Εθνικό Μετσόβιο Πολυτεχνείο
Σχολή Εφαρμοσμένων Μαθηματικών και Φυσικών Επιστημών
Τομέας Φυσικής

Study of black hole solutions in non-linear theories of gravity and electromagnetism

Μελέτη λύσεων μελανών οπών σε μη γραμμικές θεωρίες βαρύτητας και ηλεκτρομαγνητισμού

Διδακτορική Διατριβή

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Publications

During my PhD studies i published the following articles:

1. T. K, E. Papantonopoulos, Z. Y. Tang and B. Wang,"Black holes of (2+1)-dimensional $f(R)$ gravity coupled to a scalar field," Phys. Rev. D **103** (2021) no.6, 064063 [1]
2. T. K, E. Papantonopoulos, Z. Y. Tang and B. Wang,"Exact black hole solutions with a conformally coupled scalar field and dynamic Ricci curvature in f(R) gravity theories," Eur. Phys. J. C **81** (2021) no.10, 897 [2]
3. T. K, E. Papantonopoulos and C. Vlachos, "f(R) gravity wormholes sourced by a phantom scalar field," Phys. Rev. D **105** (2022) no.2, 024006 [3]
4. T. K, E. Papantonopoulos, Z. Y. Tang and B. Wang, "(2+1)-dimensional black holes in f(R,ϕ) gravity," Phys. Rev. D **105** (2022) no.4, 044038 [4]
5. T. K, G. Koutsoumbas, A. Machattou and E. Papantonopoulos, "Magnetically charged Euler-Heisenberg black holes with scalar hair," Phys. Rev. D **106** (2022) no.10, 104006 [5]
6. T. K, E. Papantonopoulos, Z. Y. Tang and B. Wang, "Rotating (2+1)-dimensional black holes in Einstein-Maxwell-dilaton theory," Phys. Rev. D **107** (2023) no.2, 024043 [6]
7. T. K, N. E. Mavromatos and E. Papantonopoulos, "Regular compact objects with scalar hair," Phys. Rev. D **108** (2023) no.2, 024001 [7]
8. T. K, G. Koutsoumbas and E. Papantonopoulos, "Black holes with scalar hair in three dimensions," Phys. Rev. D **107** (2023) no.12, 124047 [8]
9. D. P. Theodosopoulos, T. K, G. Koutsoumbas and E. Papantonopoulos, "Motion of particles around a magnetically charged Euler-Heisenberg black hole with scalar hair and the Event Horizon Telescope," [arXiv:2311.02740 [gr-qc]] [9]
10. S. Kiorpelidi, T. K, G. Koutsoumbas and E. Papantonopoulos, "Scalarization of the Reissner-Nordström black hole with higher derivative gauge field corrections," Phys. Rev. D **109** (2024) no.2, 024033 doi:10.1103/PhysRevD.109.024033 [10]
11. A. Bakopoulos, T. K, N. E. Mavromatos, T. Nakas and E. Papantonopoulos, "Exact black holes in string-inspired Euler-Heisenberg theory," [arXiv:2402.12459 [hep-th]] [11]
12. A. Bakopoulos, N. Chatzifotis and T. Karakasis, "Thermodynamics of black holes featuring primary scalar hair," [arXiv:2404.07522 [hep-th]] [12]

The works [1, 4, 5, 8, 11] will be presented in this thesis.

Our roots reaching deeper,
like daggers into flesh
Our wounds sing of freedom,
bleeding with each breath
These eyes are not blind,
I drink the insane flow
My naked heart forgives me!

"*Demons*",
Pelle Åhman

Ευχαριστίες

Σε αυτό το σημείο, θα ήθελα να ευχαριστήσω τους ανθρώπους που συνέβαλαν στην ολοκλήρωση της παρούσας διδακτορικής διατριβής.

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Abstract

In the present thesis, we examine black hole solutions in non-linear theories of gravity and electromagnetism and we analyze their properties.

At first, we examine $f(R)$ modified gravity theories and elucidate why studying them is significant. $f(R)$ gravity theories are capable of describing the early and later universe on cosmological scales much more satisfactorily than Einstein's General Theory of Relativity (GR). Considering $f(R)$ theories as the gravitational model to work with, we choose scalar fields for matter and energy fields. The rationale behind this choice is the no-hair theorem. According to this theorem, the process of gravitational collapse, which forms black holes, is a violent physical process that destroys the internal degrees of freedom previously describing the collapsing star, and consequently, the conservation laws not related to precise symmetries. Motivated by the potential violation of the no-hair theorem in $f(R)$ theories, we study black hole solutions in $f(R)$ models coupled to scalar fields in three and four spacetime dimensions. Our thermodynamic analysis shows that our solutions are preferred over those of GR due to higher entropy and temperature on the black hole horizon.

We continue with the investigation of black hole solutions in nonlinear electromagnetic theories. Such theories are predicted by fundamental physics theories, such as string theories. Specifically, we focus on the Euler-Heisenberg (EH) theory, which modifies Maxwell's linear electromagnetic theory. This theory is used to describe photon-photon scattering at the quantum level, and on an astrophysical level, it can potentially describe primordial black holes in the early universe when the intensity of electromagnetic fields was stronger. Our results show that nonlinear electromagnetism contributes to the violation of the no-hair theorem, while simultaneously, our solutions are thermodynamically preferred, as they have greater entropy on their event horizon. Additionally, the spacetimes we find respect the energy conditions, which imposes astrophysical relevance to our solutions, as they could describe black holes in the early universe when the power of electromagnetic fields was stronger.

Περίληψη της Διατριβής

Στη διδακτορική διατριβή με τίτλο "Μελέτη λύσεων μελανών οπών σε μη γραμμικές θεωρίες βαρύτητας και ηλεκτρομαγνητισμού" πραγματευόμαστε την εύρεση λύσεων μελανών οπών σε μη γραμμικές θεωρίες βαρύτητας και ηλεκτρομαγνητισμού και την ανάλυση των ιδιοτήτων τους.

Στο πρώτο μέρος της διατριβής εξετάζουμε τις $f(R)$ τροποποιημένες θεωρίες βαρύτητας και εξηγούμε τους λόγους για τους οποίους η μελέτη τους είναι σημαντική. Οι $f(R)$ θεωρίες βαρύτητας είναι ικανές να περιγράψουν σε κοσμολογικές κλίμακες το πρώιμο και μεταγενέστερο σύμπαν πολύ πιο ικανοποιητικά από τη Γενική Θεωρία της Σχετικότητας (ΓΘΣ) του Einstein. Έχοντας λοιπόν τις $f(R)$ θεωρίες ως το βαρυτικό μοντέλο με το οποίο θα δουλέψουμε, επιλέγουμε βαθμωτά πεδία για τα πεδία ύλης και ενέργειας. Ο λόγος πίσω από αυτή τη θεώρηση είναι το θεώρημα εξάλειψης ιχνών (no-hair theorem). Σύμφωνα με αυτό το θεώρημα, η διαδικασία της βαρυτικής κατάρρευσης, η οποία και δημιουργεί τις μελανές οπές, είναι μια βίαιη φυσική διαδικασία που καταστρέφει τους εσωτερικούς βαθμούς ελευθερίας που προτέρως περιέγραφαν το υπό κατάρρευση άστρο, και κατά συνέπεια τους νόμους διατήρησης που δε σχετίζονται με ακριβείς συμμετρίες. Με κίνητρο την ενδεχόμενη παραβίαση του θεωρήματος εξάλειψης ιχνών στις $f(R)$ θεωρίες, μελετάμε λύσεις μελανών οπών σε $f(R)$ μοντέλα, συζευγμένων με βαθμωτά πεδία σε τρεις και τέσσερις χωροχρονικές διαστάσεις. Η θερμοδυναμική ανάλυση μας έδειξε πως οι λύσεις μας είναι προτιμητέες σε σχέση με αυτές της ΓΘΣ λόγω μεγαλύτερης εντροπίας και θερμοκρασίας πάνω στον ορίζοντα γεγονότων της μελανής οπής.

Στο δεύτερο μέρος αυτής της διατριβής πραγματευόμαστε λύσεις μελανών οπών σε μη γραμμικές θεωρίες ηλεκτρομαγνητισμού. Τέτοιες θεωρίες προβλέπονται από θεμελιώδεις θεωρίες φυσικής, όπως επί παραδείγματι οι θεωρίες χορδών. Συγκεκριμένα, ασχολούμαστε με την θεωρία Euler-Heisenberg (EH), η οποία και τροποποιεί τη γραμμική θεωρία ηλεκτρομαγνητισμού του Maxwell. Αυτή η θεωρία χρησιμοποιείται για να περιγράψει τη σκέδαση φωτονίων με φωτόνια σε επίπεδο κβαντικής φυσικής, ενώ σε αστροφυσικό επίπεδο, μπορεί εν δυνάμει να περιγράφει αρχέγονες μελανές οπές στο πρώιμο σύμπαν, όταν η ενταση των ηλεκτρομαγνητικών πεδίων ήταν ισχυρότερη. Τα απολέσματά μας, μας δείχνουν πως ο μη γραμμικός ηλεκτρομαγνητισμός συμβάλει στην παραβίαση του θεωρήματος εξάλειψης ιχνών, ενώ ταυτόχρονα οι λύσεις μας είναι θερμοδυναμικά προτημητέες, καθότι έχουν μεγαλύτερη εντροπία στον ορίζοντα γεγονότων τους. Επιπρόσθετα, οι χωρόχρονοι τους οποίους βρίσκουμε σέβονται τις ενεργειακές συνθήκες, γεγονός που επιβάλει την αστροφυσική μελέτη των λύσεών μας, καθότι αυτές θα μπορούσαν να περιγράφουν μαύρες τρύπες στο πρώιμο σύμπαν, όταν η ισχύς των ηλεκτρομαγνητικών πεδίων ήταν ισχυρότερη.

Εκτενής Περίληψη Στα Ελληνικά

Στην παρούσα διδακτορική διατριβή πραγματευόμαστε λύσεις μελανών οπών σε μη γραμμικές θεωρίες βαρύτητας και ηλεκτρομαγνητισμού συζευγμένες με βαθμωτά πεδία. Θα ξεκινήσουμε την εισαγωγή μας, αφολυ πρώτα αναφερθούμε στο θεώρημα εξάλειψης ιχνών (no hair theorem). Σύμφωνα με το θεώρημα εξάλειψης ιχνών, οι μαύρες τρύπες θα περιγράφονται μόνο από τρεις παραμέτρους, οι οποίες και μπορούν να μετρηθούν ασυμπτωτικά με νόμους διατήρησης. Πρόεκειται για τη μάζα τους, την τροχιακή στροφορμή τους και το ηλεκτρομαγνητικό τους φορτίο. Έτσι οι μαύρες τρύπες, δεν κουβαλούν κάποιο hair (δεν έχουν "μαλλιά", εδώ η λέξη hair υποδηλώνει την απώλεια περισσοτέρων χαρακτηριστικών/παραμέτρων που περιγράφουν μια μαύρη τρύπα). Η λογική πίσω από αυτή τη θεώρηση εγκειται στο γεγονός ότι, οι μαύρες τρύπες σχηματίζονται μέσω της βαρυτικής κατάρρευσης, η οποία, είναι μια τόσο βίαιη διαδικασία και καταστρέφει όλους τους νόμους διατήρησης που δε σχετίζονται με ακριβείς συμμετρίες. Επί παραδείγματι, η χημική δομή, η ατομική δομή, ο βαρυνικός αριθμός κλπ, δε διατηρούνται κατά την κατάρρευση και οι μόνες παράμετροι που επιβιώνουν είναι η μάζα, η τροχιακή στροφορμή και το ηλεκτρομαγνητικό φορτίο. Έτσι λοιπόν, θα μπορούσε να ισχυριστεί κανείς, ότι, οι μαύρες τρύπες είναι απίστευτα απλά φυσικά συστήματα! Δυο μη περιστρεφόμενες και αφόρτιστες μαύρες τρύπες με ίδια μάζα, είναι απόλυτα όμοιες! (Φυσικά, η πραγματικότητα δεν ειναι αυτή. Η ανακάλυψη πως οι μαύρες τρύπες ικανοποιούν κάποιους νόμους μηχανικής παρόμοιους με αυτούς της θερμοδυναμικής και μάλιστα σε ημικλασικό επίπεδο, ικανοποιούν τους νόμους της θερμοδυναμικής άλλαξε κατά πολύ το τοπίο έρευνας.) Αυτό δε θα συνέβαινε αν είχαμε ένα οποιοδήποτε άλλο συμπαγές αντικείμενο, όπως για παράδειγμα ένα αστέρι. Σε δύο αστέρια με την ίδια μάζα, αυτή μπορεί να μην είναι με τον ίδιο τρόπο κατανεμημένη και να έχουμε στο ένα αστέρι περιοχές πυκνότερες, εν συγκρίσει με το άλλο!

Οι πρώτες προσπάθειες εύρεσης μελανών οπών με hair αφορούσαν μη Αβελιανά πεδία, ωστόσο τα τελευταία χρόνια, το ενδιαφέρον της επιστημονικής κοινότητας έχει στραφεί προς τα βαθμωτά πεδία. Αυτή η μετατόπιση ενδιαφέροντος, είναι συνέπεια της ανακάλυψης της σκοτεινής ενέργειας η οποία είναι πιθανό να μπορεί να μοντελοποιηθεί μέσω βαθμωτών πεδίων. Εν προκειμένω, αν αυτό το πεδίο παραβιάζει τις ενεργειακές συνθήκες ή συζεύγνειται μη ελάχιστα με τη βαρύτητα, τότε τα no-hair θεωρήματα παραβιάζονται. Το αποτέλεσμα είναι ότι πλέον κανείς αναζητά λύσεις με εξωτικά βαθμωτά πεδία όπως πεδία φαντάσματα (με αρνητική κινητική ενέγεια), πεδία Galileon, λύσεις της θεωρίας Horndeski αλλά και επεκτάσεων αυτής. Στην παρούσια διατριβή, θα μελετήσουμε λύσεις μελανών οπών συζευγμένων με βαθμωτά πεδία σε μη γραμμικές θεωρίες βαρύτητας και ηλεκτρομαγνητισμού, με το κίνητρο για τη μελέτη τους να έχει στη βάση του την κοσμολογία αλλά και πιο θεμελιώδεις φυσικές θεωρίες, όπως οι θεωρίες χορδών.

Συγκεκριμένα θα θεωρήσουμε $f(R)$ τροποποιημένες θεωρίες βαρύτητας, οι οποίες γενικεύουν τη γενική θεωρία της σχετικότητας (ΓΘΣ) του Αινστάιν. Το συναρτησιακό δράσης αυτών των θεωριών δίνεται από τη σχέση

$$S = \int d^4x\sqrt{-g}f(R) \ ,$$

όπου η συνάρτηση $f(R)$ είναι μια αναλυτική συνάρτηση του βαθμωτού του Ricci R, ενώ η επιλογή $f(R) = R$ μας δίνει τη ΓΘΣ. Επί παραδείγματι, μερικές γνωστές μη γραμμικές θεωρίες βαρύτητας ειναι το μοντέλο του Starobinski [13]

$$S = \int d^4x\sqrt{-g}(R + \alpha R^2) \ ,$$

με σημαντικές προβλέψεις για το μοντέλο του κοσμικού πληθωρισμού, με την παράμετρο α να έχει μονάδες $[\text{μήκος}]^2$, αλλά και το μοντέλο του Woodard [14], το οποίο εμπεριέχει μη γραμμικούς όρους του βαθμωτού του Ricci με τη μορφή

$$S = \int d^4x\sqrt{-g}\left(R + \frac{\beta}{R}\right) \ ,$$

το οποίο περιγράφει το χρονικά μεταγενέστερο σύμπαν. Στο μοντέλο του Woodard, η παράμετρος β έχει μονάδες $[\text{μήκος}]^{-4}$. Οι Οι εξισώσεις πεδίου που προκύπτουν απαιτώντας τον μηδενισμό της μεταβολής πρώτης τάξεως του συναρτησιακού της δράσης δίνονται από τη σχέση

$$f' R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(R) + g_{\mu\nu} \Box f' - \nabla_\mu \nabla_\nu f' = 0 \; ,$$

όπου $f' \equiv df(R)/dR$. Μπορεί κανείς να εξάγει την ανωτέρω εξίσωση ως ακολούθως:

Αρχικά, για τη μεταβολή της ορίζουσας της μετρικής έχουμε ότι

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \; .$$

Συνεπώς, μεταβάλλοντας συνολικά τη δράση έχουμε

$$\delta \int d^4x \sqrt{-g}\,(f(R)) = \int d^4x \left(-\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu} f(R) + \sqrt{-g}\delta f(R) \right) =$$
$$\int d^4x \sqrt{-g} \left(-\frac{1}{2} g_{\mu\nu}\delta g^{\mu\nu} f(R) + \frac{df(R)}{dR}\delta R \right) \; .$$

Η μεταβολή του βαθμωτού του Ricci δίνεται από τη σχέση

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + \delta R_{\mu\nu} g^{\mu\nu} \; .$$

Χρησιμοποιώντας την ταυτότητα του Palatini μπορούμε να γράψουμε τον όρο που εμπεριέχει τη μεταβολή του τανυστή του Ricci ως

$$\delta R_{\mu\nu} g^{\mu\nu} f'(R) = f'(R)\big[g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu\big]\delta g^{\mu\nu} \; ,$$

όπου $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. Επομένως, δημιουργώντας ολικά διαφορικά και ακυρώνοντας τους επιφανειακούς όρους, έχουμε

$$\int d^4x \sqrt{-g} f'(R)[g_{\mu\nu} g^{ab} \nabla_a \nabla_b \delta g^{\mu\nu}] =$$

$$\int d^4x \sqrt{-g} f'(R)[g_{\mu\nu} \nabla^b \nabla_b \delta g^{\mu\nu}] =$$

$$\int d^4x \sqrt{-g} \nabla^b [f'(R) g_{\mu\nu} \nabla_b \delta g^{\mu\nu}] - \int d^4x \sqrt{-g} \nabla^b [f'(R)] g_{\mu\nu} \nabla_b \delta g^{\mu\nu} =$$

$$-\int d^4x \sqrt{-g} \nabla^b [f'(R)] g_{\mu\nu} \nabla_b \delta g^{\mu\nu} =$$

$$-\int d^4x \sqrt{-g} \nabla_b [\nabla^b f'(R) g_{\mu\nu} \delta g^{\mu\nu}] + \int d^4x \sqrt{-g} g_{\mu\nu} \nabla_b [\nabla^b f'(R)] \delta g^{\mu\nu} =$$

$$\int d^4x \sqrt{-g} g_{\mu\nu} \nabla_b [\nabla^b f'(R)] \delta g^{\mu\nu} =$$

$$\int d^4x \sqrt{-g} \delta g^{\mu\nu} [g_{\mu\nu} g^{ab} \nabla_b \nabla_a f'(R)] \; .$$

Για τον όρο $\int d^4x \sqrt{-g} f'(R) \nabla_\mu \nabla_\nu \delta g^{\mu\nu}$ εχουμε:

$$\int d^4x\sqrt{-g}f'(R)[\nabla_\mu\nabla_\nu\delta g^{\mu\nu}] =$$

$$\int d^4x\sqrt{-g}\nabla_\mu[f'(R)\nabla_\nu\delta g^{\mu\nu}] - \int d^4x\sqrt{-g}\nabla_\mu[f'(R)]\nabla_\nu\delta g^{\mu\nu} =$$

$$-\int d^4x\sqrt{-g}\nabla_\mu[f'(R)]\nabla_\nu\delta g^{\mu\nu} =$$

$$-\int d^4x\sqrt{-g}\nabla_\nu[\nabla_\mu f'(R)\delta g^{\mu\nu}] + \int d^4x\sqrt{-g}\delta g^{\mu\nu}\nabla_\nu[\nabla_\mu f'(R)] =$$

$$\int d^4x\sqrt{-g}\delta g^{\mu\nu}\nabla_\nu[\nabla_\mu f'(R)] \; .$$

Επομένως, συγκεντρώνοντας όλους τους όρους μαζί έχουμε

$$\delta S = \int d^4x\sqrt{-g}\delta g^{\mu\nu}\Big[f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + g_{\mu\nu}\Box f'(R) - \nabla_\mu\nabla_\nu f'(R)\Big] = 0$$

Για να μηδενίζεται ανωτέρω ποσότητα για αυθαίρετες μεταβολές της μετρικής θα πρέπει

$$f'R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + g_{\mu\nu}\Box f' - \nabla_\mu\nabla_\nu f' = 0 \; .$$

Ένα μεγάλο μέρος της παρούσας διατριβής είναι αφιερωμένο στην επίλυση της ανωτέρω εξίσωσης παρουσία ύλης και συγκεκριμένα ενός βαθμωτού πεδίου, το οποίο θα πάιζει το ρόλο της ύλης στο σύστημά μας. Θα θεωρήσουμε λοιπόν $f(R)$ θεωρίες βαρύτητας συζευγμένες με βαθμωτά πεδία σε τρεις και τέσσερις χωροχρονικές διαστάσεις. Προτού αναλύσουμε εκτενώς αυτές τις περιπτώσεις αξίζει να αναφερθούμε στις θεωρίες βαρύτητας σε τρεις χωροχρονικές διαστάσεις, αλλά και στις λύσεις μελανών οπων συζευγμένων με βαθμωτά πεδία στη ΓΘΣ.

Για το λόγο αυτό θα ξεκινήσουμε από μια απλή λύση η οποία θα δίνεται από τη δράση

$$\mathcal{S} = \int d^4x\,\sqrt{-g}\left[\frac{R}{2} - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - V(\phi)\right] \; . \tag{1}$$

Μια λύση αυτής της δράσης δίνεται από τις εξής σχέσεις [15]

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + a(r)^2 d\Omega^2 \; , \tag{2}$$

$$F(r) = 1 + \chi\left(2r(\nu+r)\ln\left(\frac{\nu+r}{r}\right) - \nu(\nu+2r)\right) \; , \tag{3}$$

$$a(r) = \sqrt{r(r+\nu)} \; , \tag{4}$$

$$\phi(r) = \frac{1}{\sqrt{2}}\ln\left(1+\frac{\nu}{r}\right) \; , \tag{5}$$

$$V(\phi) = 6\chi\sinh\left(\sqrt{2}\phi\right) - 2\sqrt{2}\chi\phi\left(\cosh\left(\sqrt{2}\phi\right)+2\right) \; , \tag{6}$$

όπου χ είναι μια σταθερά της θεωρίας και ν είναι μια σταθερά ολοκλήρωσης, η οποία και παίζει το ρόλο του βαθμωτού φορτίου καθώς καθορίζει την ασυμπτωτική συμπεριφορά του βαθμωτού πεδίου σε μεγάλες αποστάσεις. Η συνάρτηση $F(r)$ συμπεριφέρεται ως

$$F(r\to\infty) \sim 1 - \frac{\nu^3\chi}{3r} + \frac{\nu^4\chi}{6r^2} - \frac{\nu^5\chi}{10r^3} + \frac{\nu^6\chi}{15r^4} - \frac{\nu^7\chi}{21r^5} + O\left(\left(\frac{1}{r}\right)^6\right) \; , \tag{7}$$

$$F(r\to 0) \sim \left(1-\nu^2\chi\right) - 2r(\chi(\nu - \nu\ln(\nu) + \nu\ln(r))) + O\left(r^2\ln r\right) \; , \tag{8}$$

και άρα θα περιγράφει μια γνήσια μελανή οπή όταν $\left(1-\nu^2\chi\right)<0$ στο διάστημα $0<r<\infty$.

Τώρα θεωρούμε τη Χαμιλτονιανή μορφή της θεωρίας μας μέσα από τη δράση

$$\mathcal{I}=\int d^3xdt\left(\pi^{ij}\dot{g}_{ij}-NH-N_iH^i\right)+\mathcal{B}\ . \tag{9}$$

Εδώ, $\mathcal{B}$ είναι ένας επιφανειακός όρος. Θα χρησιμοποιήσουμε επιχειρήματα κβαντικής βαρύτητας, και θα χρησιμοποιήσουμε το βαρυτικό ολοκλήρωμα διαδρομών, σύμφωνα με το οποίο η δράση θα πρέπει να περιέχει μόνο πρώτες παραγώγους της μετρικής [16]. Ο χωρόχρονος μας είναι στατικός (δεν υπάρχει εξάρτηση από το χρόνο t και δεν έχουμε θεωρήσει περιστροφή). Για το λόγο αυτό θα θεωρήσουμε την ελαττωμένη Χαμιλτονιανή, η οποία θα δίνεται από τη σχέση

$$\mathcal{I}=-\int d^3xdtNH+\mathcal{B}\ . \tag{10}$$

Θα θεωρήσουμε τώρα την Ευκλείδια δράση η οποία και σχετίζεται με τη Χαμιλτονιανή δράση σε Λορέντζιαν γραφή, σύμφωνα με τη σχέση

$$\mathcal{I}_E=-i\mathcal{I}\ . \tag{11}$$

Η Ευκλείδια μετρική, χρησιμοποιώντας την αποσύνθεση ADM, θα δίνεται από τη σχέση,

$$ds^2=F(r)N(r)^2d\tau^2+\frac{dr^2}{F(r)}+a(r)^2d\Omega^2\ , \tag{12}$$

όπου $\tau=it$ και $0<\tau<\beta$. Εδώ το τ είναι περιοδικό με περίοδο β. Αυτή η περιοδικότητα προέρχεται από το γεγονός ότι στην Ευκλείδια γραφή της, η μετρική καταστευάζεται από το γινόμενο δύο σφαιρών $\mathfrak{S}^2\times\mathfrak{S}^2$. Συνεπώς, για να μπορούν οι συντεταγμένες να καλύπτουν όλο τον χωρόχρονο, θα πρέπει να μεταχειριστούμε το τ σαν περιοδική συντεταγμένη. Οι υπόλοιπες συντεταγμένες παίρνουν τις συνήθεις τιμές $0\leq\varphi<2\pi$, $0\leq\theta\leq\pi$, $r>r_h$. Πραγματοποιώντας τις ολοκληρώσεις παίρνουμε

$$\mathcal{I}_E=4\pi\beta\int_{r_h}^{\infty}N(r)a(r)^2N(r)H(r)dr+\mathcal{B}_E\ . \tag{13}$$

Χρησιμοποιώντας την Ευκλείδια μετρική και το γεγονός πως $N(r)H(r)=-L$ όπου το L υποδηλώνει την Λαγκράντζιαν του συστήματός μας, καταλήγουμε στην ακόλουθη Ευκλείδια δράση

$$\mathcal{I}_E=4\pi\beta\int_{r_h}^{\infty}\frac{1}{2}N\left(2a\left(a'F'+2Fa''\right)+a^2\left(F\left(\phi'\right)^2+2V\right)+2F\left(a'\right)^2-2\right)dr+\mathcal{B}_E, \tag{14}$$

όπου και πραγματοποιήσαμε αρκετές ολοκληρώσεις κατά παράγοντες και ακυρώσαμε τους επιφανειακούς όρους. Τώρα, θα πρέπει να μεταβάλουμε την Ευκλείδια δράση, ως προς τα δυναμικά της πεδία, δηλαδή τα N,ϕ,a,F ώστε να πάρουμε τις πεδιακές εξισώσεις. Για να το επιτύχουμε αυτό θα πρέπει να ακυρώσουμε διάφορους επιφανειακούς όρους. Ωστόσο, προκειμένουν η Ευκλείδια δράση να αποτελεί ένα πραγματικό ακρότατο όταν ισχύουν οι πεδιακές εξισώσεις, θα πρέπει να βεβαιωθούμε ότι πράγματι

$$\delta\mathcal{I}_E=0\ . \tag{15}$$

Ο ρόλος του επιφανειακού όρου $\mathcal{B}_E$ είναι να πάρει την κατάλληλη μορφή ώστε πράγματι $\delta\mathcal{I}_E=0$. Ξεκιναμε λοιπόν με τη συνάρτηση N και βρίσκουμε

$$2a\left(a'F'+2Fa''\right)+a^2\left(F\left(\phi'\right)^2+2V\right)+2F\left(a'\right)^2-2=0\ , \tag{16}$$

το οποίο και σημαίνει ότι στο Χαμιλτονιανό φορμαλισμό, η Ευκλείδια δράση θα δίνεται αποκλειστικά από τον επιφανειακό όρο όταν ισχύουν οι εξισώσεις πεδίου,

Μεταβάλλοντας ως προς το ϕ παίρνουμε

$$a\left(a\left(N\left(-F'\phi' - F\phi'' + \frac{V'(r)}{\phi'}\right) - F\phi' N'(r)\right) - 2FNa'\phi'\right) = 0\ , \tag{17}$$

όπου ακυρώθηκε ο επιφανειακός όρος

$$\frac{d}{dr}\left(a^2 NF\phi'\delta\phi\right)\ . \tag{18}$$

Μεταβολή ως προς το F δίνει

$$2a'' + a\left(\phi'\right)^2 = 0\ , \tag{19}$$

όπου ακυρθώκε ο επιφανειακός όρος

$$\frac{d}{dr}\left(aNa'\delta F\right)\ . \tag{20}$$

Τέλος, μεταβολή ως προς το a δίνει

$$2a'\left(NF' + FN'\right) + 2FNa'' + a\left(3F'N' + N\left(F''(r) + F\left(\phi'\right)^2 + 2V\right) + 2FN''\right) = 0\ , \tag{21}$$

όπου και ακυρώθηκαν διάφοροι επιφανειακοί όροι.

$$\frac{d}{dr}\left(FN2a'\delta a\right)\ , \tag{22}$$

$$\frac{d}{dr}\left(aNF'\delta\alpha\right)\ , \tag{23}$$

$$\frac{d}{dr}\left(2aFN\delta a'\right)\ , \tag{24}$$

$$-\frac{d}{dr}\left(\frac{d}{dr}\left(2aFN\right)\delta a\right)\ . \tag{25}$$

Τώρα, η λύση που δίνεται από τις εξισώσεις (2)-(6) ικανοποιεί όλες τις πεδιακές εξισώσεις, συνεπώς μας έμειναν μόνο οι επιφανειακοί όροι. Όλοι οι επιφανειακοί όροι μαζί μας δίνουν

$$4\pi\beta\Big(-\delta a\left(2FNa' + 2aNF' + 2aFN'\right) + 2\delta aFNa' + a\delta FNa' + a^2\delta\phi FN\phi' + a\delta aNF' + 2a\delta a'FN\Big)\Big|_{r_h}^{\infty} + \delta\mathcal{B}_E = 0 \tag{26}$$

Για να προχωρήσουμε θα πρέπει να γνωρίζουμε τη μεταβολή των πεδίων στο άπειρο και πάνω στον ορίζοντα. Από τη στιγμή που ήδη γνωρίζουμε τη λύση, μπορούμε εύκολα να το βρούμε αυτό, και η μεταβολή στο άπειρο θα δίνεται από

$$\delta F = -\frac{\delta\nu\nu^2\chi}{r}\ , \tag{27}$$

$$\delta a = \delta\nu\left(\frac{1}{2} - \frac{\nu}{4r}\right)\ , \tag{28}$$

$$\delta\phi = \frac{\delta\nu}{\sqrt{2}r}\ . \tag{29}$$

Στον ορίζοντα θα έχουμε

$$F(r)|_{r=r_h} = F(r_h) + F'(r)|_{r=r_h}(r - r_h) = F'(r)|_{r=r_h}(r - r_h) = \frac{4\pi}{\beta}(r - r_h) \;, \qquad (30)$$

$$\delta F = -\frac{4\pi}{\beta}(\delta r_h) \;, \qquad (31)$$

$$\delta a|_{r=r_h} = \delta a(r_h) - a'(r_h)\delta r_h \;, \qquad (32)$$

$$\delta\phi|_{r=r_h} = \delta\phi(r_h) - \phi'(r_h)\delta r_h \;, \qquad (33)$$

όπου χρησιμοποιήθηκε το γεγονός ότι για να αποφεύγεται η κωνική ιδιομορφία πάνω στον ορίζοντα το τ θα πρέπει να είναι περιοδικό με περίοδο β η οποία και θα σχετίζεται με το F' μέσω της σχέσης

$$\frac{1}{\beta} \equiv T = \frac{F'(r_h)}{4\pi} \;. \qquad (34)$$

Για λόγους ευκολίας θα αποσυνθέσουμε τη μεταβολή του επιφανειακού όρου σε δύο όρους έναν στο άπειρο και έναν πάνω στον ορίζοντα

$$\delta\mathcal{B}_E = \delta\mathcal{B}_E(\infty) + \delta\mathcal{B}_E(r_h) \;. \qquad (35)$$

Η συνεισφορά στο άπειρο και στον ορίζοντα θα είναι

$$-4\left(\pi\beta\delta\nu\nu^2\chi\right) + \mathcal{O}\left(\frac{1}{r}\right) + \delta\mathcal{B}_E(\infty) - (-16\pi^2\delta a a) + \delta\mathcal{B}_E(r_h) = 0 \;. \qquad (36)$$

Συνεπώς θα έχουμε

$$\delta\mathcal{B}_E(\infty) = 4\left(\pi\beta\delta\nu\nu^2\chi\right) \to \mathcal{B}_E(\infty) = 4\pi\beta\chi\frac{\nu^3}{3} \;, \qquad (37)$$

$$\delta\mathcal{B}_E(r_h) = -16\pi^2\delta a a \to \mathcal{B}_E(r_h) = -2\pi\mathcal{A}(r_h) \;, \qquad (38)$$

όπου χρησιμοποιήθηκε το γεγονός ότι δουλεύουμε στο μεγαλοκανονικό σύνολο, συνεπώς θεωρούμε τη θερμοκρασία αμετάβλητη και επιπλέον $\mathcal{A}(r_h) = 4\pi a(r_h)^2$. Τελικά, η Ευκλείδια δράση θα δίνεται από την ακόλουθη σχέση

$$\mathcal{I}_E = \mathcal{B}_E(\infty) + \mathcal{B}_E(r_h) = 4\pi\beta\chi\frac{\nu}{3} - 2\pi\mathcal{A}(r_h) \;. \qquad (39)$$

Ωστόσο, η Ευκλείδια δράση θα σχετίζεται με την ελεύθερη ενέργεια $\mathcal{F}$ του μεγαλοκανονικό συνόλου, μέσω των σχέσεων

$$\mathcal{I}_E = \beta\mathcal{F} = \beta\mathcal{M} - \mathcal{S} \;, \qquad (40)$$

όπου $\mathcal{M}$, $\mathcal{S}$ είναι η μάζα και η εντροπία της μελανής πής. Συνεπώς, συγκρίνοντας μπορούμε να ταυτοποιήσουμε τη μάζα και την εντροπία της μελανής πής σύμφωνα με τις ακόλουθες σχέσεις.

$$\mathcal{M} = 4\pi\chi\frac{\nu^3}{3} \;, \qquad (41)$$

$$\mathcal{S} = 2\pi\mathcal{A}(r_h) \;. \qquad (42)$$

Αξίζει να σημειώσουμε πως η παράμετρος ν εμφανίζεται στην έκφραση της μάζας της μαύρης τρύπας και συνεπώς το βαθμωτό πεδίο ντύνει τη μελανή οπή με ένα δευτερεύον hair.

Περνάμε τώρα στη βαρύτητα στις τρεις χωροχρονικές διαστάσεις, ένα θέμα που θα πραγματευτούμε εκτενώς σε αυτή τη διατριβή.

Το συναρτησιακό δράσης της ΓΘΣ σε τρεις χωροχρονικές διαστάσεις δίνεται από τη σχέση

$$S = \int d^3x\sqrt{-g}R \; .$$

Οι εξισώσεις πεδίου που προκύπτουν από αυτή τη δράση είναι

$$R_{\mu\nu} = 0 = R \; ,$$

δηλαδή μηδενίζεται και ο τανυστής του Ricci, καθώς και το αντίστοιχο βαθμωτό. Ωστόσο, στις τρεις χωροχρονικές διαστάσεις ο τανυστής του Weyl μηδενίζεται ταυτοτικά και κατά συνέπεια, ο τανυστής του Riemann που εμπεριέχει όλη την πληροφορία για τη γεωμετρία του χωρόχρονου, μπορεί να γραφεί συναρτήσει του τανυστή του Ricci και του αντίστοιχου βαθμωτού ως

$$R_{\alpha\beta\gamma\delta} = 2\left(g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\gamma}R_{\delta]\alpha}\right) - Rg_{\alpha[\gamma}g_{\delta]\beta} \; ,$$

επομένως

$$R_{\alpha\beta\gamma\delta} = 0 \; ,$$

οπότε συμπεραίνουμε ότι, δε μπορεί να υπάρξει μια μη τετριμμένη (μη επίπεδη) γεωμετρία στις τρεις χωροχρονικές διαστάσεις και άρα δε μπορεί να υπάρξει μια λύση μαύρης τρύπας, εν αντιθέσει με την περίπτωση στις τεσσερις χωροχρονικές διαστάσεις, όπου και συναντάμε τη λύση του Schwarzchild. Η φυσική υπόθεση που έχει γίνει στο σύστημα και δεν επιτρέπει την παρουσία μιας λύσης μαύρης τρύπας, είναι η απουσία ύλης. Εισάγοντας λοιπόν μια κοσμολογική σταθερά, θεωρώντας δηλαδή το συναρτησιακό δράσης

$$S = \int d^3x\sqrt{-g}(R - 2\Lambda) \; ,$$

μπορεί κανείς λύνοντας τις εξισώσεις του Αινσταιν να καταλήξει στην λύση των Banados, Teitelboim και Zanelli (BTZ)

$$ds^2 = -b(r)dt^2 + b(r)^{-1}dr^2 + r^2\Big(u(r)dt + d\theta\Big)^2 \tag{43}$$

με τις σχετικές συναρτήσεις να δίνονται από τις σχέσεις

$$b(r) = \frac{J^2}{4r^2} - M + \frac{r^2}{l^2}, \tag{44}$$

$$u(r) = -\frac{J}{2r^2}. \tag{45}$$

όπου τα M, J είναι η μάζα και η στροφορμή της μαύρης τρύπας.

Προτού θεωρήσουμε τροποποιημένες θεωρίες βαρύτητας και τις λύσεις αυτών, όμως, καλό θα ήταν να κάνουμε μια απλή εισαγωγή στο θέμα, και έτσι, στο κεφάλαιο 2 λύνουμε τις εξισώσεις του Αινσταιν παρουσία ενός αυτοαλληλεπιδρώντος βαθμωτού πεδίου ϕ στις τρεις χωροχρονικές διαστάσεις που προκύπτουν από τη δράση [8], με σκοπό να βρούμε hairy λύσεις μελανών οπών.

$$S = \frac{1}{8\pi}\int d^3x\sqrt{-g}\left\{\frac{R}{2} - \frac{1}{2}\partial^\alpha\phi\partial_\alpha\phi - V(\phi)\right\} \; . \tag{46}$$

Η λύση στην οποία καταλήγουμε είναι η ακόλουθη

$$ds^2 = -h(r)dt^2 + dr^2/h(r) + r^2 d\theta^2, \tag{47}$$
$$\phi(r) = A/r \ , \tag{48}$$
$$h(r) = r^2\left(\frac{A^2\Lambda - 2\mathcal{M}}{A^2} - e^{-\frac{A^2}{2r^2}}\left(\frac{A^2\Lambda - 2\mathcal{M}}{A^2} + \Lambda\right)\right), \tag{49}$$
$$b(r) = \frac{2r^2 e^{\frac{A^2}{2r^2}}\left(\mathcal{M} - A^2\Lambda\right) - r^2 e^{\frac{A^2}{r^2}}\left(2\mathcal{M} - A^2\Lambda\right)}{A^2} \ , \tag{50}$$
$$V(r) = \frac{e^{\frac{A^2}{r^2}}\left(A^2 - 2r^2\right)\left(A^2\Lambda - 2\mathcal{M}\right)}{2A^2r^2} + \frac{2e^{\frac{A^2}{2r^2}}\left(A^2\Lambda - \mathcal{M}\right)}{A^2}, \tag{51}$$
$$V(\phi) = e^{\frac{\phi^2}{2}}(2\Lambda - 2q) + e^{\phi^2}\left(\frac{\Lambda\phi^2}{2} - \Lambda - q\phi^2 + 2q\right) \ , \tag{52}$$
$$q = \mathcal{M}/A^2 \ . \tag{53}$$

Η παράμετρος A είναι η παράμετρος που καθορίζει την ισχύ του βαθμωτού πεδίου και $\mathcal{M}$ είναι η μάζα της μαύρης τρύπας. Όπως μπορεί κανείς να παρατηρήσει, το δυναμικό της θεωρίας μας έχει ένα ολικό μέγιστο το οποίο βρίσκεται στο κενό της θεωρίας $V(\phi = 0) = 2\Lambda$ και συνδέεται με την κοσμολογική σταθερά. Ο χωρόχρονος είναι ασυμπτωτικά Anti de Sitter (AdS), ενώ το δυναμικό περιέχει, εκτός από την κοσμολογική σταθερά, εμπεριέχει την παράμετρο q η οποία και δίνει φιξαρισμένους λόγους της μάζας της μαυρης τρύπας ως προς το "φορτίο" A του βαθμωτού πεδίου ϕ. Πριν περάσουμε στη θερμοδυναμική θεώρηση, αξίζει να σχολιάσουμε τη συμπεριφορά του τανυστή ορμής-ενέργειας της θεωρίας μας μέσα από τις ενεργειακές συνθήκες, Συγκεκριμένα, η πυκνότητα ενέργειας, η ακτινική πίεση και το άθροισμά τους, θα δίνονται από τις σχέσεις

$$\rho = \mathcal{T} + V = \frac{b\phi'^2}{2} + V = \frac{e^{\frac{A^2}{2r^2}}\left(A^2 - 2r^2\right)\left(A^2 + \ell^2\mathcal{M}\right) - e^{\frac{A^2}{r^2}}(A^2 - r^2)\left(A^2 + 2\ell^2\mathcal{M}\right)}{A^2r^2\ell^2} , \tag{54}$$
$$p_r = \mathcal{T} - V = b\phi'^2/2 - V = \frac{e^{\frac{A^2}{2r^2}}\left(A^2 + 2r^2\right)\left(A^2 + \ell^2\mathcal{M}\right) - r^2 e^{\frac{A^2}{r^2}}\left(A^2 + 2\ell^2\mathcal{M}\right)}{A^2r^2\ell^2} \ , \tag{55}$$
$$\rho + p_r = 2\mathcal{T} = b\phi'^2 = \frac{2e^{\frac{A^2}{2r^2}}\left(A^2 + \ell^2\mathcal{M}\right) - e^{\frac{A^2}{r^2}}\left(A^2 + 2\ell^2\mathcal{M}\right)}{r^2\ell^2} \ , \tag{56}$$

όπου $\mathcal{T} = b(r)\phi'^2/2$ είναι η κινητική ενέργεια του βαθμωτού πεδίου. Απεικονίζοντας τώρα αυτές τις ποσότητες στο ακόλουθο σχήμα 1 μπορούμε να δούμε πως η πυκνότητα ενέργειας είναι πάντοτε αρνητική και αυτό συμβαίνει γιατί το βαθμωτό δυναμικό είναι πολύ αρνητικό και κερδίζει τη θετική συνεισφορά του κινητικού όρου. Η ακτινική πίεση και το άθροισμα της με την πυκνότητα ενέργειας ωστόσο είναι θετικές ποσότητας από τον ορίζοντα της μελανής οπής μέχρι και την ασυμπτωτική περιοχή.

Για να μελετήσουμε τη θερμοδυναμική του συστήματος χρησιμοποιήσαμε την Ευκλείδεια μέθοδο, η οποία χρησιμοποιήθηκε για πρώτη φορά από τον Hawking [17]. Σύμφωνα με την Ευκλείδεια μέθοδο, η ελεύθερη ενέργεια είναι ανάλογη της Ευκλείδειας δράσης, στην προσέγγιση σαγματικού σημείου όταν ικανοποιούνται οι πεδιακές εξισώσεις, ή αλλιώς

$$\mathcal{I}_E = \beta\mathcal{F} \ ,$$

όπου $\mathcal{I}_E$ είναι η Ευκλείδεια δράση, β είναι η περίοδος του Ευκλείδιου περιοδικού χρόνου η οποία συνδέεται με το αντίστροφο της θερμοκρασίας της μαύρης τρύπας και $\mathcal{F}$ είναι η ελέυθερη ενέργεια του συστήματος.

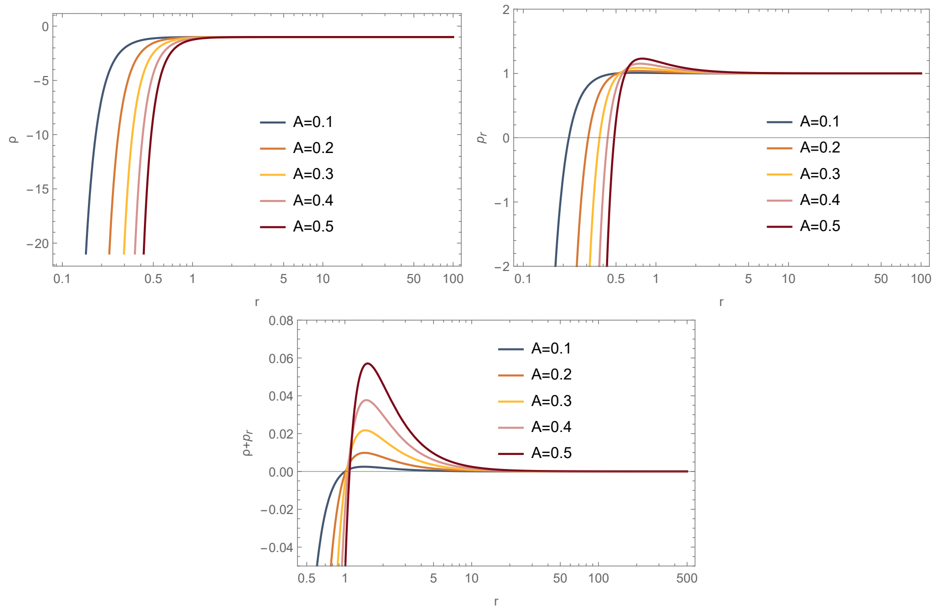

Figure 1: Η πυκνότητα ενέργειας ρ, η ακτινική πίεση p_r και το άθροισμά τους $\rho + p_r$ για $\mathcal{M} = \ell = 1$ αλλάζοντας το βαθμωτό φορτίο A.

Για λόγους πληρότητας θα παρουσιάσουμε με μερικές πράξεις τη μέθοδο. Αρχικά θα θεωρήσουμε την ακόλουθη οικογένεια μετρικών σε Ευκλείδεια υπογραφή

$$ds^2 = N(r)^2 h(r) d\tau^2 + \frac{dr^2}{h(r)s(r)^2} + r^2 d\theta^2 \ .$$

Οι συντεταγμένες μπορούν να πάρουν τιμές σύμφωνα με τις ακόλουθες σχέσεις $0 \leq \tau < \beta,\ r_+ \leq r < \infty,\ 0 \leq \theta < 2\pi$. Εδώ το τ είναι ο Ευκλείδειος χρόνος, ο οποίος είναι περιοδικός με περίοδο β, η οποία προκειμένου να αποφευχθεί μια κωνική ιδιομορφία στον ορίζοντα της μελανής οπής $h(r_+) = 0$, θα είναι αντιστρόφως ανάλογη της θερμοκρασίας και θα δίνεται από τη σχέση

$$T = \frac{1}{\beta} = \frac{N(r)h'(r)s(r)}{4\pi} \ . \tag{57}$$

Θα θεωρήσουμε την Χαμιλτονιανή δράση της θεωρίας μας με τη μορφή

$$\mathcal{H} = \int \left(\pi^{ij} \dot{g}_{ij} + p\dot{\phi} - NH - N^i H_i \right) d^2x dt + \mathcal{B}_H \ . \tag{58}$$

Μιας και η λύση είναι στατική και σφαιρικά συμμετρική, ενώ και το πεδίο είναι μόνο ακτινικά εξαρτημένο, είμαστε στη θέση να θεωρήσουμε την ακόλουθη περιορισμένη Χαμιλτονιανή

$$\mathcal{H} = -\int d^2x dt NH + \mathcal{B}_H \ . \tag{59}$$

Η ποσότητα NH είναι πρακτικά η Λαγκραντζιανή πυκνότητα $NH = -\sqrt{-g}\mathcal{L}$. Συνεπώς έχουμε να αντιμετωπίσουμε το ακόλουθο πρόβλημα μεταβολων

$$\mathcal{I}_E = 2\pi\beta \int dr NH + \mathcal{B}_E \ , \tag{60}$$

όπου φυσικά το $\mathcal{B}_E$ είναι ένας επιφανειακός όρος. Για να έχουμε ένα καλά ορισμένο πρόβλημα μεταβολών θα πρέπει ο επιφανειακός όρος $\mathcal{B}_E$ να είναι τέτοιος ώστε να ακυρώνει τους επιφανειακούς όρους που θα δημιουργηθούν καθώς μεταβάλουμε κάθε συνάρτηση. Μεταβάλλοντας τώρα τη δράση ως προς τα δυναμικά πεδία N, s, h, ϕ υπολογίζουμε τις ακόλουθεις εξισώσεις πεδίου

$$s\left(s\left(h' + hr\left(\phi'\right)^2\right) + 2hs'\right) + 2rV = 0\ , \tag{61}$$

$$s\left(Nr\left(\phi'\right)^2 - N'\right) + Ns' = 0\ , \tag{62}$$

$$N\left(-h' + hr\left(\phi'\right)^2 - \frac{2rV}{s^2}\right) - 2hN' = 0\ , \tag{63}$$

$$N\left(s\phi'\left(\phi'\left(s\left(rh' + h\right) + hrs'\right) + hrs\phi''\right) - rV'\right) + hrs^2N'\left(\phi'\right)^2 = 0\ . \tag{64}$$

Για να υπολογίσουμε αυτές τις εξισώσεις, ακυρώσαμε τους ακόλουθους επιφανειακούς όρους

$$\left(\frac{1}{4}\beta\delta hN + \frac{1}{8}\beta r\delta\phi hNs\phi' + \frac{1}{8}\beta\delta hNs\right)\Big|_{r_+}^{\infty} \tag{65}$$

Η λύση που δώσαμε πιο πάνω ικανοποιεί αυτές τις εξισώσεις με $N(r) = \text{constant}$, το οποίο, χωρίς βλαβη της γενικότητας μπορεί να τεθεί ίσο με 1, και επίσης $s(r) = e^{A^2/2r^2}$. Έχοντας τη λύση των πεδιακών εξισώσεων, είναι εύκολο να υπολογίσει κανείς τις μεταβολές των πεδίων στα σύνορα του Ευκλείδιου χωρόχρονου, στον ορίζοντα της μαύρης τρύπας και στο άπειρο.

Στο άπειρο έχουμε

$$\delta\phi = \delta A/r\ , \tag{66}$$

$$\delta h = -\frac{2A\delta A\left(8q\ell^2 + 1\right)}{\ell^2}\ , \tag{67}$$

$$\delta s = A\delta A/r^2\ , \tag{68}$$

όπου κάναμε ξεκάθαρο πως η μόνη παράμετρος η οποία επιτρέπεται να μεταβληθεί είναι η παράμετρος A. Συνεπώς, ο επιφανειακός όρος στο άπειρο δίνεται από

$$A\left(-2\beta\delta Aq - \frac{\beta\delta A}{8\ell^2}\right) + \delta\mathcal{B}(\infty) = 0\ , \tag{69}$$

όπου διαχωρίσαμε τη μεταβολή του επιφανειακού όρου $\mathcal{B}$ σε δυό μέρη, ένα στο άπειρο και ένα στον ορίζοντα της μαύρης τρύπας. Θεωρώντας τώρα πως δουλεύουμε στο μεγαλοκανονικό σύνολο, κρατώντας τη θερμοκρασία σταθερή, μπορούμε να ολοκληρώσουμε και να πάρουμε ότι

$$\mathcal{B}(\infty) = A^2\beta q + \frac{A^2\beta}{16\ell^2} = \beta\mathcal{M} + \frac{A^2\beta}{16\ell^2}\ . \tag{70}$$

Στον ορίζοντα της μαύρης τρύπας έχουμε ότι

$$\delta h = 0 - h'\delta r_+\ , \tag{71}$$

$$\delta s = \delta s(r_+) - s'\delta r_+\ , \tag{72}$$

$$\delta\phi = \delta\phi(r_+) - \phi'\delta r_+\ . \tag{73}$$

Χρησιμοποιώντας την εξίσωση για τη θερμοκρασία και το γεγονός ότι $h(r_+) = 0$, ο επιφανειακός όρος στον ορίζοντα υπολογίζεται ως

$$\frac{\pi\delta r_+}{2} + \delta\mathcal{B}(r_+) = 0 \to \mathcal{B}(r_+) = -\frac{\mathcal{A}(r_+)}{4}\ , \tag{74}$$

όπου $\mathcal{A}(r_+) = 2\pi r_+$ είναι το εμβαδό της μαύρης τρύπας. Στο μεγαλοκανονικό σύνολο η Ευκλείδεια δράση σχετίζεται με την ελεύθερη ενέργεια

$$\mathcal{I}_E = \beta\mathcal{F} = \beta\mathfrak{M} - \mathcal{S} \ , \tag{75}$$

όπου τα $\mathfrak{M}$, $\mathcal{S}$ είναι η εσωτερική ενέργεια και η εντροπία του συστήματος. Η δική μας Ευκλείδεια δράση θα δίνεται μόνο από τους επιφανειακούς όρους και άρα

$$\mathcal{I}_E = \beta\mathcal{M} + \frac{A^2\beta}{16\ell^2} - \frac{\mathcal{A}(r_+)}{4} \ , \tag{76}$$

οπότε μπορούμε να ταυτοποιήσουμε

$$\mathfrak{M} = \mathcal{M} + \frac{A^2}{16\ell^2} \ , \tag{77}$$

$$\mathcal{S} = \frac{\mathcal{A}(r_+)}{4} \ . \tag{78}$$

Εν κατακλείδι, η μαύρη τρύπα που βρήκαμε θα ικανοποιεί τον πρώτο νόμο της θερμοδυναμικής από κατασκευής

$$\delta\mathfrak{M} = T\delta\mathcal{S} \ . \tag{79}$$

Όπως μπορεί κανείς να δει, η εσωτερική ενέργεια του συστήματος θα δίνεται από τη μάζα της μαύρης τρύπας αλλά και από το φορτίο του βαθμωτού πεδίου A. Το βαθμωτό πεδίο θα ντύνει τη μελανή οπή με ένα δευτερεύον hair, καθότι, η μάζα της μαύρης τρύπας θα δίνεται με τη βοήθεια της σταθεράς της θεωρίας q. Επιπρόσθετα εξετάζοντας την εντροπία και τη θερμοκρασία της μαύρης τρύπας αλλά και τη θερμοχβρητικότητα είδαμε ότι όλες οι ποσότητες ειναι θετικές και κατά συνέπεια, η μαύρη τρύπα μπορεί να έρθει σε θερμική ισορροπία με ένα λουτρό θερμότητας. Στη συνέχεια, θεωρήσαμε και περιστρεφόμενες μελανές οπές με την κατάλληλη "μαντεψιά" για τη μορφή της χωροχρονικής μετρικής. Καταλήξαμε στο συμπέρασμα ότι αυτές οι περιστρεφόμενες μελανές οπές θα περιγράφονται από συγκεκριμένους λόγους μάζας προς τροχιακή στροφορμή. Εν κατακλείδι, και οι περιστρεφόμενες μελανές οπές ειναι θερμικά ευσταθείς, λόγω θετικής θερμοχωρητικότητας.

Στο δεύτερο μέρος αυτης της διατριβής, στα κεφάλαια 3, 4, **??** θα θεωρήσουμε $f(R)$ θεωρίες βαρύτητας συζευγμένες με βαθμωτά πεδία. Στο πρώτο εξ'αυτών, θα ασχοληθούμε με μια τρισδιάστατη $f(R)$ θεωρία με ένα ελάχιστα συζευγμένο αυτο-αλληλεπιδρών βαθμωτό πεδίο, δηλαδή [1]

$$S = \int d^3x\sqrt{-g}\left\{\frac{1}{2\kappa}f(R) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right\} \ . \tag{80}$$

Οι πεδιακές εξισώσεις που προκύπτουν από την ανωτέρω θεωρία δίνονται από

$$f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + g_{\mu\nu}\Box f_R - \nabla_\mu\nabla_\nu f_R = \kappa T_{\mu\nu} \ , \tag{81}$$

όπου $f'(R) = f_R$ και ο τανυστής ορμής ενέργειας $T_{\mu\nu}$ δίνεται από τη σχέση

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - g_{\mu\nu}V(\phi) \ . \tag{82}$$

Η εξίσωση Klein-Gordon δίνεται από τη σχέση

$$\Box\phi = \frac{dV}{d\phi} \ . \tag{83}$$

Για τη μορφή του χωρόχρονου επιλέγουμε

$$ds^2 = -b(r)dt^2 + \frac{1}{b(r)}dr^2 + r^2 d\theta^2 \ . \tag{84}$$

Εδώ θα πρέπει να τονιστεί, πως η ύπαρξη ενός βαθμωτού δυναμικού είναι απαραίτητη για την παραβίαση του θεωρήματος εξάλειψης ιχνών, καθώς χωρίς αυτό, η εξίσωση Klein-Gordon μπορεί να γραφεί ως

$$\Box\phi = b(r)\phi''(r) + \phi'(r)\Big(b'(r) + \frac{b(r)}{r}\Big) = 0 \ , \tag{85}$$

η οποία μπορεί και να γραφεί σαν ένα ολικό διαφορικό αν ολοκληρώσουμε μία φορά

$$b(r)\phi'(r)r = C \ , \tag{86}$$

όπου C είναι μια σταθερά ολοκήρωσης. Για να έχουμε ένα χωρόχρονο μελανής οπής, θα πρέπει να υπάρχει ένα πεπερασμένο $r = r_H$ τέτοιο ώστε $b(r_H) = 0$. Σε αυτή την περίπτωση θα πρέπει $C = 0$ το οποίο θα επιβάλει $b(r) = 0$ παντού για κάθε $r > 0$ το οποίο δεν έχει φυσική σημασία, ή το βαθμωτό πεδίο να είναι μια σταθερά $\phi(r) = c$. Σε αυτό το συλλογισμό, έχουμε κάνει την υπόθεση πως η παράγωγος του βαθμωτού πεδίου (συγκεκριμένα η νόρμα του κινητικού όρου, που είναι και η ποσότητα της θεωρίας) θα παραμένει πεπερασμένη πάνω στον ορίζοντα γεγονότων. Αυτή η συμπεριφορά είναι αναμενόμενη φυσικά και δεν εξαρτάται από τη μορφή για το στοιχείο μήκους της γεωμετρίας που έχουμε επιλέξει. Σύμφωνα με το θεώρημα εξάλειψης ιχνών [18] πολλαπλασιάζοντας με ϕ και ολοκληρώνοντας στο εξωτερικό της μελανής οπής παίρνουμε

$$\int d^3x\sqrt{-g}\big(\phi\Box\phi\big) \approx \int d^3x\sqrt{-g}\nabla^\mu\phi\nabla_\mu\phi = 0 \ , \tag{87}$$

όπου το σύμβολο $\approx$ δηλώνει ισότητα χωρίς να ληφθούν υπόψη επιφανειακοί όροι, οι οποίοι ακυρώθηκαν θεωρώντας πως το πεδίο πέφτει γρήγορα σε μεγάλες αποστάσεις. Από την ανωτέρω εξίσωση, μπορεί κανείς να αποφανθεί πως το βαθμωτό πεδίο θα πρέπει να είναι σταθερό.

Για να επιλύσουμε τις εξισώσεις κίνησης, θα επιλέξουμε μια μορφή για το βαθμωτό πεδίο, και συγκεκριμένα

$$\phi(r) = \sqrt{\frac{A}{r+B}} \ , \tag{88}$$

όπου τα A, B είναι δυο σταθερές ολοκλήρωσης που καθορίζουν τη συμπεριφορά του βαθμωτού πεδίου. Οι εξισώσεις κίνησης μπορούν να ολοκληρωθούν ως

$$f_R(r) = c_1 + c_2 r - \int\int \phi'(r)^2 dr dr \ , \tag{89}$$

$$b(r) = c_3 r^2 - r^2 \int \frac{K}{r^3 f_R(r)} dr \tag{90}$$

όπου τα c_1, c_2, c_3 and K είναι σταθερές ολοκλήρωσης, ενώ ολοκληρώνοντας την εξίσωση για το βαθμωτό πεδίο παίρνουμε

$$V(r) = V_0 + \int \frac{rb'(r)\phi'(r)^2 + rb(r)\phi'(r)\phi''(r) + b(r)\phi'(r)^2}{r} dr \ , \tag{91}$$

όπου ο δείκτης $_R$ δηλώνει παραγώγιση ως προς το βαθμωτό του Ricci. Μπορεί κανείς να δει ότι για ένα τετριμένο βαθμωτό πεδίο και για $c_2 = 0$ παίρνουμε πίσω τη ΓΘΣ και τη λύση BTZ. Συνεπώς, η σταθερά c_1 σχετίζεται με την ισχύ της σταθεράς συζευξης της βαρύτητας και η σταθερά c_2 με πιθανές γεωμετρικές διορθώσεις που εμπεριέχονται στις $f(R)$ θεωρίες βαρύτητας. Για τη συγκεκριμένη μορφή του βαθμωτού πεδίου, μπορούμε να υπολογίσουμε τη συνάρτηση της μετρικής $b(r)$ ως

$$b(r) = c_3 r^2 - \frac{4BK}{A-8B} - \frac{8AKr}{(A-8B)^2} - \frac{64AKr^2}{(A-8B)^3}\ln\left(\frac{8(B+r)-A}{r}\right) , \tag{92}$$

όπου μπορεί κανείς να δει ότι όταν οι σταθερές A, B ικανοποιούν τη συνθήκη $0 < A < 8B$, περιγράφει τη γεωμετρία μας μαύρης τρύπας σε έναν ασυμπτωτικά AdS χωρόχρονο, με τον ορίζοντα της να μεγαλώνει καθώς η παράμετρος A αυξάνεται καλύπτοντας την κεντρική ιδιομορφία στο $r = 0$.

Έχουμε αφήσει τις συναρτήσεις της θεωρίας να υπολογιστούν από τις υπόλοιπες συναρτήσεις, δηλαδή έχουμε βρει τη θεωρία η οποία μπορεί να υποστηρίξει τη λύση μας. Μπορούμε να υπολογίσουμε τη θεωρία $f(R)$ ασυμπτωτικά για μικρά και μεγάλα r

$$f(R) \simeq R + 2c_3 - \frac{384AK\ln(2)}{(A-8B)^3} = R - 2\Lambda_{\text{eff}} , \qquad r \to \infty , \tag{93}$$

$$f(R) \simeq R\left(1 - \frac{A}{8B}\right) , \qquad r \to 0 . \tag{94}$$

Μπορούμε να δούμε ότι, σε πρώτη τάξη, σε μεγάλες αποστάσεις παίρνουμε πίσω την ΓΘΣ παρουσία μιας κοσμολογικής σταθεράς, ενώ κοντά στην ιδιομορφία, όπου και το βαθμωτό πεδίο παίρνει τη μέγιστη τιμή του, οι σταθερές του βαθμωτού πεδίου επεισέρχονται στον όρο της ΓΘΣ τροποποιώντας τη σταθερά σύζευξης της βαρύτητας από 1 σε $1 - A/8B$ (έχουμε θέσει $\kappa = 8\pi G_3 = 1$). Στο ακόλουθο διάγραμμα, απεικονίζουμε όλες τις φυσικές ποσότητες της λύσης μας

Μπορούμε να δούμε από το διάγραμμα 4 πως καθώς αυξάνεται η παράμετρος A μεγαλώνει και ο ορίζοντας της μελανής οπής, ενώ το βαθμωτό Ricci από σταθερό που είναι για την περίπτωση της BTZ λύσης γίνεται πλέον δυναμικό. Το βαθμωτό δυναμικό απειρίζεται κοντά στην ιδιομορφία ενώ παρουσιάζει και ένα πηγάδι, φαινόμενα τα οποία είναι συνήθη για τις AdS μελανές οπές με scalar hair.

Υπολογίζοντας την εντροπία της μαύρης τρύπας, είδαμε ότι η λύση μας έχει μεγαλύτερη εντροπία από τη BTZ λύση, καθώς αυξάνουμε τη σταθερά A του βαθμωτού πεδίου. Η εντροπία υπολογισμένη πάνω στον ορίζοντα γεγονότων απεικονίζεται στο ακόλουθο σχήμα 3 όπου μπορούμε και να δούμε ότι πράγματα οι λύσεις της μη γραμμικής βαρύτητας θα κουβαλούν περισσότερη εντροπία από τη BTZ περίπτωση.

Ακολούθως, στο επόμενο κεφάλαιο, 4 θεωρήσαμε πάλι μια τροποποιημένη θεωρία βαρύτητας σε τρεις χωροχρονικές διαστάσεις, εν προκειμένω [4]

$$S = \frac{1}{2}\int d^3x\sqrt{-g}\Big\{f(R) - \partial_\mu\phi\partial^\mu\phi - \frac{1}{8}R\phi^2 - 2V(\phi)\Big\} , \tag{95}$$

όπου εδώ έχουμε θεωρήσει μια μη τετριμμένη σύζευξη μεταξύ του βαθμωτού πεδίου μέσω του όρου $R\phi^2$. Η ανωτέρω θεωρία για τις επιλογές $f(R) = R - 2\Lambda$, $V(\phi) = 0$ δέχεται την (πλέον διάσημη) λύση [19]. Πρόκειται για μια πολύ σημαντική λύση, καθώς το τετραδιάστατο ξαδερφάκι της η λύση των Bocharova–Bronnikov–Melnikov–Bekenstein (BBMB) [20] παρουσιάζει σημαντικά μαθηματικά και φυσικά προβλήματα, όπως απειρισμούς του βαθμωτού πεδίου στον ορίζοντα, αλλά και της εντροπίας, μηδενισμός της θερμοκρασίας, αστάθεια κάτω από διαταραχές, για να αναφέρουμε μερικά. Θα τα συζητήσουμε εκτενώς παρακάτω, καθότι η τρισδιάστατη λύση δεν υποφέρει από τα προβλήματα της τετραδιάστατης λύσης, μια και έχει πεπερασμένες και θετικές θερμοκρασία και εντροπία, ενώ ισχύει και ο πρώτος νόμος της θερμοδυναμικής. Η μόνη ελεύθερη παράμετρος του συστήματος είναι η μάζα της μαύρης τρύπας και δε μπορούμε να σβήσουμε το βαθμωτό πεδίο κρατώντας αμετάβλητη τη μάζα, συνεπώς η μαύρη τρύπα κουβαλά δευτερέυον hair.

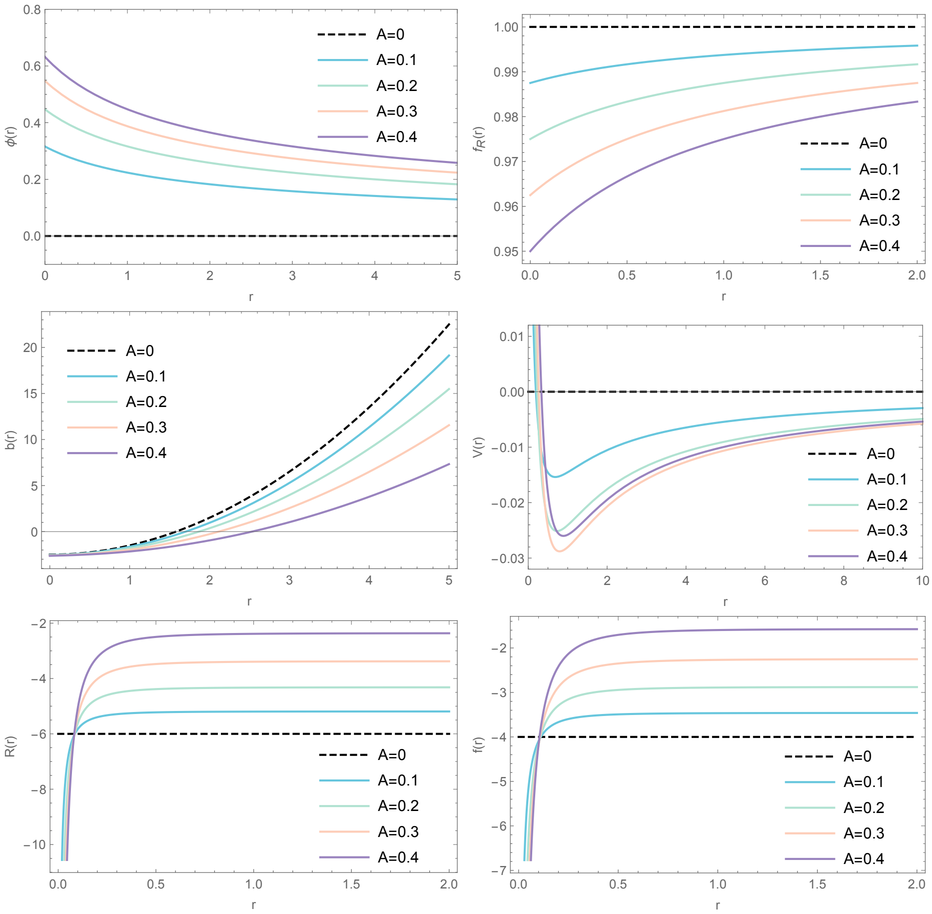

Figure 2: Όλες οι φυσικές ποσότητες για διαφορετικά A έχοντας θέσει $B = 1$, $K = -5$ και $c_3 = 1$.

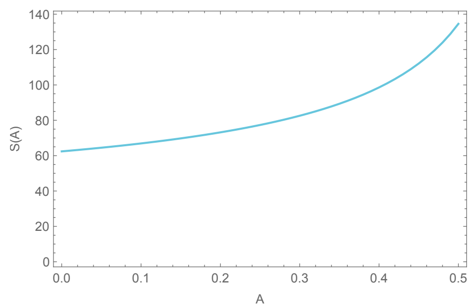

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Figure 3: Η εντροπία υπολογισμένη πάνω στον ορίζοντα γεγονότων σα συνάρτηση της παραμέτρου A, όπου οι υπόλοιπες σταθερές έχουν φιξαριστεί ως $B = 1$, $K = -5$ και $c_3 = 1$.

Σκοπός μας είναι να δούμε πως συμπεριφέρεται μια παρόμοια λύση στα πλαίσια μιας $f(R)$ θεωρίας. Για το λόγο αυτό, για να λύσουμε τις πεδιακές εξισώσεις, κάναμε την υπόθεση πως όταν σβήσουμε την παράμετρο που είναι υπεύθυνη για τη μη γραμμική βαρύτητα, παίρνουμε πίσω τη λύση [19]. Η λύση του συστήματος δίνεται από

$$\phi(r) = \sqrt{A/(r+B)}\ , \tag{96}$$

$$f_R(r) = 1 + \alpha r\ , \tag{97}$$

$$b(r) = -\frac{3B^2}{l^2(\alpha B+1)^2} - \frac{2B^3}{l^2 r(\alpha B+1)} + \frac{6\alpha B^2 r}{l^2(\alpha B+1)^3} + r^2\left(\frac{1}{l^2}\frac{6\alpha^2 B^2}{l^2(\alpha B+1)^4}\ln\left(\frac{r}{\alpha l(B+r)+l}\right)\right)\ , \tag{98}$$

$$f(R(r\to 0)) \sim R - \frac{12\alpha^2 B^2}{l^2(\alpha B+1)^3}\ln(R) + C_0\ , \tag{99}$$

$$f(R(r\to\infty)) \sim R - \frac{4B\left(6\Lambda_{\text{eff}} - R\right)^{3/4}}{3^{3/4}\sqrt{\alpha B l}} + C_\infty\ , \tag{100}$$

ενώ το βαθμωτό δυναμικό υπολογίζεται ως

$$V(\phi) = \frac{\alpha B}{512 l^2(\alpha B+1)^4\left(8\alpha B+\phi^2\right)}\Bigg(\phi^2(\alpha B+1)(3072\alpha^2B^2 + \phi^6(\alpha B(\alpha B+5)-2) + 8\alpha B\phi^4(\alpha B+1)(\alpha B+4) + 192\alpha B\phi^2(\alpha B+1)) + 6\alpha B\left(8\alpha B+\phi^2\right)\left(\left(512\alpha B+\phi^6\right)\ln\left(\frac{B\left(-\phi^2+8\right)}{l\left(8\alpha B+\phi^2\right)}\right) + 512\alpha B\ln(\alpha l)\right)\Bigg)\ . \tag{101}$$

Η μορφή της θεωρίας $f(R)$ που υποστηρίζει αυτή τη λύση μπορεί να υπολογιστεί μόνο ασυμπτωτικά και οι ανωτέρω $f(R)$ θεωρίας εμπεριέχουν μια σταθερά ολοκλήρωσης. Αυτό που παρατηρήσαμε είναι ότι η παράμετρος α, η οποία επιτρέπει την ύπαρξη των μη γραμμικών παραμέτρων, οδηγεί σε έναν μη άιχνο τανυστή ορμής ενέργειας για το βαθμωτό πεδίο, ενώ κάνει και το βαθμωτό του Ricci δυναμικό από σταθερό. Επιπρόσθετα, καθώς η παράμετρος α αυξάνεται, μικραίνει ο ορίζοντας της μελανής οπής. Στο ακόλουθο διάγραμμα 4 στο οποίο και απεικονίζονται οι φυσικές ποσότητες της λύσης μας. Ενδιαφέρον έχει να παρατηρήσει κανείς πως ο ορίζοντας μικραίνει καθώς μεγαλώνουμε την παράμετρο της μη γραμμικής βαρύτητας, ενώ παράλληλα το ίχνος του τανυστή ορμής ενέργειας από μηδέν που είναι για την περίπτωση της ΓΘΣ γίνεται δυναμικό για ένα μη μηδενικό α.

Υπολογίζοντας τώρα τις διάφορες θερμοδυναμικές ποσότητας παρατηρήσαμε επίσης ενδιαφέρουσα συμπεριφορά. Αρχικά, η λύση μη γραμμικής βαρύτητας μπορεί να είναι θερμότερη και θερμοδυναμικά προτιμητέα εν συγκρίσει με τη λύση [19]. Ωστόσο δε μπορεί να είναι και τα δυό ταυτόχρονα. Καθώς αυξάνεται η παράμετρος μη γραμμικής βαρύτητας, έπειτα από μια κρίσιμη τιμή, η θερμοκρασία Hawking ελαττώνεται, ενώ η εντροπία στον ορίζοντα μεγαλώνει. Η θερμοκρασία θα δίνεται από τη σχέση

$$T_H = \frac{b'(r_h)}{4\pi} = \frac{3B^2(B+r_h)}{2\pi l^2 r_h^2(\alpha B+\alpha r_h+1)}\ , \tag{102}$$

όπου χρησιμοποιήσαμε τη σχέση $b(r_h) = 0$. Όπως κανείς θα περίμενε παίρνουμε τη θερμοκρασία της λύσης [19] όταν $\alpha \to 0$. Στο διάγραμμα 5, απεικονίζουμε τη θερμοκρασία T_H συναρτήσει της παραμέτρου της μη γραμμικής βαρύτητας α υπολογισμένη πάνω στον ορίζοντα γεγονότων. Καθώς το α αυξάνεται, η θερμοκρασία μικραίνει λίγο, στη συνέχεια αυξάνεται φτάνοντας στη μέγιστη τιμή της

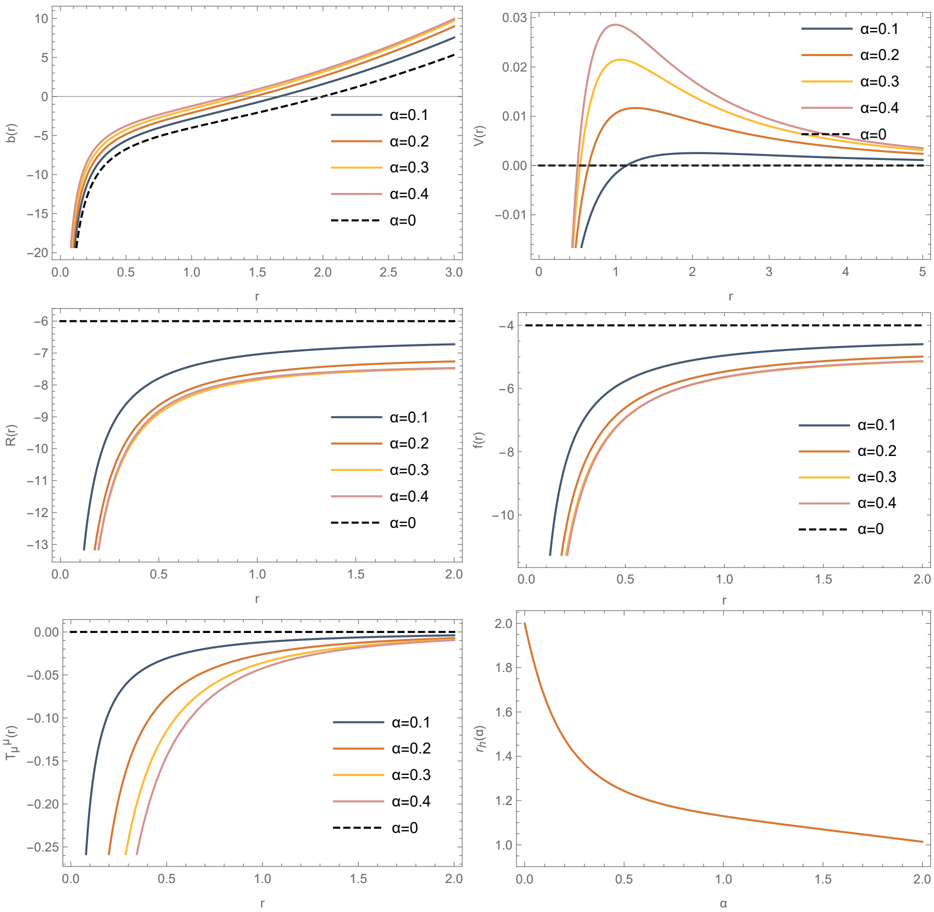


Figure 4: Οι συναρτήσεις $b(r), V(r), R(r), f(r)$ and $T_\mu{}^\mu(r)$ απεικονίζονται συναρτήσει του r για διαφορετικές τιμές του α, ενώ στο τελευταίο διάγραμμα κάτω δεξιά, ο ορίζοντας γεγονότων r_h απεικονίζεται σα συνάρτηση του α. Σε όλα τα διαγράμματα έχουμε θέσει $B = l = 1$.

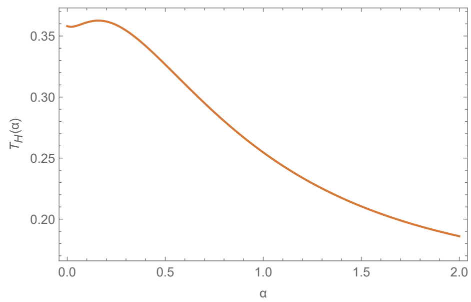


Figure 5: Η θερμοκρασία σα συνάρτηση του α,υπολογισμένη πάνω στον ορίζοντα γεγονότων αφού έχουμε θέσει $B = l = 1$.

και στη συνέχεια μικραίνει και πάλι. Συνεπώς, οι λύσεις της μη γραμμικής βαρύτητας είναι εν γένει ψυχρότερες.

Περνάμε τώρα στην εντροπία. Χρησιμοποιώντας τον τύπο του Wald [21, 22], μπορούμε να υπολογίσουμε την εντροπία της μελανής οπής για τη θεωρία μας ως

$$S = -\frac{1}{4}\int d\theta \sqrt{r_h^2}\left(\frac{\partial \mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}}\right)\Big|_{r=r_h} \hat{\varepsilon}_{\alpha\beta}\hat{\varepsilon}_{\gamma\delta} \ , \tag{103}$$

όπου$\hat{\varepsilon}_{\alpha\beta}$ είναι το binormal διάνυσμα πάνω στον ορίζοντα [23], $\mathcal{L}$ είναι η Λαγκραντζιαν της θεωρίας και

$$\frac{\partial \mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}}\Big|_{r=r_h} = \frac{1}{2}\left(\frac{f_R(r_h)}{2} - \frac{1}{16}\phi(r_h)^2\right)\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\beta\gamma}g^{\alpha\delta}\right) \ . \tag{104}$$

Εν τέλει καταλήγουμε στο

$$S = \pi r_h\left(\frac{f_R(r_h)}{2} - \frac{1}{16}\phi(r_h)^2\right) = \frac{\mathcal{A}}{4} f_{R_{\text{total}}}(r_h) \ . \tag{105}$$

Αντικαιστωντας την ποσότητα $f_{R_{\text{total}}}$, παίρνουμε

$$S = \frac{1}{2}\pi r_h\left(1 + \alpha r_h - \frac{B}{B + r_h}\right) \ , \tag{106}$$

Εδώ, ο ορίζοντας r_h μεταβάλλεται καθώς το B αλλάζει αλλά θα επηρεάζεται και από τις σταθερές l and α. Κανείς μπορεί να υποθέσει ότι αφού $\alpha > 0$, οι $f(R)$ μελανές οπές έχουν μεγαλύτερη εντροπία από αυτές του [19]. Ωστόσο, πρέπει να λάβουμε υπόψη πως οι μελανές οπές [19] έχουν μεγαλύτερο ορίζοντα, όπως μπορεί κανείς να δει από τη συνάρτηση $b(r)$ στο διάγραμμα Fig. 4. Χρησιμοποιώντας τον τύπο της εντροπίας και τη συνθήκη του ορίζοντα $b(r_h) = 0$, απεικονίζουμε την εντροπία υπολογισμένη πάνω στον ορίζοντα σα συνάρτηση της παραμέτρου α στο διάγραμμα 6. Είναι εμφανές ότι, οι λύσεις των $f(R)$ θεωριών θα κρύβουν μέσα τους λιγότερη πληροφορία απότι αυτές της ΓΘΣ.

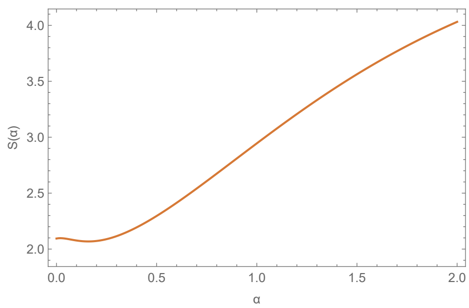


Figure 6: Η εντροπία S στον ορίζοντα γεγονότων ως συνάρτηση της παραμέτρου α, όπου έχουμε θέσει $B = l = 1$.

Επιπρόσθετα, υπολογίζοντας τη μάζα της μαύρης τρύπας, είδαμε ότι αυτή είναι μηδέν, συνεπώς η μαύρη τρύπα είναι άμαζη. Αυτή δεν είναι και η μοναδική τέτοια περίπτωση στη βιβλιογραφία, καθότι, υπάρχουν αρκετά παρόμοια παραδείγματα παραγωγής άμαζων μελανών οπών, που συνοδεύονται από τη θραύση μιας συμμετρίας, όπως αυτή της σύμμορφης συμμετρίας. Στη δική μας περίπτωση η παράμετρος α οδηγεί στη θραύση της σύμμορφης συμμετρίας και μάλιστα αποδείξαμε πως, η συνάρτηση της μετρικής μπορεί να γραφεί σαν το άθροισμα των δυο επιμέρους συναρτήσεων, η μια εκ των οποίων είναι πλήρως καθορισμένη από την παράμετρο α καθώς αν θέσουμε $\alpha = 0$ αυτή θα μηδενιστεί, δηλαδή, έχουμε $b(r) = b(r)_{\alpha,\phi} + b(r)_{GR,\alpha,\phi}$, όπου

$$b(r)_{GR,\alpha,\phi} = -\frac{3B^2}{l^2(\alpha B+1)^2} - \frac{2B^3}{l^2 r(\alpha B+1)} + \frac{r^2}{l^2} \ , \tag{107}$$

$$b(r)_{\alpha,\phi} = \frac{6\alpha B^2 r}{l^2(\alpha B+1)^3} + r^2 \frac{6\alpha^2 B^2}{l^2(\alpha B+1)^4} \ln\left(\frac{r}{\alpha l(B+r)+l}\right) \ . \tag{108}$$

Είναι ξεκάθαρο πως αν θέσουμε $\alpha = 0$ στην $b(r)_{\alpha,\phi}$, η συνάρτηση θα μηδενιστή, ενώ η $b(r)_{GR,\alpha,\phi}$ θα μας δώσει τη λύση [19]. Μπορούμε να δούμε ότι η συνάρτηση $b(r)_{GR,\alpha,\phi}$ εμπεριέχει έναν όρο μάζας

$$M_{GR,\alpha,\phi} = \frac{3B^2}{l^2(\alpha B+1)^2} \ , \tag{109}$$

ενώ η συνάρτηση $b(r)_{\alpha,\phi}$ δίνει έναν όρο μαζας

$$M_{\alpha,\phi} = -\frac{3B^2}{l^2(\alpha B+1)^2} \ , \tag{110}$$

ο οποίος είναι αντίθετος από τον όρο μάζας που η συνάρτηση $b(r)_{GR,\alpha,\phi}$ γεννά. Το γεγονός αυτό εγείρει σημαντικά ερωτήματα για τη θερμοδυναμική φύση αυτής της μελανής οπής.

Στο τρίτο μέρος της παρούσας διατριβής, μελετάμε λύσεις μελανών οπών σε μη γραμμικές θεωρίες ηλεκτρομαγνητισμού και κυρίως με τη θεωρία των Euler και Heisenberg (EH) [24], η Lagrangian της οποίας δίνεται από τη σχέση

$$\mathcal{L}_{\rm EH} \sim -\mathcal{F}^2 + \alpha\mathcal{F}^4 + \beta\mathcal{Q}^2 \ ,$$

όπου τα α, β είναι σταθερές με μονάδες [μήκος]^2 και $\mathcal{F}^2 = F_{\mu\nu}F^{\mu\nu}$, $\mathcal{Q} = \epsilon_{ijkl}F^{ij}F^{kl}$ με τον τανυστή του Maxwell να δίνεται από τη σχέση $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ όπου το A_μ είναι το πεδίο βαθμίδας του ηλεκτρομαγνητισμού. Η συγκεκριμένη θεωρία, σε κβαντομηχανικό επίπεδο, περιγράφει τη σκέδαση φωτονίων με φωτόνια. Σε επίπεδο βαρύτητας μια τέτοια θεωρία μπορεί να περιγράφει ακριβέστερα μαύρες τρύπες στο πρώιμο σύμπαν όπου η ένταση των ηλεκτρομαγνητικών πεδίων θεωρείται συγκρίσιμη με αυτή των βαρυτικών. Επιπρόσθετα, τέτοιες θεωρίες μη γραμμικές ως προς το βαθμωτό $\mathcal{F}$ προβλέπονται από πιο θεμελιώδεις φυσικές θεωρίες. Τέτοιοι όροι προβλέπονται από τη μη διαγώνια (επιτρέποντας δηλαδή την ύπαρξη ενός ηλεκτρομαγνητικού πεδίου στη χαμηλότερη διάσταση) διαστατική ελάττωση (τύπου Kaluza-Klein) Lovelock θεωριών αλλά και από θεωρίες χορδών.

Στις θεωρίες χορδών, οι ανώτεροι όροι του βαμθωτού του Maxwell $\mathcal{F}^2$ μπορούν να γραφούν σε κλειστή μορφή και συγκεκριμένα σε μια έκφραση που εμπεριέχει όλους τους όρους του αναπτύγματος, τη Lagrangian των Born και Infeld (BI) [25]. Σε τέτοιες θεωρίες, η Lagrangian των BI έρχεται από τη 10-διάστατη υπερχορδή μετά από συμπαγοποίηση πάνω στην κατάλληλη τρισδιάστατη βράνη. Σε αυτά τα μοντέλα, η BI θεωρία είναι πάντα συζευγμένη με το αντίστροφο της παραμέτρου ανάπτυξης βρόγχων της χορδής $g_s = e^\phi$ όπου το ϕ είναι το πεδίο dilaton (ένα βαθμωτό πεδίο). Η BI δράση στις 4 χωροχρονικές διαστάσεις μπορεί να γραφεί ως

$$\mathcal{S}_{\rm BI} = -\mathcal{T}_4^2 \int d^4x\,\sqrt{-g^J}\,e^{-\phi}\sqrt{1 + \frac{1}{2\,\mathcal{T}_4^2}\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu} - \frac{1}{16\,\mathcal{T}_4^4}\left(\mathcal{F}_{\mu\nu}\,\widetilde{\mathcal{F}}^{\mu\nu}\right)^2} \tag{111}$$

όπου $\widetilde{\mathcal{F}}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\,\mathcal{F}^{\rho\sigma}$ και $\mathcal{T}_4 = \frac{1}{2\pi\alpha'} = \frac{M_s^2}{2\pi}$, όπου M_s είναι η κλίμακα μάζας της χορδής η οποία ειναι εν γένει διαφορετική από τη μάζα Planck. Μπορεί κανείς εν γένει να θεωρήσει την δράση BI εξαρχής σα μια ενεργό θεωρία μη-γραμμικού ηλεκτρομαγνητισμού, οπότε και σε αυτή την περίπτωση, η παράμετρος $\mathcal{T}_4$ δε σχετίζεται με τη θεωρία χορδών αλλά είναι μια διαστατική παράμετρος η οποία μπορεί να προσδιοριστεί από το εκάστοτε σύστημα που θεωρεί κανείς.

Πραγματοποιώντας τώρα ενα ανάπτυγμα της BI δράσης σε όρους $\mathcal{T}_4^{-1}$ μπορεί κανείς να δει ότι η BI θεωρία ταυτίζεται με την EH θεωρία

$$\mathcal{S}_{\rm BI} = \int d^4x\sqrt{-g^J}\,e^{-\phi}\Big[-\mathcal{T}_4^2\,I_2 - \mathcal{T}_4^4\,I_4\Big(1+\mathcal{O}(\mathcal{F}^2)\Big)\Big]\,,$$
$$I_2 = \frac{1}{4\mathcal{T}_4^2}\,\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu}\,,\quad I_4 = -\frac{1}{8\,\mathcal{T}_4^4}\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\nu\rho}\,\mathcal{F}_{\rho\lambda}\,\mathcal{F}^{\lambda\mu} + \frac{1}{32\,\mathcal{T}_4^4}\Big(\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu}\Big)^2\,. \tag{112}$$

Έτσι αγνοόντας προς στιγμή το πεδίο dilaton, μπορεί κανείς να δει ότι η ανωτέρω θεωρία ταυτίζεται με την EH θεωρία με

$$\mathcal{L}_{\rm EH} = c_1\,\Big(\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu}\Big)^2 + c_2\,\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\nu\rho}\,\mathcal{F}_{\rho\lambda}\,\mathcal{F}^{\lambda\mu}\,, \tag{113}$$

όπου [26, 27]

$$c_1 = -\frac{1}{32\mathcal{T}_4^2}\,,\quad c_2 = \frac{1}{8\,\mathcal{T}_4^2}\,. \tag{114}$$

Θεωρώντας μη δυναμικό πεδίο dilaton και επίπεδο χωροχρόνο Minkowski η παράμετρος $\mathcal{T}_4^2$ μπορεί να περιοριστεί μέσω της φυσικής των επιταχυντών και συγκεκριμένα μέσω σκέδασης φωτονίων με φωτόνια, για την οποία υπάρχουν ξεκάθαρες πειραματικές ενδείξεις στα πειράματα του LHC ([28, 29, 30]). Οι μελέτες σκέδασης φωτονίων με φωτόνια [26] μπορούν να θέσουν ένα κάτω όριο στην παράμετρο $\mathcal{T}_4 \geq 100$ GeV. Στην περίπτωση της θεωρίας χορδών, αυτό θα οδηγούσε σε ένα κατώτατο όριο για τη κλίμακα μάζας της χορδής $M_s \geq 0.25$ TeV.

Αν κανείς τώρα λάβει υπόψη του τη δυναμική του βαρυτικού πεδίου και του πεδίου dilaton, μπορεί κανείς να γράψει την δράση BI ως

$$\mathcal{S} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big[\mathcal{R} - 2\nabla^\mu\phi\nabla_\mu\phi\Big] - \int d^4x\,\sqrt{-g}\,e^{-\phi}\Big[\mathcal{T}_4^2\,I_2^{\rm E} + \mathcal{T}_4^4\,e^{-4\phi}\,I_4^{\rm E}\Big] + \ldots\,, \tag{115}$$

στο σύστημα Einstein (το σύστημα στο οποίο η βαρύτητα είναι ελάχιστα συζευγμένη με τα πεδία, δηλαδή μετά τη μεταβολή της δράσης μπορούμε να γράψουμε $G_{\mu\nu} \sim T_{\mu\nu}$).

Μπορεί κανείς επίσης να θεωρήσει μια θεωρία με υψηλότερους όρους ηλεκτρομαγνητισμού, σε ενεργές θεωρίες χαμηλών ενεργειών που προέρχονται όμως από κλειστές θεωρίες χορδών, όπως η ετεροτική χορδή. Αυτό πρακτικά σημαίνει, ότι κανείς πραγματοποιεί ένα ανάπτυγμα στην παράμετρο της ισχύς σύζευξης της χορδής α'. Σε τετοιες θεωρίες, δε μπορεί κανείς να θεωρήσει την δράση των BI, καθότι τα πεδία βαθμίδας δε μπορούν να γραφούν σαν έναν όρο που αν αναπτυχθεί περιέχει όλους τους επιμέρους όρους όπως η BI θεωρία. Παρόλο που ενεργές θεωρίες σε βρόγχους χορδής δεν είναι γνωστές σε αναλυτική μορφή μπορούμε να θεωρήσουμε [31]:

$$S = \int d^4x\,\sqrt{-\widehat{g}}\,\Big(\frac{1}{\alpha'}\,B_g(\Phi)\,\widehat{\mathcal{R}} + \frac{1}{\alpha'}\,B_\Phi(\Phi)\,\Big[\widehat{\Box}\Phi - 4\widehat{\nabla}_\mu\Phi\,\widehat{\nabla}^\mu\Phi\Big] - \frac{B_F(\Phi)}{4}\widehat{F}_{\mu\nu}\,\widehat{F}^{\mu\nu} - B_\psi(\Phi)\overline{\widehat{\psi}}\,\widehat{\not{D}}\widehat{\psi} + \ldots\Big)\,, \tag{116}$$

στο σύστημα αναφοράς της χορδής όπου το σύμβολο $\widehat{(\ldots)}$ δηλώνει άθροιση με τη μετρική στο σύστημα αναφοράς της χορδής $\widehat{g}_{\mu\nu}$, $F_{\mu\nu}$ δηλώνει τον τανυστή του Maxwell, D_μ είναι η συναλλοίωτη παράγωγος της βαθμίδας, ψ είναι τα φερμονικά πεδία και το $\ldots$ δηλώνει άλλα πεδία ύλης αλλά και έναν άπειρο αριθμό όρων στο α'. Εδώ, οι ποσότητες $B_i(\Phi), i = g, \Phi, F, \psi$ είναι συναρτήσεις που περιέχουν μόνο το πεδίο dilaton και όχι παραγώγους του, και συγκεκριμένα δυνάμεις της συνάρτησης συζευξης της χορδής $g_s = \exp(\Phi)$, $g_s^{-\chi}$ και μπορύν να γραφούν στη μορφή

$$B_i(\Phi) = e^{-2\Phi} + c_0^{(i)} + c_1^{(i)} e^{2\Phi} + \cdots + c_{2n}^{(i)} e^{2n\Phi} + \ldots , \tag{117}$$

οπότε και πρόκειται πρακτικά για ένα ανάπτυγμα του τετραγώνου της συνάρτησης σύζευξης της χορδής. Μάλιστα, ο πρώτος όρος σε αυτό το ανάπτυγμα για $\chi = 2$ δίνει τον όρο μηδενικής τάξης στο όριο χαμηλών ενεργειών της ετεροτικής θεωρίας χορδών. Επιπρόσθετα, αυτός ο όρος εμφανίζεται και στην μη διαγώνια διαστατική ελάττωση της ΓΘΣ από τις πέντε χωροχρονικές διαστάσεις στις τέσσερις χωροχρονικές διαστάσεις με τη μέθοδο των Kaluza-Klein. Είναι εξόχως σημαντικό να τονίσουμε πως η ανωτέρω έκφραση περιέχει και έναν σταθερό όρο για $\chi = 0$, τον οποίο ωστόσο δε μπορύμε να προσδιορίσουμε καθώς πρόκειται για ένα ζήτημα το οποίο εξαρτάται σε μεγάλο βαθμό από τη θεωρία χορδών που θεωρεί κανείς. Συνεπώς, περνώντας τώρα στο σύστημα Einstein, και αγνοώντας όλα τα υπόλοιπα πεδία πέρα από το βαρυτικό, το πεδίο dilaton και το πεδίο βαθμίδας μπορούμε να γράψουμε τη δράση

$$\mathcal{S} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \Big[\mathcal{R} - 2\nabla^\mu \phi \nabla_\mu \phi \Big] - \int d^4x \, \sqrt{-g} \, B_{F^2}(\phi) \left[\mathcal{T}_4^2 \, I_2^{\mathrm{E}} \right] - \int d^4x \, \sqrt{-g} \, \mathcal{T}_4^4 \, B_{F^4}(\phi) \, I_4^{\mathrm{E}} \Big] + \ldots , \tag{118}$$

όπου οι συναρτήσεις $B_{F^i}(\phi)$, $i = 2, 4$ επιδέχονται ένα ανάπτυγμα στη συνάρτηση σύζευξης στης χορδής

$$B_{F^i}(\phi) = \sum_\chi g_s^{-\chi} c_\chi^{(F^i)} , \qquad i = 1, 2 , \qquad g_s = \exp(\phi) . \tag{119}$$

Στις ετεροτικές χορδές οι όροι ΕΗ δε μπορούν να γραφούν σε κλειστή μορφή και γι'αυτό μια γενικότερη δράση μπορεί να θεωρηθεί,

$$\mathcal{S} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \Big[\mathcal{R} - 2\nabla^\mu \phi \nabla_\mu \phi \Big] - \int d^4x \, \sqrt{-g} \, B_{F^2}(\phi) \left[\mathcal{T}_4^2 \, I_2^{\mathrm{E}} \right] - \int d^4x \sqrt{-g} \, \mathcal{T}_4^4 B_{F^4}(\phi) \, \mathcal{L}_{\mathrm{EH}} + \ldots , \tag{120}$$

όπου το $(\ldots)$ περιλαμβάνει δυναμικά για το πεδίο dilaton, η ακριβής μορφή των οποίων δεν είναι δυνατό να βρεθεί, και οι συναρτήσεις $B_{F^2}(\phi)$, $B_{F^4}(\phi)$ σε αυτή την περίπτωση δύνονται από ένα ανάπτυγμα άρτιων όρων της συνάρτησης συζευξης της χορδής. Στο τελευταίο κεφάλαιο αυτής της διατριβής 6 θα μελετήσουμε εκτενώς την ανωτέρω δράση, και συγκεκριμένα θα αφήσουμε τις παραμέτρους c_i αυθαίρετες και πιο συγκεκριμένα, θα θεωρήσουμε τη δράση

$$\mathcal{S} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \Big[\mathcal{R} - 2\nabla^\mu \phi \nabla_\mu \phi - e^{-2\phi} \mathcal{F}^2 - f(\phi) \big(2\alpha \mathcal{F}^\alpha{}_\beta \mathcal{F}^\beta{}_\gamma \mathcal{F}^\gamma{}_\delta \mathcal{F}^\delta{}_\alpha - \beta \mathcal{F}^4 \big) \Big] . \tag{121}$$

Συγκεκριμένα, η συνάρτηση $f(\phi)$ θα πρέπει να περιέχει άρτιες δυνάμες της συνάρτησης σύζευξης της χορδής, δηλαδή να έχει τη μορφή της 117. Θα λύσουμε τις εξισώσεις για $f(\phi) = -\frac{3}{2}(g_s^{-2} + g_s^2) - 2$, ωστόσο προτού συζητήσουμε αναλυτικά αυτό το κεφάλαιο, θα μιλήσουμε για το κεφάλαιο που προηγείται αυτού 5. Η αναλυτική αυτή εισαγωγή στις μη γραμμικές θεωρίες ηλεκτρομαγνητισμού είναι βασική, προκειμένου να αναλύσουμε αυτά τα κεφάλαια.

Έχοντας λοιπόν δικαιολογήσει το ότι οι μη γραμμικές θεωρίες ηλεκτρομαγνητισμού συζευγμένες με βαθμωτά πεδία είναι σημαντικό να μελετηθούν καθότι προέρχονται από θεμελιώδεις θεωρίες, θα ξεκινήσουμε από την απλή Lagrangian ενός βαθμωτού πεδίου, ελάχιστα συζευγμένο με τη βαρύτητα και το μη γραμμικό ηλεκτρομαγνητισμό, στη δράση [6]

$$S = \int d^4x\sqrt{-g}\mathcal{L} = \int d^4x\sqrt{-g}\left(R - \partial^\mu\phi\partial_\mu\phi - 2V(\phi) - P + \alpha P^2 + \beta Q^2\right) \ . \tag{122}$$

Για να λύσουμε τις πεδιακές εξισώσεις, θα υποθέσουμε τη μορφή της μετρικής

$$ds^2 = -b(r)dt^2 + b(r)^{-1}dr^2 + b_1(r)^2 d\Omega^2 \ , \tag{123}$$

ενώ για το πεδίο βαθμίδας θα επιτρέψουμε μόνο ακτινικά μαγνητικά πεδία με την επιλογή

$$A_\mu = (0, 0, 0, Q_m\cos\theta) \ , \tag{124}$$

ενώ θα αφήσουμε το βαθμωτό δυναμικό να υπολογιστεί από τις πεδιακές εξισώσεις. Συνήθως κανείς υποθέτει μια μορφή για το δυναμικό και στη συνέχεια λύνει τις εξισώσεις πεδίου, ωστόσο, αυτό είναι ένα μαθηματικά δύσκολο πρόβλημα, αφενώς καθότι σύμφωνα με το θεώρημα εξάλλειψης ιχνών, ένα θετικό δυναμικό δε μπορεί να υποστηρίξει μια hairy μαύρη τρύπα, αφετέρου, η μορφή των πεδιακών εξισώσεων είναι τέτοια που επιλέγοντας ένα δυναμικό δε μπορούμε να ολοκληρώσουμε αναλυτικά τις υπόλοιπες διαφορικές εξισώσεις. Έτσι, κανείς κάνει συνήθως μια θεώρηση, για παράδειγμα διαλέγοντας μια μορφή για το βαθμωτό πεδίο, η οποία θα πρέπει να είναι τέτοια ώστε το πεδίο να είναι καλά ορισμένο για κάθε $r > 0$, όπως κάναμε και εμείς εδώ, με αποτέλεσμα να μπορούν οι εξισώσεις να ολοκληρωθούν. Εν προκειμένω, η λύση των πεδιακών εξισώσεων είναι η

$$\phi(r) = \frac{1}{\sqrt{2}}\ln\left(1 + \frac{\nu}{r}\right) \ , \tag{125}$$

$$b_1(r) = \sqrt{r(\nu + r)} \ , \tag{126}$$

ενώ η συνάρτηση της μετρικής είναι

$$\begin{aligned} b(r) = {} & c_1 r(\nu + r) + \frac{(2r - c_2)(\nu + 2r) - 4Q_m^2}{\nu^2} + \frac{8\alpha Q_m^4\left(-\nu^2 + 12r^2 + 12\nu r\right)\left(\nu^2 + 3r^2 + 3\nu r\right)}{3\nu^6 r^2(\nu + r)^2} \\ & + \frac{2}{\nu^8}\ln\left(\frac{r}{\nu + r}\right)\Big(- \nu^5 r(c_2 + \nu)(\nu + r) - 2Q_m^2 r(\nu + r)\left(\nu^4 - 24\alpha Q_m^2\right)\ln\left(\frac{r}{\nu + r}\right) \\ & + 48\alpha\nu Q_m^4(\nu + 2r) - 2\nu^5 Q_m^2(\nu + 2r)\Big), \end{aligned} \tag{127}$$

όπου c_1, c_2 είναι δυο σταθερές ολοκήρωσης και ν είναι μια σταθερά που καθορίζει τη συμπεριφορά του βαθμωτού πεδίου, το λεγόμενο βαθμωτό "φορτίο" (scalar charge). Υπολογίζοντας κανείς τη συμπεριφορά της μετρικής σε μεγάλες αποστάσεις, μπορεί να δει πως η σταθερά c_1 σχετίζεται με το αν ο χωρόχρονος είναι ασυμπτωτικά επίπεδος, ή (A)dS, και άρα με την κοσμολογική σταθερά, ενώ η σταθερά c_2 σχετίζεται με τη μάζα της μαύρης τρύπας. Η συνεισφορά του μη γραμμικού ηλεκτρομαγνητισμού είναι πολύ σημαντική στο καθεστώς ισχυρών πεδίων καθώς καθορίζει τη μορφή του χωροχρόνου κοντά στην ιδιομορφία όπως και θα περιμέναμε καθότι τέτοιες διορθώσεις είναι κβαντικής φύσεως, επομένως η συνεισφορά τους θα είναι πολύ σημαντική σε εκείνες τις περιοχές που η κβαντομηχανική γίνεται σημαντική (όπως η ιδιομορφία). Υπολογίζοντας συγκεκριμένα το τετράγωνο του τανυστή του Riemann παίρνουμε

$$R_{\mu\nu\chi\psi}R^{\mu\nu\chi\psi}(r \to 0) \sim \frac{304\alpha^2 Q_m^8}{\nu^8 r^8} - \frac{7520\left(\alpha^2 Q_m^8\right)}{3\nu^9 r^7} + \mathcal{O}\left(\frac{1}{r}\right)^6 \ , \tag{128}$$

όπου είναι εμφανές πως ο ισχυρότερος όρος στην ιδιομορφία προέρχεται από τις διορώσεις του μη γραμμικού ηλεκτρομαγνητισμού.

Όπως προείπαμε, αφήσαμε το βαθμωτό δυναμικό να υπολογιστεί από τις πεδιακές εξισώσεις και η μορφή του είναι η ακόλουθη

$$\begin{aligned}V(\phi) =&\frac{1}{3}\Bigg(2\left(\psi(458\alpha\psi-27)+24\psi\phi^2(1-24\alpha\psi)+\Lambda_{\text{eff}}-36\sqrt{2}\chi\phi\right)\\&+\cosh\left(\sqrt{2}\phi\right)\left(8\psi(6-71\alpha\psi)+24\psi\phi^2(1-24\alpha\psi)+\Lambda_{\text{eff}}-36\sqrt{2}\chi\phi\right)\\&+36\sinh\left(\sqrt{2}\phi\right)\left(2\sqrt{2}\psi\phi(24\alpha\psi-1)+3\chi\right)-4\alpha\psi^2\left(\cosh\left(4\sqrt{2}\phi\right)-14\cosh\left(3\sqrt{2}\phi\right)\right)\\&+2\psi(3-200\alpha\psi)\cosh\left(2\sqrt{2}\phi\right)\Bigg)\,. \end{aligned} \tag{129}$$

Το δυναμικό περιλαμβάνει τριγωνομετρικές συναρτήσεις του βαθμωτού πεδίου ϕ, αλλά και τέσσερις επιπλέον σταθερές. Την κοσμολογική σταθερά, Λ_{eff}, την σταθερά $\chi \equiv m/\nu^3$ η οποία δίνει το λόγο της μάζας της μαύρης τρύπας ως προς το βαθμωτό φορτίο, τη σταθερά $\psi \equiv Q_m^2/\nu^4$ η οποία δίνει το λόγο του μαγνητικού φορτίου προς το βαθμωτό φορτίο της μαύρης τρύπας αλλά και την ΕΗ σταθερά α. Συνεπώς, η μοναδική ελεύθερη παράμετρος του συστήματος είναι το βαθμωτό φορτίο ν της μαύρης τρύπας. Συμπερασματικά, αυτού του είδους οι μαύρες τρύπες περιγράφονται από ένα περιορισμένο παραμετρικό χώρο και ο πρώτος νόμος της θερμοδυναμικής στον οποίο υπακούν δίνεται σχηματικά από τη σχέση

$$\delta\mathcal{M} \sim \mathcal{T}\delta\mathcal{S} + \Phi\delta\nu\,. \tag{130}$$

Ας συζητήσουμε αναλυτικά την ανωτέρω πρόταση, καθώς πρόκειται για κάτι που απουσιάζει από την βιβλιογραφία. Θα χρησιμοποιήσουμε και εδώ τη Χαμιλτονιανή εκδοχή της δράσης μας. Αφού χρησιμοποιούμε μόνο μαγνητικά φορτία, το πεδίο του ηλεκτρομαγνητισμού δε θα έχει ορμή και συνεπώς αφού ο χωρόχρονός μας είναι στατικός και σφαιρικά συμμετρικός, χρησιμοποιώντας το ακόλουθο ansatz για τη μετρική (σε Ευκλείδεια γραφή, έχουμε δηλαδή κάνει τον χρόνο φανταστικό)

$$ds^2 = N(R)^2B(R)d\tau^2 + \frac{dR^2W(R)^2}{B(R)} + R^2d\Omega^2\,, \tag{131}$$

η Χαμιλτονιανή δράση της θεωρίας μας σε Ευκλείδια μορφή είναι

$$\mathcal{I}_E = -\int d\tau \int d^3x NH + \mathcal{B}_E\,, \tag{132}$$

όπου $\mathcal{B}_E$ είναι ένας επιφανειακός όρος ώστε να έχουμε ένα καλώς ορισμένο πρόβλημα μεταβολών $\delta\mathcal{I}_E = 0$, ενώ για το μαγνητικό πεδίο επιλέγουμε τη μορφή $A_\mu = Z(\theta)d\varphi$. Εδω, οι συντεταγμένες παίρνουν τιμές ως εξής: $0 \leq \tau \leq \beta$, $R_h \leq R < \infty$, ενώ τα θ, φ είναι οι σύνηθεις, αζιμουθιακή και αντίστοιχα ζενίθια, γωνίες. Για να αποφευχθεί μια κωνική ιδιομορφία στον ορίζοντα της μελανής οπής θα πρέπει ο Ευκλείδειος χρόνος να είναι περιοδικός με περίοδο

$$\beta \equiv 1/T = \frac{4\pi W(R)}{N(R)B'(R)}\Big|_{R_h}\,, \tag{133}$$

όπου T θα ειναι η θερμοκρασία της μαύρης τρύπας. Εδώ, αποκλειστικά και μόνο για υπολογιστική ευκολία, χρησιμοποιούμε ένα νέο σύστημα συντεταγμένων στο οποίο, η συνάρτηση $b_1(r) \equiv R$ παίζει το λόγο της ακτινικής συντεταγμένης. Μεταβάλοντας τώρα την (132) ως προς τις άγνωστες συναρτήσεις

N, Z, B, ϕ, W παίρνουμε τις εξής εξισώσεις

$$
\begin{aligned}
&2\left(R^7WB' + W^3\left(R^8V + R^4\csc^2(\theta)\left(Z'\right)^2 - R^6 - 2\alpha\csc^4(\theta)\left(Z'\right)^4\right)\right)\\
&+ BR^6\left(W\left(R^2\left(\phi'\right)^2 + 2\right) - 4RW'\right) = 0\ , && (134)\\
&\cot(\theta)Z'(\theta) - Z''(\theta) = 0\ , && (135)\\
&-2WN' + NRW\left(\phi'\right)^2 - 2NW' = 0\ , && (136)\\
&\left(\phi'\right)^2\left(NRWB' + B\left(RWN' - NRW' + 2NW\right)\right) + BNRW\phi'\phi'' + N(-R)W^3V' = 0\ , && (137)\\
&2N\left(R^7B' + W^2\left(R^8V + R^4\csc^2(\theta)\left(Z'\right)^2 - R^6 - 2\alpha\csc^4(\theta)\left(Z'\right)^4\right)\right) +\\
&BR^6\left(2N + 4RN' + N\left(-R^2\right)\left(\phi'\right)^2\right) = 0\ . && (138)
\end{aligned}
$$

Για να βρούμε αυτές τις εξισώσεις, ακυρώσαμε κάποιους επιφανειακούς όρους, συγκεκριμένα τους

$$\left(\frac{8\pi\beta B\delta\phi NR^2\phi'}{W} - \frac{16\pi\beta B\delta WNR}{W^2} + \frac{8\pi\beta\delta BNR}{W}\right)\Bigg|_{R_h}^{\infty}\ , \tag{139}$$

ενώ ακυρώσαμε και επιφανειακούς όρους με αζιμουθιακή εξάρτηση

$$\int dR\left(\frac{8\pi\beta\delta ZNW\csc(\theta)Z'\left(R^4 - 4\alpha\csc^2(\theta)\left(Z'\right)^2\right)}{R^6}\right)\Bigg|_{\theta=0}^{\theta=\pi}\ . \tag{140}$$

Λύνοντας τις ανωτέρω εξισώσεις παίρνουμε τη λύση που αναφέραμε πιο πάνω στις (124),(125),(127),(129) για $r = \frac{1}{2}\left(\sqrt{\nu^2 + 4R^2} - \nu\right)$, ενώ βρίσκουμε πως $N(R) =$ constant, το οποίο χωρίς βλάβη της γενικότητας μπορούμε να θέσουμε ίσο με 1 και $W(R)^2 = 4R^2/(\nu^2 + 4R^2)$. Συνεπώς, η εξίσωση (140) γίνεται

$$\int dR\left(\frac{8\pi\beta\delta ZNW\csc(\theta)Z'\left(R^4 - 4\alpha\csc^2(\theta)\left(Z'\right)^2\right)}{R^6}\right)\Bigg|_{\theta=0}^{\theta=\pi} = 16\pi\beta Q_m\int dR\frac{W(R)\left(R^4 - 4\alpha Q_m^2\right)}{R^6}\delta Q_m\ . \tag{141}$$

Η δράση (132) θα δίνεται απολύτως από τον επιφανειακό όρο , καθώς οι ανωτέρω εξισώσεις επιβάλλουν το μηδενισμό της υπό ολοκλήρωσης ποσότητας. Τώρα, υπενθυμίζοντας πως η μόνη παράμετρος η οποία θα μπορεί να μεταβάλλεται είναι το ν καθώς το βαθμωτό δυναμικό επιβάλει οι μαύρες τρύπες να έχουν πεπερασμένους λόγους μάζας προς βαθμωτό φορτίο και μαγνητικό προς βαθμωτό φορτίο, δηλαδή οι λόγοι $\chi \equiv m/\nu^3$, $\psi \equiv Q_m^2/\nu^4$ είναι φιξαρισμένοι από τη θεωρία, θα έχουμε

$$\delta m = 3\nu^2\delta\nu\chi\ , \tag{142}$$

$$\delta Q_m = 2\delta\nu\nu\sqrt{\psi}\ . \tag{143}$$

Οι μεταβολές των πεδίων στο άπειρο γίνονται

$$\delta B = \delta\nu\left(\frac{4\nu^3\psi}{R^2} - \frac{6\nu^2\chi}{R}\right)\ , \tag{144}$$

$$\delta W = -\frac{\delta\nu\nu}{4R^2}\ , \tag{145}$$

$$\delta\phi = \delta\nu\left(\frac{1}{\sqrt{2}R} - \frac{\nu^2}{8\sqrt{2}R^3}\right)\ , \tag{146}$$

ενώ στον ορίζοντα έχουμε

$$\delta B|_{R_h} = -B'(R_h)\delta R_h \; , \tag{147}$$

$$\delta\phi|_{R_h} = \delta\phi(R_h) - \phi'(R_h)\delta(R_h) \; , \tag{148}$$

$$\delta W|_{R_h} = \delta W(R_h) - W'(R_h)\delta(R_h) \; . \tag{149}$$

Από τη στιγμή που η δράση μας θα δίνεται αποκλειστικά από τον επιφανειακό όρο $\mathcal{I}_E = \mathcal{B}_E$, σκοπός μας είναι να υπολογίσουμε τον επιφανειακό όρο, ώστε να μπορέσουμε να συνδέσουμε την Ευκλείδεια δράση με την ελεύθερη ενέργεια του μεγαλοκανονικού συστήματος, μέσω της προσέγγισης σαγματικού σημείου. Δηλαδή,

$$\mathcal{I}_E \equiv \mathcal{B}_E = \beta\mathcal{F} = \beta\mathcal{M} - \mathcal{S} - \beta\Phi_m Q_m \; , \tag{150}$$

όπου $\mathcal{M}, \mathcal{S}, \Phi_m$ είναι η μάζα, η εντροπία και το μαγνητοστατικό δυναμικό της μαύρης τρύπας. Σύμφωνα με τον Hawking [17], η τιμή της Ευκλείδιας δράσης θα δίνει τις θερμοδυναμικές ιδιότητες της μελανής οπής, για ένα καλά ορισμένο πρόβλημα μεταβολών. δηλαδή όταν, $\delta\mathcal{I}_E = 0$. Η μεταβολή δηλαδή του επιφανειακού όρου $\delta\mathcal{B}_E$ θα πρέπει να είναι τέτοια ώστε να ακυρώνει τους επιφανειακούς όρους που δημιουργήθηκαν όταν υπολογίζαμε τις εξισώσεις κίνησης. Για ευκολία, διαχωρίζουμε τη μεταβολή του επιφανειακού όρου σε δυό μέρη, ένα στο άπειρο και ένα στον ορίζοντα της μαύρης τρύπας,

$$\delta\mathcal{B}_E = \delta\mathcal{B}_E(\infty) + \delta\mathcal{B}_E(R_h) \; . \tag{151}$$

Υπολογίζοντας τώρα την (139) στο άπειρο, μαζί με τη μεταβολή του επιφανειακού μας όρου, βρίσκουμε πως αυτή θα δίνει έναν πεπερασμένο όρο της μορφής

$$-48\pi\beta\delta\nu\nu^2\chi + \delta\mathcal{B}_E(\infty) = 0 \; , \tag{152}$$

ενώ, στον ορίζοντα γεγονότων βρίσκουμε,

$$32\pi^2\delta R_h R_h - 16\pi\beta\delta Q_m Q_m \int dR \frac{W(R)\left(R^4 - 4\alpha Q_m^2\right)}{R^6}\bigg|_{R_h} + \delta\mathcal{B}_E(R_h) = 0 \; , \tag{153}$$

η οποία έκφραση μπορεί αναλόγως να γραφτεί ως

$$4\pi\delta A - \beta\delta Q_m\Phi_m + \delta\mathcal{B}_E(R_h) = 0 \; , \tag{154}$$

όπου έχουμε θέσει $A = 4\pi R_h^2$ και $\Phi_m = -16\pi \int dR W(R)\left(R^4 - 4\alpha Q_m^2\right)/R^6\big|_{R_h}$. Θεωρώντας το μεγαλοκανονικό σύνολο, κρατώντας δηλαδή σταθερή τη θερμοκρασία και το μαγνητοστατικό δυναμικό, μπορούμε να ολοκληρώσουμε τις δύο ανωτέρω σχέσεις και να πάρουμε

$$\mathcal{B}_E(\infty) = 16\pi\beta\nu^3\chi \; , \tag{155}$$

$$\mathcal{B}_E(R_h) = -4\pi A - \beta Q_m\Phi_m \; , \tag{156}$$

επομένως, η τιμή της Ευκλείδειας δράσης όταν οι εξισώσεις κίνησης ισχύουν δίνεται από

$$\mathcal{I}_E = 16\pi\beta\nu^3\chi - 4\pi A - \beta Q_m\Phi_m \; , \tag{157}$$

και συνεπώς, συγκίνοντας με την (150) έχουμε

$$\mathcal{M} = 16\pi\nu^3\chi \; , \tag{158}$$

$$\mathcal{S} = 4\pi A \; , \tag{159}$$

ως τη διατηρούμενη μάζα και εντροπία της μελανής οπής αντίστοιχα, ενώ θυμίζουμε πως $Q_m = \sqrt{\psi}\nu^2$. Τελικά, ο πρώτος νόμος της θερμοδυναμικής που θα υπακούει αυτή η μαύρη τρύπα θα δίνεται από

$$\delta\mathcal{M} = T\delta\mathcal{S} + 2\nu\sqrt{\psi}\Phi_m\delta\nu \ . \tag{160}$$

Ένα ενδιαφέρον χαρακτηριστικό που παρατηρήσαμε είναι το γεγονός ότι, η παράμετρος α επεισέρχεται στην έκφραση του δυναμικού. Όπως έχουμε αναφέρει, ένα θετικό δυναμικό δεν δύναται να παραβιάσει το θεώρημα εξάλειψης ιχνών σύμφωνα με τον Bekenstein [18]. Για την παραβίαση του θεωρήματος ένα αρνητικό δυναμικό είναι απαραίτητο. Απεικονίζοντας στο ακόλουθο διάγραμμα την αδιάστατη ποσότητα $V(\phi)/\chi$ συναρτήσει του ϕ, μπορούμε να δούμε πως η παράμετρος α του μη γραμμικού ηλεκτρομαγνητισμού συμβάλλει στην παραβίαση του θεωρήματος εξάλειψης ιχνών καθώς τοποθετεί το βαθμωτό δυναμικό στην αρνητική περιοχή σε όλο το χωρόχρονο, τόσο κοντά στην ιδιομορφία $\phi \gg 1$, όσο και κοντά στο κενό της θεωρίας $r \to \infty$, $\phi \to 0$. Ένα ακόμα χαρακτηριστό του δυναμικού είναι ότι για

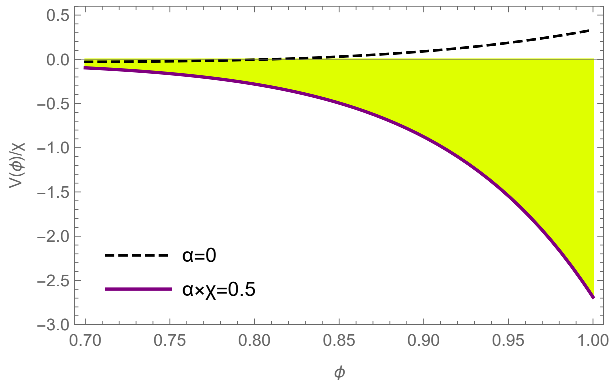


Figure 7: Η αδιάστατη ποσότητα $V(\phi)/\chi$ συναρτήσει του ϕ, για μελανές οπές με γραμμικό $\alpha = 0$ και μη γραμμικό $\alpha \times \chi = 0.5$ ηλεκτρομαγνητισμό σε ασυμπτωτικά επίπεδο χωροχρόνο.

ασυμπτωτικά επίπεδο χωροχρόνο, το δυναμικό συμπεριφέρεται ως ϕ^5. Αυτό είναι ένα γενικό χαρακτηριστικό των δυναμικών που παραβιάζουν το θεώρημα εξάλειψης ιχνών σε ασυμπτωτικά επίπεδο χωροχρόνο και μπορούμε να το δούμε ως ακολούθως. Αρχικά υποθέτουμε ότι μπορούμε να αναπτύξουμε το δυναμικό στο κενό της θεωρίας σε μορφή Taylor $V(\phi) \sim \zeta\phi^n$, όπου, το ζ είναι μια σταθερά και n ένας ακέραιος αριθμός. Επομένως, η εξίσωση του βαθμωτού πεδίου για έναν ασυμπτωτικά επίπεδο χωροχρόνο $g_{tt} = g_{rr} = 1$ σε σφαιρική συμμετρία παίρνει τη μορφή

$$\zeta(-n)\phi(r)^{n-1} + \frac{2\phi'(r)}{r} + \phi''(r) = 0 \ . \tag{161}$$

Για $n = 2$ παίρνουμε ένα Yukawa πεδίο $\phi \sim \frac{e^{-\sqrt{2}\sqrt{\zeta}r}}{r}$, ενώ αν επιθυμούμε το πεδίο να συμπεριφέρεται ασυμπτωτικά σαν $1/r$ τότε $n = 5$. Συνεπώς, ένα δυναμικό που θα μπορει να υποστηρίξει μια hairy μαύρη τρύπα, με ένα βαθμωτό πεδίο που ασυμπτωτικά συμπεριφέρεται σαν $1/r$ θα πρέπει στο κενό της θεωρίας να συμπεριφέρεται σαν ϕ^5. Αυτές οι μαύρες τρύπες μπορούν να περιγράφονται από έναν, δύο ή και τρεις ορίζοντες, για συγκεκριμένους συνδυασμούς τον παραμέτρων. Αξίζει να αναφερθούμε στην περίπτωση των τριών οριζόντων καθώς σε αυτή την περίπτωση έχουμε το σενάριο της ύπαρξης μιας μαύρης τρύπας μέσα σε μαύρη τρύπα, καθώς ο εσώτατος και ο εξώτατος ορίζοντας θα είναι ορίζοντες γεγονότων. Οι ασυμπτωτικά AdS μαύρες τρύπες παρουσιάζουν μια μετάβαση φάσης αλά Hawking και Page, με τις μικρές μαύρες τρύπες να είναι θερμοδυναμικά ασταθείς καθώς έχουν αρνητική θερμοχωρητικότητα, ενώ οι μεγάλες μαύρες τρύπες είναι θερμοδυναμικά ευσταθείς. Επιπρόσθετα, οι ενεργειακές συνθήκες παραβιάζονται από το βαθμωτό πεδίο, όπως κανείς θα περίμενε, αφού το δυναμικό της θεωρίας είναι τέτοιο ώστε να παραβιάζει το θεώρημα εξάλειψης ιχνών. Στο άρθρο [9] μελετήσαμε τις τροχές έμαζων και άμαζων σωματιδίων σε αυτό το χωρόχρονο. Είδαμε πως μπορούμε να έχουμε ευσταθείς κυκλικές τροχιές, αλλά και πλανητικές και ασταθείς τροχιές και για τα έμαζα και άμαζα σωματίδα. Συγκεκριμένα, τα άμαζα σωματίδια δεν κινούνται στον χωροχρόνο της μελανής οπής, αλλά κινούνται σε μια τροποποιημένη γεωμετρία, λόγω της αλληλεπίδρασης φωτονίων-φωτονίων που επιβάλλει ο μη γραμμικός ηλεκτρομαγνητισμός. Εν κατακλείδι, παρατηρήσαμε πως στο καθεστώς ισχυρών πεδίων (για μεγάλα μαγνητικά φορτία) τα αποτελέσματά μας είναι σε συμφωνία ενός διαστήματος εμπιστοσύνης με τα αποτελέσματα του Τηλεσκοπίου Ορίζοντα Γεγονότων.

Στο τελευταίο κεφάλαιο της παρούσας διατριβής 6, μελετήσαμε τη δράση 120, και λύσαμε τις πεδιακές εξισώσεις για τη συνάρτηση σύζευξης [11]

$$f(\phi) = -\frac{3}{2}(g_s^{-2} + g_s^2) - 2\,, \tag{162}$$

όπου $g_s = \exp(\phi)$ είναι η συνάρτηση σύζευξης του πεδίου dilaton. Προηγουμένως δικαιολογήσαμε επαρκώς αυτή την επιλογή για τη συνάρτηση σύζευξης, η οποία περιέχει και έναν σταθερό όρο, τον οποίο δε μπορούμε να προσδιορίσουμε επακριβώς, καθώς πρόκειται για ένα πρόβλημα το οποίο βασίζεται στην εκάστοτε θεωρία χορδών που μελετάει κανείς. Λύνοντας τις εξισώσεις βρίσκουμε το ακόλουθο στοιχείο μήκους

$$ds^2 = -B(R)dt^2 + \frac{[W(R)]^2 dR^2}{B(R)} + R^2 d\Omega^2\,, \tag{163}$$

όπου

$$B(R) = 1 - \frac{4M^2}{Q_m^2 + \sqrt{Q_m^4 + 4M^2R^2}} - \frac{2(\alpha-\beta)Q_m^4}{R^6}\,, \tag{164}$$

$$[W(R)]^2 = \frac{4M^2R^2}{Q_m^4 + 4M^2R^2}\,, \tag{165}$$

$$\phi(R) = -\frac{1}{2}\ln\left(\frac{\sqrt{Q_m^4 + 4M^2R^2} - Q_m^2}{\sqrt{Q_m^4 + 4M^2R^2} + Q_m^2}\right). \tag{166}$$

Για το πεδίο βαθμίδας επιλέξαμε μόνο ακτινικά μαγνητικά πεδία. Ασυμπτωτικά η συνάρτηση $B(R)$ συμπεριφέρεται σαν

$$B(R\to+\infty) = 1 - \frac{2M}{R} + \frac{Q_m^2}{R^2} - \frac{Q_m^4}{4MR^3} + \frac{Q_m^8}{64M^3R^5} - \frac{2(\alpha-\beta)Q_m^4}{R^6} + \mathcal{O}(1/R^7)\,, \tag{167}$$

$$B(R\to 0) = -\frac{2(\alpha-\beta)Q_m^4}{R^6} + \left(1 - \frac{2M^2}{Q_m^2}\right) + \mathcal{O}(R^2)\,. \tag{168}$$

όπου είναι ξεκάθαρο πως (όπως θα περιμέναμε) οι διορθώσεις του μη γραμμικού ηλεκτρομαγνητισμού είναι πολύ σημαντικές κοντά στην ιδιομορφία, ενώ σε μεγάλες αποστάσεις, ο χωρόχρονος δε μπορεί να ξεχωρίσει από μια μαγνητικά φορτισμένη μαύρη τρύπα. Όταν το πρόσημο του όρου $\alpha-\beta$ είναι θετικό οι όροι του μη γραμμικού ηλεκτρομαγνητισμού προσδίδουν μια ελκτική δύναμη στο χωροχρόνο, ενώ όταν $\alpha-\beta<0$ προσδίδουν μια απωστική δύναμη. Όταν $\alpha-\beta>0$ η μαύρη τρύπα θα έχει μεγαλύτερη ακτίνα για τον ορίζοντα γεγονότων από ότι η GMGHS [32, 33] μαύρη τρύπα και συνεπώς θα είναι θερμοδυναμικά προτιμητέα. Ένα επιπλέον πολύ ενδιαφέρον χαρακτηριστικό είναι το γεγονός ότι για $\alpha - \beta > 0$ καθώς το μαγνητικό φορτίο αυξάνει, ο ορίζοντας μικραίνει μέχρι ένα συγκεκριμένο όριο, ενώ από μια τιμή του Q_m και πάνω ο ορίζοντας μεγαλώνει. Άρα θα υπάρχει πάντα μια τιμή του ορίζοντα γεγονότων η οποία θα αντιστοιχεί σε δύο διαφορετικά μαγνητικά φορτία Q_m. Δηλαδή, η δράση 120 θα παράγει μαύρες τρύπες που μπορεί να έχουν τον ίδιο ορίζοντα γεγονότων, για διαφορετικές τιμές των φορτίων της μαύρης τρύπας.

Για να το οπτικοποιήσουμε αυτό, στο ακόλουθο σχήμα απεικονίζουμε τη συνάρτηση της μετρικής για διάφορες τιμές του λόγου του μαγνητικού φορτίο προς τη μάζα της μαύρης τρύπας. Είναι ξεκάθαρο πως

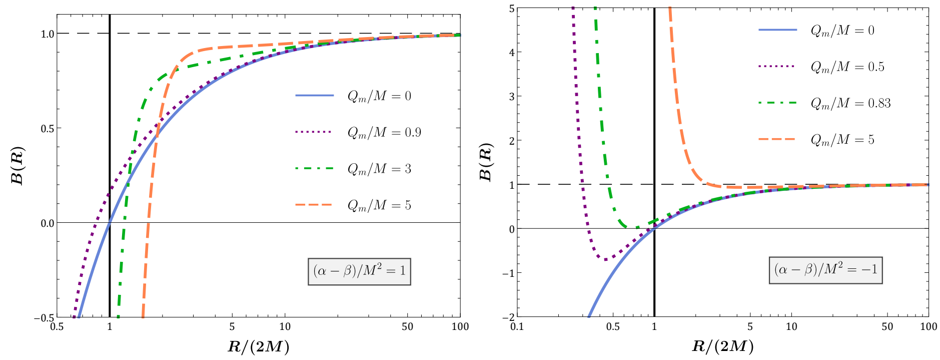


Figure 8: Η συνάρτηση $B(R)$ για $\alpha - \beta > 0$ (αριστερά) και $\alpha - \beta < 0$ (δεξια).

οι λύσεις με $\alpha - \beta > 0$ συμπεριφέρονται παρόμοια με τη Schwarzchild λύση και καθώς μεγαλώνει ο λόγος του μαγνητικού φορτίου προς τη μάζα της μελανής οπής, ο ορίζοντας μεγαλώνει. Στην περίπτωση που $\alpha - \beta < 0$ μπορούμε να έχουμε έως και δύο ορίζοντες ενώ καθώς ο λόγος του μαγνητικού φορτίου προς τη μάζα της μελανής οπής αυξάνει παίρνουμε γυμνές ιδιομορφίες όπως στην περίπτωση της Reissner-Nordstrom μαύρης τρύπας.

Στη συνέχεια απεικονίζουμε την ενδιαφέρουσα περίπτωση που μπορεί κανείς να παρατηρήσει στην περίπτωση $\alpha - \beta > 0$ στην οποία και μπορούμε να πάρουμε τον ίδιο ορίζοντα γεγονότων για διαφορετικές τιμές του λόγου του μαγνητικού φορτίου προς τη μάζα της μελανής οπής. Το σενάριο απεικονίζεται στο ακόλουθο διάγραμμα

Συνεπώς, η θεωρία μας είναι ικανή να δώσει μαύρες τρύπες με την ίδια ακτίνα ορίζοντα για διαφορετικές τιμές του λόγου του μαγνητικού φορτίου προς τη μάζα τους. Ωστόσο, όπως θα δούμε παρακάτω, μπορούμε να ξεχωρίσουμε αυτές τις λύσεις υπολογίζοντας τη θερμοκρασία τους.

Περνάμε τώρα στην ανάλυση του τανυστή ορμής ενέργειας που δίνει αυτές τις μελανές οπές συζητώντας τις ενεργειακές συνθήκες. Στο φυσικό σύστημα συντεταγμένων, (t, R, θ, φ) ο τανυστής ορμής-ενέργειας περιγράφεται από ένα ανισότροπο ρευστό και σε συναλλοίωτη μορφή μπορεί να γραφεί ως

$$T^{\mu\nu} = (\rho_E + p_\theta)u^\mu u^\nu + (p_R - p_\theta)n^\mu n^\nu + p_\theta g^{\mu\nu} \,. \tag{169}$$

Στα ανωτέρω ρ_E είναι η πυκνότητα ενέργειας του ρευστού όπως μετράται από έναν παρατηρητή ο οποίος κινείται μαζί με το ρευστό p_R είναι η ακτινική πίεση, p_θ είναι η επιφανειακή πίεση, ενώ u^μ και n^μ είναι η χρονοειδής τετραταχύτητα και ένα χωροειδές μοναδιαίο διάνυσμα κάθετο στο u^μ και στις γωνιακές κατευθύνσεις. Τα τετραδιανύσματα u^μ και n^μικανοποιούν τις συνθήκες:

$$u^\mu = u(R)\,\delta^\mu_0 \,, \quad u^\mu u^\nu g_{\mu\nu} = -1 \,, \tag{170}$$

$$n^\mu = n(R)\,\delta^\mu_1 \,, \quad n^\mu n^\nu g_{\mu\nu} = 1 \,. \tag{171}$$

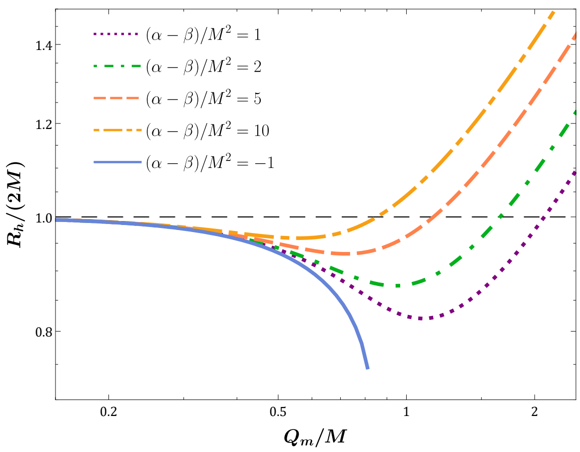


Figure 9: Ο ορίζοντας γεγονότων συναρτήσει του λόγου του μαγνητικού φορτίου προς τη μάζα της μελανής οπής.

Για τη θεωρία μας, μπορεί κανείς να υπολογίσει

$$\rho_E = -T^t{}_t = \frac{B(R)}{[W(R)]^2}\left(\frac{d\phi}{dR}\right)^2 + \frac{Q_m^2}{R^4}\,e^{-2\phi} + \frac{2(\alpha-\beta)Q_m^4}{R^8}\,f(\phi)\,, \tag{172}$$

$$p_R = T^R{}_R = \frac{B(R)}{[W(R)]^2}\left(\frac{d\phi}{dR}\right)^2 - \frac{Q_m^2}{R^4}\,e^{-2\phi} - \frac{2(\alpha-\beta)Q_m^4}{R^8}\,f(\phi)\,, \tag{173}$$

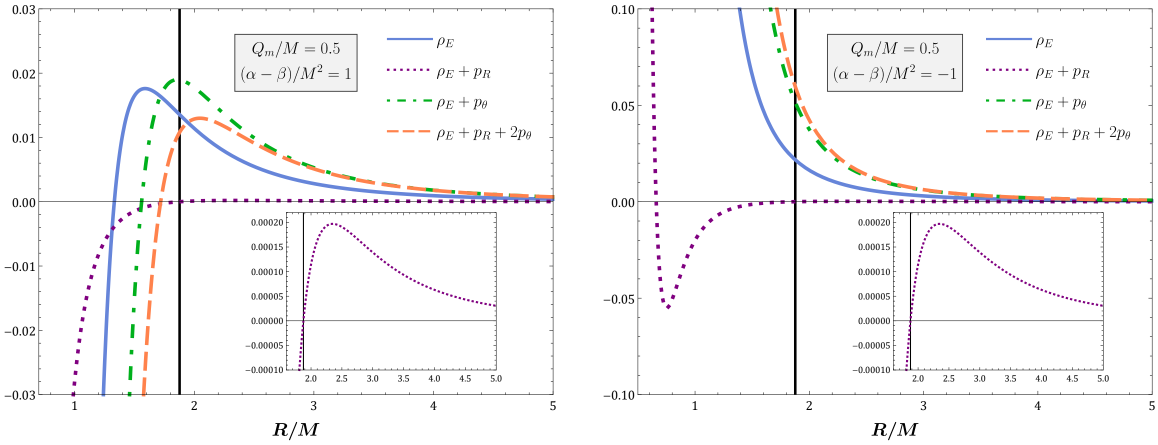
$$p_\theta = T^\theta{}_\theta = T^\varphi{}_\varphi = -\frac{B(R)}{[W(R)]^2}\left(\frac{d\phi}{dR}\right)^2 + \frac{Q_m^2}{R^4}\,e^{-2\phi} + \frac{6(\alpha-\beta)Q_m^4}{R^8}\,f(\phi)\,. \tag{174}$$

Για το ανισότροπο ρευστό, οι ενεργειακές συνθήκες αποκτούν τις ακόλουθες σχέσεις

- Φωτοειδής ενεργειακή συνθήκη (NEC): $\rho_E + p_R \geq 0 \quad \& \quad \rho_E + p_\theta \geq 0\,,$
- Ασθενής ενεργειακή συνθήκη (WEC): $\text{NEC} \quad \& \quad \rho_E \geq 0\,,$
- Ισχυρή ενεργειακή συνθήκη (SEC): $\text{NEC} \quad \& \quad \rho_E + p_R + 2p_\theta \geq 0\,.$

Στα προηγούμενα διαγράμματα απεικονίσαμε τις ενεργειακές συνθήκες, και συγκεκριμένα τις ποσότητες ρ_E, $\rho_E + p_R$, $\rho_E + p_\theta$, και $\rho_E + p_R + 2p_\theta$ ως συναρτήσεις του λόγου R/M, ενώ επιλέξαμε $Q_m/M = 0.5$, ενώ η αδιάστατη ποσότητα της διαφοράς των σταθερών της θεωρίας $(\alpha-\beta)/M^2$ παίρνει τις τιμές 1 και -1, αντίστοιχα. Είναι εμφανές από τα διαγράμματα πως οι ανωτέρω ποσότητες παραμένουν θετικές πάνω στον ορίζοντα γεγονότων της μαύρης τρύπας αλλά και έξω από αυτή, και συνεπώς όλες οι ενεργειακές συνθήκες ικανοποιούνται.

Αυτά τα αποτελέσματα συνεπώς, δηλώνουν πως μπορούμε να παραβιάσουμε τη σύγχρονη εκδοχή του θεωρήματος εξάλειψης ιχνών [34, 35] στο πνεύμα του [36]. Συνεπώς, μπορούμε να έχουμε ένα δυναμικό πεδίο dilaton έξω από μια μελανή οπή, και συνεπώς δευτερεύον hair χωρίς να παραβιάζουμε τις ενεργειακές συνθήκες. Αυτό το φαινόμενο, μπορεί να γίνει κατανοητό ως μια συνέπεια του γεγονότος πως ο τανυστής ορμής ενέργειας της θεωρίας μας είναι τέτοιος ώστε η επιφανειακή πίεση $(p_\theta = T^\theta_\theta)$



(a) (b)

Figure 10: Οι ενεργειακές συνθήκες για $Q_m/M = 0.5$ και (a) $\alpha - \beta > 0$ (b) $\alpha - \beta < 0$. Οι κάθετες μαύρες γραμμές απεικονίζουν τον ορίζοντα γεγονότων.

υπερισχύει απέναντι στην ακτινική πίεση ($p_R = T^R_R$) έξω από τον ορίζοντα. Συνεπώς η ακόλουθη ποσότητα είναι θετική έξω από τον ορίζοντα της μαύρης τρύπας

$$\mathcal{G} - \mathcal{J} = T^\theta_\theta - T^R_R > 0\,, \tag{175}$$

όπου $\mathcal{G} = \rho_E + T^\theta_\theta$ και $\mathcal{J} \equiv \rho_E + T^R_R$. Κανείς μπορεί να δει πως η ανωτέρω σχέση προέρχεται από τη φωτοειδή ενεργειακή συνθήκη. Συνεπώς, η λύση μας αποτελεί ένα πολύ απλό παράδειγμα παραβίασης του no-hair θεωρήματος στο πλαίσιο μιας θεωρίας που εμπεριέχει όρους πρώτης τάξης διόρθωσης στη σταθερά σύζευξης της χορδής.

Περνάμε τώρα στη θερμοδυναμική μελέτη του συστήματός μας. Συγκεκριμένα χρησιμοποιήσαμε επιχειρήματα κβαντικής βαρύτητας και την Ευκλείδεια δράση. Στο πλαίσιο της προσέγγισης σαγματικού σημείου, μπορεί κανείς να υποστηρίξει πως η Ευκλείδια δράση συνεισφέρει τα μέγιστα και άρα αυτή θα σχετίζεται με την ελεύθερη ενέργεια του στατιστικού συνόλου. Εν προκειμένω, για να υπολογίσουμε την Ευκλείδεια δράση, θεωρήσαμε το στοιχείο μήκους

$$ds^2 = [N(R)]^2 B(R) d\tau^2 + \frac{[W(R)]^2 dR^2}{B(R)} + R^2 d\Omega^2\,, \tag{176}$$

ενώ θέσαμε για το τετραδιάνυσμα του πεδίου βαθμίδας

$$\mathcal{A}_\mu = (0, 0, 0, A(\theta))\,.$$

Εδώ, ο Ευκλείδειος είναι περιοδικός και συγκεκριμένα παίρνει τιμές μεταξύ $0 \leq \tau \leq \beta_\tau$, ενώ, η ακτινική συνιστώσα $R \in [R_h, +\infty)$. Για να υπολογίσουμε τη θερμοκρασία πραγματοποιούμε έναν υπολογισμό παρόμοιο με αυτόν που έγινε σε ένα προηγούμενο άρθρο μας [7]. Για το σκοπό αυτό, αρχικά αγνοούμε το γωνιακό κομμάτι και πραγματοποιούμε ένα ανάπτυγμα σε σειρά γύρω από τον ορίζοντα. Συνεπώς έχουμε πλέον ένα διδιάστατο στοιχείο μήκους, το οποίο είναι το στοιχείο μήκους του διδιάστατου χώρου και εκπεφρασμένο σε πολικές συντεταγμένες δίνεται από τη σχέση $dS = d\hat{R}^2 + \hat{R}^2 d\Theta^2$. Συνεπώς έχουμε

$$d\hat{R}^2 = \frac{W(R_h)^2}{B'(R_h)(R - R_h)} dR^2 \,, \tag{177}$$

$$B'(R_h)(R - R_h) d\tau^2 = \hat{R}^2 d\Theta^2 \,. \tag{178}$$

Η συντεταγμένη Θ είναι περιοδική με περιοδικότητα 2π το οποίο σημαίνει πως το τ θα είναι επίσης περιοδικό με μια περίοδο β_τ που θα δίνεται από τη σχέση:

$$\beta_\tau = \frac{1}{T} = \left.\frac{4\pi W(R)}{N(R)B'(R)}\right|_{R_h} \,, \tag{179}$$

όπου T είναι η θερμοκρασία της μελανής οπής. Για λόγους αυτοσυνέπειας, ελέγξαμε πως η θερμοκρασία δεν εξαρτάται από την επιλογή του συστήματος συντεταγμένων καθώς πρόκειται για μια βαθμωτή ποσότητα.

Εισαγάγοντας τώρα έναν επιφανειακό όρο στη θεωρία μας τον $\mathcal{B}_E$ έτσι ώστε να έχουμε ένα καλώς ορισμένο πρόβλημα μεταβολών $\delta\mathcal{I}_E = 0$, μπορούμε να υπολογίσουμε την Ευκλείδεια δράση συναρτήσει των αγνώστων συναρτήσεων του συστήματός μας ως

$$\mathcal{I}_E = \frac{2\pi\beta_\tau}{16\pi} \int_0^\pi d\theta \int_{R_h}^\infty dR \left[-NR^2 W \sin\theta \, \mathcal{L}(R,\theta)\right] + \mathcal{B}_E \,. \tag{180}$$

Εδώ, το $\mathcal{L}$ υποδηλώνει τη Lagrangian του συστήματος η οποία είναι συνάρτηση των συντεταγμένων R, θ. Αφού ακυρώσουμε επιφανειακούς όρους, η Ευκλείδεια δράση παίρνει τη μορφή

$$\mathcal{I}_E = \beta_\tau \int_0^\pi d\theta \int_{R_h}^\infty dR \, \hat{\mathcal{L}}(Q^i, \partial_\mu Q^i) + \mathcal{B}_E \,, \tag{181}$$

όπου το $Q^i = \{N(R), W(R), B(R), \phi(R), A(\theta)\}$ και το $\hat{\mathcal{L}}(Q^i, \partial_\mu Q^i)$ θα δίνονται από

$$\begin{aligned}\hat{\mathcal{L}}(Q^i, \partial_\mu Q^i) = \frac{N\sin\theta}{4W^2R^6}\Bigg[&WR^7B' + W^3\left(2f(\phi)(\alpha-\beta)\frac{(\partial_\theta A)^4}{\sin^4\theta} + e^{-2\phi}R^4\frac{(\partial_\theta A)^2}{\sin^2\theta} - R^6\right)\\ &+ BR^6\left(WR^2\phi'^2 - 2RW' + W\right)\Bigg] \,.\end{aligned} \tag{182}$$

Σύμφωνα με τον ADM φορμαλισμό, πρέπει να μεταβάλουμε την ανωτέρω Ευκλείδεια δράση ως προς καθένα από τα δυναμικά πεδία Q^i για να πάρουμε τις πεδιακές εξισώσεις. Πραγματοποιώντας τη μεταβολή παίρνουμε τις εξισώσεις Euler-Lagrange

$$\frac{\partial\hat{\mathcal{L}}}{\partial Q^i} - \partial_\mu\left(\frac{\partial\hat{\mathcal{L}}}{\partial(\partial_\mu Q^i)}\right) = 0 \,. \tag{183}$$

Επιβάλλοντας την ανωτέρω εξίσωση για το δυναμικό πεδίο $Q^1 = N(R)$ μπορούμε να δούμε ότι ο πρώτος όρος μηδενίζεται ταυτοτικά, ενώ $\partial\hat{\mathcal{L}}/\partial N = \hat{\mathcal{L}}/N$. Συνεπώς, η εξίσωση για το $N(R)$ υποδηλώνει πως $\hat{\mathcal{L}} = 0$ το οποίο με τη σειρά του επιβάλει $\mathcal{I}_E = \mathcal{B}_E$. Συνεπώς, όταν ισχύουν οι πεδιακές εξισώσεις (on-shell) όλη η πληροφορία για τη φυσική από την Ευκλείδεια δράση θα κωδικοποιηθεί στον επιφανειακό όρο. Υπολογίζοντας τώρα και τις υπόλοιπες πεδιακές εξισώσεις, μπορεί κανείς να δείξει πως η συνάρτηση N είναι μια σταθερή συνάρτηση την οποία μπορούμε να θέσουμε ίση με τη μονάδα χωρίς βλάβη της γενικότητας.

Σημαντικό είναι να τονίσουμε πως κατά την εξαγωγή των πεδιακών εξισώσεων ακυρώσαμε κάποιους επιφανειακούς όρους και συγκεκριμένα τους

$$\beta_\tau \left(\frac{R}{2W}\delta B + \frac{2BR^2\phi'}{W}\,\delta\phi - \frac{BR}{W^2}\,\delta W \right) \Bigg|_{R_h}^{\infty} , \tag{184}$$

και

$$\beta_\tau \int_{R_h}^{\infty} dR \left(\frac{We^{-2\phi}(\partial_\theta A)}{2R^2\sin\theta} + \frac{2(\alpha-\beta)Wf(\phi)(\partial_\theta A)^3}{R^6\sin^3\theta} \right) \delta A \Bigg|_{\theta=0}^{\theta=\pi} . \tag{185}$$

Η μεταβολή του επιφανειακού όρου $\delta\mathcal{B}_E$ θα είναι τέτοια έτσι ώστε να έχουμε ένα καλά ορισμένο πρόβλημα μεταβολών $\delta\mathcal{I}_E = 0$. Η μεταβολή του πεδίου βαθμίδας θα δίνεται από $\delta A = (\delta Q_m)\cos\theta$, ακαι αντικαθστώντας πίσω στην ανωτέρω εξίσωση μπορούμε απευθείας να ολοκληρώσουμε

$$\beta_\tau \, \frac{Q_m \left\{ \sqrt{4M^2R_h^2 + Q_m^4}\left[R_h^4 - 4(\alpha-\beta)Q_m^2\right] - R_h^4 Q_m^2 \right\}}{2MR_h^6}\,\delta Q_m . \tag{186}$$

Η μεταβολή των υπόλοιπων δυναμικών πεδίων σε μεγάλες αποστάσεις θα δίνεται από

$$\delta B = -\frac{2\delta M}{R} + \mathcal{O}\left(\frac{1}{R^2}\right) , \tag{187}$$

$$\delta\phi = \frac{Q_m}{MR}\delta Q_m - \frac{Q_m^2}{2M^2R}\delta M + \mathcal{O}\left(\frac{1}{R^3}\right) , \tag{188}$$

$$\delta W = \frac{Q_m^4}{4M^3R^2}\delta M - \frac{Q_m^3}{2M^2R^2}\delta Q_m + \mathcal{O}\left(\frac{1}{R^3}\right) , \tag{189}$$

ενώ στον ορίζοντα θα έχουμε

$$\delta B|_{R_h} = -B'(R_h)\delta R_h , \tag{190}$$

$$\delta\phi|_{R_h} = \delta\phi(R_h) - \phi'(R_h)\delta(R_h) , \tag{191}$$

$$\delta W|_{R_h} = \delta W(R_h) - W'(R_h)\delta(R_h) . \tag{192}$$

Σημειώνουμε πως οι παράμετροι α, β είναι σταθερές της θεωρίας και δεν επιτρέπεται η μεταβολή τους, ενώ η μάζα M και το μαγνητικό φορτίο Q_m είναι σταθερές ολοκλήρωσης και η μεταβολή τους επιτρέπεται.

Όπως προ-αναφέραμε, για να έχουμε ένα καλώς ορισμένο πρόβλημα μεταβολών θα πρέπει $\delta\mathcal{I}_E = 0$. Για τη διευκόλυνσή μας θα πρέπει να αποσυνθέσουμε ση μεταβολή του επιφανειακού όρου σε δυο κομμάτια, ένα στο άπειρο και ένα στον ορίζοντα, δηλαδή

$$\delta\mathcal{B}_E = \delta\mathcal{B}_E(\infty) + \delta\mathcal{B}_E(R_h) . \tag{193}$$

Υπολογίζοντας τη συνεισφορά των επιφανειακών ορων στο άπειρο και θεωρώντας τη μεταβολή του $\delta\mathcal{B}(\infty)$ βρίσκουμε πως ένας όρος μηδενικής τάξης επιβιώνει και συνεπώς έχουμε

$$\frac{\beta_\tau}{2}R\left(\delta B - 2\delta W\right) + \delta\mathcal{B}_E(\infty) = 0 \Rightarrow \tag{194}$$

$$\delta\mathcal{B}_E(\infty) = \beta_\tau\,\delta M . \tag{195}$$

Από την άλλη μεριά, στον ορίζοντα γεγονότων έχουμε (λαμβάνοντας υπόψη και τη συνεισφορά του μαγνητοστατικού δυναμικού)

$$2\pi R_h \delta R_h + \beta_\tau \frac{Q_m \left\{ \sqrt{4M^2 R_h^2 + Q_m^4} \left[R_h^4 - 4(\alpha - \beta) Q_m^2 \right] - R_h^4 Q_m^2 \right\}}{2MR_h^6} \delta Q_m + \delta \mathcal{B}_E(R_h) = 0 \ , \quad (196)$$

το οποίο μπορεί να γραφεί ισοδύναμα ως

$$\frac{\delta A}{4} + \beta_\tau \Phi_m \delta Q_m + \delta \mathcal{B}_E(R_h) = 0 \ , \quad (197)$$

όπου χρησιμοποιήσαμε το γεγονός πως η επιφάνεια της μαύρης τρύπας θα δίνεται από $A = 4\pi R_h^2$, ενώ ορίσαμε το μαγνητοστατικό δυναμικό ως εξής

$$\Phi_m = \frac{Q_m \left\{ \sqrt{4M^2 R_h^2 + Q_m^4} \left[R_h^4 - 4(\alpha - \beta) Q_m^2 \right] - R_h^4 Q_m^2 \right\}}{2MR_h^6} \ . \quad (198)$$

Θεωρώντας τώρα το μεγαλοκανονικό σύνολο, κρατάμε τη θερμοκρασία σταθερή και το μαγνητοστατικό δυναμικό του συστήματος και συνεπώς, μπορούμε να ολοκληρώσουμε τις ανωτέρω σχέσεις, καταλήγοντας στα

$$\mathcal{B}_E(\infty) = \beta_\tau M \ , \quad (199)$$

$$\mathcal{B}_E(R_h) = -\frac{A}{4} - \beta_\tau \Phi_m Q_m \ . \quad (200)$$

Συνεπώς, η τιμή της Ευκλείδειας δράσης θα δίνεται από την ακόλουθη σχέση

$$\mathcal{I}_E = \beta_\tau M - \frac{A}{4} - \beta_\tau \Phi_m Q_m \ , \quad (201)$$

και από τη στιγμή που η Ευκλείδεια δράση θα σχετίζεται με την ελεύθερη ενέργεια $\mathcal{G}$ του συστήματος μέσα από τη σχέση $\mathcal{I}_E = \beta_\tau \mathcal{G} = \beta_\tau \mathcal{M} - S - \beta_\tau \Phi_m Q_m$ μπορούμε να ταυτοποιήσουμε, μέσω σύγκρισης την διατηρούμενη μάζα της μαύρης τρύπας και την εντροπία της μελανής οπής ως ακολούθως

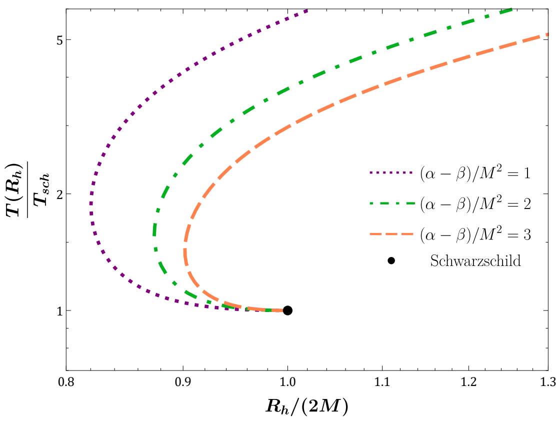
$$\mathcal{M} = M \ , \quad (202)$$

$$S = A/4 \ . \quad (203)$$

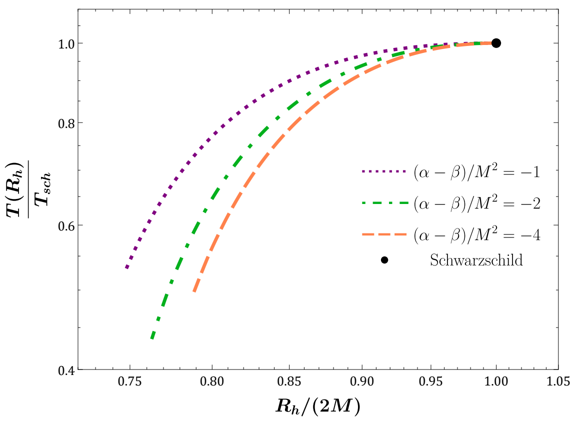
Τέλος, ο πρώτος νόμος της θερμοδυναμικής θα ικανοποιείται από κατασκευής και θα λαμβάνει τη μορφή

$$\delta M = T\delta S + \Phi_m \delta Q_m \ . \quad (204)$$

Περνάμε πλέον στην ανάλυση της θερμοκρασίας της μελανής οπής. Στο διάγραμμα 11, απεικονίζουμε τη θερμοκρασία της μελανής οπής συναρτήσει της αδιάστατης ποσότητας $R_h/(2M)$. Ταυτόχρονα, έχουμε κάνει αδιάστατη τη θερμοκρασία των λύσεών μας, διαιρώντας τη με τη θερμοκρασία της λύσης Schwarzchild. Τα διαγράμματα 11 υπολογίστηκαν σύμφωνα με την ακόλουθη λογική: Για καθεμιά τιμή της αδιάστατης ποσότητας $(\alpha - \beta)/M^2$ και του λόγου Q_m/M, υπολογίζουμε αριθμητικά την τιμή του λόγου $R_h/(2M)$. Χρησιμοποιώντας τώρα την έκφραση για την περιοδικότητα του Ευκλείδιου χρόνου, υπολογίζουμε τη θερμοκρασία της μελανής τρύπας για κάθε τέτοιο λόγο. Τέλος για τις διάφορες τιμές της ποσότητας $(\alpha - \beta)/M^2$ απεικονίζουμε τα διαγράμματα από τη λίστα $\{R_h/(2M), T(R_h)/T_{sch}\}$.



(a)



(b)

Figure 11: The black-hole temperature for (a) attractive and (b) repulsive higher-order electromagnetic contributions, with varying values of the magnetic charge (Q_m), while keeping the mass (M) the same. The axes in both figures are logarithmic.

Χρησιμοποιήσαμε την ίδια παράμετρο μάζας M για τον υπολογισμό της θερμοκρασίας της μελανής οπής Schwarzschild T_{sch}. Στα δύο αυτά διαγράμματα έχουμε συμπεριλάβει με μια κουκίδα την οικογένεια λύσεων της μελανής οπής Schwarzschild, η οποία κουκίδα θα βρίσκεται στο σημείο (1,1) κάτι που μόνο τυχαίο δεν είναι καθώς, ο ορίζοντάς της αφενώς δίνεται από την ακτίνα $R_h = 2M$ και η θερμοκρασία από την τιμή $T = 1/(8\pi M)$.

Από τα διαγράμματα της θερμοκρασίας, μπορεί να γίνει προφανές πως στην περίπτωση που $\alpha - \beta > 0$ έχουμε δύο διαφορετικούς κλάδους λύσεων. Στον πρώτο κλάδο έχουμε μελανές οπές οι οποίες είναι θερμικά ασταθείς καθώς θερμαίνονται καθώς μικραίνουν, σε αυτό τον κλάδο βρίσκεται και η μελανή οπή Schwarzschild. Επιπρόσθετα, έχουμε και ένα δεύτερο κλαδο μελανών οπών, οι οποίες θα μπορούν να έρθουν σε θερμική ισορροπία με το λουτρό θερμότητας, καθώς όσο ο ορίζοντάς τους μικραίνει, αυτές θα ψυχονται και συνεπώς θα έχουν θετική ειδική θερμοχωρητικότητα. Όπως επίσης φαίνεται από το διάγραμμα, οι μελανές οπές που θα έχουν την ίδια ακτίνα ορίζοντα γεγονότων για δυο διαφορετικές τιμές του λόγου μάζας προς μαγνητικό φορτίο θα ανήκουν σε διαφορετικούς θερμοδυναμικούς κλάδους και συνεπώς, μπορούμε να ξεχωρίσουμε τις μελανές οπές υπολογίζοντας τη θερμοκρασία τους. Από την άλλη μεριά, στην περίπτωση $\alpha - \beta < 0$ οι μαύρες τρύπες μπορεί να είναι θερμικά ευσταθείς με το λουτρό θερμότητας καθώς ψύχονται όσο ο ορίζοντάς τους μικραίνει.

Ας συζητήσουμε σχετικά σύντομα την ύπαρξη ασυμπτωτικά (A)dS λύσεων. Ακολουθώντας το [37], εισαγάγουμε ένα βαθμωτό δυναμικό στη θεωρία μας μέσω του όρου $\mathfrak{V}(\phi)$ και πλέον θεωρούμε την

$$\mathcal{S} = \int d^4x\sqrt{-g}\Big(\mathcal{R} - 2\nabla^\mu\phi\nabla_\mu\phi - e^{-2\phi}\mathcal{F}^2 + f(\phi)\big(- 2\alpha\mathcal{F}^\alpha_{\ \beta}\mathcal{F}^\beta_{\ \gamma}\mathcal{F}^\gamma_{\ \delta}\mathcal{F}^\delta_{\ \alpha} + \beta\mathcal{F}^4\big) - \mathfrak{V}(\phi)\Big) \ , \quad (205)$$

με το δυναμικό $\mathfrak{V}(\phi)$ να δίνεται από τη σχέση

$$\mathfrak{V}(\phi) = \frac{1}{3}\Lambda e^{-2\phi} + \frac{1}{3}\Lambda e^{2\phi} + \frac{4\Lambda}{3} = \frac{2}{3}\Lambda(\cosh(2\phi) + 2) \ , \quad (206)$$

και μπορούμε να υπολογίσουμε τη συνάρτηση της μετρικής ως

$$B(r) = 1 - \frac{2M}{r} - \frac{2(\alpha-\beta)Q_m^4}{r^3\left(r - \frac{Q_m^2}{M}\right)^3} - \frac{1}{3}\Lambda r\left(r - \frac{Q_m^2}{M}\right) \ , \quad (207)$$

ενώ τα $\phi(r), R(r)$ παραμένουν αμετάβλητα. Αξίζει να σημειωθεί πως τα δυναμικά $\mathfrak{V}(\phi)$ και $f(\phi)$ είναι σχεδόν ταυτόσημα και πρόκειται για δυναμικά τύπου Liouville [38].

Θεωρώντας τώρα πως ο όρος σύζευξης μεταξύ του dilaton και του όρου του Maxwell είναι της μορφής $e^{-2\gamma\phi}$ μπορούμε να πάρουμε την ίδια γεωμετρία με την περίπτωση που $\gamma = 1$ για τη μορφή της συνάρτησης $f(\phi)$

$$f(\phi) = -3\cosh(2\phi) - 2 - \frac{e^{2\phi}\left(\left(e^{-2\phi}\right)^{\gamma-1} - 1\right)Q_m^6}{2M^4\left(e^{2\phi} - 1\right)^4(\alpha - \beta)}\,. \tag{208}$$

Σε αυτή την περίπτωση, ο λόγος του μαγνητικού φορτίου προς τη μάζα της μελανής τρύπας μπορεί να πάρει τιμές οι οποίες θα είναι φιξαρισμένες από τη θεωρία. Συνεπώς, τέτοιες μαύρες οπές μπορούν να περιγράφονται από ένα περιορισμένο παραμετρικό χώρο, αφού η θεωρία θα μετατρέπει τα δύο ανεξάρτητα hair της οπής σε ένα. Σαφέστατα, θα ήταν προτιμότερο να επιτρέπεται η αλλαγή της μορφής του πεδίου για διαφορετικές συναρτήσεις, ωστόσο δεν καταφέραμε να βρούμε ακριβείς λύσεις για αυτό το σενάριο, συνεπώς θα πρέπει κανείς να αναζητήσει αριθμητικές λύσεις.

Εν κατακλειδι, οι μαύρες τρύπες που περιγράφονται στο κεφάλαιο 6 αποτελούν μια γενίκευση των μελανών οπών GMGHS [32, 33], για μια συγκεκριμένη συνάρτηση σύζευξης η οποία έχει θεμελιώδη προέλευση. Οι λύσεις που παρουσιάζονται, μπορεί να είναι θερμοδυναμικά ευσταθείς, θερμοδυναμικά προτιμητέες και βαρυτικά ευσταθείς, ενώ σέβονται και τις ενεργειακές συνθήκες. Η μελέτη τέτοιων λύσεων είναι σημαντική στα πλαίσια κβαντικών διορθώσεων των γραμμικών θεωριών του ηλεκτρομαγνητισμού. Παράλληλα, σύμφωνα με το άρθρο [39] οι λύσεις μας συμφωνούν με τα παρατηρησιακά δεδομένα ανεξαρτήτως των τιμών των σταθερών της θεωρίας α, β, και μαζί με το γεγονός πως σέβονται και τις ενεργειακές συνθήκες και είναι θερμικά και βαρυτικά ευσταθείς, καθιστά τη δική μας οικογένεια μελανών οπών έναν πιθανό υποψήφιο μαύρης τρύπας στο πρώιμο σύμπαν όταν και η ισχύς των ηλεκτρομαγνητικών πεδίων ήταν ισχυρότερη.

Contents

Part I

Introduction: $f(R)$ Gravity, Black Holes Coupled to Scalar Fields And String Inspired Non-Linear Electrodynamics

Chapter 1

Introduction

1.1 $f(R)$ Gravity

General Relativity (GR) [40] is a widely accepted theory of gravity and it provides the framework for the description of strong gravitational interactions. For weak gravitational fields, GR reduces to Newtonian gravity. The theory has been tested several times over the last $100+$ years of its existence, with many experimental successes, starting from the perihelion of Mercury, the existence of neutron stars, black holes, gravitational waves [41] with black hole shadows [42] being the latest experimental success. For extensive discussions of the subject i refer the reader to the following textbooks [43, 44, 45, 46, 47]. However, GR is thought to be inadequete in the microscopic scales of particles and high energies. In all other scales, GR has been tested and remains the golden standard. The Einstein field equation reads

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \ , \tag{1.1}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}, R$ are the Ricci tensor and the Ricci scalar respectively, $\kappa = 8\pi G/c^2$ (i will mostly use $c = 1$ throughout the thesis) is Einstein's constant and $T_{\mu\nu}$ is the energy-momentum tensor. The left hand side of this equation concerns the geometry, while the right hand side concerns the matter fields. Since the establishment of GR as the standard gravitational theory many alternatives have been proposed. For a discussion i refer the reader to [48] and references therein. An interesting approach was presented by Sakharov [49], which showed that, since fluctuations in spacetime itself lead to higher-power corrections to the Einstein equations, the Einstein-Hilbert action which describes GR is just a first order approximation of a more complicated theory. Some years later, Stelle showed [50] that these theories are renormalizable at the one loop level, hence avoiding dangerous singularities.

The development of observational cosmology shows that the universe has undergone two phases of cosmic acceleration. To match the observations, scalar fields have been considered as candidates for these situations, modifying in this way the right hand side of the energy-momentum tensor of the Einstein equation. However, one can also modify the left hand side of this equation, changing in this way the theory of gravitation. To describe the system under consideration, we will use the Lagrangian formalism. The Lagrangian for the Einstein equation (1.1) is given by

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G} + \mathcal{L}_M\right) \ , \tag{1.2}$$

where $\mathcal{L}_M$ is the matter Lagrangian that gives rise to the energy-momentum tensor. By applying calculus of variations with respect to the fields to the action (1.2) we obtain the Einstein equation (1.1). To modify the theory of gravitation, i.e the left hand side of Einstein equation, we will replace R with $f(R)$ in the action (1.2), namely

$$S = \int d^4x\sqrt{-g}\left(\frac{f(R)}{16\pi G} + \mathcal{L}_M\right) . \tag{1.3}$$

The simplest $f(R)$ one might consider is $f(R) = R - 2\Lambda$ where Λ is the cosmological constant, so this is basically GR and this $f(R)$ describes a linear theory of gravity. Non-linear models have been used to describe the early and late cosmological history of our Universe [51]-[13]. In particular, following the recent cosmological observational results the $f(R)$ gravity cosmological models were used to explain the deceleration-acceleration transition. This requirement imposed constraints on the $f(R)$ models allowing viable choices of $f(R)$ [52]. These theories exclude contributions from any curvature invariants other than R and they avoid the Ostrogradski instability [53] which usually is present in higher derivative theories [14]. For early times, the model

$$f(R) = R + \alpha R^2 \qquad \alpha > 0 , \tag{1.4}$$

known as the Starobinsky inflation model [13] has been used due to the fact that leads to the accelerated expansion of the universe because of the non-linear αR^2 gravity term. Dark energy models from $f(R)$ theories have also been used to realize the late-time acceleration, a well known model is [54, 55]

$$f(R) = R - \frac{\alpha}{R^n} \qquad \alpha, n > 0 . \tag{1.5}$$

We will now discuss some properties of $f(R)$ theories and their equivalence to Brans-Dicke theory [56].

Keeping $f(R)$ general, the field equations that arise after varying the action (1.3) with respect to the metric $g^{\mu\nu}$ are given by

$$f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\, f_R = \kappa T_{\mu\nu} , \tag{1.6}$$

where $f_R \equiv df(R)/dR$ and $\Box$ is the D'Alambert operator with respect to the metric and $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta g_{\mu\nu}} . \tag{1.7}$$

Tracing the field equation (1.6) we obtain

$$3\Box f_R + f_R R - 2f(R) = \kappa T . \tag{1.8}$$

Einstein's theory corresponds to $f(R) = R, f_R = 1$ and we obtain that $R = -\kappa T$, so that the Ricci scalar is directly determined by matter, while for a general $f(R)$ theory $\Box f_R \neq 0$, which means that there is a scalar degree of freedom that propagates (the box operator $\Box$ in general implies a field propagation), dubbed "scalaron" $\varphi \equiv f_R$ and the trace equation, determines the behavior of the scalaron.

The $f(R)$ theories may also be recasted in a type of Brans-Dicke theory [56] via suitable redefinitions of fields. We can recast the action of $f(R)$ gravity (ignoring the matter part) (1.3) in the following manner

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(f(\chi) + \frac{df(\chi)}{d\chi}(R - \chi)\right) , \tag{1.9}$$

where χ is an auxiliary field. By variation with respect to χ we obtain

$$\frac{d^2 f(\chi)}{d\chi^2}(R-\chi)=0\ , \tag{1.10}$$

and under the assumption that $\frac{d^2 f(\chi)}{d\chi^2} \neq 0$ we obtain $R=\chi$, hence the actions (1.3) and (1.9) are equivalent. By defining

$$\varphi \equiv \frac{df(\chi)}{d\chi} \tag{1.11}$$

the action (1.9) reads

$$S=\int d^4x\sqrt{-g}\left(\frac{\varphi R}{2\kappa}-U(\varphi)\right)\ , \tag{1.12}$$

where $U(\varphi)$ is a field potential given by

$$U(\varphi)=\frac{\chi(\varphi)\varphi-f\left(\chi(\varphi)\right)}{2\kappa}\ . \tag{1.13}$$

The action of the Brans-Dicke theory is [56]

$$S_{\text{BD}}=\int d^4x\sqrt{-g}\left(\frac{\varphi R}{2}-\frac{\omega_{\text{BD}}}{2\varphi}\nabla^\mu\varphi\nabla_\mu\varphi-U(\varphi)\right)\ , \tag{1.14}$$

where ω_{BD} is the dimensionless Brans-Dicke parameter. We can see that the two action (1.14) and (1.12) are equivalent for $\omega_{\text{BD}}=0$ in the unit system $\kappa=1$. Hence, the two theories describe the same physics in that unit system. However, observations have shown that ω_{BD} is given by a large number [57].

1.1.1 Conformal Transformation

It is possible to write the action for $f(R)$ gravity in the Einstein frame (in a way that after variation we will obtain the standard form of Einstein equations $G_{\mu\nu}\equiv$ fields). In view of this statement, we perform a conformal re-scaling (transformation) of the metric

$$\hat{g_{\mu\nu}}=\Omega^2 g_{\mu\nu}\ . \tag{1.15}$$

The hat represents quantities in the Einstein frame, while Ω^2 is the conformal factor. The Ricci scalar in the Jordan frame is related to the Einstein frame in the following way

$$R=\Omega^2\left(\hat{R}+6\hat{\Box}\omega-6\partial^\alpha\omega\partial_\alpha\omega\right)\ , \tag{1.16}$$

where $\omega\equiv\ln\Omega$. We can also rewrite the action for $f(R)$ theory in the following way

$$S=\int d^4\sqrt{-g}\left(\frac{1}{2\kappa}f_R R-U\right)\ , \qquad U=\frac{f_R R-f(R)}{2\kappa}\ . \tag{1.17}$$

Using the conformal transformation, we can rewrite the above action as

$$S=\int d^4x\sqrt{-\hat{g}}\left(f_R\Omega^{-2}\left(\hat{R}+6\hat{\Box}\omega-6g^{\hat{\alpha\beta}}\partial_\alpha\omega\partial_\beta\omega\right)-\Omega^{-4}U\right)\ , \tag{1.18}$$

where we've used that $\sqrt{-g} = \Omega^{-4}\sqrt{-\hat{g}}$. The $\hat{\Box}\omega$ term will vanish, since it is a total derivative, if we perform the integration. The Einstein frame corresponds to the situation where the action is linear in $\hat{R}$ which constraints f_R

$$f_R \equiv \Omega^2 > 0 \ .$$

We introduce a new scalar field ϕ given by

$$\sqrt{\kappa}\phi = \sqrt{\frac{3}{2}}\ln f_R \ , \tag{1.19}$$

so that ω turns up being

$$\omega = \sqrt{\frac{\kappa}{6}}\phi \ . \tag{1.20}$$

Finally the action will read

$$S = \int d^4x\sqrt{-\hat{g}}\Big\{\frac{\hat{R}}{2\kappa} - \frac{1}{2}\partial^a\phi\partial_a\phi - V(\phi)\Big\} \ , \tag{1.21}$$

where

$$V(\phi) = \frac{U}{f_R^2} = \frac{Rf_R - f}{2\kappa f_R^2} \ .$$

Finally, the conformal factor is found to be

$$\Omega^2 = f_R = \exp\left(\sqrt{2\kappa/3}\phi\right) \ .$$

Having showed that the $f(R)$ gravity field equations are equivalent to GR plus a scalar field we will proceed, by discussing some black hole solutions in $f(R)$ theories.

1.2 Black holes in $f(R)$ Gravity

Black hole solutions in $f(R)$ theories have been found and they either are deviations of the known black hole solutions of General Relativity, or they possess new properties that should be investigated. Static and spherically symmetric black hole solutions were derived in $(3+1)$ and $(2+1)$ dimensions [58, 59, 60, 61], while in [62, 63, 64, 65, 66, 67] charged and rotating solutions were discussed. Static and spherically symmetric black hole solutions were investigated with constant curvature, with and without electric charge and cosmological constant in [68, 69].

Solutions with dynamical curvature in vacuum [1]can be obtained in the following way. We consider $f(R)$ gravity in the absence of matter, i.e the action (1.3) with $\mathcal{L}_M = 0$, hence, the resulting field equation ends up being (1.6). To solve the tensorial field equation, we have to impose a metric ansatz. For a spherically symmetric spacetime, one can consider

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2 \ , \tag{1.22}$$

where $A(r), B(r)$ are two unknown metric functions that should be determined from the field equation. However, having two unknown functions, results in very complicated equations. Moreover, one has three unknown functions $f(R), A(r), B(r)$, but the resulting field equations yield two independent differential equations, hence one of the functions has to be given by hand. Recently, in [70], two metric functions have been considered, while the derivative of the $f(R)$ model, $f_R(r)$ has been fixed in terms of the radial co-ordinate and analytic results have been obtained in terms of the Heun functions. However, the results are complicated and we will not reproduce them here. We will assume that $B(r) = A^{-1}(r)$. This restricts the dynamics of the system under consideration and the spacetime is not the most general one. However, the static and spherically symmetric solution in vacuum and electro-vaccum (Schwarzchild and Reissner-Nordström (RN)) of GR satisfy this condition, and in order to obtain simple results we will use this gauge. From the field equations, we can obtain the $f(R)$ theory as

$$f(R) = c_1 R + c_2 \int^R r(R) dR + C \ , \tag{1.23}$$

where c_1 is related to Newton's effective constant and GR, c_2 is related to geometric corrections that are encoded in $f(R)$ gravity, while C is a constant with units $[L]^{-2}$, being related to the cosmological constant. We will ignote C and give some values to the integration constants to obtain the metric function $A(r)$. At first we set $c_1 = 1, c_2 = 0$ and the equation for the metric function reads

$$\frac{1}{2} r^2 A''(r) - A(r) + 1 = 0 \ ,$$

with solution

$$A(r) = 1 + \frac{c_3}{r} + c_4 r^2 \ , \tag{1.24}$$

which is the Schwarzchild-(A)dS solution and the Ricci scalar as well as the $f(R)$ theory read

$$R(r) = -12 c_4 \ , \tag{1.25}$$

$$f(R) = -6 c_4 \ . \tag{1.26}$$

Now if we set $c_4 = -\dfrac{\Lambda}{3}$ we have

$$R(r) = 4\Lambda \ , \tag{1.27}$$

$$f(R) = 2\Lambda \ , \tag{1.28}$$

[1]Here vacuum denotes the absence of fields. Since we consider $f(R)$ theory, the true vacuum is a non-trivial concept.

which describes (A)dS spacetime, depending on the sign of the cosmological constant. We now set $c_1 = 1$ and keep c_2 arbitrary, hence we allow for geometric corrections. The equation for the metric function $A(r)$ will now read

$$c_2 r^2 A'(r) + (c_2 r + 1)\left(r^2 A''(r) + 2\right) - 2A(r)(2c_2 r + 1) = 0 \ ,$$

which we can integrate to obtain $A(r)$. The result is complicated, but adjusting the integration constants, we find that

$$A(r) = \frac{1}{2} + \frac{1}{3c_2 r} + r^2\left(\frac{3c_2^2}{2} + c_4\right) \ , \tag{1.29}$$

where the mass term $\mathcal{O}(r^{-1})$ is given by the geometric correction c_2 and in order to have a positive mass term (and a black hole in the absence of cosmological constant) we have to impose $c_2 < 0$. We can see that there exists an effective cosmological constant term given my the $\mathcal{O}(r^2)$ term of the metric function

$$\Lambda_{\text{eff}} = -3\left(\frac{3c_2^2}{2} + c_4\right) \ , \tag{1.30}$$

which we will ignore by setting $c_4 = -3c_2^2/2$ for the following discussion of thermodynamics, in order to have a simple analytic expression of the event horizon, Finally the metric function will read

$$A(r) = \frac{1}{2} - \frac{2M}{r} \ , \tag{1.31}$$

where we have redefined $c_2 = -1/6M$. Now, the $f(R)$ theory will read

$$R(r) = \frac{1}{r^2} \ , \tag{1.32}$$

$$f(r) = \frac{1}{r^2} - \frac{1}{3Mr} \ , \tag{1.33}$$

$$f_R(r) = 1 - \frac{r}{6M} \ , \tag{1.34}$$

$$f(R) = R - \frac{1}{3M}\sqrt{R} \ . \tag{1.35}$$

We can see that a square root correction is introduced via c_2, that depends on the mass of the black hole. The resultant $f(R)$ model does avoid the tachyonic instability since

$$\frac{d^2 f(R)}{dR^2} = \frac{f_R'(r)}{R'(r)} = \frac{r^3}{12M} > 0. \tag{1.36}$$

The black hole horizon is located at $A(r_h) = 0 \to r_{\text{h}} = 4M$. Calculating the temperature $T(r_h)$, the entropy $S(r_h)$, quasi-local energy $E(r_h)$ and the Gibbs free energy $G(r_h)$ [71, 72, 66, 73, 74], we find that

$$T(r_h) = \frac{A'(r)}{4\pi}\bigg|_{r_h} = \frac{1}{32\pi M} \ , \tag{1.37}$$

$$S(r_h) = \pi r_h^2 f_R(r_h) = \frac{16\pi M^2}{3} \ , \tag{1.38}$$

$$E(r_h) = \frac{1}{4}\int \left(r^2(f(r) - f_R(r)R(r)) + 2f_R(r)\right) dr\bigg|_{r_h} = M \ , \tag{1.39}$$

$$G(r_h) = E(r_h) - S(r_h)T(r_h) = 5M/6 \ . \tag{1.40}$$

First Law of Thermodynamics from $f(R)$ Gravity Field Equations

We can obtain the aforementioned relations from the Einstein equations, by going to a frame where the $T^t_{\ t}$ component of the energy momentum tensor will be related to the energy density of matter. Hence the $f(R)$ field equations 1.6 can be written as

$$G^{\mu}_{\ \nu} = f_R^{-1}\left\{\frac{1}{2}g^{\mu}_{\ \nu}\left(f(R) - f_R R\right) + \left(\nabla^{\mu}\nabla_{\nu} - g^{\mu}_{\ \nu}\Box\right) f_R\right\} . \tag{1.41}$$

Now the tt and rr components of Einstein equations read

$$\frac{rA' + A - 1}{r^2} = -\frac{A'f'}{2f} + \frac{2A'}{r} + \frac{A''}{2} - \frac{2Af'}{rf} + \frac{A}{r^2} + \frac{F}{2f} - \frac{1}{r^2} ,$$

where we have take into account that $f''_R(r) = 0$. At the horizon we have the constraint $A = 0$, $A' \neq 0$, hence the above relation reads

$$2f_R + (f - Rf_R)r_h^2 = 4T\pi r_h^2\left(f'_R + \frac{2f_R}{r_h}\right) , \tag{1.42}$$

where all functions are evaluated at the black hole horizon and we have used the definition of temperature $T = A'/4\pi$. Now it is easy to verify that

$$S = \frac{\mathcal{A}}{4}f_R = \pi r_h^2 f_R , \tag{1.43}$$

$$dS = 2\pi r_h dr_h f_R + f'_R dr_h \pi r_h^2 , \tag{1.44}$$

where we have defined S as the entropy at the black hole horizon and $\mathcal{A}$ is the area of the black hole horizon. Further defining

$$dE = \frac{1}{4}\Big(2f_R + (f - Rf_R)r^2\Big)dr_h , \tag{1.45}$$

as the variation of a modified Misner-Sharp energy and also

$$E(r_h) = \frac{1}{4}\int \left(r^2(f(r) - f_R(r)R(r)) + 2f_R(r)\right) dr\Big|_{r=r_h} , \tag{1.46}$$

we have the first law of thermodynamics

$$dE = TdS . \tag{1.47}$$

Note here that we assumed that the horizon radius is allowed to vary. All quantities are well defined and positive. Another aspect of the obtained black holes is that they possess an angle deficit. To see this we will perform a co-ordinate transformation. The metric in Schwarzchild co-ordinates reads

$$ds^2 = -\left(\frac{1}{2} - \frac{2M}{r}\right)dt^2 + \left(\frac{1}{2} - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 . \tag{1.48}$$

We now set

$$t \to \tau\sqrt{2} , \tag{1.49}$$

$$r \to \rho/\sqrt{2} . \tag{1.50}$$

Now the metric in the new co-ordinates reads

$$ds^2 = -\left(1-\frac{4M}{\rho}\right)d\tau^2 + \left(1-\frac{4M}{\rho}\right)^{-1}d\rho^2 + \frac{\rho^2}{2}d\Omega^2 \ , \quad (1.51)$$

$$= -\left(1-\frac{4M}{\rho}\right)d\tau^2 + \left(1-\frac{4M}{\rho}\right)^{-1}d\rho^2 + \rho^2\left(1-\frac{\delta\varphi}{2\pi}\right)d\Omega^2 \ , \quad (1.52)$$

where $\delta\varphi = \pi$ is the angle deficit.

Having discussed a simple example of black-hole solutions in $f(R)$ theories and some of its properties, we will proceed in order to investigate black-hole solutions coupled to scalar fields.

1.3 Black hole solutions coupled to scalar fields

1.3.1 General comments and some particular scalar potentials that give hairy black hole spacetimes

We will at first discuss a little bit the no-scalar hair theorem possed by Bekenstein [18]. To do so, we begin with a very simple scalar field theory consisting only of a pure kinetic term for the scalar field, namely

$$\mathcal{S} = \int d^4x\,\sqrt{-g}\left[\frac{R}{2} - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi\right] \ , \quad (1.53)$$

By variation with respect to the fields one can obtain the Einstein and Klein-Gordon equations,

$$G_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\kappa\phi\nabla^\kappa\phi \ , \quad (1.54)$$

$$\Box\phi = 0 \ . \quad (1.55)$$

Assuming the existence of an asymptotically flat, static and spherically symmetric black hole solution, the metric element can be written as

$$ds^2 = -g_{tt}^2(r)dt^2 + \frac{dr^2}{g_{rr}^2(r)} + r^2 d\Omega^2 \ . \quad (1.56)$$

Under the above ansatz, the Klein-Gordon equation yields a first integral,

$$g_{tt}(r)g_{rr}(r)\phi'(r)r^2 = C \ , \quad (1.57)$$

where C is an integral constant. A black hole spacetime possesses an event horizon at a finite $r_h > 0$. The black hole horizon is defined as an one-way hypersurface where nothing can escape. Considering radial null geodesics i.e $ds^2 = 0, \ \theta =$ constant $= \varphi$, we can obtain for the velocity of photons that

$$\frac{dr}{dt} = \pm|g_{tt}(r)g_{rr}(r)| \ . \quad (1.58)$$

At large distances, the velocity becomes equal to 1 and photons travel at their usual speed of light, while in the case of a horizon, a far-away observer will see the photons freezed there and hence their velocity seems to be zero, a scenario that can be achieved with the vanishing of $g_{tt}(r)$ or $g_{rr}(r)$ at the horizon radius r_h. An event horizon is also defined as the locus of points satisfying $\nabla_\alpha r\nabla^\alpha r = 0$, while in the case of static and spherically symmetric spacetimes, this will also yield that $g_{tt} = 0$, so the event horizon will coincide with the Killing horizon. Consequently, (1.57) imposes that $C = 0$, under the assumption of finiteness of $\phi'(r_h)$. As a result we have that

$$g_{tt}(r)g_{rr}(r)\phi'(r)r^2 = 0 \,, \tag{1.59}$$

which yields that either one of the metric functions vanishes for any $r > r_h$, which is meaningless or that $\phi'(r) = 0 \to \phi \to$ constant. As a result the theory (1.53) will naturally yield the Schwarzchild black hole due to Birkhoff's theorem. Of course, this could also be seen by multiplying the wave equation with ϕ and integrating over the exterior black hole region

$$\int d^4x\sqrt{-g}\phi\Box\phi = 0 \to \int d^4x\sqrt{-g}\nabla^\mu\phi\nabla_\mu\phi = 0 \,, \tag{1.60}$$

where we dropped a total derivative term which vanishes under the assumption of a sufficiently fast-decaying scalar field. Hence, the only way to make the above integral zero is $\phi'(r) = 0 \to \phi =$ constant, and therefore no non-trivial degree of freedom sourced by a minimally coupled scalar field can survive outside the black hole horizon. In conclusion, self-interactions or non-minimal couplings should be present.

Let us now discuss a little bit the easiest way to evade the no-scalar hair theorem, which is by considering self interactions for the scalar field in the form of a scalar potential and comment on the existing bibliography. In the presence of a potential the above action has to be supplemented with the potential term $V(\phi)$, in the action

$$\mathcal{S} = \int d^4x\,\sqrt{-g}\left[\frac{R}{2} - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - V(\phi)\right] \,. \tag{1.61}$$

One of the first black hole solutions that a scalar potential has been used to evade the no-scalar-hair theorem is the MTZ black hole [75], with a scalar potential given by

$$V(\phi) = \frac{1}{2}\Lambda\cosh(2\phi) - \frac{\Lambda}{2} \,. \tag{1.62}$$

As we will see this might be the simplest potential in order to circumvent the no scalar hair theorem. This solution can be conformally maped to the Jordan frame, where the potential takes the simple form of a quartic one $V \sim \tilde{\phi}^4$. The solution generated by this potential is described by one integration constant, the mass of the black hole, and the scalar field does not introduce a length scale responsible for its behavior. If one wants to abandon conformal invariance in the Jordan frame then the potential reads [76]

$$V(\phi) = -\frac{3}{l^2}\cosh\left(\sqrt{\frac{2}{3}}\phi\right) + \frac{3}{l^2} + \frac{g}{l^2}\left(\frac{1}{8}\sinh\left(\sqrt{6}\phi\right) + \frac{9}{8}\sinh\left(\sqrt{\frac{2}{3}}\phi\right) - \sqrt{\frac{3}{2}}\phi\cosh\left(\sqrt{\frac{2}{3}}\phi\right)\right) \,, \tag{1.63}$$

where g is a dimensionless constant which is responsible for the breaking of conformal invariance in the Jordan frame. It is clear that when $g \to 0$ we obtain the MTZ potential. The parameter g is a secondary hair of the solution since it enters the conserved mass of the black hole, but is not a scale introduced by the scalar field. In this scenario, again the only integration constant is the mass of the black hole. One might be tempted to consider charged black hole solutions with the inclusion of a Maxwell term in the Lagrangian and in this case the potential has to be modified as [77]

$$V(\phi) = \frac{3}{8l^2} + \frac{\cosh(2\phi)}{2l^2} + \frac{\cosh(4\phi)}{8l^2} \,. \tag{1.64}$$

Here the solution is again described by only one integration constant being the electric charge, with the black hole being massless. The conformal invariance is also broken in the Jordan frame.

We have so far seen that the potentials are linear combinations of inverse trigonometric (exponential) functions in all three cases and the scalar field does not introduce a length scale. In the MTZ case, the scalar field backreacts to the metric and dresses the black hole with a secondary scalar hair, since there is no way to keep the mass fixed and simultaneously make the scalar field vanish, while in the second case the scalar field backreacts to the metric, with the scalar field theory dressing the black hole again with a secondary scalar hair. What will happen if we want to introduce a new length scale in the scalar sector of the theory, that controls the behavior of the scalar field, in the sense that keeping the mass fixed we can make the scalar field vanish and possibly in an asymptotically flat spacetime? Such an attempt is presented in [15], with a potential given by

$$V(\phi)=\Lambda\left(-\cosh\left(\sqrt{2}\phi\right)\right)-2\Lambda+\chi\left(6\sinh\left(\sqrt{2}\phi\right)-2\sqrt{2}\phi\left(\cosh\left(\sqrt{2}\phi\right)+2\right)\right)\ , \tag{1.65}$$

where $\chi=M/\nu^3$, where M corresponds to the black hole mass and ν to the scalar field parameter, that controls the far-field behavior of the scalar field, namely the scalar charge. We should note that black hole solutions exist also when $\Lambda=0$. A similar potential might also be found in [78, 79]. In the attempt to derive these solutions, the authors fix the form of the scalar field and solve the corresponding field equations in order to reconstruct the theory. This "potential engineering" procedure is actually a well acclaimed technique to tackle physical problems, see for example references [80, 81] for the application of this technique in cosmological inflation. However, the corresponding solution ends up being described by only one integration constant and χ which is the model parameter. In addition, these potentials arise in the context of supergravity [82]. We should mention here that these solutions cannot reduce continuously (at the level of the line element) to the Schwarzchild AdS black holes by taking the limit of $\nu\to 0$, since this would imply the vanishing of the black hole mass. Of course the theory (1.65) will give the corresponding Schwarzchild-AdS black holes at the scalar vacuum $\phi=0$. In conclusion, these spacetimes can be completely characterized by one integration constant, either M or ν and the constant of the theory χ and hence the scalar field does not introduce a new scale.

From the above discussion it is clear that the exact form of the potential under consideration plays a crucial role in the resulting black hole spacetime. Let us check whether the family of potentials that yield hairy black holes, share a similar behavior near the vacuum of the theory. In an asymptotically flat spacetime we have that $g_{tt}=1=g_{rr}$ in the line element (1.56). Near the vacuum of the theory $\phi\to 0$ we will assume that the scalar potential is analytic and can be expanded as $V(\phi\to 0)\sim\zeta\phi^n$, where ζ is a constant and n denotes the leading order term. Then the Klein-Gordon equation becomes

$$\zeta(-n)\phi(r)^{n-1}+\frac{2\phi'(r)}{r}+\phi''(r)=0\ . \tag{1.66}$$

For $n=2$ we obtain a massive scalar and the field has the Yukawa fall-off $\phi\sim\frac{e^{-\sqrt{2}\sqrt{\zeta}r}}{r}$. If we want a fall-off behavior for the scalar field as $1/r$ (in order to define a scalar charge as the scale controlling the $1/r$ term), then one can see that this corresponds to $n=5$, and the potential is $V(\phi)\sim\zeta\phi^5$ at the scalar vacuum. One can check that indeed after setting $\Lambda=0$ in the potential (1.65) the above argument holds. Moreover, the thermodynamic nature of this kind of hairy black holes is also of interest. Since χ is a constant of the theory, it is not allowed to vary. As a result, the produced compact object can lose mass through Hawking evaporation, but it also has to lose its scalar hair in order for the aforementioned ratio $\chi=M/\nu^3$ to be constant throughout the process. In addition, charged black holes of this kind also have also been found [83, 5]. In these solutions, besides χ one has to introduce another constant of the theory ψ, which gives the electromagnetic to scalar charge ratio, $\psi = Q^2/\nu^4$. As a result, the metric functions of these spacetimes are described by $g_{\mu\nu} = g_{\mu\nu}(r, \chi, \psi, \Lambda, \nu)$, where it is clear that the only integration constant allowed to vary is the scalar charge. Again, the black hole might lose mass through Hawking evaporation, as well as scalar charge, but in these cases, it must also be striped off its electromagnetic charge. However, the key point here is to note that the only constant that is allowed to vary is the mass because of the relation between the mass of the black hole and the scalar hair from the theory. If $\chi = M/\nu^3$ then

$$\delta M = 3\chi\nu^2\delta\nu \ , \tag{1.67}$$

since $\delta\chi = 0$ which will give a relation between the variation of the horizon of the black hole and the scalar charge. Consequently, these type of theories will satisfy a first law of the form

$$dM = TdS \ ,$$

since the only primary hair carried by the black hole will be the mass, even if they are charged.

In the Appendix A.0.1 we explore a simple, asymptotically flat black hole solution and discuss its thermodynamics in detail, presenting all calculations.

1.3.2 Conformal scalar hair

Let me now dive deeper in the hairy black hole scenarios.

The first attempts begun in the early 70ies with the works of Bocharova, Bronnikov, Melnikov and Bekenstein and the result is the BBMB black hole [20]. The action of the BBMB black hole, contains the Ricci scalar, a kinetic term for the scalar field and a conformal coupling between the scalar field and curvature, namely

$$S = \int d^4x\sqrt{-g}\Big[\frac{R}{2} - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - \frac{1}{12}R\phi^2\Big] \ . \tag{1.68}$$

The scalar field is conformally coupled to gravity, thus the trace of the resulting energy momentum tensor will be zero. Varying with respect to the fields we obtain the Einstein and Klein-Gordon equation:

$$G_{\mu\nu} = T_{\mu\nu} \tag{1.69}$$

$$\Box\phi = \frac{1}{6}R\phi \tag{1.70}$$

where

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi + \frac{1}{6}\big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}\big)\phi^2 \tag{1.71}$$

Indeed if we trace $T_{\mu\nu}$ and use the Klein-Gordon:

$$\begin{aligned}
g^{\mu\nu}T_{\mu\nu} &= g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - g^{\mu\nu}\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{6}g^{\mu\nu}\left[g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\nabla_\beta - \nabla_\mu\nabla_\nu + G_{\mu\nu}\right]\phi^2 \\
&= g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 4\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{4}{6}g^{\alpha\beta}\nabla_\alpha\nabla_\beta(\phi^2) - \frac{1}{6}g^{\mu\nu}\nabla_\mu\nabla_\nu(\phi^2) + \frac{1}{6}g^{\mu\nu}[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R]\phi^2 \\
&= -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g^{\mu\nu}\nabla_\mu\nabla_\nu(\phi^2) + \frac{1}{6}[R - \frac{1}{2}4R]\phi^2 \\
&= -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g^{\mu\nu}\nabla_\mu\nabla_\nu(\phi^2) - \frac{1}{6}R\phi^2
\end{aligned}$$

$$
\begin{aligned}
&= -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g^{\mu\nu}[2\partial_\mu\phi\partial_\nu\phi + 2\phi\nabla_\mu\nabla_\nu\phi] - \frac{1}{6}R\phi^2 \\
&= \phi g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{1}{6}R\phi^2 \\
&= \phi[\nabla^\nu\nabla_\nu\phi - \frac{1}{6}R\phi] \\
&= \phi[\frac{1}{6}R\phi - \frac{1}{6}R\phi] = 0
\end{aligned}
$$

Now, the equations become:

$$R = 0 \ , \tag{1.72}$$

$$\Box\phi = 0 \ . \tag{1.73}$$

Solving these equations alongside the tensorial Einstein equation we find that

$$ds^2 = -b(r)dt^2 + b(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ , \tag{1.74}$$

$$b(r) = \left(1 - \frac{m}{r}\right)^2 \ , \tag{1.75}$$

$$\phi(r) = \pm\sqrt{6}\frac{m}{r-m} \ . \tag{1.76}$$

where m is the mass of the black hole. We can see that the scalar field diverges at the event horizon of the black hole which lies at $r_h = m$. This divergence has been argued by Bekenstein that is not pathological [20]. However, it was also shown by Bronnikov [84] that this solution is unstable under scalar perturbations and a thermodynamical analysis of the black hole shows that this solution is pathological. The temperature will be given by

$$T(r_h) = \frac{b'(r_h)}{4\pi} = \frac{1}{4\pi}\frac{2m(r-m)}{r^3}\bigg|_{r=r_h=m} = 0 \ . \tag{1.77}$$

The entropy on the other side will be given by

$$S(r_h) = \pi r_h^2\left(1 - \frac{1}{6}\phi(r_h)^2\right) \to \infty \ . \tag{1.78}$$

The divergence of the scalar field at the event horizon results in the divergence of the entropy at the event horizon. As a result the produced compact object does not have a conventional thermodynamic interpretation. Also, according to the zeroth law of black hole thermodynamics a black hole should have non-zero surface gravity κ at the event horizon, which is given by

$$k^a\nabla_a k^b = \kappa k^b \ , \tag{1.79}$$

where k^a is a properly normalized Killing vector. The temperature may also be given in terms of the surface gravity

$$T(r_h) = \frac{\kappa}{2\pi}\bigg|_{r=r_h=m} \ . \tag{1.80}$$

The black hole temperature can also be obtained from the "Euclidean trick". We consider a point r that is very close to the horizon $r = r_h + \epsilon$, where ϵ is very small. Now Taylor-expanding we get that the metric function $b(r)$ near r_h asymptotes to

$$b(r \to r_h) \sim b'(r_h)(r - r_h) = b'(r_h)\epsilon \ . \tag{1.81}$$

Now ignoring the angular part of the metric and performing a Wick rotation $t \to i\tau$ we obtain

$$ds^2 = b'(r_h)\epsilon d\tau^2 + \frac{dr^2}{b'(r_h)\epsilon} \; . \tag{1.82}$$

To recover the familiar geometry of the 2−sphere, which has a conical defect at $R \to 0$ so to keep the metric regular one has to impose periodicity in $\Delta\Theta \sim 2\pi$

$$ds_1^2 = dR^2 + R^2 d\Theta^2 \; , \tag{1.83}$$

we have to define

$$\frac{dr^2}{b'(r_h)\epsilon} = dR^2 \quad \& \quad b'(r_h)\epsilon d\tau^2 = R^2 d\Theta^2 \; ,$$

and after integrations we obtain

$$R = 2\sqrt{\frac{(r - r_h)}{b'(r_h)}} \quad \& \quad \Theta = \frac{b'(r_h)}{2}\tau \; . \tag{1.84}$$

The fact that Θ is periodic with period 2π and given that τ is also related to Θ, τ should also be periodic with period β. Now we have

$$\frac{\Theta}{\tau} = \frac{b'(r_h)}{2} \to \frac{b'(r_h)}{2} = \frac{2\pi}{\beta} \to \frac{b'(r_h)}{4\pi} = \frac{1}{\beta} \equiv T \; . \tag{1.85}$$

A vanishing temperature also occurs in the case of extremal black holes. For example in the case of the extremal Reissner-Nordström (RN) solution which happens for $Q = M$ and the resulting metric is the same with the BBMB black hole, a zero temperature also happens, but the entropy is still finite and given by the area law.

A generalization of the BBMB black hole solution was presented in [85]. Consider the action

$$S = \int d^4x\sqrt{-g}\left(\frac{R - 2\Lambda}{2} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}R\phi^2 - \alpha\phi^4\right) \; . \tag{1.86}$$

The matter part of the action is invariant under conformal transformations

$$g_{\mu\nu} \to \Omega(x)^2 g_{\mu\nu}, \qquad \phi \to \Omega(x)^{-1}\phi \; . \tag{1.87}$$

As a result, the energy-momentum tensor is trace-less and in the presence of the cosmological constant the scalar curvature is

$$R = 4\Lambda \; . \tag{1.88}$$

With the BBMB black hole ansatz, the solution becomes

$$\phi(r) = \pm\sqrt{6}\frac{M}{r - M} \; , \tag{1.89}$$

$$b(r) = \left(1 - \frac{M}{r}\right)^2 - \frac{\Lambda r^2}{3} \; . \tag{1.90}$$

In order to respect the conformal invariance, the parameter α is specified, so the solution exists only for $\alpha = -\Lambda/18$. There are three horizons, the inner r_i, event r_h and cosmological horizon r_c

and all possible divergences of the curvature invariants, the metric function and the scalar field are hidden behind the event horizon. We note that we cannot have a black hole solution for an AdS spacetime, since the metric ansatz will be always positive.

We can see here, that the scalar field does not diverge at the event horizon of the black hole, which is located at

$$r_h = \frac{1}{2}\left(l - \sqrt{l(l-4M)}\right) \,, \tag{1.91}$$

the positive cosmological constant shifted the position of the event horizon.

The thermodynamics of this solution have been discussed in [86]. The temperature of the black hole at the event horizon is given by

$$T(r_h) = \frac{1}{2\pi l}\sqrt{1 - \frac{4M}{l}} \,, \tag{1.92}$$

where l is the dS radius $\Lambda = 3/l^2$, while the temperature at the cosmological horizon is given by

$$T(r_c) = -\frac{1}{2\pi l}\sqrt{1 - \frac{4M}{l}} \,, \tag{1.93}$$

which is the opposite of the temperature of the event horizon. The entropy at the black hole horizon is negative,

$$S(r_h) = \pi(-l)\sqrt{l(l-4M)} \tag{1.94}$$

while the entropy at the cosmological horizon is equal to

$$S(r_c) = -S(r_h) \,. \tag{1.95}$$

The black hole solution discussed in [85] is a generalization of the BBMB black hole solution in the presence of a positive cosmological constant. This modification allows the scalar field to be finite at the event horizon dressing the black hole with secondary scalar hair [87], but still the thermodynamic properties of the solution indicate that the produced compact object does not have a conventional thermodynamic behavior. In an attempt to understand better the thermodynamical properties of the solution a charge was introduced to the theory [85, 86].

1.3.3 Minimally coupled scalar hair in Anti de Sitter

Regarding now the minimally coupled case, one of the first exact results was given by Martinez, Troncoso and Zanelli [75]. Considering the action

$$S = \int d^4x\sqrt{-g}\Big\{\frac{R+6l^{-2}}{2} - \frac{1}{2}\partial^\alpha\phi\partial_a\phi - V(\phi)\Big\} \,, \tag{1.96}$$

and the scalar potential

$$V(\phi) = -\frac{6}{l^2}\sinh^2\left(\frac{\phi}{\sqrt{6}}\right) \,, \tag{1.97}$$

an exact black hole solution was obtained, with the scalar field being regular at the event horizon of the black hole. The scalar field is given by

$$\phi(r) = \sqrt{6}\,\mathrm{Arctanh}\,\frac{\mu}{r+\mu} \,, \tag{1.98}$$

where μ is a scalar charge. The potential has a global maxima for $\phi = 0$ and a mass term given by:

$$m^2 = V''(\phi = 0) = -\frac{2}{l^2} \tag{1.99}$$

which satisfies the Breitenlohner-Friedman bound [88, 89]that ensures the perturbative stability of AdS spacetime, while the geometry of the spatial 2-section is hyperbolic. The electrically charged case was later considered [77], where it was found that the resulting black hole is massless due to the fact that the contributions from the gravitational part of the action to the mass, get canceled by the contributions of the matter part of the action. Solutions with minimally coupled scalar field where also considered in [76], where a generalization of the MTZ black hole was investigated, with a scalar potential that breaks the conformal invariance in the Einstein frame.

1.3.4 Horndeski classes

Regarding the non-minimally coupled scenarios, the literature is also rich, with black holes in the Horndeski scenario being discussed in [90, 91, 92, 93], with time dependent scalar fields being introduced in [94]. Furthermore, bi-metric theories have also provided the framework to discuss hairy black holes [95]. We will briefly discuss the solutions of Horndeski theory reported at [90, 92]. We consider the action:

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{2} - \frac{1}{2}\left(g^{\mu\nu} - zG^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi\right) \tag{1.100}$$

where we consider Einstein's gravity and a scalar field which besides, its usual kinetic energy term, is coupled to the Einstein tensor and z is the coupling constant. This is a Horndeski theory [96] and we expect second order differential equations for the equations of motion. This model has been at first considered for cosmology, since the addition of such a term results to an accelerated expansion without the need of a scalar potential. We will discuss here some local solutions of this model.

Varying with respect to the fields, the field equations are obtained

$$\begin{aligned} G_{\mu\nu} &= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi \\ &- z\Big(- \nabla_\mu\nabla_\nu\phi\Box\phi + \nabla_\alpha(\nabla_\mu\phi)\nabla^\alpha(\nabla_\nu\phi) + R_{\mu\nu\alpha\beta}\nabla^\alpha\phi\nabla^\beta\phi - \frac{1}{2}R\nabla_\mu\phi\nabla_\nu\phi + 2\nabla^\alpha\phi R_{\alpha(\mu}\nabla_{\nu)}\phi \\ &- \frac{1}{2}G_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi + g_{\mu\nu}\Big(- R^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + \frac{1}{2}\big(\Box\phi\big)^2 - \frac{1}{2}\nabla_\alpha\nabla_\beta\phi\nabla^\alpha\nabla^\beta\phi\Big)\Big) \ , \end{aligned} \tag{1.101}$$

$$\Big(g^{\mu\nu} - zG^{\mu\nu}\Big)\nabla_\mu\nabla_\nu\phi = 0 \ . \tag{1.102}$$

We will consider the following metric ansatz

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2d\Omega^2 \ , \tag{1.103}$$

where $d\Omega^2$ is the 2-sphere line element. Since $\nabla g = 0$ and $\nabla_\mu G^{\mu\nu} = 0$ because of the Bianchi identity, we can rewrite the Klein-Gordon equation as

$$\nabla_\mu\Big\{\big(g^{\mu\nu} - zG^{\mu\nu}\big)\nabla_\nu\phi\Big\} = 0 \ . \tag{1.104}$$

Using, $\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}V^\mu\big)$, integrating once, setting the integration constant to zero and considering $\phi'(r) \neq 0$ since this will yield trivial solutions, the Klein-Gordon equation will take the form

$$\frac{rf'(r)}{f(r)} - \frac{r^2h(r)}{z} - h(r) + 1 = 0\ . \tag{1.105}$$

Einstein's equations are rather complicated but we will give them for completeness:

$$h(r)\left(z\phi'(r)\left(\phi'(r)+4r\phi''(r)\right)-2rh'(r)\right)-3rzh'(r)\phi'(r)^2+h(r)^2\left(\left(r^2+z\right)\phi'(r)^2+2\right)-2h(r)^3=0\ , \tag{1.106}$$

$$\phi'(r)^2\left(f(r)\left(h(r)\left(r^2+z\right)-3z\right)-3rzf'(r)\right)+2h(r)\left(f(r)(h(r)-1)-rf'(r)\right)=0\ , \tag{1.107}$$

$$\begin{aligned}&-f(r)(2h(r)(f'(r)(z\phi'(r)(\phi'(r)+2r\phi''(r))-rh'(r))+rzf''(r)\phi'(r)^2)-3rzf'(r)h'(r)\phi'(r)^2\\&+4h(r)^2(f'(r)+rf''(r)))+rh(r)f'(r)^2(2h(r)+z\phi'(r)^2)+f(r)^2(4h(r)(h'(r)-2z\phi'(r)\phi''(r))+\\&6zh'(r)\phi'(r)^2-4rh(r)^2\phi'(r)^2)=0\ ,\end{aligned} \tag{1.108}$$

which are the $tt, rr, \theta\theta$ equations respectively. We solve for $h(r)$ from the Klein-Gordon

$$h(r) = \frac{z\left(rf'(r)+f(r)\right)}{f(r)\left(r^2+z\right)}\ . \tag{1.109}$$

Now substituting back to tt and $\theta\theta$, we obtain a relation for $\phi'(r)^2$

$$\phi'(r)^2 = -\frac{r^2\left(rf'(r)+f(r)\right)}{f(r)\left(r^2+z\right)^2}\ , \tag{1.110}$$

and from rr a differential equation for $f(r)$ can be found

$$rf(r)\left(rf'(r)+f(r)\right)\left(r\left(3r^2z+r^4+2z^2\right)f''(r)+2z\left(3r^2+2z\right)f'(r)-2r^3f(r)\right)=0\ , \tag{1.111}$$

which has a solution

$$f(r) = \frac{c_1}{r} + \frac{c_2r^2}{3} + \frac{c_2z^{3/2}\tan^{-1}\left(\frac{r}{\sqrt{z}}\right)}{r} + 3c_2z\ . \tag{1.112}$$

The obtained configurations satisfy all components of Einstein's equations and the Klein-Gordon. The asymptotic expressions at zero and at infinity are

$$f(r\to 0) \sim \frac{c_1}{r} - \frac{c_2r^6}{7z^2} + \frac{c_2r^4}{5z} + 4c_2z\ , \tag{1.113}$$

$$f(r\to\infty) \sim \frac{c_1 + \frac{\pi c_2}{2\left(\frac{1}{z}\right)^{3/2}}}{r} - \frac{c_2z^4}{5r^6} + \frac{c_2z^3}{3r^4} - \frac{c_2z^2}{r^2} + \frac{c_2r^2}{3} + 3c_2z\ . \tag{1.114}$$

We will modify $f(r)$ in order to match Schwarzchild solution at small distances. Setting $c_1 = -2m$ and $c_2 = \frac{1}{4z}$ and $f(r)$ becomes

$$f(r) = +\frac{3}{4} - \frac{2m}{r} + \frac{r^2}{12z} + \frac{\sqrt{z}\tan^{-1}\left(\frac{r}{\sqrt{z}}\right)}{4r}\ , \tag{1.115}$$

while now the asymptotic expressions read

$$f(r \to 0) \sim 1 - \frac{2m}{r} + \frac{r^4}{20z^2} + \mathcal{O}\left(r^6\right) \,, \tag{1.116}$$

$$f(r \to \infty) \sim \frac{3}{4} + \frac{r^2}{12z} + \frac{\frac{\pi}{8\sqrt{1/z}}}{r} - \frac{2m}{r} - \frac{z}{4r^2} + \mathcal{O}\left(\left(\frac{1}{r}\right)^4\right) \,. \tag{1.117}$$

We can see that z acts as an effective cosmological constant term. Considering that $z > 0$, the metric at infinity behaves similar to the Schrwarzchild-AdS solution. Imposing $m > 0$, the metric has only one root which indicates the position of the black hole horizon. We present plots for the metric function $f(r)$ and for the squared derivative of the scalar field.

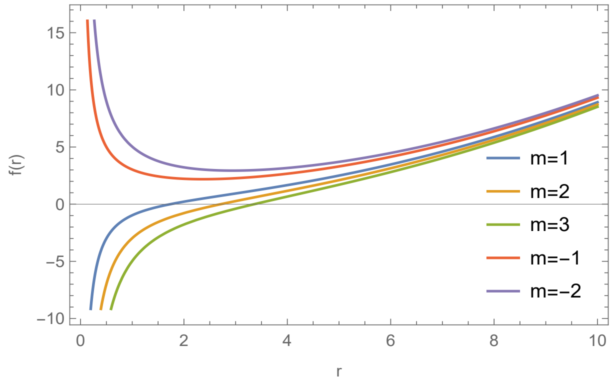


Figure 1.1: The metric function $f(r)$. Here we fix $z = 1$ and change m.

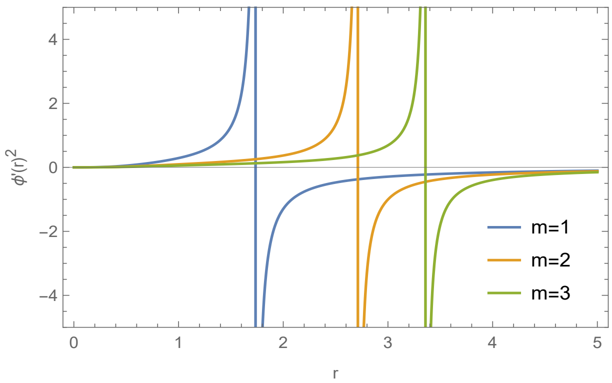


Figure 1.2: The derivative of the scalar field squared $\phi'(r)^2$ while changing m ($z = 1$).

We can see that the derivative of the scalar field blows at the event horizon of the black hole. Moreover $\phi'(r)^2$ is negative outside of the horizon and the scalar field behaves as a ghost, since $f(r) > 0$ outside the horizon while inside the horizon, the scalar field behaves as a regular one, since $f(r) < 0$.

The Kretschmann scalar is divergent at the origin $r \to 0$. It's expression is complicated but we'll give a plot.

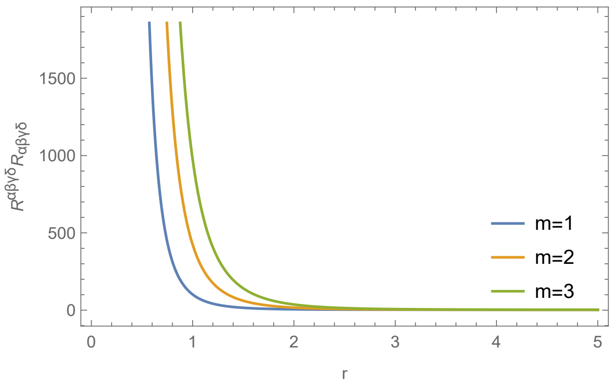


Figure 1.3: The Kretschmann scalar while changing m ($z = 1$).

The temperature is given by

$$T = \frac{1}{\beta}, \tag{1.118}$$

where $\beta = 2\pi/\kappa$, where

$$\kappa = \frac{1}{2}\frac{(-g_{tt})'}{\sqrt{-g_{tt}g_{rr}}}\bigg|_{r=r_h} . \tag{1.119}$$

Then, the temperature at the horizon can be obtained:

$$T(r_h) = \frac{r_h^2 + 2z}{8\pi z r_h} , \tag{1.120}$$

where, r_h is the position of the black hole horizon. The temperature is always positive since $z > 0$. In the limit of $z \to \infty$ we recover the temperature of the Schwarzchild black hole.

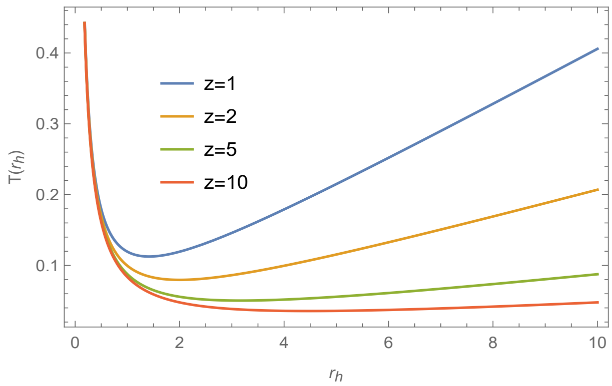


Figure 1.4: The temperature while changing z.

The temperature has a minimum value. We compute the derivative of the temperature with respect to the horizon

$$T'(r_h) = \frac{r_h^2 - 2z}{8\pi r_h^2 z} .$$

It has a root located at $r_0 = \sqrt{2z}$. The second derivative is positive at this point $T''(r_0) = \frac{1}{4\sqrt{2}\pi z^{3/2}}$ meaning that r_0 is a total minima and the value of the minima is $T(r_0) = \frac{1}{2\sqrt{2}\pi\sqrt{z}} = \frac{0.11254}{\sqrt{z}}$.

1.4 Outline of the thesis

The subject of this doctoral thesis is the study of black hole solutions in modified theories of gravity, and in particular, the study of black holes coupled to scalar fields in the context of $f(R)$ theory of gravity and string-inspired theories. Both $f(R)$ gravity and string inspired theories are well acclaimed frameworks in theoretical physics. $f(R)$ theories are mostly used to describe cosmological scenarios in the early and late universe, while string theories attempt to unify gravitational interactions with other fundamental forces. In any case, modifications of linear theories, such as GR or Maxwell's theory, will inevitably play a crucial role in the strong field regime.

This thesis is separated into two parts. In the first part we examine black hole solutions in $f(R)$ gravity theories coupled to scalar fields. We have showed that the $f(R)$ theory is mathematically equivalent to GR plus a scalar field. In our case, the scalar field is introduced *ad hoc* and is not of fundamental origin. Our motivation is to consider departures from GR with the help of $f(R)$ gravity, keeping the matter content of some well known GR solutions the same, in order to make comparisons between the GR case and the $f(R)$ case.

In the first three chapters 2, 3, 4 we deal with the context of a minimally and non-minimally coupled scalar field in $(2+1)$ dimensions. $(2+1)$ dimensional General Relativity (GR) has gained a lot of interest over the decades, since the introduction of a negative cosmological constant results to a black hole solution, the BTZ black hole named after Bañados, Teitelboim, and Zanelli [97]. This came as a surprise in the scientific community, since in (2+1) dimensions, the Weyl tensor $C_{\alpha\beta\gamma\delta}$ vanishes by definition, in the absence of matter the Einstein equation reads

$$R_{\mu\nu} = 0 = R \, , \tag{1.121}$$

where $R_{\mu\nu}, R$ denote the Ricci tensor and the Ricci scalar. Now, since the Riemann tensor $R_{\alpha\beta\gamma\delta}$ contains all information about the geometry and is given by

$$R_{\alpha\beta\gamma\delta} = 2\left(g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\gamma}R_{\delta]\alpha}\right) - Rg_{\alpha[\gamma}g_{\delta]\beta} - C_{\alpha\beta\gamma\delta} \, , \tag{1.122}$$

we can deduce that

$$R_{\alpha\beta\gamma\delta} = 0 \, , \tag{1.123}$$

and hence no non-trivial geometry can be formed. Moreover, now the Riemann curvature tensor which will contain all information about the curvature of spacetime may be written in terms of the energy momentum tensor and its trace, and there will be no pure geometrical contribution to the Riemann tensor, which basically means that in three spacetime dimensions, the notion of gravitational mass does not exist (a gravitational propagating degree of freedom, the graviton). However the inclusion of a negative cosmological constant in the action

$$S = \int d^3x\sqrt{-g}\left(R + 2l^{-2}\right) \, , \tag{1.124}$$

where l is the Anti de Sitter (AdS) radius, gives non-trivial contributions to the Einstein tensor, and as a result, the Ricci scalar and Ricci tensor are no more equal to zero but proportional to the cosmological constant

$$G_{\mu\nu} - l^{-2}g_{\mu\nu} = 0 \, , \tag{1.125}$$

$$R = -\frac{6}{l^2} \, . \tag{1.126}$$

By solving the field equation one finds the following line element

$$ds^2 = -b(r)dt^2 + b(r)^{-1}dr^2 + r^2\Big(u(r)dt + d\theta\Big)^2 \tag{1.127}$$

with

$$b(r) = \frac{J^2}{4r^2} - M + \frac{r^2}{l^2}, \tag{1.128}$$

$$u(r) = -\frac{J}{2r^2}. \tag{1.129}$$

where M, J are the mass and the angular momentum of the black hole. Of course the static case refers to $J = 0$. However, we have said that there is no gravitating mass in three dimensions. This raises a question about the nature of the integration constant M. To discuss this, let us perform some more calculations and consider for simplicity the static case. The action is

$$S = \int d^3x\sqrt{-g}\mathcal{L} = \int d^3x\sqrt{-g}(R - 2\Lambda) \ . \tag{1.130}$$

The above action may be written in Hamiltonian form as

$$\mathcal{I} = \int d^2x dt \left(\pi^{ij}\dot{g}_{ij} - NH - N_i H^i\right) \ , \tag{1.131}$$

where N is the lapse function, N^i is the shift vector and H, H^i are Hamiltonian constraints. We will take that $t_1 \leq t \leq t_2$ and $0 \leq \phi \leq 2\pi$, while $r \geq r_h$. We will use the Euclidean line element

$$ds^2 = N(r)^2F(r)d\tau^2 + \frac{dr^2}{F(r)} + r^2d\phi^2 \ . \tag{1.132}$$

Now the Hamiltonian action is

$$\mathcal{I} = -2\pi(t_2 - t_1)\int_{r=r_h}^{\infty} N(r)rN(r)H(r)dr \ , \tag{1.133}$$

where here $N(r)H(r) = -\mathcal{L}$. We will work with Euclidean periodic time $\tau = it$, with a period of β Moreover, the Euclidean action $\mathcal{I}_E$ is related to the Lorentzian action $\mathcal{I}_L$ via $\mathcal{I}_E = -i\mathcal{I}_L$. By applying these in (1.133) we have that

$$\mathcal{I}_E = 2\pi(\tau_2 - \tau_1)\int_{r=r_h}^{\infty} N(r)rN(r)H(r)dr = 2\pi\beta\int_{r=r_h}^{\infty} N(r)rN(r)H(r)dr \tag{1.134}$$

We now compute the quantity

$$N(r)rN(r)H(r) = -\mathcal{L}N(r)r = rN(r)(2\Lambda - R(r)) \ , \tag{1.135}$$

for the Euclidean line element (1.132). This calculation yields

$$3rF'(r)N'(r) + 2N(r)F'(r) + rN(r)F''(r) + 2F(r)N'(r) + 2rF(r)N''(r) + 2\Lambda rN(r) \ , \tag{1.136}$$

which up to boundary terms this is equal to

$$N(r)\left(F'(r) + 2\Lambda r\right) \ . \tag{1.137}$$

Now the action becomes

$$\mathcal{I}_E = 2\pi\beta \int_{r=r_h}^{\infty} N(r)\left(F'(r) + 2\Lambda r\right) dr + \mathcal{B}\ , \tag{1.138}$$

where $\mathcal{B}$ is a boundary term. We now want $\delta\mathcal{I}_E = 0$ to hold. This will give the equations of motion that give the BTZ black hole, while the variation of $\mathcal{B}$ will be such that it will cancel the boundary terms that will arise when we vary. Now, variation with respect to F gives

$$N'(r) = 0 \rightarrow N(r) = 1\ , \tag{1.139}$$

while variation with respect to N yields

$$2r\Lambda + F'(r) = 0 \rightarrow F(r) = -r^2\Lambda - M\ . \tag{1.140}$$

This is the stationary BTZ black hole. To find these equations we have cancelled a boundary term which was

$$\frac{d}{dr}\left(N(r)\delta F\right)\ . \tag{1.141}$$

Going back to our action, we have that

$$\delta\mathcal{I}_E = 2\pi\beta\left(N(r)\delta F(r)\right)\Big|_{r_h}^{\infty} + \delta\mathcal{B} = 0\ , \tag{1.142}$$

in order to have a well defined variational principle. Evaluating the variations at infinity we have

$$\delta F(\infty) = -\delta M\ , \tag{1.143}$$

while at the horizon we obtain

$$\delta F(r_h) = -F'(r_h)\delta r_h\ . \tag{1.144}$$

As a result, the boundary term has to be such that

$$\delta\mathcal{I}_E = 0 \rightarrow 2\pi\beta\delta F(\infty) - 2\pi\beta\delta F(r_h) + \delta\mathcal{B} = 0 \rightarrow$$

$$-2\pi\beta\delta M + \delta\mathcal{B}(\infty) + 2\pi\beta F'(r_h)\delta r_h + \delta\mathcal{B}(r_h) = 0\ ,$$

and therefore,

$$\delta\mathcal{B}(\infty) = 2\pi\beta\delta M \tag{1.145}$$

while

$$\delta\mathcal{B}(r_h) = -2\pi\beta F'(r_h)\delta r_h \tag{1.146}$$

and hence

$$\mathcal{B}(\infty) = 2\pi\beta M\ , \tag{1.147}$$

$$\delta\mathcal{B}(r_h) = -2\pi\beta\frac{4\pi}{\beta}\delta r_h \rightarrow \mathcal{B}(r_h) = -4\pi A_h\ , \tag{1.148}$$

where we have set that $F'(r_h) = 4\pi/(\beta))$ in order to avoid conical singularities at the horizon in the Euclidean space. We calculated the Euclidean action. It reads

$$\mathcal{I}_E = \mathcal{B}(\infty) + \mathcal{B}(r_h) = 2\pi\beta M - 4\pi A_h \tag{1.149}$$

However, the Euclidean action is related to the free energy of the thermodynamic system in the Grand Canonical Ensemble (fixed temperature) as

$$\mathcal{I} = \beta\mathcal{M} - \mathcal{S} \, , \tag{1.150}$$

and now we can compute the conserved black hole mass as well as entropy at the event horizon of the black hole as

$$\mathcal{M} = 2\pi M \tag{1.151}$$

and

$$\mathcal{S} = 4\pi A_h \, . \tag{1.152}$$

Our calculation agrees with that of the BTZ paper [98], which has a factor of $1/2\pi$ in their action which if taken into account would yield that $\mathcal{M} = M$ and $\mathcal{S} = 4\pi r_h$. Hence by using gravitational path integral arguments one can see that M will be the total thermodynamic mass-energy contained in the $(2+1)$ dimensional spacetime.

With the inclusion of Maxwell electrodynamics

$$S_{\text{ED}} = -\frac{1}{4}\int d^3x\sqrt{-g}F_{\mu\nu}F^{\mu\nu} \, , \tag{1.153}$$

where $F_{\mu\nu}$ the antisymmetric Faraday tensor, we find the charged BTZ black hole with a lapse function $b(r)$

$$b(r) = -M + \frac{r^2}{l^2} - 2Q^2 \ln\frac{r}{l} \, , \tag{1.154}$$

while the Maxwell field will read

$$A_\mu = (A_t(r), 0, 0) = \left(-Q\ln\frac{r}{l}, 0, 0\right) \, . \tag{1.155}$$

The charged rotating solution is a more subtle case [99, 100, 101, 102].

Before addressing modified theories of gravity, in the second chapter 2 we set the stage by discussing an exact, asymptotically AdS black hole solutions dressed with a scalar hair in $(2+1)$ spacetime dimensions. We find that such a discussion is important, because a simple exact spacetime of a simple minimally coupled to gravity scalar field theory was missing from the literature. We discuss, in detail, the implications of the scalar hair on the resulting spacetime, evaluating the thermodynamic quantities, the energy conditions and we also investigate rotating solutions.

In chapter 3 of this thesis we will dress the BTZ black hole with a scalar hair, in the context of $f(R)$ gravity [1]. By considering the theory

$$S = \int d^3x\sqrt{-g}\left\{\frac{1}{2\kappa}f(R) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right\} \, , \tag{1.156}$$

solving the field equations we will find that the Newton's constant is modified near the origin, due to the presence and strength of the scalar field there, while at large distances, the effects of the scalar field are (almost) negligable and the uncharged and non-rotating BTZ black hole is recovered. We find that the scalar potential includes a mass term that satisfies the Breitenlohner-Freedman bound in three dimensions [88, 89], ensuring the stability of AdS spacetime under scalar perturbations, hence our solution is stable. Moreover, the novel black hole solution possesses a larger radius for the event horizon of the black hole than the corresponding BTZ black hole, and as a result our solution is thermodynamically preffered, having higher entropy at the event horizon. To find this solution we fixed the scalar field function with a particular form, however, since the physics of our new black hole depends on the asymptotic behavior of matter, a scalar field and potential that satisfy the conditions

$$\phi\,(r \to \infty) = 0\;, \quad V\,(r \to \infty) = 0\;, \quad V\big|_{\phi=0} = 0\;, \tag{1.157}$$

(matter vanishes at space infinity) are expected to yield black hole solutions with the same properties.

In chapter 4 we consider a scalar field non-minimally coupled to gravity [4]. The non-minimal coupling term is given by

$$S_{\rm nmc} = -\int d^3x\sqrt{-g}\xi R\phi^2\;, \tag{1.158}$$

where ξ is a constant that expresses the strength of the coupling between matter and gravity. There is a critical value for this constant, which for a d-dimensional spacetime is given by

$$\xi = \frac{d-2}{4(d-1)} \tag{1.159}$$

and in the case of $d = 2+1$, $\xi = 1/8$. This particular value corresponds to a conformally coupled theory, which means that the action will be invariant under conformal transformations. In general, a theory

$$S = \int d\mathrm{Vol}\mathcal{L} \tag{1.160}$$

is conformally invariant when a conformal rescaling of the form

$$g_{\mu\nu} \to \Omega^2(x)g_{\mu\nu}\;, \tag{1.161}$$

will transform the Lagrangian density $\mathcal{L}$ as

$$\mathcal{L} \to \Omega(x)^{-4}\mathcal{L}\;, \tag{1.162}$$

and since the volume changes as

$$\mathrm{Vol} = \Omega(x)^4\mathrm{Vol}\;, \tag{1.163}$$

the action-theory S is preserved invariant.

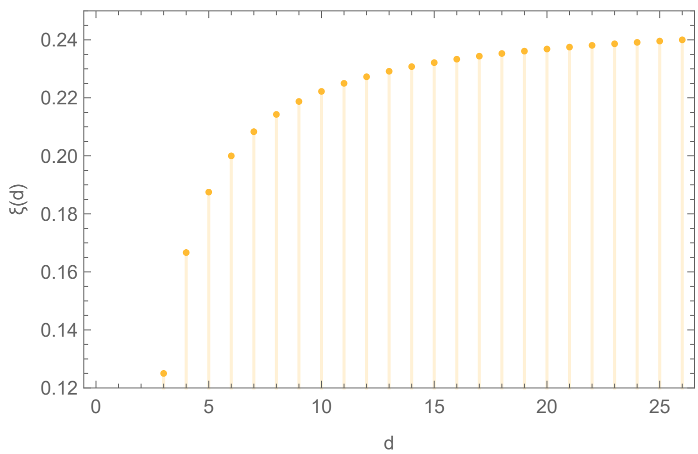


Figure 1.5: The conformal coupling constant $\xi = (d-2)\,/\,(4(d-1))$ for integer spacetime dimensions starting from $d = 3$, up to $d = 26$. For $d = 3+1 = 4$, $\xi = 1/6$.

For $\xi = 1/8$ there exists a well-known black hole solution with a scalar field regular anywhere presented in [19], where the action reads

$$S = \int d^3x\sqrt{-g}\Big\{\frac{R + 2l^{-2}}{\kappa} - \partial_\mu\phi\partial^\mu\phi - \frac{1}{8}R\phi^2\Big\} \ . \tag{1.164}$$

The theory is conformally invariant and as a result the energy-momentum tensor is trace-less, therefore the Ricci scalar is proportional to the negative cosmological constant

$$R = -\frac{6}{l^2} \ . \tag{1.165}$$

The solution of the field equations yields (for $\kappa = 1$)

$$ds^2 = -b(r)dt^2 + b(r)^{-1}dr^2 + r^2d\theta^2 \ , \tag{1.166}$$

$$b(r) = \frac{r^2}{l^2} + \frac{B^2(-2B - 3r)}{l^2 r} \ , \tag{1.167}$$

$$\phi(r) = \sqrt{\frac{8B}{r + B}} \ . \tag{1.168}$$

The matter field, dresses the black hole with a scalar hair of secondary type [87], since the scalar charge B is related to the mass of the black hole via

$$M = \frac{3B^2}{8l^2} \ . \tag{1.169}$$

For completeness, the explicit calculation of the thermodynamic quantities is given in the Appendix A.0.2.

We are interested in extending this theory by replacing R with a general $f(R)$ function, which will eventually introduce a scale (a gravitational one) in the theory, hence a scalar potential is also required to counterbalance this scale. Our theory is given by

$$S = \int d^3x\sqrt{-g}\Big\{f(R) - \partial_\mu\phi\partial^\mu\phi - \frac{1}{8}R\phi^2 - 2V(\phi)\Big\} \ . \tag{1.170}$$

To solve the field equations we assumed that the scalar field takes the particular form of the GR case [19], so the $f(R)$ model will be given by

$$f_R(r) = 1 + \alpha r \rightarrow f(R) = R + \alpha\int^R r(R)dR + C \ , \tag{1.171}$$

where α is the gravitational scale with units $[L]^{-1}$ that will allow for non trivial corrections to Einstein's theory and C is a constant with units $[L]^{-2}$ being related to the cosmological constant. The subscript $_R$ denotes differentiation with respect to the Ricci scalar $df(R)/dR \equiv f_R$. The black hole solution that we obtain has a smaller radius for the event horizon and for larger values of α, the horizon is getting smaller, while for $\alpha = 0$ we smoothly go back to the GR black hole solution [19], therefore, we can compare our novel solution with the one of GR. Calculating the trace of the energy-momentum tensor, we find that the trace is dynamical, meaning that the theory is not invariant under conformal transformations and α breaks the invariance, hence our theory has a scale. The resultant scalar potential has a mass term which is determinded by the non-minimcal coupling ans is given by

$$m_\phi^2 = \frac{1}{8}R \sim \frac{1}{8}6\Lambda_{\text{eff}} = \frac{3}{4}\Lambda_{\text{eff}} \ , \tag{1.172}$$

which satisfies the Breitenlohner-Freedman bound in three dimensions [88, 89], ensuring the stability of AdS spacetime under scalar perturbations where Λ_{eff} is an effective cosmological constant the $f(R)$ theory and non-minimal coupling term generates.

The resulting $f(R)$ model (which, due to the complicated form of the Ricci scalar, we can only calculate asymptotically) is free of ghost and tachyonic instabilities, since it satisfies

$$\frac{df(R)}{dR} > 0 \quad \& \quad \frac{d^2 f(R)}{dR^2} > 0 \ . \tag{1.173}$$

Calculating the thermodynamic quantities of our solution, we find that the $f(R)$ black hole has lower temperature and higher entropy for most values of α, therefore, our solution is thermodynamically preferred over the GR one for most values of α. The interesting part of our novel solution, is that the mass of the black hole is zero. We attribute this fact to the breaking of the conformal invariance, as we show in detail. Similar behaviors have also been found in the literature, when the conformal invariance is broken [77, 76].

In a similar work, which however is not a part of this thesis [2], but it will be mentioned for completeness we considered $(3+1)$-dimensional $f(R)$ gravity in the presence of a scalar field non-minimally coupled to gravity in the action

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left(f(R) - \partial^\mu\phi\partial_\mu\phi - \frac{1}{6}R\phi^2 - 2V(\phi)\right) \ , \tag{1.174}$$

To solve the field equations we fixed the $f(R)$ theory in some particular ways, namely,

$$f(R) = R - 2\alpha\sqrt{R} \ , \tag{1.175}$$

$$f(R) = R - 2\Lambda - 2\alpha\sqrt{R - 4\Lambda} \ . \tag{1.176}$$

The first of these models corresponds to asymptotically flat spacetime (but with a deficit), while the second to asymptotically (A)dS spacetime. To derive exact solutions of the field equations in $f(R)$ gravity is a very difficult procedure. The main motivation to consider these models is that, these models have been used to derive exact black hole solutions in vacuum and coupled to linear, conformally invariant Maxwell electrodynamics [58, 59, 103, 66, 104, 105]. From the cosmological point of view it has been pointed out, that, models of the form $f(R) = R + \alpha R^{-n}$ for $\alpha < 0, \ -1 < n < 0$ are cosmologically unacceptable [106]. In any case, considering this scenario allows for finite scalar field at the event horizon of the black hole, as we discuss in detail. Therefore, we considered them to derive exact black hole solutions with a conformally coupled scalar field. Solving the field equations, we found that indeed the scalar potential, which is determined from the scalar equation of motion preserves the conformal invariance, since it is given by

$$V(\phi) \sim \phi^4 \ . \tag{1.177}$$

Calculating the temperature of this black hole, we find that it is always positive and proportional to the gravitational scale α, while the entropy is always negative, because it receives non trivial corrections due to the square root correction and the non-minimal coupling. For our theory, according to [21, 22, 23] the entropy will be given by

$$S(r_h) = -\frac{1}{4}\oint d^2x\sqrt{h}\left(\frac{\partial\mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}}\right)\Bigg|_H \hat{\varepsilon}_{\alpha\beta}\hat{\varepsilon}_{\gamma\delta} \ , \tag{1.178}$$

where $\mathcal{L}$ corresponds to the Lagrangian of the theory, h is the induced metric on H, $\hat{\varepsilon}_{\alpha\beta}$ is the binormal to the horizon surface H (an antisymmetric quantity satisfying $\hat{\varepsilon}_{\alpha\beta}\hat{\varepsilon}^{\alpha\beta} = -2$) and r_h is the position of the event horizon. The entropy can be found to be

$$S(r_h) = \frac{\mathcal{A}}{4}\left(f_R(r_h) - \frac{1}{6}\phi(r_h)^2\right) \,, \tag{1.179}$$

where $\mathcal{A} = 4\pi r_h^2$ is the surface of the black hole. We will provide a proof for completeness in the Appendix A.0.3.

Plugging in the explicit expressions we can see that the entropy is negative. As we have already discussed, the entropy of theories where a non-minimal coupling between gravity and scalar fields exists may be non-physical.

To cure this problem, we introduced linear electrodynamics. Solving the field equations, we found that, due to the conformal invariance, the electric charge does not appear in the metric function, however it appears in the scalar field function resulting in positive entropy when particular relations hold, as we discussed in detail. This is also the case in GR, as we have already discussed in 1.3.2. However, we found that the introduction of the gravitational scale α results in a finite scalar field at the event horizon of the black hole. In 1.3.2 we saw that this is the case if we consider a positive cosmological constant.

We will now introduce the second part of this thesis, which is based on non-linear electrodynamics (NED). in regions with strong gravitational fields, such as those near black holes, traditional linear theories may break down. Non-linear electrodynamics becomes important in these strong field regimes, where the intensity of electromagnetic fields can become comparable to the strength of gravitational fields. Studying how non-linearities affect the behavior of electromagnetic fields in these regimes is crucial for understanding the physics of objects like black holes, neutron stars, and other astrophysical phenomena. Moreover, non-linear electrodynamics is expected to lead to phenomena that are absent in linear theories. In the early universe for example, when energy densities were extremely high, the interplay between gravitational and electromagnetic fields was significant. NED can be crucial in modeling the behavior of these fields during cosmological evolution. Consequently, NED allows us to investigate how electromagnetic interactions influenced the dynamics of the early universe and whether non-linear effects played a role in the formation of cosmic structures. Understanding these cosmological implications helps build a more complete picture of the evolution of the universe. For a review on non-linear electrodynamics and its applications, see [107] and references therein.

One of the particular aspects of string/brane-induced non-linear electrodynamics effects is that the higher order in the Maxwell tensor can be combined into an all-order expression, the so-called Born-Infeld (BI) Lagrangian [108, 109, 25, 110, 111, 112], as a result of re-summation of open string excitations (attached to, e.g., 3-brane worlds in the D-brane extension of string theory, in which case the world-volume of ($d = 3$)-brane leads to the DiracBI (DBI) action (see [113, 114, 115] and references therein). In such models, the BI electrodynamics in four spacetime dimensions originates from the higher ($d = 10$)-dimensional superstring action upon either compactification or appropriate restriction on a $3d$-brane volume. It is important to note that in all such string-inspired models the BI Lagrangian couples to the inverse of the open-string coupling $g_s = e^{\phi}$, where ϕ is the (dimensionless) dilaton field, so the corresponding four-dimensional action in a curved four-dimensional background metric (in the Jordan or σ-model frame), $g^J_{\mu\nu}$, reads

$$\mathcal{S}^J_{\rm BI} = -\mathcal{T}_4^2 \int d^4x\, e^{-\phi} \sqrt{{\rm Det}\Big(- g^J_{\mu\nu} + \mathcal{T}_4^{-1}\, \mathcal{F}_{\mu\nu}\Big)}\,, \tag{1.180}$$

where $\mathcal{F}_{\mu\nu}$ is the Maxwell tensor $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$, and $\mathcal{T}_4 = \frac{1}{2\pi\alpha'} = \frac{M_s^2}{2\pi}$, is the (open) string tension, with $\alpha' = M_s^{-2}$ the Regge slope (M_s the string mass scale, which in general is different from the four-dimensional Planck scale). In four space-time dimensions, the determinant can be expanded to yield

$$\mathcal{S}_{\rm BI} = -\mathcal{T}_4^2 \int d^4x\, \sqrt{-g^J}\, e^{-\phi} \sqrt{1 + \frac{1}{2\,\mathcal{T}_4^2}\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu} - \frac{1}{16\,\mathcal{T}_4^4}\Big(\mathcal{F}_{\mu\nu}\,\widetilde{\mathcal{F}}^{\mu\nu}\Big)^2} \tag{1.181}$$

where $\widetilde{\mathcal{F}}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\,\mathcal{F}^{\rho\sigma}$ is the dual of the Maxwell tensor, with $\varepsilon_{\mu\nu\rho\sigma}$ the Levi-Civita fully antisymmetric symbol in curved spacetime with metric $g^J_{\mu\nu}$. Expanding the (square root in the) four-dimensional BI action (1.181) in inverse powers of the BI parameter $\mathcal{T}_4$, leads to effective dimension 8 (and higher) operators in the effective field theory, which make contact with the generic Euler-Heisenberg (EH) NED [112, 26, 27]:

$$\begin{aligned}
\mathcal{S}_{\rm BI} &= \int d^4x \sqrt{-g^J}\, e^{-\phi}\Big[- \mathcal{T}_4^2\, I_2 - \mathcal{T}_4^4\, I_4\Big(1 + \mathcal{O}(\mathcal{F}^2)\Big)\Big]\,, \\
I_2 &= \frac{1}{4\mathcal{T}_4^2}\,\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu}\,, \quad I_4 = -\frac{1}{8\,\mathcal{T}_4^4}\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\nu\rho}\,\mathcal{F}_{\rho\lambda}\,\mathcal{F}^{\lambda\mu} + \frac{1}{32\,\mathcal{T}_4^4}\Big(\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu}\Big)^2 .
\end{aligned} \tag{1.182}$$

Hence, ignoring for the moment the dilaton, to fourth order in the field strength $\mathcal{F}_{\mu\nu}$ one obtains (up to the dilaton factors) a special case of the generic EH NED with dimension 8 operators, with Lagrangian:

$$\mathcal{L}_{\rm EH} = c_1\Big(\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu}\Big)^2 + c_2\,\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\nu\rho}\,\mathcal{F}_{\rho\lambda}\,\mathcal{F}^{\lambda\mu}\,, \tag{1.183}$$

where the BI Lagrangian corresponds to [26, 27]

$$c_1 = -\frac{1}{32\mathcal{T}_4^2}\,, \quad c_2 = \frac{1}{8\,\mathcal{T}_4^2}\,. \tag{1.184}$$

The reader should notice that the ratio of $c_2/c_1 = -4$ exactly, which is a characteristic prediction of the BI theory.

Phenomenologically, assuming a constant dilaton and flat Minkowski spacetime, the BI parameter $\mathcal{T}_4^2$ can be constrained in collider physics, via light-by-light scattering, for which there is clear experimental evidence these days at LHC experiments (see [28, 29, 30]). Such light-by-light scattering studies [26] can place a lower bound on the BI parameter $\mathcal{T}_4 > 100$ GeV. In the case of string theory, this would lead to a (weak) lower bound of the string mass scale $M_s > 0.25$ TeV. Notably, extra dimension collider (LHC) searches place currently this bound to $M_s > \mathcal{O}(10)$ TeV. Forecasts for much larger values of the lower bounds of the BI parameter in future colliders, in particular FCC, have been given in [27]. Embedding the BI (or more generally Euler-Heisenberg) theory into curved spacetime, and fully incorporating the dilaton effects, leads to a whole new area of tests of NED by employing the entire machinery of modern gravitational experiments technology.

The BI action $\mathcal{S}_{\rm BI}$ (1.180) and (1.181) in curved background metrics can be augmented, at an effective field theory level, by including the dynamics of the gravitational ($g^J_{\mu\nu}$) and dilaton (ϕ) fields. In this respect, we recall that the D-brane action is by construction in the so-called Jordan (or σ-model) frame. Passing into the Einstein frame in four dimensions, via the transformation of the metric: $g^J_{\mu\nu} \to g_{\mu\nu} = e^{-2\phi} g^J_{\mu\nu}$, we write for the pertinent gravitational action (in geometrized units $c = G = 1$, in which we work from now on):

$$\mathcal{S} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \Big[\mathcal{R} - 2\nabla^\mu \phi \nabla_\mu \phi\Big] - \int d^4x \, \sqrt{-g}\, e^{-\phi} \Big[\mathcal{T}_4^2 \, I_2^{\rm E} + \mathcal{T}_4^4 \, e^{-4\phi} \, I_4^{\rm E}\Big] + \dots \,, \qquad (1.185)$$

where the quantities $I_i^{\rm E}$, $i = 2, 4$ are given by the corresponding ones in (1.182), but the indices contraction is made by the Einstein-frame metric $g_{\mu\nu}$.

Departing from the case of the brane DBI action (1.180), one may consider higher-order (in derivatives, that is in a Regge slope α' expression) electromagnetic terms in effective low energy field theories stemming only from closed strings, e.g. the heterotic string [116]. In such theories, unlike the DBI brane or open-string case, there is no re-summation in closed form of the gauge terms. Nonetheless, some authors have generalised the BI effective action in a curved $(3+1)$-dimensional spacetime, by considering the following form of dilaton couplings to the electromagnetic fields in a BI NED setting [117, 118, 119]:

$$\begin{aligned} \mathcal{S} &= \frac{1}{16\pi} \int d^4x \sqrt{-g} \Big[\mathcal{R} - 2\nabla^\mu \phi \nabla_\mu \phi + \mathcal{L}_{\rm BI}\Big] \,, \\ \mathcal{L}_{\rm BI} &= 4\beta_{\rm BI}\, e^{2\gamma\phi} \left(1 - \sqrt{1 + \frac{e^{-4\gamma\phi}}{2\beta_{\rm BI}} \mathcal{F}^2 - \frac{e^{-8\gamma\phi}}{16\,\beta_{\rm BI}^2} (\mathcal{F}\widetilde{\mathcal{F}})^2 } \,\right) \end{aligned} \qquad (1.186)$$

where the notation has been defined above, γ defines the dilaton coupling, and now $\beta_{\rm BI}$ plays the role of the generalised BI parameter, with mass dimensions $+2$ (which is identified with $\mathcal{T}_4^2$ in the case of strings, in which case, to match with the corresponding $\mathcal{O}(\alpha')$ Maxwell terms of the heterotic-string effective action [116], $e^{-2\phi}\mathcal{F}^2$, one should fix $\gamma = 1$).

The above considerations deal with tree-level in string loops, that is first quantized actions on world-sheet with trivial topology ($2d$ sphere for closed string sectors, and disc for open one). In general, string loop effective actions are not known in closed form. In simplified phenomenological scenarios such effective actions can be expressed in the generic form, e.g. in the closed string sector in the string (or σ-model frame with metric $\widehat{g}_{\mu\nu}$ in $(3+1)$-dimensions, after string compactification) [31]:

$$S = \int d^4x \, \sqrt{-\widehat{g}} \, \Big(\frac{1}{\alpha'} \, B_g(\Phi) \, \widehat{\mathcal{R}} + \frac{1}{\alpha'} \, B_\Phi(\Phi) \, \Big[\widehat{\Box}\Phi - 4 \widehat{\nabla}_\mu \Phi \, \widehat{\nabla}^\mu \Phi \Big] - \frac{B_F(\Phi)}{4} \widehat{F}_{\mu\nu} \, \widehat{F}^{\mu\nu} - B_\psi(\Phi) \overline{\widehat{\psi}} \, \widehat{\not{D}} \widehat{\psi} + \dots \Big) \,, \qquad (1.187)$$

where the $\widehat{(...)}$ symbol above a tensorial quantity implies contraction of the world-indices with the string-frame metric $\widehat{g}_{\mu\nu}$, $F_{\mu\nu}$ denotes the field strength of the gauge field, D_μ is the gauge covariant derivative, ψ are fermionic matter fields and the $\dots$ denote other matter fields as well as (an infinity of) higher-derivative (higher order in α') terms, The quantities $B_i(\Phi), i = g, \Phi, F, \psi$ are non-derivative functions of the dilaton which arise from summing over (closed) world-sheet topologies, that is these functions involve powers of the string coupling $g_s = \exp(\Phi)$ of the form $g_s^{-\chi}$, where $\chi = 2 - 2N$ where N denotes the number of handles, is the genus of the world-sheet surface (sphere has $N = 0$, torus (one string loop) $= 0$ etc. Thus,

$$B_i(\Phi) = e^{-2\Phi} + c_0^{(i)} + c_1^{(i)} e^{2\Phi} + \dots + c_{2n}^{(i)} \, e^{2n\Phi} + \dots \,, \qquad (1.188)$$

where the constant quantities c_i pertain to effects of string loops, so that the expressions (1.188) involve a power series in the square of the string coupling $g_s^2 = \exp(2\Phi)$. The first term on the right-hand side of (1.188) leads to the standard closed string expression for the gauge field Maxwell terms in the low-energy effective action, $e^{-2\phi}\mathcal{F}^2$ for standard dilaton kinetic term normalization in the Einstein frame [116, 120, 121]).[2]

Passing to the Einstein frame, via appropriate redefinitions of [31]: the metric $\widehat{g}_{\mu\nu} \to g_{\mu\nu} = C\, B_g(\Phi)\widehat{g}_{\mu\nu}$, where C are numerical normalization constants, the dilaton $\Phi \to \phi = \int d\Phi \sqrt{\frac{3}{4}\Big(\frac{B_g'}{B_g}\Big)^2 + 2(\frac{B_\Phi'}{B_\Phi} + \frac{B_\Phi}{B_g}})$, where the prime denotes $d/d\Phi$, and the fermionic matter fields, $\widehat{\psi} \to \psi = C^{-3/4}\, B_g^{-3/4}\, B_\psi^{1/2}\, \widehat{\psi}$, leaving the gauge fields as they are, yields the effective action:

$$S = \int d^4x\, \sqrt{-g}\frac{1}{16\pi\overline{\mathrm{G}}}\, R - \frac{1}{8\pi\overline{\mathrm{G}}} + S_{\rm matter}\,,$$
$$S_{\rm matter} = \int d^4x\, \sqrt{-g}\Big[- \overline{\psi}\not{D}\psi - \frac{1}{4}B_F(\Phi)F_{\mu\nu}\,F^{\mu\nu} + \dots \Big]\,, \qquad (1.189)$$

where the reader should notice the potential existence (depending on the specific type of string theory considered) of constant (dilaton-independent) terms involving F^2 terms (*cf.* the $c_0^{(F)}$ terms on the right-hand side of (1.188)).

In case we consider more general theories involving a combination of closed and open strings (the latter attached, e.g. to brane universe)s, for which one obtains effective actions in the Einstein frame that include both closed- and open-string sectors (the latter leading to DBI terms of the form appearing in the second integral on the right-hand side of (1.185)), then, the inclusion of string loops can lead, following similar arguments to the closed-string case (1.189), to generalised situations, in which the (string-loop corrected) effective action acquires the form in the Einstein frame [122]: [3]

$$\mathcal{S} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big[\mathcal{R} - 2\nabla^\mu\phi\nabla_\mu\phi\Big] - \int d^4x\,\sqrt{-g}\,B_{F^2}(\phi)\,\Big[\mathcal{T}_4^2\, I_2^{\rm E}\Big] - \int d^4x\,\sqrt{-g}\,\mathcal{T}_4^4\,B_{F^4}(\phi)\,I_4^{\rm E}\Big] + \dots\,, \qquad (1.191)$$

where the functions $B_{F^i}(\phi),\ i = 2, 4$ admit a power series expansion in the string coupling, summing

[2]We note for completion that a similar factor accompanies the quadratic gravitational curvature (Gauss-Bonnet (GB)) terms in the action at string-loop tree level. This is a remnant of the corresponding situation of the ten-dimensional target-spacetime heterotic-string effective field theory action, which in the extra (compact) dimensional sector leads to the celebrated anomaly cancellation by equating the extra-dimensional (non-Abelian) gauge with the corresponding quadratic-curvature gravitational GB terms $\int d^6x\sqrt{-G^{(6)}}e^{-2\Phi}\Big({\rm Tr}\mathbf{F}^2 - \mathbf{R}^2_{\rm GB}\Big) \to 0$ (with the Tr being a group-index trace), which leads to the Heterotic string selecting the $E_8 \times E_8$ gauge group as the unique target-space group before compactification to (3+1)-dimensions [120, 121].

[3]Indeed, if only Abelian gauge fields are considered then only open world-sheet surfaces are taken into account, in order to evaluate the pertinent contribution to the effective action, as discussed explicitly in [122] where it was shown that the loop corrected effective action acquires the form in the sigma-model frame (ignoring antisymmetric tensor fields contributions, which are of no interest in the present discussion):

$$S = \int d^4x\sqrt{-\widehat{g}}\,\alpha'^{-2}e^{-2\phi}\,\Big[- \frac{3}{2}\alpha'\,(\widehat{\mathcal{R}} + 4(\partial\phi)^2 + \dots] + d_1\,e^{-\phi}\,\sqrt{{\rm det}\Big(\widehat{g}_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\Big)} + d_2 + d_3 + d_4\,e^{\phi} + \dots\,, \qquad (1.190)$$

where $d_i,\, i = 1, 2, \dots$, denote finite parts of the dilaton tadpoles, and the dots denote contributions from higher derivative corrections, as well as higher string loops (that is higher powers of the string coupling $g_s = \exp(\phi)$. Passing onto the appropriate Einstein frame leads to actions of the form (1.191).

up terms of the generic form

$$B_{F^i}(\phi) = \sum_\chi g_s^{-\chi} c_\chi^{(F^i)} , \qquad i = 1, 2 , \qquad g_s = \exp(\phi) , \tag{1.192}$$

where $\chi = 2 - 2N - N_H$, with N_H the number of holes (or boundaries) (eg disc has genus $\chi = 1$, since $N = 0$, $N_H = 1$ etc), where finite parts of dilaton tadpoles contribute to the coefficients $c_\chi^{(F^i)}$.

In heterotic strings, which do not involve branes, the higher derivative EH electrodynamics terms do not appear in a closed BDI form. In that case a more general action, involving EH terms, after summation over string loops, might then be considered, in the Einstein frame:

$$\mathcal{S} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \Big[\mathcal{R} - 2\nabla^\mu \phi \nabla_\mu \phi \Big] - \int d^4x \sqrt{-g}\, B_{F^2}(\phi) \left[\mathcal{T}_4^2\, I_2^{\rm E} \right] - \int d^4x \sqrt{-g}\, \mathcal{T}_4^4 B_{F^4}(\phi)\, L_{\rm EH} + \dots , \tag{1.193}$$

where the ellipsis $(\dots)$ includes possible string-loop generated dilaton-potential terms, whose precise form is not known at present, as this is a highly string-model-dependent issue, and the functions $B_{F^2}(\phi)$, $B_{F^4}(\phi)$ in this case are given by power series expansions of even powers of the string coupling, of the form (1.188), as only closed world-sheet surfaces are involved. The Euler-Heisenberg Lagrangian $L_{\rm EH}$ is given by (1.183), but the coefficients $c_i, i = 1, 2$ no longer satisfy (1.184), given that the DBI action no longer describes the electromagnetic self-interactions in closed form.

As a result, NED is a well motivated field of research as it arises in more fundamental theories, as we have already discussed. For this reason we will consider different versions of the following action

$$\mathcal{S} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \Big[\mathcal{R} - 2\nabla^\mu\phi\nabla_\mu\phi - V_{\rm scalar}(\phi) - F(\phi)\mathcal{F}^2 - f(\phi)\big(2\alpha \mathcal{F}^\alpha{}_\beta \mathcal{F}^\beta{}_\gamma \mathcal{F}^\gamma{}_\delta \mathcal{F}^\delta{}_\alpha - \beta\mathcal{F}^4\big)\Big] , \tag{1.194}$$

solve the resulting field equations in various scenarios and discuss their properties. At first we will consider that $F(\phi) = 1 = f(\phi)$ in the following action functional

$$S = \int d^4x \sqrt{-g}\mathcal{L} = \int d^4x \sqrt{-g} \left(\frac{R}{2} - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi) - P + \alpha P^2 + \beta Q^2 \right) , \tag{1.195}$$

which is basically the EH theory with a minially coupled scalar field. This is discussed in chapter 5 and the pertinent paper is [5]. To solve the field equations, we introduced pure magnetic fields via the one-form (in spherical symmetry)

$$A_\mu = (0, 0, 0, K(\theta)) . \tag{1.196}$$

In this case the scalar $Q^2 \sim E \times B$ in the above action turns out to be zero, since it vanishes if we do not consider dyons. Then the Maxwell equation obtained by variation with respect to A_μ yields

$$\nabla^\mu (F_{\mu\nu} - 2\alpha P F_{\mu\nu}) = 0 . \tag{1.197}$$

For $K(\theta) = Q_m \cos\theta$, these equations are trivially satisfied. In the Appendix A.0.4 we prove this result. This simple form of the electromagnetic one-form enabled us to obtain an exact black hole solution, dressed with a secondary scalar hair in the context of non-linear electrodynamics. In chapter 5 we will discuss in great detail the influence of each of the parameters of the black hole solution. For now, let us comment on the scalar potential. We found that the potential supporting such solution is

$$V(\phi) = \frac{1}{3\nu^8}\Bigg(\nu^8\Lambda_{\text{eff}}\left(\cosh\left(\sqrt{2}\phi\right)+2\right)-36m\nu^5\left(\sqrt{2}\phi\left(\cosh\left(\sqrt{2}\phi\right)+2\right)-3\sinh\left(\sqrt{2}\phi\right)\right)-4\alpha Q_m^4 *$$
$$\left(288\phi^2+2\left(72\phi^2+71\right)\cosh\left(\sqrt{2}\phi\right)-432\sqrt{2}\phi\sinh\left(\sqrt{2}\phi\right)+100\cosh\left(2\sqrt{2}\phi\right)-14\cosh\left(3\sqrt{2}\phi\right)+\cosh\left(4\sqrt{2}\phi\right)\right.$$
$$\left.-229\right)+6\nu^4Q_m^2\left(8\phi^2+4\left(\phi^2+2\right)\cosh\left(\sqrt{2}\phi\right)-12\sqrt{2}\phi\sinh\left(\sqrt{2}\phi\right)+\cosh\left(2\sqrt{2}\phi\right)-9\right)\Bigg)\,. \tag{1.198}$$

While not entirely clear in this form, the scalar potential is actually independent of the black hole paramters. By defining $\chi = m/\nu^3$, $\psi = Q_m^2/\nu^2$, the potential will contain only fundamental consants fixed by the theory and consiquently, both the mass and the magnetic charge are allowed to vary, consistently in order to keep the afforementioned ratios constant. These types of potentials are actually important in black hole physics, since they provide a controlled way to violate the no-hair theorem and they have their root in the context of supergravity.

When discussing the nature of the black hole, we came across an interesting scenario, that of many horizons. Due to the asymptotic nature of the metric function near the origin and at large distances, we at first established that our solution always describes a black hole with at least one horizon in asymptotically AdS or flat spacetime. However, we have found that the black hole might have up to three horizons depending on the values of the parameters, introducing in this way the notion of a black hole inside of a black hole, since the smallest and largest roots of the metric function will denote event horizons.

Furthermore, we discussed the thermodynamics of the solution. At first we calculated the temperature of the black hole when it possesses only one horizon and we found that in the flat case the temperature decreases as the event horizon radius is getting larger, or one can see it from the opposite side, the temperature is getting larger as the black hole shrinks meaning that when the event horizon ceases to exist, the black hole evaporated away. This should be also the case in any black hole spacetime. For example, in the Schwarzchild scenario, a textbook calculation unveils that

$$T = \frac{1}{8\pi M} = \frac{1}{4\pi r_h}\,, \tag{1.199}$$

where one can clearly see the aforementioned discussion. However, this thermal evaporation will take very long periods of time. The Stefan-Boltzmann law states that for an emitter or a black body, the energy emitted per unit surface area per unit time is proportional to the temperature to the fourth power and in terms of black hole physics this is

$$\frac{dM}{dtA_h} \sim T^4 \to \frac{dM}{dt} \sim T^2 = M^{-2} \to t \sim M^3\,, \tag{1.200}$$

where $A_h = 16\pi M$ is the area of the Schwarzchild black hole which is a back-of-the-envelop calculation to see that for astrophysical black holes, this is indeed a huge amount of time!

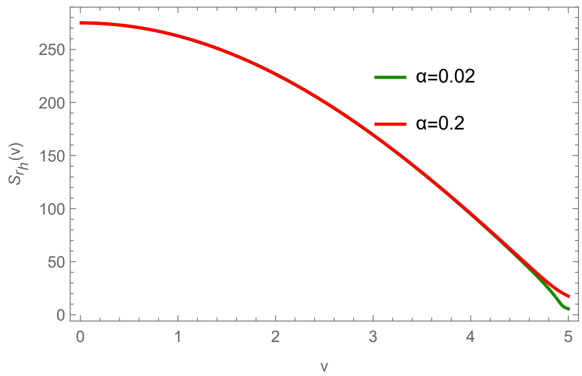


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Figure 1.6: The Bekenstein-Hawking entropy as a function of the scalar charge ν for the asymptotically flat case, having set $Q_m = 0.2$, $m = 1$ for two different values of the non-linear electrodynamics parameter.

We proceeded by calculating the entropy of the black hole by using the Wald formula. Since we are dealing with General Relativity, we show that the entropy will be given by the Bekenstein-Hawking area law

$$S \sim A_h \, . \tag{1.201}$$

In order to see the effect of the black hole parameters on the entropy we plot the entropy as a function of the scalar charge ν in Figure 1.6. We can see that the hairless black hole is a total maxima in the phase space of the parameters that affect the entropy. The non-linear electrodynamics parameter α does not affect the entropy significantly. The fact that the entropy of the hairy black holes is smaller has its root in the fact that the scalar hair results in more dense black hole solutions, in the astrophysical nomenclature this black hole is a “compact object" since it is more dense than its hairless counterpart.

Chapter 2

A simple example of a hairy black hole in $(2+1)$ spacetime dimensions.

Before we dwell further into the modified theories, let me discuss a very simple example of a hairy black hole in three spacetime dimensions. This chapter is based on the paper [8].

We consider a simple model of a scalar field minimally coupled to gravity in three dimensions in the action

$$S = \frac{1}{8\pi}\int d^3x\sqrt{-g}\left\{\frac{R}{2} - \frac{1}{2}\partial^\alpha\phi\partial_\alpha\phi - V(\phi)\right\} \ , \tag{2.1}$$

which consists of the Ricci scalar and a self interacting scalar field minimally coupled to gravity. By variation of this action we obtain the field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} \ , \tag{2.2}$$

$$\Box\phi - \frac{\partial V}{\partial\phi} = 0 \ , \tag{2.3}$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi - g_{\mu\nu}V(\phi) \ . \tag{2.4}$$

To solve the field equations we assume that the scalar field has a coulomb-like form as in the four dimensional theory [123], with the solution first appearing in [87]. We find that the scalar field dresses the black hole with secondary hair, with the scalar charge appearing in the conserved black hole mass. The Null Energy Condition is violated inside the event horizon, a feature also appearing in the four-dimensional sibling [123], and we point out that this is a global feature in any spacetime having a vanishing $1/g_{rr}(r_+) = 0$ at the event horizon of the black hole, where only radial dependence on the scalar field is assumed. At large distances, the solution reduces to the BTZ black hole, while the scalar potential admits an even power law expansion, with the mass term being above the Breitenlohner-Freedman bound.

We consider a $(2+1)$-dimensional metric ansatz of the form

$$ds^2 = -h(r)dt^2 + \frac{1}{b(r)}dr^2 + r^2d\theta^2 \ , \tag{2.5}$$

where $h(r), b(r)$ are the two unknown metric functions to be found solving the field equations. Since the metric only depends on r, we will consider that ϕ is only r-dependent, hence $\phi = \phi(r), V = V(r)$.

The $tt, rr, \theta\theta$ components of the Einstein field equation and the Klein-Gordon equation read

$$b'(r) + rb(r)\phi'(r)^2 + 2rV(r) = 0 \ , \tag{2.6}$$

$$b(r)\left(\frac{h'(r)}{rh(r)} - \phi'(r)^2\right) + 2V(r) = 0 \ , \tag{2.7}$$

$$-b(r)h'(r)^2 + 2h(r)^2\left(b(r)\phi'(r)^2 + 2V(r)\right) + h(r)\left(b'(r)h'(r) + 2b(r)h''(r)\right) = 0 \ , \tag{2.8}$$

$$\frac{1}{2}b'(r)\phi'(r) + b(r)\left(\left(\frac{h'(r)}{2h(r)} + \frac{1}{r}\right)\phi'(r) + \phi''(r)\right) - \frac{V'(r)}{\phi'(r)} = 0 \ . \tag{2.9}$$

Using the Bianchi identity one can prove that (2.9) can be obtained from the Einstein equations. Hence, we have a system of with three independent equations in four unknown functions. As a result one of the unknowns has to be fixed *ad hoc*. However, if one fixes the scalar potential from the begining in (2.1), then the system can be, at least in principle, integrated, since one will have three unknown functions and three equations. Hence, we fix the form of the scalar field as

$$\phi(r) = \frac{A}{r} \ , \tag{2.10}$$

where A is a scalar length scale that controls the behaviour of the scalar field, which we will call scalar charge. Note that the scalar charge has been defined as the term controlling the $\mathcal{O}(r^{-1})$ term of the scalar field at infinity in four dimensions and we will use the same terminology in our work. We have checked that using $\phi(r) = A/r^n$ with $n > 0$, exact results can be obtained for different n, however for $n = 1$ we can find simple exact solutions.

There is a pole in the scalar field function for $r \to 0$, however this will not be a problem because of the presence of a curvature singularity at $r = 0$ as we will discuss, thence the solution will be valid for $r > 0$. Then, we find that

$$b(r) = \frac{c_1 r^2}{A^2 c_3^2} e^{\frac{A^2}{2r^2}}\left(A^2 c_2 e^{\frac{A^2}{2r^2}} + c_3\right) \ , \tag{2.11}$$

$$h(r) = r^2\left(\frac{c_3}{A^2} e^{-\frac{A^2}{2r^2}} + c_2\right) \ , \tag{2.12}$$

$$V(r) = \frac{c_1 c_2}{2c_3^2 r^2} e^{\frac{A^2}{r^2}}\left(A^2 - 2r^2\right) - \frac{c_1}{A^2 c_3} e^{\frac{A^2}{2r^2}} \ , \tag{2.13}$$

where c_1, c_2, c_3 are constants of integration to be determined from the boundary conditions. Note here that at large distances, the solution asymptotes to

$$b(r \to \infty) \sim \frac{c_1 r^2}{c_3^2}\left(\frac{c_3}{A^2} + c_2\right) + \frac{c_1\left(2A^2 c_2 + c_3\right)}{2c_3^2} + \frac{A^2 c_1\left(4A^2 c_2 + c_3\right)}{8c_3^2 r^2} + \mathcal{O}\left(\left(\frac{1}{r}\right)^4\right) \tag{2.14}$$

$$h(r \to \infty) \sim +\frac{A^2 c_3}{8r^2} + \frac{c_3 r^2}{A^2} + c_2 r^2 - \frac{c_3}{2} + \mathcal{O}\left(\left(\frac{1}{r}\right)^4\right) \ , \tag{2.15}$$

$$V(r \to \infty) \sim -\frac{c_1}{c_3^2}\left(\frac{c_3}{A^2} + c_2\right) - \frac{A^2 c_1 c_2 + c_1 c_3}{2c_3^2 r^2} + \mathcal{O}\left(\left(\frac{1}{r}\right)^4\right) \ . \tag{2.16}$$

From the above relations we can see that we have some $\mathcal{O}(r^2)$ terms that survive at large distances which we can identify as cosmological constant terms, that are generated by the vacuum of the scalar field theory and depend on A. For $c_1 = A^4(c_2 + \Lambda)^2, c_3 = -A^2(c_2 + \Lambda)$, where Λ is an effective cosmological constant, we rewrite the asymptotic expressions of the metric functions at large distances

$$b(r\to\infty)\sim-\Lambda r^2+\frac{1}{2}A^2(c_2-\Lambda)+\frac{A^4(3c_2-\Lambda)}{8r^2}+\mathcal{O}\left(\left(\frac{1}{r}\right)^4\right)\ , \tag{2.17}$$

$$h(r\to\infty)\sim-\Lambda r^2+\frac{1}{2}A^2(c_2+\Lambda)-\frac{A^4(c_2+\Lambda)}{8r^2}+\mathcal{O}\left(\left(\frac{1}{r}\right)^4\right)\ . \tag{2.18}$$

To compute the mass of the black hole, we will use the quasi-local method [124]. The quasi-local energy at a finite distance r_0 is defined as

$$E(r_0)=2\left(\sqrt{b_0(r_0)}-\sqrt{b(r_0)}\right)\ , \tag{2.19}$$

where b_0 determines the zero of the energy (which we take to be the pure AdS space-time $b_0(r)=-\Lambda r^2$) and the quasi-local mass at r_0 can be obtained as

$$m(r_0)=\sqrt{h(r_0)}E(r_0)\ . \tag{2.20}$$

Now, the ADM mass of the black hole can be read off by taking the limit at $r_0\to\infty$

$$\mathcal{M}=-\frac{1}{2}A^2\left(c_2-\Lambda\right)\ . \tag{2.21}$$

Setting $c_2=\dfrac{A^2\Lambda-2\mathcal{M}}{A^2}$ we can rewrite the solution in terms of the black hole mass $\mathcal{M}$, the scalar charge A and the effective cosmological constant Λ, which are the parameters of our solution

$$h(r)=r^2\left(\frac{A^2\Lambda-2\mathcal{M}}{A^2}-e^{-\frac{A^2}{2r^2}}\left(\frac{A^2\Lambda-2\mathcal{M}}{A^2}+\Lambda\right)\right), \tag{2.22}$$

$$b(r)=\frac{2r^2e^{\frac{A^2}{2r^2}}\left(\mathcal{M}-A^2\Lambda\right)-r^2e^{\frac{A^2}{r^2}}\left(2\mathcal{M}-A^2\Lambda\right)}{A^2}\ , \tag{2.23}$$

$$V(r)=\frac{e^{\frac{A^2}{r^2}}\left(A^2-2r^2\right)\left(A^2\Lambda-2\mathcal{M}\right)}{2A^2r^2}+\frac{2e^{\frac{A^2}{2r^2}}\left(A^2\Lambda-\mathcal{M}\right)}{A^2}, \tag{2.24}$$

$$V(\phi)=\frac{2e^{\frac{\phi^2}{2}}\left(A^2\Lambda-\mathcal{M}\right)}{A^2}+\frac{e^{\phi^2}\left(\phi^2-2\right)\left(A^2\Lambda-2\mathcal{M}\right)}{2A^2}\ . \tag{2.25}$$

Their asymptotic expressions now read

$$h(r\to\infty)\sim-\Lambda r^2+\left(A^2\Lambda-\mathcal{M}\right)-\frac{A^2\left(A^2\Lambda-\mathcal{M}\right)}{4r^2}+\mathcal{O}\left(\left(\frac{1}{r}\right)^4\right)\ , \tag{2.26}$$

$$b(r\to\infty)\sim-\Lambda r^2-\mathcal{M}+\frac{A^4\Lambda-3A^2\mathcal{M}}{4r^2}+\mathcal{O}\left(\left(\frac{1}{r}\right)^4\right)\ , \tag{2.27}$$

$$V(r\to\infty)\sim\Lambda+\frac{A^2\Lambda}{2r^2}+\frac{A^2\left(A^2\Lambda-\mathcal{M}\right)}{4r^4}+\frac{3A^6\Lambda-5A^4\mathcal{M}}{24r^6}+\mathcal{O}\left(\left(\frac{1}{r}\right)^8\right)\ , \tag{2.28}$$

$$V(\phi\to0)\sim\Lambda+\frac{\Lambda\phi^2}{2}+\phi^4\left(\frac{\Lambda}{4}-\frac{\mathcal{M}}{4A^2}\right)+\phi^6\left(\frac{\Lambda}{8}-\frac{5\mathcal{M}}{3A^2}\right)+\mathcal{O}\left(\phi^8\right)\ . \tag{2.29}$$

We can see that at large distances, the solution resembles the BTZ black hole, while corrections in the structure of space-time appear as $\mathcal{O}\left(\left(\frac{1}{r}\right)^s\right)$ terms, where $s \geq 2$ which are supported by the existence of the scalar field. Moreover, the scalar field dresses the black hole with secondary scalar hair, since the conserved mass is given by the scalar charge in addition to an integration constant. The potential has a mass term given by $m^2 = V''(\phi = 0) = \Lambda$, which is above the Breitenlohner-Freedman bound in three dimensions [88, 89] and for small ϕ (large r), the potential admits an even power series expansion. It is also invariant under the substitution $\phi \to -\phi$. The potential (2.25) contains both the mass and the scalar charge of the black hole space-time. However this should not happen. As a result we have to find a way to render the potential independent of the black hole mass and scalar charge. By inspection we can see that we can define the conserved mass to scalar charge ratio q as

$$q = \frac{\mathcal{M}}{A^2}\,, \tag{2.30}$$

and now the potential will be

$$V(\phi) = e^{\frac{\phi^2}{2}}(2\Lambda - 2q) + e^{\phi^2}\left(\frac{\Lambda\phi^2}{2} - \Lambda - q\phi^2 + 2q\right)\,, \tag{2.31}$$

$$V(\phi \to 0) \sim \Lambda + \frac{\Lambda\phi^2}{2} + \frac{\Lambda - q}{4}\phi^4 + \frac{3\Lambda - 5q}{24}\phi^6 + \mathcal{O}\left(\phi^8\right) \tag{2.32}$$

where now q is the parameter of our theory. Now the potential is general enough and the theory (2.1) can yield black holes with different masses and scalar charges. The mass of the resulting compact object might reduce through Hawking evaporation for example, but a varying mass implies a varying scalar charge, so that their ratio q is constant. Therefore, from a field theory point-of-view we can argue that the scalar charge A is kind of a thermodynamic variable, since it has to vary when $\mathcal{M}$ is changing. In Fig. 2.1 we plot the scalar potential as a function of r and ϕ, where we can see that the potential is always negative in order to support the hairy structure and to violate the no-hair theorem. Moreover from the plot of $V(\phi)$ it is clear that the theory contains a global maximum located at $V_{\text{max}}(\phi) = \Lambda$. We have also checked that the on-shell action is constant at large distances. The $1/g_{rr}$ component has two roots given by

$$r_{\pm} = \pm A\left(\ln\left(\frac{4\left(A^2\Lambda - \mathcal{M}\right)^2}{\left(A^2\Lambda - 2\mathcal{M}\right)^2}\right)\right)^{-1/2}\,. \tag{2.33}$$

We work on the solution in AdS space-time, the horizon being r_+. From now on, we will set $\Lambda = -\ell^{-2}$, where ℓ denotes the AdS radius. In addition, note that there always exist a horizon when $\mathcal{M}$ and A are positive and the scalar field does not imply any bound for the existence of a horizon. In the limit of small scalar charge A, we obtain, at zero order, the BTZ black hole

$$h(r) \sim r^2\left(\frac{1}{\ell^2} - \frac{\mathcal{M}}{r^2}\right) + A^2r^2\left(\frac{\mathcal{M}}{4r^4} - \frac{1}{r^2\ell^2}\right) + \mathcal{O}\left(A^4\right)\,, \tag{2.34}$$

$$b(r) \sim r^2\left(\frac{1}{\ell^2} - \frac{\mathcal{M}}{r^2}\right) - \frac{3A^2\mathcal{M}}{4r^2} + \mathcal{O}\left(A^4\right)\,, \tag{2.35}$$

$$V(r) \sim -\frac{1}{\ell^2} - \frac{A^2\left(2r^2 + \ell^2\mathcal{M}\right)}{4\left(r^4\ell^2\right)} - \frac{A^4\left(3r^2 + 20\ell^2\mathcal{M}\right)}{12\left(r^6\ell^2\right)} + \mathcal{O}\left(A^6\right)\,. \tag{2.36}$$

This is related to the asymptotic nature of the scalar field. In our case, the scalar field decays fast enough ($\mathcal{O}(r^{-1})$) and hence its impact on the conserved mass is mild, in the sense that the mass

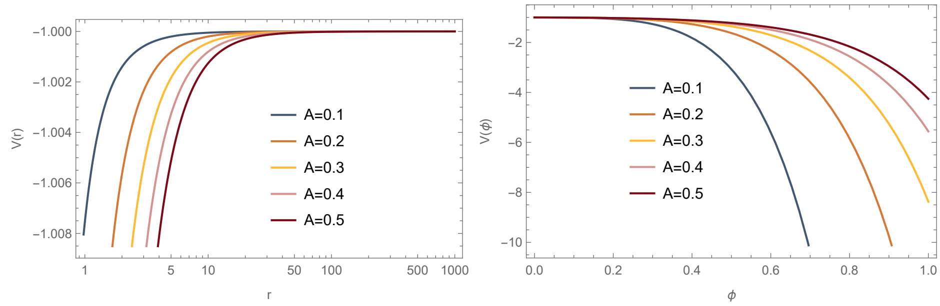


Figure 2.1: The potential $V(r)$ and $V(\phi)$ for $\mathcal{M} = -\Lambda = 1$, while changing the scalar charge A.

is not completely determined by the scalar hair parameter A. As a result, sending ϕ to zero we can obtain the massive BTZ black hole. We note that this is not the case when ϕ falls like $\mathcal{O}(r^{-1/2})$ [19, 125, 126, 127, 128, 129] where the mass is given explicitly in terms of the scalar hair parameter. In our case, the mass (2.21) depends on the scalar hair A but also on an independent integration constant c_2. Therefore, the integration constant can take any particular value and we can always have a massive black hole solution when A approaches zero and the scalar field vanishes. Recently another solution appeared [130], where the scalar field falls faster at infinity ($\mathcal{O}(r^{-1})$ as in our case) and the no-hair limit in this case is also well-defined and the BTZ black hole is obtained in the limit of vanishing scalar field.

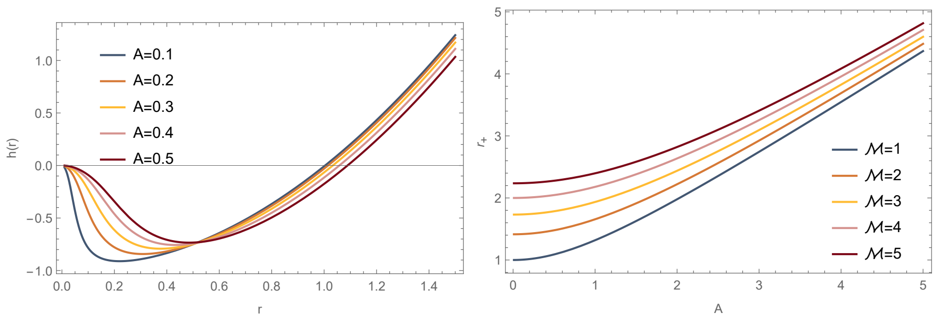


Figure 2.2: Left: $h(r)$ versus r for $\mathcal{M} = \ell = 1$, while changing the scalar charge A. Right: r_+ as a function of A, while changing the mass of the black hole for $\ell = 1$.

In Fig. 2.2 we plot $h(r)$ as a function of r for different scalar charges and r_+ as a function of the scalar charge A and we can see that as A grows, the bigger the event horizon radius becomes. We should also note that the horizon r_+ is also a root of $h(r)$ even though $h(r) \neq b(r)$ (in fact $h(r)/b(r) = e^{-\frac{A^2}{r^2}}$), therefore, the black hole has the same causal structure as the static BTZ black hole, where inside the horizon we still have one time and two position coordinates. There exists a singularity at the origin as can be seen by calculating the Kretschmann scalar, which is plotted in Fig. 2.3. Its expression is complicated, but by checking the limits we can see that it is divergent at the origin, while regular for any other $r > 0$ and at large distances is related to the cosmological constant

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}(r \to \infty) \sim \frac{12}{\ell^4} + \frac{8A^2}{r^2\ell^4} + \mathcal{O}\left(\left(\frac{1}{r}\right)^4\right) . \tag{2.37}$$

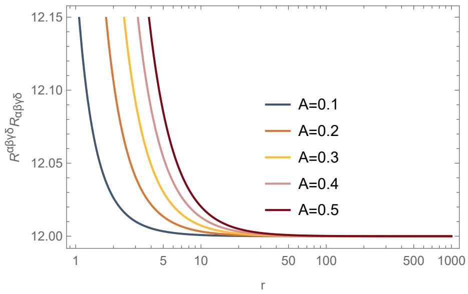


Figure 2.3: The Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}(r)$ for $\mathcal{M} = -\Lambda = 1/\ell^2 = 1$ while changing A.

2.0.1 Energy Conditions

In this subsection we will discuss the energy conditions of the obtained space-time. For this reason we rewrite the Einstein field equation as

$$G_{\mu}^{\ \nu} = T_{\mu}^{\ \nu} \ . \tag{2.38}$$

In this frame of reference, we can identify $T_t^{\ t} = -\rho$, $T_r^{\ r} = p_r$, $T_\theta^{\ \theta} = p_t = -\rho$ being the energy density, the radial pressure and the transverse pressure respectively. The energy conditions are obtained from these expressions. The weak energy condition (WEC) states that given a time-like vector field t^a, the quantity $T_{ab}t^at^b$ is positive, $T_{ab}t^at^b \geq 0 \rightarrow \rho > 0$. The null energy condition (NEC) states that $T_{ab}l^al^b \geq 0 \rightarrow \rho + p_r > 0$, where $l^al_a = 0$, so that the geometry will have a focussing effect on null geodesics. The explicit expressions read

$$\rho = \mathcal{T} + V = b(r)\phi'^2/2 + V(r) = \frac{e^{\frac{A^2}{2r^2}}\left(A^2 - 2r^2\right)\left(A^2 + \ell^2\mathcal{M}\right) - e^{\frac{A^2}{r^2}}\left(A^2 - r^2\right)\left(A^2 + 2\ell^2\mathcal{M}\right)}{A^2r^2\ell^2} , \tag{2.39}$$

$$p_r = \mathcal{T} - V = b(r)\phi'^2/2 - V(r) = \frac{e^{\frac{A^2}{2r^2}}\left(A^2 + 2r^2\right)\left(A^2 + \ell^2\mathcal{M}\right) - r^2e^{\frac{A^2}{r^2}}\left(A^2 + 2\ell^2\mathcal{M}\right)}{A^2r^2\ell^2} , \tag{2.40}$$

$$\rho + p_r = 2\mathcal{T} = b(r)\phi'(r)^2 = \frac{2e^{\frac{A^2}{2r^2}}\left(A^2 + \ell^2\mathcal{M}\right) - e^{\frac{A^2}{r^2}}\left(A^2 + 2\ell^2\mathcal{M}\right)}{r^2\ell^2} , \tag{2.41}$$

where $\mathcal{T} = b(r)\phi'^2/2$ is the kinetic energy of the scalar field. The energy density ρ is negative inside and on the black hole horizon. Inside the horizon, $b(r)$ is negative, ϕ'^2 is always positive and the potential is negative everywhere as we can see from Fig. 2.1, resulting in negative energy density inside the black hole. On the horizon, we have $b(r_+) = 0$, hence the contribution from the kinetic energy of the scalar field vanishes and the potential makes the energy density negative. Outside of the black hole horizon $b(r)$ is positive, but the effect of the potential energy is stronger than the kinetic energy, resulting in negative energy density everywhere, as we can see in Fig. 2.4. At large distances, the kinetic energy asymptotes as

$$\mathcal{T}(r \rightarrow \infty) \sim \frac{A^2}{2r^2\ell^2} - \frac{A^2\mathcal{M}}{2r^4} + \mathcal{O}\left(\left(\frac{1}{r}\right)^6\right) , \tag{2.42}$$

while the expression for the potential energy in the limit $r \to +\infty$ is given in (2.28). The leading order term in the kinetic energy is positive and since all constants $A, \ell, \mathcal{M}$ are finite constants, the kinetic energy is positive for large r. We can see in (2.28) that the potential will cancel this positive contribution of the kinetic energy, therefore the sum $\rho = \mathcal{T} + V$ will be always negative. It is known that too negative a potential might threaten the WEC of a regular scalar field. In our case the potential is the quantity that violates WEC. For the NEC, we can see that at the event horizon r_+, we have $\rho + p_{r_+} = 0$ due to the fact that $b(r_+) = 0$. Outside and on the horizon the NEC is satisfied, while inside the event horizon the NEC is violated. This is a common feature of black hole space-times that arise from an action that consists of the Ricci scalar of Einstein's gravity and a simple non-minimally self interacting scalar field in arbitrary dimensions that satisfies $g^{rr}(r_+) = 0, \phi = \phi(r)$, and not a peculiar case of our model. This behaviour is indeed present in the four-dimensional case [123, 131]. In Fig. 2.4 we plot the energy density (WEC) and the sum of the energy density and radial pressure (NEC) of our black hole in order to illustrate the discussion above. In FIG. 2.4 we also plot the radial pressure. We can see that the radial pressure is negative for some region inside the black hole horizon, while at the horizon and outside of the horizon, the radial pressure is positive.

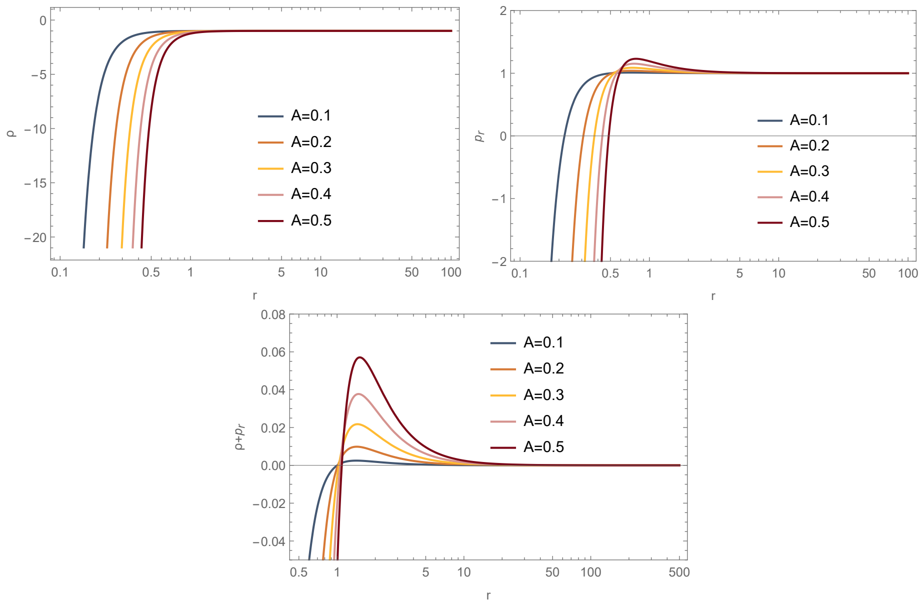


Figure 2.4: The energy density ρ, the radial pressure p_r and the radial pressure $\rho + p_r$ for $\mathcal{M} = \ell = 1$ while changing the scalar charge A.

2.0.2 Thermodynamics

In this subsection we will discuss the thermodynamics of the black hole solution. We begin with the temperature. To derive the black hole temperature, at first we perform a Wick rotation and move to imaginary time $t \to i\tau$ where τ will now be periodic, the period of which we have to find in order to specify the temperature. We will now ignore the angular part of the space-time metric and thence we are left with

$$ds^2 = h(r)d\tau^2 + \frac{1}{b(r)}dr^2 \; . \tag{2.43}$$

We now expand the metric functions near the horizon

$$h(r \to r_+) = h(r_+) + h'(r_+)(r - r_+) + ... = h'(r_+)(r - r_+) \ , \tag{2.44}$$

$$b(r \to r_+) = b(r_+) + b'(r_+)(r - r_+) + ... = b'(r_+)(r - r_+) \ , \tag{2.45}$$

and the reduced space-time element reads

$$ds^2 = h'(r_+)(r - r_+)d\tau^2 + \frac{1}{b'(r_+)(r - r_+)}dr^2 \ . \tag{2.46}$$

Even if it is not clear at this point, the above line element describes a cone in Euclidean space and has a conical singularity at the tip $r \to 0$, unless we fix the period of τ in a particular way. Therefore, we will now compare this line element with the line element of two-dimensional flat space-time in polar coordinates that reads

$$dS^2 = dR^2 + R^2 d\Theta^2 \ , \tag{2.47}$$

where Θ is periodic of period $T_\Theta = 2\pi$ and we will treat τ as an angular coordinate, in order for the space-time to be truly Euclidean. By setting $ds^2 = dS^2$ we can relate the two radial coordinates

$$dR^2 = \frac{1}{b'(r_+)(r - r_+)}dr^2 \ , \tag{2.48}$$

which by integration will yield the relation

$$R = 2\sqrt{\frac{r - r_+}{b'(r_+)}} \ , \tag{2.49}$$

and now we are left with the angular coordinates

$$h'(r_+)(r - r_+)d\tau^2 = R^2 d\Theta^2 \ , \tag{2.50}$$

which again by integration yields

$$\Theta = \frac{\sqrt{h'(r_+)b'(r_+)}}{2}\tau \to \frac{\Theta}{\tau} = \frac{\sqrt{h'(r_+)b'(r_+)}}{2} \ . \tag{2.51}$$

Θ is periodic with $T_\Theta = 2\pi$ and by denoting β the period of τ we have

$$\beta = \frac{4\pi}{\sqrt{h'(r_+)b'(r_+)}} \to \mathfrak{T} \equiv \frac{1}{\beta} = \frac{\sqrt{h'(r_+)b'(r_+)}}{4\pi} \ , \tag{2.52}$$

which is the temperature of our black hole space-time. Substituting the functions we find

$$\mathfrak{T}(r_+) = \frac{\left(A^2 + 2r_+^2\right)\left(A^2 + \ell^2\mathcal{M}\right) - r_+^2 e^{\frac{A^2}{2r_+^2}}\left(A^2 + 2\ell^2\mathcal{M}\right)}{2\pi A^2 r_+ \ell^2} \ , \tag{2.53}$$

and substituting the horizon radius we can express the temperature as a function of the black hole mass

$$\mathfrak{T}(\mathcal{M}) = \frac{\left(A^2 + \ell^2\mathcal{M}\right)\sqrt{\ln\left(A^2 + \ell^2\mathcal{M}\right) - \ln\left(A^2 + 2\ell^2\mathcal{M}\right) + \ln(2)}}{\sqrt{2}\pi A\ell^2} \ . \tag{2.54}$$

The temperature is always real and positive. In Fig. 2.5 we plot the temperature of the black hole. We can see that the temperature increases as the mass of the black hole is growing. Moreover, the

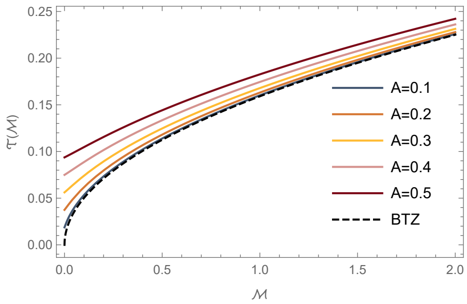


Figure 2.5: The temperature of the black hole for $\ell = 1$ as a function of $\mathcal{M}$, while changing the scalar charge.

temperature is non-zero and finite for zero mass, which happens because a horizon exists even for the massless case, given by $r_+(\mathcal{M} = 0) = A/(2\ln 2)$. For small A, it becomes

$$\mathfrak{T}(r_+) \sim \frac{r_+}{2\pi\ell^2} + \frac{A^2\left(2r_+^2 - \ell^2\mathcal{M}\right)}{8\pi r_+^3\ell^2} + \mathcal{O}\left(A^4\right) \ , \tag{2.55}$$

while the BTZ temperature corresponds to $\mathfrak{T}_{\text{BTZ}} = r_+/(2\pi\ell^2) = \sqrt{\mathcal{M}}/2\pi\ell$. As a result, the hairy black hole possesses a larger temperature at the event horizon. Using the Wald formula [21, 22], we can calculate the entropy of the black hole as

$$\mathcal{S} = -2\pi \int d\theta \sqrt{r_+^2} \left(\frac{\partial\mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}}\right)\Bigg|_{r=r_+} \hat{\varepsilon}_{\alpha\beta}\hat{\varepsilon}_{\gamma\delta} \ , \tag{2.56}$$

where $\hat{\varepsilon}_{\alpha\beta}$ is the bi-normal to the horizon surface [23], $\mathcal{L}$ is the Lagrangian of the theory $\mathcal{L} = (8\pi)^{-1}\left(R/2 - \partial^\alpha\phi\partial_\alpha\phi/2 - V(\phi)\right)$, and

$$\frac{\partial\mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}}\Bigg|_{r=r_+} = \frac{1}{2}\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\beta\gamma}g^{\alpha\delta}\right) \ , \tag{2.57}$$

hence the entropy will be given by the Bekenstein-Hawking area law

$$\mathcal{S} = \frac{\mathcal{A}}{4} = \frac{\pi r_+}{2} \ , \tag{2.58}$$

where $\mathcal{A} = 2\pi r_+$ is the circumference of the three-dimensional black hole. The hairy black holes possess a larger event horizon radius in comparison to the BTZ black hole which has an event horizon at $r_+ = \ell\sqrt{\mathcal{M}}$

$$r_+ \sim \ell\sqrt{\mathcal{M}} + \frac{3A^2}{8\ell\sqrt{\mathcal{M}}} + \mathcal{O}\left(A^3\right) \ , \tag{2.59}$$

hence they are thermodynamically preferred over the BTZ black hole, having higher entropy, when the scalar charge is small. The black hole space-time is thermally stable in the canonical ensemble, which can be seen by evaluating the heat capacity

$$C(r_+) = T\frac{dS}{dT}\Bigg|_{r=r_+} = \frac{\pi A}{2\sqrt{2}\sqrt{\ln\left(A^2 + \ell^2\mathcal{M}\right) - \ln\left(A^2 + 2\ell^2\mathcal{M}\right) + \ln(2)}} \ , \tag{2.60}$$

which is always positive. For negligible scalar hair we obtain the heat capacity of the BTZ black hole $C = \pi r_+/2 = \pi\ell\sqrt{\mathcal{M}}/2$.

Now, lets discuss the first law of thermodynamics. To do so, we will consider the Euclidean class of metrics described by

$$ds^2 = N(r)^2 h(r) d\tau^2 + \frac{dr^2}{h(r)s(r)^2} + r^2 d\theta^2 \ . \tag{2.61}$$

The coordinates here range as $0 \leq \tau < \beta,\ r_+ \leq r < \infty,\ 0 \leq \theta < 2\pi$. Here τ is the Euclidean time which is periodic with period β in order to avoid a conical singularity at the event horizon of the black hole where $h(r_+) = 0$. This periodicity is related to the temperature of the black hole spacetime as

$$T = \frac{1}{\beta} = \frac{N(r)h'(r)s(r)}{4\pi} \ . \tag{2.62}$$

We will consider the Hamiltonian version of the action now and we will write

$$\mathcal{H} = \int \left(\pi^{ij}\dot{g}_{ij} + p\dot{\phi} - NH - N^i H_i\right) d^2x dt + \mathcal{B}_H \ . \tag{2.63}$$

ince this solution is static and spherically symmetric, we can consider a *reduced Hamiltonian*

$$\mathcal{H} = -\int d^2x dt NH + \mathcal{B}_H \ . \tag{2.64}$$

The Euclidean action is related to the Lorentzian action via

$$\mathcal{I}_e = -i\mathcal{I} \ . \tag{2.65}$$

$$\mathcal{I}_E = 2\pi\beta \int dr NH + \mathcal{B}_E \ . \tag{2.66}$$

In order to have a well defined variational principle we have to include a boundary term $\mathcal{B}_E$ which will take care of the boundary terms that we will cancel in order to obtain the field equations. Therefore, we will consider the following total action

$$\mathcal{I}_E = \mathcal{I}_e + \mathcal{B}_E \ . \tag{2.67}$$

By variation with respect to the fields we obtain the following equations

$$s\left(s\left(h' + hr\left(\phi'\right)^2\right) + 2hs'\right) + 2rV = 0 \ , \tag{2.68}$$

$$s\left(Nr\left(\phi'\right)^2 - N'\right) + Ns' = 0 \ , \tag{2.69}$$

$$N\left(-h' + hr\left(\phi'\right)^2 - \frac{2rV}{s^2}\right) - 2hN' = 0 \ , \tag{2.70}$$

$$N\left(s\phi'\left(\phi'\left(s\left(rh' + h\right) + hrs'\right) + hrs\phi''\right) - rV'\right) + hrs^2 N'\left(\phi'\right)^2 = 0 \ . \tag{2.71}$$

In order to obtain these equations, we have cancelled several boundary terms. These boundary terms combined are

$$\left(\frac{1}{4}\beta\delta hN + \frac{1}{8}\beta r\delta\phi hNs\phi' + \frac{1}{8}\beta\delta hNs\right)\Big|_{r_+}^{\infty} \tag{2.72}$$

The solution reported previously satisfies these equations with $N(r)$ = constant which we can set without loss of generality equal to 1 and $s(r) = e^{A^2/2r^2}$. Having the solution, its easy to compute the variation of the fields at infinity and at the horizon. At infinity we have

$$\delta\phi = \delta A/r \ , \tag{2.73}$$

$$\delta h = -\frac{2A\delta A\left(8q\ell^2+1\right)}{\ell^2} \ , \tag{2.74}$$

$$\delta s = A\delta A/r^2 \ , \tag{2.75}$$

where we have made explicit that the only parameter that can vary is the scalar charge A. Now, the boundary term at infinity takes the value

$$A\left(-2\beta\delta Aq - \frac{\beta\delta A}{8\ell^2}\right) + \delta\mathcal{B}(\infty) = 0 \ , \tag{2.76}$$

where we split the variation of the boundary term $\mathcal{B}$ into two pieces, one at the horizon and one at infinity for simplicity. As we result, considering the Grand Canonical Ensemble and therefore keeping the temperature fixed, we obtain the boundary term at infinity

$$\mathcal{B}(\infty) = A^2\beta q + \frac{A^2\beta}{16\ell^2} = \beta\mathcal{M} + \frac{A^2\beta}{16\ell^2} \ . \tag{2.77}$$

Now, at the event horizon of the black hole we have

$$\delta h = 0 - h'\delta r_+ \ , \tag{2.78}$$

$$\delta s = \delta s(r_+) - s'\delta r_+ \ , \tag{2.79}$$

$$\delta\phi = \delta\phi(r_+) - \phi'\delta r_+ \ . \tag{2.80}$$

Now, using the formula for the temperature (2.62) and the fact that $h(r_+) = 0$ the boundary term at the horizon reads

$$\frac{\pi\delta r_+}{2} + \delta\mathcal{B}(r_+) = 0 \rightarrow \mathcal{B}(r_+) = -\frac{\mathcal{A}(r_+)}{4} \ . \tag{2.81}$$

Now, we have a well defined variational procedure that will yield $\delta\mathcal{I}_E = 0$. In the Grand Canonical Ensemble, the Euclidean action is related to the free energy of the black hole solution:

$$\mathcal{I}_E = \beta\mathcal{F} = \beta\mathfrak{M} - \mathcal{S} \ . \tag{2.82}$$

Our Euclidean action is just given by the boundary terms

$$\mathcal{I}_E = \beta\mathcal{M} + \frac{A^2\beta}{16\ell^2} - \frac{\mathcal{A}(r_+)}{4} \tag{2.83}$$

and hence we can identify

$$\mathfrak{M} = \mathcal{M} + \frac{A^2}{16\ell^2} \ , \tag{2.84}$$

$$\mathcal{S} = \frac{\mathcal{A}(r_+)}{4} \ . \tag{2.85}$$

As a result the first law of thermodynamics holds in a modified version due to the strong back-reaction of the scalar field to the spacetime metric as

$$\delta\mathfrak{M} = T\delta\mathcal{S} \ . \tag{2.86}$$

2.1 Rotating Black Hole Solutions

To discuss rotating solutions, we impose the metric ansatz

$$ds^2 = -h(r)dt^2 + \frac{1}{b(r)}dr^2 + r^2\left(d\theta + u(r)dt\right)^2 \ , \tag{2.87}$$

where we have introduced the angular shift function $u(r)$. Inserting this ansatz in the field equations (2.2),(2.3) we obtain the following solution

$$u(r) = \frac{J}{A^2}e^{-\frac{A^2}{2r^2}} - \frac{J}{A^2} \ , \tag{2.88}$$

$$b(r) = \frac{r^2\left(2e^{\frac{A^2}{2r^2}}\left(A^2\ell^2\mathcal{M} + A^4 - J^2\ell^2\right) - e^{\frac{A^2}{r^2}}\left(2A^2\ell^2\mathcal{M} + A^4 - J^2\ell^2\right) + J^2\ell^2\right)}{A^4\ell^2} \ , \tag{2.89}$$

$$h(r) = e^{-A^2/r^2}b(r) \ , \tag{2.90}$$

$$V(r) = -\frac{8r^4e^{\frac{A^2}{2r^2}}\left(A^2\ell^2\mathcal{M} + A^4 - J^2\ell^2\right) + 2r^2e^{\frac{A^2}{r^2}}\left(A^2 - 2r^2\right)\left(2A^2\ell^2\mathcal{M} + A^4 - J^2\ell^2\right)}{4A^4r^4\ell^2} + \frac{J^2\ell^2\left(2A^2r^2 + A^4 + 4r^4\right)}{4A^4r^4\ell^2} \ , \tag{2.91}$$

$$V(\phi) = -\frac{8e^{\frac{\phi^2}{2}}\left(A^2\ell^2\mathcal{M} + A^4 - J^2\ell^2\right) + 2e^{\phi^2}\left(\phi^2 - 2\right)\left(2A^2\ell^2\mathcal{M} + A^4 - J^2\ell^2\right) + J^2\ell^2\left(\phi^4 + 2\phi^2 + 4\right)}{4A^4\ell^2} \ , \tag{2.92}$$

while the scalar field is the same as (2.10) and J is the angular momentum of the black hole. We used the quasi-local method to derive the angular momentum and the conserved black hole mass [124]

$$J = \lim_{r_0\to\infty}\frac{\sqrt{b(r_0)}u'(r_0)r_0^3}{\sqrt{h(r_0)}} \ , \tag{2.93}$$

$$\mathcal{M} = \lim_{r_0\to\infty}\left(\sqrt{h(r_0)}E(r_0) - Ju(r_0)\right) \ . \tag{2.94}$$

The asymptotic expressions at large distances yield

$$u(r\to\infty) \sim -\frac{J}{2r^2} + \frac{A^2J}{8r^4} - \frac{A^4J}{48r^6} + \frac{A^6J}{384r^8} + \mathcal{O}\left(\left(\frac{1}{r}\right)^{10}\right) \ , \tag{2.95}$$

$$h(r\to\infty) \sim \frac{r^2}{\ell^2} + \left(-\frac{A^2}{\ell^2} - \mathcal{M}\right) + \frac{\frac{A^4}{\ell^2} + A^2\mathcal{M} + J^2}{4r^2} + \mathcal{O}\left(\left(\frac{1}{r}\right)^{4}\right) \ , \tag{2.96}$$

$$b(r\to\infty) \sim \frac{r^2}{\ell^2} - \mathcal{M} + \frac{-\frac{A^4}{\ell^2} - 3A^2\mathcal{M} + J^2}{4r^2} + \mathcal{O}\left(\left(\frac{1}{r}\right)^{4}\right) \ , \tag{2.97}$$

$$V(r\to\infty) \sim -\frac{1}{\ell^2} - \frac{A^2}{2r^2\ell^2} - \frac{A^2\ell^2\mathcal{M} + A^4}{4r^4\ell^2} + \mathcal{O}\left(\left(\frac{1}{r}\right)^{6}\right) \ , \tag{2.98}$$

$$V(\phi\to 0) \sim -\frac{1}{\ell^2} - \frac{\phi^2}{2\ell^2} + \phi^4\left(-\frac{\mathcal{M}}{4A^2} - \frac{1}{4\ell^2}\right) + \frac{1}{24}\phi^6\left(\frac{3J^2}{A^4} - \frac{5\mathcal{M}}{A^2} - \frac{3}{\ell^2}\right) + \mathcal{O}\left(\phi^8\right) \tag{2.99}$$

We can see that at large distances we obtain a solution similar to the rotating BTZ black hole, with changes in the structure of space-time being related to A, and in the small hair case, we also obtain the rotating BTZ black hole. The horizon is given by $b(r) = 0$ [132, 133], which is also a root of $h(r)$. $b(r)$ has two roots, which can be computed analytically but these expressions are lengthy, so we will not given them here. The existence of horizons provides bounds for the angular momentum of the black hole. Therefore, black holes can only exist when

$$A > 0 \,\&\, \ell > 0 \,\&\, \mathcal{M} > 0 \,\&\, \left(J^2 \leq \frac{A^4}{\ell^2} + 2A^2\mathcal{M} \text{ or } \frac{A^4}{\ell^2} + 2A^2\mathcal{M} < J^2 \leq \frac{\left(A^2 + \ell^2\mathcal{M}\right)^2}{\ell^2} \right) . \quad (2.100)$$

In the cases where the inequalities are saturated ($J^2 = A^4/\ell^2 + 2A^2\mathcal{M}$ or $J^2 = \left(A^2 + \ell^2\mathcal{M}\right)^2/\ell^2$) in the above expressions we have black holes with a single event horizon, while in any other case the black holes develop two horizons, an inner and an event horizon. The rotating solution admits a region of space-time where the Killing field $\partial/\partial t$ is space - like, which transforms into the condition

$$g_{tt} > 0 \rightarrow -h(r) + r^2 u(r)^2 > 0 \, . \quad (2.101)$$

It is clear that at the event horizon, the previous condition holds, and also for some region outside of the horizon [133]. In Fig. 2.6 we plot the scalar potential where it is obvious that between the inner and event horizon, negative potential wells are developed, while as we increase the mass parameter, the wells become deeper. In addition, it is clear form both figures, that in the rotating case, the vaccum of the field theory represents a local maximum, and not a global one. One should also note the fact that the potential energy is bounded from below, which will play a crucial role in the stability of the system. Moreover, in Fig. 2.7 we plot the metric function $b(r)$ and g_{tt} in order to study the geometry. From Fig. 2.7 (left) we can see that the black hole develops two horizons, while for the

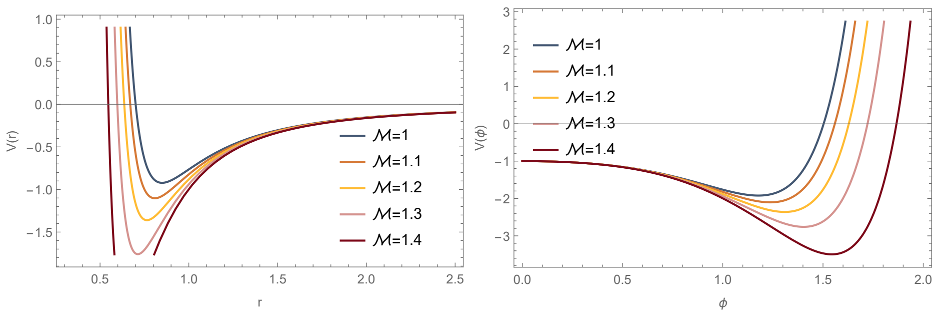


Figure 2.6: $V(r)$ having set $\ell = A = 1$, $J = 2$, while varying the mass parameter.

degenerate case $J^2 = \left(A^2 + \ell^2\mathcal{M}\right)^2/\ell^2$, the horizons coincide. The g_{tt} component is positive at the horizon and for some region after the horizon, denoting the presence of an ergo-region, as in the BTZ black hole [97, 98, 134]. The scalar potential depends on the conserved black hole charges. We can eliminate these charges by introducing a new constant χ, such as

$$\chi = \frac{J^2}{A^4} \, , \quad (2.102)$$

besides $q = \mathcal{M}/A^2$. Now the potential will read

$$V(\phi) = e^{\phi^2}\left(\phi^2\left(-q + \frac{\chi}{2} - \frac{1}{2\ell^2}\right) + 2q - \chi + \frac{1}{\ell^2}\right) + e^{\frac{\phi^2}{2}}\left(-2q + 2\chi - \frac{2}{\ell^2}\right) - \chi - \frac{\chi\phi^4}{4} - \frac{\chi\phi^2}{2} \, , \quad (2.103)$$

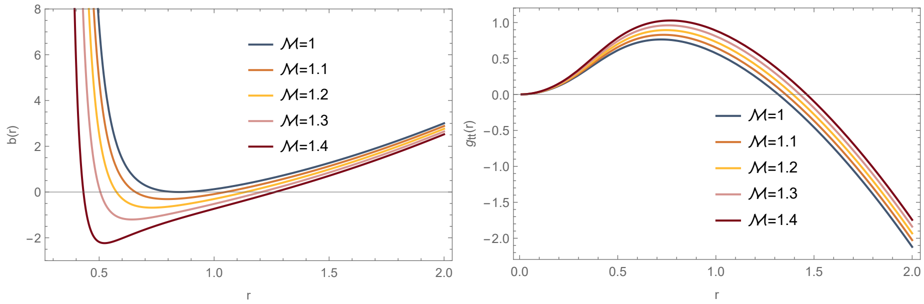


Figure 2.7: $b(r)$ and $g_{tt}(r)$ having set $\ell = A = 1$, $J = 2$, while varying the mass parameter.

$$V(\phi \to 0) \sim -\frac{1}{\ell^2} - \frac{\phi^2}{2\ell^2} + \frac{\phi^4\left(-q\ell^2 - 1\right)}{4\ell^2} + \frac{\phi^6\left(-5q\ell^2 + 3\chi\ell^2 - 3\right)}{24\ell^2} + \mathcal{O}\left(\phi^8\right) \ . \tag{2.104}$$

Consequently, our theory will give black hole space-times with a fixed angular momentum to conserved mass ratio given by

$$\frac{\chi}{q^2} = \frac{J^2}{\mathcal{M}^2} \ . \tag{2.105}$$

Both the conserved mass and the angular momentum are allowed to vary, in a consistent way, so the ratio $J^2/\mathcal{M}^2$ will always be constant. To compute the Hawking temperature we will use the concept of surface gravity, which is defined by

$$\kappa = \sqrt{-\frac{1}{2}\nabla_\mu X_\nu \nabla^\mu X^\nu} \ , \tag{2.106}$$

where X^μ is a Killing vector field of our space-time defined as $X^\mu = (1, 0, \Omega)$, where Ω is the angular velocity at the horizon of the black hole, defined as

$$\Omega = -\frac{g_{t\theta}}{g_{\theta\theta}}\bigg|_{r=r+} \ . \tag{2.107}$$

Evaluating the surface gravity we find

$$\kappa = \frac{1}{2}\sqrt{-\frac{b(r)\left(h(r)\left(ru'(r) + 2(u(r)+\Omega)\right)^2 - \left(h'(r) - r^2(u(r)+\Omega)u'(r)\right)^2\right)}{h(r)}}\Bigg|_{r=r+} \ . \tag{2.108}$$

Now, we can find the temperature as

$$\mathfrak{T}(r_+) = \frac{\kappa}{2\pi} = \frac{1}{4\pi}\sqrt{b(r)\left(\frac{h'(r)^2}{h(r)} - r^2u'(r)^2\right)}\Bigg|_{r=r+} = \frac{A^4 e^{\frac{A^2}{r_+^2}} - J^2\ell^2\left(e^{\frac{A^2}{2r_+^2}} - 1\right)^2}{4\pi A^2 r_+ \ell^2\left(e^{\frac{A^2}{r_+^2}} - e^{\frac{A^2}{2r_+^2}}\right)} \ , \tag{2.109}$$

which of course reduces to the temperature of the BTZ black hole, when A approaches 0. The entropy is given by the same formula as in the non-rotating case (2.58). The heat capacity is found to be

$$C(r_+) = \pi r_+^3 \left(2\left(A^2\left(J\ell\left(\frac{1}{J\ell - e^{\frac{A^2}{2r_+^2}}(A^2+J\ell)} + \frac{1}{e^{\frac{A^2}{2r_+^2}}(A^2-J\ell)+J\ell}\right)-3\right)+r_+^2\right)\right)^{-1}, \tag{2.110}$$

which for small A reduces to the heat capacity of the rotating BTZ black hole

$$C(r_+, A\to 0) = \frac{1}{6}\pi r_+\left(\frac{16r_+^4}{3J^2\ell^2+4r_+^4}-1\right) > 0\,. \tag{2.111}$$

Due to the complexity of (2.100) in order for the existence of horizons, we will perform a numerical analysis for the thermodynamics of this solution and plot the temperature, the entropy, and the heat capacity as functions of the black hole mass $\mathcal{M}$ for appropriate values of A, J, ℓ. In Fig. 2.8 we fix $A = 0.5, \ell = 1, J = 2$ and plot the corresponding thermodynamic quantities for the allowed values of mass. It is clear all thermodynamic quantities are positive and growing with the increase of mass,

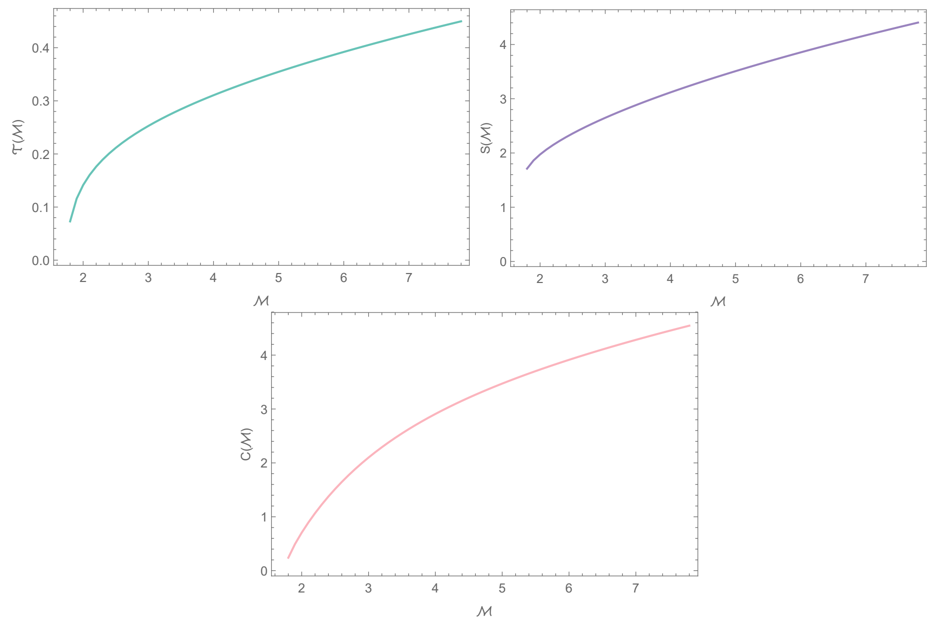


Figure 2.8: The temperature $\mathfrak{T}$, the entropy S and the heat capacity C as functions of the black hole mass

in accordance with the BTZ black hole case. Moreover, these black holes are thermally stable in the canonical ensemble and do not develop any phase transition. The system under consideration is complicated and therefore we cannot present simple calculations to show that the internal energy of the black hole do not coincide, however we do not expect a different behaviour from the non-rotating case.

Part II

Black Holes in $f(R, \phi)$ Theories

Chapter 3

Black holes of $(2+1)$ dimensional $f(R)$ gravity coupled to a scalar field

In this chapter we consider $(2+1)$-dimensional $f(R)$ gravity and a scalar field minimally coupled to gravity. Solving the field equations, we find that, at large distances where the scalar field is weak, the BTZ black hole is obtained, while at small distances, the scalar field brings strong corrections to the dynamics of the system, resulting at a new family of black hole solutions. The temperature and the entropy of the black hole are calculated and compared to the BTZ black hole case. We find that due to the larger event horizon radius of the $f(R)$ black hole, the entropy is larger in comparison to the BTZ case, hence the $f(R)$ black holes are thermodynamically preferred. This chapter is based on [1].

3.1 The setup-derivation of the field equations

We will consider the $f(R)$ gravity theory with a scalar field minimally coupled to gravity in the presence of a self-interacting potential. Varying this action we will look for hairy black hole solutions. We will show that if this scalar field decouples, we recover $f(R)$ gravity. First we will consider the case in which the scalar field does not have self-interactions.

3.1.1 Without self-interacting potential

Consider the action

$$S = \int d^3x\sqrt{-g}\left\{\frac{1}{2\kappa}f(R) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right\} \ , \tag{3.1}$$

where κ is the Newton gravitational constant $\kappa = 8\pi G$. The Einstein equations read

$$f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + g_{\mu\nu}\Box f_R - \nabla_\mu\nabla_\nu f_R = \kappa T_{\mu\nu} \ , \tag{3.2}$$

where $f'(R) = f_R$ and the energy-momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \ . \tag{3.3}$$

The Klein-Gordon equation reads

$$\Box\phi = 0 \, . \tag{3.4}$$

We consider a spherically symmetric ansatz for the metric

$$ds^2 = -b(r)dt^2 + \frac{1}{b(r)}dr^2 + r^2 d\theta^2 \, . \tag{3.5}$$

For the metric above, the Klein-Gordon equation becomes

$$\Box\phi = b(r)\phi''(r) + \phi'(r)\Big(b'(r) + \frac{b(r)}{r}\Big) = 0 \, , \tag{3.6}$$

and takes the form of a total derivative

$$b(r)\phi'(r)r = C \, , \tag{3.7}$$

where C is a constant of integration. In order to have a black hole, we require at the horizon to have $r = r_H \to b(r_H) = 0$. Then, $C = 0$. This means that either $b(r) = 0$ for any $r > 0$ and no geometry can be formed, or the scalar field is constant $\phi(r) = c$. We indeed expected this behaviour, which cannot be cured with the addition of a second degree of freedom in the metric (3.5). From the no-hair theorem [18] we know that the scalar field should satisfy its equation of motion for the black hole geometry, thus if we multiply the Klein-Gordon equation by ϕ and integrate over the black hole region we have

$$\int d^3x\sqrt{-g}\big(\phi\Box\phi\big) \approx \int d^3x\sqrt{-g}\nabla^\mu\phi\nabla_\mu\phi = 0 \, , \tag{3.8}$$

where $\approx$ means equality modulo total derivative terms. From equation (3.8) one can see that the scalar field is constant.

3.1.2 With self-interacting potential

We shown that if the matter does not have self-interactions then there are no hairy black holes in the $f(R)$ gravity. We then have to introduce self-interactions for the scalar field. Consider the action

$$S = \int d^3x\sqrt{-g}\left\{\frac{1}{2\kappa}f(R) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right\} \, . \tag{3.9}$$

The scalar field and the scalar potential obey the following conditions

$$\phi\,(r\to\infty) = 0 \, , \quad V\,(r\to\infty) = 0 \, , \quad V\big|_{\phi=0} = 0 \, . \tag{3.10}$$

Varying the action (3.9) using the metric ansatz (3.5) we get the $tt, rr, \theta\theta$ components of Einstein's equations (for $\kappa = 1$) and the Klein-Gordon equation

$$r\left(b'(r)f_R'(r) - f_R(r)b''(r) - f\,(r) + b(r)\left(2f_R''(r) + \phi'(r)^2\right) + 2V(\phi)\right) - f_R(r)b'(r) + 2b(r)f_R'(r) = 0 \, , \tag{3.11}$$

$$b(r)\left(r\left(-b'(r)f_R'(r) + f_R(r)b''(r) + f\,(r) + b(r)\phi'(r)^2 - 2V(\phi)\right) + f_R(r)b'(r) - 2b(r)f_R'(r)\right) = 0 \, , \tag{3.12}$$

$$-r\left(2b'(r)f_R'(r) + b(r)\left(2f_R''(r) + \phi'(r)^2\right) + 2V(\phi)\right) + 2f_R(r)b'(r) + rf\,(r) = 0 \, , \tag{3.13}$$

$$\frac{\left(rb'(r) + b(r)\right)\phi'(r)}{r} + b(r)\phi''(r) - \frac{V'(r)}{\phi'(r)} = 0 \, . \tag{3.14}$$

The Ricci Curvature for the metric (3.5) reads

$$R(r) = -\frac{2b'(r)}{r} - b''(r)\ . \tag{3.15}$$

From (3.11) and (3.12) equations we obtain the relation between $f_R(r)$ and $\phi(r)$

$$f_R''(r) + \phi'(r)^2 = 0\ , \tag{3.16}$$

while the (3.11) and (3.13) equations yield the relation between the metric finction $b(r)$ and $f_R(r)$

$$(2b(r) - rb'(r))\, f_R'(r) + f_R(r)\,(b'(r) - rb''(r)) = 0\ . \tag{3.17}$$

Both equations (3.16), (3.17) can be immediately integrated to yield

$$f_R(r) = c_1 + c_2 r - \int\int \phi'(r)^2 dr dr\ , \tag{3.18}$$

$$b(r) = c_3 r^2 - r^2 \int \frac{K}{r^3 f_R(r)} dr \tag{3.19}$$

where c_1, c_2, c_3 and K are constants of integration. We can also integrate the Klein-Gordon equation

$$V(r) = V_0 + \int \frac{rb'(r)\phi'(r)^2 + rb(r)\phi'(r)\phi''(r) + b(r)\phi'(r)^2}{r} dr\ . \tag{3.20}$$

Equation (3.18) is the central equation of this work. First of all, we recover General Relativity for the vanishing of scalar field and for $c_1 = 1, c_2 = 0$. We stress the fact that in $f(R)$ gravity we are able to derive non-trivial configurations for the scalar field with one degree of freedom as can be seen in the metric (3.5). This is not the case in the context of General Relativity, as it is discussed in [135]. There we can see that a second degree of freedom (equation (4) in [135]) must be added for the existence of non-trivial solutions for the scalar field. Here, the fact of non-linear gravity makes $f_R \neq const.$, and therefore we can have a one degree of freedom metric. The integration constants c_1 and c_2 have physical meaning. c_1 is related with the Einstein-Hilbert term, while c_2 is related to possible (if $c_2 \neq 0$) geometric corrections to General Relativity that are encoded in $f(R)$ gravity. The last term of this equation is related directly to the scalar field. This means that the matter not only modifies the curvature scalar R but also the gravitational model $f(R)$.

3.2 Black hole solutions

In this section we will discuss the cases where $c_1 = 1, c_2 = 0$ and $c_1 = c_2 = 0$ for a given scalar field configuration. For the second case to satisfy observational and thermodynamical constraints we will introduce a phantom scalar field and we will reconstruct the $f(R)$ theory, looking for black hole solutions.

3.2.1 $c_1 = 1, c_2 = 0$

Equations (3.18), (3.19) and (3.20) are three independent equations for the four unknown functions of our system, f_R, ϕ, V, b, hence we have the freedom to fix one of them and solve for the others. We fix the scalar field configuration as

$$\phi(r) = \sqrt{\frac{A}{r+B}}\ , \tag{3.21}$$

where A and B are some constants with unit $[L]$, the scalar charges. We now obtain from equation (3.18) $f_R(r)$

$$f_R(r) = 1 - \frac{A}{8(B+r)} \,, \tag{3.22}$$

where we have set $c_2 = 0$ and $c_1 = 1$. Therefore, we expect that, at least in principle, a pure Einstein-Hilbert term will be generated if we integrate f_R with respect to the Ricci scalar.
Now, from equation (3.19) we obtain the metric function

$$b(r) = c_3 r^2 - \frac{4BK}{A-8B} - \frac{8AKr}{(A-8B)^2} - \frac{64AKr^2}{(A-8B)^3} \ln\left(\frac{8(B+r)-A}{r}\right) \,. \tag{3.23}$$

The metric function is always continuous for positive r when the scalar charges satisfy $0 < A < 8B$. Here we show its asymptotic behaviors at the origin and space infinity

$$b(r \to 0) = -\frac{4BK}{A-8B} - \frac{8AKr}{(A-8B)^2} + c_3 r^2 + \frac{64AKr^2}{(A-8B)^3} \ln\left(-\frac{r}{A-8B}\right) + \mathcal{O}(r^3) \,, \tag{3.24}$$

$$b(r \to \infty) = \frac{K}{2} + \frac{AK}{24r} - r^2 \Lambda_{\text{eff}} + \mathcal{O}(r^{-2}) \,, \tag{3.25}$$

where the effective cosmological constant of this solution is generated from the equations can be read off

$$\Lambda_{\text{eff}} = -c_3 + \frac{192AK\ln(2)}{(A-8B)^3} \,. \tag{3.26}$$

It is important to discuss the asympotic behaviours of the metric function. At large distances, we can see that we obtain the BTZ black hole where the scalar charges appear in the effective cosmological constant of the solution. Corrections in the structure of the metric appear as $\mathcal{O}(r^{-n})$ (where $n \geq 1$) terms and are completely supported by the scalar field. At small distances we can see that the metric function has a completely different behaviour from the BTZ black hole. Besides the constant and $\mathcal{O}(r^2)$ terms there are present $\mathcal{O}(r)$ and $\mathcal{O}(r^2\ln(r))$ terms that have an impact on the metric for small r. Our findings are in agreement with the work [136] where in four dimensions Schwarzchild black holes are obtained at infinity with a scalarized mass term while at small distances a rich structure of black holes is unveiled. This is expected since at small distances the Ricci curvature becomes strong and therefore changing the form of spacetime. The Ricci scalar and the Kretschmann scalar are both divergent at the origin

$$R\,(r \to 0) = \frac{16AK}{r(A-8B)^2} + \mathcal{O}\left(\ln r\right) \,, \tag{3.27}$$

$$K\,(r \to 0) = \frac{128K^2A^2}{r^2(A-8B)^4} + \mathcal{O}\left(\frac{1}{r}\ln r\right) \,, \tag{3.28}$$

indicating a singularity at $r = 0$. As a consistency check for $A = 0$ we indeed obtain the BTZ [97] black hole solution

$$b(r) = c_3 r^2 + \frac{K}{2} \,, \tag{3.29}$$

which means that for vanishing scalar field we go back to General Relativity. Hence the solution (3.23) can be regarded as a scalarized version of the BTZ black hole in the context of $f(R)$ gravity.

Now we solve the expression of the potential from the Klein-Gordon equation

$$V(r) = \frac{1}{8AB^2(A-8B)^3(B+r)^3}$$
$$\Big(B(4A^4(-B^2(K-18c_3r^2)+36B^3c_3r+12B^4c_3-4BKr-2Kr^2)-64A^3B(r^2(9B^2c_3+K)$$
$$+Br(18B^2c_3+K)+6B^4c_3)+256A^2B(B(6r^2(B^2c_3+K)+2Br(6B^2c_3+5K)+4B^4c_3+3B^2K)+$$
$$30K\ln(2)(B+r)^3)-A^5Bc_3(2B^2+6Br+3r^2)+64BK(-A^3(2B^2+6Br+3r^2)\ln(\frac{r}{8(B+r)-A})$$
$$-8(5A^2-32AB+64B^2)(B+r)^3\ln(8(B+r)-A))-4096AB^2K(B+r)^2(12\ln(2)(B+r)+B)$$
$$+98304B^3K\ln(2)(B+r)^3)-8A^2K(A^2-32AB+64B^2)(B+r)^3\ln(r)+8K(A-8B)^4(B+r)^3\ln(B+r)\Big)\,, \tag{3.30}$$

the asymptotic behaviors of which are

$$V(r\to 0) = -\frac{K\ln(r)}{B^2(A-8B)} + \mathcal{O}(r^0)\,, \tag{3.31}$$
$$V(r\to\infty) = \frac{3A\left(24A^2Bc_3 - A^3c_3 - 192A\left(B^2c_3 - K\ln(2)\right) + 512B^3c_3\right)}{8r(A-8B)^3} + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{3.32}$$

To ensure that the potential vanishes at space infinity, we need to set the integration constant V_0 at (3.20) equal to

$$V_0 = \frac{192K\ln 2\left(5A^2-32AB+64B^2\right)}{A(A-8B)^3}\,. \tag{3.33}$$

In addition, there is a mass term in the potential that has the same sign with the effective cosmological constant

$$m^2 = V''(\phi=0) = \frac{3}{4}\left(\frac{192AK\ln(2)}{(A-8B)^3} - c_3\right) = \frac{3}{4}\Lambda_{\text{eff}}\,, \tag{3.34}$$

which satisfies the Breitenlohner-Freedman bound in three dimensions [88, 89], ensuring the stability of AdS spacetime under perturbations if we are working in the AdS spacetime.

Substituting the obtained configurations into one of the Einstein equations we can solve for $f(r)$

$$f(r) = \frac{1}{AB^2r(A-8B)^3(A-8(B+r))}$$
$$\Big[B(192BKr\ln(2)(5A^2-32AB+64B^2)(A-8(B+r))+A(A-8B)^2(16Bc_3r^2(A-8B)-2Bc_3r(A-8B)^2$$
$$+8Kr(A+8B)-AK(A-8B)))+A^2Kr(-(A^2-32AB+64B^2))\ln(r)(A-8(B+r))+Kr(8(B+r)-A)$$
$$\Big(64B^2((5A^2-32AB+64B^2)\ln(8(B+r)-A)+2A^2\ln(\frac{r}{8(B+r)-A}))-(A-8B)^4\ln(B+r)\Big)\Big]. \tag{3.35}$$

On the other side, the Ricci scalar can be calculated from the metric function

$$R(r) = \frac{16AK\left(-36r(A-8B)+(A-8B)^2+192r^2\right)}{r(A-8B)^2(A-8(B+r))^2} + \frac{384AK}{(A-8B)^3}\ln\left(\frac{8(B+r)-A}{r}\right) - 6c_3. \tag{3.36}$$

As one can see it is difficult to invert the Ricci scalar and solve the exact form of $f(R)$, though we have the expressions of $R(r)$, $f(r)$ and $f_R(r)$. Nevertheless we can still obtain the asymptotic $f(R)$ forms by studying their asymptotic behaviors

$$f(r\to\infty) = -\frac{AK(A-8B)}{128r^4} + \frac{768AK\ln(2)}{(A-8B)^3} - 4c_3 + \mathcal{O}\left(\frac{1}{r^5}\right) , \tag{3.37}$$

$$R(r\to\infty) = -\frac{AK(A-8B)}{128r^4} + \frac{1152AK\ln(2)}{(A-8B)^3} - 6c_3 + \mathcal{O}\left(\frac{1}{r^5}\right) , \tag{3.38}$$

$$f(r\to 0) = -\frac{2AK}{(A-8B)Br} + \mathcal{O}(\ln r) , \tag{3.39}$$

$$R(r\to 0) = \frac{16AK}{r(A-8B)^2} + \mathcal{O}(\ln r) , \tag{3.40}$$

which leads to

$$f(R) \simeq R + 2c_3 - \frac{384AK\ln(2)}{(A-8B)^3} = R - 2\Lambda_{\text{eff}} , \qquad r\to\infty , \tag{3.41}$$

$$f(R) \simeq R\left(1-\frac{A}{8B}\right) , \qquad r\to 0 . \tag{3.42}$$

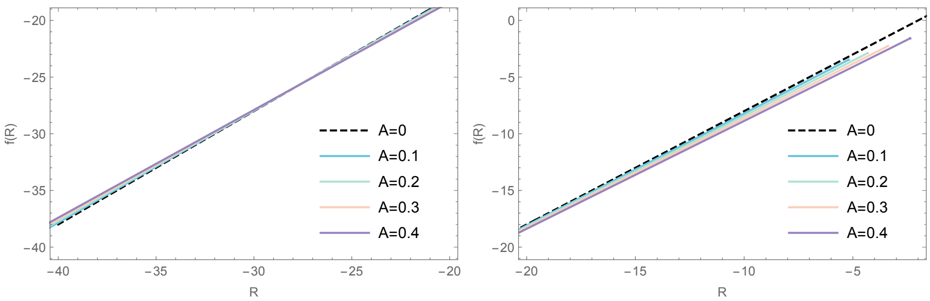


Figure 3.1: The $f(R)$ function. The black dashed line represents the Einstein Gravity $f(R) = R - 2\Lambda_{\text{eff}}$, where other parameters have been fixed as $B = 1$, $K = -5$ and $c_3 = 1$.

The fact that the Ricci scalar contains logarithmic terms prevents us from obtaining the non-linear corrections near the origin, where we expect the modified part of the $f(R)$ model to be stronger, since it is supported by the existence of the scalar field and the scalar field takes its maximum value for $r = 0 \to \phi(0) = \sqrt{A/B}$. To avoid the tachyonic instability, we check the Dolgov-Kawasaki stability ctiterion [61] which states that the second derivative of the gravitational model f_{RR} must be always positive [51, 137, 138]. Using the chain rule

$$f_{RR} = \frac{df_R(R)}{dR} = \frac{df_R(r)}{dr}\frac{dr}{dR} = \frac{f_R'(r)}{R'(r)} = -\frac{r^2(A-8(B+r))^3}{128K(A-8B)(B+r)^2} , \tag{3.43}$$

we can see that the above expression is always positive for $K < 0$ when the continuity condition $0 < A < 8B$ is considered. So far we have not imposed any condition on c_3, therefore the spacetime

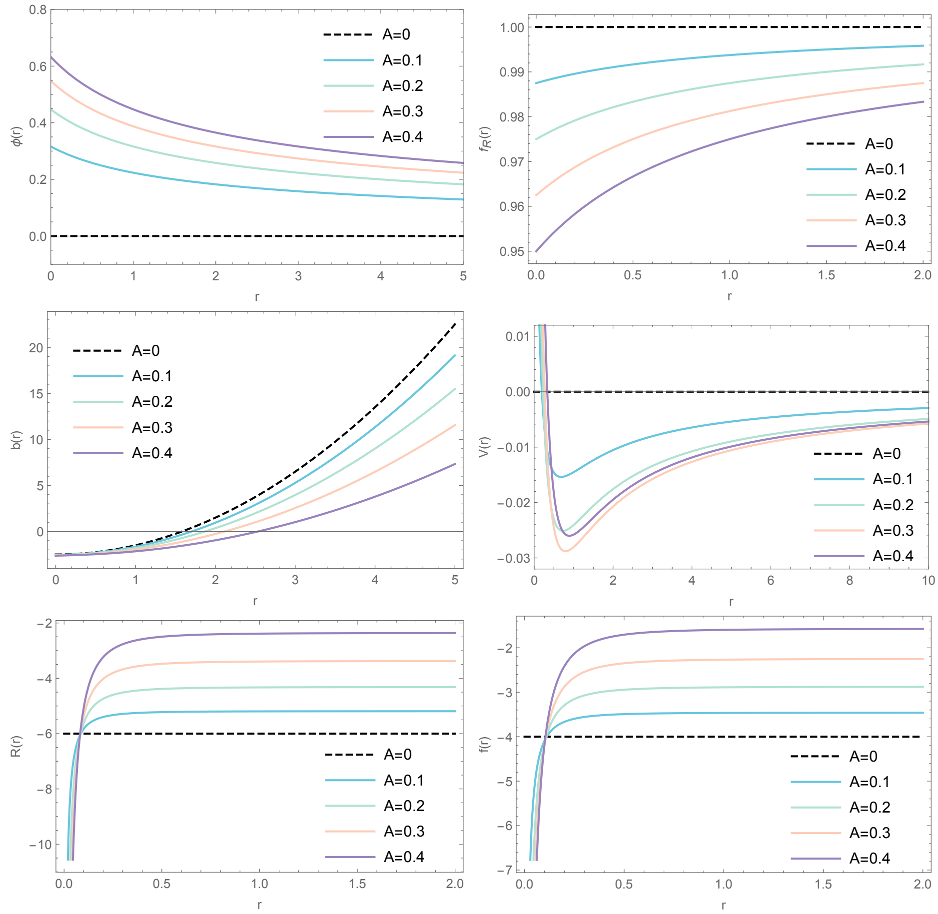


Figure 3.2: All the physical quantities of the AdS black holes are plotted with different scalar charges A, where other parameters have been fixed as $B = 1$, $K = -5$ and $c_3 = 1$.

might be asympotically AdS or dS depending on the value of parameter c_3

$$c_3 > \frac{192AK\ln(2)}{(A-8B)^3} > 0 \quad \text{asympotically AdS}, \tag{3.44}$$

$$c_3 < \frac{192AK\ln(2)}{(A-8B)^3} \quad \text{asympotically dS}. \tag{3.45}$$

We can prove that the metric function has at most one root, which can not describe a dS black hole. For the asympotically AdS spacetime, the condition $K < 0$ gives an AdS black hole solution while the condition $K > 0$ gives the pure AdS spacetime with a naked singularity at origin. For the asympotically dS spacetime, the condition $K > 0$ gives a pure dS spacetime with a cosmological horizon. Therefore pure AdS or dS spacetime described by this solution suffers from the tachyonic instability, only AdS black holes can survive from this instability. We plot all the physical quantities of the AdS black holes in FIG. 3.2 and FIG. 3.1. In FIG. 4.2 we plot the metric function, the potential, the scalar field, the Ricci scalar, the $f(r)$ and f_R functions along with the $A = 0$ (BTZ black hole) case in order to compare them. In FIG. 3.1 we plot the $f(R)$ model along with $f(R) = R - 2\Lambda_{\text{eff}}$ in order to compare our model with Einstein's Gravity. For FIG. 3.1 we used the expression for the Ricci scalar (3.36) for the horizontal axes and the expression for $f(r)$ (3.35) for the vertical axes.

From FIG. 4.2 and FIG. 3.1 we can see that the existence of scalar charge A makes the solution deviate from the GR solution, and the stronger the scalar charge is, the larger it deviates. The figure of the metric function shows that the hairy solution with stronger scalar charge has larger radius of the event horizon, while its influence on the curvature is qualitative, from constant to dynamic, with a divergence appearing at origin. The scalar charge also modifies the $f(R)$ model and the potential to support such hairy structures, where the potential develops a well near the origin to trap the scalar field providing the right matter concentration for a hairy black hole to be formed. For the $f(R)$ model, the scalar charge only sets aside a small distance with the Einstein Gravity while the slope changes little, indicating our $f(R)$ model is very close to Einstein Gravity. We can see that even slight deviations from General Relativity can support hairy structures. The asymptotic expressions (3.41) (3.42) tell us that at large scale the scalar field only modifies the effective cosmological constant while at small scale the slope of $f(R)$ can also be modified, which agrees with the figure of $f(R)$.

Next we study the thermodynamics of this solution. The Hawking temperature and Bekenstein-Hawking entropy are defined as [71, 73]

$$T(r_+) = \frac{b'(r_+)}{4\pi} = \frac{2K(B+r_+)}{\pi r_+(A-8(B+r_+))}, \tag{3.46}$$

$$S(r_+) = \frac{\mathcal{A}}{4G} f_R(r_+) = 4\pi^2 r_+ f_R(r_+) = 4\pi^2 r_+ \left(1 - \frac{A}{8(B+r_+)}\right), \tag{3.47}$$

where r_+ is the radius of the event horizon of the AdS black hole and $A = 2\pi r_+$ is the area of the event horizon, where the gravitational constant G equals $1/8\pi$ since we've set $8\pi G = 1$. Here in the first expression we have already used r_+ to replace the parameter c_3. It is clear that the Hawking temperature and Bekenstein-Hawking entropy are both positive for $K < 0$ and $0 < A < 8B$. We present their figures in FIG. 3.3. FIG. 3.3 shows that for the same radius of the event horizon, the hairy black hole solution owns higher Hawking temperature but lower Bekenstein-Hawking entropy. However, with fixed parameters B, c_3 and K, the hairy black hole solution has larger radius of the event horizon, therefore, we plot the entropy inside the event horizon as a function of the scalar charge A in FIG. 3.4 to confirm if the hairy solution is thermodynamically preferred or not. The fact is that hairy black hole solution is thermodynamically preferred, which owns higher entropy than its corresponding GR solution, BTZ black hole, and the entropy grows with the increase of the scalar charge A. It can be easily understood that the participation of the scalar field gains more entropy for the black hole.

3.2.2 Exact Black Hole Solution with Phantom Hair

In the previous section, we have set $c_1 = 1$ and $c_2 = 0$, therefore the $f(R)$ model consists of the pure Einstein-Hilbert term and corrections that arise from the existence of the scalar field. We have shown that with the vanishing of scalar field, we obtain the well known results of General Relativity, the BTZ black hole [98].

We will now discuss the possibillity that the scalar field, purely supports the $f(R)$ model by setting $c_1 = c_2 = 0$. From equation (3.18) we can see that due to the $\mathcal{O}(r^{-n})$ (where $n > 0$) nature of the scalar field and the double integration, there will be regions where $f_R < 0$. For example for our scalar profile (3.21) the f_R turns out to be

$$f_R(r) = -\frac{A}{8(B+r)} \ , \tag{3.48}$$

which is always negative for $A, B > 0$. With this form of f_R one can derive an exact hairy black hole solution similar to a hairy BTZ black hole which however has negative entropy as can be seen from the relation (3.47).

It is clear that a sign reversal of $f(R)$ can fix the negative entropy problem. As a result, the sign reversal of other terms in the action is also required, which leads to a phantom scalar field instead of the regular one. This comes in agrement with recent observational results which they require that at the early universe to explain the equation of state $w < -1$ phantom energy is needed to support the cosmological evolution [139, 140, 141]. As it will be shown in the following, in the pure $f(R)$ gravity theory the curvature acquires non-linear correction terms which makes the curvature stronger as it is expected in the early universe. Hence, we consider the following action

$$S = \int d^3x\sqrt{-g}\left\{\frac{1}{2\kappa}f(R) + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right\} \ , \tag{3.49}$$

which is the action (3.9) but the kinetic energy of the scalar field comes with the positive sign which corresponds to a phantom scalar field instead of the regular one. Under the same metric ansatz (3.5), equation (3.16) now becomes

$$f_R''(r) - \phi'(r)^2 = 0 \ , \tag{3.50}$$

and by integration

$$f_R(r) = \int\int \phi'(r)^2 dr dr \ , \tag{3.51}$$

having set $c_1 = 0$ and $c_2 = 0$. With the same profile of the scalar field, the solution of this action becomes

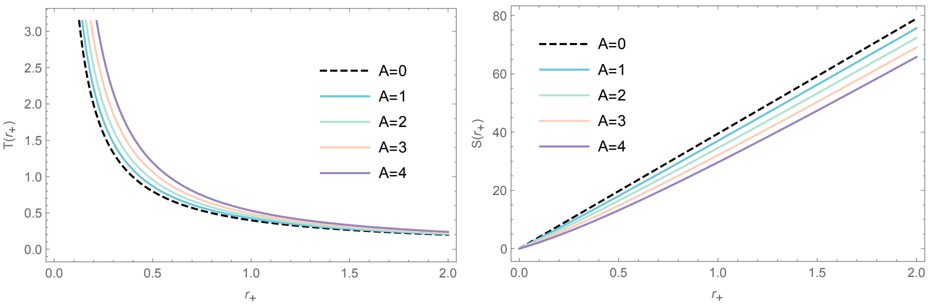


Figure 3.3: The Hawking temperature and Bekenstein-Hawking entropy are plotted with different scalar charges A, where other parameters have been fixed as $B = 1$ and $K = -5$.

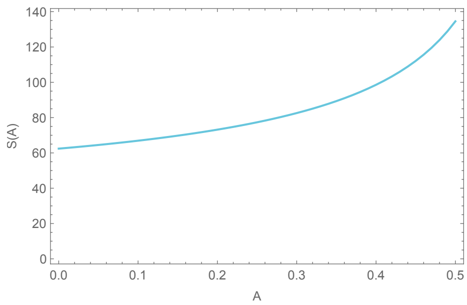


Figure 3.4: The Bekenstein-Hawking entropy as a function of the scalar charge A, where other parameters have been fixed as $B = 1$, $K = -5$ and $c_3 = 1$.

$$\phi(r) = \sqrt{\frac{A}{B+r}}\,, \tag{3.52}$$

$$f_R(r) = \frac{A}{8(B+r)}\,, \tag{3.53}$$

$$b(r) = \frac{4BK}{A} + \frac{8Kr}{A} - \Lambda r^2\,, \tag{3.54}$$

$$R(r) = 6\Lambda - \frac{16K}{Ar}\,, \tag{3.55}$$

$$V(r) = \frac{B(AB\Lambda + 4K)}{8(B+r)^3} - \frac{3AB\Lambda + 8K}{8B(B+r)} - \frac{K}{B^2}\ln\left(\frac{B+r}{r}\right), \tag{3.56}$$

$$f(r) = -\frac{2K}{Br} + \frac{2K}{B^2}\ln\left(\frac{B+r}{r}\right), \tag{3.57}$$

$$f(R) = \frac{AR}{8B} - \frac{3A\Lambda}{4B} + \frac{2K}{B^2}\ln\left(\frac{6AB\Lambda - ABR + 16K}{16K}\right), \tag{3.58}$$

$$V(\phi) = -\frac{K\phi^2}{AB} - \frac{3\Lambda\phi^2}{8} + \frac{B^2\Lambda\phi^6}{8A^2} + \frac{BK\phi^6}{2A^3} + \frac{K}{B^2}\ln\left(\frac{A}{A-B\phi^2}\right). \tag{3.59}$$

The $f(R)$ model avoids the afforementioned tachyonic instability when $f_{RR} > 0$, and for the obtained $f(R)$ function we have

$$f_{RR} = -\frac{A^2 r^2}{128K(B+r)^2} > 0 \Rightarrow K < 0\,. \tag{3.60}$$

For a particular combination of the scalar charges: $B = A/8$, the $f(R)$ model is simplified and takes the form:

$$f(R) = R - 6\Lambda + \frac{128K}{A^2}\ln\left(1 - \frac{A^2(R-6\Lambda)}{128K}\right) \tag{3.61}$$

The metric function (3.54) as we can see, is similar to the BTZ black hole with the addition of a $\mathcal{O}(r)$ term because of the presence of the scalar field, and this term gives Ricci scalar its dynamical behaviour. The potential satisfies the conditions

$$V(r \to \infty) = V(\phi \to 0) = 0\,, \tag{3.62}$$

and also $V'(\phi = 0) = 0$. It has a mass term which is given by

$$m^2 = V''(\phi = 0) = -\frac{3}{4}\Lambda\,. \tag{3.63}$$

The metric function for $\Lambda = -1/l^2$ (AdS spacetime) and for $A, B > 0$ has a positive root, since $K < 0$. For $\Lambda = 1/l^2$ (dS spacetime) the metric function is always negative provided for $A, B > 0$ and $K < 0$, therefore we will discuss only the AdS case.

The horizon is located at

$$r_+ = \frac{2l\left(\sqrt{K\left(4Kl^2 - AB\right)} - 2Kl\right)}{A}\,, \tag{3.64}$$

where we have set $\Lambda = -1/l^2$. As we can see, in this $f(R)$ gravity theory we have a hairy black hole supported by a phantom scalar field.

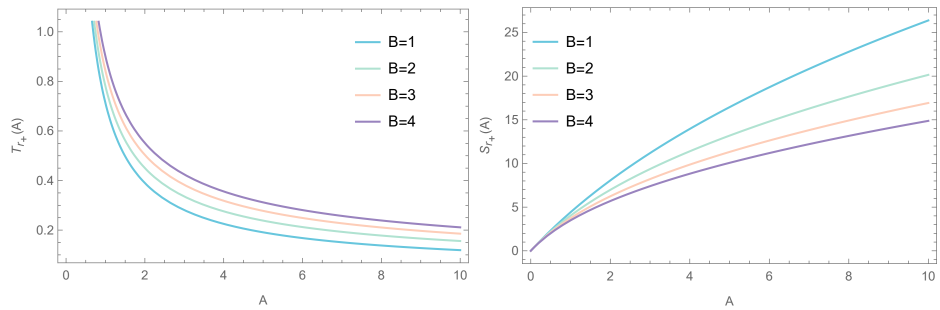


Figure 3.6: The temperature and the entropy at the horizon of the black hole, as functions of the scalar charge A while changing scalar charge B.

In FIG. 3.5 we show the behaviour of the metric function $b(r)$, the potential $V(r)$, the dynamical Ricci scalar $R(r)$ and the $f(R)$ function. As can be seen in the case of $B = A/8$, the scalar charge A plays an important role on the behaviour of the above functions. For example if the scalar charge A is getting smaller values the radius of the horizon of the black hole is getting larger. This means that even a small distribution of phantom matter can support a hairy black hole.

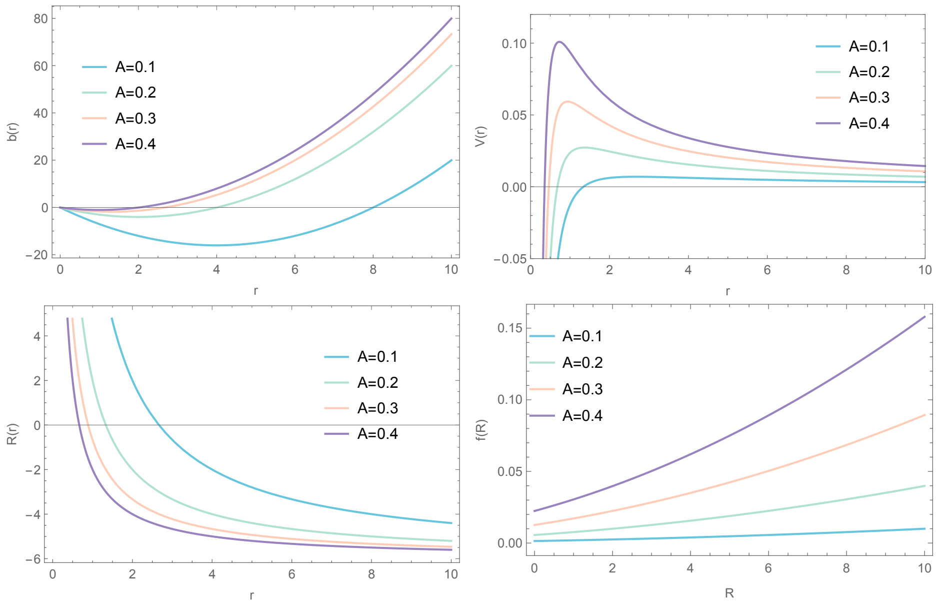


Figure 3.5: We plot the metric function, the potential, the Ricci scalar and the $f(R)$ function of the phantom black hole for different scalar charge A, where other parameters have been fixed as $B = A/8$, $K = -1$ and $\Lambda = -1$.

Looking at the thermodynamic properties of the model the Hawking temperature at the horizon is given by

$$T(r_+) = \frac{2K}{\pi A} + \frac{r_+}{2\pi l^2} = \frac{\sqrt{K\left(4Kl^2 - AB\right)}}{\pi A l} \,, \tag{3.65}$$

which is always positive for $A, B > 0$ and $K < 0$, while the Bekenstein-Hawking entropy is given by

$$S(r_+) = \frac{\mathcal{A}}{4G} f_R(r_+) = 4\pi^2 r_+ f_R(r_+) = \frac{A\pi^2 r_+}{2(B + r_+)} = -\frac{\pi^2 AKl}{\sqrt{K\left(4Kl^2 - AB\right)}} > 0 \,. \tag{3.66}$$

For the thermodynamic behaviour of the hairy black hole we can see from FIG. 3.6 that for larger scalar charge A we are getting smaller temperatures, while the entropy has the opposite behaviour.

3.3 Conclusions

In this section, we considered $(2+1)$-dimensional $f(R)$ gravity with a self interacting scalar field as a matter field. Without specifying the form of the $f(R)$ function we derived the field equations and we showed that the $f(R)$ model has a direct contribution from the scalar field. At first we considered the case, where $f_R(r) = 1 - \int\int \phi'(r)^2 dr dr$, which indicates that if we integrate with respect to the Ricci scalar we will obtain a pure Einstein-Hilbert term and another term that depends on the scalar field. The asymptotic analysis of the metric function unveiled the physical meaning of our results. At infinity we obtain a scalarized BTZ black hole. The scalar charges appear in the effective cosmological constant that is generated from the equations. Corrections in the form of spacetime appear as $\mathcal{O}(r^{-n})$ (where $n \geq 1$) terms that depend purely on the scalar charges. At the origin we obtain a different solution from the BTZ black hole, where $\mathcal{O}(r)$ and $\mathcal{O}(r^2 \ln(r))$ terms change the form of spacetime.

The scalar curvature is dynamical and due to its complexity it was difficult to obtain an exact form of the $f(R)$ function. Using asymptotic approximations, we show that the scalar charges make our theory to deviate form Einstein's Gravity. In the obtained results we considered the Dolgov-Kawasaki stability ctiterion [61] to ensure that our theory avoids tachyonic instabilities [51, 137, 138]. We then calculated the Bekenstein-Hawking entropy and the Hawking temperature of the solution and we showed that the hairy solution is thermodynamically preferred since it has higher entropy.

We then considered a pure $f(R)$ theory supported by the scalar field. We showed that thermodynamic and observational constraints require that the pure $f(R)$ theory should be builded with a phantom scalar field. The black hole solution we found has a metric function which is similar to the BTZ solution with the addition of a $\mathcal{O}(r)$ term. The scalar charge is the one that determines the behaviour of the solution. For bigger scalar charge, the horizon radius is getting smaller meaning that the black hole is formed closer to the origin. The $\mathcal{O}(r)$ term is the one that gives to the Ricci scalar its dynamical behaviour. The obtained $f(R)$ model is free from tachyonic instabilities. We computed the Hawking temperature and the Bekenstein-Hawking entropy to find out that they are both positive, with the temperature getting smaller with the increase of the scalar charge while the entropy behaves the opposite way, growing with the increase of the scalar charge.

In the $f(R)$ gravity theories if a conformal transformation is applied from the original Jordan frame to the Einstein frame then, a new scalar field appears which is coupled minimally to the conformal metric and also a scalar potential is generated. The resulted theory can be considered as a scalar-tensor theory with a geometric (gravitational) scalar field. Then it was shown in [142, 143], that this geometric scalar field cannot dress a $f(R)$ black hole with hair. On the other hand on cosmological grounds, it was shown in [52] that dark energy can be considered as a geometrical fluid that adds to the conventional stress-energy tensor, which means that the determination of the dark energy equation of state depends on the understanding of which $f(R)$ theory better fits current data. In our study we have introduced real matter parameterized by a scalar field coupled to gravity, therefore, it would be interesting to study the interplay of the geometric scalar field with the matter scalar field and see what are their implications to cosmology. However, to study this effect we have to extend this work to a study of $(3+1)$-dimensional $f(R)$ gravity theories. The main difficulty of constructing such theories is the complexity of their resulting equations. Nevertheless, even numerically we can get important information of how matter is coupled to $f(R)$ gravity and what are the cosmological implications.

It would be interesting to extent this theory including an electromagnetic field. In three dimensions the electric charge makes a contribution to the Ricci scalar, therefore we expect, like in the BTZ black hole, to find a charged hairy black hole in $f(R)$ gravity. One could also study the properties of the boundary CFT, consider a rotationally symmetric metric anstaz to find rotating hairy black holes or study hairy axially symmetric solutions from hairy spherically symmetric solutions [144].

Chapter 4

$(2+1)$ Dimensional Black Holes in $f(R,\phi)$ Gravity

In this chapter, we extend the theory we considered previously, adding in the action a direct coupling between matter and curvature of the form $\xi R\phi^2$, where $\xi = 1/8$ the conformal coupling factor. We solve the field equations and compare our results with a well known hairy black hole solution of the $(2+1)$-dimensional Einstein-Conformal scalar equations. We also derive the temperature, the entropy and the conserved mass of the novel black hole, to find that our solution might be thermodynamically preferred and is massless due to the contribution of non-linear gravity to the mass of the black hole. This chapter is based on [4].

4.1 Conformal $(2+1)$ dimensional black hole

We begin with a brief review of the conformally dressed black hole derived in [19]. The action of the theory consists of the Ricci scalar, a negative cosmological constant, and a conformally coupled scalar field, namely

$$S = \frac{1}{2}\int d^3x\sqrt{-g}\Big\{\frac{R+2l^{-2}}{\kappa} - \partial_\mu\phi\partial^\mu\phi - \frac{1}{8}R\phi^2\Big\}\ , \tag{4.1}$$

where we will use $\kappa = 8\pi G = 1$ for simplicity throughout the paper. By variation one can obtain the Einstein equation and the Klein-Gordon equation

$$G_{\mu\nu} - g_{\mu\nu}l^{-2} = T_{\mu\nu}\ , \tag{4.2}$$

$$\Box\phi - \frac{1}{8}R\phi = 0\ , \tag{4.3}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ and the energy-momentum tensor is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi + \frac{1}{8}\Big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}\Big)\phi^2\ . \tag{4.4}$$

One can prove that by virtue of equation (4.16) the energy-momentum tensor is traceless so that we have a constant Ricci scalar

$$R = -\frac{6}{l^2}\ . \tag{4.5}$$

We assume the metric ansatz with $g_{tt}g_{rr} = -1$

$$ds^2 = -b(r)dt^2 + b(r)^{-1}dr^2 + r^2 d\theta^2 \ , \tag{4.6}$$

where $b(r)$ is the only degree of freedom and it can be obtained from the Ricci scalar (1.165)

$$b(r) = \frac{r^2}{l^2} - \frac{c_1}{r} + c_2 \ , \tag{4.7}$$

while from the tt and rr components of the Einstein equation we can get the scalar field

$$\phi(r) = \frac{1}{\sqrt{c_3 r + c_4}} \ . \tag{4.8}$$

Substituting them into the $\theta\theta$ component of the Einstein equation and together with the Klein-Gordon equation (4.16) we have

$$b(r) = \frac{r^2}{l^2} + \frac{B^2(-2B-3r)}{l^2 r} \ , \tag{4.9}$$

$$\phi(r) = \sqrt{\frac{8B}{r+B}} \ , \tag{4.10}$$

where $B = c_4/c_3 > 0$.

This solution is regular for any positive r, except for a singularity at the origin $r = 0$, as can be seen from the divergence of the Kretschmann scalar $K \equiv R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$. The metric function $b(r)$ has only one root which gives the radius of the event horizon $r_h = 2B$. The scalar field does not diverge at the event horizon r_h like the BBMB black hole [20] because of the presence of a negative cosmological constant.

The Hawking temperature [17] is given by the Euclidean trick ($t \to -i\tau$)

$$T_H = \frac{b'(r_h)}{4\pi} = \frac{9B}{8\pi l^2} \ . \tag{4.11}$$

Using Wald's formula [21], the entropy at the event horizon can also be obtained as

$$S = \frac{\mathcal{A}}{4}\left(1 - \frac{1}{8}\phi(r_h)^2\right) = \frac{\pi r_h}{3} = \frac{2\pi B}{3} \ , \tag{4.12}$$

where $\mathcal{A} = 2\pi r_h$ is the horizon area.

The entropy acquires another term, besides the GR one, that depends on the scalar field, which comes from the non-minimal coupling between matter and curvature. As a result, the entropy is smaller than the corresponding BTZ black hole entropy [98] which is $\pi r_h/2$, but is positive and finite, while the entropy of the BBMB black hole is infinite due to the divergence of the scalar field at the event horizon. The conserved black hole mass can be obtained by using the first law of thermodynamics

$$dM = TdS \to M = \int T(r_h)S'(r_h)dr_h = \frac{3r_h^2}{32l^2} = \frac{3B^2}{8l^2} \ . \tag{4.13}$$

All thermodynamic quantities grow with the increase of the scalar charge B (or r_h), in agreement with those obtained from the Hamiltonian formalism [19].

The scalar field dresses the black hole with a secondary scalar hair, since its charge B is not an independent conserved quantity, as it is related to the conserved mass of the black hole.

4.2 $f(R)$ Gravity Black Hole Solution

In this Section we extend the conformal black hole solution [19] described in the previous Section in $f(R)$ gravity by replacing the Einstein-Hilbert term R with the $f(R)$ function and endowing the scalar field with a self-interacting potential $V(\phi)$ in the action

$$S = \frac{1}{2}\int d^3x\sqrt{-g}\Big\{f(R) - \partial_\mu\phi\partial^\mu\phi - \frac{1}{8}R\phi^2 - 2V(\phi)\Big\}\ , \tag{4.14}$$

The field equations that arise from this action are

$$I_{\mu\nu} \equiv \qquad f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + g_{\mu\nu}\Box f_R - \nabla_\mu\nabla_\nu f_R = T_{\mu\nu}\ , \tag{4.15}$$

$$\Box\phi - \frac{1}{8}R\phi - V'(\phi) = 0\ , \tag{4.16}$$

where $f_R \equiv \dfrac{df(R)}{dR}$ and the energy momentum tensor becomes

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi + \frac{1}{8}\Big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}\Big)\phi^2 - g_{\mu\nu}V(\phi)\ . \tag{4.17}$$

The trace of Einstein equation (4.15) gives

$$I^\mu_\mu \equiv 2f_R R - 3f(R) + 4\Box f_R = \phi\Box\phi - R\phi^2/8 - 6V(\phi)\ . \tag{4.18}$$

Assuming the same metric ansatz (4.6), the $tt, rr, \theta\theta$ components of the Einstein equation and the Klein-Gordon equation take the form

$$tt : 2r\left(b'\left(\phi\phi' - 4f_R'\right) + 4f_R b'' - 2b\left(4f_R'' + \phi'^2 - \phi\phi''\right) + 4f - 8V\right) + b'\left(8f_R + \phi^2\right) - 16bf_R' + 4b\phi\phi' = 0\ , \tag{4.19}$$

$$rr : 2r\left(b'\left(\phi\phi' - 4f_R'\right) + 4f_R b'' + 4b\phi'^2 + 4f - 8V\right) + b'\left(8f_R + \phi^2\right) - 16bf_R' + 4b\phi\phi' = 0\ , \tag{4.20}$$

$$\theta\theta : r\left(4b'\left(\phi\phi' - 4f_R'\right) + \phi^2 b'' - 4b\left(4f_R'' + \phi'^2 - \phi\phi''\right) + 8f - 16V\right) + 16f_R b' = 0\ , \tag{4.21}$$

$$\text{KG: } b'(r)\phi'(r) + \frac{\phi(r)\left(2b'(r) + rb''(r)\right)}{8r} + \frac{b(r)\phi'(r)}{r} + b(r)\phi''(r) - \frac{V'(r)}{\phi'(r)} = 0\ , \tag{4.22}$$

also the trace (4.18) becomes

$$I^\mu_\mu \equiv -32rb'f_R' + 32f_R b' + 16rf_R b'' + 8r\phi b'\phi' + 2\phi^2 b' + r\phi^2 b'' - 32bf_R' - 32rbf_R'' + 8b\phi\phi' + 8rb\phi\phi'' + 24rf - 48rV = 0\ . \tag{4.23}$$

The Klein-Gordon equation can be obtained by taking the covariant derivative of Einstein's equation [2]. Therefore, we have a system of three independent equations with four unknown functions: the $f(R)$ function, the potential $V(\phi)$, the scalar field $\phi(r)$ and the metric function $b(r)$. We will leave the potential undetermined and solve it from the field equations. We will then check the trace of the energy-momentum tensor. A vanishing trace will indicate that the matter field is conformally coupled to gravity and a scale (if any) is counterbalanced in the action. From equations (4.19) and (4.20) we can obtain the relation between the gravitational function $f_R(r)$ and the scalar field $\phi(r)$

$$4f_R''(r) + 3\phi'(r)^2 - \phi(r)\phi''(r) = 0\ . \tag{4.24}$$

We can immediately integrate this equation for $f_R(r)$

$$f_R(r) = s + \alpha r + \int\int\frac{1}{4}\left(\phi(r)\phi''(r) - 3\phi'(r)^2\right)drdr\ , \tag{4.25}$$

where s and α are constants of integration. The constant s is the coefficient of the Einstein-Hilbert term, α is related to geometric corrections to Einstein gravity that are encoded in $f(R)$ theories and the last term is generated from the scalar field. It shows that the scalar field gives an immediate modification to the $f(R)$ model if the integrand does not equal zero. To simplify the equations we consider the integrand to be vanishing, i.e. $f''_R(r) = 0$, which gives the profile of the scalar field as $\phi(r) = \sqrt{A/(r+B)}$ and $f_R(r) = s + \alpha r$. Also, in order to make it comparable with the GR case [19] we use $A = 8B$ and $s = 1$, then the scalar field becomes same with (1.168) and

$$f_R(r) = 1 + \alpha r \; . \tag{4.26}$$

We can immediately integrate $f_R(r)$ with respect to Ricci scalar to obtain the general form of the $f(R)$ theory

$$f_R(r) = 1 + \alpha r \to f(R) = R + \alpha \int^R r(R) dR + C \; , \tag{4.27}$$

where C is an integration constant with the unit $[L]^{-2}$, related to the cosmological constant. This expression shows that a geometric correction term appears in addition to the Einstein-Hilbert term, while the scalar field does not appear immediately in the $f(R)$ model as happens in [1].

Then we can solve the metric function as

$$b(r) = -\frac{3B^2}{l^2(\alpha B+1)^2} - \frac{2B^3}{l^2 r(\alpha B+1)} + \frac{6\alpha B^2 r}{l^2(\alpha B+1)^3} + r^2\left(\frac{1}{l^2} + \frac{6\alpha^2 B^2}{l^2(\alpha B+1)^4}\ln\left(\frac{r}{\alpha l(B+r)+l}\right)\right) \; , \tag{4.28}$$

where l is the AdS radius that appears as an integration constant. We can see that the metric function is well behaved for any $r > 0$ if we constrain the parameters B, α to be positive. For this reason we will impose that $\alpha, B > 0$. At large distances, the metric function asymptotes to

$$b(r \to \infty) \sim -\Lambda_{\text{eff}} r^2 - \frac{2B^2}{\alpha l^2 r} + \mathcal{O}(r^{-2}) \; , \tag{4.29}$$

where the effective cosmological constant that the $f(R)$ theory and the non-minimal coupling generate is given by

$$\Lambda_{\text{eff}} = -\left(\frac{1}{l^2} - \frac{6\alpha^2 B^2 \ln(\alpha l)}{l^2(\alpha B+1)^4}\right) \; . \tag{4.30}$$

For vanishing scalar charge B, we obtain pure AdS spacetime and we will also consider that $1 - 6\alpha^2 B^2 \ln(\alpha l)/(\alpha B+1)^4 > 0$ in order to have an asymptotically AdS spacetime, so we can compare our solution with [19]. Now, we can obtain the potential from the Klein-Gordon equation

$$V(r) = \frac{B^3}{l^2(\alpha B+1)^4}\left(\frac{6\alpha^2\left(B^2+\alpha(B+r)^3\right)}{(B+r)^3}\ln\left(\frac{r}{\alpha l(B+r)+l}\right) + \frac{3\left(\alpha^2-\alpha^4 B^2\right)}{B+r} + \frac{3\alpha^3(\alpha B+1)^2}{\alpha(B+r)+1}\right.$$
$$\left. + \frac{6\alpha^2 B(\alpha B+1)}{(B+r)^2} + \frac{\alpha B(\alpha B+1)(\alpha B(\alpha B+5)-2)}{(B+r)^3} + 6\alpha^3\ln\alpha l\right) \; , \tag{4.31}$$

which vanishes at spatial infinity and as a function of ϕ reads

$$V(\phi) = \frac{\alpha B}{512 l^2(\alpha B+1)^4\left(8\alpha B+\phi^2\right)}\Bigg(\phi^2(\alpha B+1)(3072\alpha^2 B^2+\phi^6(\alpha B(\alpha B+5)-2)+8\alpha B\phi^4(\alpha B+1)(\alpha B+4)$$
$$+192\alpha B\phi^2(\alpha B+1)) + 6\alpha B\left(8\alpha B+\phi^2\right)\left(\left(512\alpha B+\phi^6\right)\ln\left(\frac{B\left(-\phi^2+8\right)}{l\left(8\alpha B+\phi^2\right)}\right) + 512\alpha B\ln(\alpha l)\right)\Bigg) \; . \tag{4.32}$$

The Ricci scalar can be obtained from the metric function

$$R(r)=-\frac{6}{l^2}-\frac{6\alpha B^2\left(\alpha\left(2\alpha B^2+9\alpha Br+4B+6\alpha r^2+9r\right)+2\right)}{l^2r(\alpha B+1)^3(\alpha(B+r)+1)^2}-\frac{36\alpha^2B^2}{l^2(\alpha B+1)^4}\ln\left(\frac{r}{\alpha l(B+r)+l}\right)\ , \tag{4.33}$$

while the function $f(r)$ yields

$$f(r)=-\frac{4}{l^2}+\frac{6\alpha B^2\left(\alpha\left(\alpha B^2(3\alpha r-2)+B(\alpha r(2\alpha r-3)-4)-2r(2\alpha r+3)\right)-2\right)}{l^2r(\alpha B+1)^3(\alpha(B+r)+1)^2}+ \\ \frac{12\alpha^2B^2}{l^2(\alpha B+1)^4}\left((\alpha B-2)\ln\left(\frac{r}{\alpha l(B+r)+l}\right)+\alpha B\ln(\alpha l)\right)\ , \tag{4.34}$$

The Ricci scalar is divergent at origin and related to the AdS scale at infinity. The Kretschmann scalar behaves as

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}(r\to 0)\sim\frac{24B^6}{l^4r^6(\alpha B+1)^2}+\mathcal{O}\left(\frac{1}{r^4}\right)\ , \tag{4.35}$$

near the origin, which is also divergent at $r=0$, indicating a physical singularity. It is clear that we cannot invert the Ricci scalar and solve it for r, to substitute back to the $f(r)$ function in order to obtain $f(R)$. However, we can use asymptotics to have a feeling of the curvature model at the origin and at infinity. The asymptotic expressions of the Ricci scalar near the origin and at infinity are respectively

$$R(r\to 0)\ \sim\ -\frac{12\alpha B^2}{l^2r(\alpha B+1)^3}+\mathcal{O}(\ln(r))\ , \tag{4.36}$$

$$R(r\to\infty)\ \sim\ 6\Lambda_{\text{eff}}-\frac{3B^2}{\alpha^2l^2r^4}+\mathcal{O}(r^{-5})\ , \tag{4.37}$$

so the $f(R)$ function near the origin and at large distances yields respectively

$$f(R(r\to 0))\ \sim\ R-\frac{12\alpha^2B^2}{l^2(\alpha B+1)^3}\ln(R)\ , \tag{4.38}$$

$$f(R(r\to\infty))\ \sim\ R-\frac{4B\left(6\Lambda_{\text{eff}}-R\right)^{3/4}}{3^{3/4}\sqrt{\alpha B l}}\ , \tag{4.39}$$

up to a constant of integration. The argument of the ln tern is not dimensionless, but this expression is an approximation.

In FIG.4.1 we plot $f(R(r))$ as a function of $R(r)$ to see how our $f(R)$ deviates from the GR case $f(R)=R+2l^{-2}$. We can see that for stronger α, our theory deviates more from GR.

To check if the resultant $f(R,\phi)$ theory is free of ghost and tachyonic instabilities [51, 137, 145, 138, 146] we need to confirm if the following relations hold respectively

$$f_{R_{\text{total}}}>0\to f_{R_{\text{total}}}=f_{R_{\text{gravity}}}+f_{R_{\text{matter}}}=1+\alpha r-\frac{1}{8}\phi(r)^2=+1+\alpha r-\frac{B}{B+r}=r\left(\alpha+\frac{1}{B+r}\right) \tag{4.40}$$

$$f_{RR}(r)>0\to f_{RR}(r)=\frac{f_R'(r)_{\text{total}}}{R'(r)}=\frac{l^2r^2(\alpha(B+r)+1)^3\left(\alpha(B+r)^2+B\right)}{12\alpha B^2(B+r)^2}>0\ . \tag{4.41}$$

The two above relations hold for $B>0$ and $\alpha>0$ and in this case the resultant $f(R,\phi)$ theory is free of ghosts and avoids the tachyonic instability. The fact that $f_{R_{\text{total}}}>0$ also ensures that

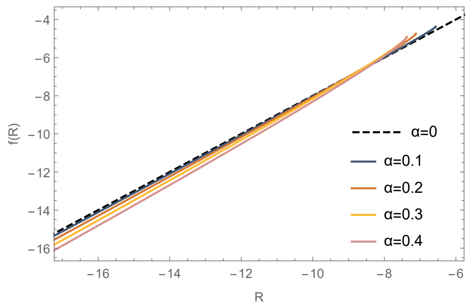


Figure 4.1: $f(R(r))$ as a function of $R(r)$ for different values of α, where we have set $B = l = 1$.

the entropy is positive [72, 73, 71, 147] and our solutions may possess a higher entropy than the corresponding GR counterpart [19] for a fixed horizon value, meaning that the $f(R)$ black holes may be thermodynamically preferred over the GR one [19].

Expanding the metric function $b(r)$, $f(r)$ and $R(r)$ near $\alpha \to 0$ we find that

$$b(r) = \frac{-3B^2 r - 2B^3 + r^3}{l^2 r} + \frac{2\alpha\left(3B^2r^2 + 3B^3 r + B^4\right)}{l^2 r} + \mathcal{O}\left(\alpha^2\right) \,, \tag{4.42}$$

$$f(r) = -\frac{4}{l^2} - \frac{12\alpha B^2}{l^2 r} + \mathcal{O}\left(\alpha^2\right) \,, \tag{4.43}$$

$$R(r) = -\frac{6}{l^2} - \frac{12\alpha B^2}{l^2 r} + \mathcal{O}\left(\alpha^2\right) \,, \tag{4.44}$$

where, as expected, at zeroth order we obtain the GR black hole [19] and the curvature functions $f(r), R(r)$ become dynamical due to the gravitational scale α. The trace of the resultant energy-momentum tensor is dynamical

$$T_{\mu}^{\ \mu} = -\frac{3\alpha^2 B^3}{2r(\alpha B+1)^4(\alpha l(B+r)+l)^2}\Bigg((\alpha B+1)(\alpha(2\alpha B^2 + B(9\alpha r+4) + 3r(2\alpha r+3)) + 2) + 6\alpha r(\alpha(B+r)+1)^2(\ln(\frac{r}{\alpha l(B+r)+l}) + \ln(\alpha l))\Bigg) \,, \tag{4.45}$$

which indicates that the theory is not conformally invariant and possesses a scale that breaks the conformal invariance. This scale is the geometric correction parameter α. For vanishing α, the trace of the energy momentum tensor vanishes as expected since $\alpha \to 0$ gives the GR case [19]. We present some plots of $b(r), R(r), f(r), V(r), T_{\mu}^{\ \mu}$ in Fig. 4.2, in order to better understand our solution alongside the $\alpha = 0$ case which corresponds to GR [19]. We can see that the modified gravity parameter affects the dynamics of the curvature related functions, while the GR black hole [19] admits a larger horizon radius in comparison with the modified gravity one. We also plot the horizon radius as a function of α. The fact that the larger the deviation from the GR solution [19] is, the smaller the horizon radius becomes, is in agreement with the $(3+1)$-dimensional case [2]. Next, we will briefly discuss scalar perturbations of the obtained spacetime. For this reason, we consider a massless test scalar field ϕ_0 that satisfies its equation of motion [148],

$$\Box\phi_0 = 0 \,. \tag{4.46}$$

Transforming the scalar field as $\phi_0 = r^{-1/2}\varphi_0 e^{-i\omega_0 t}$, the Klein-Gordon equation takes the form of a Schrodinger-like one

$$\frac{d^2\varphi_0}{dr_*^2} + (\omega_0^2 - V_{\text{eff}})\varphi_0 = 0 \ , \tag{4.47}$$

where we expressed this equation using the tortoise coordinate $r_* = \int dr b(r)^{-1}$. The resulting effective potential is complicated, however its asymptotic expression is

$$V_{\text{eff}}(r \to \infty) \sim \frac{3\Lambda_{\text{eff}}^2 r^2}{4} + \mathcal{O}\left(\frac{1}{r^2}\right) \ , \tag{4.48}$$

meaning that there is an AdS boundary at infinity constraining the matter fields, regardless of the modified gravity parameter α and the effect it has on Λ_{eff}. We checked that, the inclusion of a mass term for the test scalar field does not change the behavior of the resulting effective potential at large distances. Also, no potential well is formed near the horizon of the black hole for both the massive and massless case, as can be confirmed from FIG. 4.3, where we plot the effective potential of the massless case, meaning that the test scalar particles are not trapped near the black hole, so, as a result, the spacetime is stable under massless and massive scalar perturbations.

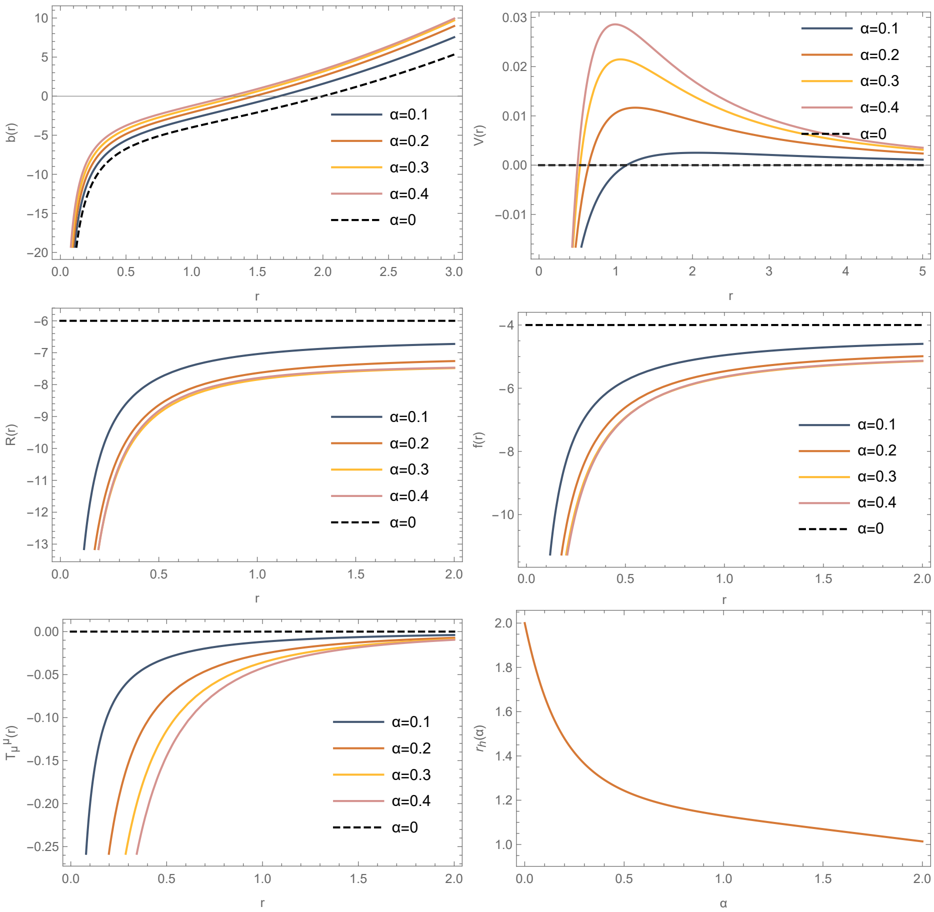


Figure 4.2: The functions $b(r), V(r), R(r), f(r)$ and $T_\mu{}^\mu(r)$ are plotted versus r with different values of α, while in the last panel, the radius of the event horizon r_h is plotted as a function of α. In all the figures we have set $B = l = 1$.

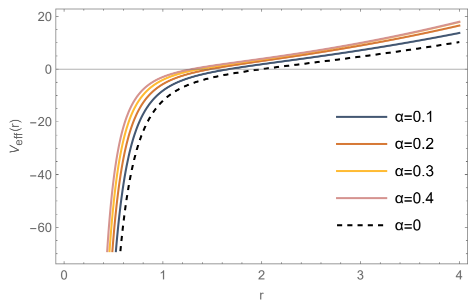


Figure 4.3: The effective potential $V_{\text{eff}}(r)$ for the massless test scalar particles as a function of r for different values of α, where we have set $B = l = 1$.

4.3 Thermodynamics

In this section we will study the thermodynamics of the extended black hole solution in $f(R)$ gravity, including Hawking temperature, entropy and the conserved mass.

4.3.1 Hawking temperature

The Hawking temperature can be calculated as

$$T_H = \frac{b'(r_h)}{4\pi} = \frac{3B^2(B+r_h)}{2\pi l^2 r_h^2(\alpha B + \alpha r_h + 1)} , \tag{4.49}$$

where the relation $b(r_h) = 0$ has been used. As expected, it can reduce to the Hawking temperature (4.11) in conformal dressed black hole case [19] when $\alpha \to 0$.

In FIG. 4.4, we plot the Hawking temperature T_H as a function of α. With the increase of α, the Hawking temperature of the black hole first decreases slightly, then grows up to a maximum, finally descends until approaching zero which can be seen from the expression (4.49).

4.3.2 Entropy

Using Wald's formula [21, 22], we can calculate the entropy of the black hole in $f(R)$ gravity with a non-minimal coupling as

$$S = -\frac{1}{4}\int d\theta \sqrt{r_h^2}\left(\frac{\partial \mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}}\right)\Big|_{r=r_h} \hat{\varepsilon}_{\alpha\beta}\hat{\varepsilon}_{\gamma\delta} , \tag{4.50}$$

where $\hat{\varepsilon}_{\alpha\beta}$ is the binormal to the horizon surface [23], $\mathcal{L}$ is the Lagrangian of the theory, and

$$\frac{\partial \mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}}\Big|_{r=r_h} = \frac{1}{2}\left(\frac{f_R(r_h)}{2} - \frac{1}{16}\phi(r_h)^2\right)\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\beta\gamma}g^{\alpha\delta}\right) . \tag{4.51}$$

Finally the formula of the entropy for our theory can be obtained

$$S = \pi r_h\left(\frac{f_R(r_h)}{2} - \frac{1}{16}\phi(r_h)^2\right) = \frac{\mathcal{A}}{4} f_{R_{\text{total}}}(r_h) . \tag{4.52}$$

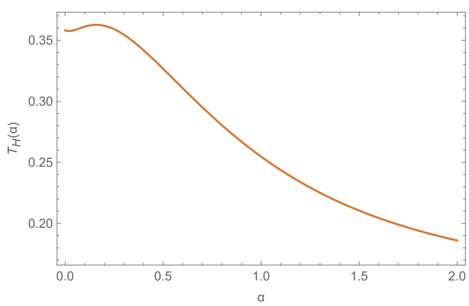


Figure 4.4: The Hawking temperature is plotted as a function of α, where we have set $B = l = 1$.

Substituting the explicit expression for $f_{R_{\text{total}}}$, we have

$$S = \frac{1}{2}\pi r_h \left(1 + \alpha r_h - \frac{B}{B + r_h}\right) , \tag{4.53}$$

In fact, here r_h is also changing with the choices of B, l and α. One might deduce that since $\alpha > 0$, the $f(R)$ black holes have higher entropy than the conformal ones [19]. However, we have to keep in mind that the conformal case [19] has a larger radius for the event horizon as can be seen from the metric function $b(r)$ in Fig. 4.2. Using the relations (4.53) and $b(r_h) = 0$, we can plot the entropy at the event horizon as a function of α in FIG. 4.5.

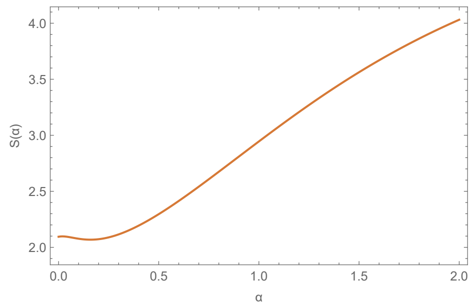


Figure 4.5: The entropy S at the event horizon is plotted as a function of α, where we have set $B = l = 1$.

With the increase of α, the entropy first decreases a little bit to a minimum value, then grows up always. Therefore, for most values of α, the entropy of the $f(R)$ black hole is higher than the corresponding conformal $(2+1)$-dimensional black hole [19], indicating that our solution is thermodynamically preferred for most cases. It is worth to mention that the conformal case ($\alpha = 0$) is a local maximum of the entropy with respect to α.

4.3.3 Conserved Mass

For a D-dimensional spacetime manifold $\mathcal{M}$, which is topologically the product of a spacelike hypersurface and a real line interval $\Sigma \times \mathcal{I}$, the total quasi-local energy is defined as [149, 150]

$$E = \int_{\mathcal{B}} d^{D-2}x\sqrt{\sigma}\varepsilon , \tag{4.54}$$

where $\mathcal{B} \equiv \partial\Sigma$ is the $(D-2)$-dimensional boundary, σ is the determinant of the induced metric σ_{ab} on B, and ε is the energy density.

The boundary $\partial\mathcal{M}$ consists of initial and final spacelike hypersurfaces t' and t'' respectively, and a timelike hypersurface $\mathcal{T} = \mathcal{B}\times\mathcal{I}$ joining them. The $(D-1)$-metric γ_{ij} on $\mathcal{T}$ can be written according to the ADM decomposition as

$$\gamma_{ij}dx^i dx^j = -N^2dt^2 + \sigma_{ab}\left(dx^a + V^a dt\right)\left(dx^b + V^b dt\right) . \tag{4.55}$$

The conserved charge associated with a Killing vector field ξ^i is defined as [149, 150]

$$Q_\xi = \int_{\mathcal{B}} d^{D-2}x\sqrt{\sigma}\left(\varepsilon u^i + j^i\right)\xi_i , \tag{4.56}$$

where u^i is the unit normal to spacelike hypersurfaces t' or t'', and j^i is the momentum density.

We first calculate the quasi-local energy inside the spacelike hypersurface $r = r_0 = const.$

$$E = \int_{\mathcal{B}} d^{D-2}x\sqrt{\sigma}\varepsilon = \int_{\mathcal{B}} d^{D-2}x\sqrt{\sigma}\left(k - \varepsilon_0\right) \ , \tag{4.57}$$

where $k = -\sqrt{b(r_0)}/r_0$ is the trace of the extrinsic curvature and ϵ_0 is the vacuum energy density.

For $r_0 \to \infty$, we have the global quasi-local energy

$$E(r_0) = -\frac{2\pi r_0}{l}\sqrt{\frac{6\alpha^2 B^2 \ln\left(\frac{1}{\alpha l}\right)}{(\alpha B+1)^4} + 1} - 2\pi r_0 \varepsilon_0(r_0) + \mathcal{O}\left(\frac{1}{r_0^2}\right) \ . \tag{4.58}$$

To make it finite, the vacuum energy density has to be

$$\varepsilon_0(r_0) = -\frac{1}{l}\sqrt{\frac{6\alpha^2 B^2 \ln\left(\frac{1}{\alpha l}\right)}{(\alpha B+1)^4} + 1} \ , \tag{4.59}$$

then the global quasi-local energy becomes zero.

The conserved mass can be further calculated as

$$\begin{aligned} M &= -\int_{\mathcal{B}} d^{D-2}x\sqrt{\sigma}\varepsilon u_i \xi^i \\ &= \lim_{r_0\to\infty} E(r_0)\sqrt{b(r_0)} \\ &= \lim_{r_0\to\infty} \frac{2\pi B^2}{\alpha l^2 r_0} - \frac{3\pi B^2}{2r_0^2\alpha^2 l^2} + \mathcal{O}\left(\frac{1}{r_0^4}\right) \\ &= 0 \ , \end{aligned} \tag{4.60}$$

which, however, turns out to be zero.

The fact that the conserved mass is zero has its root in the $f(R)$ theory. It is known that the conserved mass is related to the constant term in the metric function that survives in the asymptotic expansion at infinity when one is dealing with (A)dS spacetime in $(2+1)$ dimensions. We can split the metric function in two parts. A part that is not completely supported by the gravitational scale α denoted by $b(r)_{GR,\alpha,\phi}$ and a part that is completely supported by α, i.e, when we turn off α these terms will vanish, denoted $b(r)_{\alpha,\phi}$. We have $b(r) = b(r)_{\alpha,\phi} + b(r)_{GR,\alpha,\phi}$, where

$$b(r)_{GR,\alpha,\phi} = -\frac{3B^2}{l^2(\alpha B+1)^2} - \frac{2B^3}{l^2 r(\alpha B+1)} + \frac{r^2}{l^2} \ , \tag{4.61}$$

$$b(r)_{\alpha,\phi} = \frac{6\alpha B^2 r}{l^2(\alpha B+1)^3} + r^2\frac{6\alpha^2 B^2}{l^2(\alpha B+1)^4}\ln\left(\frac{r}{\alpha l(B+r)+l}\right) \ . \tag{4.62}$$

It is clear that by setting $\alpha = 0$ in $b(r)_{\alpha,\phi}$, the term will vanish, while $b(r)_{GR,\alpha,\phi}$ will yield the conformal black hole solution [19]. It can be seen that the $b(r)_{GR,\alpha,\phi}$ part contains a term that is related to the mass of the black hole

$$M_{GR,\alpha,\phi} = \frac{3B^2}{l^2(\alpha B+1)^2} \ , \tag{4.63}$$

while expanding the $b(r)_{\alpha,\phi}$ term at infinity, we find that the constant term will be related to the mass of the black hole reads

$$M_{\alpha,\phi} = -\frac{3B^2}{l^2(\alpha B+1)^2} \ , \tag{4.64}$$

which is the opposite of the mass term the $b(r)_{GR,\alpha,\phi}$ term generates. Hence the term that exists because of the $f(R)$ function in the metric (4.28), $b(r)_{\alpha,\phi}$ yields a massless black hole, and one can argue that the $f(R)$ theory that satisfies $f_R(r) = s + \alpha r$ yields black holes with no mass. In fact, if one ignores the scalar field and considers only

$$S = \int d^3x\sqrt{-g}f(R)\ , \tag{4.65}$$

with our metric ansatz (4.6) the field equations will naturally yield $f_R(r) = s+\alpha r$, where a logarithmic term that depends on α will cancel the mass the other terms generate yielding massless black holes. For this reason, a more general metric ansatz has to be considered that will yield different profiles for $f_R(r)$, as is indeed recently done in [151]. However, in our case, since we are interested in comparing the $f(R)$ black hole with the GR one [19], we cannot consider a more general metric ansatz, as the metric specifies the form of the scalar field, which further specifies $f_R(r)$ as can be seen in (4.25).

The parameter α which provides a gravitational correction term to the Ricci scalar term in our $f(R)$ theory, breaks the conformal invariance of the GR case presented in [19]. In the case of GR [19] the mass of the black hole depends on the scalar charge B. In our theory in the metric function (4.6) both the gravitational parameter and the scalar charge are present and except the mass term there is another term which is proportional to r^2 which appears in the metric function because of the presence of both the scalar field and the gravitational scale α. Considering the expansions of $b(r)_{\alpha,\phi}, b(r)_{GR,\alpha,\phi}$ at infinity in (4.61) and (4.62) we can say that the massless black hole is a result of the cancellation from the scalar field and the gravitational field contributions. A similar behavior was found in [77]. Breaking the conformal invariance of the action of the MTZ black hole in the Einstein frame through a particular scalar potential, a massless black hole was found and this was attributed to the cancellation of gravitational and scalar field contributions.

4.4 Conclusions

In this chapter, we considered $f(R)$ gravity theory and matter in the form of a self-interacting, non-minimally coupled scalar field. Solving the field equations we found that $f_R(r) = \frac{df(R)}{dR} = 1 + \alpha r$ where α is a non-linear correction term of the Ricci scalar R, having dimensions of inverse length. If $\alpha = 0$ we go back to GR recovering the theory of a conformally coupled scalar field to gravity, discussed in [19]. The parameter α introduces a gravitational scale that breaks the conformal invariance. Calculating the exact forms of the derivatives of $f(R)$ function we deduced that $f_{R_{\text{total}}} > 0$ and $f_{RR} > 0$ which makes our theory free of ghost and tachyonic instabilities. We also calculated the conserved mass of the black hole and interestingly we found that the black hole is massless due to the cancellation of gravitational and scalar field contributions to the mass term. We attributed this effect to the breaking of the conformal invariance due to the presence of the gravitational parameter α.

We also studied the thermodynamics of the extended black hole solution in $f(R)$ gravity, including Hawking temperature and the entropy. With the increase of α, the Hawking temperature of the black hole first decreases slightly, then grows up to a maximum, finally descends until approaching zero, while the entropy first decreases to a minimum value, then grows up with the increase of α. Besides, the entropy of the black hole is higher than the corresponding conformal $(2 + 1)$-dimensional black hole [19] for most values of α, indicating that our solution is thermodynamically preferred for most cases. We also briefly discussed the stability of the obtained spacetime under massive and massless scalar perturbations and deduced that the obtained solution is stable under both types of perturbations.

A possible extension of this work is to perform a detailed thermodynamical analysis to examine the validity of the first law of thermodynamics, as well as the thermodynamical stability and possible phase transitions of the obtained black hole solution. One can also introduce a linear Maxwell field in the action and study the interplay of the gravitational parameter α and the charge Q on the conformal invariance of the theory and see their effects on the black hole solution. With the addition of electric charge one can also study possible thermodynamical critical behaviors, pointing out how the gravitational scale α affects thermodynamics. Rotating solutions might also be considered. The properties of the resultant conformal field theory could also be studied. The stability of the obtained spacetime may be investigated, as well as the geodesic motion of particles around the black hole solution and how the gravitational parameter α affects the motion.

Part III

Black Hole Solutions in String-Inspired Non-Linear Electrodynamics with Dynamical Dilaton Fields

Chapter 5

Magnetically Charged Euler-Heisenberg Black Holes with Scalar Hair

In this chapter, we study the Einstein-Euler-Heisenberg theory in the presence of a self interacting scalar field, minimally coupled to gravity. We solve analytically the field equations for the magnetically charged case and we obtain novel magnetically charged hairy black holes. The scalar field dresses the black hole with a secondary scalar hair. The hairy black hole develops three horizons when Euler-Heisenberg parameter and the magnetic charge are small and the horizon radius is getting large when the scalar charge and the gravitational mass are large. The presence of matter and the magnetic field outside the horizon of the black hole increases the temperature only for small black holes. Calculating the heat capacity we show that the asymptotically AdS Euler-Heisenberg hairy black hole undergoes a second order phase transition and then it is stabilized. Also the weak energy condition is violated for the asymptotically AdS Euler-Heisenberg hairy black hole.

5.1 Introduction

The Euler-Heisenberg Lagrangian of electrodynamics was at first considered in 1936 [24]. The Euler-Heinsenberg theory is a more accurate classical approximation of QED than Maxwell's theory, when the fields have high intensity. The vacuum is treated as a specific type of medium, and the properties of polarization and magnetization are determined by the clouds of virtual charges surrounding the real currents and charges [152]. A way to detect the effect of the Euler-Heisenberg theory has been proposed in [153]. Since the Euler-Heisenberg theory has interesting physical features, it was a natural consequence to couple the Euler-Heineberg Lagrangian to the Ricci scalar via the volume element to search for black hole solutions. The first black hole solution to the Euler-Heisenberg electrodynamics was derived in [154], where analytical solutions were obtained for the magnetically charged case, also discussing electric charges and dyons. Electrically charged black holes were considered in [155] and [156], while in [156] the geodesic structure was the main study of the paper. In [157] motions of charged particles around the Euler-Heisenberg AdS black hole were studied. The thermodynamics of these black holes were studied in [158, 159], while the quasinormal modes were calculated in [160]. Rotating black holes were found in [161, 162], while the

Euler-Heisenberg Lagrangian was introduced along with modified gravity theories in [163, 164, 165] and the corresponding black holes were analyzed.

In this work, we generalize the Einstein-Euler-Heisenberg black holes of [154] by introducing a self interacting scalar field, minimally coupled to gravity. By assuming only magnetic charges, we integrate analytically the field equations and discuss the corresponding solutions. Electric charge is considered to be negligible near the black hole horizon, due to the presence of plasma (electrically charged particles result in electrically conductive plasma) around astrophysical black holes that neutralizes any electric charge carried by the black hole. The upper bounds of electric charge an astrophysical black hole can carry we refer to [166]. However, magnetically charged black holes cannot be neutralized with ordinary matter [167]. When the Euler-Heisenberg parameter vanishes we obtain novel magnetically charged hairy black holes, while when the scalar charge vanish, we get the solution of [154] and when both the Euler-Heisenberg parameter and the magnetic charge vanishes we go back to the well known hairy black hole solution of [15]. The scalar field dresses the black hole with secondary scalar hair, since the scalar charge is related to the mass parameter, while the scalar potential is negative in order to support the hairy structure and it possesses a mass term that satisfies the Breitenlohner-Friedman bound that ensures the perturbative stability of the AdS spacetime. The black hole horizon shrinks as the magnitude of the scalar field is getting larger, while is getting larger as the gravitational mass is increasing. Calculating thermodynamical quantities we find that the temperature develops a minima in the AdS case signalizing in this way a second order phase transition, while the scalar field gains entropy for the black hole by the addition of a linear term in the entropy and hence the hairy black holes are thermodynamically preferred. Calculating the weak energy condition we find that it is violated in the case of asymptotically AdS spacetime.

The work is organized as follows. In Section 5.2 we set up the theory, derive the solution and discuss the effect of the scalar field on the black hole. In Section 5.3 we write down some limiting behaviors of the obtained black hole solution. In Section 2.1 we discuss the thermodynamical properties, while in Section 5.5 we investigate the energy conditions and finally in Section 5.6 we conclude.

5.2 Black hole solutions

We consider the Euler-Heisenberg action in the presence of a scalar field

$$S=\int d^4x\sqrt{-g}\mathcal{L}=\int d^4x\sqrt{-g}\left(R-\partial^{\mu}\phi\partial_{\mu}\phi-2V(\phi)-P+\alpha P^2+\beta Q^2\right)\ , \tag{5.1}$$

where $\mathcal{L}$ denotes the Lagrangian of the theory, $P=F_{\mu\nu}F^{\mu\nu},\ Q=\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$,$F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$ is the Faraday tensor (field strength) and $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita tensor that satisfies

$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma}=-24\ . \tag{5.2}$$

The field equations are

$$G_{\mu\nu} = T_{\mu\nu} \equiv T^{\phi}_{\mu\nu} + T^{EM}_{\mu\nu} \ , \tag{5.3}$$

$$\Box\phi = \frac{dV}{d\phi} \ , \tag{5.4}$$

$$\nabla_\mu \left(F^{\mu\nu} - 2\alpha P F^{\mu\nu} - 2\beta Q \epsilon^{\mu\nu\xi\eta} F_{\xi\eta}\right) = 0 \ , \tag{5.5}$$

$$T^{\phi}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2} g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi - g_{\mu\nu}V(\phi) \ , \tag{5.6}$$

$$T^{EM}_{\mu\nu} = 2F_{\mu\rho}F_\nu{}^\rho + \frac{1}{2} g_{\mu\nu}\left(-P + \alpha P^2 + \beta Q^2\right) - 4\alpha P F_{\mu\rho}F^\rho_\nu - 8\beta Q \epsilon_{\mu\zeta\eta\rho}F^{\zeta\eta}F^\rho_\nu \ . \tag{5.7}$$

We consider the following spherically symmetric ansatz for the spacetime metric

$$ds^2 = -b(r)dt^2 + b(r)^{-1}dr^2 + b_1(r)^2 d\Omega^2 \ , \tag{5.8}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ which allows us to consider the following electromagnetic ansatz for the four-vector A_μ

$$A_\mu = (\mathcal{A}(r), 0, 0, Q_m \cos\theta) \ , \tag{5.9}$$

where Q_m is the magnetic charge of the black hole and the magnetic part of the four vector will be null at the equatorial plane. Under these ansaetze, the scalar quantities P, Q that enter the field equations read

$$P = \frac{2Q_m^2}{b_1(r)^4} - 2\mathcal{A}'(r)^2 \ , \tag{5.10}$$

$$Q = -\frac{8Q_m\mathcal{A}'(r)}{b_1(r)^2} \ , \tag{5.11}$$

where it is clear that Q will vanish if we do not consider dyons (both electric and magnetic charges).

The system of the field equations (5.3)-(5.7) admits an exact magnetically charged solution given by

$$\mathcal{A}(r) = 0 \ , \tag{5.12}$$

$$\phi(r) = \frac{1}{\sqrt{2}} \ln\left(1 + \frac{\nu}{r}\right) \ , \tag{5.13}$$

$$b_1(r) = \sqrt{r(\nu + r)} \ , \tag{5.14}$$

while the metric function $b(r)$ is obtained as

$$b(r) = c_1 r(\nu+r) + \frac{(2r - c_2)(\nu + 2r) - 4Q_m^2}{\nu^2} + \frac{8\alpha Q_m^4\left(-\nu^2 + 12r^2 + 12\nu r\right)\left(\nu^2 + 3r^2 + 3\nu r\right)}{3\nu^6 r^2(\nu + r)^2} + \frac{2}{\nu^8}\ln\left(\frac{r}{\nu + r}\right) * \left(-\nu^5 r(c_2 + \nu)(\nu + r) - 2Q_m^2 r(\nu + r)\left(\nu^4 - 24\alpha Q_m^2\right)\ln\left(\frac{r}{\nu + r}\right) + 48\alpha\nu Q_m^4(\nu + 2r) - 2\nu^5 Q_m^2(\nu + 2r)\right) , \tag{5.15}$$

where c_1, c_2 are constants of integration and ν is the scalar charge, also a constant of integration which determines the behavior of the scalar field. For a well behaved scalar field we will impose $\nu > 0$. At large distances, the metric function asymptotes to

$$b(r \to \infty) \sim 1 + \frac{-c_2 - \nu}{3r} + \frac{c_2\nu + \nu^2 + 6Q_m^2}{6r^2} + r^2\left(c_1 + \frac{4}{\nu^2}\right) + \frac{r\left(c_1\nu^2 + 4\right)}{\nu} - \frac{\nu\left(c_2\nu + \nu^2 + 10Q_m^2\right)}{10r^3} + \mathcal{O}\left(\left(\frac{1}{r}\right)^4\right) . \tag{5.16}$$

We can see that the scalar charge ν is introducing a new scale in the theory which leads to the appearance of an effective cosmological constant. Also the generated mass term is given by both an integration constant and the scalar charge ν, hence the scalar field dresses the black hole with a secondary scalar hair. By redefining the integration constants, the asymptotic relation yields

$$b(r\to\infty)\sim 1-\frac{2m}{r}+\frac{m\nu+Q_m^2}{r^2}-\frac{\Lambda_{\text{eff}}r^2}{3}-\frac{1}{3}r(\Lambda_{\text{eff}}\nu)-\frac{\nu\left(3m\nu+5Q_m^2\right)}{5r^3}+\mathcal{O}\left(\left(\frac{1}{r}\right)^4\right)\ , \quad (5.17)$$

where we have set $m=\dfrac{c_2+\nu}{6}$ and $\Lambda_{\text{eff}}=-\left(3c_1+\frac{12}{\nu^2}\right)$. For small r the metric behaves as

$$b(r\to 0)\sim -\frac{8\alpha Q_m^4}{3\nu^4 r^2}+\mathcal{O}(r^{-1})\ , \quad (5.18)$$

from which we can deduce that the solution always describes a black hole at least in asymprotically flat or AdS spacetime, due to the fact that $b(r)$ is continuous and changes sign in the range $0<r<\infty$. In the small scalar hair case ($\nu\to 0$), the metric function yields

$$b(r)=\left(1-\frac{2m}{r}+\frac{Q_m^2}{r^2}-\frac{2\alpha Q_m^4}{5r^6}-\frac{\Lambda_{\text{eff}}r^2}{3}\right)+\nu\left(\frac{m}{r^2}+\frac{6\alpha Q_m^4}{5r^7}-\frac{Q_m^2}{r^3}-\frac{\Lambda_{\text{eff}}r}{3}\right)+\mathcal{O}\left(\nu^2\right)\ . \quad (5.19)$$

Using the metric function (5.15) and the scalar field function (5.13) from the system of the field equations (5.3)-(5.7) we can specify the scalar potential

$$\begin{aligned}V(\phi)=\frac{1}{3\nu^8}\Bigg(&\nu^8\Lambda_{\text{eff}}\left(\cosh\left(\sqrt{2}\phi\right)+2\right)-36m\nu^5\left(\sqrt{2}\phi\left(\cosh\left(\sqrt{2}\phi\right)+2\right)-3\sinh\left(\sqrt{2}\phi\right)\right)-4\alpha Q_m^4 *\\ &\left(288\phi^2+2\left(72\phi^2+71\right)\cosh\left(\sqrt{2}\phi\right)-432\sqrt{2}\phi\sinh\left(\sqrt{2}\phi\right)+100\cosh\left(2\sqrt{2}\phi\right)-14\cosh\left(3\sqrt{2}\phi\right)+\cosh\left(4\sqrt{2}\phi\right)-229\right)\\ &+6\nu^4Q_m^2\left(8\phi^2+4\left(\phi^2+2\right)\cosh\left(\sqrt{2}\phi\right)-12\sqrt{2}\phi\sinh\left(\sqrt{2}\phi\right)+\cosh\left(2\sqrt{2}\phi\right)-9\right)\Bigg)\ . \end{aligned} \quad (5.20)$$

For small ϕ we have

$$V(\phi)\sim\Lambda_{\text{eff}}+\frac{\phi^2\Lambda_{\text{eff}}}{3}+\frac{\phi^4\Lambda_{\text{eff}}}{18}-\frac{4\left(\sqrt{2}m\right)\phi^5}{5\nu^3}+\mathcal{O}\left(\phi^6\right)\ . \quad (5.21)$$

We can also express the potential as a function of r

$$\begin{aligned}V(r)=&\frac{1}{6\nu^8r^4(\nu+r)^4}\\ &\Bigg(6\nu^6Q_m^2r^2(\nu+r)^2\left(\nu^2+12r^2+12\nu r\right)+\nu^6r^3(\nu+r)^3\left(108m(\nu+2r)+\Lambda_{\text{eff}}\nu^2\left(\nu^2+6r^2+6\nu r\right)\right)\\ &-4\alpha\nu^2Q_m^4\left(\nu^6+1332\nu^2r^4+504\nu^3r^3+30\nu^4r^2+1296\nu r^5+432r^6-6\nu^5r\right)+\\ &12r^3(\nu+r)^3\ln\left(\frac{r}{\nu+r}\right)\left(3m\nu^5\left(\nu^2+6r^2+6\nu r\right)+Q_m^2\left(\nu^2+6r^2+6\nu r\right)\left(\nu^4-24\alpha Q_m^2\right)\ln\left(\frac{r}{\nu+r}\right)\right.\\ &\left.-144\alpha\nu Q_m^4(\nu+2r)+6\nu^5Q_m^2(\nu+2r)\right)\Bigg)\ , \end{aligned} \quad (5.22)$$

and for small scalar hair ($\nu \to 0$):

$$V(r) = \Lambda_{\text{eff}} + \frac{\nu^2 \left(25r^8\Lambda_{\text{eff}} - 18\alpha Q_m^4 + 25r^4Q_m^2 - 30mr^5\right)}{150r^{10}} + \mathcal{O}\left(\nu^3\right) . \tag{5.23}$$

As expected, at zeroth order one obtains the cosmological constant. Its asymptotic behavior at large distances reads

$$V(r \to \infty) \sim \Lambda_{\text{eff}} + \frac{\Lambda_{\text{eff}}\nu^2}{6r^2} - \frac{\Lambda_{\text{eff}}\nu^3}{6r^3} + \mathcal{O}\left(\left(\frac{1}{r}\right)^4\right) . \tag{5.24}$$

There is also a mass term in the potential

$$m^2 = V''(\phi = 0) = \frac{2}{3}\Lambda_{\text{eff}} , \tag{5.25}$$

which in the case of AdS spacetime is negative and the scalar field is a tachyon, however in this case it still respects the Breitenlohner-Friedman bound that ensures the perturbative stability of the AdS spacetime [88]. The Kretschmann scalar is singular at the origin

$$R_{\mu\nu\chi\psi}R^{\mu\nu\chi\psi}(r \to 0) \sim \frac{304\alpha^2Q_m^8}{\nu^8r^8} - \frac{7520\left(\alpha^2Q_m^8\right)}{3\nu^9r^7} + \mathcal{O}\left(\frac{1}{r}\right)^6 , \tag{5.26}$$

while it is regular for any $r > 0$ and at infinity its behavior is

$$R_{\mu\nu\chi\psi}R^{\mu\nu\chi\psi}(r \to \infty) \sim \frac{8\Lambda_{\text{eff}}^2}{3} + \frac{2\Lambda_{\text{eff}}^2\nu^2}{3r^2} + \mathcal{O}\left(\left(\frac{1}{r}\right)^3\right) , \tag{5.27}$$

$$R_{\mu\nu\chi\psi}R^{\mu\nu\chi\psi}_{\Lambda_{\text{eff}}=0}(r \to \infty) \sim \frac{48m^2}{r^6} - \frac{8\left(18\nu m^2 + \nu^2 m + 12mQ_m^2\right)}{r^7} + \mathcal{O}\left(\frac{1}{r}\right)^8 . \tag{5.28}$$

Thus the solution is valid for any $r > 0$ and describes a black hole in asymptotically (A)dS or flat spacetime for appropriate relations between the parameters. We will focus on the AdS case in order to make comparisons with the uncharged AdS hairy black hole and the flat case which is also of great interest. As can be seen from the definition of Λ_{eff} in order to obtain a flat spacetime the scale introduced by the presence of the scalar field has to be canceled by the integration constant c_1. In Fig. 5.1 we plot the metric function $b(r)$ and the potential $V(r)$ for the asymptotically AdS and flat spacetimes for a fixed scalar charge while changing α. The $\alpha = 0$ case differs in structure with the $\alpha \neq 0$ cases, having an inner and an event horizon. The Euler-Heisenberg parameter α does not affect the horizon radius of the black hole as we can see. Moreover, the potentials are negative in order to support the hairy structure and violate the no-hair theorem. It is worth noting that, as we can see from the figures, α acts in favour of the no-hair theorem, since the existence of α ensures a negative potential everywhere, while for $\alpha = 0$ there is a small region where the potential can be positive. In Fig. 5.2 we also plot $b(r), V(r)$ while chancing ν having set $\alpha = 0.5$. We can see that for bigger ν (stronger scalar field) the black hole horizon radius is smaller. We also evaluate numerically and plot in Fig. 5.3 the horizon radius as a function of ν for both AdS and flat cases to visualise how the horizon changes as a function of the scalar charge.

To have a better understanding of the hairy Euler-Heisenberg black hole we found, we studied the horizon structure of the various solutions and their dependence on the gravitational mass. In Fig. 5.4 we show the dependence of the horizons on the gravitational mass for $\Lambda_{\text{eff}} = 0$. The Euler-Heisenberg parameter is small ($\alpha = 0.05$) and the magnetic field is fairly large, $Q_m = 5$. We find that, in the interval of black hole masses between 5.5 and 8.0 (or 8.5) there are three possible solutions

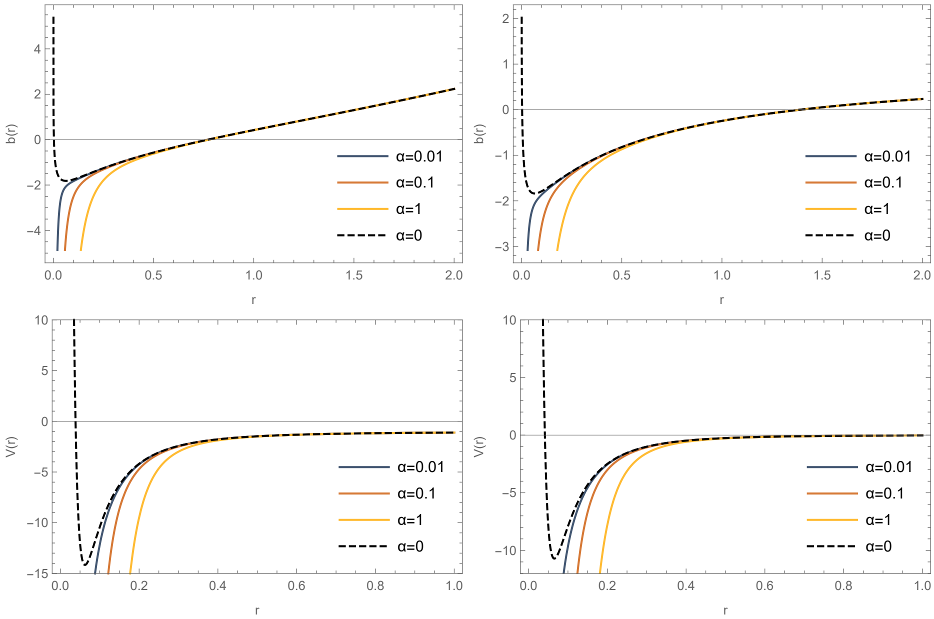


Figure 5.1: Hairy black hole configurations for asymptotically AdS $\Lambda_{\text{eff}} = -1$ (left) and flat $\Lambda_{\text{eff}} = 0$ (right) spacetimes, where we have fixed $m = 1, Q_m = 0.5, \nu = 1$, while changing the Euler-Heisenberg parameter α.

of the metric function, signalling one outer the two inner horizons. The influence of the scalar field is not large. The extreme solutions, where we get just two horizons correspond to vanishing temperatures.

In Fig. 5.5 we show the horizon structures for asymptotically AdS spaces with ($\Lambda_{\text{eff}} = -1$). The Euler-Heisenberg parameter is small ($\alpha = 0.05$) and the magnetic field is fairly large, $Q_m = 5$. We find that, in the interval of black hole masses between 8.5 and 9.0, there are three possible roots again. The influence of the scalar field is not large here either. We observe that the range of variation of the horizons is considerably smaller than the previous case: it varies between 1.0 and 3.5, which is an order of magnitude smaller than before. The extreme solutions, where we get just two solutions correspond to vanishing temperatures.

Finally in Fig. 5.6 we show once more the behaviour of the horizons for a large scalar charge ($\nu = 10$). It contains both the asymptotically flat case and the asymptotically AdS case. We see the remarkable characteristic that the horizon (just one solution for each m) starts off with very small values, while, when m is large enough, it jumps to a much larger value. That is a large black hole get suddenly large horizon values. In addition, the horizon radii for $\Lambda_{\text{eff}} = -1$ are almost one order of magnitude smaller than the ones for $\Lambda_{\text{eff}} = 0$.

To summarise our results: The structure of three roots, as well as the existence of points with $T = 0$, appears when a is small. To have this behaviour of the horizons, the magnetic charge Q_m should be large enough. The structure disappears when Λ_{eff} takes on large negative values. The horizon radius is getting large when the scalar charge and the gravitational mass are large.

The no-hair theorem by Bekenstein, states that for an asymptotically flat spacetime, a positive definite potential cannot violate the no-hair theorem. For this reason we multiply the Klein-Gordon

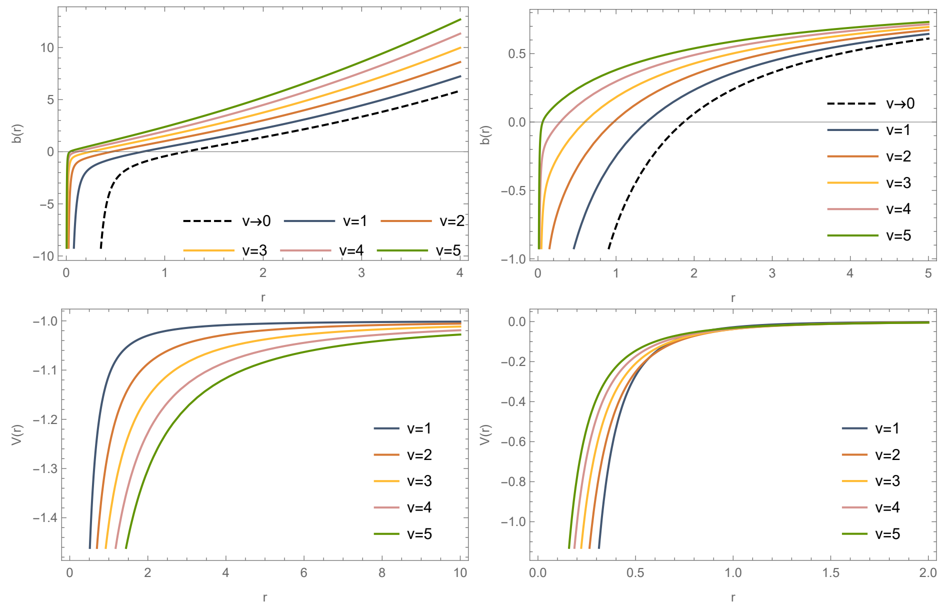


Figure 5.2: Hairy black hole configurations for asymptotically AdS $\Lambda_{\text{eff}} = -1$ (left) and flat $\Lambda_{\text{eff}} = 0$ (right) space-times, where we have fixed $m = 1, Q_m = 0.5, \alpha = 0.5$, while changing the scalar charge ν.

equation (5.4) by $V(\phi)$ and we integrate over the black hole exterior region

$$\int d^4x\sqrt{-g}\Big(V(\phi)\Box\phi - V(\phi)V'(\phi)\Big) = 0 \rightarrow \int d^4x\sqrt{-g}\Big(\nabla_\mu\left(V(\phi)\nabla^\mu\phi\right) - V'(\phi)\nabla_\mu\phi\nabla^\mu\phi - V(\phi)V'(\phi)\Big) = 0\,. \tag{5.29}$$

For an asymptotically flat spacetime, we can ignore the first term which is a total derivative and this relation becomes

$$\int d^4x\sqrt{-g}V'(\phi)\Big(\nabla_\mu\phi\nabla^\mu\phi + V(\phi)\Big) = 0\,. \tag{5.30}$$

It is clear that the kinetic term above is always positive outside the black hole region. In order for the integral to be zero, we want a negative potential in order to counterbalance the positive kinetic term, which will result in a zero area between the curve of the integrand and the r axis. The presence of the scalar field introduces a matter distribution outside the horizon of the black hole. The condition (5.30) guarantees that the kinetic energy of the scalar field has to counterbalance the potential energy of the scalar field in order to have a stable matter distribution outside the horizon of the black hole. Therefore we have to find regions of spacetime where the potentials are negative to violate the no-hair theorem and to support the hairy structure.

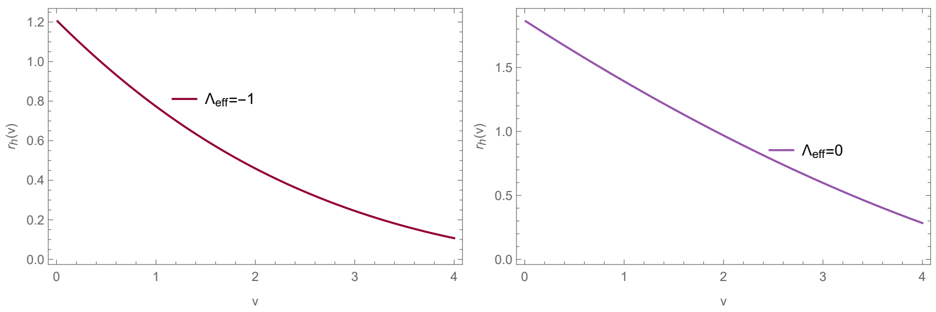


Figure 5.3: The horizon radius r_h as a function of ν having set $m = 1, Q_m = 0.5, \alpha = 0.5$, for asymptotically AdS $\Lambda_{\text{eff}} = -1$ (left) and flat $\Lambda_{\text{eff}} = 0$ (right) spacetimes.

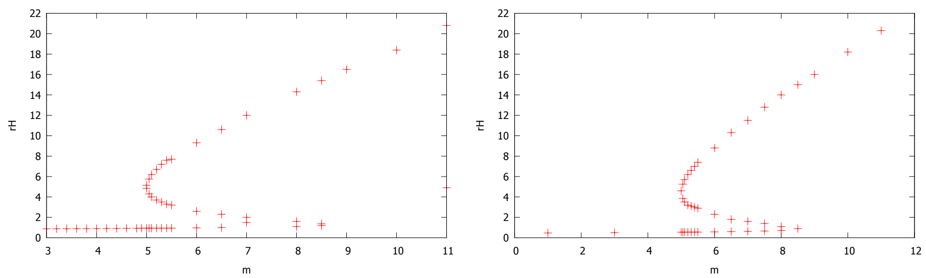


Figure 5.4: Horizons versus Black hole mass. The asymptotically flat case ($\Lambda_{\text{eff}} = 0$) is depicted. The Euler-Heisenberg parameter α equals 0.05. Left panel: No scalar field ($\nu = 0$). Right panel: Small scalar field ($\nu = 1$).

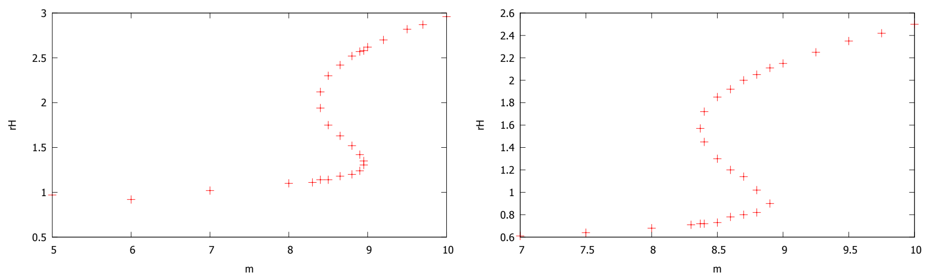


Figure 5.5: Horizons versus Black hole mass. The asymptotically flat case ($\Lambda_{\text{eff}} = -1$) is depicted. The Euler-Heisenberg parameter α equals 0.05. Left panel: No scalar field ($\nu = 0$). Right panel: Small scalar field ($\nu = 1$).

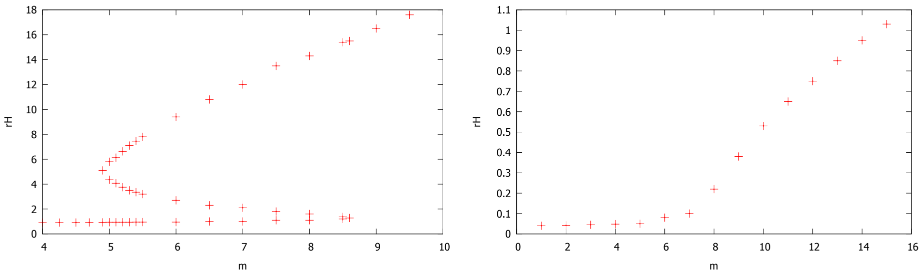


Figure 5.6: Horizons versus Black Hole mass. Large scalar field charge ($\nu = 10$). Left panel: Asymptotically flat case ($\Lambda_{\text{eff}} = 0$). Right panel: Asymptotically AdS case ($\Lambda_{\text{eff}} = -1$).

5.3 Special Cases for Black Hole Solutions

In this section we will present special cases for the black hole solutions we found in the previous section depending on the choice of the parameters.

For the case $\nu \to 0$ we have the Euler-Heisenberg black hole [154]

$$b(r) = 1 - \frac{2m}{r} - \frac{2\alpha Q_m^4}{5r^6} + \frac{Q_m^2}{r^2} - \frac{\Lambda_{\text{eff}} r^2}{3} \,, \tag{5.31}$$

while the potential gives the cosmological constant $V = \Lambda_{\text{eff}}$. As it is expected, because the scalar field is decoupled, if we set the Euler-Heisenberg parameter equal to zero we can obtain the (A)dS RN spacetime, the magnetically charged RN spacetime by also setting $\Lambda_{\text{eff}} = \alpha = 0$, and the Schwarzschild one by further imposing $Q_m = 0$.

For the case $\alpha = 0$ we obtain novel magnetically charged hairy black hole solutions where the metric function is given by

$$\begin{aligned} b(r) = 1 &- \frac{4Q_m^2}{\nu^2} - \frac{6m(\nu+2r)}{\nu^2} - \frac{1}{3}\Lambda_{\text{eff}} r(\nu+r) \\ &- \frac{4}{\nu^4}\ln\left(\frac{r}{\nu+r}\right)\left(3m\nu r(\nu+r) + \nu Q_m^2(\nu+2r) + Q_m^2 r(\nu+r)\ln\left(\frac{r}{\nu+r}\right)\right) \,, \end{aligned} \tag{5.32}$$

while the potential will be given by

$$\begin{aligned} V(\phi) = \frac{1}{3\nu^8}\Bigg(&\nu^8\Lambda_{\text{eff}}\left(\cosh\left(\sqrt{2}\phi\right)+2\right) - 36m\nu^5\left(\sqrt{2}\phi\left(\cosh\left(\sqrt{2}\phi\right)+2\right) - 3\sinh\left(\sqrt{2}\phi\right)\right) \\ &+ 6\nu^4 Q_m^2\left(8\phi^2 + 4\left(\phi^2+2\right)\cosh\left(\sqrt{2}\phi\right) - 12\sqrt{2}\phi\sinh\left(\sqrt{2}\phi\right) + \cosh\left(2\sqrt{2}\phi\right) - 9\right)\Bigg) \,. \end{aligned} \tag{5.33}$$

For the case $\alpha = Q_m = 0$ we turn back to the well known asymptotically AdS black hole solutions with a scalar hair [15] where the metric function will be given by

$$b(r) = 1 - \frac{1}{3}r\Lambda_{\text{eff}}(\nu+r) - \frac{6m(\nu+2r)}{\nu^2} - \frac{12mr}{\nu^3}(\nu+r)\ln\left(\frac{r}{\nu+r}\right) \,, \tag{5.34}$$

with potential

$$V(\phi) = \frac{1}{3}\Lambda_{\text{eff}}\left(\cosh\left(\sqrt{2}\phi\right)+2\right) - \frac{12m\left(\sqrt{2}\phi\left(\cosh\left(\sqrt{2}\phi\right)+2\right) - 3\sinh\left(\sqrt{2}\phi\right)\right)}{\nu^3} \,. \tag{5.35}$$

5.4 Thermodynamics

In this Section we will discuss the thermodynamical properties of the hairy black hole solution. We will study the temperature first. To do so we perform a Wick rotation $t \to i\tau$, and move to Euclidean time. Imposing periodicity of the Euclidean time we can obtain the black hole temperature as

$$T(r_h) = \frac{b'(r_h)}{4\pi} = -\frac{1}{12\pi\nu^5 r_h^3(\nu+r)^3\left(\nu(\nu+2r_h)+2r_h(\nu+r_h)\ln\left(\frac{r_h}{\nu+r_h}\right)\right)}$$
$$\Bigg(-8\alpha\nu^2 Q_m^4\left(\nu^2-6r_h^2-3\nu r_h\right)\left(2\nu^2+6r_h^2+9\nu r_h\right)+12\nu^6 Q_m^2 r_h^2(\nu+r_h)^2+\nu^6 r_h^3\left(\Lambda_{\text{eff}}\nu^2+12\right)(\nu+r_h)^3$$
$$+6r_h(\nu+r_h)\ln\left(\frac{r_h}{\nu+r_h}\right)\Bigg(\nu(\nu+2r_h)\left(8\alpha Q_m^4\left(-\nu^2+6r_h^2+6\nu r_h\right)+\nu^4 r_h^2(\nu+r_h)^2\right)$$
$$-2Q_m^2 r_h^2(\nu+r_h)^2\left(\nu^4-24\alpha Q_m^2\right)\ln\left(\frac{r_h}{\nu+r_h}\right)\Bigg)\Bigg), \quad (5.36)$$

where we have already substituted the mass parameter using the horizon condition $b(r_h)=0$ and r_h denotes the event horizon. For small black holes, the Euler-Heisenberg parameter α plays a decisive role since

$$T(r_h \ll 1) \sim \frac{4\alpha Q_m^4}{3\pi\nu^4 r_h^3} + \mathcal{O}\left(\frac{\ln(r_h)}{r_h^2}\right) , \quad (5.37)$$

while for large black holes the effect of α is negligible

$$T(r_h \gg 1) \sim -\frac{\Lambda_{\text{eff}} r_h}{4\pi} - \frac{\Lambda_{\text{eff}}\nu}{8\pi} + \frac{\Lambda_{\text{eff}}\nu^2+20}{80\pi r_h} + \mathcal{O}\left(\frac{1}{r_h^2}\right) . \quad (5.38)$$

We can see this behaviour in Fig. 5.7 where we plot the temperature of the black hole while changing the Euler-Heisenberg parameter α. We observe that α increases the temperature in the case of small black holes.

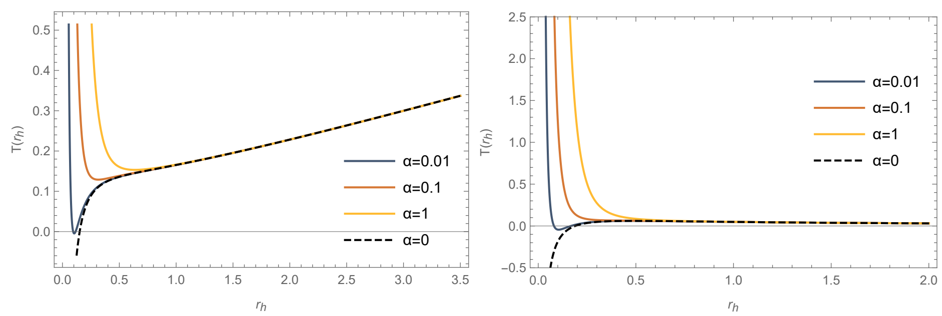


Figure 5.7: The temperature of the hairy black hole configurations for asymptotically AdS $\Lambda_{\text{eff}}=-1$ (left) and flat $\Lambda_{\text{eff}}=0$ (right) spacetimes, where we have fixed $Q_m=0.5, \nu=1$, while changing the Euler-Heisenberg parameter α.

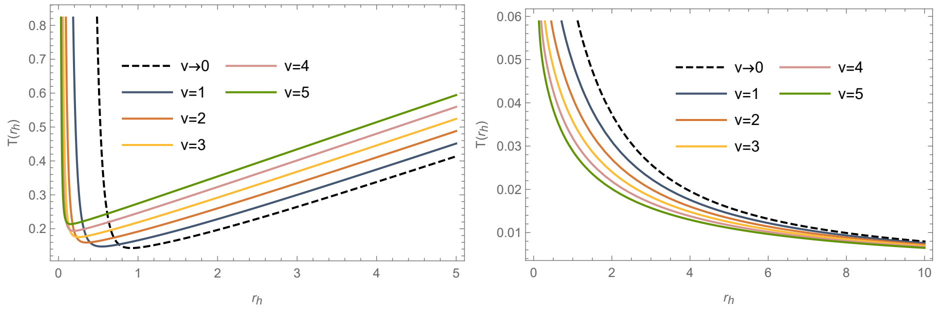


Figure 5.8: The temperature of the hairy black hole configurations for asymptotically AdS $\Lambda_{\text{eff}} = -1$ (left) and flat $\Lambda_{\text{eff}} = 0$ (right) spacetimes, where we have fixed $Q_m = 0.5, \alpha = 0.5,$ while changing the scalar charge ν.

In Fig. 5.8 we plot the black hole temperature having fixed the Euler-Heisenberg parameter $\alpha = 0.5$, while we vary the scalar charge of the solution both for asymptotically AdS and flat cases. The temperature of the asymptotically AdS case develops a minimum which can be obtained numerically. For example for $Q_m = 0.5, \alpha = 0.5, \nu = 1, \Lambda_{\text{eff}} = -1$ we find that $T'(r_h^{min}) = 0 \to r_h^{min} = 0.543748$ which corresponds to $T(r_h^{min}) = 0.147635$.

The entropy of the black hole may be obtained using Wald's formula [21] which for our action reads

$$S(r_h) = -2\pi \oint d^2x\sqrt{h} \left(\frac{\partial \mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}}\right)\Bigg|_{r=r_h} \hat{\epsilon}_{\alpha\beta}\hat{\epsilon}_{\gamma\delta} \ , \tag{5.39}$$

where $\hat{\epsilon}_{\alpha\beta}$ the binormal to the horizon surface normalized to satisfy $\hat{\epsilon}_{\alpha\beta}\hat{\epsilon}^{\alpha\beta} = -2$ and h is the induced metric on the horizon. Since the only quantity in the Lagrangian that involves the Riemann tensor is the Ricci scalar, we can obtain the standard Bekenstein-Hawking area law [168]

$$S(r_h) = 2\pi\mathcal{A} \ , \tag{5.40}$$

where $\mathcal{A} = 4\pi[b_1(r_h)]^2$ is the area of the black hole. Hence

$$S(r_h) = 8\pi^2 r_h(r_h + \nu) \ , \tag{5.41}$$

with the scalar charge appearing in the entropy, resulting in higher entropy in comparison with the non-hairy black hole, since $\nu > 0$. However, one has to keep in mind that the hairy black holes posses a smaller event horizon radius when compared to the Euler-Heisenberg black hole.

To study the possibility of phase transitions, we will calculate the heat capacity. A positive heat capacity indicates that the black hole is thermodynamically stable. Non-stable black holes, may undergo a phase transition in order to be stabilized. Phase transitions occur at the points where the heat capacity vanishes or diverges. A vanishing point in the heat capacity indicates a first order phase transition, while a divergence point indicates a second order phase transition. The first order phase transition occurs at high Gibbs anergy and it does not change the favored configuration while a second order phase transition occurs at lower Gibbs anergy and allows the coexistence of two configurations.

The heat capacity is given by

$$C(r_h) = \frac{\partial m}{\partial T}\Big|_{r=r_h} = \frac{m'(r_h)}{T'(r_h)} \ , \tag{5.42}$$

where $m(r_h)$ is the mass as a function of the event horizon of the black hole, obtained from the relation $b(r_h) = 0$. The explicit expression is too complicated to be given here. For AdS spacetime, for large r_h, the heat capacity is positive, since

$$C(r_h \gg 1) \sim 2\pi r_h^2 + 2\pi\nu r_h + \frac{\pi\left(\nu^2\Lambda_{\text{eff}} - 20\right)}{5\Lambda_{\text{eff}}} + \mathcal{O}\left(\frac{1}{r_h^2}\right) , \tag{5.43}$$

and the AdS black holes are stable. However, in order to see if the black hole undergoes a phase transition before it gets stabilized we will plot the heat capacity in Fig. 5.9. The fact is that the asymptotically AdS Euler-Heisenberg hairy black holes undergo a second order phase transition and then they are stabilized. The phase transition point occurs at the minima of temperature. For a flat spacetime, the heat capacity asymptotes to

$$C(r_h \gg 1) \sim -2\pi r_h^2 - 2(\pi\nu) r_h + \mathcal{O}\left(\frac{1}{r_h^2}\right) , \tag{5.44}$$

where we can see that the flat black holes are thermodynamically unstable and there exists no phase transition to make the black holes stable as we can deduce from Fig. 5.9.

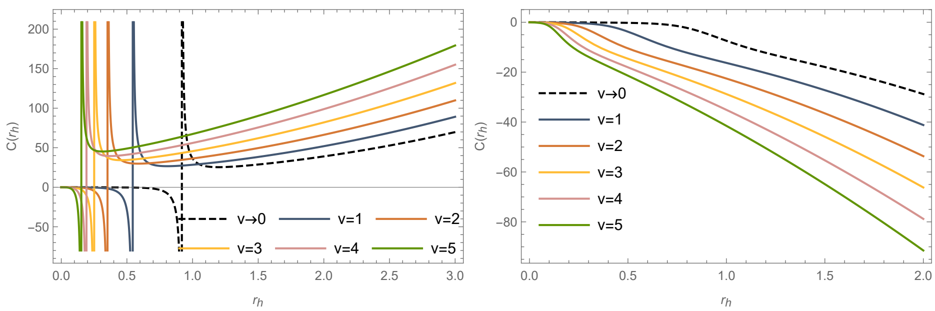


Figure 5.9: The heat capacity for asymptotically AdS $\Lambda_{\text{eff}} = -1$ and flat $\Lambda_{\text{eff}} = 0$ spacetimes, where we have fixed $Q_m = 0.5, \alpha = 0.5$, while changing the scalar charge ν.

5.5 Energy Conditions

In this Section we will discuss the energy conditions of the hairy black hole. For this reason, we will use the Einstein equation in the appropriate reference frame, where we can identify the energy density, the radial and tangential pressure as

$$G^{\mu}{}_{\nu} = T^{\mu}{}_{\nu} , \tag{5.45}$$
$$\rho = -T^{t}{}_{t} , \tag{5.46}$$
$$p_r = T^{r}{}_{r} , \tag{5.47}$$
$$p_\theta = p_\varphi = T^{\theta}{}_{\theta} . \tag{5.48}$$

The weak energy condition (WEC) states that given a timelike vector field t^a, the quantity $T_{ab}t^a t^b$ is positive, i.e $T_{ab}t^a t^b \geq 0 \rightarrow \rho \geq 0$. The null energy condition (NEC) states that $T_{ab}l^a l^b \geq 0 \rightarrow$

$\rho + p_r > 0$, where $l^a l_a = 0$, so that the geometry will have a focusing effect on null geodesics. For the energy momentum tensor of the scalar field, we have

$$\rho^\phi = \frac{1}{2} b(r) \phi'(r)^2 + V(r) = -p^\phi_\theta \ , \tag{5.49}$$

$$p^\phi_r = \frac{1}{2} b(r) \phi'(r)^2 - V(r) \ , \tag{5.50}$$

while for the energy momentum tensor of the Euler-Heisenberg theory we obtain

$$\rho^{EM} = -\frac{2\alpha Q_m^4}{b_1(r)^8} + \frac{Q_m^2 \left(1 - 4(\alpha - 32\beta)\mathcal{A}'(r)^2\right)}{b_1(r)^4} + \frac{4\beta Q_m \mathcal{A}'(r)}{b_1(r)^2} + 6\alpha \mathcal{A}'(r)^4 + \mathcal{A}'(r)^2 = -p_r^{EM} \tag{5.51}$$

$$p_\theta^{EM} = -\frac{6\alpha Q_m^4}{b_1(r)^8} + \frac{Q_m^2 \left(4(\alpha - 32\beta)\mathcal{A}'(r)^2 + 1\right)}{b_1(r)^4} - \frac{4\beta Q_m \mathcal{A}'(r)}{b_1(r)^2} + 2\alpha \mathcal{A}'(r)^4 + \mathcal{A}'(r)^2 \ . \tag{5.52}$$

We will at first discuss the NEC, which implies $\rho + p_r \geq 0$. By adding the energy densities and radial pressures, we have

$$\rho + p_r = \rho^\phi + \rho^{EM} + p^\phi_r + p^{EM}_r = \rho^\phi - p^{EM}_r + p^\phi_r + p^{EM}_r = b(r)\phi'(r)^2 \ . \tag{5.53}$$

First of all $\phi'(r)^2 > 0$ for any $r > 0$. $b(r)$ is negative inside the black hole, resulting in the violation of the NEC, zero at the event horizon resulting to $\rho + p_r = 0$, while after the event horizon $b(r)$ is positive, hence, the NEC is protected, regardless of the asymptotic nature of spacetime. For the contribution of the scalar field to the total energy density, we can see that inside the event horizon, where $b(r) < 0$, the WEC is violated by the scalar field, since $V(r)$ is also negative i.e $V(r) < 0$ regardless of the asymptotic nature of spacetime. On the event horizon $b(r_h) = 0$ and since $V(r_h) < 0$ the WEC is also violated. Caution must be given for the contribution of the scalar field to the energy density in the causal region of the black hole i.e $r > r_h$. For asymptotically AdS spacetimes, outside of the event horizon we have $b(r) > 0$ and $V(r) < 0$, however, the scalar potential is too negative, hence as we can see in Fig. 5.10, the scalar field part of the energy momentum tensor will always violate the WEC. For the asymptotically flat case, at large distances, the kinetic energy of the scalar field $\mathcal{T}(r) = b(r)\phi'(r)^2/2$ asymptotes to

$$\mathcal{T}(r \to \infty) \sim \frac{\nu^2}{4r^4} + \frac{-5\nu^3 - 7m\nu^2}{10r^5} + \mathcal{O}\left(\left(\frac{1}{r}\right)^6\right) \ , \tag{5.54}$$

while the potential behaves as

$$V(r \to \infty) \sim -\frac{m\nu^2}{5r^5} + \mathcal{O}\left(\left(\frac{1}{r}\right)^6\right) \ . \tag{5.55}$$

It is clear that their sum $\mathcal{T}(r) + V(r)$ will be positive at large distances, since the kinetic energy surpasses the contribution of the potential. It is therefore evident that for a region outside of the black hole horizon $\rho^\phi > 0$. The electromagnetic part of the energy density yields

$$\rho^{EM}(r) = \frac{Q_m^2}{r^2(\nu + r)^2} - \frac{2\alpha Q_m^4}{r^4(\nu + r)^4} \ . \tag{5.56}$$

There will be regions of negative energy density due to the Euler-Heisenberg modified electromagnetism parameter α. We plot $\rho^{EM}(r)$ in Fig. 5.10 where we can see that $\rho^{EM}(r)$ can be positive, however it does not contribute much in the total energy density, hence $\rho < 0$ everywhere and the WEC is violated in the case of AdS spacetime. However in the asymptotically flat case, it is obvious from (5.54), (5.55) and FIG. 5.10, that, for a region outside of the event horizon to infinity $\rho > 0$ and the WEC holds.

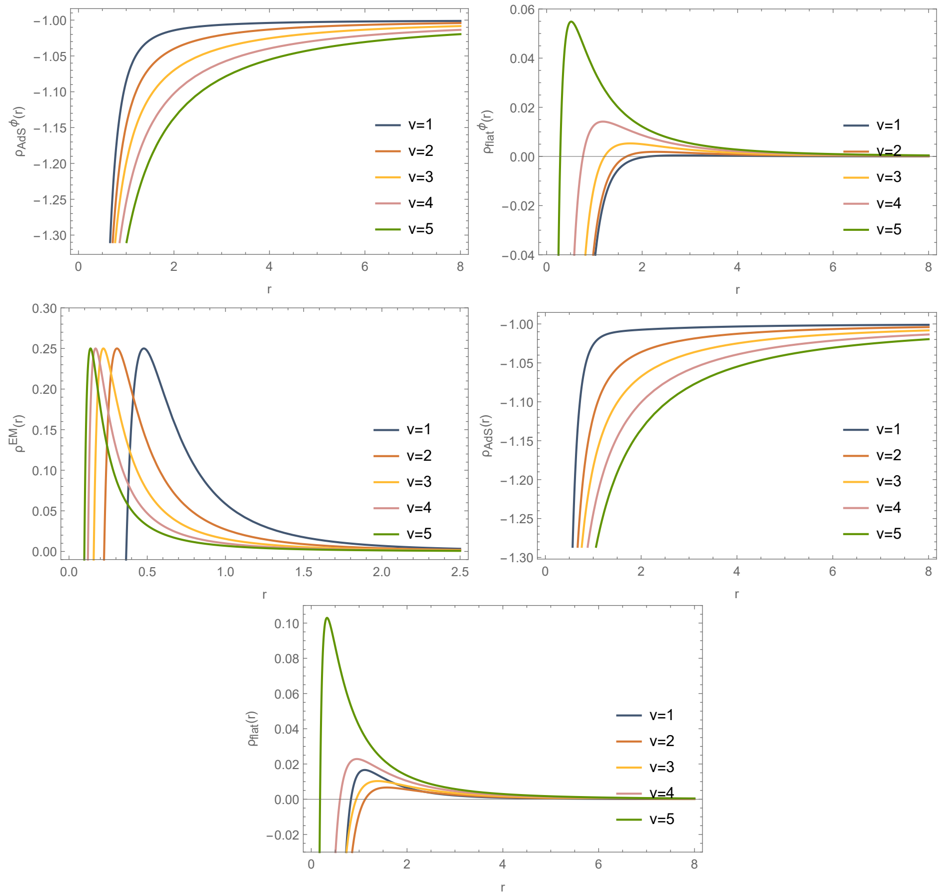


Figure 5.10: Several energy densities are plotted. In the first row, we have the energy density of the scalar field for AdS (left) and flat spacetimes (right). In the second row, we plot the energy density of the electromagnetic part of the energy momentum tensor (left) and the total energy density for AdS spacetime (right). In the third row we plot the total energy density of the asymptotically flat case. For the AdS cases we have set $\Lambda_{\text{eff}} = -1$, for the flat spacetimes $\Lambda_{\text{eff}} = 0$, while we have fixed $m = 1, Q_m = 0.5, \alpha = 0.5$, and we vary the scalar charge ν.

5.6 Conclusions

We studied the Einstein-Euler-Heisenberg theory in the presence of a minimally coupled to gravity, self interacting scalar field. We solved analytically the field equations and, assuming an electromagnetic field with magnetic charge, we obtained novel magnetically charged hairy black holes. The scalar field dresses the black hole with secondary scalar hair, while the scalar potential is negative in order to support the hairy structure and it possesses a mass term that satisfies the Breitenlohner-Friedman bound that ensures the perturbative stability of the AdS spacetime. The presence of the scalar charge is introducing a new scale in the theory which leads to the appearance of an effective cosmological constant. The hairy black hole develops three horizons when Euler-Heisenberg parameter and the magnetic charge Q_m are small and the horizon radius is getting large when the scalar charge and the gravitational mass are large.

We also studied the thermodynamics of the hairy Euler-Heisenberg black hole. We found that the presence of matter outside the horizon of the black hole increases the temperature only for small black holes. Also we found the same behaviour for the magnetic field, it increases the temperature only for small black holes. Calculating the heat capacity we found that the asymptotically AdS Euler-Heisenberg hairy black hole undergoes a second order phase transition and then it is stabilized. The phase transition point occurs at the minimum of the temperature while the scalar field gains entropy for the black hole by the addition of a linear term in the entropy and hence the hairy black holes are thermodynamically preferred.

We found that the WEC is violated on the horizon of the hairy Euler-Heisenberg black hole. For asymptotically AdS spacetimes, outside of the event horizon the scalar field part of the energy momentum tensor will always violate the WEC. However in the asymptotically flat case, we found that for a region outside of the event horizon to infinity the WEC holds.

It would be interesting to extend this work to the case that the scalar field is magnetically charged. Then we expect that the magnetized scalar field will interact with the magnetic field, so that the magnetized scalar charge, the magnetic charge and the Euler-Heisenberg parameter will play a decisive role in the structure and properties of the magnetized hairy Euler-Heisenberg black hole. It would also be of interest to study the shadow of the obtained spacetime and to constrain the modified Euler-Heisenberg parameter along with the scalar charge from the results of the Event Horizon Telescope [42] in the astrophysical scenario $Q_m \ll m$. In [169] it was found that considering Maxwell electrodynamics there is a threshold value for the electric charge Q, above which any value of the scalar charge is allowed. It would be worth investigating the same possibility in our case.

Chapter 6

Exact black holes in string-inspired Euler-Heisenberg theory

In this chapter, we embark on a comprehensive exploration of a gravitational theory extending the classical Euler-Heisenberg (EH) electrodynamics coupled to a non-trivial dilaton field. Our motivation for this study stems from the rich theoretical landscape it promises, building upon the established framework of self-gravitating dilaton-linear-electrodynamics. This extension allows us to delve into intriguing phenomena, notably exemplified by the Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GMGHS) black hole [32, 33], a significant exact solution within this domain. In our investigation, we examine the intricacies of our proposed model and unravel its associated black-hole solution in detail. One of our key insights lies in the strategic assumption of a specific profile governing the dilaton coupling to the Euler-Heisenberg terms. This choice results in an exact analytic black-hole solution, facilitating a straightforward examination of its physical characteristics.

Having the solution at hand, we then commence a rigorous analysis encompassing various facets of our model's implications. This includes a thorough examination of the geodesics of massive test particles within the black-hole spacetime, followed by a meticulous scrutiny of the energy conditions. Subsequently, we delve into the thermodynamic aspects of the black hole, computing the relevant thermodynamic quantities, such as the temperature, the entropy, and the magnetic potential (Φ_m), to demonstrate the validity of the first law of thermodynamics. Moreover, within the parameter space of solutions, we unveil the existence of pairs consisting of two distinct black holes characterized by different ratios Q_m/M, both more compact than the Schwarzschild solution yet sharing identical horizon radii. Intriguingly, despite their geometric similarity, a thermodynamic analysis reveals clear distinctions, with one black hole exhibiting thermodynamic stability while its *doppelgänger* proves to be thermodynamically unstable. Additionally, we explore the radial stability of the black-hole solution under linear perturbations and also its scalar quasi-normal modes, shedding light on its potential as an astrophysical entity. Furthermore, we extend our discussions to encompass other solutions and extensions of our model theory, including asymptotically (Anti-)de Sitter (AdS) spacetimes and more general dilaton couplings, providing a comprehensive overview of the theoretical landscape. In conclusion, our work offers a thorough investigation into the gravitational theory of non-linear EH electrodynamics coupled to a non-trivial dilaton field, unraveling a plethora of intriguing phenomena and paving the way for further exploration and theoretical advancements in this domain.

The structure of the current chapter is the following: In the next section 6.1, we discuss our model and its associated black hole solution. By assuming a specific profile for the dilaton coupling to the EH terms, in such a way that, additionally to non-trivial dilaton couplings, one has also dilaton-independent EH terms, we demonstrate the possibility of studying analytically the corresponding black-hole solution. In section 6.2 we first discuss the geodesics of test particles in such black hole spacetimes, and then demonstrate the satisfaction of the energy conditions for appropriate sets of the parameters of the solution. In section 6.3, we study the thermodynamics of the black hole, and show explicitly, by computing the relevant thermodynamical quantities, that the first law of thermodynamics is satisfied in a coordinate-independent way, as should have been expected. In the parameter space of solutions, it is possible to obtain two distinct black holes with different ratios Q_m/M that are more compact than the Schwarzschild solution and share the same horizon radius. However, these black holes even though they have the same horizon radius, from a thermodynamic point of view, are quite distinguishable, since the solution with a greater value for the ratio Q_m/M is thermodynamically stable, while its doppelgänger with a lower value for the ratio Q_m/M is thermodynamically unstable. In section 6.4, we demonstrate the radial stability of the black-hole solution under linear perturbations, and study its scalar quasi-normal models, which provide insights into its properties as a potential astrophysical object. Other solutions of (extensions of) our model theory (6.1), including asymptotically (Anti-)de Sitter (AdS) spacetimes, as well as solutions corresponding to more general couplings $\exp(-2\gamma\phi)$, of the dilaton to the Maxwell term in the action, rather than the $\gamma = 1$ in closed strings, are discussed in section 6.5. Finally, conclusions and outlook are given in section 6.6.

6.1 Theory And Solution

In the geometrised unit system ($c = G = 1$), the Einstein-frame action functional that we will occupy us in this chapter is a simplified version of (1.193) and reads

$$\mathcal{S} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big[\mathcal{R} - 2\nabla^\mu\phi\nabla_\mu\phi - e^{-2\phi}\mathcal{F}^2 - f(\phi)\big(2\alpha\mathcal{F}^\alpha_{\ \beta}\mathcal{F}^\beta_{\ \gamma}\mathcal{F}^\gamma_{\ \delta}\mathcal{F}^\delta_{\ \alpha} - \beta\mathcal{F}^4\big)\Big]\,. \tag{6.1}$$

Such a field theoretic gravitational actions also arises as part of a non-diagonal reduction of the Gauss-Bonnet action [170] and admits the GMGHS black hole [32, 33] as an exact solution when $f(\phi) = 0$. In (6.1), $\mathcal{R}$ is the Ricci scalar, $\mathcal{F}^2 \equiv \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \sim \mathbf{E^2} - \mathbf{B^2}$ is the usual Faraday scalar, and $\mathcal{F}^4 \equiv \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}$, where $\mathcal{F}_{\mu\nu}$ stands for the usual field strength $\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$ and α, β are coupling constants of the theory, with dimensions (length)^2, which in our discussion are treated phenomenologically. The scalar field ϕ and the associated scalar function $f(\phi)$ are both dimensionless.[1] For the moment we do not consider a potential for the dilaton, but only its non-linear interactions with the EH terms. The addition of a pure dilaton potential $\mathfrak{V}(\phi)$ can lead to interesting alternative solutions, including a cosmological constant, which we discuss in section 6.5.

[1]The reader should be reminded at this stage that in the special case of (open)string/brane-inspired BI theory at tree-level in string loops, the function $f(\phi) \sim e^{-5\phi}$ (*cf.* (1.185)), however in such a case the Maxwell term $\mathcal{F}^2$ in (6.1) should be accompanied by the inverse of the open string coupling, *ie.* $e^{-\phi}$, instead of $e^{-2\phi}$ that appears in (6.1). On the other hand, in the heterotic-string-inspired model (6.1) can be mapped to the model (1.186), upon choosing $\gamma = 1$, and $\mathcal{F}\widetilde{\mathcal{F}} = 0$, that is concentrating on magnetically charged black holes only (in which case, the function $f(\phi) \sim e^{-6\phi}$). However, as we have already stressed, and we shall argue below, it is crucial for an analytic treatment of the black-hole solution to have a dilaton-independent term in $f(\phi)$, which, as we have argued in the previous section, can be induced by considering higher-order string loop corrections in the underlying string-theory model.

The field equations emanating from (6.1) are of the following form

$$G_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi + 2e^{-2\phi}\left(\mathcal{F}_\mu{}^\alpha\mathcal{F}_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}\mathcal{F}^2\right) +$$
$$f(\phi)\left\{8\alpha\mathcal{F}_\mu{}^\alpha\mathcal{F}_\nu{}^\beta\mathcal{F}_\alpha{}^\eta\mathcal{F}_{\beta\eta} - \alpha g_{\mu\nu}\mathcal{F}^\alpha_{\ \beta}\mathcal{F}^\beta_{\ \gamma}\mathcal{F}^\gamma_{\ \delta}\mathcal{F}^\delta_{\ \alpha} - 4\beta\mathcal{F}_\mu{}^\xi\mathcal{F}_{\nu\xi}\mathcal{F}^2 + \frac{1}{2}g_{\mu\nu}\beta\mathcal{F}^4\right\}\,, \tag{6.2}$$

$$4\Box\phi = -2e^{-2\phi}\mathcal{F}^2 + \frac{df(\phi)}{d\phi}\left(2\alpha\mathcal{F}^\alpha_{\ \beta}\mathcal{F}^\beta_{\ \gamma}\mathcal{F}^\gamma_{\ \delta}\mathcal{F}^\delta_{\ \alpha} - \beta\mathcal{F}^4\right)\,, \tag{6.3}$$

$$\partial_\mu\left\{\sqrt{-g}\left[4\mathcal{F}^{\mu\nu}\left(2\beta f(\phi)\mathcal{F}^2 - e^{-2\phi}\right) - 16\alpha\mathcal{F}^\mu{}_\kappa\mathcal{F}^\kappa{}_\lambda\mathcal{F}^{\nu\lambda}\right]\right\} = 0\,. \tag{6.4}$$

By taking into account the higher-order electromagnetic invariants $\mathcal{F}^4$ and $\mathcal{F}^\alpha_{\ \beta}\mathcal{F}^\beta_{\ \gamma}\mathcal{F}^\gamma_{\ \delta}\mathcal{F}^\delta_{\ \alpha}$, we are interested in extending the GMGHS solution [32, 33]. To do so, we introduce the most general spherically symmetric metric ansatz in the form

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + [R(r)]^2 d\Omega^2\,, \tag{6.5}$$

where $B(r), R(r)$ are two unknown functions to be determined from the field equations, while $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$.[2] Moreover, we consider both electric and magnetic charges, via the following four-vector, which is compatible with spherical symmetry,

$$\mathcal{A}_\mu = (V(r), 0, 0, Q_m\cos\theta)\,, \tag{6.6}$$

where Q_m stands for the magnetic charge carried by the black hole. This ansatz for the electromagnetic field solves by construction the φ component of the Maxwell equations iff one considers that the scalar field inherits the spacetime symmetries, namely $\phi \equiv \phi(r)$. Interestingly, one can see that the combination

$$2\alpha\mathcal{F}^\alpha_{\ \beta}\mathcal{F}^\beta_{\ \gamma}\mathcal{F}^\gamma_{\ \delta}\mathcal{F}^\delta_{\ \alpha} - \beta\mathcal{F}^4 = \frac{4(\alpha-\beta)Q_m^4}{[R(r)]^8} + \frac{8\beta Q_m^2[V'(r)]^2}{[R(r)]^4} + 4(\alpha-\beta)[V'(r)]^4\,, \tag{6.7}$$

will vanish if one does not consider both electric and magnetic configurations in the case of $\alpha = \beta$. In the above, prime denotes derivation with respect to r. Maxwell's equation is very difficult to be integrated for the dyonic case and as a result we will consider pure magnetic fields, that is $V(r) = 0$. Consequently, both these non-linear electrodynamics terms will contribute iff $\alpha \neq \beta$. We will begin our analysis for the scalar free scenario $\phi = 0, f(\phi = 0) = 1$, for which the solution reads

$$B(r) = 1 - \frac{2M}{r} + \frac{Q_m^2}{r^2} + \frac{2(\alpha-\beta)Q_m^4}{5r^6}\,, \tag{6.8}$$

and $R(r) = r$. This solution resembles the Einstein-Euler-Heisenberg black hole [154]. The interesting thing to notice in (6.8) is that the non-linear electromagnetic terms $\mathcal{F}^\alpha_{\ \beta}\mathcal{F}^\beta_{\ \gamma}\mathcal{F}^\gamma_{\ \delta}\mathcal{F}^\delta_{\ \alpha}$ and $\mathcal{F}^4$ affect the spacetime geometry in a similar way. It is solely the values of the coupling constants α and β that determine whether this contribution survives or not. Note that in the case of $\alpha = \beta$ the higher-order electromagnetic term does not contribute at all. However, in the case where $\alpha \neq \beta$, we notice that depending on the signs of the parameters α and β, the non-linear electromagnetic

[2] Note that throughout this article, φ will always denote the azimuthal coordinate, while ϕ will always denote the scalar field.

terms can act either attractively or repulsively. Black holes with a scalar hair in the Euler-Heisenberg theory have been discussed in [5], and it was found that the scalar hair results in a more compact black hole (having a smaller radius for the event horizon) when compared to the non-hairy Einstein-Euler-Heisenberg black hole.

Let us now assume a non-trivial profile for the coupling function $f(\phi)$. In particular, we consider

$$f(\phi) = -[3\cosh(2\phi) + 2] \equiv -\frac{1}{2}\left(3e^{-2\phi} + 3e^{2\phi} + 4\right) . \tag{6.9}$$

Notice here that the coupling function $f(\phi)$ contains the dilatonic coupling $e^{2\xi\phi}$ with $\xi = \pm 1$ as well as a constant (dilaton independent) term. At this point the reader is invited, for completion, to compare such couplings with the string-loop corrected coupling functions $B_{F^4}(\phi)$, in the framework of string-inspired models (1.193), discussed in the introduction of this thesis. In such a stringy context, the exponential dilaton terms in the coupling function (6.9) can be written as $f(\phi) = -\frac{3}{2}(g_s^{-2} + g_s^2) - 2$, where $g_s = \exp(\phi)$ is the string coupling. As discussed in the introduction of this thesis, the g_s^{-2} is the standard tree-level dilaton-Maxwell term coupling [116, 120, 121], while the g_s^2 indicates two-string-loop corrections (genus-$\chi = 2$ world-sheet surfaces). The crucial, for our subsequent discussion, dilaton-independent term in $f(\phi)$ might be the result of appropriate combinations of higher-string-loop corrections in the Einstein-frame effective action.

It is now straightforward to solve the field equations of (6.1), with (6.9), in order to determine the geometry of the spacetime and the functional expression for the scalar field. By doing so, one obtains a simple exact, magnetically charged black-hole solution, for which it holds that

$$B(r) = 1 - \frac{2M}{r} - \frac{2(\alpha-\beta)Q_m^4}{r^3\left(r - \frac{Q_m^2}{M}\right)^3} , \qquad [R(r)]^2 = r\left(r - \frac{Q_m^2}{M}\right) , \tag{6.10}$$

$$\phi(r) = -\frac{1}{2}\ln\left(1 - \frac{Q_m^2}{Mr}\right) , \qquad \mathcal{A}_\mu = (0, 0, 0, Q_m\cos\theta) . \tag{6.11}$$

We observe that in this case, for $\alpha = \beta$ we obtain the GHS solution [33], while the radial coordinate $r \in (Q_m^2/M, +\infty)$ in order to have $R \in (0, +\infty)$. In this case, it is also intriguing to observe that the sign of the combination $\alpha - \beta$ among the coupling constants determines whether the higher-order electromagnetic terms in the theory will contribute attractively or not.

To obtain a better understanding of the spacetime geometry, one may express the line element (6.5) in terms of the physical coordinate system with R playing the role of the radial coordinate. By doing so, one finds that

$$ds^2 = -B(R)dt^2 + \frac{[W(R)]^2 dR^2}{B(R)} + R^2 d\Omega^2 , \tag{6.12}$$

with functions $B(R)$, $W(R)$, and $\phi(R)$ being given by

$$B(R) = 1 - \frac{4M^2}{Q_m^2 + \sqrt{Q_m^4 + 4M^2R^2}} - \frac{2(\alpha-\beta)Q_m^4}{R^6} , \tag{6.13}$$

$$[W(R)]^2 = \frac{4M^2R^2}{Q_m^4 + 4M^2R^2} , \tag{6.14}$$

$$\phi(R) = -\frac{1}{2}\ln\left(\frac{\sqrt{Q_m^4 + 4M^2R^2} - Q_m^2}{\sqrt{Q_m^4 + 4M^2R^2} + Q_m^2}\right) . \tag{6.15}$$

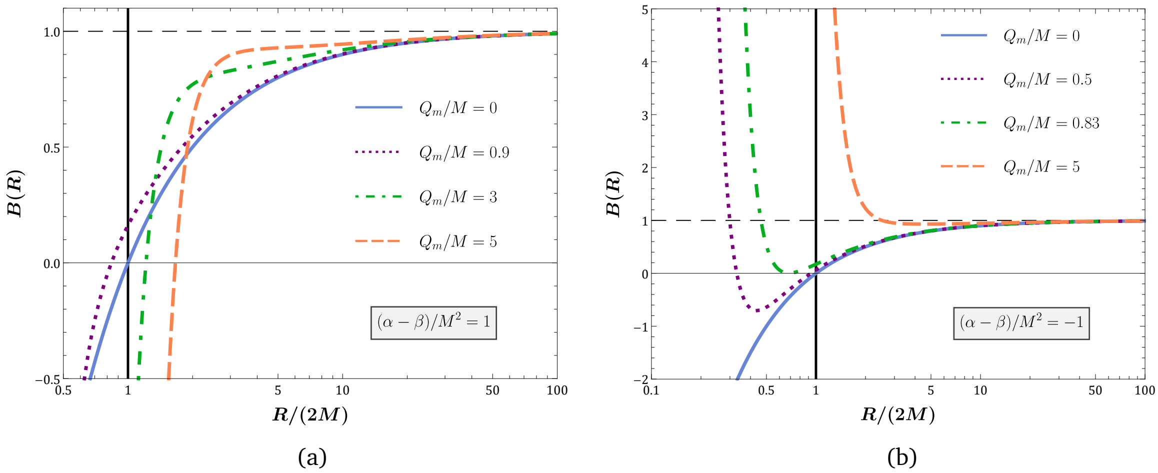


Figure 6.1: The metric function $B(R)$ in terms of $R/(2M)$ for various values of the parameter Q_m/M and (a) $(\alpha-\beta)/M^2 = 1$, (b) $(\alpha-\beta)/M^2 = -1$. All parameters are dimensionless, and the horizontal axis in both figures is logarithmic.

In the physical coordinate system (t, R, θ, φ), one can verify that the curvature invariant quantities $\mathcal{R}$, $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$, and $\mathcal{R}_{\alpha\beta\gamma\delta}\mathcal{R}^{\alpha\beta\gamma\delta}$ possess a single spacetime singularity residing at $R = 0$, while the function $B(R)$ satisfies the following expansions

$$B(R \to +\infty) = 1 - \frac{2M}{R} + \frac{Q_m^2}{R^2} - \frac{Q_m^4}{4MR^3} + \frac{Q_m^8}{64M^3R^5} - \frac{2(\alpha-\beta)Q_m^4}{R^6} + \mathcal{O}(1/R^7)\,, \tag{6.16}$$

$$B(R \to 0) = -\frac{2(\alpha-\beta)Q_m^4}{R^6} + \left(1 - \frac{2M^2}{Q_m^2}\right) + \mathcal{O}(R^2)\,. \tag{6.17}$$

From (6.16), it becomes apparent that the spacetime (6.12) is practically indistinguishable from that of a magnetically charged Reissner-Nordström black hole for an observer at infinity, with the parameter M corresponding to the ADM mass of the solution. However, an observer much closer to the black hole (6.12) would perceive a completely different picture. Indeed these quantum-gravity corrections are important near the singularity, since the geometry there is determined by their behavior.

The radial null-trajectories for the spacetime (6.12), lead to the relation

$$\frac{dt}{dR} = \pm \frac{2MR}{\sqrt{Q_m^4 + 4M^2R^2}} \frac{1}{|B(R)|}\,, \tag{6.18}$$

which by its turn means that the roots of the function $B(R_h)$ correspond to black-hole horizons. In Fig. 6.1, one can observe the behavior of the metric function $B(R)$ in terms of the dimensionless quantity $R/(2M)$. We see that the solution (6.12) describes a black hole with a single horizon when $(\alpha-\beta)/M^2 = 1$, while for $(\alpha-\beta)/M^2 = -1$ the black-hole horizons can range from two to none. It is essential to note that the previous assertion holds in general for $(\alpha-\beta)/M^2$ being either greater or lower than zero. Analysis of Fig. 6.1a reveals that a positive value for the combination $(\alpha-\beta)/M^2$ results in black-hole solutions featuring a single horizon. To facilitate comparison, we have also included the Schwarzschild solution which can be obtained by simply setting $Q_m = 0$. One can

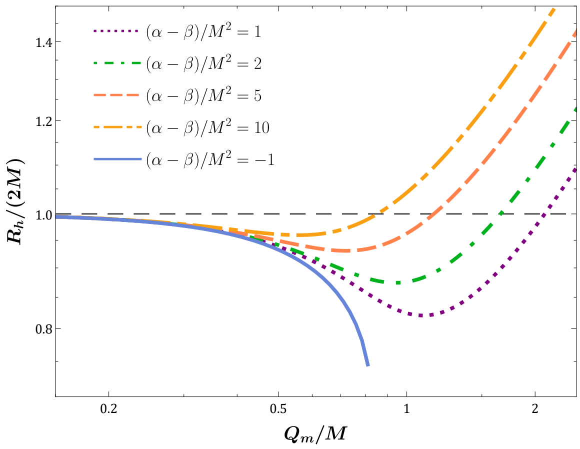


Figure 6.2: The ratio $R_h/(2M)$ in terms of the ratio Q_m/M for various values of the dimensionless parameter $(\alpha-\beta)/M^2$. Both axes are logarithmic.

readily observe that within our theory's solution spectrum, black holes can exhibit either greater compactness or sparsity relative to the Schwarzschild solution. In astrophysical scenarios where Q_m is relatively small compared to the mass, our solution appears more compact. Conversely, when the fraction $(\alpha-\beta)/M^2$ takes a negative value, the solutions range from black holes with two horizons to naked singularities. The transition from one class of solutions to the other occurs continuously as the magnetic charge Q_m increases, as depicted in Fig. 6.1b. Consequently, in this scenario, there always exists a specific value for the ratio Q_m/M that renders the black hole extremal, meaning the inner and outer horizons coincide.

It is crucial to highlight here the intriguing behavior observed in the realm of single-horizon black-hole solutions, for which $\alpha-\beta > 0$. Specifically, there exists a minimum value for the ratio $R_h/(2M)$, which is below unity, resulting in more compact black holes compared to the Schwarzschild solution. Starting from $Q_m = 0$ (Schwarzschild) and increasing the magnetic charge, the resulting black holes become progressively more compact until reaching the point where $R_h/(2M)$ attains its minimum value. Beyond this point, further increase in the ratio Q_m/M causes $R_h/(2M)$ to rise again, eventually reaching $R_h/(2M) = 1$, albeit now with $Q_m \neq 0$. Subsequently, any additional increase in the ratio Q_m/M yields a solution more sparse than the Schwarzschild counterpart. This particular behavior is elucidated by analyzing Fig. 6.2, where the relationship between the ratio of the black hole horizon (R_h) to twice the black hole mass ($2M$) and the ratio Q_m/M is depicted for various values of the dimensionless parameters $(\alpha-\beta)/M^2$. Conversely, it is observed that when $\alpha-\beta < 0$, the outer horizon radius of the resulting black holes is consistently smaller than that of the corresponding Schwarzschild black hole with the same mass. Furthermore, it is important to note that in this scenario, the graph reaches a termination point. This occurs because, beyond a certain threshold of the ratio Q_m/M (which is less than unity), there is a significant transition in the nature of the compact object. Specifically, the object transitions from being an extremal black hole to a naked singularity. Consequently, for this particular choice of parameters, there is no horizon to be depicted. These observations are further corroborated by the findings depicted in Fig. 6.1b. Returning now to the case $\alpha-\beta > 0$, the discovery of black-hole solutions sharing identical horizon radii yet varying in the ratios Q_m/M unveils a realm of *doppelgänger black holes* within the framework of theory (6.1). While it is typical to find black holes stemming from different theoretical paradigms with shared horizon radii but differing physical attributes such as mass, electric charge, or secondary scalar hair, such occurrences are notably rare when considering black holes that arise from the same theory. Even more remarkable is the fact that these two doppelgänger black holes, despite having identical horizon radii, exhibit distinguishable thermodynamic behaviors. One is thermodynamically stable while the other is unstable. This distinctive feature is thoroughly explored in Section 6.3.

6.2 Geodesics and energy conditions

6.2.1 Geodesics

In this subsection, we will examine the geodesic curves of massive particles and the effective gravitational potential generated by the spacetime geometry given by eqs. (6.5) and (6.10). We choose to work with the (t, r, θ, φ) coordinate system, as it facilitates a straightforward derivation of the effective gravitational potential V_{eff} through a well-established procedure. This will help us to better comprehend the geometry of the aforementioned black hole solutions. To do so, we introduce the effective Lagrangian

$$2L_{\text{eff}} = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -B(r)\dot{t}^2 + \frac{\dot{r}^2}{B(r)} + [R(r)]^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) , \tag{6.19}$$

the Euler-Lagrange equations of which yield the geodesic equations. In the above, τ is an affine parameter of motion which can be identified with the proper time of a particle, dot denotes derivation with respect to τ, while $2L_{\text{eff}} = -1$ corresponds to massive particles which follow a timelike path. Note that massless particles will not follow the geodesics induced by the geometry $g_{\mu\nu}$, instead they will follow the geodesics induced by an effective geometry that accounts for photon-photon interactions, introduced by the non-linear electromagnetic terms in our action. Upon inspecting the Lagrangian (6.19), it becomes evident that there is no explicit dependence on the coordinates (t, φ). As a result, the Euler-Lagrange equations for t and φ yield two conserved quantities: the energy E and the angular momentum J of the particle under consideration, respectively. Hence, we have

$$E = B(r)\dot{t} , \tag{6.20}$$

$$J = [R(r)]^2\sin^2\theta\,\dot{\varphi} . \tag{6.21}$$

The equation of motion for θ reads

$$[R(r)]^2\,\ddot{\theta} + 2R(r)R'(r)\,\dot{r}\dot{\theta} - J\,\frac{\cos\theta}{\sin\theta} = 0 , \tag{6.22}$$

and by choosing $\theta = \pi/2$ ($\dot{\theta} = 0$), the particles stay fixed at the equatorial plane. Now plugging these results back to (6.19) we obtain the radial equation of motion

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = \frac{1}{2}E^2, \tag{6.23}$$

with the effective potential induced by the geometry being

$$V_{\text{eff}}(r) = \frac{B(r)}{2}\left(1 + \frac{J^2}{[R(r)]^2}\right) , \tag{6.24}$$

and the functions $B(r)$ and $R(r)$ given by (6.10). As we have already mentioned in the previous section, the radial coordinate r ranges from Q_m^2/M to plus infinity because the physical radial coordinate $R \in (0, +\infty)$.

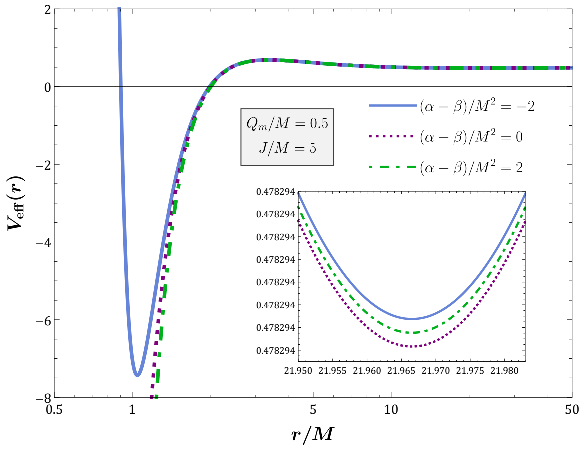


Figure 6.3: The effective potential $V_{\rm eff}$ in terms of the quantity r/M for $Q_m/M = 0.5$, $J/M = 5$, and $(\alpha-\beta)/M^2 = \{-2, 0, 2\}$. All parameters are dimensionless, and the horizontal axis is logarithmic.

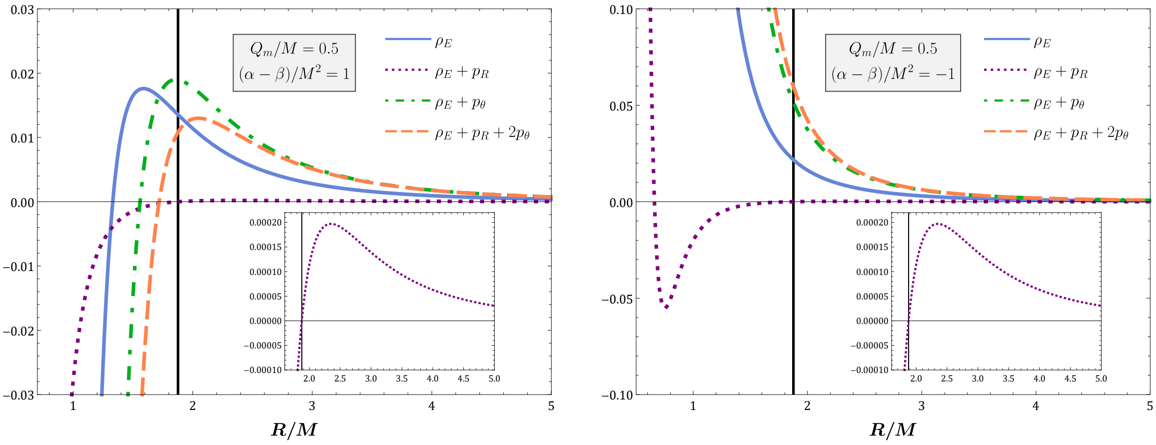
In Fig. 6.3, we depict the behavior of the effective potential $V_{\rm eff}$ in terms of the dimensionless parameter r/M, considering three distinct values for the fraction $(\alpha-\beta)/M^2$. Upon close examination, it becomes evident that the scenarios where $\alpha=\beta$ and $(\alpha-\beta)/M^2 = 2$ share a strikingly similar pattern in the effective potential. In both cases, the potential curve features one maximum and one minimum value, corresponding to unstable and stable circular orbits, respectively. On the other hand, in the case where $(\alpha-\beta)/M^2 = -2$, an additional minimum emerges, exhibiting local behavior that closely resembles the Newtonian potential. To understand the origin of this difference, we have to examine the expansion of the potential $V_{\rm eff}$ in the limit $r \to Q_m^2/M$, where one can verify that

$$V_{\rm eff}(r \to Q_m^2/M) = -\frac{J^2 M^4(\alpha-\beta)}{Q_m^4\left(r-\frac{Q_m^2}{M}\right)^4} + \frac{M^3(\alpha-\beta)\left(4J^2M^2 - Q_m^4\right)}{Q_m^6\left(r-\frac{Q_m^2}{M}\right)^3} + \mathcal{O}\left[\left(r-\frac{Q_m^2}{M}\right)^{-2}\right]. \quad (6.25)$$

We observe that the first term, which dominates in this particular regime, depends explicitly on the sign of the quantity $\alpha-\beta$. When $\alpha-\beta>0$, the potential tends toward negative infinity, whereas for $\alpha-\beta<0$, the potential tends toward positive infinity. This alignment precisely mirrors our observations in Fig. 6.3. Finally, from Fig. 6.3, it is also clear that for $r/M > 2$, the effective potential in all cases exhibits the same profile, independently of the relative values of the coupling constants α and β. This can be naively understood through the expansion of the potential at infinity, which is of the following form

$$V_{\rm eff}(r \to +\infty) = \frac{1}{2} - \frac{M}{r} + \frac{J^2}{2r^2} + \left(\frac{Q_m^2}{2M^2} - 1\right)\left[\frac{J^2 M}{r^3} + \frac{J^2 Q_m^2}{r^4} + \frac{J^2 Q_m^4}{M\, r^4} + \frac{J^2 Q_m^6}{M^2 r^4}\right] - \frac{(\alpha-\beta)Q_m^4}{r^6} + \mathcal{O}\left(\frac{1}{r^7}\right). \quad (6.26)$$

It is obvious that in the asymptotic regime, the coupling constants α and β cease to influence the potential profile, as their first contribution comes into play only in the seventh term of the expansion. Consequently, even at medium distances, we anticipate that beyond a certain point, the coupling constants will have negligible impact on the potential's behavior.



(a) (b)

Figure 6.4: The energy conditions for $Q_m/M = 0.5$ with (a) a positive and (b) a negative value assigned to the dimensionless quantity $(\alpha - \beta)/M^2$. The vertical lines correspond to the horizon of the black hole determined by these parameters.

6.2.2 Energy Conditions

We will now turn our attention to the energy conditions associated with the stress-energy tensor of our theory. In the physical coordinate system (t, R, θ, φ) the stress-energy tensor is described by an anisotropic fluid which in a covariant form can be written as

$$T^{\mu\nu} = (\rho_E + p_\theta)u^\mu u^\nu + (p_R - p_\theta)n^\mu n^\nu + p_\theta g^{\mu\nu} \,. \tag{6.27}$$

In the above, ρ_E is the energy density of the fluid measured by a comoving observer with the fluid, p_R is its radial pressure, p_θ is its tangential pressure, while u^μ and n^μ are its timelike four-velocity and a spacelike unit vector orthogonal to u^μ and also to both angular directions. The four-vectors u^μ and n^μ satisfy the following relations:

$$u^\mu = u(R)\,\delta^\mu_0 \,, \quad u^\mu u^\nu g_{\mu\nu} = -1 \,, \tag{6.28}$$

$$n^\mu = n(R)\,\delta^\mu_1 \,, \quad n^\mu n^\nu g_{\mu\nu} = 1 \,. \tag{6.29}$$

Given eqs. (6.2), (6.9), (6.12)-(6.15), and (6.27)-(6.29), one can readily compute that

$$\rho_E = -T^t{}_t = \frac{B(R)}{[W(R)]^2}\left(\frac{d\phi}{dR}\right)^2 + \frac{Q_m^2}{R^4}\,e^{-2\phi} + \frac{2(\alpha-\beta)Q_m^4}{R^8}\,f(\phi) \,, \tag{6.30}$$

$$p_R = T^R{}_R = \frac{B(R)}{[W(R)]^2}\left(\frac{d\phi}{dR}\right)^2 - \frac{Q_m^2}{R^4}\,e^{-2\phi} - \frac{2(\alpha-\beta)Q_m^4}{R^8}\,f(\phi) \,, \tag{6.31}$$

$$p_\theta = T^\theta{}_\theta = T^\varphi{}_\varphi = -\frac{B(R)}{[W(R)]^2}\left(\frac{d\phi}{dR}\right)^2 + \frac{Q_m^2}{R^4}\,e^{-2\phi} + \frac{6(\alpha-\beta)Q_m^4}{R^8}\,f(\phi) \,. \tag{6.32}$$

For the anisotropic fluid of (6.27), the energy conditions take the following expression:

- Null Energy Conditions (NEC): $\rho_E + p_R \geq 0 \quad \& \quad \rho_E + p_\theta \geq 0\,,$
- Weak Energy Conditions (WEC): NEC $\quad \& \quad \rho_E \geq 0\,,$
- Strong Energy Conditions (SEC): NEC $\quad \& \quad \rho_E + p_R + 2p_\theta \geq 0\,.$

In Figs. 6.4a and 6.4b, we illustrate the graphs of the quantities ρ_E, $\rho_E{+}p_R$, $\rho_E{+}p_\theta$, and $\rho_E{+}p_R{+}2p_\theta$ each plotted against the dimensionless parameter R/M. The free parameters of our model and solution have been chosen to be $Q_m/M = 0.5$, while the combination $(\alpha - \beta)/M^2$ takes values of 1 and -1, respectively. It is evident from Fig. 6.4 that all the aforementioned quantities maintain positive values within the causal region of spacetime and as a result, all energy conditions are satisfied.

These results imply, therefore, the existence of a dilaton hair in the black hole's exterior, while the energy conditions are satisfied, thereby leading to a bypass of the pertinent (modern version of the) no-hair theorems [34, 35] in the spirit of [36]. The situation can be understood as a consequence of the fact that the stress-energy tensor of our theory (6.1), with (6.9), is such that the tangential component of the pressure ($p_\theta = T^\theta_{\ \theta}$) dominates over its radial one ($p_R = T^R_{\ R}$) (in the (t, R, θ, ϕ) coordinate system), outside the horizon. That is, the following quantity is positive in the exterior region of the black hole,

$$\mathcal{G} - \mathcal{J} = T^\theta_{\ \theta} - T^R_{\ R} > 0\,, \tag{6.33}$$

where $\mathcal{G} = \rho_E + T^\theta_{\ \theta}$ and $\mathcal{J} \equiv \rho_E + T^R_{\ R}$. Note that the condition (6.33) follows from NEC. As discussed in detail in [36], the quantity $2\mathcal{G}/R$ is the *effective gradient pressure force*, and its positivity (i.e. that of $\mathcal{G}$, since $R > 0$) explains in a physical way the existence of scalar hair in the black-hole's exterior, without any violation of the energy conditions. The validity of the condition (6.33) can also be explicitly checked in our model from Eqs. (6.31) and (6.32). Thus, the exact black hole solution of the self-gravitating scalar-EH (non-linear) electrodynamics examined in this paper constitutes another explicit example of the general considerations of [36] for bypassing the no-hair theorem without any violation of the energy conditions.

6.3 Thermodynamic analysis

In this section, we will discuss the thermodynamics of both the GMGHS and our black-hole solution by considering their Euclidean actions. We will consider the Grand Canonical Ensemble and enclose the black hole spacetime in a cavity with a large radius r_c. In the Grand Canonical Ensemble, the black hole is allowed to exchange energy/mass and charge with its environment, so these two quantities are allowed to flow in and out through the boundary keeping the temperature and the magnetostatic potential of the boundary fixed. This effectively means that $T(r_h) = T(r_c)$ and $\Phi_m(r_h) = \Phi_m(r_c)$ and the system black hole-cavity is in thermodynamic equilibrium. Note that T is the black-hole temperature and Φ_m is the magnetostatic potential. The quantum partition function for the system is then given by

$$\mathcal{Z} = \int d[g^{(L)}_{\mu\nu}, \psi] e^{i\mathcal{S}(g^{(L)}_{\mu\nu}, \psi)} = \int d[g^{(E)}_{\mu\nu}, \psi] e^{-\mathcal{I}_E(g^{(E)}_{\mu\nu}, \psi)}\,, \tag{6.34}$$

where $\mathcal{S}$ is the Lorentzian action, $\mathcal{I}_E$ is the Euclidean action and ψ denotes all other possible fields included besides the metric tensor. The two actions are related via $\mathcal{I}_E = -i\mathcal{S}$ [171]. The quantity $g^{(L)}_{\mu\nu}$ is the Lorentzian metric with signature $(-+++)$, which corresponds to a $\mathbb{R}^{3,1}$ spacetime, while

$g_{\mu\nu}^{(E)}$ is the Euclidean metric with signature $(++++)$, which is obtained from the Lorentzian one by performing a Wick rotation [172] of the time coordinate ($\tau = it$). In the standard Matsubara formalism of finite-temperature systems, the Euclidean metric corresponds to a space $\mathbb{R}^3 \times S^1_{\beta_\tau}$, where the radius β_τ of the S^1 is the inverse temperature T^{-1} in units of the Boltzman factor $k_B = 1$. Hence, the second integral in (6.34) is evaluated over all possible field configurations that have an imaginary time τ with period β_τ. From the partition function, using standard thermodynamic relations one can obtain the Free Energy $\mathcal{G}$ of the system as

$$\mathcal{G} = -\frac{1}{\beta_\tau} \ln \mathcal{Z} \, . \tag{6.35}$$

By using the saddle point approximation (Laplace's method) we will consider that the classical action contributes the most and as a result we may drop the integral in the partition function $\mathcal{Z}$. Then the Euclidean action $\mathcal{I}_E$ can be related to the free energy evaluated on shell through the following relation

$$\mathcal{I}_E = \mathcal{G}\beta_\tau \, . \tag{6.36}$$

Having the expression of the free energy for the black hole solution, we will compare it with the free energy of the grand canonical ensemble in order to extract the mass (internal energy), the entropy, and the magnetostatic potential of the black hole. For more information in the discussion that follows, we refer the reader to the original work of Gibbons and Hawking [16].

6.3.1 GMGHS black hole

We start our analysis with the thermodynamics of the GMGHS solution. The Euclidean action, including the appropriate boundary terms, is given by

$$\mathcal{I}_E = -\frac{1}{16\pi} \int_\Sigma d^4x \sqrt{g} \left(\mathcal{R} - 2\nabla_\alpha\phi\nabla^\alpha\phi - e^{-2\phi}\mathcal{F}^2 \right) - \frac{1}{8\pi} \int_{\partial\Sigma} (K - K_0) \, d^3x \sqrt{h} \, . \tag{6.37}$$

In the Euclidean signature, the GMGHS black hole is described by the following metric:

$$ds^2 = B(r)d\tau^2 + \frac{dr^2}{B(r)} + [R(r)]^2 d\Omega^2 \, , \tag{6.38}$$

where $B(r) = 1 - 2M/r$, while $R(r)$ has the same form as in eq. (6.10). In this coordinate system, the Euclidean time coordinate is periodic and takes values in the range $0 \le \tau \le \beta_\tau$. For the derivation of the thermodynamic quantities, we assume that we have enclosed the black hole in a large cavity with radius r_c. Therefore, the radial coordinate takes values in $r_h \le r < r_c$. Finally, the two angular coordinates take their usual values. The boundary term K represents the trace of the extrinsic curvature, which in our case reads

$$K = \nabla^\alpha n_\alpha = \frac{2R'(r)\sqrt{B(r)}}{R(r)} + \frac{B'(r)}{2\sqrt{B(r)}} \, , \tag{6.39}$$

where $n_\alpha = \sqrt{1/B(r)}\,\delta^r_\alpha$ is a normalized spacelike vector field. The K term in the above hypersurface integral represents the Gibbons-Hawking-York boundary term, ensuring a well-defined variational principle. The second boundary term K_0 serves as a subtraction term to render the action finite for flat space (in the absence of the black hole). For flat space, K_0 equals $2/r$, obtained by setting $B(r) = 1$ and $R(r) = r$ in the above relation. Utilizing these relations, one can readily compute the Euclidean action (6.37) to be

$$\mathcal{I}_E = \frac{\beta_\tau M}{2} - \frac{\beta_\tau Q_m^2}{4M} \,. \tag{6.40}$$

In the above, we have used that the horizon radius is given by $r_h = 2M$. In the Grand Canonical Ensemble, the Euclidean action is identified with the free energy of the thermodynamic system as $\mathcal{I}_E = \beta_\tau \mathcal{G}$, thus, we can rewrite (6.40) as

$$\mathcal{I}_E = \beta_\tau M - \beta_\tau \Phi_m Q_m - S \,, \tag{6.41}$$

where S is the entropy and Φ_m is the magnetostatic potential, $\Phi_m = Q_m/r_h$. For the derivation of the above equation, we have used the fact that $\beta_\tau = 8\pi M \equiv 1/T$ with T being the temperature of the black hole. By combining now eqs. (6.40) and (6.41) we can evaluate the black-hole entropy S, which is given by the following relation

$$S = 2\pi M \left(2M - \frac{Q_m^2}{M}\right) = \pi[R(r_h)]^2 = \frac{A}{4}, \tag{6.42}$$

where A denotes the horizon area. It is evident that in this case, the entropy function has the well-known form of the Bekenstein-Hawking entropy. For validation, the same result may also be obtained using Wald's formula or even using the Arnowitt-Deser-Misner (ADM) formalism [173]. For a comprehensive analysis of the ADM formalism, readers are directed to [174]. Additionally, for its explicit application in black-hole solutions, we refer the interested reader to [75]. In the subsequent subsection, we will utilize the ADM formalism for the thermodynamic analysis of our black-hole solution.

The inclusion of the Gibbons-Hawking-York boundary term ensures that the Euclidean action attains an extremum within the class of fields considered here, $\delta\mathcal{I}_E = 0$. As a result, it is evident that the first law of thermodynamics in the Grand Canonical Ensemble (keeping the temperature and the magnetic potential fixed) takes the form

$$\delta M = T\delta S + \Phi_m \delta Q_m \,, \tag{6.43}$$

derived from (6.41), and holds *by construction*. The first law is also evident by taking the variation of the entropy with respect to the primary black-hole charges. The temperature of this black hole is the same as that of the Schwarzschild black hole, as pointed out in [175], since the Euclidean continuation does not care about the angular part. Consequently, the heat capacity C for constant charge will also be negative, $C = -1/(8\pi T^2)$; hence, these types of black holes cannot reach thermal equilibrium.

6.3.2 Black hole with non-linear electrodynamics

We will now focus on our black-hole solution, emanating from the action (6.1) and characterized by the line element (6.12)-(6.14). The scalar and the gauge fields are of the form $\phi = \phi(R)$—with R being the physical radial coordinate—and $\mathcal{A}_\mu = (0, 0, 0, A(\theta))$, respectively. In this case, to determine the thermodynamic quantities associated with the resulting black-hole solution, we will make use of the Euclidean signature and also utilize the ADM formalism [173, 174]. Hence, we consider the line element of the form

$$ds^2 = [N(R)]^2 B(R) d\tau^2 + \frac{[W(R)]^2 dR^2}{B(R)} + R^2 d\Omega^2 \,, \tag{6.44}$$

where the Euclidean time takes values in the range $0 \leq \tau \leq \beta_\tau$, while the radial coordinate $R \in [R_h, +\infty)$. To obtain the temperature, that is the period of the Matsubara frequency τ, in our case, we follow the calculation of [7]. To this end, we first ignore the angular part of the line element and perform a series expansion near the horizon. Thus, we are left with a two-dimensional line element which is compared with the line element of two-dimensional space expressed in polar coordinates $dS = d\hat{R}^2 + \hat{R}^2 d\Theta^2$. By doing so, we obtain

$$d\hat{R}^2 = \frac{W(R_h)^2}{B'(R_h)(R - R_h)} dR^2 \ , \tag{6.45}$$

$$B'(R_h)(R - R_h) d\tau^2 = \hat{R}^2 d\Theta^2 \ . \tag{6.46}$$

The coordinate Θ is periodic with a period 2π which implies that τ is also a periodic coordinate with a period β_τ given by:

$$\beta_\tau = \frac{1}{T} = \left. \frac{4\pi W(R)}{N(R) B'(R)} \right|_{R_h} \ , \tag{6.47}$$

where T is the temperature of the black hole. For completeness, we also remark at this point that we have also checked that, as expected, the temperature will take on the same values at the event horizon regardless of the coordinate system we are using (r or R).

The Euclidean action is related to the Lorentzian action via $\mathcal{I}_E = -i\mathcal{S}$ and we will consider the following variational problem which basically consists of the theory (6.1) alongside a boundary term denoted by $\mathcal{B}_E$ which we will consider in order to have a well-defined variational principle $\delta \mathcal{I}_E = 0$. Thus, we have

$$\mathcal{I}_E = \frac{2\pi\beta_\tau}{16\pi} \int_0^\pi d\theta \int_{R_h}^\infty dR \left[-N R^2 W \sin\theta \, \mathcal{L}(R, \theta) \right] + \mathcal{B}_E \ . \tag{6.48}$$

Here $\mathcal{L}$ denotes the Lagrangian of the theory which is a function of R, θ coordinates. After canceling total derivatives, the Euclidean action reads

$$\mathcal{I}_E = \beta_\tau \int_0^\pi d\theta \int_{R_h}^\infty dR \, \hat{\mathcal{L}}(Q^i, \partial_\mu Q^i) + \mathcal{B}_E \ , \tag{6.49}$$

with $Q^i = \{N(R), W(R), B(R), \phi(R), A(\theta)\}$ and $\hat{\mathcal{L}}(Q^i, \partial_\mu Q^i)$ given by

$$\hat{\mathcal{L}}(Q^i, \partial_\mu Q^i) = \frac{N \sin\theta}{4W^2 R^6} \Bigg[W R^7 B' + W^3 \left(2f(\phi)(\alpha - \beta) \frac{(\partial_\theta A)^4}{\sin^4\theta} + e^{-2\phi} R^4 \frac{(\partial_\theta A)^2}{\sin^2\theta} - R^6 \right) + B R^6 \left(W R^2 \phi'^2 - 2RW' + W \right) \Bigg] . \tag{6.50}$$

Following the ADM formalism, we have to vary the above Euclidean action with respect to each one of the dynamical fields Q^i to obtain the field equations. By doing so, we obtain none other than the well-known Euler-Lagrange equations, namely

$$\frac{\partial \hat{\mathcal{L}}}{\partial Q^i} - \partial_\mu \left(\frac{\partial \hat{\mathcal{L}}}{\partial(\partial_\mu Q^i)} \right) = 0 \, . \tag{6.51}$$

Let us now apply the above equation for the dynamical field $Q^1 = N(R)$. Upon substituting the expression of $\hat{\mathcal{L}}$ from (6.50) into (6.51), we find that the second term vanishes identically, while $\partial \hat{\mathcal{L}} / \partial N = \hat{\mathcal{L}} / N$. As a result, the equation for $N(R)$ indicates that $\hat{\mathcal{L}} = 0$, which in turn implies

$\mathcal{I}_E = \mathcal{B}_E$. This outcome is anticipated in the ADM formalism, where the metric construction (6.44) is specifically tailored to yield this result. Additionally, by solving the field equations (6.51) for all dynamical fields Q^i, one can determine the unknown functions and verify that the resulting solution is the one obtained in Sec. 6.1 with line element (6.12)-(6.14), alongside a constant N which without loss of generality we may set equal to 1. It is important to mention at this point, that during the derivation of the Euler-Lagrange equations, certain boundary terms were omitted. These terms are of the following form

$$\beta_\tau \left(\frac{R}{2W}\delta B + \frac{2BR^2\phi'}{W}\,\delta\phi - \frac{BR}{W^2}\,\delta W \right)\Bigg|_{R_h}^{\infty} , \tag{6.52}$$

and

$$\beta_\tau \int_{R_h}^{\infty} dR \left(\frac{We^{-2\phi}(\partial_\theta A)}{2R^2\sin\theta} + \frac{2(\alpha-\beta)Wf(\phi)(\partial_\theta A)^3}{R^6\sin^3\theta} \right)\delta A\Bigg|_{\theta=0}^{\theta=\pi} . \tag{6.53}$$

The variation of the boundary term $\delta\mathcal{B}_E$ will account for the neglected boundary terms, ensuring the attainment of a well-defined variational principle $\delta\mathcal{I}_E = 0$. Utilizing the fact that the variation of A yields $\delta A = (\delta Q_m)\cos\theta$, and substituting the expressions for the functions in (6.53), one can integrate and derive the following expression:

$$\beta_\tau\,\frac{Q_m\left\{\sqrt{4M^2R_h^2+Q_m^4}\left[R_h^4-4(\alpha-\beta)Q_m^2\right]-R_h^4Q_m^2\right\}}{2MR_h^6}\,\delta Q_m . \tag{6.54}$$

Now, the variation of the dynamical fields at infinity yield

$$\delta B = -\frac{2\delta M}{R} + \mathcal{O}\left(\frac{1}{R^2}\right) , \tag{6.55}$$

$$\delta\phi = \frac{Q_m}{MR}\delta Q_m - \frac{Q_m^2}{2M^2R}\delta M + \mathcal{O}\left(\frac{1}{R^3}\right) , \tag{6.56}$$

$$\delta W = \frac{Q_m^4}{4M^3R^2}\delta M - \frac{Q_m^3}{2M^2R^2}\delta Q_m + \mathcal{O}\left(\frac{1}{R^3}\right) , \tag{6.57}$$

while at the horizon we have that

$$\delta B|_{R_h} = -B'(R_h)\delta R_h , \tag{6.58}$$

$$\delta\phi|_{R_h} = \delta\phi(R_h) - \phi'(R_h)\delta(R_h) , \tag{6.59}$$

$$\delta W|_{R_h} = \delta W(R_h) - W'(R_h)\delta(R_h) . \tag{6.60}$$

Note that the parameters α, β are fixed by the theory and thus not allowed to vary, while M and Q_m are pure integration constants allowed in the variation.

As previously mentioned, to ensure a well-defined variational procedure, it is desirable to have $\delta\mathcal{I}_E = 0$. For clarity and convenience, we will partition the variation of the boundary term $\delta\mathcal{B}_E$ into two components: one at infinity and another at the event horizon, expressed as:

$$\delta\mathcal{B}_E = \delta\mathcal{B}_E(\infty) + \delta\mathcal{B}_E(R_h) . \tag{6.61}$$

Evaluating now (6.52) at infinity and considering the variation of the boundary term at infinity we find that a zeroth order contribution survives, which according to the variations of the fields leads to

$$\frac{\beta_\tau}{2} R\left(\delta B - 2\delta W\right) + \delta\mathcal{B}_E(\infty) = 0 \Rightarrow \tag{6.62}$$

$$\delta\mathcal{B}_E(\infty) = \beta_\tau\, \delta M \ . \tag{6.63}$$

On the other hand, eq. (6.52) at the horizon, alongside the variation of the boundary term at the horizon and (6.54) results in

$$2\pi R_h \delta R_h + \beta_\tau \, \frac{Q_m \left\{ \sqrt{4M^2R_h^2 + Q_m^4}\left[R_h^4 - 4(\alpha-\beta)Q_m^2\right] - R_h^4 Q_m^2 \right\}}{2MR_h^6}\, \delta Q_m + \delta\mathcal{B}_E(R_h) = 0 \ , \tag{6.64}$$

which might be written equivalently as

$$\frac{\delta A}{4} + \beta_\tau \Phi_m \delta Q_m + \delta\mathcal{B}_E(R_h) = 0 \ , \tag{6.65}$$

where we have used the fact that the area of the black hole is given by $A = 4\pi R_h^2$ and we have defined the magnetic potential as

$$\Phi_m = \frac{Q_m \left\{ \sqrt{4M^2R_h^2 + Q_m^4}\left[R_h^4 - 4(\alpha-\beta)Q_m^2\right] - R_h^4 Q_m^2 \right\}}{2MR_h^6} \ . \tag{6.66}$$

Considering now that we are dealing with the Grand Canonical Ensemble, we keep the temperature and the magnetic potential of the system fixed and as a result we can drop the variations to obtain

$$\mathcal{B}_E(\infty) = \beta_\tau M \ , \tag{6.67}$$

$$\mathcal{B}_E(R_h) = -\frac{A}{4} - \beta_\tau \Phi_m Q_m \ . \tag{6.68}$$

Therefore, the value of the Euclidean action is given by

$$\mathcal{I}_E = \beta_\tau M - \frac{A}{4} - \beta_\tau \Phi_m Q_m \ , \tag{6.69}$$

and since the Euclidean action is related to the free energy $\mathcal{G}$ of the system via $\mathcal{I}_E = \beta_\tau \mathcal{G} = \beta_\tau \mathcal{M} - S - \beta_\tau \Phi_m Q_m$ we can identify, by comparison the conserved black hole mass and the entropy of the black hole as

$$\mathcal{M} = M \ , \tag{6.70}$$

$$S = A/4 \ . \tag{6.71}$$

Finally, the First Law of Thermodynamics (6.43) holds *by construction* as in the GMGHS black hole.

With the confirmation that the black-hole thermodynamic quantities in our case adhere to the standard relations, we can now proceed to analyze the black hole's temperature. In Fig. 6.5, we depict the black-hole temperature as a function of the dimensionless quantity $R_h/(2M)$. Notice that the temperature is scaled by the temperature of the Schwarzschild black-hole to form a dimensionless quantity, ensuring its independence from the chosen unit system. In Figs. 6.5a and 6.5b, we explore the effects of the higher-order electromagnetic contributions on black-hole temperature, considering fixed (yet distinct) values for the coupling constants α and β, along with varying magnetic charge (Q_m) values, but maintaining the same value for the black-hole mass. Both Figs. 6.5a

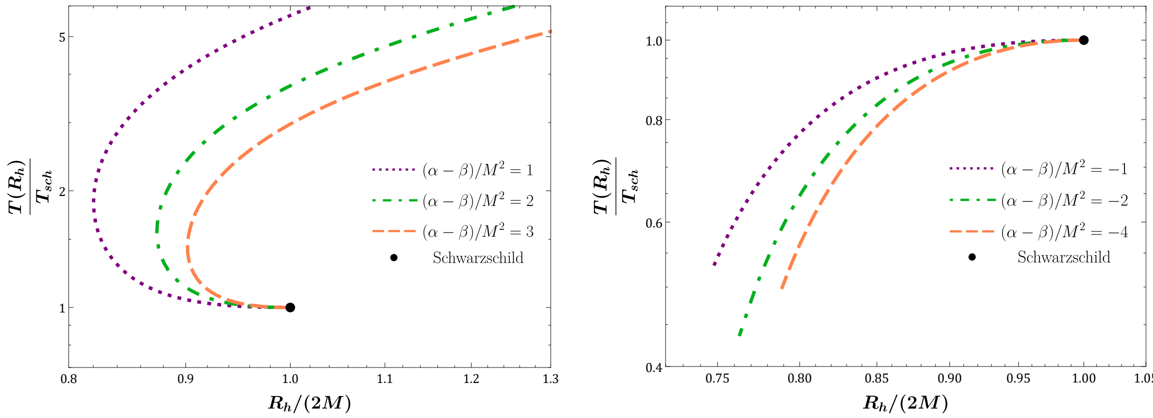


Figure 6.5: The black-hole temperature for (a) attractive and (b) repulsive higher-order electromagnetic contributions, with varying values of the magnetic charge (Q_m), while keeping the mass (M) the same. The axes in both figures are logarithmic.

and 6.5b were generated using the following procedure: For each value $(\alpha-\beta)/M^2$ and the ratio Q_m/M, we numerically evaluate the value of the ratio $R_h/(2M)$ using eq. (6.13). Subsequently, employing equation (6.47), we calculate the temperature of the black hole for each parameter pair. Finally, for each $(\alpha-\beta)/M^2$ we plot the points from the list $\{R_h/(2M), T(R_h)/T_{sch}\}$. Note that we use the same mass parameter M for the temperature calculation of the Schwarzschild black hole T_{sch}. In both figures, we have also incorporated a distinctive dot symbolizing a constant value for the quantity $T(R_h)/T_{sch}$, irrespective of the ratio $R_h/(2M)$. Apparently, this is not coincidental, as it mirrors the characteristics of the Schwarzschild black hole, where the horizon radius precisely equals $2M$ and its temperature is determined by the established formula $T = 1/(8\pi M)$.

Focusing now on the thermodynamical characteristics of our solution, we observe that regardless of the value $(\alpha-\beta)/M^2$, for $Q_m = 0$, our solution reduces to the Schwarzschild black hole and therefore all graphs in Fig. 6.5 have as a starting point the Schwarzschild point. However, when we depart from this limit, we notice that for $\alpha-\beta > 0$, as illustrated in Fig. 6.5a, the temperature of the resulting black holes consistently surpasses that of the Schwarzschild black hole, whereas for $\alpha-\beta < 0$ (Fig. 6.5b), the opposite effect occurs. Furthermore, this temperature increase, in the $\alpha-\beta > 0$ scenario, is independent of whether the black hole under examination possesses a smaller or larger horizon radius compared to the corresponding Schwarzschild black hole. As previously observed in Fig. 6.2 and discussed in Sec. 6.1, it becomes evident in Fig. 6.5a that there are consistently pairs of black-hole solutions, more compact than the Schwarzschild solution, that share the same horizon radius R_h but with different ratios Q_m/M. However, now we see that although these solutions possess the same horizon radius, their temperatures differ significantly. This can be understood through the relation (6.47) where it is evident that the formula determining the temperature of a black hole depends on the first derivative of the function $B(R)$. This means that black-hole solutions which for different ratios Q_m/M result in the same horizon radius R_h through the equation $B(R_h) = 0$, their temperatures are not necessarily the same since $B'(R_h)$ could differ in these two cases.

Moreover, we can deduce the thermodynamic stability of these black holes by examining how the temperature changes with a change in the mass. In Fig. 6.5a it is evident that there are two distinct branches of black-hole solutions. In the first branch we have black-hole solutions that get colder as the mass is decreasing, while in the second branch we have black holes that are getting hotter as the mass is getting smaller. This implies that the heat capacity $C \equiv dM/dT$ for the first branch is positive since both dM, dT are negative and the black holes are thermally stable, while the second branch, for which the temperature rises with the decrease of mass, exhibits negative heat capacity and are thermally unstable. Notice also the fact that the Schwarzchild black hole lies in the second (unstable) branch which is a well-known result. Furthermore, the parameter space of these black holes exhibits a point where $dT = 0$ indicating the divergence of the heat capacity and as a result a phase transition from hot to cold black holes. In Fig. 6.5b, we can see that as the black holes lose mass they get colder which implies that they are thermally stable since they possess positive heat capacity. These results are in agreement with the studies in [176], where the black holes are viewed as defects in the thermodynamical spacetime [177].

In the next section, we proceed to study the stability of the black holes from a linear-perturbation point of view, which, in general, is distinct from the thermodynamic stability. Indeed, as we shall demonstrate explicitly below, such a linearised stability analysis does not necessarily imply thermodynamical stability, in the sense that the thermodynamically unstable branches found above exhibit stability under linear perturbations.

6.4 Linear Perturbations

6.4.1 Radial Stability

In this section, we investigate the stability of our solution under radial perturbations. For simplicity, we focus on linear and radial perturbations. Therefore, we use the following ansatz:

$$ds^2 = -P(R,t)dt^2 + Q(R,t)dR^2 + R^2 d\Omega^2, \qquad A = a_0(R,t)dt + Q_m \cos\theta \, d\varphi, \qquad \phi = \phi(R,t), \quad (6.72)$$

where

$$P(R,t) = B(R)\left[1 - \epsilon\, e^{-i\omega t} h_1(R)\right], \qquad Q(R,t) = \frac{1}{H(R)} + \epsilon\, e^{-i\omega t} h_2(R), \quad (6.73)$$

$$\phi(R,t) = \phi(R) + \epsilon\, e^{-i\omega t}\phi_1(R), \qquad a_0(R,t) = \epsilon\, e^{-i\omega t} a_0(R). \quad (6.74)$$

For the stability analysis, it is more convenient to work within the physical coordinate system; hence, in the above equations $B(R)$, $\phi(R)$ and $H(R) = B(R)/W^2(R)$ are given in eqs. (6.13–6.15), and correspond to the background/unperturbed spacetime. Note that the radial perturbations are associated with the $L = 0$ perturbation [3] in the even sector of the gravitational perturbations. Therefore, in the electromagnetic part, only the electric-type perturbations contribute, as the magnetic-type corresponds to the odd sector [181, 182]. The dimensionless constant ϵ determines the order of the perturbation. Finally, ω specifies the decomposition of the modes with fixed energy.

A direct calculation reveals that both the spacetime and the matter field perturbations are determined from the function ϕ_1. Consequently, the investigation of the system is simplified to a single equation for the perturbation of the scalar field. This specific equation can be expressed in the conventional Schrödinger form:

$$\frac{d^2\Psi(r^*)}{dr^{*2}} + \left[\omega^2 - \mathcal{V}(R)\right]\Psi(r^*) = 0, \quad (6.75)$$

[3] L is the "angular momentum" index in the spherical harmonics function $Y_L^{M_L}(\theta,\varphi)$. For more information about the decomposition of the perturbations in spherical harmonics, see [178, 179, 180, 181].

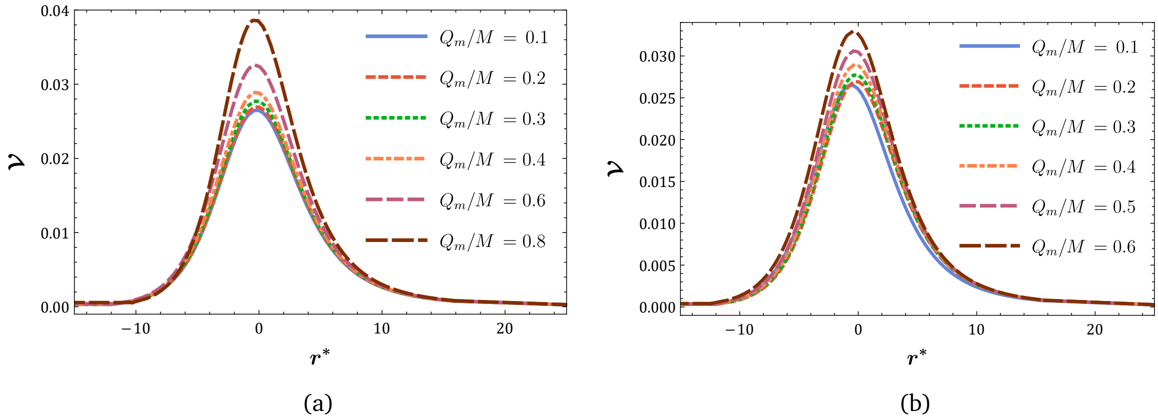


Figure 6.6: The graph of the effective potential $\mathcal{V}$ in terms of r^* for various values of the parameter Q_m/M and (a) $(\alpha-\beta)/M^2 = 1$, (b) $(\alpha-\beta)/M^2 = -1$.

where we have defined a new perturbation function as $\Psi \equiv R\phi_1$ and the tortoise coordinate is $dr^* \equiv \frac{dR}{\sqrt{B(R)H(R)}}$. The potential of the Schrödinger equation has the following form

$$\mathcal{V}(R) = \frac{HB' + 2BR\phi'\left[\phi'\left(R\left(H' + HR\phi'^2\right) + 4H\right) + 3HR\phi''\right] + BH'}{2R} + \frac{B(\alpha-\beta)Q_m^4\left(R\dot{f}\phi' + \ddot{f}\right)}{R^8} + \frac{B\gamma e^{-2\phi}Q_m^2\left(2 - R\phi'\right)}{R^4}\,. \tag{6.76}$$

Considering our emphasis on the stability of black-hole solutions, there is no need to solve eq. (6.75). The time evolution factor, $\exp(-i\omega t)$, simplifies the task, requiring us only to ascertain whether the frequency, ω, is purely imaginary or not. In the scenario where the frequency ω is purely imaginary, the mode undergoes exponential growth due to the presence of the term $\exp(-i\omega t)$ making the solution unstable. Therefore, a negative eigenvalue, $\omega^2 < 0$, that signifies an unstable mode, corresponds to a bound state in the Schrödinger equation (6.75). A general result in quantum physics is that for a potential that vanishes in both asymptotic regions, has a barrier form, and is positive definite. Therefore, eq. (6.75) does not exhibit bound states. In Fig. 6.6, we depict the potential of the Schrödinger equation for two families of solutions. The first one corresponds to $(\alpha-\beta)/M^2 = 1$, while the second one corresponds to $(\alpha-\beta)/M^2 = -1$. By carefully examining the parametric space of the solutions, we deduce that the potential always takes a form similar to the potentials in Fig. 6.6. Therefore, we conclude that our solutions are stable under radial perturbations.

Although radial stability is a strong indication regarding the stability of a particular solution, a more careful and general perturbation analysis has to be performed to extract a stronger result, which, however, lies beyond the scope of this work. Moreover, as shown in the previous section 6.3, linear stability does not necessarily imply thermodynamical stability for the black hole, in the sense that the latter, although linearly stable, nonetheless it possesses thermodynamically unstable branches.

6.4.2 Scalar Quasi-Normal Modes

Quasi-normal modes (QNMs) play a crucial role in the study of black holes and other astrophysical objects [183, 184, 185]. These modes represent the characteristic vibrations or oscillations of a black hole after a perturbation, such as a gravitational wave or a scattering event. QNMs are characterized by complex frequencies, i.e. eigenvalues of the Schrödinger equation, consisting of a real part and an imaginary part. The real part corresponds to the oscillation frequency, while the imaginary part reflects the damping or decay of the mode. The study of QNMs provides valuable insights into the nature and properties of black holes, offering a unique window into their internal dynamics.

For simplicity, we will consider the propagation of a test scalar field Φ in the background of our black hole and extract its QNMs. We begin our analysis from the following action functional for the scalar field,

$$S = \int d^4x\sqrt{-g}\left[\nabla^\mu\Phi\nabla_\mu\Phi + m^2\Phi^2\right], \tag{6.77}$$

where m is the mass of the test scalar field. The variation of the above action with respect to the scalar field yields the Klein-Gordon equation in the black hole background

$$\left(\Box - m^2\right)\Phi = 0\,. \tag{6.78}$$

Note that the test scalar field Φ, as a perturbation field, does not back-react on the spacetime metric and is a function of all spacetime coordinates $\Phi = \Phi(t, R, \theta, \varphi)$. For clarity, we choose to work in the physical coordinate system. Therefore, the background metric is given by eq. (6.12). We can apply the separation of variables as follows

$$\Phi(t, R, \theta, \varphi) = e^{-i\omega t}Y_L^{M_L}(\theta, \varphi)\frac{\Psi(R)}{R}\ , \tag{6.79}$$

where $Y_L^{M_L}(\theta, \varphi)$ represents the spherical harmonics function. By using the tortoise coordinate, $dr^* = \frac{dR}{\sqrt{B(R)H(R)}}$, one can rewrite the perturbation equation in a Schrödinger form as:

$$\frac{d^2\Psi(r^*)}{dr^{*2}} + \left[\omega^2 - V(R)\right]\Psi(r^*) = 0, \tag{6.80}$$

where $V(R)$ is the effective potential of the Schrödinger equation and is given by

$$V = \frac{HB' + BH'}{2R} + \frac{B}{R^2}L(L+1) + m^2B. \tag{6.81}$$

The background metric functions B and $H = B/W^2$ are given in eqs. (6.13–6.14).

In the pursuit of calculating the QNMs, the WKB (Wentzel-Kramers-Brillouin) approximation stands as a valuable method. Particularly useful in the context of wave-like phenomena, the WKB approximation provides an efficient and semi-classical approach to estimating the complex frequencies associated with QNMs. By treating the Schrödinger-like equation governing the perturbations as a semiclassical wave equation, the WKB method allows for the determination of QNM frequencies without the need for an exact solution. The WKB method was initially developed for quantum mechanical problems; however, Schutz and Will were the first to apply this method to the problem of scattering around black holes [186]. Later, Iyer and Will extended this approach to the third WKB order beyond the eikonal approximation [187], and Konoplya further advanced it to the sixth order

	$(\alpha-\beta)/M^2=0.01$	$(\alpha-\beta)/M^2=-1$	$(\alpha-\beta)/M^2=1$
$Q_m/M=0.01$	$0.329438-0.096255\,i$	$0.329438-0.096255\,i$	$0.329438-0.096255\,i$
$Q_m/M=0.3$	$0.333767-0.096712\,i$	$0.333767-0.096728\,i$	$0.333767-0.096696\,i$
$Q_m/M=0.6$	$0.348235-0.098171\,i$	$0.348226-0.098595\,i$	$0.348243-0.097727\,i$

Table 6.1: The dimensionless $L=1$, $n=0$ quasinormal modes ($M\omega$) for $m=0$.

	$(\alpha-\beta)/M^2=0.01$	$(\alpha-\beta)/M^2=-1$	$(\alpha-\beta)/M^2=1$
$Q_m/M=0.01$	$0.401632-0.050195\,i$	$0.401632-0.050195\,i$	$0.401632-0.050195\,i$
$Q_m/M=0.3$	$0.404334-0.052296\,i$	$0.404335-0.052294\,i$	$0.404332-0.052298\,i$
$Q_m/M=0.6$	$0.413700-0.058602\,i$	$0.413733-0.058522\,i$	$0.413667-0.058679\,i$

Table 6.2: The $L=1$, $n=0$ dimensionless quasinormal modes ($M\omega$) for $m/M=0.4$.

[188, 185]. Interestingly, the sixth order yields a relative error of approximately two orders of magnitude lower than that of the third WKB order [188, 185]. However, for simplicity, in this work, we will employ the first-order WKB approximation, in which the QNM frequencies are obtained from the solution of the following equation

$$n+\frac{1}{2}=-i\left[\frac{\omega^2-V(r^*)}{\sqrt{-2V''(r^*)}}\right]_{r*=r^*_{\max}}, \qquad n\in\mathbb{Z}^{\geq}. \tag{6.82}$$

The expression in the right-hand part of the above equation is evaluated at the maximum of the potential $r^*_{\max}$ while n is the overtone number of the QNMs.

In Tables 6.1 and 6.2, we present the dimensionless QNM frequencies, denoted by $(M\omega)$, for two distinct scenarios: when $m=0$ and $m/M=0.4$, respectively. Notably, as Q_m approaches 0, our solution converges to the Schwarzschild black hole, irrespective of the $(\alpha-\beta)/M^2$ parameter. This convergence is evident in the first row of both tables, where $Q_m/M=0.01$, as the QNM values remain constant across varying $(\alpha-\beta)/M^2$. Furthermore, as $(\alpha-\beta)/M^2$ tends toward 0, our solution adopts the characteristics of the GMGHS black hole. Consequently, the QNMs in the first row of both tables, specifically when $(\alpha-\beta)/M^2=0.01$, align with those of the GMGHS black hole. The subsequent rows in the tables provide additional insights into the characteristics of our solution. For instance, in the second and third rows, where $Q_m/M=0.3$ and 0.6, respectively, we observe a systematic variation in the QNM values with changes in both Q_m/M and $(\alpha-\beta)/M^2$. This behavior highlights the sensitivity of the QNM frequencies to the parameters characterizing the black hole solution. Furthermore, by comparing these results to the Schwarzschild and GMGHS cases, we discern how our solution deviates from these benchmark scenarios. Additionally, the tables reveal intriguing patterns in the imaginary parts of the QNMs. For varying Q_m/M and $(\alpha-\beta)/M^2$, the imaginary parts exhibit non-trivial changes, reflecting the impact of the black hole's charge and the parameter $(\alpha-\beta)/M^2$ on the damping behavior of the perturbations.

6.5 Other Solutions

6.5.1 Asymptotically (A)dS spacetimes

Let us now, briefly discuss asymptotically (A)dS spacetimes. Following [37], introducing a scalar potential $\mathfrak{V}(\phi)$ in the action and considering

$$\mathcal{S} = \int d^4x\sqrt{-g}\Big(\mathcal{R} - 2\nabla^\mu\phi\nabla_\mu\phi - e^{-2\phi}\mathcal{F}^2 + f(\phi)\big(-2\alpha\mathcal{F}^\alpha{}_\beta\mathcal{F}^\beta{}_\gamma\mathcal{F}^\gamma{}_\delta\mathcal{F}^\delta{}_\alpha + \beta\mathcal{F}^4\big) - \mathfrak{V}(\phi)\Big)\ , \quad (6.83)$$

with a $\mathfrak{V}(\phi)$ of the form

$$\mathfrak{V}(\phi) = \frac{1}{3}\Lambda e^{-2\phi} + \frac{1}{3}\Lambda e^{2\phi} + \frac{4\Lambda}{3} = \frac{2}{3}\Lambda(\cosh(2\phi)+2)\ , \quad (6.84)$$

we can obtain $B(r)$ as

$$B(r) = 1 - \frac{2M}{r} - \frac{2(\alpha-\beta)Q_m^4}{r^3\left(r-\frac{Q_m^2}{M}\right)3} - \frac{1}{3}\Lambda r\left(r - \frac{Q_m^2}{M}\right)\ , \quad (6.85)$$

while $\phi(r), R(r)$ will remain the same. Note here that the potentials $\mathfrak{V}(\phi)$ and $f(\phi)$ are almost identical, and they are both Liouville-type potentials [38].[4]

6.5.2 Solutions for general γ

Assuming that the coupling term between the dilaton and the Maxwell term is of the form $e^{-2\gamma\phi}$ we can obtain the same geometry with the $\gamma = 1$ case with the coupling function $f(\phi)$ now being given by

$$f(\phi) = -3\cosh(2\phi) - 2 - \frac{e^{2\phi}\left(\left(e^{-2\phi}\right)^{\gamma-1} - 1\right)Q_m^6}{2M^4\left(e^{2\phi}-1\right)^4(\alpha-\beta)}\ . \quad (6.86)$$

In this case, the charge-to-mass ratio is fixed by the theory. As a result, such black holes are described by a constrained phase space of free parameters, since this situation reduces the number of primary black hole hairs from two to one. A more physical result would be to let the form of the dilaton field to be affected by the change of the coupling function with the Maxwell term, however, we were not able to derive exact results in this case, so one has to employ numerical techniques. Such endeavors may be undertaken in subsequent works.

6.6 Conclusions

In the quest to comprehend gravitational phenomena and the nature of gravity itself, the theoretical exploration of black holes stands as a pivotal frontier. The predictions of General Relativity (GR) are in good agreement with current observations related to black holes. This is attributed to the large mass of the observed objects and therefore their large horizon radius and small horizon curvature. Additionally, a plethora of cosmological observations indicates instances where GR exhibits limitations, with the most notable challenges being the Dark Energy Problem and GR's inability to account

[4]Nonetheless, we should notice that the presence of a positive cosmological constant term in the gravitational effective actions is problematic within the context of string/brane-inspired quantum gravity theories, due to the so-called swampland conjecture [189, 190, 191]. This issue is open at present, and its study goes beyond the purpose of the current article.

for the inflationary epoch in our universe. Therefore, the validity of General Relativity is expected to come under scrutiny in extreme conditions. General Relativity is commonly acknowledged as an effective theory applicable only within the realm of low energies. Consequently, such observations motivate us to explore modified gravitational theories, especially in extreme conditions where GR's validity may be compromised. Among these theoretical frameworks, modifications originating from String Theory, particularly the heterotic string theory, emerge as leading contenders. Notably, String Theory offers insights into high-order corrections, ranging from the Gauss-Bonnet term to non-linear electromagnetic effects, and provides a rich avenue for exploring the behavior of black holes under diverse conditions.

One intriguing aspect of string/brane-induced non-linear electrodynamics is the emergence of the Born-Infeld (BI) Lagrangian, which encapsulates higher-order corrections to Maxwell's theory. This Lagrangian arises from the resummation of open string excitations, particularly in the context of D-brane worlds in string theory. The coupling of the BI Lagrangian to the dilaton field in curved spacetime leads to an effective four-dimensional action, offering a novel perspective on electromagnetic interactions in the presence of gravity. Furthermore, considerations of higher-order electromagnetic terms, originating from closed string sectors, broaden the theoretical landscape. The inclusion of string loops leads to generalized effective actions, incorporating both closed and open string contributions, and potentially revealing novel phenomena beyond conventional electromagnetic frameworks. Departing from traditional electromagnetic theories, the exploration of non-linear electrodynamics within the context of black hole solutions offers a rich avenue for understanding strong-field regimes and cosmological implications. Non-linear effects become crucial in regions with intense gravitational fields, such as those near black holes, shedding light on phenomena absent in linear theories. Moreover, non-linear electrodynamics holds relevance for early universe cosmology, where the interplay between gravitational and electromagnetic fields played a significant role.

In this chapter, we considered a string-inspired theory that involves a scalar field ϕ coupled to the electromagnetic field via a non-linear function $f(\phi)$. The action encompasses higher-order electromagnetic invariants, contributing to the field equations and leading to novel black hole solutions. Furthermore, we investigated the impact of a non-trivial coupling function $f(\phi)$, considering a specific functional form motivated by string-inspired models. The resulting exact, magnetically charged black hole solution revealed significant departures from the classical General Relativity predictions, with the scalar field and the electromagnetic field configurations exhibiting non-trivial behavior. We explored the implications of different coupling constants α and β on the spacetime geometry and electromagnetic field configurations. The solutions obtained exhibit intriguing features, including dependence on the sign of $\alpha - \beta$ which determines whether the higher-order electromagnetic terms contribute attractively or repulsively to the spacetime geometry. Additionally, we examined the horizon structure of the black hole solutions, observing transitions from single to multiple horizons and even to naked singularities as the parameters varied. Notably, the compactness of the black holes relative to the Schwarzschild solution depended on the the magnetic charge to mass ratio. Our findings suggest a rich interplay between the scalar field, electromagnetic field, and spacetime geometry, highlighting the potential implications of such theories in astrophysical contexts, and the search for potential signatures of string theory in black hole physics, at least those signatures that can be manifested through effective string-inspired field theory models. It goes without saying, however, that the present work does not deal with a detailed experimental sensitivity analysis of such objects, which still remains to be done.

The examination of geodesics and energy conditions delves into the intricate dynamics of particles moving within the spacetime geometry described by the black hole solutions under investigation. By analyzing the geodesic equations we unveil the behavior of massive particles in the vicinity of these black holes, elucidating the role of the effective gravitational potential. Notably, the effective potential exhibits distinct features depending on the relative values of the coupling constants, offering insights into the stability and nature of orbits around the black holes. Additionally, the examination of energy conditions associated with the effective stress-energy tensor reveals intriguing properties of the spacetime, indicating the existence of a dilaton hair in the black hole's exterior while satisfying all energy conditions. This observation challenges the traditional no-hair theorems, underscoring the nuanced interplay between gravitational theories, non-linear electrodynamics, and scalar fields in modified theories. Our thermodynamic analysis provided valuable insights into the properties of black holes in both the GMGHS solution and our black-hole solution with non-linear electrodynamics. Our analysis allows for the extraction of important thermodynamic quantities such as mass, entropy, magnetostatic potential, and the extraction of the first Law of Thermodynamics. Notably, the entropy of both black hole is consistent with the Bekenstein-Hawking entropy formula. By examining the behavior of the temperature, we concluded that when the non-linear electrodynamics terms act attractively, there exist two distinct branches of black holes, one that is getting colder as the mass is decreasing and therefore is thermally stable, and another one that is getting hotter as the mass is decreasing which is thermally unstable. On the other hand, when the non-linear electrodynamics terms have a repulsive effect, the black holes are getting colder as the mass is decreasing and as a result are thermally stable.

Finally, our analysis of linear perturbations and scalar quasi-normal modes (QNMs) provides valuable insights into the stability and dynamic behavior of black hole solutions with non-linear electrodynamics. Through a rigorous investigation of radial stability, we demonstrated that our black hole solutions remain stable under linear and radial perturbations. This finding underscores the robustness of our black hole solutions against radial perturbations, supporting their viability as physically meaningful configurations within the framework of non-linear electrodynamics. Furthermore, our examination of scalar QNMs yielded intriguing results regarding the characteristic vibrations and oscillations of the black hole spacetime. Moreover, the analysis of scalar QNMs revealed the intricate interplay between the black hole parameters, such as charge and $(\alpha-\beta)/M^2$, and the frequency and damping behavior of perturbations. By systematically varying these parameters, we observed distinct patterns in the QNM frequencies, indicating the sensitivity of the black hole's dynamic properties to its intrinsic characteristics. Notably, our results exhibited convergence to the Schwarzschild black hole in the limit of vanishing charge and alignment with the GMGHS black hole in specific parameter regimes. These observations highlight the rich phenomenology associated with black holes in our string-inspired theory.

Chapter 7

Concluding Remarks

In the present thesis, we investigated black hole solutions and their properties in modified theories of gravity and electrodynamics. Such theories are well established frameworks, in investigating the effects of gravity and electrodynamics in the strong field regime, where the intensity of the fields is strong. In such a scenario linear theories such as GR and Maxwell's theory of electrodynamics break down.

In the first part of this thesis, we presented in detail black hole solutions coupled to scalar fields, $f(R)$ gravity theories and non-linear theories of electrodynamics that stem from string theories. We discussed about well known solutions of $f(R)$ theories, presented some of their properties and shed light to some problematic behaviors they present. The no-scalar hair theorem has been also scrutinized and we discussed several ways that this theorem might be bypassed, by the inclusion of scalar potentials for the scalar field, as well as non-minimal couplings between the scalar field and gravity. The no-scalar hair theorem states that black holes are described by parameters that can be measured by an observer at infinity and therefore are associated to a Gauss law, such as the mass, the electromagnetic charge and the angular momentum. We reviewed same problematic aspects of these scenarios such as the fact that in the conformal coupling case, the scalar field blows up at the horizon of the black hole, a scenario that may be remedied with the inclusion of a positive cosmological constant only to find out that such a black hole possesses negative entropy at the event horizon, due to the fact that the entropy acquires an addition term because of the gravity-matter coupling. In addition, the potentials that may be used to bypass the no-scalar hair theorem are not well motivated from a high-energy physics point of view, but can be viewed as toy models, in order to investigate the behavior of hairy black holes.

We then moved on to the main part of this thesis by introducing scalar fields as matter fields in $f(R)$ gravity we came to some interesting conclusions. First of all, the three dimensional black holes might be thermodynamically preferred when compared to their GR counterparts, as they possess higher entropy at the event horizon of the black hole. Effects of non-linear gravity are important in the region where the scalar field is strong, i.e near the origin. On the other in the conformal-breaking case of chapter 4 the produced black hole is massless, an effect we have correlated to the parameter α that breaks the conformal invariance, by showing that the part of the metric that is completely supported by α will yield negative mass contributions.

In the final part of this thesis we discussed black hole solutions in the context of non-linear theories of electrodynamics and in particular versions of the Euler-Heisenberg theory. This theory arises naturally in the context of dimensional reduction of the ten dimensional heterotic string theory. At first we considered a simple scalar field theory of a self-interacting scalar field minimally coupled to gravity. Imposing a scalar field that is regular everywhere, in the scenario of pure radial magnetic fields, we obtained an exact magnetically charged hairy black hole solution and reconstructed the scalar potential. The black hole carries secondary charge since the black hole mass is supported by the scalar field. In addition we found that the potential restricts the free parameters of the theory since it allows for fixed mass to scalar charge and magnetic to scalar charge ratios. Such a scenario is also present in the linear electrodynamics case.

In the final chapter, we investigated a similar theory. We considered couplings of the dilaton field with the Euler-Heisenberg electrodynamics. Here the coupling function is of fundamental nature, since it contains the dilatonic coupling $e^{\pm 2\phi}$. The form of the coupling function is in particular $f(\phi) = -\frac{3}{2}(g_s^{-2} + g_s^2) - 2$, where $g_s = \exp(\phi)$ is the string coupling. As we discussed in the pertinent chapter and in the introduction, the g_s^{-2} is the standard tree-level dilaton-Maxwell term coupling [116, 120, 121], while the g_s^2 indicates two-string-loop corrections (genus-$\chi = 2$ world-sheet surfaces). The crucial, for our subsequent discussion, dilaton-independent term in $f(\phi)$ might be the result of appropriate combinations of higher-string-loop corrections in the Einstein-frame effective action. For this coupling function, we obtained exact magnetically charged black hole solutions with a non-trivial dilaton. We found that the non-linear electrodynamics term will be either of attractive or repulsive nature. When they act attractively, we found that there will be black hole solutions with the same radius for the event horizon but for different masses and charges and such compact objects cannot be therefore identified by their horizon. However, these objects will have different thermodynamic properties. Evaluating the thermodynamic quantities we found that in the case where the non-linear terms act attractively we have two branches of black hole solutions. In the first branch we have cold black hole solutions that are thermally unstable, while in the second branch we have hot black holes that are thermally stable. By examining the energy conditions, we found that our theory respects them, while a stability analysis showed that the black holes are stable against radial perturbations.

In conclusion, in this thesis, we obtain several novel black hole solutions in $f(R, \phi)$ theories and non-linear theories of electrodynamics. We discussed their thermodynamic aspects and we came to some interesting conclusions. The results obtain indicate that non-linear theories posses far more interesting phenomenology in regards to the compact objects they can host, when compared to linear theories such as GR and Maxwell's theory.

Appendices

Appendix A

More Calculations

A.0.1 A very simple asymptotically flat hairy black hole solution: Extended discussion and calculations

Let us now elaborate a bit more on the minimally coupled case, since a scalar potential is the easiest way to evade the no scalar hair theorem. Consider the action of 1.61. A solution of this action is [15]

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + a(r)^2 d\Omega^2 \ , \tag{A.1}$$

$$F(r) = 1 + \chi\left(2r(\nu + r)\ln\left(\frac{\nu + r}{r}\right) - \nu(\nu + 2r)\right) \ , \tag{A.2}$$

$$a(r) = \sqrt{r(r+\nu)} \ , \tag{A.3}$$

$$\phi(r) = \frac{1}{\sqrt{2}}\ln\left(1 + \frac{\nu}{r}\right) \ , \tag{A.4}$$

$$V(\phi) = 6\chi\sinh\left(\sqrt{2}\phi\right) - 2\sqrt{2}\chi\phi\left(\cosh\left(\sqrt{2}\phi\right) + 2\right) \ , \tag{A.5}$$

where χ is a constant of the theory and ν an integration constant which plays the role of the scalar charge, since it controls the $1/r$ behavior of the scalar field at large distances. $F(r)$ asymptotes as

$$F(r\to\infty) \sim 1 - \frac{\nu^3\chi}{3r} + \frac{\nu^4\chi}{6r^2} - \frac{\nu^5\chi}{10r^3} + \frac{\nu^6\chi}{15r^4} - \frac{\nu^7\chi}{21r^5} + O\left(\left(\frac{1}{r}\right)^6\right) \ , \tag{A.6}$$

$$F(r\to 0) \sim \left(1 - \nu^2\chi\right) - 2r(\chi(\nu - \nu\ln(\nu) + \nu\ln(r))) + O\left(r^2\ln r\right) \ , \tag{A.7}$$

and hence describes a genuine black hole spacetime for $\left(1-\nu^2\chi\right) < 0$ in the interval $0 < r < \infty$.

Now, we consider the Hamiltonian version of General Relativity, whose action is

$$\mathcal{I} = \int d^3x dt\left(\pi^{ij}\dot{g}_{ij} - NH - N_i H^i\right) + \mathcal{B} \ . \tag{A.8}$$

Here $\mathcal{B}$ is a boundary term. We are going to make use of quantum gravity arguments, and use the Gravitational Path Integral approach therefore, as required by the Path Integral, the action has to contain only first derivatives of the metric [16]. The spacetime we consider is static (no dependence on t and is irrotational). Hence we can consider the reduced action principle

$$\mathcal{I} = -\int d^3x dt N H + \mathcal{B} \; . \tag{A.9}$$

We will consider the Euclidean action which is related to the Hamiltonian action via

$$\mathcal{I}_E = -i\mathcal{I} \; . \tag{A.10}$$

The Euclidean metric reads

$$ds^2 = F(r)N(r)^2 d\tau^2 + \frac{dr^2}{F(r)} + a(r)^2 d\Omega^2 \; , \tag{A.11}$$

where $\tau = it$ and $0 < \tau < \beta$. Here τ is periodic with period β. This comes from the fact that the Euclidean version of the metric is basically the space constructed by the product of two two-spheres $\mathfrak{S}^2 \times \mathfrak{S}^2$. Hence in order to cover the whole space, we have to treat τ as periodic. The other coordinates range as $0 \leq \varphi < 2\pi$, $0 \leq \theta \leq \pi$, $r > r_h$. Now performing the integrations in the action we are left with

$$\mathcal{I}_E = 4\pi\beta \int_{r_h}^{\infty} N(r)a(r)^2 N(r)H(r)dr + \mathcal{B}_E \; . \tag{A.12}$$

By using the Euclidean metric and the fact that $N(r)H(r) = -L$ where L stands for the Lagrangian of the theory, we obtain the Euclidean action

$$\mathcal{I}_E = 4\pi\beta \int_{r_h}^{\infty} \frac{1}{2} N \left(2a \left(a'F' + 2Fa'' \right) + a^2 \left(F \left(\phi' \right)^2 + 2V \right) + 2F \left(a' \right)^2 - 2 \right) dr + \mathcal{B}_E, \tag{A.13}$$

where several integrations by part were performed. Now, we need to vary the Euclidean action with respect to the fields N, ϕ, a, F in order to obtain the equations of motion. To do so we will cancel (at first) some boundary terms. However, in order for the Euclidean action to attain a true extremum within the class of the fields considered we have to make sure that

$$\delta\mathcal{I}_E = 0 \; . \tag{A.14}$$

The role of the boundary term $\mathcal{B}_E$ is to make sure that the variation of the Euclidean action indeed will vanish. We now begin the variations, starting from N. This is trivial and we obtain

$$2a \left(a'F' + 2Fa'' \right) + a^2 \left(F \left(\phi' \right)^2 + 2V \right) + 2F \left(a' \right)^2 - 2 = 0 \; , \tag{A.15}$$

which means that in the Hamiltonian formulation, the Euclidean action is the boundary term and everything is encoded there. Here primes denote derivative with respect to r. Variation with respect to ϕ yields

$$a \left(a \left(N \left(-F'\phi' - F\phi'' + \frac{V'(r)}{\phi'} \right) - F\phi' N'(r) \right) - 2FNa'\phi' \right) = 0 \; , \tag{A.16}$$

and i cancelled the boundary term

$$\frac{d}{dr} \left(a^2 N F \phi' \delta\phi \right) \; . \tag{A.17}$$

Variation with respect to F reads

$$2a'' + a \left(\phi' \right)^2 = 0 \; , \tag{A.18}$$

and i cancelled the boundary term

$$\frac{d}{dr}\left(aNa'\delta F\right) \ . \tag{A.19}$$

Finally variation with respect to a reads

$$2a'\left(NF'+FN'\right)+2FNa''+a\left(3F'N'+N\left(F''(r)+F\left(\phi'\right)^2+2V\right)+2FN''\right)=0\ , \tag{A.20}$$

and here several boundary terms have been cancelled

$$\frac{d}{dr}\left(FN2a'\delta a\right)\ , \tag{A.21}$$

$$\frac{d}{dr}\left(aNF'\delta\alpha\right)\ , \tag{A.22}$$

$$\frac{d}{dr}\left(2aFN\delta a'\right)\ , \tag{A.23}$$

$$-\frac{d}{dr}\left(\frac{d}{dr}\left(2aFN\right)\delta a\right)\ . \tag{A.24}$$

Now, the solution reported in equations (A.1)-(A.5) satisfies the obtained equations of motion. So we are left with the boundary terms. All the boundary terms together now read

$$4\pi\beta\Big(-\delta a\left(2FNa'+2aNF'+2aFN'\right)+2\delta aFNa'+a\delta FNa' \\ +a^2\delta\phi FN\phi'+a\delta aNF'+2a\delta a'FN\Big)\Big|_{r_h}^{\infty}+\delta\mathcal{B}_E=0 \tag{A.25}$$

To proceed, we need to know the variations of the fields at infinity and at the horizon. Since we already know the solution this is easy and their variation at infinity reads

$$\delta F=-\frac{\delta\nu\nu^2\chi}{r}\ , \tag{A.26}$$

$$\delta a=\delta\nu\left(\frac{1}{2}-\frac{\nu}{4r}\right)\ , \tag{A.27}$$

$$\delta\phi=\frac{\delta\nu}{\sqrt{2}r}\ . \tag{A.28}$$

At the event horizon we have that

$$F(r)|_{r=r_h}=F(r_h)+F'(r)|_{r=r_h}(r-r_h)=F'(r)|_{r=r_h}(r-r_h)=\frac{4\pi}{\beta}(r-r_h)\ , \tag{A.29}$$

$$\delta F=-\frac{4\pi}{\beta}(\delta r_h)\ , \tag{A.30}$$

$$\delta a|_{r=r_h}=\delta a(r_h)-a'(r_h)\delta r_h\ , \tag{A.31}$$

$$\delta\phi|_{r=r_h}=\delta\phi(r_h)-\phi'(r_h)\delta r_h\ , \tag{A.32}$$

where i have used the fact that in order to avoid a conical singularity at the horizon τ is periodic with period β which is related to F' via

$$\frac{1}{\beta}\equiv T=\frac{F'(r_h)}{4\pi}\ . \tag{A.33}$$

For our convenience, lets break the variation of the boundary term into two pieces, one at infinity and another one at the horizon

$$\delta\mathcal{B}_E = \delta\mathcal{B}_E(\infty) + \delta\mathcal{B}_E(r_h) \ . \tag{A.34}$$

Now, the contributions at infinity and at the horizon reads

$$-4\left(\pi\beta\delta\nu\nu^2\chi\right) + \mathcal{O}\left(\frac{1}{r}\right) + \delta\mathcal{B}_E(\infty) - (-16\pi^2\delta a a) + \delta\mathcal{B}_E(r_h) = 0 \ . \tag{A.35}$$

As a result we have

$$\delta\mathcal{B}_E(\infty) = 4\left(\pi\beta\delta\nu\nu^2\chi\right) \to \mathcal{B}_E(\infty) = 4\pi\beta\chi\frac{\nu^3}{3} \ , \tag{A.36}$$

$$\delta\mathcal{B}_E(r_h) = -16\pi^2\delta a a \to \mathcal{B}_E(r_h) = -2\pi\mathcal{A}(r_h) \ , \tag{A.37}$$

where i used the fact that we're keeping the temperature (β) fixed (Grand Canonical Ensemble) and that $\mathcal{A}(r_h) = 4\pi a(r_h)^2$. As a result, the Euclidean action now reads

$$\mathcal{I}_E = \mathcal{B}_E(\infty) + \mathcal{B}_E(r_h) = 4\pi\beta\chi\frac{\nu}{3} - 2\pi\mathcal{A}(r_h) \ . \tag{A.38}$$

However, the Euclidean action is related to the free energy $\mathcal{F}$ in the Grand Canonical Ensemble as

$$\mathcal{I}_E = \beta\mathcal{F} = \beta\mathcal{M} - \mathcal{S} \ , \tag{A.39}$$

where $\mathcal{M}, \ \mathcal{S}$ are the mass and entropy of the black hole. Therefore, by comparison we can obtain the conserved mass and entropy of the black hole as

$$\mathcal{M} = 4\pi\chi\frac{\nu^3}{3} \ , \tag{A.40}$$

$$\mathcal{S} = 2\pi\mathcal{A}(r_h) \ . \tag{A.41}$$

Note here that the parameter ν appears in the mass of the black hole and therefore, the scalar hair is secondary since the the scalar charge ν is related to the mass of the black hole. The π factors in these expressions appear due to the fact that we have set $\kappa = 8\pi G = 1 \to G = 1/8\pi$.

A.0.2 Thermodynamics of the $(2+1)$ dimensional black hole with conformal scalar hair.

Here, we will discuss the thermodynamics of the black hole presented in [19] using the Euclidean Path Integral approach. To do so, we will consider the Hamiltonian version of this action, namely

$$\mathcal{H} = \int\left(\pi^{ij}\dot{g}_{ij} + p\dot{\phi} - NH - N^i H_i\right) d^2x dt + \mathcal{B}_H \ . \tag{A.42}$$

Here $\mathcal{B}$ is a boundary term. Since this solution is static and spherically symmetric, we can consider a *reduced Hamiltonian*

$$\mathcal{H} = -\int d^2x dt N H + \mathcal{B}_H \ . \tag{A.43}$$

We will consider the following line element

$$ds^2 = -N(r)^2 b(r) dt^2 + \frac{dr^2}{b(r)} + r^2 d\theta^2 \ , \tag{A.44}$$

where the coordinates range as $t_1 < t < t_2,\ 0 < r < \infty,\ 0 \leq \theta < 2\pi$. In the Euclidean Path Integral approach, the Euclidean action is identified with the free energy $\mathcal{F}$ in the grand canonical ensemble via

$$\mathcal{I}_E = \beta \mathcal{F} \ , \tag{A.45}$$

where β is the periodicity of the Euclidean imaginary time. The Euclidean action is related to the Hamiltonian action via $\mathcal{I}_E = -i\mathcal{H}$, making the time periodic $t \to i\tau$ with period β, and performing the time and angular integrations we end up with

$$\mathcal{I}_E = 2\pi\beta \int drNH + \mathcal{B}_E \ . \tag{A.46}$$

In order for the Euclidean spacetime to be regular at the event horizon of the black hole, find that β is given by

$$\beta = \frac{4\pi}{b'(r_h)} \equiv 1/T \ , \tag{A.47}$$

and is related to the inverse black-hole temperature. Now, equation (A.46) takes the form

$$\mathcal{I}_E = -\int dr \frac{\pi\beta N\left(\zeta^2 l^2 b' + \zeta\left(l^2\left(b'\left(r\zeta'-1\right)+2b\left(\zeta'+r\zeta''\right)\right)+2r\right) - 2bl^2 r\left(\zeta'\right)^2\right)}{\zeta\kappa l^2} + \mathcal{B}_E \ . \tag{A.48}$$

Here, we have made the substitution $\zeta \equiv \kappa\phi^2/8$ in order to much the definition of the original work [19]. Primes denote derivation with respect to the radial coordinate r. Now, varying with respect to the dynamical fields N, b, ζ we obtain the equations

$$\zeta^2 l^2 b' + \zeta\left(l^2\left(b'\left(r\zeta'-1\right)+2b\left(\zeta'+r\zeta''\right)\right)+2r\right) - 2bl^2 r\left(\zeta'\right)^2 = 0 \ , \tag{A.49}$$

$$\zeta^2 N' + \zeta\left(N'\left(r\zeta'-1\right) - Nr\zeta''\right) + 2Nr\left(\zeta'\right)^2 = 0 \ , \tag{A.50}$$

$$\begin{aligned}&\zeta\left(N'\left(3\zeta r b' + 2b\left(\zeta+2r\zeta'\right)\right) + 2b\zeta r N''\right)\\ &+ N\left(\zeta^2\left(rb''+2b'\right) + 4\zeta\left(\zeta'\left(rb'+b\right)+br\zeta''\right) - 2br\left(\zeta'\right)^2\right) = 0 \ .\end{aligned} \tag{A.51}$$

These equations are solved by the solution reported in [19], and furthermore, now the Euclidean action will be given solely by the boundary term $\mathcal{I}_E = \mathcal{B}_E$. To derive the aforementioned field equations we ignored (for the moment) some boundary terms. In order to have a well defined variational procedure $\delta\mathcal{I}_E = 0$, the variation of the boundary term $\delta\mathcal{B}_E$ has to be such that it cancels these boundary terms. Consequently, the variation of the full action becomes

$$\left.\frac{\pi\beta\left(\delta\zeta\zeta N r b' + \zeta N\left(-2br\delta\zeta' + \delta b(-\zeta) + \delta b\right) + Nr\zeta'(4b\delta\zeta - \delta b\zeta) + 2b\delta\zeta\zeta r N'\right)}{\zeta\kappa}\right|_{r_h}^{\infty} + \delta\mathcal{B}_E = 0 \ . \tag{A.52}$$

For convenience we will split the variation of the boundary into two terms, one at infinity and another one at the horizon:

$$\delta\mathcal{B}_E = \delta\mathcal{B}_E(\infty) + \delta\mathcal{B}_E(r_h) \ . \tag{A.53}$$

The variations of the fields at infinity read

$$\delta\zeta = \left(\frac{1}{r} - \frac{2B}{r^2}\right)\delta B + \mathcal{O}(r^{-3}) \ , \tag{A.54}$$

$$\delta\zeta' = \left(\frac{4B}{r^3} - \frac{1}{r^2}\right)\delta B + \mathcal{O}(r^{-4}) \ , \tag{A.55}$$

$$\delta b = \left(-\frac{6B^2}{l^2 r} - \frac{6B}{l^2}\right)\delta B \ . \tag{A.56}$$

Now evaluating the boundary term at the boundary we obtain

$$-\frac{6(\pi\beta B\delta B)}{\kappa l^2} - \frac{12\left(\pi\beta B^2\delta B\right)}{r\left(\kappa l^2\right)} + \frac{12\pi\beta B^3\delta B}{\kappa l^2 r^2} + \mathcal{O}\left(\left(\frac{1}{r}\right)^3\right) + \delta\mathcal{B}(\infty) = 0 \ . \tag{A.57}$$

At the horizons the variation of the fields reads

$$\delta b|_{r_h} = -b'(r_h)\delta r_h = -\frac{4\pi}{\beta}\delta r_h \ , \tag{A.58}$$

$$\delta\zeta|_{r_h} = \delta\zeta(r_h) - \zeta'(r_h)\delta r_h \ . \tag{A.59}$$

At the horizon (A.52) reads

$$\frac{8\pi^2\delta r_h}{3\kappa} + \delta\mathcal{B}_E(r_h) = 0 \ . \tag{A.60}$$

As a result, the variation of the Euclidean action finally reads

$$\delta\mathcal{I}_E = 0 \to -\frac{6(\pi\beta B\delta B)}{\kappa l^2} + \delta\mathcal{B}(\infty) + \frac{8\pi^2\delta r_h}{3\kappa} + \delta\mathcal{B}_E(r_h) = 0 \ . \tag{A.61}$$

Considering the Grand Canonical Ensemble (keeping the temperature fixed) we find that in order for the action to attain a true extremum when the field equations hold, the boundary terms read

$$\mathcal{B}_E(\infty) = \frac{3(\pi\beta B^2)}{\kappa l^2} \ , \tag{A.62}$$

$$\mathcal{B}_E(r_h) = \frac{8\pi^2 r_h}{3\kappa} \ . \tag{A.63}$$

In conclusion, since the action is solely given by the boundary terms and related to the free energy of the solution in the Grand Canonical Ensemble $\mathcal{I}_E = \beta\mathcal{F} = \beta\mathcal{M} - \mathcal{S}$, where $\mathcal{M}, \mathcal{S}$ are the mass and the entropy of the black hole we have

$$\mathcal{I}_E = \beta\mathcal{F} = \frac{3(\pi\beta B^2)}{\kappa l^2} - \frac{8\pi^2 r_h}{3\kappa} \ , \tag{A.64}$$

we can identify

$$\mathcal{M} = \frac{3(\pi B^2)}{\kappa l^2} \ , \tag{A.65}$$

$$\mathcal{S} = \frac{8\pi^2 r_h}{3\kappa} \ , \tag{A.66}$$

as the conserved mass and entropy of the black hole respectively. As a final comment, we note that the first law of thermodynamics

$$\delta\mathcal{M} = T\delta\mathcal{S} \ , \tag{A.67}$$

holds by construction.

A.0.3 Calculation of Entropy from Wald's Formula

The Lagrangian of the theory 1.174 is

$$\mathcal{L} = \frac{f(R)}{2} - \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi - \frac{1}{12}R\phi^2 - V(\phi) \ . \tag{A.68}$$

The quantities that involve parts of the Riemann tensor are the gravitational term $f(R)$ and the non-minimal coupling term $R\phi^2$. Hence, the derivative of the other terms in the Lagrangian with respect to the Riemann tensor wil be zero. Therefore

$$\frac{\partial\mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}} = \frac{\partial}{\partial R_{\alpha\beta\gamma\delta}}\left(\frac{f(R)}{2} - \frac{1}{12}R\phi^2\right) \ .$$

By applying the chain rule we can see that

$$\frac{\partial\mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}} = \frac{1}{2}f_R\frac{\partial R}{\partial R_{\alpha\beta\gamma\delta}} - \frac{1}{12}\phi^2\frac{\partial R}{\partial R_{\alpha\beta\gamma\delta}} \ ,$$

so we only have to calculate $\dfrac{\partial R}{\partial R_{\alpha\beta\gamma\delta}}$.

The Ricci scalar, in terms of the Riemann tensor is defined as

$$R = g^{\mu\kappa}g^{\nu\lambda}R_{\mu\nu\kappa\lambda} \ , \tag{A.69}$$

and by taking the derivative we have

$$\frac{\partial}{\partial R_{\alpha\beta\gamma\delta}}\left(g^{\mu\kappa}g^{\nu\lambda}R_{\mu\nu\kappa\lambda}\right) = g^{\mu\kappa}g^{\nu\lambda}\frac{\partial R_{\mu\nu\kappa\lambda}}{\partial R_{\alpha\beta\gamma\delta}} = \frac{1}{2}g^{\mu\kappa}g^{\nu\lambda}\frac{\partial}{\partial R_{\alpha\beta\gamma\delta}}\left(R_{\mu\nu\kappa\lambda} - R_{\nu\mu\kappa\lambda}\right) \ ,$$

since the Riemann tensor has to satisfy

$$R_{\mu\nu\kappa\lambda} = -R_{\nu\mu\kappa\lambda} \ . \tag{A.70}$$

We now can calculate the derivatives

$$\frac{1}{2}g^{\mu\kappa}g^{\nu\lambda}\frac{\partial}{\partial R_{\alpha\beta\gamma\delta}}\left(R_{\mu\nu\kappa\lambda} - R_{\nu\mu\kappa\lambda}\right) = \frac{1}{2}g^{\mu\kappa}g^{\nu\lambda}\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\delta^{\gamma}_{\kappa}\delta^{\delta}_{\lambda} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}\delta^{\gamma}_{\kappa}\delta^{\delta}_{\lambda}\right) = \frac{1}{2}\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\beta\gamma}g^{\alpha\delta}\right) \ .$$

In total, the derivative of the Lagrangian with respect to the Riemann tensor multiplied with the binormals wil now read

$$\frac{\partial\mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}}\hat{\varepsilon}_{\alpha\beta}\hat{\varepsilon}_{\gamma\delta} = \frac{1}{2}\left(\frac{f_R}{2} - \frac{1}{12}\phi^2\right)\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\beta\gamma}g^{\alpha\delta}\right)\hat{\varepsilon}_{\alpha\beta}\hat{\varepsilon}_{\gamma\delta} = -\left(f_R - \frac{1}{6}\phi^2\right) \tag{A.71}$$

Now the full expression for the entropy will be

$$S(r_h) = \frac{1}{4}\left(f_R(r_h) - \frac{1}{6}\phi(r_h)^2\right)\sqrt{r_h^4}\int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\varphi = \frac{4\pi r_h^2}{4}\left(f_R(r_h) - \frac{1}{6}\phi(r_h)^2\right) = \frac{\mathcal{A}}{4}\left(f_R(r_h) - \frac{1}{6}\phi(r_h)^2\right) \ . \tag{A.72}$$

A.0.4 Magnetically Charged Black Holes

Let's discuss magnetically charged black holes. Under the assumption $A_\mu = Q_m \cos\theta d\varphi$, the only non-zero components of the Faraday tensor are $F_{\theta\varphi} = -F_{\varphi\theta} = -Q_m \sin\theta$. As a result the electromagnetic invariant $P \equiv F_{\mu\nu}F^{\mu\nu}$ gives

$$F_{\mu\nu}F^{\mu\nu} = g^{\omega\mu}g^{\chi\nu}F_{\mu\nu}F_{\omega\chi} = g^{\theta\theta}g^{\varphi\varphi}F_{\theta\varphi}F_{\theta\varphi} + g^{\varphi\varphi}g^{\theta\theta}F_{\varphi\theta}F_{\varphi\theta} = \frac{2Q_m^{\ 2}}{g_{\theta\theta}^4} ,$$

independent of θ, φ. Now, Maxwell's equation yields

$$\nabla^\mu F_{\mu\nu} - 2\alpha\nabla^\mu PF_{\mu\nu} - 2\alpha P\nabla^\mu F_{\mu\nu} = \nabla^\mu F_{\mu\nu} - 2\alpha\partial^\mu PF_{\mu\nu} - 2\alpha P\nabla^\mu F_{\mu\nu} = 0 ,$$

where we have changed the covariant derivative with a partial since P is a coordinate independent quantity (scalar). The second term now is

$$-2\alpha g^{\alpha\mu}\partial_\alpha PF_{\mu\nu} = -2\alpha g^{\theta\theta}\partial_\theta PF_{\theta\nu} - 2\alpha g^{\varphi\varphi}\partial_\varphi PF_{\varphi\nu} = 0 ,$$

since P is independent of θ, φ. Eventually, Maxwell's equation ends up being

$$\nabla^\mu F_{\mu\nu} - 2\alpha P\nabla^\mu F_{\mu\nu} = 0 , \tag{A.73}$$

hence, if we satisfy $\nabla^\mu F_{\mu\nu} = 0$ we can satisfy the above equation. This choice for A_μ satisfies $\nabla^\mu F_{\mu\nu} = 0$ and here's why. We have

$$\nabla^\mu F_{\mu\nu} = 0 \to g^{\alpha\mu}\left(\partial_\alpha F_{\mu\nu} - \Gamma^\kappa_{\alpha\mu}F_{\kappa\nu} - \Gamma^\lambda_{\alpha\nu}F_{\mu\lambda}\right) = 0 \to g^{\theta\theta}\left(\partial_\theta F_{\theta\nu} - \Gamma^\kappa_{\theta\theta}F_{\kappa\nu} - \Gamma^\lambda_{\theta\nu}F_{\theta\lambda}\right) = 0 ,$$

and remembering that the only non-zero component of the Faraday tensor is $F_{\theta\varphi} = -F_{\varphi\theta}$ we have

$$g^{\theta\theta}\left(\partial_\theta F_{\theta\varphi} - \Gamma^\varphi_{\theta\theta}F_{\varphi\nu} - \Gamma^\varphi_{\theta\nu}F_{\theta\varphi}\right) = 0 \to \partial_\theta F_{\theta\varphi} - \Gamma^\varphi_{\theta\theta}F_{\varphi\theta} - \Gamma^\varphi_{\theta\theta}F_{\theta\varphi} = 0 \to \partial_\theta F_{\theta\varphi} - \Gamma^\varphi_{\theta\theta}F_{\theta\varphi} = 0 \to ,$$

$$-Q_m\cos\theta - (-Q_m)\sin\theta\cot\theta = 0 \to 0 = 0 ,$$

hence $\nabla^\mu F_{\mu\nu} = 0$ for $A_\mu = (0, 0, 0, Q_m \cos\theta)$.

Just for completeness, we would like to comment that any non-linear electrodynamics theory that contains purely magnetic fields and depends on the Lorentz invariant P admits such magnetic solutions. To see this, consider

$$S = \int d^4x\sqrt{-g}\big(P - \mathcal{L}(P)\big) , \tag{A.74}$$

with $\mathcal{L}(P)$ containing non-linear terms that depend only on P. The variation with respect to A_μ yields

$$\nabla^\mu\big(F_{\mu\nu} - \mathcal{L}_P(P)F_{\mu\nu}\big) = 0 , \tag{A.75}$$

where the subscript denotes differentiation with respect to the argument. Now since $\mathcal{L}_P$ will be a function of P only, and since for magnetic cases P is a function of r only, the one-form $A_\mu = (0, 0, 0, Q_m\cos\theta)$ will satisfy the aforementioned field equation.

To convince the reader that the aforementioned ansatz will give a Coulomb-like radial magnetic field, we remind that the magnetic vector will be given by [192]

$$B_r = -\frac{1}{2}\epsilon_{irkl}F^{kl} . \tag{A.76}$$

Now this evaluates as

$$-\frac{1}{2}\epsilon_{tr\theta\varphi}F^{\theta\varphi} - \frac{1}{2}\epsilon_{tr\varphi\theta}F^{\varphi\theta} = -\frac{1}{2}\Big(\sqrt{-g}g^{\theta\theta}g^{\varphi\varphi}(\partial_\theta A_\varphi - \partial_\varphi A_\theta) - \sqrt{-g}g^{\varphi\varphi}g^{\theta\theta}(-\partial_\theta A_\varphi + \partial_\varphi A_\theta)\Big) \ .$$

Now, $A_\theta = 0$ since this will give rise to angular dependent magnetic fields. Therefore,

$$-\sqrt{-g}g^{\theta\theta}g^{\varphi\varphi}\partial_\theta A_\varphi = -\sqrt{-g}g^{\theta\theta}g^{\varphi\varphi}(-Q_m \sin\theta) \sim \frac{Q_m}{g_{\theta\theta}} \ ,$$

where we made the assumption that $g_{tt} = g^{rr}$. It is consequently clear that this ansatz will give Coulomb-like radial magnetic fields.

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