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Magnetic monopoles in the Standard Model

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

του Κωνσταντίνου Κόλλια

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Magnetic Monopoles in the Standard Model

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The goal of this thesis is to present a thorough introduction to Magnetic Monopoles in Field Theories, ultimately leading to examining the Standard Model and its topological structure. We start by showing the classical mechanics of a magnetic monopole and its basic quantum mechanical properties inducing a quantization condition on the electric charges allowed. Next, after clarifying the necessary formalism needed in terms of Homotopy groups and Spontaneous Symmetry Breaking Mechanism, we present the 't Hooft-Polyakov monopole and its generalization the Julia-Zee dyon that live in the $SU(2)$ gauge theory with an adjoint Higgs. The last part involves the discussion of a new and unexpected kind of monopole- the Cho-Maison monopole- that is supposed to live in the Standard Model. Emphasis is given on the fine details and ambiguities of the Cho-Maison monopole that have not yet been clarified by theory.

Contents

I. INTRODUCTION

The magnetic monopole is an eluding and fascinating entity that has troubled physicists from long ago. In fact, the first reference to such objects comes way back from 1269 (unbelievable, yet true!) when Peter Peregrinus shared in his letter [2] his observations on magnets including two poles and that bisections of magnets fail to isolate them. Centuries later, when Electromagnetism was a well formulated theory and it was understood that it need not contain any magnetic monopoles, theoretical calculations were made based on their hypothetical existence, clarifying concepts that could be tested by experiment [13].

However, the subject merely started taking the form it has today, when Dirac announced [9] in 1931 that the consistency of quantum mechanics mandated that if an isolated magnetic pole exists, then all electric charges in nature must be quantized. This shook the world and magnetic monopoles were a really discussed topic at that time. But, the complete lack of evidence for their existence, by subsequent and even today's experiments has shook some of the interest away. Moreover, the quantization condition for the electric charges is acquired today by means of Grand Unified Theories whose Spontaneous Symmetry Breaking generates the Electromagnetic $U(1)$ group.

So, magnetic monopoles have not been observed in nature and they are no longer that needed to predict the quantization conditions for the electric charges. Why are we even bothered with them? It is because the formalism of field theories allows topological excitations and Spontaneous Symmetry Breaking of some Unified Theories generates magnetic monopoles as topological particles! It turns out that the two methods of acquiring quantization conditions are, in fact, closely related!

This bring us to the structure of this thesis. In the first chapter (II), we discuss the properties of a monopole in classical mechanics. More explicitly, we examine its non relativistic scattering and take a look at a potential for the field of a monopole. We discuss about the *Dirac string* and how it can be manipulated using gauge transformations and lastly we talk about the *Dual invariance* of Maxwell's electromagnetism, which is one of the key ideas that inspired the craft of a magnetic monopole. Indeed, one can show that the $O(2)$ symmetry in electric and magnetic fields is carried on to the sources if one allows magnetic singularities. In chapter (III), we prove the *Dirac quantization condition* in the context of quantum mechanics by mandating the Dirac string is not observable. After that, we elaborate on a formalism by *Wu* and Yang that allows us to describe the magnetic monopole without having to deal with the peculiarities of the Dirac string at all and still predicts the Dirac quantization condition. Chapter (IV) is a mathematical interlude introducing us to the concept of *Homotopy groups*, giving the necessary definitions and properties while highlighting results that will be of great use and importance later.

Chapters (V),(VI),(VII) serve us introductory chapters to the formalism needed to present the *'t Hooft-Polyakov monopole* of the $SU(2)$ theory. Specifically, *Gauge Field theories* are elaborated, firstly in the Abelian case and secondly in the non-Abelian case. The analysis is followed by the marvellous mechanism of *Spontaneous Symmetry Breaking* in various theories, ultimately leading to the *Higgs mechanism* in gauge theories. Chapter (VII) is the second "topological" discussion. This time we are involved with what we call *solitons* and represent a new class of solutions of the classical equations of motion for certain theories. We present the one dimensional case where we meet the *kink*. The two dimensional case brings us against the *vortex* whose stabilization requires *Derrick's theorem* and notions from Symmetry Breaking patterns.

Chapter (VIII) is the quintessence of this project, presenting the 't Hooft-Polyakov monopole, a time-independent, stable topological excitation with unit magnetic charge. This monopole lives in the $SU(2)$ Georgi-Glashow model, a predecessor to the Weinberg-Salam model invented as an attempt to unify the weak interaction and electromagnetism. It can be shown via the second homotopy group of the vacuum configuration that there can be non trivial maps between those two, discritized by a winding number. This means that one can now determine a whole new class of solutions to the classical equations characterized by their different winding numbers. It, also, turns out that those winding numbers (that are integers) describe the magnetic charge carried by each solution.

Chapter (IX) introduce us to the Winding Numbers in a more formal way. They are manifestations of the properties of some Homotopy groups corresponding to the Field theories examined. We also give various examples of different cases where winding numbers appearing, ending with the case describing the Georgi-Glashow model. Chapter (X) concludes the discussion on the 't Hooft-Polyakov monopole, giving each place to the grand finale, the Cho-Maison monopole in chapter (XI). Here, we extract the differential equations that the functions of the 't Hooft-Polyakov ansatz need to satisfy and we also present a twist to the monopole problem with the *Julia-Zee ansatz*, describing a dyon.

In this chapter, we present the *Cho-Maison monopole* in the Standard Model, which is in the frontier of research even today, especially with the MoEDAL experiment running and its soon to be implemented upgrades and extensions. We start from a brief presentation of the $SU(2) \times U(1)$ Weinberg-Salam model and we introduce the Cho-Maison ansatz. The procedure follows much that of the 't Hooft-Polyakov ansatz, that means we derive the system of O.D.E.'s for the ansatz and we show the magnetic charge of the dyon. There are some fine lines and ambiguities on the existence of an electroweak monopole, since topological arguments had discouraged physicists to search them. Cho and Maison [6] dodge those problems, but complications with gauge fixing and energy functional divergences arise, which we discuss in a more informal way.

The style of this presentation is not rigorous, but I try to leave no blind spots and no stones unturned. This means I present all the calculations in the simplest and most complete way possible, while elaborating ideas and critical points in a more intuitive manner that I personally enjoy and understand. I give lots of attention on the physical meaning and cultivation of a physical instinct is an important consideration of the way I am presenting certain subjects besides raw mathematical calculations which , despite their inherent beauty, are rendered useless without the physical context implemented.

A quick note on the formalism, I am using normal letters for quantities representing numbers (not necessarily scalar quantities) and bold ones for matrices of any dimensions (including 1×n and their transpose). This means that A shall be a number, and $\mathbf{A}$ a non-number quantity that I will be specifying in each occasion. Let's say for now, it represents 4-vector. Since, each component of that 4-vector is itself a number (but not a scalar!), I will be denoting them as A^{μ}, where $\mu = 0, 1, 2, 3$. Also, note that I use lowercase letters of the Greek alphabet for Minkowski indices and lowercase letters of the English alphabet for Euclidean ones. For cases, when I need to deal with a vector whose components are themselves matrices, I shall resort to omitting the index when referring to the vector itself, while including it to specify which one of its components I am referring to and retaining the bold-symbol since we are talking for non-number quantities. For example, consider an array $\left(\boldsymbol{\tau}^1, \boldsymbol{\tau}^2, \boldsymbol{\tau}^{\mathbf{3}}\right)$, where $\boldsymbol{\tau}^a$ are the Pauli matrices. In these cases, if I want to refer to the whole vector, I will simply write $\boldsymbol{\tau}$.

As a last word, the concept of magnetic monopoles is indeed a fascinating and beautiful one and I hope you come to believe the same. I also hope this thesis contributes even the slightest to that feat.

Now, with all the introductory out of the way, let us embark on a long, wild and magical journey.

II. THE MAGNETIC MONOPOLE IN CLASSICAL MECHANICS

In classical mechanics, it is sufficient to define the magnetic monopole as a point-like particle with a radial Coulomb magnetic field, then all the peculiarities of such a particle will be manifest in the interactions between electrically charged matter and the magnetic monopole[31].

A. Nonrelativistic Scattering

We consider a magnetic charge g as the source of a static Coulomb-like magnetic field, sitting at the origin.

$$\mathbf{B} = g\frac{\mathbf{r}}{r^3}. \tag{1}$$

Then, we suppose an electrically charged particle of charge e moves towards the monopole. The equations of motion for the moving particle are given by the *Lorentz force*[1]

$$m\frac{d^2\mathbf{r}}{dt^2} = e[\mathbf{v} \times \mathbf{B}] = \frac{eg}{r^3}\left[\frac{d\mathbf{r}}{dt} \times \mathbf{r}\right]. \tag{2}$$

To proceed we seek the integrals of motion.
Scalar multiplication with the vector velocity $\mathbf{v}$ yields the conservation of kinetic energy.

$$m\mathbf{v} \cdot \frac{d^2\mathbf{r}}{dt^2} = e\mathbf{v} \cdot [\mathbf{v} \times \mathbf{B}] = 0 \tag{3}$$

The LHS upon noticing $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ can be written as

$$\frac{m}{2}\frac{d}{dt}[\mathbf{v} \cdot \mathbf{v}] = 0 \tag{4}$$

So the *kinetic energy* is constant

$$E = \frac{1}{2}m|\mathbf{v}|^2 = constant \tag{5}$$

Now scalar multiplication of (2) with $\mathbf{r}$ yields

$$\mathbf{r} \cdot \frac{d^2\mathbf{r}}{dt^2} = \mathbf{r} \cdot \frac{d}{dt}[\frac{d\mathbf{r}}{dt}] = \frac{d}{dt}[\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}] - [\frac{d\mathbf{r}}{dt}]^2 = 0 \tag{6}$$

because RHS of (2) times $\mathbf{r}$ vanishes trivially.

Equation (6) can be integrated twice yielding

$$r = \sqrt{v^2t^2 + b^2} \tag{7}$$

and

$$\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = \frac{1}{2}\frac{dr^2}{dt} = v^2 t \tag{8}$$

where v is the norm of $\mathbf{v}$.
From the form of (7) we come to the conclusion that there is no closed orbit in the charge-monopole system.

---

[1] We will be working in c=1 units.

The electrically charged particle comes from infinity up to a minimum distance b, which we call *impact parameter*, and returns to infinity. This property of the "reflection" of the electric charge is called the *magnetic mirror effect*[2].

A very interesting feature of the system is that ordinary angular momentum is not conserved, which at first glance might appear surprising due to the rotational invariance of the problem[3]. Being a bit more careful, we notice, however, that the force field is not central and therefore mechanical angular momentum should not be conserved. However, one can deduce a similar quantity that is indeed conserved[4].

To proceed *ordinary angular momentum* is defined as

$$\widetilde{\mathbf{L}} = \mathbf{r} \times m\mathbf{v} \tag{9}$$

and we recall the cross-product identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \tag{10}$$

which we will be of great use very shortly.

So, we now take the cross product of $\mathbf{r}$ with (2)

$$\begin{aligned} m\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} &= \frac{eg}{r^3}(\mathbf{r} \times \widetilde{\mathbf{L}}). \\ \frac{d}{dt}\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) - \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} &= -\frac{eg}{mr^3}(\widetilde{\mathbf{L}} \times \mathbf{r}) \\ \frac{d\widetilde{\mathbf{L}}}{dt} &= \frac{eg}{mr^3}(\widetilde{\mathbf{L}} \times \mathbf{r}) \end{aligned} \tag{11}$$

Now it is evident that (9) is not conserved. Its norm is, however, as it can be easily seen from scalar product of (11) with $\widetilde{\mathbf{L}}$.

$$\widetilde{L} \equiv |\widetilde{\mathbf{L}}| = mvb \tag{12}$$

Now if only we could write the RHS of (11) as a total time derivative then we would define a modified conserved angular momentum!

We need to establish some relations for this first. Notice that

$$\begin{aligned} \frac{d}{dt}\frac{\mathbf{r}}{\sqrt{\mathbf{r}\cdot\mathbf{r}}} &= \frac{\frac{d\mathbf{r}}{dt}\sqrt{\mathbf{r}\cdot\mathbf{r}} - \frac{\frac{d\mathbf{r}}{dt}\cdot\mathbf{r}}{\sqrt{\mathbf{r}\cdot\mathbf{r}}}}{\mathbf{r}\cdot\mathbf{r}} \\ &= \frac{r^2\frac{d\mathbf{r}}{dt} - \left(\frac{d\mathbf{r}}{dt}\cdot\mathbf{r}\right)r}{r^3} \end{aligned} \tag{13}$$

While this expression might seem to have appeared out of the blue, revisiting (11) RHS will reveal something quite comforting.

$$\frac{eg}{mr^3}(\widetilde{\mathbf{L}} \times \mathbf{r}) = \frac{eg}{mr^3}\left[\left(\mathbf{r} \times m\frac{d\mathbf{r}}{dt}\right) \times \mathbf{r}\right] = \frac{eg}{r^3}\left[\frac{d\mathbf{r}}{dt}r^2 - r\left(\frac{d\mathbf{r}}{dt}\cdot\mathbf{r}\right)\right] \tag{14}$$

We have at last managed to prove that RHS of (11) is a total time derivative, so now we can define

$$\mathbf{L} = \widetilde{\mathbf{L}} - eg\hat{\mathbf{r}} \tag{15}$$

---

[2] This magnetic mirror effect manifests itself in Earth's dipole field when charged cosmic particles get trapped in it and oscillate between poles creating the aurora. While the 'trapping' effect cannot be explained with our monopole analysis, the reflection of a cosmic particle when it comes to close to one of the Earth's poles is completely analogous to the mirroring of a charged particle that moves towards a magnetic monopole.

[3] Remember that Noether's theorem states that rotational invariance is associated with a conserved angular momentum.

[4] Noether would never disappoint us like this.

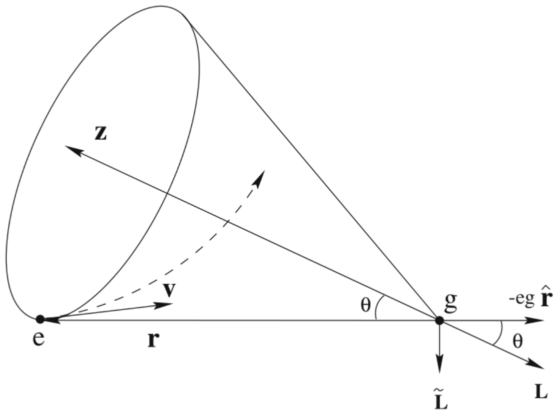

FIG. 1: Motion of electrically charged particle in a magnetic monopole field[31]

where $\hat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$.

Its norm is

$$L^2 = \tilde{L}^2 + (eg)^2 = (mub)^2 + (eg)^2 \tag{16}$$

Now this new term that joined in the definition of angular momentum seems a little peculiar at first. It turns out though that it originates from a non-trivial field contribution to the mechanical angular momentum.

Indeed the electromagnetic angular momentum for the system is given by

$$\widetilde{\mathbf{L}}_{eg} = \frac{1}{4\pi} \int d^3r' \left[\mathbf{r}' \times (\mathbf{E} \times \mathbf{B})\right] \tag{17}$$

For our smooth proceeding we mention the following identity

$$\frac{\mathbf{E} - (\mathbf{E} \cdot \mathbf{r})\mathbf{r}}{r} = (\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{r} \tag{18}$$

which holds for an arbitrary vector $\mathbf{E}$.

Now using (10) on (18) we get

$$\begin{aligned} \widetilde{\mathbf{L}}_{\text{eg}} &= \frac{1}{4\pi} \int d^3r' \left[\mathbf{E} \left(\mathbf{r}' \cdot \mathbf{B}\right) - \mathbf{B} \left(\mathbf{r}' \cdot \mathbf{E}\right)\right] \\ &= \frac{1}{4\pi} \int d^3r' \left[\mathbf{E} \cdot g \frac{\mathbf{r}' \cdot \mathbf{r}'}{r^3} - g \frac{\mathbf{r}'}{r'^3} \left(\mathbf{E} \cdot \mathbf{r}'\right)\right] \\ &= \frac{g}{4\pi} \int (\mathbf{E} \cdot \boldsymbol{\nabla}')\hat{\mathbf{r}}' d^3r'. \\ &= \frac{g}{4\pi} \oint \hat{r}'(\mathbf{E} \cdot d\mathbf{a}) - \int \left(\boldsymbol{\nabla}' \cdot \mathbf{E}\right) \hat{\mathbf{r}}' d^3r' \end{aligned} \tag{19}$$

Now the surface integral vanishes since we can make the surface approach infinity where the electric field approaches 0.

As for the volume integral we will invoke Gauss' Law $\left(\boldsymbol{\nabla}' \cdot \mathbf{E}\right) = 4\pi e \delta^{(3)} \left(\mathbf{r} - \mathbf{r}'\right)$ And we end up with

$$\widetilde{\mathbf{L}}_{eg} = -\frac{g}{4\pi} \int 4\pi e \delta^{(3)} \left(r - r'\right) d^3\mathbf{r}' = -eg\hat{\mathbf{r}} \tag{20}$$

So it turns out that the conserved angular momentum is the sum of the mechanical angular momentum with the Electromagnetic field angular momentum!

Now that we have calculated all the needed quantities, we are able to describe qualitatively and quantitavely the charged particle's trajectory in the magnetic monopole field.
Note that

$$|\mathbf{L}\cdot\hat{\mathbf{r}}| = eg = \text{ const.} \tag{21}$$

from which we can conclude that the trajectory lies on the surface of a cone whose symmetry axis is $-\mathbf{L}$, as seen from FIG.1.

The cone's angle can be deduced from simple geometrical arguments yielding

$$\cot\theta = \frac{eg}{|\widetilde{\mathbf{L}}|} = \frac{eg}{mub} \tag{22}$$

In much the same way the ordinary angular momentum $\widetilde{\mathbf{L}}$ is precessing on the surface of a different cone with the same axis.

$$\mathbf{L}\cdot\widetilde{\mathbf{L}} = \widetilde{\mathbf{L}}^2 = (mub)^2 = \text{ const.} \tag{23}$$

We sum up the most important results from this section in the from of norms and scalar products:

B	$g\frac{r}{r^3}$
r	$\sqrt{v^2t^2+b^2}$
$\widetilde{L}$	mub
L_{eg}	eg
L	$\sqrt{(mub)^2+(eg)^2}$
$\mathbf{L}\cdot\mathbf{r}$	$-eg$
$\mathbf{L}\cdot\widetilde{\mathbf{L}}$	$(mub)^2$

TABLE I: Some important results from this chapter

B. Potential of a Monopole Field

We have sufficiently described the classical physics of a monopole, but if we want ourselves to have a consistent theory and taking into consideration that it should, at some point, be quantized, then the next step in our analysis should be the search of a *potential* for such a field.

Considering the *Helmholtz decomposition theorem* which states that any sufficiently smooth, rapidly decaying vector field in three dimensions can be resolved into the sum of an irrotational (curl-free) vector field and a solenoidal (divergence-free) vector field, and also writing down the Maxwell equations in the presence only of a static magnetic monopole field source g

$$\begin{aligned} &\boldsymbol{\nabla} \times \mathbf{B} = 0 \\ &\boldsymbol{\nabla} \cdot \mathbf{B} = 4\pi\rho_m \\ &\rho_m = g\delta^{(3)}(\mathbf{r}). \end{aligned} \tag{24}$$

one could think that since $\mathbf{B}$ is no longer divergentless but its curl vanishes, a scalar potential ,whose gradient returns the magnetic field, could be of use [15]. Indeed, such a potential can be constructed as

$$\Phi = \frac{g}{r^2} \tag{25}$$

However, remembering that the interaction Lagrangian of an electric charge in an external magnetic field is

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 + e\dot{\mathbf{r}} \cdot \mathbf{A} \tag{26}$$

we realize that (25) is of no use.

It seems our only way out is to brute force a vector potential[5] so our goal is finding a vector potential $\mathbf{A}$ satisfying

$$\boldsymbol{\nabla} \times \mathbf{A} = g\frac{\mathbf{r}}{r^3} \tag{27}$$

Before setting on our journey to constructing such a potential, we are going on a small interlude trip proving (26) yields the correct equations of motion.

The Euler-Lagrange equations in vector notation are:

$$\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}}\right) = 0 \tag{28}$$

Calculating the first term:

$$\frac{\partial L}{\partial \mathbf{r}} = \boldsymbol{\nabla}(e\dot{\mathbf{r}} \cdot \mathbf{A}) = e(\dot{\mathbf{r}} \times (\boldsymbol{\nabla} \times \mathbf{A}) + (\dot{r} \cdot \boldsymbol{\nabla})\mathbf{A})$$

The second partial derivative:

$$\frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + e\mathbf{A}$$

The total time derivative:

$$\frac{d}{dt}(m\dot{\mathbf{r}} + e\mathbf{A}) = m\ddot{\mathbf{r}} + e\frac{d\mathbf{A}}{dt} = m\ddot{\mathbf{r}} + e\frac{\partial A}{\partial t} + e(\dot{\mathbf{r}} \cdot \boldsymbol{\nabla})\mathbf{A}$$

---

[5] It is not actually so much of a brute force technique rather than a consequence of the mathematics of fibre bundles on gauge groups which we will not demonstrate.

Substituting in E-L (28):

$$
\begin{aligned}
m\frac{d^2\mathbf{r}}{dt^2} &= e\dot{\mathbf{r}} \times (\boldsymbol{\nabla} \times \mathbf{A}) + e(\dot{\mathbf{r}} \cdot \boldsymbol{\nabla})\mathbf{A} - e(\dot{\mathbf{r}} \cdot \boldsymbol{\nabla})\mathbf{A} \\
\Rightarrow m\frac{d^2\mathbf{r}}{dt^2} &= e\dot{\mathbf{r}} \times \mathbf{B}.
\end{aligned}
$$

Now that we have proved our Lagrangian yields the correct E.o.M. we can return to the search of a vector potential for the monopole problem.

Employing the spherical symmetry of the problem we can deduce that the vector potential should be written as[6]

$$\mathbf{A}(\mathbf{r}) = A(\theta)\boldsymbol{\nabla}\varphi \tag{29}$$

Now it is time to choose the function $A(\theta)$.
Dirac was first in this task with $A(\theta) = -g(1 + \cos\theta)$. After some straightforward calculations we arrive at

$$
\begin{aligned}
\boldsymbol{\nabla}\varphi &= \left(-\tfrac{\sin\varphi}{r\sin\theta},\ \tfrac{\cos\varphi}{r\sin\theta},\ 0 \right), \\
\mathbf{A}(\mathbf{r}) &= \left(g\tfrac{1+\cos\theta}{r\sin\theta}\sin\varphi,\ -g\tfrac{1+\cos\theta}{r\sin\theta}\cos\varphi,\ 0 \right)
\end{aligned}
\tag{30}
$$

The vector potential can be written also in covariant form

$$\mathbf{A}(\mathbf{r}) = \frac{g}{r}\frac{[\mathbf{r} \times \mathbf{n}]}{r - (\mathbf{r} \cdot \mathbf{n})} \tag{31}$$

where $\mathbf{n}$ is the unit vector in the z axis.
The vector potential (31) is named the *Dirac Potential*.

Now for a quick test of our newly acquired potential, the magnetic field $\mathbf{B}$ in x direction, for example, turns out to be

$$
\begin{aligned}
B_x &= \partial_z\left(\frac{gx}{r(r-z)}\right) = gx\partial_z\left(\frac{1}{x^2+y^2+z^2 - z\sqrt{x^2+y^2+z^2}}\right) \\
&= -gx\frac{2z - \sqrt{x^2+y^2+z^2} - z\frac{z}{\sqrt{x^2+y^2+z^2}}}{r^2\left(r-z\right)^2} \\
&= -gx\frac{\frac{2zr - r^2 - z^2}{r}}{r^2(r-z)^2} = \frac{gx}{r^3}
\end{aligned}
\tag{32}
$$

Everything seems to be working perfectly, until we notice that (31) is singular along the whole positive z axis! Our forceful requirement of the existence of a vector potenital seems to be punching back. Indeed in the vicinity of that area the potential takes values $\mathbf{A} \sim -2g\boldsymbol{\nabla}\varphi$, which is the potential of a singular string of magnetic flux.

The potential (31) can actually be written in the form of a pure gauge.

$$\mathbf{A}(\mathbf{r}) = -g(1+\cos\theta)\boldsymbol{\nabla}\varphi = (1+\cos\theta)\frac{i}{e}U^{-1}\boldsymbol{\nabla}U \tag{33}$$

where $U = e^{-ieg\varphi}$ is the $U(1)$ group element. This gauge transformation however is singular.

It is recommended for our calculations to use a regularized potential

$$\mathbf{A}_R(\mathbf{r}, \varepsilon) = \frac{g}{R}\frac{[\mathbf{r} \times \mathbf{n}]}{R - (\mathbf{r} \cdot \mathbf{n})} \tag{34}$$

[6] Plugging (29) into the curl operator $\nabla \times \mathbf{A} = \boldsymbol{\nabla}A(\theta) \times \boldsymbol{\nabla}\varphi + A(\theta)\boldsymbol{\nabla} \times \varphi$ the first term is in the $\hat{\mathbf{r}}$ direction while the second vanishes. That's definitely a good sign for the imposed spherical symmetry of the problem.

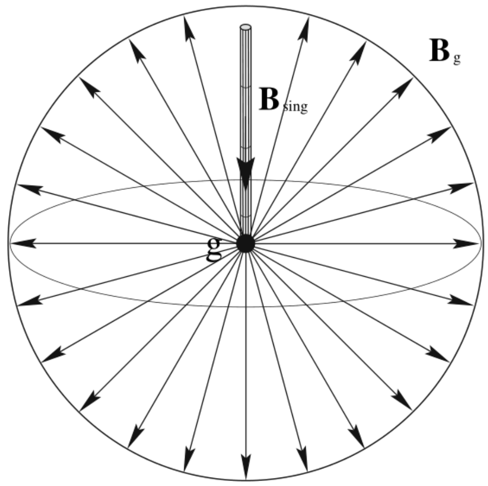

FIG. 2: Graphical interpretation of the field lines of the Dirac potential

where $R = \sqrt{x^2+y^2+z^2+\varepsilon^2}$.

The magnetic field from such an expression is

$$\mathbf{B}_R(\mathbf{r},\varepsilon) = g\frac{\mathbf{r}}{R^3} - g\varepsilon^2\left(\frac{\mathbf{n}}{R^3[R-(\mathbf{r}\cdot\mathbf{n})]} + \frac{\mathbf{n}}{R^2[R-(\mathbf{r}\cdot\mathbf{n})]^2}\right) \tag{35}$$

and using cartesian coordinates while taking the limit of $\varepsilon^2 \to 0$

$$\mathbf{B}_R(\mathbf{r},\varepsilon) \sim g\frac{\mathbf{r}}{r^3} - 2g\varepsilon^2\mathbf{n}\theta(z)\left(\frac{1}{r^2\left(x^2+y^2+\varepsilon^2\right)} + \frac{2}{\left(x^2+y^2+\varepsilon^2\right)^2}\right) \tag{36}$$

which upon the limiting procedure yields

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_g + \mathbf{B}_{\text{sing}} \; = g\frac{\mathbf{r}}{r^3} - 4g\pi\mathbf{n}\theta(z)\delta(x)\delta(y) \tag{37}$$

The second term yielding the delta function can be seen from integrating

$$f(x,y)|_{\epsilon\ll} = \frac{1}{r^2\left(x^2+y^2+\epsilon^2\right)} + \frac{2}{\left(x^2+y^2+\epsilon^2\right)^2} \tag{38}$$

on the whole xy plane.

$$\iint_{\mathbb{R}^2} dxdy\left[\frac{1}{\left(x^2+y^2+z^2\right)\left(x^2+y^2+\varepsilon^2\right)} + \frac{2}{\left(x^2+y^2+\varepsilon^2\right)^2}\right]$$

$$\overset{\text{Polar}}{=} \int_0^{2\rho} d\phi \int_0^\infty \rho d\rho\left[\frac{1}{\left(\rho^2+z^2\right)\left(p^2+\varepsilon^2\right)} + \frac{2}{\left(\rho^2+\varepsilon^2\right)^2}\right]$$

$$\begin{aligned}
&\overset{w=\rho^2}{=} 2\pi\left[\frac{1}{2\left(z^2-\varepsilon^2\right)}\int_0^\infty\left(\frac{1}{w+\varepsilon^2} - \frac{1}{w+z^2}\right)dw + \int_0^\infty \frac{1}{\left(w+\varepsilon^2\right)^2}dw\right] \\
&= \frac{\pi}{z^2-\varepsilon^2}\ln\left(\frac{w+\varepsilon^2}{w+z^2}\right)\Big|_0^\infty - 2\pi\frac{1}{w+\varepsilon^2}\Big|_0^\infty \\
&= \pi\frac{\ln\left(\frac{\varepsilon}{z}\right)}{z^2-\varepsilon^2} + 2\pi\frac{1}{\varepsilon^2}
\end{aligned}$$

Taking the limit

$$\lim_{\varepsilon^2\to 0} \varepsilon^2 \iint_{\mathbb{R}^2} f(x,y)dxdy = \left[2\pi\varepsilon^2\left(\frac{\ln\varepsilon - \ln z}{z2-\varepsilon^2}\right) + 2\pi\right]\Big|_{\varepsilon\to 0}$$
$$= 2\pi.$$

But

$$\lim_{\varepsilon^2\to 0} \varepsilon^2 f(x,y) = \lim_{\varepsilon^2\to 0} \frac{\varepsilon^2}{r^2\left(x^2+y^2+\varepsilon^2\right)} + \frac{2\varepsilon^2}{\left(x^2+y^2+\varepsilon^2\right)^2}$$

So it has to be

$$\lim_{\varepsilon\to 0} \varepsilon^2 f(x,y) = 2\pi\delta(x)\delta(y)$$

Thereby proving (37) from (36).

Using the magnetic field from (37) to calculate the magnetic flux, the flux paradox is now resolved.

$$\begin{aligned}\Phi_{\text{tot}} &= \oint d\sigma\mathbf{B} = g\left(\oint d\sigma\frac{\mathbf{r}}{r^3} - 4\pi\oint d\sigma\mathbf{n}\theta(z)\delta(x)\delta(y)\right) \\ &= 4g\pi - 4g\pi = 0.\end{aligned} \tag{39}$$

So our potential (34) corresponds in fact to an infinitely thin magnetic rod with its one pole at the origin and the other at a point in infinity, rather than a single magnetic pole.

We can cast the vector potential (34) in another form with a rather interesting physical interpretation. That is $\mathbf{A}$ can be written as:

$$\mathbf{A} = \frac{1}{4\pi}\int d^3x'\left[\boldsymbol{\nabla}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\times\mathbf{B}_{\text{sing}}\left(\mathbf{r}\right)\right] = g\int\frac{(\mathbf{r}-\mathbf{r}')\times d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{40}$$

where the integral in the RHS is taken along the *Dirac string.*[7] The proof of this assertion follows:

$$\begin{aligned}\mathbf{A} &= g\int\frac{\mathbf{r}\times d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = g\int\frac{(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}})\times dz'\hat{\mathbf{k}}}{|\mathbf{r}-\mathbf{z}'|^3} \\ &= g(\mathbf{r}\times\hat{\mathbf{n}})\int_0^\infty\frac{dz'}{|\mathbf{r}-\mathbf{z}'|^3} = g(\mathbf{r}\times\hat{\mathbf{n}})\int_0^\infty\frac{dz'}{\left[x^2+y^2+(z-z')^2\right]^{3/2}}\end{aligned}$$

For the calculation of $\int_0^\infty\frac{dz'}{\left[x^2+y^2+(z-z')^2\right]^{3/2}}$, just substitute $u=\frac{z-z'}{\sqrt{x^2+y^2}}$

---

[7] That is the line segment that $\mathbf{A}$ is singular.

$$
\begin{aligned}
&\int_0^\infty \frac{dz'}{(x^2+y^2)^{3/2}\left[1+\frac{(z-z')^2}{x^2+y^2}\right]^{3/2}} \\
&= \frac{1}{x^2+y^2}\int_{-\frac{z}{\sqrt{x^2+y^2}}}^{\infty} \frac{du}{(1+u^2)^{3/2}} \\
&= \frac{1}{x^2+y^2}\frac{u}{\sqrt{u^2+1}}\Bigg|_{-\frac{z}{\sqrt{x^2+y^2}}}^{+\infty} \\
&= \frac{1}{x^2+y^2}\left(1+\frac{1}{\sqrt{1+\frac{z^2}{x^2+y^2}}}\right) \\
&= \frac{1}{x^2+y^2}\left(1+\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \\
&= \frac{\left(z+\sqrt{x^2+y^2+z^2}\right)}{\sqrt{x^2+y^2+z^2}\,(x^2+y^2)} \times \frac{z-\sqrt{x^2+y^2+z^2}}{z-\sqrt{x^2+y^2+z^2}} \\
&= \frac{1}{\mathbf{r}[\mathbf{r}-(\mathbf{r}\cdot\hat{\mathbf{n}})]}
\end{aligned}
$$

Substituting back to $\mathbf{A}$ calculation retrieved the Dirac potential (31), finishing or proof.

We have yet to decipher the physical meaning of (40). In fact, it can be seen as the infinite sum of magnetic dipoles $gd\mathbf{r}'$ located along the string.

C. Gauge transformations and the Dirac string

Previously, we encountered Dirac's potential which upon taking its curl yielded the monopole magnetic field along with a mysterious extra field along its singularity. So what is this Dirac string? It certainly isn't something we expected and the goal of this chapter is to present the conditions to render it unobservable and therefore unphysical.

At this point we recall that $\mathbf{A}$ is not uniquely defined. It is actually defined up to a gradient or in group theory terms up to a $U(1)$ gauge transformation $U(\mathbf{r}) = e^{ie\lambda(\mathbf{r})}$.

$$
\mathbf{A} \to \mathbf{A}' = \mathbf{A} - \frac{i}{e}U^{-1}\boldsymbol{\nabla}U = \mathbf{A} + \boldsymbol{\nabla}\lambda(\mathbf{r}) \tag{41}
$$

The question that arises here is if any function $\lambda(\mathbf{r})$ is eligible to take part in a gauge transformation. Well, usually we use single valued functions, but nothing prevents us from using multi-valued ones. The gradient of such a function is single-valued itself except of a line of singularity along the line that separates the different sheets of the multi-valued function. Thus such a function would generate new singular terms of the potential $\mathbf{A}$. Indeed the change in magnetic flux from the gauge transformation can be calculated as follows:

$$
\Delta\Phi = \int_\sigma d^2S\hat{\mathbf{n}}_S\cdot(\mathbf{B}'-\mathbf{B}) = \int_\sigma d^2S\hat{\mathbf{n}}_S\cdot[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\lambda)] = \oint d\mathbf{l}\cdot\boldsymbol{\nabla}\lambda \tag{42}
$$

which assures us that only single valued gauge transformations do not affect the magnetic flux.

Experimenting a little with the power of multi-valued gauge transformations, it is tempting to try $U = \exp\{2ieg\varphi\}$.

Then the Dirac potential (31) changes as follows:

$$
\begin{aligned}
\mathbf{A}^S \to \mathbf{A}^S - \frac{i}{e}e^{-2ieg\varphi}\boldsymbol{\nabla}e^{2ieg\varphi} &= -\frac{g}{r}\frac{1+\cos\theta}{\sin\theta}\hat{\mathbf{e}}_\varphi + \frac{2g}{r\sin\theta}\hat{\mathbf{e}}_\varphi \\
&= \frac{g}{r}\frac{1-\cos\theta}{\sin\theta}\hat{\mathbf{e}}_\varphi \equiv \mathbf{A}^N.
\end{aligned} \tag{43}
$$

Potential Name	Equation	Gauge transformation from Dirac (South)
Dirac (South)	$-\frac{g}{r}\frac{1-\cos\theta}{\sin\theta}\hat{\mathbf{e}}_\phi$	
Dirac (North)	$\frac{g}{r}\frac{1+\cos\theta}{\sin\theta}\hat{\mathbf{e}}_\phi$	$\lambda = 2ieg\phi$
Schwinger	$\frac{g}{r}\frac{\cos\theta}{\sin\theta}\hat{\mathbf{e}}_\phi$	$\lambda = ieg\phi$
Banderet	$-\frac{g}{r}\phi\sin\theta\hat{\mathbf{e}}_\phi$	$\lambda = -ieg(1+\cos\theta)\phi$

TABLE II: Equivalent Magnetic Monopole Potentials

The gauge transformation we considered results in the appearance of a $4\pi g$ flux along the z axis. Therefore the new potential has its positive z singularity cancelled but a new one on the negative z axis arises.

Thus the string itself is as it has been rotated. Indeed we can construct more elaborate gauge transformations that rotate the string on to an arbitrary direction unit vector $\hat{\mathbf{n}}$.

Concluding this short chapter, we showed that gauge transformations which by definition leave the physics of electromagnetism invariant rotate the Dirac string. So the Dirac string has to be unphysical.

In TABLE II, we present some of the most known potentials to describe the magnetic monopole and their relationships between them. The last one (the Banderet potential) is really an exotic one, since it is easy to see that its singularity does not resemble a string anymore, but rather a part of a plane!

D. Dual Invariance in Classical Electromagnetism

One of the key ideas that led to the magnetic monopole theory is the dual invariance of Maxwell's equation. Indeed in vacuum the 4 equations:

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{E} &= 0\\ \boldsymbol{\nabla}\times\mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} &= 0\\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0\\ \boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} &= 0\end{aligned} \tag{44}$$

exhibit invariance under $O(2)$ symmetry group transformations:

$$G : \begin{cases}\mathbf{E}\to\mathbf{E}\cos\theta - \mathbf{B}\sin\theta\\ \mathbf{B}\to\mathbf{E}\sin\theta + \mathbf{B}\cos\theta\end{cases} \tag{45}$$

The parameter-angle θ in these transformations is named *dual*.

In covariant form these equations are written as:

$$\partial_\mu F^{\mu\nu} = 0, \quad \partial_\mu \widetilde{F}^{\mu\nu} = 0 \tag{46}$$

where the electromagnetic field strength tensor and its dual are

$$\begin{aligned}F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu\\ \widetilde{F}_{\mu\nu} &= \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = \varepsilon_{\mu\nu\rho\sigma}\partial^\rho A^\sigma\end{aligned} \tag{47}$$

and the electric and magnetic fields are to be found from

$$\begin{aligned}E_i = E^i = F_{0i} = -F^{0i} &= -\frac{1}{2}\varepsilon_{0ijk}\tilde{F}^{jk} = \frac{1}{2}\varepsilon^{0ijk}\widetilde{F}_{jk}\\ B_i = B^i = \tilde{F}_{0i} = -\widetilde{F}^{0i} &= \frac{1}{2}\varepsilon_{0ijk}F^{jk} = -\frac{1}{2}\varepsilon^{0ijk}F_{jk}\end{aligned} \tag{48}$$

It is evident that the set of equations (46) exhibit the exact same $O(2)$ symmetry

$$G : \begin{cases} F_{\mu\nu} \to F_{\mu\nu}\cos\theta - \widetilde{F}_{\mu\nu}\sin\theta \\ \widetilde{F}^{\mu\nu} \to F^{\mu\nu}\sin\theta + \widetilde{F}^{\mu\nu}\cos\theta \end{cases} \tag{49}$$

In the advent of magnetic monopoles this $O(2)$ symmetry would be manifest even when not in vacuum provided the corresponding 4-currents are rotated as well.

The generalised Maxwell equations are:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho_e \\ \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \mathbf{j}_e \\ \nabla \cdot \mathbf{B} &= \rho_m \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= -\mathbf{j}_m \end{aligned} \tag{50}$$

The sources are transformed similarly to the fields. That is:

$$\begin{aligned} \rho_e &\to \rho_e \cos\theta - \rho_m \sin\theta \\ \rho_m &\to \rho_e \sin\theta + \rho_m \cos\theta \end{aligned} . \tag{51}$$

and their currents

$$\begin{aligned} \mathbf{j}_e &\to \mathbf{j}_e \cos\theta - \mathbf{j}_m \sin\theta \\ \mathbf{j}_m &\to \mathbf{j}_e \sin\theta + \mathbf{j}_m \cos\theta \end{aligned} . \tag{52}$$

The invariance of the equations of electrodynamics under duality transformations shows that it is a matter of convention to speak of a particle possessing an electric charge, but not magnetic charge. The only meaningful question is whether *all* particles have the same ratio of magnetic to electric charge. If they do, we can perform a duality transformation and choose a suitable dual angle so that $\rho_m = 0$ and $\mathbf{j}_m = 0$, returning the equations to their usual form.

As for the transformation properties of ρ_m and $\mathbf{j}_m$ under rotations, spatial inversion and time reversal, from the known behavior of $\mathbf{E}$ and $\mathbf{B}$ from (50) we can deduce that[17]:

- ρ_m is a pseudoscalar density, odd under time reversal, and
- $\mathbf{j}_m$ is a pseudovector density, even under time reversal.

Since the symmetries of ρ_m are opposite to those of ρ_e, it is a necessary consequence of the existence of a particle with both magnetic and electric charges[8] that space inversion and time reversal are no longer valid symmetries of the laws of physics.

One last thing left to explore on dual transformations is of course the behaviour of the electromagnetic lagrangian under it.

$$L_0 = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} \tag{53}$$

$$\begin{aligned} L_0 &\to -\frac{1}{4e^2}\left(F_{\mu\nu}\cos\theta + \tilde{F}_{\mu\nu}\sin\theta\right)\left(F^{\mu\nu}\cos\theta + \tilde{F}^{\mu\nu}\sin\theta\right) \\ &= -\frac{1}{4e^2}\left(F_{\mu\nu}F^{\mu\nu}\cos^2\theta + \tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu}\sin^2\theta + 2F_{\mu\nu}\tilde{F}^{\mu\nu}\sin\theta\cos\theta\right) \\ &= -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}\left(\cos^2\theta - \sin^2\theta\right) - \frac{1}{4e^2}F_{\mu\nu}\tilde{F}^{\mu\nu}\sin(2\theta) \\ &= -\frac{1}{4e^2}\left(F_{\mu\nu}F^{\mu\nu}\cos(2\theta) - F_{\mu\nu}\tilde{F}^{\mu\nu}\sin(2\theta)\right). \end{aligned} \tag{54}$$

---

[8] A dyon to be specific.

where we have used

$$\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\sigma}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = \frac{1}{4}(-2)\left(\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma} - \delta^{\beta}_{\rho}\delta^{\alpha}_{\sigma}\right)F^{\rho\sigma}F_{\alpha\beta}$$
$$= -F^{\rho\sigma}F_{\rho\sigma}.$$

At first glance (53) seems to be non-invariant under dual transformations. However, further calculation of the second terms reveals that it is a total divergence and therefore does not contribute to the equations of motion.

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}F_{\mu\nu}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)$$
$$= 2\varepsilon^{\mu\nu\alpha\beta}\partial_\mu A_\nu \partial_\alpha A_\beta = 2\varepsilon^{\mu\nu\alpha\beta}\left[\partial_\mu\left(A_\nu\partial_\alpha A_\beta\right) - A_\nu\partial_\mu\partial_\alpha A_\beta\right]$$
$$= 2\varepsilon^{\mu\nu\alpha\beta}\partial_\mu\left(A_\nu\partial_\alpha A_\beta\right)$$

where we took advantage of the total antisymmetry of the indices and the commutativity of partial derivatives.

III. THE MAGNETIC MONOPOLE IN QUANTUM MECHANICS

A. Charge Quantization Condition

In the quantum mechanical scheme of things, the fundamental quantity of the gauge theory of electromagnetism that defines the interactions is the potential $\mathbf{A}$. This is described by the action of the covariant derivative[9] on the wave function as:

$$\mathbf{D}\psi(\mathbf{r}) \equiv [\boldsymbol{\nabla} - ie\mathbf{A}(\mathbf{r})]\psi(\mathbf{r}) \tag{55}$$

Under $U(1)$ transformations the wave function changes as: $\psi(\mathbf{r}) \to U\psi(\mathbf{r}) = e^{ie\lambda(\mathbf{r})}\psi(\mathbf{r})$ while the covariant derivative:

$$\begin{aligned} &\mathbf{D}\psi(\mathbf{r}) \to [\boldsymbol{\nabla} - ie\mathbf{A}(\mathbf{r}) - ie\boldsymbol{\nabla}\lambda(\mathbf{r})]e^{ie\lambda(\mathbf{r})}\psi(\mathbf{r}) \\ &= e^{ie\lambda(\mathbf{r})}\boldsymbol{\nabla}\psi(\mathbf{r}) + ie\boldsymbol{\nabla}\lambda(\mathbf{r})\psi(\mathbf{r}) - ie\mathbf{A}(\mathbf{r})e^{ie\lambda(\mathbf{r})}\psi(\mathbf{r}) - ie\boldsymbol{\nabla}\lambda(\mathbf{r})e^{ie\lambda(\mathbf{r})}\psi(\mathbf{r}) \\ &= e^{ie\lambda(\mathbf{r})}[\boldsymbol{\nabla} - ie\mathbf{A}(\mathbf{r})]\psi(\mathbf{r}) = e^{ie\lambda(\mathbf{r})}\mathbf{D}\psi(\mathbf{r}) \end{aligned} \tag{56}$$

The Lagrangian (26) changes under a gauge transformation as:

$$\begin{aligned} L &\to \frac{1}{2}m\dot{\mathbf{r}}^2 + e\dot{\mathbf{r}}\mathbf{A} + e\dot{\mathbf{r}}\boldsymbol{\nabla}\lambda(\mathbf{r}). \\ &= L + e\dot{\mathbf{r}}\boldsymbol{\nabla}\lambda(\mathbf{r}) = L + \frac{d}{dt}[e\lambda(\mathbf{r})] \end{aligned} \tag{57}$$

and the corresponding action:

$$S = \int_0^T dL \Rightarrow S + e\lambda(\mathbf{r})|_0^T \tag{58}$$

We want e^{iS} to be a gauge invariant quantity. Therefore, the change in action should be an integer multiple of 2π. So choosing $\lambda(\mathbf{r}) = 2g\phi$ we arrive at:

$$\begin{aligned} S_{\text{int}} &= e\int_0^T \dot{\mathbf{r}} \cdot \boldsymbol{\nabla}\lambda dt = e\oint_l d\mathbf{x} \cdot \boldsymbol{\nabla}\lambda \\ \delta S_{\text{int}} &= e\oint d\mathbf{x} \cdot \boldsymbol{\nabla}(2g\phi) = 4\pi ge \end{aligned} \tag{59}$$

So mandating the change in action is $2\pi n$ we get

$$eg = \frac{n}{2} \tag{60}$$

which is the famous *Dirac quantization condition*[9].

This is one of the most fascinating relations in monopole theory. That is because it explains why the electric charge is quantized. If just one magnetic monopole was found in the universe then all electric charges would have to be integer multiples of the smallest charge.

Another interesting thing to note is that, unlike most other cases of quantization which we are familiar with, (60) has nothing to do with the eigenvalues of a quantum mechanical operator. The reasoning behind this is that this mysterious quantization is of topological origin[10].

---

[9] The form of the covariant derivative shouldn't be that surprising to us, considering the conjugate momentum is $\pi = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + e\mathbf{A}$ and we replace $m\dot{\mathbf{r}}$ with $-i\boldsymbol{\nabla}$

[10] More on that later.

Dirac's quantization condition can be derived in many ways, the simplest of which stems from (20).

The total angular momentum of the system is given by:

$$\mathbf{L} = [\mathbf{r} \times \boldsymbol{\pi}] - eg\hat{\mathbf{r}} = [\mathbf{r} \times (\mathbf{p} - e\mathbf{A})] - eg\hat{\mathbf{r}} = \widetilde{\mathbf{L}} - e[\mathbf{r} \times \mathbf{A}] - eg\hat{\mathbf{r}} \tag{61}$$

Demanding the total angular momentum components satisfy standard commutation relations, the electromagnetic angular momentum has to take half integer values. One other interesting take on (60) is that upon taking both charges (electric and magnetic) to be minimum, corresponding to $n = 1$, and bringing back the constants we have set to 1 we get:

$$\frac{g}{e} = \frac{\hbar c}{2e^2} = \frac{1}{2\alpha} \approx \frac{137}{2} \tag{62}$$

That is the Coulomb force between two magnetic monopoles would be approximately $\left(\frac{137}{2}\right)^2 \approx 4700$ stronger than the exact same force between two electrons, making their detection less likely. Moreover, the coupling constant g grows too large and perturbation theory becomes unusable.

B. Abelian Wu-Yang Monopole

We recall that vector potential $\mathbf{A}$ comes together with a tormenting line singularity that can be oriented in different directions with suitable gauge transformations. The field strength tensor is defined everywhere with no singularities[11]. Hence, we speculate that there is a mathematical description that is free from non-physical singularities of any kind.

There is in fact an idea from T.T. Wu and C.N. Yang in 1975 that manages to accomplish just that[34]. Two basic observation are needed beforehand to come up with such an idea. Firstly, notice the Dirac string can be set to an arbitrary direction of our choice. And secondly, we are not obligated by any means to use a single potential to describe the whole space.

We therefore separate $\mathbb{R}^3/\{0\}$ into two hemispheres R^N and R^S and we assign each one of them a vector potential that its Dirac string is located on the direction of the other hemisphere, thereby making the strings undetectable in our formalism.

In terms of expressions:

$$\begin{cases} \mathbf{A}^N = g\frac{1-\cos\theta}{r\sin\theta}\hat{\mathbf{e}}_\varphi & \Longrightarrow 0 \leq \theta < \frac{\pi}{2} + \frac{\varepsilon}{2} : R^N \\ \mathbf{A}^S = -g\frac{1+\cos\theta}{r\sin\theta}\hat{\mathbf{e}}_\varphi & \Longrightarrow \frac{\pi}{2} - \frac{\varepsilon}{2} < \theta \leq \pi : R^S \end{cases} \tag{63}$$

We can take ε to be arbitrarily small so the intersection region $R^N \cap R^S$ of the two potentials becomes the equator. The potentials we used of course did not come out of the blue. The Southern hemisphere potential is the Dirac potential itself while the Northern one is the same with its string reflected just as we showed in (43).

From this setup we can derive (again) the Dirac quantization condition in a more elegant way.

Let us consider a charged particle moving along the equator. The quantum mechanical wave function picks up a phase factor

$$e\int_0^T dt\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}) = e\oint_l d\mathbf{r} \cdot \mathbf{A}(\mathbf{r})$$

---

[11] Apart from the obvious physical singularity where the monopole is placed. So for the sake of brevity we say that our manifold is $\mathbb{R}^3/\{0\}$, i.e. the whole space apart from a point we have extracted. This is going to be relevant later since it drastically changes the topology of the space.

In the above expression both vector potentials $\mathbf{A}$ are equally applicable, since they are both valid in this region! Calculating the exact same quantity for each one yields:

$$e\oint_l d\mathbf{r}\cdot\mathbf{A}^N(\mathbf{r}) = e\int_{R^N} d\mathbf{s}\cdot\left[\boldsymbol{\nabla}\times\mathbf{A}^N\right] = e\int_{R^N} d\mathbf{s}\cdot\mathbf{B},$$
$$e\oint_l d\mathbf{r}\cdot\mathbf{A}^S(\mathbf{r}) = -e\int_{R^S} d\mathbf{s}\cdot\left[\boldsymbol{\nabla}\times\mathbf{A}^S\right] = -e\int_{R^S} d\mathbf{s}\cdot\mathbf{B},$$

where the minus sign stems from the different orientation of the differential surface elements $d\mathbf{s}$ after the application of Stoke's theorem.

Thus in the overlap region the action is defined up to a term:

$$\Delta S = e\int_{R^N\cap R^S} d\mathbf{s}\cdot\mathbf{B} = e\int_V d^3r\boldsymbol{\nabla}\cdot\mathbf{B} = 4\pi eg \tag{64}$$

where we applied Gauss' theorem in collaboration with Maxwell's equation for the divergence of B. That ambiguity must be unobservable and therefore an integer multiple of 2π, since the physical amplitude depends on $\sim \exp\{iS\}$. Therefore we arrive at (60) again:

$$\Delta S = 4\pi eg = 2\pi n, \quad eg = \frac{n}{2}, \quad n\in\mathbb{Z}$$

The same equation can be extracted by noticing that in the overlap region the wavefunction are connected via a $U(1)$ transformation as

$$\psi^S = U\psi^N = e^{2ieg\varphi}\psi^N$$

After a complete rotation along the equator the azimuthal angle increases from 0 to 2π.

$$\varphi^S(0) = \varphi^N(0), \quad \varphi^S(2\pi) = e^{4\pi ieg}\varphi^N(2\pi)$$

However, because of the single-valuedness of the wave function the exponential in the RHS must be an integer multiple of 2π, thereby yielding the Dirac quantization condition once more.

IV. TOPOLOGICAL CONSIDERATIONS I: HOMOTOPY THEORY

We mentioned earlier that (60), namely the Dirac quantization condition is of topological origin. It turns out that this integer n is a winding number and it represents the number of times $U(\phi)$ (our gauge transformation) covers the group $U(1)_{em}$. It is the fact that the winding number must be an integer that makes the magnetic and electric charge quantized. That winding number is strongly related with a group named the 1st homotopy group or fundamental group of S^1 which is the equivalent manifold of the $U(1)$ group. In the following chapter we present the basics of the homotopy groups of relevance to our discussion[23].

A. Fundamental Definitions

1. Paths and Loops

The general idea of homotopy groups is to set the fundamentals of characterising topological spaces with respect to the contractibility or not of loops.

We define a **path** as a map from $\alpha : I = [0,1] \to X$ with an initial point x_0 and an end point x_1 if $a(0) = x_0$ and $a(1) = x_1$. Now if the initial point and the end point are the same we got ourselves a **loop**.

The **constant path** is defined as $c_x : I \to X$ with $c_x(s) = x, s \in I$.

We can endow now the set of paths with some algebraic structure. We define the product of two paths as follows: Let $\alpha, \beta : I \to X$ be paths with $\alpha(1) = \beta(0)$. Their product is denoted as $\alpha * \beta$ is a path in X and is defined as

$$\alpha * \beta(s) = \begin{cases} \alpha(2s) & 0 \le s \le \frac{1}{2} \\ \beta(2s-1) & \frac{1}{2} \le s \le 1 \end{cases} \tag{65}$$

We note that the parameter s runs twice as fast and covers the path α meeting its endpoint and immediately after traces path β.

Now to define the inverse path: Let $\alpha : I \to X$ be a path from x_0 to x_1. The inverse path is defined as:

$$\alpha^{-1}(s) \equiv \alpha(1-s) \quad s \in I \tag{66}$$

We notice that the inverse path is just α just traced backwards. One may be tempted to assume that we have ourselves a group structure now, but it is easy to see that $\alpha * \alpha^{-1} \neq c_x$.

We need the concept of homotopy[12] to define just that group structure needed.

2. Homotopy: An equivalence relation

The algebraic structure of paths and loops is not as useful as it is. For example the product of a path and its inverse is not the constant path. For that reason we are inspired to define an equivalence relation that identifies paths that can be continuously deformed to one another and those exact equivalence classes formed will be proved to admit a group structure.

From now own we will be referring to close paths (i.e. loops) since they are of more interest to us.

Now let α, β be loops atx_0, $\alpha, \beta : I \to X$. They are said to be **homotopic**, written $\alpha \sim \beta$, if there exists a continuous map $F : I \times I \to X$ such that

$$\begin{gathered} F(s,0) = \alpha(s), \quad F(s,1) = \beta(s) \quad \forall s \in I \\ F(0,t) = F(1,t) = x_0 \quad \forall t \in I. \end{gathered} \tag{67}$$

[12] A special kind of equivalence relation

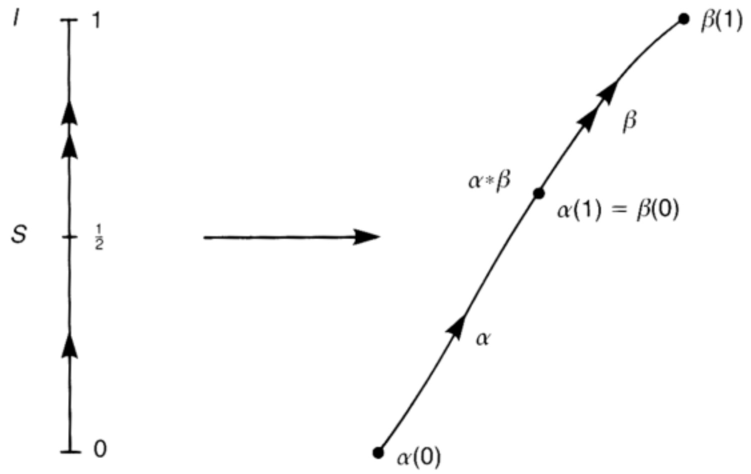

FIG. 3: The product of two paths

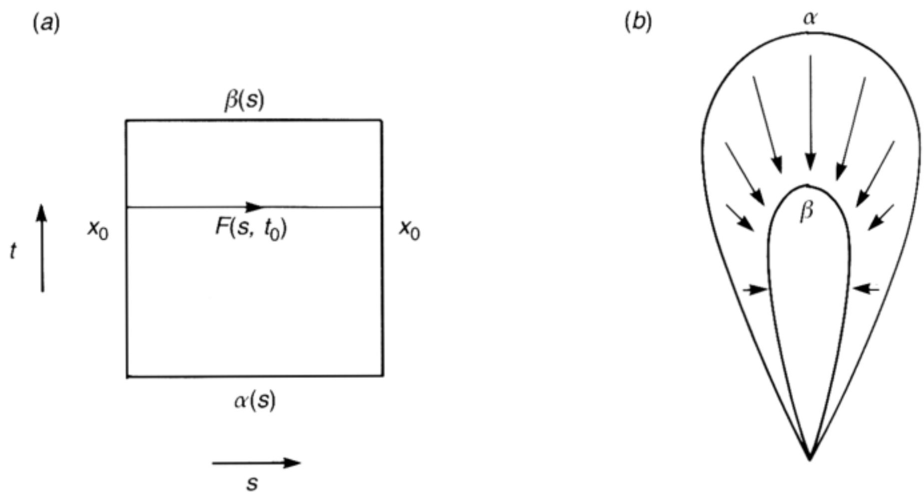

FIG. 4: ((a)Homotopy F interoplating between α and β, (b) Continuous deformation of one loop to anothen

The connecting map F is called a **homotopy** between α and β "Homotopic to" is an equivalence relation. This means it enjoys the properties of symmetry (If $\alpha \sim \beta$ then $\beta \sim \alpha$), reflectivity ($\alpha \sim \alpha$) and transitivity (If $\alpha \sim \beta$ and $\beta \sim \gamma$ then $\alpha \sim \gamma$).

3. Fundamental Groups

The equivalence class of loops is denoted [a] and is called the **homotopy class** of α.

The product between homotopy classes is defined very naturally as

$$[\alpha] * [\beta] = [\alpha * \beta]. \tag{68}$$

Let X be a topological space. The set of homotopy classes of loops at $x_0 \in X$ is denoted by $\pi_1(X, x_0)$ and is called the **fundamental group** (or the first homotopy group) of X at x_0.

It can be proven that the fundamental group has a group structure. Moreover, if X is an arcwise connected topological space (which all of the topological spaces to our interest are) then the fundamental group at a point of the topological space is isomorphic to every other point in space and therefore we can omit referring to a certain point.

B. Homotopic invariance of fundamental groups

We define homotopy equivalence and state afterwards that topological spaces of the same homotopy type share their fundamental groups, a remarkable property making them one of the most important topological invariants that characterise topological spaces.

Let X and Y be topological spaces. X and Y are of the same homotopy type, written as $X \simeq Y$, if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$. The map f is called the homotopy equivalence and g, its homotopy inverse.

If a topological space is homotopic to a point, we say it is contractible and its fundamental group is trivial.

For example, $\mathbb{R}^n$ is contractible and therefore this means its fundamental group is trivial. We can also understand this by noting that every loop in $\mathbb{R}^n$ is contractible to a point and therefore we have only one equivalence class of loops that can only play the role of the identity element in the fundamental group. So,

$$\pi_1 (\mathbb{R}^n) \cong \{e\} \tag{69}$$

Without getting ourselves more involved, we present with no rigorous proof the fundamental or higher homotopy groups of spaces which are of interest to us.

$$\pi_1 \left(S^1\right) \cong \mathbb{Z} \tag{70}$$

Although the proof of such claim is certainly not trivial, the intuition needed is easily acquired. Suppose we encircle a cylinder with a plastic band. If it encircles the cylinder m times, it cannot be continuously deformed to encircle it $n \neq m$ times. If an elastic band encircles the cylinder first n times and then m times, it encircles it $n + m$ times in total. A generalisation of this relation in higher homotopy groups is

$$\pi_1 (S^n) = 0 \tag{71}$$

which is easy to understand since any closed loop is deformable to a point in every n-sphere except the circle.

For higher homotopy groups we get:

$$\pi_n (S^n, x_0) \cong \mathbb{Z} \tag{72}$$

This will be of very big significance to us, especially since the group $SU(2)$ is identified with a 3-sphere.

That is easy to see since $SU(2)$ is the group that consists of 2×2 unitary matrices with $detU = 1$. Now written in the basis of Pauli matrices[13]:

$$\begin{aligned} &U = b_0 + ib_i\sigma^i \\ &b_0^2 + b_1^2 + b_2^2 + b_3^2 = 1 \end{aligned} \tag{73}$$

As for the Special Orthogonal groups $SO(n)$, they are identified with Real Projective planes $\mathbb{R}P^n$.

The Real Projective plane $\mathbb{R}P^n$ can be understood as S^n with its antipodal points identified. We can further understand why this space corresponds to $SO(n)$, since the latter describes rotations of a vector $\hat{\mathbf{n}}$ in $\mathbb{R}^n$, but a

[13] We will prove this decomposition later in equation (261) on page 59

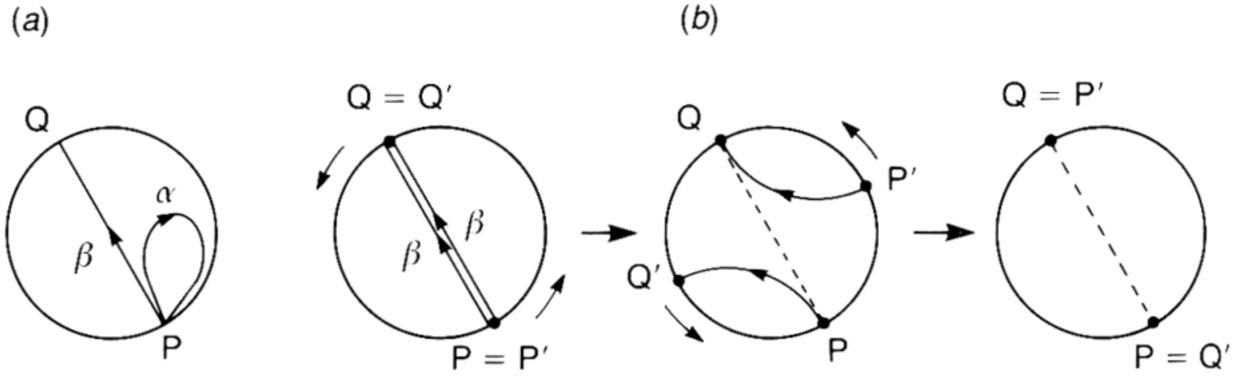

FIG. 5: (a)α is a trivial loop, while β cannot be shrunk to a point, (b)$\beta * \beta$ however, with the help of the identification of the antipodal points of the circle can indeed be shrunk to a point. Hence, we have two equivalence classes and our fundamental group is $\mathbb{Z}_2$

rotation of π degrees and one of $-\pi$ yields the same result. For higher homotopy groups ($n > 1$) it can be shown that they share the same with the corresponding n-sphere:

$$\pi_n\left(\mathbb{R}P^m\right) \cong \pi_n\left(S^m\right) \tag{74}$$

But as for the fundamental groups, we have

$$\pi_1\left(\mathbb{R}P^3\right) \cong \mathbb{Z}_2 \tag{75}$$

and

$$\pi_1\left(\mathbb{R}P^2\right) \cong \left(x; x^2\right) \cong \mathbb{Z}_2 \tag{76}$$

for the lowest dimensional real projective planes. $\mathbb{Z}_2$ is the cyclic group consisting of only two elements. The explanation to (76) is better understood via FIG.5

In physics, we should state though that we do not use homotopy groups to classify topological spaces in general. On the contrary, they are most usually employed on the classification of maps from the sphere at infinity to some vacuum manifold

$$\{f: \quad S^n \to \mathcal{M} \mid f\left(p_0\right) = m_0\}$$

We identify functions that are homotopically equivalent and then the set of equivalent classes produces a group. This group turns out to be $\pi_n\left(\mathcal{M}\right)$.

V. GAUGE FIELD THEORIES

A. Abelian Gauge Field Theory

One usually meets a *Gauge theory* when attempting to upgrade a global symmetry of the Lagrangian into a local one. The simplest example is Dirac's Lagrangian

$$L_D = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi \tag{77}$$

This Lagrangian obviously possesses a global $U(1)$ symmetry, with the field transformations given as:

$$\begin{aligned}\Psi &\to \Psi' = e^{-iq\phi}\Psi \\ \bar{\Psi} &\to \bar{\Psi}' = \bar{\Psi}e^{iq\phi}\end{aligned} \tag{78}$$

The associated conserved Noether current can be found as:

$$j^{\mu}=\frac{\partial L}{\partial(\partial_{\mu}\Psi)}\frac{\partial(\delta\Psi)}{\partial\phi}=i\bar{\Psi}\gamma^{\mu}(-iq)\Psi=q\bar{\Psi}\gamma^{\mu}\Psi \tag{79}$$

Upon trying to employ a local $U(1)$ transformation, namely $e^{iq\phi(x)}$, the mass term stays invariant while the kinetic term is not, because of $\left[\partial_{\mu},e^{iq\phi(x)}\right]\neq 0$ In fact the Lagrangian (77) changes as:

$$\begin{aligned}L_D\to L_D'=i\bar{\Psi}e^{+iq\phi(x)}\gamma^{\mu}\partial_{\mu}\left[e^{-iq\phi(x)}\Psi\right]-m\bar{\Psi}e^{+iq\phi(x)}e^{-iq\phi(x)}\psi\\ =i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi+q\partial_{\mu}\phi(x)\bar{\Psi}\Psi-m\bar{\Psi}\Psi.\end{aligned} \tag{80}$$

We could effectively cancel the extra term in the Lagrangian by introducing a new field that interacts with the fermions as

$$\mathcal{V}_{\text{photon-electron}}=-q\bar{\Psi}\gamma^{\mu}A_{\mu}\Psi \tag{81}$$

and postulating its transformation properties under $U(1)$ local transformations to be:

$$A_{\mu}\to A_{\mu}+\partial_{\mu}\phi(x) \tag{82}$$

This can also be interpreted as the substitution of the partial derivative by a *covariant derivative*:

$$\partial_{\mu}\to D_{\mu}\equiv\partial_{\mu}+iqA_{\mu} \tag{83}$$

that is the suitable object to define the canonically conjugate momentum and reminds us of (55)[14]
Including finally the electromagnetic term in the Lagrangian we arrive at the *QED* Lagrangian:

$$L_{QED}=i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi-q\bar{\Psi}\gamma^{\mu}A_{\mu}\Psi-m\bar{\Psi}\Psi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}=i\bar{\Psi}\not{D}\Psi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{84}$$

An interesting note to take is that upon applying the Euler-Lagrange equations (147) for the electromagnetic potential yields a source term:

$$\partial_{\nu}F^{\mu\nu}=j^{\mu} \tag{85}$$

which is precisely the conserved Noether current (79)!

That is, (79) plays a dual role, both as a conserved Noether current and also as our familiar *electromagnetic current.*

B. Non-Abelian Gauge Field theories

We can generalize all the above for non-Abelian gauge groups.

Guided from the Abelian group case, we define the *non-Abelian covariant* derivative as:

$$\mathbf{D}_{\mu}\equiv\partial_{\mu}\mathbf{1}-igA_{\mu}^{i}\mathbf{T}^{i}\equiv\partial_{\mu}\mathbf{1}-ig\mathbf{A}_{\mu} \tag{86}$$

where we defined $\mathbf{A}_{\mu}=A_{\mu}^{i}\mathbf{T}^{i}$ and $\mathbf{T^{i}}$ are the $dim(G)$ independent generators of the non-Abelian Lie group.

The generators satisfy these expressions:

$$\left[\mathbf{T}^{i},\mathbf{T}^{j}\right]=if^{ijk}\mathbf{T}^{k},\quad \operatorname{Tr}\left(\mathbf{T}^{i}\mathbf{T}^{j}\right)=\frac{1}{2}\delta^{ij} \tag{87}$$

where f^{ijk} are the group structure constants and the second relation serves as a normalization for the generators.

---

[14] That is, quantum mechanically following the minimal coupling perscription, it reproduces the correct Lorentz force as we shown earlier.

The gauge transformation is defined as:

$$\mathbf{U} = e^{ig\theta^i \mathbf{T}^i} \tag{88}$$

with $i = 1, 2, ..., dim(G)$.

We postulate the transformation law of the covariant derivative[15]

$$\mathbf{D}_\mu \to \mathbf{D}'_\mu = \mathbf{U} \cdot \mathbf{D}_\mu \cdot \mathbf{U}^{-1}, \quad \mathbf{U} = e^{ig\theta^i T^i}, \mathbf{U}^\dagger = \mathbf{U}^{-1} \tag{89}$$

Now we attempt to find $\mathbf{A}_\mu$'s transformation properties.
Starting from the postulate that $\mathbf{D}_\mu \mathbf{\Psi}$ transforms in the same way as Ψ, where Ψ is an arbitrary test function[16],we get:

$$\begin{aligned}
&(\mathbf{D}_\mu \mathbf{\Psi})' = \mathbf{U} \left(\partial_\mu \mathbf{1} - ig\mathbf{A}_\mu\right) \mathbf{\Psi} \\
&\left(\partial_\mu - ig\mathbf{A}'_\mu\right) \mathbf{U}\mathbf{\Psi} = \mathbf{U} \left(\partial_\mu - ig\mathbf{A}_\mu\right) \mathbf{\Psi} \\
&\mathbf{U}\partial_\mu \mathbf{\Psi} + \left(\partial_\mu \mathbf{U}\right) \mathbf{\Psi} - ig\mathbf{A}'_\mu \mathbf{U}\mathbf{\Psi} = \mathbf{U}\partial_\mu \mathbf{\Psi} - ig\mathbf{U}\mathbf{A}_\mu \mathbf{\Psi} \\
&\mathbf{A}_\mu = \mathbf{U}^\dagger \mathbf{A}'_\mu \mathbf{U} + \frac{i}{g} \mathbf{U}^\dagger \partial_\mu \mathbf{U} \\
&\mathbf{A}'_\mu = -\frac{i}{g} (\partial_\mu \mathbf{U}) \mathbf{U}^\dagger + \mathbf{U}\mathbf{A}_\mu \mathbf{U}^\dagger.
\end{aligned} \tag{90}$$

The next step is to define a *field strength tensor.*

For the Abelian case, we have:

$$\frac{i}{q} \left([D_\mu, D_\nu]\right) \Psi = -\left(F_{\mu\nu}\right) \Psi \tag{91}$$

and relying on this fact again, we are going to calculate the commutator of the non-Abelian covariant derivatives and postulate the result be proportional to the non-Abelian field strength tensor.

$$\begin{aligned}
&[\mathbf{D}_\mu, \mathbf{D}_\nu] \mathbf{\Psi} = \left[\left(\partial_\mu + iq\mathbf{A}_\mu\right)\left(\partial_\nu + iq\mathbf{A}_\nu\right) - \left(\partial_\nu + iq\mathbf{A}_\nu\right)\left(\partial_\mu + iq\mathbf{A}_\mu\right)\right] \mathbf{\Psi} \\
&= \partial_\mu \partial \nu \mathbf{\Psi} + iq(\partial_\mu \mathbf{A}_\nu)\Psi + iq\mathbf{A}_\nu \partial_\mu \mathbf{\Psi} + iq\mathbf{A}_\mu \partial_\nu \mathbf{\Psi} - q^2 \mathbf{A}_\mu \mathbf{A}_\nu \mathbf{\Psi} - [\mu \longleftrightarrow \nu] \\
&iq \left[\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + iq \left[\mathbf{A}_\mu, \mathbf{A}_\nu\right]\right]
\end{aligned} \tag{92}$$

So we define the field strength tensor in non-Abelian gauge theories as:

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + iq \left[\mathbf{A}_\mu, \mathbf{A}_\nu\right] \tag{93}$$

or in component form:

$$\begin{aligned}
&F^\alpha_{\mu\nu} T^a = \partial_\mu A^\alpha_\nu T^\alpha - \partial_\nu A^\alpha_\mu T^\alpha + iq \left[A^b_\mu T^b, A^c_\nu T^c\right] \\
&F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu - qf^{abc} A^b_\mu A^c_\nu
\end{aligned} \tag{94}$$

As a result the gauge part of the Lagrangian is written as:

$$L_{\text{gauge}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \tag{95}$$

---

[15] Inspired from the Abelian case, of course.
[16] For ease of calculations, we suppose $\mathbf{\Psi}$ is in the fundamental representation.

or using the properties of the generators (87) :

$$L_{\text{gauge}} = -\frac{1}{2}\,\text{Tr}\left(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right) \tag{96}$$

We take a moment now to prove that this term is, of course, gauge invariant. We start from the form (96) and use (92) so that upon transformation (89) we get:

$$\begin{aligned}
L_{\text{gauge}} &\to +\frac{1}{2q^2}\,\text{Tr}\left[\mathbf{D}'_\mu,\mathbf{D}'_\nu\right]\left[\mathbf{D}^{\nu\prime},\mathbf{D}^{\mu\prime}\right] \\
&= +\frac{1}{2q^2}\,\text{Tr}\left[\mathbf{U}\mathbf{D}_\mu\mathbf{U}^{-1},\mathbf{U}\mathbf{D}_\nu\mathbf{U}^{-1}\right]\left[\mathbf{U}\mathbf{D}^\mu\mathbf{U}^{-1},\mathbf{U}\mathbf{D}^\nu\mathbf{U}^{-1}\right] \\
&= +\frac{1}{2q^2}\,\text{Tr}\left[\mathbf{U}\left[\mathbf{D}_\mu,\mathbf{D}_\nu\right]\mathbf{U}^{-1}\mathbf{U}\left[\mathbf{D}^\mu,\mathbf{D}^\nu\right]\mathbf{U}^{-1}\right] \\
&= \frac{1}{2q^2}\,\text{Tr}\left[\mathbf{U}\left[\mathbf{D}_\mu,\mathbf{D}_\nu\right]\left[\mathbf{D}^\mu,\mathbf{D}^\nu\right]\mathbf{U}^{-1}\right] = -\frac{1}{2}\,\text{Tr}\left[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right]
\end{aligned} \tag{97}$$

where in the last step we used the cyclic property of the Tr trace operator.

Last but not least, it is useful to see how the covariant derivative acts on a test function on the *adjoint* representation. In fact we will prove that if $\mathbf{\Psi} = \Psi^a T^a$ then

$$\mathbf{D}_\mu\mathbf{\Psi} = \partial_\mu\mathbf{\Psi} + iq\left[\mathbf{A}_\mu,\mathbf{\Psi}\right] \tag{98}$$

or in component form

$$D_\mu\Psi^a = \partial_\mu\Psi^a - qf^{abc}A^b_\mu\Psi^c \tag{99}$$

Our motivation will be that the result of the covariant derivative when acted upon a test function in adjoint representation will have to retain the transformation properties of the adjoint representation.

i.e. If $\mathbf{\Psi}' = \mathbf{U}\mathbf{\Psi}\mathbf{U}^{-1}$ then $(\mathbf{D}_\mu\mathbf{\Psi})' = \mathbf{U}\mathbf{D}_\mu\mathbf{\Psi}\mathbf{U}^{-1}$. We will now explore the various transformation properties of RHS of (98) and prove it is the right expression to satisfy the aforementioned relationship.

For the first term:

$$\begin{aligned}
\partial_\mu\mathbf{\Psi} &\to \partial_\mu\left(\mathbf{U}\mathbf{\Psi}\mathbf{U}^{-1}\right) = \left(\partial_\mu\mathbf{U}\right)\mathbf{\Psi}\mathbf{U}^{-1} + \mathbf{U}\left(\partial_\mu\mathbf{\Psi}\right)\mathbf{U}^{-1} + \mathbf{U}\Phi\left(\partial_\mu\mathbf{U}^{-1}\right) \\
&= \mathbf{U}\left(\partial_\mu\mathbf{\Psi}\right)\mathbf{U}^{-1} + \left(\partial_\mu\mathbf{U}\right)\mathbf{U}^{-1}\left(\mathbf{U}\mathbf{\Psi}\mathbf{U}^{-1}\right) + \left(\mathbf{U}\mathbf{\Psi}\mathbf{U}^{-1}\right)\mathbf{U}\left(\partial_\mu\mathbf{U}^{-1}\right) \\
&= \mathbf{U}\left(\partial_\mu\mathbf{\Psi}\right)\mathbf{U}^{-1} + \left(\partial_\mu\mathbf{U}\right)\mathbf{U}^{-1}\left(\mathbf{U}\mathbf{\Psi}\mathbf{U}^{-1}\right) - \left(\mathbf{U}\mathbf{\Psi}\mathbf{U}^{-1}\right)\left(\partial_\mu\mathbf{U}\right)\mathbf{U}^{-1} \\
&= \mathbf{U}\partial_\mu\mathbf{\Psi}\mathbf{U}^{-1} + \left[\left(\partial_\mu\mathbf{U}\right)\mathbf{U}^{-1},\mathbf{U}\mathbf{\Psi}\mathbf{U}^{-1}\right]
\end{aligned} \tag{100}$$

where we used

$$\mathbf{U}\partial_\mu\mathbf{U}^{-1} = \partial_\mu(\mathbf{U}\mathbf{U}^{-1}) - (\partial_\mu\mathbf{U})\mathbf{U}^{-1} = -(\partial_\mu\mathbf{U})\mathbf{U}^{-1} \tag{101}$$

The second term yields:

$$iq\left[\mathbf{A}_\mu,\mathbf{\Psi}\right] \to iq\left[\mathbf{U}\mathbf{A}_\mu\mathbf{U}^{-1},\mathbf{U}\mathbf{\Psi}\mathbf{U}^{-1}\right] - \left[(\partial_\mu)\mathbf{U}^\dagger,\mathbf{U}\mathbf{\Psi}\mathbf{U}^\dagger\right] \tag{102}$$

where we used (90).

We note now that by adding this to terms we get some cancellations, explicitely:

$$\mathbf{D}_\mu\mathbf{\Psi} = \mathbf{D}_\mu\mathbf{\Psi} + iq\left[\mathbf{A}_\mu,\mathbf{\Psi}\right] = \mathbf{U}\left(\mathbf{D}_\mu\mathbf{\Psi}\right)\mathbf{U}^{-1} + iq\mathbf{U}\left[\mathbf{A}_\mu,\mathbf{\Psi}\right]\mathbf{U}^{-1} = \mathbf{U}\left(\mathbf{D}_\mu\mathbf{\Psi}\right)\mathbf{U}^{-1} \tag{103}$$

Thereby, validating (98).

VI. SPONTANEOUS SYMMETRY BREAKING

A. A $\mathbb{Z}_2$ symmetric Lagrangian

All these seemingly discrete independent topics from homotopy groups to gauge field theories really set the stage for the t'Hooft-Polyakov Monopole, which we are going to present later. For now, there is one piece left to be put into the puzzle and that is the discussion of *Spontaneous Symmetry Breaking* (or SSB for short) and the Higgs Mechanism, which we move forward to introduce following Blundell and Lancaster [19].

We consider a real scalar field Lagrangian density.

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - U(\phi) \tag{104}$$

where $U(\phi)$ is the potential energy density.

Now we consider that the potential is given by:

$$U(\phi) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{105}$$

The potential has a manifest symmetry $\phi(x) \to -\phi(x)$. Assuming μ^2 is positive, the potential has a minimum at $\phi = 0$, which corresponds to the vacuum state.
That is, because its Hamiltonian obtained by the Legendre transformation on (104)

$$H = \frac{1}{2}\left(\partial_0\phi\right)\left(\partial_0\phi\right) + \frac{1}{2}\left(\partial_i\phi\right)\left(\partial_i\phi\right) + U(\phi) \tag{106}$$

is minimised only if the field is constant on a minimum of the potential[17] U. The excitations of such field with $\mu^2 > 0$ are massive *phions* with mass $m = \mu$.

We, now, turn to a very interesting possibility. What if the sign of the μ^2 term in the potential was swapped? Then the minimization of the potential

$$U(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{107}$$

would yield:

$$\begin{aligned} \frac{\partial U}{\partial \phi} &= 0 \\ 0 &= -\mu^2\phi + \frac{\lambda}{3!}\phi^3 \end{aligned} \tag{108}$$

The solutions to (108) are of course $(0, \pm\sqrt{\frac{6\mu^2}{\lambda}})$. A simple check of the sign of the second derivative for these points:

$$\frac{\partial^2 U}{\partial \phi^2} = -\mu^2 + \frac{\lambda\phi^2}{2} \tag{109}$$

yields $-\mu^2$ for $\phi_0 = 0$ and $+2\mu^2$ for $\phi_0 = \pm\sqrt{\frac{6\mu^2}{\lambda}}$. So we end up with two new minima of the potential. The system will spontaneously choose one and the symmetry $\phi \to -\phi$ is spontaneously broken.

[17] Because its kinetic terms are positive definite

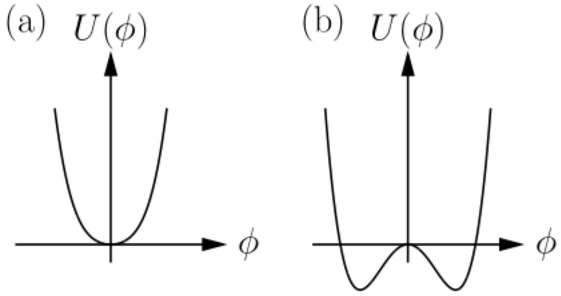

FIG. 6: (a)$U(\phi)$ for $\mu^2 > 0$, (b)$U(\phi)$ for $\mu^2 < 0$ exhibiting minima at $\pm\sqrt{6\mu^2/\lambda}$

To find the new vacuum we Taylor-expand the potential (let's say around $+\phi_0$):

$$\begin{aligned} U(\phi-\phi_0) &= U(\phi_0) + \left(\frac{\partial U}{\partial \phi}\right)_{\phi_0} (\phi-\phi_0) + \frac{1}{2!}\left(\frac{\partial^2 U}{\partial \phi^2}\right)_{\phi_0} (\phi-\phi_0)^2 + \dots \\ &= U(\phi_0) + \mu^2 (\phi-\phi_0)^2 + \dots, \end{aligned} \tag{110}$$

So we can write the Lagrangian in terms of $\phi' = \phi - \phi_0$:

$$L = \frac{1}{2}(\partial\phi')^2 - \mu^2\phi'^2 + O(\phi'^3) \tag{111}$$

We conclude this example by noticing that the breaking of symmetry is a property of the ground state of the system acquiring a non-trivial *vacuum expectation value*[18] (VEV from now on) and that the excitation ϕ' is characterised by mass $m' = \sqrt{2}\mu$.

B. A $SO(2)$ symmetric Lagrangian

Let us now consider the effects of spontaneous symmetry breaking on a Lagrangian with a continuous global symmetry.

$$L = \frac{1}{2}\left[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right] + \frac{\mu^2}{2}\left(\phi_1^2+\phi_2^2\right) - \frac{\lambda}{4!}\left(\phi_1^2+\phi_2^2\right)^2 \tag{112}$$

The Lagrangian (112) exhibits, of course, an internal global $SO(2)$ symmetry of spacetime independent rotations around the $\phi_1 - \phi_2$ plane. The minima of the potential this time are located on a circle on the $\phi_1 - \phi_2$ plane with radius[19] $\frac{6\mu^2}{\lambda}$. Let's break the symmetry by choosing a particular vacuum. The easiest for our calculations is $(\phi_1, \phi_2) = \left(+\sqrt{\frac{6\mu^2}{\lambda}}, 0\right)$ but any other point in the circle would be equivalent. Now expanding with $\phi_1' = \phi_1 - \sqrt{\frac{6\mu^2}{\lambda}}$ and $\phi_2' = \phi_2$ yields:

$$L = \frac{1}{2}\left[(\partial\phi_1')^2 + (\partial\phi_2')^2\right] - \mu^2(\phi_1')^2 + O\left(\phi'^3\right) \tag{113}$$

because $\partial^2 U/\partial\phi_1^2 = 2\mu^2$ and $\partial^2 U/\partial\phi_2^2 = 0$. We end up with the usual $m = \sqrt{2}\mu$ mass for the field ϕ_1 but it seems ϕ_2 is completely massless! This can be explained since excitations in the ϕ_1 direction cost energy, while an infinitesimal displacement in ϕ_2 directions is free of charge, which perfectly suits the explanation of a massless excitation.

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[18] In this case $\phi_0 = (\frac{6\mu^2}{\lambda})^{1/2}$
[19] To verify set $x = \phi_1^2 + \phi_2^2$ and differentiate with respect to x

Perhaps more illuminating is the case of a complex scalar field with internal $U(1)$ symmetry[20]. The Lagrangian of the theory is:

$$L = (\partial^\mu \psi)^\dagger (\partial_\mu \psi) + \mu^2 \psi^\dagger \psi - \lambda \left(\psi^\dagger \psi\right)^2 \tag{114}$$

which is of course invariant to transformations $\psi \to \psi \mathrm{e}^{\mathrm{i}\alpha}$.

It convenient to express the ψ field in a "polar" form $\psi(x) = \varrho(x)\mathrm{e}^{\mathrm{i}\theta(x)}$. Substituting into (114) we end up with:

$$L = (\partial_\mu \varrho)^2 + \varrho^2 (\partial_\mu \theta)^2 + \mu^2 \varrho^2 - \lambda \varrho^4 \tag{115}$$

And now the equivalent $U(1)$ transformation is $\varrho \to \varrho$ and $\theta \to \theta + \alpha$. Let us break the symmetry by choosing a preferred vacuum state (arbitrarily, but with ease of calculations eluding in mind). The set of minima is on the complex circle of radius $\varrho = \sqrt{\frac{\mu^2}{2\lambda}}$ and arbitrary θ. We proceed to choose $\varrho_0 = \sqrt{\frac{\mu^2}{2\lambda}}$ and $\theta_0 = 0$ and express our new fields as $\varrho' = \varrho - \varrho_0$ and $\theta' = \theta - \theta_0$. Warning! Serious algebraic massacre to follow:

$$\begin{aligned}
L &= (\partial_\mu \varrho)^2 + \varrho^2 (\partial_\mu \theta)^2 + \mu^2 \varrho^2 - \lambda \varrho^4 \\
&= (\partial_\mu \varrho')^2 + \left(\varrho' + \sqrt{\frac{\mu^2}{2\lambda}}\right)^2 (\partial_\mu \theta')^2 + \mu^2 \left(\varrho' + \sqrt{\frac{\mu^2}{2\lambda}}\right)^2 \\
&= (\partial_\mu \varrho')^2 + \left[\varrho'^2 + 2\left(\frac{\mu^2}{2\lambda}\right)\varrho'\right](\partial_\mu \theta')^2 + \frac{\mu^2}{2\lambda}(\partial_\mu \theta')^2 \\
&+ \mu^2 \varrho'^2 + \frac{\mu^4}{2\lambda} + 2\frac{\mu^3}{\sqrt{2\lambda}}\varrho' - \lambda\left(\varrho' + \sqrt{\frac{\mu^2}{2\lambda}}\right)^4 \\
&= (\partial_\mu \varrho')^2 + \left[\varrho'^2 + 2\left(\frac{\mu^2}{2\lambda}\right)\varrho'\right](\partial_\mu \theta')^2 + \frac{\mu^2}{2\lambda}(\partial_\mu \theta')^2 \\
&+ \mu^2 \varrho'^2 + \frac{\mu^4}{2\lambda} + 2\frac{\mu^3}{\sqrt{2\lambda}}\varrho' \\
&- \lambda \varrho'^4 - \lambda \frac{\mu^4}{4\lambda^2} - 4\lambda \varrho' \frac{\mu^2}{2\lambda}\sqrt{\frac{\mu^2}{2\lambda}} - 4\lambda \varrho'^3 \sqrt{\frac{\mu^2}{2\lambda}} \\
&- 6\lambda \varrho'^2 \frac{\mu^2}{2\lambda}
\end{aligned} \tag{116}$$

We notice that terms in the first power of ϱ' vanish and that θ' field is massless while ϱ' is massive with $m = \sqrt{2}\mu$. The Lagrangian (116) after some cleaning and tidying[21] can be cast into

$$\begin{aligned}
L = & \left(\tfrac{\mu^2}{2\lambda}\right)(\partial_\mu \theta')^2 && (\theta'\text{-field terms }) \\
& + (\partial_\mu \varrho')^2 - 2\mu^2 \varrho'^2 - 4\left(\tfrac{\mu^2 \lambda}{2}\right)^{\frac{1}{2}} \varrho'^3 - \lambda \varrho'^4 && (\varrho'\text{-field terms }) \\
& + \left[\varrho'^2 + \left(\tfrac{2\mu^2}{\lambda}\right)^{\frac{1}{2}} \varrho'\right](\partial_\mu \theta')^2 + \ldots && (\text{ interaction terms })
\end{aligned} \tag{117}$$

The occurance of a massless excitation θ is a manifestation of the *Goldstone theorem* which says: For every generator of the continuous symmetry group that undergoes spontaneous symmetry breaking there exists a massless excitation in the spectrum called *Goldstone boson.*

[20] Of course $U(1)$ is isomorphic to $SO(2)$, so the physics will remain the same.
[21] Ignoring constant terms, interactions and grouping same field terms.

C. Gauge Theory Symmetry Breaking - Higgs Mechanism

An extraordinary feature emerges when breaking a symmetry in a gauge theory and to showcase this we limit ourselves to the simplest example: the gauged complex scalar field theory.

The theory's Lagrangian is:

$$L = \left(\partial^\mu \psi^\dagger - \mathrm{i}qA^\mu \psi^\dagger\right)\left(\partial_\mu \psi + \mathrm{i}qA_\mu \psi\right) + \mu^2 \psi^\dagger \psi - \lambda\left(\psi^\dagger \psi\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{118}$$

which is symmetric under local $U(1)$ transformations via $\psi \to \psi e^{\mathrm{i}\alpha(x)}$ as long as we transform also $A_\mu \to A_\mu - \frac{1}{q}\partial_\mu \alpha(x)$. We will explore what happens when we break symmetry. Working in polars, the ground state take a unique phase angle $\theta(x) = \theta_0$ for all x. We are no longer permitted to change the phase of the ground state at different values of x (local symmetry), neither the phase of the ground state for the entire system (global symmetry). Now we are to determine the excitations. The Lagrangian is gauge invariant, so we will be able to perform gauge transformations to simplify the physics if needed.

The lagrangian (118) is much like (115). The most serious change is the inclusion of the covariant derivative. Writing the field ψ in polars as before, we get:

$$\partial_\mu \psi + \mathrm{i}qA_\mu \psi = \left(\partial_\mu \varrho\right) e^{\mathrm{i}\theta} + \mathrm{i}\left(\partial_\mu \theta + qA_\mu\right)\varrho e^{\mathrm{i}\theta} \tag{119}$$

Notice now that A_μ appears as:

$$A_\mu + \frac{1}{q}\partial_\mu \theta \equiv C_\mu \tag{120}$$

So it is better to start working with C_μ, to simplify some expressions for example the first term in the Lagrangian (118) becomes:

$$\begin{aligned}
&\left(\partial^\mu \psi^\dagger - iqA^\mu \psi\right)\left(\partial_\mu \psi + iqA_\mu\right) = \\
&\left[\left(\partial^\mu_\varrho \varrho e^{-i\theta} - iC^\mu e^{-i\theta}\right]\left[\left(\partial_\mu \varrho\right) e^{i\theta} + iC_\mu \varrho e^{i\theta}\right] = \\
&\partial^\mu \varrho \partial_\mu \varrho + \varrho^2 C^\mu C_\mu
\end{aligned} \tag{121}$$

Of course the replacement (120) is a gauge invariant one since the field strength tensor remains the same.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu C_\nu - \partial_\nu C_\mu \tag{122}$$

Lagrangian (118) is now cast into[22]:

$$L = \left(\partial_\mu \varrho\right)^2 + \varrho^2 q^2 C^2 + \mu^2 \varrho^2 - \lambda \varrho^4 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{123}$$

Comparing with (115) we notice that field θ has vanished into thin air. Breaking the symmetry, we consider again the minimum on $\varrho_0 = \sqrt{\frac{\mu^2}{2\lambda}}$ and $\theta_0 = 0$. Now to reveal the excitations above the ground state we expand in terms of $\frac{\chi}{\sqrt{2}} = \varrho - \varrho_0$ and we obtain the same way as before:

$$\begin{aligned}
\mathcal{L} =& \frac{1}{2}\left(\partial_\mu \chi\right)^2 - \mu^2 \chi^2 - \sqrt{\lambda}\mu \chi^3 - \frac{\lambda}{4}\chi^4 \\
&- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{M^2}{2}C^2 \\
&+ q^2\left(\frac{\mu^2}{\lambda}\right)^{\frac{1}{2}} \chi C^2 + \frac{1}{2}q^2 \chi^2 C^2 + \ldots,
\end{aligned} \tag{124}$$

---

[22] Note we have not yet broken the symmetry.

where $M = q\sqrt{\frac{\mu^2}{\lambda}}$ and we ignored constants and interaction terms.
The massive scalar excitation ϱ is called a *Higgs Boson*.

The big surprise here apart from the complete disappearance of the θ field is that the theory now contains a massive vector field C_μ. It is as the Goldstone boson has been eaten by the gauge field and grown massive.

It is important to note however that the degrees of freedom have remained the same:

$$\begin{pmatrix} 2\times \text{ massive scalar particles} \\ 2\times \text{ massless photon particles} \end{pmatrix} \to \begin{pmatrix} 1\times \text{ massive scalar particles} \\ 3\times \text{ massive vector particles} \end{pmatrix}$$

since the mass term breaks the gauge invariance (which reduces the d.o.f. by two) but the equations of motion from a Proca Lagrangian[23] introduce one constaint[24] leaving us with the three vector particles.

The disappearance of the θ Goldstone boson is attributed to the substitution (126) which is in fact a gauge transformation. The Goldstone boson entered the Lagrangian in a way that with a suitable gauge transformation could be removed. It must have been then, what we call a *pure gauge*.

---

[23] That is the Lagrangian of a massive vector field.

[24] From $L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{M^2}{2}C^\mu C\mu$ Euler-Lagrange equations yield $\partial_\mu F^{\mu\nu} + M^2 C^\nu = 0$. Taking the partial derivative ∂_ν of the above yields the non-trivial constaint $\partial_\mu C^\mu = 0$, which is a forced Lorentz-gauge condition.

VII. TOPOLOGICAL CONSIDERATIONS II: SOLITONS

In conventional *Quantum Field Theory*, we expand around the classical field vacuum assuming the fields are independent of space and time. While this is true for the majority of the effects we consider, there are some excited states which stem from the non-trivial topology of the unperturbed state. Such effects turn out to be proportional to the coupling constant in some negative power and thus no perturbative technique would ever reveal such things.

A crucial aspect of all these solutions is going to be the manifold of vacua for the theory. If there is no non-linear term, there is nothing non-trivial and therefore we have to take the interaction term seriously and add it to the action.

Consider the Φ^4 theory in $(1+1)$ dimensions:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{4}\lambda \Phi^4 \tag{125}$$

The equations of motion are:

$$\Box \Phi + \lambda \Phi^3 = 0 \tag{126}$$

where $\Box = \partial_\mu \partial^\mu$.

Our goal in the search of topological excitations is finding solutions to the equations of motion with finite energy. Let's check the energy functional.

The energy momentum tensor is:

$$\begin{aligned} T^\mu_\nu &= \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)}\left(\partial_\nu \Phi\right) - g^{\mu\nu}\mathcal{L} \\ &= \partial^\mu \Phi \partial_\nu \Phi - \frac{1}{2}\left(\partial_\mu \Phi\right)^2 \delta^\mu_\nu + \frac{\lambda}{4}\Phi^4 \delta^\mu_\nu \end{aligned} \tag{127}$$

The (0,0)-component gives the energy density, so the energy functional is

$$E[\Phi] = \int_{-\infty}^{\infty} dx \left(\frac{1}{2}\left(\partial_1 \Phi\right)^2 + \frac{1}{2}\left(\partial_0 \Phi\right)^2 + \frac{\lambda}{4}\Phi^4\right) \tag{128}$$

The energy is clearly positive definite. The energy is zero only if $\Phi = 0$, which corresponds to our classical vacuum. In this case, we have only one classical vacuum and while we can find other solution apart from $\Phi = 0$, they are all deformable to the trivial one. If we want to find topological excitations in our theory, we have to change our strategy.

A. The Kink

This time we consider the Lagrangian $(1+1)$:

$$\mathcal{L} = \frac{1}{2}\partial^\mu \Phi \partial_\mu \Phi - V(\Phi) \tag{129}$$

with

$$V(\Phi) = \frac{1}{4}\lambda\left(\Phi^2 - u^2\right)^2 \tag{130}$$

where we recognize our symmetry breaking potential from equation (107) with a constant to lift up the minima of the potential to coincide with 0 and some different coefficients which do not change the physics of course.

This theory has two vacua which we can diagrammatically see from FIG.6. These correspond to the classical field configurations $\Phi = u$ and $\Phi = -u$. After expanding around one of the minima, we easily find that the particle's mass is $m = (2\lambda)^{1/2}\, u$. However, the whole point of this chapter is to illustrate that other objects can live in the potential other than particles. These are stationary configurations[25] of the field whose energy density goes to 0 at the spatial infinities but does something non-trivial in between.

In $(1+1)$ dimensions, Φ and u are dimensionless and λ has dimensions of mass squared. In the weak coupling regime of the quantum theory, we have $\lambda \ll m^2$.

The case of one spatial dimension is interesting because the boundary consists of two points $x = -\infty$ and $x = +\infty$. This topology of the spatial boundary is mirrored by the topology of the space of vacuum field configurations, which also consists of two points $\Phi = -v$ and $\Phi = v$. In each vacuum, *both* spatial boundary points are mapped onto the *same* field value (either $-v$ or $+v$). This we recognize as a *trivial map*. More interesting for our searches is the identity map, where $x = -\infty$ is mapped to $\Phi = -u$ and $x = +\infty$ to $\Phi = +u$. The field must smoothly interpolate between $\Phi = -u$ at $x = -\infty$ and $\Phi = u$ at $x = \infty$ and this requires energy. We might say that the fields at $\pm\infty$ live in different vacua. If we are lucky enough and such configuration costs a finite amount of energy, then we have ourselves a topological excitation.

To introduce some notation, we write down our boundary conditions

$$\lim_{x\to\pm\infty} \Phi(x) = \pm u \tag{131}$$

The energy functional can be extracted from (128) substituting the Φ^4 term with our new potential (130).

$$E = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}\dot{\Phi}^2 + \frac{1}{2}\Phi'^2 + V(\Phi)\right] \tag{132}$$

where primes denote a spatial differentiation and dots a temporal one.

We rewrite now the energy functional (132) with a malicious trick

$$E = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}\left(\Phi' - \sqrt{2V(\Phi)}\right)^2 + \sqrt{2V(\Phi)}\Phi'\right] \tag{133}$$

where we ignored time derivatives, since we seek time independent solutions.

The calculation can be carried out easily up to:

$$\begin{aligned} E &= \int_{-\infty}^{+\infty} dx \frac{1}{2}\left(\Phi' - \sqrt{2V(\Phi)}\right)^2 + \int_{-v}^{+v} \sqrt{2V(\Phi)} d\Phi \\ &= \int_{-\infty}^{+\infty} dx \frac{1}{2}\left(\Phi' - \sqrt{2V(\Phi)}\right)^2 + \frac{2}{3}\left(m^2/\lambda\right) m \end{aligned} \tag{134}$$

The minimum possible energy is evidently

$$M = \frac{2}{3}\left(m^2/\lambda\right) m \tag{135}$$

which is much heavier than the particle excitation of the weakly coupled theory. Notice, also, its mass is inverse proportional to the coupling constant verifying that perturbation theory would fail to reveal such solutions.

We can require the first term to vanish in the energy functional and find the minimum energy solution by solving:

$$\Phi' = \sqrt{2V(\Phi)} \tag{136}$$

This is rather trivial but we should point out that this equation is compatible with the equations of motion for the field which can be seen by differentiation:

[25] That is $\partial_0 \Phi = 0$

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\partial \Phi}{\partial x}\right)^2 = V \\
\Rightarrow &\frac{1}{2}2\frac{\partial \Phi}{\partial x}\left(\frac{\partial x}{\partial \Phi}\frac{\partial}{\partial x}\right)\frac{\partial \Phi}{\partial x} = \frac{\partial V}{\partial \Phi} \\
\Rightarrow &\frac{\partial \Phi}{\partial x}\frac{\partial x}{\partial \Phi} \quad \frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial V}{\partial \Phi} \\
\Rightarrow &\frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial V}{\partial \Phi}
\end{aligned} \tag{137}$$

We proceed in solving (136):

$$\begin{aligned}
&\frac{d\Phi}{\sqrt{2V(\Phi)}} = dx \\
\Rightarrow &\sqrt{\frac{2}{\lambda}}\frac{d\Phi}{u^2-\Phi^2} = dx \\
\Rightarrow &\frac{d(\Phi/u)}{1-\left(\frac{\Phi}{u}\right)^2} = \sqrt{\frac{\lambda}{2}}udx \\
\Rightarrow &\int \frac{dY}{1-Y^2} = \frac{u\,(x-x_0)}{2} \\
\Rightarrow &\frac{dY}{2}\left(\frac{1}{1-Y}+\frac{1}{1+Y}\right) = \frac{m\,(x-x_0)}{2} \\
\Rightarrow &\ln\left(\frac{1+Y}{|1-Y|}\right) = m\,(x-x_0)
\end{aligned} \tag{138}$$

Before we start solving for Y, we point that we remove the absolute value leaving the signs unchanged. The rigorous method would be to solve for each case of the sign separately. Doing so would yield in the case of changing the signs of the absolute value that the solution has a singularity which we ultimately want to avoid for the sake of finite energy.

$$\begin{aligned}
\Rightarrow \quad &\frac{1+Y}{1-Y} = e^{m(x-x_0)} \Rightarrow 1+Y = e^{m(x-x_0)}(1-Y) \\
\Rightarrow \quad &Y\left(1+e^{u(x-x_0)}\right) = e^{m(x-x_0)}-1 \\
\Rightarrow \quad &\Phi = u\frac{e^{u(x-x_0)}-1}{e^{u(x-x_0)}+1} = u\tanh\left(\frac{m}{2}\,(x-x_0)\right).
\end{aligned} \tag{139}$$

where x_0 is just a constant of integration and acts as a point where the energy is localised around.

This solution is what we call a *soliton*. It might seem to stretch a little too much in the field space to exhibit particle properties, but nobody actually cares about the localization of the field configuration. It is the energy configuration that matters in that aspect.

The energy density is given again by:

$$\begin{aligned}
\mathcal{E}(x) &= \frac{1}{2}\phi'^2 + U(\phi) \\
&= \frac{m^2u^2}{8\cosh^4\left(\frac{m}{2}\,(x-x_0)\right)} + \frac{\lambda}{4}\left(u^2-u^2\tanh^2\left(\frac{m}{2}\,(x-x_0)\right)\right)^2 \\
&= \frac{m^2u^2}{8\cosh^4\left(\frac{m}{2}\,(x-x_0)\right)} + \frac{\lambda u^4}{4}\left(1-\tanh^2\left(\frac{m}{2}\,(x-x_0)\right)^2\right) \\
&= \frac{2\lambda u^4}{8\cosh^4\left(\frac{m}{2}\,(x-x_0)\right)} + \frac{\lambda u^4}{4}\cosh^{-4}\left(\frac{m}{2}\,(x-x_0)\right)^2 \\
&= \frac{\lambda u^4}{2}\cosh^{-4}\left(\frac{m}{2}\,(x-x_0)\right)
\end{aligned} \tag{140}$$

To get the solution for all times can boost this solution,

$$x - x_0 \to \gamma \left(x - x_0 - vt\right),$$

where $\gamma \equiv \left(1 - v^2\right)^{-1/2}$

Boosting the field

$$\Phi \to \pm u \tanh \left[\frac{\gamma m}{2}\left(x - x_0 - vt\right)\right]$$

with an energy density

$$\mathcal{E} = \frac{\gamma \lambda u^2}{2} \operatorname{sech}^4 \left[\frac{\gamma m}{2}\left(x - x_0 - vt\right)\right]$$

Now for a quick check that everything is okay, we will be integrating the boosted energy density over space expecting to return γ times (135).

$$E = \int_{-\infty}^{+\infty} \mathcal{E} dx = \int_{-\infty}^{+\infty} \gamma \frac{\lambda u^4}{2} \operatorname{sech}^4 \left(\frac{m}{2}\left(x - x_0\right)\right) dx$$

Substituting $y = \frac{m}{2}\left(x - x_0\right)$

$$= \gamma \frac{\lambda u^4}{2} \frac{2}{m} \int_{-\infty}^{+\infty} \operatorname{sech}^4 y dy = \gamma \frac{\lambda u^4}{m} \int_{-\infty}^{+\infty} \operatorname{sech}^2 y \left(1 - \tanh^2 y\right) dy$$

We are also prepared for substitution $\tanh y = w$

$$= \gamma \frac{\lambda u^4}{m} \int_{-1}^{+1} \left(1 - w^2\right) dw = \gamma \frac{\lambda u^4}{m} \frac{4}{3} = \gamma \frac{2}{3} \frac{m^2}{\lambda} m \tag{141}$$

Exactly as expected!

The kink is of finite energy, can be boosted to an arbitrary velocity and transferred to an arbitrary point. It, thus, behaves very much like a particle. In addition, the kink is stable: any attempt to remove the kink involves lifting a (semi)infinite length of field from one potential minima to another and that would cost an infinite amount of energy. More mathematically, we say that the kink is not *deformable*. One way to remove the kink, would be via annihilation with an antikink (that corresponds to minus the solution we found previously for the kink).

We say two solutions are topologically equivalent if there exists a continuous transformation from one to another without passing through a barrier of infinite action.

For two solutions, $f_1(x)$ and $f_2(x)$, a continuous deformation, parameterized by $w \in [0,1]$, between the two solutions is a continuous function, $F(w,x)$, such that $F(0,x) = f_1(x)$ and $F(1,x) = f_2(x)$.

We will check if the kink and antikink solutions are topologically equivalent[26].

We introduce a generic transformation from one to the other

$$\Phi\left(w, x\right) = F\left(w\right) u \tanh \left(\frac{m}{2}\left(x - x_0\right)\right) \tag{142}$$

with $F(0) = -1$ and $F(1) = 1$.

[26] Of course, we expect they are not.

The action functional is given

$$\begin{aligned} S(w) &= \int dx \frac{1}{2}\Phi'^2 + \frac{\lambda}{4}\left(\Phi^2 - \frac{m^2}{\lambda}\right)^2 \\ &= \frac{m^4}{4\lambda}\int dx F^2 \frac{1}{\cosh^4 y} + \left((F^2 \tanh^2\left(\frac{m}{2}(x-x_0)\right) - 1\right)^2 \end{aligned} \tag{143}$$

While the first term is convergent, the second one is divergent if $F^2 \neq 1$. However, since $F(0) = -1$ and $F(1) = 1$ and F is continuous, there is at least one $w \in (0,1)$ with $F \neq 1$ and so we conclude that the kink and antikink solutions are topologically inequivalent.

To distinguish between those two we endow them with conserved quantity we call *kink-charge.* A single kink should carry $Q_T = 1$ and a single antikink $Q_= - 1$. In order to come up with a sensible recipe for finding the kink, we define the *kink-current*[27]:

$$J^\mu_{\mathrm{T}} = \frac{1}{2u}\varepsilon^{\mu\nu}\partial_\nu \Phi, \tag{144}$$

Notice that this current is conserved by construction. The expression for the charge of the kink is via the usual trick

$$\begin{aligned} Q_{\mathrm{T}} &= \int_{-\infty}^{\infty} \mathrm{d}x J^0 = \frac{1}{2u}\int_{-\infty}^{\infty} \mathrm{d}x \frac{\partial \Phi}{\partial x} \\ &= \frac{1}{2u}[\Phi(\infty) - \Phi(-\infty)] = 1, \end{aligned} \tag{145}$$

whereas for the antikink we have $Q_T = -1$. We call the kink-charge a **topological charge**, explaining the subscript. Its existence is independent of the geometry of spacetime. That geometry is encoded in $g^{\mu\nu}$ which makes no appearance in our current. In contrast, our current has its indices summed via the antisymmetric symbol Levi-Civita $\varepsilon^{\mu\nu}$, which turns out to be a general feature of topological objects.

B. Derrick's theorem

This discussion will follow the lecture notes from [10]. Such theorem prohibits stable time-independent topological excitations for scalar fields depending on the spatial dimensionality of the manifold.

Consider the Lagrangian L of an N-component scalar field, $\phi(\mathbf{x}_i, t)$ in n spatial dimensions:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^i \partial^\mu \Phi^i - U\left(\Phi^i\right) \tag{146}$$

Applying the Euler-Lagrange equations, we acquire:

$$\begin{aligned} \partial_\mu \frac{\partial L}{\partial\left(\partial_\mu \Phi^i\right)} - \frac{\partial L}{\partial \Phi^i} &= 0 \\ \partial_\mu \partial^\mu \Phi^i &= \frac{\partial U}{\partial \Phi^i} \end{aligned} \tag{147}$$

The Hamiltonian of the system, considering time-independent solution is:

$$E = \underbrace{\int d^n x \frac{1}{2}\partial_j \Phi^i \partial_j \Phi^i}_{\equiv E_1[\Phi^i]} + \underbrace{\int d^n x U\left(\Phi^i\right)}_{\equiv E_2[\Phi^i]} \tag{148}$$

where the j summation implies only spatial indices.

[27] Note that this current is different to Noether currents since it owes nothing to the existence of a symmetry in the Lagrangian. We will meet a same kind of current later, when examining the t'Hooft-Polyakov monopole!

Applying a rescaling to the spatial components of the field, we get:

$$\Phi^i_\lambda(\mathbf{x}) \equiv \Phi^i_1(\lambda\mathbf{x}) \tag{149}$$

which are not in general solutions of (147).

The total energy for these configurations is:

$$\begin{aligned} E_{(\lambda)} &= \int d^n x \left(\frac{1}{2}\partial_j \Phi^i_1(\lambda\mathbf{x}) \partial_j \Phi^i_1(\lambda\mathbf{x}) + U\left[\Phi^i_1(\lambda\mathbf{x})\right] \right) \\ &= \int d^n y \lambda^{-n} \left(\frac{1}{2}\lambda^2 \frac{\partial}{\partial y^a}\Phi^i(\mathbf{y}) \frac{\partial}{\partial y^a}\Phi^i(\mathbf{y}) + U\left[\Phi_1(\mathbf{y})\right] \right) \\ &= \left(\lambda^{2-n} E_1 + \lambda^{-n} E_2\right) \end{aligned} \tag{150}$$

Now Derrick's argument is that (150) must exhibit an extremum at $\lambda = 1$

$$\frac{dE_{(\lambda)}}{d\lambda} = (2-n)\lambda^{1-n} E_1 - n\lambda^{-n-1} E_2 = 0 \Rightarrow E_2 = \frac{2-n}{n} E_1 \tag{151}$$

For $n = 1$, we find that $E_1 = E_2$, which is a virial theorem analogue.

For $n > 1$, we get $E_2 \leq 0$, which is certainly not physical and therefore we conclude that stable scalar topological excitation can be exhibited with only one spatial dimension.

It would certainly be really uninteresting, if we could not have topological excitations in more than one spatial dimensions. The problems lies in the divergent kinetic term as we will showcase next. We will also consider ways to dodge the limitations imposed from Derrick's theorem taking advantage that it only refers to scalar fields.

C. The Vortex

Having found a soliton that is localised in one spatial dimension, we turn our attention to $(2+1)$ spacetime. However, Derrick and his theorem guarantee us that not everything will go as smoothly as before. The spatial boundary in two dimensions has the topology of circle S^1. There is no nontrivial map from a circle to two points, since continuity of the map requires the whole circle to be mapped to one of the two points, so the real scalar field will not do the job this time. There do exist, though, nontrivial maps from a circle to a circle and they are in fact labeled by an integer representing the structure of the first homotopy group[28] of S^1 (70).

We consider a complex scalar field $\Phi(x)$:

$$\mathcal{L} = \partial^\mu \Phi^\dagger \partial_\mu \Phi - V(\Phi) \tag{152}$$

with our usual symmetry breaking potential

$$V(\Phi) = \frac{1}{4}\lambda \left(\Phi^\dagger \Phi - u^2\right)^2 \tag{153}$$

The vacuum field configuration at spatial infinity is:

$$\Phi = u e^{in\varphi} \tag{154}$$

where is the integer endowed by the first homotopy group of S^1 or as is commonly known, the *winding number*[29].

---

[28] For more, visit chapter IV B.
[29] More on that in chapter IX A.

To search for a finite energy solution with nonzero winding number, we start with the ansatz

$$\Phi\left(r,\varphi\right) = uh(r)e^{in\varphi} \tag{155}$$

where $h(r)$ is a real-valued function with $h(\infty) = 1$. We also have $h(0) = 0$ so that we avoid divergences of the gradient at $r = 0$.

$$\boldsymbol{\nabla}\Phi = u\left[h'(r)\hat{\mathbf{r}} + inr^{-1}h(r)\hat{\phi}\right]e^{in\varphi} \tag{156}$$

The kinetic energy is proportional to

$$\int d^2x|\boldsymbol{\nabla}\Phi|^2 \sim 2\pi n^2u^2\int^{\infty}\frac{dr}{r} \tag{157}$$

which unfortunately diverges logarithmically[30].

It's time to get a little sneaky though and outmaneuver Derrick's theorem. You see, Derrick's theorem involves only scalar fields, but nobody forbids us to introduce gauge ones too. Note that the Lagrangian (152) posseses a global $U\left(1\right)$ symmetry, so gauging it yields

$$\mathcal{L} = \left(D^{\mu}\Phi\right)^{\dagger}D_{\mu}\Phi - V(\Phi) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{158}$$

where

$$D_{\mu}\Phi = \partial_{\mu}\Phi - ieA_{\mu}\Phi \tag{159}$$

and $V(\Phi)$ still given by (153). The gauge symmetry is spontaneously broken and we end up with a massive scalar particle of $m_s = \sqrt{\lambda}u$ and a massive vector particle of $m_{\text{V}} = eu$ in complete analogy with chapter VI C.

The gradient energy density of the scalar field is now

$$|\mathbf{D}\Phi|^2 = |(\boldsymbol{\nabla} - ie\mathbf{A})\Phi|^2 \tag{160}$$

Here, we are presented with the chance to choose wisely the vector potential $\mathbf{A}$ so that we cancel the badly behaved term of the kinetic energy.

We know that for $\Phi = u$, there is no divergence, since it does not depend on the azimuthal angle φ. We should also have $A_{\mu} = 0$ in this vacuum. Now we notice that to go from $\Phi = u$ to $\Phi = ue^{in\varphi}$ we are no more than a gauge transformation away. We have to transform A_{μ}, however, via (90).
For $r \to \infty$,

$$\begin{aligned}\lim_{r\to\infty}\mathbf{A}(r,\phi) &= \frac{i}{e}U\boldsymbol{\nabla}U^{\dagger}\\ &= \frac{n}{er}\hat{\phi}\end{aligned} \tag{161}$$

Before the transformation, we had $D_{\mu}\Phi = 0$. This is also true in our new gauge (since 0 cannot be altered by a multiplicative transformation).

For $n \neq 0$, the gauge transformation $U = e^{in\varphi}$ is *large*. This means that it cannot be smoothly deformed to $U = 1$. This implies that we also cannot extend it from $r = \infty$ to the interior of space without meeting an *obstruction*, a point where $U(r,\varphi)$ is ill-defined[31]. Near the obstruction, the fields Φ and $\mathbf{A}$ must deviate from a gauge transformation of

[30] This is something we ultimately expected as it was predicted by Derrick's theorem from the previous chapter.
[31] In this case, $r = 0$ does exactly that job.

a vacuum. This deviation costs energy and results in a soliton [[32]].

We expand our previous ansantz (155) to include the vector potential $\mathbf{A}$ too.

$$\begin{aligned} \Phi(r,\varphi) &= uh(r)U(\varphi), \\ \mathbf{A}(r,\varphi) &= \frac{i}{e}a(r)U(\varphi)\boldsymbol{\nabla}U^{\dagger}(\varphi) \end{aligned} \tag{162}$$

where $U = e^{in\varphi}$ and we have the following boundary conditions:

$$\begin{aligned} h(0) = 0 \quad a(0) = 0 \\ h(\infty) = 1 \quad a(\infty) = 1 \end{aligned} \tag{163}$$

to approach a large gauge transformation at $r = \infty$ and to avoid the blowing up of the kinetic energy at $r = 0$. For $n = 1$, the soliton is called *Nielsen-Olesen vortex*[25].

The non-zero potential yields a perpendicular magnetic field

$$\begin{aligned} \mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \\ &= \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_{\varphi}) - \frac{\partial}{\partial \varphi}A_r\right)\hat{\mathbf{z}} \\ &= \frac{n}{e}\frac{a'(r)}{r}\hat{\mathbf{z}} \end{aligned} \tag{164}$$

which results in a magnetic flux

$$\begin{aligned} \Phi_M &= \int d\mathbf{S}\cdot\mathbf{B} \\ &= \lim_{r\to\infty}\int d\boldsymbol{\ell}\cdot\mathbf{A} \\ &= \frac{i}{e}\lim_{r\to\infty}a(r)\int_0^{2\pi} d\phi U\partial_{\varphi}U^{\dagger} \\ &= \frac{2\pi n}{e} \end{aligned} \tag{165}$$

where we used Stoke's theorem to transition from first to second line and next we used (161). We see that vortices carry quantized magnetic flux!

The energy of the soliton is

$$E = \int d^2x \left[|(\boldsymbol{\nabla} - ie\mathbf{A})\Phi|^2 + V(\Phi) + \frac{1}{2}\mathbf{B}^2\right] \tag{166}$$

Substituting our ansatz (162), we get

$$E = 2\pi u^2 \int_0^{\infty} drr\left[h'^2 + \frac{n^2}{r^2}(a-1)^2h^2 + \frac{1}{4}\lambda u^2\left(h^2-1\right)^2 + \frac{n^2}{e^2u^2r^2}a'^2\right] \tag{167}$$

We switch to a dimensionless radial variable $\xi \equiv eur = m_{\mathrm{V}}r$ and define $\beta^2 \equiv \lambda/e^2 = m_{\mathrm{S}}^2/m_{\mathrm{V}}^2$. The energy functional reads

$$E = 2\pi u^2 \int_0^{\infty} d\xi\xi\left[h'^2 + \frac{n^2}{\xi^2}(a-1)^2h^2 + \frac{1}{4}\beta^2\left(h^2-1\right)^2 + \frac{n^2}{\xi^2}a'^2\right] \tag{168}$$

Either by substituting ansatz (162) into the equations of motion, or by applying the variational principle on the energy functional, we get the equations that h, a satisfy.

Here, we follow the variational principle:

$$\begin{aligned}
&\frac{\partial \mathcal{E}}{\partial h} - \frac{d}{d\xi}\left(\frac{\partial \mathcal{E}}{\partial h'}\right) = 0 \\
&2\frac{n^2}{\xi}(a-1)^2 h + \frac{1}{2}\beta^2\left(h^2-1\right)2h - \frac{d}{d\xi}\left(2\xi h'\right) = 0 \\
&h'' + \frac{h'}{\xi} - \frac{n^2 h}{\xi^2}(1-a)^2 + \frac{1}{2}\beta^2\left(1-h^2\right)h = 0
\end{aligned} \tag{169}$$

and

$$\begin{aligned}
&\frac{\partial \mathcal{E}}{\partial a} - \frac{d}{d\xi}\left(\frac{\partial \mathcal{E}}{\partial a'}\right) = 0 \\
&\frac{n^2}{\xi}h^2 2(a-1) - \frac{d}{d\xi}\left(\frac{2n^2 a'}{\xi}\right) = 0 \\
&a'' - \frac{a'}{\xi} + (1-a)h^2 = 0
\end{aligned} \tag{170}$$

where $\mathcal{E}$ is the integrand in E.

Equations (169),(170) along with boundary conditions (163) do not have analytical solutions. However, we can determine their behaviour for small and large values of ξ.

For $\xi \ll 1$, a and h are small due to their boundary conditions and we can ignore them to second order. We are left with an equation for h:

$$h'' + h'/\xi - n^2 h/\xi^2 = 0 \tag{171}$$

Plugging in an ansatz $h \sim \xi^\nu$

$$\left(\nu^2 - n^2\right)\xi^{\nu-2} = 0 \tag{172}$$

So, we have $h \sim \xi^n$ for $\xi \ll 1$ (because $\nu = -n$ does not satisfy the boundary condition at $\xi = 0$). As for a in the small ξ regime

$$a'' - a'/\xi = 0 \tag{173}$$

Using the ansatz $a \sim \xi^\alpha$ and substituting

$$\left(\alpha^2 - 2\alpha\right)\xi^{\alpha-2} = 0 \tag{174}$$

we get $\alpha = 2$ (because $\alpha = 0$ does not satisfy the boundary condition) and thus $a \sim \xi^2$ for $\xi \ll 1$.

For $\xi \gg 1$, let $a = 1 - A$ and $h = 1 - H$, with A and H both $\ll 1$. Then, (170) becomes

$$-A'' + A = 0 \tag{175}$$

and the solution that vanishes at $\xi = \infty$ is $A \sim e^{-\xi}$.

As for H, we get

$$-H'' + \beta^2 F = 0 \tag{176}$$

The solution that vanishes at $\rho \to \infty$ is $H \sim e^{-\beta\xi}$.

However, if $\beta > 2$ then actually it is the third term in equation (169) that dominates at large ξ, since $(1-a)^2 = A^2 \sim e^{-2\xi}$ while the remaining terms go like $e^{-\beta\xi}$. Hence, for $\beta > 2$, we must have $H \sim e^{-2\xi}$ to achieve appropriate cancellations at large ξ.

The next step in line is to examine a topological object localised in $(3+1)$ spacetime. Remarkably, this object turns out to be a *magnetic monopole*!

VIII. THE T'HOOFT-POLYAKOV MONOPOLE

A. Georgi-Glashow Model

For this section we are going to present the $SU(2)$ Georgi-Glashow model.

We start with a Higgs triplet in adjoint representation

$$\mathbf{\Phi} = \Phi^a \mathbf{T}^a \tag{177}$$

where $\mathbf{T}^a = \frac{\sigma^a}{2}$ are the $SU(2)$ generators. They satisfy the Lie algebra:

$$\left[\mathbf{T}^a, \mathbf{T}^b\right] = i\varepsilon_{abc}\mathbf{T}^c \tag{178}$$

and the normalization condition

$$\mathrm{Tr}\left(\mathbf{T}^a\mathbf{T}^b\right) = \frac{1}{2}\delta_{ab} \tag{179}$$

Based on our previous analysis the Lagrangian with the corresponding local symmetry is

$$\begin{aligned} L &= -\frac{1}{2}\,\mathrm{Tr}\left(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right) + \mathrm{Tr}\left(\mathbf{D}_\mu\mathbf{\Phi}\mathbf{D}^\mu\mathbf{\Phi}\right) - V(\mathbf{\Phi}) \\ L &= -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{1}{2}\left(D_\mu\Phi^a\right)\left(D^\mu\Phi^a\right) - V(\Phi) \end{aligned} \tag{180}$$

The covariant derivative is

$$\mathbf{D}_\mu = \partial_\mu\mathbb{I} + ie\mathbf{A}_\mu \tag{181}$$

with $\mathbf{A}_\mu = A^a_\mu T^a$. It acts on the scalar field like

$$\begin{aligned} \mathbf{D}_\mu\mathbf{\Phi} &= \partial_\mu\mathbf{\Phi} + ie\left[\mathbf{A}_\mu, \mathbf{\Phi}\right] \\ D_\mu\Phi^a &= \partial_\mu\Phi^a - e\varepsilon_{abc}A^b_\mu\Phi^c \end{aligned} \tag{182}$$

The Higgs potential is

$$V(\Phi) = \frac{\lambda}{4}\left(\Phi^a\Phi^a - u^2\right)^2 \tag{183}$$

and the Field strength tensor is defined

$$\begin{aligned} \mathbf{F}_{\mu\nu} &= \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + ie\left[\mathbf{A}_\mu, \mathbf{A}_\nu\right] = \frac{1}{ie}\left[\mathbf{D}_\mu, \mathbf{D}_\nu\right] \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e\varepsilon_{abc}A^b_\mu A^c_\nu \end{aligned} \tag{184}$$

all of which stems directly from our analysis on Gauge field theories.

We proceed now to derive the equations of motion.

For the field Φ^a we have from Lagrangian (180):

$$\begin{aligned} &\frac{\partial L}{\partial\Phi^a} - \partial_\mu\frac{\partial L}{\partial\left(\partial_\mu\Phi^a\right)} = 0 \\ &-\frac{\partial V}{\partial\Phi^a} - D^\mu\Phi^\lambda e\varepsilon_{\lambda ba}A^b_\mu - \partial_\mu D^\mu\Phi^a = 0 \\ &\partial_\mu D^\mu\Phi^a + e\varepsilon_{\lambda ba}A^b_\mu D^\mu\Phi^\lambda = -\frac{\partial V}{\partial\Phi^a} \\ &\partial_\mu D^\mu\Phi^\alpha - e\varepsilon_{ab\lambda}A^b_\mu D^\mu\Phi^\lambda = -\lambda\Phi^a\left(\Phi^b\Phi^b - u^2\right) \\ &D_\mu D^\mu\Phi^a = -\lambda\Phi^a\left(\Phi^b\Phi^b - u^2\right) \end{aligned} \tag{185}$$

where we used $\varepsilon_{\lambda\beta a} = -\varepsilon_{a\beta\lambda}$.

Now for the gauge field A^a_μ:

$$\frac{\partial L}{\partial A^a_\mu} - \partial_\nu \frac{\partial L}{\partial\left(\partial_\nu A^a_\mu\right)} = 0 \tag{186}$$

We are going to calculate each term separately:

$$\begin{aligned}
&\frac{\partial L}{\partial A^a_\mu} = \frac{\partial}{\partial A^\alpha_\mu}\left(-\frac{1}{4}F^\beta_{\rho\sigma}F^{\mu\sigma\beta} + \frac{1}{2}D_\rho\Phi^\beta D^\rho\Phi^\beta - V(\Phi)\right). \\
&= -\frac{1}{2}F^{\rho\sigma\beta}\frac{\partial}{\partial A^a_\mu}F^\beta_{\rho\sigma} + D^\rho\Phi^\beta\frac{\partial}{\partial A^a_\mu}D_\rho\Phi^\beta \\
&= -\frac{1}{2}F^{\rho\sigma\beta}\frac{\partial}{\partial A^a_\mu}\left(\partial_\rho A^\beta_\sigma - \partial_\sigma A^\beta_\rho - e\varepsilon_{\beta\gamma\delta}A^\gamma_\rho A^\delta_\sigma\right) \\
&+ D^\rho\Phi^\beta\frac{\partial}{\partial A^\alpha_\mu}\left(\partial_\rho\Phi^\beta - e\varepsilon_{\beta\gamma\delta}A^\beta_p\Phi^\delta\right)
\end{aligned} \tag{187}$$

For the first term of (187) we have:

$$\begin{aligned}
&= -\frac{1}{2}F^{\rho\sigma\beta}\left(-e\varepsilon_{\beta\gamma\delta}\delta^{a\gamma}\delta^\rho_\mu A^\delta_\sigma - e\varepsilon_{\beta\gamma\delta}A^\gamma_\rho\delta^\mu_\sigma\delta^{a\delta}\right) \\
&= -\frac{1}{2}F^{\rho\sigma\beta}\left(-e\varepsilon_{\beta a\delta}A^\delta_\sigma\delta^\mu_\rho - e\varepsilon_{\beta\gamma a}A^\gamma_\rho\delta^\mu_\sigma\right). \\
&= +\frac{e}{2}F^{\mu\sigma\beta}\varepsilon_{\beta a\delta}A^\delta_\sigma + \frac{e}{2}\varepsilon_{\beta\gamma a}A^\gamma_\rho F^{\rho\mu\beta}\Big) \\
&= \frac{e}{2}F^{\mu\sigma\beta}\varepsilon_{\beta a\delta}A^\delta_\sigma + \frac{e}{2}\varepsilon_{\beta\delta a}A^\delta_\sigma F^{\sigma\mu\beta} \\
&= eF^{\mu\sigma\beta}\varepsilon_{\beta a\delta}A^\delta_\sigma
\end{aligned}$$

where we renamed some dummy indices. For the second term of (187):

$$\begin{aligned}
&D^\rho\Phi^\beta\frac{\partial}{\partial A^\alpha_\rho}\left(\partial_\rho\Phi^\beta - e\varepsilon_{\beta\gamma\delta}A^\gamma_\rho\Phi^\delta\right) \\
&\quad = D^\rho\Phi^\beta(-e)\varepsilon_{\beta a\delta}\Phi^\delta\delta^\mu_\rho = -e\varepsilon_{\beta\alpha\delta}(D^\mu\Phi^\beta)\Phi^\delta
\end{aligned}$$

For the second term of E-L equations (186):

$$\partial_\nu\frac{\partial L}{\partial(\partial_\nu A^a_\mu)} = -\partial_\nu F^{\nu\mu a}$$

since the above calculation differs not from the procedure of extracting the equations of motion from standard electrodynamics, which we assume the reader is familiar with.

Now substituting all the above in (186) we obtain:

$$\begin{aligned}
&\frac{\partial L}{\partial A^a_\mu} - \partial_\nu\frac{\partial L}{\partial\left(\partial_\nu A^a_\mu\right)} = 0 \\
&eF^{\mu\sigma\beta}\varepsilon_{\beta a\delta}A^\delta_\sigma - e\varepsilon_{\beta a\delta}\left(D^\mu\Phi^\beta\right)\Phi^\delta = -\partial_\nu F^{\nu\mu\alpha} \\
&\partial_\nu F^{\nu\mu\alpha} + e\varepsilon_{a\delta\beta}F^{\mu\nu\beta}A^\delta_\mu = -e\varepsilon_{a\beta\delta}\left(D^\mu\Phi^\beta\right)\Phi^\delta \\
&D_\nu F^{\nu\mu a} = -e\varepsilon_{a\beta\delta}\left(D^\mu\Phi^\beta\right)\Phi^\delta
\end{aligned} \tag{188}$$

A second equation complements (186) and that is:

$$D^{\nu}\widetilde{F}^{a}_{\mu\nu} \equiv 0 \tag{189}$$

[32] which is equivalent to a *Bianchi* identity:

$$(D_{\mu}F_{\nu\kappa})^{a} + (D_{\kappa}F_{\mu\nu})^{a} + (D_{\nu}F_{\kappa\mu})^{a} = 0 \tag{190}$$

which is also equivalent to a *Jacobi* identity of the covariant derivatives[33], using (92):

$$[D_{\mu}, [D_{\nu}, D_{\kappa}]] + [D_{\kappa}, [D_{\mu}, D_{\nu}]] + [D_{\nu}, [D_{\kappa}, D_{\mu}]] = 0 \tag{191}$$

where we used $[D_{\mu}, F^{a}_{\nu\kappa}] = D_{\mu}F^{a}_{\nu\kappa}$.

The aforementioned commutator results upon acting it on a test function as follows:

$$\begin{aligned} \frac{1}{ie}\left[D_{\mu}, [D_{\nu}, D_{\lambda}]^{a}\right]\psi &= [D_{\mu}, F^{a}_{\nu\lambda}]\,\psi \\ &= D_{\mu}\left(F^{a}_{\nu\lambda}\psi\right) - F^{a}_{\nu\lambda}D_{\mu}\psi \\ &= D_{\mu}F^{a}_{\nu\lambda}\psi + F_{\nu\lambda}D_{\mu}\psi - F^{a}_{\nu\lambda}D_{\mu}\psi \\ &= D_{\mu}F^{a}_{\nu\lambda}\psi. \end{aligned} \tag{192}$$

while the Jacobi identity (191) holds trivially for commutators.

The connection between (189) and (190) can be seen by taking the latter and contracting with $\varepsilon^{\lambda\mu\nu\kappa}$:

$$\begin{aligned} &\varepsilon^{\lambda\mu\nu\kappa}D_{\mu}F^{a}_{\nu\kappa} + \varepsilon^{\lambda\mu\nu\kappa}\left(D_{\kappa}F_{\mu\nu}\right)^{a} + \varepsilon^{\lambda\mu\nu\kappa}\left(D_{\nu}F_{\kappa\mu}\right)^{a} = 0 \\ &3\varepsilon^{\lambda\mu\nu\kappa}\left(D_{\mu}F_{\nu\kappa}\right)^{a} = 0 \\ &D_{\mu}\varepsilon^{\lambda\mu\nu\kappa}F^{a}_{\nu\kappa} = 0 \\ &D_{\mu}\widetilde{F}^{\lambda\mu} = 0 \end{aligned} \tag{193}$$

B. SSB of the *SU(2)* Georgi-Glashow model

The next step towards *SSB* is finding solutions that minimize the energy. Thus follows the calculation of the symmetric energy momentum tensor:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}L}{\delta g^{\mu\nu}} \tag{194}$$

where $g = \det\left(g^{\mu\nu}\right)$ is the determinant of the *Minkowski* metric.

$$\begin{aligned} T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}L}{\delta g^{\mu\nu}} &= \frac{2}{\sqrt{-g}}\left(\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}L + \sqrt{-g}\frac{\delta L}{\delta g^{\mu\nu}}\right) \\ &= \frac{2}{\sqrt{-g}}\left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}L + \sqrt{-g}\left(-\frac{1}{2}F^{a}_{\mu\rho}F^{a\rho}_{\nu} + \frac{1}{2}\left(D_{\mu}\Phi^{a}\right)\left(D_{\nu}\Phi^{a}\right)\right)\right) \end{aligned}$$

Substituting $\det(g^{\mu\nu}) = -1$ we get:

$$T_{\mu\nu} = -F^{a}_{\mu\rho}F^{a\rho}_{\nu} + \left(D_{\mu}\Phi^{a}\right)\left(D_{\nu}\Phi^{a}\right) - g_{\mu\nu}L \tag{195}$$

---

[32] where $\widetilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.
[33] which is also rather funny to me, since these are two identities I always mix up the names of.

We only really care about the T^{00} component that represents the total energy density of the system. So the total energy is given by the volume integral of T^{00}:

$$E = \int d^3x \left(-F^a_{0i}F^{ai}_0 + (D_0\Phi^a)(D_0\Phi^a) - g_{00}L\right) \tag{196}$$

Now for some algrebraic massage to put (196) in a more conventient form:

$$\begin{aligned} -F^a_{0i}F^{ai}_0 + (D_0\Phi^a)(D_0\Phi^a) + \tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \tfrac{1}{2}(D_\mu\Phi^a)(D^\mu\Phi^a) + V(\Phi) = \\ -F^a_{0i}F^{ai}_0 + \tfrac{1}{4}\left(F^a_{0i}F^{a0i} + F^a_{i0}F^{ai0} + F^a_{ij}F^{aij}\right) + \tfrac{1}{2}\left((D_0\Phi^a)\left(D^0\Phi^a\right) - (D_i\Phi^a)\left(D^i\Phi^a\right)\right) + V(\Phi) = \\ -\tfrac{1}{2}F^a_{0i}F^{a0i} + \tfrac{1}{4}F^a_{ij}F^{aij} + \tfrac{1}{2}\left((D_0\Phi^a)\left(D^0\Phi^a\right) + (D_i\Phi^a)(D_i\Phi^a)\right) + V(\Phi) \end{aligned}$$

What we end up is:

$$E = \int d^3x \left(\frac{1}{2}E^a_iE^a_i + \frac{1}{2}B^a_iB^a_i + \frac{1}{2}\left((D_0\Phi^a)(D_0\Phi^a) + (D_i\Phi^a)(D_i\Phi^a)\right) + V(\Phi)\right) \tag{197}$$

where we defined

$$E^a_n \equiv F^a_{0n} \quad \text{and} \quad B^a_n \equiv \frac{1}{2}\varepsilon_{nmk}F^a_{mk} \tag{198}$$

the "color" electric and magnetic fields.

The minimization conditions for the *static*[34] Hamiltonian are:

$$\Phi^a\Phi^a = v^2, \quad F^a_{mn} = 0, \quad D_n\phi^a = 0 \tag{199}$$

where the "static" condition also implies $E^a_n = 0$. To determine the particle spectrum in the new vacuum, we consider a small time independent perturbation on the scalar Higgs field. Because of the internal $SU(2)$ symmetry we can orient the VEV in any direction in isospace. For clarity's sake, we choose to align the Higgs VEV with the 3-direction in isospin space.

$$\Phi^a = \begin{pmatrix} 0 \\ 0 \\ u+\rho \end{pmatrix} \tag{200}$$

where ρ is the perturbation we considered and u the vacuum expectation value as of (199).

The Higgs potential becomes after substitution of (200):

$$\begin{aligned} V(\Phi) = \frac{\lambda}{4}\left(\Phi^a\Phi^a - u^2\right)^2 &\to \frac{\lambda}{4}\left((u+\rho)(u+\rho) - u^2\right)^2 \\ &= \frac{\lambda}{4}\left(u^2 + 2\rho u + \rho^2 - u^2\right)^2 \\ &= \frac{\lambda}{4}\left(4u^2\rho^2 + 4u\rho^3 + \rho^4\right) \end{aligned} \tag{201}$$

The covariant derivative:

$$\begin{aligned} \begin{pmatrix} D_\mu\Phi^1 \\ D_\mu\Phi^2 \\ D_\mu\Phi^3 \end{pmatrix} = \begin{pmatrix} \partial_\mu\Phi^1 - e\varepsilon_{1bc}A^b_\mu\Phi^c \\ \partial_\mu\Phi^2 - e\varepsilon_{2bc}A^b_\mu\Phi^c \\ \partial_\mu\Phi^3 - e\varepsilon_{3bc}A^b_\mu\Phi^c \end{pmatrix} = \begin{pmatrix} -e\varepsilon_{123}A^2_\mu\Phi^3 \\ -e\varepsilon_{213}A^1_\mu\Phi^3 \\ \partial_\mu\Phi^3 \end{pmatrix} \\ \begin{pmatrix} D_\mu\Phi^1 \\ D_\mu\Phi^2 \\ D_\mu\Phi^3 \end{pmatrix} = \begin{pmatrix} -eA^2_\mu(u+\rho) \\ eA^1_\mu(u+\rho) \\ \partial_\mu\rho \end{pmatrix} \end{aligned} \tag{202}$$

---

[34] That is we deny temporal dependence on the fields.

So the kinetic term yields:

$$(D_\mu \Phi^a)(D^\mu \Phi^a) = (\partial_\mu \rho)(\partial^\mu \rho) + e^2(u+\rho)^2 \left(A^1_\mu A^{\mu 1} + A^2_\mu A^{\mu 2}\right) \tag{203}$$

And the Lagrangian acquires the form:

$$L = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \frac{e^2 u^2}{2}\left(A^1_\mu A^{\mu 1} + A^2_\mu A^{\mu 2}\right) + \frac{1}{2}(\partial_i \rho)(\partial_i \rho) - \frac{1}{2}(\sqrt{2\lambda} u)^2 \rho^2 + \dots \tag{204}$$

It is explicit that this Lagrangian includes two massive vector particels of $M_W = eu$, one massive scalar with $M_H = \sqrt{2\lambda} u$ and one massless vector particle which we identify with the photon that corresponds to the unbroken $U(1)$ subgroup of $SU(2)$. This $U(1)$ describes the invariance of the Lagrangian with respect to rotations on the axis defined by the VEV of Φ^a. Its generator is $\frac{\Phi^a T^a}{u}$ and we assign it as the electromagnetic charge operator. Thus, the covariant derivative (135) can be written now:

$$L = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \frac{e^2 u^2}{2}\left(A^1_i A^{i 1} + A^2_i A^{i 2}\right) + \frac{1}{2}(\partial_i \rho)(\partial_i \rho) - \frac{1}{2}(\sqrt{2\lambda} u)^2 \rho^2 + \dots \tag{205}$$

where we defined the "electromagnetic projection" of the gauge potential.

C. Topological classification of the solutions

The particle spectrum of the *SU(2)* Georgi-Glashow model is far richer than it may seem at first glance. And that is because there exist stable soliton-like solutions to the classical equations of motion with a finite energy along the spatial asymptotic[31]. All this fuss about Derrick's theorem and how to dodge its limitations is about to pay back.

The solutions to the classical equations of motions essentially map the vacuum manifold $\mathcal{M} = S^2_{\text{vac}}$ to the boundary of the 3-dimensional space, which, in fact, is also a sphere[35]. These maps are characterised by an integer *winding number* $n = \pm 0, 1, 2, ...$, which describes the number of times the vacuum sphere is covered by a single turn around the spatial sphere.

It is crucial to note that the behaviour of Φ on the spatial asymptotic could separate the solutions in different classes. For example (200) corresponds to the trivial mapping with a winding number $n = 0$. We could also consider other kinds of spatial asympotic behaviours, even ones where the isospace directions of the Φ field are functions of the spatial directions. One could argue that since the trivial configuration corresponds to the absolute minimum of the energy functional then other such solutions would be unstable. However, any attempt to deform continuously the fields to the trivial vacuum, the energy functional would explode to infinity. In other words, all the different topological sectors are separated by infinite energy barriers. So, topology saves the day again[36].

To construct such non-trivial vacuum solutions, we again assign that $|\Phi| = u$ in the spatial asymptotic. This time though, we suppose that the isovector of the scalar field is directed in isospace along the direction of the radius vector along the spatial asymptotic.

Behold now, the mighty "hedgehog" solution! [16],[27]

$$\Phi^a \underset{r\to\infty}{\longrightarrow} \frac{v r^a}{r} \tag{206}$$

Now a single turn around the spatial boundary manifold results in a single closed path that rotates once in the vacuum sphere and thus its winding number is $n = 1$. Solutions with different winding numbers belong in different topological sectors. The winding number is a topological invariant and therefore continuous transformations cannot connect there areas.

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[35] Remember the discussion on homotopy groups and their use in physics and especially equation (72).
[36] and the stability of soliton solutions in our system

To acquire a non-trivial topological excitation, we require that our solution belongs in a homotopic class other than [I] , where the identity element belongs, has finite energy and approaches the hedgehog asymptotic conditions (206).

So apart from (206), in order to retain finite energy, we should also have

$$D_i\Phi^a \to 0 \tag{207}$$

in the spatial asymptotic. From the above condition we can calculate the form of the gauge potential:

$$\begin{aligned}
\partial_i\left(\frac{r^a}{r}\right) - e\varepsilon_{abc}A_i^b\frac{r^c}{r} &= 0\\
\frac{\delta_{ai}r^2 - r_a r_i}{r^3} &= e\varepsilon_{abc}A_i^b\frac{r^c}{r}\\
(\delta_{ai}\delta_{ck} - \delta_{ak}\delta_{ic})\frac{r_c r_k}{r^3} &= e\varepsilon_{abc}A_i^b\frac{r_c}{r}\\
\varepsilon_{acb}\varepsilon_{bik}\frac{r_c r_k}{r^3} &= e\varepsilon_{abc}A_i^b\frac{r_c}{r}
\end{aligned}$$

yielding

$$A_i^a = \varepsilon_{ani}\frac{r_n}{er^2} \tag{208}$$

The "color" magnetic field will be given by:

$$B_i^a = \frac{1}{2}\varepsilon_{ijk}F_{jk}^a = \frac{1}{2}\varepsilon_{ijk}\left(\partial_j A_k^a - \partial_k A_j^a - e\varepsilon_{abc}A_j^b A_k^c\right) = \varepsilon_{ijk}\left(\partial_j A_k^a - \frac{e}{2}\varepsilon_{abc}A_j^b A_k^c\right) \tag{209}$$

D. Interlude: Relationship between Dirac and t'Hooft monopoles

We can convince ourselves that we are taking a step in the right direction by noticing how the Dirac potential (31) is related to the one we ended up with now[37].

Of course Dirac's potential is a $U(1)$ potential and to find the correspondence of it with (208) we must first submerge it into *SU(2)* via:

$$\mathbf{A}_0 = \mathbf{A}_r = \mathbf{A}_\theta = 0, \quad \mathbf{A}_\phi = \mathbf{T}_3\left(-\frac{g}{r}\right)\left(\frac{1-\cos\theta}{\sin\theta}\right) \tag{210}$$

where $\mathbf{A}_\mu = A_\mu^a T^a$. So, we have aligned the Dirac potential in the 3-direction in isospace[38].

Considering , also, our scalar field with a *VEV* in the same direction:

$$\mathbf{\Phi} = u\mathbf{T}^3 \tag{211}$$

we perform a *SU(2)* transformation of the form:

$$\mathbf{U} = \begin{pmatrix} \cos\theta/2 & -\mathrm{e}^{-\mathrm{i}\phi}\sin\theta/2 \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta/2 & \cos\theta/2 \end{pmatrix} \tag{212}$$

where (212) is a specific case of the general *SU(2)* transformation parametrized with Euler angles:

$$\begin{aligned}
\mathbf{U} &= \mathrm{e}^{i\alpha\mathbf{T}_3}e^{i\beta\mathbf{T}_2}e^{i\gamma\mathbf{T}_3}\\
&= \begin{pmatrix} \cos\beta/2e^{i(\alpha+\gamma)/2} & \sin\beta/2e^{i(-\gamma+\alpha)/2} \\ -\sin\beta/2e^{i(\gamma-\alpha)/2} & \cos\beta/2e^{-i(\gamma+\alpha)/2} \end{pmatrix}
\end{aligned} \tag{213}$$

---

[37] We borrow this discussion from [29]
[38] Notice $\mathbf{A}_\phi$ has only a $\mathbf{T}^3$ component.

by setting $\gamma = -\alpha = \phi, \beta = -\theta$. We recall the transformation properties of $\mathbf{A}_\mu$ from (90):

$$\mathbf{A}'_\mu = \mathbf{U}\mathbf{A}_\mu\mathbf{U}^{-1} + \frac{\mathrm{i}}{e}\mathbf{U}\partial_\mu\mathbf{U}^{-1} \tag{214}$$

Straightforward differentiation gives:

$$\begin{aligned}
\partial_r\mathbf{U}^{-1} &= 0,\\
\partial_\theta\mathbf{U}^{-1} &= \frac{1}{2r}\begin{pmatrix} -\sin\theta/2 & \mathrm{e}^{-\mathrm{i}\phi}\cos\theta/2 \\ -\mathrm{e}^{\mathrm{i}\phi}\cos\theta/2 & -\sin\theta/2 \end{pmatrix},\\
\partial_\phi\mathbf{U}^{-1} &= \frac{-\mathrm{i}}{r\sin\theta}\begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta/2 \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta/2 & 0 \end{pmatrix}.
\end{aligned}$$

It is easy to see by applying the transformation (90):

$$\mathbf{A}'_0 = \mathbf{A}'_r = 0 \tag{215}$$

As for the θ, ϕ components:

$$\begin{aligned}
\mathbf{A}'_\theta &= \frac{i}{e}\begin{pmatrix} \cos(\theta/2) & -e^{-i\phi}\sin(\theta/2) \\ +e^{i\phi}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}\frac{1}{2r}\begin{pmatrix} -\sin(\theta/2) & -i\phi \\ -e^{i\phi}\cos(\theta/2) & -\sin(\theta/2) \end{pmatrix}\\
=&\frac{i}{2er}\begin{pmatrix} 0 & e^{-i\phi}(\sin^2(\theta/2)+\cos^2(\theta/2)) \\ -e^{i\phi}(\sin^2(\theta/2)+\cos^2(\theta/2)) & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \frac{i}{2er}\begin{pmatrix} 0 & -\cos\phi - i\sin\phi \\ -\cos\phi - i\sin\phi & 0 \end{pmatrix} = \frac{1}{2er}\begin{pmatrix} 0 & si\phi + i\cos\phi \\ \sin\phi - i\cos\phi & 0 \end{pmatrix}\\
&= \frac{(\sin\phi\mathbf{T}_1 - \cos\phi\mathbf{T}_2)}{\mathrm{er}}
\end{aligned} \tag{216}$$

Similarly for $\mathbf{A}_\phi$ using $g = \frac{1}{e}$, basic trigonometric identities and a lot of patience, we get:

$$\begin{aligned}
\mathbf{A}'_\phi &= \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\left(-\frac{g}{r}\right)\frac{1-\cos\theta}{\sin\theta}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \\ -e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\\
&+\frac{i(-i)}{er\sin\theta}\begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 0 & e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & 0 \end{pmatrix}\\
&= -\frac{g}{2r}\frac{1-\cos\theta}{\sin\theta}\begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix}\\
&+\frac{1}{er\sin\theta}\begin{pmatrix} -\sin^2\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}\\
&= -\frac{g}{2r}\frac{1-\cos\theta}{\sin\theta}\begin{pmatrix} \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} & 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{-i\phi} \\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{\phi} & \sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2} \end{pmatrix} + \frac{g}{2r\sin\theta}\begin{pmatrix} -2\sin^2\frac{\theta}{2} & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 2\sin^2\frac{\theta}{2} \end{pmatrix}\\
&= -\frac{g}{2r\sin\theta}\begin{pmatrix} \cos\theta - \cos^2\theta & e^{-i\phi}\sin\theta(1-\cos\theta) \\ e^{i\phi}\sin\theta(1-\cos\theta) & \cos^2\theta - \cos\theta \end{pmatrix} + \frac{g}{2r\sin\theta}\begin{pmatrix} -2\sin^2\frac{\theta}{2} & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 2\sin^2\frac{\theta}{2} \end{pmatrix}.\\
&= \frac{g}{2r\sin\theta}\begin{pmatrix} \cos^2\theta - \cos\theta - 2\sin^2\frac{\theta}{2} & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & 2\sin^2\frac{\theta}{2} + \cos\theta - \cos^2\theta \end{pmatrix} = \frac{g}{2r}\begin{pmatrix} -\sin\theta & \cos\theta e^{-i\phi} \\ \cos\theta e^{i\phi} & +\sin\theta \end{pmatrix}\\
&= \frac{1}{er}\left(\mathbf{T}_1\cos\theta\cos\phi + \mathbf{T}_2\cos\theta\sin\phi - \mathbf{T}_3\sin\theta\right)
\end{aligned} \tag{217}$$

where we acknowledge that $2\sin^2\theta - \cos^2\theta + \cos\theta$ is just a really weird way to say $\sin\theta$.

We are in position now to calculate the Cartesian components of $\mathbf{A}$. We are going to show this only for the x component but the rest follow similarly:

$$\begin{aligned}\mathbf{A}'_x &= \mathbf{A}_{r'}\cos\phi\sin\theta + \mathbf{A}'_\theta\cos\phi\cos\theta - \mathbf{A}'_\phi\sin\phi \\ &= \frac{1}{er}\left[\mathbf{T}_1\sin\phi\cos\phi\cos\theta - \mathbf{T}_2\cos^2\phi\cos\theta - \mathbf{T}_1\cos\theta\cos\phi\sin\phi - \mathbf{T}_2\cos\theta\sin^2\phi + \mathbf{T}_3\sin\theta\sin\phi\right] \\ &= \frac{1}{er}\left[\mathbf{T}_2(-\cos\theta) + \mathbf{T}_3\sin\theta\sin\phi\right] = \frac{1}{er}\left[\mathbf{T}_2\left(\frac{-z}{r}\right) + \mathbf{T}_3\left(\frac{y}{r}\right)\right]\end{aligned} \tag{218}$$

which is just the 1-group component of $\mathbf{A}$ from (208) Our efforts were not in vain. After this long calculation, we managed to prove that a specific singular *SU(2)* transformation can connect the t'Hooft and Dirac potentials by embedding the latter in the larger group of *SU(2)*. But it doesn't end here. Witness what the exact same transformation has to say about the scalar field Φ:

$$\begin{aligned}\mathbf{\Phi}' = \mathbf{U\Phi U}^{-1} &= \begin{pmatrix}\cos\frac{\theta}{2} & -e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}\frac{u}{2} & 0 \\ 0 & -\frac{u}{2}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \\ -e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix} = \\ &= \frac{u}{2}\begin{pmatrix}\cos\frac{\theta}{2} & -e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2}\end{pmatrix} \\ &= \frac{u}{2}\begin{pmatrix}\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} & +2\sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{-i\phi} \\ 2\sin\frac{\theta}{2}\cos^2 e^{-i\phi} & \sin^2\frac{\theta}{2} - \cos\frac{\theta}{2}\end{pmatrix} = \frac{u}{2}\begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta\end{pmatrix}\end{aligned} \tag{219}$$

Decomposing it in *SU(2)* generators:

$$\mathbf{\Phi}' = u(\sin\theta\cos\phi\mathbf{T}_1 + \sin\theta\sin\phi\mathbf{T}_2 + \cos\theta\mathbf{T}_3) \tag{220}$$

Now it is obvious that our gauge transformation changed the homotopy class of the configuration[39]. It is as the transformation transferred the responsibility for the monopole from the Dirac term to the Higgs sector. Before the transformation, the Higgs map is trivial, but the Dirac potential carries the magnetic monopole. After the transformation, however, the "Brouwer" degree of the 'hedgehog' gauge changes and the gauge potential is free of any singularities. The magnetic charge remains the same of course in all of this procedure as a topological invariant. We can gain some better insight in this interchange between the Higgs and gauge sector soon, when we define the *generalised electromagnetic field strength tensor.*

E. The Magnetic Charge

From the form of the gauge potential at spatial infinity we can calculate the "color" magnetic field:

$$B_i^a = \frac{1}{2}\varepsilon_{ijk}F_{jk}^a = \frac{1}{2}\varepsilon_{ijk}\left(\partial_j A_k^a - \partial_k A_j^a - e\varepsilon_{abc}A_j^b A_k^c\right) = \varepsilon_{ijk}\left(\partial_j A_k^a - \frac{e}{2}\varepsilon_{abc}A_j^b A_k^c\right) \tag{221}$$

The first term can be handled as[40]

$$\begin{aligned}\varepsilon_{ijk}\left(\partial_j A_k^a\right) &= \frac{1}{e}\varepsilon_{ijk}\varepsilon_{ank}\partial_j\left(\frac{r_n}{r^2}\right) \\ &= \frac{1}{e}\left(\delta_{ia}\delta_{jn} - \delta_{in}\delta_{ja}\right)\left[\frac{\delta_{jn}}{r^2} - \frac{2r_j r_n}{r^4}\right] \\ &= \frac{1}{e}\left(\delta_{ia}\frac{3}{r^2} - \delta_{ia}\frac{2}{r^2} - \delta_{ia}\frac{1}{r^2} + \frac{2r_a r_i}{r^4}\right) \\ &\Rightarrow \varepsilon_{ijk}\left(\partial_j A_k^a\right) = \frac{2r_a r_i}{r^4 e}\end{aligned}$$

---

[39] Of course that is because we used a singular transformation. A regular one could not have accomplished such a feat in any way.
[40] Reminder that we have defined : $\varepsilon_{0123} = -1, \varepsilon_{123} = +1$

While the second term is:

$$
\begin{aligned}
-\frac{e}{2}\varepsilon_{ijk}\varepsilon_{abc}A_j^bA_k^c &= -\frac{1}{2e}\varepsilon_{ijk}\varepsilon_{abc}\varepsilon_{bnj}\varepsilon_{cmk}\frac{r_nr_m}{r^4}\\
&= \frac{1}{2e}\varepsilon_{ikj}\varepsilon_{jbn}\varepsilon_{abc}\varepsilon_{cmk}\frac{r_nr_m}{r^4}\\
&= \frac{1}{2e}\left(\delta_{ib}\delta_{kn}-\delta_{in}\delta_{kb}\right)\left(\delta_{am}\delta_{bk}-\delta_{ak}\delta_{bm}\right)\frac{r_nr_m}{r^4}\\
&= \frac{1}{2e}\left(\delta_{am}\delta_{in}-\delta_{an}\delta_{im}-3\delta_{in}\delta_{am}+\delta_{in}\delta_{am}\right)\frac{r_nr_m}{r^4}\\
&= \frac{1}{2e}\left(2\delta_{am}\delta_{in}-\delta_{an}\delta_{im}-3\delta_{in}\delta_{am}\right)\frac{r_nr_m}{r^4}\\
&= \frac{1}{2e}\left(2\frac{r_ir_a}{r^4}-\frac{r_ar_i}{r^4}-3\frac{r_ir_a}{r^4}\right)\\
&\Rightarrow -\frac{e}{2}\varepsilon_{ijk}\varepsilon_{abc}A_j^bA_k^c = -\frac{r_ir_a}{er^4}
\end{aligned}
$$

where we repeatedly used

$$\varepsilon_{ijk}\varepsilon^{pqk} = \delta_i{}^p\delta_j{}^q - \delta_i{}^q\delta_j{}^p \tag{222}$$

Substituting the previous results in (221):

$$B_i^a = \frac{r_ir_a}{er^4} \tag{223}$$

When the vacuum configuration is $\Phi^a = u\delta^{a3}$, the electromagnetic field strength tensor is given by $F_{\mu\nu} = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$. We want to generalise the field strength tensor for more complicated scalar field configurations in a gauge-invariant way so that it reduces to the regular $F^{\mu\nu}$ when the Higgs vacuum configuration is trivial. Thus, we define[41]:

$$\mathcal{F}_{\mu\nu} = \hat{\Phi}^a F_{\mu\nu}^a + e^{-1}\varepsilon^{abc}\hat{\Phi}^a\left(D_\mu\hat{\Phi}\right)^b\left(D_\nu\hat{\Phi}\right)^c \tag{224}$$

where $\hat{\Phi}^a = \frac{\Phi^a}{|\Phi|}$. We can, in fact, use this definition of the electromagnetic field strength everywhere in space where $|\Phi| \neq 0$.[42]

To prove its gauge invariance it is much more convenient to use this form of (224):

$$\mathcal{F}_{\mu\nu} = 2\,\mathrm{Tr}\left\{\hat{\mathbf{\Phi}}\mathbf{F}_{\mu\nu} - \frac{i}{2e}\hat{\mathbf{\Phi}}\mathbf{D}_\mu\hat{\mathbf{\Phi}}\mathbf{D}_\nu\hat{\mathbf{\Phi}}\right\} \tag{225}$$

To prove their equivalence we just need to utilize the following relationships for the *SU(2)* generators:

$$\mathrm{Tr}\left(\mathbf{T}^a\mathbf{T}^b\right) = \frac{1}{2}\delta^{ab}, \mathrm{Tr}\left(\mathbf{T}^a\mathbf{T}^b\mathbf{T}^c\right) = \frac{i}{4}\varepsilon^{abc} \tag{226}$$

and remember that $\hat{\mathbf{\Phi}} = \frac{\Phi^a}{|\Phi|}\mathbf{T}^a$ and similarly for $\mathbf{F}_{\mu\nu}$. Now for the gauge invariance we invoke the transformation of $\hat{\mathbf{\Phi}}$ that is in the adjoint representation: $\hat{\Phi}' = \mathbf{U\Phi U}^{-1}$ and the same for the covariant derivative: $\boldsymbol{D}'_\mu = \mathbf{U}\boldsymbol{D}_\mu\mathbf{U}^{-1}$,

$$
\begin{aligned}
&\mathcal{F}'_{\mu\nu} = 2\,\mathrm{Tr}\left\{\hat{\mathbf{\Phi}}'\mathbf{F}'_{\mu\nu} - \frac{i}{2e}\hat{\mathbf{\Phi}}\mathbf{D}_\mu\hat{\mathbf{\Phi}}\mathbf{D}_\nu\hat{\mathbf{\Phi}}\right\}\\
&=2\,\mathrm{Tr}\left\{\mathbf{U}\hat{\mathbf{\Phi}}\mathbf{U}^\dagger\mathbf{U}\frac{[\mathbf{D}_\mu,\mathbf{D}_\nu]}{ie}\mathbf{U}^\dagger - \frac{i}{2e}\mathbf{U}\hat{\Phi}\mathbf{U}^\dagger\mathbf{U}\mathbf{D}_\mu\hat{\mathbf{\Phi}}\mathbf{U}^\dagger\mathbf{U}\mathbf{D}_\nu\mathbf{U}^\dagger\right\}\\
&=2\,\mathrm{Tr}\left\{\hat{\mathbf{\Phi}}\left[\mathbf{D}_\mu,\mathbf{D}_\nu\right] - \frac{i}{2e}\hat{\mathbf{\Phi}}\mathbf{D}_\mu\hat{\mathbf{\Phi}}\mathbf{D}_\nu\hat{\mathbf{\Phi}}\right\} = \mathcal{F}_{\mu\nu}
\end{aligned}
\tag{227}
$$

[41] Here following [32]. Later, we will present another approach that will get us a little bit further.
[42] If $|\Phi| = 0$ the *SU(2)* symmetry is unbroken and there is no gauge-invariant way to pick up a component of $F_{\mu\nu}^a$ in the first place.

where we also used the cyclic property of the Tr trace operator.
Substituting (182) and (184) in (224) it is possible to rewrite it as:

$$\mathcal{F}_{\mu\nu} = \partial_\mu \left(\hat{\Phi}^a A^a_\nu\right) - \partial_\nu \left(\hat{\Phi}^a A^a_\mu\right) + e^{-1}\varepsilon^{abc}\hat{\Phi}^a \partial_\mu \hat{\Phi}^b \partial_\nu \hat{\Phi}^c \tag{228}$$

We start by expanding substituting the aforementioned equations:

$$\begin{aligned}\mathcal{F}_{\mu\nu} &= \hat{\Phi}^a F_{\mu\nu}{}^a + e^{-1}\varepsilon^{abc}\hat{\Phi}^a D_\mu \hat{\Phi}^b D_\nu \hat{\Phi}^c \\ &= \hat{\Phi}^a \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e\varepsilon^{abc} A^b_\mu A^c_\nu\right) + \frac{\Phi^a}{e}\varepsilon^{abc}\left(\partial_\mu \hat{\Phi}^b - e\varepsilon^{bde} A^d_\mu \hat{\Phi}^e\right)\left(\partial_\nu \hat{\Phi}^c - e\varepsilon^{cfg} A^f_\nu \hat{\Phi}^g\right)\end{aligned}$$

Then we expand the terms in the brackets:

$$\begin{aligned}&= \hat{\Phi}^a \partial_\mu A^a_\nu - \hat{\Phi}^a \partial_\nu A^a_\mu - \hat{\Phi}^a e\varepsilon^{abc} A^b_\mu A^c_\nu + \frac{\Phi^a}{e}\varepsilon^{abc}\partial_\mu \hat{\Phi}^b \partial_\nu \hat{\Phi}^c + e\varepsilon^{abc}\varepsilon^{bde}\varepsilon^{cfg} A^d_\mu \hat{\Phi}^e A^f_\nu \hat{\Phi}^g \hat{\Phi}^a \\ &- \varepsilon^{abc}\varepsilon^{bde} A^d_\mu \Phi^e \partial_\nu \hat{\Phi}^c \hat{\Phi}^a - \varepsilon^{abc}\varepsilon^{cfg} A^f_\nu \hat{\Phi}^g \partial_\mu \hat{\Phi}^b \hat{\Phi}^a\end{aligned}$$

We now employ $\varepsilon_{ijk}\varepsilon^{pqk} = \delta_i{}^p \delta_j{}^q - \delta_i{}^q \delta_j{}^p$

$$\begin{aligned}&= \hat{\Phi}^a \partial_\mu A^a_\nu - \hat{\Phi}^a \partial_\nu A^a_\mu - e\hat{\Phi}^a \varepsilon^{abc} A^b_\mu A^c_\nu + \frac{1}{e}\varepsilon^{abc}\partial_\mu \hat{\Phi}^b \partial_\nu \hat{\Phi}^c \Phi^a \\ &+ e\varepsilon^{bde}\left(\delta^{af}\delta^{bg} - \delta^{ag}\delta^{bf}\right) A^d_\mu + \hat{\Phi}^e A^f_\nu \hat{\Phi}^g \Phi^a + e\left(\delta^{ad}\delta^{ce} - \delta^{ae}\delta^{dc}\right) A^d_\mu + \hat{\Phi}^e \partial_\nu \hat{\Phi}^c \Phi^a \\ &- \left(\delta^{af}\delta^{bg} - \delta^{ag}\delta^{bf}\right) A^f_\nu \hat{\Phi}^g \partial \hat{\Phi}^b \Phi^a\end{aligned}$$

Afterwards we contract the Kronecker delta's:

$$\begin{aligned}=&\hat{\Phi}^a \partial_\mu A^a_\nu - \hat{\Phi}^a \partial_\nu A^a_\nu - e\hat{\Phi}^a \varepsilon^{abc} A^b_\mu A^c_\nu + \frac{1}{e}\partial_\mu \hat{\Phi}^b \partial_\nu \hat{\Phi}^c \Phi^a \\ &+ e\varepsilon^{bde}\left(A^a_\nu \hat{\Phi}^b A^d_\mu \hat{\Phi}^e - A^d_\mu \hat{\Phi}^e A^b_\nu \hat{\Phi}^a\right)\Phi^a + \Phi^a A^a_\mu \hat{\Phi}^c \partial_\nu \hat{\Phi}^c - A^c_\mu \hat{\Phi}^a \partial_\nu \hat{\Phi}^c \Phi^a - A^a_\nu \hat{\Phi}^b \partial_\mu \hat{\Phi}^b \hat{\Phi}^a + A^b_\nu \hat{\Phi}^a \partial_\mu \hat{\Phi}^b \hat{\Phi}^a\end{aligned}$$

First good news of the day some terms cancel due to being products between a symmetric and antisymmetric tensor.

We now group the appropriate terms and rename their indices for their manipulation to be smooth.

$$\begin{aligned}&= \hat{\Phi}^a \partial_\mu A^a_\nu - \hat{\Phi}^a \partial_\nu A^a_\mu - e\hat{\Phi}^a \varepsilon^{abc} A^b_\mu A^c_\nu + \frac{1}{e}\partial_\mu \hat{\Phi}^b \partial_\nu \hat{\Phi}^c \hat{\Phi}^a \\ &- e\varepsilon^{bde} A^d_\mu A^b_\nu \hat{\Phi}^a \hat{\Phi}^a \hat{\Phi}^c + A^a_m u + \partial_\nu \left(\frac{\hat{\Phi}^c \hat{\Phi}^c}{2}\right)\hat{\Phi}^\alpha_a - A^c_\mu \left(\hat{\Phi}^\alpha \hat{\Phi}^\alpha\right)\partial_\nu \hat{\Phi}^c \\ &- A^a_\nu \partial_\mu \left(\frac{\hat{\Phi}^b \hat{\Phi}^b}{2}\right)\hat{\Phi}^a + A^b_\nu \left(\hat{\Phi}^a \hat{\Phi}^a\right)\partial_\mu \hat{\Phi}^b\end{aligned}$$

We have also prepared ourselves to use $\hat{\Phi}^a \hat{\Phi}^a = 1$.

$$\begin{aligned}&= \hat{\Phi}^a \partial_\mu A^a_\nu + A^a_\nu \partial_\mu \hat{\Phi}^a - \hat{\Phi}^a \partial_\nu A^a_\mu - A^a_\mu \partial_\nu \hat{\Phi}^a + \frac{1}{e}\partial_\mu \hat{\Phi}^b \partial_\nu \hat{\Phi}^c \hat{\Phi}^a \\ &- e\hat{\Phi}^a \varepsilon^{abc} A^b_\mu A^c_\nu - e\hat{\Phi}^e \varepsilon^{ebd} A^b_\nu A^d_\mu\end{aligned}$$

Here we are, ready to taste the victorious result of what was more of a battle than a mere calculation.

$$\mathcal{F}_{\mu\nu} = \partial_\mu \left(A^a_\nu \hat{\Phi}^a\right) - \partial_\nu \left(A^a_\mu \hat{\Phi}^a\right) + \frac{1}{e}\varepsilon_{abc}\partial_\mu \hat{\Phi}^b \partial_\nu \hat{\Phi}^c \hat{\Phi}^a. \tag{229}$$

In this form the generalised electromagnetic field strength tensor is comprised of two parts that are not separately gauge invariant. Of course, the total tensor is trivially invariant as we have shown earlier. So it is that the two parts transform in a way that the changes cancel each other out. Thus, a singular gauge transformation that can change the homotopy class of the configuration and manipulate or even annihilate the Dirac string, 'transfers' the magnetic charge between the topological properties of the Higgs and gauge fields.

It is obvious that in the trivial topological sector[43] where $\Phi^a = (0 \quad 0 \quad u)$, $\mathcal{F}_{\mu\nu}$ reduces to the usual $F_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$ we all now and love. In that case, the Bianchi identity is trivially satisfied and there is no place for magnetic monopoles here. What happens when we choose a non-trivial vacuum field configuration, we shall soon see, but first we will present another interesting way to arrive at (229) with some rather useful insight to it.

F. Generalised Electromagnetic Field Strength Tensor: An alternative approach

Returning to (229), we define the electromagnetic potential as the projection of the gauge potential on Φ^a as:

$$A^{\text{em}}_\mu = A^a_\mu \hat{\Phi}^a \tag{230}$$

Remembering also the condition that the covariant derivative on the spatial boundary is 0:

$$D_\mu \Phi^a = 0 \tag{231}$$

We propose that [8]:

$$A^a_\mu = A^{\text{em}}_\mu \hat{\Phi}^a + e^{-1}\varepsilon_{abc}\hat{\Phi}^b \partial_\mu \hat{\Phi}^c \tag{232}$$

is a general solution[44] to (231) and satisfies (230). We can readily test those claims.

Multiplying both sides of (232) with Φ^a and making use of $\Phi^a\Phi^a = u^2$, we easily retrieve (230).

For the verification of the second equation:

$$\begin{aligned}
&\partial_\mu \Phi^a - e\varepsilon_{abc}\frac{\Phi^b A^{\text{em}}_\mu}{u}\Phi^c - e\varepsilon_{abc}\frac{1}{u^2 e}\varepsilon_{bde}\Phi^c\Phi^d\partial_\mu\Phi^e = 0\\
&\partial_\mu \Phi^a - 0 + \frac{1}{u^2}\left(\delta^{ad}\delta^{ce} - \delta^{ae}\delta^{cd}\right)\Phi^c\Phi^d\partial_\mu\Phi^e = 0\\
&\partial_\mu \Phi^a + \frac{1}{u^2}\Phi^a\partial_\mu\Phi^c\Phi^c - \frac{1}{u^2}\Phi^c\Phi^c\partial\mu\Phi^a = 0
\end{aligned} \tag{233}$$

Inserting our newly acquired relation for the gauge potential into the field tensor will yield:

$$\begin{aligned}
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e\varepsilon_{abc}A^b_\mu A^c_\nu\\
F^a_{\mu\nu} &= \partial_\mu\left(\frac{\Phi^a}{u}A^{\text{em}}_\nu + \frac{1}{u^2 e}\varepsilon_{abc}\Phi^b\partial_\nu\Phi^c\right)\\
&- \partial_\nu\left(\frac{\Phi^a}{u}A^{\text{em}}_\mu + \frac{1}{u^2 e}\varepsilon_{abc}\Phi^b\partial_\mu\Phi^c\right)\\
&- e\varepsilon_{abc}\left(\frac{\Phi^b}{u}A^{\text{em}}_\mu + \frac{1}{u^2 e}\varepsilon_{bde}\Phi^d\partial_\mu\Phi^e\right)\left(\frac{\Phi^c}{u}A^{\text{em}}_\nu + \frac{1}{u^2 e}\varepsilon_{cfg}\Phi^f\partial_\nu\Phi^g\right)
\end{aligned} \tag{234}$$

Now that's a lot off terms, so we better start working on them one by one[45]:

$$\begin{aligned}
\partial_\mu\left(\hat{\Phi}^a A^{\text{em}}_\nu + \frac{1}{e}\varepsilon_{abc}\hat{\Phi}^b\partial_\nu\hat{\Phi}^c\right) &= \frac{1}{e}\varepsilon_{abc}\partial_\mu\left(\hat{\Phi}^b\partial_\nu\hat{\Phi}^c\right) + \partial_\mu\left(\hat{\Phi}^a A^{\text{em}}_\nu\right)\\
&= \frac{1}{e}\varepsilon_{abc}\left[\left(\partial_\mu\hat{\Phi}^b\right)\left(\partial_\nu\hat{\Phi}^c\right) + \hat{\Phi}^b\partial_\mu\partial_\nu\hat{\Phi}^c\right] + \partial_\mu\left(\hat{\Phi}^a A^{\text{em}}_\nu\right)
\end{aligned}$$

---

[43] with a regular gauge potential
[44] In fact, A^{em}_μ can be an arbitrary vector potential. We just identify it with the potential of electromagnetism.
[45] Meanwhile, we switch to $\hat{\Phi}^a$ notation to ease the formalism a little bit.

Similarly for the second term:

$$
\begin{aligned}
-\partial_\nu\left(\hat{\Phi}^a A_\mu^{\mathrm{em}}+\frac{1}{e}\varepsilon_{abc}\hat{\Phi}^b\partial_\mu\hat{\Phi}^c\right) &= -\frac{1}{e}\varepsilon_{abc}\partial_\nu\left(\hat{\Phi}^b\partial_\mu\hat{\Phi}^c\right)-\partial_\nu\left(\Phi^a A_\mu^{\mathrm{em}}\right)\\
&= \frac{1}{e}\varepsilon_{abc}\left[-\left(\partial_\nu\hat{\Phi}^b\right)\left(\partial_\mu\hat{\Phi}^c\right)-\hat{\Phi}^b\partial_\nu\partial_\mu\hat{\Phi}^c\right]-\partial_\nu\left(\hat{\Phi}^a A_\mu^{\mathrm{em}}\right)\\
&= \frac{1}{e}\varepsilon_{abc}\left[\left(\partial_\nu\hat{\Phi}^c\right)\left(\partial_\mu\hat{\Phi}^b\right)-\hat{\Phi}^b\partial_\mu\partial_\nu\hat{\Phi}^c\right]-\partial_\nu\left(\hat{\Phi}^a A_\mu^{\mathrm{em}}\right)
\end{aligned}
$$

Gathering our first two results:

$$
\begin{gathered}
\partial_\mu\left(\hat{\Phi}^a A_\nu^{\mathrm{em}}+\frac{1}{e}\varepsilon_{abc}\hat{\Phi}^b\partial_\nu\hat{\Phi}^c\right)-\partial_\nu\left(\hat{\Phi}^a A_\mu^{\mathrm{em}}+\frac{1}{e}\varepsilon_{abc}\hat{\Phi}^b\partial_\mu\hat{\Phi}^c\right)=\\
\frac{2}{e}\varepsilon_{abc}\left(\partial_\mu\hat{\Phi}^b\right)\left(\partial_\nu\hat{\Phi}^c\right)+\partial_\mu\left(\hat{\Phi}^a A_\nu^{\mathrm{em}}\right)-\partial_\nu\left(\hat{\Phi}^a A_\mu^{\mathrm{em}}\right)
\end{gathered}
$$

Antisymmetry annihilates the next term:

$$
-e\varepsilon_{abc}\hat{\Phi}^b\hat{\Phi}^c A_\mu^{\mathrm{em}}A_\nu^{\mathrm{em}}=0
$$

Proceeding:

$$
\begin{aligned}
-e\varepsilon_{abc}\frac{1}{e^2}\varepsilon_{bde}\varepsilon_{cfg}\hat{\Phi}^d\partial_\mu\hat{\Phi}^e\hat{\Phi}^f\partial_\nu\hat{\Phi}^g &= -\frac{1}{e}\varepsilon_{bde}\left[\left(\delta_{af}\delta_{bg}-\delta_{ag}\delta_{bf}\right)\hat{\Phi}^d\partial_\mu\hat{\Phi}^e\hat{\Phi}^f\partial_\nu\hat{\Phi}^g\right]\\
&= -\frac{1}{e}\varepsilon_{bde}\left(\hat{\Phi}^d\partial_\mu\hat{\Phi}^e\hat{\Phi}^a\partial_\nu\hat{\Phi}^b-\hat{\Phi}^d\partial_\mu\hat{\Phi}^e\hat{\Phi}^b\partial_\nu\hat{\Phi}^a\right)\\
&= -\frac{1}{e}\varepsilon_{bde}\left[\hat{\Phi}^d\partial_\mu\hat{\Phi}^e\left(\hat{\Phi}^a\partial_\nu\hat{\Phi}^b-\hat{\Phi}^b\partial_\nu\hat{\Phi}^a\right)\right]\\
&= -\frac{1}{e}\varepsilon_{bde}\hat{\Phi}^d\partial_\mu\hat{\Phi}^e\partial_\nu\hat{\Phi}^b\hat{\Phi}^a+\frac{1}{e}\varepsilon_{bde}\hat{\Phi}^d\hat{\Phi}^b\partial_\mu\hat{\Phi}^e\partial_\nu\hat{\Phi}^a\\
&= -\frac{1}{e}\varepsilon_{bde}\hat{\Phi}^d\partial_\mu\hat{\Phi}^e\partial_\nu\hat{\Phi}^b\hat{\Phi}^a
\end{aligned}
$$

And for the cross-terms:

$$
\begin{aligned}
-\varepsilon_{abc}\left(\varepsilon_{bde}\hat{\Phi}^d\partial_\mu\hat{\Phi}^e\hat{\Phi}^c A_\nu^{\mathrm{em}}\right) &= \left(\delta_{de}\delta_{ac}-\delta_{da}\delta_{ec}\right)\hat{\Phi}^d\partial_\mu\hat{\Phi}^e\hat{\Phi}^c A_\nu^{\mathrm{em}}\\
&= \left[\hat{\Phi}^a\left(\partial_\mu\hat{\Phi}^c\right)\hat{\Phi}^c A_\nu^{\mathrm{em}}-\hat{\Phi}^c\left(\partial_\mu\hat{\Phi}^a\right)\hat{\Phi}^c A_\nu^{\mathrm{em}}\right]\\
&= \left[\partial_\mu\left(\frac{\hat{\Phi}^c\hat{\Phi}^c}{2}\right)\hat{\Phi}^a A_\nu^{\mathrm{em}}-\left(\partial_\mu\hat{\Phi}^a\right)\hat{\Phi}^c\hat{\Phi}^c A_\nu^{\mathrm{em}}\right]\\
&= -\left(\partial_\mu\hat{\Phi}^a\right)A_\nu^{\mathrm{em}}
\end{aligned}
$$

Similarly the last terms falls too, giving a total of the cross terms:

$$
\begin{gathered}
-\varepsilon_{abc}\left(\varepsilon_{bde}\hat{\Phi}^d\partial_\mu\hat{\Phi}^e\hat{\Phi}^c A_\nu^{\mathrm{em}}\right)-\varepsilon_{abc}\left(\varepsilon_{cfg}\hat{\Phi}^f\partial_\mu\hat{\Phi}^g\hat{\Phi}^b A_\mu^{\mathrm{em}}\right)=\\
\left(\partial_\nu\hat{\Phi}^a\right)A_\mu^{\mathrm{em}}-\left(\partial_\mu\hat{\Phi}^a\right)A_\nu^{\mathrm{em}}
\end{gathered}
$$

Substituting all of the above into (234)

$$
F_{\mu\nu}^a=\left(\partial_\mu A_\nu^{\mathrm{em}}-\partial_\nu A_\mu^{\mathrm{em}}\right)\hat{\Phi}^a+\frac{2}{e}\varepsilon_{abc}\partial_\mu\hat{\Phi}^b\partial_\nu\hat{\Phi}^c-\frac{1}{e}\varepsilon_{dbc}\hat{\Phi}^d\partial_\mu\hat{\Phi}^b\partial_\nu\hat{\Phi}^c\hat{\Phi}^a \tag{235}
$$

And so performing an 'electromagnetic' projection by multiplying with $\hat{\Phi}^a$ we get the modified electromagnetic tensor (again):

$$
\mathcal{F}_{\mu\nu}=F_{\mu\nu}^a\hat{\Phi}^a=\partial_\mu A_\nu^{\mathrm{em}}-\partial_\nu A_\mu^{\mathrm{em}}+\frac{1}{e}\varepsilon_{abc}\partial_\mu\hat{\Phi}^b\partial_\nu\hat{\Phi}^c\hat{\Phi}^a \tag{236}
$$

We previously calculated the 'color' magnetic field, which gave us some good intuition about the existence of the magnetic monopole in our model. We are now in position to calculate the magnetic field as it stems from the definition of the field strength tensor (229) with the help of the 'hedgehog' solution (206) and the *SU(2)* gauge potential we calculated before in (208).

It is easy to see that[46]:

$$A_n^{\text{em}} = A_n^a \hat{\Phi}^a = \varepsilon_{ani} \frac{r^n r^a}{r^2} = 0 \tag{237}$$

because of the skew symmetry in n, a indices. So, only the 'Higgs' sector contributes. We continue in the calculation of $\mathcal{F}_{jk}$:

$$\begin{aligned}
\mathcal{F}_{jk} =& \frac{1}{e}\varepsilon_{abc}\frac{r^a}{r}\partial_j\left(\frac{r^b}{r}\right)\partial_k\left(\frac{r^c}{r}\right) \\
=& \frac{1}{e}\varepsilon_{abc}\frac{r^a}{r}\frac{r^2\delta^{bj} - r^j r^b}{r^3}\frac{r^2\delta^{kc} - r^k r^c}{r^3} \\
=& \frac{1}{e}\varepsilon_{abc}\frac{r^a}{r^7}\left(r^2\delta^{bj} - r^j r^b\right)\left(r^2\delta^{kc} - r^k r^c\right) \\
=& \frac{1}{e}\varepsilon_{abc}\left(r^4 r^a \delta^{bj}\delta^{kc} - r^2 r^a \delta^{bj} r^k r^c - r^2 r^a \delta^{kc} r^j r^b + r^a r^b r^c r^j r^k\right) \\
=& \frac{1}{e}\varepsilon_{abc}\frac{1}{r^3} r^a \delta^{bj}\partial^{kc} = \varepsilon_{ajk}\frac{r^a}{er^3}
\end{aligned} \tag{238}$$

where we used :

$$\partial_j\left(\frac{r^b}{r}\right) = \frac{r^2\delta^{bj} - r^j r^b}{r^3} \tag{239}$$

The magnetic field easily follows now

$$B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk} = \frac{r^a}{2er^3}\underbrace{\varepsilon_{ijk}\varepsilon_{ajk}}_{2\delta_{ai}} = \frac{r^i}{er^3} \tag{240}$$

which is exactly the field of a magnetic monopole.

G. The conserved magnetic current

We will abstain from the specific 'hedgehog' solution and delve into the general properties of the modified E/M tensor (229). More specifically, we are going to examine the dual tensor and its equation of motion:

$$\begin{aligned}
\partial_\mu \widetilde{\mathcal{F}}^{\mu\nu} &= \partial_\mu\left(\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\rho\sigma}\right) \\
&= \partial_\mu\left[\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\left(\partial_\rho A_\sigma^{\text{em}} - \partial_\sigma A_\rho^{\text{em}}\right)\right] + \partial_\mu\left[\frac{1}{2e}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abc}\hat{\Phi}^a\left(\partial_\rho\hat{\Phi}^b\right)\left(\partial_\sigma\hat{\Phi}^c\right)\right] \\
&= \frac{1}{2e}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abc}\partial_\mu\left(\hat{\Phi}^a\left(\partial_\rho\hat{\Phi}^b\right)\left(\partial_\sigma\hat{\Phi}^c\right)\right)
\end{aligned} \tag{241}$$

We can also write this in a more symmetrical form.[47]

$$\partial_\mu \widetilde{\mathcal{F}}^{\mu\nu} = \frac{1}{2e}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abc}(\partial_\mu\hat{\Phi}^a)(\partial_\rho\hat{\Phi}^b)(\partial_\sigma\hat{\Phi}^c) \tag{242}$$

---

[46] Here we calculate, only the spatial components of the gauge potential, since those are relevant in the construction of the magnetic field. We also use $\hat{\Phi}^a = \frac{r^a}{r}$.

[47] But it's really just for the looks. We will be almost exclusively be using the definition (241).

We see that the Bianchi identities are not satisfied! That's a really solid indication of the existence of magnetic charge in our theory. Not only that, but it is also conserved.

We define the *magnetic* (or toplogical) *current*:

$$k^{\nu} = \frac{1}{2e}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abc}\partial_{\mu}\left(\hat{\Phi}^{a}\left(\partial_{\rho}\hat{\Phi}^{b}\right)\left(\partial_{\sigma}\hat{\Phi}^{c}\right)\right) \tag{243}$$

Its conservation is trivial:

$$\partial_{\nu}k^{\nu} = \frac{1}{2e}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abc}\partial_{\nu}\partial_{\mu}\left(\hat{\Phi}^{a}\left(\partial_{\rho}\hat{\Phi}^{b}\right)\left(\partial_{\sigma}\hat{\Phi}^{c}\right)\right) = 0 \tag{244}$$

due to the antisymmetry of Levi-Civita and the permutation of partial derivatives.

We note that this is conserved current does not originate from some internal symmetry and Noether's theorem. It is conserved by its very definition and its origin its strictly topological, as we shall demonstrate soon.

Of course, every conserved current comes with a conserved charge and this case is no exception.

The 0-component of the magnetic current is:

$$\begin{aligned} k_0 &= -\frac{1}{2e}\varepsilon_{0mnk}\varepsilon_{abc}\partial^{m}\left(\hat{\Phi}^{a}\left(\partial^{n}\hat{\Phi}^{b}\right)\left(\partial^{k}\hat{\Phi}^{c}\right)\right) \\ &= \frac{1}{2e}\left(-\varepsilon_{0mnk}\right)\varepsilon_{abc}\partial^{m}\left(\hat{\Phi}^{a}\left(\partial^{n}\hat{\Phi}^{b}\right)\left(\partial^{k}\hat{\Phi}^{c}\right)\right) \\ &= \frac{1}{2e}\varepsilon_{mnk}\varepsilon_{abc}\partial_{m}\left(\hat{\Phi}^{a}\left(\partial_{n}\hat{\Phi}^{b}\right)\left(\partial_{k}\hat{\Phi}^{c}\right)\right) \end{aligned} \tag{245}$$

where we took into account $\varepsilon_{0123} = -1, \varepsilon_{123} = +1$. The magnetic charge is, then, given by integration in the whole space:

$$\begin{aligned} g = \int d^3x k_0 &= \frac{1}{2e}\int d^3x\varepsilon_{mnk}\varepsilon_{abc}\partial_{m}\left(\hat{\Phi}^{a}\left(\partial_{n}\hat{\Phi}^{b}\right)\left(\partial_{k}\hat{\Phi}^{c}\right)\right) \\ g &= \frac{1}{2e}\int dS_m\varepsilon_{mnk}\varepsilon_{abc}\left(\hat{\Phi}^{a}\left(\partial_{n}\hat{\Phi}^{b}\right)\left(\partial_{k}\hat{\Phi}^{c}\right)\right) \end{aligned} \tag{246}$$

We can get some intuition on how this integral should evaluate[48]. For a spherically symmetric field[49] each component of the integral dS_1, dS_2, dS_3 has no reason to differ from each other. The only change that can occur is via the Levi-Civita symbol ε_{mnk} which will evaluate $\varepsilon_{1nk}, \varepsilon_{2nk}$ and ε_{3nk} Naively now, one could argue thatε_{2nk} should cancel the ε_{1nk} because the latter will result in the same terms as the first only with different sign (due to 2 being on the first index of the Levi-Civita tensor). So we would be left with an integral like:

$$g = \frac{1}{2e}\int dS_3\varepsilon_{3nk}\varepsilon_{abc}\left(\hat{\Phi^{a}}\left(\partial_{n}\hat{\Phi}^{b}\right)\left(\partial_{k}\hat{\Phi}^{c}\right)\right) = \frac{1}{2e}\int dS\varepsilon_{nk}\varepsilon_{abc}\left(\hat{\Phi^{a}}\left(\partial_{n}\hat{\Phi}^{b}\right)\left(\partial_{k}\hat{\Phi}^{c}\right)\right) \tag{247}$$

with $n, k = 1, 2$.

Enough waving our hands, in order to believe anything of that blabbering we have to get a little more rigorous.

We start with a parametrization ξ_u, with $u = 1, 2$ since that is the minimum number of coordinates needed to describe the surface of the sphere S^2:

$$\begin{aligned} \partial_n\hat{\Phi}^a &= \frac{\partial\xi^u}{\partial r^n}\frac{\partial\hat{\Phi}^a}{\partial\xi^u} \\ dS_m &= \frac{1}{2}\varepsilon_{mij}\varepsilon_{uv}\frac{\partial r^i}{\partial\xi^u}\frac{\partial r^j}{\partial\xi^v}d^2\xi \end{aligned} \tag{248}$$

[48] Friendly reminder that: $\mathrm{d}S_{\kappa} = \mathrm{d}S^{ij} = \mathrm{d}x^i\ \mathrm{d}x^j$
[49] Like our beloved hedgehog.

Then (246) becomes:

$$
\begin{aligned}
g &= \frac{1}{2e}\int dS_m \varepsilon_{mnk}\varepsilon_{abc}\left(\hat{\Phi}^a\left(\partial_n\hat{\Phi}^b\right)\left(\partial_k\hat{\Phi}^c\right)\right) \\
&= \frac{1}{2e}\int \frac{1}{2}\varepsilon_{mij}\varepsilon_{uv}\frac{\partial r^i}{\partial \xi^u}\frac{\partial r^j}{\partial \xi^v}\varepsilon_{mnk}\varepsilon_{abc}\left(\hat{\Phi}^a\left(\frac{\partial \xi^{u'}}{\partial r^n}\frac{\partial \hat{\Phi}^b}{\partial \xi^{u'}}\right)\left(\frac{\partial \xi^{v'}}{\partial r^k}\frac{\partial \hat{\Phi}^c}{\partial \xi^{v'}}\right)\right)d^2\xi \\
&= \frac{1}{2e}\int \frac{1}{2}(\delta_{in}\delta_{jk}-\delta_{ik}\delta jn)\varepsilon_{uv}\frac{\partial r^i}{\partial \xi^u}\frac{\partial r^j}{\partial \xi^v}\varepsilon_{abc}\left(\hat{\Phi}^a\left(\frac{\partial \xi^{u'}}{\partial r^n}\frac{\partial \hat{\Phi}^b}{\partial \xi^{u'}}\right)\left(\frac{\partial \xi^{v'}}{\partial r^k}\frac{\partial \hat{\Phi}^c}{\partial \xi^{v'}}\right)\right)d^2\xi \\
&= \frac{1}{2e}\int \frac{1}{2}\left[\left(\frac{\partial r^n}{\partial \xi^u}\frac{\partial r^k}{\partial \xi^v}-\frac{\partial r^k}{\partial \xi^u}\frac{\partial r^n}{\partial \xi^v}\right)\frac{\partial \xi^{u'}}{\partial r^n}\frac{\partial \xi^{v'}}{\partial r^k}\right]\left(\hat{\Phi}^a\frac{\partial \hat{\Phi}^b}{\partial \xi^{u'}}\frac{\partial \hat{\Phi}^c}{\partial \xi^{v'}}\right)\varepsilon_{abc}\varepsilon_{uv}d^2\xi \\
&= \frac{1}{2e}\int \frac{1}{2}\left[\delta_{uu'}\delta_{vv'}-\delta_{uv'}\delta_{vu'}\right]\left(\hat{\Phi}^a\frac{\partial \hat{\Phi}^b}{\partial \xi^{u'}}\frac{\partial \hat{\Phi}^c}{\partial \xi^{v'}}\right)\varepsilon_{abc}\varepsilon_{uv}d^2\xi \\
&= \frac{1}{2e}\int \frac{1}{2}\left(\hat{\Phi}^a\frac{\partial \hat{\Phi}^b}{\partial \xi^{u}}\frac{\partial \hat{\Phi}^c}{\partial \xi^{v}}-\hat{\Phi}^a\frac{\partial \hat{\Phi}^b}{\partial \xi^{v}}\frac{\partial \hat{\Phi}^c}{\partial \xi^{u}}\right)\varepsilon_{abc}\varepsilon_{uv}d^2\xi \\
&= \frac{1}{e}\int \frac{1}{2}\varepsilon_{abc}\varepsilon_{uv}\hat{\Phi}^a\partial_u\hat{\Phi}^b\partial_v\hat{\Phi}^c d^2\xi
\end{aligned}
\tag{249}
$$

and we successfully arrived at (247). Before we proceed, it is imminent to return to topology temporarily and talk a little bit more specifically about winding numbers.

IX. TOPOLOGICAL CONSIDERATIONS III: WINDING NUMBERS

Intuitively speaking, the *winding number* of a closed curve around a given point is the numbers of times this certain curve travels clockwise around that point and it is represented (more often than not) by an integer number.

In topology, we use the winding number to describe the degree of a continuous mapping (also mentioned often as Brouwer degree). Despite the existence of a general definition, it is perhaps simpler and more useful to examine the case of an S^n to S^n map.

Let $f : S^n \to S^n$ be a continuous map. Then, f also induces a homomorphism $f_* : H_n(S^n) \to H_n(S^n)$ where $H_n(\cdot)$ is the n-th homology group[50]. Now considering $H_n(S^n) \cong \mathbb{Z}$ (much like the n-th homotopy group[51]), then we see that f_* must be of some sort $f_* : x \mapsto \alpha x$ where $\alpha \in \mathbb{Z}$. That integer α is what we call *degree* or *winding number* of f.
Below we are going to examine the definitions and properties of various winding numbers on certain S^n manifolds and illustrate some some simple examples to build intuition and correspondence with what is needed in our case. The examples are drawn almost exclusively from Srednicki's Solitons and Instantons chapters [32].

A. S^1 Winding Number

The winding number is a topological constant and sometimes it coincides with other constants of topological nature. For example, for a circle S^1, we know its first homotopy group is $\pi_1(S^1) = Z$. The winding number for a map between two circles $U : S^1 \to S^1$ is defined :

$$n = \frac{i}{2\pi}\int_0^{2\pi} d\theta U \frac{dU^\dagger}{d\theta} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{d\alpha}{d\theta} \tag{250}$$

where $U = e^{i\alpha(\theta)}$.

Well, of course, we know that U is a representation of the $U(1)$ abelian group and hence the function α has to be of the form $\alpha(\theta) = n\theta + \delta$. and substitution in the above integral verifies it.

Although the meaning of the winding number in S^1 is clear now, it is advisable that we cast the above integral in a similar form to establish some correspondence with higher order spheres (and especially S^2).

Noting that we can parametrize S^1 on the R^2 with a unit vector $\hat{\mathbf{e}}(\mathbf{r}) = (\cos\alpha(\theta), \sin\alpha(\theta))$, then we can write (250) as:

$$n = \frac{1}{4\pi}\int_0^{2\pi} d\theta \varepsilon^{ab} \hat{e}^a \partial_\theta \hat{e}^b \tag{251}$$

Now, the winding number of a map from S^2 to S^2 is given by:

$$n = \frac{1}{8\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \varepsilon^{abc} \varepsilon_{ij} \hat{e}^a \partial_i \hat{e}^b \partial_j \hat{e}^c \tag{252}$$

which comes as no surprise[52], since it is an immediate generalisation of (251).

Just for reference purposes, we give here the general definition of the winding number for a parametrization $\gamma : \mathbb{R}^n \supseteq D \to \mathbb{R}^{n+1}$:

$$n = \int_D \frac{1}{\|\gamma(\mathbf{x})\|^{n+1}} \det\left(\partial_1\gamma(\mathbf{x}), \ldots, \partial_i\gamma(\mathbf{x}), \ldots, \partial_n\gamma(\mathbf{x}), \gamma(\mathbf{x})\right) d\mathbf{x} \tag{253}$$

---

[50] Homology groups are closely related to homotopy groups we mentioned earlier in the sense that they categorize "holes" in topological spaces. Vaguely, they are groups that are made of n-cycles that are not n-boundaries. They will not be needed any more in our discussion, so we will not go into any more details.
[51] See (72)
[52] One should also note at this point the similarity with (249), so this whole discussion is not in vain after all.

but we are going to mess ourselves only with definitions (251) and (252).

So far we have continuously claimed that winding numbers are topological invariants, i.e. they remain the same under continuous transformations. It is now the time to prove so.

We are going to start with the S^1 winding number and study its behaviour under infinitesimal transformations. Of course, every smooth transformation can be made by compounding infinitesimal ones, so it will also prove to be invariant under smooth transformations. The definition of use to us will be (250) remembering that $U^\dagger U = 1$.

We consider an infinitesimal transformation of U: $U \to U + \delta U$: then using

$$U^\dagger U = 1$$

we get

$$\delta U^\dagger U + U^\dagger \delta U = 0$$

Now solving for $\delta U^\dagger$:

$$\delta U^\dagger = -U^{\dagger 2}\delta U$$

Next up, we consider the variation of the integrand in (250):

$$\begin{aligned}
\delta\left(U\partial_\theta U^\dagger\right) &= \delta U\partial_\theta U^\dagger + U\partial_\theta\delta U^\dagger \\
&= \delta U\partial_\theta U^\dagger + U\partial_\theta\left(-U^{\dagger 2}\delta U\right) \\
&= \delta U\partial_\theta U^\dagger + U\left[-2U^\dagger\left(\partial_\theta U^\dagger\right)\delta U - U^{\dagger 2}\partial_\theta\delta U\right] \\
&= \delta U\partial_\theta U^\dagger + \left[-2\left(\partial_\theta U^\dagger\right)\delta U - U^\dagger\partial_\theta\delta U\right] \\
&= -\left(\partial_\theta U^\dagger\right)\delta U - U^\dagger\partial_\theta\delta U \\
&= -\partial_\theta\left(U^\dagger\delta U\right).
\end{aligned}$$

We are now in position to calculate the variation of the winding number:

$$\delta n = \frac{i}{2\pi}\int_0^{2\pi} d\theta\partial_\theta\left(U^\dagger\delta U\right) = \frac{i}{2\pi}U^\dagger\delta U\Big|_{\theta=0}^{\theta=2\pi} = 0 \tag{254}$$

since U is continuous and $\theta = 0$ is identified with $\theta = 2\pi$ which finishes the proof of our assertion.

As a next task, we are going to show that the group product of two maps $U_n(\theta)$,$U_k(\theta)$ with winding number n, k respectively is a map $U_{n+k}(\theta)$ with a winding number $n + k$.

First we deform $U_n(\theta)$ so that it equals 1 in the interval $\theta \in [0, \pi]$.

Similarly we deform $U_k(\theta)$, so that it equals 1 in the interval $\theta \in [\pi, 2\pi]$ then their winding numbers being invariant under such smooth transformations will retain the same value but will now be given by:

$$\begin{aligned}
k &= \frac{i}{2\pi}\int_0^{\pi} d\theta U_k\partial_\theta U_k^\dagger \\
n &= \frac{i}{2\pi}\int_\pi^{2\pi} d\theta U_n\partial_\theta U_n^\dagger
\end{aligned} \tag{255}$$

So the winding number of U_{n+k} will be

$$\begin{aligned}
n + k &= \frac{i}{2\pi}\int_0^{2\pi} d\theta U_n U_k\partial_\theta\left(U_n U_k\right)^\dagger \\
&= \frac{i}{2\pi}\int_0^{\pi} d\theta U_k\partial_\theta U_k^\dagger + \frac{i}{2\pi}\int_\pi^{2\pi} d_\theta U_n\partial_\theta U_k^\dagger = n + k
\end{aligned} \tag{256}$$

since in $\theta \in [0, \pi]$ we have $U_n = 1$ and $\partial_\theta U_n = 0$ and similarly in $\theta \in [\pi, 2\pi]$ it is $U_k = 1$ and $\partial_\theta U_k = 0$.

B. S^3 Winding Number

Before involving ourselves with the S^2 case, we are going to shed some light on the S^3 winding number or *Pontryagin index.* It is given by

$$n = \frac{-1}{24\pi^2} \int d^3x \varepsilon^{ijk} \operatorname{Tr}\left[\left(\mathbf{U}\partial_i \mathbf{U}^\dagger\right) \left(\mathbf{U}\partial_j \mathbf{U}^\dagger\right) \left(\mathbf{U}\partial_k \mathbf{U}^\dagger\right) \right] \tag{257}$$

where $\mathbf{U} \in SU(2)$.

The group $SU(2)$ has the topology of a 3-sphere. This can be seen by writing a general 2x2 matrix:

$$\mathbf{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{258}$$

Then

$$\mathbf{U}^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \tag{259}$$

and its inverse is

$$\mathbf{U}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{260}$$

Elements of our group have unit determinant so we can omit the denominator in the above relationship. Now, we just have to identify $\mathbf{U}^\dagger = \mathbf{U}^{-1}$ and acquire some relationships between the matrix elements. That is $d = a^*$ and $c = -b^*$. Now, writing a, b in complex form $a = a_4 + ia_3$ and $b = a_2 + ia_1$ where $a_\mu \in \mathcal{R}$, we get

$$\mathbf{U} = \begin{pmatrix} a_4 + ia_3 & i\left(a_1 - ia_2\right) \\ i\left(a_1 + ia_2\right) & a_4 - ia_3 \end{pmatrix} = a_4 + i\vec{a} \cdot \vec{\sigma} \tag{261}$$

and putting the determinant to 1 yields

$$\det \mathbf{U} = a_\mu a_\mu = 1 \tag{262}$$

which is just a 3-sphere equation for the coefficients of a, b.

Now that we understand how the $SU(2)$ group gets involved, we will prove invariance of the winding number under infinitesimal (and therefore smooth) transformations.

Similarly to the S^1 case consider

$$\mathbf{U} \to \mathbf{U} + \delta\mathbf{U} \tag{263}$$

then using $\mathbf{U}^\dagger\mathbf{U} = 1$ we get $\delta\mathbf{U}^\dagger = -\mathbf{U}^\dagger\delta\mathbf{U}\mathbf{U}^\dagger$, where the only difference is that $\mathbf{U}$'s now do not commute.

The variation of each term entering the trace of (257) is

$$\begin{aligned} \delta\left(\mathbf{U}\partial_k\mathbf{U}^\dagger\right) &= \delta\mathbf{U}\partial_k\mathbf{U}^\dagger + \mathbf{U}\partial_k\delta\mathbf{U}^\dagger \\ &= \delta\mathbf{U}\partial_k\mathbf{U}^\dagger - \mathbf{U}\partial_k\left(\mathbf{U}^\dagger\delta\mathbf{U}\mathbf{U}^\dagger\right) \\ &= \delta\mathbf{U}\partial_k\mathbf{U}^\dagger - \mathbf{U}\partial_k\mathbf{U}^\dagger\delta\mathbf{U}\mathbf{U}^\dagger - \mathbf{U}\mathbf{U}^\dagger\partial_k\delta\mathbf{U}\mathbf{U}^\dagger - \mathbf{U}\mathbf{U}^\dagger\delta\mathbf{U}\partial_k\mathbf{U}^\dagger \\ &= -\mathbf{U}\partial_k\mathbf{U}^\dagger\delta\mathbf{U}\mathbf{U}^\dagger - \mathbf{U}\mathbf{U}^\dagger\partial_k\delta\mathbf{U}\mathbf{U}^\dagger \\ &= -\mathbf{U}\left(\partial_k\mathbf{U}^\dagger\delta\mathbf{U} + \mathbf{U}^\dagger\partial_k\delta\mathbf{U}\right)\mathbf{U}^\dagger \\ &= -\mathbf{U}\partial_k\left(\mathbf{U}^\dagger\delta\mathbf{U}\right)\mathbf{U}^\dagger \end{aligned} \tag{264}$$

However, the variations of $\mathbf{U}\partial_i\mathbf{U}^\dagger, \mathbf{U}\partial_j\mathbf{U}^\dagger$, and $\mathbf{U}\partial_k\mathbf{U}^\dagger$ contribute equally to δn after cyclic permutations of the trace.

We, thus, have

$$\begin{aligned}&\varepsilon^{ijk}\,\mathrm{Tr}\left[\left(\mathbf{U}\partial_i\mathbf{U}^\dagger\right)\left(\mathbf{U}\partial_j\mathbf{U}^\dagger\right)\delta\left(\mathbf{U}\partial_k\mathbf{U}^\dagger\right)\right]\\ &\quad=-\varepsilon^{ijk}\,\mathrm{Tr}\left[\left(\mathbf{U}\partial_i\mathbf{U}^\dagger\right)\left(\mathbf{U}\partial_j\mathbf{U}^\dagger\right)\mathbf{U}\partial_k\left(\mathbf{U}^\dagger\delta\mathbf{U}\right)\mathbf{U}^\dagger\right]\\ &\quad=-\varepsilon^{ijk}\,\mathrm{Tr}\left[\partial_i\mathbf{U}^\dagger\mathbf{U}\partial_j\mathbf{U}^\dagger\mathbf{U}\partial_k\left(\mathbf{U}^\dagger\delta\mathbf{U}\right)\right].\end{aligned} \tag{265}$$

We used the cyclic property of the trace and $\mathbf{U}^\dagger\mathbf{U}=1$ to get the last line. After integrating ∂_k by parts, terms with two derivatives acting on a single $\mathbf{U}^\dagger$ vanish when contracted with ε^{ijk}. The remaining terms are

$$\begin{aligned}&-\varepsilon^{ijk}\,\mathrm{Tr}\left[\partial_i\mathbf{U}^\dagger\mathbf{U}\partial_j\mathbf{U}^\dagger\mathbf{U}\partial_k\left(\mathbf{U}^\dagger\delta\mathbf{U}\right)\right]\\ &\quad=+\varepsilon^{ijk}\left(\mathrm{Tr}\left[\partial_i\mathbf{U}^\dagger\partial_k\mathbf{U}\partial_j\mathbf{U}^\dagger\delta\mathbf{U}\right]+\mathrm{Tr}\left[\partial_i\mathbf{U}^\dagger\mathbf{U}\partial_j\mathbf{U}^\dagger\partial_k\mathbf{U}\mathbf{U}^\dagger\delta\mathbf{U}\right]\right),\end{aligned} \tag{266}$$

where we used $\mathbf{U}\mathbf{U}^\dagger=1$ in the first term. In the second term, we now use $\mathbf{U}\partial_j\mathbf{U}^\dagger=-\partial_j\mathbf{U}\mathbf{U}^\dagger$ and $\partial_k\mathbf{U}\mathbf{U}^\dagger=-\mathbf{U}\partial_k\mathbf{U}^\dagger$. followed by $\mathbf{U}^\dagger\mathbf{U}=1$, to get

$$\begin{aligned}&-\varepsilon^{ijk}\,\mathrm{Tr}\left[\partial_i\mathbf{U}^\dagger\mathbf{U}\partial_j\mathbf{U}^\dagger\mathbf{U}\partial_k\left(\mathbf{U}^\dagger\delta\mathbf{U}\right)\right]\\ &\quad=+\varepsilon^{ijk}\left(\mathrm{Tr}\left[\partial_i\mathbf{U}^\dagger\partial_k\mathbf{U}\partial_j\mathbf{U}^\dagger\delta\mathbf{U}\right]+\mathrm{Tr}\left[\partial_i\mathbf{U}^\dagger\partial_j\mathbf{U}\partial_k\mathbf{U}^\dagger\delta\mathbf{U}\right]\right)\end{aligned} \tag{267}$$

The two terms are now symmetric on $j\leftrightarrow k$, and so cancel when contracted with ε^{ijk} and so we have

$$\delta n=0 \tag{268}$$

C. S^2 Winding Number

Now we are going to prove the invariance of the winding number under smooth transformations for the S^2 case.

We will use (252) for that purpose. In fact, we shall rewrite it replacing $\hat{e}$ with $\hat{\Phi}$ to establish correspondence with the t'Hooft-Polyakov monopole.

$$n=\frac{1}{8\pi}\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\varepsilon^{abc}\varepsilon_{ij}\hat{\Phi}^a\partial_i\hat{\Phi}^b\partial_j\hat{\Phi}^c \tag{269}$$

Using

$$\varepsilon^{ij}\varepsilon^{abc}\hat{\varphi}^a\partial_i\hat{\varphi}^b\partial_j\hat{\varphi}^c=2\varepsilon^{abc}\hat{\varphi}^a\partial_\theta\hat{\varphi}^b\partial_\phi\hat{\varphi}^c \tag{270}$$

we also get the equivalent form

$$n=\frac{1}{4\pi}\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\varepsilon^{abc}\hat{\Phi}^a\partial_1\hat{\Phi}^b\partial_2\hat{\Phi}^c$$

We follow much the same steps as before. Starting from:

$$\hat{\mathbf{\Phi}}\cdot\hat{\mathbf{\Phi}}=1$$

we have

$$\delta(\hat{\mathbf{\Phi}}\cdot\hat{\mathbf{\Phi}})=2\hat{\mathbf{\Phi}}\cdot\delta\hat{\mathbf{\Phi}}=0$$

and likewise

$$\partial_i(\hat{\mathbf{\Phi}}\cdot\hat{\mathbf{\Phi}})=2\hat{\mathbf{\Phi}}\cdot\partial_i\hat{\mathbf{\Phi}}=0$$

Since $\delta\hat{\mathbf{\Phi}},\partial_1\hat{\mathbf{\Phi}}$, and $\partial_2\hat{\mathbf{\Phi}}$ are all orthogonal to $\hat{\mathbf{\Phi}}$, it is

$$\left(\partial_1\hat{\mathbf{\Phi}}\times\partial_2\hat{\mathbf{\Phi}}\right)\cdot\delta\hat{\mathbf{\Phi}}=0$$

and so in tensorial form:

$$\varepsilon^{abc}\delta\hat{\Phi}^a\partial_1\hat{\Phi}^b\partial_2\hat{\Phi}^c = 0$$

It also holds for arbitrary partial derivative indices, because if they are the same contraction with Levi-Civita vanishes them and we have only two choices.

$$\varepsilon^{abc}\delta\hat{\Phi}^a\partial_i\hat{\Phi}^b\partial_j\hat{\Phi}^c = 0$$

We are going to use this result right afterwards.

The variation of the integrand of (270) yields:

$$\delta\left(\hat{\Phi}^a\partial_i\hat{\Phi}^b\partial_j\hat{\Phi}^c\right) = \left(\delta\hat{\Phi}^a\right)\partial_i\hat{\Phi}^b\partial_j\hat{\Phi}^c + \hat{\Phi}^a\left(\partial_i\delta\hat{\Phi}^b\right)\partial_j\hat{\Phi}^c + \hat{\Phi}^a\partial_i\hat{\Phi}^b\left(\partial_j\delta\hat{\Phi}^c\right)$$

The first term when contracted with ε^{abc} will vanish. We manipulate the other two terms as follows:

$$\begin{gathered}\hat{\Phi}^a\left(\partial_i\delta\hat{\Phi}^b\right)\partial_j\hat{\Phi}^c + \hat{\Phi}^a\partial_i\hat{\Phi}^b\partial_j\hat{\Phi}^c = \\ \partial_i\left(\hat{\Phi}^a\delta\hat{\Phi}^b\partial_j\hat{\Phi}^c\right) - \delta\hat{\Phi}^b\partial_i\hat{\Phi}^a\partial_j\hat{\Phi}^c - \hat{\Phi}^a\delta\hat{\Phi}^b\partial_i\partial_j\hat{\Phi}^c \\ +\partial_j\left(\hat{\Phi}^a\partial_i\hat{\Phi}^b\delta\hat{\Phi}^c\right) - \delta\hat{\Phi}^c\partial_j\hat{\Phi}^a\partial_i\hat{\Phi}^b - \hat{\Phi}^a\delta\hat{\Phi}^c\partial_i\partial_j\hat{\Phi}^b\end{gathered}$$

Now the total derivatives will yield 0 upon integration due to their single-valuedness and identification of the integration points. The middle terms of each line disappear due to contraction with ε^{abc} and the last terms vanish by contraction with ε_{ij} and the commutativity of partial derivatives.

Thus, we can safely conclude that:

$$\delta n = 0$$

Equation (252) is useful for proving properties and stuff, but when it comes to actually calculating a winding number given a map of some sort, the many contractions between indices make it kind of tedious. Therefore, we will try to cast (252) in a more practical form for applications [12].

We start by an arbitrary parametrization of the sphere S^2:

$$\hat{e}(\mathbf{r}) = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha) \tag{271}$$

where $\alpha(\theta,\phi)$ and $\beta(\theta,\phi)$ are functions of the polar and azimuthal angles and we use $\hat{e}$ insted of $\hat{\Phi}$ temporarily.

Before substituting (271) into (252), we first write the terms of the contractions explicitly:

$$\begin{aligned}n &= \frac{1}{8\pi}\varepsilon_{ij}\varepsilon_{abc}\int d\theta d\phi \hat{e}^a\partial_i\hat{e}^b\partial_j\hat{e}^c = \\ &= \frac{1}{8\pi}\varepsilon_{abc}2\int d\theta d\phi \hat{e}^a\partial_\theta\hat{e}^b\partial_\phi\hat{e}^c = \\ &= \frac{1}{4\pi}\varepsilon_{abc}\int d\theta d\phi \hat{e}^1\left(\partial_\theta\hat{e}^2\partial_\phi\hat{e}^3 - \partial_\theta\hat{e}^3\partial_\phi\hat{e}^2\right) + \hat{e}^2\left(\partial_\theta\hat{e}^3\partial_\phi\hat{e}^1 - \partial_\theta\hat{e}^1\partial_\phi\hat{e}^3\right) \\ &+ \hat{e}^3\left(\partial_\theta\hat{e}^1\partial_\phi\hat{e}^2 - \partial_\phi\hat{e}^1\partial_\theta\hat{e}^2\right)\end{aligned}$$

At this point we substitute (271), so fasten your seat belts, it's going to be a bumpy ride.

$$n = \frac{1}{4\pi}\int d\theta d\phi \sin\alpha\cos\beta\left(\cos\alpha\sin\beta\frac{\partial\alpha}{\partial\theta} + \sin\alpha\cos\beta\frac{\partial\beta}{\partial\theta}\right)(-\sin\alpha)\frac{\partial\alpha}{\partial\phi}$$
$$- \sin\alpha\cos\beta\left(\cos\alpha\sin\beta\frac{\partial\alpha}{\partial\phi} + \sin\alpha\cos\beta\frac{\partial\beta}{\partial\phi}\right)(-\sin\alpha)\frac{\partial\alpha}{\partial\phi}$$
$$+ \sin\alpha\sin\beta\left(-\sin\alpha\frac{\partial\alpha}{\partial\theta}\left(\cos\alpha\cos\beta\frac{\partial\alpha}{\partial\phi} - \sin\alpha\sin\beta\frac{\partial\beta}{\partial\phi}\right) + \sin\alpha\frac{\partial\alpha}{\partial\phi}\left(\cos\alpha\cos\beta\frac{\partial\alpha}{\partial\theta} - \sin\alpha\sin\beta\frac{\partial\beta}{\partial\theta}\right)\right)$$
$$+ \cos\alpha\left(\left(\cos\alpha\cos\beta\frac{\partial\alpha}{\partial\theta} - \sin\alpha\sin\beta\frac{\partial\beta}{\partial\theta}\right)\left(\cos\alpha\sin\beta\frac{\partial\alpha}{\partial\phi} + \sin\alpha\cos\beta\frac{\partial\beta}{\partial\phi}\right)\right.$$
$$- \cos\alpha\left(\cos\alpha\cos\beta\frac{\partial\alpha}{\partial\alpha} - \sin\alpha\sin\beta\frac{\partial\beta}{\partial\phi}\right)\left(\cos\alpha\sin\beta\frac{\partial\alpha}{\partial\theta} + \sin\alpha\cos\beta\frac{\partial\beta}{\partial\phi}\right)$$

$$n = \frac{1}{4\pi}\int d\theta d\phi\Big[-\sin^2\alpha\cos\alpha\sin\beta\cos\beta\frac{\partial\alpha}{\partial\theta}\frac{\partial\alpha}{\partial\phi} - \sin^3\alpha\cos^2\beta\frac{\partial\beta}{\partial\theta}\frac{\partial\alpha}{\partial\phi} + \sin^2\alpha\cos\alpha\sin\beta\cos\beta\frac{\partial\alpha}{\partial\theta}\frac{\partial\alpha}{\partial\phi}$$
$$+ \sin^3\alpha\cos^2\beta\frac{\partial\alpha}{\partial\theta}\frac{\partial\beta}{\partial\phi} - \sin^2\alpha\cos\alpha\sin\beta\cos\beta\frac{\partial\alpha}{\partial\theta}\frac{\partial\alpha}{\partial\phi} + \sin^3\alpha\sin^2\beta\frac{\partial\alpha}{\partial\theta}\frac{\partial\beta}{\partial\phi} + \sin^2\alpha\cos\alpha\sin\beta\cos\beta\frac{\partial\alpha}{\partial\theta}\frac{\partial\alpha}{\partial\phi}$$
$$- \sin^3\alpha\sin^2\beta\frac{\partial\beta}{\partial\theta}\frac{\partial\alpha}{\partial\phi} + \cos^3\alpha\sin\beta\cos\beta\frac{\partial\alpha}{\partial\theta}\frac{\partial\alpha}{\partial\phi} - \sin\alpha\cos^2\alpha\sin^2\beta\frac{\partial\beta}{\partial\theta}\frac{\partial\alpha}{\partial\phi} + \sin\alpha\cos^2\alpha\cos^2\beta\frac{\partial\alpha}{\partial\theta}\frac{\partial\theta}{\partial\phi}$$
$$- \sin^2\alpha\cos\alpha\sin\beta\cos\beta\frac{\partial\beta}{\partial\theta}\frac{\partial\beta}{\partial\phi} - \cos^3\alpha\sin\beta\cos\beta\frac{\partial\alpha}{\partial\theta}\frac{\partial\alpha}{\partial\phi} + \sin\alpha\cos^2\alpha\sin^2\beta\frac{\partial\alpha}{\partial\theta}\frac{\partial\beta}{\partial\phi} - \sin^2\alpha\cos^2\alpha\cos^2\beta\frac{\partial\beta}{\partial\theta}\frac{\partial\alpha}{\partial\phi}$$
$$+ \sin^2\alpha\cos\alpha\sin\beta\cos\beta\frac{\partial\beta}{\partial\theta}\frac{\partial\beta}{\partial\phi}$$
$$= \frac{1}{4\pi}\int d\theta d\phi\left(-\sin^3\alpha\cos^2\beta - \sin^3\alpha\sin^2\beta - \sin\alpha\cos^2\alpha\sin^2\beta - \sin\alpha\cos^2\alpha\cos^2 b\right)\frac{\partial\beta}{\partial\theta}\frac{\partial\alpha}{\partial\phi}$$
$$+ \left(\sin^3\alpha\cos^2\beta + \sin^3\alpha\sin^2\beta + \sin^2\alpha\cos^2\alpha\cos^2\beta + \sin\alpha\cos^2\alpha\sin^2\beta\right)\frac{\partial\alpha}{\partial\theta}\frac{\partial\beta}{\partial\phi}$$
$$= \frac{1}{4\pi}\int d\theta d\phi\sin a\left(\frac{\partial\alpha}{\partial\theta}\frac{\partial\beta}{\partial\phi} - \frac{\partial\beta}{\partial\theta}\frac{\partial\alpha}{\partial\phi}\right)$$

Finally, we made it! We can now shamelessly brag at the dinner table that:

$$n = \frac{1}{4\pi}\int d\theta d\phi\sin a\left(\frac{\partial\alpha}{\partial\theta}\frac{\partial\beta}{\partial\phi} - \frac{\partial\beta}{\partial\theta}\frac{\partial\alpha}{\partial\phi}\right) \tag{272}$$

Let's make use of this equation right here and now.

X. THE T'HOOFT-POLYAKOV MONOPOLE: REVISITED

We will calculate the winding number of the following field configuration in the spatial asymptotic:

$$\hat{\Phi} = (\sin\theta\cos m\phi, \sin\theta\sin m\phi, \cos\theta) \tag{273}$$

with $m \in \mathbb{Z}$. Using (272), we easily get by substituting $\alpha = \theta$ and $\beta = m\phi$:

$$n = \frac{1}{4\pi}\int d\theta d\phi \sin\theta \left(\frac{\partial\theta}{\partial\theta}\frac{\partial(m\phi)}{\partial\phi} - \frac{\partial(m\phi)}{\partial\theta}\frac{\partial\theta}{\partial\phi}\right) = \frac{1}{4\pi}\int d\theta d\phi \sin\theta m = m. \tag{274}$$

The alert reader might have noticed that for $m = 1$, we recover the t'Hooft-Polyakov ansatz for the symmetry breaking pattern of the $\mathbf{\Phi}$ field (206), since

$$\hat{\mathbf{\Phi}} = \frac{\mathbf{r}}{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{275}$$

Returning now to (249) and comparing with (269), we can identify the result of the integral of the magnetic charge as:

$$g = \frac{4\pi n}{e} \tag{276}$$

which is the t'Hooft-Polyakov analogue of the Dirac quantization condition.

To make the correspondence explicit, we need to redefine $g' = 4\pi g$, since in the Dirac case, we started with

$$\nabla\cdot\mathbf{B} = 4\pi g\delta^3(\mathbf{r})$$

Then (276) becomes:

$$g' = \frac{n}{e} \tag{277}$$

We notice that the t'Hooft-Polyakov monopole appears to be twice as large as the Dirac one, when comparing with (60).

While the Dirac monopole was introduced into our theory by hand, the t'Hooft-Polyakov comes up naturally after the energy minimization condition on symmetry breaking $D_\mu\Phi^a \to 0$. Our next step will be to find the solution for the fields in the whole space and identify the mass of the monopole.

A. The t'Hooft-Polyakov ansatz

We showed that asympotically the conditions (206) and (207) yield a monopole magnetic field. We now proceed to find the full field configurations. Considering static solutions, we are only left with an $SO(3)$ spatial symmetry from the full Poincaré group. We also have an $SO(3)$ symmetry in Φ isospace. Thus, we can make the ansatz[53] [16],[27]:

$$\Phi^a = u\frac{r^a}{r}H(r), \quad A_n^a = \varepsilon_{amn}\frac{r^m}{er}[1-K(r)], \quad A_0^a = 0 \tag{278}$$

We can find the functions H and K from the equations of motion. However, it proves much simpler to exploit the fact that the monopole configuration corresponds to a minimum in the energy functional.

We rewrite the energy functional[54] (197):

$$M = \int d^3x \left(\frac{1}{2}E_i^aE_i^a + \frac{1}{2}B_i^aB_i^a + \frac{1}{2}\left(D_i\Phi^a\right)\left(D_i\Phi^a\right) + V(\Phi)\right) \tag{279}$$

---

[53] Notice the similarities with chapter VII C

[54] which we now call M, since it corresponds to the mass of the monopole.

We now proceed in calculating separately every term in our functional with ansatz (278).

The simplest one to rewrite is the scalar potential V:

$$
\begin{aligned}
V(\Phi) &= \frac{\lambda}{4}\left(\Phi^a\Phi^a - u^2\right)^2 \\
&= \frac{\lambda}{4}\left(u^2\frac{r^a r^a}{r^2}H^2(r) - u^2\right)^2 \\
\Rightarrow V(\Phi) &= \frac{\lambda}{4}u^4\left(H^2(r) - 1\right)^2
\end{aligned}
\tag{280}
$$

The covariant derivative term is:

$$
\begin{aligned}
D_i\Phi^a &= \partial_i\Phi^a - e\varepsilon_{abc}A_i^b\Phi^c \\
&= \partial_i\left(u\frac{r^a}{r}H(r)\right) - e\varepsilon_{abc}\varepsilon_{bni}\frac{r^n}{r}\frac{1-K(r)}{er}u\frac{r^c}{r}H(r) \\
&= u\left(\frac{\delta_{ai}}{r} - \frac{r^a r^i}{r^3}\right)H(r) + u\frac{r^a r^i}{r^2}H'(r) + \left(\delta_{an}\delta_{ci} - \delta_{ai}\delta_{nc}\right)\frac{r^n r^c}{r^3}[1-K(r)]H(r) \\
&= u\frac{\delta_{ai}}{r}H(r) + u\frac{r^a r^i}{r^2}\left(H'(r) - \frac{H(r)}{r}\right) + u\frac{r_a r_i}{r^3}[1-K(r)]H(r) - u\frac{\delta_{ai}[1-K(r)]H(r)}{r} \\
&= u\frac{\delta_{ai}}{r}\left(H(r) - [1-K(r)]H(r)\right) + u\frac{r^a r^i}{r^2}\left(H'(r) - \frac{H(r)}{r} + \frac{[1-K(r)]H(r)}{r}\right) \\
\Rightarrow D_i\Phi^a &= \frac{u\delta_{ai}}{r}KH + \frac{ur^a r^i}{r^2}\left(H' - \frac{KH}{r}\right)
\end{aligned}
\tag{281}
$$

Now performing the contraction of two such terms:

$$
\begin{aligned}
\left(D_i\Phi^a\right)\left(D_i\Phi^a\right) &= \frac{3u^2}{r^2}K^2H^2 + u^2\left(H' - \frac{KH}{r}\right)^2 + 2\frac{u^2}{r^3}r^2\left(KHH' - \frac{K^2H^2}{r}\right) \\
&= \frac{3u^2}{r^2}K^2H^2 + u^2H'^2 - \frac{2KHH'}{r}u^2 + \frac{u^2}{r^2}K^2H^2 + \frac{2KHH'}{r}u^2 - 2\frac{u^2}{r^2}K^2H^2 \\
&= 2\frac{u^2}{r^2}K^2H^2 + u^2H'^2 \\
\Rightarrow \frac{1}{2}\left(D_i\Phi^a\right)\left(D_i\Phi^a\right) &= \frac{u^2}{r^2}K^2H^2 + \frac{u^2H'^2}{2}
\end{aligned}
\tag{282}
$$

For the 'color' magnetic field:

$$
\begin{aligned}
B_i^a &= \frac{1}{2}\varepsilon_{ijk}F_{jk}^a \\
&= \varepsilon_{ijk}\left(\partial_j A_k^a - \frac{e}{2}\varepsilon_{abc}A_j^b A_k^c\right) \\
&= \varepsilon_{ijk}\partial_j\left(\varepsilon_{ank}\frac{r^n}{r}\frac{1-K}{er}\right) - \frac{e}{2}\varepsilon_{abc}\varepsilon_{ijk}\varepsilon_{bmj}\varepsilon_{clk}\frac{r^m r^l}{r^2}\left(\frac{1-K}{er}\right)^2 \\
&= \varepsilon_{ijk}\varepsilon_{ank}\left(\frac{\delta_{nj}er^2 - r^n e2r^j}{e^2r^4}(1-K) - \frac{r^n}{er^2}K'\frac{r^j}{r}\right) + \frac{e}{2}\left(\delta_{ib}\delta_{mk} - \delta_{im}\delta_{bk}\right)\left(\delta_{la}\delta_{kb} - \delta_{lb}\delta_{ka}\right)\frac{r^m r^l}{r^2}\left(\frac{1-K}{er}\right)^2 \\
&= \left(\delta_{ai}\delta_{nj} - \delta_{ni}\delta_{aj}\right) - \left[\left(\frac{\delta_{nj}}{er^2} - \frac{2r^n r^j}{er^4}\right)(1-K) - \frac{r^n r^j}{er^3}K'\right] - \frac{e}{2}\left(\delta_{im}\delta_{al} + \delta_{il}\delta_{am}\right)\frac{r^m r^l}{r^2}\left(\frac{1-K}{er}\right)^2 \\
&= \left[\left(\frac{3\delta_{ia}}{er^2} - \frac{2\delta_{ia}}{er^2}\right)(1-K) - \frac{\delta_{ia}}{er}K' - \left(\frac{\delta_{ia}}{er^2} - \frac{2r^a r^i}{er^4}\right)(1-K) + \frac{r^a r^i}{er^3}K'\right] - \frac{r^a r^i}{er^4}(1-K)^2 \\
&= \frac{2r^a r^i}{er^4}(1-K) + \left(\frac{r^a r^i}{er^3} - \frac{\delta_{ai}}{er}\right)K' - \frac{r^a r^i}{er^4}(1-K)^2 \\
&= \frac{r^a r^i}{er^4}\left(2 - 2K + rK' - 1 + 2K - K^2\right) - \frac{\delta_{ai}}{er}K' \\
&\Rightarrow B_i^a = \frac{r^a r^i}{er^4}\left(rK' - K^2 + 1\right) - \frac{\delta_{ai}}{er}K'
\end{aligned}
\tag{283}
$$

The 'color' magnetic field squared:

$$
\begin{aligned}
\left(B_i^a\right)\left(B_i^a\right) &= \frac{\left(rK' - K^2 + 1\right)^2}{e^2r^4} + \frac{3K'^2}{e^2r^2} - 2\frac{\delta_{ai}}{er}\frac{r^a r^i}{er^4}K'\left(rK' - K^2 + 1\right) \\
&= \frac{\left(rK' - K^2 + 1\right)^2}{e^2r^4} + \frac{3K'^2}{e^2r^2} - \frac{2K'^2}{e^2r^2} - \frac{2\left(1-K^2\right)K'}{e^2r^3} \\
&= \frac{r^2K'^2}{e^2r^4} + \frac{2rK'\left(1-K^2\right)}{e^2r^4} + \frac{\left(1-K^2\right)^2}{e^2r^4} + \frac{K'^2}{e^2r^2} - \frac{2\left(1-K^2\right)K'}{e^2r^3} \\
&\Rightarrow \frac{1}{2}\left(B_i^a\right)\left(B_i^a\right) = \frac{K'^2}{e^2r^2} + \frac{\left(K^2-1\right)^2}{2e^2r^4}
\end{aligned}
\tag{284}
$$

This is a good place to sum up the boundary conditions for the functions H, K.

$$
\begin{aligned}
&K(r) \to 1,\ H(r) \to 0 \quad \text{as} \quad r \to 0 \\
&K(r) \to 0,\ H(r) \to 1 \quad \text{as} \quad r \to \infty
\end{aligned}
\tag{285}
$$

as for their derivatives we have

$$
K'(r) \to 0,\ H'(r) \to 0 \quad \text{as} \quad r \to \infty \tag{286}
$$

Taking now the limits of (283) and (281) for $r \to \infty$ and supposing the fields and derivatives vanish strongly enough to cancel mononomial terms, we can establish back their spatial asymptotic forms we calculated earlier (223),(207).

$$
B_i^a = \frac{r^a r^i}{er^4}\left(rK' - K^2 + 1\right) - \frac{\delta_{ai}}{er}K' \underset{r\to\infty}{\longrightarrow} \frac{r^a r^i}{er^4} \tag{287}
$$

$$
D_i\Phi^a = \frac{u\delta_{ai}}{r}KH + \frac{ur^a r^i}{r^2}\left(H' - \frac{KH}{r}\right) \underset{r\to\infty}{\longrightarrow} 0 \tag{288}
$$

We can also verify the existence of magnetic charge in our system. Starting from the magnetic flux integral we have:

$$
g = \int dS_i B_i = \int dS_i B_i^a \frac{\Phi^a}{u} = \int d^3x D_i\left(B_i^a\frac{\Phi^a}{u}\right) = \frac{4\pi}{u}\int r^2 dr B_i^a D_i\Phi^a \tag{289}
$$

where we used $D_i B_i^a = 0$. This stems from the Bianchi identity (190):

$$\varepsilon^{\lambda\mu\nu\kappa}(D_\mu F^a_{\nu\kappa}) = 0$$

Choosing $\lambda = 0$ forces the rest of the indices to become spatial. Thus, we are left with:

$$\varepsilon^{ijk} D_i F^a_{jk} = D_i \left(\varepsilon^{ijk} F^a_{jk}\right) = D_i B_i^a = 0 \tag{290}$$

Note also that,

$$\begin{aligned} D_\mu \left(E_i^a \Phi^a\right) &= D_\mu \left(E_i^a\right) \Phi^a + E_i^a D_\mu \Phi^a \\ &= \left(\partial_\mu E_i^a\right) \Phi^a - e\varepsilon^{abc} A_\mu^b E_i^c \Phi^a + E_i^a \left(\partial_\mu \Phi^a\right) - e\varepsilon^{abc} A_\mu^b \Phi^c E_i^a \\ &= \partial_\mu \left(E_i^a \Phi^a\right) - e\varepsilon^{abc} \Phi^a A_\mu^b E_i^c + e\varepsilon^{cba} \Phi^c A_\mu^b E_i^a \\ &= \partial_\mu \left(E_i^a \Phi^a\right) \end{aligned} \tag{291}$$

Here E_i^a is an arbitrary fundamental representation field, not to be confused with the color electric field. This explains why we were able to substitute directly the covariant derivative from Stoke's theorem instead of the normal derivative[55]. So back to our integral, we substitute what we have calculated up to now, namely the covariant derivative and the magnetic field

$$\begin{aligned} g &= \frac{4\pi}{u} \int r^2 dr B_i^a D_i \Phi^a \\ &= \frac{4\pi}{u} \int r^2 dr \left[\frac{r^a r^i}{er^4} \left(rK' - K^2 + 1\right) - \frac{\delta_{ai}}{er} K'\right] \left[\frac{u\delta_{ai}}{r} KH + \frac{ur^a r^i}{r^2} \left(H' - \frac{KH}{r}\right)\right] \\ &= \frac{4\pi}{e} \int dr \left[\frac{KH\left(rK' - K^2 + 1\right)}{r} + \frac{\left(rK' - K^2 + 1\right)\left(rH' - KH\right)}{r} - 3KK'H - K'\left(rH' - KH\right)\right] \\ &= \frac{4\pi}{e} \int dr \left[\frac{\left(rK' - K^2 + 1\right) rH'}{r} - K'\left(2KH + rH'\right)\right] \\ &= \frac{4\pi}{e} \int dr \left[\left(1 - K^2\right) H' - 2KK'H\right] \\ &= \frac{4\pi}{e} \int dr \frac{d}{dr} \left[\left(1 - K^2\right) H\right] \end{aligned} \tag{292}$$

which by virtue of the boundary conditions (285) yields the magnetic charge of our monopole:

$$g = \frac{4\pi}{e}$$

We can also shortly check that (289) coincides with definition (246) for the magnetic charge. This can be done by noticing:

$$\begin{aligned} g = \int dS_i B_i &= \int dS_i B_i^a \frac{\Phi^a}{u} = \int dS_i \frac{1}{2} \varepsilon_{ijk} F^a_{jk} \frac{\Phi^a}{u} \\ &\frac{1}{2u} \varepsilon_{ijk} \int dS_i \left[\left(\partial_j A_k^{\text{em}} - \partial_k A_j^{\text{em}}\right) + \frac{\varepsilon_{abc}}{eu^2} \partial_j \Phi^b \partial_k \Phi^c \Phi^a\right] \\ &= \frac{1}{2eu^3} \varepsilon_{ijk} \varepsilon_{abc} \int dS_i \partial_j \Phi^b \partial_k \Phi^c \Phi^a \end{aligned}$$

Substituting the above in the mass functional (279) we end up with

$$M = 4\pi \int r^2 dr \left(\frac{K'^2}{e^2 r^2} + \frac{\left(K^2 - 1\right)^2}{2e^2 r^4} + \frac{u^2}{r^2} K^2 H^2 + \frac{u^2 H'^2}{2} + \frac{\lambda}{4} u^4 \left(H^2 - 1\right)^2\right) \tag{293}$$

---

[55] It is required, however, that the group contracted fields are on the same representation.

where we employed spherical coordinates. We also omitted E_i^a, since in the static regime, the fields have no temporal dependence and A_0^a is 0.

It is rather useful to perform a change of variables on the mass functional

$$\begin{aligned} \xi = uer &\to r = \frac{\xi}{ue} \\ d\xi = uedr &\to \frac{d}{dr} = ue\frac{d}{d\xi} \\ r^2 dr &\to \frac{\xi^2}{u^3 e^3} d\xi \end{aligned}$$

Then (293) becomes

$$\begin{aligned} M &= 4\pi \int \frac{\xi^2}{u^3 e^3} d\xi \left(\frac{u^4 e^2}{\xi^2} \left(\frac{dK}{d\xi} \right)^2 + \frac{u^4 e^2}{2} \left(\frac{dH}{d\xi} \right)^2 + \frac{u^4 e^2}{2\xi^4} \left(K^2 - 1 \right)^2 + \frac{u^4 e^2}{\xi^2} K^2 H^2 + u^4 e^2 \frac{\beta^2}{2} \left(H^2 - 1 \right)^2 \right) \\ M &= \frac{4\pi u}{e} \int \xi^2 d\xi \left(\frac{K'^2}{\xi^2} + \frac{1}{2} H'^2 + \frac{\left(K^2 - 1 \right)^2}{2\xi^4} + \frac{K^2 H^2}{\xi^2} + \frac{\beta^2}{2} \left(H^2 - 1 \right)^2 \right) \end{aligned} \tag{294}$$

where we also defined the parameter β as half of the ratio between the mass of the Higgs boson and the massive vector bosons of our model:

$$\beta = \frac{1}{2} \frac{M_H}{M_W} = \frac{1}{2} \frac{u\sqrt{2\lambda}}{eu} \to \beta^2 = \frac{\lambda}{2e^2}$$

where the masses have been drawn from (204).

We now apply the Euler-Lagrange equations for each of the functions K, H to minimize the mass functional:

$$\begin{aligned} &\frac{\partial \mathcal{M}}{\partial K} - \frac{d}{d\xi} \frac{\partial \mathcal{M}}{\partial K'} = 0 \\ \Rightarrow &\frac{2\left(K^2 - 1 \right) 2K}{2\xi^2} + 2KH^2 - \frac{d}{d\xi} \left(2K' \right) = 0 \\ \Rightarrow &K'' = \frac{\left(K^2 - 1 \right) K}{\xi^2} + KH^2 \end{aligned} \tag{295}$$

As for the H field:

$$\begin{aligned} &\frac{\partial \mathcal{M}}{\partial H} - \frac{d}{d\xi} \frac{\partial \mathcal{M}}{\partial H'} = 0 \\ \Rightarrow &2HK^2 + \beta^2 \left(H^2 - 1 \right) - \frac{d}{d\xi} \left(2H'\xi^2 \right) = 0 \\ \Rightarrow &H'' = \frac{2HK^2}{\xi^2} - \frac{2H'}{\xi} + 2\beta^2 H \left(H^2 - 1 \right) \end{aligned} \tag{296}$$

where $\mathcal{M}$ is the integrand in M.

The system of the coupled non-linear ordinary differential equations (295),(296) cannot in general be solved analytically[56]. We will take a short break now and talk about the *Julia-Zee dyon*, a generalization of the t'Hooft-Polyakov monopole with a twist. It is endowed with electric charge too!

[56] Yet, we will cover the $\beta = 0$ solution in due time.

B. The Julia-Zee dyon

As we mentioned earlier a dyon is particle that possesses both magnetic and electric charges. The Georgi-Glashow model we are currently studying admits such solutions. We just have to alter the t'Hooft-Polyakov ansatz (278) a little bit so that we allow a non-zero temporal components for the gauge potential.
The *Julia-Zee ansatz* [18] we will be using for this section is

$$\begin{aligned} \Phi^a &= u\frac{r^a}{r}H(r) \\ A_i^a &= \varepsilon_{ani}\frac{r^n}{r}\left(\frac{1-K(r)}{er}\right) \\ A_0^a &= u\frac{r^a}{r}J(r) \end{aligned} \tag{297}$$

Not a great deal has changed from the previous section. We are just to calculate the "color" electric field contribution and see how this changes the equations of motion for our fields.

So,

$$\begin{aligned} E_i^a = F_{0i}^a &= \partial_0 A_i^a - \partial_i A_0^a - e\varepsilon_{abc}A_0^b A_i^c \\ &= -\partial_i\left(u\frac{r^a}{r}J(r)\right) - e\varepsilon_{abc}u\frac{r^b}{r}J(r)\varepsilon_{cni}\frac{r^n}{r}\left(\frac{1-K(r)}{er}\right) \\ &= -u\frac{\delta_{ai}r^2 - r^a r^i}{r^3}J - u\frac{r^a}{r}J'\frac{r^i}{r} - ue\delta_{ni}^{ab}\frac{r^b r^n}{r^2}J\left(\frac{1-K}{er}\right) \\ &= -u\left[\frac{\delta_{ai}}{r}J - \frac{r^a r^i}{r^3}J + \frac{r^a r^i}{r^2}J' + \frac{r^a r^i}{r^3}J(1-K) - \frac{\delta_{ai}}{r}J(1-K)\right] \\ &= -u\left[\frac{r^a r^i}{r^2}\left(J' - \frac{JK}{r}\right) + \frac{\delta_{ai}}{r}JK\right] \end{aligned} \tag{298}$$

Now squaring it,

$$\begin{aligned} \left(E_i^a\right)\left(E_i^a\right) &= u^2\left[\frac{\left(JK - rJ'\right)^2}{r^2} + \frac{3J^2K^2}{r^2} - \frac{2\left(JK - rJ'\right)JK}{r^2}\right] \\ &= \frac{u^2}{r^2}\left[\left(JK - rJ'\right)\left(JK - rJ' - 2JK\right) + 3J^2K^2\right] \\ &= \frac{u^2}{r^2}\left[-\left(JK - rJ'\right)\left(JK + rJ'\right) + 3J^2K^2\right] \\ &= \frac{u^2}{r^2}\left[2J^2K^2 + r^2J'^2\right] \end{aligned} \tag{299}$$

Of course we do not want our fields to be singular, so we impose specific boundary conditions on J to avoid such problems.

$$\begin{aligned} &J \to 0, \quad r \to 0 \\ &J \to -C, \quad r \to \infty \end{aligned} \tag{300}$$

where C is an arbitrary constant and J has to approach 0 at least like r^2.

The mass functional in the Julia-Zee case will be:

$$M = 4\pi\int r^2 dr\left(U^2\frac{J^2K^2}{r^2} + u^2\frac{J'^2}{2} + \frac{K'^2}{e^2r^2} + \frac{\left(K^2-1\right)^2}{2e^2r^4} + \frac{u^2}{r^2}K^2H^2 + \frac{u^2H'^2}{2} + \frac{\lambda}{4}u^4\left(H^2-1\right)^2\right) \tag{301}$$

Performing our usual regularising substitution: $\xi = uer$, we get

$$M = \frac{4\pi u}{e}\int \xi^2 d\xi\left(\frac{J^2K^2}{\xi^2} + \frac{J'^2}{2} + \frac{K'^2}{\xi^2} + \frac{1}{2}H'^2 + \frac{\left(K^2-1\right)^2}{2\xi^4} + \frac{K^2H^2}{\xi^2} + \frac{\beta^2}{2}\left(H^2-1\right)^2\right) \tag{302}$$

Applying the Euler-Lagrange equations on each of the functions J, H, K we get

$$\begin{aligned} \frac{\partial \mathcal{M}}{\partial J} - \frac{d}{d\xi}\left(\frac{\partial \mathcal{M}}{\partial J'}\right) = 0 \\ 2JK^2 - J''\xi^2 - 2J'\xi = 0 \\ J'' = \frac{2JK^2}{\xi^2} - \frac{2J'}{\xi} \end{aligned} \tag{303}$$

where $\mathcal{M}$ is the integrand in M. For H, we have

$$\begin{aligned} \frac{\partial \mathcal{M}}{\partial H} - \frac{d}{d\xi}\left(\frac{\partial \mathcal{M}}{\partial H'}\right) = 0 \\ 2HK^2 + 2\beta^2\left(H^2 - 1\right)H\xi^2 - H''\xi^2 - 2H'\xi = 0 \\ H'' = \frac{2HK^2}{\xi^2} - \frac{2H'}{\xi} + 2\beta^2 H\left(H^2 - 1\right) \end{aligned} \tag{304}$$

And last but not least

$$\begin{aligned} \frac{\partial \mathcal{M}}{\partial K} - \frac{d}{d\xi}\left(\frac{\partial \mathcal{M}}{\partial K'}\right) = 0 \\ 2J^2K + \frac{2\left(K^2 - 1\right)2K}{2\xi^2} + 2KH^2 - 2K'' = 0 \\ K'' = KJ^2 + \frac{\left(K^2 - 1\right)K}{\xi^2} + KH^2 \end{aligned} \tag{305}$$

The magnetic charge of the system is the same as in the t'Hooft-Polyakov ansatz. Here, we can, however, calculate also the electric charge of the dyon.

$$q = \int dS_i E_i = \int dS_i E_i^a \frac{\Phi^a}{u} = \int d^3x D_i\left(E_i^a \frac{\Phi^a}{u}\right) = \frac{4\pi}{u}\int r^2 dr E_i^a D_i \Phi^a \tag{306}$$

Here,we invoked the Stoke's theorem again and we used $D_i E_i^a = 0$. We can deduce this by employing the fields' equations of motion (188) as follows:

$$\begin{aligned} D_i E_i^a = D_i F_{0i}^a &= -e\varepsilon_{abc}\Phi^b D_0 \Phi^c \\ &= -e\varepsilon_{abc}\left(\partial_0 \Phi^c - e\varepsilon_{cde} A_0^d \Phi^e\right) \\ &= e^2 \varepsilon_{abc}\Phi^b \varepsilon_{cde} u^2 \frac{r^d r^e}{r^2} J(r)H(r) = 0 \end{aligned}$$

At this point, we can calculate the electric charge contained in the system. We will use the J O.D.E.(303) and the relationship between the fundamental units of electric and magnetic charge in our system[57]

$$\begin{aligned}
q &= \frac{4\pi}{u}\int r^2 dr\left[u\left(\frac{r^a r_i}{r^3}\left(JK - rJ'\right) - \frac{\delta_{ai}}{r}JK\right)\left(\frac{u\delta_{ai}}{r}KH + \frac{ur^a r^i}{r^2}\left(H' - \frac{KH}{r}\right)\right)\right] \\
&= \frac{4\pi}{u}\int u^2 r^2 dr\left[\frac{r^2}{r^4}\left(JK - rJ'\right)KH + \frac{r^4}{r^6}\left(rH' - KH\right)\left(JK - rJ'\right) - \frac{3}{r^2}JK^2H - \frac{r^2}{r^4}\left(rH' - KH\right)JK\right] \\
&= 4\pi u\int dr\left[\left(JK - rJ'\right)\left(rH' - KH + KH\right) - 3JK^2H - rH'JK + JK^2H\right] \\
&= 4\pi u\int dr\left[rH'JK - r^2J'H' - rH'JK - 2JK^2H\right] \\
&= -4\pi u\int dr\left[2JK^2H + r^2J'H'\right] \\
&= -4\pi u\int \frac{d\xi}{ue}\left[2JK^2H + \frac{\xi^2}{u^2e^2}u^2e^2J'H'\right] \\
&= -\frac{4\pi}{e}\int d\xi\left[2JK^2H + \xi^2J'H'\right] \\
&= -\frac{4\pi}{e}\int d\xi\left[\xi^2J''H + 2\xi J'H + \xi^2J'H'\right] \\
&= -\frac{4\pi}{e}\int d\xi\left[\left(\xi^2\right)'J'H + \xi^2\left(J'H\right)'\right] \\
&= -\frac{4\pi}{e}\int d\xi\left[\xi^2\left(J'H\right)\right]' = -\frac{4\pi}{e}[\xi^2J']|_{\xi=+\infty}
\end{aligned} \tag{307}$$

We have to pay special attention to the last step. Taking the O.D.E. (303) for J we have:

$$\begin{aligned}
J'' &= \frac{2JK^2}{\xi^2} - \frac{2J'}{\xi} \\
\Rightarrow \quad & \xi^2J' + 2\xi J' = 2JK^2 \\
\Rightarrow \quad & \left(\xi^2J'\right)' = 2JK^2 \\
\Rightarrow \int_0^\infty & \left(\xi^2J'\right)' d\xi = 2\int_0^\infty JK^2 d\xi \\
\Rightarrow \xi^2J'\Big|_{\xi=+\infty} &= 2\int_0^\infty JK^2 d\xi
\end{aligned} \tag{308}$$

We can now confidently state the final result [18]:

$$q = -\frac{8\pi}{e}\int_0^\infty JK^2 d\xi \tag{309}$$

On the classical level, there is no reason for the electric charge to be quantized, unlike the magnetic charge, so the value of (309) remains arbitrary. At this point, we should also note the similarity between the Higgs triplet and the 0-component of the A field. In our ansatz (297) the two fields are parallel in isospace and one can consider the latter as a second triplet field of scalars. This is called the *Julia-Zee correspondence* $\phi^a \rightleftharpoons A_0^a$.

C. The BPS limit

The *Bogomolny-Prasad-Sommerfeld* limit sets a minimum for the monopole or dyon mass [28]. Taking the *formal* limit[58] of the coupling constant $\lambda \to 0$, we have the following inequality for the mass functional (293)

$$M \geq \frac{1}{2}\int d^3x\left[\left(E_i^a\right)\left(E_i^a\right) + \left(B_i^a\right)\left(B_i^a\right) + \left(D_i\Phi^a\right)\left(D_i\Phi^a\right)\right] \tag{310}$$

---

[57] That is $g = \frac{4\pi}{e}$

[58] The limit is formal, because we need a non-zero potential to fix the value of Φ at infinity.

Now, we perform a little devilish trick[59]. We introduce an arbitrary angle θ on the mass functional

$$M \geq \frac{1}{2}\int d^3x \left[(E_i^a)(E_i^a) + (B_i^a)(B_i^a) + \left(\sin^2\theta + \cos^2\theta\right)(D_i\Phi^a)(D_i\Phi^a)\right] \tag{311}$$

Completing the square in each term yields:

$$M \geq \frac{1}{2}\int d^3x \left[(E_i^a) - \sin\theta\,(D_i\Phi^a)\right]^2 + \left[(B_i^a) - \cos\theta\,(D_i\Phi^a)\right]^2 + \int d^3x \left[\sin\theta\,(E_i^a)(D_i\Phi^a) + \cos\theta\,(B_i^a)(D_i\Phi^a)\right] \tag{312}$$

Of course, the first two terms are clearly positive and the following ones are just the electric and magnetic charge of the system repsectively.

We, thus, get a *Bogomolny* bound for the mass of the dyon

$$\begin{gathered} M \geq \int d^3x \left[\sin\theta\,(E_i^a)(D_i\Phi^a) + \cos\theta\,(B_i^a)(D_i\Phi^a)\right] = \sin\theta\int d^3x\,(E_i^a)(D_i\Phi^a) + \cos\theta\int d^3x\,(B_i^a)(D_i\Phi^a) \\ \Rightarrow E \geq uq\sin\theta + ug\cos\theta \end{gathered} \tag{313}$$

If we want to saturate that bound, we need to set the squared terms to zero

$$\begin{aligned} E_i^a &= \sin\theta\,(D_i\Phi^a) \\ B_i^a &= \cos\theta\,(D_i\Phi^a) \end{aligned} \tag{314}$$

This way, we arrive at the *BPS* equations. The minimum of the energy functional with respect to the parameter θ is easily found to be

$$\tan\theta = \frac{q}{g} \tag{315}$$

Rewriting the integral (309) as

$$\frac{q}{g} = -2\int_0^\infty JK^2 d\xi \tag{316}$$

we realize that it is strongly connected to the duality principle, identified with the tangent of the *dual angle* Using some basic trigonometric identities

$$\theta = \arctan\frac{q}{g} \Rightarrow \begin{cases} \sin\left(\arctan\frac{q}{g}\right) = \frac{\frac{q}{g}}{\sqrt{1+\left(\frac{q}{g}\right)^2}} = \frac{q}{\sqrt{q^2+g^2}} \\ \cos\left(\arctan\frac{q}{g}\right) = \frac{1}{\sqrt{1+\left(\frac{q}{g}\right)^2}} = \frac{g}{\sqrt{q^2+g^2}} \end{cases} \tag{317}$$

we arrive at the final form of the BPS bound

$$M \geq u\sqrt{q^2+g^2} \tag{318}$$

Let's concentrate on the monopole configuration for now by setting $q = 0$ or $\theta = 0$. Then using $M_{\mathrm{W}} = ev$ and $\alpha = e^2/4\pi$, we can write the Bogomolny bound as

$$M \geq \frac{M_{\mathrm{W}}}{\alpha}|n| \tag{319}$$

since $\alpha \ll 1$ the monopole is much heavier that the W boson.

Also, in our case the BPS equations include just

$$B_i^a = (D_i\Phi^a) \tag{320}$$

[59] This is the same trick used earlier in chapter VII A to find its exact solution.

Substituting the t'Hooft-Polyakov ansatz (278) and equating same tensor components we get

$$\begin{aligned} -\frac{\delta_{ai}}{er}K' &= \frac{u\delta_{ai}}{r}KH \\ K' &= -ueKH \\ ue\frac{dK}{d\xi} &= -ueKH \\ \frac{dK}{d\xi} &= -KH \end{aligned} \tag{321}$$

and

$$\begin{aligned} &\frac{r^a r^i}{er^4}\left(rK' - K^2 + 1\right) = \frac{ur^a r^i}{r^2}\left(H' - \frac{KH}{r}\right) \\ &\left(rK' - K^2 + 1\right) = uer\left(rH' - KH\right) \\ &\left(\xi K' - K^2 + 1\right) = \xi\left(\xi H' - KH\right) \\ &\xi\frac{dK}{d\xi} - K^2 + 1 = \xi^2\frac{dH}{d\xi} - \xi KH \\ &\xi^2\frac{dH}{d\xi} = 1 - K^2 \end{aligned} \tag{322}$$

where we made use of the first differential equation to make first derivative terms of K go away in the second one. We have arrived at a system of non-linear differential equations. This system can be solved analytically along with the asymptotic behavior of the corresponding functions (285).

Starting from (321), we can solve[60] for K.

$$\begin{aligned} K' &= -KH \Rightarrow \\ \frac{K'}{K} &= -H \Rightarrow \\ (\ln K)' &= -H \Rightarrow \\ K &= e^{-\int H d\xi'} \end{aligned} \tag{323}$$

We have also absorbed the integration constant into the indefinite integral of H.

We now substitute (323) into (322)

$$\xi^2 H' = 1 - e^{2\int H d\xi'} \tag{324}$$

Differentiating

$$2\xi H' + \xi^2 H'' = -e^{2\int H d\xi'} 2H \tag{325}$$

Substituting (324) into the above equation

$$\begin{aligned} 2\xi H' + \xi^2 H'' &= \left(1 - \xi^2 H'\right) 2H \\ H'' + \frac{2}{\xi}H' + 2HH' - \frac{2}{\xi^2}H &= 0 \end{aligned} \tag{326}$$

We have arrived at a second order non-linear differential equation in H. There is no standard method in solving these and a little bit of luck is required. Here, we try guessing that the solution will contain an inverse mononomial of

[60] where we use primes for the $\frac{d}{d\xi}$ operator now.

arbitrary order.

$$
\begin{aligned}
H &= G - \frac{c}{\xi^n} \\
H' &= G' + \frac{cn}{\xi^{n+1}} \\
H'' &= G'' + \frac{Cn(n+1)}{\xi^{n+2}}
\end{aligned}
\tag{327}
$$

Substituting this in (326) yields

$$
G'' - \frac{cn(n+1)}{\xi} + \frac{2}{\xi}G' + \frac{2cn}{\xi^{n+2}} + 2GG' - \frac{2c^2n}{\xi^{2n+1}} - \frac{2c}{\xi^n}G' + \frac{2cn}{\xi^{n+1}}G - \frac{2}{\xi^2}G + \frac{2c}{\xi^{n+2}} = 0 \tag{328}
$$

At this point, we notice that the choice $c = n = 1$ really saves us from a lot of trouble. Indeed, the resulting differential equation magically reduces to

$$
G' + G^2 = C^2 \tag{329}
$$

still nonlinear in its entirety, but integrable nevertheless.

Separating variables

$$
\int \frac{dG}{C^2 - G^2} = \int d\xi
$$

Now, factorizing the denominator and splitting the fractions

$$
\xi = \int \frac{1}{2C}\left[\frac{1}{C-G} + \frac{1}{C+G}\right] dG
$$

Integrating

$$
2C\xi = \ln\frac{C+G}{|C-G|} + D \tag{330}
$$

The next step is to solve for G, which is rather trivial. There is one concern though and that is the absolute value in the denominator. The rigorous way would be to take distinct cases for the sign of the absolute value and work that way. However, we remember that $H = G - \frac{1}{\xi}$ and our ultimate goal is to derive a solution that minimizes the energy functional and if our function has such a severe singularity, ain't no way we will be achieving that. If we want to at least have a chance, we should hope that G will induce another singularity in the same point that cancels the one we are already burdened with. The only way at least retain some chances is choosing $C \leq G$ (at least this choice leaves room for our function G to grow infinite).

After all this, solving for G gets us

$$
G = \frac{1 + \exp(2C\xi - D)}{1 - \exp(2C\xi - D)} = \coth\left(2\xi - \frac{D}{2}\right) \tag{331}
$$

Employing the boundary conditions (285) H must go to 0 as ξ goes to 0.

Using a power series expansion for the hyperbolic cotangent

$$
\begin{aligned}
\coth x &= \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n} x^{2n-1}}{(2n)!} \\
&= \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \frac{x^7}{4725} + \cdots
\end{aligned}
\tag{332}
$$

where B_{2n} denotes the Bernoulli numbers.

The series converges for $0 < |x| < \pi$. So taking the limit to 0

$$\lim_{\xi\to 0} H = \lim_{\xi\to 0}\left(\coth\left(C\xi - \frac{D}{2}\right) - \frac{1}{\xi}\right) = \lim_{\xi\to 0}\left(\frac{1}{C\xi - \frac{D}{2}} + \frac{C\xi - \frac{D}{2}}{3} + \ldots - \frac{1}{\xi}\right) = 0. \tag{333}$$

we deduce that $D = 0$ and $C = 1$.

So

$$H = \coth\xi - \frac{1}{\xi} \tag{334}$$

Now we turn our attention to K and equation (323)

$$\begin{aligned} K &= e^{-\int_\xi H d\xi'} = e^{-\int\left(\coth\xi' - \frac{1}{\xi}\right)d\xi'} = e^{-\int\left(\frac{\cosh\xi'}{\sinh\xi'} - \frac{1}{\xi'}\right)d\xi'} \\ &= e^{-\ln\sinh\xi + \ln\xi + c} = c\frac{\xi}{\sinh\xi} \end{aligned} \tag{335}$$

The constant c has to be 1 for the asymptotic conditions for K to be resolved.

Note, also, that the solution of the first order BPS equations automatically satisfy the Euler-Lagrange O.D.E's we got earlier (295),(296) with $\beta = 0$. For example, substituting (322) into (296) yields the second order equation we just solved (326)

$$\begin{aligned} H'' &= \frac{2H}{\xi^2}\left(1 - \xi^2 H'\right) - \frac{2H'}{\xi} \\ H'' &= \frac{2H}{\xi^2} - 2HH' - \frac{2H'}{\xi} \\ H'' &+ \frac{2H'}{\xi} + 2HH' - \frac{2H}{\xi^2} = 0 \end{aligned}$$

The compatibility of the K differential equations can be checked by direct substitution of the solution into (295).

The BPS equation along with the Bianchi identity yield:

$$\begin{aligned} D_n\Phi^a &= B_n^a \\ D_n D_n\Phi^a &= D_n B_n^a = 0 \end{aligned} \tag{336}$$

As a result

$$\begin{aligned} &\left(D_n\Phi^a\right)\left(D_n\Phi^a\right) = D_n\left(\Phi^a D_n\Phi^a\right) = \\ &= \partial_n\left(\Phi^a D_n\Phi^a\right) = \partial_n\Phi^a D_n\Phi^a + \Phi^a\partial_n D_n\Phi^a \\ &= \partial_n\Phi^a\partial_n\Phi^a - e\varepsilon_{abc}A_n^b\Phi^c\partial_n\Phi^a + \Phi^a\partial_n\partial_n\Phi^a - \Phi^a e\varepsilon_{abc}A_n^b\partial_n\Phi^c - e\varepsilon_{abc}\partial_n A_n^b\Phi^a\Phi^c \\ &= \partial_n\Phi^a\partial_n\Phi^a + \Phi^a\partial_n\partial_n\Phi^a \\ &= \frac{1}{2}\partial_n\partial_n\left(\Phi^a\Phi^a\right) \end{aligned} \tag{337}$$

where we used the Bianchi identity (290), that the covariant derivative of a product of same representation fields is just its partial derivative (291) and the skew symmetry of certain indices.

With this in mind we can show that the energy of the monopole in the BPS limit is independent of the properties of the gauge field and can be calculated from the Higgs field alone.

In the BPS limit (313) along with (314) becomes

$$\begin{aligned} M &= \int d^3x \left(D_n\Phi^a\right)\left(D_n\Phi^a\right) = 2\pi \int r^2 dr \partial_n \partial_n \left(\Phi^a\Phi^a\right) \\ &= 2\pi u^2 \int r^2 dr \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} H\right) \\ &= \frac{2\pi u}{e} \int d\xi \frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial}{\partial \xi} H\right) \\ &= \frac{2\pi u}{e}\xi^2 \left(\frac{1}{\xi^2} - \frac{1}{\sinh^2 \xi}\right)\Big|_0^{+\infty} \\ &= \frac{2\pi u}{e} \lim_{\xi \to 0} \frac{\xi^2}{\sinh^2 f} = \frac{4\pi u}{e}. \end{aligned} \tag{338}$$

where we used the solution for H (334), verifying the Bogomolny bound for the 't Hooft ansatz.

Srednicki, once wrote in his book [32]: "Alas, the Georgi-Glashow model, which has monopole solutions, is not in accord with nature, while the Standard Model, which is in accord with nature, does not have monopole solutions. This is because, in the Standard Model, electric charge is a linear combination of an SU(2) generator and the U(1) hypercharge generator. Nothing prevents us from introducing an SU(2) singlet field with an arbitrarily small hypercharge. Such a field would have an arbitrarily small electric charge (in units of e), and then the Dirac charge quantization condition would preclude the existence of magnetic monopoles."

This is also the main argument -presented in an intuitive way- why no one has searched for monopole solutions in the Weinberg-Salam model. Things have changed recently, however, when Y.M. Cho and D. Maison had the "audacity" to attempt constructing a soliton solution with dyon properties in the Standard Model. The solution they presented in [7], exhibits the monopole properties, in a very much elegant way, in what appears to be a mixture of the Dirac and 't Hooft-Polyakov monopoles.

XI. CHO-MAISON MONOPOLE

In this section, we proceed with a brief outline of the Weinberg-Salam model, along with the topology of the Standard model, paving the way to eventually present the Cho-Maison solution.

A. Weinberg-Salam Model

Trying to unify the electromagnetic and weak forces, the first model to be proposed was that of Georgi and Glashow[61]. However, the *SSB* of this model, despite exhibiting two massive gauge bosons and one massless which one could identify with the photon, it failed to predict the existence of the neutral Z boson. So the Georgi-Glashow model had to give its place to its successor, the Weinberg-Salam model.

This model is based on the spontaneous symmetry breaking of the gauge group $SU(2)\times U(1)$ to $U(1)$. Rather intuitively, we can already guess that this model has the potential to incorporate the Z boson, since the $U(1)$ addition to the Georgi-Glashow model, offers lavishly one more generator to work with.
To construct the Weinberg-Salam lagrangian density (ignoring the fermionic sector), we couple the $SU(2)\times U(1)$ Yang-Mills field to a complex Higgs doublet.

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \left(\mathbf{D}_\mu \mathbf{\Phi}\right)^\dagger \mathbf{D}^\mu \mathbf{\Phi} - \frac{\lambda}{2}\left(\mathbf{\Phi}^\dagger\mathbf{\Phi} - u^2\right)^2 \tag{339}$$

where

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu \mathrm{W}^a_\nu - \partial_\nu \mathrm{W}^a_\mu + g\varepsilon^{ajk}\mathrm{W}^j_\mu \mathrm{W}^k_\nu \\ G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \\ \mathbf{D}_\mu &= \partial_\mu - i\frac{g}{2}\tau^a \mathrm{W}^a_\mu - i\frac{g'}{2}B_\mu \end{aligned} \tag{340}$$

The *spontaneous symmetry breaking* of this model can be described by giving a non trivial *vacuum expectation value* on the complex Higgs doublet[62].

$$\langle 0|\mathbf{\Phi}|0\rangle = \begin{pmatrix} 0 \\ u \end{pmatrix} \tag{341}$$

Exploiting the local $SU(2)$ symmetry of the model, we can align the direction of the $SU(2)$ internal degrees of freedom[63], by performing independent transformations in different space-time points.

An arbitrary $\mathbf{\Phi}$ excitation can be written as:

$$\mathbf{\Phi}(x) = e^{-i\theta^i(x)\frac{\tau^i}{2}} \begin{pmatrix} 0 \\ u + \frac{\sigma(x)}{\sqrt{2}} \end{pmatrix} \tag{342}$$

The θ^i excitations can be gauged away via a suitable $SU(2)$ gauge transformation

$$\mathbf{\Phi} \to \mathbf{\Phi'}(x) = e^{-ig\alpha^i(x)\frac{\tau^i}{2}}\mathbf{\Phi} \tag{343}$$

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[61] See section VIII A
[62] Only scalar quantities can acquire non trivial *VEV* because of the Lorentz invariance that needs to be preserved
[63] or " weak isospin" as is often called

choosing $\alpha^i = -\frac{\theta^i}{g}$.
So we are left with

$$\mathbf{\Phi}(x) = \begin{pmatrix} 0 \\ u + \frac{\sigma(x)}{\sqrt{2}} \end{pmatrix} \tag{344}$$

We say that the Goldstone bosons have been "swallowed" by the longitudal components of the $SU(2)$ vector bosons. The σ excitations are called "Higgs bosons" in a completely analogous spirit to subsection VI C.

To determine the particle spectrum in the symmetry broken phase, we substitute (344) into the Lagrangian (339). To inspect the results qualitatively, it is useful to calculate the covariant derivative of $\mathbf{\Phi}$ keeping terms of up to order one in fields[64].

$$\begin{aligned} \mathbf{D}_\mu \mathbf{\Phi} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \partial_\mu \sigma \end{pmatrix} - \frac{i}{2} u \begin{pmatrix} gW^3_\mu + g' B_\mu & gW^1_\mu - igW^2_\mu \\ gW^1_\mu + igW^2_\mu & gW^3_\mu - g' B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ u \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \partial_\mu \sigma \end{pmatrix} - \frac{i}{2} \begin{pmatrix} gW^1_\mu - igW^2_\mu \\ gW^3_\mu - g' B_\mu \end{pmatrix} \end{aligned} \tag{345}$$

Calculating the square of it

$$(\mathbf{D}_\mu \mathbf{\Phi})^\dagger \mathbf{D}^\mu \mathbf{\Phi} = \left[\tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & \partial_\mu \sigma \end{pmatrix} + \tfrac{i}{2} \begin{pmatrix} gW^1_\mu + igW^2_\mu & gW^3_\mu - g' B_\mu \end{pmatrix} \right] \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \partial_\mu \sigma \end{pmatrix} - \frac{i}{2} u \begin{pmatrix} gW^1_\mu - igW^2_\mu \\ gW^3_\mu - g' B_\mu \end{pmatrix} \right]$$

$$= \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + \frac{u^2}{4} g^2 \left(\left(W^1_\mu\right)^2 + \left(W^2_\mu\right)^2 \right) + \frac{u^2}{4} \left(gW^3_\mu - g' B_\mu \right)^2$$

So, the Lagrangian (339) contains quadratic terms of the form

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \sigma\right) \left(\partial^\mu \sigma\right) - \frac{1}{2} m_H^2 \sigma^2 + \frac{1}{4} g^2 u^2 \left(W^1_\mu W^{1\mu} + W^2_\mu W^{2\mu} \right) + \frac{u^2}{4} \left(gW^3_\mu - g' B_\mu \right) \left(gW^{3\mu} - g' B^\mu \right) +$$

(higher order terms) + (kinetic terms for $\mathbf{W}, \mathrm{B}$ fields)

Although the σ, W^1, W^2 have explicit mass terms in the Lagrangian, W^3, B appear mixed. To unmix them, we perform a field rotation in field space. This rotation is described by the following field redefinition:

$$\begin{aligned} A_\mu &= \cos\theta_W B_\mu + \sin\theta_W W^3_\mu \\ Z_\mu &= -\sin\theta_W B_\mu + \cos\theta_W W^3_\mu \end{aligned} \tag{346}$$

with $\tan\theta_W = \frac{g'}{g}$.

The Lagrangian (without including interaction terms) now becomes

$$\begin{aligned} \mathcal{L}_{\text{quadratic}} &= \frac{1}{2} \left(\partial_\mu \sigma\right) \left(\partial^\mu \sigma\right) - \mu^2 \sigma^2 \\ &- \frac{1}{4} \left(\partial_\mu W^1_\nu - \partial_\nu W^1_\mu \right) \left(\partial^\mu W^{1\nu} - \partial^\nu W^{1\mu} \right) + \frac{1}{8} g^2 f^2 W^1_\mu W^{1\mu} \\ &- \frac{1}{4} \left(\partial_\mu W^2_\nu - \partial_\nu W^2_\mu \right) \left(\partial^\mu W^{2\nu} - \partial^\nu W^{2\mu} \right) + \frac{1}{8} g^2 f^2 W^2_\mu W^{2\mu} \\ &- \frac{1}{4} \left(\partial_\mu Z_\nu - \partial_\nu Z_\mu \right) \left(\partial^\mu Z^\nu - \partial^\nu Z^\mu \right) + \frac{1}{8} f^2 \left(g^2 + g'^2 \right) Z_\mu Z^\mu \\ &- \frac{1}{4} \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right). \end{aligned} \tag{347}$$

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[64] This will give us the quadratic terms in the Lagrangian.

We can now read off the masses of each boson

$$M_1 = M_2 = \frac{fg}{2} \equiv M_W, \quad M_Z = \frac{f}{2}\left(g^2 + g'^2\right)^{\frac{1}{2}} = \frac{M_1}{\cos\theta_W} \tag{348}$$

There is, also, a rather easy way to calculate the σ mass term starting from the potential with the help of (344), keeping only quadratic terms

$$\begin{aligned} V &= -\frac{\lambda}{2}\left(\mathbf{\Phi}^\dagger\mathbf{\Phi} - u^2\right)^2 = -\frac{\lambda}{2}\left(\begin{pmatrix} 0 & u + \frac{\sigma}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 \\ u + \frac{\sigma}{\sqrt{2}} \end{pmatrix} - u^2\right)^2 \\ &= -\frac{\lambda}{2}\left(\left(u + \frac{\sigma}{\sqrt{2}}\right)^2 - u^2\right)^2 = -\frac{\lambda}{2}\left(\left(u + \frac{\sigma}{\sqrt{2}} - \lambda u\right)\left(u + \frac{\sigma}{\sqrt{2}} + u\right)\right)^2 \\ &= -\frac{\lambda}{2}\frac{\sigma^2}{2}\left(2u + \frac{\sigma}{\sqrt{2}}\right)^2 \overset{\text{second-order in } \sigma}{=} -\frac{1}{2}(\sqrt{2\lambda}u)^2\sigma^2 \end{aligned} \tag{349}$$

Thus, we colnclude that the Higgs boson excitation comes equipped with a mass $m_H = \sqrt{2\lambda}u$, via the mechanism of *Sponaneous Symmetry Breaking*.

Inspecting the form of the Lagrangian now, we can come to the following colnclusions, regarding the degrees of freedom in the *SSB* of the Weinberg-Salam Model.

Before *SSB*	D.o.F	After *SSB*	D.o.F
3 massless SU(2) gauge bosons	6	3 massive gauge bosons	9
1 massless U(1) gauge boson	2	1 massless gauge boson	2
1 compplex Higgs doublet	4	1 Higgs boson	1

TABLE III: Degrees of Freedom in the *Spontaneous Symmetry Breaking* of the Weinberg-Salam model

As in our previous examples of *SSB* the Goldstone modes of this model have been swallowed by the gauge bosons which now acquire mass. We, also, have a massless gauge boson A_μ, which in the context of the Standard Model is understood as the photon, the carrier of the electromagnetic force.

To understand better the symmetry breaking patterns, we observe the masslessness of A_μ comes from the choice of our vacuum condition, which happens to be preserved by the generator $\frac{1}{2}\left(1 + \boldsymbol{\tau}^3\right)$.

Explicitly

$$\frac{1}{2}\left(1 + \boldsymbol{\tau}^3\right)\langle 0|\Phi|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ u \end{pmatrix}$$

We can write the covariant derivative from (340) using the A_μ, Z_μ fields as

$$\mathbf{D}_\mu = \left(\partial_\mu + ig\sin\theta_W\frac{\left(1 + \boldsymbol{\tau}^3\right)}{2}A_\mu + \frac{ig}{\cos\theta_W}\left(\frac{\boldsymbol{\tau}^3}{2} - \sin^2\theta_W\frac{\left(1 + \boldsymbol{\tau}^3\right)}{2}\right)Z_\mu + \ldots\right) \tag{350}$$

We can verify this by direct substitution of the field redefinitions (346) into the covariant derivative

$$\begin{aligned}
\mathbf{D}_\mu &= \partial_\mu + ig\sin\theta_W \frac{1+\boldsymbol{\tau}^3}{2} A_\mu + \frac{ig}{\cos\theta_W}\left(\frac{\boldsymbol{\tau}^3}{2} - \sin^2\theta_W \frac{1+\boldsymbol{\tau}^3}{2}\right) Z_\mu + \dots \\
&= \partial_\mu + ig\sin\theta_W \frac{1+\boldsymbol{\tau}^3}{2}\left(\cos\theta_W B_\mu + \sin\theta_W W_\mu^3\right) \\
&+ \frac{ig}{\cos\theta_W}\left(\frac{\boldsymbol{\tau}^3}{2} - \sin^2\theta_W \frac{1+\boldsymbol{\tau}^3}{2}\right)\left(-\sin\theta_W B_\mu + \cos\theta_W W_\mu^3\right) + \dots \\
&= \partial_\mu + ig\frac{1+\boldsymbol{\tau}^3}{2}\sin\theta_W\cos\theta_W B_\mu + ig\frac{1+\boldsymbol{\tau}^3}{2}\sin^2\theta_W W_\mu^3 - ig\frac{\boldsymbol{\tau}^3}{2}\sin\theta_W\frac{1}{\cos\theta_W}B_\mu + \dots \\
&+ ig\frac{\boldsymbol{\tau}^3}{2}W_\mu^3 + ig\frac{\sin^3\theta_W}{\cos\theta_W}\frac{1+\tau^3}{2}B_\mu - ig\sin^2\theta_W\frac{1+\tau^3}{2}W_\mu^3 + \dots \\
&= \partial_\mu + ig\frac{\boldsymbol{\tau}^3}{2}W_\mu^3 + ig\frac{1+\boldsymbol{\tau}^3}{2}\left(\sin\theta_W\cos\theta_W + \frac{\sin^3\theta_W}{\cos\theta_W}\right)B_\mu - ig\frac{\boldsymbol{\tau}^3}{2}\frac{\sin\theta_W}{\cos\theta_W}B_\mu + \dots \\
&= \partial_\mu + ig\frac{\boldsymbol{\tau}^3}{2}W_\mu^3 + ig\tan\theta_W\frac{1+\boldsymbol{\tau}^3}{2}\left(\cos^2\theta_W + \sin^2\theta_W\right)B_m u - ig\frac{\boldsymbol{\tau}^3}{2}\tan\theta_W B_\mu + \dots \\
&= \partial_\mu + ig\frac{\boldsymbol{\tau}^3}{2}W_\mu^3 + i\frac{g'}{2}B_\mu + \dots
\end{aligned} \tag{351}$$

From (350), we can deduce that the fundamental unit of charge is

$$e = g\sin\theta_W \tag{352}$$

Therefore, in the above example, we have the following *SSB* pattern

$$SU(2)\times U(1) \to U_{\text{em}}(1)$$

B. Topological Argument

In the case of the Weinberg-Salam model, it was generally believed that there do not exist non-trivial toplogical excitations with monopole properties. That is because the quotient space $SU(2)\times U(1)/U_{\text{em}}(1)$ has a trivial second homotopy group.
Hoever, Cho and Maison argued in [7] that the Weinberg-Salam model can be viewed as a $CP(1)$ theory, which in fact has $\pi_2(CP(1)) = \mathbb{Z}$.
More specifically, it is the ξ field, which is to be defined at (356), that is to be regarded as the $CP(1)$ field.

But first things first. What is $CP(1)$? Well, it means *Complex Projective Space* of dimension 1 and is usually referred to as the *Riemann Sphere*. The Riemann sphere represents the complex plane plus a point at infinity $\mathbf{C}\cup\{\infty\}$. The topological structure that arises is that of a sphere and can be imagined as though the complex plane is wrapped around an S^2 via a one-point compactification. Seeing, also the $CP(1)$ as the Riemann sphere, it is very much justified why $\pi_2(CP(1)) = \mathbb{Z}$ as we have earlier discussed in IV B.

As a sneak peak and to elaborate the topological argument better, we present that the ξ field is defined as the normalized Higgs doublet from the Weinberg-Salam model.

$$\boldsymbol{\Phi} = \frac{1}{\sqrt{2}}\rho\xi \quad \left(\rho^2 = 2\boldsymbol{\Phi}^\dagger\boldsymbol{\Phi}, \quad \xi^\dagger\xi = 1\right)$$

The spherical topology of ξ might not be obvious at first site, but using ξ and with the help of Pauli matrices, we can create a triplet field

$$\hat{\boldsymbol{\Phi}}^a = \xi^\dagger\boldsymbol{\tau}^a\xi$$

It can be verified using the first Fierz identity of (361)

$$\boldsymbol{\Phi}^a\boldsymbol{\Phi}^a\sum_a = \left(\xi^\dagger\boldsymbol{\tau}^a\xi\right)^2 = \left(\xi^\dagger\xi\right)^2 = 1$$

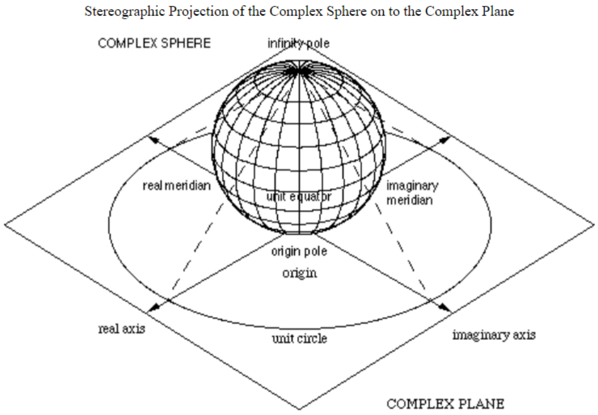

FIG. 7: Image from [1]

and now it is clear that $\hat{\mathbf{\Phi}}$ lives on a two-sphere.

It might also seem weird that we utterly ignored the ρ field, but as we will show in (357), the conserved currents of the Electroweak model will prove to depend merely proportionally to ρ and all the non-trivial stuff stem from $\hat{\mathbf{\Phi}}$. This can inferred from a decomposition of the kinetic term of the Higgs field in our models' Lagrangian using ρ and ξ. This decomposition is presented at [20] and is as follows

$$\begin{aligned}\mathcal{L} &= -\left|D_\mu \mathbf{\Phi}\right|^2 + \frac{\lambda}{2}\left(\mathbf{\Phi}^\dagger \mathbf{\Phi} - \frac{\mu^2}{\lambda}\right)^2 - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} \\ &= -\frac{1}{2}\left(\partial_\mu \rho\right)^2 - \frac{\rho^2}{2}\left|D_\mu \xi\right|^2 - \frac{\lambda}{8}\left(\rho^2 - \rho_0^2\right)^2 - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}\end{aligned}$$

Here, it also ξ that retains the covariant derivatives and ρ and ρ enters multiplicatively.

C. Cho-Maison ansatz

In the following, we will present *analytically* the results discussed in [7].

First, consider the Lagrangian (339)

$$\begin{aligned}\mathcal{L} &= -\left|\hat{\mathbf{D}}_\mu \boldsymbol{\phi}\right|^2 - \frac{\lambda}{2}\left(\boldsymbol{\phi}^\dagger \boldsymbol{\phi} - \frac{\mu^2}{\lambda}\right)^2 - \frac{1}{4}\left(\boldsymbol{F}_{\mu\nu}\right)^2 - \frac{1}{4}\left(G_{\mu\nu}\right)^2, \\ \hat{\mathbf{D}}_\mu \boldsymbol{\phi} &= \left(\partial_\mu - i\frac{g}{2}\boldsymbol{\tau}\cdot\boldsymbol{A}_\mu - i\frac{g'}{2}B_\mu\right)\boldsymbol{\phi} \\ &= \left(\mathbf{D}_\mu - i\frac{g'}{2}B_\mu\right)\boldsymbol{\phi},\end{aligned} \tag{353}$$

where we changed the symbol of the Higgs doublet from $\mathbf{\Phi}$ to $\boldsymbol{\phi}$ and $\mathbf{W}$ bosons to $\mathbf{A}$, in order for the formalism to match that in [7]. We also defined the $SU(2)\times U(1)$ covariant derivative and the raw $SU(2)$ covariant derivative in the lines above.

The equations of motion for the Weinberg-Salam model can be computed by minimizing the action functional via the Euler-Lagrange equation on the Lagrangian.

$$\begin{aligned}&\hat{\mathbf{D}}_\mu\left(\hat{\mathbf{D}}_\mu \boldsymbol{\phi}\right) = \lambda\left(\boldsymbol{\phi}^\dagger\boldsymbol{\phi} - \frac{\mu^2}{\lambda}\right)\boldsymbol{\phi}, \\ &\mathbf{D}_\mu \boldsymbol{F}_{\mu\nu} = -\boldsymbol{j}_\nu = i\frac{g}{2}\left[\boldsymbol{\phi}^\dagger\boldsymbol{\tau}\left(\hat{\mathbf{D}}_\nu\boldsymbol{\phi}\right) - \left(\hat{\mathbf{D}}_\nu\boldsymbol{\phi}\right)^\dagger\boldsymbol{\tau}\boldsymbol{\phi}\right], \\ &\partial_\mu G_{\mu\nu} = -k_\nu = i\frac{g'}{2}\left[\boldsymbol{\phi}^\dagger\left(\hat{\mathbf{D}}_\nu\boldsymbol{\phi}\right) - \left(\hat{\mathbf{D}}_\nu\boldsymbol{\phi}\right)^\dagger\boldsymbol{\phi}\right].\end{aligned} \tag{354}$$

Extracting the equation of ϕ

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial \boldsymbol{\phi}^\dagger} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \boldsymbol{\phi}^\dagger\right)} = 0 \Rightarrow \\
&-\left(+ig\frac{\boldsymbol{\tau}^\alpha}{2}A_\mu^\alpha + i\frac{g'}{2}B_\mu\right)\boldsymbol{\phi} - \frac{\lambda}{2}2\left(\boldsymbol{\phi}^\dagger\boldsymbol{\phi} - \frac{\mu^2}{\lambda}\right)\boldsymbol{\phi} \\
&+\partial_\mu \hat{\mathbf{D}}^\mu \boldsymbol{\phi} = 0 \Rightarrow \\
&\hat{\mathbf{D}}_\mu \hat{\mathbf{D}}^\mu \boldsymbol{\phi} = \lambda\left(\boldsymbol{\phi}^\dagger\boldsymbol{\phi} - \frac{\mu^2}{\lambda}\right)\boldsymbol{\phi}
\end{aligned}$$

For the $SU(2)$ Yang-Mills field, we have the following:

$$\frac{\partial F_{\mu\nu}^a}{\partial\left(\partial_\lambda A_k^d\right)} = \delta_{\lambda\mu}\delta^{da}\delta_{k\nu} - \delta^{da}\delta_{\lambda\nu}\delta_{k\mu}$$

$$\frac{\partial\left(\hat{\mathbf{D}}_\mu \boldsymbol{\phi}\right)}{\partial A_k^d} = \frac{\partial}{\partial A_k^d}\left(\partial_\mu - i\frac{g}{2}\boldsymbol{\tau}^a A_\mu^a - i\frac{g'}{2}B_\mu\right)\phi = -i\frac{g}{2}\boldsymbol{\tau}^a\delta^{da}\delta_{k\mu}\boldsymbol{\phi}$$

and

$$\frac{\partial F_{\mu\nu}^a}{\partial A_k^d} = -g\varepsilon^{abc}\left(\delta^{bd}\delta_{k\mu}A_\nu^c + A_\mu^b\delta^{cd}\delta_{k\nu}\right)$$

Using the above and the Euler-Lagrange equations, we get

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial A_k^d} - \partial_\lambda \frac{\partial \mathcal{L}}{\partial\left(\partial_\lambda A_k^d\right)} = 0 \\
&\mathrm{Re}\left\{\hat{\mathbf{D}}^\mu \boldsymbol{\phi}^\dagger(-i)\frac{g}{2}\boldsymbol{\tau}^d\delta_{k\mu}\boldsymbol{\phi}\right\} - \frac{1}{2}F^{\mu\nu a}\left[-g\varepsilon^{abc}\left(\delta^{bd}\delta_{k\mu}A_\nu^c + A_\mu^b\delta^{cd}\partial_{kv}\right)\right] \\
&-\partial_\lambda\left[-\frac{1}{2}F^{\mu\nu a}\left(\delta_{\lambda\mu}\delta^{da}\delta_{kv} - \delta^{da}\delta_{\lambda\nu}\delta_{k\mu}\right)\right] = 0 \\
&D_\lambda F^{\lambda kd} = i\frac{g}{2}\left(\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^d\hat{\mathbf{D}}^k\boldsymbol{\phi} - (\hat{\mathbf{D}}^k\boldsymbol{\phi})^\dagger\boldsymbol{\tau}^d\boldsymbol{\phi}\right)
\end{aligned} \tag{355}$$

The E.o.M. for the $U(1)$ is much similar to the one we did now, but simpler.

Now, setting

$$\begin{aligned}
\boldsymbol{\phi} &= \frac{1}{\sqrt{2}}\rho\xi \quad \left(\rho^2 = 2\boldsymbol{\phi}^\dagger\boldsymbol{\phi}, \quad \xi^\dagger\xi = 1\right) \\
\hat{\boldsymbol{\phi}} &= \xi^\dagger\boldsymbol{\tau}\xi \\
A_\mu &= \hat{\boldsymbol{\phi}}\cdot\boldsymbol{A}_\mu \\
C_\mu &= i\xi^\dagger\partial_\mu\xi
\end{aligned} \tag{356}$$

The currents can be rewritten

$$\begin{aligned}
\boldsymbol{j}_\mu &= -\frac{g\rho^2}{2}\left[\frac{g}{2}\boldsymbol{A}_\mu + \left(\frac{g'}{2}B_\mu + C_\mu\right)\hat{\boldsymbol{\phi}} + \frac{1}{2}\hat{\boldsymbol{\phi}}\times\partial_\mu\hat{\boldsymbol{\phi}}\right], \\
k_\mu &= -\frac{g'\rho^2}{2}\left(\frac{g}{2}A_\mu + \frac{g'}{2}B_\mu + C_\mu\right) = \frac{g'}{g}\left(\hat{\boldsymbol{\phi}}\cdot\boldsymbol{j}_\mu\right).
\end{aligned} \tag{357}$$

It is easy to verify the second equation

$$
\begin{aligned}
k_\mu &= -2\frac{g'}{2}\,\mathrm{Im}\left\{\frac{1}{\sqrt{2}}\rho\xi^\dagger\hat{\mathbf{D}}_\nu\left(\frac{1}{\sqrt{2}}\rho\xi\right)\right\} \\
&= \frac{g'}{2}\,\mathrm{Im}\left\{\rho\xi^\dagger\left(\partial_\nu - i\frac{g}{2}\boldsymbol{\tau}^a A^a_\nu - i\frac{g'}{2}B_\nu\right)\rho\xi\right\} \\
&= \frac{g'}{2}\,\mathrm{Im}\left\{\rho\xi^\dagger\left(\partial_\nu\rho\right)\xi + \rho^2\xi^\dagger\partial_\nu\xi - i\frac{g\rho^2}{2}A^a_\nu\xi^\dagger\boldsymbol{\tau}^a\xi - i\frac{g'}{2}B_\nu\rho^2\xi^\dagger\xi\right\} \\
&= \frac{g'}{2}\,\mathrm{Im}\left\{\rho\partial_\nu\rho + \rho^2(-i)C_\nu - i\frac{g\rho^2}{2}A^a_\nu\hat{\boldsymbol{\phi}}^a - i\frac{g'}{2}B_\nu\rho^2\right\} \\
&= -\frac{g'}{2}\rho^2\,\mathrm{Im}\left\{C_\nu + i\frac{g}{2}A_\nu + i\frac{g'}{2}B_\nu\right\} \\
&= -\frac{g'}{2}\rho^2\left(\frac{g}{2}A_\nu + \frac{g'}{2}B_\nu + C_\nu\right)
\end{aligned}
\tag{358}
$$

where we used that $C_\nu \in \mathbb{R}$.
This can be seen as follows using $\xi^\dagger\xi = 1$

$$
C_\nu = i\xi^\dagger\partial_\nu\xi = -i\partial_\nu\xi^\dagger\xi \tag{359}
$$

We showed that C_ν is equal to its complex conjugate and as a result it has to be real.
We can also verify that

$$
\begin{aligned}
k_\mu = \frac{g'}{g}\left(\hat{\boldsymbol{\phi}}\cdot\mathbf{j}_\mu\right) = \frac{g'}{g}\left(-\frac{g\rho^2}{2}\right)\left[\frac{g}{2}\hat{\boldsymbol{\phi}}\cdot\mathbf{A}_\mu + \left(\frac{g'}{2}B_\mu + C_\mu\right)\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} + \frac{1}{2}\hat{\boldsymbol{\phi}}\cdot(\hat{\boldsymbol{\phi}}\times\partial_\mu\hat{\boldsymbol{\phi}})\right] \\
= -\frac{g'\rho^2}{2}\left[\frac{g}{2}A_\mu + \frac{g'}{2}B_\mu + C_\mu\right]
\end{aligned}
\tag{360}
$$

The verification of $\mathbf{j}_\mu$ comes as follows.
First, we need to introduce some Fierz identities [24],[26].

$$
\begin{aligned}
&\sum_i\left(\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^i\boldsymbol{\phi}\right)^2 = \left(\boldsymbol{\phi}^\dagger\boldsymbol{\phi}\right)^2, \\
&\sum_i\left(\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^i\boldsymbol{\phi}\right)\left(\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^i\partial_\mu\boldsymbol{\phi}\right) = \left(\boldsymbol{\phi}^\dagger\boldsymbol{\phi}\right)\left(\boldsymbol{\phi}^\dagger\overleftrightarrow{\partial_\mu}\boldsymbol{\phi}\right), \\
&\left(\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^i\boldsymbol{\phi}\right)\left(\boldsymbol{\phi}^\dagger\overleftrightarrow{\partial_\mu}\boldsymbol{\phi}\right) - \left(\boldsymbol{\phi}^\dagger\boldsymbol{\phi}\right)\left(\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^i\overleftrightarrow{\partial_\mu}\boldsymbol{\phi}\right) = i\left(\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^i\boldsymbol{\phi}\right)\partial_\mu\left(\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^k\boldsymbol{\phi}\right)\varepsilon^{ijk},\ \text{etc.}
\end{aligned}
\tag{361}
$$

where

$$
\boldsymbol{\phi}^\dagger\overleftrightarrow{\partial_\mu}\boldsymbol{\phi} = \boldsymbol{\phi}^\dagger\partial_\mu\boldsymbol{\phi} - (\partial_\mu\boldsymbol{\phi})^\dagger\boldsymbol{\phi} \tag{362}
$$

Then starting from the RHS of the second equation from (354), we have

$$
\begin{aligned}
& i\frac{g}{2}\left[\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^i\hat{\mathbf{D}}_\nu\boldsymbol{\phi}-(\hat{\mathbf{D}}\boldsymbol{\phi})^\dagger\boldsymbol{\tau}^i\boldsymbol{\phi}\right]= \\
&= i\frac{g}{4}\left[\rho\xi^\dagger\boldsymbol{\tau}^i\left(\partial_\nu(\rho\xi)-i\frac{g}{2}\boldsymbol{\tau}^j A^j_\nu\rho\xi - i\frac{g'}{2}B_\nu\rho\xi\right)\right. \\
&\left.-\left(\partial_\nu(\rho\xi)-i\frac{g}{2}\boldsymbol{\tau}^j A^j_\nu\rho\xi-i\frac{g'}{2}B_\nu\rho\xi\right)^\dagger\boldsymbol{\tau}^i\rho\xi\right]= && \text{Expanding the covariant derivative} \\
&= \frac{ig\rho^2}{4}\left(\xi^\dagger\boldsymbol{\tau}^i\frac{\partial_\nu\rho}{\rho}\xi+\xi^\dagger\boldsymbol{\tau}^i\partial_\nu\xi-i\frac{g}{2}A^j_\nu\xi^\dagger\left\{\boldsymbol{\tau}^i,\boldsymbol{\tau}^j\right\}\xi-g'B_\nu\xi^\dagger\boldsymbol{\tau}^i\xi\right) && \text{Use anticommutator of Pauli matrices} \\
&\left.-\xi^\dagger\boldsymbol{\tau}^i\frac{\partial_\nu\rho}{\rho}\xi-\partial_\nu\xi^\dagger\boldsymbol{\tau}^i\xi\right) \\
&= \frac{g\rho^2}{2}\left(\frac{i}{2}\left(\xi^\dagger\boldsymbol{\tau}^i\partial_\nu\xi-\partial_\nu\xi^\dagger\boldsymbol{\tau}^i\xi\right)+\frac{g}{2}A^i_\nu+\frac{g'}{2}B_\nu\hat{\phi}^i\right) \\
&= \frac{g\rho^2}{2}\left(\frac{g}{2}A^i_\nu+\frac{g'}{2}B_\nu\hat{\phi}^i+\frac{i}{2}\left(\xi^\dagger\boldsymbol{\tau}_i\overleftrightarrow{\partial_\nu\xi}\right)\right) && \text{Exploit } \xi^\dagger\xi=1 \\
&= \frac{g\rho^2}{2}\left(\frac{g}{2}A^i_\nu+\frac{g'}{2}B_\nu\hat{\phi}^i+\frac{i}{2}\left(\xi^\dagger\xi\right)\left(\xi^\dagger\boldsymbol{\tau}^i\overleftrightarrow{\partial_\nu\xi}\right)\right) && \text{Use 3rd Fierz identity} \\
&= \frac{g\rho^2}{2}\left(\frac{g}{2}A^i_\nu+\frac{g'}{2}B_\nu\hat{\phi}^i+\frac{1}{2}\varepsilon^{ijk}\left(\xi^\dagger\boldsymbol{\tau}^j\xi\right)\partial_\nu\left(\xi^\dagger\boldsymbol{\tau}^k\xi\right)+\frac{i}{2}\left(\xi^\dagger\boldsymbol{\tau}^i\xi\right)\left(\xi^\dagger\overleftrightarrow{\partial_\nu\xi}\right)\right) && \text{Remember } \hat{\phi}^a=\xi^\dagger\boldsymbol{\tau}^a\xi \\
&= \frac{g\rho^2}{2}\left(\frac{g}{2}A^i_\nu+\frac{g'}{2}B_\nu\hat{\phi}^i+\frac{1}{2}\hat{\phi}\times\partial_\nu\hat{\phi}\Big|_i+\frac{i}{2}\xi^\dagger\boldsymbol{\tau}^i\xi\left(\xi^\dagger\partial_\nu\xi-\partial_\nu\xi^\dagger\xi\right)\right) && \text{Use (359)} \\
&= \frac{g\rho^2}{2}\left(\frac{g}{2}A^i_\nu+\frac{g'}{2}B_\nu\hat{\phi}^i+\frac{1}{2}\hat{\phi}\times\partial_\nu\hat{\phi}\Big|_i+C_\nu\hat{\phi}^i\right).
\end{aligned}
\tag{363}
$$

We are now ready to introduce the Cho-Maison ansatz

$$
\begin{aligned}
\rho &= \rho(r) \\
\xi &= i\begin{pmatrix}\sin(\theta/2)e^{-i\varphi}\\ -\cos(\theta/2)\end{pmatrix}, \\
\hat{\boldsymbol{\phi}} &= \xi^\dagger\boldsymbol{\tau}\xi=-\hat{r}, \\
\boldsymbol{A}_\mu &= \frac{1}{g}A(r)\partial_\mu t\hat{\boldsymbol{\phi}}+\frac{1}{g}(f(r)-1)\hat{\boldsymbol{\phi}}\times\partial_\mu\hat{\boldsymbol{\phi}}, \\
B_\mu &= -\frac{1}{g'}B(r)\partial_\mu t-\frac{1}{g'}(1-\cos\theta)\partial_\mu\varphi,
\end{aligned}
\tag{364}
$$

What are we to do with this ansatz? As in the 't Hooft-Polyakov (see section X A one or in the Julia-Zee (see section X B), we will derive a system of differential equations for the functions of the ansatz.
This can be achieved in two equiavlent ways. We can either substitute the ansatz in the equations of motion or calculate the energy functional in terms of the unknown functions and minimize the energy functional.
We will use the second way, as it is much easier to deal with a scalar quantity (i.e. the Hamiltonian) rather than all those scary vectors and matrices.

However, before we get going, in order to showcase the first way, as well, we are going to see what we can get from the $U(1)$ gauge field E.o.M.

Substituting the ansatz (364) in the E.o.M. for the $U(1)$ gauge field, we get

$$
\begin{aligned}
\partial^\mu G_{\mu\nu} &= \frac{g'\rho^2}{2}\left(\frac{g}{2}A^a_\mu\hat{\phi}^a+\frac{g'}{2}B_\mu+C_\mu\right) \\
\Rightarrow\quad \partial_\mu\partial^\mu B_\nu-\partial_\nu\partial^\mu B_\mu &= \frac{g'\rho^2}{2}\left(\frac{g}{2}A^a_\mu\left(-\hat{r}^a\right)+\frac{g'}{2}B_\mu+C_\mu\right)
\end{aligned}
\tag{365}
$$

To proceed we need to calculate each term in (365).
For the d'Alambertian of B_ν

$$\begin{aligned}\Box B_\nu &= -\frac{1}{g'},\Box B\partial_\nu t - \frac{1}{g'}\Box\left[(1-\cos\theta)\partial_\nu\varphi\right]\\ &= \frac{1}{g'}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B\right)\partial_\nu t - \frac{1}{g'}\partial^\mu\left(\sin\theta\partial_\mu\theta\partial_\nu\varphi + (1-\cos\theta)\partial_\mu\partial_\nu\varphi\right)\end{aligned} \tag{366}$$

where we employed the d'Alambertian operator in spherical coordinates in the first term. Doing the same for the second term

$$\begin{aligned}&= \frac{1}{g'}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B\right)\partial_\nu t - \frac{1}{g'}\left[\cos\theta(\nabla\theta)^2\partial_\nu\varphi + \sin\theta\nabla^2\theta\partial_\nu\varphi + \sin\theta\partial_\mu\theta\partial^\mu\partial_\nu\varphi\right]\\ &- \frac{1}{g'}\left(\sin\theta\,\partial_\mu\theta\,\partial^\mu\partial_\nu\varphi + (1-\cos\theta)\Box\partial_\nu\varphi\right)\\ &= \frac{1}{g'}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B\right)\partial_\nu t - \frac{1}{g'}\left[\frac{\cos\theta}{r^2}\partial_\nu\varphi - \sin\theta\frac{1}{r^2\sin\theta}\cos\theta\right]\\ &= \frac{1}{g'}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B\right)\partial_\nu t\end{aligned} \tag{367}$$

where we ignored terms of the form $\partial_\mu\theta\partial^\mu\partial_\nu\varphi$ and $\Box\partial_\nu\varphi$, because they are zero, of course.

For the divergence of B_μ

$$\partial_\mu B^\mu = -\frac{1}{g}\partial^\mu B\partial_\mu t - \frac{1}{g}B\partial^\mu\partial_\mu t - \frac{1}{g'}\partial^\mu(1-\cos\theta)\partial_\mu\varphi - \frac{1}{g}(1-\cos\theta)\partial_\mu\partial_\mu\varphi = 0 \tag{368}$$

For the right hand side of (365)

$$\begin{aligned}&A^a_\mu\frac{r^a}{r} = \frac{1}{g}A\partial_\mu t\\ &C_t = 0, C_r = 0\\ &C_\theta = i\xi^\dagger\partial_\theta\xi = \frac{i}{2r}(-i)\begin{pmatrix}\sin(\frac{\theta}{2})e^{i\varphi} & -\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)e^{-i\varphi}\\ \sin\left(\frac{\theta}{2}\right)\end{pmatrix} = 0.\end{aligned} \tag{369}$$

and

$$\begin{aligned}C_\phi &= i\xi^\dagger\partial_\varphi\xi = \frac{i}{r\sin\theta}(-i)\begin{pmatrix}\sin(\frac{\theta}{2})e^{i\varphi} & -\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}\sin\left(\frac{\theta}{2}\right)e^{-i\varphi}\\ 0\end{pmatrix}\\ &= \frac{\tan\left(\frac{\theta}{2}\right)}{2r}\end{aligned} \tag{370}$$

Substituting all the above equations from (367) to (370) into (365), we get

$$\frac{1}{g'}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B\right)\partial_v t = \frac{g'\rho^2}{2}\left(\frac{g}{2}\frac{1}{g}A\partial_v t - \frac{g'}{2}\frac{1}{g'}B\partial_v t - \frac{g'}{2}\frac{1}{g'}\left(1-\cos\theta\right)\partial_\nu\varphi + C_\nu\right) \tag{371}$$

Now, it is easy to see that all the choices for ν lead to trivially satisfied equations[65], apart from $\nu = t$, which yields

$$\ddot{B} + \frac{2}{r}\dot{B} = \frac{g'^2}{4}\rho^2(B-A) \tag{372}$$

[65] Maybe $\nu = \varphi$ needs a little more work, but basic trigonometry suffices.

This is how we extract equations for our ansatz functions from the equations of motion. Now, for the rest of them, we are going to turn our attention to the Hamiltonian of the system.

We can extract the Hamiltonian density from the (00) component of the energy momentum tensor.

$$\mathcal{H} = T_{00} = -G_{0\rho}G^{0\rho} - F^a_{0\rho}F^{\rho a}_0 + 2\left(\mathbf{D}_0\boldsymbol{\phi}\right)^\dagger \mathbf{D}_0\boldsymbol{\phi} - g^{00}\mathcal{L} \tag{373}$$

The Hamiltonian of the system is just the integral over space of (373). Defining the non-Abelian electric and magnetic fields as

$$E^a_i = F^a_{0i} \quad B^a_i = \frac{1}{2}\varepsilon_{ijk}F^{jka} \tag{374}$$

and their hypercharge counterparts

$$E_{Yi} = G_{0i} \quad B_{Yi} = \frac{1}{2}\varepsilon_{ijk}G^{jk} \tag{375}$$

We can write the full Hamiltonian as

$$\begin{aligned} H = \int d^3x\mathcal{H} = \int d^3x \Bigg\{ &\frac{1}{2}\left(E^a_i\right)^2 + \frac{1}{2}\left(B^a_i\right)^2 + \frac{1}{2}E^2_{Yi} + \frac{1}{2}B^2_{Y_i} + \left(\mathbf{D}_0\boldsymbol{\phi}\right)^\dagger \mathbf{D}_0\boldsymbol{\phi} \\ &+ \left(\mathbf{D}_i\boldsymbol{\phi}\right)^\dagger\left(\mathbf{D}^i\boldsymbol{\phi}\right) + \frac{\lambda}{2}\left(\boldsymbol{\phi}^\dagger\boldsymbol{\phi} - \frac{\mu^2}{\lambda}\right)^2\Bigg\} \end{aligned} \tag{376}$$

The process up to now bears much resemblance to the 't Hooft-Polyakov and Julia-Zee case,we discussed earlier.

The next step is to substitute the ansatz (364) into our Hamiltonian (376). This will prove to be a long procedure requiring patience and care. To make it easier to ourselves, we are going to treat each term of the Hamiltonian separately.

Starting with what to be the simplest term to calculate, the Higgs potential.

$$V(\boldsymbol{\phi}) = \frac{\lambda}{2}\left(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\right)^2 \tag{377}$$

Now for the $U_Y(1)$ fields. We have

$$E_{Y_i} = G_{0i} = \partial_0 B_i - \partial_i B_0 = -\partial_i B_0 = -\partial_i\left(-\frac{1}{g'}B_{(r)}\right) = \frac{1}{g'}\partial_i B_{(r)} \tag{378}$$

where temporal derivatives of B_i vanish.
Its square is

$$E_{Y_i}E^i_Y = \frac{1}{g'^2}\partial_i B\partial^i B = \frac{1}{g'^2}(\nabla B)^2 = \frac{\dot{B}^2}{g'^2} \tag{379}$$

where $\dot{B}$ indicated differentiation with respect to r.

Next up is the magnetic $U_Y(1)$ contribution.

$$B_{Yi} = \frac{1}{2}\varepsilon_{ijk}G_{jk} = \varepsilon_{ijk}\partial_j B_k = -\frac{\sin\theta}{g'}\varepsilon_{ijk}\partial_j\theta\,\partial_m\varphi \tag{380}$$

We know the drill now. We square it.

$$
\begin{aligned}
B_{Y_i} B_Y^i &= \frac{\sin^2\theta}{g'^2} \varepsilon_{ijk}\varepsilon_{ilm} \partial_j\theta\,\partial_k\varphi\,\partial_l\theta\,\partial_\mu\varphi && \text{Substituting the ansatz} \\
&= \frac{\sin^2\theta}{g'^2} \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right) \partial_j\theta\,\partial_k\varphi\,\partial_l\theta\,\partial_m\varphi && \text{Using what we should have learned by heart by now (222)} \\
&= \frac{\sin^2\theta}{g'^2} \left[\left(\partial_l\theta\,\partial_m\varphi\right)^2 - \left(\partial_m\theta\,\partial^m\varphi\right)^2 \right] && \text{Contracting with the Kronecker delta's} \\
&= \frac{\sin^2\theta}{g'^2} \left(\nabla\theta\right)^2 \left(\nabla\varphi\right)^2 && \text{The second term in the brackets above contributes nothing} \\
&= \frac{1}{g'^2 r^4} && \text{Well, that certainly looks monopoleish}
\end{aligned}
\tag{381}
$$

Next station, our favourite $SU(2)$ gauge fields.
The "color" electric field is

$$
\begin{aligned}
E_i^a &= F_{0i}^a = \partial_0 A_i^a - \partial_i A_0^a + g\varepsilon_{abc} A_0^b A_i^c && \text{Definition} \\
&= \frac{1}{g}\partial_i A(r) \frac{\hat{r}^a}{r} + \frac{1}{g} A(r) \partial_i \left(\frac{r^a}{r}\right) + g\varepsilon_{abc} \frac{1}{g} A(r) \frac{r^b}{r} \frac{1}{g} (f-1) \varepsilon_{cde} \frac{r^d}{r} \partial_i \left(\frac{r^e}{r}\right) && \text{Ansatz sub} \\
&= \frac{1}{g}\partial_i A \frac{\hat{r}^a}{r} + \frac{1}{g} A \frac{r^2\delta^{ia} - r^i r^a}{r^3} - \frac{1}{g}\left(\delta_{ad}\delta_{be} + \delta_{ae}\delta_{bd}\right) A(f-1) \frac{r^b r^d}{r^2} \frac{r^a\delta^{ie} - r^i r^e}{r^3} && \text{Using (222),(239)} \\
&= \frac{1}{g}\partial_i A \frac{r^a}{r} + \frac{1}{g} A \frac{r^2\delta^{ia} - r^i r^a}{r^3} - \frac{A(f-1)}{g} \frac{r^a r^i - r^2\delta^{ia}}{r^3} && \text{Symmetry washed off some terms} \\
&= \frac{1}{g}\partial_i A \frac{r^a}{r} + \frac{1}{g} A\, f \frac{r^2\delta^{ia} - r^i r^a}{r^3}
\end{aligned}
\tag{382}
$$

Let's square it now!

$$
\begin{aligned}
E_i^a E_i^a &= \frac{1}{g^2}\left[\partial_i A \partial^i A + \frac{A^2 f^2}{r^4} \left(r^4 \left(\delta^{ia}\right)^2 + r^4 - 2r^2\delta^{ia} r^i r^a \right) + 2A\,\partial_i A\, f \frac{r^2 r^i - r^i r^2}{r^2} \right] \\
&= \frac{1}{g^2}\left(\dot{A}^2 + 2\frac{A^2 f^2}{r^2} \right)
\end{aligned}
\tag{383}
$$

At this point, we should notice the similarity with the Julia-Zee "color" field contribution (299).

The stakes are higher now. We will calculate the contribution from the "color" magnetic field.

$$
\begin{aligned}
B_i^a &= \frac{1}{2}\varepsilon_{ijk}\left(\partial_j A_k^a - \partial_k A_j^a + g\varepsilon_{abc} A_j^b A_k^c \right) \\
&= \varepsilon_{ijk} \partial_j A_k^a + \frac{g}{2} \varepsilon_{abc}\varepsilon_{ijk} A_j^b A_k^c \\
&= \frac{1}{g}\partial_j f \varepsilon_{ijk}\varepsilon_{abc} \frac{r^b}{r} \partial_k \left(\frac{r^c}{r}\right) + \varepsilon_{ijk}\varepsilon_{abc} \frac{f-1}{g} \partial_j \left(\frac{r^b}{r}\right) \partial_k \left(\frac{r^c}{r}\right) \\
&+ \frac{g}{2}\varepsilon_{abc}\varepsilon_{ijk} A_j^b A_k^c
\end{aligned}
\tag{384}
$$

As you can see, the expressions are starting to bite. We will treat each term in (384) separately, starting from

$$
\begin{aligned}
\frac{\varepsilon_{ijk}\varepsilon_{abc}}{g} \partial_j f \frac{r^b}{r} \partial_k \left(\frac{r^c}{r}\right) &= \frac{\varepsilon_{ijk}\varepsilon_{abc}}{g} \partial_j f \frac{r^b}{r} \frac{r^2\delta^{kc} - r^k r^c}{r^3} \\
&= \frac{\partial_j f}{g r^2} \varepsilon_{ijc}\varepsilon_{abc} r^b \\
&= \frac{\partial_j f}{g r^2} \left(\partial_{ia}\partial_{jb} - \partial_{ib}\partial_{ja} \right) r^b \\
&= \frac{\left(\partial_b f\right) r^b \delta^{ia} - \partial_a f r^i}{g r^2}
\end{aligned}
\tag{385}
$$

The second term is messier.

$$
\begin{aligned}
\varepsilon_{ijk}\varepsilon_{abc}\frac{f-1}{g}\partial_j\left(\frac{r^b}{r}\right)\partial_k\left(\frac{r^c}{r}\right) &= \frac{1}{r^2}\varepsilon_{ijk}\varepsilon_{abc}\frac{f-1}{g}\frac{r^2\delta^{bj}-r^br^j}{r^2}\frac{r^2\delta^{kc}-r^kr^c}{r^2} \\
&= \frac{1}{r^6}\varepsilon_{ijk}\varepsilon_{abc}\frac{f-1}{g}\left(r^4\delta^{bj}\delta^{kc}+r^br^jr^kr^c-r^br^jr^2\delta^{kc}-r^kr^c\delta^{bj}\right) \\
&= \frac{1}{r^2}\varepsilon_{ijk}\varepsilon_{ajk}\frac{f-1}{g}-\frac{r^br^j}{r^4}\frac{f-1}{g}\varepsilon_{ijc}\varepsilon_{abc}-\frac{r^kr^c}{r^4}\frac{f-1}{g}\varepsilon_{ijk}\varepsilon_{ajc} \\
&= \frac{1}{r^2}(\delta_{ia}\delta_{jj}-\delta_{ij}\delta_{aj})\frac{f-1}{g}-\frac{2}{r}\frac{r^br^j}{r^2}\frac{f-1}{g}(\delta_{ia}\delta_{jb}-\delta_{ib}\delta_{ja}) \\
&= \frac{2}{r^2}\delta_{ia}\frac{f-1}{g}-2\frac{f-1}{g}(\delta_{ia})+\frac{2}{r^2}\frac{r^ar^i}{r^2}\frac{f-1}{g}
\end{aligned}
\tag{386}
$$

And the last one is of course the worst.

$$
\begin{aligned}
\frac{g}{2}\varepsilon_{abc}\varepsilon_{ijk}A_j^bA_k^c &= \varepsilon_{abc}\varepsilon_{ijk}\frac{1}{2gr^2}(f-1)^2\frac{r^d}{r}\partial_j\left(\frac{r^e}{r}\right)\varepsilon_{bde}\frac{r^f}{r}\partial_k\left(\frac{r^g}{r}\right)\varepsilon_{cfg} \\
&= \frac{1}{2gr^2}\frac{(f-1)^2r^dr^f}{r^2}\frac{r^2\delta^{je}-r^jr^e}{r^2}\frac{r^2\delta^{kg}-r^kr^g}{r^2}\varepsilon_{abc}\varepsilon_{ijk}\varepsilon_{bde}\varepsilon_{cfg} \\
&= \frac{1}{r^2}\frac{(f-1)^2}{2g}\frac{r^dr^f}{r^2}\varepsilon_{abc}\varepsilon_{ijk}\varepsilon_{bdj}\varepsilon_{cfk} \\
&= \frac{1}{2gr^2}\frac{(f-1)^2r^dr^f}{r^2}(\delta_{aj}\delta_{cd}-\delta_{ad}\delta_{cj})(\delta_{ic}\delta_{jf}-\delta_{if}\delta_{cj}) \\
&= \frac{1}{r^2}\frac{(f-1)^2}{2g}\frac{r^dr^f}{r^2}(\delta_{aj}\delta_{cd}\delta_{ic}\delta_{jf}-\delta_{aj}\delta_{cd}\delta_{if}\delta_{cj}-\delta_{ad}\delta_{cj}\delta_{ic}\delta_{jf}+\delta_{ad}\delta_{cj}\delta_{if}\delta_{cj}) \\
&= \frac{(f-1)^2}{2gr^2}\left(\frac{r^ir^a}{r^2}-\frac{r^ar^i}{r^2}-\frac{r^ar^i}{r^2}+3\frac{r^ar^i}{r^2}\right) \\
&= \frac{1}{r^4}\frac{(f-1)^2}{g}r^ar^i
\end{aligned}
\tag{387}
$$

After all is said and done [66], we can write for B_i^a

$$
\begin{aligned}
B_i^a &= \frac{(\partial_bf)\,r^b\delta^{ia}-\partial_afr^i}{gr^2}+2\frac{r^ar^i}{r^4}\frac{f-1}{g}+\frac{r^ar^i}{r^4}\frac{(f-1)^2}{g} \\
&= \frac{r\dot{f}\delta^{ia}}{gr^2}-\frac{\partial_afr^i}{gr^2}+\frac{f^2-1}{g}\frac{r^ar^i}{r^4}
\end{aligned}
\tag{388}
$$

And we square it,

$$
\begin{aligned}
B_i^aB_i^a &= \left(\frac{r\dot{f}\delta^{ia}}{gr^2}-\frac{\partial_afr^i}{gr^2}+\frac{f^2-1}{g}\frac{r^ar^i}{r^4}\right)^2 \\
&= \frac{r^2f^2\left(\delta^{ia}\right)^2}{g^2r^4}+\frac{(\partial_af)(\partial^af)\,r^2}{g^2r^4}+\frac{\left(f^2-1\right)^2}{g^2r^8}r^4 \\
&= 2\frac{r\dot{f}\delta^{ia}}{gr^2}\frac{\left(f^2-1\right)}{g}\frac{r^ar^i}{r^4}-2\frac{rf\delta^{ia}}{gr^2}\frac{\partial_afr^i}{gr^2}-2\frac{\partial_\alpha fr^i}{r^2}\frac{f^2-1}{g}\frac{r^ar^i}{r^4} \\
&= 3\frac{\dot{f}^2}{g^2r^2}+\frac{\dot{f}^2}{g^2r^2}+\frac{\left(f^2-1\right)^2}{g^2r^4}+2\frac{f\left(f^2-1\right)}{g^2r^3}-2\frac{\dot{f}^2}{g^2r^2}-2\frac{f\left(f^2-1\right)}{g^2r^3} \\
&= 2\frac{\dot{f}^2}{g^2r^2}+\frac{\left(f^2-1\right)^2}{g^2r^4}
\end{aligned}
\tag{389}
$$

---

[66] and also substituting (385),(386),(387) into (384)

Next up, we are going to calculate the covariant derivatives of $\boldsymbol{\phi}$, but, first, we are going to prepare a little.

$$\mathbf{D}_0\boldsymbol{\phi} = \left(\partial_0 - i\frac{g}{2}\tau^a A_0^a - i\frac{g'}{2}B_0\right)\boldsymbol{\phi} = \frac{i}{2\sqrt{2}}B\rho\xi + i\frac{A}{2\sqrt{2}}\tau^a\frac{r^a}{r}\rho\xi \tag{390}$$

Notice now that

$$\begin{aligned}\tau^a\frac{r^a}{r} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta\cos\varphi + \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}\sin\theta\sin\varphi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta \\ &= \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}\end{aligned} \tag{391}$$

When this acts on ξ, we get

$$\begin{aligned} i\begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\cdot\begin{pmatrix} \sin(\theta/2)e^{-i\varphi} \\ -\cos(\theta/2) \end{pmatrix} &= \begin{bmatrix} \left(\cos\theta\sin\frac{\theta}{2} - \sin\theta\cos\frac{\theta}{2}\right)e^{-i\varphi} \\ \left(\sin\theta\sin\frac{\theta}{2} + \cos\theta\cos\frac{\theta}{2}\right) \end{bmatrix} \\ &= \begin{bmatrix} \left(\cos^2\frac{\theta}{2}\sin\frac{\theta}{2} - \sin^3\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\right)e^{-i\varphi} \\ 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} + \cos^3\frac{\theta}{2} - \sin^2\frac{\theta}{2}\cos\frac{\theta}{2} \end{bmatrix} \\ &= \begin{bmatrix} -\sin\frac{\theta}{2}e^{-i\phi} \\ \cos\frac{\theta}{2} \end{bmatrix} = -\xi \end{aligned} \tag{392}$$

Therefore,

$$\mathbf{D}_0\boldsymbol{\phi} = \frac{iB}{2\sqrt{2}}\rho\xi - \frac{iA}{2\sqrt{2}}\rho\xi \tag{393}$$

And

$$\begin{aligned}\mathbf{D}_0\boldsymbol{\phi}^\dagger\mathbf{D}_0\boldsymbol{\phi} &= \left(\frac{-iB}{2\sqrt{2}}\rho\xi^\dagger + \frac{iA}{2\sqrt{2}}\rho\xi^\dagger\right)\left(\frac{iB}{2\sqrt{2}}\rho\xi - \frac{iA}{2\sqrt{2}}\rho\xi\right) \\ &= \frac{B^2\rho^2}{8} + \frac{A^2\rho^2}{8} - 2\frac{AB\rho^2}{8} = \frac{(A-B)^2\rho^2}{8}\end{aligned} \tag{394}$$

One last term remains and that is the spatial covariant derivative.

$$\mathbf{D}_i\boldsymbol{\phi} = \left(\partial_i - i\frac{g}{2}A_i^a\boldsymbol{\tau}^a - i\frac{g'}{2}B_i\right)\frac{1}{\sqrt{2}}\rho\xi. \tag{395}$$

We also need its daggered form.

$$(\mathbf{D}_i\boldsymbol{\phi})^\dagger = \boldsymbol{\phi}^\dagger\left(\overleftarrow{\partial}_i + i\frac{g}{2}\boldsymbol{\tau}^a A_i^a + i\frac{g'}{2}B_i\right) \tag{396}$$

The contraction of the two terms yields

$$\begin{aligned}(\mathbf{D}_i\boldsymbol{\phi})^\dagger(\mathbf{D}_i\boldsymbol{\phi}) &= \left(\partial_i\boldsymbol{\phi}^\dagger + i\frac{g}{2}\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^a A_i^a + i\frac{g'}{2}\boldsymbol{\phi}^\dagger B_i\right)\left(\partial_i\boldsymbol{\phi} - i\frac{g}{2}\tau^a A_i^a\boldsymbol{\phi} - i\frac{g'}{2}B_i\boldsymbol{\phi}\right) \\ =&\partial_i\boldsymbol{\phi}^\dagger\partial^i\boldsymbol{\phi} - i\frac{g}{2}\partial_i\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^a A_i^a\boldsymbol{\phi} - i\frac{g'}{2}B_i\partial_i\boldsymbol{\phi}^\dagger\boldsymbol{\phi} + i\frac{g}{2}\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^a A_i^a\partial_i\boldsymbol{\phi} + \frac{g^2}{4}\boldsymbol{\phi}^\dagger\left(\boldsymbol{\tau}^a A_i^a\right)^2\boldsymbol{\phi} \\ &+ \frac{gg'}{4}\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^a A_i^a B_i\boldsymbol{\phi} + i\frac{g'}{2}\boldsymbol{\phi}^\dagger B_i\partial_i\boldsymbol{\phi} + \frac{gg'}{4}\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^a A_i^a B_i\boldsymbol{\phi} + \frac{g'^2}{4}B_i^2\boldsymbol{\phi}^\dagger\boldsymbol{\phi} \\ &= \partial_i\boldsymbol{\phi}^\dagger\partial^i\boldsymbol{\phi} + i\frac{g}{2}\left(\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^a A_i^a\partial_i\boldsymbol{\phi} - \partial_i\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^a A_i^a\boldsymbol{\phi}\right) - i\frac{g'}{2}B_i\left(\partial_i\boldsymbol{\phi}^\dagger\boldsymbol{\phi} - \boldsymbol{\phi}^\dagger\partial_i\boldsymbol{\phi}\right) \\ &+ \frac{gg'}{2}\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^a A_i^a B_i\boldsymbol{\phi} + \frac{g^2}{4}\boldsymbol{\phi}^\dagger\left(\boldsymbol{\tau}^a A_i^a\right)^2\boldsymbol{\phi} + \frac{g'^2}{4}B_i^2\boldsymbol{\phi}^\dagger\boldsymbol{\phi}\end{aligned} \tag{397}$$

Now that is a lot of terms. Just as before we are going to treat them one by one.

$$
\begin{aligned}
\boldsymbol{\nabla}\boldsymbol{\phi} &= \frac{\partial \boldsymbol{\phi}}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial \boldsymbol{\phi}}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial \boldsymbol{\phi}}{\partial \varphi}\hat{\boldsymbol{\varphi}} \\
&= \frac{1}{\sqrt{2}}\dot{\xi}\hat{\mathbf{r}} + \frac{1}{r\sqrt{2}}\rho\frac{1}{2}\begin{pmatrix} \cos(\theta/2)e^{-i\varphi} \\ \sin(\theta/2) \end{pmatrix} + \frac{1}{r\sin\theta}\frac{1}{\sqrt{2}}\rho(-i)\begin{pmatrix} \sin(\theta/2)e^{-i\varphi} \\ 0 \end{pmatrix}
\end{aligned} \tag{398}
$$

Calculating the first term of (397)

$$
\begin{aligned}
\partial_i \boldsymbol{\phi}^\dagger \partial^i \boldsymbol{\phi} = \boldsymbol{\nabla}\boldsymbol{\phi}^\dagger \boldsymbol{\nabla}\boldsymbol{\phi} &= \frac{1}{2}\dot{\rho}^2 + \frac{1}{8r^2}\rho^2 \left(\cos\left(\frac{\theta}{2}\right)e^{+i\varphi} \quad \sin\left(\frac{\theta}{2}\right)\right)\begin{pmatrix} \cos\left(\frac{\theta}{2}\right)e^{-i\varphi} \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix} + \frac{\rho^2}{2r^2\sin^2\theta}\sin^2\frac{\theta}{2} \\
&= \frac{1}{2}\dot{\rho}^2 + \frac{1}{8r^2}\rho^2\left(\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\right) + \frac{\rho^2}{2r^2}\frac{\sin^2\theta/2}{\sin^2\theta} \\
&= \frac{1}{2}\dot{\rho}^2 + \frac{\rho^2}{8r^2}\left(1+\tan^2\theta/2\right)
\end{aligned} \tag{399}
$$

For our next term

$$
\begin{aligned}
\frac{g'^2}{4}B^iB^i\boldsymbol{\phi}^\dagger\boldsymbol{\phi} &= \frac{g^{2\prime}\rho^2}{8}\frac{1}{g^{2\prime}}(1-\cos\theta)^2(\boldsymbol{\nabla}\boldsymbol{\phi})^2 \\
&= \frac{\rho^2}{8}4\sin^2\frac{\theta}{2}\frac{1}{r^2\sin^2\theta} \\
&= \frac{\rho^2}{2r^2}\frac{\sin^4\theta/2}{\sin^2\theta} \\
&= \frac{\rho^2}{8r^2}\tan^2(\theta/2)
\end{aligned} \tag{400}
$$

Next up in line is

$$
\begin{aligned}
\frac{g^2}{4}\boldsymbol{\phi}^\dagger\left(\boldsymbol{\tau}^a A_i^a\right)^2\boldsymbol{\phi} &= \frac{g^2}{4}\boldsymbol{\phi}^\dagger(\boldsymbol{\tau}\cdot\mathbf{A})(\boldsymbol{\tau}\cdot\mathbf{A})\boldsymbol{\phi} \\
&= \frac{g^2}{4}\boldsymbol{\phi}^\dagger\left[(\mathbf{A}\cdot\mathbf{A})\,\mathbf{I} + i\,(\mathbf{A}\times\mathbf{A})\,\boldsymbol{\tau}\right]\boldsymbol{\phi} && \text{Pauli Matrices identity} \\
&= \frac{g^2}{4}A_i^aA_i^a\boldsymbol{\phi}^\dagger\boldsymbol{\phi} = \frac{g^2}{8}A_i^aA_i^a\rho^2 \\
&= \frac{g^2}{8}\frac{1}{g^2}(f-1)^2\rho^2\left(\hat{r}^b\partial_i\hat{r}^c\right)\left(\hat{r}^d\partial_i\hat{r}^e\right)\varepsilon_{abc}\varepsilon_{ade} \\
&= \frac{(f-1)^2\rho^2}{8}\varepsilon_{abc}\varepsilon_{ade}\frac{r^b}{r}\frac{r^2\delta^{ic}-r^ir^c}{r^3}\frac{r^d}{r}\frac{r^2\delta^{ie}-r^ir^e}{r^3} && \text{S-A terms vanish} \\
&= \frac{(f-1)^2\rho^2}{8}\left(\delta_{bd}\delta_{ce}-\delta_{be}\delta_{cd}\right)\frac{r^br^d}{r^4}\delta^{ce} \\
&= \frac{(f-1)^2\rho^2}{8r^2}(3-1) = \frac{(f-1)^2\rho^2}{4r^2}
\end{aligned} \tag{401}
$$

where we used the following identity of the Pauli matrices

$$
(\mathbf{a}\cdot\boldsymbol{\tau})(\mathbf{b}\cdot\boldsymbol{\tau}) = (\mathbf{a}\cdot\boldsymbol{b})\mathbf{I} + i(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\tau} \tag{402}
$$

For the following term

$$
\begin{aligned}
\frac{gg'}{4}\rho^2\xi^\dagger\boldsymbol{\tau}^a A_i^a B_i\xi &= \frac{gg'}{4}\rho^2\xi^\dagger\tau^a\frac{1}{g}(f-1)\frac{r^b}{r}\varepsilon_{abc}\partial_i\left(\frac{r^c}{r}\right)\left(-\frac{1}{g'}\right)(1-\cos\theta)\partial_i\varphi\xi \\
&= -\frac{1}{4}\rho^2(f-1)\left(\xi^\dagger\boldsymbol{\tau}^a\xi\right)(1-\cos\theta)\frac{r^b\delta^{ic}}{r}\partial_i\varphi\varepsilon_{abc} && \text{Notice } \hat{\boldsymbol{\phi}} \text{ definition} \\
&= -\frac{(f-1)(1-\cos\theta)}{4}\rho^2\hat{\boldsymbol{\phi}}^a\varepsilon_{abc}\frac{r^b}{r}\partial_c\varphi && \text{Parallel vector cross product} \\
&= 0
\end{aligned}
\tag{403}
$$

Bringing in the next one

$$
\begin{aligned}
i\frac{g'}{2}B_i\left(\partial_i\boldsymbol{\phi}^\dagger\boldsymbol{\phi}-\boldsymbol{\phi}^\dagger\partial_i\boldsymbol{\phi}\right) &= i2\frac{g'}{2}i\,\mathrm{Im}\left\{\left(-\frac{1}{g'}\right)(1-\cos\theta)\partial_i\varphi\,\partial^i\boldsymbol{\phi}^\dagger\boldsymbol{\phi}\right\} \\
&= -i(1-\cos\theta)i\,\mathrm{Im}\left\{\boldsymbol{\nabla}\varphi\,\boldsymbol{\nabla}\boldsymbol{\phi}^\dagger\Phi\right\} \\
&= (1-\cos\theta)\,\mathrm{Im}\left\{\frac{1}{r\sin\theta}\hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial\boldsymbol{\phi}^\dagger}{\partial\varphi}\hat{\boldsymbol{\varphi}}\boldsymbol{\phi}\right\} \\
&= 1\frac{1}{2}(1-\cos\theta)\frac{\rho^2}{r^2\sin^2\theta}\,\mathrm{Im}\left\{\left(\frac{\partial}{\partial\varphi}\xi^\dagger\right)\xi\right\} \\
&= \frac{1}{2r^2}(1-\cos\theta)\frac{\rho^2}{2\sin^2\theta}\,\mathrm{Im}\left\{\begin{pmatrix}+\sin(\theta/2)e^{-i\varphi} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta/2)e^{-i\varphi}\\ \cos(\theta/2)\end{pmatrix}(-i)\right\} \\
&= \frac{\sin^2(\theta/2)}{4r^2\sin^2(\theta/2)\cos^2(\theta/2)}\rho^2\left(-\sin^2(\theta/2)\right) \\
&= -\frac{\rho^2}{r^2}\frac{\tan^2(\theta/2)}{4}.
\end{aligned}
\tag{404}
$$

In this calculation one should notice that $\boldsymbol{\nabla}$ only contributes its azimuthal part to the Higgs field, due to the presence of $\boldsymbol{\varphi}$.

We are almost done. Proceeding, now, to the final term.

$$
\begin{aligned}
&\frac{ig}{2}\left(\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^a A_i^a\partial_i\boldsymbol{\phi}-\partial_i\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^a A_i^a\boldsymbol{\phi}\right) \\
&= \frac{ig}{2}2i\,\mathrm{Im}\left\{\boldsymbol{\phi}^\dagger\boldsymbol{\tau}^a A_i^a\boldsymbol{\phi}\right\} && \text{Cleaning up with Im} \\
&= -g\,\mathrm{Im}\left\{\frac{1}{2}\rho\xi^\dagger\boldsymbol{\tau}^a A_i^a\left(\partial_i\rho\xi+\rho\partial_i\xi\right)\right\} && \text{Expanding the derivative} \\
&= -g\frac{1}{2}\rho\partial_i\rho\,\mathrm{Im}\left\{\left(\xi^\dagger\boldsymbol{\tau}^a\xi\right)A_i^a\right\}-\boldsymbol{\phi}\,\mathrm{Im}\left\{\frac{\rho^2}{2}\xi^\dagger\boldsymbol{\tau}^a\partial^i\xi A_i^a\right\} && \text{First term here vanishes after ansatz use} \\
&= -\frac{(f-1)\rho^2}{2}\,\mathrm{Im}\left\{\xi^\dagger\boldsymbol{\tau}^a\partial^i\xi^\dagger\hat{r}^b\partial_i\hat{r}^c\varepsilon_{abc}\right\} && \text{Ansatz} \\
&= -\frac{g}{4}\,\mathrm{Im}\left\{\left(-\hat{r}^a\right)(f-1)\hat{r}^b\partial_i\hat{r}^c\varepsilon_{abc}\right\} \\
&= -\frac{(f-1)\rho^2}{2}\,\mathrm{Im}\left\{\xi^\dagger\boldsymbol{\tau}^a\partial^i\xi\hat{r}^b\frac{\delta^{ic}-\hat{r}^i\hat{r}^c}{r}\varepsilon_{abc}\right\} && \hat{r}^i\hat{r}^c\text{does not contribute} \\
&= -\frac{(f-1)\rho^2}{2r}\,\mathrm{Im}\left\{\xi^\dagger\boldsymbol{\tau}^a\partial^i\xi\hat{r}^b\varepsilon_{abi}\right\} \\
&= -\frac{(f-1)\rho^2}{2r}\,\mathrm{Im}\left\{\xi^\dagger(\boldsymbol{\tau}\times\mathbf{r})\cdot\boldsymbol{\nabla}\xi\right\} && \boldsymbol{\tau}\text{ denotes a vector with components }\boldsymbol{\tau}^a
\end{aligned}
\tag{405}
$$

To calculate the cross product, it is advisable to use spherical coordinates in order to prepare for the dot product with $\boldsymbol{\nabla}\xi$.

Therefore, we introduce the Pauli matrices in spherical coordinates [22].

$$\begin{aligned}
\boldsymbol{\tau}^r &= \cos\varphi\sin\theta\boldsymbol{\tau}^1 + \sin\theta\sin\varphi\boldsymbol{\tau}^2 + \cos\theta\boldsymbol{\tau}^3 = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{+i\varphi} & -\cos\theta \end{pmatrix} \\
\boldsymbol{\tau}^\theta &= \cos\theta\cos\varphi\boldsymbol{\tau}^1 + \cos\theta\sin\varphi\tau^2 - \sin\theta\boldsymbol{\tau}^3 = \begin{pmatrix} -\sin\theta & \cos\theta e^{-i\varphi} \\ \cos\theta e^{i\varphi} & \sin\theta \end{pmatrix} \\
\boldsymbol{\tau}^\varphi &= -\sin\varphi\sin\varphi\boldsymbol{\tau}^1 + \sin\varphi\cos\varphi\boldsymbol{\tau}^2 = \sin\varphi\begin{pmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{pmatrix}
\end{aligned} \tag{406}$$

With these in our weaponry, we take on (405) where we left it.

$$\begin{aligned}
&-\frac{(f-1)\rho^2}{2r}\operatorname{Im}\left\{\xi^\dagger \begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\varphi}} \\ \boldsymbol{\tau}^r & \boldsymbol{\tau}^\theta & \boldsymbol{\tau}^\varphi \\ 1 & 0 & 0 \end{vmatrix} \left(\frac{\partial\xi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\xi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\xi}{\partial\varphi}\hat{\boldsymbol{\varphi}}\right)\right\} \\
&= -\frac{(f-1)\rho^2}{2r}\operatorname{Im}\left\{\xi^\dagger\left(\boldsymbol{\tau}^\varphi\hat{\boldsymbol{\theta}} - \boldsymbol{\tau}^\theta\hat{\boldsymbol{\varphi}}\right)\left(\frac{1}{r}\frac{\partial\xi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\xi}{\partial_\varphi}\hat{\boldsymbol{\varphi}}\right)\right\}
\end{aligned} \tag{407}$$

$$\begin{aligned}
&= -\frac{(f-1)\rho^2}{2r}\operatorname{Im}\left\{\xi^\dagger\frac{1}{r}\boldsymbol{\tau}^\varphi\frac{\partial\xi}{\partial\theta} - \frac{1}{r\sin\theta}\xi^\dagger\boldsymbol{\tau}^\theta\frac{\partial\xi}{\partial\varphi}\right\} \\
&= -\frac{(f-1)\rho^2}{2r}\operatorname{Im}\left\{\frac{1}{2r}\begin{pmatrix}\sin\frac{\theta}{2}e^{+i\varphi} & -\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\varphi} \\ \sin\frac{\theta}{2}\end{pmatrix}\right. \\
&\left. + \frac{i}{r\sin\theta}\begin{pmatrix}\sin\frac{\theta}{2}e^{+i\varphi} & -\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix} -\sin\theta & \cos\theta e^{-i\varphi} \\ \cos\theta e^{+i\varphi} & \sin\theta \end{pmatrix}\begin{pmatrix}\sin\frac{\theta}{2}e^{-i\varphi} \\ 0\end{pmatrix}\right\} \\
&= -\frac{(f-1)\rho^2}{2r}\operatorname{Im}\left\{\frac{1}{2r}\begin{pmatrix}\sin\frac{\theta}{2}e^{i\varphi} & -\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}-ie^{-i\varphi}\sin\frac{\theta}{2} \\ i\cos\frac{\theta}{2}\end{pmatrix}\right. \\
&\left. + \frac{i}{r\sin\theta}\begin{pmatrix}\sin\frac{\theta}{2}e^{+i\varphi} & -\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}-\sin\theta\sin\frac{\theta}{2}e^{-i\varphi} \\ \cos\theta\sin\frac{\theta}{2}\end{pmatrix}\right\} \\
&= -\frac{(f-1)\rho^2}{2r}\operatorname{Im}\left\{\frac{i}{2r}\left(-\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\right) + \frac{i}{r\sin\theta}\left(-\sin\theta\sin^2\frac{\theta}{2} - \frac{\cos\theta}{2}\sin\theta\right)\right. \\
&= \frac{(f-1)\rho^2}{2r}\left[\frac{1}{2r} + \frac{1}{r}\left(\sin^2\frac{\theta}{2} + \frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{2}\right)\right] \\
&= \frac{(f-1)\rho^2}{2r^2}.
\end{aligned}$$

Adding all equations from (399) to (XI C) gives us

$$\mathbf{D}_i\boldsymbol{\phi}^\dagger\mathbf{D}^i\boldsymbol{\phi} = \frac{\dot{\rho}^2}{2} + \frac{f^2\rho^2}{4r} \tag{408}$$

Summing all our results up to now before plugging them into our Hamiltonian.

V	$\frac{\lambda}{2}\left(\frac{\rho^2}{2}-\frac{r^2}{\lambda}\right)^2$
$B_{Y_i}B_Y^i$	$\frac{1}{g'^2r^4}$
$E_{Y_i}E_Y^i$	$\frac{\dot{B}^2}{g^2}$
$B_i^aB^{ia}$	$2\frac{\dot{f}^2}{g^2r^2}+\frac{\left(f^2-1\right)^2}{g^2r^4}$
$E_i^aE^{ia}$	$\frac{\dot{A}^2}{g^2}+2\frac{A^2f^2}{g^2r^2}$
$\mathbf{D}_0\boldsymbol{\phi}^\dagger\mathbf{D}^0\boldsymbol{\phi}$	$\frac{(A-B)^2\rho^2}{8}$
$\mathbf{D}_i\boldsymbol{\phi}^\dagger\mathbf{D}^i\boldsymbol{\phi}$	$\frac{\dot{\rho}^2}{2}+\frac{f^2\rho^2}{4r}$

At long last, our Hamiltonian (376) is

$$H=\int d^3x\mathcal{H}=4\pi\int_0^\infty dr\frac{r^2\dot{A}^2}{2g^2}+\frac{A^2f^2}{g^2}+\frac{\dot{f}^2}{g^2}+\frac{\left(f^2-1\right)^2}{2g^2r^2}+\frac{\dot{B}^2r^2}{2g'^2}+\frac{1}{2g'^2r^2}$$
$$+\frac{(A-B)^2\rho^2}{8}r^2+\frac{\dot{\rho}^2}{2}r^2+\frac{f^2\rho^2}{4}+\frac{\lambda}{2}\left(\frac{\rho^2}{2}-\frac{\mu^2}{\lambda}\right)^2r^2 \tag{409}$$

At this point one should notice that our Hamiltonian contains an incurable diverging term $\frac{1}{2g'^2r^2}$. That term comes from the contribution of the Hypercharge field to the Hamiltonian which possesses via the ansatz a point like singularity at the origin. The blow up of the Hamiltonian should definitily alert us and it is urgent to find a way to regularize it. There have been numerous attempts to construct a finite energy Electroweak dyon in the context of Beyond the Standard Model physics. For more, see (Cho et al., 1997, [6]), (Mavromatos et al., 2018, [21]) and (Ellis et al., 2016,[11]).

Nevertheless, we turn a blind eye on this for now and continue minimizing the Hamiltonian by applying the Euler-Lagrange equations on each of the variable functions living in there.
This is for B.

$$\begin{aligned}
&\frac{\partial\mathcal{H}}{\partial B}-\frac{d}{dr}\frac{\partial\mathcal{H}}{\partial B}=0\\
&\Rightarrow\frac{(B-A)p^2}{4}r^2-\frac{d}{dr}\left(\frac{\dot{B}r^2}{g'^2}\right)=0\Rightarrow\\
&\frac{(B-A)\rho^2r^2}{4}-2r\frac{\dot{B}}{g'^2}-r^2\frac{\ddot{B}}{g^2}=0\Rightarrow\\
&\ddot{B}+\frac{2}{r}\dot{B}-\frac{(B-A)\rho^2g^2}{4}=0
\end{aligned} \tag{410}$$

For A, we have

$$\begin{aligned}
&\frac{\partial\mathcal{H}}{\partial A}-\frac{d}{dr}\frac{\partial\mathcal{H}}{\partial A}=0\Rightarrow\\
&\frac{2f^2}{g^2}A+\frac{(A-B)\rho^2r^2}{4}-\frac{d}{dr}\left(\frac{r^2\dot{A}}{g^2}\right)=0\Rightarrow\\
&\frac{2f^2}{g^2}A+\frac{(A-B)\rho^2r^2}{4}-2r\frac{1}{g^2}A-\frac{r^2\ddot{A}}{g^2}=0\\
&\Rightarrow\ddot{A}+\frac{2}{r}\dot{A}-\frac{2f^2A}{r^2}-\frac{(A-B)\rho^2g^2}{4}=0.
\end{aligned} \tag{411}$$

Euler-Lagrange equations on f yield

$$\begin{aligned}
&\frac{\partial \mathcal{H}}{\partial f} - \frac{d}{dr}\frac{\partial \mathcal{H}}{\partial f} = 0 \Rightarrow \\
&\frac{2A^2}{g^2} f + \frac{\left(f^2-1\right)}{g^2 r^2} 2f + \frac{\rho^2}{2} f - \frac{d}{dr}\left(\frac{2f}{g^2}\right) = 0 \Rightarrow \\
&\ddot{f} - A^2 f - \frac{\left(f^2-1\right) f}{r^2} - \frac{g^2 \rho^2 f}{4} f = 0.
\end{aligned} \tag{412}$$

and finally for ρ

$$\begin{aligned}
&\frac{\partial \mathcal{H}}{\partial \rho} - \frac{d}{dr}\frac{\partial \mathcal{H}}{\partial \rho} = 0 \Rightarrow \\
&\frac{(A-B)^2 \rho}{4} r^2 + \frac{f^2 \rho}{2} + \lambda r^2 \left(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\right)\rho - \frac{d}{dr}\left(p r^2\right) = 0 \Rightarrow \\
&\ddot{\rho} + \frac{2}{r}\rho - \lambda\left(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\right)\rho - \frac{f^2 \rho}{2r^2} - \frac{(A-B)^2 \rho}{4} = 0.
\end{aligned} \tag{413}$$

This is the system of differential equations whose solution describes the Cho-Maison monopole. There is no analytic solution for the given system and progress can by made only numerically. We show, however, as mentioned in [7] the boundary conditions and asymptotic behaviours of the functions.

The boundary conditions ensuring a regular solutions in the $SU(2)$ sector are

$$\begin{aligned}
&f(0) = 1, \quad \rho(0) = 0, \quad A(0) = 0, \quad B(0) = b_0, \\
&f(\infty) = 0, \rho(\infty) = \rho_0, A(\infty) = A_0, B(\infty) = B_0,
\end{aligned} \tag{414}$$

The asymptotic behaviour near the origin is

$$\begin{aligned}
&f \simeq 1 + \alpha_1 r^2 + \cdots, \\
&\rho \simeq \beta_1 r^\delta + \cdots, \\
&A \simeq a_1 r + \cdots, \\
&B \simeq b_0 + b_1 r + \cdots,
\end{aligned} \tag{415}$$

with $\delta = \frac{1+\sqrt{3}}{2}$. And at infinity we have

$$\begin{aligned}
&f \simeq f_1 \exp(-\kappa r) + \cdots, \\
&\rho \simeq \rho_0 + \rho_1 \frac{\exp(-\sqrt{2}\mu r)}{r} + \cdots, \\
&A \simeq A_0 + \frac{A_1}{r} + \cdots, \\
&B \simeq A + B_1 \frac{\exp(-\nu r)}{r} + \cdots,
\end{aligned} \tag{416}$$

where $\rho_0 = \sqrt{2/\lambda}\mu, \kappa = \sqrt{(g\rho_0)^2/4 - A_0^2}$, and $\nu = \sqrt{(g^2+g'^2)}\rho_0/2$.

D. Charge Properties

In this section, we are going to show that indeed the Cho-Maison ansatz (364) exhibits the magnetic properties advertised.

For this, we turn to the unitary gauge, which will simplify the calculations a lot, via the following gauge transformation.

$$\xi \longrightarrow U\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad U = -i \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2)e^{-i\varphi} \\ \sin(\theta/2)e^{i\varphi} & -\cos(\theta/2) \end{pmatrix} \tag{417}$$

Notice the remarkable similarity of this gauge transformation with the one in chapter VIII D!
It is easy to verify that in this gauge

$$\hat{\boldsymbol{\phi}}^{\boldsymbol{a}} = \delta^{a3} \tag{418}$$

But, what happens to the $SU(2)$ gauge potential?
Well, to find out we have to perform the gauge transformation.

$$\mathbf{A}'_\mu = \mathbf{U}\mathbf{A}_\mu\mathbf{U}^{-1} - \frac{2i}{g}\left(\partial_\mu \mathbf{U}\right)\mathbf{U}^{-1} \tag{419}$$

where the factor of 2 is included because we work in a basis of Pauli matrices $\boldsymbol{\tau}$ instead of $\frac{\boldsymbol{\tau}}{\mathbf{2}}$.

As always, we will work out each term separately.

$$\begin{aligned} \mathbf{A}_\mu = A^a_\mu \boldsymbol{\tau}^a &= \frac{1}{g}A\partial_\mu t\, \hat{r}^a\boldsymbol{\tau}^a + \frac{f-1}{g}\hat{r}^b\partial_\mu(\hat{r}^c)\varepsilon^{abc}\boldsymbol{\tau}^a \\ &= \frac{1}{g}A\partial_\mu t\boldsymbol{\tau}^r + \frac{f-1}{g}\hat{r}^b\frac{\delta^{\mu c} - \hat{r}^\mu\hat{r}^c}{r}\varepsilon^{abc}\boldsymbol{\tau}^a \\ &= \frac{1}{g}A\partial_\mu t\boldsymbol{\tau}^r + \frac{f-1}{gr}\boldsymbol{\tau}^a r^b\varepsilon^{ab\mu} \\ &= \frac{1}{g}A\partial_\mu t\boldsymbol{\tau}^r + \frac{f-1}{gr}\left(-\boldsymbol{\tau}^\theta\delta^{\mu\varphi} + \boldsymbol{\tau}^\varphi\delta^{\mu\theta}\right) \end{aligned} \tag{420}$$

where we used spherical coordinates, exploiting the fact that $\hat{r}^a$ has only an r-component.

The transformation of the temporal component yields

$$\begin{aligned} \frac{1}{g}A\partial_\mu t\mathbf{U}\boldsymbol{\tau}^r\mathbf{U}^{-1} &= \frac{1}{g}A\partial_\mu t\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\varphi} \\ \sin\frac{\theta}{2}e^{+i\varphi} & -\cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{+i\varphi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\varphi} \\ \sin\frac{\theta}{2}e^{+i\varphi} & -\cos\frac{\theta}{2} \end{pmatrix} \\ &= \frac{1}{g}A\partial_\mu t\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{g}A\partial_\mu t\boldsymbol{\tau}^3 \end{aligned} \tag{421}$$

The θ component

$$\frac{f-1}{gr}\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\varphi} \\ \sin\frac{\theta}{2}e^{+i\varphi} & -\cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} -\sin\theta & \cos e^{-i\varphi} \\ \cos\theta e^{+i\varphi} & \sin\theta \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\varphi} \\ \sin\frac{\theta}{2}e^{+i\varphi} & -\cos\frac{\theta}{2} \end{pmatrix} = -\frac{f-1}{gr}\begin{pmatrix} 0 & e^{-i\varphi} \\ e^{+i\varphi} & 0 \end{pmatrix} \tag{422}$$

and the φ term

$$\frac{f-1}{gr}\mathbf{U}\boldsymbol{\tau}^\varphi\mathbf{U}^{-1} = -\frac{f-1}{gr}\boldsymbol{\tau}^\varphi \tag{423}$$

The additive terms of the transformation

$$
\begin{aligned}
&\partial_r \mathbf{U} = 0 \\
&(\partial_\theta \mathbf{U})\,\mathbf{U}^{-1} = \frac{1}{2r}\begin{pmatrix} -\sin\frac{\theta}{2} & \cos\frac{\theta}{2}e^{-i\varphi} \\ \cos\frac{\theta}{2}e^{i\varphi} & \sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\varphi} \\ \sin\frac{\theta}{2}e^{i\varphi} & -\cos\frac{\theta}{2} \end{pmatrix} \\
&= \frac{1}{2r}\begin{pmatrix} 0 & -e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix} \\
&(\partial_\varphi \mathbf{U})\,\mathbf{U}^{-1} = \frac{1}{r\sin\theta}\begin{pmatrix} 0 & -i\sin\frac{\theta}{2}e^{-i\varphi} \\ i\sin\frac{\theta}{2}e^{i\varphi} & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\varphi} \\ \sin\frac{\theta}{2}e^{i\varphi} & -\cos\frac{\theta}{2} \end{pmatrix} \\
&= \frac{i}{2r}\begin{pmatrix} -\tan\frac{\theta}{2} & -e^{-i\varphi} \\ e^{i\varphi} & \tan\frac{\theta}{2} \end{pmatrix}
\end{aligned}
\tag{424}
$$

And the resulting potential post-gauge transformation is

$$
\boldsymbol{A}_\mu \longrightarrow \frac{1}{g}\begin{pmatrix} (\sin\varphi\partial_\mu\theta + \sin\theta\cos\varphi\partial_\mu\varphi)\, f(r) \\ (-\cos\varphi\partial_\mu\theta + \sin\theta\sin\varphi\partial_\mu\varphi)\, f(r) \\ -A(r)\partial_\mu t - (1-\cos\theta)\partial_\mu\varphi \end{pmatrix}
\tag{425}
$$

From this, we need

$$
\begin{aligned}
A^3_\mu &= -\frac{1}{g}A(r)\partial_\mu t - \frac{1}{g}(1-\cos\theta)\partial_\mu\varphi, \\
B_\mu &= -\frac{1}{g'}B(r)\partial_\mu t - \frac{1}{g'}(1-\cos\theta)\partial_\mu\varphi
\end{aligned}
\tag{426}
$$

Introducing now the electromagnetic potential $\mathcal{A}_\mu$ and neutral potential $\mathcal{Z}_\mu$, via the Weinberg mixing angle θ_W

$$
\begin{aligned}
\begin{pmatrix} \mathcal{A}_\mu \\ \mathcal{Z}_\mu \end{pmatrix} &= \begin{pmatrix} \cos\theta_\mathrm{w} & \sin\theta_\mathrm{w} \\ -\sin\theta_\mathrm{w} & \cos\theta_\mathrm{w} \end{pmatrix}\begin{pmatrix} B_\mu \\ A^3_\mu \end{pmatrix} \\
&= \frac{1}{\sqrt{g^2+g'^2}}\begin{pmatrix} g & g' \\ -g' & g \end{pmatrix}\begin{pmatrix} B_\mu \\ A^3_\mu \end{pmatrix}
\end{aligned}
\tag{427}
$$

we get

$$
\begin{aligned}
\mathcal{A}_\mu &= -e\left(\frac{1}{g^2}A + \frac{1}{g'^2}B\right)\partial_\mu t - \frac{1}{e}(1-\cos\theta)\partial_\mu\varphi \\
\mathcal{Z}_\mu &= \frac{e}{gg'}(B-A)\partial_\mu t
\end{aligned}
\tag{428}
$$

with

$$
e = \frac{gg'}{\sqrt{g^2+g'^2}}.
\tag{429}
$$

We now almost ready to calculate the electric charge of our solution.

$$
q_e = \oint d\mathbf{S}\cdot\mathbf{E} = \int dS^i E_i = \int dS^i \mathcal{F}_{0i}
\tag{430}
$$

but

$$
\mathcal{F}_{\mu\nu} = \partial\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu = -e\left[\frac{1}{g^2}\dot{A} + \frac{1}{g^{2\prime}}\dot{B}\right](\hat{r}_\mu\partial_\nu t - \hat{r}_v\partial_\mu t) - \frac{1}{e}\sin\theta\,(\partial_\mu\theta\,\partial_\nu\varphi - \partial_\nu\theta\,\partial_\mu\varphi)
\tag{431}
$$

so

$$\mathcal{F}_{0i} = e\left[\frac{\dot{A}}{g^2} + \frac{\dot{B}}{g'^2}\right]\hat{r}_i \tag{432}$$

and of course

$$q_e = e\oint dS_i\hat{r}^i\left(\frac{\dot{A}}{g^2} + \frac{\dot{B}}{g'^2}\right) = 4\pi e\left[r^2\left(\frac{\dot{A}}{g^2} + \frac{\dot{B}}{g'^2}\right)\right]\Bigg|_{r=\infty} \tag{433}$$

As for the magnetic charge

$$q_m = \int dS_i B^i = -\frac{1}{2}\int dS^i\varepsilon_{ijk}\mathcal{F}^{jk} = \int dS^i\frac{\hat{r}^i}{er^2} = \frac{4\pi}{e} \tag{434}$$

appears to be twice as large as the fundamental Dirac charge from the Dirac quantization condition.

As for the corresponding charges for the neutral potential, we have

$$\mathcal{G}_{\mu\nu} = \partial_\mu\mathcal{Z}_\nu - \partial_\nu\mathcal{Z}_\mu = \frac{e}{gg'}\left(\dot{B} - \dot{A}\right)(\hat{r}_\mu\partial_\nu t - \hat{r}_\nu\partial_\mu t) \tag{435}$$

It is evident that

$$\mathcal{G}_{ij} = 0 \tag{436}$$

so it does not possess a magnetic charge in the neutral sector, while its electric counterpart is

$$\begin{aligned} \mathcal{F}_{0i} &= -\frac{e}{gg'}\left(\dot{B} - \dot{A}\right)\hat{r}_i \\ q_{Ze} &= -\frac{4\pi e}{gg'}\left(r^2\dot{B} - r^2\dot{A}\right)\Big|_{r=\infty} = 0 \\ q_{Zm} &= 0 \end{aligned} \tag{437}$$

by virtue of the asymptotic behaviours of the functions A, B.

E. Gauge Invariance

While everything worked smoothly, proving the charges of our construction, one cannot ignore that we fixed the gauge to achieve our malicious goals. Proving that this method can be altered accordingly and cast in a gauge invariant way is a whole another story that certainly deserves our attention. One such attempt is presented at [5] by none other that Cho himself, while interesting ideas are presented by Savvidy [30].

In the following section, I shall be showing my take on the gauge invariance of the model, which admittedly has a lot of problems, but nevertheless succeeds on reproducing the results of the Cho-Maison ansatz in another gauge.

Let's say that we do not go into the unitary gauge, but rather stay in the radial/hedgehog gauge given by (364). Then, it is easy to see that

$$\begin{aligned} \hat{\phi}^a &= \xi^a\boldsymbol{\tau}^a\xi = -\hat{r}^a \\ \phi &= \rho\xi \\ A^a_\mu &= \frac{1}{g}A\partial_\mu t\,\hat{r}^a + \frac{1}{g}(f-1)\varepsilon_{abc}\hat{r}^b\partial_\mu\hat{r}^c. \\ A_\mu &= A^a_\mu\hat{\phi}^a = -\frac{1}{g}A\partial_\mu t \end{aligned} \tag{438}$$

Now for the first sketchy step. Inspired from the 't Hooft-Polyakov tensor, we write the Abelianisation of the $SU(2)$ tensor as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{g}\varepsilon_{abc}\partial_\mu\hat{\phi}^b\partial_\nu\hat{\phi}^c\hat{\phi}^a \tag{439}$$

This seems to have come out of nowhere, but I think it grants the Abelian $F_{\mu\nu}$ the gauge invariance needed.

The second sketchy step of this derivation has to do with the unmixing with the Weinberg angle. In standard Weinberg-Salam procedures, we are used to see this implemented via a rotation on the potentials like (427). One can see, however, that if (427) holds, so does

$$\begin{pmatrix} \mathcal{F}_{\mu\nu} \\ \mathcal{G}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathrm{w}} & \sin\theta_{\mathrm{w}} \\ -\sin\theta_{\mathrm{w}} & \cos\theta_{\mathrm{w}} \end{pmatrix} \begin{pmatrix} G_{\mu\nu} \\ F_{\mu\nu} \end{pmatrix} \tag{440}$$

Here, we are using the unmodified version of the Abelian projection of course. We can verify as

$$\begin{aligned} &\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu = && \text{By definition} \\ &\partial_\mu \left(\cos\theta_W B_\nu + \sin\theta_W A^3_\nu\right) - \partial_\nu \left(\cos\theta_W B_\mu + \sin\theta_W A^3_\mu\right) = && \text{By employing (427)} \\ &\cos\theta_W \left(\partial_\mu B_\nu - \partial_\nu B_\mu\right) + \sin\theta_W \left(\partial_\mu A^3_\nu - \partial_\nu A^3_\mu\right) = && \text{Rearranging} \\ &\cos\theta_W G_{\mu\nu} + \sin\theta_W F_{\mu\nu} \end{aligned} \tag{441}$$

Similarly for $\mathcal{G}_{\mu\nu}$

$$\begin{aligned} &\mathcal{G}_{\mu\nu} = \partial_\mu \mathcal{Z}_\nu - \partial_\nu \mathcal{Z}_\mu = && \text{By definition} \\ &\partial_\mu \left(-\sin\theta_W B_\nu + \cos\theta_W A^3_\nu\right) - \partial_\nu \left(-\sin\theta_W B_\mu + \cos\theta_W A^3_\mu\right) = && \text{By employing (427)} \\ &-\sin\theta_W \left(\partial_\mu B_\nu - \partial_\nu B_\mu\right) + \cos\theta_W \left(\partial_\mu A^3_\nu - \partial_\nu A^3_\mu\right) = && \text{Rearranging} \\ &-\sin\theta_W G_{\mu\nu} + \cos\theta_W F_{\mu\nu} \end{aligned} \tag{442}$$

Now with the modified (439), (427) does not imply in any way (440). However, the inverse is true. If (440) holds, (427) is satisfied automatically.

Let us check how this works.

$$\begin{pmatrix} \mathcal{F}_{\mu\nu} \\ \mathcal{G}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} G_{\mu\nu} \\ F_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos\theta_W G_{\mu\nu} + \sin\theta_W F_{\mu\nu} \\ -\sin\theta_W G_{\mu\nu} + \cos\theta_W F_{\mu\nu} \end{pmatrix} \tag{443}$$

$$= \begin{pmatrix} \partial_\mu \left(\cos\theta_W B_\nu + \sin\theta_W A_\nu\right) - \partial_\nu \left(\cos\theta_W B_\mu + \sin\theta_W A_\mu\right) + \frac{g'}{g\sqrt{g^2+g'^2}} \varepsilon_{abc} \partial_\mu \hat{\phi}^b \partial_\nu \hat{\phi}^c \hat{\phi}^a \\ \partial_\mu \left(-\sin\theta_W B_\nu + \cos\theta_W A_\nu\right) - \partial_\nu \left(-\sin\theta_W B_\mu + \cos\theta_W A_\mu\right) + \frac{1}{\sqrt{g^2+g'^2}} \varepsilon_{abc} \partial_\mu \hat{\phi}^b \partial_\nu \hat{\phi}^c \hat{\phi}^a \end{pmatrix}$$

Now,we can see that both neutral and electromagnetic tensor have been endowed with a 'topological' kind of term reminiscent of the topological current from the 't Hooft-Polyakov tensor (229). Also, noticing the argument of the partial derivatives, we see that (427) is forced automatically, which is a good thing for our trick. If (427) was violated, we would be doing something wrong. Of course, employing indirectly via (443) is not necessarily the right way. One could argue, for example, that only (427) holds and the tensors are obtained via direct calculation rather than rotating. Everything involved in this calculation should be taken with great caution and critical thinking and the physical meaning (or physical meaninglessness) of it all is subject to discussion. Having warned you all of the imminent dangers, we proceed with our calculation and let all the brave or irrational ones follow along![67]

To calculate the electric and magnetic content of $\mathcal{F}_{\mu\nu}, \mathcal{G}_{\mu\nu}$, we need to calculate first their respective 4-potentials and plug them in, while also not forgetting the topological current contribution.

---

[67] The notion of the Weinberg rotation in terms of tensors is also found in [24] and [33]. The latter discusses the general independent of gauge form of an electromagnetic tensor for the broken symmetry phase of the electroweak model.

$$
\begin{aligned}
\mathcal{A}_\mu &= \cos\theta_W B_\mu + \sin\theta_W A_\mu \\
&= -e\left(\frac{1}{g^2}A + \frac{1}{g'^2}B\right)\partial_\mu t - \frac{e}{g'^2}(1-\cos\theta)\partial_\mu\varphi \\
\mathcal{Z}_\mu &= -\sin\theta_W B_\mu + \cos\theta_W A_\mu \\
&= \frac{e}{gg'}(B-A)\partial_\mu t + \frac{e}{gg'}(1-\cos\theta)\partial_\mu\varphi,
\end{aligned}
\tag{444}
$$

Comparing with (428), we see that both potentials now have monopole like terms. This might appear as problematic, especially for $\mathcal{Z}_\mu$, whose field we expect to exhibit no charges, but the topological terms we added, are here to save the day! Notice, also that electrically everything remains the same as in the unitary gauge.

We need

$$
\begin{aligned}
\mathcal{F}_{jk} &= \partial_j\mathcal{A}_k - \partial_k\mathcal{A}_j - \frac{e}{g^2}\varepsilon_{abc}\partial_j\hat{r}^b\partial_k\hat{r}^c\hat{r}^a \\
&= -\frac{e}{g'^2}\sin\theta\,(\partial_j\theta\,\partial_k\varphi - \partial_k\theta\,\partial_j\varphi) - \frac{e}{g^2}\frac{1}{r^2}\varepsilon_{abc}\hat{r}^a\delta^{jb}\delta^{kc}
\end{aligned}
$$

So

$$
\begin{aligned}
\varepsilon_{ijk}\mathcal{F}_{jk} &= -2\frac{e}{g'^2}\sin\theta\;\left.(\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\varphi)\right|^i - \frac{e}{g^2}\frac{\hat{r}^a}{r^2}\varepsilon_{ajk}\varepsilon_{ijk} \\
&= -\frac{e}{g'^2}2\sin\theta\;\left.\left(\frac{1}{r}\hat{\boldsymbol{\theta}}\times\frac{1}{r\sin\phi}\hat{\boldsymbol{\varphi}}\right)\right|^i - \frac{e}{g^2}\frac{\hat{r}^a}{r^2}\left(\delta_{ai}\delta_{jj} - \delta_{aj}\delta_{ji}\right) \\
&= -2\frac{e}{r^2}\left(\frac{1}{g^2} + \frac{1}{g'^2}\right)\hat{r}^i = -2\frac{e}{r^2}\left(\frac{g'^2+g^2}{(gg')^2}\right)\hat{r}^i = -\frac{2}{er^2}\hat{r}^i
\end{aligned}
$$

We are now in position to calculate the magnetic charge of the electromagnetic tensor in the 'hedgehog' gauge.

$$
q_m = \int -\frac{1}{2}\varepsilon_{ijk}\mathcal{F}^{jk}dS_i = \frac{4\pi}{e}
\tag{445}
$$

The magnetic content of $\mathcal{G}_{\mu\nu}$ can be found as

$$
\begin{aligned}
\mathcal{G}_{jk} =& \partial_j\mathcal{Z}_k - \partial_k\mathcal{Z}_j + \frac{1}{\sqrt{g^2+g'^2}}\varepsilon_{abc}\partial_j\hat{\phi}^b\,\partial_k\hat{\phi}^c\hat{\phi}^a \\
=& \frac{1}{\sqrt{g^2+g'^2}}\sin\theta\,(\partial_j\theta\,\partial_k\phi - \partial_k\theta\,\partial_j\phi) - \frac{1}{\sqrt{g^2+g'^2}}\varepsilon_{abc}\partial_j\hat{r}^b\,\partial_k\hat{r}^c\hat{r}^a \\
=& \frac{1}{\sqrt{g^2+g'^2}}\sin\theta\,(\partial_j\theta\,\partial_k\phi - \partial_k\theta\,\partial_j\phi) - \frac{1}{\sqrt{g^2+g'^2}}\frac{r^a}{r^2}\varepsilon_{ajk}
\end{aligned}
\tag{446}
$$

And upon contraction with Levi-Civita,

$$
\varepsilon_{ijk}\mathcal{G}^{jk} = \frac{1}{\sqrt{g^2+g'^2}}2\sin\theta\,\left.(\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\phi)\right|^i - \frac{2}{\sqrt{g^2+g'^2}}\frac{\hat{r}^i}{r^2} = 0
\tag{447}
$$

Therefore, $\mathcal{G}_{\mu\nu}$ possesses no magnetic charge. We conclude that this method yields the same results in the hedgehog gauge as well as in the unitary gauge (since derivatives of $\hat{\phi}^a$ vanish there.

F. Clarifications and Discussion

This one chapter shall serve as a closure to the Cho-Maison monopole discussion. We will present some peculiarities with $SU(2)$ Yang-Mills field equation of motion in the Standard Model (354) that is seemingly not manifestly gauge invariant. Based on this and discussions from Restricted Gauge Field theory [4], we will show the group transformational properties of $\hat{\phi}^a$ and the Abelianization of the Weinberg-Salam theory [5].

1. *Gauge Properties of $SU(2)$ Yang-Mills field in Weinberg-Salam model*

Remember (354) in vector form states

$$D_\mu F^a_{\mu\nu} = -j^a_\nu = i\frac{g}{2}\left[\boldsymbol{\phi}^\dagger \boldsymbol{\tau}^a \left(\hat{D}_\nu \boldsymbol{\phi}\right) - \left(\hat{D}_\nu \boldsymbol{\phi}\right)^\dagger \boldsymbol{\tau}^a \boldsymbol{\phi}\right]$$

In adjoint representation

$$\mathbf{D}_\mu (F^a_{\mu\nu}\boldsymbol{\tau}^a) = -\mathbf{j}_\nu = i\frac{g}{2}\left[\boldsymbol{\phi}^\dagger \boldsymbol{\tau}^a \left(\hat{D}_\nu \boldsymbol{\phi}\right) - \left(\hat{D}_\nu \boldsymbol{\phi}\right)^\dagger \boldsymbol{\tau}^a \boldsymbol{\phi}\right]\boldsymbol{\tau}^a$$

Now it easier to perform a gauge transformation.

$$\mathbf{U}\mathbf{D}_\mu F^a_{\mu\nu}\boldsymbol{\tau}^a \mathbf{U}^\dagger = -\boldsymbol{j}_\nu = i\frac{g}{2}\left[\boldsymbol{\phi}^\dagger \mathbf{U}^\dagger \boldsymbol{\tau}^a \mathbf{U}\left(\hat{D}_\nu \boldsymbol{\phi}\right) - \left(\hat{D}_\nu \boldsymbol{\phi}\right)^\dagger \mathbf{U}^\dagger \boldsymbol{\tau}^a \mathbf{U}\boldsymbol{\phi}\right]\boldsymbol{\tau}^a$$

At first glance, the two sides of our equation seem to behave differently, which would ruin our day and definitely the Standard Model. Thankfully, this is not the case here, because we can prove that the two sides indeed transform the same way, but closer inspection is needed. It is also helpful to work with infinitesimal gauge transformations here, approximating $\mathbf{U}$ with

$$\mathbf{U} = \exp\left(i\frac{g}{2}\theta^i \boldsymbol{\tau}^i\right) \approx 1 + i\theta^i \boldsymbol{\tau}^i \tag{448}$$

We can show for the left hand side of (354) and in general for any adjoint representation quantity that

$$\begin{aligned}
&\mathbf{D}_\mu F^{\alpha\prime}_{\mu\nu}\boldsymbol{\tau}^\alpha = \mathbf{U}\mathbf{D}_\mu F^\alpha_{\mu\nu}\boldsymbol{\tau}^\alpha \mathbf{U}^\dagger && \text{Adjoint rep transformation law}\\
&= \left(1 + i\theta^\beta \boldsymbol{\tau}^\beta\right)\left(\mathbf{D}_\mu F^\alpha_{\mu\nu}\boldsymbol{\tau}^\alpha\right)\left(1 - i\theta^\beta \boldsymbol{\tau}^\beta\right) && \text{Infinitesimal transform}\\
&= \mathbf{D}_\mu F^\alpha_{\mu\nu}\boldsymbol{\tau}^\alpha + i\theta^\beta \mathbf{D}_\mu F^\alpha_{\mu\nu}\left(\boldsymbol{\tau}^\beta \boldsymbol{\tau}^a - \boldsymbol{\tau}^\alpha \boldsymbol{\tau}^\beta\right) && \text{Keeping up to second order terms}\\
&= \mathbf{D}_\mu F^\alpha_{\mu\nu}\boldsymbol{\tau}^\alpha + i\theta^\beta \mathbf{D}_\mu F^\alpha_{\mu\nu}\left[\boldsymbol{\tau}^\beta, \boldsymbol{\tau}^\alpha\right] &&\\
&= \mathbf{D}_\mu F^\alpha_{\mu\nu}\boldsymbol{\tau}^\alpha + i\theta^\beta F^\alpha_{\mu\nu} i\varepsilon^{\beta\alpha\gamma}\boldsymbol{\tau}^\gamma && \text{Rearranging-Renaming indices}\\
&= \left(\mathbf{D}_\mu F^\alpha_{\mu\nu} + i\theta^\beta \mathbf{D}_\mu F^\gamma_{\mu\nu} i\varepsilon^{\beta\gamma\alpha}\right)\tau^\alpha &&
\end{aligned} \tag{449}$$

At this point, one can notice from the first and last row of the above calculation that

$$\mathbf{D}_\mu F^{\alpha'}_{\mu\nu} = \left(\delta^{\alpha\gamma} + i\theta^\beta \varepsilon^{\beta\gamma\alpha}\right)\mathbf{D}_\mu F^\gamma_{\mu\nu} \tag{450}$$

and setting

$$\left(\mathbf{T}^\beta\right)_{\gamma\alpha} = \varepsilon^{\beta\gamma\alpha} \tag{451}$$

we get

$$\mathbf{D}_\mu F^{\alpha'}_{\mu\nu} = \mathbf{U}_{3\times 3}\mathbf{D}_\mu F^\alpha_{\mu\nu} \tag{452}$$

where

$$\mathbf{U}_{3\times 3} = \exp\left(i\theta^\beta \mathbf{T}^\beta\right)$$

We see that adjoint representation quantities can also be transformed as 'vectors' under the transformation generated by the structure constants!

Now, we are going to perform a similar procedure on the R.H.S of (354), but for simplicity we are going to bother only with the first term, since the other one has the same transformational properties.

$$\begin{aligned}
&\boldsymbol{\phi}^\dagger \mathbf{U}^\dagger \boldsymbol{\tau}^\alpha \mathbf{U}\mathbf{D}_\nu \boldsymbol{\phi}\boldsymbol{\tau}^\alpha && \text{Second term of (354)}\\
=&\boldsymbol{\phi}^\dagger \left(1 - i\theta^\beta \boldsymbol{\tau}^\beta\right) \boldsymbol{\tau}^\alpha \left(1 + i\theta^\beta \tau^\beta\right) \mathbf{D}_\nu \boldsymbol{\phi}\boldsymbol{\tau}^\alpha && \text{Infinitesimal transform}\\
=&\boldsymbol{\phi}^\dagger \boldsymbol{\tau}^a \mathbf{D}_\nu \boldsymbol{\phi}\boldsymbol{\tau}^\alpha + i\theta^\beta \boldsymbol{\phi}^\dagger \left(\boldsymbol{\tau}^\alpha \boldsymbol{\tau}^\beta - \boldsymbol{\tau}^\beta \boldsymbol{\tau}^\alpha\right) \mathbf{D}_\nu \boldsymbol{\phi}^\alpha \boldsymbol{\tau}^\alpha && \text{Keeping up to second order terms}\\
=&\boldsymbol{\phi}^\dagger \boldsymbol{\tau}^\alpha \mathbf{D}_\nu \boldsymbol{\phi}\boldsymbol{\tau}^\alpha + i\theta^\beta \boldsymbol{\phi}^\dagger \left[\boldsymbol{\tau}^\alpha, \boldsymbol{\tau}^\beta\right] \mathbf{D}_\nu \boldsymbol{\phi}\boldsymbol{\tau}^\alpha\\
=&\boldsymbol{\phi}^\dagger \boldsymbol{\tau}^\alpha \mathbf{D}_\nu \boldsymbol{\phi}\boldsymbol{\tau}^\alpha + i\theta^\beta \boldsymbol{\phi}^\dagger i\varepsilon^{\alpha\beta\gamma} \boldsymbol{\tau}^\gamma \mathbf{D}_\nu \boldsymbol{\phi}\boldsymbol{\tau}^\alpha\\
=&\boldsymbol{\phi}^\dagger \boldsymbol{\tau}^\alpha \mathbf{D}_\nu \boldsymbol{\phi}\boldsymbol{\tau}^\alpha + i\theta^\beta \boldsymbol{\phi}^\dagger \boldsymbol{\tau}^\gamma \mathbf{D}_\nu \boldsymbol{\phi}\, i\varepsilon^{\alpha\beta\gamma} \boldsymbol{\tau}^\alpha && \text{Renaming-Rearranging}\\
=&\left(\boldsymbol{\phi}^\dagger \boldsymbol{\tau}^\alpha \mathbf{D}_\nu \boldsymbol{\phi} + i\theta^\beta \left(\boldsymbol{\phi}^\dagger \boldsymbol{\tau}^\gamma \mathbf{D}_\nu \phi\right) i\varepsilon^{\alpha\beta\gamma}\right) \boldsymbol{\tau}^\alpha
\end{aligned} \tag{453}$$

Now comparing the last line of (453) with the last one from (449), it becomes evident that (354) holds indeed in every gauge!

2. *Transformational properties of* $\hat{\boldsymbol{\phi}}$

This brings us to the next topic. We remind that

$$\hat{\phi}^a = \xi^\dagger \boldsymbol{\tau}^a \xi \tag{454}$$

which is a triplet. Now a gauge transformation on this makes it look like

$$\hat{\phi}^{a\prime} = \xi^\dagger \mathbf{U}^\dagger \boldsymbol{\tau}^a \mathbf{U} \xi \tag{455}$$

which is in the limbo between an adjoint triplet and just something weird. We are going to show that it is in fact an adjoint triplet. One can notice that the structure of is the same with (453) and as a result, a similar analysis reveals that it indeed transforms as in adjoint representation.
There is also another interesting way to deduce this and that is to check its covariant derivative.
First, we should state the following identity for fundamental representation objects $\boldsymbol{\psi}$

$$\mathbf{D}_\mu \left(\boldsymbol{\psi}^\dagger \boldsymbol{\psi}\right) = \left(\mathbf{D}_\mu \boldsymbol{\psi}\right)^\dagger \boldsymbol{\psi} + \boldsymbol{\psi}^\dagger \mathbf{D}_\mu \boldsymbol{\psi} = \partial_\mu \left(\boldsymbol{\psi}^\dagger \boldsymbol{\psi}\right) \tag{456}$$

which can be verified be simple calculations. The only remark here is that the covariant derivative goes under the dagger operator when performing the multiplication rule and that is to guarantee that our quantity is a group scalar as it should.
Inspired from this, we will calculate the covariant derivative of $\hat{\boldsymbol{\phi}}$.

$$\begin{aligned}
\mathbf{D}_\mu \phi^\alpha &= \left(\mathbf{D}_\mu \xi\right)^\dagger \boldsymbol{\tau}^\alpha \xi + \xi^\dagger \boldsymbol{\tau}^\alpha \mathbf{D}_\mu \xi\\
&= \partial_\mu \xi^\dagger \boldsymbol{\tau}^a \xi + i\frac{g}{2} A^\beta_\mu \xi^\dagger \boldsymbol{\tau}^\beta \boldsymbol{\tau}^\alpha \xi + \xi^\dagger \boldsymbol{\tau}^\alpha \mathbf{D}_\mu \xi - i\frac{g}{2} A^b_\mu \xi^\dagger \boldsymbol{\tau}^\alpha \boldsymbol{\tau}^b \xi\\
&= \partial_\mu \left(\xi^\dagger \boldsymbol{\tau}^\alpha \xi\right) - i\frac{g}{2} A^\beta_\mu \xi^\dagger \left[\boldsymbol{\tau}^\alpha, \boldsymbol{\tau}^\beta\right] \xi\\
&= \partial_\mu \phi^\alpha - i\frac{g}{2} A^\beta_\mu i\varepsilon^{\alpha\beta\gamma} \xi^\dagger \boldsymbol{\tau}^\gamma \xi\\
&= \partial_\mu \phi^\alpha + \frac{g}{2} \varepsilon^{\alpha\beta\gamma} A^\beta_\mu \phi^\gamma
\end{aligned} \tag{457}$$

which is the covariant derivative of an adjoint representation quantity as we have shown in (99)!
This is indeed really, since now that we have our hands on an SU(2) triplet (which is also trivially invariant under U(1) transformations), we can impose the condition of the vanishing covariant derivative at infinity much like in the 't Hooft-Polyakov case of chapter VIII F and create the modified SU(2) Abelian projection tensor in the Weinberg-Salam model (439), instead of summoning it through the mystical arts of guessing.

XII. FINITE ENERGY ELECTROWEAK MONOPOLE

A. Born-Infeld Regularization of the Cho-Maison Monopole

We return for now to the problem we met at (409). As you may remember the energy of the Cho-Maison monopole has an incurable divergence because of the singular nature of the hypercharge monopole field.

We can write (409) separating the divergent part with everything else.

$$
\begin{aligned}
&H = H_0 + H_1 \\
&H_0 = 4\pi \int_0^\infty \frac{1}{2g'^2 r^2} dr \\
&H_1 = \int d^3x \mathcal{H} = 4\pi \int_0^\infty dr \frac{r^2 \dot{A}^2}{2g^2} + \frac{A^2 f^2}{g^2} + \frac{\dot{f}^2}{g^2} + \frac{\left(f^2-1\right)^2}{2g^2 r^2} + \frac{\dot{B}^2 r^2}{2g'^2} \\
&+ \frac{(A-B)^2 \rho^2}{8} r^2 + \frac{\dot{\rho}^2}{2} r^2 + \frac{f^2 \rho^2}{4} + \frac{\lambda}{2}\left(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\right)^2 r^2
\end{aligned} \tag{458}
$$

The divergence of the energy functional casts serious doubts on the existence of a consistent monopole solution in the Weinberg-Salam theory and calls for an urgent extension of the model if one wants to acquire a finite energy solution.

Here, we will examine the Born-Infeld extension of the hypercharge sector as proposed by Arunasalam and Kobakhidze [3] and continued by Mavromatos and Sarkar [21]. The modification proposed requires only the substitution of the hypercharge field kinetic term with a non-linear Born-Infeld term as shown below

$$
\mathcal{L}_{BI} = -\left(\mathbf{D}_\mu \boldsymbol{\phi}\right)^\dagger \left(\mathbf{D}^\mu \boldsymbol{\phi}\right) - \frac{\lambda}{2}\left(\boldsymbol{\phi}^\dagger \boldsymbol{\phi} - \frac{\mu^2}{\lambda}\right)^2 - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a} + \beta^2 \left(1 - \sqrt{1 + \frac{1}{2\beta^2} G_{\mu\nu} G^{\mu\nu} - \frac{1}{16\beta^4}\left(G_{\mu\nu} \widetilde{G}^{\mu\nu}\right)^2}\right) \tag{459}
$$

The formalism is the same with the Lagrangian of the Weinberg-Salam model from (353). The only additions we have to clarify are $\widetilde{G}_{\mu\nu} = 1/2 \varepsilon_{\mu\nu\rho\sigma} G^{\rho\sigma}$, which the dual tensor of $G_{\mu\nu}$ and the parameter β, which is generally connected to the string mass scale[68], but here it is sufficient to consider it a free parameter to be identified by the experiment.

Having our Lagrangian, the next step is to acquire the equations of motion. As one may expect, they do not differ that much from the ones we derived before on the Cho-Maison monopole. In fact, only the one related to the hypercharge field is modified. We do not do the derivation here, but one can intuitively understand that upon differentiation the square root moves to the denominator and on the numerator remains a function of $G_{\mu\nu}$.

$$
\begin{aligned}
&\hat{\mathbf{D}}_\mu \left(\hat{\mathbf{D}}_\mu \phi\right) = \lambda \left(\phi^\dagger \phi - \frac{\mu^2}{\lambda}\right)\phi, \\
&\mathbf{D}_\mu \boldsymbol{F}_{\mu\nu} = -\boldsymbol{j}_\nu = i\frac{g}{2}\left[\phi^\dagger \boldsymbol{\tau} \left(\hat{\mathbf{D}}_\nu \phi\right) - \left(\hat{\mathbf{D}}_\nu \phi\right)^\dagger \boldsymbol{\tau} \phi\right], \\
&\partial_\mu \left[\frac{G^{\mu\nu} - \frac{1}{4\beta^2}\left(G_{\alpha\beta}\tilde{G}^{\alpha\beta}\right)\widetilde{G}^{\mu\nu}}{\sqrt{1 + \frac{1}{2\beta^2} G_{\alpha\beta} G^{\alpha\beta} - \frac{1}{16\beta^4}\left(G_{\alpha\beta}\tilde{G}^{\alpha\beta}\right)^2}}\right] = -k_\nu = i\frac{g'}{2}\left[\phi^\dagger \left(\hat{\mathbf{D}}_\nu \phi\right) - \left(\hat{\mathbf{D}}_\nu \phi\right)^\dagger \phi\right].
\end{aligned} \tag{460}
$$

---

[68] The Born-Infeld parameter β has dimensions of $[\text{mass}]^2$. The ESM Lagrangian reduces formally to the SM Lagrangian for $\beta \to \infty$. In the context of microscopic string theory models, the parameter $\sqrt{\beta} \propto M_s$, the string mass scale [21].

And after the equations of motion are settled alright, we propose an ansatz, which in this case is very much the same as the Cho-Maison (364).

$$\begin{aligned}
\phi &= \frac{1}{\sqrt{2}}\rho\xi \\
\xi &= i\begin{pmatrix} \sin(\theta/2)e^{-i\varphi} \\ -\cos(\theta/2) \end{pmatrix}, \\
\hat{\boldsymbol{\phi}} &= \xi^{\dagger}\boldsymbol{\tau}\xi = -\hat{\boldsymbol{r}}, \\
\boldsymbol{A}_{\mu} &= \frac{1}{g}A(r)\partial_{\mu}t\hat{\boldsymbol{\phi}} + \frac{1}{g}(f(r)-1)\hat{\boldsymbol{\phi}}\times\partial_{\mu}\hat{\boldsymbol{\phi}}, \\
B_{\mu} &= -\frac{1}{g'}B(r)\partial_{\mu}t - \frac{1}{g'}(1-\cos\theta)\partial_{\mu}\varphi,
\end{aligned} \tag{461}$$

The solution indeed preserves the same magnetic charge, which can be shown in a very similar manner to the Cho-Maison case.

We can see this briefly by switching to the unitary gauge

$$\xi \quad \to U\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ with } \quad U = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2)e^{-i\varphi} \\ -\sin(\theta/2)e^{i\varphi} & \cos(\theta/2) \end{pmatrix} \tag{462}$$

and

$$A_{\mu} \longrightarrow \frac{1}{g}\begin{pmatrix} (\sin\varphi\partial_{\mu}\theta + \sin\theta\cos\varphi\partial_{\mu}\varphi)\, f(r) \\ (-\cos\varphi\partial_{\mu}\theta + \sin\theta\sin\varphi\partial_{\mu}\varphi)\, f(r) \\ -A(r)\partial_{\mu}t - (1-\cos\theta)\partial_{\mu}\varphi \end{pmatrix} \tag{463}$$

All the same as in chapter XI D, nothing new to see here. Rotating to the physical fields

$$\begin{aligned}
\begin{pmatrix} \mathcal{A}_{\mu} \\ \mathcal{Z}_{\mu} \end{pmatrix} &= \begin{pmatrix} \cos\theta_{\mathrm{w}} & \sin\theta_{\mathrm{w}} \\ -\sin\theta_{\mathrm{w}} & \cos\theta_{\mathrm{w}} \end{pmatrix}\begin{pmatrix} B_{\mu} \\ A^3_{\mu} \end{pmatrix} \\
&= \frac{1}{\sqrt{g^2+g'^2}}\begin{pmatrix} g & g' \\ -g' & g \end{pmatrix}\begin{pmatrix} B_{\mu} \\ A^3_{\mu} \end{pmatrix}
\end{aligned} \tag{464}$$

we get

$$\begin{aligned}
\mathcal{A}_{\mu} &= -e\left(\frac{1}{g^2}A + \frac{1}{g'^2}B\right)\partial_{\mu}t - \frac{1}{e}(1-\cos\theta)\partial_{\mu}\varphi \\
\mathcal{Z}_{\mu} &= \frac{e}{gg'}(B-A)\partial_{\mu}t
\end{aligned} \tag{465}$$

with

$$e = g\sin(\theta_W) = \frac{gg'}{\sqrt{g^2+(g')^2}} \tag{466}$$

.

Now for the monopole solution, one needs to set $A(r) = B(r) = 0$. Then

$$\begin{aligned}
\mathcal{A}_{\mu} &= -\frac{1}{e}(1-\cos\theta)\partial_{\mu}\varphi \\
\mathcal{Z}_{\mu} &= 0
\end{aligned} \tag{467}$$

It is now evident that the solution represent a dressed Dirac-like monopole whose magnetic charge is twice the fundamental Dirac charge[69].

$$q_m = \frac{4\pi}{e} = \frac{4\pi\sqrt{g^2 + (g')^2}}{gg'} \tag{468}$$

Our final step here is of course to show that the solution also has a finite energy!

Before that, however, we should also take into account if the Cho-Maison ansatz (with A=B=0) still satisfies the modified equation of motion for the hypercharge field. Thankfully, we have done the same analysis in the Cho-Maison case[70] and at least the RHS remains the same, which means that for vanishing A and B, it equals 0.

The hypercharge magnetic field is $B_{Yi} = \frac{q_m}{r^3} r^i$. Therefore $G_{\mu\nu}$ has only spatial components.

$$G_{ij} \propto \varepsilon_{ijk} B_Y^k$$

We do not have an electric field, so

$$G_{\alpha\beta} G^{\alpha\beta} \propto B_{Yi}^2$$

and $G_{\alpha\beta}\widetilde{G}^{\alpha\beta} = G_{\alpha\beta}\varepsilon^{\alpha\beta\gamma\delta}G^{\gamma\delta}$ but for $G_{\alpha\beta}$ to be non-vanishing α, β must be spatial and so must be γ and δ. In that case, though at least one index has to be repeated once and the term is cancelled by the Levi-Civita tensor. So the LHS is of the form

$$\partial_i \left[\frac{\varepsilon^{ijk} B_{Yk}(r)}{\sqrt{1 + \frac{1}{2\beta^2} G_{\alpha\beta} G^{\alpha\beta}}} \right]$$

Considering that only differentiation with respect to r could yield a non trivial result and that only the r-component of the magnetic field could contribute, we see that again the Levi-Civita tensor stands in the way leading a vanishing result for the LHS as well.
For completeness, we state that the rest of the O.D.E's from the ansatz substitution into the equations of motion remain intact and are exactly the same as in the Cho-Maison case.
Now we are ready to tackle the energy functional. Remember that all the terms except for the kinetic one of the hypercharge field remain unchanged. Also, since we examine a time independent solution the appearance of such term in the Hamiltonian will be the same as in the Lagrangian with a reversed sign.

$$\begin{aligned} &H = H_0 + H_1 \\ &H_1 = \int d^3x \mathcal{H} = 4\pi \int_0^\infty dr \frac{\dot{f}^2}{g^2} + \frac{\left(f^2-1\right)^2}{2g^2 r^2} + \frac{\dot{\rho}^2}{2} r^2 + \frac{f^2\rho^2}{4} + \frac{\lambda}{2}\left(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\right)^2 r^2 \end{aligned} \tag{469}$$

while the (not anymore) singular part is

$$\begin{aligned} H_0 &= 4\pi \int_0^\infty r^2 dr \left\{ \sqrt{1 + \frac{1}{2\beta^2} G_{\mu\nu}G^{\mu\nu} - \frac{1}{16\beta^4}\left(G_{\mu\nu}\widetilde{G}^{\mu\nu}\right)^2} - \beta^2 \right\} \\ &= 4\pi \int_0^\infty \beta^2 r^2 dr \left\{ \sqrt{1 + \frac{1}{2\beta^2} 2 B_{Yi} B_Y^i} - 1 \right\} && \text{Using results from above} \\ &= 4\pi \int_0^\infty \beta^2 r^2 dr \left\{ \sqrt{1 + \frac{1}{(g)^2 b^2 r^4}} - 1 \right\} && \text{Brushing up} \\ &= 4\pi \left(\frac{\beta}{g'^3}\right)^{1/2} \int_0^\infty dx \left[\sqrt{1+x^4} - x^2\right] && \text{Via } x = \sqrt{g'\beta} r \text{ substitution} \end{aligned} \tag{470}$$

[69] That is $q_m = 2\pi/e$

[70] See equations (365)-(372).

Now that's an interesting integral and who are we to back away from such a challenge.

$$
\begin{aligned}
&\int_0^\infty \left(\sqrt{1+x^4}-x^2\right)dx = && \text{Multiply with the conjugate}\\
&\int_0^\infty \frac{1}{\sqrt{1+x^4}+x^2}dx = && \text{Set } x^4=\sinh^2 y\\
&\int_0^\infty \frac{dy}{\cosh y+\sinh y}\frac{\cosh y}{2\sqrt{\sinh y}} = && \text{Use } \cosh y+\sinh y = e^{-y}\\
&\int_0^\infty dy e^{-y}(\sqrt{\sinh y})' = && \text{Perform partial integration}\\
&\sqrt{\sinh y}e^{-y}\Big|_0^\infty + \int_0^\infty \sqrt{\sinh y}e^{-y}dy = && \text{Both limits go to zero}\\
&\lim_{y\to\infty}\sqrt{\frac{e^y+-e^{-y}}{2}}e^{-y} - 0 + \int_0^\infty \sqrt{\sinh y}e^{-y}dy = && \text{Now substitute } y=\ln u\\
&\int_1^\infty \sqrt{\frac{u-\frac{1}{u}}{2}}\cdot\frac{1}{u^2}du && \text{And } w=1/u\\
&= \frac{1}{\sqrt{2}}\int_0^1 dw\sqrt{\frac{1}{w}-w} = &&\\
&= \frac{1}{\sqrt{2}}\int_0^1 dw\sqrt{\frac{1-w^2}{w}} = && \text{Set } z=w^2 \qquad (471)\\
&\frac{1}{2\sqrt{2}}\int_0^1 \sqrt{1-z}\frac{1}{z^{1/4}}\frac{dz}{\sqrt{z}} &&\\
&= \frac{1}{2\sqrt{2}}\int_0^1 (1-z)^{1/2}z^{-3/4}dz && \text{Beta function definition appears}\\
&= \frac{1}{2\sqrt{2}}B\left(\frac{3}{2},\frac{1}{4}\right) && \text{Expand with } B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\\
&= \frac{\Gamma(3/2)\Gamma(1/4)}{4\sqrt{2}\Gamma(7/4)} &&\\
&= \frac{\frac{1}{2}\Gamma(1/2)\Gamma(1/4)}{4\sqrt{2}\frac{3}{4}\Gamma(3/4)} && \text{Use } \Gamma(x+1)=x\,\Gamma(x) \text{ and } \Gamma(1/2)=\sqrt{\pi}\\
&= \frac{\sqrt{\pi}}{6\sqrt{2}}\frac{\Gamma(1/4)}{\Gamma(3/4)} && \text{Employ Legendre Duplication Formula}\\
&= \frac{\sqrt{\pi}\sin\left(\frac{3\pi}{4}\right)\Gamma^2(1/4)}{6\sqrt{2}\pi} &&\\
&= \frac{\Gamma^2(1/4)}{6\sqrt{\pi}} &&
\end{aligned}
$$

Solving analytically the equations for the ansatz functions one can now determine the mass of the monopoole with respect to the free parameter β.

Using the value of g' in the Standard Model ($g'=0.357$) the ex-singular part is calculated as

$$H_1 \approx 72.8\sqrt{\beta}$$

H_0 is already finite readily from the Cho-Maison case and its value depends on the type of solution or approximating method used and turns out to be of order 4-10 TeV [5],[21].

In Mavromatos and Sarkar [21] using their interpolating function solution, they obtain

$$H_0 = 7617 \text{ GeV}$$

thus giving a monopole mass as a function of the parameter β in the form of

$$H_{\text{total}}^{\text{mono}} = \left(72.81\sqrt{\frac{\beta}{(\text{GeV})^2}} + 7617\right) \text{GeV}$$

Now, some remarks for the closure of this chapter. In our model, up until now, we have considered the β merely as a parameter that is to be constrained by the experiment and that it refers only to the hypercharge part of the Standard Model. However, considering low-energy field models derived from limits of string theories, we expect the non-linear Born-Infeld behaviour to be inherited to the whole $SU(2) \times U(1)$ gauge group of the SM and that definitely calls for further research. It is indeed reported in [14] that in an $SU(2)$ Born-Infeld model, monopole solution stop existing below some critical value of the Born Infeld parameter β.

Conclusions

Finishing our discussion on the Cho-Maison monopole, it is time we drew some conclusions. We have shown the interconnection between the existence of topological or not magnetic monopoles and the quantization of the electric charge, thus providing a beautiful explanation of such quantization. For this reasoning to hold, however, it is mandatory that a magnetic monopole is found in the physical world, but experimental evidence are so far non existent. It is, therefore, evident that experiment and theory collaborate paving the way to the truth. Of course, nature - and thus the experiment - is the one to have the last word on everything we propose, but monopole theories especially those possibly applicable to the physical world (meaning in the SM or unified versions) must be brought under careful consideration. The Cho-Maison monopole is one of them but the problems associated with it pose serious questions on each existence. In this thesis, we tackled the gauge invariance of the monopole and provided a gauge invariant way of generating solutions in contrast with the unitary gauge chosen by Cho and Maison. We also merely scratched on ways of regularizing its infinite energy, an issue which calls for urgent resolution. Further research is also required on Born-Infeld models incorporating the whole gauge group of the Standard Model.

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