =$$

$$g = \frac{1}{2eu^3}\oint dS_\lambda\epsilon^{\lambda\mu\nu}\epsilon_{ijk}\phi^i\partial_\mu\phi^j\partial_\nu\phi^k$$

Where in the last line we used Stoke's theorem and the integral has been taken over surface S^2 on the spatial asymptotic. We can parameterise this surface with coordinates ξ^a for $a = 1, 2$. Then it holds that

$$\partial_\nu\phi^k = \frac{\partial\xi^a}{\partial r^\nu}\frac{\partial\phi^k}{\partial\xi^a}$$

$$dS = det(\frac{\partial r}{\partial\xi})d^2\xi \Rightarrow dS_\lambda = \frac{1}{2!}\epsilon_{\lambda mn}\epsilon_{ab}\frac{\partial r^m}{\partial\xi^a}\frac{\partial r^n}{\partial\xi^b}d^2\xi$$

Then by also letting $\hat{\phi}^a = \phi^a/u$ we get for the magnetic charge:

$$g = \frac{1}{2e}\oint dS_\lambda\epsilon^{\lambda\mu\nu}\epsilon_{ijk}\hat{\phi}^i\partial_\mu\hat{\phi}^j\partial_\nu\hat{\phi}^k = \frac{1}{4e}\oint d^2\xi\epsilon_{\lambda mn}\epsilon_{uv}\frac{\partial r^m}{\partial\xi^u}\frac{\partial r^n}{\partial\xi^v}\epsilon^{\lambda\mu\nu}\epsilon_{ijk}\frac{\partial\xi^b}{\partial r^\mu}\hat{\phi}^i\frac{\partial\hat{\phi}^j}{\partial\xi^b}\frac{\partial\xi^a}{\partial r^\nu}\frac{\partial\hat{\phi}^k}{\partial\xi^a} \Rightarrow$$

$$g = \frac{1}{4e}\oint d^2\xi(\delta^\mu_m\delta^\nu_n - \delta^\mu_n\delta^\nu_m)\epsilon_{uv}\frac{\partial r^m}{\partial\xi^u}\frac{\partial r^n}{\partial\xi^v}\epsilon_{ijk}\frac{\partial\xi^b}{\partial r^\mu}\hat{\phi}^i\frac{\partial\hat{\phi}^j}{\partial\xi^b}\frac{\partial\xi^a}{\partial r^\nu}\frac{\partial\hat{\phi}^k}{\partial\xi^a} \Rightarrow$$

$$g = \frac{1}{4e}\oint d^2\xi\epsilon_{uv}\epsilon_{ijk}(\frac{\partial r^\mu}{\partial\xi^u}\frac{\partial r^\nu}{\partial\xi^v}\frac{\partial\xi^b}{\partial r^\mu}\frac{\partial\xi^a}{\partial r^\nu} - \frac{\partial r^\nu}{\partial\xi^u}\frac{\partial r^\mu}{\partial\xi^v}\frac{\partial\xi^b}{\partial r^\mu}\frac{\partial\xi^a}{\partial r^\nu})\hat{\phi}^i\frac{\partial\hat{\phi}^j}{\partial\xi^b}\frac{\partial\hat{\phi}^k}{\partial\xi^a} \Rightarrow$$

$$g = \frac{2}{4e}\oint d^2\xi\epsilon_{ba}\epsilon_{ijk}\hat{\phi}^i\frac{\partial\hat{\phi}^j}{\partial\xi^b}\frac{\partial\hat{\phi}^k}{\partial\xi^a} = \frac{1}{e}\oint d^2\xi\frac{1}{2}\epsilon_{ba}\epsilon_{ijk}\hat{\phi}^i\frac{\partial\hat{\phi}^j}{\partial\xi^b}\frac{\partial\hat{\phi}^k}{\partial\xi^a} \Rightarrow$$

$$g = \frac{1}{e}\oint d^2\xi\sqrt{\mathbf{g}} = \frac{4\pi n}{e}, \quad n \in \mathbb{Z} \tag{4.46}$$

Where $\mathbf{g}$ is the determinant of the metric of S^2 on the spatial asymptotic. The integer n originates from the number of times isovector $\hat{\phi}$ covers the sphere S^2_{vac}. As for the 4π factor it is clear that originates from the integration of the unit sphere S^2_{vac}. This condition is the non Abelian anologue of the Dirac quantization condition (3.14).

$$eg = 4\pi n, \quad n \in \mathbb{Z} \tag{4.47}$$

4.6. 't Hooft-Polyakov Ansatz

The asymptotic solutions (4.38), (4.39) are boundary conditions that solutions of (4.26) must respect. Again we consider static fields and since the system of equations possesses spherical symmetry, 't Hooft and Polyakov suggest a solution of the form:

$$\vec{\phi} = uH(r)\hat{r} \Rightarrow$$

$$\phi^a = u\frac{r^a}{r}H(r) \tag{4.48}$$

$$\vec{A}_n = \frac{1}{e}[1 - K(r)]\hat{r} \times \partial_n \hat{r} \Rightarrow$$

$$A^a_n = \epsilon_{amn}\frac{r^m}{er^2}[1 - K(r)] \tag{4.49}$$

$$A^a_0 = 0 \tag{4.50}$$

This is called **'t Hooft Polyakov ansatz**. As $r \to \infty$ the functions $H(r), K(r)$ must satisfy:

$$H(r) \xrightarrow{r\to\infty} 1 \tag{4.51}$$

$$K(r) \xrightarrow{r\to\infty} 0 \tag{4.52}$$

$$H(r) \xrightarrow{r\to 0} 0 \tag{4.53}$$

$$K(r) \xrightarrow{r\to 0} 1 \tag{4.54}$$

So that boundary conditions (4.38), (4.39) hold at infinity and the fields are regular at $r = 0$. We could try to substitute these solutions to equations of motions (4.26), but we will try something more convenient. We are going to substitute them to the energy functional (4.30) and we will consider monopole solution that corresponds to a local minimum of (4.30). We get:

$$V(\phi^a\phi^a) = \frac{\lambda}{4}[\phi^a\phi^a - u^2]^2 = \frac{\lambda}{4}[u^2H^2 - u^2]^2 = \frac{\lambda u^4}{4}[H^2 - 1]^2 \tag{4.55}$$

$$E^a_n = F^a_{0n} = \partial_0 A^a_n - \partial_n A^a_0 - e\epsilon^{abc}A^b_0 A^c_n = 0 \tag{4.56}$$

$$B^a_i = \frac{1}{2}\epsilon_{ijk}F^{ajk} = \frac{1}{2}\epsilon_{ijk}(\partial^j A^{ak} - \partial^k A^{aj} - e\epsilon^{abc}A^{bj}A^{ck}) \Rightarrow$$

$$B^a_i = \frac{1}{2}\epsilon_{ijk}(\frac{2}{er^2}(1-K)\epsilon^{ajk} - \frac{1}{er^4}[2(1-K) + rK'][\epsilon^{amk}r_m r^j - r_m r^k \epsilon^{amj}] - \frac{1}{er^4}\epsilon^{cab}\epsilon^{bmj}\epsilon^{cnk}(1-K)^2 r_m r_n) \Rightarrow$$

$$B^a_i = 2\delta^a_i(1-K)\frac{1}{er^2} - \frac{1}{er^4}[2(1-K) + rK'][r^2\delta^a_i - r_i r^a] - \frac{1}{er^4}(1 - 2K + K^2)r_i r^a \Rightarrow$$

$$B^a_i = \frac{2}{er^4}r_i r^a - \frac{2K}{er^4}r_i r^a - \frac{K'}{er}\delta^a_i + \frac{K'}{er^3}r_i r^a - \frac{1}{er^4}r_i r^a + \frac{2K}{er^4}r_i r^a - \frac{K^2}{er^4}r_i r^a$$

$$B_i^a = \frac{1}{er^4}[1 + rK' - K^2]r_i r^a - \delta_i^a \frac{K'}{er} \Rightarrow$$

$$B_i^a B_i^a = \frac{1}{e^2r^4}[1 + r^2(K')^2 + K^4 - 2K^2 - 2rK^2K' + 2rK'] + \frac{3(K')^2}{e^2r^2} - 2\frac{K' + r(K')^2 - K^2K'}{e^2r^3} \Rightarrow$$

$$B_i^a B_i^a = \frac{1 - 2K^2 + K^4}{e^2r^4} + 2\frac{(K')^2}{e^2r^2} \Rightarrow$$

$$B_i^a B_i^a = \frac{(1 - K^2)^2}{e^2r^4} + \frac{2(K')^2}{e^2r^2} \tag{4.57}$$

$$(D_i)^a_b \phi^b = \partial_i(\phi^a) + eA_i^c \epsilon_{cab}\phi^b = \frac{uH}{r^3}[\delta_i^a r^2 - r_i r^a] + \frac{ur^a r_i}{r^2}H' + \epsilon_{cmi}\epsilon_{cab}\frac{r^m r^b}{r^3}uH(1 - K) \Rightarrow$$

$$(D_i)^a_b \phi^b = \frac{uH}{r^3}[\delta_i^a r^2 - r_i r^a] + \frac{ur^a r_i}{r^2}H' + [r^a r_i - r^2\delta_i^a][\frac{uH}{r^3} - \frac{uHK}{r^3}] \Rightarrow$$

$$(D_i)^a_b \phi^b = \frac{ur^a r_i}{r^2}[H' - \frac{HK}{r}] + \frac{uHK}{r}\delta_i^a$$

$$D_i\phi D_i\phi = u^2[(H')^2 + \frac{(HK)^2}{r^2} - 2\frac{H}{r}KH'] + \frac{3u^2}{r^2}H^2K^2 + \frac{u^2}{r^2}(\frac{2HKH'}{r} - 2(\frac{HK}{r})^2) \Rightarrow$$

$$D_i\phi D_i\phi = u^2(H')^2 + \frac{2u^2}{r^2}(HK)^2 \tag{4.58}$$

Then we substitute (4.55),(4.56), (4.57), (7.67) into (4.30):

$$E = \int d^3x[\frac{(1 - K^2)^2}{2e^2r^4} + \frac{(K')^2}{e^2r^2} + \frac{u^2}{2}(H')^2 + \frac{u^2}{r^2}(HK)^2 + \frac{\lambda u^4}{4}[H^2 - 1]^2] \Rightarrow$$

$$E = 4\pi\int_0^\infty dr[\frac{(1 - K^2)^2}{2e^2r^2} + \frac{(K')^2}{e^2} + \frac{u^2r^2}{2}(H')^2 + u^2(HK)^2 + \frac{\lambda u^4 r^2}{4}[H^2 - 1]^2] \tag{4.59}$$

We let the energy density as:

$$\mathcal{H} = \frac{(1 - K^2)^2}{2e^2r^2} + \frac{(K')^2}{e^2} + \frac{u^2r^2}{2}(H')^2 + u^2(HK)^2 + \frac{\lambda u^4 r^2}{4}[H^2 - 1]^2 \tag{4.60}$$

The Euler-Lagrange equations that correspond to the local minimum of the energy of the system are:

$$\frac{\partial\mathcal{H}}{\partial H} = \frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial H'}) \tag{4.61}$$

$$\frac{\partial\mathcal{H}}{\partial K} = \frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial K'}) \tag{4.62}$$

Then H and K satisfy the following differential equations:

$$\frac{\partial\mathcal{H}}{\partial H} = 2u^2K^2H + \lambda u^4 r^2 H[H^2 - 1]$$

$$\frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial H'}) = u^2r^2H'' + 2u^2rH'$$

$$\Rightarrow H'' = 2K^2 H\frac{1}{r^2} - \frac{2}{r}H' + \lambda u^2 H[H^2 - 1] \tag{4.63}$$

$$\frac{\partial \mathcal{H}}{\partial K} = -2K\frac{1-K^2}{e^2 r^2} + 2u^2 H^2 K$$

$$\frac{d}{dr}(\frac{\partial \mathcal{H}}{\partial K}) = 2\frac{K''}{e^2}$$

$$\Rightarrow K'' = \frac{K(K^2-1)}{r^2} + e^2 u^2 H^2 K \tag{4.64}$$

Now let $\xi = uer$ and the equations (4.63), (4.64) become by noting that:

$$\frac{d}{dr} = eu\frac{d}{d\xi} \qquad\qquad \frac{d^2}{dr^2} = e^2 u^2 \frac{d^2}{d\xi^2}$$

And we get:

$$\frac{d^2 K}{d\xi^2} = \frac{K(K^2-1)}{\xi^2} + H^2 K \tag{4.65}$$

$$\frac{d^2 H}{d\xi^2} = \frac{2K^2 H}{\xi^2} - \frac{2}{\xi}\frac{dH}{d\xi} + \frac{\lambda^2}{e^2}H(H^2-1) \tag{4.66}$$

Equations (4.66) and (4.65) can be solved only numerical. This does not hold in the case of $\lambda = 0$, which corresponds to the massless Higgs. This situation is called **Bogomol'nyi limit**. We will study this limit later on. In figure 6

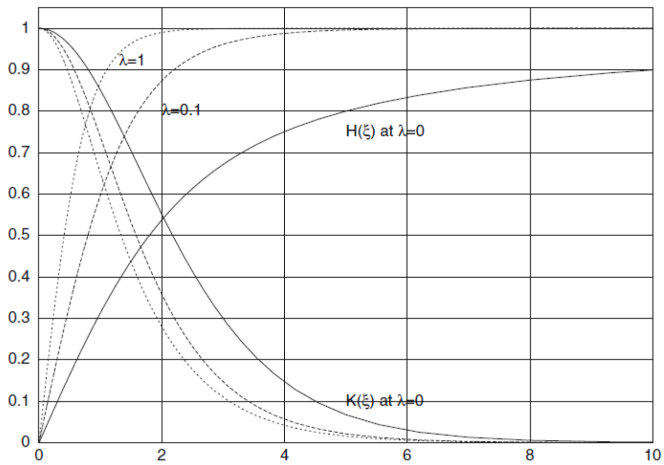

FIG. 6: Profile functions $K(\xi)$ and $H(\xi)/\xi$ are shown for the 't Hooft– Polyakov monopole at $\lambda = 0$, $\lambda = 0.1$ and $\lambda = 1$ [3]

we see that the functions $H(\xi)$, $K(\xi)$ approach their asymptotic values quite fast. The gauge fields A^a_μ approach its asymptotic value outside a core with radius of order R_c. This defines the core of the of monopole. Then we can estimate [14] the scale of the energy (4.59) as:

$$E = E_{mag} + E_s \sim \frac{4\pi}{e^2}[\frac{1}{R_c} + u^2 e^2 R_c] \tag{4.67}$$

Where we have ignored the Higg's potential dependence from the energy, since we will see in (4 4.9), that the energy of the configuration does not depend sensitively on the coupling λ. The first term describes the energy of the magnetic field inside the core and the second the one the energy of the scalar field inside the core. By minimizing this expression we obtain the value of R_c:

$$\frac{dE}{dR_c} = 0 \Rightarrow$$

$$R_c = (ue)^{-1} = \frac{1}{M_W} \sim 10^{-28} cm \tag{4.68}$$

Note that the energy of the system (4.59) indeed tends to zero as $r \to \infty$, which is the vacuum value. We can also check if the system contains magnetic charge by calculating the magnetic flux:

$$g = \oint dS_i B_i = \oint dS_i \frac{\phi^a B_i^a}{u} = \frac{1}{u} \int d^3x (B_i^a \partial_i \phi^a + \phi^a \partial_i B_i^a)$$

We can use the formula (4.11):

$$\phi^a \partial_i B_i^a = -\frac{e}{2} \phi^a \epsilon^{abc} \epsilon_{ijk} (\partial_i A^{bj} A^{ck} + A^{bj} \partial_i A^{ck}) = e \epsilon_{ijk} \epsilon_{cab} \partial_i A_j^a A_k^c \phi^b = (\epsilon^{0ijk} \partial_i A_j^a) e A_k^c \epsilon_{cab} \phi^b = e B_k^a A_k^c \epsilon_{cab} \phi^b$$

Thus we get:

$$g = \frac{1}{u} \int d^3x B_i^a D_i \phi^a \tag{4.69}$$

We continue and we get:

$$g = \frac{1}{u} \int d^3x B_i^a D_i \phi^a = \frac{1}{u} \int d^3x [\frac{1}{er^4} [1 + rK' - K^2] r_i r^a - \delta_i^a \frac{K'}{er}] [\frac{ur^a r_i}{r^2} [H' - \frac{HK}{r}] + \frac{uHK}{r} \delta_i^a] \Rightarrow$$

$$g = \frac{4\pi}{e} \int_0^\infty dr [(1 + rK' - K^2)(H' - \frac{HK}{r}) + \frac{HK}{r}(1 + rK' - K^2) - K'r(H' - \frac{HK}{r}) - 3K'KH] \Rightarrow$$

$$g = \frac{4\pi}{e} \int_0^\infty dr [H' - \frac{HK}{r} + rK'H' - K'KH - K^2H' + \frac{HK^3}{r} + \frac{HK}{r} + HKK' - \frac{HK^3}{r} - K'H'r + HKK' - 3KK'H] \Rightarrow$$

$$g = \frac{4\pi}{e} \int_0^\infty dr [H'(1 - K^2) - 2KK'H] = \frac{4\pi}{e} \int_0^\infty dr \frac{d}{dr} [H(1 - K^2)] = \frac{4\pi}{e} [H(1 - K^2)]_{r=0}^{r=\infty} = \frac{4\pi}{e}$$

Thus 't Hooft-Polyakov ansatz provides a solution that contains magnetic charge and satisfies the appropriate asymptotic behavior. Notice that the magnetic charge in this configurations is not fixed by an integer, but it corresponds to $n = 1$ in (4.47). On the other hand, at the boundary of the configurations, charges have a similar quantization condition like the Dirac quantization (3.14). This is because at large distances the $SU(2)$ symmetry is broken to $U(1)_{em}$ and the magnetic soliton resembles a Dirac monopole. But at distances smaller than the wavelength of the W bosons inside the core of the monopole the $SU(2)$ symmetry is restored and we get a pure GUT magnetic soliton.

4.7. Julia-Zee Dyon

In the 't Hooft Polyakov ansatz, we let $A_0^a = 0$. This can be generalised to a non zero time component of the vector field by setting it:

$$A_0^a = \frac{1}{e} V(r) \hat{r}^a \tag{4.70}$$

Then we can write the solution in covariant form:

$$\vec{A}_\mu = \frac{1}{e} V(r) \partial_\mu t \hat{r} + \frac{1}{e} [1 - K(r)] \hat{r} \times \partial_\mu \hat{r} \tag{4.71}$$

This field configuration is a non-Abelian dyon, a configuration which has both electric and magnetic charges. The non-Abelian electric field is given by:

$$E_i^a = F_{0i}^a = \partial_0 A_i^a - \partial_i A_0^a - e\epsilon_{abc}A_0^b A_i^c \Rightarrow$$

$$E_i^a = -\frac{V}{er}\delta_i^a - \frac{\hat{r}^a\hat{r}_i}{er}(rV' - V) - \frac{1}{er}V(1-K)(\hat{r}^a\hat{r}_i - \delta_i^a)$$

$$E_i^a = -\frac{1}{e}\frac{VK}{r}\delta_i^a + \frac{1}{er}[VK - rV']\hat{r}^a\hat{r}_i \tag{4.72}$$

$$\frac{1}{2}E_i^a E_i^a = \frac{1}{2r^2e^2}[r^2(V')^2 + 2V^2K^2]$$

The energy functional (4.30) becomes:

$$E = 4\pi\int_0^\infty dr[\frac{1}{2e^2}(2V^2K^2 + r^2(V')^2) + \frac{(1-K^2)^2}{2e^2r^2} + \frac{(K')^2}{e^2} + \frac{u^2r^2}{2}(H')^2 + u^2(HK)^2 + \frac{\lambda u^4 r^2}{4}[H^2-1]^2] \tag{4.73}$$

With energy density:

$$\mathcal{H} = \frac{1}{e^2}(V^2K^2 + \frac{r^2(V')^2}{2}) + \frac{(1-K^2)^2}{2e^2r^2} + \frac{(K')^2}{e^2} + \frac{u^2r^2}{2}(H')^2 + u^2(HK)^2 + \frac{\lambda u^4r^2}{4}[H^2-1]^2 \tag{4.74}$$

From the energy functional we can estimate the radius of the core of the dyon and since electric terms will give the same behavior as the scalar kinetic term. Thus we expect a core of radius $R_c \sim 10^{-28}cm$.

The equation of motion for $H(\xi)$ again is given by (4.66) after the change of variable $\xi = eur$. For $K(r)$ and $V(r)$ we get:

$$\frac{\partial\mathcal{H}}{\partial K} = 2KV^2\frac{1}{e^2} - 2K\frac{1-K^2}{e^2r^2} + 2u^2H^2K \qquad \frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial K'}) = 2\frac{K''}{e^2}$$

$$\xrightarrow{\xi=eur} K'' = K(H^2+V^2) + \frac{K(K^2-1)}{\xi^2} \tag{4.75}$$

$$\frac{\partial\mathcal{H}}{\partial V} = 2\frac{1}{e^2}VK^2 \qquad \frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial V'}) = \frac{1}{e^2}(2rV' + r^2V'')$$

$$\xrightarrow{\xi=eur} V'' = \frac{2VK^2}{\xi^2} - \frac{2V'}{\xi} \tag{4.76}$$

This system of equations can be solved numerically. Note that the function $V(r)$ has an asymptotic behavior:

$$V(r) \xrightarrow{r\to 0} 0 \qquad V(r) \xrightarrow{r\to\infty} C \tag{4.77}$$

Where the constant C is associated with the electric charge as we will see. Note that from equation of motion (4.26) for the gauge field, we can obtain by letting $\mu = 0$ and ν spatial:

$$D_iF_{0i}^a = -e\epsilon_{abc}\phi^b D^0\phi^c \Rightarrow D_iE_i^a = -\frac{u^2}{e}\epsilon_{abc}H^2(r)V(r)\frac{r^b}{r}\frac{r^a}{r}\frac{r^c}{r} = 0 \Rightarrow$$

$$D_iE_i^a = 0 \tag{4.78}$$

$$\Rightarrow \phi^a\partial_iE_i^a = eE_i^a\epsilon_{cab}A_i^c\phi^b$$

Just as with magnetic charge we can calculate the electric charge from the Gauss law:

$$q = \frac{1}{u}\oint dS_n E_n^a \phi^a = \frac{1}{u}\int d^3x(\phi^a \partial_n E_n^a + E_n^a \partial_n \phi^a)$$

$$q = \frac{1}{u}\int d^3x E_n^a D_n \phi^a \tag{4.79}$$

Where we have make use of (4.78). We have:

$$q = \frac{4\pi}{u}\int_0^\infty dr r^2[-\frac{1}{e}\frac{VK}{r}\delta_i^a + \frac{1}{er}[VK - rV']\hat{r}^a\hat{r}_i][u\hat{r}^a\hat{r}_i[H' - \frac{HK}{r}] + \frac{uHK}{r}\delta_i^a] \Rightarrow$$

$$q = \frac{4\pi}{e}\int_0^\infty dr r^2[-\frac{VK}{r}\delta_i^a + \frac{1}{r}[VK - rV']\hat{r}^a\hat{r}_i][\hat{r}^a\hat{r}_i(H' - \frac{HK}{r}) + \frac{HK}{r}\delta_i^a] \Rightarrow$$

$$q = -\frac{4\pi}{e}\int_0^\infty dr[2HVK^2 + r^2 H'V'] \Rightarrow$$

$$q = -\frac{4\pi}{e}\int_0^\infty d\xi[2HVK^2 + \xi^2 H'V'] \Rightarrow$$

$$\xrightarrow{(4.76)} q = -\frac{4\pi}{e}\int_0^\infty d\xi[\xi^2 HV'' + 2\xi V'H + \xi^2 V'H'] = -\frac{4\pi}{e}\int_0^\infty d\xi[\xi^2(HV')' + (\xi^2)'V'H] = -\frac{4\pi}{e}\int_0^\infty d\xi\frac{d[\xi^2 HV']}{d\xi}$$

To compute this note that (4.76)

$$\xi^2 V'' = 2VK^2 - 2V'\xi \Rightarrow \int_0^\infty d\xi V''\xi^2 = 2\int_0^\infty d\xi VK^2 - 2\int_0^\infty d\xi\xi V' \Rightarrow$$

$$V'\xi^2|_{\xi=\infty} - \int_0^\infty d\xi 2\xi V' = \int_0^\infty d\xi 2VK^2 - 2\int_0^\infty d\xi\xi V' \Rightarrow$$

$$V'H\xi^2|_{\xi=\infty} = \int_0^\infty d\xi 2VK^2 = -A$$

Thus we get:

$$q = \frac{4\pi A}{e} = gA \tag{4.80}$$

Where g is the magnetic charge of the configuration. Notice that for $A = 0$, electric charge vanishes and as $r \to \infty$, $V \to 0$. This suggests that for $V = 0 \Rightarrow A_0^a = 0$ the configuration of the system describes a magnetic monopole. Also note that A is an arbitrary parameter and it is not quantized on the classical level unlike the magnetic charge quantization, which has topological origins. Anti-dyon solution is also possibly since we could let $A \to -A$, or we could perform a gauge transformation that changes $A_\mu^0 \to -A_\mu^0$. Finally, A_0^a is parallel to the Higgs triplet ϕ^a and thus we can consider it as an additional triplet of scalar fields. This is called **Julia-Zee correspondence** $\phi^a \Leftrightarrow A_0^a$.

4.8. The Bogomol'nyi Limit

Let's try to calculate a lower bound for the dyon and magnetic monopole mass. Our starting point is the energy of static configuration (4.30) and we write in a general form:

$$E = \int d^3x[\frac{1}{2}(E_i^a E_i^a + B_i^a B_i^a + D_i\phi^a D_i\phi^a) + \frac{\lambda}{4}(\phi^a\phi^a - u^2)^2] \Rightarrow$$

$$E=\int d^3x[\frac{1}{2}(E_i^aE_i^a+B_i^aB_i^a+cos^2(a)D_i\phi^aD_i\phi^a+sin^2(a)D_i\phi^aD_i\phi^a)+\frac{\lambda}{4}(\phi^a\phi^a-u^2)^2\Rightarrow$$

$$E=\int d^3x\frac{1}{2}[(E_i^a-sin(a)D_i\phi^a)^2+(B_i^a-cos(a)D_i\phi^a)^2]+sin(a)\int d^3xE_i^aD_i\phi^a+cos(a)\int d^3xB_i^aD_i\phi^a+\int d^3x\frac{\lambda}{4}(\phi^a\phi^a-u^2)^2 \tag{4.81}$$

Where a is an arbitrary real number. We have mention that the differential equations (4.66) and (4.65) have analytical solution only for $\lambda = 0$. The potential then vanishes and the minimum of (4.81) occurs for:

$$E_i^a = sin(a)D_i\phi^a \qquad B_i^a = cos(a)D_i\phi^a \tag{4.82}$$

Since then the scalar field vanishes. These are the **Bogomol'nyi–Prasad–Sommerfield (BPS)** equations. Substitute these equations to (4.81) and we get:

$$E = sin(a)\int d^3xE_i^aD_i\phi^a + cos(a)\int d^3xB_i^aD_i\phi^a = usin(a)q + ucos(a)g \Rightarrow$$

$$E \geq u(sin(a)q + cos(a)g) \tag{4.83}$$

Where we have used (4.79) and (4.69). Equation (4.83) provides a lower bound for the energy of the configurations. As a function of a, (4.83) has a minimum for:

$$\frac{dE}{da} = 0 \Rightarrow tan(a) = \frac{q}{g} = A \tag{4.84}$$

This provides a lower bound for dyon mass:

$$M \geq u|q + ig| = u\sqrt{q^2+g^2} \tag{4.85}$$

This is known as **the Bogomol'nyi bound**. By considering (4.84) the lower bound for the dyon mass becomes:

$$M \geq u(sin(a)q + cos(a)g) = u(gtan(a)sin(a) + gcos(a)) = \frac{ug}{cos(a)} = \frac{4\pi u}{ecos(a)} = \frac{137m_W}{cos(a)}$$

Where the bound holds for a $\in [0, \pi/2)$. Equation (4.84) tell us the amount of electric and magnetic charge dyon contains. Therefore the mass of dyon depends from the electric and magnetic charge in the configuration. Note that (4.82) give:

$$E_i^a = sin(a)D_i\phi^a \Rightarrow -u\frac{JK}{r^2}\delta_i^a + u\frac{J+JK-rJ'}{r^4}r^ar_i = sin(a)[\frac{ur^ar_i}{r^2}[H' - \frac{HK}{r}] + \frac{uHK}{r}\delta_i^a] \Rightarrow$$

$$sin(a)H = -V \tag{4.86}$$

$$B_i^a = cos(a)D_i\phi^a \Rightarrow \frac{1}{er^4}[1+rK'-K^2]r^ar_i - \delta_i^a\frac{K'}{er} = cos(a)\frac{ur^ar_i}{r^2}[H' - \frac{KH}{r}] + cos(a)\frac{uHK}{r}\delta_i^a$$

$$K' = -cos(a)HKeu \qquad 1-K^2 = uer^2cos(a)H'$$

$$\xrightarrow{\xi=eur} \frac{dK}{d\xi} = -cos(a)HK \tag{4.87}$$

$$\xrightarrow{\xi=eur} \frac{dH}{d\xi} = \frac{1}{cos(a)}\frac{1-K^2}{\xi^2} \tag{4.88}$$

And this system of differential equations is solved by [10]:

$$V(\xi) = tan(a)[coth(\xi) - \frac{1}{\xi}] \tag{4.89}$$

$$K(\xi) = \frac{\xi}{sinh(\xi)} \tag{4.90}$$

$$H(\xi) = \frac{1}{cos(a)}(coth(\xi) - \frac{1}{\xi}) \tag{4.91}$$

Where the boundary conditions (4.53), (4.51), (4.54), (4.52) and (4.89) are satisfied.

Now let's investigate the lower bound of the magnetic monopole which corresponds for $C = 0 \Rightarrow a = 0$. Then the BPS equation, since $E_i^a = 0$, is given by:

$$B_i^a = D_i \phi^a \tag{4.92}$$

This equation gives

$$\frac{1}{er^4}[1 + rK' - K^2]r^a r_i - \delta_i^a \frac{K'}{er} = \frac{ur^a r_i}{r^2}[H' - \frac{KH}{r}] + \frac{uHK}{r}\delta_i^a$$

$$\Rightarrow \quad K' = -ueHK \qquad 1 + rK' - K^2 = eur^2[H' - \frac{HK}{r}]$$

$$\xrightarrow{\xi = eur} \quad \frac{dK}{d\xi} = -HK \qquad 1 + \xi\frac{dK}{d\xi} - K^2 = \xi^2[\frac{dH}{d\xi} - \frac{HK}{\xi}] \Rightarrow$$

$$\frac{dK}{d\xi} = -HK \qquad \xi^2\frac{dH}{d\xi} = 1 - K^2 \tag{4.93}$$

This system of differential equations is solved by [10]:

$$K(\xi) = \frac{\xi}{sinh(\xi)} \qquad H(\xi) = coth(\xi) - \frac{1}{\xi} \tag{4.94}$$

Where the boundary conditions (4.53), (4.51), (4.54) and (4.52) are satisfied. Notice that these solutions can be obtained if we set $a = 0$ at the (4.91) and (4.89). In figure (8) and (7) we can see that the functions $H(\xi)$ and $K(\xi)$ have the expected behavior. By setting $q = 0$ and $a = 0$, (4.85) becomes

$$M \geq ug = \frac{ue}{e^2/4\pi} = 137 m_W \tag{4.95}$$

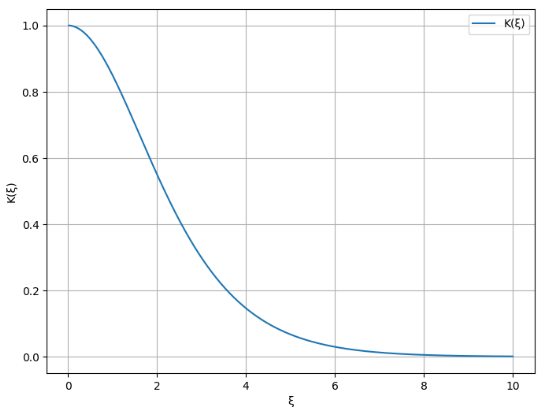

FIG. 7: Graph of $K(\xi)$

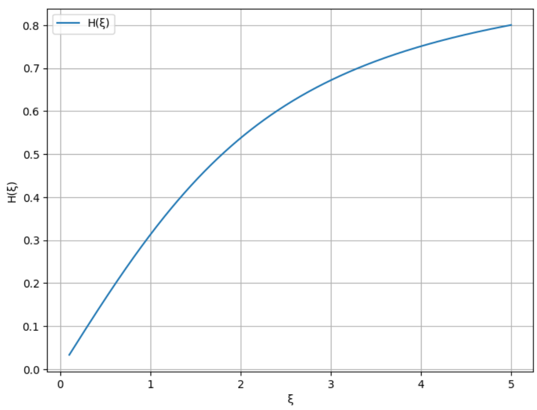

FIG. 8: Graph of $H(\xi)$

If we consider the experimental value of the weak bosons to be around $90GeV$, we get an enormous lower bound of 12 TeV. In this model the mass is too large, therefore it might explain the lack of experimental evidence so far. If this is the case, recall that an electromagnetic theory which is symmetric under $O(2)$ field transformations have one unified charge (2.21). Then the magnetic charge is expected to be observed at TeV scale, therefore a dyon configuration for example bellow this scale has a unified charge $q^2 = e^2 + g^2 \approx e^2$.

4.9. Monopole mass dependence from the Higgs coupling

BPS limits provides a lower bound for the monopole mass, since Higgs potential is turned off. We can compute correction due to the effect of λ being non-zero. This can be done by using numerical analysis techniques to solve the system of differential equations and hence calculating the mass of the monopole.

Firstly we change the variable $\xi = eur$ and let $\beta^2 = \frac{2\lambda}{e^2} = (\frac{m_H}{m_W})^2$, thus (4.59) becomes:

$$M(\beta) = \frac{4\pi u}{e}\int_0^\infty d\xi[\frac{(1-K^2)^2}{2\xi^2} + (K')^2 + \frac{\xi^2}{2}(H')^2 + (HK)^2 + \frac{\beta^2}{8}\xi^2(H^2-1)^2] = 137 m_W C(\beta) \quad (4.96)$$

Where the Bogomol'nyi bound holds:

$$C(\beta) \geq C(0) = 1 \quad (4.97)$$

Notice that:

$$\frac{dC}{d\beta} = \frac{\beta}{4}\int_0^\infty d\xi[\xi^2\beta(H^2-1)^2] > 0 \quad (4.98)$$

Which means the mass of the monopole increases as a function β.

The other extreme case where $\beta \to \infty$, the potential energy term forces the Higgs field to be frozen at its vacuum value almost everywhere. Thus at this limit $H(\xi) = 1$ for all $\xi > 0$. In this case the system of differential equations reduces to a massive Yang-Mills equation, (4.65) gives:

$$\xi^2 K''_\infty = K^3_\infty - K_\infty + \xi^2 K_\infty \quad (4.99)$$

And the energy integral (4.96) becomes:

$$C(\infty) = \int_0^\infty d\xi[\frac{(1-K^2_\infty)^2}{2\xi^2} + (K'_\infty)^2 + K^2_\infty] = 1.787 \quad (4.100)$$

Where we have used the result of [6]. Therefore for the intermediate values of β, $1 < C(\beta) < 1.787$. Indeed numerical solutions showcase this behavior in figure (9). For small values[6] of β both numerical [7] and asymptotic [6] solutions

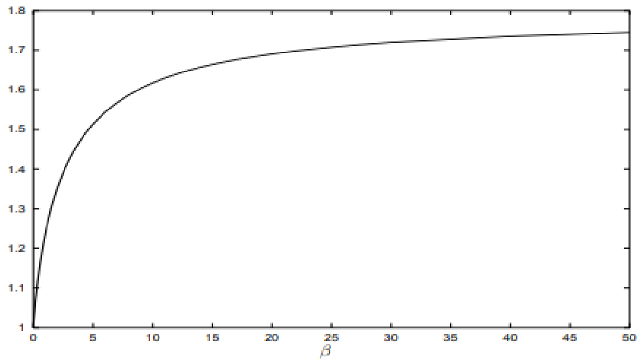

FIG. 9: $C(\beta)$ as a function of β, accurate up to 11 digits [7].

suggest an expansion for $C(\beta)$:

$$C(\beta) = 1 + \frac{\beta}{2} + \mathcal{O}(\beta^2) \quad (4.101)$$

Similarly for large values[7] of β we have an expansion of $C(\beta)$:

$$C(\beta) = 1.787 - \frac{2.228}{\beta} + \mathcal{O}(\frac{1}{\beta^2}) \quad (4.102)$$

---

[6] $(10^{-4} \leq \beta \leq 5 \times 10^{-4})$
[7] $(1000 \leq \beta \leq 2000)$

5. GRAVITATIONAL 'T HOOFT POLYAKOV MONOPOLES

In this section, we are going to discuss the gravitational effects of monopole configurations in Georgi-Glashow model. First we are going derive general properties of the monopole solution in curved space-time and then we are going to discuss the circumstances under which monopoles in such model can be viewed as black holes.

5.1. 't Hooft-Polyakov Monopoles In Curved Space-Time

The Lagrangian of the Georgi-Glashow model inside gravitational field is given by:

$$\sqrt{-g}\mathcal{L} = \sqrt{-g}[-\frac{R}{16\pi G} - \frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} + \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - \frac{\lambda}{2}(\phi^a\phi^a - u^2)^2] = \sqrt{-g}[\mathcal{L}_{gravity} + \mathcal{L}_{matter}] \tag{5.1}$$

Where R the Ricci scalar. Now our model has two sectors, the gravitational and matter sector.

$$\mathcal{L}_{gravity} = -\sqrt{-g}\frac{R}{16\pi G} \tag{5.2}$$

$$\mathcal{L}_{matter} = \sqrt{-g}[-\frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} + \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - \frac{\lambda}{4}(\phi^a\phi^a - u^2)^2] \tag{5.3}$$

In this analysis we are going to work with the static the spherical symmetric metric:

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\Omega^2 \tag{5.4}$$

Where $d\Omega$ is the infinitesimal solid angle:

$$d\Omega^2 = d\theta^2 + sin^2(\theta)d\phi^2 \tag{5.5}$$

Asymptotically we expect space-time to be flat:

$$A(\infty) = B(\infty) = 1 \tag{5.6}$$

For our convenience we let:

$$A(r) = (1 - \frac{2GM(r)}{r})^{-1} \tag{5.7}$$

Where $M(r)$ corresponds to the mass produced by the $SU(2)$ configurations. We want to generalise monopole configurations studied in the previous section to curved space-time. For this we consider the configuration in 't Hooft ansatz:

$$\phi^a = uH(r)\hat{r}^a \tag{5.8}$$

$$A_i^a = \frac{1 - K(r)}{e}[\hat{r} \times \partial_i \hat{r}]^a \tag{5.9}$$

$$A_0^a = 0 \tag{5.10}$$

With boundary conditions:

$$H(0) = 0 \qquad H(\infty) = 1 \qquad K(0) = 1 \qquad K(\infty) = 0 \tag{5.11}$$

These fields correspond to monopole configuration as we saw in the previous section. Therefore the difference with the flat case is that now the mangetic monopole causes the space-time to be curved. In order to study this generalization we will compute the various terms in our Lagrangian, which are affected by the curved-spacetime. We note that:

$$g_{00} = (g^{00})^{-1} = B(r) \qquad g_{11} = (g^{11})^{-1} = -A(r) \qquad g_{22} = (g^{22})^{-1} = -r^2 sin^2(\theta) \qquad g_{33} = (g^{33})^{-1} = -r^2 \tag{5.12}$$

From the previous section we have the components of the covariant derivative acting on the Higgs field:

$$(D_i)^a_b\phi^b = u\hat{r}^a\hat{r}_i[H' - \frac{HK}{r}] + \frac{uHK}{r}\delta^a_i \tag{5.13}$$

Write them as vectors in spherical coordinates (t, r, θ, ϕ):

$$D_1\vec{\phi} = u\hat{r}H' \qquad D_2\vec{\phi} = \frac{uHK}{r}\hat{\theta} \qquad D_3\vec{\phi} = \frac{uHK}{r}\hat{\phi} \tag{5.14}$$

Time components are zero since our solution is static. Then the kinetic Higgs term becomes:

$$\frac{1}{2}D_i\vec{\phi}\cdot D^i\vec{\phi} = \frac{1}{2}g^{ii}D^i\vec{\phi}\cdot D^i\vec{\phi} = -\frac{u^2H^{'2}}{2A} - \frac{u^2H^2K^2}{r^2} \tag{5.15}$$

The components of non-Abelian tensor are given by:

$$F^a_{ij} = \frac{K'}{er}(\epsilon_{abj}\hat{r}^b\hat{r}_i - \epsilon_{abi}\hat{r}^b\hat{r}_j) + \frac{2(1-K)}{er^2}[\epsilon_{aij} + \epsilon_{abi}\hat{r}^b\hat{r}_j - \epsilon_{abj}\hat{r}^b\hat{r}_i] - \frac{\hat{r}^a\hat{r}^m}{er^2}(K-1)^2\epsilon_{ijm} \tag{5.16}$$

The non-zero components are then:

$$\vec{F}_{12} = \frac{K'}{e}\hat{\phi} \tag{5.17}$$

$$\vec{F}_{13} = -\frac{K'}{e}sin(\theta)\hat{\theta} \tag{5.18}$$

$$\vec{F}_{23} = \frac{sin(\theta)}{e}(1-K^2)\hat{r} \tag{5.19}$$

Then we calculate the kinetic energy of the non-Abelian field:

$$-\frac{1}{4}g^{i\rho}g^{j\sigma}\vec{F}_{\rho\sigma}\cdot\vec{F}_{ij} = -\frac{1}{2}g^{11}g^{22}(\vec{F}_{12})^2 - \frac{1}{2}g^{22}g^{33}(\vec{F}_{23})^2 - \frac{1}{2}g^{11}g^{33}(\vec{F}_{13})^2 = -\frac{(K')^2}{2Ae^2r^2} - \frac{(K')^2sin^2(\theta)}{2e^2Ar^2sin^2(\theta)} - \frac{sin^2(\theta)}{2e^2r^4sin^2(\theta)}(1-K^2)^2 \Rightarrow$$

$$-\frac{1}{4}\vec{F}_{ij}\cdot\vec{F}^{ij} = -\frac{(K')^2}{Ae^2r^2} - \frac{(1-K^2)^2}{2e^2r^4} \tag{5.20}$$

The Higgs potential is equal to:

$$-\frac{\lambda}{4}(\phi^a\phi^a - u^2)^2 = -\frac{\lambda u^4}{4}(H^2-1)^2 \tag{5.21}$$

Note that:

$$\sqrt{-g} = \sqrt{-det(g_{\mu\nu})} = r^2sin(\theta)\sqrt{AB} \tag{5.22}$$

Therefore, the matter contribution to the action is:

$$S_{matter} = 2\pi\int dt\int_0^\infty r^2dr\int_0^\pi d\theta r^2sin(\theta)\sqrt{AB}[-\frac{(K')^2}{Ae^2r^2} - \frac{(1-K^2)^2}{2e^2r^4} - \frac{u^2H^{'2}}{2A} - \frac{u^2H^2K^2}{r^2} - \frac{\lambda u^4}{4}(H^2-1)^2] \Rightarrow$$

$$S_{matter} = -4\pi\int dt\int_0^\infty drr^2\sqrt{AB}\{\frac{1}{A}[\frac{(K')^2}{e^2r^2} + \frac{u^2H^{'2}}{2}] + \frac{u^2H^2K^2}{r^2} + \frac{(1-K^2)^2}{2e^2r^4} + \frac{\lambda u^4}{4}(H^2-1)^2\} = \int dt\int_0^\infty dr\mathcal{L}_{matter} \tag{5.23}$$

From the matter sector we can obtain the equations of motions for the field functions. We apply the Euler-Lagrange equations for the gauge field:

$$\frac{d}{dr}\left(\frac{\partial\mathcal{L}_{matter}}{\partial K'}\right) = \frac{\partial\mathcal{L}_{matter}}{\partial K} \Rightarrow$$

$$\frac{d}{dr}\left[\sqrt{\frac{B}{A}}\frac{K'}{e^2}\right] = \sqrt{AB}K(uH)^2 + \frac{\sqrt{AB}}{e^2r^2}K(K^2-1) \Rightarrow$$

$$\frac{1}{\sqrt{AB}}\frac{d}{dr}\left[\sqrt{\frac{B}{A}}K'\right] = K(ueH)^2 + \frac{1}{r^2}K(K^2-1) \tag{5.24}$$

And for the Higgs field:

$$\frac{d}{dr}\left(\frac{\partial\mathcal{L}_{matter}}{\partial H'}\right) = \frac{\partial\mathcal{L}_{matter}}{\partial H} \Rightarrow$$

$$\frac{1}{r^2\sqrt{AB}}\frac{d}{dr}\left[r^2H'\sqrt{\frac{B}{A}}\right] = 2\frac{HK^2}{r^2} + \lambda u^2H(H^2-1) \tag{5.25}$$

Notice in flat space-time limit; $A, B \to 1$, equations (5.25) and (5.24) reduce to (4.66) and (4.65) as it was expected.
Now we proceed with the gravity sector. The Ricci scalar of the static spherical metric is given by [29]:

$$R = \frac{B''}{AB} - \frac{B'A'}{2A^2B} - \frac{B'^2}{2B^2A} + \frac{2B'}{rAB} - \frac{2A'}{rA^2} - \frac{2}{r^2}\left(1-\frac{1}{A}\right) \tag{5.26}$$

Since the gravitational sector will give rise to equations of motion for the functions $A(r)$ and $B(r)$, we let the quantities:

$$T = \frac{(K')^2}{e^2r^2} + \frac{u^2H'^2}{2} \tag{5.27}$$

$$U = \frac{u^2H^2K^2}{r^2} + \frac{(1-K^2)^2}{2e^2r^4} + \frac{\lambda u^4}{4}(H^2-1)^2 \tag{5.28}$$

Therefore we write the action of the system:

$$S = -4\pi\int dt\int_0^\infty dr r^2\left[\frac{\sqrt{AB}}{16\pi G}\left(\frac{B''}{AB} - \frac{B'A'}{2A^2B} - \frac{B'^2}{2B^2A} + \frac{2B'}{rAB} - \frac{2A'}{rA^2} - \frac{2}{r^2}\left(1-\frac{1}{A}\right)\right) + \sqrt{AB}\left(\frac{T}{A}+U\right)\right] \tag{5.29}$$

Notice that:

$$\frac{r^2B''}{\sqrt{AB}} - \frac{r^2B'A'}{2A\sqrt{AB}} - \frac{r^2B'^2}{2B\sqrt{AB}} + \frac{2rB'}{\sqrt{AB}} = \frac{(r^2B')'}{\sqrt{AB}} - \frac{r^2(BB'A'+AB'B')}{2AB\sqrt{AB}} = \frac{\sqrt{AB}(r^2B')' - \frac{r^2BB'A'+r^2AB'B'}{2\sqrt{AB}}}{AB} \Rightarrow$$

$$\frac{r^2B''}{\sqrt{AB}} - \frac{r^2B'A'}{2A\sqrt{AB}} - \frac{r^2B'^2}{2B\sqrt{AB}} + \frac{2rB'}{\sqrt{AB}} = \frac{\sqrt{AB}(r^2B')' - r^2B'\frac{BA'+AB'}{2\sqrt{AB}}}{\sqrt{(AB)^2}} = \frac{\sqrt{AB}(r^2B')' - r^2B'\frac{(AB)'}{2\sqrt{AB}}}{\sqrt{(AB)^2}} \Rightarrow$$

$$\frac{r^2B''}{\sqrt{AB}} - \frac{r^2B'A'}{2A\sqrt{AB}} - \frac{r^2B'^2}{2B\sqrt{AB}} + \frac{2rB'}{\sqrt{AB}} = \frac{\sqrt{AB}(r^2B')' - r^2B'(\sqrt{AB})'}{\sqrt{AB}^2} = \left(\frac{r^2B'}{\sqrt{AB}}\right)'$$

Therefore these terms don't contribute, since they are written as a total derivative. We left with:

$$S = 4\pi\int dt\int_0^\infty dr r^2\left[\frac{\sqrt{AB}}{16\pi G}\left(\frac{2A'}{rA^2} + \frac{2}{r^2}\left(1-\frac{1}{A}\right)\right) - \sqrt{AB}\left(\frac{T}{A}+U\right)\right] \tag{5.30}$$

To simplify the expression we let:

$$X = \sqrt{AB} \tag{5.31}$$

$$Y = \sqrt{\frac{B}{A}} \tag{5.32}$$

From these we obtain:

$$A = \frac{X}{Y} \tag{5.33}$$

$$B = XY \tag{5.34}$$

It holds:

$$A' = \frac{X'Y - XY'}{Y^2} \tag{5.35}$$

Thus we get:

$$S = 4\pi \int dt \int_0^\infty dr[\frac{1}{16\pi G}(2r\frac{X'Y}{X} - 2rY' + 2X - 2Y) - (YT + XU)r^2] = \int dt \int_0^\infty dr\mathcal{L} \tag{5.36}$$

Now we obtain the equations of motion for X and Y, by applying the Euler-Lagrange equations:

$$\frac{d}{dr}(\frac{\partial \mathcal{L}}{\partial X'}) = \frac{\partial \mathcal{L}}{\partial X} \Rightarrow$$

$$\frac{1}{8\pi G}\frac{d}{dr}[\frac{rY}{X}] = -\frac{1}{8\pi G}\frac{rX'Y}{X^2} + \frac{1}{8\pi G} - Ur^2 \Rightarrow$$

$$\frac{d}{dr}(\frac{rY}{X}) = -\frac{rX'Y}{X^2} + 1 - 8\pi G U r^2 \Rightarrow$$

$$(rY)' = X(1 - 8\pi G r^2 U) \tag{5.37}$$

$$\frac{d}{dr}(\frac{\partial \mathcal{L}}{\partial Y'}) = \frac{\partial \mathcal{L}}{\partial Y} \Rightarrow$$

$$-\frac{1}{8\pi G} = \frac{1}{16\pi G}[\frac{2rX'}{X} - 2] - Tr^2 \Rightarrow$$

$$\frac{X'}{X} = 8\pi G T r \tag{5.38}$$

Integrate (5.38):

$$X(r) = exp[8\pi G \int_\infty^r drTr] \rightarrow$$

$$A = \frac{1}{B}exp[16\pi G \int_\infty^r drTr] = \frac{1}{B}exp[F(r)] \tag{5.39}$$

We write thus Y as:

$$Y = \sqrt{\frac{B}{A}} = \frac{1}{A} exp[\frac{F(r)}{2}] \tag{5.40}$$

Using this we obtain from (5.37):

$$(\frac{r}{A} exp(\frac{F}{2}))' = \sqrt{AB}(1 - 8\pi G r^2 U) \Rightarrow$$

$$(\frac{r}{A} exp(\frac{F}{2}))' = exp[\frac{F(r)}{2}](1 - 8\pi G r^2 U) \Rightarrow$$

$$(\frac{r}{A})' exp(\frac{F}{2}) + \frac{1}{2}\frac{r}{A}\frac{dF}{dr} exp(\frac{F}{2}) = exp(\frac{F}{2})(1 - 8\pi G r^2 U) \Rightarrow$$

$$(\frac{r}{A})' + \frac{8\pi G T r^2}{A} = (1 - 8\pi G r^2 U) \Rightarrow$$

$$(\frac{r}{A})' = 1 - 8\pi G r^2 (U + \frac{T}{A}) \tag{5.41}$$

Note that:

$$A = \frac{1}{1 - \frac{2GM(r)}{r}} \Rightarrow$$

$$\frac{r}{A} = r - 2GM(r) \tag{5.42}$$

Therefore we get:

$$1 - 2GM' = 1 - 8\pi G r^2 [U + T(1 - \frac{2GM}{r})] \Rightarrow$$

$$M' = 4\pi r^2 (T + U) - 8\pi G M T r \tag{5.43}$$

Let now:

$$P(r) = 8\pi G \int_r^\infty dr T r \tag{5.44}$$

Then we multiply (5.43) with $exp(-P(r))$ and we get:

$$M' exp(-P(r)) = 4\pi r^2 (T + U) exp(-P(r)) - 8\pi G M r exp(-P(r)) \Rightarrow$$

$$\int_0^\infty dr M' exp(-P(r)) = 4\pi \int_0^\infty dr r^2 (T + U) exp(-P(r)) - 8\pi G \int_0^\infty dr T M r exp(-P(r)) \Rightarrow$$

$$\int_0^\infty dr M' exp(-P(r)) = 4\pi \int_0^\infty dr r^2 (T + U) exp(-P(r)) - 8\pi G \int_0^\infty dr T M r exp(-P(r)) \Rightarrow$$

$$\int_0^\infty drM'exp(-P(r)) = 4\pi \int_0^\infty drr^2(T+U)exp(-P(r)) - \int_0^\infty drM[exp(-P(r))]' \Rightarrow$$

$$Mexp(-P(r))|_0^\infty = 4\pi \int_0^\infty drr^2(T+U)exp(-P(r)) \Rightarrow$$

$$M := M(\infty) = 4\pi \int_0^\infty drr^2(T+U)exp(-P(r)) + exp(-P(0))M(0) \tag{5.45}$$

When we investigate the matter field functions $K(r)$ and $H(r)$ we saw that they obtain their asymptotic behaviors for $r \to \infty$ outside the monopole core. Therefore $M := M(\infty)$ can be considered as total mass of the monopole in curved space-time. In order for (5.7) to be well defined at $r = 0$, it must $M(0) = 0$. Then by taking the limit $r \to 0$, we have $A(0) = 1$. Physically speaking this means that there is no singularity at the center of the monopole. We get then:

$$M(\infty) = 4\pi \int_0^\infty drr^2(T+U)exp(-P(r)) \tag{5.46}$$

In addition to this we rewrite (5.43):

$$M' = 4\pi r^2(T+U) - 8\pi GMTr = 4\pi r^2(T+U) - 4\pi r^2 T(1-\frac{1}{A}) = 4\pi r^2(\frac{T}{A}+U) \Rightarrow$$

$$M(\infty) = 4\pi \int_0^\infty drr^2(\frac{T}{A}+U) \tag{5.47}$$

Thus (5.46) and (5.47) describe equally $M(\infty)$. We will show now this is simply the energy of the system. The energy momentum tensor is given by (4.29)

$$T^\mu_\nu = -g^{\sigma\rho}F^{a\mu}_\rho F^a_{\sigma\nu} + D^\mu\phi^a D_\nu\phi^a - \delta^\mu_\nu \mathcal{L}_{matter} \tag{5.48}$$

The Hamiltonian of the system is given by:

$$\mathcal{H} = T^0_0 = -\mathcal{L}_{matter} = -[-\frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} + \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - \frac{\lambda}{2}(\phi^a\phi^a - u^2)^2] \Rightarrow$$

$$E = 4\pi \int_0^\infty r^2[\frac{T}{A}+U] \tag{5.49}$$

Therefore indeed (5.46) is the energy of the system. Finally let's discuss two remarks. First from (5.36) we have:

$$S = \frac{1}{2G}\int dt \int_0^\infty dr[(r\frac{X'Y}{X} - rY' + X - Y) - 8\pi G(YT+XU)r^2] \Rightarrow$$

$$L = r\frac{X'Y}{X} - rY' + X - Y - 8\pi G(YT+XU)r^2 \tag{5.50}$$

Use (5.38):

$$L = Yr^2 8\pi GT - r(Y)' + X - Y - 8\pi G(YT+XU)r^2 \Rightarrow$$

$$L = -r(Y)' + X - Y - 8\pi GXUr^2 \Rightarrow$$

$$L = -r(\sqrt{\frac{B}{A}})' + \sqrt{AB} - \sqrt{\frac{B}{A}} - \sqrt{AB}8\pi GUr^2 = -r(\frac{\sqrt{AB}}{A})' + \sqrt{AB} - \frac{\sqrt{AB}}{A} - \sqrt{AB}8\pi GUr^2 \Rightarrow$$

$$L = -r[\frac{1}{A}(\sqrt{AB})' + \sqrt{AB}(\frac{1}{A})'] + \sqrt{AB} - \frac{\sqrt{AB}}{A} - \sqrt{AB}8\pi GUr^2 = r(\sqrt{AB})' - r(\sqrt{AB})' - r\frac{1}{A}(\sqrt{AB})' - r\sqrt{AB}(\frac{1}{A})' + \sqrt{AB}(1 - \frac{1}{A})$$

$$L = r(\sqrt{AB})'(1 - \frac{1}{A}) + \sqrt{AB}(1 - \frac{1}{A}) - r\sqrt{AB}(\frac{1}{A})' - r(\sqrt{AB})' - \sqrt{AB}8\pi GUr^2 = [r(\sqrt{AB})(1 - \frac{1}{A})]' - r(\sqrt{AB})' - \sqrt{AB}8\pi GUr^2$$

From (5.38):

$$(\sqrt{AB})' = \sqrt{AB}8\pi GTr$$

And we obtain:

$$L = [r(\sqrt{AB})(1 - \frac{1}{A})]' - 8\pi GTr^2\sqrt{AB} - \sqrt{AB}8\pi GUr^2$$

The total derivative does not contribute to L and we left with:

$$L = -8\pi r^2 G\sqrt{AB}(T + U)$$

And we can multiply L with $(2G)^{-1}$:

$$\mathcal{L} = -4\pi r^2\sqrt{AB}(T + U) \leq 0 \tag{5.51}$$

This means that the action functional is manifestly negative, which means the energy functional:

$$E = -\int dt \int_0^\infty dr\mathcal{L} \tag{5.52}$$

Is positive definite and thus has a greatest lower bound E. This suggests that the system has indeed a solution.

As a second remark, consider 5.44 and notice since $T \geq 0$ then $P(r) \geq 0$. This also means that $exp(-P(r)) \geq exp(-P(0))$, since $P(\infty) = 0$. Therefore for non-singular solutions ($M(0) = 0$) obtain:

$$M \geq exp(-P(0)) \int_0^\infty dr4\pi r^2(T + U) \tag{5.53}$$

Recall that the integral above is simply the monopole mass in flat space. Then by taking the BPS limit; $\lambda = 0$:

$$M^{min}_{curved} \geq M^{min}_{flat} exp(-P(0))$$

And since $exp(-P(0)) << exp(-P(\infty)) = 1$ we conclude:

$$M^{min}_{curved} \leq M^{min}_{flat} \tag{5.54}$$

This a very interesting remark, when monopole is considered with a curved background, its mass gets smaller compared to a flat background.

5.2. 't Hooft Polyakov Monopole As A Black Hole

So far we have consider monopole solution in curved space-time with $M(0) = 0$ such that there is no singularity at $r = 0$. In order for a black hole to form, it is necessary for event horizons to form. This occurs when $A(r_h) = \infty$, with r_h being the event horizon. Solving this we can find:

$$1 - \frac{2GM(r_h)}{r_h} = 0 \Rightarrow$$

$$M(r_h) = \frac{r_h}{2G} \tag{5.55}$$

Then the differential equation (5.43) becomes:

$$M' = 4\pi r^2(T+U) - 8\pi GMTr = 4\pi r^2(T+U) - 4\pi r^2 T(1-\frac{1}{A}) = 4\pi r^2(\frac{T}{A(r_h)} + U) \Rightarrow$$

$$M'(r_h) = 4\pi r_h^2 U \tag{5.56}$$

Using this we can set the following condition at the event horizon:

$$(\frac{1}{A})'|_{r_h} = (1 - \frac{2GM(r)}{r})'|_{r_h} = -\frac{2GM'(r_h)}{r_h} + \frac{2GM(r_h)}{r_h^2} \Rightarrow$$

$$(\frac{1}{A})'|_{r_h} = \frac{1}{r_h} - 8\pi GUr_h \tag{5.57}$$

Now the equations of motion for the matter fields for $r = r_h$ give two conditions:

$$(5.25) \Rightarrow \frac{1}{r_h^2\sqrt{AB}}\frac{d}{dr}[r^2 H'\sqrt{\frac{B}{A}}]|_{r_h} = 2\frac{HK^2}{r_h^2} + \lambda u^2 H(H^2-1) \Rightarrow$$

$$\frac{1}{r_h^2\sqrt{AB}}[r_h H'\frac{d}{dr}[r\sqrt{B/A}]|_{r=r_h}] = \frac{1}{r_h^2\sqrt{AB}} r_h H'[rY]'_{r=r_h} = 2\frac{HK^2}{r_h^2} + \lambda u^2 H(H^2-1)$$

Use (5.37):

$$H'(r_h)(\frac{1}{r_H} - 8\pi G r_h U) = H'(r_h)(\frac{1}{A})'|_{r_h} = 2\frac{H(r_h)K^2(r_h)}{r_h^2} + \lambda u^2 H(r_h)(H^2(r_h)-1) \tag{5.58}$$

$$(5.24) \Rightarrow \frac{1}{\sqrt{AB}}\frac{d}{dr}[\sqrt{\frac{B}{A}}K']|_{r_h} = K(ueH)^2 + \frac{1}{r_h^2}K(K^2-1) \Rightarrow$$

$$\frac{1}{\sqrt{AB}}\frac{d}{dr}[\sqrt{\frac{B}{A}}]|_{r_h}K' = K(ueH)^2 + \frac{1}{r_h^2}K(K^2-1) \Rightarrow$$

$$\frac{1}{X}(Y'K')|_{r_h} = K(ueH)^2 + \frac{1}{r_h^2}K(K^2-1)$$

Note that at $r = r_h$:

$$Y + Y'r_h = X[1 - 8\pi G r_h^2 U] \Rightarrow$$

$$Y'|_{r_h} = X[\frac{1}{r_h} - 8\pi G r_h U]$$

Thus we obtain:

$$K'[\frac{1}{r_h} - 8\pi G r_h U] = K'(\frac{1}{A})'|_{r_h} = K(r_h)(ueH(r_h))^2 + \frac{1}{r_h^2}K(r_h)(K^2(r_h)-1) \tag{5.59}$$

These additional conditions overdetermine the solution, suggesting that for non-singular conditions events horizons don't exist.

Let's investigate singular solution, where $M(0) \neq 0$ and $1/A(0) = 0$. We expect a different mass from the singular case. Consider the asymptotic behavior of the fields, where:

$$H(\infty) = 1 \qquad K(\infty) = 0 \tag{5.60}$$

For these values we have:

$$T = 0 \qquad V = \frac{1}{2e^2r^4} \tag{5.61}$$

Then (5.43), gives:

$$M'(r) = 4\pi r^2 U = \frac{2\pi}{e^2r^2} \Rightarrow$$

$$M(r) - M(\infty) = \int\limits_0^\infty dr \frac{2\pi}{e^2r^2} = -\frac{2\pi}{e^2r} \Rightarrow$$

$$M(r) = M - \frac{2\pi}{e^2r} \tag{5.62}$$

Then (5.7) gives:

$$A(r) = (1 - \frac{2GM(r)}{r})^{-1} \Rightarrow$$

$$A(r) = (1 - \frac{r_s}{r} + \frac{4\pi G}{e^2r^2})^{-1} = (1 - \frac{2GM}{r} + \frac{r_q^2}{r^2})^{-1} \tag{5.63}$$

Where:

$$r_s = 2GM \tag{5.64}$$

$$r_q^2 = G\frac{Q_m^2}{4\pi} = \frac{G}{4\pi}(\frac{4\pi}{e})^2 = \frac{4\pi G}{e^2} \tag{5.65}$$

In addition to this (5.38) gives:

$$X' = 0 \Rightarrow \sqrt{AB} = 1 \Rightarrow$$

$$B(r) = \frac{1}{A(r)} \tag{5.66}$$

This means that the monopole singular solution corresponds Reissner-Nordstrom black hole with a magnetic charge $q_m = \frac{4\pi}{e}$. The event horizons are given by:

$$r_{\pm} = GM \pm \sqrt{G^2M^2 - r_q^2} \tag{5.67}$$

The quantity under the square root must be positive, because if it's negative we get a naked singularity. Therefore it must hold:

$$M \geq \frac{Q_m}{\sqrt{4\pi G}} \tag{5.68}$$

Therefore, there is a critical value of monopole mass $M_c = Q_m/\sqrt{4\pi G}$, for a black hole to form. We have mentioned that gravity decreases the monopole mass and is smaller than the flat case in the BPS limit, where $M_{flat}^{BPS} = 4\pi u/e$. Thus it is a good approximation to let $M_c = M_{flat}^{BPS}$. We find then:

$$\frac{4\pi u}{e} \geq \frac{Q_m}{\sqrt{4\pi G}} \Rightarrow$$

$$\frac{16\pi^2 u^2}{e^2} \geq \frac{16\pi^2}{4\pi G e^2} \Rightarrow$$

$$u^2 \geq \frac{1}{4\pi G} = \frac{M^2_{Planck}}{4\pi} \sim M^2_{Planck} \Rightarrow$$

$$u_{critical} = \frac{M_{Planck}}{2\sqrt{\pi}} \sim 10^{19} GeV \tag{5.69}$$

Let's discuss this result. For $u << M_{Planck}$, we expect gravitational effects to be weak and thus the system behaves similarly to the flat problem. On the other hand, for $u >> M_{Planck}$, recall that the monopole radius is given by $R = 1/eu$:

$$r_{\pm} >> R \tag{5.70}$$

We see that the event horizon is larger than the magnetic monopole. This means when the vacuum expectation value is much larger than the Planck mass, the event horizon of the black hole tends to 'hide' the monopole. Thus the magnetic monopole becomes a black hole. In addition to this let's see how the event horizon arises, while the monopole mass increases. In particular from 5.7, we see that when $M(r)/r$ takes its maximum value, $1/A(r)$ takes its minimum. The maximum value of mass is obtained asymptotically, therefore we consider (5.62):

$$\frac{M(r)}{r} = \frac{M}{r} - \frac{2\pi}{e^2 r^2} \Rightarrow$$

$$[\frac{M(r)}{r}]' = -\frac{M}{r_c^2} + \frac{4\pi}{e^2 r_c^3} = 0 \Rightarrow$$

$$\frac{M}{r_c^2} = \frac{4\pi}{e^2 r_c^3} \Rightarrow$$

$$r_c = \frac{4\pi}{e^2}\frac{1}{M}$$

Now we approximate $M = M(\infty) \approx \frac{4\pi u}{e}$ and we get:

$$r_c \approx \frac{1}{eu} = R \tag{5.71}$$

We see that the $1/A$ takes its minimum value approximate at the monopole radius. Its minimum value is given by:

$$\frac{1}{A(r_c)} = [1 - \frac{2GM(r_c)}{r_c}] \approx [1 - 2G(\frac{4\pi u^2}{e}e - \frac{2\pi}{e^2}u^2 e^2)] = [1 - G4\pi u^2] \Rightarrow$$

$$\frac{1}{A(r_c)} \approx 1 - G4\pi u^2 \approx 1 - \mathcal{O}(\frac{u^2}{M^2_{Planck}}) \tag{5.72}$$

We see that the minimization of $1/A(r)$ is controlled by the value of u. This is what we expected, since we already noted that the event horizon appears for $u \sim u_{critical} = \frac{1}{2\sqrt{\pi G}}$. Indeed the formula above produces the expected result:

$$\frac{1}{A(r_c)} \approx 1 - 4\pi G(\frac{1}{2\sqrt{\pi G}})^2 = 0$$

This argument also tell us that when the event horizon starts to appear for $u \approx u_{critical}$, then the event horizon is equal to the monopole radius r_c.

As a final remark the fields behave asymptotically near the event horizon outside of the monopole core. Then the solution for this region is the Reissner-Nordstrom solution. It would be unphysical for fields to behave asymptotically for finite distances. This because the proper length from the center of the monopole tends to be infinity. This is because $A(r)$ behaves as $A(r) \sim (r - r_h)^{-2}$ near r_h and thus blows up at $A(r_h)$. Then the proper length :

$$l(r) = \int_0^r dr[A(r)]^{1/2} \xrightarrow{r \to r_h} \infty \tag{5.73}$$

To sum up the discussion, we have showcase that for non-singular solutions event horizon cannot be form. Therefore we investigate singular solutions, where we showcase that event horizons can be formed for vacuum expectation values $u \geq u_{critical} \sim M_{Planck}$. For $u < u_{critical}$ the gravitational effects are weak and we expect for system to behave similarly to the flat space-time case. On the other hand for $u >> u_{critical}$, outside the monopoles core the space-time behaves like the Reissner-Nordstrom solution and the monopole is contained inside the black hole. We highlight that the solution behaves like Reissner-Nordstrom for asymptotic values of fields and as we approach to finite distances, the metric must differ. For the critical value $u \sim u_{critical}$, the event horizon of the black hole coincides with monopole radius. Finally, from this discussion it is clear that as vev increases from the critical value, the event horizon tends to expand so that monopole is contained inside the horizon.

6. ELECTROWEAK MONOPOLES

In this section we are going to review to concept of dyons in the electroweak model. We are starting with reviewing the electroweak model without matter fields and continue with topological arguments that support the existence of stable solutions within the standard model framework. Then we will introduce the Cho-Maison dyon solution which can be viewed as hybrid between 't Hooft-Polyakov monopole and Dirac monopole. Such solution is very charming, since it is widely believed that monopoles within the standard model are not possible. Despite that, there is a crucial problem; the energy of configuration is infinite. We describe two general schemes, under which the mass of Cho-Maison dyon can become finite. These schemes could be originated from some unspecified dynamics that arise from modifications of the standard model or from quantum corrections. We are starting with regularising the dyons energy by modifying the interactions of W-bosons with the electromagnetic field. Moreover, we compare the infinite energy dyon in electroweak model with finite one in the Georgi-Glashow model. After that we discuss the regularization of the hypercharge sector.

6.1. Electroweak Model

Electromagnetic and weak interactions are expected to unify at the energy scale of 246 GeV and the model which describes electroweak interactions is known as Weinberg-Salam model. Such model is a $SU_L(2) \times U_Y(1)$ Yang Mills field coupled with a complex Higgs doublet. The $SU_L(2)$ part is associated with the non-Abelian gauge field W^a_μ and $U_Y(1)$ with Abelian hypercharge gauge field B_μ. The Lagrangian of this theory is:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + (D_\mu\phi)^\dagger D^\mu\phi - \frac{\lambda}{2}(\phi^\dagger\phi - u^2)^2 \tag{6.1}$$

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{ajk}W^j_\mu W^k_\nu \tag{6.2}$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{6.3}$$

$$D_\mu = \partial_\mu - i\frac{g}{2}\vec{\sigma}\cdot\vec{W}_\mu - i\frac{g'}{2}B_\mu \tag{6.4}$$

Where we are working in the spinor representation and group elements of $SU_L(2) \times U_Y(1)$ are generated by $[\frac{\sigma^1}{2}, \frac{\sigma^2}{2}, \frac{1+\sigma^3}{2}, \frac{1-\sigma^3}{2}]$ with σ^i being the Pauli matrices. The symmetry breaking procedure for such field theory has been discussed analytically in (4 4.2) for the scalar sector and the mass of the resulting Highs field is given by (4.21). In addition to the massive Higg's boson, the non-Abelian gauge bosons also obtain mass. To showcase this let:

$$\begin{pmatrix} B_\mu \\ W^3_\mu \end{pmatrix} = \begin{pmatrix} cos(\theta_W) & -sin(\theta_W) \\ sin(\theta_W) & cos(\theta_W) \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \tag{6.5}$$

$$W_\mu^{\mp} = \frac{1}{\sqrt{2}}(W_\mu^1 \pm iW_\mu^2) \tag{6.6}$$

$$g' = gtan(\theta_W) \tag{6.7}$$

$$cos(\theta_W) = \frac{g}{\sqrt{g^2+g'^2}} \qquad sin(\theta_W) = \frac{g'}{\sqrt{g^2+g'^2}} \tag{6.8}$$

Where $\theta_W \approx \frac{\pi}{6}$ is the Weinberg angle. Then let the scalar field take the vacuum expectation value (4.17) and in the kinetic term of the scalar field we find:

$$(D_\mu\phi)^\dagger(D^\mu\phi) \ni \frac{g^2u^2}{8}(W_\mu^+W^{+\mu} + W_\mu^-W^{-\mu}) + \frac{u^2g^2}{8cos^2(\theta_W)}Z_\mu Z^\mu \tag{6.9}$$

Also note that:

$$D_\mu\phi \ni [-i\frac{g}{2}(1+\sigma^3)sin(\theta_W)A_\mu - \frac{ig}{cos(\theta_W)}(\frac{\sigma^3}{2} - sin^2(\theta_W)\frac{1+\sigma^3}{2})Z_\mu - \frac{ig}{2}(\frac{\sigma^1+i\sigma^2}{\sqrt{2}}W_\mu^+ + \frac{\sigma^1-i\sigma^2}{\sqrt{2}}W_\mu^-)]\phi$$

We identify the coupling $e = gsin(\theta_W)$. e is the fundamental electric charge associated with ubroken $U(1)$ generator $\frac{1+\sigma^3}{2}$ as we saw in (4 4.2). We can write the fundamental electric charge as a function of the couplings g and g' by using (6.8):

$$e = \frac{gg'}{\sqrt{g^2+g'^2}} \tag{6.10}$$

Recall the symmetry breaking pattern (4.18) and now in this physical context, becomes:

$$SU(2)\times U_Y(1) \xrightarrow{SSB} U_{em}(1) \tag{6.11}$$

Therefore the vector field A_μ is associated with the unbroken group is the electromagnetic field. From (6.9) we can obtain the masses of the vector bosons W^+, W^- and Z:

$$m_W = \frac{gu}{2} \tag{6.12}$$

$$m_Z = \frac{u\sqrt{g^2+g'^2}}{2} \tag{6.13}$$

$$\frac{m_W}{m_Z} = \cos(\theta_W) \tag{6.14}$$

To showcase the electric charge of vector bosons, notice in (6.4) that W_μ^a don't interact with the hypercharge field B_μ, thus they all have hypercharge zero. The standard model vacuum satisfies:

$$\frac{\sigma^3}{2}|0\rangle = 0 \tag{6.15}$$

And we write:

$$gW_\mu^a\frac{\sigma^a}{2} = \frac{g}{\sqrt{2}}(W_\mu^+\frac{\sigma^1-i\sigma^2}{2} + W_\mu^-\frac{\sigma^1+i\sigma^2}{2}) + gW_\mu^3\frac{\sigma^3}{2} = \frac{g}{\sqrt{2}}[W_\mu^+\frac{\sigma^-}{2} + W_\mu^-\frac{\sigma^+}{2}] + gW_\mu^3\frac{\sigma^3}{2}$$

Where the following commutation relation holds:

$$[\frac{\sigma^3}{2}, \frac{\sigma^\pm}{2}] = \mp\frac{\sigma^\pm}{2} \tag{6.16}$$

We identify the gauge boson isospin states:

$$W_\mu^\pm \frac{\sigma^\mp}{2} |0\rangle \qquad W_\mu^3 \frac{\sigma^3}{2} |0\rangle \tag{6.17}$$

By acting with $\sigma^3/2$ to these states we can find their isospin eignvalue:

$$\frac{\sigma^3}{2} W_\mu^\pm \frac{\sigma^\mp}{2} |0\rangle = \pm \frac{\sigma^3}{2} W^\pm [\frac{\sigma^3}{2}, \frac{\sigma^\mp}{2}] |0\rangle = \pm \frac{\sigma^3}{2} W_\mu^\pm \frac{\sigma^3}{2} \frac{\sigma^\mp}{2} |0\rangle = \pm W_\mu^\pm \frac{\sigma^\mp}{2} |0\rangle$$

$$\frac{\sigma^3}{2} W_\mu^3 \frac{\sigma^3}{2} |0\rangle = 0$$

Therefore the isospin of W^3 is $I_{W^3} = 0$ and the isospin of $W^\pm$ is $I_{W^\pm} = \pm 1$ Remember the empirical formula:

$$Q = I + \frac{Y}{2} \tag{6.18}$$

Where Y is the hypercharge and Q is the electric charge. This means that $Q_{W^3} = 0$ and $Q_{W^\pm} = \pm 1$. Note that Z is a superposition of B and W^3, and since B also has zero electric charge[8] then $Q_Z = 0$.

6.2. Topological argument

Let's investigate if stable soliton like solution are possible in the electroweak model. After symmetry breaking the broken group is $(SU(2) \times U_Y(1))/U_{em}(1) \sim S^2$. The vacuum expectation value of the complex Higgs doublet is given by the condition (4.16) and $|\phi_1|^2 + |\phi_2|^2 = u^2$. This means that vacuum manifold is S^1 and then $\pi_1(S^2) = \emptyset$. Therefore, at first look this suggest that it does not exists an asymptotic solution ϕ, that can not be continuously deformed to trivial solution and thus we can not construct a stable soliton like solution.

It turns out that Cho and Maison [11] have established that the electroweak model has the similar topogolical structure as the Georgi-Glashow model. To understand this observation consider the Higgs complex doublet $\phi \in \mathbb{C}^2$ and a map $f : C^2 \to CP^1$. In this way, the Higgs doublet plays a role a CP^1 **field**. Viewing the Higgs doublet as

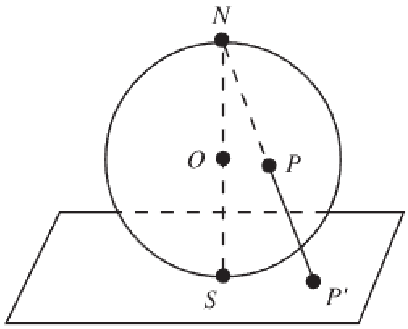

FIG. 10: Stereographic projection. This showcases the homoemorphism between S^2 and CP^1.

a CP^1 field changes the vacuum manifold since CP^1 is homeomorphic to S^2 and then $\pi_2(S^2) = \mathbb{Z}$. In (6 6.3) we will define the CP^1 field as a triplet, which can viewed as a map $S^2_\infty \to S^2_{vac}$. We conclude that such a topological argument suggest the existence of a stable soliton like solution in electroweak model.

6.3. Cho-Maison Dyon solution

We consider static solutions and the Lagrangian (6.1) becomes:

$$\mathcal{L} = -\frac{1}{4} F^a_{ij} F^{aij} - \frac{1}{4} G_{ij} G^{ij} + (D^i\phi)^\dagger D_i\phi - \frac{\lambda}{2}(\phi^\dagger\phi - u^2)^2 \tag{6.19}$$

---

[8] This holds since B has zero isospin (SU(2) singlet) and zero hypercharge.

With i, j and k being spatial indices. We can obtain the equations of motions from Euler-Lagrange equations:

$$\partial_i \frac{\partial \mathcal{L}}{\partial \partial_i \phi^\dagger} = \frac{\partial \mathcal{L}}{\partial \phi^\dagger} \Rightarrow$$

$$\partial_i D^i \phi = ig\frac{\vec{\sigma}}{2} \cdot \vec{W}_i D^i \phi + \frac{ig'}{2} B_i - \lambda(\phi^\dagger \phi - u^2)\phi \Rightarrow$$

$$D_i D^i \phi = -\lambda(\phi^\dagger \phi - u^2)\phi \tag{6.20}$$

$$\partial_i \frac{\partial \mathcal{L}}{\partial \partial_i W_j^a} = \frac{\partial \mathcal{L}}{\partial W_j^a} \Rightarrow$$

$$\partial_i F^{aji} + g\epsilon_{amc} W^{mi} F^{cji} = \frac{ig}{2}[(D^j\phi)^\dagger \sigma^a \phi - \phi^\dagger \sigma^a D^j \phi] \Rightarrow$$

$$\hat{D}_i F^{aji} = J^{aj} \tag{6.21}$$

$$\hat{D}_i = \partial_i - ig\frac{\vec{\sigma}}{2} \cdot \vec{W}_i \tag{6.22}$$

$$J^{aj} = ig[(D^j\phi)^\dagger \frac{\sigma^a}{2} \phi - \phi^\dagger \frac{\sigma^a}{2} D^j \phi] \tag{6.23}$$

$$\partial_i \frac{\partial \mathcal{L}}{\partial \partial_i B_j} = \frac{\partial \mathcal{L}}{\partial B_j} \Rightarrow$$

$$\partial_j G^{ji} = \frac{ig'}{2}[(D^i\phi)^\dagger \phi - \phi^\dagger (D^i \phi)] \Rightarrow$$

$$\partial_j G^{ji} = -K^i \tag{6.24}$$

$$K^i = \frac{ig'}{2}[(D^i\phi)^\dagger \phi - \phi^\dagger (D^i \phi)] \tag{6.25}$$

Now we let:

$$\phi = \frac{1}{\sqrt{2}}\rho\xi \qquad (\rho^2 = 2\phi^\dagger\phi, \quad \xi^\dagger\xi = 1) \tag{6.26}$$

$$\hat{\phi} = \xi^\dagger \vec{\sigma} \xi \tag{6.27}$$

$$W_i = \hat{\phi} \cdot \vec{W}_i \tag{6.28}$$

$$C_i = i\xi^\dagger \partial_i \xi \tag{6.29}$$

The equation (6.27) provides a triplet description of the CP^1 field. First let's show that $\hat{\phi}$ transforms according to the adjoint representation. The complex Higgs doublet transforms as:

$$\phi' = e^{ig\frac{\vec{\sigma}}{2}\cdot\vec{\theta} + i\frac{g'y}{2}}\phi \approx \phi - i\epsilon^a \frac{\sigma^a}{2}\phi - i\frac{\epsilon'}{2}\phi \tag{6.30}$$

If $\hat{\phi}$ is an adjoint triplet then it holds:

$$\hat{\phi}' = e^{-i\vec{T}\cdot\theta}\hat{\phi} \Rightarrow \hat{\phi}^{a'} \approx \hat{\phi}^a - g^c\epsilon^{cab}\hat{\phi}^b \tag{6.31}$$

We have:

$$\hat{\phi}^{a'} = \phi^{\dagger'}\sigma^a\phi' \Rightarrow$$

$$\hat{\phi}^a - g^c\epsilon^{cab}\hat{\phi}^b = \phi^\dagger(1 + i\epsilon^c\frac{\sigma^c}{2} + i\frac{\epsilon'}{2})\sigma^a(1 - i\epsilon^c\frac{\sigma^c}{2} - i\frac{\epsilon'}{2})\phi \Rightarrow$$

$$\hat{\phi}^a - g^c\epsilon^{cab}\hat{\phi}^b = \phi^\dagger\sigma^a\phi + \frac{i\epsilon^c}{2}\phi^\dagger\sigma^c\sigma^a\phi - \frac{i\epsilon^c}{2}\phi^\dagger\sigma^a\sigma^c\phi + ... \Rightarrow$$

$$\hat{\phi}^a - g^c\epsilon^{cab}\hat{\phi}^b = \hat{\phi}^a + \frac{i\epsilon^c}{2}\phi^\dagger(\delta^{ca} + i\epsilon^{cab}\sigma^b)\phi - \frac{i\epsilon^c}{2}\phi^\dagger(\delta^{ac} + i\epsilon^{acb}\sigma^b)\phi \Rightarrow$$

$$-g^c\epsilon^{cab}\hat{\phi}^b = -\frac{\epsilon^c}{2}\epsilon^{cab}\hat{\phi}^b + \frac{\epsilon^c}{2}\epsilon^{acb}\hat{\phi}^b = -\epsilon^c\epsilon^{cab}\hat{\phi}^b$$

Then if the infinitesimal angles are equal; $\epsilon^c = g^c$, indeed $\hat{\phi}$ is an adjoint triplet. For a soliton-like solution, just like in the case of Georgi-Glashow model, asymptotically the covariant derivative of the Higg's field is zero; $D_\mu\phi = 0$. Then it also holds asymptotically if we pick $\rho \to \rho_0 = \sqrt{2}u$:

$$D_\mu(\hat{\phi}) = D_\mu(\xi^\dagger\vec{\sigma}\xi) = \frac{2}{\rho_0}D_\mu(\phi^\dagger\vec{\sigma}\phi) = \frac{2}{\rho_0}(D_\mu\phi)^\dagger\vec{\sigma}\phi + \frac{2}{\rho_0}(\phi^\dagger\vec{\sigma}D_\mu\phi) \to 0$$

$$D_\mu\hat{\phi} \to 0 \tag{6.32}$$

We obtain asymptotically then :

$$D_\mu\hat{\phi} \to 0$$

$$\partial_\mu\hat{\phi}^a = g\epsilon_{cab}W^c_\mu\hat{\phi}^b \Rightarrow$$

$$\epsilon_{\rho ma}\hat{\phi}^m\partial_\mu\hat{\phi}^a = g\epsilon_{cab}\epsilon_{\rho ma}W^c_\mu\hat{\phi}^m\hat{\phi}^b \Rightarrow$$

$$\frac{1}{g}[\hat{\phi} \times \partial_\mu\hat{\phi}]^\rho = -gW^\rho_\mu + g(W^a_\mu\hat{\phi}^a)\hat{\phi}^\rho$$

$$\vec{W}_\mu \to \hat{\phi}W_\mu - \frac{1}{g}\hat{\phi} \times \partial_\mu\hat{\phi} \qquad W_\mu = \vec{W}_\mu \cdot \hat{\phi} \tag{6.33}$$

We are looking for dyon solution, which satisfies such asymptotic behavior. As for the $U(1)$ field, for a dyon solution we expect to have a Dirac-potential behavior. With that in mind we consider the following ansatz:

$$\vec{W}_\mu = -\frac{1}{g}W(r)\partial_\mu t\hat{r} + \frac{1}{g}(f(r) - 1)\hat{r} \times \partial_\mu\hat{r} \tag{6.34}$$

$$B_\mu = -\frac{1}{g'}B(r)\partial_\mu t - \frac{1}{g'}(1 - cos(\theta))\partial_\mu\phi \tag{6.35}$$

Where $f(\infty) = 0$ and $W(\infty) < \infty$. The ansatz above describes the most general spherical symmetric dyon of $SU(2) \times U(1)$. The second term in B_μ is familiar, since we recognise it from (3.12). Such a term produces a string singularity along the negative z-axis in ξ. 't Hooft-Polvakov monopole on the other hand does not contain Dirac string. An important note is that such monopole solution derived by working in a particular gauge. This is may be a problem, since the solution may not be gauge invariant. We conclude then that Cho-Mason monopole is hybrid between Dirac monopole and 't Hoof-Polyakov monopole. Since we have a spherical symmetric ansatz:

$$\rho = \rho(r) \tag{6.36}$$

And we are working in the radial gauge where

$$\xi = i \begin{pmatrix} sin(\theta/2)e^{-i\phi} \\ -cos(\theta/2) \end{pmatrix} \tag{6.37}$$

For such ξ, (6.27) becomes:

$$\hat{\phi}^1 = -cos(\theta/2)sin(\theta/2)(e^{i\phi} + e^{-i\phi}) = -sin(\theta)cos(\theta) = -\frac{x}{r}$$

$$\hat{\phi}^2 = -icos(\theta/2)sin(\theta/2)(e^{i\phi} - e^{-i\phi}) = -sin(\theta)sin(\phi) = -\frac{y}{r}$$

$$\hat{\phi}^3 = sin^2(\theta/2) - cos^2(\theta/2) = -cos(\theta) = -\frac{z}{r}$$

Therefore we get:

$$\hat{\phi} = \xi^\dagger \vec{\sigma} \xi = -\hat{r} \tag{6.38}$$

Notice that (6.37) is not well defined for $\theta = \pi$, since it maps $z < 0$ to $\xi = i(e^{-i\phi}, 0)$. As described in [22] we could set the field equal to zero along the negative z-axis, thus producing a vortex line. Then such string will always pull the monopole and cannot be static. This kind of monopole is called Nambu monopole. The vortex may also have a finite length and terminate some distance away on an antimonopole, then the resulting monopole-antimonopole pair will be spinning around the common center of mass. These is another interpretation of such Higgs field. We can divide the space to north and south hemispheres like Wu Yang monopole and use (6.37) for the north hemisphere and the $U(1)$ transformed $\xi' = e^{i\phi}\xi$ for the south hemisphere. We proceed with such interpretation for the field configuration.

As a final remark we write the non-Abelian current (6.23) by substituting (6.26) as:

$$J^{aj} = \frac{ig}{2}[(D^j\phi)^\dagger \sigma^a \phi - \phi^\dagger \sigma^a D^j \phi] = \frac{ig\rho^2}{4}[(\partial^j \xi)^\dagger \sigma^a \xi + \frac{ig}{2}\xi^\dagger \vec{\sigma} \cdot \vec{W}^j \sigma^a \xi + ig' B^j \xi^\dagger \sigma^a \xi - \xi^\dagger \sigma^a \partial^j \xi + \frac{ig}{2}\xi^\dagger \sigma^a \vec{\sigma} \cdot \vec{W}^j \xi] \Rightarrow$$

$$J^{aj} = \frac{-g\rho^2}{4}[-i(\partial^j\xi)^\dagger \sigma^a \xi + i\xi^\dagger \sigma^a \partial^j \xi + \frac{g}{2}\xi^\dagger(\sigma^b\sigma^a + \sigma^a\sigma^b)\xi W^{jb} + g' B^j \xi^\dagger \sigma^a \xi] \Rightarrow$$

$$J^{aj} = \frac{-g\rho^2}{4}[-i(\partial^j\xi)^\dagger \sigma^a \xi + i\xi^\dagger \sigma^a \partial^j \xi + g\xi^\dagger \delta^{ab} \xi W^{jb} + g' B^j \xi^\dagger \sigma^a \xi]$$

From [22] one can find Fierz identity:

$$\hat{\phi}^i(\xi^\dagger \partial_\mu \xi - \partial_\mu \xi^\dagger \xi) - (\xi^\dagger \sigma^i \partial_\mu \xi - \partial_\mu \xi^\dagger \sigma^i \xi) = i[\hat{\phi} \times \partial_\mu \hat{\phi}]^i \tag{6.39}$$

$$2\hat{\phi}^i C_\mu - (i\xi^\dagger \sigma^i \partial_\mu \xi - i\partial_\mu \xi^\dagger \sigma^i \xi) = -[\hat{\phi} \times \partial_\mu \hat{\phi}]^i \Rightarrow$$

$$(i\xi^\dagger \sigma^i \partial_\mu \xi - i\partial_\mu \xi^\dagger \sigma^i \xi) = \hat{\phi}^i 2C_\mu + [\hat{\phi} \times \partial_\mu \hat{\phi}]^i \tag{6.40}$$

Then the current becomes:

$$J^{aj} = \frac{-g\rho^2}{4}[2\hat{\phi}^a C^j + gW^{ja} + g' B^j \hat{\phi}^a + \epsilon_{abc}\hat{\phi}^b \partial^j \hat{\phi}^c] \Rightarrow$$

$$\vec{J}_j = \frac{-g\rho^2}{2}[\frac{g}{2}\vec{W}_j + (\frac{g'}{2}B_j + C_j)\hat{\phi} + \frac{1}{2}\hat{\phi} \times \partial_j \hat{\phi}] \tag{6.41}$$

As for the hypercharge current (6.25) by substituting (6.26) we get:

$$K^j = \frac{ig'}{2}[(D^i\phi)^\dagger \phi - \phi^\dagger (D^i \phi)] = \frac{ig'\rho^2}{4}[(\partial^j\xi)^\dagger \xi - \xi^\dagger \partial^j \xi + ig' B^j + ig\xi^\dagger \sigma^a \xi W^{ai}] = \frac{ig'\rho^2}{4}[i2i\xi^\dagger \partial^j \xi + ig' B^j + ig\xi^\dagger \sigma^a \xi W^{ai}]$$

$$K_j = \frac{-g'\rho^2}{2}[C_j + \frac{g'}{2}B_j + \frac{g}{2}W_j] = \frac{g'}{g}(\vec{J}_j \cdot \hat{\phi}) \tag{6.42}$$

6.4. Energy of electroweak Dyon

To compute the energy of the configuration we must first compute the energy momentum tensor of the electroweak model:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = -F^{a\ \mu}_{\sigma} F^{a\sigma\nu} - G^{\ \mu}_{\sigma} G^{\sigma\nu} + 2(D^\mu\phi)^\dagger D^\nu\phi - g^{\mu\nu}\mathcal{L} \tag{6.43}$$

The energy density is then:

$$\mathcal{H} = T^{00} = \frac{1}{2}F^a_{0i}F^{a0i} + \frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}G_{0i}G^{0i} + \frac{1}{4}G_{ij}G^{ij} + (D^0\phi)^\dagger D^0\phi + (D^i\phi)^\dagger D^i\phi + \frac{\lambda}{2}[\phi^\dagger\phi - u^2]^2 \tag{6.44}$$

The electric and magnetic fields associated with the non Abelian gauge field and abelia one are given by:

$$E^a_i = F^a_{0i} \qquad B^a_i = -\frac{1}{2}\epsilon_{ijk}F^{ajk} \tag{6.45}$$

$$\mathcal{E}_i = G_{0i} \qquad \mathcal{B}_i = -\frac{1}{2}\epsilon_{ijk}G^{jk} \tag{6.46}$$

And we write the energy density as:

$$\mathcal{H} = \frac{1}{2}E^a_iE^a_i + \frac{1}{2}B^a_iB^a_i + \frac{1}{2}\mathcal{E}_i\mathcal{E}_i + \frac{1}{2}\mathcal{B}_i\mathcal{B}_i + (D^0\phi)^\dagger D^0\phi + (D^i\phi)^\dagger D^i\phi + \frac{\lambda}{2}[\phi^\dagger\phi - u^2]^2 \tag{6.47}$$

We compute the field tensor of the non Abelian field:

$$F^a_{\mu\nu} = \frac{W'}{g}[\partial_\mu t\hat{r}_\nu - \partial_\nu t\hat{r}_\mu]\hat{r}^a + \frac{f'}{gr}[\epsilon_{ab\nu}\hat{r}_\mu - \epsilon_{ab\mu}\hat{r}_\nu]\hat{r}^b + \frac{2(f-1)}{gr^2}[\epsilon_{a\mu\nu} - \epsilon_{ab\nu}\hat{r}^b\hat{r}_\mu + \epsilon_{ab\mu}\hat{r}^b\hat{r}_\nu]$$

$$+\frac{Wf}{gr}[\partial_\nu t(\hat{r}^a\hat{r}_\mu - \delta^a_\mu) - \partial_\mu t(\hat{r}^a\hat{r}_\nu - \delta^a_\nu)] + \frac{(f-1)^2}{gr^2}\epsilon_{\mu\nu m}\hat{r}^a\hat{r}^m \tag{6.48}$$

Note that levi-civita symbol here takes only spatial indices $(1,2,3)$. The field tensor of the Abelian field:

$$G_{\mu\nu} = -\frac{B'}{g'}[\hat{r}_\mu\partial_\nu t - \hat{r}_\nu\partial_\mu t] - \frac{sin(\theta)}{g'}[\partial_\nu\theta\partial_\mu\phi - \partial_\mu\theta\partial_\nu\phi] \tag{6.49}$$

We can calculate E^a_i:

$$E^a_i = F^a_{0i} = \frac{1}{gr}[(rW' - Wf)\hat{r}^a\hat{r}_i + \delta^a_i Wf] \tag{6.50}$$

$$\rightarrow \frac{1}{2}E^a_iE^a_i = \frac{1}{2g^2r^2}[(rW')^2 + 2W^2f^2] \tag{6.51}$$

As for B^{ak} we have:

$$B^{ak} = \frac{1}{2}\epsilon^{kij}F^a_{ij} = \frac{1}{gr^2}[(rf' - (f^2-1))\hat{r}^a\hat{r}^k - rf'\delta^a_k] \tag{6.52}$$

$$\rightarrow \frac{1}{2}B^{ak}B^{ak} = \frac{1}{g^2r^4}[(f')^2r^2 + \frac{(f^2-1)^2}{2}] \tag{6.53}$$

We continue similarly with the Abelian field:

$$\mathcal{E}_i = G_{0i} = \frac{B'}{g'}\hat{r}_i \tag{6.54}$$

$$\rightarrow \frac{1}{2}\mathcal{E}_i\mathcal{E}_i = \frac{(B')^2}{2g'^2} \tag{6.55}$$

$$\mathcal{B}^i = -\frac{1}{2}\epsilon^{ijk}G_{jk} \Rightarrow \mathcal{B}^r = -\frac{1}{g'r^2} \tag{6.56}$$

$$\frac{1}{2}\mathcal{B}^r\mathcal{B}^r = \frac{1}{2(g')^2r^4} \tag{6.57}$$

The potential term becomes:

$$\frac{\lambda}{2}[\phi^\dagger\phi - u^2]^2 = \frac{\lambda}{2}[\frac{\rho^2}{2} - u^2]^2 \tag{6.58}$$

Now let's calculate the remaining terms:

$$D_0\phi = -i\frac{g}{2}\sigma^a W_0^a \frac{\rho\xi}{\sqrt{2}} - \frac{ig'}{2}B_0\frac{\rho\xi}{\sqrt{2}} = i\frac{\sigma^a\hat{r}^a}{2}W(r)\frac{\rho\xi}{\sqrt{2}} + \frac{i}{2}B\frac{\rho\xi}{\sqrt{2}}$$

$$(D^0\phi)^\dagger(D^0\phi) = \frac{\rho^2}{8}[W - B]^2 \tag{6.59}$$

$$D_i\phi = \partial_i\phi - ig\frac{\vec{\sigma}}{2}\cdot\vec{W}_i\phi - \frac{ig'}{2}B_i\phi \tag{6.60}$$

$$(D^i\phi)^\dagger D_i\phi = \partial^i\phi^\dagger\partial_i\phi + \frac{ig}{2}(\phi^\dagger\vec{\sigma}\cdot\vec{W}_i\partial^i\phi - \partial^i\phi^\dagger\vec{\sigma}\cdot\vec{W}_i\phi) + \frac{ig'}{2}B_i(\phi^\dagger\partial^i\phi - \partial^i\phi^\dagger\phi) + \frac{gg'}{2}B_i\phi^\dagger\vec{\sigma}\cdot\vec{W}^i\phi + \frac{g'^2}{4}B_iB^i\phi^\dagger\phi$$

$$+\frac{g^2}{4}\phi^\dagger\phi W_i^a W^{ai} \Rightarrow$$

$$(D^i\phi)^\dagger D_i\phi = \frac{1}{2}\rho^2\partial_i\xi^\dagger\partial^i\xi + \frac{1}{2}\partial_i\rho\partial^i\rho + \frac{ig}{4}(\xi^\dagger\vec{\sigma}\cdot\vec{W}_i\partial^i\xi - \partial^i\xi^\dagger\vec{\sigma}\cdot\vec{W}_i\xi) + \frac{ig'}{2}\rho^2\xi^\dagger\partial^i\xi B_i + \frac{gg'}{4}\rho^2\xi^\dagger\vec{\sigma}\cdot\vec{W}_i\xi B^i + \frac{g'^2}{8}\rho^2B_iB^i$$

$$+\frac{g^2\rho^2}{8}W_i^aW^{ai}$$

Let's calculate each term in this expression:

$$\frac{1}{2}\partial_i\rho\partial^i\rho + \frac{1}{2}\rho^2\partial_i\xi^\dagger\partial^i\xi = \frac{(\rho')^2}{2} + \frac{\rho^2}{8r^2} + \frac{\rho^2 sin^2(\frac{\theta}{2})}{2r^2sin^2\theta}$$

$$\frac{ig}{4}(\xi^\dagger\vec{\sigma}\cdot\vec{W}_i\partial^i\xi - \partial^i\xi^\dagger\vec{\sigma}\cdot\vec{W}_i\xi) = \frac{g\rho^2}{2}C_i\hat{\phi}\cdot\vec{W}^i + \frac{g\rho^2}{4}\vec{W}^i\cdot(\hat{\phi}\times\partial_i\hat{\phi}) = \frac{g\rho^2}{4r}W_i^a\epsilon_{abi}\hat{r}^b = \frac{(f-1)\rho^2}{4r^2}\epsilon_{abi}\epsilon_{ani}\hat{r}^n\hat{r}^b = \frac{f-1}{2r^2}\rho^2$$

$$\frac{ig'}{4}\rho^2\xi^\dagger\partial^i\xi B_i = \frac{i\rho^2}{2}[-isin^2(\frac{\theta}{2})\partial_i\phi][-(1-cos(\theta))\partial^i\phi] = -\frac{\rho^2 sin^4(\frac{\theta}{2})}{r^2sin^2(\theta)}$$

$$\frac{gg'}{4}\rho^2\xi^\dagger\sigma^a\xi W_i^aB^i \sim \hat{r}^a\hat{r}^n\epsilon_{ani} = 0$$

$$\frac{g'^2\rho^2}{8}B_iB^i = \frac{\rho^2}{2r^2}\frac{sin^4(\frac{\theta}{2})}{sin^2(\theta)}$$

$$\frac{g^2\rho^2}{8}W_i^aW^{ai} = \frac{\rho^2}{4r^2}(f^2 - 2f + 1)$$

Put all this together and we obtain:

$$(D^i\phi)^\dagger D_i\phi = \frac{(\rho')^2}{2} + \frac{\rho^2 f^2}{4r^2} - \frac{\rho^2}{8r^2} + \frac{\rho^2}{2r^2 sin^2(\theta)}[sin^2(\frac{\theta}{2}) - sin^4(\frac{\theta}{2})]$$

$$= \frac{(\rho')^2}{2} + \frac{\rho^2 f^2}{4r^2} - \frac{\rho^2}{8r^2} + \frac{\rho^2}{2r^2 sin^2(\theta)}[\frac{1}{2} - \frac{1}{2}cos(\theta) - \frac{1}{4} - \frac{1}{4}cos^2(\theta) + \frac{cos(\theta)}{2}]$$

$$= \frac{(\rho')^2}{2} + \frac{\rho^2 f^2}{4r^2} - \frac{\rho^2}{8r^2} + \frac{\rho^2}{2r^2 sin^2(\theta)}[\frac{1}{4} - (\frac{1}{4} - \frac{1}{4}sin^2(\theta))] \Rightarrow$$

$$(D^i\phi)^\dagger D^i\phi = \frac{(\rho')^2}{2} + \frac{f^2\rho^2}{4r^2} \tag{6.61}$$

By substituting (6.51), (6.53), (6.55), (6.57), (6.58), (6.59), (6.61) into (6.44) we get:

$$\mathcal{H} = \frac{1}{2g^2r^2}[(rW')^2 + 2Wf^2] + \frac{1}{g^2r^4}[(f')^2r^2 + \frac{(f^2-1)^2}{2}] + \frac{(B')^2}{2g'^2} + \frac{1}{2(g')^2r^4}$$

$$+\frac{\lambda}{2}[\frac{\rho^2}{2} - u^2]^2 + \frac{\rho^2}{8}[W - B]^2 + \frac{(\rho')^2}{2} + \frac{f^2\rho^2}{4r^2} \tag{6.62}$$

We obtain the energy of the configuration if we integrate this expression:

$$E = E_1 + E_2 \tag{6.63}$$

$$E_1 = \frac{2\pi}{g'^2}\int_0^\infty \frac{dr}{r^2} \tag{6.64}$$

$$E_2 = 4\pi\int_0^\infty dr[\frac{r^2(W')^2}{2g^2} + \frac{f^2W^2}{g^2} + \frac{(f')^2}{g^2} + \frac{(f^2-1)^2}{2g^2r^2}$$

$$+\frac{(rB')^2}{2g'^2} + \frac{\lambda r^2}{2}[\frac{\rho^2}{2} - u^2]^2 + r^2\frac{\rho^2}{8}[W - B]^2$$

$$+\frac{(r\rho')^2}{2} + \frac{f^2\rho^2}{4}] \tag{6.65}$$

E_2 is finite, but E_1 is infinite and originates from the point like hypercharge magnetic monopole. This means that we can not determine the dyon mass in a classical level. We will discuss in later sections how to make the energy of the dyon finite.

6.5. Equations of motion reduction

From the (6.62) we can reduce the equations of motion to a system of differential equations for $f(r)$, $\rho(r)$, $W(r)$ and $B(r)$. These function must minimise the energy of the system, since we want to obtain the mass of the dyon. The Euler langrage equations are:

$$\frac{\partial\mathcal{H}}{\partial f} = \frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial f'}) \qquad \frac{\partial\mathcal{H}}{\partial\rho} = \frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial\rho'}) \tag{6.66}$$

$$\frac{\partial\mathcal{H}}{\partial W} = \frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial W'}) \qquad \frac{\partial\mathcal{H}}{\partial B} = \frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial B'}) \tag{6.67}$$

We compute the derivatives:

$$\frac{\partial\mathcal{H}}{\partial f} = \frac{2fW^2}{g^2} + \frac{2f(f^2-1)}{g^2r^2} + \frac{f\rho^2}{2} \qquad \frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial f'}) = \frac{2f''}{g^2}$$

$$\frac{\partial\mathcal{H}}{\partial \rho} = \lambda r^2[\frac{\rho^2}{2} - u^2]\rho + \frac{r^2}{4}[W-B]^2 + \frac{\rho f^2}{2} \qquad \frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial \rho'}) = r^2\rho'' + 2r\rho'$$

$$\frac{\partial\mathcal{H}}{\partial W} = \frac{2Wf^2}{g^2} + \frac{r^2\rho^2}{4}[W-B] \qquad \frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial W}) = \frac{r^2W''}{g^2} + \frac{2rW'}{g'}$$

$$\frac{\partial\mathcal{H}}{\partial B} = -\frac{r^2\rho^2}{4}[W-B] \qquad \frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial B'}) = \frac{r^2B''}{g'^2} + \frac{2rB'}{g'^2}$$

And thus we get the following differential equations:

$$f'' - \frac{f^2-1}{r^2}f = (\frac{g^2\rho^2}{4} + W^2)f \tag{6.68}$$

$$\rho'' + \frac{2}{r}\rho' - \frac{f^2}{2r^2}\rho = \frac{1}{4}(W-B)^2\rho + \lambda(\frac{\rho^2}{2} - u^2)\rho \tag{6.69}$$

$$W'' + \frac{2}{r}W' - 2\frac{f^2}{r^2}W = \frac{g^2}{4}\rho^2(W-B) \tag{6.70}$$

$$B'' + \frac{2}{r}B' = \frac{g'^2\rho^2}{4}(B-W) \tag{6.71}$$

To integrate this system of differential equations we may choose the boundary conditions:

$$f(0)=1, \quad \rho(0)=0, \quad W(0)=0. \quad B(0)=b_0 \quad f(\infty)=0, \quad \rho(\infty)=\rho_0=\sqrt{2}u, \quad W(\infty)=W_0 \quad B(\infty)=B_0 \tag{6.72}$$

In order to investigate the asymptotic behaviors of these functions we let:

$$x = u\sqrt{\lambda}r = \frac{m_H}{\sqrt{2}}r \qquad \epsilon = \frac{g^2}{2\lambda} \qquad \epsilon' = \frac{g'^2}{2\lambda} \qquad Z(x) = B(x) - W(x) \qquad \tilde{\rho}(x) = \frac{\rho}{\rho_0} \tag{6.73}$$

And the system of differential equations becomes:

$$\tilde{\rho}''(x) + \frac{2\tilde{\rho}(x)}{x} - \frac{f^2\tilde{\rho}(x)}{2x^2} = \frac{1}{2m_H^2}Z^2(x)\tilde{\rho}(x) + (\tilde{\rho}(x)-1)\tilde{\rho}(x) \tag{6.74}$$

$$f''(x) - \frac{f^3(x)-f(x)}{x^2} = \epsilon f(x)\tilde{\rho}^2(x) + \frac{2}{m_H^2}(B(x)-Z(x))^2f(x) \tag{6.75}$$

$$Z''(x) + \frac{2}{x}Z'(x) + \frac{2f^2(x)}{x^2}(B(x)-Z(x)) = (\epsilon+\epsilon')Z\tilde{\rho}^2(x) \tag{6.76}$$

$$B''(x) + \frac{2B'(x)}{x} = \epsilon'\tilde{\rho}^2(x)Z(x) \tag{6.77}$$

We starting by investigating the behavior as $x \to 0$. Let:

$$f(x) = 1 + \delta_0(x) \qquad |\delta_0(x)| << 1 \tag{6.78}$$

$$Z(x) = b_0 + \delta_1(x) \qquad |\delta_1(x)| << 1 \tag{6.79}$$

$$B(x) = b_0 + \delta_2(x) \qquad |\delta_2(x)| << 1 \tag{6.80}$$

Then (6.74) becomes:

$$x^2\tilde{\rho}''(x) + 2x\tilde{\rho}(x) + \tilde{\rho}(x)[x^2 - \frac{1}{2}(1 + \frac{b_0^2}{2m_H^2})] = 0$$

For small b_0 we get:

$$x^2\tilde{\rho}''(x) + 2x\tilde{\rho}'(x) + \tilde{\rho}(x)(x^2 - \frac{1}{2}) = 0$$

$$x^2\tilde{\rho}(x) + 2x\tilde{\rho}(x) + (x^2 - \frac{-1+\sqrt{3}}{2}(\frac{-1+\sqrt{3}}{2} + 1))\tilde{\rho}(x) = 0$$

$$x^2\tilde{\rho}(x) + 2x\tilde{\rho}(x) + (x^2 - \delta(\delta+1))\tilde{\rho}(x) = 0 \tag{6.81}$$

Where we let $\delta = \frac{-1+\sqrt{3}}{2}$. This equation is solved by spherical Bessel functions [20] and for a regular solution around $x = 0$ we have:

$$\tilde{\rho}(x) = c_1 j_\delta(x) = c_1\frac{\sqrt{\pi}}{2}x^\delta \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\frac{(x/2)^{2m}}{\Gamma(\delta+m+3/2)} \approx c_1 x^\delta[1 - \frac{x^2}{2(2\delta+3)}] \tag{6.82}$$

Equation (6.77) becomes when $x \to 0$:

$$\delta_2''(x) + \frac{2}{x}\delta_2'(x) = \epsilon' b_0 c_1^2[1 - \frac{x^2}{4\delta+6}]^2 \Rightarrow$$

$$\delta_2(x) = \epsilon' b_0 c_1^2 x^{2\delta+2}[\frac{1}{(2\delta+3)(2\delta+2)} - \frac{x^2}{(2\delta+3)(2\delta+4)(2\delta+5)} + \mathcal{O}(x^4)] \Rightarrow$$

$$B(x) = b_0(1 + \epsilon' c_1^2 x^{2\delta+2}[\frac{1}{(2\delta+3)(2\delta+2)} - \frac{x^2}{(2\delta+3)(2\delta+4)(2\delta+5)} + \mathcal{O}(x^4)]) \tag{6.83}$$

Equation (6.75) becomes when $x \to 0$:

$$\delta_0''(x) - \frac{\delta_0(x)}{x^2} = \epsilon' b_0 c_1^2 x^{2\delta}[1 - \frac{x^2}{4\delta+6}]^2 \Rightarrow$$

$$\delta_0(x) = \frac{\epsilon c_1^2}{2}x^{2\delta+2}[\frac{1}{\delta(2\delta+3)} - \frac{x^2}{4\delta^3+20\delta^2+31\delta+15} + \mathcal{O}(x^4)] \tag{6.84}$$

Equation (6.76) becomes when $x \to 0$:

$$\delta_1'' + \frac{2}{x}\delta_1' - \frac{2\delta_1}{x^2} = (\epsilon+\epsilon')\tilde{\rho}^2 b_0 - \frac{2}{x^2}\delta_2 \Rightarrow$$

$$\delta_1(x) = z_1 x + \frac{b_0 c_1^2}{2}x^{2\delta+2}[(\epsilon+\epsilon')(\frac{1}{(2\delta+1)(\delta+2)} - \frac{x^2}{(2\delta+3)^2(\delta+3)}) - \epsilon'\frac{x^2}{(2\delta+3)(\delta+3)(2\delta^2+5\delta+3)} + \mathcal{O}(x^4)] \tag{6.85}$$

We continue with the asymptotic behaviors as $x \to \infty$. To do this we expand the functions as:

$$\tilde{\rho}(x) = 1 + \Delta_0(x), \qquad |\Delta_0(x)| << 1 \tag{6.86}$$

$$B(x) = B_0 + \Delta_1(x), \qquad |\Delta_1(x)| << 1 \tag{6.87}$$

$$Z(x) = Z_0 + \Delta_2(x), \qquad |\Delta_2(x)| << 1 \quad Z_0 = B_0 - W_0 \tag{6.88}$$

Equation (6.75) when $x \to \infty$ becomes:

$$f''(x) - (\epsilon + \frac{2W_0}{m_H^2})f(x) = 0 \Rightarrow$$

$$f(x) = f_1 exp(-kx) \qquad k^2 = \epsilon + \frac{2W_0^2}{m_H^2} \tag{6.89}$$

To simplify the calculations we set $Z_0 = 0 \to W_0 = B_0$. Then when $x \to 0$, (6.74) becomes:

$$\Delta_0''(x) + \frac{2}{x}\Delta_0'(x) - 2\Delta_0(x) = 0 \Rightarrow$$

$$\Delta_0(x) = \rho_1 \frac{exp(-\sqrt{2}x)}{x} \tag{6.90}$$

(6.76) becomes when $x \to 0$:

$$\Delta_2'' + \frac{2}{x}\Delta_2' - (\epsilon + \epsilon')\Delta_2 = 0 \Rightarrow$$

$$\Delta_2(x) = b_1 \frac{exp(-x\nu)}{x} \qquad \nu^2 = \epsilon + \epsilon' \tag{6.91}$$

6.77 becomes when $x \to \infty$:

$$\Delta_1'' + \frac{2}{x}\Delta_1'(x) = \epsilon' B_1 \frac{exp(-\nu x)}{x} \Rightarrow$$

$$\Delta_1(x) = \frac{W_1}{x} + \frac{B_1\epsilon'}{\epsilon + \epsilon'} \frac{exp(-x\sqrt{\epsilon + \epsilon'})}{x} \tag{6.92}$$

We sum up the asymptotic behaviors when $x \to \infty$:

$$\boxed{\begin{array}{l} f(x) \xrightarrow{x\to\infty} f_1 exp(-kx), \qquad k^2 = \epsilon + \dfrac{2W_0^2}{m_H^2} \\ \tilde{\rho}(x) \xrightarrow{x\to\infty} 1 + \rho_1 \dfrac{exp(-\sqrt{2}x)}{x} \\ Z(x) \xrightarrow{x\to\infty} b_1 \dfrac{exp(-\nu x)}{x}, \qquad \nu^2 = \epsilon + \epsilon' \\ B(x) \xrightarrow{x\to\infty} B_0 + \dfrac{W_1}{x} + \dfrac{B_1\epsilon'}{\nu^2}\dfrac{exp(-x\nu)}{x} \end{array}} \tag{6.93}$$

And the asymptotic behaviors around $x = 0$:

$$\boxed{\begin{array}{l} \tilde{\rho}(x) \xrightarrow{x\to 0} c_1 x^\delta [1 - \dfrac{x^2}{2(2\delta+3)} + \mathcal{O}(x^4)] \\ B(x) \xrightarrow{x\to 0} b_0 \left(1 + \dfrac{\epsilon' c_1^2}{2} x^{2\delta+2}\left[\dfrac{1}{(2\delta+3)(\delta+1)} - \dfrac{x^2}{(2\delta+3)(\delta+2)(2\delta+5)} + \mathcal{O}(x^4)\right]\right) \\ f(x) \xrightarrow{x\to 0} 1 + \dfrac{\epsilon c_1^2}{2} x^{2\delta+2}\left[\dfrac{1}{\delta(2\delta+3)} - \dfrac{x^2}{4\delta^3 + 20\delta^2 + 31\delta + 15} + \mathcal{O}(x^4)\right] \\ Z(x) \xrightarrow{x\to 0} b_0(1 + \dfrac{c_1^2}{2} x^{2\delta+2}[(\epsilon+\epsilon')(\dfrac{1}{(2\delta+1)(\delta+2)} - \dfrac{x^2}{(2\delta+3)^2(\delta+3)}) - \epsilon' \dfrac{x^2}{(2\delta+3)(\delta+3)(2\delta^2+5\delta+3)} + \mathcal{O}(x^4)] \end{array}} \tag{6.94}$$

The differential equations (6.71), (6.68) (6.70), (6.69) are very similar to the differential equations of Julia-Zee solution for Julia-Zee dyon in (44.7). Indeed the equations are identical if we let $W(r) = B(r)$. The fact that $W(r) \neq B(r)$ is a crucial difference, since this represents the neutral Z boson (6.110).

6.6. Mangetic and Electric Charge of Cho-Maison Solution

It's very important to showcase that $SU_L(2) \times U_Y(1)$ field soliton-like configurations indeed contain magnetic and electric charges. We expect asymptotically that such configuration to contain a magnetic charge, which satisfies the Dirac quantization just like we saw in Georgi-Glashow model. Asymptotically the $SU_L(2)$ field is given by (6.33). Notice that the field is not described by a particular gauge. As for the $U_Y(1)$ hypercharge field in general contains a $U_Y(1)$ magnetic charge $\tilde{g}$ and the potential is then for spatial indices:

$$B_i = \tilde{g}(1 - cos(\theta))\partial_i\phi \tag{6.95}$$

Then if we follow the analysis of Dirac-monopole and identify the coupling g' with the electric charge we find that:

$$\tilde{g}g' = \frac{n}{2} \qquad n \in \mathbb{Z} \tag{6.96}$$

And Cho-Maison dyon solution holds for $n = -2$. Asymptotically just like in Georgi-Glashow model the $SU_L(2) \times U_Y(1)$ is broken to $U_{em}(1)$ and we expect magnetic charge of the electroweak dyon to be fixed by an integer. Asymptotically the electromagnetic field tensor is given by [22]:

$$F^{em}_{jk} \to -sin(\theta_W)\vec{F}_{jk} \cdot \hat{\phi} + cos(\theta_W)G_{jk} \tag{6.97}$$

And the fields behave as:

$$\vec{F}_{jk} \cdot \hat{\phi} \to -\frac{1}{g}\hat{\phi} \cdot (\partial_j\hat{\phi} \times \partial_\nu\hat{\phi}) \tag{6.98}$$

$$G_{jk} \to \frac{n}{2g'}sin(\theta)[\partial_k\theta\partial_j\phi - \partial_j\theta\partial_k\phi] \tag{6.99}$$

The magnetic charge is given then:

$$q_m = \frac{e}{g}\int dS_i\epsilon^{ijk}\frac{1}{2}\vec{F}_{jk} \cdot \hat{\phi} + \frac{e}{g'}\int dS_i\frac{1}{2}\epsilon^{ijk}G_{jk} \Rightarrow$$

$$q_m = -\frac{e}{g^2}\int dS_i\epsilon^{ijk}\frac{1}{2}\hat{\phi} \cdot (\partial_j\hat{\phi} \times \partial_\phi\hat{\phi}) - \frac{en}{2g'^2}\int dS_i\epsilon^{ijk}sin(\theta)\partial_j\theta\partial_k\phi \Rightarrow$$

$$q_m = -\frac{4\pi e}{2g^2}n - \frac{4\pi ne}{2g'^2} \qquad n \in \mathbb{Z}$$

Where we have used the result from (4.46). Then we have:

$$q_m = -\frac{4\pi en}{2}(\frac{1}{g^2} + \frac{1}{g'^2}) \Rightarrow$$

$$q_m = -\frac{2\pi n}{e} \qquad n \in \mathbb{Z}$$

Since $n \in \mathbb{Z}$ we let $-2n \to 4n$ and thus we confirm that:

$$q_m = \frac{4\pi n}{e} \qquad n \in \mathbb{Z} \tag{6.100}$$

To determine the electric and magnetic charge of the configuration described by the Cho-Maison dyon solution we are going to work in the unitary gauge where:

$$\xi \to U\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.101}$$

$$U = -i \begin{pmatrix} cos(\theta/2) & sin(\theta/2)e^{-i\phi} \\ sin(\theta/2)e^{i\phi} & -cos(\theta/2) \end{pmatrix} \tag{6.102}$$

The non-Abelian gauge field transforms as:

$$\vec{W'}_\mu \cdot \frac{\vec{\sigma}}{2} = -\frac{i}{g}U^{-1}\partial_\mu U + U\vec{W}_\mu \cdot \frac{\vec{\sigma}}{2}U^{-1} \tag{6.103}$$

And we get:

$$\vec{W}'_\mu = \frac{1}{g}\begin{pmatrix} (sin(\phi)\partial_\mu\theta + sin(\theta)cos(\phi)\partial_\mu\phi)f(r) \\ (-cos(\phi)\partial_\mu + sin(\theta)sin(\phi)\partial_\mu\phi)f(r) \\ -W(r)\partial_\mu t - (1-cos\theta)\partial_\mu\phi \end{pmatrix} \tag{6.104}$$

In this unitary gauge we set:

$$\hat{r} \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{6.105}$$

Note that:

$$W_\mu = \frac{1}{\sqrt{2}}(W^1_\mu + iW^2_\mu) = \frac{if(r)}{g\sqrt{2}}e^{i\phi}[(\partial_\mu\theta) + isin(\theta)\partial_\mu\phi] \tag{6.106}$$

Focus on the singular component of the non Abelian gauge field:

$$W^3_\mu = -\frac{1}{g}W(r)\partial_\mu t - \frac{1}{g}(1-cos(\theta))\partial_\mu\phi \tag{6.107}$$

We can easily obtain the inverse transformation of (6.5) and by using (6.8) we get:

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \frac{1}{\sqrt{g^2+g'^2}}\begin{pmatrix} g & g' \\ -g' & g \end{pmatrix}\begin{pmatrix} B_\mu \\ W^3_\mu \end{pmatrix} \tag{6.108}$$

Then (6.107) and (6.35) become by using (6.10):

$$A_\mu = \frac{e}{g'}B_\mu + \frac{e}{g}W^3_\mu = -e[\frac{1}{g^2}W(r) + \frac{1}{g'^2}B(r)] - \frac{1}{e}(1-cos(\theta))\partial_\mu\phi \tag{6.109}$$

$$Z^\mu = -\frac{e}{g}B_\mu + \frac{e}{g'}W^3_\mu = \frac{e}{gg'}[B(r) - W(r)]\partial_\mu t \tag{6.110}$$

These are the physical fields of photons and Z-bosons. Then the electric charge of the configuration will be given by:

$$q_e = \oint d\vec{S}\cdot\vec{E} = \oint dS_iE_i = \oint dS_iF_{0i} \tag{6.111}$$

Where $F_{\mu\nu}$ is the electromagnetic field tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -e[\frac{1}{g^2}W'(r) + \frac{1}{g'^2}B'(r)][\hat{r}_\mu\partial_\nu t - \hat{r}_\nu\partial_\mu t] - \frac{1}{e}sin(\theta)[\partial_\mu\theta\partial_\nu\phi - \partial_\nu\theta\partial_\mu\phi] \tag{6.112}$$

$$F_{0i} = e[\frac{A'}{g^2} + \frac{B'}{g'^2}]\hat{r}_i \tag{6.113}$$

Thus the electric charge is:

$$q_e = e\oint dS_i\hat{r}_i[\frac{W'}{g^2} + \frac{B'}{g'^2}] = 4\pi e|\frac{r^2W'}{g^2} + \frac{r^2B'}{g'^2}|^\infty_0$$

$$q_e = \frac{4\pi}{e}|sin^2(\theta_W)r^2W' + cos^2(\theta_W)r^2B'|^{r=\infty} = \frac{4\pi}{e}W_1(sin^2(\theta_W) + cos^2(\theta_W)) = \frac{4\pi}{e}W_1 \tag{6.114}$$

Where we have used the asymptotic behaviors (6.93). As for the magnetic charge we have:

$$q_m = \oint d\vec{S} \cdot \vec{B} = \oint dS_i B_i = \frac{-1}{2}\oint dS^i \epsilon^{ijk} F_{jk} \tag{6.115}$$

We can easily compute the magnetic field in spherical coordinates:

$$B^i = -\frac{1}{2}\epsilon^{ijk}F_{jk} = \frac{sin(\theta)}{e}\epsilon^{ijk}\partial_j\theta\partial_k\phi \Rightarrow$$

$$B^r = \frac{1}{er^2} \tag{6.116}$$

And thus:

$$q_m = \frac{4\pi}{e} \tag{6.117}$$

The configuration should not contain any neutral charges. To check that consider the field tensor:

$$\mathcal{F}_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu = \frac{e}{gg'}[B' - W'](\hat{r}_\mu\partial_\nu t - \hat{r}_\nu\partial_\mu t) \tag{6.118}$$

$$\mathcal{F}_{0i} = -\frac{e}{gg'}[B' - W']\hat{r}_i \qquad \mathcal{F}_{ij} = 0 \tag{6.119}$$

And the charges are:

$$Z_e = -\frac{4\pi e}{gg'}[r^2B' - r^2W']_{r=\infty} = 0 \qquad\qquad Z_m = 0 \tag{6.120}$$

We conclude that Cho-Mason solution indeed describes a dyon, which resembles a hybrid between Dirac and 't Hooft Polyakov monopole, since it contains a singular string and it seems to be stable configuration with a magnetic topological charge. We highlight it like this, because there is a crucial problem with such solution; the energy of the configuration is infinite from the hypercharge field contribution. This suggests this dyon configuration may not be stable after all. Another important problem with this analysis is that it may not be gauge invariant, since the solution was obtained by fixing ξ in a particular gauge. If the solution is gauge dependent after all, this suggests that such configuration cannot be physical.

As final remark let's investigate the magnetic charge in the radial gauge. The gauge invariant electromagnetic tensor is given by [22]:

$$F^{em}_{jk} = sin(\theta_W)\vec{F}_{jk} \cdot \hat{\phi} + cos(\theta_W)G_{jk} \tag{6.121}$$

In the radial gauge $\hat{\phi} = -\hat{r}$ and the magnetic charge is given by:

$$q_m = -\frac{e}{g}\int dS_i\epsilon^{ijk}\frac{1}{2}\vec{F}_{jk} \cdot \hat{r} + \frac{e}{g'}\int dS_i\frac{1}{2}\epsilon^{ijk}G_{jk} \tag{6.122}$$

we get from (6.48):

$$F^a_{jk} \cdot \hat{r}^a = \frac{f^2 - 1}{gr^2}\epsilon_{jka}\hat{r}^a \tag{6.123}$$

Let's calculate the first term in the expression of the magnetic charge:

$$-\frac{e}{g}\int dS_i\frac{f^2-1}{gr^2}\hat{r}^i = -\frac{e}{g^2}\int_0^\pi d\theta\int_0^{2\pi} d\phi sin(\theta)(f^2(\infty) - 1) = \frac{4\pi e}{g}$$

The second gives:

$$\frac{e}{g'}\int dS_i \frac{1}{2}\epsilon^{ijk}G_{jk} = \frac{4\pi e}{g'^2}$$

Then magnetic charge is given by:

$$q_m = 4\pi e(\frac{1}{g^2} + \frac{1}{g'^2}) \Rightarrow$$

$$q_m = \frac{4\pi}{e} \tag{6.124}$$

The fact that the magnetic charge is consistent in both radial and unitary gauge, it's a great indication that our physical observables are gauge independent.

6.7. Electromagnetic Regularization

We are starting by adding extra electromagnetic interactions of the charged W fields with the dyon. This is the most economic way to regularise the energy of the monopole, since we are using the already existing W boson without introducing a new source.

Consider the electroweak Lagrangian (6.1) in the unitary gauge where:

$$\phi = \frac{\rho(x)}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} \tag{6.125}$$

With x being a space-time coordinate. We write:

$$D_\mu\phi^\dagger D^\mu\phi = \partial^\mu\phi^\dagger\partial_\mu\phi + \frac{ig}{2}(\phi^\dagger\vec{\sigma}\cdot\vec{W}_\mu\partial^\mu\phi - \partial^\mu\phi^\dagger\vec{\sigma}\cdot\vec{W}_\mu\phi) + \frac{ig'}{2}B_\mu(\phi^\dagger\partial^\mu\phi - \partial^\mu\phi^\dagger\phi) + \frac{gg'}{2}B_\mu\phi^\dagger\vec{\sigma}\cdot\vec{W}_\mu\phi + \frac{g'^2}{4}B_\mu B^\mu\phi^\dagger\phi$$

$$+\frac{g^2}{4}\phi^\dagger\phi W^a_\mu W^{a\mu}$$

In the unitary gauge holds:

$$\phi^\dagger\phi = \frac{1}{2}\rho^2$$

$$\partial^\mu\phi^\dagger\partial_\mu\phi = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho$$

$$\phi^\dagger\vec{\sigma}\cdot\vec{W}_\mu\partial^\mu\phi - \partial^\mu\phi^\dagger\vec{\sigma}\cdot\vec{W}_\mu\phi = \rho W^3_\mu\partial^\mu\rho - \rho W^3_\mu\partial^\mu\rho = 0$$

$$\phi^\dagger\partial^\mu\phi - \partial^\mu\phi^\dagger\phi = \frac{1}{2}\rho\partial^\mu\rho - \frac{1}{2}\partial^\mu\rho\rho = 0$$

$$\phi^\dagger\vec{\sigma}\cdot\vec{W}_\mu\phi = -\frac{\rho^2}{2}W^3_\mu$$

And we get:

$$D_\mu\phi^\dagger D^\mu\phi = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho - \frac{gg'}{4}\rho^2 B_\mu W^{3\mu} + \frac{g'^2}{8}B_\mu B^\mu\rho^2 + \frac{g^2}{8}\rho^2 W^a_\mu W^{a\mu} \tag{6.126}$$

We expand the kinetic energy of the non-Abelian field (6.2):

$$-\frac{1}{4}F^{a\mu\nu}F^{a}_{\mu\nu} = -\frac{1}{4}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)(\partial^\mu W^{a\nu} - \partial^\nu W^{a\mu}) - \frac{1}{4}2g\epsilon_{ajk}(\partial_\mu W^{a\nu} - \partial_\nu W^{a\mu})W^j_\mu W^k_\nu$$

$$-\frac{g^2}{4}(W^j_\mu W^{j\mu} W^k_\nu W^{k\nu} - W^k_\mu W^{j\mu} W^k_\nu W^{j\nu})$$

Let $\mathcal{A}_\mu = W^3_\mu$, $F_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ and $F^{\pm}_{\mu\nu} = \partial_\mu W^{\pm}_\nu - \partial_\nu W^{\pm}_\mu$ by recalling (6.6) , then the first term gives:

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}F^{-}_{\mu\nu}F^{+\mu\nu}$$

The second term gives:

$$-\frac{g}{2}\epsilon_{3jk}F^{\mu\nu}W^j_\mu W^k_\nu - \frac{g}{2}\sum_{i=1}^{2}\epsilon_{ijk}(\partial_\mu W^i_\nu - \partial_\nu W^i_\mu)W^{j\mu}W^{k\nu} =$$

$$\frac{ig}{2}F^{\mu\nu}(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu) + \frac{ig}{4}(F^-_{\mu\nu} + F^+_{\mu\nu})(W^{-\mu}\mathcal{A}^\nu - W^{+\mu}\mathcal{A}^\nu - W^{-\nu}A^\mu + W^{+\nu}\mathcal{A}^\mu) =$$

$$igF^{\mu\nu}W^+_\mu W^-_\nu + \frac{ig}{2}(F^+_{\mu\nu} + F^-_{\mu\nu})(W^{-\mu}\mathcal{A}^\nu - W^{+\mu}\mathcal{A}^\nu)$$

And the third one:

$$-\frac{g^2}{4}(W^j_\mu W^{j\mu} W^k_\nu W^{k\nu} - W^k_\mu W^{j\mu} W^k_\nu W^{j\nu}) =$$

$$\frac{g^2}{4}[-4(W^{-\mu}W^+_\mu)^2 + 2(W^-_\mu W^{+\mu})^2 + 2W^+_\mu W^{+\mu} W^-_\nu W^{-\nu}] =$$

$$\frac{g^2}{4}(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu)(W^{+\mu}W^{-\nu} - W^{-\mu}W^{+\nu})$$

Thus the non Abelian kinetic energy is:

$$-\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}F^-_{\mu\nu}F^{+\mu\nu} + igF^{\mu\nu}W^+_\mu W^-_\nu + \frac{ig}{2}(F^+_{\mu\nu} + F^-_{\mu\nu})(W^{-\mu}\mathcal{A}^\nu - W^{+\mu}\mathcal{A}^\nu) \quad (6.127)$$

The potential term becomes with (6.125)

$$V = -\frac{\lambda}{2}(\frac{\rho^2}{2} - u^2)^2 \quad (6.128)$$

Add (6.127) and (6.126)

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F^-_{\mu\nu}F^{+\mu\nu} + igF^{\mu\nu}W^+_\mu W^-_\nu + \frac{ig}{2}(F^+_{\mu\nu} + F^-_{\mu\nu})(W^{-\mu}\mathcal{A}^\nu - W^{+\mu}\mathcal{A}^\nu)$$

$$+\frac{1}{2}\partial_\mu\rho\partial^\mu\rho - \frac{gg'}{4}\rho^2 B_\mu\mathcal{A}^\mu + \frac{g'^2}{8}B_\mu B^\mu\rho^2 + \frac{g^2}{8}\rho^2\mathcal{A}_\mu\mathcal{A}^\mu + \frac{g^2}{4}\rho^2 W^+_\mu W^{-\mu} =$$

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + igF^{\mu\nu}W^+_\mu W^-_\nu + \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{4}\rho^2[g^2 W^+_\mu W^{-\mu} + (g'B_\mu - g\mathcal{A}_\mu)(g'B^\mu - g\mathcal{A}^\mu)]$$

$$+\frac{1}{2}(-\partial_\mu W^-_\nu\partial^\mu W^{+\nu} + \partial_\mu W^-_\nu\partial^\nu W^{+\mu} + \partial_\nu W^-_\mu\partial^\mu W^{+\nu} - \partial_\nu W^-_\mu\partial^\nu W^{+\mu}) + \frac{1}{2}\partial_\mu W^-_\nu(W^{-\mu}ig\mathcal{A}^\nu - W^{-\nu}ig\mathcal{A}^\mu)$$

$$-\frac{1}{2}\partial_\mu W^+_\nu(W^{+\mu}ig\mathcal{A}^\nu - W^{+\nu}ig\mathcal{A}^\mu) + \frac{1}{2}\partial_\mu W^+_\nu(W^{-\mu}ig\mathcal{A}^\nu - W^{-\nu}ig\mathcal{A}^\mu) - \frac{1}{2}\partial_\mu W^-_\nu(W^{+\mu}ig\mathcal{A}^\nu - W^{+\nu}ig\mathcal{A}^\mu) =$$

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + igF^{\mu\nu}W^+_\mu W^-_\nu + \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{4}\rho^2[g^2W^+_\mu W^{-\mu} + (g'B_\mu - g\mathcal{A}_\mu)(g'B^\mu - g\mathcal{A}^\mu)] - \frac{1}{2}|D^\mu W^{-\nu} - D^\nu W^{-\nu}|^2$$

Where the covariant derivative is defined as $D_\mu = \partial_\mu + ig\mathcal{A}_\mu$. And the electroweak Lagrangian is written as:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}\partial_\mu\rho\partial^\mu\rho - \frac{\lambda}{2}(\frac{\rho^2}{2} - u^2)^2$$

$$-\frac{1}{2}|D^\mu W^{-\nu} - D^\nu W^{-\mu}|^2 + \frac{g^2}{4}(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu)(W^{+\mu}W^{-\nu} - W^{-\mu}W^{+\nu})$$

$$+igF^{\mu\nu}W^+_\mu W^-_\nu + \frac{1}{4}\rho^2[g^2W^+_\mu W^{-\mu} + (g'B_\mu - g\mathcal{A}_\mu)(g'B^\mu - g\mathcal{A}^\mu)] \quad (6.129)$$

Notice that by working on the unitary gauge we have decompose the initial non-Abelian theory of electroweak interactions to an Abelian theory with electroweak sources. To regularise the Cho-Maison solution we introduce an extra interaction $\mathcal{L}'$:

$$\mathcal{L}' = iagF^{\mu\nu}W^+_\mu W^-_\nu + \frac{\beta^2 g^2}{4}(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu)(W^{+\mu}W^{-\nu} - W^{-\mu}W^{+\nu}) \quad (6.130)$$

Where a and β are treated as free parameters. For example such modification of the bare theory could be arise from quantum corrections, where these parameters are scale dependent. Also considering them as free parameters is easier and their final value is possible to obtain from quantum corrections near $r = 0$. We proceed by computing the energy momentum tensor of the modified theory :

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = -g^{\mu\nu}\mathcal{L} - F^{\mu\sigma}F^\nu{}_\sigma - G^{\mu\sigma}G^\nu{}_\sigma + \partial^\mu\rho\partial^\nu\rho + 4ig(1+a)F^{\mu\sigma}W^{+\nu}W^-_\sigma - (D^\mu W^+_a - D_a W^{+\mu})(D^\nu W^{-a} - D^a W^{-\nu})$$

$$g^2(1+\beta)(W^{+\mu}W^-_a - W^+_a W^{-\mu})(W^{+\nu}W^{-a} - W^{+a}W^{-\nu}) + \rho^2[g^2W^{-\mu}W^{+\nu} + \frac{1}{2}(g'B^\mu - gA^\mu)(g'B^\nu - gA^\nu)] \quad (6.131)$$

We write the solutions (6.38) in the unitary gauge where

$$W^-_\mu = (W^+_\mu)^* = \frac{i}{g}\frac{f(r)}{\sqrt{2}}e^{i\phi}(\partial_\mu\theta + isin(\theta)\partial_\mu\phi) \quad (6.132)$$

$$\mathcal{A}_\mu = W^3_\mu = -\frac{1}{g}W(r)\partial_\mu t - \frac{1}{g}(1 - cos(\theta))\partial_\mu\phi \quad (6.133)$$

And we also have

$$\rho = \rho(r) \quad (6.134)$$

Now the energy of the configuration can be obtained by integrating T^{00}. The energy expression will be similar to (6.65), with some modifications from the terms:

$$T^{00} \ni -\frac{g^2}{4}(\beta)(W^+_i W^-_j - W^-_i W^+_j)^2 - igaF_{ij}W^+_i W^-_j =$$

$$-\frac{g^2}{16}\beta(2isin(\theta)\partial_i\theta\partial_j\phi - 2isin(\theta)\partial_j\theta\partial_i\phi)^2 - 2igaF_{\theta\phi}W^+_\theta W^-_\phi =$$

$$\frac{f^4}{2g^2r^4}\beta - a\frac{f^2}{g^2r^4}$$

As for the rest of the terms, they contribute to (6.65) and the expression (6.65) by extracting $1/r^2$ term becomes:

$$E_2 = \frac{4\pi}{g^2}\int_0^\infty dr[\frac{r^2(W')^2}{2} + f^2W^2 + (f')^2 + \frac{g^2r^2}{2g'^2}B'^2 + g^2\frac{\lambda r^2}{2}[\frac{\rho^2}{2} - u^2]^2 + g^2r^2\frac{\rho^2}{8}[W-B]^2 + \frac{(r\rho')^2}{2} + \frac{g^2f^2\rho^2}{4}] \quad (6.135)$$

As for the singular energy term (6.64) with the modifications mentioned above and the term we extract from (6.65), becomes:

$$E_1 = \frac{2\pi}{g^2}\int_0^\infty \frac{dr}{r^2}[(1+\beta)f^4 - 2(a+1)f^2 + 1 + \frac{g^2}{g'^2}] \quad (6.136)$$

When $a = \beta = 0$ E_1 is singular, but it becomes regular if:

$$1 + \frac{g^2}{g'^2} + (1+\beta)f^4(0) - 2(a+1)f^2(0) = 0 \quad (6.137)$$

In addition to that the extremization of the energy functional provides us with:

$$(1+a)f(0) - (1+\beta)f^3(0) = 0 \Rightarrow$$

$$f^2(0) = \frac{1+a}{1+\beta} \quad (6.138)$$

Combine (6.137), (6.138) and we get:

$$1 + \frac{g^2}{g'^2} - 2\frac{(1+a)^2}{1+\beta} + \frac{(1+a)^2}{1+\beta} = 0 \Rightarrow$$

$$\frac{(1+a)^2}{1+\beta} = 1 + \frac{g^2}{g'^2} = \frac{1}{sin^2(\theta_W)} \quad (6.139)$$

And we get for $f(0)$

$$f(0) = \frac{1}{\sqrt{(1+a)sin^2(\theta_W)}} \quad (6.140)$$

Now the boundary condition of $f(r)$ in (6.72) provides us with:

$$f(0) = 1 \Rightarrow (1+a)sin^2(\theta_W) = 1 \Rightarrow$$

$$a = cot^2(\theta_W) \quad (6.141)$$

The parameter a depends from the initial condition of the gauge field. Next let's investigate how the equations of motion are modified. The Euler-Lagrange equations that minimise the energy functional give:

$$\frac{\delta E}{\delta f} = \frac{d}{dr}(\frac{\delta E}{\delta f'}) \Rightarrow$$

$$f'' - \frac{1}{sin^2(\theta_W)r^2}f(f^2-1) = (\frac{g^2\rho^2}{4} + W^2)f \quad (6.142)$$

The remaining equations of motion are left unchanged and given by (6.71), (6.69), (6.70). We also integrate the equations with the boundary conditions (6.72). Numerical integration of the energy functional gives [12]:

$$E = 2.922sin^2(\theta_W)\frac{4\pi}{e^2}M_W \quad (6.143)$$

We note that the integration was performed for $\lambda/g^2 = 0.5$. The energy is finite and since the differential equations are very similar to the differential equations that Cho-Maison solution satisfy. Then the solution here is expected to be similar.

6.8. Comparison to Julia-Zee Dyon

Cho-Maison solution describes a dyon field configuration in $SU(2)\times U(1)$ theory with a singularity at the center. On the other hand Julia-Zee solution describes a dyon field configuration in $SU(2)$ theory which is regular. To regularise the Cho-Maison solution is crucial the differences between these two dyons.

Notice when Z bosons are absent, (6.129) becomes:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\rho\partial^\mu\rho$$

$$-\frac{1}{2}[D^\mu W^\nu - D^\nu W^\mu]^*[D_\mu W_\nu - D_\nu W_\mu]$$

$$+igF_{\mu\nu}W^{*\mu}W^\nu + \frac{g^2}{4}(W^*_\mu W_\nu - W^*_\nu W_\mu)(W^{*\mu}W^\nu - W^{*\nu}W^\mu) - \frac{\lambda}{2}(\frac{\rho^2}{2} - u^2)^2 + \frac{1}{4}\rho^2 g^2 W^*_\mu W^\mu \tag{6.144}$$

One the other hand (4.22) after Abelianization becomes:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\partial_\mu\rho\partial^\mu\rho - \frac{1}{2}[D^\mu W^\nu - D^\nu W^\mu]^*[D_\mu W_\nu - D_\nu W_\mu]$$

$$+igF_{\mu\nu}W^{*\mu}W^\nu + \frac{g^2}{4}(W^*_\mu W_\nu - W^*_\nu W_\mu)(W^{*\mu}W^\nu - W^{*\nu}W^\mu) - \frac{\lambda}{2}(\frac{\rho^2}{2} - u^2)^2 + g^2\rho^2 W^*_\mu W^\mu \tag{6.145}$$

Notice that (6.145) is identical to (4.22). The only difference is the coupling strengths of W-boson self-interaction and Higgs interaction of W-bosons. This originates from the fact the electroweak model has two coupling constants and Georgi-Glashow model has only one coupling constant. This suggests that in spite of the fact the Cho-Maison solution has infinity energy, it is not much different from Julia-Zee dyon in Georgi-Glashow model.

Consider the Julia-Zee solution (4.71) together with (4.48):

$$\vec{\phi} = uH(r)\hat{r} = \rho(r)\hat{r} \tag{6.146}$$

$$\vec{A_\mu} = \frac{1}{e}A(r)\partial_\mu t\hat{r} + \frac{1}{e}(1 - K(r))\hat{r}\times\partial_\mu\hat{r} \tag{6.147}$$

In the unitary gauge these solutions are written as:

$$W_\mu = \frac{A^1_\mu + A^2_\mu}{\sqrt{2}} = \frac{i}{e}\frac{K(r)}{\sqrt{2}}e^{i\phi}(\partial_\mu\theta + sin(\theta)\partial_\mu\phi) \qquad \mathcal{A}_\mu = A^3_\mu = -\frac{1}{e}V(r)\partial_\mu t - \frac{1}{e}(1 - cos\theta)\partial_\mu\phi \tag{6.148}$$

This is an identical solution to Cho-Maison solution expressed in terms of the physical fields (6.106), (6.109) when $Z = 0$. In addition to that the differential equations for Julia-Zee dyon in (4 4.7) are very similar to the differential equations of Cho-Maison dyon in (6 6.5). Integration with boundary conditions mentioned in (4 4.7) gives a finite energy dyon. This confirms that Julia-Zee dyon is regularized by the function $\rho(r)$, $A(r)$ and $K(r)$. Therefore, Julia-Zee dyon is an Abelian monopole regularized by the charged vector bosons and the Higg's field, where the charged bosons add electric charge in the configuration. On the other hand Cho-Maison dyon has a non trivial dressing of Z-bosons, something that Julia-Zee dyon does not have. But notice the Z-boson plays no role in the Cho-Maison monopole[9]. This suggests that the Cho-Maison monopole could be modified to have finite energy.

---

[9] $W(r) = B(r) = 0$ in (6.109) and (6.110)

6.9. Hypercharge Regularization

When we first compute the energy of the electroweak dyon the infinity contribution comes from the hypercharge field. Then E_1 in (6.53) can be finite if quantum corrections makes the coupling $\frac{1}{g'^2}$ zero at small distances.

To do this consider the following effective Lagrangian:

$$\mathcal{L}_{eff} = (D^\mu \phi)^\dagger D_\mu \phi - \frac{\lambda}{2}(\phi^\dagger \phi - u^2)^2 - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}\epsilon(|\phi|^2)G_{\mu\nu}G^{\mu\nu} \tag{6.149}$$

This dielectric 'constant' in front of the $U_Y(1)$ kinetic term could be arise from unspecified dynamics. In this subsection we are not going to discuss an ultraviolet completion of the Standard model that might lead to such behavior, but we are going to investigate the behavior of $\epsilon(|\phi|^2)$ that could make the energy of the dyon finite. We normalise the function $\epsilon(|\phi|^2)$ as $\epsilon(|\phi|^2) \xrightarrow{\phi\to u} 1$, in order to restore the conventional normalisation of the $U(1)$ gauge field in the standard electroweak vaccum. Adding this constant does not change the differential equations of $A(r)$ and $f(r)$, and will be the same as in (6 6.5). But, we expect the differential equations of $B(r)$ and $\rho(r)$ to change due to $\epsilon(|\phi|^2) = \epsilon(\rho^2)$. To obtain the new differential equations focus on the terms in (6.44):

$$\mathcal{H} \ni \epsilon(\rho^2)\frac{1}{2}\mathcal{E}_i\mathcal{E}_i + \epsilon(\rho^2)\frac{1}{2}\mathcal{B}_i\mathcal{B}_i + (D^0\phi)^\dagger D^0\phi + (D^i\phi)^\dagger D^i\phi + \frac{\lambda}{2}(\phi^\dagger\phi - u^2)^2 \Rightarrow$$

$$\mathcal{H} \ni [r^2\frac{(B')^2}{2g'^2} + \frac{1}{g'^2r^2}]\epsilon(\rho^2) + \frac{r^2\rho^2}{8}[W-B]^2 + \frac{r^2(\rho')^2}{2} + \frac{f^2\rho^2}{4} + r^2\frac{\lambda}{2}(\frac{\rho^2}{2} - u^2)^2 \tag{6.150}$$

Then the differential equations are given by:

$$\frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial B'}) = \frac{d\mathcal{H}}{dB} \Rightarrow$$

$$r^2B''\epsilon + 2rB'\epsilon + 2\rho\rho'B'r^2\epsilon' = -\frac{r^2\rho^2g'^2}{4}(W-B) \Rightarrow$$

$$B'' + 2B'(\frac{1}{r} + \rho\rho'\frac{\epsilon'}{\epsilon}) = -\frac{\rho^2g'^2}{4\epsilon}(W-B) \tag{6.151}$$

Where $\epsilon' = \frac{d\epsilon}{d\rho^2}$. As for $\rho(r)$:

$$\frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial\rho'}) = \frac{d\mathcal{H}}{d\rho} \Rightarrow$$

$$\rho'' + \frac{2}{r}\rho' = \frac{\rho}{g'^2r^4}\epsilon' + \frac{f^2\rho}{2r^2} + \lambda\rho(\frac{\rho^2}{2} - u^2) + \frac{\rho}{4}(W-B)^2 + \epsilon'\frac{\rho B'^2}{g'^2} \Rightarrow$$

$$\rho'' + \frac{2\rho'}{r} - \frac{f^2\rho}{2r^2} = \frac{\rho}{g'^2}(\frac{1}{r^4} + B'^2)\epsilon' + \lambda\rho(\frac{\rho^2}{2} - u^2) + \frac{\rho}{4}(W-B)^2 \tag{6.152}$$

This tells us that ϵ effectively changes the $U(1)_Y$ gauge coupling g' to the 'running' coupling $\bar{g}' = g'/\sqrt{\epsilon}$. This is because with rescaling of B_μ to B_μ/g', g' changes to $g'/\sqrt{\epsilon}$. So by making $\bar{g}'$ infinite at the origin, we can regularise the energy of the configuration. In particular near the origin the singular term in (6.136) for $a = \beta = 0$ becomes:

$$E_1 \ni 2\pi\int_0^\infty \frac{dr}{\bar{g}'^2r^2} \approx \frac{\epsilon}{g'^2r^2} \sim \frac{\rho^n}{r^2} \sim r^{n\delta-2}$$

For the energy to be finite it must holds:

$$n > 2/\delta \Rightarrow$$

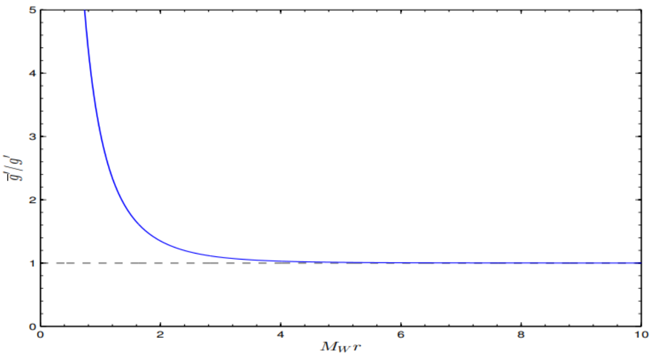

FIG. 11: Running coupling $\bar{g}'$ of $U(1)_Y$ as a function of $m_W r$.

$\epsilon(\rho)$	M [TeV]
$5(\frac{\rho}{\rho_0})^8 - 4(\frac{\rho}{\rho_0})^{10}$	6.6
$6(\frac{\rho}{\rho_0})^{10} - 5(\frac{\rho}{\rho_0})^{12}$	6.2
$8(\frac{\rho}{\rho_0})^8 - 10(\frac{\rho}{\rho_0})^{10} + 3(\frac{\rho}{\rho_0})^{12}$	6.8
$8(\frac{\rho}{\rho_0})^{14} - 7(\frac{\rho}{\rho_0})^{16}$	5.7
$-8(\frac{\rho}{\rho_0})^{14} log(\rho) + (\frac{\rho}{\rho_0})^{16}$	5.4

TABLE I: Monopole masses in TeV for various $\epsilon(\rho)$ regularisations that are consistent with both theoretical and phenomenological constraints [16].

$$n > 1 + \sqrt{3} \approx 2.732 \tag{6.153}$$

In figure (11) we observe the running coupling $\bar{g}'$ becomes divergent at the origin. This confirms that the ultraviolet regularisation of the electroweak dyon is possible. For $W = B = 0$ we can estimate the monopole energy and for $n = 8$ at (6.153) we get [13]:

$$E \approx 0.65 \frac{4\pi}{e^2} m_W \approx 7.19 TeV \tag{6.154}$$

.

However if we pick $\epsilon(\rho) \sim \rho^8$ there is an experimental problem with this choice. In particular, the effective $H\gamma\gamma$ coupling is much larger than is allowed by the LHC measurements [17]. Also such decay measurements are consistent with standard model calculations. Therefore, in addition to the theoretical constrain (6.153) we must take into account this phenomenological constraint. Further constraints of $\epsilon(\rho)$ that are consistent with both theoretical and phenomenological properties can seen in table I together with the associated monopole masses. We have seen two schemes, under which electroweak monopole obtains a finite mass. These two schemes are modification of the Lagrangian, which could arise from extension of standard model. In the next section we will see a particular extension by using Born-Infled terms.

7. BORN-INFELD MONOPOLES

In this section we are going to discuss monopole configurations in extensions of the standard model by using non-linear terms in the gauge sector. Such terms are known as Born-Infeld terms and arise naturally in the low-energy limit of strings. We are starting by briefly discussing the non-liner framework of electrodynamics, before introducing Born-Infeld electrodynamics and their relation with string theory. Then we firstly apply such modifications to electroweak model, in particular to the hypercharge sector and construct finite-energy monopole configuration. Additionally we apply these modifications non-Abelian $SO(3)$ models with monopole configurations for flat and curved backgrounds.

7.1. General Structure of Non-Linear Electrodynamics

Electromagnetism described by the Maxwell equations in (2) is used to study electric and magnetic fields in vacuum. When a material is present Maxwell equations are modified, since light itself behaves differently in various mediums. In particular fields change as $\vec{E} \to \vec{D}(\vec{E}, \vec{B})$ and $\vec{B} \to \vec{H}(\vec{E}, \vec{B})$, due to the incorporation of magnetisation and polarization. Linear materials have linear relations between fields. but other materials may have more complicated relations. As for physical laws (2.2), (2.3) remain the same and the others laws are modified as:

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{7.1}$$

$$\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \tag{7.2}$$

Where ρ and $\vec{j}$ are free sources that do not take into account the polarization charges and magnetization currents. The field tensor also changes as $F_{\mu\nu} \to G_{\mu\nu}$, where just like standard electromagnetic field tensor we get:

$$G_{0i} = D_i \tag{7.3}$$

$$G_{ij} = \epsilon_{ijk} H_k \tag{7.4}$$

And the modified physical laws become:

$$\partial_\mu G^{\nu\mu} = j_f^\nu \tag{7.5}$$

The equations above are regarded to be of limited validity, as they deal with materials, most of which do not generally lead themselves to exact analysis. Born–Infeld and in general nonlinear theories of electrodynamics in fact suggest that the above equations are as fundamental as the original Maxwell equations, if not more so. These theories do not aim to describe electromagnetism in the presence of materials but rather electromagnetism in the vacuum. The point is that in nonlinear electrodynamics the vacuum itself behaves as some kind of material. The Lagrangian of such system will be a function of the standard electromagnetic field tensor. The action is written then:

$$S = \int d^4x[\mathcal{L}(F_{\mu\nu}) + A^\mu j_\mu] \tag{7.6}$$

Since field tensor is gauge invariant then $\mathcal{L}$ is also gauge invariant.

Consider an infinitesimal variations of the field tensor $\delta F_{\mu\nu}$, then a general function M of $F_{\mu\nu}$ varies as:

$$\delta M = \frac{1}{2}\frac{\partial M}{\partial F_{\mu\nu}} \delta F_{\mu\nu}$$

Where 1/2 factor is used since $\delta F_{\mu\nu} = -\delta F_{\nu\mu}$. The variation of the action must vanish:

$$\delta S = \int d^4x[\frac{1}{2}\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} \delta F_{\mu\nu} + j^\mu \delta A_\mu] = 0 \Rightarrow$$

$$\int d^4x \frac{1}{2}\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) + \int d^4x j^\mu \delta A_\mu = 0 \Rightarrow$$

$$\int d^4x \delta A_\mu [\partial_\nu \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} + j^\mu] = 0$$

Therefore by comparing it with (7.5) we get:

$$\frac{\partial \mathcal{L}}{\partial F^{\mu\nu}} = -G_{\mu\nu} \tag{7.7}$$

Using chain-rule we can find a formula that relates $\vec{D}$ with Lagrangian $\mathcal{L}$:

$$\frac{\partial \mathcal{L}}{\partial E_i} = \frac{\partial \mathcal{L}}{\partial F_{0i}} \frac{\partial F_{0i}}{\partial E_i} = \frac{\partial \mathcal{L}}{\partial F_{0i}} = -D_i \Rightarrow$$

$$\vec{D} = -\frac{\partial \mathcal{L}}{\partial \vec{E}} \tag{7.8}$$

In a similar manner we can show that:

$$\vec{H} = \frac{\partial \mathcal{L}}{\partial \vec{B}} \tag{7.9}$$

With these formulas we can also calculate the the Hamiltonian of the system:

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu}\dot{A}_\mu - \mathcal{L} \Rightarrow$$

$$\mathcal{H} = \vec{D}\cdot\vec{E} - \mathcal{L} \tag{7.10}$$

It is easy to check that these equations hold for Maxwell equations when $\vec{E} = \vec{D}$ and $\vec{B} = \vec{H}$. So far we have not consider any particular form of $\mathcal{L}$. but we are limited since it must be both Lorentz and gauge invariant. In particular, it must be a function of :

$$s = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(E^2 - B^2) \tag{7.11}$$

$$p = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = \vec{E}\cdot\vec{B} \tag{7.12}$$

In fact these are the only two Lorentz invariant quantities we can consider.

As a final remark we should discuss when electrodynamics exhibit a non-linear behavior. This is known as **Schwinger limit** and is defined as a critical value of the electric for which the electrodynamics exhibit a non-linear behavior. The limit is defined as [27]:

$$E_c = \frac{m^2c^3}{\hbar} \tag{7.13}$$

$$B_c = \frac{m^2c^2}{\hbar} \tag{7.14}$$

Where m is the mass of a charged particle. For electric fields $E >> E_c$ the vacuum is expected to create enough virtual electron-positron pairs, causing the electromagnetism to be non-linear theory. For an electron field example, the critical values are:

$$E_c \approx 10^8 V/m \qquad B_c \approx 10^9 T \tag{7.15}$$

These are enormous values. For instance such magnetic fields are exceeded in magnetars and such electric fields are capable to accelerate a proton from rest to the maximum energy attained by protons at the LHC.

7.2. Born-Infeld Electrodynamics

We can try to modify the electromagnetic Lagrangian, by considering Lorentz invariant quantities. From a modified expression we should to recover our usual Maxwell Lagrangian in a particular limit. Such Lagrangian is the Born-Infeld Lagrangian:

$$\mathcal{L} = -\beta^2\sqrt{1 - \frac{2s}{\beta^2} - \frac{p^2}{\beta^4}} + \beta^2 = \beta^2(1 - \sqrt{1 + \frac{1}{2\beta^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16\beta^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2} \tag{7.16}$$

At the limit where $\beta \to \infty$ we have:

$$\mathcal{L} \xrightarrow{\beta\to\infty} \beta^2 - \beta^2(1 + \frac{1}{4\beta^2}F^{\mu\nu}F_{\mu\nu}) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

Of course there are other combinations, which recover Maxwell's Lagrangian, but this one has some interesting properties. One of them is that it can be written as:

$$\mathcal{L} = \beta^2[1 - \sqrt{-det(\eta + \frac{1}{\beta}F)}] \tag{7.17}$$

The η and F are the metric tensor and field tensor written as matrices. To show that this coincides with our initial Born Infeld Lagrangian we rescale $F' = \frac{F}{\beta}$ and $\mathcal{L}' = \frac{\mathcal{L}}{\beta^2}$:

$$\mathcal{L}' = 1 - \sqrt{-det(\eta + F')} \Rightarrow$$

$$\mathcal{L}' = 1 - \sqrt{-det(\eta(1 + \eta F'))} = 1 - \sqrt{-det(\eta)det(1 + \eta F'))} \Rightarrow$$

$$\mathcal{L}' = 1 - \sqrt{det(1 + \eta F'))}$$

We have:

$$1 + \eta F' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & E_1 & E_2 & E_3 \\ E_1 & 1 & B_3 & -B_2 \\ E_2 & -B_3 & 1 & B_1 \\ E_3 & B_2 & -B_1 & 1 \end{pmatrix} \Rightarrow$$

$$det(1 + \eta F') = -1 - (B^2 - E^2) + (\vec{E} \cdot \vec{B})^2 = -1 + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{16}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2$$

Therefore, this form of Born-Infeld Lagrangian is equivalent to the original one. In string theory gauge field on a D-brane are described by a similar Lagrangian:

$$\mathcal{L} = -T_0\sqrt{-det(\eta + \frac{F}{T_0})} \tag{7.18}$$

Where T_0 is the string tension, which characterises the string scale. This suggests that parameter β can be identified as the string tension.

A simply application of Born-Infeld electrodynamics is the electrostatic energy of point like charge Q. In Maxwell's theory of electromagnetism this energy is infinity. Indeed,

$$E_Q = \frac{1}{8\pi}\int d^3x E^2 = \frac{1}{8\pi}\int_0^\infty dr 4\pi r^2 \frac{Q^2}{r^4} \to \infty$$

Now let's calculate the same energy with Born-Infled Lagrangian. For an electrostatic charge we have:

$$\mathcal{L} = \beta^2(1 - \sqrt{1 - \frac{1}{\beta^2}E^2})$$

To calculate the energy of the point-like charge we have:

$$\vec{D} = -\frac{\partial \mathcal{L}}{\partial \vec{E}} = \frac{\vec{E}}{\sqrt{1 - \frac{E^2}{\beta^2}}} \Rightarrow$$

$$\vec{E} = \frac{\vec{D}}{\sqrt{1 + \frac{D^2}{\beta^2}}} \tag{7.19}$$

For a point-like charge over closed surface S holds:

$$\oint \vec{D} \cdot \vec{da} = Q \Rightarrow$$

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r} \tag{7.20}$$

Then according to (7.10) the Hamiltonian density is given by:

$$\mathcal{H} = \frac{D^2}{\sqrt{1+\frac{D^2}{\beta^2}}} - \beta^2(1 - \sqrt{1 - \frac{D^2/\beta^2}{1+\frac{D^2}{\beta^2}}}) = \frac{1 + D^2/\beta^2}{\sqrt{1+D^2/\beta^2}} - \beta^2 \Rightarrow$$

$$\mathcal{H} = \beta^2\sqrt{1+D^2/\beta^2} - \beta^2 \Rightarrow$$

$$\mathcal{H} = \beta^2\sqrt{1+\frac{Q^2}{16\pi^2\beta^2 r^4}} - \beta^2 \tag{7.21}$$

And the total energy of the point-like charge is:

$$E_Q = \int d^3x\mathcal{H} = \int_0^\infty dr 4\pi r^2(\beta^2\sqrt{1+(\frac{Q}{4\pi\beta r^2})^2} - \beta^2) = 4\pi\beta^2\int_0^\infty dr(r^2\sqrt{1+(\frac{Q}{4\pi\beta r^2})^2} - r^2)$$

Let $x = \sqrt{\frac{4\pi}{Q\beta}}r$ and we have:

$$E_Q = Q\sqrt{\frac{Q\beta}{4\pi}}\int_0^\infty dx(\sqrt{1+x^4} - x^2) \approx \frac{1}{4\pi}4.382Q^{3/2}\beta^{1/2} = \frac{1}{4\pi}4.382Q^{3/2}(T_0)^{1/4}$$

Therefore the electrostatic energy becomes finite and depends from the string tension T_0. At the limit $T_0 \to \infty$ the energy becomes infinity, thus the expression is consistent with the recovery of Maxwell's theory.

So far we have discuss the Abelian Born Infeld Lagrangian. We can generalise this description to a Yang-Mills field. Such generalization is not unique and the simplest one is given by [25]:

$$\mathcal{L} = \beta^2 tr[1 - \sqrt{-det(\eta + \frac{F}{\beta})}] = \beta^2 tr[1 - \sqrt{1 + \frac{1}{2\beta^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16\beta^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2}] \tag{7.22}$$

Where $F_{\mu\nu} = F^a_{\mu\nu}T^a$ is the non-Abelian field tensor. A second possibility is [25]:

$$\mathcal{L} = \beta^2 Str[1 - \sqrt{-det(\eta + \frac{F}{\beta})}] = \beta^2 tr[1 - \sqrt{1 + \frac{1}{2\beta^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16\beta^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2}] \tag{7.23}$$

Where we have used the symmetric trace operation:

$$Str(T_1, T_2, ..T_N) = \frac{1}{N!}\sum_p tr(T_{p(1)}T_{p(2)}...T_{p(N)}) \tag{7.24}$$

Coming next we are going to see further applications of Born-Infeld electrodynamics to monopoles. Such models can considered as standard model extensions that can be viewed as a low-energy limit of strings, or simply extensions with gauge fields exhibiting a non-linear behavior.

7.3. Electroweak Born-Infeld Monopole

In (6 6.9) we have discussed how a modified $U(1)_Y$ kinetic term in the effective Lagrangian could make the monopole mass finite without specify any dynamics that make this possible. Modifications of the theory beyond the standard model such as a non-linear Born-Infeld gauge theory could fit this description. Such theory arises as a low energy limit of strings, that effects the full standard model gauge sector, but for our purpose we will restrict to the Born-Infled hypercharge sector. Then we will investigate monopole solutions of Cho-Maison type.

Therefore, by replacing the kinetic hypercharge term in (6.1) by the Born-Infled term, we have:

$$\mathcal{L} = (D_\mu\phi)^\dagger D^\mu\phi - \frac{\lambda}{2}(\phi^\dagger\phi - u^2)^2 - \frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \beta^2(1 - \sqrt{1 + \frac{1}{2\beta^2}G_{\mu\nu}G^{\mu\nu} - \frac{1}{16\beta^4}(G_{\mu\nu}\tilde{G}^{\mu\nu})^2} \qquad (7.25)$$

In string theory models the parameter $\beta^2 = \frac{1}{2\pi a'} = T_0$ is the string tension, which sets the string energy scale. In this analysis we treat β as a free parameter that is restricted by experiment. In the limit $\beta \to \infty$ we recover (6.1) and expect the energy of the monopole to be infinite.

We can obtain the equations of motion from (7.25) and since only the hypercharge sector is modified, equations (6.20), (6.21) remaining the same and the hypercharge field satisfies:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu B_\nu} = \frac{\partial \mathcal{L}}{\partial B_\nu} \Rightarrow$$

$$\partial_\mu\left(\frac{-\frac{1}{4}\frac{\partial}{\partial_\mu B_\nu}[G_{ab}G^{ab}] + \frac{1}{16\beta^2}G_{ab}\tilde{G}^{ab}\frac{\partial}{\partial_\mu B_\nu}[G_{ab}\tilde{G}^{ab}]}{\sqrt{1 + \frac{1}{2\beta^2}G_{ab}G^{ab} - \frac{1}{16\beta^4}(G_{ab}\tilde{G}^{ab})^2}}\right) = \frac{ig'}{2}[\phi^\dagger D^\nu\phi - (D^\nu\phi)^\dagger\phi] \Rightarrow$$

$$\partial_\mu\left[\frac{G^{\mu\nu} - \frac{1}{4\beta^2}(G_{ab}\tilde{G}^{ab})\tilde{G}^{\mu\nu}}{\sqrt{1 + \frac{1}{2\beta^2}G_{ab}G^{ab} - \frac{1}{16\beta^4}(G_{ab}\tilde{G}^{ab})^2}}\right] = \frac{ig'}{2}[\phi^\dagger D^\nu\phi - (D^\nu\phi)^\dagger\phi] \qquad (7.26)$$

In this model we will consider a static Cho-Maison monopole solution. To obtain this solution we set $B(r) = 0$ in (6.35) and $W(r) = 0$ in (6.38), while working with the gauge (6.27). The solution is then

$$\phi = \frac{1}{\sqrt{2}}\rho\xi \qquad (7.27)$$

$$\vec{W}_\mu = \frac{f-1}{g}\hat{r} \times \partial_\mu\hat{r} \qquad (7.28)$$

$$B_\mu = -\frac{1}{g'}(1 - cos(\theta))\partial_\mu\phi \qquad (7.29)$$

We expect the Higgs solution ϕ and the non-Abelian $\vec{W}_\mu$ solution to satisfy the equations of motion, since they are the same as in the electroweak model. But we should check if the same holds for the Abelian B_μ. (7.26) gives if we substitute B_μ:

$$\partial_\mu\left[\frac{G^{\mu\nu}}{\sqrt{1 + \frac{1}{\beta^2 g'^2 r^4}}}\right] = \frac{ig'}{2}[\phi^\dagger D^\nu\phi - (D^\nu\phi)^\dagger\phi] \Rightarrow$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{\frac{1}{g' r^2}}{\sqrt{1 + \frac{1}{\beta^2 g'^2 r^4}}}\right] = 0 \qquad \frac{1}{r sin(\theta)}\frac{\partial}{\partial\phi}\left[-\frac{\frac{1}{g' r^2}}{\sqrt{1 + \frac{1}{\beta^2 g'^2 r^4}}}\right] = 0$$

Therefore B_μ satisfies equations of motion. The energy functional of the system is expected to be very similar to Cho-Maison dyon. In particular we must set $W = B = 0$ at (6.65) and (6.64) must change due to Born-Infeld term. Now let's focus how the Born-Infeld term effects the energy momentum tensor, we have:

$$T^{\mu\nu} \ni -g^{\mu\nu}\mathcal{L} - \frac{1}{\sqrt{1 + \frac{1}{2\beta^2}G_{\mu\nu}G^{\mu\nu} - \frac{1}{16\beta^4}(G_{\mu\nu}\tilde{G}^{\mu\nu})^2}}[G^{\nu b}G^\mu_b - \frac{1}{4\beta^2}(G^{ab}\tilde{G}_{ab})G^\mu_c\tilde{G}^{\nu c}] \Rightarrow$$

$$\mathcal{H} \ni -\beta^2[1 - \sqrt{1 + \frac{1}{2\beta^2}G_{\mu\nu}G^{\mu\nu} - \frac{1}{16\beta^4}(G_{\mu\nu}\tilde{G}^{\mu\nu})^2}] - \frac{1}{\sqrt{1 + \frac{1}{2\beta^2}G_{\mu\nu}G^{\mu\nu} - \frac{1}{16\beta^4}(G_{\mu\nu}\tilde{G}^{\mu\nu})^2}}[G^{0b}G^0_b - \frac{1}{4\beta^2}(G^{ab}\tilde{G}_{ab})G^0_c\tilde{G}^{0c}] \Rightarrow$$

$$\mathcal{H} \ni -\beta^2[1 - \sqrt{1 + \frac{1}{2\beta^2}G_{\mu\nu}G^{\mu\nu} - \frac{1}{16\beta^4}(G_{\mu\nu}\tilde{G}^{\mu\nu})^2}]$$

Where the Abelian electric fields $\mathcal{E}_i = G_{0i}$ are zero since we have static configuration. We compute now (6.49) and the only non-zero components are:

$$G_{23} = -G_{32} = \frac{1}{g'r^2} \tag{7.30}$$

We also have:

$$\tilde{G}_{23} = \frac{1}{2}\epsilon_{23\rho\sigma}G^{\rho\sigma} = 0$$

Then hypercharge sector contributes to the Hamiltonian:

$$\mathcal{H} \ni -\beta^2[1 - \sqrt{1 + \frac{1}{\beta^2 g'^2 r^4}}]$$

And the energy of the monopole is given by:

$$E = 4\pi\beta^2 \int_0^\infty dr[\sqrt{r^4 + \frac{1}{\beta^2 g'^2}} - r^2]$$

$$+4\pi \int_0^\infty dr[\frac{(f')^2}{g^2} + \frac{(f^2-1)^2}{2g^2r^2} + \frac{\lambda r^2}{2}[\frac{\rho^2}{2} - u^2]^2 + \frac{(r\rho')^2}{2} + \frac{f^2\rho^2}{4}] \tag{7.31}$$

The second term of the energy is finite just like the Cho-Maison dyon. As promised the first term which was infinite in Cho-Maison dyon, now it is finite and we can compute it analytically. We let $r = \frac{x}{\sqrt{g'\beta}}$:

$$E_1 = 4\pi\beta^2 \int_0^\infty \frac{dx}{\sqrt{g'\beta}}(\sqrt{x^4+1} - x^2)\frac{1}{g'\beta} = 4\pi\sqrt{\frac{\beta}{g'^3}}\frac{(\Gamma(1/4))^2}{6\sqrt{\pi}} \approx 15.53\sqrt{\frac{\beta}{g'^3}} \tag{7.32}$$

Notice the similarity with the electrostatic energy of the point charge. We also observe that as $\beta \to \infty$, (7.32) becomes infinite, since at this limit Cho-Maison monopole solution holds. Using standard model value $g' = 0.357$ we get:

$$E_1 \approx 72.81\sqrt{\beta} \tag{7.33}$$

As for the other term:

$$E_2 = 4\pi \int_0^\infty dr[\frac{(f')^2}{g^2} + \frac{(f^2-1)^2}{2g^2r^2} + \frac{\lambda r^2}{2}[\frac{\rho^2}{2} - u^2]^2 + \frac{(r\rho')^2}{2} + \frac{f^2\rho^2}{4}] \tag{7.34}$$

More work must be done in order to compute it.

From the energy functional we get differential equations for $f(r)$ and $\rho(r)$ by using Euler-Lagrange equations:

$$\frac{d}{dr}\frac{\delta E}{\delta f'} = \frac{\delta E}{\delta f} \qquad \frac{d}{dr}\frac{\delta E}{\delta \rho'} = \frac{\delta E}{\delta \rho} \Rightarrow$$

$$\frac{2}{g^2}f'' = \frac{2f(f^2-1)}{g^2r^2} + \frac{f\rho^2}{2} \qquad 2r\rho' + r^2\rho'' = \lambda r^2[\frac{\rho^2}{2} - u^2]\rho + \frac{f^2\rho}{2} \Rightarrow$$

$$f'' - \frac{f(f^2-1)}{r^2} = \frac{g^2 f\rho^2}{4} \tag{7.35}$$

$$\rho'' + \frac{2\rho'}{r} - \frac{f^2\rho}{2r^2} = \lambda[\frac{\rho^2}{2} - u^2]\rho \tag{7.36}$$

The functions f and ρ satisfy the boundary conditions:

$$f(0) = 1 \qquad f(\infty) = 1 \qquad \rho(0) = 0 \qquad \rho(\infty) = \rho_0 = \sqrt{2}u \tag{7.37}$$

Now let's investigate the asymptotic behaviors of $f(r)$ and $\rho(r)$. To simplify the differential equations we let:

$$\tilde{\rho} = \frac{\rho}{\rho_0} \qquad \rho_0 = \sqrt{2}u \tag{7.38}$$

$$x = \mu r \qquad \mu^2 = \frac{\lambda\rho_0^2}{2} \tag{7.39}$$

Differential equation (7.36) becomes:

$$\mu^2\tilde{\rho}'' + \mu^2\frac{2\tilde{\rho}'}{x} - \mu^2\frac{f^2\tilde{\rho}}{2x^2} = \frac{\lambda\rho_0^2}{2}(\tilde{\rho}^2 - 1)\tilde{\rho} \Rightarrow$$

$$\tilde{\rho}''(x) + \frac{2\tilde{\rho}'(x)}{x} - \frac{f^2\tilde{\rho}(x)}{2x^2} = (\tilde{\rho}^2(x) - 1)\tilde{\rho}(x) \tag{7.40}$$

As for (7.35) we get:

$$\mu^2 f'' - \mu^2\frac{f(f^2-1)}{x^2} = \frac{g^2\rho_0^2}{4}f\tilde{\rho}^2$$

Let $\epsilon = \frac{g^2}{2\lambda}$ and we get:

$$f''(x) - \frac{f(x)(f^2(x)-1)}{x^2} = \epsilon f(x)\tilde{\rho}^2(x) \tag{7.41}$$

At large distances where $x \to \infty$ we let $\tilde{\rho}(x) = 1 + \tilde{\delta}(x), \quad |\tilde{\delta}(x)| << 1$ and (7.41) becomes:

$$f'' \approx \epsilon f \Rightarrow$$

$$f(x) = f_1 e^{-\sqrt{\epsilon}x} \tag{7.42}$$

Where f_1 is a free parameter. To include subleading behavior we let $f(x) = f_1 e^{-\sqrt{\epsilon}x} + \Delta(x)$ and (7.41) gives:

$$\epsilon f_1 e^{-\sqrt{\epsilon}x} + \Delta''(x) + \frac{1}{x^2}f_1 e^{-\sqrt{\epsilon}x} \approx \epsilon f_1 e^{-\sqrt{\epsilon}x} + \epsilon\Delta \Rightarrow$$

$$\Delta''(x) - \epsilon\Delta(x) \approx -\frac{f_1 e^{-\sqrt{\epsilon}x}}{x^2} = P(x)$$

This inhomogeneous differential equations has a particular solution [20]:

$$\Delta_p(x) = -\Delta_1(x)\int_{x_0}^{x} dt\frac{P(t)\Delta_2(t)}{W(t)} + \Delta_2(x)\int_{x_0}^{x} dt\frac{P(t)\Delta_1(t)}{W(t)}$$

Where $\Delta_1(x)$ and $\Delta_2(x)$ are solutions of the homogeneous equations and $W(x) = \Delta_1(x)\Delta_2'(x) - \Delta_2'(x)\Delta_1(x)$ is the Wronskian determinant. We compute these easily:

$$\Delta_1(x) = e^{\sqrt{\epsilon}x} \qquad \Delta_2(x) = e^{-\sqrt{\epsilon}x}$$

$$W = -2\sqrt{\epsilon}$$

And we proceed as:

$$\Delta(x) = -\frac{e^{\sqrt{\epsilon}x}}{2\sqrt{\epsilon}}\int_{x_0}^{x} dt\frac{f_1 e^{-2\sqrt{\epsilon}t}}{t^2} + \frac{e^{-\sqrt{\epsilon}x}}{2\sqrt{\epsilon}}\int_{x_0}^{x} dt\frac{f_1}{t^2}$$

The first term gives:

$$\sim -e^{\sqrt{\epsilon}x}\sum_{n=2}^{\infty}\frac{(-2\sqrt{\epsilon})^n}{n!(n-1)}[x^{n-1}-x_0^{n-1}]-e^{\sqrt{\epsilon}x}ln(\frac{x}{x_0})-e^{\sqrt{\epsilon}x}(-\frac{1}{x}+\frac{1}{x_0})\to\infty$$

Thus it must be discarded, since $f(\infty)=1$. Due to the second term, we get:

$$\Delta(x)=c_1\Delta_2(x)+\Delta_p(x)=c_1e^{-\sqrt{\epsilon}x}+f_1\frac{e^{-\sqrt{\epsilon}x}}{2\sqrt{\epsilon}}(-\frac{1}{x}+\frac{1}{x_0})=(c_1+f_1\frac{1}{2\sqrt{\epsilon}x_0})e^{-\sqrt{\epsilon}x}-\frac{f_1}{2\sqrt{\epsilon}}\frac{e^{-\sqrt{\epsilon}x}}{x}$$

Thus the asymptotic $f(x)=f_\infty(x)$ behaves:

$$f_\infty(x)\approx f_1e^{-\sqrt{\epsilon}x}+c_1e^{-\sqrt{\epsilon}x}+f_1\frac{e^{-\sqrt{\epsilon}x}}{2\sqrt{\epsilon}}(-\frac{1}{x}+\frac{1}{x_0})=e^{-\sqrt{\epsilon}x}(c_1+f_1+\frac{f_1}{2\sqrt{\epsilon}x_0})-\frac{f_1}{2\sqrt{\epsilon}x}e^{-\sqrt{\epsilon}x}=d_1e^{-\sqrt{\epsilon}x}-\frac{f_1}{2\sqrt{\epsilon}x}e^{-\sqrt{\epsilon}x}\Rightarrow$$

$$f_\infty(x)=e^{-\sqrt{\epsilon}x}[d_1-\frac{f_1}{2\sqrt{\epsilon}x}(1-\frac{1}{2\sqrt{\epsilon}x}+\mathcal{O}(\frac{1}{x^2}))] \tag{7.43}$$

Now substitute $\tilde{\rho}=1+\tilde{\delta}$ in (7.40) and we get:

$$\tilde{\delta}''+\frac{2}{x}\tilde{\delta}'-\frac{f^2}{2x^2}\approx 2\tilde{\delta}\Rightarrow$$

$$\tilde{\delta}''+\frac{2}{x}\tilde{\delta}'-2\tilde{\delta}\approx\frac{f_\infty^2}{2x^2}=F(x)\Rightarrow$$

A particular solution is given again by:

$$\tilde{\delta}(x)=-\delta_1(x)\int_{x_0}^{x}dt\frac{F(t)\delta_2(t)}{W(t)}+\delta_2(x)\int_{x_0}^{x}dt\frac{F(t)\delta_1(t)}{W(t)}$$

We compute the homogeneous solutions and the Wronskian determinant.

$$\delta''+\frac{2}{x}\delta'-2\delta=0\xrightarrow{\delta=\frac{\sigma}{x}}$$

$$\sigma''-2\sigma=0\Rightarrow$$

$$\sigma_1(x)=e^{\sqrt{2}x}\qquad\sigma_2(x)=e^{-\sqrt{2}x}\Rightarrow$$

$$\delta_1(x)=\frac{e^{\sqrt{2}x}}{x}\qquad\delta_2(x)=\frac{e^{-\sqrt{2}x}}{x}$$

$$W(x)=\delta_1\delta_2'-\delta_1'\delta_2=-\frac{2\sqrt{2}}{x^2}$$

And a particular solution is:

$$\tilde{\delta}_p(x)=-\frac{d_1}{2\sqrt{2}x}[\frac{f_1}{2\sqrt{\epsilon}}exp(-2\sqrt{\epsilon}x)+\frac{\sqrt{2}f_1(\sqrt{2\epsilon}-1)+\sqrt{\epsilon}d_1}{2\sqrt{2\epsilon}(\sqrt{2\epsilon}-1)}exp(-2(\sqrt{2}-\sqrt{\epsilon})x)] \tag{7.44}$$

$$\tilde{\rho}(x)=1+c_2\delta_2(x)+\tilde{\delta}(x) \tag{7.45}$$

Now we proceed with the small asymptotic behaviors of $f(x)$ and $\tilde{\rho}(x)$. Around $x=0$ we have $f(x)=1+\Delta_0(x)$, $|\Delta_0(x)|<<1$ and (7.41) becomes:

$$\Delta_0^{''}(x)-\frac{2\Delta_0(x)}{x^2}=\epsilon\tilde{\rho}^2(x) \tag{7.46}$$

As for (7.40) we get:

$$x^2\tilde{\rho}(x) + 2x\tilde{\rho}(x) + (x^2 - \frac{1}{2})\tilde{\rho}(x) = 0 \Rightarrow$$

$$x^2\tilde{\rho}(x) + 2x\tilde{\rho}(x) + (x^2 - \delta(\delta+1))\tilde{\rho}(x) = 0 \tag{7.47}$$

Where we let $\delta = \frac{-1+\sqrt{3}}{2}$. The solution is

$$\tilde{\rho}(x) = c_1 j_\delta(x) = c_1\frac{\sqrt{\pi}}{2}x^\delta \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{(x/2)^{2m}}{\Gamma(\delta+m+3/2)} \approx c_1 x^\delta[1 - \frac{x^2}{2(2\delta+3)}] \tag{7.48}$$

Thus (7.46) becomes:

$$\Delta_0'' - \frac{2}{x^2}\Delta_0(x) = \epsilon c_1^2 x^{2\delta}[1 - \frac{x^2}{2(2\delta+3)}]^2$$

Which has a particular solution:

$$\Delta_0 = \frac{\epsilon c_1^2}{3}x^{2\delta+2}[\frac{3}{2\delta(3+2\delta)} - \frac{3x^2}{2}(15+31\delta+20\delta^2+4\delta^3)^{-1} + \mathcal{O}(x^4)] \tag{7.49}$$

We conclude that for asymptotic behaviors we have:

$$\boxed{\begin{aligned}
&f(x) \xrightarrow{x\to\infty} exp(-\sqrt{\epsilon}x)(d_1 - \frac{f_1}{2\sqrt{\epsilon}x}(1 - \frac{1}{\sqrt{\epsilon}x} + \frac{1}{2\epsilon x^2})\\
&\tilde{\rho}(x) \xrightarrow{x\to\infty} 1 + c_2\frac{e^{-\sqrt{2}x}}{x} - \frac{d_1}{2\sqrt{2}x}[\frac{f_1}{2\sqrt{\epsilon}}exp(-2\sqrt{\epsilon}x) + \frac{\sqrt{2}f_1(\sqrt{2\epsilon}-1)+\sqrt{\epsilon}d_1}{2\sqrt{2\epsilon}(\sqrt{2\epsilon}-1)}exp(-2(\sqrt{2}-\sqrt{\epsilon})x)]\\
&f(x) \xrightarrow{x\to 0} 1 + \frac{\epsilon c_1^2}{3}x^{2\delta+2}[\frac{3}{2\delta(3+2\delta)} - \frac{3x^2}{2}(15+31\delta+20\delta^2+4\delta^3)^{-1} + \mathcal{O}(x^4)]\\
&\tilde{\rho}(x) \xrightarrow{x\to 0} c_1 x^\delta[1 - \frac{x^2}{2(2\delta+3)}]
\end{aligned}} \tag{7.50}$$

Higher order small x asymptotic analysis with Pade approximation is discussed in [19] by taking into account the asymptotic behaviors derived in (7.50). This allows to numerically estimate E_2 in (7.34) and we get:

$$E_2 = 7617GeV \tag{7.51}$$

Therefore the total total energy of the configuration is given by:

$$E = (72.81\sqrt{\frac{\beta}{(GeV)^2}} + 7617)GeV \tag{7.52}$$

Born-Infeld parameter β is constrained by measurements by ATLAS as mentioned in [19]:

$$\sqrt{\beta} \geq 90GeV \tag{7.53}$$

And thus the energy of the system is expected to have o lower bound of:

$$E \geq 14.17TeV \tag{7.54}$$

Of course such bound suggests that monopoles in our model are out of detection range of LHC, but is of potential relevance to future colliders and cosmic rays.

7.4. Simply trace SO(3) Born-Infeld Monopole

Now we are going to consider the non-Abelian Lagrangian (7.22) for the $SO(3)$ group. Such gauge field is chosen to be coupled with a Higgs field in the adjoint representation. We want to investigate magnetic monopole configurations, therefore $F_{\mu\nu}\tilde{F}^{\mu\nu} = T^a T^b F_{a\mu\nu}\tilde{F}^{b\mu\nu} = -4E_i^a B^{bi} = 0$. We write then Lagrangian of our model:

$$\mathcal{L} = \frac{1}{2}(D_\mu\phi)^a(D^\mu\phi)^a - \frac{\lambda}{4}(\phi^a\phi^a - u^2)^2 + \beta^2 tr[1 - \sqrt{1 + \frac{1}{2\beta^2}F_{\mu\nu}F^{\mu\nu}}] \tag{7.55}$$

With:

$$(D_\mu\phi)^a = \partial_\mu\phi^a + ie(\vec{T}\cdot\vec{A}_\mu\phi)^a \tag{7.56}$$

Where $\vec{A}_\mu$ is the non-Abelian $SO(3)$ field. Topological arguments that we saw for the $SU(2)$ 't Hooft-monopole still hold for such model and we consider the following ansatz:

$$\vec{A}_i = \frac{1 - K(r)}{e}\hat{r}\times\partial_i\hat{r} \tag{7.57}$$

$$\vec{\phi} = uH(r)\hat{r} \tag{7.58}$$

With the following boundary conditions:

$$H(\infty) = 1 \qquad H(0) = 0 \qquad K(\infty) = 0 \qquad K(0) = 1 \tag{7.59}$$

So that asymptotically they satisfy:

$$\vec{\phi} \xrightarrow{r\to\infty} u\hat{r} \tag{7.60}$$

$$\vec{A}_i \xrightarrow{r\to\infty} -\frac{1}{e}\hat{r}\times\partial_i\hat{r} \tag{7.61}$$

These are soliton-like configurations just like those we investigate in Georgi-Glashow model, but in this case the gauge sector exhibits a non-linear behavior. The equations of motion for the non-Abelian gauge field are given by:

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial_\mu\vec{A}_\nu} = \frac{\partial\mathcal{L}}{\partial\vec{A}_\nu} \Rightarrow$$

$$\partial_\mu[\frac{F^{a\mu\nu}}{\sqrt{1 + \frac{1}{4\beta^2}F^a_{bc}F^{abc}}}] = e\epsilon^{abc}D^\nu\phi^b\phi^c \Rightarrow$$

$$\partial_\mu[\frac{\vec{F}^{\mu\nu}}{\sqrt{1 + \frac{1}{4\beta^2}\vec{F}_{bc}\cdot\vec{F}^{bc}}}] = eD^\nu\vec{\phi}\times\vec{\phi} \tag{7.62}$$

The energy momentum tensor is given by:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = -g^{\mu\nu}\mathcal{L} + (D^\mu\phi)^a(D^\nu\phi)^a - \beta^2\frac{g_{a\sigma}F^{c\mu a}F^{c\nu\sigma}}{\sqrt{1 + \frac{1}{4\beta^2}\vec{F}_{\gamma\delta}\cdot\vec{F}^{\gamma\delta}}} \tag{7.63}$$

The Hamiltonian density is given then:

$$\mathcal{H} = T^{00} = -\frac{1}{2}(D_\mu\phi)^a(D^\mu\phi)^a + \frac{\lambda}{4}(\phi^a\phi^a - u^2)^2 - \beta^2[1 - \sqrt{1 + \frac{1}{4\beta^2}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu}}] - \frac{\beta^2}{2}\frac{g_{a\sigma}\vec{F}^{0a}\cdot\vec{F}^{0\sigma}}{\sqrt{1 + \frac{1}{4\beta^2}\vec{F}_{\gamma\delta}\cdot\vec{F}^{\gamma\delta}}} \tag{7.64}$$

The non-Abelian field tensor is given by:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e\epsilon^{abc} A^b_\mu A^c_\nu =$$

$$\frac{K'}{er}(\epsilon_{ab\nu}\hat{r}^b\hat{r}_\mu - \epsilon_{ab\mu}\hat{r}^b\hat{r}_\nu) + \frac{2(K-1)}{er^2}[\epsilon_{a\mu\nu} + \epsilon_{ab\mu}\hat{r}^b\hat{r}_\nu - \epsilon_{ab\nu}\hat{r}^b\hat{r}_\mu] - \frac{\hat{r}^a\hat{r}^m}{er^2}(K-1)^2\epsilon_{\mu\nu m} \tag{7.65}$$

Therefore only spatial indices survive. The Higgs kinetic term is given by:

$$(D_\mu\phi)^a = \partial_\mu\phi^a + e\epsilon^{cab} A^c_\mu \phi^b \Rightarrow$$

$$(D_i)^a_b\phi^b = \frac{ur^a r_i}{r^2}[H' - \frac{HK}{r}] + \frac{uHK}{r}\delta^a_i \tag{7.66}$$

$$D_i\phi D_i\phi = u^2(H')^2 + \frac{2u^2}{r^2}(HK)^2 \tag{7.67}$$

To calculate the contribution of the gauge field we write the non Abelian kinetic term in terms of the color magnetic field:

$$\frac{1}{4\beta^2}\vec{F}_{ij}\cdot\vec{F}^{ij} = \frac{1}{2\beta^2}B^a_i B^a_i \tag{7.68}$$

The color magnetic field is given by:

$$B^a_i = \frac{1}{2}\epsilon_{ijk}F^{ajk} = \frac{1}{er^4}[1 + rK' - K^2]r_i r^a - \delta^a_i\frac{K'}{er} \tag{7.69}$$

And thus we get:

$$\frac{1}{4\beta^2}\vec{F}_{ij}\cdot\vec{F}^{ij} = \frac{(1-K^2)^2}{2\beta^2 e^2 r^4} + \frac{(K')^2}{\beta^2 e^2 r^2} \tag{7.70}$$

Now by using the formulas above we can calculate the energy functional of the configuration is given by:

$$E = \int d^3x\mathcal{H} = 4\pi\int_0^\infty dr r^2\{\frac{\lambda u^4}{4}(H^2-1)^2 - \beta^2(1 - \sqrt{1 + \frac{(1-K^2)^2}{2\beta^2 e^2 r^4} + \frac{K'^2}{\beta^2 e^2 r^2}}) + \frac{(uH')^2}{2} + (\frac{uHK}{r})^2\} \tag{7.71}$$

Since the configuration is static, equations of motion are given by

$$\frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial H'}) = \frac{\partial\mathcal{H}}{\partial H} \tag{7.72}$$

$$\Rightarrow H'' = \frac{2K^2H}{r^2} - \frac{2}{r}H' + \lambda u^2 H[H^2-1] \tag{7.73}$$

$$\frac{d}{dr}(\frac{\partial\mathcal{H}}{\partial K'}) = \frac{\partial\mathcal{H}}{\partial K} \tag{7.74}$$

$$\Rightarrow \frac{d}{dr}[\frac{K'}{\sqrt{1 + \frac{(1-K^2)^2}{2\beta^2 e^2 r^4} + \frac{K'^2}{\beta^2 e^2 r^2}}}] = \frac{K(K^2-1)}{r^2}\frac{1}{\sqrt{1 + \frac{(1-K^2)^2}{2\beta^2 e^2 r^4} + \frac{K'^2}{\beta^2 e^2 r^2}}} + 2K(ueH)^2$$

Let:

$$R = \sqrt{1 + \frac{(1-K^2)^2}{2\beta^2 e^2 r^4} + \frac{K'^2}{\beta^2 e^2 r^2}} \tag{7.75}$$

And we get:

$$\frac{d}{dr}(\frac{K'}{R}) = \frac{K(K^2-1)}{r^2R} + K(Heu)^2 \Rightarrow$$

$$r^2K'' - K'\frac{R'}{R}r^2 = K(K^2-1) + KR(Heu)^2r^2 \tag{7.76}$$

The non-Abelian field tensor and the Higgs field are the same as the one in Georgi-Glashow model, thus the magnetic charge of the configuration is equal to:

$$q_m = \frac{4\pi}{e} \tag{7.77}$$

Finally numerical results [25] of the differential equations showcase that there are some critical values β_c for which when $\beta \leq \beta_c$ there are no numerical solution due to high non-linearity of the equations. Moreover for these values the monopole energy is infinity. Some critical values are $\beta_c = 0.41$ for $\lambda/e^2 = 0$ and $\beta_c = 0.62$ for $\lambda = 0.5$. The origins of β_c is quite interesting. In Georgi-Glashow model the radius of the monopole core is given by $R_c = 1/M_W = 1/ue$, with M_w being the mass of the W bosons. There is another length associated with Higgs field and we let it as $R_H = 1/M_H \sim 1/\sqrt{\lambda}u$. This is simply the region for which the Higg's obtains the vacuum expectation value. We let the dimensionless parameter:

$$a = \frac{R_H}{R_c} = \frac{e}{\sqrt{\lambda}} \tag{7.78}$$

Then according to numerical solution monopole configuration arise for $a \sim 1$. Now in our current model there is an another parameter β for which R_H and R_c could in principle depend from β and $u\sqrt{\lambda}$ and the configuration minimizing the energy will result from the matching of both parameters determining the size of the monopole. In the region defined by β_c, such matching is impossible [25]. Therefore, the critical value β_c is simple the range where it is impossible to adjust the parameters of the model such that the energy is minimised.

7.5. Simply trace SO(3) Gravitational Born-Infeld Monopole

Consider the simply trace SO(3) Born-Infeld model in curved spacetime. The spacetime is described by a spherical symmetric solution of Einstein's equations and we use the parametirization [27]:

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2d\Omega^2$$

$$= e^{2\nu(r)}dt^2 - e^{2\sigma(r)}dr^2 - r^2d\Omega^2 \tag{7.79}$$

Where the the functions $B(r)$ and $A(r)$ satisfy the following conditions:

$$B(\infty) = A(\infty) = 1 \tag{7.80}$$

$$\nu(\infty) = \sigma(\infty) = 0 \tag{7.81}$$

We will consider monopole configurations and the Lagrangian of model is given by:

$$\mathcal{L} = -\frac{R}{16\pi G} + \frac{1}{2}D_\mu\vec{\phi}\cdot D^\mu\vec{\phi} + \beta^2 tr(1 - \sqrt{1+\frac{1}{2\beta^2}F_{\mu\nu}F^{\mu\nu}}) - \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi} - u^2)^2 = \mathcal{L}_g + \mathcal{L}_m \tag{7.82}$$

$$\mathcal{L}_g = -\frac{R}{16\pi G} \tag{7.83}$$

$$\mathcal{L}_m = \frac{1}{2}D_\mu\vec{\phi}\cdot D^\mu\vec{\phi} + \beta^2 tr(1 - \sqrt{1+\frac{1}{2\beta^2}F_{\mu\nu}F^{\mu\nu}}) - \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi} - u^2)^2 \tag{7.84}$$

And the action of the system is given by:

$$S = \int d^4x\sqrt{-g}\mathcal{L} \tag{7.85}$$

The field solution of the matter section is given by:

$$\vec{\phi} = uH(r)\hat{r} \tag{7.86}$$

$$\vec{A}_i = \frac{1}{g}(1-K(r))\hat{r}\times\partial_i\hat{r} \qquad \vec{A}_0 = 0 \tag{7.87}$$

With boundary conditions:

$$H(\infty) = 1 \qquad H(0) = 0 \qquad K(0) = 1 \qquad K(\infty) = 0 \tag{7.88}$$

We have describe 't Hooft monopole configuration in curved spacetime in section (5). Most of the general calculations performed in this section still hold for this case. In particular equations of motion for $A(r)$ and $B(r)$ obtained from gravity sector remain the same. The same holds for the Higgs field in the matter sector and we expect the gauge field to satisfy different equations motion. Recall from (5)

$$-\frac{1}{4}\vec{F}_{ij}\cdot\vec{F}^{ij} = -\frac{(K')^2}{Ae^2r^2} - \frac{(1-K^2)^2}{2e^2r^4} \tag{7.89}$$

Then the Born-Infeld term gives:

$$\beta^2 tr(1-\sqrt{1+\frac{1}{2\beta^2}F_{ij}F^{ij}}) = \beta^2(1-\sqrt{1+\frac{1}{4\beta^2}\vec{F}_{ij}\cdot\vec{F}^{ij}}) = \beta^2(1-\sqrt{1+\frac{(K')^2}{A\beta^2e^2r^2}+\frac{(1-K^2)^2}{2\beta^2e^2r^4}} = \beta^2(1-\mathcal{R}) \tag{7.90}$$

Where we have let:

$$\mathcal{R} = \sqrt{1+\frac{(K')^2}{A\beta^2e^2r^2}+\frac{(1-K^2)^2}{2\beta^2e^2r^4}} \tag{7.91}$$

This in order to calculate the equation of motion for the gauge field we consider the rest of the terms in the action:

$$S = \int d^4x\sqrt{-g}\mathcal{L} \ni -4\pi\int dt\int_0^\infty dr\sqrt{AB}\{u^2H^2K^2 - \beta^2r^2(1-\mathcal{R})\}$$

The equation of motion for the gauge field is given:

$$\frac{d}{dr}[\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial K'}] = \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial K} \Rightarrow$$

$$\frac{d}{dr}[\sqrt{AB}\beta^2r^2\frac{\partial\mathcal{R}}{\partial K'}] = 2\sqrt{AB}u^2H^2K + \beta^2r^2\sqrt{AB}\frac{\partial\mathcal{R}}{\partial K}$$

We have:

$$\frac{\partial\mathcal{R}}{\partial K} = \frac{2}{\mathcal{R}}\frac{K(K^2-1)}{\beta^2e^2r^4}$$

$$\frac{\partial\mathcal{R}}{\partial K'} = \frac{2}{\mathcal{R}}\frac{K'}{A\beta^2e^2r^2}$$

Then we get:

$$\frac{d}{dr}[\sqrt{\frac{B}{A}}\frac{K'}{\mathcal{R}}] = \sqrt{AB}(euH)^2K + \sqrt{AB}\frac{K(K^2-1)}{\mathcal{R}r^2} \Rightarrow$$

$$\frac{r^2}{\sqrt{AB}}\frac{d}{dr}[\sqrt{\frac{B}{A}}K'] - \frac{\mathcal{R}'}{\mathcal{R}}\frac{K'r^2}{A} = (euHr)^2K\mathcal{R} + K(K^2-1) \tag{7.92}$$

Note in that in the flat limit, where $A(\infty) = B(\infty) = 1$, we obtain the differential equation of the gauge field in (7.76), as it was expected.

From the energy momentum tensor we can obtain the Hamiltonian of the system:

$$\mathcal{H} = -\mathcal{L}_m = -[\frac{1}{2}D_\mu\vec{\phi}\cdot D^\mu\vec{\phi} + \beta^2 tr(1 - \sqrt{1 + \frac{1}{2\beta^2}F_{\mu\nu}F^{\mu\nu}}) - \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi} - u^2)^2]$$

$$= \frac{\lambda u^4}{4}(H^2-1)^2 + \frac{u^2H'^2}{2A} + \frac{u^2H^2K^2}{r^2} - \beta^2(1-\mathcal{R}) \tag{7.93}$$

The energy of the monopole configuration is given by:

$$E = 4\pi\int_0^\infty drr^2[\frac{\lambda u^4}{4}(H^2-1)^2 + \frac{u^2H'^2}{2A} + \frac{u^2H^2K^2}{r^2} - \beta^2(1-\mathcal{R})]$$

$$= 4\pi\int_0^\infty drr^2[\frac{\lambda u^4}{4}(H^2-1)^2 + \frac{u^2H'^2e^{-2\sigma(r)}}{2} + \frac{u^2H^2K^2}{r^2} - \beta^2(1 - \sqrt{1 + \frac{e^{-2\sigma(r)}(K')^2}{\beta^2e^2r^2} + \frac{(1-K^2)^2}{2\beta^2e^2r^4}})] \tag{7.94}$$

Numerical results of this system are described in [27]. Let's sum up the most important results. In this analysis we let $a^2 = 4\pi Gu$, where various solutions corresponding to different values of u, have been studied. The system exhibits a similar behavior just like the flat case, where the monopole solution cease to exist for values of β bellow the critical value $\beta_c \sim 0.1$. At last from this analysis it was found for definite value of β it exists a maximum value a_{max} of a, for which the monopole solution cease to exist for $a > a_{max}$. In particular it was found as β increases a^2_{max} decreases. The reason for such behavior is that for values $a < a_{max}$ the monopole becomes gravitationally unstable and collapses.

Conclusion

To sum up, magnetic monopoles are a theoretical concept in fundamental physics, with no experimental evidence. Despite this physicists remained stubborn about this idea, with monopoles being contained in theoretical models ever since the first exploration of Maxwell's theory. The main goal of this thesis was to present the prospect electroweak monopoles in the standard model and some of its extensions. The lack of experimental verification is a strong evidence that electroweak monopoles are only theoretical. In addition to this the theoretical description of electroweak monopole is also problematic, since it has infinity energy. It is possible that these facts could conclude the research around the topological sector of the standard model. Of course we have not discussed any dynamics of electroweak monopoles, which are very important for the experimental research in colliders. Although it seems that electroweak monopole is a speculation, it will still have a theoretical relevance, since they can be found in standard model extensions such as GUT's and other models in modern theoretical physics. This will hopefully continue to fuel physicists for further research.

Appendix

Appendix A: Homotopy theory

In this section we are going to introduce some aspects of homotopy theory. Homotopy theory in general studies continuous deformations between spaces. These deformations are maps between spaces and in physics we focus more on maps rather than spaces. In particular we will study maps between n-dimensional sphere and the vacuum manifold M,

$$\phi : S^n \to M \tag{A.1}$$

Maps are passing through a base point $m \in M$.

The set of maps that can be continuously deformed to the map ϕ is denoted as $[\phi]$. This is known as **homotopy class**.

Let $\phi, \xi : S^n \to M$ two continuous maps. If $\exists$ a continuous map $F : S^n \times I \to M$ with $I = [0, 1]$, such that $F(p, t = 0) = \phi(p)$ and $F(p, t = 1) = \xi(p) \quad \forall\, p \in S^n$, then we say $\phi \sim \xi$, meaning that ϕ is homotopic to ξ. We call $F(p, t)$ **homotopy** between ϕ and ξ. The map $F(p, t)$ is simply a way to continuously[10] deform ϕ to ξ and $\phi, \xi \in [\phi]$

Two different homotopy maps ϕ, ξ passing through a base point $m \in M$ can be multiplied via the operation:

$$\phi \circ \xi = \begin{cases} \phi(x) & x \in S^n_+ \\ \xi(x) & x \in S^n_- \end{cases} \tag{A.2}$$

Where S^n_+ stands for north hemisphere and S^n_- stands for the south hemisphere of the n dimensional sphere. The two maps are equal to the equator and whole equator is mapped to m. We can generalise this operation to homotopy classes $[\phi]$ and $[\xi]$. It can shown that this operation respects all the requirements of an abelian group and we call this group **nth homotopy group** $\pi_n(M)$. Group elements of $\pi_n(M)$ are homotopy classes.

One very important homotopy group is the first homotopy group of $U(1) \cong S^1$. It can be proved that:

$$\pi_1(S^1) = Z \tag{A.3}$$

Each homotopy class is associated with a different integer. We can think the classes as rubber bands that fit n rounds a circle. For example one that fits two rounds can not deformed into another that fits one round.
This idea can generalize to higher homotopy groups of higher dimensional spheres:

$$\pi_n(S^n) = Z \tag{A.4}$$

The first homotopy group of S^n now it can proven that is trivial:

$$\pi_1(S^n) = \emptyset \tag{A.5}$$

Since S^n is a simply connected space. This means that it has no holes and can be continuously contracted to a point, making every rubber band equivalent.
The generalization of this idea is for $m > n$:

$$\pi_n(S^m) = \emptyset \tag{A.6}$$

Appendix B: Soliton Theory

In this section we are going to introduce the idea of solitons. Solitons or solitary waves are stable field configurations of finite energy.

Our starting point is the one dimensional Sine-Gordon equation:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{b^2} sin(b\phi) = 0 \tag{B.1}$$

This is a wave equation with a non linear term, thus the superposition principle does not hold. Consider the moving solution $\phi(x, t) = f(x - ut) = f(\xi)$. We transform (B.1) by letting $\xi = x - ut$:

$$\frac{\partial^2}{\partial t^2} = u^2 \frac{\partial^2}{\partial \xi^2} \qquad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2}$$

And (B.1) becomes:

$$\frac{\partial^2 \phi}{\partial \xi^2}(1 - u^2) = \frac{1}{b^2} sin(b\phi) \tag{B.2}$$

---

[10] This is achieved via the variable $t \in I$

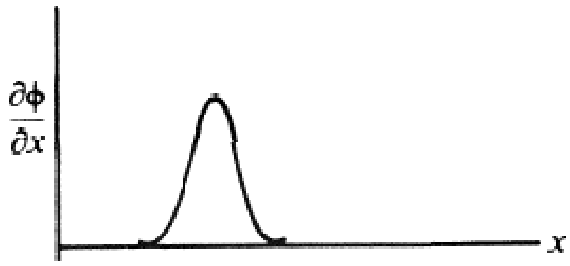

FIG. 12: Soliton wave moves in space without changing its size or shape.

This equation is solved by:

$$f(\xi) = \frac{4}{b} tan^{-1}(e^{\pm \frac{\gamma \xi}{\sqrt{b}}}), \quad \gamma = \frac{1}{\sqrt{1-u^2}} \tag{B.3}$$

Now let's investigate constant solutions. This means $\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} = 0$ and (B.1) gives:

$$sin(b\phi) = 0 \quad \Rightarrow \quad \phi = \frac{2\pi}{b} n, \quad n \in \mathbb{Z} \tag{B.4}$$

Sine-Gordon equation has a degenerate vacuum in a classical sense. It's easy to show that (B.1) can be obtained from the Lagrangian:

$$\mathcal{L} = \frac{1}{2}[(\frac{\partial \phi}{\partial t})^2 - (\frac{\partial \phi}{\partial x})^2] - \frac{1}{b^2}[1 - cos(b\phi)] \tag{B.5}$$

Where the potential $V(\phi) = \frac{1}{b^2}(1 - cos(b\phi))$ has chosen such that is equal zero for $\phi = \frac{2\pi n}{b}$. We can approximate the potential near the vacuum as:

$$V(\phi) \approx \frac{1}{b^2}(1 - 1 + b^2 \frac{\phi^2}{2} - b^4 \frac{\phi^4}{4!}) = \frac{\phi^2}{2} - \frac{b^2 \phi^4}{4!}$$

Now construct a static configuration where ϕ approaches one of the zeros of $V(\phi)$ as $\phi \to -\infty$ and different one as $\phi \to \infty$. Let's say $n = 0$ for $-\infty$ and $n = 1$ for ∞. As for the intermediate region it holds that $\frac{\partial \phi}{\partial x} \neq 0$ and $\phi \neq \frac{2\pi n}{b}$. For static configurations (B.1) gives:

$$-\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial V}{\partial \phi} = 0 \Rightarrow \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi} \frac{\partial \phi}{\partial x} = \frac{\partial V}{\partial \phi} \Rightarrow \frac{1}{2}(\frac{\partial \phi}{\partial x})^2 = V(\phi)$$

The energy of the static configuration is give by:

$$E = \int dx \mathcal{H} = -\int dx \mathcal{L} = -\int [\frac{1}{2}(\frac{\partial \phi}{\partial x})^2 + V(\phi)] dx = \int 2V(\phi) dx = 2\int_0^{2\pi/b} V \frac{dx}{d\phi} d\phi$$

$$E = \int_0^{\frac{2\pi}{b}} \sqrt{2V} d\phi = \int_0^{\frac{2\pi}{b}} d\phi \sqrt{\frac{2}{b^2}(1 - cos(b\phi))}$$

$$E = \frac{8}{b^2} \tag{B.6}$$

Thus this configuration has finite energy that is inversely proportional to the coupling constant b^2. We can visualise such configuration as an infinite horizontal string with pegs attached to it at equally spaced intervals, and connect each peg to it's neighbour with spring (coupling). The ground corresponds to the peg hanging vertically. The configuration $n = 0 \to n = 1$ described above can be seen in figure 13. From this figure we observe that such a configuration is stable and can not decay to a configuration with $E = 0$. This means for example to continuously deform $n = 1$ to $n = 0$ and obtain a configuration $n = 0 \to n = 0$. The reason for this is that it involves an infinity number of

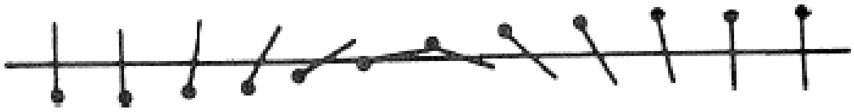

FIG. 13: Soliton solution visualisation.

pegs turning over across the horizontal line, requiring infinity energy. Thus there is infinity energy barrier between configurations with different end points. For these reason solitons are **topological objects**, since their stability is earned from topological properties of the space (boundary conditions).

We can try to generalise this idea to two spatial dimensions with polar coordinates (r, θ). Now the boundary of configurations is not the end points of a horizontal line, but a circle at infinity:

$$\phi = ae^{in\theta}, \quad r \to \infty \tag{B.7}$$

Where $n \in \mathbb{Z}$ is fixed and chosen such that ϕ is single valued. We write the potential in such a way that at the boundary (B.7) is equal to zero:

$$V(\phi^*\phi) = (a^2 - \phi^*\phi)^2 \tag{B.8}$$

And the Lagrangian is:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - V(\phi^*\phi) \tag{B.9}$$

Now for a static configuration the energy density of the system at $r \to \infty$:

$$\mathcal{H} = -\mathcal{L} = \frac{1}{2}\nabla\phi \cdot \nabla\phi + V(|\phi|^2) = \nabla\phi \cdot \nabla\phi = \frac{n^2a^2}{2r^2}$$

And the energy is

$$E = \frac{n^2a^2}{2}\int^\infty \frac{dr}{r} \to \infty$$

Such configuration has infinity energy and we can not generalise to two dimensions. In fact we can not generalise to $d > 2$. This fact is described by Derricks theorem [9]. In particular it states that any scalar field theory that respects dilation symmetry can not have stable solutions for dimension $d > 2$. It's easy to check that our one dimensional Lagrangian (B.5) is indeed invariant under dilation transformations $x \to \lambda x$.

To stabilize the configuration we add an abelian gauge field A_μ and getting the langragian:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + |(\partial_\mu + ieA_\mu)\phi|^2 - V(\phi^*\phi) \tag{B.10}$$

The energy momentum tensor is:

$$T^{\mu\nu} = -F_\sigma{}^\mu F^{\sigma\nu} + D^\mu\phi^* D^\nu\phi - g^{\mu\nu}\mathcal{L} \tag{B.11}$$

And the energy of the static system for gauge where $A_0 = 0$ is:

$$E = \int d^3x[\frac{1}{4}F^{ij}F_{ij} + \frac{1}{2}D_i\phi^* D_i\phi + (a^2 - |\phi|^2)^2] \tag{B.12}$$

Then at $r \to \infty$ energy becomes finite if:

$$D_i\phi = 0 \qquad |\phi| = a \qquad F_{ij} = 0 \tag{B.13}$$

Thus finite energy configuration is possible since at the boundary $E = 0$. We can find the form of the vector field at $r \to \infty$, by substituting (B.7) to (B.13) and we get:

$$D_i\phi = 0 \Rightarrow \frac{1}{r}in\theta + ieA_\theta = 0 \Rightarrow$$

$$A_t = 0 \qquad A_r = 0 \qquad A_\theta = -\frac{n}{er} = -\frac{i}{e}\nabla_\theta(\phi) \tag{B.14}$$

And since A is a pure gauge $F_{ij} = 0$. We can add a third dimension z such that the fields does not depend from it. Then the configuration becomes a **vortex line**. This configuration is stable if the boundary condition (B.7) can not continuously deformed to the vacuum value $|\phi| = a$. Note that vacuum manifold is S^1_{vac} since the vacuum condition is described by any element on the unit circle $\hat{\phi} = \phi/a$. So for the configuration to be stable means finding a map $\phi : S^1_\infty \to S^1_{vac}$ which is non-trivial. Indeed from homotopy theory it holds that $\pi_1(U(1)) = \pi_1(S^1) = Z$. From this discussion we see that in a field theory topology provides existence argument for soliton solution.

Acknowledgements

I would like specially thank my advisor Mr Nick Mauromatos for his genuinely kind nature with helping me understand better all these fascinating concepts of physics, and also for creating an environment where I was able to collaborate with another student, Kostas Kollias, while we were working with our projects. I also wish to thank Kostas for helping each other with our projects and I am looking forward to watch his work. In addition to that, I want to thank the school of applied mathematical and physical science in NTUA together with the national center of scientific research Demokritos, where I was able to conclude my graduate studies. At the end, I would like to thank my friends and family, who were beside me and support me during my studies.

---

[1] P. A. M. Dirac, The Theory of magnetic poles, Phys. Rev. **74**, 817 (1948). doi:10.1103/PhysRev.74.817

[2] M. Nakahara. Geometry, Topology and Physics.

[3] Yakov M. Shnir. Magnetic Monopoles.

[4] Lewis H. Ryder. Quantum Field Theory

[5] Csaba Csaki. Lecture Notes on Solitons and Instantons

[6] T.W. Kirkman and C.K. Zachos, Asymptotic analysis of the monopole structure, Phys. Rev. D24 (1981) 999

[7] P. Forg´acs, N. Obadia and S. Reuillon, Numerical and asymptotic analysis of the 't Hooft-Polyakov magnetic monopole, arXiv:hep-th/0412057

[8] G. 't Hooft, Magnetic monopoles in unified gauge theories, Nucl. Phys. B79, 276 (1974).

[9] G. H. Derrick (1964), Comments on Nonlinear Wave Equations as Models for Elementary Particles , J. Math. Phys. 5 (9): 1252–1254

[10] M. K. Prasad and Charles M. Sommerfield, Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon, Phys. Rev. Lett. 35, 760

[11] Y.M. Cho and D. Maison, Monopoles in Weinberg-Salam Model, Phys. Lett. B391, 360 (1997)

[12] Y.M. Cho and Kyoungtae Kimm, Electroweak Monopoles, arXiv:hep-th/9705213v1

[13] Kyoungtae Kimm, J. H. Yoon, and Y. M. Cho, Finite Energy Electroweak Dyon , arXiv:1305.1699v2 [hep-ph] 13 Oct 2014

[14] Y.M. Cho, Phys. Rev. D21, Restricted gauge theory, 1080 (1980)

[15] J. Preskill: Magnetic Monopoles, Ann. Rev. of Nucl. and Part. Science, 34 (1984) 461

[16] John Ellis, Nick E. Mavromatos, and Tevong You, The price of an electroweak monopole arXiv:1602.01745v2 [hep-ph] 10 Feb 2016

[17] ATLAS and CMS Collaborations, ATLAS-CONF-2015-044, CMS-PAS-HIG-15-002.

[18] Barton Zwieback, A first course in String Theory.

[19] Nick E. Mavromatos, Sarben Sarkar, Finite-energy dressed string-inspired Dirac-like monopoles, arXiv:1812.00495 [hep-ph] 3 Dec 2018

[20] Joh Mathews, R. L. Walker, Mathematical Methods of Physics.

[21] Nick E. Mavromatos, Vasiliki A. Mitsou, Magnetic monopoles revisited: Models and searches at colliders and in the Cosmos, arXiv:2005.05100v3 [hep-ph] 14 Aug 2020

[22] Y. Nambu, Nuclear Physics B130 (1977) 505-516

[23] T. T. Wu and C. N. Yang, String-like configurations in the Weinberg-Salam theory ,Nucl. Phys. B 107 (1976) 365, [doi:10.1016/0550-3213(76)90143-7].

[24] Romain Gervalle1 and Mikhail S. Volkov1, Electroweak monopoles and their stability, arXiv:2203.16590v1 [hep-th] 30 Mar 2022

[25] N. Grandi , E.F. Morenob and F.A. Schaposnik, Monopoles in non-Abelian Dirac-Born-Infeld Theory, arXiv:hep-th/9901073v1 18 Jan 1999

[26] Prasanta K. Tripathy and Fidel A. Schaposnik, Monopoles in non-Abelian Einstein-Born-Infeld Theory, arXiv:hep-th/9911065v1 10 Nov 1999

[27] Jerome Martin, Inflationary Perturbations: the Cosmological Schwinger Effect, arXiv:0704.3540 [hep-th]

[28] Kimyeong Lee, V.P.Nair, Erick J.Weinberg, Black Holes in Magnetic Monopoles, arXiv:hep-th/9911065

[29] Sean M. Carroll, Spacetime and Geometry

[30] P. van Nieuwenhuizen and David Wilkinson, Regular solution of 't Hooft's magnetic monopole model in curved space, Phys. Rev. D 13, 778

[31] Miguel E. Ortiz, Curved-space magnetic monopoles, Phys. Rev. D 45, R2586(R)