ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ

ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ
ΚΑΙ ΦΥΣΙΚΩΝ ΕΠΙΣΤΗΜΩΝ

ΣΧΟΛΗ ΜΗΧΑΝΟΛΟΓΩΝ ΜΗΧΑΝΙΚΩΝ

ΕΚΕΦΕ «ΔΗΜΟΚΡΙΤΟΣ»

ΙΝΣΤΙΤΟΥΤΟ ΝΑΝΟΕΠΙΣΤΗΜΗΣ
ΚΑΙ ΝΑΝΟΤΕΧΝΟΛΟΓΙΑΣ

ΙΝΣΤΙΤΟΥΤΟ ΠΥΡΗΝΙΚΗΣ ΚΑΙ
ΣΩΜΑΤΙΔΙΑΚΗΣ ΦΥΣΙΚΗΣ

Διατμηματικό Πρόγραμμα Μεταπτυχιακών Σπουδών
«Φυσική και Τεχνολογικές Εφαρμογές»

”ΒΑΡΥΤΙΚΑ ΚΥΜΑΤΑ ΚΑΙ Η ΑΝΙΧΝΕΥΣΗ ΤΟΥΣ”

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ
της Ιωάννου Μαρίνα

Επιβλέπων: ΓΕΩΡΓΙΟΣ ΣΑΒΒΙΔΗΣ, Καθηγητής

Αθήνα, Απρίλιος 2024

Abstract

In the present thesis, we analyse the propagation of gravitational waves in general relativity solving the dynamical equations and proving the existence of gravitational waves (GW) that are propagating with velocity of light and have two polarizations. We demonstrate that the intensity of gravitational waves is proportional to the square of the third time derivative of the quadrupole momentum of the matter and is inversely proportional to the fifth power of the light velocity. For that reason the generation of the gravitational waves requires a release of large amount of energy which can be realised during the collision and merger of massive binary systems of black holes or neutron stars. We estimated the amount of GW energy radiated from these binary systems, as well as their time of merger. It is important to note that the formulas we used for all of our calculations can only be applied for binary systems with big radii enough, so that the masses of the system can be considered as massive points. We present examples of recent experimental measurements of the intensity of gravitational waves generated by the collapse of the binary systems and describe their time evolution which consists of three stages: inspiral, merger and ring-down. In addition to the optical, radio, gamma rays and infrared astrophysics, the gravitational waves opens a new window to the investigation of the deep structure of the Universe and can provide the information about very early stages of the evolution of the Universe and of the inflation and Big Bang. We review ongoing experiments devoted to the measurements of GW and design of the future experiments allowing even better resolution of the astrophysical events in the Universe.

Contents

1 Introduction

In the present thesis, we analyse the propagation of gravitational waves in general relativity solving the dynamical equations and proving the existence of gravitational waves (GW) that are propagating with velocity of light and have two polarizations.

For the massive elementary particles for every value of s there are 2s+1 polarizations of spins, meaning that the group of symmetry is the SO(3). But for massless particles for each spin s there are only two spin polarizations -s, +s, meaning that the little symmetry group is instead SO(2). For the gravitational waves we show that they propagate with the velocity of light and that there are only two polarizations perpendicular to the direction of the propagation and therefore describe a massless particle, the graviton. Thus, the graviton, the elementary particle of gravitational interaction, is a massless particle which has two polarizations.

We also calculate the total gravitational energy radiation that the binary physical systems emit per unit time (4.1.1). Through this formula we understand the conditions of the emission of gravitational radiation. First and foremost, there must be acceleration of acceleration, since in the formula there is third time derivative of the quadrupole momentum: $D_{ij} = \int \mu(3x^i x^j - \delta^{ij} x_\kappa^2) dV$. So it is not enough for a system to accelerate in order to emit gravitational waves, as in electrodynamics, but its acceleration has to accelerate. The second remarkable point is that the gravitational radiation is proportional to $1/c^5$, in contrast to the electromagnetic radiation which is proportional to $1/c^3$, which explains why it is so difficult to generate gravitational waves and why they are much weaker and hard to detect than the electromagnetic waves.

So the generation of the gravitational waves requires a release of large amount of energy which can be realised during the collision and merger of massive binary systems of black holes or neutron stars. We estimate the amount of GW energy radiated from some examples of binary systems, for which there are recent experimental measurements, as well as their merger time and we describe their time evolution which consists of three stages: inspiral, merger and ring-down. It is important to note that the formulas we prove and use for our calculations can be applied only for big distances of the two objects of a binary system, so that they can be considered as massive points. Meaning, the time of the merger we calculate is the time the binary systems need to merge while they inspiral.

Through these examples and the corresponding calculations we do, we realise that the phase of inspiral lasts thousand of years radiating less energy, but once it comes to the merger and ring-down everything becomes more tense. These phases last only some fractions of seconds, the frequency increases and they radiate huge amount of energy. Thus, we are enabled to detect GW. Moreover, the more massive the system is, the more violent the collision and bigger the emission of radiation is. So this is why in order to detect gravitational waves (GW), we need events of significantly large binary systems merging. But since these events last only for fractions of a second, we need such detectors so that they can detect with accuracy such tricky signal. This is why a lot of effort and research have taken and still taking place throughout the years on GW detectors. We review such ongoing experiments devoted to the measurements of GW and design of the future experiments allowing even better resolution of the astrophysical events in the Universe.
In addition to the optical, radio, gamma rays and infrared astrophysics, the gravitational waves opens a new window to the investigation of the deep structure of the Universe and can provide the information about very early stages of the evolution of the Universe, the inflation and the Big Bang. Therefore, the bet for the years to come, is to construct the most efficient detectors in order to improve the studies on GW and retrieve the answers we expect we can get about the beginning of the universe.

2 Gravitational Waves

2.1 Weak Field Approximation

We consider a weak gravitational field, whose metric can be written as sum of Minkowski metric for flat space with a small perturbation $h_{\mu\nu}$ [1]:

$$\boxed{g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} \quad , \quad |h_{\mu\nu}| << 1} \quad , \quad g^{(0)}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.1.1}$$

As long as for the inverse metric $g^{\mu\nu}$ it will be:

$$g_{\mu\nu} g^{\nu\lambda} = \delta^{\lambda}_{\mu}$$

$$g^{(0)\mu\rho}(g^{(0)}_{\mu\nu} + h_{\mu\nu}) g^{\nu\lambda} = \delta^{\lambda}_{\mu} g^{(0)\mu\rho}$$

$$(\delta^{\rho}_{\nu} + h^{\rho}_{\nu}) g^{\nu\lambda} = g^{(0)\lambda\rho}$$

$$g^{\lambda\rho} + h^{\rho}_{\nu} g^{\nu\lambda} = g^{(0)\lambda\rho}$$

$$g^{\lambda\rho} = g^{(0)\lambda\rho} - h^{\rho}_{\nu} g^{\nu\lambda} \tag{2.1.2}$$

The first approximation is:

$$g^{(1)\lambda\rho} = g^{(0)\lambda\rho} - h^{\rho}_{\nu} g^{(0)\nu\lambda}$$

$$\boxed{g^{(1)\lambda\rho} = g^{(0)\lambda\rho} - h^{\lambda\rho}} \tag{2.1.3}$$

While replacing the formula (2.1.3) into the formula (2.1.2) we can take the second order approximation which is:

$$g^{(2)\lambda\rho} = g^{(0)\lambda\rho} - h^{\rho}_{\nu} g^{(1)\nu\lambda}$$

$$g^{(2)\lambda\rho} = g^{(0)\lambda\rho} - h^{\rho}_{\nu}(g^{(0)\lambda\nu} - h^{\lambda\nu})$$

$$g^{(2)\lambda\rho} = g^{(0)\lambda\rho} - h^{\lambda\rho} + h^{\rho}_{\nu} h^{\lambda\nu} \tag{2.1.4}$$

Then we calculate the determinant of the metric [2]:

$$ln\, detg_{\mu\nu} = Tr\, lng_{\mu\nu} = Tr\, ln(g^{(0)}_{\mu\nu} + h\mu\nu) = Tr\, lng^{(0)}_{\mu\nu}(\delta^{\nu}_{\lambda} + g^{(0)\nu\rho}h_{\rho\lambda}) =$$

$$= Tr\, lng^{(0)}_{\mu\nu} + Tr\, ln(\delta^{\nu}_{\lambda} + g^{(0)\nu\rho}h_{\rho\lambda}) = ln\, detg^{(0)}_{\mu\nu} + Tr\, ln(\delta^{\nu}_{\lambda} + h^{\nu}_{\lambda})$$

$$e^{ln\, detg_{\mu\nu}} = e^{ln\, detg^{(0)}_{\mu\nu} + Tr\, ln(\delta^{\nu}_{\lambda} + h^{\nu}_{\lambda})}, \quad detg_{\mu\nu} = detg^{(0)}_{\mu\nu}e^{Tr\, ln(\delta^{\nu}_{\lambda} + h^{\nu}_{\lambda})}$$

$$g = g^{(0)}(1 + Tr\, ln(\delta^{\nu}_{\lambda} + h^{\nu}_{\lambda}) + \frac{1}{2}[Tr\, ln(\delta^{\nu}_{\lambda} + h^{\nu}_{\lambda})]^2 + ...)$$

$$g = g^{(0)}(1 + h - \frac{1}{2}h^{\mu}_{\nu}h^{\nu}_{\mu} + \frac{1}{2}h^2 + ...) \tag{2.1.5}$$

where:

$$g = detg_{\mu\nu}, \quad g^{(0)} = detg^{(0)}_{\mu\nu}$$

and we used Taylor series for the exponential function:

$$e^x = 1 + x + \frac{1}{2}x^2 + ...$$

and the logarithmic:

$$Tr\, ln(1 + \hat{A}) = Tr(\hat{A} - \frac{1}{2}\hat{A}^2 + ...) = Tr\hat{A} - \frac{1}{2}Tr\hat{A}^2 + ...$$

Then we calculate for this metric the Christoffel symbols, the Riemann tensor and the Ricci tensor, neglecting powers of $h_{\mu\nu}$ higher than the first as significantly small terms, since $|h_{\mu\nu}| << 1$. We begin with the connection coefficients:

$$\left.\begin{array}{l} \Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\alpha}(\partial_{\nu}g_{\alpha\mu} + \partial_{\mu}g_{\alpha\nu} - \partial_{\alpha}g_{\mu\nu}) \\ \partial_{\rho}g_{\mu\nu} = \partial_{\rho}(g^{(0)}_{\mu\nu} + h_{\mu\nu}) = \partial_{\rho}g^{(0)}_{\mu\nu} + \partial_{\rho}h_{\mu\nu} = \partial_{\rho}h_{\mu\nu}, \quad since \quad g^{(0)}_{\mu\nu} = const. \end{array}\right\} \Rightarrow$$

$$\Rightarrow \Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}(g^{(0)\sigma\alpha} - h^{\sigma\alpha})(\partial_{\nu}h_{\alpha\mu} + \partial_{\mu}h_{\alpha\nu} - \partial_{\alpha}h_{\mu\nu})$$

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{(0)\sigma\alpha}(\partial_\nu h_{\alpha\mu} + \partial_\mu h_{\alpha\nu} - \partial_\alpha h_{\mu\nu}) - \frac{1}{2}h^{\sigma\alpha}(\partial_\nu h_{\alpha\mu} + \partial_\mu h_{\alpha\nu} - \partial_\alpha h_{\mu\nu})$$

$$\boxed{\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{(0)\sigma\alpha}(\partial_\nu h_{\alpha\mu} + \partial_\mu h_{\alpha\nu} - \partial_\alpha h_{\mu\nu})} \qquad (2.1.6)$$

since: $h^{\sigma\alpha} \cdot \partial_\nu h_{\alpha\mu} \approx 0$. Then we calculate the Riemann tensor:

$$R^{\sigma}_{\nu\lambda\rho} = \partial_\lambda(\Gamma^{\sigma}_{\rho\nu}) - \partial_\rho(\Gamma^{\sigma}_{\lambda\nu}) + \Gamma^{\gamma}_{\rho\nu}\Gamma^{\sigma}_{\lambda\gamma} - \Gamma^{\gamma}_{\lambda\nu}\Gamma^{\sigma}_{\rho\gamma} =$$

$$= \partial_\lambda[\frac{1}{2}g^{(0)\sigma\alpha}(\partial_\nu h_{\alpha\rho} + \partial_\rho h_{\alpha\nu} - \partial_\alpha h_{\rho\nu})] - \partial_\rho[\frac{1}{2}g^{(0)\sigma\alpha}(\partial_\nu h_{\alpha\lambda} + \partial_\lambda h_{\alpha\nu} - \partial_\alpha h_{\lambda\nu})] +$$

$$+ \frac{1}{2}g^{(0)\gamma\alpha}(\partial_\nu h_{\alpha\rho} + \partial_\rho h_{\alpha\nu} - \partial_\alpha h_{\rho\nu}) \cdot \frac{1}{2}g^{(0)\sigma\alpha}(\partial_\gamma h_{\alpha\lambda} + \partial_\lambda h_{\alpha\gamma} - \partial_\alpha h_{\lambda\gamma}) -$$

$$- \frac{1}{2}g^{(0)\delta\alpha}(\partial_\nu h_{\alpha\lambda} + \partial_\lambda h_{\alpha\nu} - \partial_\alpha h_{\lambda\nu}) \cdot \frac{1}{2}g^{(0)\sigma\alpha}(\partial_\delta h_{\alpha\rho} + \partial_\rho h_{\alpha\delta} - \partial_\alpha h_{\rho\delta}) =$$

$$= \frac{1}{2}g^{(0)\sigma\alpha}(\partial_\lambda\partial_\nu h_{\alpha\rho} + \cancel{\partial_\lambda\partial_\rho h_{\alpha\nu}} - \partial_\lambda\partial_\alpha h_{\rho\nu} - \partial_\rho\partial_\nu h_{\alpha\lambda} - \cancel{\partial_\rho\partial_\lambda h_{\alpha\nu}} + \partial_\rho\partial_\alpha h_{\lambda\nu}) =$$

$$= \frac{1}{2}g^{(0)\sigma\alpha}(\partial_\lambda\partial_\nu h_{\alpha\rho} - \partial_\lambda\partial_\alpha h_{\rho\nu} - \partial_\rho\partial_\nu h_{\alpha\lambda} + \partial_\rho\partial_\alpha h_{\lambda\nu})$$

$$g^{(0)}_{\sigma\mu}R^{\sigma}_{\nu\lambda\rho} = \frac{1}{2}g^{(0)}_{\sigma\mu}g^{(0)\sigma\alpha}(\partial_\lambda\partial_\nu h_{\alpha\rho} - \partial_\lambda\partial_\alpha h_{\rho\nu} - \partial_\rho\partial_\nu h_{\alpha\lambda} + \partial_\rho\partial_\alpha h_{\lambda\nu})$$

$$R_{\mu\nu\lambda\rho} = \frac{1}{2}\delta^{\alpha}_{\mu}(\partial_\lambda\partial_\nu h_{\alpha\rho} - \partial_\lambda\partial_\alpha h_{\rho\nu} - \partial_\rho\partial_\nu h_{\alpha\lambda} + \partial_\rho\partial_\alpha h_{\lambda\nu})$$

$$\boxed{R_{\mu\nu\lambda\rho} = \frac{1}{2}(\partial_\lambda\partial_\nu h_{\mu\rho} + \partial_\rho\partial_\mu h_{\lambda\nu} - \partial_\lambda\partial_\mu h_{\rho\nu} - \partial_\rho\partial_\nu h_{\mu\lambda})} \qquad (2.1.7)$$

where after the first row of calculations we eliminated the terms $\Gamma^{\gamma}_{\rho\nu}\Gamma^{\sigma}_{\lambda\gamma}$, $\Gamma^{\gamma}_{\lambda\nu}\Gamma^{\sigma}_{\rho\gamma}$ as they are of second order of h. Finally, we calculate the Ricci tensor for this metric (2.1.1):

$$g^{(0)\mu\lambda}R_{\mu\nu\lambda\rho} = \frac{1}{2}g^{(0)\mu\lambda}(\partial_\lambda\partial_\nu h_{\mu\rho} + \partial_\rho\partial_\mu h_{\lambda\nu} - \partial_\lambda\partial_\mu h_{\rho\nu} - \partial_\rho\partial_\nu h_{\mu\lambda})$$

$$R_{\nu\rho} = \frac{1}{2}(\partial_\lambda \partial_\nu h^\lambda_\rho + \partial_\rho \partial_\mu h^\mu_\nu - \partial_\lambda \partial^\lambda h_{\rho\nu} - \partial_\rho \partial_\nu h^\lambda_\lambda) \Rightarrow$$

$$\boxed{R_{\nu\rho} = \frac{1}{2}(\partial_\lambda \partial_\nu h^\lambda_\rho + \partial_\rho \partial_\mu h^\mu_\nu - \partial_\rho \partial_\nu h + \Box h_{\rho\nu})} \tag{2.1.8}$$

where we used the d' Alembertian operator: $\Box = -\partial_\lambda \partial^\lambda$ and the trace of $h_{\mu\nu} : h^\lambda_\lambda = h$.

2.2 Gauge transformations and Killing equation

The metric above (2.1.1) is not unique, there are several perturbations in agreement with the weak field approximation. So we can take a gauge transformation, such as that the curvature, hence the physical spacetime, remain unchanged and gain more restrictions about the admissible perturbations. Hence we take an infinitesimal change in the coordinates [1]:

$$\boxed{x^{\mu'} = x^\mu + \xi^\mu(x), \quad where \quad \xi^\mu(x) << 1}.$$

In these coordinates, the new metric is [2]:

$$g^{'\mu\nu}(x^{'}) = g^{\lambda\rho}(x)\frac{\partial x^{'\mu}}{\partial x^\lambda}\frac{\partial x^{'\nu}}{\partial x^\rho}, \quad g^{'\mu\nu}(x+\xi) = g^{\lambda\rho}(x)(\delta^\mu_\lambda + \frac{\partial \xi^\mu}{\partial x^\lambda})(\delta^\nu_\rho + \frac{\partial \xi^\nu}{\partial x^\rho})$$

$$g^{'\mu\nu}(x) + \xi^\lambda(x)\frac{\partial g^{'\mu\nu}(x)}{\partial x^\lambda} + .. = g^{\mu\nu}(x) + g^{\mu\rho}(x)\frac{\partial \xi^\nu}{\partial x^\rho} + g^{\lambda\nu}(x)\frac{\partial \xi^\mu}{\partial x^\lambda} + g^{\lambda\rho}(x)\frac{\partial \xi^\mu}{\partial x^\lambda}\frac{\partial \xi^\nu}{\partial x^\rho}$$

$$g^{'\mu\nu}(x) = g^{\mu\nu}(x) - \xi^\lambda(x)\frac{\partial g^{'\mu\nu}(x)}{\partial x^\lambda} + g^{\mu\rho}(x)\frac{\partial \xi^\nu}{\partial x^\rho} + g^{\lambda\nu}(x)\frac{\partial \xi^\mu}{\partial x^\lambda}$$

as a first approximation, so:

$$g^{(1)'\mu\nu}(x) = g^{\mu\nu}(x) - \xi^\lambda(x)\frac{\partial g^{\mu\nu}(x)}{\partial x^\lambda} + \xi^\sigma(x)\frac{\partial \xi^\lambda(x)}{\partial x^\sigma}\frac{\partial g^{'\mu\nu}(x)}{\partial x^\lambda} + \xi^\lambda(x)\xi^\sigma(x)\frac{\partial^2 g^{'\mu\nu}(x)}{\partial x^\sigma \partial x^\lambda} -$$

$$-\xi^\lambda(x)\frac{\partial g^{\mu\rho}(x)}{\partial x^\lambda}\frac{\partial \xi^\nu}{\partial x^\rho} - \xi^\lambda(x)g^{\mu\rho}(x)\frac{\partial^2 \xi^\nu}{\partial x^\lambda \partial x^\rho} - \xi^\sigma(x)\frac{\partial g^{\lambda\nu}(x)}{\partial x^\sigma}\frac{\partial \xi^\mu}{\partial x^\lambda} - \xi^\sigma(x)g^{\lambda\nu}(x)\frac{\partial^2 \xi^\mu}{\partial x^\sigma \partial x^\lambda} +$$

$$+g^{\mu\rho}(x)\frac{\partial\xi^{\nu}}{\partial x^{\rho}}+g^{\lambda\nu}(x)\frac{\partial\xi^{\mu}}{\partial x^{\lambda}}$$

$$g^{(1)'\mu\nu}(x)=g^{\mu\nu}(x)-\xi^{\lambda}(x)\frac{\partial g^{\mu\nu}(x)}{\partial x^{\lambda}}+g^{\mu\rho}(x)\frac{\partial\xi^{\nu}}{\partial x^{\rho}}+g^{\lambda\nu}(x)\frac{\partial\xi^{\mu}}{\partial x^{\lambda}} \qquad (2.2.1)$$

Doing some calculations we will show that the three last terms can be represented as covariant derivatives and the metric can be written in the form:

$$\boxed{g^{'\mu\nu}(x)=g^{\mu\nu}(x)+\xi^{\mu;\nu}+\xi^{\nu;\mu}} \qquad (2.2.2)$$

So let us have the analytical calculations:

$$\xi^{\mu;\nu}+\xi^{\nu;\mu}=g^{\nu\rho}\xi^{\mu}_{;\rho}+g^{\mu\rho}\xi^{\nu}_{;\rho}=g^{\nu\rho}(\frac{\partial\xi^{\mu}}{\partial x^{\rho}}+\Gamma^{\mu}_{\lambda\rho}\xi^{\lambda})+g^{\mu\rho}(\frac{\partial\xi^{\nu}}{\partial x^{\rho}}+\Gamma^{\nu}_{\lambda\rho}\xi^{\lambda})=$$

$$=g^{\nu\rho}\frac{\partial\xi^{\mu}}{\partial x^{\rho}}+g^{\mu\rho}\frac{\partial\xi^{\nu}}{\partial x^{\rho}}+A$$

where, substituting the Christoffel symbols: $\Gamma^{\mu}_{\lambda\rho}=g^{\mu\sigma}(\frac{\partial g_{\sigma\lambda}}{\partial x^{\rho}}+\frac{\partial g_{\sigma\rho}}{\partial x^{\lambda}}-\frac{\partial g_{\rho\lambda}}{\partial x^{\sigma}})$, we set as A the expression:

$$A=\frac{1}{2}g^{\nu\rho}g^{\mu\sigma}(\frac{\partial g_{\sigma\lambda}}{\partial x^{\rho}}+\frac{\partial g_{\sigma\rho}}{\partial x^{\lambda}}-\frac{\partial g_{\rho\lambda}}{\partial x^{\sigma}})\xi^{\lambda}+\frac{1}{2}g^{\mu\rho}g^{\nu\sigma}(\frac{\partial g_{\sigma\lambda}}{\partial x^{\rho}}+\frac{\partial g_{\sigma\rho}}{\partial x^{\lambda}}-\frac{\partial g_{\rho\lambda}}{\partial x^{\sigma}})\xi^{\lambda}=$$

$$=\cancel{\frac{1}{2}g^{\nu\rho}g^{\mu\sigma}\frac{\partial g_{\sigma\lambda}}{\partial x^{\rho}}}\xi^{\lambda}+\frac{1}{2}g^{\nu\rho}g^{\mu\sigma}\frac{\partial g_{\sigma\rho}}{\partial x^{\lambda}}\xi^{\lambda}-\cancel{\frac{1}{2}g^{\nu\rho}g^{\mu\sigma}\frac{\partial g_{\rho\lambda}}{\partial x^{\sigma}}\xi^{\lambda}}+\cancel{\frac{1}{2}g^{\mu\sigma}g^{\nu\rho}\frac{\partial g_{\rho\lambda}}{\partial x^{\sigma}}\xi^{\lambda}}+$$

$$+\frac{1}{2}g^{\mu\sigma}g^{\nu\rho}\frac{\partial g_{\sigma\rho}}{\partial x^{\lambda}}\xi^{\lambda}-\cancel{\frac{1}{2}g^{\mu\sigma}g^{\nu\rho}\frac{\partial g_{\sigma\lambda}}{\partial x^{\rho}}}\xi^{\lambda}=g^{\nu\rho}g^{\mu\sigma}\frac{\partial g_{\sigma\rho}}{\partial x^{\lambda}}\xi^{\lambda}=$$

$$=g^{\nu\rho}\frac{\partial}{\partial x^{\lambda}}(g^{\mu\sigma}g_{\sigma\rho})\xi^{\lambda}-g^{\nu\rho}\frac{\partial g^{\mu\sigma}}{\partial x^{\lambda}}g_{\sigma\rho}\xi^{\lambda}=g^{\nu\rho}\frac{\partial}{\partial x^{\lambda}}(\delta^{\mu}_{\rho})\xi^{\lambda}-g^{\nu\rho}\frac{\partial g^{\mu\sigma}}{\partial x^{\lambda}}g_{\sigma\rho}\xi^{\lambda}=$$

$$=-g^{\nu\rho}g_{\sigma\rho}\frac{\partial g^{\mu\sigma}}{\partial x^{\lambda}}\xi^{\lambda}=-\delta^{\nu}_{\sigma}\frac{\partial g^{\mu\sigma}}{\partial x^{\lambda}}\xi^{\lambda}=-\frac{\partial g^{\mu\nu}}{\partial x^{\lambda}}\xi^{\lambda}$$

Therefore, we have shown that the three last terms in formula (2.2.1) can be represented as covariant derivatives:

$$\xi^{\mu;\nu}+\xi^{\nu;\mu}=g^{\nu\rho}\frac{\partial\xi^{\mu}}{\partial x^{\rho}}+g^{\mu\rho}\frac{\partial\xi^{\nu}}{\partial x^{\rho}}-\frac{\partial g^{\mu\nu}}{\partial x^{\lambda}}\xi^{\lambda}$$

In order for the metric to be invariant under this transformation, from the formula (2.2.2) we get that the sum of the covariant derivatives have to be zero:

$$\boxed{\xi^{\mu;\nu} + \xi^{\nu;\mu} = 0} \tag{2.2.3}$$

The above is called "killing equation" and it determines the $\xi^\mu(x)$ for which the metric doesn't change.
The inverse metric will be:

$$\boxed{g'_{\mu\nu}(x) = g_{\mu\nu}(x) - \xi_{\mu;\nu} - \xi_{\nu;\mu}} \tag{2.2.4}$$

since: $g'_{\mu\lambda}(x) g'^{\lambda\nu}(x) = \delta^\nu_\mu$, and $g'^{\nu\lambda}(x)$:

$$g'^{\nu\lambda}(x) = g^{\nu\lambda}(x) + \xi^{\nu;\lambda} + \xi^{\lambda;\nu}$$

is of the form: $g'^{\nu\lambda}(x) = A^{\nu\lambda} + B^{\nu\lambda}$, where $A^{\nu\lambda} = g^{\nu\lambda}(x)$, $B^{\nu\lambda} = \xi^{\nu;\lambda} + \xi^{\lambda;\nu}$ and $B^{\nu\lambda} << 1$, so $g'_{\mu\lambda}(x)$ should be of the form: $g'_{\mu\lambda}(x) = A_{\mu\lambda} - B_{\mu\lambda}$, so as that: $g'_{\mu\lambda}(x) g'^{\lambda\nu}(x) = (A_{\mu\lambda} + B_{\mu\lambda}) \cdot (A^{\nu\lambda} - B^{\nu\lambda}) = A^2 - B^2 = A^2 = g^{\nu\lambda}(x) g_{\lambda\mu}(x) = \delta^\nu_\mu$.

As for the change in perturbation $h^{\mu\nu}$, we use the formula 2.1.3 for $g^{\mu\nu}$ and the same formula adjusted in the new coordinates:

$$g'^{\mu\nu} = g^{(0)\mu\nu} - h'^{\mu\nu} \tag{2.2.5}$$

where $g^{(0)\mu\nu}$ is constant so it doesn't change, for $g'^{\mu\nu}$ and we substitute them into the formula 2.2.1:

$$g'^{\mu\nu}(x) = g^{\mu\nu}(x) - \xi^\lambda(x) \frac{\partial g^{\mu\nu}(x)}{\partial x^\lambda} + g^{\mu\rho}(x) \frac{\partial \xi^\nu}{\partial x^\rho} + g^{\lambda\nu}(x) \frac{\partial \xi^\mu}{\partial x^\lambda}$$

$$\cancel{g^{(0)\mu\nu}} - h'^{\mu\nu} = \cancel{g^{(0)\mu\nu}} - h^{\mu\nu} - \xi^\lambda(x) \frac{\partial}{\partial x^\lambda}(g^{(0)\mu\nu} - h^{\mu\nu}) +$$

$$+(g^{(0)\mu\rho} - h^{\mu\rho}) \frac{\partial \xi^\nu}{\partial x^\rho} + (g^{(0)\lambda\nu} - h^{\lambda\nu}) \frac{\partial \xi^\mu}{\partial x^\lambda}$$

$$h'^{\mu\nu} = h^{\mu\nu} + \xi^\lambda(x) \frac{\partial g^{(0)\mu\nu}}{\partial x^\lambda} - \xi^\lambda(x) \frac{\partial h^{\mu\nu}}{\partial x^\lambda} - g^{(0)\mu\rho} \frac{\partial \xi^\nu}{\partial x^\rho} + h^{\mu\rho} \frac{\partial \xi^\nu}{\partial x^\rho} - g^{(0)\lambda\nu} \frac{\partial \xi^\mu}{\partial x^\lambda} + h^{\lambda\nu} \frac{\partial \xi^\mu}{\partial x^\lambda}$$

$$\boxed{h'^{\mu\nu} = h^{\mu\nu} - \frac{\partial \xi^\nu}{\partial x_\mu} - \frac{\partial \xi^\mu}{\partial x_\nu}} \tag{2.2.6}$$

So we got the perturbation of the metric under infinitesimal coordinate transformation.

2.3 Gauge fixing condition: Harmonic gauge

Since we have a system which is invariant under a gauge transformation we are going to fix the gauge through a fix gauge condition. Specifically, we will take the harmonic gauge condition $\Box x^{\mu} = 0$, which through the weak field approximation becomes (proof in the appentix A) [1]:

$$\frac{\partial \Psi^{\mu}_{\nu}}{\partial x^{\mu}} = 0, \quad where \quad \Psi^{\mu}_{\nu} = h^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} h, \quad \nu = 0,1,2,3 \tag{2.3.1}$$

which are actually four conditions (ν=0,1,2,3) and do not violate the initial condition on $h_{\mu\nu}$ being small, while:
$\Psi^{\mu}_{\mu} = h^{\mu}_{\mu} - \frac{1}{2}\delta^{\mu}_{\mu} h = h - \frac{1}{2}\cdot 4h = h - 2h = -h.$
Calculating the second derivative of $\Psi_{\mu\nu}$ [2]:

$$\frac{\partial \Psi^{\mu}_{\nu}}{\partial x^{\mu}} = \frac{\partial h^{\mu}_{\nu}}{\partial x^{\mu}} - \frac{1}{2}\delta^{\mu}_{\nu}\frac{\partial h}{\partial x^{\mu}} = 0$$

$$\frac{\partial^2 \Psi^{\mu}_{\nu}}{\partial x^{\rho}\partial x^{\mu}} = \frac{\partial^2 h^{\mu}_{\nu}}{\partial x^{\rho}\partial x^{\mu}} - \frac{1}{2}\frac{\partial^2 h}{\partial x^{\rho}\partial x^{\nu}} = 0 \tag{2.3.2}$$

and reversing the indices $\nu \leftrightarrow \rho$ we get:

$$\frac{\partial^2 h^{\mu}_{\rho}}{\partial x^{\nu}\partial x^{\mu}} - \frac{1}{2}\frac{\partial^2 h}{\partial x^{\rho}\partial x^{\nu}} = 0, \tag{2.3.3}$$

and if we sum these two formulas (2.3.2) and (2.3.3) we see that:

$$\frac{\partial^2 h^{\mu}_{\nu}}{\partial x^{\rho}\partial x^{\mu}} + \frac{\partial^2 h^{\mu}_{\rho}}{\partial x^{\nu}\partial x^{\mu}} - \frac{\partial^2 h}{\partial x^{\rho}\partial x^{\nu}} = 0 \tag{2.3.4}$$

So, eventually, through the fix gauge condition which leads to the formula (2.3.4), the Ricci tensor (2.1.8) is simplified to the following formula:

$$\boxed{R_{\nu\rho} = \frac{1}{2}\Box h_{\rho\nu}} \tag{2.3.5}$$

Finally, as the perturbation $h_{\mu\nu}$ changed through the previous gauge transformation (2.2.6), we need to examine what new restrictions appear with this new condition. So, we take the gauge fixing condition in the new coordinates of the gauge transformation. First we calculate h'^{μ}_{ν} and h' that will be needed:

$$(2.2.6) \Rightarrow g^{(0)}_{\mu\rho} h'^{\mu\nu} = g^{(0)}_{\mu\rho} h^{\mu\nu} - g^{(0)}_{\mu\rho} \frac{\partial \xi^{\nu}}{\partial x_{\mu}} - g^{(0)}_{\mu\rho} \frac{\partial \xi^{\mu}}{\partial x_{\nu}}$$

$$h'^{\nu}_{\rho} = h^{\nu}_{\rho} - \frac{\partial \xi^{\nu}}{\partial x^{\rho}} - \frac{\partial \xi_{\rho}}{\partial x_{\nu}}$$

$$\boxed{h'^{\mu}_{\nu} = h^{\mu}_{\nu} - \frac{\partial \xi^{\mu}}{\partial x^{\nu}} - \frac{\partial \xi_{\nu}}{\partial x_{\mu}}} \tag{2.3.6}$$

where in the final step we changed the indices: $\nu \to \mu$, $\rho \to \nu$. Furthermore:

$$(1.3.6) \Rightarrow h'^{\mu}_{\mu} = h^{\mu}_{\mu} - \frac{\partial \xi^{\mu}}{\partial x^{\mu}} - \frac{\partial \xi_{\mu}}{\partial x_{\mu}}$$

$$\boxed{h' = h - 2\frac{\partial \xi^{\mu}}{\partial x^{\mu}}} \tag{2.3.7}$$

So now the gauge fixing condition in the transformed coordinates will be:

$$\frac{\partial \Psi'^{\mu}_{\nu}}{\partial x^{\mu}} = \frac{\partial h'^{\mu}_{\nu}}{\partial x^{\mu}} - \frac{1}{2}\delta^{\mu}_{\nu}\frac{\partial h'}{\partial x^{\mu}} =$$

$$= \frac{\partial h^{\mu}_{\nu}}{\partial x^{\mu}} - \frac{\partial}{\partial x^{\mu}}\frac{\partial \xi^{\mu}}{\partial x^{\nu}} - \frac{\partial}{\partial x^{\mu}}\frac{\partial \xi_{\nu}}{\partial x_{\mu}} - \frac{1}{2}\delta^{\mu}_{\nu}\frac{\partial h}{\partial x^{\mu}} + \frac{1}{2}\delta^{\mu}_{\nu}\frac{\partial}{\partial x^{\mu}}\frac{\partial \xi^{\mu}}{\partial x^{\mu}} + \frac{1}{2}\delta^{\mu}_{\nu}\frac{\partial}{\partial x^{\mu}}\frac{\partial \xi_{\mu}}{\partial x_{\mu}} =$$

$$= \frac{\partial h^{\mu}_{\nu}}{\partial x^{\mu}} - \frac{1}{2}\delta^{\mu}_{\nu}\frac{\partial h}{\partial x^{\mu}} - \frac{\partial^2 \xi^{\mu}}{\partial x^{\mu}\partial x^{\nu}} - \frac{\partial^2 \xi_{\nu}}{\partial x^{\mu}\partial x_{\mu}} + \frac{1}{2}\frac{\partial^2 \xi^{\mu}}{\partial x^{\nu}\partial x^{\mu}} + \frac{1}{2}\frac{\partial^2 \xi_{\mu}}{\partial x^{\nu}\partial x_{\mu}} =$$

$$= \frac{\partial \Psi^{\mu}_{\nu}}{\partial x^{\mu}} - \frac{\partial^2 \xi^{\mu}}{\partial x^{\mu}\partial x^{\nu}} - \frac{\partial^2 \xi_{\nu}}{\partial x^{\mu}\partial x_{\mu}} + \frac{1}{2}\frac{\partial^2 \xi^{\mu}}{\partial x^{\nu}\partial x^{\mu}} + \frac{1}{2}g^{(0)\beta\mu}g^{(0)}_{\mu\lambda}\frac{\partial^2 \xi_{\mu}}{\partial x^{\nu}\partial x_{\mu}} =$$

$$= -\frac{\partial^2 \xi^{\mu}}{\partial x^{\mu}\partial x^{\nu}} - \frac{\partial^2 \xi_{\nu}}{\partial x^{\mu}\partial x_{\mu}} + \frac{1}{2}\frac{\partial^2 \xi^{\mu}}{\partial x^{\nu}\partial x^{\mu}} + \frac{1}{2}\frac{\partial^2 \xi^{\beta}}{\partial x^{\nu}\partial x^{\lambda}} =$$

$$= -\frac{\partial^2 \xi^\mu}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 \xi_\nu}{\partial x^\mu \partial x_\mu} + \frac{1}{2}\frac{\partial^2 \xi^\mu}{\partial x^\nu \partial x^\mu} + \frac{1}{2}\frac{\partial^2 \xi^\beta}{\partial x^\nu \partial x^\beta} =$$

$$= -\frac{\partial^2 \xi^\mu}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 \xi_\nu}{\partial x^\mu \partial x_\mu} + \frac{1}{2}\frac{\partial^2 \xi^\mu}{\partial x^\nu \partial x^\mu} + \frac{1}{2}\frac{\partial^2 \xi^\mu}{\partial x^\nu \partial x^\mu} =$$

$$= -\cancel{\frac{\partial^2 \xi^\mu}{\partial x^\mu \partial x^\nu}} - \frac{\partial^2 \xi_\nu}{\partial x^\mu \partial x_\mu} + \cancel{\frac{\partial^2 \xi^\mu}{\partial x^\nu \partial x^\mu}} = \Box \xi_\nu \Rightarrow$$

$$\frac{\partial \Psi'^{\mu}_{\nu}}{\partial x^\mu} = \Box \xi_\nu = 0$$

$$\boxed{\Box \xi_\nu = 0} \tag{2.3.8}$$

where $\beta = \lambda$ because $g^{(0)\beta\mu} g^{(0)}_{\mu\lambda} = \delta^\beta_\lambda$. Thus, this formula along with the killing equation (2.2.3) from the transformation of the coordinates are the two restrictions for the admissible spacetimes: "$x^\mu + \xi^\mu$", in which Ricci tensor is written in the form: $R_{\nu\rho} = \frac{1}{2}\Box h_{\rho\nu}$ (2.3.5).

2.4 Solution of the gravitational wave equation and their polarizations

Now it is the proper time to take the Einstein's equations in the void, for simplicity, and substitute the Ricci tensor we found. In the void the energy-momentum tensor is equal to zero: $T_{\mu\nu} = 0$, so from the Einstein equations [1]:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi k}{c^4} T_{\mu\nu} \tag{2.4.1}$$

as we prove in the appentix B, the Ricci tensor is equal to zero too and from the (2.3.5):

$$\Box h_{\mu\nu} = 0 \tag{2.4.2}$$

and in one dimension:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) h_{\mu\nu} = 0$$

$$h_{\mu\nu} = h_{\mu\nu}(t \pm \frac{x}{c}) \tag{2.4.3}$$

Therefore, we reached the classical wave equation and so we got to the point where the existence of gravitational waves is proved. Thus, the tiny disturbance on the metric of our spacetime is a wave which propagates with the speed of light c.
Looking more into the nature of these waves, we go back in the initiative equation (2.3.1) of gauge fixing condition: $\frac{\partial \Psi^{\mu}_{\nu}}{\partial x^{\mu}} = 0$, where Ψ^{μ}_{ν} is $\Psi^{\mu}_{\nu}(t \pm \frac{x}{c})$ as well and we can keep only the wave which propagates to the right. So (2.3.1) becomes:

$$\frac{\partial \Psi^0_\nu}{\partial x^0} + \frac{\partial \Psi^1_\nu}{\partial x^1} = 0, \quad \frac{\partial \Psi^0_\nu}{\partial (ct)} + \frac{\partial \Psi^1_\nu}{\partial x^1} = 0$$

$$\frac{1}{c}\frac{\partial \Psi^0_\nu}{\partial (t - \frac{x}{c})}\frac{\partial (t - \frac{x}{c})}{\partial t} + \frac{\partial \Psi^1_\nu}{\partial (t - \frac{x}{c})}\frac{\partial (t - \frac{x}{c})}{\partial x}, \quad \frac{1}{c}\dot{\Psi}^0_\nu - \frac{1}{c}\dot{\Psi}^1_\nu = 0$$

$$\boxed{\Psi^0_\nu - \Psi^1_\nu = 0, \quad \nu = 0, 1, 2, 3} \tag{2.4.4}$$

So from equation (2.4.4) we get four conditions on Ψ^{μ}_{ν}:

$$\begin{aligned} \Psi^0_0 - \Psi^1_0 &= 0 \\ \Psi^0_1 - \Psi^1_1 &= 0 \\ \Psi^0_2 - \Psi^1_2 &= 0 \\ \Psi^0_3 - \Psi^1_3 &= 0 \end{aligned} \tag{2.4.5}$$

Then, as we discussed in the previous two units, there is a gauge transformation:

$$x^{\mu'} = x^{\mu} + \xi^{\mu}(t - \frac{x}{c})$$

which leaves our system invariant, since:

$$\frac{\partial \Psi'^{\mu}_{\nu}}{\partial x^{\mu}} = \Box \xi_\nu = 0$$

by just setting $\Box\xi_\nu = 0$. So, using the formulas (2.3.6), (2.3.7), Ψ^μ_ν in the transformed coordinates is [2]:

$$\Psi'^\mu_\nu = h'^\mu_\nu - \frac{1}{2}\delta^\mu_\nu h' = h^\mu_\nu - \frac{\partial \xi^\mu}{\partial x^\nu} - \frac{\partial \xi_\nu}{\partial x_\mu} - \frac{1}{2}\delta^\mu_\nu (h - 2\frac{\partial \xi^\rho}{\partial x^\rho}) =$$

$$= h^\mu_\nu - \frac{1}{2}\delta^\mu_\nu h - \partial_\nu \xi^\mu - \partial^\mu \xi_\nu + \delta^\mu_\nu \partial_\rho \xi^\rho$$

$$\boxed{\Psi'^\mu_\nu = \Psi^\mu_\nu - \partial_\nu \xi^\mu - \partial^\mu \xi_\nu + \delta^\mu_\nu \partial_\rho \xi^\rho} \tag{2.4.6}$$

where $\xi^\mu = \xi^\mu(t - \frac{x}{c}) : \xi^0, \xi^1, \xi^2, \xi^3$ Thus, since ξ^μ depends only in time and x direction as well, μ=0,1 and ν=0,1, so from (2.4.6) we get another four conditions:

$$\Psi'^0_1 = \Psi^0_1 - \partial_1 \xi^0 - \partial^0 \xi_1 = \Psi^0_1 + \frac{1}{c}\dot{\xi}^0 - \frac{1}{c}\dot{\xi}_1 = 0$$

$$\Psi'^0_2 = \Psi^0_2 - \partial^0 \xi_2 = \Psi^0_2 - \frac{1}{c}\dot{\xi}_2 = 0$$

$$\Psi'^0_3 = \Psi^0_3 - \partial^0 \xi_3 = \Psi^0_3 - \frac{1}{c}\dot{\xi}_3 = 0 \tag{2.4.7}$$

$$\Psi'^2_2 + \Psi'^3_3 = \Psi^2_2 + \Psi^3_3 + 2\partial_\rho \xi^\rho = \Psi^2_2 + \Psi^3_3 + 2(\partial_0 \xi^0 + \partial_1 \xi^1) =$$

$$= \Psi^2_2 + \Psi^3_3 + \frac{2}{c}(\dot{\xi}_0 - \dot{\xi}_1) = 0$$

where we can set ero all the terms above, since we can choose accordingly the terms of $\dot{\xi}$. So eventually, we get that:

$$\Psi^0_1 = \Psi^0_2 = \Psi^0_3 = \Psi^2_2 + \Psi^3_3 = 0 \tag{2.4.8}$$

Combining this condition with the previous ones (2.4.5), we have:

$$\Psi^0_0 = \Psi^0_1 = 0, \quad \Psi^0_1 = \Psi^1_1 = 0, \quad \Psi^0_2 = \Psi^1_2 = 0, \quad \Psi^0_3 = \Psi^1_3 = 0 \tag{2.4.9}$$

Finally, we note that Ψ^μ_ν is a symmetric matrix, so at the end the matrix which describes our plane gravitational wave is the following:

$$\Psi^\mu_\nu = \begin{pmatrix} \Psi^0_0 & \Psi^0_1 & \Psi^0_2 & \Psi^0_3 \\ \Psi^1_0 & \Psi^1_1 & \Psi^1_2 & \Psi^1_3 \\ \Psi^2_0 & \Psi^2_1 & \Psi^2_2 & \Psi^2_3 \\ \Psi^3_0 & \Psi^3_1 & \Psi^3_2 & \Psi^3_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Psi^2_2 & \Psi^2_3 \\ 0 & 0 & \Psi^3_2 & \Psi^3_3 \end{pmatrix} \tag{2.4.10}$$

Thus, the only non zero components of the matrix are Ψ^2_3 and $\Psi^2_2 - \Psi^3_3$, which is obvious from the condition (2.4.6) too:

$$\Psi'^2_3 = \Psi^2_3$$

$$\Psi'^2_2 - \Psi'^3_3 = \Psi^2_2 - \Psi^3_3 + \partial_\rho \xi^\rho - \partial_\rho \xi^\rho = \Psi^2_2 - \Psi^3_3$$

so there is no way we can set these terms equal to zero. Therefore, these two are the physical degrees of freedom of our system, in contrast with the initial 16 degrees of freedom of the matrix. We see that taking into account the symmetries and the invariance of the system, we manage to find the true physical degrees of freedom, which describe its physical properties. In fact, these two degrees of freedom describe the two modes of oscillating for the gravitational waves. From the conditions (2.4.8), (2.4.9) we get that Ψ^μ_ν is traceless:

$$\Psi^0_0 + \Psi^1_1 + \Psi^2_2 + \Psi^3_3 = 0, \quad \Psi^\mu_\mu = -h = 0$$

So: $\Psi^\mu_\nu = h^\mu_\nu$ and everything we have found for Ψ^μ_ν is true for h^μ_ν as well:

$$h^\mu_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^3_3 & h^2_3 & 0 \\ 0 & h^3_2 & h^2_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^3_3 & h^2_3 & 0 \\ 0 & h^2_3 & -h^3_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.4.11}$$

where since $\Psi^2_2 + \Psi^3_3 = 0$, $h^2_2 + h^3_3 = 0$ and finally $h^3_3 = -h^2_2$ and $h^2_3 = h^3_2$ out of symmetry. So, the two different solutions for the gravitational waves are:

$$h^\mu_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^3_3 & 0 & 0 \\ 0 & 0 & -h^3_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = h^3_3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \tag{2.4.12}$$

$$h^{\mu}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & h^2_3 & 0 \\ 0 & h^2_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = h^2_3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.4.13}$$

and they correspond to the two ways we were talking before, in which the gravitational waves propagate oscillating the spacetime. So there are two polarizations, the plus and the cross polarization, where in the first case if we take a y- plane, spacetime oscillates along y and -axis, while in the second case the oscillation is rotated by 45 degrees. We will show that, exactly like electromagnetic waves, gravitational waves propagate along one axis and the oscillation of spacetime takes place in the perpendicular plane to it. We write the solution h^{μ}_{ν} in the Fourier form, since they are plane waves (2.4.3):

$$h_{\mu\nu}(x) = h_{\mu\nu}(k)e^{ik_\mu x^\mu} \tag{2.4.14}$$

Substituting $h_{\mu\nu}$ in this form into the formulas (2.3.1), (2.4.2) we will have that:

$$(2.3.1) \Rightarrow \partial_\mu h^{\mu}_{\nu}(x) - \frac{1}{2}\partial_\mu h^{\rho}_{\rho}\delta^{\mu}_{\nu} = 0$$

$$ik_\mu h_{\mu\nu}(k)e^{ik_\mu x^\mu} - \frac{1}{2}ik_\mu h^{\rho}_{\rho}(k)e^{ik_\mu x^\mu}\delta^{\mu}_{\nu} = 0$$

$$\boxed{k_\mu h_{\mu\nu}(k) = 0} \tag{2.4.15}$$

where we get that the wave vector k_μ, which shows the direction of propagation, is perpendicular to the plane of the wave's oscillation. Moreover, from the formula (2.4.2):

$$\Box h^{\mu}_{\nu} = 0, \quad \partial_\mu \partial^\mu (h^{\mu}_{\nu}(k)e^{ik_\mu x^\mu}) = 0, \quad g^{(0)\mu\nu}\partial_\nu \partial_\mu (h^{\mu}_{\nu}(k)e^{ik_\mu x^\mu}) = 0$$

$$ik_\mu h^{\mu}_{\nu}(k)g^{(0)\mu\nu}\partial_\nu e^{ik_\mu x^\mu} = 0, \quad ik^\nu h^{\mu}_{\nu}(k)\cdot ik_\mu \delta^{\mu}_{\nu} e^{ik_\mu x^\mu} = 0$$

$$-k^\nu k_\nu h^{\mu}_{\nu}(k)e^{ik_\mu x^\mu} = 0, \quad k^\nu k_\nu h^{\mu}_{\nu}(x) = 0$$

$$\boxed{k^\nu k_\nu = 0} \tag{2.4.16}$$

Consequently the wavevector k_μ is null, which is translated into the fact that the gravitational waves propagates with the speed of light and the graviton is massless:

$$k_\mu = (\frac{\omega}{c}, \vec{k}), \quad \omega = 2\pi f, \quad k = \frac{2\pi}{\lambda}$$

$$k^\mu k_\mu = 0, \quad (\frac{\omega}{c}, \vec{k})(\frac{\omega}{c}, -\vec{k}) = 0, \quad (\frac{\omega}{c})^2 - k^2 = 0, \quad \frac{\omega}{k} = c, \quad \frac{2\pi f}{\frac{2\pi}{\lambda}} = c,$$

$$\lambda f = c, \quad \upsilon = c$$

In the case we took before, where the wave propagates in x direction and still setting c=1, it will be:

$$k^\mu = (k^0, k^3, k^2, k^1) = (\omega, 0, 0, \omega)$$

$$\boxed{k^\mu = \omega(1, 0, 0, 1)} \tag{2.4.17}$$

so that:

$$k_\mu k^\mu = g^{(0)}_{\mu\nu} k^\nu k^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix} \begin{pmatrix} \omega & 0 & 0 & \omega \end{pmatrix} =$$

$$= \begin{pmatrix} \omega \\ 0 \\ 0 \\ -\omega \end{pmatrix} \begin{pmatrix} \omega & 0 & 0 & \omega \end{pmatrix} = \omega^2 - \omega^2 = 0$$

We also calculate the energy flux in a gravitational wave which is given by the terms cT^{0i} of the energy-momentum tensor, using the following formula of it [2]:

$$\boxed{T^{\mu\nu} = \frac{c^4}{32\pi k} h_\rho^{\lambda,\mu} h_\lambda^{\rho,\nu}} \tag{2.4.18}$$

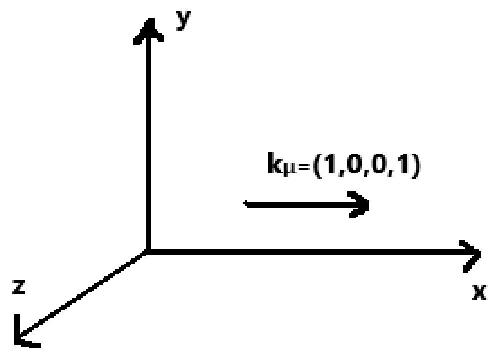

Figure 1: Propagating gravitational wave

Since we have chosen the wave to propagate through the x-axis, or equivalently the $h_{\mu\nu}$ to depend only on x,t, it is clear that the only non zero term of the energy flux will be the cT^{01} and since the only non zero components of h^{λ}_{ρ} are the $h^3_3 = -h^2_2$ and $h^2_3 = h^3_2$, there will be:

$$cT^{01} = \frac{c^5}{32\pi k} h^{\lambda,0}_{\rho} h^{\rho,1}_{\lambda} = \frac{c^5}{32\pi k}\left(\frac{\partial h^2_3}{\partial x_0}\frac{\partial h^3_2}{\partial x_1} + \frac{\partial h^3_2}{\partial x_0}\frac{\partial h^2_3}{\partial x_1} + \frac{\partial h^2_2}{\partial x_0}\frac{\partial h^2_2}{\partial x_1} + \frac{\partial h^3_3}{\partial x_0}\frac{\partial h^3_3}{\partial x_1}\right) =$$

$$= \frac{c^5}{32\pi k}\left(2\frac{\partial h^2_3}{\partial (ct)}\left(-\frac{\partial h^2_3}{\partial x}\right) + 2\frac{\partial h^2_2}{\partial (ct)}\left(-\frac{\partial h^2_2}{\partial x}\right)\right) = \frac{c^5}{16\pi k}\left(-\frac{\partial h^2_3}{\partial (t-\frac{x}{c})}\frac{\partial (t-\frac{x}{c})}{\partial (ct)}\frac{\partial h^2_3}{\partial (t-\frac{x}{c})}\frac{\partial (t-\frac{x}{c})}{\partial x} -\right.$$

$$\left. -\frac{\partial h^2_2}{\partial (t-\frac{x}{c})}\frac{\partial (t-\frac{x}{c})}{\partial (ct)}\frac{\partial h^2_2}{\partial (t-\frac{x}{c})}\frac{\partial (t-\frac{x}{c})}{\partial x}\right) = \frac{c^5}{16\pi k}\left(\frac{1}{c^2}(\dot{h}^2_3)^2 + \frac{1}{c^2}(\dot{h}^2_2)^2\right)$$

$$\boxed{cT^{01} = \frac{c^3}{16\pi k}\left((\dot{h}^2_3)^2 + \frac{1}{4}(\dot{h}^2_2 - \dot{h}^3_3)^2\right)} \qquad (2.4.19)$$

since:

$$h_{33} = -h_{22}, \quad \dot{h}_{33} = -\dot{h}_{22}, \quad -\dot{h}_{33} = \dot{h}_{22}, \quad (\dot{h}_{22} - \dot{h}_{33})^2 = (2\dot{h}_{22})^2 = 4(\dot{h}_{22})^2$$

$$(\dot{h}_{22})^2 = \frac{1}{4}(\dot{h}_{22} - \dot{h}_{33})^2.$$

Since the gravitational waves transfer energy they create around them an additional gravitational field, which is, as we can see, of second order compared to the field of the wave itself, since the energy from which is produced is of the same order. On the other hand, the action for the gravitational field is:

$$S = -\frac{c^3}{16\pi k}\int R\sqrt{-g}d^4x \tag{2.4.20}$$

where the gravitational constant is $k = 6,67 \cdot 10^{-8}\frac{cm^3}{g \cdot sec^2}$ and:

$$\frac{c^3}{16\pi k} = \frac{(3 \cdot 10^{10})^3}{16\pi \cdot 6,67 \cdot 10^{-8}}\frac{cm^3 \cdot g \cdot sec^2}{sec^3 \cdot cm^3} = 8,05 \cdot 10^{36}\frac{g}{sec}$$

Thus the dimensionality of the coupling constant is: $[\frac{c^3}{16\pi k}] = \frac{g}{sec}$ and the dimensionality of the integrant is:$[R\sqrt{g}d^4x] = \frac{1}{cm^2} * cm^4 = cm^2$, therefore the dimension of the gravitational action S is: $[S] = \frac{g}{sec} \cdot cm^2$.

3 Radiation of Gravitational Waves and their intensity

3.1 Flat Gravitational Waves and Einstein Equation

In the previous unit we ended up in the formula: $\Box h_{\mu\nu} = 0$, which proves that a gravitational field $h_{\mu\nu}$ propagates in vacuum as a wave with the speed of light. Considering some bodies moving in space and producing these gravitational waves, as we enter mass in our system we expect terms of the energy-momentum tensor to appear in the previous formula. So, using the more convenient term: $\Psi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}h$, which was defined through applying the harmonic gauge, we gradually have the following.
We have showed that:

$$R_{\mu\nu} = \frac{1}{2}g^{(0)\lambda\rho}\frac{\partial^2 h_{\mu\nu}}{\partial^\lambda \partial^\rho} = \frac{1}{2}\Box h_{\mu\nu} \quad (2.3.5)$$

Acting by operator $\Box$ on $\Psi_{\mu\nu}$ we get [2]:

$$\Box \Psi^\mu_\nu = \Box h^\mu_\nu - \frac{1}{2}\delta^\mu_\nu \Box h \tag{3.1.1}$$

Through Einstein equation, in appendix B, we have also shown that:

$$R_{\mu\nu} = \frac{8\pi k}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \quad (B.0.3)$$

$$\frac{1}{2}\Box h^\mu_\nu = \frac{8\pi k}{c^4}(T^\mu_\nu - \frac{1}{2}\delta^\mu_\nu T) \tag{3.1.2}$$

$$\frac{1}{2}\Box h = \frac{8\pi k}{c^4}(T - \frac{1}{2}\cdot 4T) = \frac{8\pi k}{c^4}(T - 2T) = -\frac{8\pi k}{c^4}T \tag{3.1.3}$$

where we substituted $R_{\mu\nu}$ from the formula (1.3.5). Combining the previous formulas (3.1.1), (3.1.2), (3.1.3), we eventually get that:

$$\frac{1}{2}\Box \Psi^\mu_\nu = \frac{8\pi k}{c^4}(T^\mu_\nu - \frac{1}{2}\delta^\mu_\nu T) - \frac{1}{2}\delta^\mu_\nu(-\frac{8\pi k}{c^4}T) + \mathrm{O}(h^2) =$$

$$= \frac{8\pi k}{c^4}T^\mu_\nu - \frac{1}{2}\cdot\frac{8\pi k}{c^4}\delta^\mu_\nu T + \frac{1}{2}\cdot\frac{8\pi k}{c^4}\delta^\mu_\nu T + \mathrm{O}(h^2)$$

$$\frac{1}{2}\Box\Psi^{\mu}_{\nu} = \frac{8\pi k}{c^4}T^{\mu}_{\nu} + \mathrm{O}(h^2) \tag{3.1.4}$$

The terms of higher order of h that we write above are originating from the corresponding terms we skipped when we calculated the Riemann and the Ricci tensors in the previous unit (2.1.7, 2.1.8), where the terms $\Gamma^{\gamma}_{\rho\nu}\Gamma^{\sigma}_{\lambda\gamma}$, $\Gamma^{\gamma}_{\lambda\nu}\Gamma^{\sigma}_{\rho\gamma}$ were eliminated as they are of second order of h. So, instead of $T_{\mu\nu}$, we use the $\tau_{\mu\nu}$, which is the energy-momentum tensor in the weak gravitational field and contains terms of $T_{\mu\nu}$ and terms of second order of h. Therefore, (3.1.4) is finally written as follows:

$$\boxed{\frac{1}{2}\Box\Psi^{\mu}_{\nu} = \frac{8\pi k}{c^4}\tau^{\mu}_{\nu}} \tag{3.1.5}$$

From this last formula, we can get that, since:

$$\frac{\partial\Psi^{\mu}_{\nu}}{\partial x^{\mu}} = 0 \quad (2.3.1)$$

will also be:

$$\boxed{\frac{\partial\tau^{\mu}_{\nu}}{\partial x^{\mu}} = 0} \tag{3.1.6}$$

For solving equation (3.1.5) we notice that is completely analogous to the corresponding one for the potential $A_{\mu}(\vec{r},t)$ of electromagnetic waves:

$$\Box A_{\mu}(\vec{r},t) = \frac{4\pi}{c}j_{\mu}(\vec{r},t)$$

the solution of which is [1]:

$$A_{\mu}(\vec{R}_0,t) = \frac{1}{c}\int \frac{1}{R}\cdot j_{\mu}(\vec{r'},t-\frac{R}{c}) \quad d\vec{r'}^{\,3} \approx \frac{1}{cR_0}\int j_{\mu}(\vec{r'},t-\frac{R_0}{c}) \quad d\vec{r'}^{\,3}$$

where $\vec{R}_0 = \vec{r'} + \vec{R}$ and if $|\vec{R}_0| \gg |\vec{r'}|$, then $|\vec{R}| = |\vec{R}_0 - \vec{r'}| \approx |\vec{R}_0|$. So, the corresponding solution for the weak gravitational field will be:

$$\Psi^{\mu}_{\nu}(\vec{r},t) \approx -\frac{4k}{R_0c^4}\int \tau^{\mu}_{\nu}(\vec{r'},t-\frac{R_0}{c}) \quad dV' \tag{3.1.7}$$

where the integration is over the volume dV' of the radiating matter which has a distribution of size L (see Fig.3) and coordinates $(\vec{r'},t-\frac{R}{c})$, with R being the distance between a random point of the matter and the observer and R_0 the position of the observer in the inertial frame or else his distance from the center of mass of the distribution (Fig.2).

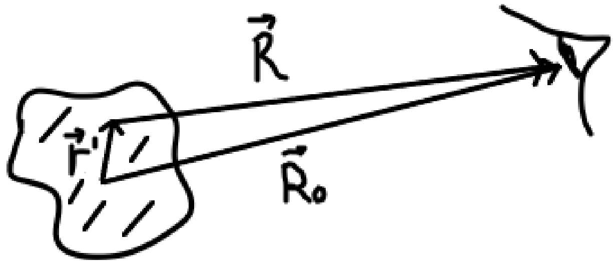

Figure 2: The geometry of the radiating matter of the gravitational waves.

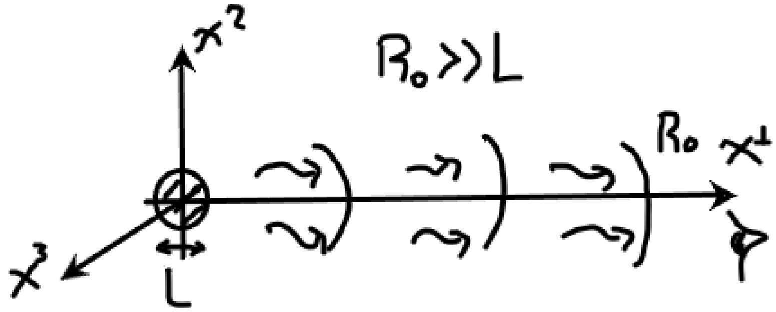

Figure 3: The radiation of the gravitational waves propagating in x^1 direction.

In order to evaluate the integral above, we use equation (3.1.6), separating the space and time components:

$$\frac{\partial \tau_{ij}}{\partial x^j} - \frac{\partial \tau_{i0}}{\partial x^0} = 0, \quad \frac{\partial \tau_{0j}}{\partial x^j} - \frac{\partial \tau_{00}}{\partial x^0} = 0 \tag{3.1.8}$$

From the first equation of (3.1.8) we get:

$$\frac{\partial}{\partial x^0}\int \tau_{i0}x^k dV = \int \frac{\partial \tau_{ij}}{\partial x^j}x^k dV = \int \frac{\partial(\tau_{ij}x^k)}{\partial x^j}dV - \int \frac{\partial x^k}{\partial x^j}\tau_{ij}dV =$$

$$= \oint_{R_0}(\tau_{ij}x^k)dS_j - \int \delta^{kj}\tau_{ij}dV = 0 - \int \tau_{ik}dV = -\int \tau_{ik}dV$$

$$\int \tau_{ik}dV = -\frac{1}{2}\frac{\partial}{\partial x^0}\int(\tau_{i0}x^k + \tau_{k0}x^i)dV \tag{3.1.9}$$

where $\oint_{R_0}(\tau_{ij}x^k)dS_j = 0$ because we consider $\tau_{ij}(R_0) = 0$. From the second equation in (3.1.8):

$$\frac{\partial}{\partial x^0}\int \tau_{00}x^i x^j dV = \int \frac{\partial \tau_{0j}}{\partial x^j}x^i x^j dV = \int \frac{\partial}{\partial x^j}(\tau_{0j}x^i x^j)dV - \int \tau_{0j}\frac{\partial x^i}{\partial x^j}x^j dV -$$

$$-\int \tau_{0j}x^i\frac{\partial x^j}{\partial x^j}dV = \oint_{R_0}(\tau_{0j}x^i x^j)dS_j - \int \tau_{0j}\delta^i_j x^j dV - \int \tau_{0j}x^i dV =$$

$$= 0 - \int \tau_{0i}x^j dV - \int \tau_{0j}x^i dV = -\int(\tau_{0j}x^j + \tau_{0j}x^i)dV, \quad so:$$

$$\frac{\partial}{\partial x^0}\int \tau_{00}x^i x^j dV = -\int(\tau_{0i}x^j + \tau_{0j}x^i)dV \tag{3.1.10}$$

From the equations (3.1.9) and (3.1.10) we finally get:

$$\boxed{\int \tau_{ik}dV = \frac{1}{2}\frac{\partial^2}{\partial x_0^2}\int \tau_{00}x^i x^j dV} \tag{3.1.11}$$

Now, τ_{00} is the energy density of our system, thus:

$$\tau_{00} = \mu c^2 = \mu(x', y', z'; t - \frac{R_0}{c})\cdot c^2 \tag{3.1.12}$$

Therefore, the weak gravitational field is:

$$\Psi^\mu_\nu = -\frac{4k}{R_0 c^4}\int \tau^\mu_\nu(t - \frac{R_0}{c}) \quad dV'$$

$$\Psi_{ij} = -\frac{4k}{R_0 c^4}\int \tau_{ij}(t-\frac{R_0}{c}) \quad dV = -\frac{4k}{R_0 c^4}\cdot\frac{1}{2}\frac{\partial^2}{\partial x_0^2}\int \tau_{00} x^i x^j dV \quad ; x_0 = ct$$

$$\Psi_{ij} = -\frac{2k}{R_0 c^4}\cdot\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int \mu c^2 x^i x^j dV = -\frac{2k}{R_0 c^4}\cdot\frac{\partial^2}{\partial t^2}\int \mu x^i x^j dV$$

$$\boxed{\Psi_{ij} = -\frac{2k}{R_0 c^4}\cdot\frac{\partial^2}{\partial t^2}\int \mu x^i x^j dV} \tag{3.1.13}$$

3.2 Energy Density Flow and the Quadrupole Momentum

It is time to introduce the quadrupole momentum tensor D_{ij} [1]:

$$\boxed{D_{ij} = \int \mu(3x^i x^j - \delta^{ij} x_\kappa^2) dV} \quad , \quad D_{ii} = 0 \tag{3.2.1}$$

We remind that in the previous unit we found the energy flow through x^1 direction to be:

$$cT^{01} = \frac{c^3}{16\pi k}(\dot{h}_{23}^2 + \frac{1}{4}(\dot{h}_{22} - \dot{h}_{33})^2) \quad (2.4.19)$$

Since for our plane wave: $h = 0$, from the formula: $\Psi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu} h$ (2.3.1) we have that:

$$\Psi_{ij} = h_{ij}, \quad i \neq j \quad and \quad \Psi_{22} - \Psi_{33} = h_{22} - h_{33}$$

which is easy to prove, as:

$$\Psi_{22} = h_{22} - \frac{1}{2}(h_{11} + h_{22} + h_{33}) = \frac{1}{2}(h_{22} - h_{11} - h_{33})$$

$$\Psi_{33} = h_{33} - \frac{1}{2}(h_{11} + h_{22} + h_{33}) = \frac{1}{2}(h_{33} - h_{11} - h_{22})$$

$$\Psi_{22} - \Psi_{33} = \frac{1}{2}(h_{22} - h_{11} - h_{33}) - \frac{1}{2}(h_{33} - h_{11} - h_{22}) =$$

$$= \frac{1}{2}(h_{22} - h_{11} - h_{33} - h_{33} + h_{11} + h_{22}) = h_{22} - h_{33}$$

So, eventually, we can write (3.1.13) as:

$$\boxed{h_{ij} = -\frac{2k}{R_0 c^4} \cdot \frac{\partial^2}{\partial t^2} \int \mu x^i x^j dV} \tag{3.2.2}$$

and now the terms h_3^2, h_2^2, h_3^3 that appear in the energy flow T^{01} (2.4.19) can be written as follows:

$$h_{23} = -\frac{2k}{R_0 c^4} \cdot \frac{\partial^2}{\partial t^2} \int \mu x^2 x^3 dV$$

$$h_{22} = -\frac{2k}{R_0 c^4} \cdot \frac{\partial^2}{\partial t^2} \int \mu x_2^2 dV$$

$$h_{33} = -\frac{2k}{R_0 c^4} \cdot \frac{\partial^2}{\partial t^2} \int \mu x_3^2 dV \quad \textit{and}$$

$$h_{22} - h_{33} = -\frac{2k}{R_0 c^4} \cdot \frac{\partial^2}{\partial t^2} \int \mu x_2^2 dV + \frac{2k}{R_0 c^4} \cdot \frac{\partial^2}{\partial t^2} \int \mu x_3^2 dV =$$

$$= -\frac{2k}{R_0 c^4} \cdot \frac{\partial^2}{\partial t^2} \int \mu (x_2^2 - x_3^2) dV$$

We now calculate the corresponding terms of D_{ij} and we compare them to the previous ones:

$$D_{23} = 3 \int \mu x^2 x^3 dV$$

$$D_{22} = \int \mu (3x_2^2 - x_1^2 - x_2^2 - x_3^2) dV = \int \mu (2x_2^2 - x_1^2 - x_3^2) dV$$

$$D_{33} = \int \mu (3x_3^2 - x_1^2 - x_2^2 - x_3^2) dV = \int \mu (2x_3^2 - x_1^2 - x_2^2) dV \quad \textit{and}$$

$$D_{22} - D_{33} = \int \mu(2x_2^2 - x_1^2 - x_3^2 - 2x_3^2 + x_1^2 + x_2^2)dV = 3\int \mu(x_2^2 - x_3^2)dV$$

So, eventually, we get that:

$$h_{23} = -\frac{2k}{3R_0c^4}\ddot{D}_{23}$$

$$h_{22} - h_{33} = -\frac{2k}{3R_0c^4}(\ddot{D}_{22} - \ddot{D}_{33})$$

Therefore, we can write the energy density flow T^{01} as a function of quadrupole momentum D_{ij} :

$$cT^{01} = \frac{c^3}{16\pi k}(\dot{h}_{23}^2 + \frac{1}{4}(\dot{h}_{22} - \dot{h}_{33})^2)$$

$$cT^{01} = \frac{c^3}{16\pi k}[(-\frac{2k}{3R_0c^4}\dddot{D}_{23})^2 + \frac{1}{4}(-\frac{2k}{3R_0c^4}(\dddot{D}_{22} - \dddot{D}_{33}))^2]$$

$$cT^{01} = \frac{c^3}{16\pi k} \cdot \frac{4k^2}{9R_0^2c^8}[\dddot{D}_{23}^2 + \frac{1}{4}(\dddot{D}_{22} - \dddot{D}_{33})^2]$$

$$\boxed{cT^{01} = \frac{k}{36\pi R_0^2c^5}[\dddot{D}_{23}^2 + \frac{1}{4}(\dddot{D}_{22} - \dddot{D}_{33})^2]} \tag{3.2.3}$$

Thus, the flow of radiation through a spherical angle $R_0^2d\Omega$ is:

$$dI = cT^{01} \cdot R_0^2d\Omega$$

$$dI = \frac{k}{36\pi c^5}[\dddot{D}_{23}^2 + \frac{1}{4}(\dddot{D}_{22} - \dddot{D}_{33})^2]d\Omega \tag{3.2.4}$$

We can notice that when a mass is spherical symmetric, so $D_{ij} = 0$, it doesn't produce gravitational radiation.

3.3 Radiation Flow and the polarizations of Flat Gravitational Waves

Speaking about the flow of radiation, we can't help bringing up the tensor polarization e_{ij} of the gravitational wave, which, as we can see from the formulas (2.4.13) and (2.4.18), is:

$$e_{ij}^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_{ij}^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.3.1}$$

with: $e_{ii} = 0$, $\quad e_{ij}^{(\lambda)} \cdot e_{ij}^{(\lambda)} = 1$ and $e_{ij} \cdot n_j = 0$, where $n_j = \frac{x_j}{|x|}$.
So, now the formula (3.2.4) takes the form:

$$dI = \frac{k}{72\pi c^5} \sum_{\lambda_1, \lambda_2} (\dddot{D}_{ij} \cdot e_{ij}^{(\lambda)})^2 d\Omega \tag{3.3.2}$$

since:

$$dI = \frac{k}{72\pi c^5} \sum_{\lambda_1, \lambda_2} (\dddot{D}_{ij} \cdot e_{ij}^{(\lambda)})^2 = \frac{k}{72\pi c^5} [(\frac{\dddot{D}_{23}}{\sqrt{2}} + \frac{\dddot{D}_{32}}{\sqrt{2}})^2 + (\frac{\dddot{D}_{33}}{\sqrt{2}} - \frac{\dddot{D}_{22}}{\sqrt{2}})^2] d\Omega =$$

$$= \frac{k}{72\pi c^5} [(\frac{2\dddot{D}_{23}}{\sqrt{2}})^2 + \frac{1}{2}(\dddot{D}_{33} - \dddot{D}_{22})^2] d\Omega = \frac{k}{72\pi c^5} [2\dddot{D}_{23}^2 + \frac{1}{2}(\dddot{D}_{33} - \dddot{D}_{22})^2] d\Omega =$$

$$= \frac{k}{72\pi c^5} \cdot 2[\dddot{D}_{23}^2 + \frac{1}{4}(\dddot{D}_{33} - \dddot{D}_{22})^2] d\Omega = \frac{k}{36\pi c^5} [\dddot{D}_{23}^2 + \frac{1}{4}(\dddot{D}_{33} - \dddot{D}_{22})^2] d\Omega$$

3.4 Average of Radiation over polarizations

The total radiation in all directions per unit time is $\frac{dI}{dt} \cdot 4\pi$ and we should average over all directions and polarizations. First, we average over polarizations, so from the formula (3.3.2):

$$\overline{dI} = \frac{k}{72\pi c^5} \sum_{\lambda_1, \lambda_2} \dddot{D}_{ij} \dddot{D}_{kl} \cdot \overline{e_{ij}^{(\lambda)} e_{kl}^{(\lambda)}} d\Omega = 2 \cdot \frac{k}{72\pi c^5} \dddot{D}_{ij} \dddot{D}_{kl} \cdot \overline{e_{ij} e_{kl}} d\Omega$$

$$\overline{dI} = \frac{k}{36\pi c^5} \dddot{D}_{ij} \dddot{D}_{kl} \cdot \overline{e_{ij} e_{kl}} d\Omega \tag{3.4.1}$$

where:

$$\overline{e_{ij} e_{kl}} = \frac{1}{4}[n_i n_j n_k n_l + (n_i n_j \delta_{kl} + n_k n_l \delta_{ij}) - (\delta_{jl} n_i n_k + \delta_{ik} n_j n_l + \delta_{jk} n_i n_l +$$

$$+\delta_{il} n_j n_k) - \delta_{ij}\delta_{kl} + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})] \quad (proof : Appendix \quad C)$$

Substituting the average of the product of polarization above, in the equation (3.4.1), we get that:

$$\overline{dI} = \frac{k}{36\pi c^5} \dddot{D}_{ij} \dddot{D}_{kl} \cdot \overline{e_{ij} e_{kl}} d\Omega =$$

$$= \frac{k}{36\pi c^5} \cdot \frac{1}{4}[\dddot{D}_{ij}\dddot{D}_{kl} n_i n_j n_k n_l + \dddot{D}_{ij}\dddot{D}_{kl} n_i n_j \delta_{kl} + \dddot{D}_{ij}\dddot{D}_{kl} n_k n_l \delta_{ij} -$$

$$-\dddot{D}_{ij}\dddot{D}_{kl}\delta_{jl} n_i n_k - \dddot{D}_{ij}\dddot{D}_{kl}\delta_{ik} n_j n_l - \dddot{D}_{ij}\dddot{D}_{kl}\delta_{jk} n_i n_l - \dddot{D}_{ij}\dddot{D}_{kl}\delta_{il} n_j n_k -$$

$$-\dddot{D}_{ij}\dddot{D}_{kl}\delta_{ij}\delta_{kl} + \dddot{D}_{ij}\dddot{D}_{kl}\delta_{ik}\delta_{jl} + \dddot{D}_{ij}\dddot{D}_{kl}\delta_{il}\delta_{jk}] d\Omega =$$

$$= \frac{k}{36\pi c^5} \cdot \frac{1}{4}[(\dddot{D}_{ij} n_i n_j)^2 + \dddot{D}_{ij}\dddot{D}_{kk} n_i n_j + \dddot{D}_{ii}\dddot{D}_{kl} n_k n_l - \dddot{D}_{il}\dddot{D}_{kl} n_i n_k -$$

$$-\dddot{D}_{kj}\dddot{D}_{kl} n_j n_l - \dddot{D}_{ik}\dddot{D}_{kl} n_i n_l - \dddot{D}_{lj}\dddot{D}_{kl} n_j n_k - \dddot{D}_{ii}\dddot{D}_{kk} + \dddot{D}_{kj}\dddot{D}_{kj} + \dddot{D}_{lj}\dddot{D}_{jl}] d\Omega =$$

$$= \frac{k}{36\pi c^5} \cdot \frac{1}{4}[(\dddot{D}_{ij} n_i n_j)^2 - 4 \cdot \dddot{D}_{il}\dddot{D}_{kl} n_i n_k + 2 \cdot \dddot{D}_{kj}\dddot{D}_{kj}], \quad so :$$

$$\overline{dI} = \frac{k}{36\pi c^5}[\frac{1}{4}(\dddot{D}_{ij} n_i n_j)^2 + \frac{1}{2} \cdot \dddot{D}_{kj}\dddot{D}_{kj} - \dddot{D}_{il}\dddot{D}_{kl} n_i n_k]$$

3.5 Average of Radiation over Directions

We still have to average over all directions n_i and from the formula above we can see that we are going to need the following average values:

$$\overline{n_i n_j} = \frac{1}{3}\delta_{ij}, \quad \overline{n_i n_j n_k n_l} = \frac{1}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad (proof: Appendix \quad D), \quad so:$$

$$\overline{dI} = \frac{k}{36\pi c^5}[\frac{1}{4}\dddot{D}_{ij}\dddot{D}_{kl}\overline{n_i n_j n_k n_l} + \frac{1}{2}\cdot\dddot{D}_{kj}\dddot{D}_{kj} - \dddot{D}_{il}\dddot{D}_{kl}\overline{n_i n_k}] =$$

$$= \frac{k}{36\pi c^5}[\frac{1}{4}\dddot{D}_{ij}\dddot{D}_{kl}\cdot\frac{1}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{1}{2}\cdot\dddot{D}_{kj}\dddot{D}_{kj} - \dddot{D}_{il}\dddot{D}_{kl}\cdot\frac{1}{3}\delta_{ik}] =$$

$$= \frac{k}{36\pi c^5}[\frac{1}{4}\cdot\frac{1}{15}(\dddot{D}_{ii}\dddot{D}_{kk} + \dddot{D}_{kj}\dddot{D}_{kj} + \dddot{D}_{lj}\dddot{D}_{lj}) + \frac{1}{2}\cdot\dddot{D}^2_{ij} - \frac{1}{3}\dddot{D}_{kl}\dddot{D}_{kl}] =$$

$$= \frac{k}{36\pi c^5}(\frac{1}{4}\cdot\frac{1}{15}\cdot 2\dddot{D}^2_{ij} + \frac{1}{2}\dddot{D}^2_{ij} - \frac{1}{3}\dddot{D}^2_{ij}) = \frac{k}{36\pi c^5}(\frac{1}{30} + \frac{1}{2} - \frac{1}{3})\dddot{D}^2_{ij}$$

$$\overline{dI} = \frac{k}{36\pi c^5}\cdot\frac{1}{5}\dddot{D}^2_{ij}$$

So the total radiation in all directions per unit time is:

$$4\pi\cdot\frac{dI}{dt} = 4\pi\cdot\frac{k}{36\pi c^5}\cdot\frac{1}{5}\dddot{D}^2_{ij}$$

$$\boxed{4\pi\cdot\frac{dI}{dt} = \frac{k}{45c^5}\cdot\dddot{D}^2_{ij}} \quad , \quad D_{ii} = 0 \quad [2] \qquad (3.5.1)$$

4 Radiation of Gravitational Waves from Binary Systems, their Orbital Decay and the Merger Time Scale

4.1 Gravitational Wave Radiation of a Binary System

The energy flow per unit time, as we just showed in the previous unit, is [1]:

$$\frac{dE}{dt} = \frac{k}{45c^5} \cdot \dddot{D}_{ij}^2 \tag{4.1.1}$$

where D_{ij} is the quadrupole momentum tensor: $D_{ij} = \int \mu(3x^i x^j - \delta^{ij} x_\kappa^2)dV$ and since μ is the density of mass: $[\mu] = \frac{g}{cm^3}$, the units of measure in the equation above are:

$$[\frac{dE}{dt}] = \frac{[k]}{[c]^5} \cdot [\dddot{D}_{ij}]^2 = \frac{cm^3}{g \cdot sec^2} \cdot \frac{sec^5}{cm^5} \cdot (\frac{\frac{g}{cm^3} \cdot cm^2 \cdot cm^3}{sec^3})^2 =$$

$$= \frac{sec^3}{g \cdot cm^2} \cdot \frac{g^2 \cdot cm^4}{sec^6} = \frac{g \cdot cm^2}{sec^2} \cdot \frac{1}{sec}$$

which is indeed the unit of energy per time. As for the order of magnitude of the energy that a system emits, we calculate the coefficient:

$$\frac{k}{45c^5} = \frac{6,67 \cdot 10^{-8}}{45 \cdot (3 \cdot 10^10)^5} = 6,1 \cdot 10^{-62} \frac{sec^3}{g \cdot cm^2}$$

We see how small it is, so we can understand why it is so hard to detect gravitational waves. To have a comparison measure, the energy of an electron in copper, since its mass is $m_e = 9 \cdot 10^{-28} g$ and its drift velocity is $\upsilon = 2,3 \cdot 10^{-3}$, is:

$$\frac{m_e \upsilon^2}{2} = 23,8 \cdot 10^{-34} \frac{g \cdot cm^2}{sec^2}$$

much bigger than the coefficient above.

We consider two bodies rotating around each other in a (x,y) plane (see Fig.4), where: $\vec{r} = \vec{r}_2 - \vec{r}_1$ and considering the reduced mass: $\mu = \frac{m_1 \cdot m_2}{m_1 + m_2}$,

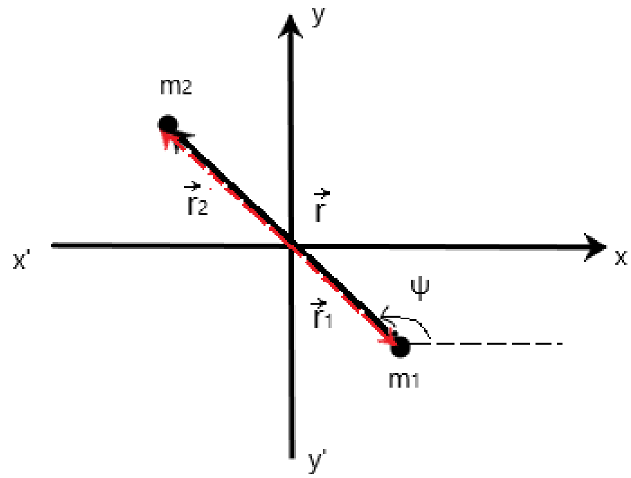

Figure 4: Two bodies rotating in (x,y) plane

we have that:

$$\vec{r}_1 = -\frac{m_2}{m_1 + m_2}\vec{r}, \quad \vec{r}_2 = \frac{m_1}{m_1 + m_2}\vec{r}$$

$$and \quad x = r \cdot cos\psi, \quad y = r \cdot sin\psi, \quad z = 0$$

We begin calculating the quadrupole momentum tensor of this system, in order to find the total radiation this emits. Here we have a system of discrete masses, so the integral in the tensor becomes summation:

$$D_{xx} = m_1(3x_1x_1 - x_1^2 - y_1^2 - z_1^2) + m_2(3x_2x_2 - x_2^2 - y_2^2 - z_2^2) =$$

$$= \frac{m_1 \cdot m_2^2}{(m_1 + m_2)^2}(2x^2 - y^2) + \frac{m_2 \cdot m_1^2}{(m_1 + m_2)^2}(2x^2 - y^2) =$$

$$= \frac{m_1 \cdot m_2^2 + m_2 \cdot m_1^2}{(m_1 + m_2)^2}(2x^2 - y^2) = \frac{m_1 \cdot m_2(m_1 + m_2)}{(m_1 + m_2)^2}(2x^2 - y^2) =$$

$$= \frac{m_1 \cdot m_2}{m_1 + m_2} \cdot r^2 (2cos^2\psi - sin^2\psi)$$

$$\boxed{D_{xx} = \mu \cdot r^2 (3cos^2\psi - 1)}$$

$$D_{yy} = m_1(3y_1y_1 - x_1^2 - y_1^2 - z_1^2) + m_2(3y_2y_2 - x_2^2 - y_2^2 - z_2^2) =$$

$$= \frac{m_1 \cdot m_2^2}{(m_1 + m_2)^2}(2y^2 - x^2) + \frac{m_2 \cdot m_1^2}{(m_1 + m_2)^2}(2y^2 - x^2) = \frac{m_1 \cdot m_2}{m_1 + m_2} \cdot r^2 (2sin^2\psi - cos^2\psi)$$

$$\boxed{D_{yy} = \mu \cdot r^2 (3sin^2\psi - 1)}$$

$$D_{xy} = 3m_1x_1y_1 + 3m_2x_2y_2 = 3\frac{m_1 \cdot m_2^2}{(m_1 + m_2)^2}xy + 3\frac{m_2 \cdot m_1^2}{(m_1 + m_2)^2}xy =$$

$$= 3\frac{m_1 \cdot m_2}{m_1 + m_2}r^2 sin\psi \cdot cos\psi$$

$$\boxed{D_{xy} = 3\mu \cdot r^2 sin\psi \cdot cos\psi}$$

$$D_{zz} = m_1(3z_1z_1 - x_1^2 - y_1^2 - z_1^2) + m_2(3z_2z_2 - x_2^2 - y_2^2 - z_2^2) =$$

$$= -\frac{m_1 \cdot m_2^2}{(m_1 + m_2)^2}(x^2 + y^2) - \frac{m_2 \cdot m_1^2}{(m_1 + m_2)^2}(x^2 + y^2) = -\frac{m_1 \cdot m_2}{m_1 + m_2} \cdot r^2 (sin^2\psi + cos^2\psi)$$

$$\boxed{D_{zz} = -\mu \cdot r^2}$$

$$D_{xz} = 3m_1x_1z_1 + 3m_2x_2z_2 = 0 = D_{yz}$$

In the simple circular motion: $r = const.$ and $\psi = \omega \cdot t$. So only ψ depends on time in the quadrupole momentum above. Moreover, taking the equation of

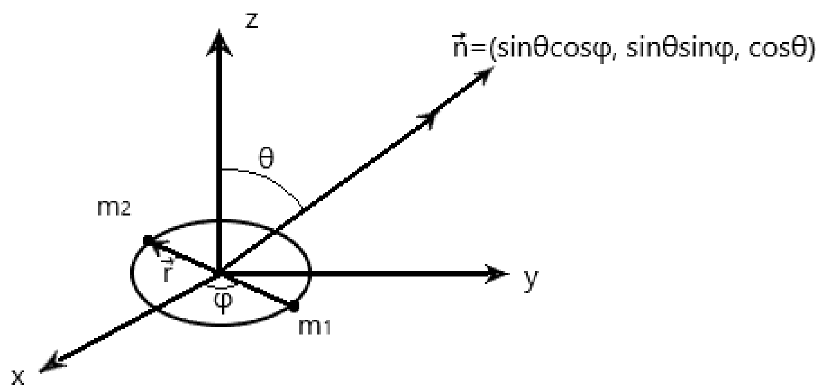

Figure 5: 3D representation of two bodies rotating in (x,y) plane

the forces of gravity between the two masses which equal to the centripetal force for their system, we get that:

$$F_{1\to 2} = F_{2\to 1} = k\frac{m_1 m_2}{r^2} = F_\kappa = \frac{\mu v^2}{r} = \frac{m_1 m_2}{m_1+m_2}\cdot\frac{\omega^2\cdot r^2}{r}$$

$$k\frac{m_1 m_2}{r^2} = \frac{m_1 m_2}{m_1+m_2}\omega^2 r, \quad \dot{\psi} = \omega = \sqrt{\frac{k(m_1+m_2)}{r^3}} \tag{4.1.2}$$

Thus, differentiating the quadrupole momentum tensor we get:

$$\dot{D}_{xx} = -6\mu\cdot r^2\omega\cdot cos\psi\cdot sin\psi, \quad \ddot{D}_{xx} = -6\mu\cdot r^2\omega^2(-sin^2\psi + cos^2\psi) =$$

$$= 6\mu\cdot r^2\omega^2(2sin^2\psi - 1), \quad \underline{\dddot{D}_{xx} = 24\mu r^2\omega^3\cdot sin\psi\cdot cos\psi}$$

$$\dot{D}_{yy} = 6\mu\cdot r^2\omega sin\psi\cdot cos\psi, \quad \ddot{D}_{yy} = 6\mu\cdot r^2\omega^2(cos^2\psi - sin^2\psi) =$$

$$= 6\mu\cdot r^2\omega^2(2cos^2\psi - 1), \quad \underline{\dddot{D}_{yy} = -24\mu r^2\omega^3\cdot sin\psi\cdot cos\psi}$$

$$\dot{D}_{xy} = 3\mu r^2\omega(cos^2\psi - sin^2\psi), \quad \ddot{D}_{xy} = 3\mu r^2\omega^2(-2cos\psi\cdot sin\psi - 2sin\psi\cdot cos\psi) =$$

$$= -12\mu r^2\omega^2 cos\psi \cdot sin\psi, \quad \underline{\dddot{D}_{xy} = 12\mu r^2\omega^3(sin^2\psi - cos^2\psi)}$$

$$\dot{D}_{zz} = 0$$

Finally, we have to take time average over the period: $T = \frac{2\pi}{\omega}$, for the squares of derivatives of quadrupole momentum before we substitute them into the formula for total radiation (4.1.1):

$$\overline{\dddot{D}^2_{xx}} = (24\mu r^2\omega^3)^2 \cdot \overline{sin^2\psi \cdot cos^2\psi} = 72\mu^2 r^4\omega^6$$

$$\overline{\dddot{D}^2_{yy}} = (-24\mu r^2\omega^3)^2\overline{sin^2\psi \cdot cos^2\psi} = 72\mu^2 r^4\omega^6$$

$$\overline{\dddot{D}^2_{xy}} = (12\mu r^2\omega^3)^2(\overline{sin^2\psi - cos^2\psi}) = 72\mu^2 r^4\omega^6$$

So eventually the total radiation the system loses while the bodies are rotating is:

$$\overline{\frac{dE}{dt}} = \frac{k}{45c^5} \cdot \overline{\dddot{D}^2_{ij}} = \frac{k}{45c^5} \cdot (\overline{\dddot{D}^2_{xx}} + \overline{\dddot{D}^2_{yy}} + \overline{\dddot{D}^2_{xy}} + \overline{\dddot{D}^2_{yx}}) =$$

$$= \frac{k}{45c^5} \cdot (\overline{\dddot{D}^2_{xx}} + \overline{\dddot{D}^2_{yy}} + 2\overline{\dddot{D}^2_{xy}}) = \frac{k}{45c^5} \cdot 288\mu^2 r^4\omega^6 = \frac{32}{5}\frac{k\mu^2 r^4\omega^6}{c^5} =$$

$$= \frac{32}{5}\frac{k\frac{m_1^2 m_2^2}{(m_1+m_2)^2} r^4 (\sqrt{\frac{k(m_1+m_2)}{r^3}})^6}{c^5} = \frac{32}{5}\frac{k\frac{m_1^2 m_2^2}{(m_1+m_2)^2} r^4 \frac{k^3(m_1+m_2)^3}{r^9}}{c^5} = \frac{32k^4 m_1^2 m_2^2(m_1+m_2)}{5c^5 r^5}$$

$$\boxed{\overline{\frac{dE}{dt}} = \frac{32k^4 m_1^2 m_2^2(m_1+m_2)}{5c^5 r^5}} \qquad (4.1.3)$$

4.2 Orbital Decay and Merger Time

From the classical Newton's law of gravity:

$$E = -k\frac{m_1 m_2}{2r}, \quad \frac{dE}{dt} = k\frac{m_1 m_2}{2r^2} \cdot \dot{r}$$

So, comparing the two relations about $\frac{dE}{dt}$ above we get that:

$$k\frac{m_1 m_2}{2r^2} \cdot \dot{r} = \frac{32k^4 m_1^2 m_2^2 (m_1+m_2)}{5c^5 r^5}, \quad \dot{r} = \frac{2r^2}{k m_1 m_2} \cdot \frac{32k^4 m_1^2 m_2^2 (m_1+m_2)}{5c^5 r^5}$$

$$\dot{r} = \frac{64k^3 m_1 m_2 (m_1+m_2)}{5c^5 r^3}, \tag{4.2.1}$$

which is the rate of orbital decay in the system of the two masses while they rotate around each other, radiating gravitational radiation and thus losing energy, hence the orbital decay.
For simplicity, we consider that the two masses are equal:

$$m_1 = m_2 = m, \quad \mu = \frac{m \cdot m}{m+m} = \frac{m^2}{2m} = \frac{m}{2}$$

Then, the total radiation per unit time becomes:

$$\frac{dE}{dt} = \frac{32k^4 m_1^2 m_2^2 (m_1+m_2)}{5c^5 r^5} = \frac{32k^4 m^4 \cdot 2m}{5c^5 r^5} = \frac{64k^4 m^5}{5c^5 r^5}$$

$$\boxed{\frac{dE}{dt} = \frac{64k^4 m^5}{5c^5 r^5}}, \tag{4.2.2}$$

$$[\frac{dE}{dt}] = \frac{[k]^4 [m]^5}{[c]^5 [r]^5} = (\frac{cm^3}{g \cdot sec^2})^4 \cdot \frac{sec^5}{cm^5} \cdot \frac{g^5}{cm^5} = \frac{cm^12}{g^4 \cdot sec^8} \cdot \frac{sec^5 \cdot g^5}{cm^10} = \frac{cm^2 \cdot g}{sec^3}.$$

Using the formula for the orbital decay (4.1.3), we can find the expected time of the merger of these two rotating masses:

$$\frac{dr}{dt} = \frac{64k^3 m_1 m_2 (m_1+m_2)}{5c^5 r^3}, \quad \int_0^r r^3 dr = \frac{64k^3 m_1 m_2 (m_1+m_2)}{5c^5} \int_0^T dt$$

$$\frac{r^4}{4} = \frac{64k^3 m_1 m_2 (m_1+m_2)}{5c^5} \cdot T, \quad T = \frac{r^4}{4} \cdot \frac{5c^5}{64k^3 m_1 m_2 (m_1+m_2)}$$

$$\boxed{T = \frac{5}{256} \cdot \frac{c^5}{k^3} \cdot \frac{r^4}{m_1 m_2 (m_1+m_2)}} \tag{4.2.3}$$

In order to understand the formulas we found above and the physics of gravitational waves, we are going to take some specific physical systems and apply the equations we found.
All these formulas above are correct as long as distance between the two masses of the binary system is much larger than their radius in order to be able to think of them as massive points as we did. So, these formulas can't be applied for merger or ring-down, since then the distance between the masses are minimal.

4.3 Application of the Equation of Total Emitting Radiation on Physical Binary Systems

- Hydrogen atom

Let us begin with the Hydrogen atom and take its radius α_0 and the masses of the particles that compose it:

$$\alpha_0 = 5,291 \cdot 10^{-9} cm, \quad m_e = 9,11 \cdot 10^{-28} g, \quad m_p = 1,67 \cdot 10^{-24} g$$

We are going to need the distance of the electron of the centre mass of the system proton-electron and thus the reduced mass as well:

$$\alpha_0^* = \frac{m_e}{\mu}\alpha_0, \quad \mu = \frac{m_e \cdot m_p}{m_e + m_p}$$

$$\alpha_0^* = \frac{m_e + m_p}{m_p}\alpha_0 = (1 + \frac{m_e}{m_p})\alpha_0 = 5,295 \cdot 10^{-9} cm$$

So, from the formula (4.2.2) the gravitational radiation of the Hydrogen electron is:

$$\frac{dE}{dt} = \frac{64k^4 m_e^5}{5c^5 {\alpha_0^*}^5} = \frac{64 \cdot (6,67 \cdot 10^{-8})^4 \cdot (9,11 \cdot 10^{-28})^5}{5 \cdot (3 \cdot 10^{10})^5 \cdot (5,29 \cdot 10^{-9})^5} = 1,58 \cdot 10^{-180} \frac{g \cdot cm^2}{sec^3}$$

As we can see the energy per unit time that is emitted while the electron is rotating around the nucleus, is extremely small. So, since we know that the energy of the Hydrogen's electron in the ground state is 1 Ry:

$$1Ry \approx 2,18 \cdot 10^{-18} J = 2,18 \cdot 10^{-18} kg \frac{m^2}{sec^2} = 2,18 \cdot 10^{-18} \cdot 10^7 g \frac{cm^2}{sec^2}$$

$$1Ry = 2,18 \cdot 10^{-11} g \frac{cm^2}{sec^2},$$

so the approximate time for the electron to collide into the nucleus will be:

$$t = \frac{1Ry}{\frac{dE}{dt}} \approx 10^{169} sec = 10^{153} billion \quad years$$

practically never.

- <u>Neutron stars binaries</u>

  Another interesting physical system is binary neutron stars. Let us take neutron stars of one solar mass which rotate around each other with radius r:

$$M_{neu} = M_{\odot} = 2 \cdot 10^{33} g, \quad r = 1,89 \cdot 10^{10} cm$$

  So, using formula (4.2.2), the time of their merger is:

$$T = \frac{5}{256} \cdot \frac{c^5}{k^3} \cdot \frac{r^4}{M_{neu} M_{neu} (M_{neu} + M_{neu})} = \frac{5}{256} \cdot \frac{c^5}{k^3} \cdot \frac{r^4}{2M_{neu}^3} =$$

$$= \frac{5}{256} \cdot \frac{(3 \cdot 10^{10})^5}{(6,67 \cdot 10^{-8})^3} \cdot \frac{(1,89 \cdot 10^{10})^4}{2 \cdot (2 \cdot 10^{33})^3} = 1,28 \cdot 10^{13} sec = 405.885 \quad years$$

  an amount of time which explains us why the merger of binary neutron stars is observable. Meanwhile, the period of their rotation, using the formula (4.1.2), is:

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{r^3}{k(M_{neu} + M_{neu})}} = 2\pi \sqrt{\frac{(1,89 \cdot 10^{10})^3}{2 \cdot 6,67 \cdot 10^{-8} \cdot 2 \cdot 10^{33}}} = 999,49 \quad sec$$

Thus, the neutron stars complete one rotation in every approximately 1000 seconds or 16,7 minutes, while the time for them to collide, when their radius is 189.000 km, is 405.885 years. A lot of white dwarfs and neutron stars exist with orbital periods in this range.
What is of the most interest here is that when the radius is reduced to 1890 km or $r = 1,89 \cdot 10^8$ cm, the collision merge time is:

$$T = 1,28 \cdot 10^5 sec = 35,55 \quad hours \approx 36 \quad hours$$

and the frequency of their rotation then is:

$$\omega = \sqrt{\frac{2kM_{neu}}{r^3}} = \sqrt{\frac{2 \cdot 6,67 \cdot 10^{-8} \cdot 2 \cdot 10^{33}}{(1,89 \cdot 10^8)^3}} = 6,29 s^{-1}, \quad f = \frac{\omega}{2\pi} = 1H$$

So, it is really remarkable that the whole process of 405.885 years takes place until the stars approach at a distance 100 times smaller and once they get there the phenomenon becomes rapid and violent. The stars complete a rotation around each other in every 1 second and within 36 hours they merge. Lastly, we calculate the emitted radiation per unit time when their radius is $r = 1,89 \cdot 10^{10} cm$ and $r = 1,89 \cdot 10^8 cm$, respectively, using the formula (4.2.2):

$$\frac{dE}{dt} = \frac{64k^4 M_{neu}^5}{5c^5 r^5} = \frac{64 \cdot (6,67 \cdot 10^{-8})^4 \cdot (2 \cdot 10^{33})^5}{5 \cdot (3 \cdot 10^{10})^5 \cdot (1,89 \cdot 10^{10})^5} = 1,38 \cdot 10^{35} \frac{g \cdot cm^2}{sec^3}$$

$$\frac{dE}{dt} = 1,38 \cdot 10^{45} \frac{g \cdot cm^2}{sec^3}$$

It is clear that as the radius of the system reduces, the amount of energy that is being released is much larger. The event is indeed more tense.

- Binary Black Hole merger GW150914

The Binary Black Hole merger GW150914 was the first astronomical observation in 2015 that confirmed the existence of gravitational waves, as the general relativity predicted a century before. This system consisted of two rotating black holes whose masses were of about:

$$M_1 = 36M_\odot, \quad M_2 = 29M_\odot$$

The event that was recorded [3], from the initial inspiral to their merger, lasted approximately 0,2 seconds and the signal we took was from 35 Hz to 250 Hz. Finally, a bigger black hole of about $M = 62M_\odot$ was formed. The radius of the system of the black holes, the time of the merger and the orbital frequency that were measured were respectively:

$$r = 350km = 35 \cdot 10^6 cm, \quad \Delta t_{merger} = 20\mu sec = 2 \cdot 10^{-5} sec, \quad f = 75Hz$$

So, using again the formulas (4.2.2) and (4.2.3), we can find ourselves

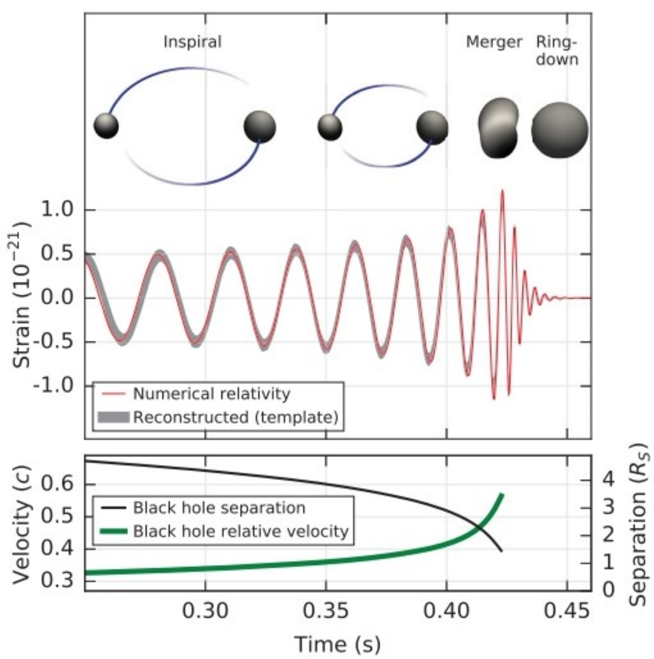

Figure 6: Binary Black Hole Merger of GW150914

the time of the merger, as well as the orbital frequency and test our calculations by comparing them:

$$\omega = \sqrt{\frac{k(M_1 + M_2)}{r^3}} = \sqrt{\frac{6,67 \cdot 10^{-8} \cdot (29+36) \cdot 2 \cdot 10^{33}}{(35 \cdot 10^6)^3}} = \sqrt{20,224 \cdot 10^4}$$

$$\omega = 4,5 \cdot 10^2 sec^{-1}, \quad f = \frac{\omega}{2\pi} = 71,57Hz$$

$$T_{merger} = \frac{5}{256} \cdot \frac{c^5}{k^3} \cdot \frac{r^4}{M_1 M_2 (M_1 + M_2)} =$$

$$= \frac{5}{256} \cdot \frac{(3 \cdot 10^{10})^5}{(6,67 \cdot 10^{-8})^3} \cdot \frac{(35 \cdot 10^6)^4}{29 \cdot 36 \cdot 65 \cdot (2 \cdot 10^{33})^3} = 4,421 \cdot 10^{-3} sec$$

We notice that the frequency is pretty accurate, the time of merger though has a significant deviation of two orders of magnitude, as expected. As we noted when we found the formula for time merger, it can be applied only when the distance of the rotating bodies is much larger than their radii, so that they can be considered as massive points. Calculating the Schwarschild radius of the black holes above, we have:

$$r_{g1} = \frac{2kM_1}{c^2} = \frac{2 \cdot 6,67 \cdot 10^{-8} \cdot 36 \cdot 2 \cdot 10^{33}}{(3 \cdot 10^{10})^2} = 10,672 \cdot 10^6 cm = 106,72km$$

$$r_{g2} = \frac{2kM_2}{c^2} = \frac{2 \cdot 6,67 \cdot 10^{-8} \cdot 29 \cdot 2 \cdot 10^{33}}{(3 \cdot 10^{10})^2} = 8,597 \cdot 10^6 cm = 85,97km$$

Hence, the radius of the system: r=350 km for which the given time of merger is $\Delta t_{merger} = 2 \cdot 10^{-5} sec$, is merely larger than the sum of the radii of the two black holes. So, then the objects are too close and applying our formula for time merger leads to inevitable, significant error.
As long as the experimental figure 6 is concerned, as we already noted, the time of merger that is depicted is approximately 0,2 sec. So, solving our formula (4.2.3) over r, we calculate the radius of the system then:

$$r = (\frac{256 \cdot k^3 M_1 M_2 (M_1 + M_2) T_{merger}}{5c^5})^{1/4} =$$

$$= (\frac{256 \cdot (6,67 \cdot 10^{-8})^3 \cdot 29 \cdot 36 \cdot 65 \cdot (2 \cdot 10^{33})^3 \cdot 0,2}{5(3 \cdot 10^{10})^5})^{1/4} = 9,077 \cdot 10^7 cm = 907,7km$$

So, approximately, when the radius of the system is almost 2,5 times bigger, the time merger is 4 orders of magnitude bigger. We ascertain that the merging becomes more and more rapid as the radius reduces and that the figure shows a bigger part of the whole phenomenon than the experimental data we took above.

In this system we can see how the much bigger masses affect the merger. The event happens much more rapidly than the merger of the binary neutron stars before, since for a radius of neutron stars $r = 1,89 \cdot 10^7 cm$, same order of magnitude with the radius $r = 9,077 \cdot 10^7 cm$ of the black holes, the time merger is calculated to be 12,8 sec in contrast with 0,2 sec here in black holes case. Moreover, the amount of radiation they emit per unit time when the radius is $r = 35 \cdot 10^6 cm$, using (4.1.3), is:

$$\frac{dE}{dt} = \frac{32k^4 M_1^2 M_2^2 (M_1 + M_2)}{5c^5 r^5} = \frac{32 \cdot (6,67 \cdot 10^{-8})^4 \cdot 29^2 \cdot 36^2 \cdot 65^2 \cdot (2 \cdot 10^{33})^5}{5 \cdot (3 \cdot 10^{10})^5 \cdot (35 \cdot 10^6)^5}$$

$$\frac{dE}{dt} = 1,46 \cdot 10^{58} \frac{g \cdot cm^2}{sec^3}$$

For the corresponding radius $r = 18,9 \cdot 10^6 cm$ the neutron stars emit per unit time:

$$\frac{dE}{dt} = 1,38 \cdot 10^{50} \frac{g \cdot cm^2}{sec^3}.$$

So, the radiation emitted per unit time from the black holes is several times larger, as well, than the corresponding one from the neutron stars. This is why we managed to detect these gravitational waves [2].

5 Running and Future GW Experiments

5.1 Cryogenic Resonant Bar Detectors

The first GW detectors that were designed by Weber were the Resonant Bar Detectors. The main part of these detectors is a massive suspended cylinder of several tons which is vibrating at a characteristic resonance frequency. The basic idea around them is that when a gravitational wave passes through them, oscillates the cylinder whose resonance frequency is close to the frequency of gravitational radiation, around 75Hz, allowing to detect the gravitational signal. The reason of suspending the cylinder is to reduce the noise and and for that the bar is also cooled in very low temperatures in order to minimiζe the thermal noise, which is one of the fundamental sources of noise for Resonant Bar Detectors. Additionally, the bigger the cylinder's mass is the better the detector's sensitivity is, as the errors minimiζe. So the next generation of such detectors aim in greater masses. Current experiments which use such detectors are ALLEGRO, AURIGA, EXPLORER, NAUTILUS, and NIOBI [4].

5.2 Michelson Laser Interferometers

These category of detectors are the contemporary, mainly used GW detectors. They are the well known Michelson-Morley Interferometers, which consist of two perpendicular vacuum optic paths with a laser placed at the one end of one of the arms. The beams from the laser at some point split through a beam splitter, placed at the point where the arms are crossed, they reflect on mirrors at the other two ends of the two arms and finally they combine again at the splitter and end in a detector at the other end of the second arm. The detector gets the signal of the two combined beams. The way this device can detect gravitational waves is that when a gravitational wave passes through, slightly stretches one arm and shortens the other, changing the lengths of the two paths and so the frequencies of the interfered beams and their interference. Though, the fluctuation is so small that the device needs to be as much as isolated from sources of noise can be and sensitive as well, in order to be able to detect the delicate gravitational waves [5]. Thus, from the beginning till now, this is the main headache of mechanics and experimentalists about the designing of such detectors, the minimiation of noise.

So, from 2002-2015 and 2023 till today, the LIGO (Laser Interferometer Gravitational Wave Observatory) detector in Louisiana and in Hanford (Washington) USA, the VIRGO in Santo Stefano a Macerata, Italy, and from 2019 and forth the KAGRA detector in Kamioka, Japan [6] also, are the Michelson Laser Interferometers that are running together and collaborate, combining their data on measuring fluctuations in their signals, checking whether they come from GWs or random noise. The LIGO-VIRGO collaboration was the one that for the first time managed to detect gravitational waves, the GW150914, which we discussed in the main text, coming from two black holes with masses of 29 and 36 solar masses merging about 1.3 billion light-years away from earth [3]. This collision released a sufficiently large amount of gravitational radiation that the detectors were able to detect.
As far as the future experiments are concerned, three basic programs are running, the ET (Einstein Telescope) in Europe, the LISA (Laser Interferometer Space Antenna) from European Space Agency (ESA) and DECIGO (DECi-hert Interferometer Gravitational wave Observatory) from Japan. The Einstein Telescope will be a ground-based detector as are the previous three detectors that are currently operating, but this one will be constructed underground in order to limit the effect of the seismic noise. The remaining characteristics that will make it a much more sensitive and improved detector are the increasing of the size of the interferometer from the 3km arm length of the Virgo detector or 4 km of the LIGO to 10km, due to its shape. The inteferometer will consist by three pieces shaping a triangle, this way its length increases a lot and easily. Moreover, new technologies are added, such as cryogenic facilities that will cool down to 10–20K the mirrors to directly reduce the thermal vibration of the test masses, new quantum technologies to reduce the fluctuations of the light, and a set of infrastructural and active noise-mitigation measures to reduce environmental perturbations [7].
The big revolution in the evolution of GW detectors comes with space-based interferometers, LISA and DECIGO. The main idea is to avoid earth originating noise, seismic, volcanos operation and others. So LISA will consist by three spacecrafts arranged in an equilateral triangle with sides 2.5 million kilometres long, moving along an Earth-like heliocentric orbit [8]. So another advantage is the huge increasing of the length of the detector, which increases its sensitivity. The DECIGO detector is designed in order to fill in the gap between the sensitive bands of frequency that LIGO and

LISA detect [9]. These future detectors are expected to operate around 2030 and 2035.

6 Conclusions

Summarising the above theoretical analysis, we stress that our main result is the calculation of the total gravitational energy radiation that the binary physical systems emits per unit time (4.1.1). Through this formula we understand the conditions of the emission of gravitational radiation. First and foremost, there must be acceleration of acceleration, since in the formula there is third time derivative of the quadrupole momentum: $D_{ij} = \int \mu(3x^i x^j - \delta^{ij} x_\kappa^2) dV$. So it is not enough for a system to accelerate in order to emit gravitational waves, as in electrodynamics, but its acceleration has to accelerate. The second remarkable point is that the gravitational radiation is proportional to $1/c^5$, in contrast to the electromagnetic radiation which is proportional to $1/c^3$, which explains why it is so difficult to generate gravitational waves and why they are much weaker and hard to detect than the electromagnetic waves. Another significant point is the polarization of the waves. For the massive elementary particles for every value of s there are 2s+1 polarizations of spins, meaning that the group of symmetry is the SO(3), but for massless particles for each spin s there are only two spin polarizations -s, +s, meaning that the little symmetry group is instead SO(2). For the gravitational waves we showed that they propagate with the velocity of light and that there are only two polarizations perpendicular to the direction of the propagation and therefore describe a massless particle, the graviton. Thus, the graviton, the elementary particle of gravitational interaction, is a massless particle which has two polarizations. We also ascertained through the examples we gave that the more massive the system is, the more violent the collision and bigger the emission of radiation is. So in order to detect gravitational waves (GW), we need events of significantly large binary systems merging. Events that last for fractions of a second, so we need such detectors so that they can detect with accuracy such tricky signal. This is why a lot of effort and research have taken and still taking place throughout the years on GW detectors. Therefore, the bet for the years to come, is to construct the most efficient detectors in order to improve the studies on GW and retrieve the answers we expect we can get about the beginning of the universe.

7 Acknowledgment

I want to deeply thank my supervisor professor George Savvidy for his valuable guidance, teaching and help. His plentiful knowledge and experience encouraged and inspired me throughout the process and without them it wouldn't be possible for me to consummate the present thesis. I want to thank him for his patience and his support. Thank you.

Appendices

A The harmonic gauge fixing condition in weak field approximation

The harmonic gauge is: $\Box x^\mu = 0$, where $\Box$ is the covariant D' Alembertian, so:

$$\Box x^\mu = 0, \quad g^{\rho\sigma}\partial_\rho\partial_\sigma x^\mu - g^{\rho\sigma}\Gamma^\lambda_{\rho\sigma}\partial_\lambda x^\mu = 0, \quad g^{\rho\sigma}\Gamma^\lambda_{\rho\sigma} = 0$$

$$(g^{(0)\rho\sigma} - h^{\rho\sigma}) * \frac{1}{2}g^{(0)\lambda\alpha}(\partial_\sigma h_{\alpha\rho} + \partial_\rho h_{\alpha\sigma} - \partial_\alpha h_{\rho\sigma}) = 0$$

$$g^{(0)\rho\sigma} * \frac{1}{2}g^{(0)\lambda\alpha}(\partial_\sigma h_{\alpha\rho} + \partial_\rho h_{\alpha\sigma} - \partial_\alpha h_{\rho\sigma}) - h^{\rho\sigma} * \frac{1}{2}g^{(0)\lambda\alpha}(\partial_\sigma h_{\alpha\rho} + \partial_\rho h_{\alpha\sigma} - \partial_\alpha h_{\rho\sigma}) = 0$$

$$g^{(0)\lambda\alpha}(\frac{1}{2}\partial_\sigma h^\sigma_\alpha + \frac{1}{2}\partial_\rho h^\rho_\alpha - \frac{1}{2}\partial_\alpha h) = 0, \quad g_{(0)\lambda\alpha}g^{(0)\lambda\alpha}(\frac{1}{2}\partial_\rho h^\rho_\alpha + \frac{1}{2}\partial_\rho h^\rho_\alpha - \frac{1}{2}\partial_\alpha h) = 0$$

$$\partial_\rho h^\rho_\alpha - \frac{1}{2}\partial_\alpha h = 0, \quad \partial_\rho h^\rho_\alpha - \frac{1}{2}g^{(0)\rho}_\alpha\partial_\rho h = 0, \quad \partial_\rho(h^\rho_\alpha - \frac{1}{2}\delta^\rho_\alpha h) = 0$$

$$\partial_\rho\Psi^\rho_\alpha = 0, \quad where \quad \Psi^\rho_\alpha = h^\rho_\alpha - \frac{1}{2}\delta^\rho_\alpha h$$

where we substituted the $g^{\rho\sigma}$ and $\Gamma^\lambda_{\rho\sigma}$ from the formulas 2.1.3, 2.1.6 respectively.

B The Ricci tensor in the vacuum

In the void the energy-momentum tensor is zero: $T_{\mu\nu} = 0$, so from the Einstein equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi k}{c^4}T_{\mu\nu} \tag{B.0.1}$$

using a generic metric $g^{\mu\nu}$, we have:

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R = \frac{8\pi k}{c^4}g^{\mu\nu}T_{\mu\nu}, \quad R - \frac{4}{2}R = \frac{8\pi k}{c^4}T$$

$$R = -\frac{8\pi k}{c^4}T \tag{B.0.2}$$

Substituting the Ricci scalar from (B.0.2) to the Einstein equation above (B.0.1):

$$R_{\mu\nu} = \frac{8\pi k}{c^4}T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R = \frac{8\pi k}{c^4}T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\frac{8\pi k}{c^4}T$$

$$R_{\mu\nu} = \frac{8\pi k}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \tag{B.0.3}$$

so, we can see from (B.0.3), that when $\boxed{T_{\mu\nu} = 0 \Rightarrow \mathrm{T} = 0 : \quad R_{\mu\nu} = 0}$ too. Thus, in the void the Ricci tensor, as well as the Ricci scalar, is equal to ero.

C Calculation of the polarizations average: $\overline{e_{ij}e_{kl}}$

Since $\overline{e_{ij}e_{kl}}$ is a normalied tensor of rank 4 to get what is equal to, it is enough to combine the unitary n_i and δ_{ij} in all possible ways having 4 indices each time:

$$\overline{e_{ij}e_{kl}} = a \cdot n_i n_j n_k n_l + b \cdot (\delta_{ij} \cdot n_k n_l + \delta_{kl} \cdot n_i n_j) + c \cdot (\delta_{ik} \cdot n_j n_l + \delta_{il} \cdot n_k n_j +$$

$$+\delta_{jl} \cdot n_k n_i + \delta_{jk} \cdot n_i n_l) + d \cdot \delta_{ij}\delta_{kl} + f \cdot (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Now we apply the properties of the tensor polarization and we begin with its trace to be zero $e_{ii} = 0$:

$$\overline{e_{ii}e_{kl}} = a \cdot n_i n_i n_k n_l + b \cdot (\delta_{ii} \cdot n_k n_l + \delta_{kl} \cdot n_i n_i) + c \cdot (\delta_{ik} \cdot n_i n_l + \delta_{il} \cdot n_k n_i +$$

$$+\delta_{il} \cdot n_k n_i + \delta_{ik} \cdot n_i n_l) + d \cdot \delta_{ii}\delta_{kl} + f \cdot (\delta_{ik}\delta_{il} + \delta_{il}\delta_{ik}) =$$

$$= a \cdot n_k n_l + b \cdot (3 n_k n_l + \delta_{kl}) + c \cdot (n_k n_l + n_k n_l + n_k n_l + n_k n_l) + 3d \cdot \delta_{kl} + 2f \cdot \delta_{kl} =$$

$$= (a + 3b + 4c) \cdot n_k n_l + (b + 3d + 2f)\delta_{kl} = 0$$

$$a + 3b + 4c = 0 \tag{C.0.1}$$

$$b + 3d + 2f = 0 \tag{C.0.2}$$

Right after that, we take the condition: $n_i \cdot e_{ij} = 0$:

$$\overline{n_i e_{ij} e_{kl}} = 0, \quad n_i \cdot \overline{e_{ij} e_{kl}} = a \cdot n_i n_i n_j n_k n_l + b \cdot (\delta_{ij} n_i n_k n_l + \delta_{kl} n_i n_i n_j) +$$

$$c \cdot (\delta_{ik} n_i n_j n_l + \delta_{il} n_i n_k n_j + \delta_{jl} n_k n_i n_i + \delta_{jk} n_i n_i n_l) + d \cdot n_i \delta_{ij} \delta_{kl} +$$

$$+ f \cdot (n_i \delta_{ik} \delta_{jl} + n_i \delta_{il} \delta_{jk}) = a \cdot n_j n_k n_l + b \cdot (n_j n_k n_l + \delta_{kl} n_j) + c \cdot (n_k n_j n_l +$$

$$+ n_l n_k n_j + \delta_{jl} n_k + \delta_{jk} n_l) + d \cdot n_j \delta_{kl} + f \cdot (n_k \delta_{jl} + n_l \delta_{jk}) =$$

$$(a + b + 2c) n_l n_k n_j + (b + d)\delta_{kl} n_j + (c + f) n_k \delta_{jl} + (c + f) n_l \delta_{jk} = 0$$

$$a + b + 2c = 0 \tag{C.0.3}$$

$$b + d = 0, \quad d = -b \tag{C.0.4}$$

$$c + f = 0, \quad f = -c \tag{C.0.5}$$

Using (C.0.4), (C.0.5), the formula (C.0.2) becomes:

$$b - 3b - 2c = 0, \quad c = -b \tag{C.0.6}$$

Likewise, from (C.0.6), (C.0.1) becomes:

$$a + 3b - 4b = 0, \quad b = a \tag{C.0.7}$$

So, eventually, for the coefficients we have that:

$$b = a; \quad c = -a; \quad d = -a; \quad f = a \tag{C.0.8}$$

The formula (C.0.3) just verifies the relations above, so we need a last condition and that will be the normaliation of the tensor: $e_{ij}e_{ij} = 1$:

$$\overline{e_{ij}e_{ij}} = a \cdot n_i n_j n_i n_j + b \cdot (\delta_{ij} n_i n_j + \delta_{ij} n_i n_j) + c \cdot (\delta_{ii} n_j n_j + \delta_{ij} n_i n_i +$$

$$+\delta_{jj} n_i n_i + \delta_{ij} n_i n_j) + d \cdot \delta_{ij}\delta_{ij} + f \cdot (\delta_{ii}\delta_{jj} + \delta_{ij}\delta_{ij}) =$$

$$= a + 2a - a \cdot (3 + 1 + 3 + 1) - 3a + a(3 \cdot 3 + 3) = 4a = 1$$

$$a = \frac{1}{4} \tag{C.0.9}$$

Thus, combining (C.0.8), (C.0.9), we finally get the $\overline{e_{ij}e_{kl}}$:

$$\overline{e_{ij}e_{kl}} = \frac{1}{4}[n_i n_j n_k n_l + (\delta_{ij} n_k n_l + \delta_{kl} n_i n_j) - (\delta_{ik} n_j n_l + \delta_{il} n_k n_j +$$

$$+\delta_{jl} n_k n_i + \delta_{jk} n_i n_l) - \delta_{ij}\delta_{kl} + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]$$

D Calculation of the space averaging: $\overline{n_i n_j}$, $\overline{n_i n_j n_k n_l}$

Starting with the average $\overline{n_i n_j}$, as it is unitary and of second rank, it will be written as a product of δ_{ij}:

$$\overline{n_i n_j} = a\delta_{ij}, \quad \overline{n_i n_i} = a\delta_{ii} = 3a = 1, \quad a = \frac{1}{3}$$

$$\overline{n_i n_j} = \frac{1}{3}\delta_{ij}$$

Likewise, the average $\overline{n_i n_j n_k n_l}$, as it is of rank 4, will be written as the sum of all combinations of $\delta_{ij}\delta_{kl}$:

$$\overline{n_i n_j n_k n_l} = a(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad \overline{n_i n_i n_k n_k} = a(\delta_{ii}\delta_{kk} + \delta_{ik}\delta_{ik} + \delta_{ik}\delta_{ik}) =$$

$$= a(3 \cdot 3 + 3 + 3) = 15a = 1, \quad a = \frac{1}{15}$$

$$\overline{n_i n_j n_k n_l} = \frac{1}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

References

[1] L. D. Landau and E. M. Lifschits. *The Classical Theory of Fields*, volume Volume 2 of *Course of Theoretical Physics*. Pergamon Press, Oxford, 1975.

[2] George Savvidis. Private lectures and notes on general relativity.

[3] B. P. Abbott, R. Abbott, T. D. Abbott, M. R. Abernathy, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. X. Adhikari, V. B. Adya, C. Affeldt, M. Agathos, K. Agatsuma, N. Aggarwal, O. D. Aguiar, L. Aiello, A. Ain, P. Ajith, B. Allen, A. Allocca, P. A. Altin, S. B. Anderson, W. G. Anderson, K. Arai, M. A. Arain, M. C. Araya, C. C. Arceneaux, J. S. Areeda, N. Arnaud, K. G. Arun, S. Ascenzi, G. Ashton, M. Ast, S. M. Aston, P. Astone, P. Aufmuth, C. Aulbert, S. Babak, P. Bacon, M. K. M. Bader, P. T. Baker, F. Baldaccini, G. Ballardin, S. W. Ballmer, J. C. Barayoga, S. E. Barclay, B. C. Barish, D. Barker, F. Barone, B. Barr, L. Barsotti, M. Barsuglia, D. Barta, J. Bartlett, M. A. Barton, I. Bartos, R. Bassiri, A. Basti, J. C. Batch, C. Baune, V. Bavigadda, M. Bazzan, B. Behnke, M. Bejger, C. Belczynski, A. S. Bell, C. J. Bell, B. K. Berger, J. Bergman, G. Bergmann, C. P. L. Berry, D. Bersanetti, A. Bertolini, J. Betzwieser, S. Bhagwat, R. Bhandare, I. A. Bilenko, G. Billingsley, J. Birch, R. Birney, O. Birnholtz, S. Biscans, A. Bisht, M. Bitossi, C. Biwer, M. A. Bizouard, J. K. Blackburn, C. D. Blair, D. G. Blair, R. M. Blair, S. Bloemen, O. Bock, T. P. Bodiya, M. Boer, G. Bogaert, C. Bogan, A. Bohe, P. Bojtos, C. Bond, F. Bondu, R. Bonnand, B. A. Boom, R. Bork, V. Boschi, S. Bose, Y. Bouffanais, A. Bozzi, C. Bradaschia, P. R. Brady, V. B. Braginsky, M. Branchesi, J. E. Brau, T. Briant, A. Brillet, M. Brinkmann, V. Brisson, P. Brockill, A. F. Brooks, D. A. Brown, D. D. Brown, N. M. Brown, C. C. Buchanan, A. Buikema, T. Bulik, H. J. Bulten, A. Buonanno, D. Buskulic, C. Buy, R. L. Byer, M. Cabero, L. Cadonati, G. Cagnoli, C. Cahillane, J. Calderón Bustillo, T. Callister, E. Calloni, J. B. Camp, K. C. Cannon, J. Cao, C. D. Capano, E. Capocasa, F. Carbognani, S. Caride, J. Casanueva Diaz, C. Casentini, S. Caudill, M. Cavaglià, F. Cavalier, R. Cavalieri, G. Cella, C. B. Cepeda, L. Cerboni Baiardi, G. Cerretani, E. Cesarini, R. Chakraborty, T. Chalermsongsak, S. J. Chamberlin, M. Chan, S. Chao, P. Charlton, E. Chassande-Mottin, H. Y. Chen, Y. Chen, C. Cheng, A. Chincarini,

A. Chiummo, H. S. Cho, M. Cho, J. H. Chow, N. Christensen, Q. Chu, S. Chua, S. Chung, G. Ciani, F. Clara, J. A. Clark, F. Cleva, E. Coccia, P.-F. Cohadon, A. Colla, C. G. Collette, L. Cominsky, M. Constancio, A. Conte, L. Conti, D. Cook, T. R. Corbitt, N. Cornish, A. Corsi, S. Cortese, C. A. Costa, M. W. Coughlin, S. B. Coughlin, J.-P. Coulon, S. T. Countryman, P. Couvares, E. E. Cowan, D. M. Coward, M. J. Cowart, D. C. Coyne, R. Coyne, K. Craig, J. D. E. Creighton, T. D. Creighton, J. Cripe, S. G. Crowder, A. M. Cruise, A. Cumming, L. Cunningham, E. Cuoco, T. Dal Canton, S. L. Danilishin, S. D'Antonio, K. Danzmann, N. S. Darman, C. F. Da Silva Costa, V. Dattilo, I. Dave, H. P. Daveloza, M. Davier, G. S. Davies, E. J. Daw, R. Day, S. De, D. DeBra, G. Debreczeni, J. Degallaix, M. De Laurentis, S. Deléglise, W. Del Pozzo, T. Denker, T. Dent, H. Dereli, V. Dergachev, R. T. DeRosa, R. De Rosa, R. DeSalvo, S. Dhurandhar, M. C. Díaz, L. Di Fiore, M. Di Giovanni, A. Di Lieto, S. Di Pace, I. Di Palma, A. Di Virgilio, G. Dojcinoski, V. Dolique, F. Donovan, K. L. Dooley, S. Doravari, R. Douglas, T. P. Downes, M. Drago, R. W. P. Drever, J. C. Driggers, Z. Du, M. Ducrot, S. E. Dwyer, T. B. Edo, M. C. Edwards, A. Effler, H.-B. Eggenstein, P. Ehrens, J. Eichholz, S. S. Eikenberry, W. Engels, R. C. Essick, T. Etzel, M. Evans, T. M. Evans, R. Everett, M. Factourovich, V. Fafone, H. Fair, S. Fairhurst, X. Fan, Q. Fang, S. Farinon, B. Farr, W. M. Farr, M. Favata, M. Fays, H. Fehrmann, M. M. Fejer, D. Feldbaum, I. Ferrante, E. C. Ferreira, F. Ferrini, F. Fidecaro, L. S. Finn, I. Fiori, D. Fiorucci, R. P. Fisher, R. Flaminio, M. Fletcher, H. Fong, J.-D. Fournier, S. Franco, S. Frasca, F. Frasconi, M. Frede, Z. Frei, A. Freise, R. Frey, V. Frey, T. T. Fricke, P. Fritschel, V. V. Frolov, P. Fulda, M. Fyffe, H. A. G. Gabbard, J. R. Gair, L. Gammaitoni, S. G. Gaonkar, F. Garufi, A. Gatto, G. Gaur, N. Gehrels, G. Gemme, B. Gendre, E. Genin, A. Gennai, J. George, L. Gergely, V. Germain, Abhirup Ghosh, Archisman Ghosh, S. Ghosh, J. A. Giaime, K. D. Giardina, A. Giazotto, K. Gill, A. Glaefke, J. R. Gleason, E. Goetz, R. Goetz, L. Gondan, G. González, J. M. Gonzalez Castro, A. Gopakumar, N. A. Gordon, M. L. Gorodetsky, S. E. Gossan, M. Gosselin, R. Gouaty, C. Graef, P. B. Graff, M. Granata, A. Grant, S. Gras, C. Gray, G. Greco, A. C. Green, R. J. S. Greenhalgh, P. Groot, H. Grote, S. Grunewald, G. M. Guidi, X. Guo, A. Gupta, M. K. Gupta, K. E. Gushwa, E. K. Gustafson, R. Gustafson, J. J. Hacker, B. R. Hall, E. D. Hall, G. Hammond, M. Haney, M. M. Hanke, J. Hanks, C. Hanna,

M. D. Hannam, J. Hanson, T. Hardwick, J. Harms, G. M. Harry, I. W. Harry, M. J. Hart, M. T. Hartman, C.-J. Haster, K. Haughian, J. Healy, J. Heefner, A. Heidmann, M. C. Heintze, G. Heinzel, H. Heitmann, P. Hello, G. Hemming, M. Hendry, I. S. Heng, J. Hennig, A. W. Heptonstall, M. Heurs, S. Hild, D. Hoak, K. A. Hodge, D. Hofman, S. E. Hollitt, K. Holt, D. E. Holz, P. Hopkins, D. J. Hosken, J. Hough, E. A. Houston, E. J. Howell, Y. M. Hu, S. Huang, E. A. Huerta, D. Huet, B. Hughey, S. Husa, S. H. Huttner, T. Huynh-Dinh, A. Idrisy, N. Indik, D. R. Ingram, R. Inta, H. N. Isa, J.-M. Isac, M. Isi, G. Islas, T. Isogai, B. R. Iyer, K. Izumi, M. B. Jacobson, T. Jacqmin, H. Jang, K. Jani, P. Jaranowski, S. Jawahar, F. Jiménez-Forteza, W. W. Johnson, N. K. Johnson-McDaniel, D. I. Jones, R. Jones, R. J. G. Jonker, L. Ju, K. Haris, C. V. Kalaghatgi, V. Kalogera, S. Kandhasamy, G. Kang, J. B. Kanner, S. Karki, M. Kasprzack, E. Katsavounidis, W. Katzman, S. Kaufer, T. Kaur, K. Kawabe, F. Kawazoe, F. Kéfélian, M. S. Kehl, D. Keitel, D. B. Kelley, W. Kells, R. Kennedy, D. G. Keppel, J. S. Key, A. Khalaidovski, F. Y. Khalili, I. Khan, S. Khan, Z. Khan, E. A. Khazanov, N. Kijbunchoo, C. Kim, J. Kim, K. Kim, Nam-Gyu Kim, Namjun Kim, Y.-M. Kim, E. J. King, P. J. King, D. L. Kinzel, J. S. Kissel, L. Kleybolte, S. Klimenko, S. M. Koehlenbeck, K. Kokeyama, S. Koley, V. Kondrashov, A. Kontos, S. Koranda, M. Korobko, W. Z. Korth, I. Kowalska, D. B. Kozak, V. Kringel, B. Krishnan, A. Królak, C. Krueger, G. Kuehn, P. Kumar, R. Kumar, L. Kuo, A. Kutynia, P. Kwee, B. D. Lackey, M. Landry, J. Lange, B. Lantz, P. D. Lasky, A. Lazzarini, C. Lazzaro, P. Leaci, S. Leavey, E. O. Lebigot, C. H. Lee, H. K. Lee, H. M. Lee, K. Lee, A. Lenon, M. Leonardi, J. R. Leong, N. Leroy, N. Letendre, Y. Levin, B. M. Levine, T. G. F. Li, A. Libson, T. B. Littenberg, N. A. Lockerbie, J. Logue, A. L. Lombardi, L. T. London, J. E. Lord, M. Lorenzini, V. Loriette, M. Lormand, G. Losurdo, J. D. Lough, C. O. Lousto, G. Lovelace, H. Lück, A. P. Lundgren, J. Luo, R. Lynch, Y. Ma, T. MacDonald, B. Machenschalk, M. MacInnis, D. M. Macleod, F. Magaña Sandoval, R. M. Magee, M. Mageswaran, E. Majorana, I. Maksimovic, V. Malvezzi, N. Man, I. Mandel, V. Mandic, V. Mangano, G. L. Mansell, M. Manske, M. Mantovani, F. Marchesoni, F. Marion, S. Márka, Z. Márka, A. S. Markosyan, E. Maros, F. Martelli, L. Martellini, I. W. Martin, R. M. Martin, D. V. Martynov, J. N. Marx, K. Mason, A. Masserot, T. J. Massinger, M. Masso-Reid, F. Matichard, L. Matone, N. Mavalvala, N. Mazumder, G. Maz-

zolo, R. McCarthy, D. E. McClelland, S. McCormick, S. C. McGuire, G. McIntyre, J. McIver, D. J. McManus, S. T. McWilliams, D. Meacher, G. D. Meadors, J. Meidam, A. Melatos, G. Mendell, D. Mendoza-Gandara, R. A. Mercer, E. Merilh, M. Merzougui, S. Meshkov, C. Messenger, C. Messick, P. M. Meyers, F. Mezzani, H. Miao, C. Michel, H. Middleton, E. E. Mikhailov, L. Milano, J. Miller, M. Millhouse, Y. Minenkov, J. Ming, S. Mirshekari, C. Mishra, S. Mitra, V. P. Mitrofanov, G. Mitselmakher, R. Mittleman, A. Moggi, M. Mohan, S. R. P. Mohapatra, M. Montani, B. C. Moore, C. J. Moore, D. Moraru, G. Moreno, S. R. Morriss, K. Mossavi, B. Mours, C. M. Mow-Lowry, C. L. Mueller, G. Mueller, A. W. Muir, Arunava Mukherjee, D. Mukherjee, S. Mukherjee, N. Mukund, A. Mullavey, J. Munch, D. J. Murphy, P. G. Murray, A. Mytidis, I. Nardecchia, L. Naticchioni, R. K. Nayak, V. Necula, K. Nedkova, G. Nelemans, M. Neri, A. Neunzert, G. Newton, T. T. Nguyen, A. B. Nielsen, S. Nissanke, A. Nitz, F. Nocera, D. Nolting, M. E. N. Normandin, L. K. Nuttall, J. Oberling, E. Ochsner, J. O'Dell, E. Oelker, G. H. Ogin, J. J. Oh, S. H. Oh, F. Ohme, M. Oliver, P. Oppermann, Richard J. Oram, B. O'Reilly, R. O'Shaughnessy, C. D. Ott, D. J. Ottaway, R. S. Ottens, H. Overmier, B. J. Owen, A. Pai, S. A. Pai, J. R. Palamos, O. Palashov, C. Palomba, A. Pal-Singh, H. Pan, Y. Pan, C. Pankow, F. Pannarale, B. C. Pant, F. Paoletti, A. Paoli, M. A. Papa, H. R. Paris, W. Parker, D. Pascucci, A. Pasqualetti, R. Passaquieti, D. Passuello, B. Patricelli, Z. Patrick, B. L. Pearlstone, M. Pedraza, R. Pedurand, L. Pekowsky, A. Pele, S. Penn, A. Perreca, H. P. Pfeiffer, M. Phelps, O. Piccinni, M. Pichot, M. Pickenpack, F. Piergiovanni, V. Pierro, G. Pillant, L. Pinard, I. M. Pinto, M. Pitkin, J. H. Poeld, R. Poggiani, P. Popolizio, A. Post, J. Powell, J. Prasad, V. Predoi, S. S. Premachandra, T. Prestegard, L. R. Price, M. Prijatelj, M. Principe, S. Privitera, R. Prix, G. A. Prodi, L. Prokhorov, O. Puncken, M. Punturo, P. Puppo, M. Pürrer, H. Qi, J. Qin, V. Quetschke, E. A. Quintero, R. Quitzow-James, F. J. Raab, D. S. Rabeling, H. Radkins, P. Raffai, S. Raja, M. Rakhmanov, C. R. Ramet, P. Rapagnani, V. Raymond, M. Razzano, V. Re, J. Read, C. M. Reed, T. Regimbau, L. Rei, S. Reid, D. H. Reitze, H. Rew, S. D. Reyes, F. Ricci, K. Riles, N. A. Robertson, R. Robie, F. Robinet, A. Rocchi, L. Rolland, J. G. Rollins, V. J. Roma, J. D. Romano, R. Romano, G. Romanov, J. H. Romie, D. Rosińska, S. Rowan, A. Rüdiger, P. Ruggi, K. Ryan, S. Sachdev, T. Sadecki, L. Sadeghian, L. Salconi, M. Saleem, F. Salemi, A. Samajdar, L. Sam-

mut, L. M. Sampson, E. J. Sanchez, V. Sandberg, B. Sandeen, G. H. Sanders, J. R. Sanders, B. Sassolas, B. S. Sathyaprakash, P. R. Saulson, O. Sauter, R. L. Savage, A. Sawadsky, P. Schale, R. Schilling, J. Schmidt, P. Schmidt, R. Schnabel, R. M. S. Schofield, A. Schönbeck, E. Schreiber, D. Schuette, B. F. Schutz, J. Scott, S. M. Scott, D. Sellers, A. S. Sengupta, D. Sentenac, V. Sequino, A. Sergeev, G. Serna, Y. Setyawati, A. Sevigny, D. A. Shaddock, T. Shaffer, S. Shah, M. S. Shahriar, M. Shaltev, Z. Shao, B. Shapiro, P. Shawhan, A. Sheperd, D. H. Shoemaker, D. M. Shoemaker, K. Siellez, X. Siemens, D. Sigg, A. D. Silva, D. Simakov, A. Singer, L. P. Singer, A. Singh, R. Singh, A. Singhal, A. M. Sintes, B. J. J. Slagmolen, J. R. Smith, M. R. Smith, N. D. Smith, R. J. E. Smith, E. J. Son, B. Sorazu, F. Sorrentino, T. Souradeep, A. K. Srivastava, A. Staley, M. Steinke, J. Steinlechner, S. Steinlechner, D. Steinmeyer, B. C. Stephens, S. P. Stevenson, R. Stone, K. A. Strain, N. Straniero, G. Stratta, N. A. Strauss, S. Strigin, R. Sturani, A. L. Stuver, T. Z. Summerscales, L. Sun, P. J. Sutton, B. L. Swinkels, M. J. Szczepańczyk, M. Tacca, D. Talukder, D. B. Tanner, M. Tápai, S. P. Tarabrin, A. Taracchini, R. Taylor, T. Theeg, M. P. Thirugnanasambandam, E. G. Thomas, M. Thomas, P. Thomas, K. A. Thorne, K. S. Thorne, E. Thrane, S. Tiwari, V. Tiwari, K. V. Tokmakov, C. Tomlinson, M. Tonelli, C. V. Torres, C. I. Torrie, D. Töyrä, F. Travasso, G. Traylor, D. Trifirò, M. C. Tringali, L. Trozzo, M. Tse, M. Turconi, D. Tuyenbayev, D. Ugolini, C. S. Unnikrishnan, A. L. Urban, S. A. Usman, H. Vahlbruch, G. Vajente, G. Valdes, M. Vallisneri, N. van Bakel, M. van Beuzekom, J. F. J. van den Brand, C. Van Den Broeck, D. C. Vander-Hyde, L. van der Schaaf, J. V. van Heijningen, A. A. van Veggel, M. Vardaro, S. Vass, M. Vasúth, R. Vaulin, A. Vecchio, G. Vedovato, J. Veitch, P. J. Veitch, K. Venkateswara, D. Verkindt, F. Vetrano, A. Viceré, S. Vinciguerra, D. J. Vine, J.-Y. Vinet, S. Vitale, T. Vo, H. Vocca, C. Vorvick, D. Voss, W. D. Vousden, S. P. Vyatchanin, A. R. Wade, L. E. Wade, M. Wade, S. J. Waldman, M. Walker, L. Wallace, S. Walsh, G. Wang, H. Wang, M. Wang, X. Wang, Y. Wang, H. Ward, R. L. Ward, J. Warner, M. Was, B. Weaver, L.-W. Wei, M. Weinert, A. J. Weinstein, R. Weiss, T. Welborn, L. Wen, P. Weßels, T. Westphal, K. Wette, J. T. Whelan, S. E. Whitcomb, D. J. White, B. F. Whiting, K. Wiesner, C. Wilkinson, P. A. Willems, L. Williams, R. D. Williams, A. R. Williamson, J. L. Willis, B. Willke, M. H. Wimmer, L. Winkelmann, W. Winkler, C. C.

Wipf, A. G. Wiseman, H. Wittel, G. Woan, J. Worden, J. L. Wright, G. Wu, J. Yablon, I. Yakushin, W. Yam, H. Yamamoto, C. C. Yancey, M. J. Yap, H. Yu, M. Yvert, A. Zadrożny, L. Zangrando, M. Zanolin, J.-P. Zendri, M. Zevin, F. Zhang, L. Zhang, M. Zhang, Y. Zhang, C. Zhao, M. Zhou, Z. Zhou, X. J. Zhu, M. E. Zucker, S. E. Zuraw, and J. Zweizig. Observation of gravitational waves from a binary black hole merger. *Phys. Rev. Lett.*, 116:061102, Feb 2016.

[4] L Ju, D G Blair, and C Zhao. Detection of gravitational waves. 63(9):1317, sep 2000.

[5] DANIEL SIGG. *GRAVITATIONAL WAVES*, pages 592–625.

[6] Davide Castelvecchi. Japan's pioneering detector set to join hunt for gravitational waves. Jan 2, 2019.

[7] Stefan Hild, Simon Chelkowski, and Andreas Freise. Pushing towards the et sensitivity using 'conventional' technology, 2008.

[8] eLISA Consortium. elisa, the first gravitational wave observatory in space, 5 December 2013.

[9] S Kawamura, M Ando, T Nakamura, K Tsubono, T Tanaka, I Funaki, N Seto, K Numata, S Sato, K Ioka, N Kanda, T Takashima, K Agatsuma, T Akutsu, T Akutsu, Koh-Suke Aoyanagi, K Arai, Y Arase, A Araya, H Asada, Y Aso, T Chiba, T Ebisuzaki, M Enoki, Y Eriguchi, M-K Fujimoto, R Fujita, M Fukushima, T Futamase, K Ganzu, T Harada, T Hashimoto, K Hayama, W Hikida, Y Himemoto, H Hirabayashi, T Hiramatsu, F-L Hong, H Horisawa, M Hosokawa, K Ichiki, T Ikegami, K T Inoue, K Ishidoshiro, H Ishihara, T Ishikawa, H Ishizaki, H Ito, Y Itoh, S Kamagasako, N Kawashima, F Kawazoe, H Kirihara, N Kishimoto, K Kiuchi, S Kobayashi, K Kohri, H Koizumi, Y Kojima, K Kokeyama, W Kokuyama, K Kotake, Y Kozai, H Kudoh, H Kunimori, H Kuninaka, K Kuroda, K i Maeda, H Matsuhara, Y Mino, O Miyakawa, S Miyoki, M Y Morimoto, T Morioka, T Morisawa, S Moriwaki, S Mukohyama, M Musha, S Nagano, I Naito, N Nakagawa, K Nakamura, H Nakano, K Nakao, S Nakasuka, Y Nakayama, E Nishida, K Nishiyama, A Nishizawa, Y Niwa, M Ohashi, N Ohishi, M Ohkawa, A Okutomi, K Onozato, K Oohara, N Sago, M Saijo, M Sakagami, S i Sakai, S Sakata, M Sasaki, T Sato, M Shibata, H Shinkai,

K Somiya, H Sotani, N Sugiyama, Y Suwa, H Tagoshi, K Takahashi, K Takahashi, T Takahashi, H Takahashi, R Takahashi, R Takahashi, A Takamori, T Takano, K Taniguchi, A Taruya, H Tashiro, M Tokuda, M Tokunari, M Toyoshima, S Tsujikawa, Y Tsunesada, K i Ueda, M Utashima, H Yamakawa, K Yamamoto, T Yamazaki, J Yokoyama, C-M Yoo, S Yoshida, and T Yoshino. The japanese space gravitational wave antenna - decigo. *Journal of Physics: Conference Series*, 122(1):012006, jul 2008.